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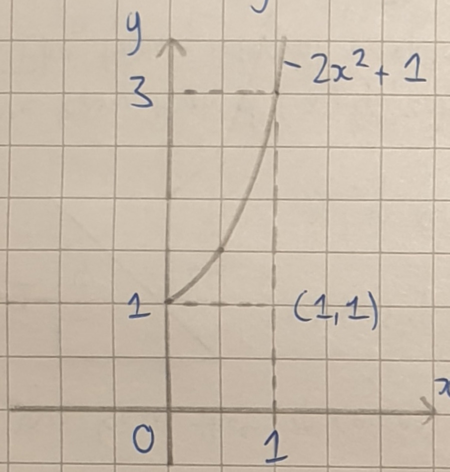
Exercise 1:

a) Reverse the order of integration for $\int_0^1 \int_0^{2x^2+1} f(x,y)\,dy\,dx$

Range: $x \in [0, 1]$, $y \in [0, 2x^2+1]$

At $x = 1 \Rightarrow y = 3$

We have: $y = 2x^2 + 1 \Rightarrow x = \sqrt{\frac{y-1}{2}} \Rightarrow y \geq 1$



$\Rightarrow$ To change order, the integral will be split into two parts

$$\Rightarrow \int_0^1 \int_0^{2x^2+1} f(x,y)\,dy\,dx$$

$$= \int_0^1 \int_0^1 f(x,y)\,dx\,dy + \int_1^3 \int_{\sqrt{\frac{y-1}{2}}}^1 f(x,y)\,dx\,dy$$

b) Compute the double integral of the function $f(x,y) = e^{x^2+y^2}$ over the top half of the disk of radius 3 centered at $(0,0)$

Polar coordinates: $x^2 + y^2 = r^2$ $\quad \mathbb{R}^2$: $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 3$

$$\iint_{\mathbb{R}} e^{x^2+y^2}\,dA = \int_0^{2\pi}\int_0^3 e^{r^2}\,r\,dr\,d\theta$$

Let $u = r^2 \Rightarrow du = 2r\,dr \Rightarrow \frac{du}{2} = r\,dr$

$$\Rightarrow \int_0^{2\pi}\int_0^9 e^u \frac{1}{2}\,du\,d\theta = \frac{1}{2}\int_0^{2\pi}\left(e^u\Big|_0^9\right)d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}(e^9 - 1)\,d\theta = \frac{1}{2}\left(e^9\theta - \theta\Big|_0^{2\pi}\right) = e^9\pi - \pi$$

The integral is $e^9\pi - \pi$

Exercise 2:

a) The planes are:

$$\begin{cases} x \geq 0 \\ y \geq 0 \\ z \geq 0 \\ z = y - x \\ x^2 + y^2 \leq 4 \end{cases} \Rightarrow \begin{cases} r\cos\theta \geq 0 \\ r\sin\theta \geq 0 \\ z \geq 0 \\ z = r\sin\theta - r\cos\theta \\ r^2 \leq 4 \end{cases} \Rightarrow \begin{cases} \cos\theta \geq 0 \\ \sin\theta \geq 0 \\ z \geq 0 \\ z = r\sin\theta - r\cos\theta \\ r \in [0, 2] \end{cases}$$

(Since r is positive)

$$\Rightarrow \begin{cases} \theta \in [-\frac{\pi}{2} ; \frac{\pi}{2}] \\ \theta \in [0 ; \pi] \\ z \geq 0 \\ \sin\theta \geq \cos\theta \text{ (since } z \geq 0 \text{ \& } r \text{ is positive)} \\ r \in [0 ; 2] \end{cases} \Rightarrow \theta \in [0 ; \frac{\pi}{2}]$$

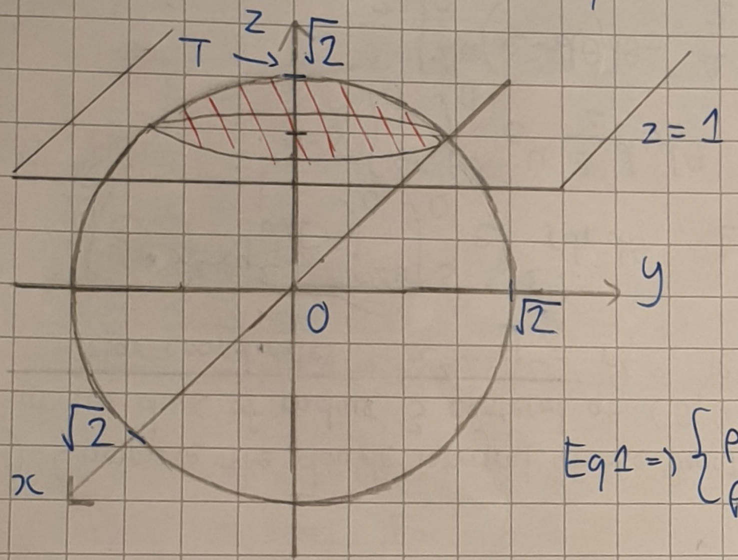
$$\Rightarrow \begin{cases} \theta \in [0 ; \frac{\pi}{2}] \\ \theta \in [\frac{\pi}{4} ; \frac{3\pi}{4}] \\ z \geq 0 \\ r \in [0 ; 2] \end{cases} \Rightarrow \begin{cases} \theta \in [\frac{\pi}{4} ; \frac{\pi}{2}] \\ z \geq 0 \\ r \in [0 ; 2] \end{cases}$$

We have: $z = r(\sin\theta - \cos\theta)$ and we have the bounds of r and θ

$\Rightarrow$ The integral: $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^2 \int_0^{r(\sin\theta - \cos\theta)} 1 \, r \, dz \, dr \, d\theta$

b) Let E be the solid region that lies inside the sphere $x^2 + y^2 + z^2 = 2$ & above $z = 1$

$d(x, y, z) = z^2$ the density



We have

$$\begin{cases} x^2 + y^2 + z^2 \leq 2 \\ z \geq 1 \end{cases} \quad (\text{Eq}1)$$

Since the region is circular

$\Rightarrow \theta = [0 ; 2\pi]$

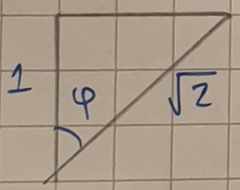
And we know that the region goes over the top point of the sphere (T).

$\Rightarrow \varphi_{min} = 0$

$$\text{Eq}1 \Rightarrow \begin{cases} \rho^2 \leq 2 \\ \rho\cos\varphi \geq 1 \end{cases} \Rightarrow \begin{cases} \rho \in [0 ; \sqrt{2}] \\ \rho \geq \sec\varphi \end{cases}$$

$\Rightarrow \rho \in [\sec\varphi, \sqrt{2}]$

We know that φ reaches its max at the circle of intersection



$\Rightarrow \cos\varphi = \frac{1}{\sqrt{2}} \Rightarrow \varphi_{max} = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

$\Rightarrow \varphi \in [0 ; \frac{\pi}{4}]$

We know that $d(x, y, z) = z^2 \Rightarrow d(\rho, \theta, \varphi) = \rho^2 \cos^2\varphi$

The tripple integral in spherical coordinates that gives the mass of E is

$$M_E = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_{\sec\varphi}^{\sqrt{2}} \underbrace{[\rho^2 \cos^2\varphi]}_{\text{density}} \underbrace{\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta}_{dV}$$

Exercise 3:

The formula that doesn't make sense as a double integral is (c) and (f) and (a)
These two formulas aren't proper because when solved to the last outside integral, it will exist as a general formula with one variabe (x or y) instead of a real definite number

b) $\int_0^1\int_0^{3x}(2x+4y)\,dy\,dx = \int_0^1\left[2xy+2y^2 \Big|_0^{3x}\right]dx$

$= \int_0^1 6x^2+18x^2\,dx = \int_0^1 24x^2\,dx$

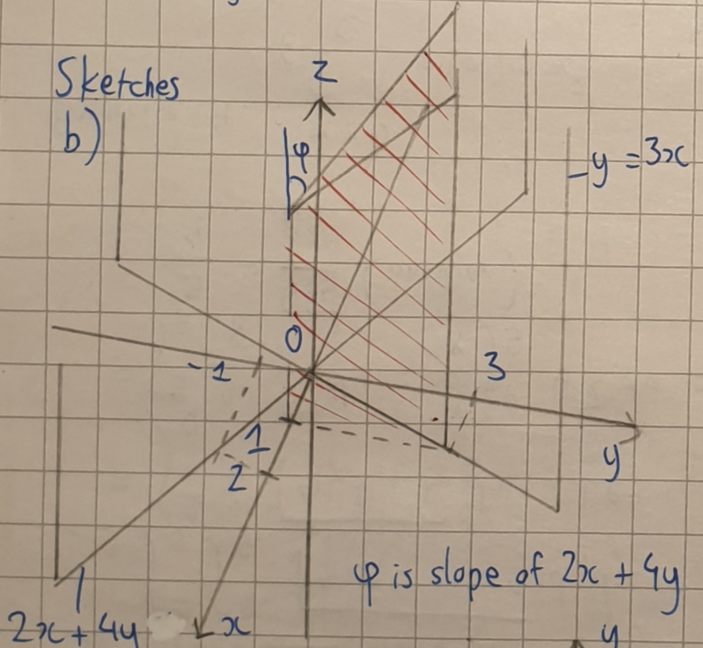
d) $\int_0^1\int_{y-1}^{1-y^2}(2x+4y)\,dx\,dy = \int_0^1\left[x^2+4yx\Big|_{y-1}^{1-y^2}\right]dy$

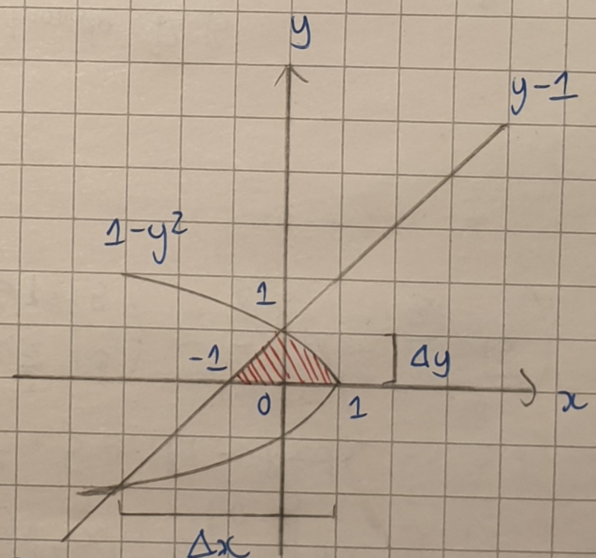
$= \int_0^1\left[(1-y^2)^2+4y(1-y^2)-(y-1)^2+4y(y-1)\right]dy$

$= \int_0^1(y^4-4y^3-7y^2+10y)\,dy$

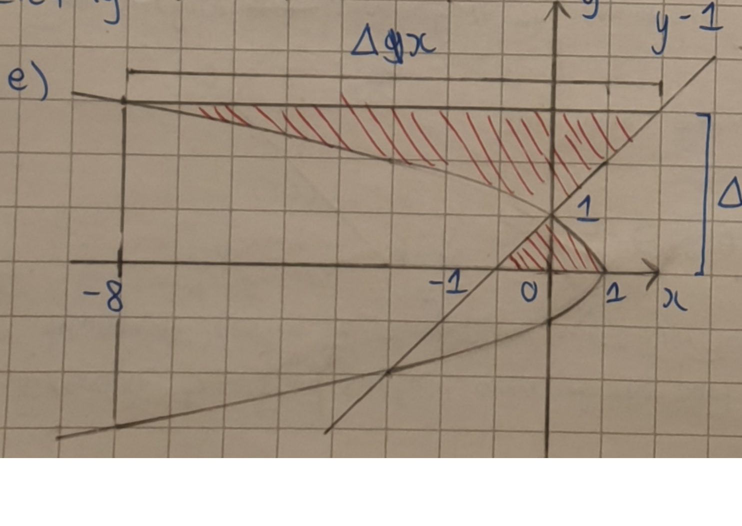
e) $\int_0^3\int_{y-1}^{1-y^2}(2x+4y)\,dx\,dy = \int_0^3(y^4-4y^3-7y^2+10y)\,dy$

Sketches



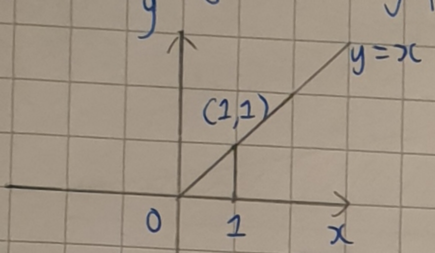


=) Region of integration: positive-cylinder shape with red base with top bounded by $2x+4y$



Since no region exists in $y=[1,3]$ the bounds are exchanged for it to exist, which results in (e) having negative integral. The base is red region. Above red is negative and below red is positive

Exercise 5: Find surface area of the part of surface $z = x^2 + 2y$ that lies above triangular region in x-y plane with vertice $(0,0)$, $(1,0)$, $(1,1)$



$\Rightarrow x \in [0,1]$, $y \in [0,x]$

We have: $f_x = 2x \quad f_y = 2$

The surface ~~are~~ lying above the region is

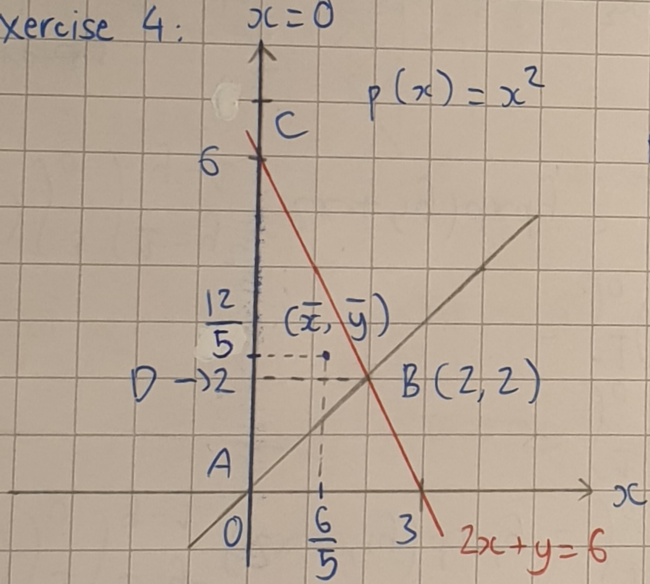
$$\int_0^1\int_0^x \sqrt{4x^2+4+1}\,dy\,dx = \int_0^1 \left[y\sqrt{4x^2+5}\,\Big|_0^x \right] dx$$

$$= \int_0^1 x\sqrt{4x^2+5}\,dx.$$ Let $u = 4x^2+5 \Rightarrow du = 8x\,dx \Rightarrow \frac{1}{8}du = x\,dx$

$$= \int_5^9 \frac{1}{8}\sqrt{u}\,du = \frac{1}{8}\int_5^9 u^{\frac{1}{2}}\,du = \frac{1}{8}\left(\frac{u^{\frac{3}{2}}}{3/2}\Big|_5^9\right) = \frac{1}{8}\left(18 - \frac{2}{3}5\sqrt{5}\right)$$

$$= \frac{9}{4} - \frac{5\sqrt{5}}{12}$$ (answer)

Exercise 4:



Mass of triangle DCB

$$M_{DCB} = \int_0^2\int_2^{-2x+6} x^2\,dy\,dx = \frac{8}{3}$$

$$M_{DBA} = \int_0^2\int_0^y x^2\,dx\,dy = \frac{4}{3}$$

$$\Rightarrow M_{ABC} = \frac{4}{3} + \frac{8}{3} = 4$$

The x - moment:

$$M_{DCB_x} = \int_0^2\int_2^{-2x+6} x^3\,dy\,dx = \frac{16}{5}$$

$$M_{DBA_x} = \int_0^2\int_0^y x^3\,dx\,dy = \frac{8}{5}$$

The y - moment:

$$M_{DCBy} = \int_0^2\int_2^{-2x+6} x^2 y\,dy\,dx = \frac{112}{15}$$

$$M_{DBAy} = \int_0^2\int_0^y x^2 y\,dx\,dy = \frac{32}{15}$$

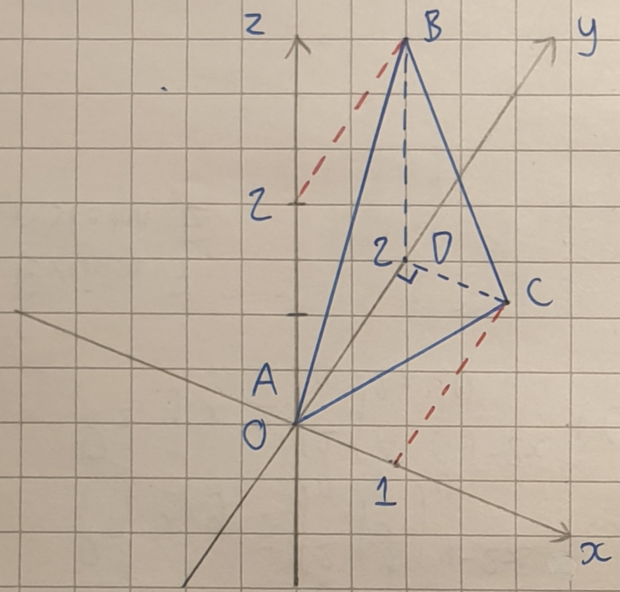
$$\Rightarrow M_x = \frac{16}{5} + \frac{8}{5} = \frac{24}{5}$$

$$\Rightarrow M_y = \frac{112}{15} + \frac{32}{15} = \frac{144}{15} \Rightarrow (\bar{x}, \bar{y}) = \left(\frac{M_x}{M_{ABC}}, \frac{M_y}{M_{ABC}}\right) = \left(\frac{6}{5}, \frac{12}{5}\right)$$

The center of mass makes sense, because the x-axis ~~sa~~ has no mass ($0^2 = 0$) and the triangle becomes heavier when it shifts horizontally to B (B has mass of $2^2 = 4$)
$\Rightarrow$ B side is heavier, but has less area than D side, so the center of mass shift to B side. Since DCB has more area than DBA, the center of mass shifts vertically to C side (lying above $y = 2$)

Exercise 6: Let E the solid bound by $z = 0$, $x = 0$, $y = 2$, $z = y - 2x$

a) Sketch the solid E



$\triangle ABD: x = 0$

$\triangle ADC: z = 0$

$\triangle BDC: y = 2$

$\triangle ABC: z = y - 2x$

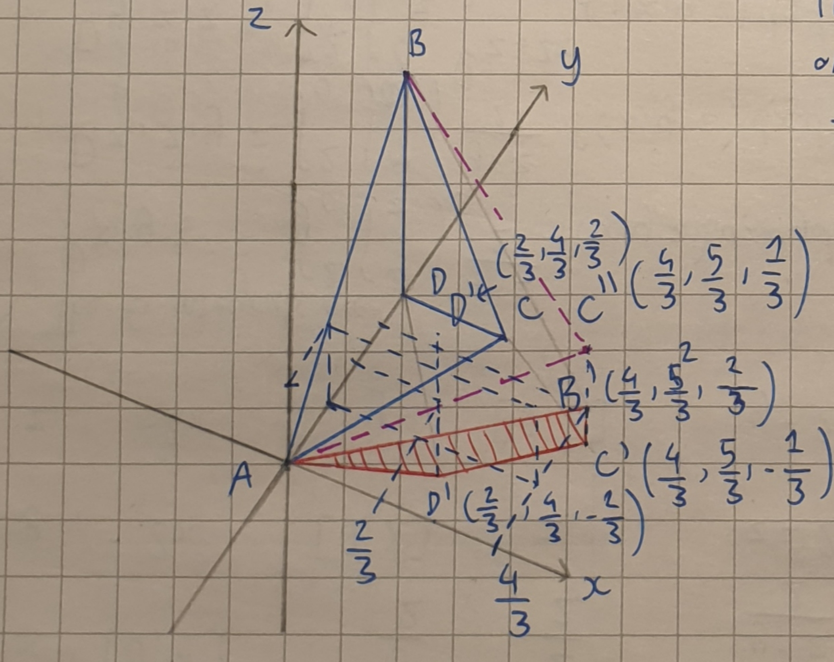
$\Rightarrow$ E is a pyramid with 4 vertex

$A = (0,0,0)$ $B = (0,2,2)$

$C = (1,2,0)$ $D = (0,2,0)$

b) $z = x - y$ and $y = x - z$ plane are two similar planes

$x = y - z \Rightarrow z = y - x$



The red region is projection of E onto $x-y$ plane and $x-z$ plane, that is $AB'C'D'$

The purple region is projection of E onto $y-z$ plane, that is triangle ABC''

c) Espress integral $\iiint_E f(x,y,z)\,dV$ as an integral in six different ways

The table of iteration

	x	y	z	multivariable
x	$[0,1]$	$\frac{y}{2}$	$-\frac{z}{2}$	$\frac{y-z}{2}$
y	$2x$	$[0,2]$	z	$z+2x$
z	$-2x$	y	$[0,2]$	$y-2x$

=> The 6 integrals are

$$D_{x,y,z} = \int_0^2 \int_z^2 \int_0^{\frac{y-z}{2}} f(x,y,z)\,dx\,dy\,dz$$

$$D_{x,z,y} = \int_0^2 \int_0^y \int_0^{\frac{y-z}{2}} f(x,y,z)\,dx\,dz\,dy$$

$$D_{y,x,z} = \int_0^2 \int_{-\frac{z}{2}}^0 \int_0^{z+2x} f(x,y,z)\,dy\,dx\,dz$$

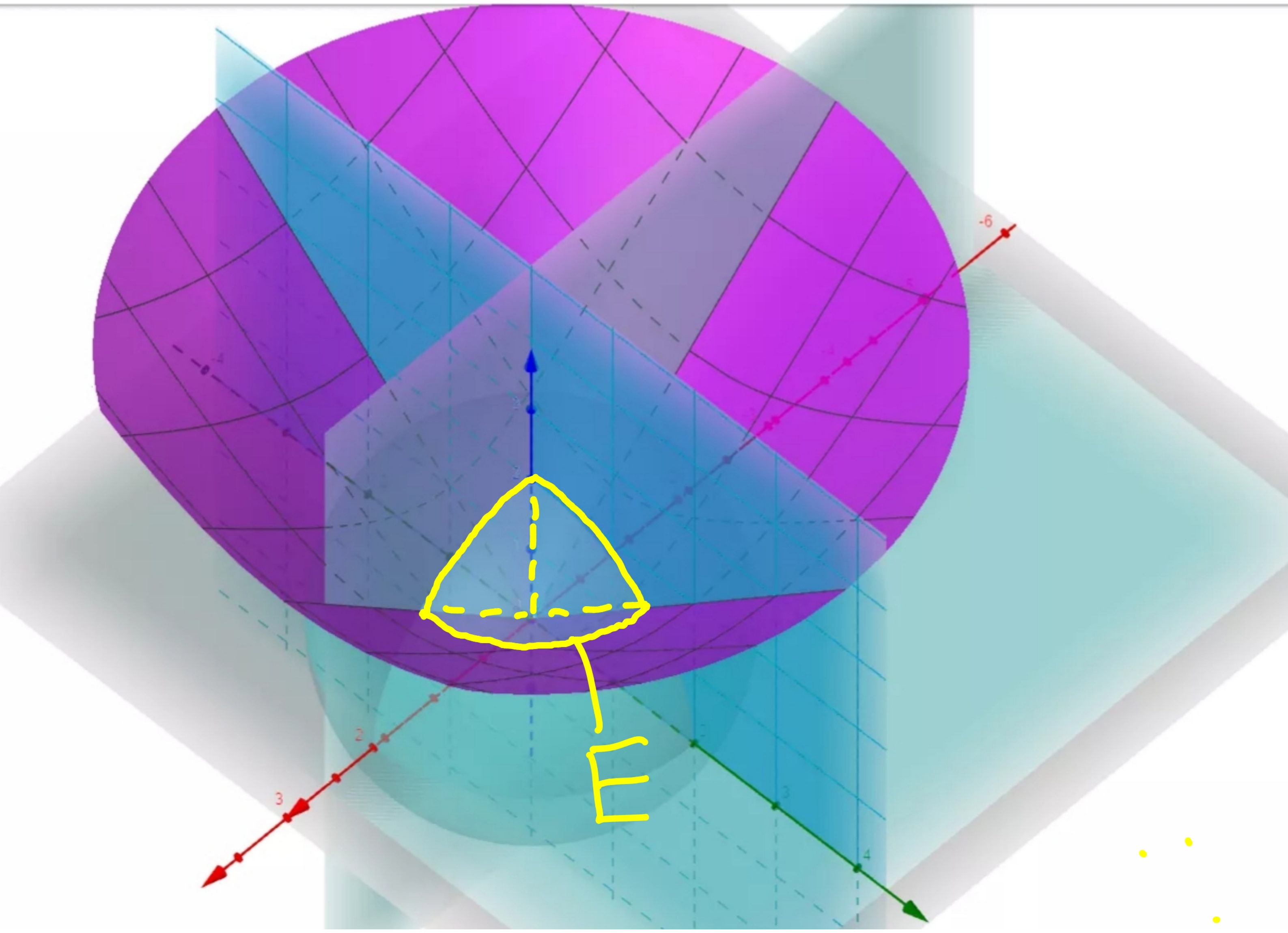
$$D_{y,z,x} = \int_0^1 \int_{-2x}^0 \int_0^{z+2x} f(x,y,z)\,dy\,dz\,dx$$

$$D_{z,x,y} = \int_0^2 \int_0^{\frac{y}{2}} \int_0^{y-2x} f(x,y,z)\,dz\,dx\,dy$$

$$D_{z,y,x} = \int_0^1 \int_{2x}^2 \int_0^{y-2x} f(x,y,z)\,dz\,dy\,dx$$

Exercise 7:

a) Sketch E


E

b) $\begin{cases} x^2+y^2+z^2 \leqslant 4 \\ -z \geqslant \sqrt{x^2+y^2} \\ x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0 \end{cases} \Rightarrow \begin{cases} r^2+z^2 \leqslant 4 \Rightarrow z \leqslant \sqrt{4-r^2} \\ z \geqslant r \\ r\cos\theta \geqslant 0,\ r\sin\theta \geqslant 0,\ z \geqslant 0 \end{cases}$

$\Rightarrow \begin{cases} r \leqslant \sqrt{4-z^2} \\ z \geqslant r \\ \theta \in [0; \frac{\pi}{2}] \\ z \geqslant 0 \end{cases} \Rightarrow \begin{cases} r \leqslant \sqrt{2} \\ r \geqslant 0 \\ \theta \in [0; \frac{\pi}{2}] \\ z \geqslant r \end{cases} \Rightarrow \begin{cases} 0 \leqslant \theta \leqslant \frac{\pi}{2} \\ 0 \leqslant r \leqslant \sqrt{2} \\ r \leqslant z \leqslant \sqrt{4-r^2} \end{cases}$

$\Rightarrow$ Triple integral: $\int_0^{\frac{\pi}{2}} \int_0^{\sqrt{2}} \int_r^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$

$= \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{2}} rz \Big|_r^{\sqrt{4-r^2}} dr\,d\theta = \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{2}} \left[r\sqrt{4-r^2} - r^2 \right] dr\,d\theta$

$= \int_0^{\frac{\pi}{2}} \frac{8-4\sqrt{2}}{3}\,d\theta = \frac{8-4\sqrt{2}}{3} \cdot \frac{\pi}{2} = \frac{4-2\sqrt{2}}{3}\pi$ (Answer)

c) $\begin{cases} x^2+y^2+z^2 \leqslant 4 \\ z \geqslant \sqrt{x^2+y^2} \\ x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0 \end{cases} \Rightarrow \begin{cases} \rho^2 \leqslant 4 \\ \cos\varphi \geqslant \sin\varphi \\ \sin\varphi\cos\theta \geqslant 0 \\ \sin\varphi\sin\theta \geqslant 0 \\ \cos\varphi \geqslant 0 \end{cases} \Rightarrow \begin{cases} \rho \in [0; 2] \\ \varphi \in [-\frac{3\pi}{4}; \frac{\pi}{4}] \\ \left[\begin{array}{l} \theta \in [0; \frac{\pi}{2}],\ \varphi \in [0; \pi] \\ \theta \in [\pi; \frac{3\pi}{2}],\ \varphi \in [-\pi; 0] \end{array}\right. \\ \varphi \in [-\frac{\pi}{2}; \frac{\pi}{2}] \end{cases}$

$\Rightarrow \begin{cases} \rho \in [0; 2] \\ \varphi \in [0; \frac{\pi}{4}] \\ \theta \in [0; \frac{\pi}{2}] \end{cases}$

$\Rightarrow$ Triple integral: $\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_0^2 \rho^2 \sin(\varphi)\,d\rho\,d\varphi\,d\theta$

$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \frac{1}{3}\rho^3 \sin\varphi \Big|_0^2 d\varphi\,d\theta = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \frac{8}{3}\sin(\varphi)\,d\varphi\,d\theta$

$= \int_0^{\frac{\pi}{2}} -\frac{8}{3}\cos(\varphi) \Big|_0^{\frac{\pi}{4}} d\theta = \int_0^{\frac{\pi}{2}} \left(\frac{8-4\sqrt{2}}{3} \right) d\theta$

$= \frac{8-4\sqrt{2}}{3} \cdot \frac{\pi}{2} = \frac{4-2\sqrt{2}}{3}\pi$ (Answer)