Foundations of Solid Mechanics

(L11: Buckling of Columns

Department of Civil Engineering
School of Engineering
Aalto University

Buckling and critical load

Whenever a member is designed, it is necessary that it satisfies specific *strength*, *deflection*, and *stability* requirement.

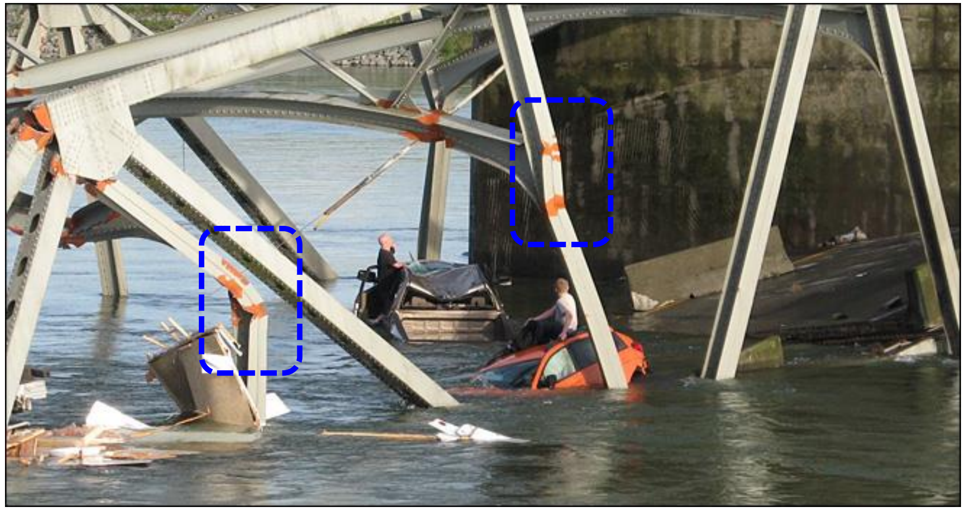
To be specific, long slender members subjected to an axial compressive load are called *columns*, and the lateral deflection that occurs is called *buckling*. The maximum axial load that a column can support when it is on the verge of buckling is called the *critical load*.

Fig. 11.1

Buckling Examples

Train tracks buckled by extreme heat

Fig. 11.2



Example-1: Stability of Structures

The storing force in spring:

$$F = k\Delta \approx \mathrm{k}\,\theta(\mathrm{L}/2)$$

Horizontal components of P:

$$P_x = P\tan\theta \approx P\theta$$

Stable equilibrium: $F > 2P_x \Rightarrow P < \dfrac{kl}{4}$

Neutral equilibrium: $F = 2P_x \Rightarrow P = \dfrac{kl}{4}$

Unstable equilibrium: $F < 2P_x \Rightarrow P > \dfrac{kl}{4}$

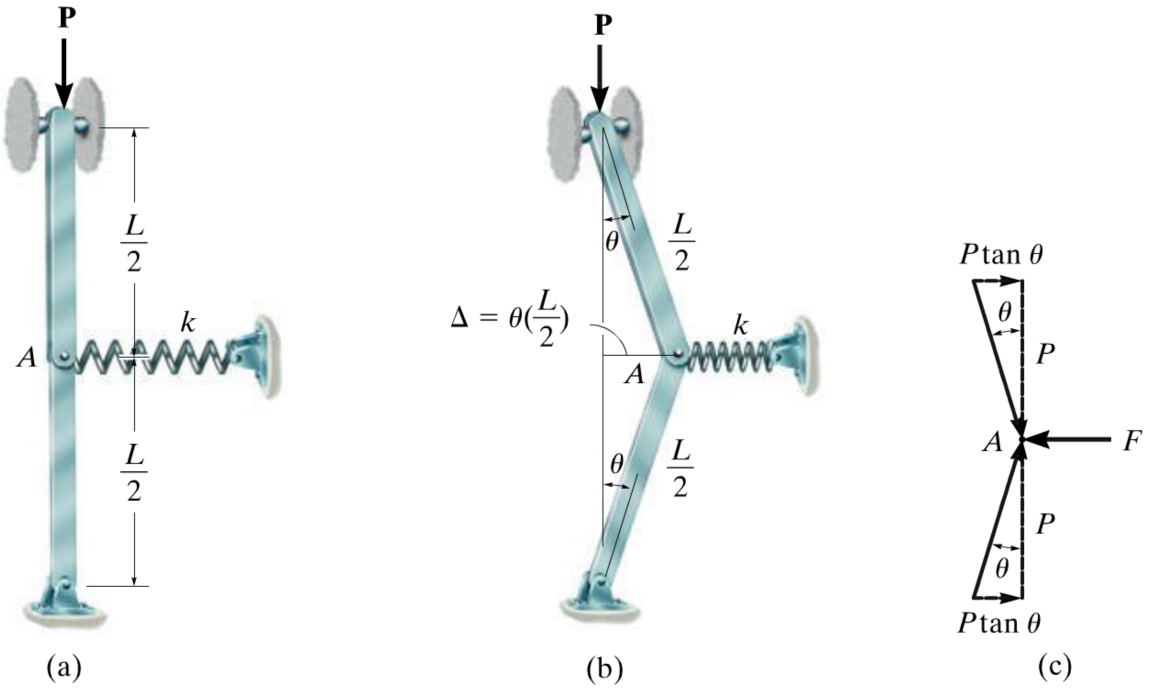
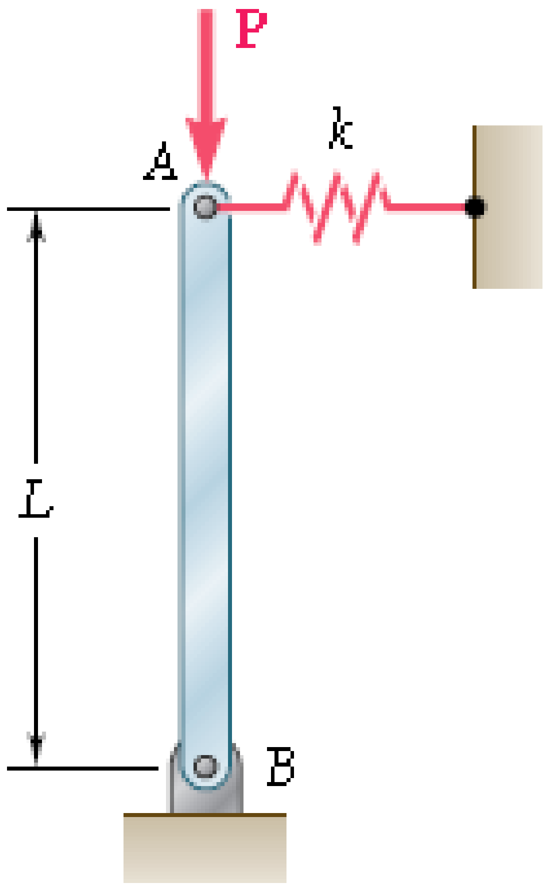


Fig. 11.3

Example-2

Knowing that the spring at A is of constant k and that the bar AB is rigid, determine the critical load P_{cr}.



Example-2

The storing force in spring:

$$F = k\Delta \approx \mathrm{kL}\,\theta$$

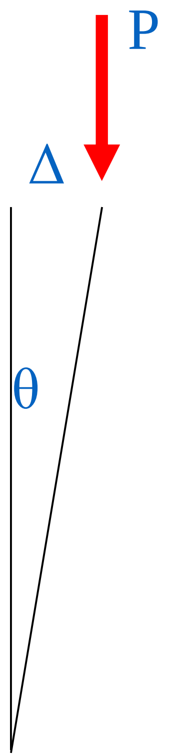
Horizontal components of P:

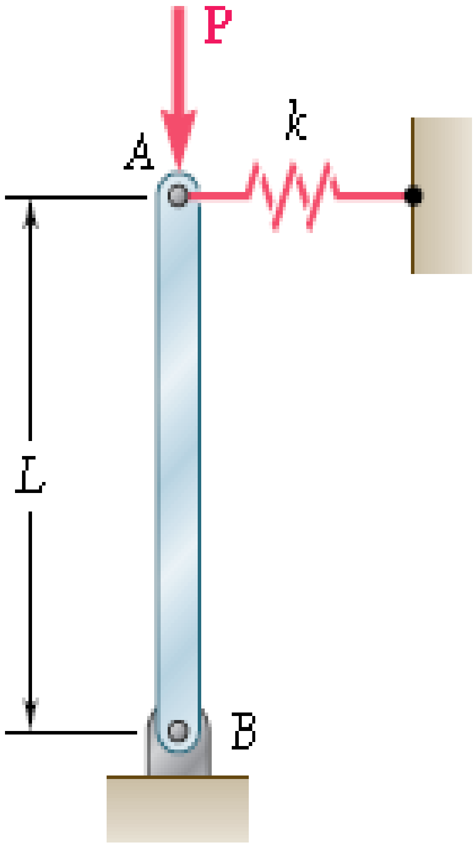
$$P_x = P\tan\theta \approx P\theta$$

Stable equilibrium: $F > P_x \Rightarrow P < kL$

Neutral equilibrium: $F = P_x \Rightarrow P = kL$

Unstable equilibrium: $F < P_x \Rightarrow P > kL$





Ideal Column with Pin Supports

Ideal column:

1) perfectly straight before loading;
2) made of homogeneous material (behavior in a linear manner);
3) the load is applied through the centroid of the cross section;
4) the column buckles or bends in a single plane.

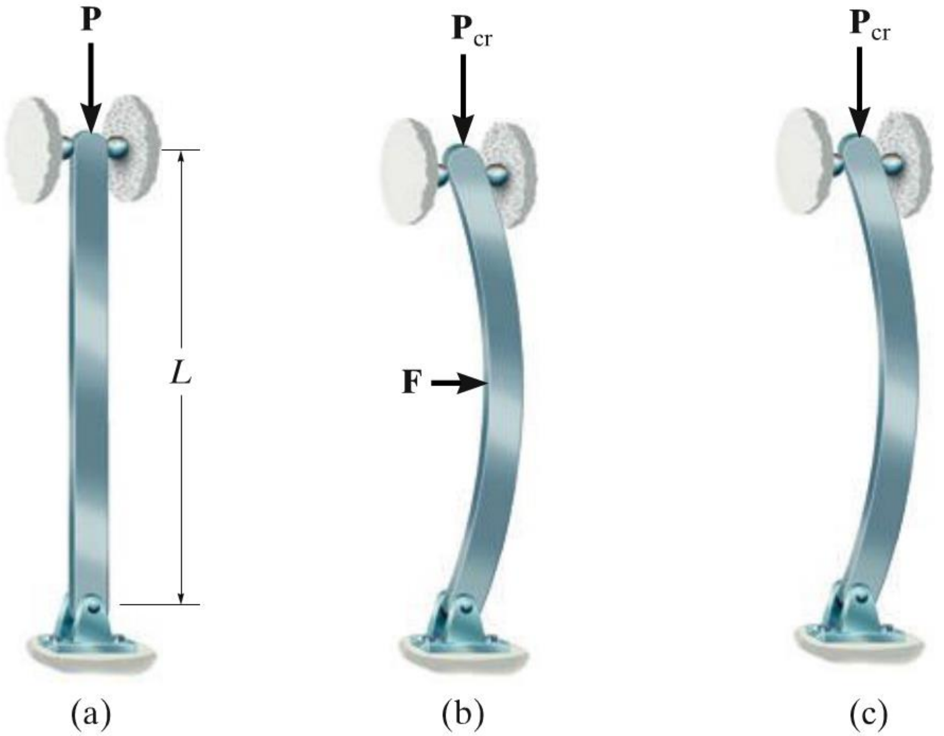


Fig. 11.4

Euler's Formula for Pin-ended Columns

A column can be considered as a beam placed in a vertical position and subjected to an axial load. It follows that the x axis will be vertical and directed downward, and the y axis horizontal and directed to the right. Considering the equilibrium of the free body AQ (Fig. 11.6b), we find that the bending moment at Q is M=Py. We write:

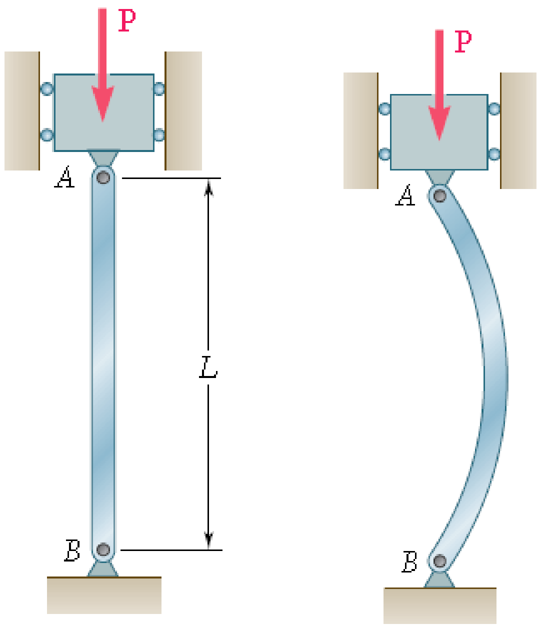


Fig. 11.5 Pin-ended column

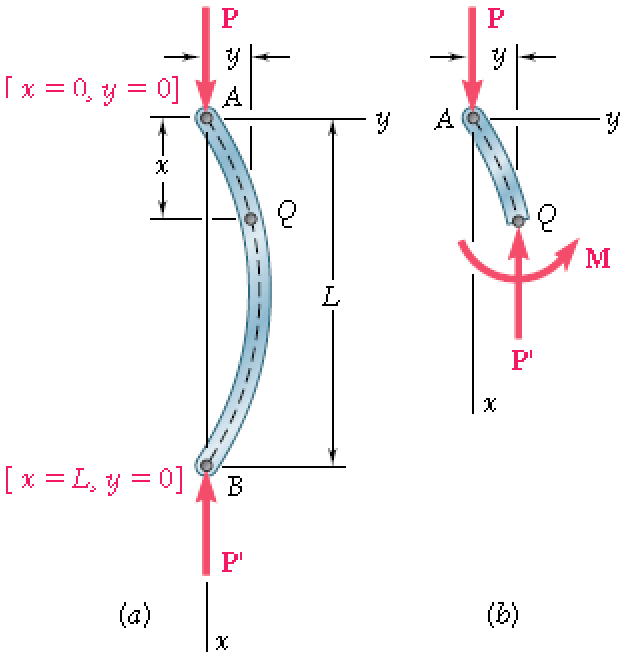


Fig. 11.6 Column in buckled position.

Euler's Formula for Pin-ended Columns

M-v relationship:

$$EI\frac{d^2y}{dx^2} = \text{-}M$$

$$M = py$$

$$EI\frac{d^2y}{dx^2} = -py$$

$$\frac{d^2y}{dx^2} + (\frac{p}{EI})\,\mathrm{y} = 0 \quad \longleftrightarrow$$

Homogeneous, second-order, linear differential equation with constant coefficients.

$\longleftrightarrow$ General solution

$$y = C_1 \sin(\sqrt{\frac{P}{EI}}x) + C_2 \cos(\sqrt{\frac{P}{EI}}x)$$

Euler's Formula for Pin-ended Columns

Boundary conditions:

$$x = 0, y = 0 \Rightarrow C_2 = 0$$

$$x = L, y = 0 \Rightarrow C_1 \sin\left(\sqrt{\frac{P}{EI}}L\right) = 0$$

$$\sin\left(\sqrt{\frac{P}{EI}}L\right) = 0 \qquad C_1 = 0 \text{ (\textit{Trivial} solution)}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\qquad\qquad y \equiv 0$$

$$\sqrt{\frac{P}{EI}}L = n\pi \Rightarrow P = \frac{n^2 p^2 EI}{L^2} \quad n = 1, 2, 3\ldots$$

Euler's Formula for Pin-ended Columns

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

where:

P_{cr} = Critical or maximum axial load on the column just before it begins to buckle. This load much not cause the stress in the column to exceed the proportional limit

E = Modulus of elasticity for the material

I = Least moment of inertia for the column's cross-sectional area

L = Unsupported length of the column, whose ends are pinned

For the purpose of design, the above equation can also be written in a more useful form by expressing $I = Ar^2$; where A is the cross-sectional area and the r is the *radius of gyration* of the cross-sectional area, thus,

$$P_{cr} = \frac{\pi^2 E(Ar^2)}{L^2} \quad \Rightarrow \quad \sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 E}{(L/r)^2}$$

Euler's Formula for Pin-ended Columns

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 E}{(L/r)^2}$$

where:

σ_{cr} = Critical stress, which is an average normal stress in the column just before the column buckles. This stress is an elastic stress and therefore

E = Modulus of elasticity for the material

L = Unsupported length of the column, whose ends are pinned

r = Smaller radius of gyration of the column, determined from $r < \sqrt{I/A}$, where I is the least moment of inertia for the column's cross-sectional area A

L/r = Slenderness ratio

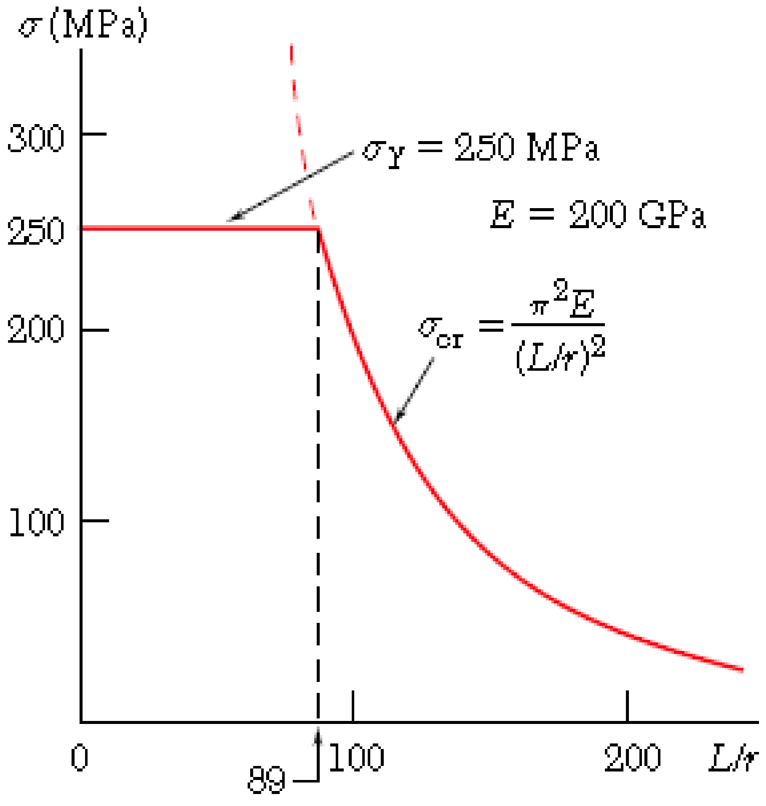


Fig. 11.7 Plot of critical stress

Extension of Euler's Formula to Columns with other End Conditions

In the case of a column with one free end A supporting a load P and one fixed end B (Fig. 11.8a), we observe that the column will behave as the upper half of a pin-connected column (Fig. 11.8b). The critical load for the column of Fig. 11.8a is thus the same as for the pin-ended column of Fig. 11.8b and can be obtained from Euler's formula by using a column length equal to twice the actual length L of the given column. We say that the effective length L_e is equal to $2L$ and substitute $L_e = 2L$ in Euler's formula:

$$P_{cr} = \frac{\pi^2 EI}{L_e^2}$$

The critical stress is found in a similar way from the formula

$$\sigma_{cr} = \frac{\pi^2 E}{(L_e/r)^2}$$

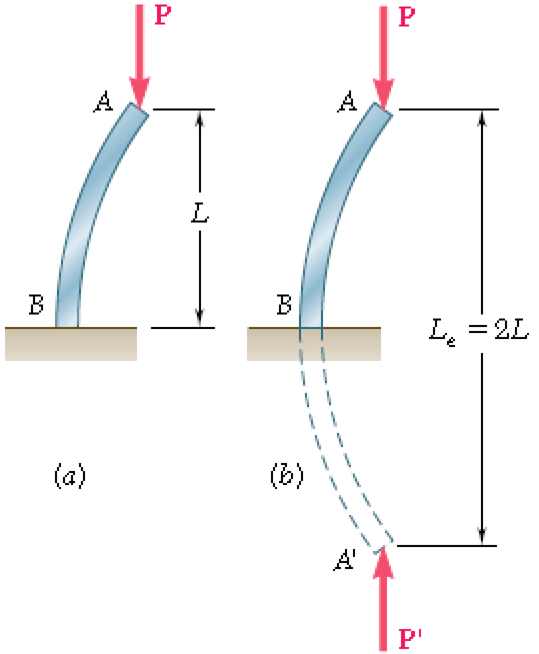


Fig. 11.8 Column with free end.

The quantity L_e/r is referred to as the effective slenderness ratio of the column and, in the case considered here, is equal to $2L/r$.

Extension of Euler's Formula to Columns with other End Conditions

Consider a column with two fixed ends. The symmetry of the supports and of the loading requires that the shear at *C* and the horizontal components of the reactions at *A* and *B* be zero. It follows that the restraints imposed upon the upper half *AC* of the column by the support at lower half *CB* are identical. *A* and *C* by the Portion AC must thus be symmetric about its midpoint *D*, and this point must be a point of inflection, where the bending moment is zero. A similar reason shows that the bending moment at the midpoint *E* of the lower half of the column must also be zero. Since the bending moment at the ends of a pin-ended column is zero, it follows that the portion *DE* of the column must behave as a pin ended column. We thus conclude that the effective length of a column with two fixed ends is $L_e = L/2$.

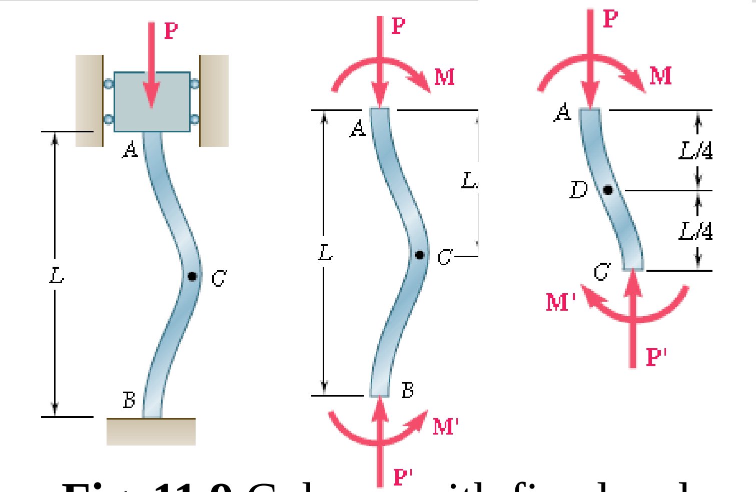


Fig. 11.9 Column with fixed ends.

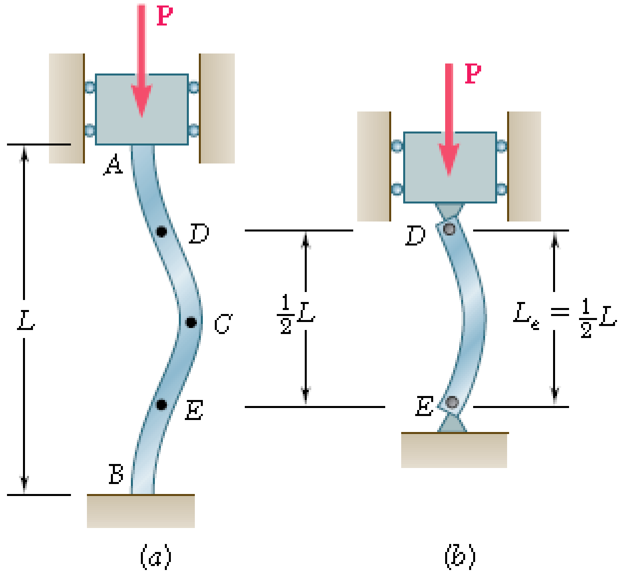


Fig. 11.10

Foundations of Solid Mechanics

Extension of Euler's Formula to Columns with other End Conditions

(a) One fixed end, one free end

(b) Both ends pinned

(c) One fixed end, one pinned end

(d) Both ends fixed

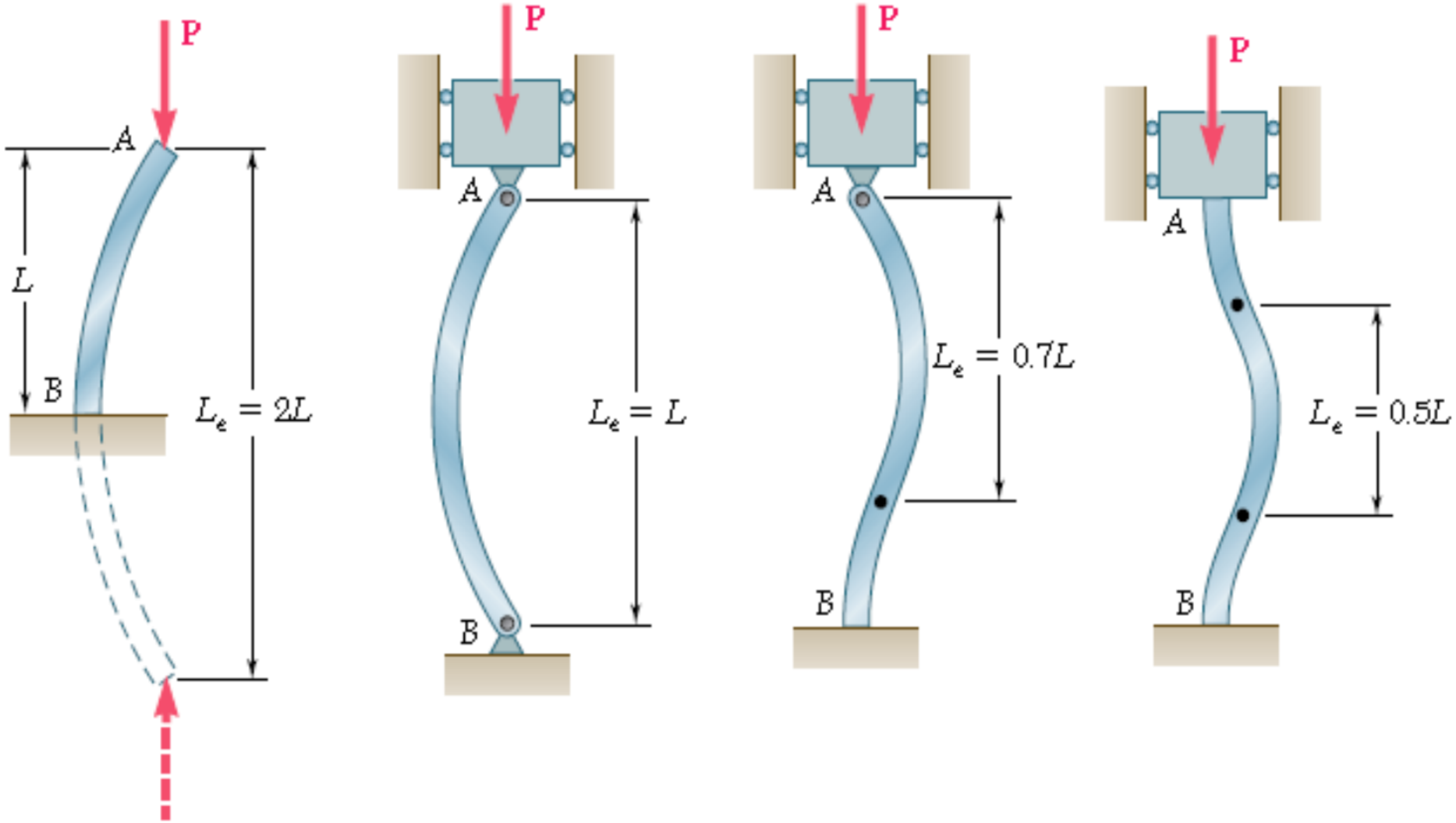
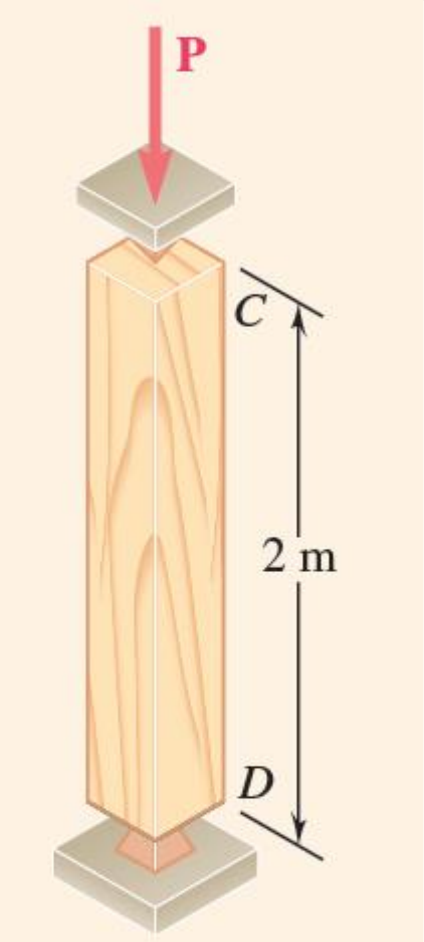


Fig. 11.11 Effective length of columns for various end conditions

Example-3

A 2-m-long pin-ended column with a square cross section is to be made of wood. Assuming E=13 GPa, σ_{all} = 12 MPa, and using a factor of safety of 2.5 to calculate Euler's critical load for buckling, determine the size of the cross section if the column is to safely support (*a*) a 100-kN load, (*b*) a 200-kN load.



Pin-ended wood column of square cross section.

a. For the 100-kN Load.

Use the given factor of safety to obtain

$$P_{cr} = 100 \times 2.5 = 250kN \qquad L = 2m \qquad E = 13GPa$$

Use Euler's formula and solve for I:

$$I = \frac{P_{cr}L^2}{\pi^2 E} = \frac{(250 \times 10^3 N)(2m)}{\pi^2(13 \times 10^9 Pa)} = 7.794 \times 10^{-6} m^4$$

For a square of side a, $I = \frac{a^4}{12}$, write

$$I = \frac{a^4}{12} = 7.794 \times 10^{-6} m^4 \qquad a = 98.3mm \approx 100mm$$

Check the value of the normal stress in the column: $\sigma = \frac{P}{A} = \frac{100kN}{(0.100m)^2} = 10MPa$

Since σ is smaller than the allowable stress, a 100×100-mm cross section is acceptable.

Example-3

b. For the 200-kN Load.

Solve again for I, but make P_{cr} = 2.5(200) = 500 kN to obtain

$$I = 15.588 \times 10^{-6} m^4$$

$$I = \frac{a^4}{12} = 15.588 \times 10^{-6} m^4 \qquad a = 116.95mm$$

The value of the normal stress is

$$\sigma = \frac{P}{A} = \frac{200kN}{(0.11695m)^2} = 14.62MPa$$

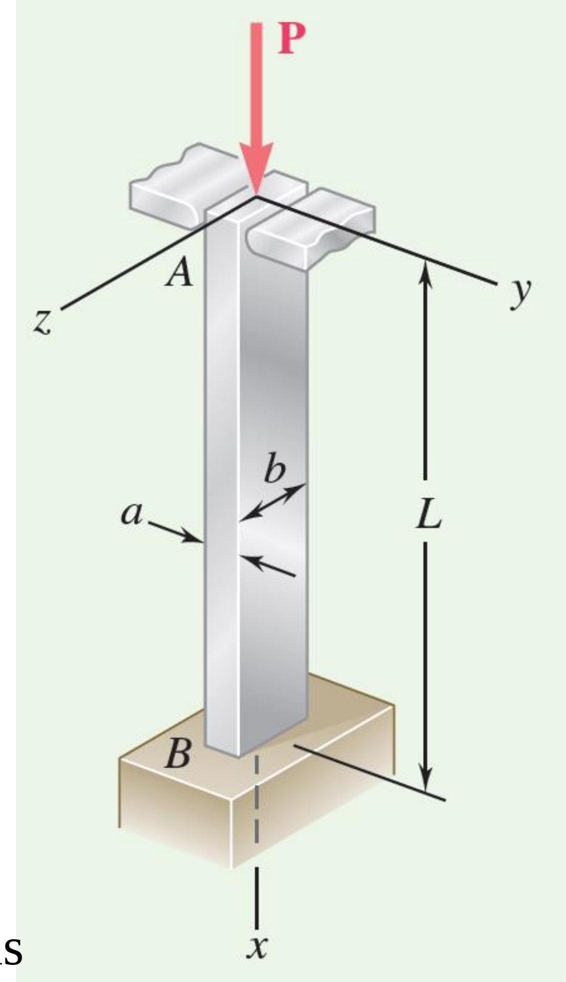
Since this is larger than the allowable stress, the dimension obtained is not acceptable, and the cross section must be selected on the basis of its resistance to compression.

$$A = \frac{P}{\sigma_{all}} = \frac{200kN}{12MPa} = 16.67 \times 10^{-3} m^2 \Rightarrow a^2 = 16.67 \times 10^{-3} m^2 \Rightarrow a = 129.1mm$$

A 130×130-mm cross section is acceptable.

Example-4 (solution-1)

An aluminium column with a length of *L* and a rectangular cross section has a fixed end *B* and supports a centric load at *A*. Two smooth and rounded fixed plates restrain end *A* from moving in one of the vertical planes of symmetry of the column but allow it to move in the other plane. Determine the ratio a/b of the two sides of the cross section corresponding to the most efficient design against buckling.



Buckling in *xy* Plane. The effective length of the column with respect to buckling in this plane is $L_e = 0.7L$. The radius of gyration r_z of the cross section is obtained by

$$I = \frac{1}{12}ba^3 \qquad A = ab$$

$$\text{since } I_z = Ar_z^2, \qquad r_z^2 = \frac{I_z}{A} = \frac{\frac{1}{12}ba^3}{ab} = \frac{a^2}{12} \qquad r_z = a/\sqrt{12}$$

The effective slenderness ratio of the column with respect to buckling in the *xy* plane is

$$\frac{l_e}{r_z} = \frac{0.7L}{a/\sqrt{12}}$$

Example-4 (solution-1)

Buckling in *xz* Plane. The effective length of the column with respect to buckling in this plane is $L_e = 2L$, and the corresponding radius of gyration is $r_y = a/\sqrt{12}$. Thus,

$$\frac{l_e}{r_y} = \frac{2L}{b/\sqrt{12}}$$

Most Efficient Design. The most efficient design is when the critical stresses corresponding to the two possible modes of buckling are equal.

$$\frac{0.7L}{a/\sqrt{12}} = \frac{2L}{b/\sqrt{12}}$$

and solving for the ratio *ab*, $\frac{a}{b} = \frac{0.7}{2}$ =0.35

Example-4 (solution-2)

Buckling in *xy* Plane. The effective length of the column with respect to buckling in this plane is $L_e = 0.7L$. The critical load of the cross section is obtained by

$$P_{cr1} = \frac{\pi^2 E I_1}{(L_{e1})^2} = \frac{\pi^2 E \frac{ba^3}{12}}{(0.7L)^2}$$

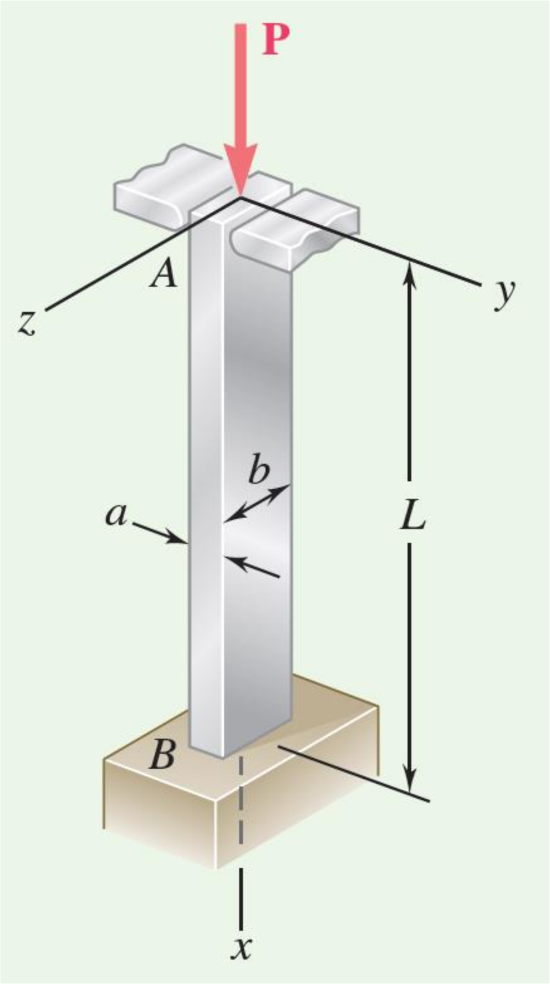
Buckling in *xz* Plane. The effective length of the column with respect to buckling in this plane is $L_e = 2L$. Thus,

$$P_{cr2} = \frac{\pi^2 E I_2}{(L_{e2})^2} = \frac{\pi^2 E \frac{ab^3}{12}}{(2L)^2}$$

Most Efficient Design. The most efficient design is when the critical loads corresponding to the two possible modes of buckling are equal.

$$\frac{\pi^2 E \frac{ba^3}{12}}{(0.7L)^2} = \frac{\pi^2 E \frac{ab^3}{12}}{(2L)^2}$$

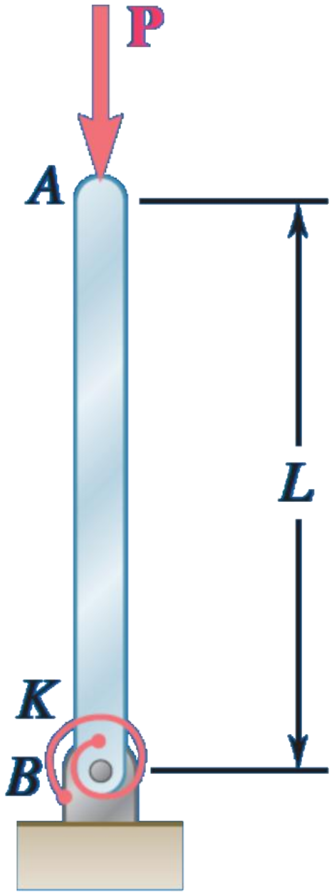
and solving for the ratio *ab*, $\frac{a}{b} = \frac{0.7}{2}$ =0.35



Foundations of Solid Mechanics

Exercise-1

Knowing that the torsional spring at B is of constant K and that the bar AB is rigid, determine the critical load P_{cr}.



$$P_{cr} = \frac{K}{L}$$

Exercise-2

Determine the largest load *P* that can be applied to the structure shown. Use E =200 GPa and consider only buckling in the plane of the structure.