Lecture 6 (May 11): Set Cover via Linear Programming (I)

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6.1 Linear Programming and Approximation

We have spent some time in the past few lectures talking about linear programming and techniques to solve linear programs. In this lecture, we would like to see how to use LPs for the purpose of approximation.

For the rest of this section, assume every optimization problem we discuss is a minimization problem. There are three basic ways that LPs can help us design and/or analyze approximation algorithms:

1. LP rounding

2. Primal-Dual approach

3. Dual fitting

We will discuss each of these in a bit more details now.

6.1.1 LP Rounding

Assume Π is our minimization problem, and we have written an ILP for it (say, with constraints $x_i \in \{0, 1\}$). We first relax the integrality constraints in order to get an *LP relaxation* of the problem (say, by setting $x_i \in [0, 1]$ instead). The optimum for this LP must be lower than the actual integral OPT (why?).

The LP OPT (a.k.a. OPT_{relax}) may be fractional. The next step is to round this solution to an integral one while only losing α factor (our desired approximation factor) in the objective function. This procedure can be deterministic or randomized.

Main challenge: ensure the feasibility of this *rounded* integral solution.

If we success, we will have:

$$\frac{ALG}{\mathrm{OPT}_{\Pi}} \leq \frac{ALG}{\mathrm{OPT}_{relax}} \leq \alpha.$$

Note: Here, we use the fractional value of the relaxed LP as our lower bound on OPT.

Figure 6.1: LP Rounding Scheme

6.1.2 The Primal-Dual Method

In this method, as the name implies, we also make use of the Dual program. We start with a feasible (trivial) solution to the dual, s_d, and an integral, but yet infeasible, solution to the primal, s_Π. One candidate for such a solution might be all primal variables set to zero.

Then, iteratively, we try to improve $obj_{dual}(s_d)$ and make s_Π "more feasible". Foreshadowing: Relaxed CSCs!

Invariant. The primal solution s_Π always remains integral.

We stop when s_Π becomes feasible, in which case our bound is $\alpha \leq \dfrac{obj_\Pi(s_\Pi)}{obj_{dual}(s_d)}$.

Note: Here, we use the objective function value of a feasible dual solution as our lower bound of OPT.

Note: In Primal-Dual algorithm, we mostly do not need to solve any LP! The process of increasing s_d (and changing s_Π) accordingly is usually combinatorial (e.g. greedy) and therefore, such algorithms are faster in practice.

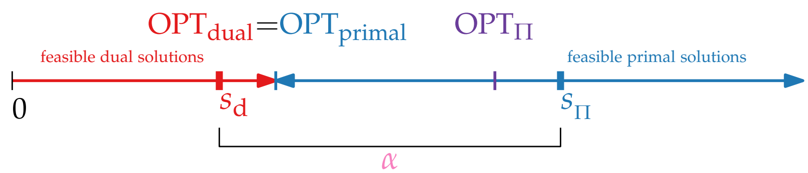

Figure 6.2: Primal-Dual Scheme

6.1.3 Dual Fitting

We use a combinatorial algorithm (e.g. greedy) to find a feasible integral primal solution s_Π and an infeasible dual solution s_d, which can "completely pay" for the primal solution. This means that $obj(s_\Pi) \leq obj(s_d)$.

Then, we scale down the variable values in s_d by a factor, say α to get a feasible solution s'_d. Note that $obj(s_d)/\alpha = obj(s'_d)$. Then, we can write

$$\frac{obj(s_\Pi)}{\alpha} \leq \frac{obj(s_d)}{\alpha} = obj(s'_d) \leq \mathrm{OPT}_{dual} \leq \mathrm{OPT}_\Pi.$$

The scaling factor α will also be the approximation factor.

Note: Again, we use the value of a feasible dual solution as the lower bound of OPT, and we bound the cost of our solution s_Π against it.

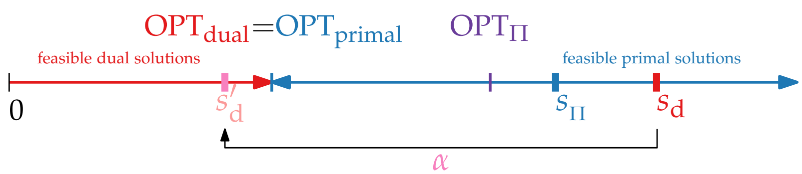

Figure 6.3: Dual Fitting Scheme

6.1.4 Integrality Gap

Every LP method that we discuss here is somewhat limited by the notion of *integrality gap*.

Definition 1 *For a minimization problem* Π *described by an ILP, the integrality gap is defined as*

$$\gamma = \sup_{I \in D_\Pi} \frac{\mathrm{OPT}_\Pi(I)}{\mathrm{OPT}_{primal}(I)}.$$

Integrality gap is a consequence of relaxing an ILP (that can potentially model an NP-hard problem) into an LP (which as we was, is in P). If an α-approximation algorithm does not receive an extra help from any source other than the LP (or as is said in the literature, if the objective value is charged only to the LP value), it is doomed to have $\alpha \geq \gamma$.

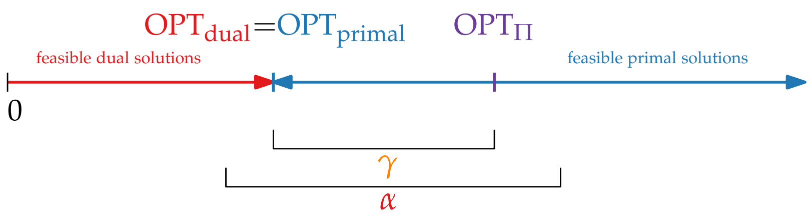

Figure 6.4: The Integrality Gap

6.2 ILP for Set Cover

Reminder:

- **Input**:
  - Ground set U
  - A collection of subsets of U, $\mathcal{S} \in 2^U$
  - A cost function $c : \mathcal{S} \to \mathbb{Q}_{\geq 0}$
- **Output**:
  - Cheapest "cover" $\mathcal{S}' \subseteq \mathcal{S}$ such that $\bigcup_{S \in \mathcal{S}'} S = U$

We write it as an ILP here:

$$\begin{array}{lllll} \text{minimize} & \sum_{S \in \mathcal{S}} c_S \cdot x_S & & & \\ \text{subject to:} & \sum_{S \ni u} x_S & \geq & 1 & \forall u \in U \\ & x_S & \in & \{0,1\} & \forall S \in \mathcal{S} \end{array}$$

Then, we relax the "integrality" constraints:

$$\begin{array}{lllll} \text{minimize} & \sum_{S \in \mathcal{S}} c_S \cdot x_S & & & \\ \text{subject to:} & \sum_{S \ni u} x_S & \geq & 1 & \forall u \in U \\ & x_S & \geq & 0 & \forall S \in \mathcal{S} \end{array}$$

Why did I write $x_S \geq 0$ instead of $x_S \in [0,1]$?

Question. Does this LP relaxation always give the OPT?

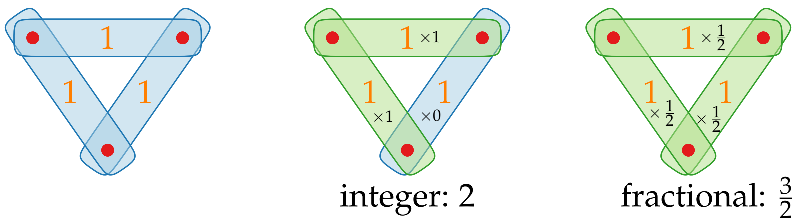

Figure 6.5: Sub-optimal Solution for Set Cover

Now, we can write the dual of the LP as follows:

$$\begin{array}{llccl} \text{maximize} & \sum_{u \in U} y_u & & & \\ \text{subject to:} & \sum_{u \in S} y_u & \leq & c_S & \forall S \in \mathcal{S} \\ & y_u & \geq & 0 & \forall u \in U \end{array}$$

6.2.1 Set Cover via LP Rounding

First, we will see the LP rounding method applied to the Set Cover problem. Assume we have computed x^*, the optimal LP solution. W.l.o.g., we assume $x^*_S \leq 1$ for all $S \in \mathcal{S}$. One way to round this solution to an integral one while guaranteeing feasibility is to round every $x^*_S > 0$ to 1. Why is this rounding not going to work?

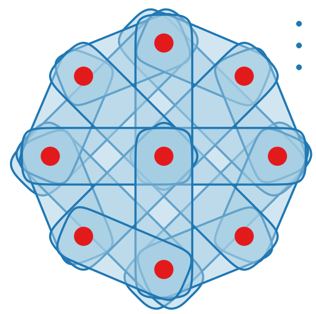

Figure 6.6: Bad example for naïve LP rounding for Set Cover

Hint: The clique example. Why $n/2$ is the optimal LP value?

So, we observe the following: For every element u, let f_u be the *frequency* of u. That is, the number of sets $S \in \mathcal{S}$ in which u is a member of. Note that in the constraint $\sum_{S \ni u} x_S \geq 1 \quad ; \forall u \in U$, the number of terms on the left-hand-side is exactly f_u. So, it implies that one x_S must be at least $1/f_u$.

We now can use this observation for rounding. Let $f = \max_{u \in U}\{f_u\}$. Now, it must be that every constraint has at least one x_S with value at least $1/f$. So, we make it the threshold for rounding. Let $\hat{x}$ is the rounded integral variable. We round according to the following rules:

$$\forall S \in \mathcal{S}: \quad \hat{x}_S = \begin{cases} 1 & ;\text{ if } x_S \geq 1/f \\ 0 & ;\text{ otherwise} \end{cases}$$

Theorem 1 *$\hat{x}$ is a feasible solution for the* Set Cover *instance, and has approximation ratio of f.*

Proof. Feasibility is immediate from the discussions we had. For the approximation ratio, note that the cost only increased for those $S \in \mathcal{S}$ whose x_S was at least $1/f$. It also increased by at most an f factor, so the contribution of their cost c_S to the optimum must have increased by at most f. ■

6.2.2 Set Cover via Primal-Dual

For the Primal-Dual (either for exact or approximate) algorithms, we need to go back to the notion of complementary slackness conditions. As a reminder, these are the exact complementary slackness condition:

Theorem 2 (Theorem 3 from Lecture 04) *Let $x = [x_1, x_2, \ldots, x_n]^T$ and $y = [y_1, y_2, \ldots, y_m]^T$ be two feasible solution for the primal and dual, respectively. Then, x and y are optimal if an only if the following conditions are met:*

- **Primal CSC**. *For each $j \in \{1, 2, \ldots, n\}$, either $x_j = 0$ or $\sum_{i=1}^{m} a_{ij} \cdot y_i = c_j$*
- **Dual CSC**. *For each $i \in \{1, 2, \ldots, m\}$, either $y_i = 0$ or $\sum_{j=1}^{n} a_{ij} \cdot x_j = b_i$*

Remember that for the optimal x and y, then we would have:

$$\sum_{j=1}^{n} c_j \cdot x_j = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} \cdot y_i \right) \cdot x_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} \cdot x_j \right) \cdot y_i = \sum_{i=1}^{m} b_i \cdot y_i.$$

We can relax these two conditions as follows:

- **Relaxed Primal CSC**. For each $j \in \{1, 2, \dots, n\}$, either $x_j = 0$ or $\frac{c_j}{\alpha} \leq \sum_{i=1}^{m} a_{ij} \cdot y_i \leq c_j$
- **Relaxed Dual CSC**. For each $i \in \{1, 2, \dots, m\}$, either $y_i = 0$ or $b_i \leq \sum_{j=1}^{n} a_{ij} \cdot x_j \leq \beta \cdot b_i$

So, if x and y satisfy the relaxed CSC, we have

$$\sum_{j=1}^{n} c_j \cdot x_j \leq \alpha\beta \sum_{i=1}^{m} b_i \cdot y_i \leq \alpha\beta \cdot \text{OPT}_{LP}.$$

which gives an approximate solution with factor $\alpha\beta$ (modulo rounding). So, for SET COVER, our goal is to define relaxed CSCs, and then fashion an integral primal solution that satisfies them. This, clearly gives us an approximation algorithm.

So, we that, let us remind ourselves about the Primal-Dual framework:

1. Start with a feasible dual and an infeasible primal (try the trivial solution)
2. Improve the feasibility of the primal, while simultaneously, improving the objective value of the dual
3. Repeat until the relaxed CSCs you defined are met, while always maintaining the integrality of the primal.

At the end of this procedure, our primal is feasible, and because of the relaxed CSCs, we obtain an approximation factor.

So now, first set out relaxed CSCs that we would like our integral feasible primal to satisfy.

Let's first consider the (non-relaxed) primal and relaxed dual CSCs for our SET COVER LP:

- **Non-relaxed Primal CSC**. For each $S \in \mathcal{S}$, either $x_S = 0$ or $\sum_{u \in S} y_u = c_S$
  - Let's call a set S for which $\sum_{u \in S} y_u = c_S$ a *critical* set.
  - Note that by the primal CSCs, the LP only chooses critical sets in its support. We can use this for making an integral solution to the LP as well.
- **Relaxed Dual CSC**. For each $u \in U$, either $y_u = 0$ or $1 \leq \sum_{S \ni u} x_S \leq f \cdot 1$
  - Here, f is defined as the last section on LP rounding.
  - Note that by the arguments of the last section, the upper bound of f trivially holds for binary values of x_S.

So, here's our full algorithm:

Algorithm 1 A Primal/Dual Approximation for SET COVER

Input: U, $\mathcal{S} \in 2^U$, and cost function c
Output: A set cover for U

$x \leftarrow 0, y \leftarrow 0$
repeat
  select an uncovered element u
  grow y_u until a set S containing u becomes critical
  pick all critical sets in the solution and update x
  mark all the elements in these sets as covered
until all elements are covered
return B

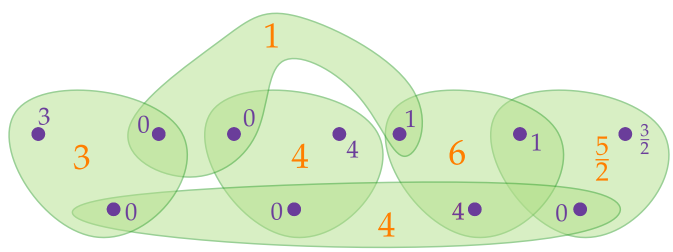

Figure 6.7: Primal-Dual approach for SET COVER

Theorem 3 *Algorithm 1 is an f-approximation for* SET COVER.

Proof. Obviously, the x we created is an integral feasible solution. Due to this, it also must satisfy the relaxed Dual CSCs. On the other hand, y was always a feasible dual solution, and by our algorithm, $x_S \neq 0 \Rightarrow \sum_{u \in S} y_u = c_S$. So, y also satisfies the Primal CSCs. The factor f follows from the previous discussions. ∎

Question. Is factor f tight for this algorithm?

The answer is yes.

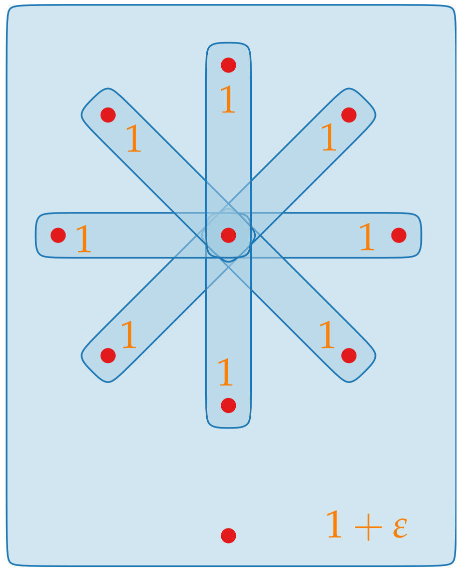

Figure 6.8: Tight example for the Primal/Dual method on Set Cover

In the next lecture, we will see the third method – the dual fitting.