Nguyen Xuan Binh 887799

Exercise 1: $f(x,y) = 4xy$. Constraint: $g(x,y)$: $g(x,y) = 4x^2+y^2 \leq 8$

We have: $\vec{\nabla} f = \langle 4y, 4x \rangle$, $\vec{\nabla} g = \langle 8x, 2y \rangle$

$\vec{\nabla} f = (0,0) \Rightarrow (x,y) = (0,0)$

When we proceed with Lagrange multiplier, we treat the constraint as an equality because we already know one critical point inside the constraint $(0,0)$

$$\Rightarrow \begin{cases} \vec{\nabla} f = \lambda \vec{\nabla} g \\ g = 8 \end{cases} \Rightarrow \begin{cases} 4y = \lambda 8x \quad (1) \\ 4x = \lambda 2y \quad (2) \\ 4x^2+y^2 = 8 \quad (3) \end{cases} \Rightarrow \begin{cases} y = 2\lambda x \quad (1) \\ 2x = \lambda y \quad (2) \\ 4x^2+y^2 = 8 \quad (3) \end{cases}$$

$(2)(1) \Rightarrow 2x = \lambda(2\lambda x) \Rightarrow x = \lambda^2 x \Rightarrow x(1-\lambda^2) = 0$

$$\Rightarrow \left[\begin{array}{l} x = 0 \\ \lambda = 1 \\ \lambda = -1 \end{array} \right.$$

- If $x = 0$, $(3) \Rightarrow y = \pm 2\sqrt{2} \Rightarrow (x,y) = (0, 2\sqrt{2}), (0, -2\sqrt{2})$
- $\lambda = 1$

$$\Rightarrow \begin{cases} y = 2x \\ 2x = y \\ 4x^2+y^2 = 8 \end{cases} \Rightarrow \begin{cases} y = 2x \\ y = 2x \\ 4x^2+4x^2 = 8 \end{cases} \Rightarrow (x,y) = (1,2), (-1,-2)$$

- $\lambda = -1$

$$\Rightarrow \begin{cases} y = -2x \\ 2x = -y \\ 4x^2+y^2 = 8 \end{cases} \Rightarrow \begin{cases} y = -2x \\ y = -2x \\ 4x^2+4x^2 = 8 \end{cases} \Rightarrow (x,y) = (1,-2), (-1,2)$$

The possible extrema values are

(x,y)	$(0,0)$	$(0,2\sqrt{2})$	$(0,-2\sqrt{2})$	$(1,2)$	$(-1,-2)$	$(1,-2)$	$(-1,2)$
$f(x,y)$	0	0	0	8	8	-8	-8

$\Rightarrow$ $f(x,y)$ reaches max at 8 at $(1,2)$ and $(-1,-2)$
$f(x,y)$ reaches min at -8 at $(1,-2)$ and $(-1,2)$

Exercise 3: Let $f(x,y) = \ln(x^2+y^3)$

$\ln(x^2+y^3)$:

$f_x = \dfrac{2x}{x^2+y^3}$

- $f_{xx} = -\dfrac{2(x^2-y^3)}{(x^2+y^3)^2}$
- $f_{xy} = f_{yx} = -\dfrac{6xy^2}{(x^2+y^3)^2}$

$f_y = \dfrac{3y^2}{x^2+y^3}$

- $f_{xy} = f_{yx} = -\dfrac{6xy^2}{(x^2+y^3)^2}$
- $f_{yy} = -\dfrac{3(y^4-2x^2y)}{(x^2+y^3)^2}$

Second order Taylor polynomial of $f(x, y)$ centered at $(1, 0)$

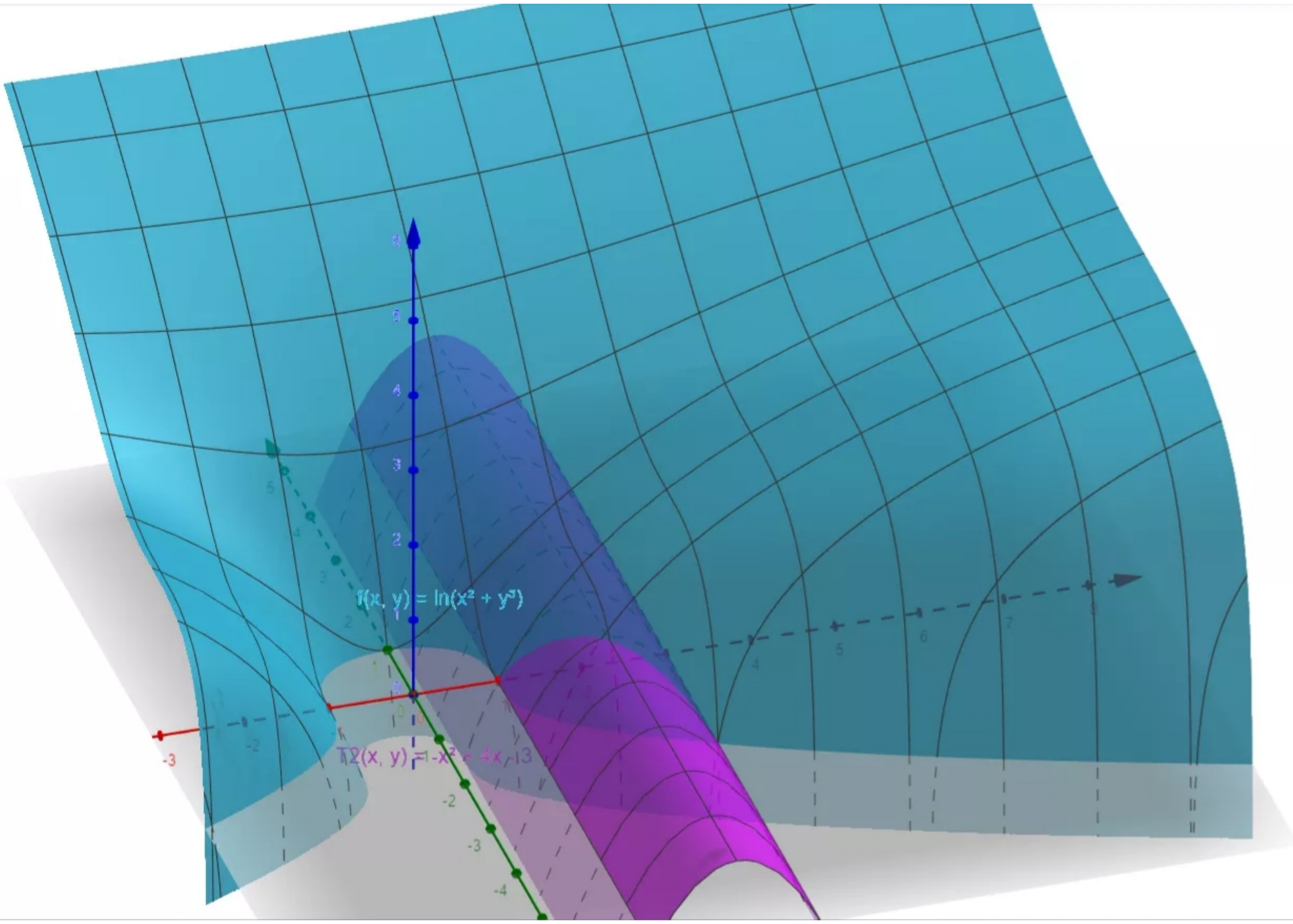
$$T_2(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$
$$+ 1/2\, f_{xx}(a,b)(x-a)^2 + 1/2\, f_{yy}(a,b)(y-b)^2 + f_{xy}(a,b)(x-a)(y-b)$$

$$\Rightarrow T_2(1,0) = f(1,0) + f_x(1,0)(x-1) + f_y(1,0)\,y$$
$$+ 1/2\, f_{xx}(1,0)(x-1)^2 + 1/2\, f_{yy}(1,0)\,y^2 + f_{xy}(1,0)(x-1)y$$

$$\Rightarrow T_2(1,0) = 0 + 2(x-1) + 0y - 1(x-1)^2 + 0y^2 + 0(x-1)y$$
$$\Rightarrow T_2(1,0) = 2(x-1) - (x-1)^2$$
$$= 2x - 2 - (x^2 - 2x + 1)$$
$$= -x^2 + 4x - 3$$

The graph below : $\ln(x^2 + y^3)$ is blue
$-x^2 + 4x - 3$ is purple

Since $-x^2 + 4x - 3$ pass through the same point $(1, 0)$, the quadratic Taylor Polynomial does approximate well near $(1, 0)$


f(x, y) = ln(x² + y²)
T2(x, y) = 1 - x² + 4x - 3

Exercise 2:

- AE 17: Find Jacobian matrix for the transformation $f(r,\theta) = (x,y)$, where

$x = r\cos\theta \qquad y = r\sin\theta$

$$\Rightarrow J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial r}(r\cos\theta) & \frac{\partial}{\partial \theta}(r\cos\theta) \\ \frac{\partial}{\partial r}(r\sin\theta) & \frac{\partial}{\partial \theta}(r\sin\theta) \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$\Rightarrow \det(J) = r\cos^2\theta + r\sin^2\theta = r(\sin^2\theta + \cos^2\theta) = r$$

- AE 18: Find Jacobian matrix for the transformation $f(R,\phi,\theta) = (x,y,z)$

$x = R\sin\phi\cos\theta$

$y = R\sin\phi\sin\theta$

$z = R\cos\phi$

$$J = \begin{bmatrix} \frac{\partial x}{\partial R} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial R} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial R} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial R}(R\sin\phi\cos\theta) & \frac{\partial}{\partial \phi}(R\sin\phi\cos\theta) & \frac{\partial}{\partial \theta}(R\sin\phi\cos\theta) \\ \frac{\partial}{\partial R}(R\sin\phi\sin\theta) & \frac{\partial}{\partial \phi}(R\sin\phi\sin\theta) & \frac{\partial}{\partial \theta}(R\sin\phi\sin\theta) \\ \frac{\partial}{\partial R}(R\cos\phi) & \frac{\partial}{\partial \phi}(R\cos\phi) & \frac{\partial}{\partial \theta}(R\cos\phi) \end{bmatrix}$$

$$\Rightarrow J = \begin{bmatrix} \sin\phi\cos\theta & R\cos\phi\cos\theta & -R\sin\phi\sin\theta \\ \sin\phi\sin\theta & R\cos\phi\sin\theta & R\sin\phi\cos\theta \\ \cos\phi & -R\sin\phi & 0 \end{bmatrix}$$

$$\Rightarrow \det(J) = \sin\phi\cos\theta(R^2\sin^2\phi\cos\theta) - R\cos\phi\cos\theta(-R\sin\phi\cos\phi\cos\theta)$$
$$- R\sin\phi\sin\theta(-R\sin^2\phi\sin\theta - R\cos^2\phi\sin\theta)$$
$$\Rightarrow \det(J) = R^2\sin\phi\cos^2\theta(\sin^2\phi + \cos^2\phi) + R^2\sin\phi\sin^2\theta(\sin^2\phi + \cos^2\phi)$$
$$\Rightarrow \det(J) = R^2\sin\phi\cos^2\theta + R^2\sin\phi\sin^2\theta$$
$$= R^2\sin\phi(\cos^2\theta + \sin^2\theta)$$
$$\Rightarrow \det(J) = R^2\sin\phi$$

Exercise 4: AE2: Find solutions of the system so that the left hand-sides of the equation vanish up to 6 decimal places, using Newton's Method

$$\begin{cases} x^2 + y^2 - 1 = 0 \\ y - e^x = 0 \end{cases}$$ There are 2 solutions

$x^2 + y^2 - 1$: $f_x = 2x$, $f_y = 2y$ $\Rightarrow J = \begin{bmatrix} 2x & 2y \\ -e^x & 1 \end{bmatrix} \Rightarrow J^{-1} =$

$y - e^x$: $g_x = -e^x$, $g_y = 1$

$$\Rightarrow J^{-1} = \begin{bmatrix} \frac{1}{2x + 2ye^x} & \frac{-2y}{2x + 2ye^x} \\ \frac{e^x}{2x + 2ye^x} & \frac{2x}{2x + 2ye^x} \end{bmatrix}$$

The formula: $x_{i+1} = x_i - J_F^{-1}(x_i)F(x_i)$

My two first initial guesses based on the graph below is (0, 1) and (-0,9, 0,4)

Untitled.m

```
clc;
format long
x = [0; 1];
f = x(1)^2 + x(2)^2 - 1;
g = x(2) - exp(x(1));
for i = 1:6
    x = x - ([2*x(1), 2*x(2); -exp(x(1)), 1])\[f; g];
    f = x(1)^2 + x(2)^2 - 1;
    g = x(2) - exp(x(1));
end
disp("The first solution is ")
disp("x = " + x(1))
disp("y = " + x(2))
disp("The values of f(x,y) = x^2 + y^2 -1 is " + f + " and g(x,y) = y - e^x is " + g)
disp(" ")

x = [-0.9; 0.4];
f = x(1)^2 + x(2)^2 - 1;
g = x(2) - exp(x(1));
for i = 1:6
    x = x - ([2*x(1), 2*x(2); -exp(x(1)), 1])\[f; g];
    f = x(1)^2 + x(2)^2 - 1;
    g = x(2) - exp(x(1));
end
disp("The second solution is ")
disp("x = " + x(1))
disp("y = " + x(2))
disp("The values of f(x,y) = x^2 + y^2 -1 is " + f + " and g(x,y) = y - e^x is " + g)
```

Command Window

```
The first solution is
x = 0
y = 1
The values of f(x,y) = x^2 + y^2 -1 is 0 and g(x,y) = y - e^x is 0

The second solution is
x = -0.91656
y = 0.39989
The values of f(x,y) = x^2 + y^2 -1 is 0 and g(x,y) = y - e^x is 0
>>
```

Exercise 5 : Bonus

Let $f(a_1, a_2, a_3, \dots a_n) = a_1 a_2 a_3 a_4 \dots a_n$

$g(a_1, a_2, a_3, \dots a_n) = a_1 + a_2 + a_3 + \dots + a_n$

$\vec{\nabla} f = \langle a_2 a_3 a_4 \dots a_n , a_1 a_3 a_4 \dots a_n , \dots a_1 a_2 a_3 \dots a_{n-1} \rangle$

$\vec{\nabla} g = \langle \underbrace{1, 1, 1, \dots 1}_{n \text{ times}} \rangle$

We have : $\vec{\nabla} f = \lambda \vec{\nabla} g \Rightarrow \begin{cases} a_2 a_3 a_4 \dots a_n = \lambda \\ a_1 a_3 a_4 \dots a_n = \lambda \\ a_1 a_2 a_3 \dots a_{n-1} = \lambda \end{cases} \Rightarrow a_1 = a_2 = a_3 = \dots a_n$

$\Rightarrow$ Maximum ~~of~~ $f(a_1, a_2, a_3 \dots a_n)$ occurs when $a_1 = a_2 = a_3 = \dots = a_n$

$\Rightarrow g(a_1, \dots a_n)_{max} = n a_1$

$\Rightarrow f(a_1, \dots a_n)_{max} = a_1^n$

$a_1 = \dfrac{a_1 + a_2 + \dots a_n}{n} \Rightarrow f(a_1, \dots a_n)_{max} = \left(\dfrac{a_1 + a_2 + \dots a_n}{n} \right)^n$

$f(a_1, \dots a_n)_{max}$ when $a_1 = a_2 = a_3 = \dots = a_n$

$\Rightarrow a_1 a_2 a_3 \dots a_n \leq \left(\dfrac{a_1 + a_2 + \dots + a_n}{n} \right)^n$

$\Rightarrow \sqrt[n]{a_1 a_2 a_3 \dots a_n} \leq \dfrac{a_1 + a_2 + \dots + a_n}{n}$ (proven)