# Local utility and multivariate risk aversion

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## Choice between multivariate risky prospects

- In view of violations of expected utility, a vast literature has emerged on other functional evaluations of risky prospects, particularly Rank Dependent Preferences (RDU).
- Simultaneously, a literature developed on the characterization of attitudes to multiple non substitutable risks and multivariate risk premia within the framework of expected utility.
- Here we characterize attitude to multivariate generalizations of standard notions of increasing risk with local utility
  - Rothschild-Stiglitz mean preserving increase in risk
  - Quiggin monotone mean preserving increase in risk

## Expected utility

- Decision maker ranks random vectors $X$ with law invariant functional $\Phi(X)=\Phi(F_X)$, with $F_X$ the cdf of $X$.
- Utility $x\mapsto U(x)$ :

$$\Phi(F)=\int U(x)dF(x)$$

## Quiggin-Yaari functional

- Distortion $x\mapsto\phi(x)$ :

$$\Phi(F)=\int \phi(t)F^{-1}(t)dt=\int \underbrace{\left(\int_{-\infty}^{x}\phi(F(s))ds\right)}_{U(x,F)}dF(x)$$

## Aumann & Serrano Riskiness

- Given $X$, the index of risk $R(X)$ is defined to be the unique positive solution (if exists) of $\mathbb{E}[\exp(-X/R(X))] = 1$
- Index of Riskiness $x \mapsto R$ :

$$R \text{ such that } \int \underbrace{\exp\left(-\frac{x}{R}\right)}_{U(x)} dF(x) = 1$$

where $U$ is CARA.

# Local utility

– Decision maker ranks random vectors $X$ with law invariant functional $\Phi(X)=\Phi(F_X)$, with $F_X$ the cdf of $X$.

– Local utility $x\mapsto U(x;F)$ is the Fréchet derivative of $\Phi$ at $F$ :

$$\Phi(F')-\Phi(F)-\int U(x;F)[dF'(x)-dF(x)]\rightarrow 0$$

- – If $\Phi$ is expected utility, local and global utilities coincide
- – If $\Phi$ is the Quiggin-Yaari functional $\Phi(X)=\int\phi(t)F_X^{-1}(t)dt$, then local utility is $U(x;F)=\int_{-\infty}^{x}\phi(F(z))dz$
- – If $\Phi$ is the Aumann-Serrano index of riskiness, the local utility $\text{cst}[1-\exp(-\alpha x)]$ is CARA

## Rothschild-Stiglitz mean preserving increase in risk

One of the most commonly used stochastic orderings to compare risky prospects is the mean preserving increase in risk (MPIR or concave ordering). Let $X$ and $Y$ be two prospects.

**Definition :**

$Y \succsim_{MPIR} X$ if $\mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)]$ for all concave utility $u$.

**Characterization :**

$*$ $Y \overset{\mathcal{L}}{=} X + Z$ with $\mathbb{E}[Z|X] = 0$ (where "$\overset{\mathcal{L}}{=}$" denotes equality in distribution).

# Local utility and MPIR

- Aversion to MPIR is equivalent to concavity of local utility (Machina 1982 for the univariate result)
- Still holds for multivariate risks :
  - $\Phi$ is MPIR averse (Schur concave) if $\Phi(\boldsymbol{Y}) \leq \Phi(\boldsymbol{X})$ when $\boldsymbol{Y}$ is an MPIR of $\boldsymbol{X}$.
  - Equivalent to concavity of $U(\boldsymbol{x}; F)$ in $\boldsymbol{x}$ for all distributions $F$

  Proof : $\Phi$ Schur concave iff $\Phi$ decreasing along all martingales $X_t$

$$\begin{aligned}
\Phi(\boldsymbol{X}_{t+dt}) - \Phi(\boldsymbol{X}_t) &= \mathbb{E}\left[U(\boldsymbol{X}_{t+dt}; F_{\boldsymbol{X}_t}) - U(\boldsymbol{X}_t; F_{\boldsymbol{X}_t})\right] \\
&= \mathbb{E}\left[\mathrm{Tr}\left(D^2 U(\boldsymbol{X}_t; F_{\boldsymbol{X}_t})\boldsymbol{\sigma}_t^\top \boldsymbol{\sigma}_t\right)\right] \quad \text{Itô} \\
&\leq 0 \ \text{ iff } \ U(\boldsymbol{x}; F) \ \text{ concave}
\end{aligned}$$

## Two shortcomings of MPIR in the theory of risk sharing

- Arrow-Pratt more risk averse decision makers do not necessarily pay more (than less risk averse ones) for a mean preserving decrease in risk.
  - Ross (Econometrica 1981)
- Partial insurance contracts offering mean preserving reduction in risk can be Pareto dominated.
  - Landsberger and Meilijson (Annals of OR 1994)

## Bickel-Lehmann dispersion order

These shortcomings are not shared by the Bickel-Lehmann dispersion order (Bickel and Lehmann 1979). Let $X$ and $Y$ have cdfs $F_X$ and $F_Y$ and quantiles $Q_X = F_X^{-1}$ and $Q_Y = F_Y^{-1}$.

**Definition :**

$Y \succsim_{Disp} X$ if $Q_Y(s) - Q_Y(s') \geq Q_X(s) - Q_X(s')$.

**Characterization** (Landsberger-Meilijson) :

$Y \overset{\mathcal{L}}{=} X + Z$ with $Z$ and $X$ comonotonic,

**Examples :**

– Normal, exponential and uniform families are dispersion ordered by the variance.

– Arrow (1970) stretches of a distribution $X \mapsto x + \alpha(X - x)$, $\alpha > 1$, are more dispersed.

# Local utility and attitude to mean preserving dispersion increase (Quiggin's monotone MPIR)

– $\Phi$ is MMPIR averse if and only if

$$\mathbb{E}\left[\frac{U'(X;F_X)}{\mathbb{E}[U'(X;F_X)]}1\{X>x\}\right]\leq\mathbb{E}\left[1\{X>x\}\right]$$

– Example : Quiggin-Yaari functional

$$\Phi(X)=\int\phi(t)F_X^{-1}(t)dt$$

with local utility is

$$U(x;F)=\int_{-\infty}^{x}\phi(F_X(z))dz$$

is MMPIR averse iff density $\phi(u)$ is stochastically dominated by the uniform (called *pessimism* by Quiggin)

## Risk sharing and dispersion

- More risk averse decision makers will always pay at least as much (as less risk averse agents) for a decrease in risk if and only if it is Bickel-Lehmann less dispersed.
  - Landsberger and Meilijson (Management Science 1994)
- Unless the uninsured position is Bickel-Lehmann more dispersed than the insured position, the existing contract can be improved so as to raise the expected utility of both parties, regardless of their (concave) utility functions.
  - Landsberger and Meilijson (Annals of OR 1994)

## Partial insurance

Consider the following insurance contract :

| | Insuree | Insurer |
|---|---|---|
| Before | $Y$ | $0$ |
| After | $X_1$ | $X_2$ |

- By construction $Y = X_1 + X_2$.
- $(X_1, X_2)$ is Pareto efficient if and only if comonotonic (Landsberger-Meilijson).
- Hence the contract is efficient iff $Y = X_1 + X_2 \succsim_{Disp} X_1$.

We generalize this result to the case of multivariate risks.

## Multivariate extension of Bickel-Lehmann and Monotone MPIR

Quantiles and comonotonicity feature in the definition and the characterization of the Bickel-Lehmann dispersion ordering. They seem to rely on the ordering on the real line.

However we can revisit these notions to provide
– Multivariate notion of comonotonicity
– Multivariate quantile definition

# Revisiting comonotonicity

- $X$ and $Y$ are comonotonic if there exists $Z$ such that $X = T_X(Z)$ and $Y = T_Y(Z)$, $T_X, T_Y$ increasing functions.
- Example :
  - If $X(\omega_i) = x_i$ and $Y(\omega_i) = y_i$, $i = 1\ldots, n$, with $x_1 \leq \ldots \leq x_n$ and $y_1 \leq \ldots \leq y_n$, then $X$ and $Y$ are comonotonic.
  - By the simple rearrangement inequality,

$$\sum_{i=1,\ldots,n} x_i y_i = \max\left\{\sum_{i=1,\ldots,n} x_i y_{\sigma(i)} : \sigma \text{ permutation}\right\}.$$

- General characterization : $X$ and $Y$ are comonotonic iff

$$\mathbb{E}[XY] = \sup\left\{\mathbb{E}[X\tilde{Y}] : \tilde{Y} \overset{\mathcal{L}}{=} Y\right\}.$$

# Revisiting the quantile function

The quantile function of a prospect $X$ is the generalized inverse of the cumulative distribution function :

$$u \mapsto Q_X(u) = \inf\{x : \mathbb{P}(X \leq x) \geq u\}$$

Equivalent characterizations :

– The quantile function $Q_X$ of a prospect $X$ is an *increasing rearrangement* of $X$,
– The quantile $Q_X(U)$ of $X$ is the version of $X$ which is comonotonic with the uniform random variable $U$ on $[0,1]$.
– $Q_X$ is the only l.s.c. **increasing function** such that

$$\mathbb{E}[Q_X(U)U] = \sup\{\mathbb{E}[\tilde{X}U]; \tilde{X} \stackrel{\mathcal{L}}{=} X\}.$$

**Multivariate $\mu$-quantiles and $\mu$-comonotonicity,** (Galichon and Henry, JET 2012)

– The univariate quantile function of a random variable $X$ is the only l.s.c. **increasing function** such that

$$\mathbb{E}[Q_X(U)U] = \sup\{\mathbb{E}[\tilde{X}U]; \tilde{X} \overset{\mathcal{L}}{=} X\}.$$

– Similarly, the $\boldsymbol{\mu}$-quantile $Q_X$ is the essentially unique **gradient of a l.s.c. convex function** (Brenier, CPAM 1991),

$$\mathbb{E}[\langle Q_{\boldsymbol{X}}(\boldsymbol{U}), \boldsymbol{U}\rangle] = \sup\{\mathbb{E}[\langle \tilde{\boldsymbol{X}}, \boldsymbol{U}\rangle]; \tilde{\boldsymbol{X}} \overset{\mathcal{L}}{=} \boldsymbol{X}\}, \text{ for some } \boldsymbol{U} \overset{\mathcal{L}}{=} \boldsymbol{\mu}.$$

– $\boldsymbol{X}$ and $\boldsymbol{Y}$ are $\boldsymbol{\mu}$-comonotonic if for some $\boldsymbol{U} \overset{\mathcal{L}}{=} \boldsymbol{\mu}$,

$$\boldsymbol{X} = Q_{\boldsymbol{X}}(\boldsymbol{U}) \text{ and } \boldsymbol{Y} = Q_{\boldsymbol{Y}}(\boldsymbol{U}),$$

namely if $\boldsymbol{X}$ and $\boldsymbol{Y}$ can be simultaneously rearranged relative to a reference distribution $\mu$.

# Example : Gaussian prospects

Suppose the baseline $\boldsymbol{U}$ is standard normal,

– $\boldsymbol{X} \sim N(\boldsymbol{0}, \Sigma_{\boldsymbol{X}})$, hence $\boldsymbol{X} = \Sigma_{\boldsymbol{X}}^{1/2} O_{\boldsymbol{X}} \boldsymbol{U}$, with $O_{\boldsymbol{X}}$ orthogonal,

– $\boldsymbol{Y} \sim N(0, \Sigma_{\boldsymbol{Y}})$, hence $\boldsymbol{Y} = \Sigma_{\boldsymbol{Y}}^{1/2} O_{\boldsymbol{Y}} \boldsymbol{U}$, with $O_{\boldsymbol{Y}}$ orthogonal,

$\mathbb{E}[\langle \tilde{\boldsymbol{X}}, \boldsymbol{U} \rangle]$ is minimized for $\tilde{\boldsymbol{X}} = \Sigma_{\boldsymbol{X}}^{1/2} \boldsymbol{U}$, so when $O_{\boldsymbol{X}}$ is the identity. Hence the generalized quantile of $\boldsymbol{X}$ relative to $\boldsymbol{U}$ is

$$Q_{\boldsymbol{X}}(\boldsymbol{U}) = \Sigma_{\boldsymbol{X}}^{1/2} \boldsymbol{U}.$$

$\boldsymbol{X}$ and $\boldsymbol{Y}$ are $N(0, \mathbb{I})$-comonotonic if $O_{\boldsymbol{X}} = O_{\boldsymbol{Y}}$ (they have the same orientation). The correlation is

$$\mathbb{E}[\boldsymbol{X}\boldsymbol{Y}^{\mathsf{T}}] = \Sigma_{\boldsymbol{X}}^{1/2} O_{\boldsymbol{X}} O_{\boldsymbol{Y}}^{\mathsf{T}} \Sigma_{\boldsymbol{Y}}^{1/2} = \Sigma_{\boldsymbol{X}}^{1/2} \Sigma_{\boldsymbol{Y}}^{1/2}.$$

# Computation of generalized quantiles

The optimal transportation map between $\boldsymbol{\mu}$ on $[0,1]^d$ and the empirical distribution relative to $(\boldsymbol{X}_1,\dots,\boldsymbol{X}_n)$ satisfies

– $\hat{Q}_{\boldsymbol{X}}(\boldsymbol{U})\in\{\boldsymbol{X}_1,\dots,\boldsymbol{X}_n\}$

– $\mu(\hat{Q}_{\boldsymbol{X}}^{-1}(\{\boldsymbol{X}_k\}))=1/n$, for each $k=1,\dots,n$

– $\hat{Q}_{\boldsymbol{X}}$ is the gradient of a convex function $V:\mathbb{R}^d\to\mathbb{R}$.

The solution for the "potential" $V$ is

$$V(\boldsymbol{u})=\max_k\{\langle\boldsymbol{u},\boldsymbol{X}_k\rangle-w_k\},$$

where $w=(w_1,\dots,w_n)'$ minimizes the convex function

$$w\mapsto\int V(\boldsymbol{u})d\boldsymbol{\mu}(\boldsymbol{u})+\sum_{k=1}^n w_k/n.$$

## $\mu$-Bickel-Lehmann dispersion order

With these multivariate extensions of quantiles and comonotonicity, we have the following multivariate extension of the Bickel-Lehmann dispersion order (Bickel and Lehmann 1979).

**Definition :**

(a) $\boldsymbol{Y} \succsim_{\mu Disp} \boldsymbol{X}$ if $Q_{\boldsymbol{Y}} - Q_{\boldsymbol{X}}$ is the gradient of a convex function.

**Characterization :**

(b) $\boldsymbol{Y} \succsim_{\mu Disp} \boldsymbol{X}$ iff $\boldsymbol{Y} \overset{\mathcal{L}}{=} \boldsymbol{X} + \boldsymbol{Z}$, where $\boldsymbol{Z}$ and $\boldsymbol{X}$ are $\boldsymbol{\mu}$-comonotonic,

The proof is based on comonotonic additivity of the $\boldsymbol{\mu}$-quantile transform, i.e., $Q_{\boldsymbol{X}+\boldsymbol{Z}} = Q_{\boldsymbol{X}} + Q_{\boldsymbol{Z}}$ when $\boldsymbol{X}$ and $\boldsymbol{Z}$ are $\boldsymbol{\mu}$-comonotonic.

# Relation with existing multivariate dispersion orders

Based on univariate characterization (c), Giovagnoli and Wynn (Stat. and Prob. Letters 1995) propose the strong dispersive order.

– $\boldsymbol{Y} \succsim_{SD} \boldsymbol{X}$ iff $\boldsymbol{Y} \overset{\mathcal{L}}{=} \psi(\boldsymbol{X})$, where $\psi$ is an expansion, i.e., if

$$\|\psi(\boldsymbol{x}) - \psi(\boldsymbol{x}')\| \geq \|\boldsymbol{x} - \boldsymbol{x}'\|.$$

This is closely related to our $\boldsymbol{\mu}$-Bickel-Lehmann ordering :

– $\boldsymbol{Y} \succsim_{SD} \boldsymbol{X}$ iff $\boldsymbol{Y} \overset{\mathcal{L}}{=} \boldsymbol{X} + \nabla V(\boldsymbol{X})$, where $V$ is a convex function.
  – The latter is equivalent to $\boldsymbol{Y} \overset{\mathcal{L}}{=} \boldsymbol{X} + \boldsymbol{Z}$, where $\boldsymbol{X}$ and $\boldsymbol{Z}$ are $c$-comonotonic (Puccetti and Scarsini (JMA 2011)),
  – It also implies $\boldsymbol{Y} \succsim_{\mu Disp} \boldsymbol{X}$ for some $\mu$, since $\boldsymbol{X}$ and $\nabla V(\boldsymbol{X})$ are $\boldsymbol{\mu}_{\boldsymbol{X}}$-comonotonic (converse not true in general).

## Partial insurance for multivariate risks

Consider the following insurance contract :

| | Insuree | Insurer |
|---|---|---|
| Before | $Y$ | 0 |
| After | $\boldsymbol{X}_1$ | $\boldsymbol{X}_2$ |

- By construction $\boldsymbol{Y} = \boldsymbol{X}_1 + \boldsymbol{X}_2$.
- $(\boldsymbol{X}_1, \boldsymbol{X}_2)$ is Pareto efficient if and only if $\boldsymbol{\mu}$-comonotonic (Carlier-Dana-Galichon JET 2012).
- Hence the contract is efficient iff $\boldsymbol{Y} = \boldsymbol{X}_1 + \boldsymbol{X}_2 \succsim_{Disp} \boldsymbol{X}_1$ (from the characterization above).

## Multivariate Quiggin-Yaari functional and risk attitude

– Given a baseline $\boldsymbol{U} \sim \boldsymbol{\mu}$, decision maker evaluates risks with the functional

$$\Phi(\boldsymbol{X}) = \mathbb{E}[Q_{\boldsymbol{X}}(\boldsymbol{U}) \cdot \phi(\boldsymbol{U})]$$

(Equivalent to monotonicity relative to stochastic dominance and comonotonic additivity of $\Phi$ - Galichon and Henry JET 2012)

– Aversion to MPIR is equivalent to $\Phi(\boldsymbol{U}) = -\boldsymbol{U}$

– Aversion to MMPIR is equivalent to $\Phi(\boldsymbol{X}) \leq \Phi(\mathbb{E}[\boldsymbol{X}])$ (obtains immediately from the comonotonicity characterization of Bickel-Lehmann dispersion)