Name________________________ Student number________________

Assignment 4 (4p)

Consider the curved beam rigidity problem on page 1-4 to 1-8 of the lecture notes. Use the curved beam equations on page 1-6 with hand calculations or Mathematica to find the analytical solution to the vertical displacement v as the function of mass m used as loading. Use the expression to deduce the coefficient a of

$$\frac{mgR^2}{EI} = a\frac{v}{R}.$$

In the expression, g is the acceleration by gravity, R is the radius , I the second moment of cross section with respect to the area centroid, and E is the Young's modulus of the rim material. The specific form above is based on dimension analysis and additional assumptions of linearity and vanishing displacement without external loading.

Solution

The first order ordinary differential equations for the curved Bernoulli beam, which is inextensible in the axial direction, are given by ($L = 3\pi R/2$)

$$\frac{dN}{dx} - \frac{1}{R}Q = 0,\ \frac{dQ}{dx} + \frac{1}{R}N = 0,\ \frac{dM}{dx} + Q = 0 \quad x \in [0, L] \text{ and } N = 0,\ Q + F = 0,\ M = 0 \quad x = L,$$

$$M = EI\frac{d\psi}{dx},\ \frac{du}{dx} - \frac{1}{R}v = 0,\ \frac{dv}{dx} + \frac{1}{R}u - \psi = 0 \ x \in [0, L] \text{ and } u = 0,\ v = 0,\ \psi = 0 \quad x = 0.$$

In a statically determined case, it is possible to first consider the equations for the stress resultants and those for displacements and rotation after that. Let us start with the equilibrium equations and boundary conditions at the free edge. Elimination of Q from the first two connected differential equations gives (notice that elimination gives a second order equation and the missing boundary condition is given by the first of the original equations)

$$\frac{d^2N}{dx^2} + \frac{1}{R^2}N = 0 \ x \in [0, L],\ N = 0 \text{ and } \frac{dN}{dx} + \frac{1}{R}F = 0 \ x = L \quad \Rightarrow \quad \frac{N}{F} = -\cos(\frac{x}{R}).$$

Then, using the first differential equation with the known solution to N

$$\frac{dN}{dx} - \frac{1}{R}Q = 0 \quad \Rightarrow \quad \frac{Q}{F} = \sin(\frac{x}{R}).$$

Finally, considering the third differential equation with the known solution to Q and the third boundary condition

$$\frac{dM}{dx}+Q=0 \quad x\in[0,L], \quad M=0 \quad x=L \quad \Rightarrow \quad \frac{M}{FR}=\cos(\frac{x}{R}).$$

When the stress resultants are known, one may consider the constitutive equation, Bernoulli constraint and the inextensibility constraint (the remaining differential equations) as first order ordinary differential equations for the displacement and rotation components. Let us start with the first of the equations with the known expression for the moment resultant:

$$EI\frac{d\psi}{dx}=FR\cos(\frac{x}{R}) \quad x\in]0,L] \quad \text{and} \quad \psi=0 \quad x=0 \quad \Rightarrow \quad \psi=\frac{FR^2}{EI}\sin(\frac{x}{R}).$$

As the remaining differential equations (constraints) are connected, one needs to eliminate either u or v. Let us eliminate u and substitute the known solution to rotation to get

$$\frac{d^2v}{dx^2}+\frac{1}{R^2}v=\frac{FR}{EI}\cos(\frac{x}{R}) \quad x\in]0,L] \quad \text{and} \quad v=0, \quad \frac{dv}{dx}=0 \quad x=0 \quad \Rightarrow \quad \frac{v}{R}=\frac{1}{2}\frac{FR^2}{EI}\frac{x}{R}\sin(\frac{x}{R}).$$

Solution to the differential equation is composed of the generic solution to the homogeneous equations and a particular solution. Solution to the axial displacement components does not matter here. Comparing the solution to the transverse component v with the expression by dimension analysis gives (notice that the solution to transverse displacement is positive upwards whereas that in the dimension analysis formula is positive downwards)

$$\frac{v}{R}=\frac{3}{4}\pi\frac{FR^2}{EI} \quad \Rightarrow \quad \frac{FR^2}{EI}=\frac{4}{3\pi}\frac{v}{R} \quad \text{so} \quad a=\frac{4}{3\pi}.$$