Nguyen Xuan Binh 887799

1) a) $D_{\vec{u}} f(1,2) = (3,4) \cdot \vec{\nabla} f = 2$

$D_{\vec{u}} f(1,2) = (5,12) \cdot \vec{\nabla} f = 1$

Let $\vec{\nabla} f = (f_x, f_y)$

$$\Rightarrow \begin{cases} 3f_x + 4f_y = 2 \\ 5f_x + 12f_y = 1 \end{cases} \Rightarrow \begin{cases} f_x = \frac{5}{4} \\ f_y = -\frac{7}{16} \end{cases}$$

$\Rightarrow$ Gradient of f at point $(1,2)$ is $\vec{\nabla} f(1,2) = \left(\frac{5}{4}, -\frac{7}{16}\right)$

b) Maximum rate of change: $D_{\vec{u}} f = \|\nabla f\| = \sqrt{\left(\frac{5}{4}\right)^2 + \left(-\frac{7}{16}\right)^2} = \frac{\sqrt{449}}{16}$

Direction of maximum rate of change

$$\vec{u} = \frac{\vec{\nabla} f}{\|\vec{\nabla} f\|} = \frac{\vec{\nabla} f}{\sqrt{499}/16} = \left(\frac{20}{\sqrt{499}}, -\frac{7}{\sqrt{499}}\right)$$

Exercise 2: $x^2 + y^2 \leq 1 \Rightarrow \begin{cases} x^2 + y^2 < 1 : \text{Interior} \\ x^2 + y^2 = 1 : \text{Boundary} \end{cases}$

$T = x^4 - 4x^2 + 2y^2$

- Interior

$$\left.\begin{array}{l} \dfrac{\partial T}{\partial x} = 4x^3 - 8x = 0 \Rightarrow x = 0, \pm\sqrt{2} \\ \dfrac{\partial T}{\partial y} = 4y = 0 \Rightarrow y = 0 \end{array}\right\} (x,y) = \begin{array}{l} (0,0) \\ (\sqrt{2}, 0)\ [\text{Omit}] \\ (-\sqrt{2}, 0)\ [\text{Omit}] \end{array}$$

Location	$(0,0)$	$\cancel{(\sqrt{2}, 0)}$	$\cancel{(-\sqrt{2}, 0)}$
Value	0	$\cancel{-4}$	$\cancel{-4}$

- $x^2 + y^2 = 1$ (Boundary) $\Rightarrow y^2 = 1 - x^2$

$\Rightarrow f(x, y(x)) = x^4 - 4x^2 + 2(1 - x^2)$

$= x^4 - 6x^2 + 2$, domain $[-1, 1]$

Critical point : $f'(x) = 4x^3 - 12x = 0 \Rightarrow x = 0, \pm\sqrt{3}$ (Omit)

$\Rightarrow y = \pm 1,$

End point : $x = \pm 1 \Rightarrow y = 0$

Value of absolute extrema

Location	(0,0)	(0, 1)	(0, -1)	(1, 0)	(-1, 0)
Value	0	2	2	-3	-3

$\Rightarrow$ Hottest points of the plate : (0, 1) and (0, -1) with temperature at 2°C
Coldest points of the plate : (1, 0) and (-1, 0) with temperature at -3°C

Exercise 3: Let $f(x, y) = \sin(xe^y) - x + 3$

a) Compute all the 1st and 2nd order partial derivatives

$$\frac{\partial f}{\partial x} = e^y \cos(xe^y) - 1 \Rightarrow \frac{\partial^2 f}{\partial x^2} = -e^{2y}\sin(xe^y)$$

$$\frac{\partial f}{\partial y} = xe^y \cos(xe^y) \Rightarrow \frac{\partial^2 f}{\partial y^2} = x(e^y\cos(e^y x) - e^{2y} x \sin(e^y x))$$

$$\Rightarrow \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = e^y(\cos(xe^y) - xe^y \sin(xe^y))$$

b) Linear approximation at (0,0)

$$L(x, y) = f(0,0) + \frac{\partial f}{\partial x}(0,0)x + \frac{\partial f}{\partial y}(0,0)y$$

$$= 3 + 0x + 0y = 3$$

$\Rightarrow L(0{,}2, 0{,}1) \approx 3$

c) If the curve of the surface is relatively smooth, the tangent plane on a point can approximate the values of very nearby points on the surface.
The plane I found in (b) happens to be a flat plane $z = 3$ orthorgonal to the z-axis, which means that every nearby points of (0, 0) can be approximated that their value is around 3, no matter their position
In fact, $f(0{,}2, 0{,}1) = 3{,}01924 \approx 3$

d) Second order polynomial

$$T_2(0{,}2, 0{,}1) = f(0,0) + f_x(0,0)(0{,}2 - 0) + f_y(0,0)(0{,}1 - 0)$$

$$+ \frac{f_{xx}(0,0)}{2}(0{,}2-0)^2 + f_{xy}(0,0)(0{,}2-0)(0{,}1-0) + \frac{f_{yy}(0,0)}{2}(0{,}1-0)^2$$

$$\Rightarrow T_2(0{,}2, 0{,}1) = 3 + 0 \cdot 0{,}2 + 0 \cdot 0{,}1 + 0 \cdot 0{,}04 + 1 \cdot 0{,}02 + 0 \cdot 0{,}01$$

$$= 3{,}02$$

Since 3,02 is closer to 3,01924 than 3, second order Taylor approximation is better than linear approximation for (0,2, 0,1) at (0,0)

e) Find a critical point of $f(x,y)$

$f_x = e^y\cos(xe^y) - 1 = 0 \Rightarrow e^y\cos(xe^y) = 1$

$f_y = xe^y\cos(xe^y) = 0 \Rightarrow \begin{cases} x = 0 \\ \cos(xe^y) = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ xe^y = \frac{\pi}{2} \end{cases}$

If $xe^y = \frac{\pi}{2} \Rightarrow e^y\cos(xe^y) = 0$, not fitting the above condition

$\Rightarrow$ only $x = 0$ satisfies

$\Rightarrow e^y\cos(0.e^y) = 1 \Rightarrow e^y = 1 \Rightarrow y = 0$

$\Rightarrow$ There is only one critical point : $(x,y) = (0,0)$

We have: $f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2 = 0.0 - 1^2 = -1 < 0$

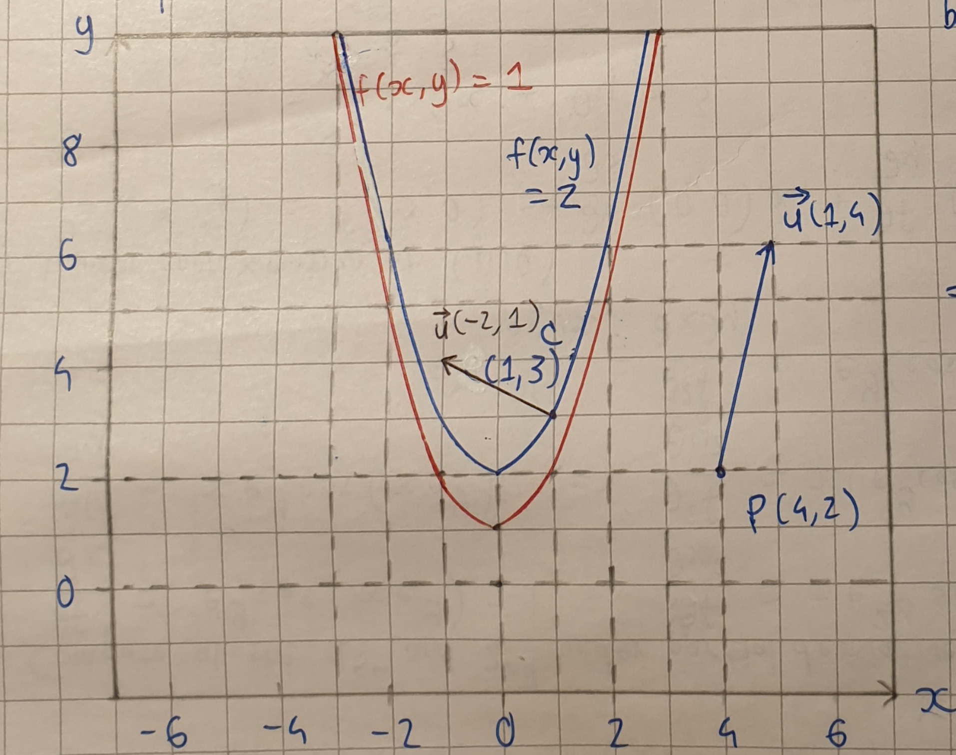
$\Rightarrow (x,y) = (0,0)$ is a saddle point

Exercise 4: Let $f(x,y) = y - x^2$

a) $f(x,y) = 1 \Rightarrow y - x^2 = 1 \Rightarrow y = x^2 + 1$

$f(x,y) = 2 \Rightarrow y - x^2 = 2 \Rightarrow y = x^2 + 2$

Contour plot



b) Let $P = (a,b)$, $\vec{u} = (m,n)$

$\frac{\partial f}{\partial x} = -2x$

$\frac{\partial f}{\partial y} = 1$

$\Rightarrow D_{\vec{u}}f(P) = \vec{\nabla}f . \vec{u}$

$= (-2x, 1).\vec{u}$

$= (-2a, 1).(m,n)$

$= -2am + n$

$D_{\vec{u}}f(P) < 0$

$\Rightarrow -2am + n < 0$

$\Rightarrow n < 2am$

$\Rightarrow \frac{n}{2} < am$

$\Rightarrow$ Choose $P(4,2)$ and $\vec{u} = (1,4)$

c) Gradient of f at point $(1,3) = (-2 \cdot 1, 1)$
$= (-2, 1)$

Maximum rate of change: $D_{\vec{u}} f(1,3) = \|\vec{\nabla} f(1,3)\| = \sqrt{(-2)^2 + 1^2} = \sqrt{5}$

Direction of maximum rate of change

$$\vec{u} = \frac{\vec{\nabla} f}{\|\vec{\nabla} f\|} = \frac{(-2, 1)}{\sqrt{5}} = \left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$$

It does look approximately correct, because $\vec{\nabla} f$ is orthogonal to tangent plane at a point

The point $(1,3)$ lies on $y = x^2 + 2 \Rightarrow y' = 2x$

Maximum direction rate of change is also the gradient vector itself

We can see that $\vec{u} = \left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$ dot product with $(1, 2)$ is 0

which means that $\vec{u} \perp y' = 2x$

Exercise 5: Let $D = \{(x,y) \mid x^2 + y^2 < 4\}$ and

$$f(x,y) = (x^2 + y^2)(\sqrt{x^2+y^2} - 1)(\sqrt{x^2+y^2} + 1)$$

We have: $x^2 + y^2 < 4$, $x^2 + y^2 \geq 0$

$\Rightarrow 0 \leq \sqrt{x^2+y^2} < 2$

Let $t = \sqrt{x^2+y^2} \Rightarrow t \in [0, 2)$

$\Rightarrow f(t) = t^2 (t-1)(t+1)$

$= t^2(t^2 - 1) = t^4 - t^2$

$\Rightarrow f'(t) = 4t^3 - 2t = 0 \Rightarrow t = \left[\begin{array}{l} 0 \\ \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \text{ (omit)} \end{array}\right.$

$\Rightarrow t = 0 \Rightarrow \sqrt{x^2+y^2} = 0$

$\Rightarrow (x,y) = (0,0) \Rightarrow f(x,y) = 0$

$t = \frac{\sqrt{2}}{2} \Rightarrow \sqrt{x^2+y^2} = \frac{\sqrt{2}}{2} \Rightarrow x^2 + y^2 = \frac{1}{2}$

$\Rightarrow (x,y)$ lies on boundary of circle centered at $(0,0)$ with radius of $\sqrt{2}/2$

$\Rightarrow f(x,y) = -\frac{1}{4}$

Suppose that $x^2 + y^2 = 4 \Rightarrow f(x,y) = 12$, bigger than 0 and $-\frac{1}{4}$

$\Rightarrow f(x,y)$ doesn't have absolute max on D since D is open boundary.

However, it does have local extrema, which is maxima 0 on $(0,0)$ and $f(x,y)$ has absolute min on D, which is $-\frac{1}{4}$