Nguyen Xuan Binh 887799 Final Exam Calculus III

Exercise 1:

Let $\gamma : [-1, 2] \to R^3$ be the parametrized curve $\gamma(t) = (3t^2, 4t^3, 3t^4)$

a) Calculate length of γ

We have: $\frac{d\gamma}{dt} = (6t, 12t^2, 12t^3) \Rightarrow \left|\frac{d\gamma}{dt}\right| = \sqrt{36t^2 + 144t^4 + 144t^6}$

$\Rightarrow \left|\frac{d\gamma}{dt}\right| = 6\sqrt{t^2 + 4t^4 + 4t^6} = 6t\sqrt{4t^4 + 4t^2 + 1}$

$= 6t\sqrt{(2t^2+1)^2} = 6t(2t^2+1) = 12t^3 + 6t$

$\Rightarrow$ Length of γ: $L = \int_{-1}^{2} \left|\frac{d\gamma}{dt}\right| dt = \int_{-1}^{2} 12t^3 + 6t \, dt$

$= 3t^4 + 3t^2 \Big|_{-1}^{2} = 54$ (answer)

b) Let F be the vector field $F(x,y,z) = (z, y, x)$. Calculate $\int_\gamma F \cdot d\vec{r}$

Construct potential function for F: $\phi(x,y,z)$

$\frac{\partial \phi}{\partial x} = z \Rightarrow \phi(x,y,z) = xz + g(y,z)$

$\frac{\partial \phi}{\partial y} = y = \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial y}(g(y,z)) \Rightarrow \frac{\partial}{\partial y} g(y,z) = y$

$\Rightarrow g(y,z) = \frac{y^2}{2} + h(z)$

$\frac{\partial \phi}{\partial z} = x = \frac{\partial}{\partial z}\left(xz + \frac{y^2}{2} + h(z)\right) = x + \frac{\partial}{\partial z}(h(z)) \Rightarrow h(z) = 0$

$\Rightarrow$ Potential function is $\phi(x,y,z) = xz + \frac{y^2}{2} + C$

We have: $\gamma(2) = (3.2^2, 4.2^3, 3.2^4) = (12, 32, 48)$

$\gamma(-1) = (3.(-1)^2; 4.(-1)^3, 3.(-1)^4) = (3, -4, 3)$

$\Rightarrow \int_\gamma F \cdot d\vec{r} = \phi(\gamma(2)) - \phi(\gamma(-1)) = (12.48 + 32^2/2) - (3.3 + (-4)^2/2)$

$= 1071$ (answer)

Exercise 2: Calculate the flux away from the origin of the vector field

$F(x,y,z) = (z, y, x)$ through the triangle with points at

$(3,0,0), (0,2,0), (0,0,1)$

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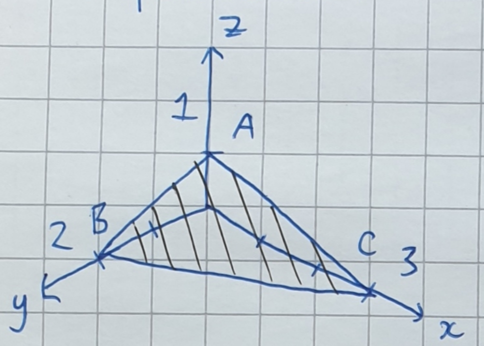
The plane in 3D:



$\vec{AB} = (0, -2, 1)$ $\quad \vec{AC} = (-3, 0, 1)$

$\Rightarrow \vec{AB} \times \vec{AC} = (-2, -3, -6)$

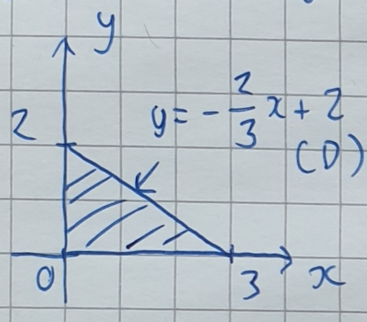
$\Rightarrow$ Normal vector to the plane is $(2, 3, 6)$

The plane's equation is $2x + 3y + 6z - 6 = 0$

$\Rightarrow G = (2x, 3y, 6z) \Rightarrow \nabla G = (2, 3, 6)$

$\Rightarrow \frac{\nabla G}{G_z} = \left(\frac{1}{3}, \frac{1}{2}, 1\right)$ pointing in the correct direction

Projection of the triangle on xy plane:



The integral: $\iint_D (z, y, x) \cdot \left(\frac{1}{3}, \frac{1}{2}, 1\right) dx\,dy$

$= \iint_D \frac{1}{3} z + \frac{1}{2} y + x \, dx\,dy = \iint_D \frac{1}{3} \cdot \frac{1}{6}(6 - 2x - 3y) + \frac{1}{2} y + x \, dx\,dy$

$= \iint_D \frac{8}{9} x + \frac{1}{3} y + \frac{1}{3} \, dx\,dy$

$= \int_0^3 \int_0^{-\frac{2}{3}x + 2} \left[\frac{8}{9} x + \frac{1}{3} y + \frac{1}{3}\right] dy\,dx$

$= \int_0^3 \frac{8}{9} xy + \frac{1}{6} y^2 + \frac{1}{3} y \Big|_0^{-\frac{2}{3}x + 2} dx = \int_0^3 \frac{8}{9} x\left(-\frac{2}{3}x + 2\right) + \frac{1}{6}\left(-\frac{2}{3}x + 2\right)^2 + \frac{1}{3}\left(-\frac{2}{3}x + 2\right) dx$

$= \int_0^3 -\frac{16}{27} x^2 + \frac{16}{9} x + \frac{2}{27} x^2 - \frac{4}{9} x + \frac{2}{3} - \frac{2}{9} x + \frac{2}{3} \, dx$

$= \int_0^3 -\frac{14}{27} x^2 + \frac{10}{9} x + \frac{4}{3} \, dx = -\frac{14}{81} x^3 + \frac{5}{9} x^2 + \frac{4}{3} x \Big|_0^3 = \frac{13}{3}$ (answer)

Exercise 3: Let $F(x, y) = (4x - 2y, 2x - 4y)$ and let γ be the positively oriented boundary curve of the set

$D = \{(x, y) \in \mathbb{R}^2 ; (x-2)^2 + (y-2)^2 \leq 4\}$

Calculate $\oint_\gamma F \cdot d\vec{r}$

We have: $\oint_\gamma F \cdot d\vec{r} = \oint_\gamma (4x - 2y)\,dx + (2x - 4y)\,dy$

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According to Green's theorem:

$$\Rightarrow \oint_\gamma F \cdot d\vec{r} = \iint_D \frac{\partial}{\partial x}(2x - 4y) - \frac{\partial}{\partial y}(4x - 2y)\, dx dy$$

$$= \iint_{(x-2)^2 + (y-2)^2 \leq 4} 2 - (-2)\, dx dy = \iint_D 4\, dx dy$$

Let $u = x - 2 \Rightarrow du = dx$; $v = y - 2 \Rightarrow dv = dy$

$\Rightarrow \oint_\gamma F \cdot d\vec{r} = \iint_{u^2 + v^2 \leq 4} 4\, du dv$. Changing to polar coordinates:

$$= \int_0^{2\pi} \int_0^2 4 r dr d\theta = \int_0^{2\pi} 2r^2 \Big|_0^2 d\theta = \int_0^{2\pi} 8 d\theta = 16\pi \text{ (answer)}$$

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Exercise 4: Let n be normal vector field pointing away from origin of parabolic shell

$S = \{(x, y, z) \in \mathbb{R}^3,\ 4x^2 + y + z^2 = 4,\ y \geq 0\}$

Let $F = \left(-z + \frac{1}{2+x},\ \arctan y,\ x + \frac{1}{4+z}\right)$

Calculate $\iint_S (\text{Curl } F) \cdot n\, dS$

Since S is a closed surface $\Rightarrow$ Applying Stoke's theorem

$\iint_S (\text{Curl } F) \cdot n\, dS = \oint_\phi F \cdot d\vec{r}$ where ϕ is the surface bounding S at $y = 0$

which is $4x^2 + z^2 = 4$

Parameterize: $\begin{cases} x = \cos(t) \\ z = -2\sin(t) \end{cases} \Rightarrow r(t) = (\cos(t), 0, -2\sin t)$

$\Rightarrow d\vec{r} = (-\sin(t), 0, -2\cos t)$

$$\Rightarrow \oint_\phi F \cdot d\vec{r} = \int_\phi \left(2\sin t + \frac{1}{2+\cos t},\ 0,\ \cos t + \frac{1}{4+2\sin t} \right)(-\sin t, 0, -2\cos t)\, dt$$

$$= \int_\phi -2\sin^2 t - \frac{\sin t}{2+\cos t} - 2\cos^2 - 2\frac{\cos t}{2+\sin t}\, dt$$

$$= -\int_0^{2\pi} \frac{\sin t}{\cos t + 2} + 2\frac{\cos t}{\sin t + 2} + 2\, dt$$

$$= \ln(\cos t + 2) - \ln(2\sin t + 4) - 2t \Big|_0^{2\pi}$$

$= -4\pi$ (answer)

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Exercise 5: f and g smooth in R^3. D is regular subset of R^3 with piecewise smooth boundary surface S. Let n denote outward unit normal field to S.

Show that $\oiint_S (f\nabla g - g\nabla f)\cdot n\, dS = \iiint_D (f\Delta g - g\Delta f)\, dV$

According to Stoke's theorem

$$\Rightarrow \oiint_S (f\nabla g - g\nabla f)\cdot n\, dS = \iiint_D \operatorname{div}(f\nabla g - g\nabla f)\, dV$$

$$= \iiint_D \operatorname{div}(f\nabla g) - \operatorname{div}(g\nabla f)\, dV = \iiint_D \nabla f\cdot\nabla g + f(\nabla\cdot\nabla g) - \nabla g\nabla f - g(\nabla\nabla f)\, dV$$

$$= \iiint_D f\nabla^2 g - g\nabla^2 f\, dV = \iiint_D f\Delta g - g\Delta f\, dV$$

(I study for this course in this period. Probably I'm taking course exam. If it is so you can ignore this exercise)