Demo

~~Board~~ Exercises 6

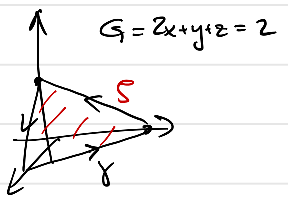
① Let γ be the boundary curve of the portion of the plane $2x+y+z=2$ in the first octant. Let γ be oriented so that it's projection on the xy-plane is oriented counter-clockwise. Calculate

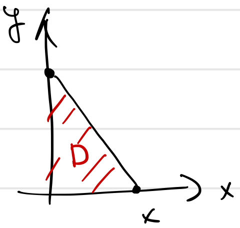
$$\oint_\gamma xz\,dx + xy\,dy + 3xz\,dz$$

<u>Solution:</u> We have $\oint_\gamma \vec{F}\cdot d\vec{r} = \iint_S \operatorname{Curl}\vec{F}\cdot\vec{N}\,dS$

First calculate $\operatorname{Curl}\vec{F}$.

$$\operatorname{Curl}\vec{F} = \begin{vmatrix} \vec{e_1} & \vec{e_2} & \vec{e_3} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz & xy & 3xz \end{vmatrix} = 0\vec{e_1} - (3z-x)\vec{e_2} + y\vec{e_3}$$





$$\iint_S \operatorname{Curl}F\cdot\vec{N}\,dS = \iint_D (0, x-3z, y)\cdot\nabla G\,dx\,dy$$

$$= \iint_D x-3z+y\,dx\,dy = \iint_D x-3(2-2x-y)+y\,dx\,dy$$

$\nabla G = (2,1,1)$

$$= \int_0^1\int_0^{2-2x} 7x+3y-6\,dy\,dx =$$

$$= \int_0^1 \left[7xy - \frac{3y^2}{2} - 6y\right]_0^{2-2x} dx =$$

$$= \int_0^1 7x(2-2x) - \frac{3}{2}(2-2x)^2 - 12 + 12x\,dx = \int_0^1 -18+38x-26x^2\,dx =$$

$$= -18 + \frac{38}{2} - \frac{26}{3} = -\frac{23}{3}$$

(2) Let $F(x,y,z) = \left(-y + x\sqrt{x^2+y^2},\ x + y\sqrt{x^2+y^2},\ z\right)$

Write the vector field in cylindrical coordinates that is find F_R, F_θ and F_z in

$$F = F_R \hat{R} + F_\theta \hat{\theta} + F_z \hat{z}.$$

Solution: We know that $[\hat{R}, \hat{\theta}, \hat{z}]$ is an orthogonal local basis where

$$\hat{R} = (\cos\theta, \sin\theta, 0)$$
$$\hat{\theta} = (-\sin\theta, \cos\theta, 0)$$
$$\hat{z} = (0,0,1)$$

Since $\begin{cases} x = R\cos\theta \\ y = R\sin\theta \\ z = z \end{cases}$ we get

$$F_R = F \cdot \hat{R} = (-R\sin\theta + R^2\cos\theta,\ R\cos\theta + R^2\sin\theta,\ z) \cdot \hat{R}$$
$$= -R\sin\theta\cos\theta + R^2\cos^2\theta + R\cos\theta\sin\theta + R^2\sin^2\theta$$
$$= R^2$$

$$F_\theta = F \cdot \hat{\theta} = (-R\sin\theta + R^2\cos\theta,\ R\cos\theta + R^2\sin\theta,\ z) \cdot \hat{\theta}$$
$$= R\sin^2\theta - R^2\cos\theta\sin\theta + R\cos^2\theta + R^2\sin\theta\cos\theta =$$
$$= R$$

$$F_z = F \cdot \hat{z} = (-R\sin\theta + R^2\cos\theta,\ R\cos\theta + R^2\sin\theta,\ z) \cdot \hat{z} = z$$

$$\Rightarrow F = R^2\hat{R} + R\hat{\theta} + z\hat{z}$$

③ Define curvilinear coordinates in xy-space via

$$\vec{r}(u,v) = (x(u,v), y(u,v)) = (u^2 - v^2, 2uv)$$

Show that this curvilinear coordinate system is orthogonal when $(x,y) \neq (0,0)$. Sketch the coordinate curves $u = u_0$ and $v = v_0$.

Solution: $\dfrac{\partial \vec{r}}{\partial u} = (2u, 2v)$ and $\dfrac{\partial \vec{r}}{\partial v} = (-2v, 2u)$

Then $h_u = \left|\dfrac{\partial \vec{r}}{\partial u}\right|$ and $h_v = \left|\dfrac{\partial \vec{r}}{\partial v}\right|$.

Also $\hat{u} = \dfrac{1}{h_u}\dfrac{\partial \vec{r}}{\partial u}$ and $\hat{v} = \dfrac{1}{h_v}\dfrac{\partial \vec{r}}{\partial v}$

Obviously $|\hat{u}| = |\hat{v}| = 1$. Also

$$\hat{u}\cdot\hat{v} = \frac{1}{h_u h_v}\frac{\partial \vec{r}}{\partial u}\cdot\frac{\partial \vec{r}}{\partial v} = \frac{1}{h_u h_v}\left(2u(-2v) + 2v\,2u\right)$$
$$= 0$$

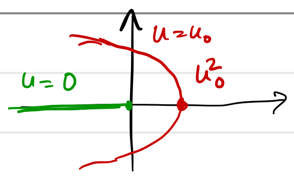
We sketch the coordinate curves.

Coordinate curve $u = u_0$

$$x = u_0^2 - v^2 \quad \text{and} \quad y = 2u_0 v$$

gives $x = u_0^2 - \dfrac{y^2}{4u_0^2}$ if $u_0 \neq 0$

$x = -v^2$ and $y = 0$ if $u_0 = 0$



Now if $v = v_0$ then

$$x = u^2 - v_0^2 \quad \text{and} \quad y = 2uv_0$$

We get

$$x = \frac{y^2}{4v_0^2} - v_0^2 \qquad \text{if } v_0 \neq 0$$

$$x = u^2 \quad y = 0 \quad \text{if } v_0 = 0$$