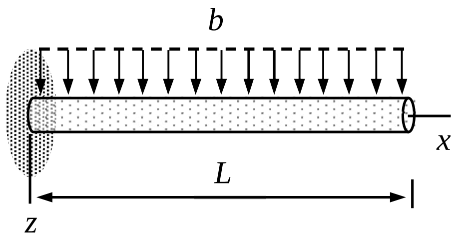
Name________________________ Student number________________

Assignment 4 (4p)



Find the stress resultants N, Q_z, M_y and the displacement and rotation components u, w, θ of the $xz-$plane cantilever beam of the figure according to the Timoshenko beam model. Problem parameters E, G, A, $S_y = 0$ and $I_{yy} = I$ are constants. Start with the generic equilibrium and constitutive equations for the beam model.

Solution

Timoshenko beam equilibrium and constitutive equations in the Cartesian $(x, y, z)-$coordinate system are

$$\begin{Bmatrix} \dfrac{dN}{dx} + b_x \\ \dfrac{dQ_y}{dx} + b_y \\ \dfrac{dQ_z}{dx} + b_z \end{Bmatrix} = 0, \quad \begin{Bmatrix} N \\ Q_y \\ Q_z \end{Bmatrix} = \begin{Bmatrix} EA\dfrac{du}{dx} - ES_z\dfrac{d\psi}{dx} + ES_y\dfrac{d\theta}{dx} \\ GA(\dfrac{dv}{dx} - \psi) - GS_y\dfrac{d\phi}{dx} \\ GA(\dfrac{dw}{dx} + \theta) + GS_z\dfrac{d\phi}{dx} \end{Bmatrix},$$

$$\begin{Bmatrix} \dfrac{dT}{dx} + c_x \\ \dfrac{dM_y}{dx} - Q_z + c_y \\ \dfrac{dM_z}{dx} + Q_y + c_z \end{Bmatrix} = 0, \quad \begin{Bmatrix} T \\ M_y \\ M_z \end{Bmatrix} = \begin{Bmatrix} -GS_y(\dfrac{dv}{dx} - \psi) + GS_z(\dfrac{dw}{dx} + \theta) + GI_{rr}\dfrac{d\phi}{dx} \\ ES_y\dfrac{du}{dx} - EI_{zy}\dfrac{d\psi}{dx} + EI_{yy}\dfrac{d\theta}{dx} \\ -ES_z\dfrac{du}{dx} + EI_{zz}\dfrac{d\psi}{dx} - EI_{yz}\dfrac{d\theta}{dx} \end{Bmatrix}.$$

In the $xz-$plane problem of the figure, geometrical properties of the cross-section are A, $S_y = 0$, and $I_{yy} = I$ and the external distributed force and moment components $b_x = b_y = 0$, $b_z = b$, and $c_x = c_y = c_z = 0$. The 3 equilibrium equations, 3 constitutive equations, and 6 boundary conditions for $N(x)$, $Q_z(x)$, $M_y(x)$, $u(x)$, $w(x)$, and $\theta(x)$ simplify to

$$\begin{Bmatrix} \dfrac{dN}{dx} \\ \dfrac{dQ_z}{dx} + b \\ \dfrac{dM_y}{dx} - Q_z \end{Bmatrix} = 0, \quad \begin{Bmatrix} N - EA\dfrac{du}{dx} \\ Q_z - GA(\dfrac{dw}{dx} + \theta) \\ M_y - EI\dfrac{d\theta}{dx} \end{Bmatrix} = 0 \text{ in } (0, L), \quad \begin{Bmatrix} N \\ Q_z \\ M_y \end{Bmatrix} = 0 \text{ at } x = L, \quad \begin{Bmatrix} u \\ w \\ \theta \end{Bmatrix} = 0 \text{ at } x = 0.$$

As stress resultants are known at the free end, one may start with the stress resultants

$$\frac{dN}{dx} = 0 \quad \text{in} \quad (0, L) \quad \text{and} \quad N = 0 \quad \text{at} \quad x = L \quad \Rightarrow \quad N(x) = 0 \, ,$$

$$\frac{dQ_z}{dx} + b = 0 \quad \text{in} \ (0, L) \quad \text{and} \quad Q_z = 0 \quad \text{at} \quad x = L \quad \Rightarrow \quad Q_z(x) = b(L - x) \, ,$$

$$\frac{dM_y}{dx} = Q_z = b(L - x) \ \text{in} \ (0, L) \quad \text{and} \quad M_y = 0 \quad \text{at} \quad x = L \quad \Rightarrow \quad M_y(x) = -\frac{1}{2} b(L - x)^2 \, ,$$

After that, solution to the displacement components follow from the constitutive equations and the remaining boundary conditions at the clamped edge:

$$\frac{du}{dx} = \frac{N}{EA} = 0 \quad \text{in} \quad (0, L) \ \text{and} \ u = 0 \quad \text{at} \quad x = 0 \quad \Rightarrow \quad u(x) = 0 \, ,$$

$$\frac{d\theta}{dx} = \frac{M_y}{EI} = -\frac{b}{2EI}(L - x)^2 \ \text{in} \ (0, L) \quad \text{and} \quad \theta = 0 \quad \text{at} \quad x = 0 \quad \Rightarrow \quad \theta(x) = -\frac{bx}{6EI}(3L^2 - 3Lx + x^2) \, ,$$

$$\frac{dw}{dx} = \frac{Q_z}{GA} - \theta = \frac{b}{GA}(L - x) + \frac{b}{2EI}(L^2 x - Lx^2 + \frac{1}{3}x^3) \ \text{in} \ (0, L) \quad \text{and} \quad w = 0 \quad \text{at} \quad x = 0 \quad \Rightarrow$$

$$w(x) = \frac{bx}{24}[\frac{1}{GA}(24L - 12x) + \frac{x}{EI}(6L^2 - 4Lx + x^2)] \, .$$

In a statically determinate case, equilibrium equation can be solved first for the stress resultants. After that, displacements follow from the constitutive equations. Often, the stress resultants are eliminated from the equilibrium equations and boundary conditions by using the constitutive equations to end up with second order diffrential equations to the displacement components only. Although, the number of equations can be reduced in this manner, finding the solution with the original set of equations is more straightforward.