MEC-E8003 Beam, plate and shell models, examples 1

1. Find the relationship between the orthonormal basis vector sets $(\vec{i},\vec{j},\vec{k})$ and $(\vec{I},\vec{J},\vec{K})$, if $\vec{I}$ has the same direction as $\vec{i}-\vec{j}+\vec{k}$ and $\vec{J}$ has the same direction as the gradient of plane $g(x,y,z)=2x+3y+z-5=0$ (hence $\vec{J}$ is the normal unit vector to the plane).

   Answer $\begin{Bmatrix} \vec{I} \\ \vec{J} \\ \vec{K} \end{Bmatrix} = \dfrac{1}{\sqrt{42}} \begin{bmatrix} \sqrt{14} & -\sqrt{14} & \sqrt{14} \\ 2\sqrt{3} & 3\sqrt{3} & \sqrt{3} \\ -4 & 1 & 5 \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}$

2. Consider the identity $\vec{\vec{I}} \times \vec{a} = \vec{a} \times \vec{\vec{I}}$ in a Cartesian $(x,y,z)-$coordinate system in which the second order unit tensor is given by $\vec{\vec{I}} = \vec{i}\vec{i} + \vec{j}\vec{j} + \vec{k}\vec{k}$. Show that the identity holds by simplifying the left and right-hand side expressions. Assume that $\vec{a} = a_x\vec{i} + a_y\vec{j}$ (just to simplify the expressions).

3. Given $\vec{\vec{a}} = a_{xx}\vec{i}\vec{i} + a_{xy}\vec{i}\vec{j} + a_{yx}\vec{j}\vec{i} + a_{yy}\vec{j}\vec{j}$ and $\vec{b} = b_x\vec{i} + b_y\vec{j}$, determine $\vec{\vec{a}}\cdot\vec{b}$, $\vec{\vec{a}}\times\vec{b}$, $\vec{b}\cdot\vec{\vec{a}}$, and $\vec{b}\times\vec{\vec{a}}$.

   Answer

   $$\vec{\vec{a}}\cdot\vec{b} = (a_{xx}b_x + a_{xy}b_y)\vec{i} + (a_{yx}b_x + a_{yy}b_y)\vec{j}, \quad \vec{\vec{a}}\times\vec{b} = (a_{xx}b_y - a_{xy}b_x)\vec{i}\vec{k} + (a_{yx}b_y - a_{yy}b_x)\vec{j}\vec{k}$$

   $$\vec{b}\cdot\vec{\vec{a}} = (b_x a_{xx} + b_y a_{yx})\vec{i} + (b_x a_{xy} + b_y a_{yy})\vec{j}, \quad \vec{b}\times\vec{\vec{a}} = (b_x a_{yx} - b_y a_{xx})\vec{k}\vec{i} + (b_x a_{yy} - b_y a_{xy})\vec{k}\vec{j}$$

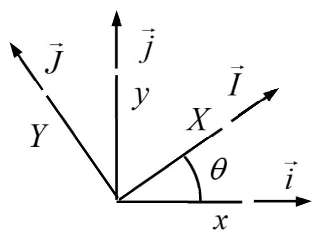
4. Calculate $\nabla\vec{r}$, $\nabla\cdot\vec{r}$ and $\nabla\times\vec{r}$ in which $\vec{r}$ is the position vector. Use the representations $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $\nabla = \vec{i}\partial/\partial x + \vec{j}\partial/\partial y + \vec{k}\partial/\partial z$ in the Cartesian $(\vec{i},\vec{j},\vec{k})$ basis.

   Answer $\nabla\vec{r} = \vec{\vec{I}}$, $\nabla\cdot\vec{r} = 3$, $\nabla\times\vec{r} = 0$

5. Derive the expressions of $\nabla\cdot\vec{u}$, $\nabla\times\vec{u}$, and $\nabla^2 u$ in Cartesian coordinate system where $\nabla = \vec{i}\partial/\partial x + \vec{j}\partial/\partial y + \vec{k}\partial/\partial z$. Vector $\vec{u}(x,y) = u_x(x,y)\vec{i} + u_y(x,y)\vec{j}$ and scalar $u(x,y)$ depend on *x* and *y* only.

   Answer $\nabla\cdot\vec{u} = \dfrac{\partial u_x}{\partial x} + \dfrac{\partial u_y}{\partial y}$, $\nabla\times\vec{u} = \vec{k}\left(\dfrac{\partial u_y}{\partial x} - \dfrac{\partial u_x}{\partial y}\right)$, $\nabla^2 u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$.

6. Let us consider a second order tensor $\vec{\vec{\varepsilon}}$ having the components ε_{xx}, ε_{xy}, ε_{yx}, ε_{yy} and ε_{XX}, ε_{XY}, ε_{YX}, ε_{YY} in the $(\vec{i},\vec{j})$ and $(\vec{I},\vec{J})$ bases, respectively. Find the relationship between the components by using the invariance of tensor quantities.



Answer $\begin{bmatrix} \varepsilon_{XX} & \varepsilon_{XY} \\ \varepsilon_{YX} & \varepsilon_{YY} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

7. Derive the component forms of the equilibrium equation $\nabla \cdot \vec{\vec{\sigma}} + \vec{f} = 0$ in plane stress case, when

$$\vec{\vec{\sigma}} = \sigma_{xx}\vec{i}\vec{i} + \sigma_{xy}\vec{i}\vec{j} + \sigma_{yx}\vec{j}\vec{i} + \sigma_{yy}\vec{j}\vec{j}\,, \quad \nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y}, \text{ and } \vec{f} = f_x\vec{i} + f_y\vec{j}$$

and the stress components depend on x and y.

Answer $\dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{yx}}{\partial y} + f_x = 0$, $\dfrac{\partial \sigma_{xy}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} + f_y = 0$

8. The small strain measure $\vec{\vec{\varepsilon}}$ is obtained as the symmetric part of displacement gradient (a tensor then). Use the definition to find the components of the strain tensor in a Cartesian coordinate system when $\nabla = \vec{i}\partial/\partial x + \vec{j}\partial/\partial y$ and $\vec{u} = u_x\vec{i} + u_y\vec{j}$.

Answer $\varepsilon_{xx} = \dfrac{\partial u_x}{\partial x}$, $\varepsilon_{yy} = \dfrac{\partial u_y}{\partial y}$, and $\varepsilon_{xy} = \varepsilon_{yx} = \dfrac{1}{2}(\dfrac{\partial u_y}{\partial x} + \dfrac{\partial u_x}{\partial y})$

9. Find the solution to the boundary value problem

$$\frac{dQ}{dx} - \rho g A = 0, \quad \frac{dM}{dx} - Q = 0, \quad M = EI\frac{d\theta}{dx}, \text{ and } Q = GA(\frac{dw}{dx} + \theta) \text{ in } (0, L),$$

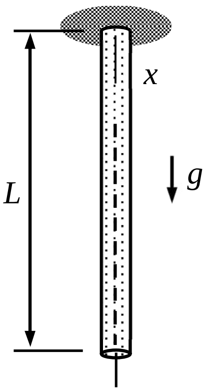
$$w = M = 0 \text{ on } \{0, L\}$$

for a simply supported beam loaded by its own weight. Cross sectional area A, second moment af area I, Young's modulus E, shear modulus G, density of the material ρ, and acceleration by gravity g are constants. Use repeated integrations.

Answer $M(x) = \dfrac{1}{2}\rho g A(x^2 - Lx)$, $Q(x) = \dfrac{1}{2}\rho g A(2x - L)$, $\theta(x) = \dfrac{\rho g A}{2EI}(\dfrac{1}{3}x^3 - \dfrac{1}{2}Lx^2 - \dfrac{1}{12}L^3)$

$$w(x) = \frac{\rho g}{2G}(x^2 - xL) - \frac{\rho g A}{2EI}(\frac{1}{12}x^4 - \frac{1}{6}Lx^3 + \frac{1}{12}L^3 x)$$

10. Consider a bar of length L loaded by its own weight. Determine the displacements at the mid-point and at the free end using the continuum model. Cross-sectional area A, acceleration by gravity g, and material properties E and ρ are constants.



Answer $u(\dfrac{L}{2}) = \dfrac{3}{8}\dfrac{\rho g L^2}{E}$, $u(L) = \dfrac{\rho g L^2}{2E}$

Find the relationship between the orthonormal right-handed basis vector sets $(\vec{i},\vec{j},\vec{k})$ and $(\vec{I},\vec{J},\vec{K})$, if $\vec{I}$ has the same direction as $\vec{i}-\vec{j}+\vec{k}$ and $\vec{J}$ has the same direction as the gradient of plane $g(x,y,z)=2x+3y+z-5=0$ (hence $\vec{J}$ is the normal unit vector to the plane).

Solution

Both sets are orthonormal, i.e., the basis vectors are orthogonal and have unit lengths. As the systems are right-handed $\vec{i}\times\vec{j}=\vec{k}$, $\vec{j}\times\vec{k}=\vec{i}$, $\vec{k}\times\vec{i}=\vec{j}$ and $\vec{I}\times\vec{J}=\vec{K}$, $\vec{J}\times\vec{K}=\vec{I}$, $\vec{K}\times\vec{I}=\vec{J}$. Vectors $\vec{i}-\vec{j}+\vec{k}$ and $\vec{J}$ have the same directions, therefore

$$\vec{I}=\frac{\vec{i}-\vec{j}+\vec{k}}{|\vec{i}-\vec{j}+\vec{k}|}=\frac{1}{\sqrt{3}}(\vec{i}-\vec{j}+\vec{k})\quad(|\vec{a}|=\sqrt{\vec{a}\cdot\vec{a}}).$$

Vector $\vec{J}$ and the gradient of $g(x,y,z)=2x+3y+z-5$ have the same directions, so

$$\nabla g=2\vec{i}+3\vec{j}+\vec{k}\quad\Rightarrow\quad\vec{J}=\frac{\nabla g}{|\nabla g|}=\frac{1}{\sqrt{14}}(2\vec{i}+3\vec{j}+\vec{k}).$$

Both coordinate systems are right-handed

$$\vec{K}=\vec{I}\times\vec{J}=\frac{1}{\sqrt{3}}(\vec{i}-\vec{j}+\vec{k})\times\frac{1}{\sqrt{14}}(2\vec{i}+3\vec{j}+\vec{k})\quad\Leftrightarrow$$

$$\vec{K}=\frac{1}{\sqrt{42}}(\vec{i}\times2\vec{i}+\vec{i}\times3\vec{j}+\vec{i}\times\vec{k}-\vec{j}\times2\vec{i}-\vec{j}\times3\vec{j}-\vec{j}\times\vec{k}+\vec{k}\times2\vec{i}+\vec{k}\times3\vec{j}+\vec{k}\times\vec{k})\quad\Leftrightarrow$$

$$\vec{K}=\frac{1}{\sqrt{42}}(3\vec{k}-\vec{j}+2\vec{k}-\vec{i}+2\vec{j}-3\vec{i})=\frac{1}{\sqrt{42}}(-4\vec{i}+\vec{j}+5\vec{k}).$$

Using the matrix notation

$$\begin{Bmatrix}\vec{I}\\\vec{J}\\\vec{K}\end{Bmatrix}=\frac{1}{\sqrt{42}}\begin{bmatrix}\sqrt{14}&-\sqrt{14}&\sqrt{14}\\2\sqrt{3}&3\sqrt{3}&\sqrt{3}\\-4&1&5\end{bmatrix}\begin{Bmatrix}\vec{i}\\\vec{j}\\\vec{k}\end{Bmatrix}.\ \leftarrow$$

Consider the identity $\vec{I}\times\vec{a}=\vec{a}\times\vec{I}$ in a Cartesian $(x,y,z)-$coordinate system in which the second order unit tensor is given by $\vec{I}=\vec{i}\vec{i}+\vec{j}\vec{j}+\vec{k}\vec{k}$. Show that the identity holds by simplifying the left and right-hand side expressions. Assume that $\vec{a}=a_x\vec{i}+a_y\vec{j}$ (just to simplify the expressions).

Solution

Let us compare the expressions on the left and right-hand sides of $\vec{I}\times\vec{a}=\vec{a}\times\vec{I}$ and use the relationships $\vec{i}\times\vec{j}=\vec{k}$, $\vec{j}\times\vec{k}=\vec{i}$, and $\vec{k}\times\vec{i}=\vec{j}$ for a right handed system.

Left hand side expression

$$\vec{I}\times\vec{a}=(\vec{i}\vec{i}+\vec{j}\vec{j}+\vec{k}\vec{k})\times(a_x\vec{i}+a_y\vec{j}) \quad \Leftrightarrow$$

$$\vec{I}\times\vec{a}=(\vec{i}\vec{i}+\vec{j}\vec{j}+\vec{k}\vec{k})\times a_x\vec{i}+(\vec{i}\vec{i}+\vec{j}\vec{j}+\vec{k}\vec{k})\times a_y\vec{j} \quad \Leftrightarrow$$

$$\vec{I}\times\vec{a}=(\vec{i}\vec{i}\times\vec{i}+\vec{j}\vec{j}\times\vec{i}+\vec{k}\vec{k}\times\vec{i})a_x+(\vec{i}\vec{i}\times\vec{j}+\vec{j}\vec{j}\times\vec{j}+\vec{k}\vec{k}\times\vec{j})a_y \quad \Leftrightarrow$$

$$\vec{I}\times\vec{a}=(\vec{k}\vec{j}-\vec{j}\vec{k})a_x+(\vec{i}\vec{k}-\vec{k}\vec{i})a_y \, . \quad \leftarrow$$

Right hand side expression

$$\vec{a}\times\vec{I}=(a_x\vec{i}+a_y\vec{j})\times(\vec{i}\vec{i}+\vec{j}\vec{j}+\vec{k}\vec{k}) \quad \Leftrightarrow$$

$$\vec{a}\times\vec{I}=a_x\vec{i}\times(\vec{i}\vec{i}+\vec{j}\vec{j}+\vec{k}\vec{k})+a_y\vec{j}\times(\vec{i}\vec{i}+\vec{j}\vec{j}+\vec{k}\vec{k}) \quad \Leftrightarrow$$

$$\vec{a}\times\vec{I}=a_x(\vec{i}\times\vec{i}\vec{i}+\vec{i}\times\vec{j}\vec{j}+\vec{i}\times\vec{k}\vec{k})+a_y(\vec{j}\times\vec{i}\vec{i}+\vec{j}\times\vec{j}\vec{j}+\vec{j}\times\vec{k}\vec{k}) \quad \Leftrightarrow$$

$$\vec{a}\times\vec{I}=a_x(\vec{k}\vec{j}-\vec{j}\vec{k})+a_y(\vec{i}\vec{k}-\vec{k}\vec{i}) \, . \quad \leftarrow$$

Given $\vec{a} = a_{xx}\vec{i}\vec{i} + a_{xy}\vec{i}\vec{j} + a_{yx}\vec{j}\vec{i} + a_{yy}\vec{j}\vec{j}$ and $\vec{b} = b_x\vec{i} + b_y\vec{j}$, determine $\vec{a}\cdot\vec{b}$, $\vec{a}\times\vec{b}$, $\vec{b}\cdot\vec{a}$, and $\vec{b}\times\vec{a}$.

Solution

In manipulation of vector expression containing vectors and tensors, it is important to remember that tensor ($\otimes$), cross ($\times$), inner ($\cdot$) products are non-commutative (order matters). The basis vectors of a curvilinear coordinate system are not constants which should be taken into account if gradient operator is a part of expression. Otherwise, simplifying an expression or finding a specific form in a given coordinate system is a straightforward (sometimes tedious) exercise. For simplicity of presentation, outer (tensor) products like $\vec{a}\otimes\vec{b}$ are denoted by $\vec{a}\vec{b}$. Otherwise, the usual rules of algebra apply.

$$\vec{a}\cdot\vec{b} = (a_{xx}\vec{i}\vec{i} + a_{xy}\vec{i}\vec{j} + a_{yx}\vec{j}\vec{i} + a_{yy}\vec{j}\vec{j})\cdot(b_x\vec{i} + b_y\vec{j}) \quad \Leftrightarrow$$

$$\vec{a}\cdot\vec{b} = (a_{xx}\vec{i}\vec{i} + a_{xy}\vec{i}\vec{j} + a_{yx}\vec{j}\vec{i} + a_{yy}\vec{j}\vec{j})\cdot b_x\vec{i} + (a_{xx}\vec{i}\vec{i} + a_{xy}\vec{i}\vec{j} + a_{yx}\vec{j}\vec{i} + a_{yy}\vec{j}\vec{j})\cdot b_y\vec{j} \quad \Leftrightarrow$$

$$\vec{a}\cdot\vec{b} = (a_{xx}\vec{i}\vec{i}\cdot\vec{i} + a_{xy}\vec{i}\vec{j}\cdot\vec{i} + a_{yx}\vec{j}\vec{i}\cdot\vec{i} + a_{yy}\vec{j}\vec{j}\cdot\vec{i})b_x + (a_{xx}\vec{i}\vec{i}\cdot\vec{j} + a_{xy}\vec{i}\vec{j}\cdot\vec{j} + a_{yx}\vec{j}\vec{i}\cdot\vec{j} + a_{yy}\vec{j}\vec{j}\cdot\vec{j})b_y \quad \Leftrightarrow$$

$$\vec{a}\cdot\vec{b} = a_{xx}b_x\vec{i} + a_{yx}b_x\vec{j} + a_{xy}b_y\vec{i} + a_{yy}b_y\vec{j} = (a_{xx}b_x + a_{xy}b_y)\vec{i} + (a_{yx}b_x + a_{yy}b_y)\vec{j}\,. \Leftarrow$$

$$\vec{a}\times\vec{b} = (a_{xx}\vec{i}\vec{i} + a_{xy}\vec{i}\vec{j} + a_{yx}\vec{j}\vec{i} + a_{yy}\vec{j}\vec{j})\times(b_x\vec{i} + b_y\vec{j}) \quad \Leftrightarrow$$

$$\vec{a}\times\vec{b} = (a_{xx}\vec{i}\vec{i} + a_{xy}\vec{i}\vec{j} + a_{yx}\vec{j}\vec{i} + a_{yy}\vec{j}\vec{j})\times b_x\vec{i} + (a_{xx}\vec{i}\vec{i} + a_{xy}\vec{i}\vec{j} + a_{yx}\vec{j}\vec{i} + a_{yy}\vec{j}\vec{j})\times b_y\vec{j} \quad \Leftrightarrow$$

$$\begin{aligned}\vec{a}\times\vec{b} = {} & (a_{xx}\vec{i}\vec{i}\times\vec{i} + a_{xy}\vec{i}\vec{j}\times\vec{i} + a_{yx}\vec{j}\vec{i}\times\vec{i} + a_{yy}\vec{j}\vec{j}\times\vec{i})b_x + \\ & (a_{xx}\vec{i}\vec{i}\times\vec{j} + a_{xy}\vec{i}\vec{j}\times\vec{j} + a_{yx}\vec{j}\vec{i}\times\vec{j} + a_{yy}\vec{j}\vec{j}\times\vec{j})b_y \quad \Leftrightarrow\end{aligned}$$

$$\vec{a}\times\vec{b} = -a_{xy}b_x\vec{i}\vec{k} - a_{yy}b_x\vec{j}\vec{k} + a_{xx}b_y\vec{i}\vec{k} + a_{yx}b_y\vec{j}\vec{k} \;= (a_{xx}b_y - a_{xy}b_x)\vec{i}\vec{k} + (a_{yx}b_y - a_{yy}b_x)\vec{j}\vec{k}\,. \Leftarrow$$

$$\vec{b}\cdot\vec{a} = (b_x\vec{i} + b_y\vec{j})\cdot(a_{xx}\vec{i}\vec{i} + a_{xy}\vec{i}\vec{j} + a_{yx}\vec{j}\vec{i} + a_{yy}\vec{j}\vec{j}) \quad \Leftrightarrow$$

$$\vec{b}\cdot\vec{a} = b_x\vec{i}\cdot(a_{xx}\vec{i}\vec{i} + a_{xy}\vec{i}\vec{j} + a_{yx}\vec{j}\vec{i} + a_{yy}\vec{j}\vec{j}) + b_y\vec{j}\cdot(a_{xx}\vec{i}\vec{i} + a_{xy}\vec{i}\vec{j} + a_{yx}\vec{j}\vec{i} + a_{yy}\vec{j}\vec{j}) \quad \Leftrightarrow$$

$$\vec{b}\cdot\vec{a} = b_x(a_{xx}\vec{i}\cdot\vec{i}\vec{i} + a_{xy}\vec{i}\cdot\vec{i}\vec{j} + a_{yx}\vec{i}\cdot\vec{j}\vec{i} + a_{yy}\vec{i}\cdot\vec{j}\vec{j}) + b_y(a_{xx}\vec{j}\cdot\vec{i}\vec{i} + a_{xy}\vec{j}\cdot\vec{i}\vec{j} + a_{yx}\vec{j}\cdot\vec{j}\vec{i} + a_{yy}\vec{j}\cdot\vec{j}\vec{j}) \quad \Leftrightarrow$$

$$\vec{b}\cdot\vec{a} = b_x a_{xx}\vec{i} + b_x a_{xy}\vec{j} + b_y a_{yx}\vec{i} + b_y a_{yy}\vec{j} = (b_x a_{xx} + b_y a_{yx})\vec{i} + (b_x a_{xy} + b_y a_{yy})\vec{j}\,. \Leftarrow$$

$$\vec{b}\times\vec{a} = (b_x\vec{i} + b_y\vec{j})\times(a_{xx}\vec{i}\vec{i} + a_{xy}\vec{i}\vec{j} + a_{yx}\vec{j}\vec{i} + a_{yy}\vec{j}\vec{j}) \quad \Leftrightarrow$$

$$\vec{b}\times\vec{a} = b_x\vec{i}\times(a_{xx}\vec{i}\vec{i} + a_{xy}\vec{i}\vec{j} + a_{yx}\vec{j}\vec{i} + a_{yy}\vec{j}\vec{j}) + b_y\vec{j}\times(a_{xx}\vec{i}\vec{i} + a_{xy}\vec{i}\vec{j} + a_{yx}\vec{j}\vec{i} + a_{yy}\vec{j}\vec{j}) \quad \Leftrightarrow$$

$$\begin{aligned}\vec{b}\times\vec{a} = {} & b_x(a_{xx}\vec{i}\times\vec{i}\vec{i} + a_{xy}\vec{i}\times\vec{i}\vec{j} + a_{yx}\vec{i}\times\vec{j}\vec{i} + a_{yy}\vec{i}\times\vec{j}\vec{j}) + \\ & b_y(a_{xx}\vec{j}\times\vec{i}\vec{i} + a_{xy}\vec{j}\times\vec{i}\vec{j} + a_{yx}\vec{j}\times\vec{j}\vec{i} + a_{yy}\vec{j}\times\vec{j}\vec{j}) \quad \Leftrightarrow\end{aligned}$$

$$\vec{b}\times\vec{a} = b_x a_{yx}\vec{k}\vec{i} + b_x a_{yy}\vec{k}\vec{j} - b_y a_{xx}\vec{k}\vec{i} - b_y a_{xy}\vec{k}\vec{j} = (b_x a_{yx} - b_y a_{xx})\vec{k}\vec{i} + (b_x a_{yy} - b_y a_{xy})\vec{k}\vec{j}\ .$$

Calculate $\nabla\vec{r}$, $\nabla\cdot\vec{r}$ and $\nabla\times\vec{r}$ in which $\vec{r}$ is the position vector. Use the representations $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$ and $\nabla=\vec{i}\partial/\partial x+\vec{j}\partial/\partial y+\vec{k}\partial/\partial z$ in the Cartesian $(\vec{i},\vec{j},\vec{k})$ basis.

Solution

In a term, gradient operator ∇ acts on everything on its right-hand side. Otherwise, the operator is treated like a vector. Manipulation of a tensor expression consist of (I) substitution of the representations, (II) term by term expansion, (III) evaluation of the terms, (IV) simplification and/or restructuring the outcome. Gradient of the position vector is a second order tensor

(I) $\nabla\vec{r}=(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z})(x\vec{i}+y\vec{j}+z\vec{k}) \quad\Leftrightarrow$

(II) $\nabla\vec{r}=\vec{i}(\frac{\partial}{\partial x}x\vec{i}+\frac{\partial}{\partial x}y\vec{j}+\frac{\partial}{\partial x}z\vec{k})+\vec{j}(\frac{\partial}{\partial y}x\vec{i}+\frac{\partial}{\partial y}y\vec{j}+\frac{\partial}{\partial y}z\vec{k})+\vec{k}(\frac{\partial}{\partial z}x\vec{i}+\frac{\partial}{\partial z}y\vec{j}+\frac{\partial}{\partial z}z\vec{k}) \quad\Leftrightarrow$

(III) $\nabla\vec{r}=\vec{i}(\vec{i}+0+0)+\vec{j}(0+\vec{j}+0)+\vec{k}(0+0+\vec{k}) \quad\Leftrightarrow$

(IV) $\nabla\vec{r}=\vec{i}\vec{i}+\vec{j}\vec{j}+\vec{k}\vec{k}=\vec{I}$.

Divergence of the position vector is a scalar

(I) $\nabla\cdot\vec{r}=(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z})\cdot(x\vec{i}+y\vec{j}+z\vec{k}) \quad\Leftrightarrow$

(II) $\nabla\cdot\vec{r}=\vec{i}\cdot(\frac{\partial}{\partial x}x\vec{i}+\frac{\partial}{\partial x}y\vec{j}+\frac{\partial}{\partial x}z\vec{k})+\vec{j}\cdot(\frac{\partial}{\partial y}x\vec{i}+\frac{\partial}{\partial y}y\vec{j}+\frac{\partial}{\partial y}z\vec{k})+\vec{k}\cdot(\frac{\partial}{\partial z}x\vec{i}+\frac{\partial}{\partial z}y\vec{j}+\frac{\partial}{\partial z}z\vec{k}) \quad\Leftrightarrow$

(III) $\nabla\cdot\vec{r}=\vec{i}\cdot(\vec{i}+0+0)+\vec{j}\cdot(0+\vec{j}+0)+\vec{k}\cdot(0+0+\vec{k}) \quad\Leftrightarrow$

(IV) $\nabla\cdot\vec{r}=\vec{i}\cdot\vec{i}+\vec{j}\cdot\vec{j}+\vec{k}\cdot\vec{k}=3$.

Curl of the position vector is a vector

(I) $\nabla\times\vec{r}=(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z})\times(x\vec{i}+y\vec{j}+z\vec{k}) \quad\Leftrightarrow$

(II) $\nabla\times\vec{r}=\vec{i}\times(\frac{\partial}{\partial x}x\vec{i}+\frac{\partial}{\partial x}y\vec{j}+\frac{\partial}{\partial x}z\vec{k})+\vec{j}\times(\frac{\partial}{\partial y}x\vec{i}+\frac{\partial}{\partial y}y\vec{j}+\frac{\partial}{\partial y}z\vec{k})+\vec{k}\times(\frac{\partial}{\partial z}x\vec{i}+\frac{\partial}{\partial z}y\vec{j}+\frac{\partial}{\partial z}z\vec{k}) \quad\Leftrightarrow$

(III) $\nabla\times\vec{r}=\vec{i}\times(\vec{i}+0+0)+\vec{j}\times(0+\vec{j}+0)+\vec{k}\times(0+0+\vec{k}) \quad\Leftrightarrow$

(IV) $\nabla\times\vec{r}=\vec{i}\times\vec{i}+\vec{j}\times\vec{j}+\vec{k}\times\vec{k}=0$.

Derive the expressions of $\nabla\cdot\vec{u}$, $\nabla\times\vec{u}$, and $\nabla^2 u$ in Cartesian coordinate system where $\nabla=\vec{i}\partial/\partial x+\vec{j}\partial/\partial y+\vec{k}\partial/\partial z$. Vector $\vec{u}(x,y)=u_x(x,y)\vec{i}+u_y(x,y)\vec{j}$ and scalar $u(x,y)$.

Solution

In manipulation of vector expression containing vectors and dyads, it is important to remember that tensor ($\otimes$), cross ($\times$), inner ($\cdot$) products are non-commutative (order matters). Otherwise, simplifying an expression or finding a specific form in a given coordinate system is a straightforward (sometimes tedious) exercise. Manipulation of a tensor expression consist of (I) substitution of the representations, (II) term-by-term expansion, (III) evaluation of the terms, (IV) simplification and/or restructuring the outcome.

Divergence of a vector (here $\nabla=\vec{i}\partial/\partial x+\vec{j}\partial/\partial y$)

(I) $\nabla\cdot\vec{u}=(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y})\cdot(u_x\vec{i}+u_y\vec{j}) \quad \Leftrightarrow$

(II) $\nabla\cdot\vec{u}=\vec{i}\cdot(\frac{\partial}{\partial x}u_x\vec{i}+\frac{\partial}{\partial x}u_y\vec{j})+\vec{j}\cdot(\frac{\partial}{\partial y}u_x\vec{i}+\frac{\partial}{\partial y}u_y\vec{j}) \quad \Leftrightarrow$

(III) $\nabla\cdot\vec{u}=\frac{\partial}{\partial x}u_x\vec{i}\cdot\vec{i}+\frac{\partial}{\partial x}u_y\vec{i}\cdot\vec{j}+\frac{\partial}{\partial y}u_x\vec{j}\cdot\vec{i}+\frac{\partial}{\partial y}u_y\vec{j}\cdot\vec{j} \quad \Leftrightarrow$

(IV) $\nabla\cdot\vec{u}=\frac{\partial}{\partial x}u_x+\frac{\partial}{\partial y}u_y$.

Curl of a vector

(I) $\nabla\times\vec{u}=(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y})\times(u_x\vec{i}+u_y\vec{j}) \quad \Leftrightarrow$

(II) $\nabla\times\vec{u}=\vec{i}\times(\frac{\partial}{\partial x}u_x\vec{i}+\frac{\partial}{\partial x}u_y\vec{j})+\vec{j}\times(\frac{\partial}{\partial y}u_x\vec{i}+\frac{\partial}{\partial y}u_y\vec{j}) \quad \Leftrightarrow$

(III) $\nabla\times\vec{u}=\frac{\partial}{\partial x}u_x\vec{i}\times\vec{i}+\frac{\partial}{\partial x}u_y\vec{i}\times\vec{j}+\frac{\partial}{\partial y}u_x\vec{j}\times\vec{i}+\frac{\partial}{\partial y}u_y\vec{j}\times\vec{j} \quad \Leftrightarrow$

(IV) $\nabla\times\vec{u}=(\frac{\partial}{\partial x}u_y-\frac{\partial}{\partial y}u_x)\vec{k}$.

Laplacian of a scalar

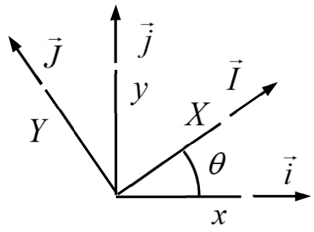
(I) $\nabla^2 u=(\nabla\cdot\nabla)u=\nabla\cdot(\nabla u)=(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y})\cdot(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y})u \quad \Leftrightarrow$

(II) $\nabla^2 u=\vec{i}\cdot(\vec{i}\frac{\partial^2 u}{\partial x^2}+\vec{j}\frac{\partial^2 u}{\partial x\partial y})+\vec{j}\cdot(\vec{i}\frac{\partial^2 u}{\partial x\partial y}+\vec{j}\frac{\partial^2 u}{\partial y^2}) \quad \Leftrightarrow$

(III) $\nabla^2 u = (1\frac{\partial^2 u}{\partial x^2} + 0\frac{\partial^2 u}{\partial x \partial y}) + (0\frac{\partial^2 u}{\partial x \partial y} + 1\frac{\partial^2 u}{\partial y^2}) \quad \Leftrightarrow$

(IV) $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$ $\Leftarrow$

Let us consider a second order tensor $\vec{\vec{\varepsilon}}$ having the components ε_{xx}, ε_{xy}, ε_{yx}, ε_{yy} and ε_{XX} , ε_{XY}, ε_{YX} , ε_{YY} in the $(\vec{i},\vec{j})$ and $(\vec{I},\vec{J})$ bases shown. Find the relationship between the components by using invariance of tensor quantities.



Solution

In mechanics tensors (vectors etc.) represent physical quantities which can be expressed in terms of any basis vector set. Components depend on the selection of the basis vectors but the quantity itself does not. According to the figure, the relationship between the $(\vec{i},\vec{j})$ and $(\vec{I},\vec{J})$ bases is given by

$$\begin{Bmatrix}\vec{I}\\ \vec{J}\end{Bmatrix}=\begin{bmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}=[F]\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix} \quad\Leftrightarrow\quad \begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}=\begin{bmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{bmatrix}\begin{Bmatrix}\vec{I}\\ \vec{J}\end{Bmatrix}=[F]^{-1}\begin{Bmatrix}\vec{I}\\ \vec{J}\end{Bmatrix}.$$

Invariace of $\vec{\vec{\varepsilon}}$ with respect to the coordinate system means that

$$\vec{\vec{\varepsilon}}=\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}\varepsilon_{xx} & \varepsilon_{xy}\\ \varepsilon_{yx} & \varepsilon_{yy}\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}=\begin{Bmatrix}\vec{I}\\ \vec{J}\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}\varepsilon_{XX} & \varepsilon_{XY}\\ \varepsilon_{YX} & \varepsilon_{YY}\end{bmatrix}\begin{Bmatrix}\vec{I}\\ \vec{J}\end{Bmatrix}.$$

Using the relationship between the basis vectors

$$\vec{\vec{\varepsilon}}=\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}\varepsilon_{xx} & \varepsilon_{xy}\\ \varepsilon_{yx} & \varepsilon_{yy}\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}=\begin{Bmatrix}\vec{I}\\ \vec{J}\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}\varepsilon_{XX} & \varepsilon_{XY}\\ \varepsilon_{YX} & \varepsilon_{YY}\end{bmatrix}\begin{Bmatrix}\vec{I}\\ \vec{J}\end{Bmatrix}=\begin{Bmatrix}\vec{I}\\ \vec{J}\end{Bmatrix}^{\mathrm{T}}[F]^{\mathrm{T}}\begin{bmatrix}\varepsilon_{XX} & \varepsilon_{XY}\\ \varepsilon_{YX} & \varepsilon_{YY}\end{bmatrix}[F]\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}.$$

Therefore, components of the two systems are related by ($[\mathrm{F}]^{-1}=[\mathrm{F}]^{\mathrm{T}}$)

$$\begin{bmatrix}\varepsilon_{xx} & \varepsilon_{xy}\\ \varepsilon_{yx} & \varepsilon_{yy}\end{bmatrix}=[F]^{\mathrm{T}}\begin{bmatrix}\varepsilon_{XX} & \varepsilon_{XY}\\ \varepsilon_{YX} & \varepsilon_{YY}\end{bmatrix}[F] \quad\Leftrightarrow\quad \begin{bmatrix}\varepsilon_{XX} & \varepsilon_{XY}\\ \varepsilon_{YX} & \varepsilon_{YY}\end{bmatrix}=[F]\begin{bmatrix}\varepsilon_{xx} & \varepsilon_{xy}\\ \varepsilon_{yx} & \varepsilon_{yy}\end{bmatrix}[F]^{\mathrm{T}}.$$

Derive the component forms of the equilibrium equation $\nabla\cdot\vec{\vec{\sigma}}+\vec{f}=0$ in plane stress case, when

$\vec{\vec{\sigma}}=\sigma_{xx}\vec{i}\vec{i}+\sigma_{xy}\vec{i}\vec{j}+\sigma_{yx}\vec{j}\vec{i}+\sigma_{yy}\vec{j}\vec{j}$, $\nabla=\vec{i}\dfrac{\partial}{\partial x}+\vec{j}\dfrac{\partial}{\partial y}$, and $\vec{f}=f_x\vec{i}+f_y\vec{j}$.

Solution

Manipulation of a tensor expression consist of (I) substitution of the representations, (II) term-by-term expansion, (III) evaluation of the terms, (IV) simplification and/or restructuring the outcome.

$$\nabla\cdot\vec{\vec{\sigma}}+\vec{f}=(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y})\cdot(\sigma_{xx}\vec{i}\vec{i}+\sigma_{xy}\vec{i}\vec{j}+\sigma_{yx}\vec{j}\vec{i}+\sigma_{yy}\vec{j}\vec{j})+f_x\vec{i}+f_y\vec{j}=0$$

$$\vec{i}\frac{\partial}{\partial x}\cdot(\sigma_{xx}\vec{i}\vec{i}+\sigma_{xy}\vec{i}\vec{j}+\sigma_{yx}\vec{j}\vec{i}+\sigma_{yy}\vec{j}\vec{j})+\vec{j}\frac{\partial}{\partial y}\cdot(\sigma_{xx}\vec{i}\vec{i}+\sigma_{xy}\vec{i}\vec{j}+\sigma_{yx}\vec{j}\vec{i}+\sigma_{yy}\vec{j}\vec{j})+f_x\vec{i}+f_y\vec{j}=0$$

Let us consider the first and second terms on the left-hand side separately

$$\vec{i}\frac{\partial}{\partial x}\cdot(\sigma_{xx}\vec{i}\vec{i}+\sigma_{xy}\vec{i}\vec{j}+\sigma_{yx}\vec{j}\vec{i}+\sigma_{yy}\vec{j}\vec{j})=$$

$$\frac{\partial\sigma_{xx}}{\partial x}\vec{i}\cdot\vec{i}\vec{i}+\frac{\partial\sigma_{xy}}{\partial x}\vec{i}\cdot\vec{i}\vec{j}+\frac{\partial\sigma_{yx}}{\partial x}\vec{i}\cdot\vec{j}\vec{i}+\frac{\partial\sigma_{yy}}{\partial x}\vec{i}\cdot\vec{j}\vec{j}=$$

$$\frac{\partial\sigma_{xx}}{\partial x}\vec{i}+\frac{\partial\sigma_{xy}}{\partial x}\vec{j}+\frac{\partial\sigma_{yx}}{\partial x}0\vec{i}+\frac{\partial\sigma_{yy}}{\partial x}0\vec{j}=\frac{\partial\sigma_{xx}}{\partial x}\vec{i}+\frac{\partial\sigma_{xy}}{\partial x}\vec{j}$$

and

$$\vec{j}\frac{\partial}{\partial y}\cdot(\sigma_{xx}\vec{i}\vec{i}+\sigma_{xy}\vec{i}\vec{j}+\sigma_{yx}\vec{j}\vec{i}+\sigma_{yy}\vec{j}\vec{j})=$$

$$\frac{\partial\sigma_{xx}}{\partial y}\vec{j}\cdot\vec{i}\vec{i}+\frac{\partial\sigma_{xy}}{\partial y}\vec{j}\cdot\vec{i}\vec{j}+\frac{\partial\sigma_{yx}}{\partial y}\vec{j}\cdot\vec{j}\vec{i}+\frac{\partial\sigma_{yy}}{\partial y}\vec{j}\cdot\vec{j}\vec{j}=$$

$$\frac{\partial\sigma_{xx}}{\partial y}0\vec{i}+\frac{\partial\sigma_{xy}}{\partial y}0\vec{j}+\frac{\partial\sigma_{yx}}{\partial y}1\vec{i}+\frac{\partial\sigma_{yy}}{\partial y}1\vec{j}=\frac{\partial\sigma_{yx}}{\partial y}\vec{i}+\frac{\partial\sigma_{yy}}{\partial y}\vec{j}\,.$$

By combining all the terms

$$\nabla\cdot\vec{\vec{\sigma}}+\vec{f}=(\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{yx}}{\partial y}+f_x)\vec{i}+(\frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+f_y)\vec{j}=0\,.$$

A vector vanishes if all its components vanish, so

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{yx}}{\partial y}+f_x=0 \quad\text{and}\quad \frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+f_y=0\,. \leftarrow$$

The small strain measure $\vec{\vec{\varepsilon}}$ is obtained as the symmetric part of displacement gradient (a tensor then). Use the definition to find the components of the strain tensor in a Cartesian coordinate system when $\nabla = \vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y$ and $\vec{u} = u_x\vec{i} + u_y\vec{j}$.

Solution

In Cartesian system, $\nabla = \vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y$ and $\vec{u} = u_x\vec{i} + u_y\vec{j}$, therefore

$$\nabla\vec{u} = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y})(u_x\vec{i} + u_y\vec{j}) = \vec{i}\vec{i}\frac{\partial u_x}{\partial x} + \vec{i}\vec{j}\frac{\partial u_y}{\partial x} + \vec{j}\vec{i}\frac{\partial u_x}{\partial y} + \vec{j}\vec{j}\frac{\partial u_y}{\partial y}) \Rightarrow$$

$$(\nabla\vec{u})_c = \vec{i}\vec{i}\frac{\partial u_x}{\partial x} + \vec{j}\vec{i}\frac{\partial u_y}{\partial x} + \vec{i}\vec{j}\frac{\partial u_x}{\partial y} + \vec{j}\vec{j}\frac{\partial u_y}{\partial y}$$

giving

$$\vec{\vec{\varepsilon}} = \frac{1}{2}[\nabla\vec{u} + (\nabla\vec{u})_c] = \vec{i}\vec{i}\frac{\partial u_x}{\partial x} + \vec{j}\vec{j}\frac{\partial u_y}{\partial y} + \vec{i}\vec{j}\frac{1}{2}(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}) + \vec{j}\vec{i}\frac{1}{2}(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}).$$

Therefor $\vec{\vec{\varepsilon}} = \begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}^{\mathrm{T}} \begin{bmatrix}\varepsilon_{xx} & \varepsilon_{xy}\\ \varepsilon_{yx} & \varepsilon_{yy}\end{bmatrix} \begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix} = \begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}^{\mathrm{T}} \begin{bmatrix}\frac{\partial u_x}{\partial x} & \frac{1}{2}(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x})\\ \frac{1}{2}(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}) & \frac{\partial u_y}{\partial y}\end{bmatrix} \begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}$ so

$$\begin{bmatrix}\varepsilon_{xx} & \varepsilon_{xy}\\ \varepsilon_{yx} & \varepsilon_{yy}\end{bmatrix} = \begin{bmatrix}\frac{\partial u_x}{\partial x} & \frac{1}{2}(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x})\\ \frac{1}{2}(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}) & \frac{\partial u_y}{\partial y}\end{bmatrix}. \leftarrow$$

Find the solution to the boundary value problem

$$\frac{dQ}{dx}-\rho gA=0\,,\ \ \frac{dM}{dx}-Q=0\,,\ M=EI\frac{d\theta}{dx},\ \text{ and }\ Q=GA(\frac{dw}{dx}+\theta)\quad \text{in}\quad (0,L)\,,$$

$$w=M=0 \quad \text{on} \quad \{0,L\}$$

for a simply supported beam loaded by its own weight. Cross sectional area A, second moment af area I, Young's modulus E, shear modulus G, density of the material ρ, and acceleration by gravity g are constants. Use repeated integrations.

Solution

The first order equation set for the beam bending

$$\frac{dQ}{dx}-\rho gA=0\,,\ \ \frac{dM}{dx}-Q=0\,,\ M=EI\frac{d\theta}{dx},\ \text{ and }\ Q=GA(\frac{dw}{dx}+\theta)\quad \text{in}\quad (0,L)\,,$$

Is composed of two equilibrium equations and constitutive equations for the bending moments and shear force (stress resultants on a cross-section). Boundary conditions

$$w=\theta=0 \quad \text{on} \quad \{0,L\}$$

describe a simple support at both ends. Let us integrate the equations one-by-one and denote the integration constants are denoted by *a*, *b*, *c*, *d* etc. Let us also apply the boundary conditions as soon as possible.

$$\frac{dQ}{dx}-\rho gA=0 \quad\Leftrightarrow\quad \frac{dQ}{dx}=\rho gA \quad\Leftrightarrow\quad Q(x)=\rho gAx+a\,,$$

$$\frac{dM}{dx}-Q=0 \quad\Rightarrow\quad \frac{dM}{dx}=\rho gAx+a \quad\Leftrightarrow\quad M(x)=\rho gA\frac{1}{2}x^2+ax+b\,,$$

According to the boundary conditions $M(0)=M(L)=0$:

$$M(0)=b=0 \ \text{ and }\ M(L)=\rho gA\frac{1}{2}L^2+aL+b=0 \quad\Leftrightarrow\quad b=0 \ \text{ and }\ a=-\rho gA\frac{1}{2}L\,.$$

Therefore $M(x)=\frac{1}{2}\rho gA(x^2-Lx)$ and $Q(x)=\frac{1}{2}\rho gA(2x-L)$. ←

Let us continue integrations with the constitutive equations

$$M=EI\frac{d\theta}{dx} \quad\Rightarrow\quad \frac{d\theta}{dx}=\frac{\rho gA}{2EI}(x^2-Lx) \quad\Leftrightarrow\quad \theta=\frac{\rho gA}{2EI}(\frac{1}{3}x^3-L\frac{1}{2}x^2)+c\,,$$

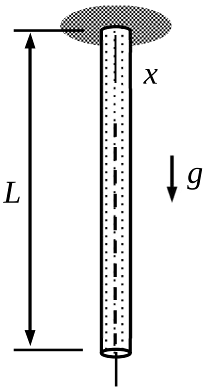
$$Q=GA(\frac{dw}{dx}+\theta) \quad\Rightarrow\quad \frac{dw}{dx}=\frac{\rho g}{2G}(2x-L)-\frac{\rho gA}{2EI}(\frac{1}{3}x^3-L\frac{1}{2}x^2)+c \quad\Leftrightarrow$$
$$w=\frac{\rho g}{2G}(x^2-xL)-\frac{\rho gA}{2EI}(\frac{1}{12}x^4-L\frac{1}{6}x^3)+cx+d\,.$$

According to the boundary conditions $w(0) = w(L) = 0$:

$$d = 0 \quad \text{and} \quad \frac{\rho g A}{24EI} L^4 + cL + d = 0 \quad \Leftrightarrow \quad d = 0 \quad \text{and} \quad c = -\frac{\rho g A}{24EI} L^3 .$$

Therefore

$$\theta(x) = \frac{\rho g A}{2EI} (\frac{1}{3} x^3 - \frac{1}{2} L x^2 - \frac{1}{12} L^3) \quad \text{and} \quad w(x) = \frac{\rho g}{2G} (x^2 - xL) - \frac{\rho g A}{2EI} (\frac{1}{12} x^4 - \frac{1}{6} L x^3 + \frac{1}{12} L^3 x) . \leftarrow$$



Consider a bar of length L loaded by its own weight. Determine the displacements at the mid-point and at the free end using the continuum model. Cross-sectional area A, acceleration by gravity g, and material properties E and ρ are constants.

Solution

In stationary case, the continuum model for the problem is given by equations

$$EA\frac{d^2u}{dx^2}+\rho Ag=0 \quad x\in]0,L[\ , \quad u=0 \quad x=0, \text{ and } \ EA\frac{du}{dx}=0 \quad x=L \, .$$

Repeated integrations in the differential equation give the generic solution containing two integration constants:

$$\frac{d^2u}{dx^2}=-\frac{\rho g}{E} \ \Rightarrow \ \frac{du}{dx}=-\frac{\rho g}{E}x+a \ \Rightarrow \ u=-\frac{\rho g}{E}\frac{1}{2}x^2+ax+b \, .$$

Then, substituting the generic solution into the boundary conditions

$$u(0)=b=0 \ \text{ and } \ \frac{du}{dx}(L)=-\frac{\rho g}{E}L+a=0 \quad \Leftrightarrow \quad a=\frac{\rho g}{E}L \ \text{ and } \ b=0 \, .$$

Solution to the problem as function of x gives also the values at the center point and end point

$$u(x)=\frac{\rho g}{E}x(L-\frac{1}{2}x) \quad \Rightarrow \quad u(\frac{L}{2})=\frac{3}{8}\frac{\rho gL^2}{E} \ \text{ and } \ u(L)=\frac{\rho gL^2}{2E} \, .$$