Foundations of Solid Mechanics

L2: Stress, Strain & Deformation under Axial Load

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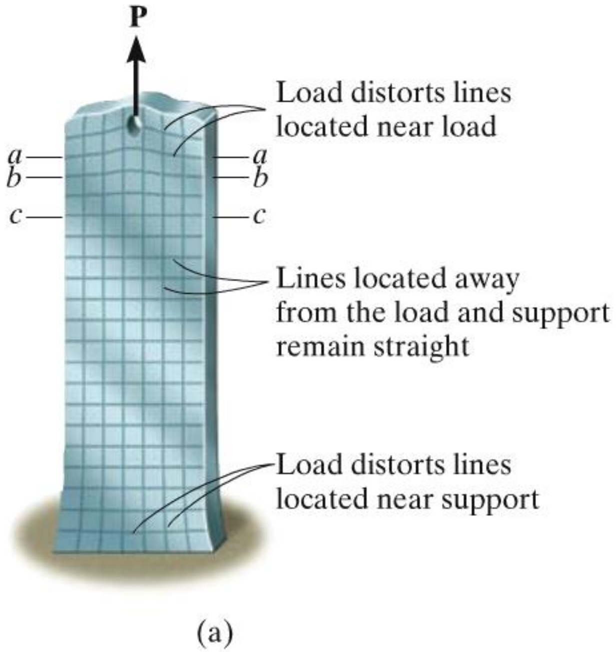
Main Contents

In this chapter, we will discuss:

- How to determine the *deformation* of axially loaded members;
- Develop a method for finding the support reactions when these reactions cannot be determined directly from the equations of equilibrium (Statically Indeterminate Axially loaded member).

Saint-Venant's Principle

Firstly noticed by the French scientist Barre de Saint-Venant in 1855. Essentially it states that *the stress and strain produced at points in a body sufficiently removed from the region of load application will be the same as the stress and strain produced by any applied loading that have the same statically equivalent resultant and are applied to the body with the same region.*



For example, if two symmetrically applied forces P/2 act on the bar, Fig.2.1c, the stress distribution at section c-c will be uniform and therefore equivalent to σ_{avg}=P/A as in Fig.2.1b.

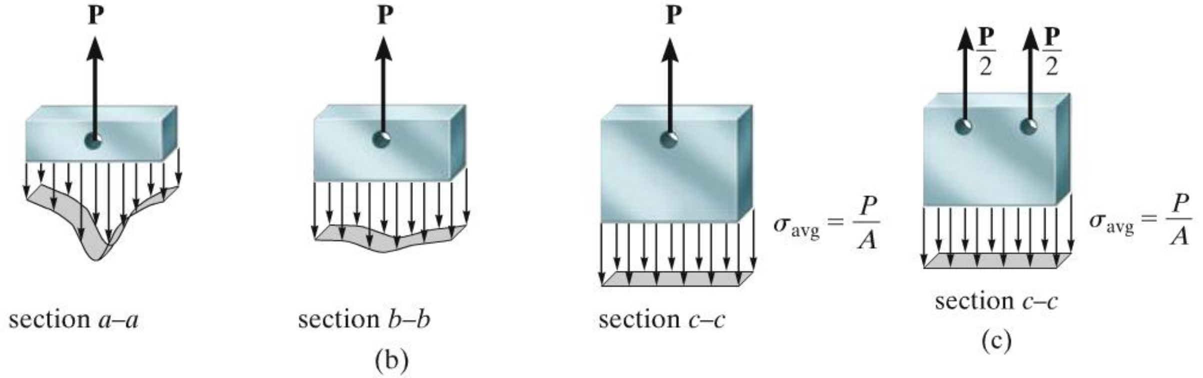


Fig.2.1

Elastic Deformation of an Axially Loaded Member

Using the method of sections, a differential element of length dx and cross-section area A(x) is isolated from the bar at the arbitrary position x. The free body of this element is shown below. The stress and strain in the element are:

$$\sigma = \frac{P(x)}{A(x)} \quad \text{and} \quad \epsilon = \frac{d\delta}{dx} \qquad (2.1)$$

Provide the stress does not exceed the proportional limit, we can apply the Hook’s law; i.e.,

$$\sigma = \varepsilon E \qquad \frac{P(x)}{A(x)} = E\left(\frac{d\delta}{dx}\right) \qquad d\delta = \frac{P(x)dx}{A(x)E} \qquad (2.2)$$

For the entire length L of the bar, we must integrate this expression to find δ. This yields:

$$\delta = \int_0^L \frac{P(x)}{A(x)E}dx \qquad (2.3)$$

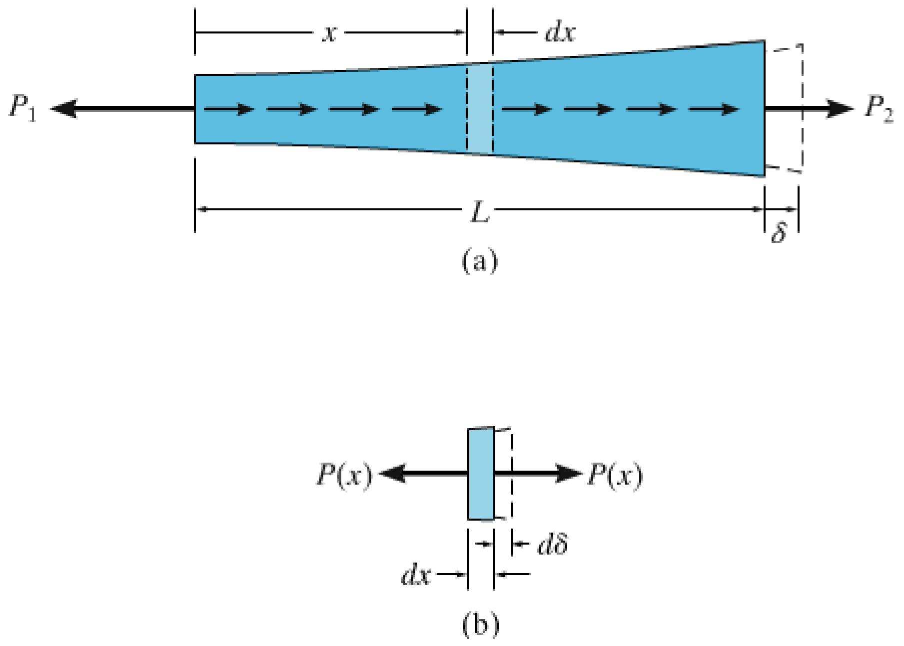


Fig.2.2

Elastic Deformation of an Axially Loaded Member

$$\delta = \int_0^L \frac{P(x)}{A(x)E}\,dx \qquad (2.3)$$

δ: displacement of one point on the bar relative to the other point
L: original length of bar
P(x): internal axial force at the section, located a distance x from one end;
A(x): cross-sectional area of bar, expressed as a function of x;
E: modulus of elasticity for the material

Constant Load and Cross-Sectional Area

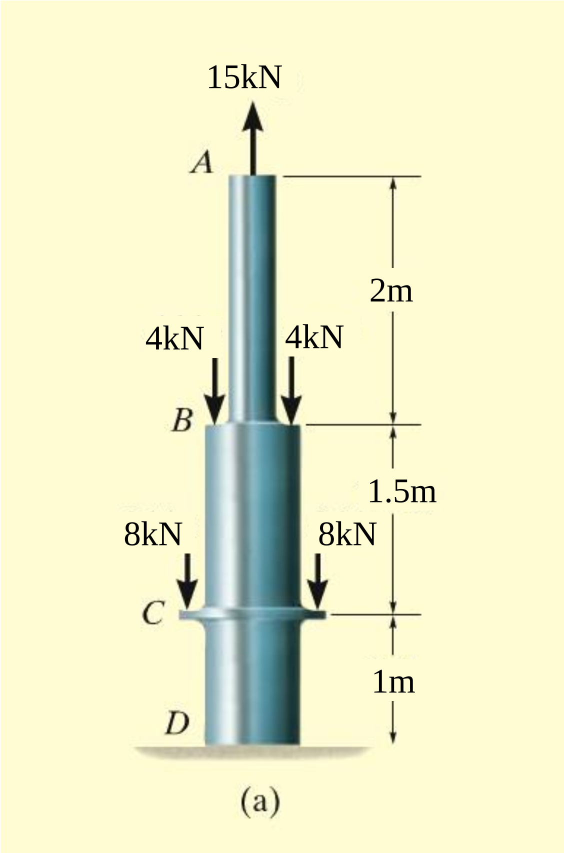
$$\delta = \frac{PL}{EA} \qquad (2.4)$$

If the bar is subjected to several different axial forces along its length, or the cross-sectional area or modulus of elasticity changes abruptly from one region of the bar to the next, the above equation can be applied to each segment of the bar where these quantities remain constant.

$$\delta = \sum \frac{PL}{EA} \qquad (2.5)$$

Example-1

The steel bar shown in below is made from two segments having cross-sectional areas of A_{AB}=1cm^2, and A_{BD}=2cm^2. Determine the vertical displacement of end A and the displacement of B relative to C. The modulus of elasticity is taken as 2.0×10^{11} Pa



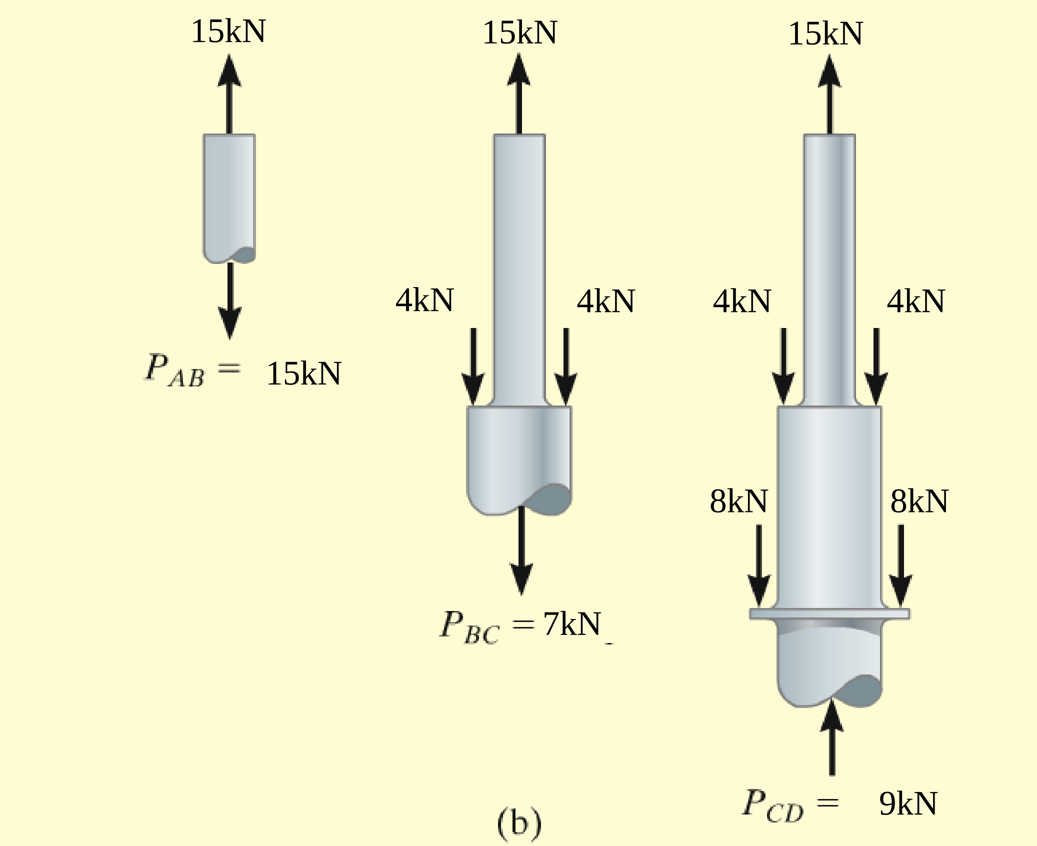


Fig.4.3

Example-1

Using the sign convention, i.e., internal tensile forces are positive and compressive forces are negative, the vertical displacement of A relative to the *fixed* support D is:

$$\delta_A = \sum \frac{PL}{EA} = \frac{(+15kN)(2m)}{(2.0 \times 10^{11}Pa)(1.0cm^2)} + \frac{(+7kN)(1.5m)}{(2.0 \times 10^{11}Pa)(2.0cm^2)} + \frac{(-9kN)(1.0m)}{(2.0 \times 10^{11}Pa)(2.0cm^2)}$$

$$= 1.54 \times 10^{-3}\mathrm{m} = 1.54mm \qquad (2.6)$$

Since the results is *positive*, the bar elongates and so the displacement at *A* is upward.

Applying the same equations between B and C, we obtain:

$$\delta_{B/C} = \frac{P_{BC}L_{BC}}{EA_{BC}} = \frac{(7kN)(1.5m)}{(2.0 \times 10^{11}Pa)(2.0cm^2)} = 2.63 \times 10^{-4}\mathrm{m} = 0.263mm \qquad (2.7)$$

Here B moves away from C, since the segment elongates

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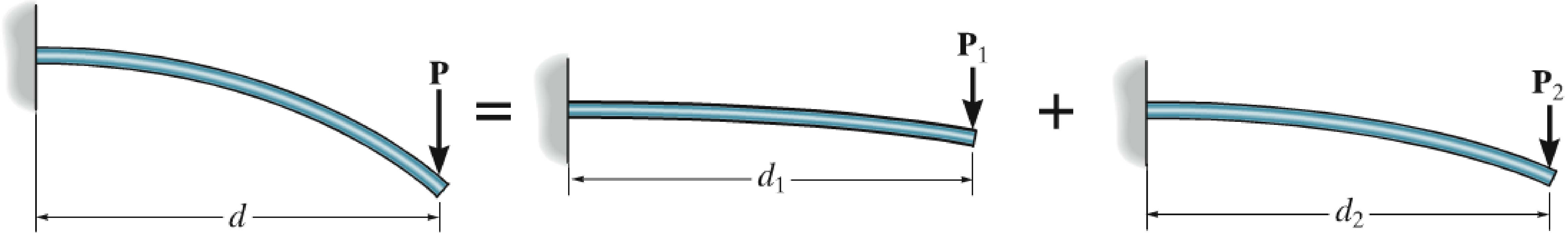
Principle of Superposition

The ***principle of superposition*** states that the resultant stress or displacement at the point can be determined by algebraically summing the stress or displacement caused by each load component applied separately to the member.

The following two conditions must be satisfied if the principle of supervision is to be applied.

1. *The loading must be linearly related to the stress or displacement that is to be determined;*
2. *The loading must not significantly change the original geometry or configuration of the member.*



(a) (b)

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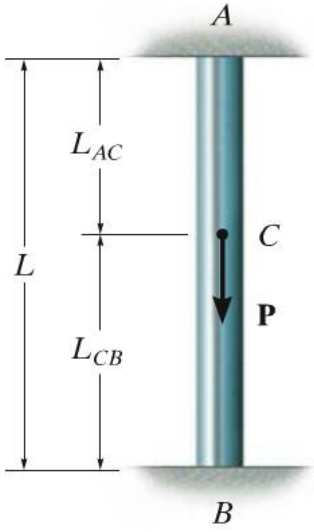
Statically Indeterminate Axially Loaded Member

Consider the bar shown, which is fixed supported at both of its ends. From the free-body diagram, equilibrium requires

$$+\uparrow \sum F = 0 \qquad\qquad F_B + F_A - P = 0 \tag{4.8}$$

This type of problem is called ***statically indeterminate***, since the equilibrium equation(s) are not sufficient to determine the two reactions on the bar.



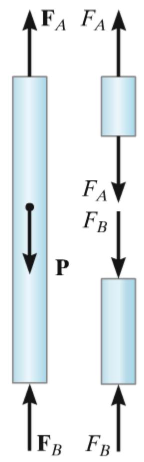
(a)

In order to establish an additional equation needed for solution, it is necessary to consider how points on the bar displace. Specifically, an equation that specifies the conditions for displacement is referred to as a ***compatibility*** or ***kinematic*** condition.

$$\frac{F_A L_{AC}}{EA} - \frac{F_B L_{CB}}{EA} = 0 \tag{4.9}$$

Therefore:

$$F_A = P\left(\frac{L_{CB}}{L}\right) \qquad\qquad F_B = P\left(\frac{L_{AC}}{L}\right) \tag{4.10}$$



(b) (c)

Thermal Stress

A change in temperature can cause a body to change its dimensions. Generally, if the temperature increases, the body will expand, whereas if the temperature decreases, it will contract. Ordinarily, this expansion or contraction is *linearly* related to the temperature increase or decrease that occurs. If this is the case, and the material is homogeneous and isotropic, it has been found from experiment that the displacement of a member having a length L can be calculated using the formula

$$\delta_T = \alpha \Delta T L \qquad (4.11)$$

where

α = A property of the material, referred to as the linear coefficient of thermal expansion.

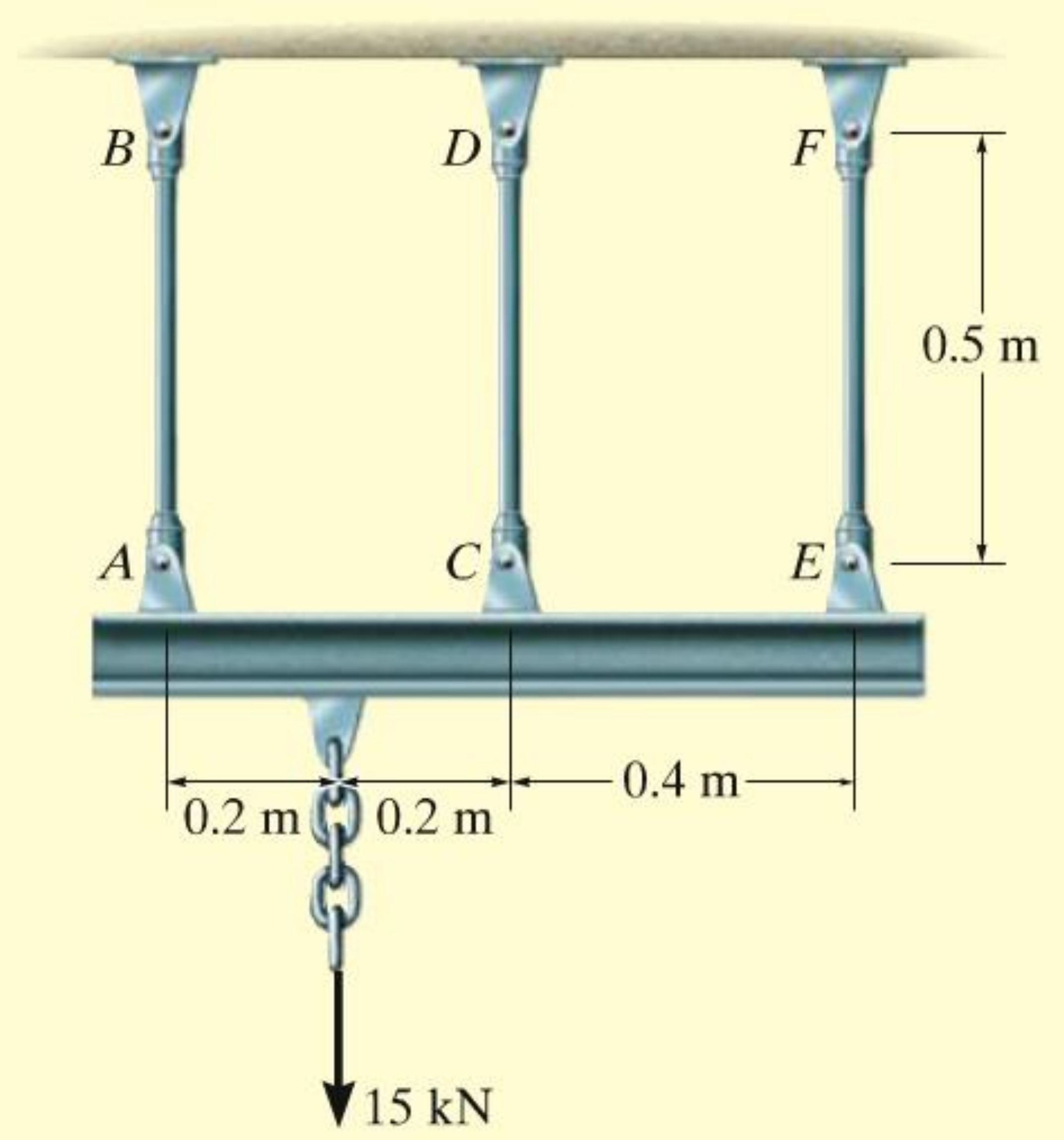
ΔT = the algebraic change in temperature of the member

L = the original length of the member

δ_T = the algebraic change in the length of the member

Example-2

The steel bars shown in below are pin connected to a rigid member. If the applied load on the member is 15kN, determine the force developed in each bar. Bars *AB* and *EF* each have a cross-sectional area of 50mm^2, and the bar *CD* has a cross-sectional area of 30mm^2.

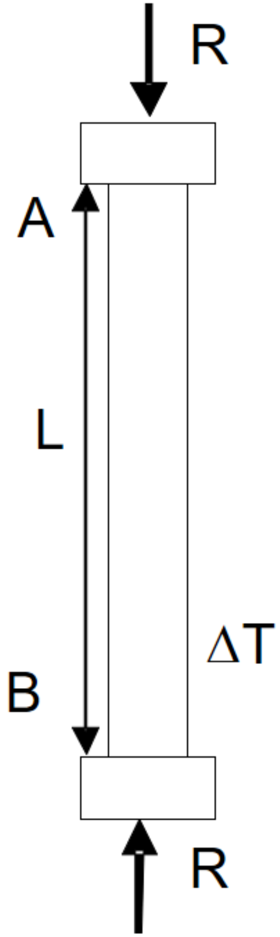


Answer: $F_A = 9.52kN$

$F_C = 3.46kN$

$F_E = 2.02kN$

Example-3

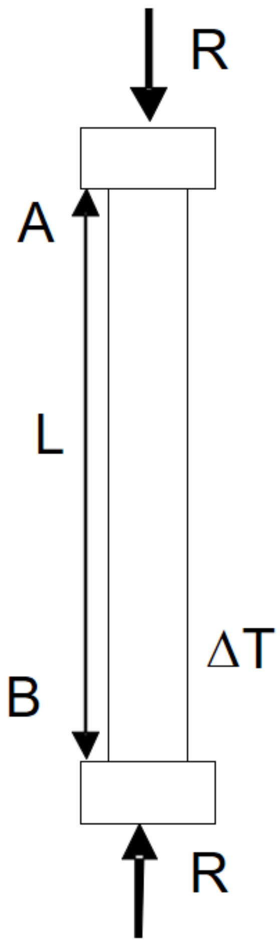


Consider a mild steel bar AB completely fixed at both ends as shown in the figure. The length of the bar is L and the cross-sectional area is A. The bar is uniformly heated up to 60 ^{0}C from the room temperature of 20 ^{0}C.

E = 220 GPa and $\alpha = 12 \times 10^{-6}\ {}^{o}C^{-1}$

Determine the maximum thermal stress developed in the bar.

Example-3



We are unable to evaluate the value of the reaction force using statics – this is a statically indeterminate problem.

The internal force is P_{AB}.

We can solve the problem by using the compliance method. To do this we remove one of the reactions and allow free expansion.

We can then apply a load R to give a displacement equal to the expansion. This force will be the required force allowing us to calculate the thermal stress.

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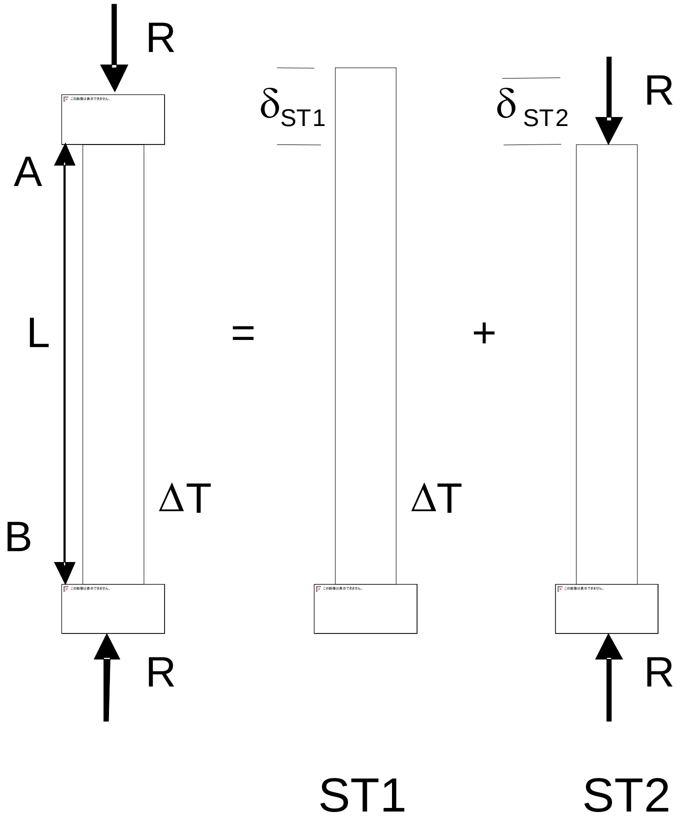
Example-3



ST1: $\delta_{ST1} = \alpha L\left(T - T_0\right) = 40\alpha L$

ST2: $\delta_{ST2} = \dfrac{RL}{EA}$

Displacement at A is zero

$$\delta_{ST1} - \delta_{ST2} = 0 \qquad \Rightarrow \quad 40\alpha L - \frac{RL}{EA} = 0$$

$$R = 40EA\alpha$$

Stress in bar $\sigma = \dfrac{\text{force}}{\text{area}} = \dfrac{R}{A} = 40E\alpha$

Notice stress is not function of area!

$$\sigma = 40 \times 220 \times 10^3 \times 12 \times 10^{-6}$$

$$\sigma = 106 \text{ MPa}$$

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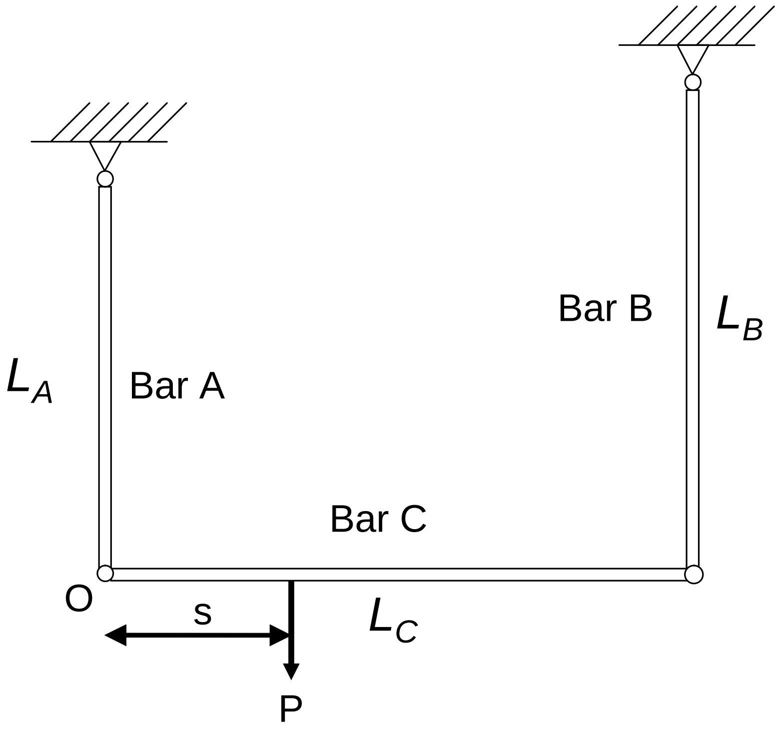
Example-4



A channel metal truss consists of two aluminium vertical bars A, L_A, and B, L_B with a horizontal steel bar C (L_C), as shown.

Bar C is rigid (i.e. no bending considered) and the bars are hinge connected. The cross sections and Young's moduli of bars A and B are A_A, A_B, E_A, and E_B, respectively

A load P is hung from the bar C at such a place that the bar ***remains horizontal***.

Determine:

(i) Axial stress in each vertical bar

(ii) Vertical displacement of bar C

(iii) Horizontal position of the external load P with respect to point O.

Example-4

Equilibrium of Bar C:
Forces P_A P_B and P

Bar A elongated by P_A by amount δ_A
Bar B elongated by P_B by amount δ_B

Equilibirum equation

$$P_A + P_B = P$$

Displacement $\delta_A = \delta_B = \delta$

$$P_A = \frac{E_A A_A L_B}{E_B A_B L_A + E_A A_A L_B} \cdot P$$

$$P_B = \frac{E_B A_B L_A}{E_A A_A L_B + E_B A_B L_A} \cdot P$$

Displacement of each bar

$$\delta_A = \frac{P_A L_A}{E A_A} \qquad \delta_B = \frac{P_B L_B}{E A_B}$$

Bar C is remains horizontal hence δ_A=δ_B

Example-4

Stress in each bar

$$\sigma_A = \frac{E_A L_B}{E_A A_A L_B + E_B A_B L_A} \cdot P$$

$$\sigma_B = \frac{E_B L_A}{E_A A_A L_B + E_B A_B L_A} \cdot P$$

Displacement of bar

$$\delta = \frac{P_A L_A}{E_A A_A} = \frac{L_A L_B}{E_B A_B L_A + E_A A_A L_B} \cdot P$$

Moments about O (LH end of C) $\quad P_B.L_C = P \times s$

$$s = \frac{P_B.L_C}{P} = P_B = \frac{E_B A_B L_A L_C}{E_A A_A L_B + E_B A_B L_A}$$