MEC-E8003 Beam, Plate and Shell models, onsite exam 12.04.2022

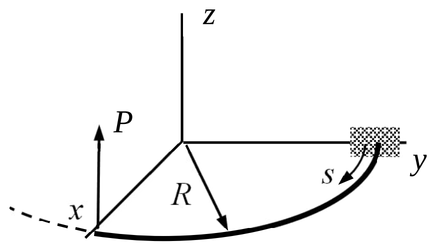
1. Use the definition $\nabla^2 = \nabla \cdot \nabla$ to derive the Laplacian operator in the cylindrical coordinate system.

2. Virtual work expression of a linearly elastic bar supported by a spring at the right end $x = L$ ($n = 1$) is given by

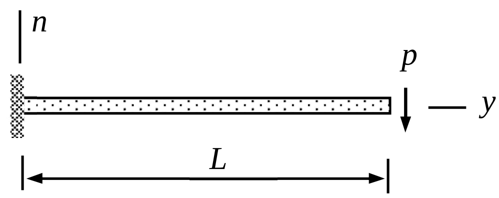
$$\delta W = \int_0^L -(EA\frac{du}{dx}\frac{d\delta u}{dx})dx + \int_0^L (b\delta u)dx - (ku\delta u)_{x=L},$$

in which $EA = EA(x)$ and k, b are constants. Displacement vanishes at the left end $x = 0$ ($n = -1$) of the bar. Find the underlying boundary value problem starting from the principle of virtual work $\delta W = 0 \; \forall \delta u \in U$. Assume that functions of U have continuous derivatives up to the second order and vanish at $x = 0$.

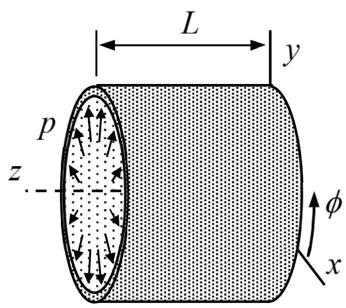
3. Consider the curved beam of the figure forming a 90-degree circular segment of radius R in the horizontal plane. Find the stress resultants $N(s)$, $Q_n(s)$, $Q_b(s)$, $T(s)$, $M_n(s)$, and $M_b(s)$. Use the equilibrium equations of the beam model in the $(s,n,b)-$coordinate system.



4. Consider the bending of a cantilever plate strip which is loaded by distributed force p [N / m] acting on the free edge. Write down the equilibrium equations, constitutive equations, and boundary conditions for the bending mode according to the Kirchhoff model. After that, solve the equations for the stress resultant, displacement, and rotation components. Thickness and length of the plate are t and L, respectively. Young's modulus E and Poisson's ratio ν are constants. Consider a plate of width H but assume that stress resultants, displacements, and rotations depend on y only.



5. A steel ring of length L, radius R, and thickness t is loaded by radial surface force p acting on the inner surface. No forces are acting on the ends. Model the ring as a cylindrical membrane, write down the equilibrium and constitutive equations, and solve for the radial displacement. Assume rotation symmetry. Young's modulus E and Poisson's ratio ν of the material are constants.



Use the definition $\nabla^2 = \nabla \cdot \nabla$ to derive the Laplacian operator in the cylindrical coordinate system.

Solution

Gradient operator and the derivatives of the basis vectors of the cylindrical (r,ϕ,z) – coordinate system are (formula collection)

$$\nabla = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \partial / \partial r \\ \partial / (r\partial\phi) \\ \partial / \partial z \end{Bmatrix} \quad \text{and} \quad \frac{\partial}{\partial\phi}\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix}.$$

1p According to the definition

$$\nabla^2 = \nabla \cdot \nabla = (\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial\phi} + \vec{e}_z \frac{\partial}{\partial z}) \cdot (\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial\phi} + \vec{e}_z \frac{\partial}{\partial z})$$

5p in which the terms

$$(\vec{e}_r \frac{\partial}{\partial r}) \cdot (\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial\phi} + \vec{e}_z \frac{\partial}{\partial z}) = \vec{e}_r \cdot (\vec{e}_r \frac{\partial^2}{\partial r^2} - \vec{e}_\phi \frac{1}{r^2}\frac{\partial}{\partial\phi} + \vec{e}_\phi \frac{1}{r}\frac{\partial^2}{\partial r\partial\phi} + \vec{e}_z \frac{\partial^2}{\partial r\partial z}) = \frac{\partial^2}{\partial r^2},$$

$$(\vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial\phi}) \cdot (\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial\phi} + \vec{e}_z \frac{\partial}{\partial z}) = (\vec{e}_\phi \frac{1}{r}) \cdot (\vec{e}_\phi \frac{\partial}{\partial r} + \vec{e}_r \frac{\partial^2}{\partial\phi\partial r} - \vec{e}_r \frac{1}{r}\frac{\partial}{\partial\phi} + \vec{e}_\phi \frac{1}{r}\frac{\partial^2}{\partial\phi^2} + \vec{e}_z \frac{\partial^2}{\partial\phi\partial z}) \quad \Leftrightarrow$$

$$(\vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial\phi}) \cdot (\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial\phi} + \vec{e}_z \frac{\partial}{\partial z}) = \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2},$$

$$(\vec{e}_z \frac{\partial}{\partial z}) \cdot (\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial\phi} + \vec{e}_z \frac{\partial}{\partial z}) = \vec{e}_z \cdot (\vec{e}_r \frac{\partial^2}{\partial z\partial r} + \vec{e}_\phi \frac{1}{r}\frac{\partial^2}{\partial z\partial\phi} + \vec{e}_z \frac{\partial^2}{\partial z^2}) = \frac{\partial^2}{\partial z^2}.$$

Combining the results for the terms gives

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2} = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}. \quad \Leftarrow$$

Virtual work expression of a linearly elastic bar supported by a spring at the right end $x=L$ $(n=1)$ is given by

$$\delta W=-\int_0^L (\frac{d\delta u}{dx}EA\frac{du}{dx})dx+\int_0^L (\delta ub)dx-(\delta uku)_{x=L},$$

in which $EA=EA(x)$ and k,b are constants. Displacement vanishes at the left end $x=0$ $(n=-1)$ of the bar. Find the underlying boundary value problem starting from the principle of virtual work $\delta W=0$ $\forall\delta u\in U$. Assume that functions of U have continuous derivatives up to the second order and vanish at $x=0$.

Solution

Fundamental theorem of calculus (integration by parts) and the fundamental lemma of variation calculus are the tools for deriving a boundary value problem starting from a virtual work expression. In the one-dimensional case, for any continuous functions a and b (or values at some point), it holds

$$\int_\Omega a\frac{db}{dx}dx=\sum_{\partial\Omega}(nab)-\int_\Omega \frac{da}{dx}bdx \quad (\text{where } n=\pm 1),$$

$$a,b\in\mathbb{R}: \qquad ab=0 \quad \forall b \quad \Leftrightarrow \quad a=0,$$

$$a,b\in C^0(\Omega): \int_\Omega abdx=0 \quad \forall b \quad \Leftrightarrow \quad a=0 \quad \text{in} \quad \Omega.$$

2p In the present case $\Omega=(0,L)$ and $\partial\Omega=\{0,L\}$. Displacement has continuous derivatives up to and including second order i.e. $u\in C^2(\Omega)$. The constraint on the function set $u=0$ at $x=0$ implies that $\delta u=0$ at $x=0$. Integration by parts gives equivalent forms (the aim is to remove the derivatives from variations in the integral over the domain)

$$\delta W=-\int_0^L (\frac{d\delta u}{dx}EA\frac{du}{dx})dx+\int_0^L (\delta ub)dx-(\delta uku)_{x=L} \quad \Leftrightarrow$$

$$\delta W=\int_0^L [\frac{d}{dx}(EA\frac{du}{dx})+b]\delta udx-[(EA\frac{du}{dx}+ku)\delta u]_{x=L}. \quad (\text{as } \delta u=0 \text{ at } x=0)$$

3p The purpose of the manipulation above was to obtain a representation which allows the use of fundamental lemma of variation calculus. According to principle of virtual work, $\delta W=0$ $\forall\delta u\in U$. Let us consider first a subset $U_0\subset U$ for which $\delta u=0$ at $x=L$ so that the boundary term vanishes. Then

$$\delta W=\int_0^L [\frac{d}{dx}(EA\frac{du}{dx})+b]\delta udx=0 \quad \delta u\in U_0\subset U$$

and the fundamental lemma of variation calculus implies that

$$\frac{d}{dx}(EA\frac{du}{dx})+b=0 \quad \text{in } (0,L).$$

Knowing this and considering the full set U, the variational equation simplifies into

$$\delta W = -[(EA\frac{du}{dx} + ku)\delta u]_{x=L} = 0 \, .$$

Then, the fundamental lemma of variation calculus implies that

$$EA\frac{du}{dx} + ku = 0 \quad \text{at} \quad x = L \, .$$

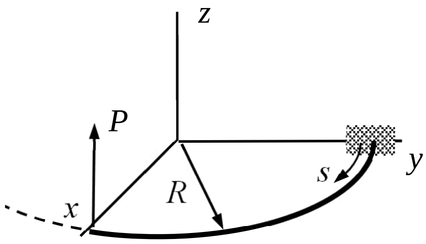
1p Finally combining the equations to form a boundary value problem (notice that the definition of the function set implies also a boundary condition):

$$\frac{d}{dx}(EA\frac{du}{dx}) + b = 0 \quad \text{in} \quad (0, L) \, ,$$

$$EA\frac{du}{dx} + ku = 0 \quad \text{at} \quad x = L \, ,$$

$$u = 0 \quad \text{at} \quad x = 0 \, .$$

Consider the curved beam of the figure forming a 90-degree circular segment of radius R in the horizontal plane. Find the stress resultants $N(s)$, $Q_n(s)$, $Q_b(s)$, $T(s)$, $M_n(s)$, and $M_b(s)$. Use the equilibrium equations of the beam model in the $(s,n,b)-$ coordinate system.



Solution

In a statically determined case, stress resultants follow from the equilibrium equations and boundary conditions at the free end of the beam (or directly from a free body diagram). In (s,n,b) coordinate system, equilibrium equations are

$$\begin{Bmatrix} N'-Q_n\kappa+b_s \\ Q_n'+N\kappa-Q_b\tau+b_n \\ Q_b'+Q_n\tau+b_b \end{Bmatrix}=0 \quad \text{and} \quad \begin{Bmatrix} T'-M_n\kappa+c_s \\ M_n'+T\kappa-M_b\tau-Q_b+c_n \\ M_b'+M_n\tau+Q_n+c_b \end{Bmatrix}=0 .$$

For a circular beam, curvature and torsion are $\kappa=1/R$ (constant) and $\tau=0$.

3p As external distributed forces and moments vanish i.e. $b_s=b_n=b_b=c_s=c_n=c_b=0$, equilibrium equations and the boundary conditions at the free end simplify to (notice that the external force acting at the free end is acting in the oppisite direction to $\vec{e}_b$)

$$\begin{Bmatrix} N'-Q_n/R \\ Q_n'+N/R \\ Q_b' \end{Bmatrix}=0 \quad \text{and} \quad \begin{Bmatrix} T'-M_n/R \\ M_n'+T/R-Q_b \\ M_b'+Q_n \end{Bmatrix}=0 \quad s\in]0,R\frac{\pi}{2}[,$$

$$\begin{Bmatrix} N \\ Q_n \\ Q_b+P \end{Bmatrix}=0 \quad \text{and} \quad \begin{Bmatrix} T \\ M_n \\ M_b \end{Bmatrix}=0 \quad s=R\frac{\pi}{2}.$$

3p Equations constitute a boundary value problem which can be solved by hand calculations without too much effort;

$$Q_b'=0 \quad s\in]0,R\frac{\pi}{2}[\quad \text{and} \quad Q_b+P=0 \quad s=R\frac{\pi}{2} \quad \Rightarrow \quad Q_b(s)=-P . \quad \leftarrow$$

Eliminating Q_n and N from the remaining two connected force equilibrium equations and using the original equations to find the missing boundary condition give

$$N''+\frac{1}{R^2}N=0 \quad s\in]0,R\frac{\pi}{2}[\quad \text{and} \quad N'=N=0 \quad s=R\frac{\pi}{2} \quad \Rightarrow \quad N(s)=0 \quad \leftarrow$$

The first equilibrium equation gives

$$Q_n(s)=0 . \quad \leftarrow$$

After that, continuing with the moment equilibrium equations with the solutions to the force equilibrium equations

$$M_b' = 0 \quad s \in]0, R\frac{\pi}{2}[\quad \text{and} \quad M_b = 0 \quad s = R\frac{\pi}{2} \quad \Rightarrow \quad M_b(s) = 0 .$$

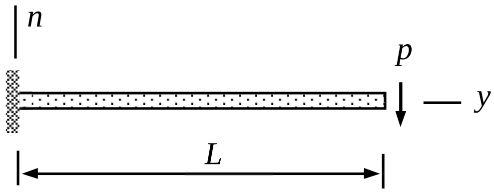
Eliminating M_n and T from the remaining two connected moment equilibrium equations and using the original equations to find the missing boundary condition gives

$$T'' + \frac{1}{R^2}T + \frac{P}{R} = 0 \quad s \in]0, R\frac{\pi}{2}[\quad \text{and} \quad T' = T = 0 \quad s = R\frac{\pi}{2} \quad \Rightarrow \quad T = PR(\sin\frac{s}{R} - 1) .$$

Knowing this, the first moment equilibrium equation gives

$$M_n(s) = RT' = RP\cos\frac{s}{R} .$$

Consider the bending of a cantilever plate strip which is loaded by distributed force p [N / m] acting on the free edge. Write down the equilibrium equations, constitutive equations, and boundary conditions for the bending mode according to the Kirchhoff model. After that, solve the equations for the stress resultant, displacement, and rotation components. Thickness and length of the plate are t and L, respectively. Young's modulus E and Poisson's ratio ν are constants. Consider a plate of width H but assume that stress resultants, displacements, and rotations depend on y only.



Solution

The starting point is the full set of Reissner-plate bending mode equations in the Cartesian $(x,y,n)-$ coordinate system.

$$\begin{Bmatrix} \dfrac{\partial Q_x}{\partial x}+\dfrac{\partial Q_y}{\partial y}+b_n \\ \dfrac{\partial M_{xx}}{\partial x}+\dfrac{\partial M_{xy}}{\partial y}-Q_x \\ \dfrac{\partial M_{yy}}{\partial y}+\dfrac{\partial M_{xy}}{\partial x}-Q_y \end{Bmatrix}=0, \quad \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix}=D\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}\begin{Bmatrix} \dfrac{\partial \theta}{\partial x} \\ -\dfrac{\partial \phi}{\partial y} \\ \dfrac{\partial \theta}{\partial y}-\dfrac{\partial \phi}{\partial x} \end{Bmatrix}, \quad \begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix}=Gt\begin{Bmatrix} \dfrac{\partial w}{\partial x}+\theta \\ \dfrac{\partial w}{\partial y}-\phi \end{Bmatrix}.$$

3p If all derivatives with respect to x vanish, the plate equations of the Cartesian $(x,y,n)-$ coordinate according to the Kirchhoff model (Kirchhoff constraint replaces the constitutive equation for the shear stress resultant) system simplify to

$$\frac{dQ_y}{dy}=0, \quad \frac{dM_{yy}}{dy}-Q_y=0, \quad M_{yy}=-D\frac{d\phi}{dy}, \quad \text{and} \quad \frac{dw}{dy}-\phi=0 \quad \text{in} \quad (0,L).$$

The boundary conditions are

$$w(0)=0, \quad \phi(0)=0, \quad M_{yy}(L)=0, \quad Q_y(L)=-p.$$

3p As the stress resultant are known at the free end, the equilibrium equations can be solved first for the stress resultants. The boundary value problems for the stress resultants give

$$\frac{dQ_y}{dy}=0 \quad y\in(0,L) \quad \text{and} \quad Q_y(L)=-p \quad \Rightarrow \quad Q_y(y)=-p, \; \leftarrow$$

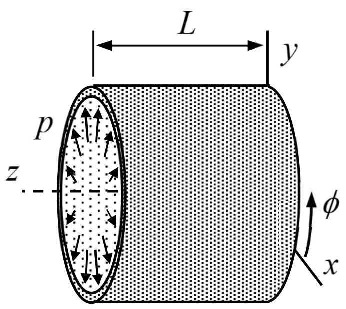
$$\frac{dM_{yy}}{dy}=Q_y=-p \quad y\in(0,L) \quad \text{and} \quad M_{yy}(L)=0 \quad \Rightarrow \quad M_{yy}(y)=-p(y-L). \; \leftarrow$$

After that, displacement and rotation follow from the constitutive equation, Kirchhoff constraint, and boundary conditions at the clamped edge

$$\frac{d\phi}{dy} = -\frac{M_{yy}}{D} = \frac{p}{D}(y-L) \quad y \in (0,L) \quad \text{and} \quad \phi(0)=0 \quad \Rightarrow \quad \phi = \frac{p}{D}(\frac{1}{2}y^2 - Ly),$$

$$\frac{dw}{dy} = \phi = \frac{p}{D}(\frac{1}{2}y^2 - Ly) \quad y \in (0,L) \quad \text{and} \quad w(0)=0 \quad \Rightarrow \quad w(y) = \frac{p}{D}(\frac{1}{6}y^3 - L\frac{1}{2}y^2).$$

A steel ring of length L, radius R, and thickness t is loaded by radial surface force p acting on the inner surface. No forces are acting on the ends. Model the ring as a cylindrical membrane, write down the equilibrium and constitutive equations, and solve for the radial displacement. Assume rotation symmetry and $u_\phi = 0$. Young's modulus E and Poisson's ratio g of the material are constants.



Solution

According to the formulae collection, equilibrium and constitutive equations of a cylindrical membrane in (z,ϕ,n) coordinates are (notice that $\vec{e}_n$ is directed inwards)

$$\begin{Bmatrix} \dfrac{1}{R}\dfrac{\partial N_{z\phi}}{\partial \phi} + \dfrac{\partial N_{zz}}{\partial z} + b_z \\[2ex] \dfrac{\partial N_{z\phi}}{\partial z} + \dfrac{1}{R}\dfrac{\partial N_{\phi\phi}}{\partial \phi} + b_\phi \\[2ex] \dfrac{1}{R} N_{\phi\phi} + b_n \end{Bmatrix} = 0, \quad \begin{Bmatrix} N_{zz} \\ N_{\phi\phi} \\ N_{z\phi} \end{Bmatrix} = \frac{tE}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u_z}{\partial z} \\[2ex] \dfrac{1}{R}(\dfrac{\partial u_\phi}{\partial \phi} - u_n) \\[2ex] \dfrac{1}{R}\dfrac{\partial u_z}{\partial \phi} + \dfrac{\partial u_\phi}{\partial z} \end{Bmatrix}.$$

4p Due to the rotation symmetry, the derivatives with respect to the angular coordinate vanish and $u_\phi = 0$. External distributed force $b_n = -p$ is due to the traction acting on the inner boundary. Therefore, the equilibrium equations and constitutive equations simplify to a set of ordinary differential equations

$$\frac{dN_{zz}}{dz} = 0, \quad \frac{dN_{z\phi}}{dz} = 0, \quad \frac{1}{R} N_{\phi\phi} - p = 0 \quad \text{in} \quad (0,L),$$

$$N_{zz} = \frac{tE}{1-\nu^2}\frac{1}{R}(R\frac{du_z}{dz} - \nu u_n), \quad N_{\phi\phi} = \frac{tE}{1-\nu^2}\frac{1}{R}(R\nu\frac{du_z}{dz} - u_n), \quad N_{z\phi} = 0 \quad \text{in} \quad (0,L),$$

As the edges are stress-free i.e.

$$N_{zz} = 0 \quad \text{and} \quad N_{z\phi} = 0 \quad \text{on} \quad \{0, L\}.$$

2p Solution to the stress resultants, as obtained from the equilibrium equations, are

$$N_{zz} = 0, \; N_{z\phi} = 0, \text{ and } N_{\phi\phi} = Rp.$$

Constitutive equations give

$$N_{zz} = \frac{tE}{1-\nu^2}\frac{1}{R}(R\frac{du_z}{dz} - \nu u_n) = 0 \quad \Rightarrow \quad \frac{du_z}{dz} = \frac{\nu}{R} u_n \quad \text{and}$$

$$Rp = N_{\phi\phi} = \frac{tE}{1-\nu^2}\frac{1}{R}(R\nu\frac{du_z}{dz} - u_n) = \frac{tE}{1-\nu^2}\frac{1}{R}(\nu^2 - 1)u_n = -\frac{tE}{R}u_n \quad \Leftrightarrow \quad u_n = -\frac{pR^2}{tE}.$$