Foundations of Solid Mechanics

L7: Analysis and Design of Beams

Department of Civil Engineering
School of Engineering
Aalto University

Design of Prismatic Beams for Bending

The design of a beam is usually controlled by the maximum absolute value $\lfloor M \rfloor_{max}$ of the bending moment that will occur in the beam. Then we can determine the maximum normal stress from:

$$\sigma_m = \frac{\lfloor M \rfloor_{max} c}{I} \qquad \sigma_m = \frac{\lfloor M \rfloor_{max}}{S}$$

where: $S = \frac{I}{c}$ is the Elastic section modulus. A safe design requires that $\sigma_m < \sigma_{all}$, where σ_{all} is the allowable stress for the material used. Substituting σ_{all} for σ_m and solving for S yields the minimum allowable value of the section modulus for the beam being designed:

$$S = \frac{\lfloor M \rfloor_{max}}{\sigma_{all}}$$

The design of common types of beams, such as timber beams of rectangular cross section and rolled-steel beams of various cross-sectional shapes, will be considered in this chapter. *A proper procedure should lead to the most economical design.*

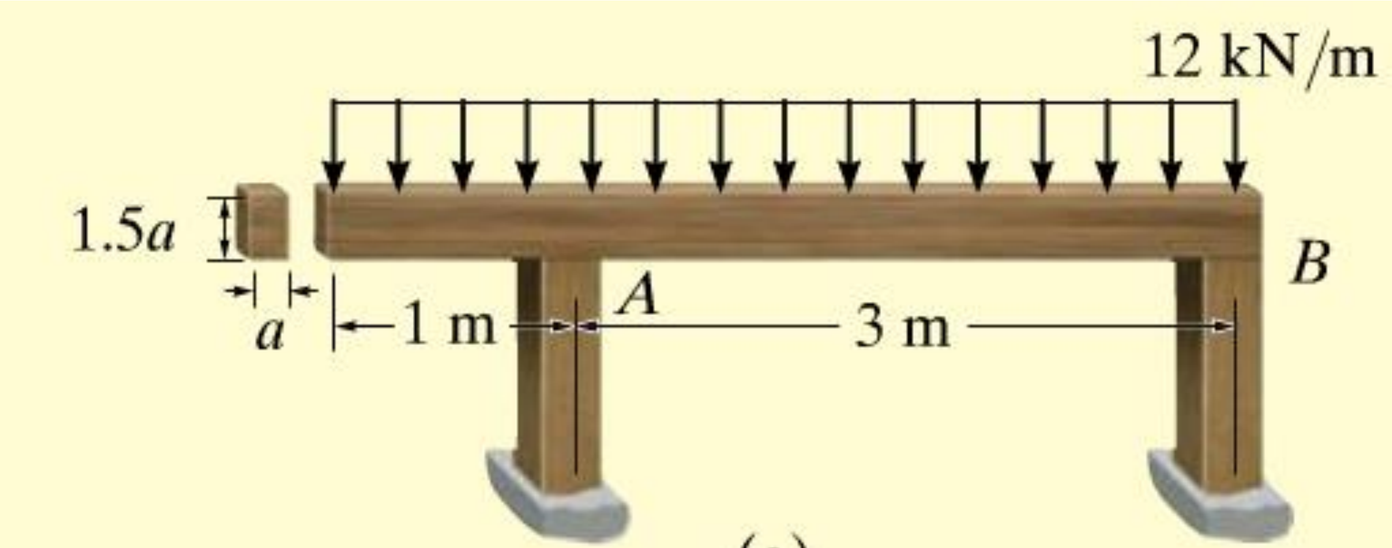
Note: All materials in this handout are used in class for educational purposes only.

Design of Prismatic Beams for Shear

Once the beam has been selected, the shear formula $\tau_{all} > VQ/It$ can then be used to check that the allowable shear stress is not exceeded. Often this requirement will not present a problem. However, if the beam is “short” and supports large concentrated loads, the shear-stress limitation may dictate the size of the problem. This limitation is particularly important in the design of wood beams, because wood tends to spit along its grain due to shear.

Example-1

The laminated wooden beam shown in below supports a uniform distribution loading of 12kN/m. If the beam have a height-to-width ratio of 1.5, determine its smallest width. The allowable bending stress is $\sigma_{all} = 9\ MPa$ and the allowable shear stress is $\tau_{all} = 0.6MPa$. Neglect the weight of the beam.

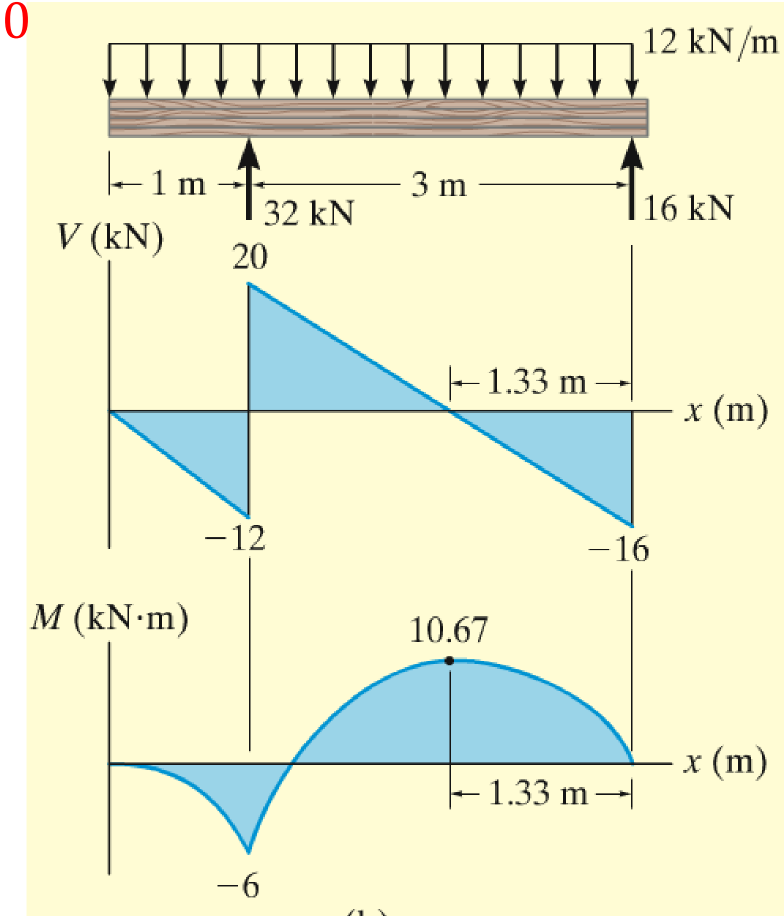


(a)

Example-1

Shear and Moment Diagrams. The support reactions at A and B have been calculated and the shear and moment diagrams are shown in the right figure. Here $V_{max} = 20kN$, $M_{max} = 10.67kN \cdot m$.

Note: how to determine the maximum bending moment? $V = 0$



(b)

Bending Stress. Applying the flexure formula,

$$S_{rep'd} = \frac{M_{max}}{\sigma_{all}} = \frac{10.67\times10^3 N\cdot m}{9\times10^6 N/m^2} = 0.00119\ m^3$$

Assuming that the width is a, then the height is 1.5a. Thus,

$$S_{rep'd} = \frac{I}{c} = 0.00119\ m^3 = \frac{\frac{1}{12}(a)(1.5a)^3}{0.75a}$$

$$a^3 = 0.003160m^3 \Rightarrow a = 0.147m$$

Shear Stress. Applying the shear formula for rectangular sections (which is a special case of $\tau_{max} = VQ/It$), we have

Example-1

$$\tau_{max} = 1.5\frac{V_{max}}{A} = 1.5\frac{20\times10^3 N}{(0.147m)(1.5\times0.147m)} = 0.929MPa > 0.6MPa$$

Since the design details fails the shear criterion, the beam must be redesigned on the basis of the shear.

$$\tau_{max} = 1.5\frac{V_{max}}{A}$$

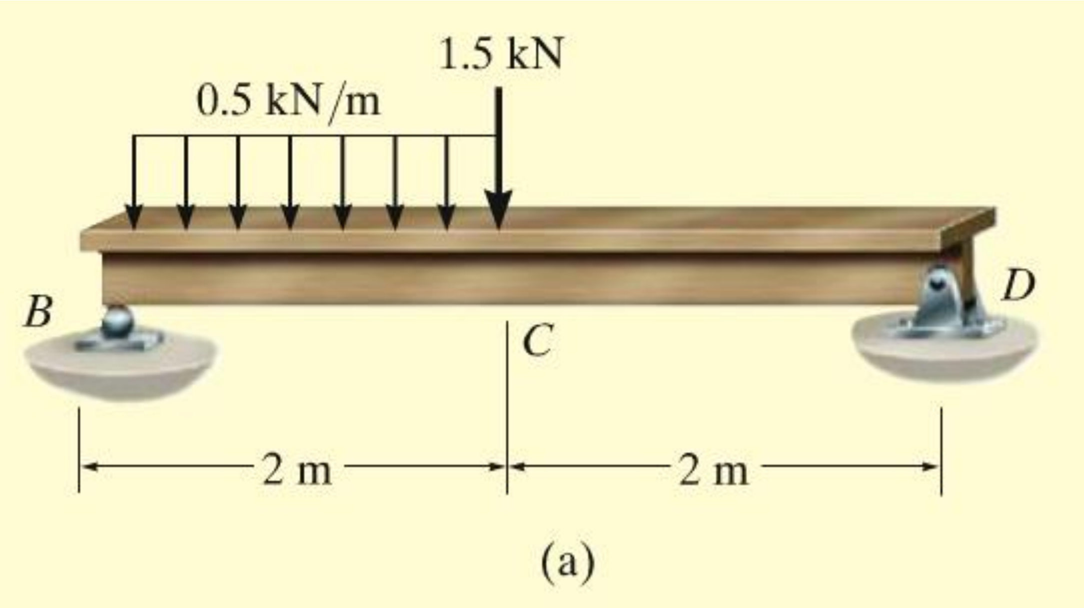
$$\Rightarrow \quad 0.6MPa = 1.5 \times \frac{20 \times 10^3 N}{(a)(1.5a)}$$

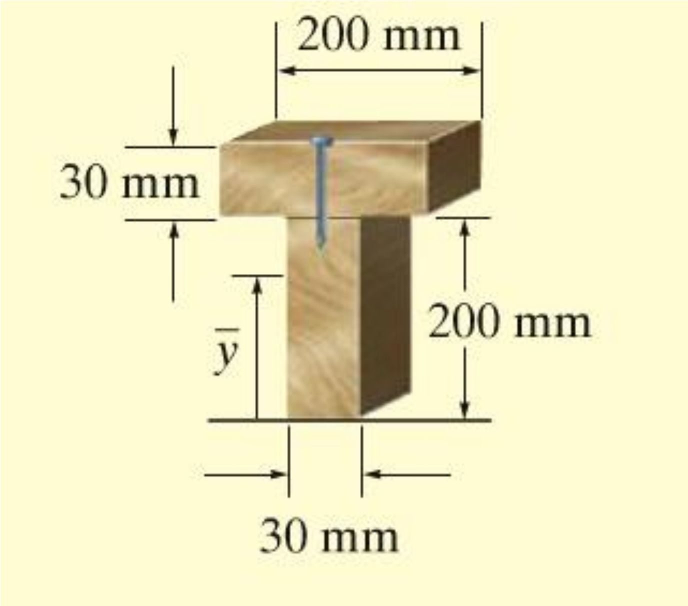
$$\Rightarrow \quad a = 0.183m = 183mm$$

This larger section will also adequately resist the normal stress.

Example-2

The wooden T-beam shown in below is made from two $200 \times 300mm$ boards. If the allowable bending stress is $\sigma_{all} = 12\ MPa$ and the allowable shear stress is $\tau_{all} = 0.8\ MPa$, determine if the beam can safely support the loading as shown. Also, specify the maximum spacing of nails needed to hold the two boards together if each nail can safely resist $1.5kN$ in shear.





Example-2

Shear and Moment Diagrams. The reactions on the beam are shown, and the shear and moment diagrams are shown in below. Here, V_{max}=1.5kN, M_{max}=2kN.m.

Bending Stress. The neutral axis (centroid) will be located from the bottom of the beam. Working in units of meters, we have

$$\bar{y} = \frac{\sum \bar{y}_i A_i}{\sum A_i} = \frac{0.03m\times0.2m\times0.1m+0.03m\times0.2m\times0.215m}{0.03m\times0.2+0.03m\times0.2m} = 0.1575m$$

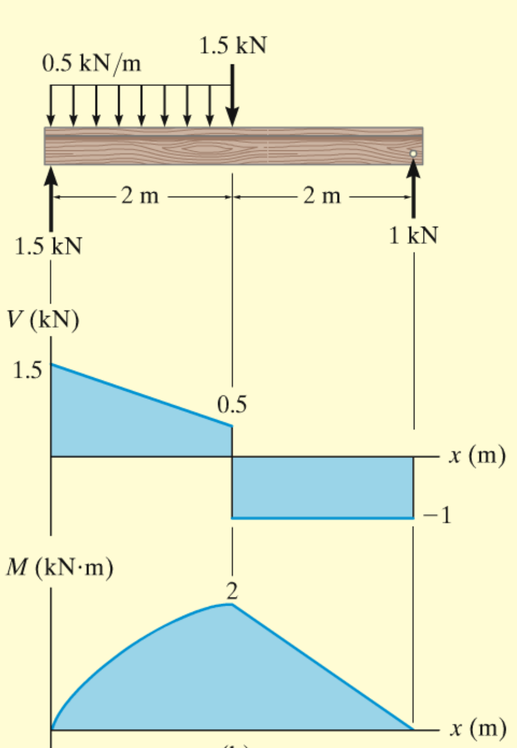
Thus

$$I = \left[\frac{1}{12}(0.03m)(0.2m)^3 + (0.03m)(0.2m)(0.1575m - 0.1m)^2\right] + \left[\frac{1}{12}(0.2m)(0.03m)^3 + (0.2m)(0.03m)(0.1575m - 0.215m)^2\right] = 60.125 \times 10^{-6} m^4$$

Since $c = 0.1575m$ (not 0.230-0.1575m=0.0725m), we require

$$\sigma_{all} = \frac{M_{max}c}{I} \Rightarrow 12MPa \geq \frac{(2\times10^3 N\cdot m)\times(0.1575\text{m})}{60.125\times10^{-6}m^4} = 5.24MPa$$

OK



(b)

Example-2

Shear Stress. Maximum shear stress in the beam depends upon the magnitude of Q and t. it occurs at the neutral axis, since Q is a maximum there and the neutral axis in in the web, where the thickness *t=0.03m* is smallest for the cross-section. For simplicity, we will use the rectangular area below the neutral axis to calculate Q, rather than a two-part composite area above this axis. We have

$$Q = \bar{y}'A' = \left(\frac{0.1575m}{2}\right)[(0.1575m)(0.03m)] = 0.372 \times 10^{-3} m^3$$

So that $$\tau_{all} \geq \frac{V_{max}Q}{It}$$

$$0.8MPa \geq \frac{(1.5\times10^3 N)(0.372\times10^{-3} m^3)}{(60.125\times10^{-6} m^4)(0.03m)} = 0.309\text{MPa}$$ OK

Example-2

Nail Spacing. From the shear diagram it is seen that the shear varies over the entire span. Since the nail spacing depends on the magnitude of shear in the beam, for simplicity (and to be conservative), we will design the spacing based on V=1.5kN for region BC and V=1kN for region CD. Since the nails join the flange to the web, we have

$$Q = \bar{y}`A` = (0.0725\text{m} - 0.015\text{m})[(0.2m)(0.03m)] = 0.345(10^{-3})m^3$$

The shear flow for each region is therefore

$$q_{BC} = \frac{V_{BC}Q}{I} = \frac{1.5(10^3)N[0.345(10^{-3})m^3]}{60.125(10^{-6})m^4} = 8.61k\,N/m$$

$$q_{CD} = \frac{V_{CD}Q}{I} = \frac{1(10^3)N[0.345(10^{-3})m^3]}{60.125(10^{-6})m^4} = 5.74k\,N/m$$

One nail can resist 1.50kN in shear, so the maximum spacing becomes

$$s_{BC} = \frac{1.5kN}{8.61k\,N/m} = 0.174m \qquad s_{CD} = \frac{1.5kN}{5.74k\,N/m} = 0.261m$$

For ease of measuring, use

$$s_{BC} = 0.15m \qquad s_{CD} = 0.25m$$

Design of Composite Beams

Beams constructed of two or more different materials are referred to as *composite beams*. For example, a beam can made of wood with straps of steel at its top and bottom. Engineers purposely design beams in this manner in order to develop a more efficient means for supporting loads.

Since the flexure formula was developed only for beams having homogeneous material, this formula cannot be applied directly to determine the normal stress in a composite beams. In this section, however, we developed a method for modifying or “transforming” a composite beam’s cross section into one made of single material. Once this has been done, the flexure formula can then be used for the stress analysis.

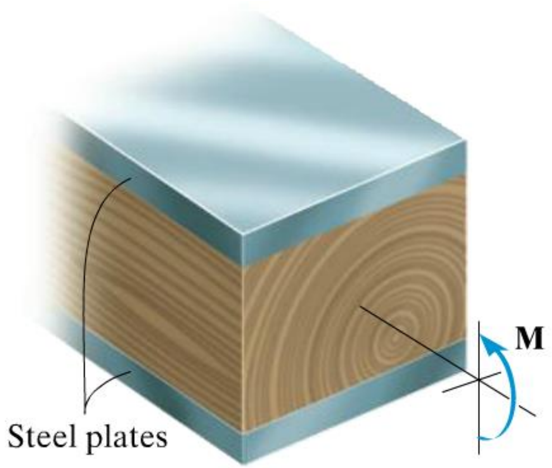


Fig.7-1

Design of Composite Beams

A simple way to solve this problem is to use the *transformed section method*, which transforms the beam into one made of a single material. For example, if the beam is thought to consist entirely of the less stiff material 2, then the cross section will look like that shown in Fig.7-2(e). Here the height h of the beam remains the same, since the strain distribution in Fig.7-2(b) must be preserved. However, the upper portion of the beam must be widened in order to carry a load equivalent to that carried by the stiffer material 1 in Fig.7-2(d). The necessary width can be determined by considering the force dF acting on an area $dA = dzdy$ of the beam in Fig.7-2(a). It is $dF = \sigma dA = (E_1\varepsilon)dzdy$. Assuming the width of a corresponding element of height dy in Fig.7-2(e) is ndz, then $dF' = \sigma' dA' = (E_2\varepsilon)ndzdy$. Equating these forces, so that they produce the same moment about the z (neutral) axis, we have

$$E_1\varepsilon dzdy = E_2\varepsilon ndzdy$$

Or

$$n = \frac{E_1}{E_2}$$

This dimensionless number n is called the transformation factor.

Design of Composite Beams

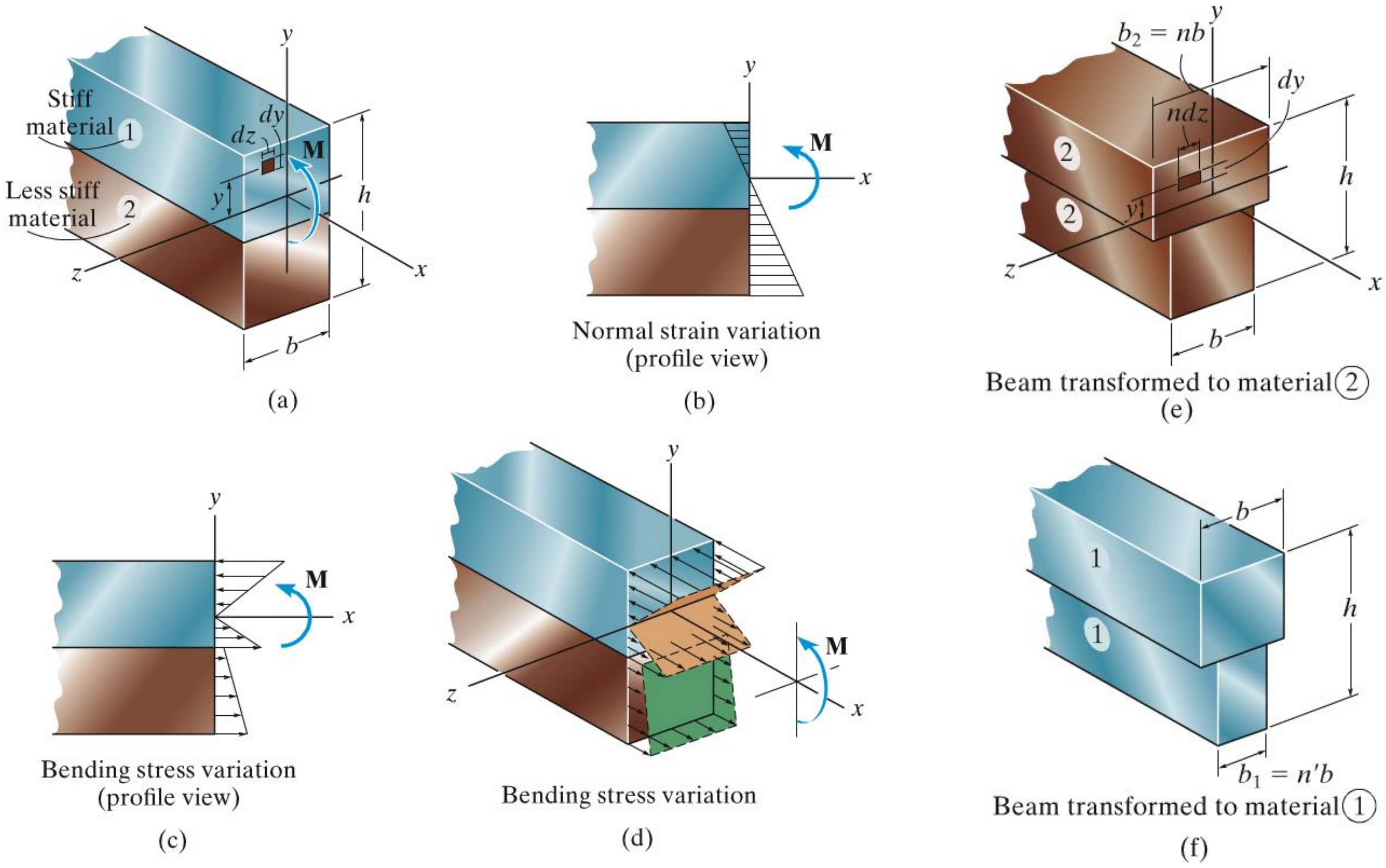


Fig.7-2

Example-3

A composite beam is made of wood and reinforced with a steel strap located on its bottom side. It has the cross-sectional area shown in below. If the beam is subjected to a bending moment of M=2kN.m, determine the normal stress at points B and C. Take E_w=12GPa and E_{st}=200GPa.

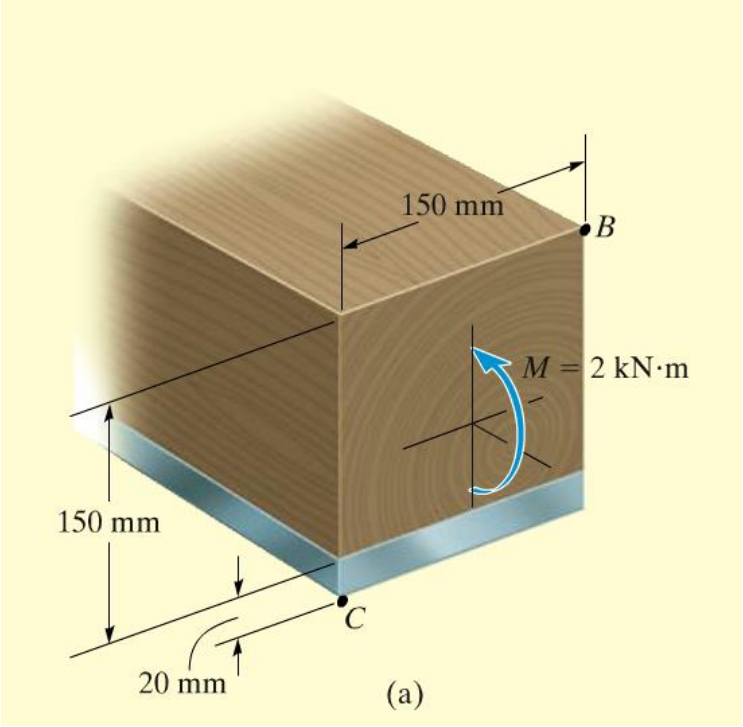
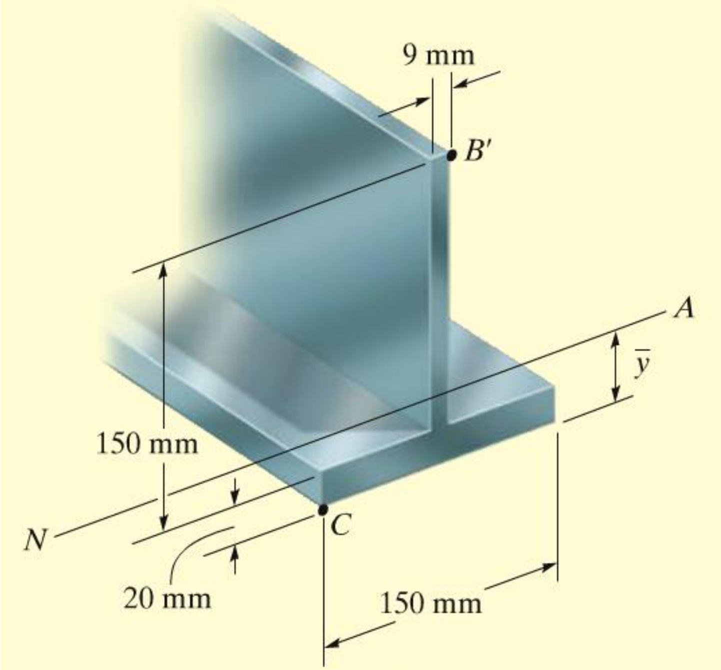


Fig.7-3

Example-3

Sectional Properties. Although the choice is arbitrary, here we will transform the section into one made entirely of steel. Since stress has greater stiffness than wood ($E_{st}>E_w$), the width of the wood must be reduced to an equivalent width for steel. Hence n must be less than one. For this to be case, $n=E_w/E_{st}$, so that

$$b_{st} = nb_w = \frac{12GPa}{200GPa} \times 150mm = 9mm$$



(b)

The location of the centroid (neutral axis), calculated from a reference axis located at the bottom of the section, is

$$\bar{y} = \frac{\sum \bar{y}_i A_i}{\sum A_i} = \frac{0.01m\times0.02m\times0.15m+0.095m\times0.009\times0.15m}{0.02m\times0.15m+0.009m\times0.15m} = 0.03638m$$

The moment of inertia about the neutral axis is therefore

$$I_{NA} = [\frac{1}{12}(0.150m)(0.02m)^3 + (0.150m)(0.02m)(0.03638m - 0.01m)^2]$$

$$+[\frac{1}{12}(0.009m)(0.150m)^3 + (0.009m)(0.150m)(0.095m - 0.03638m)^2]$$

$$= 9.358(10^{-6})m^4$$

Example-3

Normal Stress. Applying the flexure formula, the normal stress at B' and C' is

$$\sigma_{B'} = \frac{2\times10^{3}N\cdot m\times(0.170m-0.03638m)}{9.368\times10^{-6}m^{4}} = 28.56\text{MPa}$$

$$\sigma_{C} = \frac{2\times10^{3}N\cdot m\times(0.03638m)}{9.368\times10^{-6}m^{4}} = 7.78\text{MPa}$$

The normal stress distribution on the transformed (all steel) section is shown in Fig.7-4 (c).

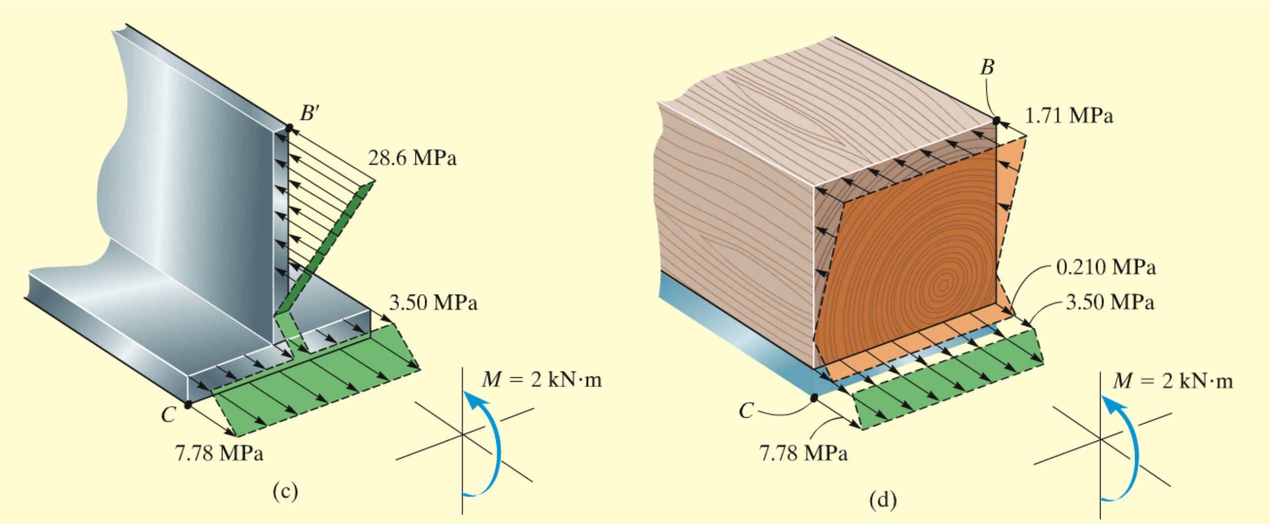


Fig.7-4 (cont.)

Example-3

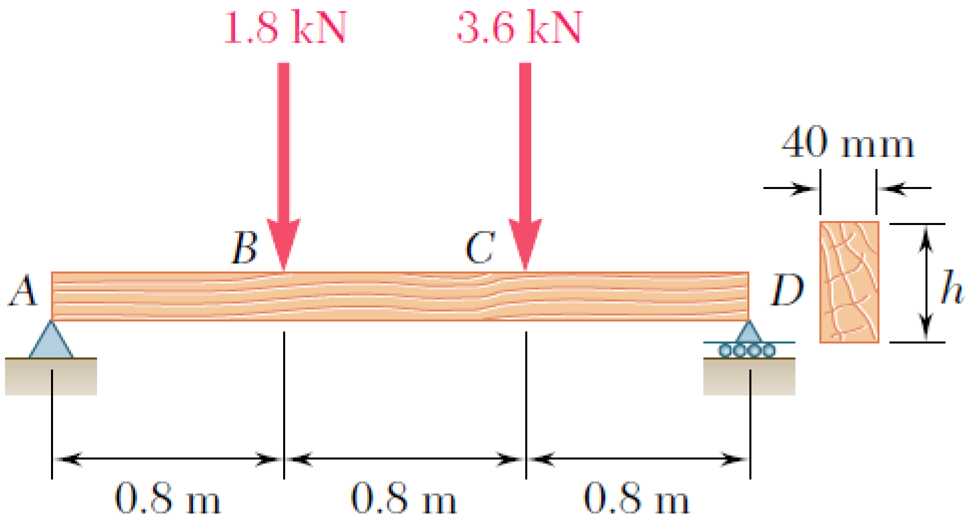
The normal stress in the wood at B is determined as:

$$\sigma_B = n\sigma_{B'} = \frac{12GPa}{200GPa} \times (28.56MPa) = 1.71MPa$$

Using these concepts, shown that the normal stress in the steel and the wood at the point where they are in contact is $\sigma_B = 3.5MPa$ and $\sigma_w = 0.21MPa$, respectively. The normal-stress distribution in the actual beam is shown in Fig.7-4 (d).

Exercise-1

For the beam and loading shown, design the cross section of the beam, knowing that the grade of timber used has an allowable normal stress of 12 MPa.



Answer: 173.2mm