Foundations of Solid Mechanics

L3: Torsion

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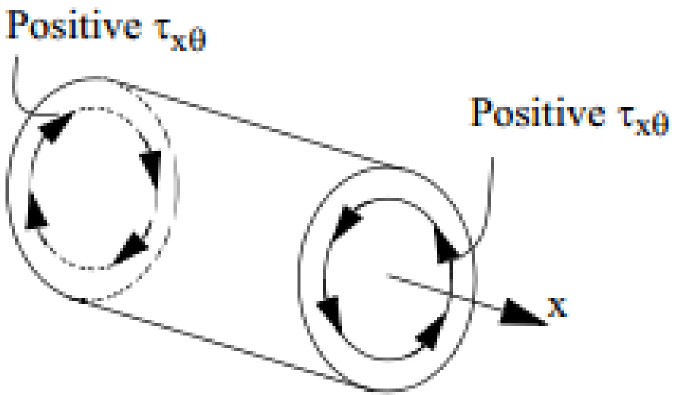
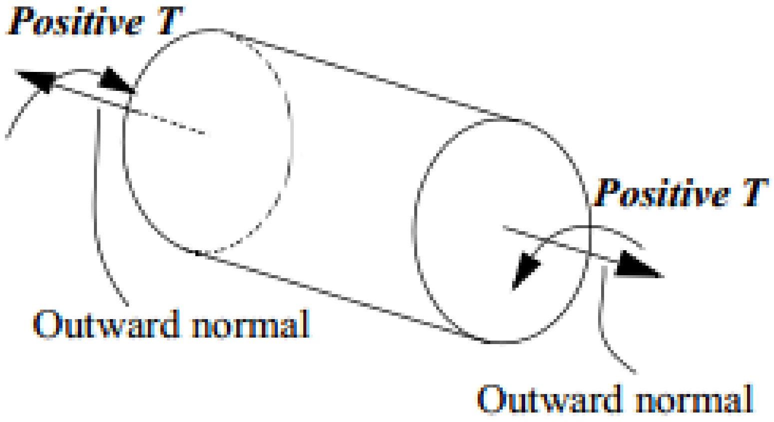
Main Contents

In this chapter, we will discuss:

- How to determine the *stress distribution* within the member and *the angle of twist* when the materials behaves in a elastic manner and when it is inelastic;
- *Statically indeterminate analysis* of shafts and tubes

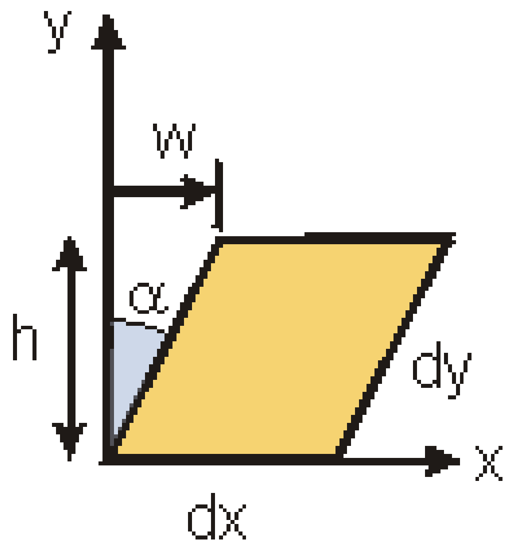
Sign convention of torsion: Internal torque is considered as positive if it is counter-clockwise with respect to the outward normal to the imaginary cut surface.





Shear Strain

Shear strain is measured as a change in angle between lines that were originally perpendicular.



Consider a 2-dimensional square element has width, dx, and height, dy. Shear deformations cause the square to change shape into a rhombus as shown at right. **Shear strain**, g, is equal to the change in right angle of a square element, α (radians). Since α is generally small, tan(α) ~ α, therefore:

$$\tan\alpha = \frac{w}{h} \qquad \text{for small } \alpha, \qquad \tan\alpha \approx \alpha = \frac{w}{h} = \gamma$$

$$\tau = \gamma G$$

τ : shear stress

γ: shear strain

G: shear modulus of elasticity or the modulus of rigidity.

Introduction

- Torsion: twisting couples with common magnitude and opposite directions along the axis of the member.

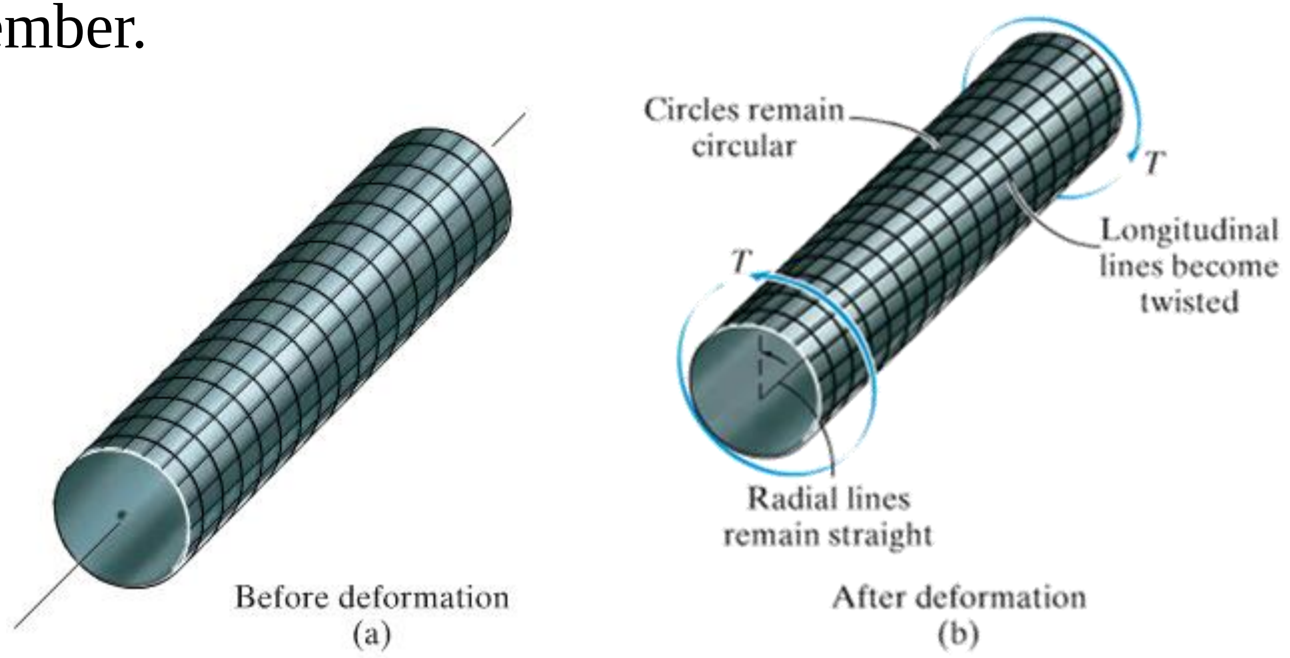


Fig.3.1

- Assumptions:
  - The length of the shaft remains unchanged.
  - The straight radial lines remain straight.
  - Cross sections remain circular

Equilibrium Equation : Circular Shaft

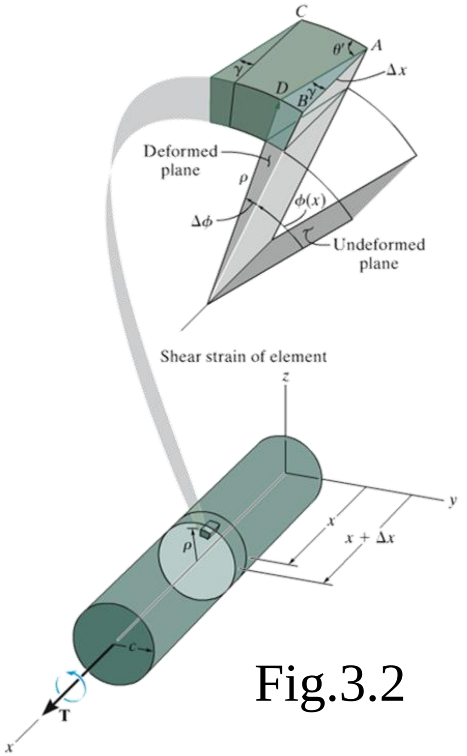
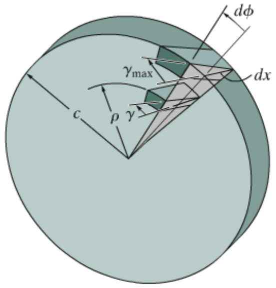


Fig.3.2



The shear strain at points on the cross section increases linearly with ρ, i.e., $\gamma = (\rho/c)\gamma_{max}$.

Fig.3.3

From the figure as shown in the left, the length of arc BD can be determined as: $BD = \rho\Delta\emptyset = \gamma\Delta x$

Therefore, if we let $\Delta x \rightarrow dx$ and $\Delta\emptyset \rightarrow d\emptyset$,

$$\gamma = \rho\frac{d\emptyset}{dx} \qquad (3\text{-}1)$$

Since dx and $d\phi$ are the same of *all elements* located at points on the cross section at x, then $d\phi/dx$ is constant over the cross section. The **Eq.3-1** states that the magnitude of the shear strain for any of these elements varies only with its radial distance ρ from the axis of the shaft. In other words, the shear strain within the shaft varies linearly along any radial line, from zero at the axis to the maximum γ_{max} at its boundary. Since $d\phi/dx = \gamma/\rho = \gamma_{max}/c$, then

$$\gamma = \left(\frac{\rho}{c}\right)\gamma_{max} \qquad (3\text{-}2)$$

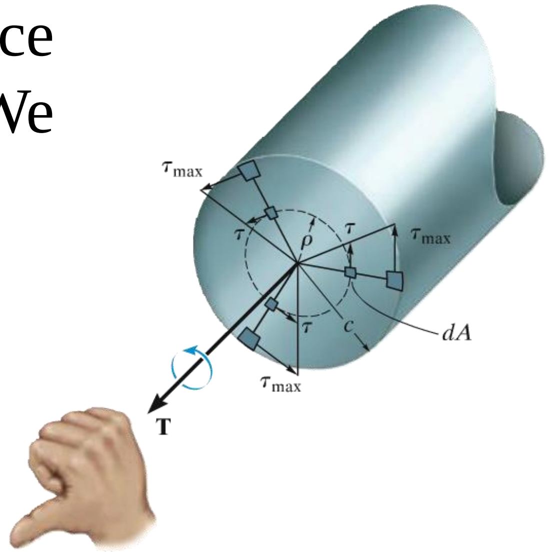
The Torsion Formula

If the material is linear-elastic, then Hooke's law applies, τ=Gγ, and consequently a linear variation in shear strain, leads to a corresponding linear variation in shear stress along any radial line on the cross section. Due to the proportionality of triangles, we can write:

$$\tau = \left(\frac{\rho}{c}\right)\tau_{max} \qquad (3\text{-}3)$$

Specifically, each element of area dA, located at ρ, is subjected to a force of $dF = \tau dA$. The torque produced by this force is dT= ρ(τdA). We therefor have for the entire cross section

$$T = \int_A \rho\left(\frac{\rho}{c}\right)\tau_{max} dA \qquad (3\text{-}4)$$



Shear stress varies linearly along each radial line of the cross section.

Fig.3.4

The integral depends only on the geometry of the shaft. It represents the polar moment of inertia of the shaft's longitudinal axis. We will symbolize its value as J, therefore:

$$\tau_{max} = \frac{Tc}{J} \qquad (3\text{-}5)$$

The Torsion Formula

$$\tau_{max} = \frac{Tc}{J} \qquad (3\text{-}5)$$

τ_{max} = the maximum shear stress in the shaft, which occurs at outer surface
T = the resultant internal torque acting at the cross section.
J = the polar moment of inertia of the cross sectional area
c = the outer radius of the shaft

The shear stress at the intermediate distance ρ can be determined from

$$\tau = \frac{T\rho}{J} \qquad (3\text{-}6)$$

Either one of the above two equations is often referred to as ***torsion formula***. Recall that it is used only if the shaft is circular and the material is homogeneous and behaves in a linear elastic manner, since the derivation is based on Hooke’s law.

Solid Shaft

If the shaft has a solid circular cross section, the polar moment of inertia J can be determined using

$$J = \int_A \rho^2 (2\pi\rho d\rho) = \int_0^c \rho^2 (2\pi\rho d\rho) = 2\pi \int_0^c \rho^3 d\rho = 2\pi\left(\frac{1}{4}\right) c^4$$

$$J = \frac{\pi}{2} c^4 \qquad (3\text{-}7)$$

What is the failure mode of a wooden shaft due to torsion?

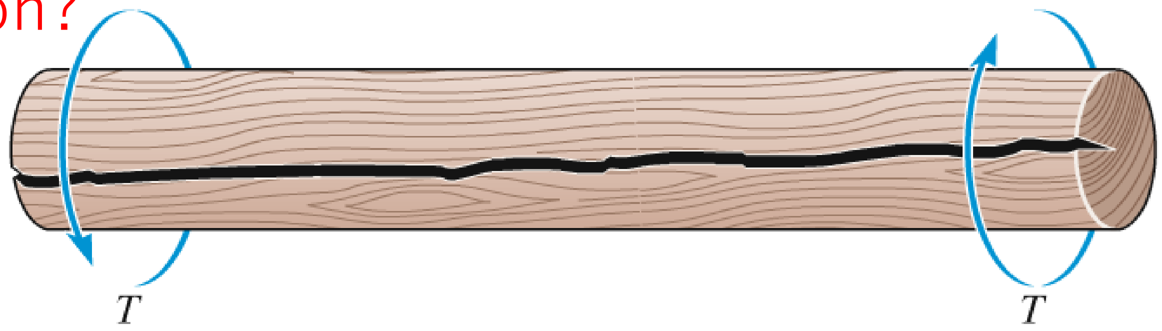


Fig.3.6 Failure of a wooden shaft due to torsion

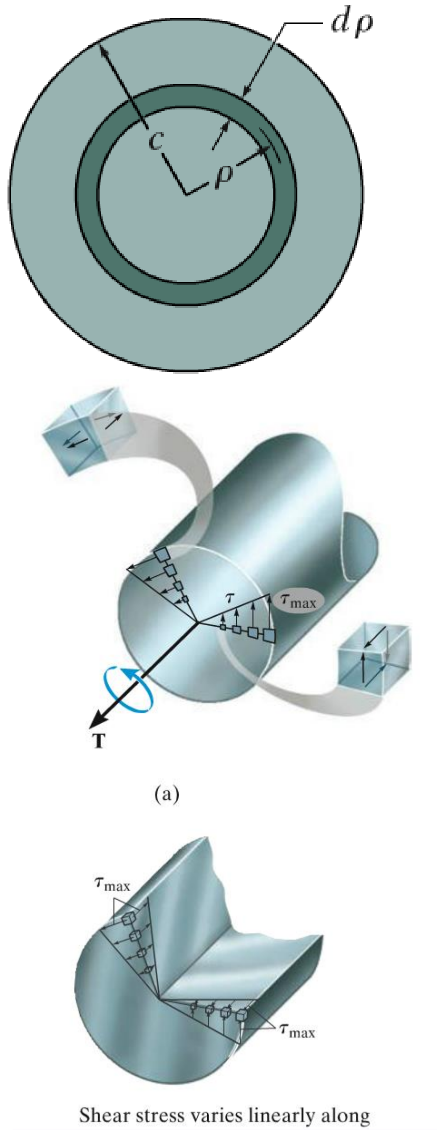


Fig.3.5

Tubular Shaft

If the shaft has a tubular cross section, with inner radius c_i and outer radius c_o, then we can determine its polar moment of inertia by subtracting J for a shaft of radius c_i from that determined for a shaft of radius c_o. the results is

$$J = \frac{\pi}{2}\left(c_o^4 - c_i^4\right) \qquad (3\text{-}8)$$

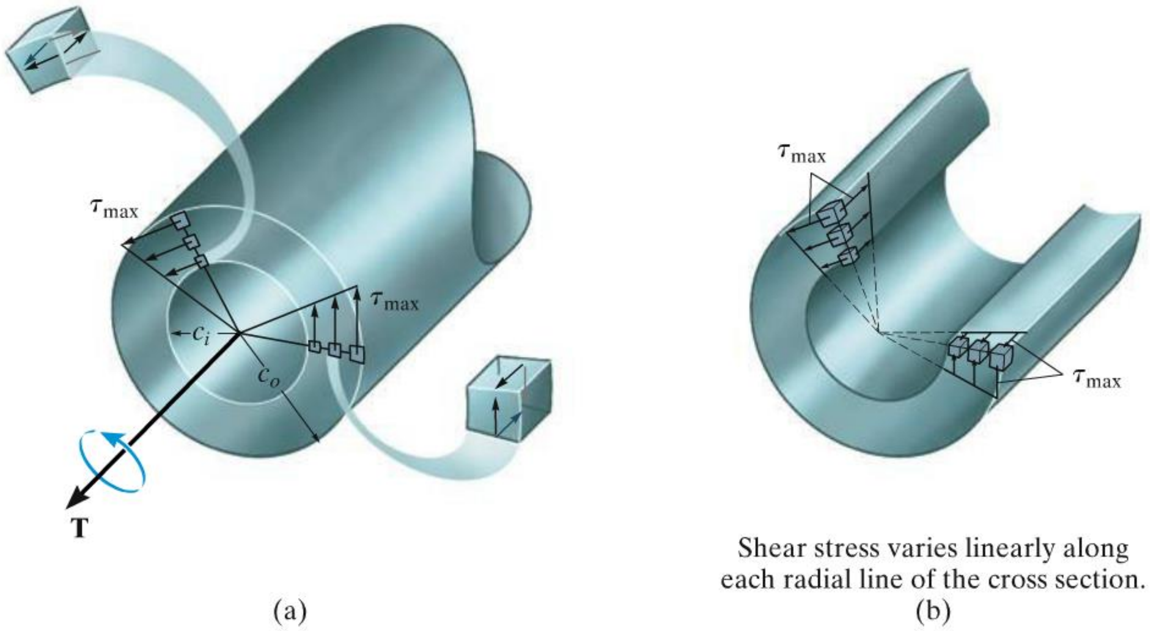


Fig.3.7

Absolute Maximum Torsional Stress

If the absolute maximum torsional stress is to be determined, then it becomes important to find the location where the ratio *Tc/J* is a maximum. In this regard, it may be helpful to show the variation of the internal torque T at each section along the axis of the shaft by drawing a ***torque diagram***, which is a plot of internal torque T versus its position x along the shaft's length. As a sign convention, T will be positive if by the right-hand rule the thumb is directed outward from the shaft when the fingers curl in the direction of twist as caused by the torque.

Angle of Twist

Using the method of sections, a differential disk of thickness dx, located at position x, is isolated from the shaft. Due to $T(x)$, the disk will twist, such that the *relative rotation* of one of its faces with respect to the other face is $d\emptyset$. As a result an element of material located at an arbitrary radius ρ within the disk will undergo a shear strain γ. The relationship between γ and $d\emptyset$ are

$$d\emptyset = \gamma \frac{dx}{\rho} \qquad (3\text{-}9)$$

Since Hooke's law, $\gamma = \tau/G$, applies and the shear stress can be expressed in terms of the applied torque using the torsion formula $\tau = T(x)\rho/J(x)$. The angle of twist for the disk is

$$d\emptyset = \frac{T(x)}{J(x)G}\, dx \qquad (3\text{-}10)$$

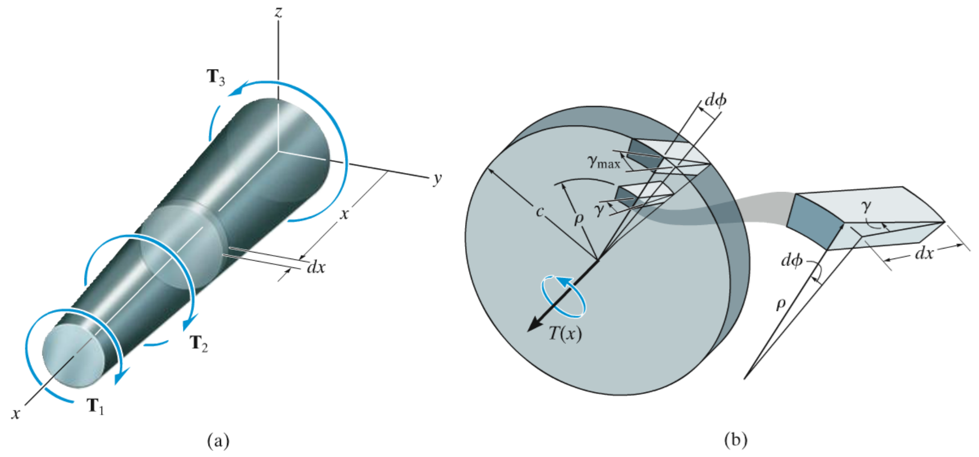


Fig.3.8

Integrating over the entire length L of the shaft, we obtain the angle of twist for the entire shaft, namely,

Angle of Twist

$$\emptyset = \int_0^L \frac{T(x)}{J(x)G}\, dx \qquad (3\text{-}11)$$

Here

$\emptyset$ = the angle of twist of one end of the shaft with respect to the other end, measured in radians

$T(x)$ = the internal torque at the arbitrary position x, found from the method of sections and the equation of moment equilibrium applied about the shaft's axis

$J(x)$ = the shaft's polar moment of inertia expressed as a function of position x

G = the shear modulus of elasticity of the material

Constant Torque and Cross-sectional Area

$$\emptyset = \frac{TL}{JG} \qquad (3\text{-}12)$$

Multiple Torques

$$\emptyset = \sum \frac{TL}{JG} \qquad (3\text{-}13)$$

Example-1

Shaft BC is hollow with inner and outer diameters of 90 mm and 120 mm, respectively. Shafts AB and CD are solid and of diameter d. For the loading shown, determine (a) the **absolute** maximum and minimum shearing stress in shaft BC, (b) the required diameter d of shafts AB and CD if the allowable shearing stress in these shafts is 65 MPa.

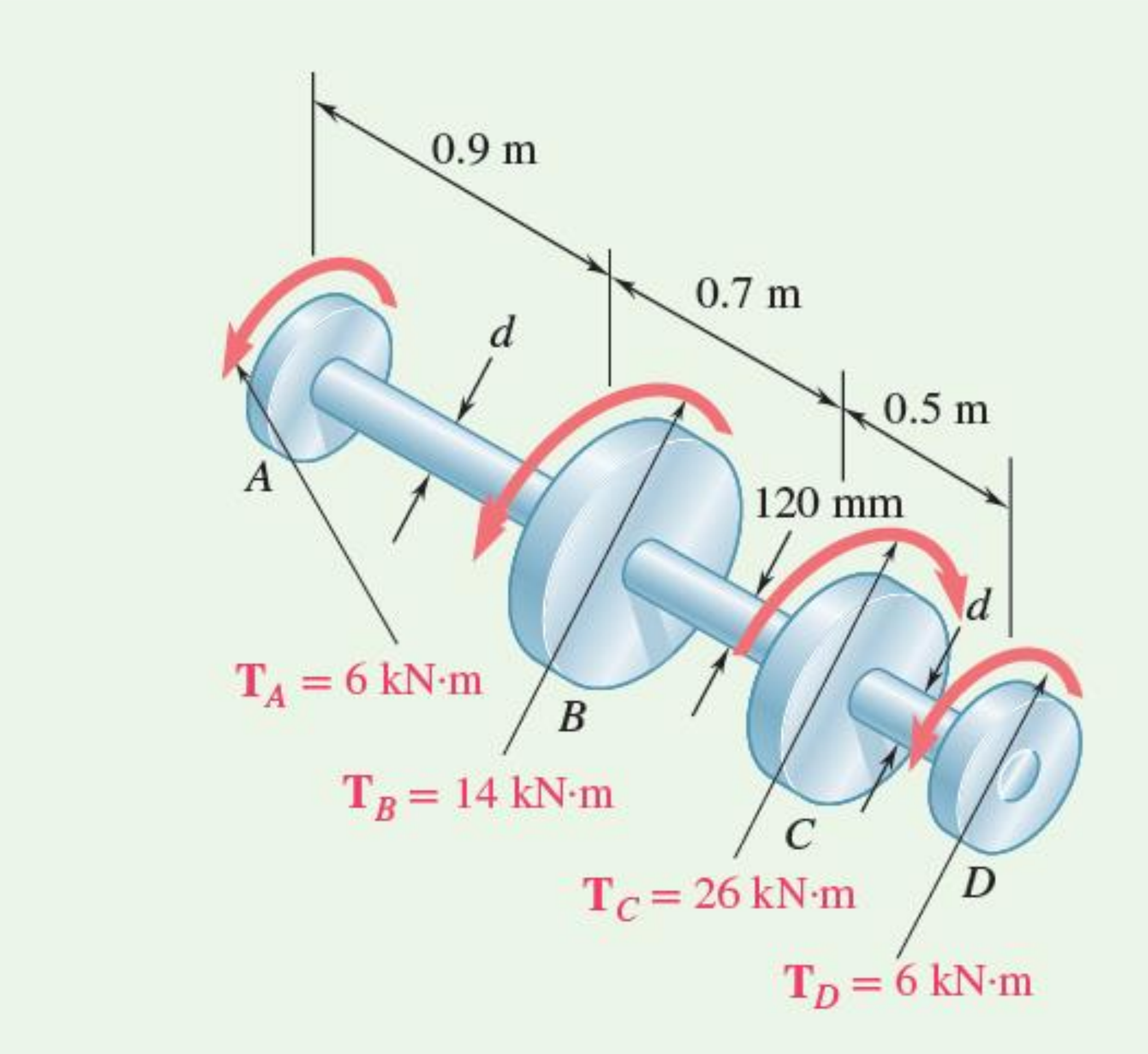


Fig.3.9

Example-1

Note: for absolute stress calculation. Otherwise, please consider the positive direction of torsion using the right-hand rule.

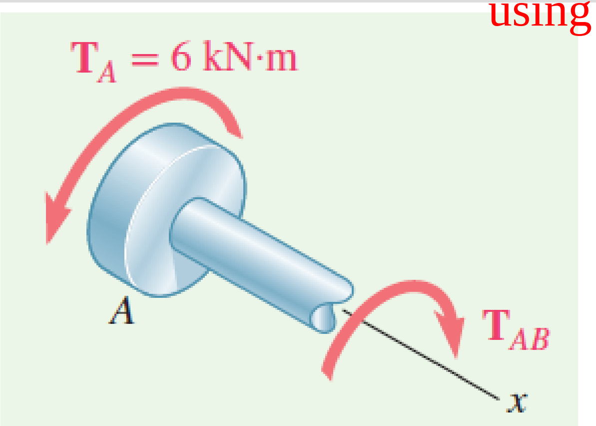


Fig.3.10 Free-body diagram for section to left of cut between A and B

For the free body shown in Fig.3.10, we write:

$\Sigma Mx = 0$: $(6\text{ kN·m}) - T_{AB} = 0$ $T_{AB} = 6$ kN·m

For the free body shown in Fig.3.11, we have

$\Sigma Mx = 0$: $(6\text{ kN·m}) + (14\text{ kN·m}) - T_{BC} = 0$ $T_{BC} = 20$ kN·m

a. Shaft *BC*. For this hollow shaft we have

$$J = \frac{\pi}{2}(c_2^4 - c_1^4) = \frac{\pi}{2}[(0.060)^4 - (0.045)^4] = 13.92 \times 10^{-6} m^4$$

Maximum Shearing Stress. On the outer surface, we have

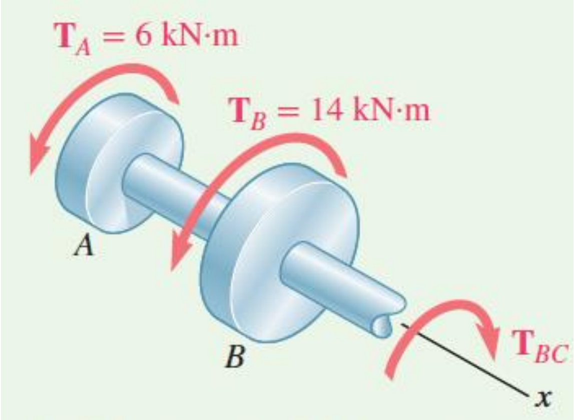


Fig.3.11 Free-body diagram for section to left of cut between B and C

$$\tau_{max} = \tau_2 = \frac{T_{BC} c_2}{J} = \frac{(20kN \cdot m)(0.06m)}{13.92 \times 10^{-6} m^4} = 86.2MPa$$

Example-1

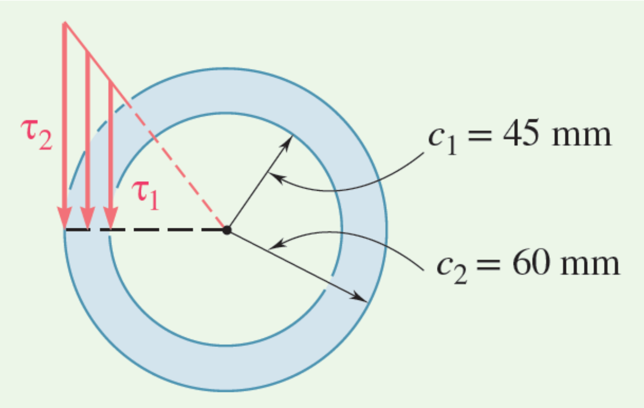


Fig.3.12 Shearing stress distribution on cross section

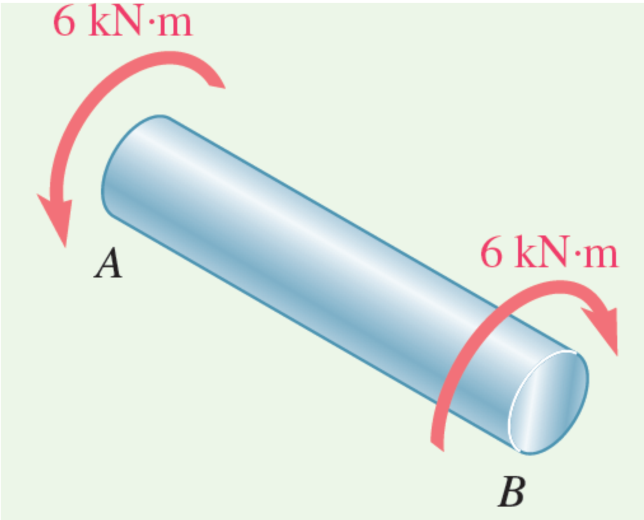


Fig.3.13 Free-body diagram of shaft portion AB

Minimum Shearing Stress. As shown in Fig. 3.12 the stresses are proportional to the distance from the axis of the shaft.

$$\frac{\tau_{min}}{\tau_{max}} = \frac{c_1}{c_2} = 86.2MPa \qquad \frac{\tau_{min}}{86.2MPa} = \frac{45mm}{60mm} \qquad \tau_{min} = 64.7MPa$$

b. Shafts *AB* and *CD*. We note that both shafts have the same torque T = 6 kN · m (Fig. 3.13). Denoting the radius of the shafts by c and knowing that τ_{all} = 65 MPa, we write

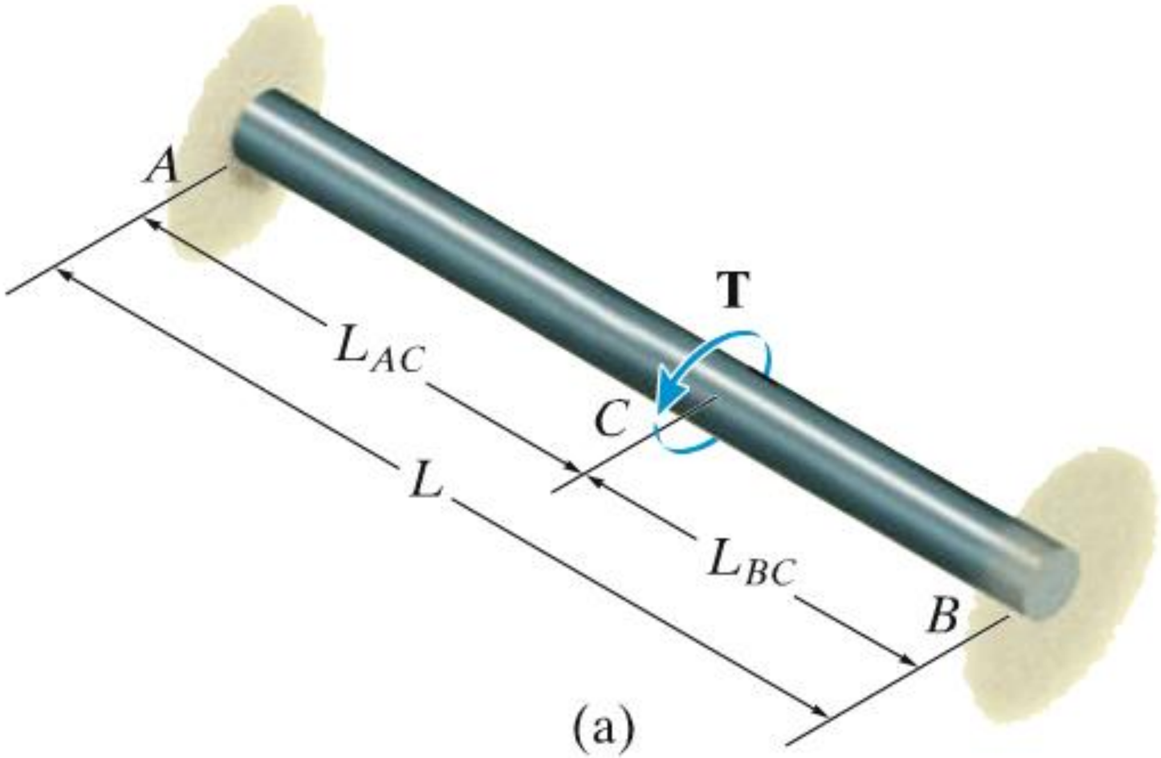
$$\tau = \frac{Tc}{J} \qquad 65MPa = \frac{(6kN \cdot m)c}{\frac{\pi}{2}c^4}$$

$$c^3 = 58.8 \times 10^{-6}m^3 \qquad c = 38.9 \times 10^{-3}m$$

$$d = 2c = 77.8 \times 10^{-3}m$$

Example-2: Statically Indeterminate Torque-loaded Members

Determine the torsional moment at A and B, respectively.



(a)

Example-2: Statically Indeterminate Torque-loaded Members

Equilibrium Equation:

$$\sum M_x = 0 \qquad T - T_A - T_B = 0$$

Compatibility or Kinematic Condition:

$$\emptyset_{A/B} = 0$$

Providing the material is linear elastic, we can apply the load-displacement relation $\phi = TL/JG$ to express the compatibility condition in terms of the unknown torques. We have:

$$\frac{T_A L_{AC}}{JG} - \frac{T_B L_{BC}}{JG} = 0$$

Solving the above two equations for the reactions, realizing that $L_{BC} + L_{AC} = L$, we get:

$$T_A = T\left(\frac{L_{BC}}{L}\right) \qquad T_B = T\left(\frac{L_{AC}}{L}\right)$$

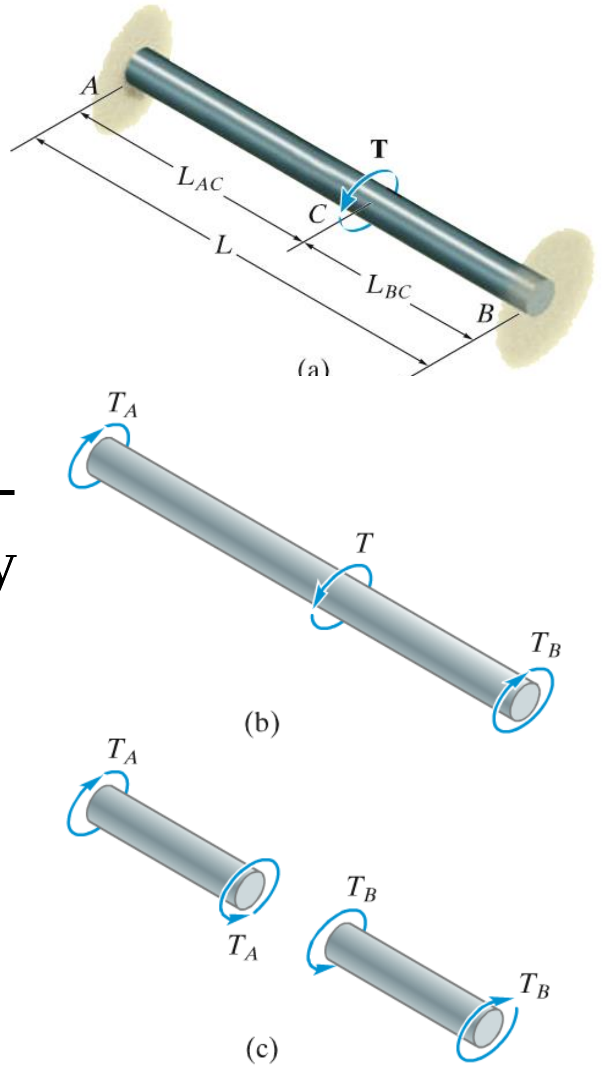
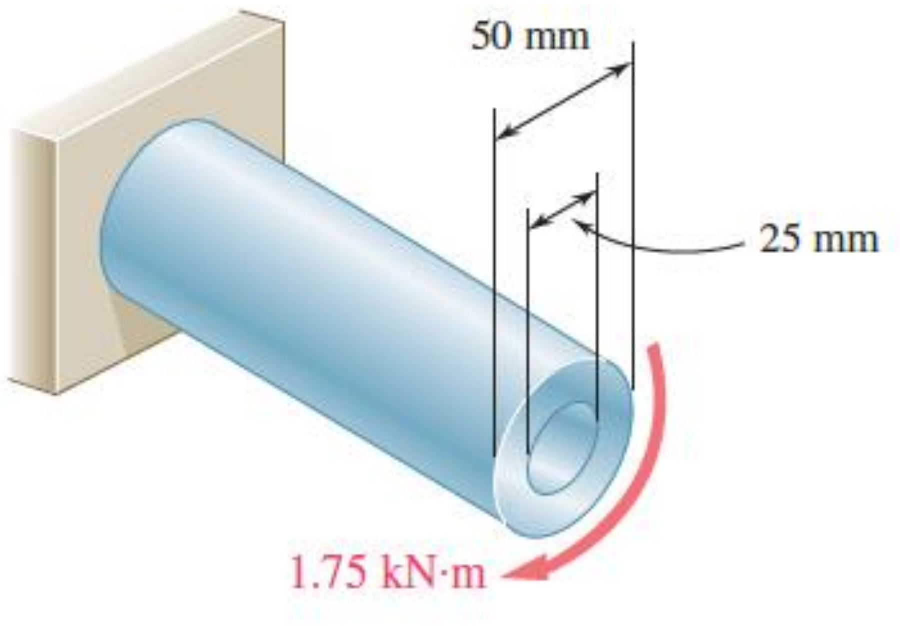


Fig.3.14

Exercise-1

A 1.75-kN·m torque is applied to the solid cylinder shown. Determine (a) the maximum shearing stress, (b) the percent of the torque carried by the inner 25-mm-diameter core.



Answer: (*a*) 71.3 MPa. (*b*) 6.25%.