Nguyen Xuan Binh 887799 Hand-in 3

Exercise 1: Calculate $\oint_\gamma x\,dy$ where γ is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
$(a > 0, b > 0)$ counterclockwise

We have: $\begin{cases} x = a\cos t \\ y = b\sin t \end{cases}$ $\left(\frac{a^2\cos^2 t}{a^2} + \frac{b^2 \sin^2 t}{b^2} = \cos^2 t + \sin^2 t = 1 \right)$

$$\Rightarrow \oint_\gamma x\,dy = \int_0^{2\pi} a\cos t \, b\cos t\,dt = \int_0^{2\pi} ab\cos^2 t\,dt$$

$$= ab \int_0^{2\pi} \frac{1}{2}(\cos 2t + 1)\,dt = \frac{1}{2} ab \int_0^{2\pi} \cos 2t + 1\,dt$$

$$= \frac{1}{2} ab \left(\frac{1}{2}\sin 2t + t \right)\Big|_0^{2\pi} = \pi ab \text{ (answer)}$$

Exercise 2: Calculate $\iint_S x\,dS$ where S is the part of $z = x^2$ above the rectangle $\{(x, y, 0) \in R^3, 0 \leq x \leq 2, 0 \leq y \leq 3\}$

We have $z = x^2 \Rightarrow \frac{\partial z}{\partial x} = 2x$ and $\frac{\partial z}{\partial y} = 0$

$$\iint_S x\,dS = \iint_A x\sqrt{(2x)^2 + 0^2 + 1}\,dA = \int_0^3 \int_0^2 x\sqrt{4x^2 + 1}\,dx\,dy$$

Let $u = 4x^2 + 1 \Rightarrow du = 8x\,dx \Rightarrow 1/8\,du = x\,dx$

$$\Rightarrow \iint_S x\,dS = \int_0^3 \int \frac{1}{8}\sqrt{u}\,du\,dy = \int_0^3 \frac{1}{8} \cdot \frac{2}{3} (4x^2 + 1)^{3/2} \Big|_0^2 dy$$

$$= \int_0^3 \frac{1}{12}(17\sqrt{17} - 1)\,dy = \frac{1}{12}(17\sqrt{17} - 1)y \Big|_0^3$$

$$= \frac{1}{4}(17\sqrt{17} - 1) \text{ (answer)}$$

Exercise 3: Calculate the flux of $F(x, y, z) = (xy, 0, -1)$ outward (away from the z-axis) through the cone $z^2 = x^2 + y^2$ where $0 \leq z \leq 1$

Since $z \geq 0 \Rightarrow z = \sqrt{x^2+y^2}$

We have: $G(x, y, z) = \sqrt{x^2+y^2} - z$

$\Rightarrow \nabla G \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1 \right)$ (point in correct direction)

$$\Rightarrow \iint_G F.n\,dS = \iint_R (xy, 0, -1) \cdot \left(\frac{x}{\sqrt{x^2+y^2}}; \frac{y}{\sqrt{x^2+y^2}}; -1 \right) dy\,dx$$

$$= \iint_R \left[\frac{x^2y}{\sqrt{x^2+y^2}} + 1 \right] dy\,dx$$ Changing to polar coordinates

$$\int_0^{2\pi}\int_0^1 \left[\frac{r^3\cos^2\theta\sin\theta}{r} + 1 \right] r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \left[r^3\cos^2\theta\sin\theta + r \right] dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^4}{4}\cos^2\theta\sin\theta + \frac{r^2}{2} \right]_0^1 d\theta = \int_0^{2\pi} \frac{1}{4}\cos^2\theta\sin\theta + \frac{1}{2}\,d\theta$$

$$= \frac{1}{4}\int_0^{2\pi} \cos^2\theta\sin\theta\,d\theta + \int_0^{2\pi} \frac{1}{2}\,d\theta = \frac{1}{4}\int_0^{2\pi}\cos^2\theta\sin\theta\,d\theta + \pi$$

Let $u = \cos\theta \Rightarrow du = -\sin\theta\,d\theta \Rightarrow -du = \sin\theta\,d\theta$

$$\Rightarrow \iint_G F.n\,dS = \frac{1}{4}\int -u^2 d\theta + \pi = \frac{-1}{4} \cdot \frac{u^3}{3} + \pi = \frac{-1}{4}\left(\frac{\cos^3\theta}{3} \right)\Bigg|_0^{2\pi} + \pi$$

$$= \frac{-1}{12} + \frac{1}{12} + \pi = \pi \text{ (answer)}$$

Exercise 4: Calculate the flux of $F(x, y, z) = (z^2, x, -3z)$ outward through the surface cut from the parabolic cylinder $z = 4 + y^2$ by the planes $x = 0$, $x = 1$, $z = 0$

We have $\operatorname{div} \vec{F} = (0, 0, -3) = 0 + 0 + -3 = -3$

According to divergence theorem, the integral is

$$\iint_S F\, dS = \iiint_E \operatorname{div} \vec{F}\, dV = \int_{-2}^{2} \int_0^1 \int_0^{4-y^2} (-3)\, dz\, dx\, dy$$

$$= \int_{-2}^{2} \int_0^1 3y^2 - 12\, dx\, dy = \int_{-2}^{2} x(3y^2 - 12) \Big|_0^1 dy$$

$$= \int_{-2}^{2} 3y^2 - 12\, dy = y^3 - 12y \Big|_{-2}^{2} = -16 - 16 = -32 \text{ (answer)}$$