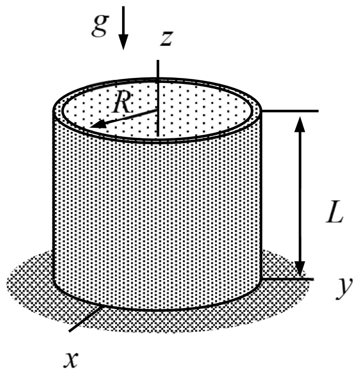
Name____________________________ Student number____________________

Assignment 4 (4p)



A cylindrical container of height L, (mid-surface) radius R, density ρ, and thickness t is loaded by its own weight. Assume rotation symmetry and use the membrane equations in (z,ϕ,n) coordinate system to find the stress resultant and the displacement components. Assume that friction between the container and floor is small and cannot constraint the transverse displacement at the contact points and rigid body motion is not possible (constrained somehow).

Solution

According to the formulae collection, equilibrium and constitutive equations of a cylindrical membrane in (z,ϕ,n) coordinates are (notice that $\vec{e}_n$ is directed inwards)

$$\begin{Bmatrix} \dfrac{1}{R}\dfrac{\partial N_{\phi z}}{\partial \phi}+\dfrac{\partial N_{zz}}{\partial z}+b_z \\ \dfrac{\partial N_{z\phi}}{\partial z}+\dfrac{1}{R}\dfrac{\partial N_{\phi\phi}}{\partial \phi}+b_\phi \\ \dfrac{1}{R}N_{\phi\phi}+b_n \end{Bmatrix}=0\,,\quad \begin{Bmatrix} N_{zz} \\ N_{\phi\phi} \\ N_{z\phi} \end{Bmatrix}=\frac{tE}{1-\nu^2}\begin{Bmatrix} \dfrac{\partial u_z}{\partial z}+\nu\dfrac{1}{R}(\dfrac{\partial u_\phi}{\partial \phi}-u_n) \\ \dfrac{1}{R}(\dfrac{\partial u_\phi}{\partial \phi}-u_n)+\nu\dfrac{\partial u_z}{\partial z} \\ \dfrac{1-\nu}{2}(\dfrac{1}{R}\dfrac{\partial u_z}{\partial \phi}+\dfrac{\partial u_\phi}{\partial z}) \end{Bmatrix},\quad N_{\phi z}=N_{z\phi}\,.$$

If the solution does not depend on ϕ, equilibrium equations of the membrane model and boundary conditions at the free end simplify to (gravity is acting in the negative direction of the $z-$axis)

$$\frac{dN_{zz}}{dz}-\rho g t=0\,,\quad \frac{dN_{z\phi}}{dz}=0\,,\ \text{ and }\ \frac{1}{R}N_{\phi\phi}=0 \quad \text{in} \quad (0,L)\,,$$

$$N_{zz}=0 \ \text{ and } \ N_{z\phi}=0 \quad \text{at} \quad z=L\,.$$

Solution to the boundary value problem for the stress resultants is given by

$$N_{zz}(z)=(z-L)\rho g t\,,\quad N_{z\phi}(z)=0\,,\quad N_{\phi\phi}(z)=0\,.$$

As the stress resultants are now known, the boundary value problem for the displacement components follows from the constitutive equations and the boundary conditions at the fixed end (in the membrane model, a boundary condition cannot be assigned to u_n)

$$(z-L)\rho g t=\frac{tE}{1-\nu^2}(\frac{du_z}{dz}-\nu\frac{1}{R}u_n)\,,\quad 0=\frac{tE}{1-\nu^2}(\nu\frac{du_z}{dz}-\frac{1}{R}u_n)\,,\ \text{ and }\ tG\frac{du_\phi}{dz}=0 \quad \text{in} \quad (0,L)\,,$$

$u_z = 0$ at $z = 0$.

The last equation implies that $u_\phi = \text{constant}$, but the constant has to be zero as a non-zero value would mean rigid body rotation around the $z-$axis so

$$u_\phi(z) = 0 .$$

Using the second equation to eliminate u_n from the first one gives the boundary value problem

$$(z-L)\rho g t = tE\frac{du_z}{dz} \quad \text{in} \quad (0,L) \quad \text{and} \quad u_z = 0 \quad \text{at} \quad z = 0 \quad \Rightarrow \quad u_z(z) = (\frac{1}{2}z^2 - Lz)\frac{\rho g}{E} .$$

Finally, substituting into the second equilibrium equation and solving for the last displacement component

$$u_n(z) = R\nu\frac{du_z}{dz} = (z-L)\nu\frac{R\rho g}{E} .$$