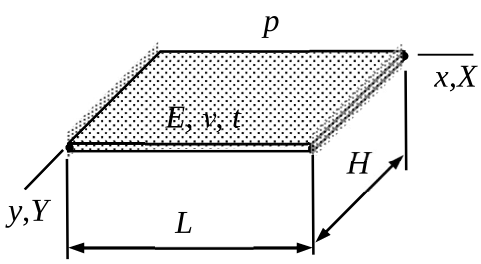
Name________________________ Student number________________

Assignment 5 (4p)

The rectangle plate shown, of thickness, length and width t, L, and H, is simply supported on edges where $x \in \{0, L\}$ and free on the remaining edges where $y \in \{0, H\}$, and loaded by pressure p acting on the upper surface. Young's modulus E and Poisson's ratio ν are constants. Determine the parameter a_0 of the approximation $w(x, y) = \mathrm{a}_0(x/L)(1-x/L)$. Use the principle of virtual work in form $\delta W = 0 \;\; \forall \delta \mathrm{a}_0 \in \mathbb{R}$ and



$$\delta W = -\int_\Omega \begin{Bmatrix} \dfrac{\partial^2 \delta w}{\partial x^2} \\ \dfrac{\partial^2 \delta w}{\partial y^2} \\ 2\dfrac{\partial^2 \delta w}{\partial x \partial y} \end{Bmatrix}^{\mathrm{T}} D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{Bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \\ 2\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} dA + \int_\Omega \delta w b_n dA .$$

Solution

Virtual work expression can also be written in the form

$$\delta W = \int_\Omega \delta w_\Omega dA = \int_\Omega (\delta w_\Omega^{\mathrm{int}} + \delta w_\Omega^{\mathrm{ext}}) dA$$

in which the virtual work densities (virtual work per unit area here) of the internal and external forces

$$\delta w_\Omega^{\mathrm{int}} = -\begin{Bmatrix} \dfrac{\partial^2 \delta w}{\partial x^2} \\ \dfrac{\partial^2 \delta w}{\partial y^2} \\ 2\dfrac{\partial^2 \delta w}{\partial x \partial y} \end{Bmatrix}^{\mathrm{T}} D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{Bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \\ 2\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} \text{ and } \delta w_\Omega^{\mathrm{ext}} = b_n \delta w .$$

Approximation to the transverse displacement and its derivatives are

$$w(x, y) = a_0 (1 - \frac{x}{L}) \frac{x}{L} \;\; \Rightarrow \;\; \frac{\partial w}{\partial x} = \frac{a_0}{L}(1 - 2\frac{x}{L}), \;\; \frac{\partial^2 w}{\partial x^2} = -2\frac{a_0}{L^2}, \text{ and } \frac{\partial w}{\partial y} = \frac{\partial^2 w}{\partial y^2} = 0 .$$

When the approximation and distributed force expression $b_n = p$ are substituted there, virtual work densities of the internal and external forces simplify to

$$\delta w_{\Omega}^{\mathrm{int}} = -\delta a_0 \frac{1}{3}\frac{t^3}{L^4}\frac{E}{1-\nu^2} a_0 \,,$$

$$\delta w_{\Omega}^{\mathrm{ext}} = \delta a_0 (1-\frac{x}{L})\frac{x}{L} p \,.$$

Virtual work expressions are integrals of the densities over the mathematical domain for the plate (midplane)

$$\delta W^{\mathrm{int}} = \int_0^H \int_0^L \delta w_{\Omega}^{\mathrm{int}} dxdy = -\delta a_0 \frac{1}{3}\frac{t^3 H}{L^3}\frac{E}{1-\nu^2} a_0 \,,$$

$$\delta W^{\mathrm{ext}} = \int_0^H \int_0^L \delta w_{\Omega}^{\mathrm{ext}} dxdy = \delta a_0 \frac{1}{6} HLp \,.$$

Virtual work expression

$$\delta W = \delta W^{\mathrm{int}} + \delta W^{\mathrm{ext}} = -\delta a_0 (\frac{1}{3}\frac{t^3 H}{L^3}\frac{E}{1-\nu^2} a_0 - \frac{1}{6} HLp) \,,$$

principle of virtual work $\delta W = 0 \;\forall \delta a_0$, and the fundamental lemma of variation calculus imply the solution

$$\frac{1}{3}\frac{t^3 H}{L^3}\frac{E}{1-\nu^2} a_0 - \frac{1}{6} HLp = 0 \quad \Leftrightarrow \quad a_0 = \frac{1}{2}(\frac{L}{t})^3 \frac{Lp}{E}(1-\nu^2) \,. \quad \Leftarrow$$