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PROBLEM SHEET 3 Exercises (Homework Problems)

Exercise 1: Find approximations M_8 and T_{16} for

$\int_0^{\pi/2} \frac{\sin x}{x} dx$, integrand = 1 at $x = 0$

▫ Trapezoid: $T_n[f, a, b] = h\left(\frac{1}{2}y_0 + y_1 + \dots + y_{n-1} + \frac{1}{2}y_n\right)$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin x}{x} dx, \quad h = \frac{\frac{\pi}{2} - 0}{16} = \frac{\pi}{32}$$

$$T_{16} = \frac{\pi}{32}\left(\frac{1}{2}\cdot 1 + \frac{\sin\left(\frac{\pi}{32}\right)}{\frac{\pi}{32}} + \frac{\sin\left(\frac{\pi}{16}\right)}{\frac{\pi}{16}} + \frac{\sin\left(\frac{3\pi}{32}\right)}{\frac{3\pi}{32}} + \frac{\sin\left(\frac{\pi}{8}\right)}{\frac{\pi}{8}}\right.$$

$$+ \frac{\sin\left(\frac{5\pi}{32}\right)}{\frac{5\pi}{32}} + \frac{\sin\left(\frac{3\pi}{16}\right)}{\frac{3}{16}\pi} + \frac{\sin\left(\frac{7\pi}{32}\right)}{\frac{7\pi}{32}} + \frac{\sin\left(\frac{\pi}{4}\right)}{\frac{\pi}{4}} + \frac{\sin\left(\frac{9\pi}{32}\right)}{\frac{9\pi}{32}}$$

$$+ \frac{\sin\left(\frac{5\pi}{16}\right)}{\frac{5\pi}{16}} + \frac{\sin\left(\frac{11\pi}{32}\right)}{\frac{11\pi}{32}} + \frac{\sin\left(\frac{3\pi}{8}\right)}{\frac{3\pi}{8}} + \frac{\sin\left(\frac{13\pi}{32}\right)}{\frac{13\pi}{32}} + \frac{\sin\left(\frac{7\pi}{16}\right)}{\frac{7\pi}{16}} + \frac{\sin\left(\frac{15\pi}{32}\right)}{\frac{15\pi}{32}}$$

$$\left. + \frac{\sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}} \cdot \frac{1}{2}\right) \approx 1,370436618$$

▫ Midpoint Rule: $M_n[f; a, b] = h\sum_{j=1}^{n} f\left(a + \left(j - \frac{1}{2}\right)h\right)$

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin x}{x} dx, \quad h = \frac{\frac{\pi}{2} - 0}{8} = \frac{\pi}{16}$$

$$M_8 = \frac{\pi}{16}\left(\frac{\sin\left(\frac{\pi}{32}\right)}{\frac{\pi}{32}} + \frac{\sin\left(\frac{3\pi}{32}\right)}{\frac{3\pi}{32}} + \frac{\sin\left(\frac{5\pi}{32}\right)}{\frac{5\pi}{32}} + \frac{\sin\left(\frac{7\pi}{32}\right)}{\frac{7\pi}{32}}\right.$$

$$\left. + \frac{\sin\left(\frac{9\pi}{32}\right)}{\frac{9\pi}{32}} + \frac{\sin\left(\frac{11\pi}{32}\right)}{\frac{11\pi}{32}} + \frac{\sin\left(\frac{13\pi}{32}\right)}{\frac{13\pi}{32}} + \frac{\sin\left(\frac{15\pi}{32}\right)}{\frac{15\pi}{32}}\right)$$

$$\Rightarrow M_8 \approx 1,371413626$$

Thứ

Ngày . . No.

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Exercise 2 [Midpoint Rule] Compute the actual error in the approximation $\int_0^1 x^2 dx \approx M_1$ and use it to show that the constant 24 in the error estimate cannot be improved

Error estimate: if $f(x)$ has second derivative continuous on $[a,b]$ and satisfies $|f''(x)| \le K$ for $\forall x \in [a,b]$ then

$$\left|\int_a^b f(x)dx - M_n\right| \le \frac{K(b-a)^2}{24n^2}$$

We have: $\int_0^1 x^2 dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$

$$M_1 = \left(0 + \left(1 - \frac{1}{2}\right)1\right)^2 \cdot (1-0) = \frac{1}{4}$$

$$\Rightarrow \left|\frac{1}{3} - \frac{1}{4}\right| \le \frac{K(1-0)^2}{24 \cdot 1^2} \Rightarrow \frac{1}{12} \le \frac{K}{24} \ (*)$$

$f(x) = x^2 \Rightarrow f''(x) = 2 \Rightarrow K \geq 2$ (requirement for K)

Assume $K = 2$. If the constant 24 gets bigger, $\frac{K}{24\uparrow}$ will be smaller than $\frac{1}{24}$ (not correct for $(*)$)

If constant 24 gets smaller $\Rightarrow \frac{K}{24\downarrow}$ is always bigger than $\frac{1}{24}$ (fitting $(*)$)

$\Rightarrow$ 24 is the ~~cons~~ largest possible constant for the assumed $K = 2$ to satisfy both $\begin{cases} \frac{K}{24} \geq \frac{1}{12} \\ K \geq 2 \end{cases}$

Of course, if K get bigger, $\frac{K}{24}$ will always $> \frac{1}{12}$, whatever how much the constant 24 gets smaller or bigger

$\Rightarrow$ The constant 24 in the error can't be improved

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Thứ
Ngày . .
No.

Exercise 3: Find the area of the region bounded by

$$y = \frac{x}{x^4+16}, \quad y = 0, \quad x = 0, \quad x = 2$$

According to the borders, the area is the integral

$$\int_0^2 \frac{x}{x^4+16}\,dx$$

Substitute $u = x^2 \Rightarrow du = 2x \Rightarrow x = du/2$

$$f(x) = \int_0^2 \frac{x}{x^4+16}\,dx = \int_0^4 \frac{du/2}{u^2+16} = \frac{1}{2}\int_0^4 \frac{1}{u^2+16}\,du$$

$$= \frac{1}{2}\int_0^4 \frac{1}{16}\cdot\frac{1}{\left(\frac{u}{4}\right)^2+1}\,du = \frac{1}{32}\int_0^4 \frac{1}{\left(\frac{u}{4}\right)^2+1}\,du$$

Let $u/4 = \tan\theta \Rightarrow \frac{1}{4}du = \sec^2\theta\,d\theta \Rightarrow du = 4\sec^2\theta\,d\theta$

$$\Rightarrow f(x) = \frac{1}{32}\int \frac{4\sec^2\theta\,d\theta}{\tan^2\theta+1} = \frac{1}{32}\int \frac{4\sec^2\theta\,d\theta}{\sec^2\theta} = \frac{1}{32}\int 4\,d\theta$$

$$\Rightarrow f(x) = \frac{1}{32}\cdot 4\theta + C = \frac{1}{8}\theta + C$$

We have: $\tan\theta = u/4 \Rightarrow \arctan\left(\frac{u}{4}\right) = \theta$

$$\Rightarrow \arctan\left(\frac{x^2}{4}\right) = \theta$$

The area: $\frac{1}{8}\arctan\left(\frac{x^2}{4}\right)\Big|_0^2 = \left[\arctan\left(\frac{2^2}{4}\right) - \arctan\left(\frac{0^2}{4}\right)\right]\cdot\frac{1}{8}$

$$= \frac{1}{8}\left(\frac{1}{4}\pi - 0\right) = \frac{1}{32}\pi$$

Thứ
Ngày . . No.

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Exercises 4: If n & m are integers, show that

(i) $\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0$, if $m \neq n$

We have: $\cos(m+n) = \cos m \cos n - \sin m \sin n$ (1)

$\cos(m-n) = \cos m \cos n + \sin m \sin n$ (2)

$\Rightarrow$ (1) + (2) $\Rightarrow 2\cos m \cos n = \cos(m+n) + \cos(m-n)$

$\Rightarrow \cos m \cos n = \frac{1}{2}\left[\cos(m+n) + \cos(m-n)\right]$

Apply to the integral, we have

$I = \int_{-\pi}^{\pi} \cos mx \cos nx \, dx$

$= \int_{-\pi}^{\pi} \frac{1}{2}\left[\cos((m+n)x) + \cos((m-n)x)\right] dx$

$= \frac{1}{2}\left[\frac{1}{m+n}\sin((m+n)x) + \frac{1}{m-n}\overset{\sin}{\cancel{\cos}}((m-n)x)\right]\Bigg|_{-\pi}^{\pi}$

As $\sin(a\pi) = 0 \ \forall a \in \mathbb{N}$ and $\sin(-a\pi) = 0 \ \forall a \in \mathbb{N}$

As $m, n \in \mathbb{N} \Rightarrow m+n$ and $m-n \in \mathbb{N}$

$\Rightarrow I = \frac{1}{2}\left[\frac{1}{m+n}\cdot 0 + \frac{1}{m-n}\cdot 0\right] - \frac{1}{2}\left[\frac{1}{m+n}\cdot 0 + \frac{1}{m-n}\cdot 0\right]$

$\Rightarrow I = 0$

Thứ
Ngày . . No.

(ii) $\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0$, if $m \neq n$

We have: $\cos(m+n) = \cos m \cos n - \sin m \sin n \quad (1)$

$\cos(m-n) = \cos m \cos n + \sin m \sin n \quad (2)$

$(2)-(1) \Rightarrow 2 \sin m \sin n = \cos(m-n) - \cos(m+n)$

$\Rightarrow \sin m \sin n = \frac{1}{2}\left[\cos(m-n) - \cos(m+n)\right]$

Apply to the integral

$I = \int_{-\pi}^{\pi} \sin mx \sin nx \, dx$

$= \int_{-\pi}^{\pi} \frac{1}{2}\left[\cos((m-n)x) - \cos((m+n)x)\right]$

$= \frac{1}{2}\left[\frac{1}{m-n}\sin((m-n)x) - \frac{1}{m+n}\sin((m+n)x)\right]_{-\pi}^{\pi}$

As explained in (i)

$\Rightarrow I = \frac{1}{2}\left[\frac{1}{m-n}.0 - \frac{1}{m+n}0\right] - \frac{1}{2}\left[\frac{1}{m-n}0 - \frac{1}{m+n}.0\right]$

$\Rightarrow I = 0$

Ngày . . No.

(iii) $\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0, \quad m \neq n$

We have: $\sin(m+n) = \sin m \cos n + \cos m \sin n \quad (1)$

$\sin(m-n) = \sin m \cos n - \cos m \sin n \quad (2)$

$(1) + (2) \Rightarrow 2 \sin m \cos n = \sin(m+n) + \sin(m-n)$

$\Rightarrow \sin m \cos n = \frac{1}{2}\left[\sin(m+n) + \sin(m-n)\right]$

Apply to the integral

$I = \int_{-\pi}^{\pi} \sin mx \cos nx \, dx$

$= \int_{-\pi}^{\pi} \frac{1}{2}\left[\sin((m+n)x) + \sin((m-n)x)\right]$

$= \frac{1}{2}\left[\frac{-1}{m+n}\cos((m+n)x) + \frac{-1}{m-n}\cos((m-n)x)\right]\Big|_{-\pi}^{\pi}$

We have: $\cos(a) = \cos(-a)$

$\Rightarrow \cos(a\pi) = \cos(-a\pi)$

$\Rightarrow I = \frac{1}{2}\left[\frac{-1}{m+n}\underbrace{\cos((m+n)\pi)}_{||} + \frac{-1}{m+n}\underbrace{\cos((m-n)\pi)}_{||}\right]$

$- \frac{1}{2}\left[\frac{-1}{m+n}\cos(-(m+n)\pi) + \frac{-1}{m+n}\cos(-(m-n)\pi)\right]$

We have: $\cos((m+n)\pi) = \cos(-(m+n)\pi)$

$\cos((m-n)\pi) = \cos(-(m+n)\pi)$

$\Rightarrow$ The first and latter part of I are equal

$\Rightarrow I = 0$