Foundations of Solid Mechanics

E9: Deflection of Beams

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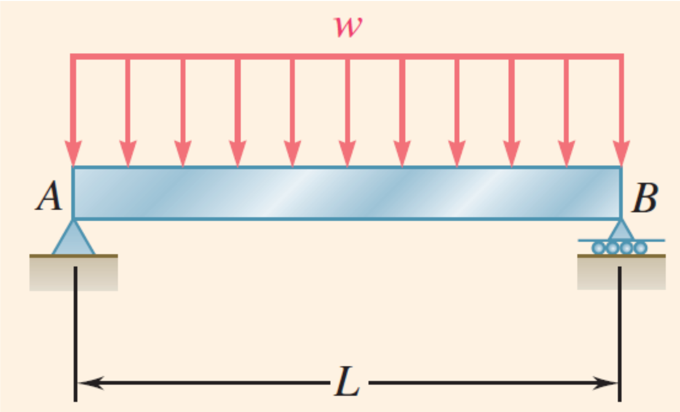
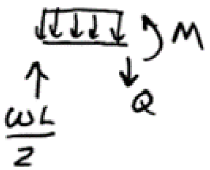
Exercise-1

1. The simply supported prismatic beam AB carries a uniformly distributed load *w* per unit length. Determine the equation of the elastic curve and the maximum deflection of the beam.



(1) determine the equation of $M(x)$



$$\circlearrowleft \sum M_x = 0 \Rightarrow M + wx\cdot\frac{x}{2} - \frac{wL}{2}x = 0$$

$$\Rightarrow M = \frac{wL}{2}x - \frac{w}{2}x^2$$

(2) determine the elastic curve

$$-EIy'' = M(x) = \frac{wL}{2}x - \frac{w}{2}x^2$$

$$EIy'' = \frac{w}{2}x^2 - \frac{wL}{2}x$$

$$EIy' = \frac{w}{6}x^3 - \frac{wL}{4}x^2 + C_1$$

- when $x = L/2$, $y' = 0 \rightarrow \frac{w}{6}\cdot\frac{L^3}{8} - \frac{wL}{4}\cdot\frac{L^2}{4} + C_1 = 0$

$$\rightarrow C_1 = \frac{1}{24}wL^3$$

$$EIy = \frac{w}{24}x^4 - \frac{wL}{12}x^3 + \frac{wL^3}{24}x + C_2$$

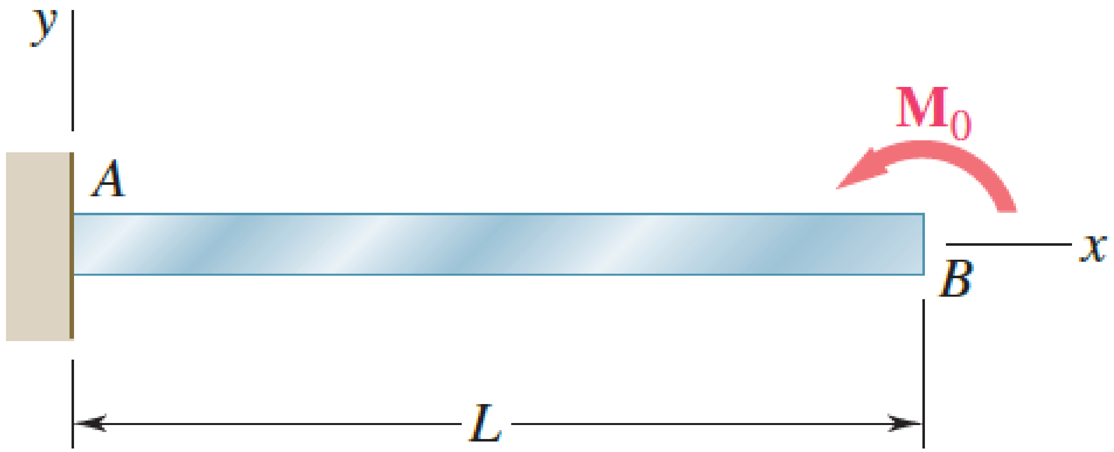
- when $x = 0$, $y = 0 \rightarrow C_2 = 0$

$$\therefore y = \frac{1}{EI}\left(\frac{w}{24}x^4 - \frac{wL}{12}x^3 + \frac{wL^3}{24}x\right)$$

Foundations of Solid Mechanics

Exercise-2

2. For the loading shown, determine (a) the equation of the elastic curve for the cantilever beam AB, (b) the deflection at the free end, (c) the slope at the free end.



$$M(x) = M_0 = -EIy''$$

$$EIy'' = -M_0$$

$$EIy' = -M_0 x + C_1$$

$$EIy = -\frac{M_0}{2}x^2 + C_1 x + C_2$$

when $x=0$, $y=0 \longrightarrow C_2 = 0$

when $x=0$, $y'=0 \longrightarrow C_1 = 0$

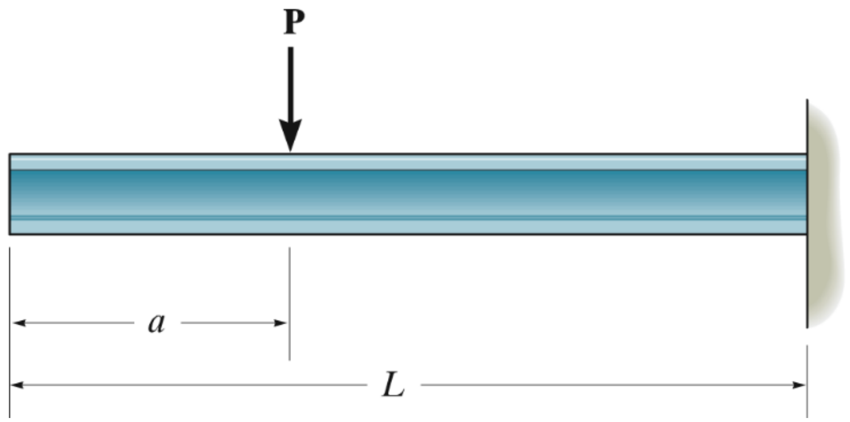
$$\therefore y = -\frac{M_0}{2EI}x^2$$

$$y_B\big|_{x=L} = -\frac{M_0}{2EI}\cdot L^2$$

$$y'_B\big|_{x=L} = -\frac{M_0}{EI}\cdot L$$

Exercise-3

3. Determine the equations of the elastic curve. EI is constant.

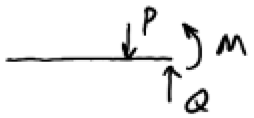


(1) determine bending moment equation

$R_B = P \qquad M_B = -P(L-a) = Pa - PL$

when $0 < x < a$, $\quad M(x) = 0$

when $a < x < L$,



$\circlearrowleft \sum M_x = 0 \rightarrow \quad M + P(x-a) = 0 \rightarrow M = Pa - Px$

(2) determine the elastic curve y

when $0 < x < a$.

$-EIy_1'' = 0 \rightarrow EIy_1' = C_1 \rightarrow EIy_1 = C_1 x + C_2$

when $a < x < L$

$-EIy_2'' = Pa - Px \longrightarrow EIy_2'' = Px - Pa$

$\rightarrow EIy_2' = \frac{P}{2}x^2 - Pax + C_3$

$\rightarrow EIy_2 = \frac{P}{6}x^3 - \frac{Pa}{2}x^2 + C_3 x + C_4$

boundary condition: $x = L.\ y_2' = 0 \rightarrow C_3 = PL(a - \frac{L}{2})$

$x = L,\ y_2 = 0 \rightarrow C_4 = \frac{PL^3}{3} - \frac{PaL^2}{2}$

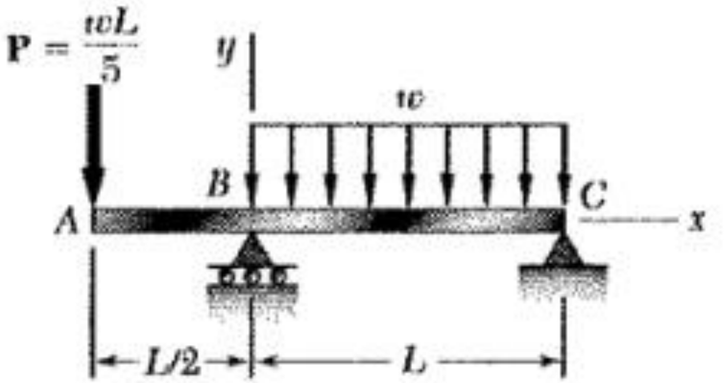
continuity condition: $x = a.\ y_1' = y_2' \rightarrow C_1 = P(-\frac{a^2}{2} + La - \frac{L^2}{2})$

$x = a.\ y_1 = y_2 \rightarrow$

$C_2 = P(-\frac{a^3}{3} + La^2 - aL^2 + \frac{L^3}{3})$

Exercise-4

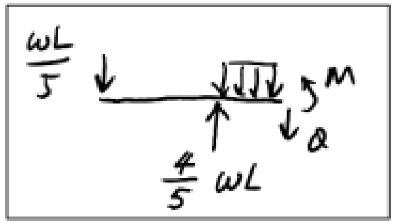
4. For the beam and loading shown, determine (a) the equation of the elastic curve for portion BC of the beam, (b) the deflection at midspan, (c) the slope at B.



(1) determine the $M(x)$

$R_B = \frac{4}{5} wL$

when $0 < x < L$



$$\circlearrowleft \Sigma M_x = 0 \rightarrow M + \frac{wL}{5}\left(x+\frac{L}{2}\right) - \frac{4}{5}wLx + \frac{wx^2}{2} = 0$$

$$M = -\frac{wx^2}{2} + \frac{3}{5}wLx - \frac{1}{10}wL^2 = -EIy''$$

$$EIy' = \int\left(\frac{wx^2}{2} - \frac{3}{5}wLx + \frac{1}{10}wL^2\right)dx$$

$$= \frac{w}{6}x^3 - \frac{3}{10}wLx^2 + \frac{1}{10}wL^2x + C_1$$

$$EIy = \frac{w}{24}x^4 - \frac{1}{10}wLx^3 + \frac{1}{20}wL^2x^2 + C_1x + C_2$$

when $x = 0$, $y = 0$, $C_2 = 0$

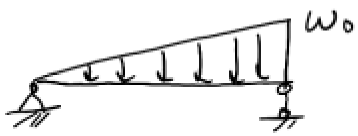
$x = L$, $y = 0$, $C_1 = \frac{1}{120}wL^4$

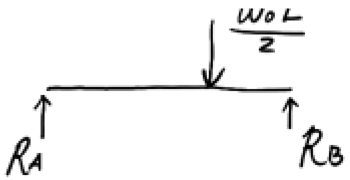
$$y = \frac{1}{EI}\left(\frac{w}{24}x^4 - \frac{wL}{10}x^3 + \frac{wL^2}{20}x^2 + \frac{wL^4}{120}x\right)$$

$$y\big|_{x=\frac{L}{2}} = \frac{13wL^4}{1920EI} \qquad y'\big|_{x=0} = \frac{wL^3}{120EI}$$

Exercise-5

5. For the beam and loading shown, express the elastic curve in terms of w_0, L, E, and I.





$R_A = \frac{w_0 L}{6}$ $\quad R_B = \frac{w_0 L}{3}$

$-EIy'' = M(x)$

$-EIy''' = Q(x)$

$-EIy'''' = q(x) = -\frac{w_0}{L}\cdot x$

$$EIy''' = \int\left(\frac{w_0}{L}x\right)dx = \frac{w_0}{2L}x^2 + C_1$$

$$EIy'' = \int\left(\frac{w_0}{2L}\cdot x^2 + C_1\right)dx = \frac{w_0}{6L}x^3 + C_1 x + C$$

$$EIy' = \frac{w_0}{24L}x^4 + \frac{C_1}{2}x^2 + C_2 x + C_3$$

$$EIy = \frac{w_0}{120L}x^5 + \frac{C_1}{6}x^3 + \frac{C_2}{2}x^2 + C_3 x + C_4$$

when $x = 0,\ y = 0 \rightarrow C_4 = 0$

shear force $x = 0,\ y''' = \frac{w_0 L}{6} \rightarrow \frac{w_0}{2L}\cdot 0 + C_1 = \frac{w_0 L}{6}$

$\rightarrow C_1 = -\frac{w_0 L}{6}$

$x = 0,\ y'' = 0 \rightarrow C_2 = 0$

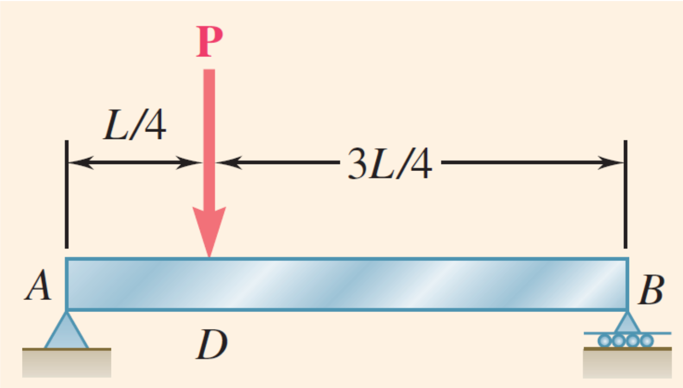
$x = L,\ y = 0 \rightarrow \frac{w_0 L^5}{120L} - \frac{w_0 L^4}{36} + C_3 L = 0$

$C_3 = \frac{7}{360}w_0 L^3$

$$y = \frac{1}{EI}\left(\frac{w_0}{120L}x^5 - \frac{w_0 L}{36}x^3 + \frac{7w_0 L^3}{360}x\right)$$

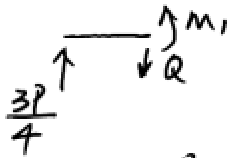
Exercise-6

6. For the prismatic beam and load shown, determine the slope and deflection at point D.



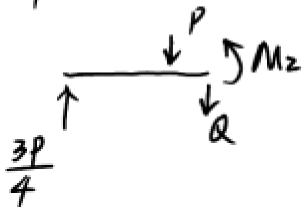
$R_A = \frac{3}{4}P \qquad R_B = \frac{P}{4}$

$0 < x < \frac{L}{4},$



$M_1 = \frac{3P}{4}x$

$\frac{L}{4} < x < \frac{3L}{4}$



$M_2 = \frac{3P}{4}x - Px + \frac{PL}{4}$

$= \frac{PL}{4} - \frac{P}{4}x$

$-EIy_1'' = M_1(x) = \frac{3P}{4}x \qquad -EIy_2'' = \frac{PL}{4} - \frac{P}{4}x$

$EIy_1' = -\frac{3}{8}Px^2 + C_1 \qquad EIy_2' = \frac{P}{8}x^2 - \frac{PL}{4}x + C_3$

$EIy_1 = -\frac{P}{8}x^3 + C_1x + C_2 \qquad EIy_2 = \frac{P}{24}x^3 - \frac{PL}{8}x^2 + C_3x + C_4$

① $x = 0 \quad y_1 = 0$ ② $x = L, \quad y_2 = 0$

③ $x = L/4. \quad y_1 = y_2$

④ $x = L/4. \quad y_1' = y_2'$

$C_2 = 0, \quad C_1 = \frac{7PL^2}{128}, \quad C_3 = \frac{11PL^2}{128}, \quad C_4 = -\frac{PL^3}{384}$

$\Rightarrow \theta_D = \frac{PL^2}{32EI} \qquad y_D = \frac{3PL^3}{256EI}$