MEC-E8003

Beam, Plate and Shell Models

2024

WEEK 9: INTRODUCTION

1 INTRODUCTION

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LEARNING OUTCOMES

Students get an overall picture about modelling in solid mechanics, use of the first principles in derivation of engineering models, and the mathematical tools used in the course. The topics are

- Modelling in solid mechanics
- First principles and concepts of solid mechanics
- Vectors and tensors
- Differential equations and boundary value problems

1.1 MODELLING IN MECHANICS

- **Crop**: Decide the boundary of structure. Interaction with surroundings need to be described in terms of known forces, moments, displacements, and rotations.

- **Idealize**: Simplify the geometry. Ignoring the details, not likely to affect the outcome, may simplify analysis a lot.

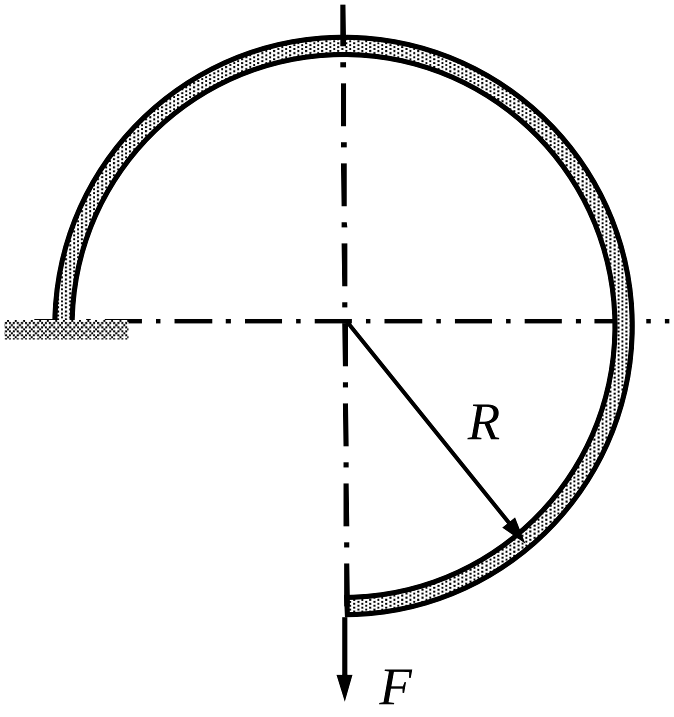
- **Parametrize**: Assign symbols to geometric and material parameter of the idealized structure. Measure or find the values needed in calculations.

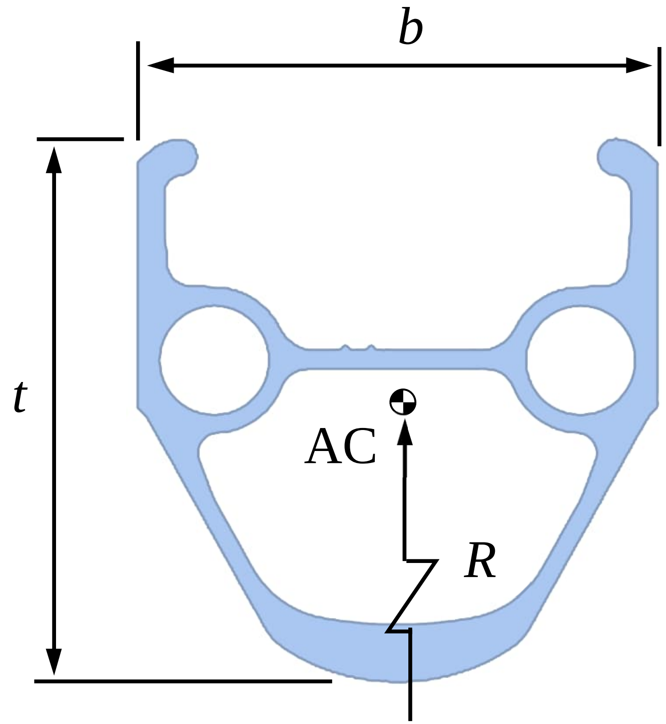
- **Model**: Write the mathematical description consisting of equilibrium equations, constitutive equations, and boundary conditions. 🡐

- **Solve**: Use an analytical or approximate method and hand calculations or Mathematica to find the solution. 🡐

RIGIDITY OF WHEEL RIM

STRUCTURE IDEALIZATION AND PARAMETRIZATION

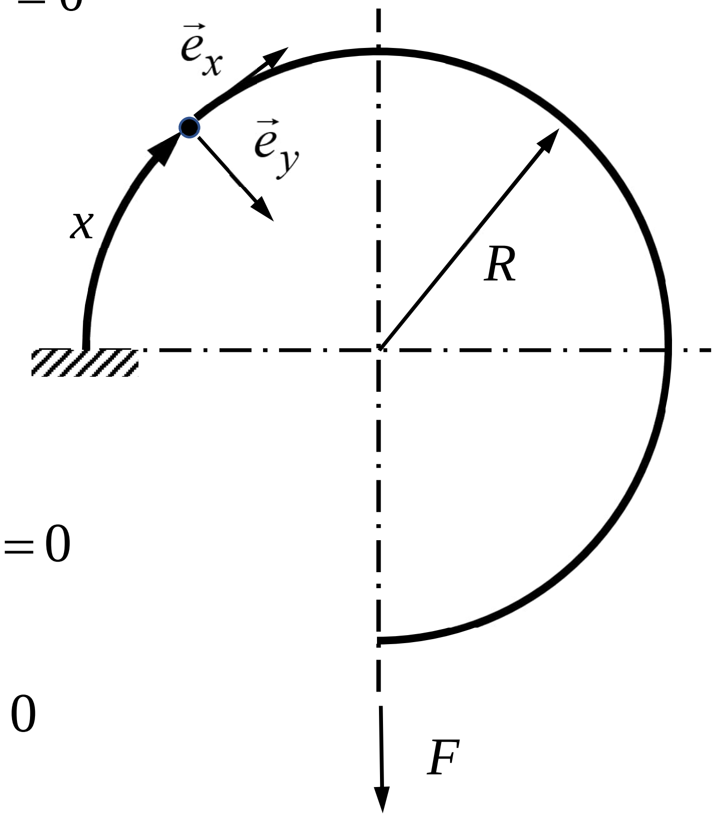




Dimension analysis with quantities E, I, R, F, and v : $\quad \dfrac{F}{ER^2} = f\left(\dfrac{v}{R}, \dfrac{I}{R^4}\right)$

CURVED BEAM EQUATIONS

Assuming a planar beam, clamping and the center of the rim on the same horizontal line, and $L = 3\pi R/2$ (curvilinear $xy-$coordinate system)



Equilibrium: $\dfrac{dN}{dx} - \dfrac{1}{R}Q = 0, \quad \dfrac{dQ}{dx} + \dfrac{1}{R}N = 0, \quad \dfrac{dM}{dx} + Q = 0$

Constitutive equation: $M = EI\dfrac{d\psi}{dx}$

Constraints: $\dfrac{du}{dx} - \dfrac{1}{R}v = 0, \quad \dfrac{dv}{dx} + \dfrac{1}{R}u - \psi = 0$

BC:s at the free end: $N(L) = 0, \quad Q(L) + F = 0, \quad M(L) = 0$

BC:s at the clamped end: $u(0) = 0, \quad v(0) = 0, \quad \psi(0) = 0$

MAVIC CXP 700C ISO 622 32H

Triangle representation based on a picture from www.mavic.com and the cross-section moment definitions: $R = 306\,\text{mm}$ and $I_{zz} = \sum_e \int_{\Omega^e} y^2 dA = 3011\,\text{mm}^4$.

MATHEMATICA SOLUTION

Mathematica can find the solution in a symbolic form. Problem description is close to its mathematical form and is composed of (ordinary) ordinary differential equations and boundary conditions.

```
equations := {
   NN'[x] - QQ[x] / R == 0,
   QQ'[x] + NN[x] / R == 0,
   MM'[x] + QQ[x] == 0,
   u'[x] - v[x] / R == 0,
   v'[x] + u[x] / R - ψ[x] == 0,
   MM[x] == E I ψ'[x],
   u[0] == 0, v[0] == 0, ψ[0] == 0,
   NN[3 Pi R / 2] == 0, QQ[3 Pi R / 2] == -FF, MM[3 Pi R / 2] == 0};

sol = DSolve[equations, {u, v, ψ, NN, QQ, MM}, x][[1]]
```

1.2 FIRST PRINCIPLES AND QUANTITIES

Balance of mass Mass of a fixed set of particles, called as a body, is constant. ←

Balance of linear momentum The rate of change of linear momentum of a body equals the external force resultant acting on the material volume. ←

Balance of angular momentum The rate of change of angular momentum of a body equals the external moment resultant acting on the material volume. ←

Balance of energy (Thermodynamics 1)

Entropy growth (Thermodynamics 2)

The balance equations in their generic forms hold for solids and fluids!

LOCAL FORMS

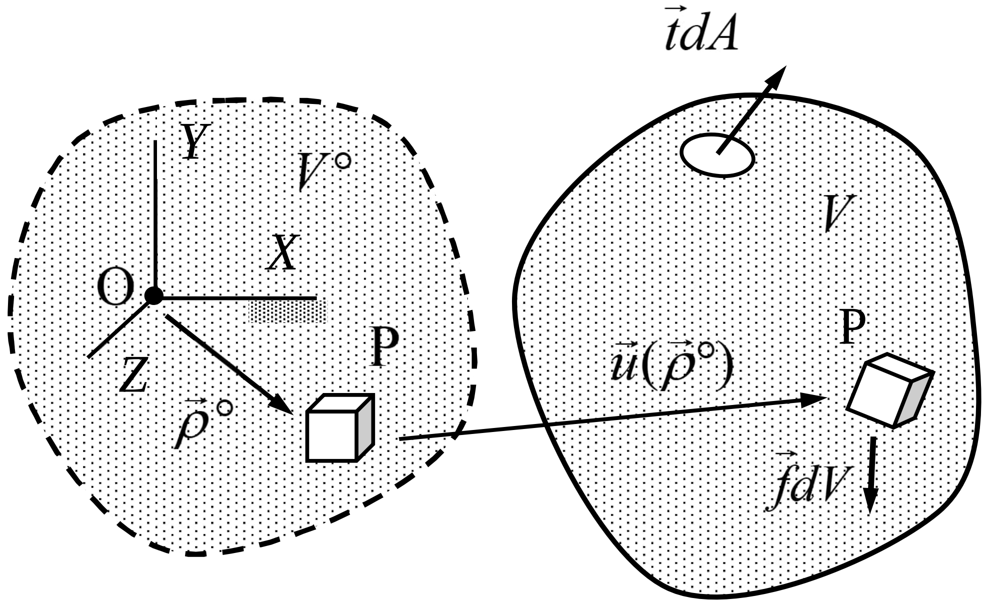
Application of the first principles to a material element inside the body or from its boundary gives the local forms:

$$\dot{m}=0 \quad : \quad \rho^{\circ}=\rho J \quad \text{in } V$$

$$\dot{\vec{p}}=\vec{F} \quad : \quad \nabla\cdot\vec{\vec{\sigma}}+\vec{f}=0 \quad \text{in } V$$

$$\dot{\vec{p}}=\vec{F} \quad : \quad \vec{\sigma}\equiv\vec{n}\cdot\vec{\vec{\sigma}}=\vec{t} \quad \text{on } \partial V_t$$

$$\dot{\vec{L}}=\vec{M} \quad : \quad \vec{\vec{\sigma}}=\vec{\vec{\sigma}}_{\mathrm{c}} \quad \text{in } V$$



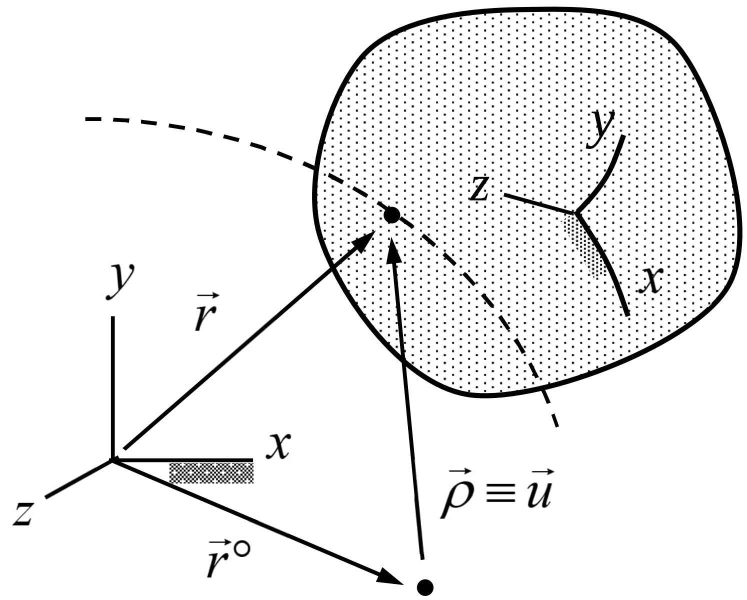
Assuming an equilibrium setting (geometry, stress, loading etc.) the local forms can be used used to find a new equilibrium setting (actually, displacements of the particles) when, e.g., external given forces are changed in some manner.

DISPLACEMENT

In Lagrangian description of solid mechanics, particles of a body are identified by their material coordinates (x, y, z). Displacement $\vec{u} = \vec{r} - \vec{r}^{\circ}$ is relative position vector of a particle initially at $\vec{r}^{\circ}$. In a Cartesian coordinate system

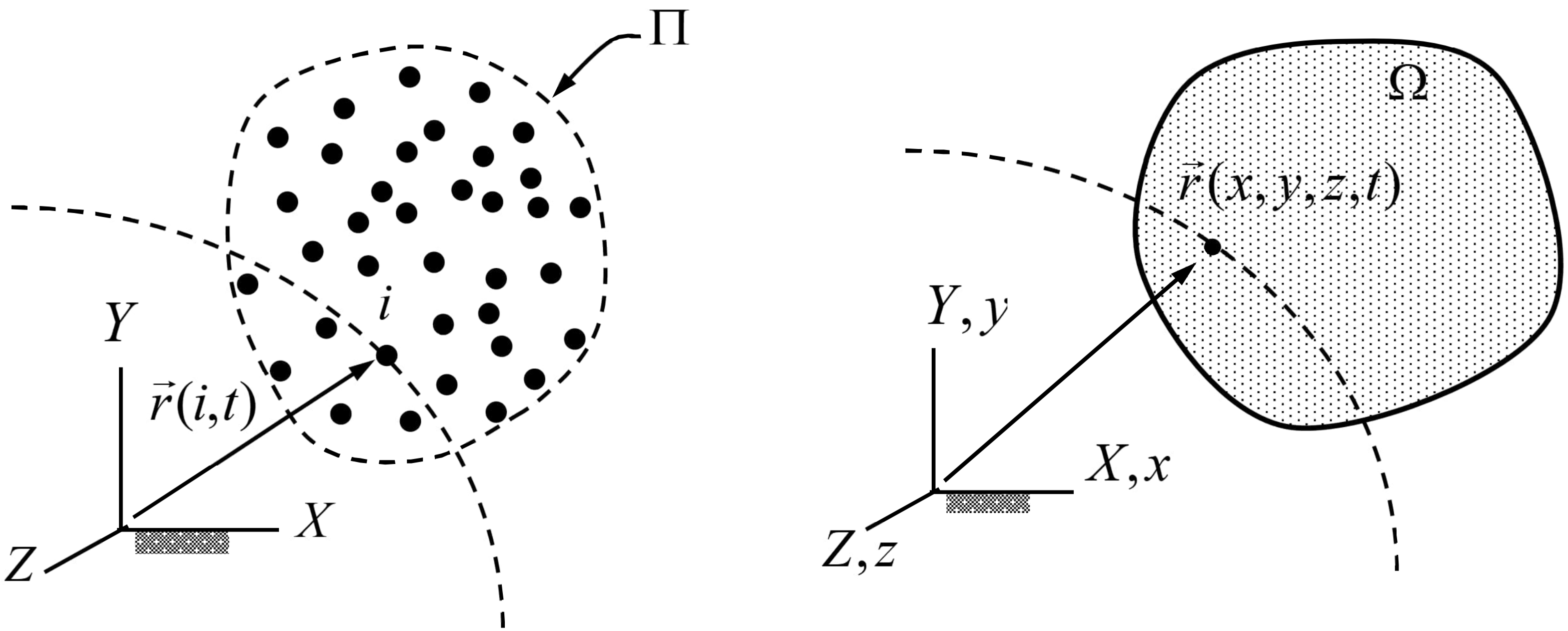
Initial $\vec{r}^{\circ} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$,

Final $\vec{r} = \vec{r}^{\circ} + \vec{u} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \left(\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} + \begin{Bmatrix} u_x(x,y,z) \\ u_y(x,y,z) \\ u_z(x,y,z) \end{Bmatrix} \right)$.



Displacement $\vec{u}(x, y, z)$ is the primary unknown of a linear elasticity problem and other quantities like strain and stress are (finally) expressed in terms of it.

In particle models, index $i \in \Pi \subset \mathbb{N}$ is used for labelling. In continuum models, material coordinates $(x, y, z) \in \Omega \subset \mathbb{R}^3$ are used for the purposes as "the particle set is too large to allow enumeration".



Time can be considered as the curve parameter for the particle paths. In stationary description, one considers only the initial position Ω° (at $t = 0$ say) and the final position Ω° (at some other instant of time) and the curve parameter can be omitted.

LINEAR STRAIN

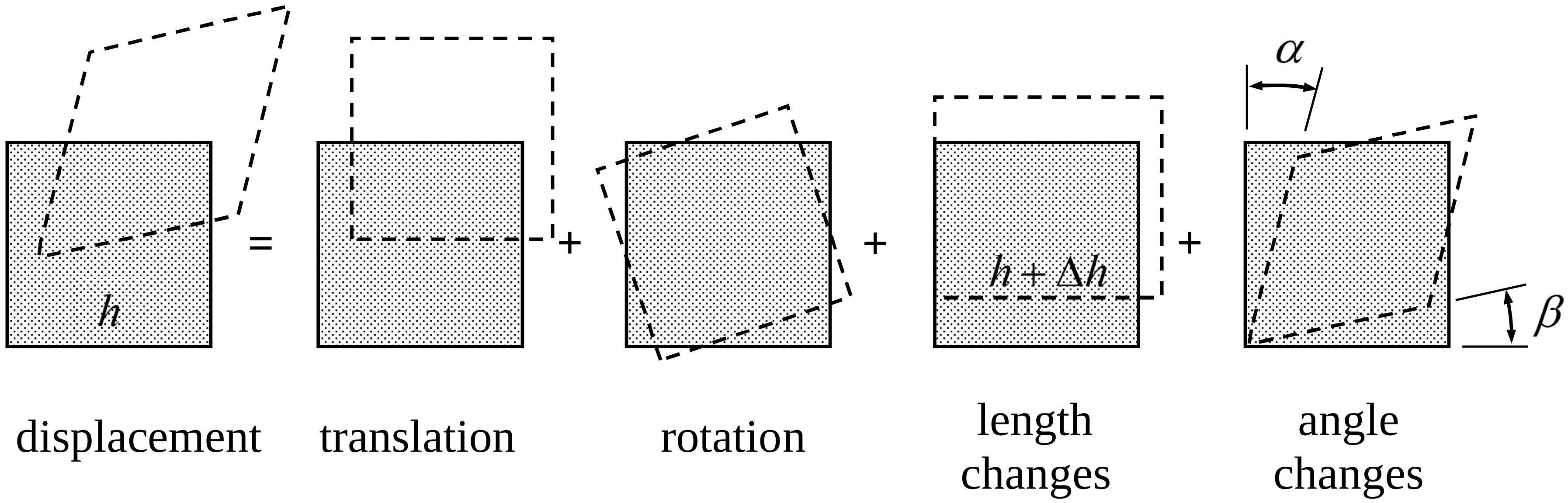
Linear strain measure $\vec{\vec{\varepsilon}} = [\nabla\vec{u} + (\nabla\vec{u})_{\mathrm{c}}]/2$ describes shape deformation of material elements. The components of the (invariant) tensor quantity depends on the selection of the coordinate system. In a Cartesian $(x, y, z)-$coordinate system

$$\vec{\vec{\varepsilon}} = \frac{1}{2}[\nabla\vec{u} + (\nabla\vec{u})_{\mathrm{c}}] = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} + \begin{Bmatrix} \vec{i}\vec{j} + \vec{j}\vec{i} \\ \vec{j}\vec{k} + \vec{k}\vec{j} \\ \vec{k}\vec{i} + \vec{i}\vec{k} \end{Bmatrix} \begin{Bmatrix} \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{Bmatrix}.$$

Normal components: $\varepsilon_{xx} = \dfrac{\partial u_x}{\partial x}, \;\; \varepsilon_{yy} = \dfrac{\partial u_y}{\partial y}, \;\; \varepsilon_{zz} = \dfrac{\partial u_z}{\partial z}$

Shear components: $\varepsilon_{xy} = \dfrac{1}{2}(\dfrac{\partial u_y}{\partial x} + \dfrac{\partial u_x}{\partial y}), \; \varepsilon_{yz} = \dfrac{1}{2}(\dfrac{\partial u_z}{\partial y} + \dfrac{\partial u_y}{\partial z}), \;\; \varepsilon_{zx} = \dfrac{1}{2}(\dfrac{\partial u_z}{\partial x} + \dfrac{\partial u_x}{\partial z})$

Displacement within a small material element consists of rigid body motion and deformation. The former can be divided into translation and rotation. The latter is caused by length and angle changes.



The geometry is described altogether by 12 parameters, of which 6 define the rigid body motion and remaining 6 the deformation (normal and shear components).

To find the expressions in terms of the displacement components, let us consider displacement within a small material element centered at $\vec{r}_0$. As the material element is assumed to be small, first two terms of the Taylor series

$$\vec{u}(\vec{r}) = \vec{u}_0 + \vec{\rho} \cdot (\nabla \vec{u})_0,$$

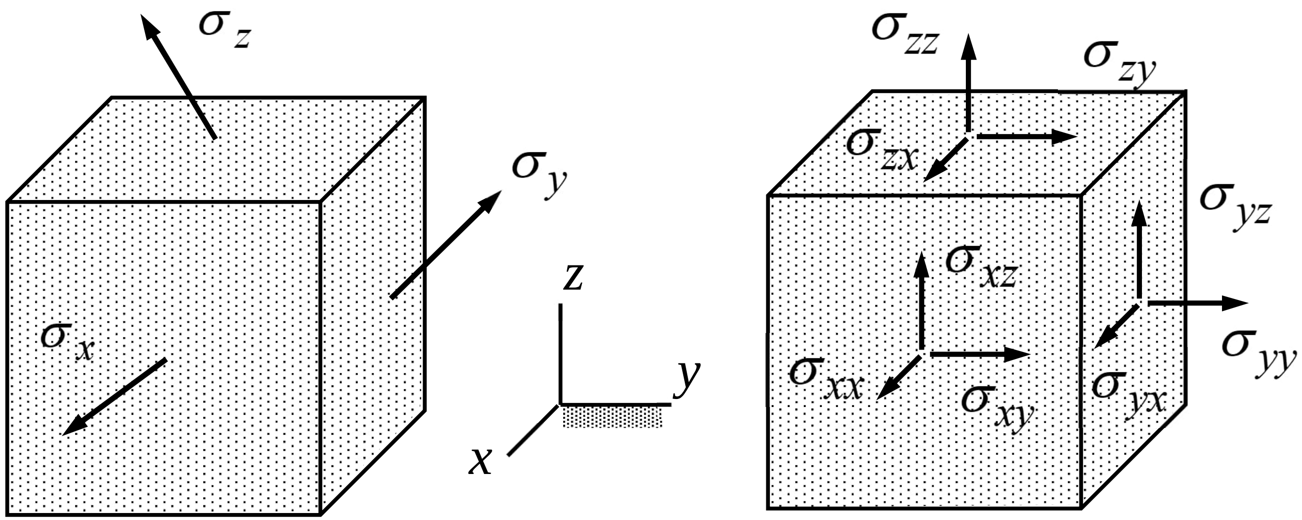
where the relative position vector $\vec{\rho} = \vec{r} - \vec{r}_0$, represent the displacement accurately enough. Division of the displacement gradient $(\nabla \vec{u})_0$ into its anti-symmetric and symmetric parts with notations $\vec{\vec{\varepsilon}} = (\nabla \vec{u})_{\mathrm{s}}$, $\vec{\vec{\theta}} = (\nabla \vec{u})_{\mathrm{u}}$ and using the concept of an associated vector $\vec{\theta}$ to an antisymmetric tensor $\vec{\vec{\theta}}$, gives

$$\vec{u}(\vec{r}) = \vec{u}_0 + \vec{\rho} \cdot \vec{\vec{\theta}}_0 + \vec{\rho} \cdot \vec{\vec{\varepsilon}}_0 = \vec{u}_0 + \vec{\theta}_0 \times \vec{\rho} + \vec{\rho} \cdot \vec{\vec{\varepsilon}}_0 \quad \text{where} \quad \vec{\vec{\varepsilon}} = (\nabla \vec{u})_{\mathrm{s}} = \tfrac{1}{2}[\nabla \vec{u} + (\nabla \vec{u})_{\mathrm{c}}].$$

The first term on the right-hand side describes translation, the second term small rigid body rotation, and the last term deformation (shape distortion) when the rotation part is small.

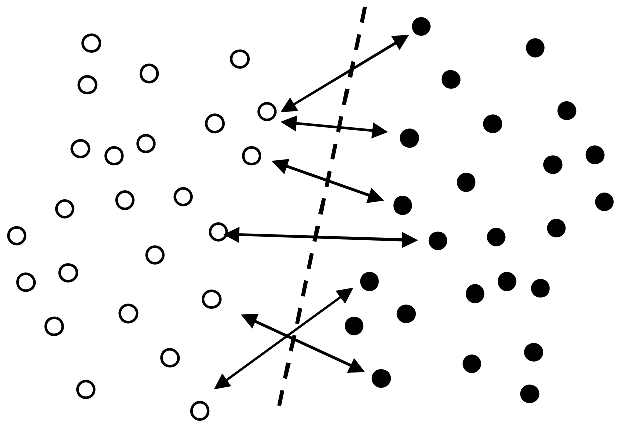
TRACTION AND STRESS

In continuum mechanics, traction $\vec{\sigma} = \Delta\vec{F} / \Delta A$ (a vector) describes the surface force between material elements of a body. Cauchy stress $\overleftrightarrow{\sigma}$ describes the surface forces acting on all edges of a material element. Traction and stress are related by $\vec{\sigma} = \vec{n} \cdot \overleftrightarrow{\sigma}$.



The first index of a stress component refers to the direction of the surface normal and the second that of the force component.

A material element is considered small compared with the size of the structure and, at the same time, large compared with the scale of the microstructure, e.g., distances between the molecules, atoms etc.



The ratio $\Delta\vec{F}/\Delta A$ of the interaction force resultant $\Delta\vec{F}$ to the area of the interaction ΔA is assumed to be constant $\vec{\sigma}$, when the area *is not too small nor too large*, therefore also $\Delta\vec{F} = \vec{\sigma}\Delta A$. Although a certain range of lower and upper limits is involved, continuum mechanics uses the relationship in form $d\vec{F} = \vec{\sigma}dA$.

The representation of the traction vectors acting on the three edges of a material element in the $(\vec{i},\vec{j},\vec{k})-$basis (directions are opposite on the opposite edges) can be expressed using the concept of stress tensor:

$$\begin{Bmatrix} \vec{\sigma}_x \\ \vec{\sigma}_y \\ \vec{\sigma}_z \end{Bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} \;\Rightarrow\; \vec{\vec{\sigma}} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vec{\sigma}_x \\ \vec{\sigma}_y \\ \vec{\sigma}_z \end{Bmatrix} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}.$$

Stress is a vector of vectors which represents the surface forces acting on all the surfaces of the material element simultaneously. In terms of the unit outward normal $\vec{n}$ of an edge, traction acting on the edge is given by $\vec{\sigma} = \vec{n} \cdot \vec{\vec{\sigma}}$ and the force the force acting on the material element through the edge

$$d\vec{F} = \vec{\sigma} dA = \vec{n} \cdot \vec{\vec{\sigma}} dA = (dA\vec{n}) \cdot \vec{\vec{\sigma}} = d\vec{A} \cdot \vec{\vec{\sigma}}.$$

LINEARLY ELASTIC MATERIAL

Material model gives a relationship between stress and strain. The generalized Hooke's laws for the isotropic and orthotropic materials can be expressed in forms:

Component: $$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} = [E] \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix}, \quad \begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = 2[G] \begin{Bmatrix} \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{Bmatrix}, \text{ and } \begin{Bmatrix} \sigma_{yx} \\ \sigma_{zy} \\ \sigma_{xz} \end{Bmatrix} = 2[G] \begin{Bmatrix} \varepsilon_{yx} \\ \varepsilon_{zy} \\ \varepsilon_{xz} \end{Bmatrix}$$

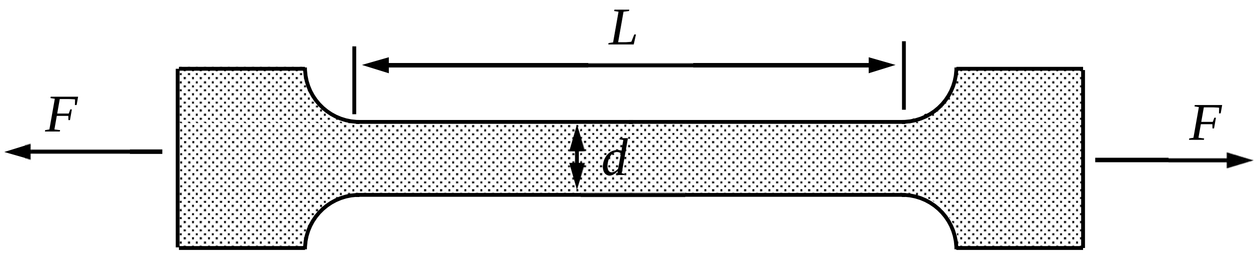
Tensor: $$\overset{\leftrightarrow}{\sigma} = \overset{\overset{\leftrightarrow}{\leftrightarrow}}{E} : \nabla \vec{u} \quad \text{where } \overset{\overset{\leftrightarrow}{\leftrightarrow}}{E} = \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix}^{\mathrm{T}} [E] \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix} + \begin{Bmatrix} \vec{i}\vec{j} + \vec{j}\vec{i} \\ \vec{j}\vec{k} + \vec{k}\vec{j} \\ \vec{k}\vec{i} + \vec{i}\vec{k} \end{Bmatrix}^{\mathrm{T}} [G] \begin{Bmatrix} \vec{i}\vec{j} + \vec{j}\vec{i} \\ \vec{j}\vec{k} + \vec{k}\vec{j} \\ \vec{k}\vec{i} + \vec{i}\vec{k} \end{Bmatrix}$$

in which the symmetric elasticity matrices $[E]$ and $[G]$ depend on material type.

Experiments indicate that length L, length change ΔL, and cross-sectional area A, diameter d, diameter change Δd, of the specimen loaded by force F are related by

$$\frac{F}{A} = E\frac{\Delta L}{L} \ ,$$

$$\frac{\Delta d}{d} = -\nu\frac{\Delta L}{L}$$



in which the coefficients E and ν (Young’s modulus and Poisson’s ratio) depend on the material.

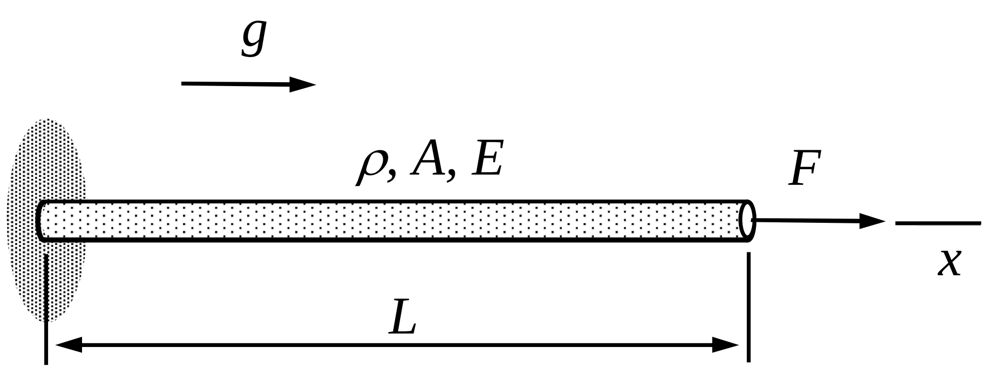
Constitutive equation brings the rigidity properties of the material into the model. The relationships between strain and stress are, basically, just compact and coherent representations of experimental data. However, the generic principles of physics restrict the set of possible linear material models.

First, a constitutive equations should be coordinate system invariant. Therefore, if a constitutive equation is known in some frame of reference, representation in some other system follows without a new set of experiments. A tensor relationship satisfies the requirement automatically

Second, a constitutive equation should be homogeneous with respect to (tensor) rank and dimension. For example, valid linear homogeneous relationships between rank 2 tensors $\overleftrightarrow{a}$ and $\overleftrightarrow{b}$ are, e.g., $\overleftrightarrow{a} = E\overleftrightarrow{b}$ and $\overleftrightarrow{a} = \overset{\leftrightarrow}{\overleftrightarrow{E}} : \overleftrightarrow{b}$ where E and $\overset{\leftrightarrow}{\overleftrightarrow{E}}$ characterize material.

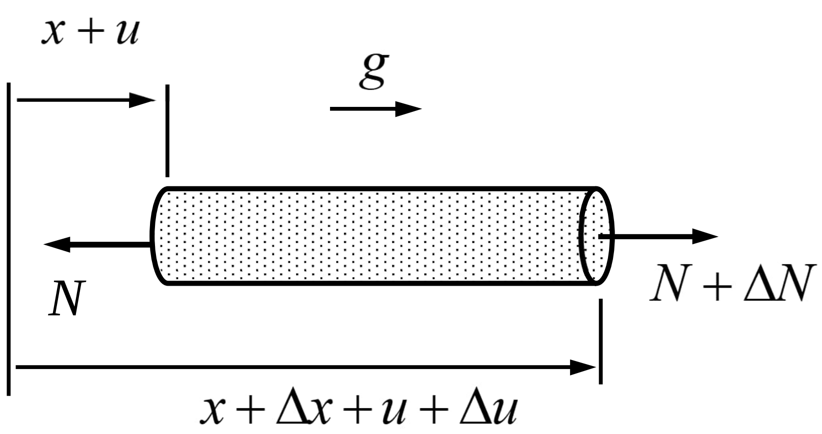
Qualitative information about material like homogeneity and isotropy restrict the form of the constitutive equations more effectively. For example, one may deduce that the number of material parameters characterizing an isotropic linearly elastic material is 2!

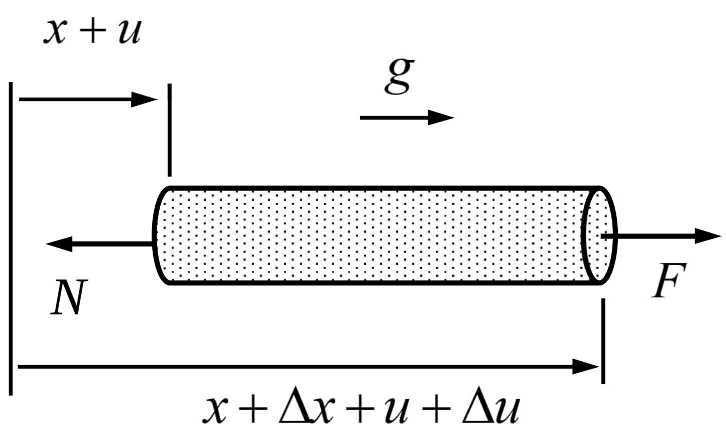
EXAMPLE Apply the first principles to a bar element inside the bar and an element at the free end to derive the differential equation and boundary condition at the right end in terms of displacement $u(x)$. Assume that the simple Hooke's law holds for the material. What is the condition at the left end?



Answer $EA\dfrac{d^2u}{dx^2}+\rho Ag=0 \quad x\in]0,L[, \quad EA\dfrac{du}{dx}=F \quad x=L,$ and $u=0 \quad x=0$

Let us apply the first principles to a material element of initial length Δx at the initial and final geometries.





The cases where the material element is inside the bar and at the right end differ.

Mass balance: $\Delta m = (\rho A)^{\circ}\Delta x = (\rho A)(\Delta x + \Delta u)$

Momentum balance $\rightarrow$: $N + \Delta N - N + g\Delta m = 0$

Momentum balance $\rightarrow$: $F - N + g\Delta m = 0$

Hooke's law: $\sigma = E\dfrac{\Delta u}{\Delta x} \;\Rightarrow\; N = EA\dfrac{\Delta u}{\Delta x}$.

The local forms follow by considering the limit $\Delta x \to 0$ and equations in terms of displacement after elimination of the stress resultant N. It is noteworthy, that the limit model assumes that $\Delta N / \Delta x$ exists also when $\Delta x \to 0$. In case of a discontinuity, like a point force P at x_0, one obtains the "jump" condition $[\![N]\!] + P = 0$ where $[\![a]\!] = \lim_{\varepsilon \to 0}[a(x_0+\varepsilon) - a(x_0-\varepsilon)]$.

1.3 VECTORS AND TENSORS

The quantities in mechanics can be classified into scalars a, vectors $\vec{a}$ and multi-vectors (vectors of vectors) $\vec{\vec{a}}$ called also as tensors of ranks 0,1, and 2.

basis vector singlet

Vector $$\vec{a} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix} = \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = a_x \vec{i} + a_y \vec{j} + a_z \vec{k} \quad \text{(rank 1 tensor)}$$

basis vector doublet

Tensor $$\vec{\vec{a}} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} a_{xx} & a_{xy} & a_{xz} \\ a_{yx} & a_{yy} & a_{yz} \\ a_{zx} & a_{zy} & a_{zz} \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = a_{xx} \vec{i}\vec{i} + a_{xy} \vec{i}\vec{j} + \ldots + a_{zz} \vec{k}\vec{k} \quad \text{(rank 2 tensor)}$$

Also, rank 4 tensors are needed. Their representations require basis vector quadruplets and 4 indices in the components.

TENSOR COMPONENTS

The multipliers of the basis vector singlets, doublets, etc. of a tensor are called as the components. The components of the first and second order tensors can be represented as column { } and square [ ] matrices:

Vector $$\vec{u} = \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} \text{ of components } \{u\} = \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix}$$

index 1 $\rightarrow$ row

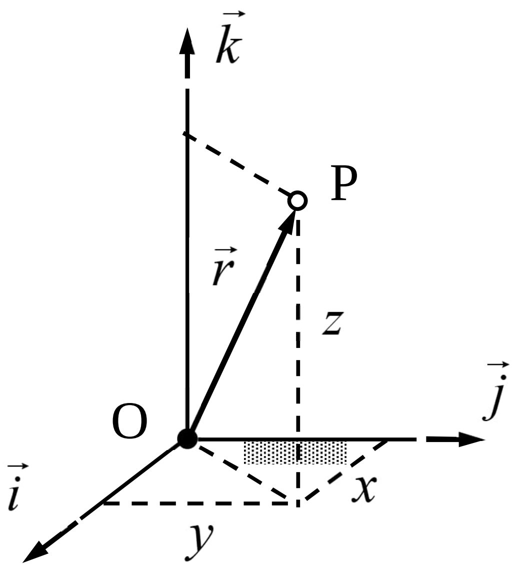
index 2 $\rightarrow$ column

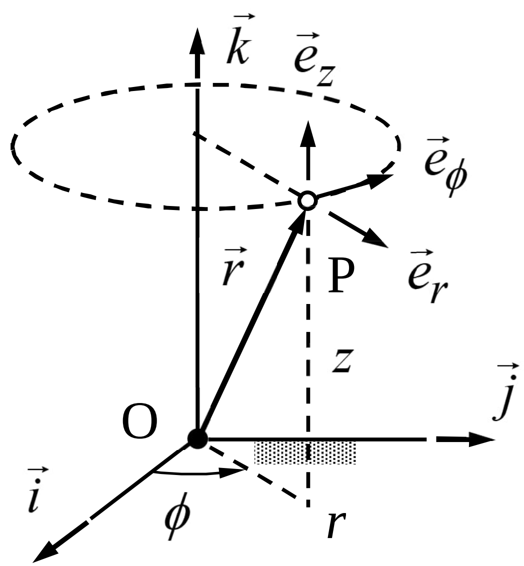
Tensor $$\vec{\vec{\sigma}} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} \text{ of components } [\sigma] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

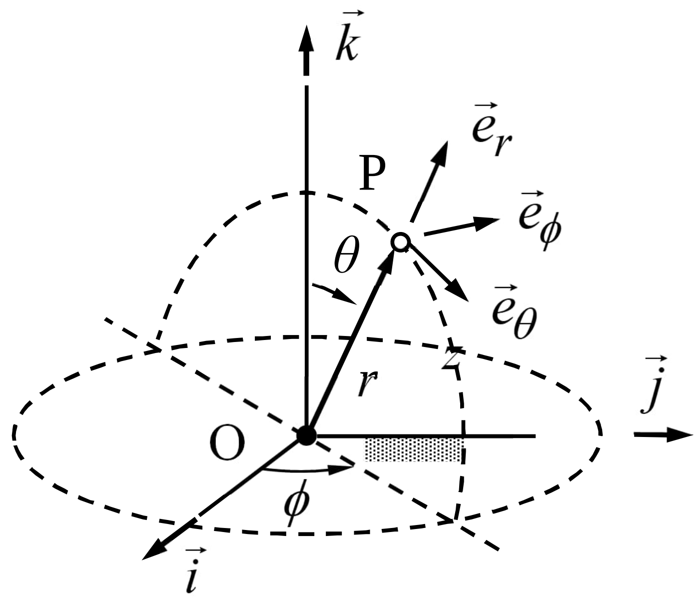
Notice. A column matrix is often called as the vector. Here, vector is a tensor of rank 1.

INVARIANCE

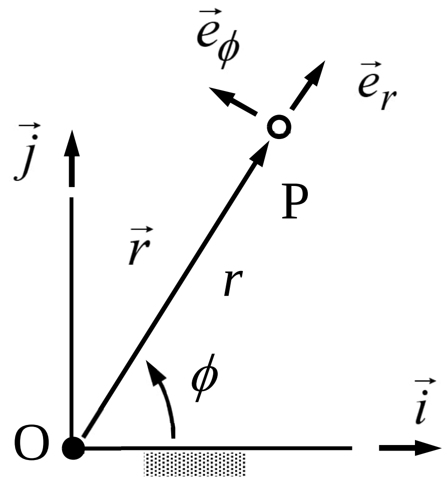
Tensor quantities are invariant with respect to coordinate system. Representation depends on the coordinate system but the tensor itself does not. Rectilinear-orthonormal (Cartesian) and curvilinear-orthonormal coordinate systems are common choices for tensor representations.







Transformation from one coordinate system to another requires the relationship between the basis vectors. Considering $\vec{a}$ of a planar case and using the relationship between the basis vectors of the Cartesian and polar coordinate systems shown ($\mathrm{c} \sim \cos$, $\mathrm{s} \sim \sin$)



$$\vec{a} = \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} a_{xx} & a_{xy} \\ a_{yx} & a_{yy} \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} \text{ and } \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} = \begin{bmatrix} \mathrm{c}\phi & -\mathrm{s}\phi \\ \mathrm{s}\phi & \mathrm{c}\phi \end{bmatrix} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} \Leftrightarrow$$

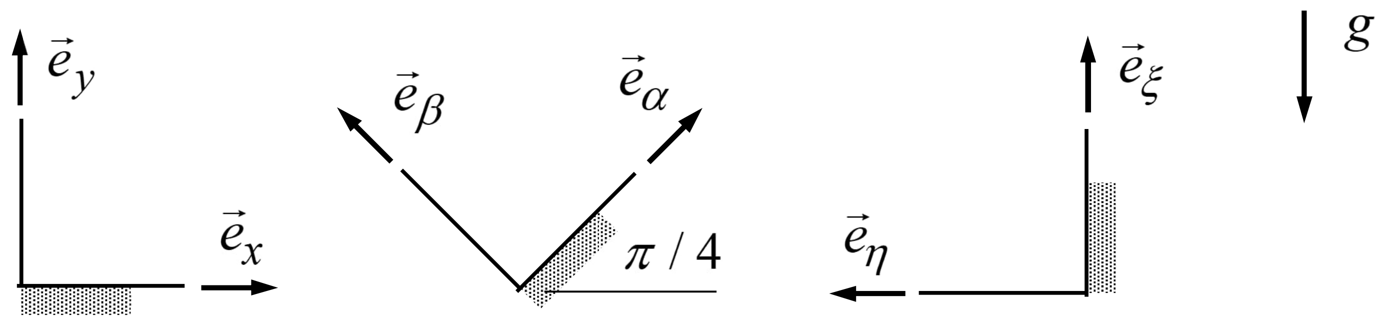
$$\vec{a} = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathrm{c}\phi & \mathrm{s}\phi \\ -\mathrm{s}\phi & \mathrm{c}\phi \end{bmatrix} \begin{bmatrix} a_{xx} & a_{xy} \\ a_{yx} & a_{yy} \end{bmatrix} \begin{bmatrix} \mathrm{c}\phi & -\mathrm{s}\phi \\ \mathrm{s}\phi & \mathrm{c}\phi \end{bmatrix} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} \Leftrightarrow$$

$$\vec{a} = \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} a_{rr} & a_{r\phi} \\ a_{\phi r} & a_{\phi\phi} \end{bmatrix} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \end{Bmatrix} \text{ where } \begin{bmatrix} a_{rr} & a_{r\phi} \\ a_{\phi r} & a_{\phi\phi} \end{bmatrix} = \begin{bmatrix} \mathrm{c}\phi & \mathrm{s}\phi \\ -\mathrm{s}\phi & \mathrm{c}\phi \end{bmatrix} \begin{bmatrix} a_{xx} & a_{xy} \\ a_{yx} & a_{yy} \end{bmatrix} \begin{bmatrix} \mathrm{c}\phi & -\mathrm{s}\phi \\ \mathrm{s}\phi & \mathrm{c}\phi \end{bmatrix}.$$

EXAMPLE Acceleration by gravity $\vec{g}$ can be represented in any of the coordinate systems of the figure starting with the known representation in one of the systems. Starting with the representation $\vec{g} = -g\vec{e}_y$ and using the relationship between the basis vectors

$$\vec{g} = -g\vec{e}_y = -g(\vec{e}_\alpha + \vec{e}_\beta)/\sqrt{2} = -g\vec{e}_\xi .$$

All these give the same direction and magnitude for the acceleration by gravity.

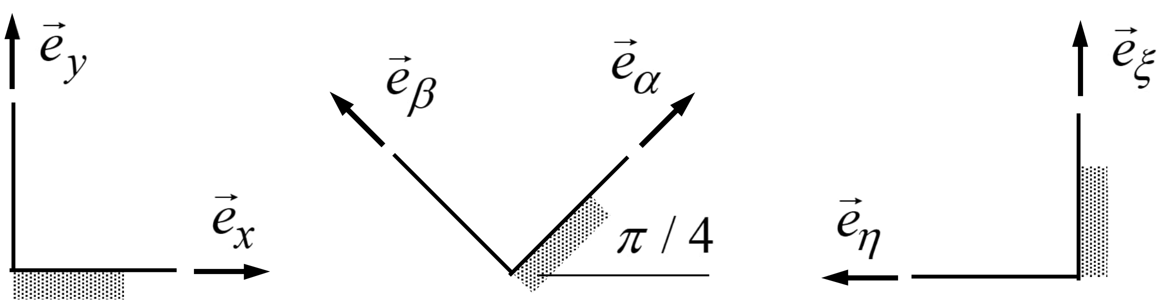


$$\vec{e}_y = \frac{1}{\sqrt{2}}(\vec{e}_\alpha + \vec{e}_\beta) \qquad \vec{e}_y = \vec{e}_\xi$$

EXAMPLE Second order tensor $\vec{\vec{a}}$ can be represented in any coordinate systems of the figure starting with the known representation $\vec{\vec{a}} = a\vec{e}_y\vec{e}_y$ in the Cartesian system. Using the relationship between the basis vectors

$$\vec{\vec{a}} = a\vec{e}_y\vec{e}_y = \frac{a}{2}(\vec{e}_\alpha\vec{e}_\alpha + \vec{e}_\alpha\vec{e}_\beta + \vec{e}_\beta\vec{e}_\alpha + \vec{e}_\beta\vec{e}_\beta) = a\vec{e}_\xi\vec{e}_\xi .$$

Graphical representation of a rank 2 tensor is not as obvious as that of a vector.



TENSOR PRODUCTS

In manipulation of an expression containing tensors, it is important to remember that tensor ($\otimes$), cross ($\times$), inner ($\cdot$) products are non-commutative (order matters). For simplicity of presentation, outer (tensor) products like $\vec{a}\otimes\vec{b}$ are denoted by $\vec{a}\vec{b}$ in MEC-E8003. Otherwise, the usual rules of vector algebra apply:

$$\vec{a}\cdot\vec{b} = a_x b_x + a_y b_y + a_z b_z,$$

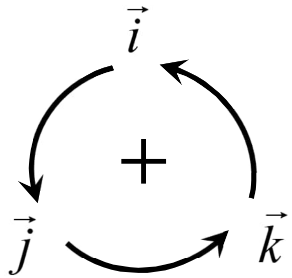
$$\vec{a}\times\vec{b} = (a_y b_z - a_z b_y)\vec{i} + (a_z b_x - a_x b_z)\vec{j} + (a_x b_y - a_y b_x)\vec{k},$$

$$\vec{a}\vec{b} = a_x b_x \vec{i}\vec{i} + a_x b_y \vec{i}\vec{j} + a_x b_z \vec{i}\vec{k} + a_y b_x \vec{j}\vec{i} + a_y b_y \vec{j}\vec{j} + a_y b_z \vec{j}\vec{k} + a_z b_x \vec{k}\vec{i} + a_z b_y \vec{k}\vec{j} + a_z b_z \vec{k}\vec{k}.$$

Calculation with tensors is straightforward although the number of terms may make manipulations somewhat tedious.

As an example, manipulations needed to find the cross-product of two vectors in a Cartesian system (orthonormal and right-handed) consists of steps

$$\vec{a}\times\vec{b} = (a_x\vec{i} + a_y\vec{j} + a_z\vec{k})\times(b_x\vec{i} + b_y\vec{j} + b_z\vec{k}) \quad \Leftrightarrow$$



$$\vec{a}\times\vec{b} = a_x b_x\vec{i}\times\vec{i} + a_x b_y\vec{i}\times\vec{j} + a_x b_z\vec{i}\times\vec{k} +$$

$$a_y b_x\vec{j}\times\vec{i} + a_y b_y\vec{j}\times\vec{j} + a_y b_z\vec{j}\times\vec{k} +$$

$$a_z b_x\vec{k}\times\vec{i} + a_z b_y\vec{k}\times\vec{j} + a_z b_z\vec{k}\times\vec{k} \quad \Rightarrow$$

$$\vec{a}\times\vec{b} = 0 + a_x b_y\vec{k} - a_x b_z\vec{j} - a_y b_x\vec{k} + 0 + a_y b_z\vec{i} + a_z b_x\vec{j} - a_z b_y\vec{i} + 0 \quad \Leftrightarrow$$

$$\vec{a}\times\vec{b} = (a_y b_z - a_z b_y)\vec{i} + (a_z b_x - a_x b_z)\vec{j} + (a_x b_y - a_y b_x)\vec{k}. \quad \leftarrow$$

The manipulations are often (but not always) easier when the components and basis vectors are arranged as matrices

$$\vec{a} = \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix} \text{ and } \vec{b} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} b_x \\ b_y \\ b_z \end{Bmatrix} = \begin{Bmatrix} b_x \\ b_y \\ b_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} \quad \Rightarrow$$

$$\vec{a} \times \vec{b} = \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix}^{\mathrm{T}} \left(\begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} \times \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^{\mathrm{T}} \right) \begin{Bmatrix} b_x \\ b_y \\ b_z \end{Bmatrix} = \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 0 & \vec{k} & -\vec{j} \\ -\vec{k} & 0 & \vec{i} \\ \vec{j} & -\vec{i} & 0 \end{bmatrix} \begin{Bmatrix} b_x \\ b_y \\ b_z \end{Bmatrix} \quad \Leftrightarrow$$

$$\vec{a} \times \vec{b} = (a_y b_z - a_z b_y)\vec{i} + (a_z b_x - a_x b_z)\vec{j} + (a_x b_y - a_y b_x)\vec{k}. \quad \Leftarrow$$

EXAMPLE The local forms of the balance laws of momentum and moment of momentum are $\nabla\cdot\vec{\vec{\sigma}}+\vec{f}=0$ and $\vec{\vec{\sigma}}=\vec{\vec{\sigma}}_{\mathrm{c}}$ (conjugate tensor). Assuming a planar case and a Cartesian coordinate system so that

$$\nabla=\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\partial/\partial x\\ \partial/\partial y\end{Bmatrix},\quad \vec{f}=\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}f_x\\ f_y\end{Bmatrix},\ \text{and } \vec{\vec{\sigma}}=\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}\sigma_{xx} & \sigma_{xy}\\ \sigma_{yx} & \sigma_{yy}\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\end{Bmatrix}$$

derive the component forms of the balance laws.

Answer $\dfrac{\partial\sigma_{xx}}{\partial x}+\dfrac{\partial\sigma_{yx}}{\partial y}+f_x=0,\quad \dfrac{\partial\sigma_{xy}}{\partial x}+\dfrac{\partial\sigma_{yy}}{\partial y}+f_y=0$ and $\sigma_{xy}=\sigma_{yx}$

In a Cartesian system, basis vectors are constants and one may transpose the gradient operator to get (transposing cannot be used with non-constant basis vectors! Why?)

$$\nabla \cdot \vec{\vec{\sigma}} + \vec{f} = \begin{Bmatrix} \partial / \partial x \\ \partial / \partial y \end{Bmatrix}^{\mathrm{T}} \left(\begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} \cdot \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}^{\mathrm{T}} \right) \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} + \begin{Bmatrix} f_x \\ f_y \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} = 0$$

$$\nabla \cdot \vec{\vec{\sigma}} + \vec{f} = \left(\begin{Bmatrix} \partial / \partial x \\ \partial / \partial y \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} + \begin{Bmatrix} f_x \\ f_y \end{Bmatrix}^{\mathrm{T}} \right) \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} = 0. \Leftarrow$$

$$\vec{\vec{\sigma}} - \vec{\vec{\sigma}}_{\mathrm{c}} = \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} - \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} = 0 \quad \Leftrightarrow$$

$$\vec{\vec{\sigma}} - \vec{\vec{\sigma}}_{\mathrm{c}} = \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 0 & \sigma_{xy} - \sigma_{yx} \\ \sigma_{yx} - \sigma_{xy} & 0 \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \end{Bmatrix} = 0. \Leftarrow$$

SOME DEFINITIONS AND IDENTITIES

Conjugate tensor $\overleftrightarrow{a}_{\mathrm{c}}$: $\overleftrightarrow{a}\cdot\vec{b}=\vec{b}\cdot\overleftrightarrow{a}_{\mathrm{c}} \quad \forall\vec{b}$

Second order identity tensor $\overleftrightarrow{I}$: $\overleftrightarrow{I}\cdot\vec{a}=\vec{a}\cdot\overleftrightarrow{I}=\vec{a} \quad \forall\vec{a}$

Fourth order identity tensor $\overleftrightarrow{\overleftrightarrow{I}}$: $\overleftrightarrow{\overleftrightarrow{I}}:\overleftrightarrow{a}=\overleftrightarrow{a}:\overleftrightarrow{\overleftrightarrow{I}}=\overleftrightarrow{a} \quad \forall\overleftrightarrow{a}$

Associated vector $\vec{a}$ **of an antisymmetric tensor** $\overleftrightarrow{a}$: $\vec{b}\cdot\overleftrightarrow{a}=\vec{a}\times\vec{b}$, when $\overleftrightarrow{a}=-\overleftrightarrow{a}_{\mathrm{c}}$

Scalar triple product $\vec{a}\cdot(\vec{b}\times\vec{c})=(\vec{a}\times\vec{b})\cdot\vec{c}$

Vector triple product $\vec{a}\times(\vec{b}\times\vec{c})=\vec{b}(\vec{a}\cdot\vec{c})-\vec{c}(\vec{a}\cdot\vec{b})$

Symmetric-antisymmetric double product $\overleftrightarrow{a}=-\overleftrightarrow{a}_{\mathrm{c}}$ and $\overleftrightarrow{b}=\overleftrightarrow{b}_{\mathrm{c}} \quad \Rightarrow \quad \overleftrightarrow{a}:\overleftrightarrow{b}=0$

Symmetric-antisymmetric division $\overleftrightarrow{a}=\overleftrightarrow{a}_s+\overleftrightarrow{a}_u=\frac{1}{2}(\overleftrightarrow{a}+\overleftrightarrow{a}_{\mathrm{c}})+\frac{1}{2}(\overleftrightarrow{a}-\overleftrightarrow{a}_{\mathrm{c}})$

1.4 DIFFERENTIAL EQUATIONS

Local forms of the balance equations imply ordinary or partial differential equations to be solved to stress and displacement components. The examples of the course apply

Trial solutions: The generic solution to ordinary homogeneous differential equations can be found (usually) with an exponential trial solution. The generic solution to a non-homogeneous equation consists of the generic solution to the homogeneous equations and a particular solution (just some solution taking care of the non-zero righthand side).

Repeated integrations: The generic solution for certain ordinary and partial differential equations can be found with simple integrations. With partial differential equations (or a set of them), “integration constants” are considered as functions of some independent variables.

$$\frac{d\psi}{dt} + k\psi = 0 \quad \Leftrightarrow \quad \psi(t) = ae^{-kt},$$

$$\frac{d^2\psi}{dt^2} + k^2\psi = 0 \quad \Leftrightarrow \quad \psi(t) = a\sin(kt) + b\cos(kt).$$

$$\frac{du}{ds} + \frac{1}{R}v = 0 \text{ and } \frac{dv}{ds} - \frac{1}{R}u = 0 \quad \Leftrightarrow \quad \frac{d^2u}{ds^2} + \frac{1}{R^2}u = 0 \quad \Leftrightarrow \quad \dots$$

$$\frac{d^2u}{ds^2} + \frac{1}{R^2}u = \frac{s}{R^2} \quad \Leftrightarrow \quad u(s) = a\sin(\frac{s}{R}) + b\cos(\frac{s}{R}) + s,$$

$$\frac{d^2u}{ds^2} + \frac{1}{R^2}u = \frac{1}{R}\cos(\frac{s}{R}) \quad \Leftrightarrow \quad u(s) = a\sin(\frac{s}{R}) + b\cos(\frac{s}{R}) + \frac{1}{2}s\sin(\frac{s}{R}),$$

$$\frac{d^2u}{ds^2} + \frac{1}{R^2}u = \frac{1}{R}\sin(\frac{s}{R}) \quad \Leftrightarrow \quad u(s) = a\sin(\frac{s}{R}) + b\cos(\frac{s}{R}) - \frac{1}{2}s\cos(\frac{s}{R}).$$

$$m\frac{d^2y}{dt^2} = -mg \quad \Leftrightarrow \quad y(t) = -\frac{1}{2}t^2 g + at + b\,,$$

$$\mu\frac{1}{r}\frac{d}{dr}(r\frac{d}{dr}v_z) = C \quad \Leftrightarrow \quad v_z(r) = \frac{C}{4\mu}r^2 + a\ln r + b\,,$$

$$(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr})(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr})w = \frac{b_n}{D} \quad \Leftrightarrow \quad w(r) = \frac{b_n}{D}\frac{r^4}{64} + a + br^2 + cr^2(1-\log r) + d\log r\,.$$

$$\frac{dp}{dx} = \mu\frac{d^2v_x}{dy^2} \quad \Leftrightarrow \quad \frac{dp}{dx} = a \;\; \text{and} \;\; \mu\frac{d^2v_x}{dy^2} = a \quad \Leftrightarrow \quad \ldots$$

$$\frac{\partial p}{\partial r} = \rho\Omega^2 r \quad \text{and} \quad \frac{\partial p}{\partial z} = -\rho g \quad \Leftrightarrow \quad p(z,r) = \frac{1}{2}\rho\Omega^2 r^2 - \rho g z + a\,.$$

$$\frac{\partial}{\partial z}N_{z\phi} + \rho g R\sin\phi = 0 \;\; \text{and} \;\; \frac{1}{R}\frac{\partial}{\partial\phi}N_{z\phi} + \frac{\partial}{\partial z}N_{zz} = 0 \quad \Leftrightarrow \quad \ldots$$

BOUNDARY VALUE PROBLEM

Boundary value problem (BVP) consists of a differential equation and additional information at the boundaries. Typically, one may know the displacement or external loading. To find the solution to BVP

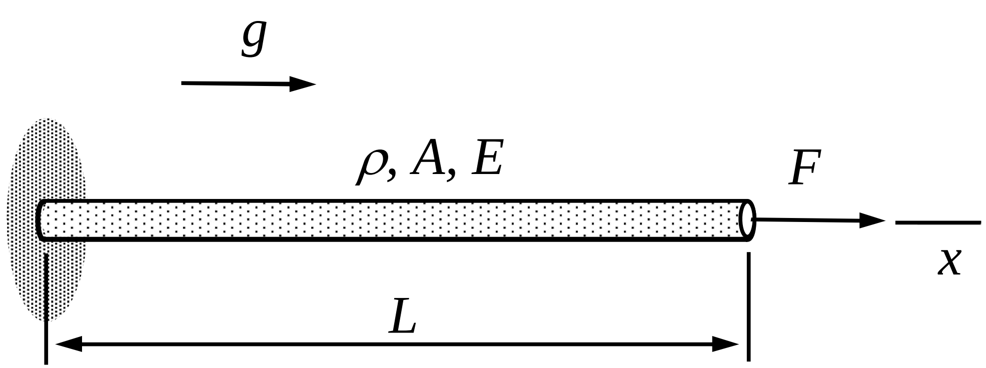
First, find the generic solution to the differential equation. Depending on the order of the equation the solution contains one or more integration constants.

Second, use the boundary conditions to express the integration constants in terms of the loading and known displacement values and substitute into the generic solution.

Solution to a BVP does not contain free parameters and, therefore stress, displacement etc. represent the quantities of a certain setting.

EXAMPLE Consider a bar of length L loaded by its own weight and point force acting at the free end. Cross-sectional area A, acceleration by gravity g, and material properties E and ρ are constants. Determine the displacement $u(x)$ from the boundary value problem

$$EA\frac{d^2u}{dx^2}+\rho Ag=0 \quad x\in]0,L[, \quad EA\frac{du}{dx}=F \quad x=L, \text{ and } u=0 \quad x=0.$$



Answer $EA\frac{d^2u}{dx^2}+\rho Ag=0 \quad x\in]0,L[, \quad EA\frac{du}{dx}=F \quad x=L, \text{ and } u=0 \quad x=0$

Let us first find the generic solution to the second order ordinary differential equation by repeated integrations

$$EA\frac{d^2u}{dx^2}+\rho Ag=0 \quad \Leftrightarrow \quad \frac{d^2u}{dx^2}=-\frac{\rho g}{E} \quad \Leftrightarrow \quad \frac{du}{dx}=-\frac{\rho g}{E}x+a \quad \Leftrightarrow \quad u=-\frac{\rho g}{2E}x^2+ax+b.$$

When the generic solution is substituted there, the two boundary conditions give

$$-A\rho gL+EAa=F \quad \text{and} \quad b=0 \quad \Leftrightarrow \quad a=\frac{\rho g}{E}L+\frac{F}{EA} \quad \text{and} \quad b=0.$$

Therefore, the displacement for the BVP takes the form

$$u(x)=\frac{\rho g}{2E}(2xL-x^2)+\frac{F}{EA}x. \quad \Leftarrow$$