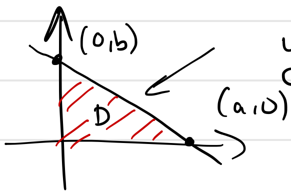
Demo Exercises 1

(1) Assume that $a>0$ and $b>0$. Calculate

$$\iint_D x-3y \, dA$$

where D is the triangle with vertices $(0,0)$, $(a,0)$ and $(0,b)$.

Solution:



$y = -\frac{b}{a}x + b$

$$\iint_D x-3y \, dA = \int_0^a \left(\int_0^{-\frac{b}{a}x+b} x-3y \, dy \right) dx =$$

$$= \int_0^a \left[xy - \frac{3y^2}{2} \right]_{y=0}^{y=-\frac{b}{a}x+b} dx =$$

$$= \int_0^a x\left(\frac{-b}{a}x+b\right) - \frac{3}{2}\left(-\frac{b}{a}x+b\right)^2 dx =$$

$$= \int_0^a -\frac{3}{2}\frac{b^2}{a^2}x^2 - \frac{b}{a}x^2 + 3\frac{b^2}{a}x + bx - \frac{3}{2}b^2 \, dx$$

$$= \left[-\frac{3b^2}{2a^2}\frac{x^3}{3} - \frac{bx^3}{3a} + \frac{3b^2x^2}{2a} + \frac{bx^2}{2} - \frac{3}{2}b^2x \right]_0^a =$$

$$= -\frac{b^2a}{2} - \frac{ba^2}{3} + \frac{3b^2a}{2} + \frac{ba^2}{2} - \frac{3b^2a}{2} =$$

$$= \left(-\frac{1}{2}+\frac{3}{2}-\frac{3}{2}\right)b^2a + \left(-\frac{1}{3}+\frac{1}{2}\right)ba^2 =$$

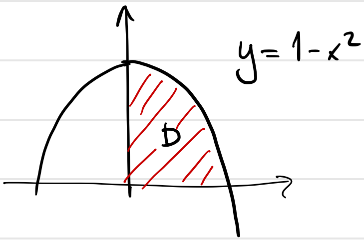
$$= -\frac{1}{2}b^2a + \frac{1}{6}ba^2 = \frac{ba}{6}(a-3b)$$

(2) Calculate

$$\iint_D \frac{x}{1+y}\, dA$$

where D is the finite region in the first quadrant bounded by the coordinate axes and the curve $y=1-x^2$.

<u>Solution</u>:



$$\iint_D \frac{x}{1+y}\, dA = \int_0^1 \left(\int_0^{1-x^2} \frac{x}{1+y}\, dy \right) dx =$$

$$= \int_0^1 \Big[x \ln|1+y| \Big]_{y=0}^{y=1-x^2} dx = \int_0^1 x \ln|2-x^2|\, dx$$

$$= \left\lceil \begin{matrix} t = 2-x^2 & t_1 = 1 \\ dt = -2x\,dx & t_0 = 2 \end{matrix} \right\rfloor = \frac{1}{2}\int_1^2 \ln t\, dt = (*)$$

$$\lceil \int 1 \cdot \ln t\, dt = t\ln t - \int t \cdot \frac{1}{t}\, dt = t\ln t - t + C \rfloor$$

$$(*) = \frac{1}{2}\Big[t\ln t - t\Big]_1^2 = \frac{1}{2}\left(2\ln 2 - 2 - \ln 1 + 1\right) =$$

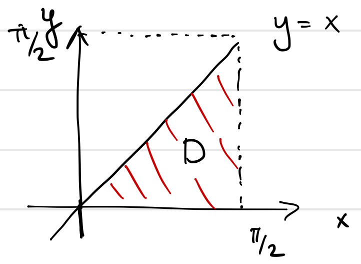
$$= \ln 2 - \frac{1}{2}$$

(3) Calculate the iterated integral

$$\int_0^{\pi/2}\left(\int_y^{\pi/2} \frac{\sin x}{x}\,dx\right)dy$$

and sketch the domain of integration.

Solution:



$$\int_0^{\pi/2}\left(\int_y^{\pi/2} \frac{\sin x}{x}\,dx\right)dy = \iint_D \frac{\sin x}{x}\,dA =$$

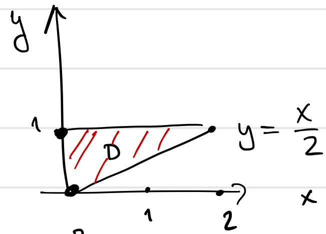
$$= \int_0^{\pi/2}\left(\int_0^{x} \frac{\sin x}{x}\,dy\right)dx = \int_0^{\pi/2}\left[\frac{y\sin x}{x}\right]_{y=0}^{y=x} dx$$

$$= \int_0^{\pi/2} \sin x\,dx = \left[-\cos x\right]_0^{\pi/2} = 1.$$

Hand-in Exercises 1

① Calculate $\iint_D x \, dA$ where D is the triangle with vertices $(0,0)$, $(2,1)$ and $(0,1)$.

Solution:



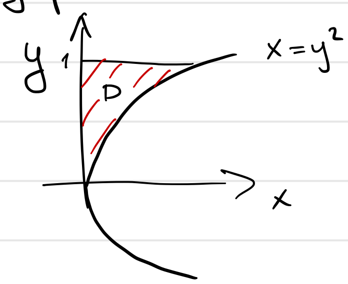
$$\iint_D x \, dA = \int_0^2 dx \int_{x/2}^1 x \, dy = \int_0^2 \left[xy\right]_{y=x/2}^{y=1} dx$$

$$= \int_0^2 x - \frac{x^2}{2} \, dx = \left[\frac{x^2}{2} - \frac{x^3}{6}\right]_0^2 =$$

$$= 2 - \frac{8}{6} = \frac{2}{3}$$

② Find the volume of the solid under $z = 1 - x^2 + y$ and above the region $0 \le y \le 1$, $0 \le x \le y^2$ in the xy-plane.

Solution



Notice that $1 - x^2 + y \ge 0$ when $(x,y) \in D$.

$$\text{Volume} = \iint_D 1-x^2+y \, dA = \int_0^1 dy \int_0^{y^2} 1-x^2+y \, dx$$

$$= \int_0^1 \left[x - \frac{x^3}{3} + yx \right]_{x=0}^{x=y^2} dy =$$

$$= \int_0^1 y^2 - \frac{y^6}{3} + y^3 \, dy = \left[\frac{y^3}{3} - \frac{y^7}{21} + \frac{y^4}{4} \right]_0^1 =$$

$$= \frac{1}{3} + \frac{1}{4} - \frac{1}{21} = \frac{28}{84} + \frac{21}{84} - \frac{4}{84} = \frac{45}{84}$$

Answer: The volume is $\frac{45}{84}$ volume units

(3) Let $a>0$. Compute $\iint_D x^2+y^2 \, dA$ where D is the disk $x^2+y^2 \le 2xa$.

Solution: $x^2+y^2 \le 2xa \iff$

$$\iff x^2 - 2xa + y^2 \le 0 \iff (x-a)^2 + y^2 - a^2 \le 0$$

$$\iff (x-a)^2 + y^2 \le a^2$$

Introduce $\begin{cases} u = x-a \\ v = y \end{cases}$ $\boxed{dudv = dxdy}$ Clearly.

$$D_{(u,v)} = \{ (u,v) \in \mathbb{R}^2 ; \ u^2+v^2 \le a^2 \}$$

$$\iint_D x^2+y^2 \, dxdy = \iint_{D_{(u,v)}} (u+a)^2 + v^2 \, dudv =$$

$$= \iint_{D_{(u,v)}} u^2+v^2 + 2au + a^2 \, dudv$$

In polar coordinates

$$\iint_{D_{(u,v)}} u^2+v^2+2au+a^2\,dudv = \int_0^{2\pi} d\theta \int_0^a (r^2+2ar\cos\theta+a^2)\,r\,dr$$

$$= \int_0^{2\pi} \left[\frac{r^4}{4} + \frac{2ar^3}{3}\cos\theta + \frac{a^2r^2}{2}\right]_0^a d\theta$$

$$= \int_0^{2\pi} \frac{a^4}{4} + \frac{2a^4}{3}\cos\theta + \frac{a^4}{2}\,d\theta = 2\pi a^4\left(\frac{1}{4}+\frac{1}{2}\right) =$$

$$= \frac{3\pi a^4}{2}$$

(4) Let $a>0$, $b>0$ and

$$D = \left\{ (x,y) \in \mathbb{R}^2 ;\ \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1 \right\}$$

Calculate $\displaystyle\iint_D \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dA$.

<u>Solution</u>: Introduce $u = \frac{x}{a}$ and $v = \frac{y}{b}$

Then $D_{(u,v)} = \{(u,v)\in\mathbb{R}^2;\ u^2+v^2\le 1\}$ and

$$dxdy = ab\,dudv$$

Therefore

$$\iint_D \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dx\,dy = ab\iint_{D(u,v)}\sqrt{1-u^2-v^2}\,du\,dv$$

Introduce polar coordinates

$$ab\iint_{D(u,v)}\sqrt{1-u^2-v^2}\,du\,dv = ab\int_0^{2\pi}\int_0^1 r\sqrt{1-r^2}\,dr\,d\theta =$$

$$= 2\pi ab\int_0^1 r\sqrt{1-r^2}\,dr = \begin{bmatrix} t = 1-r^2 & t_1 = 0 \\ dt = -2r\,dr & t_0 = 1 \end{bmatrix} =$$

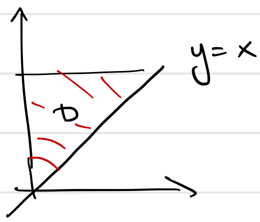
$$= -\pi ab\int_1^0 t^{1/2}\,dt = \pi ab\int_0^1 t^{1/2}\,dt =$$

$$= \pi ab\left[\frac{2t^{3/2}}{3}\right]_0^1 = \frac{2\pi ab}{3}$$

[redacted] Homework 1

(1) Find the volume of the solid under $z = 1 - x^2$ and above the region in the plane given by the inequalities $0 \le y \le 1$, $0 \le x \le y$.

Solution:



Notice that $z = 1 - x^2 \ge 0$ when $(x,y) \in D$.

$$\text{Volume} = \iint_D 1 - x^2 \, dA = \int_0^1 dy \int_0^y 1 - x^2 \, dx =$$

$$= \int_0^1 \left[x - \frac{x^3}{3} \right]_{x=0}^{x=y} dy = \int_0^1 y - \frac{y^3}{3} \, dy =$$

$$= \left[\frac{y^2}{2} - \frac{y^4}{3 \cdot 4} \right]_0^1 = \text{[redacted]} = \frac{1}{2} - \frac{1}{12} = \frac{5}{12}$$

Answer: The volume is $\frac{5}{12}$ [redacted] volume units.

(2) Calculate $\int_3^4 \int_1^2 \frac{1}{(x+y)^2} \, dy \, dx$

$$\int_3^4 \left(\int_1^2 (x+y)^{-2} dy \right) dx = \int_3^4 \left[-(x+y)^{-1} \right]_{y=1}^{y=2} dx =$$

$$= \int_3^4 \frac{1}{1+x} - \frac{1}{2+x} \, dx = \Big[\ln|x+1| - \ln|x+2| \Big]_3^4 =$$

$$= \ln 5 - \ln 6 - (\ln 4 - \ln 5) =$$

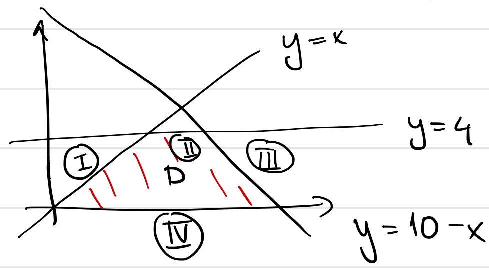
$$= \ln 25 - \ln 24 = \ln \frac{25}{24}$$

3) Write down the equations for the curve that bound the domain of integration in

$$\int_0^4 \int_y^{10-y} f(x,y)\, dx\, dy$$

Solution: We see that the domain of integration is defined by the inequalities $0 \le y \le 4$ and $y \le x \le 10-y$

So



We see that the curves (I) $y=x$, (II) $y=4$, (III) $y=10-x$ and (IV) $y=0$ bound the domain of integration.