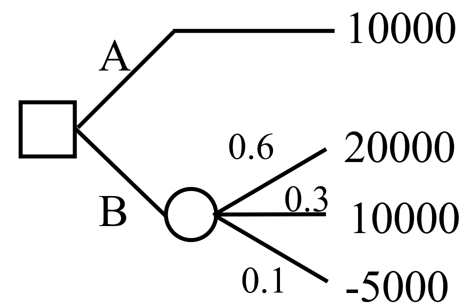
Aalto University
School of Business

Business Analytics 2 – Lecture 6: Expected Utility Theory

- *Outcomes, lotteries, compound lotteries*
- *EUT Axioms*
- *Preference representation with expected utility*
- *Computation of expected utility*
- *Uniqueness and positive affine transformations*
- *Expected utility in Decision trees and Monte Carlo*
- *Assessing utility functions*
- *EUT and choice behavior: Allais paradox and framing*

Expected Utility Theory (EUT) - Motivation

- Which alternative would you choose:
  - A: You get 10 000 € for sure
  - B: You participate in the following lottery:
    - 20 000 € p = 0.6
    - 10 000 € p = 0.3
    - -5 000 € p = 0.1
  - The EMV of B is 14 500 €, yet many people choose A because it's "less risky"



- Thus far we have compared decision alternatives with uncertain outcomes based on their expected monetary values (EMVs)
- EUT: alternatives should be compared based on their expected utility

- Learning objectives:
  - Understand that EUT is based on a set of rationality axioms
  - Ability to elicit utility functions and use expected utility to compare decision alternatives

Expected utility theory

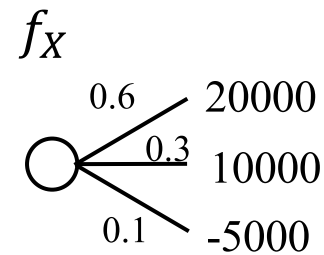
- John von Neumann and Oscar Morgenstern (1944) in Theory of Games and Economic Behaviour:
  - Axioms for preferences over alternatives with uncertain outcomes
  - If the DM follows these axioms, then she selects the alternative with the highest expected utility
  - C.f. Axioms for probability $\Rightarrow$ Rules for computing with probabilities

- Elements of EUT
  - Set of outcomes and lotteries
  - Preference relation over lotteries satisfying four axioms
  - Representation of the preference relation with expected utility

EUT: Sets of Outcomes and Lotteries

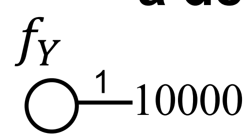
- A set of possible outcomes T
  - E.g. Revenue $T = \mathbb{R}$ euros, or demand for a product $T = \mathbb{N}$
- Set of all possible lotteries L:
  - A lottery $f \in L$ associates a probability $f(t) \in [0,1]$ with each possible outcome $t \in T$
    - Finite number of outcomes t with a positive probability $f(t) > 0$
    - Probabilities sum up to one $\sum_t f(t) = 1$
  - Note: deterministic outcomes are modelled as degenerate lotteries
- Basically lotteries are thus discrete PDFs, (decision trees with a single chance node)

a lottery



$$f_X(t) = \begin{cases} 0.6, t = 20000 \\ 0.3, t = 10000 \\ 0.1, t = -5000 \\ 0, \text{elsewhere} \end{cases}$$

a degenerate lottery



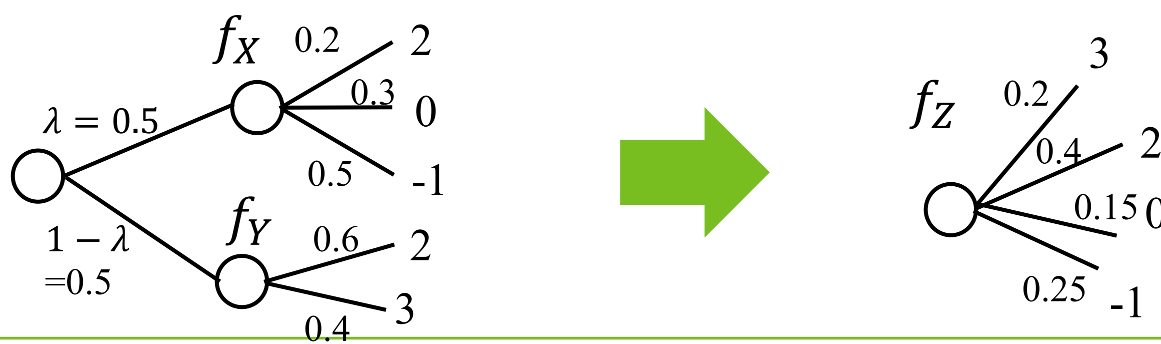
$$f_Y(t) = \begin{cases} 1, t = 10000 \\ 0, \text{elsewhere} \end{cases}$$

EUT: Compound lotteries

- Compound lottery:
  - With probability λ the outcome is some lottery $f_X \in L$ and with probability $1-\lambda$ the outcome is some other lottery $f_Y \in L$
  - This compound lottery is modelled as lottery $f_Z \in L$ defined by

$$f_Z(t) = \lambda f_X(t) + (1-\lambda) f_Y(t) \forall t \in T \ (*)$$

    - Notation: $f_Z = \lambda f_X + (1-\lambda) f_Y$
- Example:
    - You have a 50-50 chance of getting a ticket to lottery $f_X \in L$ or to lottery $f_Y \in L$



$*$) cf. $\mathrm{P(Profit = t) = P(Profit = t|Lottery\ X)P(lottery\ X)}$
$+\mathrm{P(Profit} = t|\mathrm{Lottery}\ Y)\mathrm{P(lottery}\ Y)$

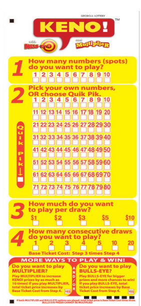
The EUT Axioms: A1-A4

- Let $\succcurlyeq$ be a relation among lotteries L
  - $f_X \succcurlyeq f_Y$ means "f_X is weakly preferred to f_Y" or "f_X is at least as good as f_Y"
    - Strict preference $f_X \succ f_Y$ defined as $\neg(f_Y \succcurlyeq f_X)$
    - Indifference $f_X \sim f_Y$ defined as $(f_X \succcurlyeq f_Y)$ and $(f_Y \succcurlyeq f_X)$

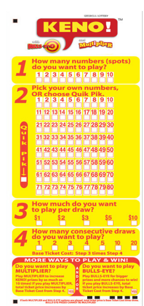
- (A1) $\succcurlyeq$ is complete
  - For any $f_X, f_Y \in L$ either $f_X \succcurlyeq f_Y$ or $f_Y \succcurlyeq f_X$ or both
  - Meaning: preference between two lotteries can always be stated – either you (i) prefer f_X, (ii) prefer f_Y, or (iii) are indifferent between the two

- (A2) $\succcurlyeq$ is transitive
  - If $f_X \succcurlyeq f_Y$ and $f_Y \succcurlyeq f_Z$, then $f_X \succcurlyeq f_Z$
  - Example:

$\succcurlyeq$



&



$\succcurlyeq$

$\rightarrow$

$\succcurlyeq$

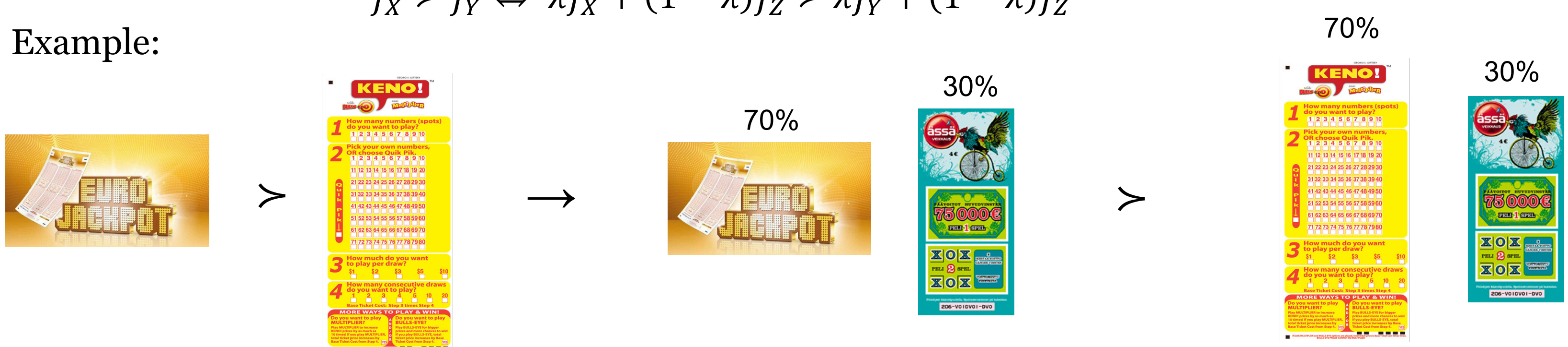
The EUT Axioms: A3 and A4

- (A3) Archimedean axiom
  - If $f_X \succ f_Y \succ f_Z$, then $\exists \lambda, \mu \in (0,1)$ such that $\lambda f_X + (1-\lambda) f_Z \succ f_Y$ and $f_Y \succ \mu f_X + (1-\mu) f_Z$
  - Example:

- (A4) Independence axiom
  - Let $\lambda \in (0,1)$. Then,

$$f_X \succ f_Y \Leftrightarrow \lambda f_X + (1-\lambda) f_Z \succ \lambda f_Y + (1-\lambda) f_Z$$

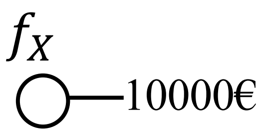
  - Example:

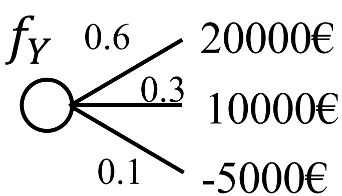


Axiom implications: Example

- Assume the DM follows the independence axiom:
  - Let $\lambda \in (0,1)$. Then, $f_X \succ f_Y \Leftrightarrow \lambda f_X + (1-\lambda) f_Z \succ \lambda f_Y + (1-\lambda) f_Z$

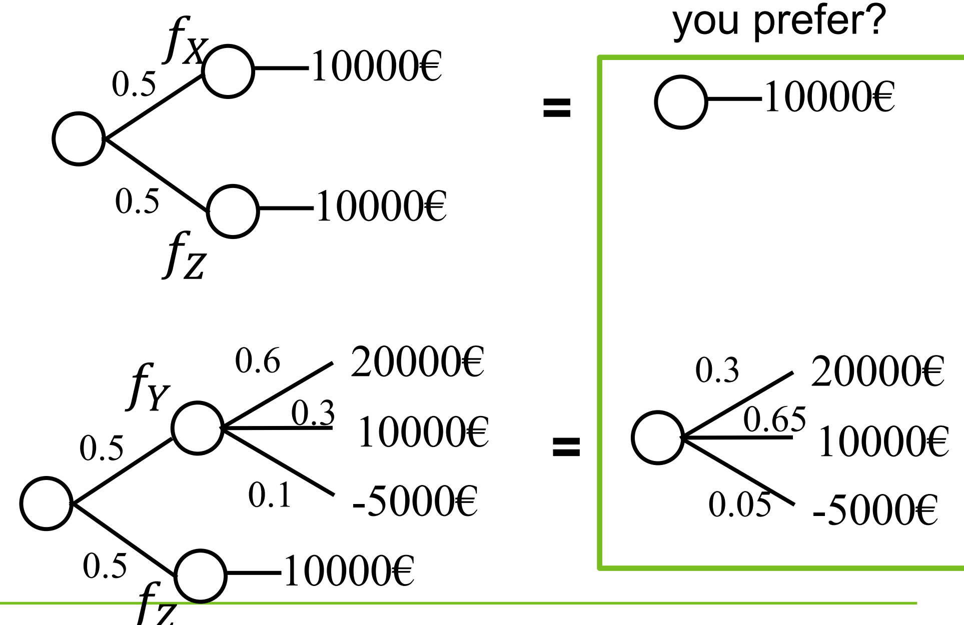
If the DM prefers lottery f_X over f_Y,...



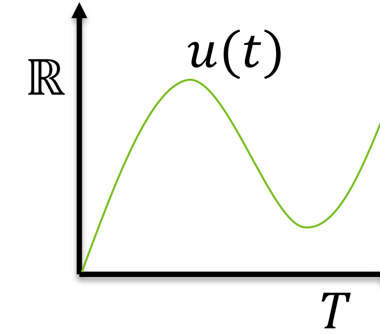


…then she must prefer the top lottery over the bottom one!

Question:
Which of these lotteries would you prefer?



Main Result: Preference representation with EU



- $\succcurlyeq$ satisfies axioms A1-A4 if and only if there exists a real-valued utility function $u: T \rightarrow \mathbb{R}$ over the set T of outcomes such that

$$f_X \succcurlyeq f_Y \Leftrightarrow \sum_{t \in T} f_X(t)u(t) \geq \sum_{t \in T} f_Y(t)u(t)$$

  – Implication: a DM following axioms A1-A4 selects the alternative with the highest expected utility

$$\mathrm{E}[u(X)] = \sum_{t \in T} f_X(t)u(t)$$

  – With more sophisticated mathematics, a similar result could be obtained for continuous distributions:
    - $f_X \succcurlyeq f_Y \Leftrightarrow E[u(X)] \geq E[u(Y)]$, where the expected utility is computed as the integral $E[u(X)] = \int f_X(t)u(t)dt$

Computation of expected utility

- Example: Joe has the following utility function for the number of oranges u(1)=2, u(2)=5, u(3)=7
  - Would he take
    - two oranges for certain (*X*) or
    - a 50-50 gamble between 1 and 3 oranges (*Y*)?

$$E[u(X)] = 1u(2) = 5$$

$$E[u(Y)] = 0.5u(1) + 0.5u(3)$$
$$= 0.5 * 2 + 0.5 * 7 = 4.5$$

- Example: Jane's utility function for profits is $u(t) = t^2$
  - Which investment would she prefer?

    X: 50-50 gamble between 3 and 5 M£?

    Y: Profits in M£ following UNI(3,5) distribution

$$E[u(X)] = 0.5u(3) + 0.5u(5)$$
$$=0.5 * 9 + 0.5 * 25 = 17$$

$$E[u(Y)] = \int_3^5 f_Y(t)\, u(t) dt = \int_3^5 \frac{1}{2} t^2 dt$$
$$= \frac{1}{6}5^3 - \frac{1}{6}3^3 = 16.3333$$

Computation of expected utility

- Question: Joe has the following utility function for the number of oranges u(1)=2, u(2)=3, u(3)=8
  - What are the expected utilities from the following alternatives?
    - two oranges for certain (X) or
    - a 50-50 gamble between 1 and 3 oranges (Y)?
- Question: Jane's utility function for profits is $u(t) = 1 - e^{-0.5t}$
  - What are the expected utilities from the following investments?

    X: 50-50 gamble between 3 and 5 M£?

    Y: Profits in M£ following UNI(3,5) distribution

Computation of expected utility

- Joe :
  - Two oranges for certain: $E[u(X)] = 1u(2) = 3$
  - 50-50 gamble between 1 and 3 oranges: $E[u(Y)] = 0.5u(1) + 0.5u(3) = 0.5 * 2 + 0.5 * 8 = 5$

- Jane's utility function for profits is $u(t) = 1 - e^{-0.5t}$
  - 50-50 gamble between 3 and 5 M£: $E[u(X)]$=0.5u(3)+0.5u(5)=0.5$*(1 - e^{-1.5})$+ 0.5$*(1 - e^{-2.5})$=0.85
  - Profits in M£ following UNI(3,5) distribution: $E[u(Y)] = \int_3^5 f_Y(t)\, u(t)dt = \int_3^5 \frac{1}{2}(1 - e^{-0.5t})dt = \frac{1}{2}(5-3) + (e^{-2.5} - e^{-1.5}) = 0.86$

Utility function is unique up to positive affine tranformations

- Example: Jane's utility function for profits is $u(t) = t^2$
  - Which investment would she prefer?

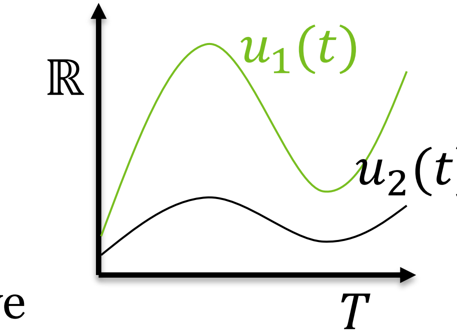
    X: 50-50 gamble between 3 and 5 M£?

    Y: Profits in M£ following UNI(3,5) distribution
  - For both investments, utilities range between 9 and 25.
  - What would the range of utilities be, if Jane's utility function was $u(t) = \frac{t^2-9}{25-9}$?

Utility function is unique up to positive affine tranformations

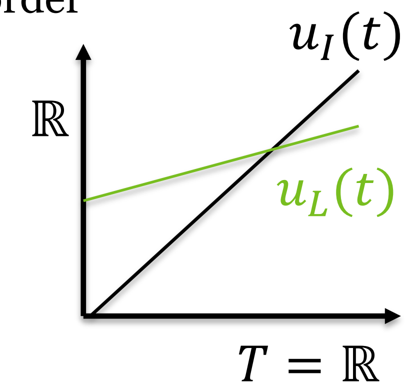
- The two utility functions $u_1(\mathrm{t})$ and $u_2(t) = \alpha u_1(t) + \beta$ $(\alpha > 0)$ establish the same preference ordering among any lotteries:

$$\mathrm{E}[u_2(X)] = E[\alpha u_1(X) + \beta] = \alpha \mathrm{E}[u_1(X)] + \beta$$



- Implications:
  1. Any linear utility function $u_L(t) = \alpha t + \beta$ $(\alpha > 0)$ is a positive affine transformation of the identity function $u_I(t) = t$
     - Hence, a linear utility function establishes the same preference order among any lotteries as expected value:

$$\mathrm{E}[u_L(X)] = E[\alpha u_I(X) + \beta] = \alpha \mathrm{E}[u_I(X)] + \beta = \alpha E[X] + \beta$$

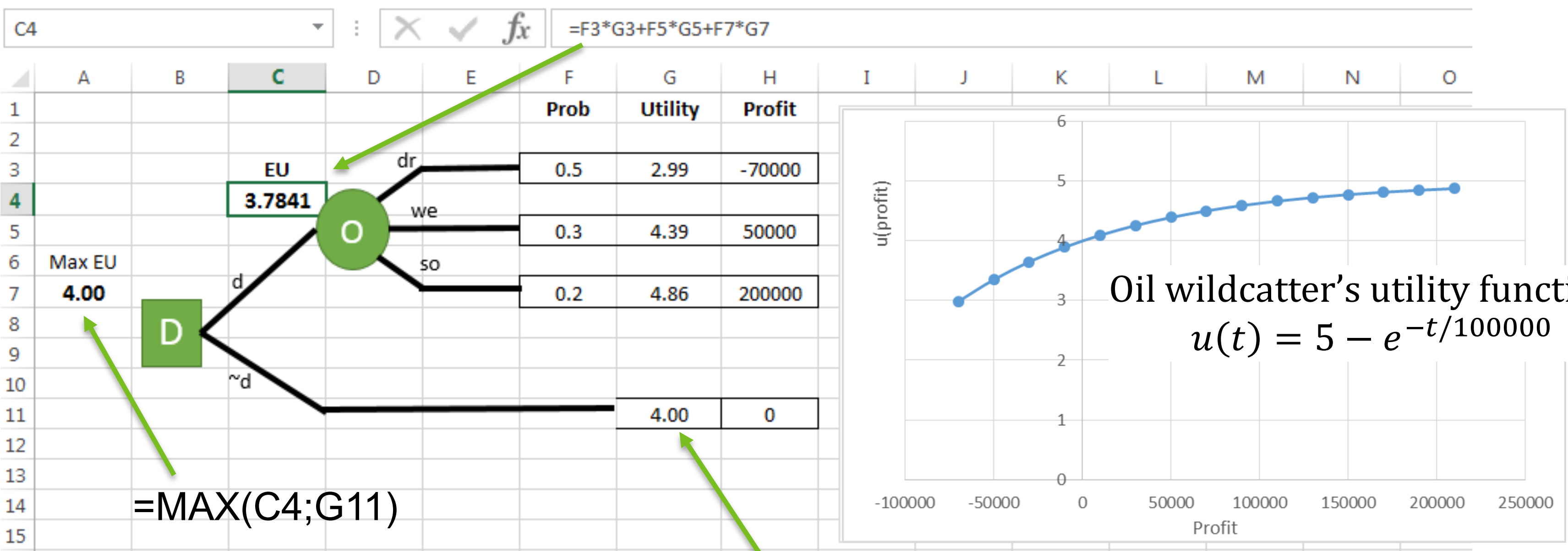


  2. Utilities for two outcomes can be chosen:
     - E.g. Scale u_1 so that $u_2(t^*) = 1$ and $u_2(t^0) = 0$:

$$u_2(t) = \frac{u_1(t) - u_1(t^0)}{u_1(t^*) - u_1(t^0)} = \underbrace{\frac{1}{u_1(t^*) - u_1(t^0)}}_{\alpha > 0} u_1(t) + \underbrace{\frac{-u_1(t^0)}{u_1(t^*) - u_1(t^0)}}_{\beta}$$

Expected Utility in Decision trees

- Go through the nodes from right to left
  - Chance node: compute expected utility
  - Decision node: select the alternative with maximum expected utility



Oil wildcatter's utility function:

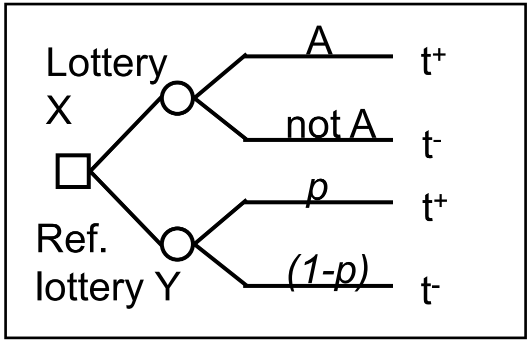
$$u(t) = 5 - e^{-t/100000}$$

=MAX(C4;G11)

=5-EXP(-H11/100000)

Reference lottery revisited

- Assume that an expected utility maximizer, whose utility function is u, uses a reference lottery to assess the probability of event A
  - p has been adjusted so that she is indifferent about which lottery to participate in:



$$E[u(X)] = E[u(Y)]$$
$$\Leftrightarrow P(A)u(t^+) + \big(1 - P(A)\big)u(t^-) = pu(t^+) + (1-p)u(t^-)$$
$$\Leftrightarrow P(A)u(t^+) - P(A)u(t^-) + u(t^-) = pu(t^+) - pu(t^-) + u(t^-)$$
$$\Leftrightarrow P(A)[u(t^+) - u(t^-)] = p[u(t^+) - u(t^-)]$$
$$\Leftrightarrow P(A) = p$$

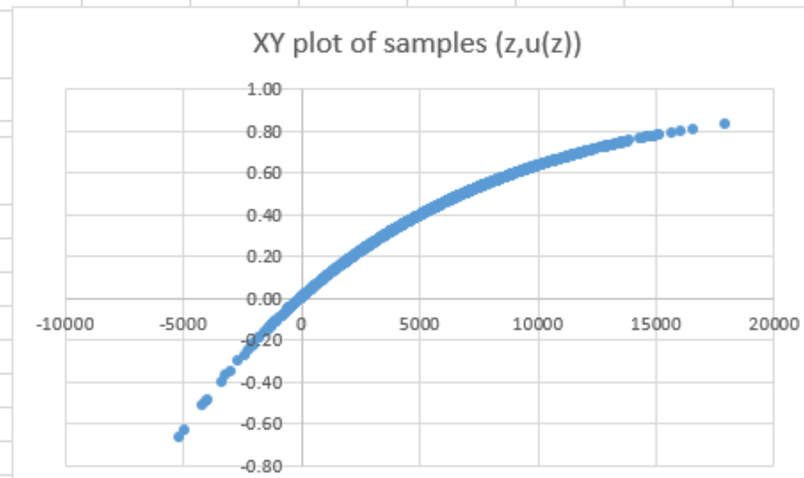
- The utility function u does not affect the result

Expected Utility in Monte Carlo

- For each sample $x_1, \dots, x_n$ of random variable X compute utility $u(x_i)$
- Mean of utility samples $u(x_1), \dots, u(x_n)$ provides an estimate for $E[u(X)]$

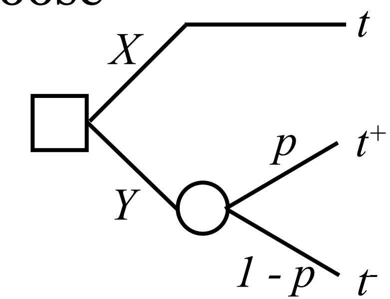
F8 =1-EXP(-E8/10000)

	B	C	D	E	F
2	Number of papers to print (q)			20000	
4	**Col. Sum / 1000**	**0.49**	**9953**	**5944**	**0.41**
5	Samp. Std	0.30	3138		
7	Sample	Random number (u)	Demand (d), units	Profit (z), euros	Utility, u(z)
8	1	0.3238919	8629	4355	0.35
9	2	0.0969499	6103	1323	0.12
10	3	0.3750874	9045	4854	0.38
11	4	0.2317308	7801	3361	0.29
12	5	0.2338982	7822	3386	0.29
13	6	0.5983368	10747	6897	0.50
14	7	0.3194225	8592	4310	0.35
15	8	0.9305541	14440	11328	0.68
16	9	0.543028	10324	6389	0.47



Utility function assessment

- Utility functions can be assessed by asking the DM to choose between a simple lottery and a certain outcome (i.e., a degenerate lottery)
  - *X*: Certain return t
  - *Y*: Return t^+ with probability p and t^- with $(1-p)$



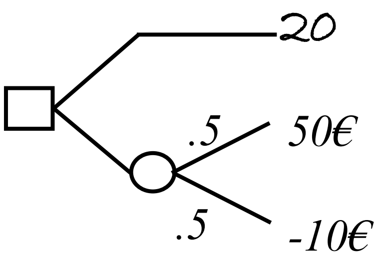
- General idea: Vary the parameters (p, t, t^-, t^+) until the DM is indifferent between X and Y:

$$E[u(X)] = E[u(Y)] \Leftrightarrow u(t) = p\,u(t^+) + (1-p)\,u(t^-)$$

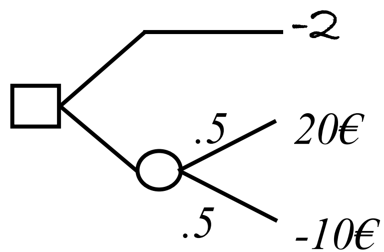
  - Repeat until sufficiently many points for the utility function have been obtained.
- Because u is unique up to positive affine transformations, utilities in two points can be chosen
  - Often the most preferred level is set to 1 and the least preferred to 0

Assessment of a utility function: The Certainty Equivalence Approach

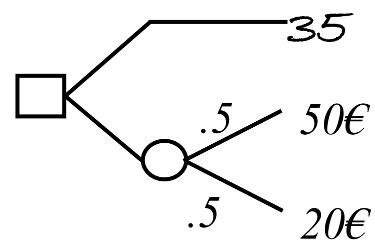
- Example: Assess utility function for the interval [−10,50] euros
  - We can fix two values so let's choose $u(-10)=0$ and $u(50)=1$



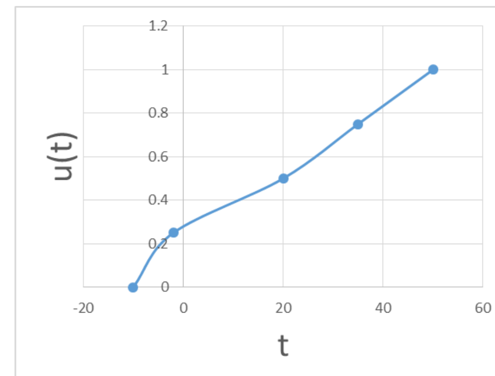
$$u(20) = 0.5u(-10) + 0.5u(50) = 0.5 * 0 + 0.5 * 1 = 0.5$$



$$u(-2) = 0.5u(-10) + 0.5u(20) = 0.5 * 0 + 0.5 * 0.5 = 0.25$$

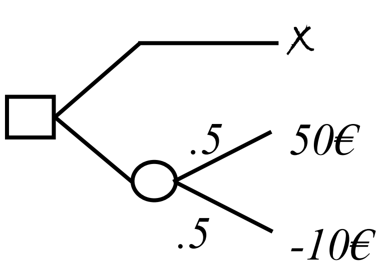


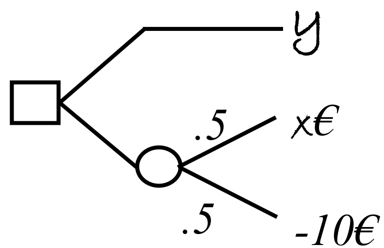
$$u(35) = 0.5u(20) + 0.5u(50) = 0.5 * 0.5 + 0.5 * 1 = 0.75$$

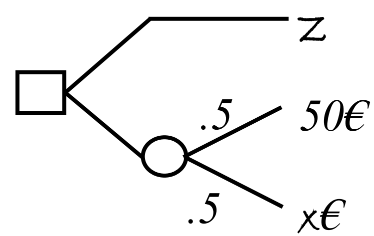


Assessment of a utility function: The Certainty Equivalence Approach

- Question: Assess your utility function for the interval [−10,50] euros by defining x,y,z (in that order) in the three decision trees below





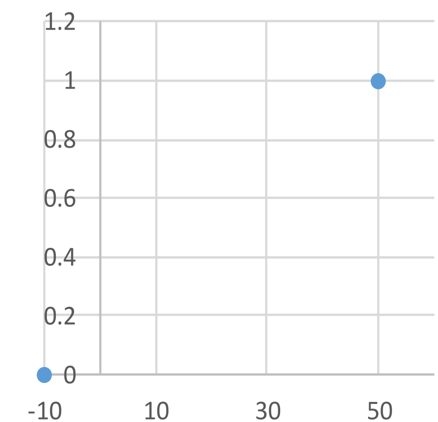


$$u(x) = 0.5u(-10) + 0.5u(50) = 0.5 * 0 + 0.5 * 1 = 0.5$$

$$u(y) = 0.5u(-10) + 0.5u(x) = 0.5 * 0 + 0.5 * 0.5 = 0.25$$

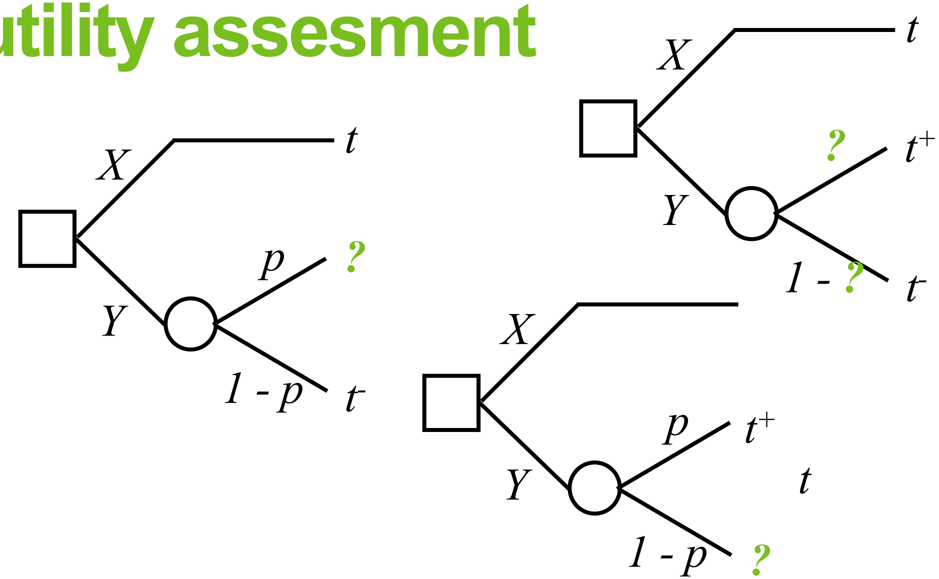
$$u(z) = 0.5u(x) + 0.5u(50) = 0.5 * 0.5 + 0.5 * 1 = 0.75$$

	t	U(t)
	-10	0
t=y		0.25
t=x		0.5
t=z		0.75
	50	1



Other approaches for utility assesment

- Probability equivalence:
  - The DM assesses p.
- Gain equivalence:
  - The DM assesses t^+
- Loss equivalence:
  - The DM assesses t^-



- Often in applications the analyst chooses a family of utility functions and then uses the above type questioning to fix the parameter(s)
  - E.g. The exponential utility function (parameter ρ)

$$u(t) = 1 - e^{-t/\rho}, \rho > 0$$

Option A		Option B
With probability **90%** : **Get $ 20** With probability **10%** : **Get $ 0**		With certainty: **Get $ 0**
		With certainty: **Get $ 1**
		With certainty: **Get $ 2**
		With certainty: **Get $ 3**
		With certainty: **Get $ 4**
		With certainty: **Get $ 5**
		With certainty: **Get $ 6**
		With certainty: **Get $ 7**
		With certainty: **Get $ 8**
		With certainty: **Get $ 9**
		With certainty: **Get $ 10**
		With certainty: **Get $ 11**
		With certainty: **Get $ 12**
		With certainty: **Get $ 13**
		With certainty: **Get $ 14**
		With certainty: **Get $ 15**
		With certainty: **Get $ 16**
		With certainty: **Get $ 17**
		With certainty: **Get $ 18**
		With certainty: **Get $ 19**
		With certainty: **Get $ 20**

Next

Figure 12: Decision screen to elicit certainty equivalents for lotteries

Eliciting utility functions in practice: “Multiple price lists”

Holt, C.A. and Laury, S.K., 2002. Risk aversion and incentive effects. *American Economic Review*, *92*(5), pp.1644-1655

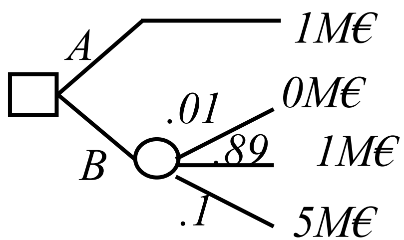
EUT for normative decision support

- In Management Science EUT is mainly used in *normative* decision support models
  - Not descriptive, i.e., describing how people select among alternatives with uncertain outcomes
  - Not predictive in the sense it would predict what alternatives people select

- The four Axioms characterize properties that are required from rational decision support
  - C.f. Probability axioms describe a rational model for uncertainty
  - They <u>are not rules</u> that people follow by instinct when choosing among alternatives with uncertain outcomes
  - Also, people may display decision inconsistency, choosing differently in the same problem presented at two separate times

Allais paradox

- Would you rather choose A or B?

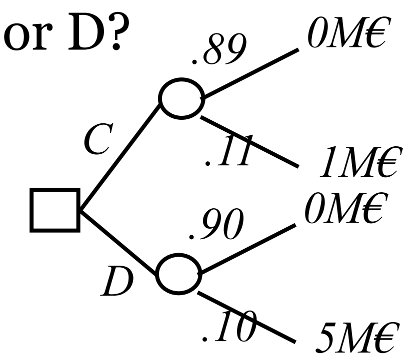


Most people choose A, hence E[u(A)]>E[u(B)]:

u(1) > 0.10 u(5) + 0.89 u(1) + 0.01 u(0)

$\Rightarrow$ $\boxed{0.11\ u(1) > 0.10\ u(5) + 0.01\ u(0)}$

- What about C or D?



Most people choose D, hence E[u(D)]>E[u(C)]:

0.10 u(5) + 0.90 u(0) > 0.11 u(1) + 0.89 u(0)

$\Rightarrow$ $\boxed{0.11\ u(1) < 0.10\ u(5) + 0.01\ u(0)}$

- Actual choice behavior not always consistent with EUT

Framing effect

- 400 people are trapped inside a cruise ship and there are two alternatives for rescue plans. Which one would you choose?