Foundations of Solid Mechanics

L9: Deflection of Beams

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Necessity

- The deflection of a beam must often be limited in order to provide integrity and stability of a structure or a machine and prevent the cracking of any attached brittle materials such as concrete or glass.
- Furthermore, code restrictions often require these members not vibrate or deflect severely in order to safely support their extended loading.
- Most importantly, though, deflections at specific points on a beam must be determined if one is to analyze those that are statically indeterminate.

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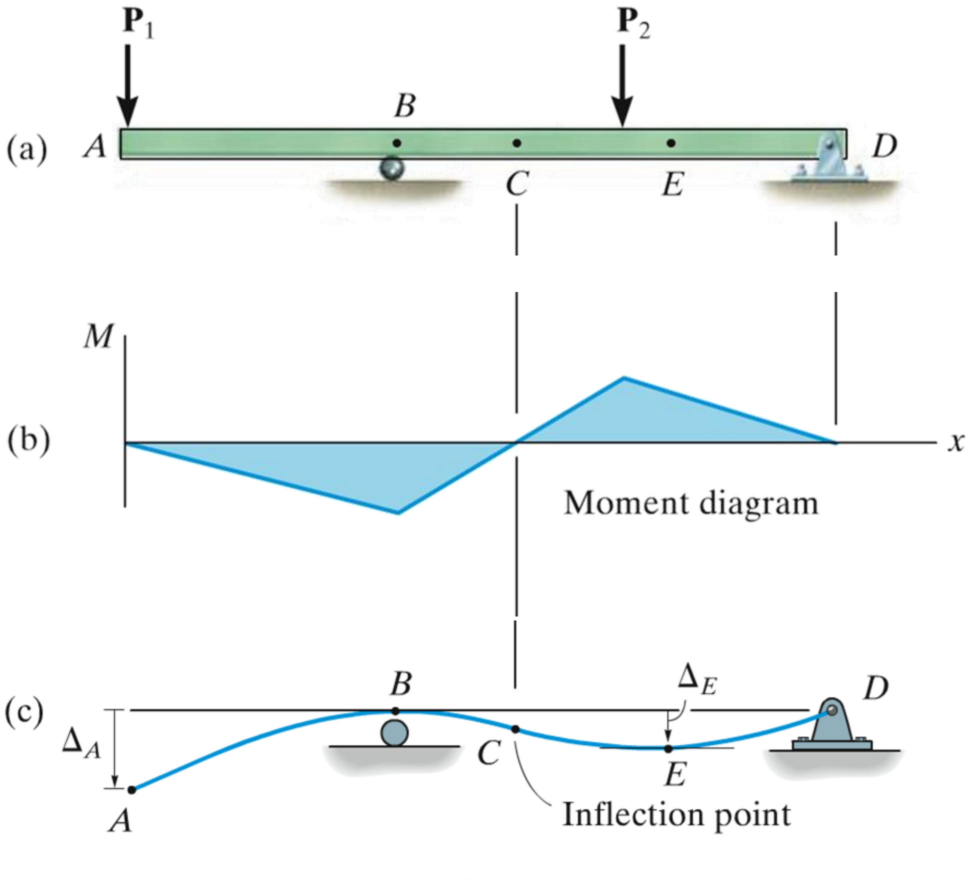
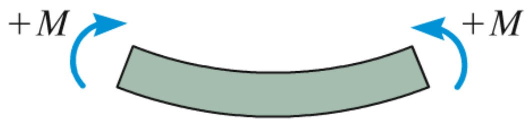
The elastic curve

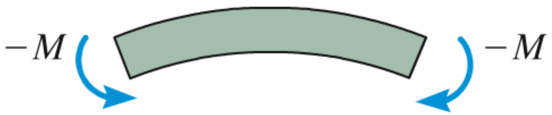
The deflection curve of the longitudinal axis that pass through the centroid of each cross-sectional area of a beam is called ***elastic curve***. The elastic curve is mainly used for visualizing and checking the computed results.



Elastic curve



Positive internal moment
concave upwards
(a)



Negative internal moment
concave downwards
(b)

Assumptions

1. Elastic behavior
2. Negligible axial loading
3. Small deformation

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Moment-Curvature Relationship

$$\varepsilon = \frac{ds' - ds}{ds}$$

$$ds = dx = \rho d\theta \quad ds' = (\rho - y)d\theta$$

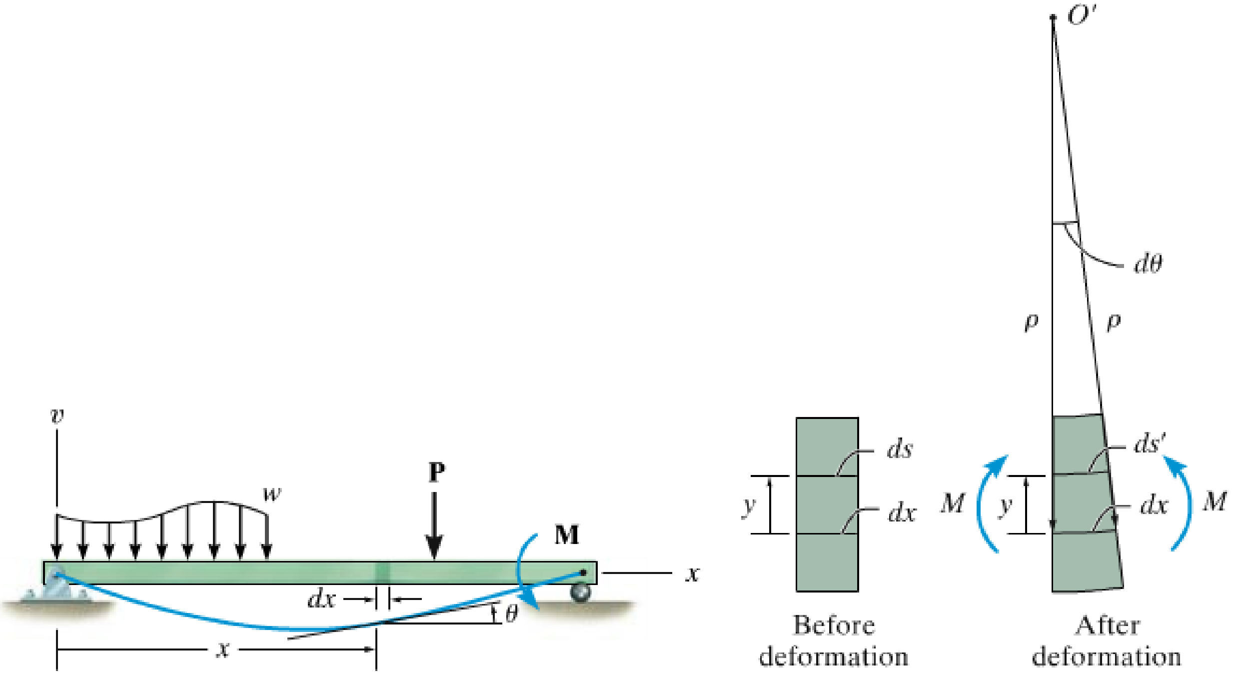
$$\varepsilon = [(\rho - y)d\theta - \rho d\theta]/\rho d\theta$$

$$\frac{1}{\rho} = -\frac{\varepsilon}{y}$$

$$\varepsilon = \frac{\sigma}{E} \qquad \sigma = \frac{My}{I}$$

$$\frac{1}{\rho} = -\frac{M}{EI}$$

The *radius of curvature* for this arc is defined as the distance ρ, which is measured from the center of curvature *O'* to *dx*.



(a) (b)

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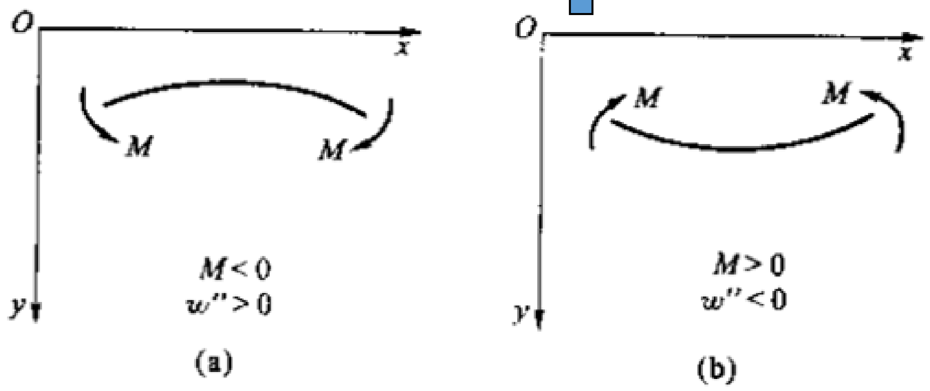
Moment-Curvature Relationship

$$\frac{1}{\rho} = \pm \frac{d^2v/dx^2}{[1+(dv/dx)^2]^{3/2}}$$

$$\frac{d^2v/dx^2}{[1+(dv/dx)^2]^{3/2}} = -\frac{M}{EI}$$



$$\boxed{\frac{d^2v}{dx^2} = -\frac{M}{EI}}$$

$$\left.\begin{aligned} -\frac{d}{dx}\left(EI\frac{d^2v}{dx^2}\right) &= V(x) \\ -\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) &= \omega(x) \end{aligned}\right\}$$

Moment-Curvature Relationship

$$\boxed{\frac{d^2v}{dx^2} = -\frac{M}{EI}} \quad \left\{ \begin{aligned} -EI\frac{d^4v}{dx^4} &= \omega(x) \\ -EI\frac{d^3v}{dx^3} &= V(x) \\ -EI\frac{d^2v}{dx^2} &= M(x) \end{aligned} \right.$$

Euler-Bernoulli Equation

It is a special case of [Timoshenko beam theory] that accounts for shear deformation and is applicable for thick beams.

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Procedure for Analysis

Elastic Curve

- Draw the beam's elastic curve. Recall that *zero slope* and *zero displacement* occur at all *fixed supports*, and *zero displacement* occurs at all *pin and roller supports*.
- Establish the x and v coordinate axes. The x axis must be parallel to the un-deflected beam and can have an origin at any point along the beam, with a positive direction either to the right or to the left.
- If several discontinuous loads are present, establish x coordinates that are valid for each region of the beam between the discontinuities. Choose these coordinates so that they will simplify subsequent algebraic work.

Load and Moment Function

- For each region in which there is an x coordinate, express the loading w or the internal moment M as a function x; in addition, please pay attention to the sign convention of the bending moment function.

Procedure for Analysis

Slope and Elastic Curve

- Provided EI is constant, apply either the load equation $-EI\ d^4v/dx^4=w(x)$, which requires four integrations to get $v=v(x)$, or the moment equation $-EI\ d^2v/dx^2=M(x)$, which requires only two integrations. For each integration it is important to include a constant of integration;
- The constants are evaluated using the *boundary conditions* for the supports and the *continuity conditions* that apply to slope and displacement at points where two functions meet. Once the constants are evaluated and substituted back into the slope and deflection equations, the slope and displacement at specific point on the elastic curve can then be determined.
- The numerical values obtained can be checked graphically by comparing them with the sketch of the displacement curve.

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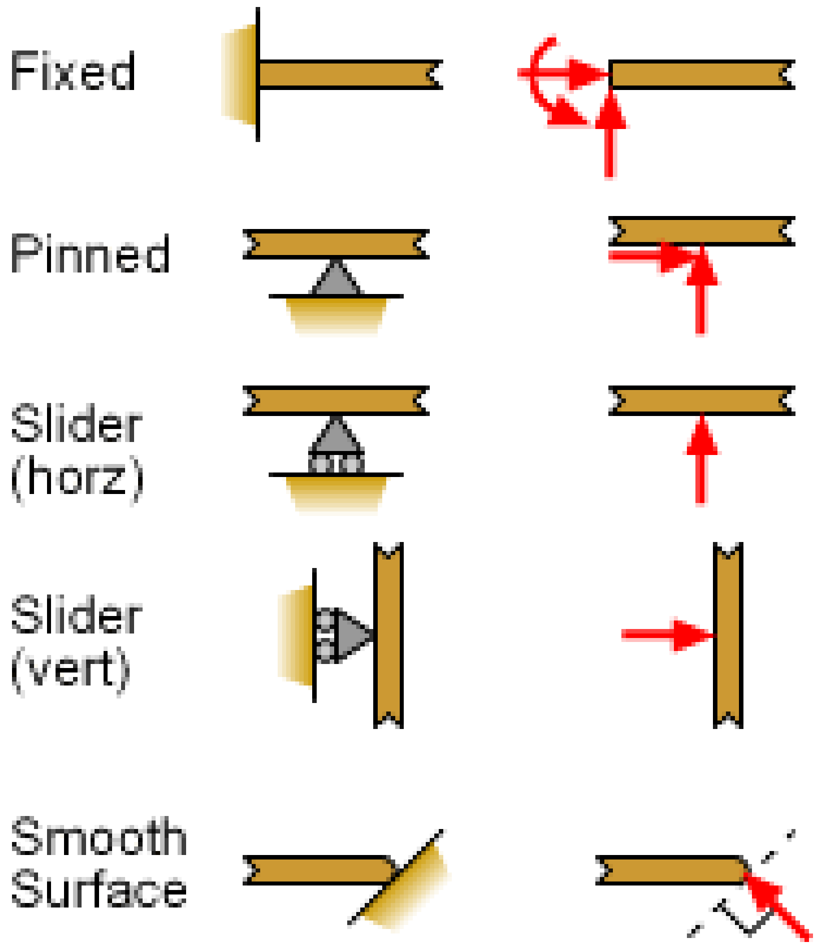
Boundary and Continuity Conditions



The fixed condition has three reactions (2 forces and 1 moment) since it cannot rotate, nor displace in either direction.

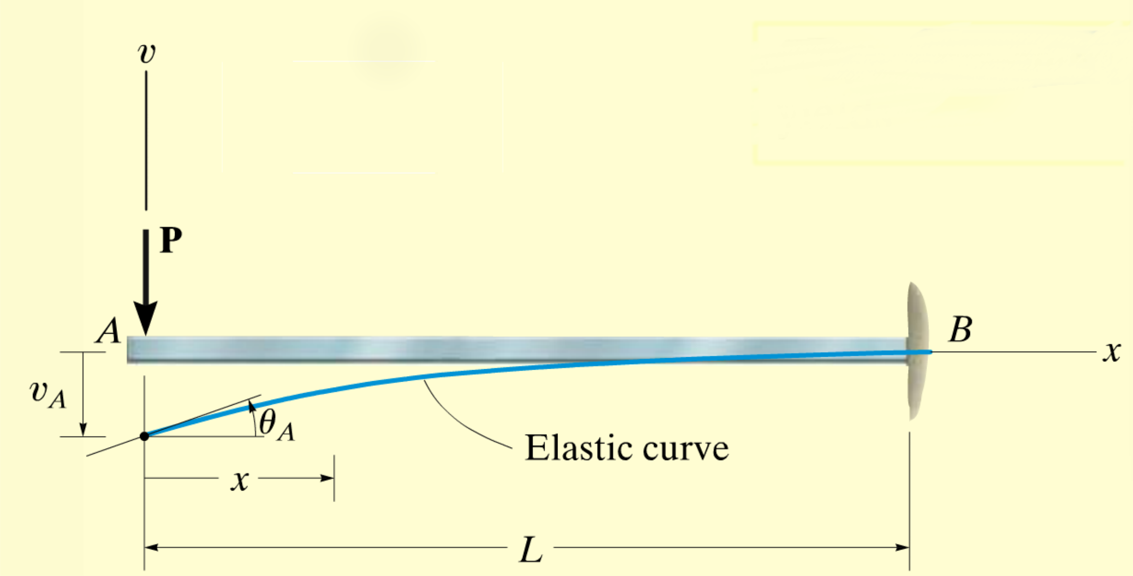
The pinned condition cannot displace but it can freely rotate. This requires only two reaction forces.

The slider condition allows the member to move in only one direction, but it can freely rotate.

A special case is when there is a smooth surface without friction. This allows the member to freely slide in the direction of the slope.

Example-1

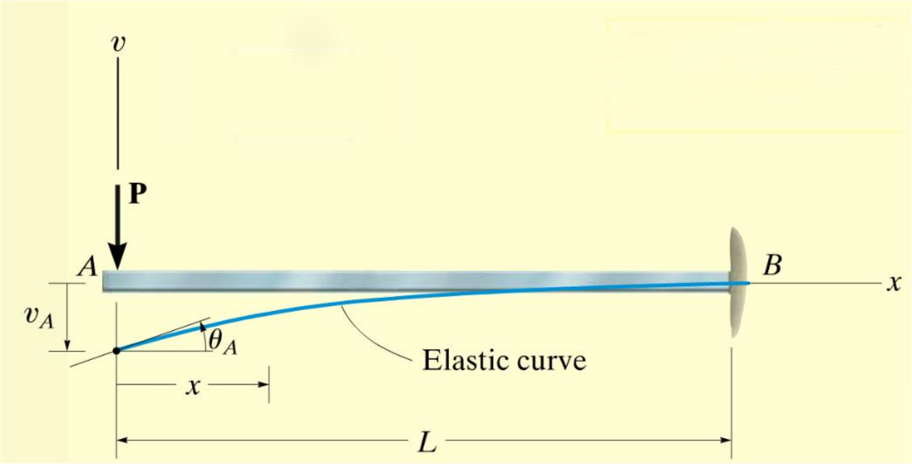
As shown below, the cantilever beam is subjected to a vertical load P at its end. Determine the equation of the elastic curve. EI is constant.



Example-1

- Elastic Curve:

The load tends to deflect the beam, as shown in the figure.



Elastic curve of the cantilever beam

- Moment Function:

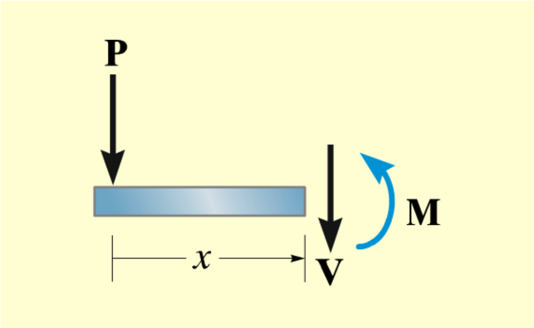
$$M = -Px$$

- Slope and Elastic Curve

$$EI\frac{d^2v}{dx^2} = Px$$

$$EI\frac{dv}{dx} = \frac{1}{2}Px^2 + C_1$$

$$EIv = \frac{1}{6}Px^3 + C_1x + C_2$$



Free body diagram

Example-1

when $x = L, dv/dx = 0$ $\quad 0 = \frac{1}{2}PL^2 + C_1$

$x = L, v = 0$

$$C_1 = -\frac{1}{2}pL^2$$

$$C_2 = +\frac{1}{3}pL^3$$

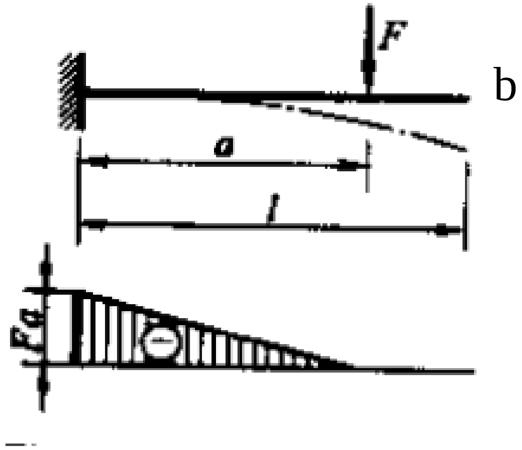
Slope curve: $\theta = \frac{P}{2EI}(-L^2 + x^2)$

Elastic curve: $v = \frac{P}{6EI}(x^3 - 3L^2x + 2L^2)$

The maximum slope and displacement occur at A(x=0), for which:

$$\theta_A = -\frac{P}{2EI}L^2 \qquad v_A = \frac{PL^3}{3EI}$$

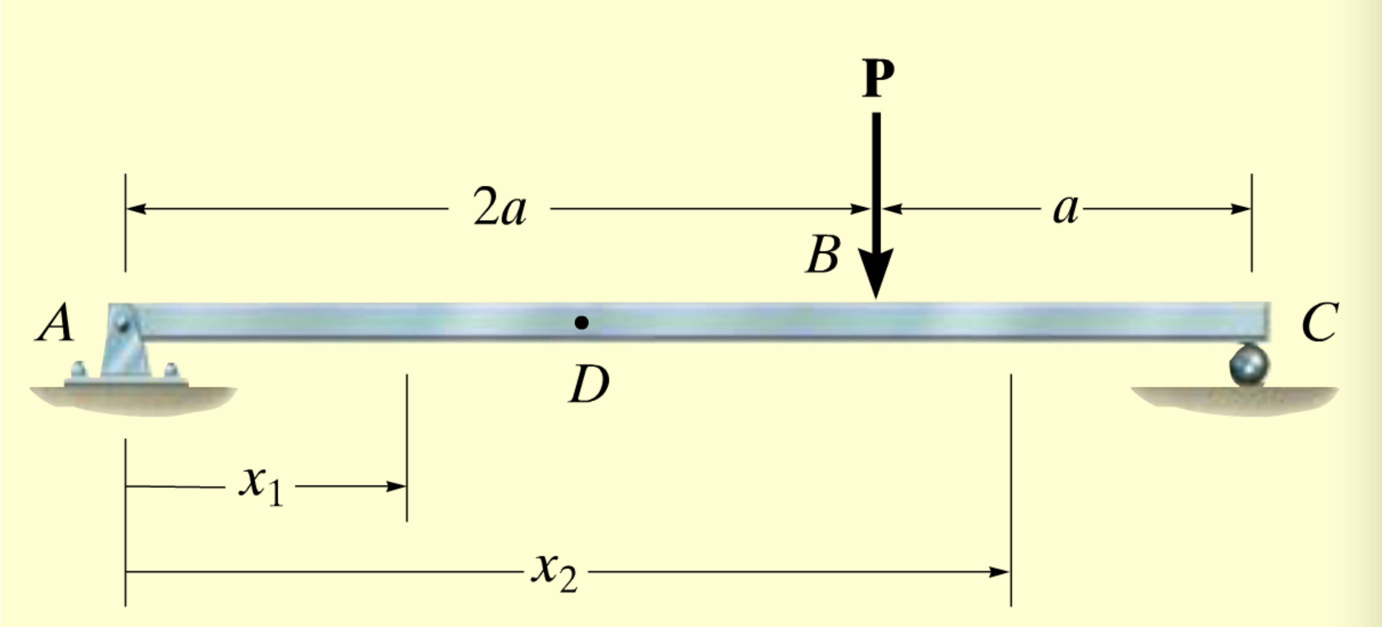
Questions…



Elastic curve	The maximum slope and displacement
$w = \frac{Fx^2}{6EI}(3a - x)$ $(0 \leqslant x \leqslant a)$ $w = \frac{Fa^2}{6EI}(3x - a)$ $(a \leqslant x \leqslant l)$	$\theta_B = \frac{Fa^2}{2EI}$ $w_B = \frac{Fa^2}{6EI}(3l - a)$

Example-2

As shown below, the simply supported beam supports the concentrated force P. Determine its elastic curve and maximum deflection. EI is constant.



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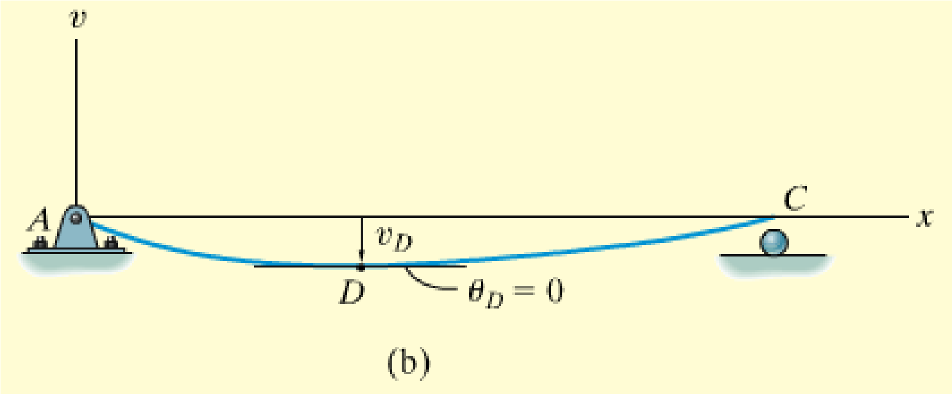
Example-2

- Elastic Curve:

  The load tends to deflect the beam, as shown in the figure.

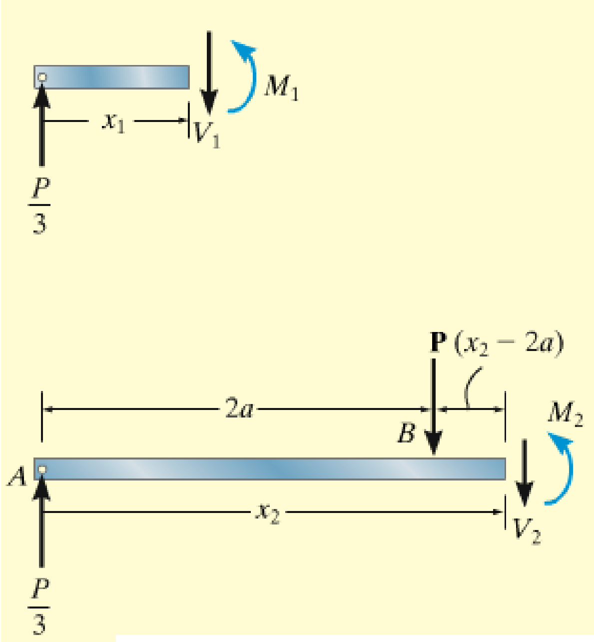


Elastic curve

- Moment Function:

$M_1 = Px_1/3 \qquad 0 \le x_1 \le 2a$

$M_2 = 2P(3a - x_2)/3 \qquad 2a \le x_2 \le 3a$



Free body diagram

- Slope and Elastic Curve

  When $0 \le x_1 \le 2a$, integrating twice yields.

$$EI\frac{d^2v_1}{dx_1^2} = -\frac{p}{3}x_1$$

$$EI\frac{dv_1}{dx_1} = -\frac{1}{6}px_1^2 + C_1$$

$$EIv_1 = -\frac{1}{18}px_1^3 + C_1x_1 + C_2$$

Example-2

Likewise, for $2a \le x_2 \le 3a,$

$$EI\frac{d^2v_2}{dx_2^2} = -\frac{2P}{3}(3a - x_2)$$

$$EI\frac{dv_2}{dx_2} = -\frac{2P}{3}(3ax_2 - \frac{1}{2}x_2^2) + C_3$$

$$EIv_2 = -\frac{2P}{3}(\frac{3}{2}a\,x_2^2 - \frac{1}{6}x_2^3) + C_3x_2 + C_4$$

Boundary conditions:

$x = 0, v_1 = 0$ ⇒ $0 = 0 + 0 + C_2$

$x = 3a, v_2 = 0$ ⇒ $0 = -\frac{2P}{3}[\frac{3}{2}a(3a)^2 - \frac{1}{6}(3a)^3] + C_3(3a) + C_4$

Continuity conditions:

$x = 2a, v_1 = v_2$ ⇒ $-\frac{1}{18}p(2a)^3 + C_1(2a) + C_2 = -\frac{2P}{3}[\frac{3}{2}a(2a)^2 - \frac{1}{6}(2a)^3] + C_3(2a) + C_4$

$x = 2a, \frac{dv_1}{dx_1} = \frac{dv_2}{dx_2}$ ⇒ $-\frac{1}{6}p(2a)^2 + C_1 = -\frac{2P}{3}(3a(2a) - \frac{1}{2}(2a)^2) + C_3$

Example-2

$$C_1 = \frac{4}{9}pa^2 \qquad C_2 = 0 \qquad C_3 = \frac{22}{9}pa^2 \qquad C_4 = -\frac{4}{3}pa^3$$

$$\theta_1 = \frac{dv_1}{dx_1} = -\frac{P}{6EI}x_1^2 + \frac{4Pa^2}{9EI}$$

$$v_1 = -\frac{P}{18EI}x_1^3 + \frac{4Pa^2}{9EI}x_1$$

$$\theta_2 = \frac{dv_2}{dx_2} = -\frac{2Pa}{EI}x_2 + \frac{P}{3EI}x_2^2 + \frac{22Pa^2}{9EI}$$

$$v_2 = -\frac{Pa}{EI}x_2^2 + \frac{P}{9EI}x_2^3 + \frac{22Pa^2}{9EI}x_2 - \frac{4Pa^3}{3EI}$$

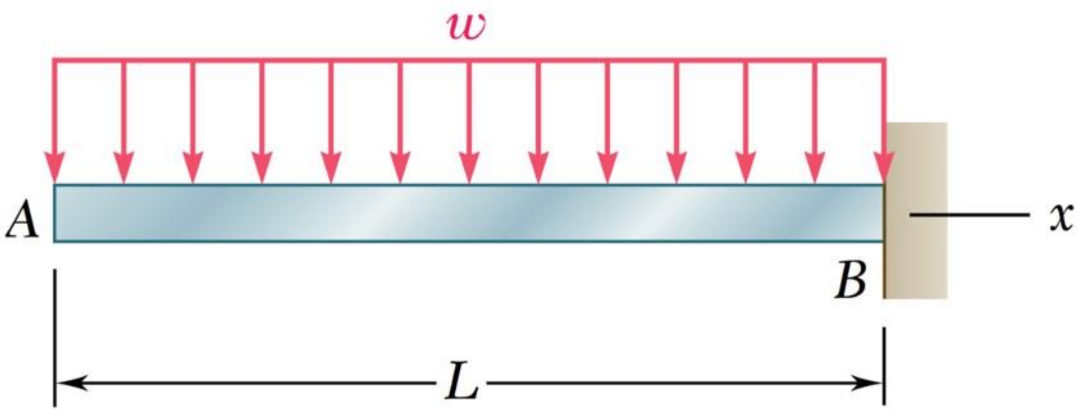
$$\theta_1 = 0$$

$$x_1 = 1.633a$$

$$v_{\max} = 0.484\frac{Pa^3}{EI}$$

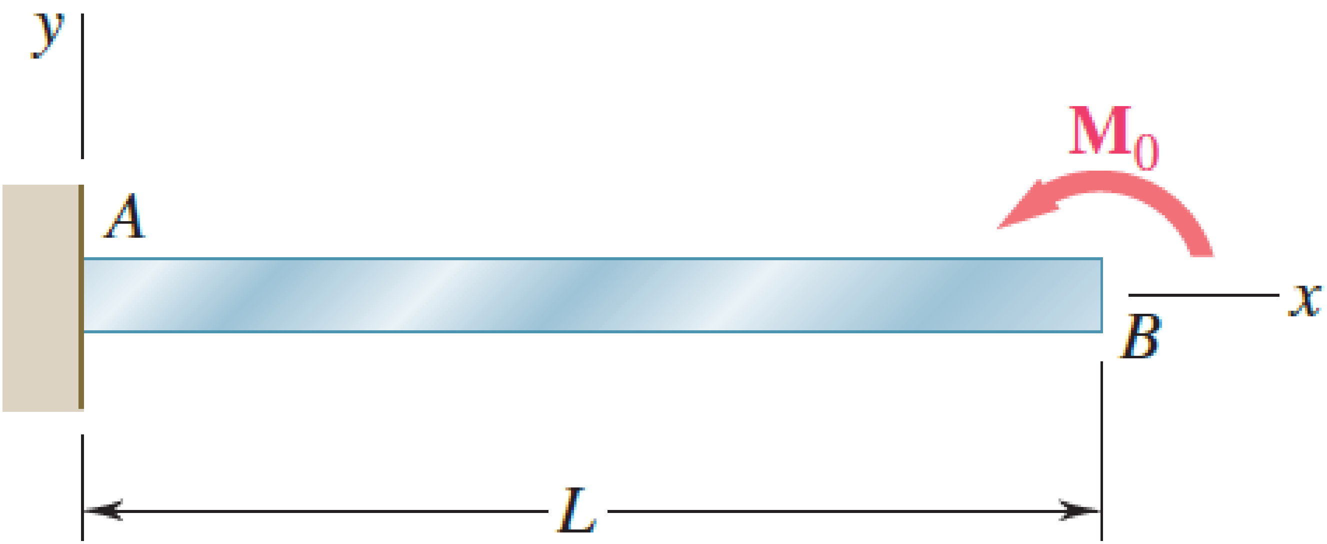
Exercise-1

For the cantilever beam and loading as shown below, determine: (a) the equation of the elastic curve for portion AB of the beam, (b) the deflection at A, and (c) the slope at A.



Exercise-2

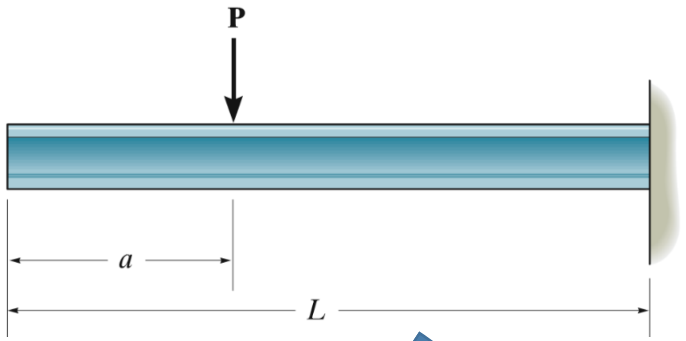
For the loading shown, determine (a) the equation of the elastic curve for the cantilever beam AB, (b) the deflection at the free end, (c) the slope at the free end.

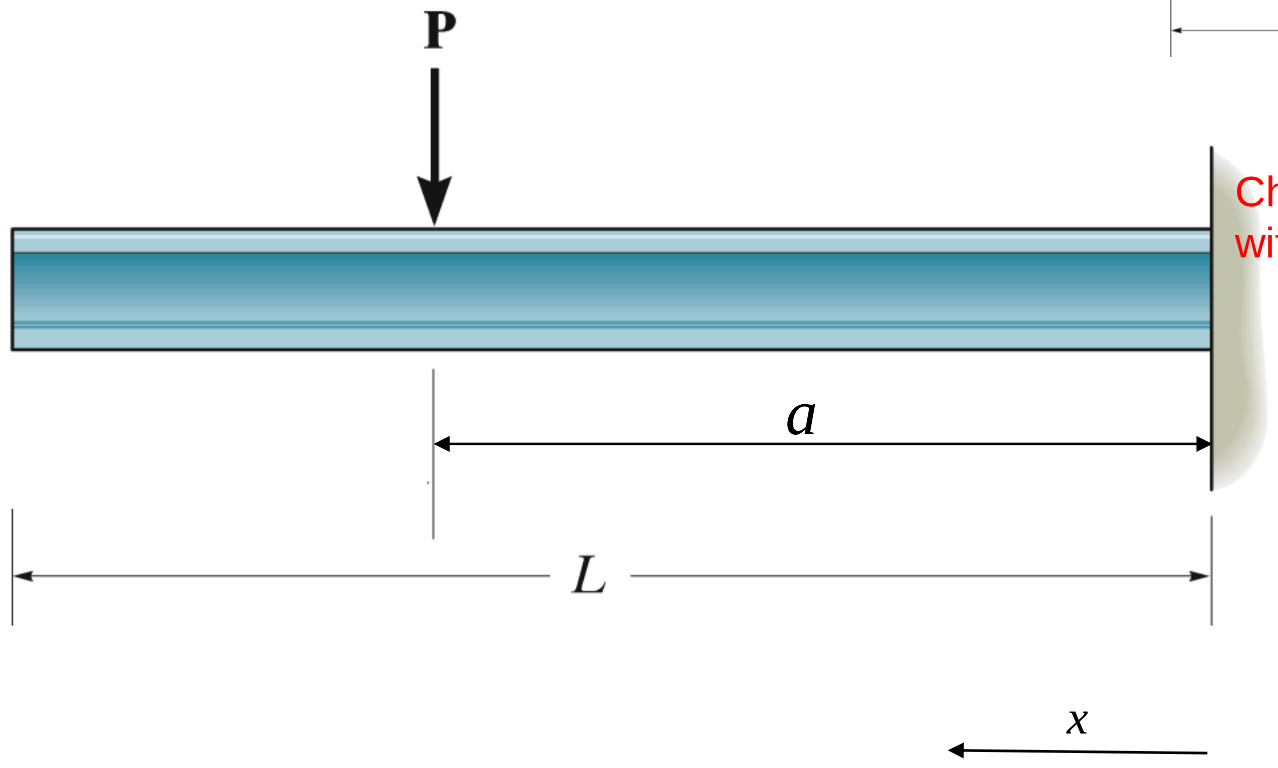


$$\theta_B = \frac{M_0 L}{EI} \qquad v_B = -\frac{M_0 L^2}{2EI}$$

Exercise-3

Determine the equations of the elastic curve. EI is constant.





Changed to compare with results in P13