Homework 4

[redacted]

(1) Let $\varphi : \mathbb{R}^3 \to \mathbb{R}$. Show that $\mathrm{Curl}(\nabla\varphi) = 0$.

<u>Solution</u>: We have $\nabla \varphi = \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z} \right)$

and

$$\mathrm{Curl}(\nabla\varphi) = \begin{vmatrix} \vec{e_1} & \vec{e_2} & \vec{e_3} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \end{vmatrix} =$$

$$= \left(\frac{\partial^2 \varphi}{\partial y \partial z} - \frac{\partial^2 \varphi}{\partial z \partial \varphi} \right) \vec{e_1} + \left(\frac{\partial^2 \varphi}{\partial z \partial x} - \frac{\partial^2 \varphi}{\partial x \partial z} \right) \vec{e_2}$$

$$+ \left(\frac{\partial^2 \varphi}{\partial x \partial y} - \frac{\partial^2 \varphi}{\partial y \partial x} \right) \vec{e_3} = \vec{0}$$

$\otimes$

(2) Assume that f and g are harmonic functions in $\mathbb{R}^n$. Show that

$$\mathrm{div}(f \nabla g - g \nabla f) = 0$$

<u>Solution</u>: We have $\mathrm{div}(\phi \vec{F}) = \nabla \phi \cdot \vec{F} + \phi (\nabla \cdot \vec{F})$

and therefore

$$\mathrm{div}(f\nabla g - g \nabla f) = \mathrm{div}(f \nabla g) - \mathrm{div}(g \nabla f) =$$

$$= \nabla f \cdot \nabla g + f(\nabla \cdot \nabla g) - \nabla g \cdot \nabla f - g(\nabla \cdot \nabla f) =$$

$$= f \Delta g - g \Delta f = 0$$

since $\Delta g = \Delta f = 0$ $\otimes$

(3) Assume that $f: \mathbb{R} \to \mathbb{R}$ is a differentiable function and $\vec{r} = (x,y,z)$. Let $r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$. Show that

$$\operatorname{div}(f(r)\vec{r}) = r f'(r) + 3f(r).$$

Solution:

First $\quad r = \sqrt{x^2+y^2+z^2} \quad$ and

$$\frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2+z^2}} = \frac{x}{r},$$

$$\frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}.$$

Therefore $\operatorname{div}(f(r)\vec{r}) = \operatorname{div}(f(r)(x,y,z)) =$

$$= \frac{\partial}{\partial x}(x f(r)) + \frac{\partial}{\partial y}(y f(r)) + \frac{\partial}{\partial z}(z f(r)) =$$

$$= f(r) + x \frac{\partial r}{\partial x} \cdot f'(r) + f(r) + y \cdot \frac{\partial r}{\partial y} \cdot f'(r)$$

$$+ f(r) + z \cdot \frac{\partial r}{\partial z} \cdot f'(r) =$$

$$= 3f(r) + \frac{x^2+y^2+z^2}{r} f'(r) =$$

$$= \frac{r^2}{r} f'(r) + 3f(r) = r f'(r) + 3f(r)$$

$\otimes$

Demo Exercises 4

(1) Find the flux of

$$F(x,y,z) = \left(\frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2}, 1 \right)$$

downward through the surface S defined parametrically by

$$\vec{r}(u,v) = (u \cos v, u \sin v, u^2)$$

$$(0 \le u \le 1, \quad 0 \le v \le 2\pi)$$

<u>Solution</u>: We use

+ or −

$$\iint_S F \cdot \vec{N}\, dS = \int_0^{2\pi} \left(\int_0^1 F(\vec{r}(u,v)) \cdot \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right) du \right) dv$$

$$\frac{\partial \vec{r}}{\partial u} = (\cos v, \sin v, 2u) \qquad \frac{\partial \vec{r}}{\partial v} = (-u \sin v, u \cos v, 0)$$

$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = (-2u^2 \cos v, -2u^2 \sin v, u)$$

Since $u \ge 0$ we take $-\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}$ to get the correct direction for $\vec{N}$.

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$$\iint_S F \cdot \vec{N}\, dS = \int_0^{2\pi}\int_0^1 \left(\frac{2u \cos v}{u^2}, \frac{2u \sin v}{u^2}, 1 \right) \cdot (2u^2 \cos v, 2u^2 \sin v, -u)$$

$$= \int_0^{2\pi} \left(\int_0^1 4u \cos^2 v + 4u \sin^2 v - u \, du \right) dv = 2\pi \int_0^1 3u \, du$$

$$= 2\pi \cdot \frac{3}{2} = 3\pi$$

(2) Calculate the flux of $F(x,y,z) = (4x, 4y, 2)$ downwards through the part of $z = x^2+y^2$ where $0 \le z \le 1$.

<u>Solution</u>: We use $\iint_S F \cdot \vec{N}\, dS = \iint_{x^2+y^2\le 1} F \cdot \frac{\nabla G}{(\partial G/\partial z)}\, dx dy$ (+ or −)

where $G(x,y,z) = x^2+y^2-z \; (=0)$.

$\nabla G = (2x, 2y, -1) \qquad \frac{\partial G}{\partial z} = -1$

$\frac{\nabla G}{\partial G/\partial z} = (-2x, -2y, 1)$

Since we are interested in the flux downwards we use $-\frac{\nabla G}{\partial G/\partial z} = (2x, 2y, -1)$

$$\iint_S F \cdot \vec{N}\, dS = \iint_{x^2+y^2\le 1} (4x, 4y, 2) \cdot (2x, 2y, -1)\, dx dy =$$

$$= \iint_{x^2+y^2\le 1} 8x^2+8y^2-2 \, dx dy \overset{\text{polar coord.}}{=}$$

$$= \int_0^{2\pi}\int_0^1 (8r^2-2) r \, dr d\theta = 2\pi \int_0^1 8r^3 - 2r \, dr =$$

$$= 2\pi \left[\frac{8r^4}{4} - r^2 \right]_0^1 = 2\pi(2-1) = 2\pi$$

(3) Let $a>0$. Calculate the flux of the vector field $F(x,y,z) = (y,-x,1)$ across the portion of the sphere $x^2+y^2+z^2=a^2$ in the first octant in the direction away from the origin.

__Solution__: We use $\displaystyle\iint_S F\cdot\vec{N}\,dS = \iint_{\substack{x^2+y^2\le a^2\\ x\ge 0,\, y\ge 0}} F\cdot \pm\frac{\nabla G}{\partial G/\partial z}\,dx\,dy$

Where $G(x,y,z) = x^2+y^2+z^2-a^2 \; (=0)$

$\nabla G = (2x, 2y, 2z) \qquad \dfrac{\partial G}{\partial z} = 2z$

$\dfrac{\nabla G}{\partial G/\partial z} = \left(\dfrac{x}{z}, \dfrac{y}{z}, 1\right)$ points in correct direction

$$\iint_S F\cdot\vec{N}\,dS = \iint_{\substack{x^2+y^2\le a^2\\ x\ge 0,\, y\ge 0}} (y,-x,-1)\cdot\left(\frac{x}{z},\frac{y}{z},1\right)dx\,dy =$$

$$= \iint_{\substack{x^2+y^2\le a^2\\ x\ge 0,\, y\ge 0}} -1\,dx\,dy = -\text{ area of } \{x^2+y^2\le a^2,\ x\ge 0, y\ge 0\}$$

$$= -\frac{\pi a^2}{4}$$

Hand-in Exercises 4

(1) Prove that

$$\mathrm{Curl}(\mathrm{Curl}\, F) = \mathrm{grad}(\mathrm{div}\, F) - (\Delta F_1, \Delta F_2, \Delta F_3)$$

for any smooth vector field $F = (F_1, F_2, F_3)$.
Here
$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

Solution: First $\mathrm{div}\, F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

and
$$\mathrm{grad}(\mathrm{div} F) = \left(\frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right), \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right), \frac{\partial}{\partial z}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) \right)$$

Therefore $\mathrm{grad}(\mathrm{div} F) - (\Delta F_1, \Delta F_2, \Delta F_3) =$
$$\left(\frac{\partial^2 F_2}{\partial x \partial y} + \frac{\partial^2 F_3}{\partial x \partial z} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_1}{\partial z^2}, \frac{\partial^2 F_1}{\partial y \partial x} + \frac{\partial^2 F_3}{\partial y \partial z} - \frac{\partial^2 F_2}{\partial x^2} - \frac{\partial^2 F_2}{\partial z^2}, \frac{\partial^2 F_1}{\partial z \partial x} + \frac{\partial^2 F_2}{\partial z \partial y} - \frac{\partial^2 F_3}{\partial x^2} - \frac{\partial^2 F_3}{\partial y^2} \right)$$

Also
$$\mathrm{Curl}\, F = \begin{vmatrix} \vec{e_1} & \vec{e_2} & \vec{e_3} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix} =$$
$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

We get $\operatorname{Curl}(\operatorname{Curl} F) =$

$$= \operatorname{Curl}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) =$$

$$= \left(\frac{\partial}{\partial y}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right), \right.$$

$$\frac{\partial}{\partial z}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right),$$

$$\left. \frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) \right) =$$

$$= \left(\frac{\partial^2 F_2}{\partial y \partial x} + \frac{\partial^2 F_3}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_1}{\partial z^2}, \frac{\partial^2 F_3}{\partial z \partial y} + \frac{\partial^2 F_1}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x^2} - \frac{\partial^2 F_2}{\partial z^2}, \frac{\partial^2 F_1}{\partial x \partial z} + \frac{\partial^2 F_2}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial x^2} - \frac{\partial^2 F_3}{\partial y^2} \right)$$

$$\Rightarrow \operatorname{Curl}(\operatorname{Curl} F) = \operatorname{grad}(\operatorname{div} F) - (\Delta F_1, \Delta F_2, \Delta F_3)$$

$\otimes$

(2) Prove that there is no vector field F such that

$$\operatorname{Curl} F = (x, y, z)$$

Proof:

We know that $\operatorname{div}(\operatorname{Curl} F) = 0$ for any vector field F. Since $\operatorname{div}(x, y, z) = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 \neq 0$ no vector field F satisfies

$$\operatorname{Curl} F = (x, y, z)$$

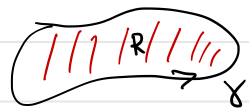
$\otimes$

(3) Calculate $\oint x^2\,dy$ where γ is the curve $(x-1)^2+y^2=1$ oriented counterclockwise.

Solution: Green's Theorem says

$$\oint_\gamma F_1\,dx + F_2\,dy = \iint_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\,dxdy$$

where



Therefore $\oint_\gamma x^2 dy = \iint_R 2x\,dxdy$ where

$R = \{(x,y)\in\mathbb{R}^2;\ (x-1)^2+y^2\le 1\}$

Put $u=x-1$ and $v=y$.

$$\oint_\gamma x^2\,dy = \iint_R 2x\,dxdy = 2\iint_{u^2+v^2\le 1} u+1\,dudv =$$

$$= 2\int_0^{2\pi}\int_0^1 (r\cos\theta+1)\,r\,drd\theta = 4\pi\int_0^1 r\,dr = 2\pi$$

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(4) The curve parametrised as $\gamma(t)=(\cos^3 t, \sin^3 t)$, $0\le t\le 2\pi$ is called an astroid. Calculate the area enclosed by the astroid.

Solution: We know that the area is

$$\oint_\gamma x\,dy = \oint_\gamma -y\,dx = \frac{1}{2}\oint_\gamma -y\,dx + x\,dy$$

Therefore the area is $\frac{1}{2}\int_\gamma -y\,dx + x\,dy =$

$$= \left\lceil \begin{array}{ll} x = \cos^3 t & dx = -3\sin t \cos^2 t\,dt \\ y = \sin^3 t & dy = 3\cos t \sin^2 t\,dt \end{array} \right\rfloor =$$

$$= \frac{1}{2}\int_0^{2\pi} 3\sin^4 t\cos^2 t + 3\cos^4 t\sin^2 t\,dt$$

$$= \frac{3}{2}\int_0^{2\pi} (\sin t\cos t)^2\,dt = \frac{3}{2}\int_0^{2\pi}\left(\frac{\sin 2t}{2}\right)^2 dt$$

$$= \frac{3}{8}\int_0^{2\pi} \sin^2 2t\,dt = \frac{3}{8}\int_0^{2\pi} \frac{1-\cos 4t}{2}\,dt$$

$$= \frac{3}{8}\cdot\frac{2\pi}{2} = \frac{3\pi}{8}.$$

$\otimes$