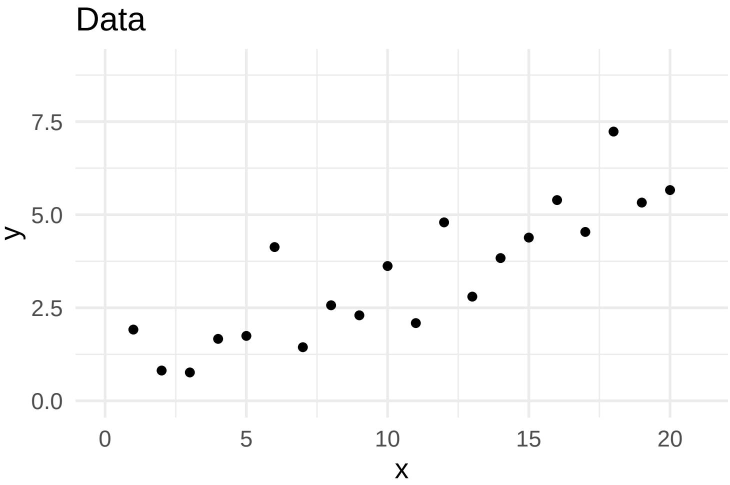
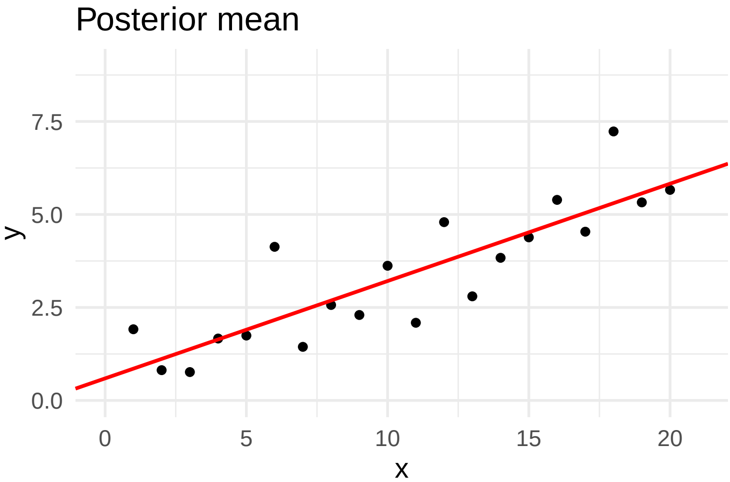
Chapter 3

- 3.1 Marginalization
- 3.2 Normal distribution with a noninformative prior (important)
- 3.3 Normal distribution with a conjugate prior (important)
- 3.4 Multinomial model (can be skipped)
- 3.5 Multivariate normal with known variance (useful for chapter 4)
- 3.6 Multivariate normal with unknown variance (glance through)
- 3.7 Bioassay example (very important, related to one of the exercises)
- 3.8 Summary (summary)

Example of uncertainty in modeling



Example of uncertainty in modeling



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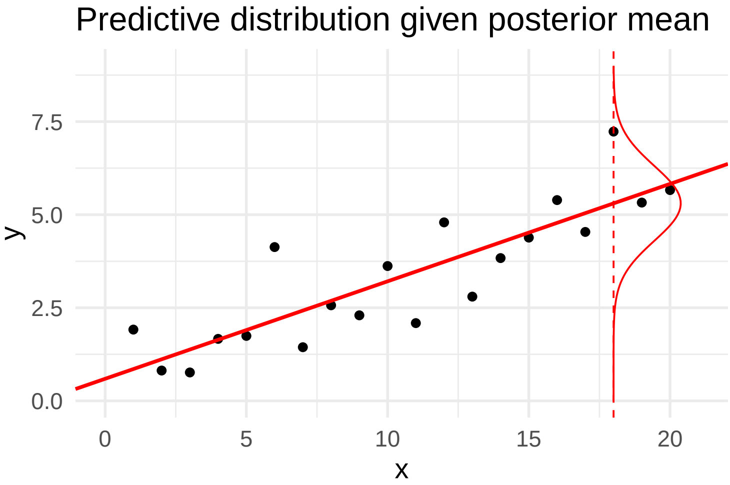

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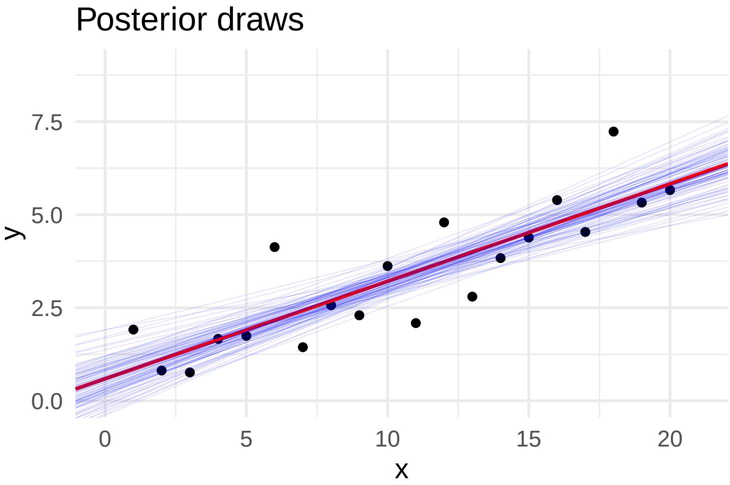
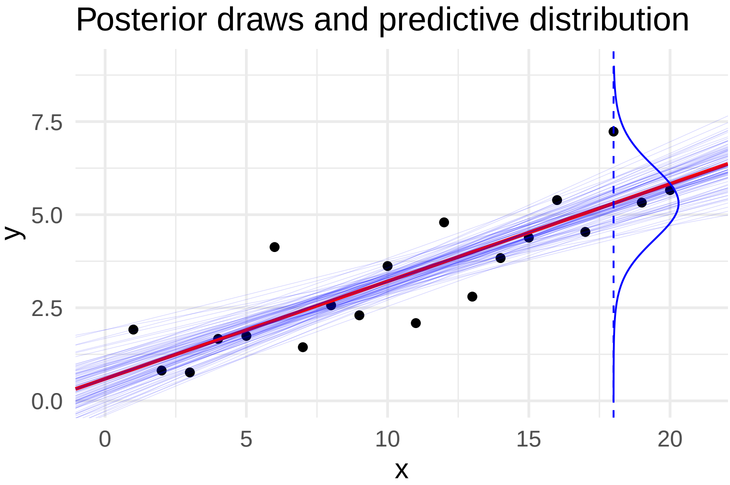
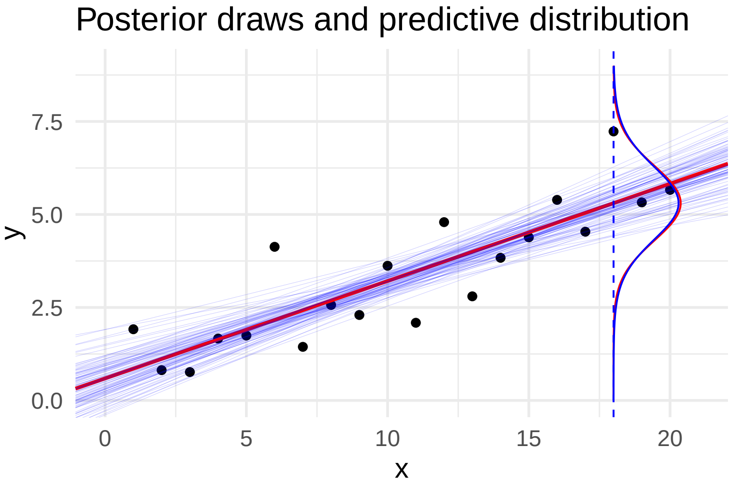


Example of uncertainty in modeling



Example of uncertainty in modeling



Monte Carlo and posterior draws

- $\theta^{(s)}$ draws from $p(\theta \mid y)$ can be used
  - for visualization

Monte Carlo and posterior draws

- $\theta^{(s)}$ draws from $p(\theta \mid y)$ can be used
  - for visualization
  - to approximate expectations (integrals)

$$E_{p(\theta|y)}[\theta] = \int \theta p(\theta \mid y) \approx \frac{1}{S} \sum_{s=1}^{S} \theta^{(s)}$$

Marginalization

- Joint distribution of parameters

$$p(\theta_1, \theta_2 \mid y) \propto p(y \mid \theta_1, \theta_2) p(\theta_1, \theta_2)$$

- Marginalization

$$p(\theta_1 \mid y) = \int p(\theta_1, \theta_2 \mid y) d\theta_2$$

$p(\theta_1 \mid y)$ is a marginal distribution

Marginalization

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- Marginalization

$$p(\theta_1 \mid y) = \int p(\theta_1, \theta_2 \mid y) d\theta_2$$

$p(\theta_1 \mid y)$ is a marginal distribution

- Monte Carlo approximation

$$p(\theta_1 \mid y) \approx \frac{1}{S} \sum_{s=1}^{S} p(\theta_1, \theta_2^{(s)} \mid y),$$

where $\theta_2^{(s)}$ are draws from $p(\theta_2 \mid y)$

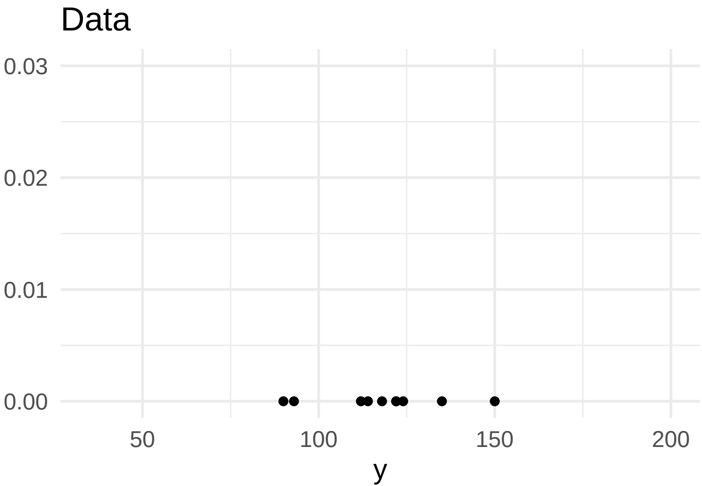
Marginalization - predictive distribution

- Marginalization over posterior distribution

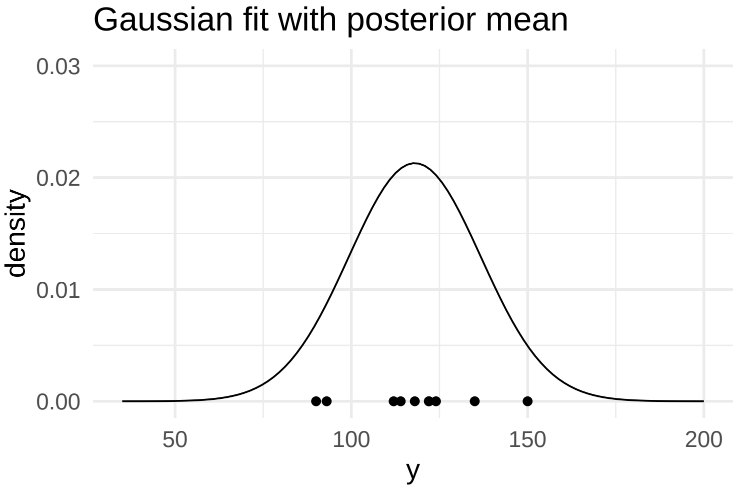
$$
\begin{aligned}
p(\tilde{y} \mid y) &= \int p(\tilde{y} \mid \theta) p(\theta \mid y) d\theta \\
&= \int p(\tilde{y}, \theta \mid y) d\theta
\end{aligned}
$$

$p(\tilde{y} \mid y)$ is a predictive distribution

Gaussian example

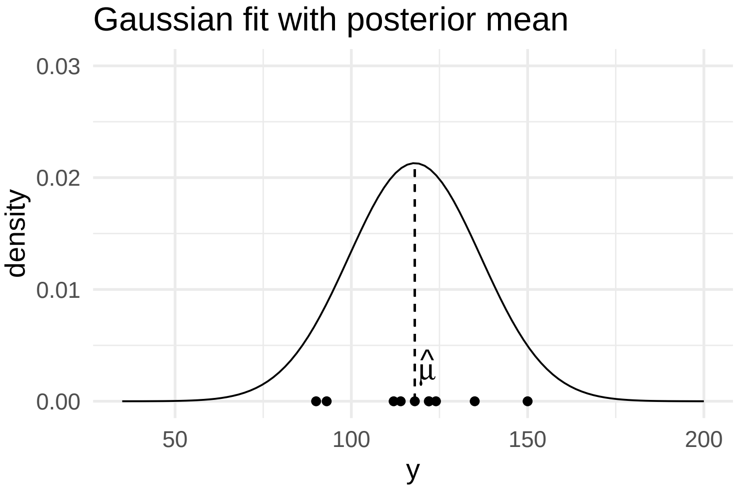


Gaussian example



$$p(y \mid \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y-\mu)^2\right)$$

Gaussian example



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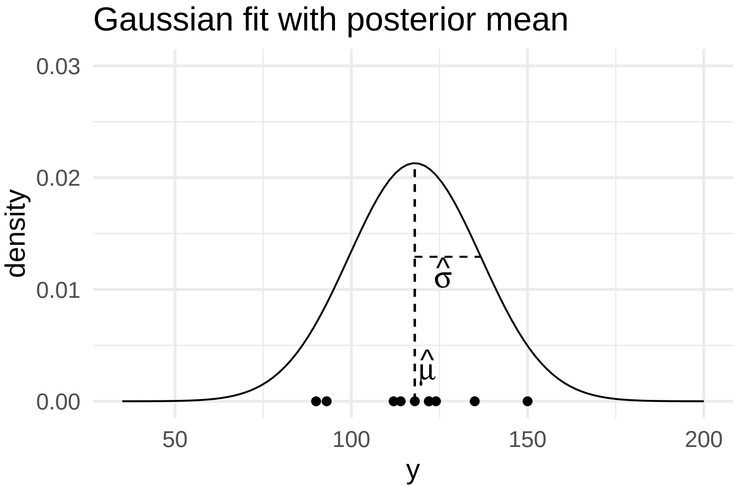

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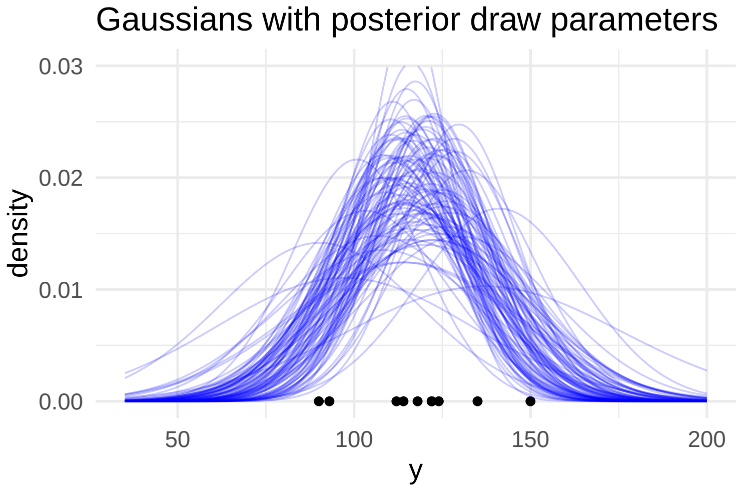
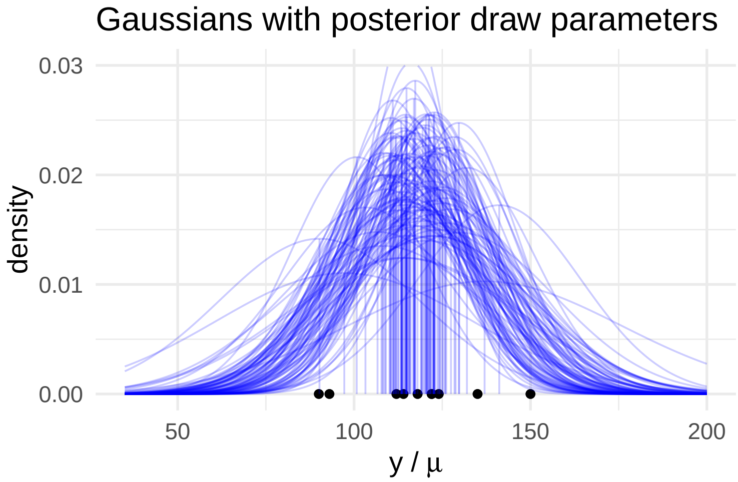


$$\color{blue}{\mu^{(s)}, \sigma^{(s)}} \sim p(\mu, \sigma \mid y)$$

Gaussian example



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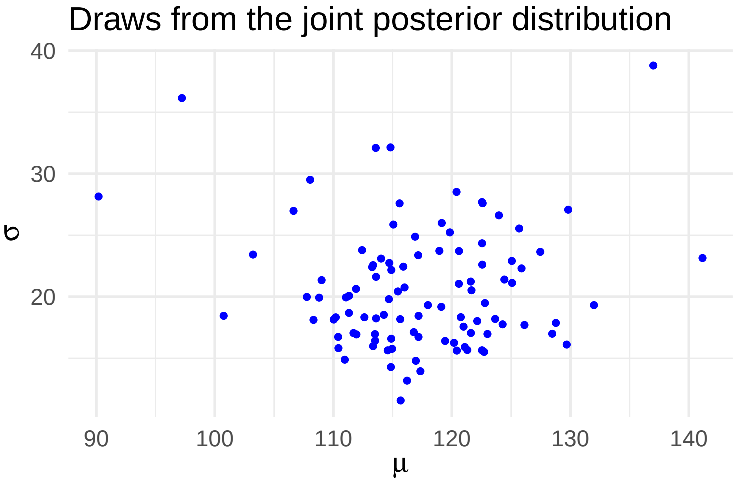
Gaussian example



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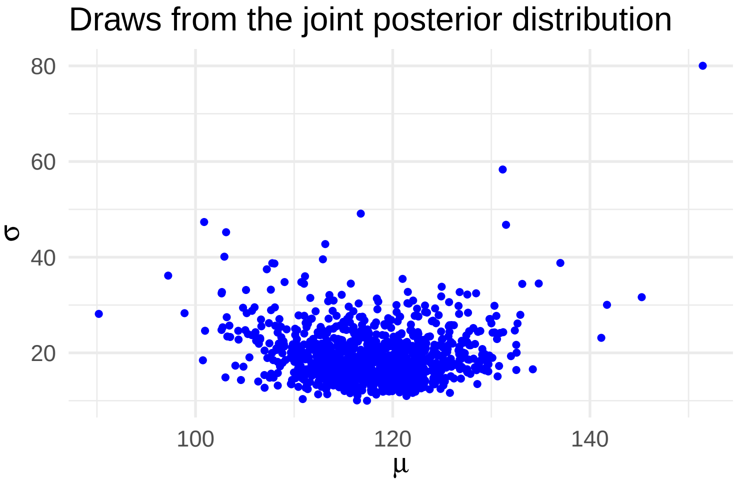

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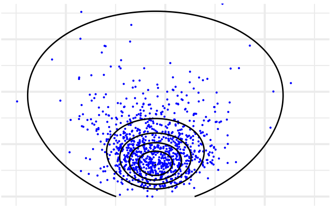
Joint posterior

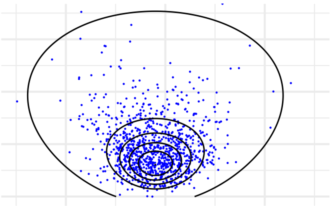
$$\color{blue}{\mu^{(s)}, \sigma^{(s)}} \sim p(\mu, \sigma \mid y)$$



Joint posterior

$$\color{blue}{\mu^{(s)}, \sigma^{(s)}} \sim p(\mu, \sigma \mid y)$$

with $p(\mu, \sigma^2) \propto \sigma^{-2}$



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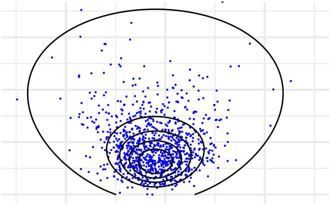

$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

with $p(\mu, \sigma^2) \propto \sigma^{-2}$

$$p(\mu, \sigma^2 \mid y) \propto \sigma^{-2} \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y_i - \mu)^2\right)$$

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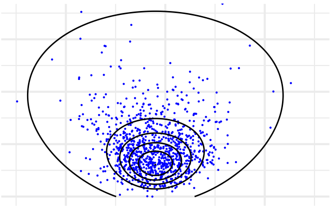
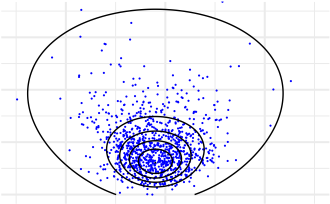


$$p(\mu, \sigma^2 \mid y) \propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \mu)^2\right)$$

Joint posterior



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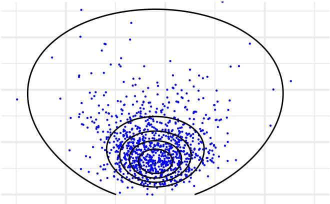
$$p(\mu, \sigma^2 \mid y) \propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \mu)^2\right)$$

$$= \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2} \left[\sum_{i=1}^{n} (y_i - \bar{y})^2 + n(\bar{y} - \mu)^2\right]\right)$$

where $\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i$

Joint posterior



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where $\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i$

$$= \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2} \left[(n-1)s^2 + n(\bar{y} - \mu)^2\right]\right)$$

where $s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (y_i - \bar{y})^2$

Gaussian - non-informative prior

$$\sum_{i=1}^{n}(y_i - \mu)^2$$

Gaussian - non-informative prior

$$\sum_{i=1}^{n}(y_i - \mu)^2$$

$$\sum_{i=1}^{n}(y_i^2 - 2y_i\mu + \mu^2)$$

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$$\sum_{i=1}^{n}(y_i^2 - 2y_i\mu + \mu^2 - \bar{y}^2 + \bar{y}^2 - 2y_i\bar{y} + 2y_i\bar{y})$$

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$$\sum_{i=1}^{n}(y_i^2 - 2y_i\bar{y} + \bar{y}^2) + \sum_{i=1}^{n}(\mu^2 - 2y_i\mu - \bar{y}^2 + 2y_i\bar{y})$$

Gaussian - non-informative prior

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$$\sum_{i=1}^{n}(y_i - \bar{y})^2 + n(\mu^2 - 2\bar{y}\mu - \bar{y}^2 + 2\bar{y}\bar{y})$$

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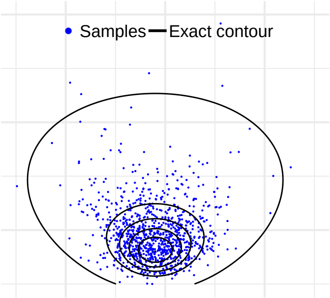
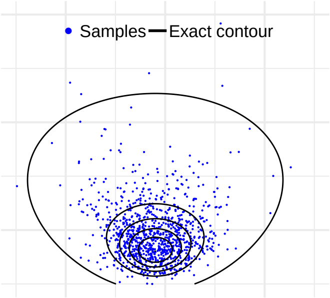
$$\sum_{i=1}^{n}(y_i - \bar{y})^2 + n(\bar{y} - \mu)^2$$

Joint posterior

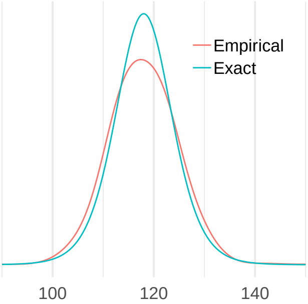


$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

Joint posterior



Marginal of mu

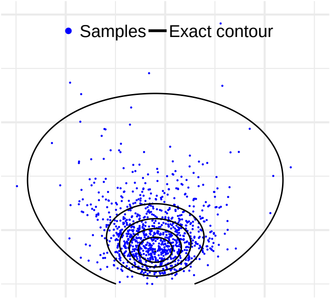


$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

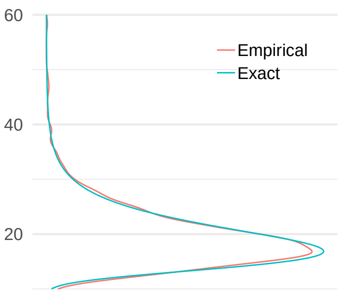
marginals

$$p(\mu \mid y) = \int p(\mu, \sigma \mid y) d\sigma$$

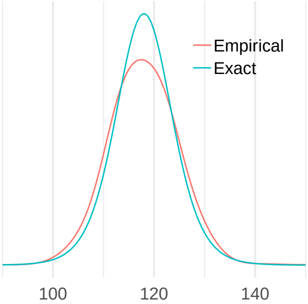
Joint posterior



Marginal of sigma



Marginal of mu



$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

marginals

$$p(\mu \mid y) = \int p(\mu, \sigma \mid y) d\sigma$$

$$p(\sigma \mid y) = \int p(\mu, \sigma \mid y) d\mu$$

Marginal posterior $p(\sigma^2 \mid y)$ (easier for σ^2 than σ)

$$p(\sigma^2 \mid y) \quad \propto \quad \int p(\mu, \sigma^2 \mid y) d\mu$$

Marginal posterior $p(\sigma^2 \mid y)$ (easier for σ^2 than σ)

$$
\begin{aligned}
p(\sigma^2 \mid y) &\propto \int p(\mu, \sigma^2 \mid y) d\mu \\
&\propto \int \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]\right) d\mu
\end{aligned}
$$

Marginal posterior $p(\sigma^2 \mid y)$ (easier for σ^2 than σ)

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&\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right) \\
&\quad \int \exp\left(-\frac{n}{2\sigma^2}(\bar{y}-\mu)^2\right) d\mu
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$$

$$\int \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y-\theta)^2\right) d\theta = 1$$

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&\quad \int \exp\left(-\frac{n}{2\sigma^2}(\bar{y}-\mu)^2\right) d\mu \\
&\qquad\qquad \int \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y-\theta)^2\right) d\theta = 1 \\
&\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right) \sqrt{2\pi\sigma^2/n}
\end{aligned}
$$

Marginal posterior $p(\sigma^2 \mid y)$ (easier for σ^2 than σ)

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\end{aligned}$$

$$\int \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y-\theta)^2\right) d\theta = 1$$

$$\begin{aligned}
&\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right) \sqrt{2\pi\sigma^2/n} \\
&\propto (\sigma^2)^{-(n+1)/2} \exp\left(-\frac{(n-1)s^2}{2\sigma^2}\right)
\end{aligned}$$

Marginal posterior $p(\sigma^2 \mid y)$ (easier for σ^2 than σ)

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\begin{aligned}
p(\sigma^2 \mid y) &\propto \int p(\mu, \sigma^2 \mid y) d\mu \\
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&\quad \int \exp\left(-\frac{n}{2\sigma^2}(\bar{y}-\mu)^2\right) d\mu \\
&\qquad\qquad \int \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y-\theta)^2\right) d\theta = 1 \\
&\propto \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right) \sqrt{2\pi\sigma^2/n} \\
&\propto (\sigma^2)^{-(n+1)/2} \exp\left(-\frac{(n-1)s^2}{2\sigma^2}\right) \\
p(\sigma^2 \mid y) &= \text{Inv-}\chi^2(\sigma^2 \mid n-1, s^2)
\end{aligned}
$$

Gaussian - non-informative prior

Known mean

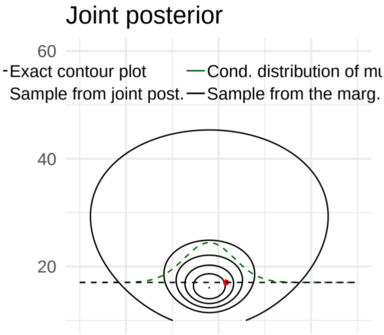
$$\sigma^2 \mid y \sim \text{Inv-}\chi^2(n, v)$$

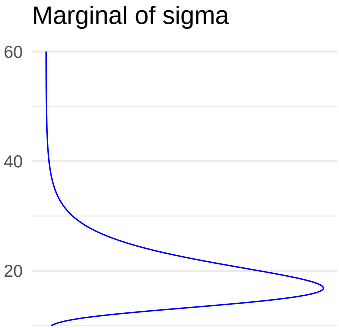
$$\text{where} \quad v = \frac{1}{n} \sum_{i=1}^{n} (y_i - \theta)^2$$

Unknown mean

$$\sigma^2 \mid y \sim \text{Inv-}\chi^2(n-1, s^2)$$

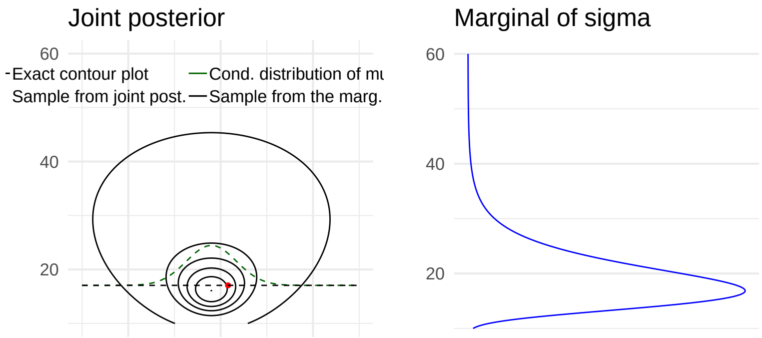
$$\text{where} \quad s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (y_i - \bar{y})^2$$





Factorization

$$p(\mu, \sigma^2 \mid y) = {\color{green} p(\mu \mid \sigma^2, y)}{\color{blue} p(\sigma^2 \mid y)}$$

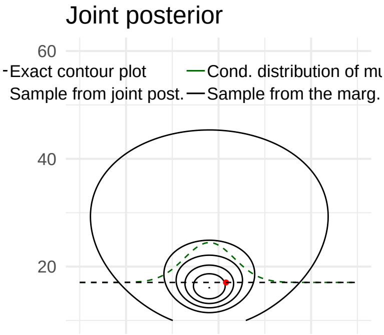


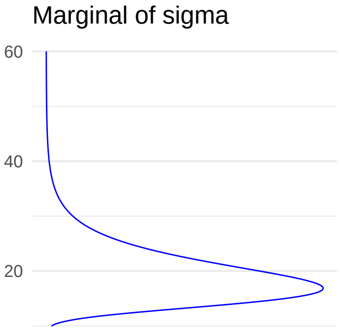
Factorization

$$p(\mu, \sigma^2 \mid y) = {\color{green} p(\mu \mid \sigma^2, y)}{\color{blue} p(\sigma^2 \mid y)}$$

$$ {\color{blue} p(\sigma^2 \mid y)} = \text{Inv-}\chi^2(\sigma^2 \mid n - 1, s^2)$$

$$(\sigma^2)^{(s)} \sim {\color{blue} p(\sigma^2 \mid y)}$$





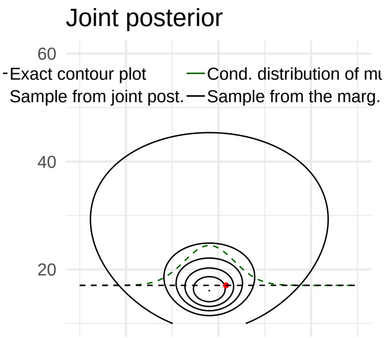
Factorization

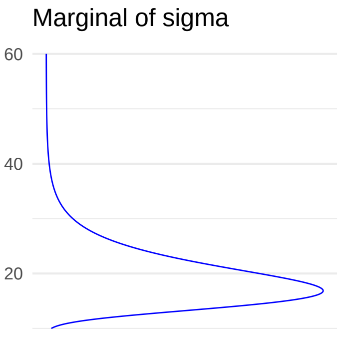
$$p(\mu, \sigma^2 \mid y) = p(\mu \mid \sigma^2, y) p(\sigma^2 \mid y)$$

$$p(\sigma^2 \mid y) = \text{Inv-}\chi^2(\sigma^2 \mid n-1, s^2)$$

$$(\sigma^2)^{(s)} \sim p(\sigma^2 \mid y)$$

$$p(\mu \mid \sigma^2, y) = \mathrm{N}(\mu \mid \bar{y}, \sigma^2/n)$$





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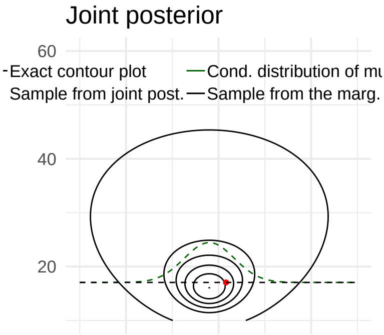

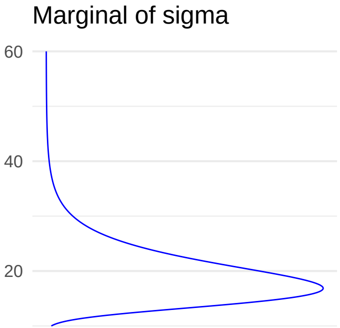

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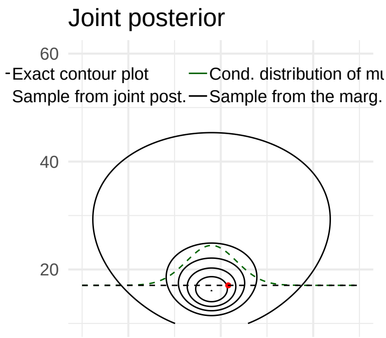

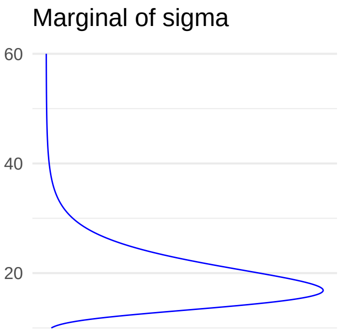

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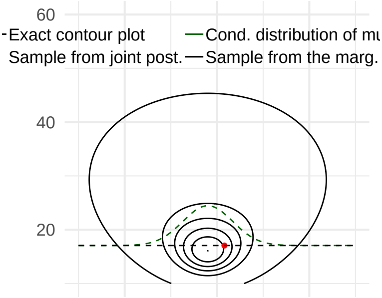
$$p(\mu \mid \sigma^2, y) = \mathrm{N}(\mu \mid \bar{y}, \sigma^2/n)$$

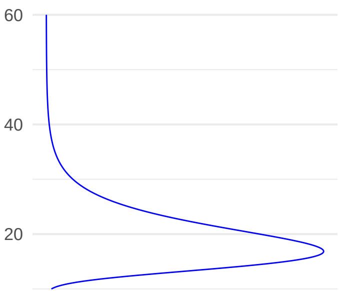
$$\mu^{(s)} \sim p(\mu \mid \sigma^2, y)$$

$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

Joint posterior

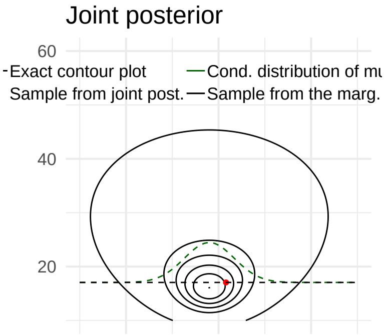


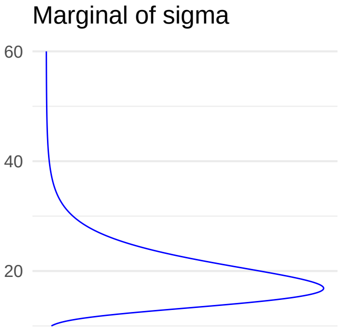
Marginal of sigma



Factorization

$$p(\mu, \sigma^2 \mid y) = p(\mu \mid \sigma^2, y) p(\sigma^2 \mid y)$$





Factorization

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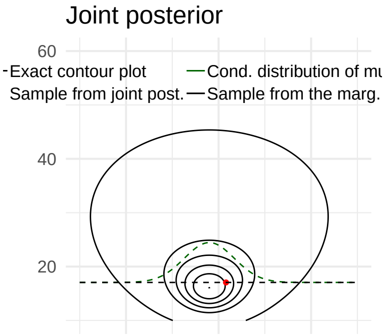

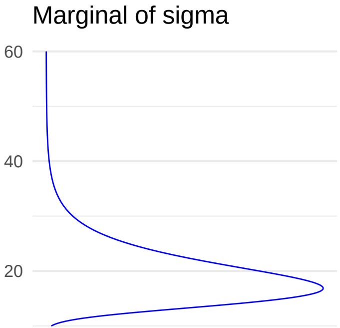

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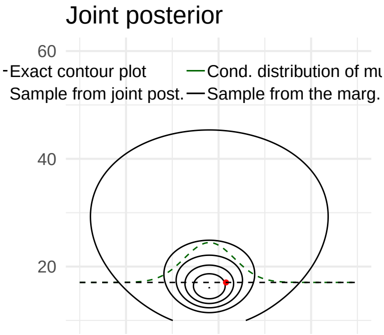

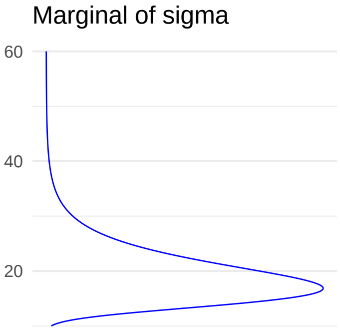

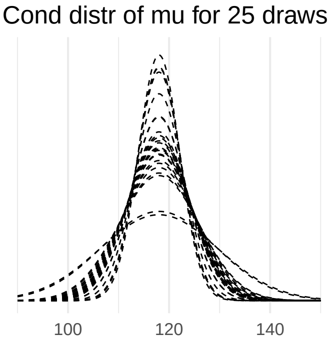

Factorization

$$p(\mu, \sigma^2 \mid y) = p(\mu \mid \sigma^2, y) p(\sigma^2 \mid y)$$

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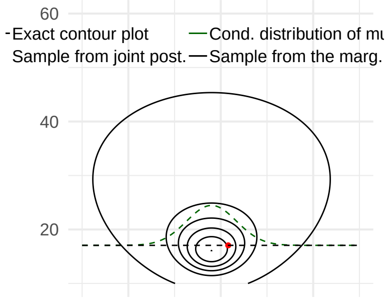
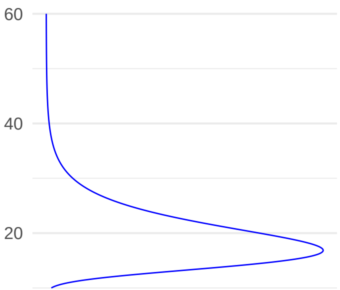
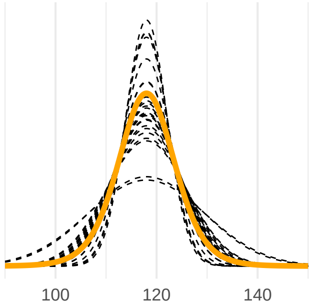
Joint posterior



Marginal of sigma



Cond distr of mu for 25 draws



Factorization

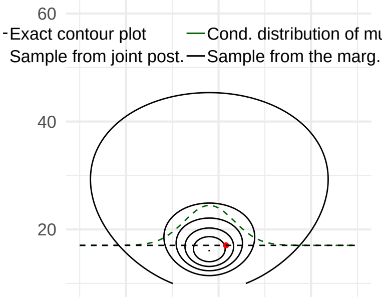
$$p(\mu, \sigma^2 \mid y) = p(\mu \mid \sigma^2, y) p(\sigma^2 \mid y)$$

$$(\sigma^2)^{(s)} \sim p(\sigma^2 \mid y)$$

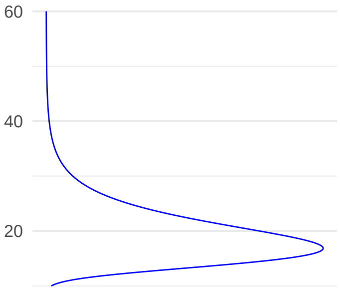
$$p(\mu \mid (\sigma^2)^{(s)}, y) = \mathrm{N}(\mu \mid \bar{y}, (\sigma^2)^{(s)}/n)$$

$$p(\mu \mid y) \approx \frac{1}{S} \sum_{s=1}^{S} \mathrm{N}(\mu \mid \bar{y}, (\sigma^2)^{(s)}/n)$$

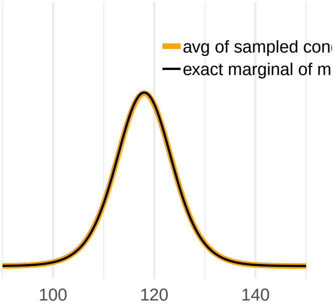
Joint posterior



Marginal of sigma



Cond. distr of mu



Factorization

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Marginal posterior $p(\mu \mid y)$

$$p(\mu \mid y) = \int_0^\infty p(\mu, \sigma^2 \mid y) d\sigma^2$$

Marginal posterior $p(\mu \mid y)$

$$
\begin{aligned}
p(\mu \mid y) &= \int_0^\infty p(\mu, \sigma^2 \mid y) d\sigma^2 \\
&\propto \int_0^\infty \sigma^{-n-2} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]\right) d\sigma^2
\end{aligned}
$$

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Transformation

$$A = (n-1)s^2 + n(\mu - \bar{y})^2$$

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$$p(\mu \mid y) = \int_0^\infty p(\mu, \sigma^2 \mid y) d\sigma^2$$
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Transformation

$$A = (n-1)s^2 + n(\mu - \bar{y})^2 \quad \text{and} \quad z = \frac{A}{2\sigma^2}$$

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Transformation

$$A = (n-1)s^2 + n(\mu - \bar{y})^2 \quad \text{and} \quad z = \frac{A}{2\sigma^2}$$

$$p(\mu \mid y) \propto A^{-n/2} \int_0^\infty z^{(n-2)/2} \exp(-z) dz$$

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$$p(\mu \mid y) \propto A^{-n/2} \int_0^\infty z^{(n-2)/2} \exp(-z) dz$$

Recognize gamma integral $\Gamma(u) = \int_0^\infty x^{u-1} \exp(-x) dx$

Marginal posterior $p(\mu \mid y)$

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Transformation

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$$\propto [(n-1)s^2 + n(\mu - \bar{y})^2]^{-n/2}$$

$$\propto \left[1 + \frac{n(\mu - \bar{y})^2}{(n-1)s^2} \right]^{-n/2}$$

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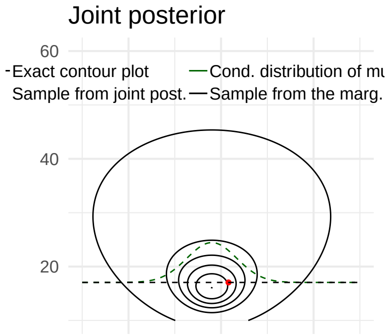
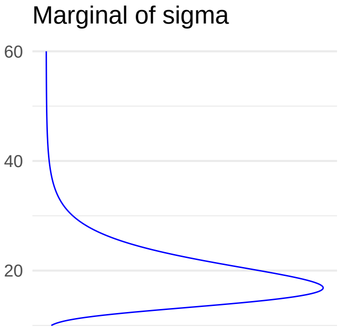
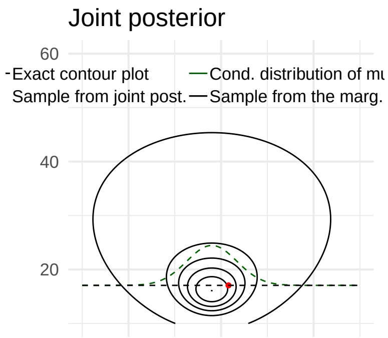
$$p(\mu \mid y) = t_{n-1}(\mu \mid \bar{y}, s^2/n) \quad \text{Student's } t$$

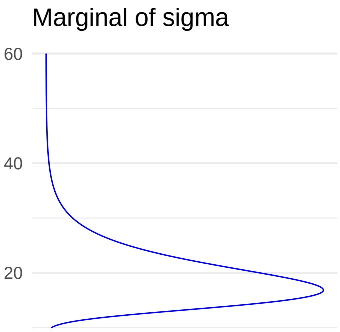




Predictive distribution for new $\tilde{y}$

$$p(\tilde{y} \mid y) = \int p(\tilde{y} \mid \mu, \sigma) p(\mu, \sigma \mid y) d\mu\sigma$$





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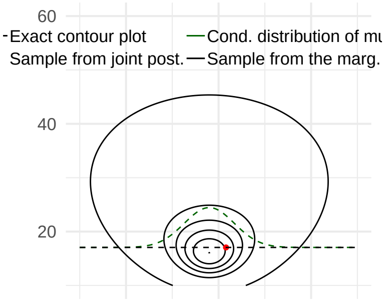
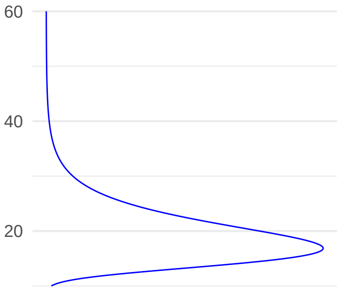
$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

Joint posterior



Marginal of sigma



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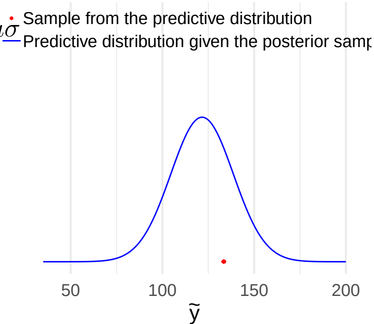
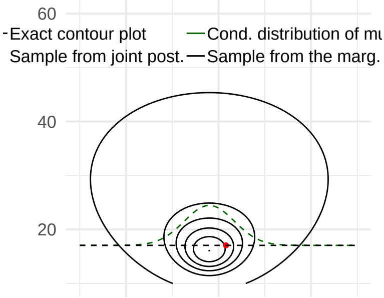
$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

$$\tilde{y}^{(s)} \sim p(\tilde{y} \mid \mu^{(s)}, \sigma^{(s)})$$

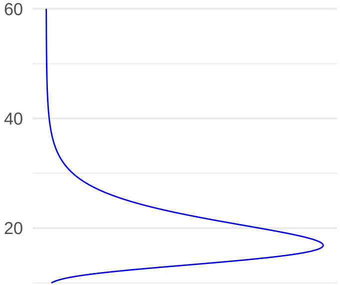
Posterior predictive distribution



Joint posterior



Marginal of sigma



Predictive distribution for new $\tilde{y}$

$$p(\tilde{y} \mid y) = \int p(\tilde{y} \mid \mu, \sigma) p(\mu, \sigma \mid y) d\mu\sigma$$

$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

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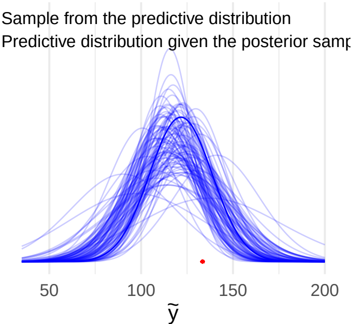
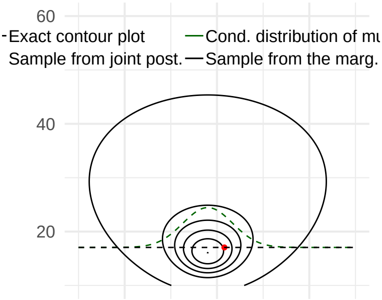
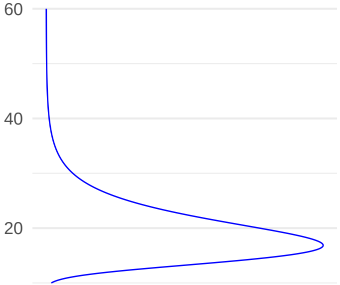
Posterior predictive distribution



Joint posterior



Marginal of sigma



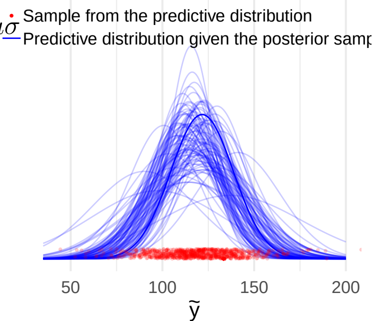
Predictive distribution for new $\tilde{y}$

$$p(\tilde{y} \mid y) = \int p(\tilde{y} \mid \mu, \sigma) p(\mu, \sigma \mid y) d\mu d\sigma$$

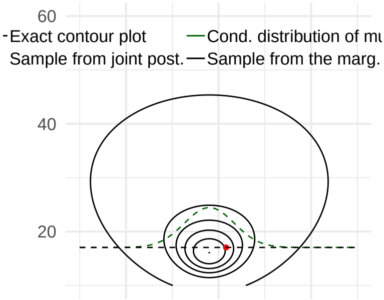
$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

$$\tilde{y}^{(s)} \sim p(\tilde{y} \mid \mu^{(s)}, \sigma^{(s)})$$

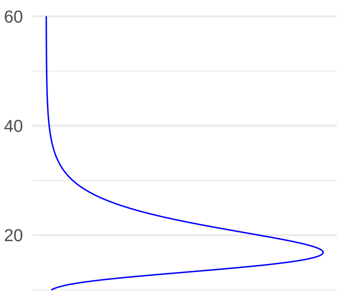
Posterior predictive distribution



Joint posterior



Marginal of sigma



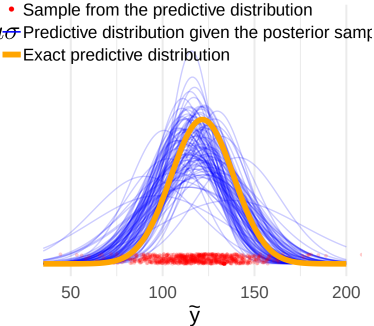
Predictive distribution for new $\tilde{y}$

$$p(\tilde{y} \mid y) = \int p(\tilde{y} \mid \mu, \sigma) p(\mu, \sigma \mid y) d\mu\sigma$$

$$\mu^{(s)}, \sigma^{(s)} \sim p(\mu, \sigma \mid y)$$

$$\tilde{y}^{(s)} \sim p(\tilde{y} \mid \mu^{(s)}, \sigma^{(s)})$$

Posterior predictive distribution



Gaussian - posterior predictive distribution

Posterior predictive distribution given known variance

$$p(\tilde{y} \mid \sigma^2, y) = \int p(\tilde{y} \mid \mu, \sigma^2) p(\mu \mid \sigma^2, y) d\mu$$

Gaussian - posterior predictive distribution

Posterior predictive distribution given known variance

$$
\begin{aligned}
p(\tilde{y} \mid \sigma^2, y) &= \int p(\tilde{y} \mid \mu, \sigma^2) p(\mu \mid \sigma^2, y) d\mu \\
&= \int \mathrm{N}(\tilde{y} \mid \mu, \sigma^2)\, \mathrm{N}(\mu \mid \bar{y}, \sigma^2/n) d\mu
\end{aligned}
$$

Gaussian - posterior predictive distribution

Posterior predictive distribution given known variance

$$
\begin{aligned}
p(\tilde{y} \mid \sigma^2, y) &= \int p(\tilde{y} \mid \mu, \sigma^2) p(\mu \mid \sigma^2, y) d\mu \\
&= \int \mathrm{N}(\tilde{y} \mid \mu, \sigma^2)\, \mathrm{N}(\mu \mid \bar{y}, \sigma^2/n) d\mu \\
&= \mathrm{N}(\tilde{y} \mid \bar{y}, (1 + \tfrac{1}{n})\sigma^2)
\end{aligned}
$$

Gaussian - posterior predictive distribution

Posterior predictive distribution given known variance

$$\begin{aligned} p(\tilde{y} \mid \sigma^2, y) &= \int p(\tilde{y} \mid \mu, \sigma^2) p(\mu \mid \sigma^2, y) d\mu \\ &= \int \mathrm{N}(\tilde{y} \mid \mu, \sigma^2)\, \mathrm{N}(\mu \mid \bar{y}, \sigma^2/n) d\mu \\ &= \mathrm{N}(\tilde{y} \mid \bar{y}, (1 + \tfrac{1}{n})\sigma^2) \end{aligned}$$

this is up to scaling factor same as $p(\mu \mid \sigma^2, y)$

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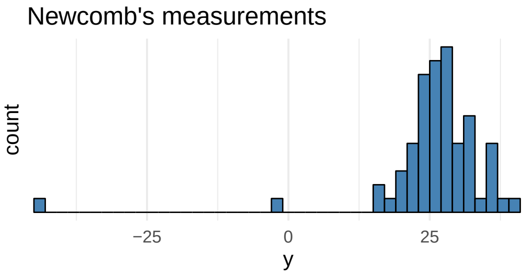
$$

this is up to scaling factor same as $p(\mu \mid \sigma^2, y)$

$$
p(\tilde{y} \mid y) = t_{n-1}(\tilde{y} \mid \bar{y}, (1 + \tfrac{1}{n}) s^2)
$$

Simon Newcomb's light of speed experiment in 1882

Newcomb measured ($n = 66$) the time required for light to travel from his laboratory on the Potomac River to a mirror at the base of the Washington Monument and back, a total distance of 7422 meters.



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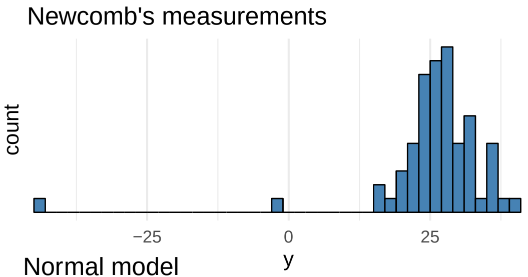

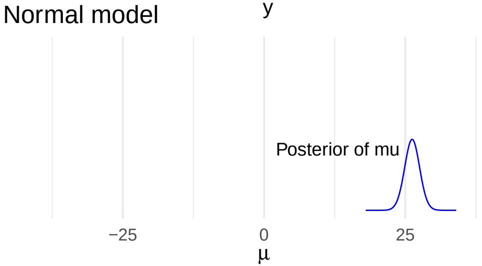

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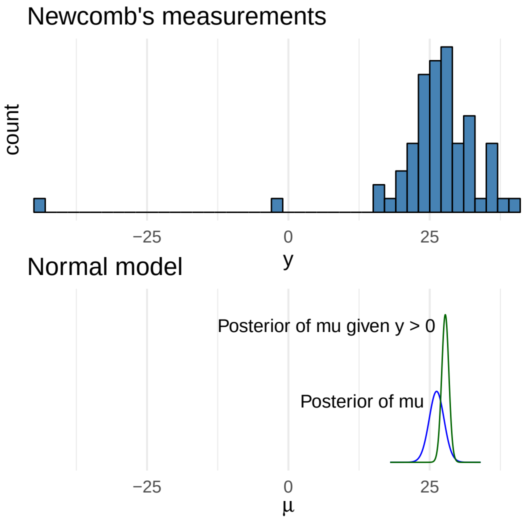

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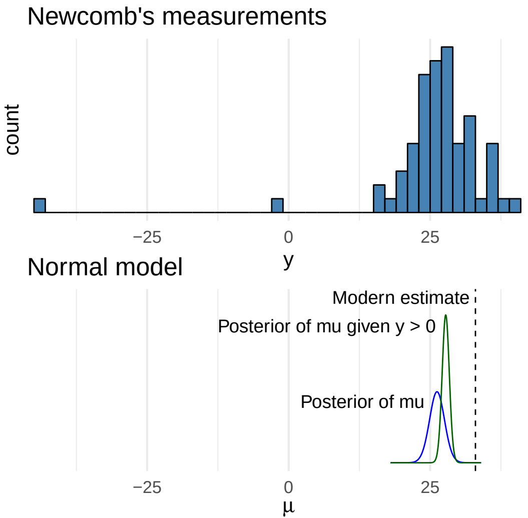

Gaussian - conjugate prior

- Conjugate prior has to have a form $p(\sigma^2)p(\mu \mid \sigma^2)$ (see the chapter notes)

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- Handy parameterization

$$\begin{aligned} \mu \mid \sigma^2 &\sim \mathrm{N}(\mu_0, \sigma^2/\kappa_0) \\ \sigma^2 &\sim \text{Inv-}\chi^2(\nu_0, \sigma_0^2) \end{aligned}$$

which can be written as

$$p(\mu, \sigma^2) = \text{N-Inv-}\chi^2(\mu_0, \sigma_0^2/\kappa_0; \nu_0, \sigma_0^2)$$

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$$p(\mu, \sigma^2) = \text{N-Inv-}\chi^2(\mu_0, \sigma_0^2/\kappa_0; \nu_0, \sigma_0^2)$$

- μ and σ^2 are a priori dependent
  - if σ^2 is large, then μ has wide prior

Gaussian - conjugate prior

Joint posterior (exercise 3.9)

$$p(\mu, \sigma^2 \mid y) = \text{N-Inv-}\chi^2(\mu_n, \sigma_n^2/\kappa_n; \nu_n, \sigma_n^2)$$

where

$$\begin{aligned}
\mu_n &= \frac{\kappa_0}{\kappa_0 + n}\mu_0 + \frac{n}{\kappa_0 + n}\bar{y} \\
\kappa_n &= \kappa_0 + n \\
\nu_n &= \nu_0 + n \\
\nu_n\sigma_n^2 &= \nu_0\sigma_0^2 + (n-1)s^2 + \frac{\kappa_0 n}{\kappa_0 + n}(\bar{y} - \mu_0)^2
\end{aligned}$$

Multinomial model for categorical data

- Extension of binomial
- Observation model

$$p(y \mid \theta) \propto \prod_{j=1}^{k} \theta_j^{y_j},$$

- BDA3 p. 69–

Multivariate Gaussian

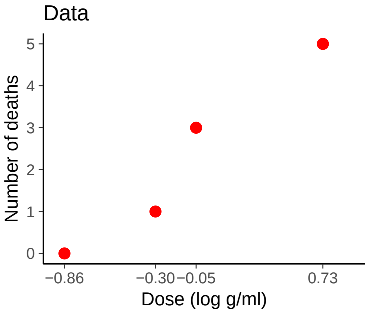
- Observation model

$$p(y \mid \mu, \Sigma) \propto \mid \Sigma \mid^{-1/2} \exp\left(-\frac{1}{2}(y-\mu)^T \Sigma^{-1}(y-\mu)\right),$$

- BDA3 p. 72–
- New recommended LKJ-prior mentioned in Appendix A, see more in Stan manual

Bioassay

Dose, x_i (log g/ml)	Number of animals, n_i	Number of deaths, y_i
-0.86	5	0
-0.30	5	1
-0.05	5	3
0.73	5	5



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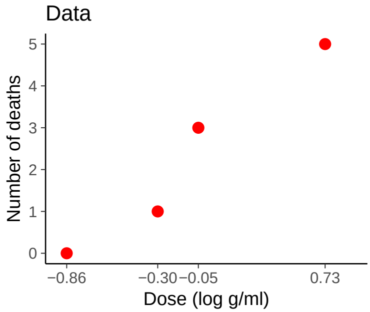

Find out lethal dose 50% (LD50)

- used to classify how hazardous chemical is
- 1984 EEC directive has 4 levels (see the chapter notes)

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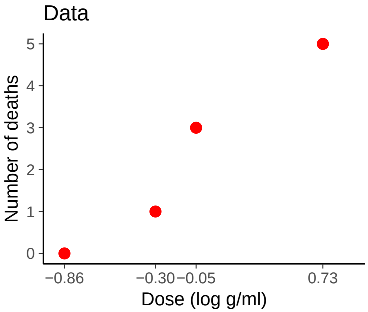

Find out lethal dose 50% (LD50)

- used to classify how hazardous chemical is
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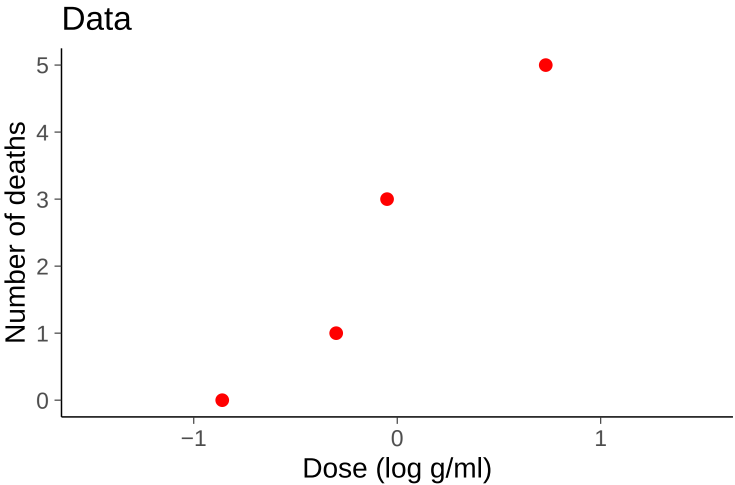
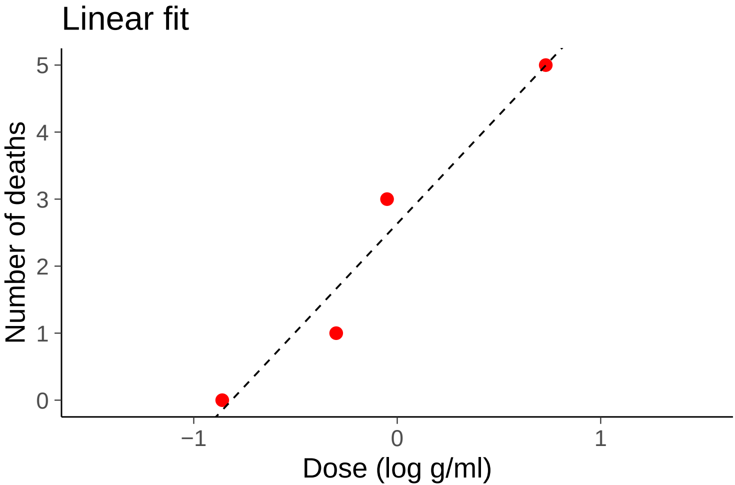
Bayesian methods help to

- reduce the number of animals needed
- easy to make sequential experiment and stop as soon as desired accuracy is obtained

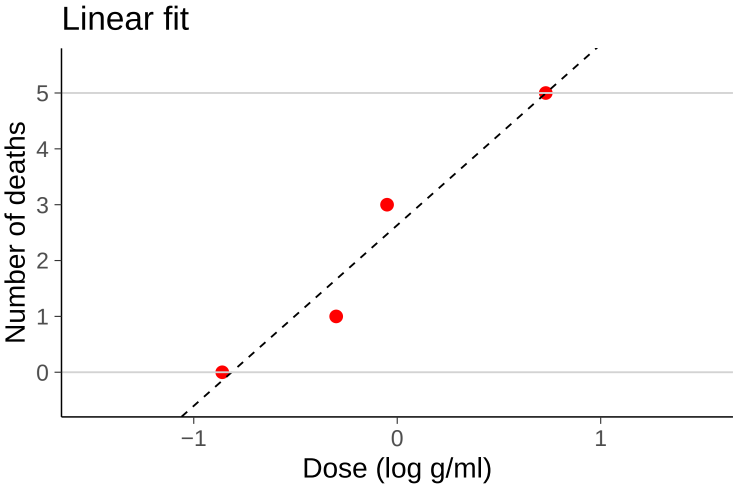
Bioassay



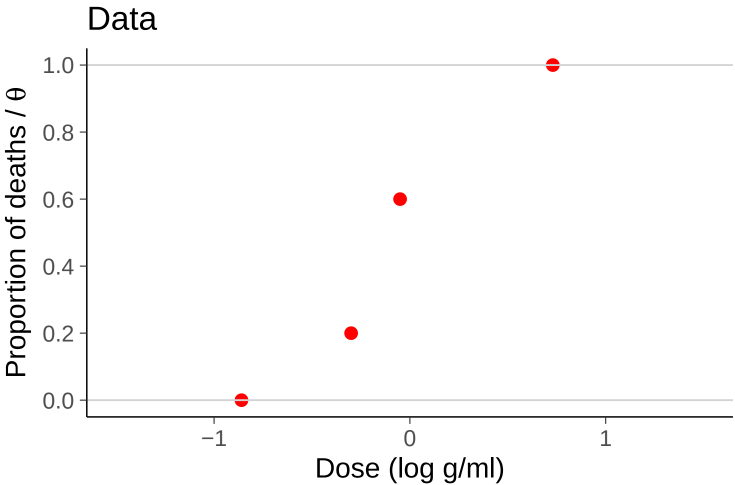
Bioassay



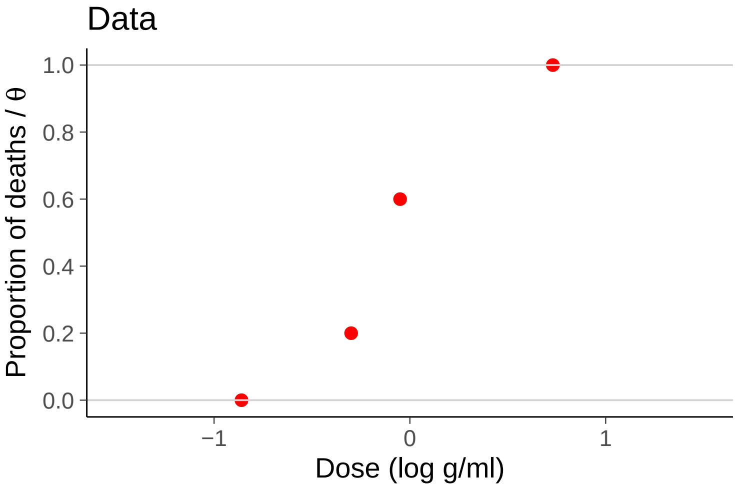
Bioassay



Bioassay



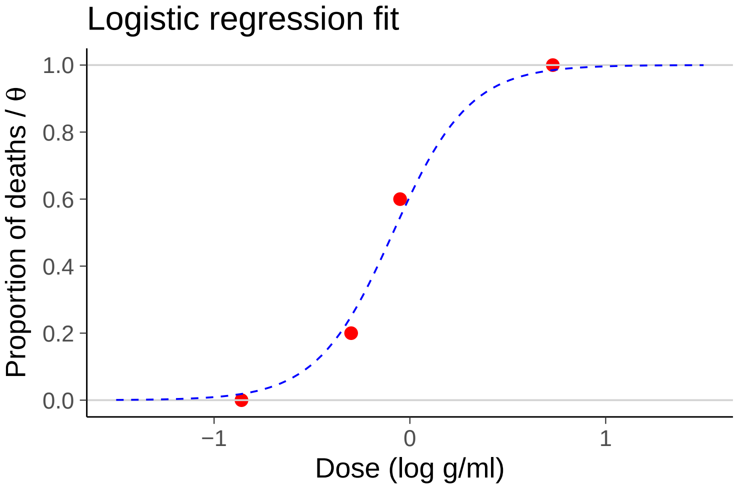
Bioassay



Binomial model

$$y_i \mid \theta_i \sim \text{Bin}(\theta_i, n_i)$$

Bioassay



Binomial model

$$y_i \mid \theta_i \sim \text{Bin}(\theta_i, n_i), \quad \text{logit}(\theta_i) = \log\left(\frac{\theta_i}{1-\theta_i}\right) = \alpha + \beta x_i$$

Bioassay

$$y_i \mid \theta_i \sim \text{Bin}(\theta_i, n_i)$$

$$\text{logit}(\theta_i) = \log\left(\frac{\theta_i}{1-\theta_i}\right)$$

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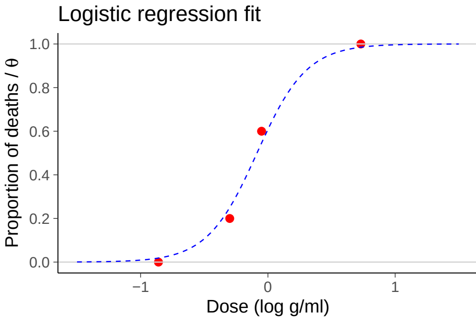

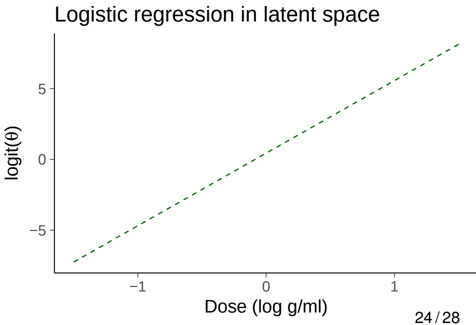

Bioassay

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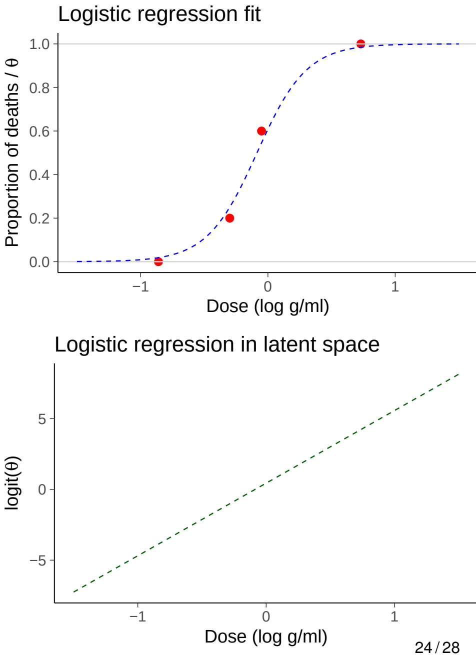
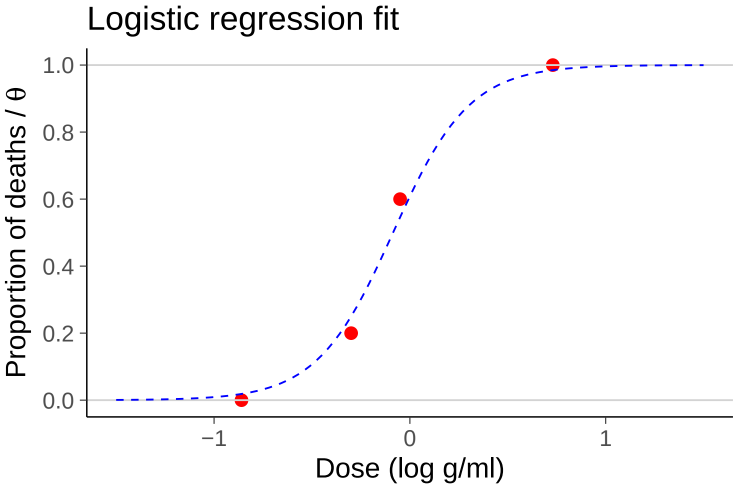
$$\text{logit}(\theta_i) = \log\left(\frac{\theta_i}{1-\theta_i}\right)$$

$$= \alpha + \beta x_i$$

$$\theta_i = \frac{1}{1+\exp(-(\alpha+\beta x_i))}$$

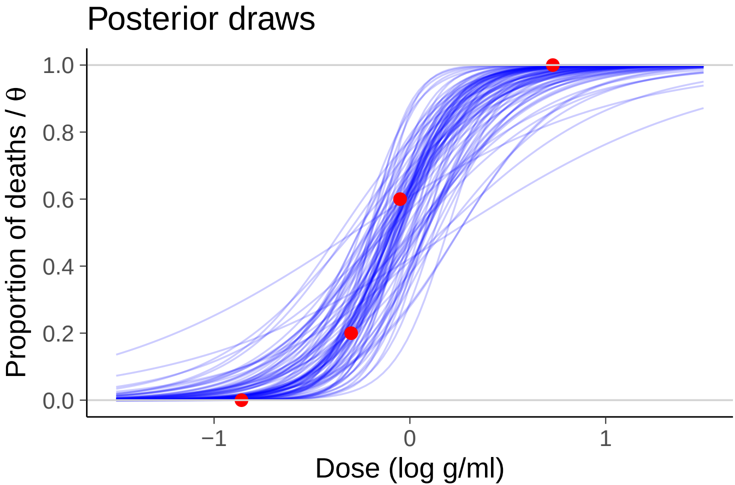


Bioassay



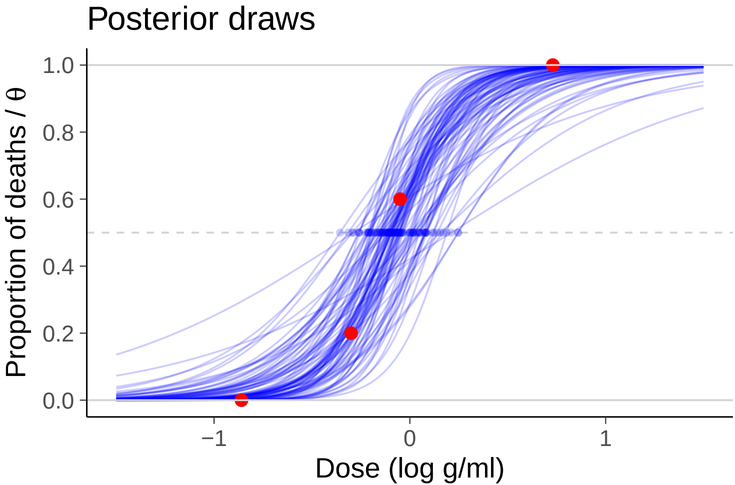
Logistic regression fit
Proportion of deaths / θ
1.0
0.8
0.6
0.4
0.2
0.0
−1
0
1
Dose (log g/ml)

Bioassay



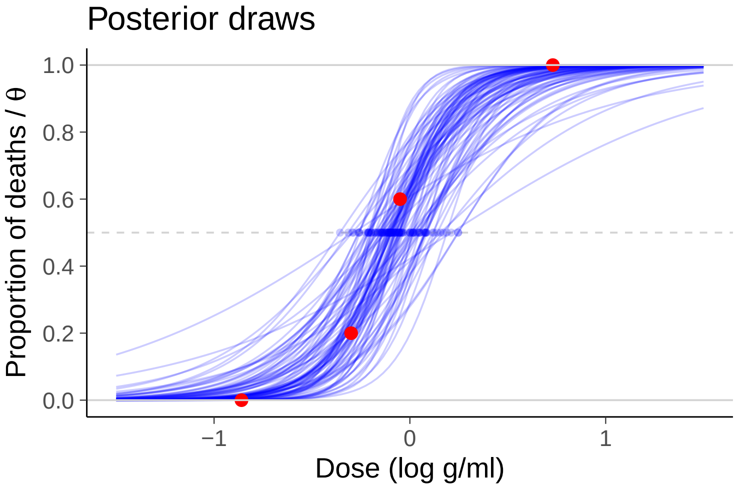
Posterior draws
Proportion of deaths / θ
1.0
0.8
0.6
0.4
0.2
0.0
−1
0
1
Dose (log g/ml)

Bioassay



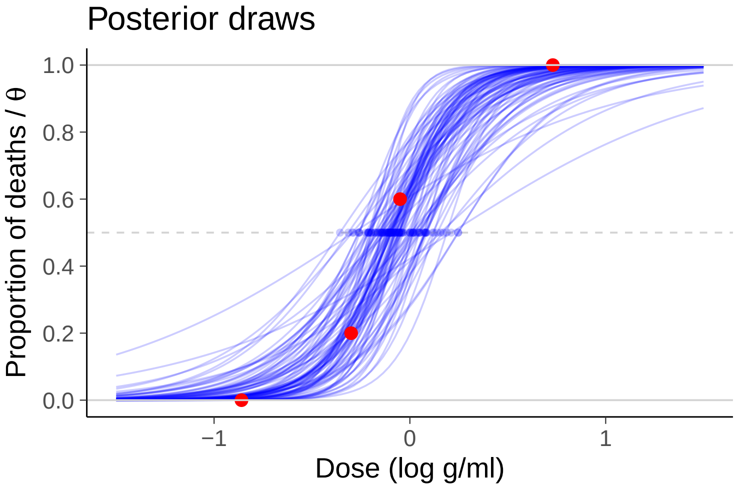
LD50: $\quad \mathrm{E}\left(\frac{y}{n}\right) = \mathrm{logit}^{-1}(\alpha + \beta x) = 0.5$

Bioassay



LD50: $\quad \mathrm{E}\left(\frac{y}{n}\right) = \mathrm{logit}^{-1}(\alpha + \beta x) = 0.5 \quad \Rightarrow \quad x_{\mathrm{LD50}} = -\alpha/\beta$

Bioassay



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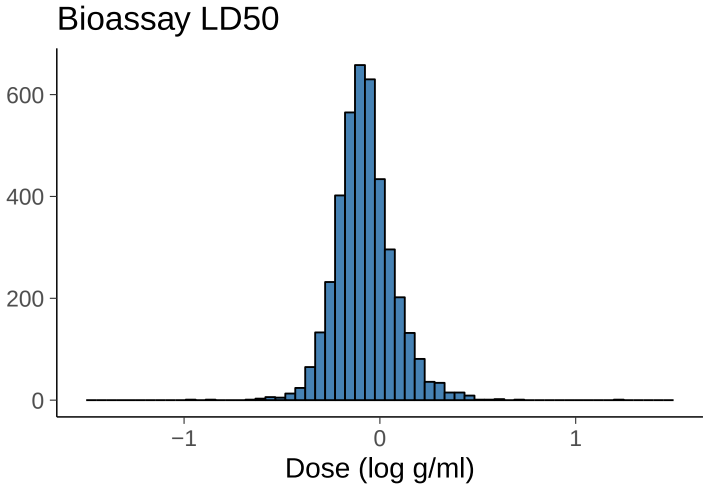
$$x^{(s)}_{\mathrm{LD50}} = -\alpha^{(s)}/\beta^{(s)}$$

Bioassay



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Bioassay posterior

Binomial model

$$y_i \mid \theta_i \sim \mathrm{Bin}(\theta_i, n_i)$$

Link function

$$\mathrm{logit}(\theta_i) = \alpha + \beta x_i$$

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Likelihood

$$p(y_i \mid \alpha, \beta, n_i, x_i) \propto \theta_i^{y_i}[1 - \theta_i]^{n_i - y_i}$$

Bioassay posterior

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$$p(y_i \mid \alpha, \beta, n_i, x_i) \propto [\mathrm{logit}^{-1}(\alpha + \beta x_i)]^{y_i} [1 - \mathrm{logit}^{-1}(\alpha + \beta x_i)]^{n_i - y_i}$$

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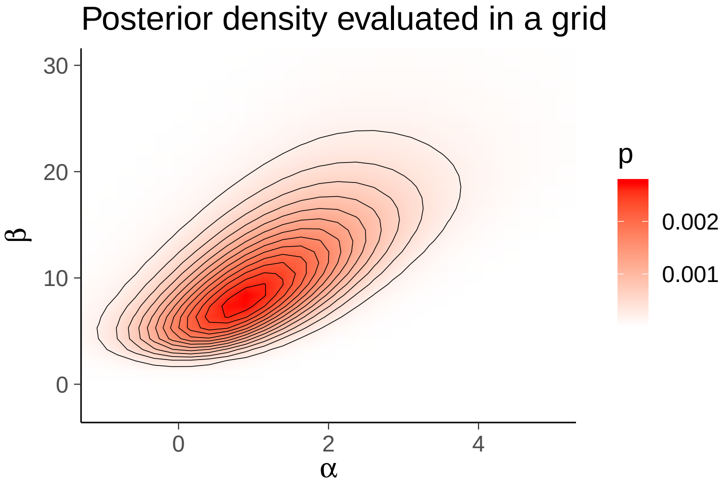
Posterior (with uniform prior on α, β)

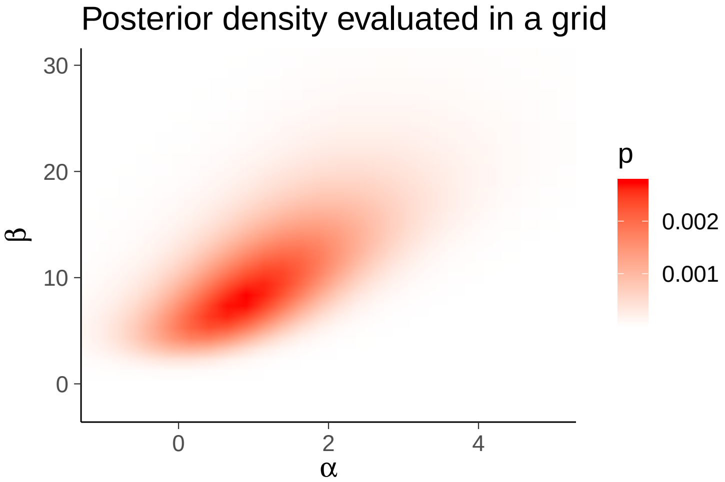
$$p(\alpha, \beta \mid y, n, x) \propto p(\alpha, \beta) \prod_{i=1}^{n} p(y_i \mid \alpha, \beta, n_i, x_i)$$

Bioassay



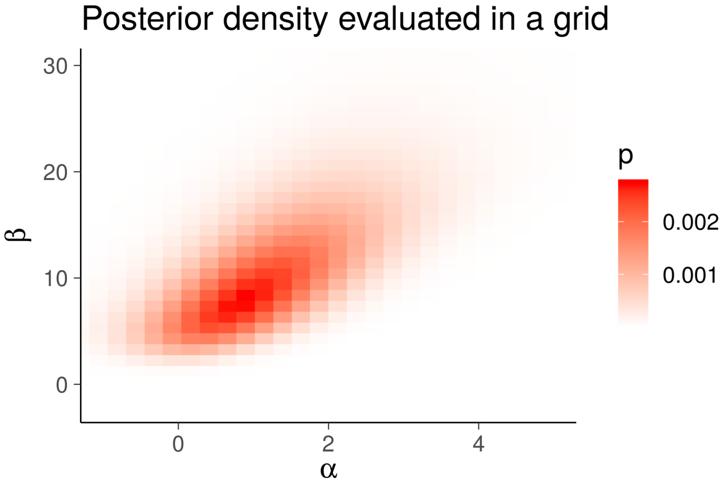
Posterior density evaluated in a grid
30
20
10
0
β
0
2
4
α
p
0.002
0.001

Bioassay



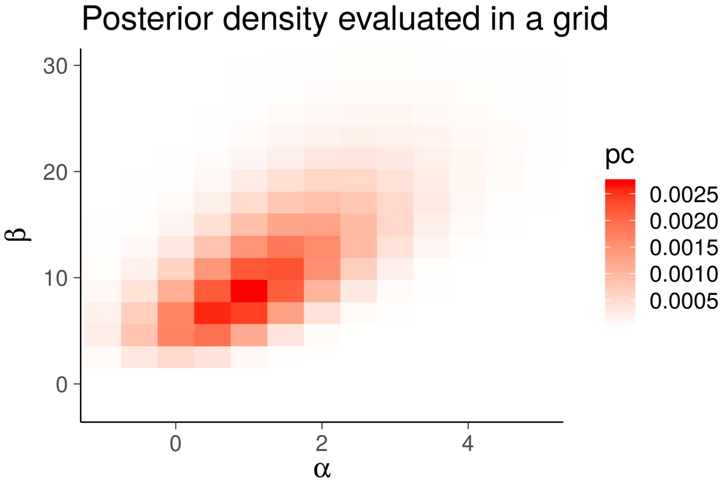
Density evaluated in grid, but plotted using interpolation

Bioassay



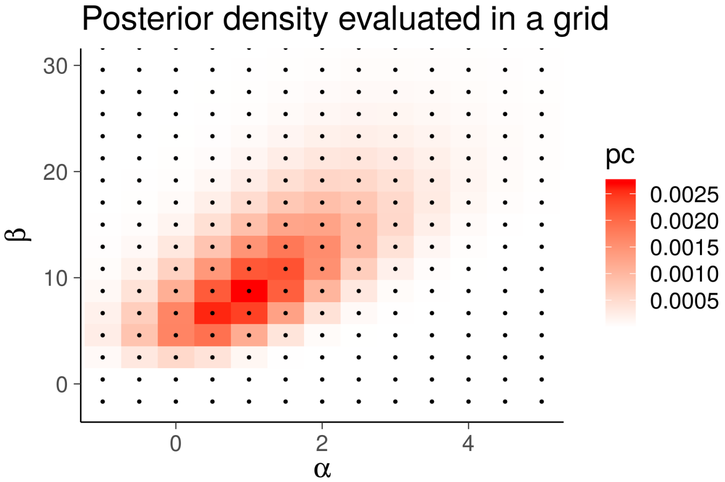
Density evaluated in grid, and plotted without interpolation

Bioassay



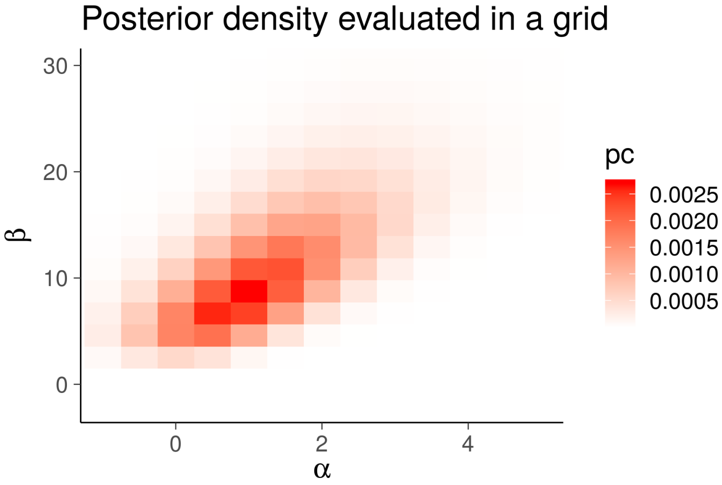
Density evaluated in a coarser grid

Bioassay



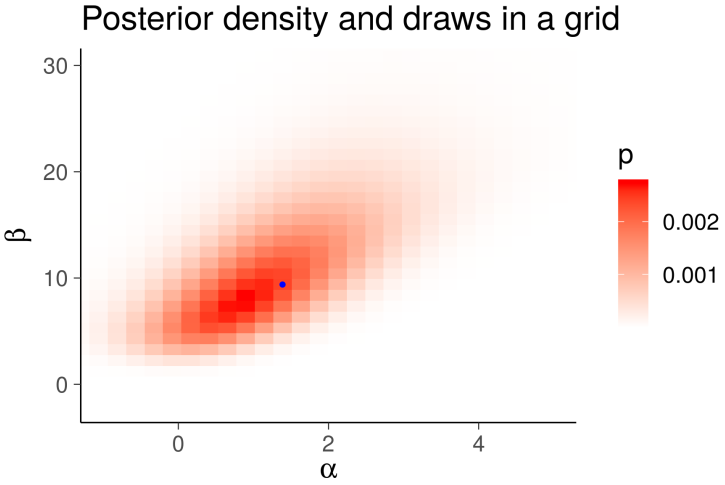
- Approximate the density as piecewise constant function
- Evaluate density in a grid over some finite region
- Density times cell area gives probability mass in each cell

Bioassay



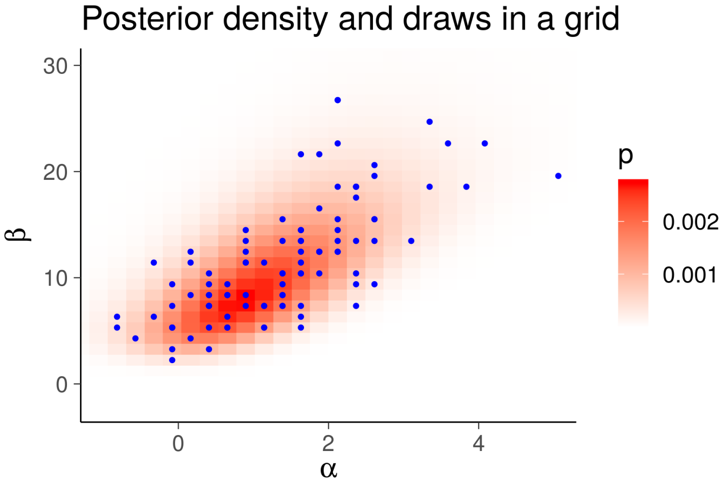
- Densities at 1, 2, and 3: 0.0027 0.0010 0.0001
- Probabilities of cells 1, 2, and 3: 0.0431 0.0166 0.0010
- Probabilities of cells sum to 1

Bioassay



- Sample according to grid cell probabilities

Bioassay



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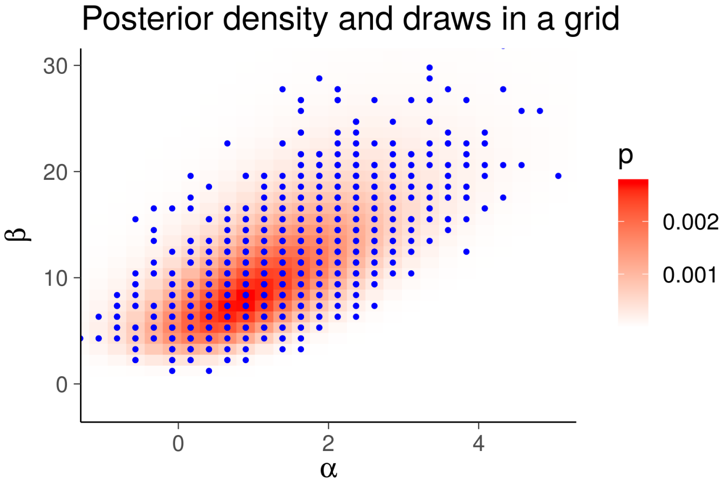
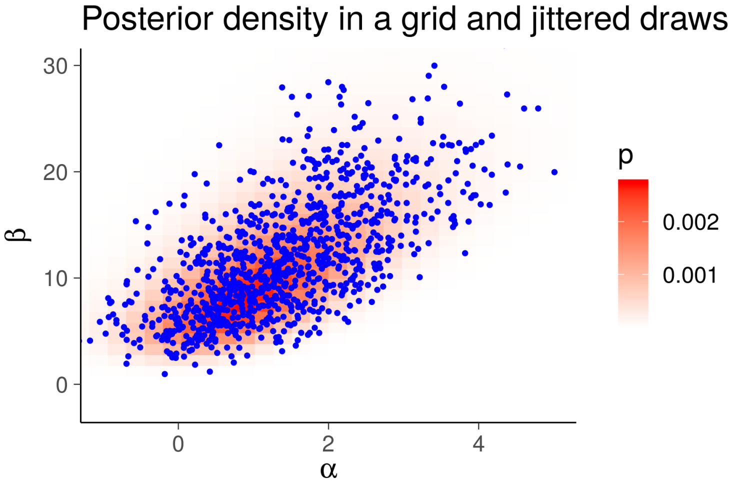


- Sample according to grid cell probabilities
- Several draws can be from the same grid cell

Bioassay



- Jitter can be added to improve visualization

Grid sampling

- Draws can be used to estimate expectations, for example

$$E[x_{\mathrm{LD50}}] = E[-\alpha/\beta] \approx \frac{1}{S}\sum_{s=1}^{S}\frac{\alpha^{(s)}}{\beta^{(s)}}$$

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- Instead of sampling, grid could be used to evaluate functions directly, for example

$$\mathrm{E}[-\alpha/\beta] \approx \sum_{t=1}^{T} w_{\mathrm{cell}}^{(t)}\frac{\alpha^{(t)}}{\beta^{(t)}},$$

where $w_{\mathrm{cell}}^{(t)}$ is the normalized probability of a grid cell t, and $\alpha^{(t)}$ and $\beta^{(t)}$ are center locations of grid cells

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- Grid sampling gets computationally too expensive in high dimensions