Aalto University
School of Business

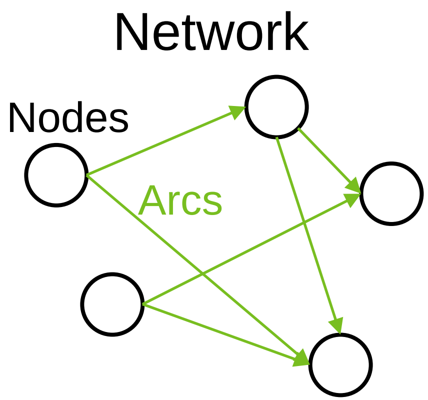
Linear programming – Distribution and Network models

- Transportation problem
- Transshipment problem
- Assignment problem

Linear Programming: Network models

- “How to distribute products from manufacturing to end-customers?”
- “How to assign workers with different skillsets to specific tasks?”

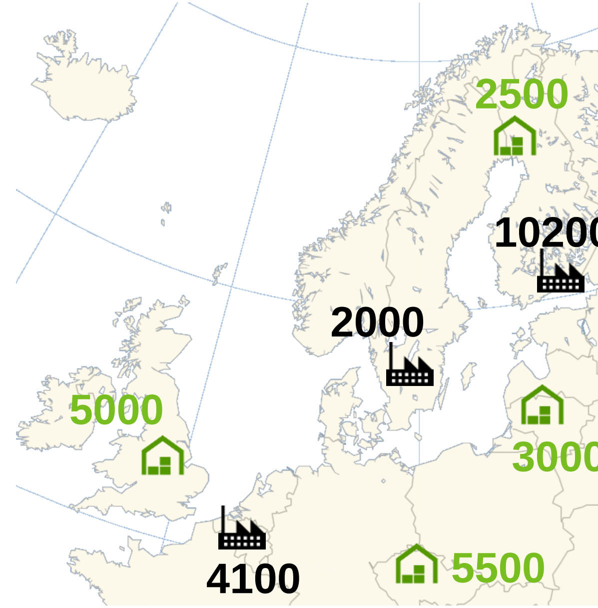
- These decision can be supported by network models
  = Linear Programming (LP) models with a special network structure



- General relationship between LP formulation and network structure
  - Decision variables = Arcs
  - Constrains = Nodes

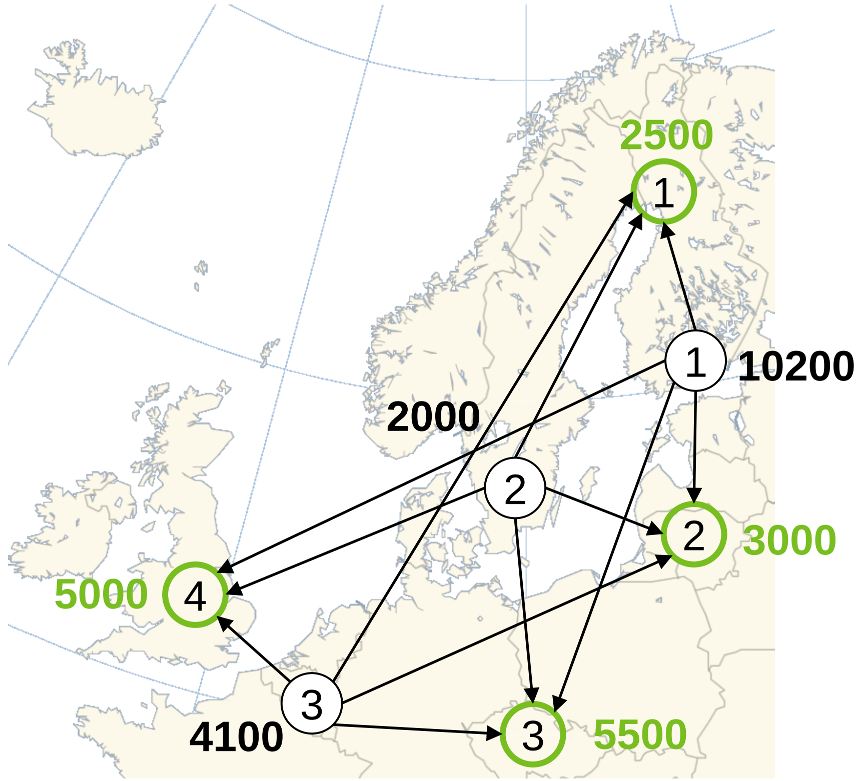
Network LP example: P&P transportation problem

- Pulp&Paper Ltd. produces cardboard at 3 mills
  - Monthly production capacities shown on the map
- From the mills the cardboard is transported to 4 warehouses that supply the customers
  - Monthly customer demand is shown on the map
- P&P wants you to build a network model to select distribution routes so that demand is satisfied with minimal transportation costs



Capacities and **demands** in tons

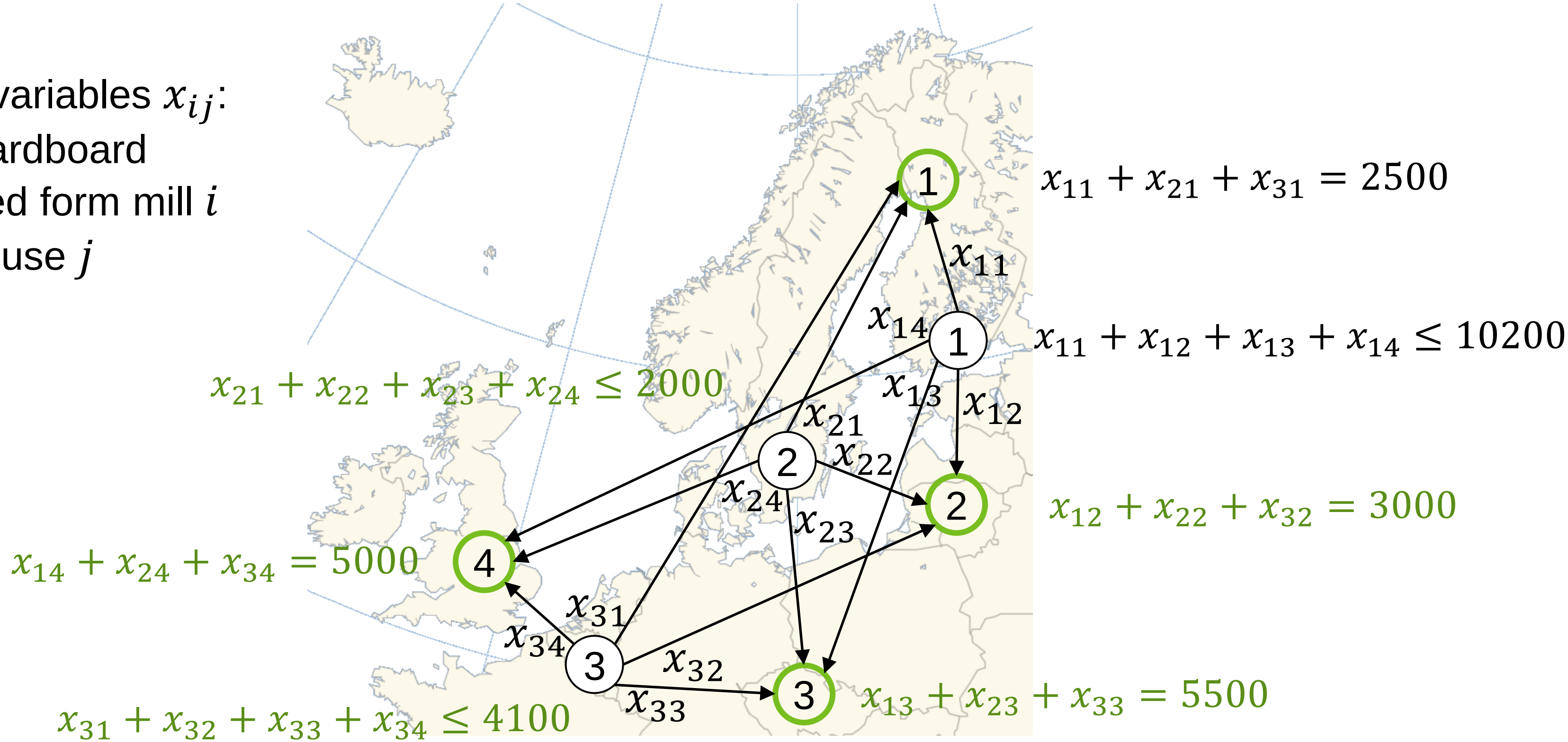
P&P transportation problem: Graphical representation



Capacities and **demands** in tons

P&P transportation problem: LP formulation

Decision variables x_{ij}:
Tons of cardboard transported form mill i to warehouse j



- **Question:** Interpret the constraints in **green**: Why are they needed?

P&P transportation problem: LP formulation (cont'd)

- Formulating the objective function requires information on unit transportation **costs** on each route (arc)

⇒LP-formulation:

$$\begin{aligned} \min\ & 53x_{11} + 95x_{12} + 136x_{13} + 160x_{14} \\ & + 60x_{21} + 120x_{22} + 132x_{23} + 140x_{24} \\ & + 210x_{31} + 190x_{32} + 89x_{33} + 71x_{34} \end{aligned}$$

Minimize total transportation costs

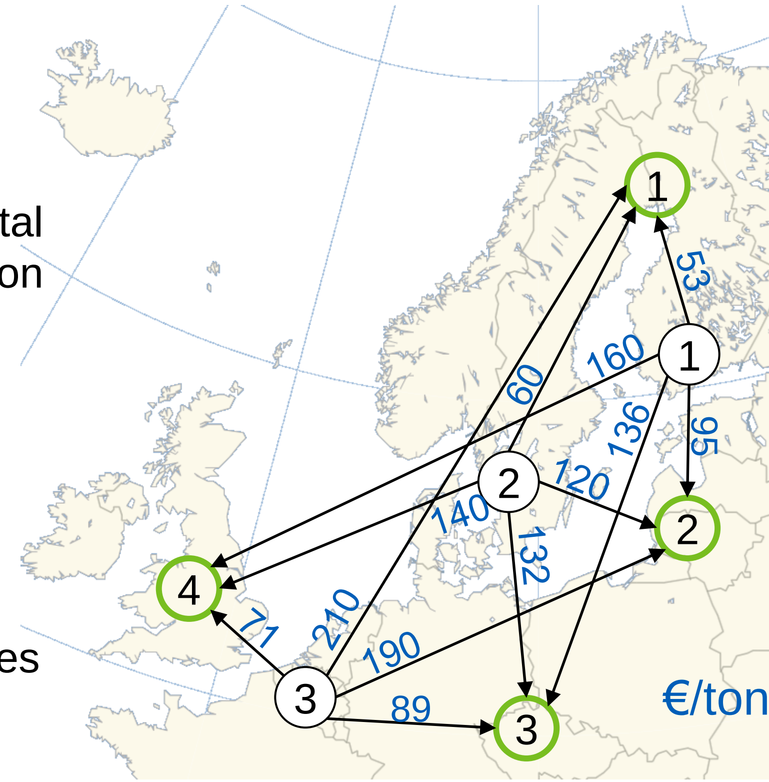
$$\begin{aligned} x_{11} + x_{21} + x_{31} &= 2500 \\ x_{12} + x_{22} + x_{32} &= 3000 \\ x_{13} + x_{23} + x_{33} &= 5500 \\ x_{14} + x_{24} + x_{34} &= 5000 \end{aligned}$$

Satisfy demand

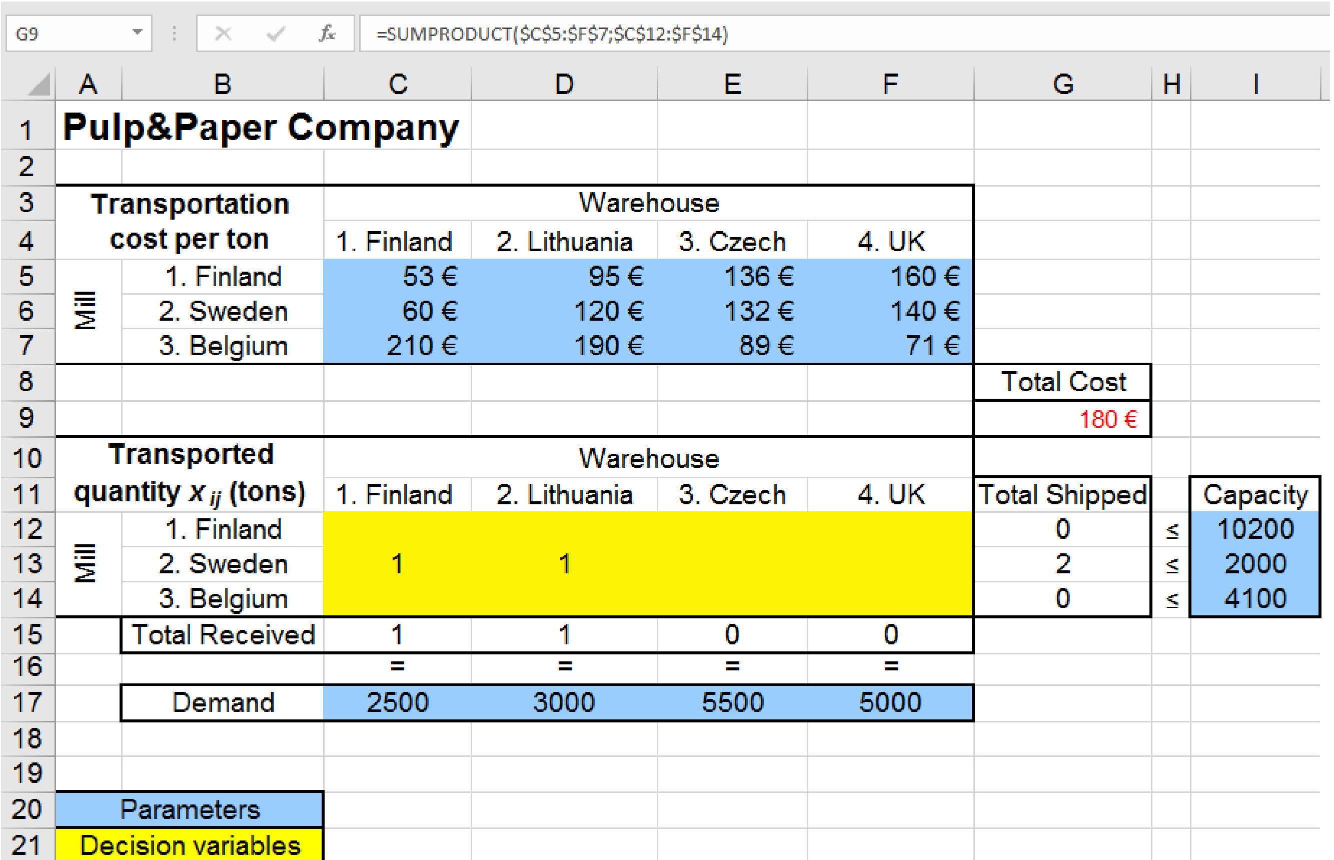
$$\begin{aligned} x_{11} + x_{12} + x_{13} + x_{14} &\leq 10200 \\ x_{21} + x_{22} + x_{23} + x_{24} &\leq 2000 \\ x_{31} + x_{32} + x_{33} + x_{34} &\leq 4100 \end{aligned}$$

Do not exceed production capacities

$$x_{ij} \geq 0, i = 1, \dots, 3, j = 1, \dots 4$$

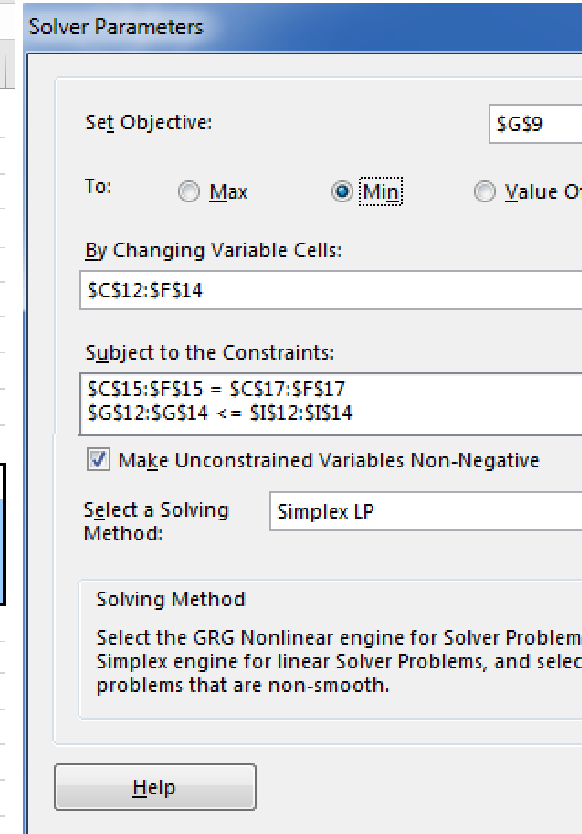


P&P transportation problem: Spreadsheet impl.



G9 =SUMPRODUCT($C$5:$F$7;$C$12:$F$14)

	A	B	C	D	E	F	G	H	I
1	**Pulp&Paper Company**								
2									
3		**Transportation**	Warehouse						
4		**cost per ton**	1. Finland	2. Lithuania	3. Czech	4. UK			
5	Mill	1. Finland	53 €	95 €	136 €	160 €			
6		2. Sweden	60 €	120 €	132 €	140 €			
7		3. Belgium	210 €	190 €	89 €	71 €			
8							Total Cost		
9							180 €		
10		**Transported**	Warehouse						
11		**quantity** x_{ij} **(tons)**	1. Finland	2. Lithuania	3. Czech	4. UK	Total Shipped		Capacity
12	Mill	1. Finland					0	≤	10200
13		2. Sweden	1	1			2	≤	2000
14		3. Belgium					0	≤	4100
15		Total Received	1	1	0	0			
16			=	=	=	=			
17		Demand	2500	3000	5500	5000			
18									
19									
20	Parameters								
21	Decision variables								



Spreadsheet model available in MyCourses

P&P transportation problem: Optimal solution

G9 =SUMPRODUCT($C$5:$F$7;$C$12:$F$14)

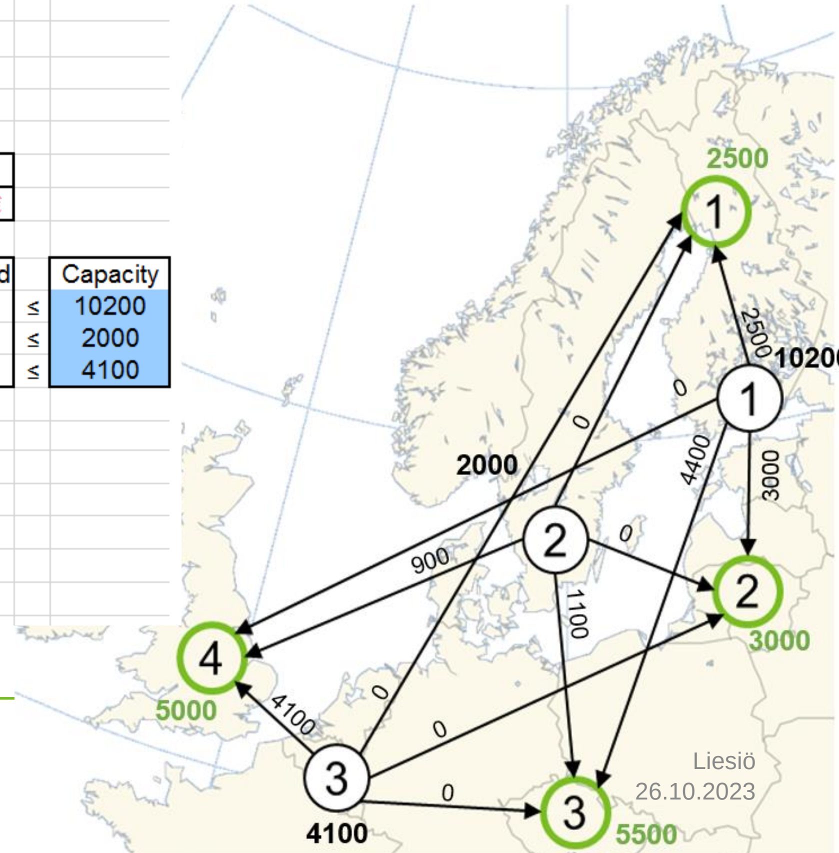
Pulp&Paper Company

Transportation cost per ton		Warehouse			
		1. Finland	2. Lithuania	3. Czech	4. UK
Mill	1. Finland	53 €	95 €	136 €	160 €
	2. Sweden	60 €	120 €	132 €	140 €
	3. Belgium	210 €	190 €	89 €	71 €

Total Cost
1 578 200 €

Transported quantity x_{ij} (tons)		Warehouse						
		1. Finland	2. Lithuania	3. Czech	4. UK	Total Shipped		Capacity
Mill	1. Finland	2500	3000	4400	0	9900	≤	10200
	2. Sweden	0	0	1100	900	2000	≤	2000
	3. Belgium	0	0	0	4100	4100	≤	4100
	Total Received	2500	3000	5500	5000			
		=	=	=	=			
	Demand	2500	3000	5500	5000			

Parameters
Decision variables



Transportation Problem: General Characteristics

- A common problem in logistics is how to transport goods from a set of sources (e.g., plants, warehouses, etc.) to a set of destinations (e.g., warehouses, customers, etc.) with minimum possible cost
- Nodes (Constraints)
  - a set of sources, each with a given supply
  - a set of destinations, each with a given demand
- Arcs (Decision variables)
  - Possible transport routes between sources and destinations, each with a shipping cost
- Objective
  - To determine how much should be shipped from each source node to each destination node so that the total transportation costs are minimized

Transportation Problem: General LP-formulation

x_{ij}: the amount shipped from supply point i to the demand point j.

$$\text{Min} \quad \Sigma_i \Sigma_j c_{ij} x_{ij}$$

$$\begin{aligned} \text{s.t.} \quad & \Sigma_j x_{ij} \leq s_i && \text{for each source } i \\ & \Sigma_i x_{ij} = d_j && \text{for each destination } j \\ & x_{ij} \geq 0 && \text{for all } i \text{ and } j \end{aligned}$$

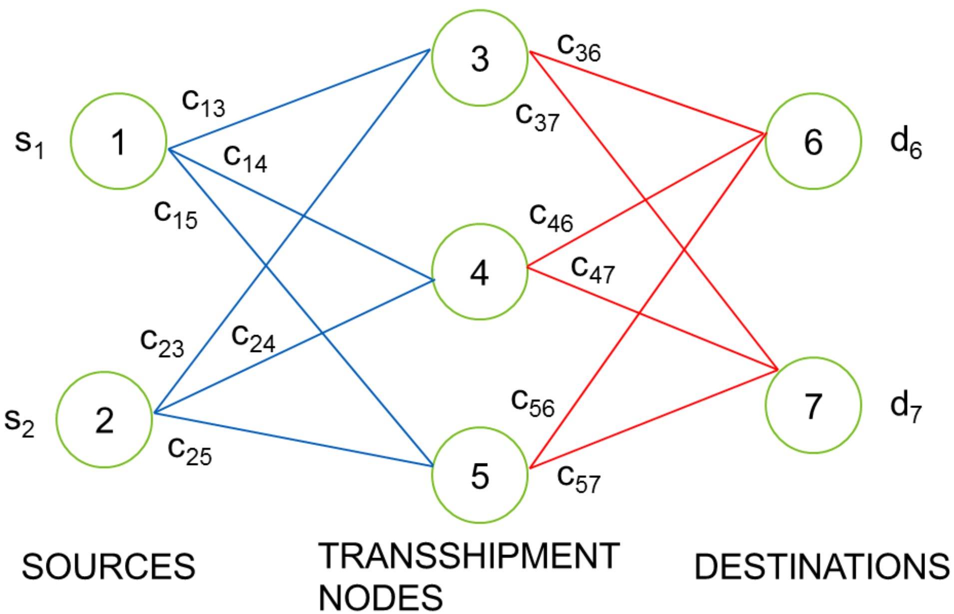
- If total supply ≥ Total demand, then not all supply will be used
- If total supply < Total demand, then the problem is infeasible
  - Redefine the problem: Satisfy the demand as much as possible at minimum cost
    - Every supply node must send its supply.
    - Every demand node receives up to its demand.

Transportation problem: Variations

- Maximize the objective function
  - c_{ij} is then the unit profit obtained by supplying j from i
- Limited route capacities
  - Capacity limitations on arcs can be handled by additional constraints, e.g.,
    - $x_{ij} \leq U_{ij}$ (max. that can be transported)
    - $x_{ij} \geq L_{ij}$ (min. that has to be transported)
- Unacceptable routes
  - It may not always be possible to use all the routes, e.g., no railroad transportation between two cities.
  - Drop the variable, corresponding to an unacceptable route, from the objective function and all constraints
    - Or limit route capacity to zero by setting U_{ij}=0

Transshipment Problem

- A transportation problem but shipment may move through intermediate nodes (transshipment nodes) before reaching a particular destination node.



$$\min \quad \begin{aligned} & c_{13}x_{13} + c_{14}x_{14} + c_{15}x_{15} \\ & + c_{23}x_{23} + c_{24}x_{24} + c_{25}x_{25} \\ & + c_{36}x_{36} + c_{37}x_{37} \\ & + c_{46}x_{46} + c_{47}x_{47} \\ & + c_{56}x_{56} + c_{57}x_{57} \end{aligned}$$

$$\left.\begin{aligned} x_{13} + x_{14} + x_{15} &\le s_1 \\ x_{23} + x_{24} + x_{25} &\le s_2 \end{aligned}\right\} \text{Sources}$$

$$\left.\begin{aligned} x_{13} + x_{23} &= x_{36} + x_{37} \\ x_{14} + x_{24} &= x_{46} + x_{47} \\ x_{15} + x_{25} &= x_{56} + x_{57} \end{aligned}\right\} \text{Trans-shipment nodes}$$

$$\left.\begin{aligned} x_{36} + x_{46} + x_{56} &= d_6 \\ x_{37} + x_{47} + x_{57} &= d_7 \end{aligned}\right\} \text{Destinations}$$

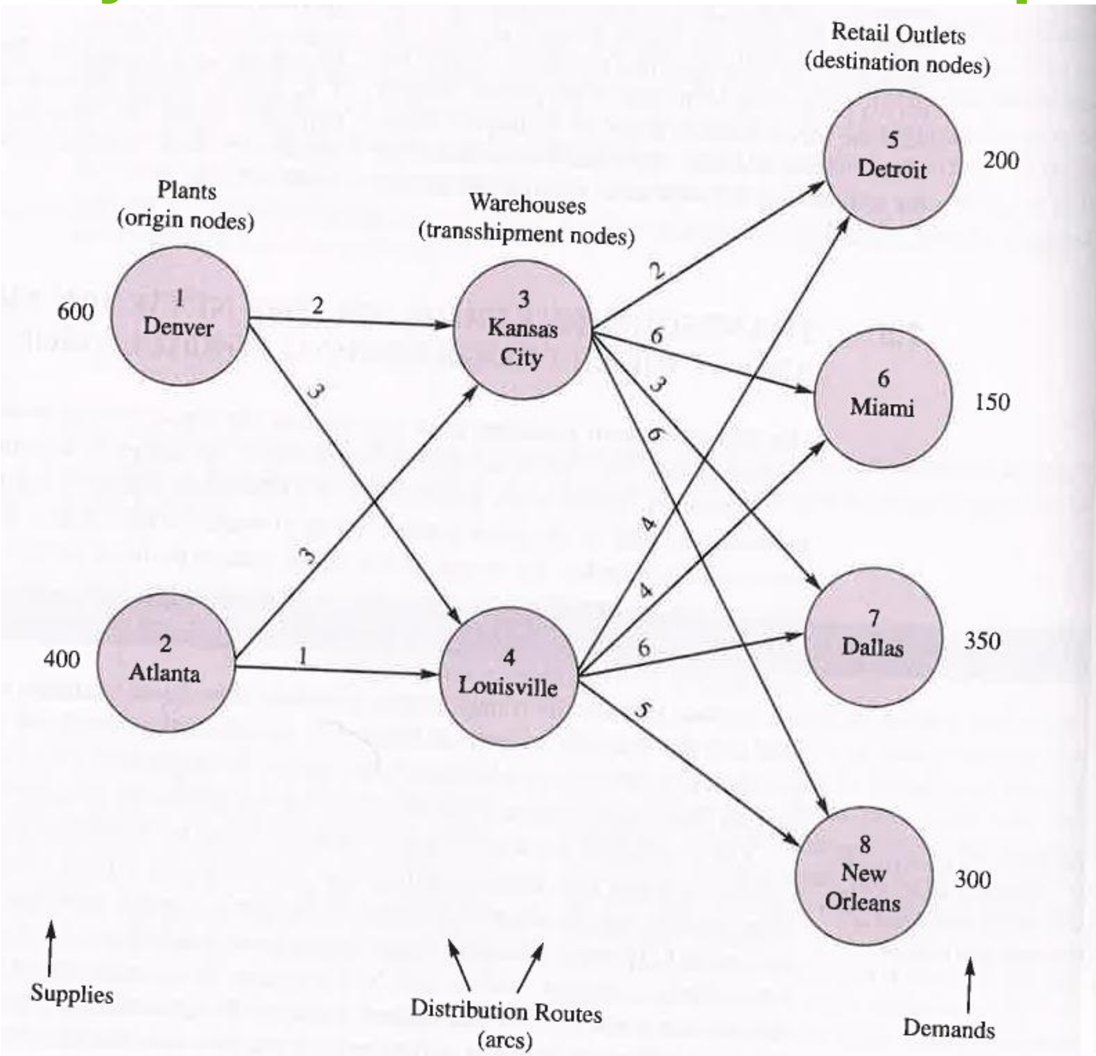
Transhipment example: Ryan Electronics

- Ryan Electronics wants to optimize its component distribution network
  - The firm holds its annual strategy weekend in Aspen
  - Production plants in Denver and Atlanta.
  - After production components are shipped to regional warehouses in Kansas City and Louisville
  - The firm's HQ is located in Denver
  - From regional warehouses the firm supplies its retail outlets in Detroit, Miami, Dallas and New Orleans

Question: Network representation

- Sources?
- Transshipment nodes?
- Destinations?

Ryan Electronics: Network representation



Question: LP formulation

- How many decision variables are needed?
- How many constraints?

Ryan Electronics: LP formulation

$$\text{Min } \overbrace{2x_{13} + 3x_{14}}^{\text{From Denver}} + \overbrace{3x_{23} + 1x_{24}}^{\text{From Atlanta}} + \overbrace{2x_{35} + 6x_{36} + 3x_{37} + 6x_{38}}^{\text{From Kansas City}} + \overbrace{4x_{45} + 4x_{46} + 6x_{47} + 5x_{48}}^{\text{From Louisville}}$$

s.t.

$$\begin{array}{rcll}
x_{13} + x_{14} & \leq & 600 & \text{Denver (1)} \\
x_{23} + x_{24} & \leq & 400 & \text{Atlanta (2)} \\
\hline
-x_{13} - x_{23} + x_{35} + x_{36} + x_{37} + x_{38} & = & 0 & \text{Kansas City (3)} \\
-x_{14} - x_{24} + x_{45} + x_{46} + x_{47} + x_{48} & = & 0 & \text{Louisville (4)} \\
\hline
x_{35} + x_{45} & = & 200 & \text{Detroit (5)} \\
x_{36} + x_{46} & = & 150 & \text{Miami (6)} \\
x_{37} + x_{47} & = & 350 & \text{Dallas (7)} \\
x_{38} + x_{48} & = & 300 & \text{New Orleans (8)}
\end{array}$$

$x_{ij} \geq 0$ for all i and j

Ryan Electronics: Spreadsheet implementation

C21 =SUMPRODUCT(D6:E7;D12:E13)+SUMPRODUCT(L6:O7;L12:O13)

Unit Cost

		Warehouse (transshipment node)				Retail outlet (destination node)			
		Kansas City	Louisville			Detroit	Miami	Dallas	New Orleans
Plant (source node)	Denver	2	3	Warehouse (transshipment node)	Kansas city	2	6	3	6
	Atlanta	3	1		Louisville	4	4	6	5

Shipment Quantity

		Warehouse (transshipment node)							Retail outlet (destination node)				
		Kansas City	Louisville	Total Shipped		Supply			Detroit	Miami	Dallas	New Orleans	Total Shipped
Plant (source node)	Denver	550	50	600	<=	600	Warehouse (transshipment node)	Kansas city	200	0	350	0	550
	Atlanta	0	400	400	<=	400		Louisville	0	150	0	300	450
	Total Received	550	450					Total Received	200	150	350	300	
									=	=	=	=	
								Demand	200	150	350	300	

Total costs

5 200

Solver Parameters

Set Objective: $C$21

To: Max Min Value Of

By Changing Variable Cells:

$D$12:$E$13;$L$12:$O$13

Subject to the Constraints:

$D$14 = $P$12
$E$14 = $P$13
$F$12:$F$13 <= $H$12:$H$13
$L$14:$O$14 = $L$16:$O$16

Make Unconstrained Variables Non-Negative

Select a Solving Method: Simplex LP

Transshipment Problem: General LP Formulation

x_{ik}, x_{kj} represents the shipment from node i to node k and from node k to node j, respectively ($i \in N_{source}, k \in N_{tran}, j \in N_{dest.}$)

Min $\Sigma_i \Sigma_k c_{ik} x_{ik} + \Sigma_k \Sigma_j c_{kj} x_{kj}$

$\Sigma_k x_{ik} \leq s_i$	for each source $i \in N_{source}$
$\Sigma_i x_{ik} - \Sigma_j x_{kj} = 0$	for each transshipment node $k \in N_{tran}$
$\Sigma_k x_{kj} = d_j$	for each destination $j \in N_{dest.}$
$x_{ik}, x_{kj} \geq 0$	for all i,j,k

Example:

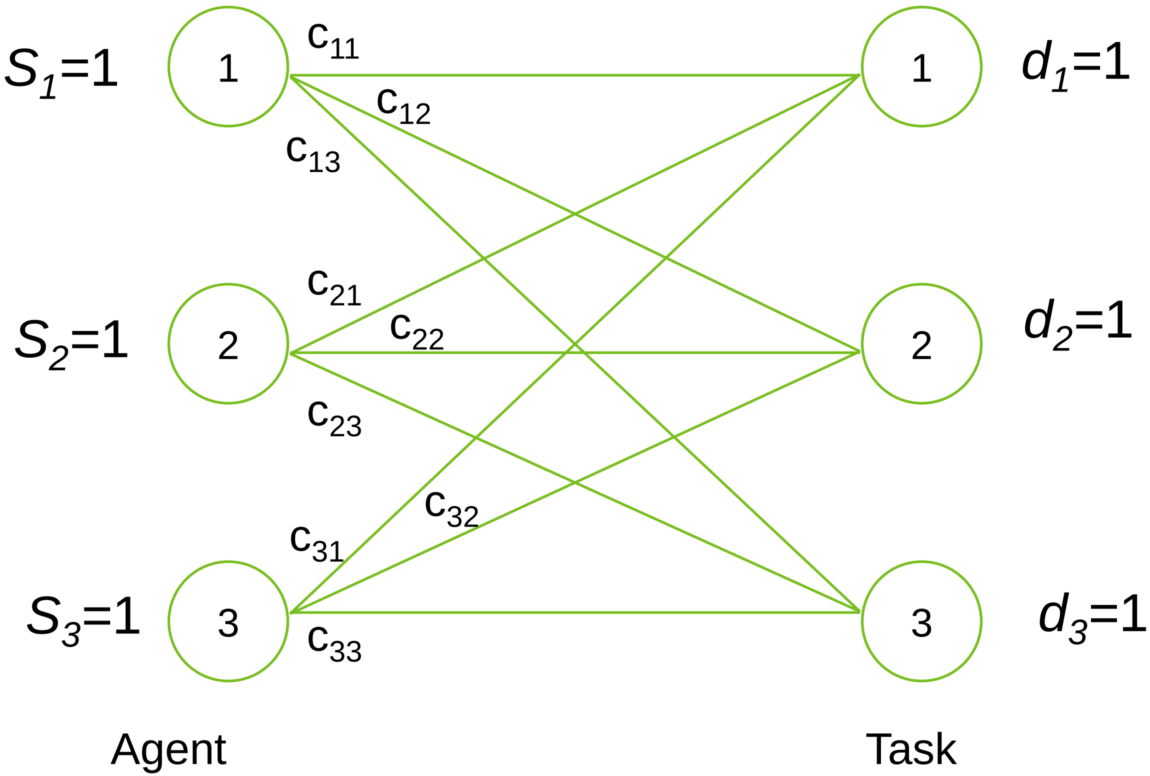
- Source nodes: $N_{source} = \{1,2\}$
- Transshipment nodes: $N_{tran} = \{3,4\}$

$$\sum_i \sum_k c_{ik} x_{ik} = \sum_{i=1}^{2} \sum_{k=3}^{4} c_{ik} x_{ik} = \sum_{i=1}^{2} (c_{i3} x_{i3} + c_{i4} x_{i4})$$
$$= (c_{13} x_{13} + c_{14} x_{14}) + (c_{23} x_{23} + c_{24} x_{24})$$

Assignment Problem

- The problem of assigning agents (people, machines) to a set of tasks is called an assignment problem
  - Problem components
    - a set of agents
    - a set of tasks
    - a cost table (cost associated with each agent performing each task)
  - Objective: Allocate agents to the tasks so that all tasks are performed at the minimum possible cost
  - A special case of a transportation problem

Assignment Problem: Network representation and general LP formulation



$$\text{Min} \quad \sum_i \sum_j c_{ij} x_{ij}$$

$$\text{s.t.} \quad \sum_j x_{ij} \le 1, \text{ for each agent } i$$

$$\sum_i x_{ij} = 1, \text{ for each task } j$$

$$x_{ij} \ge 0, \text{ for all } i \text{ and } j.$$

Assignment Problem Example: Swim team

- Help the coach of a swim team to assign swimmers to a 200-yard medley relay team
  - Four swimmers swimming 50 yards using one of the four strokes
- Data on the swimmers (time in seconds for 50 yeards):

	Backstroke	Breaststroke	Butterfly	Freestyle
Carl	37.7	43.4	33.3	29.2
Chris	32.9	33.1	28.5	26.4
David	33.8	42.2	38.9	29.6
Tony	37.0	34.7	30.4	28.5
Ken	35.4	41.8	33.6	31.1

Question: LP formulation

- How many constraints and decision variables?
- What would be the objective function?

Swim team: LP Formulation

Let $x_{ij} = 1$ if swimmer i swims stroke j; 0 otherwise
t_{ij} = time of swimmer i in stroke j

Minimize total time = $\sum_i \sum_j t_{ij} x_{ij}$

subject to

(each stroke swum) $\sum_i x_{ij} = 1$ for each stroke j

(each swimmer swims 1) $\sum_j x_{ij} \leq 1$ for each swimmer i

$x_{ij} \geq 0$ for all i and j.

Swim team: Spreadsheet Formulation

Best Times	Backstroke	Breastroke	Butterfly	Freestyle			
Carl	37.7	43.4	33.3	29.2			
Chris	32.9	33.1	28.5	26.4			
David	33.8	42.2	38.9	29.6			
Tony	37.0	34.7	30.4	28.5			
Ken	35.4	41.8	33.6	31.1			
Assignment	Backstroke	Breastroke	Butterfly	Freestyle			
Carl	0	0	0	1	1	<=	1
Chris	0	0	1	0	1	<=	1
David	1	0	0	0	1	<=	1
Tony	0	1	0	0	1	<=	1
Ken	0	0	0	0	0	<=	1
	1	1	1	1	Time	=	126.2
	=	=	=	=			
	1	1	1	1			

Optimal assignment

- Backstroke – David
- Breaststroke – Tony
- Butterfly – Chris
- Freestyle – Carl

(Ken is the towel boy)

Assignment Problem: Variations

$$\text{Min} \sum_i \sum_j c_{ij} x_{ij}$$

$$\text{s.t.} \quad \sum_j x_{ij} \leq 1 \qquad \text{for each agent } i$$

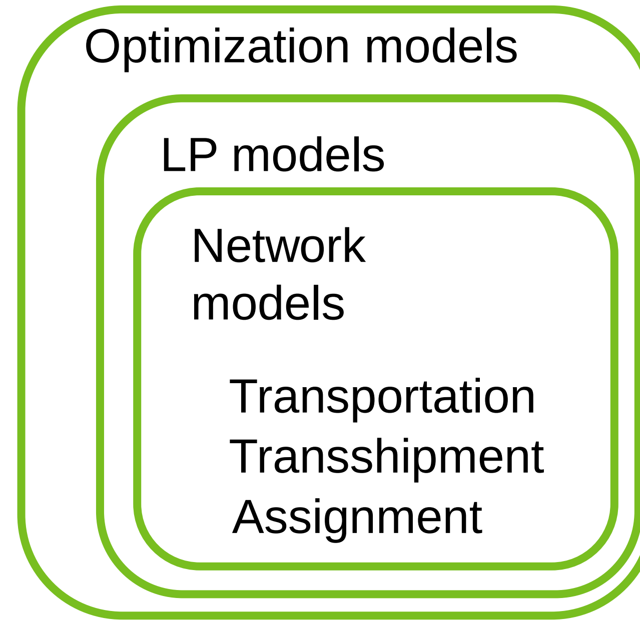
$$\sum_i x_{ij} = 1 \qquad \text{for each task } j$$

$$x_{ij} \geq 0 \qquad \text{for all } i \text{ and } j.$$

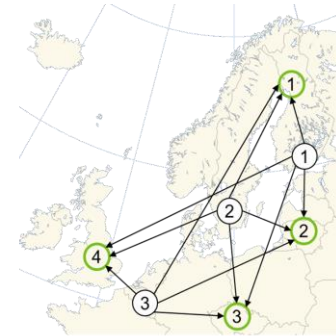
- Certain agents are unable to perform certain tasks
- There are more tasks than agents (some tasks will not be done)
- There are more agents than tasks (some agents will not work)
- An agent can be assigned to perform more than one task
- A task can be performed jointly by more than one agents

Summary: Network LP Models

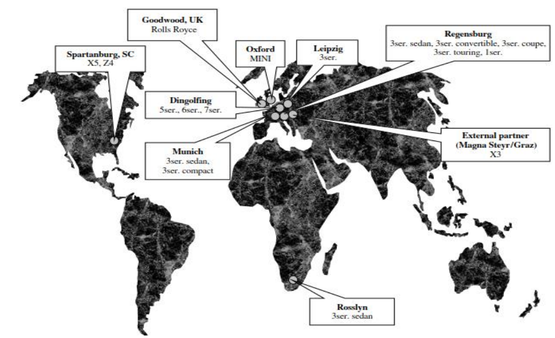
- Network Components and their LP equivalents
  - Arc = decision variable
  - Node = constraint:
    (flow in) – (flow out) {=, ≥,≤,} constant
  - Objective function = linear function of arc flows
- Properties:
  - Fast to solve (good LP solvers detect and exploit network structure)
  - Illustrative graphical representations
  - Provide integer solutions without the need for explicit *integrality constraints* used in integer linear programming



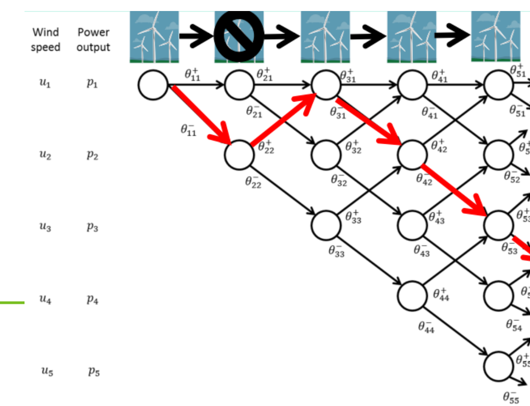
Network models: Application examples



- Sheeting network optimization at “Nordic Wood Processing Company”
  - Customized student business project



- Strategic production planning at BMW[1]
  - Optimize production allocation, supply of materials, distribution with a 12-year planning horizon



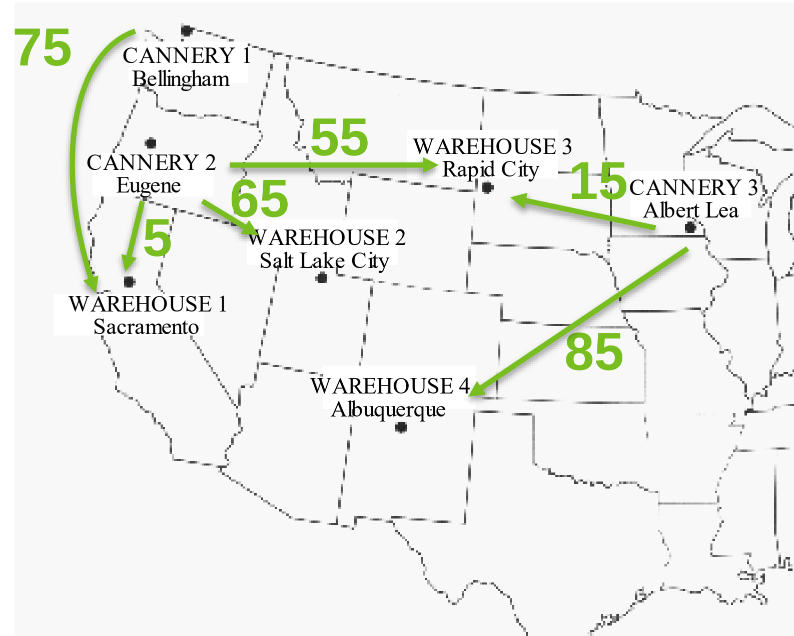
- Siting of off-shore wind farms[2]
  - Economic and environmental objectives
  - Network models used to capture wake interactions

Articles available on MyCourses:
1) Fleischmann, B., Ferber, S., Henrich, P., 2006, *Interfaces*.
2) Cranmer, A., Baker, E., Liesiö, J., Salo, A., 2018, *European Journal of Operation Research*.

Extra network LP examples: Transportation problem

P&T Company Distribution Problem

- P&T company (producer of canned peas) is unhappy with their total shipping costs
  - Peas are prepared at three canneries and shipped by truck to four distributing warehouses
  - Up to now some intuitive guidelines have been used to determine the shipment amounts from canneries to warehouses

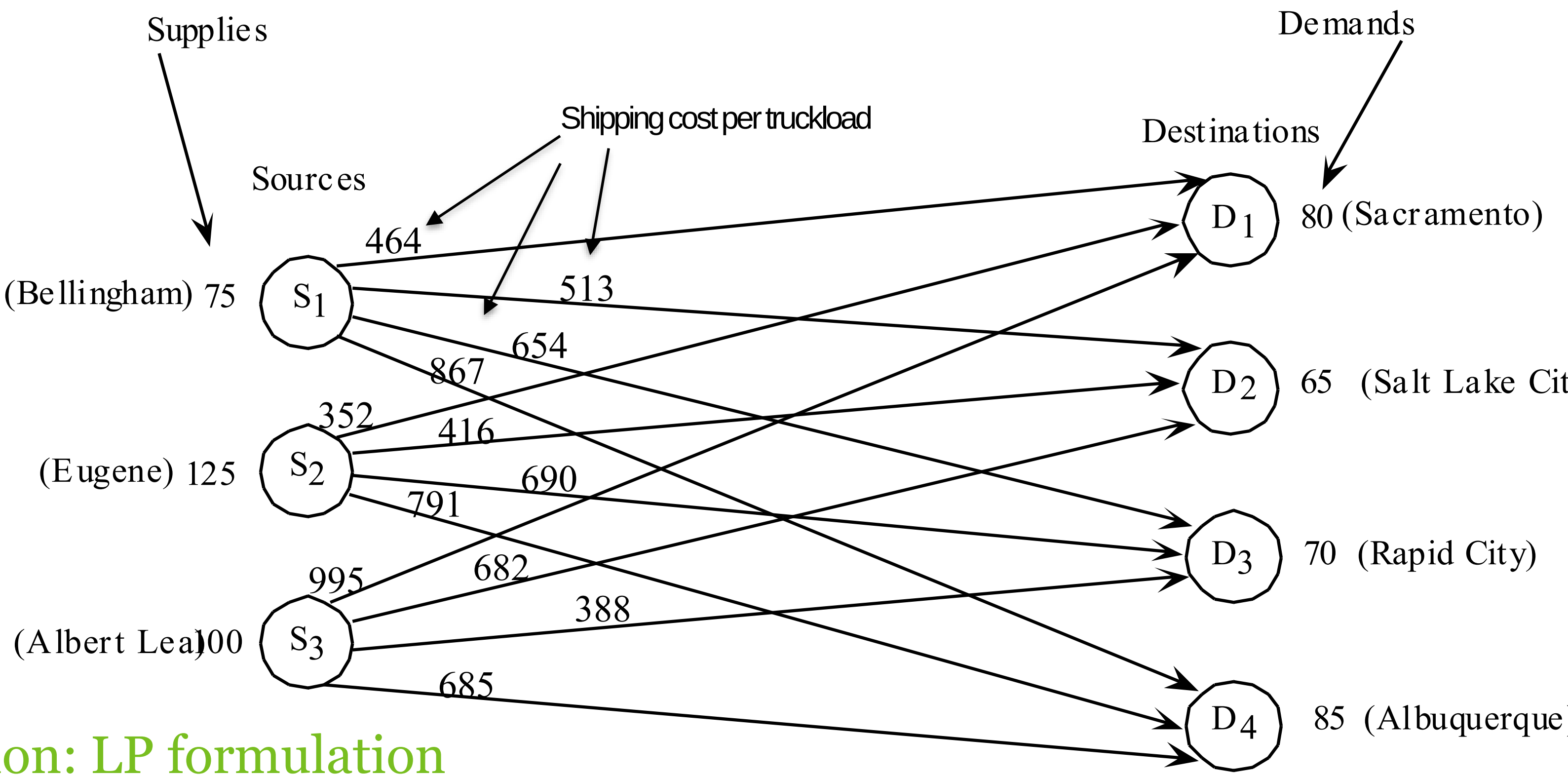


Current allocations (# of truckloads)

Question: Network representation

- What would be the
  - sources and their supplies?
  - destinations and their demands?

P&T Company: Network Representation



Question: LP formulation

- What would be the decision variables, objective function, constraints?

P&T Company: Linear Programming formulation

x_{ij}: the number of truckloads to ship from cannery i to warehouse j ($i = 1, 2, 3$; $j = 1, 2, 3, 4$)

$$\begin{aligned}\text{Minimize Cost} = {} & \$464x_{11} + \$513x_{12} + \$654x_{13} + \$867x_{14} + \$352x_{21} + \$416x_{22} + \\ & \$690x_{23} + \$791x_{24} + \$995x_{31} + \$682x_{32} + \$388x_{33} + \$685x_{34}\end{aligned}$$

subject to

Cannery 1: $x_{11} + x_{12} + x_{13} + x_{14} \le 75$

Cannery 2: $x_{21} + x_{22} + x_{23} + x_{24} \le 125$

Cannery 3: $x_{31} + x_{32} + x_{33} + x_{34} \le 100$

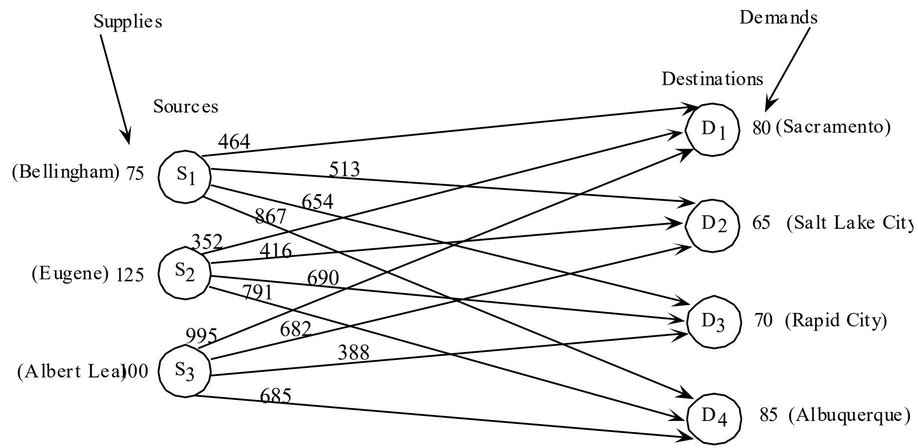
Warehouse 1: $x_{11} + x_{21} + x_{31} = 80$

Warehouse 2: $x_{12} + x_{22} + x_{32} = 65$

Warehouse 3: $x_{13} + x_{23} + x_{33} = 70$

Warehouse 4: $x_{14} + x_{24} + x_{34} = 85$

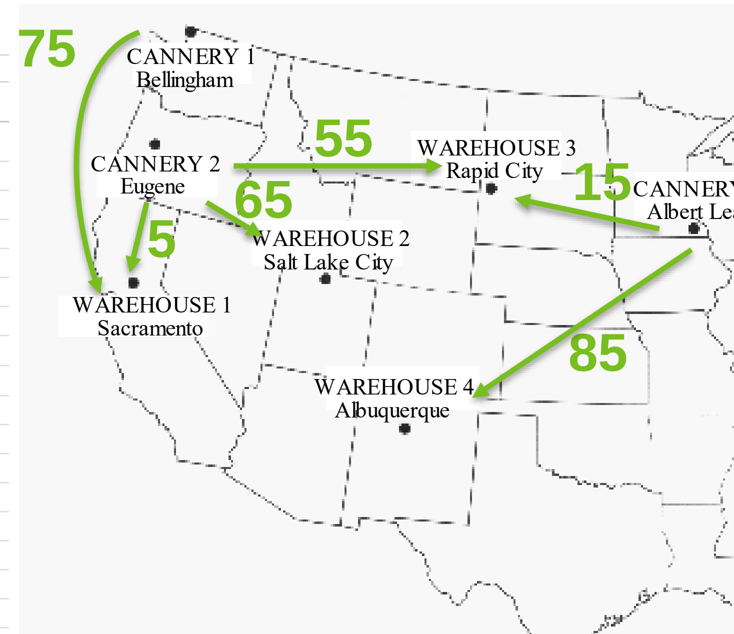
Non-negativity: $x_{ij} \ge 0$ ($i = 1, 2, 3$; $j = 1, 2, 3, 4$)



P&T Company: Spreadsheet Formulation

H9 =SUMPRODUCT($D$5:$G$7;$D$12:$G$14)

	A	B	C	D	E	F	G	H	I	J
1	**P&T Co. Distribution Problem**									
2										
3		**Unit Cost**		Destination (Warehouse)						
4				Sacramento	Salt Lake City	Rapid City	Albuquerque			
5		Source	Bellingham	$464	$513	$654	$867			
6		(Cannery)	Eugene	$352	$416	$690	$791			
7			Albert Lea	$995	$682	$388	$685			
8								Total Cost		
9								$165 595		
10		**Shipment Quantity**		Destination (Warehouse)						
11		**(Truckloads)**		Sacramento	Salt Lake City	Rapid City	Albuquerque	Total Shipped		Supply
12		Source	Bellingham	75	0	0	0	75	<=	75
13		(Cannery)	Eugene	5	65	55	0	125	<=	125
14			Albert Lea	0	0	15	85	100	<=	100
15			Total Received	80	65	70	85			
16				=	=	=	=			
17			Demand	80	65	70	85			



Current allocations (# of truckloads)

P&T Company: Optimal Solution

H9 =SUMPRODUCT($D$5:$G$7;$D$12:$G$14)

	A	B	C	D	E	F	G	H	I	J
1	**P&T Co. Distribution Problem**									
2										
3		**Unit Cost**		Destination (Warehouse)						
4				Sacramento	Salt Lake City	Rapid City	Albuquerque			
5		Source (Cannery)	Bellingham	$464	$513	$654	$867			
6			Eugene	$352	$416	$690	$791			
7			Albert Lea	$995	$682	$388	$685			
8								Total Cost		
9								$152 535		
10		**Shipment Quantity**		Destination (Warehouse)						
11		**(Truckloads)**		Sacramento	Salt Lake City	Rapid City	Albuquerque	Total Shipped		Supply
12		Source (Cannery)	Bellingham	0	20	0	55	75	<=	75
13			Eugene	80	45	0	0	125	<=	125
14			Albert Lea	0	0	70	30	100	<=	100
15			Total Received	80	65	70	85			
16				=	=	=	=			
17			Demand	80	65	70	85			

Solver Parameters

Set Objective: $H$9

To: Max Min Value Of: 0

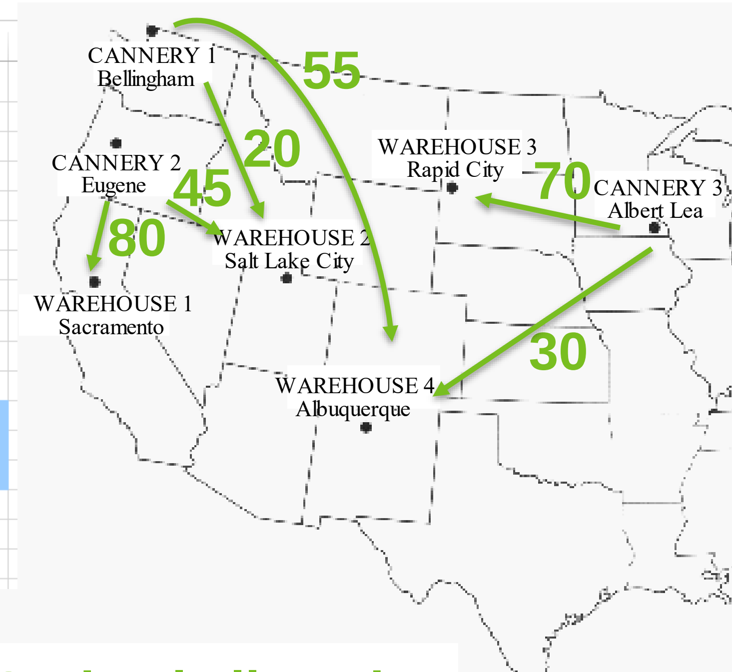
By Changing Variable Cells:

$D$12:$G$14

Subject to the Constraints:

$D$15:$G$15 = $D$17:$G$17
$H$12:$H$14 <= $J$12:$J$14

Add



Optimal allocation (# of truckloads)