

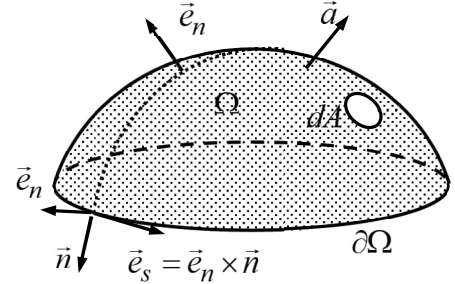
Name \_\_\_\_\_ Student number \_\_\_\_\_

## Assignment 1 (2p)

Gauss theorem implies the following integral identity for curved surfaces

$$\int_{\Omega} (\nabla_0 \cdot \vec{a} - \kappa \vec{e}_n \cdot \vec{a}) dA = \int_{\partial\Omega} (\vec{n} \cdot \vec{a}) ds$$

in which  $\kappa = \vec{k} : \vec{l} = \nabla_0 \cdot \vec{e}_n$ . Verify the integral identity in the spherical  $(\phi, \theta, n)$  coordinate system by considering vector  $\vec{a} = \theta \vec{e}_n$  and half-sphere  $\phi \in [0, 2\pi]$ ,  $\theta \in [0, \pi/2]$ , of radius  $R$  as  $\Omega$ . Derivatives of the basis vectors and the mid-surface gradient in the spherical coordinate system are



$$\frac{\partial}{\partial \phi} \begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_\theta \\ \vec{e}_n \end{Bmatrix} = \begin{Bmatrix} \sin \theta \vec{e}_n - \cos \theta \vec{e}_\theta \\ \cos \theta \vec{e}_\phi \\ -\sin \theta \vec{e}_\phi \end{Bmatrix}, \quad \frac{\partial}{\partial \theta} \begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_\theta \\ \vec{e}_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ \vec{e}_n \\ -\vec{e}_\theta \end{Bmatrix}, \quad \nabla_0 = \frac{1}{R \sin \theta} \vec{e}_\phi \frac{\partial}{\partial \phi} + \frac{1}{R} \vec{e}_\theta \frac{\partial}{\partial \theta}.$$

### Solution template

In case of a half-sphere  $\phi \in [0, 2\pi]$ ,  $\theta \in [0, \pi/2]$  of radius  $R$  and vector  $\vec{a} = \theta \vec{e}_n$ , the quantities in the integral identity take the forms

$$\nabla_0 \cdot \vec{a} = \left( \frac{1}{R \sin \theta} \vec{e}_\phi \frac{\partial}{\partial \phi} + \frac{1}{R} \vec{e}_\theta \frac{\partial}{\partial \theta} \right) \cdot \theta \vec{e}_n = -\frac{2}{R} \theta,$$

$$\kappa = \nabla_0 \cdot \vec{e}_n = \left( \frac{1}{R \sin \theta} \vec{e}_\phi \frac{\partial}{\partial \phi} + \frac{1}{R} \vec{e}_\theta \frac{\partial}{\partial \theta} \right) \cdot \vec{e}_n = -\frac{2}{R},$$

$$\vec{e}_n \cdot \vec{a} = \theta,$$

$$\vec{n} \cdot \vec{a} = 0.$$

When the expressions are substituted there, the left- and right-hand sides of the integral identity simplify to

$$\int_{\Omega} (\nabla_0 \cdot \vec{a} - \kappa \vec{e}_n \cdot \vec{a}) dA = \int_{\Omega} \left( -\frac{2}{R} \theta + \frac{2}{R} \theta \right) dA = 0,$$

$$\int_{\partial\Omega} (\vec{n} \cdot \vec{a}) ds = \int_{\partial\Omega} (0) ds = 0.$$