Nguyen Xuan Binh 887799

Thứ
Ngày . . No.

Course Exam, Oct 21, 2020
Answer Page

- Problem 1: $f'(x) = \frac{1}{2\sqrt{x}}$

- Problem 2: $T_3(f(x), \pi/4) = \frac{\sqrt{2}}{2} - \sqrt{2}\left(x - \frac{\pi}{4}\right)$

$$- \sqrt{2}\left(x - \frac{\pi}{4}\right)^2 + \frac{2\sqrt{2}}{3}\left(x - \frac{\pi}{4}\right)^3$$

$$+ O\left(\left(x - \frac{\pi}{4}\right)\right)^4$$

Lagrange remainder: $R_4(x) = \frac{2}{3}\sin\left(2\xi + \frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right)^4$

with $\xi \in \left[\frac{\pi}{4}, x\right]$

- Problem 3:

$$A = \frac{\beta - \alpha}{b - a}$$

$$B = \frac{\alpha b - a\beta}{b - a}$$

$$\int_a^b f(x)dx = \frac{\beta - \alpha}{b - a}\int_\alpha^\beta f\left(\frac{\beta - \alpha}{b - a}t + \frac{\alpha b - a\beta}{b - a}\right)dt$$

- Problem 4: $\int x^\alpha \ln x \, dx = \frac{x^{\alpha+1}}{\alpha+1}\ln x - \frac{x^{\alpha+1}}{(\alpha+1)^2}$ for all $\alpha \in R$

- Problem 5: $y(t) = \cos(4t) + \frac{1}{32}\sin(4t) - \frac{t}{8}\cos(4t)$

KOKUYO

Nguyen Xuan Binh 887799

Thứ

Ngày . .

No.

Problem 1: Find the derivative of $f(x) = \sqrt{x}$ using the definition

By definition, the function $f(x)$'s derivative is the limit

$$\lim_{x_0 \to 0} \frac{f(x+x_0) - f(x)}{x_0}$$, if this limit exists

Derivative of $f(x) = \sqrt{x}$ using the definition

$$f'(x) = \lim_{x_0 \to 0} \frac{\sqrt{x+x_0} - \sqrt{x}}{x_0}$$

$$= \lim_{x_0 \to 0} \frac{\sqrt{x+x_0} - \sqrt{x}}{x_0} \cdot \frac{\sqrt{x+x_0} + \sqrt{x}}{\sqrt{x+x_0} + \sqrt{x}}$$

$$= \lim_{x_0 \to 0} \frac{x + x_0 - x}{x_0(\sqrt{x+x_0} + \sqrt{x})}$$

$$= \lim_{x_0 \to 0} \frac{x_0}{x_0(\sqrt{x+x_0} + \sqrt{x})}$$

$$= \lim_{x_0 \to 0} \frac{1}{\sqrt{x+x_0} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x+0} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$\Rightarrow$ By definition, the derivative of $f(x)$ is $\frac{1}{2\sqrt{x}}$

Nguyen Xuan Binh 887799 [signature]

Thứ

Ngày . . No.

Problem 2: Find the Taylor Polynomial of degree three of $f(x) = \sin(2x + \pi/4)$ about the point $x = \pi/4$ and its Lagrange Remainder

The Taylor Remainder of degree 3

$$T_3(f(x), \pi/4) = \sum_{i=0}^{3} \frac{f^{(i)}(\frac{\pi}{4})}{i!}(x - \frac{\pi}{4})^i$$

$$\Rightarrow T_3(f(x), \pi/4) = \frac{\sin(3\pi/4)}{0!} + \frac{2\cos(3\pi/4)}{1!}(x - \frac{\pi}{4})$$

$$- \frac{4\sin(3\pi/4)}{2!}(x - \frac{\pi}{4})^2 - \frac{8\cos(3\pi/4)}{3!}$$

$$\Rightarrow T_3(f(x), \pi/4) = \frac{\sqrt{2}}{2} - \sqrt{2}(x - \frac{\pi}{4}) - \sqrt{2}(x - \frac{\pi}{4})^2$$

$$+ \frac{2\sqrt{2}}{3}(x - \frac{\pi}{4})^3 + O((x - \frac{\pi}{4})^4)$$

The lagrange remainder for this function is

$$R_4(x) = \frac{f^4(\xi)}{4!}(x - \frac{\pi}{4})^4$$

$$= \frac{16\sin(2\xi + \pi/4)}{4!}(x - \frac{\pi}{4})^4$$

$$= \frac{2}{3}\sin(2\xi + \frac{\pi}{4})(x - \frac{\pi}{4})^4$$

with $\xi \in [\frac{\pi}{4}, x]$

Nguyen Xuan Binh 887799

Thứ

Ngày . . No.

Problem 3: Show, that if f is a continuous function and $a < b$, then the integral $\int_a^b f(x)\,dx$ can be transformed with a substitution $x = At + B$ to the integral over the interval $[\alpha, \beta]$. Find the parameters A and B and perform the substitution

The substitution rule: if $u = g(x)$ is differentiable on an interval $[a, b]$ and f is also continuous on $[a, b]$ then

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$

We have: $g(x) = Ax + B \Rightarrow g'(x) = A$ (a constant)

$\Rightarrow \int_a^b f(x)\,dx$ can be transformed with substitution $x = At + B$ since $f(x)$ is continuous on $[a, b]$ and $g(t) = At + B$ is also differentiable on $[a, b]$

We have: $x = At + B \Rightarrow dx = A\,dt$

$$\int_a^b f(x)\,dx = A\int_{Aa+B}^{Ab+B} f(At+B)\,dt$$

$$= A\int_\alpha^\beta f(At+B)\,dt \Rightarrow \begin{cases} Ab + B = \beta \\ Aa + B = \alpha \end{cases}$$

We have: $(Ab + B) - (Aa + B) = \beta - \alpha$

$$A(b-a) = \beta - \alpha \Rightarrow A = \frac{\beta-\alpha}{b-a}$$

We have: $Aa + B = \alpha \Rightarrow \frac{\beta-\alpha}{b-a}a + B = \alpha$

$$\Rightarrow B = \frac{\alpha(b-a) - a(\beta-\alpha)}{b-a} = \frac{\alpha b - a\beta}{b-a}$$

$$\Rightarrow \int_a^b f(x)\,dx = \frac{\beta-\alpha}{b-a}\int_\alpha^\beta f\left(\frac{\beta-\alpha}{b-a}t + \frac{\alpha b - a\beta}{b-a}\right)dt$$

Nguyen Xuan Binh 887799

Thứ

Ngày . .

No.

Problem 4: Integrate $\int x^{\alpha} \ln x \, dx$ for all $\alpha \in \mathbb{R}$

Integration by parts

Let $u = \ln x \Rightarrow du = \frac{1}{x} dx$

$dv = x^{\alpha} dx \Rightarrow v = \frac{x^{\alpha+1}}{\alpha+1}$

$$\int x^{\alpha} \ln x \, dx = \frac{x^{\alpha+1}}{\alpha+1} \ln x - \int \frac{x^{\alpha+1}}{x(\alpha+1)} dx$$

$$= \frac{x^{\alpha+1}}{\alpha+1} \ln x - \int \frac{x^{\alpha} . x}{x(\alpha+1)} dx$$

$$= \frac{x^{\alpha+1}}{\alpha+1} \ln x - \int x^{\alpha} \frac{1}{\alpha+1} dx$$

$$= \frac{x^{\alpha+1}}{\alpha+1} \ln x - \frac{1}{\alpha+1} \int x^{\alpha} dx \quad \left(\frac{1}{\alpha+1} \text{ is a constant} \right)$$

$$= \frac{x^{\alpha+1}}{\alpha+1} \ln x - \frac{1}{\alpha+1} . \frac{x^{\alpha+1}}{\alpha+1}$$

$$= \frac{x^{\alpha+1}}{\alpha+1} \ln x - \frac{x^{\alpha+1}}{(\alpha+1)^2}.$$

$$\Rightarrow \int x^{\alpha} \ln x \, dx = \frac{x^{\alpha+1}}{\alpha+1} \ln x - \frac{x^{\alpha+1}}{(\alpha+1)^2} \text{ for all } \alpha \in \mathbb{R}$$

Nguyen Xuan Binh 887799

Problem 5: Find the complete solution

$$\begin{cases} y'' + 16y = \sin 4t \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$$

This is non-homogenous second order ODE

$\Rightarrow y(t) = y_0(t) + y_1(t)$

- Find $y_0(t)$: $y_0(t)$ is the general form of $y'' + 16y = 0$

Let $y = e^{rt} \Rightarrow e^{rt}(r^2 + 16) = 0$

$\Rightarrow$ Auxiliary equation: $r^2 + 16 = 0$

$\Rightarrow r = \pm 4i$

The auxiliary equation hax complete roots. The homogenous function is

$$y_0 = e^{\alpha t}(C_1 \cos(\beta t) + C_2 \sin(\beta t))$$

In this case, $\alpha = 0$, $\beta = 4$

$\Rightarrow y_0 = C_1 \cos(4t) + C_2 \sin(4t)$

- Find $y_1(t)$: $y_1(t)$ has the general form

$$y_1 = (A \sin 4t + B \cos 4t + C)t$$

$\Rightarrow y_1' = A(\sin 4t + 4t\cos 4t) + B(-4\sin 4t + \cos 4t) + C$

$\Rightarrow y_1'' = (-16A \sin 4t - 16B \cos 4t)t - (8A \sin 4t - 8B \cos 4t)$

Replacing y_1 into $y'' + 16y = \sin 4t$, we have

$\Rightarrow t(-16A \sin 4t - 16B \cos 4t) - 8A \sin 4t + 8B \cos 4t$

$+ 16t(A \sin 4t + B \cos 4t + C) = \sin 4t$

$\Rightarrow -8A \sin(4t) + 8B \cos(4t) + 16tC = \sin 4t$

$$\Rightarrow \begin{cases} -8A = 1 \\ 8B = 0 \\ 16tC = 0 \end{cases} \Rightarrow \begin{cases} A = -\frac{1}{8} \\ B = 0 \\ C = 0 \end{cases}$$

Nguyen Xuan Binh 887799

Thứ
Ngày . .
No.

$\Rightarrow y_1 = -\frac{t}{8}\cos 4t$

□ The form of $y(t) = y_0(t) + y_1(t)$

$\Rightarrow y(t) = C_1\cos(4t) + C_2\sin(4t) - \frac{t}{8}\cos(4t)$

We have $y(0) = 1$

$\Rightarrow C_1\cos(0) + C_2\sin(0) - \frac{0}{8}\cos(0) = 1$

$\Rightarrow C_1 = 1$

$\Rightarrow y(t) = \cos(4t) + C_2\sin(4t) - \frac{t}{8}\cos(4t)$

Derivative of $y(t)$

$y'(t) = -4\sin(4t) + 4C_2\cos(4t) - \frac{1}{8}\cos(4t) + \frac{1}{2}t\sin(4t)$

We have $y'(0) = 0$

$\Rightarrow 0 = 0 + 4C_2 . 1 - \frac{1}{8} + 0$

$\Rightarrow 4C_2 - \frac{1}{8} = 0 \Rightarrow C_2 = \frac{1}{32}$

The complete solution is

$y(t) = \cos(4t) + \frac{1}{32}\sin(4t) - \frac{t}{8}\cos(4t)$