Hand-in Exercises 3

(1) Calculate $\oint_\gamma x\,dy$ where γ is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (a>0 \text{ and } b>0)$$

counterclockwise.

Solution: Here we can use Green's Theorem also but we use the parametrization since we haven't talked about Green's Theorem yet.

$$\gamma(t) = \begin{cases} a\cos t \\ b\sin t \end{cases}, \quad 0 \le t \le 2\pi$$

$$\oint_\gamma x\,dy = \int_0^{2\pi} x(t)\frac{dy}{dt}\,dt = \int_0^{2\pi} a\cos t \cdot b\cos t\,dt$$

$$= ab\int_0^{2\pi} \cos^2 t\,dt = (*)$$

Since $\cos 2t = \cos^2 t - \sin^2 t = 2\cos^2 t - 1$ we get

$$(*) = ab\int_0^{2\pi} \frac{1+\cos 2t}{2}\,dt = ab\pi$$

(2) Calculate $\iint_S x \, dS$ where S is the part of $z = x^2$ above the rectangle

$$\{(x,y,0) \in \mathbb{R}^3;\ 0 \le x \le 2,\ 0 \le y \le 3\}$$

Solution:

We parametrize

$$\vec{r}(u,v) = (u, v, u^2), \quad \begin{array}{l} 0 \le u \le 2 \\ 0 \le v \le 3 \end{array}$$

$$\frac{\partial \vec{r}}{\partial u} = (1, 0, 2u) \qquad \frac{\partial \vec{r}}{\partial v} = (0, 1, 0)$$

$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \begin{vmatrix} \vec{e_1} & \vec{e_2} & \vec{e_3} \\ 1 & 0 & 2u \\ 0 & 1 & 0 \end{vmatrix} = (-2u, 0, 1)$$

$$\left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| = \sqrt{1 + 4u^2}$$

$$\iint_S x \, dS = \int_0^3 dv \int_0^2 u \sqrt{1+4u^2} \, du$$

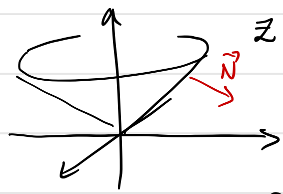
$$= 3 \int_0^2 u \sqrt{1+4u^2} \, du = \left\{ \begin{array}{ll} t = 1+4u^2 & t_2 = 17 \\ dt = 8u\,du & t_0 = 1 \end{array} \right\} =$$

$$= 3 \int_1^{17} \frac{1}{8} t^{1/2} \, dt = \frac{3}{8} \left[\frac{t^{3/2}}{3/2} \right]_1^{17} =$$

$$= \frac{1}{4} \left(17\sqrt{17} - 1 \right)$$

(3) Calculate the flux of $F(x,y,z) = (xy, 0, -1)$ outward (away from the z-axis) through the cone $z^2 = x^2 + y^2$ where $0 \le z \le 1$.

<u>Solution:</u>



$z = \sqrt{x^2+y^2}$

$\{ G(x,y,z) = x^2+y^2-z^2 = 0 \} = S$

$$\nabla G = (2x, 2y, -2z)$$

$$\frac{\nabla G}{G_z} = \frac{(2x, 2y, -2z)}{-2z} = -\left(\frac{x}{z}, \frac{y}{z}, -1\right)$$ points towards the z-axis

$$\iint_S \vec{F} \cdot \vec{N}\, dS = \iint_{x^2+y^2 \le 1} (xy, 0, -1) \cdot \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1\right) dxdy$$

$$= \iint_{x^2+y^2 \le 1} \frac{x^2 y}{\sqrt{x^2+y^2}} + 1\, dxdy \overset{\text{polar coord.}}{=}$$

$$= \int_0^{2\pi} \int_0^1 \left(\frac{r^3 \cos^2\theta \sin\theta}{r} + 1 \right) r\, dr\, d\theta =$$

$$= \int_0^{2\pi} \left[\frac{r^4}{4} \cos^2\theta \sin\theta + \frac{r^2}{2} \right]_0^1 d\theta =$$

$$= \int_0^{2\pi} \frac{1}{4} \cos^2\theta \sin\theta + \frac{1}{2}\, d\theta =$$

$$= \frac{1}{4} \int_0^{2\pi} \cos^2\theta \sin\theta\, d\theta + \pi = (*)$$

$$\int \cos^2\theta \sin\theta \, d\theta = -\frac{\cos^3\theta}{3} + C$$

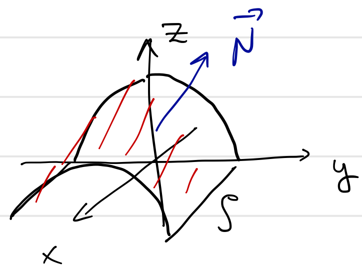
$$\Rightarrow \quad (*) = \frac{1}{4}\left[-\frac{\cos^3\theta}{3}\right]_0^{2\pi} + \pi = \pi$$

(4) Calculate the flux of

$$F(x,y,z) = (z^2, x, -3z)$$

outward through the surface cut from the parabolic cylinder $z = 4 - y^2$ by the planes $x = 0$, $x = 1$ and $z = 0$.

Solution:



$$\iint_S F \cdot N \, dS = ?$$

$$S = \{ G(x,y,z) = z + y^2 - 4 = 0 \ ; \ 0 \le x \le 1, -2 \le y \le 2 \}$$

$$\nabla G = (0, 2y, 1) \qquad \frac{\partial G}{\partial z} = 1$$

$$\iint_S F \cdot N\,dS = \int_0^1 \left(\int_{-2}^{2} (z^2, x, -3z) \cdot (0, 2y, 1)\, dy \right) dx$$

$$= \int_0^1 \left(\int_{-2}^{2} 2xy - 3z\, dy \right) dx =$$

$$= \int_0^1 \left(\int_{-2}^{2} 2xy - 3(4-y^2)\, dy \right) dx =$$

$$= \int_0^1 \left[xy^2 - 12y + y^3 \right]_{y=-2}^{y=2} dx =$$

$$= \int_0^1 4x - 4x - 12 \cdot 2 + 12(-2) + 8 - (-8)\, dx$$

$$= \int_0^1 -48 + 16\, dx = -32$$

[redacted] Homework 3

(1) Calculate $\int_\gamma x^2 ds$ where γ is the line from the origin to $(3,1,-2)$.

<u>Solution</u>: We parametrize γ as $\gamma(t) = (3t, t, -2t)$, $0 \leq t \leq 1$

$$\int_\gamma x^2 ds = \int_0^1 x(t)^2 \left| \frac{d\gamma}{dt} \right| dt$$

We have $\frac{d\gamma}{dt} = (3,1,-2)$ and $\left|\frac{d\gamma}{dt}\right| = \sqrt{9+1+4} = \sqrt{14}$

$$\int_\gamma x^2 ds = \int_0^1 9t^2 \cdot \sqrt{14}\, dt = \left[3t^3\sqrt{14}\right]_0^1 = 3\sqrt{14}.$$

(2) Let $F(x,y,z) = (y^2\cos x + z^3,\ 2y\sin x - 4,\ 3xz^2 + 2)$.
Calculate
$$\int_\gamma F \cdot d\vec{r}$$
where $\gamma(t) = (\arcsin t, 1-2t, 3t-1)$, $0 \leq t \leq 1$.

<u>Solution</u>: We check if F is conservative.
If ϕ such that $\nabla\phi = F$ then

$$\frac{\partial \phi}{\partial x} = y^2\cos x + z^3 \Rightarrow \phi(x,y,z) = y^2 \sin x + z^3 x + \alpha(y,z)$$

$$\frac{\partial \phi}{\partial y} = 2y\sin x + \frac{\partial \alpha}{\partial y} \Rightarrow \frac{\partial \alpha}{\partial y} = -4$$

$$\Rightarrow \alpha(y,z) = -4y + \beta(z)$$

$$\phi(x,y,z) = y^2 \sin x + z^3 x - 4y + \beta(z)$$

$$\frac{\partial \phi}{\partial z} = 3z^2 x + \beta'(z) \implies \beta'(z) = 2 \implies \beta(z) = 2z + C$$

$$\implies \phi(x,y,z) = y^2 \sin x + z^3 x - 4y + 2z + C$$

Note $\gamma(0) = (\arcsin 0,\ 1-2\cdot 0,\ 3\cdot 0 - 1) = (0,1,-1)$

and $\gamma(1) = (\arcsin 1,\ 1-2\cdot 1,\ 3\cdot 1 - 1) = (\frac{\pi}{2}, -1, 2)$

Therefore $\int_\gamma F \cdot d\vec{r} = \phi(\frac{\pi}{2}, -1, 2) - \phi(0, 1, -1) =$

$$= (-1)^2 \sin\frac{\pi}{2} + \frac{\pi}{2} \cdot 2^3 - 4(-1) + 2\cdot 2 -$$
$$-\left(1^2 \cdot \sin 0 + 0\cdot(-1)^3 - 4\cdot 1 + 2(-1)\right) =$$
$$= 1 + 4\pi + 4 + 4 - (-4-2) = 15 + 4\pi$$

3. Find the surface area of the part of the sphere defined as

$$S = \{(x,y,z) \in \mathbb{R}^3;\ x^2+y^2+z^2 = 2,\ x^2+y^2 \le 1, \text{ and } z \ge 0\}.$$

Solution: Surface area $= \iint_S 1 \, dS =$

$$= \iint_{x^2+y^2 \le 1} \frac{|\nabla G|}{\partial G/\partial z} \, dx\,dy \quad \text{where}$$

$G(x,y,z) = x^2+y^2+z^2 = 2$

$$\nabla G = (2x, 2y, 2z) \qquad |\nabla G| = 2\sqrt{x^2+y^2+z^2} = 2\sqrt{2}$$

$$\text{Surface area} = \iint_{x^2+y^2\le 1} \frac{2\sqrt{2}}{2z}\,dx\,dy =$$

$$= \iint_{x^2+y^2\le 1} \frac{\sqrt{2}}{\sqrt{2-x^2-y^2}}\,dx\,dy = \sqrt{2}\int_0^{2\pi}\int_0^1 \frac{r}{\sqrt{2-r^2}}\,dr\,d\theta$$

$$= \sqrt{2}\cdot 2\pi \int_0^1 \frac{r}{\sqrt{2-r^2}}\,dr = \begin{bmatrix} t = 2-r^2 & t_1 = 2-1^2 = 1 \\ dt = -2r\,dr & t_0 = 2-0^2 = 2 \end{bmatrix}$$

$$= \sqrt{2}\cdot 2\pi\left(-\frac{1}{2}\right)\int_2^1 t^{-1/2}\,dt = \pi\sqrt{2}\int_1^2 t^{-1/2}\,dt$$

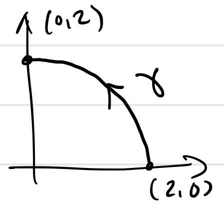
$$= \pi\sqrt{2}\left[\frac{t^{1/2}}{1/2}\right]_1^2 = 2\pi\sqrt{2}\left(\sqrt{2}-1\right) = \cancel{2\pi(4-\sqrt{2})}$$

$$2\pi\left(2-\sqrt{2}\right)$$

Demo Exercises 3

① Calculate $\int_\gamma x+y\,ds$ where γ is the part of the circle $x^2+y^2=4$ in the first quadrant from $(2,0)$ to $(0,2)$.

<u>Solution:</u> Parametrize γ



$$\gamma(t) = (2\cos t, 2\sin t)$$

$$0 \le t \le \frac{\pi}{2}$$

We have
$$\int_\gamma x+y\,ds = \int_0^{\pi/2} (x(t)+y(t)) \left|\frac{d\gamma}{dt}\right| dt$$

We have $\frac{d\gamma}{dt} = (-2\sin t, 2\cos t)$ and
$$\left|\frac{d\gamma}{dt}\right| = \sqrt{4\sin^2 t + 4\cos^2 t} = 2$$

Therefore $\int_\gamma x+y\,ds = \int_0^{\pi/2} (2\cos t + 2\sin t)\,2\,dt$

$$= 4\left(\int_0^{\pi/2} \cos t\,dt + \int_0^{\pi/2} \sin t\,dt\right) = 4(1+1) = 8.$$

(2) Calculate $\int_\gamma y\,dx + z\,dy - x\,dz$ where γ is the straight line from $(0,0,0)$ to $(1,1,1)$.

<u>Solution</u>: Parametrize the line γ.

$$\gamma(t) = (t,t,t) \ ; \ 0 \le t \le 1$$

$$\int_\gamma y\,dx + z\,dy - x\,dz = \int_0^1 \left(y(t)\frac{dx}{dt} + z(t)\frac{dy}{dt} - x(t)\frac{dz}{dt} \right) dt$$

$$= \int_0^1 (t + t - t)\,dt = \int_0^1 t\,dt = \left[\frac{t^2}{2}\right]_0^1 = \frac{1}{2}.$$

(3) Calculate the circulation of the vector field $F(x,y) = (x-y, x)$ around the unit circle $\gamma(t) = (\cos t, \sin t)$, $0 \le t \le 2\pi$.

<u>Solution</u>: We calculate $\oint_\gamma F \cdot \hat{T}\,ds =$

$$= \int_\gamma (x-y)\,dx + x\,dy =$$

$$= \int_0^{2\pi} (\cos t - \sin t)\frac{dx}{dt} + \cos t \frac{dy}{dt}\,dt =$$

$$= \int_0^{2\pi} (\cos t - \sin t)(-\sin t) + \cos t(\cos t)\,dt =$$

$$= \int_0^{2\pi} -\cos t \sin t + \sin^2 t + \cos^2 t\,dt =$$

$$= \int_0^{2\pi} 1 - \cos t \sin t\,dt =$$

$$= 2\pi - \int_0^{2\pi} \frac{\sin 2t}{2}\,dt = 2\pi$$