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PROBLEM SHEET 4 Exercises (Homework Problems)

Exercise 1: Find the area of finite plane region bounded by the curve $y = \ln x$, $y = 1$ and the tangent line to $y = \ln x$ at $x = 1$

- Tangent line to $y = \ln x$ at $x = 1$

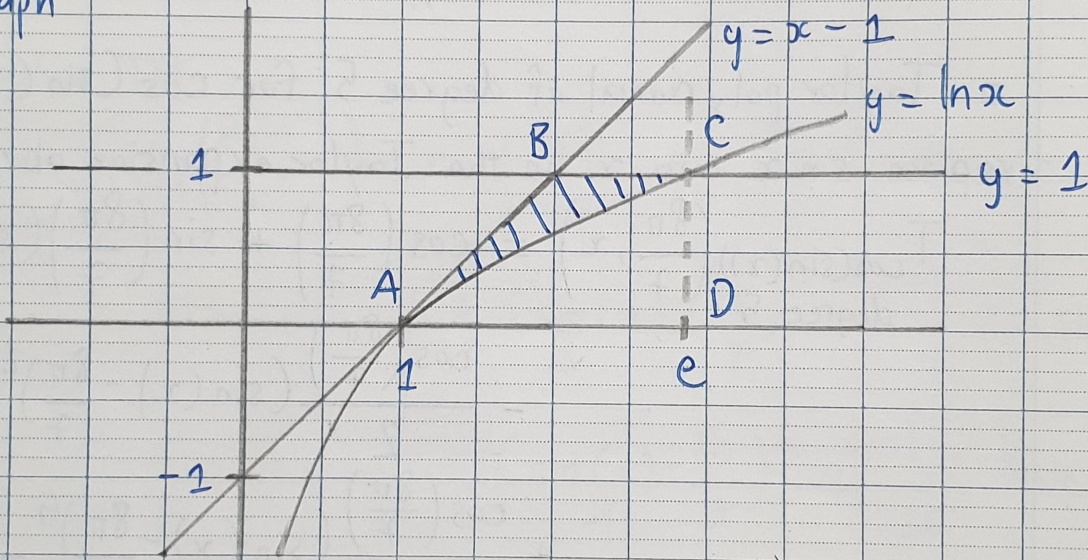
The line: $y - y_1 = m(x - x_1)$

$(\ln x)' = \frac{1}{x} \Rightarrow m = \frac{1}{1} = 1$

We have: $y - \ln 1 = 1(x - 1)$

$\Leftrightarrow \quad y = x - 1 \Rightarrow$ This is the tangent line

- Graph



- The area bounded by the three lines is the shape S_{ABC}
- Coordinate: $A(1, 0)$, $B(2, 1)$, $C(e, 1)$, $D(e, 0)$
- The area of S_{ACD} is the integral of $\ln x$ from 1 to e

$\Rightarrow S_{ACD} = \int_1^e \ln x \, dx$

Integration by part: let $u = \ln x$, $dv = dx$

$\Rightarrow du = \frac{1}{x} dx$, $v = x$

$S_{ACD} = x\ln x - \int x \cdot \frac{1}{x} dx = x\ln x - x \Big|_1^e$

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$S_{ACD} = 0 - (-1) = 1$

- Area of trapezoid ABCD

$T_{ABCD} = \frac{1}{2} 1 [(e-2) + (e-1)] = e - \frac{3}{2}$

$\Rightarrow$ Area bounded by $y = \ln x$, $y = 1$, $y = x - 1$ is

$S_{ABC} = T_{ABCD} - S_{ACD}$

$= e - \frac{3}{2} - 1$

$\Rightarrow S_{ABC} = e - \frac{5}{2} \approx 0,218281$

Exercise 3: Solve $\frac{dy}{dx} = \frac{3y-1}{x}$

$\frac{dy}{dx} = \frac{3y-1}{x} \Leftrightarrow \frac{dy}{dx} - \frac{3}{x} \cdot y = -\frac{1}{x}$

We have: $P(x) = -\frac{3}{x}$, $Q(x) = -\frac{1}{x}$

Integrating factor: $e^{\int P(x)dx} = e^{\int -\frac{3}{x} dx} = e^{-3\ln x}$

$\Leftrightarrow e^{-3\ln x} \frac{dy}{dx} + e^{-3\ln x}\left(-\frac{3y}{x}\right) = e^{-3\ln x}\left(-\frac{1}{x}\right)$

$\Leftrightarrow \frac{d}{dx}\left(e^{-3\ln x} \cdot y\right) = e^{-3\ln x}\left(-\frac{1}{x}\right)$

$\Leftrightarrow e^{-3\ln x} y = \int e^{-3\ln x}\left(-\frac{1}{x}\right) dx$

$\Leftrightarrow e^{-3\ln x} y = -\int \frac{1}{x^3} \cdot \frac{1}{x} dx$

$\Leftrightarrow e^{-3\ln x} y = -\int \frac{1}{x^4} dx + C = \frac{1}{3x^3} + C$

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$$\Leftrightarrow \frac{1}{x^3} y = \frac{1}{3x^3} + C$$

$$\Leftrightarrow \quad y = x^3 C + \frac{1}{3}$$

Exercise 4: Solve $\frac{dy}{dx} - y = e^x$

We have $P(x) = -1$, $Q(x) = e^x$

Integrating factor: $e^{\int P(x)dx} = e^{\int -1 dx} = e^{-x}$

$$\Leftrightarrow \quad e^{-x} \frac{dy}{dx} + e^{-x}(-y) = e^x . e^{-x}$$

$$\Leftrightarrow \quad \frac{d}{dx}(e^{-x} y) = 1$$

$$\Leftrightarrow \quad e^{-x} y = \int 1 dx$$

$$\Leftrightarrow \quad e^{-x} y = x + C$$

$$\Leftrightarrow \quad y = x . e^x + e^x . C$$

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Exercise 2 : If f and g are 2 functions having continuous second derivatives on the interval $[a, b]$, and if $f(a) = g(a) = f(b) = g(b) = 0$, show that

$$\int_a^b f(x)g''(x)\,dx = \int_a^b f''(x)g(x)\,dx$$

What other assumptions about the values of f and g at a and b would give the same result ?

We use integration by parts to prove

$$\int_a^b f(x)g''(x)\,dx$$

Let $u = f(x) \Rightarrow du = f'(x)dx$

$dv = g''(x)dx \Rightarrow v = g'(x)$

$$\Rightarrow \int_a^b f(x)g''(x)\,dx = f(x)g'(x)\Big|_a^b - \int_a^b f'(x)g'(x)\,dx$$

$$= -\int_a^b f'(x)g'(x)\,dx \quad (1)$$

(Because $f(a) = f(b) = 0 \Rightarrow f(x)g'(x)\big|_a^b = 0$

We continue to use integration by parts

$$-\int_a^b f'(x)g'(x)\,dx$$

Let $u = f'(x) \Rightarrow du = f''(x)dx$

$dv = g'(x)dx \Rightarrow v = g(x)$

$$\Rightarrow -\int_a^b f'(x)g'(x)\,dx = -f'(x)g(x)\Big|_a^b + \int_a^b f''(x)g(x)\,dx$$

$$= \int_a^b f''(x)g(x)\,dx \quad (2)$$

(Because $g(a) = g(b) = 0 \Rightarrow -f'(x)g(x)\big|_a^b = 0$

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From (1) & (2)

$$\Rightarrow \int_a^b f(x)g''(x)dx = \int_a^b f''(x)g(x)dx \text{ (proven)}$$

Other assumptions: According to (1) & (2), in order for the equation to be correct

$$\Rightarrow f(x)g'(x)\Big|_a^b = f'(x)g(x)\Big|_a^b$$

$$\Leftrightarrow f(b)g'(b) - f(a)g'(a) = f'(b)g(b) - f'(a)g(a)$$

$$\Leftrightarrow f(a)g'(a) - f'(a)g(a) = f(b)g'(b) - f'(b)g(b)$$