Nguyen Xuan Binh Hand-in 4

Exercise 1: Prove that

$\text{Curl}(\text{Curl } F) = \text{grad}(\text{div } F) - (\Delta F_1, \Delta F_2, \Delta F_3)$ for any smooth vector field $F = (F_1, F_2, F_3)$. Here $\Delta f = \nabla \cdot \nabla f$

We have: $\text{Curl } F = \left(\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) i ; \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) j ; \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) k \right)$

$$\Rightarrow \text{Curl}(\text{Curl } F) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} & \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} & \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{vmatrix}$$

$$\Rightarrow \text{Curl}(\text{Curl } F) = \left[\frac{\partial}{\partial y}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\right] i$$
$$+ \left[\frac{\partial}{\partial z}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right] j$$
$$+ \left[\frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\right] k$$

$$\Rightarrow \text{Curl}(\text{Curl } F) = \left[\frac{\partial^2 F_2}{\partial x \partial y} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_1}{\partial z^2} + \frac{\partial^2 F_3}{\partial x \partial z}\right] i$$
$$+ \left[\frac{\partial^2 F_3}{\partial y \partial z} - \frac{\partial^2 F_2}{\partial^2 z} - \frac{\partial^2 F_2}{\partial^2 x} + \frac{\partial^2 F_1}{\partial x \partial y}\right] j$$
$$+ \left[\frac{\partial^2 F_1}{\partial x \partial z} - \frac{\partial^2 F_3}{\partial^2 x} - \frac{\partial^2 F_3}{\partial^2 y} + \frac{\partial^2 F_2}{\partial y \partial z}\right] k$$

$$\Rightarrow \text{Curl}(\text{Curl } F) = \left(\frac{\partial^2 F_2}{\partial x \partial y} + \frac{\partial^2 F_3}{\partial x \partial z} + \frac{\partial^2 F_1^*}{\partial^2 x}\right) i - \left(\frac{\partial^2 F_1^*}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial^2 z}\right) i$$
$$+ \left(\frac{\partial^2 F_1}{\partial x \partial y} + \frac{\partial^2 F_3}{\partial y \partial z} + \frac{\partial^2 F_2}{\partial y^2}\right) j - \left(\frac{\partial^2 F_2}{\partial^2 x} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_2}{\partial^2 z}\right) j$$
$$+ \left(\frac{\partial^2 F_1}{\partial x \partial z} + \frac{\partial^2 F_2}{\partial y \partial z} + \frac{\partial^2 F_3}{\partial^2 z}\right) k - \left(\frac{\partial^2 F_3}{\partial^2 x} + \frac{\partial^2 F_3}{\partial^2 y} + \frac{\partial^2 F_3}{\partial^2 z}\right) k$$

* Both sides got added so they cancel each other out.

We have $\Delta F_1 = \frac{\partial^2 F_1}{\partial^2 x} + \frac{\partial^2 F_1}{\partial^2 y} + \frac{\partial^2 F_1}{\partial^2 z}$. The same works for ΔF_2, ΔF_3

We have : $\text{grad}(\text{div } F) = \frac{\partial}{\partial x}\left[\frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} + \frac{\partial F_1}{\partial x}\right] i + \frac{\partial}{\partial y}\left[\frac{\partial F_1}{\partial x} + \frac{\partial F_3}{\partial z} + \frac{\partial F_2}{\partial y}\right] j$

$$+ \frac{\partial}{\partial z}\left[\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right] k$$

$$\Rightarrow \text{Curl}(\text{Curl } F) = \text{grad}(\text{div } F) - \Delta F_1 - \Delta F_2 - \Delta F_3$$
$$= \text{grad}(\text{div } F) - \nabla^2 F \text{ (proven)}$$

Exercise 2: Prove that there is no vector field such that Curl $F = (x, y, z)$

We know that all vector field F satisfies $\text{div}(\text{curl } F) = 0$

We have : $\text{div}(\text{curl } F) = \text{div}(x, y, z) = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$

$$= 1 + 1 + 1 = 3$$

Since $3 \neq 0$, there is no vector field F that has curl $= (x, y, z)$

Exercise 3: Calculate $\oint_\gamma x^2 dy$ where γ is the curve $(x-1)^2+y^2=1$ counterclockwise

$\Rightarrow \gamma(t) = (\cos\theta + 1, \sin\theta)$

We have: $\oint_\gamma x\,dy = \int_0^{2\pi} (\cos\theta+1)^2 . (\cos\theta)\,d\theta$

$= \int_0^{2\pi} (\cos^2\theta + 2\cos\theta + 1)\cos\theta\,d\theta$

$= \int_0^{2\pi} (\cos^3\theta + 2\cos^2\theta + \cos\theta)\,d\theta$

$= \int_0^{2\pi} \cos^3\theta\,d\theta + \int_0^{2\pi} 2\cos^2\theta\,d\theta + \int_0^{2\pi} \cos\theta\,d\theta$

$= \left(\sin\theta - \frac{\sin^3(\theta)}{3}\right)\Big|_0^{2\pi} + \left(\theta + \frac{1}{2}\sin(2\theta)\right)\Big|_0^{2\pi} + (\sin\theta)\Big|_0^{2\pi}$

$= 0 + 2\pi + 0 = 2\pi$ (answer)

Exercise 4: The curve parameterised as $\gamma(t) = (\cos^3 t, \sin^3 t)$, $0 \leq t \leq 2\pi$ is an astroid. Calculate area enclosed by it

We have: $\gamma'(t) = (-3\cos^2 t\sin t, 3\sin^2 t\cos t)$

The area is

$A = \oint_\gamma x\,dy = \int_0^{2\pi} \cos^3 t\,(3\sin^2 t\cos t)\,dt$

$= \int_0^{2\pi} 3\cos^4 t\sin^2 t\,dt = \int_0^{2\pi} 3\cos^4 t\,(1-\cos^2 t)\,dt$

$= \int_0^{2\pi} 3\cos^4 t - 3\cos^6 t\,dt$

$= \frac{3}{32}\int_0^{2\pi} 2 + \cos 2t - 2\cos 4t - \cos 6t\,dt$

$= \frac{3}{32}\left(2t + \frac{1}{2}\sin 2t - \frac{1}{4}\sin 4t - \frac{1}{6}\sin 6t\right)\Big|_0^{2\pi}$

$= \frac{3}{8}\pi - 0 = \frac{3}{8}\pi$ (answer)