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1) Compute the limits

a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3y}{x^6+y^2}$

In order for a limit to exist, the function must approach the same value regardless of the path along which we approach $(0,0)$

- For $y = x \Rightarrow \lim\limits_{x\to0} \dfrac{x^3 \cdot x}{x^6+x^2} = \lim\limits_{x\to0} \dfrac{x^4}{x^6+x^2} = 0$
- For $y = x^3 \Rightarrow \lim\limits_{x\to0} \dfrac{x^3 \cdot x^3}{x^6+(x^3)^2} = \lim\limits_{x\to0} \dfrac{x^6}{2x^6} = \dfrac{1}{2}$

Since the limit are different for the path $y = x$ and $y = x^3$, this limit does not exist

b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{y^4}{x^2+y^2}$

We have $x^2 \geq 0 \to x^2+y^2 \geq y^2 \Rightarrow \dfrac{1}{x^2+y^2} \leq \dfrac{1}{y^2}$

We have: $0 \leq \left|\dfrac{y^4}{x^2+y^2}\right| \leq \left|\dfrac{y^4}{y^2}\right| = |y^2|$

$\lim\limits_{x\to0} 0 = 0 \qquad \lim\limits_{x\to0} |x^2| = 0$

$\Rightarrow$ By the squeeze theorem $\dfrac{y^4}{x^2+y^2} \to 0$ as $(x,y) \to (0,0)$

2) Consider the function $f(x,y) = x^3 - 3xy^2 - 6y - 1$

a) Compute all the 1[st] and 2[nd] order derivatives

$f(x,y)$

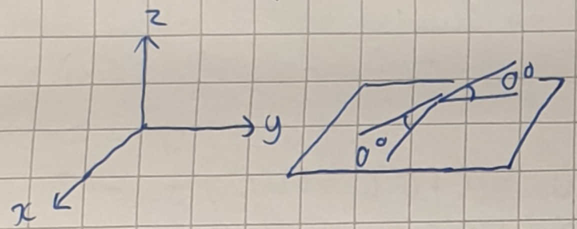
$f_x = \dfrac{\partial f}{\partial x} : 3x^2 - 3y^2$

- $f_{xx} = \dfrac{\partial^2 f}{\partial x^2} : 6x$
- $f_{xy} = \dfrac{\partial^2 f}{\partial y \partial x} : -6y$

$f_y = \dfrac{\partial f}{\partial y} : -6xy - 6$

- $f_{yy} = \dfrac{\partial^2 f}{\partial y^2} : -6x$
- $f_{yx} = \dfrac{\partial^2 f}{\partial x \partial y} : -6y$

b) Find all the points on the surface where the tangent plane is horizontal



Since the plane is horizontal, its partial derivative of both x and y direction will be 0

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$$

- $f_x = 3x^2 - 3y^2 = 0 \Rightarrow |x| = |y|$

- $f_y = -6xy - 6 = 0 \Rightarrow xy = -1$

We have $\begin{cases} |x| = |y| \\ xy = -1 \end{cases} \Rightarrow \begin{cases} (x,y) = (1,-1) \\ (x,y) = (-1,1) \end{cases}$

$f(x,y) = x^3 - 3xy^2 - 6y - 1$

$\Rightarrow f(1,-1) = 3 \qquad f(-1,1) = -5$

$\Rightarrow$ There are 2 points on the surface where tangent plane is horizontal:

$(1,-1,3)$ and $(-1,1,-5)$

c) Find the tangent plane to the surface at the point $(1,2)$

$f(1,2) = -24$

$f_x = 3x^2 - 3y^2 \Rightarrow f_x(1,2) = -9$

$f_y = -6xy - 6 \Rightarrow f_y(1,2) = -18$

The tangent plane is:

$z = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2)$

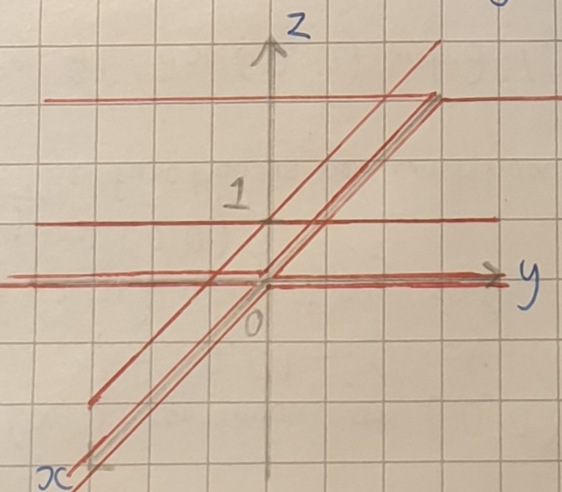
$\Rightarrow z = -24 + (-9)(x-1) + (-18)(x-2)$

$\Rightarrow z = -9x - 18y + 21$ (Answer)

3) Consider the function $f(x,y)$ defined by

$$f(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ or } y = 0 \\ 0, & \text{otherwise} \end{cases}$$

a) Sketch the surface $z = f(x,y)$



$\Rightarrow$ Plane $z = 0$ except x, y axis

$\Rightarrow$ The x, y axis is at $z = 1$

b) Is $f(x, y)$ continuous at $(0,0)$

We have : $f(0,0) = 1 \Rightarrow (0,0,1)$ is defined

$\lim_{(x,y)\to(0,0)} f(x,y)$ along $x = y = 0$. Since $f(0,0) = 1 \neq 0$ along $y = x$

$\Rightarrow$ The function is not continuous at $(0,0)$

c) Compute $\partial f/\partial x\,(0,0)$ and $\partial f/\partial y\,(0,0)$

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{1-1}{h} = 0$$

$$\left.\frac{\partial f}{\partial y}\right|_{(0,0)} = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{1-1}{h} = 0$$

d) Tangent plane equation

$$z = f(0,0) + \partial f/\partial x\,(0,0)(x-0) + \partial f/\partial y\,(0,0)(y-0)$$

$$\Rightarrow z = 1 + 0(x-0) + 0(y-0)$$

$$\Rightarrow z = 1$$

e) Choose a point $(0{,}01, 0{,}01)$ (very close to $(0,0)$)

$$\left.\frac{\partial f}{\partial x}\right|_{(0{,}01,\,0{,}01)} = \lim_{h\to 0} \frac{f(0{,}01+h,\,0{,}01) - f(0{,}01,\,0{,}01)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$\left.\frac{\partial f}{\partial y}\right|_{(0{,}01,\,0{,}01)} = \lim_{h\to 0} \frac{f(0{,}01,\,0{,}01+h) - f(0{,}01,\,0{,}01)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0$$

$\Rightarrow$ Tangent plane at $(0{,}01, 0{,}01)$

$$z = f(0{,}01, 0{,}01) + 0(x - 0{,}01) + 0(y - 0{,}01)$$

$$z = 0$$

$\Rightarrow$ The plane $z = 1$ found in 3d does not approximate well the region no matter how much smaller become because at $(0,0)$, $z = 1$ but for other points except x and y axis, the surface is at $z = 0$. The error of 1 does not improve at all

$$\left.\begin{array}{l} f(0,0) = 1 \\ f(0{,}01,\ 0{,}01) = 0 \\ f(0{,}000001;\ 0{,}000001) = 0 \end{array}\right\} \Rightarrow \text{Plane } z = 1 \text{ does not approximate the region around it}$$

f) The surface $z = f(x,y)$ does not have a tangent plane at $(0,0)$ because the surface is not continuous at $(0,0)$ and the $\lim\limits_{(x,y)\to(0,0)}$ does not exist

4) Linearization

a) Find the linearization $L(x, y)$ for function g defined by

$$g(x, y) = \frac{x}{x^2 + y^2} \text{ at point } (1, 2)$$

Linearization at a point is simply the tangent plane to the surface at the point

$$\left.\frac{\partial g}{\partial x}\right|_{(1,2)} \frac{x}{x^2+y^2} = \frac{(x^2+y^2)\left(\frac{\partial}{\partial x}(x)\right) - x\left(\frac{\partial}{\partial x}(x^2+y^2)\right)}{(x^2+y^2)^2}$$

$$= \frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2} = \frac{3}{25}$$

$$\left.\frac{\partial g}{\partial y}\right|_{(1,2)} \frac{x}{x^2+y^2} = \frac{(x^2+y^2)(\partial/\partial y(x)) - x(\partial/\partial y(x^2+y^2))}{(x^2+y^2)^2}$$

$$= \frac{0 - x(0 + 2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2} = -\frac{4}{25}$$

The tangent plane at $(1, 2)$

$$z = f(1,2) + 3/25\,(x - 1) + (-4/25)(y - 2)$$

$$= \frac{3}{25}x - \frac{4}{25}y + \frac{2}{5}$$

By linearization, the value of $g(0.8, 2.3)$ is

$$\frac{3}{25} \cdot 0{,}8 + \left(-\frac{4}{25}\right) \cdot 2{,}3 + \frac{2}{5} = 0{,}128$$

b) Calculate the data

I choose 3 points in the middle of table $(-10, 20)$ $(-10, 25)$ $(-5\ 20)$

$$\frac{\partial w}{\partial v} \approx \frac{\Delta w}{\Delta v} = \frac{-37-(-35)}{25-20} = -0{,}4$$

$$\frac{\partial w}{\partial T} \approx \frac{\Delta w}{\Delta T} = \frac{-29-(-35)}{-5-(-10)} = 1{,}2$$

- Linearization $L(v, T)$ at point $(25, -10)$

$$w = L(25, -10) - 0{,}4(v - 25) + 1{,}2(T + 10)$$

$$= -37 - 0{,}4v + 10 + 1{,}2T + 12$$

$$= -0{,}4v + 1{,}2T - 15$$

- Estimation for wind chill using $w(v, T)$

$w(25, -12) = -39{,}4$ $\quad w(23, -10) = -36{,}2$ $\quad w(23, -12) = -38{,}6$

c) We have : $f(0,0) = 5$

$$\frac{\partial f}{\partial x} \approx \frac{\Delta f}{\Delta x} = \frac{5-3}{2-0} = 1 \qquad \frac{\partial f}{\partial y} \approx \frac{\Delta f}{\Delta y} = \frac{5-3}{1-0} = 2$$

Linearization $L(x,y)$ at $(2,1)$

$$L(x,y) = f(2,1) + 1(x-2) + 2(y-1)$$
$$= 3 + 1(x-2) + 2(y-1)$$
$$= x + 2y - 1$$

Estimation

$f(2.2, 1) = 3,2 \qquad f(2, 0.8) = 2.6 \qquad f(2.2, 0.8) = 2.8$

5) Polar coordinates r, θ: $x = r\cos\theta$, $y = r\sin\theta$

$\Rightarrow f(x,y) = f(r\cos\theta, r\sin\theta) = F(r,\theta)$

Prove $\left(\frac{\partial F}{\partial r}\right)^2 + \left(\frac{1}{r}\cdot\frac{\partial F}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$

We have according to the chain rule

- $\frac{\partial F}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$
- $\frac{\partial F}{\partial \theta} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \theta} = \frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y}(r\cos\theta)$

$$\Rightarrow \left(\frac{\partial F}{\partial r}\right)^2 + \left(\frac{1}{r}\cdot\frac{\partial F}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta\right)^2 + \left[\frac{1}{r}\left(-r\sin\theta\frac{\partial f}{\partial x} + r\cos\theta\frac{\partial f}{\partial y}\right)\right]^2$$
$$= \left(\frac{\partial f}{\partial x}\cos\theta\right)^2 + \left(\frac{\partial f}{\partial y}\sin\theta\right)^2 + 2\frac{\partial f}{\partial x}\cos\theta\frac{\partial f}{\partial y}\sin\theta$$
$$+ \left(\frac{\partial f}{\partial x}\sin\theta\right)^2 + \left(\frac{\partial f}{\partial y}\cos\theta\right)^2 - 2\frac{\partial f}{\partial x}\sin\theta\frac{\partial f}{\partial y}\cos\theta$$
$$= \left(\frac{\partial f}{\partial x}\right)^2(\cos^2\theta + \sin^2\theta) + \left(\frac{\partial f}{\partial y}\right)^2(\sin^2\theta + \cos^2\theta)$$
$$= \left(\frac{\partial f}{\partial x}\right)^2\cdot 1 + \left(\frac{\partial f}{\partial y}\right)^2\cdot 1 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$$

$$\Rightarrow \left(\frac{\partial F}{\partial r}\right)^2 + \left(\frac{1}{r}\cdot\frac{\partial F}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$$

6) Given $PV = nRT$, prove $\frac{\partial P}{\partial V}\cdot\frac{\partial V}{\partial T}\cdot\frac{\partial T}{\partial P} = -1$

- $P = \frac{nRT}{V} \Rightarrow \partial P/\partial V = \frac{-nRT}{V^2}$
- $V = \frac{nRT}{P} \Rightarrow \partial V/\partial T = \frac{nR}{P}$
- $T = \frac{PV}{nR} \Rightarrow \partial T/\partial P = \frac{V}{nR}$

$$\Rightarrow \frac{\partial P}{\partial V}\cdot\frac{\partial V}{\partial T}\cdot\frac{\partial T}{\partial P} = -\frac{nRT}{V^2}\cdot\frac{nR}{P}\cdot\frac{V}{nR} = -\underbrace{\frac{nRT}{PV}}_{} = -1$$

Since $PV = nRT$

$\Rightarrow \frac{nRT}{PV} = 1$