

# MEC-E8003 Beam, Plate and Shell Models; formulae

## TENSORS

$$\vec{a} = \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix}^T \begin{Bmatrix} a_\alpha \\ a_\beta \\ a_\gamma \end{Bmatrix}, \quad \vec{a} = \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix}^T \begin{bmatrix} a_{\alpha\alpha} & a_{\alpha\beta} & a_{\alpha\gamma} \\ a_{\beta\alpha} & a_{\beta\beta} & a_{\beta\gamma} \\ a_{\gamma\alpha} & a_{\gamma\beta} & a_{\gamma\gamma} \end{bmatrix} \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix}, \text{ etc.}$$

$$\vec{I} \cdot \vec{a} = \vec{a} \cdot \vec{I} = \vec{a} \quad \forall \vec{a}, \quad \vec{I} = \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}^T \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}$$

$$\vec{I} : \vec{a} = \vec{a} : \vec{I} = \vec{a} \quad \forall \vec{a}, \quad \vec{I} = \begin{Bmatrix} \vec{ii} \\ \vec{jj} \\ \vec{kk} \end{Bmatrix}_c^T \begin{Bmatrix} \vec{ii} \\ \vec{jj} \\ \vec{kk} \end{Bmatrix} + \begin{Bmatrix} \vec{ij} \\ \vec{jk} \\ \vec{ki} \end{Bmatrix}_c^T \begin{Bmatrix} \vec{ij} \\ \vec{jk} \\ \vec{ki} \end{Bmatrix} + \begin{Bmatrix} \vec{ji} \\ \vec{kj} \\ \vec{ik} \end{Bmatrix}_c^T \begin{Bmatrix} \vec{ji} \\ \vec{kj} \\ \vec{ik} \end{Bmatrix}$$

$$\vec{a} = \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix}^T \begin{bmatrix} a_{\alpha\alpha} & a_{\alpha\beta} & a_{\alpha\gamma} \\ a_{\beta\alpha} & a_{\beta\beta} & a_{\beta\gamma} \\ a_{\gamma\alpha} & a_{\gamma\beta} & a_{\gamma\gamma} \end{bmatrix} \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix} \Leftrightarrow \vec{a}_c = \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix}^T \begin{bmatrix} a_{\alpha\alpha} & a_{\alpha\beta} & a_{\alpha\gamma} \\ a_{\beta\alpha} & a_{\beta\beta} & a_{\beta\gamma} \\ a_{\gamma\alpha} & a_{\gamma\beta} & a_{\gamma\gamma} \end{bmatrix} \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix}$$

$$\vec{a} = -\vec{a}_c \Rightarrow \vec{a} \cdot \vec{b} = \vec{a} \times \vec{b} \quad \forall \vec{b}$$

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$$

$$\vec{a} : (\nabla \vec{b})_c = \nabla \cdot (\vec{a} \cdot \vec{b}) - (\nabla \cdot \vec{a}) \cdot \vec{b}$$

## CURVILINEAR COORDINATES

$$\vec{r}(\alpha, \beta, \gamma) = x(\alpha, \beta, \gamma)\vec{i} + y(\alpha, \beta, \gamma)\vec{j} + z(\alpha, \beta, \gamma)\vec{k}$$

$$\begin{Bmatrix} \vec{h}_\alpha \\ \vec{h}_\beta \\ \vec{h}_\gamma \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \vec{r}}{\partial \alpha} \\ \frac{\partial \vec{r}}{\partial \beta} \\ \frac{\partial \vec{r}}{\partial \gamma} \end{Bmatrix} = [H] \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}, \quad \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix} = \begin{Bmatrix} \vec{h}_\alpha / h_\alpha \\ \vec{h}_\beta / h_\beta \\ \vec{h}_\gamma / h_\gamma \end{Bmatrix} = [F] \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix},$$

$$\frac{\partial}{\partial \eta} \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix} = \left( \frac{\partial}{\partial \eta} [F] \right) [F]^{-1} \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix} \quad \eta \in \{\alpha, \beta, \gamma\}, \quad \nabla = \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_\gamma \end{Bmatrix}^T ([H][F]^T)^{-1} \begin{Bmatrix} \frac{\partial}{\partial \alpha} \\ \frac{\partial}{\partial \beta} \\ \frac{\partial}{\partial \gamma} \end{Bmatrix}$$

$$\nabla = \vec{e}_\alpha \frac{1}{h_\alpha} \frac{\partial}{\partial \alpha} + \vec{e}_\beta \frac{1}{h_\beta} \frac{\partial}{\partial \beta} + \vec{e}_\gamma \frac{1}{h_\gamma} \frac{\partial}{\partial \gamma} \quad (\text{orthonormal coordinate system})$$

## CYLINDRICAL COORDINATES

$$\vec{r}(r, \phi, z) = r(\cos \phi \vec{i} + \sin \phi \vec{j}) + z \vec{k}$$

$$\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix} = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}, \quad \frac{\partial}{\partial \phi} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix} = \begin{Bmatrix} \vec{e}_\phi \\ -\vec{e}_r \\ 0 \end{Bmatrix}, \quad \frac{\partial}{\partial r} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix} = \frac{\partial}{\partial z} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix} = 0$$

$$\nabla = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r} \frac{\partial}{\partial \phi} + \vec{e}_z \frac{\partial}{\partial z}$$

## SPHERICAL COORDINATES

$$\vec{r}(\theta, \phi, r) = r(\sin \theta \cos \phi \vec{i} + \sin \theta \sin \phi \vec{j} + \cos \theta \vec{k})$$

$$\begin{Bmatrix} \vec{e}_\theta \\ \vec{e}_\phi \\ \vec{e}_r \end{Bmatrix} = \begin{bmatrix} \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \\ \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}, \quad \frac{\partial}{\partial \phi} \begin{Bmatrix} \vec{e}_\theta \\ \vec{e}_\phi \\ \vec{e}_r \end{Bmatrix} = \begin{Bmatrix} \cos \theta \vec{e}_\phi \\ -\sin \theta \vec{e}_r - \cos \theta \vec{e}_\theta \\ \sin \theta \vec{e}_\phi \end{Bmatrix}, \quad \frac{\partial}{\partial \theta} \begin{Bmatrix} \vec{e}_\theta \\ \vec{e}_\phi \\ \vec{e}_r \end{Bmatrix} = \begin{Bmatrix} -\vec{e}_r \\ 0 \\ \vec{e}_\theta \end{Bmatrix}$$

$$\frac{\partial}{\partial r} \begin{Bmatrix} \vec{e}_\theta \\ \vec{e}_\phi \\ \vec{e}_r \end{Bmatrix} = 0, \quad \nabla = \vec{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \vec{e}_\phi \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} + \vec{e}_r \frac{\partial}{\partial r}$$

## BEAM COORDINATES

$$\vec{r}(s, n, b) = \vec{r}_0(s) + n \vec{e}_n + b \vec{e}_b$$

$$\begin{Bmatrix} \vec{e}_s \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \vec{r}_0}{\partial s} \\ \frac{\partial \vec{e}_s}{\partial s} / \left| \frac{\partial \vec{e}_s}{\partial s} \right| \\ \vec{e}_s \times \vec{e}_n \end{Bmatrix}, \quad \frac{\partial}{\partial s} \begin{Bmatrix} \vec{e}_s \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix} = \begin{Bmatrix} \kappa \vec{e}_n \\ \tau \vec{e}_b - \kappa \vec{e}_s \\ -\tau \vec{e}_n \end{Bmatrix}, \quad \frac{\partial}{\partial n} \begin{Bmatrix} \vec{e}_s \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix} = \frac{\partial}{\partial b} \begin{Bmatrix} \vec{e}_s \\ \vec{e}_n \\ \vec{e}_b \end{Bmatrix} = 0$$

$$\nabla = \frac{\vec{e}_s}{1-n\kappa} \left[ \frac{\partial}{\partial s} + \tau(b \frac{\partial}{\partial n} - n \frac{\partial}{\partial b}) \right] + \vec{e}_n \frac{\partial}{\partial n} + \vec{e}_b \frac{\partial}{\partial b}$$

### SHELL COORDINATES

$$\vec{r}(\alpha, \beta, n) = \vec{r}_0(\alpha, \beta) + n\vec{e}_n$$

$$\begin{Bmatrix} \vec{h}_\alpha \\ \vec{h}_\beta \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \vec{r}_0}{\partial \alpha} \\ \frac{\partial \vec{r}_0}{\partial \beta} \end{Bmatrix}, \quad \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_n \end{Bmatrix} = \begin{Bmatrix} \vec{h}_\alpha / h_\alpha \\ \vec{h}_\beta / h_\beta \\ \vec{e}_\alpha \times \vec{e}_\beta \end{Bmatrix} = [F] \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}, \quad \frac{\partial}{\partial \eta} \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_n \end{Bmatrix} = \left( \frac{\partial}{\partial \eta} [F] \right) [F]^{-1} \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_n \end{Bmatrix} \quad \eta \in \{\alpha, \beta, n\},$$

$$\begin{Bmatrix} \frac{\partial \vec{r}}{\partial \alpha} \\ \frac{\partial \vec{r}}{\partial \beta} \\ \frac{\partial \vec{r}}{\partial n} \end{Bmatrix} = \begin{Bmatrix} \vec{h}_\alpha + n \frac{\partial \vec{e}_n}{\partial \alpha} \\ \vec{h}_\beta + n \frac{\partial \vec{e}_n}{\partial \beta} \\ \vec{e}_n \end{Bmatrix} = [H] \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}, \quad \nabla = \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_n \end{Bmatrix}^T ([H][F]^T)^{-1} \begin{Bmatrix} \frac{\partial}{\partial \alpha} \\ \frac{\partial}{\partial \beta} \\ \frac{\partial}{\partial n} \end{Bmatrix}$$

### CYLINDRICAL SHELL COORDINATES

$$\vec{r}(z, \phi, n) = (R-n)(\cos \phi \vec{i} + \sin \phi \vec{j}) + z\vec{k}$$

$$\begin{Bmatrix} \vec{e}_z \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -\sin \phi & \cos \phi & 0 \\ -\cos \phi & -\sin \phi & 0 \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}, \quad \frac{\partial}{\partial \phi} \begin{Bmatrix} \vec{e}_z \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ \vec{e}_n \\ -\vec{e}_\phi \end{Bmatrix}, \quad \frac{\partial}{\partial z} \begin{Bmatrix} \vec{e}_z \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix} = \frac{\partial}{\partial n} \begin{Bmatrix} \vec{e}_z \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix} = 0$$

$$\nabla = \vec{e}_z \frac{\partial}{\partial z} + \left( \frac{R}{R-n} \right) \frac{1}{R} \vec{e}_\phi \frac{\partial}{\partial \phi} + \vec{e}_n \frac{\partial}{\partial n}$$

### SPHERICAL SHELL COORDINATES

$$\vec{r}(\phi, \theta, n) = (R-n)(\sin \theta \cos \phi \vec{i} + \sin \theta \sin \phi \vec{j} + \cos \theta \vec{k})$$

$$\begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_\theta \\ \vec{e}_n \end{Bmatrix} = \begin{bmatrix} -\sin \phi & \cos \phi & 0 \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \theta \cos \phi & -\sin \theta \sin \phi & -\cos \theta \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}, \quad \frac{\partial}{\partial \phi} \begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_\theta \\ \vec{e}_n \end{Bmatrix} = \begin{Bmatrix} \sin \theta \vec{e}_n - \cos \theta \vec{e}_\theta \\ \cos \theta \vec{e}_\phi \\ -\sin \theta \vec{e}_\phi \end{Bmatrix}$$

$$\frac{\partial}{\partial \theta} \begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_\theta \\ \vec{e}_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ \vec{e}_n \\ -\vec{e}_\theta \end{Bmatrix}, \quad \frac{\partial}{\partial n} \begin{Bmatrix} \vec{e}_\phi \\ \vec{e}_\theta \\ \vec{e}_n \end{Bmatrix} = 0, \quad \nabla = \frac{R}{R-n} \frac{1}{R \sin \theta} \vec{e}_\phi \frac{\partial}{\partial \phi} + \frac{R}{R-n} \frac{1}{R} \vec{e}_\theta \frac{\partial}{\partial \theta} + \vec{e}_n \frac{\partial}{\partial n}$$

### CIRCULAR PLATE COORDINATES

$$\vec{r}(r, \phi, n) = r(\cos \phi \vec{i} + \sin \phi \vec{j}) + n\vec{k}$$

$$\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix} = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix}, \quad \frac{\partial}{\partial \phi} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix} = \begin{Bmatrix} \vec{e}_\phi \\ -\vec{e}_r \\ 0 \end{Bmatrix}, \quad \frac{\partial}{\partial r} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix} = \frac{\partial}{\partial n} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_n \end{Bmatrix} = 0$$

$$\nabla = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r} \frac{\partial}{\partial \phi} + \vec{e}_n \frac{\partial}{\partial n}$$

## LINEAR ELASTICITY

$$\vec{\sigma} = \vec{\vec{E}} : \vec{\varepsilon} = \vec{\vec{E}} : \nabla \vec{u}, \quad \vec{\varepsilon} = \frac{1}{2} [\nabla \vec{u} + (\nabla \vec{u})_c]$$

$$\text{Elastic material: } \vec{\vec{E}} = \begin{Bmatrix} \vec{\vec{u}} \\ \vec{\vec{v}} \\ \vec{\vec{w}} \end{Bmatrix}^T [E] \begin{Bmatrix} \vec{\vec{u}} \\ \vec{\vec{v}} \\ \vec{\vec{w}} \end{Bmatrix} + \begin{Bmatrix} \vec{\vec{u}} + \vec{\vec{v}} \\ \vec{\vec{v}} + \vec{\vec{w}} \\ \vec{\vec{u}} + \vec{\vec{w}} \end{Bmatrix}^T [G] \begin{Bmatrix} \vec{\vec{u}} + \vec{\vec{v}} \\ \vec{\vec{v}} + \vec{\vec{w}} \\ \vec{\vec{u}} + \vec{\vec{w}} \end{Bmatrix}$$

$$\text{Isotropic: } [E] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix}, \quad [G] = \begin{bmatrix} G & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix}$$

$$\text{Plane stress: } [E] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [G] = \begin{bmatrix} G & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{Beam: } [E] = \begin{bmatrix} E & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [G] = \begin{bmatrix} G & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix}$$

$$\text{Plate: } [E] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [G] = \begin{bmatrix} G & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix},$$

$$[E]_\sigma = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}, \quad [E]_\sigma^{-1} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix}, \quad D = \frac{t^3 E}{12(1-\nu^2)}$$

## PRINCIPLE OF VIRTUAL WORK

$$\delta W = \delta W^{\text{ext}} + \delta W^{\text{int}} = 0 \quad \forall \delta \vec{u} \in U \quad (\text{a function set})$$

$$\delta W = -\int_V (\vec{\sigma} : \delta \vec{\epsilon}_c) dV + \int_V (\vec{f} \cdot \delta \vec{u}) dV + \int_A (\vec{t} \cdot \delta \vec{u}) dA$$

## BEAM EQUATIONS

$$\left\{ \begin{array}{l} \frac{d\vec{F}}{ds} + \vec{b} \\ \frac{d\vec{M}}{ds} + \vec{e}_s \times \vec{F} + \vec{c} \end{array} \right\} = 0, \quad \left\{ \begin{array}{l} \vec{b} \\ \vec{c} \end{array} \right\} = \int \left\{ \begin{array}{l} \vec{f} \\ \vec{\rho} \times \vec{f} \end{array} \right\} J dA, \quad \left\{ \begin{array}{l} \vec{F} \\ \vec{M} \end{array} \right\} = \int \left\{ \begin{array}{l} \vec{t} \\ \vec{\rho} \times \vec{t} \end{array} \right\} J dA, \quad \text{where } J = 1 + n\kappa$$

$$\left\{ \begin{array}{l} \vec{F} \\ \vec{M} \end{array} \right\} = \int \left\{ \begin{array}{l} \vec{\sigma} \\ \vec{\rho} \times \vec{\sigma} \end{array} \right\} dA = \int \left[ \begin{array}{cc} \vec{E} & -\vec{E} \times \vec{\rho} \\ \vec{\rho} \times \vec{E} & -\vec{\rho} \times \vec{E} \times \vec{\rho} \end{array} \right] dA \cdot \left\{ \begin{array}{l} \frac{d\vec{u}_0}{ds} + \vec{e}_s \times \vec{\theta}_0 \\ \frac{d\vec{\theta}_0}{ds} \end{array} \right\}, \quad \text{where } \vec{E} = \vec{e}_s \cdot \ddot{\vec{E}} \cdot \vec{e}_s$$

## TIMOSHENKO BEAM $(x, y, z)$

$$\vec{u}_0 = u\vec{i} + v\vec{j} + w\vec{k}, \quad \vec{\theta}_0 = \phi\vec{i} + \theta\vec{j} + \psi\vec{k}, \quad \vec{F} = N\vec{i} + Q_y\vec{j} + Q_z\vec{k}, \quad \vec{M} = T\vec{i} + M_y\vec{j} + M_z\vec{k}$$

$$\left\{ \begin{array}{l} \frac{dN}{dx} + b_x \\ \frac{dQ_y}{dx} + b_y \\ \frac{dQ_z}{dx} + b_z \end{array} \right\} = 0, \quad \left\{ \begin{array}{l} N \\ Q_y \\ Q_z \end{array} \right\} = \left\{ \begin{array}{l} EA \frac{du}{dx} - ES_z \frac{d\psi}{dx} + ES_y \frac{d\theta}{dx} \\ GA \left( \frac{dv}{dx} - \psi \right) - GS_y \frac{d\phi}{dx} \\ GA \left( \frac{dw}{dx} + \theta \right) + GS_z \frac{d\phi}{dx} \end{array} \right\}$$

$$\left\{ \begin{array}{l} \frac{dT}{dx} + c_x \\ \frac{dM_y}{dx} - Q_z + c_y \\ \frac{dM_z}{dx} + Q_y + c_z \end{array} \right\} = 0, \quad \left\{ \begin{array}{l} T \\ M_y \\ M_z \end{array} \right\} = \left\{ \begin{array}{l} -GS_y \left( \frac{dv}{dx} - \psi \right) + GS_z \left( \frac{dw}{dx} + \theta \right) + GI_{rr} \frac{d\phi}{dx} \\ ES_y \frac{du}{dx} - EI_{zy} \frac{d\psi}{dx} + EI_{yy} \frac{d\theta}{dx} \\ -ES_z \frac{du}{dx} + EI_{zz} \frac{d\psi}{dx} - EI_{yz} \frac{d\theta}{dx} \end{array} \right\}$$

## TIMOSHENKO BEAM $(s, n, b)$

$$\vec{u} = u\vec{e}_s + v\vec{e}_n + w\vec{e}_b, \quad \vec{\theta} = \phi\vec{e}_s + \theta\vec{e}_n + \psi\vec{e}_b, \quad \vec{F} = N\vec{e}_s + Q_n\vec{e}_n + Q_b\vec{e}_b, \quad \vec{M} = T\vec{e}_s + M_n\vec{e}_n + M_b\vec{e}_b$$

$$\left\{ \begin{array}{l} \frac{dN}{ds} - Q_n\kappa + b_s \\ \frac{dQ_n}{ds} + N\kappa - Q_b\tau + b_n \\ \frac{dQ_b}{ds} + Q_n\tau + b_b \end{array} \right\} = 0, \quad \left\{ \begin{array}{l} \frac{dT}{ds} - M_n\kappa + c_s \\ \frac{dM_n}{ds} + T\kappa - M_b\tau - Q_b + c_n \\ \frac{dM_b}{ds} + M_n\tau + Q_n + c_b \end{array} \right\} = 0$$

$$\begin{Bmatrix} N \\ Q_n \\ Q_b \end{Bmatrix} = \begin{Bmatrix} EA\left(\frac{du}{ds} - v\kappa\right) + ES_n\left(\frac{d\theta}{ds} + \phi\kappa - \psi\tau\right) - ES_b\left(\frac{d\psi}{ds} + \theta\tau\right) \\ GA\left(\frac{dv}{ds} + u\kappa - w\tau - \psi\right) - GS_n\left(\frac{d\phi}{ds} - \theta\kappa\right) \\ GA\left(\frac{dw}{ds} + v\tau + \theta\right) + GS_b\left(\frac{d\phi}{ds} - \theta\kappa\right) \end{Bmatrix}$$

$$\begin{Bmatrix} T \\ M_n \\ M_b \end{Bmatrix} = \begin{Bmatrix} GS_b\left(\frac{dw}{ds} + v\tau + \theta\right) + GI_{rr}\left(\frac{d\phi}{ds} - \theta\kappa\right) - GS_n\left(\frac{dv}{ds} + u\kappa - w\tau - \psi\right) \\ ES_n\left(\frac{du}{ds} - v\kappa\right) + EI_{nn}\left(\frac{d\theta}{ds} + \phi\kappa - \psi\tau\right) - EI_{bn}\left(\frac{d\psi}{ds} + \theta\tau\right) \\ -ES_b\left(\frac{du}{ds} - v\kappa\right) - EI_{nb}\left(\frac{d\theta}{ds} + \phi\kappa - \psi\tau\right) + EI_{bb}\left(\frac{d\psi}{ds} + \theta\tau\right) \end{Bmatrix}$$

## PLATE EQUATIONS

$$\nabla \cdot \vec{F} + \vec{b} = 0, \quad \nabla \cdot \vec{M} - \vec{e}_n \cdot \vec{F} + \vec{c} = 0,$$

$$\begin{Bmatrix} \vec{F} \\ \vec{M} \end{Bmatrix} = \int \vec{\sigma} \begin{Bmatrix} 1 \\ n \end{Bmatrix} dn = \int \begin{bmatrix} 1 & n \\ n & n^2 \end{bmatrix} \vec{\bar{E}} dn : \begin{Bmatrix} \nabla_0 \vec{u}_0 + \vec{e}_n \vec{\omega}_0 \\ \nabla_0 \vec{\omega}_0 \end{Bmatrix}, \quad \vec{\omega}_0 = \vec{\theta}_0 \times \vec{e}_n$$

## REISSNER-MINDLIN PLATE $(x, y, n)$

$$\vec{F} = \int \vec{\sigma} dn = \vec{ii} N_{xx} + (\vec{ij} + \vec{ji}) N_{xy} + \vec{jj} N_{yy} + (\vec{e}_n \vec{i} + \vec{i} \vec{e}_n) Q_x + (\vec{e}_n \vec{j} + \vec{j} \vec{e}_n) Q_y$$

$$\vec{M} = \int \vec{\sigma} n dn = \vec{ii} M_{xx} + (\vec{ij} + \vec{ji}) M_{xy} + \vec{jj} M_{yy}$$

$$\begin{Bmatrix} \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} + b_x \\ \frac{\partial N_{yy}}{\partial y} + \frac{\partial N_{xy}}{\partial x} + b_y \end{Bmatrix} = 0, \quad \begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = t[E]_{\sigma} \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix}$$

$$\begin{Bmatrix} \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + b_n \\ \frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x \\ \frac{\partial M_{yy}}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y \end{Bmatrix} = 0, \quad \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \frac{t^3}{12} [E]_{\sigma} \begin{Bmatrix} \frac{\partial \theta}{\partial x} \\ -\frac{\partial \phi}{\partial y} \\ \frac{\partial \theta}{\partial y} - \frac{\partial \phi}{\partial x} \end{Bmatrix}, \quad \begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = Gt \begin{Bmatrix} \frac{\partial w}{\partial x} + \theta \\ \frac{\partial w}{\partial y} - \phi \end{Bmatrix}$$

## REISSNER-MINDLIN PLATE $(r, \phi, n)$

$$\left\{ \begin{array}{l} \frac{1}{r} \left[ \frac{\partial(rN_{rr})}{\partial r} + \frac{\partial N_{r\phi}}{\partial \phi} - N_{\phi\phi} \right] + b_r \\ \frac{1}{r} \left[ \frac{1}{r} \frac{\partial(r^2 N_{r\phi})}{\partial r} + \frac{\partial N_{\phi\phi}}{\partial \phi} \right] + b_\phi \end{array} \right\} = 0, \quad \left\{ \begin{array}{l} N_{rr} \\ N_{\phi\phi} \\ N_{r\phi} \end{array} \right\} = t[E]_\sigma \left\{ \begin{array}{l} \frac{\partial u_r}{\partial r} \\ \frac{1}{r} (u_r + \frac{\partial u_\phi}{\partial \phi}) \\ \frac{1}{r} \frac{\partial u_r}{\partial \phi} + r \frac{\partial}{\partial r} \left( \frac{u_\phi}{r} \right) \end{array} \right\}$$

$$\left\{ \begin{array}{l} \frac{1}{r} \left[ \frac{\partial(rQ_r)}{\partial r} + \frac{\partial Q_\phi}{\partial \phi} \right] + b_n \\ \frac{1}{r} \left[ \frac{\partial(rM_{rr})}{\partial r} + \frac{\partial M_{r\phi}}{\partial \phi} - M_{\phi\phi} \right] - Q_r \\ \frac{1}{r} \left[ \frac{\partial(rM_{r\phi})}{\partial r} + \frac{\partial M_{\phi\phi}}{\partial \phi} + M_{r\phi} \right] - Q_\phi \end{array} \right\} = 0, \quad \left\{ \begin{array}{l} M_{rr} \\ M_{\phi\phi} \\ M_{r\phi} \end{array} \right\} = \frac{t^3}{12} [E]_\sigma \left\{ \begin{array}{l} \frac{\partial \theta_\phi}{\partial r} \\ \frac{1}{r} (\theta_\phi - \frac{\partial \theta_r}{\partial \phi}) \\ \frac{1}{r} (\frac{\partial \theta_\phi}{\partial \phi} + \theta_r) - \frac{\partial \theta_r}{\partial r} \end{array} \right\}$$

$$\left\{ \begin{array}{l} Q_r \\ Q_\phi \end{array} \right\} = Gt \left\{ \begin{array}{l} \frac{\partial w}{\partial r} + \theta_\phi \\ \frac{1}{r} \frac{\partial w}{\partial \phi} - \theta_r \end{array} \right\}$$

### KIRCHHOFF PLATE BENDING $(r, \phi, n)$

$$\nabla_0^2 \nabla_0^2 w - \frac{b_n}{D} = 0, \quad w(r) = \frac{b_n}{D} \frac{r^4}{64} + a + br^2 + cr^2(1 - \log r) + d \log r$$

### MEMBRANE EQUATIONS IN CYLINDRICAL GEOMETRY $(z, \phi, n)$

$$\left\{ \begin{array}{l} \frac{1}{R} \frac{\partial N_{z\phi}}{\partial \phi} + \frac{\partial N_{zz}}{\partial z} + b_z \\ \frac{\partial N_{z\phi}}{\partial z} + \frac{1}{R} \frac{\partial N_{\phi\phi}}{\partial \phi} + b_\phi \\ \frac{1}{R} N_{\phi\phi} + b_n \end{array} \right\} = 0, \quad \left\{ \begin{array}{l} N_{zz} \\ N_{\phi\phi} \\ N_{z\phi} \end{array} \right\} = t[E]_\sigma \left\{ \begin{array}{l} \frac{\partial u_z}{\partial z} \\ \frac{1}{R} (\frac{\partial u_\phi}{\partial \phi} - u_n) \\ \frac{1}{R} \frac{\partial u_z}{\partial \phi} + \frac{\partial u_\phi}{\partial z} \end{array} \right\}$$

### MEMBRANE EQUATIONS IN SPHERICAL GEOMETRY $(\phi, \theta, n)$

$$\left\{ \begin{array}{l} \frac{1}{R} (\csc \theta \frac{\partial N_{\phi\phi}}{\partial \phi} + \frac{\partial N_{\phi\theta}}{\partial \theta} + 2 \cot \theta N_{\phi\theta}) + b_\phi \\ \frac{1}{R} [\csc \theta \frac{\partial N_{\phi\theta}}{\partial \phi} + \frac{\partial N_{\theta\theta}}{\partial \theta} + \cot \theta (N_{\theta\theta} - N_{\phi\phi})] + b_\theta \\ \frac{1}{R} (N_{\phi\phi} + N_{\theta\theta}) + b_n \end{array} \right\} = 0$$

$$\begin{Bmatrix} N_{\phi\phi} \\ N_{\theta\theta} \\ N_{\phi\theta} \end{Bmatrix} = t[E]_{\sigma} \begin{Bmatrix} \frac{1}{R} [\csc \theta (\cos \theta u_{\theta} + \frac{\partial u_{\phi}}{\partial \phi}) - u_n] \\ \frac{1}{R} (\csc \theta \sin \theta \frac{\partial u_{\theta}}{\partial \theta} - u_n) \\ \frac{1}{R} (\csc \theta \frac{\partial u_{\theta}}{\partial \phi} - \cot \theta u_{\phi} + \frac{\partial u_{\phi}}{\partial \theta}) \end{Bmatrix} \quad (\csc \theta = \frac{1}{\sin \theta})$$

### SHELL EQUATIONS IN CYLINDRICAL GEOMETRY $(z, \phi, n)$

$$\begin{Bmatrix} \frac{1}{R} \frac{\partial N_{\phi z}}{\partial \phi} + \frac{\partial N_{zz}}{\partial z} + b_z \\ \frac{\partial N_{z\phi}}{\partial z} + \frac{1}{R} \frac{\partial N_{\phi\phi}}{\partial \phi} - \frac{1}{R} Q_{\phi} + b_{\phi} \end{Bmatrix} = 0, \quad \begin{Bmatrix} \frac{1}{R} \frac{\partial Q_{\phi}}{\partial \phi} + \frac{\partial Q_z}{\partial z} + \frac{1}{R} N_{\phi\phi} + b_n \\ \frac{\partial M_{z\phi}}{\partial z} + \frac{1}{R} \frac{\partial M_{\phi\phi}}{\partial \phi} - \frac{1}{R} M_{\phi n} - Q_{\phi} + c_{\phi} \\ \frac{\partial M_{zz}}{\partial z} + \frac{1}{R} \frac{\partial M_{\phi z}}{\partial \phi} - Q_z + c_z \end{Bmatrix} = 0$$

$$\begin{Bmatrix} N_{zz} \\ N_{\phi\phi} \\ N_{z\phi} \\ N_{\phi z} \end{Bmatrix} = \begin{Bmatrix} \frac{tE}{1-\nu^2} [\frac{\partial u_z}{\partial z} + \nu \frac{1}{R} (\frac{\partial u_{\phi}}{\partial \phi} - u_n)] - D \frac{1}{R} \frac{\partial \theta_{\phi}}{\partial z} \\ \frac{tE}{1-\nu^2} [\frac{1}{R} (\frac{\partial u_{\phi}}{\partial \phi} - u_n) + \nu \frac{\partial u_z}{\partial z}] - D \frac{1}{R^2} \frac{\partial \theta_z}{\partial \phi} \\ Gt (\frac{1}{R} \frac{\partial u_z}{\partial \phi} + \frac{\partial u_{\phi}}{\partial z}) + \frac{1}{2} (1-\nu) D \frac{1}{R} \frac{\partial \theta_z}{\partial z} \\ Gt (\frac{1}{R} \frac{\partial u_z}{\partial \phi} + \frac{\partial u_{\phi}}{\partial z}) + \frac{1}{2} (1-\nu) D \frac{1}{R^2} \frac{\partial \theta_{\phi}}{\partial \phi} \end{Bmatrix}, \quad \begin{Bmatrix} Q_z \\ Q_{\phi} \end{Bmatrix} = tG \begin{Bmatrix} \frac{\partial u_n}{\partial z} + \theta_{\phi} \\ \frac{1}{R} (\frac{\partial u_n}{\partial \phi} + u_{\phi}) - \theta_z \end{Bmatrix}$$

$$\begin{Bmatrix} M_{zz} \\ M_{\phi\phi} \\ M_{z\phi} \\ M_{\phi z} \end{Bmatrix} = D \begin{Bmatrix} \frac{\partial \theta_{\phi}}{\partial z} - \nu \frac{1}{R} \frac{\partial \theta_z}{\partial \phi} - \frac{1}{R} \frac{\partial u_z}{\partial z} \\ \nu \frac{\partial \theta_{\phi}}{\partial z} - \frac{1}{R} \frac{\partial \theta_z}{\partial \phi} + \frac{1}{R^2} (\frac{\partial u_{\phi}}{\partial \phi} - u_n) \\ \frac{1}{2} (1-\nu) [(\frac{1}{R} \frac{\partial \theta_{\phi}}{\partial \phi} - \frac{\partial \theta_z}{\partial z}) - \frac{1}{R} \frac{\partial u_{\phi}}{\partial z}] \\ \frac{1}{2} (1-\nu) [(\frac{1}{R} \frac{\partial \theta_{\phi}}{\partial \phi} - \frac{\partial \theta_z}{\partial z}) + \frac{1}{R^2} \frac{\partial u_z}{\partial \phi}] \end{Bmatrix}, \quad M_{\phi n} = \frac{1}{2} (1-\nu) D \frac{1}{R} [\frac{1}{R} (\frac{\partial u_n}{\partial \phi} + u_{\phi}) - \theta_z]$$

### SHELL EQUATIONS IN SPHERICAL GEOMETRY $(\phi, \theta, n)$

$$\begin{Bmatrix} \frac{1}{R} (\frac{\partial}{\partial \theta} N_{\phi\theta} + \csc \theta \frac{\partial}{\partial \phi} N_{\phi\phi} + 2 \cot \theta N_{\phi\theta} - Q_{\phi}) + b_{\phi} \\ \frac{1}{R} (\frac{\partial}{\partial \theta} N_{\theta\theta} + \csc \theta \frac{\partial}{\partial \phi} N_{\phi\theta} + \cot \theta N_{\theta\theta} - \cot \theta N_{\phi\phi} - Q_{\theta}) + b_{\theta} \end{Bmatrix} = 0,$$



$$\left\{ \begin{array}{l} \frac{1}{R} \left( \frac{\partial}{\partial \theta} Q_\theta + \csc \theta \frac{\partial}{\partial \phi} Q_\phi + \cot \theta Q_\theta + N_{\theta\theta} + N_{\phi\phi} \right) + b_n \\ \frac{1}{R} \left( \frac{\partial}{\partial \theta} M_{\phi\theta} + \csc \theta \frac{\partial}{\partial \phi} M_{\phi\phi} + 2 \cot \theta M_{\phi\theta} \right) - Q_\phi + c_\phi \\ \frac{1}{R} \left( \frac{\partial}{\partial \theta} M_{\theta\theta} + \csc \theta \frac{\partial}{\partial \phi} M_{\phi\theta} + \cot \theta M_{\theta\theta} - \cot \theta M_{\phi\phi} \right) - Q_\theta + c_\theta \end{array} \right\} = 0,$$

$$\left\{ \begin{array}{l} N_{\phi\phi} \\ N_{\theta\theta} \\ N_{\phi\theta} \end{array} \right\} = \frac{Et}{1-\nu^2} \frac{1}{R} \left\{ \begin{array}{l} (u_\theta \cot \theta + \frac{\partial u_\phi}{\partial \phi} \csc \theta - u_n) + \nu \left( \frac{\partial u_\theta}{\partial \theta} - u_n \right) \\ \nu (u_\theta \cot \theta + \frac{\partial u_\phi}{\partial \phi} \csc \theta - u_n) + \left( \frac{\partial u_\theta}{\partial \theta} - u_n \right) \\ \frac{1-\nu}{2} (-u_\phi \cot \theta + \frac{\partial u_\theta}{\partial \phi} \csc \theta + \frac{\partial u_\phi}{\partial \theta}) \end{array} \right\},$$

$$\left\{ \begin{array}{l} M_{\phi\phi} \\ M_{\theta\theta} \\ M_{\phi\theta} \end{array} \right\} = D \frac{1}{R} \left\{ \begin{array}{l} -\theta_\phi \cot \theta + \frac{\partial \theta_\theta}{\partial \phi} \csc \theta - \nu \frac{\partial \theta_\phi}{\partial \theta} \\ \nu (-\theta_\phi \cot \theta + \frac{\partial \theta_\theta}{\partial \phi} \csc \theta) - \frac{\partial \theta_\phi}{\partial \theta} \\ \frac{1-\nu}{2} \left( \frac{\partial \theta_\theta}{\partial \theta} - \theta_\theta \cot \theta - \frac{\partial \theta_\phi}{\partial \phi} \csc \theta \right) \end{array} \right\}, \quad \left\{ \begin{array}{l} Q_\phi \\ Q_\theta \end{array} \right\} = tG \left\{ \begin{array}{l} \theta_\theta + \frac{1}{R} (u_\phi + \frac{\partial u_n}{\partial \phi} \csc \theta) \\ -\theta_\phi + \frac{1}{R} (u_\theta + \frac{\partial u_n}{\partial \theta}) \end{array} \right\}$$