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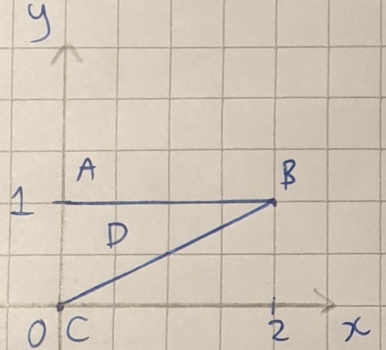
Hand-in exercises 1

1) Calculate $\iint_D x\,dA$ with D is a triangle vertex at $(0,0)$, $(2,1)$, $(0,1)$




The line BC is $y = \frac{1}{2}x$

$$\Rightarrow \begin{cases} 0 \le x \le 2 \\ \frac{1}{2}x \le y \le 1 \end{cases}$$

$$\Rightarrow \int_0^2 \int_{\frac{1}{2}x}^{1} x\,dy\,dx = \int_0^2 \left(x - \frac{1}{2}x^2\right)dx$$

$$\Rightarrow \frac{1}{2}x^2 - \frac{1}{6}x^3 \Big|_0^2 = \frac{2}{3}$$

2) Find volume of solid under $z = 1 - x^2 + y$ and above region $0 \le y \le 1$, $0 \le x \le y^2$

$$\int_0^1 \int_0^{y^2} (1 - x^2 + y)\,dx\,dy = \int_0^1 \left(x - \frac{1}{3}x^3 + yx\right)\Big|_0^{y^2} dy$$

$$= \int_0^1 y^2 + y^3 - \frac{y^6}{3}\,dy = \int \left(\frac{1}{3}y^3 + \frac{1}{4}y^4 - \frac{y^7}{21}\right)\Big|_0^1 = \frac{15}{28}$$

3) Let $a > 0$. Compute $\iint_D x^2 + y^2\,dA$ where D is $x^2 + y^2 \le 2xa$

We have : $x^2 + y^2 \le 2xa \Rightarrow x^2 - 2xa + y^2 \le 0$

$\Rightarrow x^2 - 2xa + a^2 + y^2 \le a^2$

$\Rightarrow (x-a)^2 + y^2 \le a^2$

$\Rightarrow$ D is a disk centered at $(a,0)$ having radius a

When we shift the origin to $(a,0)$, the paraboloid will be to the left of the origin $(a>0)$ by a unit

$\Rightarrow f(x,y) = (x+a)^2 + y^2 = x^2 + 2ax + a^2 + y^2 = r^2 + 2ar\cos\theta + a^2$

$\Rightarrow$ The integral: $\int_0^{2\pi} \int_0^{a} \left(r^2 + 2ar\cos\theta + a^2\right) r\,dr\,d\theta$

$$= \int_0^{2\pi} \left(\frac{a^4}{2} + \frac{a^4}{4} + 2a\cos\theta\,\frac{a^3}{3}\right)d\theta$$

$$= \pi a^4 + \frac{\pi a^4}{2}$$

4) Let $a > 0$, $b > 0$

$$D = \left\{ (x,y) \in \mathbb{R}^2 ; \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\}$$

Calculate $\iint_D \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\, dA$

Let $u = \frac{x}{a}$, $v = \frac{y}{b}$ $\Rightarrow u^2 + v^2 \leq 1$ : a circle with radius 1

$\Rightarrow x = ua$, $y = vb \Rightarrow \frac{\partial x}{\partial u} = a$, $\frac{\partial x}{\partial v} = 0$, $\frac{\partial y}{\partial u} = 0$, $\frac{\partial y}{\partial v} = b$

$$\Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$$

$$\Rightarrow \iint_D \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\, dA = \int_{-1}^{1} \int_{-\sqrt{1-v^2}}^{\sqrt{1+v^2}} \left( \sqrt{1 - u^2 - v^2} \right) ab \, du dv$$

Change to polar coordinate : $r^2 = u^2 + v^2$

$$= \int_0^{2\pi} \int_0^1 \sqrt{1 - r^2}\, ab\, r dr d\theta = \int_0^{2\pi} ab \frac{1}{3} d\theta = \frac{2\pi}{3} ab$$