CS-E400204: Approximation Algorithms **Spring (Fifth Period) 2022**

Midterm 1 (May 10): From Vertex Cover to LP

Due Date: May 19 *Due Time: 23:59 pm*

Question 1. Sender-Receiver

Let $G = (V, E)$ be a graph with non-negative edge costs $c\colon E \to \mathbb{Q}_{\geq 0}$. Let S and R be two disjoint sets of vertices, which we call *sender* and *receiver*. The Sender-Receiver problem asks for a subgraph of G with minimum cost such that every receiver is connected to at least one sender by a path.

a) Show that an *exact* solution can be found in polynomial time if $S \cup R = V$ holds.

[3 points]

b) Show that the general version of the problem is NP-hard, that is, if $R \cup S \neq V$ is also admitted. Then, give a factor-2 approximation algorithm for the general version of the problem.

[5 points]

Question 2. Min-s-t-Cut

A matrix A is called *totally unimodular* if every square submatrix has determinant 0, +1, or −1. Totally unimodular matrices are a quick way to verify that a linear program has an integral optimum. In particular, if A is totally unimodular and b is integral, then the linear program $\{\min c^\top x \mid Ax \geq b, x \geq 0\}$ has an integral optimum for any c.

Consider the Min-s–t-Cut problem from the fourth lecture:

$$\begin{array}{lll} \textbf{minimize} & \displaystyle\sum_{(u,v)\in E} c_{uv} d_{uv}\,, & \\ \textbf{subject to} & d_{uv} - p_u + p_v \geq 0 & \forall (u,v) \in E \setminus \{(t,s)\}, \\ & p_s - p_t \geq 1, & \\ & d_{uv} \geq 0 & \forall (u,v) \in E, \\ & p_u \geq 0 & \forall u \in V. \end{array}$$

Prove that the coefficient matrix A of this LP is totally unimodular and hence the problem has an integral optimum.

[4 points]

Question 3. LP relaxation for Vertex Cover

Consider the following LP relaxation for Vertex Cover on a graph $G = (V, E)$ with vertex weights $c\colon V \to \mathbb{Q}^+$:

$$\begin{array}{lll} \min & \displaystyle\sum_{v\in V} c(v) x_v & \\ \text{s.t.} & x_u + x_v \geq 1 & uv \in E \\ & x_v \geq 0 & v \in V\,. \end{array}$$

a. One can show that extreme points of a polytope are those that cannot be expressed as the convex combination of any two other points of the polytope. Show that in every extreme point solution of this relaxation, $x_v \in \{0, \frac{1}{2}, 1\}$ holds for all $v \in V$. Derive a factor-2 approximation algorithm for vertex-weighted VERTEX COVER from this property.

[4 points]

b. Give a factor-$\frac{3}{2}$ approximation algorithm for planar graphs.

[3 points]

Question 4. Feedback Vertex Set on tournaments

A *tournament* is a directed graph $G = (V, E)$ that contains exactly one of the edges (u, v) and (v, u) for each pair of vertices $u \neq v$. The FEEDBACK VERTEX SET problem asks for a smallest set of vertices whose removal from G results in an acyclic graph.

Show (with proof) a factor-3 approximation algorithm for FEEDBACK VERTEX SET on tournaments.

You can make use of the SimpleGreedySetCover(U, $\mathcal{S}$) from Assignment 02.

[6 points]

Question 5. The Museum Problem

You are the head of security of a museum that is home to some valuable artifacts. The layout of your museum has some hallways and some turns. You want to install some security cameras at some of the turns to monitor the hallways. Once a camera is installed at a turn, all the hallways leading to it are monitored. The catch is that installing a camera at any certain turn has its own cost (maybe different costs for different turns). Your objective is to make sure all the hallways are monitored while minimizing the cost of the installations.

Meanwhile, a third-party security company approaches you with an offer! They claim that they can monitor all the hallways by having guards patrol them. They charge a certain amount per hallway (maybe different charges for different hallways), but in order to convince you to accept, they guarantee that for any turn, the sum they charge for all the hallways leading to that turn will be no more than the cost of installing a camera at that turn.

Should you accepted their offer or not? Explain.

Note: Your answer should not depend on a specific layout and should be justifiable for all layouts (all instances of the problem).

[4 points]