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Exercise 1: Let $f(x,y) = ye^{x+y^2}$

a) We have: $\frac{\partial f}{\partial x} = ye^{x+y^2}$ $\frac{\partial f}{\partial y} = (2y^2+1)e^{x+y^2}$

Linear approximation of $f(-4.02, 2.05)$ around the point $(-4; 2)$

$$L(-4.02, 2.05) = f(-4;2) + f_x(-4.02, 2.05)(-4.02+4) + f_y(-4.02, 2.05)(2-2.05)$$

$$= 2e^0 + 2.05e^{0.1825}(-0.02) + 9.405e^{0.1825}(-0.05)$$

$$\approx 1.38639$$

b) We have: $f(-4.02, 2.05) \approx 2.4604$

$\Rightarrow$ The approximation is smaller than the true value $(L(x,y) < f(x,y))$

Exercise 2: Consider $f(x,y) = \frac{y^2}{x}$ $(x \neq 0)$, ellipse $2x^2+y^2 = 1$

a) We know that the gradient vector is normal to the surface at a given point

$f(x,y) = \frac{y^2}{x} \Rightarrow \frac{\partial f}{\partial x} = -\frac{y^2}{x^2}$ $\frac{\partial f}{\partial y} = \frac{2y}{x}$

$$\Rightarrow \vec{\nabla} f(a,b) = \left\langle -\frac{b^2}{a^2}; \frac{2b}{a} \right\rangle$$

ellipse: $2x^2+y^2 = 1 \Rightarrow \frac{\partial f}{\partial x} = 4x$ $\frac{\partial f}{\partial y} = 2y$

$\Rightarrow \vec{\nabla} E(a,b) = \langle 4a, 2b \rangle$ orthogonal to surface at (a,b)

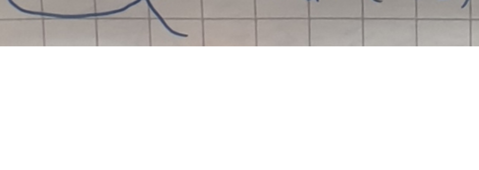
Directional derivative of $f(a,b)$ in the direction of $\vec{\nabla} E(a,b)$ is

$$D_{\vec{u}} f(a,b) = \langle 4a, 2b \rangle . \left\langle -\frac{b^2}{a^2}; \frac{2b}{a} \right\rangle$$

$$= 4a . -\frac{b^2}{a^2} + 2b . \frac{2b}{a} = -\frac{4b^2}{a} + \frac{4b^2}{a} = 0$$

b) Geometric interpretation

Since the normal vector is orthogonal to the ellipse, any point belonging to the ellipse lying on $f(x,y)$ will have their rate of change orthogonal to the surface of $f(x,y)$, rendering them having no rate of change at all



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Exercise 3: We have: $g(2) = 3$, $h(2) = 4$, $g'(2) = 5$, $h'(2) = 6$

$f(2,2) = -3$, $f(3,4) = 9$

$f_x(2,2) = 4$, $f_x(3,4) = 7$, $f_y(2,2) = 6$, $f_y(3,4) = 8$

Find derivative of $f(g(t), h(t))$ at $t = 2$

$f(g(2), h(2)) = f(3,4)$

According to the chain rule:

$$\frac{df}{dt} = \frac{\partial f}{\partial g} \cdot \frac{dg}{dt} + \frac{\partial f}{\partial h} \cdot \frac{dh}{dt}$$

At $t = 2 \Rightarrow \dfrac{df}{dt} = 7 . 5 + 8 . 6 = 83$

$\Rightarrow$ The tangent plane: $f(2;2) + f_x(2,2$

Exercise 4: Let $f(x, y, z) = x - 2y + 2z$ constrained to $g(x,y,z) = x^2 + y^2 + z^2 = 36$

a) Find extrema with the above condition

Let $g(x,y,z) = 0 \Rightarrow g(x,y,z) = x^2 + y^2 + z^2 - 36 = 0$

We have : $\frac{\partial f}{\partial x} = 1 \quad \frac{\partial f}{\partial y} = -2 \quad \frac{\partial f}{\partial z} = 2$

$\frac{\partial g}{\partial x} = 2x \quad \frac{\partial g}{\partial y} = 2y \quad \frac{\partial g}{\partial z} = 2z$

The lagrange multiplier

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 36 \end{cases} \Rightarrow \begin{cases} 1 = \lambda 2x \\ -2 = \lambda 2y \\ 2 = \lambda 2z \\ x^2 + y^2 + z^2 = 36 \end{cases} \Rightarrow \begin{cases} \lambda 2x = 1 \\ -\lambda y = 1 \\ \lambda z = 1 \\ x^2 + y^2 + z^2 = 36 \end{cases}$$

λ can't be 0 because $0.2x = 0 \Rightarrow \lambda = c \neq 0$

$$\Rightarrow \lambda 2x = -\lambda y = \lambda z \Rightarrow \left[\begin{array}{l} x = \frac{1}{2\lambda} ; \; y = -\frac{1}{\lambda} ; \; z = \frac{1}{\lambda} \; (2) \\ 2x = -y = z \; (1) \end{array} \right.$$

Consider (1) $\Rightarrow x^2 + 4x^2 + 4x^2 = 36$

$9x^2 = 36 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

$\Rightarrow$ Two extrema : $(2; -4; 4)$ & $(-2; 4; -4)$

Consider (2) $\Rightarrow \left(\frac{1}{2\lambda}\right)^2 + \left(-\frac{1}{\lambda}\right)^2 + \left(\frac{1}{\lambda}\right)^2 = 36$

$\Rightarrow \frac{1}{4\lambda^2} + \frac{1}{\lambda^2} + \frac{1}{\lambda^2} = 36$

$\Rightarrow \frac{9}{4\lambda^2} = 36 \Rightarrow 4\lambda^2 = \frac{1}{4} \Rightarrow \lambda = \pm \frac{1}{4}$

For $\lambda = \frac{1}{4} \Rightarrow (x, y, z) = (2; -4; 4)$

$\lambda = -\frac{1}{4} \Rightarrow (x, y, z) = (-2; 4; -4)$

We have : $f(2; -4; 4) = 18 \Rightarrow$ maxima

$f(-2; 4; -4) = -18 \Rightarrow$ minima

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b) Geometric interpretation

We can see that : $x - 2y + 2z = \pm 18$ are tangent planes to the sphere

$L(2; -4; 4) = 18 + 1(x-2) - 2(y+4) + 2(z-4) = 0$

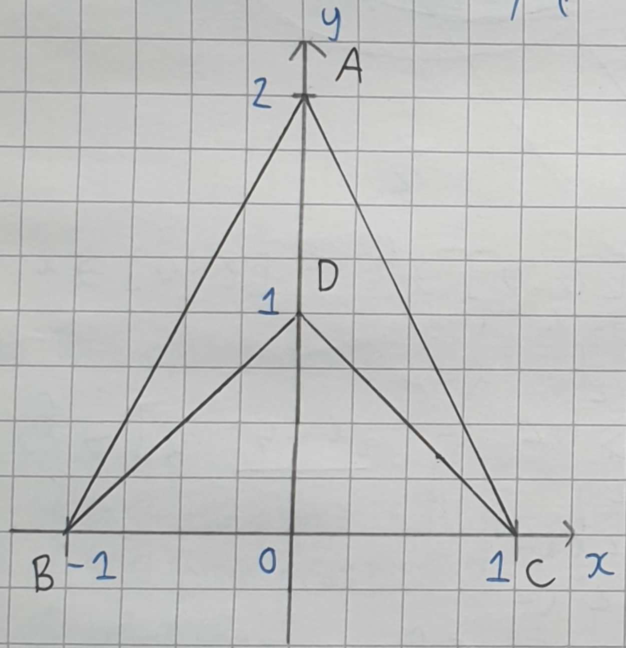
$\Rightarrow 18 + x - 2 - 2y - 8 + 2z - 8 = 0$

$\Leftrightarrow x - 2y + 2z = 0$

$\Rightarrow$ If we doesn't solve by calculus, we can reason that $f(x,y)$ can reach its extrema by shifting to the rears of the sphere and become tangent planes.

$\Rightarrow$ Calculate which value of $f(x, y, z)$ that makes it the tangent plane to sphere

Exercise 5: D has density $\rho(x,y) = 1$



a) Describe ACD

We have : $0 \leq x \leq 1$

Line passing through DC has equation

$y = -x + 1$

Line passing through AC has equation

$y = -2x + 2$

ACD lies above DC line and below AC line

$\Rightarrow -x + 1 \leq y \leq -2x + 2$

$\Rightarrow$ The set of inequalities

$$\begin{cases} 0 \leq x \leq 1 \\ -x + 1 \leq y \leq -2x + 2 \end{cases}$$

b) The mass of ACD is

$$\int_0^1 \int_{-x+1}^{-2x+2} \underbrace{1}_{\text{density}} \, dy\,dx = \int_0^1 (-2x + 2 - (-x+1))\,dx$$

$$= \int_0^1 (-x+1)\,dx = -\frac{1}{2}x^2 + x \Big|_0^1 = \frac{1}{2}$$

Since ABD is the same as ACD $\Rightarrow$ mass of ABCD is $\frac{1}{2} \times 2 = 1$

We have

$$M_x(ACD) = \int_0^1 \int_{-x+1}^{-2x+2} y\,dy\,dx = \frac{1}{2} \Rightarrow \bar{x}_{(ACD)} = \frac{M_y(ACD)}{M(ACD)} = \frac{1/6}{1/2} = \frac{1}{3}$$

$$M_y(ACD) = \int_0^1 \int_{-x+1}^{-2x+2} x\,dy\,dx = \frac{1}{6} \Rightarrow \bar{y}_{(ACD)} = \frac{M_x(ACD)}{M(ACD)} = \frac{1/2}{1/2} = 1$$

By reflection, we also know that $M_x(ABD) = \frac{1}{2}$; $M_y(ABD) = -\frac{1}{6}$

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$\Rightarrow M_x = \frac{1}{2} + \frac{1}{2} = 1 ; \quad M_y = \frac{1}{6} + \left(-\frac{1}{6}\right) = 0$

$\Rightarrow \bar{x}_{arrow} = \frac{0}{1} = 0 , \quad \bar{y}_{arrow} = \frac{1}{1} = 1$

$\Rightarrow$ Center of mass of arrow is $(0; 1)$

c) The center of mass lies inside the arrow

Exercise 6: Consider triple integral $\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{8-x^2-y^2}} 1 \, dz\,dy\,dx$

a) Sketch region of integration

$z \in [0; \sqrt{8-x^2-y^2}] \Rightarrow z \leq \sqrt{8-x^2-y^2}$

$\Rightarrow z^2 \leq 8 - x^2 - y^2$

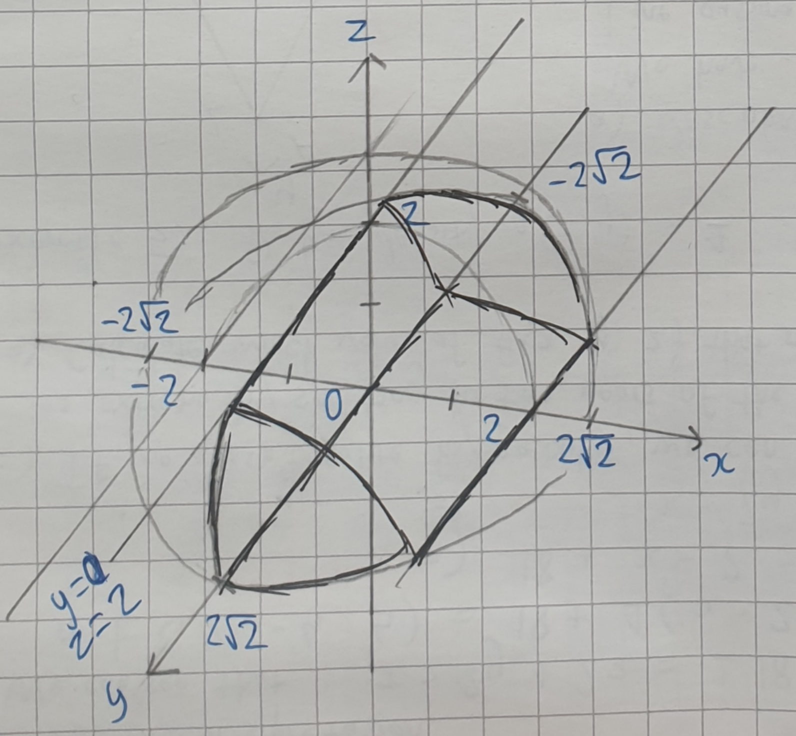
$\Rightarrow \begin{cases} z^2 + x^2 + y^2 \leq 8 \\ z \geq 0 \end{cases} \Rightarrow$ z is upper half sphere region inside the sphere radius $2\sqrt{2}$

$y \in [0; \sqrt{4-x^2}] \Rightarrow y \leq \sqrt{4-x^2}$

$\Rightarrow y^2 \leq 4 - x^2$

$\Rightarrow x^2 + y^2 \leq 4 \Rightarrow$ y is upper half cylinder along y-axis

$x \in [0; 2] \Rightarrow y \in [0; 2], \ z \in [0; 2]$



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b) $\begin{cases} 0 \leq z \leq \sqrt{8-x^2-y^2} \\ 0 \leq y \leq \sqrt{4-x^2} \\ 0 \leq x \leq 2 \end{cases} \Rightarrow \begin{cases} 0 \leq z \leq \sqrt{8-r^2} \\ 0 \leq r\sin\theta \leq \sqrt{4-r^2\cos^2\theta} \\ 0 \leq r\cos\theta \leq 2 \end{cases}$

$\Rightarrow \begin{cases} 0 \leq z \leq \sqrt{8-r^2} \\ r^2\sin^2\theta \leq 4 - r^2\cos^2\theta \\ 0 \leq r\cos\theta \leq 2 \end{cases} \Rightarrow \begin{cases} 0 \leq z \leq \sqrt{8-r^2} \\ \cancel{0 \leq r \leq 2} \quad r^2\cos^2\theta \leq r^2 \leq 4 \\ 0 \geq \cos\theta \geq \frac{2}{r} \end{cases}$

$\Rightarrow \begin{cases} 0 \leq z \leq \sqrt{8-r^2} \\ 0 \leq r \leq 2 \\ 0 \leq \theta \leq \frac{\pi}{2} \end{cases} \Rightarrow \int_0^{\frac{\pi}{2}} \int_0^2 \int_0^{\sqrt{8-r^2}} 1 \, dz\,dr\,d\theta$

c) $\int_0^{\frac{\pi}{2}} \int_0^2 \int_0^{\sqrt{8-r^2}} r\,dr\,dz\,d\theta = \int_0^{\frac{\pi}{2}} \int_0^2 r\sqrt{-r^2+8}\,dr\,dt$

$= \int_0^{\frac{\pi}{2}} \left(\frac{8(2\sqrt{2}-1)}{3} \right) d\theta$

$= \frac{4\pi(2\sqrt{2}-1)}{3}$ (Answer)

d) The value of the integral represents the volume bounded by the triple integral