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Hand-in 2

Exercise 1: Consider vector field $F(x,y,z) = \frac{-C}{(x^2+y^2+z^2)^{3/2}}(x,y,z)$
with $(x,y,z) \neq 0$

We have: $\frac{\partial \phi}{\partial x} = \frac{-Cx}{(x^2+y^2+z^2)^{\frac{3}{2}}} \Rightarrow \phi(x,y,z) = \int \frac{-Cx}{(x^2+y^2+z^2)^{\frac{3}{2}}} dx$

Let $u = x^2+y^2+z^2 \Rightarrow du = 2x dx \Rightarrow 1/2\, du = x dx$

$\Rightarrow \phi(x,y,z) = \int -\frac{1}{2} C u^{-\frac{3}{2}} du = -\frac{1}{2} C(-2u^{-\frac{1}{2}}) + f(y,z)$

$= Cu^{-\frac{1}{2}} + f(y,z) = \frac{C}{\sqrt{x^2+y^2+z^2}} + f(y,z)$

We have: $\frac{\partial \phi}{\partial y} = \frac{-Cy}{(x^2+y^2+z^2)^{3/2}} = \frac{-Cy}{(x^2+y^2+z^2)^{3/2}} + f'(y,z)$

$\Rightarrow f'(y,z) = 0 \Rightarrow f(y,z) = D$

The potential function for the vector field is $\frac{C}{\sqrt{x^2+y^2+z^2}} + D$

Exercise 2: The vector field $F(x,y,z) = \left(\frac{2x}{z}, \frac{2y}{z}, -\frac{x^2+y^2}{z^2}\right)$

We have: $\frac{\partial \phi}{\partial z} = -\frac{x^2+y^2}{z^2} \Rightarrow \phi(x,y,z) = \int -\frac{x^2+y^2}{z^2} dz = \frac{x^2+y^2}{z} + f(x,y)$

We have: $\frac{\partial \phi}{\partial x} = \frac{2x}{z} = f'(x,y) + \frac{2x}{z} \Rightarrow f'(x,y) = 0$
$\Rightarrow f(x,y) = D$

$\frac{\partial \phi}{\partial y} = \frac{2y}{z} = f'(x,y) + \frac{2y}{z} \Rightarrow f'(x,y) = 0$
$\Rightarrow f(x,y) = D$

The potential function for the vector field is $\phi(x,y,z) = \frac{x^2+y^2}{z} + D$

Exercise 3: Let $a > 0$ and $\gamma(t) = (a[t - \sin t], a[1-\cos t])$
$0 \leq t \leq 2\pi$

Calculate $\int_\gamma (2a - y)dx + x dy$

We have: $\frac{d\gamma}{dt} = (a - a\cos t,\ a\sin t)$

$$\Rightarrow \left|\frac{d\gamma}{dt}\right| = \sqrt{(a - a\cos t)^2 + (a\sin t)^2} = \sqrt{a^2 - 2a^2\cos t + a^2\cos^2 t + a^2\sin^2 t}$$

$$= \sqrt{2a^2 - 2a^2\cos t}$$ (This one is unnecessary since there's no ds)

$$\Rightarrow \int_\gamma (2a - y)\,dx + x\,dy = \int_0^{2\pi} (2a - a + a\cos t)(a - a\cos t) + (at - a\sin t)(a\sin t)\,dt$$

$$= \int_0^{2\pi} (a^2 - a^2\cos^2 t) + a^2 t\sin t - a^2\sin^2 t\,dt$$

$$= \int_0^{2\pi} a^2 t\sin t\,dt.$$ Let $u = t$, $\sin t\,dt = dv$

$\Rightarrow du = dt$, $v = -\cos t$

$$= a^2\left(-t\cos t + \int \cos t\,dt\right) = a^2(\sin t - t\cos t)\Big|_0^{2\pi} = (-2\pi - 0)a^2$$

$$= -2\pi a^2$$

Exercise 4: Calculate $\int_\gamma \frac{y\,dx - x\,dy}{y^2}$ where γ is $xy = 2$ from $(1,2)$ to $(2,1)$

We have: $xy = 2 \Rightarrow \gamma(t) = \left(t, \frac{2}{t}\right)$, $1 \leq t \leq 2$

$$\Rightarrow \frac{d\gamma}{dt} = \left(1, -\frac{2}{t^2}\right)$$

$$\Rightarrow \int_\gamma \frac{y\,dx - x\,dy}{y^2} = \int_1^2 \frac{\frac{2}{t} + \frac{2}{t}}{\left(\frac{2}{t}\right)^2}\,dt = \int_1^2 \frac{4/t}{4/t^2}\,dt = \int_1^2 t\,dt$$

$$= \frac{t^2}{2}\Big|_1^2 = \frac{3}{2}$$