Homework 5

[redacted]

① Calculate the flux of $F(x,y,z) = (x^2, xz, 3z)$ outward across the sphere $x^2+y^2+z^2=4$.

Solution: We use Gauss's Theorem

$$\operatorname{div} F = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(3z) =$$

$$= 2x + 3$$

$\iiint x \, dV = 0$ by symmetry

$$\iint_{x^2+y^2+z^2=4} F \cdot N \, dS = \iiint_{x^2+y^2+z^2 \le 4} 2x + 3 \, dV = \iiint_{x^2+y^2+z^2 \le 4} 3 \, dV$$

$= 3 \cdot$ volume of sphere with radius 2

$$= 3 \cdot \frac{4\pi \cdot 2^3}{3} = 32\pi.$$

② Calculate the flux of $F(x,y,z) = (x^2, y^2, z^2)$ outward across the boundary of the domain

$$D = \{ (x,y,z) \in \mathbb{R}^3 ;\ (x-2)^2 + y^2 + (z-3)^2 \le 9 \}$$

Solution:

$$I = \iint_{\partial D} F \cdot N \, dS = \iiint_D \operatorname{div} F \, dV$$

$$= 2 \iiint_D x + y + z \, dV$$

Let $\begin{cases} u = x-2 \\ v = y \\ w = z-3 \end{cases}$. We have $dx\,dy\,dz = du\,dv\,dw$

Therefore $I = 2\iiint_{u^2+v^2+w^2 \leq 9} u+2+v+w+3 \; du\,dv\,dw =$ $\left[\begin{array}{l}\text{Symmetry gives} \\ \iiint u\,dV = \iiint v\,dV = \\ = \iiint w\,dV = 0\end{array}\right]$

$$= 10 \iiint_{u^2+v^2+w^2 \leq 9} 1 \; dV = 10 \cdot \frac{4\pi\, 3^3}{3} = 360\pi$$

(3) Assume that $S$ is an orientable smooth surface that is the boundary of a regular domain $D$ in $\mathbb{R}^3$. Assume that $F$ is a smooth vector field on $\mathbb{R}^3$. Show that

$$\oiint_S (\text{Curl}\, F) \cdot \vec{N}\, dS = 0$$

Solution: The assumptions makes it possible to use Gauss's Theorem. We know that $\text{div}(\text{Curl}\, F) = 0$ for every smooth vector field. Therefore

$$\oiint_S \text{Curl}\, F \cdot N\, dS = \iiint_D \text{div}(\text{Curl}\, F)\, dV$$

$$= \iiint_D 0\, dV = 0$$

Hand-in Exercises 5

(1) Let $F = (xz, yz, 1)$ and

$$D = \{(x,y,z);\ x^2+y^2+z^2 \le 25,\ z \ge 3\}$$

Calculate the flux of F outwards across $\partial D$.

Solution: Gauss's Theorem says

$$\iint_{\partial D} \vec{F}\cdot\vec{N}\, dS = \iiint_D \operatorname{div}\vec{F}\, dV$$

Here $\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(1) = 2z$

So $\iint_{\partial D} \vec{F}\cdot\vec{N}\,dS = \iiint_D 2z\, dV = $ ( Cylindrical coord)

$$= \int_0^{2\pi} d\theta \iint_{\substack{r^2+z^2\le 25\\ z\ge 3}} 2z\, r\,dr\,dz = 2\pi \int_0^4 2r \int_3^{\sqrt{25-r^2}} z\, dz\, dr$$

$$= 2\pi \int_0^4 2r \left[\frac{z^2}{2}\right]_3^{\sqrt{25-r^2}} dr =$$

$$= 2\pi \int_0^4 r\,(25-r^2-9)\, dr =$$

$$= 2\pi \int_0^4 16r - r^3\, dr = 2\pi \left[8r^2 - \frac{r^4}{4}\right]_0^4 =$$

$$= 2\pi\,(8\cdot 16 - 4\cdot 16) = 2\pi\cdot 4\cdot 16 = 128\pi$$

(2) Assume that $f$ is harmonic. Assume that $D$ is a regular closed set in $\mathbb{R}^3$ bounded by a smooth orientable surface $S$ and that $\vec{N}$ is a unit normal vector field $S$ pointing outwards. Show that

$$\oiint_S \nabla f \cdot \vec{N}\, dS = 0$$

Solution:

$$\oiint_S \nabla f \cdot \vec{N}\, dS = \iiint_D \operatorname{div}(\nabla f)\, dV$$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \quad \text{and}$$

$$\operatorname{div}(\nabla f) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) =$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0.$$

$$\Longrightarrow \oiint_S \nabla f \cdot \vec{N}\, dS = 0$$

(3) Let $S$ be the boundary surface of

$$D = \{(x,y,z) \in \mathbb{R}^3;\ x^2+y^2 \leq z^2,\ 0 \leq z \leq 1\}$$

and let $\vec{N}$ the unit normal vector field to $S$ that points outward from $D$. Let $F(x,y,z) = (x^2, y^2, z^2)$ and calculate

$$\oiint_S F \cdot \vec{N}\, dS.$$

<u>Solution:</u> Gauss's Theorem gives

$$\oint\!\!\oint_S F\cdot\vec{N}\,dS = \iiint_D \operatorname{div} F\, dV$$

and $\operatorname{div} F = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2) =$

$$= 2x + 2y + 2z$$

Therefore $\oint\!\!\oint_S F\cdot\vec{N}\,dS = \iiint_D 2x+2y+2z\,dV$

$=$ (cylindrical coordinates) $=$

$$= \int_0^{2\pi}\iint_{\substack{r^2\le z^2\\ 0\le z\le 1}} (2r\cos\theta + 2r\sin\theta + 2z)\, r\,dr\,d\theta\,dz$$

$\underset{\left(\int_0^{2\pi}\cos\theta\,d\theta = \int_0^{2\pi}\sin\theta\,d\theta = 0\right)}{=}$ $2\pi\int_0^1\int_0^z 2zr\,dr\,dz =$

$$= 2\pi\int_0^1 z\,[r^2]_{r=0}^{r=z}\,dz = 2\pi\int_0^1 z^3\,dz = \frac{\pi}{2}$$

④ Let $\gamma$ be the intersection curve of $x^2+y^2+z^2=1$ and $x+y+z=0$ oriented counter clockwise (when looking from above along the z-axis). Calculate

$$\oint_\gamma (y+z)\,dx + (x+z)\,dy + (x+y)\,dz$$

Solution: Stokes's Theorem says

$$\oint_\gamma \vec{F} \cdot d\vec{r} = \iint_S \text{Curl} \vec{F} \cdot \vec{N} \, dS$$

Here $\vec{F} = (y+z, x+z, x+y)$ and

$$\text{Curl} \vec{F} = \begin{vmatrix} \vec{e_1} & \vec{e_2} & \vec{e_3} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y+z & x+z & x+y \end{vmatrix} = \left(\frac{\partial}{\partial y}(x+y) - \frac{\partial}{\partial z}(x+z)\right)\vec{e_1}$$

$$- \left(\frac{\partial}{\partial x}(x+y) - \frac{\partial}{\partial z}(y+z)\right)\vec{e_2} + \left(\frac{\partial}{\partial x}(x+z) - \frac{\partial}{\partial y}(y+z)\right)\vec{e_3}$$

$$= (1-1)\vec{e_1} - (1-1)\vec{e_2} + (1-1)\vec{e_3} = (0,0,0)$$

$$\Longrightarrow \oint_\gamma (y+z)\,dx + (x+z)\,dy + (x+y)\,dz = \iint_S \vec{0} \cdot \vec{N} \, dS = 0.$$

# Demo Exercises 5

(1) Let $\gamma$ be the positively oriented boundary curve to a square in the plane and let $F(x,y) = (xy^2, x^2y + 2x)$.

Show that $\oint_\gamma F \cdot d\vec{r}$ depends only the area of the square and not on the location.

Solution: We use Green's Theorem

$$\oint_\gamma F \cdot d\vec{r} = \iint_R \frac{\partial}{\partial x}(x^2y + 2x) - \frac{\partial}{\partial y}(xy^2)\,dxdy$$

$$= \iint_R 2xy + 2 - 2xy \, dxdy = 2\iint_R dxdy$$

$$= 2\,\text{area}\,(R)$$

(2) Let $F(x,y) = (-\sin y, x\cos y)$ and $\gamma$ be the boundary curve of

$$R = \{(x,y) \in \mathbb{R}^2;\ 0 \le x \le \pi/2,\ 0 \le y \le \pi/2\}$$

oriented counterclockwise. Calculate the circulation of $F$ along $\gamma$.

<u>Solution</u>: We use Green's Theorem

$$\oint_\gamma F \cdot d\vec{r} = \iint_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \, dA =$$

$$= \iint_R \cos y - (-\cos y) \, dA =$$

$$= \int_0^{\pi/2} \left( \int_0^{\pi/2} 2\cos y \, dx \right) dy =$$

$$= \int_0^{\pi/2} \pi \cos y \, dy = \pi$$

(3) Assume that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$

Show that $\oint_\gamma \frac{\partial f}{\partial y} dx - \frac{\partial f}{\partial x} dy = 0$ for every smooth simple curve $\gamma$ that bounds a regular closed domain.

<u>Solution</u>: We use Green's Theorem.

$$\oint_\gamma \frac{\partial f}{\partial y} dx - \frac{\partial f}{\partial x} dy = \iint_R \left( \frac{\partial}{\partial x}\left(-\frac{\partial f}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \right) dA$$

$$= - \iint \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \, dA = 0$$

(If $\gamma$ is oriented clockwise you get $-0 = 0$ :) )