Foundations of Solid Mechanics

L5: Transverse Shear and Combined Loading

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Main Contents

In this chapter, we will discuss:

- How to determine the *shear stress caused by transverse shear force*;
- How to determine the *stress caused by combined loading (axial force, bending moment, shear force and torsional moment)*.

Shear Examples

Assumption: the cross-sectional warping due to shear is small enough so that it can be neglected.

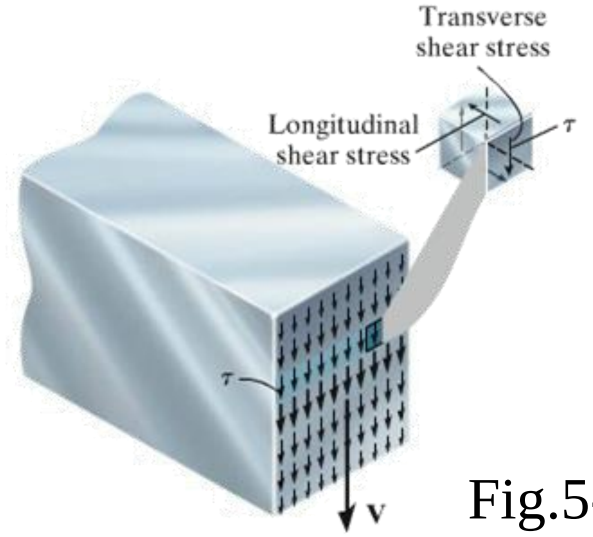


Fig.5-1

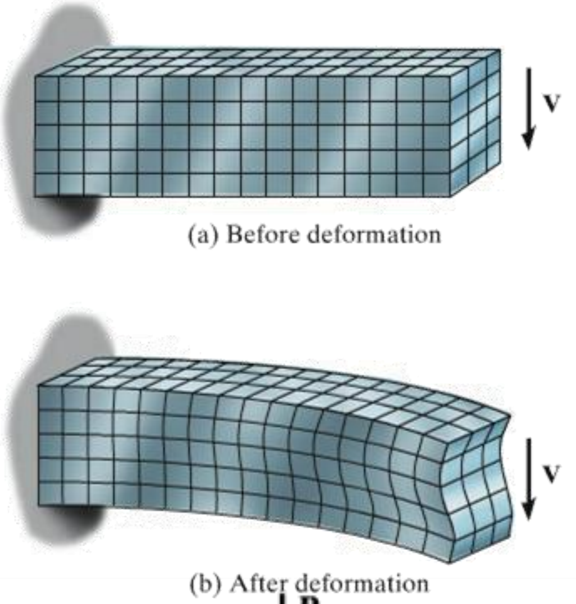


Fig.5-2

Fig.5-3

Shear Stress Equilibrium

The Fig.5-4a shows a volume element of material taken at a point on the sectional area which is subjected to a shear stress τ_{zy}. Force and moment equilibrium requires the shear stress acting this face of element to be accompanied by shear stress acting on the other three faces. To show this, we will first consider the force equilibrium in y direction:

$$\Sigma F_y = 0; \qquad \overbrace{\overbrace{\tau_{zy}}^{\text{stress}}\overbrace{(\Delta x\ \Delta y)}^{\text{area}}}^{\text{force}} - \tau'_{zy}\,\Delta x\ \Delta y = 0$$

$$\tau_{zy} = \tau'_{zy}$$

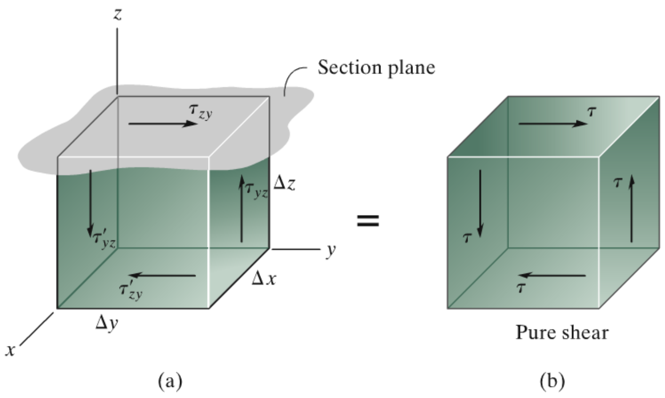


Fig.5-4

In a similar manner, force equilibrium in the z direction yields: $\tau_{yz} = \tau'_{yz}$

Finally, take the moment about x axis:

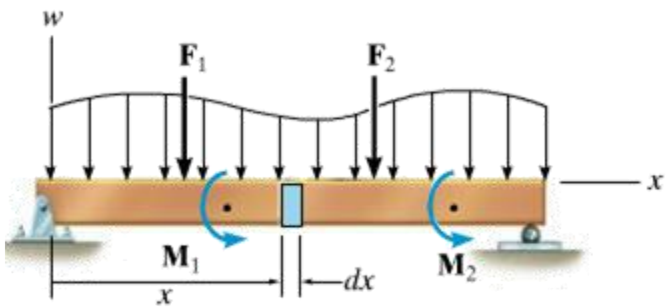
$$\Sigma M_x = 0; \qquad -\overbrace{\overbrace{\overbrace{\tau_{zy}}^{\text{stress}}\overbrace{(\Delta x\ \Delta y)}^{\text{area}}}^{\text{force}}\ \overbrace{\Delta z}^{\text{arm}}}^{\text{moment}} + \tau_{yz}(\Delta x\ \Delta z)\,\Delta y = 0$$

$$\tau_{zy} = \tau_{yz}$$

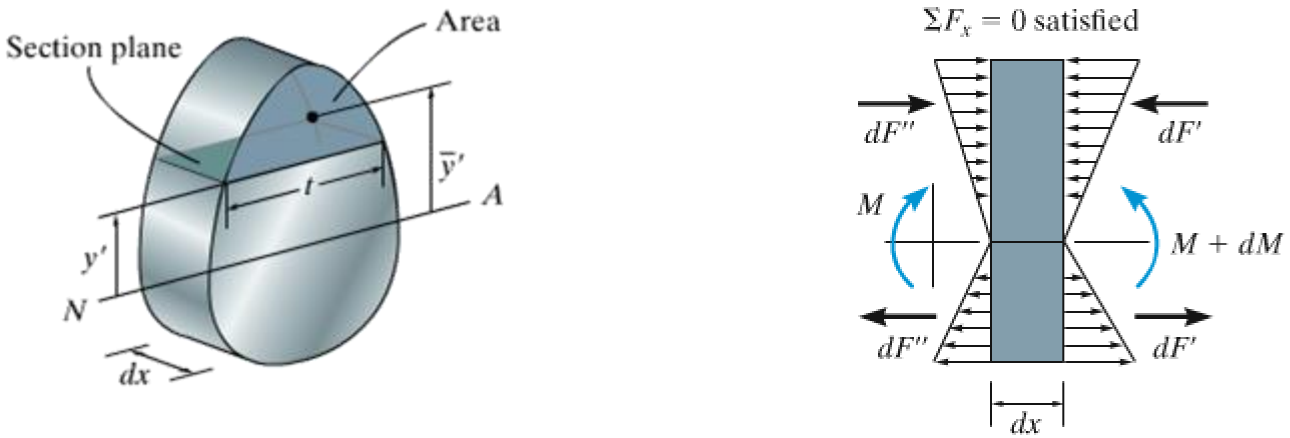
So that: $\tau_{zy} = \tau'_{zy} = \tau_{yz} = \tau'_{yz} = \tau$

Shear Stress Equilibrium

- Because the strain distribution for shear is not easily defined, as in the case of axial force, torsion, and bending, we will develop the shear formula in an indirect manner. To do this we will consider the *horizontal force equilibrium* of a portion of the element taken from the beam in Fig.5-5a. A free body diagram of this *element* is shown in Fig.5-5b. This distribution is caused by the bending moments M and $M+dM$. we have excluded the effects of V, $V+dV$, and w(x) on the free body diagram because these loading are *vertical* and will therefore not be involved in a horizontal force summation.



（a）



（b）

Fig.5-5

Shear Stress Equilibrium

Applying the equation of horizontal force equilibrium, and using the flexure formula,

$$\sum F_x = 0 \qquad \int_{A'} \sigma' dA' - \int_{A'} \sigma dA' - \tau(t dx) = 0$$

$$\int_{A'} \left(\frac{M+dM}{I}\right) y dA' - \int_{A'} \left(\frac{M}{I}\right) y dA' - \tau(t dx) = 0$$

$$\left(\frac{dM}{I}\right) \int_{A'} y dA' = \tau(t dx)$$

Solving for τ, we get $$\tau = \frac{1}{It}\left(\frac{dM}{dx}\right)\int_{A'} y dA'$$

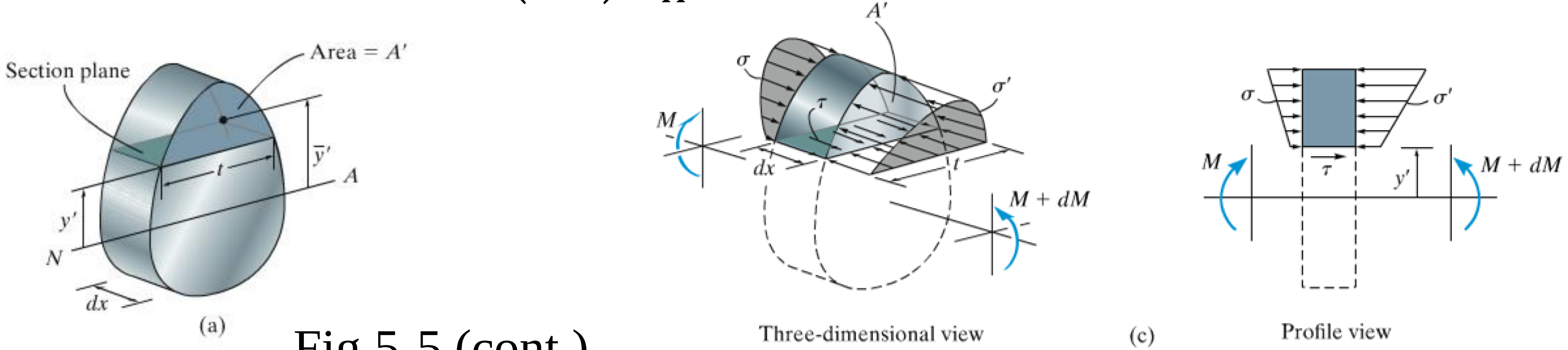


Fig.5-5 (cont.)

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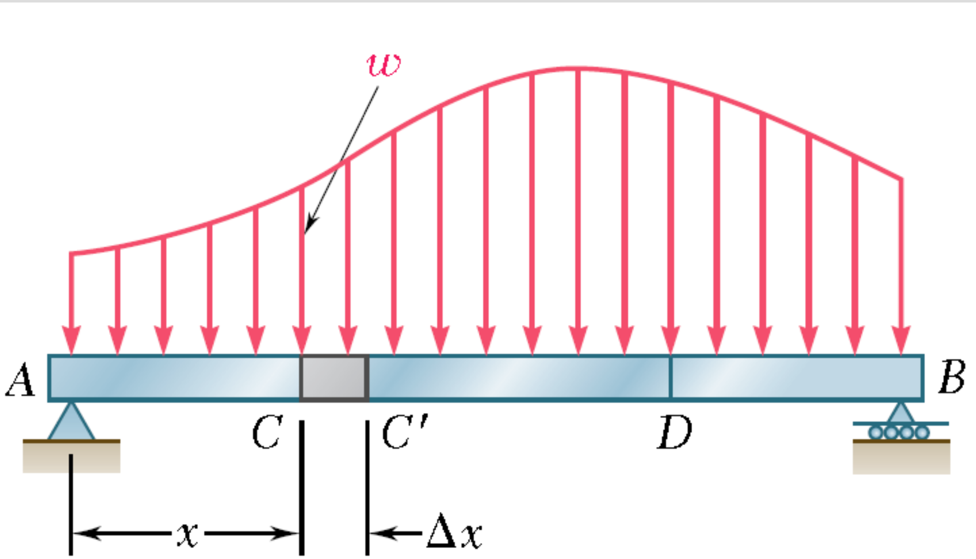
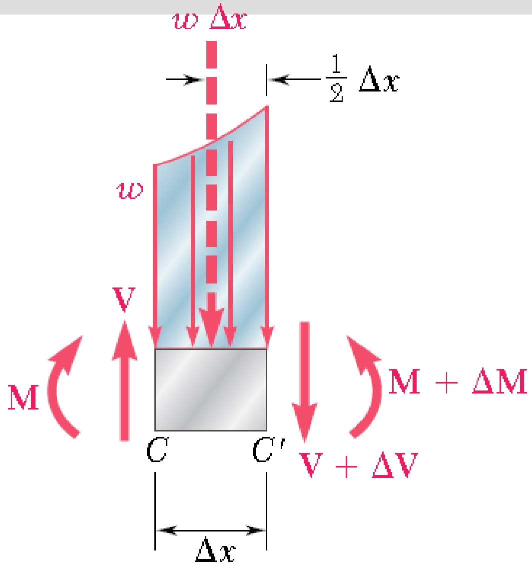
Regions of Distributed Load





$$\Sigma M_{c'} = 0 \Rightarrow (M + \Delta M) - M - V\Delta x + w\Delta x \frac{\Delta x}{2} = 0$$

$$\Rightarrow \Delta M = V\Delta x - \frac{1}{2}\omega(\Delta x)^2 \Rightarrow \frac{dM}{dx} = V$$

Shear Stress Equilibrium

This equation can be simplified by noting that *V=dM/dx*. Also, the integral represents the moment of the area *A*` about the neutral axis. We will denote this by the symbol *Q*. since the location of the centroid of the area *A*` is determined from $\bar{y} = \int_{A'} y dA' / A'$, we can also write

$$Q = \int_{A'} y dA' = \bar{y} A'$$

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Shear Stress Equilibrium

The final results is therefore:

$$\tau = \frac{VQ}{It}$$

Here:

τ = the shear stress in the member at the point located a distance y` from the neutral axis. This stress is assumed to be constant and therefore averaged across the width t of the member.

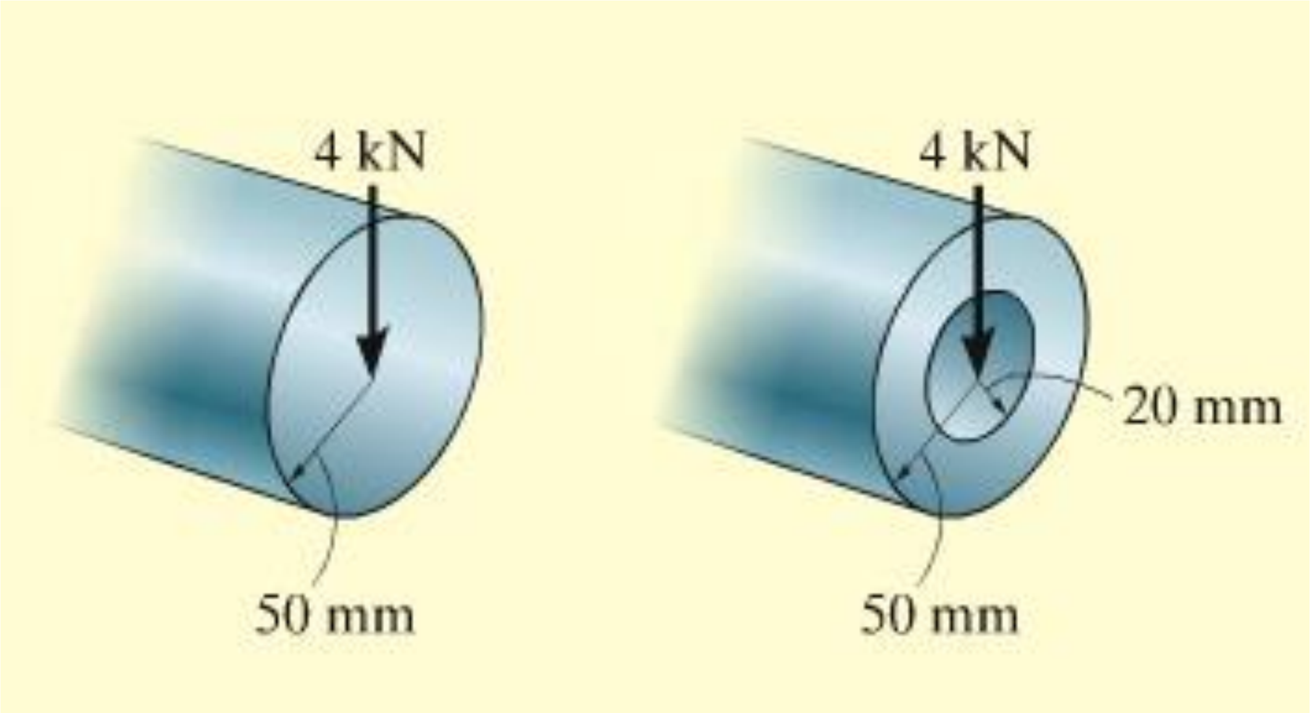
V = the internal resultant shear force, determined from the method of sections and the equations of equilibrium.

t = the width if the member's cross-sectional area, measured at the point where τ is to be determined.

$Q = \bar{y}A'$, where *A*` is the area of the top (or bottom) portion of the member's cross-sectional area, above (or below) the section plane where *t* is measured, and $\bar{y}$ is the distance from the neutral axis to the centroid of *A*`.

Example-1

The solid shaft and tube shown in below are subjected to the shear force of 4kN. Determine the shear stress acting over the diameter of each cross section.



Example-1

Sectional Properties

$$I_{solid} = \frac{1}{4}\pi c^4 = \frac{1}{4}\pi(0.05m)^4 = 4.909 \times 10^{-6}m^4$$

$$I_{tube} = \frac{1}{4}\pi(c_o^4 - c_i^4) = \frac{1}{4}\pi[(0.05m)^4 - (0.02m)^4] = 4.783 \times 10^{-6}m^4$$

$$Q_{solid} = \bar{y}`A` = \frac{4c}{3\pi}\left(\frac{\pi c^2}{2}\right) = \frac{4 \times (0.05)}{3\pi}\left(\frac{\pi(0.05)^2}{2}\right) = 83.3 \times 10^{-6}m^4$$

$$Q_{tube} = \sum \bar{y}'A' = \frac{4c_o}{3\pi}\left(\frac{\pi c_o^2}{2}\right) - \frac{4c_i}{3\pi}\left(\frac{\pi c_i^2}{2}\right) = \frac{4 \times (0.05)}{3\pi}\left(\frac{\pi \times (0.05)^2}{2}\right) - \frac{4 \times (0.02)}{3\pi}\left(\frac{\pi \times (0.02)^2}{2}\right) = 78.0 \times 10^{-6}m^4$$

Shear Stress

Applying the shear formula where t=0.1m for solid section, and t=2×0.03m=0.06m for the tube, we have

$$\tau_{solid} = \frac{VQ}{It} = \frac{4 \times 10^3 \times 83.3 \times 10^{-6}}{4.909 \times 10^{-6} \times 0.1} = 679kPa$$

$$\tau_{tube} = \frac{VQ}{It} = \frac{4 \times 10^3 \times 78.0 \times 10^{-6}}{4.783 \times 10^{-6} \times 0.06} = 1.09MPa$$

Thin-Walled Pressure Vessels

In general, "*thin wall*" refers to a vessel having an inner-radius-to-wall-thickness ratio of 10 or more ($r/t \geq 10$). Specially, when $r/t=10$ the results of a thin-wall analysis will predict a stress that is approximately 4% less than the actual maximum stress in the vessel. For larger r/t ratio this error will be even smaller.

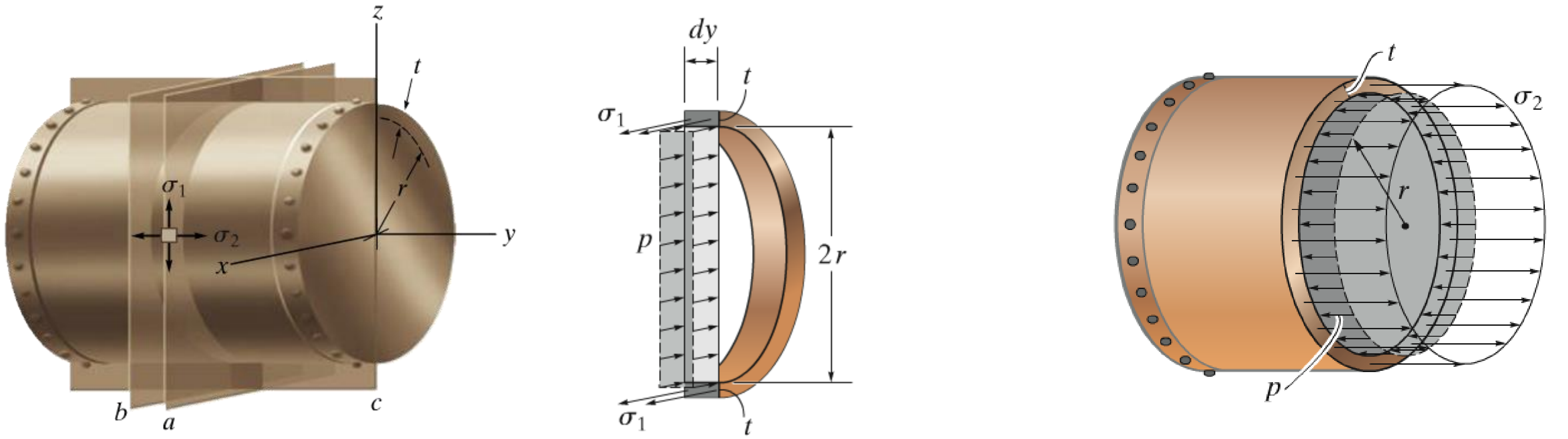
Fig.5-6

Cylindrical Vessels

- Consider the cylindrical vessel having a wall thickness t, inner radius r, and subjected to gauge pressure p that developed within the vessel by a contained gas. Due to this loading, a small element of the vessel that is sufficient removed from the ends and oriented as shown in Fig.5.7b, is subjected to normal stress σ_1 in the ***circumferential*** or ***hoop direction*** and σ_2 in the ***longitudinal or axial direction***.

For equilibrium in the x direction, we require

$$\sum F_x = 0 \Rightarrow 2[\sigma_1(tdy)] - p(2rdy) = 0 \Rightarrow \sigma_1 = \frac{pr}{t}$$



(a) (b) Fig.5-7 (c)

Cylindrical Vessels

The longitudinal stress can be determined by considering the left portion of section b of the cylinder, Fig.5-7a. As shown in Fig.5-7c, σ_2 acts on the section of contained gas. Since the mean radius is approximately equal to the vessel's inner radius, equilibrium in the y direction requires

$$\sum F_y = 0 \quad \Rightarrow \quad \sigma_2(2\pi rt) - p(\pi r^2) = 0 \quad \Rightarrow \quad \sigma_2 = \frac{pr}{2t}$$

In the above equation,

$\sigma_1, \sigma_2 =$ the normal stress in the hoop and longitudinal directions, respectively. Each is assumed to be constant throughout the wall of the cylinder, and each subjects the material to tension.

$P =$ the internal gauge pressure developed by contained gas

$r =$ the inner radius of the cylinder

$t =$ the thickness of the wall ($r/t \geq 10$)

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Spherical Vessels

- We can analyze a spherical pressure vessel in a similar manner. To do this, consider the vessel to have a wall thickness *t*, inner radius *r*, and subjected an internal gauge pressure *p*, Fig.5-8a. If the vessel is sectioned in half, the resulting free body diagram is shown in Fig.5-8b. Like the cylinder, equilibrium in the y direction requires

$$\sum F_y = 0 \quad \Rightarrow \quad \sigma_2(2\pi rt) - p(\pi r^2) = 0 \quad \Rightarrow \quad \sigma_2 = \frac{pr}{2t}$$

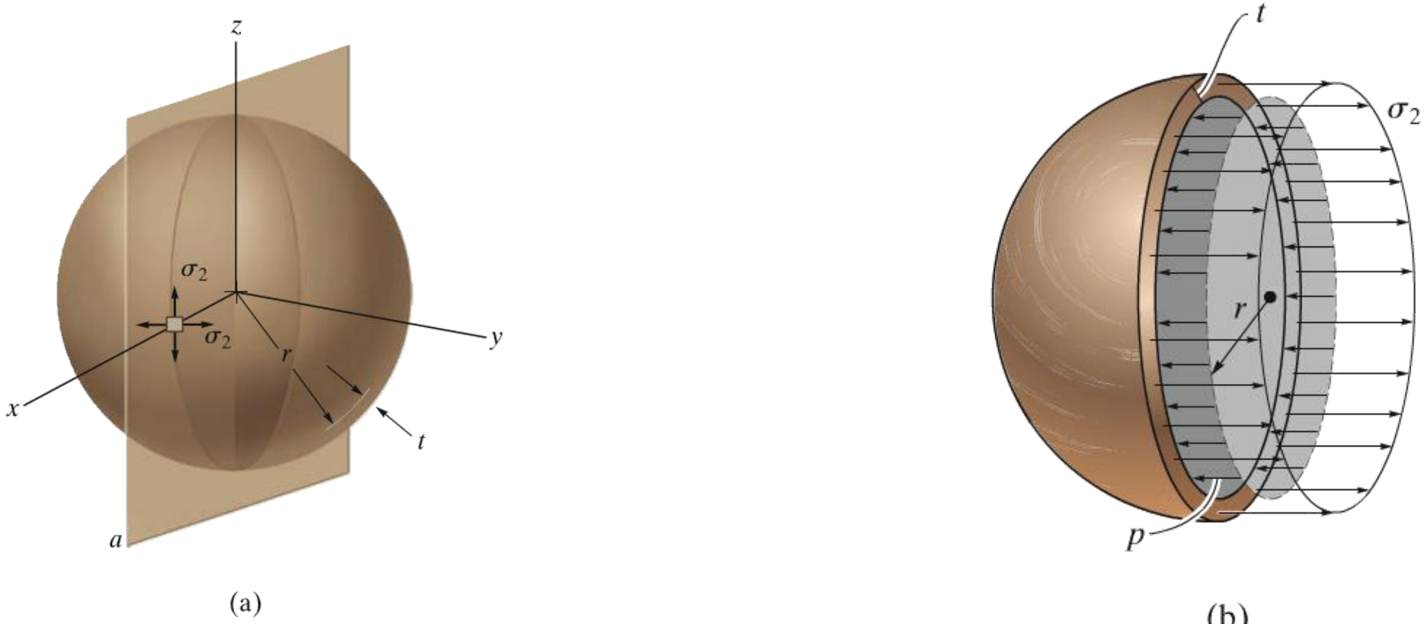

Fig.5-8

State of Stress Caused by Combined Loading

- In general condition, the cross section of a member is subjected to several loadings (axial force, shear force, bending, and torsional moment) *simultaneously*.
- When this occurs, the *methods of superposition* can be used to determine the resultant stress distribution.

- Normal Force: the internal normal force is developed by a uniform normal-stress distribution determined from $\sigma=P/A$.
- Shear Force: the internal shear force in a member is developed by shear-stress distribution determined from the shear formula, $\tau=VQ/It$.
- Bending Moment: for straight members the internal bending moment is development a normal-stress distribution that varies linearly from zero at the neutral axis to a maximum at the outer boundary of the member. This stress distribution is determined from the flexure formula, $\sigma=My/I$.
- Torsional Moment: for circular shafts and tubes the internal torsional moment is developed by a shear-stress distribution that varies linearly from the central axis of the shaft to a maximum at the shaft's outer boundary. This stress distribution is determined from the torsion formula, $\tau=T\rho/J$.

State of Stress Caused by Combined Loading

Superposition.

- Once the normal and shear stress components for each loading have been calculated, use the principle of superposition and determine the resultant normal and shear stress components.
- Represent the results on an element of material located at the point, or show the results as a distribution of stress acting over the member’s cross-sectional area。

Example-2

A force of 150N is applied to the edge of the member as shown in below. Neglect the weight of the member and determine the state of stress at points B and C.

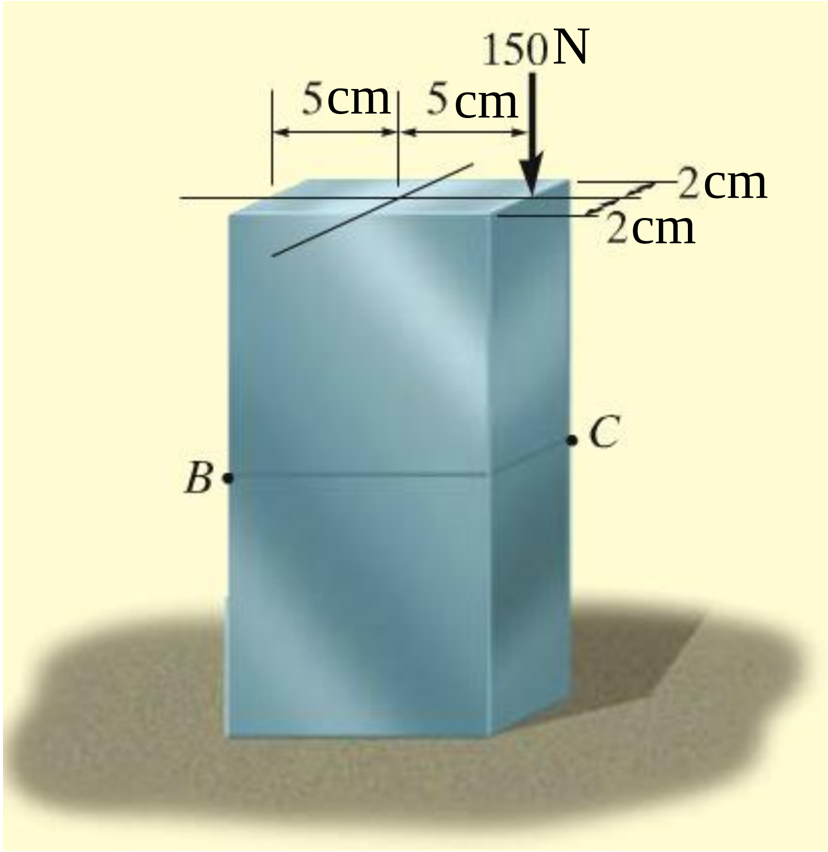

Fig.5-9

Example-2

Stress Components.

- Normal Force. The uniform normal-stress distribution due to the normal force is shown in Fig.5-10. Here

$$\sigma = \frac{P}{A} = \frac{150N}{(10cm)\times(4cm)} = 3.75\text{kPa}$$

- Bending Moment. The normal stress due to the bending moment is shown in The uniform normal-stress distribution due to the normal force is shown in Fig.5-10. The maximum stress is

$$\sigma = \frac{Mc}{I} = \frac{750N\cdot cm\times 5cm}{\frac{1}{12}(4cm)\times(10cm)^3} = 11.25\text{kPa}$$

- Superposition. If the above normal-stress distribution are added algebraically, element of material at B and C are subjected only to normal or uniaxial stress as shown in Fig.5-11. Hence:

$\sigma_B = 7.5\text{kPa (tension)}$ $\sigma_c = 15\text{kPa (compression)}$

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Example-2

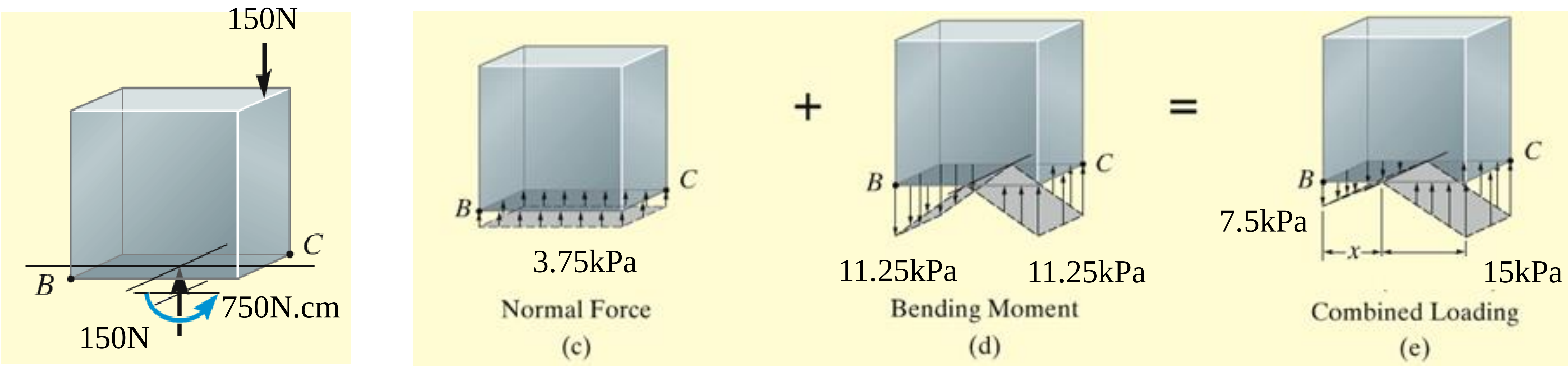

Fig.5-10

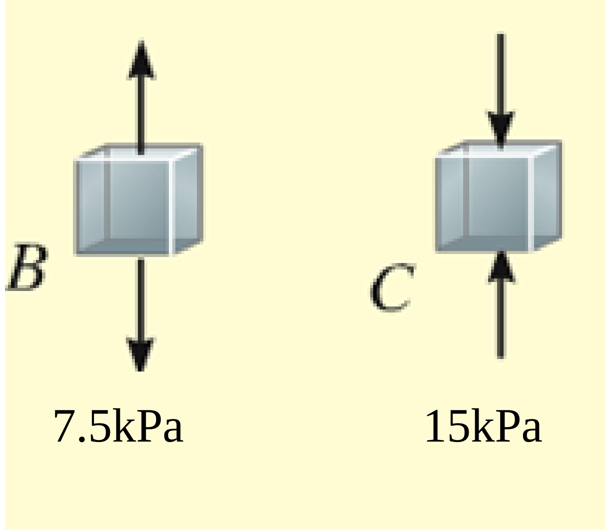

Fig.5-11

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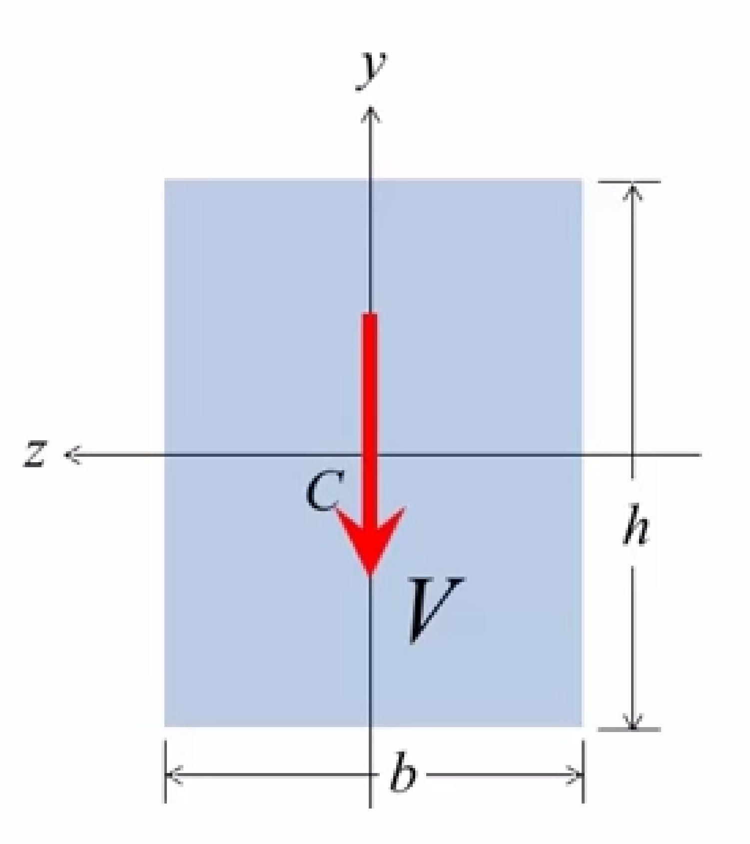
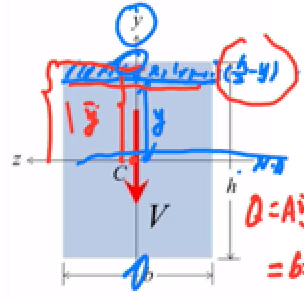
Exercise-1

For the rectangular cross-section area subjected to internal shear force V, determine the shear stress distribution as a function of y.





$$Q = A\bar{y} = b\times\left(\frac{h}{2}-y\right)\times\left(\frac{h}{2}-\frac{h}{4}+\frac{y}{2}\right)$$
$$= b\times\left(\frac{h}{2}-y\right)\times\left(\frac{h}{4}+\frac{y}{2}\right)$$
$$= \frac{b}{2}\times\left(\frac{h}{2}-y\right)\times\left(\frac{h}{2}+y\right)$$
$$= \frac{b}{2}\left(\frac{h^2}{4}-y^2\right)$$

$$\tau = \frac{V.Q}{It} = \frac{V\times\frac{\cancel{b}}{2}\left(\frac{h^2}{4}-y^2\right)}{\frac{bh^3}{12}\times \cancel{b}}$$
$$= \frac{6V}{bh^3}\left(\frac{h^2}{4}-y^2\right)$$

$$\tau_{max} = \frac{6V}{bh^3}\times\left(\frac{h^2}{4}-0\right) = 1.5\frac{V}{bh} = 1.5\,\tau_{avg}$$