Homework 2

[redacted]

① Let $F(x,y) = (e^x, e^{-x})$. Determine the integral curves for the vector field.

Solution: We study

$$\frac{dx}{e^x} = \frac{dy}{e^{-x}} \iff \int \frac{dx}{e^{2x}} = \int dy$$

We get

$$-\frac{1}{2}e^{-2x} + C = y$$

and therefore the integral curves are given by $y = C - \frac{1}{2}e^{-2x}$ for $C \in \mathbb{R}$.

② Let $F(x,y) = (2x-2y, 2y-2x)$. Show that F is conservative by constructing a potential function. Determine the equipotential curves and the integral curves for F.

Solution:

$$\frac{\partial \phi}{\partial x} = 2x - 2y \implies \phi(x,y) = x^2 - 2xy + \alpha(y)$$

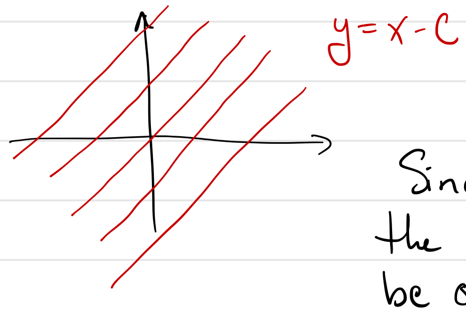
$$\implies \frac{\partial \phi}{\partial y} = -2x + \alpha'(y) \implies \alpha'(y) = 2y$$

$$\implies \alpha(y) = y^2 + C$$

$$\implies \phi(x,y) = x^2 - 2xy + y^2 = (x-y)^2$$

is a potential function.

Equipotential curves are $x - y = C$ for $C \in \mathbb{R}$



Since we know that the integral curves need to be orthogonal trajectories to the equipotential curves it is easy to guess that $y = -x + D$ gives integral curves.
We see that $F(x,y) = (2x - 2y)(1,-1)$ all are tangential to $y = -x + D$ (unless $x = y$)

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③ Let $F(x,y) = (x, 1/y)$. Where is the vector field defined? Determine the integral curves for F.

Solution:

The vector field is defined when $y \neq 0$ (That is off the x-axis)

Study
$$\int \frac{dx}{x} = \int y \, dy$$

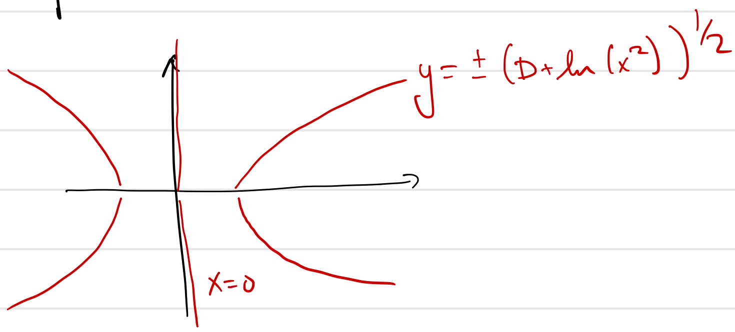
$$\Longrightarrow \quad \ln|x| = \frac{y^2}{2} - c$$

$y^2 = 2\ln|x| + D$

$y = \pm\left(D + \ln(x^2)\right)^{1/2}$ where $\ln(x^2) > D$

If $x = 0$ then we see that $F(0,y) = (0, \frac{1}{y})$

has integral curve $x=0$.



Hand-in exercises 2

(1) Consider the vector field

$$F(x,y,z) = \frac{-C}{(x^2+y^2+z^2)^{3/2}}(x,y,z)$$

for a positive constant C. This vector field is defined when $(x,y,z) \neq (0,0,0)$ and is conservative. Calculate a potential function.

Solution:

$$\frac{\partial \phi}{\partial x} = \frac{-Cx}{(x^2+y^2+z^2)^{3/2}} \Rightarrow$$

$$\Rightarrow \phi(x,y,z) = \frac{C}{(x^2+y^2+z^2)^{1/2}} + \alpha(y,z)$$

$$\Rightarrow \frac{\partial \phi}{\partial y} = -\frac{Cy}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial \alpha}{\partial y}$$

$$\Rightarrow \frac{\partial \alpha}{\partial y} = 0 \Rightarrow \alpha(y,z) = \beta(z)$$

$$\frac{\partial \phi}{\partial z} = -\frac{Cz}{(x^2+y^2+z^2)^{3/2}} + \beta'(z) \Rightarrow \beta'(z) = 0$$

$$\Rightarrow \beta(z) = \text{constant}$$

$$\Rightarrow \phi(x,y,z) = \frac{C}{\sqrt{x^2+y^2+z^2}}$$ is a potential function

(2) The vector field

$$F(x,y,z) = \left(\frac{2x}{z}, \frac{2y}{z}, -\frac{x^2+y^2}{z^2} \right) \text{ is}$$

defined when $z \neq 0$. It is conservative. Find a potential function. Describe it's equipotential surfaces.

Solution:

$$\frac{\partial \phi}{\partial x} = \frac{2x}{z} \implies \phi(x,y,z) = \frac{x^2}{z} + \alpha(y,z)$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial \alpha}{\partial y} \implies \frac{\partial \alpha}{\partial y} = \frac{2y}{z}$$

$$\implies \alpha(y,z) = \frac{y^2}{z} + \beta(z)$$

$$\implies \phi(x,y,z) = \frac{x^2+y^2}{z} + \beta(z)$$

$$\implies \frac{\partial \phi}{\partial z} = -\frac{x^2+y^2}{z^2} + \beta'(z) \implies \beta'(z) = 0$$

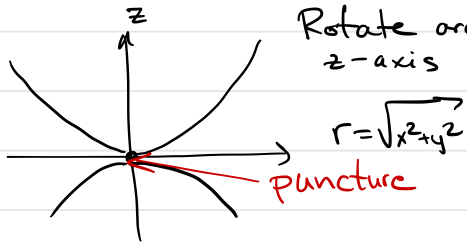
$\beta(z) =$ constant

$$\implies \phi(x,y,z) = \frac{x^2+y^2}{z} \quad \text{is a potential function.}$$

Equipotential surfaces

$$\frac{x^2+y^2}{z} = C$$

When $C \neq 0$ we get $z = \frac{1}{c}(x^2+y^2)$ which defines paraboloids



When $C = 0$ then $\frac{x^2+y^2}{z} = 0 \Longrightarrow x=y=0$

So the "surfaces" becomes two rays $\{(0,0,t);\ t>0\}$ or $\{(0,0,t);\ t<0\}$

③ Let $a>0$ and $\gamma(t) = (a(t-\sin t), a(1-\cos t))$ when $0 \leq t \leq 2\pi$. Calculate

$$\int_\gamma (2a-y)\,dx + x\,dy$$

Solution: $\int_\gamma (2a-y)\,dx + x\,dy =$

$$= \left[\begin{array}{l} dx = a(1-\cos t)\,dt \\ dy = a\sin t\,dt \end{array}\right] =$$

$$= \int_0^{2\pi} \left(2a - a(1-\cos t)\right) a(1-\cos t) + a(t-\sin t)\, a\sin t\,dt$$

$$= a^2 \int_0^{2\pi} (1+\cos t)(1-\cos t) + t\sin t - \sin^2 t\,dt =$$

$$= a^2 \int_0^{2\pi} 1 - \cos^2 t - \sin^2 t + t\sin t\,dt =$$

$$= a^2 \int_0^{2\pi} t \sin t\,dt = a^2\left(\left[-t\cos t\right]_0^{2\pi} + \int_0^{2\pi} \cos t\,dt\right)$$

$$= a^2(-2\pi)\cos(2\pi) = -2\pi a^2$$

④ Calculate
$$\int_\gamma \frac{y\,dx - x\,dy}{y^2}$$

where γ is the part of $xy=2$ that begins at $(1,2)$ and ends at $(2,1)$.

Solution: First we try to construct a potential function for $F(x,y) = \left(\frac{1}{y}, \frac{-x}{y^2}\right)$.

$$\frac{\partial \phi}{\partial x} = \frac{1}{y} \implies \phi(x,y) = \frac{x}{y} + \alpha(y)$$

$$\implies \frac{\partial \phi}{\partial y} = -\frac{x}{y^2} + \alpha'(y) \implies \alpha'(y) = 0$$

$\implies \alpha(y) = \text{constant} \implies \phi(x,y) = \frac{x}{y}$ is a potential function for $F(x,y) = \left(\frac{1}{y}, -\frac{x}{y^2}\right)$

$$\implies \int_\gamma \frac{y\,dx - x\,dy}{y^2} = \phi(2,1) - \phi(1,2)$$

$$= \frac{2}{1} - \frac{1}{2} = \frac{3}{2}$$

Alternative solution for 4

$$\int_\gamma \frac{y\,dx - x\,dy}{y^2}$$ where γ is the part of $xy=2$ starting at $(1,2)$ and ending at $(2,1)$.

We can parametrise γ using $y=\frac{2}{x}$ and $1 \le x \le 2$. Then $dy = -\frac{2}{x^2}dx$ and

$$\int_\gamma \frac{y\,dx - x\,dy}{y^2} = \int_1^2 \frac{\frac{2}{x}dx - x\left(-\frac{2}{x^2}\right)dx}{\left(\frac{2}{x}\right)^2} =$$

$$= \int_1^2 x\,dx = \left[\frac{x^2}{2}\right]_1^2 = 2 - \frac{1}{2} = \frac{3}{2}.$$

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Demo exercises 2

(1) Determine the integral curves of the vector field $F(x,y) = (y,x)$.

Solution: For integral curves we have

$$\frac{dx}{y} = \frac{dy}{x}$$

or

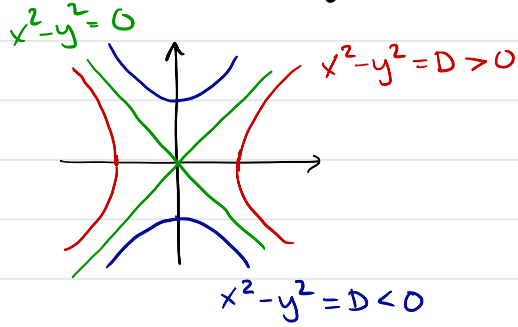
$$\int x\,dx = \int y\,dy$$

Hence

$$\frac{x^2}{2} = \frac{y^2}{2} + C$$

$$\Longrightarrow \quad x^2 - y^2 = D \; (= 2C)$$

This is a family of hyperbolas



(2) Determine if the vector field $F(x,y,z) = (y, x, z^2)$ is conservative or not by constructing a potential function or showing that none exist.

Solution: If F is conservative then there is ϕ such that $\nabla\phi = F$. We try to construct ϕ.

$$\frac{\partial \phi}{\partial x} = y \implies \phi(x,y,z) = xy + \alpha(y,z)$$

$$\frac{\partial \phi}{\partial y} = x + \frac{\partial \alpha}{\partial y} = x \implies \frac{\partial \alpha}{\partial y} = 0 \implies$$

$$\implies \alpha(y,z) = \beta(z) \implies \phi(x,y,z) = xy + \beta(z)$$

$$\frac{\partial \phi}{\partial z} = \beta'(z) \implies \beta'(z) = z^2 \implies \beta(z) = \frac{z^3}{3} + C$$

Therefore $\phi(x,y,z) = xy + \frac{z^3}{3} + C$ are potential functions for any $C \in \mathbb{R}$.

$\implies F(x,y,z) = (y, x, z^2)$ is a conservative vector field.

③ Define $F(x,y) = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2} \right)$ when $(x,y) \neq (0,0)$.

Determine whether this is a conservative vector field or not by constructing a potential function or showing that such cannot exist.

Solution: We try to construct $\phi(x,y)$ such that $\nabla \phi = F$ when $(x,y) \neq (0,0)$.

$$\frac{\partial \phi}{\partial x} = \frac{x}{x^2+y^2} \quad \text{and} \quad \frac{\partial \phi}{\partial y} = \frac{-y}{x^2+y^2}$$

$$\frac{\partial \phi}{\partial x} = \frac{x}{x^2+y^2} \implies \phi(x,y) = \frac{1}{2} \ln(x^2+y^2) + \alpha(y)$$

We also see that then

$$\frac{\partial \phi}{\partial y} = \frac{y}{x^2+y^2} + \alpha'(y)$$

$$\implies -\frac{y}{x^2+y^2} = \frac{y}{x^2+y^2} + \alpha'(y)$$

$$\implies \alpha'(y) = -\frac{2y}{x^2+y^2}$$ which gives

a contradiction since this is a function of $\underline{\underline{x \text{ and } y}}$!

$F(x,y) = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2} \right)$ is not conservative.