Distributionally Robust State Estimation

(Ph.D. Oral Defense)

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Content

Content

A motivating example

Research on state estimation is active in many fields, e.g., astronautics, robotics, reliability engineering, geodesy, power system.

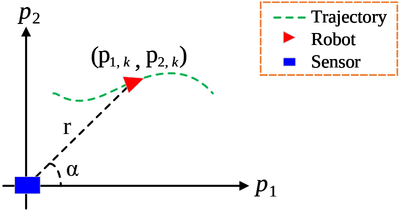

Figure: A 2-dimensional robot tracking problem.

- **Aim** to infer the real-time positions and velocities of the moving robot with observable information from a sensor.
- At time k, the sensor can capture the noisy value of $\mathbf{p}_k$, or the noisy values of some transforms of $\mathbf{p}_k$. Typical transforms include:
    - The range r.
    - The azimuth α.

A motivating example

According to basic kinematics, we have

$$\begin{bmatrix} \mathbf{p}_k \\ \boldsymbol{v}_k \end{bmatrix} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p}_{k-1} \\ \boldsymbol{v}_{k-1} \end{bmatrix} + \begin{bmatrix} \frac{\Delta t^2}{2} \\ \Delta t \end{bmatrix} \mathbf{a}_{k-1}.$$

- Δt denotes the sampling time between the time instant $k-1$ and the time instant k.
- $\boldsymbol{v}_{k-1}$ is the average velocity in-between the time instants $k-1$ and k.
- $\mathbf{a}_{k-1}$ is the average acceleration during the same time slot.
- But $\forall k$, $\mathbf{p}_k$, $\mathbf{p}_{k-1}$, $\boldsymbol{v}_{k-1}$, and $\mathbf{a}_{k-1}$ are all unknown to us.

A motivating example

The **process dynamics equation** (also known as the state evolution equation or the state transition equation)

$$\mathbf{x}_k = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \mathbf{x}_{k-1} + \begin{bmatrix} \frac{\Delta t^2}{2} \\ \Delta t \end{bmatrix} \mathbf{w}_{k-1},$$

- $\mathbf{x}_k := [\mathbf{p}_k^\top, \boldsymbol{v}_k^\top]^\top$ and $\mathbf{x}_k$ is termed the **state** vector.
- We are using a random vector $\mathbf{w}_{k-1}$ to model the unknown acceleration $\mathbf{a}_{k-1}$.
- $\mathbf{w}_{k-1}$ is the **process noise** vector.

A motivating example

The **measurement dynamics** equation (also known as the state measurement equation or the state observation equation) might be

$$\mathbf{y}_k = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_k \\ \boldsymbol{v}_k \end{bmatrix} + \mathbf{v}_k = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}_k + \mathbf{v}_k.$$

or, if a different sensor is used, it might be

$$\mathbf{y}_k = \begin{bmatrix} \mathrm{r} \\ \alpha \end{bmatrix} + \mathbf{v}_k = \begin{bmatrix} \sqrt{\mathrm{p}_{1,k}^2 + \mathrm{p}_{2,k}^2} \\ \arctan\left(\frac{\mathrm{p}_{2,k}}{\mathrm{p}_{1,k}}\right) \end{bmatrix} + \mathbf{v}_k.$$

- $\mathbf{y}_k$ is the **measurement** vector
- $\mathbf{v}_k$ is the **measurement noise** vector.

Linear Systems

A **linear system** is given as

$$\begin{cases} \mathbf{x}_k & = \boldsymbol{F}_{k-1}\mathbf{x}_{k-1} + \boldsymbol{G}_{k-1}\mathbf{w}_{k-1}, \\ \mathbf{y}_k & = \boldsymbol{H}_k\mathbf{x}_k + \mathbf{v}_k, \end{cases}$$

In the contexts of the robot tracking problem above, we specifically have

$$\boldsymbol{F}_{k-1} := \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{G}_{k-1} := \begin{bmatrix} \frac{\Delta t^2}{2} \\ \Delta t \end{bmatrix}, \quad \text{and} \quad \boldsymbol{H}_k := \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Nonlinear Systems

A **nonlinear system** is given as

$$\begin{cases} \mathbf{x}_k &= \boldsymbol{f}_k(\mathbf{x}_{k-1}, \mathbf{w}_{k-1}), \\ \mathbf{y}_k &= \boldsymbol{h}_k(\mathbf{x}_k, \mathbf{v}_k), \end{cases}$$

where $\boldsymbol{f}_k(\cdot,\cdot)$ and $\boldsymbol{h}_k(\cdot,\cdot)$ are termed the **process dynamics** function and the **measurement dynamics** function, respectively. In the contexts of the robot tracking problem above, we specifically have

$$\mathbf{x}_k = \boldsymbol{f}_k(\mathbf{x}_{k-1}, \mathbf{w}_{k-1}) := \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \mathbf{x}_{k-1} + \begin{bmatrix} \frac{\Delta t^2}{2} \\ \Delta t \end{bmatrix} \mathbf{w}_{k-1},$$

where $\boldsymbol{f}_k(\cdot,\cdot)$ degenerates to a linear form and

$$\mathbf{y}_k = \boldsymbol{h}_k(\mathbf{x}_k, \mathbf{v}_k) := \begin{bmatrix} \sqrt{\mathrm{p}_{1,k}^2 + \mathrm{p}_{2,k}^2} \\ \arctan\left(\frac{\mathrm{p}_{2,k}}{\mathrm{p}_{1,k}}\right) \end{bmatrix} + \mathbf{v}_k,$$

where $\boldsymbol{h}_k(\cdot,\cdot)$ is of a nonlinear form.

Problem Statement

- Definition: State estimation is to estimate the unknown state $\mathbf{x}_k$ based on observable information $\{\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_k\}$ and the linear/nonlinear system dynamics, for every discrete time index k.

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- Consequences: Optimal state estimator designed for the nominal model degrades or even diverges.
- Philosophy: Robust solutions insensitive to model mismatches are expected.

Literature Review

For linear systems.

- Unknown-input filters, [8] etc.:

$$\begin{cases} \mathbf{x}_k &= \boldsymbol{F}_{k-1}\mathbf{x}_{k-1} + \boldsymbol{\Gamma}_{k-1}\mathbf{d}_{k-1} + \boldsymbol{G}_{k-1}\mathbf{w}_{k-1}, \\ \mathbf{y}_k &= \boldsymbol{H}_k\mathbf{x}_k + \mathbf{v}_k, \end{cases}$$

where $\mathbf{d}_{k-1} \in \mathbb{R}^q$ is the unknown input used to describe the parameter uncertainties. Q: But need to elegantly specify $\boldsymbol{\Gamma}_{k-1}$.

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- In [16, 19] etc.,

$$\begin{cases} \mathbf{x}_k &= (\boldsymbol{F}_{k-1} + \delta\boldsymbol{F}_{k-1})\mathbf{x}_{k-1} + (\boldsymbol{G}_{k-1} + \delta\boldsymbol{G}_{k-1})\mathbf{w}_{k-1}, \\ \mathbf{y}_k &= \boldsymbol{H}_k\mathbf{x}_k + \mathbf{v}_k, \end{cases}$$

where $\delta\boldsymbol{F}_{k-1}$ and $\delta\boldsymbol{G}_{k-1}$ are used to model the perturbations imposed on the nominal system matrices $\boldsymbol{F}_{k-1}$ and $\boldsymbol{G}_{k-1}$, respectively. Q: But need to elegantly specify $\delta\boldsymbol{F}_{k-1}$ and $\delta\boldsymbol{G}_{k-1}$.

Literature Review

For linear systems (continued).

- Distributionally Robust Estimation Using Wasserstein Metric [1]:
  At each k, assuming true joint distribution of $(\mathbf{x}_k, \mathbf{y}_k) \sim \mathbb{P}$ is in

$$\mathcal{F}_{\mathbf{x}_k,\mathbf{y}_k|\mathcal{Y}_{k-1}}(\theta) := \left\{ \mathbb{P} \in \mathcal{N}_{n+m} \,\middle|\, \mathrm{W}(\mathbb{P}, \ \bar{\mathbb{P}}_{\mathbf{x}_k,\mathbf{y}_k|\mathcal{Y}_{k-1}}) \leq \theta \right\}$$

  where W defines Wasserstein distance and $\bar{\mathbb{P}}_{\mathbf{x}_k,\mathbf{y}_k|\mathcal{Y}_{k-1}}$ is the nominal distribution (which is Gaussian). $\mathcal{N}_{n+m}$: Gaussian distributions.
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- In [9, 18, 15] etc., for linear systems

  $$\begin{cases} \mathbf{x}_k & = \boldsymbol{F}_{k-1}\mathbf{x}_{k-1} + \boldsymbol{G}_{k-1}\mathbf{w}_{k-1}, \\ \mathbf{y}_k & = \boldsymbol{H}_k\mathbf{x}_k + \mathbf{v}_k, \end{cases}$$

  the distribution of $\mathbf{v}_k$ is assumed to be t-distributed, Laplacian, etc.
  Q: But cannot handle parameter uncertainties.

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- Particle filters are most popular.
- Over the years, efforts are put on designing efficient sampling and resampling techniques [4, 13, 5, 12].
- Virtually all of the past literature assume that the process dynamics and measurement dynamics are accurate.
- Q: There is no literature addressing model uncertainties and measurement outliers in particle filtering.

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  - For linear systems, the robust state estimators should be insensitive to both parameter uncertainties and measurement outliers.
  - For nonlinear systems, we focus on robustifying the particle filter so that it can be insensitive to model uncertainties and measurement outliers.
- Methodology: We leverage Distributionally Robust Optimization Theories [7] and Robust Statistics Theories [10].

Content

Aim: to estimate the hidden state vector $\mathbf{x}_k$ of a linear Markov system given the measurement set $\{\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_k\}$.

$$\begin{cases} \mathbf{x}_k &= \boldsymbol{F}_{k-1}\mathbf{x}_{k-1} + \boldsymbol{G}_{k-1}\mathbf{w}_{k-1}, \\ \mathbf{y}_k &= \boldsymbol{H}_k\mathbf{x}_k + \mathbf{v}_k, \end{cases}$$

- where k is the discrete time index.
- $\mathbf{x}_k$, $\mathbf{y}_k$, $\mathbf{w}_k$, and $\mathbf{v}_k$ have finite second moments.
- In literature, typical assumptions:
  1. $\mathbf{w}_k \sim \mathcal{N}_p(\mathbf{0}, \boldsymbol{Q}_k)$, and $\mathbf{v}_k \sim \mathcal{N}_m(\mathbf{0}, \boldsymbol{R}_k)$; Hence, no measurement outliers modeled.
  2. $\boldsymbol{Q}_k$, $\boldsymbol{R}_k$, $\boldsymbol{F}_{k-1}$, $\boldsymbol{G}_{k-1}$, and $\boldsymbol{H}_k$ are exactly known.

For every discrete time index $k = 1, 2, ...,$ let

$$\mathcal{Y}_k := (\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_k)$$

denote the measurement set.

Let

$$\mathcal{H}'_{\mathcal{Y}_k} := \left\{ \phi(\mathbf{y}_1, ..., \mathbf{y}_k) \;\middle|\; \begin{array}{l} \phi : \underbrace{\mathbb{R}^m \times \cdots \times \mathbb{R}^m}_{k} \to \mathbb{R}^n \\ \phi \text{ is Borel-measurable} \\ \int_{\mathbb{R}^{m \times k}} [\phi(\boldsymbol{Y}_k)]^\top [\phi(\boldsymbol{Y}_k)] \mathrm{d}\mathbb{P}_{\mathcal{Y}_k}(\boldsymbol{Y}_k) < \infty \end{array} \right\}.$$

Intuitively, $\mathcal{H}'_{\mathcal{Y}_k}$ contains all possible estimator of $\mathbf{x}_k$:

$$\hat{\mathbf{x}}_k = \phi(\mathbf{y}_1, ..., \mathbf{y}_k), \quad \forall k.$$

- Suppose the nominal joint state-measurement distribution at time k is $\bar{\mathbb{P}}_{\mathbf{x}_k,\mathcal{Y}_k}$. We would like to solve the following optimization problem

$$\min_{\phi \in \mathcal{H}'_{\mathcal{Y}_k}} \operatorname{Tr} \mathbb{E}[\mathbf{x}_k - \phi(\mathcal{Y}_k)][\mathbf{x}_k - \phi(\mathcal{Y}_k)]^\top,$$

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- Expectation is taken over $\bar{\mathbb{P}}_{\mathbf{x}_k,\mathcal{Y}_k}$ and $\phi(\cdot)$ is referred to as an estimator (the optimal one is called the optimal estimator).
- The optimal estimator of $\mathbf{x}_k$ in this minimum mean square error sense is $\mathbb{E}(\mathbf{x}_k|\mathcal{Y}_k)$.

- Question: What if the true joint state-measurement distribution at time k, i.e., $\mathbb{P}_{\mathbf{x}_k, \mathcal{Y}_k}$, deviates from the nominal $\bar{\mathbb{P}}_{\mathbf{x}_k, \mathcal{Y}_k}$?

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- Motivation: We study the distributionally robust counterpart:

$$\min_{\phi \in \mathcal{H}'_{\mathcal{Y}_k}} \max_{\mathbb{P} \in \mathcal{F}_{\mathbf{x}_k,\mathcal{Y}_k}(\theta)} \operatorname{Tr} \mathbb{E}[\mathbf{x}_k - \phi(\mathcal{Y}_k)][\mathbf{x}_k - \phi(\mathcal{Y}_k)]^\top,$$

Linear System Case

Problem Background, Statements, and Motivations

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  - $D(\cdot,\cdot)$ is a possible statistical similarity measure, e.g., Wasserstein distance, Kullback–Leibler divergence.
  - We do not know the true distribution, but we assume that it lies in a ball centered at the nominal distribution.

Recap: $$\min_{\phi \in \mathcal{H}'_{\mathcal{Y}_k}} \max_{\mathbb{P} \in \mathcal{F}_{\mathbf{x}_k, \mathcal{Y}_k}(\theta)} \operatorname{Tr} \mathbb{E}[\mathbf{x}_k - \phi(\mathcal{Y}_k)][\mathbf{x}_k - \phi(\mathcal{Y}_k)]^\top,$$

- Issue: What if measurements $\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_k$ sequentially arrives along k?

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- Issue: What if measurements $\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_k$ sequentially arrives along k?
- We study time-incremental (a.k.a. time-series, online) version:

$$\min_{\phi \in \mathcal{H}'_{\mathbf{y}_k}} \max_{\mathbb{P} \in \mathcal{F}_{\mathbf{x}_k, \mathbf{y}_k | \mathcal{Y}_{k-1}}(\theta)} \operatorname{Tr} \mathbb{E} \left\{ [\mathbf{x}_k - \phi(\mathbf{y}_k)][\mathbf{x}_k - \phi(\mathbf{y}_k)]^\top \middle| \mathcal{Y}_{k-1} \right\},$$

where the ambiguity set is defined as

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Recap: $$\min_{\phi \in \mathcal{H}'_{\mathbf{y}_k}} \max_{\mathbb{P} \in \mathcal{F}_{\mathbf{x}_k, \mathbf{y}_k | \mathcal{Y}_{k-1}}(\theta)} \operatorname{Tr} \mathbb{E} \left\{ [\mathbf{x}_k - \phi(\mathbf{y}_k)][\mathbf{x}_k - \phi(\mathbf{y}_k)]^\top \middle| \mathcal{Y}_{k-1} \right\},$$

- It can be proved that the min-max problem is equivalent to the max-min problem below

$$\max_{\mathbb{P} \in \mathcal{F}_{\mathbf{x}_k, \mathbf{y}_k | \mathcal{Y}_{k-1}}(\theta)} \min_{\phi \in \mathcal{H}'_{\mathbf{y}_k}} \operatorname{Tr} \mathbb{E} \left\{ [\mathbf{x}_k - \phi(\mathbf{y}_k)][\mathbf{x}_k - \phi(\mathbf{y}_k)]^\top \middle| \mathcal{Y}_{k-1} \right\},$$

  - Intuition: The objective is positive-definite quadratic (thus convex) in ϕ, and linear (thus concave) in $\mathbb{P}$.

Recap: $$\min_{\phi \in \mathcal{H}'_{\mathbf{y}_k}} \max_{\mathbb{P} \in \mathcal{F}_{\mathbf{x}_k, \mathbf{y}_k | \mathcal{Y}_{k-1}}(\theta)} \operatorname{Tr} \mathbb{E} \left\{ [\mathbf{x}_k - \phi(\mathbf{y}_k)][\mathbf{x}_k - \phi(\mathbf{y}_k)]^\top \middle| \mathcal{Y}_{k-1} \right\},$$

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$$\max_{\mathbb{P} \in \mathcal{F}_{\mathbf{x}_k, \mathbf{y}_k | \mathcal{Y}_{k-1}}(\theta)} \min_{\phi \in \mathcal{H}'_{\mathbf{y}_k}} \operatorname{Tr} \mathbb{E} \left\{ [\mathbf{x}_k - \phi(\mathbf{y}_k)][\mathbf{x}_k - \phi(\mathbf{y}_k)]^\top \middle| \mathcal{Y}_{k-1} \right\},$$

  - Intuition: The objective is positive-definite quadratic (thus convex) in ϕ, and linear (thus concave) in $\mathbb{P}$.
  - Fact: The saddle point exists.

Recap: $$\min_{\phi \in \mathcal{H}'_{\mathbf{y}_k}} \max_{\mathbb{P} \in \mathcal{F}_{\mathbf{x}_k, \mathbf{y}_k | \mathcal{Y}_{k-1}}(\theta)} \operatorname{Tr} \mathbb{E} \left\{ [\mathbf{x}_k - \phi(\mathbf{y}_k)][\mathbf{x}_k - \phi(\mathbf{y}_k)]^\top \middle| \mathcal{Y}_{k-1} \right\},$$

- It can be proved that the min-max problem is equivalent to the max-min problem below

$$\max_{\mathbb{P} \in \mathcal{F}_{\mathbf{x}_k, \mathbf{y}_k | \mathcal{Y}_{k-1}}(\theta)} \min_{\phi \in \mathcal{H}'_{\mathbf{y}_k}} \operatorname{Tr} \mathbb{E} \left\{ [\mathbf{x}_k - \phi(\mathbf{y}_k)][\mathbf{x}_k - \phi(\mathbf{y}_k)]^\top \middle| \mathcal{Y}_{k-1} \right\},$$

  - Intuition: The objective is positive-definite quadratic (thus convex) in ϕ, and linear (thus concave) in $\mathbb{P}$.
  - Fact: The saddle point exists.

- **Note**. The latter is easier to solve because for every $\mathbb{P}$, we can find the associated optimal estimator, i.e., associated conditional mean.

Recap:
$$\max_{\mathbb{P}\in\mathcal{F}_{\mathbf{x}_k,\mathbf{y}_k|\mathcal{Y}_{k-1}}(\theta)} \min_{\phi\in\mathcal{H}'_{\mathbf{y}_k}} \operatorname{Tr}\mathbb{E}\left\{[\mathbf{x}_k-\phi(\mathbf{y}_k)][\mathbf{x}_k-\phi(\mathbf{y}_k)]^\top \middle| \mathcal{Y}_{k-1}\right\},$$

- Therefore, at each k, we are inspired to **first** study a one-stage distributionally robust Bayesian estimation problem

$$\max_{\mathbb{P}\in\mathcal{F}_{\mathbf{x},\mathbf{y}}(\theta)} \min_{\phi\in\mathcal{H}'_{\mathbf{y}}} \operatorname{Tr}\mathbb{E}[\mathbf{x}-\phi(\mathbf{y})][\mathbf{x}-\phi(\mathbf{y})]^\top$$

subject to

Recap: $$\max_{\mathbb{P}\in\mathcal{F}_{\mathbf{x}_k,\mathbf{y}_k|\mathcal{Y}_{k-1}}(\theta)} \min_{\phi\in\mathcal{H}'_{\mathbf{y}_k}} \operatorname{Tr}\mathbb{E}\left\{[\mathbf{x}_k-\phi(\mathbf{y}_k)][\mathbf{x}_k-\phi(\mathbf{y}_k)]^\top \middle| \mathcal{Y}_{k-1}\right\},$$

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$$\max_{\mathbb{P}\in\mathcal{F}_{\mathbf{x},\mathbf{y}}(\theta)} \min_{\phi\in\mathcal{H}'_{\mathbf{y}}} \operatorname{Tr}\mathbb{E}[\mathbf{x}-\phi(\mathbf{y})][\mathbf{x}-\phi(\mathbf{y})]^\top$$

subject to

- the nominal prior state distribution $\bar{\mathbb{P}}_{\mathbf{x}}$,

Recap: $$\max_{\mathbb{P}\in\mathcal{F}_{\mathbf{x}_k,\mathbf{y}_k|\mathcal{Y}_{k-1}}(\theta)} \min_{\phi\in\mathcal{H}'_{\mathbf{y}_k}} \operatorname{Tr}\mathbb{E}\left\{[\mathbf{x}_k-\phi(\mathbf{y}_k)][\mathbf{x}_k-\phi(\mathbf{y}_k)]^\top \middle| \mathcal{Y}_{k-1}\right\},$$

- Therefore, at each k, we are inspired to **first** study a one-stage distributionally robust Bayesian estimation problem

$$\max_{\mathbb{P}\in\mathcal{F}_{\mathbf{x},\mathbf{y}}(\theta)} \min_{\phi\in\mathcal{H}'_{\mathbf{y}}} \operatorname{Tr}\mathbb{E}[\mathbf{x}-\phi(\mathbf{y})][\mathbf{x}-\phi(\mathbf{y})]^\top$$

subject to

- the nominal prior state distribution $\bar{\mathbb{P}}_{\mathbf{x}}$,
- the nominal conditional measurement distribution $\bar{\mathbb{P}}_{\mathbf{y}|\mathbf{x}}$,

Recap: $$\max_{\mathbb{P}\in\mathcal{F}_{\mathbf{x}_k,\mathbf{y}_k|\mathcal{Y}_{k-1}}(\theta)} \min_{\phi\in\mathcal{H}'_{\mathbf{y}_k}} \operatorname{Tr}\mathbb{E}\left\{[\mathbf{x}_k-\phi(\mathbf{y}_k)][\mathbf{x}_k-\phi(\mathbf{y}_k)]^\top \middle| \mathcal{Y}_{k-1}\right\},$$

- Therefore, at each k, we are inspired to **first** study a one-stage distributionally robust Bayesian estimation problem

$$\max_{\mathbb{P}\in\mathcal{F}_{\mathbf{x},\mathbf{y}}(\theta)} \min_{\phi\in\mathcal{H}'_{\mathbf{y}}} \operatorname{Tr}\mathbb{E}[\mathbf{x}-\phi(\mathbf{y})][\mathbf{x}-\phi(\mathbf{y})]^\top$$

subject to

- the nominal prior state distribution $\bar{\mathbb{P}}_{\mathbf{x}}$,
- the nominal conditional measurement distribution $\bar{\mathbb{P}}_{\mathbf{y}|\mathbf{x}}$,
- a properly constructed ambiguity set $\mathcal{F}_{\mathbf{x},\mathbf{y}}(\theta)$ that contains fat-tailed (marginal) distributions for $\mathbf{y}$,

Recap: $$\max_{\mathbb{P} \in \mathcal{F}_{\mathbf{x}_k, \mathbf{y}_k | \mathcal{Y}_{k-1}}(\theta)} \min_{\boldsymbol{\phi} \in \mathcal{H}'_{\mathbf{y}_k}} \operatorname{Tr} \mathbb{E} \left\{ [\mathbf{x}_k - \boldsymbol{\phi}(\mathbf{y}_k)][\mathbf{x}_k - \boldsymbol{\phi}(\mathbf{y}_k)]^\top \middle| \mathcal{Y}_{k-1} \right\},$$

- Therefore, at each k, we are inspired to **first** study a one-stage distributionally robust Bayesian estimation problem

$$\max_{\mathbb{P} \in \mathcal{F}_{\mathbf{x},\mathbf{y}}(\theta)} \min_{\boldsymbol{\phi} \in \mathcal{H}'_{\mathbf{y}}} \operatorname{Tr} \mathbb{E} [\mathbf{x} - \boldsymbol{\phi}(\mathbf{y})][\mathbf{x} - \boldsymbol{\phi}(\mathbf{y})]^\top$$

subject to

- the nominal prior state distribution $\bar{\mathbb{P}}_{\mathbf{x}}$,
- the nominal conditional measurement distribution $\bar{\mathbb{P}}_{\mathbf{y}|\mathbf{x}}$,
- a properly constructed ambiguity set $\mathcal{F}_{\mathbf{x},\mathbf{y}}(\theta)$ that contains fat-tailed (marginal) distributions for $\mathbf{y}$,
- the linear measurement equation $\mathbf{y} = \boldsymbol{H}\mathbf{x} + \mathbf{v}$.

Recap: $$\max_{\mathbb{P}\in\mathcal{F}_{\mathbf{x}_k,\mathbf{y}_k|\mathcal{Y}_{k-1}}(\theta)} \min_{\phi\in\mathcal{H}'_{\mathbf{y}_k}} \operatorname{Tr}\mathbb{E}\left\{[\mathbf{x}_k-\phi(\mathbf{y}_k)][\mathbf{x}_k-\phi(\mathbf{y}_k)]^\top \middle| \mathcal{Y}_{k-1}\right\},$$

- Therefore, at each k, we are inspired to **first** study a one-stage distributionally robust Bayesian estimation problem

$$\max_{\mathbb{P}\in\mathcal{F}_{\mathbf{x},\mathbf{y}}(\theta)} \min_{\phi\in\mathcal{H}'_{\mathbf{y}}} \operatorname{Tr}\mathbb{E}[\mathbf{x}-\phi(\mathbf{y})][\mathbf{x}-\phi(\mathbf{y})]^\top$$

  subject to
  - the nominal prior state distribution $\bar{\mathbb{P}}_{\mathbf{x}}$,
  - the nominal conditional measurement distribution $\bar{\mathbb{P}}_{\mathbf{y}|\mathbf{x}}$,
  - a properly constructed ambiguity set $\mathcal{F}_{\mathbf{x},\mathbf{y}}(\theta)$ that contains fat-tailed (marginal) distributions for $\mathbf{y}$,
  - the linear measurement equation $\mathbf{y} = \boldsymbol{H}\mathbf{x} + \mathbf{v}$.
- **Then**, by identifying $\mathbb{P}_{\mathbf{x}_k,\mathbf{y}_k|\mathcal{Y}_{k-1}}$, we can solve the distributionally robust state estimation problem.

Theorem 1

Consider the joint distribution $\mathbb{P}_{\mathbf{x},\mathbf{y}}$ *and* $\mathbf{y} = \boldsymbol{H}\mathbf{x} + \mathbf{v}$. *Suppose* $\mathbf{x} \sim \mathcal{N}_n(\boldsymbol{c_x}, \boldsymbol{\Sigma_x})$, *the mean of* $\mathbf{v}$ *is* $\boldsymbol{c_v}$, *the covariance of* $\mathbf{v}$ *is* $\boldsymbol{\Sigma_v}$, $\mathbf{x}$ *is independent of* $\mathbf{v}$, *all involved densities exist and twice continuously differentiable.*
Let $\mathbf{e} := \mathbf{y} - \boldsymbol{H}\boldsymbol{c_x} - \boldsymbol{c_v}$ *denote the innovation vector,* $\boldsymbol{S}$ *the covariance of* $\mathbf{e}$, *and* $\mathbf{u} := \boldsymbol{S}^{-1/2}\mathbf{e}$ *the diagonalized and normalized innovation.*
Then the MMSE optimal estimator $\hat{\mathbf{x}}$ *of* $\mathbf{x}$ *given* $\mathbf{y}$, *i.e.,* $\mathbb{E}(\mathbf{x}|\mathbf{y})$, *is*

$$\hat{\mathbf{x}} = \boldsymbol{c_x} + \boldsymbol{\Sigma_x}\boldsymbol{H}^\top\boldsymbol{S}^{-1/2}\left[-\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\mu}}\ln p_{\mathbf{u}}(\boldsymbol{\mu})\right]_{\boldsymbol{\mu}=\mathbf{u}}.$$

and the estimation error covariance, i.e., $\boldsymbol{P} := \mathbb{E}(\hat{\mathbf{x}} - \mathbf{x})(\hat{\mathbf{x}} - \mathbf{x})^\top$, *is*

$$\boldsymbol{P} = \boldsymbol{\Sigma_x} - \boldsymbol{\Sigma_x}\boldsymbol{H}^\top\boldsymbol{S}^{-1}\boldsymbol{H}\boldsymbol{\Sigma_x} \cdot \mathbb{E}\left[-\frac{\mathrm{d}^2}{\mathrm{d}\mu^2}\ln p_{\mathrm{u}}(\mu)\Big|_{\mu=\mathrm{u}}\right],$$

where $\boldsymbol{S} = \boldsymbol{H}\boldsymbol{\Sigma_x}\boldsymbol{H}^\top + \boldsymbol{\Sigma_v}$, $p_{\mathbf{u}}(\boldsymbol{\mu}) = p_{\mathbf{y}}(\boldsymbol{S}^{1/2}\boldsymbol{\mu} + \boldsymbol{H}\boldsymbol{c_x} + \boldsymbol{c_v}) \cdot \det(\boldsymbol{S}^{1/2})$ *is the density of* $\mathbf{u}$, $p_{\mathbf{y}}(\cdot)$ *is the density of* $\mathbf{y}$, *and* $\det(\cdot)$ *denotes the determinant of a matrix.*

Linear System Case

Solve Distributionally Robust Bayesian Estimation Problem

Recap: $$\max_{\mathbb{P}\in\mathcal{F}_{\mathbf{x},\mathbf{y}}(\theta)} \min_{\boldsymbol{\phi}\in\mathcal{H}'_{\mathbf{y}}} \operatorname{Tr}\mathbb{E}[\mathbf{x}-\boldsymbol{\phi}(\mathbf{y})][\mathbf{x}-\boldsymbol{\phi}(\mathbf{y})]^\top$$

Corollary 1 (Reducing to finding worst-case distribution)

The distributionally robust Bayesian estimation can be reformulated as

$$\max_{\mathbb{P}\in\mathcal{F}_{\mathbf{x},\mathbf{y}}(\theta)} \operatorname{Tr}\boldsymbol{P},$$

where

$$\boldsymbol{P} = \boldsymbol{\Sigma}_{\boldsymbol{x}} - \boldsymbol{\Sigma}_{\boldsymbol{x}}\boldsymbol{H}^\top(\boldsymbol{H}\boldsymbol{\Sigma}_{\boldsymbol{x}}\boldsymbol{H}^\top + \boldsymbol{\Sigma}_{\boldsymbol{v}})^{-1}\boldsymbol{H}\boldsymbol{\Sigma}_{\boldsymbol{x}} \cdot \mathbb{E}\left[-\frac{\mathrm{d}^2}{\mathrm{d}\mu^2}\ln p_{\mathrm{u}}(\mu)\bigg|_{\mu=\mathrm{u}}\right].$$

Because for every possible joint distribution $\mathbb{P}_{\mathbf{x},\mathbf{y}}$,

$$\hat{\mathbf{x}} = \boldsymbol{\phi}(\mathbf{y}) = \boldsymbol{c}_{\boldsymbol{x}} + \boldsymbol{\Sigma}_{\boldsymbol{x}}\boldsymbol{H}^\top(\boldsymbol{H}\boldsymbol{\Sigma}_{\boldsymbol{x}}\boldsymbol{H}^\top + \boldsymbol{\Sigma}_{\boldsymbol{v}})^{-1/2}\left[-\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\mu}}\ln p_{\mathbf{u}}(\boldsymbol{\mu})\right]_{\boldsymbol{\mu}=\mathbf{u}}$$

is uniquely determined.

Recap:
$$\max_{\mathbb{P}\in\mathcal{F}_{\mathbf{x},\mathbf{y}}(\theta)} \operatorname{Tr} \boldsymbol{\Sigma}_{\boldsymbol{x}} - \boldsymbol{\Sigma}_{\boldsymbol{x}}\boldsymbol{H}^\top(\boldsymbol{H}\boldsymbol{\Sigma}_{\boldsymbol{x}}\boldsymbol{H}^\top + \boldsymbol{\Sigma}_{\boldsymbol{v}})^{-1}\boldsymbol{H}\boldsymbol{\Sigma}_{\boldsymbol{x}} \cdot \mathbb{E}\left[-\frac{\mathrm{d}^2 \ln p_{\mathrm{u}}(\mu)}{\mathrm{d}\mu^2}\bigg|_{\mu=\mathrm{u}} \right]$$

- We need to define the ambiguity set $\mathcal{F}_{\mathbf{x},\mathbf{y}}(\theta)$, which contains fat-tailed distributions for $\mathbf{y}$. We call every $\mathbb{P}$ in $\mathcal{F}_{\mathbf{x},\mathbf{y}}(\theta)$ a **candidate**.

Linear System Case

Solve Distributionally Robust Bayesian Estimation Problem

Recap: $$\max_{\mathbb{P}\in\mathcal{F}_{\mathbf{x},\mathbf{y}}(\theta)} \operatorname{Tr} \boldsymbol{\Sigma}_{\boldsymbol{x}} - \boldsymbol{\Sigma}_{\boldsymbol{x}}\boldsymbol{H}^\top(\boldsymbol{H}\boldsymbol{\Sigma}_{\boldsymbol{x}}\boldsymbol{H}^\top + \boldsymbol{\Sigma}_{\boldsymbol{v}})^{-1}\boldsymbol{H}\boldsymbol{\Sigma}_{\boldsymbol{x}} \cdot \mathbb{E}\left[-\frac{\mathrm{d}^2 \ln p_{\mathrm{u}}(\mu)}{\mathrm{d}\mu^2}\bigg|_{\mu=\mathrm{u}} \right]$$

- We need to define the ambiguity set $\mathcal{F}_{\mathbf{x},\mathbf{y}}(\theta)$, which contains fat-tailed distributions for $\mathbf{y}$. We call every $\mathbb{P}$ in $\mathcal{F}_{\mathbf{x},\mathbf{y}}(\theta)$ a **candidate**.

Recap: $$\max_{\mathbb{P}\in\mathcal{F}_{\mathbf{x},\mathbf{y}}(\theta)} \operatorname{Tr} \boldsymbol{\Sigma}_{\boldsymbol{x}} - \boldsymbol{\Sigma}_{\boldsymbol{x}}\boldsymbol{H}^\top(\boldsymbol{H}\boldsymbol{\Sigma}_{\boldsymbol{x}}\boldsymbol{H}^\top + \boldsymbol{\Sigma}_{\boldsymbol{v}})^{-1}\boldsymbol{H}\boldsymbol{\Sigma}_{\boldsymbol{x}} \cdot \mathbb{E}\left[-\frac{\mathrm{d}^2 \ln p_{\mathrm{u}}(\mu)}{\mathrm{d}\mu^2}\bigg|_{\mu=\mathrm{u}} \right]$$

- We need to define the ambiguity set $\mathcal{F}_{\mathbf{x},\mathbf{y}}(\theta)$, which contains fat-tailed distributions for $\mathbf{y}$. We call every $\mathbb{P}$ in $\mathcal{F}_{\mathbf{x},\mathbf{y}}(\theta)$ a **candidate**.
- 1. Nominal $\bar{\mathbb{P}}_{\mathbf{x}} := \mathcal{N}_n(\bar{\boldsymbol{x}}, \boldsymbol{M})$ and candidate $\mathbb{P}_{\mathbf{x}} := \mathcal{D}_n(\boldsymbol{c}_{\boldsymbol{x}}, \boldsymbol{\Sigma}_{\boldsymbol{x}})$.

Recap: $$\max_{\mathbb{P}\in\mathcal{F}_{\mathbf{x},\mathbf{y}}(\theta)} \operatorname{Tr} \boldsymbol{\Sigma}_{\boldsymbol{x}} - \boldsymbol{\Sigma}_{\boldsymbol{x}} \boldsymbol{H}^\top (\boldsymbol{H}\boldsymbol{\Sigma}_{\boldsymbol{x}}\boldsymbol{H}^\top + \boldsymbol{\Sigma}_{\boldsymbol{v}})^{-1} \boldsymbol{H}\boldsymbol{\Sigma}_{\boldsymbol{x}} \cdot \mathbb{E}\left[-\left.\frac{\mathrm{d}^2 \ln p_{\mathrm{u}}(\mu)}{\mathrm{d}\mu^2}\right|_{\mu=\mathrm{u}} \right]$$

- We need to define the ambiguity set $\mathcal{F}_{\mathbf{x},\mathbf{y}}(\theta)$, which contains fat-tailed distributions for $\mathbf{y}$. We call every $\mathbb{P}$ in $\mathcal{F}_{\mathbf{x},\mathbf{y}}(\theta)$ a **candidate**.
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  1. Nominal $\bar{\mathbb{P}}_{\mathbf{x}} := \mathcal{N}_n(\bar{\boldsymbol{x}}, \boldsymbol{M})$ and candidate $\mathbb{P}_{\mathbf{x}} := \mathcal{D}_n(\boldsymbol{c}_{\boldsymbol{x}}, \boldsymbol{\Sigma}_{\boldsymbol{x}})$.
  2. Nominal $\bar{\mathbb{P}}_{\mathbf{v}} := \mathcal{N}_m(\mathbf{0}, \boldsymbol{R})$ and candidate $\mathbb{P}_{\mathbf{v}} := \mathcal{D}_m(\boldsymbol{c}_{\boldsymbol{v}}, \boldsymbol{\Sigma}_{\boldsymbol{v}})$.

Linear System Case

Solve Distributionally Robust Bayesian Estimation Problem

Recap: $$\max_{\mathbb{P}\in\mathcal{F}_{\mathbf{x},\mathbf{y}}(\theta)} \operatorname{Tr} \boldsymbol{\Sigma}_{\boldsymbol{x}} - \boldsymbol{\Sigma}_{\boldsymbol{x}} \boldsymbol{H}^\top (\boldsymbol{H}\boldsymbol{\Sigma}_{\boldsymbol{x}}\boldsymbol{H}^\top + \boldsymbol{\Sigma}_{\boldsymbol{v}})^{-1} \boldsymbol{H}\boldsymbol{\Sigma}_{\boldsymbol{x}} \cdot \mathbb{E}\left[- \frac{\mathrm{d}^2 \ln p_{\mathrm{u}}(\mu)}{\mathrm{d}\mu^2}\bigg|_{\mu=\mathrm{u}} \right]$$

- We need to define the ambiguity set $\mathcal{F}_{\mathbf{x},\mathbf{y}}(\theta)$, which contains fat-tailed distributions for $\mathbf{y}$. We call every $\mathbb{P}$ in $\mathcal{F}_{\mathbf{x},\mathbf{y}}(\theta)$ a **candidate**.
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  1. Nominal $\bar{\mathbb{P}}_{\mathbf{x}} := \mathcal{N}_n(\bar{\boldsymbol{x}}, \boldsymbol{M})$ and candidate $\mathbb{P}_{\mathbf{x}} := \mathcal{D}_n(\boldsymbol{c}_{\boldsymbol{x}}, \boldsymbol{\Sigma}_{\boldsymbol{x}})$.
  2. Nominal $\bar{\mathbb{P}}_{\mathbf{v}} := \mathcal{N}_m(\mathbf{0}, \boldsymbol{R})$ and candidate $\mathbb{P}_{\mathbf{v}} := \mathcal{D}_m(\boldsymbol{c}_{\boldsymbol{v}}, \boldsymbol{\Sigma}_{\boldsymbol{v}})$.
  3. Nominal $\bar{\mathbb{P}}_{\mathrm{u}}$ is a Standard Gaussian and candidate $\mathbb{P}_{\mathrm{u}}$.

Recap: $$\max_{\mathbb{P}\in\mathcal{F}_{\mathbf{x},\mathbf{y}}(\theta)} \operatorname{Tr} \boldsymbol{\Sigma}_{\boldsymbol{x}} - \boldsymbol{\Sigma}_{\boldsymbol{x}}\boldsymbol{H}^\top(\boldsymbol{H}\boldsymbol{\Sigma}_{\boldsymbol{x}}\boldsymbol{H}^\top + \boldsymbol{\Sigma}_{\boldsymbol{v}})^{-1}\boldsymbol{H}\boldsymbol{\Sigma}_{\boldsymbol{x}} \cdot \mathbb{E}\left[-\frac{\mathrm{d}^2 \ln p_{\mathrm{u}}(\mu)}{\mathrm{d}\mu^2}\bigg|_{\mu=\mathrm{u}} \right]$$

- We need to define the ambiguity set $\mathcal{F}_{\mathbf{x},\mathbf{y}}(\theta)$, which contains fat-tailed distributions for $\mathbf{y}$. We call every $\mathbb{P}$ in $\mathcal{F}_{\mathbf{x},\mathbf{y}}(\theta)$ a **candidate**.
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  1. Nominal $\bar{\mathbb{P}}_{\mathbf{x}} := \mathcal{N}_n(\bar{\boldsymbol{x}}, \boldsymbol{M})$ and candidate $\mathbb{P}_{\mathbf{x}} := \mathcal{D}_n(\boldsymbol{c}_{\boldsymbol{x}}, \boldsymbol{\Sigma}_{\boldsymbol{x}})$.
  2. Nominal $\bar{\mathbb{P}}_{\mathbf{v}} := \mathcal{N}_m(\mathbf{0}, \boldsymbol{R})$ and candidate $\mathbb{P}_{\mathbf{v}} := \mathcal{D}_m(\boldsymbol{c}_{\boldsymbol{v}}, \boldsymbol{\Sigma}_{\boldsymbol{v}})$.
  3. Nominal $\bar{\mathbb{P}}_{\mathrm{u}}$ is a Standard Gaussian and candidate $\mathbb{P}_{\mathrm{u}}$.
- Recall $\mathbf{u} = \boldsymbol{S}^{-1/2}(\mathbf{y} - \boldsymbol{H}\boldsymbol{c}_{\boldsymbol{x}} - \boldsymbol{c}_{\boldsymbol{v}})$ and $\mathbf{y} = (\boldsymbol{H}\boldsymbol{\Sigma}_{\boldsymbol{x}}\boldsymbol{H}^\top + \boldsymbol{\Sigma}_{\boldsymbol{v}})^{1/2}\mathbf{u} + \boldsymbol{H}\boldsymbol{c}_{\boldsymbol{x}} + \boldsymbol{c}_{\boldsymbol{v}}$

Recap: $$\max_{\mathbb{P}\in\mathcal{F}_{\mathbf{x},\mathbf{y}}(\theta)} \operatorname{Tr} \boldsymbol{\Sigma}_{\boldsymbol{x}} - \boldsymbol{\Sigma}_{\boldsymbol{x}} \boldsymbol{H}^\top (\boldsymbol{H}\boldsymbol{\Sigma}_{\boldsymbol{x}}\boldsymbol{H}^\top + \boldsymbol{\Sigma}_{\boldsymbol{v}})^{-1} \boldsymbol{H}\boldsymbol{\Sigma}_{\boldsymbol{x}} \cdot \mathbb{E}\left[-\frac{\mathrm{d}^2 \ln p_{\mathrm{u}}(\mu)}{\mathrm{d}\mu^2}\bigg|_{\mu=\mathrm{u}} \right]$$

- We need to define the ambiguity set $\mathcal{F}_{\mathbf{x},\mathbf{y}}(\theta)$, which contains fat-tailed distributions for $\mathbf{y}$. We call every $\mathbb{P}$ in $\mathcal{F}_{\mathbf{x},\mathbf{y}}(\theta)$ a **candidate**.
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  1. Nominal $\bar{\mathbb{P}}_{\mathbf{x}} := \mathcal{N}_n(\bar{\boldsymbol{x}}, \boldsymbol{M})$ and candidate $\mathbb{P}_{\mathbf{x}} := \mathcal{D}_n(\boldsymbol{c}_{\boldsymbol{x}}, \boldsymbol{\Sigma}_{\boldsymbol{x}})$.
  2. Nominal $\bar{\mathbb{P}}_{\mathbf{v}} := \mathcal{N}_m(\mathbf{0}, \boldsymbol{R})$ and candidate $\mathbb{P}_{\mathbf{v}} := \mathcal{D}_m(\boldsymbol{c}_{\boldsymbol{v}}, \boldsymbol{\Sigma}_{\boldsymbol{v}})$.
  3. Nominal $\bar{\mathbb{P}}_{\mathrm{u}}$ is a Standard Gaussian and candidate $\mathbb{P}_{\mathrm{u}}$.
- Recall $\mathbf{u} = \boldsymbol{S}^{-1/2}(\mathbf{y} - \boldsymbol{H}\boldsymbol{c}_{\boldsymbol{x}} - \boldsymbol{c}_{\boldsymbol{v}})$ and $\mathbf{y} = (\boldsymbol{H}\boldsymbol{\Sigma}_{\boldsymbol{x}}\boldsymbol{H}^\top + \boldsymbol{\Sigma}_{\boldsymbol{v}})^{1/2}\mathbf{u} + \boldsymbol{H}\boldsymbol{c}_{\boldsymbol{x}} + \boldsymbol{c}_{\boldsymbol{v}}$
- $\mathbb{P}_{\mathrm{u}}$ must be fat-tailed. Because the type of the distribution of $\mathbf{y}$ only depends on the type of $\mathbf{u}$.

Recap: $$\max_{\mathbb{P} \in \mathcal{F}_{\mathbf{x},\mathbf{y}}(\theta)} \operatorname{Tr} \boldsymbol{\Sigma}_{\boldsymbol{x}} - \boldsymbol{\Sigma}_{\boldsymbol{x}} \boldsymbol{H}^\top (\boldsymbol{H} \boldsymbol{\Sigma}_{\boldsymbol{x}} \boldsymbol{H}^\top + \boldsymbol{\Sigma}_{\boldsymbol{v}})^{-1} \boldsymbol{H} \boldsymbol{\Sigma}_{\boldsymbol{x}} \cdot \mathbb{E} \left[-\frac{\mathrm{d}^2 \ln p_{\mathrm{u}}(\mu)}{\mathrm{d}\mu^2} \bigg|_{\mu = \mathrm{u}} \right]$$

- We need to define the ambiguity set $\mathcal{F}_{\mathbf{x},\mathbf{y}}(\theta)$, which contains fat-tailed distributions for $\mathbf{y}$. We call every $\mathbb{P}$ in $\mathcal{F}_{\mathbf{x},\mathbf{y}}(\theta)$ a **candidate**.
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  1. Nominal $\bar{\mathbb{P}}_{\mathbf{x}} := \mathcal{N}_n(\bar{\boldsymbol{x}}, \boldsymbol{M})$ and candidate $\mathbb{P}_{\mathbf{x}} := \mathcal{D}_n(\boldsymbol{c}_{\boldsymbol{x}}, \boldsymbol{\Sigma}_{\boldsymbol{x}})$.
  2. Nominal $\bar{\mathbb{P}}_{\mathbf{v}} := \mathcal{N}_m(\mathbf{0}, \boldsymbol{R})$ and candidate $\mathbb{P}_{\mathbf{v}} := \mathcal{D}_m(\boldsymbol{c}_{\boldsymbol{v}}, \boldsymbol{\Sigma}_{\boldsymbol{v}})$.
  3. Nominal $\bar{\mathbb{P}}_{\mathrm{u}}$ is a Standard Gaussian and candidate $\mathbb{P}_{\mathrm{u}}$.
- Recall $\mathbf{u} = \boldsymbol{S}^{-1/2}(\mathbf{y} - \boldsymbol{H}\boldsymbol{c}_{\boldsymbol{x}} - \boldsymbol{c}_{\boldsymbol{v}})$ and $\mathbf{y} = (\boldsymbol{H}\boldsymbol{\Sigma}_{\boldsymbol{x}}\boldsymbol{H}^\top + \boldsymbol{\Sigma}_{\boldsymbol{v}})^{1/2}\mathbf{u} + \boldsymbol{H}\boldsymbol{c}_{\boldsymbol{x}} + \boldsymbol{c}_{\boldsymbol{v}}$
- $\mathbb{P}_{\mathrm{u}}$ must be fat-tailed. Because the type of the distribution of $\mathbf{y}$ only depends on the type of $\mathbf{u}$.
- Assuming candidate $\mathbb{P}_{\mathbf{x}}$ and candidate $\mathbb{P}_{\mathbf{v}}$ to be Gaussian is sufficient to describe the uncertainties in parameters $\{\bar{\boldsymbol{x}}, \mathbf{0}, \boldsymbol{M}, \boldsymbol{R}\}$.

Recap: $$\max_{\mathbb{P}\in\mathcal{F}_{\mathbf{x},\mathbf{y}}(\theta)} \operatorname{Tr} \boldsymbol{\Sigma}_{\boldsymbol{x}} - \boldsymbol{\Sigma}_{\boldsymbol{x}}\boldsymbol{H}^\top(\boldsymbol{H}\boldsymbol{\Sigma}_{\boldsymbol{x}}\boldsymbol{H}^\top + \boldsymbol{\Sigma}_{\boldsymbol{v}})^{-1}\boldsymbol{H}\boldsymbol{\Sigma}_{\boldsymbol{x}} \cdot \mathbb{E}\left[-\frac{\mathrm{d}^2 \ln p_{\mathrm{u}}(\mu)}{\mathrm{d}\mu^2}\bigg|_{\mu=\mathrm{u}} \right]$$

- We need to define the ambiguity set $\mathcal{F}_{\mathbf{x},\mathbf{y}}(\theta)$, which contains fat-tailed distributions for $\mathbf{y}$. We call every $\mathbb{P}$ in $\mathcal{F}_{\mathbf{x},\mathbf{y}}(\theta)$ a **candidate**.
- 1. Nominal $\bar{\mathbb{P}}_{\mathbf{x}} := \mathcal{N}_n(\bar{\boldsymbol{x}}, \boldsymbol{M})$ and candidate $\mathbb{P}_{\mathbf{x}} := \mathcal{D}_n(\boldsymbol{c}_{\boldsymbol{x}}, \boldsymbol{\Sigma}_{\boldsymbol{x}})$.
  2. Nominal $\bar{\mathbb{P}}_{\mathbf{v}} := \mathcal{N}_m(\mathbf{0}, \boldsymbol{R})$ and candidate $\mathbb{P}_{\mathbf{v}} := \mathcal{D}_m(\boldsymbol{c}_{\boldsymbol{v}}, \boldsymbol{\Sigma}_{\boldsymbol{v}})$.
  3. Nominal $\bar{\mathbb{P}}_{\mathrm{u}}$ is a Standard Gaussian and candidate $\mathbb{P}_{\mathrm{u}}$.
- Recall $\mathbf{u} = \boldsymbol{S}^{-1/2}(\mathbf{y} - \boldsymbol{H}\boldsymbol{c}_{\boldsymbol{x}} - \boldsymbol{c}_{\boldsymbol{v}})$ and $\mathbf{y} = (\boldsymbol{H}\boldsymbol{\Sigma}_{\boldsymbol{x}}\boldsymbol{H}^\top + \boldsymbol{\Sigma}_{\boldsymbol{v}})^{1/2}\mathbf{u} + \boldsymbol{H}\boldsymbol{c}_{\boldsymbol{x}} + \boldsymbol{c}_{\boldsymbol{v}}$
- $\mathbb{P}_{\mathrm{u}}$ must be fat-tailed. Because the type of the distribution of $\mathbf{y}$ only depends on the type of $\mathbf{u}$.
- Assuming candidate $\mathbb{P}_{\mathbf{x}}$ and candidate $\mathbb{P}_{\mathbf{v}}$ to be Gaussian is sufficient to describe the uncertainties in parameters $\{\bar{\boldsymbol{x}}, \mathbf{0}, \boldsymbol{M}, \boldsymbol{R}\}$.
  1. The uncertain counterpart of $\{\bar{\boldsymbol{x}}, \mathbf{0}, \boldsymbol{M}, \boldsymbol{R}\}$ is $\{\boldsymbol{c}_{\boldsymbol{x}}, \boldsymbol{c}_{\boldsymbol{v}}, \boldsymbol{\Sigma}_{\boldsymbol{x}}, \boldsymbol{\Sigma}_{\boldsymbol{v}}\}$.

Ambiguity sets for $\mathbb{P}_{\mathbf{x}}$ and $\mathbb{P}_{\mathbf{v}}$: mainly accounts for parameter uncertainties
Moment-based set, Kullback–Leibler divergence, Wasserstein distance, etc., can be used.

Example. Wasserstein distance

$$\mathcal{F}_{\mathbf{x}}(\theta_{\boldsymbol{x}}) = \left\{ \mathbb{P}_{\mathbf{x}} = \mathcal{N}_n(\boldsymbol{c}_{\boldsymbol{x}}, \boldsymbol{\Sigma}_{\boldsymbol{x}}) \,\middle|\, \mathrm{W}(\mathbb{P}_{\mathbf{x}} \| \bar{\mathbb{P}}_{\mathbf{x}}) \leq \theta_{\boldsymbol{x}} \right\},$$

$$\mathcal{F}_{\mathbf{v}}(\theta_{\boldsymbol{v}}) = \left\{ \mathbb{P}_{\mathbf{v}} = \mathcal{N}_m(\boldsymbol{c}_{\boldsymbol{v}}, \boldsymbol{\Sigma}_{\boldsymbol{v}}) \,\middle|\, \mathrm{W}(\mathbb{P}_{\mathbf{v}} \| \bar{\mathbb{P}}_{\mathbf{v}}) \leq \theta_{\boldsymbol{v}} \right\}.$$

where $\mathrm{W}(\cdot, \cdot)$ denotes the Wasserstein metric and under Gaussianity assumption, the type-2 Wasserstein distance is given as

$$\mathrm{W}(\mathbb{P}_{\mathbf{x}}, \bar{\mathbb{P}}_{\mathbf{x}}) = \sqrt{\|\boldsymbol{c}_{\boldsymbol{x}} - \bar{\boldsymbol{x}}\|^2 + \mathrm{Tr}[\boldsymbol{\Sigma}_{\boldsymbol{x}} + \boldsymbol{M} - 2(\boldsymbol{M}^{\frac{1}{2}} \boldsymbol{\Sigma}_{\boldsymbol{x}} \boldsymbol{M}^{\frac{1}{2}})^{\frac{1}{2}}]}.$$

Ambiguity sets for $\mathbb{P}_{\mathrm{u}}$: mainly accounts for measurement outliers.
ϵ-contamination set, ϵ-normal set, etc., can be used.

Example. **ϵ-contamination set**.

$$\mathcal{F}_{\mathrm{u}}(\epsilon) = \left\{ \mathbb{P}_{\mathrm{u}} \in \mathcal{P}(\mathbb{R}) \;\middle|\; \begin{array}{l} F_{\mathrm{u}}(\mu) = \mathbb{P}_{\mathrm{u}}(\mathrm{u} \leq \mu) \\ F_{\mathrm{u}}(\mu) = (1-\epsilon)\Phi(\mu) + \epsilon H(\mu) \\ H(\mu) = 1 - H(-\mu), \quad H(\mu) \text{ is a cumulative on } \mathbb{R} \end{array} \right\}.$$

Intuitively, it means with probability ϵ, say 5%, a transformed measurement u is from the contamination distribution $H(\cdot)$, and with probability $1-\epsilon$, u is from the Gaussian distribution $\Phi(\cdot)$.

Recap: $$\max_{\mathbb{P} \in \mathcal{F}_{\mathbf{x},\mathbf{y}}(\theta)} \operatorname{Tr} \boldsymbol{\Sigma}_{\boldsymbol{x}} - \boldsymbol{\Sigma}_{\boldsymbol{x}} \boldsymbol{H}^\top (\boldsymbol{H} \boldsymbol{\Sigma}_{\boldsymbol{x}} \boldsymbol{H}^\top + \boldsymbol{\Sigma}_{\boldsymbol{v}})^{-1} \boldsymbol{H} \boldsymbol{\Sigma}_{\boldsymbol{x}} \cdot \mathbb{E}\left[- \left.\frac{\mathrm{d}^2 \ln p_{\mathrm{u}}(\mu)}{\mathrm{d}\mu^2}\right|_{\mu=\mathrm{u}} \right]$$

- Hence, we solve it independently and sequentially, i.e., solving the innermost first and the outermost last.

$$\max_{\mathbb{P}_{\mathbf{x}} \in \mathcal{F}_{\mathbf{x}}(\theta_{\boldsymbol{x}})} \max_{\mathbb{P}_{\mathbf{v}} \in \mathcal{F}_{\mathbf{v}}(\theta_{\boldsymbol{v}})} \max_{\mathbb{P}_{\mathrm{u}} \in \mathcal{F}_{\mathrm{u}}(\epsilon)} \operatorname{Tr} \boldsymbol{P}.$$

i.e., (due to parameterizations of distributions)

$$\max_{\boldsymbol{\Sigma}_{\boldsymbol{x}}} \max_{\boldsymbol{\Sigma}_{\boldsymbol{v}}} \max_{i_\mu} \operatorname{Tr} \boldsymbol{P}$$

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- First, we solve the innermost sub-problem over i_μ. Note that $\boldsymbol{\Sigma}_{\boldsymbol{x}}\boldsymbol{H}^\top(\boldsymbol{H}\boldsymbol{\Sigma}_{\boldsymbol{x}}\boldsymbol{H}^\top + \boldsymbol{\Sigma}_{\boldsymbol{v}})^{-1}\boldsymbol{H}\boldsymbol{\Sigma}_{\boldsymbol{x}} \succeq \mathbf{0}$ because $\boldsymbol{\Sigma}_{\boldsymbol{x}} \in \mathbb{S}^n_+$ and $\boldsymbol{\Sigma}_{\boldsymbol{v}} \in \mathbb{S}^m_{++}$.

Linear System Case

Solve Distributionally Robust Bayesian Estimation Problem

Recap: $$\max_{\mathbb{P}\in\mathcal{F}_{\mathbf{x},\mathbf{y}}(\theta)} \operatorname{Tr} \boldsymbol{\Sigma}_{\boldsymbol{x}} - \boldsymbol{\Sigma}_{\boldsymbol{x}}\boldsymbol{H}^{\top}(\boldsymbol{H}\boldsymbol{\Sigma}_{\boldsymbol{x}}\boldsymbol{H}^{\top} + \boldsymbol{\Sigma}_{\boldsymbol{v}})^{-1}\boldsymbol{H}\boldsymbol{\Sigma}_{\boldsymbol{x}} \cdot \mathbb{E}\left[-\frac{\mathrm{d}^2 \ln p_{\mathrm{u}}(\mu)}{\mathrm{d}\mu^2}\bigg|_{\mu=\mathrm{u}} \right]$$

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- The non-negative and minimal i_{μ} maximizes $\operatorname{Tr} \boldsymbol{P}$.

Lemma 1 (Huber 1964 [11])

The functional optimization over the ϵ-contamination ambiguity set

$$\min_{p(\mu)} \mathbb{E}\left[-\frac{\mathrm{d}^2}{\mathrm{d}\mu^2}\ln p(\mu)\Big|_{\mu=\mathrm{u}}\right] \quad \textit{s.t.} \quad \begin{cases} p(\mu) = \dfrac{\mathrm{d}F_{\mathrm{u}}(\mu)}{\mathrm{d}\mu} \\ F_{\mathrm{u}}(\mu) = (1-\epsilon)\Phi(\mu) + \epsilon H(\mu) \\ H(\mu) = 1 - H(-\mu) \end{cases}$$

is solved by the Laplacian-tailed least-favorable distribution

$$p(\mu) = \begin{cases} (1-\epsilon)\frac{1}{\sqrt{2\pi}}e^{K\mu+\frac{1}{2}K^2}, & \mu \le -K \\ (1-\epsilon)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\mu^2}, & |\mu| \le K \\ (1-\epsilon)\frac{1}{\sqrt{2\pi}}e^{-K\mu+\frac{1}{2}K^2}, & \mu \ge K, \end{cases}$$

where $K \in \mathbb{R}_+$ is implicitly defined by ϵ: $\int_{-K}^{K} p(\mu)dt + \frac{2p(K)}{K} = 1$.
Furthermore, $\min i_\mu = \min \mathbb{E}\left[-\frac{d^2}{d\mu^2}\ln p(\mu)\right] = (1-\epsilon)[1-2\Phi(-K)]$.

Theorem 2

The distributionally robust Bayesian estimation is equivalent to

$$\max_{\boldsymbol{\Sigma}_{\boldsymbol{x}}} \max_{\boldsymbol{\Sigma}_{\boldsymbol{v}}} \operatorname{Tr}\left[\boldsymbol{\Sigma}_{\boldsymbol{x}} - \boldsymbol{\Sigma}_{\boldsymbol{x}}\boldsymbol{H}^\top(\boldsymbol{H}\boldsymbol{\Sigma}_{\boldsymbol{x}}\boldsymbol{H}^\top + \boldsymbol{\Sigma}_{\boldsymbol{v}})^{-1}\boldsymbol{H}\boldsymbol{\Sigma}_{\boldsymbol{x}} \cdot i_\mu^{\min}\right],$$

where $i_\mu^{\min} := \min i_\mu := \min \mathbb{E}\left[-\frac{d^2}{d\mu^2}\ln p(\mu)\right]$ *is a constant defined in the Lemma above. Besides,* $0 \leq i_\mu^{\min} \leq 1$.

Linear System Case

Solve Distributionally Robust Bayesian Estimation Problem

After solving the inner-most optimization over $p(\mu)$, we next we solve the outer sub-problems over $\boldsymbol{\Sigma}_{\boldsymbol{x}}$ and $\boldsymbol{\Sigma}_{\boldsymbol{v}}$

$$\max_{\boldsymbol{\Sigma}_{\boldsymbol{x}}} \ \max_{\boldsymbol{\Sigma}_{\boldsymbol{v}}} \operatorname{Tr}\left[\boldsymbol{\Sigma}_{\boldsymbol{x}} - \boldsymbol{\Sigma}_{\boldsymbol{x}} \boldsymbol{H}^\top (\boldsymbol{H} \boldsymbol{\Sigma}_{\boldsymbol{x}} \boldsymbol{H}^\top + \boldsymbol{\Sigma}_{\boldsymbol{v}})^{-1} \boldsymbol{H} \boldsymbol{\Sigma}_{\boldsymbol{x}} \cdot i_\mu^{\min}\right],$$

Under Wasserstein ambiguities of $\mathcal{F}_{\mathbf{x}}(\theta_{\boldsymbol{x}})$ and $\mathcal{F}_{\mathbf{v}}(\theta_{\boldsymbol{v}})$, we have

$$\max_{\boldsymbol{\Sigma}_{\boldsymbol{x}}} \ \max_{\boldsymbol{\Sigma}_{\boldsymbol{v}}} \operatorname{Tr}\left[\boldsymbol{\Sigma}_{\boldsymbol{x}} - \boldsymbol{\Sigma}_{\boldsymbol{x}} \boldsymbol{H}^\top (\boldsymbol{H} \boldsymbol{\Sigma}_{\boldsymbol{x}} \boldsymbol{H}^\top + \boldsymbol{\Sigma}_{\boldsymbol{v}})^{-1} \boldsymbol{H} \boldsymbol{\Sigma}_{\boldsymbol{x}} \cdot i_\mu^{\min}\right],$$

subject to

$$\begin{cases} \sqrt{\operatorname{Tr}\left[\boldsymbol{\Sigma}_{\boldsymbol{x}} + \boldsymbol{M} - 2\left(\boldsymbol{M}^{\frac{1}{2}} \boldsymbol{\Sigma}_{\boldsymbol{x}} \boldsymbol{M}^{\frac{1}{2}}\right)^{\frac{1}{2}}\right]} \le \theta_{\boldsymbol{x}} \\ \sqrt{\operatorname{Tr}\left[\boldsymbol{\Sigma}_{\boldsymbol{v}} + \boldsymbol{R} - 2\left(\boldsymbol{R}^{\frac{1}{2}} \boldsymbol{\Sigma}_{\boldsymbol{v}} \boldsymbol{R}^{\frac{1}{2}}\right)^{\frac{1}{2}}\right]} \le \theta_{\boldsymbol{v}} \\ \boldsymbol{\Sigma}_{\boldsymbol{x}} \succeq \mathbf{0} \\ \boldsymbol{\Sigma}_{\boldsymbol{v}} \succ \mathbf{0}. \end{cases}$$

Linear System Case

Solve Distributionally Robust Bayesian Estimation Problem

Theorem 3 (Under Wasserstein Ambiguity)

Suppose $\boldsymbol{R} \succ \mathbf{0}$. It can be reformulated as a linear SDP

$$\max_{\boldsymbol{\Sigma_x}, \boldsymbol{\Sigma_v}, \boldsymbol{V_x}, \boldsymbol{V_v}, \boldsymbol{U}} \operatorname{Tr}\left[\boldsymbol{\Sigma_x} - i_{\mu}^{\min} \cdot \boldsymbol{U}\right],$$

$$s.t. \quad \begin{cases} \begin{bmatrix} \boldsymbol{U} & \boldsymbol{\Sigma_x H}^\top \\ \boldsymbol{H\Sigma_x} & \boldsymbol{H\Sigma_x H}^\top + \boldsymbol{\Sigma_v} \end{bmatrix} \succeq \mathbf{0} \\ \operatorname{Tr}\left[\boldsymbol{\Sigma_x} + \boldsymbol{M} - 2\boldsymbol{V_x}\right] \le \theta_{\boldsymbol{x}}^2 \\ \begin{bmatrix} \boldsymbol{M}^{\frac{1}{2}} \boldsymbol{\Sigma_x} \boldsymbol{M}^{\frac{1}{2}} & \boldsymbol{V_x} \\ \boldsymbol{V_x} & \boldsymbol{I} \end{bmatrix} \succeq \mathbf{0} \\ \operatorname{Tr}\left[\boldsymbol{\Sigma_v} + \boldsymbol{R} - 2\boldsymbol{V_v}\right] \le \theta_{\boldsymbol{v}}^2 \\ \begin{bmatrix} \boldsymbol{R}^{\frac{1}{2}} \boldsymbol{\Sigma_v} \boldsymbol{R}^{\frac{1}{2}} & \boldsymbol{V_v} \\ \boldsymbol{V_v} & \boldsymbol{I} \end{bmatrix} \succeq \mathbf{0} \\ \boldsymbol{\Sigma_x} \succeq \mathbf{0}, \boldsymbol{\Sigma_v} \succ \mathbf{0}, \boldsymbol{V_x} \succeq \mathbf{0}, \boldsymbol{V_v} \succeq \mathbf{0}, \boldsymbol{U} \succeq \mathbf{0}. \end{cases}$$

Under moment-based ambiguities of $\mathcal{F}_{\mathbf{x}}(\theta_{\boldsymbol{x}})$ and $\mathcal{F}_{\mathbf{v}}(\theta_{\boldsymbol{v}})$, we have

$$\max_{\boldsymbol{\Sigma}_{\boldsymbol{x}}} \max_{\boldsymbol{\Sigma}_{\boldsymbol{v}}} \operatorname{Tr}\left[\boldsymbol{\Sigma}_{\boldsymbol{x}} - \boldsymbol{\Sigma}_{\boldsymbol{x}}\boldsymbol{H}^\top(\boldsymbol{H}\boldsymbol{\Sigma}_{\boldsymbol{x}}\boldsymbol{H}^\top + \boldsymbol{\Sigma}_{\boldsymbol{v}})^{-1}\boldsymbol{H}\boldsymbol{\Sigma}_{\boldsymbol{x}} \cdot \dot{i}_{\mu}^{\min}\right],$$

subject to

$$\begin{cases} \boldsymbol{\Sigma}_{\boldsymbol{x}} \preceq \theta_{2,\boldsymbol{x}}\boldsymbol{M} \\ \boldsymbol{\Sigma}_{\boldsymbol{x}} \succeq \theta_{1,\boldsymbol{x}}\boldsymbol{M} \\ \boldsymbol{\Sigma}_{\boldsymbol{v}} \preceq \theta_{2,\boldsymbol{v}}\boldsymbol{R} \\ \boldsymbol{\Sigma}_{\boldsymbol{v}} \succeq \theta_{1,\boldsymbol{v}}\boldsymbol{R} \succ \boldsymbol{0} \\ \boldsymbol{\Sigma}_{\boldsymbol{x}} \succeq \boldsymbol{0} \\ \boldsymbol{\Sigma}_{\boldsymbol{v}} \succ \boldsymbol{0}. \end{cases}$$

Theorem 4 (Under Moment-Based Ambiguity)

It is analytically solved by $\boldsymbol{\Sigma}_{\boldsymbol{x}} = \theta_{2,\boldsymbol{x}}\boldsymbol{M}$ *and* $\boldsymbol{\Sigma}_{\boldsymbol{v}} = \theta_{2,\boldsymbol{v}}\boldsymbol{R}$.

Linear System Case

Solve Distributionally Robust Bayesian Estimation Problem

Recap: $$\max_{\mathbb{P}\in\mathcal{F}_{\mathbf{x},\mathbf{y}}(\theta)} \min_{\boldsymbol{\phi}\in\mathcal{H}'_{\mathbf{y}}} \operatorname{Tr}\mathbb{E}[\mathbf{x}-\boldsymbol{\phi}(\mathbf{y})][\mathbf{x}-\boldsymbol{\phi}(\mathbf{y})]^\top$$

Theorem 5 (Solution to Distributionally Robust Bayesian Estimation)

Optimal Estimator.

$$\hat{\mathbf{x}} = \bar{\boldsymbol{x}} + \boldsymbol{\Sigma}_{\boldsymbol{x}}^{*}\boldsymbol{H}^\top \boldsymbol{S}^{*-1/2}\cdot \boldsymbol{\psi}[\boldsymbol{S}^{*-1/2}(\mathbf{y}-\boldsymbol{H}\bar{\boldsymbol{x}})],$$

where $\boldsymbol{S}^* := \boldsymbol{H}\boldsymbol{\Sigma}_{\boldsymbol{x}}^{*}\boldsymbol{H}^\top + \boldsymbol{\Sigma}_{\boldsymbol{v}}^{*}$ *where* $\boldsymbol{\Sigma}_{\boldsymbol{x}}^{*}$ *and* $\boldsymbol{\Sigma}_{\boldsymbol{v}}^{*}$ *are optimal solutions of nonlinear SDPs associated with the Wasserstein metric or the moment-based set.*

If the ϵ*-contamination ambiguity set is used,* $\boldsymbol{\psi}(\boldsymbol{\mu})$ *is entry-wise identical and for each entry*

$$\psi(\mu) = \begin{cases} -K, & \mu \le -K \\ \mu, & |\mu| \le K \\ K, & \mu \ge K. \end{cases}$$

Whenever a measurement $\mathbf{y} = \boldsymbol{y}$ *is large, the value of* $\psi(\cdot)$ *is limited to* $\pm K$.

Linear System Case

Solve Distributionally Robust Bayesian Estimation Problem

Recap:
$$\begin{cases} \mathbf{x}_k & = \boldsymbol{F}_{k-1}\mathbf{x}_{k-1} + \boldsymbol{G}_{k-1}\mathbf{w}_{k-1}, \\ \mathbf{y}_k & = \boldsymbol{H}_k\mathbf{x}_k + \mathbf{v}_k, \end{cases}$$

Theorem 6 (Solution to Distributionally Robust State Estimation)

Optimal Recursive State Estimator.

$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \boldsymbol{\Sigma}^*_{\boldsymbol{x},k}\boldsymbol{H}_k^\top \boldsymbol{S}_k^{*-1/2} \cdot \psi[\boldsymbol{S}_k^{*-1/2}(\mathbf{y}_k - \boldsymbol{H}_k\hat{\mathbf{x}}_{k|k-1})],$$

where

$$\hat{\mathbf{x}}_{k|k-1} = \boldsymbol{F}_{k-1}\hat{\mathbf{x}}_{k-1|k-1},$$

and

$$\boldsymbol{S}_k^* := \boldsymbol{H}_k\boldsymbol{\Sigma}^*_{\boldsymbol{x},k}\boldsymbol{H}_k^\top + \boldsymbol{\Sigma}^*_{\boldsymbol{v},k};$$

$\psi(\cdot)$, $\boldsymbol{\Sigma}^*_{\boldsymbol{x},k}$, *and* $\boldsymbol{\Sigma}^*_{\boldsymbol{v},k}$ *are defined in Theorem above.*

In the nominal case, the distributionally robust state estimator degenerates to the Kalman filter: e.g., $\psi(\mu) = \mu$ (no longer outlier treatment).

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4. The robust estimator is nonlinear: there exists a nonlinear function to limit the influence that a large-valued measurement can bring to the estimator.

Content

We consider a state estimation problem for nonlinear systems.

$$\begin{cases} \mathbf{x}_k & = \boldsymbol{f}_k(\mathbf{x}_{k-1}, \mathbf{w}_{k-1}) \\ \mathbf{y}_k & = \boldsymbol{h}_k(\mathbf{x}_k, \mathbf{v}_k) \end{cases}$$

- $k = 1, 2, 3, ...$ denote discrete time index.
- $\mathbf{x}_k \in \mathbb{R}^n$ is hidden state, $\mathbf{y}_k \in \mathbb{R}^m$ is measurement, $\mathbf{w}_{k-1} \in \mathbb{R}^p$ is process noise, $\mathbf{v}_k \in \mathbb{R}^q$ is measurement noise.
- $\boldsymbol{f}_k(\cdot, \cdot)$ is process dynamics, and $\boldsymbol{h}_k(\cdot, \cdot)$ is measurement dynamics.
- **Assume**: $\mathbf{x}_k$, $\mathbf{y}_k$, $\mathbf{w}_k$, and $\mathbf{v}_k$ have finite second moments; $\mathbf{w}_k$ and $\mathbf{v}_k$ have known distributions.
- **Task**: estimate/infer the hidden $\mathbf{x}_k$ based on measurement sequence $\mathcal{Y}_k := \{\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_k\}$.

Issues regarding the state estimation for the nonlinear system.

1. **Issue 1**: Typically, we assume **nominal** forms of nonlinear mappings $\boldsymbol{f}_k(\cdot,\cdot)$ and $\boldsymbol{h}_k(\cdot,\cdot)$, and **nominal** types and parameters of the distributions of $\mathbf{w}_{k-1}$ and $\mathbf{v}_k$ are exactly true. However, in practice, they might be uncertain.

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   - Only easy for additive and multiplicative measurement noises.
   - **Aim:** Therefore, a general likelihood evaluation method is expected.
3. **Issue 3**: What if measurement outliers exist? How to treat them?

Nonlinear System Case

Problem Background, Statements, and Motivations

Recall the Bayesian estimation procedure (n.b., $\boldsymbol{Y}_{k-1} := \{\boldsymbol{y}_1, \boldsymbol{y}_2, ..., \boldsymbol{y}_{k-1}\}$):

A-Priori Step: $p\left(\boldsymbol{x}_k \mid \boldsymbol{Y}_{k-1}\right) = \int_{\boldsymbol{x}_{k-1}} p\left(\boldsymbol{x}_k \mid \boldsymbol{x}_{k-1}\right) p\left(\boldsymbol{x}_{k-1} \mid \boldsymbol{Y}_{k-1}\right) d\boldsymbol{x}_{k-1}$

A-Posteriori Step: $p\left(\boldsymbol{x}_k \mid \boldsymbol{Y}_k\right) \propto p\left(\boldsymbol{y}_k \mid \boldsymbol{x}_k\right) \cdot p\left(\boldsymbol{x}_k \mid \boldsymbol{Y}_{k-1}\right)$

Handle Issue 1:

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Nonlinear System Case

Problem Background, Statements, and Motivations

Recall the Bayesian estimation procedure (n.b., $\boldsymbol{Y}_{k-1} := \{\boldsymbol{y}_1, \boldsymbol{y}_2, ..., \boldsymbol{y}_{k-1}\}$):

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- Find worst-case prior state distribution $p^*(\boldsymbol{x}_k \mid \boldsymbol{Y}_{k-1})$ and worst-case likelihood distribution $p^*(\boldsymbol{y}_k \mid \boldsymbol{x}_k)$. "Worst-case" scenario defined by "entropy".

Handle Issue 2: The maximum-entropy scheme can serve as a general likelihood evaluation method.

Handle Issue 3: Evaluating likelihoods of all prior state particles at the given measurement. If the **largest** likelihood (of all prior state particles) is smaller than a threshold (e.g., 5%), we treat this measurement as an outlier because none of these prior state particle can possibly generate this measurement.

1. **On Worst-Case Prior Distribution**: Given the particle-represented nominal $\hat{p}\left(\boldsymbol{x}_k \mid \boldsymbol{Y}_{k-1}\right)$, find a **max**imum-**ent**ropy distribution near it. Note that $\hat{p}(\cdot|\cdot)$ is supported on $\{\boldsymbol{x}^i\}_{i\in[N]}$.

Continuous: If the maxent is continous.

$$
\begin{array}{cc}
\max\limits_{p(\boldsymbol{x})\in L^1} & \displaystyle\int -p(\boldsymbol{x}) \ln p(\boldsymbol{x}) d\boldsymbol{x} \\
\textit{s.t.} & \left\{ \begin{array}{ll} D(p, \hat{p}) & \le \theta \\ \int p(\boldsymbol{x}) d\boldsymbol{x} & = 1 \end{array} \right.
\end{array}
$$

Discrete: If the maxent is discrete and supported on $\{\boldsymbol{x}^j\}_{j\in[M]}$ (not necessarily the same to $\{\boldsymbol{x}^i\}_{i\in[N]}$ but usually can be).

$$
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\max\limits_{p(\boldsymbol{x})\in l^1} & \displaystyle\sum_j -p(\boldsymbol{x}^j) \ln p(\boldsymbol{x}^j) \\
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$$

2. **On Worst-Case Likelihood Distribution**: Given the particle-represented nominal $\hat{p}\left(\boldsymbol{y} \mid \boldsymbol{x}^j\right), \forall j \in [M]$, find a maxent distribution near it. Note that $\hat{p}(\cdot|\cdot)$ is supported on $\{\boldsymbol{y}^r|\boldsymbol{x}^j\}_{r\in[R]}, \forall j \in [M]$.

Continuous: If the maxent is continuous.

$$
\begin{array}{cc}
\displaystyle\max_{p_{\mathbf{y}|\boldsymbol{x}^j}(\boldsymbol{y})\in L^1} & \displaystyle\int -p_{\mathbf{y}|\boldsymbol{x}^j}(\boldsymbol{y}) \ln p_{\mathbf{y}|\boldsymbol{x}^j}(\boldsymbol{y}) d\boldsymbol{y} \\
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Discrete: If the maxent is discrete and supported on $\{\boldsymbol{y}^t|\boldsymbol{x}^j\}_{t\in[T]}$, $\forall j \in [M]$ (not necessarily the same to $\{\boldsymbol{y}^r|\boldsymbol{x}^j\}_{r\in[R]}$ but usually can be).

$$
\begin{array}{cc}
\displaystyle\max_{p_{\mathbf{y}|\boldsymbol{x}^j}(\boldsymbol{y})\in l^1} & \displaystyle\sum_t -p_{\mathbf{y}|\boldsymbol{x}^j}(\boldsymbol{y}^t) \ln p_{\mathbf{y}|\boldsymbol{x}^j}(\boldsymbol{y}^t) \\
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\end{array}
$$

One Example:

Theorem 7 (Continuous Case Under Wasserstein)

The continuous maximum entropy distribution in Wasserstein ball is

$$p(\boldsymbol{x}) = \exp\left\{-v_0 \min_{i \in [N]} \left\{\|\boldsymbol{x} - \boldsymbol{x}^i\| - \lambda_i\right\} - v_1 - 1\right\}$$

where $v_0 \in \mathbb{R}^1$, $v_1 \in \mathbb{R}^1$, *and* $\lambda_i \in \mathbb{R}^1, \forall i$ *solve the following convex and smooth problem (n.b., almost-everywhere smooth in terms of* λ_i*; non-smooth only on zero-measure boundaries):*

$$\begin{aligned} \min_{v_0, v_1, \boldsymbol{\lambda}} \quad & v_0 \cdot (\theta - \textstyle\sum_{i=1}^{N} \lambda_i q_i) + v_1 + \\ & \int \exp\left\{-v_0 \min_{i \in [N]} \left\{\|\boldsymbol{x} - \boldsymbol{x}^i\| - \lambda_i\right\} - v_1 - 1\right\} d\boldsymbol{x} \\ \text{s.t.} \quad & v_0 \geq 0, \end{aligned}$$

where $\boldsymbol{\lambda} := [\lambda_1, \lambda_2, ..., \lambda_N]^\top$.

Projected Gradient Descent to solve the minimization sub-problem.

Main Points in This Section:

We propose to use maximum entropy prior/likelihood distributions to realize the distributionally robust state estimation for nonlinear systems.

Make main use of prior if likelihood is uncertain; make main use of likelihood if prior is uncertain.

Specifically, the three steps to robustify the particle filter:

1. Calculate worst-case prior state distribution.
2. Evaluate worst-case likelihood.
3. Outlier identification and treatment.

Content

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   - It is better to take into consideration uncertainties from immediate sources where uncertainties occur. For example, for the model:

$$\begin{cases} \mathbf{x}_k &= (\boldsymbol{F}_{k-1} + \delta \boldsymbol{F}_{k-1})\mathbf{x}_{k-1} + (\boldsymbol{G}_{k-1} + \delta \boldsymbol{G}_{k-1})\mathbf{w}_{k-1}, \\ \mathbf{y}_k &= \boldsymbol{H}_k \mathbf{x}_k + \mathbf{v}_k, \end{cases}$$

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Distributionally Robust State Estimation

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     - Distributionally robust solutions are supplementary, not dominating. It is extremely useful when $\delta\boldsymbol{F}_{k-1}$ and $\delta\boldsymbol{G}_{k-1}$ cannot be directly modeled.
   - Why not directly constrain parameters?, e.g., $D(\boldsymbol{F}_k, \hat{\boldsymbol{F}}_k) \leq \theta$. It raises a matrix optimization problem that is even not a SDP. Tractability is a big issue!

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   - One should carefully (and pragmatically) tune this parameter to achieve good performances for their specific real problems.

Content

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    - In some special cases it can be analytically (i.e., efficiently) solved.

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5. We show: Gaussian approximation state estimators are distributionally robust.
    - Fixing second moments, maximum entropy distributions are Gaussian.

Thank You!

Content

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