Foundations of Solid Mechanics

L4: Bending

Department of Civil Engineering
School of Engineering
Aalto University

Shear and Bending Diagrams

- In order to properly design a beam, it therefore becomes necessary to determine the *maximum* shear and bending moment in the beam. One way to do this is to express V and M as functions of their arbitrary position x along the beam's axis. These *shear and moment functions* can then be plotted and represented by graphs called ***shear and moment diagrams***.

- Draw the shear and moment diagram for the beam shown in below.

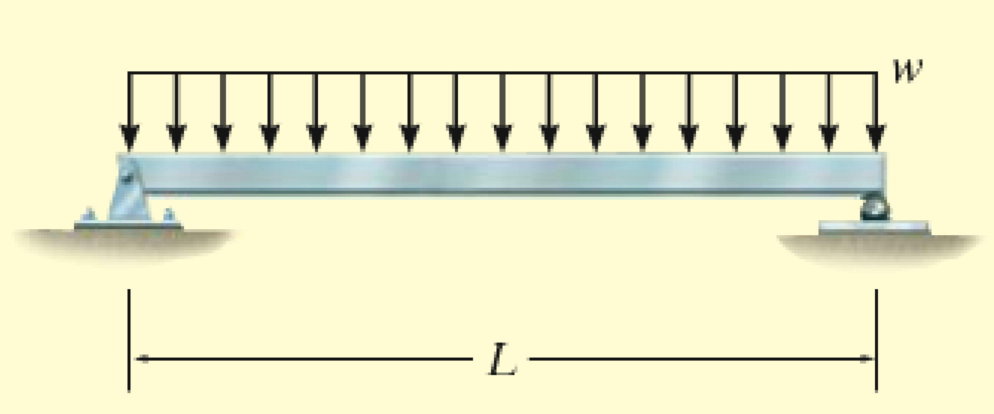
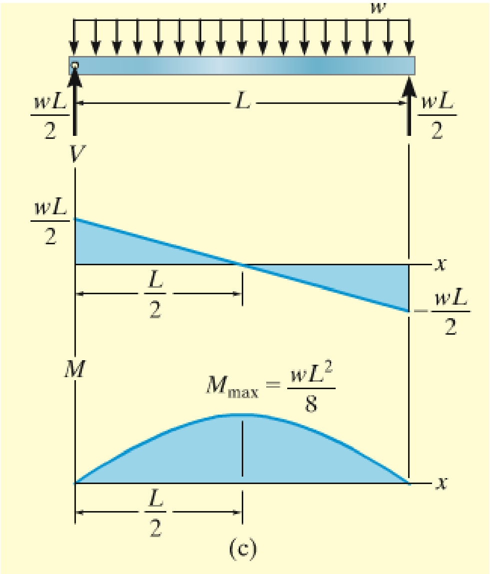


Fig.4.1



Bending Deformation of a Straight Member

- The bending moment causes the material within the *bottom portion* of the bar to *stretch* and the materials within the *top portion* to *compress*. Consequently, between these two regions there must be a surface, called the *neutral surface*, in which longitudinal fibers of the material will not undergo a change in length.

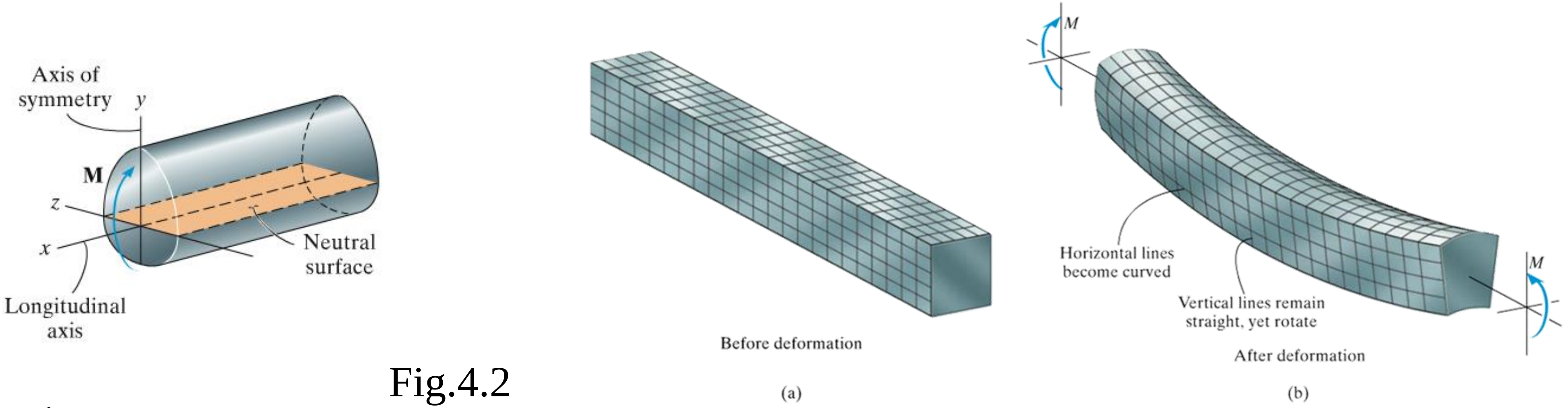


Fig.4.2

- Assumptions:
  - The length of longitudinal axis in the neutral surface remains unchanged.
  - All *cross sections* of the beam *remain plane* and perpendicular to the longitudinal axis during the deformation.
  - *Any deformation* of the cross section within its plane will be *neglected*.

Foundations of Solid Mechanics

Bending Deformation of a Straight Member

- Notice that any line segment Δx, located on the neutral surface, does not change its length, whereas any line segment Δs, located at the arbitrary distance y above the neutral surface, will contract and become Δs`, after deformation. By definition, the normal strain along Δs is determined:

$$\varepsilon = \lim_{\Delta s \to \infty} \frac{\Delta s' - \Delta s}{\Delta s} \qquad (1)$$

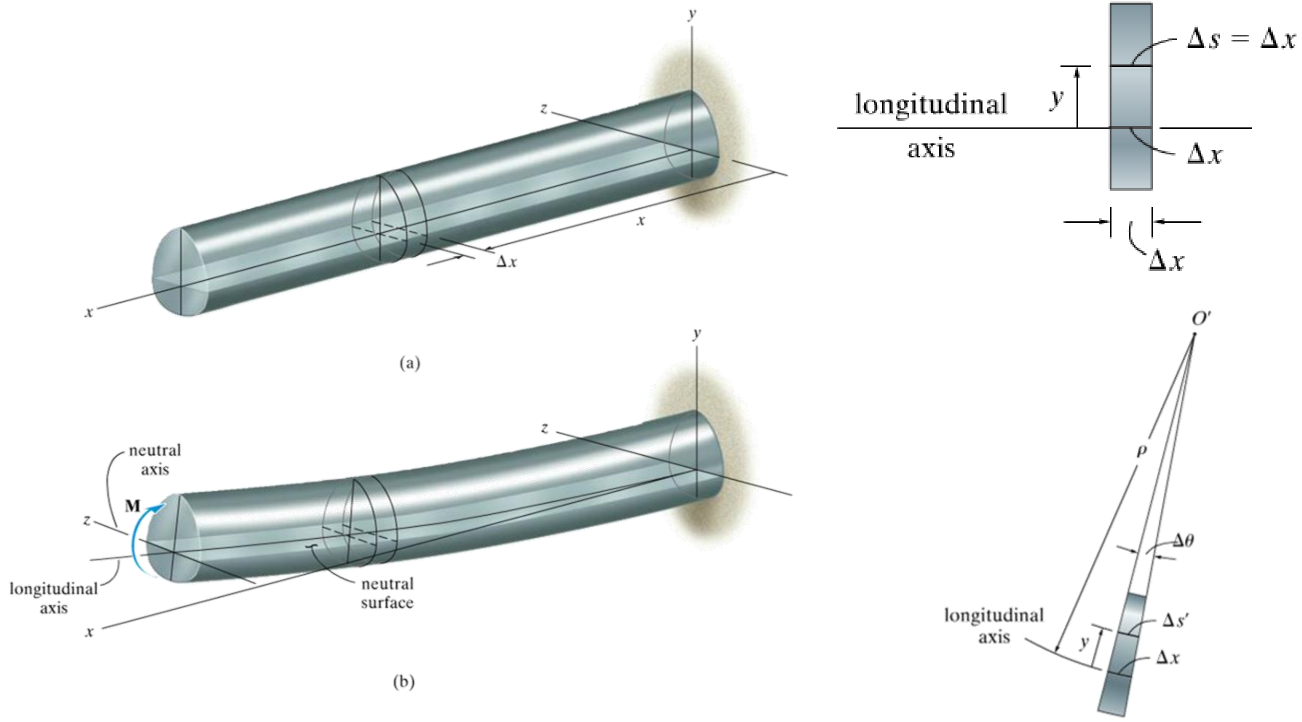


Fig.4.3

Foundations of Solid Mechanics

Bending Deformation of a Straight Member

- Before deformation: $\Delta s = \Delta x$
- After deformation, Δx has a radius of curvature ρ, with center of curvature at point O`. since $\Delta\theta$ defines the angle between the sides of the element, $\Delta x = \Delta s = \rho\Delta\theta$. In the same manner, the deformed length of Δs becomes $\Delta s' = (\rho - y)\Delta\theta$. Substituting into the above equation, we get:

$$\varepsilon = \lim_{\Delta\theta\to 0} \frac{(\rho - y)\Delta\theta - \rho\Delta\theta}{\rho\Delta\theta} \qquad \text{or} \qquad \varepsilon = -\frac{y}{\rho} \qquad (2)$$

This results indicate that the longitudinal normal strain of any element within the beam depends on its location y on the cross section and the radius of curvature of the beam's longitudinal axis at the point.

The maximum strain occurs at the outermost fiber, locates a distance of $y = c$ from the neutral axis. Then:

$$\frac{\varepsilon}{\varepsilon_{max}} = \frac{y/\rho}{c/\rho} \qquad \text{So that:} \qquad \varepsilon = \frac{y}{c}\varepsilon_{max} \qquad (3)$$

The Flexure Formula

- Assumptions:

- Materials behaves in a linear-elastic manner and therefore a linear variation of normal strain, then must be result of a linear variation in normal stress. Hence:

$$\sigma = \frac{y}{c}\sigma_{max} \tag{4}$$

- As the resultant force produced by the stress distribution over the cross-sectional area must be equal to zero. Noting that the force $dF=\sigma dA$ acts on the arbitrary element dA, we require:

$$F_R = \sum F_x$$

$$= \int_A dF = \int_A \sigma dA = \int_A \frac{y}{c}\sigma_{max} dA = \frac{\sigma_{max}}{c}\int_A y dA \tag{5}$$

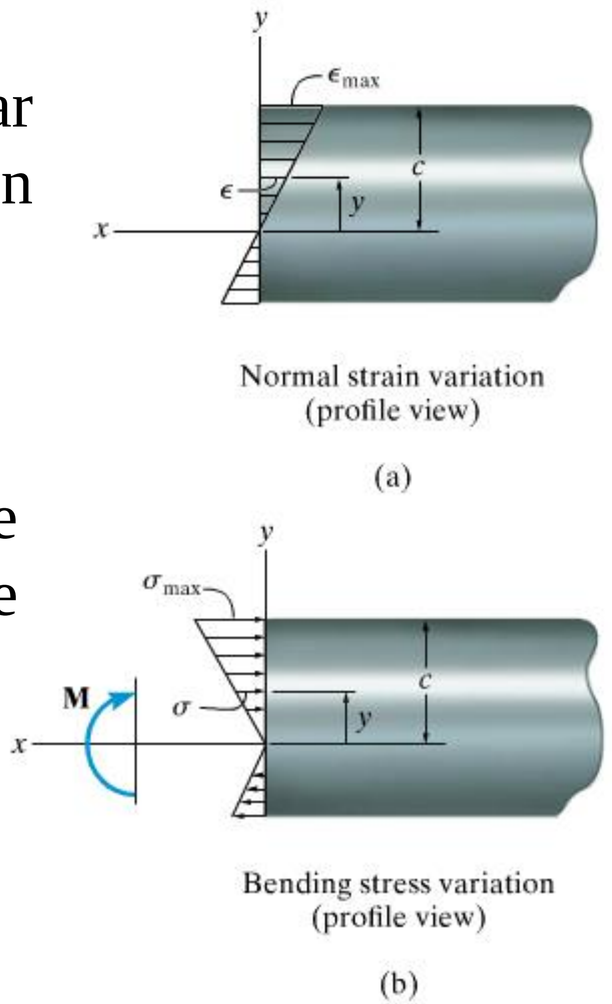


Fig.4.4

The Flexure Formula

Since σ_{max}/c is not equal to zero, then:

$$\int_A y dA = 0 \qquad (6)$$

This condition can only be satisfied if the neutral axis is also the horizontal **centroid** axis for the cross-sectional area. Consequently, once the centroid for the member's cross-sectional area is determined, the location of the neutral axis is known.

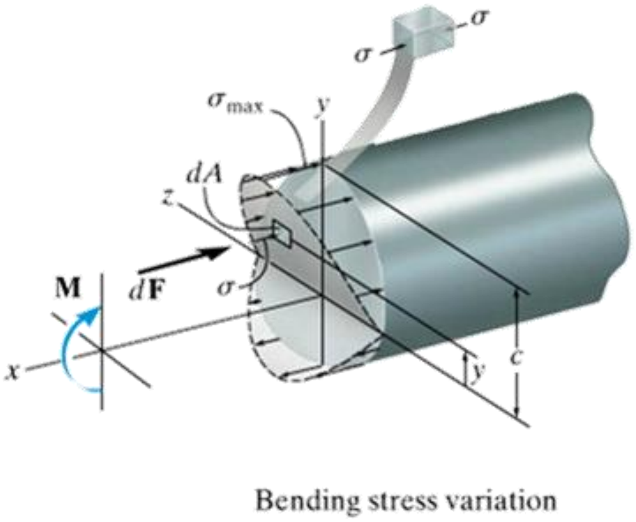


Fig.4.5

We can determine the stress in the beam from the requirement that the resultant internal moment M must be equal to the moment produced by the stress distribution about the neutral axis. The moment of dF about the neutral axis is $dM = ydF$. Since $dF = \sigma dA$, we have for the entire cross section,

$$M = \int_A ydF = \int_A y(\sigma dA) = \int_A y\left(\frac{y}{c}\sigma_{max}\right) dA \quad \Rightarrow \quad M = \frac{\sigma_{max}}{c}\int_A y^2 dA$$

The Flexure Formula

The integral represents the moment of inertia if the cross-sectional area about the neutral axis. We will symbolize its value as I. Hence:

$$\sigma_{max} = \frac{Mc}{I} \qquad (7)$$

Here:

$\sigma_{max} =$ the maximum normal stress in the member, which occurs at a point on the cross-sectional area farthest away from the neutral axis.

$M =$ the resultant internal moment, determined from the method of sections and the equation of equilibrium, and calculated about the neutral axis of the cross section

$c =$ the perpendicular distance from the neutral axis to a point farthest away from the neutral axis. This is here σ_{max} acts

$I =$ the moment of inertia of the cross-sectional area about the neutral axis.

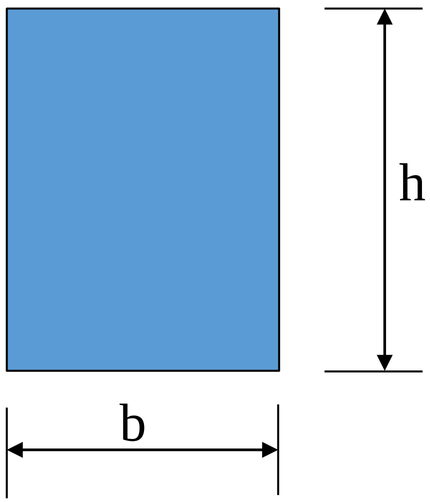
Since $\sigma_{max}/c = \sigma/y$, the normal stress at the intermediate distance y can be determined from the equation:

$$\sigma = \frac{My}{I} \qquad (8)$$

The Area Moment of Inertia

For rectangular cross-section:

$$I_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} by^2 dy = \frac{bh^3}{12}$$



For circular cross-section:

$$I_x = I_y = \frac{\pi d^4}{64}$$

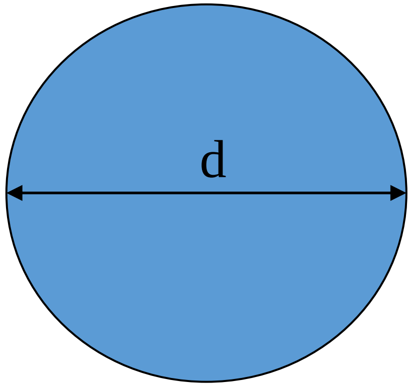


Fig.4.6

Example-1

Draw the shear and moment diagrams for the beam shown in Figure.

Support Reactions.

R_A=5.75 kN R_C=34.25 kN

Shear and Moment Functions.

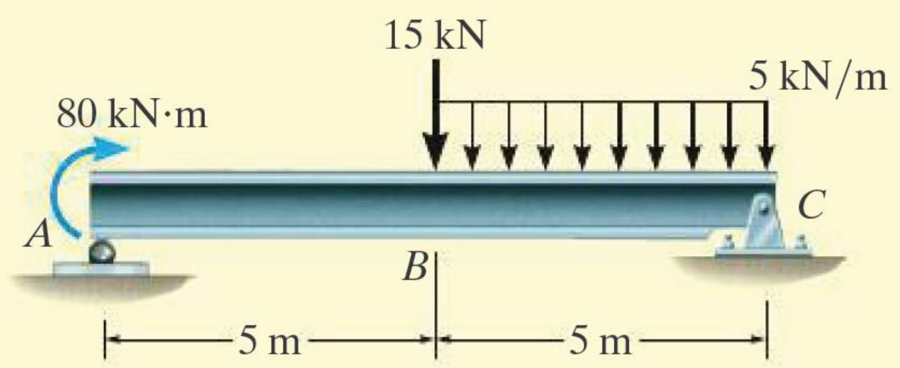
$0 \leq x_1 < 5$ m, Fig. 6–7*b*:

$$+\uparrow \Sigma F_y = 0; \qquad 5.75 \text{ kN} - V = 0$$

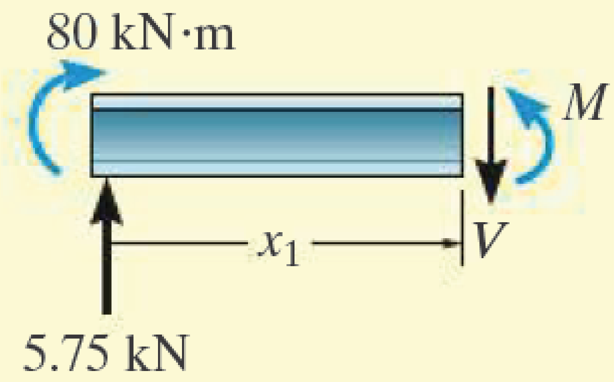
$$V = 5.75 \text{ kN} \qquad (1)$$

$$\circlearrowleft + \Sigma M = 0; \qquad -80 \text{ kN}\cdot\text{m} - 5.75 \text{ kN}\, x_1 + M = 0$$

$$M = (5.75x_1 + 80) \text{ kN}\cdot\text{m} \qquad (2)$$

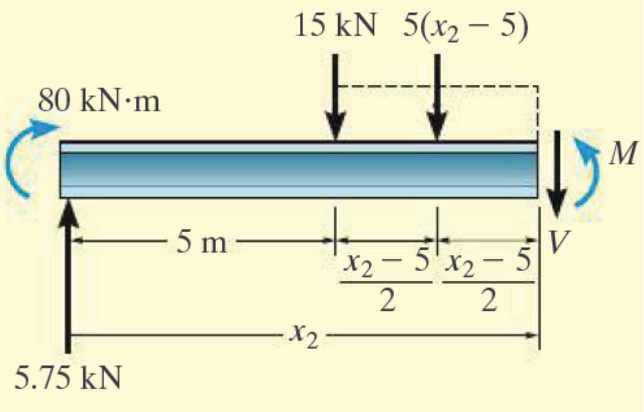


(a)



(b)

Example-2



(c)

$5 \text{ m} < x_2 \leq 10 \text{ m}$, Fig. 6–7*c*:

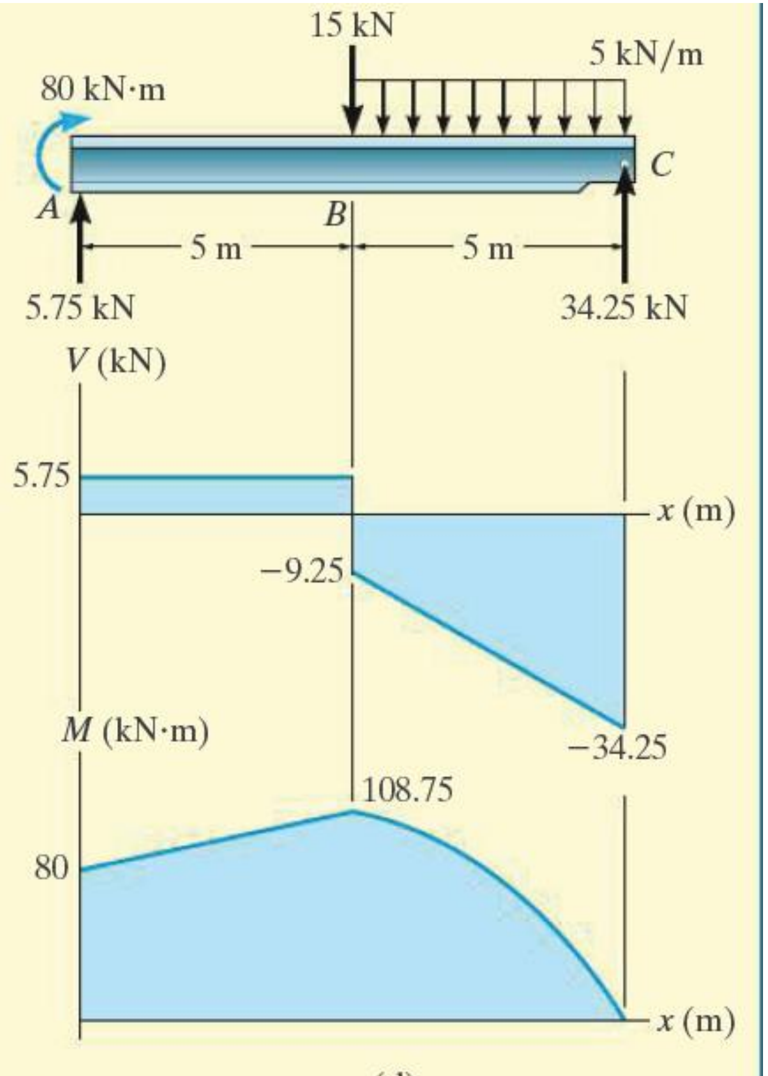
$$+\uparrow \Sigma F_y = 0; \qquad 5.75 \text{ kN} - 15 \text{ kN} - 5 \text{ kN/m}(x_2 - 5 \text{ m}) - V = 0$$

$$V = (15.75 - 5x_2) \text{ kN} \qquad (3)$$

$$\circlearrowleft + \Sigma M = 0; \qquad -80 \text{ kN}\cdot\text{m} - 5.75 \text{ kN}\, x_2 + 15 \text{ kN}(x_2 - 5 \text{ m})$$

$$+ 5 \text{ kN/m}(x_2 - 5 \text{ m})\left(\frac{x_2 - 5 \text{ m}}{2}\right) + M = 0$$

$$M = (-2.5x_2^2 + 15.75x_2 + 92.5) \text{ kN}\cdot\text{m} \qquad (4)$$

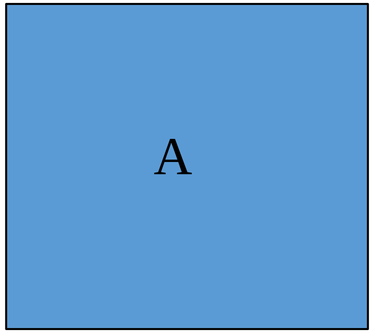


(d)

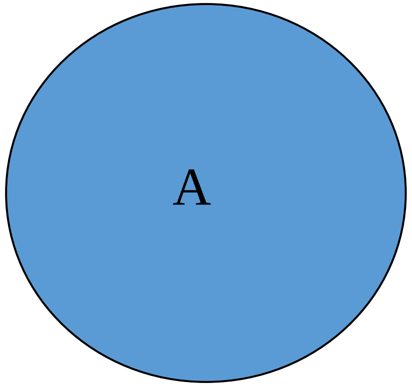
Foundations of Solid Mechanics

Exercise-1

Beam-1 and Beam-2 have the same cross sectional area of A but different cross-section shapes (square cross section in beam-1 and circular cross section in beam-2). Which beam has a larger I?



(a) Cross section in beam-1

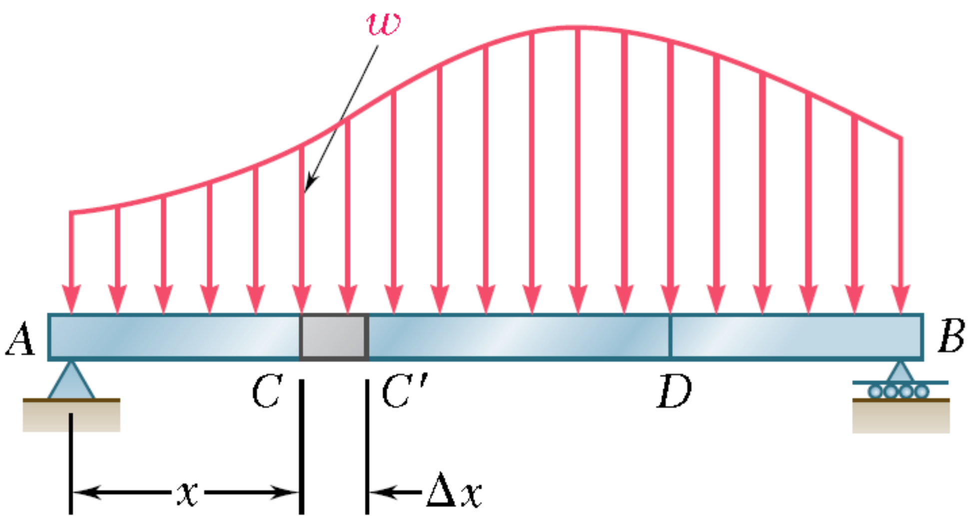


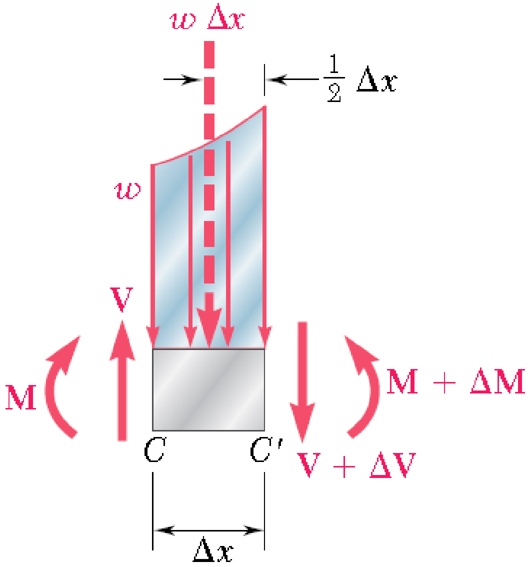
(b) Cross section in beam-2

Relations among Load, Shear and Bending moment

Foundations of Solid Mechanics

Regions of Distributed Load





Consider a beam subject to bending and transverse shear. At some distance along the x direction further consider a short length Δx. Over this length the bending moment increases by ΔM and the shear force increases by ΔV.

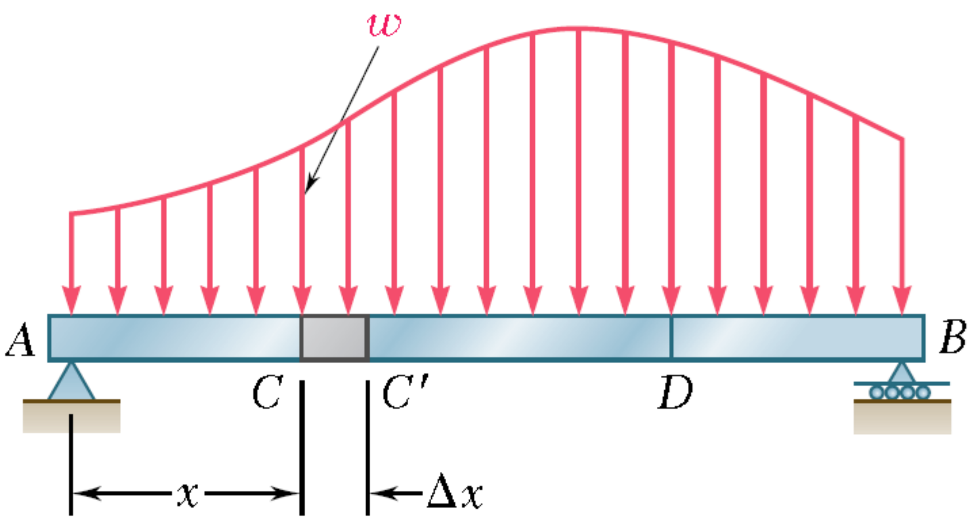
$$+\uparrow \sum F_y = 0 \Rightarrow V - \omega \Delta x = V + \Delta V$$

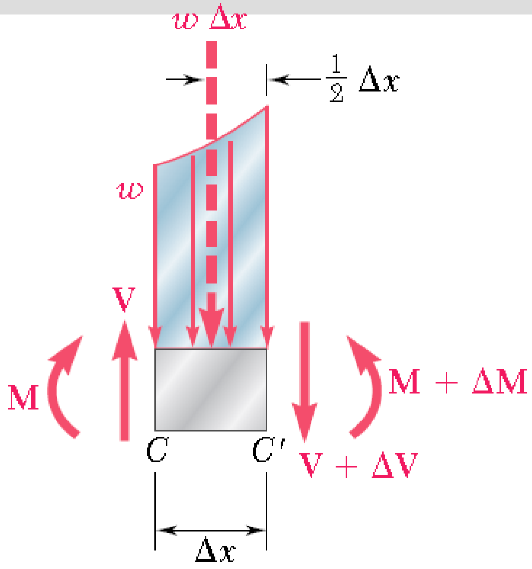
$$\Rightarrow \frac{dV}{dx} = -\omega$$

Integrating between points C and D: $V_D - V_C = -\int_{x_C}^{x_D} \omega dx$

V_D-V_C=-(area under load curve between C and D)

Regions of Distributed Load





$$\Sigma M_{c'} = 0 \Rightarrow (M + \Delta M) - M - V\Delta x + w\Delta x \frac{\Delta x}{2} = 0$$

$$\Rightarrow \Delta M = V\Delta x - \frac{1}{2}\omega(\Delta x)^2 \Rightarrow \frac{dM}{dx} = V$$

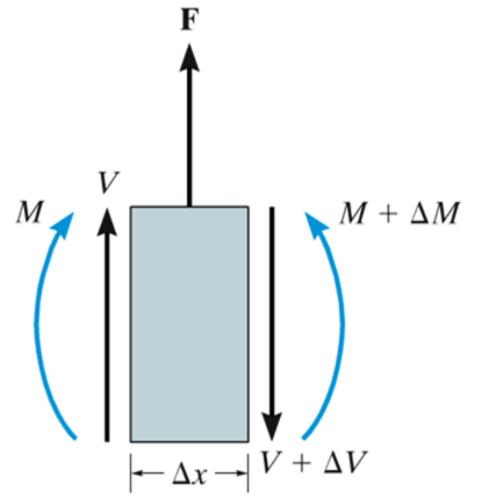
Integrating between points C and D: $M_D - M_C = \int_{x_C}^{x_D} V\,dx$

M_D-M_C=-(area under shear force between C and D)

Regions of Concentrated Force and Moment

$$+\uparrow \sum F_y = 0 \quad \Rightarrow \quad V + F = V + \Delta V$$

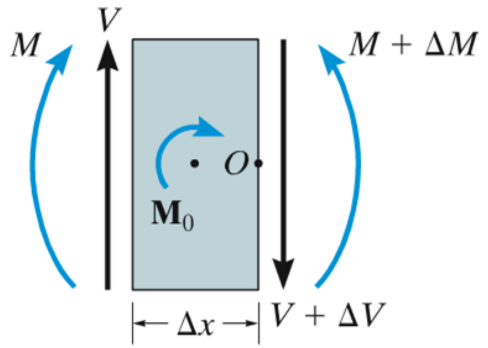
$$\Rightarrow \quad \Delta V = F$$



Thus, when F acts upwards on the beam, ΔV is positive so the shear will jump "upwards". Otherwise, if F acts downwards, the jump will be downward.

$$\sum M = 0 \quad \Rightarrow \quad M + M_o + V\Delta x = M + \Delta M$$

$$\Delta x \to 0 \quad \Rightarrow \quad \Delta M = M_o$$



(b)

Thus, when M_O is applied clockwise, ΔM is positive so the moment diagram will jump "upwards". Otherwise, if M_O acts counterclockwise, the jump will be downward.

Foundations of Solid Mechanics

Procedure for Analysis

Support Reactions

- Determine the support reactions and resolve the forces acting on the beam into components that are perpendicular and parallel to the beam's axis.

Shear Force Diagrams (SFD)

- Establish the V and x axes and plot the known value of the shear force at two *ends* of the beams;
- Notice how the values of the distributed load vary along the beam, and realize that each of these values indicate the way the shear diagram will slope ($dV/dx=w$). Here w is positive when its acts upward.
- If a numerical value of the shear is to be determined at a point, one can find this value either by using the method of sections and the equation of force equilibrium, or by using $\Delta V = \int \omega(x)$, which states that the change in the shear between any two points is equal to *the area under the load diagram* between the two points.

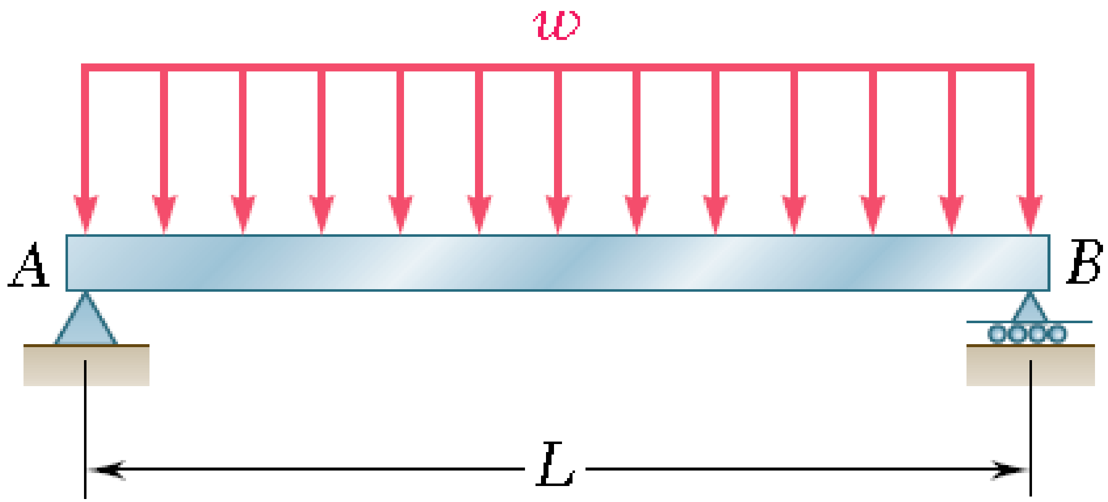
Procedure for Analysis

Bending Moment Diagrams (BMD)

- Establish the M and x axes and plot the known value of the moment at the *ends* of the beams;
- Notice how the values of the shear diagram vary along the beam, and realize that each of these values indicate the way the moment diagram will slope ($dM/dx=V$).
- At the point where the shear is zero, $dM/dx=0$, and therefore this would be a point of maximum or minimum moment.
- If a numerical value of the shear is to be determined at a point, one can find this value either by using the method of sections and the equation of force equilibrium, or by using , which states that the *change in moment* between any two points is equal to *the area under the shear diagram* between the two points.
- Since w(x) must be integrated to obtain ΔV, and V(x) is integrated to obtain M(x), then if w(x) is a curve of degree n, V(x) will be a curve of degree n+1, and M(x) will be a curve of degree n+2. for example, if w(x) is uniform, V(x) will be linear, and M(x) will be parabolic,

Examples-2

Draw the shear and bending-moment diagrams for the simply supported beam shown in below and determine the maximum value of the bending moment.



Example-2 (Method-1)

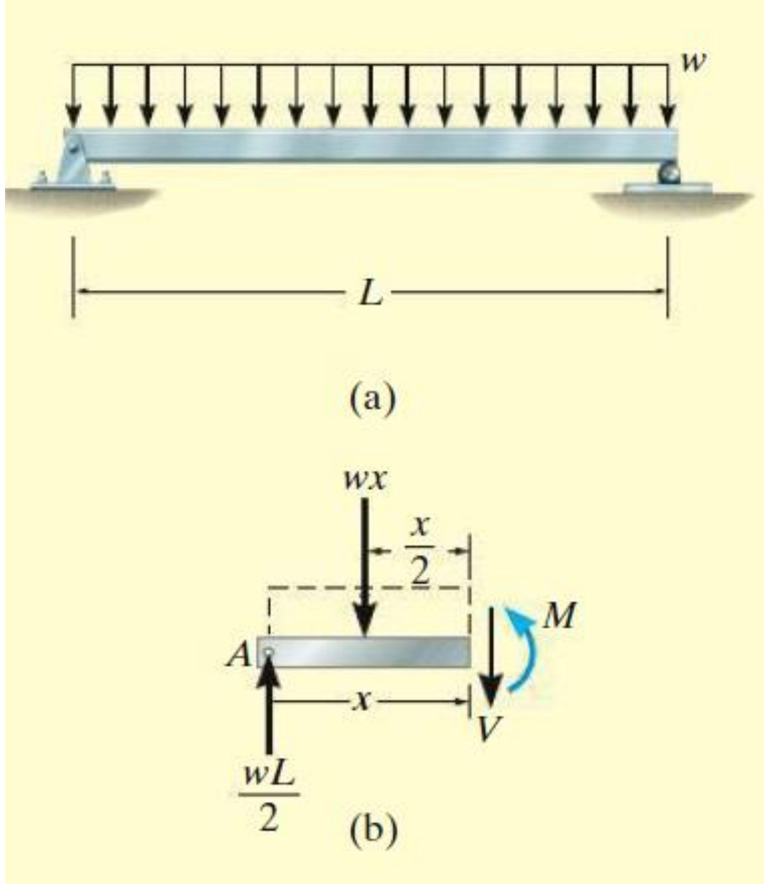
Draw the shear and moment diagrams for the beam shown in Figure.

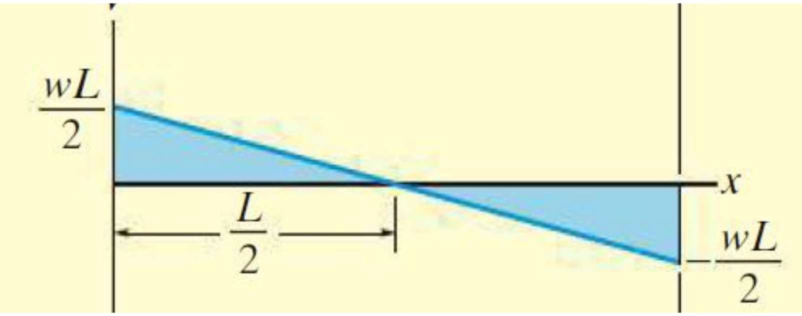
$$+\uparrow \Sigma F_y = 0; \qquad \frac{wL}{2} - wx - V = 0$$

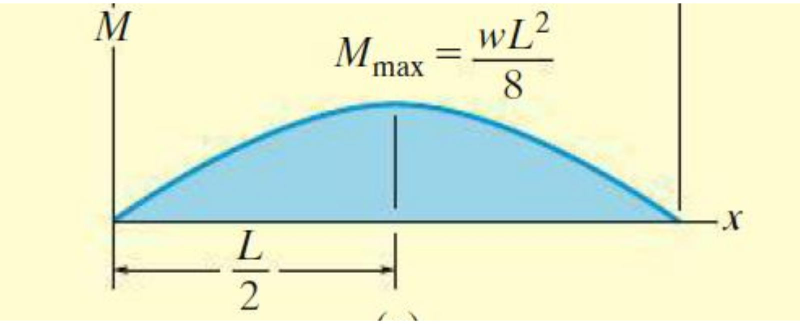
$$V = w\left(\frac{L}{2} - x\right) \qquad (1)$$

$$\circlearrowleft + \Sigma M = 0; \qquad -\left(\frac{wL}{2}\right)x + (wx)\left(\frac{x}{2}\right) + M = 0$$

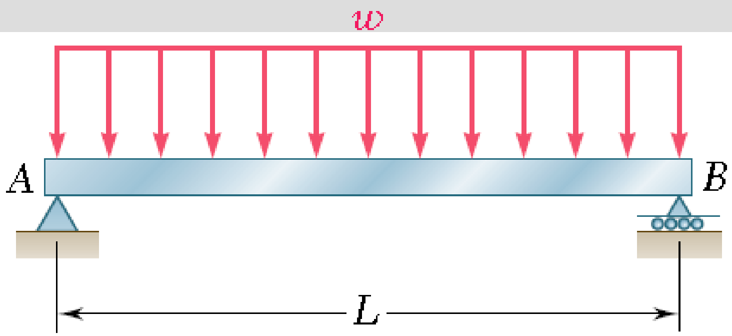
$$M = \frac{w}{2}(Lx - x^2) \qquad (2)$$

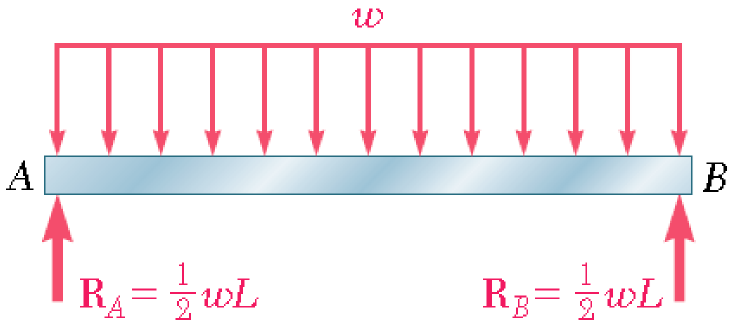






Examples-2 (Method-2)





- Reaction force:

$$R_A = R_B = \frac{wl}{2}$$

- Shear force:

$$V - V_A = -\int_0^x w\,dx = -wx$$

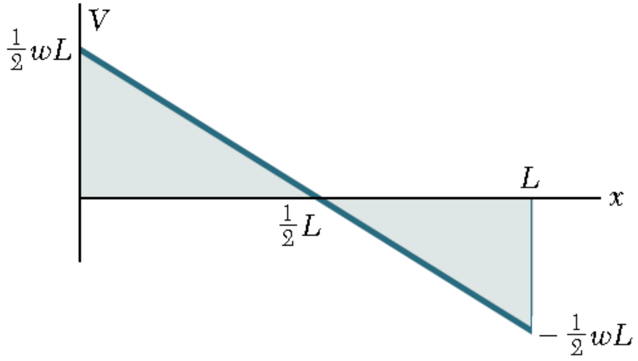
$$V = V_A - wx = \frac{1}{2}wl - wx = w(\frac{1}{2}l - x)$$

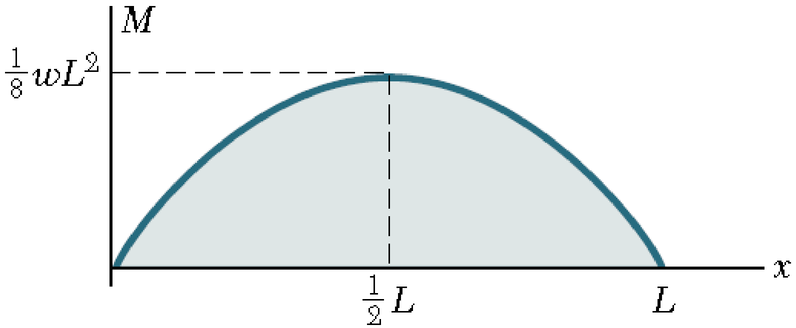
- Bending moment

$$M - M_A = \int_0^x V dx$$

$$M = \int_0^x \omega(\frac{1}{2}l - x)dx = \frac{1}{2}\omega(lx - x^2)$$

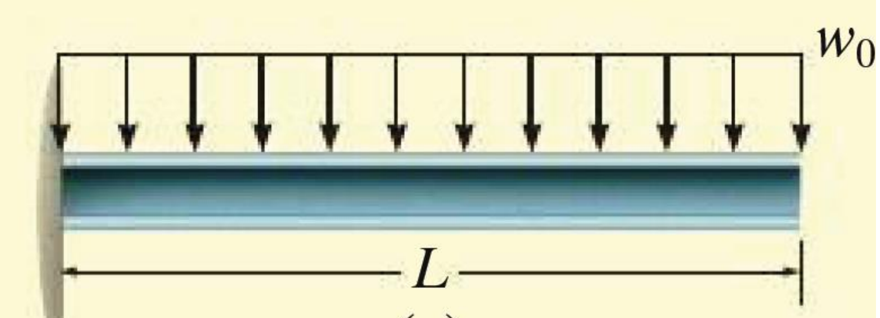
- Shear and Moment Diagrams:



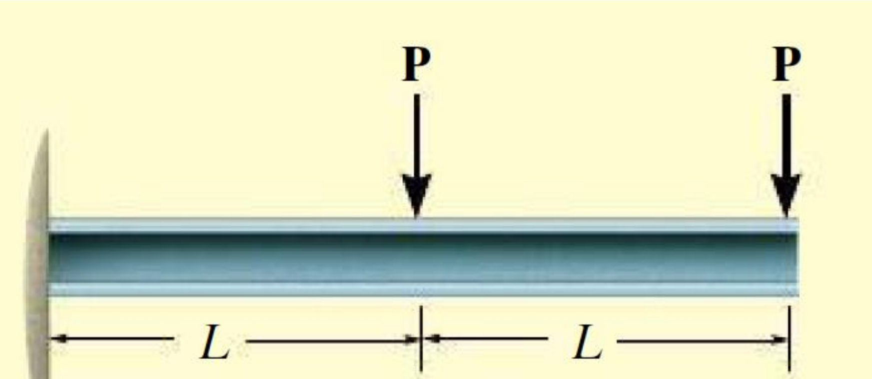


Exercise-2

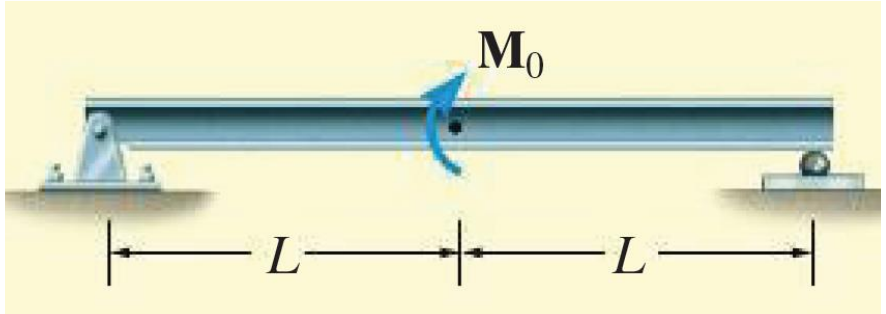
Draw the shear and moment diagrams for the beam shown in Figure.



(a)



(b)



(c)