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### Assignment 3 (2p)

Determine  $\text{tr}[S] \equiv \vec{I} : \vec{S}$ ,  $\vec{S} : \vec{S}$ ,  $\vec{S} : \vec{S}_c$ ,  $\vec{A} \cdot \vec{S}$  ja  $\vec{S} \cdot \vec{T}$  when the components of tensors  $\vec{A}$ ,  $\vec{S}$  and  $\vec{T}$  in the orthonormal  $(\vec{i}, \vec{j}, \vec{k})$  basis are

$$\{A\} = \begin{Bmatrix} 0 \\ -1 \\ 1 \end{Bmatrix}, [S] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } [T] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

#### Solution template

The inner products of the basis vectors  $\vec{i} \cdot \vec{i} = 1$ ,  $\vec{j} \cdot \vec{j} = 1$  and  $\vec{k} \cdot \vec{k} = 1$  all the other combinations giving zeros. The double inner product should be treated just as two inner products by keeping the positions of the multiplication operator with respect to vectors. Therefore, in the vector identity  $\vec{a}\vec{b} : \vec{c}\vec{d} = (\vec{b} \cdot \vec{c})(\vec{a} \cdot \vec{d})$ , the first inner product between  $\vec{b}$  and  $\vec{c}$  produces a scalar which can be moved in front of the expression. What remains is the inner product between  $\vec{a}$  and  $\vec{d}$ .

In conjugate  $\vec{S}_c$  to  $\vec{S}$  the component matrix is transposed which corresponds to order change in all the dyads  $\vec{a}\vec{b} \rightarrow \vec{b}\vec{a}$   $\vec{a}, \vec{b} \in \{\vec{i}, \vec{j}, \vec{k}\}$  of the tensor representation. Representations of  $\vec{A}$ ,  $\vec{S}$ ,  $\vec{S}_c$ ,  $\vec{T}$  and the second order unit tensor  $\vec{I}$  in  $(\vec{i}, \vec{j}, \vec{k})$  – basis

$$\vec{A} = \begin{Bmatrix} 0 \\ -1 \\ 1 \end{Bmatrix}^T \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = -\vec{j} + \vec{k},$$

$$\vec{S} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = \vec{ii} + \vec{ki} + \vec{kk},$$

$$\vec{S}_c = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}^T \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = \vec{ii} + \vec{ik} + \vec{kk},$$

$$\vec{T} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^T \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = \vec{jk} + \vec{ik} + \vec{ki},$$

$$\vec{I} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = \vec{ii} + \vec{jj} + \vec{kk}.$$

Evaluation of a tensor product expressions consist of (I) substitution of the representations, (II) term-by-term expansion, (III) evaluation of the terms, (IV) simplification and/or restructuring the outcome.

Double inner product  $\text{tr}[S] \equiv \vec{I} : \vec{S}$  produces a scalar

$$(I) \quad \text{tr}[S] \equiv \vec{I} : \vec{S} = (\vec{ii} + \vec{jj} + \vec{kk}) : (\vec{ii} + \vec{ki} + \vec{kk}) \Leftrightarrow$$

$$(II) \quad \text{tr}[S] = \vec{ii} : \vec{ii} + \vec{ii} : \vec{ki} + \vec{ii} : \vec{kk} + \vec{jj} : \vec{ii} + \vec{jj} : \vec{ki} + \vec{jj} : \vec{kk} + \vec{kk} : \vec{ii} + \vec{kk} : \vec{ki} + \vec{kk} : \vec{kk} \Leftrightarrow$$

$$(III) \quad \text{tr}[S] = 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 1 \Leftrightarrow$$

$$(IV) \quad \text{tr}[S] = 2. \quad \leftarrow$$

Double inner product  $\vec{S} : \vec{S}$  produces a scalar

$$(I) \quad \vec{S} : \vec{S} = (\vec{ii} + \vec{ki} + \vec{kk}) : (\vec{ii} + \vec{ki} + \vec{kk}) \Leftrightarrow$$

$$(II) \quad \vec{S} : \vec{S} = \vec{ii} : \vec{ii} + \vec{ii} : \vec{ki} + \vec{ii} : \vec{kk} + \vec{ki} : \vec{ii} + \vec{ki} : \vec{ki} + \vec{ki} : \vec{kk} + \vec{kk} : \vec{ii} + \vec{kk} : \vec{ki} + \vec{kk} : \vec{kk} \Leftrightarrow$$

$$(III) \quad \vec{S} : \vec{S} = 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 1 \Leftrightarrow$$

$$(IV) \quad \vec{S} : \vec{S} = 2. \quad \leftarrow$$

Double inner product  $\vec{S} : \vec{S}_c$  produces a scalar

$$(I) \quad \vec{S} : \vec{S}_c = (\vec{ii} + \vec{ki} + \vec{kk}) : (\vec{ii} + \vec{ik} + \vec{kk}) \Leftrightarrow$$

$$(II) \quad \vec{S} : \vec{S}_c = \vec{ii} : \vec{ii} + \vec{ii} : \vec{ik} + \vec{ii} : \vec{kk} + \vec{ki} : \vec{ii} + \vec{ki} : \vec{ik} + \vec{ki} : \vec{kk} + \vec{kk} : \vec{ii} + \vec{kk} : \vec{ik} + \vec{kk} : \vec{kk} \Leftrightarrow$$

$$(III) \quad \vec{S} : \vec{S}_c = 1 + 0 + 0 + 0 + 1 + 0 + 0 + 0 + 1 \Leftrightarrow$$

$$(IV) \quad \vec{S} : \vec{S}_c = 3. \quad \leftarrow$$

Inner product  $\vec{A} \cdot \vec{S}$  produces a vector

$$(I) \quad \vec{A} \cdot \vec{S} = (-\vec{j} + \vec{k}) \cdot (\vec{ii} + \vec{ki} + \vec{kk}) \Leftrightarrow$$

$$(II) \quad \vec{A} \cdot \vec{S} = -\vec{j} \cdot \vec{ii} - \vec{j} \cdot \vec{ki} - \vec{j} \cdot \vec{kk} + \vec{k} \cdot \vec{ii} + \vec{k} \cdot \vec{ki} + \vec{k} \cdot \vec{kk} \Leftrightarrow$$

$$(III) \quad \vec{A} \cdot \vec{S} = -0\vec{i} - 0\vec{i} - 0\vec{k} + 0\vec{i} + 1\vec{i} + 1\vec{k} \quad \Leftrightarrow$$

$$(IV) \quad \vec{A} \cdot \vec{S} = \vec{i} + \vec{k} . \quad \leftarrow$$

Inner product  $\vec{S} \cdot \vec{T}$  produces a second order tensor

$$(I) \quad \vec{S} \cdot \vec{T} = (\vec{ii} + \vec{ki} + \vec{k\vec{k}}) \cdot (\vec{jj} + \vec{i\vec{k}} + \vec{k\vec{i}}) \quad \Leftrightarrow$$

$$(II) \quad \vec{S} \cdot \vec{T} = \vec{ii} \cdot \vec{jj} + \vec{ii} \cdot \vec{i\vec{k}} + \vec{ii} \cdot \vec{k\vec{i}} + \vec{ki} \cdot \vec{jj} + \vec{ki} \cdot \vec{i\vec{k}} + \vec{ki} \cdot \vec{k\vec{i}} + \vec{k\vec{k}} \cdot \vec{jj} + \vec{k\vec{k}} \cdot \vec{i\vec{k}} + \vec{k\vec{k}} \cdot \vec{k\vec{i}} \quad \Leftrightarrow$$

$$(III) \quad \vec{S} \cdot \vec{T} = \vec{i}0\vec{j} + \vec{i}1\vec{k} + \vec{i}0\vec{i} + \vec{k}0\vec{j} + \vec{k}1\vec{k} + \vec{k}0\vec{i} + \vec{k}0\vec{j} + \vec{k}0\vec{k} + \vec{k}1\vec{i} \quad \Leftrightarrow$$

$$(IV) \quad \vec{S} \cdot \vec{T} = \vec{i\vec{k}} + \vec{k\vec{k}} + \vec{k\vec{i}} . \quad \leftarrow$$