

Name _____ Student number _____

Assignment 3

Derive the component form of $\nabla \cdot \vec{\sigma} + \vec{f} = 0$ in the polar coordinate system. Assume that the components of stress do not depend on angle ϕ . In the polar coordinate system, the component forms of stress, external force, and gradient operator, and derivatives of the basis vectors are given by

$$\vec{\sigma} = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^T \begin{bmatrix} \sigma_{rr} & \sigma_{r\phi} \\ \sigma_{\phi r} & \sigma_{\phi\phi} \end{bmatrix} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}, \quad \vec{f} = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^T \begin{Bmatrix} f_r \\ f_\phi \end{Bmatrix}, \quad \nabla = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^T \begin{Bmatrix} \partial / \partial r \\ \partial / (r \partial \phi) \end{Bmatrix},$$

$$\frac{\partial}{\partial \phi} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} = \begin{Bmatrix} \vec{e}_\phi \\ -\vec{e}_r \end{Bmatrix}, \quad \frac{\partial}{\partial r} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} = 0.$$

Solution template

In manipulation of vector expression containing vectors and tensors, it is important to remember that tensor (\otimes), cross (\times), inner (\cdot) products are non-commutative (order may matter). The basis vectors of a curvilinear coordinate system are not constants which should be taken into account if gradient operator is a part of expression. Otherwise, simplifying an expression or finding a specific form in a given coordinate system is a straightforward (sometimes tedious) exercise. For simplicity of presentation, outer (tensor) products like $\vec{a} \otimes \vec{b}$ are denoted by $\vec{a}\vec{b}$. Otherwise, the usual rules of algebra apply: Gradient operator ∇ acts on everything on its right-hand side, the operator is treated like a vector etc.

The task is to simplify the vector equation

$$\nabla \cdot \vec{\sigma} + \vec{f} = (\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{\partial}{r \partial \phi}) \cdot (\sigma_{rr} \vec{e}_r \vec{e}_r + \sigma_{r\phi} \vec{e}_r \vec{e}_\phi + \sigma_{\phi r} \vec{e}_\phi \vec{e}_r + \sigma_{\phi\phi} \vec{e}_\phi \vec{e}_\phi) + (f_r \vec{e}_r + f_\phi \vec{e}_\phi) = 0$$

to see the component forms. Let us consider the effect of the first term of the displacement gradient to stress

$$\vec{e}_r \frac{\partial}{\partial r} \cdot (\sigma_{rr} \vec{e}_r \vec{e}_r + \sigma_{r\phi} \vec{e}_r \vec{e}_\phi + \sigma_{\phi r} \vec{e}_\phi \vec{e}_r + \sigma_{\phi\phi} \vec{e}_\phi \vec{e}_\phi) =$$

$$\vec{e}_r \cdot \left(\frac{\partial \sigma_{rr}}{\partial r} \vec{e}_r \vec{e}_r + \frac{\partial \sigma_{r\phi}}{\partial r} \vec{e}_r \vec{e}_\phi + \frac{\partial \sigma_{\phi r}}{\partial r} \vec{e}_\phi \vec{e}_r + \frac{\partial \sigma_{\phi\phi}}{\partial r} \vec{e}_\phi \vec{e}_\phi \right) =$$

$$\vec{e}_r \left(\frac{\partial \sigma_{rr}}{\partial r} \right) + \vec{e}_\phi \left(\frac{\partial \sigma_{r\phi}}{\partial r} \right).$$

Then the same for the second term of the displacement gradient. As the stress components do not depend on ϕ (by assumption)

$$\vec{e}_\phi \frac{1}{r} \frac{\partial}{\partial \phi} \cdot (\sigma_{rr} \vec{e}_r \vec{e}_r + \sigma_{r\phi} \vec{e}_r \vec{e}_\phi + \sigma_{\phi r} \vec{e}_\phi \vec{e}_r + \sigma_{\phi\phi} \vec{e}_\phi \vec{e}_\phi) =$$

$$\vec{e}_\phi \frac{1}{r} \cdot (\sigma_{rr} \vec{e}_\phi \vec{e}_r + \sigma_{rr} \vec{e}_r \vec{e}_\phi + \sigma_{r\phi} \vec{e}_\phi \vec{e}_\phi - \sigma_{r\phi} \vec{e}_r \vec{e}_r - \sigma_{\phi r} \vec{e}_r \vec{e}_r + \sigma_{\phi r} \vec{e}_\phi \vec{e}_\phi - \sigma_{\phi\phi} \vec{e}_r \vec{e}_\phi - \sigma_{\phi\phi} \vec{e}_\phi \vec{e}_r) =$$

$$\vec{e}_r \left(\frac{1}{r} \sigma_{rr} - \frac{1}{r} \sigma_{\phi\phi} \right) + \vec{e}_\phi \left(\frac{1}{r} \sigma_{r\phi} + \frac{1}{r} \sigma_{\phi r} \right).$$

Finally combining everything

$$\nabla \cdot \vec{\sigma} + \vec{f} = \vec{e}_r \left(\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\phi\phi}}{r} + f_r \right) + \vec{e}_\phi \left(\frac{\partial \sigma_{r\phi}}{\partial r} + \frac{\sigma_{r\phi} + \sigma_{\phi r}}{r} + f_\phi \right) = 0$$

Therefore, the two equilibrium equations are given by

$$\left\{ \begin{array}{l} \frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\phi\phi}}{r} + f_r \\ \frac{\partial \sigma_{r\phi}}{\partial r} + \frac{\sigma_{r\phi} + \sigma_{\phi r}}{r} + f_\phi \end{array} \right\} = 0. \quad \leftarrow$$