Aalto University
School of Business

Business Analytics 2 – Lecture 7: Modelling Risk Preferences

- *EUT: Certainty equivalent, risk premium, convex and concave utility functions*
- *First-degree Stochastic dominance and its connection to EUT*
- *Second-degree Stochastic dominance and its connection to EUT*
- *Properties of FSD and SSD*
- *Risk-measures: VaR and CVaR and their connection to stochastic dominance*

What are risk preferences?

- Risk: Possibility of loss (or some other less preferred outcome)
  - Risk is characterized both by the probability and magnitude of loss

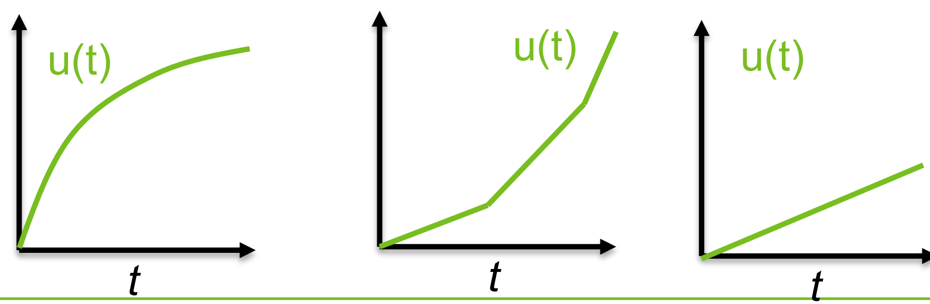
- Risk preferences
  - “How does the riskiness of a decision alternative affect its desirability?”
  - Exact definition depends on which model is used
    - Only the concept of risk-neutrality is general
    - Risk-neutral = Optimize only expected (monetary) value, riskiness is not a factor

- Learning outcomes:
  - Ability to use EUT, stochastic dominance and risk measures to compare decision alternatives
  - Understand the relationship between these different models

Assumptions for this lecture

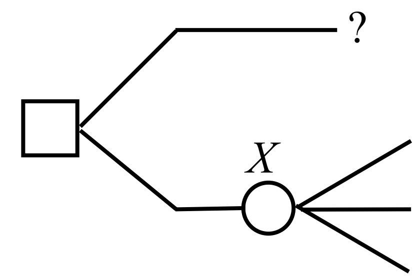
- Definition of risk preferences requires that outcomes T are quantitative and preferences among them are monotonic
  - E.g. Profits, costs, lives saved, etc.
  - Monotonic: either more preferred to less, or less preferred to more

- In this lecture we assume the set of outcomes is such that more is preferred to less
  - $u \in U^0$, where U^0 is the set of all strictly increasing functions on T



Certainty Equivalent in EUT

- **Definition:** Certainty Equivalent CE[X] of a random variable X is an outcome in T such that

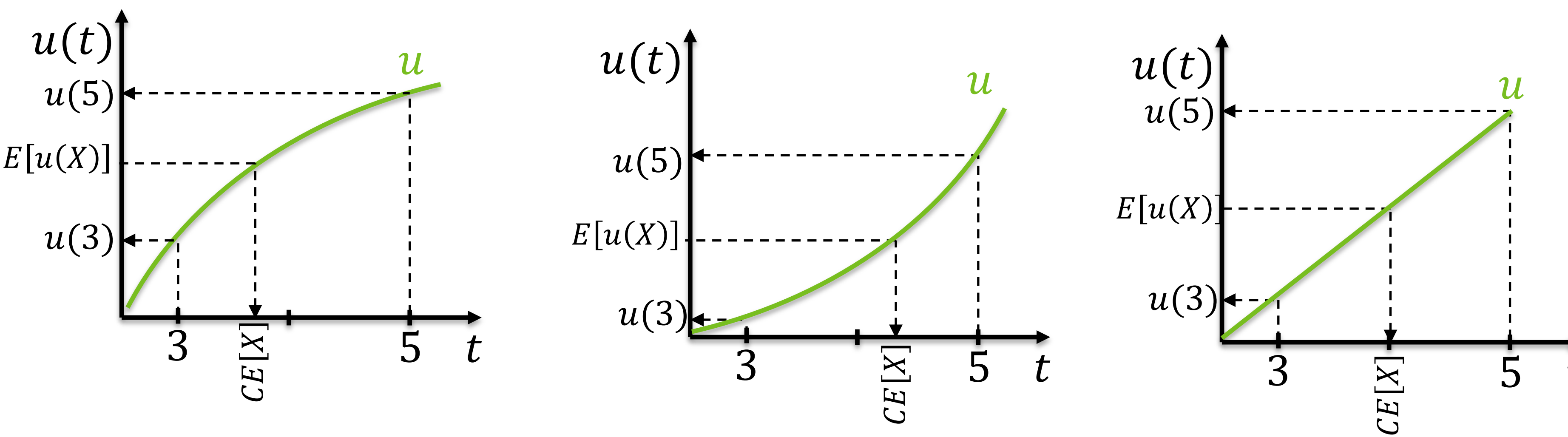


$$u(\mathrm{CE}[X]) = \mathrm{E}[u(X)]$$

  – Alternative definition: CE[X]=u^{-1}(E[u(X)])
  – DM is indifferent between alternative X and the certain outcome CE[X]
    - Note u(CE[X])=E[u(CE[X])] since CE[X] is an outcome, not a random variable
  – CE[X] depends on both the DM's preferences (u) and the uncertainty in the decision alternative (distribution of X)
    - E.g. "My CE for roulette is different from your CE for roulette"
    - E.g. "My CE for roulette is different from my CE for one-armed bandit"

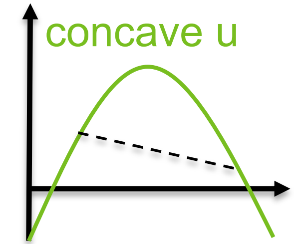
EUT Certainty Equivalent - Example

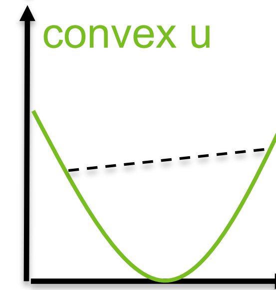
- Consider a decision alternative X with $f_X(3)=.5$ and $f_X(5)=.5$ (and thus E[X]=4) and three DMs with the below utility functions
- Compute each DM's certainty equivalent for X



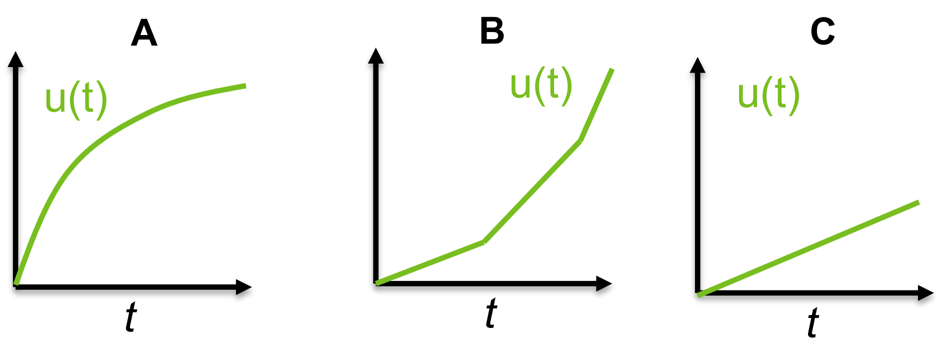
- The shape of the utility functions seems to determine if CE[X] is above, below, or equal to E[X]. Is this a general result?

Convex and Concave functions





- **Definition.** u is concave if for any t_1, t_2:
  $$\lambda u(t_1) + (1-\lambda)u(t_2) \le u(\lambda t_1 + (1-\lambda)t_2)\ \forall \lambda \in [0,1]$$
  - I.e. $u''(t) \le 0\ \forall t \in T$ if second derivative exists
- **Definition.** u is convex if for any t_1, t_2:
  $$\lambda u(t_2) + (1-\lambda)u(t_1) \ge u(\lambda t_1 + (1-\lambda)t_2)\ \forall \lambda \in [0,1]$$
  - I.e. $u''(t) \ge 0\ \forall t \in T$ if second derivative exists
- Question: Which of these functions are concave, which are convex?



∀ means "for all"

Jensen's inequality

- Jensen has shown: For any random variable X, if function u is
  I. convex, then $\mathrm{E}[u(X)] \geq u\,(\mathrm{E}[X])$
  II. concave, then $\mathrm{E}[u(X)] \leq u(\mathrm{E}[X])$

- How to use these inequalities?
  - Map both sides of the inequalities through $u^{-1}(.)$
    - Allowed since u is monotonic (we assume more is preferred to less) and thus u^{-1} is also monotonic

u concave

$$\Rightarrow E[u(X)] \leq u(E[X])$$
$$\Leftrightarrow\; u^{-1}(E[u(X)]) \leq u^{-1}\big(u(E[X])\big)$$
$$\Leftrightarrow CE[X] \leq E[X]$$

u convex

$$\Rightarrow E[u(X)] \geq u(E[X])$$
$$\Leftrightarrow\; u^{-1}(E[u(X)]) \geq u^{-1}\big(u(E[X])\big)$$
$$\Leftrightarrow CE[X] \geq E[X]$$

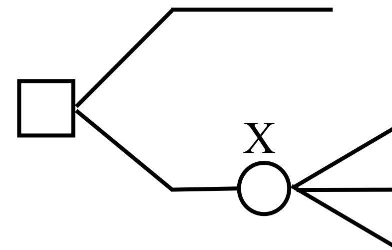
- Alternative definition: $\mathrm{CE}[X]=u^{-1}(\mathrm{E}[u(X)])$

Risk-attitudes in EUT

I. u is linear iff CE[X]=E[X] for all X

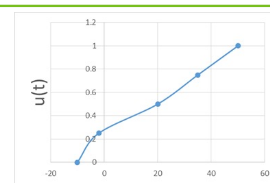
II. u is concave iff CE[X]≤E[X] for all X

III. u is convex iff CE[X]≥E[X] for all X



- A DM with a linear utility function is called risk-neutral
  - She is indifferent between the uncertain outcome X and a certain outcome equal to E[X]
- A DM with a concave (not linear) utility function is called risk-averse
  - She takes a certain outcome smaller than E[X] rather than the uncertain outcome X
- A DM with a convex (not linear) utility function is called risk-seeking
  - She requires a certain outcome greater than E[X] not to choose the uncertain outcome X

Note a DM can have a utility function that does not fall into any of these three categories.



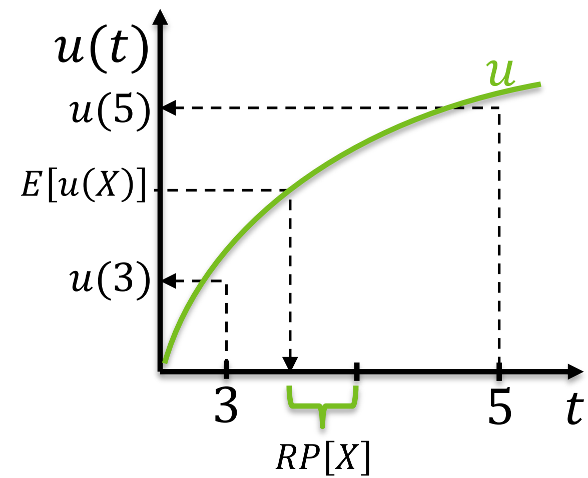
Risk Premium in EUT

- **Definition.** Risk premium for r.v. X is RP[X]=E[X]-CE[X]
  - RP[X] depends on both the DM's preferences (u) and the uncertainty in the decision alternative (X)
  - RP[X] is the premium the DM requires on the expected value to change certain outcome CE[X] to uncertain outcome X

I. u is linear iff RP[X]=0 for all X

II. u is concave iff RP[X]≥0 for all X

III. u is convex iff RP[X]≤0 for all X



Computing CE and RP

1. Compute E[u(X)]
2. Solve u^{-1}( )
3. Compute CE[X]=u^{-1}(E[u(X)])
4. Compute RP[X]=E[X]-CE[X]

- Step 1: see EUT slides
- Steps 2-3: alternatively, you can solve CE[X] numerically from the equation u(CE[X])=E[u(X)]
  - Trial and error
  - Excel Solver

- Example: Jane's $u(t) = t^2$ and investment's profits Y~UNI(3,5)

1. $E[u(Y)] = \int_3^5 f_Y(t)\,u(t)dt = 16.33$
2. $v = u(t) = t^2 \Leftrightarrow t = u^{-1}(v) = \sqrt{v}$
3. $CE[Y] = u^{-1}(16.3) = \sqrt{16.3}$=4.04
4. $\mathrm{RP}[Y] = 4 - 4.04 = -0.04$

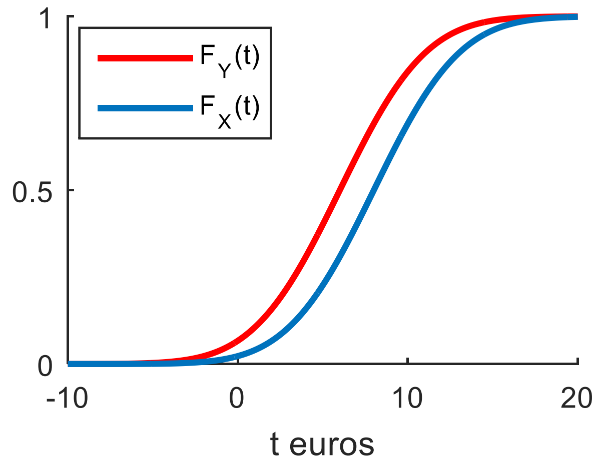
Computing CE and RP

Question: Jane's utility function for profits is $u(t) = 1 - e^{-0.5t}$. Her expected utility for an investment with profits following distribution UNI(3,5) (in M£) is 0.86. What is Jane's certainty equivalent for this investment?

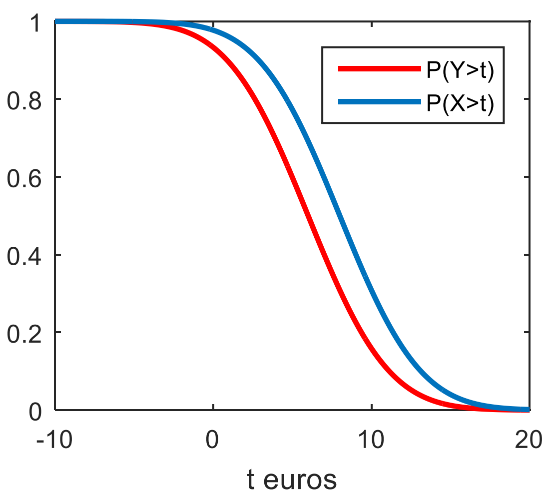
What is the risk premium?

Stochastic Dominance - Motivation

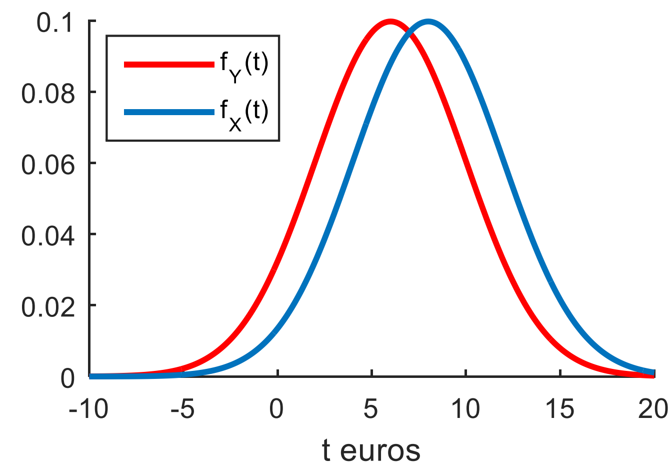
- Question: Would you choose *X* or *Y*?



$=1 - F_{(\cdot)}(t)$



$=\frac{d}{dt} F_{(\cdot)}(t)$

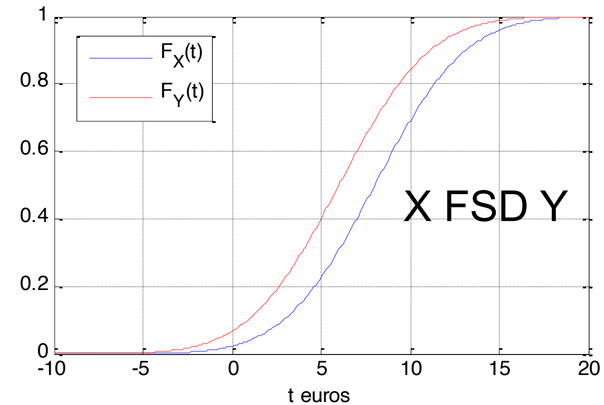


First-degree Stochastic Dominance

- **Definition:** X dominates Y in sense of First-degree Stochastic Dominance if

$$F_X(t) \leq F_Y(t) \text{ for all } t \in T$$

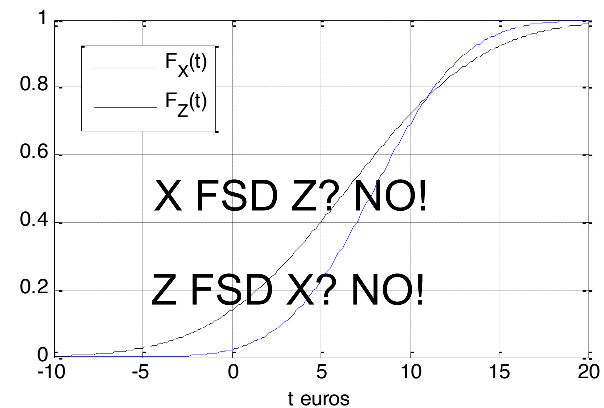
  - Denoted X FSD Y



- Is there a connection to EUT?
- **Result:** X FSD Y, if and only if

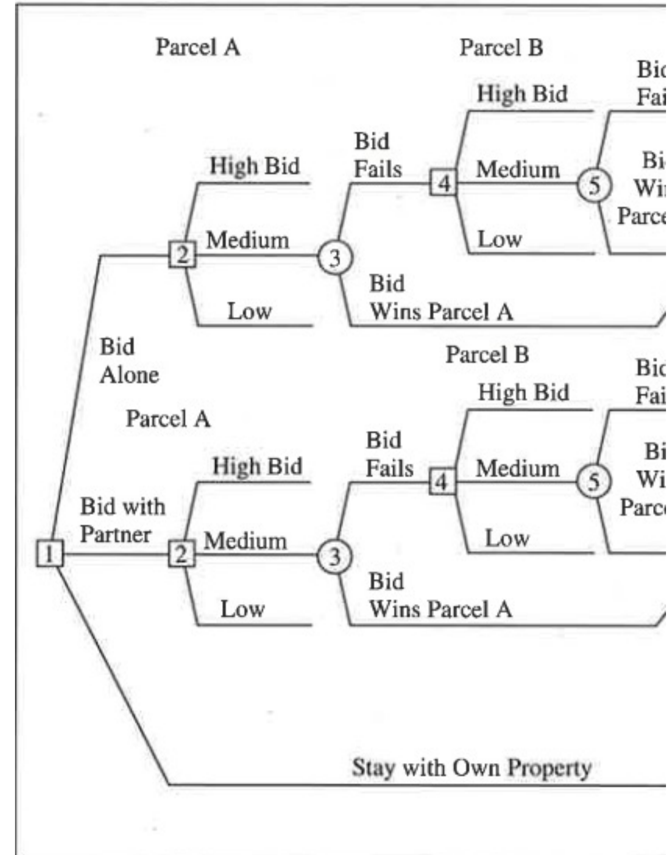
$$E[u(X)] \geq E[u(Y)] \text{ for all } u \in U^0$$

  - U^o is the set of all strictly increasing functions
  - If an alternative is strictly dominated in sense of FSD, then any DM who prefers more to less would not choose it.



FSD: Mining Example

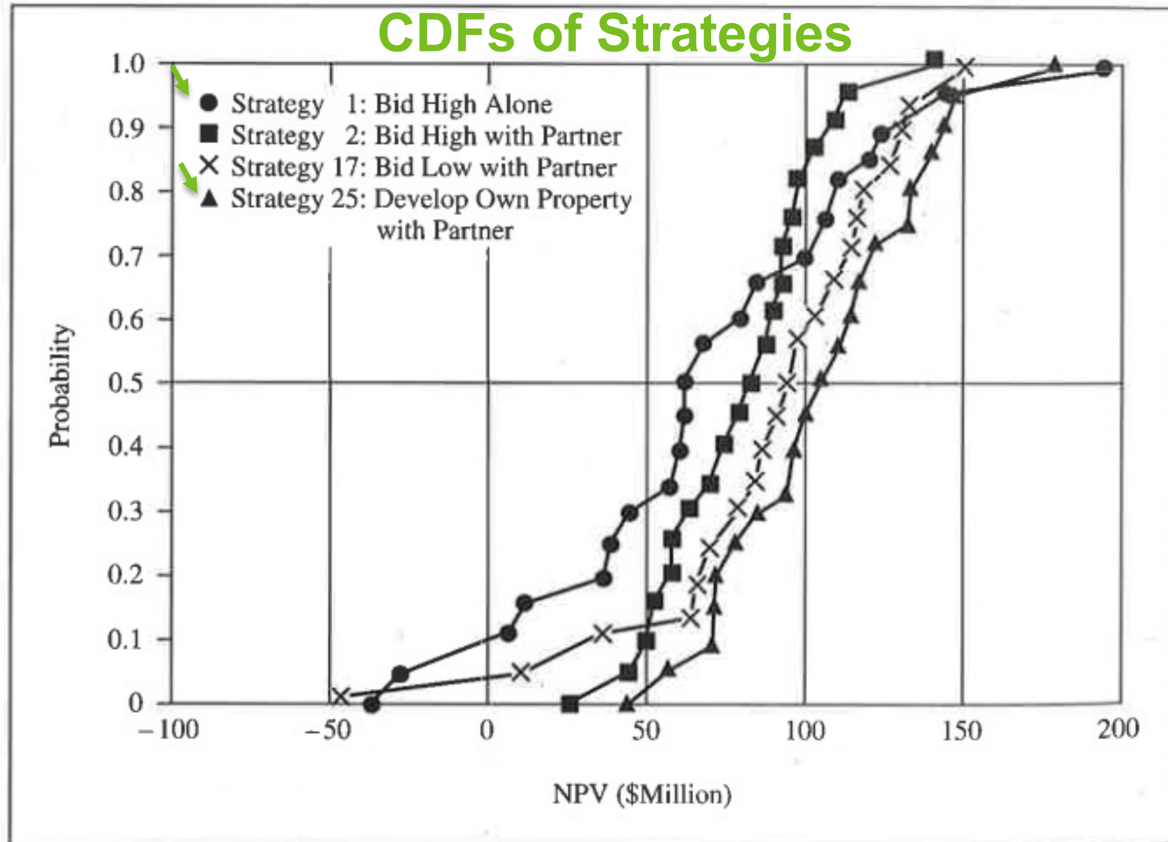
- Mining firm has an opportunity to bid on two separate parcels of land
- Decision on
  - How much to bid?
  - Bid alone or with a partner?
  - How to develop the site if the bid were successful?
- Overall commitment some $500 million
- Large decision tree model built to obtain CDFs of different strategies (decision alternatives)



Source: Hax and Wiig (1977): “The use of Decision Analysis in a Capital Investment Problem” In Bell, Keeney, Raiffa (eds.): Conflicting Objectives in Decisions. Wiley.

FSD: Mining Example (Cont'd)

- Assume the company prefers a larger NPV to a smaller one.
- Which strategies would you recommend?



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Second-degree Stochastic Dominance

- What if we also knew that the DM was risk averse or risk neutral?
- **Result:**

$$E[u(X)] \geq E[u(Y)] \forall u \in U^{ccv} \Leftrightarrow \int_{-\infty}^{z} [F_X(t) - F_Y(t)]\, dt \leq 0 \;\forall z \in T,$$

where $U^{ccv} = \{u \in U^0 | u \text{ is concave}\}$, i.e., the set of increasing concave utility functions

- This result motivates naming the above integral:
- **Definition:** X dominates Y in sense of Second-degree Stochastic Dominance if

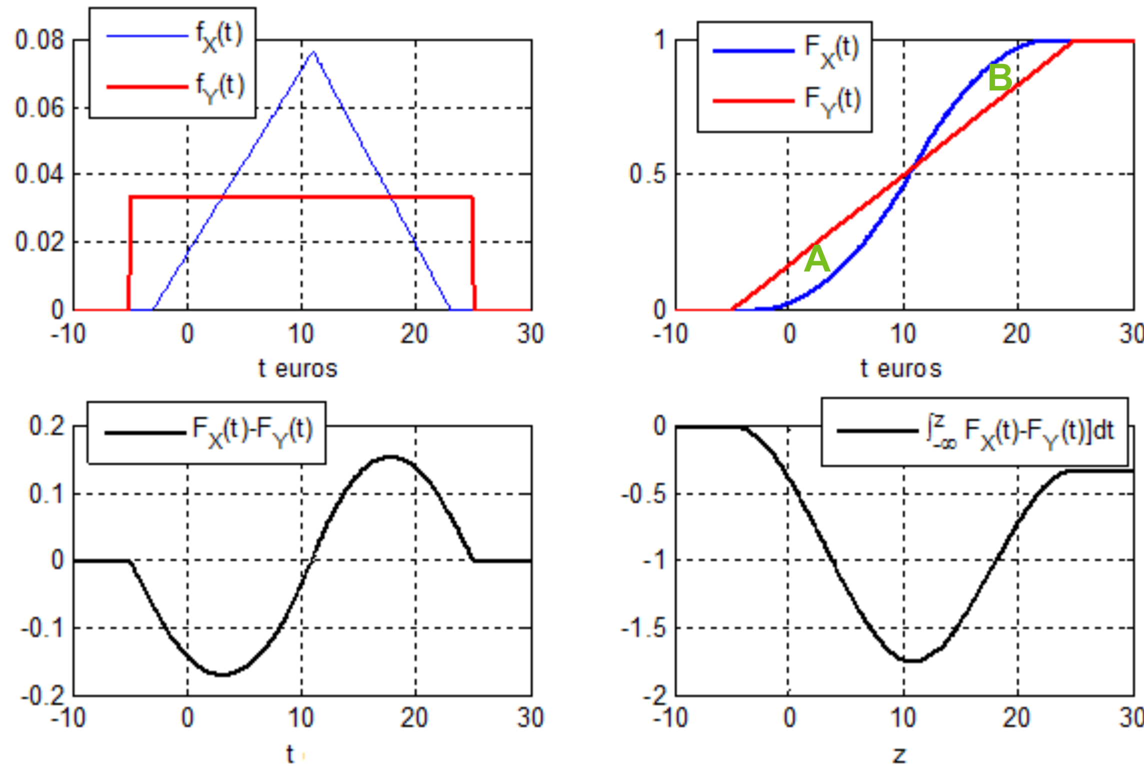
$$\int_{-\infty}^{z} [F_X(t) - F_Y(t)]\, dt \leq 0 \;\forall z \in T$$

  - Denoted X SSD Y

SSD: Graphical Interpretation

$$\int_{-\infty}^{z} [F_X(t) - F_Y(t)]\, dt \ \leq 0 \ \forall z \in T$$

- The integral calculates the area between the horizontal axis and $F_X(t) - F_Y(t)$ up to point z
- If it is negative for all z then X SSD Y

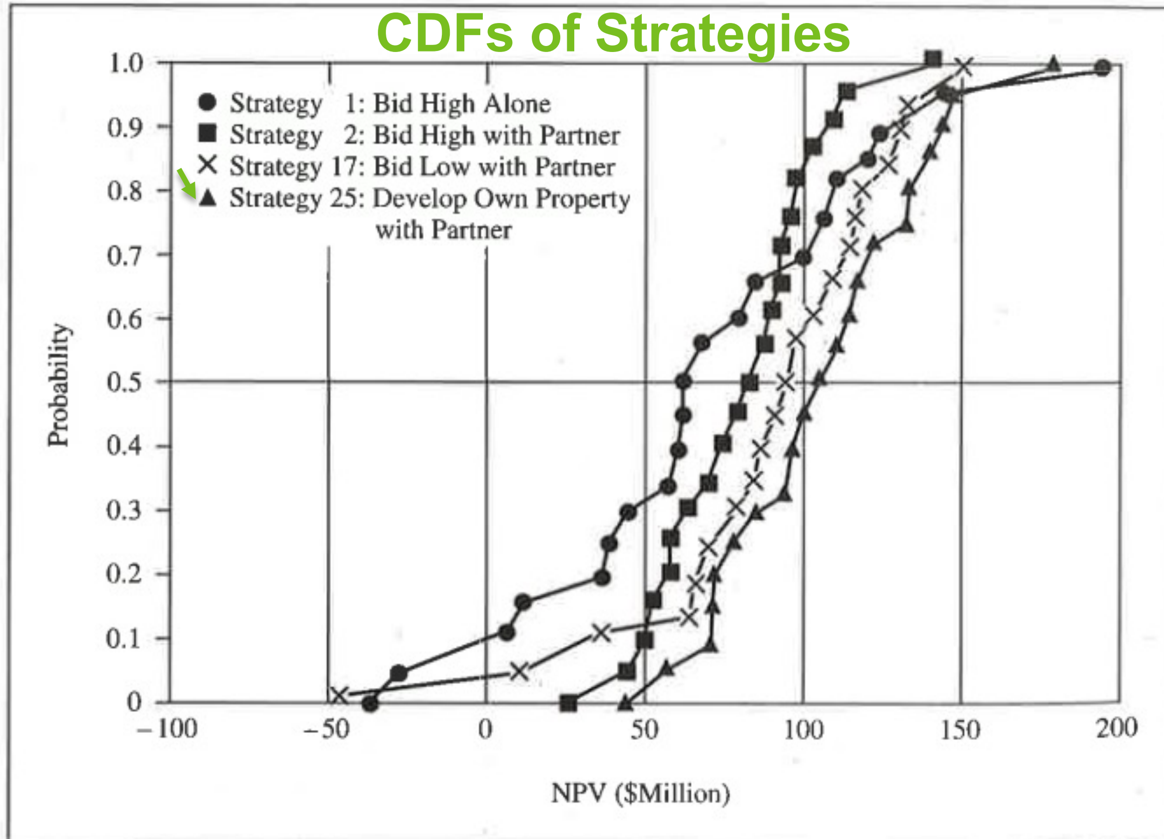


- This Example:
  - X SSD Y because area A is bigger than area B, and A is left of B

$$\int_{-\infty}^{z} [F_X(t) - F_Y(t)]\, dt = \int_{-\infty}^{z} F_X(t)\, dt - \int_{-\infty}^{z} F_Y(t)\, dt$$

SSD: Mining Example Revisited

- Assume the mining company is either risk-averse or risk neutral.
- Which strategies would you recommend?



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Properties of FSD and SSD

$U^{ccv} = \{u \in U^0 | u \text{ is concave}\}$

- Both FSD and SSD are transitive:
  - If X FSD Y and Y FSD Z, then X FSD Z
    - Why? Take any t. Then $F_X(t) \leq F_Y(t) \leq F_Z(t)$.
  - If X SSD Y and Y SSD Z, then X SSD Z
    - Why? Take any $u \in U^{ccv}$. Then $E[u(X)] - E[u(Z)] \geq E[u(Y)] - E[u(Z)] \geq 0$

- FSD implies SSD:
  - If X FSD Y, then X SSD Y
    - Why? Take any $u \in U^{ccv}$. Then $u \in U^0$ and since X FSD Y, we have $E[u(X)] \geq E[u(Y)]$.

Risk-measures

- Risk measure is a function that maps each decision alternative (random variable) to a single number describing its risk
  - A non-EUT based approach for modelling risk
  - Needs to be used together with EMV to produce decision recommendations:
    - **Risk constraint**: Among alternatives whose risk is below some predetermined threshold, selected the one with maximum EMV
    - **Risk minimization**: Among alternatives whose EMV is above some predetermined threshold, select the one with the minimum risk
    - **Efficient frontier**: Identify decision alternatives that are efficient, i.e. no other alternative provides a greater EMV with smaller risk

- Example: Variance $\mathrm{Var}[X] = E[(X - E[X])^2]$
  - The higher the variance, the higher the risk
  - Punishes for the possibility of positive surprise (i.e., outcomes better than E[X])
    - Not a good measure for risk without additional distribution assumptions

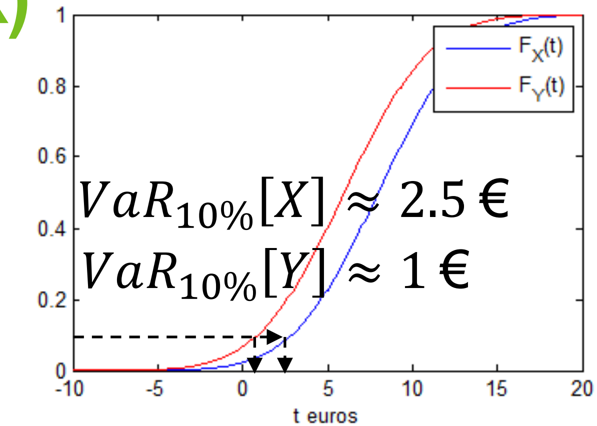
Risk measures: Value-at-Risk (VaR)

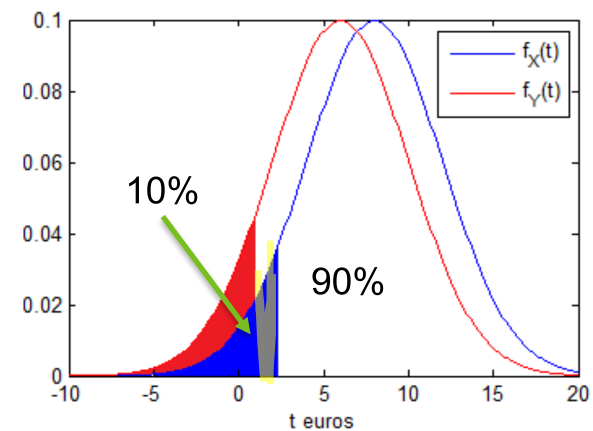
- Value-At-Risk: $\text{VaR}_\alpha[X]$
  - $\text{VaR}_\alpha[X]$ describes an outcome such that probability of an equal or worse outcome is α:

$$\int_{-\infty}^{VaR_\alpha[X]} f_X(t)dt = F_X(VaR_\alpha[X]) = \alpha$$

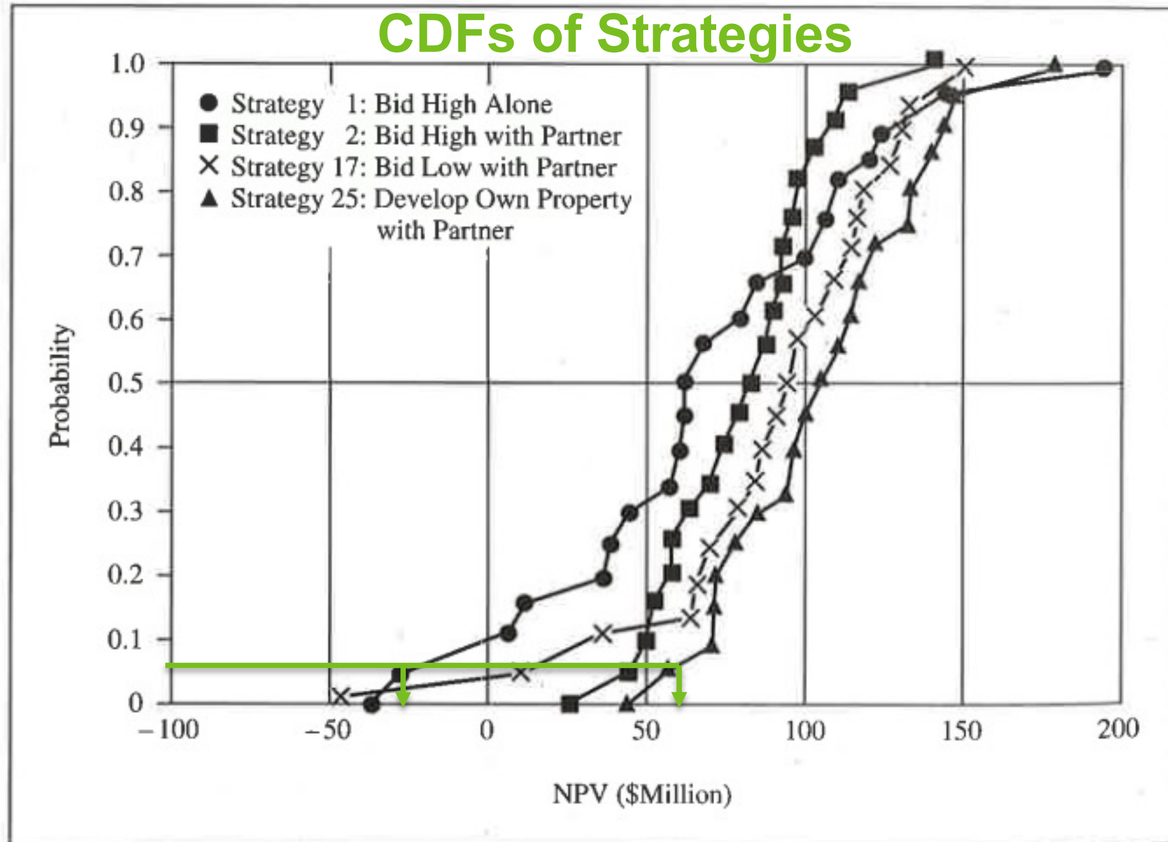
  - Higher VaR means smaller risk
    - Warning! When applied to loss distribution higher VaR means higher risk
  - Common values for α are 1%, 5% and 10%
  - Actually a family of risk measures:
    - E.g. $\text{VaR}_{10\%}[.]$ and $\text{VaR}_{5\%}[.]$ are different measures
  - Problem: The length of the tail is not taken into account

Profits





Mining Example Revisited

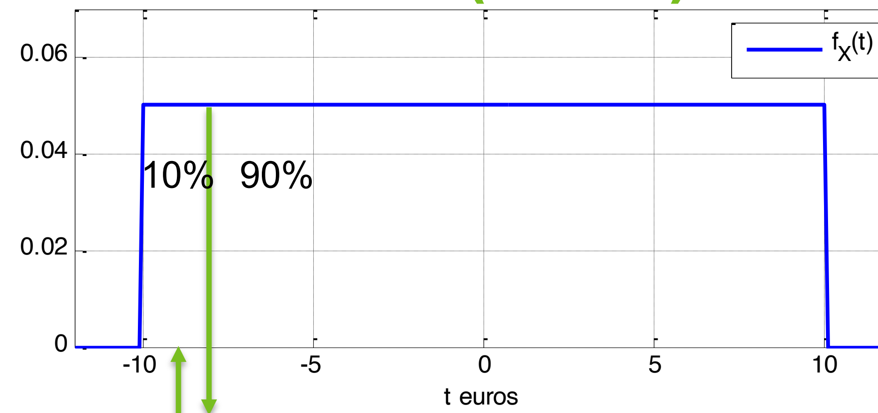


- Assess $VaR_{5\%}$ for
  - ● Strategy 1
  - ▲ Strategy 25

Source: Hax and Wiig (1977): “The use of Decision Analysis in a Capital Investment Problem” In Bell, Keeney, Raiffa (eds.): Conflicting Objectives in Decisions. Wiley.

Risk measures: Conditional Value-at-Risk (CVaR)

- Conditional Value-At-Risk: $\text{CVaR}_\alpha[X]$
  - Describes the expected outcome when the outcome is equal to or worse than $\text{VaR}_\alpha[X]$:
    $CVaR_\alpha[X] = E[X|X \leq VaR_\alpha[X]]$
  - Higher CVaR means smaller risk
    Note: For losses higher CVaR, higher risk



$VaR_{10\%}[X] = -8\,€$

$CVaR_{10\%}[X] = -9\,€$

- Computation of $E[X|X \leq VaR_\alpha[X]]$ for discrete and continuous r.v. X:

$$E[X|X \leq VaR_\alpha[X]] = \sum_{t \leq VaR_\alpha[X]} t\,\frac{f_X(t)}{\alpha} \qquad E[X|X \leq VaR_\alpha[X]] = \int_{-\infty}^{VaR_\alpha[X]} t\,\frac{f_X(t)}{\alpha}\,dt$$

  - Note: $\alpha = \text{P}(X \leq VaR_\alpha[X])$

Risk measures: Computation of VaR and CVaR

- If the inverse CDF of X is well defined, VaR can be obtained from
$$VaR_{\alpha}[X] = F_X^{-1}(\alpha)$$
  - For instance, the inverse of the CDF of a normal distribution is given by the Excel function norm.inv

- With discrete random variables, VaR and CVaR are not always well defined for small values of α
  - Example:
    - $VaR_{10\%}[X]=1$
    - $CVaR_{10\%}[X] = [.06(-10) + 0.02(-5) + 0.02(1)]/0.1 = -6.8$
    - But what is $VaR_{5\%}[X]$ or $CVaR_{5\%}[X]$?

t	20	-10	10	1	-5
$f_X(t)$	0.4	0.06	0.5	0.02	0.02

Risk Measures: Var and CVaR with Monte Carlo

H24

	A	B	C	D	E	F	G	H	I
2		Number of papers to print (q)			12000				
4		**Average**	**0.50**	**9942**	**7837**	**1811.15**			
5		Samp. Std	0.29	2983				**VaR-10%**	**3729.71**
7		Sample	Random number (u)	Demand (d), units	Profit (z), euros	Profits below VaR			
8		1	0.334432	8717	6860	above			
9		2	0.0103486	3060	72	71.53067			
10		3	0.8713102	13398	10800	above			
11		4	0.4111918	9327	7592	above			
12		5	0.8636455	13291	10800	above			
13		6	0.8767255	13476	10800	above			
14		7	0.066804	5500	3000	2999.912			
15		8	0.4703005	9776	8132	above			
16		9	0.7439743	11967	10760	above			
17		10	0.4836635	9877	8253	above			
18		11	0.1951152	7422	5307	above			
19		12	0.763508	12153	10800	above			
20		13	0.3182713	8582	6699	above			
21		14	0.7126954	11684	10421	above			
22		15	0.0458006	4939	2327	2326.759			

CVaR 10%

=PERCENTILE(E8:E1007;0.1)

=IF(E8<=I5;E8;"above")

Note! 1000 samples is not a lot since only 1/10 is used to estimate 10% VaR and CVaR

Risk Measures: Var and CVaR with Monte Carlo

VaR and CVaR in profit simulation (Lecture 3)

In [16]:
```
# GENERATE THE SIMULATED DATA
N <- 100000
c1 <- sample(c(43,44,44,45,45,45,45,46,46,47),N,replace=T)
c2 <- runif(N,80,100)
D <- rnorm(N, mean=15000, sd=4500)

profit <- (249-c1-c2)*D-10^6
profit <- profit/1000 # express profit in T€

# DETERMINE VaR 10%
VaR10 <- quantile(profit, .1)[[1]]

# DETERMINE CVAR 10%
CVaR10 <- mean(profit[profit<=VaR10]) # conditional mean
```

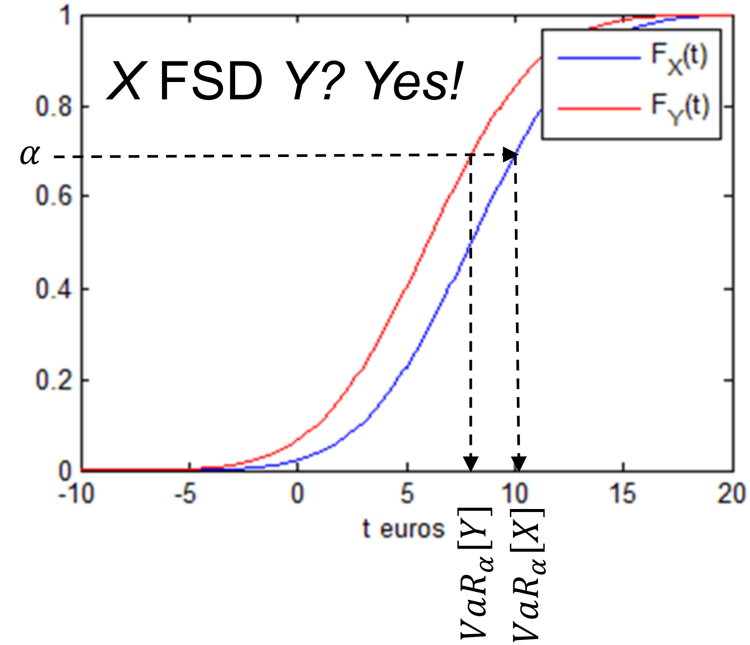
Note! 1000 samples is not a lot since only 1/10 is used to estimate 10% VaR and CVaR

Linking Risk Measures to EUT

- No direct link, but via stochastic dominance:

- **Result:** X FSD Y if and only if $VaR_{\alpha}[X] \geq VaR_{\alpha}[Y]\ \forall\alpha \in [0,1]$
  – The dominating alternative is less risky no matter which VaR_{α} measure is used

- **Result:** X SSD Y if and only if $CVaR_{\alpha}[X] \geq CVaR_{\alpha}[Y]\ \forall\alpha \in [0,1]$
  – The dominating alternative is less risky no matter which $CVaR_{\alpha}$ measure is used



Challenges with Risk Measures

- Which measure to use?
- Which α to use in VaR and CVaR?
- How to combine EMV and Risk measure values into overall performance measure for each alternative?

- If answers to these questions are given from the outside, then use of risk measures can be easy, beneficial or even mandatory
  - Outside = Industry standard, regulation, legislation, etc.