MEC-E8003 Beam, plate and shell models, examples 6

1. Curvilinear (s, α, n) – coordinate system of a shell is defined by mapping

$$\vec{r}(s, \alpha, n) = (R + n)\cos(\frac{s}{R})\vec{i} + (R + n)\sin(\frac{s}{R})\vec{j} + \alpha L \vec{k}$$

in which R and L are some constants, and n is the transverse coordinate. Derive in detail the expressions for the basis vectors, the non-zero basic vector derivatives, gradient operator, and the curvature tensor of mid-surface.

2. Derive the component forms of cylindrical shell moment equilibrium equations in the (z, ϕ, n) coordinate system starting from the invariant form $(\nabla_0 \cdot \vec{M} - \kappa \vec{e}_n \cdot \vec{M} - \vec{e}_n \cdot \vec{F} + \vec{c}) \times \vec{e}_n = 0$
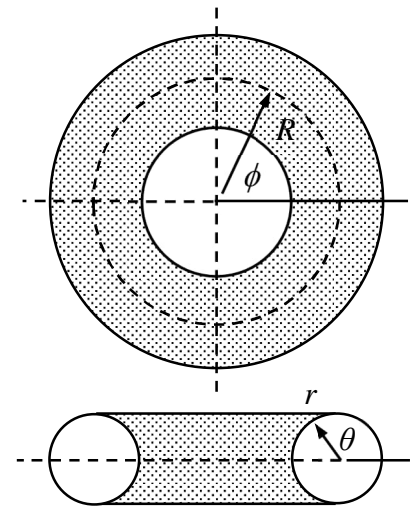
Answer
$$\begin{Bmatrix} \dfrac{\partial M_{zz}}{\partial z} + \dfrac{1}{R}\dfrac{\partial M_{\phi z}}{\partial \phi} - Q_z + c_z \\[4mm] \dfrac{\partial M_{z\phi}}{\partial z} + \dfrac{1}{R}\dfrac{\partial M_{\phi\phi}}{\partial \phi} - \dfrac{1}{R}M_{\phi n} - Q_\phi + c_\phi \end{Bmatrix} = 0$$

3. A thin walled cylindrical body of length L, (mid-surface) radius R, and thickness t is subjected to shear loading τt $[\tau t] = $ N/m at the free end $z = L$ as shown in the figure. Assuming rotation symmetry, use the membrane equations in (z, ϕ, n) coordinate system to derive the relationship between the moment resultant T (in the direction of z-axis) of the shear loading and the angle of rotation of the free end defined by $\theta = u_\phi / R$.
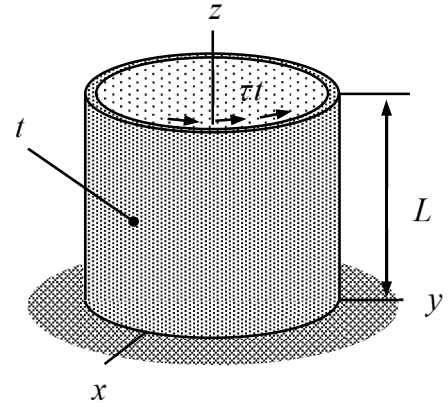
Answer $T = \dfrac{2\pi R^3 t}{L} G\theta$

4. Consider a torus shaped balloon under the loading caused by inner pressure difference Δp relative to the ambient pressure. Use (ϕ, θ, n) coordinate system, assume rotation symmetry with respect to ϕ, and solve for the stress resultant components from the equilibrium equations:

$$\begin{cases} \dfrac{1}{r}\dfrac{\partial N_{\phi\theta}}{\partial\theta}+\dfrac{1}{R+r\cos\theta}[\dfrac{\partial N_{\phi\phi}}{\partial\phi}-2N_{\phi\theta}\sin\theta]+b_{\phi} \\[4mm] \dfrac{1}{r}\dfrac{\partial N_{\theta\theta}}{\partial\theta}+\dfrac{1}{R+r\cos\theta}[\dfrac{\partial N_{\phi\theta}}{\partial\phi}+(N_{\phi\phi}-N_{\theta\theta})\sin\theta]+b_{\theta} \\[4mm] -\dfrac{1}{r}N_{\theta\theta}-\dfrac{1}{R+r\cos\theta}\cos\theta N_{\phi\phi}+b_{n} \end{cases}=0$$

Answer $N_{\theta\theta}=r\Delta p\dfrac{2R+r\cos\theta}{2R+2r\cos\theta}$, $N_{\phi\phi}=\dfrac{r\Delta p}{2}$

5. Displacement and rotation in rigid body motion are $\vec{u}=\vec{U}+\vec{\Omega}\times\vec{r}_0$ and $\vec{\Omega}$ in which $\vec{U}$ and $\vec{\Omega}$ are constant vectors in the Cartesian (x,y,z) coordinate system. Calculate the cylindrical shell stress resultant components $M_{z\phi}$ and $M_{\phi z}$ in rigid body mode $\Omega_x\neq0$ and $U_x=U_y=U_z=\Omega_y=\Omega_z=0$.

Answer $M_{z\phi}=M_{\phi z}=0$

6. A steel ring of length L, radius R, and thickness t is loaded by radial surface force p acting on the inner surface. No forces are acting on the ends. Model the ring as a cylindrical membrane, write down the equilibrium and constitutive equations, and solve for the radial displacement. Assume rotation symmetry. Young's modulus E and Poisson's ratio ν of the material are constants.
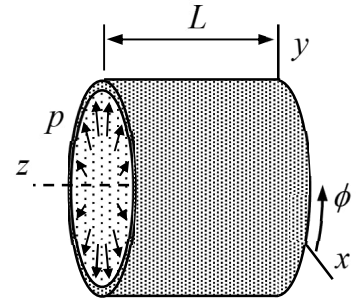
Answer $u_n=-\dfrac{R^2}{tE}p$

7. Consider a simply supported (long) circular cylindrical shell of radius R, thickness t, and filled with liquid of density ρ in cylindrical (z,ϕ,n) coordinates. Determine the mid-surface stress resultants $N_{\phi\phi}$, $N_{z\phi}$ and N_{zz} by assuming that there are no axial forces at the ends of the shell and bending deformation is negligible. (J.N.Reddy: Example 11.3.1)
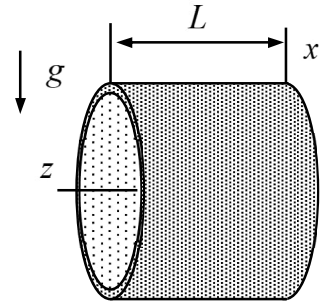
Answer $N_{\phi\phi}=-R\Delta p-\rho gR^2\cos\phi$, $N_{z\phi}=\rho gR(\dfrac{1}{2}L-z)\sin\phi+A$, $N_{zz}=\rho g\dfrac{1}{2}(z^2-zL)\cos\phi$

8. Consider a cylindrical shell of radius R, subjected to bending moment $M_{zz}=\underline{M}$ and shearing force $Q_z=\underline{Q}$ at the end $z=L$. The other end $z=0$ is clamped. Assuming rotational symmetry, derive the boundary value problem of Kirchhoff type for deflection $u_n(z)$. Start with the component forms of the Reissner-Mindlin (type) shell equations in cylindrical (z,ϕ,n) coordinates.

Answer $(1-\frac{a^2}{12})\frac{d^4 u_n}{dz^4}+\nu\frac{2}{R^2}\frac{d^2 u_n}{dz^2}+\frac{tE}{DR^2}u_n=0$ in $(0,L)$, $u_n=\frac{du_n}{dz}=0$ at $z=0$,

$$(1-\frac{1}{12}a^2)\frac{d^3 u_n}{dz^3}+\nu\frac{1}{R^2}\frac{du_n}{dz}+\frac{Q}{d}=0, \quad (1-\frac{1}{12}a^2)\frac{d^2 u_n}{dz^2}+\nu\frac{1}{R^2}u_n+\frac{M}{d}=0 \text{ at } z=L.$$

9. Consider a circular cylindrical shell of radius R, subjected to bending moment $M_{zz}=\underline{M}$ and shearing force $Q_z=\underline{Q}$ at the end $z=L$. The other end $z=0$ is clamped. Assuming rotational symmetry, derive the boundary value problem of Reissner-Mindlin type for deflection $u_n(z)$ and rotation $\theta_\phi(z)$.
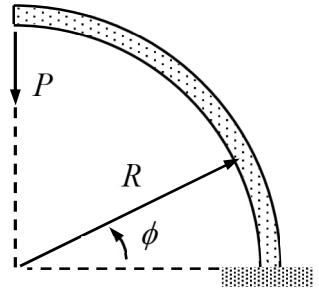
Answer $(Gt+\nu D\frac{1}{R^2})\frac{d\theta_\phi}{dz}+Gt\frac{d^2 u_n}{dz^2}-\frac{Et}{R^2}u_n=0$ in $(0,L)$,

$$D[(1-\frac{1}{12}a^2)\frac{d^2\theta_\phi}{dz^2}-\nu\frac{1}{R^2}\frac{du_n}{dz}]-Gt(\theta_\phi+\frac{du_n}{dz})=0 \text{ in } (0,L),$$

$$Gt(\theta_\phi+\frac{du_n}{dz})-\underline{Q}=0, \quad D[(1-\frac{1}{12}a^2)\frac{d\theta_\phi}{dz}-\nu\frac{1}{R^2}u_n]-\underline{M}=0 \text{ at } z=L,$$

$$u_n=0, \ \theta_\phi=0 \quad \text{at } z=0.$$

10. A strip of cylindrical shell is loaded by shear force P ($[P]=$ N/m) at the free end. Write down the boundary value problem of first order ordinary differential equations consisting of the equilibrium and constitutive equations according to the Kirchhoff theory. Thickness t, width H, and the material parameters E, ν are constants. Assume that the solution depends on ϕ only.

Answer $\left\{\begin{array}{l} \dfrac{1}{R}\dfrac{dN_{\phi\phi}}{d\phi}-\dfrac{1}{R}Q_\phi \\[2mm] \dfrac{1}{R}\dfrac{dQ_\phi}{d\phi}+\dfrac{1}{R}N_{\phi\phi} \\[2mm] \dfrac{1}{R}\dfrac{dM_{\phi\phi}}{d\phi}-Q_\phi \end{array}\right\}=0, \ \left\{\begin{array}{l} N_{\phi\phi}-\dfrac{tE}{1-\nu^2}\dfrac{1}{R}(\dfrac{du_\phi}{d\phi}-u_n)+D\dfrac{1}{R^2}\dfrac{d\theta_z}{d\phi} \\[2mm] M_{\phi\phi}-D[-\dfrac{1}{R}\dfrac{d\theta_z}{d\phi}+\dfrac{1}{R^2}(\dfrac{du_\phi}{d\phi}-u_n)] \\[2mm] \dfrac{1}{R}(\dfrac{du_n}{d\phi}+u_\phi)-\theta_z \end{array}\right\}=0$ in $(0,\dfrac{\pi}{2})$.

$\left\{\begin{array}{l} N_{\phi\phi} \\ Q_\phi-P \\ M_{\phi\phi} \end{array}\right\}=0$ at $\phi=\dfrac{\pi}{2}$, $\left\{\begin{array}{l} u_\phi \\ u_n \\ \theta_z \end{array}\right\}=0$ at $\phi=0$.

Curvilinear (s, α, n) – coordinate system of a shell is defined by mapping

$$\vec{r}(s, \alpha, n) = (R+n)\cos(\frac{s}{R})\vec{i} + (R+n)\sin(\frac{s}{R})\vec{j} + \alpha L \vec{k}$$

in which R and L are some constants, and n is the transverse coordinate. Derive in detail the expressions for the basis vectors, the non-zero basic vector derivatives, gradient operator, and the curvature tensor of mid-surface.

Solution

Let us start with the partial derivatives with respect to the curvilinear coordinates

$$\begin{Bmatrix} \vec{h}_s \\ \vec{h}_\alpha \\ \vec{h}_n \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial \vec{r}}{\partial s} \\ \dfrac{\partial \vec{r}}{\partial \alpha} \\ \dfrac{\partial \vec{r}}{\partial n} \end{Bmatrix} = \begin{bmatrix} -\dfrac{R+n}{R}\sin(\dfrac{s}{R}) & \dfrac{R+n}{R}\cos(\dfrac{s}{R}) & 0 \\ 0 & 0 & L \\ \cos(\dfrac{s}{R}) & \sin(\dfrac{s}{R}) & 0 \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = [H] \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}$$

Scaling factor are the length of partial derivatives $h_s = 1 + n/R$, $h_\alpha = L$, and $h_n = 1$. Therefore, the basis vectors are

$$\begin{Bmatrix} \vec{e}_s \\ \vec{e}_\alpha \\ \vec{e}_n \end{Bmatrix} = \begin{Bmatrix} \vec{h}_s / h_s \\ \vec{h}_\alpha / h_\alpha \\ \vec{h}_n / h_n \end{Bmatrix} = \begin{bmatrix} -\sin(\dfrac{s}{R}) & \cos(\dfrac{s}{R}) & 0 \\ 0 & 0 & 1 \\ \cos(\dfrac{s}{R}) & \sin(\dfrac{s}{R}) & 0 \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = [F] \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}. \quad \leftarrow$$

As the coordinate systems are orthonormal $[F]^{-1} = [F]^{T}$ and the non-zero expressions for the partial derivatives

$$\frac{\partial}{\partial s} \begin{Bmatrix} \vec{e}_s \\ \vec{e}_\alpha \\ \vec{e}_n \end{Bmatrix} = \frac{1}{R} \begin{bmatrix} -\cos(\dfrac{s}{R}) & -\sin(\dfrac{s}{R}) & 0 \\ 0 & 0 & 0 \\ -\sin(\dfrac{s}{R}) & \cos(\dfrac{s}{R}) & 0 \end{bmatrix} \begin{bmatrix} -\sin(\dfrac{s}{R}) & 0 & \cos(\dfrac{s}{R}) \\ \cos(\dfrac{s}{R}) & 0 & \sin(\dfrac{s}{R}) \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} \vec{e}_s \\ \vec{e}_\alpha \\ \vec{e}_n \end{Bmatrix} = \frac{1}{R} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \vec{e}_s \\ \vec{e}_\alpha \\ \vec{e}_n \end{Bmatrix}$$

$$\frac{\partial \vec{e}_s}{\partial s} = -\frac{1}{R}\vec{e}_n \quad \text{and} \quad \frac{\partial \vec{e}_n}{\partial s} = \frac{1}{R}\vec{e}_s. \quad \leftarrow$$

Gradient operator follows from the definition ($[H][F]^{T} = \mathrm{diag}(\{h_s, h_\alpha, h_n\})$)

$$\nabla = \left\{\begin{array}{c}\vec{e}_s \\ \vec{e}_\alpha \\ \vec{e}_n\end{array}\right\}^{\mathrm{T}} [F]^{-\mathrm{T}}[H]^{-1}\left\{\begin{array}{c}\dfrac{\partial}{\partial s} \\[2mm] \dfrac{\partial}{\partial \alpha} \\[2mm] \dfrac{\partial}{\partial n}\end{array}\right\} = \left\{\begin{array}{c}\vec{e}_s \\ \vec{e}_\alpha \\ \vec{e}_n\end{array}\right\}^{\mathrm{T}} ([H][F]^{\mathrm{T}})^{-1}\left\{\begin{array}{c}\dfrac{\partial}{\partial s} \\[2mm] \dfrac{\partial}{\partial \alpha} \\[2mm] \dfrac{\partial}{\partial n}\end{array}\right\} \quad \Rightarrow$$

$$\nabla = \vec{e}_s \frac{R}{R+n}\frac{\partial}{\partial s} + \vec{e}_\alpha \frac{1}{L}\frac{\partial}{\partial \alpha} + \vec{e}_n \frac{\partial}{\partial n}. \quad \leftarrow$$

Curvature tensor follows from definition at the mid-surface $n = 0$

$$\nabla \vec{e}_n = (\vec{e}_s \frac{R}{R+n}\frac{\partial}{\partial s} + \vec{e}_\alpha \frac{1}{L}\frac{\partial}{\partial \alpha} + \vec{e}_n \frac{\partial}{\partial n})\vec{e}_n = \frac{1}{R+n}\vec{e}_s\vec{e}_s \quad \Rightarrow \quad \ddot{\kappa} = (\nabla\vec{e}_n)_{\mathrm{c}} = \frac{1}{R}\vec{e}_s\vec{e}_s. \quad \leftarrow$$

Derive the component forms of cylindrical shell moment equilibrium equations in the (z, ϕ, n) coordinate system starting from the invariant form $(\nabla_0 \cdot \vec{\vec{M}} - \kappa \vec{e}_n \cdot \vec{\vec{M}} - \vec{e}_n \cdot \vec{\vec{F}} + \vec{c}) \times \vec{e}_n = 0$.

Solution

Component representations of the quantities in the equilibrium equation are (notice that the transverse normal components are missing)

$$\vec{\vec{F}} = N_{zz} \vec{e}_z \vec{e}_z + N_{z\phi} \vec{e}_z \vec{e}_\phi + N_{\phi z} \vec{e}_\phi \vec{e}_z + N_{\phi\phi} \vec{e}_\phi \vec{e}_\phi + Q_z \vec{e}_z \vec{e}_n + Q_z \vec{e}_n \vec{e}_z + Q_\phi \vec{e}_\phi \vec{e}_n + Q_\phi \vec{e}_n \vec{e}_\phi,$$

$$\vec{\vec{M}} = M_{zz} \vec{e}_z \vec{e}_z + M_{z\phi} \vec{e}_z \vec{e}_\phi + M_{\phi z} \vec{e}_\phi \vec{e}_z + M_{\phi\phi} \vec{e}_\phi \vec{e}_\phi + M_{zn} \vec{e}_z \vec{e}_n + M_{nz} \vec{e}_n \vec{e}_z + M_{\phi n} \vec{e}_\phi \vec{e}_n + M_{n\phi} \vec{e}_n \vec{e}_\phi,$$

$$\vec{b} = b_z \vec{e}_z + b_\phi \vec{e}_\phi + b_n \vec{e}_n, \text{ and } \nabla_0 = \vec{e}_z \frac{\partial}{\partial z} + \vec{e}_\phi \frac{1}{R} \frac{\partial}{\partial \phi}.$$

In the shell model, the stress resultants may not be symmetric although the stress $\vec{\vec{\sigma}}$ always is. Derivatives of the basis vectors, the unit tensor and curvature tensor are

$$\frac{\partial}{\partial \phi} \vec{e}_\phi = \vec{e}_n, \quad \frac{\partial}{\partial \phi} \vec{e}_n = -\vec{e}_\phi, \quad \vec{\vec{I}} = \vec{e}_z \vec{e}_z + \vec{e}_\phi \vec{e}_\phi + \vec{e}_n \vec{e}_n,$$

$$\vec{\vec{\kappa}} = (\nabla_0 \vec{e}_n)_c = (\vec{e}_z \frac{\partial \vec{e}_n}{\partial z} + \vec{e}_\phi \frac{1}{R} \frac{\partial \vec{e}_n}{\partial \phi})_c = -\vec{e}_\phi \vec{e}_\phi \frac{1}{R} \quad \Rightarrow \quad \kappa = \vec{\vec{I}} : \vec{\vec{\kappa}} = -\frac{1}{R}.$$

Let us consider the mid-surface and transverse parts of the moment separately to shorten the expressions, First the mid-surface part

$$\nabla_0 \cdot \vec{\vec{M}}_m = (\vec{e}_z \frac{\partial}{\partial z} + \vec{e}_\phi \frac{1}{R} \frac{\partial}{\partial \phi}) \cdot (M_{zz} \vec{e}_z \vec{e}_z + M_{z\phi} \vec{e}_z \vec{e}_\phi + M_{\phi z} \vec{e}_\phi \vec{e}_z + M_{\phi\phi} \vec{e}_\phi \vec{e}_\phi) \text{ where}$$

$$(\vec{e}_z \frac{\partial}{\partial z}) \cdot (M_{zz} \vec{e}_z \vec{e}_z + M_{z\phi} \vec{e}_z \vec{e}_\phi + M_{\phi z} \vec{e}_\phi \vec{e}_z + M_{\phi\phi} \vec{e}_\phi \vec{e}_\phi) = \frac{\partial M_{zz}}{\partial z} \vec{e}_z + \frac{\partial M_{z\phi}}{\partial z} \vec{e}_\phi,$$

$$(\vec{e}_\phi \frac{1}{R} \frac{\partial}{\partial \phi}) \cdot (M_{zz} \vec{e}_z \vec{e}_z + M_{z\phi} \vec{e}_z \vec{e}_\phi + M_{\phi z} \vec{e}_\phi \vec{e}_z + M_{\phi\phi} \vec{e}_\phi \vec{e}_\phi) = \frac{1}{R} \frac{\partial M_{\phi z}}{\partial \phi} \vec{e}_z + \frac{1}{R} \frac{\partial M_{\phi\phi}}{\partial \phi} \vec{e}_\phi + \frac{1}{R} M_{\phi\phi} \vec{e}_n \quad \Rightarrow$$

$$\nabla_0 \cdot \vec{\vec{M}}_m = (\frac{\partial M_{zz}}{\partial z} + \frac{1}{R} \frac{\partial M_{\phi z}}{\partial \phi}) \vec{e}_z + (\frac{\partial M_{z\phi}}{\partial z} + \frac{1}{R} \frac{\partial M_{\phi\phi}}{\partial \phi}) \vec{e}_\phi + \frac{1}{R} M_{\phi\phi} \vec{e}_n.$$

Then, the transverse part

$$\nabla_0 \cdot \vec{\vec{M}}_t = (\vec{e}_z \frac{\partial}{\partial z} + \vec{e}_\phi \frac{1}{R} \frac{\partial}{\partial \phi}) \cdot (M_{zn} \vec{e}_z \vec{e}_n + M_{\phi n} \vec{e}_\phi \vec{e}_n + M_{nz} \vec{e}_n \vec{e}_z + M_{n\phi} \vec{e}_n \vec{e}_\phi) \text{ where}$$

$$(\vec{e}_z \frac{\partial}{\partial z}) \cdot (M_{zn} \vec{e}_z \vec{e}_n + M_{\phi n} \vec{e}_\phi \vec{e}_n + M_{nz} \vec{e}_n \vec{e}_z + M_{n\phi} \vec{e}_n \vec{e}_\phi) = \frac{\partial M_{zn}}{\partial z} \vec{e}_n,$$

$$(\vec{e}_\phi \frac{1}{R}\frac{\partial}{\partial\phi}) \cdot (M_{zn}\vec{e}_z\vec{e}_n + M_{\phi n}\vec{e}_\phi\vec{e}_n + M_{nz}\vec{e}_n\vec{e}_z + M_{n\phi}\vec{e}_n\vec{e}_\phi) = \frac{1}{R}(\frac{\partial M_{\phi n}}{\partial\phi}\vec{e}_n - M_{\phi n}\vec{e}_\phi - M_{nz}\vec{e}_z - M_{n\phi}\vec{e}_\phi).$$

$$\nabla_0 \cdot \vec{\vec{M}}_t = (\frac{\partial M_{zn}}{\partial z} + \frac{1}{R}\frac{\partial M_{\phi n}}{\partial\phi})\vec{e}_n - \frac{1}{R}M_{nz}\vec{e}_z - \frac{1}{R}(M_{n\phi} + M_{\phi n})\vec{e}_\phi.$$

The second term of the equilibrium equation simplifies to

$$\kappa\vec{e}_n \cdot \vec{\vec{M}} = -\frac{1}{R}(M_{nz}\vec{e}_z + M_{n\phi}\vec{e}_\phi).$$

Third term

$$\vec{e}_n \cdot \vec{\vec{F}} = Q_z\vec{e}_z + Q_\phi\vec{e}_\phi$$

Finally, combining the terms (all terms in the normal direction vanish due to the cross product with $\vec{e}_n$)

$$\begin{Bmatrix} \dfrac{\partial M_{zz}}{\partial z} + \dfrac{1}{R}\dfrac{\partial M_{\phi z}}{\partial\phi} - Q_z + c_z \\[3mm] \dfrac{\partial M_{z\phi}}{\partial z} + \dfrac{1}{R}\dfrac{\partial M_{\phi\phi}}{\partial\phi} - \dfrac{1}{R}M_{\phi n} - Q_\phi + c_\phi \end{Bmatrix} = 0. \quad \leftarrow$$

A thin walled cylindrical body of length L, (mid-surface) radius R, and thickness t is subjected to shear loading τt $[\tau t] = $ N/m at the free end $z = L$ as shown in the figure. Assuming rotation symmetry, use the membrane equations in (z, ϕ, n) coordinate system to derive the relationship between the moment resultant T of the shear loading and the angle of rotation of the free end defined by $\theta = u_\phi / R$.
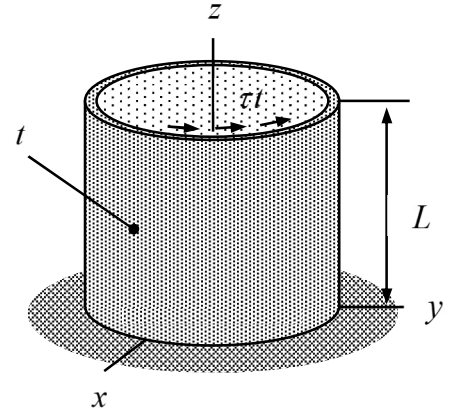
Solution

As the solution does not depend on ϕ, equilibrium equations of the membrane model and boundary conditions at the free end simplify to (a cylindrical membrane $z -$ strip problem)

$$\frac{dN_{zz}}{dz} = 0, \quad \frac{dN_{z\phi}}{dz} = 0, \quad \frac{1}{R}N_{\phi\phi} = 0 \quad \text{in} \ (0, L),$$

$N_{zz} = 0$ and $N_{z\phi} = \tau t$ at $z = L$.

Solution to the boundary value problem for the stress resultants is given by

$$N_{zz} = N_{\phi\phi} = 0 \quad \text{and} \quad N_{z\phi}(z) = \tau t. \quad \leftarrow$$

Knowing the stress resultants, the boundary value problem for the displacement components follows from the constitutive equations and the boundary conditions at the fixed end (in the membrane model, a boundary condition cannot be assigned to u_n)

$$\frac{tE}{1-v^2}(\frac{du_z}{dz} - v\frac{1}{R}u_n) = 0, \quad \frac{tE}{1-v^2}(v\frac{du_z}{dz} - \frac{1}{R}u_n) = 0, \quad \text{and} \quad tG\frac{du_\phi}{dz} = \tau t \quad \text{in} \ (0, L),$$

$u_z = 0, \ u_\phi = 0$ at $z = 0$.

Solution to the boundary value problem is given by

$$u_z = u_n = 0 \quad \text{and} \quad u_\phi(z) = \frac{\tau}{G}z.$$

Moment resultant of the shear loading

$$T = \int_0^{2\pi} t\tau R(Rd\phi) = 2\pi R^2 t\tau \quad \Rightarrow \quad \tau = \frac{T}{2\pi R^2 t}.$$
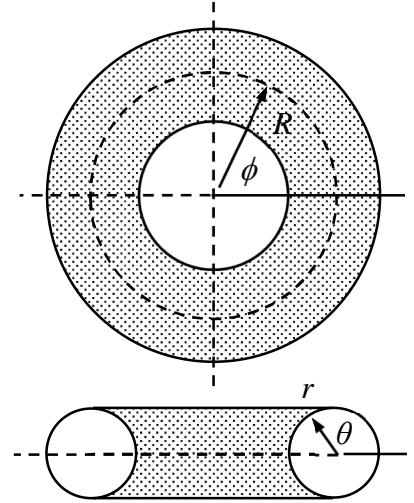
Therefore, at the free end

$$u_\phi = \frac{\tau}{G}L = \frac{L}{2\pi R^2 tG}T = R\theta \quad \Rightarrow \quad T = \frac{2\pi R^3 t}{L}G\theta. \quad \leftarrow$$

The polar moment predicted here is $I_p = 2\pi R^3 t$ whereas the exact is $I_p = \dfrac{1}{2}\pi Rt(4R^2 + t^2)$.

Consider a torus shaped balloon under the loading caused by inner pressure difference Δp relative to the ambient pressure. Use (ϕ, θ, n) coordinate system, assume rotation symmetry with respect to ϕ, and solve for the stress resultant components from the equilibrium equations:

$$\begin{cases} \dfrac{1}{r}\dfrac{\partial N_{\phi\theta}}{\partial \theta} + \dfrac{1}{R+r\cos\theta}[\dfrac{\partial N_{\phi\phi}}{\partial \phi} - 2N_{\phi\theta}\sin\theta] + b_\phi \\[3mm] \dfrac{1}{r}\dfrac{\partial N_{\theta\theta}}{\partial \theta} + \dfrac{1}{R+r\cos\theta}[\dfrac{\partial N_{\phi\theta}}{\partial \phi} + (N_{\phi\phi} - N_{\theta\theta})\sin\theta] + b_\theta \\[3mm] -\dfrac{1}{r}N_{\theta\theta} - \dfrac{1}{R+r\cos\theta}\cos\theta N_{\phi\phi} + b_n \end{cases} = 0 .$$

Solution

As the solution should be independent of ϕ, partial derivatives with respect to ϕ vanish and the equilibrium equations of torus geometry simplify to ordinary diffrential equations. In toroidal system $\vec{e}_n$ is directed outwards and therefore $b_n = \Delta p$:

$$\dfrac{dN_{\phi\theta}}{d\theta} - \dfrac{2r\sin\theta}{R+r\cos\theta}N_{\phi\theta} = 0 \quad \Rightarrow \quad N_{\phi\theta} = 0 \quad \text{(clearly a solution)},$$

$$\dfrac{dN_{\theta\theta}}{d\theta} + \dfrac{r\sin\theta}{R+r\cos\theta}(N_{\phi\phi} - N_{\theta\theta}) = 0 ,$$

$$N_{\theta\theta} + \dfrac{r\cos\theta}{R+r\cos\theta}N_{\phi\phi} - r\Delta p = 0 \quad \Leftrightarrow \quad \dfrac{r\cos\theta}{R+r\cos\theta}N_{\phi\phi} = r\Delta p - N_{\theta\theta} .$$

Eliminating $N_{\phi\phi}$ from the last two equations gives

$$\dfrac{dN_{\theta\theta}}{d\theta} + \tan\theta(\dfrac{r\cos\theta}{R+r\cos\theta}N_{\phi\phi}) - \dfrac{r\sin\theta}{R+r\cos\theta}N_{\theta\theta} = 0 \quad \Leftrightarrow$$

$$\dfrac{dN_{\theta\theta}}{d\theta} + \tan\theta(r\Delta p - N_{\theta\theta}) - \dfrac{r\sin\theta}{R+r\cos\theta}N_{\theta\theta} = 0 \quad \Leftrightarrow$$

$$\dfrac{dN_{\theta\theta}}{d\theta} - (\dfrac{R+2r\cos\theta}{R+r\cos\theta})\tan\theta N_{\theta\theta} + r\Delta p\tan\theta = 0 .$$

Solution to the equation can be obtained by using an integrating factor. Let us write the differential equation in form

$$e^A\dfrac{dN_{\theta\theta}}{d\theta} + e^A\dfrac{dA}{d\theta}N_{\theta\theta} + e^A r\Delta p\tan\theta = \dfrac{d}{d\theta}(e^A N_{\theta\theta}) + e^A r\Delta p\tan\theta = 0 ,$$

$$\frac{dA}{d\theta} = -\left(\frac{R + 2r\cos\theta}{R + r\cos\theta}\right)\tan\theta \quad \Leftrightarrow \quad A = \log(R\cos\theta + r\cos^2\theta) \quad \Leftrightarrow \quad e^A = \cos\theta(R + r\cos\theta).$$

Continuing with the other equation of the set (integration constant is zero as stress resultant should vanish when $\Delta p = 0$)

$$\frac{d}{d\theta}(e^A N_{\theta\theta}) = -e^A r\Delta p \tan\theta = -\sin\theta(R + r\cos\theta)r\Delta p \quad \Rightarrow$$

$$e^A N_{\theta\theta} = \Delta p\, r\cos\theta(R + \frac{1}{2}r\cos\theta) \quad \Rightarrow$$

$$N_{\theta\theta} = r\Delta p\,\frac{2R + r\cos\theta}{2R + 2r\cos\theta} \quad \text{and} \quad N_{\phi\phi} = \frac{r\Delta p}{2}. \quad \leftarrow$$

Displacement and rotation in rigid body motion are $\vec{u} = \vec{U} + \vec{\Omega} \times \vec{r}_0$ and $\vec{\Omega}$ in which $\vec{U}$ and $\vec{\Omega}$ are constant vectors in the Cartesian (x, y, z) coordinate system. Calculate the cylindrical shell stress resultant components $M_{z\phi}$ and $M_{\phi z}$ in rigid body mode $\Omega_x \neq 0$ and $U_x = U_y = U_z = \Omega_y = \Omega_z = 0$.

Solution

The representations of the quantities in the cylindrical (z, ϕ, n) coordinate system can be obtained from the relationship between the basis vectors of the Cartesian and cylindrical system

$$
\begin{Bmatrix} \vec{e}_z \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -\sin\phi & \cos\phi & 0 \\ -\cos\phi & -\sin\phi & 0 \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} \quad \Leftrightarrow \quad \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -\sin\phi & \cos\phi & 0 \\ -\cos\phi & -\sin\phi & 0 \end{bmatrix}^T \begin{Bmatrix} \vec{e}_z \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix}.
$$

Therefore

$$
\vec{U} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^T \begin{Bmatrix} U_x \\ U_y \\ U_z \end{Bmatrix} = \begin{Bmatrix} \vec{e}_z \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix}^T \begin{bmatrix} 0 & 0 & 1 \\ -\sin\phi & \cos\phi & 0 \\ -\cos\phi & -\sin\phi & 0 \end{bmatrix} \begin{Bmatrix} U_x \\ U_y \\ U_z \end{Bmatrix} = \begin{Bmatrix} \vec{e}_z \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix}^T \begin{Bmatrix} U_z \\ U_y \cos\phi - U_x \sin\phi \\ -U_x \cos\phi - U_y \sin\phi \end{Bmatrix},
$$

$$
\vec{\Omega} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^T \begin{Bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{Bmatrix} = \begin{Bmatrix} \vec{e}_z \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix}^T \begin{bmatrix} 0 & 0 & 1 \\ -\sin\phi & \cos\phi & 0 \\ -\cos\phi & -\sin\phi & 0 \end{bmatrix} \begin{Bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{Bmatrix} = \begin{Bmatrix} \vec{e}_z \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix}^T \begin{Bmatrix} \Omega_z \\ \Omega_y \cos\phi - \Omega_x \sin\phi \\ -\Omega_x \cos\phi - \Omega_y \sin\phi \end{Bmatrix},
$$

$$
\vec{r}_0 = \vec{e}_z z - \vec{e}_n R .
$$

The displacement and rotation components due to rigid body motion are (rotation of a typical line segment aligned in the normal direction does not have a component in the normal direction)

$$
\vec{u} = \vec{U} + \vec{\Omega} \times \vec{r}_0 = \begin{Bmatrix} \vec{e}_z \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix}^T \begin{Bmatrix} U_z + R\Omega_x \sin\phi - R\Omega_y \cos\phi \\ U_y \cos\phi - U_x \sin\phi + R\Omega_z - z\Omega_x \cos\phi - z\Omega_y \sin\phi \\ -U_x \cos\phi - U_y \sin\phi + z\Omega_x \sin\phi - z\Omega_y \cos\phi \end{Bmatrix},
$$

$$
\vec{\theta} = \begin{Bmatrix} \vec{e}_z \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix}^T \begin{Bmatrix} \Omega_z \\ \Omega_y \cos\phi - \Omega_x \sin\phi \\ 0 \end{Bmatrix}.
$$

Let us consider the rigid body mode obtained with $\Omega_x \neq 0$ and $U_x = U_y = U_z = \Omega_y = \Omega_z = 0$ and substitute the components obtained into the constitutive equations for $M_{z\phi}$ and $M_{\phi z}$ of the formulae collection:

$$u_z = \Omega_x R \sin \phi, \quad u_\phi = -\Omega_x z \cos \phi, \quad u_n = \Omega_x z \sin \phi, \quad \theta_z = 0, \text{ and } \quad \theta_\phi = -\Omega_x \sin \phi \quad \Rightarrow$$

$$M_{z\phi} = D \frac{1-\nu}{2} \left(-\frac{\partial \theta_z}{\partial z} + \frac{1}{R} \frac{\partial \theta_\phi}{\partial \phi} - \frac{1}{R} \frac{\partial u_\phi}{\partial z} \right) = D \frac{1-\nu}{2} \Omega_x \left(-\frac{1}{R} \cos \phi + \frac{1}{R} \cos \phi \right) = 0. \quad \leftarrow$$

$$M_{\phi z} = D \frac{1-\nu}{2} \left(-\frac{\partial \theta_z}{\partial z} + \frac{1}{R} \frac{\partial \theta_\phi}{\partial \phi} + \frac{1}{R^2} \frac{\partial u_z}{\partial \phi} \right) = D \frac{1-\nu}{2} \Omega_x \left(-\frac{1}{R} \cos \phi + \frac{1}{R} \cos \phi \right) = 0. \quad \leftarrow$$

A steel ring of length L, radius R, and thickness t is loaded by radial surface force p acting on the inner surface. No forces are acting on the ends. Model the ring as a cylindrical membrane, write down the equilibrium and constitutive equations, and solve for the radial displacement. Assume rotation symmetry and $u_\phi = 0$. Young's modulus E and Poisson's ratio g of the material are constants.
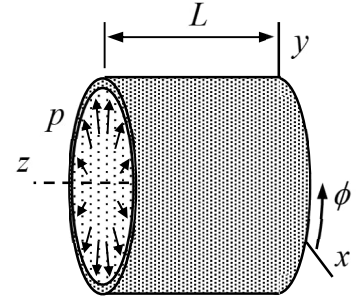
Solution

According to the formulae collection, equilibrium and constitutive equations of a cylindrical membrane in (z, ϕ, n) coordinates are (notice that $\vec{e}_n$ is directed inwards)

$$\begin{cases} \dfrac{1}{R}\dfrac{\partial N_{z\phi}}{\partial \phi} + \dfrac{\partial N_{zz}}{\partial z} + b_z \\[2mm] \dfrac{\partial N_{z\phi}}{\partial z} + \dfrac{1}{R}\dfrac{\partial N_{\phi\phi}}{\partial \phi} + b_\phi \\[2mm] \dfrac{1}{R}N_{\phi\phi} + b_n \end{cases} = 0, \quad \begin{Bmatrix} N_{zz} \\ N_{\phi\phi} \\ N_{z\phi} \end{Bmatrix} = \dfrac{tE}{1-v^2} \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-v) \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u_z}{\partial z} \\[2mm] \dfrac{1}{R}(\dfrac{\partial u_\phi}{\partial \phi} - u_n) \\[2mm] \dfrac{1}{R}\dfrac{\partial u_z}{\partial \phi} + \dfrac{\partial u_\phi}{\partial z} \end{Bmatrix} .$$

Due to the rotation symmetry, the derivatives with respect to the angular coordinate vanish and $u_\phi = 0$. External distributed force $b_n = -p$ is due to the traction acting on the inner boundary. Therefore, the equilibrium equations and constitutive equations simplify to a set of ordinary differential equations

$$\dfrac{dN_{zz}}{dz} = 0, \quad \dfrac{dN_{z\phi}}{dz} = 0, \quad \dfrac{1}{R}N_{\phi\phi} - p = 0 \quad \text{in } (0,L),$$

$$N_{zz} = \dfrac{tE}{1-v^2}\dfrac{1}{R}(R\dfrac{du_z}{dz} - vu_n), \quad N_{\phi\phi} = \dfrac{tE}{1-v^2}\dfrac{1}{R}(Rv\dfrac{du_z}{dz} - u_n), \quad N_{z\phi} = 0 \quad \text{in } (0,L),$$

As the edges are stress-free i.e.

$N_{zz} = 0$ and $N_{z\phi} = 0$ on $\{0, L\}$.

Solution to the stress resultants, as obtained from the equilibrium equations, are

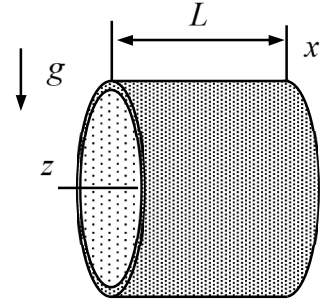$N_{zz} = 0$, $N_{z\phi} = 0$, and $N_{\phi\phi} = Rp$.

Constitutive equations give

$$N_{zz} = \dfrac{tE}{1-v^2}\dfrac{1}{R}(R\dfrac{du_z}{dz} - vu_n) = 0 \quad \Rightarrow \quad \dfrac{du_z}{dz} = \dfrac{v}{R}u_n \quad \text{and}$$

$$Rp = N_{\phi\phi} = \dfrac{tE}{1-v^2}\dfrac{1}{R}(Rv\dfrac{du_z}{dz} - u_n) = \dfrac{tE}{1-v^2}\dfrac{1}{R}(v^2 - 1)u_n = -\dfrac{tE}{R}u_n \quad \Leftrightarrow \quad u_n = -\dfrac{pR^2}{tE}. \quad \textcolor{red}{\leftarrow}$$

Consider a simply supported (long) circular cylindrical shell of radius R, thickness t, and filled with liquid of density ρ in cylindrical (z, ϕ, n) – coordinates. Determine the mid-surface stress resultants $N_{\phi\phi}$, $N_{z\phi}$ and N_{zz} by assuming that there are no axial forces at the ends of the shell and bending deformation is negligible. (J.N.Reddy: Example 11.3.1)

Solution

The membrane equations of the cylindrical coordinate system are (formulae collection)

$$\frac{1}{R}\frac{\partial N_{z\phi}}{\partial \phi} + \frac{\partial N_{zz}}{\partial z} + b_z = 0, \quad \frac{\partial N_{z\phi}}{\partial z} + \frac{1}{R}\frac{\partial N_{\phi\phi}}{\partial \phi} + b_\phi = 0, \text{ and } \frac{1}{R}N_{\phi\phi} + b_n = 0.$$

Definition of the external distributed force (let us assume that $t/R \ll 1$ so that $J \approx 1$ to simplify the setting somewhat)
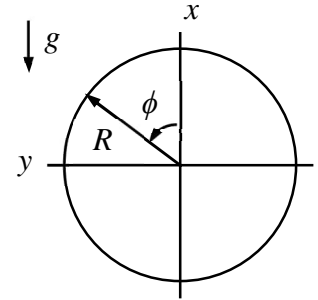
$$\vec{b} = \int \vec{f}dn + \sum \vec{t}$$

takes into account the volume forces acting on the body and tractions acting on the outer and inner surfaces. In the present case $\vec{f} = 0$ and the traction part is due to the hydrostatic pressure of the liquid inside the cylinder. Therefore

$$\vec{b} = (p_{out} - p_{in})\vec{e}_n.$$

The hydrostatic pressure inside $p_{in} = p_0 - \rho gx = p_0 - \rho gR\cos\phi$ gives

$$p_{out} - p_{in} = p_{out} - (p_0 - \rho gR\cos\phi) = \Delta p + \rho gR\cos\phi$$

in which $\Delta p = p_{out} - p_0$ is a constant. The equations to be solved become (notice that $N_{zz}(z,\phi)$. $N_{\phi\phi}(z,\phi)$, $N_{z\phi}(z,\phi)$ and direct integration of a partial differential equation involves unknown functions instead of integration constants)

$$\frac{1}{R}N_{\phi\phi} + \Delta p + \rho gR\cos\phi = 0 \quad \Leftrightarrow \quad N_{\phi\phi}(z,\phi) = -R\Delta p - \rho gR^2\cos\phi,$$

$$\frac{\partial N_{z\phi}}{\partial z} + \frac{1}{R}\frac{\partial N_{\phi\phi}}{\partial \phi} = 0 \quad \Rightarrow \quad \frac{\partial N_{z\phi}}{\partial z} + \rho gR\sin\phi = 0 \quad \Leftrightarrow \quad N_{z\phi}(z,\phi) = -\rho gRz\sin\phi + A(\phi),$$

$$\frac{1}{R}\frac{\partial N_{z\phi}}{\partial \phi} + \frac{\partial N_{zz}}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial N_{zz}}{\partial z} = -\frac{1}{R}\frac{\partial N_{z\phi}}{\partial \phi} \quad \Rightarrow \quad \frac{\partial N_{zz}}{\partial z} = \rho gz\cos\phi - \frac{1}{R}A'(\phi) \quad \Leftrightarrow$$

$$N_{zz}(z,\phi) = \rho g\frac{1}{2}z^2\cos\phi - z\frac{1}{R}A'(\phi) + B(\phi).$$

In the solution, $A(\phi)$ and $B(\phi)$ are arbitrary functions subjected to $A(\phi) = A(2\pi + \phi)$ and $B(\phi) = B(2\pi + \phi)$ $\forall \phi$ (periodicity) as the domain is closed in the $\phi-$direction. Also, according to the assumption, N_{zz} vanishes at the ends. Therefore

$$N_{zz}(z,\phi) = \rho g \frac{1}{2} z^2 \cos\phi - z \frac{1}{R} A'(\phi) + B(\phi) = 0 \qquad z \in \{0, L\} \quad \Rightarrow$$

$$B(\phi) = 0 \ \text{ and } \ \rho g \frac{1}{2} L^2 \cos\phi - L \frac{1}{R} A'(\phi) + B(\phi) = 0 \quad \Rightarrow \quad A'(\phi) = \rho g R \frac{1}{2} L \cos\phi \quad \Leftrightarrow$$

$$A(\phi) = \rho g R \frac{1}{2} L \sin\phi + A \ \text{ (a constant now).}$$

Solution to force resultants becomes

$$N_{\phi\phi} = -R\Delta p_0 - \rho g R^2 \cos\phi, \qquad \leftarrow$$

$$N_{z\phi} = \rho g R (\frac{1}{2} L - z) \sin\phi + A, \qquad \leftarrow$$

$$N_{zz} = \rho g \frac{1}{2} (z^2 - zL) \cos\phi \qquad \leftarrow$$

in which pressure difference Δp and integration constant A cannot be determined with the information given.

Consider a cylindrical shell of radius R, subjected to bending moment $M_{zz} = \underline{M}$ and shearing force $Q_z = \underline{Q}$ at the end $z = L$. The other end $z = 0$ is clamped. Assuming rotational symmetry, derive the boundary value problem of Kirchhoff type for deflection $u_n(z)$. Start with the component forms of the Reissner-Mindlin (type) shell equations in cylindrical (z, ϕ, n) coordinates.

Solution

Cylindrical shell Reissner-Mindlin equilibrium and constitutive equations in (z, ϕ, n)-coordinate system are given in the formulae collection. Under the assumption of rotational symmetry, derivatives with respect to ϕ vanish and $u_\phi = 0$. As the loading is through the boundary conditions, the equilibrium equations of cylindrical shell in (z, ϕ, n) – coordinate system simplify to

$$\frac{dN_{zz}}{dz} = 0, \quad \frac{dM_{zz}}{dz} - Q_z = 0, \text{ and } \frac{dQ_z}{dz} + \frac{1}{R}N_{\phi\phi} = 0.$$

Constitutive equations for the stress resultant components simplify to (notice that the constitutive equation for the shear force is replaced by the Kirchhoff constraint $du_n / dz + \theta_\phi = 0$ which is used to eliminate rotation θ_ϕ from the constitutive equations)

$$N_{zz} = \frac{tE}{1-v^2}(\frac{du_z}{dz} - v\frac{1}{R}u_n) + D\frac{1}{R}\frac{d^2 u_n}{dz^2},$$

$$N_{\phi\phi} = \frac{Et}{1-v^2}(v\frac{du_z}{dz} - \frac{1}{R}u_n), \text{ and } M_{zz} = -D(\frac{d^2 u_n}{dz^2} + \frac{1}{R}\frac{du_z}{dz}).$$

Force equilibrium equation in the axial boundary condition, constitutive equation for the axial stress resultant and the boundary condition at the free edge give

$$\frac{dN_{zz}}{dz} = 0 \text{ in } (0, L) \text{ and } N_{zz}(L) = 0 \implies N_{zz} = 0, \text{ therefore}$$

$$N_{zz} = \frac{tE}{1-v^2}(\frac{du_z}{dz} - v\frac{1}{R}u_n) + D\frac{1}{R}\frac{d^2 u_n}{dz^2} = 0 \implies \frac{du_z}{dz} = v\frac{1}{R}u_n - \frac{1-v^2}{tE}D\frac{1}{R}\frac{d^2 u_n}{dz^2}.$$

giving with notation $a = t / R$

$$N_{\phi\phi} = \frac{Et}{1-v^2}(v\frac{du_z}{dz} - \frac{1}{R}u_n) = \frac{Et}{1-v^2}[(v^2 - 1)\frac{1}{R}u_n - v\frac{1-v^2}{tE}D\frac{1}{R}\frac{d^2 u_n}{dz^2})] = -\frac{Et}{R}u_n - vD\frac{1}{R}\frac{d^2 u_n}{dz^2},$$

$$M_{zz} = -D(\frac{d^2 u_n}{dz^2} + v\frac{1}{R^2}u_n - \frac{1-v^2}{tE}D\frac{1}{R^2}\frac{d^2 u_n}{dz^2}) = -D[(1 - \frac{1}{12}a^2)\frac{d^2 u_n}{dz^2} + v\frac{1}{R^2}u_n].$$

The moment equilibrium equation is used next to eliminate the shear force from the remaining equilibrium equation to get

$$\frac{d^2 M_{zz}}{dz^2} + \frac{1}{R} N_{\phi\phi} = -D[(1 - \frac{a^2}{12}) \frac{d^4 u_n}{dz^4} + v \frac{2}{R^2} \frac{d^2 u_n}{dz^2}] - \frac{tE}{R^2} u_n = 0 \quad \text{in} \quad (0, L). \quad \leftarrow$$

Boundary conditions at the loaded end take the forms

$$Q_z - \underline{Q} = \frac{dM_{zz}}{dz} - \underline{Q} = -D[(1 - \frac{1}{12} a^2) \frac{d^3 u_n}{dz^3} + v \frac{1}{R^2} \frac{du_n}{dz}] - \underline{Q} = 0 \quad \text{and}$$

$$M_{zz} - \underline{M} = -D[(1 - \frac{1}{12} a^2) \frac{d^2 u_n}{dz^2} + v \frac{1}{R^2} u_n] - \underline{M} = 0 \quad \text{at} \quad z = L, \quad \leftarrow$$

and those for the clamped end

$$u_n = 0, \quad \frac{du_n}{dz} = 0 \quad \text{at} \quad z = 0. \quad \leftarrow$$

Consider a circular cylindrical shell of radius R, subjected to uniform bending moment $M_{zz} = \underline{M}$ and shearing force $Q_z = \underline{Q}$ at the end $z = L$. The other end $z = 0$ is clamped. Assuming rotational symmetry, derive the boundary value problem of Reissner-Mindlin type for deflection $u_n(z)$ and rotation $\theta_\phi(z)$.

Solution

As derivatives with respect to the angular coordinate vanish, the equilibrium equations of cylindrical shell in (z, ϕ, n) coordinate system simplify to

$$\frac{dN_{zz}}{dz} = 0, \quad \frac{dM_{zz}}{dz} - Q_z = 0, \text{ and } \frac{dQ_z}{dz} + \frac{1}{R} N_{\phi\phi} = 0.$$

Constitutive equations for the stress resultant components in the equilibrium equations simplify to

$$N_{zz} = \frac{tE}{1-v^2}(\frac{du_z}{dz} - v\frac{1}{R}u_n) - D\frac{1}{R}\frac{d\theta_\phi}{dz}, \quad N_{\phi\phi} = \frac{Et}{1-v^2}(v\frac{du_z}{dz} - \frac{1}{R}u_n),$$

$$Q_z = Gt(\theta_\phi + \frac{du_n}{dz}), \text{ and } M_{zz} = D(\frac{d\theta_\phi}{dz} - \frac{1}{R}\frac{du_z}{dz}).$$

Force equilibrium equation in the axial boundary condition, constitutive equation for the axial stress resultant and the boundary condition at the free edge give

$$\frac{dN_{zz}}{dz} = 0 \text{ in } (0, L) \text{ and } N_{zz}(L) = 0 \implies N_{zz}(z) = 0, \text{ therefore}$$

$$N_{zz} = \frac{tE}{1-v^2}(\frac{du_z}{dz} - v\frac{1}{R}u_n) - D\frac{1}{R}\frac{d\theta_\phi}{dz} = 0 \implies \frac{du_z}{dz} = v\frac{1}{R}u_n + \frac{1-v^2}{tE}D\frac{1}{R}\frac{d\theta_\phi}{dz}.$$

With the relationship, the constitutive equation for $N_{\phi\phi}$ and N_{zz} simplify to ($a = t/R$)

$$N_{\phi\phi} = \frac{Et}{1-v^2}(v\frac{du_z}{dz} - \frac{1}{R}u_n) = \frac{Et}{1-v^2}[(v^2-1)\frac{1}{R}u_n + v\frac{1-v^2}{tE}D\frac{1}{R}\frac{d\theta_\phi}{dz})] = -\frac{Et}{R}u_n + vD\frac{1}{R}\frac{d\theta_\phi}{dz},$$

$$M_{zz} = D(\frac{d\theta_\phi}{dz} - v\frac{1}{R^2}u_n - \frac{1-v^2}{tE}D\frac{1}{R^2}\frac{d\theta_\phi}{dz}) = D[(1-\frac{1}{12}a^2)\frac{d\theta_\phi}{dz} - v\frac{1}{R^2}u_n].$$

When the constitutive equations are substituted there, equilibrium equations in terms of u_n and θ_ϕ take the forms

$$\frac{dQ_z}{dz} + \frac{1}{R}N_{\phi\phi} = (Gt + vD\frac{1}{R^2})\frac{d\theta_\phi}{dz} + Gt\frac{d^2u_n}{dz^2} - \frac{Et}{R^2}u_n = 0 \text{ in } (0, L), \quad \color{red}{\leftarrow}$$

$$\frac{dM_{zz}}{dz} - Q_z = D[(1-\frac{1}{12}a^2)\frac{d^2\theta_\phi}{dz^2} - v\frac{1}{R^2}\frac{du_n}{dz}] - Gt(\theta_\phi + \frac{du_n}{dz}) = 0 \text{ in } (0, L). \quad \color{red}{\leftarrow}$$

Boundary conditions at the ends are

$$Q_z - \underline{Q} = Gt(\theta_\phi + \frac{du_n}{dz}) - \underline{Q} = 0 \quad \text{at} \quad z = L, \quad \leftarrow$$

$$M_{zz} - \underline{M} = D[(1 - \frac{1}{12}a^2)\frac{d\theta_\phi}{dz} - v\frac{1}{R^2}u_n] - \underline{M} = 0 \quad \text{at} \quad z = L \quad \leftarrow$$

$$u_n = 0, \ \theta_\phi = 0 \quad \text{at} \quad z = 0. \quad \leftarrow$$

A strip of cylindrical shell is loaded by shear force P ($[P] = $ N/m) at the free end. Write down the boundary value problem of first order ordinary differential equations consisting of the equilibrium and constitutive equations according to the Kirchhoff theory. Thickness t, width H, and the material parameters E, v are constants. Assume that the solution depends on ϕ only.
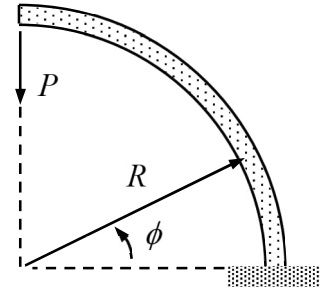
Solution

Equilibrium and constitutive equations of cylindrical shell in (z, ϕ, n) – coordinate system are given in the formulae collection. In a shell strip problem, it is enough to consider the force equilibrium equations in the plane of the figure and the moment equilibrium in the normal direction of the plane and constitutive equations for the stress resultants (appearing in the equilibrium equations). In the Kirchhoff model, the constitutive equation for the shear force is replaced by the Kirchhoff constraint. Also, derivatives with respect to z vanish. Therefore the differential equations and constitutive equations simplify to

$$
\left\{
\begin{array}{c}
\dfrac{1}{R}\dfrac{dN_{\phi\phi}}{d\phi} - \dfrac{1}{R}Q_{\phi} \\[3mm]
\dfrac{1}{R}\dfrac{dQ_{\phi}}{d\phi} + \dfrac{1}{R}N_{\phi\phi} \\[3mm]
\dfrac{1}{R}\dfrac{dM_{\phi\phi}}{d\phi} - \dfrac{1}{R}M_{\phi n} - Q_{\phi}
\end{array}
\right\} = 0 \quad \text{and} \quad
\left\{
\begin{array}{c}
N_{\phi\phi} - \dfrac{tE}{1-v^2}\dfrac{1}{R}(\dfrac{du_{\phi}}{d\phi} - u_n) + D\dfrac{1}{R^2}\dfrac{d\theta_z}{d\phi} \\[3mm]
M_{\phi\phi} - D[-\dfrac{1}{R}\dfrac{d\theta_z}{d\phi} + \dfrac{1}{R^2}(\dfrac{du_{\phi}}{d\phi} - u_n)] \\[3mm]
\dfrac{1}{R}(\dfrac{du_n}{d\phi} + u_{\phi}) - \theta_z
\end{array}
\right\} = 0 \quad \text{in} \quad (0, \dfrac{\pi}{2}). \quad \text{←}
$$

where $M_{n\phi} = 0$ due to the Kirchhoff constraint. The boundary conditions are

$$
\left\{
\begin{array}{c}
N_{\phi\phi} \\
Q_{\phi} - P \\
M_{\phi\phi}
\end{array}
\right\} = 0 \quad \text{at} \quad \phi = \dfrac{\pi}{2} \quad \text{and} \quad
\left\{
\begin{array}{c}
u_{\phi} \\
u_n \\
\theta_z
\end{array}
\right\} = 0 \quad \text{at} \quad \phi = 0. \quad \text{←}
$$

As the force resultants are known on one edge, equilibrium equations can be solved for the force resultants. Knowing these, displacements and rotation follow from the constitutive equations.