HW 4: Due Tue Feb 5th 2020

1. Even though it is nearly winter you have just fired up the outdoor grill to cook your sausages. Say the grill plate is a square 60cm×60cm. The temperature of the plate is $T(x, y) = 200e^{(-x^2-3y^2)/10}$ where x and y are measured in cm from the center of the plate. An ant accidentally falls and lands on the plate at the point $(1, 1)$. The ant now has very hot feet and needs to walk in the direction that it can cool down fastest. You need to figure out this direction.

   (a) Where is the plate the hottest and what is the temperature at this point.

   (b) What is the approximate temperature at the edge of the plate. Answer without doing any fancy calculations.

   (c) Sketch some level curves of T, including the level curve passing through the point $(1, 1)$.

   (d) From the sketch, in which direction approximately should the ant walk. Give you answer in terms of a vector.

   (e) Using the gradient vector to find the exact direction the ant should walk. Compare you answer to part (d).

   (f) If the ant starts to walk in the direction $\langle 2, 1\rangle$, then find the rate of change of temperature that the ant experiences.

2. Suppose that at the point $(1, 2)$, the directional derivative of the function $z = f(x, y)$ in the direction $2\mathbf{i} + 3\mathbf{j}$ is equal to 5 and the directional derivative of $z = f(x, y)$ in the direction $-4\mathbf{i} + \mathbf{j}$ is equal to 4.

   (a) Determine the gradient of f at the point $(1, 2)$.

   (b) What is the maximum rate of change of the function f at the point $(1, 2)$?

3. A flat circular plate has the shape of the region $x^2+y^2 \leq 1$. The plate (including the boundary $x^2+y^2 = 1$) is heated so that the temperature T at any point (x, y) is given by $T = x^4-4x^2+2y^2$. Locate the hottest and coldest points of the plate and determine the temperature at each of these points.

4. A rectangle with length L and width W is cut into four smaller rectangles by two lines parallel to the sides.

   (a) Find the minimum value of the sum of the squares of the areas of the smaller rectangles.

   (b) Show that the maximum of the sum of the squares of the areas occurs when the cutting lines correspond to sides of the rectangle (so that there is only one rectangle).

5. Determine the rectangular solid R with maximal volume subject to the following constraints:

   - All the 12 edges are combined 60 cm long.
   - Its surface area is 144 cm^2.

   Hint: Show first that exactly two of the sides are equal.

6. Let $a_1, a_2, \ldots, a_n$ be positive numbers

   (a) Find the maximum value of the expression $a_1x_1 + a_2x_2 + \ldots + a_nx_n$, if the variables $x_1, x_2, \ldots, x_n$ are restricted so that the sum of their squares is one.

   (b) What is the minimum value of $a_1x_1 + a_2x_2 + \ldots + a_nx_n$ in this case?

   (c) Give an interpretation of your results using the dot product in the case $n = 3$.

7. **BONUS QUESTION** - This exercise will help you understand the second derivative test, and in particular where the condition $f_{xx}f_{yy} - f_{xy}^2 > 0$.

   (a) Consider the function
   $$g(h) = f(a + hu_1, b + hu_2)$$
   where f is a function whose partial derivatives of all orders are continuous. Recall from calc 2 that $g(h)$ has the Taylor series,
   $$g(h) = g(0) + g'(0)h + \frac{g''(0)}{2!}h^2 + \text{higher order terms.}$$
   Use the chain rule to calculate $g'(0)$ and $g''(0)$.

   (b) A critical point is a point where the vector $\underline{\nabla f} = \underline{0}$. Show that at a critical point (a, b), for small values of h,
   $$f(a + hu_1, b + hu_2) - f(a, b) \simeq u_1^2 f_{xx}(a, b) + 2u_1u_2 f_{xy}(a, b) + u_2^2 f_{yy}(a, b)$$

   (c) Now let
   $$\underline{u} = \frac{< k, 1 >}{\sqrt{1 + k^2}}$$
   Substitute this vector into the RHS of the expression you have in question 2. Observe that this expression is quadratic in k.

   (d) If we impose the condition
   $$f_{xx}f_{yy} - (f_{xy})^2 > 0$$
   what does this say about the quadratic term in the previous question? What if
   $$f_{xx}f_{yy} - (f_{xy})^2 < 0 \ ?$$
   What are the implications of these conditions for the local extremum at the point (a, b)?

   (e) Finally what does it tell us about the local extremum at (a, b) if
   $$f_{xx}f_{yy} - (f_{xy})^2 = 0 \ ?$$

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