1) Calculate the following derivatives

a) $\frac{d}{dx}(\sin^4(2x)) = 4\sin^3(2x) \cdot \frac{d}{dx}(\sin 2x)$

$= 4\sin^3(2x) \cdot \cos 2x \cdot (2x)' = 8\sin^3(2x)\cos(2x)$

b) $\frac{d}{dt}(t^3 \cdot e^{-2t}) = (t^3)' \cdot e^{-2t} + t^3 \cdot (e^{-2t})'$

$= 3t^2 \cdot e^{-2t} + t^3 \cdot e^{-2t} \cdot (-2t)'$

$= 3t^2 \cdot e^{-2t} + t^3 \cdot e^{-2t} \cdot (-2) = t^2 \cdot e^{-2t}(3 - 2t)$

c) $\frac{d}{du}(u\sqrt{1+u^2} + \ln(u + \sqrt{1+u^2})) = \frac{d}{du}(u\sqrt{1+u^2}) + \frac{d}{du}(\ln(u + \sqrt{1+u^2}))$

$= u'\sqrt{1+u^2} + u \cdot (\sqrt{1+u^2})' + \frac{1}{u+\sqrt{1+u^2}} \cdot (u + \sqrt{1+u^2})'$

$= \sqrt{1+u^2} + u \cdot \frac{1}{2\sqrt{1+u^2}} \cdot 2u + \frac{1}{u+\sqrt{1+u^2}} \cdot (1 + \frac{1}{2\sqrt{1+u^2}} \cdot 2u)$

$= \sqrt{1+u^2} + \frac{u^2}{\sqrt{1+u^2}} + \frac{1}{u+\sqrt{1+u^2}} \cdot \frac{u+\sqrt{1+u^2}}{\sqrt{1+u^2}}$

$= \frac{2u^2+1}{\sqrt{1+u^2}} + \frac{1}{\sqrt{1+u^2}} = \frac{2(u^2+1)}{\sqrt{1+u^2}} = 2\sqrt{1+u^2}$

2) Calculate the following integrals

a) $\int (x^3 + x^{1/3})dx = \frac{x^4}{4} + \frac{x^{4/3}}{\frac{4}{3}} + C = \frac{1}{4}x^4 + \frac{3}{4}x^{\frac{4}{3}} + C$

b) $\int_0^{\frac{\pi}{2}} \cos^3(t)\sin^2(t)dt = \int_0^{\frac{\pi}{2}} \sin^2(t) \cdot \cos^2(t)\cos(t)dt$

$= \int_0^{\frac{\pi}{2}} \sin^2(t) \cdot (1 - \sin^2(t)) \cdot \cos(t)dt$

Substitute $u = \sin(t)$ and $du = \cos(t)dt$

$= \int_0^{\frac{\pi}{2}} u^2(1-u^2)du = \int_0^{\frac{\pi}{2}}(u^2 - u^4)du = \frac{u^3}{3} - \frac{u^5}{5}\Big|_0^{\frac{\pi}{2}}$

$= \frac{\sin^3(t)}{3} - \frac{\sin^5(t)}{5}\Big|_0^{\frac{\pi}{2}} = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$

c) $\int_0^\infty ye^{-y}dy$

Let $u = y$ $\quad dv = e^{-y}dy$

$\Rightarrow du = dy \quad v = -e^{-y}$

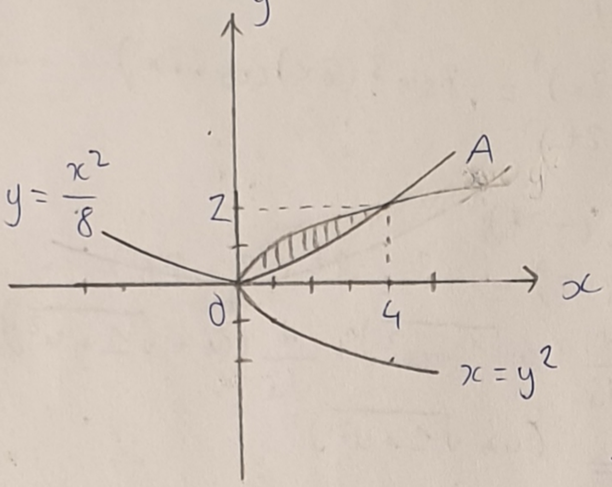
$\Rightarrow \int_0^\infty ye^{-y}dy = -ye^{-y}\Big|_0^\infty + \int_0^\infty e^{-y}dy = -ye^{-y} - e^{-y}\Big|_0^\infty$

$= -e^{-y}(y+1)\Big|_0^\infty = \left[\lim_{y\to\infty} -e^{-y}(y+1)\right] + 1 = \lim_{y\to\infty}(-e^{-y} \cdot y - e^{-y}) + 1$

$= \lim_{y\to\infty}\left(-\frac{1}{e^y} \cdot y - \frac{1}{e^y}\right) + 1 = 0 + 1 = 1$

(e^y grows faster than y when $y \to \infty$ so $-\frac{1}{e^y} \cdot y$ will approach 0 as $y \to \infty$)

3) Sketch the region bounded by 2 parabolos $x = y^2$ and $y = \frac{x^2}{8}$, and calculate its area



$x = y^2 \Rightarrow y = \pm\sqrt{x}$

Only $y = \sqrt{x}$ intersects with $y = \frac{x^2}{8}$

We have the intersection of $\sqrt{x}$ and $\frac{x^2}{8}$

$$\sqrt{x} = \frac{x^2}{8} \Rightarrow \left[\begin{array}{l} x = 0, y = 0 \\ x = 4, y = 2 \end{array}\right.$$

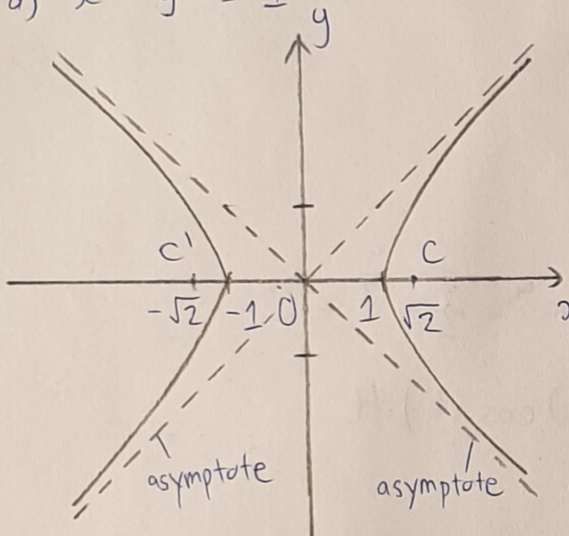
We have: $S_{region} = \int_0^4 \sqrt{x}\,dx - \int_0^4 \frac{x^2}{8}\,dx$

$$\Rightarrow S_{region} = \frac{2}{3}x^{\frac{3}{2}}\Big|_0^4 - \frac{x^3}{24}\Big|_0^4$$

$$\Rightarrow S_{region} = \frac{16}{3} - \frac{8}{3} = \frac{8}{3}$$

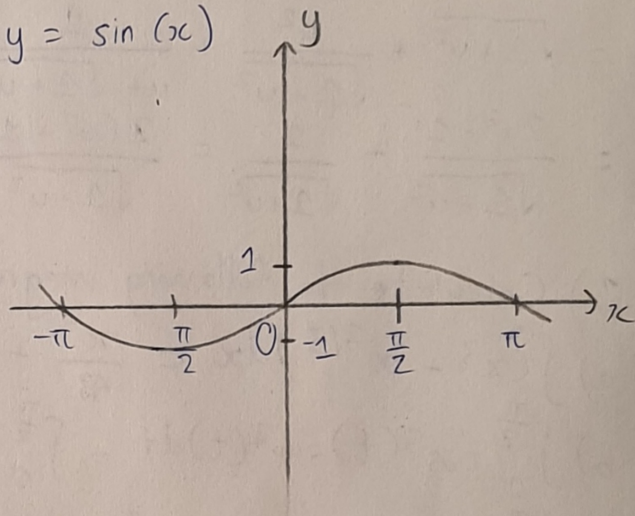
4) Sketch the curves

a) $x^2 - y^2 = 1$



This curve is called hyperbola
C & C' are focuses

b) $y = \sin(x)$

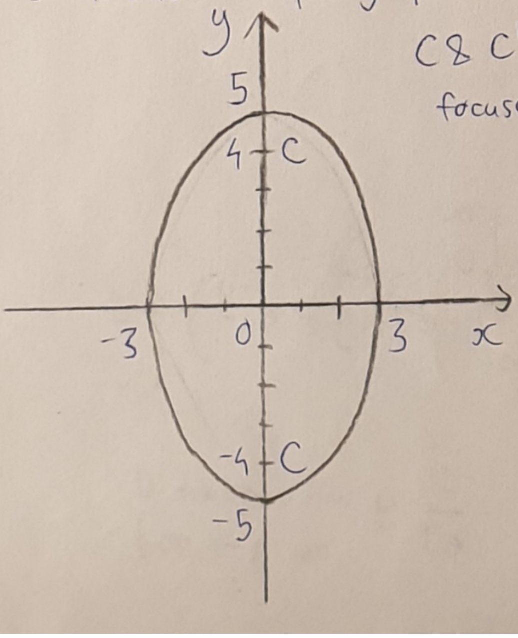


c) $x(t) = 3\cos(t)$, $y(t) = 5\sin(t)$, $0 \leq t \leq 2\pi$

We have: $\frac{x(t)}{3} = \cos(t)$, $\frac{y(t)}{5} = \sin(t)$

$$\Rightarrow \frac{x^2(t)}{9} + \frac{y^2(t)}{25} = \cos^2(t) + \sin^2(t) = 1$$

$\Rightarrow$ This is an ellipse graph



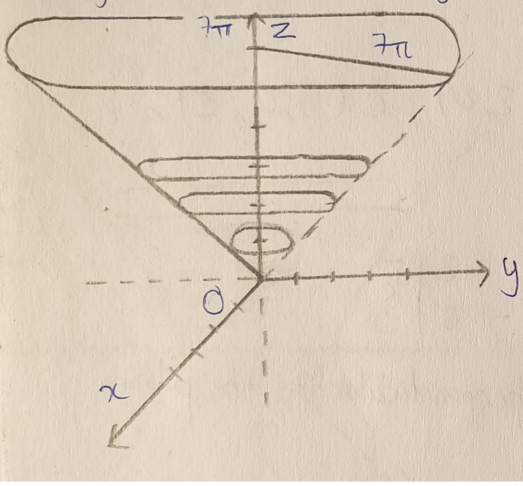
C & C' are focuses

d) $x(t) = t\cos(t)$, $y(t) = t\sin(t)$, $z(t) = t$ for $0 \leq t \leq 7\pi$

We have : $x^2(t) + y^2(t) = t^2(\cos^2(t) + \sin^2(t)) = t^2$

$z^2(t) = t^2$

$\Rightarrow$ $x^2 + y^2 = z^2$ $\rightarrow$ it is equation of circle on xy-plane with changing radius z along the z axis, creating 2 cones. As $0 \leq t \leq 7\pi$, the cone will have ~~height~~ radius of 7π at its base



5) Find equation of line passing through the points $(0, 1, 1)$ and $(3, 4, -1)$

Slope vector: $\vec{u} = (3-0, 4-1, -1-1) = (3, 3, -2)$

$\Rightarrow \vec{r}(t) = (0, 1, 1) + t(3, 3, -2)$

Parametric equation of the line

$$\begin{cases} x(t) = 3t \\ y(t) = 3t + 1 \quad (\text{with } t \in \mathbb{R}) \\ z(t) = -2t + 1 \end{cases}$$

6) Find equation of the plane passing through $A(1,0,0)$, $B(0,2,0)$, $C(0,0,1)$.

Sketch the plane

We have: $\vec{AB} = (-1, 2, 0)$ $\vec{AC} = (-1, 0, 1)$

$$\Rightarrow \vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ -1 & 2 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} \vec{i} - \begin{vmatrix} -1 & 0 \\ -1 & 1 \end{vmatrix} \vec{j} + \begin{vmatrix} -1 & 2 \\ -1 & 0 \end{vmatrix} \vec{k}$$

$= 2\vec{i} + \vec{j} + 2\vec{k} \Rightarrow \vec{u}(2, 1, 2)$ is perpendicular to the plane

Equation: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

with $\vec{u} = (2, 1, 2) = (a, b, c)$ and $A(1, 0, 0) = (x_0, y_0, z_0)$

$\Rightarrow 2(x - 1) + 1(y - 0) + 2(z - 0) = 0$

$\Rightarrow 2x + y + 2z - 2 = 0$ (Answer)

7) Using limits, state definites of the following

a) Derivative of function $f(x)$ at point A

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

b) Integral of function $g(x)$ on closed interval $[a, b]$

$$\int_a^b g(x)\,dx = \lim_{n \to \infty} \sum_{k=1}^{n} f\left(a + \left(\frac{b-a}{n}\right)k\right) \cdot \frac{b-a}{n}$$

(Sum of infinitely thin rectangles under the curve)