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Exercise 1: Let $F(x,y) = \left(-y + x\sqrt{x^2+y^2},\ x + y\sqrt{x^2+y^2}\right)$

a) Write the vector field in polar coordinates

In polar coordinates, we have $\begin{cases} \hat{r} = (\cos\theta, \sin\theta) \\ \hat{\theta} = (-\sin\theta, \cos\theta) \end{cases} \Rightarrow \hat{r} \cdot \hat{\theta} = 0$

We have: $x = R\cos\theta$, $y = R\sin\theta$

$\Rightarrow F(R,\theta) = (-R\sin\theta + R^2\cos\theta,\ R\cos\theta + R^2\sin\theta)$

We have: $F_R = F \cdot \hat{r} = (-R\sin\theta + R^2\cos\theta,\ R\cos\theta + R^2\sin\theta) \cdot (\cos\theta, \sin\theta)$

$= -R\sin\theta\cos\theta + R^2\cos^2\theta + R\sin\theta\cos\theta + R^2\sin^2\theta$

$= R^2(\sin^2\theta + \cos^2\theta) = R^2$

$$F_\theta = F\cdot\hat{\theta} = (-R\sin\theta + R^2\cos\theta,\ R\cos\theta + R^2\sin\theta)\cdot(-\sin\theta,\ \cos\theta)$$
$$= R\sin^2\theta - R^2\sin\theta\cos\theta + R\cos^2\theta + R^2\sin\theta\cos\theta$$
$$= R(\sin^2\theta + \cos^2\theta) = R$$

$\Rightarrow$ The vector field in polar coordinates is $F = F_r\hat{r} + F_\theta\hat{\theta} = R^2\hat{r} + R\hat{\theta}$

b) Calculate div F in polar coordinates

~~We have:~~ $\cancel{\dfrac{\partial F}{\partial r} = (-\sin\theta + 2R\cos\theta,\ \cos\theta + 2R\sin\theta)}$

$\cancel{\dfrac{\partial F}{\partial \theta} = (-R\cos\theta - R^2\sin\theta,\ -R\sin\theta + R^2\cos\theta)}$

$h_r = 1 \qquad h_\theta = R$

$$\Rightarrow \operatorname{div} F = \frac{1}{h_r h_\theta}\left(\frac{\partial}{\partial r}(h_\theta F_r) + \frac{\partial}{\partial\theta}(h_r F_\theta)\right)$$
$$= \frac{1}{R}\left(\frac{\partial}{\partial r}(R^3) + \frac{\partial}{\partial\theta}(R)\right) = \frac{1}{R}(3R^2) = 3R$$

Exercise 2: Define curvilinear coordinates in xy-space via $\vec{r}(u,v) = (x(u,v), y(u,v)) = (u^2 - v^2, 2uv)$ is orthogonal when $(x,y) \neq (0,0)$

a) We have: $\begin{cases} x = u^2 - v^2 = (-u)^2 - (-v)^2 \\ y = 2uv = 2(-u)(-v) \end{cases}$

Since (x,y) corresponds to both (u,v) and $(-u,-v)$

$\Rightarrow \vec{r} : \mathbb{R}^2 \to \mathbb{R}^2 \setminus (0,0)$ is not bijective

b) What are the scale factors for this coordinate change?

We have: $\dfrac{\partial\vec{r}}{\partial u} = (2u, 2v) \quad \dfrac{\partial\vec{r}}{\partial v} = (-2v, 2u)$

$$\Rightarrow h_u = \left|\frac{\partial\vec{r}}{\partial u}\right| = \sqrt{(2u)^2 + (2v)^2} = 2\sqrt{u^2+v^2}$$
$$h_v = \left|\frac{\partial\vec{r}}{\partial v}\right| = \sqrt{(-2v)^2 + (2u)^2} = 2\sqrt{u^2+v^2}$$

$\Rightarrow$ Scale factors are $h_u = h_v = 2\sqrt{u^2+v^2}$

Exercise 3: Let $F(r,\theta,z) = r^2\hat{r} + r\hat{\theta} + z\hat{z}$ in cylindrical coordinates

a) Calculate $\operatorname{div} F$

The scale factors: $h_R = h_z = 1$, $h_\theta = R$

$$\Rightarrow \operatorname{div} F = \frac{1}{h_R h_z h_\theta}\left(\frac{\partial}{\partial R}(h_z h_\theta F_r) + \frac{\partial}{\partial z}(h_R h_\theta F_z) + \frac{\partial}{\partial\theta}(h_R h_z F_\theta)\right)$$

$$= \frac{1}{R}\left(\frac{\partial}{\partial R}(R.R^2) + \frac{\partial}{\partial z}(Rz) + \frac{\partial}{\partial\theta}(R)\right)$$

$$= \frac{1}{R}\left(3R^2 + R\right) = 3R + 1 \text{ (answer)}$$

b) Calculate $\operatorname{Curl} F$

$$\operatorname{Curl} F = \frac{1}{h_R h_z h_\theta}\begin{vmatrix} h_R\hat{r} & h_\theta\hat{\theta} & h_z\hat{z} \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ h_R F_R & h_\theta F_\theta & h_z F_z \end{vmatrix} = \frac{1}{R}\begin{vmatrix} \hat{r} & R\hat{\theta} & \hat{z} \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ R^2 & R^2 & z \end{vmatrix}$$

$$\Rightarrow \operatorname{Curl} F = \frac{1}{R}\left[\hat{r}\left(\frac{\partial}{\partial\theta}(z) - \frac{\partial}{\partial z}(R^2)\right) - R\hat{\theta}\left(\frac{\partial}{\partial R}(z) - \frac{\partial}{\partial z}(R^2)\right) + \hat{z}\left(\frac{\partial}{\partial R}(R^2) - \frac{\partial}{\partial\theta}(R^2)\right)\right]$$

$$\operatorname{Curl} F = \frac{1}{R}\left(0\hat{r} - R\hat{\theta}.0 + \hat{z}(2R)\right) = 0\hat{r} + 0\hat{\theta} + 2\hat{z}$$

$$= 2\hat{z} \text{ (answer)}$$

Exercise 4: Let $f(R,\phi,\theta)$ be a function given in spherical coordinates in $\mathbb{R}^3$. Deduce a formula for $\Delta f = \operatorname{div}(\nabla f)$ in spherical coordinates

We have: $h_R = 1$, $h_\varphi = R$, $h_\theta = R\sin\varphi$

(In this exercise, θ is the angle of rotation of from x-axis and φ is the angle of rotation from z-axis)

$$\Rightarrow \begin{cases} x = r\sin\varphi\cos\theta \\ y = r\sin\varphi\sin\theta \\ z = r\cos\varphi \end{cases}$$

$$\nabla f = \frac{1}{h_R}\cdot\frac{\partial f}{\partial R}\cdot\hat{u} + \frac{1}{h_\theta}\cdot\frac{\partial f}{\partial\theta}\cdot\hat{\theta} + \frac{1}{h_\varphi}\cdot\frac{\partial f}{\partial\varphi}\cdot\hat{\varphi} = \underbrace{\frac{\partial f}{\partial R}}_{F_R}\hat{r} + \underbrace{\frac{1}{R\sin\varphi}\frac{\partial f}{\partial\theta}}_{F_\theta}\hat{\theta} + \underbrace{\frac{1}{R}\cdot\frac{\partial f}{\partial\varphi}}_{F_\varphi}\hat{\varphi}$$

$$\Rightarrow \operatorname{div}(\nabla f) = \frac{1}{h_R h_\theta h_\varphi}\left(\frac{\partial}{\partial R}(h_\theta h_\varphi F_R) + \frac{\partial}{\partial\theta}(h_R h_\varphi F_\theta) + \frac{\partial}{\partial\varphi}(h_R h_\theta F_\varphi)\right)$$

$$\operatorname{div}(\nabla f) = \frac{1}{R^2\sin\varphi}\left(\frac{\partial}{\partial R}\left(R^2\sin\varphi\frac{\partial f}{\partial R}\right) + \frac{\partial}{\partial\theta}\left(R.\frac{1}{R\sin\varphi}\frac{\partial f}{\partial\theta}\right) + \frac{\partial}{\partial\varphi}\left(R\sin\varphi\,\frac{1}{R}\cdot\frac{\partial f}{\partial\varphi}\right)\right)$$

$$\underbrace{\operatorname{div}(\nabla f)}_{\Delta f} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial f}{\partial R}\right) + \frac{1}{R^2\sin^2\varphi}\cdot\frac{\partial^2 f}{\partial\theta^2} + \frac{1}{R^2\sin\varphi}\cdot\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial f}{\partial\varphi}\right) \text{ (answer)}$$