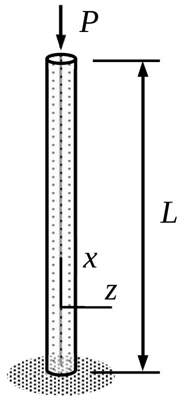
MEC-E8003 Beam, Plate and Shell models, exam 03.06.2024

1. Derive the expressions of linear strain components ε_{rr}, $\varepsilon_{r\phi}$, $\varepsilon_{\phi r}$ and $\varepsilon_{\phi\phi}$ of the polar coordinate system. Use the displacement representation $\vec{u} = u_r \vec{e}_r + u_\phi \vec{e}_\phi$ where the components depend on the polar coordinates r and ϕ. Use definitions

$$\vec{\varepsilon} = \frac{1}{2}[\nabla\vec{u} + (\nabla\vec{u})_{\mathrm{c}}],\ \nabla = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{\partial}{r\partial\phi},\ \frac{\partial}{\partial\phi}\begin{Bmatrix}\vec{e}_r \\ \vec{e}_\phi\end{Bmatrix} = \begin{Bmatrix}\vec{e}_\phi \\ -\vec{e}_r\end{Bmatrix},\ \frac{\partial}{\partial r}\begin{Bmatrix}\vec{e}_r \\ \vec{e}_\phi\end{Bmatrix} = 0.$$

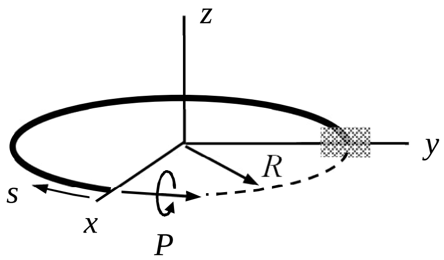
2. When displacement is confined to the $xz-$plane, the virtual work expression of a slender Bernoulli beam (figure) is given by



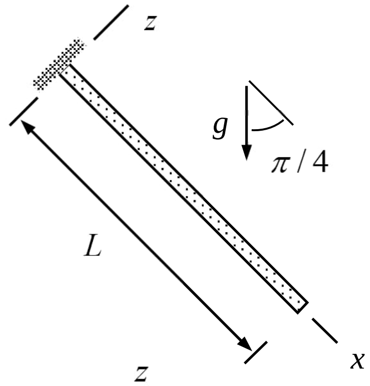
$$\delta W = -\int_0^L \left(\frac{d^2\delta w}{dx^2} EI \frac{d^2 w}{dx^2}\right)dx + P\int_0^L \left(\frac{d\delta w}{dx}\frac{dw}{dx}\right)dx.$$

Deduce in detail the underlying differential equation and boundary conditions implied by the principle of virtual work and the fundamental lemma of variation calculus.

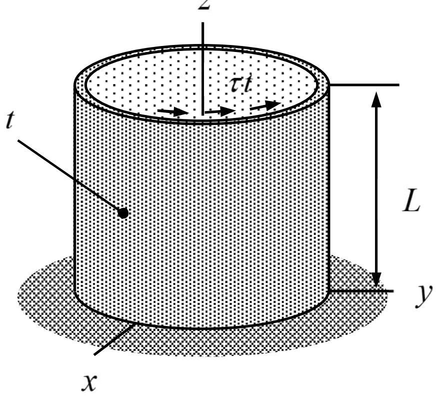
3. Consider a curved beam forming ¾ of a full circle of radius R in the horizontal plane. The given torque of magnitude P is acting on the free end as shown. Write down the equilibrium equations and boundary conditions for the *stress resultants* and solve the equations for $N(s)$, $Q_n(s)$, $Q_b(s)$, $T(s)$, $M_n(s)$, and $M_b(s)$.



4. Consider a cantilever Reissner-Mindlin plate strip (long in the $y-$direction) loaded by its own weight. Assuming that the solution is independent of y, determine the first order ordinary differential equations and the boundary conditions giving $N_{xx} = N(x)$, $Q_x = Q(x)$, $M_{xx} = M(x)$, $u(x)$, $w(x)$ and $\theta(x)$ as solutions. Thickness of the plate t, density ρ, Young's modulus E, and Poisson's ratio ν are constants.



5. A thin walled cylindrical body of length L, (mid-surface) radius R, and thickness t is subjected to shear loading τt $[\tau t] = \mathrm{N/m}$ at the free end $z = L$ as shown in the figure. Assuming rotation symmetry, use the membrane equations in (z, ϕ, n) coordinate system to derive the relationship between the moment resultant T (in the direction of z-axis) of the shear loading and the angle of rotation of the free end defined by $\theta = u_\phi / R$.



Derive the expressions of linear strain components ε_{rr}, $\varepsilon_{r\phi}$, $\varepsilon_{\phi r}$ and $\varepsilon_{\phi\phi}$ of the polar coordinate system. Use the displacement representation $\vec{u} = u_r \vec{e}_r + u_\phi \vec{e}_\phi$ where the components depend on the polar coordinates r and ϕ. Use definitions

$$\vec{\varepsilon} = \frac{1}{2}[\nabla\vec{u} + (\nabla\vec{u})_{\mathrm{c}}], \; \nabla = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{\partial}{r\partial\phi}, \; \frac{\partial}{\partial\phi}\begin{Bmatrix}\vec{e}_r \\ \vec{e}_\phi\end{Bmatrix} = \begin{Bmatrix}\vec{e}_\phi \\ -\vec{e}_r\end{Bmatrix}, \; \frac{\partial}{\partial r}\begin{Bmatrix}\vec{e}_r \\ \vec{e}_\phi\end{Bmatrix} = 0 .$$

Solution

In manipulation of vector expression containing vectors and tensors, it is important to remember that tensor ($\otimes$), cross ($\times$), inner ($\cdot$) products are non-commutative (order may matter). The basis vectors of a curvilinear coordinate system are not constants which should be taken into account if gradient operator is a part of expression. Otherwise, simplifying an expression or finding a specific form in a given coordinate system is a straightforward (sometimes tedious) exercise. For simplicity of presentation, outer (tensor) products like $\vec{a} \otimes \vec{b}$ are denoted by $\vec{a}\vec{b}$. Otherwise, the usual rules of algebra apply: Gradient operator ∇ acts on everything on its right hand side, the operator is treated like a vector etc.

Let us start with the gradient of displacement (an outer product). Substitute first the representations in the polar coordinate system

$$\nabla\vec{u} = (\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{\partial}{r\partial\phi})(u_r \vec{e}_r + u_\phi \vec{e}_\phi) .$$

4p Then expand to have a term-by-term representation. Keep the order of the basis vectors and the position of derivatives

$$\nabla\vec{u} = \vec{e}_r \frac{\partial}{\partial r}(u_r \vec{e}_r) + \vec{e}_r \frac{\partial}{\partial r}(u_\phi \vec{e}_\phi) + \vec{e}_\phi \frac{\partial}{r\partial\phi}(u_r \vec{e}_r) + \vec{e}_\phi \frac{\partial}{r\partial\phi}(u_\phi \vec{e}_\phi)$$

Use the derivative rule of products. Notice that the basis vectors are not constants and may have non-zero derivatives

$$\nabla\vec{u} = \vec{e}_r (\frac{\partial u_r}{\partial r}\vec{e}_r + u_r \frac{\partial \vec{e}_r}{\partial r}) + \vec{e}_r (\frac{\partial u_\phi}{\partial r}\vec{e}_\phi + u_\phi \frac{\partial \vec{e}_\phi}{\partial r}) + \vec{e}_\phi (\frac{\partial u_r}{r\partial\phi}\vec{e}_r + u_r \frac{\partial \vec{e}_r}{r\partial\phi}) + \vec{e}_\phi (\frac{\partial u_\phi}{r\partial\phi}\vec{e}_\phi + u_\phi \frac{\partial \vec{e}_\phi}{r\partial\phi}) .$$

Substitute the derivatives of the basis vectors

$$\nabla\vec{u} = \vec{e}_r (\frac{\partial u_r}{\partial r}\vec{e}_r) + \vec{e}_r (\frac{\partial u_\phi}{\partial r}\vec{e}_\phi) + \vec{e}_\phi (\frac{\partial u_r}{r\partial\phi}\vec{e}_r + \frac{u_r}{r}\vec{e}_\phi) + \vec{e}_\phi (\frac{\partial u_\phi}{r\partial\phi}\vec{e}_\phi - \frac{u_\phi}{r}\vec{e}_r) .$$

Combine the terms having the same pair of basis vectors (order matters so terms containing $\vec{e}_\phi\vec{e}_r$ and $\vec{e}_r\vec{e}_\phi$ cannot be combined)

$$\nabla\vec{u} = \frac{\partial u_r}{\partial r}\vec{e}_r\vec{e}_r + \frac{\partial u_\phi}{\partial r}\vec{e}_r\vec{e}_\phi + (\frac{\partial u_r}{r\partial\phi} - \frac{u_\phi}{r})\vec{e}_\phi\vec{e}_r + (\frac{u_r}{r} + \frac{\partial u_\phi}{r\partial\phi})\vec{e}_\phi\vec{e}_\phi .$$

Conjugate of a second order tensor can be obtained by swapping the basis vectors in all the pairs. Conjugate is a kind of transpose and can also be obtained by transposing the matrix of the component representation.

$$(\nabla\vec{u})_{\mathrm{c}} = \frac{\partial u_r}{\partial r}\vec{e}_r\vec{e}_r + (\frac{\partial u_r}{r\partial\phi} - \frac{u_\phi}{r})\vec{e}_r\vec{e}_\phi + \frac{\partial u_\phi}{\partial r}\vec{e}_\phi\vec{e}_r + (\frac{u_r}{r} + \frac{\partial u_\phi}{r\partial\phi})\vec{e}_\phi\vec{e}_\phi$$

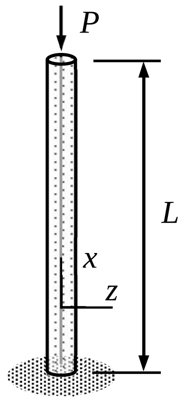
2p Finally using the definition $\vec{\vec{\varepsilon}} = \frac{1}{2}[\nabla\vec{u} + (\nabla\vec{u})_{\mathrm{c}}]$

$$\vec{\vec{\varepsilon}} = \frac{\partial u_r}{\partial r}\vec{e}_r\vec{e}_r + \frac{1}{2}(\frac{\partial u_r}{r\partial\phi} - \frac{u_\phi}{r} + \frac{\partial u_\phi}{\partial r})(\vec{e}_r\vec{e}_\phi + \vec{e}_\phi\vec{e}_r) + (\frac{\partial u_\phi}{r\partial\phi} + \frac{u_r}{r})\vec{e}_\phi\vec{e}_\phi \, .$$

In the components of strain ε_{rr}, $\varepsilon_{r\phi}$, $\varepsilon_{\phi r}$ and $\varepsilon_{\phi\phi}$, indices are in the same order as the indices in the basis vector pairs. Hence

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\phi\phi} = \frac{\partial u_\phi}{r\partial\phi} + \frac{u_r}{r}, \quad \varepsilon_{r\phi} = \varepsilon_{\phi r} = \frac{1}{2}(\frac{\partial u_r}{r\partial\phi} - \frac{u_\phi}{r} + \frac{\partial u_\phi}{\partial r}) \, .$$

When displacement is confined to the $xz-$plane, the virtual work expression of a slender Bernoulli beam (figure) is given by



$$\delta W=-\int_0^L \left(\frac{d^2\delta w}{dx^2}EI\frac{d^2 w}{dx^2}\right)dx+P\int_0^L \left(\frac{d\delta w}{dx}\frac{dw}{dx}\right)dx\,.$$

Deduce in detail the underlying differential equation and boundary conditions implied by the principle of virtual work and the fundamental lemma of variation calculus.

Solution

2p Integration by parts gives an equivalent but a more convenient form (assuming continuity up to and including second derivatives)

$$\delta W=-\int_0^L \left(\frac{d^2\delta w}{dx^2}EI\frac{d^2 w}{dx^2}\right)dx+P\int_0^L \left(\frac{d\delta w}{dx}\frac{dw}{dx}\right)dx \quad \Leftrightarrow \quad (P \text{ is a constant})$$

$$\delta W=-\int_0^L \delta w\left(EI\frac{d^4 w}{dx^4}+P\frac{d^2 w}{dx^2}\right)dx+\sum_{\{0,L\}} n\delta w\left(EI\frac{d^3 w}{dx^3}+P\frac{dw}{dx}\right)-\sum_{\{0,L\}} n\frac{d\delta w}{dx}\left(EI\frac{d^2 w}{dx^2}\right).$$

2p According to principle of virtual work $\delta W=0\ \forall\delta w$. Let us consider first the subset of variations for which $\delta w=0$ and $d\delta w/dx=0$ on $\{0,L\}$. The fundamental lemma of variation calculus implies

$$EI\frac{d^4 w}{dx^4}+P\frac{d^2 w}{dx^2}=0 \quad \text{in} \quad (0,L).$$

Let us consider then the subset of variations for which $d\delta w/dx=0$ on $\{0,L\}$. Knowing the condition above, the fundamental lemma of variation calculus implies

$$EI\frac{d^3 w}{dx^3}+P\frac{dw}{dx}=0 \quad \text{or} \quad w-\underline{w}=0 \quad \text{on} \quad \{0,L\}.$$

Finally, let us consider the subset of variations for which $\delta w=0$ on $\{0,L\}$. Knowing the previous results, the fundamental lemma of variation calculus implies

$$EI\frac{d^2 w}{dx^2}=0 \quad \text{or} \quad \frac{dw}{dx}-\underline{\theta}=0 \quad \text{on} \quad \{0,L\}.$$

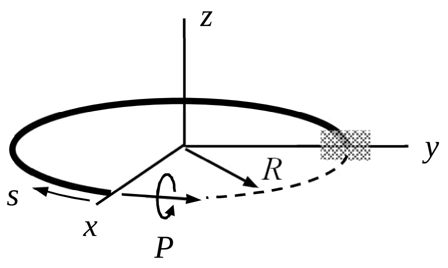
2p For the problem of the figure, one obtains

$$EI\frac{d^4 w}{dx^4}+P\frac{d^2 w}{dx^2}=0 \quad \text{in} \quad (0,L), \quad \Leftarrow$$

$$EI\frac{d^3 w}{dx^3}+P\frac{dw}{dx}=0 \quad \text{and} \quad EI\frac{d^2 w}{dx^2}=0 \quad \text{at } x=L, \quad \Leftarrow$$

$$w = 0 \quad \text{and} \quad \frac{dw}{dx} = 0 \quad \text{at} \quad x = 0 \,.$$

Consider a curved beam forming ¾ of a full circle of radius R in the horizontal plane. Torque of magnitude P is acting on the free end as shown. Write down the boundary value problem for *stress resultants* and solve the equations for $N(s)$, $Q_n(s)$, $Q_b(s)$, $T(s)$, $M_n(s)$, and $M_b(s)$.



Solution

3p In the geometry of the figure $\tau = 0$, $\kappa = 1/R$. External distributed forces and moments vanish. Therefore, the curved beam equilibrium equations of the formulae collection simplify to

$$\begin{Bmatrix} N' - Q_n / R \\ Q_n' + N / R \\ Q_b' \end{Bmatrix} = 0 \quad \text{and} \quad \begin{Bmatrix} T' - M_n / R \\ M_n' + T / R - Q_b \\ M_b' + Q_n \end{Bmatrix} = 0 \quad s \in (0, L) \quad \text{where } L = \frac{3}{2}\pi R .$$

Boundary conditions at $s = 0$ are (notice the unit outward normal to the solution domain $n = -1$, $\vec{e}_s$ is pointing to the direction of s, and the component of the given moment on $\vec{e}_s$ is negative)

$$\begin{Bmatrix} N \\ Q_n \\ Q_b \end{Bmatrix} = 0 \quad \text{and} \quad \begin{Bmatrix} -T + P \\ M_n \\ M_b \end{Bmatrix} = 0 \quad s = 0 .$$

3p Solution to the boundary values problem for Q_b

$$Q_b' = 0 \quad s \in (0, L) \quad \text{and} \quad Q_b = 0 \quad s = 0 \quad \Rightarrow \quad Q_b(s) = 0 . \;\leftarrow$$

Solution to the connected boundary value problems for Q_n and N

$$N' - \frac{1}{R} Q_n = 0 , \; Q_n' + \frac{1}{R} N = 0 \quad s \in (0, L) , \qquad Q_n = 0 \text{ and } N = 0 \quad s = 0 \quad \Rightarrow$$

$$N'' + \frac{1}{R^2} N = 0 \quad s \in (0, L) \quad \text{and} \quad N = 0 , \;\; N' = 0 \;\; \text{at} \;\; s = 0 \;\; \Rightarrow$$

$$N(s) = 0 \quad \text{and} \quad Q_n(s) = 0 . \;\leftarrow$$

Solution to the boundary value problem for M_b

$$M_b' = 0 \quad s \in (0, L) \quad \text{and} \quad M_b = 0 \quad s = 0 \quad \Rightarrow \quad M_b(s) = 0 . \;\leftarrow$$

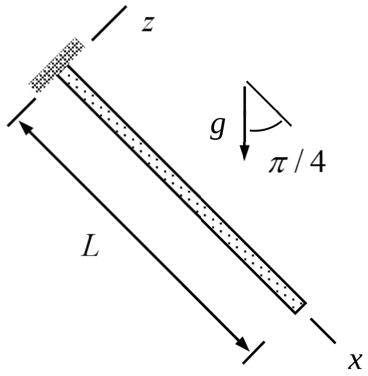
Solution to the connected boundary value problem for M_n and T

$$T' - \frac{1}{R} M_n = 0 \quad \text{and} \quad M_n' + \frac{1}{R} T = 0 \quad s \in (0, L) , \qquad T = P \quad \text{and} \quad M_n = 0 \quad s = 0 \quad \Rightarrow$$

$$RT'' + \frac{1}{R}T = 0 \quad s \in (0, L), \quad T = P \;\text{ and }\; T' = 0 \quad \Rightarrow$$

$$T(s) = P\cos\left(\frac{s}{R}\right) \;\text{ and }\; M_n(s) = -P\sin\left(\frac{s}{R}\right).$$

Consider a cantilever Reissner-Mindlin plate strip (long in the $y-$direction) loaded by its own weight. Assuming that the solution is independent of y, determine the first order ordinary differential equations and the boundary conditions giving $N_{xx} = N(x)$, $Q_x = Q(x)$, $M_{xx} = M(x)$, $u(x)$, $w(x)$ and $\theta(x)$ as solutions. Thickness of the plate t, density ρ, Young's modulus E, and Poisson's ratio ν are constants.



Solution

Equilibrium and constitutive equations of the thin-slab and bending modes are

$$\begin{Bmatrix} \dfrac{\partial N_{xx}}{\partial x} + \dfrac{\partial N_{xy}}{\partial y} + b_x \\ \dfrac{\partial N_{yy}}{\partial y} + \dfrac{\partial N_{xy}}{\partial x} + b_y \end{Bmatrix} = 0, \quad \begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = t[E]_\sigma \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} = \frac{tE}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix},$$

$$\begin{Bmatrix} \dfrac{\partial Q_x}{\partial x} + \dfrac{\partial Q_y}{\partial y} + b_n \\ \dfrac{\partial M_{xx}}{\partial x} + \dfrac{\partial M_{xy}}{\partial y} - Q_x \\ \dfrac{\partial M_{yy}}{\partial y} + \dfrac{\partial M_{xy}}{\partial x} - Q_y \end{Bmatrix} = 0, \quad \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \theta}{\partial x} \\ -\dfrac{\partial \phi}{\partial y} \\ \dfrac{\partial \theta}{\partial y} - \dfrac{\partial \phi}{\partial x} \end{Bmatrix}, \quad \begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = Gt \begin{Bmatrix} \dfrac{\partial w}{\partial x} + \theta \\ \dfrac{\partial w}{\partial y} - \phi \end{Bmatrix}.$$

4p Derivatives with respect to y vanish, $b_x = \rho g t / \sqrt{2}$, and $b_n = -\rho g t / \sqrt{2}$. The Reissner-Mindlin plate equations of the planar problem simplify to

$$\frac{dN}{dx} + \frac{\rho g t}{\sqrt{2}} = 0, \quad \frac{dQ}{dx} - \frac{\rho g t}{\sqrt{2}} = 0, \quad \frac{dM}{dx} - Q = 0 \quad \text{in} \quad (0, L), \quad \leftarrow$$

$$N = \frac{tE}{1-\nu^2}\frac{du}{dx}, \quad Q = Gt(\frac{dw}{dx} + \theta), \quad M = D\frac{d\theta}{dx} \quad \text{in} \quad (0, L), \quad \leftarrow$$

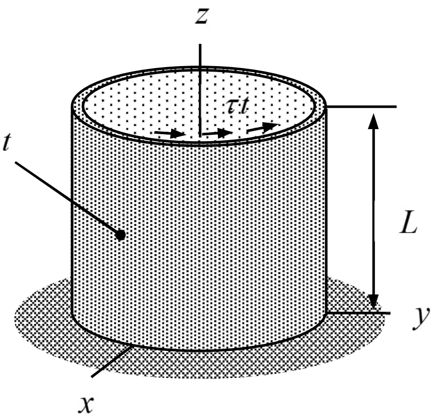
2p Boundary conditions can be deduced from the figure:

$$u = 0, \quad w = 0, \quad \theta = 0 \quad \text{at} \quad x = 0, \quad \leftarrow$$

$$N = 0, \quad M = 0, \quad Q = 0 \quad \text{at} \quad x = L. \quad \leftarrow$$

Solution to equations can be obtained by considering the equilibrium equations and the boundary conditions at the free end first. After that, solutions to the displacement components follow from the constitutive equations and the boundary conditions at the clamped edge.

A thin walled cylindrical body of length L, (mid-surface) radius R, and thickness t is subjected to shear loading τt $[\tau t] = \text{N/m}$ at the free end $z = L$ as shown in the figure. Assuming rotation symmetry, use the membrane equations in (z,ϕ,n) coordinate system to derive the relationship between the moment resultant T of the shear loading and the angle of rotation of the free end defined by $\theta = u_\phi / R$.



Solution

2p As the solution does not depend on ϕ, equilibrium equations of the membrane model and boundary conditions at the free end simplify to (a cylindrical membrane $z-$strip problem)

$$\frac{dN_{zz}}{dz} = 0\,, \quad \frac{dN_{z\phi}}{dz} = 0\,, \quad \frac{1}{R} N_{\phi\phi} = 0 \quad \text{in} \quad (0,L)\,,$$

$$N_{zz} = 0 \text{ and } N_{z\phi} = \tau t \quad \text{at} \quad z = L\,.$$

Solution to the boundary value problem for the stress resultants is given by

$$N_{zz} = N_{\phi\phi} = 0 \text{ and } N_{z\phi}(z) = \tau t\,. \leftarrow$$

2p Knowing the stress resultants, the boundary value problem for the displacement components follows from the constitutive equations and the boundary conditions at the fixed end (in the membrane model, a boundary condition cannot be assigned to u_n)

$$\frac{tE}{1-\nu^2}\left(\frac{du_z}{dz} - \nu \frac{1}{R} u_n\right) = 0\,, \quad \frac{tE}{1-\nu^2}\left(\nu \frac{du_z}{dz} - \frac{1}{R} u_n\right) = 0\,, \text{ and } tG \frac{du_\phi}{dz} = \tau t \quad \text{in} \quad (0,L)\,,$$

$$u_z = 0\,, \quad u_\phi = 0 \quad \text{at} \quad z = 0\,.$$

Solution to the boundary value problem is given by

$$u_z = u_n = 0 \text{ and } u_\phi(z) = \frac{\tau}{G} z\,.$$

2p Moment resultant of the shear loading

$$T = \int_0^{2\pi} t\tau R(Rd\phi) = 2\pi R^2 t\tau \quad \Rightarrow \quad \tau = \frac{T}{2\pi R^2 t}\,.$$

Therefore, at the free end

$$u_\phi = \frac{\tau}{G} L = \frac{L}{2\pi R^2 tG} T = R\theta \quad \Rightarrow \quad T = \frac{2\pi R^3 t}{L} G\theta\,. \leftarrow$$

The polar moment predicted here is $I_p = 2\pi R^3 t$ whereas the exact is $I_p = \frac{1}{2}\pi Rt(4R^2 + t^2)$.