Name__________________________ Student number__________________

Assignment 3 (4p)

Derive the component forms of cylindrical shell force equilibrium equations in the (z,ϕ,n) coordinate system starting from the invariant form $\nabla_0 \cdot \vec{\vec{F}} - \kappa \vec{e}_n \cdot \vec{\vec{F}} + \vec{b} = 0$. The force resultant representations and kinematic quantities of the cylindrical shell (z,ϕ,n) coordinate system are

$$\vec{\vec{F}} = N_{zz}\vec{e}_z\vec{e}_z + N_{z\phi}\vec{e}_z\vec{e}_\phi + N_{\phi z}\vec{e}_\phi\vec{e}_z + N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi + Q_z\vec{e}_z\vec{e}_n + Q_z\vec{e}_n\vec{e}_z + Q_\phi\vec{e}_\phi\vec{e}_n + Q_\phi\vec{e}_n\vec{e}_\phi \,,$$

$$\vec{b} = b_z\vec{e}_z + b_\phi\vec{e}_\phi + b_n\vec{e}_n \,,\quad \nabla_0 = \vec{e}_z\frac{\partial}{\partial z} + \vec{e}_\phi\frac{1}{R}\frac{\partial}{\partial\phi} \,,\quad \frac{\partial}{\partial\phi}\vec{e}_\phi = \vec{e}_n \,,\quad \frac{\partial}{\partial\phi}\vec{e}_n = -\vec{e}_\phi \,,\quad \vec{\vec{I}} = \vec{e}_z\vec{e}_z + \vec{e}_\phi\vec{e}_\phi + \vec{e}_n\vec{e}_n \,.$$

Solution

In the shell model, the stress resultants may not be symmetric. Definition $\vec{\vec{\kappa}} = (\nabla_0\vec{e}_n)_{\mathrm{c}}$ gives the curvature tensor

$$\vec{\vec{\kappa}} = (\nabla_0\vec{e}_n)_{\mathrm{c}} = [(\vec{e}_z\frac{\partial}{\partial z} + \vec{e}_\phi\frac{1}{R}\frac{\partial}{\partial\phi})\vec{e}_n]_{\mathrm{c}} = (\vec{e}_z\frac{\partial\vec{e}_n}{\partial z} + \vec{e}_\phi\frac{1}{R}\frac{\partial\vec{e}_n}{\partial\phi})_{\mathrm{c}} = -\vec{e}_\phi\vec{e}_\phi\frac{1}{R} \quad\Rightarrow\quad \kappa = \vec{\vec{I}} : \vec{\vec{\kappa}} = -\frac{1}{R} \,.$$

Let us consider the mid-surface (membrane) and shear parts of $\vec{\vec{F}} = \vec{\vec{N}} + \vec{Q}\vec{e}_n + \vec{e}_n\vec{Q}$ separately. First the membrane mode term

$$\nabla_0 \cdot \vec{\vec{N}} = (\vec{e}_z\frac{\partial}{\partial z} + \vec{e}_\phi\frac{1}{R}\frac{\partial}{\partial\phi}) \cdot (N_{zz}\vec{e}_z\vec{e}_z + N_{z\phi}\vec{e}_z\vec{e}_\phi + N_{\phi z}\vec{e}_\phi\vec{e}_z + N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi) \,,\quad \text{where}$$

$$(\vec{e}_z\frac{\partial}{\partial z}) \cdot (N_{zz}\vec{e}_z\vec{e}_z + N_{z\phi}\vec{e}_z\vec{e}_\phi + N_{\phi z}\vec{e}_\phi\vec{e}_z + N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi) = \frac{\partial N_{zz}}{\partial z}\vec{e}_z + \frac{\partial N_{z\phi}}{\partial z}\vec{e}_\phi \quad \text{and}$$

$$(\vec{e}_\phi\frac{1}{R}\frac{\partial}{\partial\phi}) \cdot (N_{zz}\vec{e}_z\vec{e}_z + N_{z\phi}\vec{e}_z\vec{e}_\phi + N_{\phi z}\vec{e}_\phi\vec{e}_z + N_{\phi\phi}\vec{e}_\phi\vec{e}_\phi) = \frac{1}{R}\frac{\partial N_{\phi z}}{\partial\phi}\vec{e}_z + \frac{1}{R}\frac{\partial N_{\phi\phi}}{\partial\phi}\vec{e}_\phi + \frac{1}{R}N_{\phi\phi}\vec{e}_n \,.$$

Altogether

$$\nabla_0 \cdot \vec{\vec{N}} = (\frac{\partial N_{zz}}{\partial z} + \frac{1}{R}\frac{\partial N_{\phi z}}{\partial\phi})\vec{e}_z + (\frac{\partial N_{z\phi}}{\partial z} + \frac{1}{R}\frac{\partial N_{\phi\phi}}{\partial\phi})\vec{e}_\phi + \frac{1}{R}N_{\phi\phi}\vec{e}_n \,.$$

Then, the shear part associated with the bending mode

$$\nabla_0 \cdot (\vec{Q}\vec{e}_n + \vec{e}_n\vec{Q}) = (\vec{e}_z\frac{\partial}{\partial z} + \vec{e}_\phi\frac{1}{R}\frac{\partial}{\partial\phi}) \cdot (Q_z\vec{e}_z\vec{e}_n + Q_\phi\vec{e}_\phi\vec{e}_n + Q_z\vec{e}_n\vec{e}_z + Q_\phi\vec{e}_n\vec{e}_\phi) \,,\quad \text{where}$$

$$(\vec{e}_z\frac{\partial}{\partial z}) \cdot (Q_z\vec{e}_z\vec{e}_n + Q_\phi\vec{e}_\phi\vec{e}_n + Q_z\vec{e}_n\vec{e}_z + Q_\phi\vec{e}_n\vec{e}_\phi) = \frac{\partial Q_z}{\partial z}\vec{e}_n \quad \text{and}$$

$$(\vec{e}_\phi \frac{1}{R}\frac{\partial}{\partial\phi})\cdot(Q_z\vec{e}_z\vec{e}_n + Q_\phi\vec{e}_\phi\vec{e}_n + Q_z\vec{e}_n\vec{e}_z + Q_\phi\vec{e}_n\vec{e}_\phi) = \frac{1}{R}(\frac{\partial Q_\phi}{\partial\phi}\vec{e}_n - Q_\phi\vec{e}_\phi - Q_z\vec{e}_z - Q_\phi\vec{e}_\phi) .$$

Altogether

$$\nabla_0\cdot(\vec{Q}\vec{e}_n + \vec{e}_n\vec{Q}) = -\frac{1}{R}Q_z\vec{e}_z - 2\frac{1}{R}Q_\phi\vec{e}_\phi + (\frac{\partial Q_z}{\partial z} + \frac{1}{R}\frac{\partial Q_\phi}{\partial\phi})\vec{e}_n = -\frac{1}{R}\vec{Q} - \frac{1}{R}Q_\phi\vec{e}_\phi + (\frac{\partial Q_z}{\partial z} + \frac{1}{R}\frac{\partial Q_\phi}{\partial\phi})\vec{e}_n .$$

The second term of the equilibrium equation simplifies to

$$\kappa\vec{e}_n\cdot\vec{\vec{F}} = -\frac{1}{R}(Q_z\vec{e}_z + Q_\phi\vec{e}_\phi) = -\frac{1}{R}\vec{Q} .$$

Therefore, combining the terms

$$\nabla_0\cdot\vec{\vec{F}} - \kappa\vec{e}_n\cdot\vec{\vec{F}} + \vec{b} = \begin{Bmatrix}\vec{e}_z\\ \vec{e}_\phi\\ \vec{e}_n\end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \frac{\partial N_{zz}}{\partial z} + \frac{1}{R}\frac{\partial N_{\phi z}}{\partial\phi} + b_z \\ \frac{\partial N_{z\phi}}{\partial z} + \frac{1}{R}\frac{\partial N_{\phi\phi}}{\partial\phi} - \frac{1}{R}Q_\phi + b_\phi \\ \frac{\partial Q_z}{\partial z} + \frac{1}{R}N_{\phi\phi} + \frac{1}{R}\frac{\partial Q_\phi}{\partial\phi} + b_n \end{Bmatrix} = 0 . \Leftarrow$$