Name________________________ Student number________________

Assignment 4

Derive the expressions of linear strain components ε_{rr}, $\varepsilon_{r\phi}$, $\varepsilon_{\phi r}$ and $\varepsilon_{\phi\phi}$ of the polar coordinate system. Use the displacement representation $\vec{u} = u_r \vec{e}_r + u_\phi \vec{e}_\phi$ where the components depend on the polar coordinates r and ϕ. Use definitions

$$\vec{\varepsilon} = \frac{1}{2}[\nabla\vec{u} + (\nabla\vec{u})_{\mathrm{c}}], \; \nabla = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{\partial}{r\partial\phi}, \; \frac{\partial}{\partial\phi}\begin{Bmatrix}\vec{e}_r \\ \vec{e}_\phi\end{Bmatrix} = \begin{Bmatrix}\vec{e}_\phi \\ -\vec{e}_r\end{Bmatrix}, \; \frac{\partial}{\partial r}\begin{Bmatrix}\vec{e}_r \\ \vec{e}_\phi\end{Bmatrix} = 0 .$$

Solution template

In manipulation of vector expression containing vectors and tensors, it is important to remember that tensor ($\otimes$), cross ($\times$), inner ($\cdot$) products are non-commutative (order may matter). The basis vectors of a curvilinear coordinate system are not constants which should be taken into account if gradient operator is a part of expression. Otherwise, simplifying an expression or finding a specific form in a given coordinate system is a straightforward (sometimes tedious) exercise. For simplicity of presentation, outer (tensor) products like $\vec{a} \otimes \vec{b}$ are denoted by $\vec{a}\vec{b}$. Otherwise, the usual rules of algebra apply: Gradient operator ∇ acts on everything on its right hand side, the operator is treated like a vector etc.

Let us start with the gradient of displacement (an outer product). Substitute first the representations in the polar coordinate system

$$\nabla\vec{u} = (\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{\partial}{r\partial\phi})(u_r \vec{e}_r + u_\phi \vec{e}_\phi) .$$

Then expand to have a term-by-term representation. Keep the order of the basis vectors and the position of derivatives

$$\nabla\vec{u} = \vec{e}_r \frac{\partial}{\partial r}(u_r \vec{e}_r) + \vec{e}_r \frac{\partial}{\partial r}(u_\phi \vec{e}_\phi) + \vec{e}_\phi \frac{\partial}{r\partial\phi}(u_r \vec{e}_r) + \vec{e}_\phi \frac{\partial}{r\partial\phi}(u_\phi \vec{e}_\phi)$$

Use the derivative rule of products. Notice that the basis vectors are not constants and may have non-zero derivatives

$$\nabla\vec{u} = \vec{e}_r(\frac{\partial u_r}{\partial r}\vec{e}_r + u_r \frac{\partial \vec{e}_r}{\partial r}) + \vec{e}_r(\frac{\partial u_\phi}{\partial r}\vec{e}_\phi + u_\phi \frac{\partial \vec{e}_\phi}{\partial r}) + \vec{e}_\phi(\frac{\partial u_r}{r\partial\phi}\vec{e}_r + u_r \frac{\partial \vec{e}_r}{r\partial\phi}) + \vec{e}_\phi(\frac{\partial u_\phi}{r\partial\phi}\vec{e}_\phi + u_\phi \frac{\partial \vec{e}_\phi}{r\partial\phi}) .$$

Substitute the derivatives of the basis vectors

$$\nabla\vec{u} = \vec{e}_r(\frac{\partial u_r}{\partial r}\vec{e}_r) + \vec{e}_r(\frac{\partial u_\phi}{\partial r}\vec{e}_\phi) + \vec{e}_\phi(\frac{\partial u_r}{r\partial\phi}\vec{e}_r + \frac{u_r}{r}\vec{e}_\phi) + \vec{e}_\phi(\frac{\partial u_\phi}{r\partial\phi}\vec{e}_\phi - \frac{u_\phi}{r}\vec{e}_r) .$$

Combine the terms having the same pair of basis vectors (order matters so terms containing $\vec{e}_\phi\vec{e}_r$ and $\vec{e}_r\vec{e}_\phi$ cannot be combined)

$$\nabla\vec{u} = \frac{\partial u_r}{\partial r}\vec{e}_r\vec{e}_r + \frac{\partial u_\phi}{\partial r}\vec{e}_r\vec{e}_\phi + (\frac{\partial u_r}{r\partial\phi} - \frac{u_\phi}{r})\vec{e}_\phi\vec{e}_r + (\frac{u_r}{r} + \frac{\partial u_\phi}{r\partial\phi})\vec{e}_\phi\vec{e}_\phi .$$

Conjugate of a second order tensor can be obtained by swapping the basis vectors in all the pairs. Conjugate is a kind of transpose and can also be obtained by transposing the matrix of the component representation.

$$(\nabla\vec{u})_c = \frac{\partial u_r}{\partial r}\vec{e}_r\vec{e}_r + (\frac{\partial u_r}{r\partial\phi} - \frac{u_\phi}{r})\vec{e}_r\vec{e}_\phi + \frac{\partial u_\phi}{\partial r}\vec{e}_\phi\vec{e}_r + (\frac{u_r}{r} + \frac{\partial u_\phi}{r\partial\phi})\vec{e}_\phi\vec{e}_\phi$$

Finally using the definition $\vec{\vec{\varepsilon}} = \frac{1}{2}\left[\nabla\vec{u} + (\nabla\vec{u})_c\right]$

$$\vec{\vec{\varepsilon}} = \frac{\partial u_r}{\partial r}\vec{e}_r\vec{e}_r + \frac{1}{2}(\frac{\partial u_r}{r\partial\phi} - \frac{u_\phi}{r} + \frac{\partial u_\phi}{\partial r})(\vec{e}_r\vec{e}_\phi + \vec{e}_\phi\vec{e}_r) + (\frac{\partial u_\phi}{r\partial\phi} + \frac{u_r}{r})\vec{e}_\phi\vec{e}_\phi .$$

In the components of strain ε_{rr}, $\varepsilon_{r\phi}$, $\varepsilon_{\phi r}$ and $\varepsilon_{\phi\phi}$, indices are in the same order as the indices in the basis vector pairs. Hence

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\phi\phi} = \frac{\partial u_\phi}{r\partial\phi} + \frac{u_r}{r}, \quad \varepsilon_{r\phi} = \varepsilon_{\phi r} = \frac{1}{2}(\frac{\partial u_r}{r\partial\phi} - \frac{u_\phi}{r} + \frac{\partial u_\phi}{\partial r}) .$$