Aalto University
School of Business

Business Analytics 2 – Lecture 8: Multi-Attribute Utility Theory

- MAUT: EUT with multidimensional outcomes
- Preference Independence of attributes
- The additive multiattribute utility function
- Assessing attribute-specific utility functions
- Assessing weights: The tradeoff approach
- Supplementary material on preference assumptions in MAUT: Preference, Utility, and Additive Independence

Multi-attribute Utility Theory (MAUT) - Motivation

- Many problems have multiple objectives:
  - Planning the national budget
    - improve social security, reduce debt, cut taxes, build national defense
  - Planning an advertising campaign
    - reach, expenses, target groups
  - Designing a distribution system
    - minimize transportation costs, minimize CO_2 emissions
  - Planning an investment portfolio
    - maximize expected returns, minimize risk

- MAUT is EUT applied to multi-objective problems
  - Attribute: a measure for the achievement of an objective (="criterion" also)

From EUT to MAUT

- The set of possible outcomes T is multidimensional, denoted by A:

$$A = A_1 \times \cdots \times A_n$$

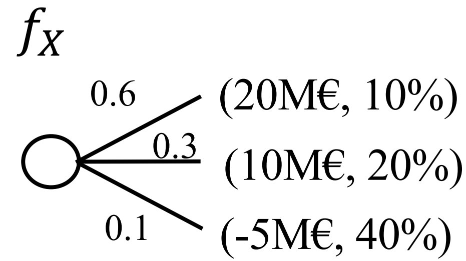
  - Examples of attributes:
    - revenue ($)
    - CO_2-reduction
    - Social responsibility
- Set of all possible lotteries L:
  - A lottery $f \in L$ associates a probability $f(x) \in [0,1]$ with each possible outcome

$$x = (x_1, \ldots x_n) \in A$$

  - Deterministic outcomes are modelled as degenerate lotteries

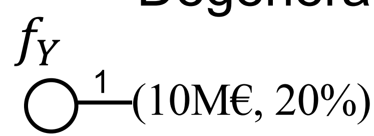
$n = 2$ attributes:
A_1: Net present value;
A_2: market share

Lottery



$$f_X(x) = \begin{cases} 0.6, x = (20,0.1) \\ 0.3, x = (10,0.2) \\ 0.1, x = (-5,0.4) \\ 0, \text{elsewhere} \end{cases}$$

Degenerate Lottery



$$f_Y(x) = \begin{cases} 1, x = (10,0.2) \\ 0, \text{elsewhere} \end{cases}$$

Multi-attribute Preference representation with EU

- $\succcurlyeq$ satisfies axioms A1-A4 if and only if there exists a real-valued utility function over the set of outcomes $u: A \rightarrow \mathbb{R}$ such that

$$f_X \succcurlyeq f_Y \Leftrightarrow \sum_{x \in A} f_X(x)u(x) \geq \sum_{x \in A} f_Y(x)u(x)$$

where $x = (x_1, \ldots, x_n)$ and $\sum_{x \in A} f_X(x)u(x)$ is the expected utility

- We say u represents preferences $\succcurlyeq$
- Remember:
  - For continuous random variables, EU is computed as an integral
  - u is unique up to positive affine transformations

$u: A \rightarrow \mathbb{R}$

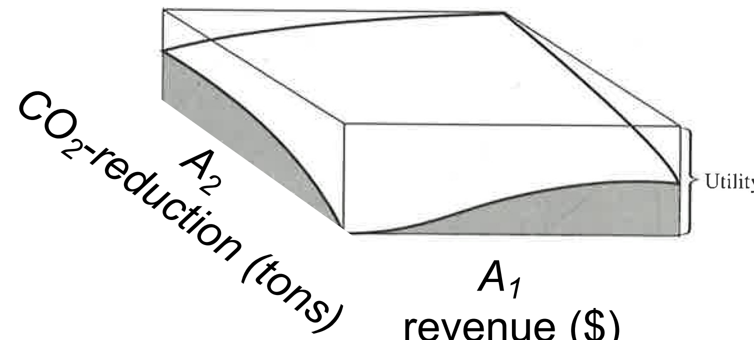
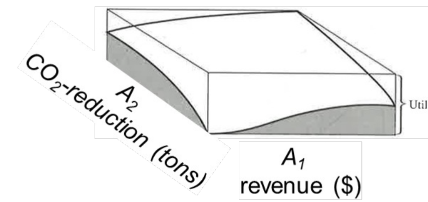


Figure: Clemen (1996). Making Hard Decisions: An introduction to decision analysis

EUT vs. MAUT

- Since EUT does not require outcomes to be unidimensional, in principle EUT contains MAUT
- But the assessment of $u: A_1 \times \cdots \times A_n \to \mathbb{R}$ is much more complicated than assessment of $u: T \to \mathbb{R}$

$u: A \to \mathbb{R}$



- In fact MAUT is mainly a collection of methods and models to decompose the assessment of u into two parts
  1. Assessing the attribute-specific utility functions $u_i: A_i \to \mathbb{R}, i = 1, \dots, n$
  2. Choosing a functional form to aggregate $u_1, \dots, u_n$ to overall utility u
  - Step 1 is similar to assessing unidimensional utility functions
  - For Step 2, we have to make some assumptions about preferences among the multiple attributes

Preference Independence (PI)

- **Definition.** A Subset of attributes $S \subset \{A_1, \dots, A_n\}$ is **PI** if the preference order of degenerate lotteries that differ only on attributes in S does not depend on the levels of the rest of the attributes $\bar{S}$, i.e.,
$$(x_S^I, x_{\bar{S}}) \succcurlyeq (x_S^{II}, x_{\bar{S}}) \Rightarrow (x_S^I, x'_{\bar{S}}) \succcurlyeq (x_S^{II}, x'_{\bar{S}}) \forall x'_{\bar{S}}$$

- **Example:** Consider an investment selection problem with three attributes: Profits, Market share, and CO_2 reduction. Is S={Profits} preference independent?
  - $\bar{S}$ ={Market Share, CO_2 reduction}
  - Take arbitrary *a* and *b* such that (*a*M€, 10%, 1ton) $\succcurlyeq$ (*b*M€, 10%, 1ton).
  - Then, is it true that (*a*M€, *c*%, *d* tons) $\succcurlyeq$ (*b*M€, *c*%, *d* tons) for any *c* and *d*?
  - It the answer is yes, then {Profits} is PI
- Note! The definition of the most and least preferred levels $x_i^*, x_i^0 \in A_i$ for attribute $\{A_i\}$ is possible only if the attribute is PI.

Preference Independence (PI): Meal example

- Consider choosing a meal with three attributes:
  - Wine: A_1={red, white}
  - Main: A_2= {beef, fish}
  - Side: A_3= {potatoes, rice}
  - The DM states that:
    1. "I prefer red wine to white wine": (red, b, c) $\succ$ (white, b, c) $\forall b \in A_2, c \in A_3$
    2. "I prefer beef to fish": (a, beef, c) $\succ$ (a, fish, c) $\forall a \in A_1, c \in A_3$
    3. "I prefer potatoes to rice": (a, b, potatoes) $\succ$ (a, b, rice) $\forall a \in A_1, b \in A_2$
    4. (red, beef, rice) $\succ$ (white, beef, potatoes)
    5. (red, fish, rice) $\prec$ (white, fish, potatoes)

E.g. vector
(white, fish, rice) is a meal

- Questions:
  - Which of the attributes {wine}, {main} and {side} are PI?
  - Is the subset of attributes {wine, side} PI?

Additive Multiattribute Utility Function

- **Theorem.** If <u>each</u> subset of attributes is PI and some other assumptions* hold, then preferences $\succcurlyeq$ are represented by an additive MAU function

$$u(x) = \sum_{i=1}^{n} w_i u_i(x_i) + u(x^0), \text{where}$$

  - $u_i(x_i) = \frac{u(x_1^0, x_2^0, \ldots, x_i, \ldots, x_n^0) - u(x^0)}{u(x_1^0, x_2^0, \ldots, x_i^*, \ldots, x_n^0) - u(x^0)} \in [0,1]$ is the attribute-specific utility function for A_i
  - $w_i = [u(x_1^0, x_2^0, \ldots, x_i^*, \ldots, x_n^0) - u(x^0)] > 0$ is the "importance" weight for attribute A_i
  - $u(x^*) - u(x^0) = \sum_{i=1}^{n} w_i$

Notation:

Vector of attribute levels (an alternative): $x = (x_1, \ldots, x_n)$

All attributes at the most preferred level: $x^* = (x_1^*, \ldots, x_n^*)$

All attributes at the least preferred level: $x^0 = (x_1^0, \ldots, x_n^0)$

*) See supplementary slides at the end of the presentation

Standard Scaling of the Additive MAU function

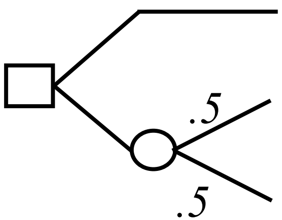
- Remember from EUT: Utilities for two outcomes can be fixed
- The theorem on the previous slide left the numerical values for $u(x^0)$ and $u(x^*)$ unspecified
  - Any values can be selected (as long as $u(x^0) < u(x^*)$)
  - Usually $u(x^0) = 0$ and $u(x^*) = 1$, which leads to

$$u(x) = \sum_{i=1}^{n} w_i u_i(x_i)\,, \text{where} \sum_{i=1}^{n} w_i = 1$$

  - It would be possible to choose some other multi-attribute outcomes $x^+, x^- \in A$, $x^+ \succ x^-$, and scale the utility function so that $u(x^+) = 1$ and $u(x^-) = 0$
    - E.g. In some applications we might want the status-quo alternative (or its CE) to have zero utility to help communicate the model results

Assessing Attribute-specific Utility Functions

- Use same techniques as with unidimensional utility functions
  - Certainty equivalent, probability equivalent, etc; Scale so that $u_i(x_i^0) = 0$ and $u_i(x_i^*) = 1$ for each attribute i.
  - Also direct "scoring" often applied in practice
- What about the levels of other attributes ?
  - Fixed at the same level in every outcome (cf. 4€)
    - Usually not explicitly shown to the DM
  - Results do not depend on what this level is because we are assuming that preferences can be represented with the additive utility function
    - E.g., utility of apples with the additive utility function:



$$u(x_1,4) = 0.5u(50,4) + 0.5u(-10,4)$$
$$\Leftrightarrow w_1u_1(x_1) + w_2u_2(4) = 0.5w_1u_1(50) + 0.5w_2u_2(4) + 0.5w_1u_1(-10) + 0.5w_2u_2(4)$$
$$\Leftrightarrow w_1u_1(x_1) = 0.5w_1u_1(50) + 0.5w_1u_1(-10)$$
$$\Leftrightarrow u_1(x_1) = 0.5u_1(50) + 0.5u_1(-10)$$

(The amount of euros does not matter!)

Example: Choosing a Software Supplier

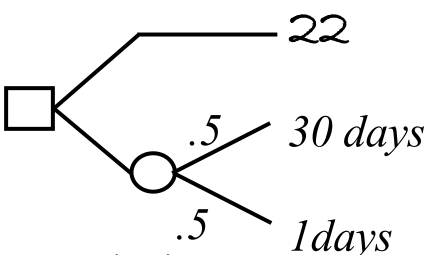
- Step 1: Generate objectives
  - Minimize cost, Minimize Delay, Maximize quality
- Step 2: Develop attributes to measure the achievement of the objectives
  - Cost obtained from suppliers' offers
  - A numerical evaluation of delay
  - A verbal evaluation based on recommendations is used to measure quality

i	Name	A_i	x_i^0	x_i^*
1	Cost	[10,40] k€	40	10
2	Delay	{1,2,…,30} days	30	1
3	Quality	{fair, good, excellent}	fair	exc.

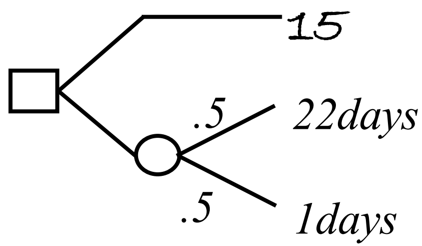
Example: Choosing a Software Supplier (Cont'd)

i	Name	A_i	x_i^0	x_i^*
1	Cost	[10,40] €	40	10
2	Delay	{1,2,…,30} days	30	1
3	Quality	{fair, good, excellent}	fair	exc.

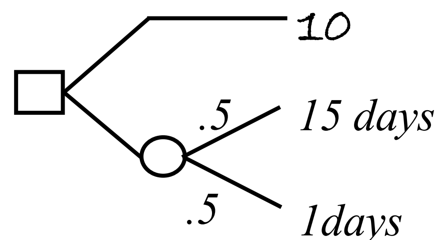
- Step 3: Assess attribute-specific utility functions
  - Quality: Direct assessment
    - u_3(fair)=0, u_3(good)=0.4 and u_3(exc.)=1
  - Cost: Linear utility function
    - $u_1(x_1) = \frac{40-x_1}{30}$
  - Delay: Assessment with CE approach



$u_2(22)$
$= 0.5u_2(1) + 0.5u_2(30)$
$= 0.5 * 1 + 0.5 * 0$
$= 0.5$



$u_2(15)$
$= 0.5u_2(1) + 0.5u_2(22)$
$= 0.5 * 1 + 0.5 * 0.5$
$= 0.75$



$u_2(10)$
$= 0.5u_2(1) + 0.5u_2(15)$
$= 0.5 * 1 + 0.5 * 0.75$
$= 0.875$

Example: Choosing a Software Supplier (Cont'd)

Attribute weights (wi)		
Cost	Delay	Quality

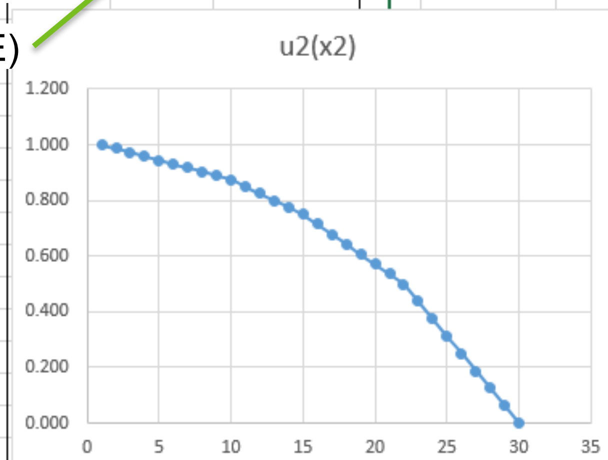
	Attribute-specific performances			Attribute-specific utilities		
	Cost	Delay	Quality	Cost	Delay	Quality
Supplier	x1	x2	x3	u1(x1)	u2(x2)	u3(x3)
x^0	40	30	fair	0.00	0.00	0.00
x^*	10	1	excellent	1.00	1.00	1.00
Dummy	25	20	good	0.50	0.57	0.40

=(40-B11)/30

=VLOOKUP(C11;$O$2:$P$31;2;FALSE)

=VLOOKUP(D11;$L$14:$M$16;2;FALSE)

x3	u3(x3)
fair	0
good	0.4
excellent	1



x2	u2(x2)
1	1.000
2	0.986
3	0.972
4	0.958
5	0.944
6	0.931
7	0.917
8	0.903
9	0.889
10	0.875
11	0.850
12	0.825
13	0.800
14	0.775
15	0.750
16	0.714
17	0.679
18	0.643
19	0.607
20	0.571
21	0.536
22	0.500
23	0.438
24	0.375
25	0.313
26	0.250
27	0.188
28	0.125
29	0.063
30	0.000

Assessing attribute weights: Tradeoff-weighting

- First, ask the DM to establish a preference order for n hypothetical alternatives $(x_1^0, \dots, x_i^*, \dots, x_n^0)$, $i=1,\dots,n$
  - ith attribute has the most preferred outcome; others the least preferred
- Assume this results in the order

$$(x_1^*, x_2^0, \dots, x_n^0) \succcurlyeq (x_1^0, x_2^*, x_3^0 \dots, x_n^0) \succcurlyeq \cdots \succcurlyeq (x_1^0, \dots, x_i^*, \dots, x_n^0) \succcurlyeq \cdots \succcurlyeq (x_1^0, \dots, x_{n-1}^0, x_n^*)$$

- Then, for each $i=2,\dots,n$ ask the DM to define $x_i \in A_i$ such that

$$(\dots, x_i, x_{i+1}^0, \dots) \sim (\dots, x_i^0, x_{i+1}^*, \dots)$$

She is indifferent between the two alternatives

  - For each i this results in linear equation

$$u(\dots, x_i, x_{i+1}^0, \dots) = u(\dots, x_i^0, x_{i+1}^*, \dots)$$
$$\Leftrightarrow w_i u_i(x_i) = w_{i+1} u_{i+1}(x_{i+1}^*)$$
$$\Leftrightarrow w_i u_i(x_i) = w_{i+1}$$

Other terms of the sum cancel out

$u_i(x_i)$ is known

- These n-1 equations together with $\sum w_i = 1$ are sufficient to solve the values on the n unknowns $w_1, \dots, w_n$

Example: SW supplier (Cont'd)

i	Name	A_i	x_i^0	x_i^*
1	Cost	[10,40] €	40	10
2	Delay	{1,2,…,30} days	30	1
3	Quality	{fair, good, excellent}	fair	exc.

- Step 4: Assess attribute weights
  - Assume the DM states: $(40,1,\text{fair}) \succcurlyeq (10,30,\text{fair}) \succcurlyeq (40,30,\text{exc.})$
  - Choose delay $x_2 \in \{1,\ldots,30\}$ such that $(40,x_2,x_3) \sim (10,30,x_3)$
  - Answer *x_2=8* gives

$$w_1u_1(40) + w_2u_2(8) + w_3u_3(x_3) = w_1u_1(10) + w_2u_2(30) + w_3u_3(x_3)$$
$$w_2u_2(8) = w_1$$
$$\Leftrightarrow w_2 \cdot 0.9 = w_1$$

Value of x_3 does not matter

  - Choose cost $x_1 \in [10,40]$ such that $(x_1,x_2,\text{fair}) \sim (40,x_2,\text{excl.})$
  - Answer *x_1=20* gives

$$w_1u_1(20) + w_2u_2(x_2) + w_3u_3(\text{fair}) = w_1u_1(40) + w_2u_2(x_2) + w_3u_3(\text{excl.})$$
$$w_1u_1(20) = w_3$$
$$\Leftrightarrow w_1 \cdot \frac{2}{3} = w_3$$

Value of x_2 does not matter

Values for $u_1(20)$ and $u_2(8)$ obtained from the attribute-specific utility functions

Example: Choosing a Software Supplier (Cont'd)

- Step 4: Assess attribute weights (cont'd)

$$w_2 \cdot 0.9 = w_1$$
$$w_1 \cdot \frac{2}{3} = w_3$$
$$\sum_i w_i = 1$$

i	Name	A_i	x_i^0	x_i^*
1	Cost	[10,40] €	40	10
2	Delay	{1,2,…,30} days	30	1
3	Quality	{fair, good, excellent}	fair	exc.

$$\Rightarrow 1 = w_1 + w_2 + w_3 = w_1 + \frac{w_1}{0.9} + \frac{2w_1}{3} \approx 2.774 w_1 \Rightarrow w_1 \approx 0.36$$
$$\Rightarrow w_2 = \frac{w_1}{0.9} \approx 0.40$$
$$\Rightarrow w_3 = \frac{2w_1}{3} \approx 0.24$$

Example: Choosing a Software Supplier (Cont'd)

QUARTILE | =$F$4*F9+$G$4*G9+$H$4*H9

Attribute weights (wi)		
Cost	Delay	Quality
0.36	0.40	0.24

Sum of weights: 1.00

	Attribute-specific performances			Attribute-specific utilities			Overall utility
	Cost	Delay	Quality	Cost	Delay	Quality	
Supplier	x1	x2	x3	u1(x1)	u2(x2)	u3(x3)	u(x1,x2,x3)
x^0	40	30	fair	0.00	0.00	0.00	G9+$H$4*H9
x^*	10	1	excellent	1.00	1.00	1.00	1.00
Tradeoff1a	40	8	fair	0.00	0.90	0.00	0.36
Tradeoff1b	10	30	fair	1.00	0.00	0.00	0.36
Tradeoff2a	20	1	fair	0.67	1.00	0.00	0.64
Tradeoff2b	40	1	excellent	0.00	1.00	1.00	0.64

$$u(x) = \sum_{i=1}^{3} w_i u_i(x_i)$$

x3	u3(x3)
fair	0
good	0.4
excellent	1

x2	u2(x2)
1	1.000
2	0.986
9	0.889
10	0.875
11	0.850
12	0.825
13	0.800
14	0.775
15	0.750
16	0.714
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24	0.375
25	0.313
26	0.250
27	0.188
28	0.125
29	0.063
30	0.000

MAUT: Decision recommendations

- Consider there are m decision alternatives $x^j = (x_1^j, x_2^j, \dots, x_n^j), j=1,\dots,m$
- Alternatives are ranked based on their expected utilities $E\left[u\left(x^j\right)\right]$
  - If there are no uncertainties then

$$E\left[u\left(x^j\right)\right] = u(x^j) = \sum_i w_i u_i(x_i^j)$$

  - If there are uncertainties then x^j is a random variable with some PDF $f_{x^j}(x)$ and the expected utility is computed as

$$E\left[u\left(x^j\right)\right] = \sum_{x \in A} f_{x^j}(x)\, u(x) = \sum_{x \in A} f_{x^j}(x) \sum_i w_i u_i(x)$$

    - Integral for continuous r.v:s
- In a decision tree MAU is used just like unidimensional utility

Example: Choosing a Software Supplier (Cont'd)

- Step 5: Producing decision recommendations
  - Assume there are three possible suppliers
  - Supplier 1: Expensive, fair quality, has the software ready

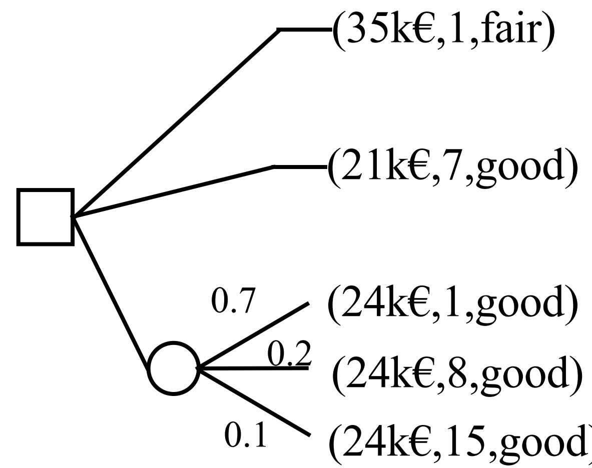
    $$x^1 = (35, 1, \text{fair})$$

  - Supplier 2: Cheap, good quality, software available in 1 week

    $$x^2 = (21, 7, \text{good})$$

  - Supplier 3: Moderate price, good quality, there is a 20% chance of 1-week delay and 10% chance of 2-week delay

$$x^3 = (24, \tilde{x}^3{}_2, \text{good}), f_{\tilde{x}^3{}_2}(x) = \begin{cases} 0.7, \text{if } x = (24, 1, \text{good}) \\ 0.2, \text{if } x = (24, 8, \text{good}) \\ 0.1, \text{if } x = (24, 15, \text{good}) \end{cases}$$



Example: Choosing a Software Supplier (Cont'd)

Attribute weights (wi)			
Cost	Delay	Quality	
0.36	0.40	0.24	1.00

	Attribute-specific performances			Attribute-specific utilities				
	Cost	Delay	Quality	Cost	Delay	Quality	Overall utility	
Supplier	x1	x2	x3	u1(x1)	u2(x2)	u3(x3)	u(x1,x2,x3)	
x^0	40	30	fair	0.00	0.00	0.00	0.00	
x^*	10	1	excellent	1.00	1.00	1.00	1.00	
Tradeoff1a	40	8	fair	0.00	0.90	0.00	0.36	
Tradeoff1b	10	30	fair	1.00	0.00	0.00	0.36	
Tradeoff2a	20	1	fair	0.67	1.00	0.00	0.64	
Tradeoff2b	40	1	excellent	0.00	1.00	1.00	0.64	
Supplier 1	35	1	fair	0.17	1.00	0.00	0.46	
Supplier 2	21	7	good	0.63	0.92	0.40	0.69	
Supplier 3 (s1)	24	1	good	0.53	1.00	0.40	0.69	
Supplier 3 (s2)	24	8	good	0.53	0.90	0.40	0.65	0.67
Supplier 3 (s3)	24	15	good	0.53	0.75	0.40	0.59	

$E[u(x^1)]$

$E[u(x^2)]$

$E[u(x^3)]$

=0.7*J19+0.2*J20+0.1*J21

Supplementary material on preference assumptions in MAUT/MAVT

Preference Independence (PI)

- **Definition.** A Subset of attributes $S \subset \{A_1, \ldots, A_n\}$ is **PI** if the preference order of degenerate lotteries that differ only on attributes in S does not depend on the levels of the rest of the attributes $\bar{S}$, i.e.,

$$(x_S^I, x_{\bar{S}}) \succcurlyeq (x_S^{II}, x_{\bar{S}}) \Rightarrow (x_S^I, x'_{\bar{S}}) \succcurlyeq (x_S^{II}, x'_{\bar{S}}) \forall x'_{\bar{S}}$$

- **Example:** Consider an investment selection problem with three attributes: Profits, Market share, and CO_2 reduction.
  - Is S={Profits} preference independent?
    - $\bar{S}$ ={Market Share, CO_2 reduction}
    - Take arbitrary a and b such that (aM€, 10%, 1ton) $\succcurlyeq$ (bM€, 10%, 1ton).
    - Then, is it true that (aM€, c%, d tons) $\succcurlyeq$ (bM€, c%, d tons) for any c and d?
    - It the answer is yes, then {Profits} is PI
- Note! The definition of the most and least preferred levels $x_i^*, x_i^0 \in A_i$ for attribute $\{A_i\}$ is possible only if the attribute is PI.

Mutual Preference Independence

- Example (Cont'd).
  - Is S={Profits, Market Share} preference independent?
    - $\bar{S}$ ={CO_2 reduction}
    - Assume (10M€,11%,10 tons) $\succcurlyeq$ (5M€,12%,10 tons)?
    - Then, does it hold for any a that (10M€,11%,a tons) $\succcurlyeq$ (5M€,12%,a tons)?
    - It the answer is yes then attribute set {Profits, Market Share} is PI
    - However, it might be that, for instance, (10M€,11%,0 tons) $\preccurlyeq$ (5M€,12%,0 tons), in which case {Profits, Market Share} is not PI.
      - E.g. "If the investment does not contribute to the environmental objective, it becomes more important to establish a stronger market share to survive the PR blowback"

- **Definition.** If every subset S $\subset \{A_1, \ldots, A_n\}$ is PI then we say the attributes $A_1, \ldots, A_n$ are **mutually preference independent (MPI)**

PI and MPI: Meal example

- Consider choosing a meal with three attributes: wine: A1={red, white}, dish; A2= {beef, fish}, side dish: A3= {potato, rice}
- The DM states ”I prefer (i) red wine to white, (ii) beef to fish and (iii) potato to rise.
  - Since these statements do not depend on the levels of attributes, they imply that each attribute $\{A_i\}$ is PI
- The DM also has the following preferences:
  - (red, beef, rice) $\succ$ (white, beef, potato)
  - (red, fish, rice) $\prec$ (white, fish, potato)

For any $a \in A_1, b \in A_2, c \in A_3$
(red, *b, c*) $\succ$ (white, *b, c*)
(*a*, beef, *c*) $\succ$ (*a*, fish, *c*)
(*a, b*, potato) $\succ$ (*a, b,* rice)

Question: Are the attributes MPI?

- The subset {wine, side dish} is not PI, and therefore the attributes are not mutually preference indep.

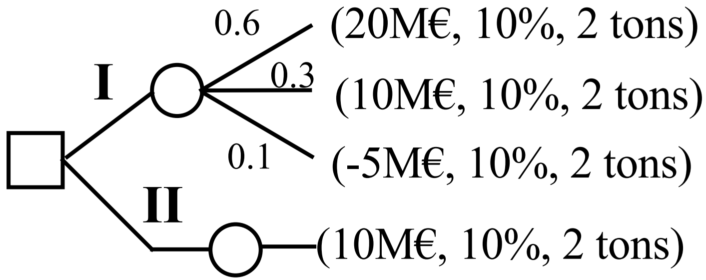
Utility Independence (UI)

- **Definition.** Subset of attributes $S \subset \{A_1, \dots, A_n\}$ is **UI** if the preference order of lotteries that have equal certain outcomes on attributes $\bar{S}$ does not depend on the level of these outcomes, i.e.,

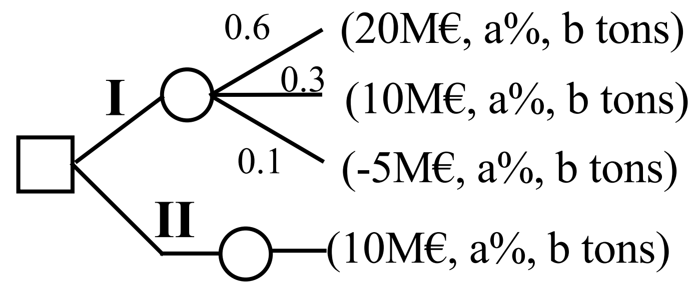
$$(\tilde{\boldsymbol{x}}_S^I, x_{\bar{S}}) \succcurlyeq (\tilde{\boldsymbol{x}}_S^{II}, x_{\bar{S}}) \Rightarrow (\tilde{\boldsymbol{x}}_S^I, x'_{\bar{S}}) \succcurlyeq (\tilde{\boldsymbol{x}}_S^{II}, x'_{\bar{S}}) \forall x'_{\bar{S}}$$

  - Example:

Assume DM prefers **I**:



If {Profits} is UI, then she prefers **I** for any *a* and *b:*

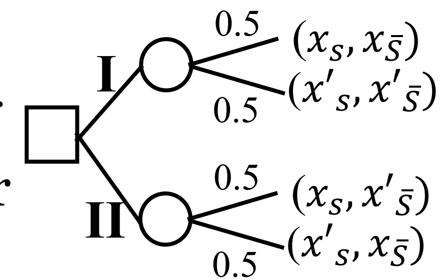


However, for a small market share (a), she may be more risk-averse about profits and choose II

➔ {Profits} not UI.

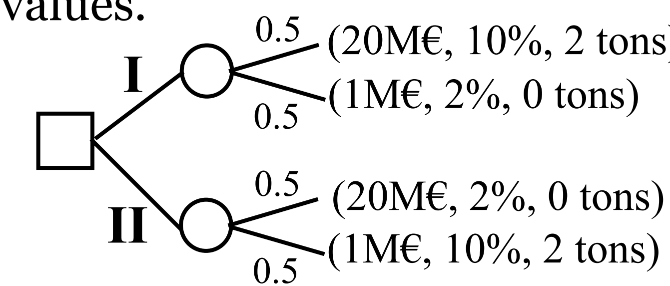
- **Definition.** If every subset $S \subset \{A_1, \dots, A_n\}$ is UI then we ww say the attributes $A_1, \dots, A_n$ are **mutually utility independent (MUI)**

Additive independence (AI)



- **Definition.** Subset of attributes S $\subset \{A_1, \dots, A_n\}$ is **AI** if the DM is indifferent between these lotteries I and II for any $x, x' \in A$.

- Example:
  - {Profits} is AI if the DM is indifferent between I and II
  - However, she might prefer II since it does not include an outcome where all attributes have very poor values. In this case {Profits} is not AI.



- AI is the strongest of the three preference assumptions:
  - Let S $\subset \{A_1, \dots, A_n\}$. Then S is AI $\Rightarrow$ S is UI $\Rightarrow$ S is PI

Additive Multiattribute Utility Function

- **Theorem.** The attributes are MPI and one attribute is AI iff preferences $\succcurlyeq$ are represented by an **additive** MAU function

$$u(x) = \sum_{i=1}^{n} w_i u_i(x_i) + u(x^0), \text{where}$$

  - $u_i(x_i) = \frac{u(x_1^0, x_2^0, \ldots, x_i, \ldots, x_n^0) - u(x^0)}{u(x_1^0, x_2^0, \ldots, x_i^*, \ldots, x_n^0) - u(x^0)} \in [0,1]$ is the attribute specific utility function for A_i
  - $w_i = [u(x_1^0, x_2^0, \ldots, x_i^*, \ldots, x_n^0) - u(x^0)] > 0$ is the "importance" weight for attribute A_i
  - $u(x^*) - u(x^0) = \sum_{i=1}^{n} w_i$

  – Sketch of proof:
  - If an attribute is AI it is also UI. This together with the MPI assumption implies the attributes are MUI. This implies that u is either multiplicative or additive. The AI assumption then implies it must be additive.

Notation:
$x = (x_1, \ldots, x_n)$, $x^* = (x_1^*, \ldots, x_n^*)$, $x^0 = (x_1^0, \ldots, x_n^0)$

What if preference assumptions do not hold?

- More complicated utility functions are available (not covered here)
  - If attributes mutually UI (but no attribute is AI) then preferences represented by the multiplicative utility function

$$u(x) = \frac{\prod_{i=1}^{n}[1 + kw_i u_i(x_i)]}{k} - \frac{1}{k}$$

    - One additional parameter k to assess
  - If each attribute is UI then preferences are represented by the multilinear utility function

$$u(x) = \sum_{I \subseteq \{1,\dots,n\}} w_I \prod_{i \in I} u_i(x_i) \prod_{i \notin I} (1 - u_i(x_i))$$

    - 2^n parameters to assess $(w_I, I \subseteq \{1, \dots, n\})$
- Common observation from practical applications: Additive or multiplicative multi-attribute utility usually sufficient.
  - May however require restructuring attributes