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Exercise 1: Let $F = (xz, yz, 1)$ and $D = \{(x, y, z); x^2 + y^2 + z^2 \leq 25, z \geq 3\}$

D is a closed cap region so divergence theorem can be applied

We have: $\nabla \cdot F = \operatorname{div} F = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(1)$

$= z + z = 2z$

$\Rightarrow \iint_S \vec{F} . \vec{n} \, dS = \iiint_D 2z \, dV$

Changing to polar coordinates:

$$\begin{cases} 0 \leq \theta \leq 2\pi \\ 0 \leq r \leq \sqrt{25 - z^2} \\ 3 \leq z \leq \sqrt{25 - r^2} \end{cases}$$

$\Rightarrow \sqrt{25 - z^2}$ is maximized when z_{min}

$\Rightarrow 0 \leq r \leq \sqrt{25 - 3^2} = 4$

$$\Rightarrow \iiint_D 2z\, dV = \int_0^{2\pi}\int_0^4\int_3^{\sqrt{25-r^2}} 2z\, r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^4 z^2 r \Big|_3^{\sqrt{25-r^2}} dr\, d\theta = \int_0^{2\pi}\int_0^4 [r(25 - r^2) - 9r]\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^4 [25r - r^3 - 9r]\, dr\, d\theta = \int_0^{2\pi} \frac{25}{2}r^2 - \frac{r^4}{4} - \frac{9}{2}r^2 \Big|_0^4 d\theta$$

$$= \int_0^{2\pi} 64\, d\theta = 64\theta \Big|_0^{2\pi} = 128\pi \text{ (answer)}$$

Exercise 2: Assume that $f(x, y, z)$ is harmonic (that is $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$)

Assume that D is closed. Show that $\oiint \nabla f \cdot \vec{N}\, dS = 0$

Since D is closed $\Rightarrow \oiint \nabla f \cdot \vec{N}\, dS = \iiint_D \operatorname{div}(\nabla f)\, dV$

We have: $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

$$\operatorname{div}(\nabla f) = \nabla \cdot \nabla f = \frac{\partial}{\partial x}\cdot\frac{\partial f}{\partial x} + \frac{\partial}{\partial y}\cdot\frac{\partial f}{\partial y} + \frac{\partial}{\partial z}\cdot\frac{\partial f}{\partial z}$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \Delta f = 0$$

$$\Rightarrow \oiint \nabla f \cdot \vec{N}\, dS = \iiint_D \operatorname{div}(\nabla f)\, dV = \iiint_D \nabla f\, dV = \iiint_D 0\, dV = 0 \text{ (proven)}$$

Exercise 3: Let S be the boundary surface of

$$D = \{(x, y, z) \in \mathbb{R}^3,\ x^2 + y^2 \leq z^2,\ 0 \leq z \leq 1\}$$

Let $F(x, y, z) = (x^2, y^2, z^2)$ and calculate $\oiint_S F \cdot \vec{N}\, dS$

Since D is a cone limited by $0 \leq z \leq 1 \Rightarrow D$ is a closed region

We have: $\operatorname{div} F = 2x + 2y + 2z$

$$\oiint_S F \cdot \vec{N}\, dS = \iiint_D 2x + 2y + 2z\, dV$$

Changing to polar coordinates

$$\int_0^{2\pi}\int_0^1\int_0^z 2(r\cos\theta + r\sin\theta + z)\, r\,dr\,dz\,d\theta$$

$$= \int_0^{2\pi}\int_0^1\int_0^z 2r^2\cos\theta + 2r^2\sin\theta + 2zr\,dr\,dz\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \frac{2}{3}r^3\cos\theta + \frac{2}{3}r^3\sin\theta + r^2 z\Big|_0^z dz\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \frac{2}{3}z^3\cos\theta + \frac{2}{3}z^3\sin\theta + z^3\,dz\,d\theta = \int_0^{2\pi} \frac{1}{6}z^4\cos\theta + \frac{1}{6}z^4\sin\theta + \frac{z^4}{4}\Big|_0^1 d\theta$$

$$= \int_0^{2\pi} \frac{1}{6}\cos\theta + \frac{1}{6}\sin\theta + \frac{1}{4}\,d\theta = \frac{1}{6}\sin\theta + \left(\frac{-1}{6}\cos\theta\right) + \frac{1}{4}\theta\Big|_0^{2\pi}$$

$$= \left(-\frac{1}{6} + \frac{1}{2}\pi\right) - \left(-\frac{1}{6}\right) = \frac{1}{2}\pi \text{ (answer)}$$

Exercise 4: Let γ be the intersection curve of $x^2+y^2+z^2=1$ and $x+y+z=0$ oriented clockwise. Calculate

$$\oint_\gamma (y+z)dx + (x+z)dy + (x+y)dz$$

We will apply Stokes' theorem to solve

$$\oint_\gamma (y+z)dx + (x+z)dy + (x+y)dz = \oint_\gamma (y+z, x+z, x+y)\cdot dr$$

$$= \iint_S \mathrm{Curl}(\langle y+z, x+z, x+y\rangle)\cdot \vec{N}\,dS$$

We have $\mathrm{Curl}(\langle y+z, x+z, x+y\rangle) = (0,0,0)$

$$\Rightarrow \oint_\gamma (y+z, x+z, x+y)\cdot dr = \iint_S 0\cdot\vec{N}\,dS = 0$$