PROBLEM SHEET 5 Exercises (Homework Problems)

⇒ Exercise 1: Use Euler's method with step sizes (a) $h = 0,2$, (b) $h = 0,1$ to approximate $y(2)$ given that $y' = xe^{-y}$ and $y(0) = 0$

Euler's method: $y_{n+1} = y_n + hf(x_n, y_n)$

a) $h = 0,2$

$x_1 = 0 + 0,2 = 0,2 \Rightarrow y_1 = y_0 + hf(x_0, y_0)$
$= 0 + 0,2 . 0 = 0$

$x_2 = 0,2 + 0,2 = 0,4 \Rightarrow y_2 = y_1 + hf(x_1, y_1)$
$= 0 + 0,2 . 0,4e^{-0} = 0,08$

$x_3 = 0,4 + 0,2 = 0,6 \Rightarrow y_3 = y_2 + hf(\cancel{y_2}\, x_2, y_2)$
$= 0,08 + 0,2 . 0,6e^{-0,08}$

....

$x_{10} = 1,8 + 0,2 = 2 \Rightarrow y_{10} = y_9 + hf(x_9, y_9)$
$\simeq 0,93247 + 0,2 . 0,70844 \simeq 1,07159$

⇒ Using Euler's method at $h = 0,2$, we approximate $y(2)$ to be $1,07159$

b) $h = 0,1$

$x_1 = 0 + 0,1 = 0,1 \Rightarrow y_1 = y_0 + hf(x_0, y_0)$
$= 0 + 0,1 . 0 = 0$

$x_2 = 0,1 + 0,1 = 0,2 \Rightarrow y_2 = y_1 + hf(x_1, y_1)$
$= 0 + 0,1 . 0,1e^{-0} = 0,01$

$x_3 = 0,2 + 0,1 = 0,3 \Rightarrow y_3 = y_2 + hf(x_2, y_2)$
$= 0,01 + 0,1 . 0,2 . e^{-0,01}$

....

$x_{20} = 1,9 + 0,1 = 2 \Rightarrow y_{20} = y_{19} + hf(y_{19}, x_{19})$
$\approx 1,01798 + 0,1 . (0,68651) \approx 1,08663$

⇒ At $h = 0,1$, we approximate $y(2)$ to be $1,08663$

Exercise 2: Find the complete solution

$$\begin{cases} y'' + 4y = 0 \\ y(0) = 2 \\ y'(0) = -5 \end{cases}$$

- $y'' + 4y = 0$. Let $y = e^{rx}$

$\Rightarrow e^{rx}(r^2 + 4) = 0 \Rightarrow$ Auxiliary equation: $r^2 + 4 = 0$

$\Rightarrow r = \pm 2i$

- When the zeros of the auxiliary number are complex numbers, the homogenous function is

$$y_H = e^{ax}(C_1 \sin \beta x + C_2 \cos \beta x)$$

As $r = \pm 2i \Rightarrow \alpha = 0$, $\beta = 2$

$$y_H = C_1 \sin 2x + C_2 \cos 2x$$

- We have $y(0) = 2$

$\Rightarrow 2 = C_1 \sin 0 + C_2 \cos 0$

$\Rightarrow C_2 = 2$

$\Rightarrow y_H = C_1 \sin 2x + 2 \cos 2x$

- Derivative of y_H

$$y_H' = 2C_1 \cancel{\sin} \cos 2x - 4 \sin 2x$$

We have $y'(0) = 5$

$\Rightarrow 5 = 2C_1 \cos 0 - 4 \sin 0$

$\Rightarrow C_1 = \dfrac{5}{2}$

The complete solution is

$$y_H = \frac{5}{2} \sin 2x + 2 \cos 2x$$

Exercise 3 : Find the complete solution

$$\begin{cases} y'' + 4y' + 3y = 0 \\ y(3) = 1 \\ y'(3) = 0 \end{cases}$$

▫ $y'' + 4y' + 3y = 0$. Let $y = e^{rx}$

$\Rightarrow e^{rx}(r^2 + 4r + 3) = 0 \Rightarrow$ Auxiliary equation : $r^2 + 4r + 3 = 0$

$\Rightarrow r = -1, r = -3$

▫ The auxiliary equation has two roots. The homogenous function is

$\Rightarrow y_H = C_1 e^{-x} + C_2 e^{-3x}$

We have $y(3) = 1$

$\Rightarrow 1 = C_1 e^{-3} + C_2 e^{-9}$ (1)

▫ Derivative of y_H

$y'_H = -C_1 e^{-x} - 3C_2 e^{-3x}$

We have $y'(3) = 0$

$\Rightarrow 0 = -C_1 e^{-3} - 3C_2 e^{-9}$ (2)

We take (1) + (2)

$\Rightarrow 1 = -2C_2 e^{-9}$

$\Rightarrow C_2 = -\frac{1}{2} \cdot \frac{1}{e^{-9}} = -\frac{e^9}{2}$ (3)

Replace (3) into (1), we have

$1 = C_1 e^{-3} - \frac{e^9}{2} e^{-9}$

$\Rightarrow C_1 e^{-3} = \frac{3}{2} \Rightarrow C_1 = \frac{3}{2} e^3$

The complete solution is

$y_H = \frac{3}{2} e^3 e^{-x} - \frac{e^9}{2} e^{-3x} = \frac{3}{2} e^{3-x} - \frac{1}{2} e^{9-3x}$

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Exercise 4: By using the change of dependent variable

$$u(x) = c - k^2 y(x)$$

Solve the initial-value problem $\begin{cases} y''(x) = c - k^2 y(x) \\ y(0) = a \\ y'(0) = b \end{cases}$

- We have:

$$\begin{cases} u(x) = c - k^2 y(x) \\ u'(x) = -k^2 y'(x) \\ u''(x) = -k^2 y''(x) \end{cases} \Rightarrow \begin{cases} u(0) = c - k^2 y(0) = c - k^2 a \quad (1) \\ u'(0) = -k^2 y'(0) = -k^2 b \quad (2) \\ u''(x) = -k^2 u(x) \quad (3) \end{cases}$$

From (3) $\Rightarrow u''(x) + k^2 u(x) = 0$

Let $u(x) = e^{rx}$

$\Rightarrow e^{rx}(r^2 + k^2) = 0 \Rightarrow$ Auxiliary equation: $r^2 + k^2 = 0$

$\Rightarrow r^2 = -k^2 \Rightarrow r = \pm ki$

- The auxiliary equation has complex roots. The homogenous function is

$$u(x) = e^{\alpha x}(C_1 \cos(\beta x) + C_2 \sin(\beta x))$$

In this case, $\alpha = 0$, $\beta = k$

$\Rightarrow u(x) = C_1 \cos(kx) + C_2 \sin(kx)$

From (1)

$\Rightarrow u(0) = C_1 . 1 + 0 = c - k^2 a \Rightarrow C_1 = c - k^2 a$

Replace $C_1 = c - k^2 a$ into $u(x)$

$$u(x) = (c - k^2 a)\cos(kx) + C_2 \sin(kx)$$

Derivative of $u(x)$

$$u'(x) = -k(c - k^2 a)\sin(kx) + kC_2 \cos(kx)$$

From (2)

$\Rightarrow u'(0) = 0 + kC_2 . 1 = -k^2 b$

$\Rightarrow C_2 = -kb$

- $u(x)$ in its complete solution

$$u(x) = (c - k^2 a)\cos(kx) - kb\sin(kx)$$

The substitution: $u(x) = c - k^2 y(x)$

$$\Rightarrow y(x) = \frac{c - u(x)}{k^2} = \frac{c - (c - k^2 a)\cos(kx) + kb\sin(kx)}{k^2}$$

$$y(x) = \frac{c - c.\cos(kx) + k^2 a\cos(kx) + kb\sin(kx)}{k^2}$$

$$y(x) = \frac{c(1 - \cos(kx))}{k^2} + a\cos(kx) + \frac{b}{k}\sin(kx)$$

The complete solution of $y(x)$ is

$$y(x) = \frac{c}{k^2}(1 - \cos(kx)) + a\cos(kx) + \frac{b}{k}\sin(kx)$$