Aalto University
School of Business

Business Analytics 2 – Lecture 9: Multi-Attribute Value Theory

- Multiattribute value theory (MAVT): the additive value function and its assessment
- SWING weighting
- Interpreting attributes' importance
- MAUT/MAVT comparison
- Value trees
- Supplementary material on preference assumptions in MAUT and MAVT: Preference, Utility, Additive and Difference Independence

Last time

- Multiattribute Utility Theory (MAUT): how to represent preference relations between alternatives with uncertain outcomes on multiple objectives?
- Main result: under certain assumptions, preferences $\succcurlyeq$ are represented by an additive MAU function:

$$u(x) = \sum_{i=1}^{n} w_i u_i(x_i)\text{, where} \sum_{i=1}^{n} w_i = 1, u(x^0) = 0 \text{ and } u(x^*) = 1$$

- Steps in a MAUT process:
  1. Define objectives + attributes to measure the attainment of these objectives
  2. Elicit attribute-specific utility functions and scale them so that $u_i(x_i^0) = 0$ and $u_i(x_i^*) = 1$
  3. Elicit attribute weights w_i using the tradeoff method
  4. Compute the expected multiattribute utilities for the decision alternatives

$$E\left[u(x^j)\right] = \sum_{x \in A} f_{x^j}(x)\, u(x) = \sum_{x \in A} f_{x^j}(x) \sum_i w_i u_i(x)$$

Multi-attribute value theory (MAVT) – MAUT's deterministic sister

- MAUT has a single preference relation for uncertain outcomes, but MAVT has two preference relations:
  - $\succcurlyeq_V$ among certain outcomes (exactly like $\succcurlyeq$ when applied to degenerate lotteries)
    - E.g., (10M,10%) $\succcurlyeq_V$ (5M,12%): ”The first outcome is preferred to the latter one”
  - $\succcurlyeq_{VD}$ among differences between certain outcomes (strength of preference)
    - E.g., [(10M€,7%) ←(5M€,7%)] $\succcurlyeq_{VD}$ [(10M€,9%) ←(5 €M,9%)]: ”An increase in profits from 5 to 10M€ is more valuable when the market share is 7% than when the market share is 9%”
- MAV function v represents preferences $\succcurlyeq_V$, $\succcurlyeq_{VD}$ iff

$$v(x) \geq v(x') \Leftrightarrow x \succcurlyeq_V x'$$

$$v(x) - v(x') \geq v(x'') - v(x''') \Leftrightarrow x \leftarrow x' \succcurlyeq_{VD} x'' \leftarrow x'''$$

  - v is unique up to positive affine transformations

MAVT: Additive Multi-attribute Value Function

- **Theorem.** If each subset of the attributes is PI and some other assumptions* hold, then preferences $\succcurlyeq_V$, $\succcurlyeq_{VD}$ are represented by an **additive** MAV function

$$v(x) = \sum_{i=1}^{n} w_i v_i(x_i) + v(x^0), \text{where}$$

  - $v_i(x_i) = \frac{v(x_1^0, x_2^0, \dots, x_i, \dots, x_n^0) - v(x^0)}{v(x_1^0, x_2^0, \dots, x_i^*, \dots, x_n^0) - v(x^0)} \in [0,1]$ is the attribute-specific *value* function for A_i
  - $w_i = v(x_1^0, x_2^0, \dots, x_i^*, \dots, x_n^0) - v(x^0) > 0$ is the importance weight for attribute A_i
  - $v(x^*) - v(x^0) = \sum_{i=1}^{n} w_i$

Notation: $x = (x_1, \dots, x_n)$, $x^* = (x_1^*, \dots, x_n^*)$, $x^0 = (x_1^0, \dots, x_n^0)$

*) See supplementary slides at the end of the presentation

Steps in a MAVT process

1. Define objectives + attributes to measure the attainment of these objectives (as in MAUT)
2. Elicit attribute-specific value functions and scale them so that $v_i(x_i^0) = 0$ and $v_i(x_i^*) = 1$
3. Elicit attribute weights w_i such that $\sum_{i=1}^{n} w_i = 1$
4. Compute the multiattribute value for the decision alternatives

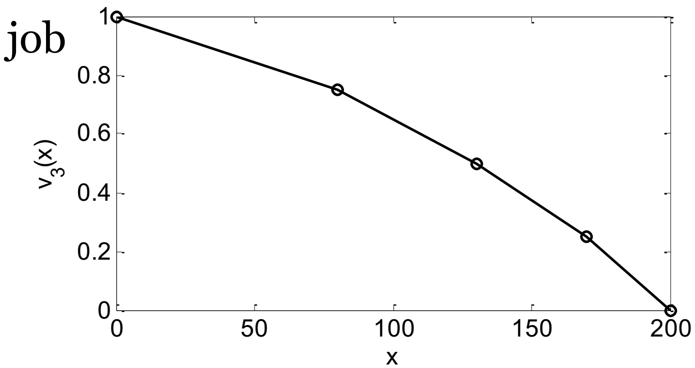
$$v(x^j) = \sum_i w_i v_i(x_i^j)$$

MAVT: Assessing attribute-specific value functions

- Attribute-specific value functions are assessed by comparing <u>differences</u> between alternatives (not lotteries as in MAUT)
- E.g., bisection method:
  - Ask the DM to assess level $x_{0.5} \in [x_i^0, x_i^*]$ such that she is indifferent between change $x_{0.5} \leftarrow x_i^0$ and change $x_i^* \leftarrow x_{0.5}$.
  - Then, ask her to assess levels $x_{0.25}$ and $x_{0.75}$ such that she is indifferent between
    - changes $x_{0.25} \leftarrow x_i^0$ and $x_{0.5} \leftarrow x_{0.25}$, and
    - changes $x_{0.75} \leftarrow x_{0.5}$ and $x_i^* \leftarrow x_{0.75}$.
  - Continue until sufficiently many points have been obtained
    - Use, e.g, linear interpolation between elicited points if needed
  - The value function can be obtained by fixing $v_i(x_i^0)$ and $v_i(x_i^*)$ at, e.g., 0 and 1

MAVT: Assessing attribute-specific value functions

- Example: consider a situation where the value of a job opportunity is characterized by four attributes
  - Salary: A_1=[1000€/month, 6000€/month]
  - Holiday: A_2 =[2 weeks/year, 8 weeks/year]
  - Travel: A_3 =[0 days/year, 200 days/year]
  - Fit with interests: A_4 ={poor, fair, good, excellent}



- Assessing the attribute-specific value function of ”Travel”:
  - ”What would be the number $x_{0.5}$ of traveling days such that you would be indifferent between a decrease from 200 to $x_{0.5}$ days a year and a decrease from $x_{0.5}$ to zero days a year?” (Answer e.g., ”130”)
  - ”What would be the number $x_{0.25}$ of traveling days such that you would be indifferent between a decrease from 200 to $x_{0.25}$ days a year and a decrease from $x_{0.25}$ to 130 days a year?” (Answer e.g., ”170”)
  - ”What would be the number $x_{0.75}$ of traveling days such that you would be indifferent between a decrease from 130 to $x_{0.75}$ days a year and a decrease from $x_{0.75}$ to zero days a year?” (Answer e.g., ”80”)

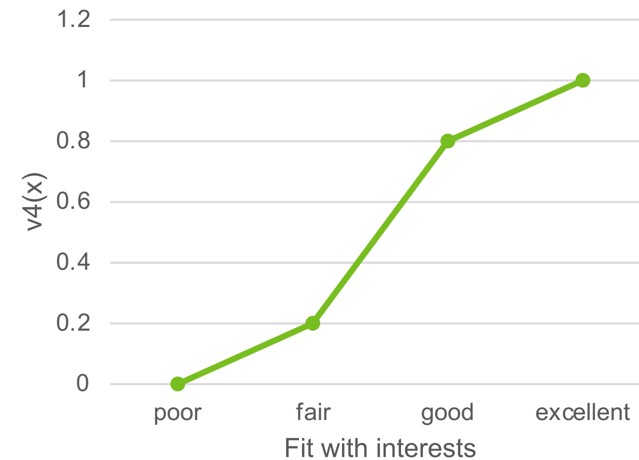
$$v_3(130) - v_3(200) = v_3(0) - v_3(130) \Rightarrow$$
$$v_3(130) = \frac{v_3(0) + v_3(200)}{2} = 0.5$$

$$v_3(170) - v_3(200) = v_3(130) - v_3(170) \Rightarrow$$
$$v_3(170) = \frac{v_3(130) + v_3(200)}{2} = 0.25$$

$$v_3(80) - v_3(130) = v_3(0) - v_3(80) \Rightarrow$$
$$v_3(80) = \frac{v_3(0) + v_3(130)}{2} = 0.75$$

MAVT: Assessing attribute-specific value functions

- Indifference methods (such as the bisection method) are the gold standard and should be used whenever possible
- Nevertheless, such methods cannot be used when the measurement scale is discrete
  - E.g., Fit with interest: A_4 ={poor, fair, good, excellent}
- In such cases, the value function must be assessed ”directly”
- E.g.,
  - Assume that the value of ”Poor fit with interests” is 0 and the value of ”Excellent fit with interests” is 1. What is the value of ”Fair fit with interests”? (Answer e.g., 0.2)
  - How about good fit? (Answer e.g., 0.8).



MAVT: Assessing attribute weights

- The tradeoff method is the gold standard - works basically just like in MAUT (cf. slides 14-16 on lecture 8)
- Other popular methods include SWING, which can also be used in MAUT:

1. Think of an alternative that has the least preferred performance level on each attribute $x^0 = (x_1^0, \ldots, x_n^0)$
2. Choose the attribute A_i that you would first like to change to the most preferred performance level. Give a rating of W_i =100 points to this attribute.
   - MAVT interpretation: For which attribute is the change $x_i^0 \rightarrow x_i^*$ the most valuable?
   - MAUT interp.: Which of the alternatives $(x_1^0, \ldots, x_i^*, \ldots, x_n^0), i = 1, \ldots, n$ is most preferred?

MAVT: Assessing attribute weights

3. Again, assume that all attributes are on the least preferred performance level. Choose the next attribute you would like to change to the most preferred level. Give a rating W_i between 0 and 100 that reflects this improvement compared to first improvement in Step 2.
   - MAVT interpretation: $v(x_1^0, \dots, x_i^*, \dots, x_n^0) \sim W_i$
   - MAUT interpretation: $u(x_1^0, \dots, x_i^*, \dots, x_n^0) \sim W_i$
4. Repeat step 3 until all attributes have been rated.
5. Obtain normalized weights through $w_i = W_i / \sum_{j=1}^{n} W_j$

SWING Example

- Example: Choosing a car
  - Attributes: Life span, Price, Color

$x^0 = (x_1^0, x_2^0, x_3^0)$

(x_1^*, x_2^0, x_3^0)

Attribute Swung from Worst to Best	Consequence to Compare	Rank	W_j Rate	w_j Weight
(Benchmark)	6 years, $17,000, red	4	0	
Life span	12 years, $17,000, red	2	75	0.405 = 75/185
Price	6 years, $8000, red	1	100	0.541 = 100/185
Color	6 years, $17,000, yellow	3	10	0.054 = 10/185
		Total	185	1.000

(x_1^0, x_2^*, x_3^0)

(x_1^0, x_2^0, x_3^*)

Clemen (1996). Making Hard Decisions: An introduction to decision analysis

About attribute's "importance"

- Importance is an extremely vague concept:
  - "Increasing profits is more important than reducing CO_2 emissions!"
  - "So you would not accept a 1€ reduction in profits if it eliminated all CO_2 emissions?"

- Hence, in MAUT (MAVT) the attribute weight w_i does not describe any universal "importance" of the ith attribute
  - Attribute weight only captures the increase in utility (value) when the attribute-specific outcome is changes from the worst level to the best

$$w_i = u(x_1^0, \dots, x_i^*, \dots, x_n^0) - u(x^0) \quad (w_i = v(x_1^0, \dots, x_i^*, \dots, x_n^0) - v(x^0))$$

- When assessing weights from the DM
  - Do not use the term "importance"
  - Use verbal descriptions that explicitly link w_i to levels x_i^0, x_i^* on attribute scale A_i
  - Between analysts in a specific problem with fixed attribute scales the statement "Attribute i is more important than j" means $w_i \geq w_j$

Comparison: MAUT vs. MAVT

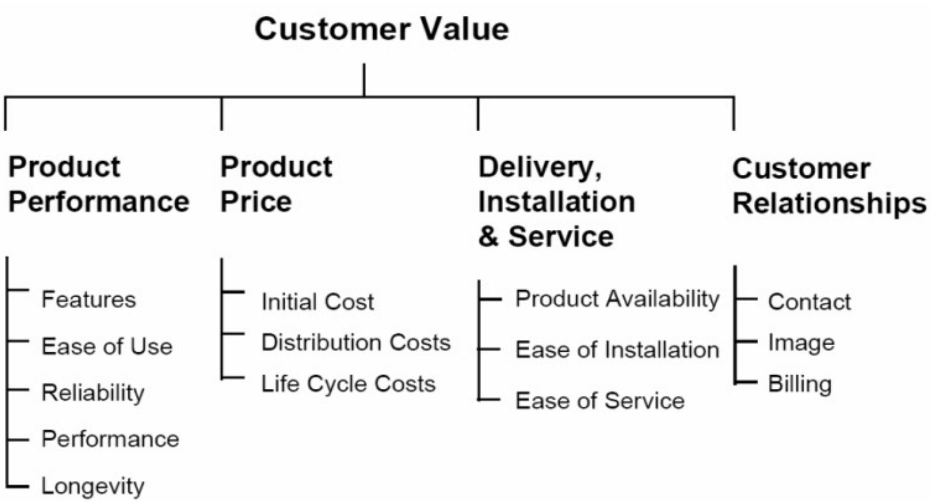
- MAVT: does not model risk preferences; models decreasing marginal value
  - e.g. change in salary from 3k€ to 4k€ is preferred to a change from 4k€ to 5k€
- MAUT: does not distinguish between risk-preferences and non-constant marginal value: both are "hidden" in the utility function
  - E.g. If CE of 50-50 gamble between 3k€ and 5k€ salary is 3.9k€, is this the result of risk aversion or decreasing marginal value of salary?

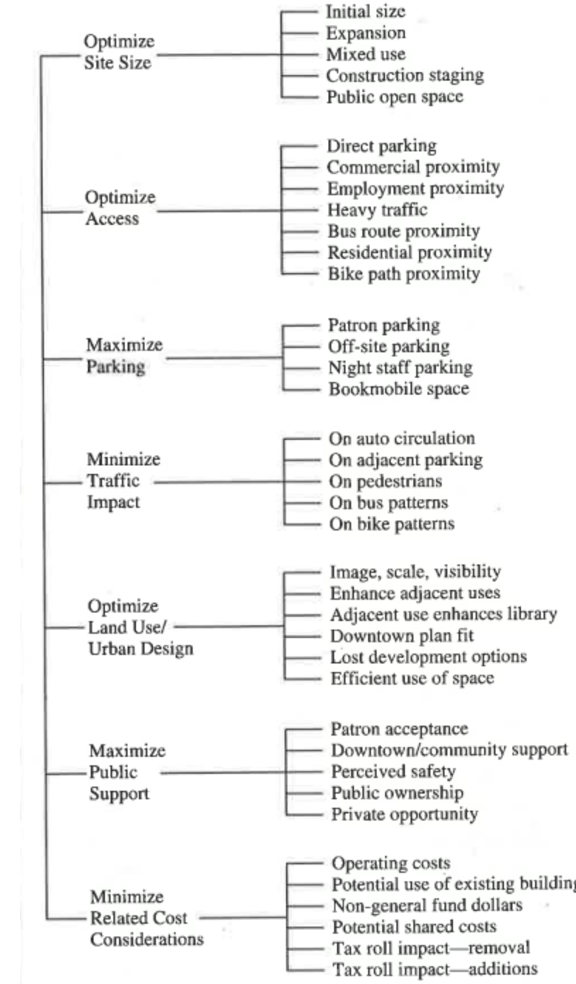
- There exists literature on the theoretical and empirical relationship between value and utility

- In an application it is not always clear which model is used, since different model elements suggest different theoretical basis
  - Use of lottery elicitation question suggests MAUT
  - Comparison of changes between outcomes suggests MAVT
  - Asking the DM to assess strength of preference suggests MAVT
  - Computation of expectations over the preference model suggests MAUT

Value trees

- Often objectives and attributes are visualized as "Value trees"
  - Also called "objective hierarchies"
  - May sometimes include alternatives in the bottom level





Supplementary material on preference assumptions in MAUT/MAVT

Preference Independence (PI)

- **Definition.** A Subset of attributes $S \subset \{A_1, \ldots, A_n\}$ is **PI** if the preference order between alternatives that differ only on attributes in S does not depend on levels of the rest of the attributes $\bar{S}$, i.e.,
$$(x_S^I, x_{\bar{S}}) \succcurlyeq (x_S^{II}, x_{\bar{S}}) \Rightarrow (x_S^I, x'_{\bar{S}}) \succcurlyeq (x_S^{II}, x'_{\bar{S}}) \forall x'_{\bar{S}}$$
- **Example:** Consider an investment selection with three attributes NPV, Market share, and CO_2 reduction.
  - Is S={Profits} preference independent?
    - $\bar{S}$ ={Market Share, CO_2 reduction}
    - Take arbitrary a and b such that (aM€,10%,1ton) $\succcurlyeq$ (bM€,10%,1ton).
    - Then, does it hold for any c and d that (aM€,c%,d tons) $\succcurlyeq$ (bM€,c%,d tons)?
    - It the answer is yes then {Profits} is PI
- Note! Only if an attribute $\{A_i\}$ is PI, it is possible to define its most and least preferred levels $x_i^*, x_i^0 \in A_i$

Mutual Preference Independence

- Example (Cont'd).
  - Is S={Profits, Market Share} preference independent?
    - $\bar{S}$ ={CO_2 reduction}
    - Assume (10M€,11%,10 tons) $\succcurlyeq$ (5M€,12%,10 tons)?
    - Then, does it hold for any a that (10M€,11%,a tons) $\succcurlyeq$ (5M€,12%,a tons)?
    - It the answer is yes, then attribute set {Profits, Market Share} is PI
    - However, it might be that, for instance, (10M€,11%,0 tons) $\preccurlyeq$ (5M€,12%,0 tons), in which case {Profits, Market Share} is not PI.
      - E.g. "If the investment does not contribute to the environmental objective, it becomes more important to establish a stronger market share to survive the PR blowback"

- **Definition.** If every subset S $\subset \{A_1, \dots, A_n\}$ is PI then we say the attributes $A_1, \dots, A_n$ are **mutually preference independent (MPI)**

PI and MPI: Meal example

- Consider choosing a meal with three attributes: wine: A1={red, white}, dish; A2= {beef, fish}, side dish: A3= {potato, rice}
- The DM states ”I prefer (i) red wine to white, (ii) beef to fish and (iii) potato to rise.
  - Since these statements do not depend on the levels of attributes, they imply that each attribute $\{A_i\}$ is PI
- The DM also has the following preferences:
    - (red, beef, rice) $\succ$ (white, beef, potato)
    - (red, fish, rice) $\prec$ (white, fish, potato)

For any $a \in A_1, b \in A_2, c \in A_3$
(red, *b, c*) $\succ$ (white, *b, c*)
(*a*, beef, *c*) $\succ$ (*a*, fish, *c*)
(*a, b*, potato) $\succ$ (*a, b,* rice)

Question: Are the attributes MPI?

- The subset {wine, side dish} is not PI, and therefore the attributes are not mutually preference indep.

Multi-attribute value theory (MAVT)

- **Definition.** Attribute S={A_i} is difference independent (**DI**) if
$$(x_S^I, x_{\bar{S}}) \leftarrow (x_S^{II}, x_{\bar{S}}) \sim_{VD} (x_S^I, x'_{\bar{S}}) \leftarrow (x_S^{II}, x'_{\bar{S}}) \forall x'_{\bar{S}}$$
  - E.g. ”An increase in profits from 5 to 10M € is equally preferred for any level of market share”
- **Theorem.** The attributes are MPI (based on $\succcurlyeq_V$) and one attribute is DI iff preferences $\succcurlyeq_V$, $\succcurlyeq_{VD}$ are represented by an **additive** MAV function
$$v(x) = \sum_{i=1}^{n} w_i v_i(x_i) + v(x^0), \text{where}$$
  - $v_i(x_i) = \frac{v(x_1^0, x_2^0, \dots, x_i, \dots, x_n^0) - v(x^0)}{v(x_1^0, x_2^0, \dots, x_i^*, \dots, x_n^0) - v(x^0)} \in [0,1]$ is the attribute-specific *value* function for A_i
  - $w_i = v(x_1^0, x_2^0, \dots, x_i^*, \dots, x_n^0) - v(x^0) > 0$ is the importance weight for attribute A_i
  - $v(x^*) - v(x^0) = \sum_{i=1}^{n} w_i$