Assignment 1 (2p)
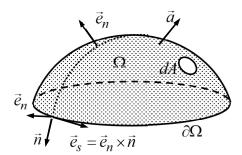
Gauss theorem implies the following integral identity for curved surfaces

$$\int_\Omega (\nabla_0 \cdot \vec{a} - \kappa\vec{e}_n \cdot \vec{a})dA = \int_{\partial\Omega} (\vec{n}\cdot\vec{a})ds$$

in which $\kappa = \vec{\kappa}:\vec{I} = \nabla_0 \cdot \vec{e}_n$. Verify the integral identity in the spherical (ϕ,θ,n) coordinate system by considering vector $\vec{a} = \theta\vec{e}_n$ and half-sphere $\phi\in[0,2\pi]$, $\theta\in[0,\pi/2]$, of radius R as Ω. Derivatives of the basis vectors and the mid-surface gradient in the spherical coordinate system are

$$\frac{\partial}{\partial\phi}\begin{Bmatrix}\vec{e}_\phi\\\vec{e}_\theta\\\vec{e}_n\end{Bmatrix} = \begin{Bmatrix}\sin\theta\vec{e}_n - \cos\theta\vec{e}_\theta\\\cos\theta\vec{e}_\phi\\-\sin\theta\vec{e}_\phi\end{Bmatrix}, \quad \frac{\partial}{\partial\theta}\begin{Bmatrix}\vec{e}_\phi\\\vec{e}_\theta\\\vec{e}_n\end{Bmatrix} = \begin{Bmatrix}0\\\vec{e}_n\\-\vec{e}_\theta\end{Bmatrix}, \quad \nabla_0 = \frac{1}{R\sin\theta}\vec{e}_\phi\frac{\partial}{\partial\phi} + \frac{1}{R}\vec{e}_\theta\frac{\partial}{\partial\theta}.$$

Solution template

In case of a half-sphere $\phi\in[0,2\pi]$, $\theta\in[0,\pi/2]$ of radius R and vector $\vec{a} = \theta\vec{e}_n$, the quantities in the integral identity take the forms

$$\nabla_0 \cdot \vec{a} = (\frac{1}{R\sin\theta}\vec{e}_\phi\frac{\partial}{\partial\phi} + \frac{1}{R}\vec{e}_\theta\frac{\partial}{\partial\theta})\cdot\theta\vec{e}_n = \underline{\hspace{3cm}},$$

$$\kappa = \nabla_0 \cdot \vec{e}_n = (\frac{1}{R\sin\theta}\vec{e}_\phi\frac{\partial}{\partial\phi} + \frac{1}{R}\vec{e}_\theta\frac{\partial}{\partial\theta})\cdot\vec{e}_n = \underline{\hspace{3cm}},$$

$$\vec{e}_n \cdot \vec{a} = \underline{\hspace{3cm}},$$

$$\vec{n}\cdot\vec{a} = \underline{\hspace{3cm}}.$$

When the expressions are substituted there, the left- and right-hand sides of the integral identity simplify to

$$\int_\Omega (\nabla_0 \cdot \vec{a} - \kappa\vec{e}_n \cdot \vec{a})dA = \int_\Omega (\underline{\hspace{3cm}})dA = \underline{\hspace{3cm}},$$

$$\int_{\partial\Omega} (\vec{n}\cdot\vec{a})ds = \int_{\partial\Omega} (\underline{\hspace{3cm}})ds = \underline{\hspace{3cm}}.$$