Foundations of Solid Mechanics

L6: Properties of Geometric Sections

Department of Civil Engineering
School of Engineering
Aalto University

Main Contents

In this chapter, we will discuss:

- How to determine the *properties of geometric sections*;
- Fundamentals of beam analysis and design.

Geometric Properties

Type of property:	**Definition:**
1. Cross section area A	Axial stress σ and shear stress τ
2. Centroid C	Center of mass (Neutral Axis)
3. Moment of Inertia I	Bending stress and deflection
4. Polar Moment of Inertia J	Torsion stress τ
5. Section Modulus Q	Shear stress τ
6. Radius of Gyration r	Column slenderness $r = \sqrt{I/A}$

First Moment of an Area; Centroid of an Area

Consider an area A located in the xy plane (Fig. 6-1). Denoting by x and y the coordinates of an element of area dA, we define *the first moment of the area A with respect to the x axis as the integral*

$$Q_x = \int_A y dA \qquad (1)$$

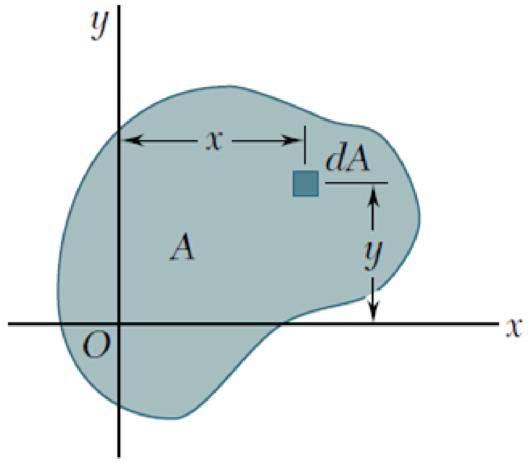

Fig. 6-1

Similarly, the first moment of the area A with respect to the y axis is defined as the integral

$$Q_y = \int_A x dA \qquad (2)$$

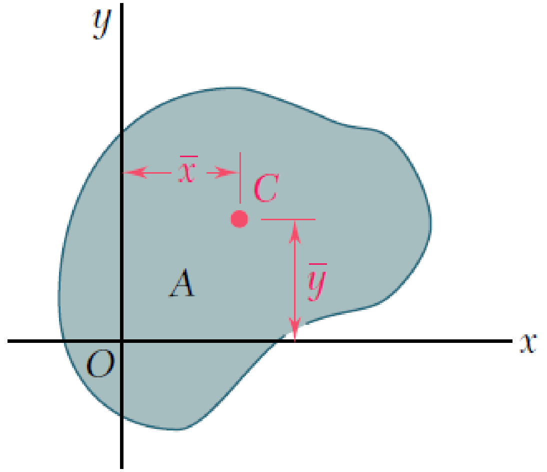

Fig. 6-2

Each of these integrals may be positive, zero, or negative. The centroid of the area A is defined as the point C of coordinates $\bar{x}$ and $\bar{y}$.

First Moment of an Area; Centroid of an Area

$$\int_A x\,dA = A\bar{x} \qquad \int_A y\,dA = A\bar{y} \qquad (3)$$

Comparing Eqs. (1) and (2) with Eqs. (3), we note that the first moments of the area A can be expressed as the products of the area and of the coordinates of its centroid:

$$Q_x = A\bar{y} \qquad Q_y = A\bar{x} \qquad (4)$$

When an area possesses an axis of symmetry, the first moment of the area with respect to that axis is zero.

When an area possesses a *center of symmetry O*, the first moment of the area about any axis through O is zero.

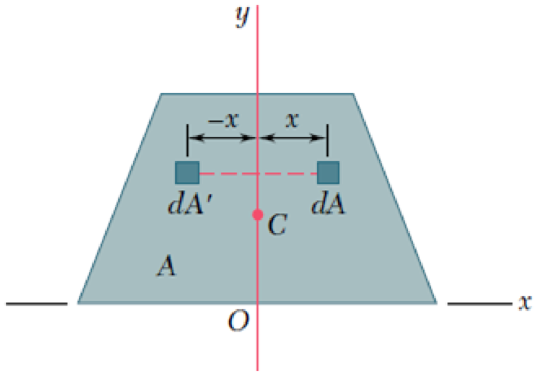

Fig. 6-3

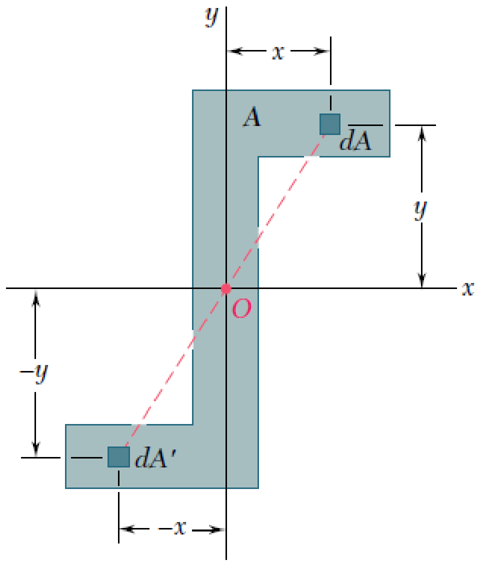

Fig. 6-4

First Moment and Centroid of a Composite Area

Consider an area A, such as the trapezoidal area shown in Fig. 6.5, which may be divided into simple geometric shapes. As we saw in the preceding section, the first moment Q_x of the area with respect to the x axis is represented by the integral $\int ydA$, which extends over the entire area A. dividing A into its component pares A_1, A_2, A_3, we write

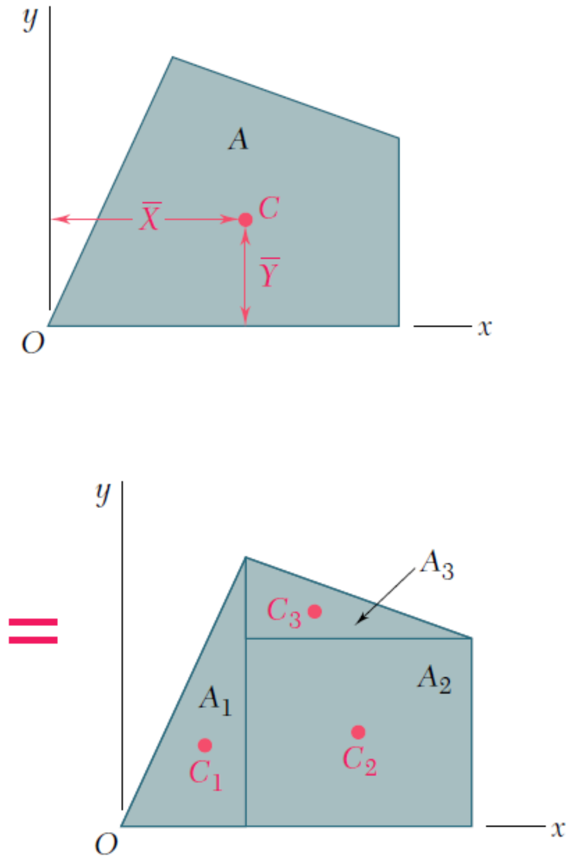

Fig. 6-5

$$Q_x = \int_A ydA = \int_{A_1} ydA + \int_{A_2} ydA + \int_{A_3} ydA$$

or $$Q_x = A_1\bar{y}_1 + A_2\bar{y}_2 + A_3\bar{y}_3 \qquad (5)$$

Then we have $$Q_x = \sum A_i\bar{y}_i \qquad Q_y = \sum A_i\bar{x}_i$$

To obtain the coordinates $\bar{X}$ and $\bar{Y}$ of the centroid C of the composite are A,

$$\bar{X} = \frac{\sum A_i\bar{x}_i}{\sum A_i} \qquad \bar{Y} = \frac{\sum A_i\bar{y}_i}{\sum A_i} \qquad (6)$$

Second Moment, or Moment Inertia of an Area

Consider again an area A located in the xy plane (Fig. 6-1) and the element of area dA of coordinates x and y. The *second moment*, or *moment of inertia*, of the area A with respect to the x axis, and the second moment, or moment of inertia, of A with respect to the y axis are defined, respectively, as

$$I_x = \int_A y^2 dA \qquad I_y = \int_A x^2 dA \tag{7}$$

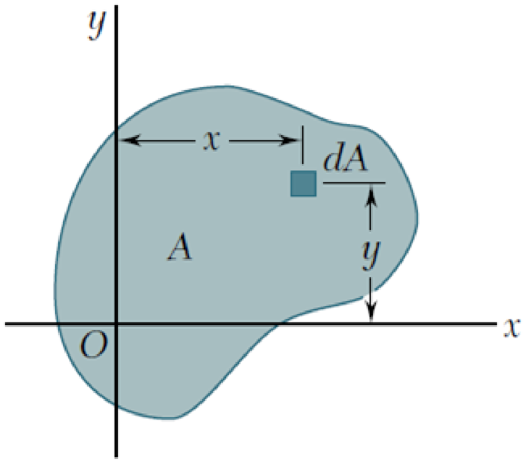

Fig. 6-1

These integrals are referred to as *rectangular moments of inertia*, since they are computed from the rectangular coordinates of the element dA. While each integral is actually a double integral, it is possible in many applications to select elements of area dA in the shape of thin horizontal or vertical strips, and thus reduce the computations to integrations in a single variable.

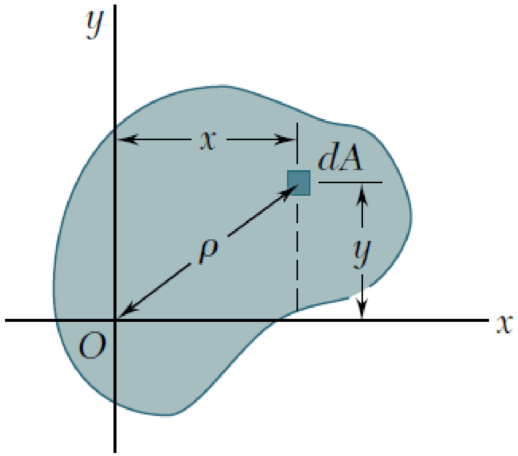

Fig. 6-6

Foundations of Solid Mechanics

Second Moment, or Moment Inertia of an Area

We now define the polar moment of inertia of the area A with respect to point *O* as the integral

$$J_o = \int_A \rho^2 dA \qquad (8)$$

where ρ is the distance from *O* to the element *dA*. While this integral is again a double integral, it is possible in the case of a circular area to select elements of area *dA* in the shape of thin circular rings, and thus reduce the computation of J_O to a single integration.

The moments of inertia of an area are positive quantities.

An important relation may be established between the polar moment of inertia J_O of a given area and the rectangular moments of inertia I_x and I_y of the same area. Noting that $\rho^2 = x^2 + y^2$, we write

$$J_o = \int_A \rho^2 dA = \int_A (x^2 + y^2) dA = \int_A x^2 dA += \int_A y^2 dA$$

or $$J_o = I_x + I_y \qquad (9)$$

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Second Moment, or Moment Inertia of an Area

The *radius of gyration* of an area A with respect to the x axis is defined as the quantity r_x, that satisfies the relation

$$I_x = r_x^2 A \tag{10}$$

where I_x is the moment of inertia of A with respect to the x axis. Solving Eq. (10) for r_x, we have

$$r_x = \sqrt{\frac{I_x}{A}} \tag{11}$$

In a similar way, we define the radii of gyration with respect to the y axis and the origin O. We write

$$I_y = r_y^2 A \qquad r_y = \sqrt{\frac{I_y}{A}} \tag{12}$$

$$I_o = r_o^2 A \qquad r_o = \sqrt{\frac{I_o}{A}} \tag{13}$$

Substituting for J_O, I_x, and I_y in terms of the corresponding radii of gyration in Eq. (9), we observe that

$$r_o^2 = r_x^2 + r_y^2 \tag{14}$$

Parallel-Axis Theorem

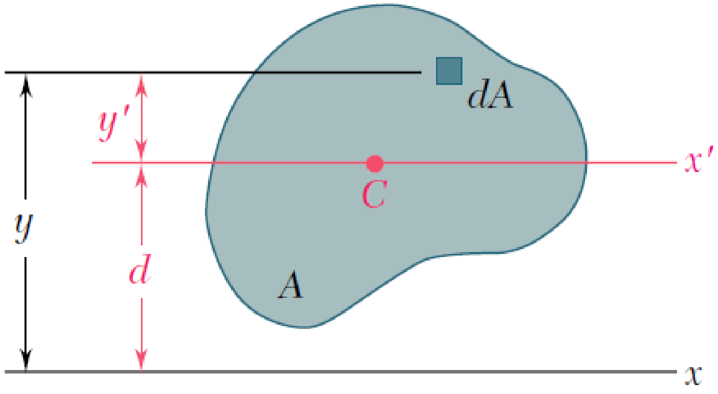


Fig. 6-7

Consider the moment of inertia I_x of an area A with respect to an arbitrary x axis. Denoting by y the distance from an element of area dA to that axis, we recall that

$$I_x = \int_A y^2 dA \tag{15}$$

Let us now draw the *centroid* x' *axis*, i.e., the axis parallel to the x axis which passes through the centroid C of the area. Denoting by y' the distance from the element dA to that axis, we write $y = y' + d$, where d is the distance between the two axes. Substituting for y in the integral representing I_x, we write

$$I_x = \int_A y^2 dA = \int_A (y' + d)^2 dA = \int_A {y'}^2 dA + 2d\int_A y' dA + d^2 \int_A dA \tag{16}$$

The first integral in Eq. (16) represents the moment of inertia I_x of the area with respect to the centroid x' axis. The second integral represents the first moment $Q_{x'}$ of the area with respect to the x' axis and is equal to zero, since the centroid C of the area is located on that axis.

Parallel-Axis Theorem

$$Q_{x'} = A\overline{y'} = A \times 0 = 0$$

Finally, we observe that the last integral in Eq. (16) is equal to the total area *A*. We have, therefore,

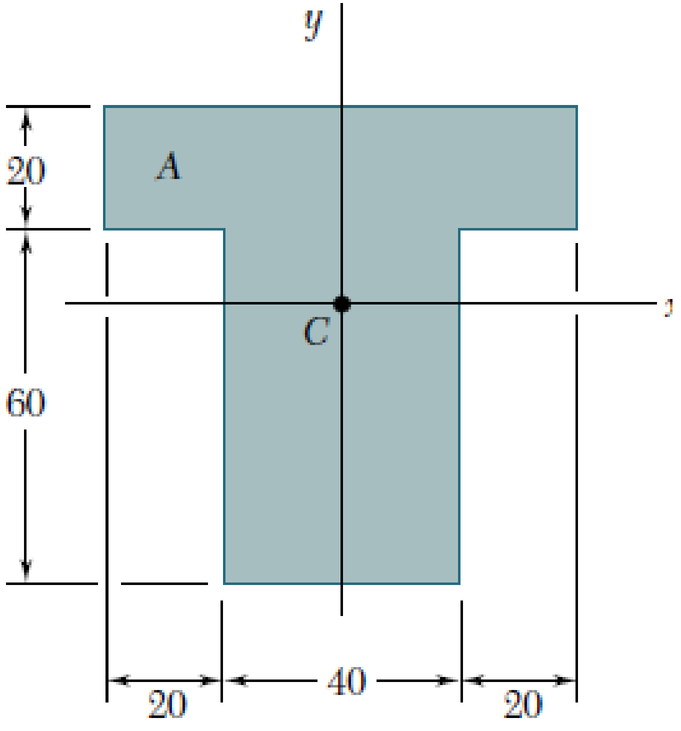
$$I_x = \bar{I}_{x'} + Ad^2 \qquad (17)$$

A similar formula may be derived, which relates the polar moment of inertia J_O of an area with respect to an arbitrary point *O* and the polar moment of inertia JC of the same area with respect to its centroid *C*. Denoting by *d* the distance between *O* and *C*, we write

$$J_o = \bar{J_c} + Ad^2 \qquad (18)$$

Example-1

Determine the moment of inertia I_x of the area shown with respect to the centroidal x axis.

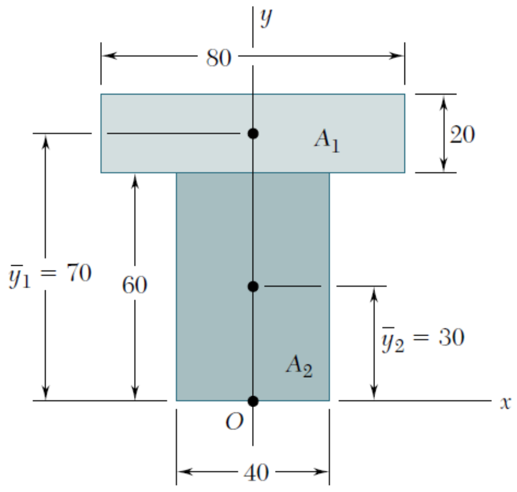


Example-1

Location of Centroid. The centroid C of the area must first be located. Selecting the coordinate axes shown in the figure, we note that the centroid C must be located on the y axis, since this axis is an axis of symmetry; thus, $\bar{X} = 0$. Dividing A into its component parts A_1 and A_2, we determine the ordinate Y of the centroid.

$$\bar{Y} = \frac{\sum A_i \bar{y}_i}{\sum A_i} = \frac{20 \times 80 \times 70 + 40 \times 60 \times 30}{20 \times 80 + 40 \times 60} = \frac{184 \times 10^3 mm^3}{4 \times 10^3 mm^2} = 46mm$$

Computation of Moment of Inertia. We divide the area A into the two rectangular areas A_1 and A_2, and compute the moment of inertia of each area with respect to the x axis.



Dimensions in mm

Rectangular Area A_1.

$$(\bar{I}_{x'})_1 = \frac{1}{12} bh^3 = \frac{1}{12} \times (80mm) \times (20mm)^3 = 53.3 \times 10^3 mm^4$$

$$(I_x)_1 = (\bar{I}_{x'})_1 + A_1 d_1^2 = 53.3 \times 10^3 + (80 \times 20) \times 24^2 = 975 \times 10^3 mm^4$$

Example-1

Rectangular Area A_2.

$$(\bar{I}_{x'})_2 = \frac{1}{12}bh^3 = \frac{1}{12} \times (40mm) \times (60mm)^3 = 720 \times 10^3 mm^4$$

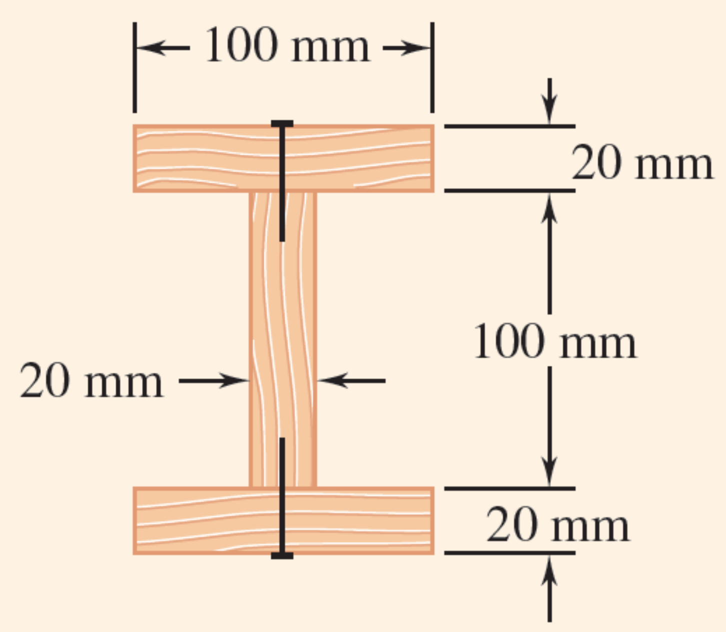
$$(I_x)_2 = (\bar{I}_{x'})_2 + A_2 d_2^2 = 720 \times 10^3 + (40 \times 60) \times 16^2 = 1334 \times 10^3 mm^4$$

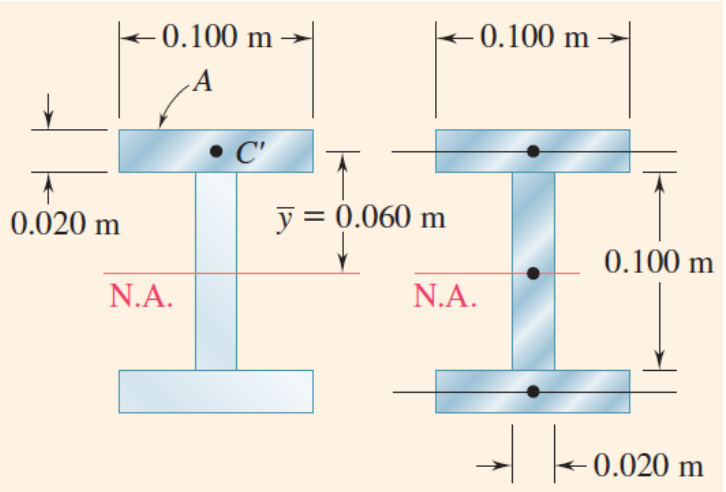
Entire Area *A*. Adding the values computed for the moments of inertia of A_1 and A_2 with respect to the x axis, we obtain the moment of inertia Ix of the entire area:

$$\bar{I}_x = (I_x)_1 + (I_x)_2 = 975 \times 10^3 + 1334 \times 10^3 = 2.31 \times 10^6 mm^4$$

Example-2

A beam is made of three planks, 20 by 100 mm in cross section, and nailed together. Knowing that the spacing between nails is 25 mm and the vertical shear in the beam is V = 500 N, determine the shearing force in each nail





$$Q = A\bar{y} = (0.020m \times 0.100m)(0.060m) = 120 \times 10^{-6} m^3$$

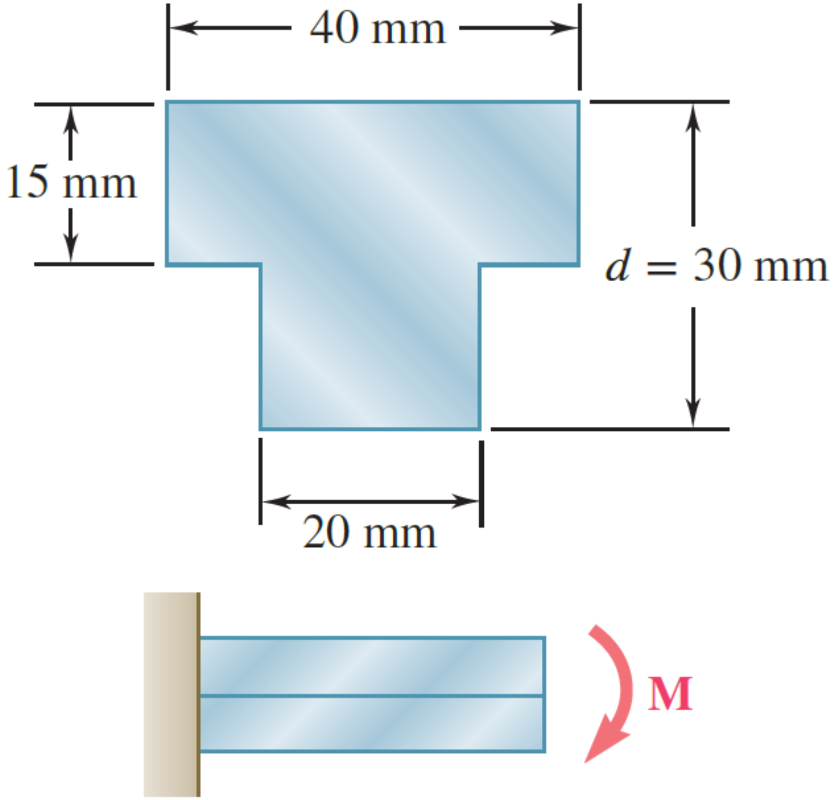
$$I = \frac{1}{12}(0.020m)(0.100m)^3 + 2\left[\frac{1}{12}(0.100m)(0.020m)^3 + (0.020m \times 0.100m)(0.060m)^2\right]$$
$$= 16.20 \times 10^{-6} m^3$$

$$\tau = \frac{VQ}{It} = \frac{(500N)(120 \times 10^{-6} m^3)}{(16.20 \times 10^{-6} m^3) \times t}$$

$$F = \tau \times t \times (0.025m) = 92.6N$$

Exercise-1

The beam shown is made of a nylon for which the allowable stress is 24 MPa in tension and 30 MPa in compression. Determine the largest couple *M* that can be applied to the beam.



Exercise-2

A cast-iron machine part is acted upon by the 3 kN·m couple shown. Knowing that E = 165 GPa and neglecting the effect of fillets, determine the maximum tensile and compressive stresses in the casting.