Foundations of Solid Mechanics

E10: Statically Indeterminate Beams

Department of Civil Engineering
School of Engineering
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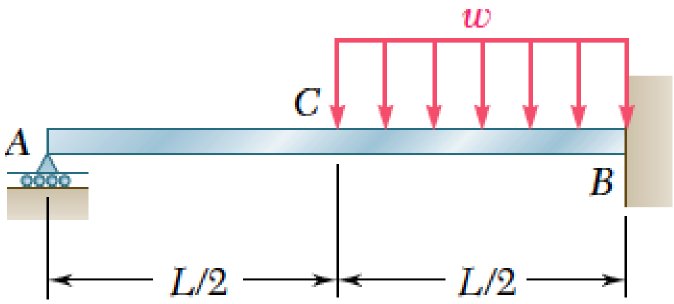
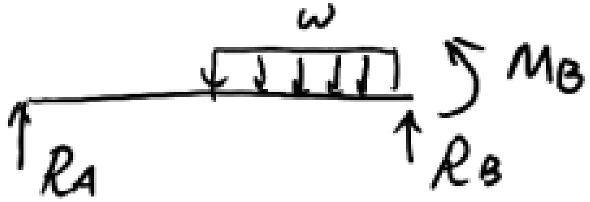
Foundations of Solid Mechanics

Exercise-1

1. Determine the reaction at the roller support and draw the bending moment diagram for the beam and loading shown.

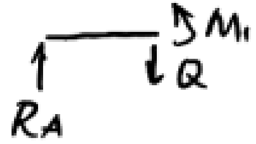




$$\uparrow \Sigma F_y = 0 \Rightarrow R_A + R_B - wL/2 = 0$$

$$\circlearrowright \Sigma M_B = 0 \Rightarrow -R_A \cdot L + \frac{wL}{2} \cdot \frac{L}{4} + M_B = 0$$

when $0 < x < \frac{L}{2}$,



$$M_1 = R_A x$$

when $\frac{L}{2} < x < L$,

$$M_2 = R_A x - \frac{w}{2}\left(x - \frac{L}{2}\right)^2$$

$$EIy_1'' = -M_1 = -R_A x$$

$$EIy_1' = -\frac{R_A}{2} x^2 + C_1$$

$$EIy_1 = -\frac{R_A}{6} x^3 + C_1 x + C_2$$

$$EIy_2'' = \frac{w}{2}\left(x - \frac{L}{2}\right)^2 - R_A x$$

$$EIy_2' = \frac{w}{6}\left(x - \frac{L}{2}\right)^3 - \frac{R_A}{2} x^2 + C_3$$

$$EIy_2 = \frac{w}{24}\left(x - \frac{L}{2}\right)^4 - \frac{R_A}{6} x^3 + C_3 x + C_4$$

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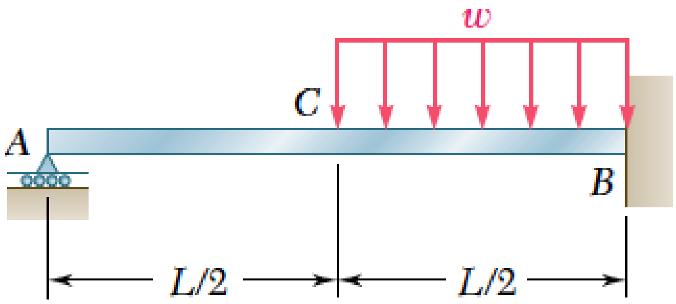
Exercise-1

1. Determine the reaction at the roller support and draw the bending moment diagram for the beam and loading shown.



$R_A = 7wL/128 \uparrow$; $M_C = 0.02734wL^2$,
$M = 0.02884wL^2$ at $x = 0.555\ L$.
$M_B = -0.07031wL^2$,

boundary conditions

$x = 0, \quad y_1 = 0 \quad \cdots\cdots ①$

$x = L, \quad y_2' = 0 \quad \cdots\cdots ②$

$x = L, \quad y_2 = 0 \quad \cdots\cdots ③$

continuity conditions

$x = \frac{L}{2}, \quad y_1' = y_2' \quad \cdots\cdots ④$

$x = \frac{L}{2}, \quad y_1 = y_2 \quad \cdots\cdots ⑤$

$① \rightarrow C_2 = 0$

$② \rightarrow C_3 = \frac{R_A}{2}L^2 - \frac{w}{48}L^3$

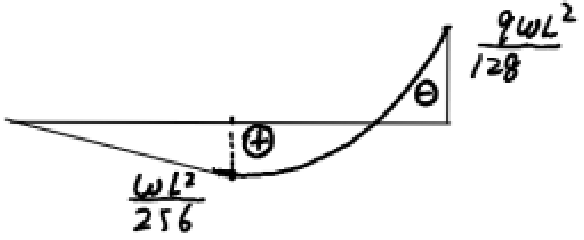
$③ \rightarrow C_4 = \frac{7wL^4}{384} - \frac{R_A}{3}L^3$

$④ \rightarrow C_1 = C_3 = \frac{R_A}{2}L^2 - \frac{w}{48}L^3$

$⑤ \rightarrow C_4 = 0 = \frac{7wL^4}{384} - \frac{R_A}{3}L^3$

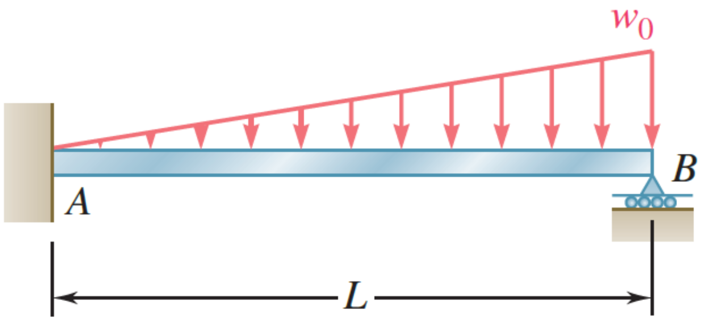
$\downarrow$

$R_A = \frac{7}{128}wL, \quad M_B = -\frac{9wL^2}{128}$

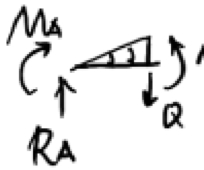


Exercise-2

2. For the beam and loading shown, determine the reaction at the roller support.



Method 1 : regular way



$$\circlearrowleft \Sigma M_x = 0 \Rightarrow -M_A - R_A x + \frac{w_0 x^3}{6L} + M = 0$$

$$\Rightarrow M = -\frac{w_0 x^3}{6L} + R_A x + M_A$$

$$\therefore -EIy'' = M = R_A x + M_A - \frac{w_0 x^3}{6L}$$

$$EIy' = \frac{w_0}{24L}x^4 - \frac{R_A}{2}x^2 - M_A x + C_1$$

$$EIy = \frac{w_0}{120L}x^5 - \frac{R_A}{6}x^3 - \frac{M_A}{2}x^2 + C_1 x + C_2$$

$$\begin{cases} x=0 & y'=0 & \rightarrow C_1 = 0 \\ x=0 & y=0 & \rightarrow C_2 = 0 \\ x=L & y=0 & \rightarrow \frac{w_0}{120}L^4 - \frac{R_A}{6}L^3 - \frac{M_A}{2}L^2 = 0 \end{cases}$$

$$M_A = \frac{w_0}{60}L^2 - \frac{R_A}{3}L \quad \cdots\cdots ①$$

- $\circlearrowleft \Sigma M_B = 0 \Rightarrow \frac{w_0 L}{2} \cdot \frac{L}{3} - R_A L - M_A = 0$

$$\rightarrow M_A = \frac{w_0 L}{6} - R_A L \quad \cdots\cdots ②$$

Combine ① and ②,

$$\frac{w_0 L}{6} - R_A L = \frac{w_0 L^2}{60} - \frac{R_A}{3}L \rightarrow R_A = \frac{9}{40}w_0 L$$

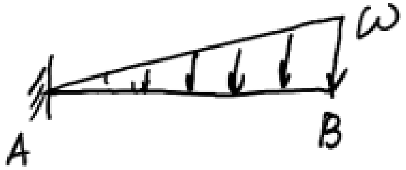
$$\uparrow \Sigma F_y = 0 \rightarrow -\frac{w_0 L}{2} + R_A + R_B = 0$$

$$\rightarrow R_B = \frac{w_0 L}{2} - R_A = \frac{w_0 L}{2} - \frac{9}{40}w_0 L = \frac{11}{40}w_0 L$$

Exercise-2

2. For the beam and loading shown, determine the reaction at the roller support.

Method 2: Principal of superposition



$$\longrightarrow R_A = \frac{wL}{2},\quad M_A = -\frac{wL^2}{3}$$

$$q(x) = -\frac{w_0}{L}\cdot x = -EIy''''$$

$$EIy''' = \frac{w_0}{2L}x^2 + C_1$$

$$EIy'' = \frac{w_0}{6L}x^3 + C_1x + C_2$$

$$EIy' = \frac{w_0}{24L}x^4 + \frac{C_1}{2}x^2 + C_2x + C_3$$

$$EIy = \frac{w_0}{120L}x^5 + \frac{C_1}{6}x^3 + \frac{C_2}{2}x^2 + C_3x + C_4$$

boundary conditions:

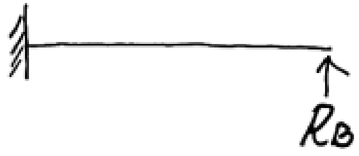
$x=0,\ y'=0 \longrightarrow C_3 = 0$

$x=0,\ y=0 \longrightarrow C_4 = 0$

$x=0,\ -EIy'' = M_A \longrightarrow C_2 = -M_A = \frac{wL^2}{3}$

$x=0,\ -EIy''' = Q_A = R_A \rightarrow C_1 = -R_A = -\frac{wL}{2}$

$$\therefore\ y\big|_{x=L} = \frac{1}{EI}\left(\frac{w_0}{120L}\cdot L^5 - \frac{wL}{12}\cdot L^3 + \frac{wL^4}{6}\right) = \frac{11}{120}\frac{wL^4}{EI}$$

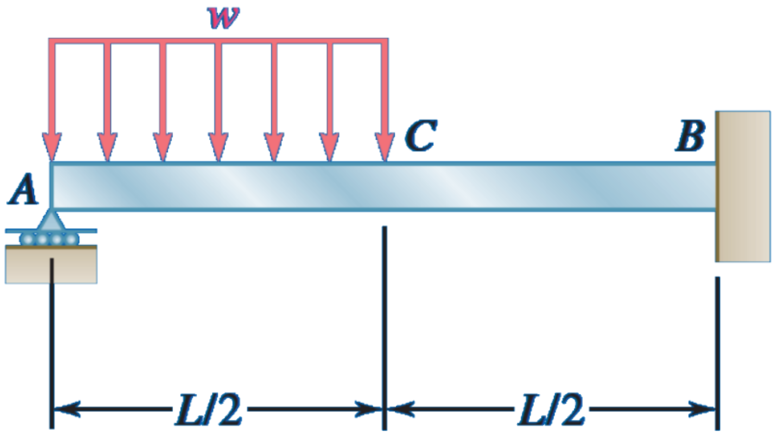


$$\longrightarrow \delta_B = \frac{R_B L^3}{3EI} = y\big|_{x=L} = \frac{11wL^4}{120EI}$$

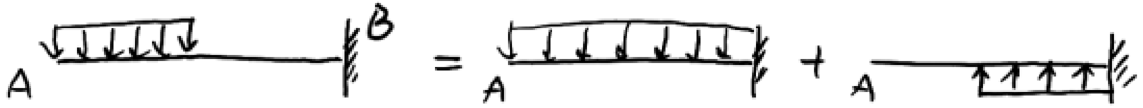
$$\longrightarrow R_B = \frac{11}{40}wL$$

Exercise-3

3. Determine the reaction at the roller support and draw the bending moment diagram for the beam and loading shown.



Principle of superposition



I = II + III

$$y_{A,I} = y_{A,II} + y_{A,III}$$

$$y_{A,II} = \frac{wL^4}{8EI} \ (\downarrow)$$

$$y_{A,III} = -\frac{w(L/2)^4}{8EI} - \theta_{C,III}\cdot\frac{L}{2}$$

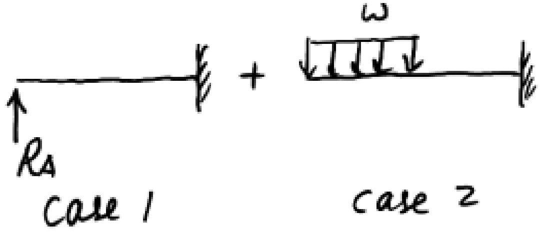
$$= -\frac{w(L/2)^4}{8EI} - \frac{w(L/2)^3}{6EI}\cdot\frac{L}{2} = -\frac{7}{24}\cdot\frac{wL^4}{EI}\cdot\frac{1}{16}$$

$$\therefore\ y_{A,I} = y_{A,II} + y_{A,III}$$

$$= \frac{wL^4}{EI}\left(\frac{1}{8} - \frac{7}{24\times16}\right) = \frac{41\,wL^4}{384\,EI}$$

$$\frac{R_A\cdot L^3}{3EI} = y_{A,I} = \frac{41\,wL^4}{384\,EI} \longrightarrow R_A = \frac{41\,wL}{128}$$

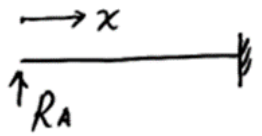
Real BDM =



Exercise-3

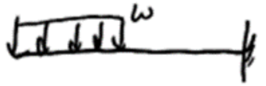
3. Determine the reaction at the roller support and draw the bending moment diagram for the beam and loading shown.

case 1 :



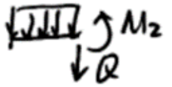
$M_1(x) = R_A x$

case 2 :



$M_2(x)$

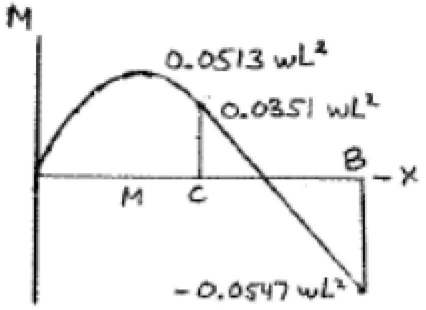
when $0 < x < L/2$



$M_2(x) = -\dfrac{wx^2}{2}$

when $\dfrac{L}{2} < x < L$

$$M_2(x) = -\frac{wL}{2}\cdot\left(x - \frac{L}{4}\right) = \frac{wL^2}{8} - \frac{wL}{2}x$$



$$M(x) = \begin{cases} \dfrac{41wL}{128}\cdot x - \dfrac{w}{2}x^2 & \left(0 < x < \dfrac{L}{2}\right) \\[2ex] \dfrac{wL^2}{8} - \dfrac{23wL}{128}x & \left(\dfrac{L}{2} < x < L\right) \end{cases}$$

$$M_C = M\Big|_{x=\frac{L}{2}} \left.\begin{aligned} &= \frac{41wL^2}{256} - \frac{w}{2}\cdot\frac{L^2}{4} \\ &= \frac{wL^2}{8} - \frac{23wL^2}{256} \end{aligned}\right\} = \frac{9wL^2}{256} = 0.035wL^2$$

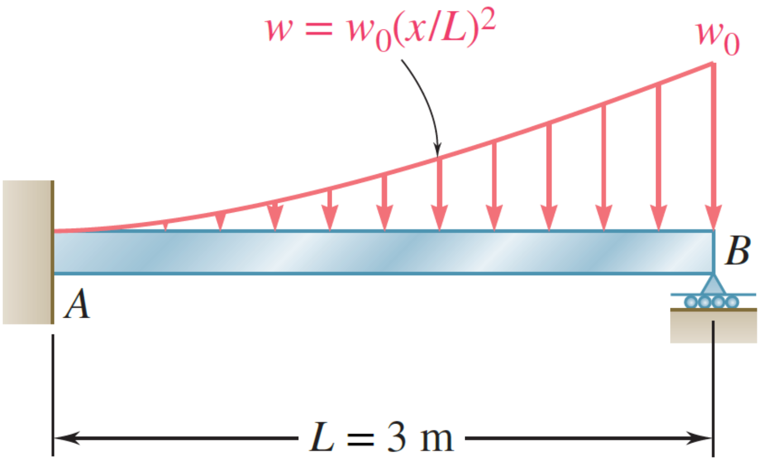
$$M_B = M\Big|_{x=L} = -\frac{7}{128}wL^2 = -0.055\,wL^2$$

$$\frac{dM(x)}{dx} = \frac{41wL}{128} - wx = 0 \Rightarrow x = \frac{41}{128}L = 0.32L$$

$$M_m\Big|_{x=\frac{41}{128}L} = \frac{1681}{32768}wL^2 = 0.0513\,wL^2$$

Exercise-4

4. For the beam shown, determine the reaction at the roller support when $w_0 = 65$ kN/m.



$$-EIy'''' = w(x) = -w_0\left(\frac{x}{L}\right)^2 = -\frac{w_0}{L^2}x^2$$

$$EIy''' = \frac{w_0}{3L^2}\cdot x^3 + C_1$$

$$EIy'' = \frac{w_0}{12L^2}x^4 + C_1 x + C_2$$

$$EIy' = \frac{w_0}{60L^2}x^5 + \frac{C_1}{2}x^2 + C_2 x + C_3$$

$$EIy = \frac{w_0}{360L^2}x^6 + \frac{C_1}{6}x^3 + \frac{C_2}{2}x^2 + C_3 x + C_4$$

boundary conditions:

$x=0, \quad y'=0 \rightarrow C_3 = 0$

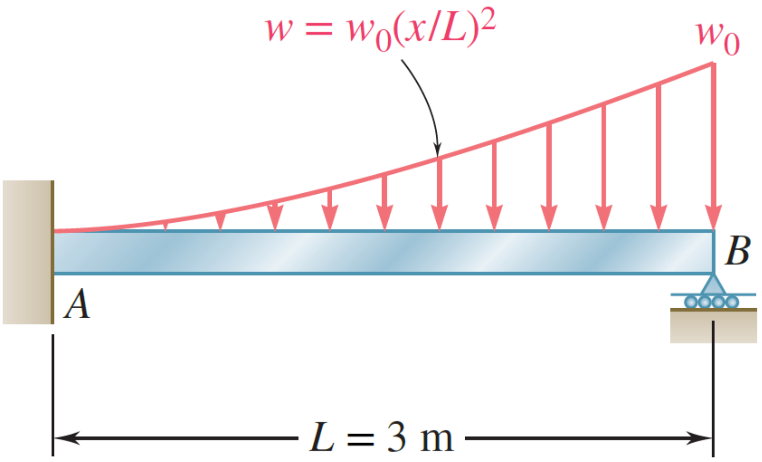
$x=0, \quad y = 0 \rightarrow C_4 = 0$

$x=0, \quad -EIy'' = M_A \rightarrow C_2 = \frac{w_0 L^2}{4}$

$x=0, \quad -EIy''' = Q_A \rightarrow C_1 = -\frac{w_0 L}{3}$

Exercise-4

4. For the beam shown, determine the reaction at the roller support when $w_0 = 65$ kN/m.



$$Q_A = \int_0^L w(x)\,dx = \int_0^L w_0 \frac{x^2}{L^2}\,dx = \frac{w_0}{3L^2} x^3 \Big|_0^L = \frac{w_0 L}{3}$$

$$M_A = -\int_0^L w(x)\cdot x \cdot dx = -\int_0^L w_0 \frac{x^3}{L^2}\,dx = -\frac{w_0}{4L^2} x^4 \Big|_0^L = -\frac{w_0 L^2}{4}$$

$$y\big|_{x=L} = \frac{1}{EI}\left(\frac{w_0}{360L^2}\cdot L^6 - \frac{w_0 L^4}{18} + \frac{w_0 L^4}{8}\right) = \frac{13\, w_0 L^4}{180\, EI}$$

$$\frac{R_B \cdot L^3}{3EI} = y\big|_{x=L} \Rightarrow R_B = \frac{39 \cdot w_0 L}{180}$$