Name________________________ Student number________________

Assignment 4 (4p)

Derive the component forms of the membrane equilibrium equation $\nabla\cdot\vec{\vec{N}}+\vec{b}=0$ in the spherical shell coordinate system. Assume external loading $\vec{b}=-\Delta p\vec{e}_n$ due to a pressure difference between the inner and outer surfaces. Also assume rotation symmetry with respect to both angular coordinates so that $\vec{\vec{N}}=N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi+N_{\theta\theta}\vec{e}_\theta\vec{e}_\theta$ where stress components $N_{\phi\phi}$ and $N_{\theta\theta}$ are constants. The basis vector derivatives and gradient of the spherical shell coordinate system are given by

$$\frac{\partial}{\partial\phi}\begin{Bmatrix}\vec{e}_\phi\\ \vec{e}_\theta\\ \vec{e}_n\end{Bmatrix}=\begin{Bmatrix}\sin\theta\vec{e}_n-\cos\theta\vec{e}_\theta\\ \cos\theta\vec{e}_\phi\\ -\sin\theta\,\vec{e}_\phi\end{Bmatrix},\ \frac{\partial}{\partial\theta}\begin{Bmatrix}\vec{e}_\phi\\ \vec{e}_\theta\\ \vec{e}_n\end{Bmatrix}=\begin{Bmatrix}0\\ \vec{e}_n\\ -\vec{e}_\theta\end{Bmatrix},\ \nabla=\frac{1}{R\sin\theta}\vec{e}_\phi\frac{\partial}{\partial\phi}+\frac{1}{R}\vec{e}_\theta\frac{\partial}{\partial\theta}.$$

Solution

Membrane model is the thin-slab model in curved geometry. Equilibrium equations differ from the thin-slab ones as the non-constant basis vectors brings additional terms. Therefore, a membrane may take also external forces in the transverse direction (like pressure difference between the outer and inner surfaces of a balloon). The task is to simplify the vector equation

$$\nabla\cdot\vec{\vec{N}}-\vec{b}=\frac{1}{R}\left(\frac{1}{\sin\theta}\vec{e}_\phi\frac{\partial}{\partial\phi}+\vec{e}_\theta\frac{\partial}{\partial\theta}\right)\cdot(N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi+N_{\theta\theta}\vec{e}_\theta\vec{e}_\theta)-\Delta p\vec{e}_n=0$$

to see the component forms. Let us start with the divergence of stress

$$\frac{1}{R\sin\theta}\vec{e}_\phi\frac{\partial}{\partial\phi}\cdot N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi=\frac{1}{R\sin\theta}\vec{e}_\phi\cdot\left(\frac{\partial N_{\phi\phi}}{\partial\phi}\vec{e}_\phi\vec{e}_\phi+N_{\phi\phi}\frac{\partial\vec{e}_\phi}{\partial\phi}\vec{e}_\phi+N_{\phi\phi}\vec{e}_\phi\frac{\partial\vec{e}_\phi}{\partial\phi}\right)=\frac{1}{R}N_{\phi\phi}(\vec{e}_n-\cot\theta\vec{e}_\theta),$$

$$\frac{1}{R}\frac{1}{\sin\theta}\vec{e}_\phi\frac{\partial}{\partial\phi}\cdot N_{\theta\theta}\vec{e}_\theta\vec{e}_\theta=\frac{1}{R}\frac{1}{\sin\theta}\vec{e}_\phi\cdot\left(\frac{\partial N_{\theta\theta}}{\partial\phi}\vec{e}_\theta\vec{e}_\theta+N_{\theta\theta}\frac{\partial\vec{e}_\theta}{\partial\phi}\vec{e}_\theta+\frac{\partial}{\partial\phi}N_{\theta\theta}\vec{e}_\theta\frac{\partial\vec{e}_\theta}{\partial\phi}\right)=\frac{1}{R}N_{\theta\theta}\cot\theta\vec{e}_\theta,$$

$$\frac{1}{R}\vec{e}_\theta\frac{\partial}{\partial\theta}\cdot(N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi)=\frac{1}{R}\vec{e}_\theta\cdot\left(\frac{\partial N_{\phi\phi}}{\partial\theta}\vec{e}_\phi\vec{e}_\phi+N_{\phi\phi}\frac{\partial\vec{e}_\phi}{\partial\theta}\vec{e}_\phi+N_{\phi\phi}\vec{e}_\phi\frac{\partial\vec{e}_\phi}{\partial\theta}\right)=0,$$

$$\frac{1}{R}\vec{e}_\theta\frac{\partial}{\partial\theta}\cdot(N_{\theta\theta}\vec{e}_\theta\vec{e}_\theta)=\frac{1}{R}\vec{e}_\theta\cdot\left(\frac{\partial N_{\theta\theta}}{\partial\theta}\vec{e}_\theta\vec{e}_\theta+N_{\theta\theta}\vec{e}_n\vec{e}_\theta+N_{\theta\theta}\vec{e}_\theta\vec{e}_n\right)=\frac{1}{R}N_{\theta\theta}\vec{e}_n.$$

Finally combining everything

$$\nabla\cdot\vec{\vec{N}}-\vec{b}=\frac{1}{R}N_{\phi\phi}(\vec{e}_n-\cot\theta\vec{e}_\theta)+\frac{1}{R}N_{\theta\theta}\cot\theta\vec{e}_\theta+\frac{1}{R}N_{\theta\theta}\vec{e}_n-\Delta p\vec{e}_n=0\quad\Leftrightarrow$$

$$\nabla \cdot \overleftrightarrow{N} - \vec{b} = [\frac{1}{R}(N_{\theta\theta} - N_{\phi\phi})\cot\theta]\vec{e}_\theta + [\frac{1}{R}(N_{\phi\phi} + N_{\theta\theta}) - \Delta p]\vec{e}_n = 0 \quad \Leftrightarrow$$

$$N_{\theta\theta} - N_{\phi\phi} = 0 \quad \text{and} \quad \frac{1}{R}(N_{\phi\phi} + N_{\theta\theta}) - \Delta p = 0\,.$$

Therefore, solution to the stress resultants is

$$N_{\phi\phi} = N_{\theta\theta} = \frac{R\Delta p}{2}\,.$$