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Exercise 1: Let $\varphi: R^3 \to R$. Show that $\text{Curl}(\nabla\varphi) = 0$

We have: $\nabla\varphi = \frac{\partial\varphi}{\partial x}\vec{i} + \frac{\partial\varphi}{\partial y}\vec{j} + \frac{\partial\varphi}{\partial z}\vec{k}$

$$\Rightarrow \text{Curl}(\nabla\varphi) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} & \frac{\partial\varphi}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}\left(\frac{\partial\varphi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial\varphi}{\partial y}\right)\right)\vec{i} + \left(\frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial\varphi}{\partial x}\right)\right)\vec{j}$$

$$+ \left(\frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial\varphi}{\partial x}\right)\right)\vec{k}$$

$$= \left(\frac{\partial^2\varphi}{\partial y\partial z} - \frac{\partial^2\varphi}{\partial y\partial z}\right)\vec{i} + \left(\frac{\partial^2\varphi}{\partial x\partial z} - \frac{\partial^2\varphi}{\partial x\partial z}\right)\vec{j} + \left(\frac{\partial^2\varphi}{\partial x\partial y} - \frac{\partial^2\varphi}{\partial x\partial y}\right)\vec{k}$$

$$= 0\vec{i} + 0\vec{j} + 0\vec{k}$$

$$= 0$$

Since $\varphi$ is $R^3 \to R$, it is twice continuously differentiable $\Rightarrow$ Their second derivatives are independent of the order.

Example of $\varphi$ is $f(x, y, z) = x^2 + y^2 + z^2$

Exercise 2 Assume that f and g are harmonic functions ($\nabla.\nabla f = \Delta f = 0$)

Show that $\text{div}(f\nabla g - g\nabla f) = 0$

We have: $\text{div}(f) = \nabla \cdot f$

$$\Rightarrow \text{div}(f\nabla g - g\nabla f) = \nabla\cdot(f\nabla g - g\nabla f)$$
$$= \nabla\cdot f\nabla g - \nabla\cdot g\nabla f$$
$$= f\nabla^2 g - g\nabla^2 f$$

Since f and g are harmonic ($\nabla^2 f = \nabla^2 g = 0$)

$$\Rightarrow f\nabla^2 g - g\nabla^2 f = f.0 - g.0 = 0$$

Exercise 3: Assume that $f: R \to R$ is an differentiable function and $\vec{r} = (x, y, z)$

Let $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$. Show that

$$\operatorname{div}(f(r)\vec{r}) = rf'(r) + 3f(r)$$

We have: $r = \sqrt{x^2+y^2+z^2} \Rightarrow \frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot 2x = \frac{x}{r}$

Similarly we have $\frac{\partial r}{\partial y} = \frac{y}{r}$ ; $\frac{\partial r}{\partial z} = \frac{z}{r}$

$$\Rightarrow \nabla r = \left(\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z}\right) = \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) = \frac{1}{r}(x, y, z)$$

$$= \frac{1}{r} \cdot \vec{r} = \frac{\vec{r}}{r}$$

Expanding: $\operatorname{div}(f(r)\vec{r}) = \nabla \cdot (f(r)\vec{r})$

$= f(r)\nabla \cdot \vec{r} + \nabla(f(r)) \cdot \vec{r}$ (product rule)

We have: $\nabla \cdot \vec{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$

$= 1 + 1 + 1 = 3$

$\Rightarrow \operatorname{div}(f(r)\vec{r}) = \nabla(f(r)) \cdot \vec{r} + 3f(r)$

$= [f'(r)\, \nabla(r)] \cdot \vec{r} + 3f(r)$

(Chain rule) $= f'(r)(\nabla(r) \cdot \vec{r}) + 3f(r)$

We have: $\nabla(r) \cdot \vec{r} = \frac{\vec{r}}{r} \cdot \vec{r} = \frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r}$

$$= \frac{x^2+y^2+z^2}{r} = \frac{r^2}{r} = r$$

$\Rightarrow \operatorname{div}(f(r)\vec{r}) = rf'(r) + 3f(r)$ (proven)