Foundations of Solid Mechanics

L10: Statically Indeterminate Beam

Department of Civil Engineering
School of Engineering
Aalto University

Statically Determinate and Indeterminate Structures

Statically determinate structures

No. of Equilibrium Equations = No. of unknowns

In this situation we can determine the unknown forces by using the principles of statics to determine the unknown forces, e.g. drawing free body diagrams and solving equilibrium equations.

Statically indeterminate structures

We cannot determine all the unknown forces using the principles of statics.

No. of Equilibrium Equations < No. of unknowns

System is statically indeterminate because we either have **too many members** or **over stiff support conditions** giving too many reaction forces. To solve we need extra information. This information comes from the geometric characteristics of deformation during loading which gives additional equations.

Statically Determinate and Indeterminate Structures

For this case, the following three basic concepts must be satisfied:

Equilibrium Conditions

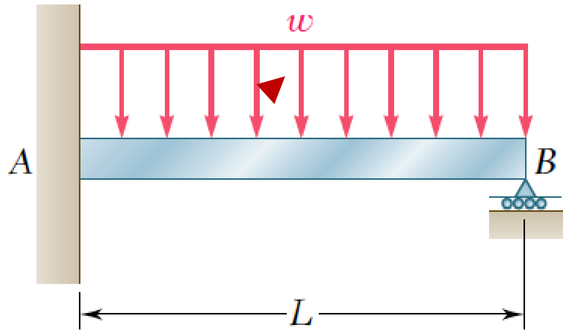
Geometric Compatibility

Constitutive Relations

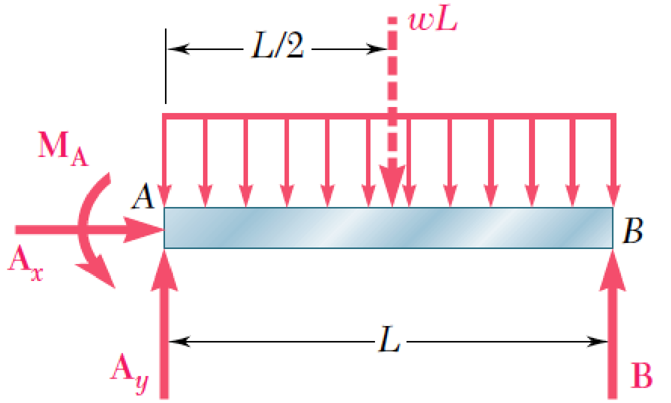
Statically Indeterminate Beams

The prismatic beam AB, which has a fixed end at A and is supported by a roller at B. Reactions involve four unknowns, while only three equilibrium equations are available, namely

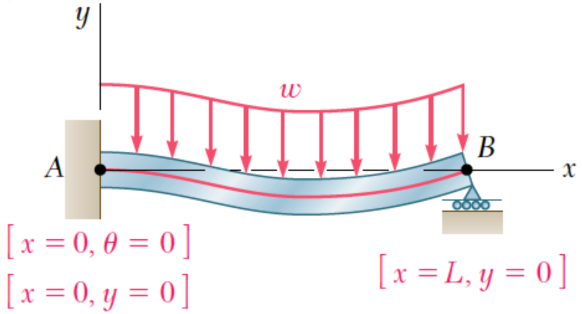
$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum M(x) = 0$$



(a)



(b)



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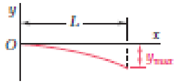
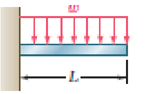
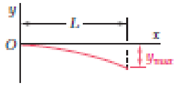
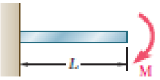
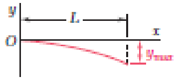
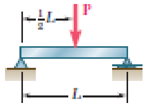
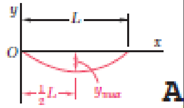
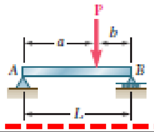
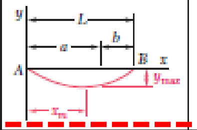
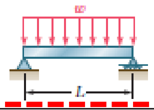
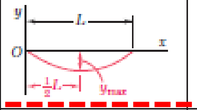
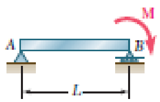
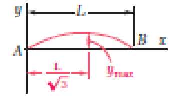
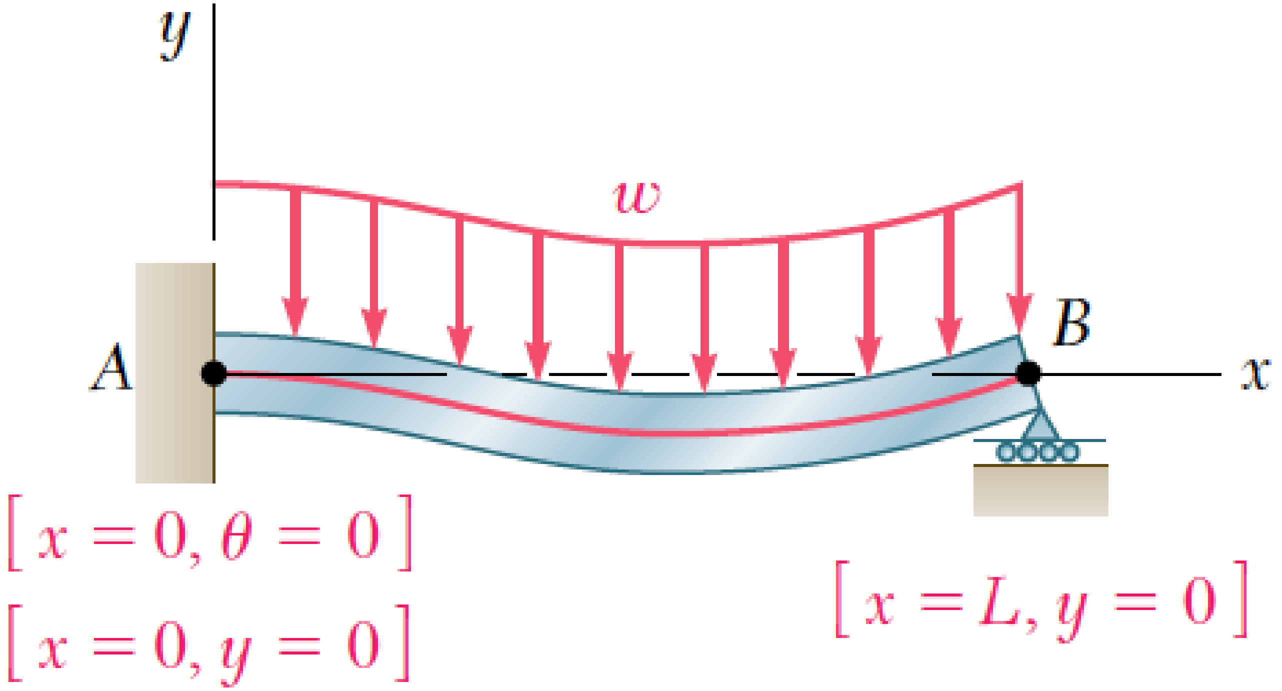
Deflection of Beams

APPENDIX D Beam Deflections and Slopes

Beam and Loading	Elastic Curve	Maximum Deflection	Slope at End	Equation of Elastic Curve
1 		$-\frac{PL^3}{3EI}$	$-\frac{PL^2}{2EI}$	$y = \frac{P}{6EI}(x^3 - 3Lx^2)$
2 		$-\frac{wL^4}{8EI}$	$-\frac{wL^3}{6EI}$	$y = -\frac{w}{24EI}(x^4 - 4Lx^3 + 6L^2x^2)$
3 		$-\frac{ML^2}{2EI}$	$-\frac{ML}{EI}$	$y = -\frac{M}{2EI}x^2$
4 		$-\frac{PL^3}{48EI}$	$\pm\frac{PL^2}{16EI}$	For $x \le \frac{1}{2}L$: $y = \frac{P}{48EI}(4x^3 - 3L^2x)$
5 		For $a > b$: $-\frac{Pb(L^2-b^2)^{3/2}}{9\sqrt{3}EIL}$ at $x_m = \sqrt{\frac{L^2-b^2}{3}}$	$\theta_A = -\frac{Pb(L^2-b^2)}{6EIL}$ $\theta_B = +\frac{Pa(L^2-a^2)}{6EIL}$	For $x < a$: $y = \frac{Pb}{6EIL}[x^3 - (L^2-b^2)x]$ For $x = a$: $y = -\frac{Pa^2b^2}{3EIL}$
6 		$-\frac{5wL^4}{384EI}$	$\pm\frac{wL^3}{24EI}$	$y = -\frac{w}{24EI}(x^4 - 2Lx^3 + L^3x)$
7 		$\frac{ML^2}{9\sqrt{3}EI}$	$\theta_A = +\frac{ML}{6EI}$ $\theta_B = -\frac{ML}{3EI}$	$y = -\frac{M}{6EIL}(x^3 - L^2x)$

Example-1

Since only A_x can be determined from these equations, we conclude that the beam is *statically indeterminate*.



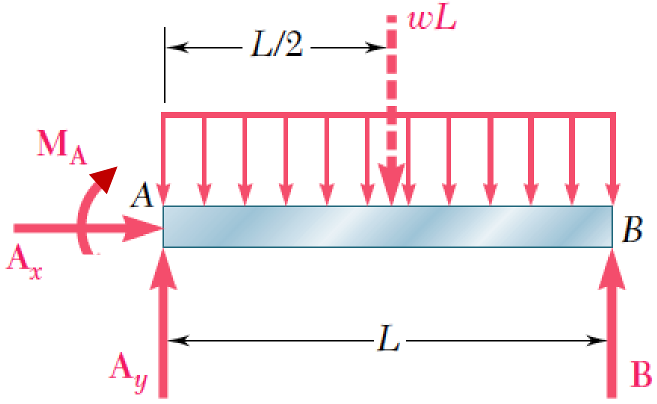
Example-1

Equilibrium equations:

$$\sum F_x = 0 \qquad A_x = 0$$

$$\sum F_y = 0 \qquad A_y + B = 0$$

$$\sum M_A = 0 \qquad M_A - B \times L + \frac{1}{2}\omega L^2 = 0$$



(b)

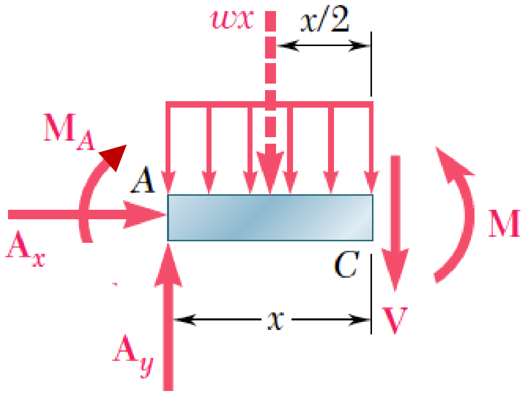
Equation of elastic curve:

$$\sum M_C = 0 \qquad M_A - \frac{1}{2}\omega x^2 + A_y x - M = 0$$

$$EI\frac{d^2v}{dx^2} = -\frac{1}{2}\omega x^2 + A_y x + M_A$$

$$EI\theta = EI\frac{dy}{dx} = -\frac{1}{6}\omega x^3 + \frac{1}{2}A_y x^2 + M_A x + C_1$$

$$EIy = -\frac{1}{24}\omega x^4 + \frac{1}{6}A_y x^3 + \frac{1}{2}M_A x + C_1 x + C_2$$



Example-1

Equilibrium equations:

$x = 0, \theta = 0 \qquad C_1 = 0$

$x = 0, y = 0 \qquad C_2 = 0$

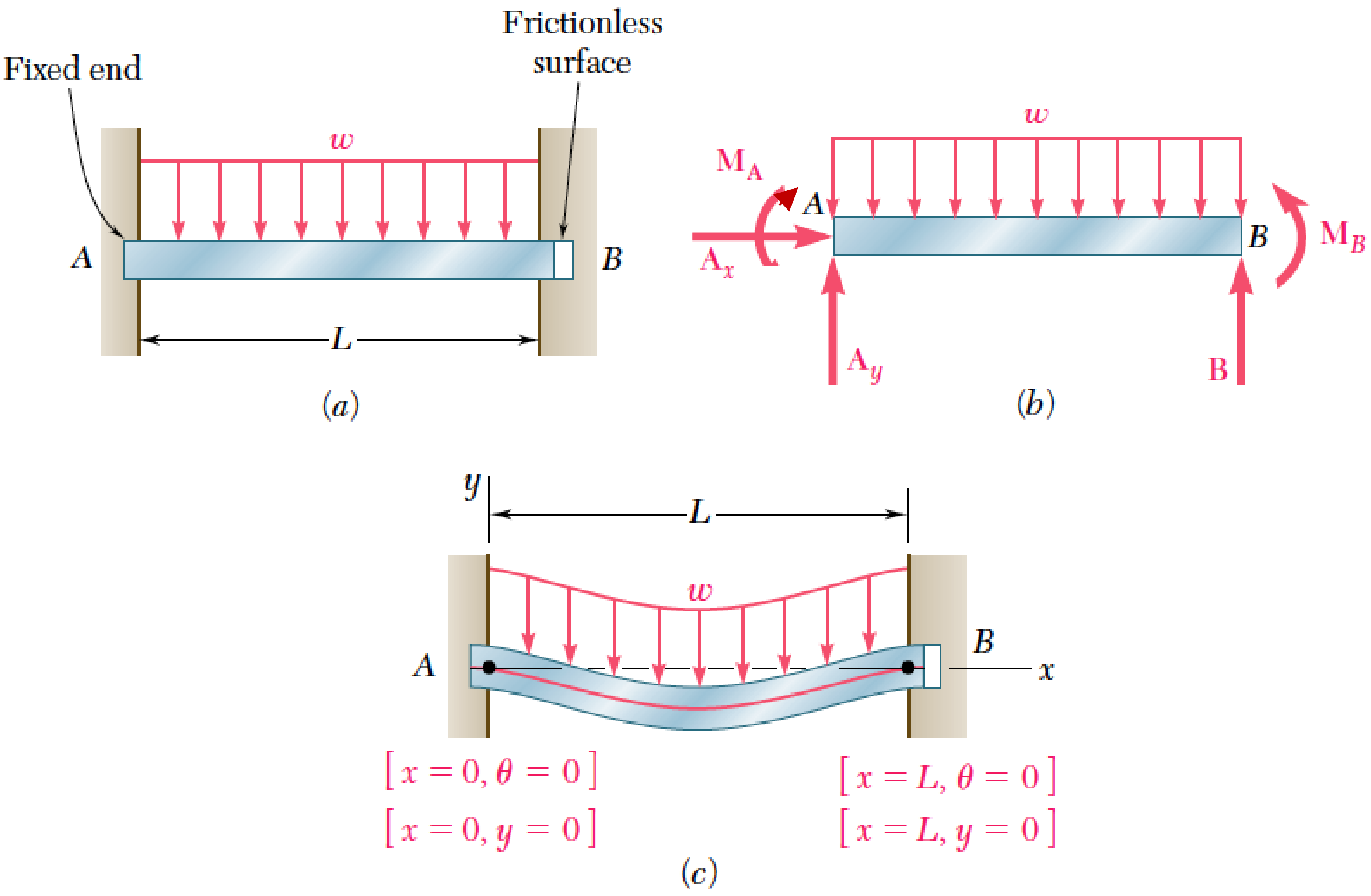
$x = L, y = 0$

$$\Rightarrow \quad EIy = -\frac{1}{24}\omega x^4 + \frac{1}{6}A_y x^3 + \frac{1}{2}M_A x$$

$$\Rightarrow \quad 0 = -\frac{1}{24}\omega L^2 + \frac{1}{6}A_y L^3 + \frac{1}{2}M_A$$

$$A_x = 0 \qquad A_y = \frac{5}{8}\omega L \qquad M_A = -\frac{1}{8}\omega L^2 \qquad B = \frac{3}{8}\omega L$$

Example-2



Beam statically indeterminate to the second degree.

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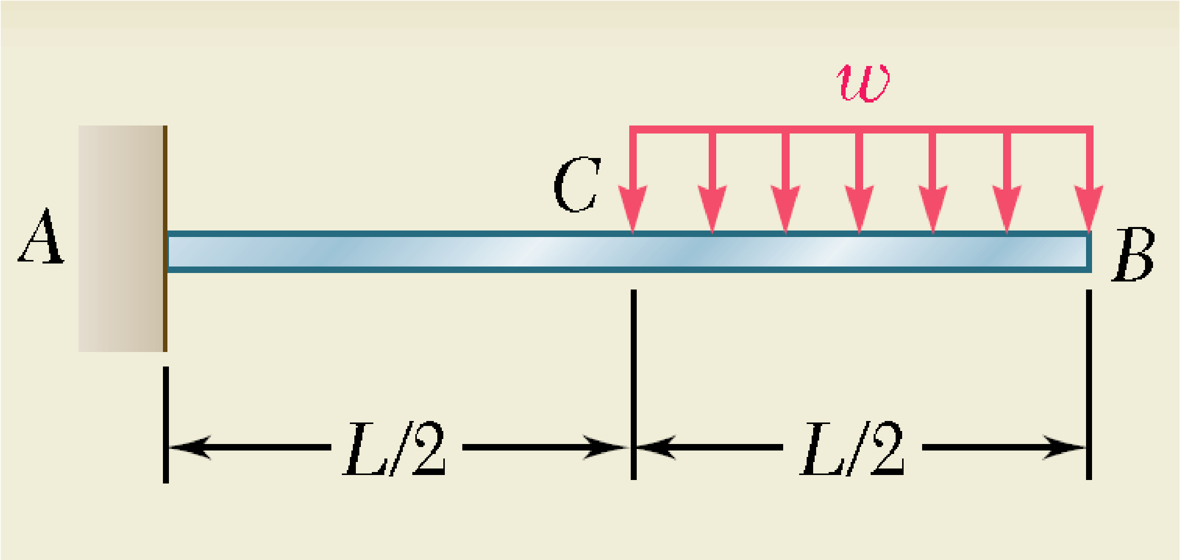
Example-3: Principle of Superposition

For the beam and loading shown, determine the slope and deflection at point B.



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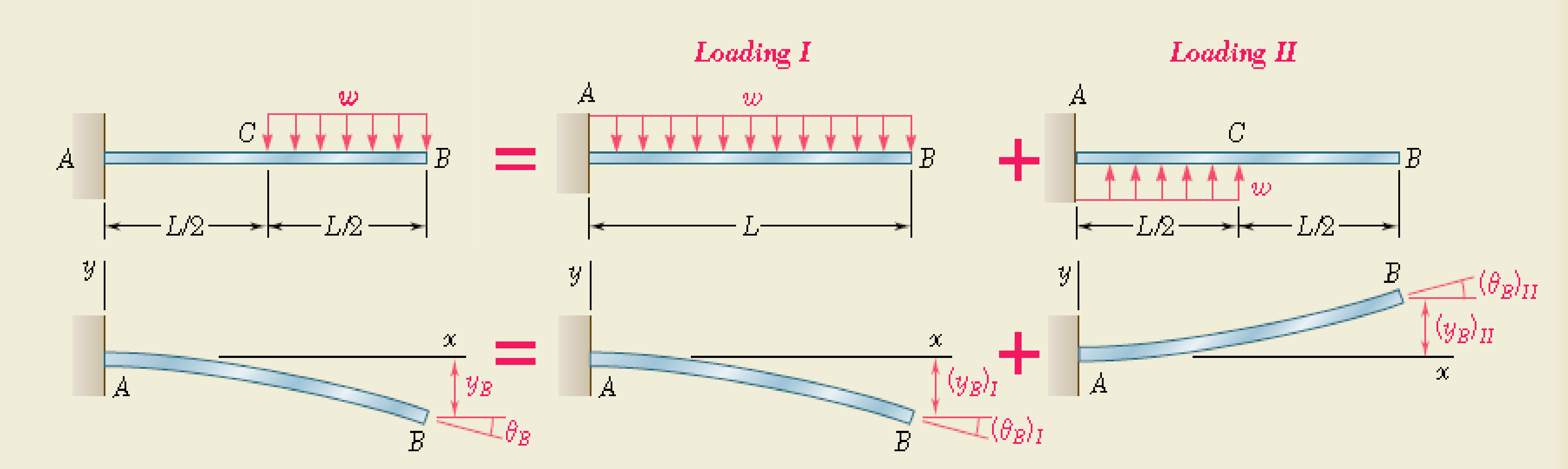
Example-3: Principle of Superposition



Loading I

$$(\theta_B)_I = \frac{\omega L^3}{6EI} \qquad (y_B)_I = \frac{\omega L^4}{8EI}$$

Loading II

$$(\theta_C)_{II} = -\frac{\omega (L/2)^3}{6EI} = -\frac{\omega L^3}{48EI}$$

$$(y_C)_{II} = -\frac{\omega (L/2)^4}{8EI} = -\frac{\omega L^4}{128EI}$$

Exanple-3: Principle of Superposition

In portion CB, the bending moment for loading II is zero and thus the elastic curve is a straight line.

$$(\theta_B)_{II} = (\theta_C)_{II} = -\frac{\omega L^3}{48EI}$$

$$(y_B)_{II} = (y_C)_{II} + (\theta_C)_{II}\left(\frac{L}{2}\right) = -\frac{\omega L^4}{128EI} - \frac{\omega L^3}{48EI}\left(\frac{L}{2}\right) = -\frac{7\omega L^4}{384EI}$$

Slope at Point B

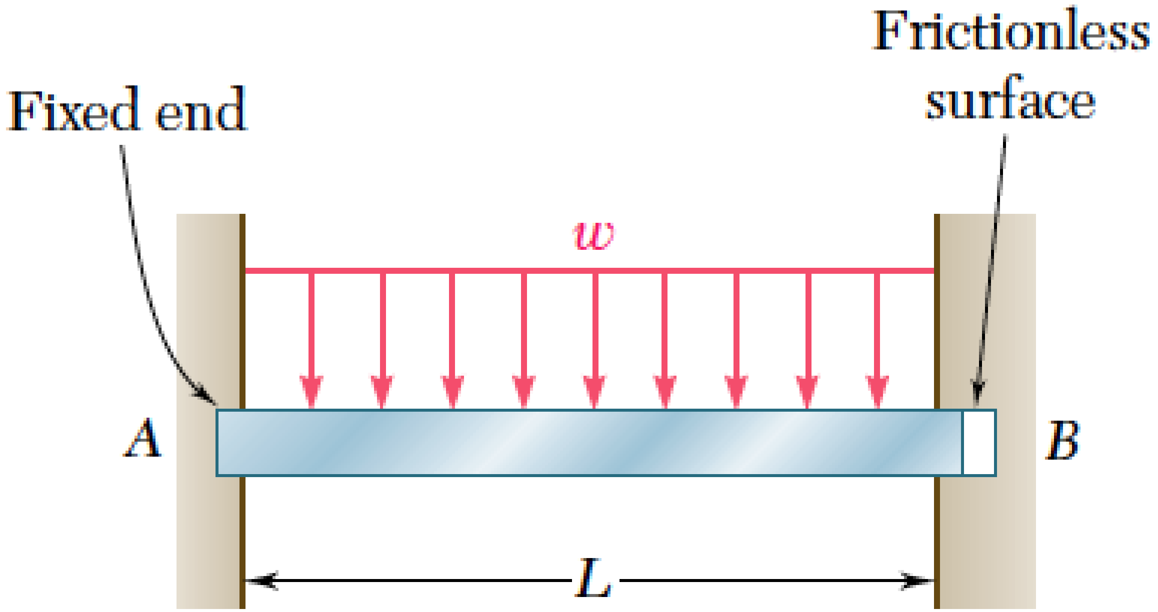
$$\theta_B = (\theta_B)_I + (\theta_B)_{II} = \frac{\omega L^3}{6EI} - \frac{\omega L^3}{48EI} = \frac{7\omega L^3}{48EI}$$

Deflection at Point B

$$y_B = (y_B)_I + (y_B)_{II} = \frac{\omega L^4}{8EI} - \frac{7\omega L^4}{384EI} = \frac{41\omega L^4}{384EI}$$

Principle of Superposition

For the uniform beam AB, (a) determine the reaction at A, (b) derive the equation of the elastic curve, (c) determine the slope at A.



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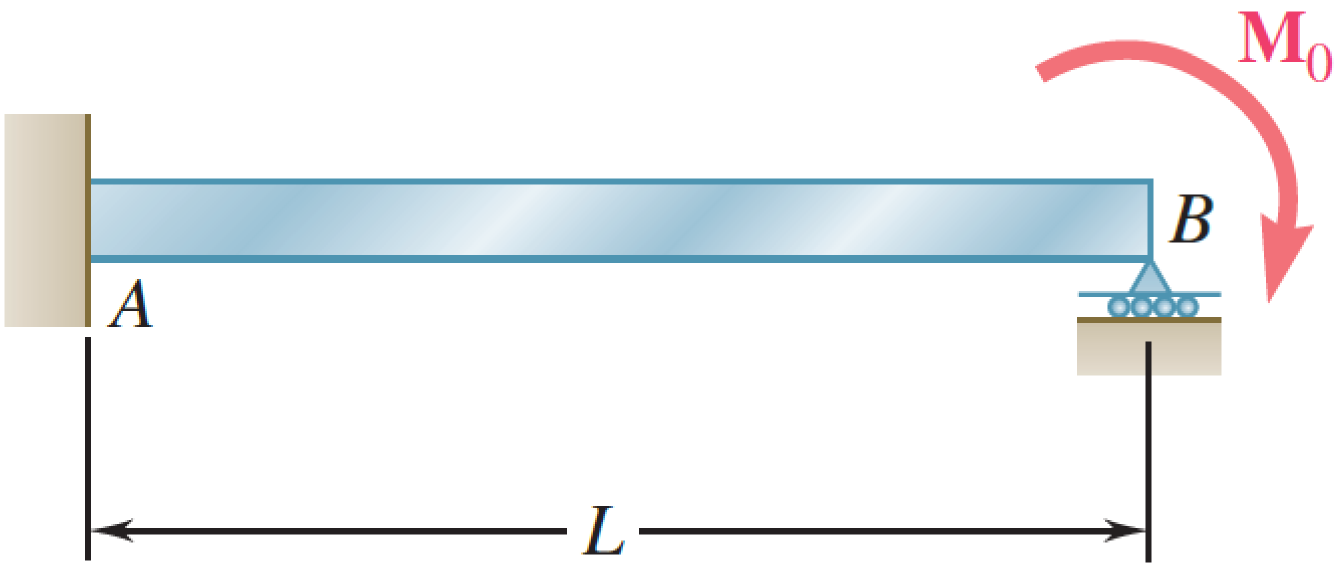
Exercise-1

For the beam and loading shown, determine the reaction at the roller support.



Exercise-2

Determine the reaction at the roller support and draw the bending moment diagram for the beam and loading shown.