Foundations of Solid Mechanics

L8: Plane-Stress Transformation & Mohr’s Circle

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Introduction

Our discussion of the stress transformation will deal mainly with plane stress, i.e., with a situation in which two of the faces of the cubic element are free of any stress. If the *z* axis is chosen perpendicular to these faces, we have $\sigma_z = \tau_{zx} = \tau_{zy} = 0$, and the only remaining stress components are σ_x, σ_y, and τ_{xy} (Fig. 8.2).

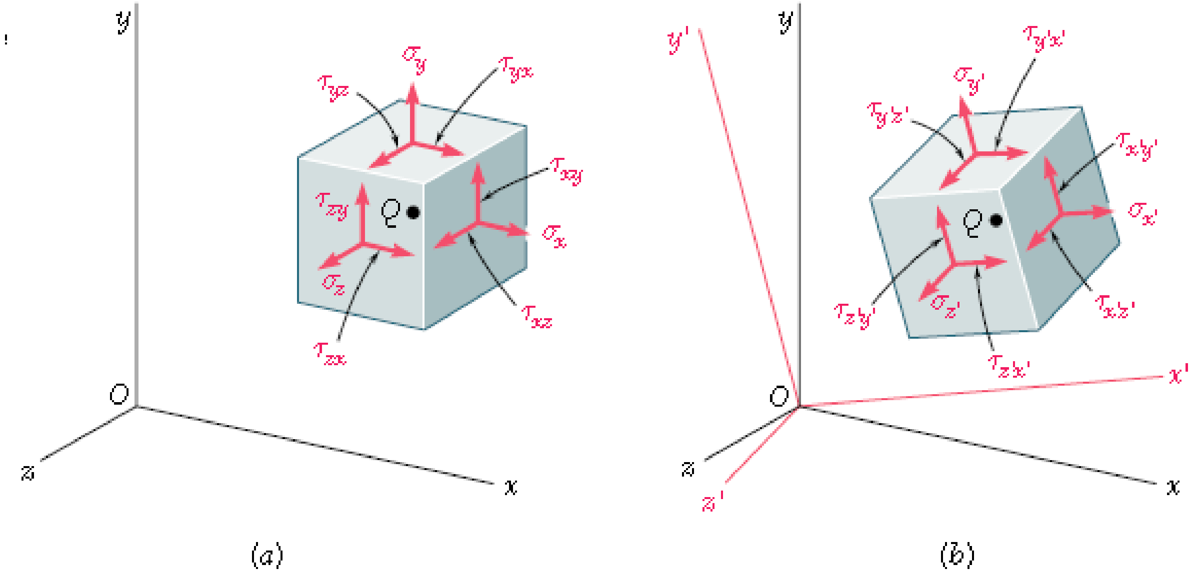


Fig. 8.1 General state of stress at a point.

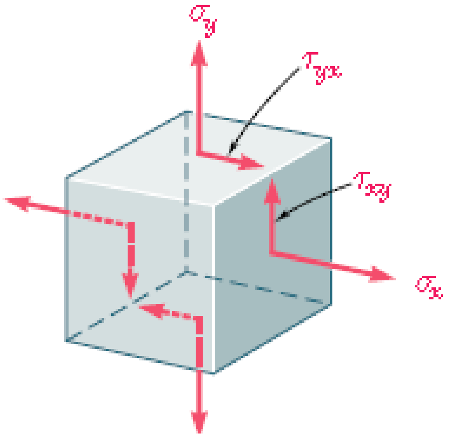


Fig. 8.2 Plane stress

Note: All materials in this handout are used in class for educational purposes only.

Introduction

Such a situation occurs in a thin plate subjected to forces acting in the mid-plane of the plate (Fig.8.3). It also occurs on the free surface of a structural element or machine component, i.e., at any point of the surface of that element or component that is not subjected to an external force (Fig.8.4).

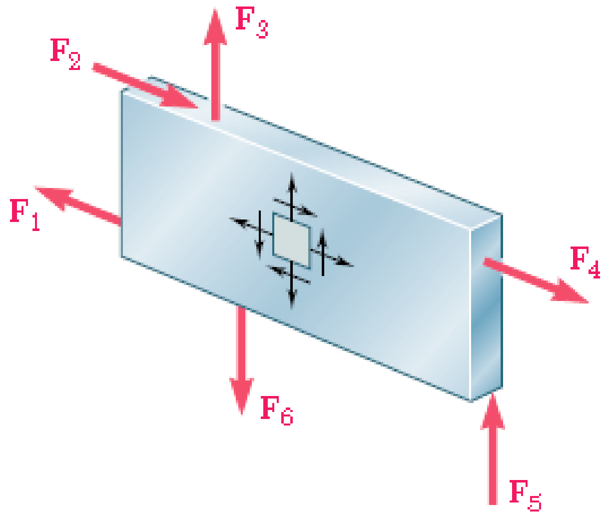


Fig. 8.3 Example of plane stress.

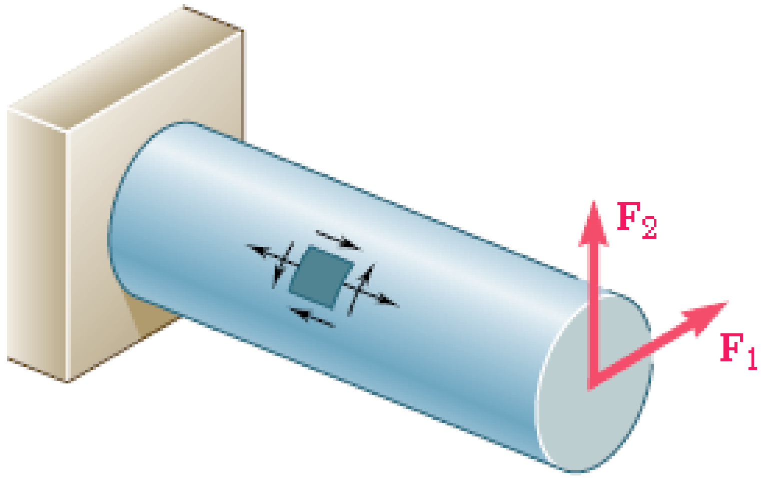


Fig. 8.4 Example of plane stress.

Transformation of Plane Stress

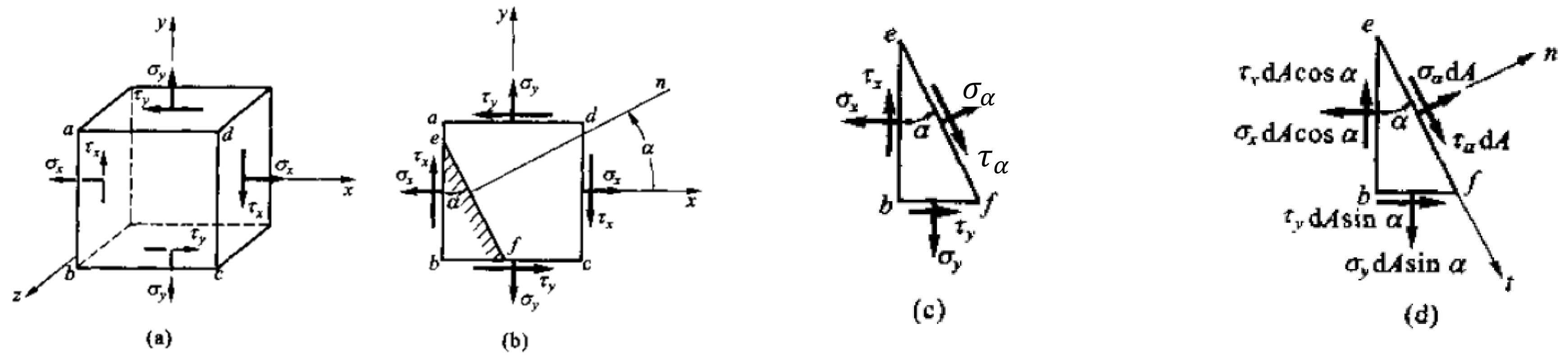


Fig. 8.5

$$\sum F_{x'} = 0 \qquad \sigma_\alpha \mathrm{dA} - \sigma_x(\mathrm{dA}\cos\alpha)\cos\alpha + \tau_x(\mathrm{dA}\cos\alpha)\sin\alpha - \sigma_y(\mathrm{dA}\sin\alpha)\sin\alpha + \tau_y(\mathrm{dA}\sin\alpha)\cos\alpha = 0$$

$$\sum F_{y'} = 0 \qquad \tau_\alpha \mathrm{dA} - \sigma_x(\mathrm{dA}\cos\alpha)\sin\alpha - \tau_x(\mathrm{dA}\cos\alpha)\cos\alpha + \sigma_y(\mathrm{dA}\sin\alpha)\cos\alpha + \tau_y(\mathrm{dA}\sin\alpha)\sin\alpha = 0$$

Solving the first equation for σ_α and the second for τ_α, we have

$$\sigma_\alpha = \sigma_x\cos^2\alpha + \sigma_y\sin^2\alpha - 2\tau_x\sin\alpha\cos\alpha$$

$$\tau_\alpha = (\sigma_x - \sigma_y)\sin\alpha\cos\alpha + \sigma_x(\cos^2\alpha - \sin^2\alpha)$$

Transformation of Plane Stress

Recalling the trigonometric relations

$$\sin 2\alpha = 2\sin\alpha\cos\alpha \qquad \cos 2\alpha = \cos^2\alpha\text{-}\sin^2\alpha$$

$$\cos^2\alpha = \frac{1+\cos 2\alpha}{2} \qquad \sin^2\alpha = \frac{1-\cos 2\alpha}{2}$$

$$\boxed{\begin{aligned}
\sigma_{x'} &= \sigma_\alpha = \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\alpha - \tau_x \sin 2\alpha \\
\tau_{x'y'} &= \tau_\alpha = \frac{\sigma_x-\sigma_y}{2}\sin 2\alpha + \tau_x \cos 2\alpha \\
\sigma_{y'} &= \sigma_{\alpha+90^\circ} = \frac{\sigma_x+\sigma_y}{2} - \frac{\sigma_x-\sigma_y}{2}\cos 2\alpha + \tau_x \sin 2\alpha
\end{aligned}}$$

$$\sigma_{x'} + \sigma_{y'} = \sigma_x + \sigma_y$$

Principal Stresses, Maximum Shear Stress

$$\left(\sigma_\alpha - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_\alpha^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_x^2$$

Setting $\quad \sigma_{ave} = \dfrac{\sigma_x + \sigma_y}{2} \qquad R = \sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_x^2}$

$$\Rightarrow \quad (\sigma_\alpha - \sigma_{ave})^2 + \tau_\alpha^2 = R^2$$

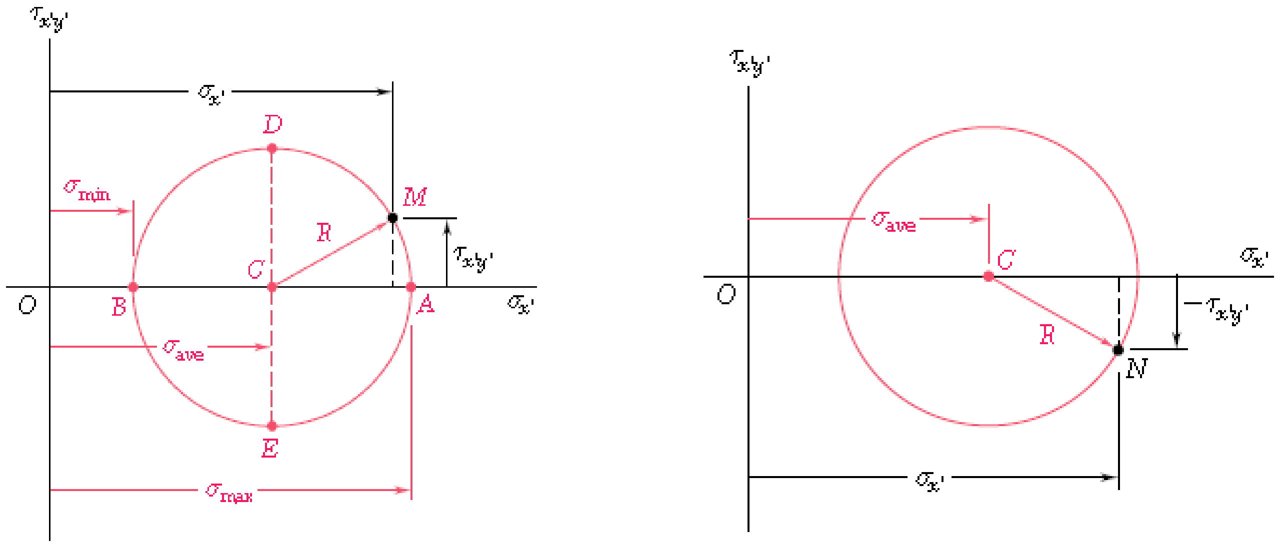


Fig.8.6 Mohr's Circle

Principal Stresses, Maximum Shear Stress

Thus, the values α_ρ of the parameter α which correspond to points A and B can be obtained by setting $\tau_{x'y'}$. We write:

$$\tan 2\alpha_\rho = -\frac{2\tau_x}{\sigma_x - \sigma_y}$$

$$\sigma_{max} = \sigma_{ave} + R \qquad \text{and} \qquad \sigma_{min} = \sigma_{ave} - R$$

Substituting for σ_{ave} and R, we write

$$\sigma_{max,min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_x^2}$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_x^2} \qquad \tan 2\alpha_s = \frac{\sigma_x - \sigma_y}{2\tau_x}$$

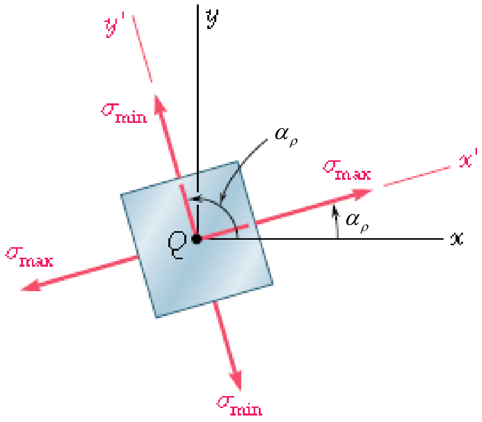


Fig. 8.7 Principal stresses

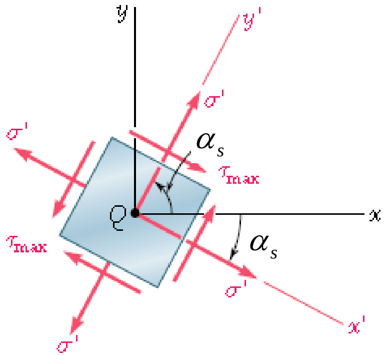


Fig. 8.8 Maximum shearing stress.

Example-1

For the state of plane stress shown in below, determine (a) the principal planes, (b) the principal stresses, (c) the maximum shearing stress and the corresponding normal stress.

(a) Principal Planes.

Following the usual sign convention, we write the stress components as

$$\sigma_x = +50MPa \qquad \sigma_y = -10MPa \qquad \tau_x = -40MPa$$

Then we have

$$\tan 2\alpha_\rho = -\frac{2\tau_x}{\sigma_x - \sigma_y} = -\frac{2\times(-40)}{50-(-10)} = \frac{4}{3}$$

$$2\alpha_\rho = 53.1^\circ \quad and \qquad 53.1^\circ + 180^\circ = 233.1^\circ$$

$$\alpha_\rho = 26.6^\circ \quad and \qquad 116.6^\circ$$

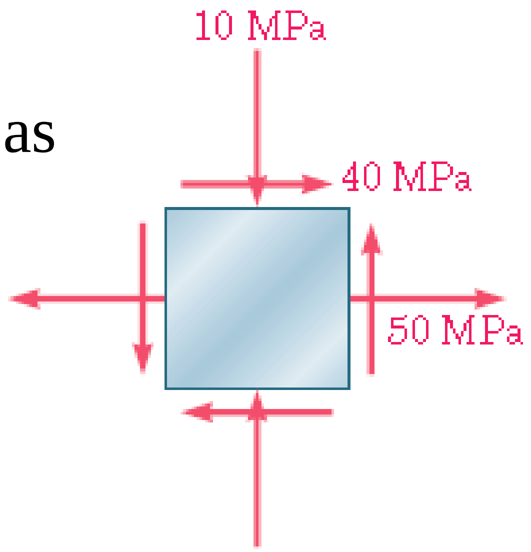


Fig. 8.9

Example-1

(b) Principal Stresses

$$\sigma_{max,min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_x^2} = 20 \pm \sqrt{(30)^2 + (-40)^2}$$

$$\sigma_{max} = 20 + 50 = 70MPa$$

$$\sigma_{min} = 20 - 50 = -30MPa$$

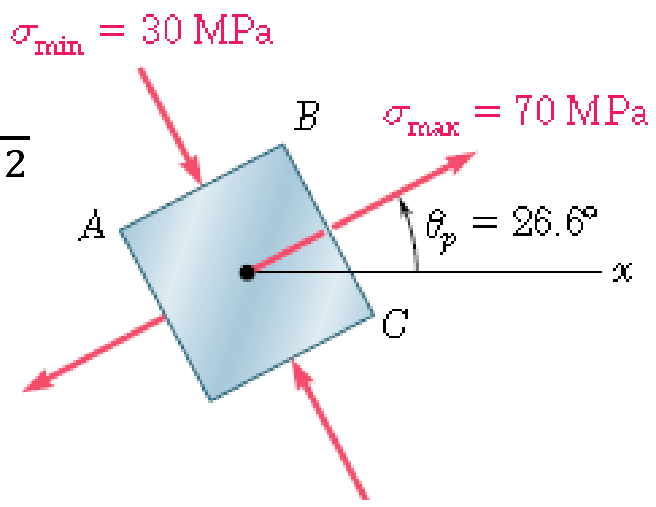


Fig. 8.10

The principal planes and principal stresses are sketched in Fig. 8.10. Making $\alpha_\rho = 26.6^{\circ}$, we check that the normal stress exerted on face BC of the element is the maximum stress:

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\alpha - \tau_x \sin 2\alpha$$

$$= \frac{50 - 10}{2} + \frac{50 + 10}{2} \cos 53.1^{\circ} - 40 \sin 53.1^{\circ} = 70MPa = \sigma_{max}$$

Example-1

(c) Maximum Shearing Stress.

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_x^2} = \sqrt{(30)^2 + (40)^2} = 50MPa$$

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = \frac{50 - 10}{2} = 20MPa$$

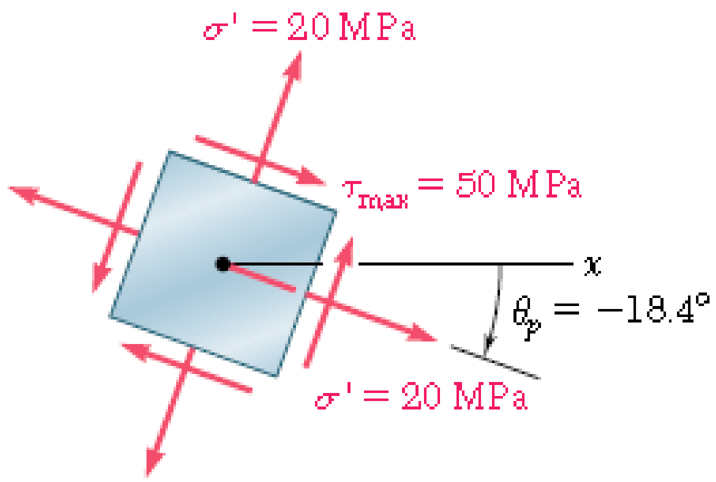


Fig. 8.11

General State of Stress

As shown below, the most general state of stress at a given point *Q* may be represented by six components. Three of these components, σ_x, σ_y, and σ_z, define the normal stresses exerted on the faces of a small cubic element centered at *Q* and of the same orientation as the coordinate axes, and the other three, τ_{xy}, τ_{yz}, and τ_{zx}, the components of the shearing stresses on the same element.

For principal stresses: $\sigma_1 > \sigma_2 > \sigma_3$

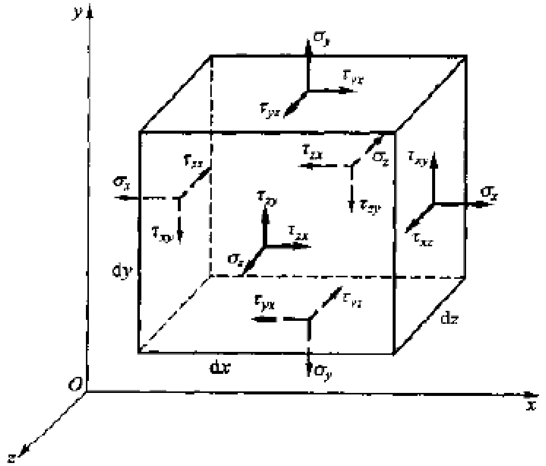


Fig.8.12 General state of stress at a point.

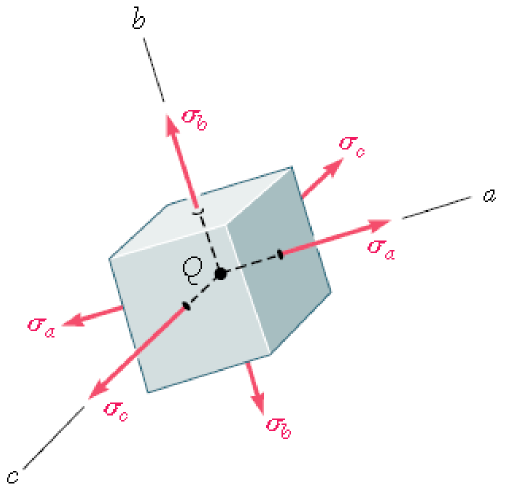


Fig.8.13 Principle stresses

Application of Mohr’s Circle to the 3-D Analysis of Stress

- If the element is rotated about one of the principal axes at *Q*, say the *c* axis (Fig.8.14), the corresponding transformation of stress can be analyzed by means of Mohr’s circle as if it were a transformation of plane stress.

- Indeed, the shearing stresses exerted on the faces perpendicular to the *c* axis remain equal to zero, and the normal stress σ_c is perpendicular to the plane *ab* in which the transformation takes place and, thus, does not affect this transformation.

- We therefore use the circle of diameter *AB* to determine the normal and shearing stresses exerted on the faces of the element as it is rotated about the *c* axis (Fig. 8.15). Similarly, circles of diameter *BC* and *CA* can be used to determine the stresses on the element as it is rotated about the *a* and *b* axes, respectively. While our analysis will be limited to rotations about the principal axes, it could be shown that any other transformation of axes would lead to stresses represented in Fig. 8.15 by a point located within the shaded area.

Application of Mohr's Circle to the 3-D Analysis of Stress

- Thus, the radius of the largest of the three circles yields the maximum value of the shearing stress at point *Q*. Noting that the diameter of that circle is equal to the difference between σ_{max} and σ_{min}, we write

$$\tau_{max} = \frac{|\sigma_{max} - \sigma_{min}|}{2} \qquad \left(\text{or } \tau_{max} = \frac{|\sigma_1 - \sigma_3|}{2}\right)$$

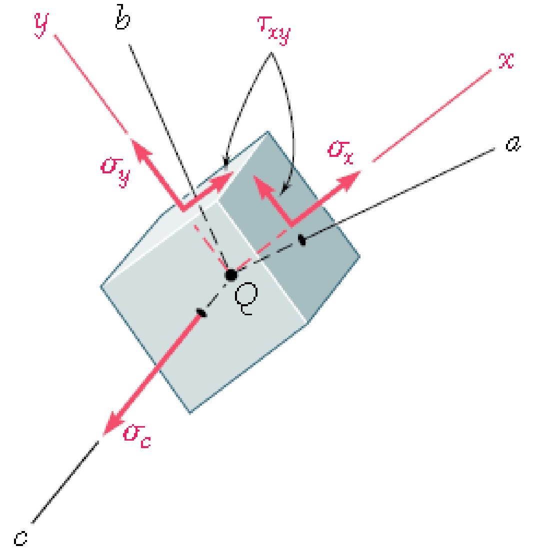


Fig.8.14

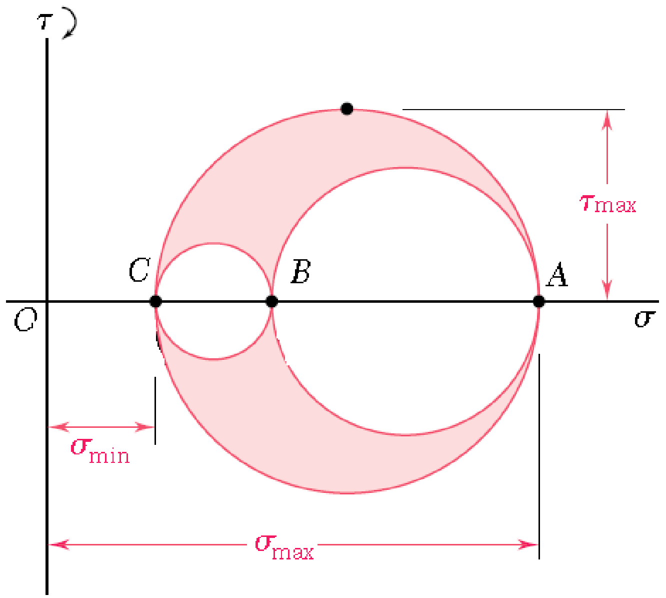


Fig.8.15 Mohr's circles for general state of stress.

Theory of Failure (Isotropic Yield Criteria)

- Fracture Criteria for Brittle Materials
  - Maximum Principal Stress Theory
  - Maximum Principal Strain Theory

- Fracture Criteria for Ductile Materials
  - Maximum-Shearing-Stress Theory
  - Maximum-Distortion-Energy Theory

Maximum Principal Stress Theory

- According to this criterion, a given structural component fails when the maximum stress in that component reaches the yield strength σ_y obtained from the tensile test of a specimen of the same material.

- Thus, the structural component will be safe as long as the first principal stresses σ_1 less than σ_y

$$\boxed{\sigma_1 \leq \sigma_y}$$

- However, this criterion is not effective for triaxial compression state as the maximum stress is smaller than 0.

Maximum Principal Strain Theory

- According to this criterion, a given structural component fails when the maximum strain in that component reaches the ultimate strain ε_U of the brittle material.
- The ultimate strength can be obtained for the material test. If the material can be treated as linear until failure, then we have

$$\varepsilon_1 = \varepsilon_y = \frac{\sigma_y}{E}$$

For state of principle stress : $\varepsilon_1 = \frac{1}{E}\left(\sigma_1 - \nu\left(\sigma_2 + \sigma_3\right)\right)$

Thus $\boxed{\frac{1}{E}\left(\sigma_1 - \nu\left(\sigma_2 + \sigma_3\right)\right) \le \frac{\sigma_y}{E}}$ or $\boxed{\sigma_1 - \nu\left(\sigma_2 + \sigma_3\right) \le \sigma_y}$

Maximum-Shearing-Stress Theory

- This criterion is based on the observation that yield in ductile materials is caused by slippage of the material along oblique surfaces and is due primarily to shearing Stresses.
- According to this criterion, a given structural component is safe as long as the maximum value τ_{max} of the shearing stress in that component remains smaller than the ultimate shearing stress obtained in a tensile-test.

Ultimate shearing stress in a tensile-test: $\tau_y = \frac{1}{2}\sigma_y$

For state of principle stress $\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3)$

When failure: $\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3) \le \tau_y = \frac{1}{2}\sigma_y$

Thus $\boxed{\sigma_1 - \sigma_3 \le \sigma_y}$

Distortion Energy Theory (Von Mises Yield Criterion)

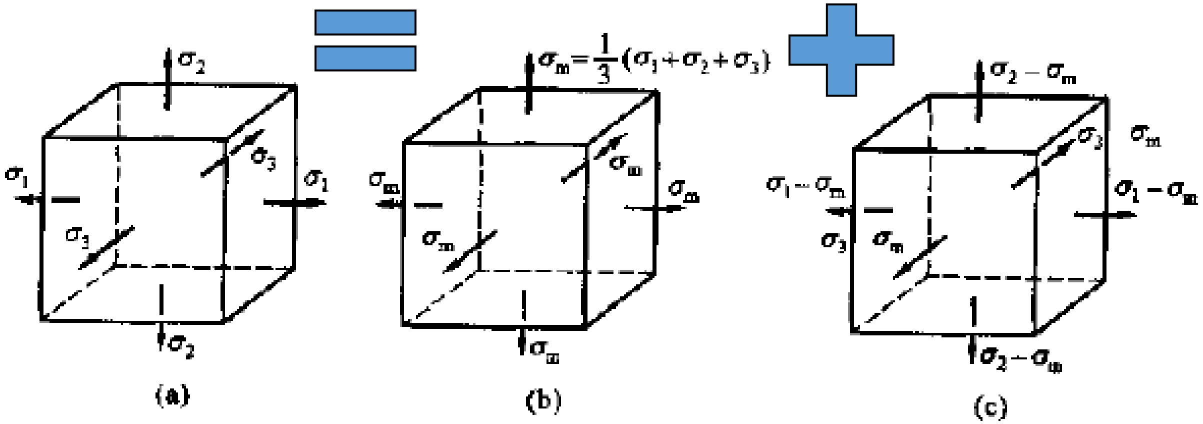
- This theory proposes that the total strain energy can be separated into two components: the volumetric (hydrostatic) strain energy and the shape (distortion or shear) strain energy. It is proposed that yield occurs when the distortion component exceeds that at the yield point for a simple tensile test. This theory is also known as the *von Mises yield criterion*.

- For the state of principle stress, the strain energy:

$$v_\varepsilon = \frac{1}{2}(\sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_3\varepsilon_3) = \frac{1}{2E}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right)$$

Strain Energy=Volume Change Energy + Distortion Energy

$$v_m = \frac{1}{2E}\left(\sigma_m^2 + \sigma_m^2 + \sigma_m^2 - 2\mu(\sigma_m^2 + \sigma_m^2 + \sigma_m^2)\right) = \frac{3(1-2\mu)}{2E}\sigma_m^2 = \frac{1-2\mu}{6E}(\sigma_1 + \sigma_2 + \sigma_3)^2$$



Foundations of Solid Mechanics

Distortion Energy Theory (Von Mises Yield Criterion)

$$v_m = v_\varepsilon - v_m = \frac{1+\mu}{6E}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]$$

For the material test, $\sigma_1 = \sigma_y, \sigma_2 = \sigma_3 = 0$

$$v_{du} = \frac{1+\mu}{6E} \times 2\sigma_y^2$$

Thus, the yield criterion:

$$\frac{1+\mu}{6E}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right] \le \frac{1+\mu}{6E} \times 2\sigma_y^2$$

$$\boxed{\sqrt{\frac{1}{2}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]} \le \sigma_y}$$

Exercise-2

The state of plane stress shown occurs at a critical point of a steel machine component. As a result of several tensile tests, the tensile yield strength is $\sigma_y = 250$ MPa for the grade of steel used. Determine the factor of safety with respect to yield using the maximum-shearing-stress criterion.