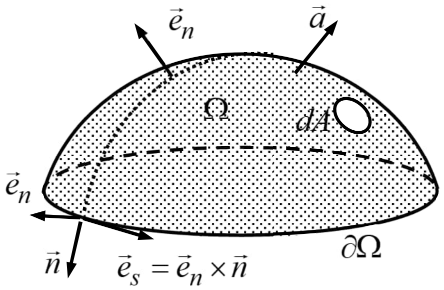
Name________________________ Student number________________

Assignment 1 (2p)

Gauss theorem implies the following integral identity for curved surfaces



$$\int_{\Omega} (\nabla_0 \cdot \vec{a} - \kappa \vec{e}_n \cdot \vec{a}) dA = \int_{\partial\Omega} (\vec{n} \cdot \vec{a}) ds$$

in which $\kappa = \vec{\kappa} : \vec{I} = \nabla_0 \cdot \vec{e}_n$. Verify the integral identity in the spherical (ϕ, θ, n) coordinate system by considering vector $\vec{a} = \theta \vec{e}_n$ and half-sphere $\phi \in [0, 2\pi]$, $\theta \in [0, \pi/2]$, of radius R as Ω. Derivatives of the basis vectors and the mid-surface gradient in the spherical coordinate system are

$$\frac{\partial}{\partial \phi} \begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_\theta \\ \vec{e}_n \end{Bmatrix} = \begin{Bmatrix} \sin\theta \vec{e}_n - \cos\theta \vec{e}_\theta \\ \cos\theta \vec{e}_\phi \\ -\sin\theta \vec{e}_\phi \end{Bmatrix}, \quad \frac{\partial}{\partial \theta} \begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_\theta \\ \vec{e}_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ \vec{e}_n \\ -\vec{e}_\theta \end{Bmatrix}, \quad \nabla_0 = \frac{1}{R\sin\theta} \vec{e}_\phi \frac{\partial}{\partial \phi} + \frac{1}{R} \vec{e}_\theta \frac{\partial}{\partial \theta}.$$

Solution template

In case of a half-sphere $\phi \in [0, 2\pi]$, $\theta \in [0, \pi/2]$ of radius R and vector $\vec{a} = \theta \vec{e}_n$, the quantities in the integral identity take the forms

$$\nabla_0 \cdot \vec{a} = \left(\frac{1}{R\sin\theta} \vec{e}_\phi \frac{\partial}{\partial \phi} + \frac{1}{R} \vec{e}_\theta \frac{\partial}{\partial \theta}\right) \cdot \theta \vec{e}_n = -\frac{2}{R}\theta,$$

$$\kappa = \nabla_0 \cdot \vec{e}_n = \left(\frac{1}{R\sin\theta} \vec{e}_\phi \frac{\partial}{\partial \phi} + \frac{1}{R} \vec{e}_\theta \frac{\partial}{\partial \theta}\right) \cdot \vec{e}_n = -\frac{2}{R},$$

$$\vec{e}_n \cdot \vec{a} = \theta,$$

$$\vec{n} \cdot \vec{a} = 0.$$

When the expressions are substituted there, the left- and right-hand sides of the integral identity simplify to

$$\int_{\Omega} (\nabla_0 \cdot \vec{a} - \kappa \vec{e}_n \cdot \vec{a}) dA = \int_{\Omega} \left(-\frac{2}{R}\theta + \frac{2}{R}\theta\right) dA = 0,$$

$$\int_{\partial\Omega} (\vec{n} \cdot \vec{a}) ds = \int_{\partial\Omega} (0) ds = 0.$$