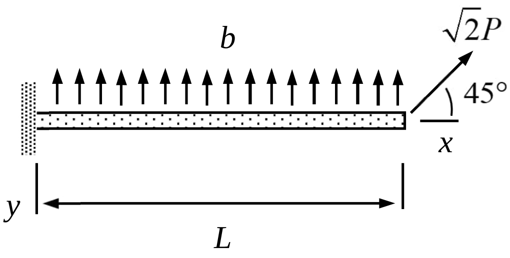
Name________________________ Student number________________

Assignment 3 (4p)

Consider the $xy-$plane beam of length L shown. Material properties E and G, cross-section properties A, I are constants, and $S=0$. Write down the boundary value problem according to the Bernoulli beam model in terms of axial displacement $u(x)$ and transverse displacement $v(x)$. Start with the generic equilibrium and constitutive equations of the Timoshenko beam model



$$\begin{Bmatrix} \dfrac{dN}{dx}+b_x \\ \dfrac{dQ_y}{dx}+b_y \\ \dfrac{dQ_z}{dx}+b_z \end{Bmatrix}=0,\quad \begin{Bmatrix} N \\ Q_y \\ Q_z \end{Bmatrix}=\begin{Bmatrix} EA\dfrac{du}{dx}-ES_z\dfrac{d\psi}{dx}+ES_y\dfrac{d\theta}{dx} \\ GA(\dfrac{dv}{dx}-\psi)-GS_y\dfrac{d\phi}{dx} \\ GA(\dfrac{dw}{dx}+\theta)+GS_z\dfrac{d\phi}{dx} \end{Bmatrix}$$

$$\begin{Bmatrix} \dfrac{dT}{dx}+c_x \\ \dfrac{dM_y}{dx}-Q_z+c_y \\ \dfrac{dM_z}{dx}+Q_y+c_z \end{Bmatrix}=0,\quad \begin{Bmatrix} T \\ M_y \\ M_z \end{Bmatrix}=\begin{Bmatrix} -GS_y(\dfrac{dv}{dx}-\psi)+GS_z(\dfrac{dw}{dx}+\theta)+GI_{rr}\dfrac{d\phi}{dx} \\ ES_y\dfrac{du}{dx}-EI_{zy}\dfrac{d\psi}{dx}+EI_{yy}\dfrac{d\theta}{dx} \\ -ES_z\dfrac{du}{dx}+EI_{zz}\dfrac{d\psi}{dx}-EI_{yz}\dfrac{d\theta}{dx} \end{Bmatrix}$$

Solution

Timoshenko beam equations boil down to the Bernoulli beam equations when the Bernoulli constraints $dv/dx-\psi=0$ and $dw/dx+\theta=0$ are applied there. If $S_y=S_z=I_{yz}=0$ one may just replace the constitutive equations for the shear stress resultants by the Bernoulli constraints to get the Bernoulli model equilibrium and constitutive equations (elimination of rotations and bending moments gives the well-known fourth order beam bending equations of textbooks)

In $xy-$plane problem, the non-zero displacements and rotations are u, v, and ψ. Geometrical properties of the cross-section are A, $S_z=S=0$, $I_{zz}=I$. External distributed forces are $b_x=0$, $b_y=-b$, $b_z=0$, $c_x=c_y=c_z=0$. With these selections, equilibrium equations, constitutive equations, and boundary conditions of the planar problem take the forms

$$\begin{Bmatrix} \dfrac{dN}{dx} \\ \dfrac{dQ_y}{dx} - b \\ \dfrac{dM_z}{dx} + Q_y \end{Bmatrix} = 0 \quad \text{and} \quad \begin{Bmatrix} N \\ 0 \\ M_z \end{Bmatrix} = \begin{Bmatrix} EA\dfrac{du}{dx} \\ \dfrac{dv}{dx} - \psi \\ EI_{zz}\dfrac{d\psi}{dx} \end{Bmatrix} \quad \text{in} \quad (0, L),$$

$$\begin{Bmatrix} u \\ v \\ \psi \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{at} \quad x = 0 \quad \text{and} \quad \begin{Bmatrix} N - P \\ Q_y + P \\ M_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{at} \quad x = L.$$

To get a boundary value problem in terms of the axial displacement $u(x)$, transverse displacement $v(x)$, stress resultants and rotation are eliminated to end up with

$$EA\frac{d^2u}{dx^2} = 0 \quad \text{in} \quad (0, L), \quad EA\frac{du}{dx} - P = 0 \quad \text{at} \quad x = L, \quad \text{and} \quad u = 0 \quad \text{at} \quad x = 0. \quad \Leftarrow$$

$$\frac{d^2}{dx^2}(EI_{zz}\frac{d^2v}{dx^2}) + b = 0 \quad \text{in} \quad (0, L), \quad -\frac{d}{dx}(EI_{zz}\frac{d^2v}{dx^2}) + P = 0 \quad \text{at} \quad x = L, \quad EI_{zz}\frac{d^2v}{dx^2} = 0 \quad \text{at} \quad x = L,$$

$$\frac{dv}{dx} = 0 \quad \text{at} \quad x = 0, \quad \text{and} \quad v = 0 \quad \text{at} \quad x = 0. \quad \Leftarrow$$