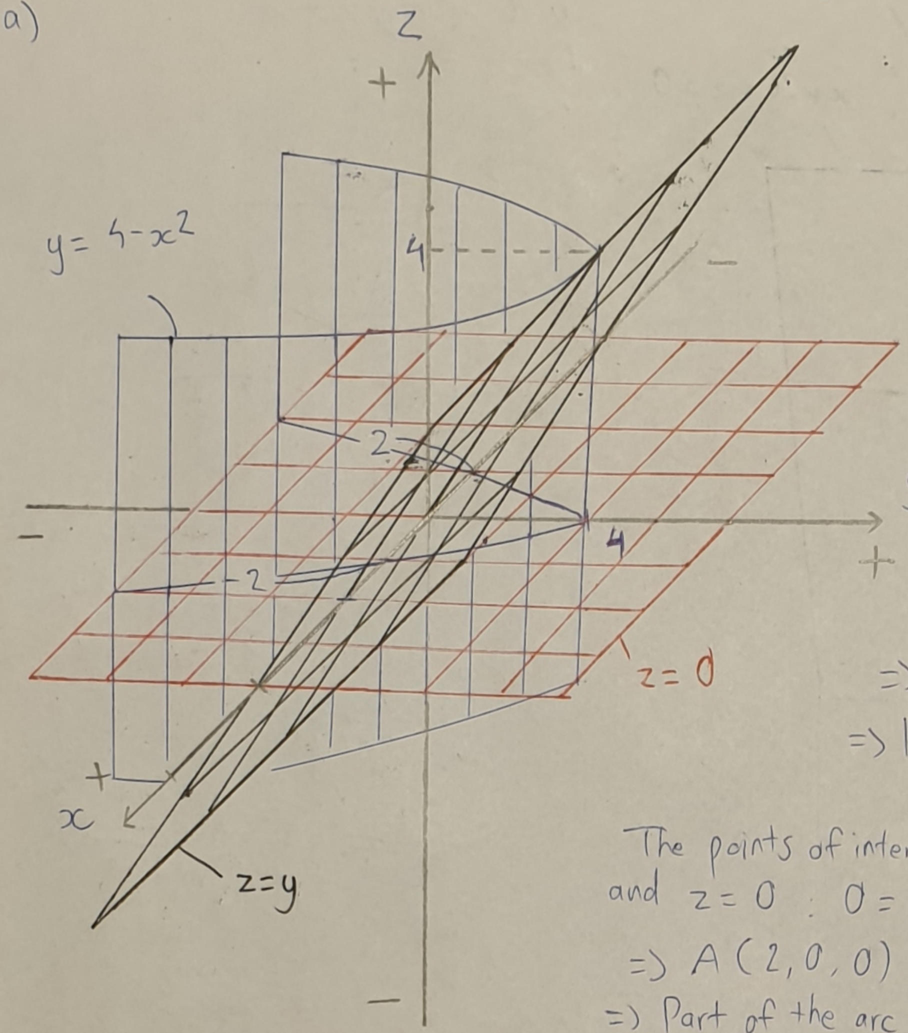
Nguyen Xuan Binh 887799

1) Consider the three surfaces $z = 0$, $z = y$ and $y = 4 - x^2$

a)



b) Find parametric equation of the curve of intersection of plane $z = y$ and $y = 4 - x^2$

Let $x = t$

$\Rightarrow y = z = 4 - t^2$

The parametric equation is

$\vec{r}(t) = \langle t, (4 - t^2), (4 - t^2) \rangle$

c) Find the arc length of the part of the curve that lies above xy-plane

We have

$\vec{r}(t) = \langle t, (4 - t^2), (4 - t^2) \rangle$

$\Rightarrow \vec{r}'(t) = \langle 1, -2t, -2t \rangle$

$\Rightarrow \|\vec{r}'(t)\| = \sqrt{1 + 4t^2 + 4t^2}$

$= \sqrt{1 + 8t^2}$

The points of intersection of $z = y$ and $y = 4 - x^2$ and $z = 0$: $0 = 4 - x^2 \Rightarrow x = \pm 2$

$\Rightarrow A(2, 0, 0)$ and $B(-2, 0, 0)$

$\Rightarrow$ Part of the arc that lies above xy-plane is from A to B or for $t = -2$ to $t = 2$

$\Rightarrow$ Arc length: $\int_{-2}^{2} \|\vec{r}'(t)\| dt = \int_{-2}^{2} \sqrt{1 + 8t^2}\, dt$

$\int \sqrt{1 + 8t^2}\, dt = 2\sqrt{2} \int \sqrt{t^2 + \left(\frac{\sqrt{2}}{4}\right)^2}$

Let $t = \frac{\sqrt{2}}{4}\tan\theta \Rightarrow dt = \frac{\sqrt{2}}{4}\sec^2\theta\, d\theta \Rightarrow 2\sqrt{2}\int \sqrt{\left(\frac{\sqrt{2}}{4}\right)^2 + \left(\frac{1}{8}\tan^2\theta\right)}\, \frac{\sqrt{2}}{4}\sec^2\theta\, d\theta$

$= \frac{\sqrt{2}}{4}\int \sqrt{\sec^2\theta}\, \sec^2\theta\, d\theta = \frac{\sqrt{2}}{4}\int \sec^3\theta\, d\theta$

$= \frac{\sqrt{2}}{4} \cdot \left(\frac{1}{2}\left[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right]\right) + C$

$= \frac{\sqrt{2}}{8}\left[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right] + C$

$t = 2 \Rightarrow \frac{\sqrt{2}}{4}\tan\theta = 2 \Rightarrow \theta = \arctan(\sqrt{32})$

$t = -2 \Rightarrow \frac{\sqrt{2}}{4}\tan\theta = -2 \Rightarrow \theta = -\arctan(\sqrt{32})$

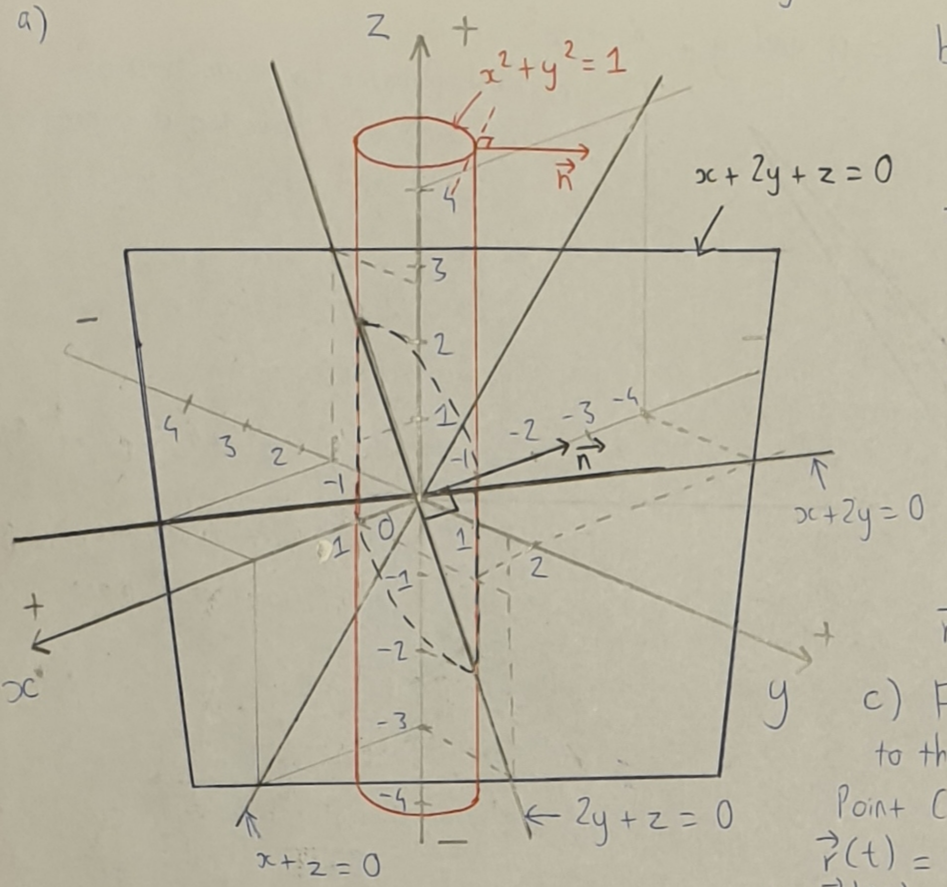
$\Rightarrow \frac{\sqrt{2}}{4}\int_{-\arctan\sqrt{32}}^{\arctan\sqrt{32}} \sec^3\theta\, d\theta = \frac{\sqrt{2}}{8}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right)\Big|_{-\arctan\sqrt{32}}^{\arctan\sqrt{32}}$

$= \frac{\sqrt{2}}{8}\Big[\sqrt{32}\left(\sec(\arctan\sqrt{32}) - \sec(\arctan\sqrt{32})\right) + \ln|\sec(\arctan\sqrt{32}) + \sqrt{32}|$
$+ \ln|\sec(-\arctan\sqrt{32}| - \sqrt{32}|\Big)$

$\approx 12{,}35$ (Length of the curve)

2) Consider the intersection of the surfaces $x^2+y^2=1$ and $x+2y+z=0$

a)



b) Find parametric equation of the curve of intersection

Let $x=t$

$\Rightarrow t^2+y^2=1$

$y^2=1-t^2$

$\Rightarrow y=\sqrt{1-t^2}$

We have : $x+2y+z=0$

$\Rightarrow t+2\sqrt{1-t^2}+z=0$

$\Rightarrow z=-t-2\sqrt{1-t^2}$

The parametric equation is

$\vec{r}(t)=\langle t, \sqrt{1-t^2}, -t-2\sqrt{1-t^2}\rangle$

c) Find the equation of the tangent line to the curve at point $(0,1,-2)$

Point $(0,1,-2) \Rightarrow t=0$

$\vec{r}(t)=\langle t, \sqrt{1-t^2}, -t-2\sqrt{1-t^2}\rangle$

$\Rightarrow \vec{r}\,'(t)=\langle 1, -\dfrac{t}{\sqrt{1-t^2}}, -1+\dfrac{2t}{\sqrt{1-t^2}}\rangle$

At $t=0 \Rightarrow \vec{r}\,'(0)=\langle 1,0,-1\rangle$

The tangent line pass through $(0,1,-2)$ and has direction of $\langle 1,0,-1\rangle$

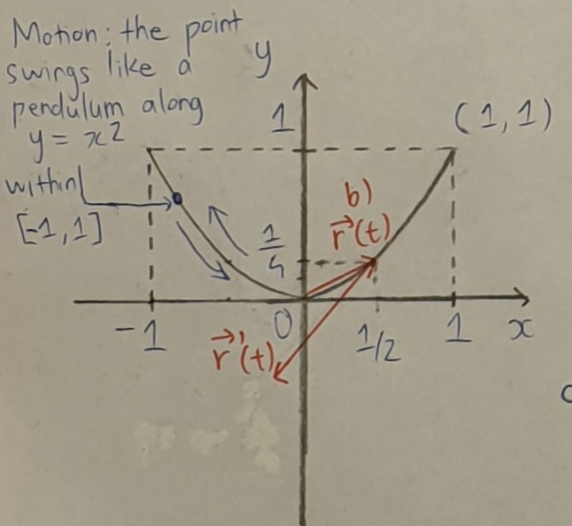
$\Rightarrow$ Parametric equation of tangent line $\begin{cases} x=0+1t \\ y=1+0t \\ z=-2+-1t \end{cases} \Rightarrow \begin{cases} x=t \\ y=1 \\ z=-t-2 \end{cases}$

3) Consider the parametric curve $x(t)=\cos(t)$, $y(t)=\cos^2 t$ for $-\infty<t<\infty$

a) For $t\in[-\infty,\infty]$, $\cos(t)\in[-1,1] \to \cos^2(t)\in[0,1]$

$\left.\begin{matrix} x(t)=\cos(t) \\ y(t)=\cos^2(t) \end{matrix}\right\} \Rightarrow y=x^2 < \left.\begin{matrix} \text{Domain } [-1,1] \\ \text{Range } [0,1] \end{matrix}\right\} \Rightarrow$ limited parabol

Motion : parabol $y=x^2$ from $[-1,1]$



b) Tangent vector to $(\frac{1}{2}, \frac{1}{4})$

$y=x^2 \Rightarrow \vec{r}(t)=\langle\cos(t), \cos^2 t\rangle$, $t\in(-\infty,+\infty)$

$\Rightarrow \vec{r}\,'(t)=\langle-\sin(t), -\sin(2t)\rangle$, $t\in(-\infty,+\infty)$

Point $(\frac{1}{2}, \frac{1}{4}) \Rightarrow t=\frac{1}{3}\pi \Rightarrow \vec{r}\,'(\frac{1}{3}\pi)=(-\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2})$

Parameterization of tangent vector $\begin{cases} x=\frac{1}{2}-\frac{\sqrt{3}}{2}t \\ y=\frac{1}{4}-\frac{\sqrt{3}}{2}t \end{cases}$

From $t=1$ to $t=\frac{1}{3}\pi$, the direction of motion is moving down from $(1,1)$ along $y=x^2$ to $(\frac{1}{2}, \frac{1}{4})$

c) Tangent vector to $(1,1)$

Point $(1,1) \Rightarrow t=0 \Rightarrow \vec{r}\,'(0)=\langle-\sin(0), -\sin(2.0)\rangle$

$\Rightarrow \vec{r}\,'(0)=\langle 0,0\rangle$

This does not make sense because the tangent vector is a zero vector. At $(1,1)$, the point changes its direction and thus $(1,1)$ is a sharp point

$\Rightarrow$ The curve is not smooth at $(1,1)$

Motion : $t\in(0,+\infty)$: the point runs from $(1,1)$ down to $(0,0)$ and swing to $(-1,1)$

$t\to(0,-\infty)$: the point runs from $(1,1)$ $(0,0)$ and swing to $(-1,1)$

d) Find the length of the curve

We have $\vec{r}'(t) = (1; 2t)$

$\Rightarrow \|\vec{r}'(t)\| = \sqrt{1+4t^2}$

Length of curve: $\int_{-1}^{1} \sqrt{1+4t^2}\,dt$ Substitute $t = \frac{1}{2}\tan\theta \Rightarrow dt = \frac{1}{2}\sec^2\theta\, d\theta$

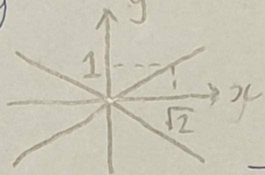
$\Rightarrow \int_{-1}^{1}\sqrt{1+4t^2} = \frac{1}{2}\int_{-\tan^{-1}(2)}^{\tan^{-1}(2)} \sec^3\theta\, d\theta = \frac{1}{4}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right)\Big|_{-\tan^{-1}(2)}^{\tan^{-1}(2)}$

$\approx 1{,}4789 - (-1{,}4789) \approx 2{,}9578$ (Answer)

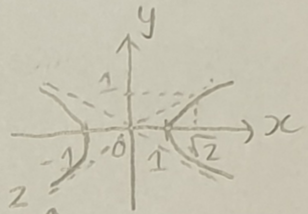
4) Consider the function $z = f(x,y) = x^2 - 2y^2$

a) Horizontal traces

$z = 0 \Rightarrow x^2 = 2y^2 \Rightarrow |x| = \sqrt{2}|y| \rightarrow$

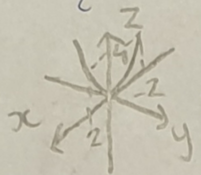


$z = 1 \Rightarrow x^2 - 2y^2 = 1 \Rightarrow \frac{x^2}{2} - y^2 = \frac{1}{2}$ $\rightarrow$

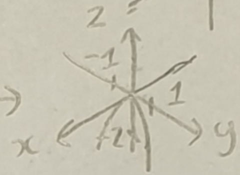


Vertical traces

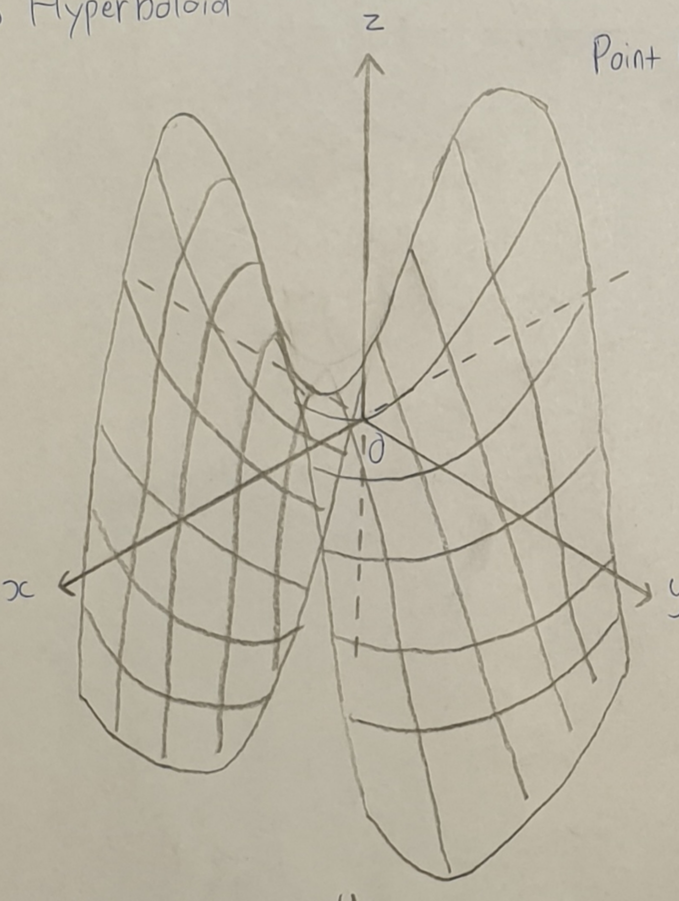
$y = 0, \Rightarrow z = x^2 \rightarrow$



$x = 0 \Rightarrow z = -2y^2 \rightarrow$



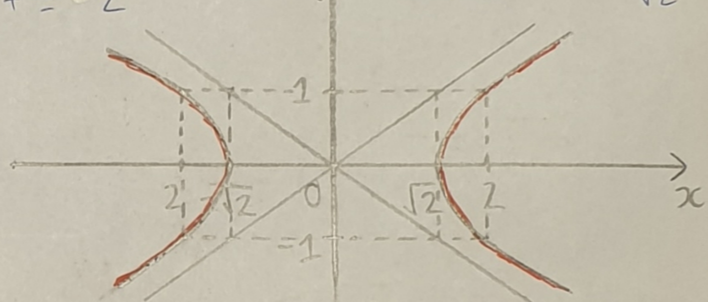
$\Rightarrow$ Hyperboloid

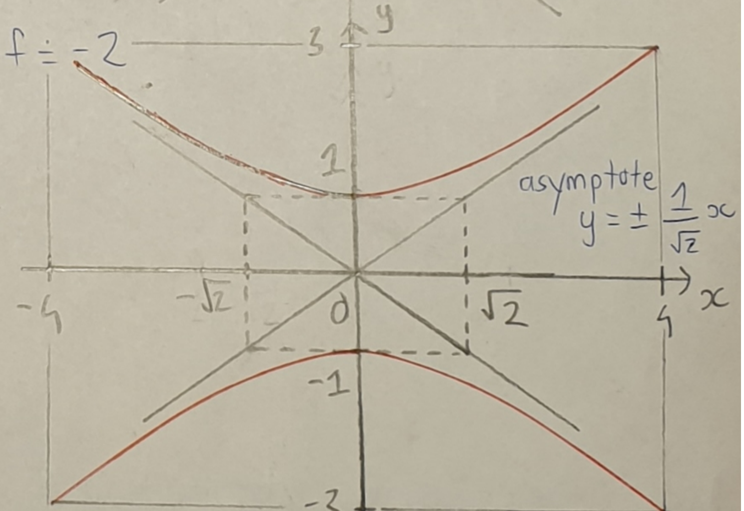


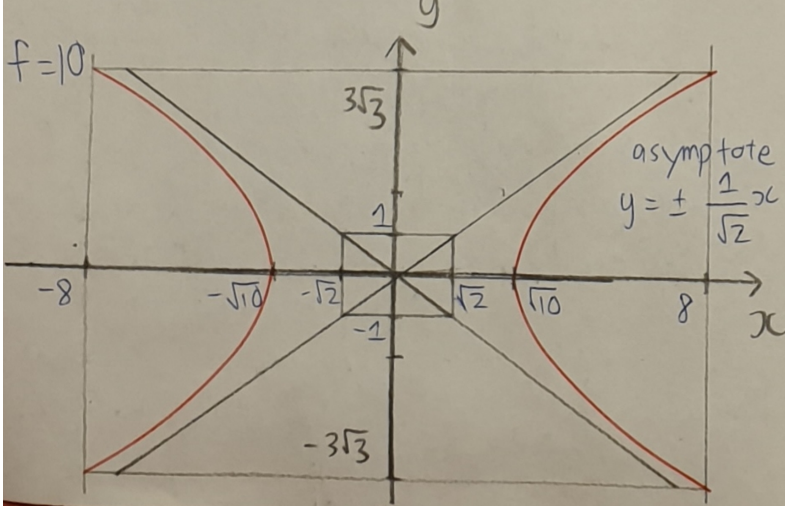
Point $(0,0,0)$ is the center of the saddle

asymptote $y = \pm\frac{1}{\sqrt{2}}x$

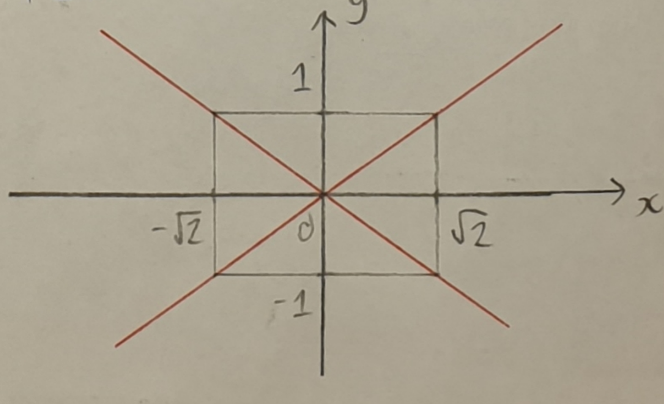
b) $f = 2$







f = 0



6) Let $f(x)$ be a function defined $\forall x \in \mathbb{R}$. Is it possible for two different level curves to intersect? (If $a \neq b$, can level curves $f = a$ and $f = b$ intersect?

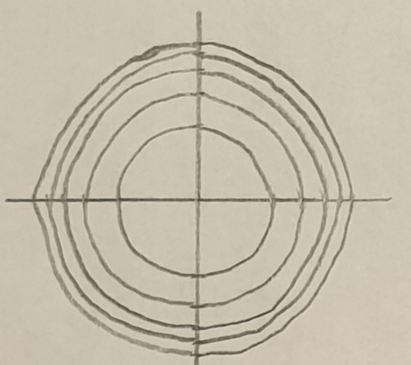
Let $f(x_1, y_1) = a$ and $f(x_2, y_2) = b$

$$\begin{cases} a \neq b \\ f(x_1, y_1) \text{ intersects } f(x_2, y_2) \end{cases} \Rightarrow \begin{cases} a \neq b \\ \text{There exists a point on both } f(x_1, y_1) \text{ and } f(x_2, y_2) \end{cases}$$

$$\Rightarrow \begin{cases} a \neq b \\ a = b \end{cases} \text{ (Contradicting)}$$

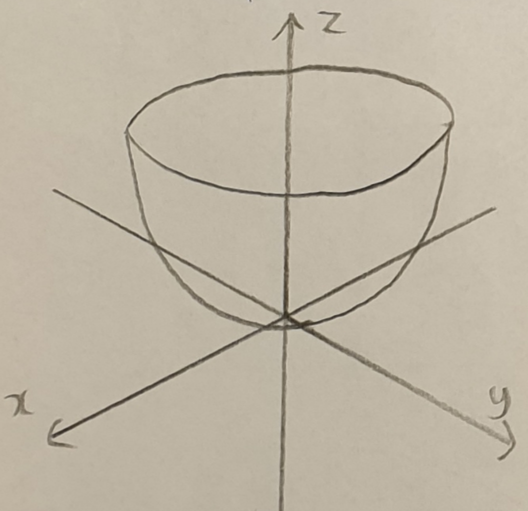
$\Rightarrow$ It is impossible for two different level curves of the same function to intersect

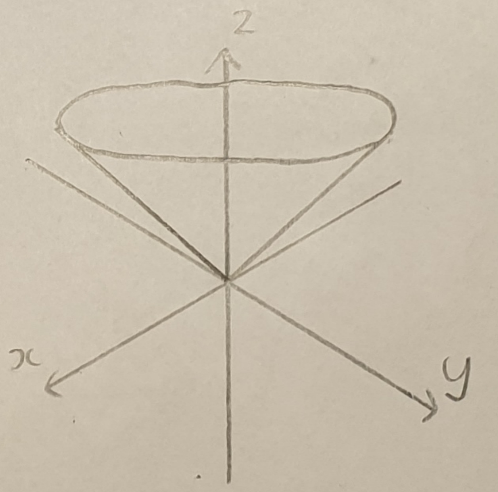
5) Sets of level curves



This ~~cur~~ level curves is a paraboloid

This level curves is a cone





Reason: the circles of the contour map are not spaced evenly because the height of the paraboloid increases exponentially fast (parabola). The further a way the circle from the axis, the smaller its radius increases

Reason: the circles of the contour map are spaced evenly because the height of the cone increases linearly ($y = ax + b$). No matter how far away the circle from the axis, its radius will always increase by the same amount