Name________________________ Student number________________

Assignment 3

Derive the component form of $\nabla\cdot\vec{\sigma}+\vec{f}=0$ in the polar coordinate system. Assume that the components of stress do not depend on angle ϕ. In the polar coordinate system, the component forms of stress, external force, and gradient operator, and derivatives of the basis vectors are given by

$$\vec{\sigma}=\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}\sigma_{rr} & \sigma_{r\phi}\\ \sigma_{\phi r} & \sigma_{\phi\phi}\end{bmatrix}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix},\quad \vec{f}=\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}f_r\\ f_\phi\end{Bmatrix},\quad \nabla=\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\partial/\partial r\\ \partial/(r\partial\phi)\end{Bmatrix},$$

$$\frac{\partial}{\partial\phi}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix}=\begin{Bmatrix}\vec{e}_\phi\\ -\vec{e}_r\end{Bmatrix},\quad \frac{\partial}{\partial r}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix}=0.$$

Solution template

In manipulation of vector expression containing vectors and tensors, it is important to remember that tensor ($\otimes$), cross ($\times$), inner ($\cdot$) products are non-commutative (order may matter). The basis vectors of a curvilinear coordinate system are not constants which should be taken into account if gradient operator is a part of expression. Otherwise, simplifying an expression or finding a specific form in a given coordinate system is a straightforward (sometimes tedious) exercise. For simplicity of presentation, outer (tensor) products like $\vec{a}\otimes\vec{b}$ are denoted by $\vec{a}\vec{b}$. Otherwise, the usual rules of algebra apply: Gradient operator ∇ acts on everything on its right-hand side, the operator is treated like a vector etc.

The task is to simplify the vector equation

$$\nabla\cdot\vec{\sigma}+\vec{f}=(\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\phi\frac{\partial}{r\partial\phi})\cdot(\sigma_{rr}\vec{e}_r\vec{e}_r+\sigma_{r\phi}\vec{e}_r\vec{e}_\phi+\sigma_{\phi r}\vec{e}_\phi\vec{e}_r+\sigma_{\phi\phi}\vec{e}_\phi\vec{e}_\phi)+(f_r\vec{e}_r+f_\phi\vec{e}_\phi)=0$$

to see the component forms. Let us consider the effect of the first term of the displacement gradient to stress

$$\vec{e}_r\frac{\partial}{\partial r}\cdot(\sigma_{rr}\vec{e}_r\vec{e}_r+\sigma_{r\phi}\vec{e}_r\vec{e}_\phi+\sigma_{\phi r}\vec{e}_\phi\vec{e}_r+\sigma_{\phi\phi}\vec{e}_\phi\vec{e}_\phi)=$$

$\vec{e}_r\cdot($____$)=$

$\vec{e}_r($________________$)+\vec{e}_\phi($________________$)$.

Then the same for the second term of the displacement gradient. As the stress components do not depend on ϕ (by assumption)

$$\vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial \phi}\cdot(\sigma_{rr}\vec{e}_r\vec{e}_r + \sigma_{r\phi}\vec{e}_r\vec{e}_\phi + \sigma_{\phi r}\vec{e}_\phi\vec{e}_r + \sigma_{\phi\phi}\vec{e}_\phi\vec{e}_\phi) =$$

$$\vec{e}_\phi \frac{1}{r}\cdot(\underline{\hspace{30em}}) =$$

$$\vec{e}_r(\underline{\hspace{9em}}) + \vec{e}_\phi(\underline{\hspace{9em}}) .$$

Finally combining everything

$$\nabla\cdot\vec{\vec{\sigma}} + \vec{f} = \vec{e}_r(\underline{\hspace{10em}}) + \vec{e}_\phi(\underline{\hspace{10em}}) = 0$$

Therefore, the two equilibrium equations are given by

$$\begin{Bmatrix} \underline{\hspace{14em}} \\ \underline{\hspace{14em}} \end{Bmatrix} = 0 .$$