Name________________________ Student number________________

Assignment 5 (4p)

Derive the component forms of the membrane equilibrium equations $\nabla\cdot\vec{\vec{N}}+\vec{b}=0$ in the cylindrical shell (z,ϕ,n)-coordinate system. Use the stress resultant, external force, and gradient representations

$$\vec{\vec{N}}=\begin{Bmatrix}\vec{e}_z\\ \vec{e}_\phi\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}N_{zz} & N_{z\phi}\\ N_{z\phi} & N_{\phi\phi}\end{bmatrix}\begin{Bmatrix}\vec{e}_z\\ \vec{e}_\phi\end{Bmatrix},\ \vec{b}=\begin{Bmatrix}\vec{e}_z\\ \vec{e}_\phi\\ \vec{e}_n\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}b_z\\ b_\phi\\ b_n\end{Bmatrix},\ \nabla=\begin{Bmatrix}\vec{e}_z\\ \vec{e}_\phi\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\partial/\partial z\\ \partial/(R\partial\phi)\end{Bmatrix}.$$

The non-zero basis vector derivatives are $\dfrac{\partial}{\partial\phi}\vec{e}_\phi=\vec{e}_n$ and $\dfrac{\partial}{\partial\phi}\vec{e}_n=-\vec{e}_\phi$.

Solution

Membrane model is the thin-slab model in curved geometry. Equilibrium equations differ from the thin-slab ones as the non-constant basis vectors bring additional terms. Therefore, a membrane may take also external forces in the transverse direction (like pressure difference between the outer and inner surfaces of a balloon). The task is to simplify the vector equation

$$\nabla\cdot\vec{\vec{N}}+\vec{b}=(\vec{e}_z\frac{\partial}{\partial z}+\vec{e}_\phi\frac{1}{R}\frac{\partial}{\partial\phi})\cdot(N_{zz}\vec{e}_z\vec{e}_z+N_{z\phi}\vec{e}_z\vec{e}_\phi+N_{z\phi}\vec{e}_\phi\vec{e}_z+N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi)+b_z\vec{e}_z+b_\phi\vec{e}_\phi+b_n\vec{e}_n=0$$

to see the component forms. Let us start with the divergence of stress and consider the term in two parts (to avoid lengthy expressions)

$$(\vec{e}_z\frac{\partial}{\partial z})\cdot(N_{zz}\vec{e}_z\vec{e}_z+N_{z\phi}\vec{e}_z\vec{e}_\phi+N_{z\phi}\vec{e}_\phi\vec{e}_z+N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi)=\frac{\partial N_{zz}}{\partial z}\vec{e}_z+\frac{\partial N_{z\phi}}{\partial z}\vec{e}_\phi,$$

$$(\vec{e}_\phi\frac{1}{R}\frac{\partial}{\partial\phi})\cdot(N_{zz}\vec{e}_z\vec{e}_z+N_{z\phi}\vec{e}_z\vec{e}_\phi+N_{z\phi}\vec{e}_\phi\vec{e}_z+N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi)=\frac{1}{R}\frac{\partial N_{z\phi}}{\partial\phi}\vec{e}_z+\frac{1}{R}\frac{\partial N_{\phi\phi}}{\partial\phi}\vec{e}_\phi+\frac{1}{R}N_{\phi\phi}\vec{e}_n.$$

After that, combining the terms

$$\nabla\cdot\vec{\vec{N}}+\vec{b}=\begin{Bmatrix}\vec{e}_z\\ \vec{e}_\phi\\ \vec{e}_n\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\dfrac{\partial N_{zz}}{\partial z}+\dfrac{1}{R}\dfrac{\partial N_{z\phi}}{\partial\phi}+b_z\\ \dfrac{\partial N_{z\phi}}{\partial z}+\dfrac{1}{R}\dfrac{\partial N_{\phi\phi}}{\partial\phi}+b_\phi\\ \dfrac{1}{R}N_{\phi\phi}+b_n\end{Bmatrix}=0.$$