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Tests of fundamental physics using LIGO gravitational wave observations

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Abstract. Gravitational waves are allowed to propagate in $D-1$ large spatial dimensions. The far-zone gravitational waveform of a binary system of two point masses in circular orbit is obtained at quadrupole order, in the adiabatic approximation, for D even. The D-dimensional quadrupole energy loss formula is derived and found to be in agreement with previous literature (Cardoso et al., 2003), and the D-dimensional quadrupole angular momentum loss formula is given. The description introduces a distance scale possibly related to the size of the extra dimensions, and the resulting waveform is found to strongly vary with this scale. It is found that such a model is ruled out with current gravitational wave observations.

Personal Statement

In the first month, I was reading the books by Misner, Thorne and Wheeler and by Dirac in order to refresh my scarce knowledge of GR. I also started on papers by Peters and Mathews, while meeting my supervisor once a week to clarify and stratify certain concepts, mostly relating to energy and angular momentum of the gravitational field and gauge invariance.

I then looked into the shortwave formalism, and decided to ignore large-scale curvature effects. I also looked into how the waveform is obtained to quadrupole order. Meanwhile, in the second month I was trying to find out how to model the presence of higher dimensions, and was still trying to understand some results in Peters' paper, especially for the angular momentum loss formula.

Due to these delays, I only found prior to the Christmas holiday that there can be no bound stable orbits in D $>$ 4-dimensional Schwarzchild spacetimes. However, the model would proceed, as it should be possible to describe propagation effects on large scales coupled with a screening mechanism.

I continued over the holiday to learn about Noether's theorem for classical fields so as to find an understandable way to obtain the angular momentum loss of the field. This was what was missing after finding Cardoso's derivation of the D-dimensional quadrupole energy loss formula. I also learned the methods to solve the D-dimensional wave equation.

Continuing in January, I spent a significant amount of time unable to obtain the angular momentum formula that would reduce to the correct limit, and trying to understand whether I needed both the external and internal (spin) angular momentum, as well as looking into the group theory description of GW, which did not end up being much help.

Over the next few weeks, in February-March, I began working on the report and trying to formulate a self-contained derivation. I obtained the circular orbit result and spent a long time struggling with the elliptical derivation, while also looking for a different derivation for the angular momentum current.

Towards the end of the project, I spent my time contemplating two issues - what to make of the result, and what is to be done next. I was spending most of the time researching for similar results and writing the report.

Contents

1. Introduction

The first detection of a gravitational wave (GW) signal (Abbott et al., 2016) due to a binary black hole merger and the five subsequent detections are one of the most exciting developments of the last 3 years in astronomy. Over the last 3 years, not only has this fundamental prediction of General Relativity (GR) been repeatedly confirmed, but this involved the first direct observation a binary neutron star merger (Abbott et al., 2017).

The importance of this new window on our universe cannot be overstated, and, as a minute overview of but a few of the stimulating research directions, the GW astronomy programme promises to

- Provide precision tests of strong-field gravity through theory-agnostic parametrisations (Yunes & Pretorius, 2009) or specific modifications (Sperhake et al., 2017).
- Give insight into the behaviour of compact nuclear matter through probing the neutron star equation of state (Annala et al., 2017).
- Complement other large-scale structure probes in the inference of parameters, both through resolved sources (Del Pozzo, 2012) and through cross-correlation of number count or weak lensing power spectra with a GW background (Regimbau, 2011).
- Separate between black holes and other compact objects that one could envisage (Carballo-Rubio, 2018) that would appear different through purely GW effects, such as echoes (Cardoso & Pani, 2017).

This has been permitted by (and pushes forward) advances such as

- Full numerical solution of the Einstein field equations (Sperhake, 2015) for both binary compact object mergers, and GW sources such as supernova core collapse.
- Self-force formalism (see Poisson et al. (2011) for a review) for extreme mass ratio inspirals, such as plunges into supermassive black holes.
- Novel applications of statistical inference techniques for highly-noised data, such as time-frequency clustering (Gair & Jones, 2007) or deep filtering (George & Huerta, 2018), for example.
- Post-Newtonian expansions (see Blanchet (2014) for a review), double expansions (Blanchet et al., 2008), and the EOB formalism (Buonanno & Damour, 1999), making the connection between the intermediate stages of the inspiral and the late/merger stages.

A brief story of GW

It is perhaps not an overstatement that GR is neither a conceptually nor mathematically straightforward theory, and the first four decades of research in the domain of GW were stalled by the fact that prominent researchers, including Einstein, were convinced that the plane wave solutions are not physical (Denson Hill & Nurowski, 2016). A general consensus was only reached closer to 1960s, when a rigorous plane wave solution in analogy by symmetry classification to the electromagnetic case was presented (Bondi et al., 1959), and when it was shown that the plane waves indeed carry energy (Bondi, 1957). Finally coming to grips with GW led to the next fifty years of fruitful research in the field.

Results obtained for the leading order (quadrupolar) radiation from a Keplerian orbit (Peters & Mathews, 1963) and the proposal of laser interferometry as a possible detection route (Gertsenshtein & Pustovoit, 1962) in the following decade added further hope to the feasibility of such studies. This gave way to the first (indirect) proof of the existence of GW from the observations of the Hulse-Taylor binary (Taylor et al., 1979). From here on it was clear that this programme needs to be moved forward, and the previously mentioned theoretical developments, together with the contribution of ingenuous experimentalists and engineers, and with an enormous amount of work, too sizeable to mention briefly here, led to us having 3 operational ground-based detectors today, with several more in the planning stage, and a confirmed space-based detector, LISA, planned to start operating in mid-2030s (Marx et al., 2011).

Gravity and the other forces

There has been an ongoing quest through the 20th century, which continues in the 21st, to find a unified description of the fundamental forces. The canonical quantisation approach, applied to gravity, leads to a nonrenormalisable theory (Deser, 2000). It may also seem unnatural to separate spacetime from the fields that live on it, but this is how we study quantum theories, and there have been many advancements made over the recent years in this direction with regards to incorporating gravity (Pullin, 2003), with a recent experimental proposal to determine whether gravity has a quantum nature to it (Bose et al., 2017).

An alternative approach is to include matter in the geometry, also known as space-time matter (STM) theories. This idea can be dated back to Nordstrom (Nordström, 1914), who proposed to unify electromagnetism with gravity this way, by allowing for an extra spatial dimension. This was continued in Kaluza's work (Kaluza, 1921) showing that the electromagnetic field can exist in our 4-dimensional world without having to explicitly add it as the stress-energy tensor in Einstein's equations, but by solving instead the vacuum equations in 5 dimensions. The immediate shortfalls were the requirement of the cylinder condition, that 4-dimensional quantities did not depend on the 5th coordinate, and the lack of explanation for the unobservability of this fifth dimension.

This thinking was in line with what Einstein expected the theory to shape into (Einstein & Rosen, 1935), and it inspired Misner's geometrostatics (Misner, 1963), paving way for prescribing initial conditions on spacetimes with multiple black holes. The previously mentioned shortfalls were amended by Klein (Klein, 1926a; Klein, 1926b), who suggested the dimensions were compact, and thus their effects were unobservable up to a certain energy scale.

Kaluza's construction creates a massless vector and a massless scalar. What about other matter? For a small extra dimension of circular topology, adding a 5-dimensional massless scalar leads to there appearing an infinite number of massive scalars in 4 dimensions, with local U(1) gauge invariance achieved by minimally coupling them to EM field (Overduin & Wesson, 1997). The charges of these scalars are reasonable, however, the masses exceed by many orders of magnitude any standard model (SM) particles. To include the full SM gauge group, it becomes much more complicated, and the original idea that matter comes from geometry does not seem as viable anymore.

This problem was tackled with a novel mathematical framework, string theory. Although ridding of some pains of QFT such as ultraviolet divergences, its consistent description necessarily requires a large, model-dependent number of extra dimensions (Satheeshkumar & Suresh, 2006). What connects the aforementioned theories is that they picture a so-called brane-world, with gravity propagating in all D dimensions, while the other fundamental forces are constrained to the 4-dimensional "bulk" (Rubakov, 2001). The consideration is not limited to compact dimensions, as compactification only serves to hide unobserved effects. There are interesting consquences of large, or even infinite, extra dimensions (Visser, 1985; Dvali et al., 2000), or large extra dimensions in the early universe, which, when shrinking, could have caused ours to inflate (Levin, 1995). Testing for the presence of extra dimensions is where GW come in.

GW provide the perfect playground for tests of effects due to extra-dimensional scenarios. The alterations in the waves can manifest themselves in various ways, as changes of the amplitude decay with distance (Pardo et al., 2018), changes in the propagation velocity, changes in the phase-time relationship (Deffayet & Menou, 2007), as additional propagating modes that may also affect differently the two polarisations (Alesci & Montani, 2005; Andriot & Lucena Gómez, 2017), or as even more peculiar effects like diffraction radiation (Cardoso et al., 2006).

1.1. Aims and outline

Experimentally, no deviations are seen from the inverse square law at either the laboratory scale (Hoskins et al., 1985; Hoyle et al., 2001) or at solar system scales (Will, 2014). In fact, one couldn't have stable bound orbits if the Schwarzchild metric were also modified (Tangherlini, 1963). Yet it is difficult to stop one from imagining that gravity works differently on the largest scales, especially given the complete lack of understanding of the density component that causes the accelerated expansion of the universe.

The primary aim of this work is to provide a derivation, starting from the field equations, of the leading order effect on the gravitational waveform from a Keplerian binary in the context of a modification where the gravitational wave can propagate in D dimensions. It is expected that the greater rate of energy leakage due to GW may be responsible for incorrect parameter constraints and population estimates due to degeneracy of waveforms, especially since severe misconstraints could have already been placed without the need for any such modifications (Creswell et al., 2018).

No self-consistent theory is given - the orbits of the inspiralling binary are considered to follow the well-known Kepler's law, while the dynamical potentials travel in $D-1$ spatial dimensions. However, the calculation is general and should be portable. What is different here from similar work, such as that of Pardo et al. (2018), is that energy and angular momentum loss over a hypersphere are considered in addition to the modification of the amplitude/luminosity distance relationship.

The work proceeds as follows: in §2, the field equations are presented, non-dynamical degrees of freedom and redundancies in coordinate choice are isolated, the equations are linearised and the smallness of the amplitude of GW is argued for with Newtonian gravity. Finally, the canonical stress-energy and angular momentum tensors are introduced. In §3, these results are applied to the inspiralling binary. The D-dimensional wave equation is solved, the energy and angular momentum loss rates are found for this binary, and the waveform for the circular case is given. The results are discussed in §4.

2. Background theory

Spacetime is a 4-dimensional topological manifold with a Lorentzian metric and a torsionless connection compatible with the metric. General Relativity is the theory which makes spacetime dynamical. Spacetime then determines the motion of test particles, which follow paths of shortest proper time (geodesics). To find the set of equations that dictate the dynamics of spacetime, Einstein was guided by two key principles:

- General covariance - the choice of coordinates should be arbitrary in the description of the laws of physics.
- Equivalence - in a locally inertial frame the laws of physics are the same as in a Minkowski background.

These, and the fact that the dynamics of particles in the weak field static case should restore Newton's law of gravity, are incorporated by construction in the Einstein Field Equations (EFE):

$$R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi T_{ab} \tag{1}$$

Indices on tensors from the start of the latin alphabet (a, b, c,...) denote abstract "slots" and this is the first and the last time they are used, to emphasize general covariance. Indices from the greek alphabet (μ, ν, ρ,...) will denote the spacetime components and indices from the middle of the alphabet (i, j, k,...) will denote spatial components in the coordinate basis. From §2.5 onwards, they will run from 0 to $D-1$, and from 1 to $D-1$, respectively, and no meaning will be assigned to components other than those living in our 4 dimensions. Where spatial indices are used, no distinction will be made between up/down indices.

The field equations are a set of 10 second-order partial differential equations solved for the metric $g_{\mu\nu}(x)$, which locally measures the distance between two spacetime events, and therefore allows one to calculate the geodesics. Given this information, one could make the claim "give me 20 pieces of initial data - 10 initial data for $g_{\mu\nu}(t,\mathbf{x})|_{t=t_0}$ and 10 initial data for $\partial_t g_{\mu\nu}(t,\mathbf{x})|_{t=t_0}$, and I will directly integrate these equations". That would not be the best approach, as the previously mentioned gauge invariance means the field equations do not uniquely determine the metric.

An important quality demanded of the right-hand side of the equations is the covariant conservation of stress-energy

$$\nabla_\mu T^{\mu\nu} = 0 \tag{2}$$

which is separately a property of the left-hand side due to the Bianchi identities. For the LHS, this identity can be written as

$$\partial_0 G^{0\nu} = -\partial_i G^{i\nu} - \Gamma^{\nu}{}_{\mu\lambda}G^{\mu\lambda} - \Gamma^{\mu}{}_{\mu\lambda}G^{\lambda\nu} \tag{3}$$

with $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$. As the highest time derivative on the RHS of eqn. 3 is second order, $G^{0\nu}$ can at most contain a first time derivative of the metric, hence the 4 EFE for $G^{0\nu}$ do not describe the dynamical degrees of freedom, but rather place constraints on the initial data (Weinberg (1972) §7.5). One is then left with 6 equations for 10 unknowns. Coordinate freedom allows one to use up another 4 initial data in order to place constraints on the coordinates. This will yield the 4 remaining equations for the second derivatives of the metric and leave the EFE describing 2 dynamical degrees of freedom (DOF), or, in D dimensions, $N_D = [D(D-3)]/2$ dynamical DOF.

In general, without sufficiently restrictive symmetry conditions, these equations are analytically intractable due to their nonlinear nature (see Kramer & Schmutzer (1980) for a categorisation of solutions, for example). However, one can linearise the EFE, and this is where the gravitational wave appears.

2.1. Linearised gravity

To linearise the field equations, the metric is expanded around a Minkowski background, and this ansatz is substituted into the EFE,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{4}$$

The Minkowski background is chosen to simplify calculations, as the indices of the perturbation are lowered and raised using the background metric, with correction terms being next order in the amplitude of the perturbation. This treatment ignores the large-scale curvature of the background due to the GW themselves, which have a non-vanishing stress energy tensor given in §2.5 that acts as a source term in the EFE. Also ignored are the associated propagation effects, as these are second-order in the amplitude, which is shown to be sufficiently small in §2.4. A more detailed discussion of these can be found in §35 of Misner et al. (1973).

In the harmonic gauge, one has that $\bar{h}_{\mu\nu}{}^{,\nu} = 0$ with $\bar{h}_{\mu\nu} = h_{\mu\nu} - \eta_{\mu\nu}h/2$, and $h = \eta^{\mu\nu}h_{\mu\nu}$ being the trace of the perturbation. The EFE are then

$$\Box\bar{h}_{\mu\nu} = -16\pi S_{\mu\nu} \tag{5}$$

where the full stress-energy tensor $S_{\mu\nu} = T_{\mu\nu} + X^{(2+)}_{\mu\nu}$ includes the matter stress-energy $T_{\mu\nu}$ and terms of order h^2 and greater, $X^{(2+)}_{\mu\nu}$. A non-covariant conservation of stress-energy remains, $S_{\mu\nu}{}^{,\mu} = 0$. In the linearised theory, the quantities carrying indices, Lorentz tensors, will simply be referred to as tensors.

A key result, often taken for granted, is the fact that GW are unaffected by dust (Ehlers et al., 1987), and that when travelling through a dissipative fluid, they are attenuated on a timescale proportional to the inverse viscosity (Anile & Pirronello, 1978), hence allowing one to consider them to propagate freely through the universe.

2.2. Coordinate constraints

Using the equivalence of passive and active transformations, it will be shown how to impose the harmonic gauge to 1st order in h, with a sketch derivation of how this could be done to arbitrary order (for the result for an arbitrary perturbation around a background spacetime in full mathematical rigour, see Bruni et al. (1997)).

Consider a relabelling of coordinates $x^\mu \to x'^\mu$ (a passive transformation) with the new labels defined by $x'^\mu = x^\mu + n^\mu(x)$. It is equivalent to changing the components of tensors in the old chart, specifically for the metric these new components $g'_{\mu\nu}$ in the old chart are defined by requiring that an infinitesimal interval between two events remain the same

$$g'_{\mu\nu}(x)dx^\mu dx^\nu = g_{\alpha\beta}(\tilde{x})d\tilde{x}^\alpha d\tilde{x}^\beta \tag{6}$$

where $\tilde{x}$ is the point, which, after relabelling, corresponds to the point that was previously labelled x, $\tilde{x}'^\mu = \tilde{x}^\mu + n^\mu(\tilde{x}) = x^\mu$. Applying the transformation law under change of coordinates to the RHS, one finds

$$\eta'_{\mu\nu} + h'_{\mu\nu}(x) = \eta_{\alpha\beta}\frac{\partial\tilde{x}^\alpha}{\partial x^\mu}\frac{\partial\tilde{x}^\beta}{\partial x^\nu} + h_{\alpha\beta}(\tilde{x}^\mu(x))\frac{\partial\tilde{x}^\alpha}{\partial x^\mu}\frac{\partial\tilde{x}^\beta}{\partial x^\nu} \tag{7}$$

To get $\tilde{x}^\mu(x)$, one needs $A^\mu_\nu = \frac{\partial\tilde{x}^\mu}{\partial x^\nu}$, which is obtained by matrix inversion, symbolically as $A = (1+T)^{-1}$ where $T^\mu{}_\nu = n^\mu{}_{,\nu}$. Then integrating A and imposing the condition by which $\tilde{x}$ is defined should yield $\tilde{x}^\mu(x)$ to all orders in n.

To first order in n, this inversion results in $A^\mu{}_\nu = \delta^\mu_\nu - n^\mu{}_{,\nu}$ and $\tilde{x}^\mu(x) = x^\mu - n^\mu(x)$. Requiring that, for the new components of the perturbation, $h'_{\mu\nu}{}^{,\mu} - \frac{1}{2}h'_{,\nu} = 0$ (harmonic condition) gives

$$\Box n_\nu = h_{\mu\nu}{}^{,\mu} - \frac{1}{2}h_{,\nu} \tag{8}$$

as the coordinate transformation necessary to impose the gauge to first order. Of the 10 dynamical DOF, this condition leaves only 6. Variations in the coordinates that obey $\Box n_\mu = 0$, hence $n_\mu(x) = A_\mu e^{ik\cdot x}$, are still allowed after fixing the gauge, and the choice of n_μ such that $\partial_\mu n^\mu = h/2$ is convenient because it makes the perturbation traceless. The remaining 3 constraints can be used to impose the transverse condition, $\bar{h}_{\mu 0} = 0$. The persistence of the TT gauge can be ensured during propagation as shown in Appendix B of Butcher et al. (2010).

The transverse-traceless (TT) components are the only ones that cannot be made to vanish by a coordinate transformation, as the Riemann tensor is gauge invariant

$$R_{j0k0} = -\frac{1}{2}h^{TT}_{jk,00} \tag{9}$$

2.3. The Kepler orbit

The Kepler orbit traces out an ellipse of semimajor axis a and eccentricity e with period $T = 2\pi\sqrt{a^3/GM}$ with $M = m_1 + m_2$. The EOM are due to each particle acting as a test mass in the other's instantaneous Schwarzchild spacetime, neglecting the $1/r^3$ term in the radial potential.

$$r_1 = \frac{\mu r}{m_1}[cos(\phi)\mathbf{e}_1 + sin(\phi)\mathbf{e}_2] \quad r_2 = \frac{\mu r}{m_2}[-cos(\phi)\mathbf{e}_1 - sin(\phi)\mathbf{e}_2] \tag{10}$$

where $\mu = (m_1 m_2)/(m_1 + m_2)$. The radius is, in terms of true anomaly ϕ,

$$r = \frac{a(1-e^2)}{1 + e\ cos(\phi)} \tag{11}$$

Throughout this work, numerical values are given for a typical binary of two neutron stars at a distance of 0.1 Gpc from the observer, with the components of equal mass, 1.4 $M_\odot$. The binary will have entered our ground based detectors' band at a components' separation of 1000 km, at an orbital frequency $\simeq$ 20 Hz.

2.4. Scalar GW

The previously claimed smallness of the perturbation can be inferred from considering the scalar wave in Newtonian gravity (Schutz, 1984). Starting from the Poisson equation for the Newtonian potential,

$$\nabla^2\phi(\mathbf{x}, t) = 4\pi\rho(\mathbf{x}, t), \tag{12}$$

for which the solution is found to be, using Green's function methods,

$$\phi(\mathbf{x}, t) = -\int d^3x' \frac{\rho(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} \tag{13}$$

Making the modification to a "relativistic" potential by adding a retardation effect, $t \to t - |\mathbf{x} - \mathbf{x}'|$, the equation

$$\phi_R(\mathbf{x}, t) = -\int d^3x' \frac{\rho(\mathbf{x}', t - |\mathbf{x} - \mathbf{x}'|)}{|\mathbf{x} - \mathbf{x}'|} \tag{14}$$

becomes a solution to the wave equation with a source. In the far zone one can drop the $O(1/r^2)$ terms and work to second order in x'/λ, the wavelength of the wave, leading to

$$\phi_R(\mathbf{x}, t) = -\frac{1}{r}\int d^3x' \left[\rho(\mathbf{x}', t_0) + \dot{\rho}(\mathbf{x}', t_0)(\mathbf{n}\cdot\mathbf{x}') + \frac{1}{2}\ddot{\rho}(\mathbf{x}', t_0)(\mathbf{n}\cdot\mathbf{x}')^2\right] \tag{15}$$

After relating the first term in the expansion to the total mass of the source, and the second term, using the continuity equation, to the total momentum of the source, the relativistic field is written as

$$\phi_R(\mathbf{x}, t) = -\frac{M}{r} + \frac{\mathbf{n}\cdot\mathbf{P}}{r} - \frac{\ddot{D}_{ij}(t) n_i n_j}{2r} \tag{16}$$

where the quadrupole D_{ij} is defined as

$$D_{ij}(t) = \int d^3x' \rho(\mathbf{x}', t) x'_i x'_j \tag{17}$$

and the "gravitational wave" h is defined as the dimensionless quantity that is time-varying part of ϕ_R/c^2. One thus finds that the first dynamical contribution is proportional to the quadrupole,

$$h = -\frac{G}{2rc^4} \ddot{D}_{ij} n_i n_j \tag{18}$$

Now $\ddot{D}$ can be shown to be proportional to v^2 where v is a typical velocity of the sources, and this is bounded, due to the virial theorem, by the interaction potential inside the system ϕ_{int}, leading to $\ddot{D} \simeq 2M\phi_{int}$ and therefore, for the example binary,

$$h \simeq \frac{\phi_N \phi_{int}}{c^4} \simeq \frac{GM}{Rc^4} \frac{GM}{r} \simeq 6 \times 10^{-24} \tag{19}$$

The result matches the analogous calculation in GR (see §3.3) up to a numerical prefactor. However, an important difference is that the observer can only probe the transverse-traceless to its line of sight part of the source's quadrupole, implying that for spherically symmetric sources there is no radiation in GR, and the quadrupole should be replaced by its trace-removed version.

2.5. Energy and angular momentum of GW

In order to describe the changes in the orbit due to gravitational waves, and thus determine the evolution of the waveform, it is necessary to determine the rates of loss of energy and angular momentum due to gravitational radiation.

Here the energy and angular momentum exchanges between the binary and the field are considered globally. This is done using the description of a massless spin-2 field in a Minkowski background. Starting from the Einstein-Hilbert action (Maggiore (2008) §2),

$$S_{EH} = \frac{1}{16\pi G^{(D)}} \int d^D x \ R\sqrt{-g} \tag{20}$$

with the gravitational constant in D dimensions, $G^{(D)}$, defined so as to keep the units of the vacuum EFE invariant, $[G/c^2] = 1 \ m^{D-3} \ kg^{-1}$ (Zwiebach (2009) §3.8).

Expanded to second order in h, the action becomes

$$S_{EH} = -\frac{1}{64\pi} \int d^D x \ \left[h_{\alpha\beta,\mu} h^{\alpha\beta,\mu} - h_{,\mu} h^{,\mu} + 2h^{\mu\nu}{}_{,\mu} h_{,\nu} - 2h^{\mu\nu}{}_{,\mu} h^{\rho}{}_{\nu,\rho} \right] \tag{21}$$

For this field, the canonical stress-energy tensor for spacetime translations is

$$\tau^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu h_{\alpha\beta})} \partial^\nu h_{\alpha\beta} - \eta^{\mu\nu} \mathcal{L} \tag{22}$$

where the first index corresponds to the charge/currents and the second to each translation. Fortunately, it just so happens to be symmetric in indices, and if it were not, there is a procedure to put it into symmetric form (Belinfante, 1940).

This stress-energy tensor is only defined in a spatially averaged sense if one considers that is is the stress-energy contribution due to gravity, the $X^{(2)}_{\mu\nu}$ term in eqn. 5, while one considers the oscillatory part as the perturbation to the geometry at second order.

Using that, one can set $\langle \mathcal{L} \rangle = 0$ by integrating by parts and using the vacuum linearised EFE. The stress-energy tensor is then, for spacetime translations, in the Lorentz gauge

$$\tau^{\mu\nu} = \frac{1}{32\pi} \left\langle \bar{h}_{\alpha\beta}{}^{,\mu} \bar{h}^{\alpha\beta,\nu} + \frac{1}{2-D} \bar{h}^{,\mu} \bar{h}^{,\nu} \right\rangle \tag{23}$$

thus giving the rate of energy loss as

$$\frac{dE}{dt} = \int \tau_{i0} dS^i = \frac{1}{32\pi} \int d\Omega_{D-2}\, r^{D-2}\, n^i \left\langle \bar{h}_{\alpha\beta,i} \bar{h}^{\alpha\beta}{}_{,0} + \frac{1}{2-D} \bar{h}_{,i} \bar{h}_{,0} \right\rangle \tag{24}$$

Similarly, considering the infinitesimal action of the Lorentz group on the field,

$$\left(1 + w^{\alpha\beta} M_{\alpha\beta}\right) h_{\mu\nu}(x) = \left(\delta^\alpha_\mu + w^\alpha{}_\mu\right) \left(\delta^\beta_\nu + w^\beta{}_\nu\right) h_{\alpha\beta}(\Lambda^{-1} x) \tag{25}$$

where $w^{\alpha\beta}$ are the parameters and $M_{\alpha\beta}$ the generators, the Noether current is found to be, keeping the dependence on the parameters of the infinitesimal transformation,

$$j^\sigma = w^{\alpha\beta} \left[\frac{\partial \mathcal{L}}{\partial(\partial_\sigma h_{\mu\nu})} \left(\frac{1}{2}(x_\alpha \partial_\beta - x_\beta \partial_\alpha) h_{\mu\nu} - \eta_{\beta\mu} h_{\alpha\nu} - \eta_{\beta\nu} h_{\mu\alpha} \right) - \frac{1}{2}(\delta^\sigma_\beta x_\alpha - \delta^\sigma_\alpha x_\beta) \mathcal{L} \right] \tag{26}$$

Specialising to spatial rotations such that $w^{ij} = \epsilon^{ijk} n_k \delta\theta$ gives the angular momentum loss of the field as

$$\frac{dL_i}{dt} = \frac{1}{32\pi} \epsilon_{ikl} \int d\Omega_{D-2}\, r^{D-2} \left\langle 2\mathcal{P}\bar{h}_{ak} \mathcal{P}\bar{h}_{al,0} - x_k \bar{h}^{\mu\nu,0} \bar{h}_{\mu\nu,l} - \frac{1}{2} x_k \bar{h}_{,0} \bar{h}_{,l} \right\rangle \tag{27}$$

Where the TT projector $\mathcal{P}$ is defined as (see also the appendix)

$$A^{TT}_{ij} = \mathcal{P} A_{ij} = P_{ijkl}(\mathbf{n}) A_{kl} \tag{28}$$

The surface integrals in equations 27 and 24 are to be evaluated at sufficiently far away where the terms that scale as higher inverse orders of distance to the source can be neglected.

At this point the reader may be worried why the projector is introduced only in parts of eqn. 27. The answer is because it is this and only this exact combination of terms that leads to the correct angular momentum loss equation for the binary, given in eqn. 44, where the correctness is justified by comparison to the energy loss formula, and the fact that it restores the right formula for $D = 4$.

3. Leading order waveform of a binary orbit

In what follows, the linearised EFE are solved with a binary as a source in order to obtain the waveform evolution. The point particles in bound orbits under consideration appear in the stress-energy tensor as dust, with $T_{\mu\nu} = \rho U_\mu U_\nu$, where U_μ is the four-velocity with unit norm (Dirac, 1996). In fact, only the component $T_{00} = \rho(\mathbf{x}, t) = \sum_a m_a \delta^{D-1}(\mathbf{x} - \mathbf{x}_a(t))$ will be the necessary in the quadrupole approximation. Internal structure can be safely ignored, as it only enters the equations of motion, for compact bodies, at 5PN order (Kopeikin & Vlasov, 2006). Furthermore, the averaging in the energy and angular momentum loss formulae will be over the period of one orbit, as in the adiabatic approximation it is assumed that the changes in the orbital parameters over one orbit are negligible.

3.1. Solving the wave equation

In a universe occupied only by a distant observer and a binary bound system of point masses, outside the source zone take plane wave solutions to eqn. 5,

$$\bar{h}_{\mu\nu}(\mathbf{x}, t) = \int \frac{d\omega}{2\pi} e_{\mu\nu}(\mathbf{x}, \omega) e^{-i\omega t} \tag{29}$$

Analogously decomposing the RHS of eqn. 5, it remains to solve the $D-1$ dimensional Helmholtz equation for $e_{\mu\nu}$

$$\left(\nabla^2 - (-i\omega)^2\right) e_{\mu\nu}(\mathbf{x}, \omega) = 16\pi \tilde{T}_{\mu\nu}(\mathbf{x}, \omega) \tag{30}$$

the result for its Green's function is given in §22.4 of Hassani (2002) as

$$G(\mathbf{r}, \omega) = \frac{i\pi/2}{(2\pi)^{(D-1)/2}} \left(\frac{\omega}{r}\right)^{(D-3)/2} H^{(1)}_{(D-3)/2}(\omega r) \tag{31}$$

Using the asymptotic expansion (Abramowitz & Stegun, 1965) as $\omega r \to \infty$ for $H^{(1)}_\nu(\omega r)$

$$H^{(1)}_{(D-3)/2}(\omega r) = \left(\frac{2}{\pi r \omega}\right)^{1/2} e^{i\omega r} e^{(-i\pi/4)(D-2)} \tag{32}$$

the solution (for outgoing waves) is found to be

$$e_{\mu\nu}(\mathbf{x}, \omega) = e^{(-i\pi/4)(D-4)} \frac{8\pi\omega^{(D-4)/2}}{(2\pi)^{(D-2)/2}} \int d^{D-1}x' \; e^{i\omega|\mathbf{x}-\mathbf{x}'|} \frac{\tilde{T}_{\mu\nu}(\mathbf{x}', \omega)}{|\mathbf{x}-\mathbf{x}'|^{(D-2)/2}} \tag{33}$$

The distinction between three distance scales now needs to be made, x', the typical size of the binary orbit, λ, the wavelength of the GW, and $R = |\mathbf{x}|$, the distance from the source. Eqn. 33 is only valid for $\omega R \gg 1$, and thus only for the dynamic part of the perturbation. Expanding the quantities $|\mathbf{x} - \mathbf{x}'|$ in the limit $\omega x' \ll 1$ and $x'/R \ll 1$, to next to leading order in both, one obtains

$$\bar{h}_{\mu\nu}(\mathbf{x}, t) = e^{(-i\pi/4)(D-4)} \frac{8\pi}{(2\pi R)^{(D-2)/2}} \int \frac{d\omega}{2\pi} e^{-i\omega(t-R)} \omega^{(D-4)/2} \int d^{D-1}x' \Bigg(1 - \\ i\omega \mathbf{n}\cdot\mathbf{x}' - \frac{\omega^2}{2}(\mathbf{n}\cdot\mathbf{x}')^2 + \frac{D-2}{2}\frac{\mathbf{n}\cdot\mathbf{x}'}{R} - \frac{1}{2} i\omega \frac{(\mathbf{n}\cdot\mathbf{x}')^2}{R} + \frac{D-1}{2} i\omega \frac{\mathbf{x}'^2}{R}\Bigg) \tilde{T}_{\mu\nu}(\mathbf{x}', \omega) \tag{34}$$

Then, given the linearised stress-energy conservation for the source $T_{\mu\nu}{}^{,\mu} = 0 + O(h^2)$, one identifies the time-independent quantities

$$M = \int d^{D-1}x' \; T_{00} \quad P_i = \int d^{D-1}x' \; T_{0i} \quad X_i = \int d^{D-1}x' \; T_{00}x'_i \tag{35}$$

allowing one to write the dynamical components of the waveform, in the previously mentioned limits, in terms of the quadrupole only (here the terms given are what appeared in the $d^{D-1}x'$ integral of eqn. 34)

$$\bar{h}_{00} \sim \int dt' e^{i\omega t'} \left(-\frac{\omega^2}{2} n_i n_j D_{ij}(t') - i\omega \frac{D-1}{2r} n_i n_j D_{ij}(t') \right) \tag{36}$$

$$\bar{h}_{ok} \sim \int dt' e^{i\omega t'} \left(\frac{\omega^2}{2} n_i D_{ik}(t') + i\omega \frac{D-2}{4r} n_i D_{ik}(t') \right) \tag{37}$$

$$\bar{h}_{kl} \sim \int dt' e^{i\omega t'} \; \frac{-\omega^2}{2} D_{kl}(t') \tag{38}$$

3.2. Kepler orbit relations

The quadrupole for the system is

$$D_{ij} = \frac{\mu r^2}{2} Q_{ij} \tag{39}$$

with

$$Q_{ij} = \begin{bmatrix} cos(2\phi) & sin(2\phi) & 0 \\ sin(2\phi) & -cos(2\phi) & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{40}$$

Now exploiting the periodicity to write the quadrupole as a Fourier series with the fundamental angular frequency $\Omega = 2\pi/T$,

$$D_{ij} = \sum_n D^n_{ij} e^{-in\Omega t} \tag{41}$$

where the Fourier components are given by

$$D^n_{ij} = \frac{\pi}{T} \int_{-T/2}^{T/2} dt\ e^{in\Omega t} D_{ij} \tag{42}$$

Next substituting eqn. 41 into the waveform and using the results in eqns. 24 and 27 one obtains the energy and angular momentum losses from the periodic source,

$$\frac{dE}{dt} = k_D \sum_n (-in\Omega)^{D+2} |D^n_{ij}|^2 \tag{43}$$

$$\frac{dL_j}{dt} = 2k_D \epsilon_{jkl} \sum_n (-in\Omega)^{D+1} D^{n*}_{km} D^n_{lm} \tag{44}$$

with k_D being a constant prefactor

$$k_D = \frac{2^{2-D} \pi^{(5-D)/2} D(D-3)}{\Gamma((D-1)/2)(D-2)(D+1)} \tag{45}$$

and the waveform, to leading order in the quadrupole approximation, seen by an observer at $\mathbf{x}$ in the wave zone

$$h_{ij}(\mathbf{x}, t) = \frac{4\pi G^{(D)}}{(2\pi R)^{(D-2)/2}\ c^{(D+4)/2}} \sum_m (m\Omega)^{D/2} \mathcal{P} D^m_{ij} e^{-im\Omega(t-R/c)} \tag{46}$$

3.3. The circular case

In the circular case, one can explicitly write out the fourier components of the quadrupole tensor. This gives for the energy loss

$$\frac{dE}{dt} = k_D \mu^2 r^4 \omega^{D+2} \left(2^D + (-2)^D\right) \tag{47}$$

which vanishes for D odd, so the solutions following are for D even only. Using $E = -GM\mu/2r$, one finds the solution to the decay of the orbital radius

$$r(t) = r_0 \left(1 - \frac{t}{t_c}\right)^{\frac{2}{3D-4}} \tag{48}$$

where $r_0 = r(t = 0)$ and t_c is the coalescence time,

$$t_c = \frac{1}{(3D-4) k_D 2^{D+1}} \left(\frac{r_0}{r_s}\right)^{D/2} \left(\frac{r_0}{r_N}\right)^{D-4} \frac{r_0^2 c}{G\mu} \tag{49}$$

with the scale r_N set by $G^{(D)} = G r_N^{D-4}$, and r_s the Schwarzchild radius of M. Then, using that $\omega = \sqrt{G^{(4)} M / r^3}$ one obtains for the time evolution of the phase

$$\phi(t) - \phi_0 = \int_0^t dt' \; \omega(t') = -\omega_0 t_c \frac{4-3D}{7-3D} \left(1 - \frac{t}{t_c}\right)^{\frac{7-3D}{4-3D}} \tag{50}$$

with $\omega_0 = d\phi/dt|_{t=0} = \sqrt{G^{(4)} M / r_0^3}$. Now using these results in eqn. 46, one obtains for the circular inspiral waveform:

$$h_{ij}(\mathbf{x}, t) = 4\pi^{\frac{4-D}{2}} \left(\frac{r_N}{R}\right)^{\frac{D-2}{2}} \left(\frac{r_N}{r_0}\right)^{\frac{D-6}{2}} \left(\frac{r_s}{r_0}\right)^{\frac{D}{4}} \frac{G\mu}{r_0 c^2} \left(\frac{\omega(t)}{\omega_0}\right)^{\frac{3D-8}{6}} \mathcal{P} Q_{ij}(t) \tag{51}$$

where the t on the RHS is the retarded time.

This waveform reduces correctly to the $D = 4$ case, but are the other even-dimensional cases worthy of consideration at this point? Indeed, already for $D = 6$, considering the example binary, in order to match the inspiral timescale, the scale r_N needs to be set to 10^7 m. For any larger values of r_N, the inspiral timescales become unrealistically short. However, the amplitude is now weakened by many orders of magnitude compared to the $D = 4$ case. This can be amended by bringing the binary closer, from 100 Mpc to an uncomfortable distance of $\simeq 4000$ A.U. Starting at a component distance of $\simeq 500$ km, for these inputs, both binaries will then take about 2 minutes to coalesce. Figure 1 shows the chirping behaviour for $\simeq 0.4$ seconds until coalescence.

How far can one go making predictions with this circular waveform? For the case $D = 4$, the relative change in the orbital radius over one period at 500 km components' separation is 0.02 % and becomes 3 % at component's separation of 50 km, hence the adiabatic approximation holds quite well. The ratio $\omega r/c$, on the other hand, is much larger, 10 % and 27 %, respectively, implying that the last 2 minutes of the inspiral are already in the regime where the neglect of terms of order $(\omega r)^2$ gives inaccuracies at the order of 10 % at this order in the approximation.

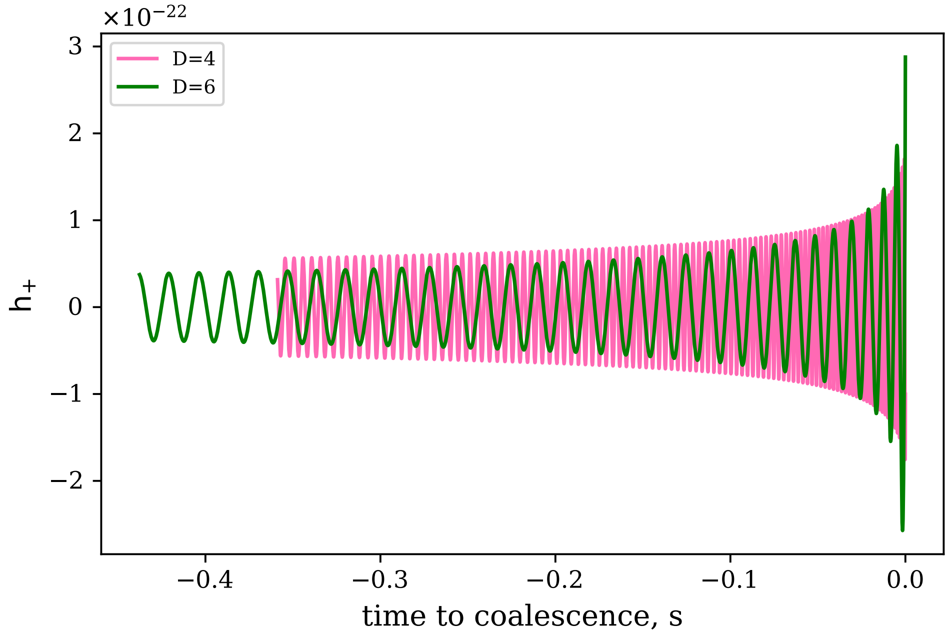

Figure 1. Comparison of the chirping behaviour of the circular binary inspiral waveforms in $D = 4$ and $D = 6$. This plot is only illustrative, to show that, for a very specific choice of length scale r_N distance to the binary R, the amplitudes can be made similar, but the frequency evolution of the already-matched waveforms rules out any other even-dimensional case but $D = 4$.

4. Discussion

The far-zone gravitational waveform due to a binary system of two point masses in circular orbit, under the assumption of $D-1$ large spatial dimensions, is obtained. As the orbit is expected to rapidly circularise due to GW emission (Peters, 1964), the circular case is a proof-of-concept, showing that the model is not viable as it stands. The derivation of the elliptical case remains to be completed as further work, although the enhancement factor due to ellipticity to the amplitude would not amend the situation much.

The D-dimensional quadrupole (energy loss) formula is derived, and is in agreement with Cardoso et al. (2003), and the D-dimensional angular momentum loss formula is a new result, to the best of the author's knowledge. It would be interesting to investigate the case of D odd, where a more careful analysis is required due to the breaking down of Huygens' principle (Barvinsky & Solodukhin, 2003), in order to compare to $D=5$, but if the behaviour seen is general, then it seems that large extra dimensions are ruled out. What was not said at any point is what large means in this context, and what is the size of these extra dimensions considered. This is related to the interpretation of the scale r_N that appeared in the solutions for D other than 4. It could be viewed as the volume of the extra dimensions if the Einstein-Hilbert action is written as

$$S_{EH} = \frac{1}{16\pi G^{(D)}} \int d^4x \; d^{D-4}y \; R\sqrt{-g} = \frac{1}{16\pi G} \int d^4x \; R\sqrt{-g} \tag{52}$$

with $G^{(D)} = GV^{(D-4)}$ if the cylinder condition is assumed to hold. This is similar to the analysis of Qiang et al. (2016), and restores the EFE in 4D. This way of thinking, however, seems incompatible with the idea of allowing gravity to propagate in the extra dimensions. It would also be interesting to see what the effect of a Yukawa-type transition between the two constants would be.

Additional degrees of freedom appearing in the metric in extra dimensions were neglected in the present analysis, however, they have been shown to be potentially detectable already with current ground-based detectors (Andriot & Lucena Gómez, 2017), and it should be clear by now that one should consider GW propagation on large scales as through 4 dimensions, possibly allowing for higher-dimensional effects at the source. The optimal way to approach this may be from the effective field theory (EFT) point of view, where the "graviton" is split into more than one type, with different propagation properties (Goldberger & Rothstein, 2006). This approach of EFT of non-relativistic gravity for extended objects provides a cleaner separation of distance scales, and allows one to use known field theory techniques such as regularisation of divergent integrals, which appear when back-scattered waves hitting the orbit are considered, for example. It has been successfully used to derive well-known PN corrections, such as the EIH Lagrangian (Kol & Smolkin, 2008) and the D-dimensional energy loss quadrupole formula (Cardoso et al., 2008). It may well be that the extra-dimensional signal is hiding in a modification to the black hole spacetime, implying that the regime of interest for extra-dimensional scenarios will be in the extreme-mass-ratio inspirals, or post-merger ringdowns.

With regards to the stress-energy tensor question, having used the canonical Noether currents, the reader may have little belief as to whether it makes any sense at all to work from the Lagrangian, as one can always add boundary terms to the Lagrangian that vanish in the action. It would be more desirable to obtain it otherwise, by varying the action directly, for example. Moreover, one would like to make at least a locally co-ordinate invariant conserved current, but it is not clear how to apply the standard "covariantisation" procedure to the linearised EH action.

It has also been re-iterated in the literature that a local description of energy exchange between the field and a detector is impossible due to the equivalence principle - the gravitational field and its first derivatives can always be made to vanish in a locally inertial frame. However, a local description of the energy and momentum exchange is necessary if we are to take the famous "rod with beads" argument for truth given that the size of such a detector would be much smaller than the typical wavelength of the radiation. Such a local description was given by Butcher et al. (2010). In addition to that, it was shown that, under a minimal set of constraints, the harmonic gauge was required to obtain a result, and any further gauge freedom was removed for plane waves. It would therefore be interesting, in future work, to obtain energy and angular momentum loss formulae without using spatial or temporal averaging to see if the results obtained for the waveform are dependent on this.

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Appendix

TT projector

The transverse-traceless symmetric projector $\mathcal{P}$ in the direction of unit 3-vector $\mathbf{n}$ is defined by

$$A_{ij}^{TT} = \mathcal{P}A_{ij} = P_{ijkl}(\mathbf{n})A_{kl} \tag{53}$$

The result is a transverse projector as it is built from vector transverse projectors, $P_{ij}(\mathbf{n}) = \delta_{ij} - n_i n_j$,

$$P_{ijkl} = aP_{ij}P_{kl} + bP_{il}P_{jk} + cP_{ik}P_{jl} \tag{54}$$

By imposing the conditions that it is symmetric in the final two indices (acts on symmetric matrices), that the contraction on the first two indices is 0 (projects to traceless), and that it is a projector, $P_{ijkl}P_{klmn} = P_{ijmn}$, one has that

$$P_{ijkl} = \frac{1}{2}\left(P_{ik}P_{jl} + P_{il}P_{jk} - \frac{2}{D-2}P_{ij}P_{kl}\right) \tag{55}$$

Spherical integrals

$$\int d\Omega_{D-2}\ n_i n_j = \frac{\Omega_{D-2}}{D-1}\delta_{ij} \tag{56}$$

$$\int d\Omega_{D-2}\ n_i n_j n_k n_l = \frac{\Omega_{D-2}}{(D-1)(D+1)}\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \tag{57}$$

$$\int d\Omega_{D-2}\ n_{i_1}n_{i_2}...n_{i_{2m}} = \frac{\Omega_{D-2}}{(D-1)(D+1)...(D+2m-3)}\left(\delta_{i_1 i_2}\delta_{i_3 i_4}...\delta_{i_{2m-1}i_{2m}} + ...\right) \tag{58}$$

$$\Omega_{D-2} = \frac{2\pi^{(D-1)/2}}{\Gamma[(D-1/)2]} \tag{59}$$

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