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## Assignment 4

Derive the expressions of linear strain components  $\varepsilon_{rr}$ ,  $\varepsilon_{r\phi}$ ,  $\varepsilon_{\phi r}$  and  $\varepsilon_{\phi\phi}$  of the polar coordinate system. Use the displacement representation  $\vec{u} = u_r \vec{e}_r + u_\phi \vec{e}_\phi$  where the components depend on the polar coordinates  $r$  and  $\phi$ . Use definitions

$$\vec{\varepsilon} = \frac{1}{2} [\nabla \vec{u} + (\nabla \vec{u})^c], \quad \nabla = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{\partial}{r \partial \phi}, \quad \frac{\partial}{\partial \phi} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} = \begin{Bmatrix} \vec{e}_\phi \\ -\vec{e}_r \end{Bmatrix}, \quad \frac{\partial}{\partial r} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} = 0.$$

### Solution template

In manipulation of vector expression containing vectors and tensors, it is important to remember that tensor ( $\otimes$ ), cross ( $\times$ ), inner ( $\cdot$ ) products are non-commutative (order may matter). The basis vectors of a curvilinear coordinate system are not constants which should be taken into account if gradient operator is a part of expression. Otherwise, simplifying an expression or finding a specific form in a given coordinate system is a straightforward (sometimes tedious) exercise. For simplicity of presentation, outer (tensor) products like  $\vec{a} \otimes \vec{b}$  are denoted by  $\vec{a} \vec{b}$ . Otherwise, the usual rules of algebra apply: Gradient operator  $\nabla$  acts on everything on its right hand side, the operator is treated like a vector etc.

Let us start with the gradient of displacement (an outer product). Substitute first the representations in the polar coordinate system

$$\nabla \vec{u} = (\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{\partial}{r \partial \phi})(u_r \vec{e}_r + u_\phi \vec{e}_\phi).$$

Then expand to have a term-by-term representation. Keep the order of the basis vectors and the position of derivatives

$$\nabla \vec{u} = \vec{e}_r \frac{\partial}{\partial r} (u_r \vec{e}_r) + \vec{e}_r \frac{\partial}{\partial r} (u_\phi \vec{e}_\phi) + \vec{e}_\phi \frac{\partial}{r \partial \phi} (u_r \vec{e}_r) + \vec{e}_\phi \frac{\partial}{r \partial \phi} (u_\phi \vec{e}_\phi)$$

Use the derivative rule of products. Notice that the basis vectors are not constants and may have non-zero derivatives

$$\nabla \vec{u} = \vec{e}_r \left( \frac{\partial u_r}{\partial r} \vec{e}_r + u_r \frac{\partial \vec{e}_r}{\partial r} \right) + \vec{e}_r \left( \frac{\partial u_\phi}{\partial r} \vec{e}_\phi + u_\phi \frac{\partial \vec{e}_\phi}{\partial r} \right) + \vec{e}_\phi \left( \frac{\partial u_r}{r \partial \phi} \vec{e}_r + u_r \frac{\partial \vec{e}_r}{r \partial \phi} \right) + \vec{e}_\phi \left( \frac{\partial u_\phi}{r \partial \phi} \vec{e}_\phi + u_\phi \frac{\partial \vec{e}_\phi}{r \partial \phi} \right).$$

Substitute the derivatives of the basis vectors

$$\nabla \vec{u} = \vec{e}_r \left( \frac{\partial u_r}{\partial r} \vec{e}_r \right) + \vec{e}_r \left( \frac{\partial u_\phi}{\partial r} \vec{e}_\phi \right) + \vec{e}_\phi \left( \frac{\partial u_r}{r \partial \phi} \vec{e}_r + \frac{u_r}{r} \vec{e}_\phi \right) + \vec{e}_\phi \left( \frac{\partial u_\phi}{r \partial \phi} \vec{e}_\phi - \frac{u_\phi}{r} \vec{e}_r \right).$$

Combine the terms having the same pair of basis vectors (order matters so terms containing  $\vec{e}_\phi \vec{e}_r$  and  $\vec{e}_r \vec{e}_\phi$  cannot be combined)

$$\nabla \vec{u} = \frac{\partial u_r}{\partial r} \vec{e}_r \vec{e}_r + \frac{\partial u_\phi}{\partial r} \vec{e}_r \vec{e}_\phi + \left( \frac{\partial u_r}{r \partial \phi} - \frac{u_\phi}{r} \right) \vec{e}_\phi \vec{e}_r + \left( \frac{u_r}{r} + \frac{\partial u_\phi}{r \partial \phi} \right) \vec{e}_\phi \vec{e}_\phi.$$

Conjugate of a second order tensor can be obtained by swapping the basis vectors in all the pairs. Conjugate is a kind of transpose and can also be obtained by transposing the matrix of the component representation.

$$(\nabla \vec{u})_c = \frac{\partial u_r}{\partial r} \vec{e}_r \vec{e}_r + \left( \frac{\partial u_r}{r \partial \phi} - \frac{u_\phi}{r} \right) \vec{e}_r \vec{e}_\phi + \frac{\partial u_\phi}{\partial r} \vec{e}_\phi \vec{e}_r + \left( \frac{u_r}{r} + \frac{\partial u_\phi}{r \partial \phi} \right) \vec{e}_\phi \vec{e}_\phi$$

Finally using the definition  $\vec{\varepsilon} = \frac{1}{2} [\nabla \vec{u} + (\nabla \vec{u})_c]$

$$\vec{\varepsilon} = \frac{\partial u_r}{\partial r} \vec{e}_r \vec{e}_r + \frac{1}{2} \left( \frac{\partial u_r}{r \partial \phi} - \frac{u_\phi}{r} + \frac{\partial u_\phi}{\partial r} \right) (\vec{e}_r \vec{e}_\phi + \vec{e}_\phi \vec{e}_r) + \left( \frac{\partial u_\phi}{r \partial \phi} + \frac{u_r}{r} \right) \vec{e}_\phi \vec{e}_\phi.$$

In the components of strain  $\varepsilon_{rr}$ ,  $\varepsilon_{r\phi}$ ,  $\varepsilon_{\phi r}$  and  $\varepsilon_{\phi\phi}$ , indices are in the same order as the indices in the basis vector pairs. Hence

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\phi\phi} = \frac{\partial u_\phi}{r \partial \phi} + \frac{u_r}{r}, \quad \varepsilon_{r\phi} = \varepsilon_{\phi r} = \frac{1}{2} \left( \frac{\partial u_r}{r \partial \phi} - \frac{u_\phi}{r} + \frac{\partial u_\phi}{\partial r} \right). \quad \leftarrow$$