MEC-E8003 Beam, Plate, and Shell models; Schedule 2024

Week	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Orientation							
9		**14:15-15:30** Lecture 1 (R003/F239a) **15:30-16:00** Assignment 1 (R003/F239a) **16:15-17:00** Calculation hours (Zoom) **23:55 DL** Assignment 1 (MyCourses)	**14:15-15:30** Lecture 2 (R003/F239a) **15:30-16:00** Assignment 2 (R003/F239a) **16:15-17:00** Calculation hours (Zoom) **23:55 DL** Assignment 2 (MyCourses)	**12:15-13:30** Calculation examples (R003/F239a) **13:30-14:00** Assignments 3,4,5 (R003/F239a) **14:15-16:00** Calculation hours (R003/F239a)	Experiment 9-12 and 14-16 Puumiehenkuja 5L		**23:55 DL** Assignments 3,4,5 (MyCourses)
Lectures and exercises							
10		**14:15-15:30** Lecture 1 (R003/F239a) **15:30-16:00** Assignment 1 (R003/F239a) **16:15-17:00** Calculation hours (Zoom) **23:55 DL** Assignment 1 (MyCourses)	**14:15-15:30** Lecture 2 (R003/F239a) **15:30-16:00** Assignment 2 (R003/F239a) **16:15-17:00** Calculation hours (Zoom) **23:55 DL** Assignment 2 (MyCourses)	**12:15-13:30** Calculation examples (R003/F239a) **13:30-14:00** Assignments 3,4,5 (R003/F239a) **14:15-16:00** Calculation hours (R003/F239a)			**23:55 DL** Assignments 3,4,5 (MyCourses)
11		**14:15-15:30** Lecture 1 (R003/F239a) **15:30-16:00** Assignment 1 (R003/F239a) **16:15-17:00** Calculation hours (Zoom) **23:55 DL** Assignment 1 (MyCourses)	**14:15-15:30** Lecture 2 (R003/F239a) **15:30-16:00** Assignment 2 (R003/F239a) **16:15-17:00** Calculation hours (Zoom) **23:55 DL** Assignment 2 (MyCourses)	**10:15-11:30** Calculation examples (R003/F239a) **11:30-12:00** Assignments 3,4,5 (R003/F239a) **14:15-16:00** Calculation hours (R001/U1)			**23:55 DL** Assignments 3,4,5 (MyCourses)
12		**14:15-15:30** Lecture 1 (R003/F239a) **15:30-16:00**	**14:15-15:30** Lecture 2 (R003/F239a) **15:30-16:00**	**12:15-13:30** Calculation examples (R003/F239a)			**23:55 DL** Assignments 3,4,5 (MyCourses)

		Assignment 1 (R003/F239a) **16:15-17:00** Calculation hours (Zoom) **23:55 DL** Assignment 1 (MyCourses)	Assignment 2 (R003/F239a) **16:15-17:00** Calculation hours (Zoom) **23:55 DL** Assignment 2 (MyCourses)	**13:30-14:00** Assignments 3,4,5 (R003/F239a) **14:15-16:00** Calculation Hours (R003/F239a)			
13		**14:15-15:30** Lecture 1 (R003/F239a) **15:30-16:00** Assignment 1 (R003/F239a) **16:15-17:00** Calculation hours (Zoom) **23:55 DL** Assignment 1 (MyCourses)	**14:15-15:30** Lecture 2 (R003/F239a) **15:30-16:00** Assignment 2 (R003/F239a) **16:15-17:00** Calculation hours (Zoom) **23:55 DL** Assignment 2 (MyCourses)				
14				**10:15-11:30** Calculation examples (R003/F239a) **11:30-12:00** Assignments 3,4,5 (R003/F239a) **14:15-16:00** Calculation hours (R003/F239a)			**23:55 DL** Assignments 3,4,5 (MyCourses)
15		**14:15-15:30** Lecture 1 (R003/F239a) **15:30-16:00** Assignment 1 (R003/F239a) **16:15-17:00** Calculation hours (Zoom) **23:55 DL** Assignment 1 (MyCourses)	**14:15-15:30** Lecture 2 (R003/F239a) **15:30-16:00** Assignment 2 (R003/F239a) **16:15-17:00** Calculation hours (Zoom) **23:55 DL** Assignment 2 (MyCourses)	**12:15-13:30** Calculation examples (R003/F239a) **13:30-14:00** Assignments 3,4,5 (R003/F239a) **14:15-16:00** Calculation hours (R003/F239a)			**23:55 DL** Assignments 3,4,5 (MyCourses)
Exams							

16		**13:00-17:00** Final exam (R008/215) (MyCourses)					

MEC-E8003 Beam, Plate and Shell Models; formulae

TENSORS

$$\vec{a}=\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_\gamma\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}a_\alpha\\ a_\beta\\ a_\gamma\end{Bmatrix},\quad \vec{\vec{a}}=\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_\gamma\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}a_{\alpha\alpha} & a_{\alpha\beta} & a_{\alpha\gamma}\\ a_{\beta\alpha} & a_{\beta\beta} & a_{\beta\gamma}\\ a_{\gamma\alpha} & a_{\gamma\beta} & a_{\gamma\gamma}\end{bmatrix}\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_\gamma\end{Bmatrix},\ \text{etc.}$$

$$\vec{\vec{I}}\cdot\vec{a}=\vec{a}\cdot\vec{\vec{I}}=\vec{a}\quad\forall\vec{a},\quad \vec{\vec{I}}=\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}$$

$$\vec{\vec{\vec{\vec{I}}}}:\vec{\vec{a}}=\vec{\vec{a}}:\vec{\vec{\vec{\vec{I}}}}=\vec{\vec{a}}\quad\forall\vec{\vec{a}},\quad \vec{\vec{\vec{\vec{I}}}}=\begin{Bmatrix}\vec{i}\vec{i}\\ \vec{j}\vec{j}\\ \vec{k}\vec{k}\end{Bmatrix}_{\mathrm{c}}^{\mathrm{T}}\begin{Bmatrix}\vec{i}\vec{i}\\ \vec{j}\vec{j}\\ \vec{k}\vec{k}\end{Bmatrix}+\begin{Bmatrix}\vec{i}\vec{j}\\ \vec{j}\vec{k}\\ \vec{k}\vec{i}\end{Bmatrix}_{\mathrm{c}}^{\mathrm{T}}\begin{Bmatrix}\vec{i}\vec{j}\\ \vec{j}\vec{k}\\ \vec{k}\vec{i}\end{Bmatrix}+\begin{Bmatrix}\vec{j}\vec{i}\\ \vec{k}\vec{j}\\ \vec{i}\vec{k}\end{Bmatrix}_{\mathrm{c}}^{\mathrm{T}}\begin{Bmatrix}\vec{j}\vec{i}\\ \vec{k}\vec{j}\\ \vec{i}\vec{k}\end{Bmatrix}$$

$$\vec{\vec{a}}=\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_\gamma\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}a_{\alpha\alpha} & a_{\alpha\beta} & a_{\alpha\gamma}\\ a_{\beta\alpha} & a_{\beta\beta} & a_{\beta\gamma}\\ a_{\gamma\alpha} & a_{\gamma\beta} & a_{\gamma\gamma}\end{bmatrix}\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_\gamma\end{Bmatrix}\ \Leftrightarrow\ \vec{\vec{a}}_{\mathrm{c}}=\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_\gamma\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}a_{\alpha\alpha} & a_{\alpha\beta} & a_{\alpha\gamma}\\ a_{\beta\alpha} & a_{\beta\beta} & a_{\beta\gamma}\\ a_{\gamma\alpha} & a_{\gamma\beta} & a_{\gamma\gamma}\end{bmatrix}^{\mathrm{T}}\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_\gamma\end{Bmatrix}$$

$$\vec{\vec{a}}=-\vec{\vec{a}}_{\mathrm{c}}\ \Rightarrow\ \vec{\vec{a}}\cdot\vec{b}=\vec{a}\times\vec{b}\quad\forall\vec{b}$$

$$\vec{a}\cdot(\vec{b}\times\vec{c})=(\vec{a}\times\vec{b})\cdot\vec{c}$$

$$\vec{a}\times(\vec{b}\times\vec{c})=\vec{b}(\vec{a}\cdot\vec{c})-\vec{c}(\vec{a}\cdot\vec{b})$$

$$\vec{\vec{a}}:(\nabla\vec{b})_{\mathrm{c}}=\nabla\cdot(\vec{\vec{a}}\cdot\vec{b})-(\nabla\cdot\vec{\vec{a}})\cdot\vec{b}$$

CURVILINEAR COORDINATES

$$\vec{r}(\alpha,\beta,\gamma)=x(\alpha,\beta,\gamma)\vec{i}+y(\alpha,\beta,\gamma)\vec{j}+z(\alpha,\beta,\gamma)\vec{k}$$

$$\begin{Bmatrix}\vec{h}_\alpha\\ \vec{h}_\beta\\ \vec{h}_\gamma\end{Bmatrix}=\begin{Bmatrix}\dfrac{\partial\vec{r}}{\partial\alpha}\\ \dfrac{\partial\vec{r}}{\partial\beta}\\ \dfrac{\partial\vec{r}}{\partial\gamma}\end{Bmatrix}=[H]\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix},\quad \begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_\gamma\end{Bmatrix}=\begin{Bmatrix}\vec{h}_\alpha/h_\alpha\\ \vec{h}_\beta/h_\beta\\ \vec{h}_\gamma/h_\gamma\end{Bmatrix}=[F]\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix},$$

$$\frac{\partial}{\partial\eta}\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_\gamma\end{Bmatrix} = (\frac{\partial}{\partial\eta}[F])[F]^{-1}\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_\gamma\end{Bmatrix} \quad \eta\in\{\alpha,\beta,\gamma\}, \quad \nabla = \begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_\gamma\end{Bmatrix}^{\mathrm{T}} ([H][F]^{\mathrm{T}})^{-1}\begin{Bmatrix}\dfrac{\partial}{\partial\alpha}\\ \dfrac{\partial}{\partial\beta}\\ \dfrac{\partial}{\partial\gamma}\end{Bmatrix}$$

$$\nabla = \vec{e}_\alpha\frac{1}{h_\alpha}\frac{\partial}{\partial\alpha} + \vec{e}_\beta\frac{1}{h_\beta}\frac{\partial}{\partial\beta} + \vec{e}_\gamma\frac{1}{h_\gamma}\frac{\partial}{\partial\gamma} \quad \text{(orthonormal coordinate system)}$$

CYLINDRICAL COORDINATES

$$\vec{r}(r,\phi,z) = r(\cos\phi\vec{i} + \sin\phi\vec{j}) + z\vec{k}$$

$$\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_z\end{Bmatrix} = \begin{bmatrix}\mathrm{c}\phi & \mathrm{s}\phi & 0\\ -\mathrm{s}\phi & \mathrm{c}\phi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}, \quad \frac{\partial}{\partial\phi}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_z\end{Bmatrix} = \begin{Bmatrix}\vec{e}_\phi\\ -\vec{e}_r\\ 0\end{Bmatrix}, \quad \frac{\partial}{\partial r}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_z\end{Bmatrix} = \frac{\partial}{\partial z}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_z\end{Bmatrix} = 0$$

$$\nabla = \vec{e}_r\frac{\partial}{\partial r} + \vec{e}_\phi\frac{1}{r}\frac{\partial}{\partial\phi} + \vec{e}_z\frac{\partial}{\partial z}$$

SPHERICAL COORDINATES

$$\vec{r}(\theta,\phi,r) = r(\sin\theta\cos\phi\vec{i} + \sin\theta\sin\phi\vec{j} + \cos\theta\vec{k})$$

$$\begin{Bmatrix}\vec{e}_\theta\\ \vec{e}_\phi\\ \vec{e}_r\end{Bmatrix} = \begin{bmatrix}\mathrm{c}\theta\mathrm{c}\phi & \mathrm{c}\theta\mathrm{s}\phi & -\mathrm{s}\theta\\ -\mathrm{s}\phi & \mathrm{c}\phi & 0\\ \mathrm{s}\theta\mathrm{c}\phi & \mathrm{s}\theta\mathrm{s}\phi & \mathrm{c}\theta\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}, \quad \frac{\partial}{\partial\phi}\begin{Bmatrix}\vec{e}_\theta\\ \vec{e}_\phi\\ \vec{e}_r\end{Bmatrix} = \begin{Bmatrix}\mathrm{c}\theta\,\vec{e}_\phi\\ -\mathrm{s}\theta\vec{e}_r - \mathrm{c}\theta\vec{e}_\theta\\ \mathrm{s}\theta\vec{e}_\phi\end{Bmatrix}, \quad \frac{\partial}{\partial\theta}\begin{Bmatrix}\vec{e}_\theta\\ \vec{e}_\phi\\ \vec{e}_r\end{Bmatrix} = \begin{Bmatrix}-\vec{e}_r\\ 0\\ \vec{e}_\theta\end{Bmatrix}$$

$$\frac{\partial}{\partial r}\begin{Bmatrix}\vec{e}_\theta\\ \vec{e}_\phi\\ \vec{e}_r\end{Bmatrix} = 0, \quad \nabla = \vec{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \vec{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} + \vec{e}_r\frac{\partial}{\partial r}$$

BEAM COORDINATES

$$\vec{r}(s,n,b) = \vec{r}_0(s) + n\vec{e}_n + b\vec{e}_b$$

$$\begin{Bmatrix}\vec{e}_s\\ \vec{e}_n\\ \vec{e}_b\end{Bmatrix} = \begin{Bmatrix}\dfrac{\partial\vec{r}_0}{\partial s}\\ \dfrac{\partial\vec{e}_s}{\partial s}/|\dfrac{\partial\vec{e}_s}{\partial s}|\\ \vec{e}_s\times\vec{e}_n\end{Bmatrix}, \quad \frac{\partial}{\partial s}\begin{Bmatrix}\vec{e}_s\\ \vec{e}_n\\ \vec{e}_b\end{Bmatrix} = \begin{Bmatrix}\kappa\vec{e}_n\\ \tau\vec{e}_b - \kappa\vec{e}_s\\ -\tau\vec{e}_n\end{Bmatrix}, \quad \frac{\partial}{\partial n}\begin{Bmatrix}\vec{e}_s\\ \vec{e}_n\\ \vec{e}_b\end{Bmatrix} = \frac{\partial}{\partial b}\begin{Bmatrix}\vec{e}_s\\ \vec{e}_n\\ \vec{e}_b\end{Bmatrix} = 0$$

$$\nabla = \frac{\vec{e}_s}{1-n\kappa}[\frac{\partial}{\partial s} + \tau(b\frac{\partial}{\partial n} - n\frac{\partial}{\partial b})] + \vec{e}_n\frac{\partial}{\partial n} + \vec{e}_b\frac{\partial}{\partial b}$$

SHELL COORDINATES

$$\vec{r}(\alpha,\beta,n) = \vec{r}_0(\alpha,\beta) + n\vec{e}_n$$

$$\begin{Bmatrix} \vec{h}_\alpha \\ \vec{h}_\beta \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial \vec{r}_0}{\partial \alpha} \\ \dfrac{\partial \vec{r}_0}{\partial \beta} \end{Bmatrix}, \quad \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_n \end{Bmatrix} = \begin{Bmatrix} \vec{h}_\alpha / h_\alpha \\ \vec{h}_\beta / h_\beta \\ \vec{e}_\alpha \times \vec{e}_\beta \end{Bmatrix} = [F]\begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}, \quad \frac{\partial}{\partial \eta}\begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_n \end{Bmatrix} = (\frac{\partial}{\partial \eta}[F])[F]^{-1}\begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_n \end{Bmatrix} \quad \eta \in \{\alpha,\beta,n\},$$

$$\begin{Bmatrix} \dfrac{\partial \vec{r}}{\partial \alpha} \\ \dfrac{\partial \vec{r}}{\partial \beta} \\ \dfrac{\partial \vec{r}}{\partial n} \end{Bmatrix} = \begin{Bmatrix} \vec{h}_\alpha + n\dfrac{\partial \vec{e}_n}{\partial \alpha} \\ \vec{h}_\beta + n\dfrac{\partial \vec{e}_n}{\partial \beta} \\ \vec{e}_n \end{Bmatrix} = [H]\begin{Bmatrix} \vec{i} \\ \vec{j} \\ k \end{Bmatrix}, \quad \nabla = \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_n \end{Bmatrix}^{\mathrm{T}} ([H][F]^{\mathrm{T}})^{-1}\begin{Bmatrix} \dfrac{\partial}{\partial \alpha} \\ \dfrac{\partial}{\partial \beta} \\ \dfrac{\partial}{\partial n} \end{Bmatrix}$$

CYLINDRICAL SHELL COORDINATES

$$\vec{r}(z,\phi,n) = (R-n)(\cos\phi\vec{i} + \sin\phi\vec{j}) + z\vec{k}$$

$$\begin{Bmatrix} \vec{e}_z \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -\sin\phi & \cos\phi & 0 \\ -\cos\phi & -\sin\phi & 0 \end{bmatrix}\begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}, \quad \frac{\partial}{\partial \phi}\begin{Bmatrix} \vec{e}_z \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ \vec{e}_n \\ -\vec{e}_\phi \end{Bmatrix}, \quad \frac{\partial}{\partial z}\begin{Bmatrix} \vec{e}_z \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix} = \frac{\partial}{\partial n}\begin{Bmatrix} \vec{e}_z \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix} = 0$$

$$\nabla = \vec{e}_z\frac{\partial}{\partial z} + (\frac{R}{R-n})\frac{1}{R}\vec{e}_\phi\frac{\partial}{\partial \phi} + \vec{e}_n\frac{\partial}{\partial n}$$

SPHERICAL SHELL COORDINATES

$$\vec{r}(\phi,\theta,n) = (R-n)(\sin\theta\cos\phi\vec{i} + \sin\theta\sin\phi\vec{j} + \cos\theta\vec{k})$$

$$\begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_\theta \\ \vec{e}_n \end{Bmatrix} = \begin{bmatrix} -\sin\phi & \cos\phi & 0 \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta\cos\phi & -\sin\theta\sin\phi & -\cos\theta \end{bmatrix}\begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}, \quad \frac{\partial}{\partial \phi}\begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_\theta \\ \vec{e}_n \end{Bmatrix} = \begin{Bmatrix} \sin\theta\vec{e}_n - \cos\theta\vec{e}_\theta \\ \cos\theta\vec{e}_\phi \\ -\sin\theta\,\vec{e}_\phi \end{Bmatrix}$$

$$\frac{\partial}{\partial \theta}\begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_\theta \\ \vec{e}_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ \vec{e}_n \\ -\vec{e}_\theta \end{Bmatrix}, \quad \frac{\partial}{\partial n}\begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_\theta \\ \vec{e}_n \end{Bmatrix} = 0, \quad \nabla = \frac{R}{R-n}\frac{1}{R\sin\theta}\vec{e}_\phi\frac{\partial}{\partial \phi} + \frac{R}{R-n}\frac{1}{R}\vec{e}_\theta\frac{\partial}{\partial \theta} + \vec{e}_n\frac{\partial}{\partial n}$$

CIRCULAR PLATE COORDINATES

$$\vec{r}(r,\phi,n) = r(\cos\phi\vec{i} + \sin\phi\vec{j}) + n\vec{k}$$

$$\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}, \quad \frac{\partial}{\partial\phi} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix} = \begin{Bmatrix} \vec{e}_\phi \\ -\vec{e}_r \\ 0 \end{Bmatrix}, \quad \frac{\partial}{\partial r} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix} = \frac{\partial}{\partial n} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix} = 0$$

$$\nabla = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial\phi} + \vec{e}_n \frac{\partial}{\partial n}$$

LINEAR ELASTICITY

$$\vec{\sigma} = \vec{\vec{E}} : \vec{\varepsilon} = \vec{\vec{E}} : \nabla\vec{u}, \quad \vec{\varepsilon} = \frac{1}{2}[\nabla\vec{u} + (\nabla\vec{u})_{\mathrm{c}}]$$

Elastic material: $\vec{\vec{E}} = \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix}^{\mathrm{T}} [E] \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix} + \begin{Bmatrix} \vec{i}\vec{j} + \vec{j}\vec{i} \\ \vec{j}\vec{k} + \vec{k}\vec{j} \\ \vec{k}\vec{i} + \vec{i}\vec{k} \end{Bmatrix}^{\mathrm{T}} [G] \begin{Bmatrix} \vec{i}\vec{j} + \vec{j}\vec{i} \\ \vec{j}\vec{k} + \vec{k}\vec{j} \\ \vec{k}\vec{i} + \vec{i}\vec{k} \end{Bmatrix}$

Isotropic: $[E] = \dfrac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix}, \ [G] = \begin{bmatrix} G & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix}$

Plane stress: $[E] = \dfrac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ [G] = \begin{bmatrix} G & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Beam: $[E] = \begin{bmatrix} E & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ [G] = \begin{bmatrix} G & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix}$

Plate: $[E] = \dfrac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ [G] = \begin{bmatrix} G & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix},$

$$[E]_\sigma = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}, \quad [E]_\sigma^{-1} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix}, \quad D = \frac{t^3 E}{12(1-\nu^2)}$$

PRINCIPLE OF VIRTUAL WORK

$$\delta W = \delta W^{\text{ext}} + \delta W^{\text{int}} = 0 \quad \forall \delta \vec{u} \in U \quad \text{(a function set)}$$

$$\delta W = -\int_V (\vec{\sigma} : \delta \vec{\varepsilon}_{\text{c}}) dV + \int_V (\vec{f} \cdot \delta \vec{u}) dV + \int_A (\vec{t} \cdot \delta \vec{u}) dA$$

BEAM EQUATIONS

$$\begin{Bmatrix} \dfrac{d\vec{F}}{ds} + \vec{b} \\ \dfrac{d\vec{M}}{ds} + \vec{e}_s \times \vec{F} + \vec{c} \end{Bmatrix} = 0, \quad \begin{Bmatrix} \vec{b} \\ \vec{c} \end{Bmatrix} = \int \begin{Bmatrix} \vec{f} \\ \vec{\rho} \times \vec{f} \end{Bmatrix} J dA, \text{ where } J = 1 + n\kappa$$

$$\begin{Bmatrix} \vec{F} \\ \vec{M} \end{Bmatrix} = \int \begin{Bmatrix} \vec{\sigma} \\ \vec{\rho} \times \vec{\sigma} \end{Bmatrix} dA = \int \begin{bmatrix} \vec{E} & -\vec{E} \times \vec{\rho} \\ \vec{\rho} \times \vec{E} & -\vec{\rho} \times \vec{E} \times \vec{\rho} \end{bmatrix} dA \cdot \begin{Bmatrix} \dfrac{d\vec{u}_0}{ds} + \vec{e}_s \times \vec{\theta}_0 \\ \dfrac{d\vec{\theta}_0}{ds} \end{Bmatrix}, \text{ where } \vec{E} = \vec{e}_s \cdot \vec{\vec{E}} \cdot \vec{e}_s$$

TIMOSHENKO BEAM (x, y, z)

$$\vec{u}_0 = u\vec{i} + v\vec{j} + w\vec{k}, \ \vec{\theta}_0 = \phi\vec{i} + \theta\vec{j} + \psi\vec{k}, \ \vec{F} = N\vec{i} + Q_y\vec{j} + Q_z\vec{k}, \ \vec{M} = T\vec{i} + M_y\vec{j} + M_z\vec{k}$$

$$\begin{Bmatrix} \dfrac{dN}{dx} + b_x \\ \dfrac{dQ_y}{dx} + b_y \\ \dfrac{dQ_z}{dx} + b_z \end{Bmatrix} = 0, \quad \begin{Bmatrix} N \\ Q_y \\ Q_z \end{Bmatrix} = \begin{Bmatrix} EA\dfrac{du}{dx} - ES_z\dfrac{d\psi}{dx} + ES_y\dfrac{d\theta}{dx} \\ GA(\dfrac{dv}{dx} - \psi) - GS_y\dfrac{d\phi}{dx} \\ GA(\dfrac{dw}{dx} + \theta) + GS_z\dfrac{d\phi}{dx} \end{Bmatrix}$$

$$\begin{Bmatrix} \dfrac{dT}{dx} + c_x \\ \dfrac{dM_y}{dx} - Q_z + c_y \\ \dfrac{dM_z}{dx} + Q_y + c_z \end{Bmatrix} = 0, \quad \begin{Bmatrix} T \\ M_y \\ M_z \end{Bmatrix} = \begin{Bmatrix} -GS_y(\dfrac{dv}{dx} - \psi) + GS_z(\dfrac{dw}{dx} + \theta) + GI_{rr}\dfrac{d\phi}{dx} \\ ES_y\dfrac{du}{dx} - EI_{zy}\dfrac{d\psi}{dx} + EI_{yy}\dfrac{d\theta}{dx} \\ -ES_z\dfrac{du}{dx} + EI_{zz}\dfrac{d\psi}{dx} - EI_{yz}\dfrac{d\theta}{dx} \end{Bmatrix}$$

TIMOSHENKO BEAM (s, n, b)

$$\vec{u} = u\vec{e}_s + v\vec{e}_n + w\vec{e}_b, \ \vec{\theta} = \phi\vec{e}_s + \theta\vec{e}_n + \psi\vec{e}_b, \vec{F} = N\vec{e}_s + Q_n\vec{e}_n + Q_b\vec{e}_b, \ \vec{M} = T\vec{e}_s + M_n\vec{e}_n + M_b\vec{e}_b$$

$$\begin{Bmatrix} \dfrac{dN}{ds} - Q_n\kappa + b_s \\ \dfrac{dQ_n}{ds} + N\kappa - Q_b\tau + b_n \\ \dfrac{dQ_b}{ds} + Q_n\tau + b_b \end{Bmatrix} = 0, \quad \begin{Bmatrix} \dfrac{dT}{ds} - M_n\kappa + c_s \\ \dfrac{dM_n}{ds} + T\kappa - M_b\tau - Q_b + c_n \\ \dfrac{dM_b}{ds} + M_n\tau + Q_n + c_b \end{Bmatrix} = 0$$

$$\begin{Bmatrix} N \\ Q_n \\ Q_b \end{Bmatrix} = \begin{Bmatrix} EA(\frac{du}{ds} - v\kappa) + ES_n(\frac{d\theta}{ds} + \phi\kappa - \psi\tau) - ES_b(\frac{d\psi}{ds} + \theta\tau) \\ GA(\frac{dv}{ds} + u\kappa - w\tau - \psi) - GS_n(\frac{d\phi}{ds} - \theta\kappa) \\ GA(\frac{dw}{ds} + v\tau + \theta) + GS_b(\frac{d\phi}{ds} - \theta\kappa) \end{Bmatrix}$$

$$\begin{Bmatrix} T \\ M_n \\ M_b \end{Bmatrix} = \begin{Bmatrix} GS_b(\frac{dw}{ds} + v\tau + \theta) + GI_{rr}(\frac{d\phi}{ds} - \theta\kappa) - GS_n(\frac{dv}{ds} + u\kappa - w\tau - \psi) \\ ES_n(\frac{du}{ds} - v\kappa) + EI_{nn}(\frac{d\theta}{ds} + \phi\kappa - \psi\tau) - EI_{bn}(\frac{d\psi}{ds} + \theta\tau) \\ -ES_b(\frac{du}{ds} - v\kappa) - EI_{nb}(\frac{d\theta}{ds} + \phi\kappa - \psi\tau) + EI_{bb}(\frac{d\psi}{ds} + \theta\tau) \end{Bmatrix}$$

PLATE EQUATIONS

$$\nabla_0 \cdot \vec{\vec{F}} + \vec{b} = 0\,,\ (\nabla_0 \cdot \vec{\vec{M}} - \vec{e}_n \cdot \vec{\vec{F}} + \vec{c}) \times \vec{e}_n = 0\,,\ \begin{Bmatrix} \vec{b} \\ \vec{c} \end{Bmatrix} = \int \vec{f} \begin{Bmatrix} 1 \\ n \end{Bmatrix} dn$$

$$\begin{Bmatrix} \vec{\vec{F}} \\ \vec{\vec{M}} \end{Bmatrix} = \int \vec{\vec{\sigma}} \begin{Bmatrix} 1 \\ n \end{Bmatrix} dn = \int \begin{bmatrix} 1 & n \\ n & n^2 \end{bmatrix} \vec{\vec{\vec{\vec{E}}}} dn : \begin{Bmatrix} \vec{\vec{\varepsilon}} \\ \vec{\vec{\kappa}} \end{Bmatrix} = \begin{bmatrix} \vec{\vec{\vec{\vec{A}}}} & \vec{\vec{\vec{\vec{C}}}} \\ \vec{\vec{\vec{\vec{C}}}} & \vec{\vec{\vec{\vec{B}}}} \end{bmatrix} : \begin{Bmatrix} \nabla_0 \vec{u}_0 + \vec{e}_n \vec{\omega}_0 \\ \nabla_0 \vec{\omega}_0 \end{Bmatrix},\ \vec{\omega}_0 = \vec{\theta}_0 \times \vec{e}_n$$

REISSNER-MINDLIN PLATE (x, y, n)

$$\vec{\vec{F}} = \int \vec{\vec{\sigma}} dn = \vec{i}\vec{i} N_{xx} + (\vec{i}\vec{j} + \vec{j}\vec{i}) N_{xy} + \vec{j}\vec{j} N_{yy} + (\vec{e}_n \vec{i} + \vec{i}\vec{e}_n) Q_x + (\vec{e}_n \vec{j} + \vec{j}\vec{e}_n) Q_y$$

$$\vec{\vec{M}} = \int \vec{\vec{\sigma}} n dn = \vec{i}\vec{i} M_{xx} + (\vec{i}\vec{j} + \vec{j}\vec{i}) M_{xy} + \vec{j}\vec{j} M_{yy}$$

$$\begin{Bmatrix} \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} + b_x \\ \frac{\partial N_{yy}}{\partial y} + \frac{\partial N_{xy}}{\partial x} + b_y \end{Bmatrix} = 0\,,\ \begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = t[E]_\sigma \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix}$$

$$\begin{Bmatrix} \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + b_n \\ \frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x \\ \frac{\partial M_{yy}}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y \end{Bmatrix} = 0\,,\ \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \frac{t^3}{12}[E]_\sigma \begin{Bmatrix} \frac{\partial \theta}{\partial x} \\ -\frac{\partial \phi}{\partial y} \\ \frac{\partial \theta}{\partial y} - \frac{\partial \phi}{\partial x} \end{Bmatrix},\ \begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = Gt \begin{Bmatrix} \frac{\partial w}{\partial x} + \theta \\ \frac{\partial w}{\partial y} - \phi \end{Bmatrix}$$

REISSNER-MINDLIN PLATE (r,ϕ,n)

$$\begin{Bmatrix} \frac{1}{r}[\frac{\partial(rN_{rr})}{\partial r}+\frac{\partial N_{r\phi}}{\partial \phi}-N_{\phi\phi}]+b_r \\ \frac{1}{r}[\frac{1}{r}\frac{\partial(r^2N_{r\phi})}{\partial r}+\frac{\partial N_{\phi\phi}}{\partial \phi}]+b_\phi \end{Bmatrix}=0\,,\ \begin{Bmatrix} N_{rr} \\ N_{\phi\phi} \\ N_{r\phi} \end{Bmatrix}=t\left[E\right]_\sigma \begin{Bmatrix} \frac{\partial u_r}{\partial r} \\ \frac{1}{r}(u_r+\frac{\partial u_\phi}{\partial \phi}) \\ \frac{1}{r}\frac{\partial u_r}{\partial \phi}+r\frac{\partial}{\partial r}(\frac{u_\phi}{r}) \end{Bmatrix}$$

$$\begin{Bmatrix} \frac{1}{r}[\frac{\partial(rQ_r)}{\partial r}+\frac{\partial Q_\phi}{\partial \phi}]+b_n \\ \frac{1}{r}[\frac{\partial(rM_{rr})}{\partial r}+\frac{\partial M_{r\phi}}{\partial \phi}-M_{\phi\phi}]-Q_r \\ \frac{1}{r}[\frac{\partial(rM_{r\phi})}{\partial r}+\frac{\partial M_{\phi\phi}}{\partial \phi}+M_{r\phi}]-Q_\phi \end{Bmatrix}=0\,,\ \begin{Bmatrix} M_{rr} \\ M_{\phi\phi} \\ M_{r\phi} \end{Bmatrix}=\frac{t^3}{12}\left[E\right]_\sigma \begin{Bmatrix} \frac{\partial \theta_\phi}{\partial r} \\ \frac{1}{r}(\theta_\phi-\frac{\partial \theta_r}{\partial \phi}) \\ \frac{1}{r}(\frac{\partial \theta_\phi}{\partial \phi}+\theta_r)-\frac{\partial \theta_r}{\partial r} \end{Bmatrix}$$

$$\begin{Bmatrix} Q_r \\ Q_\phi \end{Bmatrix}=Gt\begin{Bmatrix} \frac{\partial w}{\partial r}+\theta_\phi \\ \frac{1}{r}\frac{\partial w}{\partial \phi}-\theta_r \end{Bmatrix}$$

KIRCHHOFF PLATE BENDING (r,ϕ,n)

$$\nabla_0^2\nabla_0^2 w-\frac{b_n}{D}=0\,,\quad w(r)=\frac{b_n}{D}\frac{r^4}{64}+a+br^2+cr^2(1-\log r)+d\log r$$

SHELL EQUATIONS

$$(\nabla_0-\kappa\vec{e}_n)\cdot\vec{F}+\vec{b}=0\,,\ (\nabla_0-\kappa\vec{e}_n)\cdot\vec{M}-\vec{e}_n\cdot\vec{F}+\vec{c})\times\vec{e}_n=0\,,\quad \begin{Bmatrix}\vec{b}\\ \vec{c}\end{Bmatrix}=\int \vec{f}\begin{Bmatrix}1\\ n\end{Bmatrix}Jdn+\sum \vec{t}\begin{Bmatrix}1\\ n\end{Bmatrix}J$$

$$\begin{Bmatrix}\vec{F}\\ \vec{M}\end{Bmatrix}=\int \begin{Bmatrix}1\\ n\end{Bmatrix}J\vec{D}_{\mathrm{c}}\cdot\vec{\sigma}dn=\int \begin{bmatrix}1 & n\\ n & n^2\end{bmatrix}(\vec{D}_{\mathrm{c}}\cdot\vec{\vec{E}}\cdot\vec{D}J)dn:\begin{Bmatrix}\vec{\varepsilon}\\ \vec{\kappa}\end{Bmatrix}=\begin{bmatrix}\vec{\vec{A}} & \vec{\vec{C}}\\ \vec{\vec{C}} & \vec{\vec{B}}\end{bmatrix}:\begin{Bmatrix}\nabla_0\vec{u}_0+\vec{e}_n\vec{\omega}_0\\ \nabla_0\vec{\omega}_0\end{Bmatrix},$$

$$\vec{\omega}_0=\vec{\theta}_0\times\vec{e}_n$$

MEMBRANE EQUATIONS IN CYLINDRICAL GEOMETRY (z,ϕ,n)

$$\begin{Bmatrix} \frac{1}{R}\frac{\partial N_{z\phi}}{\partial \phi}+\frac{\partial N_{zz}}{\partial z}+b_z \\ \frac{\partial N_{z\phi}}{\partial z}+\frac{1}{R}\frac{\partial N_{\phi\phi}}{\partial \phi}+b_\phi \\ \frac{1}{R}N_{\phi\phi}+b_n \end{Bmatrix}=0\,,\quad \begin{Bmatrix} N_{zz} \\ N_{\phi\phi} \\ N_{z\phi} \end{Bmatrix}=t\left[E\right]_\sigma \begin{Bmatrix} \frac{\partial u_z}{\partial z} \\ \frac{1}{R}(\frac{\partial u_\phi}{\partial \phi}-u_n) \\ \frac{1}{R}\frac{\partial u_z}{\partial \phi}+\frac{\partial u_\phi}{\partial z} \end{Bmatrix}$$

MEMBRANE EQUATIONS IN SPHERICAL GEOMETRY (ϕ,θ,n)

$$\begin{Bmatrix} \frac{1}{R}(\csc\theta\frac{\partial N_{\phi\phi}}{\partial \phi}+\frac{\partial N_{\phi\theta}}{\partial \theta}+2\cot\theta N_{\phi\theta})+b_\phi \\ \frac{1}{R}[\csc\theta\frac{\partial N_{\phi\theta}}{\partial \phi}+\frac{\partial N_{\theta\theta}}{\partial \theta}+\cot\theta(N_{\theta\theta}-N_{\phi\phi})]+b_\theta \\ \frac{1}{R}(N_{\phi\phi}+N_{\theta\theta})+b_n \end{Bmatrix}=0$$

$$\begin{Bmatrix} N_{\phi\phi} \\ N_{\theta\theta} \\ N_{\phi\theta} \end{Bmatrix}=t\left[E\right]_\sigma \begin{Bmatrix} \frac{1}{R}[\csc\theta(\cos\theta u_\theta+\frac{\partial u_\phi}{\partial \phi})-u_n] \\ \frac{1}{R}(\csc\theta\sin\theta\frac{\partial u_\theta}{\partial \theta}-u_n) \\ \frac{1}{R}(\csc\theta\frac{\partial u_\theta}{\partial \phi}-\cot\theta u_\phi+\frac{\partial u_\phi}{\partial \theta}) \end{Bmatrix}\quad (\csc\theta=\frac{1}{\sin\theta})$$

SHELL EQUATIONS IN CYLINDRICAL GEOMETRY (z,ϕ,n)

$$\begin{Bmatrix} \frac{1}{R}\frac{\partial N_{\phi z}}{\partial \phi}+\frac{\partial N_{zz}}{\partial z}+b_z \\ \frac{\partial N_{z\phi}}{\partial z}+\frac{1}{R}\frac{\partial N_{\phi\phi}}{\partial \phi}-\frac{1}{R}Q_\phi+b_\phi \end{Bmatrix}=0\,,\quad \begin{Bmatrix} \frac{1}{R}\frac{\partial Q_\phi}{\partial \phi}+\frac{\partial Q_z}{\partial z}+\frac{1}{R}N_{\phi\phi}+b_n \\ \frac{\partial M_{z\phi}}{\partial z}+\frac{1}{R}\frac{\partial M_{\phi\phi}}{\partial \phi}-\frac{1}{R}M_{\phi n}-Q_\phi+c_\phi \\ \frac{\partial M_{zz}}{\partial z}+\frac{1}{R}\frac{\partial M_{\phi z}}{\partial \phi}-Q_z+c_z \end{Bmatrix}=0$$

$$\begin{Bmatrix} N_{zz} \\ N_{\phi\phi} \\ N_{z\phi} \\ N_{\phi z} \end{Bmatrix}=\begin{Bmatrix} \frac{tE}{1-\nu^2}[\frac{\partial u_z}{\partial z}+\nu\frac{1}{R}(\frac{\partial u_\phi}{\partial \phi}-u_n)]-D\frac{1}{R}\frac{\partial \theta_\phi}{\partial z} \\ \frac{tE}{1-\nu^2}[\frac{1}{R}(\frac{\partial u_\phi}{\partial \phi}-u_n)+\nu\frac{\partial u_z}{\partial z}]-D\frac{1}{R^2}\frac{\partial \theta_z}{\partial \phi} \\ Gt(\frac{1}{R}\frac{\partial u_z}{\partial \phi}+\frac{\partial u_\phi}{\partial z})+\frac{1}{2}(1-\nu)D\frac{1}{R}\frac{\partial \theta_z}{\partial z} \\ Gt(\frac{1}{R}\frac{\partial u_z}{\partial \phi}+\frac{\partial u_\phi}{\partial z})+\frac{1}{2}(1-\nu)D\frac{1}{R^2}\frac{\partial \theta_\phi}{\partial \phi} \end{Bmatrix},\quad \begin{Bmatrix} Q_z \\ Q_\phi \end{Bmatrix}=tG\begin{Bmatrix} \frac{\partial u_n}{\partial z}+\theta_\phi \\ \frac{1}{R}(\frac{\partial u_n}{\partial \phi}+u_\phi)-\theta_z \end{Bmatrix}$$

$$\begin{Bmatrix} M_{zz} \\ M_{\phi\phi} \\ M_{z\phi} \\ M_{\phi z} \end{Bmatrix} = D \begin{Bmatrix} \dfrac{\partial\theta_\phi}{\partial z} - \nu \dfrac{1}{R}\dfrac{\partial\theta_z}{\partial\phi} - \dfrac{1}{R}\dfrac{\partial u_z}{\partial z} \\ \nu \dfrac{\partial\theta_\phi}{\partial z} - \dfrac{1}{R}\dfrac{\partial\theta_z}{\partial\phi} + \dfrac{1}{R^2}(\dfrac{\partial u_\phi}{\partial\phi} - u_n) \\ \dfrac{1}{2}(1-\nu)[(\dfrac{1}{R}\dfrac{\partial\theta_\phi}{\partial\phi} - \dfrac{\partial\theta_z}{\partial z}) - \dfrac{1}{R}\dfrac{\partial u_\phi}{\partial z}] \\ \dfrac{1}{2}(1-\nu)[(\dfrac{1}{R}\dfrac{\partial\theta_\phi}{\partial\phi} - \dfrac{\partial\theta_z}{\partial z}) + \dfrac{1}{R^2}\dfrac{\partial u_z}{\partial\phi}] \end{Bmatrix}, \quad M_{\phi n} = \frac{1}{2}(1-\nu) D \frac{1}{R}[\frac{1}{R}(\frac{\partial u_n}{\partial\phi} + u_\phi) - \theta_z]$$

SHELL EQUATIONS IN SPHERICAL GEOMETRY (ϕ, θ, n)

$$\begin{Bmatrix} \dfrac{1}{R}(\dfrac{\partial}{\partial\theta}N_{\phi\theta} + \csc\theta\dfrac{\partial}{\partial\phi}N_{\phi\phi} + 2\cot\theta N_{\phi\theta} - Q_\phi) + b_\phi \\ \dfrac{1}{R}(\dfrac{\partial}{\partial\theta}N_{\theta\theta} + \csc\theta\dfrac{\partial}{\partial\phi}N_{\phi\theta} + \cot\theta N_{\theta\theta} - \cot\theta N_{\phi\phi} - Q_\theta) + b_\theta \end{Bmatrix} = 0,$$

$$\begin{Bmatrix} \dfrac{1}{R}(\dfrac{\partial}{\partial\theta}Q_\theta + \csc\theta\dfrac{\partial}{\partial\phi}Q_\phi + \cot\theta Q_\theta + N_{\theta\theta} + N_{\phi\phi}) + b_n \\ \dfrac{1}{R}(\dfrac{\partial}{\partial\theta}M_{\phi\theta} + \csc\theta\dfrac{\partial}{\partial\phi}M_{\phi\phi} + 2\cot\theta M_{\phi\theta}) - Q_\phi + c_\phi \\ \dfrac{1}{R}(\dfrac{\partial}{\partial\theta}M_{\theta\theta} + \csc\theta\dfrac{\partial}{\partial\phi}M_{\phi\theta} + \cot\theta M_{\theta\theta} - \cot\theta M_{\phi\phi}) - Q_\theta + c_\theta \end{Bmatrix} = 0,$$

$$\begin{Bmatrix} N_{\phi\phi} \\ N_{\theta\theta} \\ N_{\phi\theta} \end{Bmatrix} = \frac{Et}{1-\nu^2}\frac{1}{R} \begin{Bmatrix} (u_\theta\cot\theta + \dfrac{\partial u_\phi}{\partial\phi}\csc\theta - u_n) + \nu(\dfrac{\partial u_\theta}{\partial\theta} - u_n) \\ \nu(u_\theta\cot\theta + \dfrac{\partial u_\phi}{\partial\phi}\csc\theta - u_n) + (\dfrac{\partial u_\theta}{\partial\theta} - u_n) \\ \dfrac{1-\nu}{2}(-u_\phi\cot\theta + \dfrac{\partial u_\theta}{\partial\phi}\csc\theta + \dfrac{\partial u_\phi}{\partial\theta}) \end{Bmatrix},$$

$$\begin{Bmatrix} M_{\phi\phi} \\ M_{\theta\theta} \\ M_{\phi\theta} \end{Bmatrix} = D\frac{1}{R} \begin{Bmatrix} -\theta_\phi\cot\theta + \dfrac{\partial\theta_\theta}{\partial\phi}\csc\theta - \nu\dfrac{\partial\theta_\phi}{\partial\theta} \\ \nu(-\theta_\phi\cot\theta + \dfrac{\partial\theta_\theta}{\partial\phi}\csc\theta) - \dfrac{\partial\theta_\phi}{\partial\theta} \\ \dfrac{1-\nu}{2}(\dfrac{\partial\theta_\theta}{\partial\theta} - \theta_\theta\cot\theta - \dfrac{\partial\theta_\phi}{\partial\phi}\csc\theta) \end{Bmatrix}, \quad \begin{Bmatrix} Q_\phi \\ Q_\theta \end{Bmatrix} = tG \begin{Bmatrix} \theta_\theta + \dfrac{1}{R}(u_\phi + \dfrac{\partial u_n}{\partial\phi}\csc\theta) \\ -\theta_\phi + \dfrac{1}{R}(u_\theta + \dfrac{\partial u_n}{\partial\theta}) \end{Bmatrix}$$

MEC-E8003

Beam, Plate and Shell Models

2024

WEEK 9: INTRODUCTION

1 INTRODUCTION

LEARNING OUTCOMES

Students get an overall picture about modelling in solid mechanics, use of the first principles in derivation of engineering models, and the mathematical tools used in the course. The topics are

- □ Modelling in solid mechanics
- □ First principles and concepts of solid mechanics
- □ Vectors and tensors
- □ Differential equations and boundary value problems

1.1 MODELLING IN MECHANICS

- **Crop**: Decide the boundary of structure. Interaction with surroundings need to be described in terms of known forces, moments, displacements, and rotations.

- **Idealize**: Simplify the geometry. Ignoring the details, not likely to affect the outcome, may simplify analysis a lot.

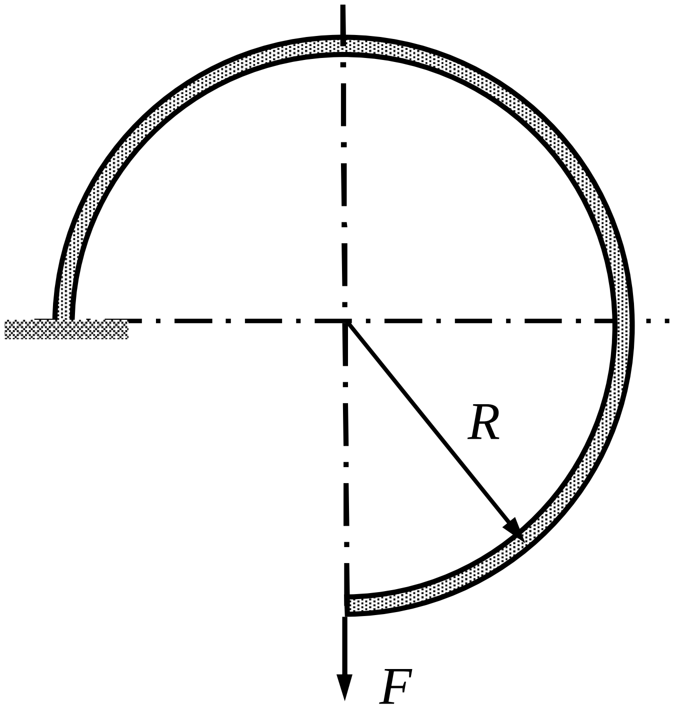
- **Parametrize**: Assign symbols to geometric and material parameter of the idealized structure. Measure or find the values needed in calculations.

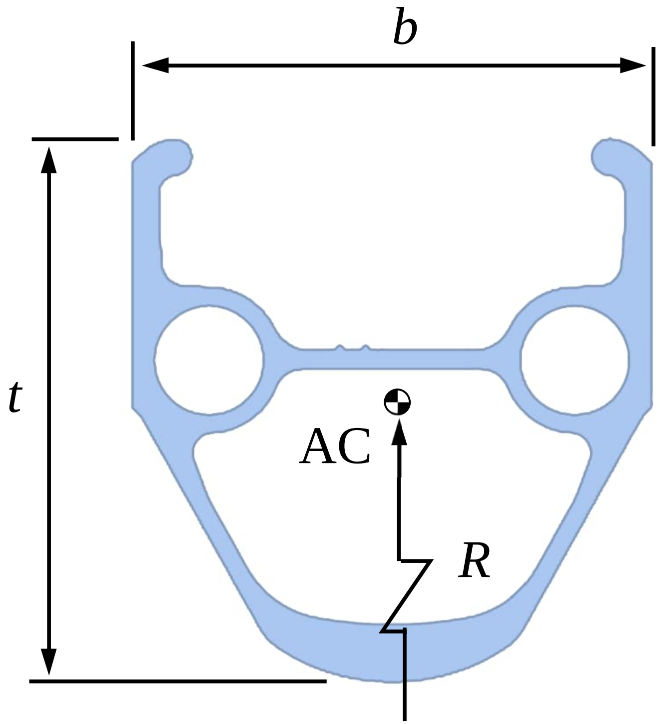
- **Model**: Write the mathematical description consisting of equilibrium equations, constitutive equations, and boundary conditions. ←

- **Solve**: Use an analytical or approximate method and hand calculations or Mathematica to find the solution. ←

RIGIDITY OF WHEEL RIM


STRUCTURE IDEALIZATION AND PARAMETRIZATION

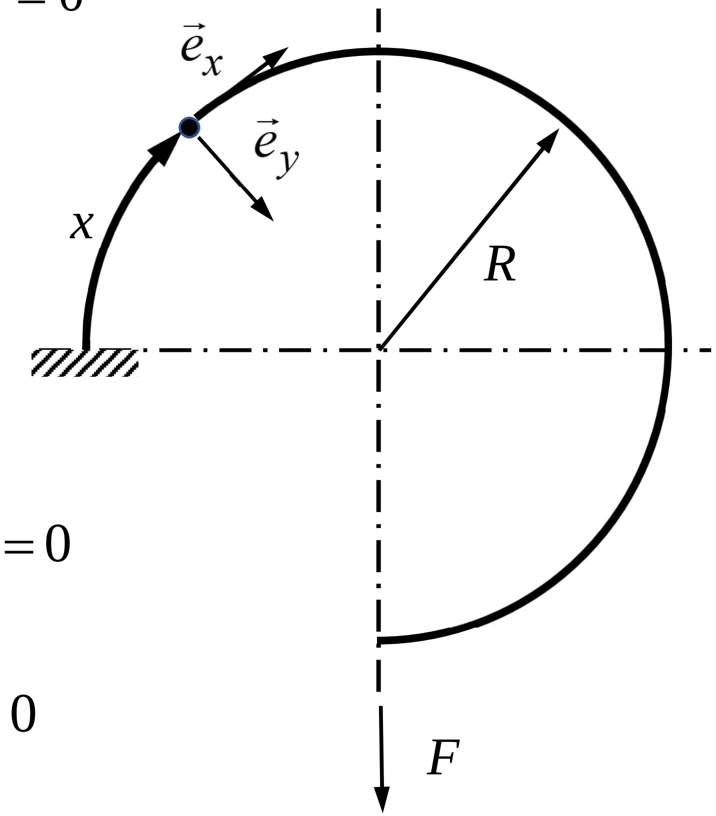




Dimension analysis with quantities E, I, R, F, and v : $$\frac{F}{ER^2} = f\left(\frac{v}{R}, \frac{I}{R^4}\right)$$

CURVED BEAM EQUATIONS

Assuming a planar beam, clamping and the center of the rim on the same horizontal line, and $L = 3\pi R/2$ (curvilinear xy−coordinate system)



Equilibrium: $\dfrac{dN}{dx} - \dfrac{1}{R}Q = 0$, $\dfrac{dQ}{dx} + \dfrac{1}{R}N = 0$, $\dfrac{dM}{dx} + Q = 0$

Constitutive equation: $M = EI\dfrac{d\psi}{dx}$

Constraints: $\dfrac{du}{dx} - \dfrac{1}{R}v = 0$, $\dfrac{dv}{dx} + \dfrac{1}{R}u - \psi = 0$

BC:s at the free end: $N(L) = 0$, $Q(L) + F = 0$, $M(L) = 0$

BC:s at the clamped end: $u(0) = 0$, $v(0) = 0$, $\psi(0) = 0$

MAVIC CXP 700C ISO 622 32H

Triangle representation based on a picture from www.mavic.com and the cross-section moment definitions: $R = 306\,\text{mm}$ and $I_{zz} = \sum_e \int_{\Omega^e} y^2 dA = 3011\text{mm}^4$.

MATHEMATICA SOLUTION

Mathematica can find the solution in a symbolic form. Problem description is close to its mathematical form and is composed of (ordinary) ordinary differential equations and boundary conditions.

```
equations := {
   NN'[x] - QQ[x] / R == 0,
   QQ'[x] + NN[x] / R == 0,
   MM'[x] + QQ[x] == 0,
   u'[x] - v[x] / R == 0,
   v'[x] + u[x] / R - ψ[x] == 0,
   MM[x] == E I ψ'[x],
   u[0] == 0, v[0] == 0, ψ[0] == 0,
   NN[3 Pi R / 2] == 0, QQ[3 Pi R / 2] == -FF, MM[3 Pi R / 2] == 0};

sol = DSolve[equations, {u, v, ψ, NN, QQ, MM}, x]⟦1⟧
```

1.2 FIRST PRINCIPLES AND QUANTITIES

Balance of mass Mass of a fixed set of particles, called as a body, is constant. 🡐

Balance of linear momentum The rate of change of linear momentum of a body equals the external force resultant acting on the material volume. 🡐

Balance of angular momentum The rate of change of angular momentum of a body equals the external moment resultant acting on the material volume. 🡐

Balance of energy (Thermodynamics 1)

Entropy growth (Thermodynamics 2)

The balance equations in their generic forms hold for solids and fluids!

LOCAL FORMS

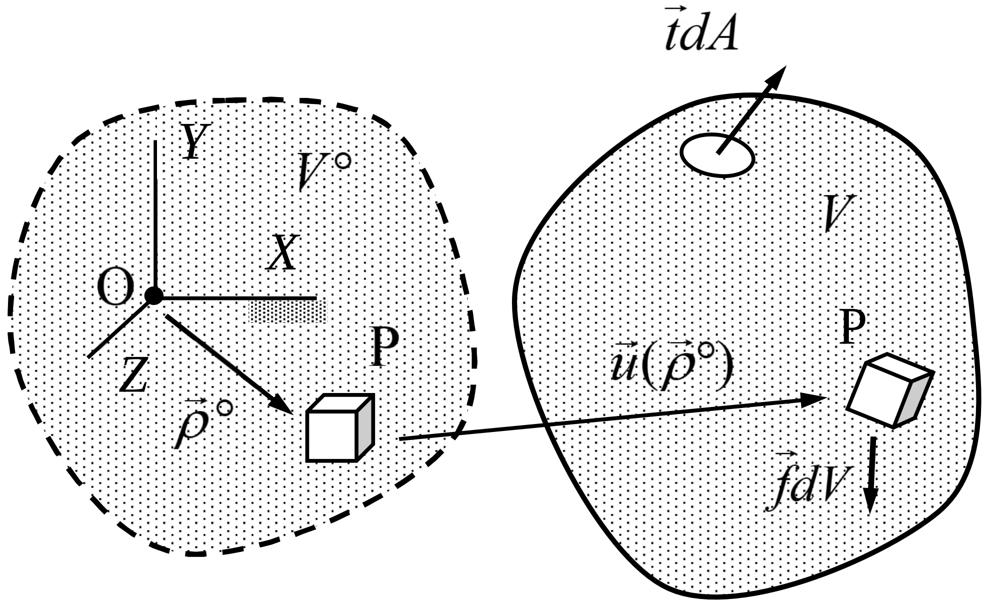
Application of the first principles to a material element inside the body or from its boundary gives the local forms:

$$\dot{m}=0 \quad : \quad \rho^{\circ}=\rho J \quad \text{in } V$$

$$\dot{\vec{p}}=\vec{F} \quad : \quad \nabla\cdot\vec{\vec{\sigma}}+\vec{f}=0 \quad \text{in } V$$

$$\dot{\vec{p}}=\vec{F} \quad : \quad \vec{\sigma}\equiv\vec{n}\cdot\vec{\vec{\sigma}}=\vec{t} \quad \text{on } \partial V_t$$

$$\dot{\vec{L}}=\vec{M} \quad : \quad \vec{\vec{\sigma}}=\vec{\vec{\sigma}}_{\mathrm{c}} \quad \text{in } V$$



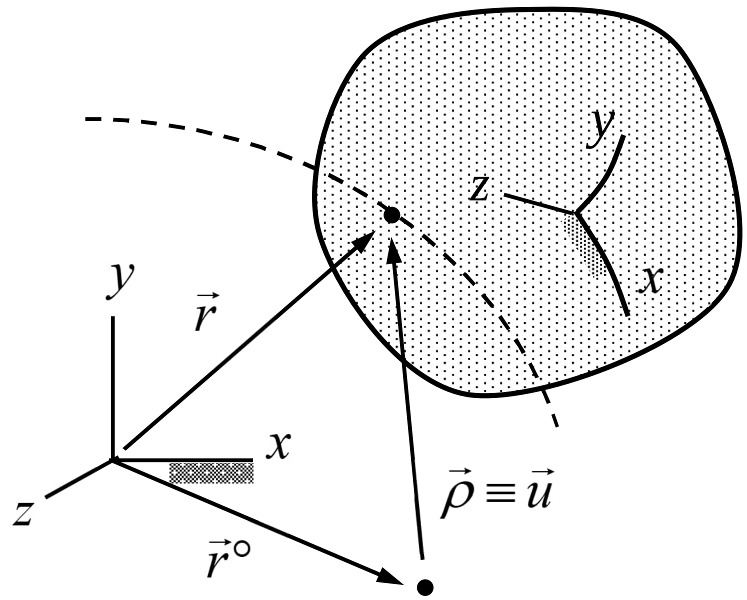
Assuming an equilibrium setting (geometry, stress, loading etc.) the local forms can be used used to find a new equilibrium setting (actually, displacements of the particles) when, e.g., external given forces are changed in some manner.

DISPLACEMENT

In Lagrangian description of solid mechanics, particles of a body are identified by their material coordinates (x, y, z). Displacement $\vec{u} = \vec{r} - \vec{r}^{\circ}$ is relative position vector of a particle initially at $\vec{r}^{\circ}$. In a Cartesian coordinate system

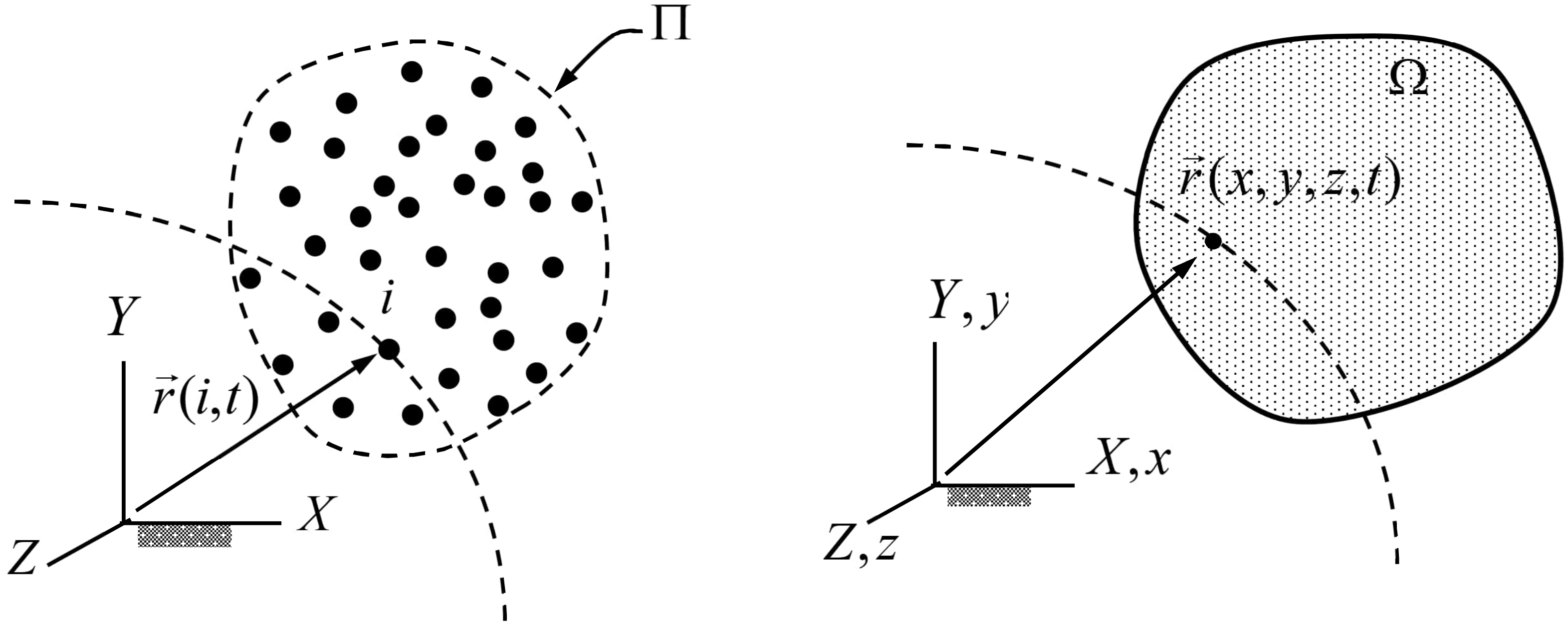
Initial $\vec{r}^{\circ} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix},$

Final $\vec{r} = \vec{r}^{\circ} + \vec{u} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \left(\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} + \begin{Bmatrix} u_x(x,y,z) \\ u_y(x,y,z) \\ u_z(x,y,z) \end{Bmatrix} \right).$



Displacement $\vec{u}(x, y, z)$ is the primary unknown of a linear elasticity problem and other quantities like strain and stress are (finally) expressed in terms of it.

In particle models, index $i \in \Pi \subset \mathbb{N}$ is used for labelling. In continuum models, material coordinates $(x, y, z) \in \Omega \subset \mathbb{R}^3$ are used for the purposes as “the particle set is too large to allow enumeration”.



Time can be considered as the curve parameter for the particle paths. In stationary description, one considers only the initial position Ω° (at $t = 0$ say) and the final position Ω° (at some other instant of time) and the curve parameter can be omitted.

LINEAR STRAIN

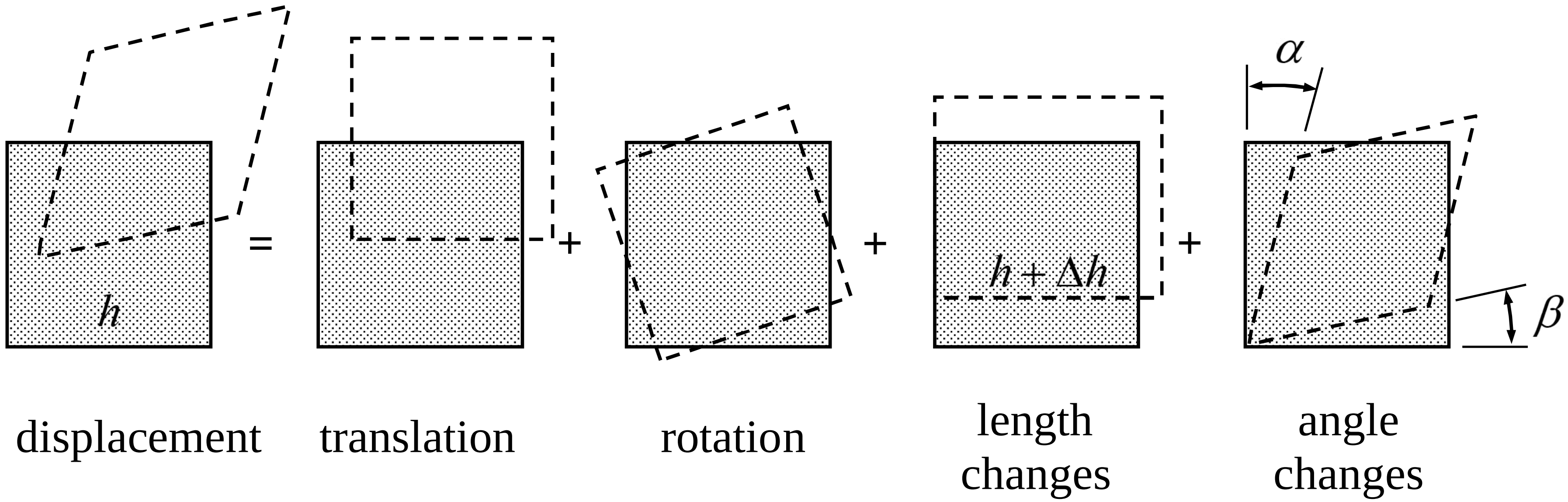
Linear strain measure $\overset{\leftrightarrow}{\varepsilon} = [\nabla\vec{u} + (\nabla\vec{u})_{\mathrm{c}}]/2$ describes shape deformation of material elements. The components of the (invariant) tensor quantity depends on the selection of the coordinate system. In a Cartesian $(x, y, z)-$coordinate system

$$\overset{\leftrightarrow}{\varepsilon} = \frac{1}{2}[\nabla\vec{u} + (\nabla\vec{u})_{\mathrm{c}}] = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} + \begin{Bmatrix} \vec{i}\vec{j} + \vec{j}\vec{i} \\ \vec{j}\vec{k} + \vec{k}\vec{j} \\ \vec{k}\vec{i} + \vec{i}\vec{k} \end{Bmatrix} \begin{Bmatrix} \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{Bmatrix}.$$

Normal components: $\varepsilon_{xx} = \dfrac{\partial u_x}{\partial x}, \ \varepsilon_{yy} = \dfrac{\partial u_y}{\partial y}, \ \varepsilon_{zz} = \dfrac{\partial u_z}{\partial z}$

Shear components: $\varepsilon_{xy} = \dfrac{1}{2}(\dfrac{\partial u_y}{\partial x} + \dfrac{\partial u_x}{\partial y}), \ \varepsilon_{yz} = \dfrac{1}{2}(\dfrac{\partial u_z}{\partial y} + \dfrac{\partial u_y}{\partial z}), \ \varepsilon_{zx} = \dfrac{1}{2}(\dfrac{\partial u_z}{\partial x} + \dfrac{\partial u_x}{\partial z})$

Displacement within a small material element consists of rigid body motion and deformation. The former can be divided into translation and rotation. The latter is caused by length and angle changes.



The geometry is described altogether by 12 parameters, of which 6 define the rigid body motion and remaining 6 the deformation (normal and shear components).

To find the expressions in terms of the displacement components, let us consider displacement within a small material element centered at $\vec{r}_0$. As the material element is assumed to be small, first two terms of the Taylor series

$$\vec{u}(\vec{r}) = \vec{u}_0 + \vec{\rho} \cdot (\nabla \vec{u})_0,$$

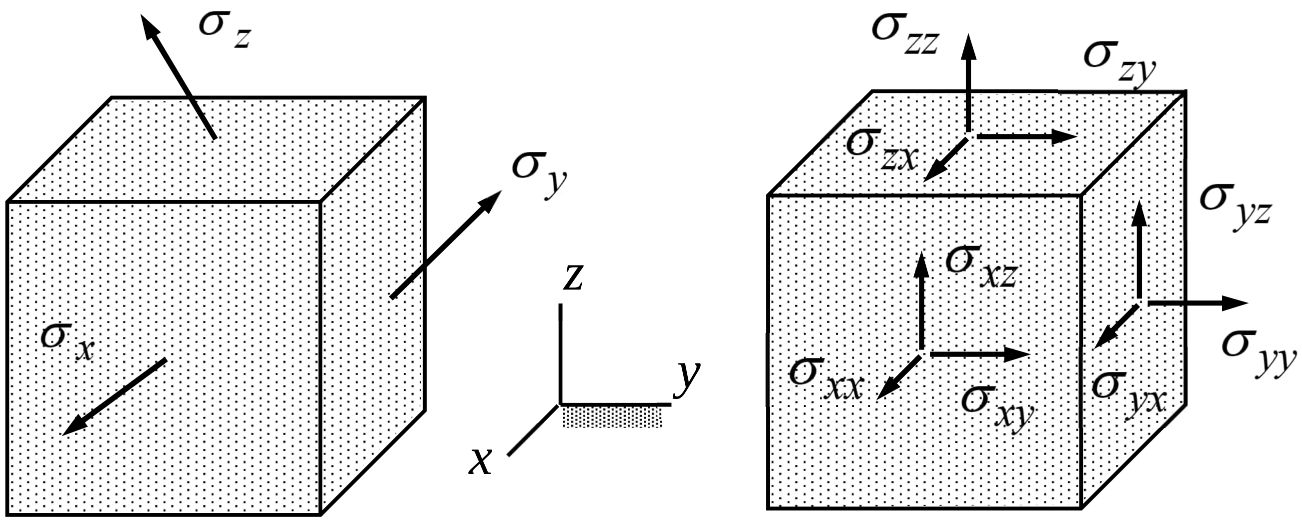
where the relative position vector $\vec{\rho} = \vec{r} - \vec{r}_0$, represent the displacement accurately enough. Division of the displacement gradient $(\nabla \vec{u})_0$ into its anti-symmetric and symmetric parts with notations $\vec{\vec{\varepsilon}} = (\nabla \vec{u})_{\mathrm{s}}$, $\vec{\vec{\theta}} = (\nabla \vec{u})_{\mathrm{u}}$ and using the concept of an associated vector $\vec{\theta}$ to an antisymmetric tensor $\vec{\vec{\theta}}$, gives

$$\vec{u}(\vec{r}) = \vec{u}_0 + \vec{\rho} \cdot \vec{\vec{\theta}}_0 + \vec{\rho} \cdot \vec{\vec{\varepsilon}}_0 = \vec{u}_0 + \vec{\theta}_0 \times \vec{\rho} + \vec{\rho} \cdot \vec{\vec{\varepsilon}}_0 \quad \text{where} \quad \vec{\vec{\varepsilon}} = (\nabla \vec{u})_{\mathrm{s}} = \tfrac{1}{2}[\nabla \vec{u} + (\nabla \vec{u})_{\mathrm{c}}].$$

The first term on the right-hand side describes translation, the second term small rigid body rotation, and the last term deformation (shape distortion) when the rotation part is small.

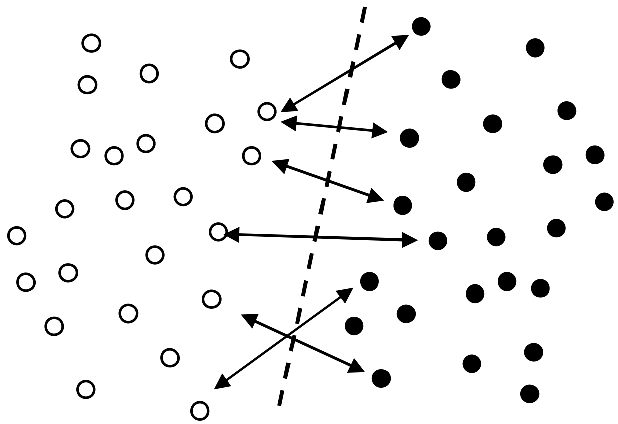
TRACTION AND STRESS

In continuum mechanics, traction $\vec{\sigma} = \Delta\vec{F} / \Delta A$ (a vector) describes the surface force between material elements of a body. Cauchy stress $\overleftrightarrow{\sigma}$ describes the surface forces acting on all edges of a material element. Traction and stress are related by $\vec{\sigma} = \vec{n} \cdot \overleftrightarrow{\sigma}$.



The first index of a stress component refers to the direction of the surface normal and the second that of the force component.

A material element is considered small compared with the size of the structure and, at the same time, large compared with the scale of the microstructure, e.g., distances between the molecules, atoms etc.



The ratio $\Delta\vec{F}/\Delta A$ of the interaction force resultant $\Delta\vec{F}$ to the area of the interaction ΔA is assumed to be constant $\vec{\sigma}$, when the area *is not too small nor too large*, therefore also $\Delta\vec{F} = \vec{\sigma}\Delta A$. Although a certain range of lower and upper limits is involved, continuum mechanics uses the relationship in form $d\vec{F} = \vec{\sigma} dA$.

The representation of the traction vectors acting on the three edges of a material element in the $(\vec{i},\vec{j},\vec{k})-$basis (directions are opposite on the opposite edges) can be expressed using the concept of stress tensor:

$$\begin{Bmatrix}\vec{\sigma}_x\\ \vec{\sigma}_y\\ \vec{\sigma}_z\end{Bmatrix}=\begin{bmatrix}\sigma_{xx} & \sigma_{xy} & \sigma_{xz}\\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz}\\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix} \;\Rightarrow\; \vec{\vec{\sigma}}=\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\vec{\sigma}_x\\ \vec{\sigma}_y\\ \vec{\sigma}_z\end{Bmatrix}=\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}\sigma_{xx} & \sigma_{xy} & \sigma_{xz}\\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz}\\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}.$$

Stress is a vector of vectors which represents the surface forces acting on all the surfaces of the material element simultaneously. In terms of the unit outward normal $\vec{n}$ of an edge, traction acting on the edge is given by $\vec{\sigma}=\vec{n}\cdot\vec{\vec{\sigma}}$ and the force the force acting on the material element through the edge

$$d\vec{F}=\vec{\sigma}dA=\vec{n}\cdot\vec{\vec{\sigma}}dA=(dA\vec{n})\cdot\vec{\vec{\sigma}}=d\vec{A}\cdot\vec{\vec{\sigma}}.$$

LINEARLY ELASTIC MATERIAL

Material model gives a relationship between stress and strain. The generalized Hooke's laws for the isotropic and orthotropic materials can be expressed in forms:

Component: $$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\end{Bmatrix} = [E]\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\end{Bmatrix},\quad \begin{Bmatrix}\sigma_{xy}\\ \sigma_{yz}\\ \sigma_{zx}\end{Bmatrix} = 2[G]\begin{Bmatrix}\varepsilon_{xy}\\ \varepsilon_{yz}\\ \varepsilon_{zx}\end{Bmatrix},\ \text{and}\ \begin{Bmatrix}\sigma_{yx}\\ \sigma_{zy}\\ \sigma_{xz}\end{Bmatrix} = 2[G]\begin{Bmatrix}\varepsilon_{yx}\\ \varepsilon_{zy}\\ \varepsilon_{xz}\end{Bmatrix}$$

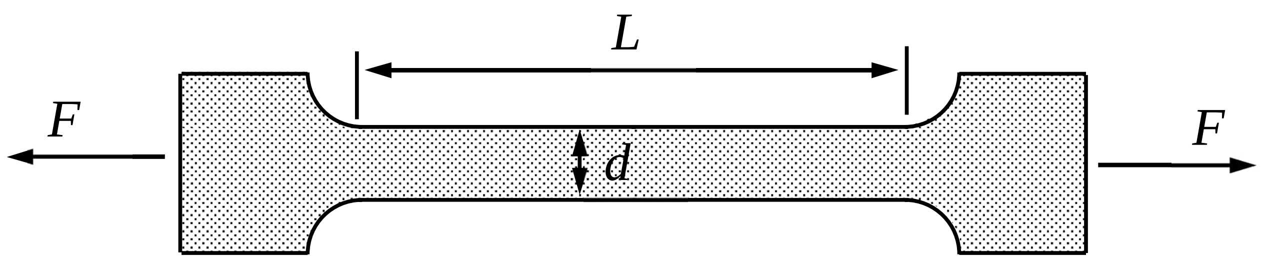
Tensor: $$\vec{\vec{\sigma}} = \vec{\vec{E}} : \nabla\vec{u} \quad \text{where } \vec{\vec{E}} = \begin{Bmatrix}\vec{i}\vec{i}\\ \vec{j}\vec{j}\\ \vec{k}\vec{k}\end{Bmatrix}^{\mathrm{T}} [E]\begin{Bmatrix}\vec{i}\vec{i}\\ \vec{j}\vec{j}\\ \vec{k}\vec{k}\end{Bmatrix} + \begin{Bmatrix}\vec{i}\vec{j}+\vec{j}\vec{i}\\ \vec{j}\vec{k}+\vec{k}\vec{j}\\ \vec{k}\vec{i}+\vec{i}\vec{k}\end{Bmatrix}^{\mathrm{T}} [G]\begin{Bmatrix}\vec{i}\vec{j}+\vec{j}\vec{i}\\ \vec{j}\vec{k}+\vec{k}\vec{j}\\ \vec{k}\vec{i}+\vec{i}\vec{k}\end{Bmatrix}$$

in which the symmetric elasticity matrices $[E]$ and $[G]$ depend on material type.

Experiments indicate that length L, length change ΔL, and cross-sectional area A, diameter d, diameter change Δd, of the specimen loaded by force F are related by

$$\frac{F}{A} = E\frac{\Delta L}{L} \ ,$$

$$\frac{\Delta d}{d} = -\nu \frac{\Delta L}{L}$$



in which the coefficients E and ν (Young's modulus and Poisson's ratio) depend on the material.

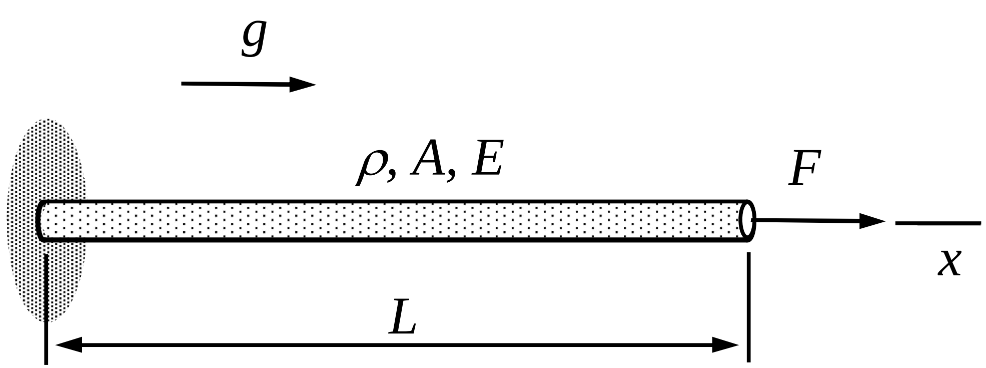
Constitutive equation brings the rigidity properties of the material into the model. The relationships between strain and stress are, basically, just compact and coherent representations of experimental data. However, the generic principles of physics restrict the set of possible linear material models.

First, a constitutive equations should be coordinate system invariant. Therefore, if a constitutive equation is known in some frame of reference, representation in some other system follows without a new set of experiments. A tensor relationship satisfies the requirement automatically

Second, a constitutive equation should be homogeneous with respect to (tensor) rank and dimension. For example, valid linear homogeneous relationships between rank 2 tensors $\overleftrightarrow{a}$ and $\overleftrightarrow{b}$ are, e.g., $\overleftrightarrow{a} = E\overleftrightarrow{b}$ and $\overleftrightarrow{a} = \overleftrightarrow{\overleftrightarrow{E}} : \overleftrightarrow{b}$ where E and $\overleftrightarrow{\overleftrightarrow{E}}$ characterize material.

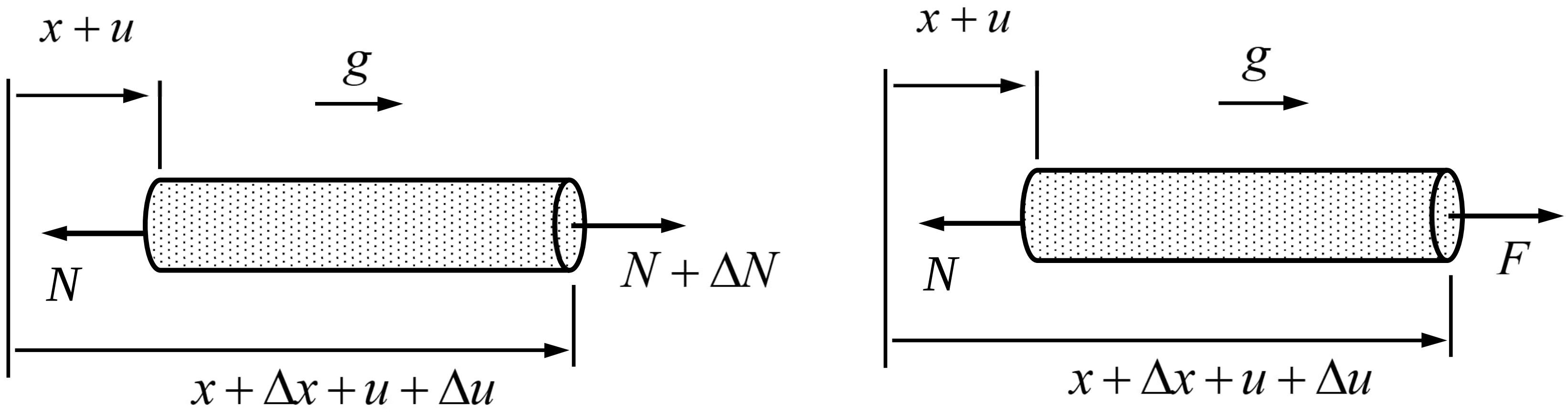
Qualitative information about material like homogeneity and isotropy restrict the form of the constitutive equations more effectively. For example, one may deduce that the number of material parameters characterizing an isotropic linearly elastic material is 2!

EXAMPLE Apply the first principles to a bar element inside the bar and an element at the free end to derive the differential equation and boundary condition at the right end in terms of displacement $u(x)$. Assume that the simple Hooke's law holds for the material. What is the condition at the left end?



Answer $EA\dfrac{d^2u}{dx^2}+\rho Ag=0 \quad x\in]0,L[, \quad EA\dfrac{du}{dx}=F \quad x=L,$ and $u=0 \quad x=0$

Let us apply the first principles to a material element of initial length Δx at the initial and final geometries.



The cases where the material element is inside the bar and at the right end differ.

Mass balance: $\Delta m = (\rho A)^{\circ} \Delta x = (\rho A)(\Delta x + \Delta u)$

Momentum balance $\rightarrow$: $N + \Delta N - N + g\Delta m = 0$

Momentum balance $\rightarrow$: $F - N + g\Delta m = 0$

Hooke's law: $\sigma = E\dfrac{\Delta u}{\Delta x} \;\Rightarrow\; N = EA\dfrac{\Delta u}{\Delta x}$.

The local forms follow by considering the limit $\Delta x \to 0$ and equations in terms of displacement after elimination of the stress resultant N. It is noteworthy, that the limit model assumes that $\Delta N / \Delta x$ exists also when $\Delta x \to 0$. In case of a discontinuity, like a point force P at x_0, one obtains the "jump" condition $[\![N]\!] + P = 0$ where $[\![a]\!] = \lim_{\varepsilon \to 0}[a(x_0 + \varepsilon) - a(x_0 - \varepsilon)]$.

1.3 VECTORS AND TENSORS

The quantities in mechanics can be classified into scalars a, vectors $\vec{a}$ and multi-vectors (vectors of vectors) $\vec{\vec{a}}$ called also as tensors of ranks 0,1, and 2.

basis vector singlet

Vector $$\vec{a} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix} = \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = a_x \vec{i} + a_y \vec{j} + a_z \vec{k} \quad \text{(rank 1 tensor)}$$

basis vector doublet

Tensor $$\vec{\vec{a}} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} a_{xx} & a_{xy} & a_{xz} \\ a_{yx} & a_{yy} & a_{yz} \\ a_{zx} & a_{zy} & a_{zz} \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = a_{xx} \vec{i}\vec{i} + a_{xy} \vec{i}\vec{j} + \ldots + a_{zz} \vec{k}\vec{k} \quad \text{(rank 2 tensor)}$$

Also, rank 4 tensors are needed. Their representations require basis vector quadruplets and 4 indices in the components.

TENSOR COMPONENTS

The multipliers of the basis vector singlets, doublets, etc. of a tensor are called as the components. The components of the first and second order tensors can be represented as column { } and square [ ] matrices:

Vector $\vec{u} = \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix}$ of components $\{u\} = \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix}$

index 1 $\rightarrow$ row

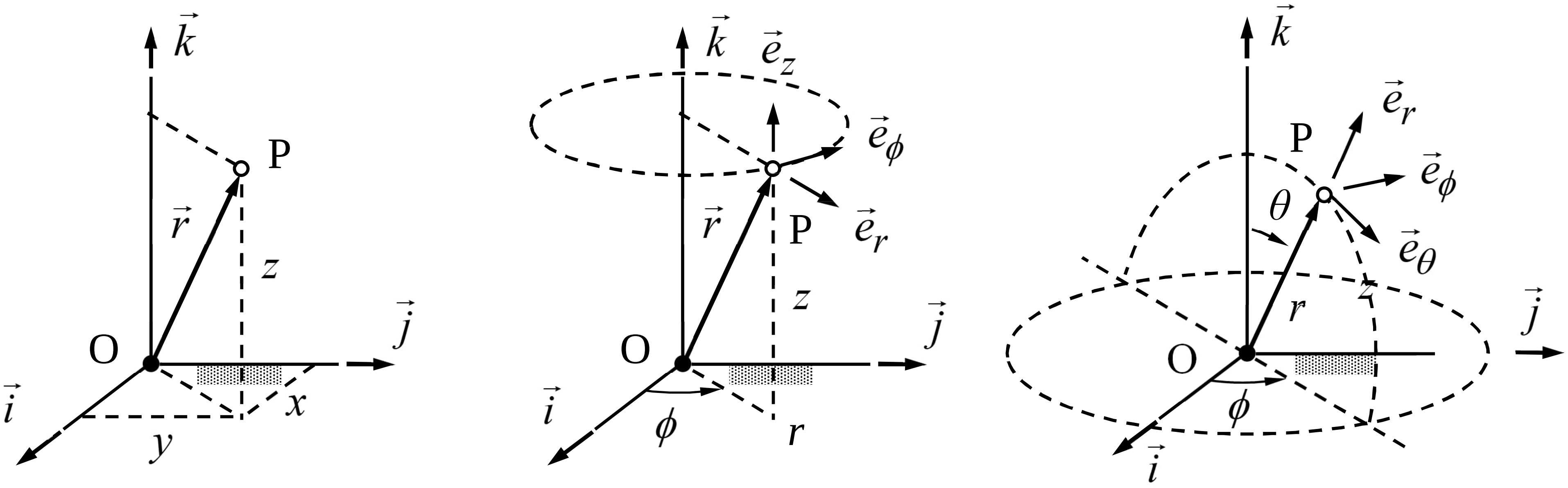
index 2 $\rightarrow$ column

Tensor $\vec{\vec{\sigma}} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}$ of components $[\sigma] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$

Notice. A column matrix is often called as the vector. Here, vector is a tensor of rank 1.

INVARIANCE

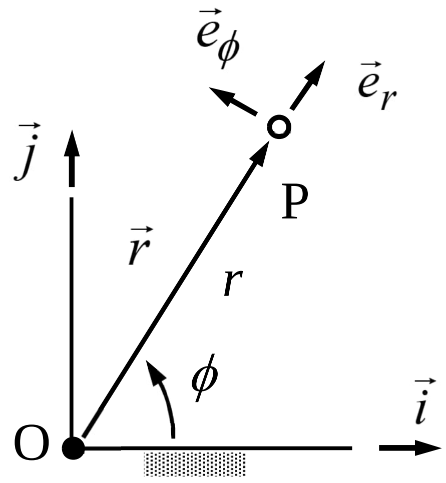
Tensor quantities are invariant with respect to coordinate system. Representation depends on the coordinate system but the tensor itself does not. Rectilinear-orthonormal (Cartesian) and curvilinear-orthonormal coordinate systems are common choices for tensor representations.



Transformation from one coordinate system to another requires the relationship between the basis vectors. Considering $\vec{a}$ of a planar case and using the relationship between the basis vectors of the Cartesian and polar coordinate systems shown ($\mathrm{c} \sim \cos$, $\mathrm{s} \sim \sin$)

$$\vec{a} = \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} a_{xx} & a_{xy} \\ a_{yx} & a_{yy} \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} \text{ and } \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} = \begin{bmatrix} \mathrm{c}\phi & -\mathrm{s}\phi \\ \mathrm{s}\phi & \mathrm{c}\phi \end{bmatrix} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} \Leftrightarrow$$

$$\vec{a} = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathrm{c}\phi & \mathrm{s}\phi \\ -\mathrm{s}\phi & \mathrm{c}\phi \end{bmatrix} \begin{bmatrix} a_{xx} & a_{xy} \\ a_{yx} & a_{yy} \end{bmatrix} \begin{bmatrix} \mathrm{c}\phi & -\mathrm{s}\phi \\ \mathrm{s}\phi & \mathrm{c}\phi \end{bmatrix} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} \Leftrightarrow$$

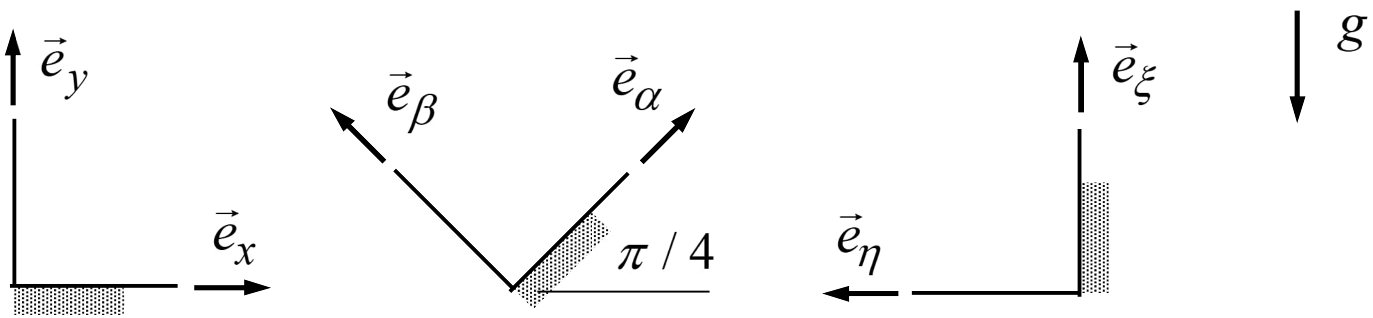


$$\vec{a} = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} a_{rr} & a_{r\phi} \\ a_{\phi r} & a_{\phi\phi} \end{bmatrix} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} \text{ where } \begin{bmatrix} a_{rr} & a_{r\phi} \\ a_{\phi r} & a_{\phi\phi} \end{bmatrix} = \begin{bmatrix} \mathrm{c}\phi & \mathrm{s}\phi \\ -\mathrm{s}\phi & \mathrm{c}\phi \end{bmatrix} \begin{bmatrix} a_{xx} & a_{xy} \\ a_{yx} & a_{yy} \end{bmatrix} \begin{bmatrix} \mathrm{c}\phi & -\mathrm{s}\phi \\ \mathrm{s}\phi & \mathrm{c}\phi \end{bmatrix}.$$

EXAMPLE Acceleration by gravity $\vec{g}$ can be represented in any of the coordinate systems of the figure starting with the known representation in one of the systems. Starting with the representation $\vec{g} = -g\vec{e}_y$ and using the relationship between the basis vectors

$$\vec{g} = -g\vec{e}_y = -g(\vec{e}_\alpha + \vec{e}_\beta)/\sqrt{2} = -g\vec{e}_\xi .$$

All these give the same direction and magnitude for the acceleration by gravity.

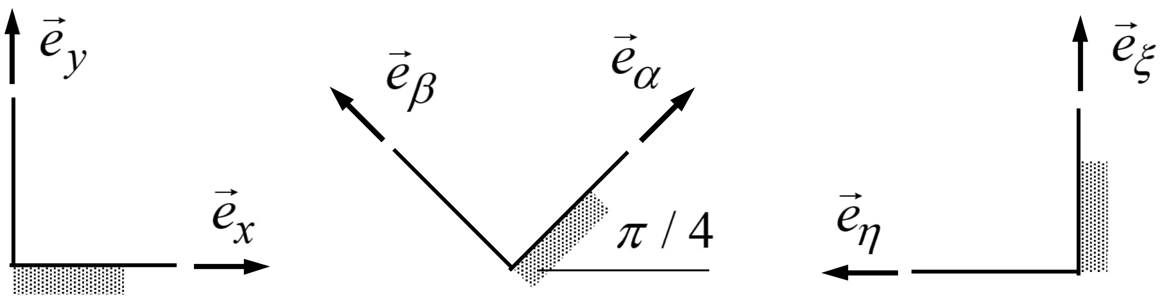


$$\vec{e}_y = \frac{1}{\sqrt{2}}(\vec{e}_\alpha + \vec{e}_\beta) \qquad \vec{e}_y = \vec{e}_\xi$$

EXAMPLE Second order tensor $\vec{\vec{a}}$ can be represented in any coordinate systems of the figure starting with the known representation $\vec{\vec{a}} = a\vec{e}_y\vec{e}_y$ in the Cartesian system. Using the relationship between the basis vectors

$$\vec{\vec{a}} = a\vec{e}_y\vec{e}_y = \frac{a}{2}(\vec{e}_\alpha\vec{e}_\alpha + \vec{e}_\alpha\vec{e}_\beta + \vec{e}_\beta\vec{e}_\alpha + \vec{e}_\beta\vec{e}_\beta) = a\vec{e}_\xi\vec{e}_\xi .$$

Graphical representation of a rank 2 tensor is not as obvious as that of a vector.



TENSOR PRODUCTS

In manipulation of an expression containing tensors, it is important to remember that tensor ($\otimes$), cross ($\times$), inner ($\cdot$) products are non-commutative (order matters). For simplicity of presentation, outer (tensor) products like $\vec{a}\otimes\vec{b}$ are denoted by $\vec{a}\vec{b}$ in MEC-E8003. Otherwise, the usual rules of vector algebra apply:

$$\vec{a}\cdot\vec{b} = a_x b_x + a_y b_y + a_z b_z,$$

$$\vec{a}\times\vec{b} = (a_y b_z - a_z b_y)\vec{i} + (a_z b_x - a_x b_z)\vec{j} + (a_x b_y - a_y b_x)\vec{k},$$

$$\vec{a}\vec{b} = a_x b_x \vec{i}\vec{i} + a_x b_y \vec{i}\vec{j} + a_x b_z \vec{i}\vec{k} + a_y b_x \vec{j}\vec{i} + a_y b_y \vec{j}\vec{j} + a_y b_z \vec{j}\vec{k} + a_z b_x \vec{k}\vec{i} + a_z b_y \vec{k}\vec{j} + a_z b_z \vec{k}\vec{k}.$$

Calculation with tensors is straightforward although the number of terms may make manipulations somewhat tedious.

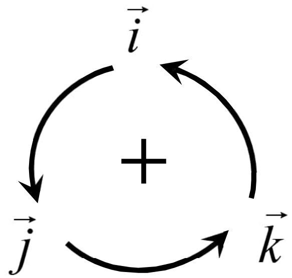
As an example, manipulations needed to find the cross-product of two vectors in a Cartesian system (orthonormal and right-handed) consists of steps

$$\vec{a}\times\vec{b} = (a_x\vec{i} + a_y\vec{j} + a_z\vec{k})\times(b_x\vec{i} + b_y\vec{j} + b_z\vec{k}) \quad \Leftrightarrow$$

$$\vec{a}\times\vec{b} = a_x b_x \vec{i}\times\vec{i} + a_x b_y \vec{i}\times\vec{j} + a_x b_z \vec{i}\times\vec{k} +$$

$$a_y b_x \vec{j}\times\vec{i} + a_y b_y \vec{j}\times\vec{j} + a_y b_z \vec{j}\times\vec{k} +$$

$$a_z b_x \vec{k}\times\vec{i} + a_z b_y \vec{k}\times\vec{j} + a_z b_z \vec{k}\times\vec{k} \quad \Rightarrow$$



$$\vec{a}\times\vec{b} = 0 + a_x b_y \vec{k} - a_x b_z \vec{j} - a_y b_x \vec{k} + 0 + a_y b_z \vec{i} + a_z b_x \vec{j} - a_z b_y \vec{i} + 0 \quad \Leftrightarrow$$

$$\vec{a}\times\vec{b} = (a_y b_z - a_z b_y)\vec{i} + (a_z b_x - a_x b_z)\vec{j} + (a_x b_y - a_y b_x)\vec{k}. \quad \boldsymbol{\leftarrow}$$

The manipulations are often (but not always) easier when the components and basis vectors are arranged as matrices

$$\vec{a} = \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix} \text{ and } \vec{b} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} b_x \\ b_y \\ b_z \end{Bmatrix} = \begin{Bmatrix} b_x \\ b_y \\ b_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} \quad \Rightarrow$$

$$\vec{a} \times \vec{b} = \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix}^{\mathrm{T}} \left(\begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} \times \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \right) \begin{Bmatrix} b_x \\ b_y \\ b_z \end{Bmatrix} = \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 0 & \vec{k} & -\vec{j} \\ -\vec{k} & 0 & \vec{i} \\ \vec{j} & -\vec{i} & 0 \end{bmatrix} \begin{Bmatrix} b_x \\ b_y \\ b_z \end{Bmatrix} \quad \Leftrightarrow$$

$$\vec{a} \times \vec{b} = (a_y b_z - a_z b_y)\vec{i} + (a_z b_x - a_x b_z)\vec{j} + (a_x b_y - a_y b_x)\vec{k}. \quad \Leftarrow$$

EXAMPLE The local forms of the balance laws of momentum and moment of momentum are $\nabla \cdot \vec{\vec{\sigma}} + \vec{f} = 0$ and $\vec{\vec{\sigma}} = \vec{\vec{\sigma}}_{\mathrm{c}}$ (conjugate tensor). Assuming a planar case and a Cartesian coordinate system so that

$$\nabla = \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \partial / \partial x \\ \partial / \partial y \end{Bmatrix}, \quad \vec{f} = \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix}, \text{ and } \vec{\vec{\sigma}} = \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}$$

derive the component forms of the balance laws.

Answer $\dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{yx}}{\partial y} + f_x = 0, \quad \dfrac{\partial \sigma_{xy}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} + f_y = 0$ and $\sigma_{xy} = \sigma_{yx}$

In a Cartesian system, basis vectors are constants and one may transpose the gradient operator to get (transposing cannot be used with non-constant basis vectors! Why?)

$$\nabla\cdot\vec{\vec{\sigma}}+\vec{f}=\begin{Bmatrix}\partial/\partial x\\ \partial/\partial y\end{Bmatrix}^{\mathrm{T}}\left(\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}\cdot\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}^{\mathrm{T}}\right)\begin{bmatrix}\sigma_{xx} & \sigma_{xy}\\ \sigma_{yx} & \sigma_{yy}\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}+\begin{Bmatrix}f_x\\ f_y\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}=0$$

$$\nabla\cdot\vec{\vec{\sigma}}+\vec{f}=\left(\begin{Bmatrix}\partial/\partial x\\ \partial/\partial y\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}\sigma_{xx} & \sigma_{xy}\\ \sigma_{yx} & \sigma_{yy}\end{bmatrix}+\begin{Bmatrix}f_x\\ f_y\end{Bmatrix}^{\mathrm{T}}\right)\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}=0. \Leftarrow$$

$$\vec{\vec{\sigma}}-\vec{\vec{\sigma}}_{\mathrm{c}}=\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}\sigma_{xx} & \sigma_{xy}\\ \sigma_{yx} & \sigma_{yy}\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}-\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}\sigma_{xx} & \sigma_{xy}\\ \sigma_{yx} & \sigma_{yy}\end{bmatrix}^{\mathrm{T}}\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}=0 \quad\Leftrightarrow$$

$$\vec{\vec{\sigma}}-\vec{\vec{\sigma}}_{\mathrm{c}}=\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}0 & \sigma_{xy}-\sigma_{yx}\\ \sigma_{yx}-\sigma_{xy} & 0\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}=0. \Leftarrow$$

SOME DEFINITIONS AND IDENTITIES

Conjugate tensor $\overleftrightarrow{a}_{\mathrm{c}}$: $\overleftrightarrow{a} \cdot \vec{b} = \vec{b} \cdot \overleftrightarrow{a}_{\mathrm{c}} \quad \forall \vec{b}$

Second order identity tensor $\overleftrightarrow{I}$: $\overleftrightarrow{I} \cdot \vec{a} = \vec{a} \cdot \overleftrightarrow{I} = \vec{a} \quad \forall \vec{a}$

Fourth order identity tensor $\overset{\leftrightarrow}{\overleftrightarrow{I}}$: $\overset{\leftrightarrow}{\overleftrightarrow{I}} : \overleftrightarrow{a} = \overleftrightarrow{a} : \overset{\leftrightarrow}{\overleftrightarrow{I}} = \overleftrightarrow{a} \quad \forall \overleftrightarrow{a}$

Associated vector $\vec{a}$ of an antisymmetric tensor $\overleftrightarrow{a}$: $\vec{b} \cdot \overleftrightarrow{a} = \vec{a} \times \vec{b}$, when $\overleftrightarrow{a} = -\overleftrightarrow{a}_{\mathrm{c}}$

Scalar triple product $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$

Vector triple product $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$

Symmetric-antisymmetric double product $\overleftrightarrow{a} = -\overleftrightarrow{a}_{\mathrm{c}}$ and $\overleftrightarrow{b} = \overleftrightarrow{b}_{\mathrm{c}} \quad \Rightarrow \quad \overleftrightarrow{a} : \overleftrightarrow{b} = 0$

Symmetric-antisymmetric division $\overleftrightarrow{a} = \overleftrightarrow{a}_s + \overleftrightarrow{a}_u = \frac{1}{2}(\overleftrightarrow{a} + \overleftrightarrow{a}_{\mathrm{c}}) + \frac{1}{2}(\overleftrightarrow{a} - \overleftrightarrow{a}_{\mathrm{c}})$

1.4 DIFFERENTIAL EQUATIONS

Local forms of the balance equations imply ordinary or partial differential equations to be solved to stress and displacement components. The examples of the course apply

Trial solutions: The generic solution to ordinary homogeneous differential equations can be found (usually) with an exponential trial solution. The generic solution to a non-homogeneous equation consists of the generic solution to the homogeneous equations and a particular solution (just some solution taking care of the non-zero righthand side).

Repeated integrations: The generic solution for certain ordinary and partial differential equations can be found with simple integrations. With partial differential equations (or a set of them), “integration constants” are considered as functions of some independent variables.

$$\frac{d\psi}{dt} + k\psi = 0 \quad \Leftrightarrow \quad \psi(t) = ae^{-kt},$$

$$\frac{d^2\psi}{dt^2} + k^2\psi = 0 \quad \Leftrightarrow \quad \psi(t) = a\sin(kt) + b\cos(kt).$$

$$\frac{du}{ds} + \frac{1}{R}v = 0 \;\text{ and }\; \frac{dv}{ds} - \frac{1}{R}u = 0 \quad \Leftrightarrow \quad \frac{d^2u}{ds^2} + \frac{1}{R^2}u = 0 \quad \Leftrightarrow \quad \ldots$$

$$\frac{d^2u}{ds^2} + \frac{1}{R^2}u = \frac{s}{R^2} \quad \Leftrightarrow \quad u(s) = a\sin(\frac{s}{R}) + b\cos(\frac{s}{R}) + s,$$

$$\frac{d^2u}{ds^2} + \frac{1}{R^2}u = \frac{1}{R}\cos(\frac{s}{R}) \quad \Leftrightarrow \quad u(s) = a\sin(\frac{s}{R}) + b\cos(\frac{s}{R}) + \frac{1}{2}s\sin(\frac{s}{R}),$$

$$\frac{d^2u}{ds^2} + \frac{1}{R^2}u = \frac{1}{R}\sin(\frac{s}{R}) \quad \Leftrightarrow \quad u(s) = a\sin(\frac{s}{R}) + b\cos(\frac{s}{R}) - \frac{1}{2}s\cos(\frac{s}{R}).$$

$$m\frac{d^2y}{dt^2} = -mg \quad \Leftrightarrow \quad y(t) = -\frac{1}{2}t^2 g + at + b,$$

$$\mu\frac{1}{r}\frac{d}{dr}(r\frac{d}{dr}v_z) = C \quad \Leftrightarrow \quad v_z(r) = \frac{C}{4\mu}r^2 + a\ln r + b,$$

$$(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr})(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr})w = \frac{b_n}{D} \quad \Leftrightarrow \quad w(r) = \frac{b_n}{D}\frac{r^4}{64} + a + br^2 + cr^2(1-\log r) + d\log r.$$

$$\frac{dp}{dx} = \mu\frac{d^2v_x}{dy^2} \quad \Leftrightarrow \quad \frac{dp}{dx} = a \text{ and } \mu\frac{d^2v_x}{dy^2} = a \quad \Leftrightarrow \quad \dots$$

$$\frac{\partial p}{\partial r} = \rho\Omega^2 r \text{ and } \frac{\partial p}{\partial z} = -\rho g \quad \Leftrightarrow \quad p(z,r) = \frac{1}{2}\rho\Omega^2 r^2 - \rho g z + a.$$

$$\frac{\partial}{\partial z}N_{z\phi} + \rho g R\sin\phi = 0 \text{ and } \frac{1}{R}\frac{\partial}{\partial\phi}N_{z\phi} + \frac{\partial}{\partial z}N_{zz} = 0 \quad \Leftrightarrow \quad \dots$$

BOUNDARY VALUE PROBLEM

Boundary value problem (BVP) consists of a differential equation and additional information at the boundaries. Typically, one may know the displacement or external loading. To find the solution to BVP

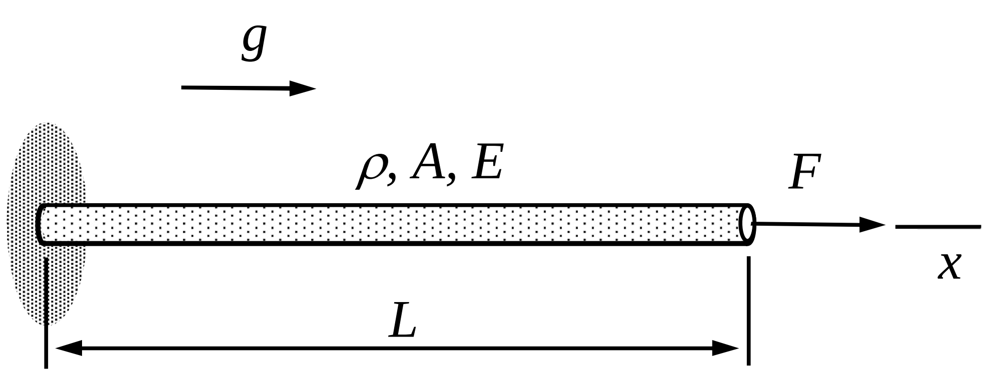
First, find the generic solution to the differential equation. Depending on the order of the equation the solution contains one or more integration constants.

Second, use the boundary conditions to express the integration constants in terms of the loading and known displacement values and substitute into the generic solution.

Solution to a BVP does not contain free parameters and, therefore stress, displacement etc. represent the quantities of a certain setting.

EXAMPLE Consider a bar of length L loaded by its own weight and point force acting at the free end. Cross-sectional area A, acceleration by gravity g, and material properties E and ρ are constants. Determine the displacement $u(x)$ from the boundary value problem

$$EA\frac{d^2u}{dx^2}+\rho Ag=0 \quad x\in]0,L[, \quad EA\frac{du}{dx}=F \quad x=L, \text{ and } u=0 \quad x=0.$$



Answer $EA\dfrac{d^2u}{dx^2}+\rho Ag=0 \quad x\in]0,L[, \quad EA\dfrac{du}{dx}=F \quad x=L,$ and $u=0 \quad x=0$

Let us first find the generic solution to the second order ordinary differential equation by repeated integrations

$$EA\frac{d^2u}{dx^2}+\rho Ag=0 \quad \Leftrightarrow \quad \frac{d^2u}{dx^2}=-\frac{\rho g}{E} \quad \Leftrightarrow \quad \frac{du}{dx}=-\frac{\rho g}{E}x+a \quad \Leftrightarrow \quad u=-\frac{\rho g}{2E}x^2+ax+b.$$

When the generic solution is substituted there, the two boundary conditions give

$$-A\rho gL+EAa=F \quad \text{and} \quad b=0 \quad \Leftrightarrow \quad a=\frac{\rho g}{E}L+\frac{F}{EA} \quad \text{and} \quad b=0.$$

Therefore, the displacement for the BVP takes the form

$$u(x)=\frac{\rho g}{2E}(2xL-x^2)+\frac{F}{EA}x.$$

MEC-E8003 Beam, plate and shell models, examples 1

1. Find the relationship between the orthonormal basis vector sets $(\vec{i},\vec{j},\vec{k})$ and $(\vec{I},\vec{J},\vec{K})$, if $\vec{I}$ has the same direction as $\vec{i}-\vec{j}+\vec{k}$ and $\vec{J}$ has the same direction as the gradient of plane $g(x,y,z)=2x+3y+z-5=0$ (hence $\vec{J}$ is the normal unit vector to the plane).

Answer $\begin{Bmatrix} \vec{I} \\ \vec{J} \\ \vec{K} \end{Bmatrix} = \dfrac{1}{\sqrt{42}} \begin{bmatrix} \sqrt{14} & -\sqrt{14} & \sqrt{14} \\ 2\sqrt{3} & 3\sqrt{3} & \sqrt{3} \\ -4 & 1 & 5 \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}$

2. Consider the identity $\vec{I}\times\vec{a}=\vec{a}\times\vec{I}$ in a Cartesian $(x,y,z)-$coordinate system in which the second order unit tensor is given by $\vec{I}=\vec{i}\vec{i}+\vec{j}\vec{j}+\vec{k}\vec{k}$. Show that the identity holds by simplifying the left and right-hand side expressions. Assume that $\vec{a}=a_x\vec{i}+a_y\vec{j}$ (just to simplify the expressions).

3. Given $\vec{a}=a_{xx}\vec{i}\vec{i}+a_{xy}\vec{i}\vec{j}+a_{yx}\vec{j}\vec{i}+a_{yy}\vec{j}\vec{j}$ and $\vec{b}=b_x\vec{i}+b_y\vec{j}$, determine $\vec{a}\cdot\vec{b}$, $\vec{a}\times\vec{b}$, $\vec{b}\cdot\vec{a}$, and $\vec{b}\times\vec{a}$.

Answer

$\vec{a}\cdot\vec{b}=(a_{xx}b_x+a_{xy}b_y)\vec{i}+(a_{yx}b_x+a_{yy}b_y)\vec{j}$, $\quad \vec{a}\times\vec{b}=(a_{xx}b_y-a_{xy}b_x)\vec{i}\vec{k}+(a_{yx}b_y-a_{yy}b_x)\vec{j}\vec{k}$

$\vec{b}\cdot\vec{a}=(b_x a_{xx}+b_y a_{yx})\vec{i}+(b_x a_{xy}+b_y a_{yy})\vec{j}$, $\quad \vec{b}\times\vec{a}=(b_x a_{yx}-b_y a_{xx})\vec{k}\vec{i}+(b_x a_{yy}-b_y a_{xy})\vec{k}\vec{j}$

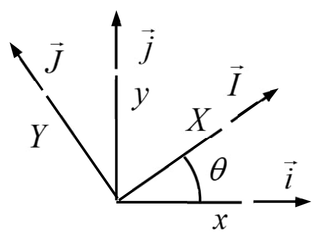
4. Calculate $\nabla\vec{r}$, $\nabla\cdot\vec{r}$ and $\nabla\times\vec{r}$ in which $\vec{r}$ is the position vector. Use the representations $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$ and $\nabla=\vec{i}\partial/\partial x+\vec{j}\partial/\partial y+\vec{k}\partial/\partial z$ in the Cartesian $(\vec{i},\vec{j},\vec{k})$ basis.

Answer $\nabla\vec{r}=\vec{I}$, $\nabla\cdot\vec{r}=3$, $\nabla\times\vec{r}=0$

5. Derive the expressions of $\nabla\cdot\vec{u}$, $\nabla\times\vec{u}$, and $\nabla^2 u$ in Cartesian coordinate system where $\nabla=\vec{i}\partial/\partial x+\vec{j}\partial/\partial y+\vec{k}\partial/\partial z$. Vector $\vec{u}(x,y)=u_x(x,y)\vec{i}+u_y(x,y)\vec{j}$ and scalar $u(x,y)$ depend on *x* and *y* only.

Answer $\nabla\cdot\vec{u}=\dfrac{\partial u_x}{\partial x}+\dfrac{\partial u_y}{\partial y}$, $\nabla\times\vec{u}=\vec{k}\left(\dfrac{\partial u_y}{\partial x}-\dfrac{\partial u_x}{\partial y}\right)$, $\nabla^2 u=\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}$.

6. Let us consider a second order tensor $\vec{\varepsilon}$ having the components ε_{xx}, ε_{xy}, ε_{yx}, ε_{yy} and ε_{XX}, ε_{XY}, ε_{YX}, ε_{YY} in the $(\vec{i},\vec{j})$ and $(\vec{I},\vec{J})$ bases, respectively. Find the relationship between the components by using the invariance of tensor quantities.



Answer $\begin{bmatrix} \varepsilon_{XX} & \varepsilon_{XY} \\ \varepsilon_{YX} & \varepsilon_{YY} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

7. Derive the component forms of the equilibrium equation $\nabla\cdot\vec{\vec{\sigma}}+\vec{f}=0$ in plane stress case, when

$$\vec{\vec{\sigma}}=\sigma_{xx}\vec{i}\vec{i}+\sigma_{xy}\vec{i}\vec{j}+\sigma_{yx}\vec{j}\vec{i}+\sigma_{yy}\vec{j}\vec{j}\,,\ \nabla=\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}, \text{ and } \vec{f}=f_x\vec{i}+f_y\vec{j}$$

and the stress components depend on x and y.

Answer $\dfrac{\partial\sigma_{xx}}{\partial x}+\dfrac{\partial\sigma_{yx}}{\partial y}+f_x=0$, $\dfrac{\partial\sigma_{xy}}{\partial x}+\dfrac{\partial\sigma_{yy}}{\partial y}+f_y=0$

8. The small strain measure $\vec{\vec{\varepsilon}}$ is obtained as the symmetric part of displacement gradient (a tensor then). Use the definition to find the components of the strain tensor in a Cartesian coordinate system when $\nabla=\vec{i}\partial/\partial x+\vec{j}\partial/\partial y$ and $\vec{u}=u_x\vec{i}+u_y\vec{j}$.

Answer $\varepsilon_{xx}=\dfrac{\partial u_x}{\partial x}$, $\varepsilon_{yy}=\dfrac{\partial u_y}{\partial y}$, and $\varepsilon_{xy}=\varepsilon_{yx}=\dfrac{1}{2}(\dfrac{\partial u_y}{\partial x}+\dfrac{\partial u_x}{\partial y})$

9. Find the solution to the boundary value problem

$$\frac{dQ}{dx}-\rho gA=0,\ \frac{dM}{dx}-Q=0,\ M=EI\frac{d\theta}{dx},\ \text{and}\ Q=GA(\frac{dw}{dx}+\theta)\ \text{ in } (0,L),$$

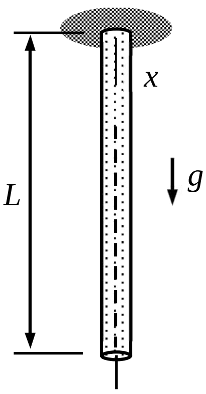
$$w=M=0 \text{ on } \{0,L\}$$

for a simply supported beam loaded by its own weight. Cross sectional area A, second moment af area I, Young's modulus E, shear modulus G, density of the material ρ, and acceleration by gravity g are constants. Use repeated integrations.

Answer $M(x)=\dfrac{1}{2}\rho gA(x^2-Lx)$, $Q(x)=\dfrac{1}{2}\rho gA(2x-L)$, $\theta(x)=\dfrac{\rho gA}{2EI}(\dfrac{1}{3}x^3-\dfrac{1}{2}Lx^2-\dfrac{1}{12}L^3)$

$$w(x)=\frac{\rho g}{2G}(x^2-xL)-\frac{\rho gA}{2EI}(\frac{1}{12}x^4-\frac{1}{6}Lx^3+\frac{1}{12}L^3x)$$

10. Consider a bar of length L loaded by its own weight. Determine the displacements at the mid-point and at the free end using the continuum model. Cross-sectional area A, acceleration by gravity g, and material properties E and ρ are constants.



Answer $u(\dfrac{L}{2})=\dfrac{3}{8}\dfrac{\rho gL^2}{E}$, $u(L)=\dfrac{\rho gL^2}{2E}$

Find the relationship between the orthonormal right-handed basis vector sets $(\vec{i},\vec{j},\vec{k})$ and $(\vec{I},\vec{J},\vec{K})$, if $\vec{I}$ has the same direction as $\vec{i}-\vec{j}+\vec{k}$ and $\vec{J}$ has the same direction as the gradient of plane $g(x,y,z)=2x+3y+z-5=0$ (hence $\vec{J}$ is the normal unit vector to the plane).

Solution

Both sets are orthonormal, i.e., the basis vectors are orthogonal and have unit lengths. As the systems are right-handed $\vec{i}\times\vec{j}=\vec{k}$, $\vec{j}\times\vec{k}=\vec{i}$, $\vec{k}\times\vec{i}=\vec{j}$ and $\vec{I}\times\vec{J}=\vec{K}$, $\vec{J}\times\vec{K}=\vec{I}$, $\vec{K}\times\vec{I}=\vec{J}$. Vectors $\vec{i}-\vec{j}+\vec{k}$ and $\vec{J}$ have the same directions, therefore

$$\vec{I}=\frac{\vec{i}-\vec{j}+\vec{k}}{|\vec{i}-\vec{j}+\vec{k}|}=\frac{1}{\sqrt{3}}(\vec{i}-\vec{j}+\vec{k}) \quad (|\vec{a}|=\sqrt{\vec{a}\cdot\vec{a}}).$$

Vector $\vec{J}$ and the gradient of $g(x,y,z)=2x+3y+z-5$ have the same directions, so

$$\nabla g=2\vec{i}+3\vec{j}+\vec{k} \quad \Rightarrow \quad \vec{J}=\frac{\nabla g}{|\nabla g|}=\frac{1}{\sqrt{14}}(2\vec{i}+3\vec{j}+\vec{k}).$$

Both coordinate systems are right-handed

$$\vec{K}=\vec{I}\times\vec{J}=\frac{1}{\sqrt{3}}(\vec{i}-\vec{j}+\vec{k})\times\frac{1}{\sqrt{14}}(2\vec{i}+3\vec{j}+\vec{k}) \quad \Leftrightarrow$$

$$\vec{K}=\frac{1}{\sqrt{42}}(\vec{i}\times2\vec{i}+\vec{i}\times3\vec{j}+\vec{i}\times\vec{k}-\vec{j}\times2\vec{i}-\vec{j}\times3\vec{j}-\vec{j}\times\vec{k}+\vec{k}\times2\vec{i}+\vec{k}\times3\vec{j}+\vec{k}\times\vec{k}) \quad \Leftrightarrow$$

$$\vec{K}=\frac{1}{\sqrt{42}}(3\vec{k}-\vec{j}+2\vec{k}-\vec{i}+2\vec{j}-3\vec{i})=\frac{1}{\sqrt{42}}(-4\vec{i}+\vec{j}+5\vec{k}).$$

Using the matrix notation

$$\begin{Bmatrix}\vec{I}\\ \vec{J}\\ \vec{K}\end{Bmatrix}=\frac{1}{\sqrt{42}}\begin{bmatrix}\sqrt{14} & -\sqrt{14} & \sqrt{14}\\ 2\sqrt{3} & 3\sqrt{3} & \sqrt{3}\\ -4 & 1 & 5\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}.$$

Consider the identity $\vec{I} \times \vec{a} = \vec{a} \times \vec{I}$ in a Cartesian $(x, y, z)-$coordinate system in which the second order unit tensor is given by $\vec{I} = \vec{i}\vec{i} + \vec{j}\vec{j} + \vec{k}\vec{k}$. Show that the identity holds by simplifying the left and right-hand side expressions. Assume that $\vec{a} = a_x\vec{i} + a_y\vec{j}$ (just to simplify the expressions).

Solution

Let us compare the expressions on the left and right-hand sides of $\vec{I} \times \vec{a} = \vec{a} \times \vec{I}$ and use the relationships $\vec{i} \times \vec{j} = \vec{k}$, $\vec{j} \times \vec{k} = \vec{i}$, and $\vec{k} \times \vec{i} = \vec{j}$ for a right handed system.

Left hand side expression

$$\vec{I} \times \vec{a} = (\vec{i}\vec{i} + \vec{j}\vec{j} + \vec{k}\vec{k}) \times (a_x\vec{i} + a_y\vec{j}) \quad \Leftrightarrow$$

$$\vec{I} \times \vec{a} = (\vec{i}\vec{i} + \vec{j}\vec{j} + \vec{k}\vec{k}) \times a_x\vec{i} + (\vec{i}\vec{i} + \vec{j}\vec{j} + \vec{k}\vec{k}) \times a_y\vec{j} \quad \Leftrightarrow$$

$$\vec{I} \times \vec{a} = (\vec{i}\vec{i} \times \vec{i} + \vec{j}\vec{j} \times \vec{i} + \vec{k}\vec{k} \times \vec{i})a_x + (\vec{i}\vec{i} \times \vec{j} + \vec{j}\vec{j} \times \vec{j} + \vec{k}\vec{k} \times \vec{j})a_y \quad \Leftrightarrow$$

$$\vec{I} \times \vec{a} = (\vec{k}\vec{j} - \vec{j}\vec{k})a_x + (\vec{i}\vec{k} - \vec{k}\vec{i})a_y\,. \quad \leftarrow$$

Right hand side expression

$$\vec{a} \times \vec{I} = (a_x\vec{i} + a_y\vec{j}) \times (\vec{i}\vec{i} + \vec{j}\vec{j} + \vec{k}\vec{k}) \quad \Leftrightarrow$$

$$\vec{a} \times \vec{I} = a_x\vec{i} \times (\vec{i}\vec{i} + \vec{j}\vec{j} + \vec{k}\vec{k}) + a_y\vec{j} \times (\vec{i}\vec{i} + \vec{j}\vec{j} + \vec{k}\vec{k}) \quad \Leftrightarrow$$

$$\vec{a} \times \vec{I} = a_x(\vec{i} \times \vec{i}\vec{i} + \vec{i} \times \vec{j}\vec{j} + \vec{i} \times \vec{k}\vec{k}) + a_y(\vec{j} \times \vec{i}\vec{i} + \vec{j} \times \vec{j}\vec{j} + \vec{j} \times \vec{k}\vec{k}) \quad \Leftrightarrow$$

$$\vec{a} \times \vec{I} = a_x(\vec{k}\vec{j} - \vec{j}\vec{k}) + a_y(\vec{i}\vec{k} - \vec{k}\vec{i})\,. \quad \leftarrow$$

Given $\vec{a} = a_{xx}\vec{i}\vec{i} + a_{xy}\vec{i}\vec{j} + a_{yx}\vec{j}\vec{i} + a_{yy}\vec{j}\vec{j}$ and $\vec{b} = b_x\vec{i} + b_y\vec{j}$, determine $\vec{a}\cdot\vec{b}$, $\vec{a}\times\vec{b}$, $\vec{b}\cdot\vec{a}$, and $\vec{b}\times\vec{a}$.

Solution

In manipulation of vector expression containing vectors and tensors, it is important to remember that tensor ($\otimes$), cross ($\times$), inner ($\cdot$) products are non-commutative (order matters). The basis vectors of a curvilinear coordinate system are not constants which should be taken into account if gradient operator is a part of expression. Otherwise, simplifying an expression or finding a specific form in a given coordinate system is a straightforward (sometimes tedious) exercise. For simplicity of presentation, outer (tensor) products like $\vec{a}\otimes\vec{b}$ are denoted by $\vec{a}\vec{b}$. Otherwise, the usual rules of algebra apply.

$$\vec{a}\cdot\vec{b} = (a_{xx}\vec{i}\vec{i} + a_{xy}\vec{i}\vec{j} + a_{yx}\vec{j}\vec{i} + a_{yy}\vec{j}\vec{j})\cdot(b_x\vec{i} + b_y\vec{j}) \quad \Leftrightarrow$$

$$\vec{a}\cdot\vec{b} = (a_{xx}\vec{i}\vec{i} + a_{xy}\vec{i}\vec{j} + a_{yx}\vec{j}\vec{i} + a_{yy}\vec{j}\vec{j})\cdot b_x\vec{i} + (a_{xx}\vec{i}\vec{i} + a_{xy}\vec{i}\vec{j} + a_{yx}\vec{j}\vec{i} + a_{yy}\vec{j}\vec{j})\cdot b_y\vec{j} \quad \Leftrightarrow$$

$$\vec{a}\cdot\vec{b} = (a_{xx}\vec{i}\vec{i}\cdot\vec{i} + a_{xy}\vec{i}\vec{j}\cdot\vec{i} + a_{yx}\vec{j}\vec{i}\cdot\vec{i} + a_{yy}\vec{j}\vec{j}\cdot\vec{i})b_x + (a_{xx}\vec{i}\vec{i}\cdot\vec{j} + a_{xy}\vec{i}\vec{j}\cdot\vec{j} + a_{yx}\vec{j}\vec{i}\cdot\vec{j} + a_{yy}\vec{j}\vec{j}\cdot\vec{j})b_y \quad \Leftrightarrow$$

$$\vec{a}\cdot\vec{b} = a_{xx}b_x\vec{i} + a_{yx}b_x\vec{j} + a_{xy}b_y\vec{i} + a_{yy}b_y\vec{j} = (a_{xx}b_x + a_{xy}b_y)\vec{i} + (a_{yx}b_x + a_{yy}b_y)\vec{j}.$$

$$\vec{a}\times\vec{b} = (a_{xx}\vec{i}\vec{i} + a_{xy}\vec{i}\vec{j} + a_{yx}\vec{j}\vec{i} + a_{yy}\vec{j}\vec{j})\times(b_x\vec{i} + b_y\vec{j}) \quad \Leftrightarrow$$

$$\vec{a}\times\vec{b} = (a_{xx}\vec{i}\vec{i} + a_{xy}\vec{i}\vec{j} + a_{yx}\vec{j}\vec{i} + a_{yy}\vec{j}\vec{j})\times b_x\vec{i} + (a_{xx}\vec{i}\vec{i} + a_{xy}\vec{i}\vec{j} + a_{yx}\vec{j}\vec{i} + a_{yy}\vec{j}\vec{j})\times b_y\vec{j} \quad \Leftrightarrow$$

$$\begin{aligned}\vec{a}\times\vec{b} = {} & (a_{xx}\vec{i}\vec{i}\times\vec{i} + a_{xy}\vec{i}\vec{j}\times\vec{i} + a_{yx}\vec{j}\vec{i}\times\vec{i} + a_{yy}\vec{j}\vec{j}\times\vec{i})b_x + \\ & (a_{xx}\vec{i}\vec{i}\times\vec{j} + a_{xy}\vec{i}\vec{j}\times\vec{j} + a_{yx}\vec{j}\vec{i}\times\vec{j} + a_{yy}\vec{j}\vec{j}\times\vec{j})b_y \quad \Leftrightarrow\end{aligned}$$

$$\vec{a}\times\vec{b} = -a_{xy}b_x\vec{i}\vec{k} - a_{yy}b_x\vec{j}\vec{k} + a_{xx}b_y\vec{i}\vec{k} + a_{yx}b_y\vec{j}\vec{k} = (a_{xx}b_y - a_{xy}b_x)\vec{i}\vec{k} + (a_{yx}b_y - a_{yy}b_x)\vec{j}\vec{k}.$$

$$\vec{b}\cdot\vec{a} = (b_x\vec{i} + b_y\vec{j})\cdot(a_{xx}\vec{i}\vec{i} + a_{xy}\vec{i}\vec{j} + a_{yx}\vec{j}\vec{i} + a_{yy}\vec{j}\vec{j}) \quad \Leftrightarrow$$

$$\vec{b}\cdot\vec{a} = b_x\vec{i}\cdot(a_{xx}\vec{i}\vec{i} + a_{xy}\vec{i}\vec{j} + a_{yx}\vec{j}\vec{i} + a_{yy}\vec{j}\vec{j}) + b_y\vec{j}\cdot(a_{xx}\vec{i}\vec{i} + a_{xy}\vec{i}\vec{j} + a_{yx}\vec{j}\vec{i} + a_{yy}\vec{j}\vec{j}) \quad \Leftrightarrow$$

$$\vec{b}\cdot\vec{a} = b_x(a_{xx}\vec{i}\cdot\vec{i}\vec{i} + a_{xy}\vec{i}\cdot\vec{i}\vec{j} + a_{yx}\vec{i}\cdot\vec{j}\vec{i} + a_{yy}\vec{i}\cdot\vec{j}\vec{j}) + b_y(a_{xx}\vec{j}\cdot\vec{i}\vec{i} + a_{xy}\vec{j}\cdot\vec{i}\vec{j} + a_{yx}\vec{j}\cdot\vec{j}\vec{i} + a_{yy}\vec{j}\cdot\vec{j}\vec{j}) \quad \Leftrightarrow$$

$$\vec{b}\cdot\vec{a} = b_x a_{xx}\vec{i} + b_x a_{xy}\vec{j} + b_y a_{yx}\vec{i} + b_y a_{yy}\vec{j} = (b_x a_{xx} + b_y a_{yx})\vec{i} + (b_x a_{xy} + b_y a_{yy})\vec{j}.$$

$$\vec{b}\times\vec{a} = (b_x\vec{i} + b_y\vec{j})\times(a_{xx}\vec{i}\vec{i} + a_{xy}\vec{i}\vec{j} + a_{yx}\vec{j}\vec{i} + a_{yy}\vec{j}\vec{j}) \quad \Leftrightarrow$$

$$\vec{b}\times\vec{a} = b_x\vec{i}\times(a_{xx}\vec{i}\vec{i} + a_{xy}\vec{i}\vec{j} + a_{yx}\vec{j}\vec{i} + a_{yy}\vec{j}\vec{j}) + b_y\vec{j}\times(a_{xx}\vec{i}\vec{i} + a_{xy}\vec{i}\vec{j} + a_{yx}\vec{j}\vec{i} + a_{yy}\vec{j}\vec{j}) \quad \Leftrightarrow$$

$$\vec{b}\times\vec{a} = b_x(a_{xx}\vec{i}\times\vec{i}\vec{i} + a_{xy}\vec{i}\times\vec{i}\vec{j} + a_{yx}\vec{i}\times\vec{j}\vec{i} + a_{yy}\vec{i}\times\vec{j}\vec{j}) +$$
$$b_y(a_{xx}\vec{j}\times\vec{i}\vec{i} + a_{xy}\vec{j}\times\vec{i}\vec{j} + a_{yx}\vec{j}\times\vec{j}\vec{i} + a_{yy}\vec{j}\times\vec{j}\vec{j}) \quad \Leftrightarrow$$

$$\vec{b}\times\vec{a} = b_x a_{yx}\vec{k}\vec{i} + b_x a_{yy}\vec{k}\vec{j} - b_y a_{xx}\vec{k}\vec{i} - b_y a_{xy}\vec{k}\vec{j} = (b_x a_{yx} - b_y a_{xx})\vec{k}\vec{i} + (b_x a_{yy} - b_y a_{xy})\vec{k}\vec{j} \;.$$

Calculate $\nabla\vec{r}$, $\nabla\cdot\vec{r}$ and $\nabla\times\vec{r}$ in which $\vec{r}$ is the position vector. Use the representations $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$ and $\nabla=\vec{i}\,\partial/\partial x+\vec{j}\,\partial/\partial y+\vec{k}\,\partial/\partial z$ in the Cartesian $(\vec{i},\vec{j},\vec{k})$ basis.

Solution

In a term, gradient operator ∇ acts on everything on its right-hand side. Otherwise, the operator is treated like a vector. Manipulation of a tensor expression consist of (I) substitution of the representations, (II) term by term expansion, (III) evaluation of the terms, (IV) simplification and/or restructuring the outcome. Gradient of the position vector is a second order tensor

(I) $\nabla\vec{r}=(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z})(x\vec{i}+y\vec{j}+z\vec{k}) \quad \Leftrightarrow$

(II) $\nabla\vec{r}=\vec{i}(\frac{\partial}{\partial x}x\vec{i}+\frac{\partial}{\partial x}y\vec{j}+\frac{\partial}{\partial x}z\vec{k})+\vec{j}(\frac{\partial}{\partial y}x\vec{i}+\frac{\partial}{\partial y}y\vec{j}+\frac{\partial}{\partial y}z\vec{k})+\vec{k}(\frac{\partial}{\partial z}x\vec{i}+\frac{\partial}{\partial z}y\vec{j}+\frac{\partial}{\partial z}z\vec{k}) \quad \Leftrightarrow$

(III) $\nabla\vec{r}=\vec{i}(\vec{i}+0+0)+\vec{j}(0+\vec{j}+0)+\vec{k}(0+0+\vec{k}) \quad \Leftrightarrow$

(IV) $\nabla\vec{r}=\vec{i}\vec{i}+\vec{j}\vec{j}+\vec{k}\vec{k}=\vec{I}$.

Divergence of the position vector is a scalar

(I) $\nabla\cdot\vec{r}=(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z})\cdot(x\vec{i}+y\vec{j}+z\vec{k}) \quad \Leftrightarrow$

(II) $\nabla\cdot\vec{r}=\vec{i}\cdot(\frac{\partial}{\partial x}x\vec{i}+\frac{\partial}{\partial x}y\vec{j}+\frac{\partial}{\partial x}z\vec{k})+\vec{j}\cdot(\frac{\partial}{\partial y}x\vec{i}+\frac{\partial}{\partial y}y\vec{j}+\frac{\partial}{\partial y}z\vec{k})+\vec{k}\cdot(\frac{\partial}{\partial z}x\vec{i}+\frac{\partial}{\partial z}y\vec{j}+\frac{\partial}{\partial z}z\vec{k}) \quad \Leftrightarrow$

(III) $\nabla\cdot\vec{r}=\vec{i}\cdot(\vec{i}+0+0)+\vec{j}\cdot(0+\vec{j}+0)+\vec{k}\cdot(0+0+\vec{k}) \quad \Leftrightarrow$

(IV) $\nabla\cdot\vec{r}=\vec{i}\cdot\vec{i}+\vec{j}\cdot\vec{j}+\vec{k}\cdot\vec{k}=3$.

Curl of the position vector is a vector

(I) $\nabla\times\vec{r}=(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z})\times(x\vec{i}+y\vec{j}+z\vec{k}) \quad \Leftrightarrow$

(II) $\nabla\times\vec{r}=\vec{i}\times(\frac{\partial}{\partial x}x\vec{i}+\frac{\partial}{\partial x}y\vec{j}+\frac{\partial}{\partial x}z\vec{k})+\vec{j}\times(\frac{\partial}{\partial y}x\vec{i}+\frac{\partial}{\partial y}y\vec{j}+\frac{\partial}{\partial y}z\vec{k})+\vec{k}\times(\frac{\partial}{\partial z}x\vec{i}+\frac{\partial}{\partial z}y\vec{j}+\frac{\partial}{\partial z}z\vec{k}) \quad \Leftrightarrow$

(III) $\nabla\times\vec{r}=\vec{i}\times(\vec{i}+0+0)+\vec{j}\times(0+\vec{j}+0)+\vec{k}\times(0+0+\vec{k}) \quad \Leftrightarrow$

(IV) $\nabla\times\vec{r}=\vec{i}\times\vec{i}+\vec{j}\times\vec{j}+\vec{k}\times\vec{k}=0$.

Derive the expressions of $\nabla \cdot \vec{u}$, $\nabla \times \vec{u}$, and $\nabla^2 u$ in Cartesian coordinate system where $\nabla = \vec{i}\partial/\partial x + \vec{j}\partial/\partial y + \vec{k}\partial/\partial z$. Vector $\vec{u}(x,y) = u_x(x,y)\vec{i} + u_y(x,y)\vec{j}$ and scalar $u(x,y)$.

Solution

In manipulation of vector expression containing vectors and dyads, it is important to remember that tensor ($\otimes$), cross ($\times$), inner ($\cdot$) products are non-commutative (order matters). Otherwise, simplifying an expression or finding a specific form in a given coordinate system is a straightforward (sometimes tedious) exercise. Manipulation of a tensor expression consist of (I) substitution of the representations, (II) term-by-term expansion, (III) evaluation of the terms, (IV) simplification and/or restructuring the outcome.

Divergence of a vector (here $\nabla = \vec{i}\partial/\partial x + \vec{j}\partial/\partial y$)

(I) $\nabla \cdot \vec{u} = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y}) \cdot (u_x\vec{i} + u_y\vec{j}) \quad \Leftrightarrow$

(II) $\nabla \cdot \vec{u} = \vec{i} \cdot (\frac{\partial}{\partial x}u_x\vec{i} + \frac{\partial}{\partial x}u_y\vec{j}) + \vec{j} \cdot (\frac{\partial}{\partial y}u_x\vec{i} + \frac{\partial}{\partial y}u_y\vec{j}) \quad \Leftrightarrow$

(III) $\nabla \cdot \vec{u} = \frac{\partial}{\partial x}u_x\vec{i} \cdot \vec{i} + \frac{\partial}{\partial x}u_y\vec{i} \cdot \vec{j} + \frac{\partial}{\partial y}u_x\vec{j} \cdot \vec{i} + \frac{\partial}{\partial y}u_y\vec{j} \cdot \vec{j} \quad \Leftrightarrow$

(IV) $\nabla \cdot \vec{u} = \frac{\partial}{\partial x}u_x + \frac{\partial}{\partial y}u_y$.

Curl of a vector

(I) $\nabla \times \vec{u} = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y}) \times (u_x\vec{i} + u_y\vec{j}) \quad \Leftrightarrow$

(II) $\nabla \times \vec{u} = \vec{i} \times (\frac{\partial}{\partial x}u_x\vec{i} + \frac{\partial}{\partial x}u_y\vec{j}) + \vec{j} \times (\frac{\partial}{\partial y}u_x\vec{i} + \frac{\partial}{\partial y}u_y\vec{j}) \quad \Leftrightarrow$

(III) $\nabla \times \vec{u} = \frac{\partial}{\partial x}u_x\vec{i} \times \vec{i} + \frac{\partial}{\partial x}u_y\vec{i} \times \vec{j} + \frac{\partial}{\partial y}u_x\vec{j} \times \vec{i} + \frac{\partial}{\partial y}u_y\vec{j} \times \vec{j} \quad \Leftrightarrow$

(IV) $\nabla \times \vec{u} = (\frac{\partial}{\partial x}u_y - \frac{\partial}{\partial y}u_x)\vec{k}$.

Laplacian of a scalar

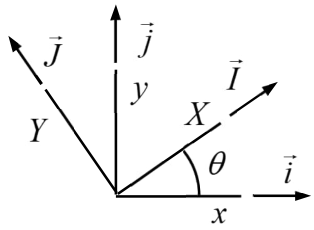
(I) $\nabla^2 u = (\nabla \cdot \nabla)u = \nabla \cdot (\nabla u) = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y}) \cdot (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y})u \quad \Leftrightarrow$

(II) $\nabla^2 u = \vec{i} \cdot (\vec{i}\frac{\partial^2 u}{\partial x^2} + \vec{j}\frac{\partial^2 u}{\partial x \partial y}) + \vec{j} \cdot (\vec{i}\frac{\partial^2 u}{\partial x \partial y} + \vec{j}\frac{\partial^2 u}{\partial y^2}) \quad \Leftrightarrow$

(III) $\nabla^2 u = (1\frac{\partial^2 u}{\partial x^2} + 0\frac{\partial^2 u}{\partial x \partial y}) + (0\frac{\partial^2 u}{\partial x \partial y} + 1\frac{\partial^2 u}{\partial y^2}) \quad \Leftrightarrow$

(IV) $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$. $\Leftarrow$

Let us consider a second order tensor $\vec{\vec{\varepsilon}}$ having the components ε_{xx}, ε_{xy}, ε_{yx}, ε_{yy} and ε_{XX}, ε_{XY}, ε_{YX}, ε_{YY} in the $(\vec{i}, \vec{j})$ and $(\vec{I}, \vec{J})$ bases shown. Find the relationship between the components by using invariance of tensor quantities.



Solution

In mechanics tensors (vectors etc.) represent physical quantities which can be expressed in terms of any basis vector set. Components depend on the selection of the basis vectors but the quantity itself does not. According to the figure, the relationship between the $(\vec{i}, \vec{j})$ and $(\vec{I}, \vec{J})$ bases is given by

$$\begin{Bmatrix} \vec{I} \\ \vec{J} \end{Bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} = [F] \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} \quad \Leftrightarrow \quad \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} \vec{I} \\ \vec{J} \end{Bmatrix} = [F]^{-1} \begin{Bmatrix} \vec{I} \\ \vec{J} \end{Bmatrix}.$$

Invariace of $\vec{\vec{\varepsilon}}$ with respect to the coordinate system means that

$$\vec{\vec{\varepsilon}} = \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} = \begin{Bmatrix} \vec{I} \\ \vec{J} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \varepsilon_{XX} & \varepsilon_{XY} \\ \varepsilon_{YX} & \varepsilon_{YY} \end{bmatrix} \begin{Bmatrix} \vec{I} \\ \vec{J} \end{Bmatrix}.$$

Using the relationship between the basis vectors

$$\vec{\vec{\varepsilon}} = \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} = \begin{Bmatrix} \vec{I} \\ \vec{J} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \varepsilon_{XX} & \varepsilon_{XY} \\ \varepsilon_{YX} & \varepsilon_{YY} \end{bmatrix} \begin{Bmatrix} \vec{I} \\ \vec{J} \end{Bmatrix} = \begin{Bmatrix} \vec{I} \\ \vec{J} \end{Bmatrix}^{\mathrm{T}} [F]^{\mathrm{T}} \begin{bmatrix} \varepsilon_{XX} & \varepsilon_{XY} \\ \varepsilon_{YX} & \varepsilon_{YY} \end{bmatrix} [F] \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}.$$

Therefore, components of the two systems are related by $([\mathrm{F}]^{-1} = [\mathrm{F}]^{\mathrm{T}})$

$$\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} = [F]^{\mathrm{T}} \begin{bmatrix} \varepsilon_{XX} & \varepsilon_{XY} \\ \varepsilon_{YX} & \varepsilon_{YY} \end{bmatrix} [F] \quad \Leftrightarrow \quad \begin{bmatrix} \varepsilon_{XX} & \varepsilon_{XY} \\ \varepsilon_{YX} & \varepsilon_{YY} \end{bmatrix} = [F] \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} [F]^{\mathrm{T}}.$$

Derive the component forms of the equilibrium equation $\nabla\cdot\vec{\vec{\sigma}}+\vec{f}=0$ in plane stress case, when

$\vec{\vec{\sigma}}=\sigma_{xx}\vec{i}\vec{i}+\sigma_{xy}\vec{i}\vec{j}+\sigma_{yx}\vec{j}\vec{i}+\sigma_{yy}\vec{j}\vec{j}$, $\nabla=\vec{i}\dfrac{\partial}{\partial x}+\vec{j}\dfrac{\partial}{\partial y}$, and $\vec{f}=f_x\vec{i}+f_y\vec{j}$.

Solution

Manipulation of a tensor expression consist of (I) substitution of the representations, (II) term-by-term expansion, (III) evaluation of the terms, (IV) simplification and/or restructuring the outcome.

$$\nabla\cdot\vec{\vec{\sigma}}+\vec{f}=(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y})\cdot(\sigma_{xx}\vec{i}\vec{i}+\sigma_{xy}\vec{i}\vec{j}+\sigma_{yx}\vec{j}\vec{i}+\sigma_{yy}\vec{j}\vec{j})+f_x\vec{i}+f_y\vec{j}=0$$

$$\vec{i}\frac{\partial}{\partial x}\cdot(\sigma_{xx}\vec{i}\vec{i}+\sigma_{xy}\vec{i}\vec{j}+\sigma_{yx}\vec{j}\vec{i}+\sigma_{yy}\vec{j}\vec{j})+\vec{j}\frac{\partial}{\partial y}\cdot(\sigma_{xx}\vec{i}\vec{i}+\sigma_{xy}\vec{i}\vec{j}+\sigma_{yx}\vec{j}\vec{i}+\sigma_{yy}\vec{j}\vec{j})+f_x\vec{i}+f_y\vec{j}=0$$

Let us consider the first and second terms on the left-hand side separately

$$\vec{i}\frac{\partial}{\partial x}\cdot(\sigma_{xx}\vec{i}\vec{i}+\sigma_{xy}\vec{i}\vec{j}+\sigma_{yx}\vec{j}\vec{i}+\sigma_{yy}\vec{j}\vec{j})=$$

$$\frac{\partial\sigma_{xx}}{\partial x}\vec{i}\cdot\vec{i}\vec{i}+\frac{\partial\sigma_{xy}}{\partial x}\vec{i}\cdot\vec{i}\vec{j}+\frac{\partial\sigma_{yx}}{\partial x}\vec{i}\cdot\vec{j}\vec{i}+\frac{\partial\sigma_{yy}}{\partial x}\vec{i}\cdot\vec{j}\vec{j}=$$

$$\frac{\partial\sigma_{xx}}{\partial x}\vec{i}+\frac{\partial\sigma_{xy}}{\partial x}\vec{j}+\frac{\partial\sigma_{yx}}{\partial x}0\vec{i}+\frac{\partial\sigma_{yy}}{\partial x}0\vec{j}=\frac{\partial\sigma_{xx}}{\partial x}\vec{i}+\frac{\partial\sigma_{xy}}{\partial x}\vec{j}$$

and

$$\vec{j}\frac{\partial}{\partial y}\cdot(\sigma_{xx}\vec{i}\vec{i}+\sigma_{xy}\vec{i}\vec{j}+\sigma_{yx}\vec{j}\vec{i}+\sigma_{yy}\vec{j}\vec{j})=$$

$$\frac{\partial\sigma_{xx}}{\partial y}\vec{j}\cdot\vec{i}\vec{i}+\frac{\partial\sigma_{xy}}{\partial y}\vec{j}\cdot\vec{i}\vec{j}+\frac{\partial\sigma_{yx}}{\partial y}\vec{j}\cdot\vec{j}\vec{i}+\frac{\partial\sigma_{yy}}{\partial y}\vec{j}\cdot\vec{j}\vec{j}=$$

$$\frac{\partial\sigma_{xx}}{\partial y}0\vec{i}+\frac{\partial\sigma_{xy}}{\partial y}0\vec{j}+\frac{\partial\sigma_{yx}}{\partial y}1\vec{i}+\frac{\partial\sigma_{yy}}{\partial y}1\vec{j}=\frac{\partial\sigma_{yx}}{\partial y}\vec{i}+\frac{\partial\sigma_{yy}}{\partial y}\vec{j}\ .$$

By combining all the terms

$$\nabla\cdot\vec{\vec{\sigma}}+\vec{f}=(\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{yx}}{\partial y}+f_x)\vec{i}+(\frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+f_y)\vec{j}=0\ .$$

A vector vanishes if all its components vanish, so

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{yx}}{\partial y}+f_x=0\quad\text{and}\quad\frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+f_y=0\ .\ \leftarrow$$

The small strain measure $\vec{\vec{\varepsilon}}$ is obtained as the symmetric part of displacement gradient (a tensor then). Use the definition to find the components of the strain tensor in a Cartesian coordinate system when $\nabla = \vec{i}\partial/\partial x + \vec{j}\partial/\partial y$ and $\vec{u} = u_x\vec{i} + u_y\vec{j}$.

Solution

In Cartesian system, $\nabla = \vec{i}\partial/\partial x + \vec{j}\partial/\partial y$ and $\vec{u} = u_x\vec{i} + u_y\vec{j}$, therefore

$$\nabla\vec{u} = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y})(u_x\vec{i} + u_y\vec{j}) = \vec{i}\vec{i}\frac{\partial u_x}{\partial x} + \vec{i}\vec{j}\frac{\partial u_y}{\partial x} + \vec{j}\vec{i}\frac{\partial u_x}{\partial y} + \vec{j}\vec{j}\frac{\partial u_y}{\partial y}) \quad \Rightarrow$$

$$(\nabla\vec{u})_{\mathrm{c}} = \vec{i}\vec{i}\frac{\partial u_x}{\partial x} + \vec{j}\vec{i}\frac{\partial u_y}{\partial x} + \vec{i}\vec{j}\frac{\partial u_x}{\partial y} + \vec{j}\vec{j}\frac{\partial u_y}{\partial y}$$

giving

$$\vec{\vec{\varepsilon}} = \frac{1}{2}[\nabla\vec{u} + (\nabla\vec{u})_{\mathrm{c}}] = \vec{i}\vec{i}\frac{\partial u_x}{\partial x} + \vec{j}\vec{j}\frac{\partial u_y}{\partial y} + \vec{i}\vec{j}\frac{1}{2}(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}) + \vec{j}\vec{i}\frac{1}{2}(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}).$$

Therefor $\vec{\vec{\varepsilon}} = \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} = \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{1}{2}(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}) \\ \frac{1}{2}(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}) & \frac{\partial u_y}{\partial y} \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}$ so

$$\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{1}{2}(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}) \\ \frac{1}{2}(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}) & \frac{\partial u_y}{\partial y} \end{bmatrix}. \leftarrow$$

Find the solution to the boundary value problem

$$\frac{dQ}{dx}-\rho gA=0, \quad \frac{dM}{dx}-Q=0, \quad M=EI\frac{d\theta}{dx}, \text{ and } \quad Q=GA(\frac{dw}{dx}+\theta) \quad \text{in} \quad (0,L),$$

$$w=M=0 \quad \text{on} \quad \{0,L\}$$

for a simply supported beam loaded by its own weight. Cross sectional area A, second moment af area I, Young's modulus E, shear modulus G, density of the material ρ, and acceleration by gravity g are constants. Use repeated integrations.

Solution

The first order equation set for the beam bending

$$\frac{dQ}{dx}-\rho gA=0, \quad \frac{dM}{dx}-Q=0, \quad M=EI\frac{d\theta}{dx}, \text{ and } \quad Q=GA(\frac{dw}{dx}+\theta) \quad \text{in} \quad (0,L),$$

Is composed of two equilibrium equations and constitutive equations for the bending moments and shear force (stress resultants on a cross-section). Boundary conditions

$$w=\theta=0 \quad \text{on} \quad \{0,L\}$$

describe a simple support at both ends. Let us integrate the equations one-by-one and denote the integration constants are denoted by *a*, *b*, *c*, *d* etc. Let us also apply the boundary conditions as soon as possible.

$$\frac{dQ}{dx}-\rho gA=0 \quad \Leftrightarrow \quad \frac{dQ}{dx}=\rho gA \quad \Leftrightarrow \quad Q(x)=\rho gAx+a,$$

$$\frac{dM}{dx}-Q=0 \quad \Rightarrow \quad \frac{dM}{dx}=\rho gAx+a \quad \Leftrightarrow \quad M(x)=\rho gA\frac{1}{2}x^2+ax+b,$$

According to the boundary conditions $M(0)=M(L)=0$:

$$M(0)=b=0 \text{ and } M(L)=\rho gA\frac{1}{2}L^2+aL+b=0 \quad \Leftrightarrow \quad b=0 \text{ and } a=-\rho gA\frac{1}{2}L.$$

Therefore $M(x)=\frac{1}{2}\rho gA(x^2-Lx)$ and $Q(x)=\frac{1}{2}\rho gA(2x-L)$. ←

Let us continue integrations with the constitutive equations

$$M=EI\frac{d\theta}{dx} \quad \Rightarrow \quad \frac{d\theta}{dx}=\frac{\rho gA}{2EI}(x^2-Lx) \quad \Leftrightarrow \quad \theta=\frac{\rho gA}{2EI}(\frac{1}{3}x^3-L\frac{1}{2}x^2)+c,$$

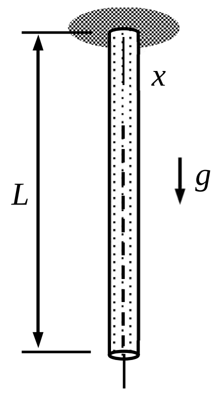
$$Q=GA(\frac{dw}{dx}+\theta) \quad \Rightarrow \quad \frac{dw}{dx}=\frac{\rho g}{2G}(2x-L)-\frac{\rho gA}{2EI}(\frac{1}{3}x^3-L\frac{1}{2}x^2)+c \quad \Leftrightarrow$$
$$w=\frac{\rho g}{2G}(x^2-xL)-\frac{\rho gA}{2EI}(\frac{1}{12}x^4-L\frac{1}{6}x^3)+cx+d.$$

According to the boundary conditions $w(0) = w(L) = 0$:

$$d = 0 \quad \text{and} \quad \frac{\rho g A}{24EI} L^4 + cL + d = 0 \quad \Leftrightarrow \quad d = 0 \quad \text{and} \quad c = -\frac{\rho g A}{24EI} L^3 .$$

Therefore

$$\theta(x) = \frac{\rho g A}{2EI} (\frac{1}{3} x^3 - \frac{1}{2} L x^2 - \frac{1}{12} L^3) \quad \text{and} \quad w(x) = \frac{\rho g}{2G} (x^2 - xL) - \frac{\rho g A}{2EI} (\frac{1}{12} x^4 - \frac{1}{6} L x^3 + \frac{1}{12} L^3 x) . \;\Leftarrow$$



Consider a bar of length L loaded by its own weight. Determine the displacements at the mid-point and at the free end using the continuum model. Cross-sectional area A, acceleration by gravity g, and material properties E and ρ are constants.

Solution

In stationary case, the continuum model for the problem is given by equations

$$EA\frac{d^2u}{dx^2}+\rho Ag=0 \quad x\in]0,L[\;,\quad u=0 \quad x=0, \text{ and } \quad EA\frac{du}{dx}=0 \quad x=L.$$

Repeated integrations in the differential equation give the generic solution containing two integration constants:

$$\frac{d^2u}{dx^2}=-\frac{\rho g}{E} \;\Rightarrow\; \frac{du}{dx}=-\frac{\rho g}{E}x+a \;\Rightarrow\; u=-\frac{\rho g}{E}\frac{1}{2}x^2+ax+b\,.$$

Then, substituting the generic solution into the boundary conditions

$$u(0)=b=0 \text{ and } \frac{du}{dx}(L)=-\frac{\rho g}{E}L+a=0 \quad\Leftrightarrow\quad a=\frac{\rho g}{E}L \text{ and } b=0.$$

Solution to the problem as function of x gives also the values at the center point and end point

$$u(x)=\frac{\rho g}{E}x(L-\frac{1}{2}x) \quad\Rightarrow\quad u(\frac{L}{2})=\frac{3}{8}\frac{\rho gL^2}{E} \text{ and } u(L)=\frac{\rho gL^2}{2E}\,.$$

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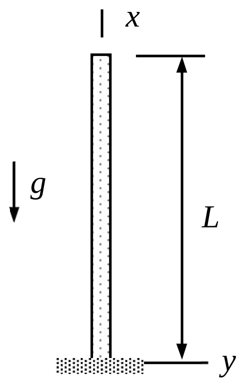
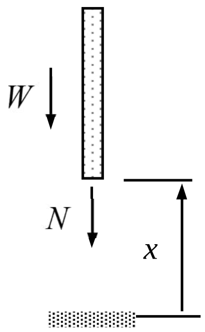
Assignment 1 (2p)



The column of the figure is loaded by its own weight. Determine stress σ_{xx}, strain ε_{xx} and displacement u_x as functions of x. Cross-sectional area A and density ρ of the material are constants. Assume that stress and strain are related by Hooke's law $\sigma_{xx} = E\varepsilon_{xx}$.

Solution template

Let us start with the axial force N by considering the equilibrium of the column part shown



Weight of the column part $W = \rho Ag(L-x)$

Equilibrium equation $N + W = 0$

Axial force $N = \rho Ag(x-L)$

Stress at x follows from definition "force divided by the area" as directed area and force are aligned in the present problem.

Stress $\sigma_{xx} = \rho g(x-L)$.

Strain at x follows from the stress-strain relationship $\sigma_{xx} = E\varepsilon_{xx}$.

Strain $\varepsilon_{xx} = \dfrac{\rho g}{E}(x-L)$.

Displacement of the column at x follows from the definition of strain (strain-displacement relationship) $\varepsilon_{xx} = du_x / dx$ to be considered as an ordinary first order differential equation to displacement u_x. Let the integration constant be C.

Generic solution to displacement $u_x = \dfrac{\rho g}{E}(\dfrac{1}{2}x^2 - Lx) + C$

Displacement is known to vanish at $x = 0$. Elimination the integration constant by using the boundary condition $u_x(0) = 0$ gives the displacement for the problem.

Displacement $u_x = \dfrac{\rho g}{E}(\dfrac{1}{2}x^2 - Lx)$.

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Assignment 2 (2p)

Find the displacement $u(x)$ of a bar of length L using the boundary value problem

$$EA\frac{d^2u}{dx^2}+\rho Ag=0 \quad x\in]0,L[\,, \quad u(0)=u(L)=0$$

given by the continuum model. Assume that the cross-sectional area A, Young's modulus E of the material, density ρ of the material, and acceleration by gravity g are constants.

Solution template

First, repeated integrations with the differential equation are used to find the generic solution. Let the integration constants be a and b:

$$\frac{d^2u}{dx^2}=-\frac{\rho Ag}{EA} \quad \Rightarrow \quad \frac{du}{dx}=-\frac{\rho Ag}{EA}x+a \quad \Rightarrow \quad u(x)=-\frac{\rho Ag}{EA}\frac{1}{2}x^2+ax+b\,.$$

Second, boundary conditions are used to find the values of the integration constants a and b:

$$u(0)=b=0 \quad \text{and} \quad u(L)=-\frac{\rho Ag}{EA}\frac{1}{2}L^2+aL+b=0 \quad \Rightarrow \quad b=0 \quad \text{and} \quad a=\frac{\rho Ag}{EA}\frac{1}{2}L$$

Finally, the values of the integration constants are substituted into the generic solution to get the solution:

$$u(x)=\frac{\rho Ag}{EA}\frac{1}{2}x(L-x)\,. \leftarrow$$

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Assignment 3 (2p)

Determine $\mathrm{tr}[S] \equiv \vec{\vec{I}} : \vec{\vec{S}}$, $\vec{\vec{S}} : \vec{\vec{S}}$, $\vec{\vec{S}} : \vec{\vec{S}}_{\mathrm{c}}$, $\vec{A} \cdot \vec{\vec{S}}$ ja $\vec{\vec{S}} \cdot \vec{\vec{T}}$ when the components of tensors $\vec{A}$, $\vec{\vec{S}}$ and $\vec{\vec{T}}$ in the orthonormal $(\vec{i}, \vec{j}, \vec{k})$ basis are

$$\{A\} = \begin{Bmatrix} 0 \\ -1 \\ 1 \end{Bmatrix}, \; [S] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } [T] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Solution template

The inner products of the basis vectors $\vec{i} \cdot \vec{i} = 1$, $\vec{j} \cdot \vec{j} = 1$ and $\vec{k} \cdot \vec{k} = 1$ all the other combinations giving zeros. The double inner product should be treated just as two inner products by keeping the positions of the multiplication operator with respect to vectors. Therefore, in the vector identity $\vec{a}\vec{b} : \vec{c}\vec{d} = (\vec{b} \cdot \vec{c})(\vec{a} \cdot \vec{d})$, the first inner product between $\vec{b}$ and $\vec{c}$ produces a scalar which can be moved in front of the expression. What remains is the inner product between $\vec{a}$ and $\vec{d}$.

In conjugate $\vec{\vec{S}}_{\mathrm{c}}$ to $\vec{\vec{S}}$ the component matrix is transposed which corresponds to order change in all the dyads $\vec{a}\vec{b} \to \vec{b}\vec{a}$ $\vec{a}, \vec{b} \in \{\vec{i}, \vec{j}, \vec{k}\}$ of the tensor representation. Representations of $\vec{A}$, $\vec{\vec{S}}$, $\vec{\vec{S}}_{\mathrm{c}}$, $\vec{\vec{T}}$ and the second order unit tensor $\vec{\vec{I}}$ in $(\vec{i}, \vec{j}, \vec{k})$ – basis

$$\vec{A} = \begin{Bmatrix} 0 \\ -1 \\ 1 \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = -\vec{j} + \vec{k},$$

$$\vec{\vec{S}} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = \vec{i}\vec{i} + \vec{k}\vec{i} + \vec{k}\vec{k},$$

$$\vec{\vec{S}}_{\mathrm{c}} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = \vec{i}\vec{i} + \vec{i}\vec{k} + \vec{k}\vec{k},$$

$$\vec{\vec{T}} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = \vec{j}\vec{j} + \vec{i}\vec{k} + \vec{k}\vec{i},$$

$$\vec{\vec{I}} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = \vec{i}\vec{i} + \vec{j}\vec{j} + \vec{k}\vec{k} \, .$$

Evaluation of a tensor product expressions consist of (I) substitution of the representations, (II) term-by-term expansion, (III) evaluation of the terms, (IV) simplification and/or restructuring the outcome.

Double inner product $\mathrm{tr}[S] \equiv \vec{\vec{I}} : \vec{\vec{S}}$ produces a scalar

(I) $\quad \mathrm{tr}[S] \equiv \vec{\vec{I}} : \vec{\vec{S}} = (\vec{i}\vec{i} + \vec{j}\vec{j} + \vec{k}\vec{k}) : (\vec{i}\vec{i} + \vec{k}\vec{i} + \vec{k}\vec{k}) \quad \Leftrightarrow$

(II) $\quad \mathrm{tr}[S] = \vec{i}\vec{i} : \vec{i}\vec{i} + \vec{i}\vec{i} : \vec{k}\vec{i} + \vec{i}\vec{i} : \vec{k}\vec{k} + \vec{j}\vec{j} : \vec{i}\vec{i} + \vec{j}\vec{j} : \vec{k}\vec{i} + \vec{j}\vec{j} : \vec{k}\vec{k} + \vec{k}\vec{k} : \vec{i}\vec{i} + \vec{k}\vec{k} : \vec{k}\vec{i} + \vec{k}\vec{k} : \vec{k}\vec{k} \quad \Leftrightarrow$

(III) $\quad \mathrm{tr}[S] = 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 1 \quad \Leftrightarrow$

(IV) $\quad \mathrm{tr}[S] = 2 \, .$

Double inner product $\vec{\vec{S}} : \vec{\vec{S}}$ produces a scalar

(I) $\quad \vec{\vec{S}} : \vec{\vec{S}} = (\vec{i}\vec{i} + \vec{k}\vec{i} + \vec{k}\vec{k}) : (\vec{i}\vec{i} + \vec{k}\vec{i} + \vec{k}\vec{k}) \quad \Leftrightarrow$

(II) $\quad \vec{\vec{S}} : \vec{\vec{S}} = \vec{i}\vec{i} : \vec{i}\vec{i} + \vec{i}\vec{i} : \vec{k}\vec{i} + \vec{i}\vec{i} : \vec{k}\vec{k} + \vec{k}\vec{i} : \vec{i}\vec{i} + \vec{k}\vec{i} : \vec{k}\vec{i} + \vec{k}\vec{i} : \vec{k}\vec{k} + \vec{k}\vec{k} : \vec{i}\vec{i} + \vec{k}\vec{k} : \vec{k}\vec{i} + \vec{k}\vec{k} : \vec{k}\vec{k} \quad \Leftrightarrow$

(III) $\quad \vec{\vec{S}} : \vec{\vec{S}} = 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 1 \quad \Leftrightarrow$

(IV) $\quad \vec{\vec{S}} : \vec{\vec{S}} = 2 \, .$

Double inner product $\vec{\vec{S}} : \vec{\vec{S}}_{\mathrm{c}}$ produces a scalar

(I) $\quad \vec{\vec{S}} : \vec{\vec{S}}_{\mathrm{c}} = (\vec{i}\vec{i} + \vec{k}\vec{i} + \vec{k}\vec{k}) : (\vec{i}\vec{i} + \vec{i}\vec{k} + \vec{k}\vec{k}) \quad \Leftrightarrow$

(II) $\quad \vec{\vec{S}} : \vec{\vec{S}}_{\mathrm{c}} = \vec{i}\vec{i} : \vec{i}\vec{i} + \vec{i}\vec{i} : \vec{i}\vec{k} + \vec{i}\vec{i} : \vec{k}\vec{k} + \vec{k}\vec{i} : \vec{i}\vec{i} + \vec{k}\vec{i} : \vec{i}\vec{k} + \vec{k}\vec{i} : \vec{k}\vec{k} + \vec{k}\vec{k} : \vec{i}\vec{i} + \vec{k}\vec{k} : \vec{i}\vec{k} + \vec{k}\vec{k} : \vec{k}\vec{k} \quad \Leftrightarrow$

(III) $\quad \vec{\vec{S}} : \vec{\vec{S}}_{\mathrm{c}} = 1 + 0 + 0 + 0 + 1 + 0 + 0 + 0 + 1 \quad \Leftrightarrow$

(IV) $\quad \vec{\vec{S}} : \vec{\vec{S}}_{\mathrm{c}} = 3 \, .$

Inner product $\vec{A} \cdot \vec{\vec{S}}$ produces a vector

(I) $\quad \vec{A} \cdot \vec{\vec{S}} = (-\vec{j} + \vec{k}) \cdot (\vec{i}\vec{i} + \vec{k}\vec{i} + \vec{k}\vec{k}) \quad \Leftrightarrow$

(II) $\quad \vec{A} \cdot \vec{\vec{S}} = -\vec{j} \cdot \vec{i}\vec{i} - \vec{j} \cdot \vec{k}\vec{i} - \vec{j} \cdot \vec{k}\vec{k} + \vec{k} \cdot \vec{i}\vec{i} + \vec{k} \cdot \vec{k}\vec{i} + \vec{k} \cdot \vec{k}\vec{k} \quad \Leftrightarrow$

(III) $\vec{A}\cdot\vec{\vec{S}} = -0\vec{i} - 0\vec{i} - 0\vec{k} + 0\vec{i} + 1\vec{i} + 1\vec{k} \quad \Leftrightarrow$

(IV) $\vec{A}\cdot\vec{\vec{S}} = \vec{i} + \vec{k}$.

Inner product $\vec{\vec{S}}\cdot\vec{\vec{T}}$ produces a second order tensor

(I) $\vec{\vec{S}}\cdot\vec{\vec{T}} = (\vec{i}\vec{i} + \vec{k}\vec{i} + \vec{k}\vec{k})\cdot(\vec{j}\vec{j} + \vec{i}\vec{k} + \vec{k}\vec{i}) \quad \Leftrightarrow$

(II) $\vec{\vec{S}}\cdot\vec{\vec{T}} = \vec{i}\vec{i}\cdot\vec{j}\vec{j} + \vec{i}\vec{i}\cdot\vec{i}\vec{k} + \vec{i}\vec{i}\cdot\vec{k}\vec{i} + \vec{k}\vec{i}\cdot\vec{j}\vec{j} + \vec{k}\vec{i}\cdot\vec{i}\vec{k} + \vec{k}\vec{i}\cdot\vec{k}\vec{i} + \vec{k}\vec{k}\cdot\vec{j}\vec{j} + \vec{k}\vec{k}\cdot\vec{i}\vec{k} + \vec{k}\vec{k}\cdot\vec{k}\vec{i} \quad \Leftrightarrow$

(III) $\vec{\vec{S}}\cdot\vec{\vec{T}} = \vec{i}\,0\vec{j} + \vec{i}\,1\vec{k} + \vec{i}\,0\vec{i} + \vec{k}\,0\vec{j} + \vec{k}\,1\vec{k} + \vec{k}\,0\vec{i} + \vec{k}\,0\vec{j} + \vec{k}\,0\vec{k} + \vec{k}\,1\vec{i} \quad \Leftrightarrow$

(IV) $\vec{\vec{S}}\cdot\vec{\vec{T}} = \vec{i}\vec{k} + \vec{k}\vec{k} + \vec{k}\vec{i}$.

Name________________________ Student number________________

Assignment 4 (4p)

Consider the curved beam rigidity problem on page 1-4 to 1-8 of the lecture notes. Use the curved beam equations on page 1-6 with hand calculations or Mathematica to find the analytical solution to the vertical displacement v as the function of mass m used as loading. Use the expression to deduce the coefficient a of

$$\frac{mgR^2}{EI} = a\frac{v}{R}.$$

In the expression, g is the acceleration by gravity, R is the radius , I the second moment of cross section with respect to the area centroid, and E is the Young's modulus of the rim material. The specific form above is based on dimension analysis and additional assumptions of linearity and vanishing displacement without external loading.

Solution

The first order ordinary differential equations for the curved Bernoulli beam, which is inextensible in the axial direction, are given by ($L = 3\pi R/2$)

$$\frac{dN}{dx} - \frac{1}{R}Q = 0, \quad \frac{dQ}{dx} + \frac{1}{R}N = 0, \quad \frac{dM}{dx} + Q = 0 \quad x \in [0, L] \text{ and } N = 0, \quad Q + F = 0, \quad M = 0 \quad x = L,$$

$$M = EI\frac{d\psi}{dx}, \quad \frac{du}{dx} - \frac{1}{R}v = 0, \quad \frac{dv}{dx} + \frac{1}{R}u - \psi = 0 \quad x \in [0, L] \text{ and } u = 0, \quad v = 0, \quad \psi = 0 \quad x = 0.$$

In a statically determined case, it is possible to first consider the equations for the stress resultants and those for displacements and rotation after that. Let us start with the equilibrium equations and boundary conditions at the free edge. Elimination of Q from the first two connected differential equations gives (notice that elimination gives a second order equation and the missing boundary condition is given by the first of the original equations)

$$\frac{d^2N}{dx^2} + \frac{1}{R^2}N = 0 \quad x \in [0, L], \quad N = 0 \text{ and } \frac{dN}{dx} + \frac{1}{R}F = 0 \quad x = L \quad \Rightarrow \quad \frac{N}{F} = -\cos\left(\frac{x}{R}\right).$$

Then, using the first differential equation with the known solution to N

$$\frac{dN}{dx} - \frac{1}{R}Q = 0 \quad \Rightarrow \quad \frac{Q}{F} = \sin\left(\frac{x}{R}\right).$$

Finally, considering the third differential equation with the known solution to Q and the third boundary condition

$$\frac{dM}{dx}+Q=0 \quad x\in[0,L], \quad M=0 \quad x=L \quad \Rightarrow \quad \frac{M}{FR}=\cos(\frac{x}{R}).$$

When the stress resultants are known, one may consider the constitutive equation, Bernoulli constraint and the inextensibility constraint (the remaining differential equations) as first order ordinary differential equations for the displacement and rotation components. Let us start with the first of the equations with the known expression for the moment resultant:

$$EI\frac{d\psi}{dx}=FR\cos(\frac{x}{R}) \quad x\in]0,L] \quad \text{and} \quad \psi=0 \quad x=0 \quad \Rightarrow \quad \psi=\frac{FR^2}{EI}\sin(\frac{x}{R}).$$

As the remaining differential equations (constraints) are connected, one needs to eliminate either u or v. Let us eliminate u and substitute the known solution to rotation to get

$$\frac{d^2v}{dx^2}+\frac{1}{R^2}v=\frac{FR}{EI}\cos(\frac{x}{R}) \quad x\in]0,L] \quad \text{and} \quad v=0, \quad \frac{dv}{dx}=0 \quad x=0 \quad \Rightarrow \quad \frac{v}{R}=\frac{1}{2}\frac{FR^2}{EI}\frac{x}{R}\sin(\frac{x}{R}).$$

Solution to the differential equation is composed of the generic solution to the homogeneous equations and a particular solution. Solution to the axial displacement components does not matter here. Comparing the solution to the transverse component v with the expression by dimension analysis gives (notice that the solution to transverse displacement is positive upwards whereas that in the dimension analysis formula is positive downwards)

$$\frac{v}{R}=\frac{3}{4}\pi\frac{FR^2}{EI} \quad \Rightarrow \quad \frac{FR^2}{EI}=\frac{4}{3\pi}\frac{v}{R} \quad \text{so} \quad a=\frac{4}{3\pi}.$$

Name________________________ Student number________________

Assignment 5 (4p)

Consider the curved beam rigidity problem on pages 1-4 to 1-8 of the lecture notes. Measure the displacement v of the loading point as the function of mass m used as loading. Thereafter, use the mass-displacement data to find the coefficient of relationship

$$\frac{mgR^2}{EI} = a\frac{v}{R}.$$

The values of the geometrical and material parameters are $E = 70\,\text{GPa}$, $R = 306\,\text{mm}$, $I = 3011\,\text{mm}^4$ and $g = 9.81\,\text{m/s}^2$.

Experiment: The set-up is located in Puumiehenkuja 5L (Konemiehentie side of the building). The hall is open 9-12 and 14-16 on Fri 01.03.2024. Place a mass on the loading tray and record the displacement shown on the laptop display. Gather enough mass-displacement data to find the coefficient a reliably. For example, you may repeat a measurement with certain loading several times to reduce the effect of random error by averaging etc. You may also consider different loading sequences (like increasing and decreasing the mass) to minimize the possible friction effects in the set-up.

Solution

Let us use notations $\underline{m} = mgR^2/(EI)$ and $\underline{v} = v/R$ for the dimensionless mass and displacement, respectively. To find the coefficient a of relationship, one may use the least-squares method giving the value of a as the minimizer of function

$$\Pi(a) = \frac{1}{2}\sum (a\underline{u}_i - \underline{m}_i)^2,$$

where the sum is over all the measured mass-displacement values. The method looks for a parameter value giving as good as possible overall match to the data. For a perfect fit with the best value of a, function $\Pi = 0$ so the value of Π at the minimum point is a measure of the quality of the fit.

At the minimum point, derivative of $\Pi(a)$ vanishes. Therefore, one obtains

$$\frac{d\Pi(a)}{da} = \sum \underline{u}_i(a\underline{u}_i - \underline{m}_i) = 0 \quad \Rightarrow \quad a = \frac{\sum \underline{u}_i\underline{m}_i}{\sum \underline{u}_i\underline{u}_i}.$$

From this point on, it is convenient to use Mathematica, Matlab, Excel or some other computational tool. The first step is to transform the measured data into dimensionless form using the definitions and the given values $E = 70\,\text{GPa}$, $R = 306\,\text{mm}$, $I = 3011\,\text{mm}^4$, and $g = 9.81\,\text{m/s}^2$

m [kg]	v [mm]	$100\, mgR^2 / (EI)$	$100\, v / R$
0	0.00	0.00	0.00
0.7	1.89	0.305	0.617
1.2	3.49	0.523	1.140
1.5	5.12	0.634	1.673
2	6.20	0.872	2.026
2.7	8.32	1.177	2.719
3.0	9.74	1.307	3.184

With the data in the table $a = \dfrac{\sum \underline{u}_i \underline{m}_i}{\sum \underline{u}_i \underline{u}_i} = 0.421$.

MEC-E8003

Beam, Plate and Shell Models

2024

WEEK 10: KINEMATICS

2 KINEMATICS

LEARNING OUTCOMES

Students are able to solve the weekly lecture problems, home problems, and exercise problems on kinematics:

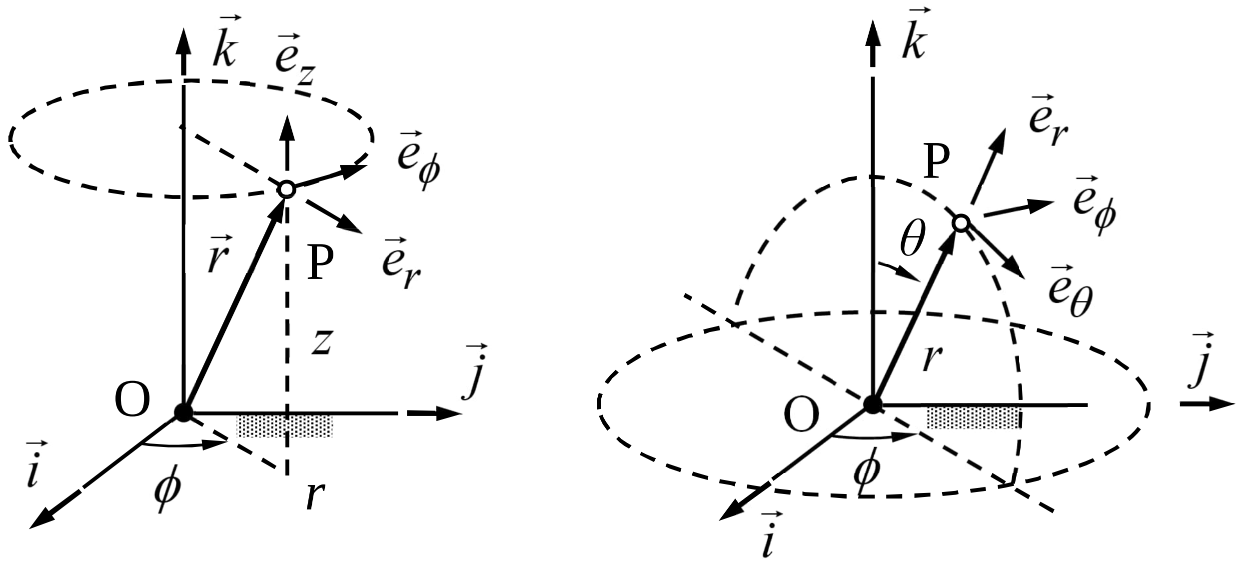
- ☐ Material coordinate system. Vectors, basis vector derivatives, and gradient operator in the polar, cylindrical, and spherical material coordinate systems.

- ☐ Basis vectors, basis vector derivatives, and gradient operator in the beam and shell material coordinate systems.

- ☐ Curvature of curves and surfaces.

2.1 COORDINATE SYSTEMS

In solid mechanics, particles of a body (a closed system of particles) are identified by coordinates of the initial geometry. Equilibrium equations etc. can be written for any selection of the material coordinates, but a clever selection may simplify the setting.



A Cartesian (x, y, z) coordinate system with known derivatives of the basis vector, gradient operator etc. is always needed as a reference system.

CURVILINEAR MATERIAL COORDINATE SYSTEM

Position vector: $\vec{r}(\alpha,\beta,\gamma) = \begin{Bmatrix} x(\alpha,\beta,\gamma) \\ y(\alpha,\beta,\gamma) \\ z(\alpha,\beta,\gamma) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}, \qquad \begin{Bmatrix} \vec{h}_\alpha \\ \vec{h}_\beta \\ \vec{h}_\gamma \end{Bmatrix} = \begin{Bmatrix} \partial\vec{r}/\partial\alpha \\ \partial\vec{r}/\partial\beta \\ \partial\vec{r}/\partial\gamma \end{Bmatrix} = \left[H\right] \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}$

Basis vectors: $\begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix} = \begin{Bmatrix} \vec{h}_\alpha / h_\alpha \\ \vec{h}_\beta / h_\beta \\ \vec{h}_\gamma / h_\gamma \end{Bmatrix} = \left[F\right] \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}, \qquad \frac{\partial}{\partial\eta} \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix} = (\frac{\partial}{\partial\eta}\left[F\right])\left[F\right]^{-1} \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix}$

Gradient: $\nabla = \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix}^{\mathrm{T}} \left[F\right]^{-\mathrm{T}} \left[H\right]^{-1} \begin{Bmatrix} \partial/\partial\alpha \\ \partial/\partial\beta \\ \partial/\partial\gamma \end{Bmatrix},$

BASIS VECTORS

The basis vectors of a Cartesian coordinate system are constants. Starting with the position vector $\vec{r}(\alpha,\beta,\gamma) = x(\alpha,\beta,\gamma)\vec{i} + y(\alpha,\beta,\gamma)\vec{j} + z(\alpha,\beta,\gamma)\vec{k}$ of particle (α,β,γ) and using definitions $\vec{h}_\alpha = \partial\vec{r} / \partial\alpha$, $\vec{h}_\beta = \partial\vec{r} / \partial\beta$, $\vec{h}_\gamma = \partial\vec{r} / \partial\gamma$ (h_α, h_β, h_γ are the lengths or, later, the scaling coefficients)

Basis vectors: $\begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix} = \begin{Bmatrix} \vec{h}_\alpha / h_\alpha \\ \vec{h}_\beta / h_\beta \\ \vec{h}_\gamma / h_\gamma \end{Bmatrix} = [F] \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} \quad \Leftrightarrow \quad \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = [F]^{-1} \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix}.$

Basis vector derivatives: $\dfrac{\partial}{\partial\eta} \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix} = (\dfrac{\partial}{\partial\eta}[F])[F]^{-1} \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix} \quad \eta \in \{\alpha,\beta,\gamma\}$

The starting point is the position vector of a material point in the reference system $\vec{r}(\alpha,\beta,\gamma) = x(\alpha,\beta,\gamma)\vec{i} + y(\alpha,\beta,\gamma)\vec{j} + z(\alpha,\beta,\gamma)\vec{k}$ expressed in terms of (α,β,γ) identifying the particles. Basis vectors of the curvilinear $(\alpha,\beta,\gamma)-$ system

$$\begin{Bmatrix} \vec{h}_\alpha \\ \vec{h}_\beta \\ \vec{h}_\gamma \end{Bmatrix} = \begin{Bmatrix} \partial\vec{r}/\partial\alpha \\ \partial\vec{r}/\partial\beta \\ \partial\vec{r}/\partial\gamma \end{Bmatrix} = \left[H\right]\begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} \text{ and } \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix} = \begin{Bmatrix} \vec{h}_\alpha / h_\alpha \\ \vec{h}_\beta / h_\beta \\ \vec{h}_\gamma / h_\gamma \end{Bmatrix} = \left[F\right]\begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} \Leftrightarrow \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = \left[F\right]^{-1}\begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix}$$

where $h_\alpha = \left|\vec{h}_\alpha\right| \equiv \sqrt{\vec{h}_\alpha \cdot \vec{h}_\alpha}$, $h_\beta = \left|\vec{h}_\beta\right|$, and $h_\gamma = \left|\vec{h}_\gamma\right|$ are the scaling coefficients.

As basis vectors of the Cartesian reference coordinate system are constants, the derivatives of the basis vectors of the curvilinear (α,β,γ) coordinate system with respect to $\eta \in \{\alpha,\beta,\gamma\}$ become

$$\frac{\partial}{\partial\eta}\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_\gamma\end{Bmatrix}=\frac{\partial}{\partial\eta}\left[F\right]\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}=\left(\frac{\partial}{\partial\eta}\left[F\right]\right)\left[F\right]^{-1}\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_\gamma\end{Bmatrix}. \quad \Leftarrow$$

In the last form, the relationship between the basis vectors is used the other way around to have the derivatives in the basis of the curvilinear system.

GRADIENT OPERATOR

As a vector, gradient is invariant with respect to the coordinate system. Selection of the material coordinates and the related basis vectors affect, however, the representation

$$(x,y,z): \quad \nabla = \begin{Bmatrix} \vec{e}_x \\ \vec{e}_y \\ \vec{e}_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \partial / \partial x \\ \partial / \partial y \\ \partial / \partial z \end{Bmatrix} = \vec{e}_x \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y} + \vec{e}_z \frac{\partial}{\partial z} \ ,$$

$$(\alpha,\beta,\gamma): \ \nabla = \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix}^{\mathrm{T}} ([H][F]^{\mathrm{T}})^{-1} \begin{Bmatrix} \partial / \partial \alpha \\ \partial / \partial \beta \\ \partial / \partial \gamma \end{Bmatrix} \text{ where } [H] = \begin{bmatrix} \partial x / \partial \alpha & \partial y / \partial \alpha & \partial z / \partial \alpha \\ \partial x / \partial \beta & \partial y / \partial \beta & \partial z / \partial \beta \\ \partial x / \partial \gamma & \partial y / \partial \gamma & \partial z / \partial \gamma \end{bmatrix}.$$

Notice that $[F]$ and $[H]$ differ only in the scaling of the rows!

Using the chain rule, the relationships between coordinates and basis vectors and the (coordinate system) invariance of the gradient operator (it is a vector)

$$\begin{Bmatrix} \partial/\partial\alpha \\ \partial/\partial\beta \\ \partial/\partial\gamma \end{Bmatrix} = \begin{bmatrix} \partial x/\partial\alpha & \partial y/\partial\alpha & \partial z/\partial\alpha \\ \partial x/\partial\beta & \partial y/\partial\beta & \partial z/\partial\beta \\ \partial x/\partial\gamma & \partial y/\partial\gamma & \partial z/\partial\gamma \end{bmatrix} \begin{Bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{Bmatrix} = [H] \begin{Bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{Bmatrix} \Rightarrow$$

$$\nabla = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{Bmatrix} = \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix}^{\mathrm{T}} [F]^{-\mathrm{T}} [H]^{-1} \begin{Bmatrix} \partial/\partial\alpha \\ \partial/\partial\beta \\ \partial/\partial\gamma \end{Bmatrix} \Leftrightarrow$$

$$\nabla = \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix}^{\mathrm{T}} [F]^{-\mathrm{T}} [H]^{-1} \begin{Bmatrix} \partial/\partial\alpha \\ \partial/\partial\beta \\ \partial/\partial\gamma \end{Bmatrix} = \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix}^{\mathrm{T}} ([H][F]^{\mathrm{T}})^{-1} \begin{Bmatrix} \partial/\partial\alpha \\ \partial/\partial\beta \\ \partial/\partial\gamma \end{Bmatrix}. \Leftarrow$$

Matrices $[H]$ and $[F]$ differ only in scaling of the rows. Denoting the diagonal matrix of the scaling coefficient by $[h] = diag\{h_\alpha, h_\beta, h_\gamma\}$ it holds $[h][F] = [H]$. Further, in an orthonormal coordinate system $[F]^{-1} = [F]^{\mathrm{T}}$ so

$$\nabla = \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix}^{\mathrm{T}} ([H][F]^{\mathrm{T}})^{-1} \begin{Bmatrix} \partial / \partial\alpha \\ \partial / \partial\beta \\ \partial / \partial\gamma \end{Bmatrix} = \vec{e}_\alpha \frac{1}{h_\alpha}\frac{\partial}{\partial\alpha} + \vec{e}_\beta \frac{1}{h_\beta}\frac{\partial}{\partial\beta} + \vec{e}_\gamma \frac{1}{h_\gamma}\frac{\partial}{\partial\gamma} \; .$$

Therefore, it is enough to know the scaling coefficients.

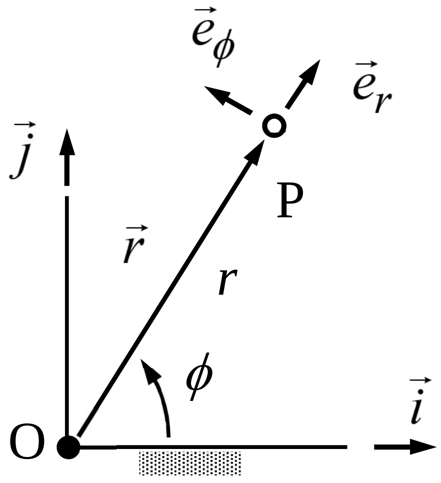
POLAR COORDINATES (r,ϕ)

In a curvilinear rectangular Polar coordinate system, a particle is identified by its distance r from the origin and angle ϕ from a chosen line. Basis vectors, their derivatives, and the gradient operator are given by mapping $\vec{r}(r,\phi) = r\cos\phi\vec{i} + r\sin\phi\vec{j}$:

$$\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} = [F] \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix},$$

$$\frac{\partial}{\partial\phi} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} = \begin{Bmatrix} \vec{e}_\phi \\ -\vec{e}_r \end{Bmatrix} \quad \text{(otherwise zeros)},$$

$$\nabla = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} ([H][F]^{\mathrm{T}})^{-1} \begin{Bmatrix} \partial/\partial r \\ \partial/\partial\phi \end{Bmatrix} = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial\phi}.$$



The derivatives follow from the generic expression or in a more clear manner from steps (just to emphasize the idea)

$$\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} = [F] \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} \Leftrightarrow \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} = [F]^{-1} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} \Rightarrow$$

$$\frac{\partial}{\partial\phi} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} = (\frac{\partial}{\partial\phi} \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}) \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} = \begin{bmatrix} -\sin\phi & \cos\phi \\ -\cos\phi & -\sin\phi \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} \quad \Rightarrow$$

$$\frac{\partial}{\partial\phi} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} = \begin{bmatrix} -\sin\phi & \cos\phi \\ -\cos\phi & -\sin\phi \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} = \begin{Bmatrix} \vec{e}_\phi \\ -\vec{e}_r \end{Bmatrix}. \quad \Leftarrow$$

In writing the gradient expression, one needs the relationships between basis and partial derivatives in a Cartesian and polar coordinate systems:

$$\begin{Bmatrix} \partial/\partial r \\ \partial/\partial\phi \end{Bmatrix} = \begin{bmatrix} \partial x/\partial r & \partial y/\partial r \\ \partial x/\partial\phi & \partial y/\partial\phi \end{bmatrix} \begin{Bmatrix} \partial/\partial x \\ \partial/\partial y \end{Bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -r\sin\phi & r\cos\phi \end{bmatrix} \begin{Bmatrix} \partial/\partial x \\ \partial/\partial y \end{Bmatrix} = [H] \begin{Bmatrix} \partial/\partial x \\ \partial/\partial y \end{Bmatrix}.$$

Using the vector (operator) invariance with respect to the coordinate system

$$\nabla = \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \partial/\partial x \\ \partial/\partial y \end{Bmatrix} = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} ([H][F]^{\mathrm{T}})^{-1} \begin{Bmatrix} \partial/\partial r \\ \partial/\partial\phi \end{Bmatrix} \Rightarrow$$

$$\nabla = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} \left(\begin{bmatrix} \cos\phi & \sin\phi \\ -r\sin\phi & r\cos\phi \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \right)^{-1} \begin{Bmatrix} \partial/\partial r \\ \partial/\partial\phi \end{Bmatrix} \Leftrightarrow$$

$$\nabla = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & 0 \\ 0 & 1/r \end{bmatrix} \begin{Bmatrix} \partial/\partial r \\ \partial/\partial\phi \end{Bmatrix} = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial\phi}. \quad \leftarrow$$

EXAMPLE Derive the component forms of the balance law $\nabla\cdot\vec{\vec{\sigma}}+\vec{f}=0$ in the polar coordinate system when stress and distributed force

$$\vec{\vec{\sigma}}=\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}\sigma_{rr} & \sigma_{r\phi}\\ \sigma_{\phi r} & \sigma_{\phi\phi}\end{bmatrix}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix} \text{ and } \vec{f}=\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}f_r\\ f_\phi\end{Bmatrix},$$

respectively. Derivatives of the basis vectors and the gradient operator of the polar coordinate system are

$$\frac{\partial}{\partial\phi}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix}=\begin{Bmatrix}\vec{e}_\phi\\ -\vec{e}_r\end{Bmatrix} \text{ and } \nabla=\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\partial/\partial r\\ \partial/(r\partial\phi)\end{Bmatrix}=\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\phi\frac{1}{r}\frac{\partial}{\partial\phi}.$$

Answer $\frac{1}{r}[\frac{\partial(r\sigma_{rr})}{\partial r}+\frac{\partial\sigma_{\phi r}}{\partial\phi}-\sigma_{\phi\phi}]+f_r=0$ and $\frac{1}{r}[\frac{\partial(r\sigma_{r\phi})}{\partial r}+\frac{\partial\sigma_{\phi\phi}}{\partial\phi}+\sigma_{\phi r}]+f_\phi=0$

In polar coordinate system basis vectors depend on the angular coordinate. First, let us expand the stress divergence and consider the terms one-by-one by keeping the order of the basis vectors and position of the inner product:

$$\nabla\cdot\vec{\vec{\sigma}} = (\vec{e}_r\frac{\partial}{\partial r} + \vec{e}_\phi\frac{1}{r}\frac{\partial}{\partial\phi})\cdot(\sigma_{rr}\vec{e}_r\vec{e}_r + \sigma_{r\phi}\vec{e}_r\vec{e}_\phi + \sigma_{\phi r}\vec{e}_\phi\vec{e}_r + \sigma_{\phi\phi}\vec{e}_\phi\vec{e}_\phi) \quad \Leftrightarrow$$

$$\nabla\cdot\vec{\vec{\sigma}} = \vec{e}_r\cdot\frac{\partial}{\partial r}\sigma_{rr}\vec{e}_r\vec{e}_r + \vec{e}_r\cdot\frac{\partial}{\partial r}\sigma_{r\phi}\vec{e}_r\vec{e}_\phi + \vec{e}_r\cdot\frac{\partial}{\partial r}\sigma_{\phi r}\vec{e}_\phi\vec{e}_r + \vec{e}_r\cdot\frac{\partial}{\partial r}\sigma_{\phi\phi}\vec{e}_\phi\vec{e}_\phi +$$

$$\vec{e}_\phi\cdot\frac{1}{r}\frac{\partial}{\partial\phi}\sigma_{rr}\vec{e}_r\vec{e}_r + \vec{e}_\phi\cdot\frac{1}{r}\frac{\partial}{\partial\phi}\sigma_{r\phi}\vec{e}_r\vec{e}_\phi + \vec{e}_\phi\cdot\frac{1}{r}\frac{\partial}{\partial\phi}\sigma_{\phi r}\vec{e}_\phi\vec{e}_r + \vec{e}_\phi\cdot\frac{1}{r}\frac{\partial}{\partial\phi}\sigma_{\phi\phi}\vec{e}_\phi\vec{e}_\phi\ .$$

Next, by considering the terms one-by-one by keeping the order of the basis vectors, position of the inner product, and taking into account the non-zero derivatives $\partial\vec{e}_r / \partial\phi = \vec{e}_\phi$ and $\partial\vec{e}_\phi / \partial\phi = -\vec{e}_r$,

$$\vec{e}_r \cdot \frac{\partial}{\partial r}\sigma_{rr}\vec{e}_r\vec{e}_r = \frac{\partial \sigma_{rr}}{\partial r}(\vec{e}_r \cdot \vec{e}_r)\vec{e}_r = \frac{\partial \sigma_{rr}}{\partial r}\vec{e}_r, \qquad \text{(basis vectors are orthonormal)}$$

$$\vec{e}_r \cdot \frac{\partial}{\partial r}\sigma_{r\phi}\vec{e}_r\vec{e}_\phi = \frac{\partial \sigma_{r\phi}}{\partial r}(\vec{e}_r \cdot \vec{e}_r)\vec{e}_\phi = \frac{\partial \sigma_{r\phi}}{\partial r}\vec{e}_\phi,$$

$$\vec{e}_r \cdot \frac{\partial}{\partial r}\sigma_{\phi r}\vec{e}_\phi\vec{e}_r = \frac{\partial \sigma_{\phi r}}{\partial r}(\vec{e}_r \cdot \vec{e}_\phi)\vec{e}_r = 0,$$

$$\vec{e}_r \cdot \frac{\partial}{\partial r}\sigma_{\phi\phi}\vec{e}_\phi\vec{e}_\phi = \frac{\partial \sigma_{\phi\phi}}{\partial r}(\vec{e}_r \cdot \vec{e}_\phi)\vec{e}_\phi = 0,$$

$$\vec{e}_\phi \cdot \frac{1}{r}\frac{\partial}{\partial \phi}\sigma_{rr}\vec{e}_r\vec{e}_r = \frac{1}{r}\frac{\partial \sigma_{rr}}{\partial \phi}(\vec{e}_\phi \cdot \vec{e}_r)\vec{e}_r + \frac{1}{r}\sigma_{rr}(\vec{e}_\phi \cdot \frac{\partial \vec{e}_r}{\partial \phi})\vec{e}_r + \frac{1}{r}\sigma_{rr}(\vec{e}_\phi \cdot \vec{e}_r)\frac{\partial \vec{e}_r}{\partial \phi} = \frac{1}{r}\sigma_{rr}\vec{e}_r,$$

$$\vec{e}_\phi \cdot \frac{1}{r}\frac{\partial}{\partial \phi}\sigma_{r\phi}\vec{e}_r\vec{e}_\phi = \frac{1}{r}\frac{\partial \sigma_{r\phi}}{\partial \phi}(\vec{e}_\phi \cdot \vec{e}_r)\vec{e}_\phi + \frac{1}{r}\sigma_{r\phi}(\vec{e}_\phi \cdot \frac{\partial \vec{e}_r}{\partial \phi})\vec{e}_\phi + \frac{1}{r}\sigma_{r\phi}(\vec{e}_\phi \cdot \vec{e}_r)\frac{\partial \vec{e}_\phi}{\partial \phi} = \frac{1}{r}\sigma_{r\phi}\vec{e}_\phi,$$

$$\vec{e}_\phi \cdot \frac{1}{r}\frac{\partial}{\partial\phi}\sigma_{\phi r}\vec{e}_\phi\vec{e}_r = \frac{1}{r}\frac{\partial\sigma_{\phi r}}{\partial\phi}(\vec{e}_\phi \cdot \vec{e}_\phi)\vec{e}_r + \frac{1}{r}\sigma_{\phi r}(\vec{e}_\phi \cdot \frac{\partial\vec{e}_\phi}{\partial\phi})\vec{e}_r + \frac{1}{r}\sigma_{\phi r}(\vec{e}_\phi \cdot \vec{e}_\phi)\frac{\partial\vec{e}_r}{\partial\phi}$$

$$= \frac{1}{r}\frac{\partial\sigma_{\phi r}}{\partial\phi}\vec{e}_r + \frac{1}{r}\sigma_{\phi r}\vec{e}_\phi,$$

$$\vec{e}_\phi \cdot \frac{\partial}{r\partial\phi}\sigma_{\phi\phi}\vec{e}_\phi\vec{e}_\phi = \frac{1}{r}\frac{\partial\sigma_{\phi\phi}}{\partial\phi}(\vec{e}_\phi \cdot \vec{e}_\phi)\vec{e}_\phi + \frac{1}{r}\sigma_{\phi\phi}(\vec{e}_\phi \cdot \frac{\partial\vec{e}_\phi}{\partial\phi})\vec{e}_\phi + \frac{1}{r}\sigma_{\phi\phi}(\vec{e}_\phi \cdot \vec{e}_\phi)\frac{\partial\vec{e}_\phi}{\partial\phi}$$

$$= \frac{1}{r}\frac{\partial\sigma_{\phi\phi}}{\partial\phi}\vec{e}_\phi - \frac{1}{r}\sigma_{\phi\phi}\vec{e}_r,$$

Finally, by combining the terms

$$\nabla \cdot \vec{\vec{\sigma}} = \frac{\partial\sigma_{rr}}{\partial r}\vec{e}_r + \frac{\partial\sigma_{r\phi}}{\partial r}\vec{e}_\phi + \frac{1}{r}\sigma_{rr}\vec{e}_r + \frac{1}{r}\sigma_{r\phi}\vec{e}_\phi + \frac{1}{r}\frac{\partial\sigma_{\phi r}}{\partial\phi}\vec{e}_r + \frac{1}{r}\sigma_{\phi r}\vec{e}_\phi + \frac{1}{r}\frac{\partial\sigma_{\phi\phi}}{\partial\phi}\vec{e}_\phi - \frac{1}{r}\sigma_{\phi\phi}\vec{e}_r$$

$$\nabla\cdot\vec{\vec{\sigma}}=\frac{1}{r}(r\frac{\partial\sigma_{rr}}{\partial r}+\sigma_{rr}+\frac{\partial\sigma_{\phi r}}{\partial\phi}-\sigma_{\phi\phi})\vec{e}_r+\frac{1}{r}(r\frac{\partial\sigma_{r\phi}}{\partial r}+\sigma_{r\phi}+\sigma_{\phi r}+\frac{\partial\sigma_{\phi\phi}}{\partial\phi})\vec{e}_\phi \quad \Leftrightarrow$$

$$\nabla\cdot\vec{\vec{\sigma}}=\frac{1}{r}[\frac{\partial(r\sigma_{rr})}{\partial r}+\frac{\partial\sigma_{\phi r}}{\partial\phi}-\sigma_{\phi\phi}]\vec{e}_r+\frac{1}{r}[\frac{\partial(r\sigma_{r\phi})}{\partial r}+\frac{\partial\sigma_{\phi\phi}}{\partial\phi}+\sigma_{\phi r}]\vec{e}_\phi .$$

With the distributed force $\vec{f}=f_r\vec{e}_r+f_\phi\vec{e}_\phi$, the local form of the momentum balance $\nabla\cdot\vec{\vec{\sigma}}+\vec{f}=0$ in the polar coordinate system

$$\frac{1}{r}[\frac{\partial(r\sigma_{rr})}{\partial r}+\frac{\partial\sigma_{\phi r}}{\partial\phi}-\sigma_{\phi\phi}+rf_r]\vec{e}_r+\frac{1}{r}[\frac{\partial(r\sigma_{r\phi})}{\partial r}+\frac{\partial\sigma_{\phi\phi}}{\partial\phi}+\sigma_{\phi r}+rf_\phi]\vec{e}_\phi=0.$$

EXAMPLE The small strain measure is obtained as the symmetric part of displacement gradient (a tensor then). Use the definition to find the components of the strain tensor (a) in Cartesian coordinate system and (b) in the polar system.

Answer (a) $\varepsilon_{xx} = \dfrac{\partial u_x}{\partial x}$, $\varepsilon_{yy} = \dfrac{\partial u_y}{\partial y}$, and $\varepsilon_{xy} = \varepsilon_{yx} = \dfrac{1}{2}(\dfrac{\partial u_y}{\partial x} + \dfrac{\partial u_x}{\partial y})$

(b) $\varepsilon_{rr} = \dfrac{\partial u_r}{\partial r}$, $\varepsilon_{\phi\phi} = \dfrac{1}{r}(\dfrac{\partial u_\phi}{\partial \phi} + u_r)$, and $\varepsilon_{r\phi} = \varepsilon_{\phi r} = \dfrac{1}{2}(\dfrac{1}{r}\dfrac{\partial u_r}{\partial \phi} - \dfrac{u_\phi}{r} + \dfrac{\partial u_\phi}{\partial r})$

In Cartesian system, $\nabla = \vec{i}\partial/\partial x + \vec{j}\partial/\partial y$ and $\vec{u} = u_x\vec{i} + u_y\vec{j}$, therefore

$$\nabla\vec{u} = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y})(u_x\vec{i} + u_y\vec{j}) = \vec{i}\vec{i}\frac{\partial u_x}{\partial x} + \vec{i}\vec{j}\frac{\partial u_y}{\partial x} + \vec{j}\vec{i}\frac{\partial u_x}{\partial y} + \vec{j}\vec{j}\frac{\partial u_y}{\partial y}) \;\Rightarrow$$

$$(\nabla\vec{u})_{\mathrm{c}} = \vec{i}\vec{i}\frac{\partial u_x}{\partial x} + \vec{j}\vec{i}\frac{\partial u_y}{\partial x} + \vec{i}\vec{j}\frac{\partial u_x}{\partial y} + \vec{j}\vec{j}\frac{\partial u_y}{\partial y}$$

giving

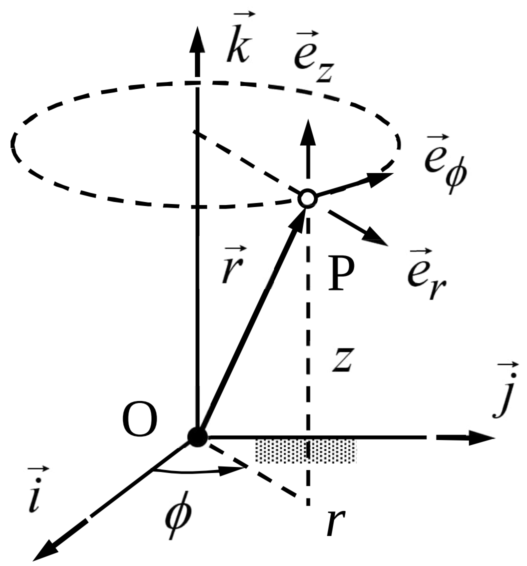
$$\vec{\vec{\varepsilon}} = \frac{1}{2}[\nabla\vec{u} + (\nabla\vec{u})_{\mathrm{c}}] = \vec{i}\vec{i}\frac{\partial u_x}{\partial x} + \vec{j}\vec{j}\frac{\partial u_y}{\partial y} + \vec{i}\vec{j}\frac{1}{2}(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}) + \vec{j}\vec{i}\frac{1}{2}(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}).$$

In polar coordinates $\vec{u} = u_r\vec{e}_r + u_\phi\vec{e}_\phi$, and $\nabla = \vec{e}_r\partial/\partial r + \vec{e}_\phi\partial/(r\partial\phi)$, $\partial\vec{e}_r/\partial\phi = \vec{e}_\phi$ and $\partial\vec{e}_\phi/\partial\phi = -\vec{e}_r$. Otherwise, calculation follows the steps used with the Cartesian coordinate system (one of the exercise problems).

CYLINDRICAL COORDINATES (r,ϕ,z)

A particle is identified by its distance r from the z-axis origin, angle ϕ from the x-axis and distance z from the xy-plane. Mapping $\vec{r} = r\cos\phi\vec{i} + r\sin\phi\vec{j} + z\vec{k}$ gives

$$\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix},$$



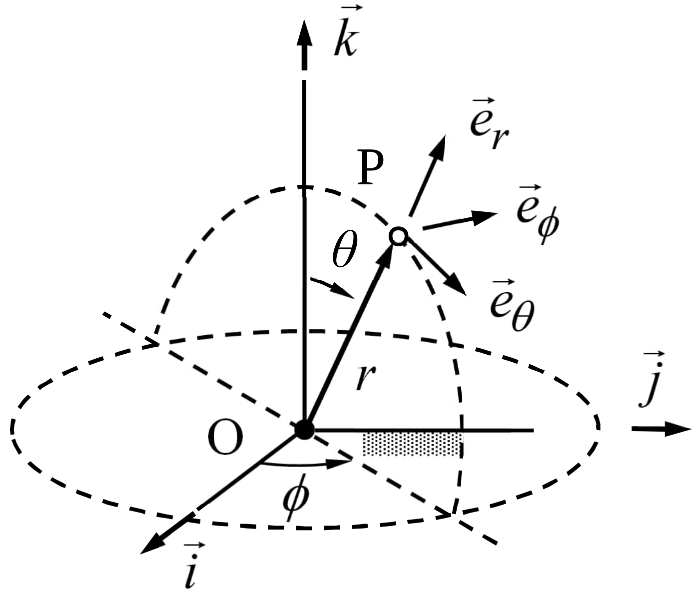
$$\frac{\partial}{\partial\phi}\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix} \text{ otherwise zeros,}$$

$$\nabla = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial\phi} + \vec{e}_z \frac{\partial}{\partial z}.$$

SPHERICAL COORDINATES (θ,ϕ,r)

A particle is identified by its distance r, angle ϕ from the x-axis, and angle θ from the z-axis. Mapping $\vec{r}(\theta,\phi,r)=r(\sin\theta\cos\phi\vec{i}+\sin\theta\sin\phi\vec{j}+\cos\theta\vec{k})$, gives

$$\begin{Bmatrix}\vec{e}_\theta\\ \vec{e}_\phi\\ \vec{e}_r\end{Bmatrix}=\begin{bmatrix}\cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta\\ -\sin\phi & \cos\phi & 0\\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix},$$



$$\frac{\partial}{\partial\phi}\begin{Bmatrix}\vec{e}_\theta\\ \vec{e}_\phi\\ \vec{e}_r\end{Bmatrix}=\begin{Bmatrix}\cos\theta\vec{e}_\phi\\ -\sin\theta\vec{e}_r-\cos\theta\vec{e}_\theta\\ \sin\theta\vec{e}_\phi\end{Bmatrix},\ \frac{\partial}{\partial\theta}\begin{Bmatrix}\vec{e}_\theta\\ \vec{e}_\phi\\ \vec{e}_r\end{Bmatrix}=\begin{Bmatrix}-\vec{e}_r\\ 0\\ \vec{e}_\theta\end{Bmatrix},$$

$$\nabla=\vec{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}+\vec{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}+\vec{e}_r\frac{\partial}{\partial r}.$$

According to the generic recipe (here $\mathrm{c} \sim \cos$ and $\mathrm{s} \sim \sin$)

$$\frac{\partial}{\partial r}\begin{Bmatrix} \vec{e}_\theta \\ \vec{e}_\phi \\ \vec{e}_r \end{Bmatrix} = (\frac{\partial}{\partial r}[F])[F]^{-1}\begin{Bmatrix} \vec{e}_\theta \\ \vec{e}_\phi \\ \vec{e}_r \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix},$$

$$\frac{\partial}{\partial \phi}\begin{Bmatrix} \vec{e}_\theta \\ \vec{e}_\phi \\ \vec{e}_r \end{Bmatrix} = (\frac{\partial}{\partial \phi}[F])[F]^{-1}\begin{Bmatrix} \vec{e}_\theta \\ \vec{e}_\phi \\ \vec{e}_r \end{Bmatrix} = \begin{bmatrix} 0 & \mathrm{c}\theta & 0 \\ -\mathrm{c}\theta & 0 & -\mathrm{s}\theta \\ 0 & \mathrm{s}\theta & 0 \end{bmatrix}\begin{Bmatrix} \vec{e}_\theta \\ \vec{e}_\phi \\ \vec{e}_r \end{Bmatrix} = \begin{Bmatrix} \mathrm{c}\theta\,\vec{e}_\phi \\ -\mathrm{s}\theta\vec{e}_r - \mathrm{c}\theta\vec{e}_\theta \\ \mathrm{s}\theta\vec{e}_\phi \end{Bmatrix},$$

$$\frac{\partial}{\partial \theta}\begin{Bmatrix} \vec{e}_\theta \\ \vec{e}_\phi \\ \vec{e}_r \end{Bmatrix} = (\frac{\partial}{\partial \theta}[F])[F]^{-1}\begin{Bmatrix} \vec{e}_\theta \\ \vec{e}_\phi \\ \vec{e}_r \end{Bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \vec{e}_\theta \\ \vec{e}_\phi \\ \vec{e}_r \end{Bmatrix} = \begin{Bmatrix} -\vec{e}_r \\ 0 \\ \vec{e}_\theta \end{Bmatrix}. \Leftarrow$$

2.2 CURVES AND SURFACES

The solution domain Ω of an engineering (beam or plate) model has usually lower dimension than the body $V \in \mathbb{R}^3$. The representation of the domain embedded in $\mathbb{R}^3$ may be mid-line or mid-curve of curve of beam or mid-plane or mid-surface of plate)

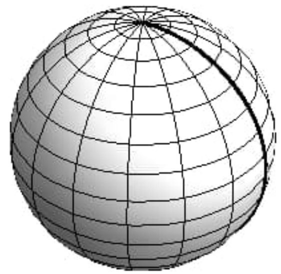
Curve: $\vec{r}_0(\alpha) = x(\alpha)\vec{i} + y(\alpha)\vec{j} + z(\alpha)\vec{k} \qquad \alpha \in \Omega \subset \mathbb{R}$ 1 parameter

Surface: $\vec{r}_0(\alpha,\beta) = x(\alpha,\beta)\vec{i} + y(\alpha,\beta)\vec{j} + z(\alpha,\beta)\vec{k} \qquad (\alpha,\beta) \in \Omega \subset \mathbb{R}^2$ 2 parameters

Shape of a mid-curve is defined by a one-parameter mapping and a mid-surface by a two-parameter mapping. In MEC-E8003, the coordinate curves of surfaces (defined by constant values of α or β**)** are assumed to be orthogonal (just to simplify the setting).

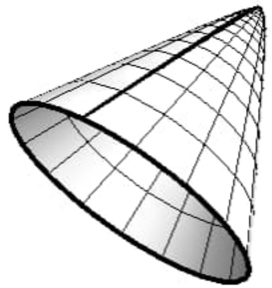
SOME MAPPINGS

Coil $\vec{r}_0(\phi) = \vec{i}R\cos\phi + \vec{j}R\sin\phi + \vec{k}h\dfrac{\phi}{2\pi},$



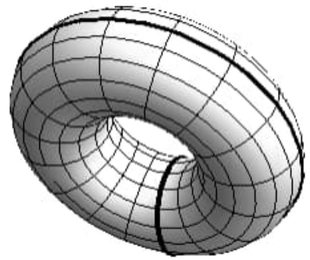
Cylinder $\vec{r}_0(z,\phi) = R(\vec{i}\cos\phi + \vec{j}\sin\phi) + \vec{k}z$

Cone $\vec{r}_0(z,s) = R(z)(\vec{i}\cos\phi + \vec{j}\sin\phi) + \vec{k}z$



Sphere $\vec{r}_0(\phi,\theta) = R(\vec{i}\sin\theta\cos\phi + \vec{j}\sin\theta\sin\phi + \vec{k}\cos\theta)$

Ellipsoid $\vec{r}_0(\phi,\theta) = R(\vec{i}\sin\theta\cos\phi + \vec{j}\sin\theta\sin\phi + \varepsilon\vec{k}\cos\theta)$



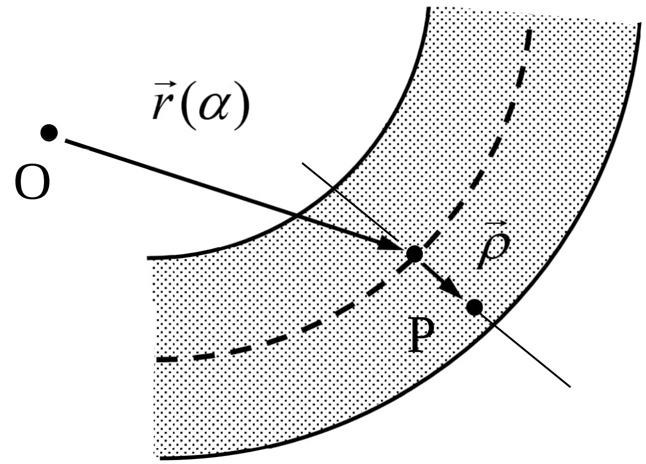
Hyperboloid $\vec{r}_0(\phi,\theta) = R(\vec{i}\sinh\theta\cos\phi + \vec{j}\sinh\theta\sin\phi - \varepsilon\vec{k}\cosh\theta)$

Torus $\vec{r}_0(\phi,\theta) = \vec{i}\cos\phi(R + r\cos\theta) + \vec{j}\sin\phi(R + r\cos\theta) + \vec{k}r\sin\theta$

Mid-curve or mid-surface mapping identifies the particles on the mid-curve or mid-surface. Identification of all particles (P in the figure) of a thin body requires also the relative position vector $\vec{\rho}$:

Beam mapping: $\vec{r}(\alpha,n,b)=\vec{r}_0(\alpha)+\overbrace{n\vec{e}_n(\alpha)+b\vec{e}_b(\alpha)}^{\text{relative}}$

Shell mapping: $\vec{r}(\alpha,\beta,n)=\vec{r}_0(\alpha,\beta)+\underbrace{n\vec{e}_n(\alpha,\beta)}_{\text{relative}}$



The mapping for the mid-curve or surface is used to define the basis vectors. In MEC-E8003 basis vectors are orthonormal to keep the setting as simple as possible (curved geometry induces some complications anyway)!

BEAM COORDINATES (s,n,b)

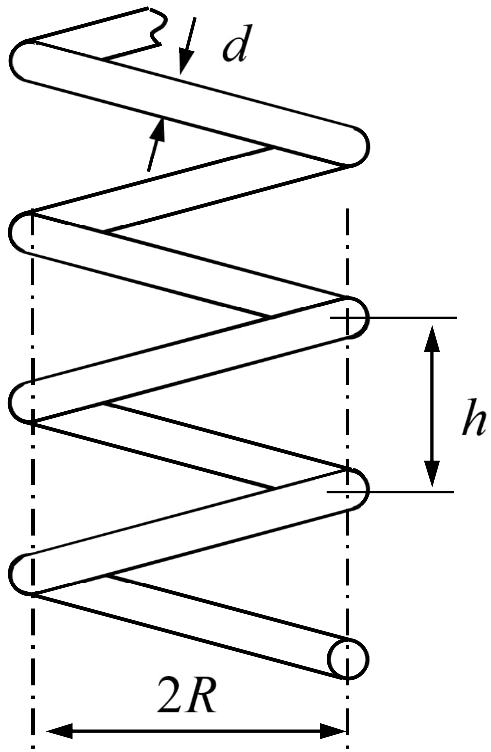
Particle is identified by distance s along the mid-curve and distances (n,b) from the curve.

Mapping $\vec{r}(s,n,b)=\vec{r}_0(s)+n\vec{e}_n(s)+b\vec{e}_b(s)$ gives

$$\begin{Bmatrix}\vec{e}_s\\ \vec{e}_n\\ \vec{e}_b\end{Bmatrix}=\begin{Bmatrix}\partial\vec{r}_0/\partial s\\ (\partial\vec{e}_s/\partial s)/\left|\partial\vec{e}_s/\partial s\right|\\ \vec{e}_s\times\vec{e}_n\end{Bmatrix}=[F]\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix},\quad \begin{Bmatrix}\partial\vec{r}/\partial s\\ \partial\vec{r}/\partial n\\ \partial\vec{r}/\partial b\end{Bmatrix}=[H]\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}$$

$$\frac{\partial}{\partial s}\begin{Bmatrix}\vec{e}_s\\ \vec{e}_n\\ \vec{e}_b\end{Bmatrix}=\begin{Bmatrix}\kappa\vec{e}_n\\ -\kappa\vec{e}_s+\tau\vec{e}_b\\ -\tau\vec{e}_n\end{Bmatrix},$$

$$\nabla=\frac{1}{1-n\kappa}\vec{e}_s\left[\frac{\partial}{\partial s}+\tau\left(b\frac{\partial}{\partial n}-n\frac{\partial}{\partial b}\right)\right]+\vec{e}_n\frac{\partial}{\partial n}+\vec{e}_b\frac{\partial}{\partial b}.$$



Beam (s,n,b) coordinate system is curvilinear and orthonormal. Therefore the matrix of the basis vector derivatives is anti-symmetric (why?) and expressible in form (Serret-Frenet formulas in literature)

$$\frac{\partial}{\partial s}\begin{Bmatrix}\vec{e}_s\\ \vec{e}_n\\ \vec{e}_b\end{Bmatrix}=(\frac{\partial}{\partial s}[F])[F]^{-1}\begin{Bmatrix}\vec{e}_s\\ \vec{e}_n\\ \vec{e}_b\end{Bmatrix}=\begin{bmatrix}0 & \kappa_b & -\kappa_n\\ -\kappa_b & 0 & \kappa_s\\ \kappa_n & -\kappa_s & 0\end{bmatrix}\begin{Bmatrix}\vec{e}_s\\ \vec{e}_n\\ \vec{e}_b\end{Bmatrix}=\begin{Bmatrix}\kappa\vec{e}_n\\ -\kappa\vec{e}_s+\tau\vec{e}_b\\ -\tau\vec{e}_n\end{Bmatrix}$$

containing geometrical quantities $\kappa_s=\tau$, $\kappa_n=0$, and $\kappa_b=\kappa=1/R$. Explicit expressions for *curvature* κ and *torsion* τ require an explicit form of $\vec{r}_0(s)$ or $[F]$.

The gradient operator at a generic point *in terms of the basis vectors at the mid-curve* is based on $\vec{r}=\vec{r}_0+n\vec{e}_n+b\vec{e}_b$ the beam is given by ($[H]$ follows from $\vec{r}$ and $[F]$ follows from $\vec{r}_0$

$$\begin{Bmatrix} \partial\vec{r}/\partial s \\ \partial\vec{r}/\partial n \\ \partial\vec{r}/\partial b \end{Bmatrix} = \begin{bmatrix} 1-\kappa n & -\tau b & \tau n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \vec{e}_s \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix} = \begin{bmatrix} 1-\kappa n & -\tau b & \tau n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} [F] \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = [H] \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}.$$

As the curvilinear coordinate system is orthonormal so $[F]^{\mathrm{T}} = [F]^{-1}$, the generic formula for gradient gives

$$\nabla = \begin{Bmatrix} \vec{e}_s \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix}^{\mathrm{T}} \left([H][F]^{\mathrm{T}}\right)^{-1} \begin{Bmatrix} \partial/\partial s \\ \partial/\partial n \\ \partial/\partial b \end{Bmatrix} = \begin{Bmatrix} \vec{e}_s \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1-\kappa n & -\tau b & \tau n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{Bmatrix} \partial/\partial s \\ \partial/\partial n \\ \partial/\partial b \end{Bmatrix}.$$

SHELL COORDINATES (α,β,n)

A particle is identified by mid-surface position (α,β) (generalized coordinates) and distance n in the normal direction. Mapping $\vec{r}(\alpha,\beta,n)=\vec{r}_0(\alpha,\beta)+n\vec{e}_n(\alpha,\beta)$ gives

$$\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_n\end{Bmatrix}=\begin{Bmatrix}(\partial\vec{r}_0/\partial\alpha)/\left|\partial\vec{r}_0/\partial\alpha\right|\\ (\partial\vec{r}_0/\partial\beta)/\left|\partial\vec{r}_0/\partial\beta\right|\\ \vec{e}_\alpha\times\vec{e}_\beta\end{Bmatrix}=[F]\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix},\quad \begin{Bmatrix}\partial\vec{r}/\partial\alpha\\ \partial\vec{r}/\partial\beta\\ \partial\vec{r}/\partial n\end{Bmatrix}=[H]\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}$$

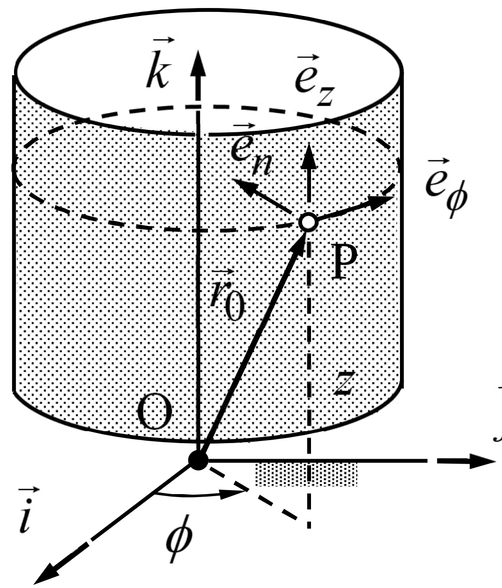
$$\frac{\partial}{\partial\eta}\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_n\end{Bmatrix}=(\frac{\partial}{\partial\eta}[F])[F]^{-1}\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_n\end{Bmatrix}\ \ \eta\in\{\alpha,\beta,n\}\ \text{ and }\ \nabla=\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_n\end{Bmatrix}^{\mathrm{T}}([H][F]^{\mathrm{T}})^{-1}\begin{Bmatrix}\partial/\partial\alpha\\ \partial/\partial\beta\\ \partial/\partial n\end{Bmatrix}$$

In MEC-E8003, the mapping $\vec{r}_0(\alpha,\beta)$ is restricted by orthogonality condition $\vec{e}_\alpha\cdot\vec{e}_\beta=0$.

CYLINDRICAL SHELL (z,ϕ,n)

A particle is identified by mid-surface coordinates (z,ϕ) and distance n in the normal direction (inwards). Mid-surface mapping $\vec{r}_0(\phi,z)=\vec{i}R\cos\phi+\vec{j}R\sin\phi+\vec{k}z$ gives

$$\begin{Bmatrix}\vec{e}_z\\ \vec{e}_\phi\\ \vec{e}_n\end{Bmatrix}=\begin{bmatrix}0 & 0 & 1\\ -\sin\phi & \cos\phi & 0\\ -\cos\phi & -\sin\phi & 0\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}=[F]\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix},$$



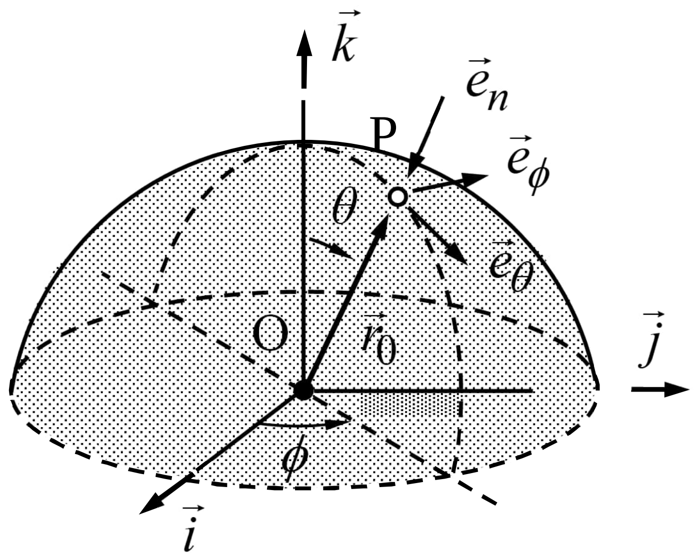
$$\frac{\partial}{\partial\phi}\begin{Bmatrix}\vec{e}_z\\ \vec{e}_\phi\\ \vec{e}_n\end{Bmatrix}=\begin{Bmatrix}0\\ \vec{e}_n\\ -\vec{e}_\phi\end{Bmatrix}\ \text{zeros otherwise,}$$

$$\nabla=\vec{e}_z\frac{\partial}{\partial z}+\frac{R}{R-n}\frac{1}{R}\vec{e}_\phi\frac{\partial}{\partial\phi}+\vec{e}_n\frac{\partial}{\partial n}.$$

SPHERICAL SHELL (ϕ, θ, n)

A particle is identified by mid-surface position (ϕ, θ) and distance n in the normal direction (inwards). Mid-surface mapping $\vec{r}_0(\phi, \theta) = R(\vec{i}\sin\theta\cos\phi + \vec{j}\sin\theta\sin\phi + \vec{k}\cos\theta)$:

$$\begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_\theta \\ \vec{e}_n \end{Bmatrix} = \begin{bmatrix} -\sin\phi & \cos\phi & 0 \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta\cos\phi & -\sin\theta\sin\phi & -\cos\theta \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix},$$



$$\frac{\partial}{\partial\phi}\begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_\theta \\ \vec{e}_n \end{Bmatrix} = \begin{Bmatrix} \sin\theta\vec{e}_n - \cos\theta\vec{e}_\theta \\ \cos\theta\vec{e}_\phi \\ -\sin\theta\vec{e}_\phi \end{Bmatrix}, \quad \frac{\partial}{\partial\theta}\begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_\theta \\ \vec{e}_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ \vec{e}_n \\ -\vec{e}_\theta \end{Bmatrix},$$

$$\nabla = \frac{R}{R-n}\frac{1}{R\sin\theta}\vec{e}_\phi\frac{\partial}{\partial\phi} + \frac{R}{R-n}\frac{1}{R}\vec{e}_\theta\frac{\partial}{\partial\theta} + \vec{e}_n\frac{\partial}{\partial n}.$$

2.3 CURVATURE

Curvature is the amount by which surface or curve embedded in $\mathbb{R}^3$ deviates from being *flat* or *straight*. The radius of curvature $R = 1/\kappa$ of a curve is given by the best fitting circle. Curvature of a surface at a point depends on the direction of a curve through that point.

Curvature: $\overleftrightarrow{\kappa}_c = \nabla_0 \vec{e}_n$ Gradient at the mid-curve or mid-surface !

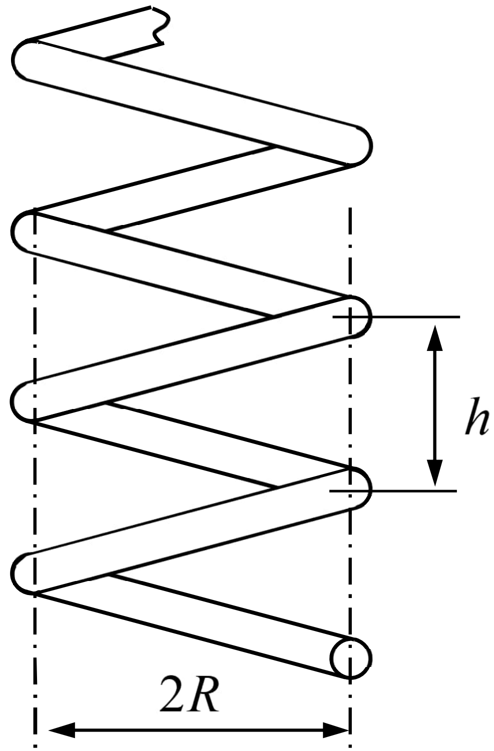
Principal curvatures: $(\kappa_1, \vec{n}_1)$ and $(\kappa_2, \vec{n}_2)$ such that $\overleftrightarrow{\kappa} \cdot \vec{n} = \kappa \vec{n}$

Gaussian curvature: $K = \det[\kappa] = \kappa_1 \kappa_2$ Curvature measure!

Mean curvature: $H = \dfrac{1}{2} \nabla \cdot \vec{e}_n = \dfrac{1}{2} \overleftrightarrow{I} : \overleftrightarrow{\kappa} = \dfrac{1}{2}(\kappa_1 + \kappa_2)$ Another curvature measure!

CURVATURE AND TORSION OF BEAM

The radius of curvature $R = 1/\kappa$ at a point is given by the best fitting circle. Torsion τ describes the rate of rotation of $\vec{e}_n$ and $\vec{e}_b$ around the mid-curve (change of rotation angle divided by change of the mid-curve coordinate s)



Circular: $\kappa = \dfrac{1}{R}$ and $\tau = 0$

Twisted beam: $\kappa = 0$ and $\tau = \dfrac{2\pi}{h}$

Coil: $\kappa = \dfrac{R}{h^2 + R^2}$ and $\tau = \dfrac{h}{h^2 + R^2}$

A circular beam of radius R has zero torsion. The basis vectors of (x, y, z) and (s, n, b) coordinate systems differ by rotation with respect to the normal direction to the plane of circle (z here). With distance s along the mid-curve and $\phi = s/R$

$$\begin{Bmatrix} \vec{e}_s \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix} = \begin{bmatrix} -\sin\phi & \cos\phi & 0 \\ -\cos\phi & -\sin\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} \quad \Rightarrow \quad \kappa = \vec{e}_n \cdot \frac{\partial}{\partial s} \vec{e}_s = \frac{1}{R}. \quad \Leftarrow$$

A twisted beam has zero curvature. The basis vectors of (x, y, z) and (s, n, b) differ by rotation along the $x-$axis. With notation $\omega = 2\phi / h$

$$\begin{Bmatrix} \vec{e}_s \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega s & \sin\omega s \\ 0 & -\sin\omega s & \cos\omega s \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} \quad \Rightarrow \quad \tau = \vec{e}_b \cdot \frac{\partial}{\partial s} \vec{e}_n = \frac{2\pi}{h}. \quad \Leftarrow$$

EXAMPLE A planar curve in $xy-$ plane is defined by mapping

(a) $\vec{r}_0(\alpha) = x(\alpha)\vec{i} + y(\alpha)\vec{j}$ (generic parametric form of a planar curve)

(b) $\vec{r}_0(x) = x\vec{i} + y(x)\vec{j}$

Derive the curvature tensor $\overleftrightarrow{\kappa}_{\mathrm{c}} = \nabla_0 \vec{e}_n$.

Answer: (a) $\overleftrightarrow{\kappa} = \vec{e}_\alpha \vec{e}_\alpha \dfrac{y'x'' - x'y''}{(x'^2 + y'^2)^{3/2}}$ (b) $\overleftrightarrow{\kappa} = -\vec{e}_x \vec{e}_x \dfrac{y''}{(1 + y'^2)^{3/2}}$

To use the definition, one needs the derivatives of the basis vectors and also the gradient operator of the curvilinear (α, n, b) system at $n = b = 0$. With the Lagrange's notation of derivative with respect to α (choose $\vec{e}_b = \vec{k}$ so $\vec{e}_n = \vec{k} \times \vec{e}_\alpha$)

$$\begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix} = \begin{bmatrix} x'/h_\alpha & y'/h_\alpha & 0 \\ -y'/h_\alpha & x'/h_\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = [F] \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}, \text{ where } h_\alpha = \sqrt{x'^2 + y'^2}.$$

The derivatives of the basis vectors follow from the generic expression

$$\frac{\partial}{\partial \alpha} \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix} = (\frac{\partial}{\partial \alpha}[F])[F]^{-1} \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix} = h_n \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix}, \text{ where } h_n = \frac{y'x'' - x'y''}{x'^2 + y'^2}.$$

At the mid-curve, where $n = b = 0$, the gradient operator for a curvilinear orthonormal coordinate system of a beam simplifies to ($\kappa = h_n / h_\alpha$)

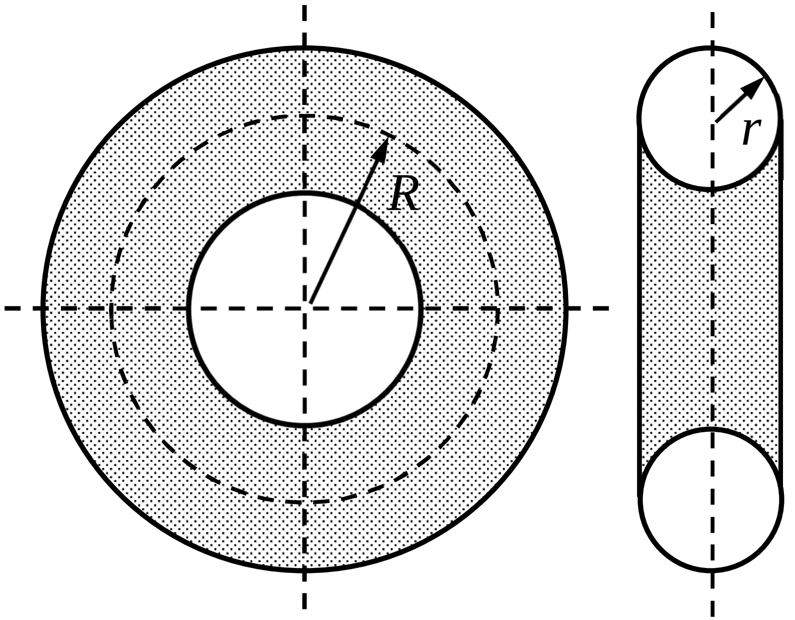
$$\nabla = \vec{e}_\alpha \frac{1}{h_\alpha} \frac{\partial}{\partial \alpha} + \vec{e}_n \frac{\partial}{\partial n} + \vec{e}_b \frac{\partial}{\partial b} \quad \Rightarrow$$

$$\overset{\leftrightarrow}{\kappa}_{\mathrm{c}} = \nabla \vec{e}_n = \vec{e}_\alpha \frac{1}{h_\alpha} \frac{\partial \vec{e}_n}{\partial \alpha} = \vec{e}_\alpha \vec{e}_\alpha \frac{h_n}{h_\alpha} = \vec{e}_\alpha \vec{e}_\alpha \frac{y'x'' - x'y''}{(x'^2 + y'^2)^{3/2}} \quad \Leftrightarrow$$

$$\overset{\leftrightarrow}{\kappa} = \vec{e}_\alpha \vec{e}_\alpha \kappa \quad \text{where} \quad \kappa = \frac{y'x'' - x'y''}{(x'^2 + y'^2)^{3/2}}.$$

EXAMPLE Consider torus surface (donut) having distance R from the center of the tube to the center of the torus and radius r of the tube. Derive the basis vectors, basis vector derivatives, gradient expression, and curvature in (ϕ, θ, n) coordinate system. The mapping defining the geometry, $\phi \in [0, 2\pi]$ and $\theta \in [0, 2\pi]$, is

$$\vec{r}_0(\phi, \theta) = (R + r\cos\theta)(\vec{i}\cos\phi + \vec{j}\sin\phi) + \vec{k} r \sin\theta .$$



Answer:

$$\nabla = \frac{1}{R + (n+r)\cos\theta} \vec{e}_\phi \frac{\partial}{\partial\phi} + \vec{e}_\theta \frac{1}{n+r} \frac{\partial}{\partial\theta} + \vec{e}_n \frac{\partial}{\partial n}\ , \quad \vec{\vec{\kappa}} = \vec{e}_\phi \vec{e}_\phi \frac{\cos\theta}{R + (n+r)\cos\theta} + \vec{e}_\theta \vec{e}_\theta \frac{1}{n+r}$$

Let us start with the relationship between the basis vectors. Definitions give

$$\begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_\theta \\ \vec{e}_n \end{Bmatrix} = \begin{bmatrix} -\sin\phi & \cos\phi & 0 \\ -\cos\phi\sin\theta & -\sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & \sin\theta \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = [F] \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}.$$

Since the basis is orthonormal i.e. $[F]^{-1} = [F]^{\mathrm{T}}$, the derivatives of the basis vectors are given by

antisymmetric!

$$\frac{\partial}{\partial\phi} \begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_\theta \\ \vec{e}_n \end{Bmatrix} = (\frac{\partial}{\partial\phi}[F])[F]^{-1} \begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_\theta \\ \vec{e}_n \end{Bmatrix} = \begin{bmatrix} 0 & \sin\theta & -\cos\theta \\ -\sin\theta & 0 & 0 \\ \cos\theta & 0 & 0 \end{bmatrix} \begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_\theta \\ \vec{e}_n \end{Bmatrix}$$

$$\frac{\partial}{\partial\theta}\begin{Bmatrix}\vec{e}_\phi\\ \vec{e}_\theta\\ \vec{e}_n\end{Bmatrix}=(\frac{\partial}{\partial\theta}[F])[F]^{-1}\begin{Bmatrix}\vec{e}_\phi\\ \vec{e}_\theta\\ \vec{e}_n\end{Bmatrix}=\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix}\begin{Bmatrix}\vec{e}_\phi\\ \vec{e}_\theta\\ \vec{e}_n\end{Bmatrix},\quad \text{and } \frac{\partial}{\partial n}\begin{Bmatrix}\vec{e}_\phi\\ \vec{e}_\theta\\ \vec{e}_n\end{Bmatrix}=0.$$

The gradient expression in concerned with a generic material point so that the mapping between the curvilinear (ϕ,θ,n) coordinate system and the reference (x,y,z) coordinate system is written as $\vec{r}=\vec{r}_0+n\vec{e}_n$ (the mapping needs to define positions of all the particles of body not just those on the mid-surface). With $\vec{h}_\phi=\partial\vec{r}/\partial\phi$ etc.

$$\begin{Bmatrix}\vec{h}_\phi\\ \vec{h}_\theta\\ \vec{h}_n\end{Bmatrix}=\begin{bmatrix}-[R+(n+r)\cos\theta]\sin\phi & [R+(n+r)\cos\theta]\cos\phi & 0\\ -(n+r)\cos\phi\sin\theta & -(n+r)\sin\theta\sin\phi & (n+r)\cos\theta\\ \cos\theta\cos\phi & \cos\theta\sin\phi & \sin\theta\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}=[H]\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}$$

The generic formula for the gradient operator gives (Mathematica is handy in this step)

$$\nabla = \begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_\theta \\ \vec{e}_n \end{Bmatrix}^{\mathrm{T}} ([H][F]^{\mathrm{T}})^{-1} \begin{Bmatrix} \partial / \partial\phi \\ \partial / \partial\theta \\ \partial / \partial n \end{Bmatrix} = \frac{1}{R+(n+r)\cos\theta}\vec{e}_\phi \frac{\partial}{\partial\phi} + \vec{e}_\theta \frac{1}{n+r}\frac{\partial}{\partial\theta} + \vec{e}_n \frac{\partial}{\partial n} \ .$$

Finally, curvature of the torus geometry becomes (at the mid-surface $n = 0$)

$$\nabla_0 \vec{e}_n = \frac{1}{R + r\cos\theta}\vec{e}_\phi \frac{\partial \vec{e}_n}{\partial\phi} + \vec{e}_\theta \frac{1}{r}\frac{\partial \vec{e}_n}{\partial\theta} + \vec{e}_n \frac{\partial \vec{e}_n}{\partial n} \quad \Rightarrow$$

$$\vec{\kappa} = (\nabla_0 \vec{e}_n)_{\mathrm{c}} = \frac{\cos\theta}{R + r\cos\theta}\vec{e}_\phi\vec{e}_\phi + \vec{e}_\theta\vec{e}_\theta \frac{1}{r} \ .$$

MEC-E8003 Beam, plate and shell models, examples 2

1. Use the definition $\nabla^2 = \nabla \cdot \nabla$ to derive the Laplacian operator in the polar coordinate system.

   Answer $\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}$

2. Given the strain components ε_{xx}, ε_{yy}, ε_{xy}, and ε_{yx} in a Cartesian $(x, y)-$coordinate system, derive the strain components ε_{rr}, $\varepsilon_{\phi\phi}$, $\varepsilon_{r\phi}$, and $\varepsilon_{\phi r}$ of the polar $(r,\phi)-$coordinate system (in terms of the Cartesian components). Use the invariance of tensor quantities.

   Answer $\begin{bmatrix} \varepsilon_{rr} & \varepsilon_{r\phi} \\ \varepsilon_{\phi r} & \varepsilon_{\phi\phi} \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$

3. Calculate $\nabla\vec{r}$, $\nabla\cdot\vec{r}$ and $\nabla\times\vec{r}$ in which $\vec{r}$ is the position vector. Use the representations of the cylindrical coordinate system

   $$\vec{r} = \begin{Bmatrix} r \\ 0 \\ z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix} = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} r \\ 0 \\ z \end{Bmatrix}, \quad \nabla = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \partial/\partial r \\ \partial/(r\partial\phi) \\ \partial/\partial z \end{Bmatrix} \text{ and } \frac{\partial}{\partial\phi}\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix} = \begin{Bmatrix} \vec{e}_\phi \\ -\vec{e}_r \\ 0 \end{Bmatrix} \text{ (zeros otherwise).}$$

   Answer $\nabla\vec{r} = \vec{I}$, $\nabla\cdot\vec{r} = 3$, $\nabla\times\vec{r} = 0$

4. Use the definition $\nabla^2 = \nabla \cdot \nabla$ to derive the Laplacian operator in the cylindrical coordinate system.

   Answer $\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$

5. Derive the expressions of $\nabla\cdot\vec{u}$, $\nabla\times\vec{u}$, and $\nabla^2 u$ in the polar coordinate system. Vector $\vec{u}(r,\phi) = u_r\vec{e}_r + u_\phi\vec{e}_\phi$ and scalar $u(r,\phi)$ depend on the polar coordinates r and ϕ. In the polar coordinate system

   $$\nabla = \vec{e}_r\frac{\partial}{\partial r} + \vec{e}_\phi\frac{\partial}{r\partial\phi}, \quad \frac{\partial}{\partial\phi}\vec{e}_r = \vec{e}_\phi, \quad \frac{\partial}{\partial\phi}\vec{e}_\phi = -\vec{e}_r, \quad \text{(and } \vec{e}_r\times\vec{e}_\phi = \vec{k}\text{)}.$$

   Answer

   $$\nabla\cdot\vec{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\phi}{\partial\phi}, \quad \nabla\times\vec{u} = (\frac{1}{r}\frac{\partial}{\partial r}(ru_\phi) - \frac{1}{r}\frac{\partial u_r}{\partial\phi})\vec{k}, \quad \nabla^2 u = (\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2})u$$

6. In the beam coordinate system and planar case, the displacement assumption of a curved Timoshenko beam model is $\vec{u}=\vec{u}_0+\vec{\theta}_0\times\vec{\rho}$, where $\vec{u}_0=u(s)\vec{e}_s+v(s)\vec{e}_n$, $\vec{\theta}_0=\psi(s)\vec{e}_b$, and $\vec{\rho}=n\vec{e}_n$. Derive the small strain component expressions ε_{ss} and $\varepsilon_{sn}=\varepsilon_{ns}$ using

$$\vec{\varepsilon}=\frac{1}{2}\left[\nabla\vec{u}+(\nabla\vec{u})_{\mathrm{c}}\right],\ \nabla=\vec{e}_s\frac{1}{1-n/R}\frac{\partial}{\partial s}+\vec{e}_n\frac{\partial}{\partial n},\ \frac{\partial}{\partial s}\begin{Bmatrix}\vec{e}_s\\ \vec{e}_n\end{Bmatrix}=\frac{1}{R}\begin{Bmatrix}\vec{e}_n\\ -\vec{e}_s\end{Bmatrix},\ \text{and}\ \frac{\partial}{\partial n}\begin{Bmatrix}\vec{e}_s\\ \vec{e}_n\end{Bmatrix}=0.$$

Assume that curvature $\kappa=1/R$ is constant.

Answer $\varepsilon_{ss}=\dfrac{R}{R-n}(u'-\psi'n-\dfrac{v}{R})$, $\varepsilon_{sn}=\varepsilon_{ns}=\dfrac{1}{2}\dfrac{R}{R-n}(\dfrac{1}{R}u+v'-\psi)$

7. Mapping $\vec{r}(r,\phi,z)=r\cos\phi\vec{i}+r\sin\phi\vec{j}+z\vec{k}$ defines the cylindrical $(r,\phi,z)-$coordinate system. Use (in detail) the generic formula

$$\frac{\partial}{\partial\eta}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_z\end{Bmatrix}=(\frac{\partial}{\partial\eta}[F])[F]^{-1}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_z\end{Bmatrix},\ \text{where}\ \eta\in\{r,\phi,z\}$$

to find the derivatives of the basis vectors.

Answer $\dfrac{\partial}{\partial\phi}\vec{e}_r=\vec{e}_\phi$, $\dfrac{\partial}{\partial\phi}\vec{e}_\phi=-\vec{e}_r$ (zeros otherwise)

8. Derive the gradient expression of the spherical $(\theta,\phi,r)-$coordinate system, when the mapping defining the coordinate system is given by $\vec{r}(\theta,\phi,r)=r(\sin\theta\cos\phi\vec{i}+\sin\theta\sin\phi\vec{j}+\cos\theta\vec{k})$.

Answer $\nabla=\vec{e}_\theta\dfrac{1}{r}\dfrac{\partial}{\partial\theta}+\vec{e}_\phi\dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\phi}+\vec{e}_r\dfrac{\partial}{\partial r}$

9. Compute the derivatives of the basis vectors, gradient operator, and curvature for the cylindrical shell geometry with the mid-surface representation $\vec{r}_0(\phi,z)=R(\cos\phi\vec{i}+\sin\phi\vec{j})+z\vec{k}$ in terms of coordinates (ϕ,z). Notice that the order of the coordinates differs from that of the lecture notes, which affects, e.g., direction of $\vec{e}_n$.

Answer $\dfrac{\partial\vec{e}_\phi}{\partial\phi}=-\vec{e}_n$, $\dfrac{\partial\vec{e}_n}{\partial\phi}=\vec{e}_\phi$, $\nabla=(\dfrac{R}{R+n})\dfrac{1}{R}\vec{e}_\phi\dfrac{\partial}{\partial\phi}+\vec{e}_z\dfrac{\partial}{\partial z}+\vec{e}_n\dfrac{\partial}{\partial n}$, $\vec{\kappa}=\vec{e}_\phi\vec{e}_\phi\dfrac{1}{R+n}$

10. Consider the mid-surface mapping $\vec{r}_0(r,\phi)=r^2\cos(2\phi)\vec{i}+r^2\sin(2\phi)\vec{j}$ of shell. Compute the expression of the basis vector derivatives and gradient operator ∇. Is the mid-surface defined by the mapping flat or curved?

Answer $\dfrac{\partial\vec{e}_r}{\partial\phi}=2\vec{e}_\phi$, $\dfrac{\partial\vec{e}_\phi}{\partial\phi}=-2\vec{e}_r$, $\nabla=\vec{e}_r\dfrac{1}{2r}\dfrac{\partial}{\partial r}+\vec{e}_\phi\dfrac{1}{2r^2}\dfrac{\partial}{\partial\phi}+\vec{e}_n\dfrac{\partial}{\partial n}$

Use the definition $\nabla^2 = \nabla \cdot \nabla$ to derive the Laplacian operator in the polar coordinate system.

Solution

Gradient operator and the derivatives of the basis vectors of the polar $(r,\phi)-$coordinate system are

$$\nabla = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \partial / \partial r \\ \partial / (r \partial \phi) \end{Bmatrix} \text{ and } \frac{\partial}{\partial \phi} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}.$$

According to the definition

$$\nabla^2 = \nabla \cdot \nabla = (\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r} \frac{\partial}{\partial \phi}) \cdot (\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r} \frac{\partial}{\partial \phi})$$

in which the terms

$$(\vec{e}_r \frac{\partial}{\partial r}) \cdot (\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r} \frac{\partial}{\partial \phi}) = \vec{e}_r \cdot (\vec{e}_r \frac{\partial^2}{\partial r^2} - \vec{e}_\phi \frac{1}{r^2} \frac{\partial}{\partial \phi} + \vec{e}_\phi \frac{1}{r} \frac{\partial^2}{\partial r \partial \phi}) = \frac{\partial^2}{\partial r^2},$$

$$(\vec{e}_\phi \frac{1}{r} \frac{\partial}{\partial \phi}) \cdot (\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r} \frac{\partial}{\partial \phi}) = (\vec{e}_\phi \frac{1}{r}) \cdot (\vec{e}_\phi \frac{\partial}{\partial r} + \vec{e}_r \frac{\partial^2}{\partial \phi \partial r} - \vec{e}_r \frac{1}{r} \frac{\partial}{\partial \phi} + \vec{e}_\phi \frac{1}{r} \frac{\partial^2}{\partial \phi^2}) = \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2}$$

Combining the results for the terms gives

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} = \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial}{\partial r}) + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2}. \quad \leftarrow$$

Given the strain components ε_{xx}, ε_{yy}, ε_{xy}, and ε_{yx} in a Cartesian $(x,y)-$coordinate system, derive the strain components ε_{rr}, $\varepsilon_{\phi\phi}$, $\varepsilon_{r\phi}$, and $\varepsilon_{\phi r}$ of the polar $(r,\phi)-$coordinate system (in terms of the Cartesian components). Use the invariance of tensor quantities.

Solution

In mechanics tensors (vectors etc.) represent physical quantities which can be expressed in terms of any basis vector set. Components depend on the selection of the basis vectors but the quantity itself does not. With notation $\vec{e}_x \equiv \vec{i}$ and $\vec{e}_y \equiv \vec{j}$, the relationship between the Cartesian and polar basis is

$$\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix}=\begin{bmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{bmatrix}\begin{Bmatrix}\vec{e}_x\\ \vec{e}_y\end{Bmatrix}=[F]\begin{Bmatrix}\vec{e}_x\\ \vec{e}_y\end{Bmatrix} \quad\Leftrightarrow\quad \begin{Bmatrix}\vec{e}_x\\ \vec{e}_y\end{Bmatrix}=\begin{bmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{bmatrix}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix}=[F]^{-1}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix}$$

and invariance of $\vec{\vec{\varepsilon}}$ with respect to the coordinate system means that

$$\vec{\vec{\varepsilon}}=\varepsilon_{xx}\vec{e}_x\vec{e}_x+\varepsilon_{xy}\vec{e}_x\vec{e}_y+\varepsilon_{yx}\vec{e}_y\vec{e}_x+\varepsilon_{yy}\vec{e}_y\vec{e}_y=\varepsilon_{rr}\vec{e}_r\vec{e}_r+\varepsilon_{r\phi}\vec{e}_r\vec{e}_\phi+\varepsilon_{\phi r}\vec{e}_\phi\vec{e}_r+\varepsilon_{\phi\phi}\vec{e}_\phi\vec{e}_\phi\,.$$

By substituting $\vec{e}_x = \mathrm{c}\phi\vec{e}_r - \mathrm{s}\phi\vec{e}_\phi$ and $\vec{e}_y = \mathrm{s}\phi\vec{e}_r + \mathrm{c}\phi\vec{e}_\phi$ into the representation in the Cartesian $(x,y)-$system

$$\vec{\vec{\varepsilon}}=\varepsilon_{xx}\vec{e}_x\vec{e}_x+\varepsilon_{xy}\vec{e}_x\vec{e}_y+\varepsilon_{yx}\vec{e}_y\vec{e}_x+\varepsilon_{yy}\vec{e}_y\vec{e}_y \quad\Rightarrow$$

$$\vec{\vec{\varepsilon}}=\varepsilon_{xx}(\mathrm{c}\phi\vec{e}_r-\mathrm{s}\phi\vec{e}_\phi)(\mathrm{c}\phi\vec{e}_r-\mathrm{s}\phi\vec{e}_\phi)+\varepsilon_{xy}(\mathrm{c}\phi\vec{e}_r-\mathrm{s}\phi\vec{e}_\phi)(\mathrm{s}\phi\vec{e}_r+\mathrm{c}\phi\vec{e}_\phi)+$$

$$\varepsilon_{yx}(\mathrm{s}\phi\vec{e}_r+\mathrm{c}\phi\vec{e}_\phi)(\mathrm{c}\phi\vec{e}_r-\mathrm{s}\phi\vec{e}_\phi)+\varepsilon_{yy}(\mathrm{s}\phi\vec{e}_r+\mathrm{c}\phi\vec{e}_\phi)(\mathrm{s}\phi\vec{e}_r+\mathrm{c}\phi\vec{e}_\phi) \quad\Leftrightarrow$$

$$\vec{\vec{\varepsilon}}=\varepsilon_{xx}(\mathrm{c}^2\phi\vec{e}_r\vec{e}_r-\mathrm{c}\phi\mathrm{s}\phi\vec{e}_r\vec{e}_\phi-\mathrm{s}\phi\mathrm{c}\phi\vec{e}_\phi\vec{e}_r+\mathrm{s}^2\phi\vec{e}_\phi\vec{e}_\phi)+\varepsilon_{xy}(\mathrm{c}\phi\mathrm{s}\phi\vec{e}_r\vec{e}_r+\mathrm{c}^2\phi\vec{e}_r\vec{e}_\phi-\mathrm{s}^2\phi\vec{e}_\phi\vec{e}_r-\mathrm{s}\phi\mathrm{c}\phi\vec{e}_\phi\vec{e}_\phi)+$$

$$\varepsilon_{yx}(\mathrm{s}\phi\mathrm{c}\phi\vec{e}_r\vec{e}_r-\mathrm{s}^2\phi\vec{e}_r\vec{e}_\phi+\mathrm{c}^2\phi\vec{e}_\phi\vec{e}_r-\mathrm{c}\phi\mathrm{s}\phi\vec{e}_\phi\vec{e}_\phi)+\varepsilon_{yy}(\mathrm{s}^2\phi\vec{e}_r\vec{e}_r+\mathrm{s}\phi\mathrm{c}\phi\vec{e}_r\vec{e}_\phi+\mathrm{c}\phi\mathrm{s}\phi\vec{e}_\phi\vec{e}_r+\mathrm{c}^2\phi\vec{e}_\phi\vec{e}_\phi)\,.$$

and after collecting the components

$$\vec{\vec{\varepsilon}}=(\varepsilon_{xx}\mathrm{c}^2\phi+\varepsilon_{xy}\mathrm{c}\phi\mathrm{s}\phi+\varepsilon_{yx}\mathrm{s}\phi\mathrm{c}\phi+\varepsilon_{yy}\mathrm{s}^2\phi)\vec{e}_r\vec{e}_r+(-\varepsilon_{xx}\mathrm{c}\phi\mathrm{s}\phi+\varepsilon_{xy}\mathrm{c}^2\phi-\varepsilon_{yx}\mathrm{s}^2\phi+\varepsilon_{yy}\mathrm{s}\phi\mathrm{c}\phi)\vec{e}_r\vec{e}_\phi+$$

$$(-\varepsilon_{xx}\mathrm{s}\phi\mathrm{c}\phi-\varepsilon_{xy}\mathrm{s}^2\phi+\varepsilon_{yx}\mathrm{c}^2\phi+\varepsilon_{yy}\mathrm{c}\phi\mathrm{s}\phi)\vec{e}_\phi\vec{e}_r+(\varepsilon_{xx}\mathrm{s}^2\phi-\varepsilon_{xy}\mathrm{s}\phi\mathrm{c}\phi-\varepsilon_{yx}\mathrm{c}\phi\mathrm{s}\phi+\varepsilon_{yy}\mathrm{c}^2\phi)\vec{e}_\phi\vec{e}_\phi$$

$$=\varepsilon_{rr}\vec{e}_r\vec{e}_r+\varepsilon_{r\phi}\vec{e}_r\vec{e}_\phi+\varepsilon_{\phi r}\vec{e}_\phi\vec{e}_r+\varepsilon_{\phi\phi}\vec{e}_\phi\vec{e}_\phi$$

Comparison of the representations gives

$$\begin{Bmatrix}\varepsilon_{rr}\\ \varepsilon_{r\phi}\\ \varepsilon_{\phi r}\\ \varepsilon_{\phi\phi}\end{Bmatrix}=\begin{bmatrix}\cos^2\phi & \cos\phi\sin\phi & \cos\phi\sin\phi & \sin^2\phi\\ -\cos\phi\sin\phi & \cos^2\phi & -\sin^2\phi & \cos\phi\sin\phi\\ -\cos\phi\sin\phi & -\sin^2\phi & \cos^2\phi & \cos\phi\sin\phi\\ \sin^2\phi & -\cos\phi\sin\phi & -\cos\phi\sin\phi & \cos^2\phi\end{bmatrix}\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{xy}\\ \varepsilon_{yx}\\ \varepsilon_{yy}\end{Bmatrix}. \leftarrow$$

Alternatively, one may work with matrices (a more convenient way for vectors and second order tensors)

$$\vec{\varepsilon} = \begin{Bmatrix} \vec{e}_x \\ \vec{e}_y \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} \begin{Bmatrix} \vec{e}_x \\ \vec{e}_y \end{Bmatrix} = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} [\mathrm{F}]^{-\mathrm{T}} \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} [\mathrm{F}]^{-1} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_y \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \varepsilon_{rr} & \varepsilon_{r\phi} \\ \varepsilon_{\phi r} & \varepsilon_{\phi\phi} \end{bmatrix} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}.$$

Therefore, components of the two systems are related by

$$\begin{bmatrix} \varepsilon_{rr} & \varepsilon_{r\phi} \\ \varepsilon_{\phi r} & \varepsilon_{\phi\phi} \end{bmatrix} = [\mathrm{F}]^{-\mathrm{T}} \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} [\mathrm{F}]^{-1} \quad ([\mathrm{F}]^{-\mathrm{T}} = [\mathrm{F}] \text{ and } [\mathrm{F}]^{-1} = [\mathrm{F}]^{\mathrm{T}} \text{ here})$$

giving

$$\begin{bmatrix} \varepsilon_{rr} & \varepsilon_{r\phi} \\ \varepsilon_{\phi r} & \varepsilon_{\phi\phi} \end{bmatrix} = [\mathrm{F}] \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} [\mathrm{F}]^{\mathrm{T}} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}. \leftarrow$$

Calculate $\nabla\vec{r}$, $\nabla\cdot\vec{r}$ and $\nabla\times\vec{r}$ in which $\vec{r}$ is the position vector. Use the representations of the cylindrical coordinate system

$$\vec{r}=\begin{Bmatrix} r \\ 0 \\ z \end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix}=\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix} r \\ 0 \\ z \end{Bmatrix},\quad \nabla=\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix} \partial/\partial r \\ \partial/(r\partial\phi) \\ \partial/\partial z \end{Bmatrix} \text{ and } \frac{\partial}{\partial\phi}\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix}=\begin{Bmatrix} \vec{e}_\phi \\ -\vec{e}_r \\ 0 \end{Bmatrix} \text{ (zeros otherwise).}$$

Solution

In a term, gradient operator ∇ acts on everything on its right-hand side. Otherwise, the operator is treated like a vector (if the basis vectors are not constants, the derivative operators should be after the basis vectors)

$$\vec{r}=\begin{Bmatrix} r \\ 0 \\ z \end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix}=\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix} r \\ 0 \\ z \end{Bmatrix}=r\vec{e}_r+z\vec{e}_z\,,$$

$$\nabla=\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix} \partial/\partial r \\ \partial/(r\partial\phi) \\ \partial/\partial z \end{Bmatrix}=\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\phi\frac{\partial}{r\partial\phi}+\vec{e}_z\frac{\partial}{\partial z}\,.$$

Manipulation of a tensor expression consist of (I) substitution of the representations, (II) term by term expansion, (III) evaluation of the terms, (IV) simplification and/or restructuring the outcome. Gradient of the position vector is a second order tensor

(I) $\nabla\vec{r}=(\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\phi\frac{\partial}{r\partial\phi}+\vec{e}_z\frac{\partial}{\partial z})(r\vec{e}_r+z\vec{e}_z)\quad\Leftrightarrow$

(II) $\nabla\vec{r}=\vec{e}_r\frac{\partial}{\partial r}r\vec{e}_r+\vec{e}_r\frac{\partial}{\partial r}z\vec{e}_z+\vec{e}_\phi\frac{\partial}{r\partial\phi}r\vec{e}_r+\vec{e}_\phi\frac{\partial}{r\partial\phi}z\vec{e}_z+\vec{e}_z\frac{\partial}{\partial z}r\vec{e}_r+\vec{e}_z\frac{\partial}{\partial z}z\vec{e}_z\quad\Leftrightarrow$

(III) $\nabla\vec{r}=\vec{e}_r\vec{e}_r+0+\vec{e}_\phi\vec{e}_\phi+0+0+\vec{e}_z\vec{e}_z\quad\Leftrightarrow$

(IV) $\nabla\vec{r}=\vec{e}_r\vec{e}_r+\vec{e}_\phi\vec{e}_\phi+\vec{e}_z\vec{e}_z=\vec{\vec{I}}\,.$ ←

Divergence of the position vector is a scalar

(I) $\nabla\cdot\vec{r}=(\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\phi\frac{\partial}{r\partial\phi}+\vec{e}_z\frac{\partial}{\partial z})\cdot(r\vec{e}_r+z\vec{e}_z)\quad\Leftrightarrow$

(II) $\nabla\cdot\vec{r}=\vec{e}_r\frac{\partial}{\partial r}\cdot r\vec{e}_r+\vec{e}_r\frac{\partial}{\partial r}\cdot z\vec{e}_z+\vec{e}_\phi\frac{\partial}{r\partial\phi}\cdot r\vec{e}_r+\vec{e}_\phi\frac{\partial}{r\partial\phi}\cdot z\vec{e}_z+\vec{e}_z\frac{\partial}{\partial z}\cdot r\vec{e}_r+\vec{e}_z\frac{\partial}{\partial z}\cdot z\vec{e}_z\quad\Leftrightarrow$

(III) $\nabla\cdot\vec{r}=1+0+1+0+0+1\quad\Leftrightarrow$

(IV) $\nabla \cdot \vec{r} = 3$.

Curl of the position vector is a vector

(I) $\nabla \times \vec{r} = (\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{\partial}{r\partial \phi} + \vec{e}_z \frac{\partial}{\partial z}) \times (r\vec{e}_r + z\vec{e}_z) \quad \Leftrightarrow$

(II) $\nabla \times \vec{r} = \vec{e}_r \frac{\partial}{\partial r} \times r\vec{e}_r + \vec{e}_r \frac{\partial}{\partial r} \times z\vec{e}_z + \vec{e}_\phi \frac{\partial}{r\partial \phi} \times r\vec{e}_r + \vec{e}_\phi \frac{\partial}{r\partial \phi} \times z\vec{e}_z + \vec{e}_z \frac{\partial}{\partial z} \times r\vec{e}_r + \vec{e}_z \frac{\partial}{\partial z} \times z\vec{e}_z \quad \Leftrightarrow$

(III) $\nabla \times \vec{r} = 0 + 0 + 0 + 0 + 0 + 0 \quad \Leftrightarrow$

(IV) $\nabla \times \vec{r} = 0$.

Use the definition $\nabla^2 = \nabla \cdot \nabla$ to derive the Laplacian operator in the cylindrical coordinate system.

Solution

Gradient operator and the derivatives of the basis vectors of the cylindrical (r,ϕ,z) – coordinate system are (formula collection)

$$\nabla = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \partial / \partial r \\ \partial / (r\partial\phi) \\ \partial / \partial z \end{Bmatrix} \quad \text{and} \quad \frac{\partial}{\partial \phi}\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix}.$$

According to the definition

$$\nabla^2 = \nabla \cdot \nabla = (\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial \phi} + \vec{e}_z \frac{\partial}{\partial z}) \cdot (\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial \phi} + \vec{e}_z \frac{\partial}{\partial z})$$

in which the terms

$$(\vec{e}_r \frac{\partial}{\partial r}) \cdot (\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial \phi} + \vec{e}_z \frac{\partial}{\partial z}) = \vec{e}_r \cdot (\vec{e}_r \frac{\partial^2}{\partial r^2} - \vec{e}_\phi \frac{1}{r^2}\frac{\partial}{\partial \phi} + \vec{e}_\phi \frac{1}{r}\frac{\partial^2}{\partial r \partial \phi} + \vec{e}_z \frac{\partial^2}{\partial r \partial z}) = \frac{\partial^2}{\partial r^2},$$

$$(\vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial \phi}) \cdot (\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial \phi} + \vec{e}_z \frac{\partial}{\partial z}) = (\vec{e}_\phi \frac{1}{r}) \cdot (\vec{e}_\phi \frac{\partial}{\partial r} + \vec{e}_r \frac{\partial^2}{\partial \phi \partial r} - \vec{e}_r \frac{1}{r}\frac{\partial}{\partial \phi} + \vec{e}_\phi \frac{1}{r}\frac{\partial^2}{\partial \phi^2} + \vec{e}_z \frac{\partial^2}{\partial \phi \partial z}) \quad \Leftrightarrow$$

$$(\vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial \phi}) \cdot (\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial \phi} + \vec{e}_z \frac{\partial}{\partial z}) = \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2},$$

$$(\vec{e}_z \frac{\partial}{\partial z}) \cdot (\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial \phi} + \vec{e}_z \frac{\partial}{\partial z}) = \vec{e}_z \cdot (\vec{e}_r \frac{\partial^2}{\partial z \partial r} + \vec{e}_\phi \frac{1}{r}\frac{\partial^2}{\partial z \partial \phi} + \vec{e}_z \frac{\partial^2}{\partial z^2}) = \frac{\partial^2}{\partial z^2}.$$

Combining the results for the terms gives

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}.$$

Derive the expressions of $\nabla\cdot\vec{u}$, $\nabla\times\vec{u}$, and $\nabla^2 u$ in the polar coordinate system. Vector $\vec{u}(r,\phi)=u_r\vec{e}_r+u_\phi\vec{e}_\phi$ and scalar $u(r,\phi)$ depend on the polar coordinates r and ϕ. In the polar coordinate system

$$\nabla=\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\phi\frac{\partial}{r\partial\phi},\quad \frac{\partial}{\partial\phi}\vec{e}_r=\vec{e}_\phi,\quad \frac{\partial}{\partial\phi}\vec{e}_\phi=-\vec{e}_r,\quad \text{(and } \vec{e}_r\times\vec{e}_\phi=\vec{k}\text{).}$$

Solution

In manipulation of vector expression containing vectors and dyads, it is important to remember that tensor ($\otimes$), cross ($\times$), inner ($\cdot$) products are non-commutative (order matters). The basis vectors of a curvilinear coordinate system are not constants which should be taken into account if gradient operator is part of the expression. Otherwise, simplifying an expression or finding a specific form in a given coordinate system is a straightforward (sometimes tedious) exercise. Manipulation of a tensor expression consist of (I) substitution of the representations, (II) term-by-term expansion, (III) evaluation of the terms, (IV) simplification and/or restructuring the outcome.

Divergence of a vector

(I) $$\nabla\cdot\vec{u}=(\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\phi\frac{\partial}{r\partial\phi})\cdot(u_r\vec{e}_r+u_\phi\vec{e}_\phi)\quad\Leftrightarrow$$

(II) $$\nabla\cdot\vec{u}=\vec{e}_r\cdot\frac{\partial}{\partial r}(u_r\vec{e}_r)+\vec{e}_r\cdot\frac{\partial}{\partial r}(u_\phi\vec{e}_\phi)+\vec{e}_\phi\cdot\frac{\partial}{r\partial\phi}(u_r\vec{e}_r)+\vec{e}_\phi\cdot\frac{\partial}{r\partial\phi}(u_\phi\vec{e}_\phi)\quad\Leftrightarrow$$

(III) $$\nabla\cdot\vec{u}=\vec{e}_r\cdot(\frac{\partial u_r}{\partial r}\vec{e}_r+u_r\frac{\partial\vec{e}_r}{\partial r})+\vec{e}_r\cdot(\frac{\partial u_\phi}{\partial r}\vec{e}_\phi+u_\phi\frac{\partial\vec{e}_\phi}{\partial r})+\vec{e}_\phi\cdot(\frac{\partial u_r}{r\partial\phi}\vec{e}_r+\frac{u_r}{r}\frac{\partial\vec{e}_r}{\partial\phi})+$$

$$\vec{e}_\phi\cdot(\frac{\partial u_\phi}{r\partial\phi}\vec{e}_\phi+\frac{u_\phi}{r}\frac{\partial\vec{e}_\phi}{\partial\phi})\quad\Leftrightarrow$$

$$\nabla\cdot\vec{u}=\vec{e}_r\cdot\frac{\partial u_r}{\partial r}\vec{e}_r+\vec{e}_r\cdot\frac{\partial u_\phi}{\partial r}\vec{e}_\phi+\vec{e}_\phi\cdot(\frac{\partial u_r}{r\partial\phi}\vec{e}_r+\frac{u_r}{r}\vec{e}_\phi)+\vec{e}_\phi\cdot(\frac{\partial u_\phi}{r\partial\phi}\vec{e}_\phi-\frac{u_\phi}{r}\vec{e}_r)\quad\Leftrightarrow$$

$$\nabla\cdot\vec{u}=\frac{\partial u_r}{\partial r}+u_r\frac{1}{r}+\frac{\partial u_\phi}{r\partial\phi}\quad\Leftrightarrow$$

(IV) $$\nabla\cdot\vec{u}=\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{1}{r}\frac{\partial u_\phi}{\partial\phi}.$$

Curl of a vector

(I) $$\nabla\times\vec{u}=(\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\phi\frac{\partial}{r\partial\phi})\times(u_r\vec{e}_r+u_\phi\vec{e}_\phi)\quad\Leftrightarrow$$

(II) $$\nabla\times\vec{u}=\vec{e}_r\frac{\partial}{\partial r}\times u_r\vec{e}_r+\vec{e}_r\frac{\partial}{\partial r}\times u_\phi\vec{e}_\phi+\vec{e}_\phi\frac{\partial}{r\partial\phi}\times u_r\vec{e}_r+\vec{e}_\phi\frac{\partial}{r\partial\phi}\times u_\phi\vec{e}_\phi\quad\Leftrightarrow$$

(III) $$\nabla\times\vec{u}=\vec{e}_r\times\left(\frac{\partial u_r}{\partial r}\vec{e}_r+u_r\frac{\partial\vec{e}_r}{\partial r}\right)+\vec{e}_r\times\left(\frac{\partial u_\phi}{\partial r}\vec{e}_\phi+u_\phi\frac{\partial\vec{e}_\phi}{\partial r}\right)+\vec{e}_\phi\times\left(\frac{\partial u_r}{r\partial\phi}\vec{e}_r+\frac{u_r}{r}\frac{\partial\vec{e}_r}{\partial\phi}\right)+$$

$$\vec{e}_\phi\times\left(\frac{\partial u_\phi}{r\partial\phi}\vec{e}_\phi+\frac{u_\phi}{r}\frac{\partial\vec{e}_\phi}{\partial\phi}\right)\quad\Leftrightarrow$$

$$\nabla\times\vec{u}=\vec{e}_r\times\frac{\partial u_r}{\partial r}\vec{e}_r+\vec{e}_r\times\frac{\partial u_\phi}{\partial r}\vec{e}_\phi+\vec{e}_\phi\times\left(\frac{\partial u_r}{r\partial\phi}\vec{e}_r+\frac{u_r}{r}\vec{e}_\phi\right)+\vec{e}_\phi\times\left(\frac{\partial u_\phi}{r\partial\phi}\vec{e}_\phi-\frac{u_\phi}{r}\vec{e}_r\right)\quad\Leftrightarrow$$

$$\nabla\times\vec{u}=\frac{\partial u_\phi}{\partial r}\vec{k}-\frac{\partial u_r}{r\partial\phi}\vec{k}+\frac{u_\phi}{r}\vec{k}\quad\Leftrightarrow$$

(IV) $$\nabla\times\vec{u}=\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_\phi)-\frac{1}{r}\frac{\partial u_r}{\partial\phi}\right)\vec{k}\,.$$

Laplacian of a scalar

(I) $$\nabla^2u=(\nabla\cdot\nabla)u=\left(\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\phi\frac{\partial}{r\partial\phi}\right)\cdot\left(\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\phi\frac{\partial}{r\partial\phi}\right)u\quad\Leftrightarrow$$

(II) $$\nabla^2u=\vec{e}_r\cdot\frac{\partial}{\partial r}\left(\vec{e}_r\frac{\partial}{\partial r}\right)u+\vec{e}_\phi\cdot\frac{\partial}{r\partial\phi}\left(\vec{e}_r\frac{\partial}{\partial r}\right)u+\vec{e}_r\cdot\frac{\partial}{\partial r}\left(\vec{e}_\phi\frac{\partial}{r\partial\phi}\right)u+\vec{e}_\phi\cdot\frac{\partial}{r\partial\phi}\left(\vec{e}_\phi\frac{\partial}{r\partial\phi}\right)u\quad\Leftrightarrow$$

$$\nabla^2u=\vec{e}_r\cdot\left(\frac{\partial\vec{e}_r}{\partial r}\frac{\partial}{\partial r}+\vec{e}_r\frac{\partial^2}{\partial r^2}\right)u+\vec{e}_\phi\cdot\left(\frac{\partial\vec{e}_r}{r\partial\phi}\frac{\partial}{\partial r}+\vec{e}_r\frac{1}{r}\frac{\partial^2}{\partial\phi\partial r}\right)u+$$

$$\vec{e}_r\cdot\left(\frac{\partial\vec{e}_\phi}{\partial r}\frac{\partial}{r\partial\phi}-\vec{e}_\phi\frac{1}{r^2}\frac{\partial}{\partial\phi}+\vec{e}_\phi\frac{1}{r}\frac{\partial^2}{\partial r\partial\phi}\right)u+\vec{e}_\phi\cdot\left(\frac{\partial\vec{e}_\phi}{r\partial\phi}\frac{\partial}{r\partial\phi}+\vec{e}_\phi\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\right)u\quad\Leftrightarrow$$

$$\nabla^2u=\vec{e}_r\cdot\left(\vec{e}_r\frac{\partial^2}{\partial r^2}\right)u+\vec{e}_\phi\cdot\left(\frac{1}{r}\vec{e}_\phi\frac{\partial}{\partial r}+\vec{e}_r\frac{1}{r}\frac{\partial^2}{\partial\phi\partial r}\right)u+\vec{e}_r\cdot\left(-\vec{e}_\phi\frac{1}{r^2}\frac{\partial}{\partial\phi}+\vec{e}_\phi\frac{1}{r}\frac{\partial^2}{\partial r\partial\phi}\right)u+$$

$$\vec{e}_\phi\cdot\left(-\frac{1}{r}\vec{e}_r\frac{\partial}{r\partial\phi}+\vec{e}_\phi\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\right)u\quad\Leftrightarrow$$

$$\nabla^2u=\frac{\partial^2}{\partial r^2}u+\frac{1}{r}\frac{\partial}{\partial r}u+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}u\quad\Leftrightarrow$$

(IV) $$\nabla^2u=\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\right)u\,.$$

In the beam coordinate system and planar case, the displacement assumption of a curved Timoshenko beam model is $\vec{u} = \vec{u}_0 + \vec{\theta}_0 \times \vec{\rho}$, where $\vec{u}_0 = u(s)\vec{e}_s + v(s)\vec{e}_n$, $\vec{\theta}_0 = \psi(s)\vec{e}_b$, and $\vec{\rho} = n\vec{e}_n$. Derive the small strain component expressions ε_{ss} and $\varepsilon_{sn} = \varepsilon_{ns}$ using

$$\vec{\varepsilon} = \frac{1}{2}[\nabla\vec{u} + (\nabla\vec{u})_{\mathrm{c}}], \ \nabla = \vec{e}_s \frac{1}{1-n/R}\frac{\partial}{\partial s} + \vec{e}_n \frac{\partial}{\partial n}, \ \frac{\partial}{\partial s}\begin{Bmatrix}\vec{e}_s \\ \vec{e}_n\end{Bmatrix} = \frac{1}{R}\begin{Bmatrix}\vec{e}_n \\ -\vec{e}_s\end{Bmatrix}, \text{ and } \frac{\partial}{\partial n}\begin{Bmatrix}\vec{e}_s \\ \vec{e}_n\end{Bmatrix} = 0.$$

Assume that curvature $\kappa = 1/R$ is constant.

Solution

The curved beam coordinates are distance s measured along the mid-curve (identifying the particles along the mid-curve), and n,b identifying the particles away from the mid-curve. The $(s,n,b)-$ system is orthonormal and right-handed. The gradient expression of the coordinate system

$$\nabla = \vec{e}_s \frac{R}{R-n}\frac{\partial}{\partial s} + \vec{e}_n \frac{\partial}{\partial n}$$

is available in the formula collection. When the expressions of the cross-section translation, cross-section rotation and the relative position vector are substituted there, the displacement expression takes the form

$$\vec{u} = \vec{u}_0 + \vec{\theta}_0 \times \vec{\rho} = u(s)\vec{e}_s + v(s)\vec{e}_n + \psi(s)\vec{e}_b \times n\vec{e}_n = (u - \psi n)\vec{e}_s + v\vec{e}_n .$$

Displacement gradient is given by (Lagrange’s notation for derivative with respect to s)

$$\nabla\vec{u} = (\vec{e}_s \frac{R}{R-n}\frac{\partial}{\partial s} + \vec{e}_n \frac{\partial}{\partial n})[(u-\psi n)\vec{e}_s + v\vec{e}_n] \quad \Leftrightarrow$$

$$\nabla\vec{u} = \vec{e}_s \frac{R}{R-n}\frac{\partial}{\partial s}(u-\psi n)\vec{e}_s + \vec{e}_s \frac{R}{R-n}\frac{\partial}{\partial s}(v\vec{e}_n) + (\vec{e}_n \frac{\partial}{\partial n})(u-\psi n)\vec{e}_s + \vec{e}_n \frac{\partial}{\partial n} v\vec{e}_n \quad \Leftrightarrow$$

$$\nabla\vec{u} = \vec{e}_s \frac{R}{R-n}(u'-\psi' n)\vec{e}_s + \vec{e}_s \frac{R}{R-n}(u-\psi n)\vec{e}_s' + \vec{e}_s \frac{R}{R-n} v'\vec{e}_n + \vec{e}_s \frac{R}{R-n} v\vec{e}_n' - \vec{e}_n \psi \vec{e}_s \quad \Rightarrow$$

$$\nabla\vec{u} = \vec{e}_s \frac{R}{R-n}(u'-\psi' n)\vec{e}_s + \vec{e}_s \frac{R}{R-n}(u-\psi n)\frac{1}{R}\vec{e}_n + \vec{e}_s \frac{R}{R-n} v'\vec{e}_n - \vec{e}_s \frac{R}{R-n} v \frac{1}{R}\vec{e}_s - \vec{e}_n \psi \vec{e}_s \quad \Leftrightarrow$$

$$\nabla\vec{u} = [\frac{R}{R-n}(u'-\psi' n) - \frac{1}{R-n}v]\vec{e}_s\vec{e}_s + [\frac{1}{R-n}(u-\psi n) + \frac{R}{R-n}v']\vec{e}_s\vec{e}_n - \psi\vec{e}_n\vec{e}_s .$$

In conjugate tensor, order of the basis vectors is changed

$$(\nabla\vec{u})_{\mathrm{c}} = [\frac{R}{R-n}(u'-\psi' n) - \frac{1}{R-n}v]\vec{e}_s\vec{e}_s + [\frac{1}{R-n}(u-\psi n) + \frac{R}{R-n}v']\vec{e}_n\vec{e}_s - \psi\vec{e}_s\vec{e}_n .$$

Definition of the small (linear) strains $\vec{\varepsilon} = \frac{1}{2}[\nabla\vec{u} + (\nabla\vec{u})_{\mathrm{c}}]$ gives first

$$\ddot{\varepsilon} = \frac{1}{2}[\frac{R}{R-n}(u'-\psi'n) - \frac{1}{R-n}v]\vec{e}_s\vec{e}_s + \frac{1}{2}[\frac{1}{R-n}(u-\psi n) + \frac{R}{R-n}v']\vec{e}_s\vec{e}_n - \frac{1}{2}\psi\vec{e}_n\vec{e}_s +$$

$$\frac{1}{2}[\frac{R}{R-n}(u'-\psi'n) - \frac{1}{R-n}v]\vec{e}_s\vec{e}_s + \frac{1}{2}[\frac{1}{R-n}(u-\psi n) + \frac{R}{R-n}v']\vec{e}_n\vec{e}_s - \frac{1}{2}\psi\vec{e}_s\vec{e}_n \quad \Leftrightarrow$$

$$\ddot{\varepsilon} = [\frac{R}{R-n}(u'-\psi'n) - \frac{1}{R-n}v]\vec{e}_s\vec{e}_s + \frac{1}{2}[\frac{1}{R-n}(u-\psi n) + \frac{R}{R-n}v' - \psi](\vec{e}_s\vec{e}_n + \vec{e}_n\vec{e}_s) .$$

Hence as the components of strain are the multipliers of the basis vector pairs with the same order of indices

$$\varepsilon_{ss} = \frac{R}{R-n}(u'-\psi'n - \frac{1}{R}v) \quad \text{and} \quad \varepsilon_{sn} = \varepsilon_{ns} = \frac{1}{2}\frac{R}{R-n}(\frac{1}{R}u + v' - \psi) .$$

Mapping $\vec{r}(r,\phi,z)=r\cos\phi\vec{i}+r\sin\phi\vec{j}+z\vec{k}$ defines the cylindrical $(r,\phi,z)-$coordinate system. Use (in detail) the generic formula

$$\frac{\partial}{\partial\eta}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_z\end{Bmatrix}=(\frac{\partial}{\partial\eta}[F])[F]^{-1}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_z\end{Bmatrix}, \text{ where } \eta\in\{r,\phi,z\}$$

to find the derivatives of the basis vectors.

Solution

Basis vectors of the $(r,\phi,z)-$coordinate system are obtained as derivatives of the position vector $\vec{r}(r,\phi,z)=r\cos\phi\vec{i}+r\sin\phi\vec{j}+z\vec{k}$ with respect to $(r,\phi,z)-$coordinates. As the position vector is expressed in terms of the constants basis vectors of a Cartesian system, the outcome is a relationship between the basis vectors of the spherical and Cartesian systems (i.e. the matrix $[F]$ needed in the generic expression for the basis vector derivatives)

$$\frac{\partial}{\partial r}\vec{r}=\cos\phi\vec{i}+\sin\phi\vec{j} \quad\Rightarrow\quad \left|\frac{\partial}{\partial r}\vec{r}\right|=1,$$

$$\frac{\partial}{\partial\phi}\vec{r}=-r\sin\phi\vec{i}+r\cos\phi\vec{j} \quad\Rightarrow\quad \left|\frac{\partial}{\partial\phi}\vec{r}\right|=r,$$

$$\frac{\partial}{\partial z}\vec{r}=\vec{k} \quad\Rightarrow\quad \left|\frac{\partial}{\partial z}\vec{r}\right|=1.$$

The relationship between the basis vectors can be written as

$$\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_z\end{Bmatrix}=\begin{Bmatrix}(\partial\vec{r}/\partial r)/|\partial\vec{r}/\partial r|\\ (\partial\vec{r}/\partial\phi)/|\partial\vec{r}/\partial\phi|\\ (\partial\vec{r}/\partial z)/|\partial\vec{r}/\partial z|\end{Bmatrix}=\begin{bmatrix}\cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}=[F]\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}$$

in which the matrix satisfies $[F]^{-1}=[F]^{\mathrm{T}}$. The generic formula for the partial derivatives of the basis vectors gives now

$$\frac{\partial}{\partial r}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_z\end{Bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_z\end{Bmatrix}=\begin{Bmatrix}0\\ 0\\ 0\end{Bmatrix}, \leftarrow$$

$$\frac{\partial}{\partial\phi}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_z\end{Bmatrix}=\begin{bmatrix}-\sin\phi & \cos\phi & 0\\ -\cos\phi & -\sin\phi & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_z\end{Bmatrix}=\begin{bmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_z\end{Bmatrix}=\begin{Bmatrix}\vec{e}_\phi\\ -\vec{e}_r\\ 0\end{Bmatrix}, \leftarrow$$

and

$$\frac{\partial}{\partial z}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_z\end{Bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}\cos\phi&-\sin\phi&0\\ \sin\phi&\cos\phi&0\\0&0&1\end{bmatrix}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_z\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\end{Bmatrix}.$$

Derive the gradient expression of the spherical $(\theta,\phi,r)-$coordinate system, when the mapping defining the coordinate system is given by $\vec{r}(\theta,\phi,r)=r(\sin\theta\cos\phi\vec{i}+\sin\theta\sin\phi\vec{j}+\cos\theta\vec{k})$.

Solution

According to the generic recipe (formulae collection)

$$\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_\gamma\end{Bmatrix}=\begin{Bmatrix}(\partial\vec{r}/\partial\alpha)/|\partial\vec{r}/\partial\alpha|\\ (\partial\vec{r}/\partial\beta)/|\partial\vec{r}/\partial\beta|\\ (\partial\vec{r}/\partial\gamma)/|\partial\vec{r}/\partial\gamma|\end{Bmatrix}=[F]\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix},\quad \frac{\partial}{\partial\eta}\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_\gamma\end{Bmatrix}=(\frac{\partial}{\partial\eta}[F])[F]^{-1}\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_\gamma\end{Bmatrix}\quad \eta\in\{\alpha,\beta,\gamma\},$$

$$\nabla=\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_\gamma\end{Bmatrix}^{\mathrm{T}}[F]^{-\mathrm{T}}[H]^{-1}\begin{Bmatrix}\partial/\partial\alpha\\ \partial/\partial\beta\\ \partial/\partial\gamma\end{Bmatrix}\quad\text{where } [H]=\begin{bmatrix}\partial r_x/\partial\alpha & \partial r_y/\partial\alpha & \partial r_z/\partial\alpha\\ \partial r_x/\partial\beta & \partial r_y/\partial\beta & \partial r_z/\partial\beta\\ \partial r_x/\partial\gamma & \partial r_y/\partial\gamma & \partial r_z/\partial\gamma\end{bmatrix},$$

in which $\alpha=\theta$, $\beta=\phi$, and $\gamma=r$ in the present case. Matrices $[F]$ and $[H]$ depend on the mapping

$$\vec{r}(\theta,\phi,r)=r(\sin\theta\cos\phi\vec{i}+\sin\theta\sin\phi\vec{j}+\cos\theta\vec{k})=r_x(\theta,\phi,r)\vec{i}+r_y(\theta,\phi,r)\vec{j}+r_z(\theta,\phi,r)\vec{k}\,.$$

By definition

$$\vec{e}_\theta=\frac{\partial\vec{r}}{\partial\theta}/\left|\frac{\partial\vec{r}}{\partial\theta}\right|=\cos\theta\cos\phi\vec{i}+\cos\theta\sin\phi\vec{j}-\sin\theta\vec{k}\,,$$

$$\vec{e}_\phi=\frac{\partial\vec{r}}{\partial\phi}/\left|\frac{\partial\vec{r}}{\partial\phi}\right|=-\sin\phi\vec{i}+\cos\phi\vec{j}\,,$$

$$\vec{e}_r=\frac{\partial\vec{r}}{\partial r}/\left|\frac{\partial\vec{r}}{\partial r}\right|=\sin\theta\cos\phi\vec{i}+\sin\theta\sin\phi\vec{j}+\cos\theta\vec{k}$$

and therefore

$$\begin{Bmatrix}\vec{e}_\theta\\ \vec{e}_\phi\\ \vec{e}_r\end{Bmatrix}=\begin{bmatrix}\mathrm{c}\theta\mathrm{c}\phi & \mathrm{c}\theta\mathrm{s}\phi & -\mathrm{s}\theta\\ -\mathrm{s}\phi & \mathrm{c}\phi & 0\\ \mathrm{s}\theta\mathrm{c}\phi & \mathrm{s}\theta\mathrm{s}\phi & \mathrm{c}\theta\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}=[F]\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}\text{ so } [F]=\begin{bmatrix}\mathrm{c}\theta\mathrm{c}\phi & \mathrm{c}\theta\mathrm{s}\phi & -\mathrm{s}\theta\\ -\mathrm{s}\phi & \mathrm{c}\phi & 0\\ \mathrm{s}\theta\mathrm{c}\phi & \mathrm{s}\theta\mathrm{s}\phi & \mathrm{c}\theta\end{bmatrix}.$$

According to the mapping, the relationship between the components of the position vector in the Cartesian and cylindrical systems are $r_x=r\sin\theta\cos\phi$, $r_y=r\sin\theta\sin\phi$, and $r_z=r\cos\theta$

$$[H]=\begin{bmatrix}\partial r_x/\partial\theta & \partial r_y/\partial\theta & \partial r_z/\partial\theta\\ \partial r_x/\partial\phi & \partial r_y/\partial\phi & \partial r_z/\partial\phi\\ \partial r_x/\partial r & \partial r_y/\partial r & \partial r_z/\partial r\end{bmatrix}=\begin{bmatrix}r\mathrm{c}\theta\mathrm{c}\phi & r\mathrm{c}\theta\mathrm{s}\phi & -r\mathrm{s}\theta\\ -r\mathrm{s}\theta s\phi & r\mathrm{s}\theta\mathrm{c}\phi & 0\\ \mathrm{s}\theta\mathrm{c}\phi & \mathrm{s}\theta\mathrm{s}\phi & \mathrm{c}\theta\end{bmatrix}.$$

Gradient follows now from the generic recipe

$$\nabla = \begin{Bmatrix} \vec{e}_\theta \\ \vec{e}_\phi \\ \vec{e}_r \end{Bmatrix}^{\mathrm{T}} [F]^{-\mathrm{T}} [H]^{-1} \begin{Bmatrix} \partial / \partial\theta \\ \partial / \partial\phi \\ \partial / \partial r \end{Bmatrix} = \begin{Bmatrix} \vec{e}_\theta \\ \vec{e}_\phi \\ \vec{e}_r \end{Bmatrix}^{\mathrm{T}} ([H][F]^{\mathrm{T}})^{-1} \begin{Bmatrix} \partial / \partial\theta \\ \partial / \partial\phi \\ \partial / \partial r \end{Bmatrix}.$$

Let us calculate first the matrix inside the parenthesis

$$[H][F]^{\mathrm{T}} = \begin{bmatrix} r\mathrm{c}\theta\mathrm{c}\phi & r\mathrm{c}\theta\mathrm{s}\phi & -r\mathrm{s}\theta \\ -r\mathrm{s}\theta\mathrm{s}\phi & r\mathrm{s}\theta\mathrm{c}\phi & 0 \\ \mathrm{s}\theta\mathrm{c}\phi & \mathrm{s}\theta\mathrm{s}\phi & \mathrm{c}\theta \end{bmatrix} \begin{bmatrix} \mathrm{c}\theta\mathrm{c}\phi & -\mathrm{s}\phi & \mathrm{s}\theta\mathrm{c}\phi \\ \mathrm{c}\theta\mathrm{s}\phi & \mathrm{c}\phi & \mathrm{s}\theta\mathrm{s}\phi \\ -\mathrm{s}\theta & 0 & \mathrm{c}\theta \end{bmatrix} = \begin{bmatrix} r & 0 & 0 \\ 0 & r\mathrm{s}\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \Leftrightarrow$$

$$([H][F]^{\mathrm{T}})^{-1} = \begin{bmatrix} r & 0 & 0 \\ 0 & r\mathrm{s}\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/r & 0 & 0 \\ 0 & 1/(r\mathrm{s}\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Substituting into the gradient expression

$$\nabla = \begin{Bmatrix} \vec{e}_\theta \\ \vec{e}_\phi \\ \vec{e}_r \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1/r & 0 & 0 \\ 0 & 1/(r\mathrm{s}\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \partial / \partial\theta \\ \partial / \partial\phi \\ \partial / \partial r \end{Bmatrix} = \vec{e}_\theta \frac{1}{r}\frac{\partial}{\partial\theta} + \vec{e}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} + \vec{e}_r \frac{\partial}{\partial r}.$$

Compute the derivatives of the basis vectors, gradient operator, and curvature for the cylindrical shell geometry with the mid-surface representation $\vec{r}_0(\phi, z) = R(\cos\phi\vec{i} + \sin\phi\vec{j}) + z\vec{k}$ in terms of coordinates (ϕ, z). Notice that the order of the coordinates differs from that of the lecture notes, which affects, e.g., direction of $\vec{e}_n$.

Solution

Solution of the problems consists of two parts. The aim of the first part is to derive the derivatives of the basis vectors and the gradient operator representation of a curvilinear system (by direct calculation or from the formulae collection). The second part is just application of the curvature tensor definition

$$\vec{\kappa} = (\nabla\vec{e}_n)_{\mathrm{c}}$$

describing in a concise manner the way the coordinate system deviates from being Cartesian. The mid-surface curvature tensor corresponds to $n = 0$. Measures of $\vec{\kappa}$, like Gaussian curvature or mean curvature, describe the geometry of the mid-surface.

The relationship between the basis vectors of the Cartesian and cylindrical systems follows from definition. Notice that two of the basis vectors follow from the mid-surface mapping. The third is always normal to the surface and obtained as a vector product (see the coordinate system for shells)

$$\begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_z \\ \vec{e}_n \end{Bmatrix} = \begin{Bmatrix} (\partial\vec{r}_0 / \partial\phi) / |\partial\vec{r}_0 / \partial\phi| \\ (\partial\vec{r}_0 / \partial z) / |\partial\vec{r}_0 / \partial z| \\ \vec{e}_\phi \times \vec{e}_z \end{Bmatrix} = \begin{bmatrix} -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \\ \cos\phi & \sin\phi & 0 \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = [F] \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}.$$

Derivatives of the basis vector follow from the generic formula

$$\frac{\partial}{\partial\phi} \begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_z \\ \vec{e}_n \end{Bmatrix} = \begin{bmatrix} -\cos\phi & -\sin\phi & 0 \\ 0 & 0 & 1 \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \\ \cos\phi & \sin\phi & 0 \end{bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_z \\ \vec{e}_n \end{Bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_z \\ \vec{e}_n \end{Bmatrix}$$

the remaining being zeros. Mapping for the mid-surface is kind of labelling system for the particles (points) on that surface. To have a labelling system for all the particles (points) of a thin body, a relative position vector is also needed, so

$$\vec{r}(\phi, z, n) = \vec{r}_0(\phi, z) + n\vec{e}_n(\phi, z).$$

The relative position vector defines also the line segments perpendicular to the mid-surface (an important concept in plate theory). The Hessian of the mapping between the Cartesian and thin-body (ϕ, z, n) – coordinates takes the form

$$[H] = \begin{bmatrix} \partial r_x / \partial\phi & \partial r_y / \partial\phi & \partial r_z / \partial\phi \\ \partial r_x / \partial z & \partial r_y / \partial z & \partial r_z / \partial z \\ \partial r_x / \partial n & \partial r_y / \partial n & \partial r_z / \partial n \end{bmatrix} = \begin{bmatrix} -(n+R)\sin\phi & (n+R)\cos\phi & 0 \\ 0 & 0 & 1 \\ \cos\phi & \sin\phi & 0 \end{bmatrix}$$

having the inverse

$$[H]^{-1}=\begin{bmatrix} -\sin\phi/(n+R) & 0 & \cos\phi \\ \cos\phi/(n+R) & 0 & \sin\phi \\ 0 & 1 & 0 \end{bmatrix}.$$

The generic form of the gradient expression gives now (formulae collection)

$$\nabla=\begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_z \\ \vec{e}_n \end{Bmatrix}^{\mathrm{T}}\begin{bmatrix} -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \\ \cos\phi & \sin\phi & 0 \end{bmatrix}\begin{bmatrix} -\dfrac{\sin\phi}{n+R} & 0 & \cos\phi \\ \dfrac{\cos\phi}{n+R} & 0 & \sin\phi \\ 0 & 1 & 0 \end{bmatrix}\begin{Bmatrix} \partial/\partial\phi \\ \partial/\partial z \\ \partial/\partial n \end{Bmatrix} \Leftrightarrow$$

$$\nabla=\vec{e}_\phi\frac{1}{n+R}\frac{\partial}{\partial\phi}+\vec{e}_z\frac{\partial}{\partial z}+\vec{e}_n\frac{\partial}{\partial n}.$$

Curvature is obtained from the gradient of the normal vector

$$\vec{\kappa}_{\mathrm{c}}=\vec{e}_\phi\frac{1}{n+R}\frac{\partial\vec{e}_n}{\partial\phi}+\vec{e}_z\frac{\partial\vec{e}_n}{\partial z}+\vec{e}_n\frac{\partial\vec{e}_n}{\partial n}=\vec{e}_\phi\vec{e}_\phi\frac{1}{n+R}$$

giving at the mid-surface ($n=0$) $\vec{\kappa}=\vec{e}_\phi\vec{e}_\phi\dfrac{1}{R}$.

Consider the mid-surface mapping $\vec{r}_0(r,\phi)=r^2\cos(2\phi)\vec{i}+r^2\sin(2\phi)\vec{j}$ of shell. Compute the expression of the basis vector derivatives and gradient operator ∇. Is the mid-surface defined by the mapping flat or curved?

Solution

The relationship between the basis vectors of the Cartesian and cylindrical systems follows from definition. Notice that two of the basis vectors follow from the mid-surface mapping. The third is always normal to the surface and obtained as a vector product (see the coordinate system for shells). With the present mid-surface $(r,\phi)-$coordinates

$$\vec{r}_0(r,\phi)=r^2\cos(2\phi)\vec{i}+r^2\sin(2\phi)\vec{j} \text{ and } \begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_n\end{Bmatrix}=\begin{Bmatrix}(\partial\vec{r}_0/\partial r)/|\partial\vec{r}_0/\partial r|\\ (\partial\vec{r}_0/\partial\phi)/|\partial\vec{r}_0/\partial\phi|\\ \vec{e}_r\times\vec{e}_\phi\end{Bmatrix}=[F]\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}.$$

Expressions of the basis vectors of the curvilinear system are

$$\frac{\partial}{\partial r}\vec{r}_0=2r\cos(2\phi)\vec{i}+2r\sin(2\phi)\vec{j} \quad\Rightarrow\quad \vec{e}_r=(\frac{\partial}{\partial r}\vec{r}_0)/|\frac{\partial}{\partial r}\vec{r}_0|=\cos(2\phi)\vec{i}+\sin(2\phi)\vec{j},$$

$$\frac{\partial}{\partial\phi}\vec{r}_0=-2r^2\sin(2\phi)\vec{i}+2r^2\cos(2\phi)\vec{j} \quad\Rightarrow\quad \vec{e}_\phi=(\frac{\partial}{\partial\phi}\vec{r}_0)/|\frac{\partial}{\partial\phi}\vec{r}_0|=-\sin(2\phi)\vec{i}+\cos(2\phi)\vec{j},$$

$$\vec{e}_n=\vec{e}_r\times\vec{e}_\phi=[\cos(2\phi)\vec{i}+\sin(2\phi)\vec{j}]\times[-\sin(2\phi)\vec{i}+\cos(2\phi)\vec{j}]=\vec{k}.$$

In a more compact form

$$\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_n\end{Bmatrix}=\begin{bmatrix}\cos(2\phi) & \sin(2\phi) & 0\\ -\sin(2\phi) & \cos(2\phi) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}=[F]\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix} \text{ in which } [F]^{-1}=[F]^{\mathrm{T}}.$$

Direct use of the definition gives (just take the derivatives on both sides of the relationship above and use inverse of the same relationship to replace the basis vectors of the Cartesian system by the basis vectors of the $(r,\phi,n)-$system)

$$\frac{\partial}{\partial r}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_n\end{Bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}\cos(2\phi) & -\sin(2\phi) & 0\\ \sin(2\phi) & \cos(2\phi) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_n\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\end{Bmatrix},$$

$$\frac{\partial}{\partial\phi}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_n\end{Bmatrix}=\begin{bmatrix}-2\sin(2\phi) & 2\cos(2\phi) & 0\\ -2\cos(2\phi) & -2\sin(2\phi) & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\cos(2\phi) & -\sin(2\phi) & 0\\ \sin(2\phi) & \cos(2\phi) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_n\end{Bmatrix}=\begin{bmatrix}0&2&0\\-2&0&0\\0&0&0\end{bmatrix}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_n\end{Bmatrix},$$

$$\frac{\partial}{\partial n}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_n\end{Bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}\cos(2\phi)&-\sin(2\phi)&0\\ \sin(2\phi)&\cos(2\phi)&0\\0&0&1\end{bmatrix}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_n\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\end{Bmatrix}.$$

Gradient in the $(r,\phi,n)-$ system follows from the mapping

$$\vec{r}(r,\phi,n)=\vec{r}_0+\vec{\rho}=r^2\cos(2\phi)\vec{i}+r^2\sin(2\phi)\vec{j}+n\vec{k}$$

and the generic formula in terms of $[F]$ and $[H]$ with

$$\begin{Bmatrix}\partial\vec{r}/\partial r\\ \partial\vec{r}/\partial\phi\\ \partial\vec{r}/\partial n\end{Bmatrix}=\begin{bmatrix}2r\cos(2\phi)&2r\sin(2\phi)&0\\-2r^2\sin(2\phi)&2r^2\cos(2\phi)&0\\0&0&1\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}=[H]\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}\quad\Rightarrow$$

$$[H][F]^{\mathrm{T}}=\begin{bmatrix}2r\cos(2\phi)&2r\sin(2\phi)&0\\-2r^2\sin(2\phi)&2r^2\cos(2\phi)&0\\0&0&1\end{bmatrix}\begin{bmatrix}\cos(2\phi)&-\sin(2\phi)&0\\ \sin(2\phi)&\cos(2\phi)&0\\0&0&1\end{bmatrix}=\begin{bmatrix}2r&0&0\\0&2r^2&0\\0&0&1\end{bmatrix}\Rightarrow$$

$$\nabla=\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_n\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}2r&0&0\\0&2r^2&0\\0&0&1\end{bmatrix}^{-1}\begin{Bmatrix}\partial/\partial r\\ \partial/\partial\phi\\ \partial/\partial n\end{Bmatrix}=\vec{e}_r\frac{1}{2r}\frac{\partial}{\partial r}+\vec{e}_\phi\frac{1}{2r^2}\frac{\partial}{\partial\phi}+\vec{e}_n\frac{\partial}{\partial n}\;.$$

Curvature of the mid-surface ($n=0$)

$$\vec{\kappa}_{\mathrm{c}}=\nabla\vec{e}_n=\vec{e}_r\frac{1}{2r}\frac{\partial\vec{e}_n}{\partial r}+\vec{e}_\phi\frac{1}{2r^2}\frac{\partial\vec{e}_n}{\partial\phi}+\vec{e}_n\frac{\partial\vec{e}_n}{\partial n}=0$$

which indicates that the mid-surface is flat.

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Assignment 1

Use the polar coordinate system representations

$$\nabla = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \partial / \partial r \\ \partial / (r\partial\phi) \end{Bmatrix} = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial\phi} \quad , \quad \vec{u} = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_r \\ u_\phi \end{Bmatrix} = \vec{e}_r u_r + \vec{e}_\phi u_\phi$$

to calculate $\nabla\vec{u}$. Assume that the displacement components u_r and u_ϕ are functions of the angle ϕ only. Derivatives of the basis vectors are

$$\frac{\partial}{\partial\phi}\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} = \begin{Bmatrix} \vec{e}_\phi \\ -\vec{e}_r \end{Bmatrix} \text{ and } \frac{\partial}{\partial r}\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} = 0 \; .$$

Solution template

Evaluation of a tensor expression consist of (I) substitution of the representations, (II) term-by-term expansion, (III) evaluation of the terms, (IV) simplification and/or restructuring the outcome.

First, substitute the representations

$$\nabla\vec{u} = (\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial\phi})(\vec{e}_r u_r + \vec{e}_\phi u_\phi)$$

Second, expand

$$\nabla\vec{u} = \vec{e}_r \frac{\partial}{\partial r}(\vec{e}_r u_r) + \vec{e}_r \frac{\partial}{\partial r}(\vec{e}_\phi u_\phi) + \vec{e}_\phi \frac{\partial}{r\partial\phi}(\vec{e}_r u_r) + \vec{e}_\phi \frac{\partial}{r\partial\phi}(\vec{e}_\phi u_\phi)$$

Third, calculate the derivatives by taking into account the known expressions of the basis vector derivatives and assumption that u_r and u_ϕ are functions of the angle ϕ only.

$$\nabla\vec{u} = 0 + 0 + \vec{e}_\phi \frac{1}{r}(\vec{e}_\phi u_r + \vec{e}_r \frac{\partial u_r}{\partial\phi}) + \vec{e}_\phi \frac{1}{r}(-\vec{e}_r u_\phi + \vec{e}_\phi \frac{\partial u_\phi}{\partial\phi})$$

Fourth, combine the terms to get

$$\nabla\vec{u} = \vec{e}_\phi\vec{e}_\phi \frac{1}{r}(\frac{\partial u_\phi}{\partial\phi} + u_r) + \vec{e}_\phi\vec{e}_r \frac{1}{r}(\frac{\partial u_r}{\partial\phi} - u_\phi) \; . \quad \Leftarrow$$

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Assignment 2

In the coil beam and spherical shell geometries the gradients at the mid-curve ($n=b=0$) or at the mid-surface ($n=0$) and the basis vector derivatives are

$$\nabla_0=\begin{Bmatrix}\vec{e}_s\\ \vec{e}_n\\ \vec{e}_b\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\partial/\partial s\\ \partial/\partial n\\ \partial/\partial b\end{Bmatrix},\quad \frac{\partial}{\partial s}\begin{Bmatrix}\vec{e}_s\\ \vec{e}_n\\ \vec{e}_b\end{Bmatrix}=\frac{1}{h^2+R^2}\begin{bmatrix}0 & R & 0\\ -R & 0 & h\\ 0 & -h & 0\end{bmatrix}\begin{Bmatrix}\vec{e}_s\\ \vec{e}_n\\ \vec{e}_b\end{Bmatrix},\ \text{and}$$

$$\nabla_0=\begin{Bmatrix}\vec{e}_\phi\\ \vec{e}_\theta\\ \vec{e}_n\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\partial/(R\sin\theta\partial\phi)\\ \partial/(R\partial\theta)\\ \partial/\partial n\end{Bmatrix},\quad \frac{\partial}{\partial\phi}\begin{Bmatrix}\vec{e}_\phi\\ \vec{e}_\theta\\ \vec{e}_n\end{Bmatrix}=\begin{Bmatrix}\sin\theta\vec{e}_n-\cos\theta\vec{e}_\theta\\ \cos\theta\vec{e}_\phi\\ -\sin\theta\,\vec{e}_\phi\end{Bmatrix},\quad \frac{\partial}{\partial\theta}\begin{Bmatrix}\vec{e}_\phi\\ \vec{e}_\theta\\ \vec{e}_n\end{Bmatrix}=\begin{Bmatrix}0\\ \vec{e}_n\\ -\vec{e}_\theta\end{Bmatrix},$$

respectively. Above R, h are constants. Use the definition $\vec{\kappa}=(\nabla_0\vec{e}_n)_{\mathrm{c}}$ to find the curvature of the mid-curve of the coil beam and the mid-surface of the spherical shell.

Solution template

Since the basis vectors derivatives and the gradient expression are given, definition of the curvature can be used directly. For the coil beam

$$\nabla_0\vec{e}_n=(\vec{e}_s\frac{\partial}{\partial s}+\vec{e}_n\frac{\partial}{\partial n}+\vec{e}_b\frac{\partial}{\partial b})\vec{e}_n \quad \Rightarrow$$

$$\nabla_0\vec{e}_n=\vec{e}_s\frac{\partial\vec{e}_n}{\partial s}+0+0=-\frac{R}{R^2+h^2}\vec{e}_s\vec{e}_s+\frac{h}{R^2+h^2}\vec{e}_s\vec{e}_b \quad \Rightarrow$$

$$\vec{\kappa}=(\nabla_0\vec{e}_n)_{\mathrm{c}}=-\frac{R}{R^2+h^2}\vec{e}_s\vec{e}_s+\frac{h}{R^2+h^2}\vec{e}_b\vec{e}_s\,.$$

For the spherical shell

$$\nabla_0\vec{e}_n=(\,\vec{e}_\phi\frac{\partial}{R\sin\theta\partial\phi}+\vec{e}_\theta\frac{\partial}{R\partial\theta}+\vec{e}_n\frac{\partial}{\partial n})\vec{e}_n \quad \Rightarrow$$

$$\nabla_0\vec{e}_n=-\frac{1}{R}\vec{e}_\phi\vec{e}_\phi-\frac{1}{R}\vec{e}_\theta\vec{e}_\theta+0 \quad \Rightarrow$$

$$\vec{\kappa}=(\nabla_0\vec{e}_n)_{\mathrm{c}}=-\frac{1}{R}(\vec{e}_\phi\vec{e}_\phi+\vec{e}_\theta\vec{e}_\theta)\,.$$

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Assignment 3

Derive the component form of $\nabla\cdot\vec{\sigma}+\vec{f}=0$ in the polar coordinate system. Assume that the components of stress do not depend on angle ϕ. In the polar coordinate system, the component forms of stress, external force, and gradient operator, and derivatives of the basis vectors are given by

$$\vec{\sigma}=\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}\sigma_{rr} & \sigma_{r\phi}\\ \sigma_{\phi r} & \sigma_{\phi\phi}\end{bmatrix}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix},\quad \vec{f}=\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}f_r\\ f_\phi\end{Bmatrix},\quad \nabla=\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\partial/\partial r\\ \partial/(r\partial\phi)\end{Bmatrix},$$

$$\frac{\partial}{\partial\phi}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix}=\begin{Bmatrix}\vec{e}_\phi\\ -\vec{e}_r\end{Bmatrix},\quad \frac{\partial}{\partial r}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix}=0\,.$$

Solution template

In manipulation of vector expression containing vectors and tensors, it is important to remember that tensor ($\otimes$), cross ($\times$), inner ($\cdot$) products are non-commutative (order may matter). The basis vectors of a curvilinear coordinate system are not constants which should be taken into account if gradient operator is a part of expression. Otherwise, simplifying an expression or finding a specific form in a given coordinate system is a straightforward (sometimes tedious) exercise. For simplicity of presentation, outer (tensor) products like $\vec{a}\otimes\vec{b}$ are denoted by $\vec{a}\vec{b}$. Otherwise, the usual rules of algebra apply: Gradient operator ∇ acts on everything on its right-hand side, the operator is treated like a vector etc.

The task is to simplify the vector equation

$$\nabla\cdot\vec{\sigma}+\vec{f}=(\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\phi\frac{\partial}{r\partial\phi})\cdot(\sigma_{rr}\vec{e}_r\vec{e}_r+\sigma_{r\phi}\vec{e}_r\vec{e}_\phi+\sigma_{\phi r}\vec{e}_\phi\vec{e}_r+\sigma_{\phi\phi}\vec{e}_\phi\vec{e}_\phi)+(f_r\vec{e}_r+f_\phi\vec{e}_\phi)=0$$

to see the component forms. Let us consider the effect of the first term of the displacement gradient to stress

$$\vec{e}_r\frac{\partial}{\partial r}\cdot(\sigma_{rr}\vec{e}_r\vec{e}_r+\sigma_{r\phi}\vec{e}_r\vec{e}_\phi+\sigma_{\phi r}\vec{e}_\phi\vec{e}_r+\sigma_{\phi\phi}\vec{e}_\phi\vec{e}_\phi)=$$

$$\vec{e}_r\cdot(\frac{\partial\sigma_{rr}}{\partial r}\vec{e}_r\vec{e}_r+\frac{\partial\sigma_{r\phi}}{\partial r}\vec{e}_r\vec{e}_\phi+\frac{\partial\sigma_{\phi r}}{\partial r}\vec{e}_\phi\vec{e}_r+\frac{\partial\sigma_{\phi\phi}}{\partial r}\vec{e}_\phi\vec{e}_\phi)=$$

$$\vec{e}_r(\frac{\partial\sigma_{rr}}{\partial r})+\vec{e}_\phi(\frac{\partial\sigma_{r\phi}}{\partial r})\,.$$

Then the same for the second term of the displacement gradient. As the stress components do not depend on ϕ (by assumption)

$$\vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial\phi}\cdot(\sigma_{rr}\vec{e}_r\vec{e}_r + \sigma_{r\phi}\vec{e}_r\vec{e}_\phi + \sigma_{\phi r}\vec{e}_\phi\vec{e}_r + \sigma_{\phi\phi}\vec{e}_\phi\vec{e}_\phi) =$$

$$\vec{e}_\phi \frac{1}{r}\cdot(\sigma_{rr}\vec{e}_\phi\vec{e}_r + \sigma_{rr}\vec{e}_r\vec{e}_\phi + \sigma_{r\phi}\vec{e}_\phi\vec{e}_\phi - \sigma_{r\phi}\vec{e}_r\vec{e}_r - \sigma_{\phi r}\vec{e}_r\vec{e}_r + \sigma_{\phi r}\vec{e}_\phi\vec{e}_\phi - \sigma_{\phi\phi}\vec{e}_r\vec{e}_\phi - \sigma_{\phi\phi}\vec{e}_\phi\vec{e}_r) =$$

$$\vec{e}_r(\frac{1}{r}\sigma_{rr} - \frac{1}{r}\sigma_{\phi\phi}) + \vec{e}_\phi(\frac{1}{r}\sigma_{r\phi} + \frac{1}{r}\sigma_{\phi r}) .$$

Finally combining everything

$$\nabla\cdot\vec{\sigma} + \vec{f} = \vec{e}_r(\frac{\partial\sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\phi\phi}}{r} + f_r) + \vec{e}_\phi(\frac{\partial\sigma_{r\phi}}{\partial r} + \frac{\sigma_{r\phi} + \sigma_{\phi r}}{r} + f_\phi) = 0$$

Therefore, the two equilibrium equations are given by

$$\begin{Bmatrix} \dfrac{\partial\sigma_{rr}}{\partial r} + \dfrac{\sigma_{rr} - \sigma_{\phi\phi}}{r} + f_r \\ \dfrac{\partial\sigma_{r\phi}}{\partial r} + \dfrac{\sigma_{r\phi} + \sigma_{\phi r}}{r} + f_\phi \end{Bmatrix} = 0 . \leftarrow$$

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Assignment 4

Derive the expressions of linear strain components ε_{rr}, $\varepsilon_{r\phi}$, $\varepsilon_{\phi r}$ and $\varepsilon_{\phi\phi}$ of the polar coordinate system. Use the displacement representation $\vec{u} = u_r\vec{e}_r + u_\phi\vec{e}_\phi$ where the components depend on the polar coordinates r and ϕ. Use definitions

$$\vec{\varepsilon} = \frac{1}{2}[\nabla\vec{u} + (\nabla\vec{u})_{\mathrm{c}}],\ \nabla = \vec{e}_r\frac{\partial}{\partial r} + \vec{e}_\phi\frac{\partial}{r\partial\phi},\ \frac{\partial}{\partial\phi}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix} = \begin{Bmatrix}\vec{e}_\phi\\ -\vec{e}_r\end{Bmatrix},\ \frac{\partial}{\partial r}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix} = 0\,.$$

Solution template

In manipulation of vector expression containing vectors and tensors, it is important to remember that tensor ($\otimes$), cross ($\times$), inner ($\cdot$) products are non-commutative (order may matter). The basis vectors of a curvilinear coordinate system are not constants which should be taken into account if gradient operator is a part of expression. Otherwise, simplifying an expression or finding a specific form in a given coordinate system is a straightforward (sometimes tedious) exercise. For simplicity of presentation, outer (tensor) products like $\vec{a}\otimes\vec{b}$ are denoted by $\vec{a}\vec{b}$. Otherwise, the usual rules of algebra apply: Gradient operator ∇ acts on everything on its right hand side, the operator is treated like a vector etc.

Let us start with the gradient of displacement (an outer product). Substitute first the representations in the polar coordinate system

$$\nabla\vec{u} = (\vec{e}_r\frac{\partial}{\partial r} + \vec{e}_\phi\frac{\partial}{r\partial\phi})(u_r\vec{e}_r + u_\phi\vec{e}_\phi)\,.$$

Then expand to have a term-by-term representation. Keep the order of the basis vectors and the position of derivatives

$$\nabla\vec{u} = \vec{e}_r\frac{\partial}{\partial r}(u_r\vec{e}_r) + \vec{e}_r\frac{\partial}{\partial r}(u_\phi\vec{e}_\phi) + \vec{e}_\phi\frac{\partial}{r\partial\phi}(u_r\vec{e}_r) + \vec{e}_\phi\frac{\partial}{r\partial\phi}(u_\phi\vec{e}_\phi)$$

Use the derivative rule of products. Notice that the basis vectors are not constants and may have non-zero derivatives

$$\nabla\vec{u} = \vec{e}_r(\frac{\partial u_r}{\partial r}\vec{e}_r + u_r\frac{\partial\vec{e}_r}{\partial r}) + \vec{e}_r(\frac{\partial u_\phi}{\partial r}\vec{e}_\phi + u_\phi\frac{\partial\vec{e}_\phi}{\partial r}) + \vec{e}_\phi(\frac{\partial u_r}{r\partial\phi}\vec{e}_r + u_r\frac{\partial\vec{e}_r}{r\partial\phi}) + \vec{e}_\phi(\frac{\partial u_\phi}{r\partial\phi}\vec{e}_\phi + u_\phi\frac{\partial\vec{e}_\phi}{r\partial\phi})\ .$$

Substitute the derivatives of the basis vectors

$$\nabla\vec{u} = \vec{e}_r(\frac{\partial u_r}{\partial r}\vec{e}_r) + \vec{e}_r(\frac{\partial u_\phi}{\partial r}\vec{e}_\phi) + \vec{e}_\phi(\frac{\partial u_r}{r\partial\phi}\vec{e}_r + \frac{u_r}{r}\vec{e}_\phi) + \vec{e}_\phi(\frac{\partial u_\phi}{r\partial\phi}\vec{e}_\phi - \frac{u_\phi}{r}\vec{e}_r)\,.$$

Combine the terms having the same pair of basis vectors (order matters so terms containing $\vec{e}_\phi \vec{e}_r$ and $\vec{e}_r \vec{e}_\phi$ cannot be combined)

$$\nabla \vec{u} = \frac{\partial u_r}{\partial r} \vec{e}_r \vec{e}_r + \frac{\partial u_\phi}{\partial r} \vec{e}_r \vec{e}_\phi + \left(\frac{\partial u_r}{r \partial \phi} - \frac{u_\phi}{r}\right) \vec{e}_\phi \vec{e}_r + \left(\frac{u_r}{r} + \frac{\partial u_\phi}{r \partial \phi}\right) \vec{e}_\phi \vec{e}_\phi .$$

Conjugate of a second order tensor can be obtained by swapping the basis vectors in all the pairs. Conjugate is a kind of transpose and can also be obtained by transposing the matrix of the component representation.

$$(\nabla \vec{u})_{\mathrm{c}} = \frac{\partial u_r}{\partial r} \vec{e}_r \vec{e}_r + \left(\frac{\partial u_r}{r \partial \phi} - \frac{u_\phi}{r}\right) \vec{e}_r \vec{e}_\phi + \frac{\partial u_\phi}{\partial r} \vec{e}_\phi \vec{e}_r + \left(\frac{u_r}{r} + \frac{\partial u_\phi}{r \partial \phi}\right) \vec{e}_\phi \vec{e}_\phi$$

Finally using the definition $\vec{\vec{\varepsilon}} = \frac{1}{2}\left[\nabla \vec{u} + (\nabla \vec{u})_{\mathrm{c}}\right]$

$$\vec{\vec{\varepsilon}} = \frac{\partial u_r}{\partial r} \vec{e}_r \vec{e}_r + \frac{1}{2}\left(\frac{\partial u_r}{r \partial \phi} - \frac{u_\phi}{r} + \frac{\partial u_\phi}{\partial r}\right)(\vec{e}_r \vec{e}_\phi + \vec{e}_\phi \vec{e}_r) + \left(\frac{\partial u_\phi}{r \partial \phi} + \frac{u_r}{r}\right) \vec{e}_\phi \vec{e}_\phi .$$

In the components of strain ε_{rr}, $\varepsilon_{r\phi}$, $\varepsilon_{\phi r}$ and $\varepsilon_{\phi\phi}$, indices are in the same order as the indices in the basis vector pairs. Hence

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\phi\phi} = \frac{\partial u_\phi}{r \partial \phi} + \frac{u_r}{r}, \quad \varepsilon_{r\phi} = \varepsilon_{\phi r} = \frac{1}{2}\left(\frac{\partial u_r}{r \partial \phi} - \frac{u_\phi}{r} + \frac{\partial u_\phi}{\partial r}\right) . \leftarrow$$

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Assignment 5

Derive the gradient expressions of $(\alpha,\beta,\gamma)-$coordinate system, when the mapping defining the coordinate system is given by

$$\vec{r}(\alpha,\beta,\gamma)=(uv\alpha+\sqrt{1-u^2}\gamma)\vec{i}+\beta\vec{j}+(u\gamma-v\sqrt{1-u^2}\alpha)\vec{k}$$

in which $u\in[-1,1]$ and $v>0$ are parameters.

Solution

According to the generic recipe (formulae collection)

$$\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_\gamma\end{Bmatrix}=\begin{Bmatrix}(\partial\vec{r}/\partial\alpha)/|\partial\vec{r}/\partial\alpha|\\ (\partial\vec{r}/\partial\beta)/|\partial\vec{r}/\partial\beta|\\ (\partial\vec{r}/\partial\gamma)/|\partial\vec{r}/\partial\gamma|\end{Bmatrix}=[F]\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix},\quad \frac{\partial}{\partial\eta}\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_\gamma\end{Bmatrix}=(\frac{\partial}{\partial\eta}[F])[F]^{-1}\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_\gamma\end{Bmatrix}\quad \eta\in\{\alpha,\beta,\gamma\},$$

$$\nabla=\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_\gamma\end{Bmatrix}^{\mathrm{T}}[F]^{-\mathrm{T}}[H]^{-1}\begin{Bmatrix}\partial/\partial\alpha\\ \partial/\partial\beta\\ \partial/\partial\gamma\end{Bmatrix}\quad\text{where } [H]=\begin{bmatrix}\partial r_x/\partial\alpha & \partial r_y/\partial\alpha & \partial r_z/\partial\alpha\\ \partial r_x/\partial\beta & \partial r_y/\partial\beta & \partial r_z/\partial\beta\\ \partial r_x/\partial\gamma & \partial r_y/\partial\gamma & \partial r_z/\partial\gamma\end{bmatrix},$$

Matrices $[F]$ and $[H]$ depend on the mapping. In the present case

$$\vec{r}(\alpha,\beta,\gamma)=r_x\vec{i}+r_y\vec{j}+r_z\vec{k}=(uv\alpha+\sqrt{1-u^2}\gamma)\vec{i}+\beta\vec{j}+(u\gamma-v\sqrt{1-u^2}\alpha)\vec{k}\,.$$

By definition

$$\vec{e}_\alpha=\frac{\partial\vec{r}}{\partial\alpha}/\left|\frac{\partial\vec{r}}{\partial\alpha}\right|=(vu\vec{i}-v\sqrt{1-u^2}\vec{k})/v=u\vec{i}-\sqrt{1-u^2}\vec{k}\,,$$

$$\vec{e}_\beta=\frac{\partial\vec{r}}{\partial\beta}/\left|\frac{\partial\vec{r}}{\partial\beta}\right|=\vec{j}\,,$$

$$\vec{e}_\gamma=\frac{\partial\vec{r}}{\partial\gamma}/\left|\frac{\partial\vec{r}}{\partial\gamma}\right|=\sqrt{1-u^2}\vec{i}+u\vec{k}$$

and therefore

$$\begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix} = \begin{bmatrix} u & 0 & -\sqrt{1-u^2} \\ 0 & 1 & 0 \\ \sqrt{1-u^2} & 0 & u \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = [F] \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} \text{ so } [F] = \begin{bmatrix} u & 0 & -\sqrt{1-u^2} \\ 0 & 1 & 0 \\ \sqrt{1-u^2} & 0 & u \end{bmatrix}.$$

According to the mapping, the relationship between the components of the position vector in the Cartesian and cylindrical systems are $r_x = uv\alpha + \sqrt{1-u^2}\gamma$, $r_y = \beta$, and $r_z = u\gamma - \sqrt{1-u^2}v\alpha$

$$[H] = \begin{bmatrix} \partial r_x / \partial\alpha & \partial r_y / \partial\alpha & \partial r_z / \partial\alpha \\ \partial r_x / \partial\beta & \partial r_y / \partial\beta & \partial r_z / \partial\beta \\ \partial r_x / \partial\gamma & \partial r_y / \partial\gamma & \partial r_z / \partial\gamma \end{bmatrix} = \begin{bmatrix} vu & 0 & -v\sqrt{1-u^2} \\ 0 & 1 & 0 \\ \sqrt{1-u^2} & 0 & u \end{bmatrix}.$$

Gradient follows now from the generic recipe

$$\nabla = \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix}^{\mathrm{T}} [F]^{-\mathrm{T}} [H]^{-1} \begin{Bmatrix} \partial / \partial\alpha \\ \partial / \partial\beta \\ \partial / \partial\gamma \end{Bmatrix} = \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix}^{\mathrm{T}} ([H][F]^{\mathrm{T}})^{-1} \begin{Bmatrix} \partial / \partial\alpha \\ \partial / \partial\beta \\ \partial / \partial\gamma \end{Bmatrix}.$$

Let us calculate first the matrix inside the parenthesis

$$[H][F]^{\mathrm{T}} = \begin{bmatrix} vu & 0 & -v\sqrt{1-u^2} \\ 0 & 1 & 0 \\ \sqrt{1-u^2} & 0 & u \end{bmatrix} \begin{bmatrix} u & 0 & \sqrt{1-u^2} \\ 0 & 1 & 0 \\ -\sqrt{1-u^2} & 0 & u \end{bmatrix} = \begin{bmatrix} v & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Leftrightarrow$$

$$([H][F]^{\mathrm{T}})^{-1} = \begin{bmatrix} v & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/v & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Substituting into the gradient expression

$$\nabla = \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1/v & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \partial / \partial\alpha \\ \partial / \partial\beta \\ \partial / \partial\gamma \end{Bmatrix} = \vec{e}_\alpha \frac{1}{v}\frac{\partial}{\partial\alpha} + \vec{e}_\beta \frac{\partial}{\partial\beta} + \vec{e}_\gamma \frac{\partial}{\partial\gamma}. \quad \leftarrow$$

MEC-E8003

Beam, Plate and Shell Models

2024

WEEK 11: KINETICS

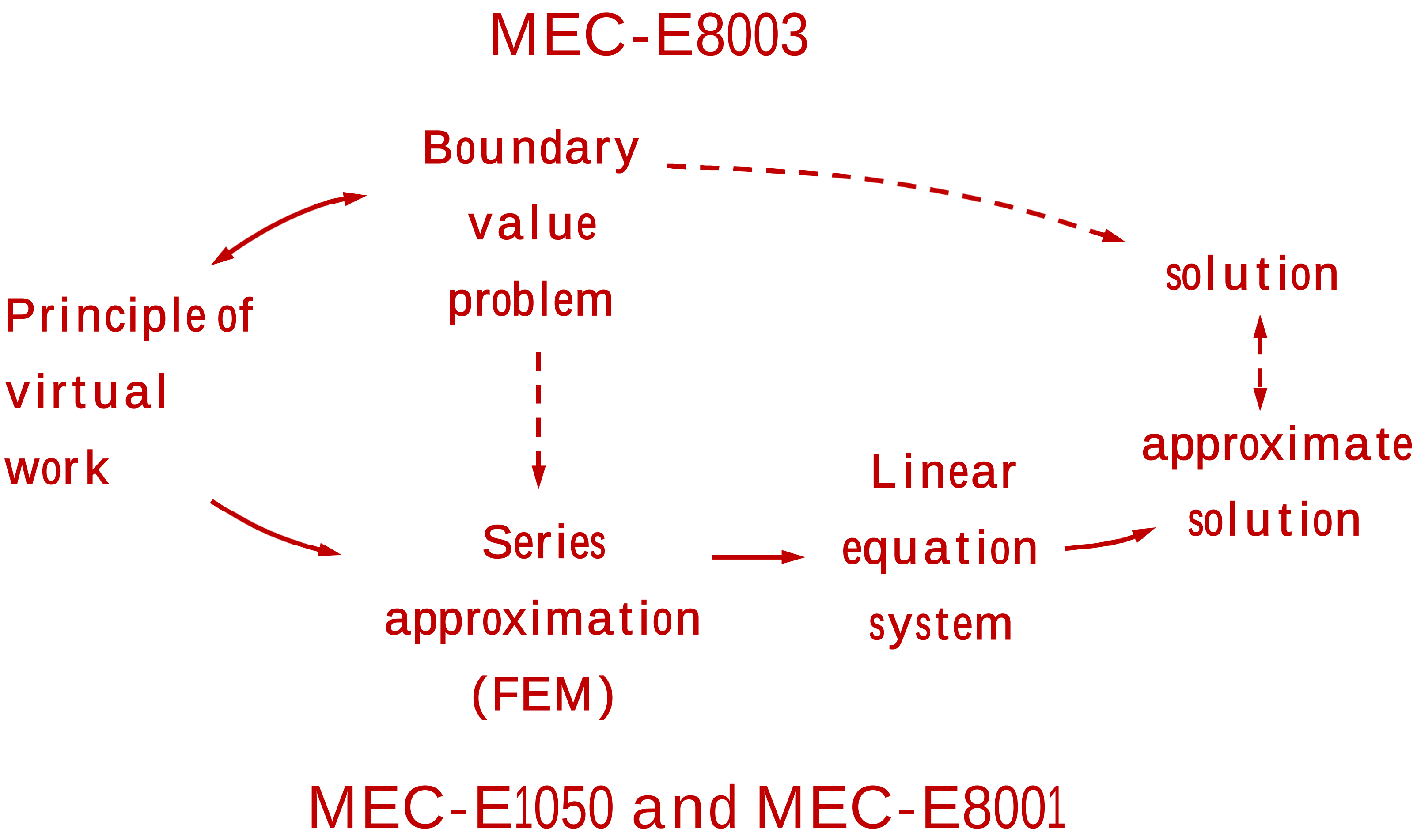
3 KINETICS

LEARNING OUTCOMES

Students can solve the weekly lecture problems, home problems, and exercise problems on kinetics:

- Quantities and equations of classical elasticity
- Constitutive equation of linearly elastic isotropic material
- Derivation of engineering models by using the principle of virtual work, integration by parts, and the fundamental lemma of variation calculus

DERIVATION OF ENGINEERING MODELS



3.1 CLASSICAL LINEAR ELASTICITY

Balance of mass (def. of a body or a material volume) Mass of a body is constant. ←

Balance of linear momentum (Newton 2) The rate of change of linear momentum within a material volume equals the external force resultant acting on the material volume. ←

Balance of angular momentum (Cor. of Newton 2) The rate of change of angular momentum within a material volume equals the external moment resultant acting on the material volume. ←

Balance of energy (Thermodynamics 1)

Entropy growth (Thermodynamics 2)

LOCAL FORMS

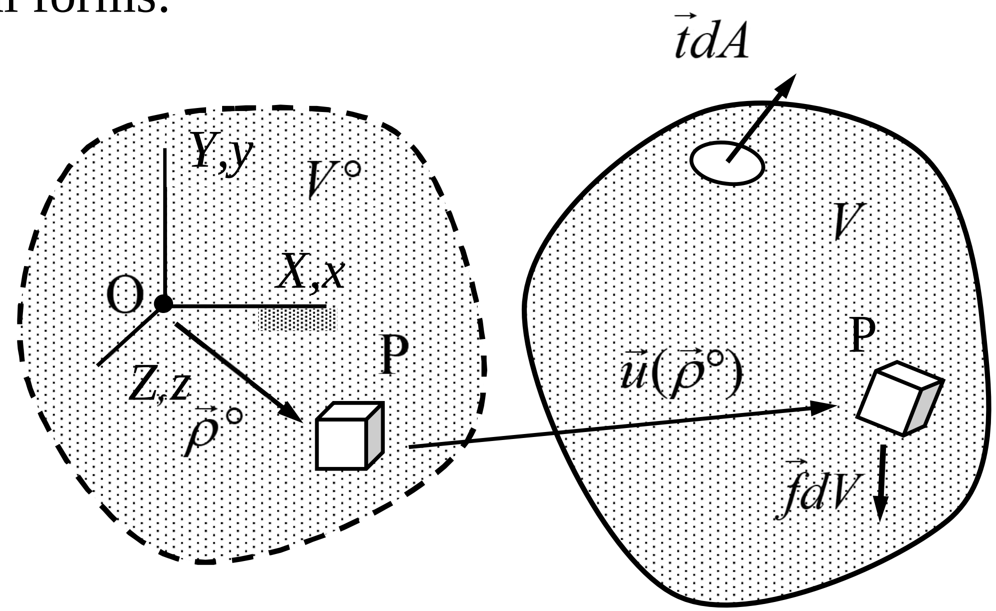
Application of the first principles to a material element inside the body or from its boundary gives the coordinate system invariant local forms:

$$\dot{m} = 0 \quad : \quad \rho^{\circ} = \rho J \quad \text{in } V$$

$$\dot{\vec{p}} = \vec{F} \quad : \quad \nabla \cdot \vec{\vec{\sigma}} + \vec{f} = 0 \quad \text{in } V$$

$$\dot{\vec{p}} = \vec{F} \quad : \quad \vec{\sigma} \equiv \vec{n} \cdot \vec{\vec{\sigma}} = \vec{t} \quad \text{on } \partial V_t$$

$$\dot{\vec{L}} = \vec{M} \quad : \quad \vec{\vec{\sigma}} = \vec{\vec{\sigma}}_{\mathrm{c}} \quad \text{in } V$$



Assuming an equilibrium setting (geometry, stress, loading etc.) the local forms can be used to find a new equilibrium setting (actually, displacements of the particles) when, e.g., external given forces are changed in some manner.

TRACTION AND STRESS

Material elements of a body interact with a surface force (force per unit area) called as the traction vector. Stress $\overset{\leftrightarrow}{\sigma}$ describes the surface forces acting on (all edges of) a material element. In a Cartesian $(x, y, z)-$coordinate system, the second order stress tensor

$$\overset{\leftrightarrow}{\sigma} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} + \begin{Bmatrix} \vec{i}\vec{j} \\ \vec{j}\vec{k} \\ \vec{k}\vec{i} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} + \begin{Bmatrix} \vec{j}\vec{i} \\ \vec{k}\vec{j} \\ \vec{i}\vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \sigma_{yx} \\ \sigma_{zy} \\ \sigma_{xz} \end{Bmatrix}$$

Traction acting on an edge of unit outward normal $\vec{n}$ is given by $\vec{\sigma} = \vec{n} \cdot \overset{\leftrightarrow}{\sigma}$ and the force (element) $d\vec{F} = \vec{\sigma} dA = \vec{n} \cdot \overset{\leftrightarrow}{\sigma} dA = d\vec{A} \cdot \overset{\leftrightarrow}{\sigma}$ where the last form uses the directed area concept $d\vec{A} = \vec{n} dA$. The representation in $(\alpha, \beta, \gamma)-$coordinate system follows by changing the basis vectors and indices of the components.

LINEAR STRAIN

Shape deformation measure of material element is the symmetric part of displacement gradient, i.e., $\vec{\vec{\varepsilon}} = [\nabla\vec{u} + (\nabla\vec{u})_{\mathrm{c}}]/2$. In a Cartesian $(x, y, z)-$coordinate system, the second order linear strain tensor

$$\vec{\vec{\varepsilon}} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} + \begin{Bmatrix} \vec{i}\vec{j} \\ \vec{j}\vec{k} \\ \vec{k}\vec{i} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{Bmatrix} + \begin{Bmatrix} \vec{j}\vec{i} \\ \vec{k}\vec{j} \\ \vec{i}\vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \varepsilon_{yx} \\ \varepsilon_{zy} \\ \varepsilon_{xz} \end{Bmatrix} \text{ where}$$

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} = \begin{Bmatrix} \partial u_x / \partial x \\ \partial u_y / \partial y \\ \partial u_z / \partial z \end{Bmatrix} \text{ and } \begin{Bmatrix} \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{yx} \\ \varepsilon_{zy} \\ \varepsilon_{xz} \end{Bmatrix} = \frac{1}{2} \begin{Bmatrix} \partial u_y / \partial x + \partial u_x / \partial y \\ \partial u_z / \partial y + \partial u_y / \partial z \\ \partial u_x / \partial z + \partial u_z / \partial x \end{Bmatrix}.$$

The representation in $(\alpha, \beta, \gamma)-$coordinate system follows from the definition.

LINEARLY ELASTIC MATERIAL

The generalized Hooke's law for an isotropic (properties do not depend on direction) and homogeneous (properties do not depend on position) can be expressed in tensor form $\vec{\vec{\sigma}} = \vec{\vec{\vec{\vec{E}}}} : \nabla \vec{u}$. In a Cartesian $(x, y, z)-$coordinate system, the fourth order elasticity tensor

$$\vec{\vec{\vec{\vec{E}}}} = \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix}^{\mathrm{T}} [E] \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix} + \begin{Bmatrix} \vec{i}\vec{j} + \vec{j}\vec{i} \\ \vec{j}\vec{k} + \vec{k}\vec{j} \\ \vec{k}\vec{i} + \vec{i}\vec{k} \end{Bmatrix}^{\mathrm{T}} [G] \begin{Bmatrix} \vec{i}\vec{j} + \vec{j}\vec{i} \\ \vec{j}\vec{k} + \vec{k}\vec{j} \\ \vec{k}\vec{i} + \vec{i}\vec{k} \end{Bmatrix}$$

depends on the 3×3 elasticity matrices $[E]$ and $[G]$ given material experiments. Representation in a $(\alpha, \beta, \gamma)-$coordinate system follows by replacing the Cartesian $(x, y, z)-$coordinate system basis vectors by their representations in terms of the basis vectors of the $(\alpha, \beta, \gamma)-$coordinate system.

The generalized Hooke's law in its component form and linear strain components (not engineering strains) according to, e.g., literature is given by

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} = [E] \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix}, \quad \begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = 2[G] \begin{Bmatrix} \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{Bmatrix}, \text{ and } \begin{Bmatrix} \sigma_{yx} \\ \sigma_{zy} \\ \sigma_{xz} \end{Bmatrix} = 2[G] \begin{Bmatrix} \varepsilon_{yx} \\ \varepsilon_{zy} \\ \varepsilon_{xz} \end{Bmatrix}.$$

Starting with the stress representation

$$\vec{\vec{\sigma}} = \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} + \begin{Bmatrix} \vec{i}\vec{j} \\ \vec{j}\vec{k} \\ \vec{k}\vec{i} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} + \begin{Bmatrix} \vec{j}\vec{i} \\ \vec{k}\vec{j} \\ \vec{i}\vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \sigma_{yx} \\ \sigma_{zy} \\ \sigma_{xz} \end{Bmatrix},$$

Using the component forms of the generalized Hooke's law (and symmetry of strain to get rid of the multiplier 2)

$$\overleftrightarrow{\sigma}=\begin{Bmatrix}\vec{i}\vec{i}\\ \vec{j}\vec{j}\\ \vec{k}\vec{k}\end{Bmatrix}^{\mathrm{T}}\left[E\right]\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\end{Bmatrix}+\begin{Bmatrix}\vec{i}\vec{j}\\ \vec{j}\vec{k}\\ \vec{k}\vec{i}\end{Bmatrix}^{\mathrm{T}}\left[G\right](\begin{Bmatrix}\varepsilon_{xy}\\ \varepsilon_{yz}\\ \varepsilon_{zx}\end{Bmatrix}+\begin{Bmatrix}\varepsilon_{yx}\\ \varepsilon_{zy}\\ \varepsilon_{xz}\end{Bmatrix})+\begin{Bmatrix}\vec{j}\vec{i}\\ \vec{k}\vec{j}\\ \vec{i}\vec{k}\end{Bmatrix}^{\mathrm{T}}\left[G\right](\begin{Bmatrix}\varepsilon_{xy}\\ \varepsilon_{yz}\\ \varepsilon_{zx}\end{Bmatrix}+\begin{Bmatrix}\varepsilon_{yx}\\ \varepsilon_{zy}\\ \varepsilon_{xz}\end{Bmatrix}).$$

Finally substituting the representations

$$\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\end{Bmatrix}=\begin{Bmatrix}\vec{i}\vec{i}\\ \vec{j}\vec{j}\\ \vec{k}\vec{k}\end{Bmatrix}:\overleftrightarrow{\varepsilon},\quad \begin{Bmatrix}\varepsilon_{xy}\\ \varepsilon_{yz}\\ \varepsilon_{zx}\end{Bmatrix}=\begin{Bmatrix}\vec{j}\vec{i}\\ \vec{k}\vec{j}\\ \vec{i}\vec{k}\end{Bmatrix}:\overleftrightarrow{\varepsilon},\ \text{and}\ \begin{Bmatrix}\varepsilon_{yx}\\ \varepsilon_{zy}\\ \varepsilon_{xz}\end{Bmatrix}=\begin{Bmatrix}\vec{i}\vec{j}\\ \vec{j}\vec{k}\\ \vec{k}\vec{i}\end{Bmatrix}:\overleftrightarrow{\varepsilon}$$

gives

$$\overleftrightarrow{\sigma}=(\begin{Bmatrix}\vec{i}\vec{i}\\ \vec{j}\vec{j}\\ \vec{k}\vec{k}\end{Bmatrix}^{\mathrm{T}}\left[E\right]\begin{Bmatrix}\vec{i}\vec{i}\\ \vec{j}\vec{j}\\ \vec{k}\vec{k}\end{Bmatrix}+\begin{Bmatrix}\vec{i}\vec{j}+\vec{j}\vec{i}\\ \vec{j}\vec{k}+\vec{k}\vec{j}\\ \vec{k}\vec{i}+\vec{i}\vec{k}\end{Bmatrix}^{\mathrm{T}}[G]\begin{Bmatrix}\vec{i}\vec{j}+\vec{j}\vec{i}\\ \vec{j}\vec{k}+\vec{k}\vec{j}\\ \vec{k}\vec{i}+\vec{i}\vec{k}\end{Bmatrix}):\overleftrightarrow{\varepsilon}\equiv\overset{\leftrightarrow}{\overleftrightarrow{E}}:\overleftrightarrow{\varepsilon}=\overset{\leftrightarrow}{\overleftrightarrow{E}}:\nabla\vec{u}\,. \quad \Leftarrow$$

CONSTITUTIVE EQUATION VARIANTS

Stress-displacement relationship of linearly elastic material model can be expressed in various equivalent forms depending on the symmetry conditions imposed on the fourth order elasticity tensor $\vec{\vec{\vec{\vec{E}}}}$:

(a) $\vec{\vec{\sigma}} = \vec{\vec{\vec{\vec{E}}}} : \frac{1}{2}[\nabla\vec{u} + (\nabla\vec{u})_{\mathrm{c}}]$ and $\vec{\vec{\sigma}} = \vec{\vec{\sigma}}_{\mathrm{c}}$ $\Leftrightarrow$

(b) $\vec{\vec{\sigma}} = \vec{\vec{\vec{\vec{E}}}} : \nabla\vec{u}$ and $\vec{\vec{\sigma}} = \vec{\vec{\sigma}}_{\mathrm{c}}$ and $\vec{\vec{\vec{\vec{E}}}} = \vec{\vec{\vec{\vec{E}}}}_{\cdot\mathrm{c}}$ $\Leftrightarrow$ Last index pair conjugate!

(c) $\vec{\vec{\sigma}} = \vec{\vec{\vec{\vec{E}}}} : \nabla\vec{u}$ and $\vec{\vec{\vec{\vec{E}}}} = \vec{\vec{\vec{\vec{E}}}}_{\cdot\mathrm{c}} = \vec{\vec{\vec{\vec{E}}}}_{\mathrm{c}\cdot} = \vec{\vec{\vec{\vec{E}}}}_{\mathrm{cc}}$ 🡐

Also, other kinetic conditions like $\sigma_{zz} = 0$ can be satisfied ‘a priori’ by the selection of elasticity tensor. The conditions of (c) are called as the minor and major symmetries.

ISOTROPIC MATERIAL

The generalized Hooke's law for an isotropic material follows with the elasticity matrices

$$[E]=E\begin{bmatrix}1 & -\nu & -\nu\\ -\nu & 1 & -\nu\\ -\nu & -\nu & 1\end{bmatrix}^{-1}=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & \nu\\ \nu & 1-\nu & \nu\\ \nu & \nu & 1-\nu\end{bmatrix},$$

$$[G]=\begin{bmatrix}G & 0 & 0\\ 0 & G & 0\\ 0 & 0 & G\end{bmatrix}=\frac{E}{2(1+\nu)}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

in which the material parameters E and ν are the Young's modulus and the Poisson's ratio, respectively, and $G=E/(2+2\nu)$ the shear modulus. Using these, one may deduce the elasticity matrices for the engineering models.

In the coordinate system invariant form $\overleftrightarrow{\sigma} = \overset{\leftrightarrow}{\overset{\leftrightarrow}{E}} : \overleftrightarrow{\varepsilon} = \overset{\leftrightarrow}{\overset{\leftrightarrow}{E}} : \nabla\vec{u}$, the elasticity tensor (satisfying the major and minor symmetries) is given by

$$\overset{\leftrightarrow}{\overset{\leftrightarrow}{E}} = \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix}^{\mathrm{T}} E \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix}^{-1} \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix} + \begin{Bmatrix} \vec{i}\vec{j} + \vec{j}\vec{i} \\ \vec{j}\vec{k} + \vec{k}\vec{j} \\ \vec{k}\vec{i} + \vec{i}\vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} G & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \vec{i}\vec{j} + \vec{j}\vec{i} \\ \vec{j}\vec{k} + \vec{k}\vec{j} \\ \vec{k}\vec{i} + \vec{i}\vec{k} \end{Bmatrix}.$$

Elasticity tensor of plate model ($\sigma_{zz} = 0$)

$$\overset{\leftrightarrow}{\overset{\leftrightarrow}{E}} = \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix}^{\mathrm{T}} \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix} + \begin{Bmatrix} \vec{i}\vec{j} + \vec{j}\vec{i} \\ \vec{j}\vec{k} + \vec{k}\vec{j} \\ \vec{k}\vec{i} + \vec{i}\vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} G & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \vec{i}\vec{j} + \vec{j}\vec{i} \\ \vec{j}\vec{k} + \vec{k}\vec{j} \\ \vec{k}\vec{i} + \vec{i}\vec{k} \end{Bmatrix}.$$

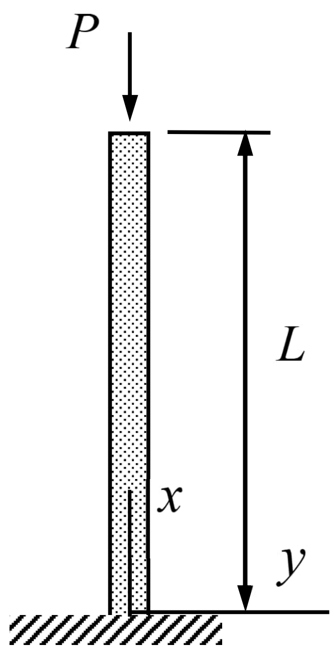
Elasticity tensor of the beam model ($\sigma_{yy} = \sigma_{zz} = 0$)

$$\overset{\leftrightarrow}{E} = \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} E & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix} + \begin{Bmatrix} \vec{i}\vec{j} + \vec{j}\vec{i} \\ \vec{j}\vec{k} + \vec{k}\vec{j} \\ \vec{k}\vec{i} + \vec{i}\vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} G & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \vec{i}\vec{j} + \vec{j}\vec{i} \\ \vec{j}\vec{k} + \vec{k}\vec{j} \\ \vec{k}\vec{i} + \vec{i}\vec{k} \end{Bmatrix}.$$

Representation in some other system can be obtained from the Cartesian $(x, y, z)-$system representation by using the relationships between the basis vectors. For example, in the cylindrical $(r, \phi, z)-$coordinate system

$$\overset{\leftrightarrow}{E} = \begin{Bmatrix} \vec{e}_r\vec{e}_r \\ \vec{e}_\phi\vec{e}_\phi \\ \vec{e}_z\vec{e}_z \end{Bmatrix}^{\mathrm{T}} [E] \begin{Bmatrix} \vec{e}_r\vec{e}_r \\ \vec{e}_\phi\vec{e}_\phi \\ \vec{e}_z\vec{e}_z \end{Bmatrix} + \begin{Bmatrix} \vec{e}_r\vec{e}_\phi + \vec{e}_\phi\vec{e}_r \\ \vec{e}_\phi\vec{e}_z + \vec{e}_z\vec{e}_\phi \\ \vec{e}_z\vec{e}_r + \vec{e}_r\vec{e}_z \end{Bmatrix}^{\mathrm{T}} [G] \begin{Bmatrix} \vec{e}_r\vec{e}_\phi + \vec{e}_\phi\vec{e}_r \\ \vec{e}_\phi\vec{e}_z + \vec{e}_z\vec{e}_\phi \\ \vec{e}_z\vec{e}_r + \vec{e}_r\vec{e}_z \end{Bmatrix}.$$

EXAMPLE The cross section of the column is square of side length h. Density ρ, Young's modulus E, and Poisson's ratio ν are constants. The column is loaded by a constant traction of magnitude P/h^2 at its free end. Determine stress $\overleftrightarrow{\sigma}$ and displacement $\vec{u}$ starting from the generic equations for linear elasticity. Assume that the transverse (to the axis) displacement is not constrained by the support.



Answer $\vec{u} = \dfrac{P}{Eh^2}(-x\vec{i} + \nu y\vec{j} + \nu z\vec{k}),\quad \overleftrightarrow{\sigma} = -\dfrac{P}{h^2}\vec{i}\vec{i}$

The component forms of the equilibrium equations and constitutive equations of a linearly elastic isotropic material in a Cartesian $(x, y, z)-$coordinate system

$$\begin{Bmatrix} \partial\sigma_{xx}/\partial x + \partial\sigma_{yx}/\partial y + \partial\sigma_{zx}/\partial z + f_x \\ \partial\sigma_{xy}/\partial x + \partial\sigma_{yy}/\partial y + \partial\sigma_{zy}/\partial z + f_y \\ \partial\sigma_{xz}/\partial x + \partial\sigma_{yz}/\partial y + \partial\sigma_{zz}/\partial z + f_z \end{Bmatrix} = 0,$$

$$\begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial w/\partial z \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix}\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix}, \text{ and } \begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = \begin{Bmatrix} \sigma_{yx} \\ \sigma_{zy} \\ \sigma_{xz} \end{Bmatrix} = G\begin{Bmatrix} \partial u/\partial y + \partial v/\partial x \\ \partial v/\partial z + \partial w/\partial y \\ \partial w/\partial x + \partial u/\partial z \end{Bmatrix}.$$

Let us assume that the only non-zero stress component $\sigma_{xx}(x)$ and displacement components $u_x = u(x)$, $u_y = v(y)$ and $u_z = w(z)$. The axial stress follows from the equilibrium equation and traction is known at the free end $x = L$. Therefore

$$\frac{d\sigma_{xx}}{dx} = 0 \quad 0 < x < L \quad \text{and} \quad \sigma_{xx}(L) = -\frac{P}{h^2} \quad \Rightarrow \quad \sigma_{xx}(x) = -\frac{P}{h^2}.$$

Generalized Hooke's law written for the uniaxial stress implies that

$$\frac{du}{dx} = \frac{\sigma_{xx}}{E} = -\frac{P}{Eh^2}, \quad \frac{dv}{dy} = -\frac{\nu}{E}\sigma_{xx} = \nu\frac{P}{Eh^2}, \quad \frac{dw}{dz} = -\frac{\nu}{E}\sigma_{xx} = \nu\frac{P}{Eh^2}.$$

Axial displacement vanishes at the support and the transverse displacement at the axis:

$$\frac{du}{dx} = -\frac{P}{Eh^2} \quad 0 < x < L \quad \text{and} \quad u(0) = 0 \quad \Rightarrow \quad u(x) = -\frac{P}{Eh^2}x, \quad \Leftarrow$$

$$\frac{dv}{dy} = \nu\frac{P}{Eh^2} \quad -\frac{1}{2}h < y < \frac{1}{2}h \quad \text{and} \quad v(0) = 0 \quad \Rightarrow \quad v(y) = \nu\frac{P}{Eh^2}y, \quad \Leftarrow$$

$$\frac{dw}{dz} = -\nu\frac{P}{Eh^2} \quad -\frac{1}{2}h < z < \frac{1}{2}h \quad \text{and} \quad w(0) = 0 \quad \Rightarrow \quad w(z) = \nu\frac{P}{Eh^2}z. \quad \Leftarrow$$

3.2 PRINCIPLE OF VIRTUAL WORK

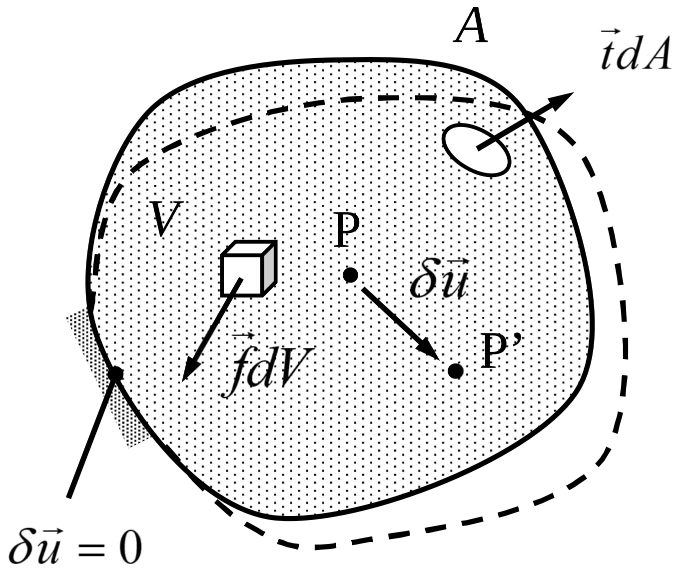
Principle of virtual work $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = 0 \quad \forall \delta\vec{u} \in U$ is just one representation of the balance laws of continuum mechanics. It is important due to its wide applicability and physical meanings of the terms.

virtual work density

$$\delta W^{\text{int}} = \int_V \delta w_V^{\text{int}} dV = -\int_V (\vec{\vec{\sigma}} : \delta\vec{\vec{\varepsilon}}_{\text{c}}) dV$$

$$\delta W_V^{\text{ext}} = \int_V \delta w_V^{\text{ext}} dV = \int_V (\vec{f} \cdot \delta\vec{u}) dV$$

$$\delta W_A^{\text{ext}} = \int_A \delta w_A^{\text{ext}} dA = \int_A (\vec{t} \cdot \delta\vec{u}) dA$$



The details of the expressions vary case by case, but the principle itself does not!

In what follows, we skip some of the technical details and assume that displacement boundary conditions are satisfied ‘a priori’. The local and variational forms of elasticity problem are equivalent, i.e., the local form implies the variational form and the other way around. Let us consider first the derivation of the variational form:

$$\left.\begin{aligned} &\nabla\cdot\vec{\vec{\sigma}}+\vec{f}=0 \quad\text{and}\quad \vec{\vec{\sigma}}=\vec{\vec{\sigma}}_{\mathrm{c}} \quad\text{in}\ V, \\ &\vec{u}-\underline{\vec{u}}=0 \quad\text{or}\quad \vec{n}\cdot\vec{\vec{\sigma}}-\vec{t}=0 \quad\text{on}\ \partial V. \end{aligned}\right\}\ \text{local form}$$

Multiplication of the momentum equation by virtual displacement $\delta\vec{u}$, integration over the solution domain, and integration by parts with $(\nabla\cdot\vec{\vec{a}})\cdot\vec{b}=\nabla\cdot(\vec{\vec{a}}\cdot\vec{b})-\vec{\vec{a}}:(\nabla\vec{b})_{\mathrm{c}}$ (selections $\vec{\vec{a}}=\vec{\vec{\sigma}}$ and $\vec{b}=\delta\vec{u}$), and division of the displacement gradient into its symmetric and anti-symmetric parts according to $\nabla\vec{u}=\vec{\vec{\varepsilon}}+\vec{\vec{\phi}}$ give

$$\int_V (\nabla\cdot\overleftrightarrow{\sigma}+\vec{f})\cdot\delta\vec{u}dV=0 \quad \forall\delta\vec{u}\in U \qquad \Rightarrow$$

$$\int_V (-\overleftrightarrow{\sigma}:\delta\overleftrightarrow{\varepsilon}_{\mathrm{c}})dV+\int_V (\vec{f}\cdot\delta\vec{u})dV+\int_{\partial V} (\vec{n}\cdot\overleftrightarrow{\sigma}\cdot\delta\vec{u})dA=0 \quad \forall\delta\vec{u}\in U\,.$$

The boundary conditions of the local form imply that either $\delta\vec{u}=0$ or $\vec{n}\cdot\overleftrightarrow{\sigma}=\vec{t}$ at all points of ∂V. Therefore, one ends up with

variational form

$$\delta W=\int_V (-\overleftrightarrow{\sigma}:\delta\overleftrightarrow{\varepsilon}_{\mathrm{c}})dV+\int_V (\vec{f}\cdot\delta\vec{u})dV+\int_{\partial V} (\vec{t}\cdot\delta\vec{u})dA=0 \quad \forall\delta\vec{u}\in U\,.$$

The derivation assumes that $\overleftrightarrow{\sigma}=\overleftrightarrow{\sigma}_{\mathrm{c}}$ (where exactly?). In practice, symmetry of stress is satisfied ‘a priori’ by the form of the constitutive equation.

In derivation to the reverse direction (with the assumption $\overleftrightarrow{\sigma}=\overleftrightarrow{\sigma}_{\mathrm{c}}$ for consistency), the starting point is the variational form. One substitutes first division $\overleftrightarrow{\varepsilon}=\nabla\vec{u}-\overleftrightarrow{\phi}$ to get

$$\delta W = \int_V [-\vec{\vec{\sigma}} : (\nabla \delta \vec{u})_{\mathrm{c}}] dV + \int_V (\vec{f} \cdot \delta \vec{u}) dV + \int_{\partial V} (\vec{t} \cdot \delta \vec{u}) dA = 0 \quad \forall \delta \vec{u} \ .$$

Integration by parts with $(\nabla \cdot \vec{a}) \cdot \vec{b} = \nabla \cdot (\vec{a} \cdot \vec{b}) - \vec{a} : (\nabla \vec{b})_{\mathrm{c}}$ (selections $\vec{a} = \vec{\vec{\sigma}}$ and $\vec{b} = \delta \vec{u}$) gives an equivalent but more convenient form

$$\delta W = \int_V (\nabla \cdot \vec{\vec{\sigma}} + \vec{f}) \cdot \delta \vec{u} dV + \int_{\partial V} (-\vec{n} \cdot \vec{\vec{\sigma}} + \vec{t}) \cdot \delta \vec{u} dA = 0 \quad \forall \delta \vec{u} \ .$$

The variational form, together with the assumed symmetry of stress and the conditions for the function set U, implies equations

$$\left.\begin{array}{l} \nabla \cdot \vec{\vec{\sigma}} + \vec{f} = 0 \quad \text{and} \quad \vec{\vec{\sigma}} - \vec{\vec{\sigma}}_{\mathrm{c}} = 0 \ \text{ in } V, \\ \vec{n} \cdot \vec{\vec{\sigma}} - \vec{t} = 0 \quad \text{or} \quad \vec{u} - \underline{\vec{u}} = 0 \quad \text{on } \partial V. \end{array}\right\} \text{The starting point}$$

BOUNDARY VALUE PROBLEM

Principle of virtual work is one of the variational forms of equations of mechanics. Given a variational form, the underlying boundary value problems follows with the steps:

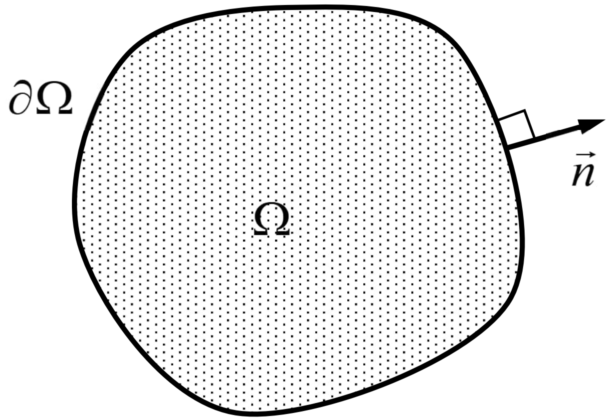
First, use integration by parts in the integral over the mathematical solution domain to remove the derivatives acting on the variations of displacement components.

Second, use the fundamental lemma of variation calculus to deduce the differential equation(s) and boundary (natural) conditions. Consider convenient subsets of possible displacement variations to deduce first the equilibrium equation and thereafter the conditions at the boundaries.

Third, deduce the additional (essential) boundary conditions using the set of displacement variations (for example, if variation of a quantity vanishes, the quantity is given).

GAUSS'S THEOREM

Divergence theorem is needed in transforming between the local and variational forms of a boundary value problem. For a continuous function $a \in C^0(\Omega)$, the fundamental theorem of calculus implies, e.g.,



1D: $\int_\Omega \frac{da}{dx} dx = \sum_{\partial\Omega} an$

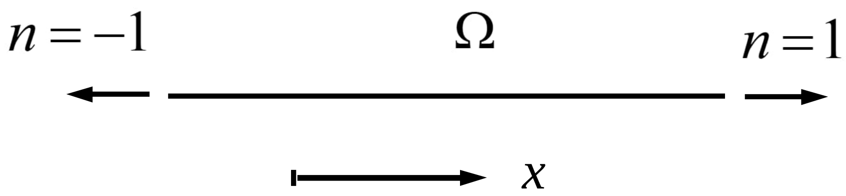
2D: $\int_\Omega (\nabla \cdot \vec{a}) dA = \int_{\partial\Omega} \vec{a} \cdot \vec{n} ds.$

3D: $\int_\Omega (\nabla \cdot \vec{a}) dV = \int_{\partial\Omega} \vec{a} \cdot \vec{n} dA.$

The generic theorem implies useful integral identities for various purposes. For example, in derivation of a boundary value problem from its variational form, one uses different selections for a or $\vec{a}$ with generic vector identities like $(\nabla \cdot \vec{a}) \cdot \vec{b} = \nabla \cdot (\vec{a} \cdot \vec{b}) - \vec{a} : (\nabla \vec{b})_c$.

In the one-dimensional case, the summing on the right-hand side is over the boundary points and the unit normal to the boundary $n = \pm 1$. The integration by parts identity

$$\int_{\Omega} a \frac{db}{dx} dx = \sum_{\partial\Omega} (nab) - \int_{\Omega} b \frac{da}{dx} dx$$



follows with selection ab of the function. Assumption of continuity is essential, and the simple form of integration by parts formula above requires modifications for, e.g., a discontinuity inside Ω. A useful integration by parts identity for several dimension

$$\int_{\Omega} \vec{a} : (\nabla \vec{b})_{\mathrm{c}} dV = \int_{\partial\Omega} (\vec{n} \cdot \vec{a} \cdot \vec{b}) dA - \int_{\Omega} (\nabla \cdot \vec{a}) \cdot \vec{b} dV$$

follows with selection $\vec{a} \cdot \vec{b}$ and use of vector identity $\nabla \cdot (\vec{a} \cdot \vec{b}) = \vec{a} : (\nabla \vec{b})_{\mathrm{c}} + (\nabla \cdot \vec{a}) \cdot \vec{b}$. The various versions of integration by parts identities will be used to move derivatives to act on certain parts of integrand.

FUNDAMENTAL LEMMA OF VARIATION CALCULUS

- $a, b \in \mathbb{R}$: $ab = 0 \quad \forall b \quad \Leftrightarrow \quad a = 0$
- $\{a\}, \{b\} \in \mathbb{R}^n$: $\{a\}^{\mathrm{T}} \{b\} = 0 \quad \forall \{b\} \quad \Leftrightarrow \quad \vec{a} = 0$
- $\vec{a}, \vec{b} \in \mathbb{R}^3$: $\vec{a} \cdot \vec{b} = 0 \quad \forall \vec{b} \quad \Leftrightarrow \quad \vec{a} = 0$
- $a, b \in C^0(\Omega)$: $\int_\Omega ab d\Omega = 0 \quad \forall b \quad \Leftrightarrow \quad a = 0 \ \text{ in } \Omega$
- $a, b \in C^2(\Omega)$: $\int_\Omega \nabla a \cdot \nabla b d\Omega = 0 \ \forall b \ \Leftrightarrow \ \nabla^2 a = 0 \text{ in } \Omega,\ a = \underline{a} \text{ or } \vec{n} \cdot \nabla a = 0 \text{ on } \partial\Omega$

In connection with principle of virtual work, b is taken to be kinematically admissible variation $\delta\vec{u}$ of displacement $\vec{u}$ (vanishes whenever $\vec{u}$ is known).

EXAMPLE Principle of virtual work for a Bernoulli beam problem is given by: find $w \in U$ such that $\forall \delta w \in U$

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = \int_{\Omega} (-\frac{d^2 \delta w}{dx^2} EI \frac{d^2 w}{dx^2} + \delta w b) dx = 0$$

in which $\Omega = (0, L)$, $U = \{w \in C^4(\Omega) : w = dw / dx = 0 \text{ at } x = 0\}$ and the bending stiffness $EI(x)$ and $b(x)$ are given. Deduce the underlying boundary value problem by using integration by parts and the fundamental lemma of variation calculus.

Answer $-\frac{d^2}{dx^2}(EI \frac{d^2 w}{dx^2}) + b = 0$ in $(0, L)$, $\frac{d}{dx}(EI \frac{d^2 w}{dx^2}) = 0$ at $x = L$,

$-EI \frac{d^2 w}{dx^2} = 0$ at $x = L$, $\frac{dw}{dx} = 0$ at $x = 0$, and $w = 0$ at $x = 0$

Integration by parts twice in the first term gives an equivalent form (notice that $\delta w \in U$ and therefore $\delta w = d\delta w / dx = 0$ at $x = 0$)

$$\delta W = \int_{\Omega} (-\frac{d^2\delta w}{dx^2} EI \frac{d^2 w}{dx^2} + \delta w b) dx \quad \Leftrightarrow$$

$$\delta W = \int_{\Omega} [\frac{d\delta w}{dx} \frac{d}{dx}(EI \frac{d^2 w}{dx^2}) + \delta w b] dx - [\frac{d\delta w}{dx}(EI \frac{d^2 w}{dx^2})]_{x=L} \quad \Leftrightarrow$$

$$\delta W = \int_{\Omega} [-\frac{d^2}{dx^2}(EI \frac{d^2 w}{dx^2}) + b] \delta w dx - [\frac{d\delta w}{dx}(EI \frac{d^2 w}{dx^2}) - \delta w \frac{d}{dx}(EI \frac{d^2 w}{dx^2})]_{x=L}.$$

According to principle of virtual work $\delta W = 0 \ \forall \delta w \in U$. Let us first consider a subset $U_0 \subset U$ for which $\delta w = d\delta w / dx = 0$ at $x = L$ so that the boundary terms vanish. The equilibrium equation follows from the fundamental lemma of variation calculus:

$$\delta W = \int_\Omega [-\frac{d^2}{dx^2}(EI\frac{d^2w}{dx^2}) + b]\delta w dx = 0 \quad \Rightarrow \quad -\frac{d^2}{dx^2}(EI\frac{d^2w}{dx^2}) + b = 0 \quad \text{in } (0, L).$$

After that, let us consider U with restriction $d\delta w / dx = 0$ first and then with $\delta w = 0$ at $x = L$ and simplify the virtual work expression by using the equilibrium equation already obtained. The natural boundary conditions follow from the fundamental lemma of variation calculus

$$\delta W = [\delta w \frac{d}{dx}(EI\frac{d^2w}{dx^2})]_{x=L} = 0 \quad \Rightarrow \quad \frac{d}{dx}(EI\frac{d^2w}{dx^2}) = 0 \quad \text{at} \quad x = L,$$

$$\delta W = -[\frac{d\delta w}{dx}(EI\frac{d^2w}{dx^2})]_{x=L} = 0 \quad \Rightarrow \quad -EI\frac{d^2w}{dx^2} = 0 \quad \text{at} \quad x = L.$$

Boundary conditions $w = dw / dx = 0$ at $x = 0$ follow from assumption $w \in U$.

3.3 DERIVATION OF ENGINEERING MODELS

First, write the virtual work expression by using the virtual work densities of an engineering model. If not available, start with the generic virtual work expression, kinematical and kinetic assumptions of the model, and integrate over the small dimensions.

Second, use the principle of virtual work, integration by parts, and the fundamental lemma of variation calculus to deduce the field equation(s) and (natural) boundary conditions in terms of stress resultants. Consider suitable subset of function space U to deduce first the equilibrium equation and thereafter the conditions at the boundaries.

Third, use the definitions of the stress resultants to derive the constitutive equations corresponding to the material model required.

DENSITY EXPRESSIONS

Virtual work densities (virtual work per unit volume or area) of the internal forces, external volume forces, and external surface forces. In a Cartesian $(x,y,z)-$coordinate system

$$\delta w_V^{\text{int}} = \delta\vec{\vec{\varepsilon}}_{\text{c}} : \vec{\vec{\sigma}} = -\begin{Bmatrix} \delta\varepsilon_{xx} \\ \delta\varepsilon_{yy} \\ \delta\varepsilon_{zz} \end{Bmatrix}^{\text{T}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} - \begin{Bmatrix} \delta\varepsilon_{xy} \\ \delta\varepsilon_{yz} \\ \delta\varepsilon_{zx} \end{Bmatrix}^{\text{T}} \begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} - \begin{Bmatrix} \delta\varepsilon_{yx} \\ \delta\varepsilon_{zy} \\ \delta\varepsilon_{xz} \end{Bmatrix}^{\text{T}} \begin{Bmatrix} \sigma_{yx} \\ \sigma_{zy} \\ \sigma_{xz} \end{Bmatrix},$$

$$\delta w_V^{\text{ext}} = \delta\vec{u} \cdot \vec{f} = \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix}^{\text{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} \text{ and } \delta w_A^{\text{ext}} = \delta\vec{u} \cdot \vec{t} = \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix}^{\text{T}} \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix}.$$

The terms of the expressions consist of work conjugate pairs of kinematic and kinetic quantities. As stress is symmetric $\vec{\vec{\sigma}} = \vec{\vec{\sigma}}_{\text{c}}$, one may write $(\delta\nabla\vec{u})_{\text{c}} : \vec{\vec{\sigma}} = \delta\vec{\vec{\varepsilon}}_{\text{c}} : \vec{\vec{\sigma}}$.

THIN BODY ASSUMPTIONS

Bar: $\vec{u}(x,y,z)=\vec{u}_0(x)$ and $\sigma_{yy}=\sigma_{zz}=\sigma_{xy}=\sigma_{yz}=\sigma_{zx}=0$

String: $\vec{u}(s,n,b)=\vec{u}_0(s)$ and $\sigma_{nn}=\sigma_{bb}=\sigma_{sn}=\sigma_{nb}=\sigma_{bs}=0$

Straight beam: $\vec{u}(x,y,z)=\vec{u}_0(x)+\vec{\theta}(x)\times\vec{\rho}(y,z)$ and $\sigma_{yy}=\sigma_{zz}=0$

Curved beam: $\vec{u}(s,n,b)=\vec{u}_0(s)+\vec{\theta}(s)\times\vec{\rho}(n,b)$ and $\sigma_{nn}=\sigma_{bb}=0$

Thin slab: $\vec{u}(x,y,z)=\vec{u}_0(x,y)$ and $\sigma_{zz}=\sigma_{yz}=\sigma_{zx}=0$

Membrane: $\vec{u}(\alpha,\beta,n)=\vec{u}_0(\alpha,\beta)$ and $\sigma_{nn}=\sigma_{\beta n}=\sigma_{n\alpha}=0$

Plate: $\vec{u}(x,y,z)=\vec{u}_0(x,y)+\vec{\theta}(x,y)\times\vec{\rho}(z)$ and $\sigma_{zz}=0$

Shell: $\vec{u}(z,s,n)=\vec{u}_0(z,s)+\vec{\theta}(z,s)\times\vec{\rho}(n)$ and $\sigma_{nn}=0$

BAR EQUATIONS

Bar is one of the loading modes of the beam model and it can be considered also as the elasticity problem in one dimension. The model assumes that displacement and stress have just axial components depending on the axial coordinate only. In a Cartesian $(x, y, z)-$ coordinate system, the bar boundary value problem is given by

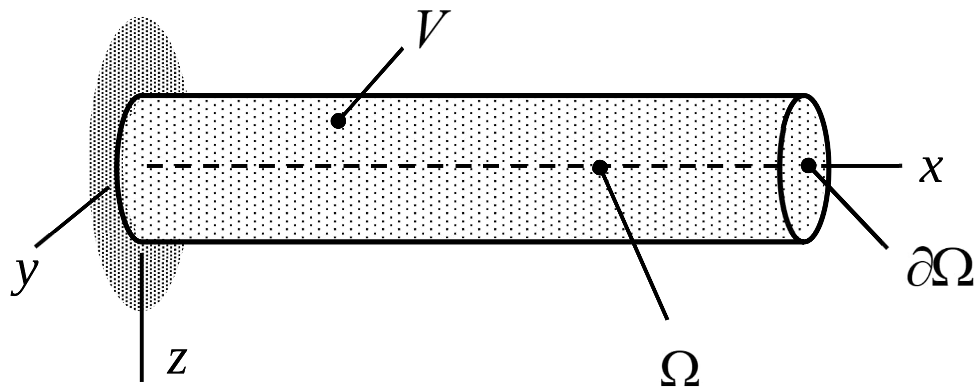
$$\frac{dN}{dx} + b = 0 \text{ in } \Omega \quad \text{and} \quad nN - \underline{F} = 0 \text{ or } u - \underline{u} = 0 \text{ on } \partial\Omega,$$

where

$$N = \int \sigma_{xx} dA, \ b = \int f_x dA, \text{ and } \underline{F} = \int t_x dA.$$

For a closed equation system (number of equations and unknown functions should match) a material model is also needed (Hooke's law).

The physical domain of the bar model is V occupied by a body althought the solution domain of the equations is the mid-line Ω. The starting point is the virtual work expression written for the physical domain.



Let us consider the steps in the Cartesian $(x, y, z)-$coordinate system for clarity. The bar model assumes that displacement and stress have just axial components depending on the axial coordinate only. Representations of stress, displacement and gradient operator are $\vec{\vec{\sigma}} = \sigma_{xx}\vec{i}\vec{i}$ and $\vec{u}(x) = u(x)\vec{i}$, $\nabla = \vec{i}\partial / \partial x + \vec{j}\partial / \partial y + \vec{k}\partial / \partial z$.

$$\delta W^{\text{int}} = -\int_V (\nabla \delta \vec{u})_{\text{c}} : \vec{\sigma} dV = -\int_\Omega \frac{d\delta u}{dx} (\int \sigma_{xx} dA) dx = -\int_\Omega \frac{d\delta u}{dx} N dx,$$

$$\delta W^{\text{ext}} = \int_V \delta\vec{u} \cdot \vec{f} dV + \int_{\partial V} \delta\vec{u} \cdot \vec{t} dA = \int_\Omega \delta u b dx + \sum_{\partial\Omega} \delta u F$$

in which (integrals over the cross-sectional area)

$$N = \int \sigma_{xx} dA, \; b = \int f_x dA, \text{ and } F = \int t_x dA.$$

According to the principle of virtual work $\delta W = 0 \;\; \forall \delta u \in U$. Integration by parts is used first to obtain a more convenient form for deducing the bar equations.

$$\delta W = -\int_\Omega (N \frac{d\delta u}{dx}) dx + \int_\Omega (b \delta u) dx + \sum_{\partial\Omega} (F \delta u) = 0 \quad \Leftrightarrow$$

$$\delta W = \int_\Omega (\frac{dN}{dx} + b) \delta u dx + \sum_{\partial\Omega} (-nN + F) \delta u = 0 \text{ in which } n = \pm 1.$$

After that, by considering a subset of variations $\delta u \in U$ with restriction $\delta u = 0$ on $\partial\Omega$ and using the fundamental lemma of variational calculus

$$\delta W = \int_{\Omega} (\frac{dN}{dx} + b)\delta u dx = 0 \quad \forall\, \delta u \in U \quad \Leftrightarrow \quad \frac{dN}{dx} + b = 0 \text{ in } \Omega.$$

By considering next $\delta u \in U$ without restrictions on the boundary (and using the equilibrium equation to get rid of the first term of the virtual work expression)

$$\delta W = \sum_{\partial\Omega} (-nN + F)\delta u = 0 \quad \forall\, \delta u \in U \quad \Leftrightarrow \quad nN - F = 0 \text{ on } \partial\Omega.$$

The boundary term vanishes also if $\delta u = 0$ on $\partial\Omega$ which implies that u is given on $\partial\Omega$. Therefore, on the boundary either $u - \underline{u} = 0$ or $nN - F = 0$ but not both. In solid mechanics, one may specify the force or displacement, but not both. The constitutive equation for an elastic material follows from the generalized Hooke's law for the bar model $\sigma_{xx} = Edu/dx$ and the definition of stress resultant

$$N = \int \sigma_{xx} dA = EA\frac{du}{dx}.$$

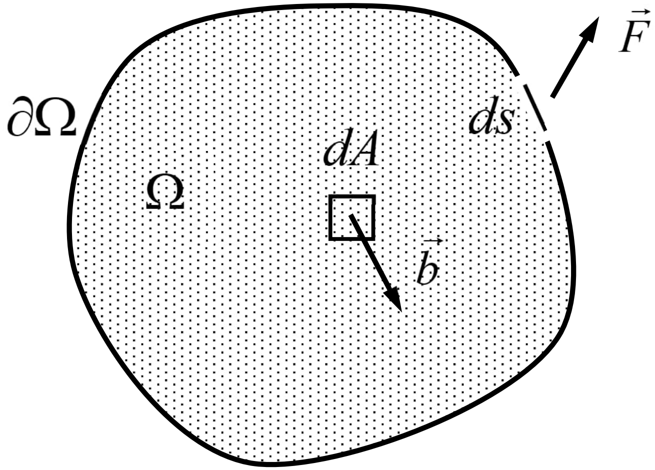
The bar model boundary value problem combines the equations

$$\left.\begin{aligned} &\frac{dN}{dx} + b = 0 \quad \text{and} \quad N = EA\frac{du}{dx} \quad \text{in} \quad \Omega, \\ &nN - F = 0 \quad \text{or} \quad u - \underline{u} = 0 \quad \text{on} \quad \partial\Omega. \end{aligned}\right\} \quad \text{Local form}$$

For an unique solution, the displacement boundary condition should be given at least on one boundary point.

THIN SLAB EQUATIONS

Thin slab model assumes that the transverse displacement (perpendicular to the mid-plane) and stress components vanish and that the quantitities do not depend on the transverse coordinate. Principle of virtual work gives



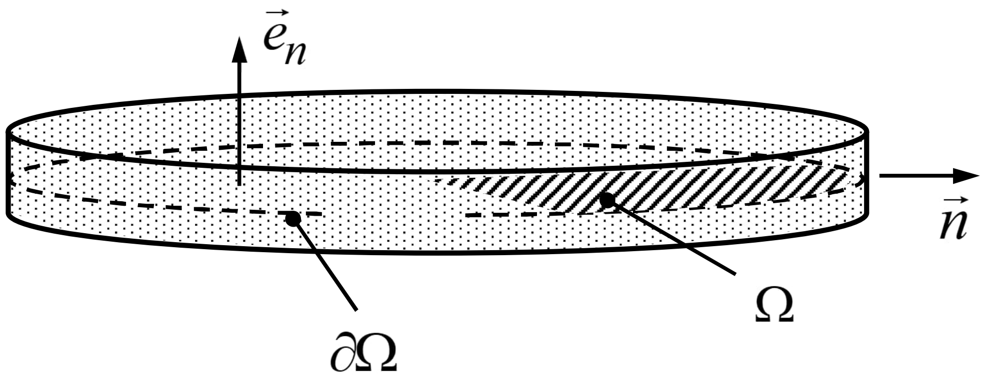
$$\nabla\cdot\overset{\leftrightarrow}{N}+\vec{b}=0 \quad \text{in} \quad \Omega\,,$$

$$\vec{n}\cdot\overset{\leftrightarrow}{N}-\vec{F}=0 \quad \text{or} \quad \vec{u}-\underline{\vec{u}}=0 \qquad \text{on} \quad \partial\Omega,$$

$$\overset{\leftrightarrow}{N}=\int \overset{\leftrightarrow}{\sigma}dn,\;\; \vec{b}=\int \vec{f}dn,\;\; \text{and}\;\; \vec{F}=\int \vec{t}dn\,.$$

Constitutive equation $f(\overset{\leftrightarrow}{N},\vec{u})=0$, which is needed for a closed system of equations, follows form a material model and the stress resultant definition. Writing a boundary value problem in detail, requires specification of the coordinate system.

The physical domain of the thin-slab model is a prismatic body althought the solution domain of the equations is the mid plane. The starting point is virtual work expression written for the physical domain.



If the external forces on the top and bottom surfaces vanish and stress is symmetric ‘a priori’, virtual work expressions of the internal and external forces simplify to (volume element $dV = dndA$ and area element on the boundary $dA = dnds$)

$$\delta W^{\text{int}} = -\int \vec{\vec{\sigma}} : \delta(\nabla\vec{u})_{\text{c}} dV = -\int_{\Omega} (\int \vec{\vec{\sigma}} dn) : \delta(\nabla\vec{u})_{\text{c}} dA = -\int_{\Omega} \vec{\vec{N}} : \delta(\nabla\vec{u})_{\text{c}} dA,$$

$$\delta W_V^{\text{ext}} = \int \vec{f} \cdot \delta\vec{u} dV = \int_\Omega (\int \vec{f} dn) \cdot \delta\vec{u} dA = \int_\Omega \vec{b} \cdot \delta\vec{u} dA,$$

$$\delta W_A^{\text{ext}} = \int \vec{t} \cdot \delta\vec{u} dA = \int_{\partial\Omega} (\int \vec{t} dn) \cdot \delta\vec{u} ds = \int_{\partial\Omega} \vec{F} \cdot \delta\vec{u} ds$$

in which the stress resultants

$\overleftrightarrow{N} = \int \overleftrightarrow{\sigma} dn$, $\vec{b} = \int \vec{f} dn$, and $\vec{F} = \int \vec{t} dn$. integrals over the thickness!

Integration by parts with the vector identity $\overleftrightarrow{a} : (\nabla\vec{b})_c = \nabla \cdot (\overleftrightarrow{a} \cdot \vec{b}) - (\nabla \cdot \overleftrightarrow{a}) \cdot \vec{b}$ in the virtual work expression gives an equivalent but more convenient form for the next step

$$\delta W = -\int_\Omega \overleftrightarrow{N} : \delta(\nabla\vec{u})_c dA + \int_\Omega \vec{b} \cdot \delta\vec{u} dA + \int_{\partial\Omega} \vec{F} \cdot \delta\vec{u} ds \quad \Leftrightarrow$$

$$\delta W = \int_\Omega (\nabla \cdot \overleftrightarrow{N} + \vec{b}) \cdot \delta\vec{u} dA + \int_{\partial\Omega} (-\vec{n} \cdot \overleftrightarrow{N} + \vec{F}) \cdot \delta\vec{u} ds.$$

Principle of virtual work and the fundamental lemma of variation calculus imply the local forms. By considering first a subset of variations $\delta\vec{u} \in U$ with restriction $\delta\vec{u} = 0$ on $\partial\Omega$

$$\delta W = \int_{\Omega} (\nabla \cdot \vec{\vec{N}} + \vec{b}) \cdot \delta\vec{u} dA = 0 \quad \forall \delta\vec{u} \in U \quad \Leftrightarrow \quad \nabla \cdot \vec{\vec{N}} + \vec{b} = 0 \text{ in } \Omega.$$

According to the equilibrium equation, the first term of the virtual work expression vanishes. Next, by considering $\delta\vec{u} \in U$ without restrictions on the boundary

$$\delta W = \int_{\partial\Omega} (-\vec{n} \cdot \vec{\vec{N}} + \vec{F}) \cdot \delta\vec{u} ds = 0 \quad \Rightarrow \quad -\vec{n} \cdot \vec{\vec{N}} + \vec{F} = 0 \text{ or } \delta\vec{u} = 0 \text{ on } \partial\Omega.$$

Vanishing of variation $\delta\vec{u} = 0$ on $\partial\Omega$ implies that displacement is given, i.e., $\vec{u} = \underline{\vec{u}}$. To be precise, one may specify a force component or the corresponding displacement component but not both. Constitutive equation $f(\vec{\vec{N}}, \vec{u}) = 0$ follows from the definition

$$\vec{\vec{N}} = \int \vec{\vec{\sigma}} dn$$

when the stress-displacement relationship for plane-stress is subsitituted there. Altogether, the boundary value problem in its coordinate system invariant form

$$\left.\begin{aligned} &\nabla \cdot \vec{\vec{N}} + \vec{b} = 0 \ \text{ and } \ f(\vec{\vec{N}}, \vec{u}) = 0 \ \text{ in } \ \Omega, \\ &\vec{n} \cdot \vec{\vec{N}} - \vec{F} = 0 \ \text{ or } \ \vec{u} - \underline{\vec{u}} = 0 \ \partial\Omega. \end{aligned}\right\}$$

Integration by parts step of derivation uses the Gauss theorem for a flat geometry which may exclude domains of non-vanishing curvature (it turns out later that the form is valid also in curved geometry).

THIN SLAB EQUATIONS IN (x, y)-COORDINATES

Component representation follows when the tensors of the equilibrium and constitutive equations are expressed in the Cartesian $(\vec{i}, \vec{j})-$basis. Assuming a linearly elastic isotropic material, equilibrium and constitutive equations take the forms,

$$\begin{Bmatrix} \dfrac{\partial N_{xx}}{\partial x} + \dfrac{\partial N_{xy}}{\partial y} + b_x \\ \dfrac{\partial N_{yy}}{\partial y} + \dfrac{\partial N_{xy}}{\partial x} + b_y \end{Bmatrix} = 0, \text{ where } \begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = t[E]_\sigma \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}.$$

Boundary conditions define usually either displacement or traction in the normal and tangential directions to the boundary.

Representations in the Cartesian system (notice that the second form of the gradient is valid only when basis vectors are constants)

$$\nabla = \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \partial / \partial x \\ \partial / \partial y \end{Bmatrix} = \begin{Bmatrix} \partial / \partial x \\ \partial / \partial y \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}, \ \overleftrightarrow{N} = \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} N_{xx} & N_{xy} \\ N_{yx} & N_{yy} \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}, \ \vec{b} = \begin{Bmatrix} b_x \\ b_y \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}$$

$$\nabla \cdot \overleftrightarrow{N} + \vec{b} = \begin{Bmatrix} \partial / \partial x \\ \partial / \partial y \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} \cdot \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} N_{xx} & N_{xy} \\ N_{yx} & N_{yy} \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} + \begin{Bmatrix} b_x \\ b_y \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} = 0$$

$$\nabla \cdot \overleftrightarrow{N} + \vec{b} = \left(\begin{Bmatrix} \partial / \partial x \\ \partial / \partial y \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} N_{xx} & N_{xy} \\ N_{yx} & N_{yy} \end{bmatrix} + \begin{Bmatrix} b_x \\ b_y \end{Bmatrix}^{\mathrm{T}} \right) \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} = 0. \quad \Leftarrow$$

A constitutive equation is needed for a closed system of equations (here the number of unknown stress components is 3, whereas the number of equations is 2. Assuming that the thin slab is made of isotropic homogeneous and linearly elastic material of thickness t (steel, aluminum etc.), stress-displacement relationship, kinematic assumption of the model, and elasticity tensor of the plane-stress case give $(N_{yx} = N_{xy})$

$$\overleftrightarrow{N} = \int \overleftrightarrow{\sigma} dn = \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{i}\vec{j} + \vec{j}\vec{i} \end{Bmatrix}^{\mathrm{T}} t[E]_{\sigma} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} \quad \text{so} \quad \begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = t[E]_{\sigma} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}. \quad \Leftarrow$$

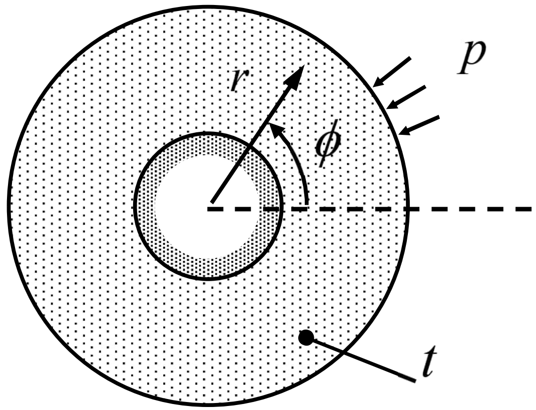
THIN SLAB EQUATIONS IN (r,ϕ)-COORDINATES

Component representation follows when the tensors of the equilibrium and constitutive equations are expressed in the polar $(\vec{e}_r,\vec{e}_\phi)-$basis. Assuming a linearly elastic isotropic material, equilibrium and constitutive equations take the forms,

$$\begin{Bmatrix} \dfrac{1}{r}[\dfrac{\partial(rN_{rr})}{\partial r}+\dfrac{\partial N_{r\phi}}{\partial\phi}-N_{\phi\phi}]+b_r \\ \dfrac{1}{r}[\dfrac{1}{r}\dfrac{\partial(r^2N_{r\phi})}{\partial r}+\dfrac{\partial N_{\phi\phi}}{\partial\phi}]+b_\phi \end{Bmatrix}=0,\ \text{ where } \begin{Bmatrix} N_{rr} \\ N_{\phi\phi} \\ N_{r\phi} \end{Bmatrix}=t\left[E\right]_\sigma \begin{Bmatrix} \dfrac{\partial u_r}{\partial r} \\ \dfrac{1}{r}(u_r+\dfrac{\partial u_\phi}{\partial\phi}) \\ \dfrac{1}{r}\dfrac{\partial u_r}{\partial\phi}+r\dfrac{\partial}{\partial r}(\dfrac{u_\phi}{r}) \end{Bmatrix}.$$

Boundary conditions define usually either displacement or traction in the normal and tangential directions to the boundary.

EXAMPLE Consider a disk $r \in [\varepsilon R, R]$ which is loaded by traction $\vec{t} = -p\vec{e}_r$ on the outer edge $r = R$ (p is constant). Assuming rotation symmetry i.e. that all quantities depend only on the distance r from the center point, find the displacement components $u_r = u(r)$ and $u_\phi = v(r)$ for a linearly elastic material when Young's modulus E and Poisson's ratio ν are constants.



Answer $$u = \frac{(\varepsilon R)^2 - r^2}{r} \frac{p}{E} \frac{1-\nu^2}{1+\nu+\varepsilon^2(1-\nu)}$$

If the displacement and stress resultant components depend only on the radial coordinate, the equilibrium equations and the constitutive equations of the polar coordinate system simplify to (here $b_r = b_\phi = 0$)

$$\frac{dN_{rr}}{dr} + \frac{1}{r}(N_{rr} - N_{\phi\phi}) = \frac{1}{r}[\frac{d}{dr}(rN_{rr}) - N_{\phi\phi}] = 0, \quad \frac{\partial N_{r\phi}}{\partial r} + \frac{2}{r}N_{r\phi} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 N_{r\phi}) = 0$$

and

$$N_{rr} = \frac{tE}{1-\nu^2}(\frac{du}{dr} + \nu\frac{u}{r}), \quad N_{\phi\phi} = \frac{tE}{1-\nu^2}(\frac{u}{r} + \nu\frac{du}{dr}), \quad N_{r\phi} = tG(\frac{dv}{dr} - \frac{v}{r}) = tGr\frac{d}{dr}(\frac{v}{r}).$$

On the inner edge $r = \varepsilon R$ displacement vanishes, i.e., $u_r \equiv u = 0$. On the outer edge $r = R$, $\vec{n} = \vec{e}_r$, $\vec{n} \cdot \vec{\vec{N}} - \vec{F} = 0$, and $\vec{F} = -pt\vec{e}_r$. These conditions give the boundary value problem,

$$\frac{1}{r}[\frac{d}{dr}(rN_{rr}) - N_{\phi\phi}] = 0, \; N_{rr} = \frac{tE}{1-\nu^2}(\frac{du}{dr} + \nu\frac{u}{r}), \quad N_{\phi\phi} = \frac{tE}{1-\nu^2}(\frac{u}{r} + \nu\frac{du}{dr}) \;\; \text{in} \;\; (\varepsilon R, R),$$

$$u = 0 \text{ at } r = \varepsilon R \quad \text{and} \quad N_{rr} = -pt \text{ at } r = R.$$

Elimination of the stress resultants from the equilibrium equation and boundary conditions gives the boundary value problem for the radial displacement component

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d(ru)}{dr}\right] = 0 \quad \text{in} \quad (\varepsilon R, R),$$

$$u = 0 \text{ at } r = \varepsilon R \quad \text{and} \quad \frac{tE}{1-\nu^2}\left(\frac{du}{dr} + \nu\frac{u}{r}\right) = -pt \text{ at } r = R.$$

The generic solution to the differential equation is $u = a/r + br$. Thereafter, the boundary conditions give the values of the integration constants and solution,

$$u = \frac{(\varepsilon R)^2 - r^2}{r}\frac{p}{E}\frac{1-\nu^2}{1+\nu+\varepsilon^2(1-\nu)}.$$

The boundary value problem for the displacement component in the angular direction (in terms of displacement component and stress resultant) is given by

$$\frac{1}{r^2}\frac{d}{dr}(r^2 N_{r\phi}) = 0 \quad \text{and} \quad N_{r\phi} = tGr\frac{d}{dr}\left(\frac{v}{r}\right) \quad \text{in} \quad (\varepsilon R, R),$$

$$v = 0 \quad \text{at} \quad r = \varepsilon R \quad \text{and} \quad N_{r\phi} = 0 \quad \text{at} \quad r = R.$$

Equilibrium equation and the condition on the outer edge imply first $N_{r\phi}(r) = 0$. After that, the constitutive equation, and the displacement boundary condition result into

$$v = 0.$$

MEC-E8003 Beam, plate and shell models, examples 3

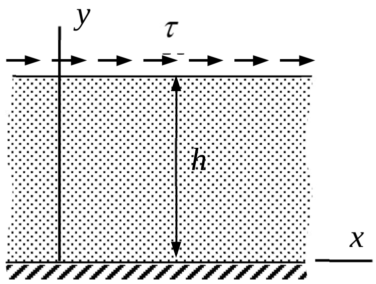
1. The elasticity matrices for *an isotropic material* are the same no matter the *orthonormal* coordinate systems. Consider the elasticity tensor of plane stress in Cartesian $(x.y)$ – and polar (r,ϕ) – coordinate systems and show that

$$\vec{\vec{E}} = \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{i}\vec{j}+\vec{j}\vec{i} \end{Bmatrix}^{\mathrm{T}} [E]_{\sigma} \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{i}\vec{j}+\vec{j}\vec{i} \end{Bmatrix} = \begin{Bmatrix} \vec{e}_r\vec{e}_r \\ \vec{e}_\phi\vec{e}_\phi \\ \vec{e}_r\vec{e}_\phi+\vec{e}_\phi\vec{e}_r \end{Bmatrix}^{\mathrm{T}} [E]_{\sigma} \begin{Bmatrix} \vec{e}_r\vec{e}_r \\ \vec{e}_\phi\vec{e}_\phi \\ \vec{e}_r\vec{e}_\phi+\vec{e}_\phi\vec{e}_r \end{Bmatrix} \text{ where}$$

$$[E]_{\sigma} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}.$$

Solution Discussed during the calculation examples session

2. External shear stress τ is acting on a layer of elastic isotropic material. Young's modulus E and Poisson's ratio ν of the material are constants. Determine stress and displacement in the layer. Assume that stress and displacement components depend on y only and that the external volume force is negligible. Use the component forms of plane stress



$$\begin{Bmatrix} \dfrac{\partial\sigma_{xx}}{\partial x}+\dfrac{\partial\sigma_{yx}}{\partial y}+f_x \\ \dfrac{\partial\sigma_{xy}}{\partial x}+\dfrac{\partial\sigma_{yy}}{\partial y}+f_y \end{Bmatrix} = 0 \,, \quad \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = [E]_{\sigma} \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y+\partial v/\partial x \end{Bmatrix}.$$

Answer $\sigma_{xx}=0$, $\sigma_{yx}=\tau$, $\sigma_{yy}=0$, $u=\dfrac{\tau}{G}y$, $v=0$,

3. Let us consider the principle of virtual work *without* 'a priori' symmetry assumption $\vec{\vec{\sigma}}=\vec{\vec{\sigma}}_{\mathrm{c}}$, when the displacement gradient is expressed as the sum of its symmetric and antisymmetric parts so that $\delta\vec{\vec{\varepsilon}}=\nabla\delta\vec{u}-\delta\vec{\vec{\phi}}$ in which $\delta\vec{\vec{\phi}}=-\delta\vec{\vec{\phi}}_{\mathrm{c}}$. Show that

$$\delta W = -\int_V (\vec{\vec{\sigma}} : \delta\vec{\vec{\varepsilon}}_{\mathrm{c}})dV + \int_V (\underline{\vec{f}}\cdot\delta\vec{u})dV + \int_{A_t} (\underline{\vec{t}}\cdot\delta\vec{u})dA = 0 \quad \forall \delta\vec{u},\delta\vec{\vec{\phi}} \,,$$

implies e.g. the balance laws of continuum mechanics $\nabla\cdot\vec{\vec{\sigma}}+\underline{\vec{f}}=0$ and $\vec{\vec{\sigma}}=\vec{\vec{\sigma}}_{\mathrm{c}}$ in V.

Solution Discussed during the calculation examples session

4. Derive the component forms of the thin slab equilibrium equation $\nabla\cdot\vec{N}+\vec{b}=0$ in the polar coordinate system.

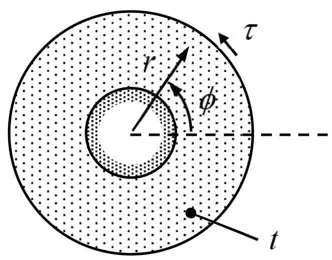
Answer $\begin{Bmatrix} \dfrac{\partial N_{rr}}{\partial r}+\dfrac{1}{r}(N_{rr}-N_{\phi\phi}+\dfrac{\partial N_{\phi r}}{\partial \phi})+b_r \\ \dfrac{\partial N_{r\phi}}{\partial r}+\dfrac{1}{r}(N_{r\phi}+N_{\phi r}+\dfrac{\partial N_{\phi\phi}}{\partial \phi})+b_\phi \end{Bmatrix}=0$

5. Derive the component form of the thin slab model constitutive equation $f(\vec{N},\vec{u})=0$ in the polar coordinate system starting from the stress resultant definition, stress-strain relationship, and elasticity tensor of the plane stress

$$\vec{N}=\int\vec{\sigma}dn\,,\quad \vec{\sigma}=\vec{\vec{E}}:\nabla\vec{u}\,,\quad \vec{\vec{E}}=\begin{Bmatrix} \vec{e}_r\vec{e}_r \\ \vec{e}_\phi\vec{e}_\phi \\ \vec{e}_r\vec{e}_\phi+\vec{e}_\phi\vec{e}_r \end{Bmatrix}^{\mathrm{T}}[E]_\sigma\begin{Bmatrix} \vec{e}_r\vec{e}_r \\ \vec{e}_\phi\vec{e}_\phi \\ \vec{e}_r\vec{e}_\phi+\vec{e}_\phi\vec{e}_r \end{Bmatrix},$$

Answer $\begin{Bmatrix} N_{rr} \\ N_{\phi\phi} \\ N_{r\phi} \end{Bmatrix}=t[E]_\sigma\begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{1}{r}(\dfrac{\partial v}{\partial \phi}+u) \\ \dfrac{\partial v}{\partial r}+\dfrac{1}{r}(\dfrac{\partial u}{\partial \phi}-v) \end{Bmatrix},\ N_{\phi r}=N_{r\phi}\,.$

6. A thin slab of inner radius $r=\varepsilon R$ and outer radius $r=R$ is loaded by tangential traction $\vec{t}=\tau\vec{e}_\phi$ on the outer edge $r=R$ (shear stress τ is constant). Assuming rotation symmetry i.e. that stress and displacement components depend only on the distance r from the center point and $u_r\equiv u=0$, solve for the stress and displacement. Material is linearly elastic and isotropic with material parameters E and ν. External distributed forces vanish. Use the component forms



$$\begin{Bmatrix} \dfrac{\partial N_{rr}}{\partial r}+\dfrac{1}{r}(N_{rr}-N_{\phi\phi}+\dfrac{\partial N_{\phi r}}{\partial \phi})+b_r \\ \dfrac{\partial N_{r\phi}}{\partial r}+\dfrac{1}{r}(N_{r\phi}+N_{\phi r}+\dfrac{\partial N_{\phi\phi}}{\partial \phi})+b_\phi \end{Bmatrix}=0\,,\ \begin{Bmatrix} N_{rr} \\ N_{\phi\phi} \\ N_{r\phi} \end{Bmatrix}=t[E]_\sigma\begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{1}{r}(\dfrac{\partial v}{\partial \phi}+u) \\ \dfrac{\partial v}{\partial r}+\dfrac{1}{r}(\dfrac{\partial u}{\partial \phi}-v) \end{Bmatrix}.$$

Answer $N_{rr}=N_{\phi\phi}=0\,,\ N_{r\phi}=\tau t\dfrac{R^2}{r^2}\,,\ \ u=0\,,\ v=-\dfrac{\tau}{2G}(\dfrac{R^2}{r}-\dfrac{r}{\varepsilon^2})$

7. Virtual work expression of a linearly elastic bar supported by a spring at the right end $x = L$ ($n = 1$) is given by

$$\delta W = \int_0^L -(EA\frac{du}{dx}\frac{d\delta u}{dx})dx + \int_0^L (b\delta u)dx - (ku\delta u)_{x=L},$$

in which $EA = EA(x)$ and k, b are constants. Displacement vanishes at the left end $x = 0$ ($n = -1$) of the bar. Find the underlying boundary value problem starting from the principle of virtual work $\delta W = 0\ \forall \delta u \in U$. Assume that functions of U have continuous derivatives up to the second order and vanish at $x = 0$.

Answer $\frac{d}{dx}(EA\frac{du}{dx}) + b = 0$ in $(0, L)$, $EA\frac{du}{dx} + ku = 0$ at $x = L$, and $u = 0$ at $x = 0$

8. Virtual work expression of a torsion bar is given by

$$\delta W = \int_0^L (-\frac{d\delta\phi}{dx}GJ\frac{d\phi}{dx} - \delta\phi k\phi + \delta\phi c)dx + (T\delta\phi)_{x=L}$$

in which $c(x)$ and T represent the given loading. Deduce *in detail* the differential equation for the rotation $\phi(x)$ and the boundary conditions implied by principle of virtual work and the fundamental lemma of variation calculus. The unknown $\phi(x)$ and the given $GJ(x)$, $k(x)$ and $c(x)$ are assumed to have continuous derivatives of all orders. In addition, $\delta\phi$ and ϕ are assumed to vanish at $x = 0$.

Answer $\frac{d}{dx}(GJ\frac{d\phi}{dx}) - k\phi + c = 0$ in $(0, L)$, $GJ\frac{d\phi}{dx} - T = 0$ at $x = L$, $\phi = 0$ at $x = 0$.

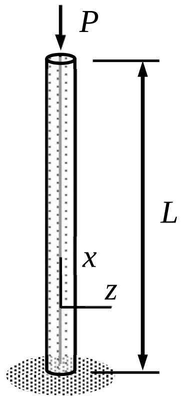
9. Virtual work expression of a Bernoulli beam, clamped at the left end $x = 0$ and loaded by force F and moment R at the right end $x = L$ of solution domain $\Omega = (0, L)$, is given by

$$\delta W = \int_0^L (M\frac{d^2\delta w}{dx^2} + b\delta w)dx + (F\delta w - R\frac{d\delta w}{dx})_{x=L}.$$

Use the principle of virtual work $\delta W = 0\ \forall \delta w \in U$ to derive the beam equilibrium equation in Ω, natural boundary conditions on $x = L$, and essential boundary conditions on $x = 0$. Functions of set U have continuous derivatives up to the fourth order in Ω. In addition, a function of U vanishes at $x = 0$ as does also its first derivative.

Answer $\frac{d^2M}{dx^2} + b = 0$ in $(0, L)$, $-\frac{dM}{dx} + F = 0$ and $M - R = 0$ at $x = L$, $w = \frac{dw}{dx} = 0$ at $x = 0$

10. When displacement is confined to the $xz-$plane, the virtual work expression of a slender Bernoulli beam (figure) is given by



$$\delta W = -\int_0^L \left(\frac{d^2\delta w}{dx^2} EI \frac{d^2 w}{dx^2}\right)dx + P\int_0^L \left(\frac{d\delta w}{dx}\frac{dw}{dx}\right)dx \,.$$

Deduce in detail the underlying differential equation and boundary conditions implied by the principle of virtual work and the fundamental lemma of variation calculus.

Answer $EI\dfrac{d^4 w}{dx^4} + P\dfrac{d^2 w}{dx^2} = 0$ in $(0,L)$, $EI\dfrac{d^3 w}{dx^3} + P\dfrac{dw}{dx} = EI\dfrac{d^2 w}{dx^2} = 0$ at $x = L$,

$w = \dfrac{dw}{dx} = 0$ at $x = 0$.

The elasticity matrices for *an isotropic material* are the same no matter the *orthonormal* coordinate systems. Consider the elasticity tensor of plane stress in Cartesian $(x.y)-$ and polar $(r,\phi)-$ coordinate systems and show that

$$\vec{\vec{E}}=\begin{Bmatrix}\vec{i}\vec{i}\\ \vec{j}\vec{j}\\ \vec{i}\vec{j}+\vec{j}\vec{i}\end{Bmatrix}^{\mathrm{T}}[E]_\sigma\begin{Bmatrix}\vec{i}\vec{i}\\ \vec{j}\vec{j}\\ \vec{i}\vec{j}+\vec{j}\vec{i}\end{Bmatrix}=\begin{Bmatrix}\vec{e}_r\vec{e}_r\\ \vec{e}_\phi\vec{e}_\phi\\ \vec{e}_r\vec{e}_\phi+\vec{e}_\phi\vec{e}_r\end{Bmatrix}^{\mathrm{T}}[E]_\sigma\begin{Bmatrix}\vec{e}_r\vec{e}_r\\ \vec{e}_\phi\vec{e}_\phi\\ \vec{e}_r\vec{e}_\phi+\vec{e}_\phi\vec{e}_r\end{Bmatrix}\text{ where}$$

$$[E]_\sigma=\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & (1-\nu)/2\end{bmatrix}.$$

Solution

Let us start with the elasticity tensor in the polar coordinate system and the relationship between the basis vectors of the Cartesian and polar coordinate systems

$$\vec{\vec{E}}=\begin{Bmatrix}\vec{e}_r\vec{e}_r\\ \vec{e}_\phi\vec{e}_\phi\\ \vec{e}_r\vec{e}_\phi+\vec{e}_\phi\vec{e}_r\end{Bmatrix}^{\mathrm{T}}[E]_\sigma\begin{Bmatrix}\vec{e}_r\vec{e}_r\\ \vec{e}_\phi\vec{e}_\phi\\ \vec{e}_r\vec{e}_\phi+\vec{e}_\phi\vec{e}_r\end{Bmatrix}\text{ and }\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix}=\begin{bmatrix}\mathrm{c}\phi & \mathrm{s}\phi\\ -\mathrm{s}\phi & \mathrm{c}\phi\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}.$$

Using

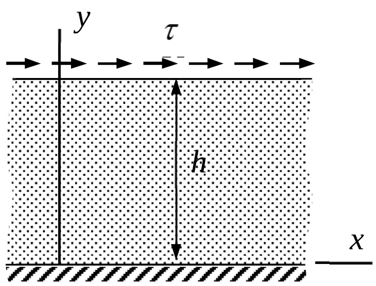
$$\begin{Bmatrix}\vec{e}_r\vec{e}_r\\ \vec{e}_\phi\vec{e}_\phi\\ \vec{e}_r\vec{e}_\phi+\vec{e}_\phi\vec{e}_r\end{Bmatrix}=\begin{bmatrix}\mathrm{c}^2\phi & \mathrm{s}^2\phi & \mathrm{c}\phi\mathrm{s}\phi\\ \mathrm{s}^2\phi & \mathrm{c}^2\phi & -\mathrm{c}\phi\mathrm{s}\phi\\ -2\mathrm{c}\phi\mathrm{s}\phi & 2\mathrm{c}\phi\mathrm{s}\phi & \mathrm{c}^2\phi-\mathrm{s}^2\phi\end{bmatrix}\begin{Bmatrix}\vec{i}\vec{i}\\ \vec{j}\vec{j}\\ \vec{i}\vec{j}+\vec{j}\vec{i}\end{Bmatrix}=[T]\begin{Bmatrix}\vec{i}\vec{i}\\ \vec{j}\vec{j}\\ \vec{i}\vec{j}+\vec{j}\vec{i}\end{Bmatrix}$$

gives the expression of the elasticity matrix in the Cartesian $(x.y)-$ coordinate system:

$$\vec{\vec{E}}=\begin{Bmatrix}\vec{i}\vec{i}\\ \vec{j}\vec{j}\\ \vec{i}\vec{j}+\vec{j}\vec{i}\end{Bmatrix}^{\mathrm{T}}[T]^{\mathrm{T}}[E]_\sigma[T]\begin{Bmatrix}\vec{i}\vec{i}\\ \vec{j}\vec{j}\\ \vec{i}\vec{j}+\vec{j}\vec{i}\end{Bmatrix}.$$

What remains is showing that $[T]^{\mathrm{T}}[E]_\sigma[T]=[E]_\sigma$. See Mathematica notebook for the simplification of the left-hand side.

External shear stress τ is acting on a layer of elastic isotropic material. Young's modulus E and Poisson's ratio ν of the material are constants. Determine stress and displacement in the layer. Assume that stress and displacement components depend on y only and that the external volume force is negligible. Use the component forms of plane stress



$$\begin{Bmatrix} \partial\sigma_{xx}/\partial x + \partial\sigma_{yx}/\partial y + f_x \\ \partial\sigma_{xy}/\partial x + \partial\sigma_{yy}/\partial y + f_y \end{Bmatrix} = 0, \quad \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix}.$$

Solution

Solution to stress and displacement follows from the equilibrium equations, constitutive equations, and boundary conditions. In the layer problem, $u_x = u(y)$, $u_y = v(y)$, $f_x = f_y = 0$. At the lower edge $u(0) = 0$ and at the upper edge $\sigma_{yx}(h) = \sigma_{xy}(h) = \tau$ (stress is symmetric).

The two equilibrium equations and three constitutive equations simplify to

Equilibrium $\dfrac{d\sigma_{yx}}{dy} = 0$ and $\dfrac{d\sigma_{yy}}{dy} = 0$

Constitutive $\sigma_{xx} = \dfrac{\nu E}{1-\nu^2}\dfrac{dv}{dy}$, $\sigma_{yy} = \dfrac{E}{1-\nu^2}\dfrac{dv}{dy}$, $\sigma_{xy} = \sigma_{yx} = \dfrac{E}{2(1+\nu)}\dfrac{du}{dy} = G\dfrac{du}{dy}$,

In this case, solution to stress and displacement follows by considering first the equilibrium equations and using, after that, the constitutive equations. Boundary value problem for the stress components are composed of the equilibrium equations and the boundary condition at the upper edge. Boundary value problems and their solutions to the stress components are

$$\frac{d\sigma_{yx}}{dy} = 0 \quad \text{in} \quad (0,h) \quad \text{and} \quad \sigma_{yx}(h) = \tau \quad \Rightarrow \quad \sigma_{yx}(y) = \tau.$$

$$\frac{d\sigma_{yy}}{dy} = 0 \quad \text{in} \quad (0,h) \quad \text{and} \quad \sigma_{yy}(h) = 0 \quad \Rightarrow \quad \sigma_{yy}(y) = 0.$$

Knowing the stress, boundary value problems for the displacement components are composed of the constitutive equations and the boundary condition on the lower edge. Boundary value problems and their solutions to the displacement components are

$$G\frac{du}{dy} = \sigma_{yx} = \tau \quad \text{in} \quad (0,h) \quad \text{and} \quad u(0) = 0 \quad \Rightarrow \quad u(y) = \frac{\tau}{G}y.$$

$$\frac{E}{1-\nu^2}\frac{dv}{dy} = \sigma_{yy} = 0 \quad \text{in} \quad (0,h) \quad \text{and} \quad v(0) = 0 \quad \Rightarrow \quad v(y) = 0.$$

Finally, the third constitutive equation, not used above, gives

$$\sigma_{xx} = \frac{\nu E}{1-\nu^2}\frac{dv}{dy} = 0\,.$$

Let us consider the principle of virtual work *without* ‘a priori’ symmetry assumption $\vec{\vec{\sigma}} = \vec{\vec{\sigma}}_\mathrm{c}$, when the displacement gradient is expressed as the sum of its symmetric and antisymmetric parts so that $\delta\vec{\vec{\varepsilon}} = \nabla\delta\vec{u} - \delta\vec{\vec{\phi}}$ in which $\delta\vec{\vec{\phi}} = -\delta\vec{\vec{\phi}}_\mathrm{c}$. Show that

$$\delta W = -\int_V (\vec{\vec{\sigma}} : \delta\vec{\vec{\varepsilon}}_\mathrm{c}) dV + \int_V (\underline{\vec{f}} \cdot \delta\vec{u}) dV + \int_{A_t} (\underline{\vec{t}} \cdot \delta\vec{u}) dA = 0 \quad \forall \delta\vec{u}, \delta\vec{\vec{\phi}} \ ,$$

Implies, e.g., the balance laws of continuum mechanics $\nabla \cdot \vec{\vec{\sigma}} + \underline{\vec{f}} = 0$ and $\vec{\vec{\sigma}} = \vec{\vec{\sigma}}_\mathrm{c}$ in V.

Solution

Symmetry of stress is just the local form of moment of momentum balance law. The condition is often satisfied ‘a priori’ but it can also be embedded in the virtual work expression form. Also now, the fundamental theorem of calculus (integration by parts) and the fundamental lemma of variation calculus are the tools for deriving a boundary value problem starting from a given virtual work expression. In addition, division of displacement gradient into its symmetric and antisymmetric parts

$$\nabla\delta\vec{u} = \frac{1}{2}[\nabla\delta\vec{u} + (\nabla\delta\vec{u})_\mathrm{c}] + \frac{1}{2}[\nabla\delta\vec{u} - (\nabla\delta\vec{u})_\mathrm{c}] = \delta\vec{\vec{\varepsilon}} + \delta\vec{\vec{\phi}}$$

is needed. According to principle of virtual work without the assumption of stress symmetry (generic form)

$$\delta W = -\int_V (\vec{\vec{\sigma}} : \delta\vec{\vec{\varepsilon}}_\mathrm{c}) dV + \int_V (\underline{\vec{f}} \cdot \delta\vec{u}) dV + \int_{A_t} (\underline{\vec{t}} \cdot \delta\vec{u}) dA = 0 \quad \forall \delta\vec{u}, \delta\vec{\vec{\phi}} \qquad \Leftrightarrow$$

$$\delta W = -\int_V \vec{\vec{\sigma}} : (\nabla\delta\vec{u})_\mathrm{c} dV + \int_V \vec{\vec{\sigma}} : \delta\vec{\vec{\phi}}_\mathrm{c} dV + \int_V (\underline{\vec{f}} \cdot \delta\vec{u}) dV + \int_{A_t} (\underline{\vec{t}} \cdot \delta\vec{u}) dA = 0 \quad \forall \delta\vec{u}, \delta\vec{\vec{\phi}} \ .$$

Vector identity $\vec{\vec{a}} : (\nabla\vec{b})_\mathrm{c} = \nabla \cdot (\vec{\vec{a}} \cdot \vec{b}) - (\nabla \cdot \vec{\vec{a}}) \cdot \vec{b}$ adapted to the present case with $\vec{\vec{a}} = \vec{\vec{\sigma}}$ and $\vec{b} = \vec{u}$ gives the form

$$\delta W = \int_V (\nabla \cdot \vec{\vec{\sigma}} + \underline{\vec{f}}) \cdot \delta\vec{u} dV - \int_V \nabla \cdot (\vec{\vec{\sigma}} \cdot \delta\vec{u}) dV + \int_V (\vec{\vec{\sigma}} : \delta\vec{\vec{\phi}}_\mathrm{c}) dV + \int_{A_t} (\underline{\vec{t}} \cdot \delta\vec{u}) dA = 0 \quad \forall \delta\vec{u}, \delta\vec{\vec{\phi}} \ .$$

Use of the divergence theorem, also known as Gauss's theorem, in the second term on the right-hand side gives first

$$\delta W = \int_V (\nabla \cdot \vec{\vec{\sigma}} + \underline{\vec{f}}) \cdot \delta\vec{u} dV - \int_A (\vec{n} \cdot \vec{\vec{\sigma}}) \cdot \delta\vec{u} dA + \int_V (\vec{\vec{\sigma}} : \delta\vec{\vec{\phi}}_\mathrm{c}) dV + \int_{A_t} (\underline{\vec{t}} \cdot \delta\vec{u}) dA = 0 \quad \forall \delta\vec{u}, \delta\vec{\vec{\phi}}$$

and after dividing the boundary into disjoint parts $A = A_u \cup A_t$ and $A_u \cap A_t = \varnothing$ and assuming that $\delta\vec{u} = 0$ on A_u

$$\delta W = \int_V (\nabla \cdot \vec{\vec{\sigma}} + \underline{\vec{f}}) \cdot \delta\vec{u} dV + \int_V (\vec{\vec{\sigma}} : \delta\vec{\vec{\phi}}) dV - \int_{A_t} (\vec{n} \cdot \vec{\vec{\sigma}} - \underline{\vec{t}}) \cdot \delta\vec{u} dA = 0 \quad \forall \delta\vec{u}, \delta\vec{\vec{\phi}} \ .$$

The purpose of the manipulation above was just to obtain a representation that allows the use of fundamental lemma of variation calculus. Selection $\delta\vec{u}=0$ on A_t vanishing and $\delta\vec{\vec{\phi}}=0$, and thereafter $\delta\vec{u}\equiv 0$ (everywhere) imply

$$\nabla\cdot\vec{\vec{\sigma}}+\underline{\vec{f}}=0 \quad \text{in} \quad V$$

$$\vec{\vec{\sigma}}-\vec{\vec{\sigma}}_{\mathrm{c}}=0 \quad \text{in} \quad V\,.$$

It is noteworthy that the latter selections implies that $\vec{\vec{\sigma}}=\vec{\vec{\sigma}}_{\mathrm{c}}$ due to the restriction $\delta\vec{\vec{\phi}}=-\delta\vec{\vec{\phi}}_{\mathrm{c}}$. Knowing the conditions above, virtual work expression simplifies to

$$\delta W=-\int_{A_t}(\vec{n}\cdot\vec{\vec{\sigma}}-\underline{\vec{t}})\cdot\delta\vec{u}dA=0 \qquad \forall\,\delta\vec{u},\delta\vec{\vec{\phi}}$$

which implies $\vec{n}\cdot\vec{\vec{\sigma}}-\underline{\vec{t}}=0$ on A_t. Taking into account the restriction on $\delta\vec{u}$ on A_u, boundary value problem takes the form

$$\left.\begin{aligned}
&\nabla\cdot\vec{\vec{\sigma}}+\underline{\vec{f}}=0 \quad \text{in} \quad V\,,\\
&\vec{\vec{\sigma}}-\vec{\vec{\sigma}}_{\mathrm{c}}=0 \quad \text{in} \quad V\,,\\
&\vec{n}\cdot\vec{\vec{\sigma}}-\underline{\vec{t}}=0 \quad \text{on} \quad A_t\,,\\
&\vec{u}-\underline{\vec{u}}=0 \quad \text{on} \quad A_u\,.
\end{aligned}\right\}$$

The first three equations are the local balance laws for momentum and moment of momentum.

Derive the component forms of the thin slab equilibrium equation $\nabla \cdot \vec{\vec{N}} + \vec{b} = 0$ in the polar coordinate system.

Solution

The component forms of stress, external force, and gradient operator of the polar coordinate system are

$$\vec{\vec{N}} = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} N_{rr} & N_{r\phi} \\ N_{\phi r} & N_{\phi\phi} \end{bmatrix} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}, \ \vec{b} = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} b_r \\ b_\phi \end{Bmatrix}, \ \nabla = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \partial / \partial r \\ \partial / (r\partial\phi) \end{Bmatrix}, \ \frac{\partial}{\partial \phi} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} = \begin{Bmatrix} \vec{e}_\phi \\ -\vec{e}_r \end{Bmatrix}.$$

Let us start with the terms of stress resultant divergence

$$\nabla \cdot \vec{\vec{N}} = (\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{\partial}{r\partial\phi}) \cdot (N_{rr}\vec{e}_r\vec{e}_r + N_{r\phi}\vec{e}_r\vec{e}_\phi + N_{\phi r}\vec{e}_\phi\vec{e}_r + N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi).$$

First term of the gradient simplifies to

$$\vec{e}_r \frac{\partial}{\partial r} \cdot (N_{rr}\vec{e}_r\vec{e}_r + N_{r\phi}\vec{e}_r\vec{e}_\phi + N_{\phi r}\vec{e}_\phi\vec{e}_r + N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi) \quad \Rightarrow$$

$$\vec{e}_r \cdot (\frac{\partial N_{rr}}{\partial r}\vec{e}_r\vec{e}_r + \frac{\partial N_{r\phi}}{\partial r}\vec{e}_r\vec{e}_\phi + \frac{\partial N_{\phi r}}{\partial r}\vec{e}_\phi\vec{e}_r + \frac{\partial N_{\phi\phi}}{\partial r}\vec{e}_\phi\vec{e}_\phi) \quad \Rightarrow$$

$$\vec{e}_r \frac{\partial}{\partial r} \cdot (N_{rr}\vec{e}_r\vec{e}_r + N_{r\phi}\vec{e}_r\vec{e}_\phi + N_{\phi r}\vec{e}_\phi\vec{e}_r + N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi) = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \dfrac{\partial N_{rr}}{\partial r} \\ \dfrac{\partial N_{r\phi}}{\partial r} \end{Bmatrix}.$$

Then the same manipulation for the second term of the displacement gradient

$$\vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial \phi} \cdot (N_{rr}\vec{e}_r\vec{e}_r + N_{r\phi}\vec{e}_r\vec{e}_\phi + N_{\phi r}\vec{e}_\phi\vec{e}_r + N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi) \quad \Rightarrow$$

$$\vec{e}_\phi \frac{1}{r} \cdot (\frac{\partial N_{rr}}{\partial \phi}\vec{e}_r\vec{e}_r + N_{rr}\vec{e}_\phi\vec{e}_r + N_{rr}\vec{e}_r\vec{e}_\phi + \frac{\partial N_{r\phi}}{\partial \phi}\vec{e}_r\vec{e}_\phi + N_{r\phi}\vec{e}_\phi\vec{e}_\phi - N_{r\phi}\vec{e}_r\vec{e}_r +$$

$$\frac{\partial N_{\phi r}}{\partial \phi}\vec{e}_\phi\vec{e}_r - N_{\phi r}\vec{e}_r\vec{e}_r + N_{\phi r}\vec{e}_\phi\vec{e}_\phi + \frac{\partial N_{\phi\phi}}{\partial \phi}\vec{e}_\phi\vec{e}_\phi - N_{\phi\phi}\vec{e}_r\vec{e}_\phi - N_{\phi\phi}\vec{e}_\phi\vec{e}_r) \qquad \Rightarrow$$

$$\vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial \phi} \cdot (N_{rr}\vec{e}_r\vec{e}_r + N_{r\phi}\vec{e}_r\vec{e}_\phi + N_{\phi r}\vec{e}_\phi\vec{e}_r + N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi) = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \dfrac{1}{r}(N_{rr} - N_{\phi\phi} + \dfrac{\partial N_{\phi r}}{\partial \phi}) \\ \dfrac{1}{r}(N_{r\phi} + N_{\phi r} + \dfrac{\partial N_{\phi\phi}}{\partial \phi}) \end{Bmatrix}.$$

Finally, by combining the terms of the divergence and external loading

$$\nabla \cdot \vec{\vec{N}} + \vec{b} = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \dfrac{\partial N_{rr}}{\partial r} + \dfrac{1}{r}(N_{rr} - N_{\phi\phi} + \dfrac{\partial N_{\phi r}}{\partial \phi}) + b_r \\ \dfrac{\partial N_{r\phi}}{\partial r} + \dfrac{1}{r}(N_{r\phi} + N_{\phi r} + \dfrac{\partial N_{\phi\phi}}{\partial \phi}) + b_\phi \end{Bmatrix} = 0 \,.$$

Derive the component form of the thin slab model constitutive equation $f(\vec{N},\vec{u})=0$ in the polar coordinate system starting from the stress resultant definition, stress-strain relationship, and elasticity tensor of the plane stress

$$\vec{N}=\int\vec{\sigma}dn\,,\quad \vec{\sigma}=\vec{\vec{E}}:\nabla\vec{u}\,,\quad \vec{\vec{E}}=\begin{Bmatrix}\vec{e}_r\vec{e}_r\\ \vec{e}_\phi\vec{e}_\phi\\ \vec{e}_r\vec{e}_\phi+\vec{e}_\phi\vec{e}_r\end{Bmatrix}^{\mathrm{T}}\frac{E}{1-\nu^2}\begin{bmatrix}1&\nu&0\\ \nu&1&0\\ 0&0&(1-\nu)/2\end{bmatrix}\begin{Bmatrix}\vec{e}_r\vec{e}_r\\ \vec{e}_\phi\vec{e}_\phi\\ \vec{e}_r\vec{e}_\phi+\vec{e}_\phi\vec{e}_r\end{Bmatrix}.$$

Solution

Polar coordinate system representations of the gradient expression, planar displacement, and the basis vector derivatives are

$$\nabla=\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\partial/\partial r\\ \partial/(r\partial\phi)\end{Bmatrix},\quad \vec{u}=\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}u_r\\ u_\phi\end{Bmatrix},\ \text{and}\ \frac{\partial}{\partial\phi}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix}=\begin{Bmatrix}\vec{e}_\phi\\ -\vec{e}_r\end{Bmatrix}.$$

Substitution into the displacement gradient gives

$$\nabla\vec{u}=(\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\phi\frac{\partial}{r\partial\phi})(u_r\vec{e}_r+u_\phi\vec{e}_\phi)\quad\Rightarrow$$

$$\nabla\vec{u}=\vec{e}_r\vec{e}_r\frac{\partial u_r}{\partial r}+\vec{e}_r\vec{e}_\phi\frac{\partial u_\phi}{\partial r}+\vec{e}_\phi\vec{e}_r\frac{1}{r}\frac{\partial u_r}{\partial\phi}+\vec{e}_\phi\vec{e}_\phi\frac{1}{r}u_r+\vec{e}_\phi\vec{e}_\phi\frac{1}{r}\frac{\partial u_\phi}{\partial\phi}-\vec{e}_\phi\vec{e}_r\frac{1}{r}u_\phi\quad\Rightarrow$$

$$\nabla\vec{u}=\vec{e}_r\vec{e}_r\frac{\partial u_r}{\partial r}+\vec{e}_\phi\vec{e}_\phi\frac{1}{r}(\frac{\partial u_\phi}{\partial\phi}+u_r)+\vec{e}_r\vec{e}_\phi\frac{\partial u_\phi}{\partial r}+\vec{e}_\phi\vec{e}_r\frac{1}{r}(\frac{\partial u_r}{\partial\phi}-u_\phi)\,.$$

The double inner product with the basis vector combinations of the elasticity tensor gives the stress expression

$$\vec{\sigma}=\vec{\vec{E}}:\nabla\vec{u}=\begin{Bmatrix}\vec{e}_r\vec{e}_r\\ \vec{e}_\phi\vec{e}_\phi\\ \vec{e}_r\vec{e}_\phi+\vec{e}_\phi\vec{e}_r\end{Bmatrix}^{\mathrm{T}}\frac{E}{1-\nu^2}\begin{bmatrix}1&\nu&0\\ \nu&1&0\\ 0&0&(1-\nu)/2\end{bmatrix}\begin{Bmatrix}\dfrac{\partial u_r}{\partial r}\\ \dfrac{1}{r}(\dfrac{\partial u_\phi}{\partial\phi}+u_r)\\ \dfrac{\partial u_\phi}{\partial r}+\dfrac{1}{r}(\dfrac{\partial u_r}{\partial\phi}-u_\phi)\end{Bmatrix}.$$

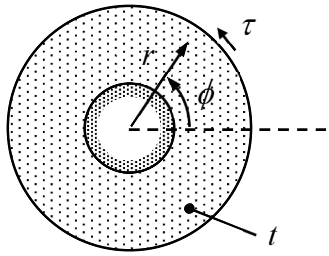
According to the definition, stress resultant is integral of stress over the thickness

$$\vec{N}=\int\vec{\sigma}dn=\begin{Bmatrix}\vec{e}_r\vec{e}_r\\ \vec{e}_\phi\vec{e}_\phi\\ \vec{e}_r\vec{e}_\phi+\vec{e}_\phi\vec{e}_r\end{Bmatrix}^{\mathrm{T}}\frac{tE}{1-\nu^2}\begin{bmatrix}1&\nu&0\\ \nu&1&0\\ 0&0&(1-\nu)/2\end{bmatrix}\begin{Bmatrix}\dfrac{\partial u_r}{\partial r}\\ \dfrac{1}{r}(\dfrac{\partial u_\phi}{\partial\phi}+u_r)\\ \dfrac{\partial u_\phi}{\partial r}+\dfrac{1}{r}(\dfrac{\partial u_r}{\partial\phi}-u_\phi)\end{Bmatrix}$$

or in the component form

$$\begin{Bmatrix} N_{rr} \\ N_{\phi\phi} \\ N_{r\phi} \end{Bmatrix} = \frac{tE}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u_r}{\partial r} \\ \dfrac{1}{r}(\dfrac{\partial u_\phi}{\partial \phi} + u_r) \\ \dfrac{\partial u_\phi}{\partial r} + \dfrac{1}{r}(\dfrac{\partial u_r}{\partial \phi} - u_\phi) \end{Bmatrix} \text{ and } N_{\phi r} = N_{r\phi} .$$

A thin slab of inner radius $r=\varepsilon R$ and outer radius $r=R$ is loaded by tangential traction $\vec{t}=\tau\vec{e}_\phi$ on the outer edge $r=R$ (shear stress τ is constant). Assuming rotation symmetry i.e. that stress and displacement components depend only on the distance r from the center point and $u_r \equiv u = 0$, solve for the stress and displacement. Material is linearly elastic and isotropic with material parameters E and ν. External distributed forces vanish. Use the component forms



$$\begin{Bmatrix} \dfrac{\partial N_{rr}}{\partial r}+\dfrac{1}{r}(N_{rr}-N_{\phi\phi}+\dfrac{\partial N_{\phi r}}{\partial \phi})+b_r \\ \dfrac{\partial N_{r\phi}}{\partial r}+\dfrac{1}{r}(N_{r\phi}+N_{\phi r}+\dfrac{\partial N_{\phi\phi}}{\partial \phi})+b_\phi \end{Bmatrix}=0, \quad \begin{Bmatrix} N_{rr} \\ N_{\phi\phi} \\ N_{r\phi} \end{Bmatrix}=\frac{tE}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{1}{r}(\dfrac{\partial v}{\partial \phi}+u) \\ \dfrac{\partial v}{\partial r}+\dfrac{1}{r}(\dfrac{\partial u}{\partial \phi}-v) \end{Bmatrix}.$$

Solution

As stress resultants and displacement components depend only on r, equilibrium equations and the constitutive equations simplify to (notice that the derivatives are ordinary ones as the quantities are known to depend on r only)

$$\frac{dN_{rr}}{dr}+\frac{1}{r}N_{rr}-\frac{1}{r}N_{\phi\phi}=0, \quad \frac{dN_{r\phi}}{dr}+\frac{2}{r}N_{r\phi}=\frac{1}{r^2}\frac{d}{dr}(r^2 N_{r\phi})=0 \quad \text{and}$$

$$N_{rr}=\frac{tE}{1-\nu^2}(\frac{du}{dr}+\nu\frac{u}{r}), \quad N_{\phi\phi}=\frac{tE}{1-\nu^2}(\frac{u}{r}+\nu\frac{du}{dr}), \quad N_{r\phi}=tG(\frac{dv}{dr}-\frac{v}{r})=tGr\frac{d}{dr}(\frac{v}{r})$$

Solution to the shear stress resultant $N_{r\phi}$ follows from boundary value problem composed of the equilibrium equation and boundary condition on the outer edge

$$\frac{1}{r^2}\frac{d}{dr}(r^2 N_{r\phi})=0 \quad r\in(\varepsilon R,R) \quad \text{and} \quad N_{r\phi}=\tau t \quad \text{at} \quad r=R \quad \Rightarrow \quad N_{r\phi}=\tau t\frac{R^2}{r^2}. \quad \leftarrow$$

After that, solution to displacement component $u_\phi = v(r)$ follows from boundary value problem composed of the constitutive equation and boundary condition on the inner edge

$$tGr\frac{d}{dr}(\frac{1}{r}v)=\tau t\frac{R^2}{r^2} \quad r\in(\varepsilon R,R) \quad \text{and} \quad v=0 \quad \text{at} \quad r=\varepsilon R \quad \Rightarrow \quad v=-\frac{\tau}{2G}(\frac{R^2}{r}-\frac{r}{\varepsilon^2}). \quad \leftarrow$$

Displacement component $u_r = u(r) = 0$ by assumption which implies that

$$N_{rr}=N_{\phi\phi}=0. \quad \leftarrow$$

The same solution follows also without the assumption $u_r = u(r) = 0$. Eliminating the stress resultants from the second equilibrium equation gives the boundary value problem (on the outer edge $N_{rr}=0$)

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{u}{r^2}=\frac{d}{dr}[\frac{1}{r}\frac{d}{dr}(ru)]=0 \quad r\in(\varepsilon R,R),$$

$$u=0 \quad \text{at} \quad r=\varepsilon R \quad \text{and} \quad \frac{du}{dr}+\nu\frac{u}{r}=0 \quad \text{at} \quad r=R \quad \Rightarrow \quad u_r=u(r)=0\,.$$

Virtual work expression of a linearly elastic bar supported by a spring at the right end $x=L$ $(n=1)$ is given by

$$\delta W=-\int_{0}^{L}(\frac{d\delta u}{dx}EA\frac{du}{dx})dx+\int_{0}^{L}(\delta ub)dx-(\delta uku)_{x=L},$$

in which $EA=EA(x)$ and k,b are constants. Displacement vanishes at the left end $x=0$ $(n=-1)$ of the bar. Find the underlying boundary value problem starting from the principle of virtual work $\delta W=0\ \forall\delta u\in U$. Assume that functions of U have continuous derivatives up to the second order and vanish at $x=0$.

Solution

Fundamental theorem of calculus (integration by parts) and the fundamental lemma of variation calculus are the tools for deriving a boundary value problem starting from a virtual work expression. In the one-dimensional case, for any continuous functions a and b (or values at some point), it holds

$$\int_{\Omega}a\frac{db}{dx}dx=\sum_{\partial\Omega}(nab)-\int_{\Omega}\frac{da}{dx}bdx \quad \text{(where } n=\pm1\text{)},$$

$$a,b\in\mathbb{R}: \qquad ab=0\quad\forall b \quad\Leftrightarrow\quad a=0,$$

$$a,b\in C^{0}(\Omega): \quad \int_{\Omega}abdx=0\quad\forall b \quad\Leftrightarrow\quad a=0 \quad\text{in}\quad \Omega.$$

In the present case $\Omega=(0,L)$ and $\partial\Omega=\{0,L\}$. Displacement has continuous derivatives up to and including second order i.e. $u\in C^{2}(\Omega)$. The constraint on the function set $u=0$ at $x=0$ implies that $\delta u=0$ at $x=0$. Integration by parts gives equivalent forms (the aim is to remove the derivatives from variations in the integral over the domain)

$$\delta W=-\int_{0}^{L}(\frac{d\delta u}{dx}EA\frac{du}{dx})dx+\int_{0}^{L}(\delta ub)dx-(\delta uku)_{x=L}\quad\Leftrightarrow$$

$$\delta W=\int_{0}^{L}[\frac{d}{dx}(EA\frac{du}{dx})+b]\delta udx-[(EA\frac{du}{dx}+ku)\delta u]_{x=L}. \quad (\text{as } \delta u=0 \text{ at } x=0)$$

The purpose of the manipulation above was to obtain a representation which allows the use of fundamental lemma of variation calculus. According to principle of virtual work, $\delta W=0$ $\forall\delta u\in U$. Let us consider first a subset $U_{0}\subset U$ for which $\delta u=0$ at $x=L$ so that the boundary term vanishes. Then

$$\delta W=\int_{0}^{L}[\frac{d}{dx}(EA\frac{du}{dx})+b]\delta udx=0\quad \delta u\in U_{0}\subset U$$

and the fundamental lemma of variation calculus implies that

$$\frac{d}{dx}(EA\frac{du}{dx})+b=0\quad\text{in } (0,L).$$

Knowing this and considering the full set U, the variational equation simplifies into

$$\delta W = -[(EA\frac{du}{dx} + ku)\delta u]_{x=L} = 0 .$$

Then, the fundamental lemma of variation calculus implies that

$$EA\frac{du}{dx} + ku = 0 \quad \text{at} \quad x = L .$$

Finally combining the equations to form a boundary value problem (notice that the definition of the function set implies also a boundary condition):

$$\frac{d}{dx}(EA\frac{du}{dx}) + b = 0 \quad \text{in} \ (0, L) ,$$

$$EA\frac{du}{dx} + ku = 0 \quad \text{at} \quad x = L ,$$

$$u = 0 \quad \text{at} \quad x = 0 .$$

Virtual work expression of a torsion bar is given by

$$\delta W=\int_0^L \left(-\frac{d\delta\phi}{dx}GJ\frac{d\phi}{dx}-\delta\phi k\phi+\delta\phi c\right)dx+(T\delta\phi)_{x=L}$$

in which $c(x)$ and T represent the given loading. Deduce *in detail* the differential equation for the rotation $\phi(x)$ and boundary conditions implied by principle of virtual work and the fundamental lemma of variation calculus. The unknown $\phi(x)$ and the given $GJ(x)$, $k(x)$ and $c(x)$ are assumed to have continuous derivatives of all orders. In addition, $\delta\phi$ and ϕ are assumed to vanish at $x=0$.

Solution

Here $\Omega=(0,L)$ and $\partial\Omega=\{0,L\}$. Rotation has continuous derivatives up to and including second order, i.e., $\phi\in U\subset C^2(\Omega)$. Function set U is constrained by $\phi=0$ at $x=0$, which implies that $\delta\phi=0$ at $x=0$. Integration by parts gives equivalent forms

$$\delta W=\int_0^L \left(-\frac{d\delta\phi}{dx}GJ\frac{d\phi}{dx}-\delta\phi k\phi+\delta\phi c\right)dx+(T\delta\phi)_{x=L}\quad\Leftrightarrow$$

$$\delta W=\int_0^L \left[\frac{d}{dx}\left(GJ\frac{d\phi}{dx}\right)-k\phi+c\right]\delta\phi dx-\sum_{\{0,L\}}\left[n\left(GJ\frac{d\phi}{dx}\right)\delta\phi\right]+(T\delta\phi)_{x=L}\ \Rightarrow$$

$$\delta W=\int_0^L \left[\frac{d}{dx}\left(GJ\frac{d\phi}{dx}\right)-k\phi+c\right]\delta\phi dx-\left[\left(GJ\frac{d\phi}{dx}-T\right)\delta\phi\right]_{x=L}\quad\text{as}\quad\delta\phi=0\ \text{at}\ x=0.$$

The purpose of the manipulation is to obtain a representation that allows the use of fundamental lemma of variation calculus.

According to principle of virtual work $\delta W=0\ \ \forall\delta\phi$. Let us consider first a subset for which $\delta\phi=0$ at $x=L$ so that the boundary term vanishes. Then

$$\delta W=\int_0^L \left[\frac{d}{dx}\left(GJ\frac{d\phi}{dx}\right)-k\phi+c\right]\delta\phi dx=0\quad\forall\delta\phi\ \text{satisfying}\ \delta\phi=0\ \text{at}\ x=L$$

and the fundamental lemma of variation calculus implies that

$$\frac{d}{dx}\left(GJ\frac{d\phi}{dx}\right)-k\phi+c=0\quad\text{in}\ (0,L).$$

Knowing this and considering the function set without the additional constraint

$$\delta W=-\left[\left(GJ\frac{d\phi}{dx}-T\right)\delta\phi\right]_{x=L}=0\qquad\forall\delta\phi.$$

The fundamental lemma of variation calculus implies now

$$GJ\frac{d\phi}{dx}-T=0\quad\text{at}\quad x=L\ .$$

Finally combining the equations to form a boundary value problem (notice that the constraint $\phi = 0$ at $x = 0$ implies a boundary condition) :

$$\frac{d}{dx}(GJ\frac{d\phi}{dx}) - k\phi + c = 0 \quad \text{in } (0, L),$$

$$GJ\frac{d\phi}{dx} - T = 0 \quad \text{at} \quad x = L,$$

$$\phi = 0 \quad \text{at} \quad x = 0.$$

Virtual work expression of a Bernoulli beam, clamped at the left end $x=0$ and loaded by force F and moment R at the right end $x=L$ of solution domain $\Omega=(0,L)$, is given by

$$\delta W=\int_0^L (\frac{d^2\delta w}{dx^2}M+\delta wb)dx+(F\delta w-R\frac{d\delta w}{dx})_{x=L} .$$

Use the principle of virtual work $\delta W=0\ \forall\delta w\in U$ to derive the beam equilibrium equation in Ω, natural boundary conditions on $x=L$, and essential boundary conditions on $x=0$. Functions of set U have continuous derivatives up to the fourth order in Ω. In addition, a function of U vanishes at $x=0$ as does also its first derivative.

Solution

In MEC-E8003, principle of virtual work is used to derive the equilibrium equation(s) in terms of the stress resultants (like shear forces and bending moments). The constitutive equation, giving the relationship between the stress resultants and kinetic quantities (like displacements and rotations), is a separate story. The mathematical tools needed in the derivation are (one-dimensional case $\Omega\subset\mathbb{R}$) $a,b\in C^0(\Omega)$

$$\int_\Omega \frac{d}{dx}(ab)dx=\sum_{\partial\Omega} nab$$, where $n=\pm 1$ is the unit outward normal to Ω (on $\partial\Omega$)

$$\int_\Omega abdx=0 \quad \forall b \quad \Leftrightarrow \quad a=0 \ \text{ in } \Omega .$$

Integration by parts once in the first term gives an equivalent form (notice that $\delta w\in U$ and therefore $\delta w=d\delta w/dx=0$ at $x=0$

$$\delta W=\int_0^L (M\frac{d^2\delta w}{dx^2}+b\delta w)dx+(F\delta w-R\frac{d\delta w}{dx})_{x=L} \quad \Leftrightarrow$$

$$\delta W=\int_0^L (-\frac{dM}{dx}\frac{d\delta w}{dx}+b\delta w)dx+[(M-R)\frac{d\delta w}{dx}]_{x=L}+(F\delta w)_{x=L} .$$

Integration by parts second time in the first term gives also an equivalent form

$$\delta W=\int_0^L (\frac{d^2 M}{dx^2}+b)\delta wdx+[(-\frac{dM}{dx}+F)\delta w]_{x=L}+[(M-R)\frac{d\delta w}{dx}]_{x=L} .$$

According to the principle of virtual work $\delta W=0\ \ \forall\delta w\in U$. Let us first consider a subset $U_0\subset U$ for which $\delta w=d\delta w/dx=0$ at $x=L$ so that the boundary terms vanish. The equilibrium equation follows from the fundamental lemma of variation calculus:

$$\delta W=\int_0^L (\frac{d^2 M}{dx^2}+b)\delta wdx=0 \quad \forall\delta w\in U_0 \qquad \Leftrightarrow \qquad \frac{d^2 M}{dx^2}+b=0 \quad \text{in} \quad (0,L).$$

Let us next consider a subset $U_0 \subset U$ for which only $d\delta w / dx = 0$ at $x = 0$ so that the last boundary term of the virtual work expression vanishes. Also, the first term can be omitted due to the equilibrium equation. The natural boundary condition follows from the fundamental lemma of variation calculus:

$$\delta W = [(-\frac{dM}{dx} + F)\delta w]_{x=L} = 0 \quad \forall \delta w \in U_0 \quad \Leftrightarrow \quad -\frac{dM}{dx} + F = 0 \quad \text{at} \quad x = L .$$

Finally, let us consider a subset $U_0 \subset U$ for which only $\delta w = 0$ at $x = L$ and use the equations already obtained to simplify the virtual work expression. The natural boundary condition follows from the fundamental lemma of variation calculus:

$$\delta W = [(M - R)\frac{d\delta w}{dx}]_{x=L} = 0 \quad \forall \delta w \in U_0 \qquad \Leftrightarrow \quad M - R = 0 \quad \text{at} \quad x = L .$$

As the last step, the essential boundary conditions follow from the problem definition (clamped). They can also partly be deduced from the definition of U . Vanishing of variation $d\delta w / dx$ and δw at $x = 0$ imply that dw / dx and w are given at $x = 0$.

A beam boundary value problem is composed of the equations implied by the principle of virtual work

$$\frac{d^2 M}{dx^2} + b = 0 \quad \text{in} \quad (0, L) . \quad \leftarrow$$

$$-\frac{dM}{dx} + F = 0 \quad \text{and} \quad M - R = 0 \quad \text{at} \quad x = L . \quad \leftarrow$$

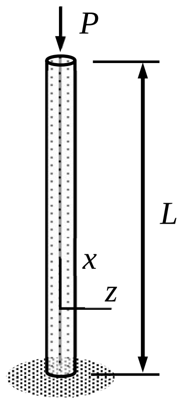
$$w = 0 \ \text{ and } \ \frac{dw}{dx} = 0 \quad \text{at} \quad x = 0 . \quad \leftarrow$$

Definition of stress resultant, stress-strain relationship, and elasticity tensor for the beam problem gives the constitutive equation

$$M = -EI\frac{d^2 w}{dx^2}$$

which is needed for a closed system.

When displacement is confined to the xz–plane, the virtual work expression of a slender Bernoulli beam (figure) is given by



$$\delta W=-\int_0^L (\frac{d^2\delta w}{dx^2}EI\frac{d^2 w}{dx^2})dx+P\int_0^L (\frac{d\delta w}{dx}\frac{dw}{dx})dx\,.$$

Deduce in detail the underlying differential equation and boundary conditions implied by the principle of virtual work and the fundamental lemma of variation calculus. Assume that have continuous derivatives up to (and including) fourth order.

Solution

Integration by parts gives an equivalent but a more convenient form (assuming continuity up to and including second derivatives)

$$\delta W=-\int_0^L (\frac{d^2\delta w}{dx^2}EI\frac{d^2 w}{dx^2})dx+P\int_0^L (\frac{d\delta w}{dx}\frac{dw}{dx})dx \quad \Leftrightarrow \quad (P \text{ is a constant})$$

$$\delta W=-\int_0^L \delta w(EI\frac{d^4 w}{dx^4}+P\frac{d^2 w}{dx^2})dx+\sum\nolimits_{\{0,L\}} n\delta w(EI\frac{d^3 w}{dx^3}+P\frac{dw}{dx})-\sum\nolimits_{\{0,L\}} n\frac{d\delta w}{dx}(EI\frac{d^2 w}{dx^2})\,.$$

According to principle of virtual work $\delta W=0\ \forall \delta w$. Let us consider first the subset of variations for which $\delta w=0$ and $d\delta w/dx=0$ on $\{0,L\}$. The fundamental lemma of variation calculus implies

$$EI\frac{d^4 w}{dx^4}+P\frac{d^2 w}{dx^2}=0 \quad \text{in} \quad (0,L)\,.$$

Let us consider then the subset of variations for which $d\delta w/dx=0$ on $\{0,L\}$. Knowing the condition above, the fundamental lemma of variation calculus implies

$$EI\frac{d^3 w}{dx^3}+P\frac{dw}{dx}=0 \quad \text{or} \quad w-\underline{w}=0 \quad \text{on} \quad \{0,L\}\,.$$

Finally, let us consider the subset of variations for which $\delta w=0$ on $\{0,L\}$. Knowing the previous results, the fundamental lemma of variation calculus implies

$$EI\frac{d^2 w}{dx^2}=0 \quad \text{or} \quad \frac{dw}{dx}-\underline{\theta}=0 \quad \text{on} \quad \{0,L\}\,.$$

For the problem of the figure, one obtains

$$EI\frac{d^4 w}{dx^4}+P\frac{d^2 w}{dx^2}=0 \quad \text{in} \quad (0,L)\,,$$

$$EI\frac{d^3w}{dx^3}+P\frac{dw}{dx}=0 \quad \text{and} \quad EI\frac{d^2w}{dx^2}=0 \quad \text{at } x=L,$$

$$w=0 \quad \text{and} \quad \frac{dw}{dx}=0 \quad \text{at } x=0.$$

Name____________________________ Student number____________________

Assignment 1 (2p)

Principle of virtual work for a bar problem is given by: find $u \in U$ such that $\forall \delta u \in U$

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = \int_0^L \left(-\frac{d\delta u}{dx} EA \frac{du}{dx} + \delta u b\right) dx + (\delta u F)_{x=L} = 0$$

in which $u(0) = \delta u(0) = 0$. Assuming that EA, b and F are given constants, deduce the underlying boundary value problem for $u(x)$. Use integration by parts in the first term and the fundamental lemma of variation calculus to deduce the implications of principle of virtual work.

Solution

Integration by parts in the first term gives an equivalent form. Notice that variation $\delta u(0) = 0$

$$\delta W = \int_0^L \left(-\frac{d\delta u}{dx} EA \frac{du}{dx} + \delta u b\right) dx + (\delta u F)_{x=L} = 0 \quad \Leftrightarrow$$

$$\delta W = \int_0^L \left(EA \frac{d^2 u}{dx^2} + b\right) \delta u dx + \left[\left(-EA \frac{du}{dx} + F\right) \delta u\right]_{x=L} = 0 .$$

According to principle of virtual work $\delta W = 0 \quad \forall \delta u \in U$. Let us first consider a subset of $U_0 \subset U$ for which $\delta u(L) = 0$ so that the boundary terms vanish. Then, the fundamental lemma of variation calculus implies that

$$\delta W = \int_0^L \left(EA \frac{d^2 u}{dx^2} + b\right) \delta u dx \quad \forall \delta u \in U_0 \quad \Leftrightarrow \quad EA \frac{d^2 u}{dx^2} + b = 0 \quad \text{in } (0, L) .$$

After that, let us consider the original set U and simplify the virtual work expression by using the equilibrium equation already obtained. Then, the fundamental lemma of variation calculus implies

$$\delta W = \left[\left(-EA \frac{du}{dx} + F\right) \delta u\right]_{x=L} = 0 \quad \forall \delta u \in U \quad \Rightarrow \quad -EA \frac{du}{dx} + F = 0 \quad \text{at } x = L .$$

Boundary value problem consist of the equations obtained and the constraint for the function set

$$EA \frac{d^2 u}{dx^2} + b = 0 \quad \text{in } (0, L), \quad \text{(differential equation)}$$

$$EA \frac{du}{dx} - F = 0 \quad \text{at } x = L \quad \text{and} \quad u = 0 \quad \text{at } x = 0 . \quad \text{(boundary conditions)}$$

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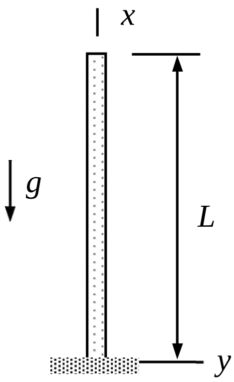
Assignment 2 (2p)

The bar of the figure is loaded by its own weight. Cross-sectional area A and density ρ of the material are constants. Use the bar model boundary value problem



$$\frac{dN}{dx}+b=0 \quad \text{and} \quad N=EA\frac{du}{dx} \quad \text{in} \quad \Omega,$$

$$nN-F=0 \quad \text{or} \quad u-\underline{u}=0 \quad \text{on} \quad \partial\Omega.$$

to determine the stress measure (force) N and displacement u.

Solution

The boundary value problem given represents a generic form which needs to be adapted to the bar shown. Let us consider the differential equations in the form given (elimination of the axial force is also possible)

$$\frac{dN}{dx}-\rho g A=0 \;\text{ in } [0,L[\quad \text{and} \quad N(L)=0,$$

$$N=EA\frac{du}{dx} \quad \text{in} \quad]0,L] \quad \text{and} \quad u(0)=0.$$

In a statically determined case, it is possible to solve for the stress resultants first. The generic solutions to the first order differential equation is obtained by integration. Thereafter, the integration constant follows from the boundary condition

$$\frac{dN}{dx}-\rho g A=0 \quad \Leftrightarrow \quad N(x)=\rho g A x+a,$$

$$N(L)=\rho g A L+a=0 \quad \Leftrightarrow \quad a=-\rho g A L.$$

Solution to axial force $\quad N(x)=\rho g A(x-L)$.

Knowing the axial force, the constitutive equation can be considered as a differential equation for the displacement to be treated in the same manner as the equation for the axial force:

$$\frac{du}{dx}=\frac{\rho g}{E}(x-L) \quad \Leftrightarrow \quad u(x)=\frac{\rho g}{E}(\frac{1}{2}x^2-Lx)+b,$$

$u(0) = b = 0$.

Solution to displacement $u(x) = \frac{\rho g}{E}(\frac{1}{2}x^2 - Lx)$.

Name____________________________ Student number____________________

Assignment 3 (4p)

Principle of virtual work for a simply supported Bernoulli beam is given by the variational problem: find $w \in U$ such that

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = \int_\Omega \left(-\frac{d^2\delta w}{dx^2} EI \frac{d^2 w}{dx^2} + \delta w b\right) dx = 0 \quad \forall \delta w \in U$$

in which $\Omega = (0, L)$, $\partial\Omega = \{0, L\}$, $U = \{w \in C^4(\Omega) : w = 0 \text{ on } x \in \{0, L\}\}$. Bending stiffness EI and the external distributed force b are constants. Deduce first the underlying boundary value problem. After that, solve the problem for the transverse displacement $w(x)$.

Solution

Integration by parts twice in the first term gives an equivalent form

$$\delta W = \int_0^L \left(-\frac{d^2\delta w}{dx^2} EI \frac{d^2 w}{dx^2} + \delta w b\right) dx \quad \Leftrightarrow$$

$$\delta W = \int_\Omega \left(\frac{d\delta w}{dx} EI \frac{d^3 w}{dx^3} + \delta w b\right) dx - \sum_{\partial\Omega} n\left(EI \frac{d^2 w}{dx^2} \frac{d\delta w}{dx}\right) \quad \Leftrightarrow$$

$$\delta W = \int_\Omega \left(-EI \frac{d^4 w}{dx^4} + b\right) \delta w dx - \sum_{\partial\Omega} n\left(EI \frac{d^2 w}{dx^2} \frac{d\delta w}{dx}\right).$$

The boundary term of the second integration by parts vanishes as $\delta w \in U$ and therefore $\delta w = 0$ on $\partial\Omega = \{0, L\}$. According to the variational problem $\delta W = 0 \ \forall \delta w \in U$.

Let us first consider a subset of $U_0 \subset U$ for which $d\delta w / dx = 0$ on $\partial\Omega = \{0, L\}$ so that the boundary terms vanish. The equilibrium equation follows from the fundamental lemma of variation calculus:

$$\delta W = \int_\Omega \left(-EI \frac{d^4 w}{dx^4} + b\right) \delta w dx = 0 \quad \Rightarrow \quad -EI \frac{d^4 w}{dx^4} + b = 0 \quad \text{in } \Omega.$$

After that, let us consider U without restriction $d\delta w / dx = 0$ on $\partial\Omega = \{0, L\}$ and simplify the virtual work expression by using the equilibrium equation already obtained. The natural boundary conditions follow from the fundamental lemma of variation calculus

$$\delta W = \sum_{\partial\Omega} n\left(EI \frac{d^2 w}{dx^2} \frac{d\delta w}{dx}\right) = 0 \quad \Rightarrow \quad EI \frac{d^2 w}{dx^2} = 0 \quad \text{on } \partial\Omega.$$

The natural boundary conditions mean that moment vanishes at the ends of a simply supported beam. The missing two boundary conditions needed for a fourth order ordinary differential equation of a Bernoulli beam follow from the definition of the function set for the transverse displacement. As all elements (functions) in U should vanish on $\partial\Omega$ and $w \in U$:

$$w = 0 \quad \text{on} \quad \partial\Omega .$$

Boundary value problem for a simply supported beam consists of the equilibrium equation and the boundary condition implied by the principle of virtual work

$$-EI\frac{d^4w}{dx^4} + b = 0 \quad \text{in } \Omega ,$$

$$EI\frac{d^2w}{dx^2} = 0 \quad \text{on} \quad \partial\Omega ,$$

$$w = 0 \quad \text{on} \quad \partial\Omega .$$

Solution to the equations of the boundary value problem can be obtained by repetitive integrations of the differential equation. After finding the generic solution containing four integration constants, boundary condition are used to find the values of the integration constants and thereby a solution satisfying all equations of the problem. First integration four times

$$\frac{d^4w}{dx^4} = \frac{b}{EI} \quad \Leftrightarrow \quad \frac{d^3w}{dx^3} = \frac{b}{EI}x + a_1 \quad \Leftrightarrow \quad \frac{d^2w}{dx^2} = \frac{b}{EI}\frac{1}{2}x^2 + a_1 x + a_2 \quad \Leftrightarrow$$

$$\frac{dw}{dx} = \frac{b}{EI}\frac{1}{6}x^3 + a_1\frac{1}{2}x^2 + a_2 x + a_3 \quad \Leftrightarrow \quad w(x) = \frac{b}{EI}\frac{1}{24}x^4 + a_1\frac{1}{6}x^3 + a_2\frac{1}{2}x^2 + a_3 x + a_4 .$$

The boundary conditions give

$$w(0) = a_4 = 0 , \; w(L) = \frac{b}{EI}\frac{1}{24}L^4 + a_1\frac{1}{6}L^3 + a_2\frac{1}{2}L^2 + a_3 L + a_4 = 0$$

$$\frac{d^2w}{dx^2}(0) = a_2 , \; \frac{b}{EI}\frac{1}{2}L^2 + a_1 L + a_2 = 0 \quad \Leftrightarrow \quad a_2 = a_4 = 0 , \; a_1 = -\frac{1}{2}\frac{bL}{EI} , \quad a_3 = \frac{1}{24}\frac{bL^3}{EI} .$$

Therefore, the solution to the transverse displacement

$$w(x) = \frac{1}{24}\frac{b}{EI}(x^4 - 2Lx^3 + L^3 x) = \frac{1}{24}\frac{bL^4}{EI}(\xi^4 - 2\xi + \xi) \quad \text{where} \quad \xi = x / L .$$

Name________________________ Student number________________

Assignment 4 (4p)

Derive the component forms of the membrane equilibrium equation $\nabla \cdot \vec{\vec{N}} + \vec{b} = 0$ in the spherical shell coordinate system. Assume external loading $\vec{b} = -\Delta p \vec{e}_n$ due to a pressure difference between the inner and outer surfaces. Also assume rotation symmetry with respect to both angular coordinates so that $\vec{\vec{N}} = N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi + N_{\theta\theta}\vec{e}_\theta\vec{e}_\theta$ where stress components $N_{\phi\phi}$ and $N_{\theta\theta}$ are constants. The basis vector derivatives and gradient of the spherical shell coordinate system are given by

$$\frac{\partial}{\partial\phi}\begin{Bmatrix}\vec{e}_\phi\\ \vec{e}_\theta\\ \vec{e}_n\end{Bmatrix} = \begin{Bmatrix}\sin\theta\vec{e}_n - \cos\theta\vec{e}_\theta\\ \cos\theta\vec{e}_\phi\\ -\sin\theta\,\vec{e}_\phi\end{Bmatrix},\ \frac{\partial}{\partial\theta}\begin{Bmatrix}\vec{e}_\phi\\ \vec{e}_\theta\\ \vec{e}_n\end{Bmatrix} = \begin{Bmatrix}0\\ \vec{e}_n\\ -\vec{e}_\theta\end{Bmatrix},\ \nabla = \frac{1}{R\sin\theta}\vec{e}_\phi\frac{\partial}{\partial\phi} + \frac{1}{R}\vec{e}_\theta\frac{\partial}{\partial\theta}.$$

Solution

Membrane model is the thin-slab model in curved geometry. Equilibrium equations differ from the thin-slab ones as the non-constant basis vectors brings additional terms. Therefore, a membrane may take also external forces in the transverse direction (like pressure difference between the outer and inner surfaces of a balloon). The task is to simplify the vector equation

$$\nabla \cdot \vec{\vec{N}} - \vec{b} = \frac{1}{R}\left(\frac{1}{\sin\theta}\vec{e}_\phi\frac{\partial}{\partial\phi} + \vec{e}_\theta\frac{\partial}{\partial\theta}\right)\cdot(N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi + N_{\theta\theta}\vec{e}_\theta\vec{e}_\theta) - \Delta p\vec{e}_n = 0$$

to see the component forms. Let us start with the divergence of stress

$$\frac{1}{R\sin\theta}\vec{e}_\phi\frac{\partial}{\partial\phi}\cdot N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi = \frac{1}{R\sin\theta}\vec{e}_\phi\cdot\left(\frac{\partial N_{\phi\phi}}{\partial\phi}\vec{e}_\phi\vec{e}_\phi + N_{\phi\phi}\frac{\partial\vec{e}_\phi}{\partial\phi}\vec{e}_\phi + N_{\phi\phi}\vec{e}_\phi\frac{\partial\vec{e}_\phi}{\partial\phi}\right) = \frac{1}{R}N_{\phi\phi}(\vec{e}_n - \cot\theta\vec{e}_\theta),$$

$$\frac{1}{R}\frac{1}{\sin\theta}\vec{e}_\phi\frac{\partial}{\partial\phi}\cdot N_{\theta\theta}\vec{e}_\theta\vec{e}_\theta = \frac{1}{R}\frac{1}{\sin\theta}\vec{e}_\phi\cdot\left(\frac{\partial N_{\theta\theta}}{\partial\phi}\vec{e}_\theta\vec{e}_\theta + N_{\theta\theta}\frac{\partial\vec{e}_\theta}{\partial\phi}\vec{e}_\theta + \frac{\partial}{\partial\phi}N_{\theta\theta}\vec{e}_\theta\frac{\partial\vec{e}_\theta}{\partial\phi}\right) = \frac{1}{R}N_{\theta\theta}\cot\theta\vec{e}_\theta,$$

$$\frac{1}{R}\vec{e}_\theta\frac{\partial}{\partial\theta}\cdot(N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi) = \frac{1}{R}\vec{e}_\theta\cdot\left(\frac{\partial N_{\phi\phi}}{\partial\theta}\vec{e}_\phi\vec{e}_\phi + N_{\phi\phi}\frac{\partial\vec{e}_\phi}{\partial\theta}\vec{e}_\phi + N_{\phi\phi}\vec{e}_\phi\frac{\partial\vec{e}_\phi}{\partial\theta}\right) = 0,$$

$$\frac{1}{R}\vec{e}_\theta\frac{\partial}{\partial\theta}\cdot(N_{\theta\theta}\vec{e}_\theta\vec{e}_\theta) = \frac{1}{R}\vec{e}_\theta\cdot\left(\frac{\partial N_{\theta\theta}}{\partial\theta}\vec{e}_\theta\vec{e}_\theta + N_{\theta\theta}\vec{e}_n\vec{e}_\theta + N_{\theta\theta}\vec{e}_\theta\vec{e}_n\right) = \frac{1}{R}N_{\theta\theta}\vec{e}_n.$$

Finally combining everything

$$\nabla \cdot \vec{\vec{N}} - \vec{b} = \frac{1}{R}N_{\phi\phi}(\vec{e}_n - \cot\theta\vec{e}_\theta) + \frac{1}{R}N_{\theta\theta}\cot\theta\vec{e}_\theta + \frac{1}{R}N_{\theta\theta}\vec{e}_n - \Delta p\vec{e}_n = 0 \quad \Leftrightarrow$$

$$\nabla \cdot \overleftrightarrow{N} - \vec{b} = [\frac{1}{R}(N_{\theta\theta} - N_{\phi\phi})\cot\theta]\vec{e}_\theta + [\frac{1}{R}(N_{\phi\phi} + N_{\theta\theta}) - \Delta p]\vec{e}_n = 0 \quad \Leftrightarrow$$

$$N_{\theta\theta} - N_{\phi\phi} = 0 \quad \text{and} \quad \frac{1}{R}(N_{\phi\phi} + N_{\theta\theta}) - \Delta p = 0 \,. \quad \Leftarrow$$

Therefore, solution to the stress resultants is

$$N_{\phi\phi} = N_{\theta\theta} = \frac{R\Delta p}{2} \,.$$

Name________________________ Student number________________

Assignment 5 (4p)

Derive the component forms of the membrane equilibrium equations $\nabla\cdot\vec{\vec{N}}+\vec{b}=0$ in the cylindrical shell (z,ϕ,n)-coordinate system. Use the stress resultant, external force, and gradient representations

$$\vec{\vec{N}}=\begin{Bmatrix}\vec{e}_z\\ \vec{e}_\phi\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}N_{zz} & N_{z\phi}\\ N_{z\phi} & N_{\phi\phi}\end{bmatrix}\begin{Bmatrix}\vec{e}_z\\ \vec{e}_\phi\end{Bmatrix},\ \vec{b}=\begin{Bmatrix}\vec{e}_z\\ \vec{e}_\phi\\ \vec{e}_n\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}b_z\\ b_\phi\\ b_n\end{Bmatrix},\ \nabla=\begin{Bmatrix}\vec{e}_z\\ \vec{e}_\phi\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\partial/\partial z\\ \partial/(R\partial\phi)\end{Bmatrix}.$$

The non-zero basis vector derivatives are $\dfrac{\partial}{\partial\phi}\vec{e}_\phi=\vec{e}_n$ and $\dfrac{\partial}{\partial\phi}\vec{e}_n=-\vec{e}_\phi$.

Solution

Membrane model is the thin-slab model in curved geometry. Equilibrium equations differ from the thin-slab ones as the non-constant basis vectors bring additional terms. Therefore, a membrane may take also external forces in the transverse direction (like pressure difference between the outer and inner surfaces of a balloon). The task is to simplify the vector equation

$$\nabla\cdot\vec{\vec{N}}+\vec{b}=(\vec{e}_z\frac{\partial}{\partial z}+\vec{e}_\phi\frac{1}{R}\frac{\partial}{\partial\phi})\cdot(N_{zz}\vec{e}_z\vec{e}_z+N_{z\phi}\vec{e}_z\vec{e}_\phi+N_{z\phi}\vec{e}_\phi\vec{e}_z+N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi)+b_z\vec{e}_z+b_\phi\vec{e}_\phi+b_n\vec{e}_n=0$$

to see the component forms. Let us start with the divergence of stress and consider the term in two parts (to avoid lengthy expressions)

$$(\vec{e}_z\frac{\partial}{\partial z})\cdot(N_{zz}\vec{e}_z\vec{e}_z+N_{z\phi}\vec{e}_z\vec{e}_\phi+N_{z\phi}\vec{e}_\phi\vec{e}_z+N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi)=\frac{\partial N_{zz}}{\partial z}\vec{e}_z+\frac{\partial N_{z\phi}}{\partial z}\vec{e}_\phi,$$

$$(\vec{e}_\phi\frac{1}{R}\frac{\partial}{\partial\phi})\cdot(N_{zz}\vec{e}_z\vec{e}_z+N_{z\phi}\vec{e}_z\vec{e}_\phi+N_{z\phi}\vec{e}_\phi\vec{e}_z+N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi)=\frac{1}{R}\frac{\partial N_{z\phi}}{\partial\phi}\vec{e}_z+\frac{1}{R}\frac{\partial N_{\phi\phi}}{\partial\phi}\vec{e}_\phi+\frac{1}{R}N_{\phi\phi}\vec{e}_n.$$

After that, combining the terms

$$\nabla\cdot\vec{\vec{N}}+\vec{b}=\begin{Bmatrix}\vec{e}_z\\ \vec{e}_\phi\\ \vec{e}_n\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\dfrac{\partial N_{zz}}{\partial z}+\dfrac{1}{R}\dfrac{\partial N_{z\phi}}{\partial\phi}+b_z\\ \dfrac{\partial N_{z\phi}}{\partial z}+\dfrac{1}{R}\dfrac{\partial N_{\phi\phi}}{\partial\phi}+b_\phi\\ \dfrac{1}{R}N_{\phi\phi}+b_n\end{Bmatrix}=0.$$

MEC-E8003

Beam, Plate and Shell Models

2024

WEEK 12: BEAMS

4 BEAM

LEARNING OUTCOMES

Students can solve the weekly lecture problems, home problems, and exercise problems on the beam model in flat and curved geometry:

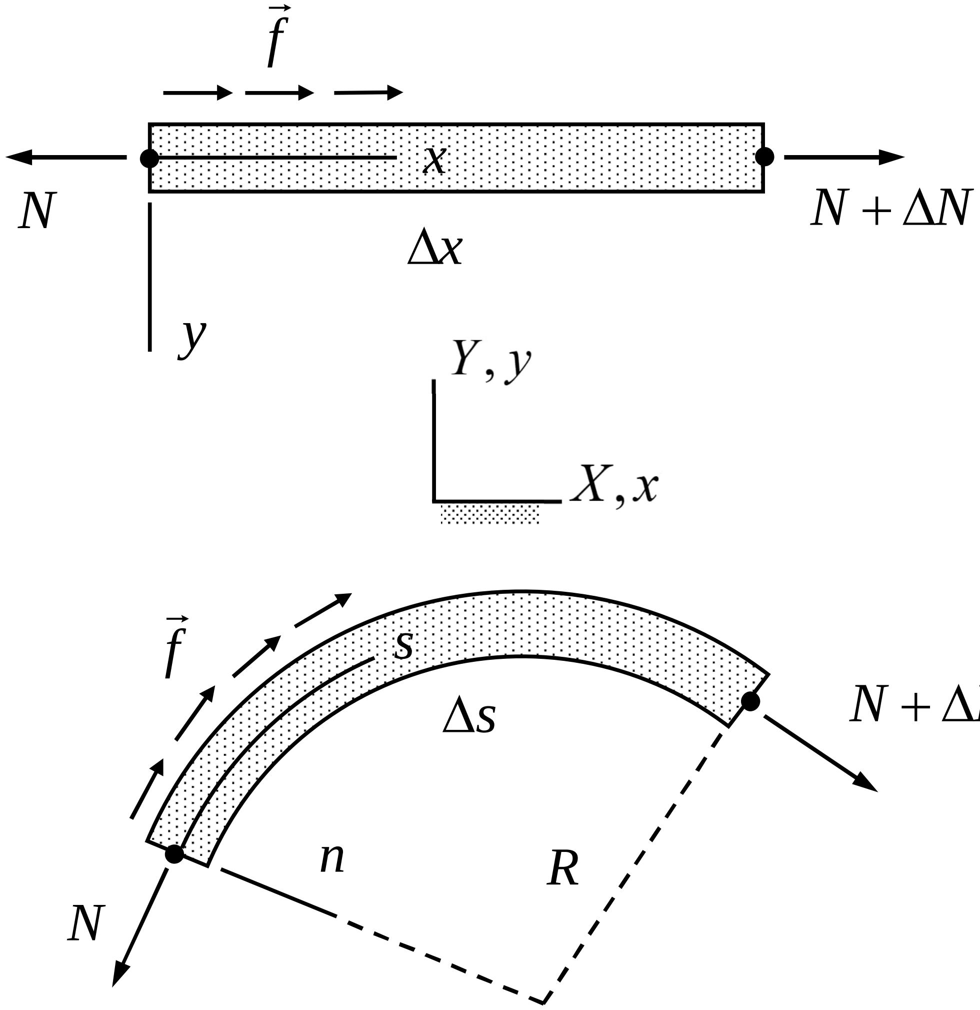
- ☐ Timoshenko and Bernoulli beam models.

- ☐ Component representations of the beam equations in $(x, y, z)-$ and $(s, n, b)-$coordinate systems.

- ☐ Derivation of the beam equations by using the principle of virtual work, integration by parts, and the fundamental lemma of variation calculus. Beam equilibrium and constitutive equations in their tensor forms.

- ☐ Kinematics, virtual work density, and constitutive equation in $(s, n, b)-$coordinates

THE CURVATURE EFFECT



$$\left.\begin{aligned} &\vec{N} = N\vec{i} \\ &\vec{f} = f_x\vec{i} + f_y\vec{j} \\ &\frac{d\vec{N}}{dx} + \vec{f} = 0 \end{aligned}\right\}$$

$$\left.\begin{aligned} &\vec{N} = N\vec{e}_s \\ &\vec{f} = f_s\vec{e}_s + f_n\vec{e}_n \\ &\frac{d\vec{N}}{ds} + \vec{f} = 0 \end{aligned}\right\}$$

The basis vectors of the material $(x, y, z)-$coordinate system are constants

$$\frac{d\vec{N}}{dx} + \vec{f} = \frac{d(N\vec{i})}{dx} + f_x\vec{i} + f_y\vec{j} = (\frac{dN}{dx} + f_x)\vec{i} + f_y\vec{j} = 0 \quad \Leftrightarrow$$

$$\frac{dN}{dx} + f_x = 0 \quad \text{and} \quad f_y = 0. \quad \leftarrow$$

The basis vectors of the material (s,n,b) coordinate system are *not* constants

$$\frac{d\vec{N}}{ds} + \vec{f} = \frac{d(N\vec{e}_s)}{ds} + f_s\vec{e}_s + f_n\vec{e}_n = (\frac{dN}{ds} + f_s)\vec{e}_s + (\frac{N}{R} + f_n)\vec{e}_n = 0 \quad \Leftrightarrow$$

$$\frac{dN}{ds} + f_s = 0 \quad \text{and} \quad \frac{N}{R} + f_n = 0\ . \quad \leftarrow$$

EXAMPLE Consider an inextensible string having constant mass per unit length (m) under its own weight. Write the equilibrium equations in the structural (x, y, z) system with the selection x as the curve parameter and show that $y - c = a\cosh[(x - b)/a]$, in which a, b, c are constants, is a solution (Catenary curve)

www.math.udel.edu/.../Chain/Demo%20015.jpg

teachers.sduhsd.k12.ca.us/.../GatewayArch.jpg

Let us write the equilibrium equations $dN/ds + f_s = 0$ and $N/R + f_n = 0$ in terms of x and y as we would like to get the solution in form $y = y(x)$. Using $y' = dy/dx$,

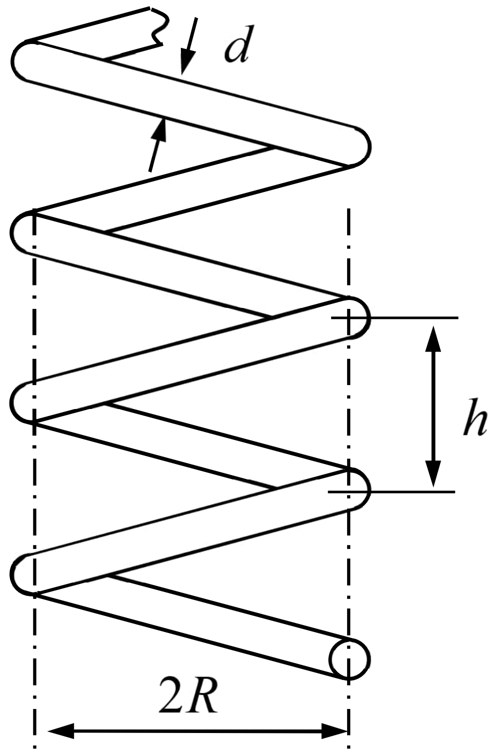
$$\frac{d}{ds} = \frac{1}{(1+y'^2)^{1/2}}\frac{d}{dx}, \quad \frac{1}{R} = \frac{y''}{(1+y'^2)^{3/2}}, \quad f_s = -\frac{mgy'}{(1+y'^2)^{1/2}}, \text{ and } f_n = -\frac{mg}{(1+y'^2)^{1/2}}$$

elimination of the forces gives the equation for geometry

$$\left(\frac{1+y'^2}{y''} - y\right)' = 0 \;\Rightarrow\; y - c = a\cosh\left(\frac{x-b}{a}\right).$$

Hence, the shape is a Catenary curve. The solution to the non-linear differential equation can be obtained by using, e.g., Mathematica.

EXAMPLE Equilibrium equations in (s,n,b) system can be used, e.g., to derive the well-known formula for the elastic spring coefficient ($\Delta F = k\Delta L$). The geometrical parameters are coil radius R, pitch h, number of coils n, and diameter d of wire. Material parameters are Young's modulus E and shear modulus G.

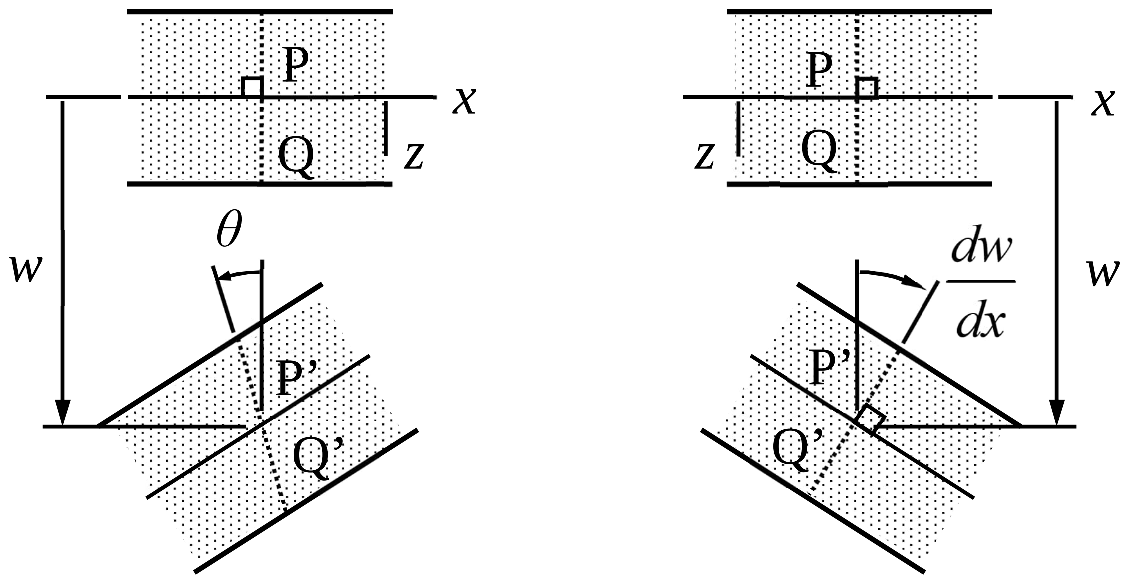


$$k = \frac{EG\pi d^4\sqrt{h^2+4\pi^2R^2}}{4n(Gd^2h^2+16Gh^2R^2+4E\pi^2d^2R^2+32E\pi^2R^4)} \quad \Rightarrow$$

$$k \approx \frac{Gd^4}{64nR^3} \quad \text{when} \quad \left(\frac{d}{R}\right)^2 \ll 1 \quad \text{and} \quad \left(\frac{h}{R}\right)^2 \ll 1$$

the formula of literature

4.1 BEAM MODEL



Timoshenko $(\vec{u}_{\mathrm{P}} = 0)$ **Bernoulli** $(\vec{u}_{\mathrm{P}} = 0)$

Normal planes to the (material) axis of beam remain planes (Timoshenko) and normal to the axis (Bernoulli) in deformation. Mathematically $\vec{u}_{\mathrm{Q}} = \vec{u}_{\mathrm{P}} + \vec{\theta} \times \vec{\rho}_{\mathrm{PQ}}$ (see any textbook on statics and/or dynamics).

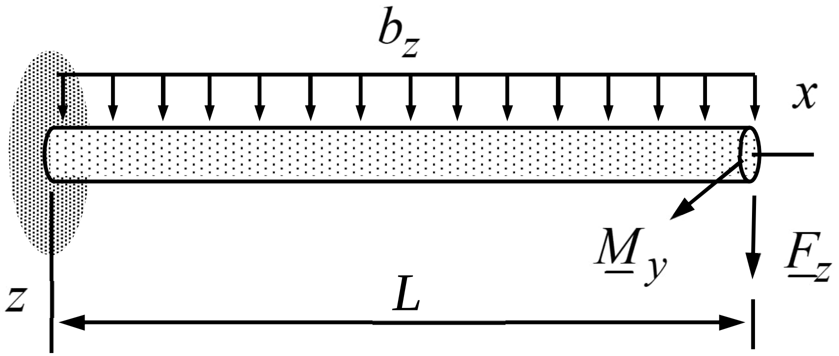
The kinematic assumption means that the normal planes to the mid-curve move as rigid bodies in deformation. In terms of displacement of the translation point $y = z = 0$ and small rotation of the cross-section, displacement of a particle identified by (x, y, z) is given by $\vec{u} = (u\vec{i} + v\vec{j} + w\vec{k}) + (\phi\vec{i} + \theta\vec{j} + \psi\vec{k}) \times (y\vec{j} + z\vec{k})$. According to the kinetic assumption of the beam model $\sigma_{zz} = \sigma_{yy} = 0$.

In the Bernoulli model, the cross-sections are assumed to remain normal planes to the mid-curve in deformation which brings the *Bernoulli constraints*

$$\gamma_{xy} = \frac{dv}{dx} - \psi = 0 \quad \text{and} \quad \gamma_{xz} = \frac{dw}{dx} + \theta = 0.$$

Due to the more severe assumptions, the modeling error of the Bernoulli model is larger than that of the Timoshenko beam model!

TIMOSHENKO BEAM BENDING $(x,z)-$plane



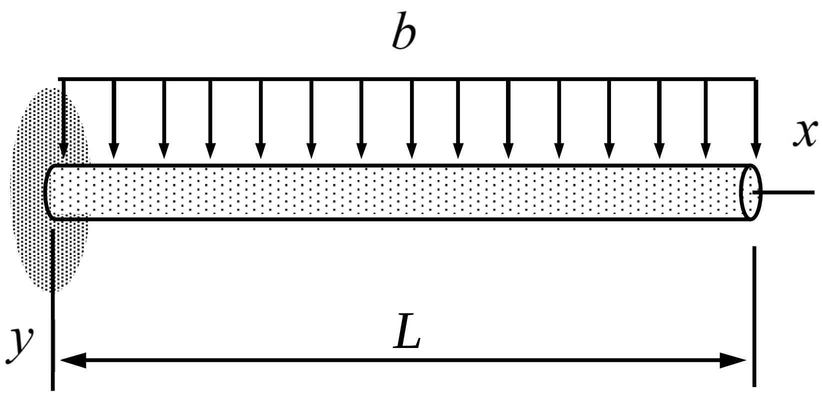
Equilibrium eqs. : $\dfrac{dQ_z}{dx}+b_z=0$ and $\dfrac{dM_y}{dx}-Q_z+c_y=0$ in $(0,L)$

Constitutive eqs. : $Q_z=GA(\dfrac{dw}{dx}+\theta)$ and $M_y=EI\dfrac{d\theta}{dx}$ in $(0,L)$

Natural boundary condition: $M_y=\underline{M}_y$ and $Q_z=\underline{F}_z$ at $x=L$

Essential boundary condition: $\theta=0$ and $w=0$ at $x=0$

EXAMPLE 4.1 Consider the beam of the figure of length L. Material properties E and G, cross-section properties A, $S=0$, I and the loading b are constants. Determine the axial displacement, deflection, and rotation at the free end according to the Timoshenko beam model.

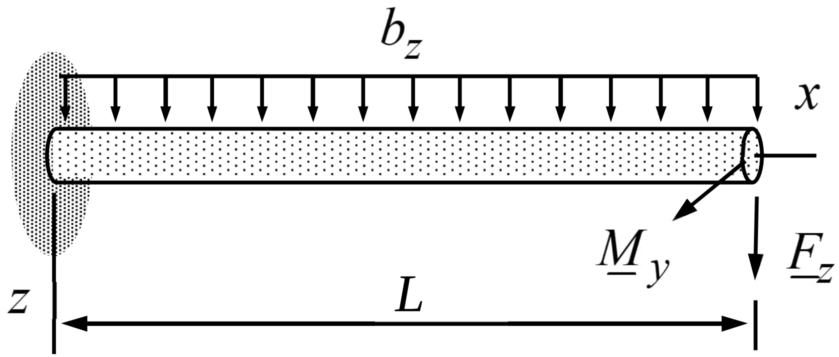


"Timoshenko effect"

$\sim 1+(t/L)^2$

Answer $u(L)=0$, $v(L)=\dfrac{bL^4}{8EI}\dfrac{4EI+GAL^2}{GAL^2}$, and $\psi(L)=\dfrac{bL^3}{6EI}$

BERNOULLI BEAM BENDING $(x, z)-$plane



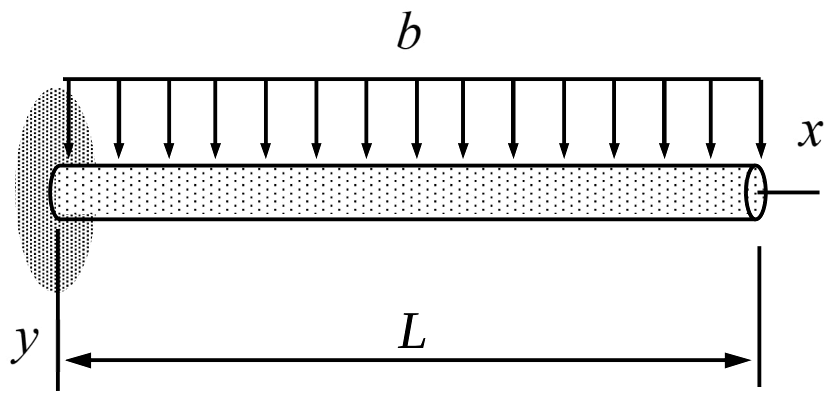
Equilibrium eqs. : $\dfrac{d^2 M_y}{dx^2} + b_z = 0$ and $Q_z = \dfrac{dM_y}{dx}$ in $(0, L)$

Constitutive eqs. : $M_y = -EI \dfrac{d^2 w}{dx^2}$ in $(0, L)$ (Bernoulli constraint $\gamma_{xz} = \dfrac{dw}{dx} + \theta = 0$)

Natural boundary condition: $M_y = \underline{M}_y$ and $Q_z = \underline{F}_z$ at $x = L$

Essential boundary condition: $w = 0$ and $\theta = -\dfrac{dw}{dx} = 0$ at $x = 0$

EXAMPLE 4.2 Consider the beam of the figure of length L. Material properties E and G, cross-section properties A, $S=0$ and I, and loading b are constants. Determine the axial displacement, deflection, and rotation at the free end according to the Bernoulli beam equations.



Answer $u(L)=0$, $v(L)=\dfrac{bL^4}{8EI}$, and $\psi(L)=\dfrac{bL^3}{6EI}$

MOMENTS OF AREA

Cross-section geometry of a beam influences the constitutive equations through the moments of area (material is assumed to be homogeneous):

Zero moment: $A = \int dA$

First moments: $S_z = \int ydA$ and $S_y = \int zdA$

Second moments: $I_{zz} = \int y^2 dA$, $I_{yy} = \int z^2 dA$, and $I_{zy} = I_{yz} = \int yzdA$

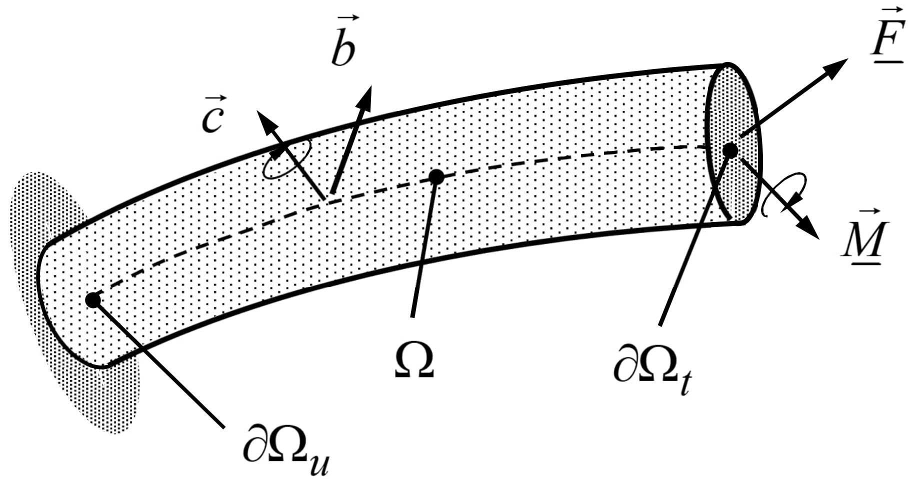
Polar moment: $I_{rr} = \int y^2 + z^2 dA = I_{zz} + I_{yy}$

The moments depend on the material coordinate system. For the simplest representation, position of the *x*-axis and orientation of the *y*-axis should result into $S_z = S_y = I_{yz} = 0$.

4.2 BEAM EQUATIONS

Virtual work expression of beam, principle of virtual work, integration by parts, and the fundamental lemma of variation calculus imply the equations:

$$\left.\begin{aligned}
&\frac{d\vec{F}}{ds}+\vec{b}=0 \quad \text{in} \quad \Omega,\\
&\frac{d\vec{M}}{ds}+\vec{e}_s\times\vec{F}+\vec{c}=0 \quad \text{in} \quad \Omega,\\
&n\vec{F}-\underline{\vec{F}}=0 \quad \text{or} \quad \vec{u}-\underline{\vec{u}}=0 \quad \text{on} \quad \partial\Omega,\\
&n\vec{M}-\underline{\vec{M}}=0 \quad \text{or} \quad \vec{\theta}-\underline{\vec{\theta}}=0 \quad \text{on} \quad \partial\Omega.
\end{aligned}\right\}$$



Constitutive equations $\vec{M}=\vec{M}(\vec{u},\vec{\theta})$, $\vec{F}=\vec{F}(\vec{u},\vec{\theta})$ (Bernoulli or Timoshenko) are needed in displacement analysis and in statically indeterminate cases!

Curvilinear (s, n, b) system represents a generic coordinate system for beams. In terms of the stress and external force resultant, virtual work densities of the beam model

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \delta\vec{\varepsilon} \\ \delta\vec{\kappa} \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \vec{F} \\ \vec{M} \end{Bmatrix}, \quad \delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta\vec{u}_0 \\ \delta\vec{\theta}_0 \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \vec{b} \\ \vec{c} \end{Bmatrix}, \quad \text{and} \quad \delta w_{\partial\Omega}^{\text{ext}} = \begin{Bmatrix} \delta\vec{u}_0 \\ \delta\vec{\theta}_0 \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \underline{\vec{F}} \\ \underline{\vec{M}} \end{Bmatrix}$$

in which the strain measures $\vec{\varepsilon} = \dfrac{d\vec{u}_0}{ds} + \vec{e}_s \times \vec{\theta}_0$ and $\vec{\kappa} = \dfrac{d\vec{\theta}_0}{ds}$.

Integration by parts in the virtual work expression $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}}$ gives a more convenient form for deducing the beam equations (the simple form of integration by parts formula applies):

$$\delta W = \int_{\Omega} -[\vec{F} \cdot (\frac{d\delta\vec{u}_0}{ds} + \vec{e}_s \times \delta\vec{\theta}_0) - \vec{M} \cdot \frac{d\delta\vec{\theta}_0}{ds}] ds +$$

$$\int_\Omega (\delta\vec{u}_0\cdot\vec{b}+\delta\vec{\theta}_0\cdot\vec{c})ds+\sum_{\partial\Omega}(\delta\vec{u}_0\cdot\underline{\vec{F}}+\delta\vec{\theta}_0\cdot\underline{\vec{M}}) \quad \Leftrightarrow$$

$$\delta W=\int_\Omega [\frac{d\vec{F}}{ds}\cdot\delta\vec{u}_0+(\frac{d\vec{M}}{ds}+\vec{e}_s\times\vec{F})\cdot\delta\vec{\theta}_0]ds-\sum_{\partial\Omega}(\delta\vec{u}_0\cdot n\vec{F}+\delta\vec{\theta}_0\cdot n\vec{M})+$$

$$\int_\Omega (\delta\vec{u}_0\cdot\vec{b}+\delta\vec{\theta}_0\cdot\vec{c})ds+\sum_{\partial\Omega}(\delta\vec{u}_0\cdot\underline{\vec{F}}+\delta\vec{\theta}_0\cdot\underline{\vec{M}}) \quad \Leftrightarrow$$

$$\delta W=\int_\Omega [(\frac{d\vec{F}}{ds}+\vec{b})\cdot\delta\vec{u}_0+(\frac{d\vec{M}}{ds}+\vec{e}_s\times\vec{F}+\vec{c})\cdot\delta\vec{\theta}_0]ds+$$

$$\sum_{\partial\Omega}[(-n\vec{F}+\underline{\vec{F}})\cdot\delta\vec{u}_0+(-n\vec{M}+\underline{\vec{M}})\cdot\delta\vec{\theta}_0].$$

According to the principle of virtual work $\delta W=0\ \forall(\delta\vec{u}_0,\delta\vec{\theta}_0)\in U$. First, if $\delta\vec{u}_0$ and $\delta\vec{\theta}_0$ are chosen to vanish on $\partial\Omega$, the fundamental lemma of variation calculus implies

$$\frac{d\vec{F}}{ds}+\vec{b}=0 \quad \text{and} \quad \frac{d\vec{M}}{ds}+\vec{e}_s\times\vec{F}+\vec{c}=0 \quad \text{in } \Omega. \quad \leftarrow$$ equilibrium equations

Second, if $\delta\vec{u}_0$ and $\delta\vec{\theta}_0$ are varied without any restrictions on the boundary (the equilibrium equations are used to simplify the virtual work expression), the fundamental lemma of variation calculus gives

$$n\vec{F} - \underline{\vec{F}} = 0 \quad \text{and} \quad n\vec{M} - \underline{\vec{M}} = 0 \quad \text{on} \quad \partial\Omega. \quad \leftarrow \qquad \text{natural boundary conditions}$$

Third, the boundary terms vanish also if $\delta\vec{u}_0 = 0$ and/or $\delta\vec{\theta}_0 = 0$ on $\partial\Omega$ by definition of U. Then one may not deduce the condition above. However, $\delta\vec{u}_0 = 0$ and $\delta\vec{\theta}_0 = 0$ on $\partial\Omega_u$ imply that $\vec{u}_0 - \underline{\vec{u}}_0 = 0$ and $\vec{\theta}_0 - \underline{\vec{\theta}}_0 = 0$ on $\partial\Omega_u$.

RESULTANT DEFINITIONS

Stress and external force resultants are integrals over the cross-section. If the kinetic assumptions are embedded in the elasticity tensor of the beam model

$$\begin{Bmatrix} \vec{F} \\ \vec{M} \end{Bmatrix} = \int \begin{Bmatrix} \vec{\sigma} \\ \vec{\rho}\times\vec{\sigma} \end{Bmatrix} dA = \int \begin{bmatrix} \overleftrightarrow{E} & -\overleftrightarrow{E}\times\vec{\rho} \\ \vec{\rho}\times\overleftrightarrow{E} & -\vec{\rho}\times\overleftrightarrow{E}\times\vec{\rho} \end{bmatrix} dA \cdot \begin{Bmatrix} \vec{\varepsilon} \\ \vec{\kappa} \end{Bmatrix} = \begin{bmatrix} \overleftrightarrow{A} & \overleftrightarrow{C} \\ \overleftrightarrow{C}_{\mathrm{c}} & \overleftrightarrow{B} \end{bmatrix} \cdot \begin{Bmatrix} \vec{\varepsilon} \\ \vec{\kappa} \end{Bmatrix},$$ constitutive equation

$$\begin{Bmatrix} \vec{b} \\ \vec{c} \end{Bmatrix} = \int \begin{Bmatrix} \vec{f} \\ \vec{\rho}\times\vec{f} \end{Bmatrix} J dA,$$ external distributed force and moment

$$\begin{Bmatrix} \underline{\vec{F}} \\ \underline{\vec{M}} \end{Bmatrix} = \int \begin{Bmatrix} \vec{t} \\ \vec{\rho}\times\vec{t} \end{Bmatrix} dA$$ external point force and moment

where $J = 1 - n\kappa$ and $\overleftrightarrow{E} = E\vec{e}_s\vec{e}_s + G\vec{e}_n\vec{e}_n + G\vec{e}_b\vec{e}_b$ for an isotropic material.

4.3 CARTESIAN COORDINATE SYSTEM

Timoshenko beam model equilibrium and constitutive equations in component forms

$$\begin{Bmatrix} \dfrac{dN}{dx} + b_x \\ \dfrac{dQ_y}{dx} + b_y \\ \dfrac{dQ_z}{dx} + b_z \end{Bmatrix} = 0, \quad \begin{Bmatrix} N \\ Q_y \\ Q_z \end{Bmatrix} = \begin{Bmatrix} EA\dfrac{du}{dx} - ES_z \dfrac{d\psi}{dx} + ES_y \dfrac{d\theta}{dx} \\ GA(\dfrac{dv}{dx} - \psi) - GS_y \dfrac{d\phi}{dx} \\ GA(\dfrac{dw}{dx} + \theta) + GS_z \dfrac{d\phi}{dx} \end{Bmatrix},$$

$$\begin{Bmatrix} \dfrac{dT}{dx} + c_x \\ \dfrac{dM_y}{dx} - Q_z + c_y \\ \dfrac{dM_z}{dx} + Q_y + c_z \end{Bmatrix} = 0, \text{ and } \begin{Bmatrix} T \\ M_y \\ M_z \end{Bmatrix} = \begin{Bmatrix} -GS_y(\dfrac{dv}{dx} - \psi) + GS_z(\dfrac{dw}{dx} + \theta) + GI_{rr} \dfrac{d\phi}{dx} \\ ES_y \dfrac{du}{dx} - EI_{zy} \dfrac{d\psi}{dx} + EI_{yy} \dfrac{d\theta}{dx} \\ -ES_z \dfrac{du}{dx} + EI_{zz} \dfrac{d\psi}{dx} - EI_{yz} \dfrac{d\theta}{dx} \end{Bmatrix}.$$

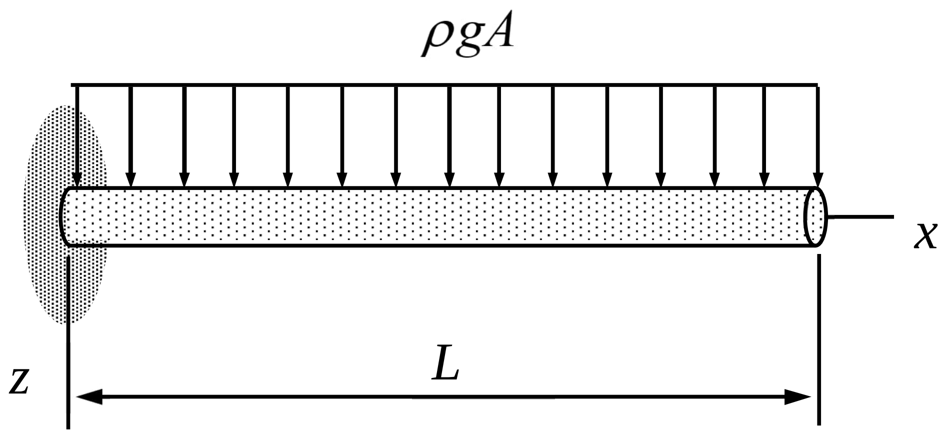
If the $x-$axis of the material coordinate system is aligned with the geometrical axis, the Cartesian system component representations of displacement, rotation, force resultant, moment resultant, elasticity tensor of beam and the relative position vector

$$\vec{u}_0=\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}u(x)\\ v(x)\\ w(x)\end{Bmatrix},\ \vec{\theta}_0=\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\phi(x)\\ \theta(x)\\ \psi(x)\end{Bmatrix},\ \vec{F}=\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}N(x)\\ Q_y(x)\\ Q_z(x)\end{Bmatrix},\ \vec{M}=\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}T(x)\\ M_y(x)\\ M_z(x)\end{Bmatrix},$$

$$\overleftrightarrow{E}=\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}E & 0 & 0\\ 0 & G & 0\\ 0 & 0 & G\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}\text{ and }\vec{\rho}=\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}0\\ y\\ z\end{Bmatrix}.$$

What remains is just finding the component representations of equilibrium and constitutive equations by substituting the expression above.

EXAMPLE 4.3 Consider a beam loaded by its own weight and clamped at its left end (figure). Determine $\vec{F}$ and $\vec{M}$ as functions of x by using the beam equations $d\vec{F}/dx+\vec{b}=0$ and $d\vec{M}/dx+\vec{i}\times\vec{F}+\vec{c}=0$ and the boundary conditions $\vec{F}=0$ and $\vec{M}=0$ at the free end.



Answer $N(x)=0,\ Q_z(x)=-\rho gA(x-L),\ M_y(x)=-\rho gA\frac{1}{2}(x-L)^2$

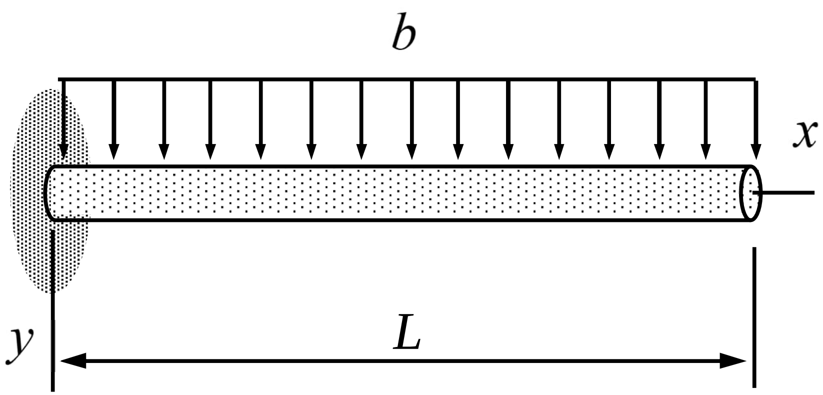
In a statically determinate case one may solve the beam equations for the stress resultants no matter the material. The non-zero loading component $b_z = \rho g A$. Equilibrium equations and the boundary condition at the free end (let us consider only the equations of the planar problem) give

$$\frac{dN}{dx} = 0 \;\text{ in }\; (0, L) \quad\text{and}\quad N(L) = 0 \quad\Rightarrow\quad N(x) = 0,$$

$$\frac{dQ_z}{dx} + \rho g A = 0 \;\text{ in }\; (0, L) \quad\text{and}\quad Q_z(L) = 0 \quad\Rightarrow\quad Q_z(x) = -\rho g A(x - L),$$

$$\frac{dM_y}{dx} + \rho g A(x - L) = 0 \;\text{ in }\; (0, L) \quad\text{and}\quad M_y(L) = 0 \quad\Rightarrow\quad M_y(x) = -\rho g A \frac{1}{2}(x - L)^2.$$

EXAMPLE Consider the beam of the figure of length L. Material properties E and G, cross-section properties A, $S=0$ and I, and loading b are constants. Determine the axial displacement, deflection, and rotation at the free end according to the Bernoulli beam equations.

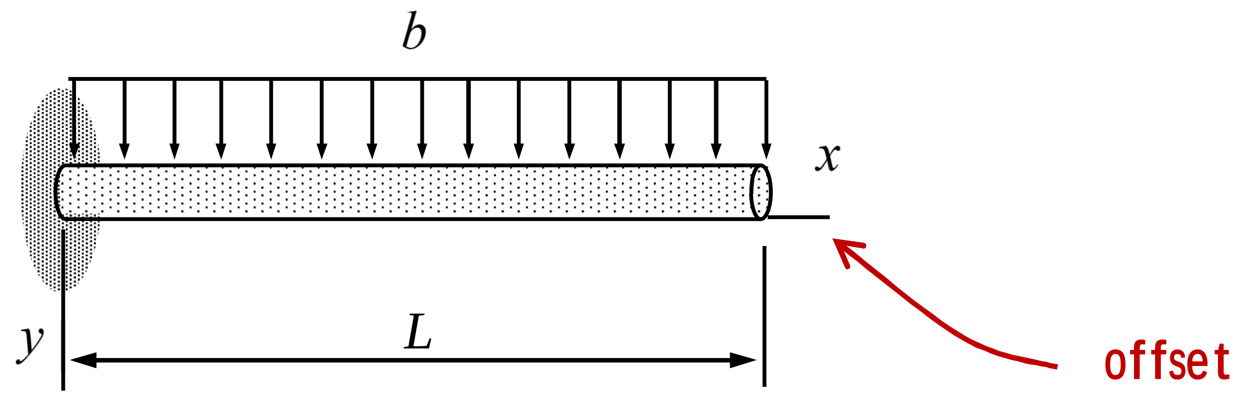


Answer (Mathematica notebook) $u(L)=0$, $v(L)=\dfrac{bL^4}{8EI}$, and $\psi(L)=\dfrac{bL^3}{6EI}$

In the Bernoulli model, Bernoulli constraints $\gamma_{xy} = dv/dx - \psi = 0$ and $\gamma_{xz} = dw/dx + \theta = 0$ are used to eliminate the rotation components θ and ψ from the constitutive equations of the Timoshenko beam model. Then shear force components Q_y and Q_z become constraint forces whose values follow from the equilibrium equations. Assuming that $S_y = S_z = I_{yz} = 0$, one may just replace the constitutive equations for Q_y and Q_z by Bernoulli constraints.

$$\begin{Bmatrix} \dfrac{dN}{dx} + b_x \\ \dfrac{dQ_y}{dx} + b_y \\ \dfrac{dQ_z}{dx} + b_z \end{Bmatrix} = 0, \quad \begin{Bmatrix} N \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} EA\dfrac{du}{dx} \\ \dfrac{dv}{dx} - \psi \\ \dfrac{dw}{dx} + \theta \end{Bmatrix}, \quad \begin{Bmatrix} \dfrac{dT}{dx} + c_x \\ \dfrac{dM_y}{dx} - Q_z + c_y \\ \dfrac{dM_z}{dx} + Q_y + c_z \end{Bmatrix} = 0, \text{ and } \begin{Bmatrix} T \\ M_y \\ M_z \end{Bmatrix} = \begin{Bmatrix} GI_{rr}\dfrac{d\phi}{dx} \\ EI_{yy}\dfrac{d\theta}{dx} \\ EI_{zz}\dfrac{d\psi}{dx} \end{Bmatrix}.$$

EXAMPLE 4.4 Consider the beam of the figure of length L. Material properties E and G, and loading b are constants. Due to the offset of the x-axis, cross-section properties are given by A, $S = -rA$ and $I + r^2 A$, in which I is the second moment with respect to the symmetry axis and r is the radius of the cross-section. Determine the axial displacement, deflection, and rotation at the free end (at the x-axis) according to the Bernoulli beam model.



Answer (Mathematica notebook) $u(L) = -\dfrac{bL^3 r}{6EI}$, $v(L) = \dfrac{bL^4}{8EI}$, and $\psi(L) = \dfrac{bL^3}{6EI}$

4.4 CURVILINEAR COORDINATE SYSTEM

Assuming that $S_n = S_b = I_{nb} = 0$, the equilibrium and constitutive equations are

$$\begin{Bmatrix} \dfrac{dN}{ds} - Q_n\kappa + b_s \\ \dfrac{dQ_n}{ds} + N\kappa - Q_b\tau + b_n \\ \dfrac{dQ_b}{ds} + Q_n\tau + b_b \end{Bmatrix} = 0, \quad \begin{Bmatrix} \dfrac{dT}{ds} - M_n\kappa + c_s \\ \dfrac{dM_n}{ds} + T\kappa - M_b\tau - Q_b + c_n \\ \dfrac{dM_b}{ds} + M_n\tau + Q_n + c_b \end{Bmatrix} = 0,$$

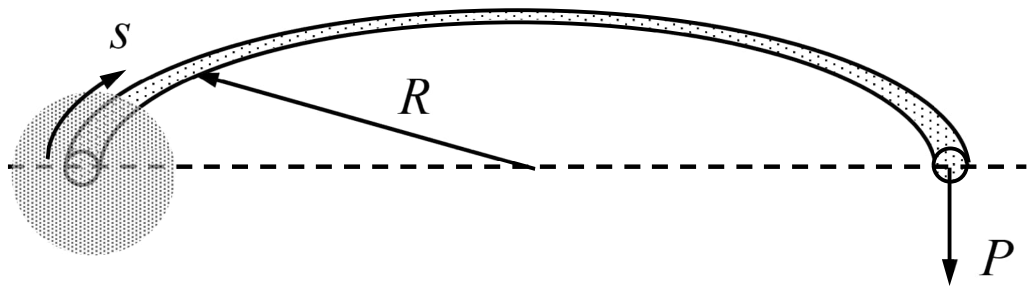
$$\begin{Bmatrix} N \\ Q_n \\ Q_b \end{Bmatrix} = \begin{Bmatrix} EA(\dfrac{du}{ds} - v\kappa) \\ GA(\dfrac{dv}{ds} + u\kappa - w\tau - \psi) \\ GA(\dfrac{dw}{ds} + v\tau + \theta) \end{Bmatrix}, \text{ and } \begin{Bmatrix} T \\ M_n \\ M_b \end{Bmatrix} = \begin{Bmatrix} GI_{rr}(\dfrac{d\phi}{ds} - \theta\kappa) \\ EI_{nn}(\dfrac{d\theta}{ds} + \phi\kappa - \psi\tau) \\ EI_{bb}(\dfrac{d\psi}{ds} + \theta\tau) \end{Bmatrix}.$$

The constitutive equations assume that the $(s,n,b)-$coordinate system is chosen so that $S_n = S_b = I_{nb} = 0$ to simplify the generic forms of the constitutive equations

$$\begin{Bmatrix} N \\ Q_n \\ Q_b \end{Bmatrix} = \begin{Bmatrix} EA(\frac{du}{ds} - v\kappa) + ES_n(\frac{d\theta}{ds} + \phi\kappa - \psi\tau) - ES_b(\frac{d\psi}{ds} + \theta\tau) \\ GA(\frac{dv}{ds} + u\kappa - w\tau - \psi) - GS_n(\frac{d\phi}{ds} - \theta\kappa) \\ GA(\frac{dw}{ds} + v\tau + \theta) + GS_b(\frac{d\phi}{ds} - \theta\kappa) \end{Bmatrix},$$

$$\begin{Bmatrix} T \\ M_n \\ M_b \end{Bmatrix} = \begin{Bmatrix} GS_b(\frac{dw}{ds} + v\tau + \theta) + GI_{rr}(\frac{d\phi}{ds} - \theta\kappa) - GS_n(\frac{dv}{ds} + u\kappa - w\tau - \psi) \\ ES_n(\frac{du}{ds} - v\kappa) + EI_{nn}(\frac{d\theta}{ds} + \phi\kappa - \psi\tau) - EI_{bn}(\frac{d\psi}{ds} + \theta\tau) \\ -ES_b(\frac{du}{ds} - v\kappa) - EI_{nb}(\frac{d\theta}{ds} + \phi\kappa - \psi\tau) + EI_{bb}(\frac{d\psi}{ds} + \theta\tau) \end{Bmatrix}.$$

EXAMPLE 4.5 Consider the planar beam loaded perpendicularly to its plane and clamped at its end as shown. Mid-curve of the beam is a half-circle of radius R. Write down the equilibrium equations of the curved beam and solve for the stress resultants as functions of s. Curvature and torsion of a circular mid-curve $\kappa = 1/R$ and $\tau = 0$ are constants.



Answer $Q_b = P$, $T = PR(1+\cos\dfrac{s}{R})$, and $M_n = -PR\sin(\dfrac{s}{R})$

Boundary conditions define the external forces or moments acting on the boundaries or their work conjugate displacements and rotations. The number of conditions need to match the number of equations of the first order representation i.e. 12. In a statically determined case, the equilibrium equations of beam and force/moment conditions at the free end suffice

equil. eqns.

$$\left.\begin{array}{lll} \dfrac{dN}{ds}-\dfrac{1}{R}Q_n=0 & \dfrac{dQ_n}{ds}+\dfrac{1}{R}N=0 & \dfrac{dQ_b}{ds}=0 \\[2ex] \dfrac{dT}{ds}-\dfrac{1}{R}M_n=0 & \dfrac{dM_n}{ds}+\dfrac{1}{R}T-Q_b=0 & \dfrac{dM_b}{ds}+Q_n=0 \end{array}\right\} \quad \text{in} \quad (0,L)$$

B.C:s

$$\left.\begin{array}{lll} N=0 & Q_n=0 & Q_b=P \\[1ex] T=0 & M_n=0 & M_b=0 \end{array}\right\} \quad \text{at} \quad s=L$$

The equations can be integrated, e.g., in the following order ($L=\pi R$). First

$$\frac{dQ_b}{ds} = 0 \quad \text{in} \quad (0, L) \quad \text{and} \quad Q_b = P \quad \text{at} \quad s = L \quad \Rightarrow \quad Q_b(s) = P,$$

then

$$\frac{dN}{ds} - \frac{1}{R} Q_n = 0, \quad \frac{dQ_n}{ds} + \frac{1}{R} N = 0 \quad \text{in} \quad (0, L) \quad \text{and} \quad N = Q_n = 0 \quad \text{at} \quad s = L \quad \Rightarrow$$

$$\frac{d^2 Q_n}{ds^2} + \frac{1}{R^2} Q_n = 0 \quad \text{in} \quad (0, L) \quad \text{and} \quad Q_n = \frac{dQ_n}{ds} = 0 \quad \text{at} \quad s = L \quad \Rightarrow \quad Q_n(s) = 0$$

$$\frac{d^2 N}{ds^2} + \frac{1}{R^2} N = 0 \quad \text{in} \quad (0, L) \quad \text{and} \quad N = \frac{dN}{ds} = 0 \quad \text{at} \quad s = L \quad \Rightarrow \quad N(s) = 0,$$

then

$$\frac{dM_b}{ds}=0 \text{ in } (0,L) \text{ and } M_b=0 \text{ at } s=L \quad \Rightarrow \quad M_b(s)=0,$$

then

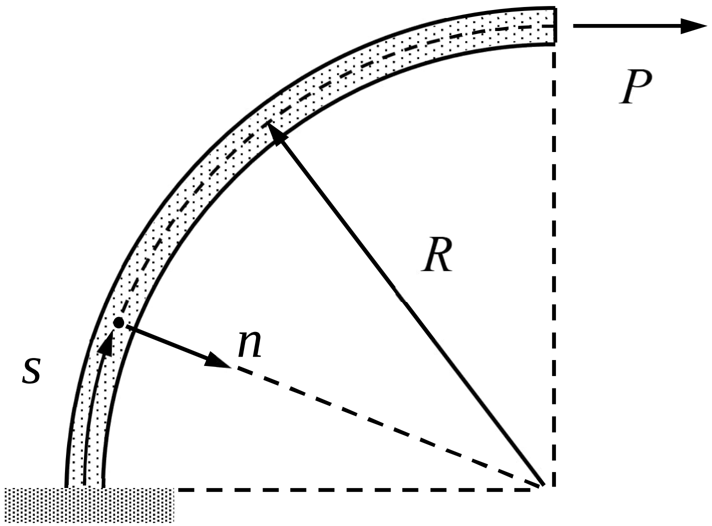
$$\frac{dT}{ds}-\frac{1}{R}M_n=0,\ \frac{dM_n}{ds}+\frac{1}{R}T-P=0 \text{ in } (0,L) \text{ and } T=M_n=0 \text{ at } s=L \quad \Rightarrow$$

$$R\frac{d^2T}{ds^2}+\frac{1}{R}T-P=0 \text{ in } (0,L) \text{ and } T=\frac{dT}{ds}=0 \text{ at } s=L \quad \Rightarrow$$

$$T=PR(\cos\frac{s}{R}+1) \quad \Rightarrow \quad M_n=-PR\sin(\frac{s}{R}).$$

When hand calculations become tedious, the Mathematica notebook of MEC-E8003 homepage helps.

EXAMPLE 4.6 Consider the curved beam shown. Determine the displacement and rotation components u, v and ψ at the free end according to the Bernoulli beam theory. The moments of cross-section A, $S=0$, and I. Material parameters E, G, curvature $\kappa=1/R$ and torsion $\tau=0$ are constants.



Answer

$$u(L)=\frac{PR}{E}\frac{I\pi+A(-8+3\pi)R^2}{4AI},\; v(L)=\frac{PR}{E}\frac{AR^2-I}{2AI},\text{ and } \psi(L)=\frac{PR^2}{EI}\frac{\pi-2}{2}$$

When writing the beam equations, it is convenient to write the equilibrium equations, constitutive equations and boundary conditions “as is” without any eliminations (notice that the constitutive equation for the shear force has been replaced by its “work conjugate” Bernoulli constraint):

$$\frac{dN}{ds} - \frac{1}{R} Q_n = 0, \quad \frac{dQ_n}{ds} + \frac{1}{R} N = 0, \text{ and } \quad \frac{dM_b}{ds} + Q_n = 0 \quad \text{in} \quad (0, L),$$

$$N = EA\left(\frac{du}{ds} - \frac{1}{R} v\right), \quad \frac{dv}{ds} - \psi + \frac{1}{R} u = 0, \quad \text{and} \quad M_b = EI \frac{d\psi}{ds} \quad \text{in} \quad (0, L),$$

$$u = 0, \quad v = 0, \quad \text{and} \quad \psi = 0 \quad \text{at} \quad s = 0,$$

$$N = P, \quad Q_n = 0, \text{ and } M_b = 0 \quad \text{at} \quad s = L.$$

In a statically determined case, it is possible to solve for the stress resultants first. Elimination is used in the connected first order equations to end up with second order non-connected equations. With $L = \pi R / 2$

$$\frac{d^2 N}{ds^2} + \frac{1}{R^2} N = 0 \text{ in } (0, L) \quad \text{and} \quad \frac{dN}{ds} = 0,\ N = P \ \text{ at } \ s = L \Rightarrow$$

$$N(s) = P \sin(\frac{s}{R}) \ , \ Q_n(s) = P \cos(\frac{s}{R}),$$

$$\frac{dM_b}{ds} + P \cos(\frac{s}{R}) = 0 \ \text{ in } (0, L) \text{ and } M_b = 0 \ \text{ at } \ s = L \Rightarrow \ M_b(s) = PR[1 - \sin(\frac{s}{R})].$$

After that, one may continue with the constitutive equations. Again, elimination is used in the connected first order equations to end up with second order non-connected equations

$$\frac{d\psi}{ds} = \frac{PR}{EI}[1 - \sin(\frac{s}{R})] \text{ in } (0, L) \text{ and } \psi = 0 \text{ at } s = 0 \Rightarrow \psi = \frac{PR^2}{EI}[\frac{s}{R} + \cos(\frac{s}{R}) - 1],$$

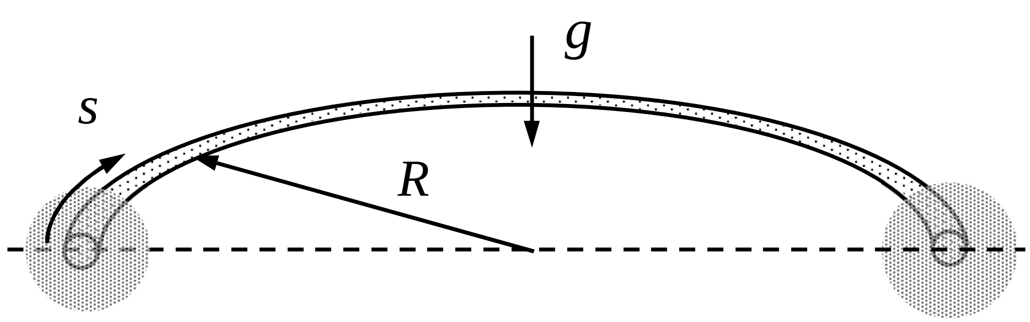
$$\frac{d^2 v}{ds^2} + \frac{1}{R^2} v = \frac{PR}{EI} - (\frac{P}{EA} + \frac{PR^2}{EI})\frac{1}{R}\sin(\frac{s}{R}) \text{ in } (0, L) \text{ and } v = 0,\ \frac{dv}{ds} = 0 \text{ at } s = 0 \Rightarrow$$

$$v(s) = \frac{PR^3}{EI} + (\frac{P}{EA} + \frac{PR^2}{EI})[\frac{s}{2}\cos(\frac{s}{R}) - \frac{R}{2}\sin(\frac{s}{R})], \quad \Leftarrow$$

$$u(s) = R(\psi - \frac{dv}{ds}) = \frac{PR^3}{EI}[\frac{s}{R} + \cos(\frac{s}{R}) - 1] + (\frac{P}{EA} + \frac{PR^2}{EI})[\frac{s}{2}\sin(\frac{s}{R})]. \quad \Leftarrow$$

Notice that the missing boundary condition for the second order problems, obtained through the elimination, is given by the original first order equations!

EXAMPLE 4.7 Consider the semi-circular beam loaded perpendicularly to its plane and clamped at its ends shown. Write down the Timoshenko beam boundary value problem and solve for the vertical deflection at the mid-point with the Mathematica notebook of the course. The cross-section is circular with properties A, $I_{nn} = I_{bb} = I$, $I_{rr} = 2I$ and $S_n = S_b = 0$. The material properties E, G, ρ and curvature $\kappa = 1/R$ are constants (torsion $\tau = 0$). Finally, consider the Bernoulli limit.



Answer $w = \dfrac{\rho A g[16G(-2+\pi)+E(-16+8\pi-4\pi^2+\pi^3)]R^4}{16EGI\pi}$

The parameters of the Timoshenko beam equations are in this case $b_s = b_n = 0$, $b_b = b = \rho A g$, and $c_s = c_n = c_b = 0$. Therefore

equil. eqns.

$$\frac{dN}{ds} - \frac{1}{R} Q_n = 0 \qquad \frac{dQ_n}{ds} + \frac{1}{R} N = 0 \qquad \frac{dQ_b}{ds} + b = 0$$

$$\frac{dT}{ds} - \frac{1}{R} M_n = 0 \qquad \frac{dM_n}{ds} + \frac{1}{R} T - Q_b = 0 \qquad \frac{dM_b}{ds} + Q_n = 0$$

in $(0, \pi R)$

const. eqns.

$$N = EA\left(\frac{du}{ds} - \frac{1}{r} v\right) \qquad Q_n = GA\left(\frac{dv}{ds} + \frac{1}{R} u - \psi\right) \qquad Q_b = GA\left(\frac{dw}{ds} + \theta\right)$$

$$T = 2GI\left(\frac{d\phi}{ds} - \frac{1}{R}\theta\right) \qquad M_n = EI\left(\frac{d\theta}{ds} + \frac{1}{R}\phi\right) \qquad M_b = EI\frac{d\psi}{ds}$$

in $(0, \pi R)$

B.C:s

$$u = 0 \qquad v = 0 \qquad w = 0$$

$$\phi = 0 \qquad \theta = 0 \qquad \psi = 0$$

$s \in \{0, \pi R\}$

In a statically non-determinate case all equations have to be solved simultaneously which may mean tedious calculations. Solution to the non-zero displacement and rotation components at the mid-point $s = \pi R / 2$ as given by the Mathematica notebook are

$$\phi(\frac{\pi R}{2}) = \rho g A R^3 \frac{(\pi - 2)(\pi E - 4G) - 4E}{4G\pi EI},$$

$$w(\frac{\pi R}{2}) = \rho g A R^4 \frac{16G(\pi - 2) + (8\pi - 4\pi^2 + \pi^3 - 16)E}{16G\pi EI}.$$

The code finds first the Timoshenko solution. After that, the Bernoulli solution is obtained by enforcing the Bernoulli constraints.

4.5 CURVED BEAM KINEMATICS

The use of arc length s as the curve parameter is convenient. Curvature $\kappa = 1/R$ and torsion τ define the basis vector derivatives.

Mapping: $\vec{r}(s,n,b) = \vec{r}_0(s) + n\vec{e}_n(s) + b\vec{e}_b(s)$

Basis: $\begin{Bmatrix} \vec{e}_s \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix} = \begin{Bmatrix} \partial\vec{r}_0 / \partial s \\ (\partial\vec{e}_s / \partial s) / \left|\partial\vec{e}_s / \partial s\right| \\ \vec{e}_s \times \vec{e}_n \end{Bmatrix}$ and $\dfrac{\partial}{\partial s}\begin{Bmatrix} \vec{e}_s \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix}\begin{Bmatrix} \vec{e}_s \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix}$

Gradient: $\nabla = \vec{e}_s \dfrac{1}{J}[\dfrac{\partial}{\partial s} + \tau(b\dfrac{\partial}{\partial n} - n\dfrac{\partial}{\partial b})] + \vec{e}_n \dfrac{\partial}{\partial n} + \vec{e}_b \dfrac{\partial}{\partial b}$, where $J = 1 - n\kappa$

Volume element: $dV = JdAds$

The intrinsic $(s,n,b)-$coordinate system is curvilinear and orthonormal. The matrix of the basis vector derivatives is anti-symmetric and expressible in the form

$$\frac{\partial}{\partial s}\begin{Bmatrix}\vec{e}_s\\ \vec{e}_n\\ \vec{e}_b\end{Bmatrix}=(\frac{\partial}{\partial s}[F])[F]^{-1}\begin{Bmatrix}\vec{e}_s\\ \vec{e}_n\\ \vec{e}_b\end{Bmatrix}=\begin{bmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{bmatrix}\begin{Bmatrix}\vec{e}_s\\ \vec{e}_n\\ \vec{e}_b\end{Bmatrix}$$

depending on torsion τ and curvature $\kappa = 1/R$. Using the generic expression of the gradient operator (the basis vectors are not orthogonal away from the axis and the simple expression based on the scaling coefficient does not apply)

$$\nabla=\begin{Bmatrix}\vec{e}_s\\ \vec{e}_n\\ \vec{e}_b\end{Bmatrix}^{\mathrm{T}}[F]^{-\mathrm{T}}[H]^{-1}\begin{Bmatrix}\partial_s\\ \partial_n\\ \partial_b\end{Bmatrix}=\frac{\vec{e}_s}{1-n\kappa}[\frac{\partial}{\partial s}+\tau(b\frac{\partial}{\partial n}-n\frac{\partial}{\partial b})]+\vec{e}_n\frac{\partial}{\partial n}+\vec{e}_b\frac{\partial}{\partial b}$$

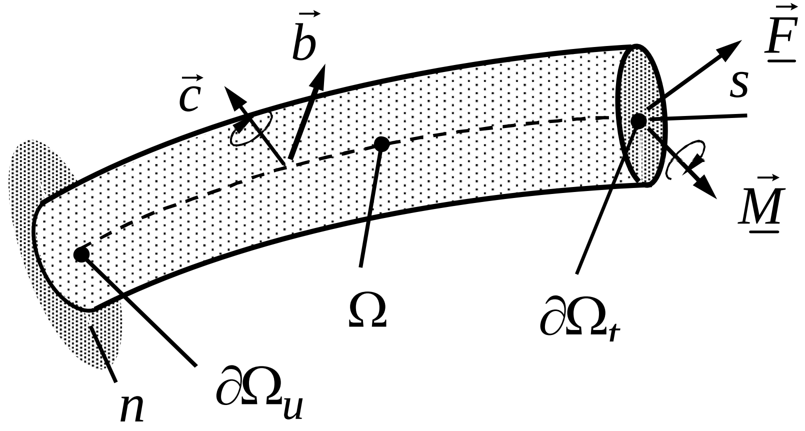
The expression of the volume element is

$$dV = \left(\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial n}\right) \cdot \frac{\partial \vec{r}}{\partial b} dndbds = [(\vec{e}_s + n\frac{d\vec{e}_n}{ds} + b\frac{d\vec{e}_b}{ds}) \times \vec{e}_n] \cdot \vec{e}_b dAds = (1 - n\kappa)dAds$$

The radius of curvature $R = 1/\kappa$ at a point is given by the best fitting circle and it is a geometric quantity. Torsion τ describes the rate by which $\vec{e}_n$ and $\vec{e}_b$ rotate around the mid-curve.

4.6 VIRTUAL WORK DENSITY

Virtual work densities can be expressed in terms of generalized forces (force, moment) and their work conjugate strains:



$$\delta w_\Omega^{\mathrm{int}} = -\begin{Bmatrix} \delta\vec{\varepsilon} \\ \delta\vec{\kappa} \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \vec{F} \\ \vec{M} \end{Bmatrix}$$

$$\delta w_\Omega^{\mathrm{ext}} = \begin{Bmatrix} \delta\vec{u}_0 \\ \delta\vec{\theta}_0 \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \vec{b} \\ \vec{c} \end{Bmatrix}, \quad \delta w_{\partial\Omega}^{\mathrm{ext}} = \begin{Bmatrix} \delta\vec{u}_0 \\ \delta\vec{\theta}_0 \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \underline{\vec{F}} \\ \underline{\vec{M}} \end{Bmatrix}$$

where

$$\begin{Bmatrix} \vec{F} \\ \vec{M} \end{Bmatrix} = \int \begin{Bmatrix} \vec{\sigma} \\ \vec{\rho}\times\vec{\sigma} \end{Bmatrix} dA, \quad \begin{Bmatrix} \vec{b} \\ \vec{c} \end{Bmatrix} = \int \begin{Bmatrix} \vec{f} \\ \vec{\rho}\times\vec{f} \end{Bmatrix} JdA, \text{ and } \begin{Bmatrix} \underline{\vec{F}} \\ \underline{\vec{M}} \end{Bmatrix} = \int \begin{Bmatrix} \vec{t} \\ \vec{\rho}\times\vec{t} \end{Bmatrix} dA.$$

In vector notation, the kinematic assumption of the beam model, gradient operator, gradient of the relative position vector and displacement gradient are ($J = 1 - n\kappa$)

$$\left.\begin{aligned}
&\vec{u}(s,n,b) = \vec{u}_0(s) + \vec{\theta}_0(s) \times \vec{\rho}(n,b) \quad \text{where} \quad \vec{\rho} = n\vec{e}_n + b\vec{e}_b, \\
&\nabla = \nabla_s + \vec{e}_n \frac{\partial}{\partial n} + \vec{e}_b \frac{\partial}{\partial b} \quad \text{where} \quad \nabla_s = \vec{e}_s \frac{1}{J}[\frac{\partial}{\partial s} + \tau(b\frac{\partial}{\partial n} - n\frac{\partial}{\partial b})], \\
&\nabla\vec{\rho} = \overleftrightarrow{I} - \frac{1}{J}\vec{e}_s\vec{e}_s \quad \text{where} \quad \overleftrightarrow{I} = \vec{e}_s\vec{e}_s + \vec{e}_n\vec{e}_n + \vec{e}_b\vec{e}_b, \\
&dV = JdAds.
\end{aligned}\right\}$$

The displacement gradient becomes ($\overleftrightarrow{I} \times \vec{\theta}$ is antisymmetric) as displacement components depend on s only

$$\nabla\vec{u} = \frac{1}{J}(\vec{e}_s \frac{d\vec{u}_0}{ds} + \vec{e}_s \frac{d\vec{\theta}_0}{ds} \times \vec{\rho}) - (\vec{\vec{I}} - \frac{1}{J}\vec{e}_s\vec{e}_s) \times \vec{\theta}_0 \quad \Leftrightarrow$$

$$\nabla\vec{u} = \frac{1}{J}\vec{e}_s(\vec{\varepsilon} + \vec{\kappa} \times \vec{\rho}) - \vec{\vec{I}} \times \vec{\theta}_0 \text{ where } \vec{\varepsilon} = \frac{d\vec{u}_0}{ds} + \vec{e}_s \times \vec{\theta}_0 \text{ and } \vec{\kappa} = \frac{d\vec{\theta}_0}{ds}.$$

Assuming the symmetry of stress $\vec{\vec{\sigma}} = \vec{\vec{\sigma}}_{\mathrm{c}}$, virtual work of internal forces per unit volume becomes (vector identity $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$ is needed in the derivation) with notation $\vec{\sigma} = \vec{e}_s \cdot \vec{\vec{\sigma}} = \vec{\vec{\sigma}} \cdot \vec{e}_s$

$$\delta w_V^{\mathrm{int}} = -\vec{\vec{\sigma}}_{\mathrm{c}} : \delta\nabla\vec{u} = -\frac{1}{J}\begin{Bmatrix} \delta\vec{\varepsilon} \\ \delta\vec{\kappa} \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \vec{\sigma} \\ \vec{\rho} \times \vec{\sigma} \end{Bmatrix}.$$

As $dV = JdAds$ and $ds = d\Omega$, the virtual work expression of internal forces takes the form

$$\delta W^{\text{int}} = \int_V \delta w_V^{\text{int}} dV = \int_\Omega - \begin{Bmatrix} \delta\vec{\varepsilon} \\ \delta\vec{\kappa} \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \vec{F} \\ \vec{M} \end{Bmatrix} d\Omega, \quad \text{where} \begin{Bmatrix} \vec{F} \\ \vec{M} \end{Bmatrix} = \int \begin{Bmatrix} \vec{\sigma} \\ \vec{\rho}\times\vec{\sigma} \end{Bmatrix} dA. \;\Leftarrow$$

If the surface forces are acting on the end surfaces only (just to simplify the derivation), the virtual works of external volume and surface forces are (vector identity $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$ and $dV = JdAd\Omega$ are used again)

$$\delta W_V^{\text{ext}} = \int_V (\delta\vec{u}\cdot\vec{f})dV = \int_\Omega \begin{Bmatrix} \delta\vec{u}_0 \\ \delta\vec{\theta}_0 \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \vec{b} \\ \vec{c} \end{Bmatrix} d\Omega, \quad \text{where} \begin{Bmatrix} \vec{b} \\ \vec{c} \end{Bmatrix} = \int \begin{Bmatrix} \vec{f} \\ \vec{\rho}\times\vec{f} \end{Bmatrix} JdA. \;\Leftarrow$$

$$\delta W_A^{\text{ext}} = \int_A (\delta\vec{u}\cdot\vec{t})dA = \sum_{\partial\Omega} \begin{Bmatrix} \delta\vec{u}_0 \\ \delta\vec{\theta}_0 \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \underline{\vec{F}} \\ \underline{\vec{M}} \end{Bmatrix}, \quad \text{where} \begin{Bmatrix} \underline{\vec{F}} \\ \underline{\vec{M}} \end{Bmatrix} = \int \begin{Bmatrix} \vec{t} \\ \vec{\rho}\times\vec{t} \end{Bmatrix} dA. \;\Leftarrow$$

4.7 CONSTITUTIVE EQUATIONS

Constitutive equations follow from the generalized Hooke's law (taking into accout the kinetic assumptions $\sigma_{nn} = \sigma_{bb} = 0$), gradient of the displacement for the beam model, and definitions of stress resultants:

$$\begin{Bmatrix} \vec{F} \\ \vec{M} \end{Bmatrix} = \int \begin{Bmatrix} \vec{\sigma} \\ \vec{\rho} \times \vec{\sigma} \end{Bmatrix} dA = \begin{bmatrix} \overset{\leftrightarrow}{A} & \overset{\leftrightarrow}{C} \\ \overset{\leftrightarrow}{C}_{\mathrm{c}} & \overset{\leftrightarrow}{B} \end{bmatrix} \cdot \begin{Bmatrix} \vec{\varepsilon} \\ \vec{\kappa} \end{Bmatrix}, \text{ where } \begin{Bmatrix} \vec{\varepsilon} \\ \vec{\kappa} \end{Bmatrix} = \begin{Bmatrix} \dfrac{d\vec{u}_0}{ds} + \vec{e}_s \times \vec{\theta}_0 \\ \dfrac{d\vec{\theta}_0}{ds} \end{Bmatrix} \text{ and}$$

$$\begin{bmatrix} \overset{\leftrightarrow}{A} & \overset{\leftrightarrow}{C} \\ \overset{\leftrightarrow}{C}_{\mathrm{c}} & \overset{\leftrightarrow}{B} \end{bmatrix} = \int \begin{bmatrix} \overset{\leftrightarrow}{E} & -\overset{\leftrightarrow}{E} \times \vec{\rho} \\ \vec{\rho} \times \overset{\leftrightarrow}{E} & -\vec{\rho} \times \overset{\leftrightarrow}{E} \times \vec{\rho} \end{bmatrix} \frac{1}{J} dA, \text{ where } \overset{\leftrightarrow}{E} = \vec{e}_s \cdot \overset{\leftrightarrow\leftrightarrow}{E} \cdot \vec{e}_s \text{ and } \vec{\rho} = n\vec{e}_n + b\vec{e}_b.$$

The three tensors $\overset{\leftrightarrow}{A}$, $\overset{\leftrightarrow}{B}$, and $\overset{\leftrightarrow}{C}$ define the constitutive equations.

Derivation uses stress resultant definitions, beam model elasticity tensor, and displacement gradient. Displacement gradient was derived earlier when discussing the virtual work expression

$$\nabla\vec{u} = \frac{1}{J}\vec{e}_s(\vec{\varepsilon} + \vec{\kappa}\times\vec{\rho}) - \overset{\leftrightarrow}{I}\times\vec{\theta}_0 \ .$$

The stress-strain relationship of a linearly elastic material with elasticity tensor $\overset{\leftrightarrow}{\overset{\leftrightarrow}{E}}$ for the beam model gives with definition $\vec{\sigma} = \vec{e}_s \cdot \overset{\leftrightarrow}{\sigma} = \overset{\leftrightarrow}{\sigma}\cdot\vec{e}_s$ (notice that $\overset{\leftrightarrow}{I}\times\vec{\theta}_0$ is antisymmetric and vanishes as elasticity tensor is symmetric with respect to the last index pair)

$$\overset{\leftrightarrow}{\sigma} = \overset{\leftrightarrow}{\overset{\leftrightarrow}{E}} : \nabla\vec{u} = \overset{\leftrightarrow}{\overset{\leftrightarrow}{E}}\cdot\vec{e}_s\cdot(\vec{\varepsilon}+\vec{\kappa}\times\vec{\rho})\frac{1}{J} \quad \Rightarrow \quad \vec{\sigma} = \vec{e}_s\cdot\overset{\leftrightarrow}{\sigma} = \overset{\leftrightarrow}{E}\cdot(\vec{\varepsilon}+\vec{\kappa}\times\vec{\rho})\frac{1}{J}$$

in which $\overset{\leftrightarrow}{E} = \vec{e}_s\cdot\overset{\leftrightarrow}{\overset{\leftrightarrow}{E}}\cdot\vec{e}_s$.

Finally, constitutive equations follow from the definitions of stress resultants. In a concise form, the constitutive equations and parameters taking into account the cross-section geometry and the material can be written as

$$\begin{Bmatrix} \vec{F} \\ \vec{M} \end{Bmatrix} = \int \begin{Bmatrix} \vec{\sigma} \\ \vec{\rho}\times\vec{\sigma} \end{Bmatrix} dA = \int \begin{Bmatrix} \overset{\leftrightarrow}{E}\cdot(\vec{\varepsilon}+\vec{\kappa}\times\vec{\rho}) \\ \vec{\rho}\times\overset{\leftrightarrow}{E}\cdot(\vec{\varepsilon}+\vec{\kappa}\times\vec{\rho}) \end{Bmatrix} \frac{1}{J} dA = \begin{bmatrix} \overset{\leftrightarrow}{A} & \overset{\leftrightarrow}{C} \\ \overset{\leftrightarrow}{C}_{\mathrm{c}} & \overset{\leftrightarrow}{B} \end{bmatrix} \cdot \begin{Bmatrix} \vec{\varepsilon} \\ \vec{\kappa} \end{Bmatrix} \text{ where}$$

may depend on s

$$\begin{bmatrix} \overset{\leftrightarrow}{A} & \overset{\leftrightarrow}{C} \\ \overset{\leftrightarrow}{C}_{\mathrm{c}} & \overset{\leftrightarrow}{B} \end{bmatrix} = \int \begin{bmatrix} \overset{\leftrightarrow}{E} & -\overset{\leftrightarrow}{E}\times\vec{\rho} \\ \vec{\rho}\times\overset{\leftrightarrow}{E} & -\vec{\rho}\times\overset{\leftrightarrow}{E}\times\vec{\rho} \end{bmatrix} \frac{1}{J} dA \quad \text{and} \quad \begin{Bmatrix} \vec{\varepsilon} \\ \vec{\kappa} \end{Bmatrix} = \begin{Bmatrix} \dfrac{d\vec{u}_0}{ds} + \vec{e}_s \times \vec{\theta}_0 \\ \dfrac{d\vec{\theta}_0}{ds} \end{Bmatrix}.$$

Assuming an isotropic material, the kinetic assumption of the beam model $\sigma_{nn} = \sigma_{bb} = 0$ gives the elasticity tensor expression

$$\overset{\leftrightarrow}{\overset{\leftrightarrow}{E}} = \begin{Bmatrix} \vec{e}_s\vec{e}_s \\ \vec{e}_n\vec{e}_n \\ \vec{e}_b\vec{e}_b \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} E & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \vec{e}_s\vec{e}_s \\ \vec{e}_n\vec{e}_n \\ \vec{e}_b\vec{e}_b \end{Bmatrix} + \begin{Bmatrix} \vec{e}_s\vec{e}_n + \vec{e}_n\vec{e}_s \\ \vec{e}_n\vec{e}_b + \vec{e}_b\vec{e}_n \\ \vec{e}_b\vec{e}_s + \vec{e}_s\vec{e}_b \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} G & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \vec{e}_s\vec{e}_n + \vec{e}_n\vec{e}_s \\ \vec{e}_n\vec{e}_b + \vec{e}_b\vec{e}_n \\ \vec{e}_b\vec{e}_s + \vec{e}_s\vec{e}_b \end{Bmatrix} \Rightarrow$$

$$\overset{\leftrightarrow}{E} = \vec{e}_s \cdot \overset{\leftrightarrow}{\overset{\leftrightarrow}{E}} \cdot \vec{e}_s = \begin{Bmatrix} \vec{e}_s \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} E & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \vec{e}_s \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix}.$$

If the material is further homogeneous and cross section geometry constant, component forms of $\overset{\leftrightarrow}{A}$, $\overset{\leftrightarrow}{B}$ and $\overset{\leftrightarrow}{C}$ of the constitutive equation take the forms

$$\overset{\leftrightarrow}{A} = \int \overset{\leftrightarrow}{E} \frac{1}{J} dA = \overset{\leftrightarrow}{E} \int \frac{1}{J} dA = \begin{Bmatrix} \vec{e}_s \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} AE & 0 & 0 \\ 0 & AG & 0 \\ 0 & 0 & AG \end{bmatrix} \begin{Bmatrix} \vec{e}_s \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix},$$

$$\overleftrightarrow{C} = -\int (\overleftrightarrow{E} \times \vec{\rho}) \frac{1}{J} dA = \begin{Bmatrix} \vec{e}_s \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 0 & S_n E & -S_b E \\ -S_n G & 0 & 0 \\ S_b G & 0 & 0 \end{bmatrix} \begin{Bmatrix} \vec{e}_s \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix},$$

$$\overleftrightarrow{B} = -\int \vec{\rho} \times \overleftrightarrow{E} \times \vec{\rho} \frac{1}{J} dA = \begin{Bmatrix} \vec{e}_s \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} (I_{nn} + I_{bb})G & 0 & 0 \\ 0 & I_{nn}E & -I_{nb}E \\ 0 & -I_{nb}E & I_{bb}E \end{bmatrix} \begin{Bmatrix} \vec{e}_s \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix}$$

in which the moments of cross sections

$$A = \int \frac{1}{J} dA,\ \ S_n = \int b \frac{1}{J} dA,\ \ S_b = \int n \frac{1}{J} dA,\ \ I_{bn} = I_{nb} = \int nb \frac{1}{J} dA,$$

$$I_{nn} = \int b^2 \frac{1}{J} dA,\ \ I_{bb} = \int n^2 \frac{1}{J} dA,\ \text{ and } I_{rr} = I_{nn} + I_{bb}.$$

depend on the geometry of the cross-section, curvature ($J = 1 - n\kappa$), and positioning of the material coordinate system.

Finally, the component representations of the constitutive equations for an isotropic and homogeneous material take the forms ($\vec{u}_0 = u\vec{e}_s + v\vec{e}_n + w\vec{e}_b,\ \vec{\theta}_0 = \phi\vec{e}_s + \theta\vec{e}_n + \psi\vec{e}_b$)

$$\begin{Bmatrix} N \\ Q_n \\ Q_b \end{Bmatrix} = \begin{Bmatrix} EA(\dfrac{du}{ds} - v\kappa) + ES_n(\dfrac{d\theta}{ds} + \phi\kappa - \psi\tau) - ES_b(\dfrac{d\psi}{ds} + \theta\tau) \\ GA(\dfrac{dv}{ds} + u\kappa - w\tau - \psi) - GS_n(\dfrac{d\phi}{ds} - \theta\kappa) \\ GA(\dfrac{dw}{ds} + v\tau + \theta) + GS_b(\dfrac{d\phi}{ds} - \theta\kappa) \end{Bmatrix}, \quad \Leftarrow$$

$$\begin{Bmatrix} T \\ M_n \\ M_b \end{Bmatrix} = \begin{Bmatrix} GS_b(\frac{dw}{ds} + v\tau + \theta) + GI_{rr}(\frac{d\phi}{ds} - \theta\kappa) - GS_n(\frac{dv}{ds} + u\kappa - w\tau - \psi) \\ ES_n(\frac{du}{ds} - v\kappa) + EI_{nn}(\frac{d\theta}{ds} + \phi\kappa - \psi\tau) - EI_{bn}(\frac{d\psi}{ds} + \theta\tau) \\ -ES_b(\frac{du}{ds} - v\kappa) - EI_{nb}(\frac{d\theta}{ds} + \phi\kappa - \psi\tau) + EI_{bb}(\frac{d\psi}{ds} + \theta\tau) \end{Bmatrix}. \leftarrow$$

The cross-section properties are constants only in case of constant curvature and cross-section geometry. If the beam is thin compared to the radius of curvature $J \approx 1$, this effect can be omitted.

EXAMPLE In curved geometry, position of the neutral axis depends on the curvature as $J = 1 - n\kappa$. If the mid-curve is placed at the geometric centroid of the cross-section and curvature $\kappa = 1/R$, the (non-zero) moments for a circular cross-section of radius r are ($\varepsilon = r/R$):

$\varepsilon = r/R$	0	1/10	2/10	3/10	4/10	5/10	6/10	7/10	8/10	9/10	1
$A/(\pi r^2)$	1	1.00	1.01	1.02	1.04	1.07	1.11	1.17	1.25	1.39	2
$S_b/(\pi r^3)$	0	0.03	0.05	0.08	0.11	0.14	0.18	0.24	0.31	0.44	1
$I_{bb}/(\frac{1}{4}\pi r^4)$	1	1.00	1.02	1.05	1.09	1.15	1.23	1.36	1.56	1.94	4
$I_{nn}/(\frac{1}{4}\pi r^4)$	1	1.00	1.01	1.02	1.03	1.05	1.07	1.10	1.15	1.21	1.33

The moments of the cross section follow from the integrals (need to be evaluated numerically for each value of $\varepsilon = r/R$,

$$A = \pi r^2 \left(\frac{1}{\pi} \int_0^{2\pi} \int_0^1 \frac{s}{1 - s\varepsilon\cos\beta} ds d\beta \right),$$

$$S_b = \pi r^3 \left(\frac{1}{\pi} \int_0^{2\pi} \int_0^1 s\cos\beta \frac{s}{1 - s\varepsilon\cos\beta} ds d\beta \right),$$

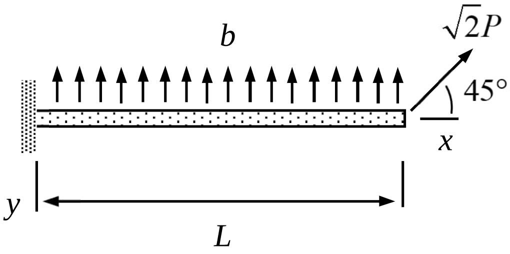
$$I_{bb} = \frac{1}{4}\pi r^4 \left(\frac{4}{\pi} \int_0^{2\pi} \int_0^1 s^2\cos^2\beta \frac{s}{1 - s\varepsilon\cos\beta} ds d\beta \right),$$

$$I_{nn} = \frac{1}{4}\pi r^4 \left(\frac{4}{\pi} \int_0^{2\pi} \int_0^1 s^2\sin^2\beta \frac{s}{1 - s\varepsilon\cos\beta} ds d\beta \right).$$

Above, $n = sr\cos\beta$ and $b = sr\sin\beta$.

MEC-E8003 Beam, plate and shell models, examples 4

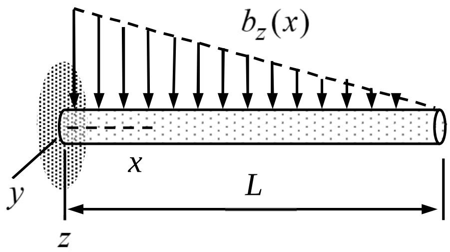
1. Consider the $xy-$plane beam of length L shown. Material properties E and G, cross-section properties A, I are constants, and $S=0$. Write down the boundary value problem according to the Timoshenko beam model in terms of the axial displacement $u(x)$, transverse displacement $v(x)$, and rotation $\psi(x)$.



Answer

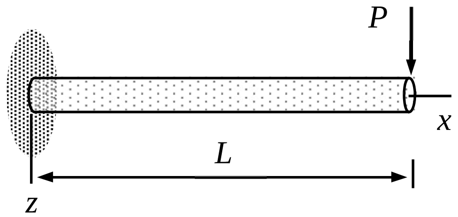
$$\begin{Bmatrix} EA\dfrac{d^2u}{dx^2} \\ GA(\dfrac{d^2v}{dx^2}-\dfrac{d\psi}{dx})-b \\ EI\dfrac{d^2\psi}{dx^2}+GA(\dfrac{dv}{dx}-\psi) \end{Bmatrix}=0 \text{ in } (0,L), \quad \begin{Bmatrix} u \\ v \\ \psi \end{Bmatrix}=0 \quad x=0, \begin{Bmatrix} EA\dfrac{du}{dx}-P \\ GA(\dfrac{dv}{dx}-\psi)+P \\ EI\dfrac{d\psi}{dx} \end{Bmatrix}=0 \quad x=L.$$

2. Find the stress resultants Q_z, M_y and the transverse displacement w of the cantilever beam shown according to the Bernoulli beam model. Problem parameters E, G, A, $S=0$ and I are constants and the distributed force $b_z=b(1-x/L)$. Start with the generic equilibrium and constitutive equations for the beam model.



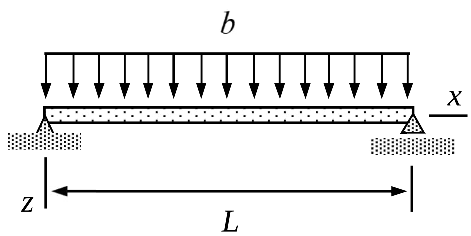
Answer $Q_z=\dfrac{b}{2L}(L-x)^2$, $M_y=-\dfrac{b}{6L}(L-x)^3$, $w=\dfrac{b}{120LEI}[L^5-(L-x)^5-5L^4x]$

3. Find the displacement and rotation of the $xz-$plane cantilever beam of the figure according to the Bernoulli beam model. Problem parameters E, G, A, $S=0$ and I are constants.



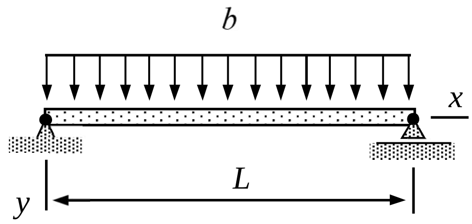
Answer $w(x)=\dfrac{P}{6EI}x^2(3L-x)$, $\theta(x)=\dfrac{P}{2EI}x(x-2L)$

4. Consider the simply supported (plane) beam of the figure of length L. Material properties E and G, cross-section properties A, $S=0$, I, and loading b are constants. Determine the deflection and rotation at the mid-point $x=L/2$ according to the Timoshenko beam model.



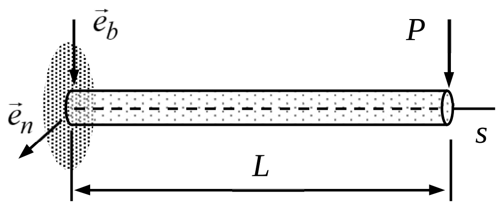
Answer $w(L/2) = \dfrac{bL^2}{8AG} + \dfrac{5bL^4}{384EI}$, $\theta(L/2) = 0$

5. Consider the simply supported xy – plane beam of length L shown. Material properties E and G, cross-section properties A, $S = 0$, I, and loading b are constants. Write down the equilibrium equations, constitutive equations, and boundary conditions according to the Bernoulli beam model. After that, solve the equations for the transverse displacement.



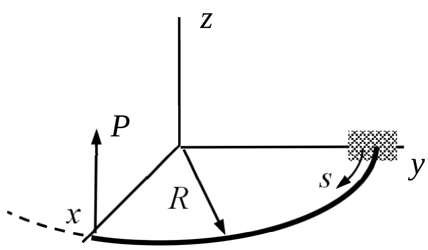
Answer $v(x) = \dfrac{b}{24EI} x(L^3 - 2Lx^2 + x^3)$

6. Consider the cantilever beam of the figure of zero curvature and torsion $\tau = 4\pi / L$. Write down the equilibrium equations and the boundary conditions at the free end in the (s,n,b) – coordinate system. Also, solve the boundary value problem for the stress resultants as functions of s.



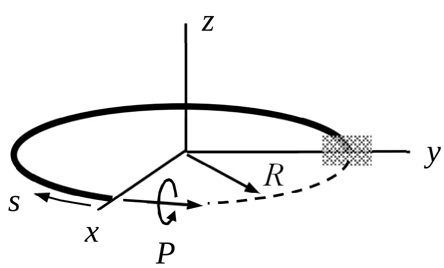
Answer $Q_n = P\sin(\tau s)$, $Q_b = P\cos(\tau s)$, $M_n = P(s-L)\cos(\tau s)$, $M_b = P(L-s)\sin(\tau s)$

7. Consider the curved beam of the figure forming a 90-degree circular segment of radius R in the horizontal plane. Find the stress resultants $N(s)$, $Q_n(s)$, $Q_b(s)$, $T(s)$, $M_n(s)$, and $M_b(s)$. Use the equilibrium equations of the beam model in the (s,n,b) – coordinate system.



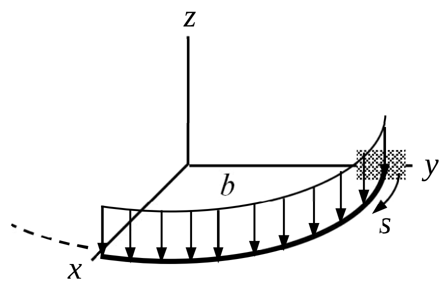
Answer $\begin{Bmatrix} N \\ Q_n \\ Q_b \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -P \end{Bmatrix}$, $\begin{Bmatrix} T \\ M_n \\ M_b \end{Bmatrix} = \begin{Bmatrix} PR[\sin(s/R) - 1] \\ RP\cos(s/R) \\ 0 \end{Bmatrix}$

8. Consider a curved beam forming ¾ of a full circle of radius R in the horizontal plane. The given torque of magnitude P is acting on the free end as shown. Write down the equilibrium equations and boundary conditions for the *stress resultants* and solve the equations for $N(s)$, $Q_n(s)$, $Q_b(s)$, $T(s)$, $M_n(s)$, and $M_b(s)$.



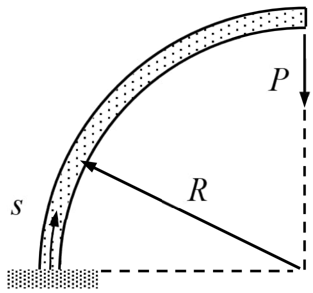
Answer $\begin{Bmatrix} N \\ Q_n \\ Q_b \end{Bmatrix} = 0$, $\begin{Bmatrix} T \\ M_n \\ M_b \end{Bmatrix} = \begin{Bmatrix} P\cos(s/R) \\ -P\sin(s/R) \\ 0 \end{Bmatrix}$

9. Consider the curved beam of the figure forming a 90-degree circular segment of radius R in the horizontal plane. Find the stress resultants $N(s)$, $Q_n(s)$, $Q_b(s)$, $T(s)$, $M_n(s)$, and $M_b(s)$. Use the equilibrium equations of the beam model in the $(s,n,b)-$coordinate system. The distributed constant load of magnitude b is acting to the negative direction of the $z-$axis.



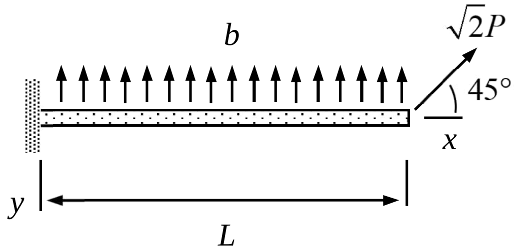
Answer $Q_b = b(L-s)$, $T = -bR^2\cos(\dfrac{s}{R}) + bR(L-s)$, $M_n = bR^2[\sin(\dfrac{s}{R}) - 1]$

10. Consider the curved planar Timoshenko beam shown in the figure. Write down the boundary value problem for $u(s)$, $v(s)$, $\psi(s)$, $N(s)$, $Q_n(s)$, and $M_b(s)$. Also, solve the equations for displacement $v(s)$ and rotation $\psi(s)$. The properties of the cross-section A, $S_b = 0$, $I_{bb} = I$, and material parameters E, G are constants. Curvature is $\kappa = 1/R$ and torsion $\tau = 0$.



Answer $v(s) = \dfrac{P}{2}(\dfrac{R^2}{EI} + \dfrac{1}{GA} + \dfrac{1}{EA})s\sin(\dfrac{s}{R})$, $\psi(s) = \dfrac{PR^2}{EI}\sin(\dfrac{s}{R})$

Consider the $xy-$plane beam of length L shown. Material properties E and G, cross-section properties A, I are constants, and $S=0$. Write down the boundary value problem according to the Timoshenko beam model in terms of the axial displacement $u(x)$, transverse displacement $v(x)$, and rotation $\psi(x)$.



Solution

Timoshenko beam equations in the Cartesian system are

$$\begin{Bmatrix} \dfrac{dN}{dx}+b_x \\ \dfrac{dQ_y}{dx}+b_y \\ \dfrac{dQ_z}{dx}+b_z \end{Bmatrix}=0,\quad \begin{Bmatrix} N \\ Q_y \\ Q_z \end{Bmatrix}=\begin{Bmatrix} EA\dfrac{du}{dx}-ES_z\dfrac{d\psi}{dx}+ES_y\dfrac{d\theta}{dx} \\ GA(\dfrac{dv}{dx}-\psi)-GS_y\dfrac{d\phi}{dx} \\ GA(\dfrac{dw}{dx}+\theta)+GS_z\dfrac{d\phi}{dx} \end{Bmatrix}$$

$$\begin{Bmatrix} \dfrac{dT}{dx}+c_x \\ \dfrac{dM_y}{dx}-Q_z+c_y \\ \dfrac{dM_z}{dx}+Q_y+c_z \end{Bmatrix}=0,\quad \begin{Bmatrix} T \\ M_y \\ M_z \end{Bmatrix}=\begin{Bmatrix} -GS_y(\dfrac{dv}{dx}-\psi)+GS_z(\dfrac{dw}{dx}+\theta)+GI_{rr}\dfrac{d\phi}{dx} \\ ES_y\dfrac{du}{dx}-EI_{zy}\dfrac{d\psi}{dx}+EI_{yy}\dfrac{d\theta}{dx} \\ -ES_z\dfrac{du}{dx}+EI_{zz}\dfrac{d\psi}{dx}-EI_{yz}\dfrac{d\theta}{dx} \end{Bmatrix}$$

In $xy-$plane problem, the non-zero displacements and rotations are u, v, and ψ. Geometrical properties of the cross-section are A, $S_z=S=0$, $I_{zz}=I$. External distributed forces are $b_x=0$, $b_y=-b$, $b_z=0$, $c_x=c_y=c_z=0$. With these selections, equilibrium equations, constitutive equations, and boundary conditions of the planar problem take the forms

$$\begin{Bmatrix} \dfrac{dN}{dx} \\ \dfrac{dQ_y}{dx}-b \\ \dfrac{dM_z}{dx}+Q_y \end{Bmatrix}=0 \quad\text{and}\quad \begin{Bmatrix} N \\ Q_y \\ M_z \end{Bmatrix}=\begin{Bmatrix} EA\dfrac{du}{dx} \\ GA(\dfrac{dv}{dx}-\psi) \\ EI_{zz}\dfrac{d\psi}{dx} \end{Bmatrix} \quad\text{in}\quad (0,L),$$

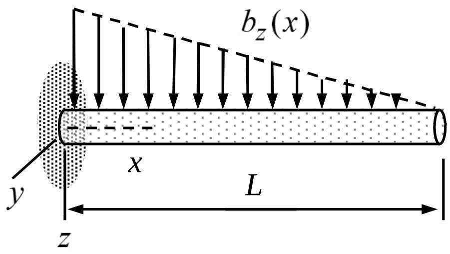
$$\begin{Bmatrix} u \\ v \\ \psi \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad\text{at}\quad x=0 \quad\text{and}\quad \begin{Bmatrix} N-P \\ Q_y+P \\ M_z \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad\text{at}\quad x=L.$$

To get a boundary value problem in terms of the axial displacement $u(x)$, transverse displacement $v(x)$, and rotation $\psi(x)$, stress resultants are eliminated to end up with

$$\begin{Bmatrix} EA\dfrac{d^2u}{dx^2} \\ GA(\dfrac{d^2v}{dx^2}-\dfrac{d\psi}{dx})-b \\ EI\dfrac{d^2\psi}{dx^2}+GA(\dfrac{dv}{dx}-\psi) \end{Bmatrix} = 0 \quad \text{in} \quad (0,L),$$

$$\begin{Bmatrix} u \\ v \\ \psi \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \text{ at } \ x=0 \quad \text{and} \quad \begin{Bmatrix} EA\dfrac{du}{dx}-P \\ GA(\dfrac{dv}{dx}-\psi)+P \\ EI\dfrac{d\psi}{dx} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \text{ at } \ x=L.$$

Find the stress resultants Q_z, M_y and the transverse displacement w of the cantilever beam shown according to the Bernoulli beam model. Problem parameters E, G, A, $S=0$ and I are constants and the distributed force $b_z = b(1-x/L)$. Start with the generic equilibrium and constitutive equations for the beam model.



Solution

Timoshenko beam equations boil down to the Bernoulli beam equations when the Bernoulli constraints $dv/dx-\psi=0$ and $dw/dx+\theta=0$ are applied there. If $S_y = S_z = I_{yz} = 0$ one may just replace the constitutive equations for the shear stress resultants by the Bernoulli constraints to get the Bernoulli model equilibrium and constitutive equations (elimination with the Bernoulli constraints and constitutive equations gives the well-known fourth order beam equation of textbooks)

$$\begin{Bmatrix} \dfrac{dN}{dx}+b_x \\ \dfrac{dQ_y}{dx}+b_y \\ \dfrac{dQ_z}{dx}+b_z \end{Bmatrix}=0,\quad \begin{Bmatrix} \dfrac{dT}{dx}+c_x \\ \dfrac{dM_y}{dx}-Q_z+c_y \\ \dfrac{dM_z}{dx}+Q_y+c_z \end{Bmatrix}=0,\quad \begin{Bmatrix} N \\ 0 \\ 0 \end{Bmatrix}=\begin{Bmatrix} EA\dfrac{du}{dx} \\ \dfrac{dv}{dx}-\psi \\ \dfrac{dw}{dx}+\theta \end{Bmatrix},\quad \begin{Bmatrix} T \\ M_y \\ M_z \end{Bmatrix}=\begin{Bmatrix} GI_{rr}\dfrac{d\phi}{dx} \\ EI_{yy}\dfrac{d\theta}{dx} \\ EI_{zz}\dfrac{d\psi}{dx} \end{Bmatrix}.$$

In the $xz-$plane problem of the figure, the non-zero displacements and rotations are u, w, and θ. The distributed forces and moments vanish except b_z. Therefore, the Bernoulli beam boundary value problem takes the form

$$\begin{Bmatrix} \dfrac{dN}{dx} \\ \dfrac{dQ_z}{dx}+b_z \\ \dfrac{dM_y}{dx}-Q_z \end{Bmatrix}=0 \;\text{ and }\; \begin{Bmatrix} N \\ 0 \\ M_y \end{Bmatrix}=\begin{Bmatrix} EA\dfrac{du}{dx} \\ \dfrac{dw}{dx}+\theta \\ EI\dfrac{d\theta}{dx} \end{Bmatrix} \;\text{ in }\; (0,L),\quad \begin{Bmatrix} N \\ Q_z \\ M_y \end{Bmatrix}=0 \;\text{ at }\; x=L,\quad \begin{Bmatrix} u \\ w \\ \theta \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \;\text{ at }\; x=0.$$

The equations for Q_z, M_y and the transverse displacement w can be solved one at a time. As the beam is statically determined, let us start with the equilibrium equations and the boundary conditions at the free end.

$$\frac{dQ_z}{dx}=-\frac{b}{L}(L-x) \;\text{ in }\; (0,L) \;\text{ and }\; Q_z(L)=0 \quad\Rightarrow\quad Q_z(x)=\frac{b}{2L}(L-x)^2,$$

$$\frac{dM_y}{dx}=Q_z=\frac{b}{2L}(L-x)^2 \;\text{ in }\; (0,L) \;\text{ and }\; M_y(L)=0 \quad\Rightarrow\quad M_y(x)=-\frac{b}{6L}(L-x)^3.$$

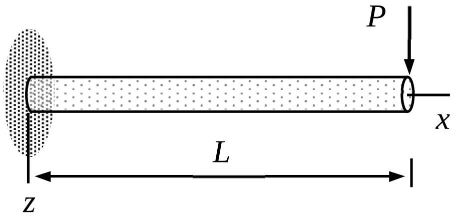
Knowing the stress resultants, displacements and rotations follow from the constitutive equations and the boundary conditions at $x=0$:

$$\frac{d\theta}{dx} = \frac{M_y}{EI} = -\frac{b}{6LEI}(L-x)^3 \quad \text{in} \quad (0,L) \quad \text{and} \quad \theta(0)=0 \quad \Rightarrow \quad \theta(x) = \frac{b}{24LEI}[(L-x)^4 - L^4],$$

$$\frac{dw}{dx} = -\theta = \frac{b}{24LEI}[(L-x)^4 - L^4] \quad \text{in} \quad (0,L) \quad \text{and} \quad w(0)=0 \quad \Rightarrow$$

$$w(x) = \frac{b}{120LEI}[L^5 - (L-x)^5 - 5L^4 x] = -\frac{bx^2}{120LEI}(-10L^3 + 10L^2 x - 5Lx^2 + x^3).$$

Find the displacement and rotation of the $xz-$plane cantilever beam of the figure according to the Bernoulli beam model. Problem parameters E, G, A $S=0$ and I are constants.



Solution

Timoshenko beam equations boil down to the Bernoulli beam equations when the Bernoulli constraints $dv/dx-\psi=0$ and $dw/dx+\theta=0$ are applied there. In practice, the constraints are used to eliminate the rotation components θ and ψ from the constitutive equations. Then the corresponding shear forces become constraint forces having no constitutive equations. If $S_y=S_z=I_{yz}=0$ (the constitutive equations for the shear forces are replaced by the Bernoulli constraints)

$$\begin{Bmatrix} \dfrac{dN}{dx}+b_x \\ \dfrac{dQ_y}{dx}+b_y \\ \dfrac{dQ_z}{dx}+b_z \end{Bmatrix}=0\,,\quad \begin{Bmatrix} N \\ 0 \\ 0 \end{Bmatrix}=\begin{Bmatrix} EA\dfrac{du}{dx} \\ \dfrac{dv}{dx}-\psi \\ \dfrac{dw}{dx}+\theta \end{Bmatrix},\quad \begin{Bmatrix} \dfrac{dT}{dx}+c_x \\ \dfrac{dM_y}{dx}-Q_z+c_y \\ \dfrac{dM_z}{dx}+Q_y+c_z \end{Bmatrix}=0\,,\quad \begin{Bmatrix} T \\ M_y \\ M_z \end{Bmatrix}=\begin{Bmatrix} GI_{rr}\dfrac{d\phi}{dx} \\ EI_{yy}\dfrac{d\theta}{dx} \\ EI_{zz}\dfrac{d\psi}{dx} \end{Bmatrix}=\begin{Bmatrix} GI_{rr}\dfrac{d\phi}{dx} \\ -EI_{yy}\dfrac{d^2w}{dx^2} \\ EI_{zz}\dfrac{d^2v}{dx^2} \end{Bmatrix}$$

giving, after elimination of the shear forces from the equilibrium equations and omitting the equations not needed in the displacement calculations, the usual forms of textbooks

$$\begin{Bmatrix} \dfrac{dN}{dx}+b_x \\ \dfrac{dT}{dx}+c_x \end{Bmatrix}=0\,,\quad \begin{Bmatrix} N \\ T \end{Bmatrix}=\begin{Bmatrix} EA\dfrac{du}{dx} \\ GI_{rr}\dfrac{d\phi}{dx} \end{Bmatrix} \text{ in } \Omega\,,\quad n\begin{Bmatrix} N \\ T \end{Bmatrix}=\begin{Bmatrix} \underline{N} \\ \underline{T} \end{Bmatrix} \text{ or } \begin{Bmatrix} u \\ \phi \end{Bmatrix}=\begin{Bmatrix} \underline{u} \\ \underline{\phi} \end{Bmatrix} \text{ on } \partial\Omega$$

$$\begin{Bmatrix} \dfrac{d^2M_z}{dx^2}+\dfrac{dc_z}{dx}-b_y \\ \dfrac{d^2M_y}{dx^2}+\dfrac{dc_y}{dx}+b_z \end{Bmatrix}=0\,,\quad \begin{Bmatrix} M_z \\ M_y \end{Bmatrix}=\begin{Bmatrix} EI_{zz}\dfrac{d^2v}{dx^2} \\ -EI_{yy}\dfrac{d^2w}{dx^2} \end{Bmatrix} \text{ in } \Omega\,,\quad \text{(altogether four conditions are needed)}$$

$$n\begin{Bmatrix} M_z \\ M_y \end{Bmatrix}=\begin{Bmatrix} \underline{M}_z \\ \underline{M}_y \end{Bmatrix} \text{ or } \begin{Bmatrix} \dfrac{dv}{dx} \\ \dfrac{dw}{dx} \end{Bmatrix}=\begin{Bmatrix} \underline{\psi} \\ -\underline{\theta} \end{Bmatrix} \text{ and } n\begin{Bmatrix} \dfrac{dM_z}{dx}+c_z \\ \dfrac{dM_y}{dx}+c_y \end{Bmatrix}=\begin{Bmatrix} -\underline{Q}_y \\ \underline{Q}_z \end{Bmatrix} \text{ or } \begin{Bmatrix} v \\ w \end{Bmatrix}=\begin{Bmatrix} \underline{v} \\ \underline{w} \end{Bmatrix} \text{ on } \partial\Omega\,.$$

In the $xz-$plane problem of the figure, the non-zero displacements and rotations are $u\,,w$ and the geometrical properties of the cross-section are A, $I_{yy}=I$. External distributed forces and moment vanish. Furthermore, axial displacement and force resultant clearly vanish. Therefore, the Bernoulli beam equations imply the boundary value problem

$$\frac{d^2M_y}{dx^2}=0\ ,\ M_y=-EI_{yy}\frac{d^2w}{dx^2}\ \text{in}\ \Omega\quad\text{and}\quad w(0)=\frac{dw}{dx}(0)=0\ ,\ M_y(L)=0,\ \text{and}\ \frac{dM_y}{dx}(L)=P$$

for the transverse displacement and the bending moment. Eliminating the bending moment gives finally

$$\frac{d^4w}{dx^4}=0\ \text{ in }\Omega,\ \ w(0)=\frac{dw}{dx}(0)=0,\ \ \frac{d^2w}{dx^2}(L)=0,\ \text{and}\ \ \frac{d^3w}{dx^3}(L)=-\frac{P}{EI}.$$

The generic solution to the differential equation $w=a+bx+cx^2+dx^3$ contains 4 parameters to be determined from the boundary conditions

$$w(0)=a=0,\ \ \frac{dw}{dx}(0)=b=0,\ \ \frac{d^2w}{dx^2}(L)=2c+6dL=0,\ \ \frac{d^3w}{dx^3}(L)=6d=-\frac{P}{EI}\quad\Rightarrow$$

$$w=\frac{P}{6EI}x^2(3L-x)\ \text{ and }\ \theta=-\frac{dw}{dx}=\frac{P}{2EI}x(x-2L).\ \text{(from the Bernoulli constraint)}$$

Alternatively, the equations can be solved easily one-by-one in their original forms as the problem is statically determined

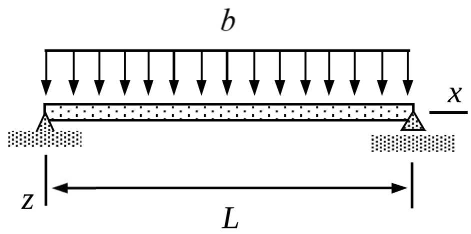
$$\frac{dQ_z}{dx}=0\ \text{ in }\ (0,L)\quad\text{and}\quad Q_z(L)=P\quad\Rightarrow\quad Q_z(x)=P,$$

$$\frac{dM_y}{dx}=Q_z=P\ \text{ in }\ (0,L)\ \text{ and }\ M_y(L)=0\quad\Rightarrow\quad M_y(x)=P(x-L),$$

$$\theta'=\frac{M_y}{EI_{yy}}=\frac{P}{EI_{yy}}(x-L)\ \text{ in }\ (0,L)\ \text{ and }\ \theta(0)=0\quad\Rightarrow\quad\theta(x)=\frac{P}{EI_{yy}}(\frac{1}{2}x^2-Lx),$$

$$\frac{dw}{dx}=-\theta=\frac{P}{EI_{yy}}(Lx-\frac{1}{2}x^2)\ \text{ in }\ (0,L)\ \text{ and }\ w(0)=0\quad\Rightarrow\quad w(0)=\frac{P}{EI_{yy}}(\frac{1}{2}Lx^2-\frac{1}{6}x^3).$$

Consider the simply supported (plane) beam of the figure of length L. Material properties E and G, cross-section properties A, $S=0$, I, and loading b are constants. Determine the deflection and rotation at the mid-point $x=L/2$ according to the Timoshenko beam model.



Solution

Beam equations of the Cartesian system

$$\begin{Bmatrix} \dfrac{dN}{dx}+b_x \\ \dfrac{dQ_y}{dx}+b_y \\ \dfrac{dQ_z}{dx}+b_z \end{Bmatrix}=0, \quad \begin{Bmatrix} N \\ Q_y \\ Q_z \end{Bmatrix}=\begin{Bmatrix} EA\dfrac{du}{dx}-ES_z\dfrac{d\psi}{dx}+ES_y\dfrac{d\theta}{dx} \\ GA(\dfrac{dv}{dx}-\psi)-GS_y\dfrac{d\phi}{dx} \\ GA(\dfrac{dw}{dx}+\theta)+GS_z\dfrac{d\phi}{dx} \end{Bmatrix}$$

$$\begin{Bmatrix} \dfrac{dT}{dx}+c_x \\ \dfrac{dM_y}{dx}-Q_z+c_y \\ \dfrac{dM_z}{dx}+Q_y+c_z \end{Bmatrix}=0, \quad \begin{Bmatrix} T \\ M_y \\ M_z \end{Bmatrix}=\begin{Bmatrix} -GS_y(\dfrac{dv}{dx}-\psi)+GS_z(\dfrac{dw}{dx}+\theta)+GI_{rr}\dfrac{d\phi}{dx} \\ ES_y\dfrac{du}{dx}-EI_{zy}\dfrac{d\psi}{dx}+EI_{yy}\dfrac{d\theta}{dx} \\ -ES_z\dfrac{du}{dx}+EI_{zz}\dfrac{d\psi}{dx}-EI_{yz}\dfrac{d\theta}{dx} \end{Bmatrix}$$

are given in the formulae collection. In $xz-$plane problem, the non-zero displacements and rotations are u, w, and θ. Geometrical properties of the cross-section are A, $S_y=S=0$, $I_{yy}=I$. External distributed forces are $b_x=0$, $b_y=0$, $b_z=b$, $c_x=c_y=c_z=0$. By taking into account the equilibrium and constitutive equations of the planar problem and the corresponding boundary conditions of the problem definition, the boundary value problem becomes

$$\begin{Bmatrix} \dfrac{dN}{dx} \\ \dfrac{dQ_z}{dx}+b \\ \dfrac{dM_y}{dx}-Q_z \end{Bmatrix}=0, \quad \begin{Bmatrix} N \\ Q_z \\ M_y \end{Bmatrix}=\begin{Bmatrix} EA\dfrac{du}{dx} \\ GA(\dfrac{dw}{dx}+\theta) \\ EI\dfrac{d\theta}{dx} \end{Bmatrix} \quad \text{in} \quad \Omega=(0,L),$$

$$\begin{Bmatrix} u \\ w \\ M_y \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{at} \quad x=0 \quad \text{and} \quad \begin{Bmatrix} N \\ w \\ M_y \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{at} \quad x=L.$$

Unless the problem is statically determined, the stress resultants are usually eliminated from equations to end up with a boundary value problem in terms of the displacement components and rotations. However, the equations can also be solved without eliminations one-by-one. First equilibrium equation and constitutive equation for the normal force

$$\frac{dN}{dx} = 0 \quad \text{in} \quad (0, L) \quad \text{and} \quad N(L) = 0 \quad \Rightarrow \quad N(x) = 0$$

$$EA\frac{du}{dx} = N = 0 \quad (0, L) \quad \text{and} \quad u(0) = 0 \quad \Rightarrow \quad u(x) = 0 \,.$$

After that, equilibrium equations for the shear force and bending moment. Notice that the integration constant for the shear force follows from the boundary conditions for the bending moment:

$$\frac{dQ_z}{dx} + b = 0 \quad \text{in} \quad (0, L) \quad \Rightarrow \quad Q_z = -bx + a \qquad (\, a = \frac{1}{2}bL \,)$$

$$\frac{dM_y}{dx} = Q_z = -bx + a \quad \text{in} \quad (0, L) \quad \text{and} \quad M_y(0) = M_y(L) = 0 \quad \Rightarrow \quad M_y(x) = \frac{1}{2}b(Lx - x^2) \,.$$

Finally, rotation and transverse displacement from the constitutive equations for the shear force and bending moment, Again, integration constant for the rotation follows from the boundary condition for displacement

$$\frac{d\theta}{dx} = \frac{M_y}{EI} \quad \text{in} \quad (0, L) \quad \Rightarrow \quad \theta(x) = \frac{b}{2EI}(L\frac{1}{2}x^2 - \frac{1}{3}x^3) + a \qquad (\, a = -\frac{bL^3}{24EI} \,)$$

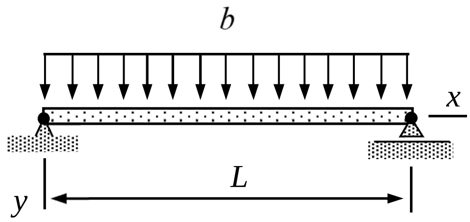
$$GA(\frac{dw}{dx} + \theta) = Q = \frac{1}{2}b(L - 2x) \quad \Rightarrow \quad w(x) = \frac{b}{2GA}(Lx - x^2) - \frac{b}{2EI}(L\frac{1}{6}x^3 - \frac{1}{12}x^4) + \frac{bL^3}{24EI}x \,.$$

$$w(x) = \frac{b}{2GA}(Lx - x^2) - \frac{b}{2EI}(L\frac{1}{6}x^3 - \frac{1}{12}x^4) + \frac{bL^3}{24EI}x \,.$$

Displacement and rotation at the center point

$$w(L/2) = \frac{bL^2}{8AG} + \frac{5bL^4}{384EI} \quad \text{and} \quad \theta(L/2) = 0 \,.$$

Consider the simply supported $xy-$plane beam of length L shown. Material properties E and G, cross-section properties A, $S=0$, I, and loading $\underline{b}$ are constants. Write down the equilibrium equations, constitutive equations, and boundary conditions according to the Bernoulli beam model. After that, solve the equations for the transverse displacement.



Solution

In $xy-$plane problem, the non-zero displacements and rotations are u, v, and ψ and the geometrical properties of the cross-section are A, $S_z=0$, $I_{zz}=I$. External distributed forces are $b_x=0$, $b_y=b$, $b_z=0$, $c_x=c_y=c_z=0$. With these, Timoshenko beam equations in the Cartesian system simplify

$$\begin{Bmatrix} \dfrac{dN}{dx} \\ \dfrac{dQ_y}{dx}+b \\ \dfrac{dM_z}{dx}+Q_y \end{Bmatrix}=0\,,\quad \begin{Bmatrix} N \\ Q_y \\ M_z \end{Bmatrix}=\begin{Bmatrix} EA\dfrac{du}{dx} \\ GA(\dfrac{dv}{dx}-\psi) \\ EI_{zz}\dfrac{d\psi}{dx} \end{Bmatrix}.$$

The equations of the Bernoulli model are obtained by replacing the constitutive equation for the shear force resultant by the corresponding Bernoulli constraint. By taking into account the boundary conditions of the problem definition, the boundary value problem becomes

$$\begin{Bmatrix} \dfrac{dN}{dx} \\ \dfrac{dQ_y}{dx}+b \\ \dfrac{dM_z}{dx}+Q_y \end{Bmatrix}=0\,,\quad \begin{Bmatrix} N \\ 0 \\ M_z \end{Bmatrix}=\begin{Bmatrix} EA\dfrac{du}{dx} \\ \dfrac{dv}{dx}-\psi \\ EI_{zz}\dfrac{d\psi}{dx} \end{Bmatrix} \text{ in } (0,L) \text{ and } \begin{Bmatrix} u \\ v \\ M_z \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \text{ at } x=0\,,\ \begin{Bmatrix} N \\ v \\ M_z \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \text{ at } x=L\,.$$

The stress resultants are usually eliminated from equations to end up with a boundary value problem in terms of the displacement components and rotations. Alternatively, the equation set can be solved in its original form. First equilibrium equation and constitutive equation for the normal force

$$\frac{dN}{dx}=0 \text{ in } (0,L) \text{ and } N(L)=0 \quad\Rightarrow\quad N(x)=0\,,$$

$$EA\frac{du}{dx}=N=0 \quad (0,L) \text{ and } \quad u(0)=0 \quad\Rightarrow\quad u(x)=0\,.$$

After that, equilibrium equations for the shear force and bending moment. Notice that the integration constant for the shear force follows from the boundary conditions for the bending moment:

$$\frac{dQ_y}{dx}+b=0 \quad \text{in} \quad (0,L) \quad \Rightarrow \quad Q_y=-bx+a \qquad (a=\frac{1}{2}bL)$$

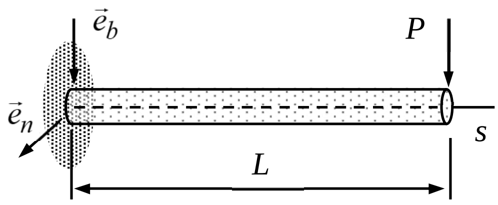
$$\frac{dM_z}{dx}=-Q_y=bx-a \quad \text{in} \quad (0,L) \quad \text{and} \quad M_z(0)=M_z(L)=0 \quad \Rightarrow \quad M_z(x)=\frac{1}{2}b(x^2-Lx)\,.$$

Finally, rotation and transverse displacement from the constitutive equations for the shear force and bending moment, Again, integration constant a for the rotation follows from the boundary condition for the displacement

$$\frac{d\psi}{dx}=\frac{M_z}{EI}=\frac{b}{2EI}(x^2-Lx) \ \text{in} \ (0,L) \quad \Rightarrow \quad \psi(x)=\frac{b}{2EI}(\frac{1}{3}x^3-L\frac{1}{2}x^2)+a \qquad (a=\frac{bL^3}{24EI})$$

$$\frac{dv}{dx}=\psi=\frac{b}{2EI}(\frac{1}{3}x^3-L\frac{1}{2}x^2)+a \Rightarrow \quad v(x)=\frac{b}{2EI}(\frac{1}{12}x^4-L\frac{1}{6}x^3)+\frac{bL^3}{24EI}x\,. \quad \leftarrow$$

Consider the cantilever beam of the figure of zero curvature and torsion $\tau = 4\pi / L$. Write down the equilibrium equations and the boundary conditions at the free end in the (s,n,b) system, and solve the boundary value problem for the stress resultants as functions of s.



Solution

In a statically determined case, stress resultants follow from the equilibrium equations and boundary conditions at the free end of the beam (or directly from a free body diagram). In (s,n,b) coordinate system, equilibrium equations are

$$\begin{Bmatrix} N' - Q_n\kappa + b_s \\ Q_n' + N\kappa - Q_b\tau + b_n \\ Q_b' + Q_n\tau + b_b \end{Bmatrix} = 0 \quad \text{and} \quad \begin{Bmatrix} T' - M_n\kappa + c_s \\ M_n' + T\kappa - M_b\tau - Q_b + c_n \\ M_b' + M_n\tau + Q_n + c_b \end{Bmatrix} = 0 .$$

In the present case $\kappa = 0$ and $\tau = 4\pi / L$ which means that the basis vector rotate two full circles around the axis when s goes from 0 to L and have the same orientations on $\partial\Omega = \{0, L\}$. As external distributed forces and moments vanish i.e. $b_s = b_n = b_b = 0$ and $c_s = c_n = c_b = 0$, equilibrium equations and the boundary conditions at the free end simplify to

$$\begin{Bmatrix} N' \\ Q_n' - Q_b\tau \\ Q_b' + Q_n\tau \end{Bmatrix} = 0 \quad \text{and} \quad \begin{Bmatrix} T' \\ M_n' - M_b\tau - Q_b \\ M_b' + M_n\tau + Q_n \end{Bmatrix} = 0 \quad \text{in} \quad \Omega = (0, L) ,$$

$$\begin{Bmatrix} N \\ Q_n \\ Q_b \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ P \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} T \\ M_n \\ M_b \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{on} \quad \partial\Omega_t = \{L\} .$$

Equations constitute a boundary value problem that can be solved by hand calculations:

$$N' = 0 \quad \text{in} \quad \Omega, \quad N(L) = 0 \quad \Rightarrow \quad N(s) = 0 . \quad \Leftarrow$$

Eliminating Q_n or Q_b from the remaining two connected force equilibrium equations and using the original equations to find the missing boundary condition gives

$$Q_n'' + Q_n\tau^2 = 0 \quad s \in \Omega , \quad Q_n(L) = 0, \quad Q_n'(L) = P\tau \quad \Rightarrow \quad (\tau = \frac{4\pi}{L})$$

$$Q_n(s) = P\sin(\tau s) \quad \text{and} \quad Q_b(s) = P\cos(\tau s) . \quad \Leftarrow$$

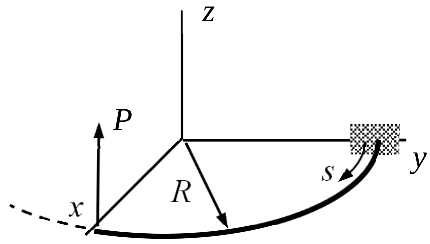
Continuing with the moment equilibrium equations in the same manner

$$T' = 0 \quad \text{in} \quad \Omega, \quad T(L) = 0 \quad \Rightarrow \quad T(s) = 0 , \quad \Leftarrow$$

$$M_b'' + M_b\tau^2 = -2P\tau\cos(\tau s) \quad \text{in} \quad \Omega\,, \quad M_b(L) = 0\,, \quad M_b'(L) = 0 \quad \Rightarrow$$

$$M_b(s) = P(L-s)\sin(\tau s) \quad \Rightarrow \quad M_n = P(s-L)\cos(\tau s)\,.$$

Consider the curved beam of the figure forming a 90-degree circular segment of radius R in the horizontal plane. Find the stress resultants $N(s)$, $Q_n(s)$, $Q_b(s)$, $T(s)$, $M_n(s)$, and $M_b(s)$. Use the equilibrium equations of the beam model in the $(s,n,b)-$ coordinate system.



Solution

In a statically determined case, stress resultants follow from the equilibrium equations and boundary conditions at the free end of the beam (or directly from a free body diagram). In (s,n,b) coordinate system, equilibrium equations are

$$\begin{Bmatrix} N' - Q_n\kappa + b_s \\ Q_n' + N\kappa - Q_b\tau + b_n \\ Q_b' + Q_n\tau + b_b \end{Bmatrix} = 0 \quad \text{and} \quad \begin{Bmatrix} T' - M_n\kappa + c_s \\ M_n' + T\kappa - M_b\tau - Q_b + c_n \\ M_b' + M_n\tau + Q_n + c_b \end{Bmatrix} = 0 .$$

For a circular beam, curvature and torsion are $\kappa = 1/R$ (constant) and $\tau = 0$.

As external distributed forces and moments vanish i.e. $b_s = b_n = b_b = c_s = c_n = c_b = 0$, equilibrium equations and the boundary conditions at the free end simplify to (notice that the external force acting at the free end is acting in the oppisite direction to $\vec{e}_b$)

$$\begin{Bmatrix} N' - Q_n / R \\ Q_n' + N / R \\ Q_b' \end{Bmatrix} = 0 \quad \text{and} \quad \begin{Bmatrix} T' - M_n / R \\ M_n' + T / R - Q_b \\ M_b' + Q_n \end{Bmatrix} = 0 \quad s \in]0, R\frac{\pi}{2}[,$$

$$\begin{Bmatrix} N \\ Q_n \\ Q_b + P \end{Bmatrix} = 0 \quad \text{and} \quad \begin{Bmatrix} T \\ M_n \\ M_b \end{Bmatrix} = 0 \quad s = R\frac{\pi}{2} .$$

Equations constitute a boundary value problem which can be solved by hand calculations without too much effort;

$$Q_b' = 0 \quad s \in]0, R\frac{\pi}{2}[\quad \text{and} \quad Q_b + P = 0 \quad s = R\frac{\pi}{2} \quad \Rightarrow \quad Q_b(s) = -P .$$

Eliminating Q_n and N from the remaining two connected force equilibrium equations and using the original equations to find the missing boundary condition give

$$N'' + \frac{1}{R^2}N = 0 \quad s \in]0, R\frac{\pi}{2}[\quad \text{and} \quad N' = N = 0 \quad s = R\frac{\pi}{2} \quad \Rightarrow \quad N(s) = 0$$

The first equilibrium equation gives

$$Q_n(s) = 0 .$$

After that, continuing with the moment equilibrium equations with the solutions to the force equilibrium equations

$$M_b' = 0 \quad s \in]0, R\frac{\pi}{2}[\quad \text{and} \quad M_b = 0 \quad s = R\frac{\pi}{2} \quad \Rightarrow \quad M_b(s) = 0 .$$

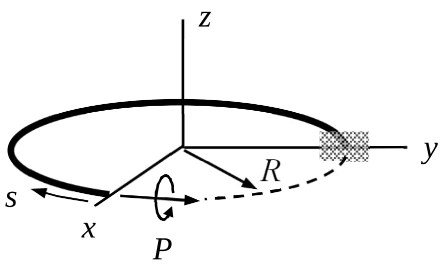
Eliminating M_n and T from the remaining two connected moment equilibrium equations and using the original equations to find the missing boundary condition gives

$$T'' + \frac{1}{R^2}T + \frac{P}{R} = 0 \quad s \in]0, R\frac{\pi}{2}[\quad \text{and} \quad T' = T = 0 \quad s = R\frac{\pi}{2} \quad \Rightarrow \quad T = PR(\sin\frac{s}{R} - 1) .$$

Knowing this, the first moment equilibrium equation gives

$$M_n(s) = RT' = RP\cos\frac{s}{R} .$$

Consider a curved beam forming ¾ of a full circle of radius R in the horizontal plane. Torque of magnitude P is acting on the free end as shown. Write down the boundary value problem for *stress resultants* and solve the equations for $N(s)$, $Q_n(s)$, $Q_b(s)$, $T(s)$, $M_n(s)$, and $M_b(s)$.



Solution

In the geometry of the figure $\tau = 0$, $\kappa = 1/R$. External distributed forces and moments vanish. Therefore the curved beam equilibrium equations of the formulae collection simplify to

$$\begin{Bmatrix} N' - Q_n / R \\ Q_n' + N / R \\ Q_b' \end{Bmatrix} = 0 \quad \text{and} \quad \begin{Bmatrix} T' - M_n / R \\ M_n' + T / R - Q_b \\ M_b' + Q_n \end{Bmatrix} = 0 \quad s \in (0, L) \quad \text{where } L = \frac{3}{2}\pi R.$$

Boundary conditions at $s = 0$ are (notice the unit outward normal to the solution domain $n = -1$, $\vec{e}_s$ is pointing to the direction of *s*, and the component of the given moment on $\vec{e}_s$ is negative)

$$\begin{Bmatrix} N \\ Q_n \\ Q_b \end{Bmatrix} = 0 \quad \text{and} \quad \begin{Bmatrix} -T + P \\ M_n \\ M_b \end{Bmatrix} = 0 \quad s = 0.$$

Solution to the boundary values problem for Q_b

$$Q_b' = 0 \quad s \in (0, L) \quad \text{and} \quad Q_b = 0 \quad s = 0 \quad \Rightarrow \quad Q_b(s) = 0.$$

Solution to the connected boundary value problems for Q_n and N

$$N' - \frac{1}{R} Q_n = 0, \; Q_n' + \frac{1}{R} N = 0 \quad s \in (0, L), \quad Q_n = 0 \text{ and } N = 0 \quad s = 0 \quad \Rightarrow$$

$$N'' + \frac{1}{R^2} N = 0 \quad s \in (0, L) \quad \text{and} \quad N = 0, \; N' = 0 \text{ at } s = 0 \quad \Rightarrow$$

$$N(s) = 0 \quad \text{and} \quad Q_n(s) = 0.$$

Solution to the boundary value problem for M_b

$$M_b' = 0 \quad s \in (0, L) \quad \text{and} \quad M_b = 0 \quad s = 0 \quad \Rightarrow \quad M_b(s) = 0.$$

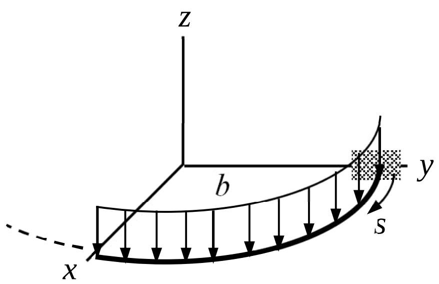
Solution to the connected boundary value problem for M_n and T

$$T' - \frac{1}{R} M_n = 0 \quad \text{and} \quad M_n' + \frac{1}{R} T = 0 \quad s \in (0, L), \quad T = P \text{ and } M_n = 0 \quad s = 0 \quad \Rightarrow$$

$$RT'' + \frac{1}{R} T = 0 \quad s \in (0, L), \quad T = P \text{ and } T' = 0 \quad \Rightarrow$$

$$T(s) = P\cos(\frac{s}{R}) \quad \text{and} \quad M_n(s) = -P\sin(\frac{s}{R}) .$$

Consider the curved beam of the figure forming a 90-degree circular segment of radius R in the horizontal plane. Find the stress resultants $N(s)$, $Q_n(s)$, $Q_b(s)$, $T(s)$, $M_n(s)$, and $M_b(s)$. Use the equilibrium equations of the beam model in the $(s,n,b)-$coordinate system. The distributed constant load of magnitude b is acting to the negative direction of the $z-$axis.



Solution

In a statically determined case, stress resultants follow from the equilibrium equations and boundary conditions at the free end of the beam (or directly from a free body diagram). In $(s,n,b)-$coordinate system, equilibrium equations are

$$\begin{Bmatrix} N' - Q_n\kappa + b_s \\ Q_n' + N\kappa - Q_b\tau + b_n \\ Q_b' + Q_n\tau + b_b \end{Bmatrix} = 0 \quad \text{and} \quad \begin{Bmatrix} T' - M_n\kappa + c_s \\ M_n' + T\kappa - M_b\tau - Q_b + c_n \\ M_b' + M_n\tau + Q_n + c_b \end{Bmatrix} = 0 .$$

For a circular beam, curvature and torsion are $\kappa = 1/R$ (constant) and $\tau = 0$. As external distributed forces and moments $b_s = b_n = c_s = c_n = c_b = 0$ and $b_b = b$, equilibrium equations and the boundary conditions at the free end simplify to (here $L = \pi R/2$)

$$\begin{Bmatrix} N' - Q_n/R \\ Q_n' + N/R \\ Q_b' + b \end{Bmatrix} = 0 \quad \text{and} \quad \begin{Bmatrix} T' - M_n/R \\ M_n' + T/R - Q_b \\ M_b' + Q_n \end{Bmatrix} = 0 \quad \text{in} \quad (0,L)$$

$$\begin{Bmatrix} N \\ Q_n \\ Q_b \end{Bmatrix} = 0 \quad \text{and} \quad \begin{Bmatrix} T \\ M_n \\ M_b \end{Bmatrix} = 0 \quad \text{at} \quad s = L .$$

Equations constitute a boundary value problem which can be solved one equation at a time by following certain order

$$Q_b' = -b \quad \text{in} \quad (0,L) \quad \text{and} \quad Q_b(L) = 0 \quad \Rightarrow \quad Q_b(s) = b(L-s) .$$

Eliminating Q_n and N from the remaining two connected force equilibrium equations and using the original equations to find the missing boundary condition gives

$$N'' + \frac{1}{R^2}N = 0 \quad \text{in} \quad (0,L) \quad \text{and} \quad N'(L) = N(L) = 0 \quad \Rightarrow \quad N(s) = 0 .$$

Knowing the result above, the first equilibrium equation gives

$$Q_n(s) = 0 .$$

After that, continuing with the moment equilibrium equations with the already known solutions to the force equilibrium equations

$$M_b' = -Q_n = 0 \quad \text{in} \quad (0,L) \quad \text{and} \quad M_b(L) = 0 \quad \Rightarrow \quad M_b(s) = 0 .$$

Eliminating M_n and T from the remaining two connected moment equilibrium equations and using the original equations to find the missing boundary condition gives $(L = \pi R / 2)$

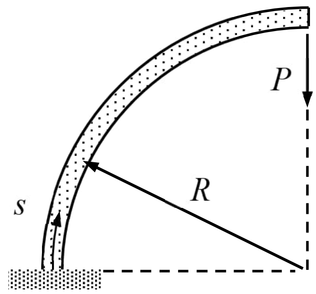
$$T'' + \frac{1}{R^2} T = \frac{1}{R} Q_b = \frac{b}{R}(L-s) \quad \text{in} \quad (0,L) \quad \text{and} \quad T'(L) = T(L) = 0 \quad \Rightarrow$$

$$T(s) = -bR^2 \cos(\frac{s}{R}) + bR(L-s) .$$

Knowing this, the first moment equilibrium equation gives

$$M_n(s) = RT' = bR^2 \sin(\frac{s}{R}) - bR^2 .$$

Consider the curved planar Timoshenko beam shown in the figure. Write down the boundary value problem for $u(s)$, $v(s)$, $\psi(s)$, $N(s)$, $Q_n(s)$, and $M_b(s)$. Also, solve the equations for displacement $v(s)$ and rotation $\psi(s)$. The properties of the cross-section A, $S_b = 0$, $I_{bb} = I$, and material parameters E, G are constants. Curvature is $\kappa = 1/R$ and torsion $\tau = 0$.



Solution

In (s,n,b) coordinate system, equilibrium equations and constitutive equations of the beam model are

$$\begin{Bmatrix} N' - Q_n\kappa + b_s \\ Q_n' + N\kappa - Q_b\tau + b_n \\ Q_b' + Q_n\tau + b_b \end{Bmatrix} = 0, \quad \begin{Bmatrix} T' - M_n\kappa + c_s \\ M_n' + T\kappa - M_b\tau - Q_b + c_n \\ M_b' + M_n\tau + Q_n + c_b \end{Bmatrix} = 0,$$

$$\begin{Bmatrix} N \\ Q_n \\ Q_b \end{Bmatrix} = \begin{Bmatrix} EA(u' - v\kappa) \\ GA(v' + u\kappa - w\tau - \psi) \\ GA(w' + v\tau + \theta) \end{Bmatrix}, \quad \begin{Bmatrix} T \\ M_n \\ M_b \end{Bmatrix} = \begin{Bmatrix} GI_{rr}(\phi' - \theta\kappa) \\ EI_{nn}(\theta' + \phi\kappa - \psi\tau) \\ EI_{bb}(\psi' + \theta\tau) \end{Bmatrix}.$$

In a planar problem, it is enough to consider force equilibrium equations in the $s-$ and $n-$ directions and moment equilibrium equation in the $b-$direction. As the present problem is statically determined, it is possible to solve for the stress resultants first ($L = \pi R/2$)

$$\begin{Bmatrix} N' - Q_n / R \\ Q_n' + N / R \\ M_b' + Q_n \end{Bmatrix} = 0 \text{ in } (0,L) \text{ and } \begin{Bmatrix} N \\ Q_n - P \\ M_b \end{Bmatrix} = 0 \text{ at } s = L.$$

Solution to the shear stress resultant Q_n can be obtained by eliminating the normal stress resultant N from the first two connected equilibrium equations. The missing boundary condition for the second order differential equation follows from the second equilibrium equations at the endpoint (there $N = 0$)

$$Q_n'' + \frac{1}{R^2} Q_n = 0 \text{ in } (0,L), \; Q_n'(L) = 0, \text{ and } Q_n(L) - P = 0 \quad \Rightarrow \quad Q_n(s) = P\sin(\frac{s}{R}).$$

Knowing the shear stress resultant, the second and third equilibrium equations give

$$Q_n' + N / R = 0 \text{ in } (0,L) \quad \Rightarrow \quad N(s) = -P\cos(\frac{s}{R}),$$

$$M_b' + Q_n = 0 \text{ in } \Omega \text{ and } M_b(L) = 0 \quad \Rightarrow \quad M_b = PR\cos(\frac{s}{R}).$$

As the force resultants are now known, displacements and rotations follow from the constitutive equations (and the corresponding boundary conditions)

$$\begin{Bmatrix} N \\ Q_n \\ M_b \end{Bmatrix} = \begin{Bmatrix} EA(u' - v / R) \\ GA(v' + u / R - \psi) \\ EI\psi' \end{Bmatrix} \text{ in } (0, L) \quad \text{and} \quad \begin{Bmatrix} u(0) \\ v(0) \\ \psi(0) \end{Bmatrix} = 0 .$$

Let us start with the last constitutive equation

$$EI\psi' = M_b = PR\cos(\frac{s}{R}) \text{ and } \psi(0) = 0 \quad \Rightarrow \quad \psi(s) = \frac{PR^2}{EI}\sin(\frac{s}{R}) .$$

Elimination of the axial displacement from the first two constitutive equations with the known expression of the rotation (the missing boundary condition follows from the second constitutive equation at $s = 0$) gives

$$v'' + \frac{1}{R^2}v = P(\frac{R^2}{EI} + \frac{1}{GA} + \frac{1}{EA})\frac{1}{R}\cos(\frac{s}{R}) , \quad v(0) = 0 , \ v'(0) = 0 \quad \Rightarrow$$

$$v(s) = \frac{P}{2}(\frac{R^2}{EI} + \frac{1}{GA} + \frac{1}{EA})s\sin(\frac{s}{R}) .$$

Finally, using the second constitutive equation

$$u(s) = (\frac{PR}{GA} + \frac{PR^3}{EI})\sin(\frac{s}{R}) - \frac{PR}{2}(\frac{R^2}{EI} + \frac{1}{GA} + \frac{1}{EA})[\sin(\frac{s}{R}) + \frac{s}{R}\cos(\frac{s}{R})] .$$

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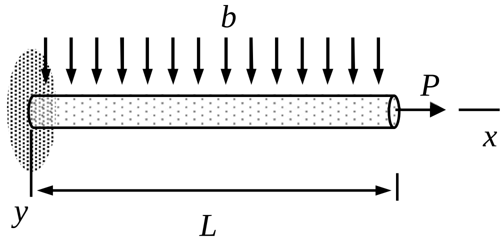
Assignment 1 (2p)



Find the stress resultants of the $xy-$plane cantilever beam of the figure. Use the beam equilibrium equations and natural boundary conditions in the Cartesian system

$$\begin{Bmatrix} \dfrac{dN}{dx}+b_x \\ \dfrac{dQ_y}{dx}+b_y \\ \dfrac{dQ_z}{dx}+b_z \end{Bmatrix}=0 \quad \text{in} \quad (0,L) \quad \text{and} \quad n\begin{Bmatrix} N \\ Q_y \\ Q_z \end{Bmatrix}-\begin{Bmatrix} \underline{N} \\ \underline{Q}_y \\ \underline{Q}_z \end{Bmatrix}=0 \quad \text{at} \quad x=L$$

$$\begin{Bmatrix} \dfrac{dT}{dx}+c_x \\ \dfrac{dM_y}{dx}-Q_z+c_y \\ \dfrac{dM_z}{dx}+Q_y+c_z \end{Bmatrix}=0 \quad \text{in} \quad (0,L) \quad \text{and} \quad n\begin{Bmatrix} T \\ M_y \\ M_z \end{Bmatrix}-\begin{Bmatrix} \underline{T} \\ \underline{M}_y \\ \underline{M}_z \end{Bmatrix}=0 \quad \text{at} \quad x=L$$

Solution

In a statically determinate case, it is possible to solve for the stress resultants from a boundary value problem consisting of the equilibrium equations and the natural boundary conditions. The three differential equations and their boundary conditions are (when written in the standard form $something=0$)

$$\frac{dN}{dx}=0 \quad \text{in} \quad (0,L) \quad \text{and} \quad N-P=0 \quad \text{at} \quad x=L$$

$$\frac{dQ_y}{dx}+b=0 \quad \text{in} \quad (0,L) \quad \text{and} \quad Q_y=0 \quad \text{at} \quad x=L$$

$$\frac{dM_z}{dx}+Q_y=0 \quad \text{in} \quad (0,L) \quad \text{and} \quad M_z=0 \quad \text{at} \quad x=L$$

Solution to the boundary value problem is

$N(x)=P$, ←

$Q_y(x)=b(L-x)$, ←

$$M_z(x) = \frac{b}{2}(L-x)^2 .$$

Use the Mathematica notebook Beam.nb of the homepage to check your solution!

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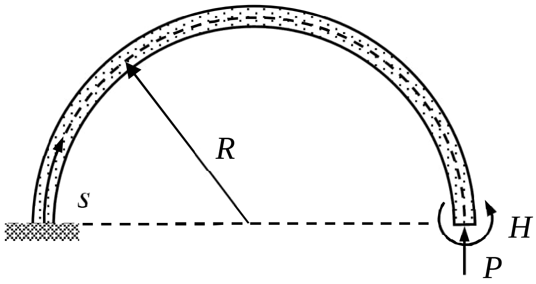
Assignment 2 (2p)



Consider a circular planar beam of radius R loaded by a point force and moment at the free end as shown in the figure. Write the equilibrium equations and boundary conditions giving as their solution stress resultant components $N(s)$, $Q_n(s)$, and $M_b(s)$. Start with the equilibrium equations and natural boundary conditions of the beam model in the curvilinear $(s,n,b)-$coordinate system ($L=\pi R$)

$$\begin{Bmatrix} \dfrac{dN}{ds}-Q_n\kappa+b_s \\ \dfrac{dQ_n}{ds}+N\kappa-Q_b\tau+b_n \\ \dfrac{dQ_b}{ds}+Q_n\tau+b_b \end{Bmatrix}=0 \quad \text{in } (0,L), \qquad n\begin{Bmatrix} N \\ Q_n \\ Q_b \end{Bmatrix}-\begin{Bmatrix} \underline{N} \\ \underline{Q}_n \\ \underline{Q}_b \end{Bmatrix}=0 \quad \text{at } s=L,$$

$$\begin{Bmatrix} \dfrac{dT}{ds}-M_n\kappa+c_s \\ \dfrac{dM_n}{ds}+T\kappa-M_b\tau-Q_b+c_n \\ \dfrac{dM_b}{ds}+M_n\tau+Q_n+c_b \end{Bmatrix}=0 \quad \text{in } (0,L), \qquad n\begin{Bmatrix} T \\ M_n \\ M_b \end{Bmatrix}-\begin{Bmatrix} \underline{T} \\ \underline{M}_n \\ \underline{M}_b \end{Bmatrix}=0 \quad \text{at } s=L.$$

Solution

In the present problem $L=\pi R$ and the other parameters in the equilibrium equations and the natural boundary conditions (planar problem)

$\tau=0$, $\kappa=\dfrac{1}{R}$, $b_s=0$, $b_n=0$, $c_b=0$

$n=1$, $\underline{N}=-P$, $\underline{Q}_n=0$, $\underline{M}_b=-H$.

In a statically determinate case, it is possible to solve for the stress resultants from a boundary value problem consisting of the equilibrium equations and the natural boundary conditions. The three differential equations of the planar problem and their boundary conditions are (when written in the standard form $something=0$)

$$\frac{dN}{ds} - \frac{Q_n}{R} = 0 \quad \text{in } (0, L) \quad \text{and} \quad N + P = 0 \quad \text{at } s = L$$

$$\frac{dQ_n}{ds} + \frac{N}{R} = 0 \quad \text{in } (0, L) \quad \text{and} \quad Q_n = 0 \quad \text{at } s = L$$

$$\frac{dM_b}{ds} + Q_n = 0 \quad \text{in } (0, L) \quad \text{and} \quad M_b + H = 0 \quad \text{at } s = L$$

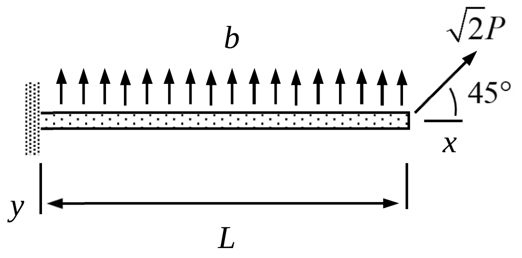
Finally, use the Mathematica notebook Beam.nb of the homepage to find the solution

$$N(s) = P\cos(\frac{s}{R}), \quad Q_n(s) = -P\sin(\frac{s}{R}), \quad M_b(s) = -H - PR - PR\cos(\frac{s}{R}).$$

Name________________________ Student number________________

Assignment 3 (4p)

Consider the $xy-$plane beam of length L shown. Material properties E and G, cross-section properties A, I are constants, and $S=0$. Write down the boundary value problem according to the Bernoulli beam model in terms of axial displacement $u(x)$ and transverse displacement $v(x)$. Start with the generic equilibrium and constitutive equations of the Timoshenko beam model



$$\begin{Bmatrix} \dfrac{dN}{dx}+b_x \\ \dfrac{dQ_y}{dx}+b_y \\ \dfrac{dQ_z}{dx}+b_z \end{Bmatrix}=0, \quad \begin{Bmatrix} N \\ Q_y \\ Q_z \end{Bmatrix}=\begin{Bmatrix} EA\dfrac{du}{dx}-ES_z\dfrac{d\psi}{dx}+ES_y\dfrac{d\theta}{dx} \\ GA(\dfrac{dv}{dx}-\psi)-GS_y\dfrac{d\phi}{dx} \\ GA(\dfrac{dw}{dx}+\theta)+GS_z\dfrac{d\phi}{dx} \end{Bmatrix}$$

$$\begin{Bmatrix} \dfrac{dT}{dx}+c_x \\ \dfrac{dM_y}{dx}-Q_z+c_y \\ \dfrac{dM_z}{dx}+Q_y+c_z \end{Bmatrix}=0, \quad \begin{Bmatrix} T \\ M_y \\ M_z \end{Bmatrix}=\begin{Bmatrix} -GS_y(\dfrac{dv}{dx}-\psi)+GS_z(\dfrac{dw}{dx}+\theta)+GI_{rr}\dfrac{d\phi}{dx} \\ ES_y\dfrac{du}{dx}-EI_{zy}\dfrac{d\psi}{dx}+EI_{yy}\dfrac{d\theta}{dx} \\ -ES_z\dfrac{du}{dx}+EI_{zz}\dfrac{d\psi}{dx}-EI_{yz}\dfrac{d\theta}{dx} \end{Bmatrix}$$

Solution

Timoshenko beam equations boil down to the Bernoulli beam equations when the Bernoulli constraints $dv/dx-\psi=0$ and $dw/dx+\theta=0$ are applied there. If $S_y=S_z=I_{yz}=0$ one may just replace the constitutive equations for the shear stress resultants by the Bernoulli constraints to get the Bernoulli model equilibrium and constitutive equations (elimination of rotations and bending moments gives the well-known fourth order beam bending equations of textbooks)

In $xy-$plane problem, the non-zero displacements and rotations are u, v, and ψ. Geometrical properties of the cross-section are A, $S_z=S=0$, $I_{zz}=I$. External distributed forces are $b_x=0$, $b_y=-b$, $b_z=0$, $c_x=c_y=c_z=0$. With these selections, equilibrium equations, constitutive equations, and boundary conditions of the planar problem take the forms

$$\begin{Bmatrix} \dfrac{dN}{dx} \\ \dfrac{dQ_y}{dx} - b \\ \dfrac{dM_z}{dx} + Q_y \end{Bmatrix} = 0 \quad \text{and} \quad \begin{Bmatrix} N \\ 0 \\ M_z \end{Bmatrix} = \begin{Bmatrix} EA\dfrac{du}{dx} \\ \dfrac{dv}{dx} - \psi \\ EI_{zz}\dfrac{d\psi}{dx} \end{Bmatrix} \quad \text{in} \quad (0, L),$$

$$\begin{Bmatrix} u \\ v \\ \psi \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{at} \quad x = 0 \quad \text{and} \quad \begin{Bmatrix} N - P \\ Q_y + P \\ M_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{at} \quad x = L.$$

To get a boundary value problem in terms of the axial displacement $u(x)$, transverse displacement $v(x)$, stress resultants and rotation are eliminated to end up with

$$EA\frac{d^2u}{dx^2} = 0 \quad \text{in} \quad (0, L), \quad EA\frac{du}{dx} - P = 0 \quad \text{at} \ x = L, \ \text{and} \quad u = 0 \quad \text{at} \quad x = 0. \quad \Leftarrow$$

$$\frac{d^2}{dx^2}(EI_{zz}\frac{d^2v}{dx^2}) + b = 0 \quad \text{in} \quad (0, L), \quad -\frac{d}{dx}(EI_{zz}\frac{d^2v}{dx^2}) + P = 0 \quad \text{at} \quad x = L, \quad EI_{zz}\frac{d^2v}{dx^2} = 0 \quad \text{at} \quad x = L,$$

$$\frac{dv}{dx} = 0 \quad \text{at} \quad x = 0, \ \text{and} \ v = 0 \quad \text{at} \quad x = 0. \quad \Leftarrow$$

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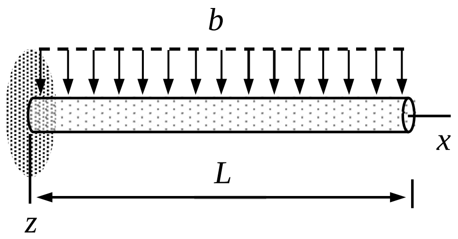
Assignment 4 (4p)

Find the stress resultants N, Q_z, M_y and the displacement and rotation components u, w, θ of the $xz-$plane cantilever beam of the figure according to the Timoshenko beam model. Problem parameters E, G, A, $S_y = 0$ and $I_{yy} = I$ are constants. Start with the generic equilibrium and constitutive equations for the beam model.



Solution

Timoshenko beam equilibrium and constitutive equations in the Cartesian $(x,y,z)-$coordinate system are

$$\begin{Bmatrix} \dfrac{dN}{dx}+b_x \\ \dfrac{dQ_y}{dx}+b_y \\ \dfrac{dQ_z}{dx}+b_z \end{Bmatrix} = 0, \quad \begin{Bmatrix} N \\ Q_y \\ Q_z \end{Bmatrix} = \begin{Bmatrix} EA\dfrac{du}{dx} - ES_z\dfrac{d\psi}{dx} + ES_y\dfrac{d\theta}{dx} \\ GA(\dfrac{dv}{dx}-\psi) - GS_y\dfrac{d\phi}{dx} \\ GA(\dfrac{dw}{dx}+\theta) + GS_z\dfrac{d\phi}{dx} \end{Bmatrix},$$

$$\begin{Bmatrix} \dfrac{dT}{dx}+c_x \\ \dfrac{dM_y}{dx}-Q_z+c_y \\ \dfrac{dM_z}{dx}+Q_y+c_z \end{Bmatrix} = 0, \quad \begin{Bmatrix} T \\ M_y \\ M_z \end{Bmatrix} = \begin{Bmatrix} -GS_y(\dfrac{dv}{dx}-\psi) + GS_z(\dfrac{dw}{dx}+\theta) + GI_{rr}\dfrac{d\phi}{dx} \\ ES_y\dfrac{du}{dx} - EI_{zy}\dfrac{d\psi}{dx} + EI_{yy}\dfrac{d\theta}{dx} \\ -ES_z\dfrac{du}{dx} + EI_{zz}\dfrac{d\psi}{dx} - EI_{yz}\dfrac{d\theta}{dx} \end{Bmatrix}.$$

In the $xz-$plane problem of the figure, geometrical properties of the cross-section are A, $S_y = 0$, and $I_{yy} = I$ and the external distributed force and moment components $b_x = b_y = 0$, $b_z = b$, and $c_x = c_y = c_z = 0$. The 3 equilibrium equations, 3 constitutive equations, and 6 boundary conditions for $N(x)$, $Q_z(x)$, $M_y(x)$, $u(x)$, $w(x)$, and $\theta(x)$ simplify to

$$\begin{Bmatrix} \dfrac{dN}{dx} \\ \dfrac{dQ_z}{dx}+b \\ \dfrac{dM_y}{dx}-Q_z \end{Bmatrix} = 0, \quad \begin{Bmatrix} N - EA\dfrac{du}{dx} \\ Q_z - GA(\dfrac{dw}{dx}+\theta) \\ M_y - EI\dfrac{d\theta}{dx} \end{Bmatrix} = 0 \text{ in } (0,L), \quad \begin{Bmatrix} N \\ Q_z \\ M_y \end{Bmatrix} = 0 \text{ at } x = L, \quad \begin{Bmatrix} u \\ w \\ \theta \end{Bmatrix} = 0 \text{ at } x = 0.$$

As stress resultants are known at the free end, one may start with the stress resultants

$$\frac{dN}{dx}=0 \quad \text{in} \quad (0,L) \quad \text{and} \quad N=0 \quad \text{at} \quad x=L \quad \Rightarrow \quad N(x)=0\,,$$

$$\frac{dQ_z}{dx}+b=0 \quad \text{in} \ (0,L) \quad \text{and} \quad Q_z=0 \quad \text{at} \quad x=L \quad \Rightarrow \quad Q_z(x)=b(L-x)\,,$$

$$\frac{dM_y}{dx}=Q_z=b(L-x) \ \text{in} \ (0,L) \quad \text{and} \quad M_y=0 \quad \text{at} \quad x=L \quad \Rightarrow \quad M_y(x)=-\frac{1}{2}b(L-x)^2\,,$$

After that, solution to the displacement components follow from the constitutive equations and the remaining boundary conditions at the clamped edge:

$$\frac{du}{dx}=\frac{N}{EA}=0 \quad \text{in} \quad (0,L) \ \text{and} \ u=0 \quad \text{at} \quad x=0 \quad \Rightarrow \quad u(x)=0\,,$$

$$\frac{d\theta}{dx}=\frac{M_y}{EI}=-\frac{b}{2EI}(L-x)^2 \ \text{in} \ (0,L) \quad \text{and} \quad \theta=0 \quad \text{at} \quad x=0 \quad \Rightarrow \quad \theta(x)=-\frac{bx}{6EI}(3L^2-3Lx+x^2)\,,$$

$$\frac{dw}{dx}=\frac{Q_z}{GA}-\theta=\frac{b}{GA}(L-x)+\frac{b}{2EI}(L^2x-Lx^2+\frac{1}{3}x^3) \ \text{in} \ (0,L) \quad \text{and} \quad w=0 \quad \text{at} \quad x=0 \quad \Rightarrow$$

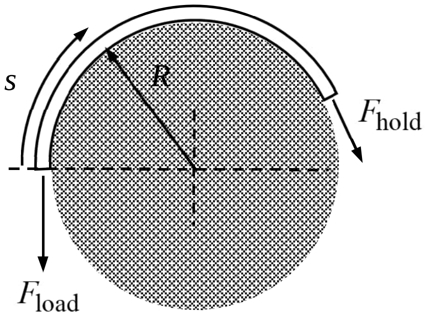
$$w(x)=\frac{bx}{24}[\frac{1}{GA}(24L-12x)+\frac{x}{EI}(6L^2-4Lx+x^2)]\,.$$

In a statically determinate case, equilibrium equation can be solved first for the stress resultants. After that, displacements follow from the constitutive equations. Often, the stress resultants are eliminated from the equilibrium equations and boundary conditions by using the constitutive equations to end up with second order diffrential equations to the displacement components only. Although, the number of equations can be reduced in this manner, finding the solution with the original set of equations is more straightforward.

Name________________________ Student number________________

Assignment 5 (4p)

Derive the range of the hold force for a rope of length L around a bollard so that equilibrium is possible. Assume that the rope is inextensible in the direction of the mid-curve and flexible with respect to bending ($Q_n = M_b = 0$). Consider the fully developed Coulomb friction when the load is about to move up or down. Start with the equilibrium equations of beam in $(s,n,b)-$system. *Hint*: External force b_n is the unknown of the problem and $b_s = \pm\mu b_n$ is opposite to the pending motion (μ is the coefficient of friction).



$$\begin{Bmatrix} \dfrac{dN}{ds} - Q_n\kappa + b_s \\ \dfrac{dQ_n}{ds} + N\kappa - Q_b\tau + b_n \\ \dfrac{dQ_b}{ds} + Q_n\tau + b_b \end{Bmatrix} = 0, \quad \begin{Bmatrix} \dfrac{dT}{ds} - M_n\kappa + c_s \\ \dfrac{dM_n}{ds} + T\kappa - M_b\tau - Q_b + c_n \\ \dfrac{dM_b}{ds} + M_n\tau + Q_n + c_b \end{Bmatrix} = 0.$$

Solution

For a circular beam, curvature and torsion are $\kappa = 1/R$ (constant) and $\tau = 0$. The distributed external force components b_n and b_s describe interaction with the bollard. Contact force in the normal direction b_n is an unknown of the problem and $b_s = \pm\mu b_n$ in which the sign should be chosen so that the friction force is opposite to the pending motion.

In the problem, $Q_n = M_b = 0$ and the equilibrium equations and the boundary condition at $s = 0$ simplify to (the remaining are of the form $0 = 0$)

$$\frac{dN}{ds} + b_s = 0 \quad \text{and} \quad N\kappa + b_n = 0 \quad s \in (0,L), \quad N(0) = F_{\text{load}}$$

Clearly, b_n needs to be negative. Therefore, assuming that the load is about to move downwards, friction acts in the direction of *s*, and

$$\frac{dN}{ds} - \mu b_n = 0 \quad \text{and} \quad N\kappa + b_n = 0 \quad s \in (0,L), \quad N(0) = F_{\text{load}} \quad \Leftrightarrow \quad N_-(s) = F_{\text{load}} \exp(-\frac{\mu}{R}s).$$

Assuming that the load is about to move upwards, friction acts in the direction opposite to *s*, and

$$\frac{dN}{ds} + \mu b_n = 0 \quad \text{and} \quad N\kappa + b_n = 0 \quad s \in (0,L), \quad N(0) = F_{\text{load}} \quad \Leftrightarrow \quad N_+(s) = F_{\text{load}} \exp(\frac{\mu}{R}s).$$

The corresponding hold forces at $s = L$

$$F_{\text{hold}}^{-} = N^{-}(L) = F_{\text{load}} \exp(-\mu \frac{L}{E}) \quad \text{and} \quad F_{\text{hold}}^{+} = N^{+}(L) = F_{\text{load}} \exp(\mu \frac{L}{E}) .$$

Slipping does not occur, if the hold force satisfies $F_{\text{hold}}^{-} \le F_{\text{hold}} \le F_{\text{hold}}^{+}$ or

$$\exp(-\mu \frac{L}{R}) \le \frac{F_{\text{hold}}}{F_{\text{load}}} \le \exp(\mu \frac{L}{R}) .$$

Assuming for example one full circle around the bollard $L/R = 2\pi$ and the friction coefficient $\mu = 1/2$, one obtains the range $F_{\text{load}} 0.043 \le F_{\text{hold}} \le 23 F_{\text{load}}$.

MEC-E8003

Beam, Plate and Shell Models

2024

WEEK 13: PLATES

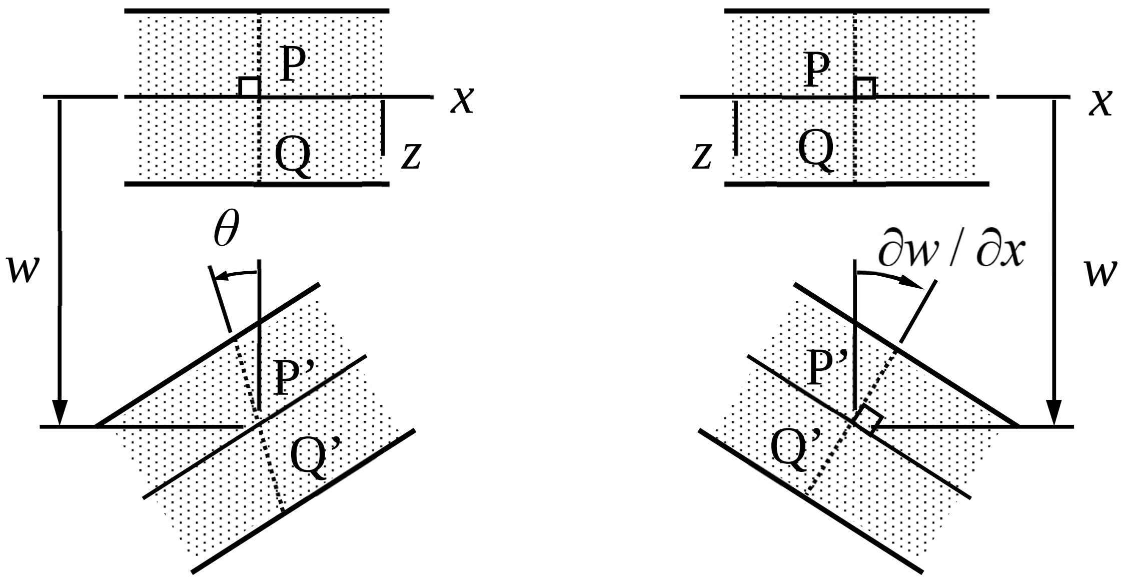
5 PLATE

LEARNING OUTCOMES

Students are able to solve the weekly lecture problems, home problems, and exercise problems on the plate model:

- ☐ Reissner-Mindlin and Kirchhoff plate models.
- ☐ Derivation of the plate equations by using the principle of virtual work, integration by parts, and the fundamental lemma of variation calculus. Plate equilibrium and constitutive equations in their tensor forms.
- ☐ Component representations of the plate equations in $(x,y,z)-$ and $(r,\phi,n)-$coordinate systems.
- ☐ Approximate series solutions to plate equations.

5.1 PLATE MODELS



Kinematic assumption: Line segments perpendicular to the mid/reference-plane remain straight in deformation (Reissner-Mindlin) and perpendicular to the mid-plane (Kirchhoff). Then, line segments move as rigid bodies according to $\vec{u} = \vec{u}_0 + \vec{\theta}_0 \times \vec{\rho}$.

Kinetic assumption: Normal stress in the thickness direction is negligible.

The kinematic assumption means that the normal line segments to the mid-plane move as rigid bodies in deformation. In terms of displacement of the translation point $z = 0$ and small rotation of the line segments, displacement of a particle (x, y, z) is given by $\vec{u} = (u\vec{i} + v\vec{j} + w\vec{k}) + (\phi\vec{i} + \theta\vec{j}) \times (z\vec{k})$ in which the translation and rotation components depend on the mid-plane position (x, y). The kinetic assumption of the plate model is $\sigma_{zz} = 0$.

In the Kirchhoff model, line segments are assumed to remain normal to the mid-plane in deformation which brings the *Kirchhoff constraints* $(\nabla w + \vec{\omega}_0 = 0,\ \ \vec{\omega}_0 = \vec{\theta}_0 \times \vec{k})$

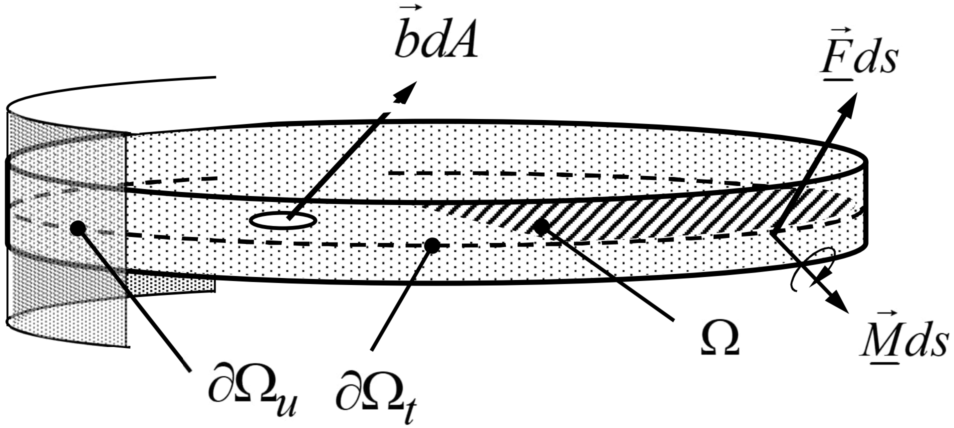
$$\gamma_{xz} = \frac{\partial w}{\partial x} + \theta = 0 \quad \text{and} \quad \gamma_{yz} = \frac{\partial w}{\partial y} - \phi = 0.$$

The modeling error in the Kirchhoff plate model is larger than that of the Reissner-Mindlin plate model!

BENDING MODE OF KIRCHHOFF PLATE

Kirchhoff model is the practical choice for the bending of thin isotropic and homogeneous simply supported plates. Assuming that the origin of the transverse axis is placed at the mid-plane, the boundary value problem for bending of a simply supported plate loaded by distributed transverse force b_n is given by

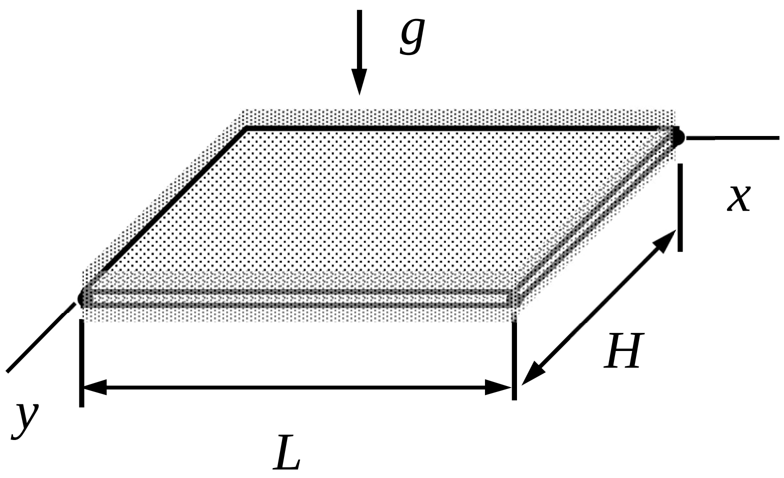
$$\nabla_0^2\nabla_0^2 w - \frac{b_n}{D} = 0 \quad \text{in} \quad \Omega,$$



$$w = 0 \quad \text{and} \quad D\frac{\partial^2 w}{\partial n^2} = 0 \quad \text{on} \quad \partial\Omega$$

in which $D = \dfrac{Et^3}{12(1-\nu^2)}$ is the bending stiffness of the plate and $\nabla = \nabla_0 + \vec{e}_n \dfrac{\partial}{\partial n}$.

EXAMPLE 5.1 Consider bending of a simply supported Kirchhoff plate in the rectangle domain $\Omega=(0,L)\times(0,H)$. Thickness t, Young's modulus E, and Poisson's ratio ν, and distributed load b in direction of $z-$axis are constants. Derive the double sine series solution of the form $w(x,y)=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} w_{ij}\sin(i\pi x/L)\sin(j\pi y/H)$.



Answer $w_{ij}=16\dfrac{b}{D}\dfrac{1}{ij\pi^6}[(\dfrac{i}{L})^2+(\dfrac{j}{H})^2]^{-2}\quad i,j\in\{1,3,5,\ldots\},\quad w_{ij}=0$ otherwise.

The double sine series satisfies the simply supported boundary conditions ‘a priori’. Elimination of the stress resultants gives the fourth order differential equation for the transverse displacement

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{b}{D} \ , \quad \text{where} \quad D = \frac{Et^3}{12(1-\nu^2)}.$$

The series solution is based on the orthogonality properties of the sine and cosine functions (like)

Cronecker delta

$$\int_0^L \sin(i\pi\frac{x}{L})\sin(j\pi\frac{x}{L})dx = \delta_{ij}\frac{L}{2} \quad \text{and} \quad \int_0^L \sin(i\pi\frac{x}{L})dx = \frac{L}{i\pi}[1-(-1)^i]$$

$$\int_0^H \sin(i\pi\frac{y}{H})\sin(j\pi\frac{y}{H})dy = \delta_{ij}\frac{H}{2} \quad \text{and} \quad \int_0^H \sin(i\pi\frac{y}{H})dy = \frac{H}{i\pi}[1-(-1)^i]$$

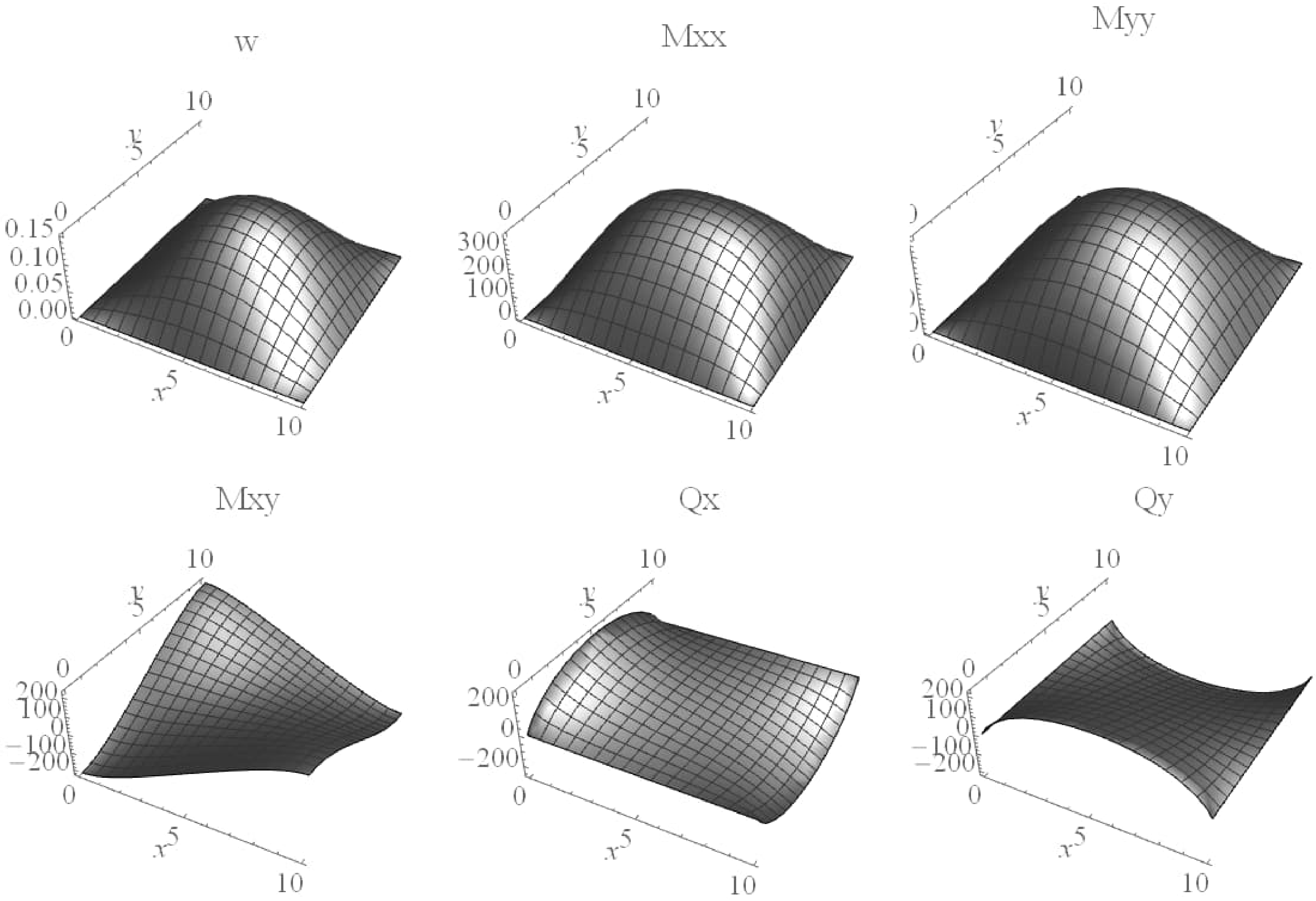
When the series approximation is substituted into the equilibrium equation, the outcome is

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} w_{ij}[(\frac{i\pi}{L})^2 + (\frac{j\pi}{H})^2]^2 \sin(i\pi\frac{x}{L})\sin(j\pi\frac{y}{H}) = \frac{b}{D}.$$

The unknown coefficient can be solved by multiplying both sides of the equation by $\sin(k\pi x/L)\sin(l\pi y/H)$, integrating over the domain $\Omega = (0,L)\times(0,H)$, and using orthogonality of sine functions:

$$w_{ij}\frac{L}{2}\frac{H}{2}[(\frac{i\pi}{L})^2 + (\frac{j\pi}{H})^2]^2 = \frac{b}{D}\frac{LH}{ij\pi^2}[1-(-1)^i][1-(-1)^j] \quad \Leftrightarrow$$

$$w_{ij} = 16\frac{b}{D}\frac{1}{ij\pi^6}\frac{1}{[(\frac{i}{L})^2 + (\frac{j}{H})^2]^2} \quad i,j \in \{1,3,5,\ldots\}, \quad w_{ij} = 0 \text{ otherwise.}$$


w
Mxx
Myy
Mxy
Qx
Qy
0.15
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0.05
0.00
300
200
100
200
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−100
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200
0
−200
x
y
0
5
10

5.2 PLATE EQUATIONS

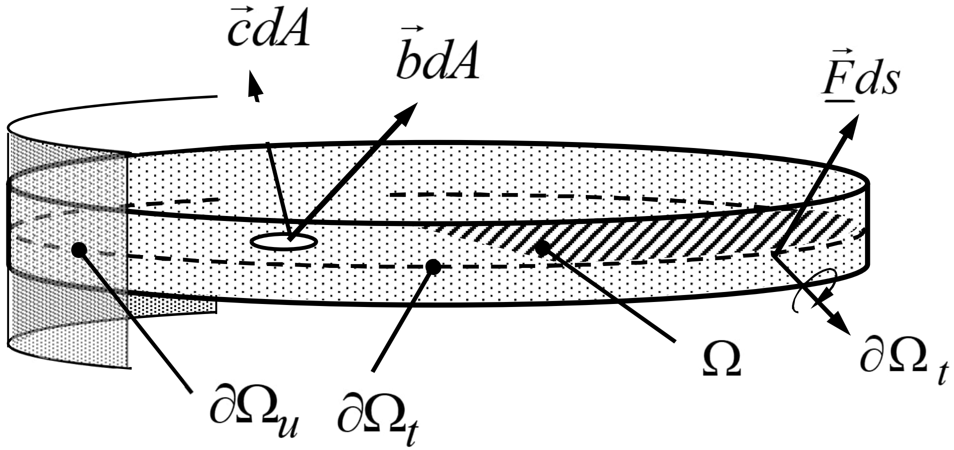
Virtual work expression of plate, principle of virtual work, integration by parts, and the fundamental lemma of variation calculus give ($\vec{e}_n$ is the normal to the mid-plane):

$$\nabla_0 \cdot \vec{\vec{F}} + \vec{b} = 0 \;\text{ in } \Omega,$$

$$\nabla_0 \cdot \vec{\vec{M}} - \vec{e}_n \cdot \vec{\vec{F}} = 0 \;\text{ in } \Omega,$$

$$\vec{n} \cdot \vec{\vec{F}} - \underline{\vec{F}} = 0 \quad \text{or} \quad \vec{u}_0 - \underline{\vec{u}}_0 = 0 \text{ on } \partial\Omega,$$

$$(\vec{n} \cdot \vec{\vec{M}} - \underline{\vec{M}}) \times \vec{e}_n = 0 \text{ or } \vec{\theta}_0 - \underline{\vec{\theta}}_0 = 0 \text{ on } \partial\Omega.$$



Stress resultants are symmetric so that $\vec{\vec{F}} = \vec{\vec{F}}_c$ and $\vec{\vec{M}} = \vec{\vec{M}}_c$. Constitutive equations $\vec{\vec{M}} = \vec{\vec{M}}(\vec{u}_0, \vec{\theta}_0)$, $\vec{\vec{F}} = \vec{\vec{F}}(\vec{u}_0, \vec{\theta}_0)$ are needed for a closed equation system!

In terms of the stress and external force resultants, virtual work densities of the plate model ($\vec{\omega}_0 = \vec{\theta}_0 \times \vec{e}_n \quad \Leftrightarrow \quad \vec{\theta}_0 = \vec{e}_n \times \vec{\omega}_0$) are given by

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \delta\vec{\vec{\varepsilon}} \\ \delta\vec{\vec{\kappa}} \end{Bmatrix}_{\text{c}}^{\text{T}} : \begin{Bmatrix} \vec{\vec{F}} \\ \vec{\vec{M}} \end{Bmatrix}, \quad \delta w_{\Omega}^{\text{ext}} = -\begin{Bmatrix} \delta\vec{u}_0 \\ \delta\vec{\omega}_0 \end{Bmatrix}^{\text{T}} \cdot \begin{Bmatrix} \vec{b} \\ \vec{c} \end{Bmatrix}, \text{ and } \delta w_{\partial\Omega}^{\text{ext}} = -\begin{Bmatrix} \delta\vec{u}_0 \\ \delta\vec{\omega}_0 \end{Bmatrix}^{\text{T}} \cdot \begin{Bmatrix} \underline{\vec{F}} \\ \underline{\vec{M}} \end{Bmatrix}$$

where the strain measures of the plate model are $\vec{\vec{\varepsilon}} = \nabla_0 \vec{u}_0 + \vec{e}_n \vec{\omega}_0$ and $\vec{\vec{\kappa}} = \nabla_0 \vec{\omega}_0$ and gradient operator $\nabla = \nabla_0 + \vec{e}_n \partial / \partial n$.

Virtual work expression of plate is obtained as integral of the density expression over the plate domain $\Omega \subset \mathbb{R}^2$ (mid-plane)

$$\delta W = -\int_{\Omega} [\vec{\vec{F}} : (\nabla_0 \delta\vec{u}_0 + \vec{e}_n \delta\vec{\omega}_0)_{\text{c}} + \vec{\vec{M}} : (\nabla_0 \delta\vec{\omega}_0)_{\text{c}}] dA +$$

$$\int_{\Omega} (\vec{b} \cdot \delta\vec{u}_0 + \vec{c} \cdot \delta\vec{\omega}_0) dA + \int_{\partial\Omega} (\underline{\vec{F}} \cdot \delta\vec{u}_0 + \underline{\vec{M}} \cdot \delta\vec{\omega}_0) ds .$$

Integration by parts gives an equivalent form (the aim is remove the derivatives acting on the variations), retaining the original rotation variable with $\vec{\omega}_0 = \vec{\theta}_0 \times \vec{e}_n$, and using the vector identity $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$ gives

$$\delta W = \int_\Omega [(\nabla_0 \cdot \overleftrightarrow{F} + \vec{b}) \cdot \delta\vec{u}_0 + (\nabla_0 \cdot \overleftrightarrow{M} - \vec{e}_n \cdot \overleftrightarrow{F} + \vec{c}) \cdot \delta\vec{\omega}_0] dA +$$

$$\int_{\partial\Omega} [(-\vec{n} \cdot \overleftrightarrow{F} + \vec{\underline{F}}) \cdot \delta\vec{u}_0 + (-\vec{n} \cdot \overleftrightarrow{M} + \vec{\underline{M}}) \cdot \delta\vec{\omega}_0] ds \qquad \Rightarrow$$

$$\delta W = \int_\Omega [(\nabla_0 \cdot \overleftrightarrow{F} + \vec{b}) \cdot \delta\vec{u}_0 - [(\nabla_0 \cdot \overleftrightarrow{M} - \vec{e}_n \cdot \overleftrightarrow{F} + \vec{c}) \times \vec{e}_n] \cdot \delta\vec{\theta}_0] dA +$$

$$\int_{\partial\Omega} [(-\vec{n} \cdot \overleftrightarrow{F} + \vec{\underline{F}}) \cdot \delta\vec{u}_0 - [(-\vec{n} \cdot \overleftrightarrow{M} + \vec{\underline{M}}) \times \vec{e}_n] \cdot \delta\vec{\theta}_0] ds.$$

Principle of virtual work and the fundamental lemma of variation calculus imply the equilibrium equations and boundary conditions

$$\left.\begin{aligned} &\nabla_0 \cdot \overleftrightarrow{F} + \vec{b} = 0 \quad \text{in } \Omega \\ &(\nabla_0 \cdot \overleftrightarrow{M} - \vec{e}_n \cdot \overleftrightarrow{F} + \vec{c}) \times \vec{e}_n = 0 \quad \text{in } \Omega \end{aligned}\right\} \text{equilibrium eqs.}$$

$$\left.\begin{aligned} &\vec{n} \cdot \overleftrightarrow{F} - \underline{\vec{F}} = 0 \quad \text{or} \quad \vec{u}_0 - \underline{\vec{u}}_0 = 0 \quad \text{on } \partial\Omega \\ &(\vec{n} \cdot \overleftrightarrow{M} - \underline{\vec{M}}) \times \vec{e}_n = 0 \quad \text{or} \quad \vec{\theta}_0 - \underline{\vec{\theta}}_0 = 0 \quad \text{on } \partial\Omega \end{aligned}\right\} \text{boundary conditions}$$

Above, underbars denote given boundary values. Boundary conditions specify either a kinematic quantity or its work conjugate kinetic (force like) quantity.

RESULTANT DEFINITIONS

Stress and external force resultants are integrals over the thickness ($\vec{\omega}_0 = \vec{\theta}_0 \times \vec{e}_n$). Stress resultant definition gives the constitutive equations:

$$\begin{Bmatrix} \vec{F} \\ \vec{M} \end{Bmatrix} = \int \vec{\sigma} \begin{Bmatrix} 1 \\ n \end{Bmatrix} dn = \int \begin{bmatrix} 1 & n \\ n & n^2 \end{bmatrix} \overset{\leftrightarrow}{\overset{\leftrightarrow}{E}} dn : \begin{Bmatrix} \overset{\leftrightarrow}{\varepsilon} \\ \overset{\leftrightarrow}{\kappa} \end{Bmatrix} = \begin{bmatrix} \overset{\leftrightarrow}{\overset{\leftrightarrow}{A}} & \overset{\leftrightarrow}{\overset{\leftrightarrow}{C}} \\ \overset{\leftrightarrow}{\overset{\leftrightarrow}{C}} & \overset{\leftrightarrow}{\overset{\leftrightarrow}{B}} \end{bmatrix} : \begin{Bmatrix} \overset{\leftrightarrow}{\varepsilon} \\ \overset{\leftrightarrow}{\kappa} \end{Bmatrix},$$

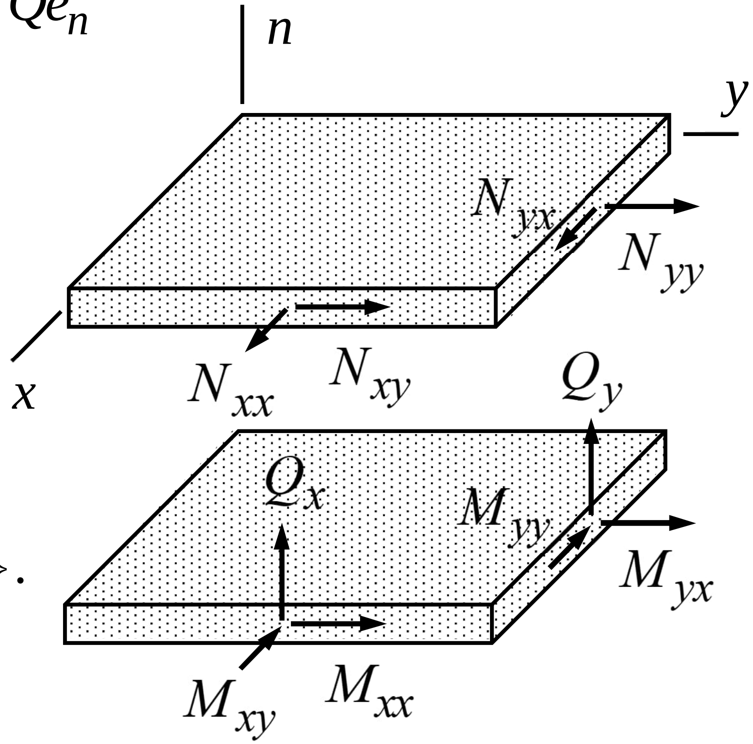
$$\begin{Bmatrix} \vec{b} \\ \vec{c} \end{Bmatrix} = \int \vec{f} \begin{Bmatrix} 1 \\ n \end{Bmatrix} dn,$$ external force and moment per unit area

$$\begin{Bmatrix} \underline{\vec{F}} \\ \underline{\vec{M}} \end{Bmatrix} = \int \vec{t} \begin{Bmatrix} 1 \\ n \end{Bmatrix} dn.$$ external force and moment per unit length

Elasticity tensor $\overset{\leftrightarrow}{\overset{\leftrightarrow}{E}}$ is assumed to satisfy the minor and major symmetries and condition $\vec{e}_n \vec{e}_n : \overset{\leftrightarrow}{\overset{\leftrightarrow}{E}} = 0$, which implies that the kinetic assumption $\sigma_{nn} = 0$ is satisfied 'a priori'.

STRESS RESULTANTS

Using the conventional notation for the components in the (x, y, n) coordinate system, assumption $\sigma_{nn} = 0$ and representation $\overset{\leftrightarrow}{F} = \overset{\leftrightarrow}{N} + \vec{e}_n \vec{Q} + \vec{Q}\vec{e}_n$



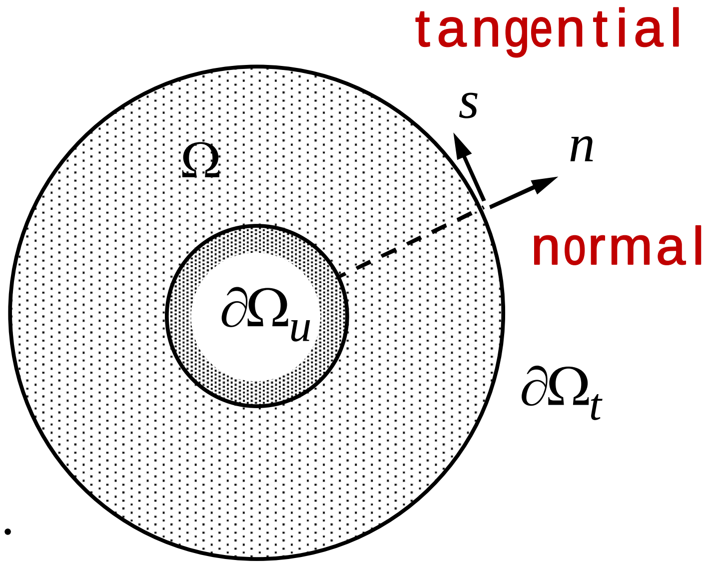
$$\overset{\leftrightarrow}{N} = \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} N_{xx} & N_{xy} \\ N_{yx} & N_{yy} \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix},$$

$$\overset{\leftrightarrow}{M} = \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} M_{xx} & M_{xy} \\ M_{yx} & M_{yy} \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}, \text{ and } \vec{Q} = \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix}.$$

The first and second indices of the components of $\overset{\leftrightarrow}{M}$ do not have the same interpretation as those of $\overset{\leftrightarrow}{\sigma}$.

BENDING MODE OF KIRCHHOFF PLATE

Equilibrium equations of the Kirchhoff plate model can be deduced from the Reissner-Mindlin equations. However, boundary conditions are somewhat tricky and they require derivation from the virtual work densities:



$$\nabla_0 \cdot \vec{Q} + b_n = 0 \ \text{ and } \ \vec{Q} = \nabla_0 \cdot \vec{M} \ \text{ in } \ \Omega,$$

$$M_{nn} - \underline{M}_n = 0 \quad \text{or} \quad \frac{\partial w}{\partial n} + \underline{\theta}_s = 0 \quad \text{on} \ \partial\Omega,$$

$$Q_n - \underline{Q} + \frac{\partial}{\partial s}(M_{ns} - \underline{M}_s) = 0 \quad \text{or} \quad w - \underline{w} = 0 \quad \text{on} \ \partial\Omega \ .$$

Constituive equation of the moment resultant $\overleftrightarrow{M} = -\overleftrightarrow{\overleftrightarrow{B}} : \nabla_0 \nabla_0 w$ follow from the Reissner-Mindlin model, Kirchhoff constraint $\vec{\omega}_0 + \nabla_0 w = 0$, and assumes that $\overleftrightarrow{\overleftrightarrow{C}} = 0$.

The equilibrium equation can be deduced from the Reissner-Mindlin equations by separating the thin-slab and bending modes of plate with $\overset{\leftrightarrow}{F} = \overset{\leftrightarrow}{N} + \vec{e}_n \vec{Q} + \vec{Q} \vec{e}_n$ and $\vec{b} = \vec{b} + \vec{e}_n b_n$ ($\vec{c} = 0$ for simplicity). Then

$$\nabla_0 \cdot \overset{\leftrightarrow}{F} + \vec{b} = 0 \ \text{ and } \ \nabla_0 \cdot \overset{\leftrightarrow}{M} - \vec{e}_n \cdot \overset{\leftrightarrow}{F} = 0 \quad \Leftrightarrow$$

$$\nabla_0 \cdot \overset{\leftrightarrow}{N} + \vec{b}_0 = 0, \quad \nabla_0 \cdot \vec{Q} + b_n = 0, \ \text{ and } \nabla_0 \cdot \overset{\leftrightarrow}{M} - \vec{Q} = 0 \quad \Rightarrow$$

$$\nabla_0 \cdot (\nabla_0 \cdot \overset{\leftrightarrow}{M}) + b_n = 0. \quad \leftarrow$$

The <u>biharmonic equation</u> for the transverse displacement of literature follows from the constitutive equation of homogeneous and isotropic material

$$\nabla_0 \cdot (\nabla_0 \cdot \overset{\leftrightarrow}{M}) + b_n = 0 \ \text{ and } \ \overset{\leftrightarrow}{M} = -\frac{t^2}{12} \overset{\leftrightarrow}{\overset{\leftrightarrow}{E}} : \nabla_0 \nabla_0 w \quad \Rightarrow \quad D \nabla_0^2 \nabla_0^2 w - b_n = 0. \quad \leftarrow$$

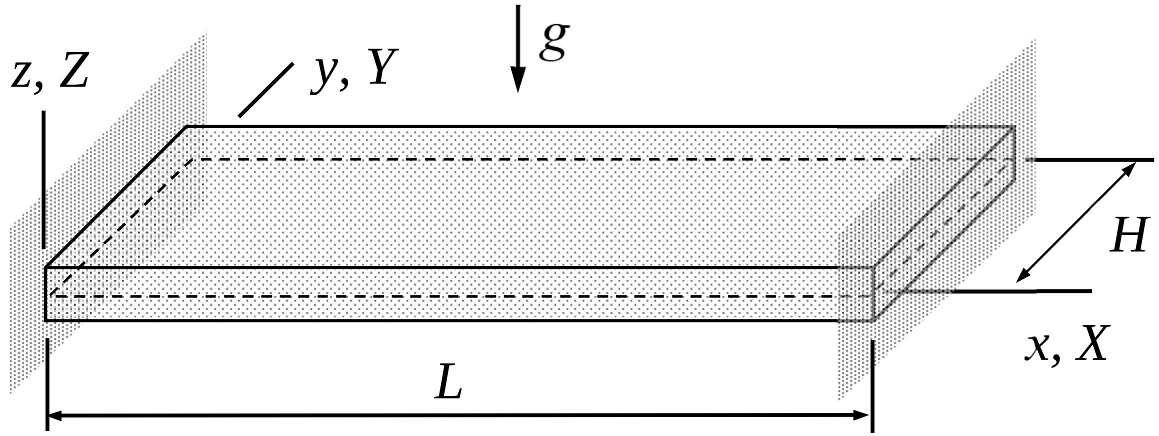
5.3 CARTESIAN COORDINATES

Reissner-Mindlin model bending mode equilibrium and constitutive equations in (x, y, n) coordinates follow from the coordinate system invariant forms.

$$\begin{Bmatrix} \dfrac{\partial Q_x}{\partial x} + \dfrac{\partial Q_y}{\partial y} + b_n \\ \dfrac{\partial M_{xx}}{\partial x} + \dfrac{\partial M_{xy}}{\partial y} - Q_x \\ \dfrac{\partial M_{yy}}{\partial y} + \dfrac{\partial M_{xy}}{\partial x} - Q_y \end{Bmatrix} = 0, \quad \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \theta}{\partial x} \\ -\dfrac{\partial \phi}{\partial y} \\ \dfrac{\partial \theta}{\partial y} - \dfrac{\partial \phi}{\partial x} \end{Bmatrix}, \text{ and}$$

$$\begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = Gt \begin{Bmatrix} \dfrac{\partial w}{\partial x} + \theta \\ \dfrac{\partial w}{\partial y} - \phi \end{Bmatrix} \text{ in } \Omega. \qquad (\text{notation } D = \frac{Et^3}{12(1-\nu^2)})$$

EXAMPLE 5.2 Consider the plate strip clamped at its ends and loaded by its own weight. Determine the deflection w and rotation θ of the plate according to the Reissner-Mindlin model. Thickness, width, and length of the plate are t, H, and L, respectively ($H \gg L$). Density ρ, Young's modulus E, and Poisson's ratio ν are constants. Assume that the stress-resultants, displacement, and rotations depend on x only.



Answer $w(x) = -\rho g(L-x)x(\frac{1}{2G} + \frac{t(L-x)x}{24D})$ and $\theta(x) = \frac{\rho g t}{12D} x(L^2 - 3Lx + 2x^2)$

According to the assumption, derivatives with respect to y vanish. The equilibrium and constitutive equations simplify to

$$\frac{dQ_x}{dx}-\rho g t=0,\ \ \frac{dM_{xx}}{dx}-Q_x=0,\ M_{xx}=D\frac{d\theta}{dx},\ \text{ and }\ Q_x=Gt(\frac{dw}{dx}+\theta)\ \ \text{ in }\ \ (0,L),$$

Boundary value problem for the transverse displacement and rotation, obtained by eliminating the stress resultants,

$$D\frac{d^2\theta}{dx^2}-Gt(\frac{dw}{dx}+\theta)=0,\ Gt(\frac{d^2w}{dx^2}+\frac{d\theta}{dx})-\rho g t=0\ \text{ in }\ (0,L),\ \ w=\theta=0\ \text{ on }\ \{0,L\}.$$

gives (use the Mathematica notebook)

$$w(x)=-\rho g(L-x)x[\frac{1}{2G}+\frac{t(L-x)x}{24D}]\ \text{ and }\ \theta(x)=\frac{\rho g t}{12D}x(L^2-3Lx+2x^2).$$

KIRCHHOFF PLATE EQUATIONS

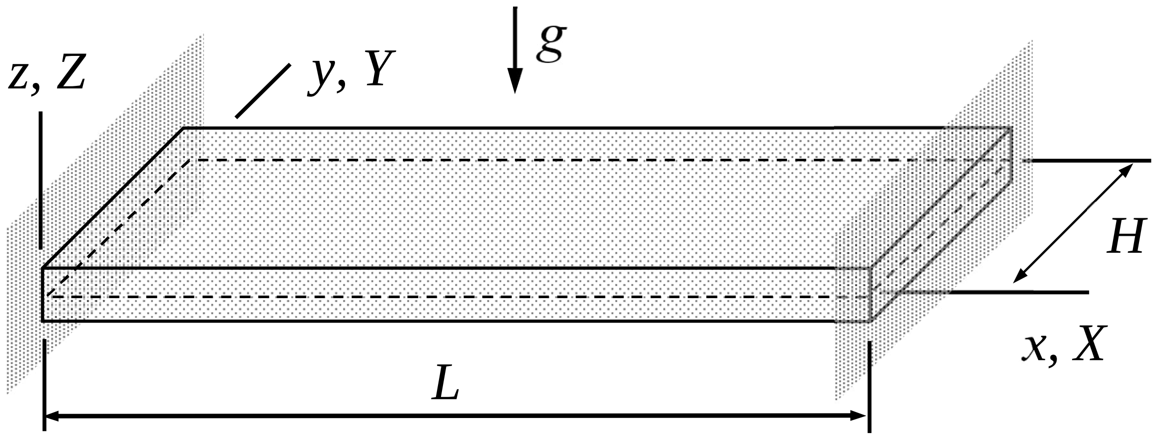
Equilibrium and constitutive equations of the bending mode according to the Kirchhoff model follow from the Reissner-Mindlin equations

$$\begin{Bmatrix} \dfrac{\partial Q_x}{\partial x} + \dfrac{\partial Q_y}{\partial y} + b_n \\ \dfrac{\partial M_{xx}}{\partial x} + \dfrac{\partial M_{xy}}{\partial y} - Q_x \\ \dfrac{\partial M_{yy}}{\partial y} + \dfrac{\partial M_{xy}}{\partial x} - Q_y \end{Bmatrix} = 0, \quad \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \theta}{\partial x} \\ -\dfrac{\partial \phi}{\partial y} \\ \dfrac{\partial \theta}{\partial y} - \dfrac{\partial \phi}{\partial x} \end{Bmatrix}, \text{ and}$$

$$\begin{Bmatrix} \dfrac{\partial w}{\partial x} + \theta \\ \dfrac{\partial w}{\partial y} - \phi \end{Bmatrix} = 0 \text{ in } \Omega.$$

Kirchhoff constraints!

EXAMPLE 5.3 Consider the plate strip clamped at its ends and loaded by its own weight. Determine the deflection w and rotation θ of the plate according to the Kirchhoff model. Thickness, width, and length of the plate are t, H, and L, respectively ($H \gg L$). Density ρ, Young's modulus E, and Poisson's ratio ν are constants. Assume that the stress-resultants, displacement, and rotations depend on x only.



Answer $w(x) = -\dfrac{\rho g t}{24D}(L-x)^2 x^2$ and $\theta(x) = -\dfrac{dw}{dx} = \dfrac{\rho g t}{12D} x(L^2 - 3Lx + 2x^2)$

According to the assumption, derivatives with respect to y vanish, and the set of partial differential equations becomes a set of ordinary differential equations. The relevant differential equations and the boundary conditions are

$$\frac{dQ_x}{dx} - \rho g t = 0, \quad \frac{dM_{xx}}{dx} - Q_x = 0, \quad M_{xx} = D\frac{d\theta}{dx}, \quad \frac{dw}{dx} + \theta = 0 \quad \text{in} \quad (0, L),$$

$$w = \theta = 0 \quad \text{on} \quad \{0, L\}.$$

Solution to w can be obtained, e.g., by eliminating the rotation and the stress resultants

$$D\frac{d^4 w}{dx^4} + \rho g t = 0 \quad \text{in} \quad (0, L) \quad \text{and} \quad w = \frac{dw}{dx} = 0 \quad \text{on} \quad \{0, L\} \quad \Rightarrow$$

$$w(x) = -\frac{\rho g t}{D}\frac{(L-x)^2 x^2}{24}. \quad \Leftarrow$$

5.4 CURVILINEAR COORDINATES

Equilibrium and constitutive equations of the bending mode according to the Reissner-Mindlin model in (r,ϕ,n) coordinates follow from the coordinate system invariant forms:

$$\begin{Bmatrix} \frac{1}{r}[\frac{\partial(rQ_r)}{\partial r}+\frac{\partial Q_\phi}{\partial\phi}]+b_n \\ \frac{1}{r}[\frac{\partial(rM_{rr})}{\partial r}+\frac{\partial M_{r\phi}}{\partial\phi}-M_{\phi\phi}]-Q_r \\ \frac{1}{r}[\frac{\partial(rM_{r\phi})}{\partial r}+\frac{\partial M_{\phi\phi}}{\partial\phi}+M_{r\phi}]-Q_\phi \end{Bmatrix}=0,\ \begin{Bmatrix} M_{rr} \\ M_{\phi\phi} \\ M_{r\phi} \end{Bmatrix}=\frac{t^3}{12}[E]_\sigma \begin{Bmatrix} \frac{\partial\theta_\phi}{\partial r} \\ \frac{1}{r}(\theta_\phi-\frac{\partial\theta_r}{\partial\phi}) \\ \frac{1}{r}(\frac{\partial\theta_\phi}{\partial\phi}+\theta_r)-\frac{\partial\theta_r}{\partial r} \end{Bmatrix},\ \text{and}$$

$$\begin{Bmatrix} Q_r \\ Q_\phi \end{Bmatrix}=Gt\begin{Bmatrix} \frac{\partial w}{\partial r}+\theta_\phi \\ \frac{1}{r}\frac{\partial w}{\partial\phi}-\theta_r \end{Bmatrix} \text{ in } \Omega. \quad (\text{notation } [E]_\sigma=\frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix})$$

The component representations of quantities in the (r,ϕ,n) coordinate system are

$$\vec{Q}=\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}Q_r\\ Q_\phi\end{Bmatrix},\ \overset{\leftrightarrow}{M}=\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}M_{rr} & M_{r\phi}\\ M_{\phi r} & M_{\phi\phi}\end{bmatrix}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix},\ \nabla_0=\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\dfrac{\partial}{\partial r}\\ \dfrac{1}{r}\dfrac{\partial}{\partial\phi}\end{Bmatrix},$$

$$\overset{\leftrightarrow}{\overset{\leftrightarrow}{E}}=\begin{Bmatrix}\vec{e}_r\vec{e}_r\\ \vec{e}_\phi\vec{e}_\phi\\ \vec{e}_n\vec{e}_n\end{Bmatrix}^{\mathrm{T}}\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}\vec{e}_r\vec{e}_r\\ \vec{e}_\phi\vec{e}_\phi\\ \vec{e}_n\vec{e}_n\end{Bmatrix}+\begin{Bmatrix}\vec{e}_r\vec{e}_\phi+\vec{e}_\phi\vec{e}_r\\ \vec{e}_\phi\vec{e}_n+\vec{e}_n\vec{e}_\phi\\ \vec{e}_n\vec{e}_r+\vec{e}_r\vec{e}_n\end{Bmatrix}^{\mathrm{T}}G\begin{Bmatrix}\vec{e}_r\vec{e}_\phi+\vec{e}_\phi\vec{e}_r\\ \vec{e}_\phi\vec{e}_n+\vec{e}_n\vec{e}_\phi\\ \vec{e}_n\vec{e}_r+\vec{e}_r\vec{e}_n\end{Bmatrix}.$$

Direct calculation (basis vectors are not constants) gives

$$\nabla_0\cdot\vec{Q}+b_n=\frac{1}{r}[\frac{\partial(rQ_r)}{\partial r}+\frac{\partial Q_\phi}{\partial\phi}]+b_n=0, \quad \Leftarrow$$

$$\nabla_0 \cdot \overleftrightarrow{M} - \vec{Q} = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} \frac{1}{r} \begin{Bmatrix} \dfrac{\partial (rM_{rr})}{\partial r} + \dfrac{\partial M_{r\phi}}{\partial \phi} - M_{\phi\phi} - rQ_r \\ \dfrac{\partial (rM_{r\phi})}{\partial r} + \dfrac{\partial M_{\phi\phi}}{\partial \phi} + M_{r\phi} - rQ_\phi \end{Bmatrix} = 0, \quad \Leftarrow$$

$$\overleftrightarrow{M} = \frac{t^3}{12} \overleftrightarrow{\overleftrightarrow{E}} : \overleftrightarrow{\kappa} = \begin{Bmatrix} \vec{e}_r \vec{e}_r \\ \vec{e}_\phi \vec{e}_\phi \\ \vec{e}_\phi \vec{e}_r + \vec{e}_r \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \theta_\phi}{\partial r} \\ \dfrac{1}{r}(\theta_\phi - \dfrac{\partial \theta_r}{\partial \phi}) \\ \dfrac{1}{r}(\dfrac{\partial \theta_\phi}{\partial \phi} + \theta_r) - \dfrac{\partial \theta_r}{\partial r} \end{Bmatrix}, \quad \Leftarrow$$

$$\vec{Q} = t\vec{e}_n \cdot \overleftrightarrow{\overleftrightarrow{E}} : \overleftrightarrow{\varepsilon} = \begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_r \end{Bmatrix}^{\mathrm{T}} Gt \begin{Bmatrix} \dfrac{1}{r}\dfrac{\partial w}{\partial \phi} - \theta_r \\ \dfrac{\partial w}{\partial r} + \theta_\phi \end{Bmatrix}. \quad \Leftarrow$$

KIRCHHOFF PLATE EQUATIONS

Equilibrium and constitutive equations of the bending mode according to the Kirchhoff model follow from the Reissner-Mindlin equations

$$\begin{Bmatrix} \frac{1}{r}[\frac{\partial(rQ_r)}{\partial r}+\frac{\partial Q_\phi}{\partial\phi}]+b_n \\ \frac{1}{r}[\frac{\partial(rM_{rr})}{\partial r}+\frac{\partial M_{r\phi}}{\partial\phi}-M_{\phi\phi}]-Q_r \\ \frac{1}{r}[\frac{\partial(rM_{r\phi})}{\partial r}+\frac{\partial M_{\phi\phi}}{\partial\phi}+M_{r\phi}]-Q_\phi \end{Bmatrix}=0, \quad \begin{Bmatrix} M_{rr} \\ M_{\phi\phi} \\ M_{r\phi} \end{Bmatrix}=\frac{t^3}{12}[E]_\sigma \begin{Bmatrix} \frac{\partial\theta_\phi}{\partial r} \\ \frac{1}{r}(\theta_\phi-\frac{\partial\theta_r}{\partial\phi}) \\ \frac{1}{r}(\frac{\partial\theta_\phi}{\partial\phi}+\theta_r)-\frac{\partial\theta_r}{\partial r} \end{Bmatrix}, \text{ and}$$

$$\begin{Bmatrix} \frac{\partial w}{\partial r}+\theta_\phi \\ \frac{1}{r}\frac{\partial w}{\partial\phi}-\theta_r \end{Bmatrix}=0.$$

Kirchhoff constraints!

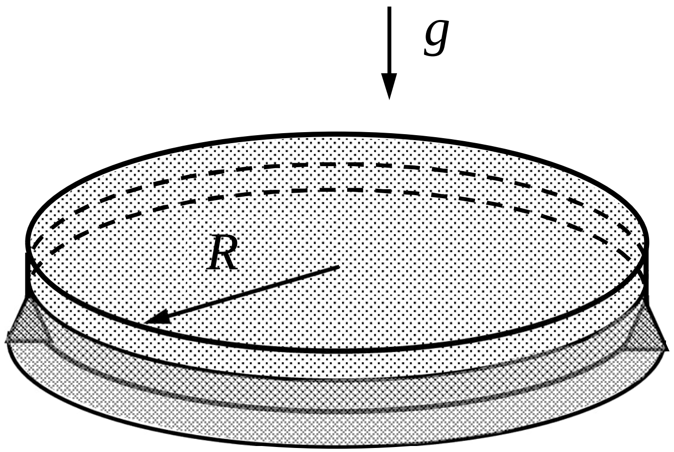
The various forms in literature follow after elimination of rotations or stress resultants in the generic forms. For example, solving for the shear forces in terms of the moment resultants and eliminating the rotations in the constitutive equations by using the Kirchhoff constraints gives first

$$\begin{Bmatrix} Q_r \\ Q_\phi \end{Bmatrix} = \frac{1}{r} \begin{Bmatrix} \dfrac{\partial}{\partial r}(rM_{rr}) + \dfrac{\partial}{\partial \phi} M_{r\phi} - M_{\phi\phi} \\ \dfrac{\partial}{\partial r}(rM_{r\phi}) + \dfrac{\partial}{\partial \phi} M_{\phi\phi} + M_{r\phi} \end{Bmatrix}, \quad \begin{Bmatrix} M_{rr} \\ M_{\phi\phi} \\ M_{r\phi} \end{Bmatrix} = -D \begin{Bmatrix} \dfrac{\partial^2 w}{\partial r^2} + \nu \dfrac{1}{r}(\dfrac{\partial w}{\partial r} + \dfrac{1}{r}\dfrac{\partial^2 w}{\partial \phi^2}) \\ \nu \dfrac{\partial^2 w}{\partial r^2} + \dfrac{1}{r}(\dfrac{\partial w}{\partial r} + \dfrac{1}{r}\dfrac{\partial^2 w}{\partial \phi^2}) \\ (1-\nu)\dfrac{\partial}{\partial r}(\dfrac{1}{r}\dfrac{\partial w}{\partial \phi}) \end{Bmatrix} \Rightarrow$$

$$\nabla_0^2 \nabla_0^2 w = [\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}][\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}]w = \frac{b_n}{D}.$$

Biharmonic equation

EXAMPLE 5.4 A simply supported circular plate of radius R is loaded by its own weight as shown in the figure. Write down the boundary value problem giving as its solution the transverse displacement. Use Kirchhoff plate equations in the polar coordinate system. Problem parameters E, ν, ρ and t are constants. Assume that w depends on the radial coordinate only.



Answer:

$$[\frac{1}{r}\frac{d}{dr}(r\frac{d}{dr})][\frac{1}{r}\frac{d}{dr}(r\frac{d}{dr})]w-\frac{b_n}{D}=0 \text{ in } (0,R),\;\; M_{rr}(R)=w(R)=-M_{rr}(0)=-Q_r(0)=0$$

Assuming rotation symmetry, the bending mode equilibrium equation and the boundary conditions of circular simply supported plate of isotropic homogeneous material simplify to

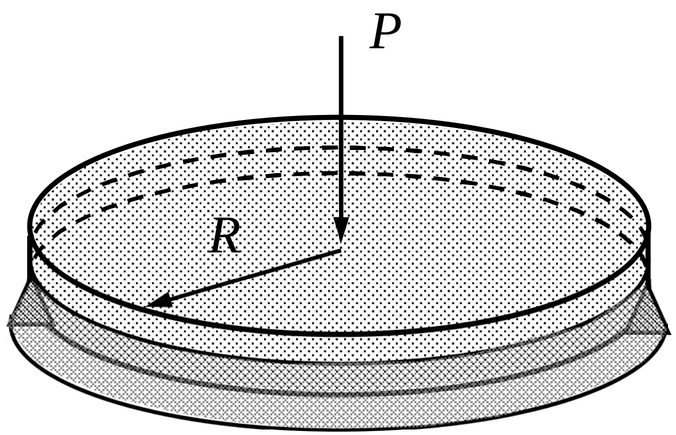
$$[\frac{1}{r}\frac{d}{dr}(r\frac{d}{dr})][\frac{1}{r}\frac{d}{dr}(r\frac{d}{dr})]w-\frac{b_n}{D}=0 \quad \text{in} \quad (0,R),$$

$$M_{rr}=-D(\frac{d^2w}{dr^2}+\nu\frac{1}{r}\frac{dw}{dr})=0 \quad \text{and} \quad w=0 \quad \text{on} \quad \{R\},$$

$$Q_r=-D(\frac{d^3w}{dr^3}+\frac{1}{r}\frac{d^2w}{dr^2}-\frac{1}{r^2}\frac{dw}{dr})=0, \quad M_{rr}=-D(\frac{d^2w}{dr^2}+\nu\frac{1}{r}\frac{dw}{dr})=0 \quad \text{on} \quad \{0\}.$$

The generic solution to the equilibrium equation in terms of integration constants a, b, c, and d (obtained by repeated integrations) is $w=a+br^2+cr^2(1-\log r)+d\log r$.

EXAMPLE 5.5 A simply supported circular plate of radius R is loaded by a point force P acting at the midpoint as shown in the figure. Determine the displacement of the plate at the midpoint by using the Kirchhoff plate model in the polar coordinate system. Problem parameters E, ν and t are constants. Assume that the solution depends on the radial coordinate only. Use the generic solution $w(r) = a + br^2 + cr^2(1-\log r) + d\log r$.



Answer: $$w(0) = -\frac{1}{16\pi}\frac{PR^2}{D}\frac{3+\nu}{1+\nu} = -\frac{3}{4\pi}\frac{PR^2}{Et^3}(3+\nu)(1-\nu)$$

Let us consider first solution on an annular domain of outer radius R which is simple supported on the outer boundary and loaded by constant distributed force $\underline{Q} = -P/(2\pi\varepsilon)$ on the inner boundary $r = \varepsilon$. Assuming rotation symmetry, the bending mode equilibrium equation and the boundary conditions simplify to

$$(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr})(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr})w = 0 \quad \text{in} \quad (\varepsilon, R),$$

$$M_{rr}(R) = 0, \;\; w(R) = 0 \;\; \text{and} \;\; -Q_r(\varepsilon) - \underline{Q} = 0, \;\; -M_{rr}(\varepsilon) = 0$$

where $Q_r = -D(\frac{d^3w}{dr^3} + \frac{1}{r}\frac{d^2w}{dr^2} - \frac{1}{r^2}\frac{dw}{dr})$ and $M_{rr} = -D(\frac{d^2w}{dr^2} + \nu\frac{1}{r}\frac{dw}{dr})$.

The generic solution to the biharmonic equation (obtained by repeated integrations) contains parameters a, b, c, and d to be determined from the boundary conditions. When the solution to the linear equations

$$-Q_r(\varepsilon) - \underline{Q} = \frac{1}{2\pi\varepsilon}(P - 8cD\pi) = 0,$$

$$-M_{rr}(\varepsilon) = -\frac{D}{\varepsilon^2}[d - d\nu + \varepsilon^2(c - c\nu - 2b - 2b\nu) + 2c\varepsilon^2(1+\nu)\log\varepsilon] = 0,$$

$$M_{rr}(R) = \frac{D}{R^2}[d - d\nu + R^2(c - c\nu - 2b - 2b\nu) + 2cR^2(1+\nu)\log R] = 0,$$

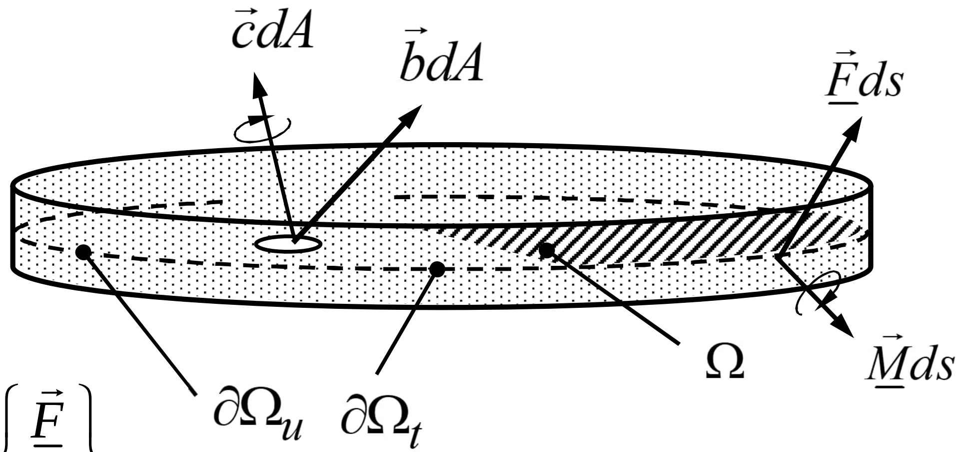
$$w(R) = a + (b+c)R^2 + (d - cR^2)\log R = 0$$

is substituted in the generic solution to the transverse displacement $w(\varepsilon)$, solution to the original problem is obtained as the limit

$$w(0) = \lim_{\varepsilon \to 0} w(\varepsilon) = -\frac{1}{16\pi}\frac{PR^2}{D}\frac{3+\nu}{1+\nu}.$$

4.5 VIRTUAL WORK DENSITIES

Virtual work expressions contain generalized forces (force and moment) corresponding to the chosen kinematic quantities:



$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \delta\vec{\vec{\varepsilon}}_{\mathrm{c}} \\ \delta\vec{\vec{\kappa}}_{\mathrm{c}} \end{Bmatrix}^{\mathrm{T}} : \begin{Bmatrix} \vec{\vec{F}} \\ \vec{\vec{M}} \end{Bmatrix},$$

$$\delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta\vec{u}_0 \\ \delta\vec{\omega}_0 \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \vec{b} \\ \vec{c} \end{Bmatrix}, \quad \delta w_{\partial\Omega}^{\text{ext}} = \begin{Bmatrix} \delta\vec{u}_0 \\ \delta\vec{\omega}_0 \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \underline{\vec{F}} \\ \underline{\vec{M}} \end{Bmatrix}$$

in which

$$\begin{Bmatrix} \vec{\vec{F}} \\ \vec{\vec{M}} \end{Bmatrix} = \int \vec{\vec{\sigma}} \begin{Bmatrix} 1 \\ n \end{Bmatrix} dn, \quad \begin{Bmatrix} \vec{b} \\ \vec{c} \end{Bmatrix} = \int \vec{f} \begin{Bmatrix} 1 \\ n \end{Bmatrix} dn, \quad \begin{Bmatrix} \underline{\vec{F}} \\ \underline{\vec{M}} \end{Bmatrix} = \int \vec{t} \begin{Bmatrix} 1 \\ n \end{Bmatrix} dn, \text{ and } \ \vec{\omega}_0 = \vec{\theta}_0 \times \vec{e}_n.$$

Straight line segments perpendicular to the mid/reference-plane remain straight in deformation. In vector notation

$$\vec{u} = \vec{u}_0 + \vec{\theta}_0 \times n\vec{e}_n = \vec{u}_0 + n\vec{\omega}_0 \quad \text{where} \quad \vec{\omega}_0 = \vec{\theta}_0 \times \vec{e}_n \qquad \Rightarrow$$

$$\nabla\vec{u} = (\nabla_0 + \vec{e}_n \frac{\partial}{\partial n})\vec{u} = \nabla_0\vec{u}_0 + \vec{e}_n\vec{\omega}_0 + n\nabla_0\vec{\omega}_0 = \vec{\vec{\varepsilon}} + n\vec{\vec{\kappa}}$$

where the strain measures of plate $\vec{\vec{\varepsilon}} = \nabla_0\vec{u}_0 + \vec{e}_n\vec{\omega}_0$ and $\vec{\vec{\kappa}} = \nabla_0\vec{\omega}_0$.

Assuming symmetry of stress $\delta w_V^{\text{int}} = -\vec{\vec{\sigma}} : (\nabla\delta\vec{u})_{\text{c}}$. Virtual work density of the plate model is obtained by integrating the virtual work density over the small dimension ($dV = dndA$)

$$\delta W_V^{\text{int}} = -\int_\Omega [\begin{Bmatrix} \delta\vec{\vec{\varepsilon}} \\ \delta\vec{\vec{\kappa}} \end{Bmatrix}_{\text{c}}^{\text{T}} : (\int \vec{\vec{\sigma}} \begin{Bmatrix} 1 \\ n \end{Bmatrix} dn)]dA = -\int_\Omega \begin{Bmatrix} \delta\vec{\vec{\varepsilon}} \\ \delta\vec{\vec{\kappa}} \end{Bmatrix}_{\text{c}}^{\text{T}} : \begin{Bmatrix} \vec{\vec{F}} \\ \vec{\vec{M}} \end{Bmatrix} dA, \quad \text{hence}$$

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \delta\overleftrightarrow{\varepsilon} \\ \delta\overleftrightarrow{\kappa} \end{Bmatrix}_{\text{c}}^{\text{T}} : \begin{Bmatrix} \overleftrightarrow{F} \\ \overleftrightarrow{M} \end{Bmatrix} \text{ and } \begin{Bmatrix} \overleftrightarrow{F} \\ \overleftrightarrow{M} \end{Bmatrix} = \int \overleftrightarrow{\sigma} \begin{Bmatrix} 1 \\ n \end{Bmatrix} dn. \quad \Leftarrow$$

Virtual work expression of external forces takes into account volume forces and surface forces acting on the body ($dV = dndA$ and $dA = dnds$).

$$\delta W_{V}^{\text{ext}} = \int_{V} \vec{f} \cdot \delta\vec{u} dV = \int_{\Omega} [\begin{Bmatrix} \delta\vec{u}_0 \\ \delta\vec{\omega}_0 \end{Bmatrix}^{\text{T}} \cdot (\int \vec{f} \begin{Bmatrix} 1 \\ n \end{Bmatrix} dn)] dA = \int_{\Omega} (\begin{Bmatrix} \delta\vec{u}_0 \\ \delta\vec{\omega}_0 \end{Bmatrix}^{\text{T}} \cdot \begin{Bmatrix} \vec{b} \\ \vec{c} \end{Bmatrix}) dA \;\Rightarrow$$

$$\delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta\vec{u}_0 \\ \delta\vec{\omega}_0 \end{Bmatrix}^{\text{T}} \cdot \begin{Bmatrix} \vec{b} \\ \vec{c} \end{Bmatrix}, \text{ where } \begin{Bmatrix} \vec{b} \\ \vec{c} \end{Bmatrix} = \int \vec{f} \begin{Bmatrix} 1 \\ n \end{Bmatrix} dn. \quad \Leftarrow$$

$$\delta W_{A}^{\text{ext}} = \int_{A} \vec{t} \cdot \delta\vec{u} dA = \int_{\partial\Omega} \begin{Bmatrix} \delta\vec{u}_0 \\ \delta\vec{\omega}_0 \end{Bmatrix}^{\text{T}} \cdot (\int \vec{t} \begin{Bmatrix} 1 \\ n \end{Bmatrix} dn) ds \;\Rightarrow$$

$$\delta w_{\partial\Omega}^{\text{ext}} = \begin{Bmatrix} \delta\vec{u}_0 \\ \delta\vec{\omega}_0 \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \underline{\vec{F}} \\ \underline{\vec{M}} \end{Bmatrix}, \text{ where } \begin{Bmatrix} \underline{\vec{F}} \\ \underline{\vec{M}} \end{Bmatrix} = \int \vec{t} \begin{Bmatrix} 1 \\ n \end{Bmatrix} dn. \Leftarrow$$

In the derivation, surface forces acting on the top and bottom surfaces of the plate have been omitted for simplicity (they may contribute to $\vec{b}$ and $\vec{c}$).

RESULTANT DEFINITIONS

Stress and external force resultants are integrals over the thickness ($\vec{\omega}_0 = \vec{\theta}_0 \times \vec{e}_n$). Stress resultant definition gives the constitutive equations:

$$\begin{Bmatrix} \vec{\vec{F}} \\ \vec{\vec{M}} \end{Bmatrix} = \int \vec{\vec{\sigma}} \begin{Bmatrix} 1 \\ n \end{Bmatrix} dn = \int \begin{bmatrix} 1 & n \\ n & n^2 \end{bmatrix} \vec{\vec{\vec{\vec{E}}}} dn : \begin{Bmatrix} \vec{\vec{\varepsilon}} \\ \vec{\vec{\kappa}} \end{Bmatrix} = \begin{bmatrix} \vec{\vec{\vec{\vec{A}}}} & \vec{\vec{\vec{\vec{C}}}} \\ \vec{\vec{\vec{\vec{C}}}} & \vec{\vec{\vec{\vec{B}}}} \end{bmatrix} : \begin{Bmatrix} \vec{\vec{\varepsilon}} \\ \vec{\vec{\kappa}} \end{Bmatrix},$$

$$\begin{Bmatrix} \vec{b} \\ \vec{c} \end{Bmatrix} = \int \vec{f} \begin{Bmatrix} 1 \\ n \end{Bmatrix} dn,$$ external force and moment per unit area

$$\begin{Bmatrix} \underline{\vec{F}} \\ \underline{\vec{M}} \end{Bmatrix} = \int \vec{t} \begin{Bmatrix} 1 \\ n \end{Bmatrix} dn.$$ external force and moment per unit length

Elasticity dyad $\vec{\vec{\vec{\vec{E}}}}$ is assumed to satisfy the minor and major symmetries and condition $\vec{e}_n \vec{e}_n : \vec{\vec{\vec{\vec{E}}}} = 0$, which implies that the kinetic assumption $\sigma_{nn} = 0$ is satisfied ‘a priori’.

4.6 KIRCHHOFF PLATE EQUATIONS

Kirchhoff plate model equilibrium and constitutive equations can be deduced from the Reissner-Mindlin ones. However, the somewhat tricky boundary conditions require a more careful consideration starting from the virtual work expression:

$$\nabla_0 \cdot \nabla_0 \cdot \overleftrightarrow{M} + b_n = 0 \quad \text{in} \quad \Omega$$

$$M_{nn} - \underline{M}_n = 0 \quad \text{or} \quad \frac{\partial w}{\partial n} + \underline{\theta}_s = 0 \quad \text{on} \quad \partial\Omega$$

$$Q_n - \underline{Q} + \frac{\partial}{\partial s}(M_{ns} - \underline{M}_s) = 0 \quad \text{or} \quad w - \underline{w} = 0 \quad \text{on} \quad \partial\Omega \,.$$

In the model, shear stress resultant is a constraint force to be solved from the moment equilibrium equation $\vec{Q} = \nabla_0 \cdot \overleftrightarrow{M}$ and constitutive equation for the moment resultant.

In the Kirchhoff model, straight line segments normal to the mid/reference-plane remain line segments and perpendicular to the mid-plane so that $\nabla_0 w + \vec{\omega}_0 = 0$ (Kirchhoff constraint). After elimination of the rotation components and second integration by parts, the Reissner-Mindlin virtual work expression takes the form

$$\delta W = \int_\Omega [(\nabla_0 \cdot \overleftrightarrow{F} + \vec{b}) \cdot \delta\vec{u} - (\nabla_0 \cdot \overleftrightarrow{M} - \vec{e}_n \cdot \overleftrightarrow{F}) \cdot \nabla_0 \delta w] dA +$$

$$\int_{\partial\Omega} [(-\vec{n} \cdot \overleftrightarrow{F} + \underline{\vec{F}}) \cdot \delta\vec{u} - (-\vec{n} \cdot \overleftrightarrow{M} + \underline{\vec{M}}) \cdot \nabla_0 \delta w) ds \quad \Leftrightarrow$$

$$\delta W = \int_\Omega [(\nabla_0 \cdot \overleftrightarrow{F} + \vec{b}) \cdot \delta\vec{u} + \nabla_0 \cdot (\nabla_0 \cdot \overleftrightarrow{M} - \vec{e}_n \cdot \overleftrightarrow{F}) \delta w] dA +$$

$$\int_{\partial\Omega} [(-\vec{n} \cdot \overleftrightarrow{F} + \underline{\vec{F}}) \cdot \delta\vec{u} - (-\vec{n} \cdot \overleftrightarrow{M} + \underline{\vec{M}}) \cdot \nabla_0 \delta w - \vec{n} \cdot (\nabla_0 \cdot \overleftrightarrow{M} - \vec{e}_n \cdot \overleftrightarrow{F}) \delta w] ds .$$

The thin-slab and bending modes can be separated by writing $\delta\vec{u}_0 = \delta\vec{v}_0 + \delta w \vec{e}_n$. Omitting the thin slab mode ($\vec{Q} = \nabla_0 \cdot \overleftrightarrow{M}$)

$$\delta W=\int_{\Omega}(\nabla_0\cdot\nabla_0\cdot\overleftrightarrow{M}+b_n)\delta w dA+$$

$$\int_{\partial\Omega}(-\vec{n}\cdot\vec{Q}+\underline{Q})\delta w ds-\int_{\partial\Omega}(-\vec{n}\cdot\overleftrightarrow{M}+\underline{\vec{M}})\cdot(\nabla_0\delta w)ds.$$

As only w and its normal derivative $\partial w/\partial n$ can be varied independently on $\partial\Omega$, some additional manipulations are needed before application of the fundamental lemma of variation calculus. Using division

$$\nabla_0 w=\vec{n}\frac{\partial w}{\partial n}+\vec{s}\frac{\partial w}{\partial s},$$

where $\vec{n}$ and $\vec{s}=\vec{e}_n\times\vec{n}$ are the unit outward normal and tangential vectors to the boundary, integration by parts in the boundary term containing $\partial w/\partial s$ with respect to s gives

$$\delta W=\int_{\Omega}(\nabla_0\cdot\nabla_0\cdot\overleftrightarrow{M}+b_n)\delta w dA-\sum_{\Pi}[\![-M_{ns}+\underline{M}_s]\!]\delta w+$$

$$\int_{\partial\Omega}[-Q_n+\underline{Q}+\frac{\partial}{\partial s}(-M_{ns}+\underline{M}_s)]\delta w ds-\int_{\partial\Omega}(-M_{nn}+\underline{M}_n)\delta\frac{\partial w}{\partial n}ds\,.$$

Integration by parts is over a closed one-dimensional domain starting and ending at the same point having opposite unit outward normal (±1). In the expression, ⟦ ⟧ denotes jump and Π is the set of points where the jump takes place (the usual integration by parts assumes continuity. A more generic form for piecewise continuity contains jump terms). The last term vanishes if the quantity inside the jump brackets is continuous or $\delta w = 0$ on $\partial\Omega$. In what follows we assume so to avoid further discussions about conditions at corners etc. when deflection w is not specified. Arranging the terms gives

$$\delta W=\int_{\Omega}(\nabla_0\cdot\nabla_0\cdot\overleftrightarrow{M}+b_n)\delta w d\Omega+$$

$$\int_{\partial\Omega}[-Q_n+\underline{Q}+\frac{\partial}{\partial s}(-M_{ns}+\underline{M}_s)]\delta w ds-\int_{\partial\Omega}(-M_{nn}+\underline{M}_n)\delta\frac{\partial w}{\partial n}ds\,.$$

Principle of virtual work and the fundamental lemma of variation calculus give

$$\nabla_0 \cdot \nabla_0 \cdot \overleftrightarrow{M} + b_n = 0 \quad \text{in} \quad \Omega \,,$$

$$M_{nn} - \underline{M}_n = 0 \quad \text{or} \quad \frac{\partial w}{\partial n} + \underline{\theta}_s = 0 \quad \text{on} \quad \partial\Omega \,,$$

$$Q_n - \underline{Q} + \frac{\partial}{\partial s}(M_{ns} - \underline{M}_s) = 0 \quad \text{or} \quad w - \underline{w} = 0 \quad \text{on} \quad \partial\Omega \,.$$

5.7 APPROXIMATE SOLUTIONS

Principle of virtual work can be used to find approximate series solutions to plate equations. An approximation satisfying the essential boundary conditions 'a priori' is just substituted into the virtual work expression by considering the coefficient of the terms of the series as the unknowns. For the plate model

$$\delta W^{\text{int}} = -\int_{\Omega} \delta \begin{Bmatrix} \nabla_0 \vec{u}_0 + \vec{e}_n \vec{\omega}_0 \\ \nabla_0 \vec{\omega}_0 \end{Bmatrix}_{\text{c}}^{\text{T}} : \begin{bmatrix} \overset{\leftrightarrow}{A} & \overset{\leftrightarrow}{C} \\ \overset{\leftrightarrow}{C} & \overset{\leftrightarrow}{B} \end{bmatrix} : \begin{Bmatrix} \nabla_0 \vec{u}_0 + \vec{e}_n \vec{\omega}_0 \\ \nabla_0 \vec{\omega}_0 \end{Bmatrix} dA,$$

$$\delta W^{\text{ext}} = \int_{\Omega} \begin{Bmatrix} \delta \vec{u}_0 \\ \delta \vec{\omega}_0 \end{Bmatrix}^{\text{T}} \cdot \begin{Bmatrix} \vec{b} \\ \vec{c} \end{Bmatrix} dA + \int_{\partial\Omega} \begin{Bmatrix} \delta \vec{u}_0 \\ \delta \vec{\omega}_0 \end{Bmatrix}^{\text{T}} \cdot \begin{Bmatrix} \underline{\vec{F}} \\ \underline{\vec{M}} \end{Bmatrix} ds.$$

Various series solution in literature and the finite element method are just particular cases of this theme.

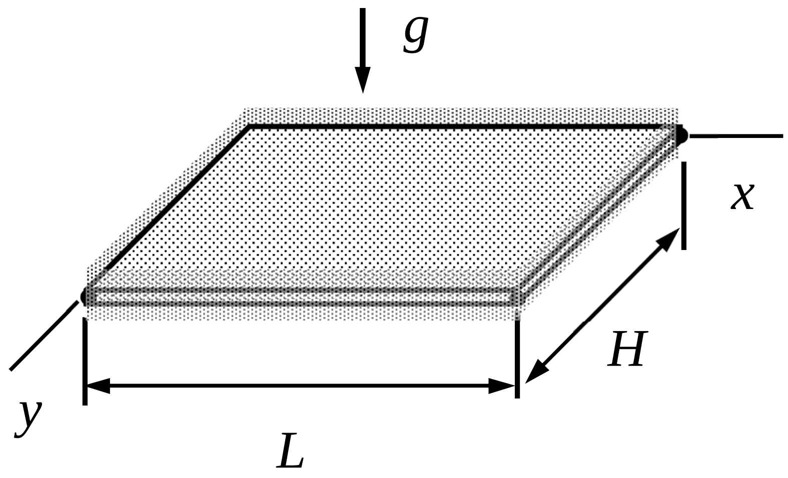
Assuming that $\overset{\leftrightarrow}{C}=0$, the thin slab and bending modes of the plate model disconnect and one may often consider the modes separately. Virtual work expression for the bending mode (Kirchhoff plate model) simplifies to

$$\delta W=-\int_{\Omega}\nabla_0\nabla_0 w:\overset{\leftrightarrow}{B}:\nabla_0\nabla_0 w dA+\int_{\Omega}\delta w b dA.$$

When written in the Cartesian (x,y,n) coordinate system

$$\delta W=-\int_{\Omega}\begin{Bmatrix}\dfrac{\partial^2\delta w}{\partial x^2}\\[2mm] \dfrac{\partial^2\delta w}{\partial y^2}\\[2mm] 2\dfrac{\partial^2\delta w}{\partial x\partial y}\end{Bmatrix}^{\mathrm{T}} D\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1}{2}(1-\nu)\end{bmatrix}\begin{Bmatrix}\dfrac{\partial^2 w}{\partial x^2}\\[2mm] \dfrac{\partial^2 w}{\partial y^2}\\[2mm] 2\dfrac{\partial^2 w}{\partial x\partial y}\end{Bmatrix}dA+\int_{\Omega}\delta w b_n dA.$$

EXAMPLE 5.6 Consider pure bending of a rectangle Kirchhoff plate $\Omega = (0, L) \times (0, H)$. Derive the series solution $w(x, y) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} w_{ij} \sin(i\pi \frac{x}{L}) \sin(j\pi \frac{y}{H})$ by considering the coefficients w_{ij} as the unknowns of the virtual work expression. Thickness t, Young's modulus E, and Poisson's ratio ν, and distributed load b in direction of $z-$axis are constants.



Answer $w_{ij} = 16 \frac{b}{D} \frac{1}{ij} / [(\frac{\pi i}{L})^2 + (\frac{\pi j}{H})^2]^2 \quad i, j \in \{1, 3, 5, \ldots\}, \quad w_{ij} = 0$ otherwise.

When the series approximation is substituted there, the virtual work expression becomes a variational expression for the unknown coefficients. Using then orthogonality of the sines and cosines on $\Omega=(0,L)\times(0,H)$, virtual work expressions of the internal and external forces boil down to

$$\delta W^{\mathrm{int}}=-\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\delta w_{ij}D\frac{LH}{4}[(\frac{i\pi}{L})^2+(\frac{j\pi}{H})^2]^2 w_{ij},$$

$$\delta W^{\mathrm{ext}}=-\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\delta w_{ij}b_{ij}, \text{ where } b_{ij}=\int_0^L\int_0^H b(x,y)\sin(i\pi\frac{x}{L})\sin(j\pi\frac{y}{H})dxdy.$$

The fundamental lemma of variation calculus implies that (here $b(x,y)=b=\rho g t$)

$$w_{ij}=b_{ij}/\left[D\frac{LH}{4}[(\frac{i\pi}{L})^2+(\frac{j\pi}{H})^2]^2\right], \text{ where } b_{ij}=4b\frac{LH}{ij\pi^2}\quad i,j\in\{1,3,5,\ldots\}.$$

MEC-E8003 Beam, plate and shell models, examples 5

1. Derive the components of the elastic isotropic Kirchhoff plate constitutive equation for bending. Consider the Cartesian (x, y, n) coordinate system and use the definitions $\vec{M} = \vec{\vec{B}} : \vec{\kappa}$ and $\vec{\kappa} = -\nabla_0 \nabla_0 w$. Cartesian coordinate system representation of the elasticity tensor is

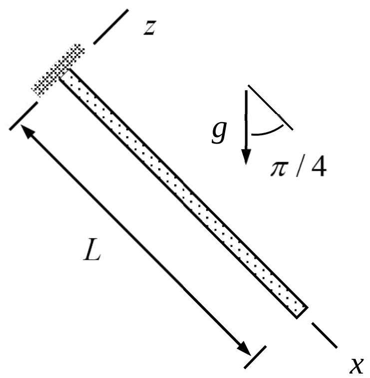
$$\vec{\vec{B}} = \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{i}\vec{j} + \vec{j}\vec{i} \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12} [E]_\sigma \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{i}\vec{j} + \vec{j}\vec{i} \end{Bmatrix} \text{ in which } [E]_\sigma = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}.$$

Answer $M_{xx} = -D(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2})$, $M_{yy} = -D(\nu \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2})$, $M_{xy} = -D(1-\nu)\frac{\partial^2 w}{\partial x \partial y}$.

2. Show that the transverse displacement of the Kirchhoff plate model satisfies the biharmonic equation $D\nabla_0^2 \nabla_0^2 w = b_n$. Start with the Reissner-Mindlin plate model equations for the bending mode:

$$\begin{Bmatrix} \dfrac{\partial Q_x}{\partial x} + \dfrac{\partial Q_y}{\partial y} + b_n \\ \dfrac{\partial M_{xx}}{\partial x} + \dfrac{\partial M_{xy}}{\partial y} - Q_x \\ \dfrac{\partial M_{yy}}{\partial y} + \dfrac{\partial M_{xy}}{\partial x} - Q_y \end{Bmatrix} = 0, \quad \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \frac{t^3}{12}[E]_\sigma \begin{Bmatrix} \dfrac{\partial \theta}{\partial x} \\ -\dfrac{\partial \phi}{\partial y} \\ \dfrac{\partial \theta}{\partial y} - \dfrac{\partial \phi}{\partial x} \end{Bmatrix}, \quad \begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = Gt \begin{Bmatrix} \dfrac{\partial w}{\partial x} + \theta \\ \dfrac{\partial w}{\partial y} - \phi \end{Bmatrix}.$$

3. Consider a cantilever Reissner-Mindlin plate strip (long in the $y-$direction) loaded by its own weight. Assuming that the solution is independent of y, determine the first order ordinary differential equations and the boundary conditions giving $N_{xx} = N(x)$, $Q_x = Q(x)$, $M_{xx} = M(x)$, $u(x)$, $w(x)$ and $\theta(x)$ as solutions. Thickness of the plate t, density ρ, Young's modulus E, and Poisson's ratio ν are constants.

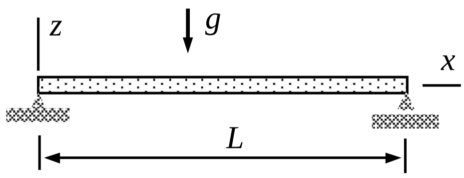


Answer

$$\frac{dN}{dx} + \frac{\rho g t}{\sqrt{2}} = 0, \quad \frac{dQ}{dx} - \frac{\rho g t}{\sqrt{2}} = 0, \quad \frac{dM}{dx} - Q = 0 \text{ in } (0, L), \qquad N = 0, \ M = 0, \ Q = 0 \text{ at } x = L.$$

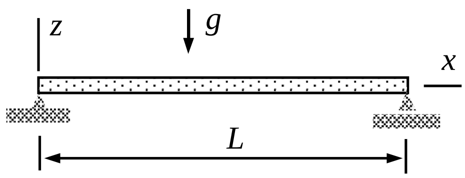
$$N = \frac{tE}{1-\nu^2}\frac{du}{dx}, \quad Q = Gt(\frac{dw}{dx} + \theta), \quad M = D\frac{d\theta}{dx} \text{ in } (0, L), \qquad u = 0, \ w = 0, \ \theta = 0 \text{ at } x = 0.$$

4. Consider a plate strip of the figure loaded by its own weight. Determine deflection w and rotation θ of the plate according to the Kirchhoff model. Thickness, and length of the plate are t and L, respectively. Density ρ, Young's modulus E, and Poisson's ratio ν are constants. Assume that stress resultants, displacements, and rotations depend on x only (consider a plate of width H).



Answer $w = -\dfrac{\rho g t L^4}{24D}[\dfrac{x}{L} - 2(\dfrac{x}{L})^3 + (\dfrac{x}{L})^4]$, $\theta = \dfrac{\rho g t L^3}{24D}[1 - 6(\dfrac{x}{L})^2 + 4(\dfrac{x}{L})^3]$

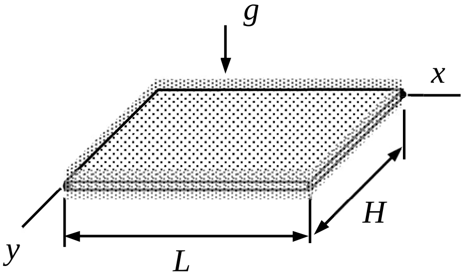
5. Consider the plate strip shown loaded by its own weight. Thickness, length and width of the plate are t, L, and H, respectively. Density ρ, Young's modulus E, and Poisson's ratio ν are constants. Find an approximation to the transverse displacement w of the plate using series $w = \mathrm{a}_0(1 - x/L)(x/L)$ (just one term of a series) in which a_0 is an unknown parameter. Use the principle of virtual work in form $\delta W = 0 \quad \forall \delta \mathrm{a}_0 \in \mathbb{R}$ and



$$\delta W = -\int_\Omega \begin{Bmatrix} \dfrac{\partial^2 \delta w}{\partial x^2} \\ \dfrac{\partial^2 \delta w}{\partial y^2} \\ 2\dfrac{\partial^2 \delta w}{\partial x \partial y} \end{Bmatrix}^{\mathrm{T}} D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{Bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \\ 2\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} dA + \int_\Omega \delta w b_n dA .$$

Answer $w = -\dfrac{\rho g t L^4}{24D}(1 - \dfrac{x}{L})\dfrac{x}{L}$

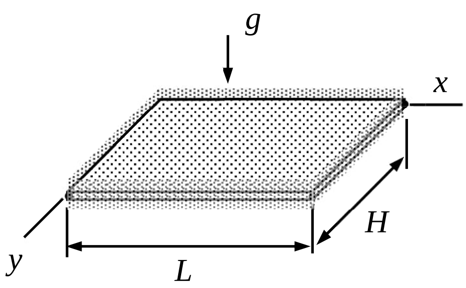
6. A simply supported rectangular plate of size $L \times H$ and thickness t is loaded by its own weight. Material parameters E, ν, and ρ are constants. Determine the displacement at the center point with $w = \mathrm{a}_0(xy/LH)(1 - x/L)(1 - y/H)$ (just one term of a series) in which a_0 is an unknown parameter. Use the principle of virtual work in form $\delta W = 0$ $\forall \delta \mathrm{a}_0 \in \mathbb{R}$. Assume that the solution does not explicitly depend on ν. Virtual work expression



$$\delta W=-\int_{\Omega}\begin{Bmatrix}\dfrac{\partial^2\delta w}{\partial x^2}\\ \dfrac{\partial^2\delta w}{\partial y^2}\\ 2\dfrac{\partial^2\delta w}{\partial x\partial y}\end{Bmatrix}^{\mathrm{T}} D\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1}{2}(1-\nu)\end{bmatrix}\begin{Bmatrix}\dfrac{\partial^2 w}{\partial x^2}\\ \dfrac{\partial^2 w}{\partial y^2}\\ 2\dfrac{\partial^2 w}{\partial x\partial y}\end{Bmatrix}dA+\int_{\Omega}\delta w b_n dA\,.$$

Answer $w=\dfrac{5}{8}\dfrac{\rho g t}{D}\dfrac{H^4L^4}{3H^4+5H^2L^2+3L^4}\dfrac{x}{L}(1-\dfrac{x}{L})\dfrac{y}{H}(1-\dfrac{y}{H})$

7. A simply supported rectangular plate of size $L\times H$ and thickness t is loaded by its own weight. Material parameters E, ν, and ρ are constants. Find an approximation to the transverse displacement by using $w=\mathrm{a}_0\sin(\pi x/L)\sin(\pi y/H)$ (just one term of a series) in which a_0 is an unknown parameter. Use the principle of virtual work in form $\delta W=0\ \ \forall\delta\mathrm{a}_0\in\mathbb{R}$ and assume that the solution does not explicitly depend on ν. Virtual work expression of the bending mode



$$\delta W=-\int_{\Omega}\begin{Bmatrix}\dfrac{\partial^2\delta w}{\partial x^2}\\ \dfrac{\partial^2\delta w}{\partial y^2}\\ 2\dfrac{\partial^2\delta w}{\partial x\partial y}\end{Bmatrix}^{\mathrm{T}} D\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1}{2}(1-\nu)\end{bmatrix}\begin{Bmatrix}\dfrac{\partial^2 w}{\partial x^2}\\ \dfrac{\partial^2 w}{\partial y^2}\\ 2\dfrac{\partial^2 w}{\partial x\partial y}\end{Bmatrix}dA+\int_{\Omega}\delta w b_n dA\,.$$

Answer $w=\dfrac{16}{\pi^6}\dfrac{\rho g t}{D}\dfrac{H^4L^4}{(H^2+L^2)^2}\sin(\pi\dfrac{x}{L})\sin(\pi\dfrac{y}{H})$

8. Find the representation $\nabla_0\cdot(\nabla_0\cdot\vec{\vec{M}})+b_n=0$ of the Kirchhoff plate bending equation in terms of components M_{rr}, $M_{r\phi}$ and $M_{\phi\phi}$ of the polar coordinate system. Assume rotation symmetry so that the moment components depend on r only.

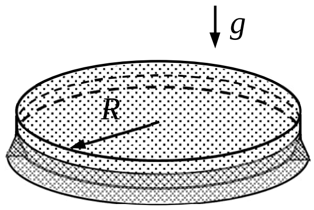
Answer $\dfrac{1}{r}\dfrac{d}{dr}[r(\dfrac{dM_{rr}}{dr}-\dfrac{1}{r}M_{\phi\phi})]+b_n=0$

9. Derive the components of the elastic isotropic Kirchhoff plate constitutive equation for bending in polar (r,ϕ,n) coordinate system. Use definitions $\vec{\vec{M}}=\vec{\vec{\vec{\vec{B}}}}:\vec{\vec{\kappa}}$, $\vec{\vec{\kappa}}=-\nabla_0\nabla_0 w$ and assume rotation symmetry $\partial w/\partial\phi=0$. The polar coordinate system representation of the bending elasticity tensor of plate model is

$$\vec{\vec{B}}=\begin{Bmatrix}\vec{e}_r\vec{e}_r\\ \vec{e}_\phi\vec{e}_\phi\\ \vec{e}_r\vec{e}_\phi+\vec{e}_\phi\vec{e}_r\end{Bmatrix}^{\mathrm{T}} D\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1}{2}(1-\nu)\end{bmatrix}\begin{Bmatrix}\vec{e}_r\vec{e}_r\\ \vec{e}_\phi\vec{e}_\phi\\ \vec{e}_r\vec{e}_\phi+\vec{e}_\phi\vec{e}_r\end{Bmatrix} \text{ in which } D=\frac{Et^3}{12(1-\nu^2)}.$$

Answer $M_{rr}=-D(\frac{d^2w}{dr^2}+\nu\frac{1}{r}\frac{dw}{dr})$, $M_{\phi\phi}=-D(\nu\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr})$, $M_{r\phi}=M_{\phi r}=0$.

10. A simply supported circular plate of radius R is loaded by its own weight as shown in the figure. Determine the displacement of the plate at the midpoint by using the Kirchhoff plate model in the polar coordinate system. Problem parameters E, ν, ρ and t are constants. Assume that the solution depends on the radial coordinate only.



Answer $w(0)=-\frac{3}{16}\frac{\rho g R^4}{Et^2}(5+\nu)(1-\nu)$

Derive the components of the elastic isotropic Kirchhoff plate constitutive equations for bending. Consider the Cartesian (x,y,n) coordinate system and use the definitions $\vec{M}=\vec{\vec{B}}:\vec{\kappa}$ and $\vec{\kappa}=-\nabla_0\nabla_0 w$. Cartesian coordinate system representation of the elasticity tensor is

$$\vec{\vec{B}}=\begin{Bmatrix}\vec{i}\vec{i}\\ \vec{j}\vec{j}\\ \vec{i}\vec{j}+\vec{j}\vec{i}\end{Bmatrix}^{\mathrm{T}} D\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1}{2}(1-\nu)\end{bmatrix}\begin{Bmatrix}\vec{i}\vec{i}\\ \vec{j}\vec{j}\\ \vec{i}\vec{j}+\vec{j}\vec{i}\end{Bmatrix} \text{ in which } D=\frac{Et^3}{12(1-\nu^2)}.$$

Solution

The (mid-plane) gradient operator of the Cartesian (x,y,n) coordinate system coordinate system gives

$$\vec{\kappa}=-\nabla_0\nabla_0 w=-(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y})(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y})w=-\vec{i}\vec{i}\frac{\partial^2 w}{\partial x^2}-(\vec{i}\vec{j}+\vec{j}\vec{i})\frac{\partial^2 w}{\partial x\partial y}-\vec{j}\vec{j}\frac{\partial^2 w}{\partial y^2} \; .$$

By using the constitutive equation and elasticity tensor

$$\vec{M}=\vec{\vec{B}}:\vec{\kappa}=-\begin{Bmatrix}\vec{i}\vec{i}\\ \vec{j}\vec{j}\\ \vec{i}\vec{j}+\vec{j}\vec{i}\end{Bmatrix}^{\mathrm{T}} D\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1}{2}(1-\nu)\end{bmatrix}\begin{Bmatrix}\dfrac{\partial^2 w}{\partial x^2}\\ \dfrac{\partial^2 w}{\partial y^2}\\ 2\dfrac{\partial^2 w}{\partial x\partial y}\end{Bmatrix}.$$

Therefore, the Cartesian coordinate system components of the bending moment constitutive equation are

$$M_{xx}=-D(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}),\; M_{yy}=-D(\nu\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}),\; M_{xy}=M_{yx}=-D(1-\nu)\frac{\partial^2 w}{\partial x\partial y}. \quad \leftarrow$$

Show that the vertical displacement $w(x,y)$ of the Kirchhoff plate model satisfies the biharmonic equation $D\nabla_0^2\nabla_0^2 w = b_n$. Start with the Reissner-Mindlin plate model equations for the bending mode:

$$\begin{Bmatrix} \dfrac{\partial Q_x}{\partial x}+\dfrac{\partial Q_y}{\partial y}+b_n \\ \dfrac{\partial M_{xx}}{\partial x}+\dfrac{\partial M_{xy}}{\partial y}-Q_x \\ \dfrac{\partial M_{yy}}{\partial y}+\dfrac{\partial M_{xy}}{\partial x}-Q_y \end{Bmatrix}=0,\quad \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix}=D\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}\begin{Bmatrix} \dfrac{\partial \theta}{\partial x} \\ -\dfrac{\partial \phi}{\partial y} \\ \dfrac{\partial \theta}{\partial y}-\dfrac{\partial \phi}{\partial x} \end{Bmatrix},\quad \begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix}=Gt\begin{Bmatrix} \dfrac{\partial w}{\partial x}+\theta \\ \dfrac{\partial w}{\partial y}-\phi \end{Bmatrix}.$$

Solution

Kirchhoff constraints are first used to write the constitutive equations in terms of the transverse displacement

$$\theta=-\frac{\partial w}{\partial x} \text{ and } \phi=\frac{\partial w}{\partial y} \quad\Rightarrow\quad \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix}=-D\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}\begin{Bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \\ 2\dfrac{\partial^2 w}{\partial x\partial y} \end{Bmatrix}=-D\begin{Bmatrix} \dfrac{\partial^2 w}{\partial x^2}+\nu\dfrac{\partial^2 w}{\partial y^2} \\ \dfrac{\partial^2 w}{\partial y^2}+\nu\dfrac{\partial^2 w}{\partial x^2} \\ (1-\nu)\dfrac{\partial^2 w}{\partial x\partial y} \end{Bmatrix}.$$

In the Kirchhoff model, shear forces Q_x and Q_y are in the role of constraint forces to be solved from the moment equations. Eliminating the shear forces from the equilibrium equation in the transverse direction by using the moment equations gives

$$\begin{Bmatrix} \dfrac{\partial Q_x}{\partial x}+\dfrac{\partial Q_y}{\partial y}+b_n \\ \dfrac{\partial M_{xx}}{\partial x}+\dfrac{\partial M_{xy}}{\partial y}-Q_x \\ \dfrac{\partial M_{yy}}{\partial y}+\dfrac{\partial M_{xy}}{\partial x}-Q_y \end{Bmatrix}=0 \quad\Rightarrow\quad \frac{\partial^2 M_{xx}}{\partial x^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}+\frac{\partial^2 M_{yy}}{\partial y^2}+b_n=0.$$

The biharmonic equation for the transverse displacement follows from the equilibrium equation above, when the constitutive equations for the moments are substituted there

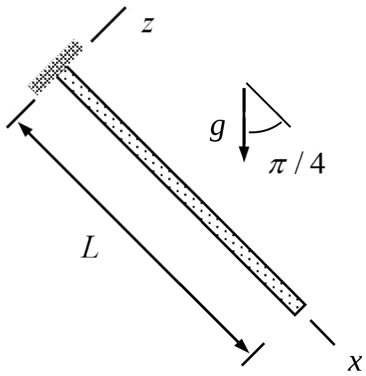
$$\frac{\partial^2 M_{xx}}{\partial x^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}+\frac{\partial^2 M_{yy}}{\partial y^2}+b_n=0 \quad\Leftrightarrow$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)+2(1-\nu)\frac{\partial^2}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)-\frac{b_n}{D}=0 \quad\Leftrightarrow$$

$$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}-\frac{b_n}{D}=0 \quad\text{or}\quad \nabla_0^2\nabla_0^2 w=\frac{b_n}{D}.$$

The last invariant form holds also, e.g., in the polar coordinate system.

Consider a cantilever Reissner-Mindlin plate strip (long in the $y-$ direction) loaded by its own weight. Assuming that the solution is independent of y, determine the first order ordinary differential equations and the boundary conditions giving $N_{xx} = N(x)$, $Q_x = Q(x)$, $M_{xx} = M(x)$, $u(x)$, $w(x)$ and $\theta(x)$ as solutions. Thickness of the plate t, density ρ, Young's modulus E, and Poisson's ratio ν are constants.



Solution

Equilibrium and constitutive equations of the thin-slab and bending modes are

$$\begin{Bmatrix} \dfrac{\partial N_{xx}}{\partial x} + \dfrac{\partial N_{xy}}{\partial y} + b_x \\ \dfrac{\partial N_{yy}}{\partial y} + \dfrac{\partial N_{xy}}{\partial x} + b_y \end{Bmatrix} = 0 , \quad \begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = t[E]_\sigma \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} = \frac{tE}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix},$$

$$\begin{Bmatrix} \dfrac{\partial Q_x}{\partial x} + \dfrac{\partial Q_y}{\partial y} + b_n \\ \dfrac{\partial M_{xx}}{\partial x} + \dfrac{\partial M_{xy}}{\partial y} - Q_x \\ \dfrac{\partial M_{yy}}{\partial y} + \dfrac{\partial M_{xy}}{\partial x} - Q_y \end{Bmatrix} = 0 , \quad \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \theta}{\partial x} \\ -\dfrac{\partial \phi}{\partial y} \\ \dfrac{\partial \theta}{\partial y} - \dfrac{\partial \phi}{\partial x} \end{Bmatrix}, \quad \begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = Gt \begin{Bmatrix} \dfrac{\partial w}{\partial x} + \theta \\ \dfrac{\partial w}{\partial y} - \phi \end{Bmatrix}.$$

Derivatives with respect to y vanish, $b_x = \rho g t / \sqrt{2}$, and $b_n = -\rho g t / \sqrt{2}$. The Reissner-Mindlin plate equations of the planar problem simplify to

$$\frac{dN}{dx} + \frac{\rho g t}{\sqrt{2}} = 0 , \quad \frac{dQ}{dx} - \frac{\rho g t}{\sqrt{2}} = 0 , \quad \frac{dM}{dx} - Q = 0 \text{ in } (0, L), \quad \Leftarrow$$

$$N = \frac{tE}{1-\nu^2} \frac{du}{dx}, \quad Q = Gt(\frac{dw}{dx} + \theta), \quad M = D\frac{d\theta}{dx} \text{ in } (0, L), \quad \Leftarrow$$

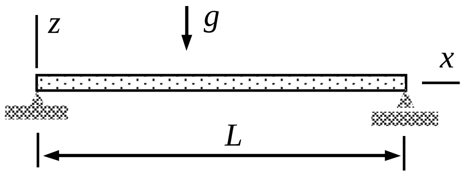
Boundary conditions can be deduced from the figure:

$$u = 0 , \quad w = 0 , \quad \theta = 0 \quad \text{at} \quad x = 0 , \quad \Leftarrow$$

$$N = 0 , \quad M = 0 , \quad Q = 0 \quad \text{at} \quad x = L . \quad \Leftarrow$$

Solution to equations can be obtained by considering the equilibrium equations and the boundary conditions at the free end first. After that, solutions to the displacement components follow from the constitutive equations and the boundary conditions at the clamped edge.

Consider a plate strip of the figure loaded by its own weight. Determine deflection w and rotation θ of the plate according to the Kirchhoff model. Thickness, and length of the plate are t and L, respectively. Density ρ, Young's modulus E, and Poisson's ratio ν are constants. Assume that stress-resultants, displacements, and rotations depend on x only (consider a plate of width H).



Solution

Assuming that all derivatives with respect to y vanish, the plate equations of the formulae collection (just the equations associated with the bending modes) are

$$\begin{Bmatrix} \dfrac{\partial Q_x}{\partial x}+\dfrac{\partial Q_y}{\partial y}+b_n \\ \dfrac{\partial M_{xx}}{\partial x}+\dfrac{\partial M_{xy}}{\partial y}-Q_x \\ \dfrac{\partial M_{yy}}{\partial y}+\dfrac{\partial M_{xy}}{\partial x}-Q_y \end{Bmatrix}=0\,,\quad \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix}=D\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}\begin{Bmatrix} \dfrac{\partial \theta}{\partial x} \\ -\dfrac{\partial \phi}{\partial y} \\ \dfrac{\partial \theta}{\partial y}-\dfrac{\partial \phi}{\partial x} \end{Bmatrix},\quad \begin{Bmatrix} \dfrac{\partial w}{\partial x}+\theta \\ \dfrac{\partial w}{\partial y}-\phi \end{Bmatrix}=0 \text{ in } \Omega\,,$$

$$Q_n-\underline{Q}+\frac{\partial}{\partial s}(M_{ns}-\underline{M}_s)=0 \text{ or } w-\underline{w}=0 \text{ on } \partial\Omega\,,$$

$$M_{nn}-\underline{M}_n=0 \text{ or } \frac{\partial w}{\partial n}+\underline{\theta}_s=0 \text{ on } \partial\Omega\,.$$

Taking into account only the equations needed and using notation $M_{xx}\equiv M$ and $Q_x \equiv Q$

$$\frac{dQ}{dx}+b_n=0\,,\ \frac{dM}{dx}-Q=0\,,\ M_{xx}=D\frac{d\theta}{dx}\,,\ \text{and}\ \frac{dw}{dx}+\theta=0 \text{ in } (0,L)\,.$$

Boundary conditions specify either displacement or shear force and bending moment or rotation. From the figure

$$M=0 \text{ and } w=0 \text{ on } \{0,L\}\,.$$

After elimination of the shear force and the bending moment, the boundary value problem for the deflection w becomes (the equation system can also be solved one equation at a time in its original form)

$$-D\frac{d^4w}{dx^4}+b_n=0 \text{ in } (0,L) \text{ and } w=\frac{d^2w}{dx^2}=0 \text{ on } (0,L)\,. \Leftarrow$$

Generic solution to the differential equation can be obtained by repeated integrations

$$\frac{d^4 w}{dx^4} = \frac{b_n}{D} \quad \Rightarrow \quad w(x) = \frac{b_n}{D}\frac{x^4}{24} + ax^3 + bx^2 + cx + d \ .$$

Boundary conditions imply that (the equations can be solved starting from the first, then using the already obtained solution in the second etc.)

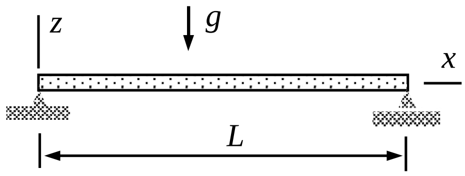
$$\begin{Bmatrix} w(0) \\ \dfrac{d^2 w}{dx^2}(0) \\ \dfrac{d^2 w}{dx^2}(L) \\ w(L) \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 6L & 2 & 0 & 0 \\ L^3 & L^2 & L & 1 \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \\ d \end{Bmatrix} + \frac{b_n L^2}{2D} \begin{Bmatrix} 0 \\ 0 \\ 1 \\ L^2/12 \end{Bmatrix} = 0 \quad \Rightarrow$$

$$d = b = 0 \ \Rightarrow \ a = -\frac{b_n L}{12D} \ \Rightarrow \ c = \frac{b_n L^3}{24D} \ .$$

Therefore, using the expressions of the coefficient in the displacement solution, expression of the distributed force $b_n = -\rho g t$, and the constraint $\theta = -dw/dx$ of the Kirchhoff model

$$w = -\frac{\rho g t L^4}{24D}[\frac{x}{L} - 2(\frac{x}{L})^3 + (\frac{x}{L})^4] \quad \Rightarrow \quad \theta = -\frac{dw}{dx} = \frac{\rho g t L^3}{24D}[1 - 6(\frac{x}{L})^2 + 4(\frac{x}{L})^3] \ .$$

Consider the plate strip shown loaded by its own weight. Thickness, length and width of the plate are t, L, and H, respectively. Density ρ, Young's modulus E, and Poisson's ratio ν are constants. Find an approximation to the transverse displacement w of the plate using series $w = \mathrm{a}_0(1 - x/L)(x/L)$ (just one term of a series) in which a_0 is an unknown parameter. Use the principle of virtual work in form $\delta W = 0 \quad \forall \delta \mathrm{a}_0 \in \mathbb{R}$ and



$$\delta W = -\int_\Omega \delta \begin{Bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ 2\partial^2 w/\partial x \partial y \end{Bmatrix}^{\mathrm{T}} D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ 2\partial^2 w/\partial x \partial y \end{Bmatrix} dA + \int_\Omega b_n \delta w dA .$$

Solution

Principle of virtual work gives a straightforward way to find approximate/series solutions to beam, plate etc. problem. First, approximation of the "right" form is substituted into the virtual work expression. The approximation is a sum of terms having multipliers to be determined. Second, principle of virtual work and the fundamental lemma of variation calculus are applied with respect to the multipliers. Finite element method, sine series solutions, etc. can be taken just particular cases of the method. Virtual work expression of the Kirchhoff plate model bending mode

$$\delta W = -\int_\Omega \delta \vec{\kappa}_{\mathrm{c}} : \vec{M} dA + \int_\Omega b_n \delta w dA$$

in terms of the transverse displacement follow when the constitutive equation $\vec{M} = \vec{\vec{B}} : \vec{\kappa}$, strain definition $\vec{\kappa} = -\nabla_0 \nabla_0 w$, and the Cartesian coordinate system representation of the elasticity tensor

$$\vec{\vec{B}} = D \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{i}\vec{j} + \vec{j}\vec{i} \end{Bmatrix} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{i}\vec{j} + \vec{j}\vec{i} \end{Bmatrix} .$$

are substituted there. Approximation to the transverse displacement (notice that the polynomial shape is known and variation of displacement is through the multiplier)

$$w = \mathrm{a}_0 (1 - \frac{x}{L})(\frac{x}{L}) \quad \Rightarrow \quad \frac{\partial^2 w}{\partial x^2} = -\mathrm{a}_0 \frac{2}{L^2} \quad \text{and} \quad \frac{\partial^2 w}{\partial x \partial y} = \frac{\partial^2 w}{\partial y^2} = 0 \quad \Rightarrow$$

$$\delta w = \delta \mathrm{a}_0 (1 - \frac{x}{L})(\frac{x}{L}) \quad \Rightarrow \quad \delta \frac{\partial^2 w}{\partial x^2} = -\delta \mathrm{a}_0 \frac{2}{L^2} \quad \text{and} \quad \delta \frac{\partial^2 w}{\partial x \partial y} = \delta \frac{\partial^2 w}{\partial y^2} = 0 .$$

When the approximation is substituted there, virtual work expression simplifies to

$$\delta W=-\int_0^H\int_0^L D(-\delta a_0\frac{2}{L^2})(-a_0\frac{2}{L^2})dxdy+\int_0^H\int_0^L b_n\delta a_0(1-\frac{x}{L})(\frac{x}{L})dxdy \quad\Rightarrow$$

$$\delta W=-\delta a_0 H(\frac{4}{L^3}Da_0+b_n\frac{1}{6})\,.$$

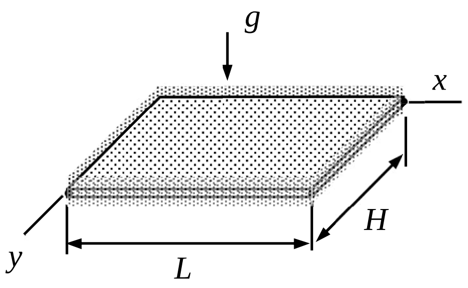
Principle of virtual work $\delta W=0\ \forall\delta a_0$ and the fundamental lemma of variation calculus give with $b_n=-\rho gt$

$$\delta a_0 HL(-\frac{4}{L^4}Da_0+b_n\frac{1}{6})=0\ \ \forall\delta a_0 \quad\Leftrightarrow\quad -\frac{4}{L^4}Da_0+b_n\frac{1}{6}=0 \quad\Rightarrow\quad a_0=\frac{L^4 b_n}{24D}=-\frac{L^4\rho gt}{24D}\,.$$

Therefore, an approximation to the transverse displacement is given by

$$w=-\frac{\rho gtL^4}{24D}(1-\frac{x}{L})\frac{x}{L}\,.$$

A simply supported rectangular plate of size $L \times H$ and thickness t is loaded by its own weight. Material parameters E, ν, and ρ are constants. Determine the displacement at the center point with $w = \mathrm{a}_0 (xy/LH)(1-x/L)(1-y/H)$ (just one term of a series) in which a_0 is an unknown parameter. Use the principle of virtual work in form $\delta W = 0 \;\; \forall \delta \mathrm{a}_0 \in \mathbb{R}$ and assume that the solution does not explicitly depend on ν. Virtual work expression of the bending mode



$$\delta W = -\int_\Omega \begin{Bmatrix} \dfrac{\partial^2 \delta w}{\partial x^2} \\ \dfrac{\partial^2 \delta w}{\partial y^2} \\ 2\dfrac{\partial^2 \delta w}{\partial x \partial y} \end{Bmatrix}^{\mathrm{T}} D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{Bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \\ 2\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} dA + \int_\Omega \delta w b_n dA \,.$$

Solution

As the solution does not depend on the Poisson's ratio ν (additional information), one may consider D and ν as the two independent material parameters of a linearly elastic material and choose a convenient value like $\nu = 1$ to simplify the calculations. Then, the virtual work expression simplifies to

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = -\int_\Omega D \begin{Bmatrix} \dfrac{\partial^2 \delta w}{\partial x^2} \\ \dfrac{\partial^2 \delta w}{\partial y^2} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \end{Bmatrix} dA + \int_\Omega b_n \delta w dA \,.$$

Approximation to the transverse displacement is chosen to be (the polynomial shape is known and variation of displacement is through the multiplier)

$$w = \mathrm{a}_0 [\frac{x}{L} - (\frac{x}{L})^2][\frac{y}{H} - (\frac{y}{H})^2] \quad \Rightarrow \quad \delta w = \delta \mathrm{a}_0 [\frac{x}{L} - (\frac{x}{L})^2][\frac{y}{H} - (\frac{y}{H})^2] \,,$$

$$\frac{\partial^2 w}{\partial x^2} = -\mathrm{a}_0 \frac{2}{L^2}[\frac{y}{H} - (\frac{y}{H})^2] \quad \Rightarrow \quad \delta \frac{\partial^2 w}{\partial x^2} = -\delta \mathrm{a}_0 \frac{2}{L^2}[\frac{y}{H} - (\frac{y}{H})^2] \,,$$

$$\frac{\partial^2 w}{\partial y^2} = -\mathrm{a}_0 \frac{2}{H^2}[\frac{x}{L} - (\frac{x}{L})^2] \quad \Rightarrow \quad \frac{\partial^2 w}{\partial y^2} = -\delta \mathrm{a}_0 \frac{2}{H^2}[\frac{x}{L} - (\frac{x}{L})^2].$$

When the approximation is substituted there, virtual work expression takes the form

$$\delta W^{\text{int}} = -\delta \mathrm{a}_0 \int_0^L \int_0^H D \begin{Bmatrix} \frac{2}{L^2}[\frac{y}{H}-(\frac{y}{H})^2] \\ \frac{2}{H^2}[\frac{x}{L}-(\frac{x}{L})^2] \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \frac{2}{L^2}[\frac{y}{H}-(\frac{y}{H})^2] \\ \frac{2}{H^2}[\frac{x}{L}-(\frac{x}{L})^2] \end{Bmatrix} dxdy \mathrm{a}_0 \quad \Rightarrow$$

$$\delta W^{\text{int}} = -\delta \mathrm{a}_0 4D(\frac{1}{L^4}\frac{HL}{30}+\frac{2}{H^2L^2}\frac{HL}{36}+\frac{1}{H^4}\frac{HL}{30})\mathrm{a}_0 ,$$

$$\delta W^{\text{ext}} = \delta \mathrm{a}_0 \int_0^L \int_0^H b_n[\frac{x}{L}-(\frac{x}{L})^2][\frac{y}{H}-(\frac{y}{H})^2]dxdy = \delta \mathrm{a}_0 b_n \frac{HL}{36} .$$

Virtual work expression

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = -\delta \mathrm{a}_0 [4D(\frac{1}{L^4}\frac{HL}{30}+\frac{2}{H^2L^2}\frac{HL}{36}+\frac{1}{H^4}\frac{HL}{30})\mathrm{a}_0 - b_n \frac{HL}{36}] .$$

Principle of virtual work $\delta W = 0 \ \forall \delta \mathrm{a}_0$ and the fundamental lemma of variation calculus imply that (here $b_n = \rho g t$)

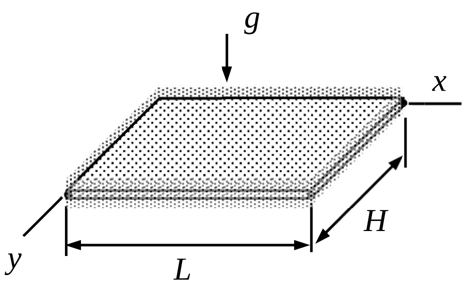
$$4D(\frac{1}{L^4}\frac{HL}{30}+\frac{2}{H^2L^2}\frac{HL}{36}+\frac{1}{H^4}\frac{HL}{30})\mathrm{a}_0 - b_n \frac{HL}{36} = 0 \quad \Leftrightarrow \quad \mathrm{a}_0 = \frac{\rho g t}{8D}\frac{5H^4L^4}{3H^4+5H^2L^2+3L^4} .$$

Therefore, approximation to the transverse displacement becomes

$$w = \frac{5\rho g t}{8D}\frac{H^4L^4}{3H^4+5H^2L^2+3L^4}\frac{x}{L}(1-\frac{x}{L})\frac{y}{H}(1-\frac{y}{H}) .$$

Notice! The double sine series solution with 100 terms in both directions gives in case of the square plate displacement $w = 0.0041 b_n L^4 / D$ at the center point. Displacement given by the one parameter approximation is $w = 0.0036 b_n L^4 / D$.

A simply supported rectangular plate of size $L \times H$ and thickness t is loaded by its own weight. Material parameters E, ν, and ρ are constants. Find an approximation to transverse displacement by using $w = \mathrm{a}_0 \sin(\pi x / L)\sin(\pi y / H)$ (just one term of a series) in which a_0 is an unknown parameter. Use the principle of virtual work in form $\delta W = 0 \;\; \forall \delta \mathrm{a}_0 \in \mathbb{R}$ and assume that the solution does not explicitly depend on ν. Virtual work expression of the bending mode



$$\delta W = -\int_\Omega \begin{Bmatrix} \dfrac{\partial^2 \delta w}{\partial x^2} \\ \dfrac{\partial^2 \delta w}{\partial y^2} \\ 2\dfrac{\partial^2 \delta w}{\partial x \partial y} \end{Bmatrix}^{\mathrm{T}} D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{Bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \\ 2\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} dA + \int_\Omega \delta w b_n dA .$$

Solution

As the solution does not depend on the Poisson's ratio ν (additional information), one may consider D and ν as the two independent material parameters of a linearly elastic material and choose a convenient value like $\nu = 1$ to simplify the calculations. Then, the virtual work expression simplifies to

$$\delta W = \delta W^{\mathrm{int}} + \delta W^{\mathrm{ext}} = -\int_\Omega D \begin{Bmatrix} \dfrac{\partial^2 \delta w}{\partial x^2} \\ \dfrac{\partial^2 \delta w}{\partial y^2} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \end{Bmatrix} dA + \int_\Omega b_n \delta w dA .$$

Principle of virtual work gives a straightforward way to find approximate/series solutions to beam, plate etc. problem. First, approximation of the "right" form is substituted into the virtual work expression. Second, principle of virtual work and the fundamental lemma of variation calculus are applied with respect to the multipliers. Approximation to the transverse displacement is chosen to be (variation of displacement is through the multiplier)

$$w = \mathrm{a}_0 \sin(\pi \frac{x}{L})\sin(\pi \frac{y}{H}) \quad \text{and} \quad \delta w = \delta \mathrm{a}_0 \sin(\pi \frac{x}{L})\sin(\pi \frac{y}{H}) ,$$

$$\frac{\partial^2 w}{\partial x^2} = -\mathrm{a}_0 \frac{\pi^2}{L^2} \sin(\pi \frac{x}{L})\sin(\pi \frac{y}{H}) \quad \text{and} \quad \delta \frac{\partial^2 w}{\partial x^2} = -\delta \mathrm{a}_0 \frac{\pi^2}{L^2} \sin(\pi \frac{x}{L})\sin(\pi \frac{y}{H}) ,$$

$$\frac{\partial^2 w}{\partial y^2} = -\mathrm{a}_0 \frac{\pi^2}{H^2} \sin(\pi \frac{x}{L})\sin(\pi \frac{y}{H}) \quad \text{and} \quad \delta \frac{\partial^2 w}{\partial y^2} = -\delta \mathrm{a}_0 \frac{\pi^2}{H^2} \sin(\pi \frac{x}{L})\sin(\pi \frac{y}{H}) .$$

Orthogonality of sines and cosines and known integrals like

$$\int_0^L \sin(i\pi\frac{x}{L})\sin(j\pi\frac{x}{L})dx = \delta_{ij}\frac{L}{2}, \quad \int_0^L \cos(i\pi\frac{x}{L})\cos(j\pi\frac{x}{L})dx = \delta_{ij}\frac{L}{2},$$

$$\int_0^L \sin(i\pi\frac{x}{L})dx = \frac{L}{\pi}\frac{1-(-1)^i}{i}$$

simplify the calculations with sinusoidal shape functions. When the approximation is substituted there, virtual work expressions of the internal and external forces take the forms

$$\delta W^{\text{int}} = -\delta \mathrm{a}_0 \int_0^L \int_0^H D \begin{Bmatrix} \frac{\pi^2}{L^2}\sin(\pi\frac{x}{L})\sin(\pi\frac{y}{H}) \\ \frac{\pi^2}{H^2}\sin(\pi\frac{x}{L})\sin(\pi\frac{y}{H}) \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \frac{\pi^2}{L^2}\sin(\pi\frac{x}{L})\sin(\pi\frac{y}{H}) \\ \frac{\pi^2}{H^2}\sin(\pi\frac{x}{L})\sin(\pi\frac{y}{H}) \end{Bmatrix} dxdy\mathrm{a}_0 \quad \Rightarrow$$

$$\delta W^{\text{int}} = -\delta \mathrm{a}_0 \int_0^L \int_0^H D\sin^2(\pi\frac{x}{L})\sin^2(\pi\frac{y}{H}) \begin{Bmatrix} \frac{\pi^2}{L^2} \\ \frac{\pi^2}{H^2} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \frac{\pi^2}{L^2} \\ \frac{\pi^2}{H^2} \end{Bmatrix} dxdy\mathrm{a}_0 \quad \Rightarrow$$

$$\delta W^{\text{int}} = -\delta \mathrm{a}_0 D\frac{LH}{4}[(\frac{\pi}{L})^4 + 2(\frac{\pi}{L})^2(\frac{\pi}{H})^2 + (\frac{\pi}{H})^4]\mathrm{a}_0 = -\delta \mathrm{a}_0 D\frac{LH}{4}[(\frac{\pi}{L})^2 + (\frac{\pi}{H})^2]^2\mathrm{a}_0,$$

$$\delta W^{\text{ext}} = \delta \mathrm{a}_0 b_n \int_0^L \sin(\pi\frac{x}{L})dx \int_0^H \sin(\pi\frac{y}{H})dy = \delta \mathrm{a}_0 b_n \frac{4LH}{\pi^2}.$$

Virtual work expression

$$\delta W = -\delta \mathrm{a}_0 D\frac{LH}{4}[(\frac{\pi}{L})^2 + (\frac{\pi}{H})^2]^2\mathrm{a}_0 + \delta \mathrm{a}_0 \frac{4LH}{\pi^2}b_n.$$

Principle of virtual work $\delta W = 0 \;\forall \delta \mathrm{a}_0$ and the fundamental lemma of variation calculus imply that (here $b_n = \rho g t$)

$$\mathrm{a}_0 = \frac{4LH}{\pi^2}\frac{b_n}{D} / D\frac{LH}{4}[(\frac{\pi}{L})^2 + (\frac{\pi}{H})^2]^2 = \frac{16}{\pi^6}\frac{H^4L^4}{(L^2+H^2)^2}\frac{b_n}{D}.$$

Therefore, approximation to the transverse displacement becomes

$$w = \frac{16}{\pi^6}\frac{H^4L^4}{(L^2+H^2)^2}\frac{b_n}{D}\sin(\pi\frac{x}{L})\sin(\pi\frac{y}{H}).$$

Derive the component form of the Kirchhoff plate equation (bending)

$$\nabla_0 \cdot (\nabla_0 \cdot \vec{\vec{M}}) + b_n = 0$$

in terms of components M_{rr}, $M_{r\phi}$ and $M_{\phi\phi}$ of the polar coordinate system. Assume rotation symmetry so that the moment components depend on r only.

Solution

Assuming symmetry $\vec{\vec{M}} = \vec{\vec{M}}_{\text{c}}$, the component representations of the planar gradient operator and moment tensor in the (r,ϕ,n) coordinate system are

$$\nabla_0 = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\text{T}} \begin{Bmatrix} \dfrac{\partial}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial}{\partial \phi} \end{Bmatrix} = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial \phi},$$

$$\vec{\vec{M}} = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\text{T}} \begin{bmatrix} M_{rr} & M_{r\phi} \\ M_{r\phi} & M_{\phi\phi} \end{bmatrix} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} = \vec{e}_r\vec{e}_r M_{rr} + \vec{e}_r\vec{e}_\phi M_{r\phi} + \vec{e}_\phi\vec{e}_r M_{r\phi} + \vec{e}_\phi\vec{e}_\phi M_{\phi\phi}.$$

First divergence of the moment tensor by considering the four terms of $\vec{\vec{M}}$ one by one (each term of $\vec{\vec{M}}$ may give rise to 6 derivative terms). The derivatives of the basis vectors are $\partial\vec{e}_r / \partial\phi = \vec{e}_\phi$ and $\partial\vec{e}_\phi / \partial\phi = -\vec{e}_r$ and $\partial M_{rr} / \partial\phi = 0$, $\partial M_{\phi\phi} / \partial\phi = 0$, and $\partial M_{r\phi} / \partial\phi = 0$ by assumption.

$$(\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial \phi}) \cdot (\vec{e}_r\vec{e}_r M_{rr}) = \vec{e}_r \frac{dM_{rr}}{dr} + \vec{e}_r \frac{1}{r} M_{rr} = \vec{e}_r \frac{1}{r}\frac{d}{dr}(rM_{rr}),$$

$$(\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial \phi}) \cdot (\vec{e}_r\vec{e}_\phi M_{r\phi}) = \vec{e}_\phi \frac{dM_{r\phi}}{dr} + \vec{e}_\phi \frac{1}{r} M_{r\phi} = \vec{e}_\phi \frac{1}{r}\frac{d}{dr}(rM_{r\phi}),$$

$$(\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial \phi}) \cdot (\vec{e}_\phi\vec{e}_r M_{r\phi}) = \vec{e}_\phi \frac{1}{r} M_{r\phi},$$

$$(\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial \phi}) \cdot (\vec{e}_\phi\vec{e}_\phi M_{\phi\phi}) = -\frac{1}{r}\vec{e}_r M_{\phi\phi}.$$

Therefore, the divergence of the moment simplifies to

$$\nabla_0 \cdot \vec{\vec{M}} = \vec{e}_r \frac{1}{r}[\frac{d}{dr}(rM_{rr}) - M_{\phi\phi}] + \vec{e}_\phi \frac{1}{r}[\frac{d}{dr}(rM_{r\phi}) + M_{r\phi}].$$

Application of the divergence operator again gives

$$(\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial \phi}) \cdot \vec{e}_r[\frac{1}{r}\frac{d}{dr}(rM_{rr}) - \frac{1}{r}M_{\phi\phi}] = \frac{d}{dr}[\frac{1}{r}\frac{d}{dr}(rM_{rr}) - \frac{1}{r}M_{\phi\phi}]) + \frac{1}{r}[\frac{1}{r}\frac{d}{dr}(rM_{rr}) - \frac{1}{r}M_{\phi\phi}],$$

$$(\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial \phi}) \cdot \vec{e}_\phi [\frac{1}{r}\frac{d}{dr}(rM_{r\phi}) + \frac{1}{r}M_{r\phi}] = 0\,.$$

Finally, combining the terms

$$\nabla_0 \cdot (\nabla_0 \cdot \overleftrightarrow{M}) + b_n = \frac{1}{r}\frac{d}{dr}[\frac{d}{dr}(rM_{rr}) - M_{\phi\phi}] + b_n = 0\,.$$

Derive the components of the elastic isotropic Kirchhoff plate constitutive equations for bending in polar (r,ϕ,n) coordinate system. Use definitions $\vec{\vec{M}}=\vec{\vec{\vec{\vec{B}}}}:\vec{\kappa}$, $\vec{\kappa}=-\nabla_0\nabla_0 w$ and assume rotation symmetry $\partial w/\partial\phi=0$. The polar coordinate system representation of the bending elasticity tensor of plate model is

$$\vec{\vec{\vec{\vec{B}}}}=\begin{Bmatrix}\vec{e}_r\vec{e}_r\\ \vec{e}_\phi\vec{e}_\phi\\ \vec{e}_r\vec{e}_\phi+\vec{e}_\phi\vec{e}_r\end{Bmatrix}^{\mathrm{T}} D\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1}{2}(1-\nu)\end{bmatrix}\begin{Bmatrix}\vec{e}_r\vec{e}_r\\ \vec{e}_\phi\vec{e}_\phi\\ \vec{e}_r\vec{e}_\phi+\vec{e}_\phi\vec{e}_r\end{Bmatrix} \text{ in which } D=\frac{Et^3}{12(1-\nu^2)}.$$

Solution

The (mid-plane) gradient operator of the polar (r,ϕ,n) coordinate system gives

$$\vec{\kappa}=-\nabla_0\nabla_0 w=-(\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\phi\frac{\partial}{r\partial\phi})(\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\phi\frac{\partial}{r\partial\phi})w \quad\Rightarrow$$

$$\vec{\kappa}=-\nabla_0\nabla_0 w=-(\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\phi\frac{\partial}{r\partial\phi})(\vec{e}_r\frac{\partial w}{\partial r})=-\vec{e}_r\vec{e}_r\frac{d^2w}{dr^2}-\vec{e}_\phi\vec{e}_\phi\frac{1}{r}\frac{dw}{dr}\ .$$

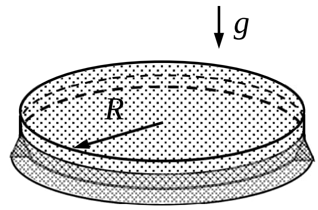
By using the constitutive equation and the elasticity tensor

$$\vec{\vec{M}}=\vec{\vec{\vec{\vec{B}}}}:\vec{\kappa}=-\begin{Bmatrix}\vec{e}_r\vec{e}_r\\ \vec{e}_\phi\vec{e}_\phi\\ \vec{e}_r\vec{e}_\phi+\vec{e}_\phi\vec{e}_r\end{Bmatrix}^{\mathrm{T}} D\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1}{2}(1-\nu)\end{bmatrix}\begin{Bmatrix}\dfrac{d^2w}{dr^2}\\ \dfrac{1}{r}\dfrac{dw}{dr}\\ 0\end{Bmatrix}.$$

Therefore, the polar coordinate system components of the bending moment constitutive equation are

$$M_{rr}=-D(\frac{d^2w}{dr^2}+\nu\frac{1}{r}\frac{dw}{dr}),\ M_{\phi\phi}=-D(\nu\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}),\ \text{and } M_{r\phi}=M_{\phi r}=0.$$

A simply supported circular plate of radius R is loaded by its own weight as shown in the figure. Determine the displacement of the plate at the midpoint by using the Kirchhoff plate model in the polar coordinate system. Problem parameters E, ν, ρ and t are constants. Assume that the solution depends on the radial coordinate only.



Solution

Under the rotation symmetry assumption, the equilibrium equation and the two constitutive equations of Kirchhoff plate bending

$$\nabla_0 \cdot (\nabla_0 \cdot \vec{M}) + b_n = \frac{1}{r}\frac{d}{dr}[\frac{d}{dr}(rM_{rr}) - M_{\phi\phi}] + b_n = 0,$$

$$M_{rr} = -D(\frac{d^2w}{dr^2} + \nu\frac{1}{r}\frac{dw}{dr}), \text{ and } M_{\phi\phi} = -D(\nu\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr})$$

Give, after elimination of the moment resultants, the equilibrium equation

$$(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr})(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr})w = \frac{b_n}{D}.$$

The boundary value problem for a simply supported circular plate of the problem

$$(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr})(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr})w = \frac{b_n}{D} \text{ in } (0,R) \quad \text{and} \quad w = M_{rr} = 0 \text{ at } r = R.$$

Repeated integrations in the equilibrium equation give

$$\frac{d}{dr}(r\frac{d}{dr}\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}w) = \frac{b_n}{D}r \quad \Rightarrow \quad r\frac{d}{dr}\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}w = \frac{b_n}{D}\frac{r^2}{2} + a \quad \Rightarrow$$

$$\frac{d}{dr}(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}w) = \frac{b_n}{D}\frac{r}{2} + \frac{a}{r} \quad \Rightarrow \quad \frac{1}{r}\frac{d}{dr}r\frac{d}{dr}w = \frac{b_n}{D}\frac{r^2}{4} + a\ln r + b \quad \Rightarrow$$

$$\frac{d}{dr}r\frac{d}{dr}w = \frac{b_n}{D}\frac{r^3}{4} + ar\ln r + br \quad \Rightarrow \quad r\frac{d}{dr}w = \frac{b_n}{D}\frac{r^4}{16} + a(-\frac{r^2}{4} + \frac{1}{2}r^2\ln r) + b\frac{r^2}{2} + c \quad \Rightarrow$$

$$\frac{d}{dr}w = \frac{b_n}{D}\frac{r^3}{16} + a(-\frac{r}{4} + \frac{1}{2}r\ln r) + b\frac{r}{2} + c\frac{1}{r} \quad \Rightarrow \quad w = \frac{b_n}{D}\frac{r^4}{64} + a\frac{1}{4}r^2(\ln r - 1) + b\frac{r^2}{4} + c\ln r + d$$

or by redefining the coefficients to get a more compact solution

$$w = \frac{b_n}{D}\frac{r^4}{64} + a + br^2 + cr^2(1 - \log r) + d\log r.$$

The generic solution contains parameters a, b, c, and d to be determined from the boundary conditions. As origin belongs to the solution domain and only the distributed load is acting on the plate, derivatives should be bounded at the origin which implies that $c = d = 0$. Boundary condition on the outer edge

$$M_{rr}(R) = -D(\frac{d^2w}{dr^2} + \nu\frac{1}{r}\frac{dw}{dr})(R) = -2bD(1+\nu) - \frac{1}{16}b_n R^2(3+\nu) = 0 \quad \Rightarrow \quad b = -\frac{1}{32}\frac{b_n R^2}{D}\frac{3+\nu}{1+\nu},$$

$$w(R) = a + bR^2 + \frac{b_n R^4}{64D} = 0 \quad \Rightarrow \quad a = \frac{b_n}{D}\frac{R^4}{64}\frac{5+\nu}{1+\nu}.$$

Displacement at the center point when $b_n = -\rho g t$

$$w(0) = \frac{b_n}{D}\frac{R^4}{64}\frac{5+\nu}{1+\nu} = -\frac{3}{16}\frac{\rho g R^4}{Et^2}(5+\nu)(1-\nu) \quad (D = \frac{t^3 E}{12(1-\nu^2)}). \leftarrow$$

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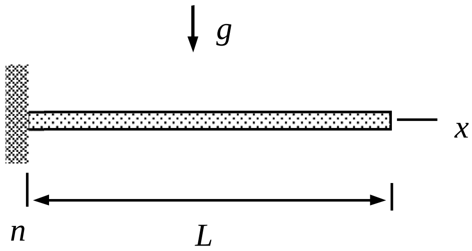
Assignment 1 (2p)

Find the stress resultants of the plate strip of length L, width H, and thickness t which is loaded by its own weight. Density of the plate material is ρ. Assume that the resultants do not depend on y. Use the plate equilibrium equations in the Cartesian system



$$\begin{Bmatrix} \dfrac{\partial N_{xx}}{\partial x}+\dfrac{\partial N_{xy}}{\partial y}+b_x \\ \dfrac{\partial N_{yy}}{\partial y}+\dfrac{\partial N_{xy}}{\partial x}+b_y \end{Bmatrix}=0 \quad \text{and} \quad \begin{Bmatrix} \dfrac{\partial Q_x}{\partial x}+\dfrac{\partial Q_y}{\partial y}+b_n \\ \dfrac{\partial M_{xx}}{\partial x}+\dfrac{\partial M_{xy}}{\partial y}-Q_x \\ \dfrac{\partial M_{yy}}{\partial y}+\dfrac{\partial M_{xy}}{\partial x}-Q_y \end{Bmatrix}=0 \quad \text{in} \quad (0,L)\times(0,H)\,.$$

Solution

In plate strip problem, the resultants do not depend on y and $Q_y = N_{xy} = M_{xy} = 0$. The differential equations that are not satisfied automatically and their boundary conditions at the free end simplify to

$$\frac{dN_{xx}}{dx}=0 \quad \text{in} \quad (0,L) \quad \text{and} \quad N_{xx}=0 \quad \text{at} \quad x=L$$

$$\frac{dQ_x}{dx}+\rho g t=0 \quad \text{in} \quad (0,L) \quad \text{and} \quad Q_x=0 \quad \text{at} \quad x=L$$

$$\frac{dM_{xx}}{dx}-Q_x=0 \quad \text{in} \quad (0,L) \quad \text{and} \quad M_{xx}=0 \quad \text{at} \quad x=L$$

Solutions to the stress resultants are

$$N_{xx}(x)=0\,,$$

$$Q_x(x)=\rho g t(L-x)\,,$$

$$M_{xx}(x)=-\frac{\rho g t}{2}(L-x)^2\,.$$

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Assignment 2 (2p)

Derive the constitutive equations of the Kirchhoff plate model associated with the bending mode in polar coordinates (r,ϕ,n). Start with the constitutive equations of the Reissner-Mindlin model

$$\begin{Bmatrix} M_{rr} \\ M_{\phi\phi} \\ M_{r\phi} \end{Bmatrix} = D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \theta_\phi}{\partial r} \\ \dfrac{1}{r}(\theta_\phi - \dfrac{\partial \theta_r}{\partial \phi}) \\ \dfrac{1}{r}(\dfrac{\partial \theta_\phi}{\partial \phi} + \theta_r) - \dfrac{\partial \theta_r}{\partial r} \end{Bmatrix} \text{ and } \begin{Bmatrix} Q_r \\ Q_\phi \end{Bmatrix} = Gt \begin{Bmatrix} \dfrac{\partial w}{\partial r} + \theta_\phi \\ \dfrac{1}{r}\dfrac{\partial w}{\partial \phi} - \theta_r \end{Bmatrix}.$$

Solution

Kirchhoff constraints for the rotations can be deduced from the constitutive equations of shear forces:

$$\begin{Bmatrix} Q_r \\ Q_\phi \end{Bmatrix} = Gt \begin{Bmatrix} \dfrac{\partial w}{\partial r} + \theta_\phi \\ \dfrac{1}{r}\dfrac{\partial w}{\partial \phi} - \theta_r \end{Bmatrix} \quad \Rightarrow \quad \begin{Bmatrix} \theta_\phi \\ \theta_r \end{Bmatrix} = \begin{Bmatrix} -\dfrac{\partial w}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial w}{\partial \phi} \end{Bmatrix}.$$

Elimination of the rotation variables from the constitutive equations of moments gives

$$\begin{Bmatrix} M_{rr} \\ M_{\phi\phi} \\ M_{r\phi} \end{Bmatrix} = D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \theta_\phi}{\partial r} \\ \dfrac{1}{r}(\theta_\phi - \dfrac{\partial \theta_r}{\partial \phi}) \\ \dfrac{1}{r}(\dfrac{\partial \theta_\phi}{\partial \phi} + \theta_r) - \dfrac{\partial \theta_r}{\partial r} \end{Bmatrix} \quad \Rightarrow$$

$$\begin{Bmatrix} M_{rr} \\ M_{\phi\phi} \\ M_{r\phi} \end{Bmatrix} = D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{Bmatrix} -\dfrac{\partial^2 w}{\partial r^2} \\ -\dfrac{1}{r}(\dfrac{\partial w}{\partial r} + \dfrac{1}{r}\dfrac{\partial^2 w}{\partial \phi^2}) \\ -2\dfrac{\partial}{\partial r}(\dfrac{1}{r}\dfrac{\partial w}{\partial \phi}) \end{Bmatrix}.$$

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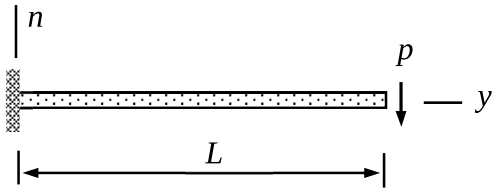
Assignment 3 (4p)

Consider the bending of a cantilever plate strip which is loaded by distributed force p [N / m] acting on the free edge. Write down the equilibrium equations, constitutive equations, and boundary conditions for the bending mode according to the Kirchhoff model. After that, solve the equations for the stress resultant, displacement, and rotation components. Thickness and length of the plate are t and L, respectively. Young's modulus E and Poisson's ratio ν are constants. Consider a plate of width H but assume that stress resultants, displacements, and rotations depend on y only.



Solution

The starting point is the full set of Reissner-plate bending mode equations in the Cartesian $(x, y, n)-$ coordinate system.

$$\begin{Bmatrix} \dfrac{\partial Q_x}{\partial x} + \dfrac{\partial Q_y}{\partial y} + b_n \\ \dfrac{\partial M_{xx}}{\partial x} + \dfrac{\partial M_{xy}}{\partial y} - Q_x \\ \dfrac{\partial M_{yy}}{\partial y} + \dfrac{\partial M_{xy}}{\partial x} - Q_y \end{Bmatrix} = 0 \,, \quad \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \theta}{\partial x} \\ -\dfrac{\partial \phi}{\partial y} \\ \dfrac{\partial \theta}{\partial y} - \dfrac{\partial \phi}{\partial x} \end{Bmatrix} , \quad \begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = Gt \begin{Bmatrix} \dfrac{\partial w}{\partial x} + \theta \\ \dfrac{\partial w}{\partial y} - \phi \end{Bmatrix} .$$

If all derivatives with respect to x vanish, the plate equations of the Cartesian $(x, y, n)-$ coordinate according to the Kirchhoff model (Kirchhoff constraint replaces the constitutive equation for the shear stress resultant) system simplify to

$$\frac{dQ_y}{dy} = 0 \,, \quad \frac{dM_{yy}}{dy} - Q_y = 0 \,, \quad M_{yy} = -D\frac{d\phi}{dy} , \quad \text{and} \quad \frac{dw}{dy} - \phi = 0 \quad \text{in} \quad (0, L) .$$

The boundary conditions are

$$w(0) = 0 \,, \quad \phi(0) = 0 \,, \quad M_{yy}(L) = 0 \,, \quad Q_y(L) = -p \,.$$

As the stress resultant are known at the free end, the equilibrium equations can be solved first for the stress resultants. The boundary value problems for the stress resultants give

$$\frac{dQ_y}{dy} = 0 \quad y \in (0, L) \quad \text{and} \quad Q_y(L) = -p \quad \Rightarrow \quad Q_y(y) = -p \,,$$

$$\frac{dM_{yy}}{dy} = Q_y = -p \quad y \in (0, L) \quad \text{and} \quad M_{yy}(L) = 0 \quad \Rightarrow \quad M_{yy}(y) = -p(y - L) \,.$$

After that, displacement and rotation follow from the constitutive equation, Kirchhoff constraint, and boundary conditions at the clamped edge

$$\frac{d\phi}{dy} = -\frac{M_{yy}}{D} = \frac{p}{D}(y - L) \quad y \in (0, L) \quad \text{and} \quad \phi(0) = 0 \quad \Rightarrow \quad \phi = \frac{p}{D}(\frac{1}{2}y^2 - Ly) \,,$$

$$\frac{dw}{dy} = \phi = \frac{p}{D}(\frac{1}{2}y^2 - Ly) \quad y \in (0, L) \quad \text{and} \quad w(0) = 0 \quad \Rightarrow \quad w(y) = \frac{p}{D}(\frac{1}{6}y^3 - L\frac{1}{2}y^2) \,.$$

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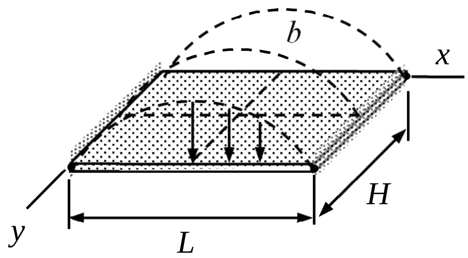
Assignment 4 (4p)

A rectangular plate of size $L \times H$ and thickness t is loaded by $b_n = b\sin(\pi x / L)$ in the transverse direction. The plate is simply supported on edges where $x \in \{0, L\}$ and free on the edges where $y \in \{0, H\}$ Assuming that the material parameters E, ν are constants, find the amplitude a_0 of transverse displacement $w = \mathrm{a}_0 \sin(\pi x / L)$ so that the bi-harmonic equation for the transverse displacement is satisfied. Start with the invariant form of the bi-harmonic equation $D\nabla_0^2\nabla_0^2 w - b_n = 0$.



Solution

The biharmonic equation for the transverse displacement follows from the generic equilibrium and constitutive equations for the Reissner-Mindlin plate bending, when the Kirchhoff constraints, constitutive equations, and moment equilibrium equations are used to express the force equilibrium in the transverse direction in terms of the transverse displacement w. Let us derive first the representation in the Cartesian (x, y, n) – coordinate system. Starting with the mid-surface gradient

$$\nabla_0 = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} \quad \Rightarrow$$

$$\nabla_0^2 = \nabla_0 \cdot \nabla_0 = (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y}) \cdot (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y}) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad \Rightarrow$$

$$\nabla_0^2\nabla_0^2 = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}.$$

When the assumed expressions for the transverse displacement and the given loading are substituted there, the bi-harmonic equation implies the condition

$$\nabla_0^2\nabla_0^2 w - \frac{b_n}{D} = [\mathrm{a}_0(\frac{\pi}{L})^4 - \frac{b}{D}]\sin(\frac{\pi x}{L}) = 0.$$

Therefore, the assumed solution satisfies the bi-harmonic equation if

$$\mathrm{a}_0 = (\frac{L}{\pi})^4 \frac{b}{D}.$$

Transverse displacement

$$w(x, y) = \left(\frac{L}{\pi}\right)^4 \frac{b}{D} \sin\left(\pi \frac{x}{L}\right)$$

satisfies the equilibrium equation and the boundary conditions and it is thus the exact solution to the plate boundary value problem.

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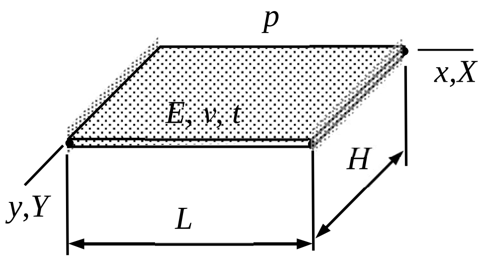
Assignment 5 (4p)

The rectangle plate shown, of thickness, length and width t, L, and H, is simply supported on edges where $x \in \{0, L\}$ and free on the remaining edges where $y \in \{0, H\}$, and loaded by pressure p acting on the upper surface. Young's modulus E and Poisson's ratio ν are constants. Determine the parameter a_0 of the approximation $w(x, y) = \mathrm{a}_0 (x/L)(1 - x/L)$. Use the principle of virtual work in form $\delta W = 0 \quad \forall \delta \mathrm{a}_0 \in \mathbb{R}$ and



$$\delta W = -\int_\Omega \begin{Bmatrix} \dfrac{\partial^2 \delta w}{\partial x^2} \\ \dfrac{\partial^2 \delta w}{\partial y^2} \\ 2\dfrac{\partial^2 \delta w}{\partial x \partial y} \end{Bmatrix}^{\mathrm{T}} D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{Bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \\ 2\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} dA + \int_\Omega \delta w b_n dA .$$

Solution

Virtual work expression can also be written in the form

$$\delta W = \int_\Omega \delta w_\Omega dA = \int_\Omega (\delta w_\Omega^{\mathrm{int}} + \delta w_\Omega^{\mathrm{ext}}) dA$$

in which the virtual work densities (virtual work per unit area here) of the internal and external forces

$$\delta w_\Omega^{\mathrm{int}} = -\begin{Bmatrix} \dfrac{\partial^2 \delta w}{\partial x^2} \\ \dfrac{\partial^2 \delta w}{\partial y^2} \\ 2\dfrac{\partial^2 \delta w}{\partial x \partial y} \end{Bmatrix}^{\mathrm{T}} D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{Bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \\ 2\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} \text{ and } \delta w_\Omega^{\mathrm{ext}} = b_n \delta w .$$

Approximation to the transverse displacement and its derivatives are

$$w(x, y) = a_0 (1 - \frac{x}{L}) \frac{x}{L} \quad \Rightarrow \quad \frac{\partial w}{\partial x} = \frac{a_0}{L}(1 - 2\frac{x}{L}), \quad \frac{\partial^2 w}{\partial x^2} = -2\frac{a_0}{L^2}, \text{ and } \frac{\partial w}{\partial y} = \frac{\partial^2 w}{\partial y^2} = 0 .$$

When the approximation and distributed force expression $b_n = p$ are substituted there, virtual work densities of the internal and external forces simplify to

$$\delta w_\Omega^{\text{int}} = -\delta a_0 \frac{1}{3}\frac{t^3}{L^4}\frac{E}{1-\nu^2} a_0 \,,$$

$$\delta w_\Omega^{\text{ext}} = \delta a_0 (1-\frac{x}{L})\frac{x}{L} p \,.$$

Virtual work expressions are integrals of the densities over the mathematical domain for the plate (midplane)

$$\delta W^{\text{int}} = \int_0^H \int_0^L \delta w_\Omega^{\text{int}} dxdy = -\delta a_0 \frac{1}{3}\frac{t^3 H}{L^3}\frac{E}{1-\nu^2} a_0 \,,$$

$$\delta W^{\text{ext}} = \int_0^H \int_0^L \delta w_\Omega^{\text{ext}} dxdy = \delta a_0 \frac{1}{6} HLp \,.$$

Virtual work expression

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = -\delta a_0 (\frac{1}{3}\frac{t^3 H}{L^3}\frac{E}{1-\nu^2} a_0 - \frac{1}{6} HLp) \,,$$

principle of virtual work $\delta W = 0 \ \forall \delta a_0$, and the fundamental lemma of variation calculus imply the solution

$$\frac{1}{3}\frac{t^3 H}{L^3}\frac{E}{1-\nu^2} a_0 - \frac{1}{6} HLp = 0 \quad \Leftrightarrow \quad a_0 = \frac{1}{2}(\frac{L}{t})^3 \frac{Lp}{E}(1-\nu^2) \,. \quad \leftarrow$$

MEC-E8003

Beam, Plate and Shell Models

2024

WEEK 15: SHELLS

6 SHELL

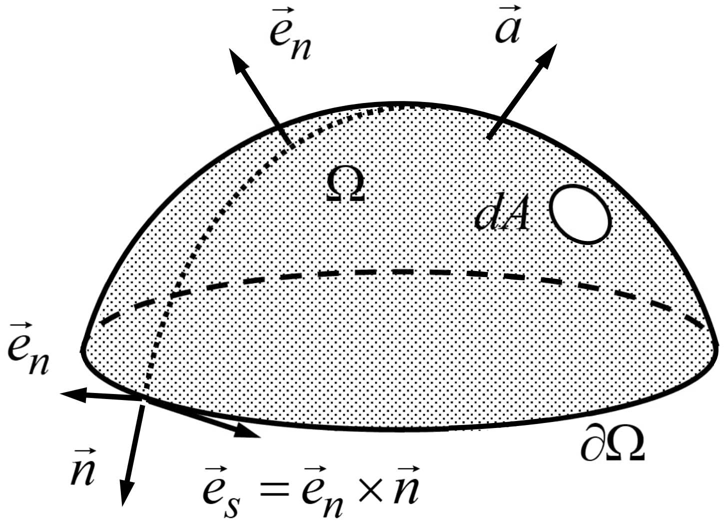
LEARNING OUTCOMES

Students are able to solve the weekly lecture problems, home problems, and exercise problems about the shell model:

- ☐ Reissner-Mindlin and Kirchhoff shell models and Kirchhoff constraints.
- ☐ Shell equilibrium and constitutive equations in their tensor forms.
- ☐ Component representations of the membrane and shell equations for cylindrical and spherical geometries
- ☐ Derivation of shell equations by using the principle of virtual work, integration by parts, and the fundamental lemma of variation calculus.

GAUSS'S THEOREM

Let us consider a vector valued function $\vec{a} \in C^0(\Omega)$ on a surface embedded in three-space $\Omega \in \mathbb{R}^3$ of the unit normal $\vec{e}_n$, outward unit normal on the boundary $\vec{n}$, and tangential line element $d\vec{s} = \vec{e}_n \times \vec{n} ds$. Then



Flat: $\int_\Omega (\nabla_0 \cdot \vec{a}) dA = \int_{\partial\Omega} (\vec{n} \cdot \vec{a}) ds$

Curved: $\int_\Omega (\nabla_0 - \kappa \vec{e}_n) \cdot \vec{a} dA = \int_{\partial\Omega} (\vec{n} \cdot \vec{a}) ds.$

In both forms, the area integral is over the surface and the boundary integral over the boundary of the surface. Term $\kappa = \nabla_0 \cdot \vec{e}_n$ is twice the mean curvature of the mid-surface or the trace of curvature tensor $\kappa = \overleftrightarrow{\kappa} : \overleftrightarrow{I}$.

Selection $\vec{a} = \overset{\leftrightarrow}{F} \cdot \delta\vec{u}$ and vector identity $\nabla \cdot (\overset{\leftrightarrow}{F} \cdot \delta\vec{u}) = (\nabla \cdot \overset{\leftrightarrow}{F}) \cdot \delta\vec{u} + \overset{\leftrightarrow}{F} : (\nabla\delta\vec{u})_{\mathrm{c}}$ gives the useful integral identity

$$\int_{\Omega} [(\nabla_0 \cdot (\overset{\leftrightarrow}{F} \cdot \delta\vec{u}) - \kappa(\vec{e}_n \cdot \overset{\leftrightarrow}{F}) \cdot \delta\vec{u}]dA = \int_{\partial\Omega} (\vec{n} \cdot \overset{\leftrightarrow}{F}) \cdot \delta\vec{u}ds \quad \Rightarrow$$

$$\int_{\Omega} \overset{\leftrightarrow}{F} : (\nabla_0\delta\vec{u})_{\mathrm{c}} dA = -\int_{\Omega} [\nabla_0 \cdot \overset{\leftrightarrow}{F} - \kappa(\vec{e}_n \cdot \overset{\leftrightarrow}{F})] \cdot \delta\vec{u}dA + \int_{\partial\Omega} (\vec{n} \cdot \overset{\leftrightarrow}{F}) \cdot \delta\vec{u}ds \quad \leftarrow$$

The last form can be taken as integration by parts formula on a curved surface. If $\kappa = 0$ or $\vec{e}_n \cdot \overset{\leftrightarrow}{F} = 0$, the usual form used already in connection with plates is obtained.

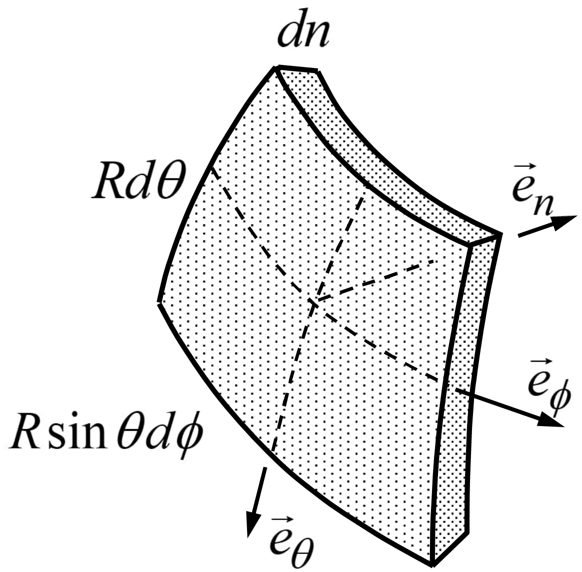
VOLUME AND AREA ELEMENTS

The integrals of the virtual work expression are always over a body. Representations of the volume and area elements consist of the mid-surface elements and scaling factors taking into account the offset in the transverse direction.

scaling factor

$$dV = \left(\frac{R-n}{R}\right)^2 dn \underbrace{(R^2 \sin\theta)\overbrace{d\phi d\theta}^{\text{domain element}}}_{\text{mid-surface area element}}$$

small dimension



The scaling factors for the area elements depend on the direction of the boundary $\vec{n}$ (the unit outward normal vector), curvature of the mid-surface $\vec{\kappa}$, and normal coordinate n.

GRADIENT REPRESENTATION

In derivation of equilibrium equations from virtual work expression of shell, gradient needs to be expressed in terms of the mid-surface gradient ∇_0, offset scaling $\overleftrightarrow{D}$, and the normal part:

scaling mid-surface part

Generic: $\nabla = \overleftrightarrow{D} \cdot (\nabla_0 + \vec{e}_n \frac{\partial}{\partial n})$ normal part

Cylindrical: $\nabla = (\vec{e}_z \vec{e}_z + \frac{R}{R-n} \vec{e}_\phi \vec{e}_\phi + \vec{e}_n \vec{e}_n) \cdot (\vec{e}_z \frac{\partial}{\partial z} + \frac{1}{R} \vec{e}_\phi \frac{\partial}{\partial \phi} + \vec{e}_n \frac{\partial}{\partial n})$

Spherical: $\nabla = (\frac{R}{R-n} \vec{e}_\phi \vec{e}_\phi + \frac{R}{R-n} \vec{e}_\theta \vec{e}_\theta + \vec{e}_n \vec{e}_n) \cdot (\frac{1}{R \sin\theta} \vec{e}_\phi \frac{\partial}{\partial \phi} + \frac{1}{R} \vec{e}_\theta \frac{\partial}{\partial \theta} + \vec{e}_n \frac{\partial}{\partial n})$

In flat geometry $\overleftrightarrow{D} = \overleftrightarrow{I}$ and in the thin body limit $(t / R \ll 1)$ $\overleftrightarrow{D} \approx \overleftrightarrow{I}$. Notice that integration by parts formula on curved surfaces is concerned with ∇_0.

EXAMPLE As a generic vector identity, Gauss's theorem is valid also when a thin body has curved mid-surface geometry. However, all parts of the boundary need to be accounted for correctly. As an example, let us consider a cylindrical body of constant thickness t of mid-surface area and line elements dA, ds and vector $\vec{a}(z,\phi)$. Then, using $\Gamma=[-t/2,t/2]$

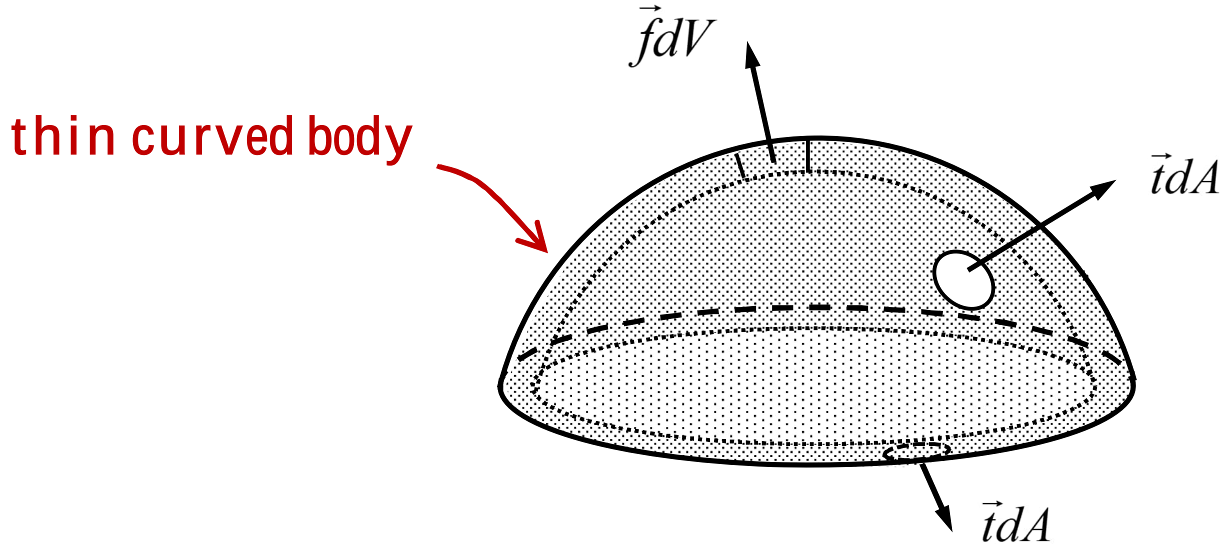
$$\int_{\Omega\times\Gamma}(\nabla\cdot\vec{a})dV=\int_{\partial\Omega\times\Gamma}(\vec{n}\cdot\vec{a})dA \quad \Rightarrow$$

$$\int_{\Omega\times\Gamma}(\nabla_0\cdot\vec{a})(1-\frac{n}{R})dndA=\int_{\partial\Omega\times\Gamma}(\vec{n}\cdot\vec{a})(1-\frac{n}{R})dnds+\sum_{\partial\Gamma}\int_{\partial\Omega}(\vec{n}\cdot\vec{a})(1-\frac{n}{R})dA \quad \Rightarrow$$

$$\int_{\Omega}(\nabla_0\cdot\vec{a})dA=\int_{\partial\Omega}(\vec{n}\cdot\vec{a})ds-\int_{\Omega}\frac{1}{R}(\vec{a}\cdot\vec{e}_n)dA.$$

In the last term on the right-hand side $1/R=-\nabla_0\cdot\vec{e}_n$. The additional term related with the curvature takes into account the different areas of the inner and outer surfaces of the cylindrical body.

6.1 SHELL MODEL

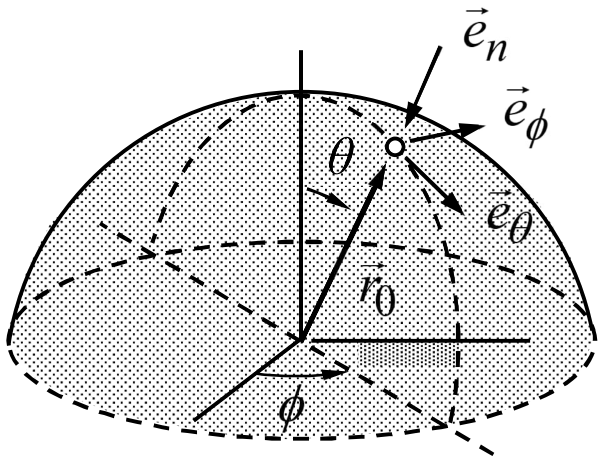


Kinematic assumption: Straight line segments perpendicular to the mid-surface remain straight in deformation (Reissner-Mindlin) or straight and perpendicular to the mid-surface (Kirchhoff) in deformation, so $\vec{u} = \vec{u}_0 + \vec{\theta}_0 \times n\vec{e}_n = \vec{u}_0 + n\vec{\omega}_0$.

Kinetic assumption: Stress component $\sigma_{nn} = 0$.

EFFECT OF CURVATURE

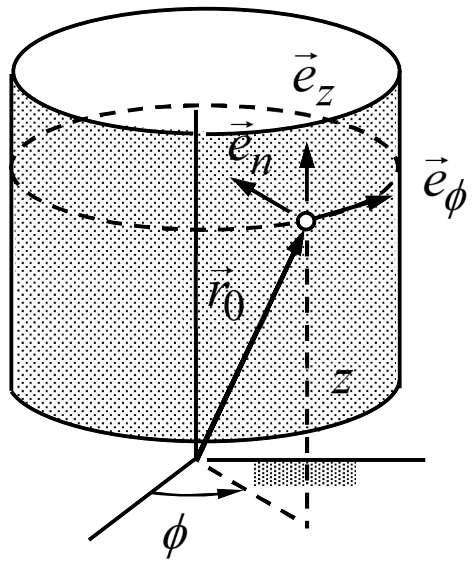
Sphere subjected to internal pressure:



$$N_{\phi\phi} = \frac{1}{2} pR \quad \text{and} \quad N_{\theta\theta} = \frac{1}{2} pR \ \Rightarrow$$

$$\overleftrightarrow{N} = \frac{1}{2} pR(\vec{e}_\phi \vec{e}_\phi + \vec{e}_\theta \vec{e}_\theta) = \frac{1}{2} pR\overleftrightarrow{I} \quad \text{(isotropic stress)}$$

Long cylinder subjected to internal pressure:



$$N_{zz} = \frac{1}{2} pR \quad \text{and} \quad N_{\phi\phi} = pR \ \Rightarrow$$

"curvature"

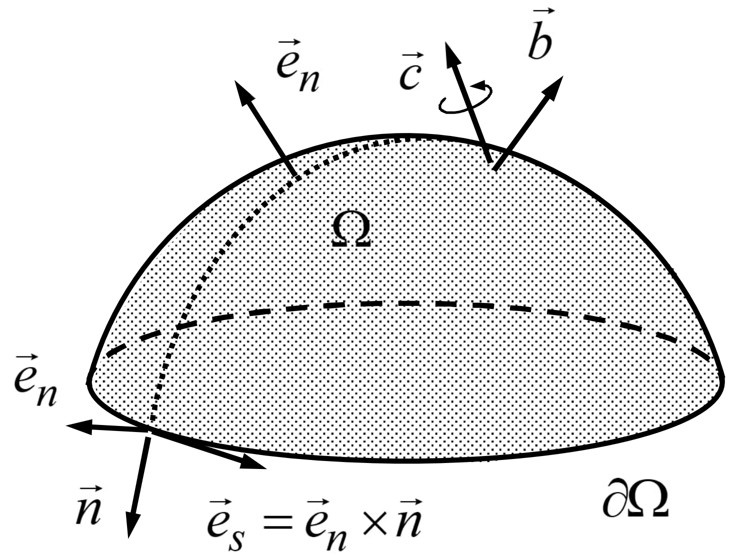
$$\overleftrightarrow{N} = \frac{1}{2} pR(\vec{e}_z \vec{e}_z + 2\vec{e}_\phi \vec{e}_\phi)$$

6.2 SHELL EQUATIONS

Virtual work expression of shell, principle of virtual work, integration by parts on curved surfaces (Kelvin-Stokes), and the fundamental lemma of variation calculus give:

$$\left.\begin{aligned} &(\nabla_0 - \kappa\vec{e}_n)\cdot\overleftrightarrow{F} + \vec{b} = 0 \\ &(\nabla_0 - \kappa\vec{e}_n)\cdot\overleftrightarrow{M} - \vec{e}_n\cdot\overleftrightarrow{F} + \vec{c})\times\vec{e}_n = 0 \end{aligned}\right\}\quad \text{in } \Omega$$

$$\left.\begin{aligned} &\vec{n}\cdot\overleftrightarrow{F} - \underline{\vec{F}} = 0 \quad \text{or} \quad \vec{u}_0 - \underline{\vec{u}}_0 = 0 \\ &(\vec{n}\cdot\overleftrightarrow{M} - \underline{\vec{M}})\times\vec{e}_n = 0 \quad \text{or} \quad \vec{\theta}_0 - \underline{\vec{\theta}}_0 = 0 \end{aligned}\right\}\quad \text{on } \partial\Omega$$



Shell and plate equations differ in the “derivative” operator. Conditions on $\partial\Omega$ need to be expressed finally with component representations in the boundary $(\vec{e}_n, \vec{n}, \vec{e}_s)$ basis.

Virtual work expression of the shell model coincides with the plate model. However, as the mid-surface is not flat, the simple Gauss theorem is replaced by a version valid on curved surfaces

$$\delta W=\int_\Omega \Big(-\begin{Bmatrix}\delta\overleftrightarrow{\varepsilon}\\ \delta\vec{\eta}\end{Bmatrix}_{\mathrm{c}}^{\mathrm{T}} : \begin{Bmatrix}\overleftrightarrow{F}\\ \overleftrightarrow{M}\end{Bmatrix}\Big)dA+\int_\Omega \begin{Bmatrix}\delta\vec{u}_0\\ \delta\vec{\omega}_0\end{Bmatrix}^{\mathrm{T}}\cdot\begin{Bmatrix}\vec{b}\\ \vec{c}\end{Bmatrix}dA+\int_{\partial\Omega}\begin{Bmatrix}\delta\vec{u}_0\\ \delta\vec{\omega}_0\end{Bmatrix}^{\mathrm{T}}\cdot\begin{Bmatrix}\underline{\vec{F}}\\ \underline{\vec{M}}\end{Bmatrix}ds.$$

Integration by parts in terms containing derivatives of the variations gives (mean curvature $\kappa=\nabla_0\cdot\vec{e}_n$) with the version of the Gauss theorem and the tensor identity $\nabla\cdot(\overleftrightarrow{b}\cdot\vec{a})=(\nabla\cdot\overleftrightarrow{b})\cdot\vec{a}+\overleftrightarrow{b}_{\mathrm{c}}:\nabla\vec{a}$ gives

$$\int_\Omega \overleftrightarrow{F}:(\nabla_0\delta\vec{u}_0)_{\mathrm{c}}\,dA=-\int_\Omega (\nabla_0\cdot\overleftrightarrow{F}-\kappa\vec{e}_n\cdot\overleftrightarrow{F})\cdot\delta\vec{u}_0 dA+\int_{\partial\Omega}(\vec{n}\cdot\overleftrightarrow{F}\cdot\delta\vec{u}_0)ds,$$

$$\int_\Omega \overleftrightarrow{M}:(\nabla_0\delta\vec{\omega}_0)_{\mathrm{c}}\,dA=-\int_\Omega (\nabla_0\cdot\overleftrightarrow{M}-\kappa\vec{e}_n\cdot\overleftrightarrow{M})\cdot\delta\vec{\omega}_0 dA+\int_{\partial\Omega}(\vec{n}\cdot\overleftrightarrow{M}\cdot\delta\vec{\omega}_0)ds$$

and thereby an equivalent but a more useful form of the virtual work expression

$$\delta W = \int_\Omega \begin{Bmatrix} (\nabla - \kappa\vec{e}_n)\cdot\vec{\vec{F}} + \vec{b} \\ (\nabla - \kappa\vec{e}_n)\cdot\vec{\vec{M}} - \vec{e}_n\cdot\vec{\vec{F}} + \vec{c} \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \delta\vec{u}_0 \\ \delta\vec{\omega}_0 \end{Bmatrix} dA - \int_{\partial\Omega} \begin{Bmatrix} \vec{n}\cdot\vec{\vec{F}} - \underline{\vec{F}} \\ \vec{n}\cdot\vec{\vec{M}} - \underline{\vec{M}} \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \delta\vec{u}_0 \\ \delta\vec{\omega}_0 \end{Bmatrix} ds.$$

When definition $\delta\vec{\omega}_0 = \delta\vec{\theta}_0 \times \vec{e}_n$ and the vector identity $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$ are used there (to recover the original rotation variable), the principle of virtual work and the fundamental lemma of variation calculus imply that (notice the terms due to curvature)

$$(\nabla_0 - \kappa\vec{e}_n)\cdot\vec{\vec{F}} + \vec{b} = 0 \quad \text{in } \Omega$$

equilibrium eqs.

$$[(\nabla_0 - \kappa\vec{e}_n)\cdot\vec{\vec{M}} - \vec{e}_n\cdot\vec{\vec{F}} + \vec{c}]\times\vec{e}_n = 0 \quad \text{in } \Omega$$

$$\vec{n}\cdot\vec{\vec{F}} - \underline{\vec{F}} = 0 \quad \text{or} \quad \vec{u}_0 - \underline{\vec{u}}_0 = 0 \quad \text{on } \partial\Omega$$

boundary conditions

$$(\vec{n}\cdot\vec{\vec{M}} - \underline{\vec{M}})\times\vec{e}_n = 0 \quad \text{or} \quad \vec{\theta}_0 - \underline{\vec{\theta}}_0 = 0 \quad \text{on } \partial\Omega$$

RESULTANT DEFINITIONS

Stress and external force resultants are integrals over the thickness ($\vec{\omega}_0 = \vec{\theta}_0 \times \vec{e}_n$). Stress resultant definition gives the constitutive equations:

$$\begin{Bmatrix} \overleftrightarrow{F} \\ \overleftrightarrow{M} \end{Bmatrix} = \begin{bmatrix} \overset{\leftrightarrow}{\overleftrightarrow{A}} & \overset{\leftrightarrow}{\overleftrightarrow{C}} \\ \overset{\leftrightarrow}{\overleftrightarrow{C}} & \overset{\leftrightarrow}{\overleftrightarrow{B}} \end{bmatrix} : \begin{Bmatrix} \overleftrightarrow{\varepsilon} \\ \overleftrightarrow{\kappa} \end{Bmatrix}, \begin{bmatrix} \overset{\leftrightarrow}{\overleftrightarrow{A}} & \overset{\leftrightarrow}{\overleftrightarrow{C}} \\ \overset{\leftrightarrow}{\overleftrightarrow{C}} & \overset{\leftrightarrow}{\overleftrightarrow{B}} \end{bmatrix} = \int \begin{bmatrix} 1 & n \\ n & n^2 \end{bmatrix} (\overleftrightarrow{D}_c \cdot \overset{\leftrightarrow}{\overleftrightarrow{E}} \cdot \overleftrightarrow{D} J)\, dn, \begin{Bmatrix} \overleftrightarrow{\varepsilon} \\ \overleftrightarrow{\kappa} \end{Bmatrix} = \begin{Bmatrix} \nabla_0 \vec{u}_0 + \vec{e}_n \vec{\omega}_0 \\ \nabla_0 \vec{\omega}_0 \end{Bmatrix}$$

$$\begin{Bmatrix} \vec{b} \\ \vec{c} \end{Bmatrix} = \int \vec{f} \begin{Bmatrix} 1 \\ n \end{Bmatrix} J dn + \sum \vec{t} \begin{Bmatrix} 1 \\ n \end{Bmatrix} J,$$

external force and moment per unit area

$$\begin{Bmatrix} \underline{\vec{F}} \\ \underline{\vec{M}} \end{Bmatrix} = \int \vec{t} \begin{Bmatrix} 1 \\ n \end{Bmatrix} J_n dn.$$

external force and moment per unit length

Elasticity tensor $\overset{\leftrightarrow}{\overleftrightarrow{E}}$ is assumed to satisfy the minor and major symmetries and condition $\vec{e}_n \vec{e}_n : \overset{\leftrightarrow}{\overleftrightarrow{E}} = 0$. For a thin shell $t\kappa \ll 1$, scaling factors $\overleftrightarrow{D} \approx \overleftrightarrow{I}$, $J \approx 1$, and $J_n \approx 1$.

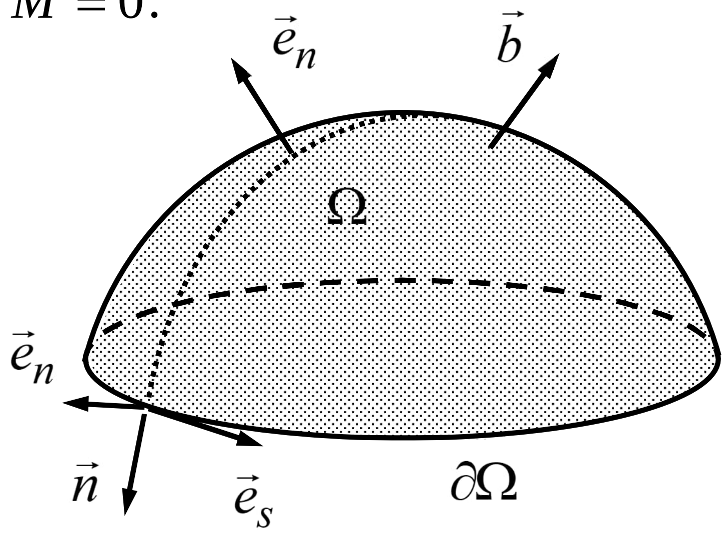
MEMBRANE EQUATIONS

Shell equations combine the thin-slab and bending modes. The membrane model, i.e., thin-slab model in curved geometry, applies to thin materials of negligible bending rigidity. The invariant forms of the shell model equilibrium and constitutive equations follow from the shell equations with the assumptions $\vec{u} = \vec{u}_0$, $\vec{Q} = 0$, $\overleftrightarrow{M} = 0$:

$$\nabla_0 \cdot \overleftrightarrow{N} + \vec{b} = 0 \quad \text{in} \quad \Omega \ ,$$

$$\overleftrightarrow{N} = \overleftrightarrow{\overleftrightarrow{A}} : \nabla_0 \vec{u}_0 \quad \text{in} \quad \Omega,$$

$$\vec{n} \cdot \overleftrightarrow{N} - \underline{\vec{N}} = 0 \quad \text{or} \quad \vec{u}_0 - \underline{\vec{u}}_0 = 0 \quad \text{on} \quad \partial\Omega.$$

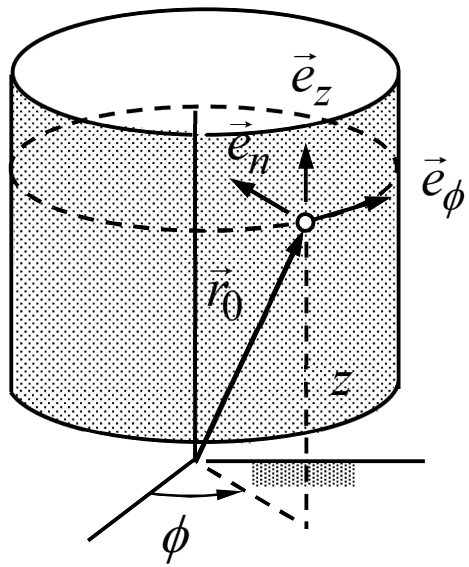


The membrane model finds use ,e.g,, in textile material, balloon, air-supported hall etc. applications.

CYLINDRICAL MEMBRANE (z,ϕ,n)

Equilibrium and constitutive equations of a cylindrical membrane follow from the coordinate system invariant forms of the membrane equations when gradient etc. are represented in (z,ϕ,n)-coordinate system:

$$\begin{Bmatrix} \dfrac{1}{R}\dfrac{\partial N_{z\phi}}{\partial\phi}+\dfrac{\partial N_{zz}}{\partial z}+b_z \\ \dfrac{\partial N_{z\phi}}{\partial z}+\dfrac{1}{R}\dfrac{\partial N_{\phi\phi}}{\partial\phi}+b_\phi \\ \dfrac{1}{R}N_{\phi\phi}+b_n \end{Bmatrix}=0, \quad \begin{Bmatrix} N_{zz} \\ N_{\phi\phi} \\ N_{z\phi} \end{Bmatrix}=t[E]_\sigma \begin{Bmatrix} \dfrac{\partial u_z}{\partial z} \\ \dfrac{1}{R}(\dfrac{\partial u_\phi}{\partial\phi}-u_n) \\ \dfrac{1}{R}\dfrac{\partial u_z}{\partial\phi}+\dfrac{\partial u_\phi}{\partial z} \end{Bmatrix}.$$



Boundary conditions on $\partial\Omega$ should be expressed in the boundary system with $\vec{n}=\vec{e}_z n_z+\vec{e}_\phi n_\phi$ and $\vec{e}_s=\vec{e}_n\times\vec{n}=\vec{e}_\phi n_r-\vec{e}_r n_\phi$.

In cylindrical geometry and (z,ϕ,n) coordinates, gradient operator takes the form

$$\nabla = \vec{e}_z \frac{\partial}{\partial z} + (\frac{R}{R-n})\frac{1}{R}\vec{e}_\phi \frac{\partial}{\partial \phi} + \vec{e}_n \frac{\partial}{\partial n} = \overleftrightarrow{D} \cdot (\nabla_0 + \vec{e}_n \frac{\partial}{\partial n}),$$

where

$$\nabla_0 = \vec{e}_z \frac{\partial}{\partial z} + \kappa \vec{e}_\phi \frac{\partial}{\partial \phi}, \;\; \overleftrightarrow{D} = \vec{e}_z\vec{e}_z + \frac{1}{1-\kappa n}\vec{e}_\phi\vec{e}_\phi + \vec{e}_n\vec{e}_n, \text{ and } \;\kappa = \frac{1}{R}.$$

Direct calculation with representations $\overleftrightarrow{N} = N_{zz}\vec{e}_z\vec{e}_z + N_{z\phi}\vec{e}_z\vec{e}_\phi + N_{\phi z}\vec{e}_\phi\vec{e}_z + N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi$, $\vec{b} = b_z\vec{e}_z + b_\phi\vec{e}_\phi + b_n\vec{e}_n$ and the known derivatives of the basis vectors gives

$$\nabla_0 \cdot \overleftrightarrow{N} + \vec{b} = \vec{e}_z(\frac{1}{R}\frac{\partial N_{\phi z}}{\partial \phi} + \frac{\partial N_{zz}}{\partial z} + b_z) + \vec{e}_\phi(\frac{\partial N_{z\phi}}{\partial z} + \frac{1}{R}\frac{\partial N_{\phi\phi}}{\partial \phi} + b_\phi) + \vec{e}_n(\frac{1}{R}N_{\phi\phi} + b_n) = 0.$$

Tensor $\overset{\leftrightarrow}{A}$ of shell constitutive equation depends on the plate model elasticity tensor $\overset{\leftrightarrow}{\overset{\leftrightarrow}{E}}$, scaling $\overset{\leftrightarrow}{D}$, and Jacobian $J = 1 - \kappa n$. Assuming a thin membrane $t / R \ll 1$ for simplicity so that $J \approx 1$ and $\overset{\leftrightarrow}{D} \approx \overset{\leftrightarrow}{I}$ (the precise expressions will be discussed later)

$$\overset{\leftrightarrow}{A} = \int (\overset{\leftrightarrow}{D}_{\mathrm{c}} \cdot \overset{\leftrightarrow}{\overset{\leftrightarrow}{E}} \cdot \overset{\leftrightarrow}{D} J) dn = \begin{Bmatrix} \vec{e}_\phi \vec{e}_\phi \\ \vec{e}_{\mathrm{z}} \vec{e}_{\mathrm{z}} \\ \vec{e}_\phi \vec{e}_{\mathrm{z}} + \vec{e}_{\mathrm{z}} \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} t [E]_\sigma \begin{Bmatrix} \vec{e}_\phi \vec{e}_\phi \\ \vec{e}_{\mathrm{z}} \vec{e}_{\mathrm{z}} \\ \vec{e}_\phi \vec{e}_{\mathrm{z}} + \vec{e}_{\mathrm{z}} \vec{e}_\phi \end{Bmatrix}.$$

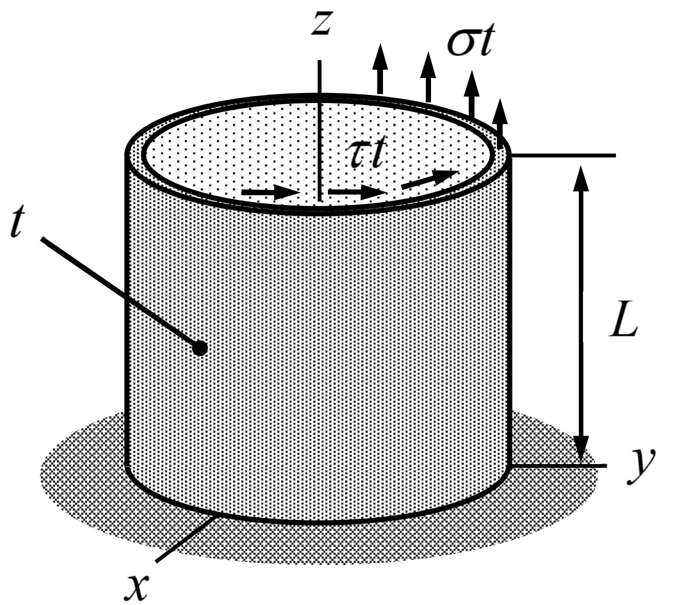
Only the translation part $\vec{u} = u_z \vec{e}_z + u_\phi \vec{e}_\phi + u_n \vec{e}_n$ of the kinematic assumption matters. Direct calculation with the known derivatives of the basis vectors gives

$$\nabla_0 \vec{u}_0 = \frac{\partial u_z}{\partial z} \vec{e}_z \vec{e}_z + \frac{\partial u_\phi}{\partial z} \vec{e}_z \vec{e}_\phi + \frac{1}{R} \frac{\partial u_z}{\partial \phi} \vec{e}_\phi \vec{e}_z + \frac{1}{R} \left(\frac{\partial u_\phi}{\partial \phi} - u_n \right) \vec{e}_\phi \vec{e}_\phi + \frac{\partial u_n}{\partial z} \vec{e}_z \vec{e}_n + \frac{1}{R} \left(u_\phi + \frac{\partial u_n}{\partial \phi} \right) \vec{e}_\phi \vec{e}_n$$

Therefore, the constitutive equation takes the form

$$\overset{\leftrightarrow}{N} = \overset{\leftrightarrow}{\overset{\leftrightarrow}{A}} : \nabla_0 \vec{u}_0 = \begin{Bmatrix} \vec{e}_\phi \vec{e}_\phi \\ \vec{e}_z \vec{e}_z \\ \vec{e}_\phi \vec{e}_z + \vec{e}_z \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} t \left[E \right]_\sigma \begin{Bmatrix} \frac{1}{R}\left(\frac{\partial u_\phi}{\partial \phi} - u_n\right) \\ \frac{\partial u_z}{\partial z} \\ \frac{\partial u_\phi}{\partial z} + \frac{1}{R}\frac{\partial u_z}{\partial \phi} \end{Bmatrix}. \quad \leftarrow$$

EXAMPLE 6.1 A thin walled cylindrical body of length L, (mid-surface) radius R, and thickness t is subjected to distributed loading $\vec{b} = b_z\vec{e}_z + b_\phi\vec{e}_\phi + b_n\vec{e}_n$ of constant components and boundary loading $\underline{\vec{F}} = t(\sigma\vec{e}_z + \tau\vec{e}_\phi)$ at the free end $z = L$. Assume rotational symmetry and use the membrane equations in (z,ϕ,n) coordinate system to solve for the mid-surface stress resultants.



Answer: $N_{zz} = \sigma t + b_z(L - z)$, $N_{z\phi} = \tau t + b_\phi(L - z)$, and $N_{\phi\phi} = -b_n R$

A rotational symmetric solution does not depend on ϕ. Then the equilibrium equations of the membrane model and the boundary conditions at the free edge simplify to

$$\frac{dN_{zz}}{dz} + b_z = 0, \quad \frac{dN_{z\phi}}{dz} + b_\phi = 0, \quad \frac{1}{R} N_{\phi\phi} + b_n = 0 \ \text{ in } (0, L)$$

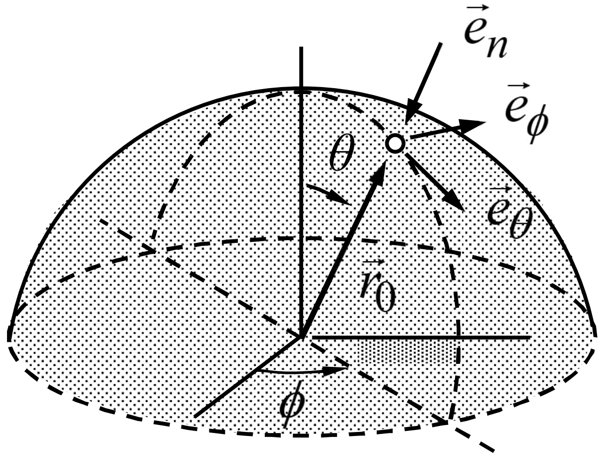
$$N_{zz} - \sigma t = 0, \ N_{z\phi} - \tau t = 0 \ \text{ at } \ z = L.$$

Solution to the boundary value problem of two ordinary first order differential equations and one algebraic equation for the stress resultants is given by

$$N_{zz} = \sigma t + b_z (L - z), \ N_{z\phi} = \tau t + b_\phi (L - z), \ \text{and} \ N_{\phi\phi} = -b_n R.$$

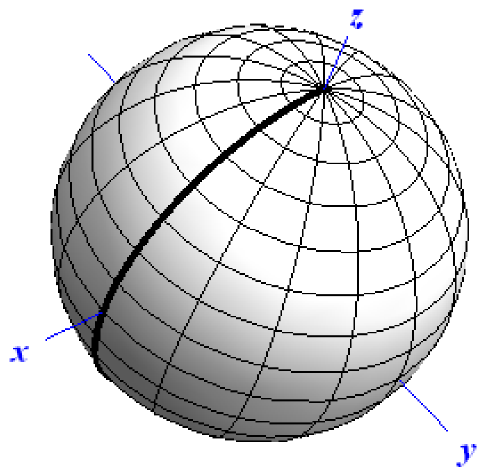
SPHERICAL MEMBRANE (ϕ,θ,n)

$$\begin{Bmatrix} \dfrac{1}{R}[\csc\theta\dfrac{\partial N_{\phi\phi}}{\partial\phi}+\dfrac{\partial N_{\theta\phi}}{\partial\theta}+\cot\theta(N_{\theta\phi}+N_{\phi\theta})]+b_{\phi} \\ \dfrac{1}{R}[\csc\theta\dfrac{\partial N_{\phi\theta}}{\partial\phi}+\dfrac{\partial N_{\theta\theta}}{\partial\theta}+\cot\theta(N_{\theta\theta}-N_{\phi\phi})]+b_{\theta} \\ \dfrac{1}{R}(N_{\phi\phi}+N_{\theta\theta})+b_{n} \end{Bmatrix}=0$$



$$\begin{Bmatrix} N_{\phi\phi} \\ N_{\theta\theta} \\ N_{\phi\theta} \end{Bmatrix}=t\left[E\right]_{\sigma}\frac{1}{R}\begin{Bmatrix} \csc\theta(\cos\theta u_{\theta}+\dfrac{\partial u_{\phi}}{\partial\phi})-u_{n} \\ \csc\theta\sin\theta\dfrac{\partial u_{\theta}}{\partial\theta}-u_{n} \\ \csc\theta\dfrac{\partial u_{\theta}}{\partial\phi}-\cot\theta u_{\phi}+\dfrac{\partial u_{\phi}}{\partial\theta} \end{Bmatrix} \text{ and } N_{\theta\phi}=N_{\phi\theta}$$

EXAMPLE 6.2 Consider a balloon in (ϕ,θ,n) coordinates under positive pressure difference $\Delta p = p_{in} - p_{out}$. Assuming a rotational symmetric solution with respect to two axes, so that all stress resultants and displacement components are independent of ϕ and θ, find the membrane stress and displacement of the surface.



Answer: $\overleftrightarrow{N} = \dfrac{\Delta pR}{2}(\vec{e}_\phi\vec{e}_\phi + \vec{e}_\theta\vec{e}_\theta)$ and $\vec{u} = -\dfrac{\Delta pR^2(1-\nu)}{2tE}\vec{e}_n$

NOTICE. Linear elasticity theory assumes an equilibrium initial geometry with $\overleftrightarrow{N}_0$, Δp_0, and R_0. The aim is to find the new equilibrium $\overleftrightarrow{N}$, Δp, and R due to the change in pressure. Here, displacement gives the change in radius due to the increase in the pressure difference.

According to the assumption, derivatives with respect to ϕ and θ vanish. The components of distributed force are $b_\phi = b_\theta = 0$ and $b_n = -\Delta p$ (n is directed inwards). Equilibrium equations ($N_{\theta\phi} = N_{\phi\theta}$) simplify to

$$2\cot\theta N_{\phi\theta} = 0, \quad \cot\theta(N_{\theta\theta} - N_{\phi\phi}) = 0, \quad N_{\theta\theta} + N_{\phi\phi} - \Delta pR = 0 \quad \text{in } \Omega \quad \Rightarrow$$

$$N_{\phi\theta} = 0 \quad \text{and} \quad N_{\theta\theta} = N_{\phi\phi} = \frac{\Delta pR}{2}.$$

With the solution above, constitutive equations give

$$\frac{\Delta pR}{2}\begin{Bmatrix}1\\1\\0\end{Bmatrix} = \frac{t}{R}[E]_\sigma \begin{bmatrix}0 & \cot\theta & -1\\ 0 & 0 & -1\\ -\cot\theta & 0 & 0\end{bmatrix}\begin{Bmatrix}u_\phi\\u_\theta\\u_n\end{Bmatrix} \Leftrightarrow \begin{Bmatrix}u_\phi\\u_\theta\\u_n\end{Bmatrix} = -\frac{\Delta pR^2(1-\nu)}{2tE}\begin{Bmatrix}0\\0\\1\end{Bmatrix}.$$

6.3 CYLINDRICAL SHELL (z,ϕ,n)

In curved geometry, the thin-slab and bending modes are always connected. In cylindrical geometry and (z,ϕ,n) coordinates, the equilibrium equations of shell take the forms to

$$\begin{Bmatrix} \dfrac{1}{R}\dfrac{\partial N_{\phi z}}{\partial \phi}+\dfrac{\partial N_{zz}}{\partial z}+b_z \\ \dfrac{\partial N_{z\phi}}{\partial z}+\dfrac{1}{R}\dfrac{\partial N_{\phi\phi}}{\partial \phi}-\dfrac{1}{R}Q_{\phi n}+b_\phi \end{Bmatrix}=0, \quad \begin{Bmatrix} \dfrac{1}{R}\dfrac{\partial Q_{\phi n}}{\partial \phi}+\dfrac{\partial Q_{zn}}{\partial z}+\dfrac{1}{R}N_{\phi\phi}+b_n \\ \dfrac{\partial M_{zz}}{\partial z}+\dfrac{1}{R}\dfrac{\partial M_{\phi z}}{\partial \phi}-Q_{nz}+c_z \\ \dfrac{\partial M_{z\phi}}{\partial z}+\dfrac{1}{R}\dfrac{\partial M_{\phi\phi}}{\partial \phi}-\dfrac{1}{R}M_{\phi n}-Q_{n\phi}+c_\phi \end{Bmatrix}=0.$$

The boundary conditions on $\partial\Omega$ need to be deduced from the generic forms for the boundary system with $\vec{n}=\vec{e}_z n_z+\vec{e}_\phi n_\phi$ and $\vec{e}_s=\vec{e}_n\times\vec{n}=\vec{e}_\phi n_z-\vec{e}_z n_\phi$. The non-zero constitutive equations for a thin shell $(t/R)^2 \ll 1)$ take the forms

continues ...

In-plane and shear force resultants

$$\begin{Bmatrix} N_{zz} \\ N_{\phi\phi} \\ N_{z\phi} \\ N_{\phi z} \end{Bmatrix} = \begin{Bmatrix} \frac{tE}{1-\nu^2}[\frac{\partial u_z}{\partial z} + \nu \frac{1}{R}(\frac{\partial u_\phi}{\partial \phi} - u_n)] - D\frac{1}{R}\frac{\partial \theta_\phi}{\partial z} \\ \frac{tE}{1-\nu^2}[\frac{1}{R}(\frac{\partial u_\phi}{\partial \phi} - u_n) + \nu \frac{\partial u_z}{\partial z}] - D\frac{1}{R^2}\frac{\partial \theta_z}{\partial \phi} \\ Gt(\frac{1}{R}\frac{\partial u_z}{\partial \phi} + \frac{\partial u_\phi}{\partial z}) + \frac{1}{2}(1-\nu)D\frac{1}{R}\frac{\partial \theta_z}{\partial z} \\ Gt(\frac{1}{R}\frac{\partial u_z}{\partial \phi} + \frac{\partial u_\phi}{\partial z}) + \frac{1}{2}(1-\nu)D\frac{1}{R^2}\frac{\partial \theta_\phi}{\partial \phi} \end{Bmatrix}, \quad \begin{Bmatrix} Q_z \\ Q_\phi \end{Bmatrix} = Gt \begin{Bmatrix} \theta_\phi + \frac{\partial u_n}{\partial z} \\ \frac{1}{R}(\frac{\partial u_n}{\partial \phi} + u_\phi) - \theta_z \end{Bmatrix}.$$

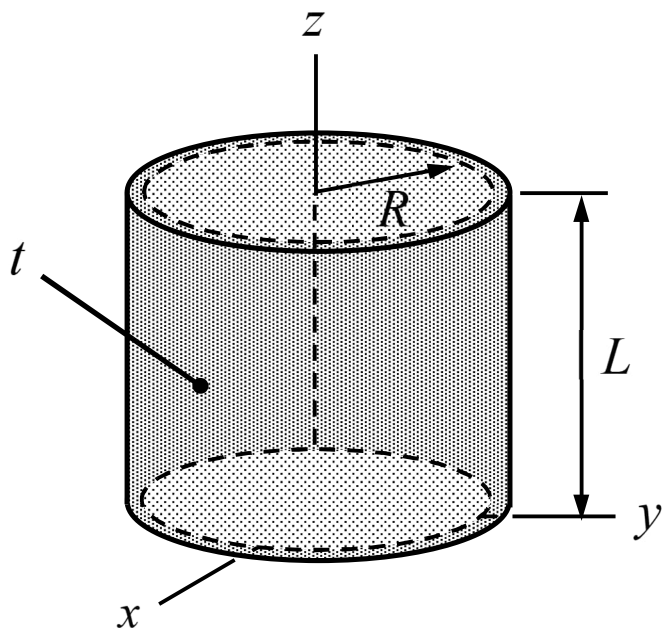
It is noteworthy that $N_{z\phi} \neq N_{\phi z}$ although $\sigma_{z\phi} = \sigma_{\phi z}$. The Kirchhoff constraints can be deduced from the shear force expressions is the same manner as with the model in flat geometry (plate). In the moment resultants

continues ...

$$\begin{Bmatrix} M_{zz} \\ M_{\phi\phi} \\ M_{z\phi} \\ M_{\phi z} \end{Bmatrix} = D \begin{Bmatrix} \dfrac{\partial \theta_\phi}{\partial z} - \nu \dfrac{1}{R}\dfrac{\partial \theta_z}{\partial \phi} - \dfrac{1}{R}\dfrac{\partial u_z}{\partial z} \\ \nu \dfrac{\partial \theta_\phi}{\partial z} - \dfrac{1}{R}\dfrac{\partial \theta_z}{\partial \phi} + \dfrac{1}{R^2}(\dfrac{\partial u_\phi}{\partial \phi} - u_n) \\ \dfrac{1}{2}(1-\nu)[(\dfrac{1}{R}\dfrac{\partial \theta_\phi}{\partial \phi} - \dfrac{\partial \theta_z}{\partial z}) - \dfrac{1}{R}\dfrac{\partial u_\phi}{\partial z}] \\ \dfrac{1}{2}(1-\nu)[(\dfrac{1}{R}\dfrac{\partial \theta_\phi}{\partial \phi} - \dfrac{\partial \theta_z}{\partial z}) + \dfrac{1}{R^2}\dfrac{\partial u_z}{\partial \phi}] \end{Bmatrix}, \; M_{\phi n} = \frac{1}{2}(1-\nu)D\frac{1}{R}[\frac{1}{R}(\frac{\partial u_n}{\partial \phi} + u_\phi) - \theta_z]$$

$M_{z\phi} \neq M_{\phi z}$ and the one needs also the expression for component $M_{\phi n}$ which is not present in the flat geometry equilibrium equations. The stress resultant definitions give expressions for all the components, but only those appearing in the equilibrium equations are needed in displacement analysis.

EXAMPLE 6.3 Consider a cylindrical container of radius R subjected to distributed force b_n due to internal excess pressure p. Assuming rigid end plates and rotation symmetry (derivatives with respect to ϕ vanish and $u_\phi = \theta_z = 0$), derive the differential equation and the boundary conditions for the transverse deflection $w(z) = u_n(z)$ according to the Kirchhoff model. Material is linearly elastic with properties E and ν. Thickness of the container wall is t.



Answer: $\dfrac{d^4 w}{dz^4} + \dfrac{2\nu}{R^2}\dfrac{d^2 w}{dz^2} + \dfrac{Et}{DR^2} w - \dfrac{1}{D}(\dfrac{\nu N}{R} + b_n) = 0$

In the Kirchhoff model, constitutive equations for the shear forces are replaced by Kirchhoff constraints. With relationship $\theta_\phi = -du_n / dz$ and the assumptions of the problem, the non-zero constitutive equations for the stress resultants simplify to

$$N_{zz} = \frac{tE}{1-\nu^2}\left(\frac{du_z}{dz} - \nu\frac{1}{R}u_n\right) + D\frac{1}{R}\frac{d^2u_n}{dz^2},$$

$$N_{\phi\phi} = \frac{tE}{1-\nu^2}\left(\nu\frac{du_z}{dz} - \frac{1}{R}u_n\right),$$

$$M_{zz} = -D\left(\frac{d^2u_n}{dz^2} + \frac{1}{R}\frac{du_z}{dz}\right) .$$

Equilibrium equations simplify to

$$\frac{dN_{zz}}{dz} = 0, \quad Q_\phi = 0 \ , \quad \frac{dQ_z}{dz} + \frac{1}{R}N_{\phi\phi} + b_n = 0, \ \text{and} \ \frac{dM_{zz}}{dz} - Q_z = 0 \ .$$

and after elimination of the shear force (using the moment equation)

$$\frac{dN_{zz}}{dz} = 0 \quad \text{and} \quad \frac{d^2 M_{zz}}{dz^2} + \frac{1}{R} N_{\phi\phi} + b_n = 0.$$

The constitutive equations for $N_{\phi\phi}$ and M_{zz} can be expressed in terms of u_n by using the equilibrium and constitutive equations for N_{zz}:

$$\frac{dN_{zz}}{dz} = 0 \quad \Rightarrow \quad N_{zz} = \frac{tE}{1-\nu^2}\left(\frac{du_z}{dz} - \nu \frac{1}{R} u_n\right) + D \frac{1}{R} \frac{d^2 u_n}{dz^2} = N = const. \quad \Rightarrow$$

$$\frac{du_z}{dz} = \nu \frac{1}{R} u_n + \frac{1-\nu^2}{tE}\left(N - D \frac{1}{R} \frac{d^2 u_n}{dz^2}\right).$$

Hence after elimination of du_z / dz and with the shorthand notation $a = t / R$

$$N_{\phi\phi} = \frac{tE}{1-\nu^2}(\nu\frac{du_z}{dz} - \frac{1}{R}u_n) = -\frac{tE}{R}u_n + \nu(N - D\frac{1}{R}\frac{d^2u_n}{dz^2}),$$

$$M_{zz} = -D(\frac{d^2u_n}{dz^2} + \frac{1}{R}\frac{du_z}{dz}) = -D[(1-\frac{a^2}{12})\frac{d^2u_n}{dz^2} + \frac{1}{R}\frac{1-\nu^2}{tE}N + \nu\frac{1}{R^2}u_n].$$

Using notation $u_n \equiv w$, equilibrium equation in the transverse direction gives

$$\frac{d^2M_{zz}}{dz^2} + \frac{1}{R}N_{\phi\phi} + b_n = -D[(1-\frac{a^2}{12})\frac{d^4w}{dz^4} + \nu\frac{2}{R^2}\frac{d^2w}{dz^2}] - \frac{tE}{R^2}w + \nu\frac{N}{R} + b_n = 0 \; . \quad \Leftarrow$$

Assuming that the end plates are rigid so that the displacement and rotation vanish at ends of the cylindrical container and $a^2 \ll 1$, the boundary value problem for the transverse displacement (positive inwards) takes the form

$$\frac{d^4 w}{dz^4} + \nu \frac{2}{R^2} \frac{d^2 w}{dz^2} + \frac{tE}{DR^2} w - \frac{1}{D} (\nu \frac{N}{R} + b_n) = 0 \quad \text{in} \;\; (0, L),$$

$$w = \frac{dw}{dz} = 0 \quad \text{on} \;\; \{0, L\}.$$

The fourth order differential equation can further be simplified by omitting the second derivative term as negligible compared to the fourth order derivative term.

6.4 SPHERICAL SHELL

In spherical geometry and (ϕ,θ,n) coordinate system, the equilibrium equations of shell simplify to

$$\left\{\begin{array}{c} \frac{1}{R}\left(\frac{\partial}{\partial\theta}N_{\theta\phi}+\csc\theta\frac{\partial}{\partial\phi}N_{\phi\phi}+2\cot\theta N_{\phi\theta}-Q_{\phi}\right)+b_{\phi} \\ \frac{1}{R}\left(\frac{\partial}{\partial\theta}N_{\theta\theta}+\csc\theta\frac{\partial}{\partial\phi}N_{\phi\theta}+\cot\theta N_{\theta\theta}-\cot\theta N_{\phi\phi}-Q_{\theta}\right)+b_{\theta} \\ \frac{1}{R}\left(\frac{\partial}{\partial\theta}Q_{\theta}+\csc\theta\frac{\partial}{\partial\phi}Q_{\phi}+\cot\theta Q_{\theta}+N_{\theta\theta}+N_{\phi\phi}\right)+b_{n} \end{array}\right\}=0,$$

$$\left\{\begin{array}{c} \frac{1}{R}\left(\frac{\partial}{\partial\theta}M_{\theta\phi}+\csc\theta\frac{\partial}{\partial\phi}M_{\phi\phi}+2\cot\theta M_{\phi\theta}\right)-Q_{\phi}+c_{\phi} \\ \frac{1}{R}\left(\frac{\partial}{\partial\theta}M_{\theta\theta}+\csc\theta\frac{\partial}{\partial\phi}M_{\phi\theta}+\cot\theta M_{\theta\theta}-\cot\theta M_{\phi\phi}\right)-Q_{\theta}+c_{\theta} \end{array}\right\}=0,$$

continues...

In-plane and shear force resultants are

$$\begin{Bmatrix} N_{\phi\phi} \\ N_{\theta\theta} \\ N_{\phi\theta} \end{Bmatrix} = \frac{Et}{1-\nu^2}\frac{1}{R}\begin{Bmatrix} (u_\theta \cot\theta + \dfrac{\partial u_\phi}{\partial \phi}\csc\theta - u_n) + \nu(\dfrac{\partial u_\theta}{\partial \theta} - u_n) \\ \nu(u_\theta \cot\theta + \dfrac{\partial u_\phi}{\partial \phi}\csc\theta - u_n) + (\dfrac{\partial u_\theta}{\partial \theta} - u_n) \\ \dfrac{1-\nu}{2}(-u_\phi \cot\theta + \dfrac{\partial u_\theta}{\partial \phi}\csc\theta + \dfrac{\partial u_\phi}{\partial \theta}) \end{Bmatrix} \text{ where } \csc\theta = \frac{1}{\sin\theta} .$$

For the spherical geometry $N_{\theta\phi} = N_{\phi\theta}$ and (the Kirchhoff constraints can be deduced from the shear force expressions in the same manner as those for the flat geometry)

$$\begin{Bmatrix} Q_\phi \\ Q_\theta \end{Bmatrix} = tG\begin{Bmatrix} \theta_\theta + \dfrac{1}{R}(u_\phi + \dfrac{\partial u_n}{\partial \phi}\csc\theta) \\ -\theta_\phi + \dfrac{1}{R}(u_\theta + \dfrac{\partial u_n}{\partial \theta}) \end{Bmatrix}.$$

continues...

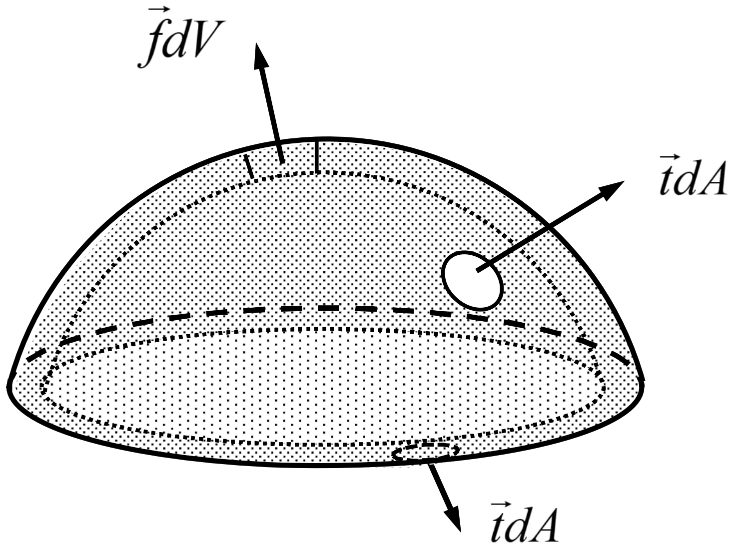
The expressions for the moment resultants of the equilibrium equations are given by

$$\begin{Bmatrix} M_{\phi\phi} \\ M_{\theta\theta} \\ M_{\phi\theta} \end{Bmatrix} = D\frac{1}{R}\begin{Bmatrix} -\theta_{\phi}\cot\theta + \dfrac{\partial\theta_{\theta}}{\partial\phi}\csc\theta - \nu\dfrac{\partial\theta_{\phi}}{\partial\theta} \\ \nu(-\theta_{\phi}\cot\theta + \dfrac{\partial\theta_{\theta}}{\partial\phi}\csc\theta) - \dfrac{\partial\theta_{\phi}}{\partial\theta} \\ \dfrac{1-\nu}{2}(\dfrac{\partial\theta_{\theta}}{\partial\theta} - \theta_{\theta}\cot\theta - \dfrac{\partial\theta_{\phi}}{\partial\phi}\csc\theta) \end{Bmatrix}.$$

Again, for the spherical geometry $M_{\theta\phi} = M_{\phi\theta}$. The stress resultant definitions give expressions for all the components, but only those appearing in the equilibrium equations are needed in displacement analysis.

6.5 VIRTUAL WORK DENSITIES

Virtual work densities of the shell model follow from the generic expression for linear elasticity theory and the kinematic and kinetic assumptions of the model. Integration over the thin dimension gives



$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \delta\overleftrightarrow{\varepsilon} \\ \delta\overleftrightarrow{\eta} \end{Bmatrix}_{\text{c}}^{\text{T}} : \begin{Bmatrix} \overleftrightarrow{F} \\ \overleftrightarrow{M} \end{Bmatrix}, \text{ where } \begin{Bmatrix} \overleftrightarrow{\varepsilon} \\ \overleftrightarrow{\eta} \end{Bmatrix} = \begin{Bmatrix} \nabla_0 \vec{u}_0 + \vec{e}_n \vec{\omega}_0 \\ \nabla_0 \vec{\omega}_0 \end{Bmatrix}$$

$$\delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta\vec{u} \\ \delta\vec{\omega} \end{Bmatrix}^{\text{T}} \cdot \begin{Bmatrix} \vec{b} \\ \vec{c} \end{Bmatrix}, \quad \delta w_{\partial\Omega}^{\text{ext}} = \begin{Bmatrix} \delta\vec{u} \\ \delta\vec{\omega} \end{Bmatrix}^{\text{T}} \cdot \begin{Bmatrix} \underline{\vec{F}} \\ \underline{\vec{M}} \end{Bmatrix}$$

in which

$$\begin{Bmatrix} \overleftrightarrow{F} \\ \overleftrightarrow{M} \end{Bmatrix} = \int \begin{Bmatrix} 1 \\ n \end{Bmatrix} J\overleftrightarrow{D}_{\text{c}} \cdot \overleftrightarrow{\sigma} dn, \quad \begin{Bmatrix} \vec{b} \\ \vec{c} \end{Bmatrix} = \int \begin{Bmatrix} 1 \\ n \end{Bmatrix} \vec{f} J dn + \sum \begin{Bmatrix} 1 \\ n \end{Bmatrix} \vec{t} J, \text{ and } \begin{Bmatrix} \underline{\vec{F}} \\ \underline{\vec{M}} \end{Bmatrix} = \int \begin{Bmatrix} 1 \\ n \end{Bmatrix} J(\vec{n}) \vec{t} dn$$

All the kinematical quantities need to be expressed in terms of the kinematical quantities of the mid-surface $\vec{u}_0$, $\vec{\theta}_0$, ∇_0 etc. With $\vec{\omega}_0 = \vec{\theta}_0 \times \vec{e}_n$, displacement gradient

$$\nabla\vec{u} = \overleftrightarrow{D}\cdot(\nabla_0 + \vec{e}_n \frac{\partial}{\partial n})(\vec{u}_0 + n\vec{\omega}_0) = \overleftrightarrow{D}\cdot(\overleftrightarrow{\varepsilon} + n\overleftrightarrow{\eta}),$$

where $\overleftrightarrow{\varepsilon} = \nabla_0\vec{u}_0 + \vec{e}_n\vec{\omega}_0$ and $\overleftrightarrow{\eta} = \nabla_0\vec{\omega}_0$ are the strain measures. With the vector identities $\vec{a}:(\overleftrightarrow{b}\cdot\vec{c}) = (\vec{a}\cdot\overleftrightarrow{b}):\vec{c}$ and $(\vec{a}\cdot\overleftrightarrow{b})_{\mathrm{c}} = \overleftrightarrow{b}_{\mathrm{c}}\cdot\vec{a}_{\mathrm{c}}$, the virtual work density of internal forces takes the form

$$\delta w_V^{\mathrm{int}} = -(\nabla\delta\vec{u})_{\mathrm{c}}:\overleftrightarrow{\sigma} = -(\delta\overleftrightarrow{\varepsilon}_{\mathrm{c}} + n\delta\overleftrightarrow{\eta}_{\mathrm{c}}):(\overleftrightarrow{D}_{\mathrm{c}}\cdot\overleftrightarrow{\sigma}) = -\begin{Bmatrix}\delta\overleftrightarrow{\varepsilon}\\ \delta\overleftrightarrow{\eta}\end{Bmatrix}_{\mathrm{c}}^{\mathrm{T}} : \begin{Bmatrix}\overleftrightarrow{D}_{\mathrm{c}}\cdot\overleftrightarrow{\sigma}\\ n\overleftrightarrow{D}_{\mathrm{c}}\cdot\overleftrightarrow{\sigma}\end{Bmatrix}.$$

The volume element can be expressed as $dV = JdndA$, in which dA is the mid-surface area element. Therefore, integration over the domain occupied by the body gives

$$\delta W^{\text{int}} = \int_{\Omega} [-\begin{Bmatrix} \delta\overleftrightarrow{\varepsilon} \\ \delta\overleftrightarrow{\eta} \end{Bmatrix}_{\text{c}}^{\text{T}} : (\int \begin{Bmatrix} 1 \\ n \end{Bmatrix} J\overleftrightarrow{D}_{\text{c}} \cdot \overleftrightarrow{\sigma} dn)] dA = \int_{\Omega} (-\begin{Bmatrix} \delta\overleftrightarrow{\varepsilon} \\ \delta\overleftrightarrow{\eta} \end{Bmatrix}_{\text{c}}^{\text{T}} : \begin{Bmatrix} \overleftrightarrow{F} \\ \overleftrightarrow{M} \end{Bmatrix}) dA \quad \Leftarrow$$

in which the stress resultants

$$\begin{Bmatrix} \overleftrightarrow{F} \\ \overleftrightarrow{M} \end{Bmatrix} = \int \begin{Bmatrix} 1 \\ n \end{Bmatrix} J\overleftrightarrow{D}_{\text{c}} \cdot \overleftrightarrow{\sigma} dn$$

are work conjugates to the strain measures. It is noteworthy that $\overleftrightarrow{F}$ and/or $\overleftrightarrow{M}$ of shell theory need not to be symmetric although the balance law of moment of momentum requires that $\overleftrightarrow{\sigma} = \overleftrightarrow{\sigma}_{\text{c}}$. Volume and area forces contribute to the virtual work of external forces.

$$\delta W^{\text{ext}} = \int_{\Omega} \vec{f} \cdot \delta\vec{u} dV + \int_{\partial\Omega} \vec{t} \cdot \delta\vec{u} dA.$$

There, the boundary contribution needs to be divided into parts coming from the outer and inner surfaces and from the edge (the sum is over the outer and inner surfaces $n = \pm t/2$)

$$\delta W_{\Omega}^{\text{ext}} = \int_{\Omega} [\begin{Bmatrix} \delta\vec{u}_0 \\ \delta\vec{\omega}_0 \end{Bmatrix}^{\mathrm{T}} \cdot (\int \begin{Bmatrix} 1 \\ n \end{Bmatrix} \vec{f} J dn) + \sum \begin{Bmatrix} 1 \\ n \end{Bmatrix} \vec{t} J)] dA \quad \Rightarrow$$

$$\delta W_{\Omega}^{\text{ext}} = \int_{\Omega} \begin{Bmatrix} \delta\vec{u}_0 \\ \delta\vec{\omega}_0 \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \vec{b} \\ \vec{c} \end{Bmatrix} dA, \quad \text{where} \quad \begin{Bmatrix} \vec{b} \\ \vec{c} \end{Bmatrix} = \int \begin{Bmatrix} 1 \\ n \end{Bmatrix} \vec{f} J dn) + \sum \begin{Bmatrix} 1 \\ n \end{Bmatrix} \vec{t} J . \quad \Leftarrow$$

$$\delta W_{\partial\Omega}^{\text{ext}} = \int_{\partial\Omega} [\begin{Bmatrix} \delta\vec{u}_0 \\ \delta\vec{\omega}_0 \end{Bmatrix}^{\mathrm{T}} \cdot (\int \begin{Bmatrix} 1 \\ n \end{Bmatrix} \vec{t} J(\vec{n}) dn)] ds \quad \Rightarrow$$

$$\delta W_{\partial\Omega}^{\text{ext}} = \int_{\partial\Omega} \begin{Bmatrix} \delta\vec{u}_0 \\ \delta\vec{\omega}_0 \end{Bmatrix}^{\mathrm{T}} \cdot \begin{Bmatrix} \underline{\vec{F}} \\ \underline{\vec{M}} \end{Bmatrix} ds, \quad \text{where} \quad \begin{Bmatrix} \underline{\vec{F}} \\ \underline{\vec{M}} \end{Bmatrix} = \int \begin{Bmatrix} 1 \\ n \end{Bmatrix} \vec{t} J(\vec{n}) dn . \quad \Leftarrow$$

6.6 CONSTITUTIVE EQUATIONS

Constitutive equations $\overleftrightarrow{F} = \overleftrightarrow{F}(\vec{u}_0, \vec{\theta}_0)$, $\overleftrightarrow{M} = \overleftrightarrow{M}(\vec{u}_0, \vec{\theta}_0)$ follow from the generalized Hooke's law, the definition of small strain, and the kinetic and kinematic assumptions of the model:

notice this!

$$\begin{Bmatrix} \overleftrightarrow{F} \\ \overleftrightarrow{M} \end{Bmatrix} = \begin{bmatrix} \overset{\leftrightarrow}{\overleftrightarrow{A}} & \overset{\leftrightarrow}{\overleftrightarrow{C}} \\ \overset{\leftrightarrow}{\overleftrightarrow{C}} & \overset{\leftrightarrow}{\overleftrightarrow{B}} \end{bmatrix} : \begin{Bmatrix} \overleftrightarrow{\varepsilon} \\ \overleftrightarrow{\eta} \end{Bmatrix}, \text{ where } \begin{bmatrix} \overset{\leftrightarrow}{\overleftrightarrow{A}} & \overset{\leftrightarrow}{\overleftrightarrow{C}} \\ \overset{\leftrightarrow}{\overleftrightarrow{C}} & \overset{\leftrightarrow}{\overleftrightarrow{B}} \end{bmatrix} = \int \begin{bmatrix} 1 & n \\ n & n^2 \end{bmatrix} (\overleftrightarrow{D}_\mathrm{c} \cdot \overset{\leftrightarrow}{\overleftrightarrow{E}} \cdot \overleftrightarrow{D} J)\, dn$$

Elasticity tensor $\overset{\leftrightarrow}{\overleftrightarrow{E}}$ is assumed to satisfy the minor and major symmetries and condition $\vec{e}_n \vec{e}_n : \overset{\leftrightarrow}{\overleftrightarrow{E}} = 0$. Elasticity tensors $\overset{\leftrightarrow}{\overleftrightarrow{A}}$, $\overset{\leftrightarrow}{\overleftrightarrow{B}}$ and $\overset{\leftrightarrow}{\overleftrightarrow{C}}$ of shell depend on the material, positioning of the mid-surface (actually the reference surface), thickness of the shell, and curvature of the mid-surface. Assuming a thin shell $\kappa t \ll 1$so that $\overleftrightarrow{D} \approx \overleftrightarrow{I}$ and $J \approx 1$, the expressions boil down to the plate expressions.

Constitutive equations follow from the stress resultant definitions when the stress expression is substituted there

$$\begin{Bmatrix} \overset{\leftrightarrow}{F} \\ \overset{\leftrightarrow}{M} \end{Bmatrix} = \int \left(\begin{Bmatrix} 1 \\ n \end{Bmatrix} J \overset{\leftrightarrow}{D}_{\mathrm{c}} \cdot \overset{\leftrightarrow}{\sigma} \right) dn.$$

The stress resultant tensors may not be symmetric even though the stress tensor always is. The displacement gradient expression was earlier found to be

$$\nabla\vec{u} = \overset{\leftrightarrow}{D} \cdot (\nabla_0 + \vec{e}_n \frac{\partial}{\partial n})(\vec{u}_0 + n\vec{\omega}_0) = \begin{Bmatrix} 1 \\ n \end{Bmatrix}^{\mathrm{T}} \overset{\leftrightarrow}{D} \cdot \begin{Bmatrix} \overset{\leftrightarrow}{\varepsilon} \\ \overset{\leftrightarrow}{\eta} \end{Bmatrix}, \text{ where } \begin{Bmatrix} \overset{\leftrightarrow}{\varepsilon} \\ \overset{\leftrightarrow}{\eta} \end{Bmatrix} = \begin{Bmatrix} \nabla_0\vec{u}_0 + \vec{e}_n\vec{\omega}_0 \\ \nabla_0\vec{\omega}_0 \end{Bmatrix}.$$

Let us assume a linearly elastic material and an elasticity tensor satisfying the minor and major symmetries and condition $\vec{e}_n\vec{e}_n : \overset{\leftrightarrow}{\overset{\leftrightarrow}{E}} = 0$. Stress-strain relationship gives (tensor identity $\overset{\leftrightarrow}{\overset{\leftrightarrow}{a}} : (\overset{\leftrightarrow}{b} \cdot \overset{\leftrightarrow}{c}) = (\overset{\leftrightarrow}{\overset{\leftrightarrow}{a}} \cdot \overset{\leftrightarrow}{b}) : \overset{\leftrightarrow}{c}$)

$$\vec{\sigma} = \vec{\vec{E}} : \nabla\vec{u} = \begin{Bmatrix} 1 \\ n \end{Bmatrix}^{\mathrm{T}} (\vec{\vec{E}} \cdot \vec{D}) : \begin{Bmatrix} \vec{\varepsilon} \\ \vec{\eta} \end{Bmatrix}.$$

The stress-resultant definition gives now expression

$$\begin{Bmatrix} \vec{F} \\ \vec{M} \end{Bmatrix} = \int (\begin{Bmatrix} 1 \\ n \end{Bmatrix} J\vec{D}_{\mathrm{c}} \cdot \vec{\sigma}) dn = \int \begin{bmatrix} 1 & n \\ n & n^2 \end{bmatrix} (J\vec{D}_{\mathrm{c}} \cdot \vec{\vec{E}} \cdot \vec{D}) dn : \begin{Bmatrix} \vec{\varepsilon} \\ \vec{\eta} \end{Bmatrix} \quad \Rightarrow$$

$$\begin{Bmatrix} \vec{F} \\ \vec{M} \end{Bmatrix} = \begin{bmatrix} \vec{\vec{A}} & \vec{\vec{C}} \\ \vec{\vec{C}} & \vec{\vec{B}} \end{bmatrix} : \begin{Bmatrix} \vec{\varepsilon} \\ \vec{\eta} \end{Bmatrix}, \text{ where } \begin{bmatrix} \vec{\vec{A}} & \vec{\vec{C}} \\ \vec{\vec{C}} & \vec{\vec{B}} \end{bmatrix} = \int \begin{bmatrix} 1 & n \\ n & n^2 \end{bmatrix} (J\vec{D}_{\mathrm{c}} \cdot \vec{\vec{E}} \cdot \vec{D}) dn. \quad \Leftarrow$$

which depends on the material properties, position of the mid-surface (actually the reference surface), thickness of the shell, and curvature of the reference surface. Without simplifications the membrane and bending modes are always connected.

SIMPLIFIED CONSTITUTIVE EXPRESSIONS

The practical expressions of constitutive equations are often simplified by omitting the "small terms". The simplified expressions of the stress resultants should

(1) vanish in rigid body motion of the shell $\vec{u} = \vec{U} + \vec{\Omega} \times \vec{r}_0$ and $\vec{\theta} = \vec{\Omega}$ in which $\vec{U}$ and $\vec{\Omega}$ are constant vectors in the Cartesian (x, y, z) coordinate system

(2) satisfy the moment equilibrium $\vec{e}_n \cdot \overleftrightarrow{F} \times \vec{e}_n + \nabla_0 \cdot (\underline{\overleftrightarrow{F}} \times \vec{r}_0) + \nabla_0 \cdot (\underline{\overleftrightarrow{M}} \times \vec{e}_n) = 0$, in which the underbars denote constants with respect to the gradient operator.

Both conditions are satisfied by the constitutive equations of spherical shell and by the cylindrical no matter the number of terms used for $g(\alpha)$ (not all simplifications of the constitutive equations satisfy conditions (1) and (2)).

The latter requirement is the local form of balance law of moment of momentum for shell (symmetry of stress $\overleftrightarrow{\sigma} = \overleftrightarrow{\sigma}_{\mathrm{c}}$ of classical elasticity is the outcome of the same law). With the equilibrium equations ($\vec{b} = \vec{c} = 0$ for simplicity), one obtains

$$\int_{\partial\Omega} (\vec{r}_0 \times \vec{F} + \vec{e}_n \times \vec{M})ds = -\int_{\Omega} [\nabla \cdot (\overleftrightarrow{F} \times \vec{r}_0 + \overleftrightarrow{M} \times \vec{e}_n) - \kappa \vec{e}_n \cdot (\overleftrightarrow{F} \times \vec{r}_0 + \overleftrightarrow{M} \times \vec{e}_n)]dA = 0 \;\Rightarrow$$

$$\int_{\partial\Omega} (\vec{r}_0 \times \vec{F} + \vec{e}_n \times \vec{M})ds = -\int_{\Omega} [\vec{e}_n \cdot \overleftrightarrow{F} \times \vec{e}_n + \nabla_0 \cdot (\underline{\overleftrightarrow{F}} \times \vec{r}_0) + \nabla_0 \cdot (\underline{\overleftrightarrow{M}} \times \vec{e}_n)]dA = 0$$

in which the underbars denote constants with respect to the gradient operator. As Ω is arbitrary, the second form implies that

$$\vec{e}_n \cdot \overleftrightarrow{F} \times \vec{e}_n + \nabla_0 \cdot (\underline{\overleftrightarrow{F}} \times \vec{r}_0) + \nabla_0 \cdot (\underline{\overleftrightarrow{M}} \times \vec{e}_n) = 0 .$$

CYLINDRICAL SHELL CONSTITUTIVE EQUATIONS

Derivation of the constitutive equations is a straightforward but somewhat tedious task. If the origin of the $n-$axis is placed at the mid-surface, constitutive equations take the forms $(F_{nn}=M_{nn}=0)$

$$F_{zz}=\frac{tE}{1-\nu^2}(\varepsilon_{zz}+\nu\varepsilon_{\phi\phi})-D\frac{1}{R}\kappa_{zz}=\frac{tE}{1-\nu^2}[\frac{\partial u_z}{\partial z}+\nu\frac{1}{R}(\frac{\partial u_\phi}{\partial\phi}-u_n)]-D\frac{1}{R}\frac{\partial\theta_\phi}{\partial z},$$

$$F_{\phi\phi}=\frac{tE}{1-\nu^2}[g\varepsilon_{\phi\phi}+\nu\varepsilon_{zz}+(g-1)R\kappa_{\phi\phi}]=\frac{tE}{1-\nu^2}[g\frac{1}{R}(\frac{\partial u_\phi}{\partial\phi}-u_n)+\nu\frac{\partial u_z}{\partial z}-(g-1)\frac{\partial\theta_z}{\partial\phi}],$$

$$F_{z\phi}=Gt(\varepsilon_{z\phi}+\varepsilon_{\phi z})-\frac{1}{2}(1-\nu)D\frac{1}{R}\kappa_{z\phi}=Gt(\frac{\partial u_\phi}{\partial z}+\frac{1}{R}\frac{\partial u_z}{\partial\phi})+\frac{1}{2}(1-\nu)D\frac{1}{R}\frac{\partial\theta_z}{\partial z},$$

$$F_{\phi z}=Gt[g\varepsilon_{\phi z}+\varepsilon_{z\phi}+(g-1)R\kappa_{\phi z}]=Gt[g\frac{1}{R}\frac{\partial u_z}{\partial\phi}+\frac{\partial u_\phi}{\partial z}+(g-1)\frac{\partial\theta_\phi}{\partial\phi}],$$

continues...

$$F_{zn} = Gt(\varepsilon_{nz} + \varepsilon_{zn}) - \frac{1}{2}(1-\nu)D\frac{1}{R}(\kappa_{nz} + \kappa_{zn}) = Gt(\theta_\phi + \frac{\partial u_n}{\partial z}),$$

$$F_{nz} = Gt(\varepsilon_{nz} + \varepsilon_{zn}) - \frac{1}{2}(1-\nu)D\frac{1}{R}(\kappa_{nz} + \kappa_{zn}) = Gt(\theta_\phi + \frac{\partial u_n}{\partial z}),$$

$$F_{\phi n} = Gt[g\varepsilon_{\phi n} + \varepsilon_{n\phi} + (g-1)R\kappa_{\phi n}] = Gtg[\frac{1}{R}(\frac{\partial u_n}{\partial \phi} + u_\phi) - \theta_z],$$

$$F_{n\phi} = Gt(\varepsilon_{n\phi} + \varepsilon_{\phi n}) - \frac{1}{2}(1-\nu)D\kappa_{n\phi} = Gt[-\theta_z + \frac{1}{R}(\frac{\partial u_n}{\partial \phi} + u_\phi)],$$

$$M_{zz} = D(\kappa_{zz} + \nu\kappa_{\phi\phi} - \frac{1}{R}\varepsilon_{zz}) = D(\frac{\partial \theta_\phi}{\partial z} - \nu\frac{1}{R}\frac{\partial \theta_z}{\partial \phi} - \frac{1}{R}\frac{\partial u_z}{\partial z}),$$

$$M_{\phi\phi} = D(f\kappa_{\phi\phi} + \nu\kappa_{zz} + f\frac{1}{R}\varepsilon_{\phi\phi}) = D[-f\frac{1}{R}\frac{\partial \theta_z}{\partial \phi} + \nu\frac{\partial \theta_\phi}{\partial z} + f\frac{1}{R^2}(\frac{\partial u_\phi}{\partial \phi} - u_n)],$$

continues...

$$M_{z\phi} = \frac{1}{2}(1-\nu)D(\kappa_{z\phi} + \kappa_{\phi z} - \frac{1}{R}\varepsilon_{z\phi}) = \frac{1}{2}(1-\nu)D[(-\frac{\partial \theta_z}{\partial z} + \frac{1}{R}\frac{\partial \theta_\phi}{\partial \phi}) - \frac{1}{R}\frac{\partial u_\phi}{\partial z}],$$

$$M_{\phi z} = \frac{1}{2}(1-\nu)D(f\kappa_{\phi z} + \kappa_{z\phi} + f\frac{1}{R}\varepsilon_{\phi z}) = \frac{1}{2}(1-\nu)D(f\frac{1}{R}\frac{\partial \theta_\phi}{\partial \phi} - \frac{\partial \theta_z}{\partial z} + f\frac{1}{R^2}\frac{\partial u_z}{\partial \phi}),$$

$$M_{zn} = \frac{1}{2}(1-\nu)D[\kappa_{zn} + \kappa_{nz} - \frac{1}{R}(\varepsilon_{zn} + \varepsilon_{nz})] = -\frac{1}{2}(1-\nu)D\frac{1}{R}(\frac{\partial u_n}{\partial z} + \theta_\phi),$$

$$M_{nz} = \frac{1}{2}(1-\nu)D[\kappa_{nz} + \kappa_{zn} - \frac{1}{R}(\varepsilon_{nz} + \varepsilon_{zn})] = -\frac{1}{2}(1-\nu)D\frac{1}{R}(\frac{\partial u_n}{\partial z} + \theta_\phi),$$

$$M_{n\phi} = \frac{1}{2}(1-\nu)D(\kappa_{n\phi} + \kappa_{\phi n} - \frac{1}{R}\varepsilon_{n\phi}) = 0,$$

$$M_{\phi n} = \frac{1}{2}(1-\nu)D(f\frac{1}{R}\varepsilon_{\phi n} + \kappa_{n\phi} + f\kappa_{\phi n}) = \frac{1}{2}(1-\nu)Df\frac{1}{R}[-\theta_z + \frac{1}{R}(\frac{\partial u_n}{\partial \phi} + u_\phi)],$$

where the functions depending on the relative thickness $a = t / R$

continues...

$$g = \frac{1}{a}\log(\frac{2+a}{2-a}) \approx 1 + \frac{a^2}{12} + \frac{a^4}{80} + \dots, \quad \text{and} \quad f = 12\frac{1}{a^2}(g-1).$$

In the simplified constitutive equations, shell is assumed to be thin in the sense that $a = t/R \ll 1$ so that the first ($g \approx 1,\ f = 0$) or the first two terms ($g \approx 1 + a^2/12,\ \ f = 1$) of g give an accurate enough representation. No matter the number of terms used, constitutive equations satisfy the moment balance of the domain element

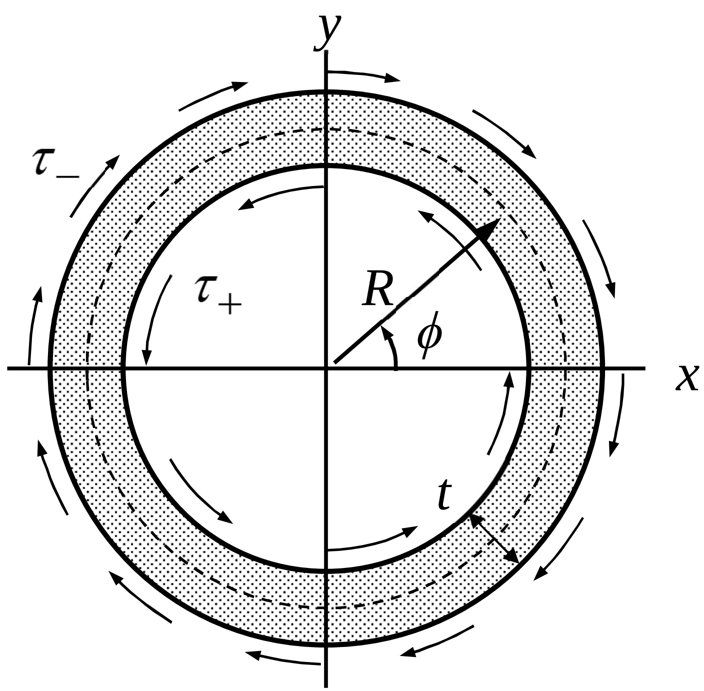
$$F_{nz} - F_{zn} = 0\ ,\ \ F_{z\phi} - F_{\phi z} + \frac{1}{R}M_{\phi z} = 0,\ \text{and}\ F_{n\phi} - F_{\phi n} + \frac{1}{R}M_{\phi n} = 0$$

'a priori'. Also, stress resultants vanish in the rigid body motion of the shell

$$\vec{u}(z,\phi,n) = \vec{U}_0 + \vec{\Omega}_0 \times \vec{r}_0 \quad \text{and} \quad \vec{\theta}(z,\phi,n) = \vec{\Omega}_0$$

in which $\vec{U}_0$ and $\vec{\Omega}_0$ are constant vectors.

EXAMPLE 6.4 Consider a cylinder subjected to shear forces acting on the inner and outer surfaces as shown. Use the Reissner-Mindlin type shell model in $(z,\phi,n)-$coordinate system to derive the expression of displacement $\vec{u}(n)$. Assume that the only non-zero displacement/rotation component θ_z is constant and that the cylinder is in equilibrium so that the shear forces per unit area satisfy $\tau = \tau_-(1+a/2)^2 = \tau_+(1-a/2)^2$ where $a = t/r$.



Answer $\vec{u} = \dfrac{\tau}{G} n\vec{e}_\phi$ when $a = \dfrac{t}{R} \ll 1$

As all other displacement/rotation components except θ_z are assumed to vanish, the equilibrium and constitutive equations ($g_\alpha \approx 1 + a^2/12$ and $f_\alpha = 1$) take the forms

$$\frac{1}{R} M_{\phi n} + Q_\phi - c_\phi = 0, \ \ Q_\phi = -Gt\theta_z, \ \text{and} \ M_{\phi n} = -Gt\frac{a^2}{12} R\theta_z.$$

The distributed force and moment follow from definition

$$\begin{Bmatrix} \vec{b} \\ \vec{c} \end{Bmatrix} = \int \vec{f} \begin{Bmatrix} 1 \\ n \end{Bmatrix} Jdn + \sum \vec{t} \begin{Bmatrix} 1 \\ n \end{Bmatrix} J$$

in which $\vec{f}$ is the external volume force (due to gravity for example) and $\vec{t}$ is the given area force acting on the outer and inner surfaces. The sum is over the coordinates $\{n_-, n_+\}$ of surfaces. Notice that – side is the outer surface and + the inner surface since n is directed inwards in (z,ϕ,n) coordinates. Here $\vec{f}=0$ and scaling coefficient expression $J=1-n/R$ for the cylindrical shell

$$\vec{c}=\sum \vec{t}nJ=(1+\frac{a}{2})(-\frac{t}{2})(-\tau_{-})\vec{e}_{\phi}+(1-\frac{a}{2})(\frac{t}{2})(\tau_{+})\vec{e}_{\phi}=R\frac{a}{1-(a/2)^{2}}\tau\vec{e}_{\phi} \ .$$

When the constitutive equations are substituted there, equilibrium equation simplifies to (assuming that $a^2 \ll 1$)

$$-Gt\theta_z - Ra\tau = 0 \ \Rightarrow \ \theta_z = -\frac{\tau}{G}.$$

Finally, using the kinematic assumption of the shell-model $\vec{u}=\theta_z\vec{e}_z \times n\vec{e}_n = -n\theta_z\vec{e}_\phi$ and therefore

$$\vec{u}=\frac{\tau}{G}n\vec{e}_{\phi}.$$

MEC-E8003 Beam, plate and shell models, examples 6

1. Curvilinear $(s,\alpha,n)-$coordinate system of a shell is defined by mapping

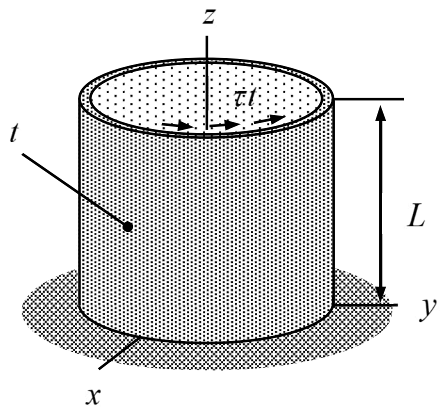
$$\vec{r}(s,\alpha,n)=(R+n)\cos(\frac{s}{R})\vec{i}+(R+n)\sin(\frac{s}{R})\vec{j}+\alpha L\vec{k}$$

in which R and L are some constants, and n is the transverse coordinate. Derive in detail the expressions for the basis vectors, the non-zero basic vector derivatives, gradient operator, and the curvature tensor of mid-surface.

2. Derive the component forms of cylindrical shell moment equilibrium equations in the (z,ϕ,n) coordinate system starting from the invariant form $(\nabla_0\cdot\vec{\vec{M}}-\kappa\vec{e}_n\cdot\vec{\vec{M}}-\vec{e}_n\cdot\vec{\vec{F}}+\vec{c})\times\vec{e}_n=0$

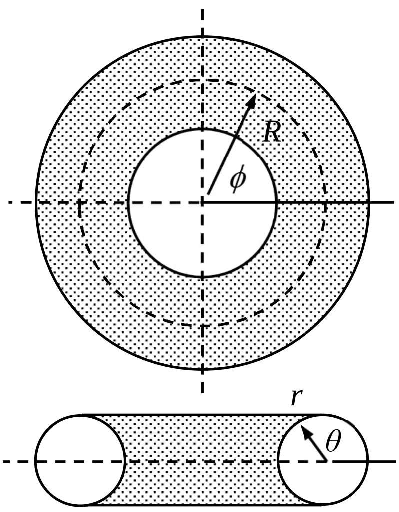
Answer $$\begin{Bmatrix} \dfrac{\partial M_{zz}}{\partial z}+\dfrac{1}{R}\dfrac{\partial M_{\phi z}}{\partial\phi}-Q_z+c_z \\ \dfrac{\partial M_{z\phi}}{\partial z}+\dfrac{1}{R}\dfrac{\partial M_{\phi\phi}}{\partial\phi}-\dfrac{1}{R}M_{\phi n}-Q_\phi+c_\phi \end{Bmatrix}=0$$

3. A thin walled cylindrical body of length *L*, (mid-surface) radius *R*, and thickness *t* is subjected to shear loading τt $[\tau t]=\text{N/m}$ at the free end $z=L$ as shown in the figure. Assuming rotation symmetry, use the membrane equations in (z,ϕ,n) coordinate system to derive the relationship between the moment resultant T (in the direction of *z*-axis) of the shear loading and the angle of rotation of the free end defined by $\theta=u_\phi/R$.



Answer $T=\dfrac{2\pi R^3 t}{L}G\theta$

4. Consider a torus shaped balloon under the loading caused by inner pressure difference Δp relative to the ambient pressure. Use (ϕ,θ,n) coordinate system, assume rotation symmetry with respect to ϕ, and solve for the stress resultant components from the equilibrium equations:



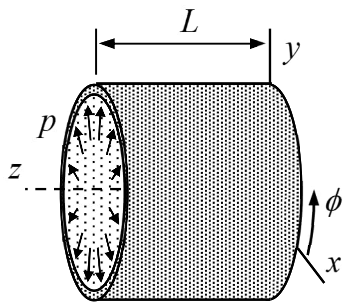
$$\left\{\begin{matrix} \dfrac{1}{r}\dfrac{\partial N_{\phi\theta}}{\partial\theta}+\dfrac{1}{R+r\cos\theta}[\dfrac{\partial N_{\phi\phi}}{\partial\phi}-2N_{\phi\theta}\sin\theta]+b_{\phi} \\ \dfrac{1}{r}\dfrac{\partial N_{\theta\theta}}{\partial\theta}+\dfrac{1}{R+r\cos\theta}[\dfrac{\partial N_{\phi\theta}}{\partial\phi}+(N_{\phi\phi}-N_{\theta\theta})\sin\theta]+b_{\theta} \\ -\dfrac{1}{r}N_{\theta\theta}-\dfrac{1}{R+r\cos\theta}\cos\theta N_{\phi\phi}+b_n \end{matrix}\right\}=0$$

Answer $N_{\theta\theta}=r\Delta p\dfrac{2R+r\cos\theta}{2R+2r\cos\theta}$, $N_{\phi\phi}=\dfrac{r\Delta p}{2}$

5. Displacement and rotation in rigid body motion are $\vec{u}=\vec{U}+\vec{\Omega}\times\vec{r}_0$ and $\vec{\Omega}$ in which $\vec{U}$ and $\vec{\Omega}$ are constant vectors in the Cartesian (x,y,z) coordinate system. Calculate the cylindrical shell stress resultant components $M_{z\phi}$ and $M_{\phi z}$ in rigid body mode $\Omega_x\neq 0$ and $U_x=U_y=U_z=\Omega_y=\Omega_z=0$.

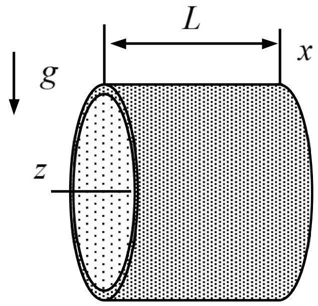
Answer $M_{z\phi}=M_{\phi z}=0$

6. A steel ring of length L, radius R, and thickness t is loaded by radial surface force p acting on the inner surface. No forces are acting on the ends. Model the ring as a cylindrical membrane, write down the equilibrium and constitutive equations, and solve for the radial displacement. Assume rotation symmetry. Young's modulus E and Poisson's ratio ν of the material are constants.



Answer $u_n=-\dfrac{R^2}{tE}p$

7. Consider a simply supported (long) circular cylindrical shell of radius R, thickness t, and filled with liquid of density ρ in cylindrical (z,ϕ,n) coordinates. Determine the mid-surface stress resultants $N_{\phi\phi}$, $N_{z\phi}$ and N_{zz} by assuming that there are no axial forces at the ends of the shell and bending deformation is negligible. (J.N.Reddy: Example 11.3.1)



Answer $N_{\phi\phi}=-R\Delta p-\rho gR^2\cos\phi$, $N_{z\phi}=\rho gR(\dfrac{1}{2}L-z)\sin\phi+A$, $N_{zz}=\rho g\dfrac{1}{2}(z^2-zL)\cos\phi$

8. Consider a cylindrical shell of radius R, subjected to bending moment $M_{zz}=\underline{M}$ and shearing force $Q_z=\underline{Q}$ at the end $z=L$. The other end $z=0$ is clamped. Assuming rotational symmetry, derive the boundary value problem of Kirchhoff type for deflection $u_n(z)$. Start with the component forms of the Reissner-Mindlin (type) shell equations in cylindrical (z,ϕ,n) coordinates.

Answer $(1-\frac{a^2}{12})\frac{d^4u_n}{dz^4}+\nu\frac{2}{R^2}\frac{d^2u_n}{dz^2}+\frac{tE}{DR^2}u_n=0$ in $(0,L)$, $u_n=\frac{du_n}{dz}=0$ at $z=0$,

$$(1-\frac{1}{12}a^2)\frac{d^3u_n}{dz^3}+\nu\frac{1}{R^2}\frac{du_n}{dz}+\frac{\underline{Q}}{d}=0\,,\ (1-\frac{1}{12}a^2)\frac{d^2u_n}{dz^2}+\nu\frac{1}{R^2}u_n+\frac{\underline{M}}{d}=0 \text{ at } z=L\,.$$

9. Consider a circular cylindrical shell of radius R, subjected to bending moment $M_{zz}=\underline{M}$ and shearing force $Q_z=\underline{Q}$ at the end $z=L$. The other end $z=0$ is clamped. Assuming rotational symmetry, derive the boundary value problem of Reissner-Mindlin type for deflection $u_n(z)$ and rotation $\theta_\phi(z)$.

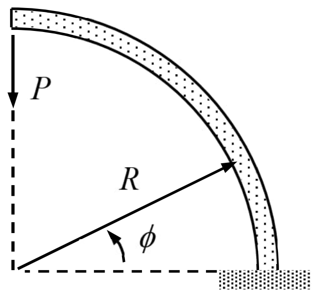
Answer $(Gt+\nu D\frac{1}{R^2})\frac{d\theta_\phi}{dz}+Gt\frac{d^2u_n}{dz^2}-\frac{Et}{R^2}u_n=0$ in $(0,L)$,

$$D[(1-\frac{1}{12}a^2)\frac{d^2\theta_\phi}{dz^2}-\nu\frac{1}{R^2}\frac{du_n}{dz}]-Gt(\theta_\phi+\frac{du_n}{dz})=0 \text{ in } (0,L)\,,$$

$$Gt(\theta_\phi+\frac{du_n}{dz})-\underline{Q}=0\,,\ D[(1-\frac{1}{12}a^2)\frac{d\theta_\phi}{dz}-\nu\frac{1}{R^2}u_n]-\underline{M}=0 \text{ at } z=L\,,$$

$$u_n=0\,,\ \theta_\phi=0 \text{ at } z=0\,.$$

10. A strip of cylindrical shell is loaded by shear force P ($[P]=\mathrm{N/m}$) at the free end. Write down the boundary value problem of first order ordinary differential equations consisting of the equilibrium and constitutive equations according to the Kirchhoff theory. Thickness t, width H, and the material parameters E, ν are constants. Assume that the solution depends on ϕ only.



Answer $$\begin{Bmatrix} \frac{1}{R}\frac{dN_{\phi\phi}}{d\phi}-\frac{1}{R}Q_\phi \\ \frac{1}{R}\frac{dQ_\phi}{d\phi}+\frac{1}{R}N_{\phi\phi} \\ \frac{1}{R}\frac{dM_{\phi\phi}}{d\phi}-Q_\phi \end{Bmatrix}=0\,,\ \begin{Bmatrix} N_{\phi\phi}-\frac{tE}{1-\nu^2}\frac{1}{R}(\frac{du_\phi}{d\phi}-u_n)+D\frac{1}{R^2}\frac{d\theta_z}{d\phi} \\ M_{\phi\phi}-D[-\frac{1}{R}\frac{d\theta_z}{d\phi}+\frac{1}{R^2}(\frac{du_\phi}{d\phi}-u_n)] \\ \frac{1}{R}(\frac{du_n}{d\phi}+u_\phi)-\theta_z \end{Bmatrix}=0 \text{ in } (0,\frac{\pi}{2})\,.$$

$$\begin{Bmatrix} N_{\phi\phi} \\ Q_\phi-P \\ M_{\phi\phi} \end{Bmatrix}=0 \text{ at } \phi=\frac{\pi}{2}\,,\quad \begin{Bmatrix} u_\phi \\ u_n \\ \theta_z \end{Bmatrix}=0 \text{ at } \phi=0\,.$$

Curvilinear (s,α,n) – coordinate system of a shell is defined by mapping

$$\vec{r}(s,\alpha,n)=(R+n)\cos(\frac{s}{R})\vec{i}+(R+n)\sin(\frac{s}{R})\vec{j}+\alpha L\vec{k}$$

in which R and L are some constants, and n is the transverse coordinate. Derive in detail the expressions for the basis vectors, the non-zero basic vector derivatives, gradient operator, and the curvature tensor of mid-surface.

Solution

Let us start with the partial derivatives with respect to the curvilinear coordinates

$$\begin{Bmatrix}\vec{h}_s\\ \vec{h}_\alpha\\ \vec{h}_n\end{Bmatrix}=\begin{Bmatrix}\dfrac{\partial\vec{r}}{\partial s}\\ \dfrac{\partial\vec{r}}{\partial\alpha}\\ \dfrac{\partial\vec{r}}{\partial n}\end{Bmatrix}=\begin{bmatrix}-\dfrac{R+n}{R}\sin(\dfrac{s}{R}) & \dfrac{R+n}{R}\cos(\dfrac{s}{R}) & 0\\ 0 & 0 & L\\ \cos(\dfrac{s}{R}) & \sin(\dfrac{s}{R}) & 0\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}=[H]\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}$$

Scaling factor are the length of partial derivatives $h_s=1+n/R$, $h_\alpha=L$, and $h_n=1$. Therefore, the basis vectors are

$$\begin{Bmatrix}\vec{e}_s\\ \vec{e}_\alpha\\ \vec{e}_n\end{Bmatrix}=\begin{Bmatrix}\vec{h}_s/h_s\\ \vec{h}_\alpha/h_\alpha\\ \vec{h}_n/h_n\end{Bmatrix}=\begin{bmatrix}-\sin(\dfrac{s}{R}) & \cos(\dfrac{s}{R}) & 0\\ 0 & 0 & 1\\ \cos(\dfrac{s}{R}) & \sin(\dfrac{s}{R}) & 0\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}=[F]\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}. \leftarrow$$

As the coordinate systems are orthonormal $[F]^{-1}=[F]^{\mathrm{T}}$ and the non-zero expressions for the partial derivatives

$$\frac{\partial}{\partial s}\begin{Bmatrix}\vec{e}_s\\ \vec{e}_\alpha\\ \vec{e}_n\end{Bmatrix}=\frac{1}{R}\begin{bmatrix}-\cos(\dfrac{s}{R}) & -\sin(\dfrac{s}{R}) & 0\\ 0 & 0 & 0\\ -\sin(\dfrac{s}{R}) & \cos(\dfrac{s}{R}) & 0\end{bmatrix}\begin{bmatrix}-\sin(\dfrac{s}{R}) & 0 & \cos(\dfrac{s}{R})\\ \cos(\dfrac{s}{R}) & 0 & \sin(\dfrac{s}{R})\\ 0 & 1 & 0\end{bmatrix}\begin{Bmatrix}\vec{e}_s\\ \vec{e}_\alpha\\ \vec{e}_n\end{Bmatrix}=\frac{1}{R}\begin{bmatrix}0 & 0 & -1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{bmatrix}\begin{Bmatrix}\vec{e}_s\\ \vec{e}_\alpha\\ \vec{e}_n\end{Bmatrix}$$

$$\frac{\partial\vec{e}_s}{\partial s}=-\frac{1}{R}\vec{e}_n \quad\text{and}\quad \frac{\partial\vec{e}_n}{\partial s}=\frac{1}{R}\vec{e}_s. \leftarrow$$

Gradient operator follows from the definition ($[H][F]^{\mathrm{T}}=\mathrm{diag}(\{h_s,h_\alpha,h_n\})$)

$$\nabla = \begin{Bmatrix} \vec{e}_s \\ \vec{e}_\alpha \\ \vec{e}_n \end{Bmatrix}^{\mathrm{T}} [F]^{-\mathrm{T}} [H]^{-1} \begin{Bmatrix} \dfrac{\partial}{\partial s} \\ \dfrac{\partial}{\partial \alpha} \\ \dfrac{\partial}{\partial n} \end{Bmatrix} = \begin{Bmatrix} \vec{e}_s \\ \vec{e}_\alpha \\ \vec{e}_n \end{Bmatrix}^{\mathrm{T}} ([H][F]^{\mathrm{T}})^{-1} \begin{Bmatrix} \dfrac{\partial}{\partial s} \\ \dfrac{\partial}{\partial \alpha} \\ \dfrac{\partial}{\partial n} \end{Bmatrix} \quad \Rightarrow$$

$$\nabla = \vec{e}_s \frac{R}{R+n} \frac{\partial}{\partial s} + \vec{e}_\alpha \frac{1}{L} \frac{\partial}{\partial \alpha} + \vec{e}_n \frac{\partial}{\partial n}. \quad \Leftarrow$$

Curvature tensor follows from definition at the mid-surface $n = 0$

$$\nabla \vec{e}_n = (\vec{e}_s \frac{R}{R+n} \frac{\partial}{\partial s} + \vec{e}_\alpha \frac{1}{L} \frac{\partial}{\partial \alpha} + \vec{e}_n \frac{\partial}{\partial n}) \vec{e}_n = \frac{1}{R+n} \vec{e}_s \vec{e}_s \quad \Rightarrow \quad \vec{\kappa} = (\nabla \vec{e}_n)_{\mathrm{c}} = \frac{1}{R} \vec{e}_s \vec{e}_s. \quad \Leftarrow$$

Derive the component forms of cylindrical shell moment equilibrium equations in the (z,ϕ,n) coordinate system starting from the invariant form $(\nabla_0\cdot\vec{\vec{M}}-\kappa\vec{e}_n\cdot\vec{\vec{M}}-\vec{e}_n\cdot\vec{\vec{F}}+\vec{c})\times\vec{e}_n=0$.

Solution

Component representations of the quantities in the equilibrium equation are (notice that the transverse normal components are missing)

$$\vec{\vec{F}}=N_{zz}\vec{e}_z\vec{e}_z+N_{z\phi}\vec{e}_z\vec{e}_\phi+N_{\phi z}\vec{e}_\phi\vec{e}_z+N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi+Q_z\vec{e}_z\vec{e}_n+Q_z\vec{e}_n\vec{e}_z+Q_\phi\vec{e}_\phi\vec{e}_n+Q_\phi\vec{e}_n\vec{e}_\phi,$$

$$\vec{\vec{M}}=M_{zz}\vec{e}_z\vec{e}_z+M_{z\phi}\vec{e}_z\vec{e}_\phi+M_{\phi z}\vec{e}_\phi\vec{e}_z+M_{\phi\phi}\vec{e}_\phi\vec{e}_\phi+M_{zn}\vec{e}_z\vec{e}_n+M_{nz}\vec{e}_n\vec{e}_z+M_{\phi n}\vec{e}_\phi\vec{e}_n+M_{n\phi}\vec{e}_n\vec{e}_\phi,$$

$$\vec{b}=b_z\vec{e}_z+b_\phi\vec{e}_\phi+b_n\vec{e}_n,\text{ and }\nabla_0=\vec{e}_z\frac{\partial}{\partial z}+\vec{e}_\phi\frac{1}{R}\frac{\partial}{\partial\phi}.$$

In the shell model, the stress resultants may not be symmetric although the stress $\vec{\vec{\sigma}}$ always is. Derivatives of the basis vectors, the unit tensor and curvature tensor are

$$\frac{\partial}{\partial\phi}\vec{e}_\phi=\vec{e}_n,\quad\frac{\partial}{\partial\phi}\vec{e}_n=-\vec{e}_\phi,\quad\vec{\vec{I}}=\vec{e}_z\vec{e}_z+\vec{e}_\phi\vec{e}_\phi+\vec{e}_n\vec{e}_n,$$

$$\vec{\vec{\kappa}}=(\nabla_0\vec{e}_n)_\mathrm{c}=(\vec{e}_z\frac{\partial\vec{e}_n}{\partial z}+\vec{e}_\phi\frac{1}{R}\frac{\partial\vec{e}_n}{\partial\phi})_\mathrm{c}=-\vec{e}_\phi\vec{e}_\phi\frac{1}{R}\quad\Rightarrow\quad\kappa=\vec{\vec{I}}:\vec{\vec{\kappa}}=-\frac{1}{R}.$$

Let us consider the mid-surface and transverse parts of the moment separately to shorten the expressions, First the mid-surface part

$$\nabla_0\cdot\vec{\vec{M}}_m=(\vec{e}_z\frac{\partial}{\partial z}+\vec{e}_\phi\frac{1}{R}\frac{\partial}{\partial\phi})\cdot(M_{zz}\vec{e}_z\vec{e}_z+M_{z\phi}\vec{e}_z\vec{e}_\phi+M_{\phi z}\vec{e}_\phi\vec{e}_z+M_{\phi\phi}\vec{e}_\phi\vec{e}_\phi)\text{ where}$$

$$(\vec{e}_z\frac{\partial}{\partial z})\cdot(M_{zz}\vec{e}_z\vec{e}_z+M_{z\phi}\vec{e}_z\vec{e}_\phi+M_{\phi z}\vec{e}_\phi\vec{e}_z+M_{\phi\phi}\vec{e}_\phi\vec{e}_\phi)=\frac{\partial M_{zz}}{\partial z}\vec{e}_z+\frac{\partial M_{z\phi}}{\partial z}\vec{e}_\phi,$$

$$(\vec{e}_\phi\frac{1}{R}\frac{\partial}{\partial\phi})\cdot(M_{zz}\vec{e}_z\vec{e}_z+M_{z\phi}\vec{e}_z\vec{e}_\phi+M_{\phi z}\vec{e}_\phi\vec{e}_z+M_{\phi\phi}\vec{e}_\phi\vec{e}_\phi)=\frac{1}{R}\frac{\partial M_{\phi z}}{\partial\phi}\vec{e}_z+\frac{1}{R}\frac{\partial M_{\phi\phi}}{\partial\phi}\vec{e}_\phi+\frac{1}{R}M_{\phi\phi}\vec{e}_n\ \Rightarrow$$

$$\nabla_0\cdot\vec{\vec{M}}_m=(\frac{\partial M_{zz}}{\partial z}+\frac{1}{R}\frac{\partial M_{\phi z}}{\partial\phi})\vec{e}_z+(\frac{\partial M_{z\phi}}{\partial z}+\frac{1}{R}\frac{\partial M_{\phi\phi}}{\partial\phi})\vec{e}_\phi+\frac{1}{R}M_{\phi\phi}\vec{e}_n.$$

Then, the transverse part

$$\nabla_0\cdot\vec{\vec{M}}_t=(\vec{e}_z\frac{\partial}{\partial z}+\vec{e}_\phi\frac{1}{R}\frac{\partial}{\partial\phi})\cdot(M_{zn}\vec{e}_z\vec{e}_n+M_{\phi n}\vec{e}_\phi\vec{e}_n+M_{nz}\vec{e}_n\vec{e}_z+M_{n\phi}\vec{e}_n\vec{e}_\phi)\text{ where}$$

$$(\vec{e}_z\frac{\partial}{\partial z})\cdot(M_{zn}\vec{e}_z\vec{e}_n+M_{\phi n}\vec{e}_\phi\vec{e}_n+M_{nz}\vec{e}_n\vec{e}_z+M_{n\phi}\vec{e}_n\vec{e}_\phi)=\frac{\partial M_{zn}}{\partial z}\vec{e}_n,$$

$$(\vec{e}_\phi \frac{1}{R}\frac{\partial}{\partial\phi})\cdot(M_{zn}\vec{e}_z\vec{e}_n + M_{\phi n}\vec{e}_\phi\vec{e}_n + M_{nz}\vec{e}_n\vec{e}_z + M_{n\phi}\vec{e}_n\vec{e}_\phi) = \frac{1}{R}(\frac{\partial M_{\phi n}}{\partial\phi}\vec{e}_n - M_{\phi n}\vec{e}_\phi - M_{nz}\vec{e}_z - M_{n\phi}\vec{e}_\phi)\,.$$

$$\nabla_0 \cdot \vec{\vec{M}}_t = (\frac{\partial M_{zn}}{\partial z} + \frac{1}{R}\frac{\partial M_{\phi n}}{\partial\phi})\vec{e}_n - \frac{1}{R}M_{nz}\vec{e}_z - \frac{1}{R}(M_{n\phi} + M_{\phi n})\vec{e}_\phi\,.$$

The second term of the equilibrium equation simplifies to

$$\kappa\vec{e}_n \cdot \vec{\vec{M}} = -\frac{1}{R}(M_{nz}\vec{e}_z + M_{n\phi}\vec{e}_\phi)\,.$$

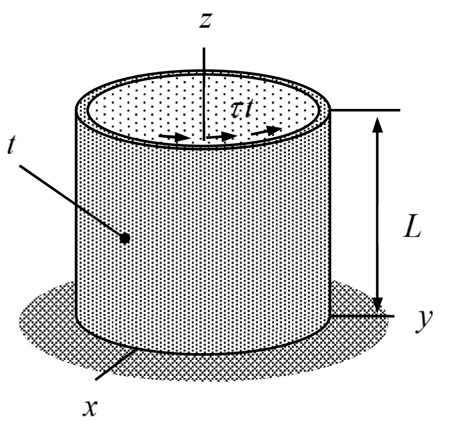
Third term

$$\vec{e}_n \cdot \vec{\vec{F}} = Q_z\vec{e}_z + Q_\phi\vec{e}_\phi$$

Finally, combining the terms (all terms in the normal direction vanish due to the cross product with $\vec{e}_n$)

$$\left\{\begin{array}{c} \dfrac{\partial M_{zz}}{\partial z} + \dfrac{1}{R}\dfrac{\partial M_{\phi z}}{\partial\phi} - Q_z + c_z \\ \dfrac{\partial M_{z\phi}}{\partial z} + \dfrac{1}{R}\dfrac{\partial M_{\phi\phi}}{\partial\phi} - \dfrac{1}{R}M_{\phi n} - Q_\phi + c_\phi \end{array}\right\} = 0\,.$$

A thin walled cylindrical body of length L, (mid-surface) radius R, and thickness t is subjected to shear loading τt $[\tau t] = \text{N/m}$ at the free end $z = L$ as shown in the figure. Assuming rotation symmetry, use the membrane equations in (z,ϕ,n) coordinate system to derive the relationship between the moment resultant T of the shear loading and the angle of rotation of the free end defined by $\theta = u_\phi / R$.



Solution

As the solution does not depend on ϕ, equilibrium equations of the membrane model and boundary conditions at the free end simplify to (a cylindrical membrane $z-$strip problem)

$$\frac{dN_{zz}}{dz} = 0\,,\quad \frac{dN_{z\phi}}{dz} = 0\,,\quad \frac{1}{R}N_{\phi\phi} = 0 \quad \text{in} \quad (0,L)\,,$$

$$N_{zz} = 0 \text{ and } N_{z\phi} = \tau t \quad \text{at} \quad z = L\,.$$

Solution to the boundary value problem for the stress resultants is given by

$$N_{zz} = N_{\phi\phi} = 0 \text{ and } N_{z\phi}(z) = \tau t\,. \leftarrow$$

Knowing the stress resultants, the boundary value problem for the displacement components follows from the constitutive equations and the boundary conditions at the fixed end (in the membrane model, a boundary condition cannot be assigned to u_n)

$$\frac{tE}{1-\nu^2}\left(\frac{du_z}{dz} - \nu\frac{1}{R}u_n\right) = 0\,,\quad \frac{tE}{1-\nu^2}\left(\nu\frac{du_z}{dz} - \frac{1}{R}u_n\right) = 0\,,\ \text{and}\ \ tG\frac{du_\phi}{dz} = \tau t \quad \text{in} \quad (0,L)\,,$$

$$u_z = 0\,,\quad u_\phi = 0 \quad \text{at} \quad z = 0\,.$$

Solution to the boundary value problem is given by

$$u_z = u_n = 0 \text{ and } u_\phi(z) = \frac{\tau}{G}z\,.$$

Moment resultant of the shear loading

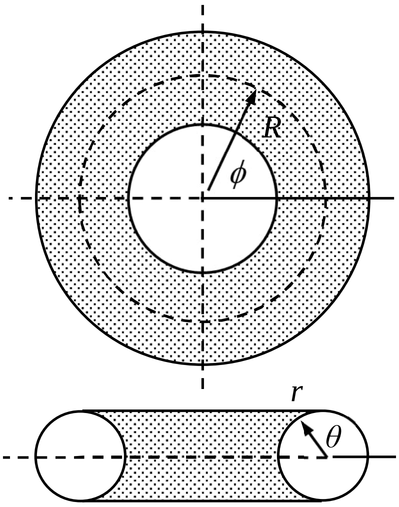
$$T = \int_0^{2\pi} t\tau R(Rd\phi) = 2\pi R^2 t\tau \quad \Rightarrow \quad \tau = \frac{T}{2\pi R^2 t}\,.$$

Therefore, at the free end

$$u_\phi = \frac{\tau}{G}L = \frac{L}{2\pi R^2 tG}T = R\theta \quad \Rightarrow \quad T = \frac{2\pi R^3 t}{L}G\theta\,. \leftarrow$$

The polar moment predicted here is $I_p = 2\pi R^3 t$ whereas the exact is $I_p = \frac{1}{2}\pi Rt(4R^2 + t^2)$.

Consider a torus shaped balloon under the loading caused by inner pressure difference Δp relative to the ambient pressure. Use (ϕ,θ,n) coordinate system, assume rotation symmetry with respect to ϕ, and solve for the stress resultant components from the equilibrium equations:



$$\begin{Bmatrix} \frac{1}{r}\frac{\partial N_{\phi\theta}}{\partial\theta}+\frac{1}{R+r\cos\theta}[\frac{\partial N_{\phi\phi}}{\partial\phi}-2N_{\phi\theta}\sin\theta]+b_{\phi} \\ \frac{1}{r}\frac{\partial N_{\theta\theta}}{\partial\theta}+\frac{1}{R+r\cos\theta}[\frac{\partial N_{\phi\theta}}{\partial\phi}+(N_{\phi\phi}-N_{\theta\theta})\sin\theta]+b_{\theta} \\ -\frac{1}{r}N_{\theta\theta}-\frac{1}{R+r\cos\theta}\cos\theta N_{\phi\phi}+b_{n} \end{Bmatrix}=0 .$$

Solution

As the solution should be independent of ϕ, partial derivatives with respect to ϕ vanish and the equilibrium equations of torus geometry simplify to ordinary diffrential equations. In toroidal system $\vec{e}_n$ is directed outwards and therefore $b_n=\Delta p$:

$$\frac{dN_{\phi\theta}}{d\theta}-\frac{2r\sin\theta}{R+r\cos\theta}N_{\phi\theta}=0 \quad \Rightarrow \quad N_{\phi\theta}=0 \quad \text{(clearly a solution)},$$

$$\frac{dN_{\theta\theta}}{d\theta}+\frac{r\sin\theta}{R+r\cos\theta}(N_{\phi\phi}-N_{\theta\theta})=0 ,$$

$$N_{\theta\theta}+\frac{r\cos\theta}{R+r\cos\theta}N_{\phi\phi}-r\Delta p=0 \quad \Leftrightarrow \quad \frac{r\cos\theta}{R+r\cos\theta}N_{\phi\phi}=r\Delta p-N_{\theta\theta} .$$

Eliminating $N_{\phi\phi}$ from the last two equations gives

$$\frac{dN_{\theta\theta}}{d\theta}+\tan\theta(\frac{r\cos\theta}{R+r\cos\theta}N_{\phi\phi})-\frac{r\sin\theta}{R+r\cos\theta}N_{\theta\theta}=0 \quad \Leftrightarrow$$

$$\frac{dN_{\theta\theta}}{d\theta}+\tan\theta(r\Delta p-N_{\theta\theta})-\frac{r\sin\theta}{R+r\cos\theta}N_{\theta\theta}=0 \quad \Leftrightarrow$$

$$\frac{dN_{\theta\theta}}{d\theta}-(\frac{R+2r\cos\theta}{R+r\cos\theta})\tan\theta N_{\theta\theta}+r\Delta p\tan\theta=0 .$$

Solution to the equation can be obtained by using an integrating factor. Let us write the differential equation in form

$$e^{A}\frac{dN_{\theta\theta}}{d\theta}+e^{A}\frac{dA}{d\theta}N_{\theta\theta}+e^{A}r\Delta p\tan\theta=\frac{d}{d\theta}(e^{A}N_{\theta\theta})+e^{A}r\Delta p\tan\theta=0 ,$$

$$\frac{dA}{d\theta} = -(\frac{R+2r\cos\theta}{R+r\cos\theta})\tan\theta \quad \Leftrightarrow \quad A = \log(R\cos\theta + r\cos^2\theta) \quad \Leftrightarrow \quad e^A = \cos\theta(R + r\cos\theta).$$

Continuing with the other equation of the set (integration constant is zero as stress resultant should vanish when $\Delta p = 0$)

$$\frac{d}{d\theta}(e^A N_{\theta\theta}) = -e^A r\Delta p\tan\theta = -\sin\theta(R + r\cos\theta)r\Delta p \quad \Rightarrow$$

$$e^A N_{\theta\theta} = \Delta pr\cos\theta(R + \frac{1}{2}r\cos\theta) \quad \Rightarrow$$

$$N_{\theta\theta} = r\Delta p\frac{2R + r\cos\theta}{2R + 2r\cos\theta} \quad \text{and} \quad N_{\phi\phi} = \frac{r\Delta p}{2}\ .$$

Displacement and rotation in rigid body motion are $\vec{u} = \vec{U} + \vec{\Omega} \times \vec{r}_0$ and $\vec{\Omega}$ in which $\vec{U}$ and $\vec{\Omega}$ are constant vectors in the Cartesian (x, y, z) coordinate system. Calculate the cylindrical shell stress resultant components $M_{z\phi}$ and $M_{\phi z}$ in rigid body mode $\Omega_x \neq 0$ and $U_x = U_y = U_z = \Omega_y = \Omega_z = 0$.

Solution

The representations of the quantities in the cylindrical (z, ϕ, n) coordinate system can be obtained from the relationship between the basis vectors of the Cartesian and cylindrical system

$$\begin{Bmatrix} \vec{e}_z \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -\sin\phi & \cos\phi & 0 \\ -\cos\phi & -\sin\phi & 0 \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} \quad \Leftrightarrow \quad \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -\sin\phi & \cos\phi & 0 \\ -\cos\phi & -\sin\phi & 0 \end{bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vec{e}_z \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix}.$$

Therefore

$$\vec{U} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} U_x \\ U_y \\ U_z \end{Bmatrix} = \begin{Bmatrix} \vec{e}_z \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 0 & 0 & 1 \\ -\sin\phi & \cos\phi & 0 \\ -\cos\phi & -\sin\phi & 0 \end{bmatrix} \begin{Bmatrix} U_x \\ U_y \\ U_z \end{Bmatrix} = \begin{Bmatrix} \vec{e}_z \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} U_z \\ U_y \cos\phi - U_x \sin\phi \\ -U_x \cos\phi - U_y \sin\phi \end{Bmatrix},$$

$$\vec{\Omega} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{Bmatrix} = \begin{Bmatrix} \vec{e}_z \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 0 & 0 & 1 \\ -\sin\phi & \cos\phi & 0 \\ -\cos\phi & -\sin\phi & 0 \end{bmatrix} \begin{Bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{Bmatrix} = \begin{Bmatrix} \vec{e}_z \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \Omega_z \\ \Omega_y \cos\phi - \Omega_x \sin\phi \\ -\Omega_x \cos\phi - \Omega_y \sin\phi \end{Bmatrix},$$

$$\vec{r}_0 = \vec{e}_z z - \vec{e}_n R .$$

The displacement and rotation components due to rigid body motion are (rotation of a typical line segment aligned in the normal direction does not have a component in the normal direction)

$$\vec{u} = \vec{U} + \vec{\Omega} \times \vec{r}_0 = \begin{Bmatrix} \vec{e}_z \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} U_z + R\Omega_x \sin\phi - R\Omega_y \cos\phi \\ U_y \cos\phi - U_x \sin\phi + R\Omega_z - z\Omega_x \cos\phi - z\Omega_y \sin\phi \\ -U_x \cos\phi - U_y \sin\phi + z\Omega_x \sin\phi - z\Omega_y \cos\phi \end{Bmatrix},$$

$$\vec{\theta} = \begin{Bmatrix} \vec{e}_z \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \Omega_z \\ \Omega_y \cos\phi - \Omega_x \sin\phi \\ 0 \end{Bmatrix}.$$

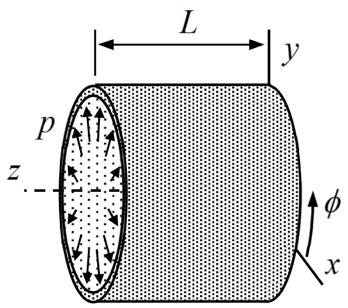
Let us consider the rigid body mode obtained with $\Omega_x \neq 0$ and $U_x = U_y = U_z = \Omega_y = \Omega_z = 0$ and substitute the components obtained into the constitutive equations for $M_{z\phi}$ and $M_{\phi z}$ of the formulae collection:

$u_z = \Omega_x R \sin\phi$, $u_\phi = -\Omega_x z \cos\phi$, $u_n = \Omega_x z \sin\phi$, $\theta_z = 0$, and $\theta_\phi = -\Omega_x \sin\phi \quad \Rightarrow$

$$M_{z\phi} = D\frac{1-\nu}{2}\left(-\frac{\partial\theta_z}{\partial z} + \frac{1}{R}\frac{\partial\theta_\phi}{\partial\phi} - \frac{1}{R}\frac{\partial u_\phi}{\partial z}\right) = D\frac{1-\nu}{2}\Omega_x\left(-\frac{1}{R}\cos\phi + \frac{1}{R}\cos\phi\right) = 0\,. \Leftarrow$$

$$M_{\phi z} = D\frac{1-\nu}{2}\left(-\frac{\partial\theta_z}{\partial z} + \frac{1}{R}\frac{\partial\theta_\phi}{\partial\phi} + \frac{1}{R^2}\frac{\partial u_z}{\partial\phi}\right) = D\frac{1-\nu}{2}\Omega_x\left(-\frac{1}{R}\cos\phi + \frac{1}{R}\cos\phi\right) = 0\,. \Leftarrow$$

A steel ring of length L, radius R, and thickness t is loaded by radial surface force p acting on the inner surface. No forces are acting on the ends. Model the ring as a cylindrical membrane, write down the equilibrium and constitutive equations, and solve for the radial displacement. Assume rotation symmetry and $u_\phi = 0$. Young's modulus E and Poisson's ratio g of the material are constants.



Solution

According to the formulae collection, equilibrium and constitutive equations of a cylindrical membrane in (z,ϕ,n) coordinates are (notice that $\vec{e}_n$ is directed inwards)

$$\begin{Bmatrix} \dfrac{1}{R}\dfrac{\partial N_{z\phi}}{\partial \phi} + \dfrac{\partial N_{zz}}{\partial z} + b_z \\[2ex] \dfrac{\partial N_{z\phi}}{\partial z} + \dfrac{1}{R}\dfrac{\partial N_{\phi\phi}}{\partial \phi} + b_\phi \\[2ex] \dfrac{1}{R} N_{\phi\phi} + b_n \end{Bmatrix} = 0 , \quad \begin{Bmatrix} N_{zz} \\ N_{\phi\phi} \\ N_{z\phi} \end{Bmatrix} = \frac{tE}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u_z}{\partial z} \\[2ex] \dfrac{1}{R}(\dfrac{\partial u_\phi}{\partial \phi} - u_n) \\[2ex] \dfrac{1}{R}\dfrac{\partial u_z}{\partial \phi} + \dfrac{\partial u_\phi}{\partial z} \end{Bmatrix} .$$

Due to the rotation symmetry, the derivatives with respect to the angular coordinate vanish and $u_\phi = 0$. External distributed force $b_n = -p$ is due to the traction acting on the inner boundary. Therefore, the equilibrium equations and constitutive equations simplify to a set of ordinary differential equations

$$\frac{dN_{zz}}{dz} = 0 , \quad \frac{dN_{z\phi}}{dz} = 0 , \quad \frac{1}{R} N_{\phi\phi} - p = 0 \quad \text{in} \quad (0, L) ,$$

$$N_{zz} = \frac{tE}{1-\nu^2}\frac{1}{R}(R\frac{du_z}{dz} - \nu u_n) , \quad N_{\phi\phi} = \frac{tE}{1-\nu^2}\frac{1}{R}(R\nu\frac{du_z}{dz} - u_n) , \quad N_{z\phi} = 0 \quad \text{in} \quad (0, L) ,$$

As the edges are stress-free i.e.

$$N_{zz} = 0 \ \text{ and } \ N_{z\phi} = 0 \quad \text{on} \quad \{0, L\} .$$

Solution to the stress resultants, as obtained from the equilibrium equations, are

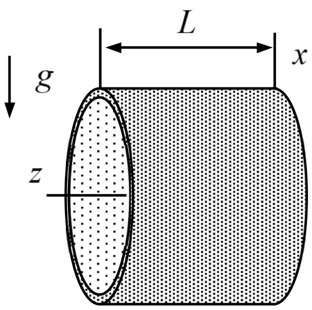
$$N_{zz} = 0 , \ N_{z\phi} = 0 , \text{ and } \ N_{\phi\phi} = Rp .$$

Constitutive equations give

$$N_{zz} = \frac{tE}{1-\nu^2}\frac{1}{R}(R\frac{du_z}{dz} - \nu u_n) = 0 \quad \Rightarrow \quad \frac{du_z}{dz} = \frac{\nu}{R} u_n \quad \text{and}$$

$$Rp = N_{\phi\phi} = \frac{tE}{1-\nu^2}\frac{1}{R}(R\nu\frac{du_z}{dz} - u_n) = \frac{tE}{1-\nu^2}\frac{1}{R}(\nu^2 - 1)u_n = -\frac{tE}{R}u_n \quad \Leftrightarrow \quad u_n = -\frac{pR^2}{tE} .$$

Consider a simply supported (long) circular cylindrical shell of radius R, thickness t, and filled with liquid of density ρ in cylindrical (z,ϕ,n) – coordinates. Determine the mid-surface stress resultants $N_{\phi\phi}$, $N_{z\phi}$ and N_{zz} by assuming that there are no axial forces at the ends of the shell and bending deformation is negligible. (J.N.Reddy: Example 11.3.1)



Solution

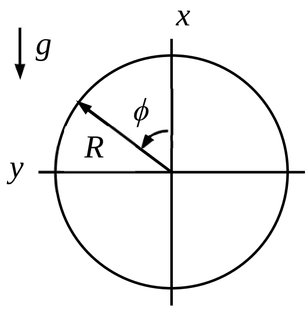
The membrane equations of the cylindrical coordinate system are (formulae collection)

$$\frac{1}{R}\frac{\partial N_{z\phi}}{\partial \phi}+\frac{\partial N_{zz}}{\partial z}+b_z=0\,,\quad \frac{\partial N_{z\phi}}{\partial z}+\frac{1}{R}\frac{\partial N_{\phi\phi}}{\partial \phi}+b_\phi=0\,,\text{ and }\quad \frac{1}{R}N_{\phi\phi}+b_n=0\,.$$

Definition of the external distributed force (let us assume that $t/R \ll 1$ so that $J \approx 1$ to simplify the setting somewhat)

$$\vec{b}=\int \vec{f}dn+\sum \vec{t}$$

takes into account the volume forces acting on the body and tractions acting on the outer and inner surfaces. In the present case $\vec{f}=0$ and the traction part is due to the hydrostatic pressure of the liquid inside the cylinder. Therefore



$$\vec{b}=(p_{out}-p_{in})\vec{e}_n\,.$$

The hydrostatic pressure inside $p_{in}=p_0-\rho g x=p_0-\rho g R\cos\phi$ gives

$$p_{out}-p_{in}=p_{out}-(p_0-\rho g R\cos\phi)=\Delta p+\rho g R\cos\phi$$

in which $\Delta p=p_{out}-p_0$ is a constant. The equations to be solved become (notice that $N_{zz}(z,\phi)$. $N_{\phi\phi}(z,\phi)$, $N_{z\phi}(z,\phi)$ and direct integration of a partial differential equation involves unknown functions instead of integration constants)

$$\frac{1}{R}N_{\phi\phi}+\Delta p+\rho g R\cos\phi=0 \quad\Leftrightarrow\quad N_{\phi\phi}(z,\phi)=-R\Delta p-\rho g R^2\cos\phi\,,$$

$$\frac{\partial N_{z\phi}}{\partial z}+\frac{1}{R}\frac{\partial N_{\phi\phi}}{\partial \phi}=0 \quad\Rightarrow\quad \frac{\partial N_{z\phi}}{\partial z}+\rho g R\sin\phi=0 \quad\Leftrightarrow\quad N_{z\phi}(z,\phi)=-\rho g R z\sin\phi+A(\phi)\,,$$

$$\frac{1}{R}\frac{\partial N_{z\phi}}{\partial \phi}+\frac{\partial N_{zz}}{\partial z}=0 \quad\Rightarrow\quad \frac{\partial N_{zz}}{\partial z}=-\frac{1}{R}\frac{\partial N_{z\phi}}{\partial \phi} \quad\Rightarrow\quad \frac{\partial N_{zz}}{\partial z}=\rho g z\cos\phi-\frac{1}{R}A'(\phi) \quad\Leftrightarrow$$

$$N_{zz}(z,\phi)=\rho g\frac{1}{2}z^2\cos\phi-z\frac{1}{R}A'(\phi)+B(\phi)\,.$$

In the solution, $A(\phi)$ and $B(\phi)$ are arbitrary functions subjected to $A(\phi)=A(2\pi+\phi)$ and $B(\phi)=B(2\pi+\phi)$ $\forall\phi$ (periodicity) as the domain is closed in the $\phi-$direction. Also, according to the assumption, N_{zz} vanishes at the ends. Therefore

$$N_{zz}(z,\phi)=\rho g\frac{1}{2}z^2\cos\phi-z\frac{1}{R}A'(\phi)+B(\phi)=0 \qquad z\in\{0,L\} \quad \Rightarrow$$

$$B(\phi)=0 \;\text{ and }\; \rho g\frac{1}{2}L^2\cos\phi-L\frac{1}{R}A'(\phi)+B(\phi)=0 \quad \Rightarrow \quad A'(\phi)=\rho gR\frac{1}{2}L\cos\phi \quad \Leftrightarrow$$

$$A(\phi)=\rho gR\frac{1}{2}L\sin\phi+A \;\; \text{(a constant now).}$$

Solution to force resultants becomes

$$N_{\phi\phi}=-R\Delta p_0-\rho gR^2\cos\phi\,,$$

$$N_{z\phi}=\rho gR(\frac{1}{2}L-z)\sin\phi+A\,,$$

$$N_{zz}=\rho g\frac{1}{2}(z^2-zL)\cos\phi$$

in which pressure difference Δp and integration constant A cannot be determined with the information given.

Consider a cylindrical shell of radius R, subjected to bending moment $M_{zz} = \underline{M}$ and shearing force $Q_z = \underline{Q}$ at the end $z = L$. The other end $z = 0$ is clamped. Assuming rotational symmetry, derive the boundary value problem of Kirchhoff type for deflection $u_n(z)$. Start with the component forms of the Reissner-Mindlin (type) shell equations in cylindrical (z,ϕ,n) coordinates.

Solution

Cylindrical shell Reissner-Mindlin equilibrium and constitutive equations in (z,ϕ,n)-coordinate system are given in the formulae collection. Under the assumption of rotational symmetry, derivatives with respect to ϕ vanish and $u_\phi = 0$. As the loading is through the boundary conditions, the equilibrium equations of cylindrical shell in (z,ϕ,n) – coordinate system simplify to

$$\frac{dN_{zz}}{dz} = 0\,, \quad \frac{dM_{zz}}{dz} - Q_z = 0\,, \text{ and } \frac{dQ_z}{dz} + \frac{1}{R}N_{\phi\phi} = 0\,.$$

Constitutive equations for the stress resultant components simplify to (notice that the constitutive equation for the shear force is replaced by the Kirchhoff constraint $du_n / dz + \theta_\phi = 0$ which is used to eliminate rotation θ_ϕ from the constitutive equations)

$$N_{zz} = \frac{tE}{1-\nu^2}(\frac{du_z}{dz} - \nu\frac{1}{R}u_n) + D\frac{1}{R}\frac{d^2u_n}{dz^2}\,,$$

$$N_{\phi\phi} = \frac{Et}{1-\nu^2}(\nu\frac{du_z}{dz} - \frac{1}{R}u_n)\,, \text{ and } M_{zz} = -D(\frac{d^2u_n}{dz^2} + \frac{1}{R}\frac{du_z}{dz})\,.$$

Force equilibrium equation in the axial boundary condition, constitutive equation for the axial stress resultant and the boundary condition at the free edge give

$$\frac{dN_{zz}}{dz} = 0 \text{ in } (0,L) \text{ and } N_{zz}(L) = 0 \quad \Rightarrow \quad N_{zz} = 0\,, \text{ therefore}$$

$$N_{zz} = \frac{tE}{1-\nu^2}(\frac{du_z}{dz} - \nu\frac{1}{R}u_n) + D\frac{1}{R}\frac{d^2u_n}{dz^2} = 0 \quad \Rightarrow \quad \frac{du_z}{dz} = \nu\frac{1}{R}u_n - \frac{1-\nu^2}{tE}D\frac{1}{R}\frac{d^2u_n}{dz^2}\,.$$

giving with notation $a = t / R$

$$N_{\phi\phi} = \frac{Et}{1-\nu^2}(\nu\frac{du_z}{dz} - \frac{1}{R}u_n) = \frac{Et}{1-\nu^2}[(\nu^2-1)\frac{1}{R}u_n - \nu\frac{1-\nu^2}{tE}D\frac{1}{R}\frac{d^2u_n}{dz^2})] = -\frac{Et}{R}u_n - \nu D\frac{1}{R}\frac{d^2u_n}{dz^2}\,,$$

$$M_{zz} = -D(\frac{d^2u_n}{dz^2} + \nu\frac{1}{R^2}u_n - \frac{1-\nu^2}{tE}D\frac{1}{R^2}\frac{d^2u_n}{dz^2}) = -D[(1-\frac{1}{12}a^2)\frac{d^2u_n}{dz^2} + \nu\frac{1}{R^2}u_n]\,.$$

The moment equilibrium equation is used next to eliminate the shear force from the remaining equilibrium equation to get

$$\frac{d^2 M_{zz}}{dz^2} + \frac{1}{R} N_{\phi\phi} = -D[(1-\frac{a^2}{12})\frac{d^4 u_n}{dz^4} + \nu \frac{2}{R^2}\frac{d^2 u_n}{dz^2}] - \frac{tE}{R^2} u_n = 0 \quad \text{in} \quad (0, L) .$$

Boundary conditions at the loaded end take the forms

$$Q_z - \underline{Q} = \frac{dM_{zz}}{dz} - \underline{Q} = -D[(1-\frac{1}{12}a^2)\frac{d^3 u_n}{dz^3} + \nu \frac{1}{R^2}\frac{du_n}{dz}] - \underline{Q} = 0 \quad \text{and}$$

$$M_{zz} - \underline{M} = -D[(1-\frac{1}{12}a^2)\frac{d^2 u_n}{dz^2} + \nu \frac{1}{R^2} u_n] - \underline{M} = 0 \quad \text{at} \quad z = L ,$$

and those for the clamped end

$$u_n = 0 , \ \frac{du_n}{dz} = 0 \quad \text{at} \quad z = 0 .$$

Consider a circular cylindrical shell of radius R, subjected to uniform bending moment $M_{zz} = \underline{M}$ and shearing force $Q_z = \underline{Q}$ at the end $z = L$. The other end $z = 0$ is clamped. Assuming rotational symmetry, derive the boundary value problem of Reissner-Mindlin type for deflection $u_n(z)$ and rotation $\theta_\phi(z)$.

Solution

As derivatives with respect to the angular coordinate vanish, the equilibrium equations of cylindrical shell in (z, ϕ, n) coordinate system simplify to

$$\frac{dN_{zz}}{dz} = 0, \quad \frac{dM_{zz}}{dz} - Q_z = 0, \text{ and } \frac{dQ_z}{dz} + \frac{1}{R} N_{\phi\phi} = 0.$$

Constitutive equations for the stress resultant components in the equilibrium equations simplify to

$$N_{zz} = \frac{tE}{1-\nu^2}\left(\frac{du_z}{dz} - \nu \frac{1}{R} u_n\right) - D\frac{1}{R}\frac{d\theta_\phi}{dz}, \quad N_{\phi\phi} = \frac{Et}{1-\nu^2}\left(\nu \frac{du_z}{dz} - \frac{1}{R} u_n\right),$$

$$Q_z = Gt\left(\theta_\phi + \frac{du_n}{dz}\right), \text{ and } M_{zz} = D\left(\frac{d\theta_\phi}{dz} - \frac{1}{R}\frac{du_z}{dz}\right).$$

Force equilibrium equation in the axial boundary condition, constitutive equation for the axial stress resultant and the boundary condition at the free edge give

$$\frac{dN_{zz}}{dz} = 0 \text{ in } (0, L) \text{ and } N_{zz}(L) = 0 \quad \Rightarrow \quad N_{zz}(z) = 0, \text{ therefore}$$

$$N_{zz} = \frac{tE}{1-\nu^2}\left(\frac{du_z}{dz} - \nu \frac{1}{R} u_n\right) - D\frac{1}{R}\frac{d\theta_\phi}{dz} = 0 \quad \Rightarrow \quad \frac{du_z}{dz} = \nu \frac{1}{R} u_n + \frac{1-\nu^2}{tE} D \frac{1}{R}\frac{d\theta_\phi}{dz}.$$

With the relationship, the constitutive equation for $N_{\phi\phi}$ and N_{zz} simplify to ($a = t / R$)

$$N_{\phi\phi} = \frac{Et}{1-\nu^2}\left(\nu \frac{du_z}{dz} - \frac{1}{R} u_n\right) = \frac{Et}{1-\nu^2}\left[(\nu^2 - 1)\frac{1}{R} u_n + \nu \frac{1-\nu^2}{tE} D \frac{1}{R}\frac{d\theta_\phi}{dz})\right] = -\frac{Et}{R} u_n + \nu D \frac{1}{R}\frac{d\theta_\phi}{dz},$$

$$M_{zz} = D\left(\frac{d\theta_\phi}{dz} - \nu \frac{1}{R^2} u_n - \frac{1-\nu^2}{tE} D \frac{1}{R^2}\frac{d\theta_\phi}{dz}\right) = D\left[\left(1 - \frac{1}{12}a^2\right)\frac{d\theta_\phi}{dz} - \nu \frac{1}{R^2} u_n\right].$$

When the constitutive equations are substituted there, equilibrium equations in terms of u_n and θ_ϕ take the forms

$$\frac{dQ_z}{dz} + \frac{1}{R} N_{\phi\phi} = \left(Gt + \nu D \frac{1}{R^2}\right)\frac{d\theta_\phi}{dz} + Gt \frac{d^2 u_n}{dz^2} - \frac{Et}{R^2} u_n = 0 \quad \text{in} \quad (0, L),$$

$$\frac{dM_{zz}}{dz} - Q_z = D\left[\left(1 - \frac{1}{12}a^2\right)\frac{d^2\theta_\phi}{dz^2} - \nu \frac{1}{R^2}\frac{du_n}{dz}\right] - Gt\left(\theta_\phi + \frac{du_n}{dz}\right) = 0 \quad \text{in} \quad (0, L).$$

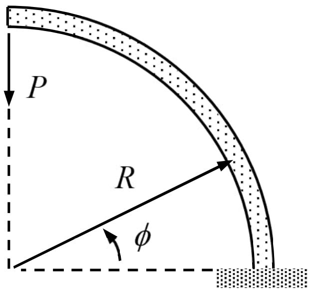
Boundary conditions at the ends are

$$Q_z - \underline{Q} = Gt(\theta_\phi + \frac{du_n}{dz}) - \underline{Q} = 0 \quad \text{at} \quad z = L\,,$$

$$M_{zz} - \underline{M} = D[(1 - \frac{1}{12}a^2)\frac{d\theta_\phi}{dz} - \nu \frac{1}{R^2}u_n] - \underline{M} = 0 \quad \text{at} \quad z = L$$

$$u_n = 0\,,\ \theta_\phi = 0 \quad \text{at} \quad z = 0\,.$$

A strip of cylindrical shell is loaded by shear force P ($[P]=\mathrm{N/m}$) at the free end. Write down the boundary value problem of first order ordinary differential equations consisting of the equilibrium and constitutive equations according to the Kirchhoff theory. Thickness t, width H, and the material parameters E, ν are constants. Assume that the solution depends on ϕ only.



Solution

Equilibrium and constitutive equations of cylindrical shell in $(z,\phi,n)-$ coordinate system are given in the formulae collection. In a shell strip problem, it is enough to consider the force equilibrium equations in the plane of the figure and the moment equilibrium in the normal direction of the plane and constitutive equations for the stress resultants (appearing in the equilibrium equations). In the Kirchhoff model, the constitutive equation for the shear force is replaced by the Kirchhoff constraint. Also, derivatives with respect to z vanish. Therefore the differential equations and constitutive equations simplify to

$$\begin{Bmatrix} \frac{1}{R}\frac{dN_{\phi\phi}}{d\phi}-\frac{1}{R}Q_{\phi} \\ \frac{1}{R}\frac{dQ_{\phi}}{d\phi}+\frac{1}{R}N_{\phi\phi} \\ \frac{1}{R}\frac{dM_{\phi\phi}}{d\phi}-\frac{1}{R}M_{\phi n}-Q_{\phi} \end{Bmatrix}=0 \quad \text{and} \quad \begin{Bmatrix} N_{\phi\phi}-\frac{tE}{1-\nu^2}\frac{1}{R}(\frac{du_{\phi}}{d\phi}-u_n)+D\frac{1}{R^2}\frac{d\theta_z}{d\phi} \\ M_{\phi\phi}-D[-\frac{1}{R}\frac{d\theta_z}{d\phi}+\frac{1}{R^2}(\frac{du_{\phi}}{d\phi}-u_n)] \\ \frac{1}{R}(\frac{du_n}{d\phi}+u_{\phi})-\theta_z \end{Bmatrix}=0 \quad \text{in} \quad (0,\frac{\pi}{2}).$$

where $M_{n\phi}=0$ due to the Kirchhoff constraint. The boundary conditions are

$$\begin{Bmatrix} N_{\phi\phi} \\ Q_{\phi}-P \\ M_{\phi\phi} \end{Bmatrix}=0 \quad \text{at} \quad \phi=\frac{\pi}{2} \quad \text{and} \quad \begin{Bmatrix} u_{\phi} \\ u_n \\ \theta_z \end{Bmatrix}=0 \quad \text{at} \quad \phi=0.$$

As the force resultants are known on one edge, equilibrium equations can be solved for the force resultants. Knowing these, displacements and rotation follow from the constitutive equations.

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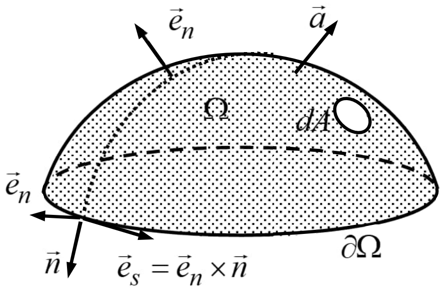
Assignment 1 (2p)

Gauss theorem implies the following integral identity for curved surfaces



$$\int_{\Omega} (\nabla_0 \cdot \vec{a} - \kappa \vec{e}_n \cdot \vec{a}) dA = \int_{\partial\Omega} (\vec{n} \cdot \vec{a}) ds$$

in which $\kappa = \vec{\kappa} : \vec{I} = \nabla_0 \cdot \vec{e}_n$. Verify the integral identity in the spherical (ϕ, θ, n) coordinate system by considering vector $\vec{a} = \theta \vec{e}_n$ and half-sphere $\phi \in [0, 2\pi]$, $\theta \in [0, \pi/2]$, of radius R as Ω. Derivatives of the basis vectors and the mid-surface gradient in the spherical coordinate system are

$$\frac{\partial}{\partial \phi} \begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_\theta \\ \vec{e}_n \end{Bmatrix} = \begin{Bmatrix} \sin\theta \vec{e}_n - \cos\theta \vec{e}_\theta \\ \cos\theta \vec{e}_\phi \\ -\sin\theta \vec{e}_\phi \end{Bmatrix}, \quad \frac{\partial}{\partial \theta} \begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_\theta \\ \vec{e}_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ \vec{e}_n \\ -\vec{e}_\theta \end{Bmatrix}, \quad \nabla_0 = \frac{1}{R \sin\theta} \vec{e}_\phi \frac{\partial}{\partial \phi} + \frac{1}{R} \vec{e}_\theta \frac{\partial}{\partial \theta}.$$

Solution template

In case of a half-sphere $\phi \in [0, 2\pi]$, $\theta \in [0, \pi/2]$ of radius R and vector $\vec{a} = \theta \vec{e}_n$, the quantities in the integral identity take the forms

$$\nabla_0 \cdot \vec{a} = (\frac{1}{R \sin\theta} \vec{e}_\phi \frac{\partial}{\partial \phi} + \frac{1}{R} \vec{e}_\theta \frac{\partial}{\partial \theta}) \cdot \theta \vec{e}_n = -\frac{2}{R}\theta,$$

$$\kappa = \nabla_0 \cdot \vec{e}_n = (\frac{1}{R \sin\theta} \vec{e}_\phi \frac{\partial}{\partial \phi} + \frac{1}{R} \vec{e}_\theta \frac{\partial}{\partial \theta}) \cdot \vec{e}_n = -\frac{2}{R},$$

$$\vec{e}_n \cdot \vec{a} = \theta,$$

$$\vec{n} \cdot \vec{a} = 0.$$

When the expressions are substituted there, the left- and right-hand sides of the integral identity simplify to

$$\int_{\Omega} (\nabla_0 \cdot \vec{a} - \kappa \vec{e}_n \cdot \vec{a}) dA = \int_{\Omega} (-\frac{2}{R}\theta + \frac{2}{R}\theta) dA = 0,$$

$$\int_{\partial\Omega} (\vec{n} \cdot \vec{a}) ds = \int_{\partial\Omega} (0) ds = 0.$$

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Assignment 2 (2p)

Derive the constitutive equations of Kirchhoff shell in cylindrical (z,ϕ,n) coordinates. Assume that all derivatives with respect to the angular coordinate ϕ and displacement and rotation components u_ϕ and θ_z vanish. The constitutive equations of Reissner-Mindlin shell in cylindrical (z,ϕ,n) coordinates are

$$\begin{Bmatrix} M_{zz} \\ M_{\phi\phi} \\ M_{z\phi} \\ M_{\phi z} \end{Bmatrix} = D \begin{Bmatrix} \frac{\partial \theta_\phi}{\partial z} - \nu \frac{1}{R}\frac{\partial \theta_z}{\partial \phi} - \frac{1}{R}\frac{\partial u_z}{\partial z} \\ \nu \frac{\partial \theta_\phi}{\partial z} - \frac{1}{R}\frac{\partial \theta_z}{\partial \phi} + \frac{1}{R^2}(\frac{\partial u_\phi}{\partial \phi} - u_n) \\ \frac{1}{2}(1-\nu)[(\frac{1}{R}\frac{\partial \theta_\phi}{\partial \phi} - \frac{\partial \theta_z}{\partial z}) - \frac{1}{R}\frac{\partial u_\phi}{\partial z}] \\ \frac{1}{2}(1-\nu)[(\frac{1}{R}\frac{\partial \theta_\phi}{\partial \phi} - \frac{\partial \theta_z}{\partial z}) + \frac{1}{R^2}\frac{\partial u_z}{\partial \phi}] \end{Bmatrix}, \quad M_{\phi n} = \frac{1}{2}(1-\nu) D \frac{1}{R}[\frac{1}{R}(\frac{\partial u_n}{\partial \phi} + u_\phi) - \theta_z],$$

$$\begin{Bmatrix} N_{zz} \\ N_{\phi\phi} \\ N_{z\phi} \\ N_{\phi z} \end{Bmatrix} = \begin{Bmatrix} \frac{tE}{1-\nu^2}[\frac{\partial u_z}{\partial z} + \nu \frac{1}{R}(\frac{\partial u_\phi}{\partial \phi} - u_n)] - D\frac{1}{R}\frac{\partial \theta_\phi}{\partial z} \\ \frac{tE}{1-\nu^2}[\frac{1}{R}(\frac{\partial u_\phi}{\partial \phi} - u_n) + \nu \frac{\partial u_z}{\partial z}] - D\frac{1}{R^2}\frac{\partial \theta_z}{\partial \phi} \\ Gt(\frac{1}{R}\frac{\partial u_z}{\partial \phi} + \frac{\partial u_\phi}{\partial z}) + \frac{1}{2}(1-\nu)D\frac{1}{R}\frac{\partial \theta_z}{\partial z} \\ Gt(\frac{1}{R}\frac{\partial u_z}{\partial \phi} + \frac{\partial u_\phi}{\partial z}) + \frac{1}{2}(1-\nu)D\frac{1}{R^2}\frac{\partial \theta_\phi}{\partial \phi} \end{Bmatrix}, \quad \begin{Bmatrix} Q_z \\ Q_\phi \end{Bmatrix} = Gt \begin{Bmatrix} \frac{\partial u_n}{\partial z} + \theta_\phi \\ \frac{1}{R}(\frac{\partial u_n}{\partial \phi} + u_\phi) - \theta_z \end{Bmatrix}.$$

Solution template

All derivatives with respect to the angular coordinate ϕ and displacement and rotation components u_ϕ and θ_z vanish. Kirchhoff constraints imply that

$$\theta_\phi = -\frac{du_n}{dz}.$$

When the Kirchhoff constraint is used to eliminate the rotation variable θ_ϕ, the constitutive equations for the non-zero stress resultants in terms of u_z and u_n simplify to

$$N_{zz} = \frac{tE}{1-\nu^2}[\frac{\partial u_z}{\partial z} + \nu \frac{1}{R}(\frac{\partial u_\phi}{\partial \phi} - u_n)] - D\frac{1}{R}\frac{\partial \theta_\phi}{\partial z} = \frac{tE}{1-\nu^2}(\frac{du_z}{dz} - \nu\frac{u_n}{R}) + D\frac{1}{R}\frac{d^2 u_n}{d^2 z},$$

$$N_{\phi\phi} = \frac{tE}{1-\nu^2}\left[\frac{1}{R}\left(\frac{\partial u_\phi}{\partial \phi} - u_n\right) + \nu\frac{\partial u_z}{\partial z}\right] - D\frac{1}{R^2}\frac{\partial \theta_z}{\partial \phi} = \frac{tE}{1-\nu^2}\left(\nu\frac{du_z}{dz} - \frac{u_n}{R}\right),$$

$$M_{zz} = D\left(\frac{\partial \theta_\phi}{\partial z} - \nu\frac{1}{R}\frac{\partial \theta_z}{\partial \phi} - \frac{1}{R}\frac{\partial u_z}{\partial z}\right) = -D\left(\frac{d^2 u_n}{dz^2} + \frac{1}{R}\frac{du_z}{dz}\right),$$

$$M_{\phi\phi} = D\left[\nu\frac{\partial \theta_\phi}{\partial z} - \frac{1}{R}\frac{\partial \theta_z}{\partial \phi} + \frac{1}{R^2}\left(\frac{\partial u_\phi}{\partial \phi} - u_n\right)\right] = -D\left(\nu\frac{d^2 u_n}{dz^2} + \frac{1}{R^2}u_n\right).$$

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Assignment 3 (4p)

Derive the component forms of cylindrical shell force equilibrium equations in the (z,ϕ,n) coordinate system starting from the invariant form $\nabla_0 \cdot \vec{\vec{F}} - \kappa \vec{e}_n \cdot \vec{\vec{F}} + \vec{b} = 0$. The force resultant representations and kinematic quantities of the cylindrical shell (z,ϕ,n) coordinate system are

$$\vec{\vec{F}} = N_{zz}\vec{e}_z\vec{e}_z + N_{z\phi}\vec{e}_z\vec{e}_\phi + N_{\phi z}\vec{e}_\phi\vec{e}_z + N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi + Q_z\vec{e}_z\vec{e}_n + Q_z\vec{e}_n\vec{e}_z + Q_\phi\vec{e}_\phi\vec{e}_n + Q_\phi\vec{e}_n\vec{e}_\phi,$$

$$\vec{b} = b_z\vec{e}_z + b_\phi\vec{e}_\phi + b_n\vec{e}_n, \quad \nabla_0 = \vec{e}_z\frac{\partial}{\partial z} + \vec{e}_\phi\frac{1}{R}\frac{\partial}{\partial\phi}, \quad \frac{\partial}{\partial\phi}\vec{e}_\phi = \vec{e}_n, \quad \frac{\partial}{\partial\phi}\vec{e}_n = -\vec{e}_\phi, \quad \vec{\vec{I}} = \vec{e}_z\vec{e}_z + \vec{e}_\phi\vec{e}_\phi + \vec{e}_n\vec{e}_n.$$

Solution

In the shell model, the stress resultants may not be symmetric. Definition $\vec{\vec{\kappa}} = (\nabla_0\vec{e}_n)_{\mathrm{c}}$ gives the curvature tensor

$$\vec{\vec{\kappa}} = (\nabla_0\vec{e}_n)_{\mathrm{c}} = [(\vec{e}_z\frac{\partial}{\partial z} + \vec{e}_\phi\frac{1}{R}\frac{\partial}{\partial\phi})\vec{e}_n]_{\mathrm{c}} = (\vec{e}_z\frac{\partial\vec{e}_n}{\partial z} + \vec{e}_\phi\frac{1}{R}\frac{\partial\vec{e}_n}{\partial\phi})_{\mathrm{c}} = -\vec{e}_\phi\vec{e}_\phi\frac{1}{R} \quad \Rightarrow \quad \kappa = \vec{\vec{I}} : \vec{\vec{\kappa}} = -\frac{1}{R}.$$

Let us consider the mid-surface (membrane) and shear parts of $\vec{\vec{F}} = \vec{\vec{N}} + \vec{Q}\vec{e}_n + \vec{e}_n\vec{Q}$ separately. First the membrane mode term

$$\nabla_0 \cdot \vec{\vec{N}} = (\vec{e}_z\frac{\partial}{\partial z} + \vec{e}_\phi\frac{1}{R}\frac{\partial}{\partial\phi}) \cdot (N_{zz}\vec{e}_z\vec{e}_z + N_{z\phi}\vec{e}_z\vec{e}_\phi + N_{\phi z}\vec{e}_\phi\vec{e}_z + N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi), \quad \text{where}$$

$$(\vec{e}_z\frac{\partial}{\partial z}) \cdot (N_{zz}\vec{e}_z\vec{e}_z + N_{z\phi}\vec{e}_z\vec{e}_\phi + N_{\phi z}\vec{e}_\phi\vec{e}_z + N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi) = \frac{\partial N_{zz}}{\partial z}\vec{e}_z + \frac{\partial N_{z\phi}}{\partial z}\vec{e}_\phi \quad \text{and}$$

$$(\vec{e}_\phi\frac{1}{R}\frac{\partial}{\partial\phi}) \cdot (N_{zz}\vec{e}_z\vec{e}_z + N_{z\phi}\vec{e}_z\vec{e}_\phi + N_{\phi z}\vec{e}_\phi\vec{e}_z + N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi) = \frac{1}{R}\frac{\partial N_{\phi z}}{\partial\phi}\vec{e}_z + \frac{1}{R}\frac{\partial N_{\phi\phi}}{\partial\phi}\vec{e}_\phi + \frac{1}{R}N_{\phi\phi}\vec{e}_n.$$

Altogether

$$\nabla_0 \cdot \vec{\vec{N}} = (\frac{\partial N_{zz}}{\partial z} + \frac{1}{R}\frac{\partial N_{\phi z}}{\partial\phi})\vec{e}_z + (\frac{\partial N_{z\phi}}{\partial z} + \frac{1}{R}\frac{\partial N_{\phi\phi}}{\partial\phi})\vec{e}_\phi + \frac{1}{R}N_{\phi\phi}\vec{e}_n.$$

Then, the shear part associated with the bending mode

$$\nabla_0 \cdot (\vec{Q}\vec{e}_n + \vec{e}_n\vec{Q}) = (\vec{e}_z\frac{\partial}{\partial z} + \vec{e}_\phi\frac{1}{R}\frac{\partial}{\partial\phi}) \cdot (Q_z\vec{e}_z\vec{e}_n + Q_\phi\vec{e}_\phi\vec{e}_n + Q_z\vec{e}_n\vec{e}_z + Q_\phi\vec{e}_n\vec{e}_\phi), \quad \text{where}$$

$$(\vec{e}_z\frac{\partial}{\partial z}) \cdot (Q_z\vec{e}_z\vec{e}_n + Q_\phi\vec{e}_\phi\vec{e}_n + Q_z\vec{e}_n\vec{e}_z + Q_\phi\vec{e}_n\vec{e}_\phi) = \frac{\partial Q_z}{\partial z}\vec{e}_n \quad \text{and}$$

$$(\vec{e}_\phi \frac{1}{R}\frac{\partial}{\partial\phi})\cdot(Q_z\vec{e}_z\vec{e}_n + Q_\phi\vec{e}_\phi\vec{e}_n + Q_z\vec{e}_n\vec{e}_z + Q_\phi\vec{e}_n\vec{e}_\phi) = \frac{1}{R}(\frac{\partial Q_\phi}{\partial\phi}\vec{e}_n - Q_\phi\vec{e}_\phi - Q_z\vec{e}_z - Q_\phi\vec{e}_\phi).$$

Altogether

$$\nabla_0\cdot(\vec{Q}\vec{e}_n + \vec{e}_n\vec{Q}) = -\frac{1}{R}Q_z\vec{e}_z - 2\frac{1}{R}Q_\phi\vec{e}_\phi + (\frac{\partial Q_z}{\partial z} + \frac{1}{R}\frac{\partial Q_\phi}{\partial\phi})\vec{e}_n = -\frac{1}{R}\vec{Q} - \frac{1}{R}Q_\phi\vec{e}_\phi + (\frac{\partial Q_z}{\partial z} + \frac{1}{R}\frac{\partial Q_\phi}{\partial\phi})\vec{e}_n.$$

The second term of the equilibrium equation simplifies to

$$\kappa\vec{e}_n\cdot\vec{\vec{F}} = -\frac{1}{R}(Q_z\vec{e}_z + Q_\phi\vec{e}_\phi) = -\frac{1}{R}\vec{Q}.$$

Therefore, combining the terms

$$\nabla_0\cdot\vec{\vec{F}} - \kappa\vec{e}_n\cdot\vec{\vec{F}} + \vec{b} = \begin{Bmatrix}\vec{e}_z\\ \vec{e}_\phi\\ \vec{e}_n\end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \dfrac{\partial N_{zz}}{\partial z} + \dfrac{1}{R}\dfrac{\partial N_{\phi z}}{\partial\phi} + b_z \\ \dfrac{\partial N_{z\phi}}{\partial z} + \dfrac{1}{R}\dfrac{\partial N_{\phi\phi}}{\partial\phi} - \dfrac{1}{R}Q_\phi + b_\phi \\ \dfrac{\partial Q_z}{\partial z} + \dfrac{1}{R}N_{\phi\phi} + \dfrac{1}{R}\dfrac{\partial Q_\phi}{\partial\phi} + b_n \end{Bmatrix} = 0. \quad \Leftarrow$$

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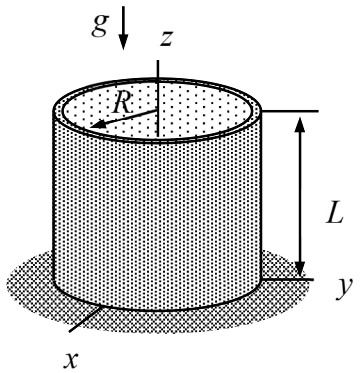
Assignment 4 (4p)



A cylindrical container of height L, (mid-surface) radius R, density ρ, and thickness t is loaded by its own weight. Assume rotation symmetry and use the membrane equations in (z,ϕ,n) coordinate system to find the stress resultant and the displacement components. Assume that friction between the container and floor is small and cannot constraint the transverse displacement at the contact points and rigid body motion is not possible (constrained somehow).

Solution

According to the formulae collection, equilibrium and constitutive equations of a cylindrical membrane in (z,ϕ,n) coordinates are (notice that $\vec{e}_n$ is directed inwards)

$$\begin{Bmatrix} \dfrac{1}{R}\dfrac{\partial N_{\phi z}}{\partial \phi}+\dfrac{\partial N_{zz}}{\partial z}+b_z \\ \dfrac{\partial N_{z\phi}}{\partial z}+\dfrac{1}{R}\dfrac{\partial N_{\phi\phi}}{\partial \phi}+b_\phi \\ \dfrac{1}{R}N_{\phi\phi}+b_n \end{Bmatrix}=0\,,\quad \begin{Bmatrix} N_{zz} \\ N_{\phi\phi} \\ N_{z\phi} \end{Bmatrix}=\frac{tE}{1-\nu^2}\begin{Bmatrix} \dfrac{\partial u_z}{\partial z}+\nu\dfrac{1}{R}(\dfrac{\partial u_\phi}{\partial \phi}-u_n) \\ \dfrac{1}{R}(\dfrac{\partial u_\phi}{\partial \phi}-u_n)+\nu\dfrac{\partial u_z}{\partial z} \\ \dfrac{1-\nu}{2}(\dfrac{1}{R}\dfrac{\partial u_z}{\partial \phi}+\dfrac{\partial u_\phi}{\partial z}) \end{Bmatrix},\quad N_{\phi z}=N_{z\phi}\,.$$

If the solution does not depend on ϕ, equilibrium equations of the membrane model and boundary conditions at the free end simplify to (gravity is acting in the negative direction of the $z-$axis)

$$\frac{dN_{zz}}{dz}-\rho g t=0\,,\quad \frac{dN_{z\phi}}{dz}=0\,,\ \text{ and }\ \frac{1}{R}N_{\phi\phi}=0 \quad \text{in} \quad (0,L)\,,$$

$$N_{zz}=0 \ \text{ and } \ N_{z\phi}=0 \ \text{ at } \ z=L\,.$$

Solution to the boundary value problem for the stress resultants is given by

$$N_{zz}(z)=(z-L)\rho g t\,,\quad N_{z\phi}(z)=0\,,\quad N_{\phi\phi}(z)=0\,. \leftarrow$$

As the stress resultants are now known, the boundary value problem for the displacement components follows from the constitutive equations and the boundary conditions at the fixed end (in the membrane model, a boundary condition cannot be assigned to u_n)

$$(z-L)\rho g t=\frac{tE}{1-\nu^2}(\frac{du_z}{dz}-\nu\frac{1}{R}u_n)\,,\quad 0=\frac{tE}{1-\nu^2}(\nu\frac{du_z}{dz}-\frac{1}{R}u_n)\,,\ \text{ and }\ tG\frac{du_\phi}{dz}=0 \quad \text{in} \quad (0,L)\,,$$

$u_z = 0$ at $z = 0$.

The last equation implies that $u_\phi =$ constant, but the constant has to be zero as a non-zero value would mean rigid body rotation around the $z-$axis so

$$u_\phi(z) = 0 .$$

Using the second equation to eliminate u_n from the first one gives the boundary value problem

$$(z-L)\rho g t = tE\frac{du_z}{dz} \quad \text{in} \quad (0,L) \quad \text{and} \quad u_z = 0 \quad \text{at} \quad z = 0 \quad \Rightarrow \quad u_z(z) = (\frac{1}{2}z^2 - Lz)\frac{\rho g}{E} .$$

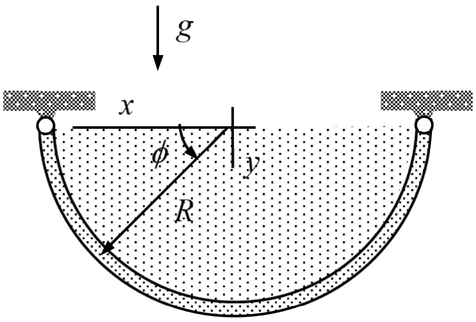
Finally, substituting into the second equilibrium equation and solving for the last displacement component

$$u_n(z) = R\nu\frac{du_z}{dz} = (z-L)\nu\frac{R\rho g}{E} .$$

Name______________________________ Student number____________________

Assignment 5 (4p)

A long rain gutter of cylindrical shape is filled with water (density ρ) and mounted with long cylindrical joints. Assuming that displacement and stress are independent in the z – coordinate, and weight of the gutter is negligible, write the boundary value problem giving as its solution the stress resultants and displacement components according to the Kirchhoff theory. Thickness t, radius R, and the material parameters E, ν are constants. Use the equilibrium and constitutive equations in the cylindrical (z,ϕ,n) coordinate system ($\vec{e}_n$ points inwards).



Solution

In shell strip problem, it is enough to consider force equilibrium equations in the plane of the figure and moment equilibrium in the normal direction of the plane and constitutive equations for the stress resultants in the equilibrium equations. In the Kirchhoff model, the constitutive equation for the shear force is replaced by the Kirchhoff constraint. Also, as solution is assumed to depend on ϕ only, all derivatives with respect to z vanish. External distributed force is due to hydrostatic pressure

$$b_n = -p = -\rho g y = -\rho g R \sin\phi \, .$$

Therefore, the equilibrium equations simplify to

$$\frac{1}{R}(\frac{dN_{\phi\phi}}{d\phi} - Q_\phi) = 0 \, , \quad \leftarrow$$

$$\frac{1}{R}(\frac{dQ_\phi}{d\phi} + N_{\phi\phi}) - \rho g R \sin\phi = 0 \, , \quad \leftarrow$$

$$\frac{1}{R}(\frac{dM_{\phi\phi}}{d\phi} - M_{\phi n}) - Q_\phi = 0 \, , \quad \leftarrow$$

and the constitutive equations (for the stress resultants in the equilibrium equations) to

$$N_{\phi\phi} = \frac{tE}{1-\nu^2}\frac{1}{R}(\frac{du_\phi}{d\phi} - u_n) - D\frac{1}{R^2}\frac{d\theta_z}{d\phi} \, , \quad \leftarrow$$

$$M_{\phi\phi} = D\frac{1}{R}[-\frac{d\theta_z}{d\phi} + \frac{1}{R}(\frac{du_\phi}{d\phi} - u_n)] \, , \quad \leftarrow$$

$$M_{\phi n} = \frac{1}{2}(1-\nu)D\frac{1}{R}[\frac{1}{R}(\frac{du_n}{d\phi}+u_\phi)-\theta_z],$$

The Kirchhoff constraint

$$\frac{1}{R}(\frac{du_n}{d\phi}+u_\phi)-\theta_z = 0$$

follows from the constitutive equations for Q_ϕ. In the Kirchhoff model, Q_ϕ is a constraint force whose value follows from the equilibrium equations. Notice that $M_{\phi n} = 0$ due to the Kirchhoff constraint. The independent variable is ϕ and the mathematical solution domain for the equations $(0,\pi)$. The work conjugates appearing in the equations are $(u_\phi, N_{\phi\phi})$, (u_n, Q_ϕ), and $(\theta_z, M_{\phi\phi})$ of which one has to specify either kinetic or kinematic quantity at all boundary points. For a simply supported rain gutter

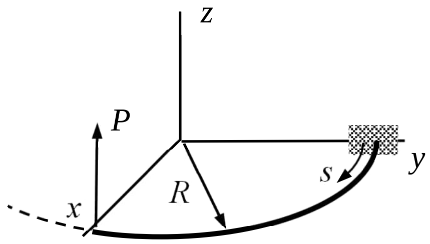
$$u_\phi = 0,\ \ u_n = 0,\ \text{and}\ M_{\phi\phi} = 0\ \ \text{on}\ \{0,\pi\}.$$

MEC-E8003 Beam, Plate and Shell models, onsite exam 16.04.2024

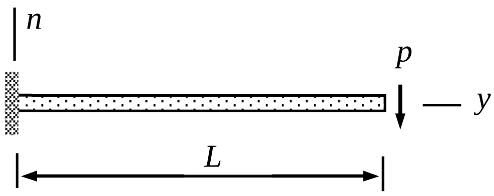
1. Consider the mid-surface mapping $\vec{r}_0(r,\phi) = r^2 \cos(2\phi)\vec{i} + r^2 \sin(2\phi)\vec{j}$ of shell. Compute the expression of the basis vector derivatives and gradient operator ∇. Is the mid-surface defined by the mapping flat or curved?

2. Derive the component forms of the thin slab equilibrium equation $\nabla \cdot \vec{N} + \vec{b} = 0$ in the polar coordinate system.

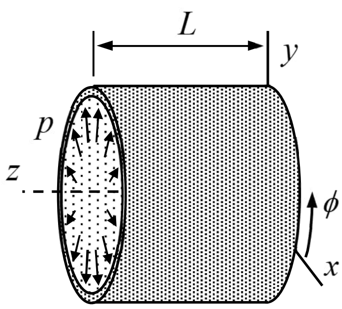
3. Consider the curved beam of the figure forming a 90-degree circular segment of radius R in the horizontal plane. Find the stress resultants $N(s)$, $Q_n(s)$, $Q_b(s)$, $T(s)$, $M_n(s)$, and $M_b(s)$. Use the equilibrium equations of the beam model in the $(s,n,b)-$coordinate system.



4. Consider the bending of a cantilever plate strip which is loaded by distributed force p $[\mathrm{N/m}]$ acting on the free edge. Write down the equilibrium equations, constitutive equations, and boundary conditions for the bending mode according to the Kirchhoff model. After that, solve the equations for the stress resultant, displacement, and rotation components. Thickness and length of the plate are t and L, respectively. Young's modulus E and Poisson's ratio ν are constants. Consider a plate of width H but assume that stress resultants, displacements, and rotations depend on y only.



5. A steel ring of length L, radius R, and thickness t is loaded by radial surface force p acting on the inner surface. No forces are acting on the ends. Model the ring as a cylindrical membrane, write down the equilibrium and constitutive equations, and solve for the radial displacement. Assume rotation symmetry. Young's modulus E and Poisson's ratio ν of the material are constants.



Consider the mid-surface mapping $\vec{r}_0(r,\phi) = r^2\cos(2\phi)\vec{i} + r^2\sin(2\phi)\vec{j}$ of shell. Compute the expression of the basis vector derivatives and gradient operator ∇. Is the mid-surface defined by the mapping flat or curved?

Solution

The relationship between the basis vectors of the Cartesian and cylindrical systems follows from definition. Notice that two of the basis vectors follow from the mid-surface mapping. The third is always normal to the surface and obtained as a vector product (see the coordinate system for shells). With the present mid-surface $(r,\phi)-$coordinates

$$\vec{r}_0(r,\phi) = r^2\cos(2\phi)\vec{i} + r^2\sin(2\phi)\vec{j} \text{ and } \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix} = \begin{Bmatrix} (\partial\vec{r}_0/\partial r)/|\partial\vec{r}_0/\partial r| \\ (\partial\vec{r}_0/\partial\phi)/|\partial\vec{r}_0/\partial\phi| \\ \vec{e}_r\times\vec{e}_\phi \end{Bmatrix} = [F]\begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}.$$

2p Expressions of the basis vectors of the curvilinear system are

$$\frac{\partial}{\partial r}\vec{r}_0 = 2r\cos(2\phi)\vec{i} + 2r\sin(2\phi)\vec{j} \quad \Rightarrow \quad \vec{e}_r = (\frac{\partial}{\partial r}\vec{r}_0)/|\frac{\partial}{\partial r}\vec{r}_0| = \cos(2\phi)\vec{i} + \sin(2\phi)\vec{j},$$

$$\frac{\partial}{\partial\phi}\vec{r}_0 = -2r^2\sin(2\phi)\vec{i} + 2r^2\cos(2\phi)\vec{j} \quad \Rightarrow \quad \vec{e}_\phi = (\frac{\partial}{\partial\phi}\vec{r}_0)/|\frac{\partial}{\partial\phi}\vec{r}_0| = -\sin(2\phi)\vec{i} + \cos(2\phi)\vec{j},$$

$$\vec{e}_n = \vec{e}_r\times\vec{e}_\phi = [\cos(2\phi)\vec{i} + \sin(2\phi)\vec{j}]\times[-\sin(2\phi)\vec{i} + \cos(2\phi)\vec{j}] = \vec{k}.$$

In a more compact form

$$\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix} = \begin{bmatrix} \cos(2\phi) & \sin(2\phi) & 0 \\ -\sin(2\phi) & \cos(2\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = [F]\begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} \quad \text{in which } [F]^{-1} = [F]^{\mathrm{T}}.$$

1p Direct use of the definition gives (just take the derivatives on both sides of the relationship above and use inverse of the same relationship to replace the basis vectors of the Cartesian system by the basis vectors of the $(r,\phi,n)-$system)

$$\frac{\partial}{\partial r}\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \cos(2\phi) & -\sin(2\phi) & 0 \\ \sin(2\phi) & \cos(2\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix},$$

$$\frac{\partial}{\partial\phi}\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix} = \begin{bmatrix} -2\sin(2\phi) & 2\cos(2\phi) & 0 \\ -2\cos(2\phi) & -2\sin(2\phi) & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \cos(2\phi) & -\sin(2\phi) & 0 \\ \sin(2\phi) & \cos(2\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix},$$

$$\frac{\partial}{\partial n}\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \cos(2\phi) & -\sin(2\phi) & 0 \\ \sin(2\phi) & \cos(2\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \leftarrow$$

2p Gradient in the $(r,\phi,n)-$ system follows from the mapping

$$\vec{r}(r,\phi,n)=\vec{r}_0+\vec{\rho}=r^2\cos(2\phi)\vec{i}+r^2\sin(2\phi)\vec{j}+n\vec{k}$$

and the generic formula in terms of $[F]$ and $[H]$ with

$$\begin{Bmatrix}\partial\vec{r}/\partial r\\ \partial\vec{r}/\partial\phi\\ \partial\vec{r}/\partial n\end{Bmatrix}=\begin{bmatrix}2r\cos(2\phi) & 2r\sin(2\phi) & 0\\ -2r^2\sin(2\phi) & 2r^2\cos(2\phi) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}=[H]\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix}\quad\Rightarrow$$

$$[H][F]^{\mathrm{T}}=\begin{bmatrix}2r\cos(2\phi) & 2r\sin(2\phi) & 0\\ -2r^2\sin(2\phi) & 2r^2\cos(2\phi) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos(2\phi) & -\sin(2\phi) & 0\\ \sin(2\phi) & \cos(2\phi) & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}2r & 0 & 0\\ 0 & 2r^2 & 0\\ 0 & 0 & 1\end{bmatrix}\Rightarrow$$

$$\nabla=\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_n\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}2r & 0 & 0\\ 0 & 2r^2 & 0\\ 0 & 0 & 1\end{bmatrix}^{-1}\begin{Bmatrix}\partial/\partial r\\ \partial/\partial\phi\\ \partial/\partial n\end{Bmatrix}=\vec{e}_r\frac{1}{2r}\frac{\partial}{\partial r}+\vec{e}_\phi\frac{1}{2r^2}\frac{\partial}{\partial\phi}+\vec{e}_n\frac{\partial}{\partial n}\ .$$

1p Curvature of the mid-surface ($n=0$)

$$\vec{\kappa}_{\mathrm{c}}=\nabla\vec{e}_n=\vec{e}_r\frac{1}{2r}\frac{\partial\vec{e}_n}{\partial r}+\vec{e}_\phi\frac{1}{2r^2}\frac{\partial\vec{e}_n}{\partial\phi}+\vec{e}_n\frac{\partial\vec{e}_n}{\partial n}=0$$

which indicates that the mid-surface is flat.

Derive the component forms of the thin slab equilibrium equation $\nabla\cdot\vec{N}+\vec{b}=0$ in the polar coordinate system.

Solution

1p The component forms of stress, external force, and gradient operator of the polar coordinate system are

$$\vec{N}=\begin{Bmatrix}\vec{e}_r\\\vec{e}_\phi\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}N_{rr} & N_{r\phi}\\N_{\phi r} & N_{\phi\phi}\end{bmatrix}\begin{Bmatrix}\vec{e}_r\\\vec{e}_\phi\end{Bmatrix},\ \vec{b}=\begin{Bmatrix}\vec{e}_r\\\vec{e}_\phi\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}b_r\\b_\phi\end{Bmatrix},\ \nabla=\begin{Bmatrix}\vec{e}_r\\\vec{e}_\phi\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\partial/\partial r\\\partial/(r\partial\phi)\end{Bmatrix},\ \frac{\partial}{\partial\phi}\begin{Bmatrix}\vec{e}_r\\\vec{e}_\phi\end{Bmatrix}=\begin{Bmatrix}\vec{e}_\phi\\-\vec{e}_r\end{Bmatrix}.$$

5p Let us start with the terms of stress resultant divergence

$$\nabla\cdot\vec{N}=(\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\phi\frac{\partial}{r\partial\phi})\cdot(N_{rr}\vec{e}_r\vec{e}_r+N_{r\phi}\vec{e}_r\vec{e}_\phi+N_{\phi r}\vec{e}_\phi\vec{e}_r+N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi).$$

First term of the gradient simplifies to

$$\vec{e}_r\frac{\partial}{\partial r}\cdot(N_{rr}\vec{e}_r\vec{e}_r+N_{r\phi}\vec{e}_r\vec{e}_\phi+N_{\phi r}\vec{e}_\phi\vec{e}_r+N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi)\quad\Rightarrow$$

$$\vec{e}_r\cdot(\frac{\partial N_{rr}}{\partial r}\vec{e}_r\vec{e}_r+\frac{\partial N_{r\phi}}{\partial r}\vec{e}_r\vec{e}_\phi+\frac{\partial N_{\phi r}}{\partial r}\vec{e}_\phi\vec{e}_r+\frac{\partial N_{\phi\phi}}{\partial r}\vec{e}_\phi\vec{e}_\phi)\quad\Rightarrow$$

$$\vec{e}_r\frac{\partial}{\partial r}\cdot(N_{rr}\vec{e}_r\vec{e}_r+N_{r\phi}\vec{e}_r\vec{e}_\phi+N_{\phi r}\vec{e}_\phi\vec{e}_r+N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi)=\begin{Bmatrix}\vec{e}_r\\\vec{e}_\phi\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\dfrac{\partial N_{rr}}{\partial r}\\\dfrac{\partial N_{r\phi}}{\partial r}\end{Bmatrix}.$$

Then the same manipulation for the second term of the displacement gradient

$$\vec{e}_\phi\frac{1}{r}\frac{\partial}{\partial\phi}\cdot(N_{rr}\vec{e}_r\vec{e}_r+N_{r\phi}\vec{e}_r\vec{e}_\phi+N_{\phi r}\vec{e}_\phi\vec{e}_r+N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi)\quad\Rightarrow$$

$$\vec{e}_\phi\frac{1}{r}\cdot(\frac{\partial N_{rr}}{\partial\phi}\vec{e}_r\vec{e}_r+N_{rr}\vec{e}_\phi\vec{e}_r+N_{rr}\vec{e}_r\vec{e}_\phi+\frac{\partial N_{r\phi}}{\partial\phi}\vec{e}_r\vec{e}_\phi+N_{r\phi}\vec{e}_\phi\vec{e}_\phi-N_{r\phi}\vec{e}_r\vec{e}_r+$$

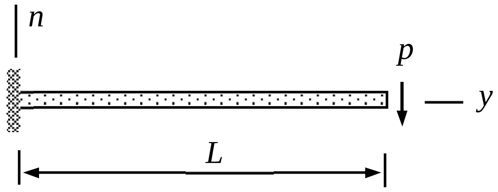
$$\frac{\partial N_{\phi r}}{\partial\phi}\vec{e}_\phi\vec{e}_r-N_{\phi r}\vec{e}_r\vec{e}_r+N_{\phi r}\vec{e}_\phi\vec{e}_\phi+\frac{\partial N_{\phi\phi}}{\partial\phi}\vec{e}_\phi\vec{e}_\phi-N_{\phi\phi}\vec{e}_r\vec{e}_\phi-N_{\phi\phi}\vec{e}_\phi\vec{e}_r)\qquad\Rightarrow$$

$$\vec{e}_\phi\frac{1}{r}\frac{\partial}{\partial\phi}\cdot(N_{rr}\vec{e}_r\vec{e}_r+N_{r\phi}\vec{e}_r\vec{e}_\phi+N_{\phi r}\vec{e}_\phi\vec{e}_r+N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi)=\begin{Bmatrix}\vec{e}_r\\\vec{e}_\phi\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\dfrac{1}{r}(N_{rr}-N_{\phi\phi}+\dfrac{\partial N_{\phi r}}{\partial\phi})\\\dfrac{1}{r}(N_{r\phi}+N_{\phi r}+\dfrac{\partial N_{\phi\phi}}{\partial\phi})\end{Bmatrix}.$$

Finally, by combining the terms of the divergence and external loading

$$\nabla \cdot \vec{\vec{N}} + \vec{b} = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \dfrac{\partial N_{rr}}{\partial r} + \dfrac{1}{r}(N_{rr} - N_{\phi\phi} + \dfrac{\partial N_{\phi r}}{\partial \phi}) + b_r \\ \dfrac{\partial N_{r\phi}}{\partial r} + \dfrac{1}{r}(N_{r\phi} + N_{\phi r} + \dfrac{\partial N_{\phi\phi}}{\partial \phi}) + b_\phi \end{Bmatrix} = 0 .$$

Consider the bending of a cantilever plate strip which is loaded by distributed force p [N / m] acting on the free edge. Write down the equilibrium equations, constitutive equations, and boundary conditions for the bending mode according to the Kirchhoff model. After that, solve the equations for the stress resultant, displacement, and rotation components. Thickness and length of the plate are t and L, respectively. Young's modulus E and Poisson's ratio ν are constants. Consider a plate of width H but assume that stress resultants, displacements, and rotations depend on y only.



Solution

The starting point is the full set of Reissner-plate bending mode equations in the Cartesian (x,y,n)– coordinate system.

$$\begin{Bmatrix} \dfrac{\partial Q_x}{\partial x}+\dfrac{\partial Q_y}{\partial y}+b_n \\ \dfrac{\partial M_{xx}}{\partial x}+\dfrac{\partial M_{xy}}{\partial y}-Q_x \\ \dfrac{\partial M_{yy}}{\partial y}+\dfrac{\partial M_{xy}}{\partial x}-Q_y \end{Bmatrix}=0\,,\quad \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix}=D\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}\begin{Bmatrix} \dfrac{\partial \theta}{\partial x} \\ -\dfrac{\partial \phi}{\partial y} \\ \dfrac{\partial \theta}{\partial y}-\dfrac{\partial \phi}{\partial x} \end{Bmatrix},\quad \begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix}=Gt\begin{Bmatrix} \dfrac{\partial w}{\partial x}+\theta \\ \dfrac{\partial w}{\partial y}-\phi \end{Bmatrix}.$$

3p If all derivatives with respect to x vanish, the plate equations of the Cartesian (x,y,n)– coordinate according to the Kirchhoff model (Kirchhoff constraint replaces the constitutive equation for the shear stress resultant) system simplify to

$$\frac{dQ_y}{dy}=0\,,\quad \frac{dM_{yy}}{dy}-Q_y=0\,,\quad M_{yy}=-D\frac{d\phi}{dy}\,,\quad \text{and}\quad \frac{dw}{dy}-\phi=0 \quad \text{in} \quad (0,L)\,.$$

The boundary conditions are

$$w(0)=0\,,\quad \phi(0)=0\,,\quad M_{yy}(L)=0\,,\quad Q_y(L)=-p\,.$$

3p As the stress resultant are known at the free end, the equilibrium equations can be solved first for the stress resultants. The boundary value problems for the stress resultants give

$$\frac{dQ_y}{dy}=0 \quad y\in(0,L) \quad \text{and} \quad Q_y(L)=-p \qquad \Rightarrow \qquad Q_y(y)=-p\,, \leftarrow$$

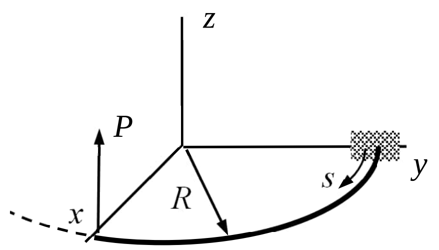
$$\frac{dM_{yy}}{dy}=Q_y=-p \quad y\in(0,L) \quad \text{and} \quad M_{yy}(L)=0 \qquad \Rightarrow \qquad M_{yy}(y)=-p(y-L)\,. \leftarrow$$

After that, displacement and rotation follow from the constitutive equation, Kirchhoff constraint, and boundary conditions at the clamped edge

$$\frac{d\phi}{dy} = -\frac{M_{yy}}{D} = \frac{p}{D}(y-L) \quad y \in (0,L) \quad \text{and} \quad \phi(0)=0 \quad \Rightarrow \quad \phi = \frac{p}{D}(\frac{1}{2}y^2 - Ly),$$

$$\frac{dw}{dy} = \phi = \frac{p}{D}(\frac{1}{2}y^2 - Ly) \quad y \in (0,L) \quad \text{and} \quad w(0)=0 \quad \Rightarrow \quad w(y) = \frac{p}{D}(\frac{1}{6}y^3 - L\frac{1}{2}y^2).$$

Consider the curved beam of the figure forming a 90-degree circular segment of radius R in the horizontal plane. Find the stress resultants $N(s)$, $Q_n(s)$, $Q_b(s)$, $T(s)$, $M_n(s)$, and $M_b(s)$. Use the equilibrium equations of the beam model in the $(s,n,b)-$ coordinate system.



Solution

In a statically determined case, stress resultants follow from the equilibrium equations and boundary conditions at the free end of the beam (or directly from a free body diagram). In (s,n,b) coordinate system, equilibrium equations are

$$\begin{Bmatrix} N'-Q_n\kappa+b_s \\ Q_n'+N\kappa-Q_b\tau+b_n \\ Q_b'+Q_n\tau+b_b \end{Bmatrix}=0 \quad \text{and} \quad \begin{Bmatrix} T'-M_n\kappa+c_s \\ M_n'+T\kappa-M_b\tau-Q_b+c_n \\ M_b'+M_n\tau+Q_n+c_b \end{Bmatrix}=0 .$$

2p For a circular beam, curvature and torsion are $\kappa=1/R$ (constant) and $\tau=0$. As external distributed forces and moments vanish i.e. $b_s=b_n=b_b=c_s=c_n=c_b=0$, equilibrium equations and the boundary conditions at the free end simplify to (notice that the external force acting at the free end is acting in the oppisite direction to $\vec{e}_b$)

$$\begin{Bmatrix} N'-Q_n/R \\ Q_n'+N/R \\ Q_b' \end{Bmatrix}=0 \quad \text{and} \quad \begin{Bmatrix} T'-M_n/R \\ M_n'+T/R-Q_b \\ M_b'+Q_n \end{Bmatrix}=0 \quad s\in]0,R\frac{\pi}{2}[,$$

$$\begin{Bmatrix} N \\ Q_n \\ Q_b+P \end{Bmatrix}=0 \quad \text{and} \quad \begin{Bmatrix} T \\ M_n \\ M_b \end{Bmatrix}=0 \quad s=R\frac{\pi}{2}.$$

4p Equations constitute a boundary value problem which can be solved by hand calculations without too much effort;

$$Q_b'=0 \quad s\in]0,R\frac{\pi}{2}[\quad \text{and} \quad Q_b+P=0 \quad s=R\frac{\pi}{2} \quad \Rightarrow \quad Q_b(s)=-P. \quad \Leftarrow$$

Eliminating Q_n and N from the remaining two connected force equilibrium equations and using the original equations to find the missing boundary condition give

$$N''+\frac{1}{R^2}N=0 \quad s\in]0,R\frac{\pi}{2}[\quad \text{and} \quad N'=N=0 \quad s=R\frac{\pi}{2} \quad \Rightarrow \quad N(s)=0 \quad \Leftarrow$$

The first equilibrium equation gives

$$Q_n(s)=0. \quad \Leftarrow$$

After that, continuing with the moment equilibrium equations with the solutions to the force equilibrium equations

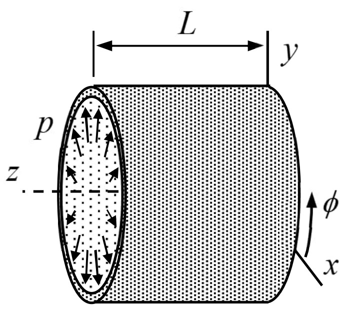
$$M_b' = 0 \quad s \in]0, R\frac{\pi}{2}[\quad \text{and} \quad M_b = 0 \quad s = R\frac{\pi}{2} \quad \Rightarrow \quad M_b(s) = 0 .$$

Eliminating M_n and T from the remaining two connected moment equilibrium equations and using the original equations to find the missing boundary condition gives

$$T'' + \frac{1}{R^2}T + \frac{P}{R} = 0 \quad s \in]0, R\frac{\pi}{2}[\quad \text{and} \quad T' = T = 0 \quad s = R\frac{\pi}{2} \quad \Rightarrow \quad T = PR(\sin\frac{s}{R} - 1) .$$

Knowing this, the first moment equilibrium equation gives

$$M_n(s) = RT' = RP\cos\frac{s}{R} .$$



A steel ring of length L, radius R, and thickness t is loaded by radial surface force p acting on the inner surface. No forces are acting on the ends. Model the ring as a cylindrical membrane, write down the equilibrium and constitutive equations, and solve for the radial displacement. Assume rotation symmetry and $u_\phi = 0$. Young's modulus E and Poisson's ratio g of the material are constants.

Solution

According to the formulae collection, equilibrium and constitutive equations of a cylindrical membrane in (z,ϕ,n) coordinates are (notice that $\vec{e}_n$ is directed inwards)

$$\begin{Bmatrix} \dfrac{1}{R}\dfrac{\partial N_{z\phi}}{\partial \phi}+\dfrac{\partial N_{zz}}{\partial z}+b_z \\ \dfrac{\partial N_{z\phi}}{\partial z}+\dfrac{1}{R}\dfrac{\partial N_{\phi\phi}}{\partial \phi}+b_\phi \\ \dfrac{1}{R}N_{\phi\phi}+b_n \end{Bmatrix}=0, \quad \begin{Bmatrix} N_{zz} \\ N_{\phi\phi} \\ N_{z\phi} \end{Bmatrix}=\frac{tE}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}\begin{Bmatrix} \dfrac{\partial u_z}{\partial z} \\ \dfrac{1}{R}(\dfrac{\partial u_\phi}{\partial \phi}-u_n) \\ \dfrac{1}{R}\dfrac{\partial u_z}{\partial \phi}+\dfrac{\partial u_\phi}{\partial z} \end{Bmatrix}.$$

3p Due to the rotation symmetry, the derivatives with respect to the angular coordinate vanish and $u_\phi = 0$. External distributed force $b_n = -p$ is due to the traction acting on the inner boundary. Therefore, the equilibrium equations and constitutive equations simplify to a set of ordinary differential equations

$$\frac{dN_{zz}}{dz}=0, \quad \frac{dN_{z\phi}}{dz}=0, \quad \frac{1}{R}N_{\phi\phi}-p=0 \quad \text{in} \quad (0,L),$$

$$N_{zz}=\frac{tE}{1-\nu^2}\frac{1}{R}(R\frac{du_z}{dz}-\nu u_n), \quad N_{\phi\phi}=\frac{tE}{1-\nu^2}\frac{1}{R}(R\nu\frac{du_z}{dz}-u_n), \quad N_{z\phi}=0 \quad \text{in} \quad (0,L),$$

As the edges are stress-free i.e.

$$N_{zz}=0 \quad \text{and} \quad N_{z\phi}=0 \quad \text{on} \quad \{0,L\}.$$

3p Solution to the stress resultants, as obtained from the equilibrium equations, are

$$N_{zz}=0, \quad N_{z\phi}=0, \text{ and } N_{\phi\phi}=Rp.$$

Constitutive equations give

$$N_{zz}=\frac{tE}{1-\nu^2}\frac{1}{R}(R\frac{du_z}{dz}-\nu u_n)=0 \quad \Rightarrow \quad \frac{du_z}{dz}=\frac{\nu}{R}u_n \quad \text{and}$$

$$Rp=N_{\phi\phi}=\frac{tE}{1-\nu^2}\frac{1}{R}(R\nu\frac{du_z}{dz}-u_n)=\frac{tE}{1-\nu^2}\frac{1}{R}(\nu^2-1)u_n=-\frac{tE}{R}u_n \quad \Leftrightarrow \quad u_n=-\frac{pR^2}{tE}. \quad \leftarrow$$

MEC-E8003 Beam, Plate and Shell models, exam 03.06.2024

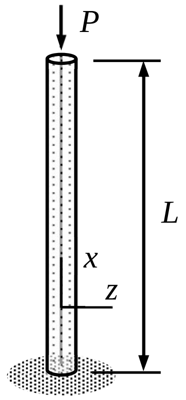
1. Derive the expressions of linear strain components ε_{rr}, $\varepsilon_{r\phi}$, $\varepsilon_{\phi r}$ and $\varepsilon_{\phi\phi}$ of the polar coordinate system. Use the displacement representation $\vec{u} = u_r \vec{e}_r + u_\phi \vec{e}_\phi$ where the components depend on the polar coordinates r and ϕ. Use definitions

$$\vec{\varepsilon} = \frac{1}{2}[\nabla\vec{u} + (\nabla\vec{u})_{\mathrm{c}}],\ \nabla = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{\partial}{r\partial\phi},\ \frac{\partial}{\partial\phi}\begin{Bmatrix}\vec{e}_r \\ \vec{e}_\phi\end{Bmatrix} = \begin{Bmatrix}\vec{e}_\phi \\ -\vec{e}_r\end{Bmatrix},\ \frac{\partial}{\partial r}\begin{Bmatrix}\vec{e}_r \\ \vec{e}_\phi\end{Bmatrix} = 0 .$$

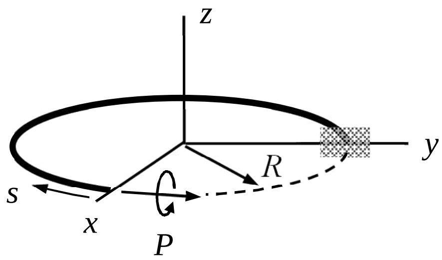
2. When displacement is confined to the xz-plane, the virtual work expression of a slender Bernoulli beam (figure) is given by

$$\delta W = -\int_0^L \left(\frac{d^2\delta w}{dx^2} EI \frac{d^2 w}{dx^2}\right)dx + P\int_0^L \left(\frac{d\delta w}{dx}\frac{dw}{dx}\right)dx .$$

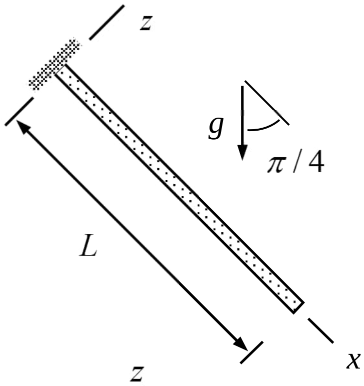
Deduce in detail the underlying differential equation and boundary conditions implied by the principle of virtual work and the fundamental lemma of variation calculus.



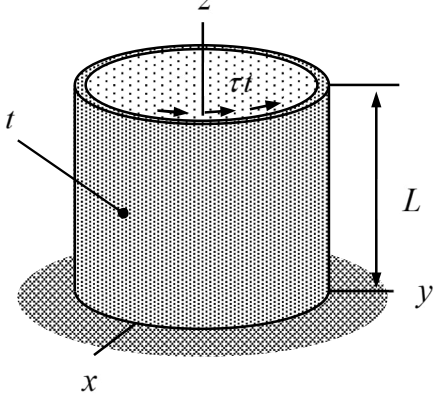
3. Consider a curved beam forming ¾ of a full circle of radius R in the horizontal plane. The given torque of magnitude P is acting on the free end as shown. Write down the equilibrium equations and boundary conditions for the *stress resultants* and solve the equations for $N(s)$, $Q_n(s)$, $Q_b(s)$, $T(s)$, $M_n(s)$, and $M_b(s)$.



4. Consider a cantilever Reissner-Mindlin plate strip (long in the y-direction) loaded by its own weight. Assuming that the solution is independent of y, determine the first order ordinary differential equations and the boundary conditions giving $N_{xx} = N(x)$, $Q_x = Q(x)$, $M_{xx} = M(x)$, $u(x)$, $w(x)$ and $\theta(x)$ as solutions. Thickness of the plate t, density ρ, Young's modulus E, and Poisson's ratio ν are constants.



5. A thin walled cylindrical body of length L, (mid-surface) radius R, and thickness t is subjected to shear loading τt $[\tau t] = \mathrm{N/m}$ at the free end $z = L$ as shown in the figure. Assuming rotation symmetry, use the membrane equations in (z, ϕ, n) coordinate system to derive the relationship between the moment resultant T (in the direction of z-axis) of the shear loading and the angle of rotation of the free end defined by $\theta = u_\phi / R$.



Derive the expressions of linear strain components ε_{rr}, $\varepsilon_{r\phi}$, $\varepsilon_{\phi r}$ and $\varepsilon_{\phi\phi}$ of the polar coordinate system. Use the displacement representation $\vec{u} = u_r \vec{e}_r + u_\phi \vec{e}_\phi$ where the components depend on the polar coordinates r and ϕ. Use definitions

$$\vec{\varepsilon} = \frac{1}{2}[\nabla\vec{u} + (\nabla\vec{u})_{\mathrm{c}}], \; \nabla = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{\partial}{r\partial\phi}, \; \frac{\partial}{\partial\phi}\begin{Bmatrix}\vec{e}_r \\ \vec{e}_\phi\end{Bmatrix} = \begin{Bmatrix}\vec{e}_\phi \\ -\vec{e}_r\end{Bmatrix}, \; \frac{\partial}{\partial r}\begin{Bmatrix}\vec{e}_r \\ \vec{e}_\phi\end{Bmatrix} = 0 .$$

Solution

In manipulation of vector expression containing vectors and tensors, it is important to remember that tensor ($\otimes$), cross ($\times$), inner ($\cdot$) products are non-commutative (order may matter). The basis vectors of a curvilinear coordinate system are not constants which should be taken into account if gradient operator is a part of expression. Otherwise, simplifying an expression or finding a specific form in a given coordinate system is a straightforward (sometimes tedious) exercise. For simplicity of presentation, outer (tensor) products like $\vec{a} \otimes \vec{b}$ are denoted by $\vec{a}\vec{b}$. Otherwise, the usual rules of algebra apply: Gradient operator ∇ acts on everything on its right hand side, the operator is treated like a vector etc.

Let us start with the gradient of displacement (an outer product). Substitute first the representations in the polar coordinate system

$$\nabla\vec{u} = (\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{\partial}{r\partial\phi})(u_r \vec{e}_r + u_\phi \vec{e}_\phi) .$$

4p Then expand to have a term-by-term representation. Keep the order of the basis vectors and the position of derivatives

$$\nabla\vec{u} = \vec{e}_r \frac{\partial}{\partial r}(u_r \vec{e}_r) + \vec{e}_r \frac{\partial}{\partial r}(u_\phi \vec{e}_\phi) + \vec{e}_\phi \frac{\partial}{r\partial\phi}(u_r \vec{e}_r) + \vec{e}_\phi \frac{\partial}{r\partial\phi}(u_\phi \vec{e}_\phi)$$

Use the derivative rule of products. Notice that the basis vectors are not constants and may have non-zero derivatives

$$\nabla\vec{u} = \vec{e}_r (\frac{\partial u_r}{\partial r}\vec{e}_r + u_r \frac{\partial \vec{e}_r}{\partial r}) + \vec{e}_r (\frac{\partial u_\phi}{\partial r}\vec{e}_\phi + u_\phi \frac{\partial \vec{e}_\phi}{\partial r}) + \vec{e}_\phi (\frac{\partial u_r}{r\partial\phi}\vec{e}_r + u_r \frac{\partial \vec{e}_r}{r\partial\phi}) + \vec{e}_\phi (\frac{\partial u_\phi}{r\partial\phi}\vec{e}_\phi + u_\phi \frac{\partial \vec{e}_\phi}{r\partial\phi}) .$$

Substitute the derivatives of the basis vectors

$$\nabla\vec{u} = \vec{e}_r (\frac{\partial u_r}{\partial r}\vec{e}_r) + \vec{e}_r (\frac{\partial u_\phi}{\partial r}\vec{e}_\phi) + \vec{e}_\phi (\frac{\partial u_r}{r\partial\phi}\vec{e}_r + \frac{u_r}{r}\vec{e}_\phi) + \vec{e}_\phi (\frac{\partial u_\phi}{r\partial\phi}\vec{e}_\phi - \frac{u_\phi}{r}\vec{e}_r) .$$

Combine the terms having the same pair of basis vectors (order matters so terms containing $\vec{e}_\phi \vec{e}_r$ and $\vec{e}_r \vec{e}_\phi$ cannot be combined)

$$\nabla\vec{u} = \frac{\partial u_r}{\partial r}\vec{e}_r \vec{e}_r + \frac{\partial u_\phi}{\partial r}\vec{e}_r \vec{e}_\phi + (\frac{\partial u_r}{r\partial\phi} - \frac{u_\phi}{r})\vec{e}_\phi \vec{e}_r + (\frac{u_r}{r} + \frac{\partial u_\phi}{r\partial\phi})\vec{e}_\phi \vec{e}_\phi .$$

Conjugate of a second order tensor can be obtained by swapping the basis vectors in all the pairs. Conjugate is a kind of transpose and can also be obtained by transposing the matrix of the component representation.

$$(\nabla\vec{u})_{\mathrm{c}} = \frac{\partial u_r}{\partial r}\vec{e}_r\vec{e}_r + (\frac{\partial u_r}{r\partial\phi} - \frac{u_\phi}{r})\vec{e}_r\vec{e}_\phi + \frac{\partial u_\phi}{\partial r}\vec{e}_\phi\vec{e}_r + (\frac{u_r}{r} + \frac{\partial u_\phi}{r\partial\phi})\vec{e}_\phi\vec{e}_\phi$$

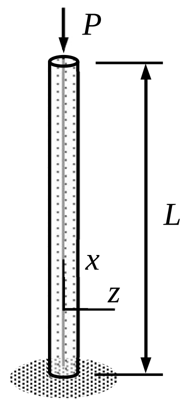
2p Finally using the definition $\vec{\varepsilon} = \frac{1}{2}\left[\nabla\vec{u} + (\nabla\vec{u})_{\mathrm{c}}\right]$

$$\vec{\varepsilon} = \frac{\partial u_r}{\partial r}\vec{e}_r\vec{e}_r + \frac{1}{2}(\frac{\partial u_r}{r\partial\phi} - \frac{u_\phi}{r} + \frac{\partial u_\phi}{\partial r})(\vec{e}_r\vec{e}_\phi + \vec{e}_\phi\vec{e}_r) + (\frac{\partial u_\phi}{r\partial\phi} + \frac{u_r}{r})\vec{e}_\phi\vec{e}_\phi\,.$$

In the components of strain ε_{rr}, $\varepsilon_{r\phi}$, $\varepsilon_{\phi r}$ and $\varepsilon_{\phi\phi}$, indices are in the same order as the indices in the basis vector pairs. Hence

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\phi\phi} = \frac{\partial u_\phi}{r\partial\phi} + \frac{u_r}{r}, \quad \varepsilon_{r\phi} = \varepsilon_{\phi r} = \frac{1}{2}(\frac{\partial u_r}{r\partial\phi} - \frac{u_\phi}{r} + \frac{\partial u_\phi}{\partial r})\,. \leftarrow$$

When displacement is confined to the $xz-$plane, the virtual work expression of a slender Bernoulli beam (figure) is given by



$$\delta W=-\int_0^L (\frac{d^2\delta w}{dx^2} EI \frac{d^2 w}{dx^2})dx+P\int_0^L (\frac{d\delta w}{dx}\frac{dw}{dx})dx\,.$$

Deduce in detail the underlying differential equation and boundary conditions implied by the principle of virtual work and the fundamental lemma of variation calculus.

Solution

2p Integration by parts gives an equivalent but a more convenient form (assuming continuity up to and including second derivatives)

$$\delta W=-\int_0^L (\frac{d^2\delta w}{dx^2} EI \frac{d^2 w}{dx^2})dx+P\int_0^L (\frac{d\delta w}{dx}\frac{dw}{dx})dx \quad \Leftrightarrow \quad (P \text{ is a constant})$$

$$\delta W=-\int_0^L \delta w(EI\frac{d^4 w}{dx^4}+P\frac{d^2 w}{dx^2})dx+\sum_{\{0,L\}} n\delta w(EI\frac{d^3 w}{dx^3}+P\frac{dw}{dx})-\sum_{\{0,L\}} n\frac{d\delta w}{dx}(EI\frac{d^2 w}{dx^2})\,.$$

2p According to principle of virtual work $\delta W=0\ \forall \delta w$. Let us consider first the subset of variations for which $\delta w=0$ and $d\delta w/dx=0$ on $\{0,L\}$. The fundamental lemma of variation calculus implies

$$EI\frac{d^4 w}{dx^4}+P\frac{d^2 w}{dx^2}=0 \quad \text{in} \quad (0,L)\,.$$

Let us consider then the subset of variations for which $d\delta w/dx=0$ on $\{0,L\}$. Knowing the condition above, the fundamental lemma of variation calculus implies

$$EI\frac{d^3 w}{dx^3}+P\frac{dw}{dx}=0 \quad \text{or} \quad w-\underline{w}=0 \quad \text{on} \quad \{0,L\}\,.$$

Finally, let us consider the subset of variations for which $\delta w=0$ on $\{0,L\}$. Knowing the previous results, the fundamental lemma of variation calculus implies

$$EI\frac{d^2 w}{dx^2}=0 \quad \text{or} \quad \frac{dw}{dx}-\underline{\theta}=0 \quad \text{on} \quad \{0,L\}\,.$$

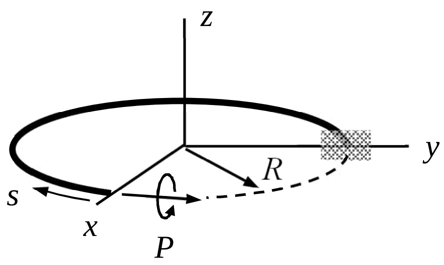
2p For the problem of the figure, one obtains

$$EI\frac{d^4 w}{dx^4}+P\frac{d^2 w}{dx^2}=0 \quad \text{in} \quad (0,L)\,,$$

$$EI\frac{d^3 w}{dx^3}+P\frac{dw}{dx}=0 \quad \text{and} \quad EI\frac{d^2 w}{dx^2}=0 \quad \text{at } x=L\,,$$

$$w = 0 \quad \text{and} \quad \frac{dw}{dx} = 0 \quad \text{at} \quad x = 0 \,.$$

Consider a curved beam forming ¾ of a full circle of radius R in the horizontal plane. Torque of magnitude P is acting on the free end as shown. Write down the boundary value problem for *stress resultants* and solve the equations for $N(s)$, $Q_n(s)$, $Q_b(s)$, $T(s)$, $M_n(s)$, and $M_b(s)$.



Solution

3p In the geometry of the figure $\tau = 0$, $\kappa = 1/R$. External distributed forces and moments vanish. Therefore, the curved beam equilibrium equations of the formulae collection simplify to

$$\begin{Bmatrix} N' - Q_n / R \\ Q_n' + N / R \\ Q_b' \end{Bmatrix} = 0 \quad \text{and} \quad \begin{Bmatrix} T' - M_n / R \\ M_n' + T / R - Q_b \\ M_b' + Q_n \end{Bmatrix} = 0 \quad s \in (0, L) \quad \text{where } L = \frac{3}{2}\pi R.$$

Boundary conditions at $s = 0$ are (notice the unit outward normal to the solution domain $n = -1$, $\vec{e}_s$ is pointing to the direction of s, and the component of the given moment on $\vec{e}_s$ is negative)

$$\begin{Bmatrix} N \\ Q_n \\ Q_b \end{Bmatrix} = 0 \quad \text{and} \quad \begin{Bmatrix} -T + P \\ M_n \\ M_b \end{Bmatrix} = 0 \quad s = 0.$$

3p Solution to the boundary values problem for Q_b

$$Q_b' = 0 \quad s \in (0, L) \quad \text{and} \quad Q_b = 0 \quad s = 0 \quad \Rightarrow \quad Q_b(s) = 0. \leftarrow$$

Solution to the connected boundary value problems for Q_n and N

$$N' - \frac{1}{R} Q_n = 0, \; Q_n' + \frac{1}{R} N = 0 \quad s \in (0, L), \qquad Q_n = 0 \text{ and } N = 0 \quad s = 0 \quad \Rightarrow$$

$$N'' + \frac{1}{R^2} N = 0 \quad s \in (0, L) \quad \text{and} \quad N = 0, \; N' = 0 \text{ at } s = 0 \quad \Rightarrow$$

$$N(s) = 0 \quad \text{and} \quad Q_n(s) = 0. \leftarrow$$

Solution to the boundary value problem for M_b

$$M_b' = 0 \quad s \in (0, L) \quad \text{and} \quad M_b = 0 \quad s = 0 \quad \Rightarrow \quad M_b(s) = 0. \leftarrow$$

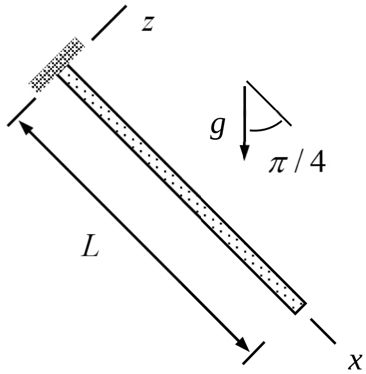
Solution to the connected boundary value problem for M_n and T

$$T' - \frac{1}{R} M_n = 0 \quad \text{and} \quad M_n' + \frac{1}{R} T = 0 \quad s \in (0, L), \qquad T = P \text{ and } M_n = 0 \quad s = 0 \quad \Rightarrow$$

$$RT'' + \frac{1}{R}T = 0 \quad s \in (0, L), \quad T = P \quad \text{and} \quad T' = 0 \quad \Rightarrow$$

$$T(s) = P\cos(\frac{s}{R}) \quad \text{and} \quad M_n(s) = -P\sin(\frac{s}{R}) \,.$$

Consider a cantilever Reissner-Mindlin plate strip (long in the $y-$direction) loaded by its own weight. Assuming that the solution is independent of y, determine the first order ordinary differential equations and the boundary conditions giving $N_{xx}=N(x)$, $Q_x=Q(x)$, $M_{xx}=M(x)$, $u(x)$, $w(x)$ and $\theta(x)$ as solutions. Thickness of the plate t, density ρ, Young's modulus E, and Poisson's ratio ν are constants.



Solution

Equilibrium and constitutive equations of the thin-slab and bending modes are

$$\begin{Bmatrix} \dfrac{\partial N_{xx}}{\partial x}+\dfrac{\partial N_{xy}}{\partial y}+b_x \\ \dfrac{\partial N_{yy}}{\partial y}+\dfrac{\partial N_{xy}}{\partial x}+b_y \end{Bmatrix}=0, \quad \begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix}=t[E]_\sigma \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x} \end{Bmatrix}=\frac{tE}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x} \end{Bmatrix},$$

$$\begin{Bmatrix} \dfrac{\partial Q_x}{\partial x}+\dfrac{\partial Q_y}{\partial y}+b_n \\ \dfrac{\partial M_{xx}}{\partial x}+\dfrac{\partial M_{xy}}{\partial y}-Q_x \\ \dfrac{\partial M_{yy}}{\partial y}+\dfrac{\partial M_{xy}}{\partial x}-Q_y \end{Bmatrix}=0, \quad \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix}=D\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}\begin{Bmatrix} \dfrac{\partial \theta}{\partial x} \\ -\dfrac{\partial \phi}{\partial y} \\ \dfrac{\partial \theta}{\partial y}-\dfrac{\partial \phi}{\partial x} \end{Bmatrix}, \quad \begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix}=Gt\begin{Bmatrix} \dfrac{\partial w}{\partial x}+\theta \\ \dfrac{\partial w}{\partial y}-\phi \end{Bmatrix}.$$

4p Derivatives with respect to y vanish, $b_x=\rho g t/\sqrt{2}$, and $b_n=-\rho g t/\sqrt{2}$. The Reissner-Mindlin plate equations of the planar problem simplify to

$$\frac{dN}{dx}+\frac{\rho g t}{\sqrt{2}}=0, \quad \frac{dQ}{dx}-\frac{\rho g t}{\sqrt{2}}=0, \quad \frac{dM}{dx}-Q=0 \quad \text{in} \quad (0,L), \quad \leftarrow$$

$$N=\frac{tE}{1-\nu^2}\frac{du}{dx}, \quad Q=Gt(\frac{dw}{dx}+\theta), \quad M=D\frac{d\theta}{dx} \quad \text{in} \quad (0,L), \quad \leftarrow$$

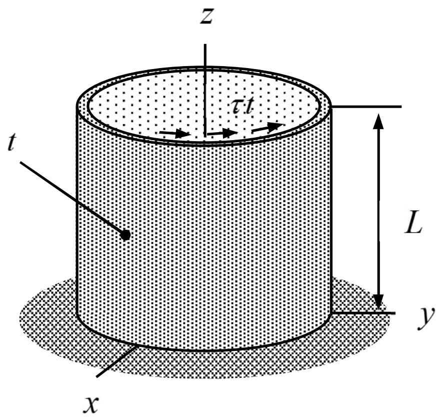
2p Boundary conditions can be deduced from the figure:

$$u=0, \quad w=0, \quad \theta=0 \quad \text{at} \quad x=0, \quad \leftarrow$$

$$N=0, \quad M=0, \quad Q=0 \quad \text{at} \quad x=L. \quad \leftarrow$$

Solution to equations can be obtained by considering the equilibrium equations and the boundary conditions at the free end first. After that, solutions to the displacement components follow from the constitutive equations and the boundary conditions at the clamped edge.

A thin walled cylindrical body of length L, (mid-surface) radius R, and thickness t is subjected to shear loading τt $[\tau t] = \text{N/m}$ at the free end $z = L$ as shown in the figure. Assuming rotation symmetry, use the membrane equations in (z, ϕ, n) coordinate system to derive the relationship between the moment resultant T of the shear loading and the angle of rotation of the free end defined by $\theta = u_\phi / R$.



Solution

2p As the solution does not depend on ϕ, equilibrium equations of the membrane model and boundary conditions at the free end simplify to (a cylindrical membrane $z-$strip problem)

$$\frac{dN_{zz}}{dz} = 0, \quad \frac{dN_{z\phi}}{dz} = 0, \quad \frac{1}{R} N_{\phi\phi} = 0 \quad \text{in} \quad (0, L),$$

$$N_{zz} = 0 \text{ and } N_{z\phi} = \tau t \quad \text{at} \quad z = L.$$

Solution to the boundary value problem for the stress resultants is given by

$$N_{zz} = N_{\phi\phi} = 0 \text{ and } N_{z\phi}(z) = \tau t. \leftarrow$$

2p Knowing the stress resultants, the boundary value problem for the displacement components follows from the constitutive equations and the boundary conditions at the fixed end (in the membrane model, a boundary condition cannot be assigned to u_n)

$$\frac{tE}{1-\nu^2}\left(\frac{du_z}{dz} - \nu \frac{1}{R} u_n\right) = 0, \quad \frac{tE}{1-\nu^2}\left(\nu \frac{du_z}{dz} - \frac{1}{R} u_n\right) = 0, \text{ and } tG \frac{du_\phi}{dz} = \tau t \quad \text{in} \quad (0, L),$$

$$u_z = 0, \quad u_\phi = 0 \quad \text{at} \quad z = 0.$$

Solution to the boundary value problem is given by

$$u_z = u_n = 0 \text{ and } u_\phi(z) = \frac{\tau}{G} z.$$

2p Moment resultant of the shear loading

$$T = \int_0^{2\pi} t\tau R(R d\phi) = 2\pi R^2 t\tau \quad \Rightarrow \quad \tau = \frac{T}{2\pi R^2 t}.$$

Therefore, at the free end

$$u_\phi = \frac{\tau}{G} L = \frac{L}{2\pi R^2 t G} T = R\theta \quad \Rightarrow \quad T = \frac{2\pi R^3 t}{L} G\theta. \leftarrow$$

The polar moment predicted here is $I_p = 2\pi R^3 t$ whereas the exact is $I_p = \frac{1}{2}\pi R t(4R^2 + t^2)$.