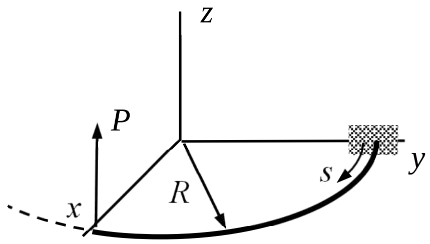
MEC-E8003 Beam, Plate and Shell models, onsite exam 16.04.2024

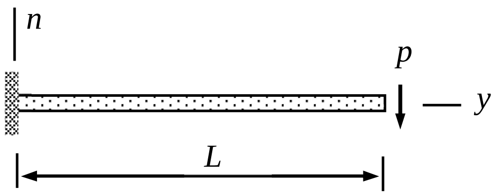
1. Consider the mid-surface mapping $\vec{r}_0(r,\phi) = r^2\cos(2\phi)\vec{i} + r^2\sin(2\phi)\vec{j}$ of shell. Compute the expression of the basis vector derivatives and gradient operator ∇. Is the mid-surface defined by the mapping flat or curved?

2. Derive the component forms of the thin slab equilibrium equation $\nabla\cdot\vec{N} + \vec{b} = 0$ in the polar coordinate system.

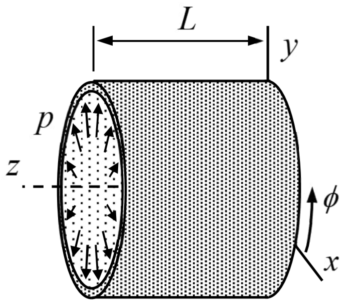
3. Consider the curved beam of the figure forming a 90-degree circular segment of radius R in the horizontal plane. Find the stress resultants $N(s)$, $Q_n(s)$, $Q_b(s)$, $T(s)$, $M_n(s)$, and $M_b(s)$. Use the equilibrium equations of the beam model in the (s,n,b) – coordinate system.



4. Consider the bending of a cantilever plate strip which is loaded by distributed force p [N / m] acting on the free edge. Write down the equilibrium equations, constitutive equations, and boundary conditions for the bending mode according to the Kirchhoff model. After that, solve the equations for the stress resultant, displacement, and rotation components. Thickness and length of the plate are t and L, respectively. Young's modulus E and Poisson's ratio ν are constants. Consider a plate of width H but assume that stress resultants, displacements, and rotations depend on y only.



5. A steel ring of length L, radius R, and thickness t is loaded by radial surface force p acting on the inner surface. No forces are acting on the ends. Model the ring as a cylindrical membrane, write down the equilibrium and constitutive equations, and solve for the radial displacement. Assume rotation symmetry. Young's modulus E and Poisson's ratio ν of the material are constants.



Consider the mid-surface mapping $\vec{r}_0(r,\phi)=r^2\cos(2\phi)\vec{i}+r^2\sin(2\phi)\vec{j}$ of shell. Compute the expression of the basis vector derivatives and gradient operator ∇. Is the mid-surface defined by the mapping flat or curved?

Solution

The relationship between the basis vectors of the Cartesian and cylindrical systems follows from definition. Notice that two of the basis vectors follow from the mid-surface mapping. The third is always normal to the surface and obtained as a vector product (see the coordinate system for shells). With the present mid-surface $(r,\phi)-$coordinates

$$\vec{r}_0(r,\phi)=r^2\cos(2\phi)\vec{i}+r^2\sin(2\phi)\vec{j} \text{ and } \begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_n\end{Bmatrix}=\begin{Bmatrix}(\partial\vec{r}_0/\partial r)/|\partial\vec{r}_0/\partial r|\\ (\partial\vec{r}_0/\partial\phi)/|\partial\vec{r}_0/\partial\phi|\\ \vec{e}_r\times\vec{e}_\phi\end{Bmatrix}=[F]\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}.$$

2p Expressions of the basis vectors of the curvilinear system are

$$\frac{\partial}{\partial r}\vec{r}_0=2r\cos(2\phi)\vec{i}+2r\sin(2\phi)\vec{j}\quad\Rightarrow\quad\vec{e}_r=(\frac{\partial}{\partial r}\vec{r}_0)/|\frac{\partial}{\partial r}\vec{r}_0|=\cos(2\phi)\vec{i}+\sin(2\phi)\vec{j},$$

$$\frac{\partial}{\partial\phi}\vec{r}_0=-2r^2\sin(2\phi)\vec{i}+2r^2\cos(2\phi)\vec{j}\quad\Rightarrow\quad\vec{e}_\phi=(\frac{\partial}{\partial\phi}\vec{r}_0)/|\frac{\partial}{\partial\phi}\vec{r}_0|=-\sin(2\phi)\vec{i}+\cos(2\phi)\vec{j},$$

$$\vec{e}_n=\vec{e}_r\times\vec{e}_\phi=[\cos(2\phi)\vec{i}+\sin(2\phi)\vec{j}]\times[-\sin(2\phi)\vec{i}+\cos(2\phi)\vec{j}]=\vec{k}.$$

In a more compact form

$$\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_n\end{Bmatrix}=\begin{bmatrix}\cos(2\phi)&\sin(2\phi)&0\\ -\sin(2\phi)&\cos(2\phi)&0\\ 0&0&1\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}=[F]\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}\quad\text{in which }[F]^{-1}=[F]^{\mathrm{T}}.$$

1p Direct use of the definition gives (just take the derivatives on both sides of the relationship above and use inverse of the same relationship to replace the basis vectors of the Cartesian system by the basis vectors of the $(r,\phi,n)-$system)

$$\frac{\partial}{\partial r}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_n\end{Bmatrix}=\begin{bmatrix}0&0&0\\ 0&0&0\\ 0&0&0\end{bmatrix}\begin{bmatrix}\cos(2\phi)&-\sin(2\phi)&0\\ \sin(2\phi)&\cos(2\phi)&0\\ 0&0&1\end{bmatrix}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_n\end{Bmatrix}=\begin{Bmatrix}0\\ 0\\ 0\end{Bmatrix},$$

$$\frac{\partial}{\partial\phi}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_n\end{Bmatrix}=\begin{bmatrix}-2\sin(2\phi)&2\cos(2\phi)&0\\ -2\cos(2\phi)&-2\sin(2\phi)&0\\ 0&0&0\end{bmatrix}\begin{bmatrix}\cos(2\phi)&-\sin(2\phi)&0\\ \sin(2\phi)&\cos(2\phi)&0\\ 0&0&1\end{bmatrix}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_n\end{Bmatrix}=\begin{bmatrix}0&2&0\\ -2&0&0\\ 0&0&0\end{bmatrix}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_n\end{Bmatrix},$$

$$\frac{\partial}{\partial n}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_n\end{Bmatrix}=\begin{bmatrix}0&0&0\\ 0&0&0\\ 0&0&0\end{bmatrix}\begin{bmatrix}\cos(2\phi)&-\sin(2\phi)&0\\ \sin(2\phi)&\cos(2\phi)&0\\ 0&0&1\end{bmatrix}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_n\end{Bmatrix}=\begin{Bmatrix}0\\ 0\\ 0\end{Bmatrix}.$$

2p Gradient in the $(r,\phi,n)-$ system follows from the mapping

$$\vec{r}(r,\phi,n)=\vec{r}_0+\vec{\rho}=r^2\cos(2\phi)\vec{i}+r^2\sin(2\phi)\vec{j}+n\vec{k}$$

and the generic formula in terms of $[F]$ and $[H]$ with

$$\begin{Bmatrix}\partial\vec{r}/\partial r\\ \partial\vec{r}/\partial\phi\\ \partial\vec{r}/\partial n\end{Bmatrix}=\begin{bmatrix}2r\cos(2\phi) & 2r\sin(2\phi) & 0\\ -2r^2\sin(2\phi) & 2r^2\cos(2\phi) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}=[H]\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}\quad\Rightarrow$$

$$[H][F]^{\mathrm{T}}=\begin{bmatrix}2r\cos(2\phi) & 2r\sin(2\phi) & 0\\ -2r^2\sin(2\phi) & 2r^2\cos(2\phi) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos(2\phi) & -\sin(2\phi) & 0\\ \sin(2\phi) & \cos(2\phi) & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}2r & 0 & 0\\ 0 & 2r^2 & 0\\ 0 & 0 & 1\end{bmatrix}\Rightarrow$$

$$\nabla=\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_n\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}2r & 0 & 0\\ 0 & 2r^2 & 0\\ 0 & 0 & 1\end{bmatrix}^{-1}\begin{Bmatrix}\partial/\partial r\\ \partial/\partial\phi\\ \partial/\partial n\end{Bmatrix}=\vec{e}_r\frac{1}{2r}\frac{\partial}{\partial r}+\vec{e}_\phi\frac{1}{2r^2}\frac{\partial}{\partial\phi}+\vec{e}_n\frac{\partial}{\partial n}\;.$$

1p Curvature of the mid-surface ($n=0$)

$$\vec{\kappa}_{\mathrm{c}}=\nabla\vec{e}_n=\vec{e}_r\frac{1}{2r}\frac{\partial\vec{e}_n}{\partial r}+\vec{e}_\phi\frac{1}{2r^2}\frac{\partial\vec{e}_n}{\partial\phi}+\vec{e}_n\frac{\partial\vec{e}_n}{\partial n}=0$$

which indicates that the mid-surface is flat.

Derive the component forms of the thin slab equilibrium equation $\nabla \cdot \vec{N} + \vec{b} = 0$ in the polar coordinate system.

Solution

1p The component forms of stress, external force, and gradient operator of the polar coordinate system are

$$\vec{N} = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} N_{rr} & N_{r\phi} \\ N_{\phi r} & N_{\phi\phi} \end{bmatrix} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}, \; \vec{b} = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} b_r \\ b_\phi \end{Bmatrix}, \; \nabla = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \partial / \partial r \\ \partial / (r\partial\phi) \end{Bmatrix}, \; \frac{\partial}{\partial\phi} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} = \begin{Bmatrix} \vec{e}_\phi \\ -\vec{e}_r \end{Bmatrix}.$$

5p Let us start with the terms of stress resultant divergence

$$\nabla \cdot \vec{N} = (\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{\partial}{r\partial\phi}) \cdot (N_{rr}\vec{e}_r\vec{e}_r + N_{r\phi}\vec{e}_r\vec{e}_\phi + N_{\phi r}\vec{e}_\phi\vec{e}_r + N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi).$$

First term of the gradient simplifies to

$$\vec{e}_r \frac{\partial}{\partial r} \cdot (N_{rr}\vec{e}_r\vec{e}_r + N_{r\phi}\vec{e}_r\vec{e}_\phi + N_{\phi r}\vec{e}_\phi\vec{e}_r + N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi) \quad \Rightarrow$$

$$\vec{e}_r \cdot (\frac{\partial N_{rr}}{\partial r}\vec{e}_r\vec{e}_r + \frac{\partial N_{r\phi}}{\partial r}\vec{e}_r\vec{e}_\phi + \frac{\partial N_{\phi r}}{\partial r}\vec{e}_\phi\vec{e}_r + \frac{\partial N_{\phi\phi}}{\partial r}\vec{e}_\phi\vec{e}_\phi) \quad \Rightarrow$$

$$\vec{e}_r \frac{\partial}{\partial r} \cdot (N_{rr}\vec{e}_r\vec{e}_r + N_{r\phi}\vec{e}_r\vec{e}_\phi + N_{\phi r}\vec{e}_\phi\vec{e}_r + N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi) = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \dfrac{\partial N_{rr}}{\partial r} \\ \dfrac{\partial N_{r\phi}}{\partial r} \end{Bmatrix}.$$

Then the same manipulation for the second term of the displacement gradient

$$\vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial\phi} \cdot (N_{rr}\vec{e}_r\vec{e}_r + N_{r\phi}\vec{e}_r\vec{e}_\phi + N_{\phi r}\vec{e}_\phi\vec{e}_r + N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi) \quad \Rightarrow$$

$$\vec{e}_\phi \frac{1}{r} \cdot (\frac{\partial N_{rr}}{\partial\phi}\vec{e}_r\vec{e}_r + N_{rr}\vec{e}_\phi\vec{e}_r + N_{rr}\vec{e}_r\vec{e}_\phi + \frac{\partial N_{r\phi}}{\partial\phi}\vec{e}_r\vec{e}_\phi + N_{r\phi}\vec{e}_\phi\vec{e}_\phi - N_{r\phi}\vec{e}_r\vec{e}_r +$$

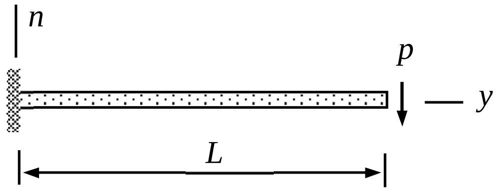
$$\frac{\partial N_{\phi r}}{\partial\phi}\vec{e}_\phi\vec{e}_r - N_{\phi r}\vec{e}_r\vec{e}_r + N_{\phi r}\vec{e}_\phi\vec{e}_\phi + \frac{\partial N_{\phi\phi}}{\partial\phi}\vec{e}_\phi\vec{e}_\phi - N_{\phi\phi}\vec{e}_r\vec{e}_\phi - N_{\phi\phi}\vec{e}_\phi\vec{e}_r) \quad \Rightarrow$$

$$\vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial\phi} \cdot (N_{rr}\vec{e}_r\vec{e}_r + N_{r\phi}\vec{e}_r\vec{e}_\phi + N_{\phi r}\vec{e}_\phi\vec{e}_r + N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi) = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \dfrac{1}{r}(N_{rr} - N_{\phi\phi} + \dfrac{\partial N_{\phi r}}{\partial\phi}) \\ \dfrac{1}{r}(N_{r\phi} + N_{\phi r} + \dfrac{\partial N_{\phi\phi}}{\partial\phi}) \end{Bmatrix}.$$

Finally, by combining the terms of the divergence and external loading

$$\nabla \cdot \vec{\vec{N}} + \vec{b} = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \dfrac{\partial N_{rr}}{\partial r} + \dfrac{1}{r}(N_{rr} - N_{\phi\phi} + \dfrac{\partial N_{\phi r}}{\partial \phi}) + b_r \\ \dfrac{\partial N_{r\phi}}{\partial r} + \dfrac{1}{r}(N_{r\phi} + N_{\phi r} + \dfrac{\partial N_{\phi\phi}}{\partial \phi}) + b_\phi \end{Bmatrix} = 0 \, .$$

Consider the bending of a cantilever plate strip which is loaded by distributed force p $[\mathrm{N/m}]$ acting on the free edge. Write down the equilibrium equations, constitutive equations, and boundary conditions for the bending mode according to the Kirchhoff model. After that, solve the equations for the stress resultant, displacement, and rotation components. Thickness and length of the plate are t and L, respectively. Young's modulus E and Poisson's ratio ν are constants. Consider a plate of width H but assume that stress resultants, displacements, and rotations depend on y only.



Solution
The starting point is the full set of Reissner-plate bending mode equations in the Cartesian (x,y,n)–coordinate system.

$$\begin{Bmatrix} \dfrac{\partial Q_x}{\partial x}+\dfrac{\partial Q_y}{\partial y}+b_n \\ \dfrac{\partial M_{xx}}{\partial x}+\dfrac{\partial M_{xy}}{\partial y}-Q_x \\ \dfrac{\partial M_{yy}}{\partial y}+\dfrac{\partial M_{xy}}{\partial x}-Q_y \end{Bmatrix}=0\,,\quad \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix}=D\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}\begin{Bmatrix} \dfrac{\partial \theta}{\partial x} \\ -\dfrac{\partial \phi}{\partial y} \\ \dfrac{\partial \theta}{\partial y}-\dfrac{\partial \phi}{\partial x} \end{Bmatrix},\quad \begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix}=Gt\begin{Bmatrix} \dfrac{\partial w}{\partial x}+\theta \\ \dfrac{\partial w}{\partial y}-\phi \end{Bmatrix}.$$

3p If all derivatives with respect to x vanish, the plate equations of the Cartesian (x,y,n)– coordinate according to the Kirchhoff model (Kirchhoff constraint replaces the constitutive equation for the shear stress resultant) system simplify to

$$\frac{dQ_y}{dy}=0\,,\quad \frac{dM_{yy}}{dy}-Q_y=0\,,\quad M_{yy}=-D\frac{d\phi}{dy}\,,\quad \text{and}\quad \frac{dw}{dy}-\phi=0 \quad \text{in} \quad (0,L)\,.$$

The boundary conditions are

$$w(0)=0\,,\quad \phi(0)=0\,,\quad M_{yy}(L)=0\,,\quad Q_y(L)=-p\,.$$

3p As the stress resultant are known at the free end, the equilibrium equations can be solved first for the stress resultants. The boundary value problems for the stress resultants give

$$\frac{dQ_y}{dy}=0 \quad y\in(0,L) \quad \text{and} \quad Q_y(L)=-p \quad \Rightarrow \quad Q_y(y)=-p\,,$$

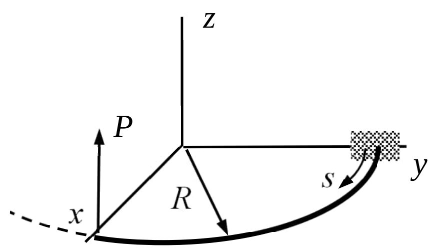
$$\frac{dM_{yy}}{dy}=Q_y=-p \quad y\in(0,L) \quad \text{and} \quad M_{yy}(L)=0 \quad \Rightarrow \quad M_{yy}(y)=-p(y-L)\,.$$

After that, displacement and rotation follow from the constitutive equation, Kirchhoff constraint, and boundary conditions at the clamped edge

$$\frac{d\phi}{dy} = -\frac{M_{yy}}{D} = \frac{p}{D}(y-L) \quad y \in (0,L) \quad \text{and} \quad \phi(0)=0 \quad \Rightarrow \quad \phi = \frac{p}{D}(\frac{1}{2}y^2 - Ly),$$

$$\frac{dw}{dy} = \phi = \frac{p}{D}(\frac{1}{2}y^2 - Ly) \quad y \in (0,L) \quad \text{and} \quad w(0)=0 \quad \Rightarrow \quad w(y) = \frac{p}{D}(\frac{1}{6}y^3 - L\frac{1}{2}y^2).$$

Consider the curved beam of the figure forming a 90-degree circular segment of radius R in the horizontal plane. Find the stress resultants $N(s)$, $Q_n(s)$, $Q_b(s)$, $T(s)$, $M_n(s)$, and $M_b(s)$. Use the equilibrium equations of the beam model in the $(s,n,b)-$ coordinate system.



Solution

In a statically determined case, stress resultants follow from the equilibrium equations and boundary conditions at the free end of the beam (or directly from a free body diagram). In (s,n,b) coordinate system, equilibrium equations are

$$\begin{Bmatrix} N' - Q_n\kappa + b_s \\ Q_n' + N\kappa - Q_b\tau + b_n \\ Q_b' + Q_n\tau + b_b \end{Bmatrix} = 0 \quad \text{and} \quad \begin{Bmatrix} T' - M_n\kappa + c_s \\ M_n' + T\kappa - M_b\tau - Q_b + c_n \\ M_b' + M_n\tau + Q_n + c_b \end{Bmatrix} = 0 .$$

2p For a circular beam, curvature and torsion are $\kappa = 1/R$ (constant) and $\tau = 0$. As external distributed forces and moments vanish i.e. $b_s = b_n = b_b = c_s = c_n = c_b = 0$, equilibrium equations and the boundary conditions at the free end simplify to (notice that the external force acting at the free end is acting in the oppisite direction to $\vec{e}_b$)

$$\begin{Bmatrix} N' - Q_n / R \\ Q_n' + N / R \\ Q_b' \end{Bmatrix} = 0 \quad \text{and} \quad \begin{Bmatrix} T' - M_n / R \\ M_n' + T / R - Q_b \\ M_b' + Q_n \end{Bmatrix} = 0 \quad s \in]0, R\frac{\pi}{2}[,$$

$$\begin{Bmatrix} N \\ Q_n \\ Q_b + P \end{Bmatrix} = 0 \quad \text{and} \quad \begin{Bmatrix} T \\ M_n \\ M_b \end{Bmatrix} = 0 \quad s = R\frac{\pi}{2} .$$

4p Equations constitute a boundary value problem which can be solved by hand calculations without too much effort;

$$Q_b' = 0 \quad s \in]0, R\frac{\pi}{2}[\quad \text{and} \quad Q_b + P = 0 \quad s = R\frac{\pi}{2} \quad \Rightarrow \quad Q_b(s) = -P . \quad \Leftarrow$$

Eliminating Q_n and N from the remaining two connected force equilibrium equations and using the original equations to find the missing boundary condition give

$$N'' + \frac{1}{R^2} N = 0 \quad s \in]0, R\frac{\pi}{2}[\quad \text{and} \quad N' = N = 0 \quad s = R\frac{\pi}{2} \quad \Rightarrow \quad N(s) = 0 \quad \Leftarrow$$

The first equilibrium equation gives

$$Q_n(s) = 0 . \quad \Leftarrow$$

After that, continuing with the moment equilibrium equations with the solutions to the force equilibrium equations

$$M_b' = 0 \quad s \in]0, R\frac{\pi}{2}[\quad \text{and} \quad M_b = 0 \quad s = R\frac{\pi}{2} \quad \Rightarrow \quad M_b(s) = 0 .$$

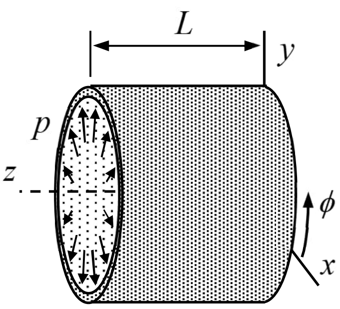
Eliminating M_n and T from the remaining two connected moment equilibrium equations and using the original equations to find the missing boundary condition gives

$$T'' + \frac{1}{R^2}T + \frac{P}{R} = 0 \quad s \in]0, R\frac{\pi}{2}[\quad \text{and} \quad T' = T = 0 \quad s = R\frac{\pi}{2} \quad \Rightarrow \quad T = PR(\sin\frac{s}{R} - 1) .$$

Knowing this, the first moment equilibrium equation gives

$$M_n(s) = RT' = RP\cos\frac{s}{R} .$$

A steel ring of length L, radius R, and thickness t is loaded by radial surface force p acting on the inner surface. No forces are acting on the ends. Model the ring as a cylindrical membrane, write down the equilibrium and constitutive equations, and solve for the radial displacement. Assume rotation symmetry and $u_\phi = 0$. Young's modulus E and Poisson's ratio g of the material are constants.



Solution

According to the formulae collection, equilibrium and constitutive equations of a cylindrical membrane in (z,ϕ,n) coordinates are (notice that $\vec{e}_n$ is directed inwards)

$$\begin{Bmatrix} \dfrac{1}{R}\dfrac{\partial N_{z\phi}}{\partial \phi}+\dfrac{\partial N_{zz}}{\partial z}+b_z \\ \dfrac{\partial N_{z\phi}}{\partial z}+\dfrac{1}{R}\dfrac{\partial N_{\phi\phi}}{\partial \phi}+b_\phi \\ \dfrac{1}{R}N_{\phi\phi}+b_n \end{Bmatrix}=0, \quad \begin{Bmatrix} N_{zz} \\ N_{\phi\phi} \\ N_{z\phi} \end{Bmatrix}=\frac{tE}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}\begin{Bmatrix} \dfrac{\partial u_z}{\partial z} \\ \dfrac{1}{R}(\dfrac{\partial u_\phi}{\partial \phi}-u_n) \\ \dfrac{1}{R}\dfrac{\partial u_z}{\partial \phi}+\dfrac{\partial u_\phi}{\partial z} \end{Bmatrix}.$$

3p Due to the rotation symmetry, the derivatives with respect to the angular coordinate vanish and $u_\phi = 0$. External distributed force $b_n = -p$ is due to the traction acting on the inner boundary. Therefore, the equilibrium equations and constitutive equations simplify to a set of ordinary differential equations

$$\frac{dN_{zz}}{dz}=0, \quad \frac{dN_{z\phi}}{dz}=0, \quad \frac{1}{R}N_{\phi\phi}-p=0 \quad \text{in} \quad (0,L),$$

$$N_{zz}=\frac{tE}{1-\nu^2}\frac{1}{R}(R\frac{du_z}{dz}-\nu u_n), \quad N_{\phi\phi}=\frac{tE}{1-\nu^2}\frac{1}{R}(R\nu\frac{du_z}{dz}-u_n), \quad N_{z\phi}=0 \quad \text{in} \quad (0,L),$$

As the edges are stress-free i.e.

$$N_{zz}=0 \text{ and } N_{z\phi}=0 \text{ on } \{0,L\}.$$

3p Solution to the stress resultants, as obtained from the equilibrium equations, are

$$N_{zz}=0, \; N_{z\phi}=0, \text{ and } N_{\phi\phi}=Rp.$$

Constitutive equations give

$$N_{zz}=\frac{tE}{1-\nu^2}\frac{1}{R}(R\frac{du_z}{dz}-\nu u_n)=0 \quad \Rightarrow \quad \frac{du_z}{dz}=\frac{\nu}{R}u_n \quad \text{and}$$

$$Rp=N_{\phi\phi}=\frac{tE}{1-\nu^2}\frac{1}{R}(R\nu\frac{du_z}{dz}-u_n)=\frac{tE}{1-\nu^2}\frac{1}{R}(\nu^2-1)u_n=-\frac{tE}{R}u_n \quad \Leftrightarrow \quad u_n=-\frac{pR^2}{tE}.$$