Name________________________ Student number________________

Assignment 5

Derive the gradient expressions of $(\alpha,\beta,\gamma)-$coordinate system, when the mapping defining the coordinate system is given by

$$\vec{r}(\alpha,\beta,\gamma)=(uv\alpha+\sqrt{1-u^2}\gamma)\vec{i}+\beta\vec{j}+(u\gamma-v\sqrt{1-u^2}\alpha)\vec{k}$$

in which $u\in[-1,1]$ and $v>0$ are parameters.

Solution

According to the generic recipe (formulae collection)

$$\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_\gamma\end{Bmatrix}=\begin{Bmatrix}(\partial\vec{r}/\partial\alpha)/|\partial\vec{r}/\partial\alpha|\\ (\partial\vec{r}/\partial\beta)/|\partial\vec{r}/\partial\beta|\\ (\partial\vec{r}/\partial\gamma)/|\partial\vec{r}/\partial\gamma|\end{Bmatrix}=[F]\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix},\quad \frac{\partial}{\partial\eta}\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_\gamma\end{Bmatrix}=(\frac{\partial}{\partial\eta}[F])[F]^{-1}\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_\gamma\end{Bmatrix}\quad \eta\in\{\alpha,\beta,\gamma\},$$

$$\nabla=\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_\gamma\end{Bmatrix}^{\mathrm{T}}[F]^{-\mathrm{T}}[H]^{-1}\begin{Bmatrix}\partial/\partial\alpha\\ \partial/\partial\beta\\ \partial/\partial\gamma\end{Bmatrix}\quad\text{where } [H]=\begin{bmatrix}\partial r_x/\partial\alpha & \partial r_y/\partial\alpha & \partial r_z/\partial\alpha\\ \partial r_x/\partial\beta & \partial r_y/\partial\beta & \partial r_z/\partial\beta\\ \partial r_x/\partial\gamma & \partial r_y/\partial\gamma & \partial r_z/\partial\gamma\end{bmatrix},$$

Matrices $[F]$ and $[H]$ depend on the mapping. In the present case

$$\vec{r}(\alpha,\beta,\gamma)=r_x\vec{i}+r_y\vec{j}+r_z\vec{k}=(uv\alpha+\sqrt{1-u^2}\gamma)\vec{i}+\beta\vec{j}+(u\gamma-v\sqrt{1-u^2}\alpha)\vec{k}\,.$$

By definition

$$\vec{e}_\alpha=\frac{\partial\vec{r}}{\partial\alpha}/\left|\frac{\partial\vec{r}}{\partial\alpha}\right|=(vu\vec{i}-v\sqrt{1-u^2}\vec{k})/v=u\vec{i}-\sqrt{1-u^2}\vec{k}\,,$$

$$\vec{e}_\beta=\frac{\partial\vec{r}}{\partial\beta}/\left|\frac{\partial\vec{r}}{\partial\beta}\right|=\vec{j}\,,$$

$$\vec{e}_\gamma=\frac{\partial\vec{r}}{\partial\gamma}/\left|\frac{\partial\vec{r}}{\partial\gamma}\right|=\sqrt{1-u^2}\vec{i}+u\vec{k}$$

and therefore

$$\begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix} = \begin{bmatrix} u & 0 & -\sqrt{1-u^2} \\ 0 & 1 & 0 \\ \sqrt{1-u^2} & 0 & u \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = [F] \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} \text{ so } [F] = \begin{bmatrix} u & 0 & -\sqrt{1-u^2} \\ 0 & 1 & 0 \\ \sqrt{1-u^2} & 0 & u \end{bmatrix}.$$

According to the mapping, the relationship between the components of the position vector in the Cartesian and cylindrical systems are $r_x = uv\alpha + \sqrt{1-u^2}\gamma$, $r_y = \beta$, and $r_z = u\gamma - \sqrt{1-u^2}v\alpha$

$$[H] = \begin{bmatrix} \partial r_x / \partial\alpha & \partial r_y / \partial\alpha & \partial r_z / \partial\alpha \\ \partial r_x / \partial\beta & \partial r_y / \partial\beta & \partial r_z / \partial\beta \\ \partial r_x / \partial\gamma & \partial r_y / \partial\gamma & \partial r_z / \partial\gamma \end{bmatrix} = \begin{bmatrix} vu & 0 & -v\sqrt{1-u^2} \\ 0 & 1 & 0 \\ \sqrt{1-u^2} & 0 & u \end{bmatrix}.$$

Gradient follows now from the generic recipe

$$\nabla = \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix}^{\mathrm{T}} [F]^{-\mathrm{T}} [H]^{-1} \begin{Bmatrix} \partial / \partial\alpha \\ \partial / \partial\beta \\ \partial / \partial\gamma \end{Bmatrix} = \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix}^{\mathrm{T}} ([H][F]^{\mathrm{T}})^{-1} \begin{Bmatrix} \partial / \partial\alpha \\ \partial / \partial\beta \\ \partial / \partial\gamma \end{Bmatrix}.$$

Let us calculate first the matrix inside the parenthesis

$$[H][F]^{\mathrm{T}} = \begin{bmatrix} vu & 0 & -v\sqrt{1-u^2} \\ 0 & 1 & 0 \\ \sqrt{1-u^2} & 0 & u \end{bmatrix} \begin{bmatrix} u & 0 & \sqrt{1-u^2} \\ 0 & 1 & 0 \\ -\sqrt{1-u^2} & 0 & u \end{bmatrix} = \begin{bmatrix} v & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \Leftrightarrow$$

$$([H][F]^{\mathrm{T}})^{-1} = \begin{bmatrix} v & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/v & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Substituting into the gradient expression

$$\nabla = \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1/v & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \partial / \partial\alpha \\ \partial / \partial\beta \\ \partial / \partial\gamma \end{Bmatrix} = \vec{e}_\alpha \frac{1}{v} \frac{\partial}{\partial\alpha} + \vec{e}_\beta \frac{\partial}{\partial\beta} + \vec{e}_\gamma \frac{\partial}{\partial\gamma}. \quad \Leftarrow$$