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Exercise 1: Calculate the flux of $F(x,y,z) = (x^2, xz, 3z)$ outward across the sphere $x^2+y^2+z^2=4$

The divergence of vector field is : $\nabla \cdot F = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} = 2x+3$

We have $\iiint_D \operatorname{div} \vec{F} \, dV = \oiint_S \vec{F}.\vec{N} \, dS = \iiint_D 2x+3 \, dV$

Changing to spherical coordinates : $dV = r^2 \sin\varphi \, dr \, d\varphi \, d\theta$

$D = \{(r,\theta,\varphi) \mid 0 \leq \theta \leq 2\pi$
$0 \leq \varphi \leq \pi, \; 0 \leq r \leq 2\}$

The outward flux across the boundary of D is

$$\oiint \vec{F}.\vec{N} \, dS = \int_0^{2\pi}\int_0^{\pi}\int_0^{2} (2r\cos\theta\sin\varphi + 3)(r^2\sin\varphi) \, dr \, d\varphi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{2} (2r^3\cos\theta\sin^2\varphi + 3r^2\sin\varphi) \, dr \, d\varphi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \left(\frac{1}{2} r^4 \cos\theta \sin^2\varphi + r^3\sin\varphi\right)\Big|_0^2 \, d\varphi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} (8\cos\theta\sin^2\varphi + 8\sin\varphi) \, d\varphi \, d\theta$$

$$= \int_0^{2\pi} \left(4\cos\theta\left(\varphi - \frac{1}{2}\sin(2\varphi)\right) - 8\cos\varphi\Big|_0^{\pi} \, d\theta\right.$$

$$= \int_0^{2\pi} (4\pi\cos\theta + 16) \, d\theta = 4\pi\sin\theta + 16\theta\Big|_0^{2\pi}$$

$= 32\pi$ (answer)

Exercise 2: Calculate the flux of $F(x,y,z) = (x^2, y^2, z^2)$ outward across the boundary of the domain

$$D = \{(x,y,z) \in R^3 ; (x-2)^2 + y^2 + (z-3)^2 \leq 9\}$$

Let $u = x-2 \Rightarrow x = u+2$; $y = v$; $w = z-3 \Rightarrow z = w+3$

The flux vector is now $F(u,v,w) = ((u+2)^2, v^2, (w+3)^2)$

$$\Rightarrow \operatorname{div} F = \frac{\partial}{\partial u}(u+2)^2 + \frac{\partial}{\partial v}(v^2) + \frac{\partial}{\partial w}(w+3)^2$$

$$= 2(u+v+w) + 10$$

The outward flux across the sphere is

$$\oiint \vec{F} . \vec{N} dS = \int_0^{2\pi}\int_0^{\pi}\int_0^{3} [2r(\sin\theta\cos\varphi + \sin\theta\sin\varphi + \cos\theta) + 10] r^2 \sin\varphi \, dr d\varphi d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{r^3}{3}[2r(\sin\theta\cos\varphi + \sin\theta\sin\varphi + \cos\theta) + 10]\sin\varphi \Big|_0^3 d\varphi d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} 9[6(\sin\theta\cos\varphi + \sin\theta\sin\varphi + \cos\theta) + 10]\sin\varphi \, d\varphi d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} (54\sin\theta\cos\varphi + 54\sin\theta\sin\varphi + 54\cos\theta + 90)\sin\varphi \, d\varphi d\theta$$

$$= \int_0^{2\pi} 27\sin\theta\sin^2\varphi + 27\sin\theta(\varphi - \frac{1}{2}\sin(2\varphi)) + (-54\cos\theta\cos\varphi) - 90\cos\varphi \Big|_0^{\pi} d\theta$$

$$= \int_0^{2\pi} 180 \, d\theta = 180\theta \Big|_0^{2\pi} = 360\pi \text{ (answer)}$$

Exercise 3: Assume that S is an orientable smooth surface that is the boundary of a regular domain D in R^3. Assume that F is a smooth vector field on R^3. Show that

$$\oiint_S (\text{Curl } F) \cdot N dS = 0$$

Using Gauss divergence theorem, we have:

$$\oiint_S (\text{Curl } F) \cdot N dS = \iiint_S \text{div}(\text{Curl } F) dV$$

Since the divergence of a curl is zero (proven last week's exercise)

$$\Rightarrow \oiint_S (\text{Curl } F) \cdot N dS = \iiint_S 0 dV = 0 \text{ (proven)}$$