1) Introductory Problems

- Intro 1: Evaluate the limit or explain why it doesn't exist

$$\lim_{h \to 0} \frac{\sqrt{4+h} - 2}{h} = f(h)$$

$$\Rightarrow \lim_{h \to 0} \frac{(\sqrt{4+h} - 2)(\sqrt{4+h} + 2)}{h(\sqrt{4+h} + 2)}$$

$$\Rightarrow \lim_{h \to 0} \frac{4 + h - 4}{h(\sqrt{4+h} + 2)} \Rightarrow \lim_{h \to 0} \frac{1}{\sqrt{4+h} + 2} = \frac{1}{\sqrt{4+0} + 2}$$

$$\Rightarrow \lim_{h \to 0} f(h) = \frac{1}{4}$$

- Intro 2:

If $2 - x^2 \leq g(x) \leq 2\cos x$ for all x, find $\lim_{x \to 0} g(x)$

There is the squeeze theorem which states that if $f(x) \leq g(x) \leq h(x)$ for all numbers, and there exists a so that $f(a) = h(a)$, so then $g(a)$ must be equal to them as well

At $x = 0 \Rightarrow f(0) = 2 - x^2 = 2 - 0^2 = 2$

$\Rightarrow h(0) = 2\cos x = 2\cos 0 = 2$

Since $f(0) = h(0) = 2$, according to squeeze theorem, $g(0)$ must equal 2 as well

Therefore, $\lim_{x \to 0} g(x) = 2$

Thứ

Ngày . . No.

- Intro 3

If $\lim_{x \to a} g(x) = M$, show that there exists a number $\delta > 0$ such that

$$0 < |x - a| < \delta \Rightarrow |g(x)| < 1 + |M|$$

Definition of limit $\lim_{x \to a} f(x) = L$

Given $\varepsilon > 0$, $\exists\, \delta > 0$ so that

$$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Apply this to $\lim_{x \to a} g(x) = M$

$$0 < |x - a| < \delta \Rightarrow |g(x) - M| < \varepsilon$$

Take $\varepsilon = 1$ in the definition of limit $\Rightarrow |g(x) - M| < 1$ (1)

We have: $|g(x)| = |g(x) - M + M|$

Applying the triangle inequality $|A + B| \leq |A| + |B|$

$\Rightarrow |g(x)| = |g(x) - M + M| \leq |g(x) - M| + |M|$ (2)

Replace (1) into (2)

$\Rightarrow |g(x)| \leq |g(x - M| + |M| < 1 + |M|$

$\Rightarrow |g(x) < 1 + |M|$ (proven)

- Intro 4: Evaluate, if possible, the limit of the sequence $\{a_n\}$

$$a_n = \sqrt{n+1} - \sqrt{n}$$

- Domain: $n \geq 0$. When $n = 0$, $a_n = 1$

$\Rightarrow \lim_{n \to 0} a_n = 1$

- The other end is $n \to +\infty$

$$\lim_{n \to +\infty} a_n = \sqrt{n+1} - \sqrt{n} = \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{(\sqrt{n+1} + \sqrt{n})}$$

$$= \frac{n + 1 - n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{(\sqrt{n+1} + \sqrt{n})}$$ → This approach $+\infty$ as $n \to +\infty$

$\Rightarrow \lim_{n \to +\infty} a_n = 0$

Thứ
Ngày . . No.

▫ Intro 5 : Use the definition of ~~derati~~ derivative to calculate

$$\frac{d}{dx}\left(\frac{x}{x^2+1}\right)\bigg|_{x=3} \quad (*)$$

By definition, the function $f(x)$ is differentiable at x_0 if the limit $\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ exists

$$(*) = \lim_{x\to 3} \frac{\frac{x}{x^2+1} - \frac{x_0}{x_0^2+1}}{x-x_0} = \lim_{x\to 3} \frac{\frac{x}{x^2+1} - \frac{3}{3^2+1}}{x-3}$$

$$= \lim_{x\to 3}\left(\frac{x}{x^2+1} - \frac{3}{10}\right)\left(\frac{1}{x-3}\right) = \lim_{x\to 3}\left(\frac{10x-3x^2-3}{10(x^2+1)(x-3)}\right)$$

$$= \lim_{x\to 3} \frac{-(x-3)(3x-1)}{10(x^2+1)(x-3)} = \lim_{x\to 3} \frac{-(3x-1)}{10(x^2+1)} = -\frac{2}{25}$$

This limit exists, so $\frac{d}{dx}\left(\frac{x}{x^2+1}\right)\bigg|_{x=3} = -\frac{2}{25}$

▫ Intro 6 : Calculate the derivative of $f(x) = x^{1/3}$ using only the definition

We have : $\frac{d}{dx} x^{1/3} = \frac{x^{\frac{1}{3}} - x_0^{\frac{1}{3}}}{x - x_0} = \frac{x^{1/3} - x_0^{1/3}}{(x^{\frac{1}{3}})^3 - (x_0^{\frac{1}{3}})^{\frac{1}{3}}}$

$$\Rightarrow \frac{d}{dx} x^{1/3} = \frac{x^{1/3} - x_0^{1/3}}{(x^{1/3} - x_0^{1/3})(x^{2/3} + (x x_0)^{1/3} + x_0^{2/3})}$$

$$\Rightarrow \frac{d}{dx} x^{1/3} = \frac{1}{x^{2/3} + (x x_0)^{1/3} + x_0^{2/3}}$$

For the derivative to exist

$$\lim_{x\to x_0} x^{1/3} = \frac{1}{x^{2/3} + x^{2/3} + x^{2/3}} = \frac{1}{3x^{2/3}} = \frac{1}{3} x^{-2/3}$$

KOKUYO

Thứ
Ngày . . No.

2) Homework Problems

E1: Evaluate the limit or explain why it doesn't exist

$$\lim_{x \to 1/2} \frac{1}{\sqrt{x - x^2}}$$. Replace $x = \frac{1}{2}$ into the function, we have

$$\lim_{x \to 1/2} \frac{1}{\sqrt{1/2 - (1/2)^2}} = 2.$$

So the limit of the function is 2

E2: Evaluate

$$\lim_{x \to 0} \frac{1}{|x-1| - |x+1|}$$

-LM|

I'll denote that the plus is approaching the number from its right and the subtract is approaching the number from its left

* If $x \to 0^+ \Rightarrow \begin{cases} x - 1 \to -1^+ \\ x + 1 \to 1^+ \end{cases} \Rightarrow \begin{cases} |x-1| \to 1^- \ (1) \\ |x+1| \to 1^+ \ (2) \end{cases}$

$(1)(2) \Rightarrow |x-1| - |x+1| \to 0^-$

Put demonitor as y. We have $\frac{1}{y}$ while $y \to 0^-$

$\Rightarrow \frac{1}{y}$ approaches $-\infty$

The same process is inversed for $x \to 0^- \Rightarrow \frac{1}{y}$ approaches $+\infty$

So the function appraoches $\pm\infty$ as $x \to 0$

KOKUYO

E3: If $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$, prove that

$$\lim_{x \to a} f(x)g(x) = L.M \quad (*)$$

With $\forall \varepsilon > 0$, $\exists \delta_1, \delta_2$ that satisfy

$$0 < |x - a| < \delta_1 \Rightarrow |f(x) - L| < \varepsilon$$

$$0 < |x - a| < \delta_2 \Rightarrow |g(x) - M| < \varepsilon$$

To prove $(*)$, we have to prove that $|f(x)g(x) - LM| < \varepsilon$ because $|f(x)g(x) - LM| \geqslant 0$, while ε is getting infinitely smaller as $\varepsilon \to 0 \Rightarrow$ when $x = a$, $\varepsilon = 0$ and thus $|f(x)g(x) - LM| = 0 \Rightarrow f(x)g(x) = LM$ when $x = a$

We have: $|f(x)g(x) - LM| = |f(x)g(x) - \overset{g}{\cancel{f}}(x)\overset{L}{\cancel{M}} + \overset{g}{\cancel{f}}(x)\overset{L}{\cancel{M}} - LM|$

$$\Leftrightarrow |f(x)g(x) - LM| = |g(x)(f(x) - L) + L(g(x) - M)|$$

Applying triangular inequality

$$\Rightarrow |f(x)g(x) - LM| \leq |g(x)(f(x) - L)| + |L(g(x) - M)|$$

$$< |g(x)|.\varepsilon + |L|.\varepsilon \quad (1)$$

We have to modify (1) by dividing or multiply ε with any positive number, As $|g(x)|$ and $|L|$ are positive number themselves

$$\Rightarrow |f(x)g(x) - LM| < |g(x)|.\frac{\varepsilon}{2|g(x)|} + |L|.\frac{\varepsilon}{2|L|}$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

$$\Rightarrow \lim_{x \to a} f(x)g(x) = LM$$

E34: Evaluate, if possible, the limit of the sequence $\{a_n\}$

$$a_n = \sqrt{n^2+n} - \sqrt{n^2-1}$$

Domain: $(-\infty; -1] \cup [1; +\infty)$

At $n = -1$, $a_n = 0$. At $n = 1$, $a_n = \sqrt{2}$

$$a_n = \sqrt{n^2+n} - \sqrt{n^2-1} \cancel{= \sqrt{n(n+1)}}$$

$$= \sqrt{n^2+n} - \sqrt{n^2-1} \; \frac{\sqrt{n^2+n} + \sqrt{n^2-1}}{\sqrt{n^2+n} + \sqrt{n^2-1}}$$

$$= \frac{n^2+n-(n^2-1)}{\sqrt{n^2+n} + \sqrt{n^2-1}} = \frac{n+1}{\sqrt{n^2+n} + \sqrt{n^2-1}}$$

Divide both nomi- & denominator by n, we have

$$a_n = \frac{1+\frac{1}{n}}{\sqrt{1+1/n} + \sqrt{1-1/n^2}}$$

For $\lim\limits_{n \to +\infty} a_n$, we have $\frac{1}{n} \to 0^+ \Rightarrow 1 + \frac{1}{n} \to 1^+$

$\Rightarrow \sqrt{1+1/n} \to 1^+$

As $n \to +\infty$, $\frac{1}{n^2}$ will be smaller than $\frac{1}{n}$.

$$\Rightarrow \sqrt{1+1/n} + \sqrt{1-1/n^2} \to 2^+$$

So $\lim\limits_{n \to +\infty} a_n = \frac{1}{2}$ (from $\frac{1}{2}^+$)

Similarly, $\lim\limits_{n \to -\infty} a_n = -\frac{1}{2}$ (from $-\frac{1}{2}^+$)

Thứ
Ngày . . No.

- E5: How should the function $g(x) = x^2 \operatorname{sgn} x$ be defined at $x = 0$ so that it is continuous there? Is it then differentiable there?

To prove that a function is continuous at $x = x_0$ we have to prove that

$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = f(x_0)$$

We have:

$$\operatorname{sgn}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

$$\Rightarrow g(x) = x^2 \operatorname{sgn}(x) = \begin{cases} -x^2, & x < 0 \\ 0, & x = 0 \\ x^2, & x > 0 \end{cases}$$

We have: $\lim_{x \to 0^-} g(x) = -0^2 = 0$

$\lim_{x \to 0^+} g(x) = 0^2 = 0$

$g(0) = 0$

$\Bigg\}$ $g(x)$ is $\Rightarrow$ continuous at $x = 0$

- Differentiability:

$$\lim_{x \to 0^{\pm}} \frac{g(x) - g(0)}{x - 0} = \frac{g(x)}{x}$$

We have $\lim_{x \to 0^+} \frac{g(x)}{x} = \frac{x^2}{x} = x = 0$

$$\lim_{x \to 0^-} \frac{g(x)}{x} = \frac{-x^2}{x} = -x = 0$$

So $\lim_{x \to 0^+} \frac{g(x)}{x}$ exists $\Rightarrow g(x) = x^2 \operatorname{sgn} x$ is differentiable at $x = 0$

E6: Calculate the derivative of $f(x) = x^{1/n}$, where n is a positive integer using the definition

$$\frac{d}{dx} f(x) = \cancel{x^{1/n}} \Rightarrow \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \text{ exists}$$

$$\Rightarrow \lim_{x \to x_0} \frac{x^{1/n} - x_0^{1/n}}{(x^{1/n})^n - (x_0^{1/n})^n}$$

$$= \lim_{x \to x_0} \frac{x^{1/n} - x_0^{1/n}}{(x^{1/n} - x_0^{1/n})\left(x^{\frac{n-1}{n}} + x^{\frac{n-2}{n}} \cdot x_0^{\frac{1}{n}} + \dots + x^{\frac{1}{n}} x_0^{\frac{n-2}{n}} + x_0^{\frac{n-1}{n}}\right)}$$

$$= \frac{1}{x^{\frac{n-1}{n}} + x^{\frac{n-2}{n}} \cdot x^{\frac{1}{n}} + \dots + x^{\frac{1}{n}} x^{\frac{n-2}{n}} + x^{\frac{n-1}{n}}}$$

$$= \frac{1}{2 \cdot \frac{n}{2} \cdot x^{\frac{n-1}{n}}} = \frac{1}{n x^{\frac{n-1}{n}}} = \frac{1}{n} \cdot \frac{1}{x^{\frac{n-1}{n}}} = \frac{1}{n} x^{-\left(\frac{n-1}{n}\right)}$$

$$\Rightarrow \frac{d}{dx} f(x) = \frac{1}{n} x^{\frac{1-n}{n}}$$