Aalto University
School of Business

Business Analytics 2 – Lecture 10: Supporting Decision Making with Optimization

- *When is optimization needed?*
- *Optimization problem: General formulation, problem types*
- *Multi-objective Optimization (MOO): Efficient solutions, solution through MAU(V)T*
- *Optimization under uncertainty: Use of scenarios*

Dear second-year bachelor's student and first-year master's student – how are you?

Please answer the AllWell? student survey that we will send to you soon.

You will receive feedback based on your responses and help us do our work better.

Thank you!

AllWell?

The survey will be open 1.2-14.2.2024

Supporting Decision Making with Optimization

- Thus far we have covered decision making problems in which the set of decision alternatives is explicitly given or can be easily generated
  - "The company can build a large or a small condominium…" (= 2 alternatives)
  - "Moreover, the company has an option to buy a market study." (= 6 alternatives)

- Optimization is needed when the decision alternatives are implicitly defined and the number of alternatives is very large or even infinite, e.g:
  - Decision alternatives = "Combinations of project portfolios that can be funded with the given budget"
  - Decision alternatives = "Production plans that are feasible with the available resources"

- Learning objective:
  - Ability to deploy optimization, EUT and MAUT to solve decision problems with uncertainties and/or multiple objectives

Optimization problem: General formulation

- The term (Mathematical) Programming also widely used
- Optimization problem with m decision variables and q constraints:

$$\max_{x} f(x_1, \dots, x_m)$$
$$g_j(x_1, \dots, x_m) \leq 0, j = 1, \dots, q \quad (1)$$
$$x_i \in M_i, i = 1, \dots, m \quad (2)$$

  - $f(\cdot)$ is the objective function
  - M_i can be $\mathbb{R}$, $\mathbb{N}$ or {0,1}, for instance
  - A solution $x = (x_1, \dots, x_m)$ is **feasible** if it satisfies all constraints (1) and (2)
  - **Feasible region** S is the set of all feasible solutions
  - Solution $x \in S$ is (globally) **optimal** if $f(x) \geq f(x') \; \forall x' \in S$
  - Implementing = and ≥ constraints:

$$g(x) \geq 0 \Leftrightarrow -g(x) \leq 0, \qquad g(x) = 0 \Leftrightarrow \begin{cases} g(x) \leq 0 \\ -g(x) \leq 0 \end{cases}$$

Optimization problem: General formulation (Cont'd)

$$\max_x f(x_1, \dots, x_m)$$
$$g_j(x_1, \dots, x_m) \leq 0, j = 1, \dots, q$$
$$x_i \in M_i, i = 1, \dots, m$$

- If function $h(\cdot)$ is monotonically
  - increasing, then $\max_x f(x)$ and $\max_x h(f(x))$ have the same optimal solution
    - e.g. $\max_x (b + af(x)) = b + a \max_x f(x)$
  - decreasing, then $\max_x f(x)$ and $\min_x h(f(x))$ have the same optimal solution
    - e.g. $\max_x f(x)$ and $\min_x -f(x)$

- Adding a constraint cannot improve the optimal objective function value
  - A new constraint can reduce the feasible region, but not add new solutions to it

LP, ILP, MILP Problems

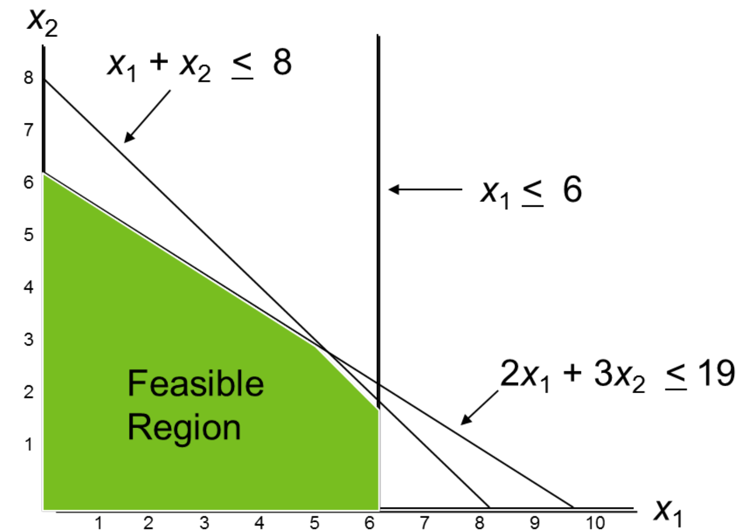
- In a Linear Programming (LP) problem the objective function and all the constraints are linear

$$\max_{x} \sum_{i=1}^{m} c_i x_i$$

$$\sum_{i=1}^{m} a_{ji} x_i - b_j \leq 0, j = 1, \dots, q$$

$$x_i \in M_i, i = 1, \dots, m \quad (2)$$

  - LP: $M_i = \mathbb{R}$ for all i
  - Integer LP (ILP): $M_i = \mathbb{Z}$ for all i
  - Binary LP (BLP): $M_i = \{0,1\}$ for all i
  - Mixed Integer LP (MILP): Both continuous and integer variables

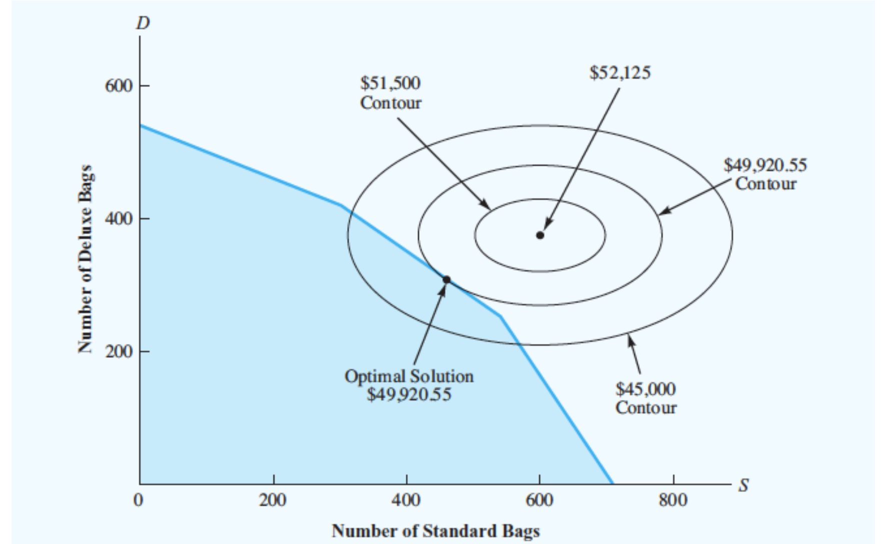


- Solution algorithms: Simplex (+ Branch and Bound if there are integer variables)
  - Fast & reliable

Non-linear programming problems

- Non-Linear Programming (NLP):
  - The objective function or at least one of the constraints is non-linear

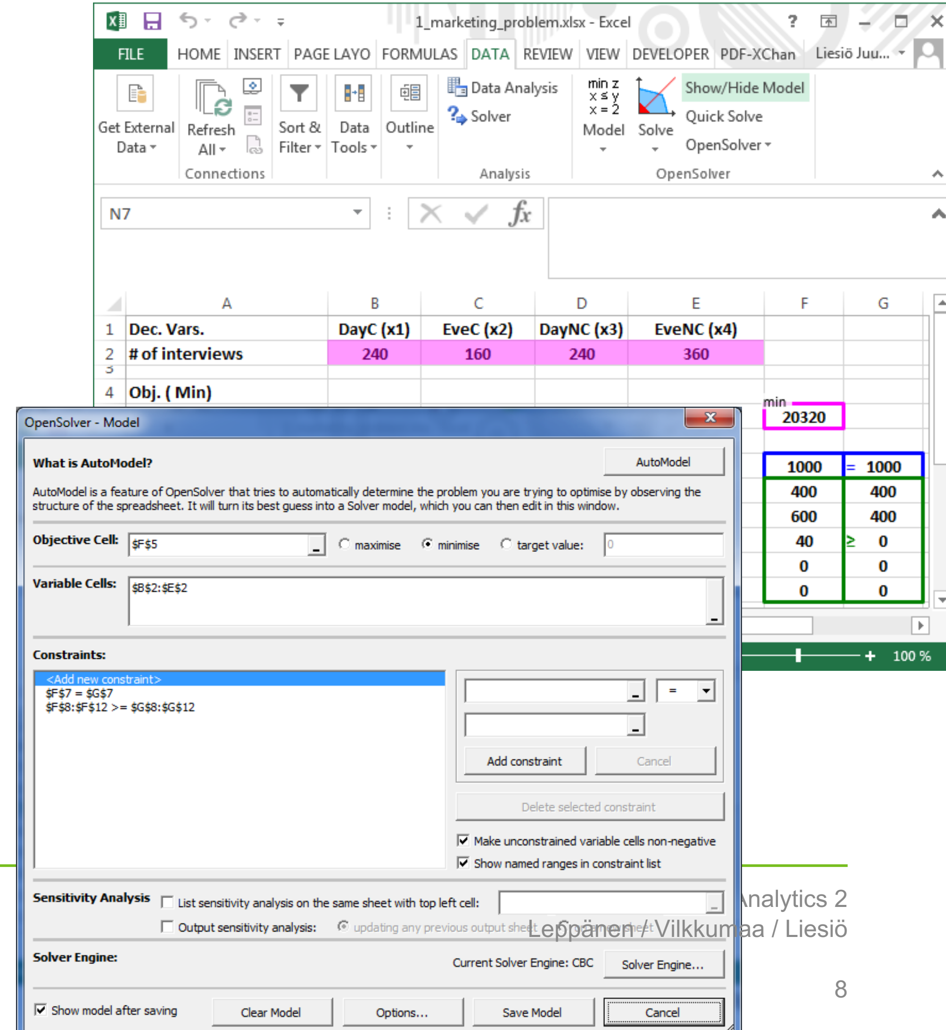
$$\max_{x} f(x_1, \dots, x_m)$$
$$g_j(x_1, \dots, x_m) \le 0, j = 1, \dots, q$$
$$x_i \in M_i, i = 1, \dots, m$$



- Terms:
  - NLP: $M_i = \mathbb{R}$ for all i
  - Integer Programming (IP): $M_i = \mathbb{Z}$ for all i
  - Mixed Integer Programming (MIP): Both continuous and integer variables
- Solution algorithms:
  - E.g. the GRG algorithm for concave/convex NLPs
  - Evolutionary algorithms (heuristic)

Spreadsheet-based optimization solvers

- Excel Solver
  - Not particularly powerful
  - Max decision variables 200
  - Max constraints 100
- Open solver offers a free alternative for Win and Mac
  - Download: http://opensolver.org/
  - No limits for the numbers of dec. variables and constraints
  - Better user interface
  - Can use state-of-the-art solvers (e.g. gurobi)



Multi-objective optimization (MOO)

- An n-objective optimization problem with q constraints

$$\max_x f_1(x_1, \dots, x_m)$$
$$\dots$$
$$\max_x f_n(x_1, \dots, x_m)$$
$$g_j(x_1, \dots, x_m) \leq 0, j = 1, \dots, q$$
$$x_i \in M_i, i = 1, \dots, m$$

  - Note: Objective $\min_x h(x)$ can be written as $\max_x f_k(x)$, where $f_k(x) = -h(x)$

- **Definition.** A feasible solution $x \in S$ to MOO problem is **efficient**, if there does not exists another feasible solution $x' \in S$ such that

  (i) $f_i(x') \geq f_i(x)$ for all $i \in \{1, \dots, n\}$

  (ii) $f_i(x') > f_i(x)$ for some $i \in \{1, \dots, n\}$

Case: Deciding on the number of districts to provide social and healthcare services in Finland

The three feasible solutions

Objective function values	4–5 areas	10–12 areas	15–19 areas
Equal availability of services	excellent	excellent	satisfactory
Productivity, effectiveness and cost control	excellent	good	weak
Applicability to handle other administrative functions	satisfactory	excellent	satisfactory
Democracy	good	excellent	excellent

source: yle.fi

Questions:

- Which solutions(s) are efficient?
- Which solution did the previous government choose?

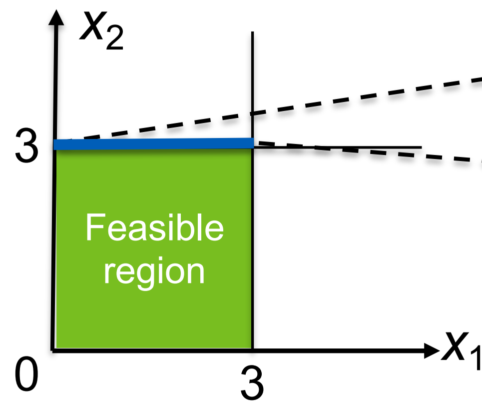
Example: Efficient solutions in MOLP

Graphical representation in the...

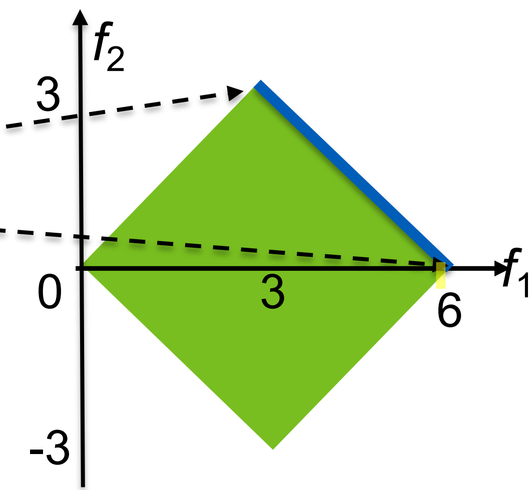
Math. formulation

$$\text{Max } f_1(x) = x_1 + x_2$$
$$\text{Max } f_2(x) = -x_1 + x_2$$
$$\text{s.t. } x_1 \leq 3$$
$$x_2 \leq 3$$
$$x_1, x_2 \geq 0$$

Decision variable space



Objective function space



MAUT/MAVT to solve MOO problems

- Which efficient solution to choose?
- Additive value or utility functions can be used to aggregate the objective functions
- Consider the MOO problem

$$\max_x f_1(x_1, \dots, x_m)$$
$$\dots$$
$$\max_x f_n(x_1, \dots, x_m)$$
$$g_j(x_1, \dots, x_m) \leq 0, j = 1, \dots, q$$
$$x_i \in M_i, i = 1, \dots, m$$

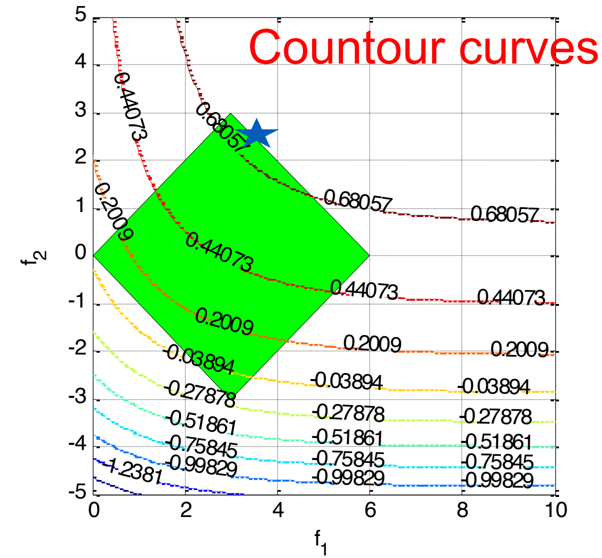
– **Result:** If x is an efficient solution to the MOO problem, there exists $w_i, u_i, i = 1, \dots, n$ such that x maximizes the additive value/utility:

$$\max_x \sum_{i=1}^{n} w_i u_i(f_i(x))$$
$$g_j(x_1, \dots, x_m) \leq 0, j = 1, \dots, q$$
$$x_i \in M_i, i = 1, \dots, m$$

MOLP Example (Con't)

$$u_1(f_1) = 1 - e^{-\frac{f_1}{2}}, w_1 = 0.6$$

$$u_2(f_2) = 1 - e^{-\frac{f_2}{3}}, w_2 = 0.4$$



Example: MAU(V)T + optimization

- Adcom Adveritsing Agency is trying to determine an advertising schedule for a German Auto Company
- The auto company has two objectives
  - Maximize the number of high-income men (HIM) who see the ad
  - Maximize the number of high-income women (HIW) who see the ad
- It can purchase advertisements during soccer games and soap operas, whose expected viewers (in millions) are listed in the table below.

Advertisement	HIM	HIW	Cost ($)
Soccer	7	3	100,000
Soap Opera	4	5	90,000

- Budget for the advertisements is $600,000.

Example: MAU(V)T + optimization (Cont'd)

Ad	HIM	HIW	Cost ($)
Soccer	7	3	100,000
Soap Opera	4	5	90,000

- Formulation as a multiple objective optimization problem

$$\max_x f_1(x_1, x_2) = 7x_1 + 4x_2$$
$$\max_x f_2(x_1, x_2) = 3x_1 + 5x_2$$
$$100x_1 + 90\, x_2 \leq 600$$
$$x_1, x_2 \geq 0, \text{integer}$$

- Question: Interpret the obj. functions, constraints and decision variables

- Formulation as a single objective NLP using MAVT
  - MAVT with decreasing marginal value of HIM/HIW viewers
  - $v_i(f_i) = 1 - \exp\left(-\frac{f_i}{30}\right)$
  - $w_1 = w_2 = 0.5$

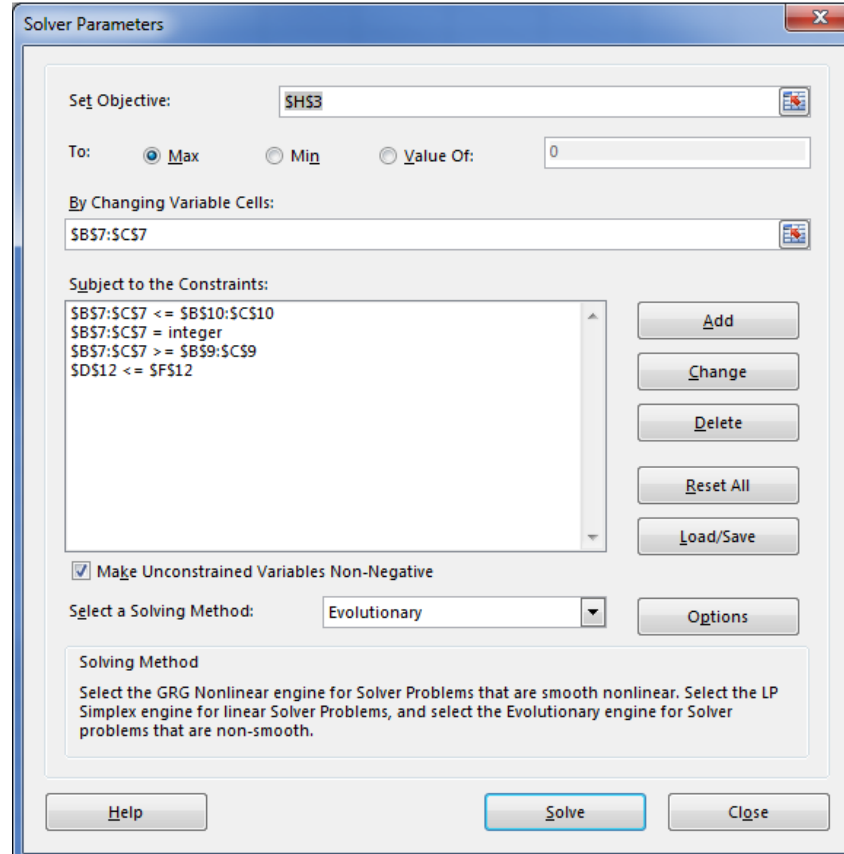
$$\max_x \sum_{i=1}^{2} w_i v_i(f_i)$$
$$f_1 = 7x_1 + 4x_2$$
$$f_2 = 3x_1 + 5x_2$$
$$100x_1 + 90\, x_2 \leq 600$$
$$x_1, x_2 \geq 0, \text{integer}$$

Example: MAU(V)T + optimization (Cont'd)

$$\max_{x} \sum_{i=1}^{2} w_i v_i(f_i)$$
$$f_1 = 7x_1 + 4x_2$$
$$f_2 = 3x_1 + 5x_2$$
$$100x_1 + 90\,x_2 \leq 600$$
$$x_1, x_2 \geq 0, \text{integer}$$

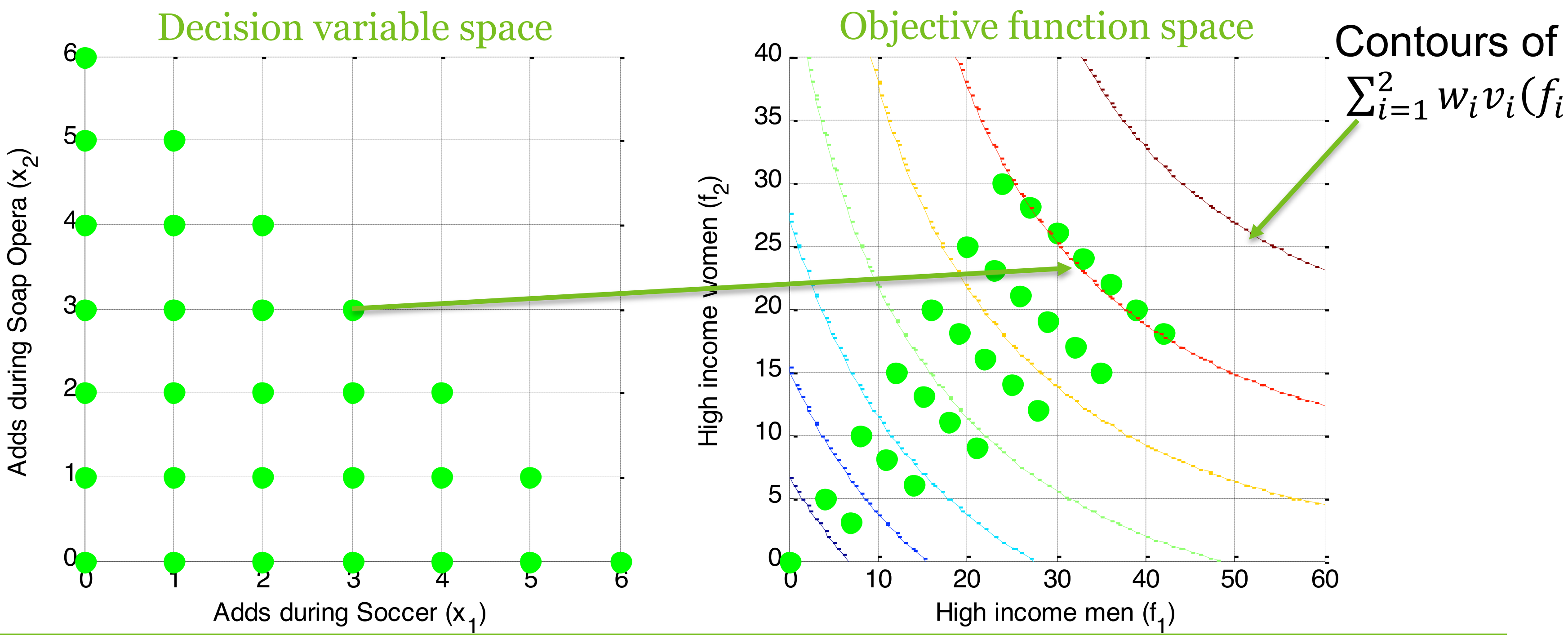
$$v_i(f_i) = 1 - \exp\left(-\frac{f_i}{30}\right)$$
$$w_1 = w_2 = 0.5$$

	A	B	C	D	E	F	G	H
1							Weighted	
2				f_i	v_i(f_i)	w_i	value	v(x)
3	HIM (i=1)	7	4	33	0.6671	0.5	0.33	**0.61**
4	HIW (i=2)	3	5	24	0.5507	0.5	0.28	
5								
6		Soccer	Soap Opera					
7	dec.var	3	3					
8								
9	low.bnd.	0	0					
10	up. bnd.	11	11					
11				Tot.cost		Budget		
12	Costs	100	90	570	<=	600		
13								



Example: MAU(V)T + optimization (Cont'd)

- Complete enumeration of all feasible solutions
  - Possible because the problem is so small



Stochastic Programming (SP)

- Some parameters of the optimization problem are uncertain and thus modelled as random variables / probability distributions
  - The objective functions and constraints are some functionals of these distributions/random variables
    - Expected monetary value, Expected utility
    - Risk measures (e.g., CVaR)
    - Probability of some event (e.g., Probability of exceeding budget max. 5%)

- Solving an SP problem is usually based on formulating it as a standard (deterministic) optimization problem (e.g. MILP, NLP)
  - General approach: Replace the continuous distribution with a finite set of **scenarios** that capture the possible outcomes of the random variable(s)

SP: Tabloid Publishing

- Each day Tabloid Publishing decides on the number of newspapers to print (q) before knowing the demand
  - Uncertain demand D: a random variable with the CDF $F_D(d)$
  - Cost of printing is 0.3€ per paper, each paper is sold at 1.2€
  - Thus, the profit is the random variable

$$Z = \begin{cases} 0.9q, \text{if } D \geq q \\ 1.2D - 0.3q, \text{if } D < q \end{cases} = -0.3q + 1.2 \cdot \min\{D, q\}$$

→Stochastic Programming problem to maximize expected profit:

$$\max_q \mathbb{E}[-0.3q + 1.2 \cdot \min\{D, q\}\ ]$$

SP: Tabloid Publishing (Cont'd)

$$\max_{q} \mathbb{E}[-0.3q + 1.2 \cdot \min\{D, q\}\ ] \qquad \textbf{(SP)}$$

Formulating an LP model to approximate this SP problem:

- Approximate D with a suitable discrete distribution with n possible outcomes $d_1, \dots, d_n$. Denote $P(D = d_i) = p_i$.*

$$\max_{q} \sum_{i=1}^{n} p_i \left(-0.3q + 1.2 \cdot \min\{d_i, q\}\right) \qquad \textbf{(NLP)}$$

- Linearize by introducing additional decision variables z_i and constraints:

$$\max_{q, z_i} \sum_{i=1}^{n} p_i\, z_i \qquad \textbf{(LP)}$$

$$z_i \leq -0.3q + 1.2d_i, i = 1, \dots, n$$

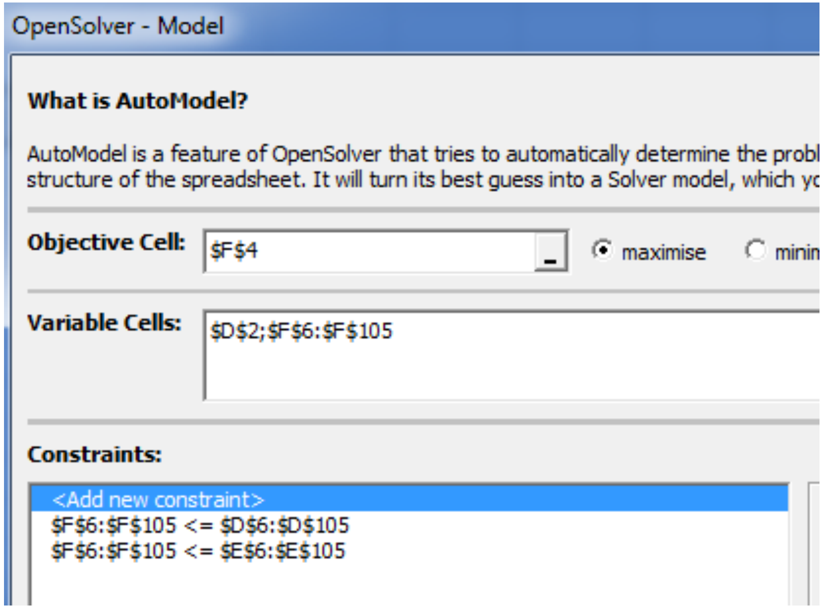
$$z_i \leq -0.3q + 1.2q, i = 1, \dots, n$$

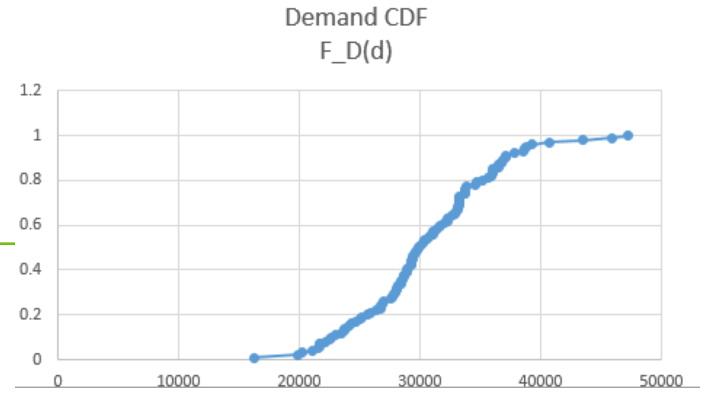
- Question: Why does the LP provide the same optimal solution as the NLP?

*) If D is continuous and $d_1, \dots, d_n$ is a sample from it (e.g. historical observations or generated by the inverse CDF method) then a reasonable assumption is that $p_i = \frac{1}{n}$ for all i

SP: Tabloid Publishing (Cont'd)

	A	B	C	D	E	F	G	H
1				q				
2				33780				
3								
4	Average	50.5	30487.62	26451.144	30402	25274.83		
5		i	d_i	-0.3q+1.2*d_i	-0.3q+1.2*q	z_i		F_D(d_i)=i/n
6		1	16238	9351.6	30402	9351.6		0.01
7		2	19934	13786.8	30402	13786.8		0.02
8		3	20224	14134.8	30402	14134.8		0.03
9		4	21103	15189.6	30402	15189.6		0.04
10		5	21510	15678	30402	15678		0.05
11		6	21625	15816	30402	15816		0.06
12		7	21698	15903.6	30402	15903.6		0.07
13		8	22204	16510.8	30402	16510.8		0.08
14		9	22513	16881.6	30402	16881.6		0.09
15		10	22623	17013.6	30402	17013.6		0.1
16		11	23014	17482.8	30402	17482.8		0.11
17		12	23534	18106.8	30402	18106.8		0.12
18		13	23761	18379.2	30402	18379.2		0.13
19		14	23785	18408	30402	18408		0.14
20		15	24143	18837.6	30402	18837.6		0.15
21		16	24319	19048.8	30402	19048.8		0.16
22		17	24639	19432.8	30402	19432.8		0.17
23		18	25032	19904.4	30402	19904.4		0.18
24		19	25163	20061.6	30402	20061.6		0.19
25		20	25664	20662.8	30402	20662.8		0.2
26		21	25854	20890.8	30402	20890.8		0.21
27		22	26358	21495.6	30402	21495.6		0.22
28		23	26706	21913.2	30402	21913.2		0.23
29		24	26755	21972	30402	21972		0.24
30		25	26906	22153.2	30402	22153.2		0.25
31		26	26953	22209.6	30402	22209.6		0.26
32		27	27617	23006.4	30402	23006.4		0.27





SP Example: R&D portfolio

- A technology company selects an R&D project portfolio from 30 project proposals, whose values depend on the success of either technology A or B
  - Limited budget (1.2M) and human resources (50 py)
- The company has
  - Constructed pessimistic (-), zero (0) and optimistic (+) scenarios for both technologies' success
  - Assessed the projects' cash flows in each scenario (CF_{ji})
  - Assessed the joint probabilities of the scenarios
- Question: Are the scenarios for the technologies' success independent?

	Technology B		
	-	0	+
Project B1	80	110	200
Project B2	60	150	190
Project B3	70	150	240
Project B4	140	160	270
Project B5a	40	220	230
Project B5b	50	170	290
Project B6	200	230	260
Project B7	40	240	310
Project B8	120	260	320
Project B9	230	270	320
Project B10	200	260	330
Project B11	60	250	360
Project B12	130	240	380
Project B13	180	370	480

	Technology A		
	-	0	+
Project A1	0	0	380
Project A2	0	0	420
Project A3	0	10	540
Investment A1-3	0	0	0
Project A4.0	80	100	520
Project A4.1	20	130	690
Project A5	70	110	360
Project A6	130	230	230
Project A7a	0	10	460
Project A7b	30	60	420
Project A8	40	40	440
Project A9	60	70	420
Project A10	60	70	450
Project A11	150	180	180
Project A12	40	190	190
Project A13	100	230	230

		Technology A			
		-	0	+	sum
Tech. B	-	**0.04**	**0.02**	**0.08**	0.14
	0	**0.04**	**0.35**	**0.20**	0.59
	+	**0.10**	**0.16**	**0.01**	0.27
	sum	0.18	0.53	0.29	1.00

SP Example - R&D portfolio: The Mathematical Model

1. $\max_{z \in \{0,1\}^{30}} \sum_{i=1}^{9} p_i \, u\left(\sum_{j=1}^{30} z_j \, CF_{ji}\right)$
2. $\sum_{j=1}^{30} z_j C_j \leq 1200$
3. $\sum_{j=1}^{30} z_j HR_j \leq 50$
4. $z_1 + z_2 + z_3 \leq 3z_4$
5. $z_5 \geq z_6$
6. $z_9 + z_{10} \leq 1$
7. $z_{21} + z_{22} \leq 1$

		Technology A		
		-	0	+
Tech. B	-	s1	s2	s3
	0	s4	s5	s6
	+	s7	s8	s9

	Cost	HR	Decision variable
Project A1	44	5	z1
Project A2	31	5	z2
Project A3	30	6	z3
Investment A1-3	80	0	z4
Project A4.0	79	6	z5
Project A4.1	85	6	z6
Project A5	142	2	z7
Project A6	121	2	z8
Project A7a	132	6	z9
Project A7b	111	8	z10
Project A8	87	6	z11
Project A9	132	9	z12
Project A10	117	8	z13
Project A11	96	6	z14
Project A12	145	8	z15
Project A13	101	2	z16
Project B1	98	2	z17
Project B2	182	4	z18
Project B3	183	6	z19
Project B4	224	2	z20
Project B5a	105	9	z21
Project B5b	157	10	z22
Project B6	177	3	z23
Project B7	139	4	z24
Project B8	184	9	z25
Project B9	157	7	z26
Project B10	254	2	z27
Project B11	224	4	z28
Project B12	224	5	z29
Project B13	331	8	z30

Question: What is the interpretation for the objective function and constraints?

SP Example - R&D portf: The Excel model

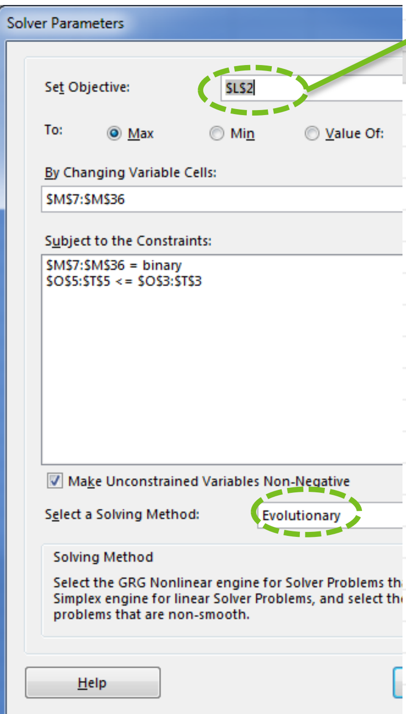
=SUMPRODUCT(H7:H36;$M$7:$M$36)

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T
1	Probability	0.04	0.02	0.08	0.04	0.35	0.2	0.1	0.16	0.01		**Expected**								
2	**Utility**	0	0	0.131	0	0	0.131	0	0	0.131		0.0381								
3	**Portfolio cash**	0	0	380	0	0	380	$36)	0	380		110.2			1200	50	0	0	1	1
4	**Scenario**	**s1**	**s2**	**s3**	**s4**	**s5**	**s6**	**s7**	**s8**	**s9**					≥	≥	≥	≥	≥	≥
5	**Technology A**	**-**	**0**	**+**	**-**	**0**	**+**	**-**	**0**	**+**			Decision		44	5	1	0	0	0
6	**Technology B**	**-**	**-**	**-**	**0**	**0**	**0**	**+**	**+**	**+**			variables		Cost	HR	A1-A3	A4	A7	B5
7	Project A1	0	0	380	0	0	380	0	0	380			1		44	5	1			
8	Project A2	0	0	420	0	0	420	0	0	420			0		31	5	1			
9	Project A3	0	10	540	0	10	540	0	10	540			0		30	6	1			
10	Investment A1-3	0	0	0	0	0	0	0	0	0			0		80	0	-3			
11	Project A4.0	80	100	520	80	100	520	80	100	520			0		79	6		-1		
12	Project A4.1	20	130	690	20	130	690	20	130	690			0		85	6		1		
13	Project A5	70	110	360	70	110	360	70	110	360			0		142	2				
14	Project A6	130	230	230	130	230	230	130	230	230			0		121	2				
15	Project A7a	0	10	460	0	10	460	0	10	460			0		132	6			1	
16	Project A7b	30	60	420	30	60	420	30	60	420			0		111	8			1	
17	Project A8	40	40	440	40	40	440	40	40	440			0		87	6				
18	Project A9	60	70	420	60	70	420	60	70	420			0		132	9				
19	Project A10	60	70	450	60	70	450	60	70	450			0		117	8				
20	Project A11	150	180	180	150	180	180	150	180	180			0		96	6				
21	Project A12	40	190	190	40	190	190	40	190	190			0		145	8				
22	Project A13	100	230	230	100	230	230	100	230	230			0		101	2				
23	Project B1	80	80	80	110	110	110	200	200	200			0		98	2				
24	Project B2	60	60	60	150	150	150	190	190	190			0		182	4				
25	Project B3	70	70	70	150	150	150	240	240	240			0		183	6				
26	Project B4	140	140	140	160	160	160	270	270	270			0		224	2				
27	Project B5a	40	40	40	220	220	220	230	230	230			0		105	9				1
28	Project B5b	50	50	50	170	170	170	290	290	290			0		157	10				1
29	Project B6	200	200	200	230	230	230	260	260	260			0		177	3				
30	Project B7	40	40	40	240	240	240	310	310	310			0		139	4				
31	Project B8	120	120	120	260	260	260	320	320	320			0		184	9				
32	Project B9	230	230	230	270	270	270	320	320	320			0		157	7				
33	Project B10	200	200	200	260	260	260	330	330	330			0		254	2				
34	Project B11	60	60	60	250	250	250	360	360	360			0		224	4				
35	Project B12	130	130	130	240	240	240	380	380	380			0		224	5				
36	Project B13	180	180	180	370	370	370	480	480	480			0		331	8				

=1-EXP(-C3/2700)

SP Example Excel model

– Maximize expected utility

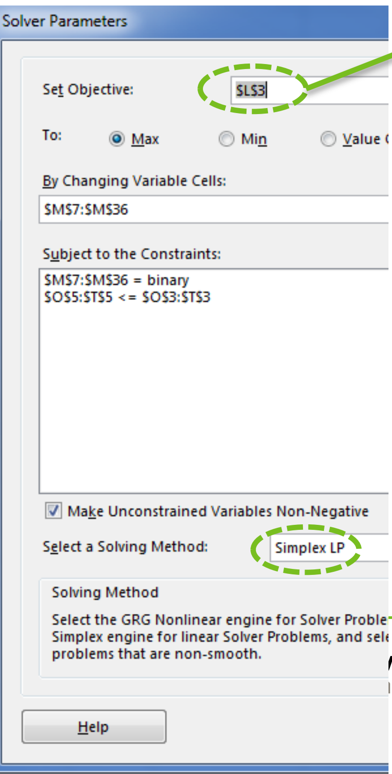


	A	B	C	D	E	F	G	H	I	J
1	Probability	0.04	0.02	0.08	0.04	0.35	0.2	0.1	0.16	0.01
2	**Utility**	0.327	0.418	0.595	0.437	0.513	0.661	0.487	0.556	0.691
3	**Portfolio cash**	1070	1460	2440	1550	1940	2920	1800	2190	3170
4	**Scenario**	**s1**	**s2**	**s3**	**s4**	**s5**	**s6**	**s7**	**s8**	**s9**
5	**Technology A**	**-**	**0**	**+**	**-**	**0**	**+**	**-**	**0**	**+**
6	**Technology B**	**-**	**-**	**-**	**0**	**0**	**0**	**+**	**+**	**+**
7	Project A1	0	0	380	0	0	380	0	0	380
8	Project A2	0	0	420	0	0	420	0	0	420
9	Project A3	0	10	540	0	10	540	0	10	540
10	Investment A1-3	0	0	0	0	0	0	0	0	0
11	Project A4.0	80	100	520	80	100	520	80	100	520
12	Project A4.1	20	130	690	20	130	690	20	130	690
13	Project A5	70	110	360	70	110	360	70	110	360
14	Project A6	130	230	230	130	230	230	130	230	230
15	Project A7a	0	10	460	0	10	460	0	10	460
16	Project A7b	30	60	420	30	60	420	30	60	420
17	Project A8	40	40	440	40	40	440	40	40	440
18	Project A9	60	70	420	60	70	420	60	70	420
19	Project A10	60	70	450	60	70	450	60	70	450
20	Project A11	150	180	180	150	180	180	150	180	180
21	Project A12	40	190	190	40	190	190	40	190	190
22	Project A13	100	230	230	100	230	230	100	230	230
23	Project B1	80	80	80	110	110	110	200	200	200
24	Project B2	60	60	60	150	150	150	190	190	190
25	Project B3	70	70	70	150	150	150	240	240	240
26	Project B4	140	140	140	160	160	160	270	270	270
27	Project B5a	40	40	40	220	220	220	230	230	230
28	Project B5b	50	50	50	170	170	170	290	290	290
29	Project B6	200	200	200	230	230	230	260	260	260
30	Project B7	40	40	40	240	240	240	310	310	310
31	Project B8	120	120	120	260	260	260	320	320	320
32	Project B9	230	230	230	270	270	270	320	320	320
33	Project B10	200	200	200	260	260	260	330	330	330
34	Project B11	60	60	60	250	250	250	360	360	360
35	Project B12	130	130	130	240	240	240	380	380	380
36	Project B13	180	180	180	370	370	370	480	480	480

	L	M	O	P	Q	R	S	T
1	**Expected**							
2	0.5425							
3	2154.3		1200	50	0	0	1	1
4			≥	≥	≥	≥	≥	≥
5		Decision	1158	47	0	0	0	1
6		variables	Cost	HR	A1-A3	A4	A7	B5
7		0	44	5	1			
8		0	31	5	1			
9		0	30	6	1			
10		0	80	0	-3			
11		1	79	6		-1		
12		1	85	6		1		
13		0	142	2				
14		1	121	2				
15		0	132	6			1	
16		0	111	8			1	
17		0	87	6				
18		0	132	9				
19		0	117	8				
20		1	96	6				
21		0	145	8				
22		1	101	2				
23		1	98	2				
24		0	182	4				
25		0	183	6				
26		0	224	2				
27		1	105	9				1
28		0	157	10				1
29		1	177	3				
30		1	139	4				
31		0	184	9				
32		1	157	7				
33		0	254	2				
34		0	224	4				
35		0	224	5				
36		0	331	8				

SP Example Excel model

- Maximize expected cash flows



	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T
1	Probability	0.04	0.02	0.08	0.04	0.35	0.2	0.1	0.16	0.01		**Expected**								
2	**Utility**	0.317	0.411	0.712	0.389	0.473	0.742	0.441	0.518	0.764		0.5420								
3	**Portfolio cash**	1030	1430	3360	1330	1730	3660	1570	1970	3900		2240.5			1200	50	0	0	1	1
4	**Scenario**	**s1**	**s2**	**s3**	**s4**	**s5**	**s6**	**s7**	**s8**	**s9**					⇓	⇓	⇓	⇓	⇓	⇓
5	**Technology A**	**-**	**0**	**+**	**-**	**0**	**+**	**-**	**0**	**+**			Decision		1194	49	-1	0	0	0
6	**Technology B**	**-**	**-**	**-**	**0**	**0**	**0**	**+**	**+**	**+**			variables		Cost	HR	A1-A3	A4	A7	B5
7	Project A1	0	0	380	0	0	380	0	0	380			0		44	5	1			
8	Project A2	0	0	420	0	0	420	0	0	420			1		31	5	1			
9	Project A3	0	10	540	0	10	540	0	10	540			1		30	6	1			
10	Investment A1-3	0	0	0	0	0	0	0	0	0			1		80	0	-3			
11	Project A4.0	80	100	520	80	100	520	80	100	520			1		79	6		-1		
12	Project A4.1	20	130	690	20	130	690	20	130	690			1		85	6		1		
13	Project A5	70	110	360	70	110	360	70	110	360			0		142	2				
14	Project A6	130	230	230	130	230	230	130	230	230			1		121	2				
15	Project A7a	0	10	460	0	10	460	0	10	460			0		132	6			1	
16	Project A7b	30	60	420	30	60	420	30	60	420			0		111	8			1	
17	Project A8	40	40	440	40	40	440	40	40	440			0		87	6				
18	Project A9	60	70	420	60	70	420	60	70	420			0		132	9				
19	Project A10	60	70	450	60	70	450	60	70	450			0		117	8				
20	Project A11	150	180	180	150	180	180	150	180	180			1		96	6				
21	Project A12	40	190	190	40	190	190	40	190	190			0		145	8				
22	Project A13	100	230	230	100	230	230	100	230	230			1		101	2				
23	Project B1	80	80	80	110	110	110	200	200	200			1		98	2				
24	Project B2	60	60	60	150	150	150	190	190	190			0		182	4				
25	Project B3	70	70	70	150	150	150	240	240	240			0		183	6				
26	Project B4	140	140	140	160	160	160	270	270	270			0		224	2				
27	Project B5a	40	40	40	220	220	220	230	230	230			0		105	9				1
28	Project B5b	50	50	50	170	170	170	290	290	290			0		157	10				1
29	Project B6	200	200	200	230	230	230	260	260	260			1		177	3				
30	Project B7	40	40	40	240	240	240	310	310	310			1		139	4				
31	Project B8	120	120	120	260	260	260	320	320	320			0		184	9				
32	Project B9	230	230	230	270	270	270	320	320	320			1		157	7				
33	Project B10	200	200	200	260	260	260	330	330	330			0		254	2				
34	Project B11	60	60	60	250	250	250	360	360	360			0		224	4				
35	Project B12	130	130	130	240	240	240	380	380	380			0		224	5				
36	Project B13	180	180	180	370	370	370	480	480	480			0		331	8				

Supplementary material on optimization problem types
(Covered on Business Analytics I)

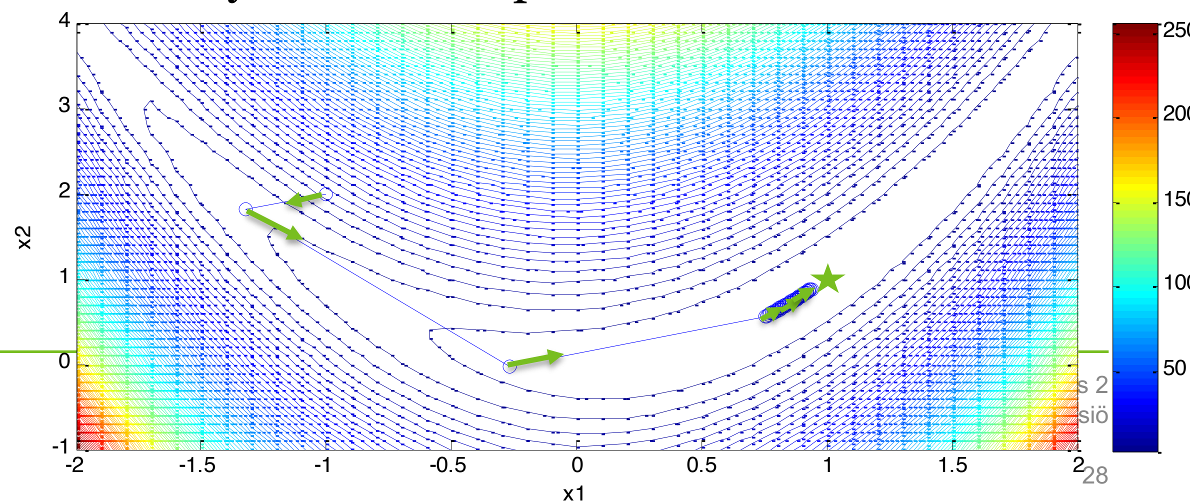
Computer solution to NLP problems

- The GRG algorithm in Solver is based on gradient search ("hill-climbing")
  - With the initial starting solution, a direction is computed that most rapidly improves the objective function value
  - Solution is moved (values of decision variables changed) to this direction until
    - a constraint boundary is encountered OR
    - the objective function value no longer improves
  - A new direction is computed with the new solution and the process is repeated until no further improvement in any direction is possible
- Example:

  min $(x_2-x_1^2)^2 + (1-x_1)^2$

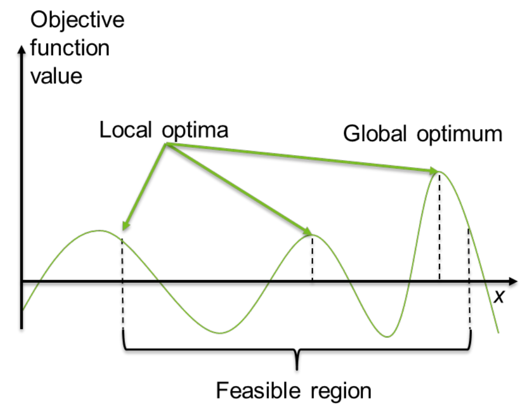
  $s.t.\ \ x_1,x_2 \geq -2$

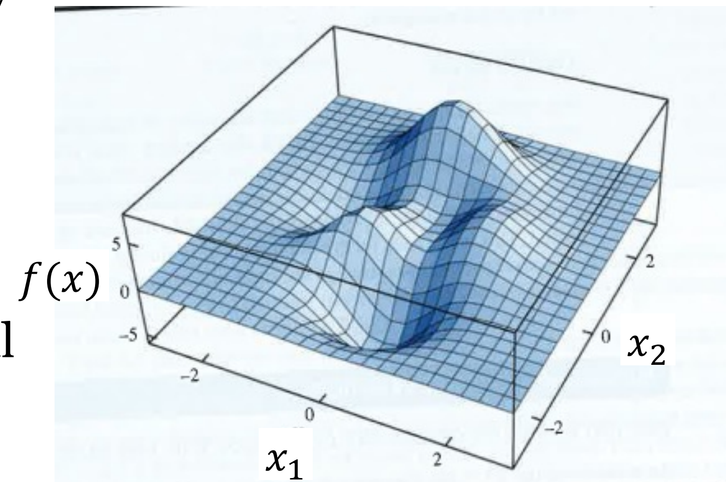
  Initial solution (-1,2)



Global and local optimal solutions

- For NLP problems we often do not have a guarantee that the optimal solution is a global
  - I.e. no other feasible solution provides a better objective function value
- Most NLP algorithms terminate when they have found a **local** optimal solution
  - I.e. a feasible solution such that all neighboring feasible solutions are worse
- Special case:
  - For “convex/concave NLPs” any local optimal solution is a global optimal solution.





Convex/Concave NLPs

- **Definition.** Convex NLP:
  - A convex[1] objective func. is minimized
  - The feasible region is convex*

Note! a maximization problem can be transformed to minimization problem:

$$\max_x f(x) = -[\min_x -f(x)]$$

Hence the two definitions are equivalent!

- **Definition.** Concave NLP:
  - A concave[1] obj. func. is maximized
  - The feasible region is convex*

- How to determine if the constraints produce a convex feasible region?
  - If the LHS function g_j of each ≤ constraint is convex*, then the feasible region is convex
  - Note: Integer valued decision variables result in a feasible region that is not convex (it is not even connected)

1) Definition: A **set** S is convex $a, b \in S \Rightarrow [\lambda a + (1-\lambda)b] \in S \forall \lambda \in [0,1]$

*) Definition of convex/concave **functions** in Lecture "Risk preferences"

Evolutionary algorithms

- If an optimization problem is non-linear but its not a convex NLP, how to solve it?
- One possibility: Evolutionary Algorithms.
- Evolutionary algorithms are **heuristic**: i.e., produce a “good” feasible solution, but no guarantees on optimality
- Idea: A large set of solutions (”population”) simulated through multiple iterations (”generations”)
  - On each iteration :
    - Solutions with best objective function value (”fitness”) are combined to produce new solutions (”reproduction”)
    - Random changes to some solutions (”mutation”)
    - Infeasible solution and those with poor objective function value (”unfit”) are deleted
- Excel solver includes an evolutionary algorithm