CS-E400204: Approximation Algorithms **Spring (Fifth Period) 2022**

Lecture 10 (May 25): Randomized Rounding for MAX-SAT

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10.1 Problem definition

The maximum satisfiability problem (MAX-SAT) is to find the value of a series of boolean variables, which maximizes the total weight of satisfied ones among the clauses of a conjunctive normal form (CNF) formula.

- Literals: either a variable x_j or its negation $\bar{x}_j$;
- Disjunctive: a clause C_i of the disjunction(or) of one or more literals, to avoid triviality, empty clause or a clause containing both x_j and $\bar{x}_j$ of a same variable is not allowed;
- Length of a disjunctive: the number of literals contained in a disjunctive clause;
- Conjunctive Normal Form (CNF): a formula Φ of the conjuction (and) of one or more disjunctive clauses;
- Input:
  - n boolean variables $x_1, \dots, x_n$;
  - m disjunctive clauses $C_1, \dots, C_m$ (or in the form of CNF);
  - the weight of these clauses $w_1, \dots, w_m$;
- Output:
  - $\forall i \in \{1, \dots, n\}$, the value of variable $x_i \leftarrow$ **true** | **false**, that maximizes $\sum_{j=1}^{m} w_j \,|\, \text{clause } C_j \text{ satisfied}$.

10.2 Unbiased randomized algorithm

Consider setting the values of all the variables by coin-flipping: assign $x_i \leftarrow$ **true** with probability $\frac{1}{2}$ and $x_i \leftarrow$ **false** with probability $\frac{1}{2}$, for $i \in \{1, \dots, n\}$.

Theorem 1 (Unbiased randomized algorithm) *This algorithm gives an expected $\frac{1}{2}$-approximation for* MAX-SAT.

Proof. Note the indicator variable y_j as 1 if clause C_j is satisfied, or 0 if not, $\forall j \in \{1, \dots, m\}$, and W is the total weight of satisfied clauses. By definition, $W = \sum_{j=1}^{m} w_j y_j$, so by the linearity of expectation,

$$E[W] = E[\sum_{j=1}^{m} w_j y_j] = \sum_{j=1}^{m} w_j E[y_j] = \sum_{j=1}^{m} w_j \cdot (1 \cdot \Pr[C_j \text{ satisfied}] + 0 \cdot \Pr[C_j \text{ not satisfied}])$$

Note l_j as the length of clause C_j, the only case of clause C_j is not satisfied is that every literal in it is not satisfied (x_i is set as **false** for a literal x_i in C_j, and x_i is set as **true** for a literal $\bar{x}_i$ in C_j), so based on unbiased randomized assignment, $\Pr[C_j \text{ not satisfied}] = 2^{-l_j}$, and $\Pr[C_j \text{ satisfied}] = 1 - \Pr[C_j \text{ not satisfied}] = 1 - 2^{-l_j}$. As an empty clause is trivial and not allowed, $l_j \geq 1$, so $\Pr[C_j \text{ satisfied}] = 1 - 2^{-l_j} \geq \frac{1}{2}$, thus,

$$E[W] = \sum_{j-1}^{m} w_j \cdot \Pr[C_j \text{ satisfied}] \geq \frac{1}{2} \sum_{j=1}^{m} w_j \geq \frac{1}{2} \cdot \text{OPT}$$

■

10.3 Derandomization via conditional expectations

The approximation factor of the unbiased randomized algorithm is expectational. How to derandomize it, or to ensure that the algorithm gives an $\frac{1}{2}$-approximation solution in any case?

Algorithm 1 A derandomized $\frac{1}{2}$-approximation algorithm for MAX-SAT

for $i \in \{1, \ldots, n\}$ **do**
  if $E[W \mid x_1, \ldots, x_{i-1}, x_i = \textbf{true}] \geq E[W \mid x_1 \ldots, x_{i-1}, x_i = \textbf{false}]$ **then**
    $x_i \leftarrow \textbf{true}$
  else
    $x_i \leftarrow \textbf{false}$
  end if
end for
return $x_1, \ldots, x_n$

Theorem 2 *The derandomized algorithm is an $\frac{1}{2}$-approximation algorithm.*

Proof. By definition of expectation,

$$E[W] = \frac{1}{2}(E[W \mid x_1 = \textbf{true}] + E[W \mid x_1 = \textbf{false}])$$

$$E[W \mid x_1, \ldots, x_{i-1}] = \frac{1}{2}(E[W \mid x_1, \ldots, x_{i-1}, x_i = \textbf{true}] + E[W \mid x_1, \ldots, x_{i-1}, x_i = \textbf{false}]), \forall i \in \{2, \ldots, n\}$$

As the larger of two values is surely no less than their average, in the first round of loop ($i = 1$), it sets x_1 as the value leading to a larger expectation $E[W \mid x_1]$, there must be $E[W \mid x_1] \geq E[W]$ at the end of the first round. Similarly, every round i after that it sets x_i as the value leading to a larger expectation $E[W \mid x_1, \ldots, x_i]$, there must be $E[W \mid x_1, \ldots, x_i] \geq E[W \mid x_1, \ldots, x_{i-1}], \forall i \in \{2, \ldots, n\}$. By induction, $E[W \mid x_1, \ldots, x_n] \geq E[W] \geq \frac{1}{2} \cdot \text{OPT}$, as the expectation $E[W \mid x_1, \ldots, x_n]$ becomes a fixed value of total weight of satisfied clauses after all boolean variables have been assigned, the derandomized algorithm has a definite approximation ratio of $\frac{1}{2}$. ■

There is one more problem about the derandomized algorithm: how to compute the conditional expectation $E[W \mid x_1, \ldots, x_i]$ after the value of $x_1, \ldots, x_i$ have been assigned?

By definition, the conditional expectation $E[W \mid x_1, \ldots, x_i] = \sum_{j=1}^{m} w_j \cdot \Pr[C_j \text{ satisfied} \mid x_1, \ldots, x_i]$. For every clause C_j, if it is already satisfied under the value assignment of $x_1, \ldots, x_i$, then of course $\Pr[C_j \text{ satisfied} \mid x_1, \ldots, x_i] = 1$. Otherwise, assume C_j contains k unassigned variables (k literals of x_j or $\bar{x}_j$ that $i < j \leq n$), the only case of C_j is not satisfied is that all of those unassigned literals are not satisfied, so $\Pr[C_j \text{ satisfied} \mid x_1, \ldots, x_i] = 1 - 2^{-k}$. Then the conditional expectation is known via all the conditional probabilities.

10.4 Randomized rounding algorithm

The other solution to MAX-SAT is interger programming. To describe the model, we introduce two other sets of variables: z_i is 1 if $x_i = \textbf{true}$, or 0 if $x_i = \textbf{false}$, and y_j is 1 if C_j is satisfied, or 0 if C_j is not satisfied. In addition, we separate every clause C_j into variables included as positive literals P_j and variables included as negated literals N_j, such as

$$C_j = \bigvee_{i \in P_j} x_i \vee \bigvee_{i \in N_j} \bar{x}_i$$

Then the integer programming model is described as

$$\begin{aligned}
&\max \sum_{j=1}^{m} w_j y_j && \\
&\texttt{s.t.} \sum_{i \in P_j} z_i + \sum_{i \in N_j} (1 - z_i) \geq y_j && \forall C_j = \bigvee_{i \in P_j} x_i \vee \bigvee_{i \in N_j} \bar{x}_i \\
&z_i \in \{0, 1\}, && i \in \{1, \ldots, n\} \\
&y_j \in \{0, 1\}, && j \in \{1, \ldots, m\}
\end{aligned}$$

The main limitation is to ensure that at least one literal is satisfied if a clause C_j is satisfied ($y_j = 1$), or all the literals is not satisfied, thus C_j is not satisfied ($y_j = 0$).

The corresponding LP relaxation is

$$\begin{aligned}
&\max \sum_{j=1}^{m} w_j y_j && \\
&\texttt{s.t.} \sum_{i \in P_j} z_i + \sum_{i \in N_j} (1 - z_i) \geq y_j && \forall C_j = \bigvee_{i \in P_j} x_i \vee \bigvee_{i \in N_j} \bar{x}_i \\
&0 \leq z_i \leq 1, && i \in \{1, \ldots, n\} \\
&0 \leq y_j \leq 1, && j \in \{1, \ldots, m\}
\end{aligned}$$

As OPT is the optimum of the integer programming model, note OPT_{LP} as the optimum of the relaxation, then obviously $\text{OPT}_{LP} \geq \text{OPT}$.

With the help of LP relaxation, there is a randomized rounding strategy that to solve the relaxation and (z^*, y^*) is the optimum, then set $x_i \leftarrow \textbf{true}$ with probability of z_i^*, and $x_i \leftarrow \textbf{false}$ with probability of $1 - z_i^*$.

Theorem 3 (Randomized rounding algorithm) *The randomized rounding strategy above gives an* $(1 - \frac{1}{e})$-*approximation algorithm of* MAX-SAT.

There are two facts required to proof it.

Definition 1 *A function* $f : \mathbb{R} \to \mathbb{R}$ *is called concave on its domain, if* $f''(x) \leq 0$ *on the entire domain.*

Fact 1 *If a function* $f(x)$ *is concave on* $[0, 1]$*, and* $f(0) = a, f(1) = a + b$*, then* $f(x) \geq a + bx, \forall x \in [0, 1]$.

Fact 2 (Arithmetic-geometric mean inequality) *For any nonnegative* $a_1, \ldots, a_k$*, there is*

$$(\prod_{i=1}^{k} a_i)^{\frac{1}{k}} \leq \frac{1}{k} \sum_{i=1}^{k} a_i$$

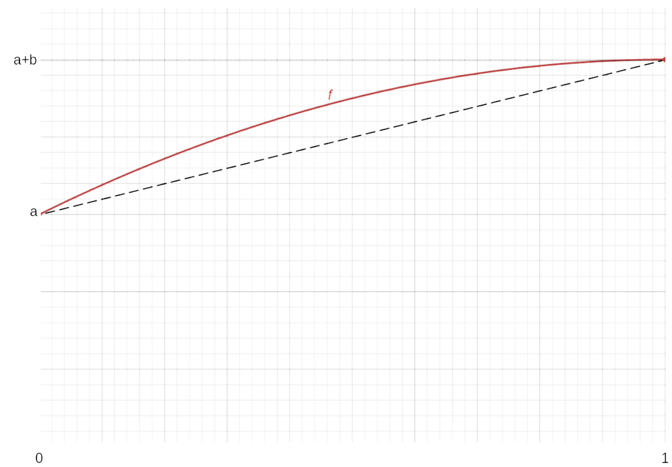

Figure 10.1: A concave function on the interval $[0, 1]$

Proof.[Randomized rounding algorithm] According to Fact 2, for every clause C_j, the probability of not satisfied under assignment by randomized rounding is

$$\begin{aligned}
\Pr[C_j \text{ not satisfied}] &= \prod_{i \in P_j} (1 - z_i^*) \prod_{i \in N_j} z_i^* \\
&\leq [\frac{1}{l_j}(\sum_{i \in P_j} (1 - z_i^*) + \sum_{i \in N_j} z_i^*)]^{l_j} \\
&= \{1 - \frac{1}{l_j}[\sum_{i \in P_j} z_i^* + \sum_{i \in N_j} (1 - z_i^*)]\}^{l_j} \\
&\leq (1 - \frac{y_j^*}{l_j})^{l_j}
\end{aligned}$$

For any specific $l \geq 1$, consider the function $g_l(u) = 1 - (1 - \frac{u}{l})^l, u \in [0, 1]$, then

$$g_l'(u) = -l(1 - \frac{u}{l})^{l-1} \cdot (-\frac{1}{l}) = (1 - \frac{u}{l})^{l-1}$$

$$g_l''(u) = (l-1)(1 - \frac{u}{l})^{l-2} \cdot (-\frac{1}{l}) = \frac{1-l}{l}(1 - \frac{u}{l})^{l-2}, l \geq 2$$

$$g_1''(u) = \frac{\mathrm{d}}{\mathrm{d}u}(1) = 0$$

When $l \geq 1, 0 \leq u \leq 1$, there must be $g_l''(u) \leq 0$, so g_l is concave on $[0, 1]$. As $g_{l_j}(0) = 0, g_{l_j}(1) = 1 - (1 -$

$\frac{1}{l_j})^{l_j}, y_j^* \in [0,1]$, according to fact 1,

$$\begin{aligned}
\Pr[C_j \text{ satisfied}] &= 1 - \Pr[C_j \text{ not satisfied}] \\
&\geq 1 - (1 - \frac{y_j^*}{l_j})^{l_j} \\
&= g_{l_j}(y_j^*) \\
&\geq g_{l_j}(0) + [g_{l_j}(1) - g_{l_j}(0)]y_j^* \\
&= [1 - (1 - \frac{1}{l_j})^{l_j}]y_j^* \\
&\geq (1 - \frac{1}{e})y_j^*
\end{aligned}$$

Therefore,

$$\begin{aligned}
E[W] &= \sum_{j=1}^{m} w_j \cdot \Pr[C_j \text{ satisfied}] \\
&\geq (1 - \frac{1}{e}) \sum_{j=1}^{m} w_j y_j^* \\
&= (1 - \frac{1}{e})\mathrm{OPT}_{LP} \\
&\geq (1 - \frac{1}{e})\mathrm{OPT}
\end{aligned}$$

this algorithm has an approximation-factor of $(1 - \frac{1}{e})$. ■

10.5 Combining both algorithms

Theorem 4 (Meta-ALG) *Let both of the unbiased randomized and the randomized rounding algorithm run, and return the better solution, it leads to a $\frac{3}{4}$-approximation algorithm.*