Name________________________ Student number________________

Assignment 3 (4p)

Principle of virtual work for a simply supported Bernoulli beam is given by the variational problem: find $w \in U$ such that

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = \int_{\Omega} (-\frac{d^2\delta w}{dx^2} EI \frac{d^2 w}{dx^2} + \delta w b) dx = 0 \quad \forall \delta w \in U$$

in which $\Omega = (0, L)$, $\partial\Omega = \{0, L\}$, $U = \{w \in C^4(\Omega) : w = 0 \text{ on } x \in \{0, L\}\}$. Bending stiffness EI and the external distributed force b are constants. Deduce first the underlying boundary value problem. After that, solve the problem for the transverse displacement $w(x)$.

Solution

Integration by parts twice in the first term gives an equivalent form

$$\delta W = \int_0^L (-\frac{d^2\delta w}{dx^2} EI \frac{d^2 w}{dx^2} + \delta w b) dx \quad \Leftrightarrow$$

$$\delta W = \int_{\Omega} (\frac{d\delta w}{dx} EI \frac{d^3 w}{dx^3} + \delta w b) dx - \sum_{\partial\Omega} n(EI \frac{d^2 w}{dx^2} \frac{d\delta w}{dx}) \quad \Leftrightarrow$$

$$\delta W = \int_{\Omega} (-EI \frac{d^4 w}{dx^4} + b) \delta w dx - \sum_{\partial\Omega} n(EI \frac{d^2 w}{dx^2} \frac{d\delta w}{dx}).$$

The boundary term of the second integration by parts vanishes as $\delta w \in U$ and therefore $\delta w = 0$ on $\partial\Omega = \{0, L\}$. According to the variational problem $\delta W = 0 \ \forall \delta w \in U$.

Let us first consider a subset of $U_0 \subset U$ for which $d\delta w / dx = 0$ on $\partial\Omega = \{0, L\}$ so that the boundary terms vanish. The equilibrium equation follows from the fundamental lemma of variation calculus:

$$\delta W = \int_{\Omega} (-EI \frac{d^4 w}{dx^4} + b) \delta w dx = 0 \quad \Rightarrow \quad -EI \frac{d^4 w}{dx^4} + b = 0 \quad \text{in } \Omega. \quad \leftarrow$$

After that, let us consider U without restriction $d\delta w / dx = 0$ on $\partial\Omega = \{0, L\}$ and simplify the virtual work expression by using the equilibrium equation already obtained. The natural boundary conditions follow from the fundamental lemma of variation calculus

$$\delta W = \sum_{\partial\Omega} n(EI \frac{d^2 w}{dx^2} \frac{d\delta w}{dx}) = 0 \quad \Rightarrow \quad EI \frac{d^2 w}{dx^2} = 0 \quad \text{on } \partial\Omega. \quad \leftarrow$$

The natural boundary conditions mean that moment vanishes at the ends of a simply supported beam. The missing two boundary conditions needed for a fourth order ordinary differential equation of a Bernoulli beam follow from the definition of the function set for the transverse displacement. As all elements (functions) in U should vanish on $\partial\Omega$ and $w \in U$:

$$w = 0 \text{ on } \partial\Omega .$$

Boundary value problem for a simply supported beam consists of the equilibrium equation and the boundary condition implied by the principle of virtual work

$$-EI\frac{d^4w}{dx^4} + b = 0 \quad \text{in } \Omega ,$$

$$EI\frac{d^2w}{dx^2} = 0 \quad \text{on } \partial\Omega ,$$

$$w = 0 \text{ on } \partial\Omega .$$

Solution to the equations of the boundary value problem can be obtained by repetitive integrations of the differential equation. After finding the generic solution containing four integration constants, boundary condition are used to find the values of the integration constants and thereby a solution satisfying all equations of the problem. First integration four times

$$\frac{d^4w}{dx^4} = \frac{b}{EI} \quad \Leftrightarrow \quad \frac{d^3w}{dx^3} = \frac{b}{EI}x + a_1 \quad \Leftrightarrow \quad \frac{d^2w}{dx^2} = \frac{b}{EI}\frac{1}{2}x^2 + a_1x + a_2 \quad \Leftrightarrow$$

$$\frac{dw}{dx} = \frac{b}{EI}\frac{1}{6}x^3 + a_1\frac{1}{2}x^2 + a_2x + a_3 \quad \Leftrightarrow \quad w(x) = \frac{b}{EI}\frac{1}{24}x^4 + a_1\frac{1}{6}x^3 + a_2\frac{1}{2}x^2 + a_3x + a_4 .$$

The boundary conditions give

$$w(0) = a_4 = 0 , \; w(L) = \frac{b}{EI}\frac{1}{24}L^4 + a_1\frac{1}{6}L^3 + a_2\frac{1}{2}L^2 + a_3L + a_4 = 0$$

$$\frac{d^2w}{dx^2}(0) = a_2 , \; \frac{b}{EI}\frac{1}{2}L^2 + a_1L + a_2 = 0 \quad \Leftrightarrow \quad a_2 = a_4 = 0 , \; a_1 = -\frac{1}{2}\frac{bL}{EI} , \; a_3 = \frac{1}{24}\frac{bL^3}{EI} .$$

Therefore, the solution to the transverse displacement

$$w(x) = \frac{1}{24}\frac{b}{EI}(x^4 - 2Lx^3 + L^3x) = \frac{1}{24}\frac{bL^4}{EI}(\xi^4 - 2\xi + \xi) \quad \text{where} \quad \xi = x / L .$$