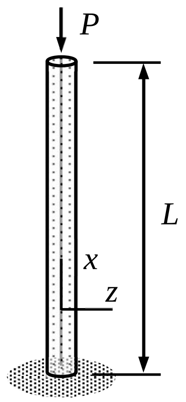
MEC-E8003 Beam, Plate and Shell models, onsite exam 08.04.2025

1. Use the definition $\nabla^2 = \nabla \cdot \nabla$ to derive the Laplacian operator in the polar coordinate system.

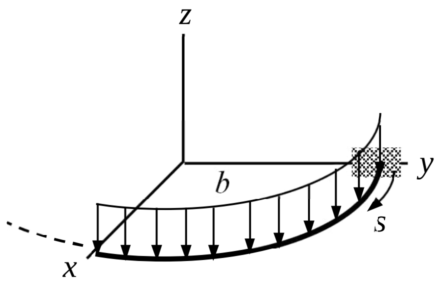
2. When displacement is confined to the $xz-$plane, the virtual work expression of a slender Bernoulli beam (figure) is given by

$$\delta W = -\int_0^L \left(\frac{d^2\delta w}{dx^2} EI \frac{d^2 w}{dx^2}\right)dx + P\int_0^L \left(\frac{d\delta w}{dx}\frac{dw}{dx}\right)dx .$$

Deduce in detail the underlying differential equation and boundary conditions implied by the principle of virtual work and the fundamental lemma of variation calculus.

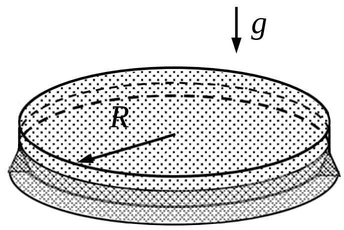


3. Consider the curved beam of the figure forming a 90-degree circular segment of radius R in the horizontal plane. Find the stress resultants $N(s)$, $Q_n(s)$, $Q_b(s)$, $T(s)$, $M_n(s)$, and $M_b(s)$. Use the equilibrium equations of the beam model in the $(s,n,b)-$coordinate system. The distributed constant load of magnitude b is acting to the negative direction of the $z-$axis.

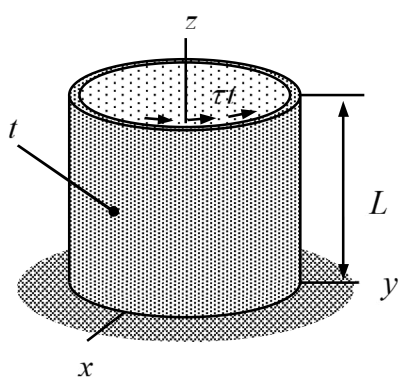


4. A simply supported circular plate of radius R is loaded by its own weight. Determine the displacement of the plate at the midpoint by using the *Reissner-Mindlin* plate model in the polar coordinate system. Problem parameters E, ν, ρ and t are constants. Assume that the solution depends on the radial coordinate only.

Hint: $r\dfrac{d}{dr}[\dfrac{1}{r}\dfrac{d}{dr}(r\theta)] = r\dfrac{d^2\theta}{dr^2} + \dfrac{d\theta}{dr} - \dfrac{1}{r}\theta$



5. A thin walled cylindrical body of length L, (mid-surface) radius R, and thickness t is subjected to shear loading τt $[\tau t] = \mathrm{N/m}$ at the free end $z = L$ as shown in the figure. Assuming rotation symmetry, use the membrane equations in (z,ϕ,n) coordinate system to derive the relationship between the moment resultant T (in the direction of z-axis) of the shear loading and the angle of rotation of the free end defined by $\theta = u_\phi / R$.



Use the definition $\nabla^2 = \nabla \cdot \nabla$ to derive the Laplacian operator in the polar coordinate system.

Solution

2p Gradient operator and the derivatives of the basis vectors of the polar (r,ϕ) – coordinate system are

$$\nabla = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \partial / \partial r \\ \partial / (r \partial \phi) \end{Bmatrix} \text{ and } \frac{\partial}{\partial \phi} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}.$$

According to the definition

$$\nabla^2 = \nabla \cdot \nabla = (\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r} \frac{\partial}{\partial \phi}) \cdot (\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r} \frac{\partial}{\partial \phi})$$

4p in which the terms

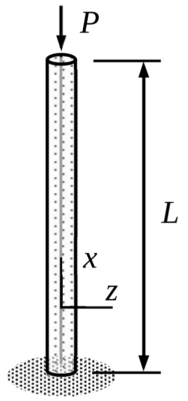
$$(\vec{e}_r \frac{\partial}{\partial r}) \cdot (\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r} \frac{\partial}{\partial \phi}) = \vec{e}_r \cdot (\vec{e}_r \frac{\partial^2}{\partial r^2} - \vec{e}_\phi \frac{1}{r^2} \frac{\partial}{\partial \phi} + \vec{e}_\phi \frac{1}{r} \frac{\partial^2}{\partial r \partial \phi}) = \frac{\partial^2}{\partial r^2},$$

$$(\vec{e}_\phi \frac{1}{r} \frac{\partial}{\partial \phi}) \cdot (\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r} \frac{\partial}{\partial \phi}) = (\vec{e}_\phi \frac{1}{r}) \cdot (\vec{e}_\phi \frac{\partial}{\partial r} + \vec{e}_r \frac{\partial^2}{\partial \phi \partial r} - \vec{e}_r \frac{1}{r} \frac{\partial}{\partial \phi} + \vec{e}_\phi \frac{1}{r} \frac{\partial^2}{\partial \phi^2}) = \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2}$$

Combining the results for the terms gives

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} = \frac{1}{r} \frac{\partial}{\partial r}(r \frac{\partial}{\partial r}) + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2}. \quad \leftarrow$$

When displacement is confined to the $xz-$plane, the virtual work expression of a slender Bernoulli beam (figure) is given by



$$\delta W = -\int_0^L \left(\frac{d^2\delta w}{dx^2} EI \frac{d^2 w}{dx^2}\right)dx + P\int_0^L \left(\frac{d\delta w}{dx}\frac{dw}{dx}\right)dx \, .$$

Deduce in detail the underlying differential equation and boundary conditions implied by the principle of virtual work and the fundamental lemma of variation calculus. Assume that have continuous derivatives up to (and including) fourth order.

Solution

2p Integration by parts gives an equivalent but a more convenient form (assuming continuity up to and including second derivatives)

$$\delta W = -\int_0^L \left(\frac{d^2\delta w}{dx^2} EI \frac{d^2 w}{dx^2}\right)dx + P\int_0^L \left(\frac{d\delta w}{dx}\frac{dw}{dx}\right)dx \quad \Leftrightarrow \quad (P \text{ is a constant})$$

$$\delta W = -\int_0^L \delta w\left(EI\frac{d^4 w}{dx^4} + P\frac{d^2 w}{dx^2}\right)dx + \sum_{\{0,L\}} n\delta w\left(EI\frac{d^3 w}{dx^3} + P\frac{dw}{dx}\right) - \sum_{\{0,L\}} n\frac{d\delta w}{dx}\left(EI\frac{d^2 w}{dx^2}\right).$$

3p According to principle of virtual work $\delta W = 0 \;\forall \delta w$. Let us consider first the subset of variations for which $\delta w = 0$ and $d\delta w / dx = 0$ on $\{0, L\}$. The fundamental lemma of variation calculus implies

$$EI\frac{d^4 w}{dx^4} + P\frac{d^2 w}{dx^2} = 0 \quad \text{in} \quad (0, L).$$

Let us consider then the subset of variations for which $d\delta w / dx = 0$ on $\{0, L\}$. Knowing the condition above, the fundamental lemma of variation calculus implies

$$EI\frac{d^3 w}{dx^3} + P\frac{dw}{dx} = 0 \quad \text{or} \quad w - \underline{w} = 0 \quad \text{on} \quad \{0, L\}.$$

Finally, let us consider the subset of variations for which $\delta w = 0$ on $\{0, L\}$. Knowing the previous results, the fundamental lemma of variation calculus implies

$$EI\frac{d^2 w}{dx^2} = 0 \quad \text{or} \quad \frac{dw}{dx} - \underline{\theta} = 0 \quad \text{on} \quad \{0, L\}.$$

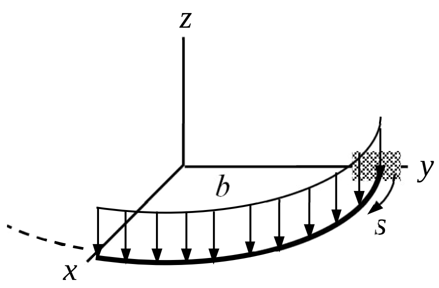
1p For the problem of the figure, one obtains

$$EI\frac{d^4 w}{dx^4} + P\frac{d^2 w}{dx^2} = 0 \quad \text{in} \quad (0, L),$$

$$EI\frac{d^3 w}{dx^3} + P\frac{dw}{dx} = 0 \quad \text{and} \quad EI\frac{d^2 w}{dx^2} = 0 \quad \text{at} \quad x = L,$$

$$w = 0 \quad \text{and} \quad \frac{dw}{dx} = 0 \quad \text{at} \quad x = 0 .$$

Consider the curved beam of the figure forming a 90-degree circular segment of radius R in the horizontal plane. Find the stress resultants $N(s)$, $Q_n(s)$, $Q_b(s)$, $T(s)$, $M_n(s)$, and $M_b(s)$. Use the equilibrium equations of the beam model in the $(s,n,b)-$coordinate system. The distributed constant load of magnitude b is acting to the negative direction of the $z-$axis.



Solution

In a statically determined case, stress resultants follow from the equilibrium equations and boundary conditions at the free end of the beam (or directly from a free body diagram). In $(s,n,b)-$coordinate system, equilibrium equations are

$$\begin{Bmatrix} N'-Q_n\kappa+b_s \\ Q_n'+N\kappa-Q_b\tau+b_n \\ Q_b'+Q_n\tau+b_b \end{Bmatrix}=0 \quad \text{and} \quad \begin{Bmatrix} T'-M_n\kappa+c_s \\ M_n'+T\kappa-M_b\tau-Q_b+c_n \\ M_b'+M_n\tau+Q_n+c_b \end{Bmatrix}=0 .$$

2p For a circular beam, curvature and torsion are $\kappa=1/R$ (constant) and $\tau=0$. As external distributed forces and moments $b_s=b_n=c_s=c_n=c_b=0$ and $b_b=b$, equilibrium equations and the boundary conditions at the free end simplify to (here $L=\pi R/2$)

$$\begin{Bmatrix} N'-Q_n/R \\ Q_n'+N/R \\ Q_b'+b \end{Bmatrix}=0 \quad \text{and} \quad \begin{Bmatrix} T'-M_n/R \\ M_n'+T/R-Q_b \\ M_b'+Q_n \end{Bmatrix}=0 \quad \text{in} \quad (0,L)$$

$$\begin{Bmatrix} N \\ Q_n \\ Q_b \end{Bmatrix}=0 \quad \text{and} \quad \begin{Bmatrix} T \\ M_n \\ M_b \end{Bmatrix}=0 \quad \text{at} \quad s=L .$$

4p Equations constitute a boundary value problem which can be solved one equation at a time by following certain order

$$Q_b'=-b \quad \text{in} \quad (0,L) \quad \text{and} \quad Q_b(L)=0 \quad \Rightarrow \quad Q_b(s)=b(L-s) . \quad \Leftarrow$$

Eliminating Q_n and N from the remaining two connected force equilibrium equations and using the original equations to find the missing boundary condition gives

$$N''+\frac{1}{R^2}N=0 \quad \text{in} \quad (0,L) \quad \text{and} \quad N'(L)=N(L)=0 \quad \Rightarrow \quad N(s)=0 . \quad \Leftarrow$$

Knowing the result above, the first equilibrium equation gives

$$Q_n(s)=0 . \quad \Leftarrow$$

After that, continuing with the moment equilibrium equations with the already known solutions to the force equilibrium equations

$$M_b' = -Q_n = 0 \quad \text{in} \quad (0, L) \quad \text{and} \quad M_b(L) = 0 \quad \Rightarrow \quad M_b(s) = 0 .$$

Eliminating M_n and T from the remaining two connected moment equilibrium equations and using the original equations to find the missing boundary condition gives $(L = \pi R / 2)$

$$T'' + \frac{1}{R^2} T = \frac{1}{R} Q_b = \frac{b}{R}(L - s) \quad \text{in} \quad (0, L) \quad \text{and} \quad T'(L) = T(L) = 0 \quad \Rightarrow$$

$$T(s) = -bR^2 \cos(\frac{s}{R}) + bR(L - s) .$$

Knowing this, the first moment equilibrium equation gives

$$M_n(s) = RT' = bR^2 \sin(\frac{s}{R}) - bR^2 .$$

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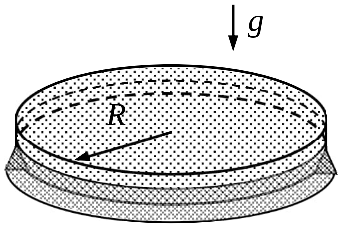


Hint: $r\dfrac{d}{dr}[\dfrac{1}{r}\dfrac{d}{dr}(r\theta)] = r\dfrac{d^2\theta}{dr^2} + \dfrac{d\theta}{dr} - \dfrac{1}{r}\theta$

Solution

1p Let us start with the Reissner-Mindlin plate bending mode equations (linearly elastic homogeneous material) for rotation symmetric case ($\theta = \theta_\phi$ and $b_n = -\rho g t$)

$$\begin{Bmatrix} \dfrac{1}{r}\dfrac{d(rQ_r)}{dr} + b_n \\ \dfrac{1}{r}[\dfrac{d(rM_{rr})}{dr} - M_{\phi\phi}] - Q_r \end{Bmatrix} = 0, \quad \begin{Bmatrix} M_{rr} \\ M_{\phi\phi} \end{Bmatrix} = D \begin{Bmatrix} \dfrac{d\theta}{dr} + \nu\dfrac{1}{r}\theta \\ \dfrac{1}{r}\theta + \nu\dfrac{d\theta}{dr} \end{Bmatrix}, \quad Q_r = Gt(\frac{dw}{dr} + \theta)$$

1p As Q_r needs to be bounded at the origin, integration of the force equilibrium equation gives

$$\frac{1}{r}\frac{d(rQ_r)}{dr} + b_n = 0 \quad \Rightarrow \quad Q_r(r) = -\frac{b_n r}{2}.$$

2p When the bending mode constitutive equations and shear force expression are substituted there, the moment equilibrium equation simplifies to

$$r\frac{d^2\theta}{dr^2} + \frac{d\theta}{dr} - \frac{1}{r}\theta = -\frac{b_n r^2}{2D} \quad \text{or} \quad r\frac{d}{dr}[\frac{1}{r}\frac{d}{dr}(r\theta)] = -\frac{b_n r^2}{2D}.$$

As rotation angle should be bounded, repeated integrations in the latter form gives

$$\theta(r) = -\frac{b_n r^3}{16D} + A\frac{r}{2}.$$

Integration constant A follows from the moment condition $M_{rr}(R) = 0$

$$\frac{d\theta}{dr} + \nu\frac{1}{r}\theta = -\frac{3b_n R^2}{16D} + A\frac{1}{2} - \nu\frac{b_n R^2}{16D} + \nu A\frac{1}{2} = 0 \quad \Leftrightarrow \quad A = \frac{b_n R^2}{8D}\frac{3+\nu}{1+\nu}.$$

Substituting in the rotation expression gives the rotation solution

$$\theta(r) = \frac{b_n}{16D}(-r^3 + rR^2\frac{3+\nu}{1+\nu}).$$

2p Using then the remaining shear force (per unit length) constitutive equation

$$\frac{dw}{dr}=\frac{Q_r}{Gt}-\theta=-\frac{b_n r}{2Gt}-\frac{b_n}{16D}(-r^3+R^2 r\frac{3+\nu}{1+\nu}) \quad \Rightarrow$$

$$w(r)=-\frac{b_n r^2}{4Gt}-\frac{b_n}{16D}(-\frac{r^4}{4}+R^2\frac{r^2}{2}\frac{3+\nu}{1+\nu})+B\,.$$

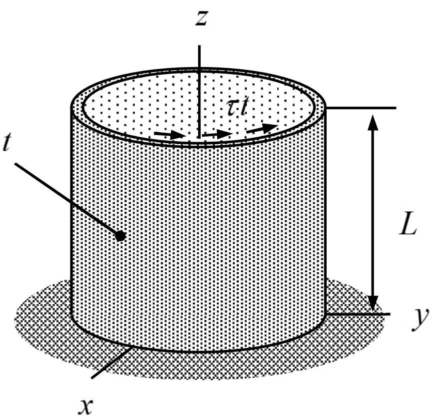
Then using the remaining boundary condition

$$w(R)=-\frac{b_n R^2}{4Gt}-\frac{b_n}{16D}(-\frac{R^4}{4}+R^2\frac{R^2}{2}\frac{3+\nu}{1+\nu})+B=0 \quad \Leftrightarrow \quad B=\frac{b_n R^2}{4Gt}+\frac{b_n R^4}{64D}\frac{5+\nu}{1+\nu}\,.$$

At the centerpoint

$$w(0)=B=\frac{b_n R^2}{4Gt}+\frac{b_n R^4}{64D}\frac{5+\nu}{1+\nu}\,. \quad \leftarrow$$

A thin walled cylindrical body of length L, (mid-surface) radius R, and thickness t is subjected to shear loading τt $[\tau t] = \text{N/m}$ at the free end $z = L$ as shown in the figure. Assuming rotation symmetry, use the membrane equations in (z, ϕ, n) coordinate system to derive the relationship between the moment resultant T of the shear loading and the angle of rotation of the free end defined by $\theta = u_\phi / R$.



Solution

2p As the solution does not depend on ϕ, equilibrium equations of the membrane model and boundary conditions at the free end simplify to (a cylindrical membrane $z-$strip problem)

$$\frac{dN_{zz}}{dz} = 0, \quad \frac{dN_{z\phi}}{dz} = 0, \quad \frac{1}{R} N_{\phi\phi} = 0 \quad \text{in} \quad (0, L),$$

$$N_{zz} = 0 \text{ and } N_{z\phi} = \tau t \quad \text{at} \quad z = L.$$

Solution to the boundary value problem for the stress resultants is given by

$$N_{zz} = N_{\phi\phi} = 0 \text{ and } N_{z\phi}(z) = \tau t.$$

2p Knowing the stress resultants, the boundary value problem for the displacement components follows from the constitutive equations and the boundary conditions at the fixed end (in the membrane model, no boundary condition can be assigned to u_n)

$$\frac{tE}{1-\nu^2}\left(\frac{du_z}{dz} - \nu \frac{1}{R} u_n\right) = 0, \quad \frac{tE}{1-\nu^2}\left(\nu \frac{du_z}{dz} - \frac{1}{R} u_n\right) = 0, \text{ and } tG \frac{du_\phi}{dz} = \tau t \quad \text{in} \quad (0, L),$$

$$u_z = 0, \quad u_\phi = 0 \quad \text{at} \quad z = 0.$$

Solution to the boundary value problem is given by

$$u_z = u_n = 0 \text{ and } u_\phi(z) = \frac{\tau}{G} z.$$

2p Moment resultant of the shear loading

$$T = \int_0^{2\pi} t\tau R (R d\phi) = 2\pi R^2 t\tau \quad \Rightarrow \quad \tau = \frac{T}{2\pi R^2 t}.$$

Therefore, at the free end

$$u_\phi = \frac{\tau}{G} L = \frac{L}{2\pi R^2 t G} T = R\theta \quad \Rightarrow \quad T = \frac{2\pi R^3 t}{L} G\theta.$$

The polar moment predicted here is $I_p = 2\pi R^3 t$ whereas the exact is $I_p = \frac{1}{2}\pi R t(4R^2 + t^2)$.