Assignment 5 (4p)

Derive the component forms of the membrane equilibrium equations $\nabla \cdot \vec{N} + \vec{b} = 0$ in the cylindrical shell (z, ϕ, n) -coordinate system. Use the stress resultant, external force, and gradient representations

$$\vec{N} = \left\{ \vec{e}_z \right\}^{\mathrm{T}} \begin{bmatrix} N_{zz} & N_{z\phi} \\ N_{z\phi} & N_{\phi\phi} \end{bmatrix} \left\{ \vec{e}_z \\ \vec{e}_\phi \right\}, \ \vec{b} = \left\{ \vec{e}_z \\ \vec{e}_\phi \\ \vec{e}_n \right\}^{\mathrm{T}} \left\{ \begin{matrix} b_z \\ b_\phi \\ b_n \end{matrix} \right\}, \ \nabla = \left\{ \vec{e}_z \\ \vec{e}_\phi \right\}^{\mathrm{T}} \left\{ \begin{matrix} \partial / \partial z \\ \partial / (R \partial \phi) \end{matrix} \right\}.$$

The non-zero basis vector derivatives are $\frac{\partial}{\partial \phi} \vec{e}_{\phi} = \vec{e}_{n}$ and $\frac{\partial}{\partial \phi} \vec{e}_{n} = -\vec{e}_{\phi}$.

Solution

Membrane model is the thin-slab model in curved geometry. Equilibrium equations differ from the thin-slab ones as the non-constant basis vectors bring additional terms. Therefore, a membrane may take also external forces in the transverse direction (like pressure difference between the outer and inner surfaces of a balloon). The task is to simplify the vector equation

$$\nabla \cdot \vec{N} + \vec{b} = (\vec{e}_z \frac{\partial}{\partial z} + \vec{e}_\phi \frac{1}{R} \frac{\partial}{\partial \phi}) \cdot (N_{zz} \vec{e}_z \vec{e}_z + N_{z\phi} \vec{e}_z \vec{e}_\phi + N_{z\phi} \vec{e}_\phi \vec{e}_z + N_{\phi\phi} \vec{e}_\phi \vec{e}_\phi) + b_z \vec{e}_z + b_\phi \vec{e}_\phi + b_n \vec{e}_n = 0$$

to see the component forms. Let us start with the divergence of stress and consider the term in two parts (to avoid lengthy expressions)

$$(\vec{e}_z \frac{\partial}{\partial z}) \cdot (N_{zz} \vec{e}_z \vec{e}_z + N_{z\phi} \vec{e}_z \vec{e}_\phi + N_{z\phi} \vec{e}_\phi \vec{e}_z + N_{\phi\phi} \vec{e}_\phi \vec{e}_\phi) = \frac{\partial N_{zz}}{\partial z} \vec{e}_z + \frac{\partial N_{z\phi}}{\partial z} \vec{e}_\phi,$$

$$(\vec{e}_{\phi}\,\frac{1}{R}\,\frac{\partial}{\partial\phi})\cdot(N_{zz}\vec{e}_{z}\vec{e}_{z}+N_{z\phi}\vec{e}_{z}\vec{e}_{\phi}+N_{z\phi}\vec{e}_{\phi}\vec{e}_{z}+N_{\phi\phi}\vec{e}_{\phi}\vec{e}_{\phi}) = \frac{1}{R}\frac{\partial N_{z\phi}}{\partial\phi}\vec{e}_{z}+\frac{1}{R}\frac{\partial N_{\phi\phi}}{\partial\phi}\vec{e}_{\phi}+\frac{1}{R}N_{\phi\phi}\vec{e}_{n}.$$

After that, combining the terms

$$\nabla \cdot \vec{N} + \vec{b} = \begin{cases} \vec{e}_z \\ \vec{e}_\phi \\ \vec{e}_n \end{cases}^{\rm T} \begin{cases} \frac{\partial N_{zz}}{\partial z} + \frac{1}{R} \frac{\partial N_{z\phi}}{\partial \phi} + b_z \\ \frac{\partial N_{z\phi}}{\partial z} + \frac{1}{R} \frac{\partial N_{\phi\phi}}{\partial \phi} + b_\phi \\ \frac{1}{R} N_{\phi\phi} + b_n \end{cases} = 0 \,. \label{eq:delta_var_var}$$