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PROBLEM SHEET 2 Exercises (Homework Problems)

Exercise 1: Write the indicated case of Taylor's formula for the given function. What is the Lagrange remainder in this case?

$$f(x) = x, \quad a = 1, \quad n = 6$$

Taylor polynomial for this case

$$T_6(x, 1) = \sum_{i=0}^{6} \frac{f^{(i)}(1)}{i!}(x-1)^i$$

$$= 1 + \frac{0}{1!}(x-1) + \frac{0}{2!}(x-1)^2 + \frac{0}{3!}(x-1)^3$$

$$+ \frac{0}{4!}(x-1)^4 + \frac{0}{5!}(x-1)^5 + \frac{0}{6!}(x-1)^6$$

$$= 1$$

Lagrange Remainder

$$f(x) = T_n(x, a) + R(x, a) \leftarrow \text{Lagrange remainder}$$

$$\Leftrightarrow 1 = 1 + R(x, a)$$

$\Rightarrow$ Lagrange Remainder $= 0$

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Exercise 2 : Find the nth-order Maclaurin polynomial of

$$f(x) = \frac{1}{(1-x)^2}$$

using the alternative definition of the Taylor polynomial

Theorem : If $f(x) = Q_n(x) + O((x-a)^{n+1})$ when x approaches a, and $Q_n(x)$ is a polynomial of degree $\leq n$

$\Rightarrow Q_n(x) = T_n(x)$

As the exercise asks for nth-order Maclaurin polynomial where $a = 0$

$\Rightarrow$ We should prove $f(x) = \frac{1}{(1-x)^2} = Q_n(x) + O(x^{n+1})$

We have : $\frac{d}{dx} \frac{1}{(1-x)} = -\frac{1}{(1-x)^2}(1-x)'$

$$= \frac{1}{(1-x)^2}$$

$\Rightarrow$ To find $Q_n(x)$ for $f(x)$, first we have to find $Q_n(x)$ for $\frac{1}{1-x}$, and then take first derivative for $Q_n(x)$ of $\frac{1}{1-x}$

Generally, we have

$$(1-x)(1+x+x^2+\dots+x^n) = 1 - x^{n+1}$$

$$\Leftrightarrow \frac{1}{1-x} - \frac{x^{n+1}}{1-x} = 1 + x + x^2 + \dots + x^n$$

$$\Leftrightarrow \underbrace{\frac{1}{1-x}}_{f(x)} = \underbrace{1 + x + x^2 + \dots + x^n}_{Q_n(x) \,=\, \sum_{i=0}^{n} x^i} + \underbrace{\frac{x^{n+1}}{1-x}}_{O(x^{n+1})}$$

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To confirm $Q_n(x) = \sum_{i=0}^{n} x^i$, we have to prove $\frac{x^{n+1}}{1-x} = O(x^{n+1})$

$\Rightarrow \exists c \geq 0, \exists \delta > 0$ so that

$0 < |x| < \delta \Rightarrow \left| \frac{x^{n+1}}{1-x} \right| \leq c|x^{n+1}|$

$\Leftrightarrow \frac{1}{|1-x|} \leq c \Leftrightarrow c|1-x| \geq 1$

~~Pick $\delta = 1/2 \Rightarrow -1/2 < |x| < 1/2 \Rightarrow 1/2 < x < 3/2$~~

~~$\Leftrightarrow 0 < 1 - x < 2$~~

~~$\Leftrightarrow c|1-x| = c(1-x)$~~

For $c|1-x| \geq 1 \Rightarrow c > 0$. Assume $c = 2$

$\Leftrightarrow 2|1-x| \geq 1 \Leftrightarrow |1-x| \geq \frac{1}{2}$

$\Leftrightarrow -\frac{1}{2} < \left[\begin{array}{l} 1-x \geq 1/2 \\ 1-x \leq -1/2 \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x \leq 1/2 \\ x \geq 3/2 \end{array}\right.$

$\Leftrightarrow |x| = [0, \frac{1}{2}] \cup [\frac{3}{2}, +\infty)$

$\Rightarrow$ Exist δ so that $0 < |x| < \delta$

Therefore: $\frac{x^{n+1}}{1-x} = O(x^{n+1}) \Rightarrow Q_n(x) = \sum_{i=0}^{n} x^i$

$\Rightarrow Q_n(x)$ of $f(x)$ is $\frac{d}{dx}\left(\sum_{i=0}^{n} x^i\right) = \sum_{i=1}^{n} i x^{i-1}$

$\Rightarrow Q_n(x)$ of $f(x)$ is: $1 + 2x + 3x^2 + 4x^3 + \ldots + n.x^{n-1}$

Maclaurin

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Exercise 3: Express the given limit as a definite integral

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n} \ln\left(1 + \frac{2i}{n}\right)$$

Riemann Sum: $S_p = \sum_{k=1}^{n} f(\xi_k) \Delta x_k$

Since $\xi_k \in I_k \Rightarrow \xi_k = a + k\Delta x$ with a as the beginning of the interval in $[a, b]$

$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n} \ln\left(1 + \frac{2i}{n}\right)$ (*) when expressed as a definite integral will be calculated by Riemann Sum

$$\Rightarrow (*) = \lim_{n\to\infty} \sum_{k=1}^{n} f(a + k\Delta x)\Delta x = \int_a^b f(x)\,dx$$

$\Rightarrow f(x) = \ln(x)$, $a = 1$, $\Delta x = 2/n$

We have $\Delta x = \frac{b-a}{n} = \frac{2}{n} \Rightarrow b - a = 2$

$\Rightarrow b - 1 = 2 \Rightarrow b = 3$

There fore, the limit equals to

$$\int_1^3 \ln(x)\,dx$$

Integration by parts: Let $u = \ln x$ and $dv = dx$

We have: $\int u\,dv = uv - \int v\,du$

$u = \ln x \Rightarrow du = \frac{1}{x}dx \quad | \quad dv = dx \Rightarrow v = x$

Therefore

$$\int_1^3 \ln x\,dx = x\ln x - \int_1^3 dx = x\ln x - x\Big|_1^3$$

$$= 3\ln 3 - 3 - (\ln 1 - 1) = 3\ln 3 - \cancel{4}\,2$$

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Exercise 4: Suppose that $a < b$ and f is continuous on $[a, b]$. Find the constant k that minimises the integral

$$\int_a^b (f(x) - k)^2 \, dx$$

To determine k that minimizes the integral, we have to set k as "x" and set $f(x)$ as a constant "k"

Let $h(f(x), k) = \int_a^b (f(x) - k)^2 \, dx$

- Take first derivative for $h(f(x), k)$ about k

$$\frac{d}{dk} h(f(x), k) = \frac{d}{dk} \int_a^b (f(x) - k)^2 \, dx$$

$$= \int_a^b (f(x)^2 - 2f(x)k + k^2) \frac{d}{dk} \, dx$$

$$= \int_a^b (-2f(x) + 2k) \, dx$$

$$= \int_a^b 2(k - f(x)) \, dx = 2 \int_a^b (k - f(x)) \, dx$$

$$= 2\left[\int_a^b k \, dx - \int_a^b f(x) \, dx\right]$$

$$= 2\left[kx \Big|_a^b - \int_a^b f(x) \, dx\right] = 2\left[k(b-a) - \int_a^b f(x) \, dx\right]$$

To find extrema for $h(f(x), k)$

$$\Rightarrow 2\left[k(b-a) - \int_a^b f(x) \, dx\right] = 0$$

$$\Rightarrow k(b-a) - \int_a^b f(x) \, dx = 0 \Rightarrow k = \frac{\int_a^b f(x) \, dx}{b-a}$$

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□ Take second derivative test to check if $k = \dfrac{\int_a^b f(x)\,dx}{b-a}$ will make $h(f(x),k)$ reaches its minimum

$$\frac{d^2}{d^2k} h(f(x),k) = \frac{d}{dk} \cdot 2\left(\int_a^b k\,dx - \int_a^b f(x)\,dx\right)$$

$$= 2\left(\int_a^b k \cdot \frac{d}{dk}\,dx - \int_a^b f(x) \cdot \frac{d}{dk}\,dx\right)$$

$$= 2\left(\int_a^b dx - \int_a^b 0\,dx\right) = 2\int_a^b dx$$

$$= 2(b-a)$$

Since $b > a \Rightarrow 2(b-a) > 0$

In second derivative test, $\dfrac{d^2}{d^2k} h(f(x),k)$ is always > 0

$\Rightarrow k = \dfrac{\int_a^b f(x)\,dx}{b-a}$ will minimizes the integral $\int_a^b (f(x)-k)^2\,dx$