

Name _____ Student number _____

Assignment 3 (4p)

Derive the component forms of cylindrical shell force equilibrium equations in the (z, ϕ, n) coordinate system starting from the invariant form $\nabla_0 \cdot \vec{F} - \kappa \vec{e}_n \cdot \vec{F} + \vec{b} = 0$. The force resultant representations and kinematic quantities of the cylindrical shell (z, ϕ, n) coordinate system are

$$\vec{F} = N_{zz} \vec{e}_z \vec{e}_z + N_{z\phi} \vec{e}_z \vec{e}_\phi + N_{\phi z} \vec{e}_\phi \vec{e}_z + N_{\phi\phi} \vec{e}_\phi \vec{e}_\phi + Q_z \vec{e}_z \vec{e}_n + Q_z \vec{e}_n \vec{e}_z + Q_\phi \vec{e}_\phi \vec{e}_n + Q_\phi \vec{e}_n \vec{e}_\phi,$$

$$\vec{b} = b_z \vec{e}_z + b_\phi \vec{e}_\phi + b_n \vec{e}_n, \quad \nabla_0 = \vec{e}_z \frac{\partial}{\partial z} + \vec{e}_\phi \frac{1}{R} \frac{\partial}{\partial \phi}, \quad \frac{\partial}{\partial \phi} \vec{e}_\phi = \vec{e}_n, \quad \frac{\partial}{\partial \phi} \vec{e}_n = -\vec{e}_\phi, \quad \vec{I} = \vec{e}_z \vec{e}_z + \vec{e}_\phi \vec{e}_\phi + \vec{e}_n \vec{e}_n.$$

Solution

In the shell model, the stress resultants may not be symmetric. Definition $\vec{\kappa} = (\nabla_0 \vec{e}_n)_c$ gives the curvature tensor

$$\vec{\kappa} = (\nabla_0 \vec{e}_n)_c = [(\vec{e}_z \frac{\partial}{\partial z} + \vec{e}_\phi \frac{1}{R} \frac{\partial}{\partial \phi}) \vec{e}_n]_c = (\vec{e}_z \frac{\partial \vec{e}_n}{\partial z} + \vec{e}_\phi \frac{1}{R} \frac{\partial \vec{e}_n}{\partial \phi})_c = -\vec{e}_\phi \vec{e}_\phi \frac{1}{R} \Rightarrow \kappa = \vec{I} : \vec{\kappa} = -\frac{1}{R}.$$

Let us consider the mid-surface (membrane) and shear parts of $\vec{F} = \vec{N} + \vec{Q} \vec{e}_n + \vec{e}_n \vec{Q}$ separately. First the membrane mode term

$$\nabla_0 \cdot \vec{N} = (\vec{e}_z \frac{\partial}{\partial z} + \vec{e}_\phi \frac{1}{R} \frac{\partial}{\partial \phi}) \cdot (N_{zz} \vec{e}_z \vec{e}_z + N_{z\phi} \vec{e}_z \vec{e}_\phi + N_{\phi z} \vec{e}_\phi \vec{e}_z + N_{\phi\phi} \vec{e}_\phi \vec{e}_\phi), \quad \text{where}$$

$$(\vec{e}_z \frac{\partial}{\partial z}) \cdot (N_{zz} \vec{e}_z \vec{e}_z + N_{z\phi} \vec{e}_z \vec{e}_\phi + N_{\phi z} \vec{e}_\phi \vec{e}_z + N_{\phi\phi} \vec{e}_\phi \vec{e}_\phi) = \frac{\partial N_{zz}}{\partial z} \vec{e}_z + \frac{\partial N_{z\phi}}{\partial z} \vec{e}_\phi \quad \text{and}$$

$$(\vec{e}_\phi \frac{1}{R} \frac{\partial}{\partial \phi}) \cdot (N_{zz} \vec{e}_z \vec{e}_z + N_{z\phi} \vec{e}_z \vec{e}_\phi + N_{\phi z} \vec{e}_\phi \vec{e}_z + N_{\phi\phi} \vec{e}_\phi \vec{e}_\phi) = \frac{1}{R} \frac{\partial N_{\phi z}}{\partial \phi} \vec{e}_z + \frac{1}{R} \frac{\partial N_{\phi\phi}}{\partial \phi} \vec{e}_\phi + \frac{1}{R} N_{\phi\phi} \vec{e}_n.$$

Altogether

$$\nabla_0 \cdot \vec{N} = \left(\frac{\partial N_{zz}}{\partial z} + \frac{1}{R} \frac{\partial N_{\phi z}}{\partial \phi} \right) \vec{e}_z + \left(\frac{\partial N_{z\phi}}{\partial z} + \frac{1}{R} \frac{\partial N_{\phi\phi}}{\partial \phi} \right) \vec{e}_\phi + \frac{1}{R} N_{\phi\phi} \vec{e}_n.$$

Then, the shear part associated with the bending mode

$$\nabla_0 \cdot (\vec{Q} \vec{e}_n + \vec{e}_n \vec{Q}) = (\vec{e}_z \frac{\partial}{\partial z} + \vec{e}_\phi \frac{1}{R} \frac{\partial}{\partial \phi}) \cdot (Q_z \vec{e}_z \vec{e}_n + Q_\phi \vec{e}_\phi \vec{e}_n + Q_z \vec{e}_n \vec{e}_z + Q_\phi \vec{e}_n \vec{e}_\phi), \quad \text{where}$$

$$(\vec{e}_z \frac{\partial}{\partial z}) \cdot (Q_z \vec{e}_z \vec{e}_n + Q_\phi \vec{e}_\phi \vec{e}_n + Q_z \vec{e}_n \vec{e}_z + Q_\phi \vec{e}_n \vec{e}_\phi) = \frac{\partial Q_z}{\partial z} \vec{e}_n \quad \text{and}$$

$$(\vec{e}_\phi \frac{1}{R} \frac{\partial}{\partial \phi}) \cdot (Q_z \vec{e}_z \vec{e}_n + Q_\phi \vec{e}_\phi \vec{e}_n + Q_z \vec{e}_n \vec{e}_z + Q_\phi \vec{e}_n \vec{e}_\phi) = \frac{1}{R} \left(\frac{\partial Q_\phi}{\partial \phi} \vec{e}_n - Q_\phi \vec{e}_\phi - Q_z \vec{e}_z - Q_\phi \vec{e}_\phi \right).$$

Altogether

$$\nabla_0 \cdot (\vec{Q} \vec{e}_n + \vec{e}_n \vec{Q}) = -\frac{1}{R} Q_z \vec{e}_z - 2 \frac{1}{R} Q_\phi \vec{e}_\phi + \left(\frac{\partial Q_z}{\partial z} + \frac{1}{R} \frac{\partial Q_\phi}{\partial \phi} \right) \vec{e}_n = -\frac{1}{R} \vec{Q} - \frac{1}{R} Q_\phi \vec{e}_\phi + \left(\frac{\partial Q_z}{\partial z} + \frac{1}{R} \frac{\partial Q_\phi}{\partial \phi} \right) \vec{e}_n.$$

The second term of the equilibrium equation simplifies to

$$\kappa \vec{e}_n \cdot \vec{F} = -\frac{1}{R} (Q_z \vec{e}_z + Q_\phi \vec{e}_\phi) = -\frac{1}{R} \vec{Q}.$$

Therefore, combining the terms

$$\nabla_0 \cdot \vec{F} - \kappa \vec{e}_n \cdot \vec{F} + \vec{b} = \begin{Bmatrix} \vec{e}_z \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix}^T \begin{Bmatrix} \frac{\partial N_{zz}}{\partial z} + \frac{1}{R} \frac{\partial N_{\phi z}}{\partial \phi} + b_z \\ \frac{\partial N_{z\phi}}{\partial z} + \frac{1}{R} \frac{\partial N_{\phi\phi}}{\partial \phi} - \frac{1}{R} Q_\phi + b_\phi \\ \frac{\partial Q_z}{\partial z} + \frac{1}{R} N_{\phi\phi} + \frac{1}{R} \frac{\partial Q_\phi}{\partial \phi} + b_n \end{Bmatrix} = 0. \quad \leftarrow$$