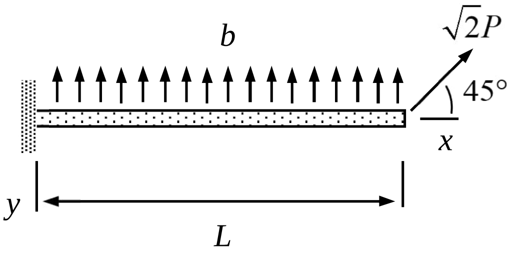
MEC-E8003 Beam, plate and shell models, examples 4

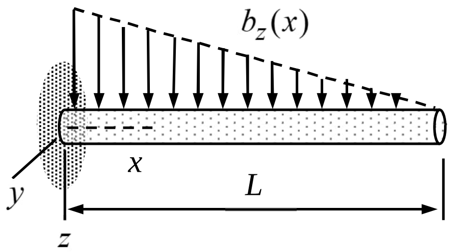
1. Consider the xy – plane beam of length L shown. Material properties E and G, cross-section properties A, I are constants, and $S=0$. Write down the boundary value problem according to the Timoshenko beam model in terms of the axial displacement $u(x)$, transverse displacement $v(x)$, and rotation $\psi(x)$.



Answer

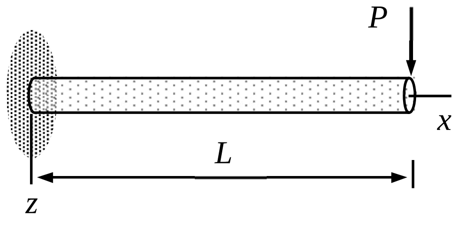
$$\begin{Bmatrix} EA\dfrac{d^2u}{dx^2} \\ GA(\dfrac{d^2v}{dx^2}-\dfrac{d\psi}{dx})-b \\ EI\dfrac{d^2\psi}{dx^2}+GA(\dfrac{dv}{dx}-\psi) \end{Bmatrix}=0 \text{ in } (0,L), \quad \begin{Bmatrix} u \\ v \\ \psi \end{Bmatrix}=0 \quad x=0, \quad \begin{Bmatrix} EA\dfrac{du}{dx}-P \\ GA(\dfrac{dv}{dx}-\psi)+P \\ EI\dfrac{d\psi}{dx} \end{Bmatrix}=0 \quad x=L.$$

2. Find the stress resultants Q_z, M_y and the transverse displacement w of the cantilever beam shown according to the Bernoulli beam model. Problem parameters E, G, A, $S=0$ and I are constants and the distributed force $b_z=b(1-x/L)$. Start with the generic equilibrium and constitutive equations for the beam model.



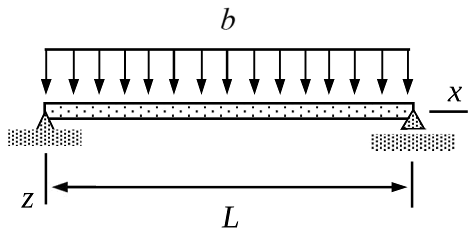
Answer $Q_z=\dfrac{b}{2L}(L-x)^2$, $M_y=-\dfrac{b}{6L}(L-x)^3$, $w=\dfrac{b}{120LEI}[L^5-(L-x)^5-5L^4x]$

3. Find the displacement and rotation of the xz – plane cantilever beam of the figure according to the Bernoulli beam model. Problem parameters E, G, A, $S=0$ and I are constants.



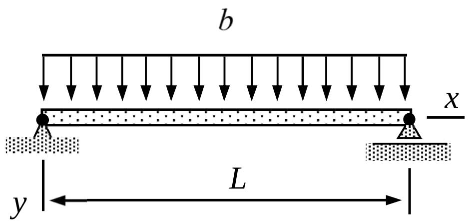
Answer $w(x)=\dfrac{P}{6EI}x^2(3L-x)$, $\theta(x)=\dfrac{P}{2EI}x(x-2L)$

4. Consider the simply supported (plane) beam of the figure of length L. Material properties E and G, cross-section properties A, $S=0$, I, and loading b are constants. Determine the deflection and rotation at the mid-point $x=L/2$ according to the Timoshenko beam model.



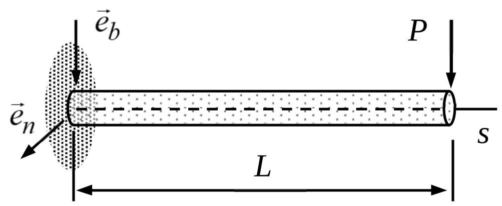
Answer $w(L/2)=\dfrac{bL^2}{8AG}+\dfrac{5bL^4}{384EI}$, $\theta(L/2)=0$

5. Consider the simply supported $xy-$plane beam of length L shown. Material properties E and G, cross-section properties A, $S=0$, I, and loading b are constants. Write down the equilibrium equations, constitutive equations, and boundary conditions according to the Bernoulli beam model. After that, solve the equations for the transverse displacement.



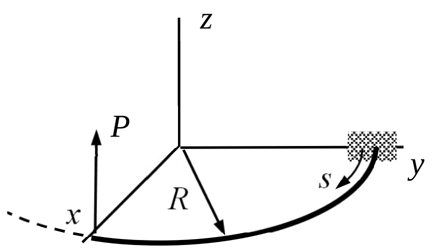
Answer $v(x)=\dfrac{b}{24EI}x(L^3-2Lx^2+x^3)$

6. Consider the cantilever beam of the figure of zero curvature and torsion $\tau=4\pi/L$. Write down the equilibrium equations and the boundary conditions at the free end in the $(s,n,b)-$ coordinate system. Also, solve the boundary value problem for the stress resultants as functions of s.



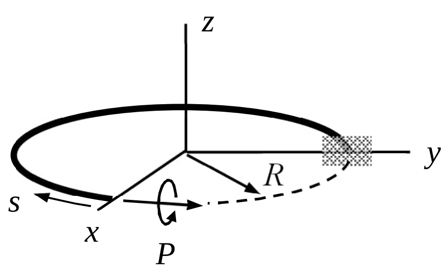
Answer $Q_n=P\sin(\tau s)$, $Q_b=P\cos(\tau s)$, $M_n=P(s-L)\cos(\tau s)$, $M_b=P(L-s)\sin(\tau s)$

7. Consider the curved beam of the figure forming a 90-degree circular segment of radius R in the horizontal plane. Find the stress resultants $N(s)$, $Q_n(s)$, $Q_b(s)$, $T(s)$, $M_n(s)$, and $M_b(s)$. Use the equilibrium equations of the beam model in the $(s,n,b)-$coordinate system.



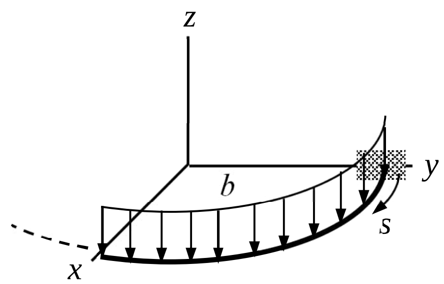
Answer $\begin{Bmatrix} N \\ Q_n \\ Q_b \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ -P \end{Bmatrix}$, $\begin{Bmatrix} T \\ M_n \\ M_b \end{Bmatrix}=\begin{Bmatrix} PR[\sin(s/R)-1] \\ RP\cos(s/R) \\ 0 \end{Bmatrix}$

8. Consider a curved beam forming ¾ of a full circle of radius R in the horizontal plane. The given torque of magnitude P is acting on the free end as shown. Write down the equilibrium equations and boundary conditions for the *stress resultants* and solve the equations for $N(s)$, $Q_n(s)$, $Q_b(s)$, $T(s)$, $M_n(s)$, and $M_b(s)$.



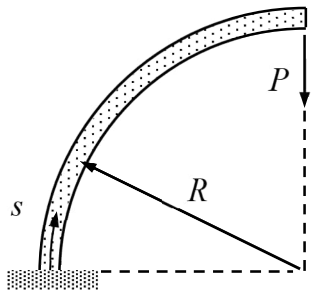
Answer $\begin{Bmatrix} N \\ Q_n \\ Q_b \end{Bmatrix}=0$, $\begin{Bmatrix} T \\ M_n \\ M_b \end{Bmatrix}=\begin{Bmatrix} P\cos(s/R) \\ -P\sin(s/R) \\ 0 \end{Bmatrix}$

9. Consider the curved beam of the figure forming a 90-degree circular segment of radius R in the horizontal plane. Find the stress resultants $N(s)$, $Q_n(s)$, $Q_b(s)$, $T(s)$, $M_n(s)$, and $M_b(s)$. Use the equilibrium equations of the beam model in the $(s,n,b)-$coordinate system. The distributed constant load of magnitude b is acting to the negative direction of the $z-$axis.



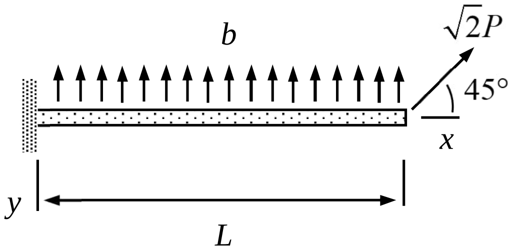
Answer $Q_b = b(L-s)$, $T = -bR^2 \cos(\frac{s}{R}) + bR(L-s)$, $M_n = bR^2[\sin(\frac{s}{R}) - 1]$

10. Consider the curved planar Timoshenko beam shown in the figure. Write down the boundary value problem for $u(s)$, $v(s)$, $\psi(s)$, $N(s)$, $Q_n(s)$, and $M_b(s)$. Also, solve the equations for displacement $v(s)$ and rotation $\psi(s)$. The properties of the cross-section A, $S_b = 0$, $I_{bb} = I$, and material parameters E, G are constants. Curvature is $\kappa = 1/R$ and torsion $\tau = 0$.



Answer $v(s) = \frac{P}{2}(\frac{R^2}{EI} + \frac{1}{GA} + \frac{1}{EA}) s \sin(\frac{s}{R})$, $\psi(s) = \frac{PR^2}{EI} \sin(\frac{s}{R})$

Consider the $xy-$plane beam of length L shown. Material properties E and G, cross-section properties A, I are constants, and $S=0$. Write down the boundary value problem according to the Timoshenko beam model in terms of the axial displacement $u(x)$, transverse displacement $v(x)$, and rotation $\psi(x)$.



Solution

Timoshenko beam equations in the Cartesian system are

$$\begin{Bmatrix} \dfrac{dN}{dx}+b_x \\ \dfrac{dQ_y}{dx}+b_y \\ \dfrac{dQ_z}{dx}+b_z \end{Bmatrix}=0,\quad \begin{Bmatrix} N \\ Q_y \\ Q_z \end{Bmatrix}=\begin{Bmatrix} EA\dfrac{du}{dx}-ES_z\dfrac{d\psi}{dx}+ES_y\dfrac{d\theta}{dx} \\ GA(\dfrac{dv}{dx}-\psi)-GS_y\dfrac{d\phi}{dx} \\ GA(\dfrac{dw}{dx}+\theta)+GS_z\dfrac{d\phi}{dx} \end{Bmatrix}$$

$$\begin{Bmatrix} \dfrac{dT}{dx}+c_x \\ \dfrac{dM_y}{dx}-Q_z+c_y \\ \dfrac{dM_z}{dx}+Q_y+c_z \end{Bmatrix}=0,\quad \begin{Bmatrix} T \\ M_y \\ M_z \end{Bmatrix}=\begin{Bmatrix} -GS_y(\dfrac{dv}{dx}-\psi)+GS_z(\dfrac{dw}{dx}+\theta)+GI_{rr}\dfrac{d\phi}{dx} \\ ES_y\dfrac{du}{dx}-EI_{zy}\dfrac{d\psi}{dx}+EI_{yy}\dfrac{d\theta}{dx} \\ -ES_z\dfrac{du}{dx}+EI_{zz}\dfrac{d\psi}{dx}-EI_{yz}\dfrac{d\theta}{dx} \end{Bmatrix}$$

In $xy-$plane problem, the non-zero displacements and rotations are u, v, and ψ. Geometrical properties of the cross-section are A, $S_z=S=0$, $I_{zz}=I$. External distributed forces are $b_x=0$, $b_y=-b$, $b_z=0$, $c_x=c_y=c_z=0$. With these selections, equilibrium equations, constitutive equations, and boundary conditions of the planar problem take the forms

$$\begin{Bmatrix} \dfrac{dN}{dx} \\ \dfrac{dQ_y}{dx}-b \\ \dfrac{dM_z}{dx}+Q_y \end{Bmatrix}=0 \quad\text{and}\quad \begin{Bmatrix} N \\ Q_y \\ M_z \end{Bmatrix}=\begin{Bmatrix} EA\dfrac{du}{dx} \\ GA(\dfrac{dv}{dx}-\psi) \\ EI_{zz}\dfrac{d\psi}{dx} \end{Bmatrix} \quad\text{in}\quad (0,L),$$

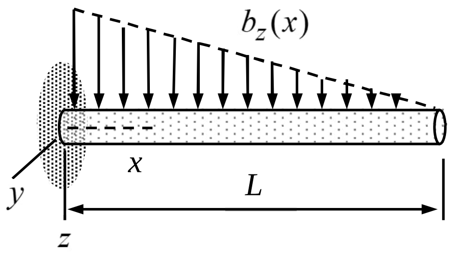
$$\begin{Bmatrix} u \\ v \\ \psi \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad\text{at}\quad x=0 \quad\text{and}\quad \begin{Bmatrix} N-P \\ Q_y+P \\ M_z \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad\text{at}\quad x=L.$$

To get a boundary value problem in terms of the axial displacement $u(x)$, transverse displacement $v(x)$, and rotation $\psi(x)$, stress resultants are eliminated to end up with

$$\begin{Bmatrix} EA\dfrac{d^2u}{dx^2} \\ GA(\dfrac{d^2v}{dx^2}-\dfrac{d\psi}{dx})-b \\ EI\dfrac{d^2\psi}{dx^2}+GA(\dfrac{dv}{dx}-\psi) \end{Bmatrix} = 0 \quad \text{in} \quad (0, L),$$

$$\begin{Bmatrix} u \\ v \\ \psi \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \text{ at } \ x=0 \quad \text{and} \quad \begin{Bmatrix} EA\dfrac{du}{dx}-P \\ GA(\dfrac{dv}{dx}-\psi)+P \\ EI\dfrac{d\psi}{dx} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \text{ at } \ x=L.$$

Find the stress resultants Q_z, M_y and the transverse displacement w of the cantilever beam shown according to the Bernoulli beam model. Problem parameters E, G, A, $S=0$ and I are constants and the distributed force $b_z = b(1-x/L)$. Start with the generic equilibrium and constitutive equations for the beam model.



Solution

Timoshenko beam equations boil down to the Bernoulli beam equations when the Bernoulli constraints $dv/dx-\psi=0$ and $dw/dx+\theta=0$ are applied there. If $S_y = S_z = I_{yz} = 0$ one may just replace the constitutive equations for the shear stress resultants by the Bernoulli constraints to get the Bernoulli model equilibrium and constitutive equations (elimination with the Bernoulli constraints and constitutive equations gives the well-known fourth order beam equation of textbooks)

$$\begin{Bmatrix} \dfrac{dN}{dx}+b_x \\ \dfrac{dQ_y}{dx}+b_y \\ \dfrac{dQ_z}{dx}+b_z \end{Bmatrix}=0,\quad \begin{Bmatrix} \dfrac{dT}{dx}+c_x \\ \dfrac{dM_y}{dx}-Q_z+c_y \\ \dfrac{dM_z}{dx}+Q_y+c_z \end{Bmatrix}=0,\quad \begin{Bmatrix} N \\ 0 \\ 0 \end{Bmatrix}=\begin{Bmatrix} EA\dfrac{du}{dx} \\ \dfrac{dv}{dx}-\psi \\ \dfrac{dw}{dx}+\theta \end{Bmatrix},\quad \begin{Bmatrix} T \\ M_y \\ M_z \end{Bmatrix}=\begin{Bmatrix} GI_{rr}\dfrac{d\phi}{dx} \\ EI_{yy}\dfrac{d\theta}{dx} \\ EI_{zz}\dfrac{d\psi}{dx} \end{Bmatrix}.$$

In the $xz-$plane problem of the figure, the non-zero displacements and rotations are u, w, and θ. The distributed forces and moments vanish except b_z. Therefore, the Bernoulli beam boundary value problem takes the form

$$\begin{Bmatrix} \dfrac{dN}{dx} \\ \dfrac{dQ_z}{dx}+b_z \\ \dfrac{dM_y}{dx}-Q_z \end{Bmatrix}=0 \quad\text{and}\quad \begin{Bmatrix} N \\ 0 \\ M_y \end{Bmatrix}=\begin{Bmatrix} EA\dfrac{du}{dx} \\ \dfrac{dw}{dx}+\theta \\ EI\dfrac{d\theta}{dx} \end{Bmatrix} \text{ in } (0,L),\quad \begin{Bmatrix} N \\ Q_z \\ M_y \end{Bmatrix}=0 \text{ at } x=L,\quad \begin{Bmatrix} u \\ w \\ \theta \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \text{ at } x=0.$$

The equations for Q_z, M_y and the transverse displacement w can be solved one at a time. As the beam is statically determined, let us start with the equilibrium equations and the boundary conditions at the free end.

$$\frac{dQ_z}{dx}=-\frac{b}{L}(L-x) \text{ in } (0,L) \text{ and } Q_z(L)=0 \quad\Rightarrow\quad Q_z(x)=\frac{b}{2L}(L-x)^2,$$

$$\frac{dM_y}{dx}=Q_z=\frac{b}{2L}(L-x)^2 \text{ in } (0,L) \text{ and } M_y(L)=0 \quad\Rightarrow\quad M_y(x)=-\frac{b}{6L}(L-x)^3.$$

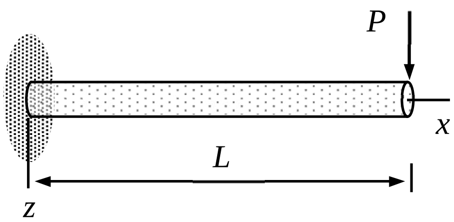
Knowing the stress resultants, displacements and rotations follow from the constitutive equations and the boundary conditions at $x=0$:

$$\frac{d\theta}{dx}=\frac{M_y}{EI}=-\frac{b}{6LEI}(L-x)^3 \quad \text{in} \quad (0,L) \quad \text{and} \quad \theta(0)=0 \quad \Rightarrow \quad \theta(x)=\frac{b}{24LEI}[(L-x)^4-L^4],$$

$$\frac{dw}{dx}=-\theta=-\frac{b}{24LEI}[(L-x)^4-L^4] \quad \text{in} \quad (0,L) \quad \text{and} \quad w(0)=0 \quad \Rightarrow$$

$$w(x)=-\frac{b}{120LEI}[L^5-(L-x)^5-5L^4x]=-\frac{bx^2}{120LEI}(-10L^3+10L^2x-5Lx^2+x^3).$$

Find the displacement and rotation of the xz–plane cantilever beam of the figure according to the Bernoulli beam model. Problem parameters E, G, A $S=0$ and I are constants.



Solution

Timoshenko beam equations boil down to the Bernoulli beam equations when the Bernoulli constraints $dv/dx-\psi=0$ and $dw/dx+\theta=0$ are applied there. In practice, the constraints are used to eliminate the rotation components θ and ψ from the constitutive equations. Then the corresponding shear forces become constraint forces having no constitutive equations. If $S_y=S_z=I_{yz}=0$ (the constitutive equations for the shear forces are replaced by the Bernoulli constraints)

$$\begin{Bmatrix} \dfrac{dN}{dx}+b_x \\ \dfrac{dQ_y}{dx}+b_y \\ \dfrac{dQ_z}{dx}+b_z \end{Bmatrix}=0\,,\quad \begin{Bmatrix} N \\ 0 \\ 0 \end{Bmatrix}=\begin{Bmatrix} EA\dfrac{du}{dx} \\ \dfrac{dv}{dx}-\psi \\ \dfrac{dw}{dx}+\theta \end{Bmatrix},\quad \begin{Bmatrix} \dfrac{dT}{dx}+c_x \\ \dfrac{dM_y}{dx}-Q_z+c_y \\ \dfrac{dM_z}{dx}+Q_y+c_z \end{Bmatrix}=0\,,\quad \begin{Bmatrix} T \\ M_y \\ M_z \end{Bmatrix}=\begin{Bmatrix} GI_{rr}\dfrac{d\phi}{dx} \\ EI_{yy}\dfrac{d\theta}{dx} \\ EI_{zz}\dfrac{d\psi}{dx} \end{Bmatrix}=\begin{Bmatrix} GI_{rr}\dfrac{d\phi}{dx} \\ -EI_{yy}\dfrac{d^2w}{dx^2} \\ EI_{zz}\dfrac{d^2v}{dx^2} \end{Bmatrix}$$

giving, after elimination of the shear forces from the equilibrium equations and omitting the equations not needed in the displacement calculations, the usual forms of textbooks

$$\begin{Bmatrix} \dfrac{dN}{dx}+b_x \\ \dfrac{dT}{dx}+c_x \end{Bmatrix}=0\,,\quad \begin{Bmatrix} N \\ T \end{Bmatrix}=\begin{Bmatrix} EA\dfrac{du}{dx} \\ GI_{rr}\dfrac{d\phi}{dx} \end{Bmatrix} \text{ in } \Omega\,,\quad n\begin{Bmatrix} N \\ T \end{Bmatrix}=\begin{Bmatrix} \underline{N} \\ \underline{T} \end{Bmatrix} \text{ or } \begin{Bmatrix} u \\ \phi \end{Bmatrix}=\begin{Bmatrix} \underline{u} \\ \underline{\phi} \end{Bmatrix} \text{ on } \partial\Omega$$

$$\begin{Bmatrix} \dfrac{d^2M_z}{dx^2}+\dfrac{dc_z}{dx}-b_y \\ \dfrac{d^2M_y}{dx^2}+\dfrac{dc_y}{dx}+b_z \end{Bmatrix}=0\,,\quad \begin{Bmatrix} M_z \\ M_y \end{Bmatrix}=\begin{Bmatrix} EI_{zz}\dfrac{d^2v}{dx^2} \\ -EI_{yy}\dfrac{d^2w}{dx^2} \end{Bmatrix} \text{ in } \Omega\,,\quad \text{(altogether four conditions are needed)}$$

$$n\begin{Bmatrix} M_z \\ M_y \end{Bmatrix}=\begin{Bmatrix} \underline{M}_z \\ \underline{M}_y \end{Bmatrix} \text{ or } \begin{Bmatrix} \dfrac{dv}{dx} \\ \dfrac{dw}{dx} \end{Bmatrix}=\begin{Bmatrix} \underline{\psi} \\ -\underline{\theta} \end{Bmatrix} \text{ and } n\begin{Bmatrix} \dfrac{dM_z}{dx}+c_z \\ \dfrac{dM_y}{dx}+c_y \end{Bmatrix}=\begin{Bmatrix} -\underline{Q}_y \\ \underline{Q}_z \end{Bmatrix} \text{ or } \begin{Bmatrix} v \\ w \end{Bmatrix}=\begin{Bmatrix} \underline{v} \\ \underline{w} \end{Bmatrix} \text{ on } \partial\Omega\,.$$

In the xz–plane problem of the figure, the non-zero displacements and rotations are $u\,,w$ and the geometrical properties of the cross-section are A, $I_{yy}=I$. External distributed forces and moment vanish. Furthermore, axial displacement and force resultant clearly vanish. Therefore, the Bernoulli beam equations imply the boundary value problem

$$\frac{d^2 M_y}{dx^2} = 0 \ , \ M_y = -EI_{yy} \frac{d^2 w}{dx^2} \text{ in } \Omega \quad \text{and} \quad w(0) = \frac{dw}{dx}(0) = 0 \ , \ M_y(L) = 0 \text{, and } \frac{dM_y}{dx}(L) = P$$

for the transverse displacement and the bending moment. Eliminating the bending moment gives finally

$$\frac{d^4 w}{dx^4} = 0 \ \text{ in } \Omega, \ \ w(0) = \frac{dw}{dx}(0) = 0, \ \ \frac{d^2 w}{dx^2}(L) = 0 \text{, and } \frac{d^3 w}{dx^3}(L) = -\frac{P}{EI}.$$

The generic solution to the differential equation $w = a + bx + cx^2 + dx^3$ contains 4 parameters to be determined from the boundary conditions

$$w(0) = a = 0, \ \frac{dw}{dx}(0) = b = 0, \ \frac{d^2 w}{dx^2}(L) = 2c + 6dL = 0, \ \frac{d^3 w}{dx^3}(L) = 6d = -\frac{P}{EI} \quad \Rightarrow$$

$$w = \frac{P}{6EI} x^2 (3L - x) \ \text{ and } \ \theta = -\frac{dw}{dx} = \frac{P}{2EI} x(x - 2L). \text{ (from the Bernoulli constraint)}$$

Alternatively, the equations can be solved easily one-by-one in their original forms as the problem is statically determined

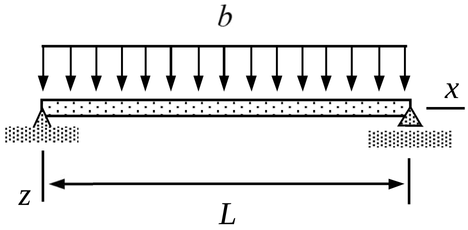
$$\frac{dQ_z}{dx} = 0 \ \text{ in } \ (0, L) \quad \text{and} \quad Q_z(L) = P \quad \Rightarrow \quad Q_z(x) = P \ ,$$

$$\frac{dM_y}{dx} = Q_z = P \ \text{ in } \ (0, L) \ \text{ and } \ M_y(L) = 0 \quad \Rightarrow \quad M_y(x) = P(x - L) \ ,$$

$$\theta' = \frac{M_y}{EI_{yy}} = \frac{P}{EI_{yy}}(x - L) \ \text{ in } \ (0, L) \ \text{ and } \ \theta(0) = 0 \quad \Rightarrow \quad \theta(x) = \frac{P}{EI_{yy}}(\frac{1}{2}x^2 - Lx) \ ,$$

$$\frac{dw}{dx} = -\theta = \frac{P}{EI_{yy}}(Lx - \frac{1}{2}x^2) \ \text{ in } \ (0, L) \ \text{ and } \ w(0) = 0 \quad \Rightarrow \quad w(0) = \frac{P}{EI_{yy}}(\frac{1}{2}Lx^2 - \frac{1}{6}x^3) \ .$$

Consider the simply supported (plane) beam of the figure of length L. Material properties E and G, cross-section properties A, $S=0$, I, and loading b are constants. Determine the deflection and rotation at the mid-point $x=L/2$ according to the Timoshenko beam model.



Solution

Beam equations of the Cartesian system

$$\begin{Bmatrix} \dfrac{dN}{dx}+b_x \\ \dfrac{dQ_y}{dx}+b_y \\ \dfrac{dQ_z}{dx}+b_z \end{Bmatrix}=0,\ \begin{Bmatrix} N \\ Q_y \\ Q_z \end{Bmatrix}=\begin{Bmatrix} EA\dfrac{du}{dx}-ES_z\dfrac{d\psi}{dx}+ES_y\dfrac{d\theta}{dx} \\ GA(\dfrac{dv}{dx}-\psi)-GS_y\dfrac{d\phi}{dx} \\ GA(\dfrac{dw}{dx}+\theta)+GS_z\dfrac{d\phi}{dx} \end{Bmatrix}$$

$$\begin{Bmatrix} \dfrac{dT}{dx}+c_x \\ \dfrac{dM_y}{dx}-Q_z+c_y \\ \dfrac{dM_z}{dx}+Q_y+c_z \end{Bmatrix}=0,\ \begin{Bmatrix} T \\ M_y \\ M_z \end{Bmatrix}=\begin{Bmatrix} -GS_y(\dfrac{dv}{dx}-\psi)+GS_z(\dfrac{dw}{dx}+\theta)+GI_{rr}\dfrac{d\phi}{dx} \\ ES_y\dfrac{du}{dx}-EI_{zy}\dfrac{d\psi}{dx}+EI_{yy}\dfrac{d\theta}{dx} \\ -ES_z\dfrac{du}{dx}+EI_{zz}\dfrac{d\psi}{dx}-EI_{yz}\dfrac{d\theta}{dx} \end{Bmatrix}$$

are given in the formulae collection. In $xz-$plane problem, the non-zero displacements and rotations are u, w, and θ. Geometrical properties of the cross-section are A, $S_y=S=0$, $I_{yy}=I$. External distributed forces are $b_x=0$, $b_y=0$, $b_z=b$, $c_x=c_y=c_z=0$. By taking into account the equilibrium and constitutive equations of the planar problem and the corresponding boundary conditions of the problem definition, the boundary value problem becomes

$$\begin{Bmatrix} \dfrac{dN}{dx} \\ \dfrac{dQ_z}{dx}+b \\ \dfrac{dM_y}{dx}-Q_z \end{Bmatrix}=0,\ \begin{Bmatrix} N \\ Q_z \\ M_y \end{Bmatrix}=\begin{Bmatrix} EA\dfrac{du}{dx} \\ GA(\dfrac{dw}{dx}+\theta) \\ EI\dfrac{d\theta}{dx} \end{Bmatrix} \quad \text{in} \quad \Omega=(0,L),$$

$$\begin{Bmatrix} u \\ w \\ M_y \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{at} \quad x=0 \quad \text{and} \quad \begin{Bmatrix} N \\ w \\ M_y \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{at} \quad x=L.$$

Unless the problem is statically determined, the stress resultants are usually eliminated from equations to end up with a boundary value problem in terms of the displacement components and rotations. However, the equations can also be solved without eliminations one-by-one. First equilibrium equation and constitutive equation for the normal force

$$\frac{dN}{dx} = 0 \quad \text{in} \quad (0, L) \quad \text{and} \quad N(L) = 0 \quad \Rightarrow \quad N(x) = 0$$

$$EA\frac{du}{dx} = N = 0 \quad (0, L) \quad \text{and} \quad u(0) = 0 \quad \Rightarrow \quad u(x) = 0 \,.$$

After that, equilibrium equations for the shear force and bending moment. Notice that the integration constant for the shear force follows from the boundary conditions for the bending moment:

$$\frac{dQ_z}{dx} + b = 0 \quad \text{in} \quad (0, L) \quad \Rightarrow \quad Q_z = -bx + a \qquad (a = \frac{1}{2} bL)$$

$$\frac{dM_y}{dx} = Q_z = -bx + a \quad \text{in} \quad (0, L) \quad \text{and} \quad M_y(0) = M_y(L) = 0 \quad \Rightarrow \quad M_y(x) = \frac{1}{2} b(Lx - x^2) \,.$$

Finally, rotation and transverse displacement from the constitutive equations for the shear force and bending moment, Again, integration constant for the rotation follows from the boundary condition for displacement

$$\frac{d\theta}{dx} = \frac{M_y}{EI} \quad \text{in} \quad (0, L) \quad \Rightarrow \quad \theta(x) = \frac{b}{2EI}(L\frac{1}{2}x^2 - \frac{1}{3}x^3) + a \qquad (a = -\frac{bL^3}{24EI})$$

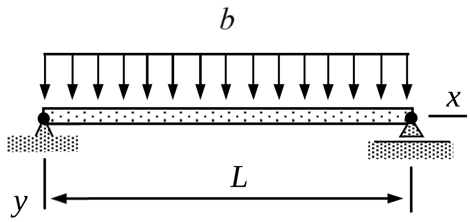
$$GA(\frac{dw}{dx} + \theta) = Q = \frac{1}{2} b(L - 2x) \quad \Rightarrow \quad w(x) = \frac{b}{2GA}(Lx - x^2) - \frac{b}{2EI}(L\frac{1}{6}x^3 - \frac{1}{12}x^4) + \frac{bL^3}{24EI} x \,.$$

$$w(x) = \frac{b}{2GA}(Lx - x^2) - \frac{b}{2EI}(L\frac{1}{6}x^3 - \frac{1}{12}x^4) + \frac{bL^3}{24EI} x \,.$$

Displacement and rotation at the center point

$$w(L/2) = \frac{bL^2}{8AG} + \frac{5bL^4}{384EI} \quad \text{and} \quad \theta(L/2) = 0 \,.$$

Consider the simply supported $xy-$plane beam of length L shown. Material properties E and G, cross-section properties A, $S=0$, I, and loading $\underline{b}$ are constants. Write down the equilibrium equations, constitutive equations, and boundary conditions according to the Bernoulli beam model. After that, solve the equations for the transverse displacement.



Solution

In $xy-$plane problem, the non-zero displacements and rotations are u, v, and ψ and the geometrical properties of the cross-section are A, $S_z=0$, $I_{zz}=I$. External distributed forces are $b_x=0$, $b_y=b$, $b_z=0$, $c_x=c_y=c_z=0$. With these, Timoshenko beam equations in the Cartesian system simplify

$$\begin{Bmatrix} \dfrac{dN}{dx} \\ \dfrac{dQ_y}{dx}+b \\ \dfrac{dM_z}{dx}+Q_y \end{Bmatrix}=0, \quad \begin{Bmatrix} N \\ Q_y \\ M_z \end{Bmatrix}=\begin{Bmatrix} EA\dfrac{du}{dx} \\ GA(\dfrac{dv}{dx}-\psi) \\ EI_{zz}\dfrac{d\psi}{dx} \end{Bmatrix}.$$

The equations of the Bernoulli model are obtained by replacing the constitutive equation for the shear force resultant by the corresponding Bernoulli constraint. By taking into account the boundary conditions of the problem definition, the boundary value problem becomes

$$\begin{Bmatrix} \dfrac{dN}{dx} \\ \dfrac{dQ_y}{dx}+b \\ \dfrac{dM_z}{dx}+Q_y \end{Bmatrix}=0, \quad \begin{Bmatrix} N \\ 0 \\ M_z \end{Bmatrix}=\begin{Bmatrix} EA\dfrac{du}{dx} \\ \dfrac{dv}{dx}-\psi \\ EI_{zz}\dfrac{d\psi}{dx} \end{Bmatrix} \text{ in } (0,L) \text{ and } \begin{Bmatrix} u \\ v \\ M_z \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \text{ at } x=0, \quad \begin{Bmatrix} N \\ v \\ M_z \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \text{ at } x=L.$$

The stress resultants are usually eliminated from equations to end up with a boundary value problem in terms of the displacement components and rotations. Alternatively, the equation set can be solved in its original form. First equilibrium equation and constitutive equation for the normal force

$$\frac{dN}{dx}=0 \text{ in } (0,L) \text{ and } N(L)=0 \quad \Rightarrow \quad N(x)=0,$$

$$EA\frac{du}{dx}=N=0 \quad (0,L) \text{ and } \quad u(0)=0 \quad \Rightarrow \quad u(x)=0.$$

After that, equilibrium equations for the shear force and bending moment. Notice that the integration constant for the shear force follows from the boundary conditions for the bending moment:

$$\frac{dQ_y}{dx} + b = 0 \quad \text{in} \quad (0,L) \quad \Rightarrow \quad Q_y = -bx + a \qquad (a = \frac{1}{2}bL)$$

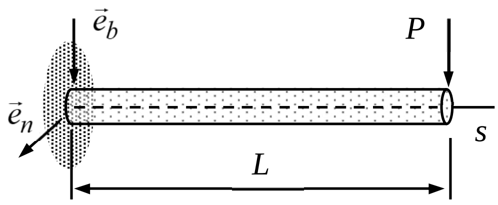
$$\frac{dM_z}{dx} = -Q_y = bx - a \quad \text{in} \quad (0,L) \quad \text{and} \quad M_z(0) = M_z(L) = 0 \quad \Rightarrow \quad M_z(x) = \frac{1}{2}b(x^2 - Lx)\,.$$

Finally, rotation and transverse displacement from the constitutive equations for the shear force and bending moment, Again, integration constant a for the rotation follows from the boundary condition for the displacement

$$\frac{d\psi}{dx} = \frac{M_z}{EI} = \frac{b}{2EI}(x^2 - Lx) \;\text{ in }\; (0,L) \quad \Rightarrow \quad \psi(x) = \frac{b}{2EI}(\frac{1}{3}x^3 - L\frac{1}{2}x^2) + a \qquad (a = \frac{bL^3}{24EI})$$

$$\frac{dv}{dx} = \psi = \frac{b}{2EI}(\frac{1}{3}x^3 - L\frac{1}{2}x^2) + a \;\Rightarrow\quad v(x) = \frac{b}{2EI}(\frac{1}{12}x^4 - L\frac{1}{6}x^3) + \frac{bL^3}{24EI}x\,.$$

Consider the cantilever beam of the figure of zero curvature and torsion $\tau = 4\pi / L$. Write down the equilibrium equations and the boundary conditions at the free end in the (s,n,b) system, and solve the boundary value problem for the stress resultants as functions of s.



Solution

In a statically determined case, stress resultants follow from the equilibrium equations and boundary conditions at the free end of the beam (or directly from a free body diagram). In (s,n,b) coordinate system, equilibrium equations are

$$\begin{Bmatrix} N' - Q_n\kappa + b_s \\ Q_n' + N\kappa - Q_b\tau + b_n \\ Q_b' + Q_n\tau + b_b \end{Bmatrix} = 0 \quad \text{and} \quad \begin{Bmatrix} T' - M_n\kappa + c_s \\ M_n' + T\kappa - M_b\tau - Q_b + c_n \\ M_b' + M_n\tau + Q_n + c_b \end{Bmatrix} = 0 .$$

In the present case $\kappa = 0$ and $\tau = 4\pi / L$ which means that the basis vector rotate two full circles around the axis when s goes from 0 to L and have the same orientations on $\partial\Omega = \{0, L\}$. As external distributed forces and moments vanish i.e. $b_s = b_n = b_b = 0$ and $c_s = c_n = c_b = 0$, equilibrium equations and the boundary conditions at the free end simplify to

$$\begin{Bmatrix} N' \\ Q_n' - Q_b\tau \\ Q_b' + Q_n\tau \end{Bmatrix} = 0 \quad \text{and} \quad \begin{Bmatrix} T' \\ M_n' - M_b\tau - Q_b \\ M_b' + M_n\tau + Q_n \end{Bmatrix} = 0 \quad \text{in} \quad \Omega = (0, L) ,$$

$$\begin{Bmatrix} N \\ Q_n \\ Q_b \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ P \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} T \\ M_n \\ M_b \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{on} \quad \partial\Omega_t = \{L\} .$$

Equations constitute a boundary value problem that can be solved by hand calculations:

$$N' = 0 \quad \text{in} \quad \Omega, \quad N(L) = 0 \quad \Rightarrow \quad N(s) = 0 . \quad \Leftarrow$$

Eliminating Q_n or Q_b from the remaining two connected force equilibrium equations and using the original equations to find the missing boundary condition gives

$$Q_n'' + Q_n\tau^2 = 0 \quad s \in \Omega , \quad Q_n(L) = 0, \quad Q_n'(L) = P\tau \quad \Rightarrow \quad (\tau = \frac{4\pi}{L})$$

$$Q_n(s) = P\sin(\tau s) \quad \text{and} \quad Q_b(s) = P\cos(\tau s) . \quad \Leftarrow$$

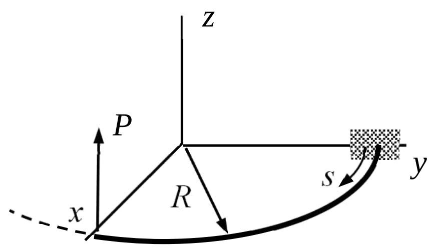
Continuing with the moment equilibrium equations in the same manner

$$T' = 0 \quad \text{in} \quad \Omega, \quad T(L) = 0 \quad \Rightarrow \quad T(s) = 0 , \quad \Leftarrow$$

$$M_b'' + M_b \tau^2 = -2P\tau\cos(\tau s) \quad \text{in} \quad \Omega\,, \quad M_b(L) = 0\,, \quad M_b'(L) = 0 \quad \Rightarrow$$

$$M_b(s) = P(L-s)\sin(\tau s) \quad \Rightarrow \quad M_n = P(s-L)\cos(\tau s)\,.$$

Consider the curved beam of the figure forming a 90-degree circular segment of radius R in the horizontal plane. Find the stress resultants $N(s)$, $Q_n(s)$, $Q_b(s)$, $T(s)$, $M_n(s)$, and $M_b(s)$. Use the equilibrium equations of the beam model in the $(s,n,b)-$coordinate system.



Solution

In a statically determined case, stress resultants follow from the equilibrium equations and boundary conditions at the free end of the beam (or directly from a free body diagram). In (s,n,b) coordinate system, equilibrium equations are

$$\begin{Bmatrix} N' - Q_n\kappa + b_s \\ Q_n' + N\kappa - Q_b\tau + b_n \\ Q_b' + Q_n\tau + b_b \end{Bmatrix} = 0 \quad \text{and} \quad \begin{Bmatrix} T' - M_n\kappa + c_s \\ M_n' + T\kappa - M_b\tau - Q_b + c_n \\ M_b' + M_n\tau + Q_n + c_b \end{Bmatrix} = 0 .$$

For a circular beam, curvature and torsion are $\kappa = 1/R$ (constant) and $\tau = 0$.

As external distributed forces and moments vanish i.e. $b_s = b_n = b_b = c_s = c_n = c_b = 0$, equilibrium equations and the boundary conditions at the free end simplify to (notice that the external force acting at the free end is acting in the oppisite direction to $\vec{e}_b$)

$$\begin{Bmatrix} N' - Q_n / R \\ Q_n' + N / R \\ Q_b' \end{Bmatrix} = 0 \quad \text{and} \quad \begin{Bmatrix} T' - M_n / R \\ M_n' + T / R - Q_b \\ M_b' + Q_n \end{Bmatrix} = 0 \quad s \in]0, R\frac{\pi}{2}[,$$

$$\begin{Bmatrix} N \\ Q_n \\ Q_b + P \end{Bmatrix} = 0 \quad \text{and} \quad \begin{Bmatrix} T \\ M_n \\ M_b \end{Bmatrix} = 0 \quad s = R\frac{\pi}{2} .$$

Equations constitute a boundary value problem which can be solved by hand calculations without too much effort;

$$Q_b' = 0 \quad s \in]0, R\frac{\pi}{2}[\quad \text{and} \quad Q_b + P = 0 \quad s = R\frac{\pi}{2} \quad \Rightarrow \quad Q_b(s) = -P .$$

Eliminating Q_n and N from the remaining two connected force equilibrium equations and using the original equations to find the missing boundary condition give

$$N'' + \frac{1}{R^2} N = 0 \quad s \in]0, R\frac{\pi}{2}[\quad \text{and} \quad N' = N = 0 \quad s = R\frac{\pi}{2} \quad \Rightarrow \quad N(s) = 0$$

The first equilibrium equation gives

$$Q_n(s) = 0 .$$

After that, continuing with the moment equilibrium equations with the solutions to the force equilibrium equations

$$M_b' = 0 \quad s \in]0, R\frac{\pi}{2}[\quad \text{and} \quad M_b = 0 \quad s = R\frac{\pi}{2} \quad \Rightarrow \quad M_b(s) = 0 \,.$$

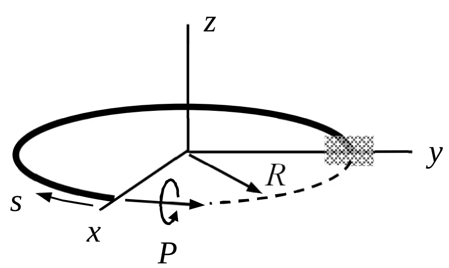
Eliminating M_n and T from the remaining two connected moment equilibrium equations and using the original equations to find the missing boundary condition gives

$$T'' + \frac{1}{R^2}T + \frac{P}{R} = 0 \quad s \in]0, R\frac{\pi}{2}[\quad \text{and} \quad T' = T = 0 \quad s = R\frac{\pi}{2} \quad \Rightarrow \quad T = PR(\sin\frac{s}{R} - 1) \,.$$

Knowing this, the first moment equilibrium equation gives

$$M_n(s) = RT' = RP\cos\frac{s}{R} \,.$$

Consider a curved beam forming ¾ of a full circle of radius R in the horizontal plane. Torque of magnitude P is acting on the free end as shown. Write down the boundary value problem for *stress resultants* and solve the equations for $N(s)$, $Q_n(s)$, $Q_b(s)$, $T(s)$, $M_n(s)$, and $M_b(s)$.



Solution

In the geometry of the figure $\tau = 0$, $\kappa = 1/R$. External distributed forces and moments vanish. Therefore the curved beam equilibrium equations of the formulae collection simplify to

$$\begin{Bmatrix} N' - Q_n / R \\ Q_n' + N / R \\ Q_b' \end{Bmatrix} = 0 \quad \text{and} \quad \begin{Bmatrix} T' - M_n / R \\ M_n' + T / R - Q_b \\ M_b' + Q_n \end{Bmatrix} = 0 \quad s \in (0, L) \quad \text{where } L = \frac{3}{2}\pi R .$$

Boundary conditions at $s = 0$ are (notice the unit outward normal to the solution domain $n = -1$, $\vec{e}_s$ is pointing to the direction of *s*, and the component of the given moment on $\vec{e}_s$ is negative)

$$\begin{Bmatrix} N \\ Q_n \\ Q_b \end{Bmatrix} = 0 \quad \text{and} \quad \begin{Bmatrix} -T + P \\ M_n \\ M_b \end{Bmatrix} = 0 \quad s = 0 .$$

Solution to the boundary values problem for Q_b

$$Q_b' = 0 \quad s \in (0, L) \quad \text{and} \quad Q_b = 0 \quad s = 0 \quad \Rightarrow \quad Q_b(s) = 0 .$$

Solution to the connected boundary value problems for Q_n and N

$$N' - \frac{1}{R} Q_n = 0 , \; Q_n' + \frac{1}{R} N = 0 \quad s \in (0, L) , \quad Q_n = 0 \text{ and } N = 0 \quad s = 0 \quad \Rightarrow$$

$$N'' + \frac{1}{R^2} N = 0 \quad s \in (0, L) \quad \text{and} \quad N = 0 , \; N' = 0 \; \text{ at } \; s = 0 \quad \Rightarrow$$

$$N(s) = 0 \quad \text{and} \quad Q_n(s) = 0 .$$

Solution to the boundary value problem for M_b

$$M_b' = 0 \quad s \in (0, L) \quad \text{and} \quad M_b = 0 \quad s = 0 \quad \Rightarrow \quad M_b(s) = 0 .$$

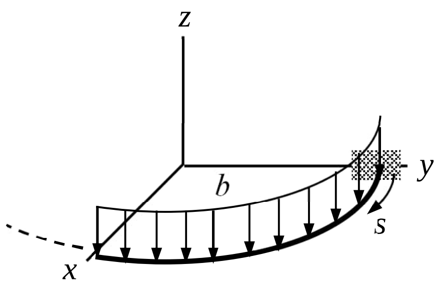
Solution to the connected boundary value problem for M_n and T

$$T' - \frac{1}{R} M_n = 0 \quad \text{and} \quad M_n' + \frac{1}{R} T = 0 \quad s \in (0, L) , \quad T = P \quad \text{and} \quad M_n = 0 \quad s = 0 \quad \Rightarrow$$

$$R T'' + \frac{1}{R} T = 0 \quad s \in (0, L) , \quad T = P \quad \text{and} \quad T' = 0 \quad \Rightarrow$$

$$T(s) = P\cos(\frac{s}{R}) \quad \text{and} \quad M_n(s) = -P\sin(\frac{s}{R}) .$$

Consider the curved beam of the figure forming a 90-degree circular segment of radius R in the horizontal plane. Find the stress resultants $N(s)$, $Q_n(s)$, $Q_b(s)$, $T(s)$, $M_n(s)$, and $M_b(s)$. Use the equilibrium equations of the beam model in the $(s,n,b)-$coordinate system. The distributed constant load of magnitude b is acting to the negative direction of the $z-$axis.



Solution

In a statically determined case, stress resultants follow from the equilibrium equations and boundary conditions at the free end of the beam (or directly from a free body diagram). In $(s,n,b)-$coordinate system, equilibrium equations are

$$\begin{Bmatrix} N' - Q_n\kappa + b_s \\ Q_n' + N\kappa - Q_b\tau + b_n \\ Q_b' + Q_n\tau + b_b \end{Bmatrix} = 0 \quad \text{and} \quad \begin{Bmatrix} T' - M_n\kappa + c_s \\ M_n' + T\kappa - M_b\tau - Q_b + c_n \\ M_b' + M_n\tau + Q_n + c_b \end{Bmatrix} = 0 .$$

For a circular beam, curvature and torsion are $\kappa = 1/R$ (constant) and $\tau = 0$. As external distributed forces and moments $b_s = b_n = c_s = c_n = c_b = 0$ and $b_b = b$, equilibrium equations and the boundary conditions at the free end simplify to (here $L = \pi R/2$)

$$\begin{Bmatrix} N' - Q_n/R \\ Q_n' + N/R \\ Q_b' + b \end{Bmatrix} = 0 \quad \text{and} \quad \begin{Bmatrix} T' - M_n/R \\ M_n' + T/R - Q_b \\ M_b' + Q_n \end{Bmatrix} = 0 \quad \text{in} \quad (0,L)$$

$$\begin{Bmatrix} N \\ Q_n \\ Q_b \end{Bmatrix} = 0 \quad \text{and} \quad \begin{Bmatrix} T \\ M_n \\ M_b \end{Bmatrix} = 0 \quad \text{at} \quad s = L .$$

Equations constitute a boundary value problem which can be solved one equation at a time by following certain order

$$Q_b' = -b \quad \text{in} \quad (0,L) \quad \text{and} \quad Q_b(L) = 0 \quad \Rightarrow \quad Q_b(s) = b(L-s). \quad \Leftarrow$$

Eliminating Q_n and N from the remaining two connected force equilibrium equations and using the original equations to find the missing boundary condition gives

$$N'' + \frac{1}{R^2}N = 0 \quad \text{in} \quad (0,L) \quad \text{and} \quad N'(L) = N(L) = 0 \quad \Rightarrow \quad N(s) = 0. \quad \Leftarrow$$

Knowing the result above, the first equilibrium equation gives

$$Q_n(s) = 0. \quad \Leftarrow$$

After that, continuing with the moment equilibrium equations with the already known solutions to the force equilibrium equations

$$M_b' = -Q_n = 0 \quad \text{in} \quad (0,L) \quad \text{and} \quad M_b(L) = 0 \quad \Rightarrow \quad M_b(s) = 0 .$$

Eliminating M_n and T from the remaining two connected moment equilibrium equations and using the original equations to find the missing boundary condition gives $(L = \pi R / 2)$

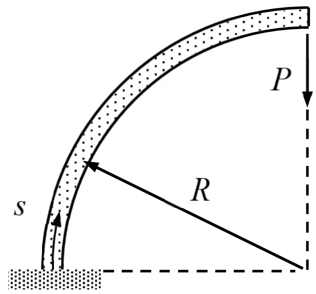
$$T'' + \frac{1}{R^2} T = \frac{1}{R} Q_b = \frac{b}{R}(L-s) \quad \text{in} \quad (0,L) \quad \text{and} \quad T'(L) = T(L) = 0 \quad \Rightarrow$$

$$T(s) = -bR^2 \cos(\frac{s}{R}) + bR(L-s) .$$

Knowing this, the first moment equilibrium equation gives

$$M_n(s) = RT' = bR^2 \sin(\frac{s}{R}) - bR^2 .$$

Consider the curved planar Timoshenko beam shown in the figure. Write down the boundary value problem for $u(s)$, $v(s)$, $\psi(s)$, $N(s)$, $Q_n(s)$, and $M_b(s)$. Also, solve the equations for displacement $v(s)$ and rotation $\psi(s)$. The properties of the cross-section A, $S_b = 0$, $I_{bb} = I$, and material parameters E, G are constants. Curvature is $\kappa = 1/R$ and torsion $\tau = 0$.



Solution

In (s,n,b) coordinate system, equilibrium equations and constitutive equations of the beam model are

$$\begin{Bmatrix} N' - Q_n\kappa + b_s \\ Q_n' + N\kappa - Q_b\tau + b_n \\ Q_b' + Q_n\tau + b_b \end{Bmatrix} = 0, \quad \begin{Bmatrix} T' - M_n\kappa + c_s \\ M_n' + T\kappa - M_b\tau - Q_b + c_n \\ M_b' + M_n\tau + Q_n + c_b \end{Bmatrix} = 0,$$

$$\begin{Bmatrix} N \\ Q_n \\ Q_b \end{Bmatrix} = \begin{Bmatrix} EA(u' - v\kappa) \\ GA(v' + u\kappa - w\tau - \psi) \\ GA(w' + v\tau + \theta) \end{Bmatrix}, \quad \begin{Bmatrix} T \\ M_n \\ M_b \end{Bmatrix} = \begin{Bmatrix} GI_{rr}(\phi' - \theta\kappa) \\ EI_{nn}(\theta' + \phi\kappa - \psi\tau) \\ EI_{bb}(\psi' + \theta\tau) \end{Bmatrix}.$$

In a planar problem, it is enough to consider force equilibrium equations in the $s-$ and $n-$ directions and moment equilibrium equation in the $b-$direction. As the present problem is statically determined, it is possible to solve for the stress resultants first ($L = \pi R/2$)

$$\begin{Bmatrix} N' - Q_n/R \\ Q_n' + N/R \\ M_b' + Q_n \end{Bmatrix} = 0 \text{ in } (0,L) \text{ and } \begin{Bmatrix} N \\ Q_n - P \\ M_b \end{Bmatrix} = 0 \text{ at } s = L.$$

Solution to the shear stress resultant Q_n can be obtained by eliminating the normal stress resultant N from the first two connected equilibrium equations. The missing boundary condition for the second order differential equation follows from the second equilibrium equations at the endpoint (there $N = 0$)

$$Q_n'' + \frac{1}{R^2}Q_n = 0 \text{ in } (0,L), \; Q_n'(L) = 0, \text{ and } Q_n(L) - P = 0 \quad \Rightarrow \quad Q_n(s) = P\sin(\frac{s}{R}). \; \Leftarrow$$

Knowing the shear stress resultant, the second and third equilibrium equations give

$$Q_n' + N/R = 0 \text{ in } (0,L) \quad \Rightarrow \quad N(s) = -P\cos(\frac{s}{R}), \; \Leftarrow$$

$$M_b' + Q_n = 0 \text{ in } \Omega \text{ and } M_b(L) = 0 \quad \Rightarrow \quad M_b = PR\cos(\frac{s}{R}). \; \Leftarrow$$

As the force resultants are now known, displacements and rotations follow from the constitutive equations (and the corresponding boundary conditions)

$$\begin{Bmatrix} N \\ Q_n \\ M_b \end{Bmatrix} = \begin{Bmatrix} EA(u' - v/R) \\ GA(v' + u/R - \psi) \\ EI\psi' \end{Bmatrix} \text{ in } (0,L) \quad \text{and} \quad \begin{Bmatrix} u(0) \\ v(0) \\ \psi(0) \end{Bmatrix} = 0 .$$

Let us start with the last constitutive equation

$$EI\psi' = M_b = PR\cos(\frac{s}{R}) \text{ and } \psi(0) = 0 \quad \Rightarrow \quad \psi(s) = \frac{PR^2}{EI}\sin(\frac{s}{R}) .$$

Elimination of the axial displacement from the first two constitutive equations with the known expression of the rotation (the missing boundary condition follows from the second constitutive equation at $s = 0$) gives

$$v'' + \frac{1}{R^2}v = P(\frac{R^2}{EI} + \frac{1}{GA} + \frac{1}{EA})\frac{1}{R}\cos(\frac{s}{R}), \quad v(0) = 0, \; v'(0) = 0 \quad \Rightarrow$$

$$v(s) = \frac{P}{2}(\frac{R^2}{EI} + \frac{1}{GA} + \frac{1}{EA})s\sin(\frac{s}{R}) .$$

Finally, using the second constitutive equation

$$u(s) = (\frac{PR}{GA} + \frac{PR^3}{EI})\sin(\frac{s}{R}) - \frac{PR}{2}(\frac{R^2}{EI} + \frac{1}{GA} + \frac{1}{EA})[\sin(\frac{s}{R}) + \frac{s}{R}\cos(\frac{s}{R})] .$$