Aalto University
School of Business

Mixed Integer Linear Programming (MILP)

- *Types of Integer Linear Programming Models*
- *Feasible regions and graphical solution*
- *LP relaxation*
- *Special 0-1 constraints*
- *Computer solution*
- *Cautionary notes on sensitivity analysis and rounding*
- *(M)ILP applications and formulations*

Integers

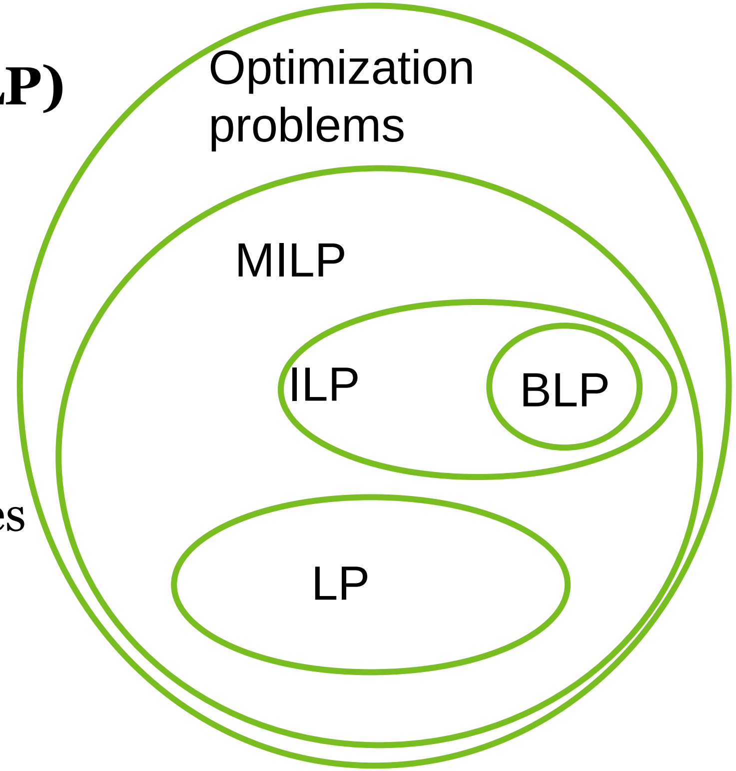
…-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5…

group often denoted with $\mathbb{Z}$,
e.g. $x \in \mathbb{Z}$

$\in$ reads “in” or “is element of the set”

Types of Integer Linear Programming Models

- **Pure Integer Linear Programming (ILP)**
  - All the decision variables are integers
- **Mixed Integer Linear Programming (MILP)**
  - Some of the decision variables are integers
- **Binary Linear Programming (BLP)**
  - Decision variables restricted to be binary values (i.e. 0 or 1)
  - Sometimes the term zero-one linear programming (ZOLP) is used
  - Pure BLP: all the decision variables binary
  - Mixed BLP: some decision variables binary



Examples of ILP problems

BLP

$$\begin{aligned} \text{Max} \quad & 3x_1 + 2x_2 \\ \text{s.t.} \quad & 3x_1 + x_2 \leq 9 \\ & x_1 + 3x_2 \leq 7 \\ & -x_1 + x_2 \leq 1 \\ & x_1, x_2 \in \{0,1\} \end{aligned}$$

(pure) ILP

$$\begin{aligned} \text{Max} \quad & 3x_1 + 2x_2 \\ \text{s.t.} \quad & 3x_1 + x_2 \leq 9 \\ & x_1 + 3x_2 \leq 7 \\ & -x_1 + x_2 \leq 1 \\ & x_1, x_2 \geq 0 \\ & x_1, x_2 \text{ integer} \end{aligned}$$

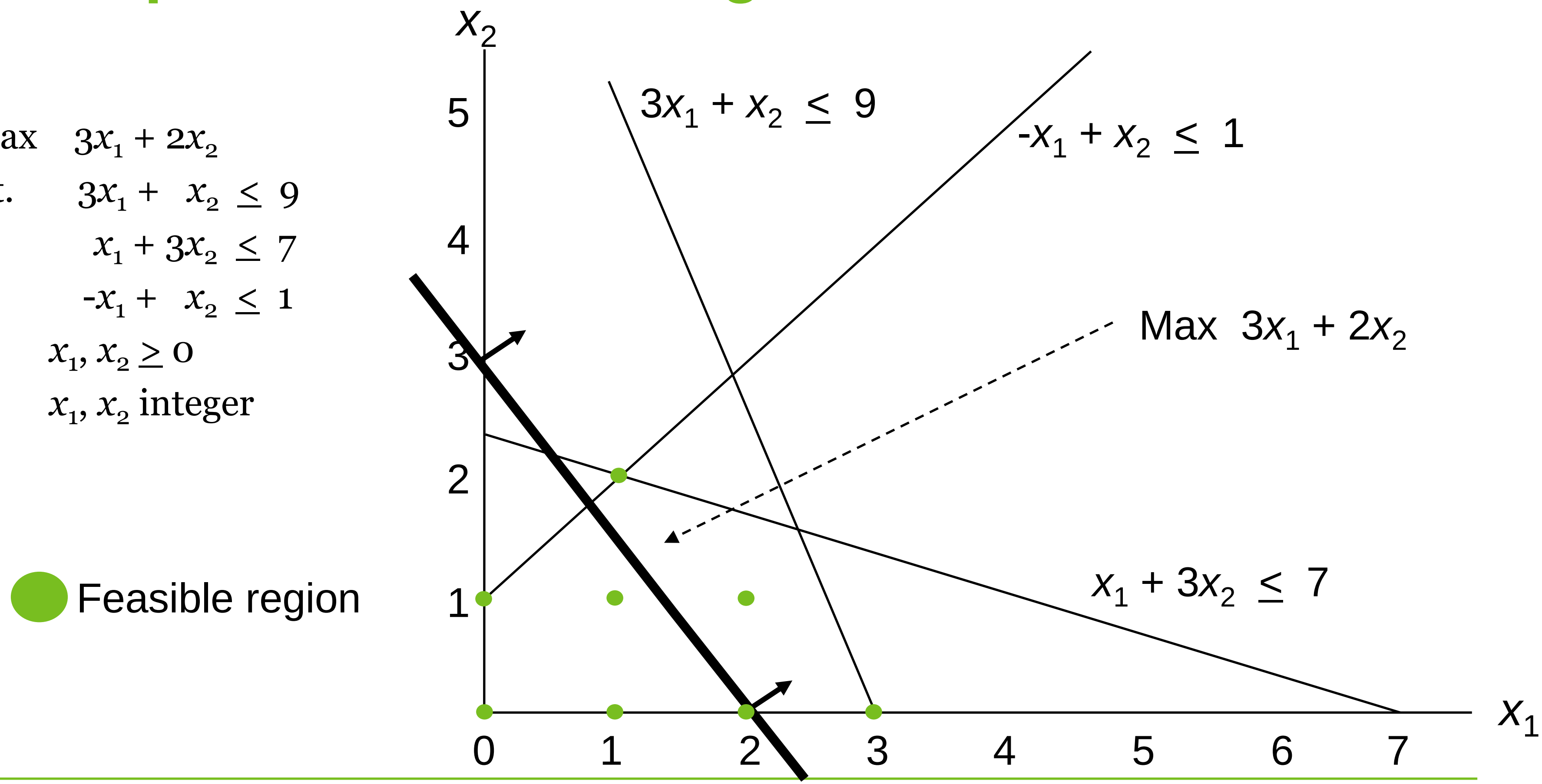
MILP

$$\begin{aligned} \text{Max} \quad & 3x_1 + 2x_2 \\ \text{s.t.} \quad & 3x_1 + x_2 \leq 9 \\ & x_1 + 3x_2 \leq 7 \\ & -x_1 + x_2 \leq 1 \\ & x_1, x_2 \geq 0 \\ & x_1 \text{ integer} \end{aligned}$$

$\in$ reads “in” or “element of the set”

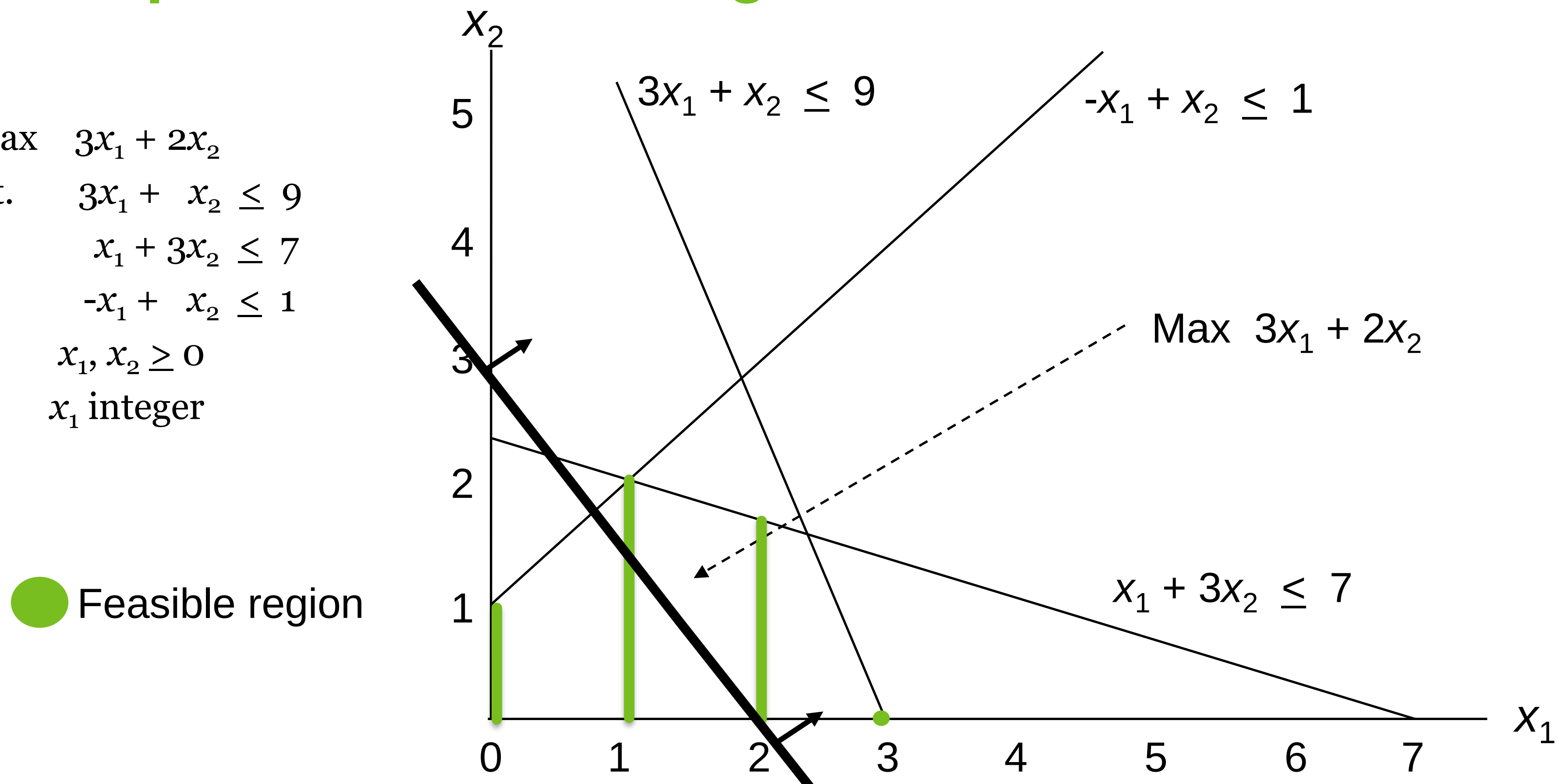
Example of the feasible region: Pure ILP

$$
\begin{aligned}
\text{Max} \quad & 3x_1 + 2x_2 \\
\text{s.t.} \quad & 3x_1 + x_2 \le 9 \\
& x_1 + 3x_2 \le 7 \\
& -x_1 + x_2 \le 1 \\
& x_1, x_2 \ge 0 \\
& x_1, x_2 \text{ integer}
\end{aligned}
$$



Example of the feasible region: MILP

$$
\begin{aligned}
\text{Max} \quad & 3x_1 + 2x_2 \\
\text{s.t.} \quad & 3x_1 + x_2 \le 9 \\
& x_1 + 3x_2 \le 7 \\
& -x_1 + x_2 \le 1 \\
& x_1, x_2 \ge 0 \\
& x_1 \text{ integer}
\end{aligned}
$$



Graphical Method for Solving MILP Problems

- Optimal solutions to MILP problems with two decision variables can be found by applying the graphical solution method for LPs
  - Caution: feasible region not equal to the LP feasible region!

Points satisfying

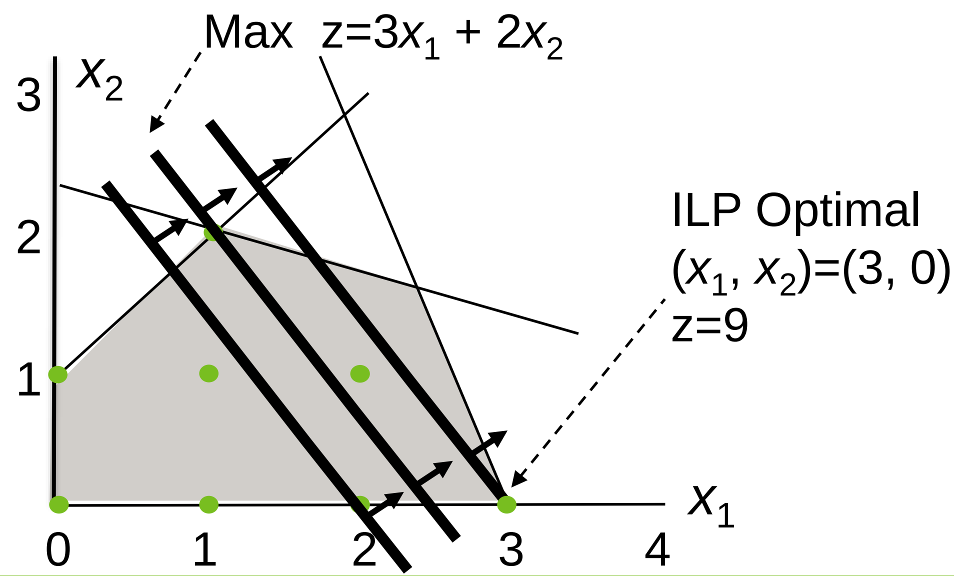
$3x_1 + x_2 \leq 9$

$x_1 + 3x_2 \leq 7$

$-x_1 + x_2 \leq 1$

$x_1, x_2 \geq 0$

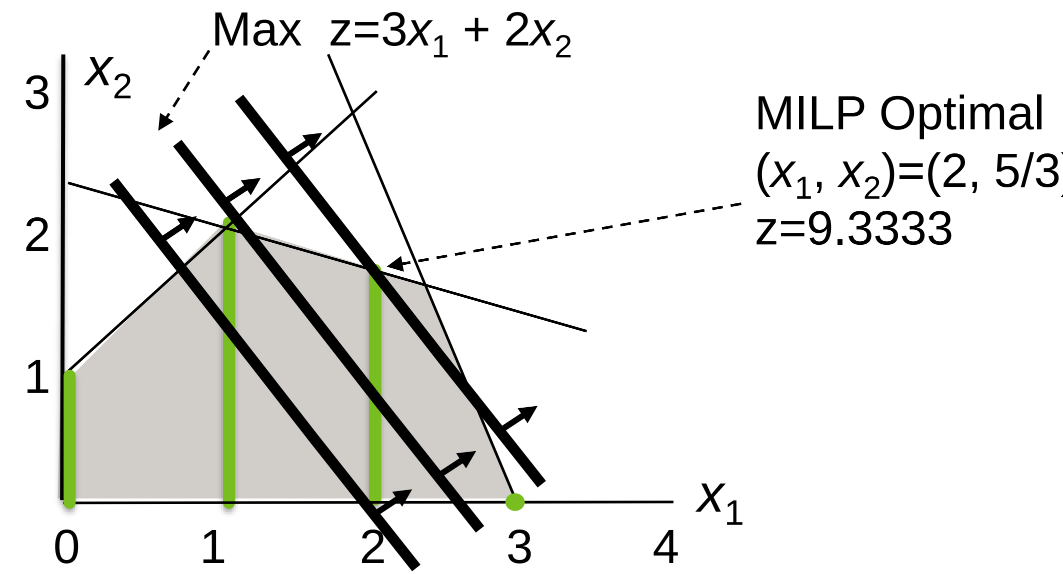
Feasible region (x_1, x_2 integer)



Graphical Method for Solving ILP Problems (Cont'd)

MILP example:

$$\begin{aligned} \text{Max} \quad & 3x_1 + 2x_2 \\ \text{s.t.} \quad & 3x_1 + x_2 \leq 9 \\ & x_1 + 3x_2 \leq 7 \\ & -x_1 + x_2 \leq 1 \\ & x_1, x_2 \geq 0 \\ & x_1 \text{ integer} \end{aligned}$$



Points satisfying
$3x_1 + x_2 \leq 9$
$x_1 + 3x_2 \leq 7$
$-x_1 + x_2 \leq 1$
$x_1, x_2 \geq 0$

Feasible region (x_1 integer)

Linear programming relaxation

- The LP relaxation of a (M)ILP problem is the LP problem obtained when all the integrality constraints are removed

(pure) ILP

$$\begin{aligned} \text{Max} \quad & 3x_1 + 2x_2 \\ \text{s.t.} \quad & 3x_1 + x_2 \leq 9 \\ & x_1 + 3x_2 \leq 7 \\ & -x_1 + x_2 \leq 1 \\ & x_1, x_2 \geq 0 \\ & x_1, x_2 \text{ integer} \end{aligned}$$

MILP

$$\begin{aligned} \text{Max} \quad & 3x_1 + 2x_2 \\ \text{s.t.} \quad & 3x_1 + x_2 \leq 9 \\ & x_1 + 3x_2 \leq 7 \\ & -x_1 + x_2 \leq 1 \\ & x_1, x_2 \geq 0 \\ & x_2 \text{ integer} \end{aligned}$$

LP relaxation

$$\begin{aligned} \text{Max} \quad & 3x_1 + 2x_2 \\ \text{s.t.} \quad & 3x_1 + x_2 \leq 9 \\ & x_1 + 3x_2 \leq 7 \\ & -x_1 + x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Linear programming relaxation (cont'd)

- The LP relaxation of a BLP problem is the LP problem obtained when all the integrality constraints are removed

BLP

$$
\begin{aligned}
\text{Max} \quad & 3x_1 + 2x_2 \\
\text{s.t.} \quad & 3x_1 + x_2 \leq 9 \\
& x_1 + 3x_2 \leq 7 \\
& -x_1 + x_2 \leq 1 \\
& x_1, x_2 \in \{0,1\}
\end{aligned}
$$

LP relaxation

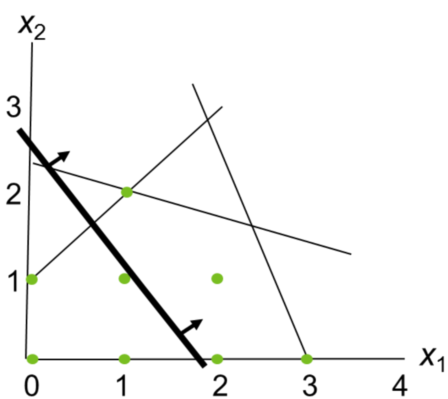
$$
\begin{aligned}
\text{Max} \quad & 3x_1 + 2x_2 \\
\text{s.t.} \quad & 3x_1 + x_2 \leq 9 \\
& x_1 + 3x_2 \leq 7 \\
& -x_1 + x_2 \leq 1 \\
& x_1 \leq 1 \\
& x_2 \leq 1 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

$\in$ reads "in" or "element of the set"

Linear programming relaxation (cont'd)

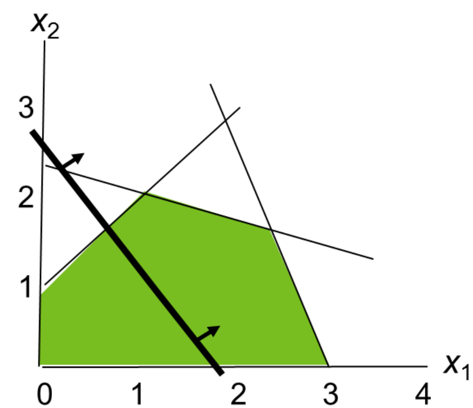
(pure) ILP

$$\begin{aligned} \text{Max} \quad & 3x_1 + 2x_2 \\ \text{s.t.} \quad & 3x_1 + x_2 \le 9 \\ & x_1 + 3x_2 \le 7 \\ & -x_1 + x_2 \le 1 \\ & x_1, x_2 \ge 0 \\ & x_1, x_2 \text{ integer} \end{aligned}$$



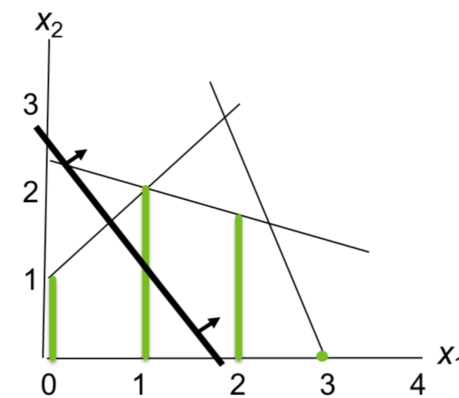
LP relaxation

$$\begin{aligned} \text{Max} \quad & 3x_1 + 2x_2 \\ \text{s.t.} \quad & 3x_1 + x_2 \le 9 \\ & x_1 + 3x_2 \le 7 \\ & -x_1 + x_2 \le 1 \\ & x_1, x_2 \ge 0 \end{aligned}$$



MILP

$$\begin{aligned} \text{Max} \quad & 3x_1 + 2x_2 \\ \text{s.t.} \quad & 3x_1 + x_2 \le 9 \\ & x_1 + 3x_2 \le 7 \\ & -x_1 + x_2 \le 1 \\ & x_1, x_2 \ge 0 \\ & x_1 \text{ integer} \end{aligned}$$



Understanding the implications of relaxing (integrality) constraints

I

$$\begin{aligned} \text{Max} \quad & 3x_1 + 2x_2 \\ 3x_1 + x_2 &\le 9 \\ -x_1 + x_2 &\le 1 \\ x_1, x_2 &\ge 0 \\ x_1, x_2 & \text{ integer} \end{aligned}$$

II

$$\begin{aligned} \text{Max} \quad & 3x_1 + 2x_2 \\ 3x_1 + x_2 &\le 9 \\ -x_1 + x_2 &\le 1 \\ x_1, x_2 &\ge 0 \end{aligned}$$

III

$$\begin{aligned} \text{Max} \quad & 3x_1 + 2x_2 \\ 3x_1 + x_2 &\le 9 \\ -x_1 + x_2 &\le 1 \\ x_1, x_2 &\in \{0,1\} \end{aligned}$$

IV

$$\begin{aligned} \text{Max} \quad & 3x_1 + 2x_2 \\ 3x_1 + x_2 &\le 9 \\ -x_1 + x_2 &\le 1 \\ x_1, x_2 &\ge 0 \\ x_2 & \text{ integer} \end{aligned}$$

Question:

- Rank problems I-IV with regard to their optimal objective function value (i.e., the highest, 2nd highest,…)

BLP example: Product portfolio selection

- Metropolitan Microwaves, an electronic appliances store, is planning to include new product lines to its selection
  - The company has identified seven new possible product lines:

	Product line	Initial investment ($)	Floor space (m^2)	Expected rate of return (%)
1	Tablets	6,000	125	8.1
2	Laptops	2,000	150	9
3	Workstations	20,000	200	11
4	VR-equipment	14,000	40	10.2
5	Phones	15,000	40	10.5
6	Video Games	2,000	20	14.1
7	Gaming consoles	32,000	100	13.2

  - The company has $45,000 to invest and 420 sq. ft. of floor space available
- A management scientist developed an integer linear programming model to support this decision, but she left for the academia and only her notes about the model remain

BLP example: Product portfolio selection (Cont'd)

"Model Notes"

$$\text{Max} \quad (6*1.081)x_1 + (2*1.09)x_2 + \ldots$$

s.t.

A. $125x_1 + 150\,x_2 + \ldots + 100\,x_7 \leq 420$

B. $6x_1 + \ldots + 32x_7 \leq 45$

C. $x_4 + x_5 \leq 1$

D. $x_6 \leq x_7$

E. $2x_3 \leq x_1 + x_2$

F. $x_1 + x_2 + \ldots + x_7 \geq 3$

$x_1, \ldots, x_7 \in \{0,1\}$

Data

	Product line	Initial investment ($)	Floor space (m^2)	Expected rate of return (%)
1	Tablets	6,000	125	8.1
2	Laptops	2,000	150	9
3	Workstations	20,000	200	11
4	VR-equipment	14,000	40	10.2
5	Phones	15,000	40	10.5
6	Video Games	2,000	20	14.1
7	Gaming consoles	32,000	100	13.2

Question

- Interpret the meaning of the decision variables and the constraints

BLP: Special 0-1 Constraints

- Since binary variables only provide two choices, they are ideal for modelling yes-or-no (go/no-go, continue/discard, etc.) decisions
- Constraints can then be used to capture logical dependencies between these decisions
- Examples:
  - $x_i = 1$, $i=1,\dots,n$ if and only if project *i* is started, otherwise $x_i = 0$
  - At most *k* out of *n* projects can be started:

    $$\Sigma_i x_i \leq k$$

  - Project *j* is conditional on project *i*:

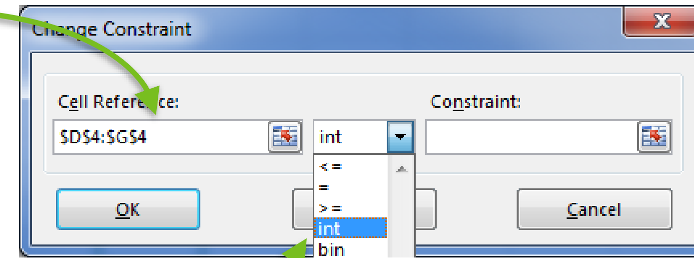
    $$x_j - x_i \leq 0$$

  - Projects *i* and *j* are mutually exclusive:

    $$x_i + x_j \leq 1$$

Computer Solution to (M)ILP problems

- In Excel Solver (2010-16)
  - Add an additional constraint that tells solver which decision variables are required to take integer ('int') or binary ('bin') values
  - Choose algorithm "Simplex LP" as before
    - This tells Solver that problem is linear, i.e. ILP, MILP, or BLP (not some non-linear optimization problem with integer variables)



- The Simplex algorithm is only for solving LP problems despite the misleading naming of algorithms used in Excel Solver
- Algorithms for (M)ILP problems: Branch-and-bound, Cutting plane,..
  - These algorithms often use the Simplex for solving sub-problems

Branch-and-bound (B&B) algorithm: Example*

ILP problem:

$$\text{Max} \quad z = 5x_1 + 4x_2$$

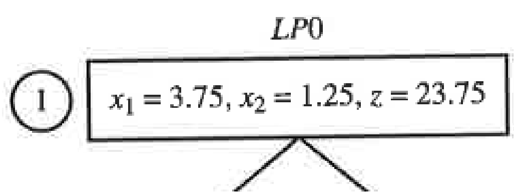
$$\text{s.t.} \quad x_1 + x_2 \leq 5$$

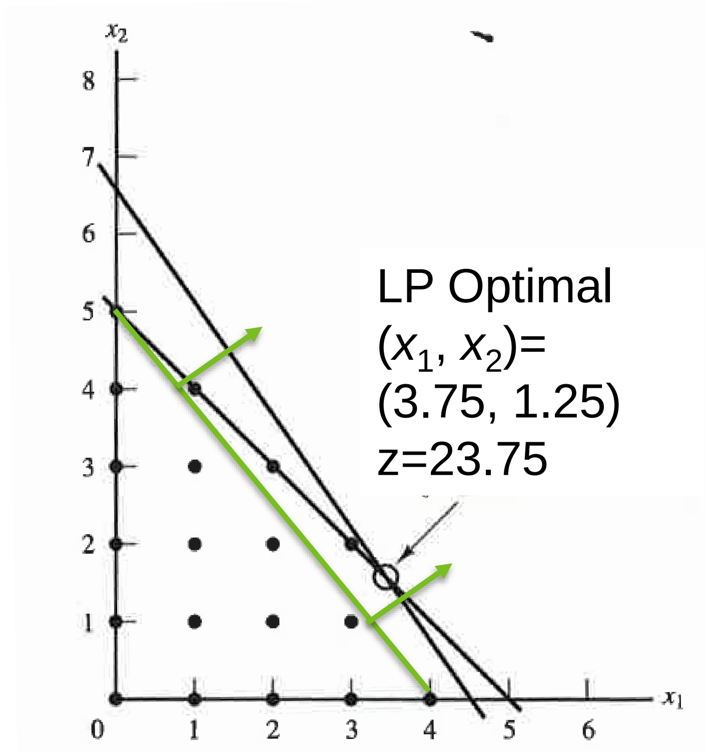
$$10x_1 + 6x_2 \leq 45$$

$$x_1, x_2 \geq 0 \text{ and integer}$$

Step 1: Bounding

- Solve the LP relaxation with Simplex
  - Gives an upper bound to the optimal value (cf. implications of relaxation)



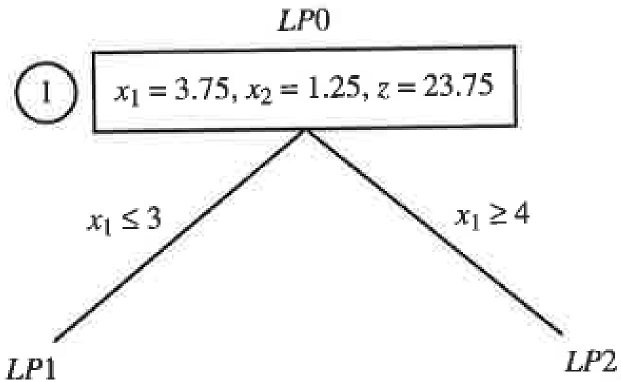


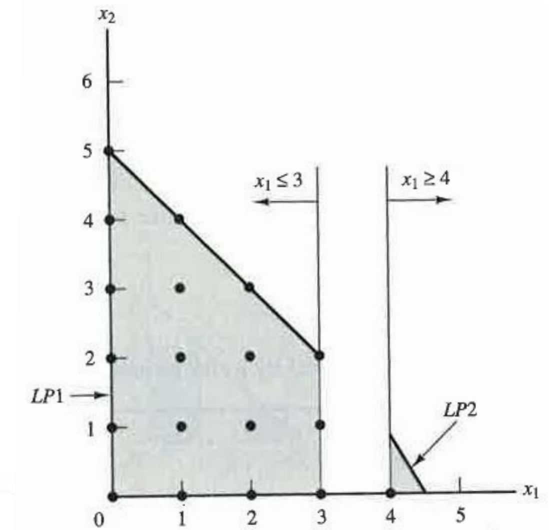
*) Example from Taha 1997: Operations research – An introduction, Prentice-Hall

Branch-and-bound (B&B) algorithm: Example

Step 2: Branching

- Add constraints on one variable that did not have an integer value in the optimal solution





LP1:

$$
\begin{array}{ll}
\text{Max} & z = 5x_1 + 4x_2 \\
\text{s.t.} & x_1 + x_2 \leq 5 \\
& 10x_1 + 6x_2 \leq 45 \\
& x_1 \leq 3 \\
& x_1, x_2 \geq 0 \\
& \text{and integer}
\end{array}
$$

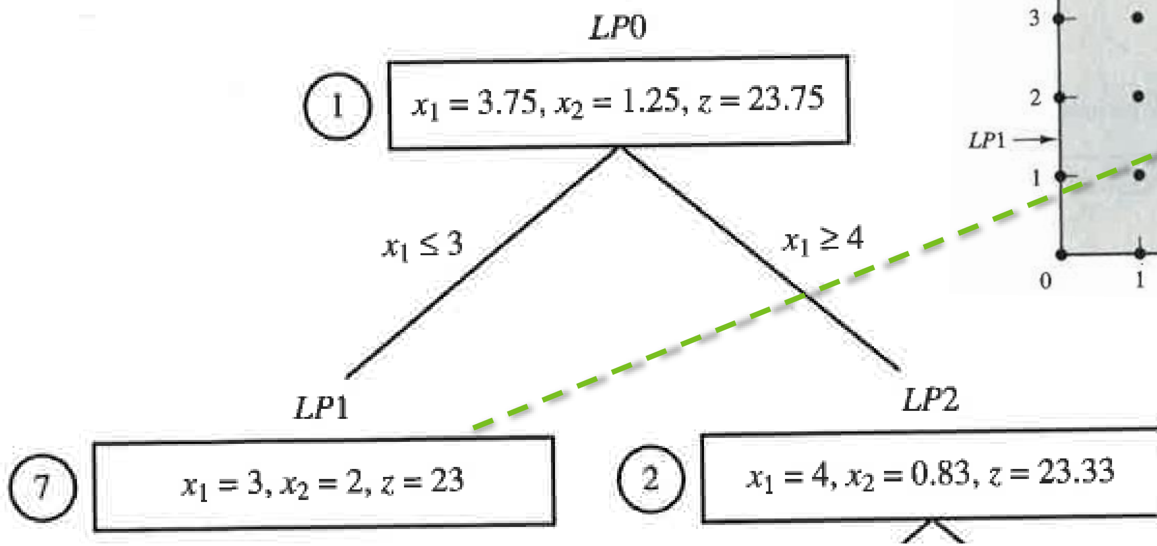
LP2:

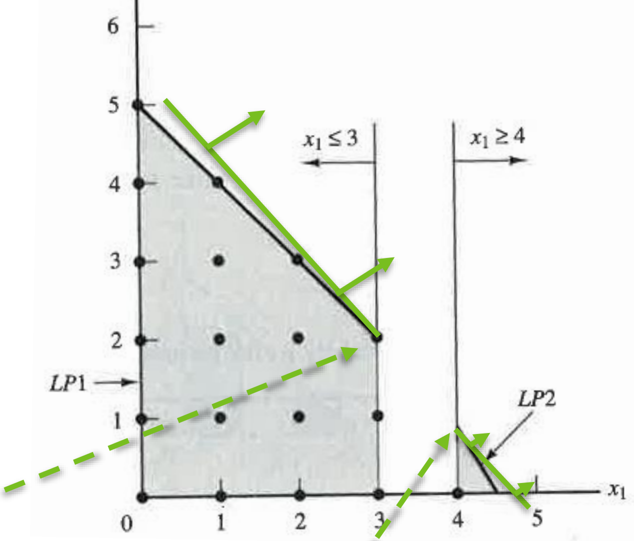
$$
\begin{array}{ll}
\text{Max} & z = 5x_1 + 4x_2 \\
\text{s.t.} & x_1 + x_2 \leq 5 \\
& 10x_1 + 6x_2 \leq 45 \\
& x_1 \geq 4,\ x_2 \geq 0 \\
& \text{and integer}
\end{array}
$$

Branch-and-bound (B&B) algorithm: Example

Step 3: Bounding

- Solve the LP relaxations in each branch

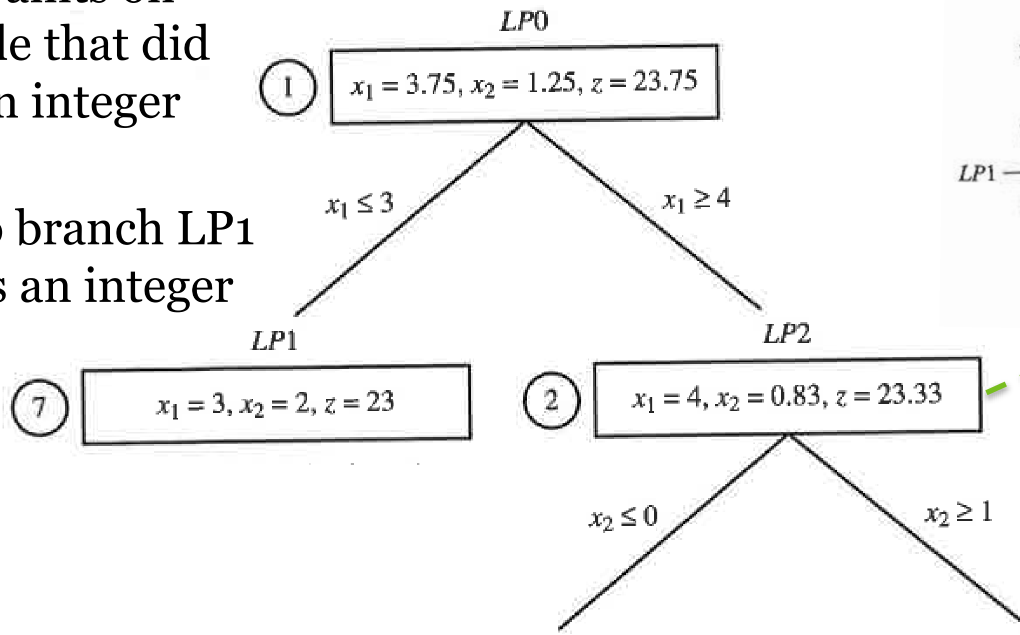


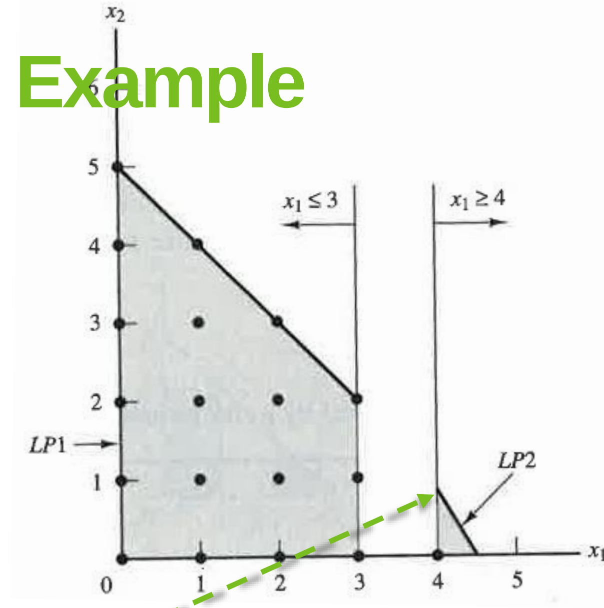


Branch-and-bound (B&B) algorithm: Example

Step 4: Branching

- Add constraints on one variable that did not have an integer value
- No need to branch LP1 since it has an integer solution!

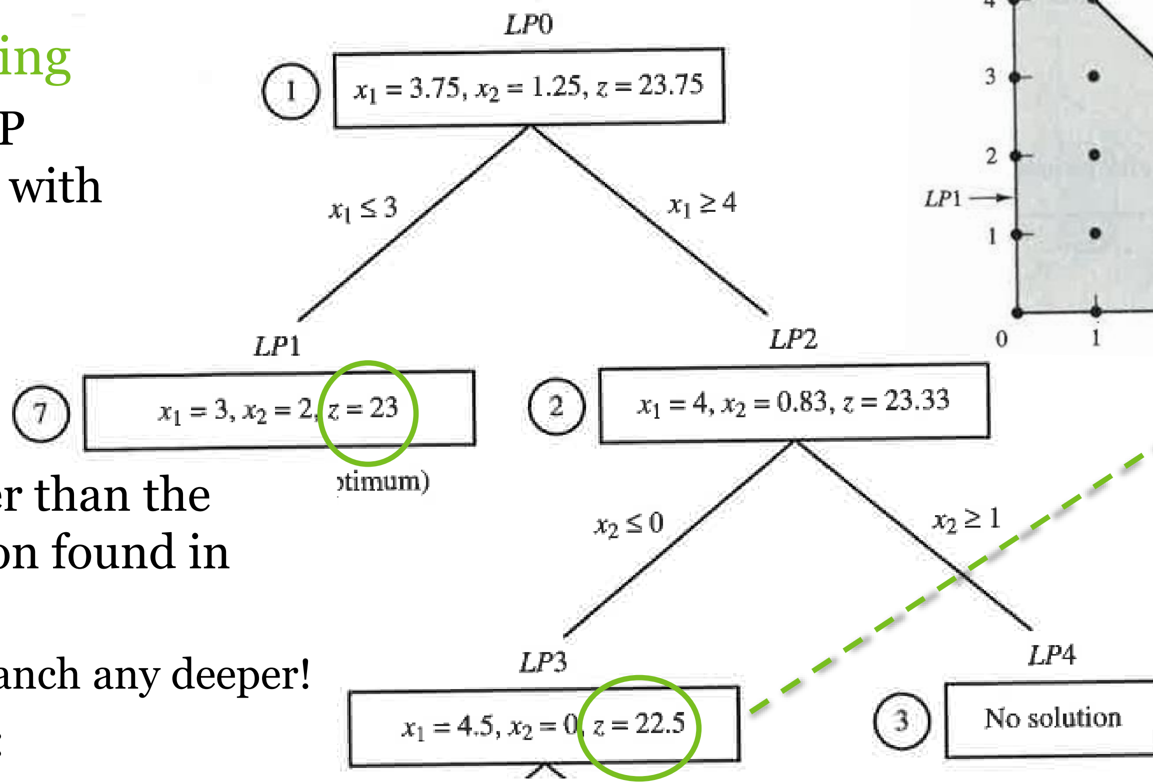


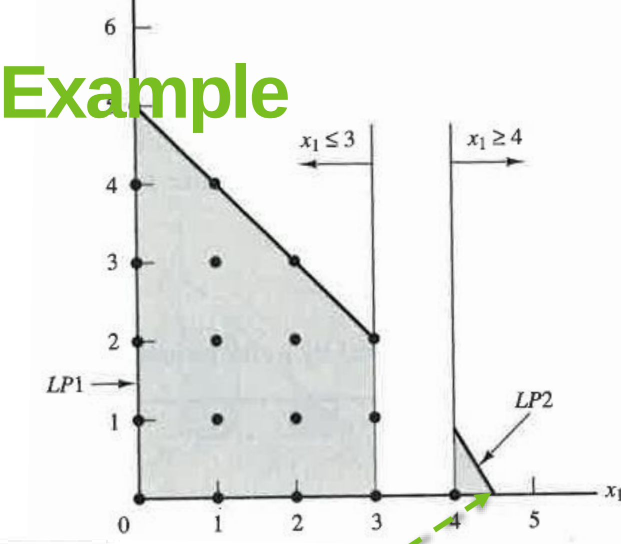


Branch-and-bound (B&B) algorithm: Example

Step 5: Bounding

- Solve the LP relaxations with Simple





(the problem is infeasible)

- Bound is lower than the integer solution found in Step 3:

  No need to branch any deeper!

- ILP optimum:

  $(x_1, x_2) = (3,2)$, $z=23$

Aalto University
School of Business

Computational complexity: LP vs. MILP

- Solving (M)ILP problems is computationally much more demanding then solving LP problems
  - It is possible that in the B&B algorithm the number of sub-problems doubles with each branching-step
  - Hence, running time of the algorithm can grow exponentially as a function of the number of integer valued decision variables

- Adding constraints to an LP problem usually makes solving it computationally more demanding

- Adding constraints to a MILP problem can make it easier to solve!
  - Think about the B&B example problem with the additional constraint $x_1 \leq 3$

BLP Example: Capital Budgeting

- Perry Construction is faced with the problem of determining which projects it should undertake over the next three years:

max $180x_A + \ldots + 80x_E$

s.t.
1. $30x_A + \ldots + 20x_E \le 70$
2. $40x_A + 8x_B \ldots + 40x_E \le 90$
3. $40x_A + 20x_C \ldots + 40x_E \le 100$
4. $x_A, \ldots, x_E \in \{0,1\}$

	Estimated	Capital Requirements		
Project	Present Value	Year1	Year2	Year3
A	180 000	30 000	40 000	40 000
B	20 000	12 000	8 000	0
C	72 000	30 000	20 000	20 000
D	25 000	15 000	10 000	24 000
E	80 000	20 000	40 000	40 000
Funds Available		70 000	90 000	100 000

Question

- Interpret the meaning of the decision variables and the constraints

BLP Example: Capital Budgeting Revisited

- Question: Help the mgmt to formulate the additional restrictions:
  1. At most three projects can be selected

     $$x_A + x_B + x_C + x_D + x_E \leq 3$$

  2. Projects A and D cannot be both selected
  3. If project C is selected, then project E must also be selected
  4. If project B or E is selected, then project A cannot be selected
  5. If project E is selected, then projects C and D must also be selected

BLP: Special 0-1 Constraints (Cont'd)

- To make sure that a logical constraint works check that:
  1. solutions that are not allowed by the problem description are infeasible
  2. solutions allowed by the problem description are feasible
- Example: 1st year capital requirement is reduced by $10,000 if at least 2 of projects C, D and E are selected (cf. a synergy)

  →New "dummy" variable x_s added to the model and 1. constraint modified:

$$30x_A + \ldots + 20x_E - 10x_S \leq 70$$

  - New constraints ensure that "project *S*" is selected if and only if at least 2 of projects C, D and E are selected:

$$x_c + x_D + x_E - 2x_s \geq 0$$
$$x_c + x_D + x_E - 2x_s \leq 1$$

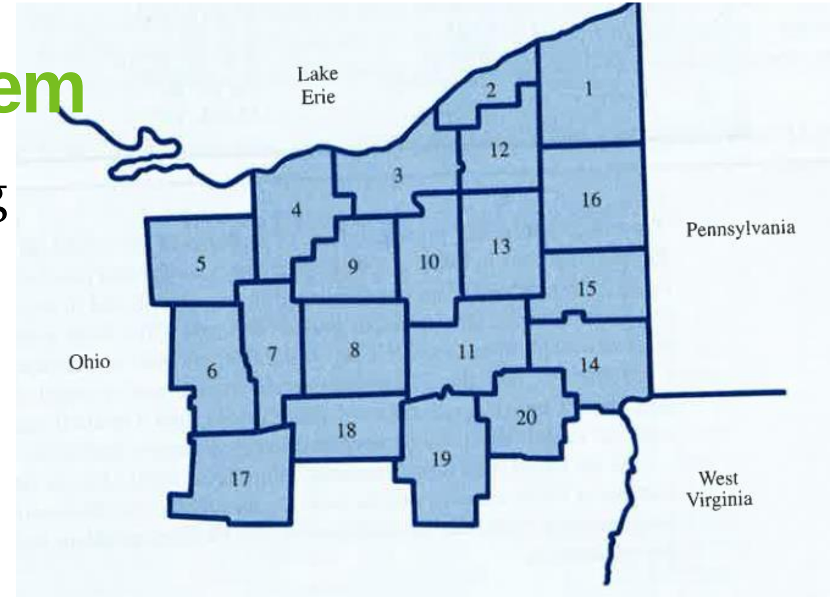
Allowed?	$x_C + x_D + x_E$	x_S	$x_C + x_D + x_E - 2x_s$	Feasible?
yes	0	0	0	yes
no	0	1	-2	no
yes	1	0	1	yes
no	1	1	-1	no
no	2	0	2	no
yes	2	1	0	yes
no	3	0	3	no
yes	3	1	1	yes

BLP example: Covering problem

- Ohio Trust Company (OTC) is expanding into 20 new counties
  - Ohio Banking law: “A branch bank can be established in a county only if an adjacent county has a Principle Place of Business (PPB)”
  - Establishing a PPB requires state’s approval so OTC seeks to establish as few as possible new PPBs

Question:

- Interpret the BLP problem
  - HINT: Decision variable $x_i = 1$ iff PPB is established in county i



$$\min x_1 + x_2 + x_3 + \cdots + x_{20}$$

(1) $x_1 + x_2 + x_{12} + x_{16} \geq 1$

(2) $x_1 + x_2 + x_3 + x_{12} \geq 1$

(3) $x_2 + x_3 + x_4 + x_9 + x_{10} + x_{12} + x_{13} \geq 1$

…

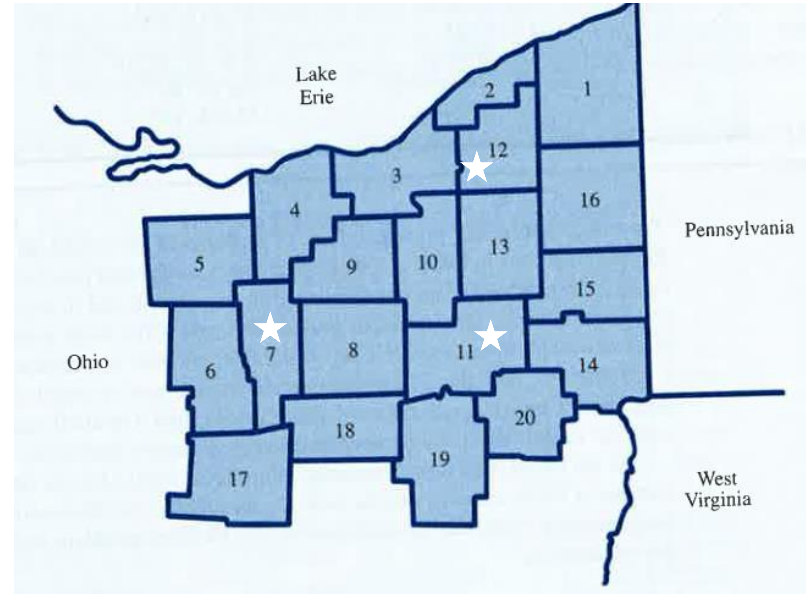
(20) $x_{11} + x_{14} + x_{19} + x_{20} \geq 1$

$$x_i \in \{0,1\}, i = 1, \dots, 20 \geq 1$$

BLP example: Covering problem (Cont'd)

- Optimal solution:

$$x_i = 1, \qquad i = 7, 11, 12$$
$$x_i = 0, \qquad i \neq 7, 11, 12$$



Question:

- Assume that the cost of establishing a PPB varies across counties
  - Denote cost in county i by c_i
- How would you modify the BLP model to minimize total costs?

$$\min c_1x_1 + c_2x_2 + c_3x_3 + \cdots + c_{20}x_{20}$$

(1) $x_1 + x_2 + x_{12} + x_{16} \geq 1$
(2) $x_1 + x_2 + x_3 + x_{12} \geq 1$
(3) $x_2 + x_3 + x_4 + x_9 + x_{10} + x_{12} + x_{13} \geq 1$
…
(20) $x_{11} + x_{14} + x_{19} + x_{20} \geq 1$

$x_i \in \{0,1\}, i = 1, \ldots, 20 \geq 1$

BLP Example: Marketing Plan

- The Supersuds Corporation is developing its next year's marketing plan for three new products.
  - Five TV spots purchased for commercials on national television networks.
  - Max 3 spots for each product
  - Each spot will feature a single product.
- How many spots should be allocated to each of the three products?

	Estimated Profits (Millions)		
Number of TV Spots	**Product 1**	**Product 2**	**Product 3**
0	$1	$2	$0
1	2	3	4
2	3	5	5
3	4	6	6

BLP Example: Marketing Plan (Cont'd)

Decision variables: x_{ij} are binary (i = 1, 2, 3; j = 1, 2, 3,4).

Maximize Profit = $1x_{11}+2x_{12}+3x_{13}+4x_{14}\ +2x_{21}+3x_{22}+5x_{23}+6x_{24}+0x_{31}+4x_{32}+5x_{33}+6x_{34}$

	Estimated Profits (Millions)		
Number of TV Spots	**Product 1**	**Product 2**	**Product 3**
0	$1	$2	$0
1	2	3	4
2	3	5	5
3	4	6	6

subject to:

Mutually Exclusive:

Product 1: $x_{11}+x_{12}+x_{13}+x_{14}= 1$

Product 2: $x_{21}+x_{22}+x_{23}+x_{24}= 1$

Product 3: $x_{31}+x_{32}+x_{33}+x_{34}= 1$

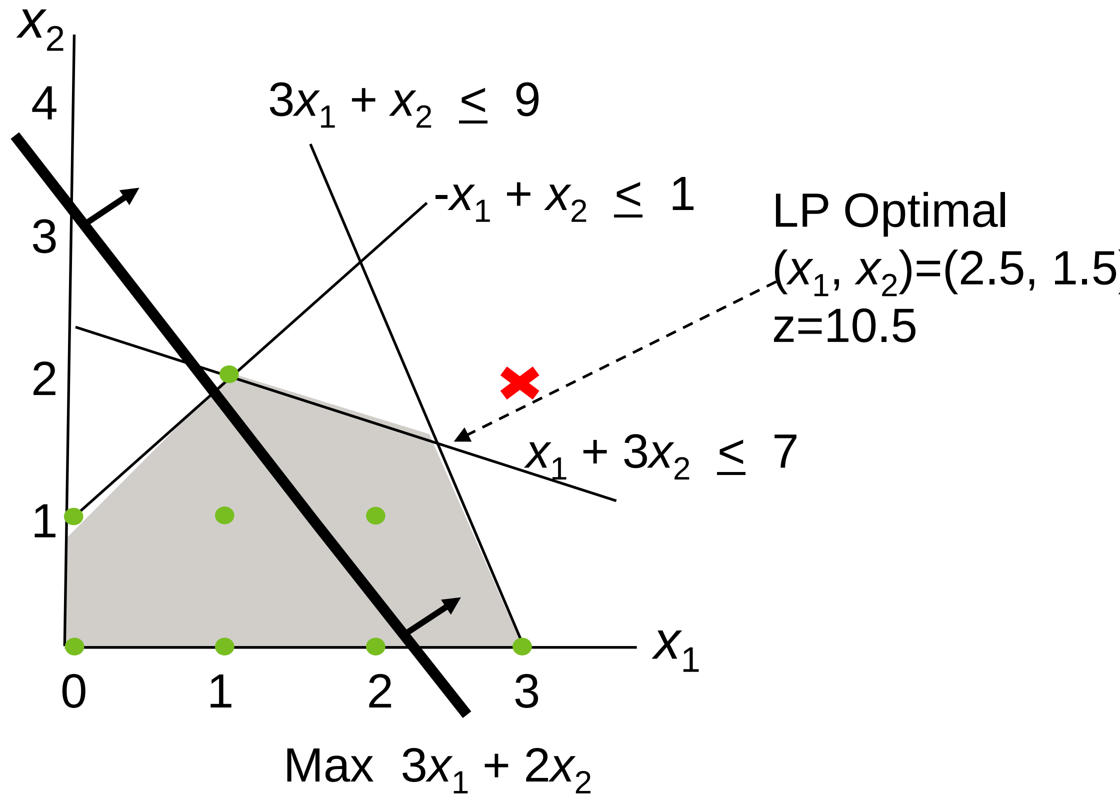
Total available spots: $1x_{12}+2x_{13}+3x_{14}\ +1x_{22}+2x_{23}+3x_{24}+1x_{32}+2x_{33}+3x_{34}\leq 5$

Other Examples of BLP Applications

- Investment Analysis
  - Should we make a certain fixed investment?
- Site Selection
  - Should a certain site be selected for the location of a new facility?
- Designing a Production and Distribution Network
  - Should a certain plant (distribution center) remain open?
  - Should a certain site be selected for a new plant (or distribution center)?
  - Should a distribution center remain open?
  - Should a certain distribution center be assigned to serve a certain market area?
- Scheduling Interrelated Activities
  - Should a certain activity begin in a certain time period?
- Airline Applications:
  - Should a certain type of airplane be assigned to a certain flight leg?
  - Should a certain sequence of flight legs be assigned to a crew?

Cautionary note about solving (M)ILP problems by "rounding"

- Trying to solve the problem by first solving the LP relaxation and then rounding-up gives an infeasible solution:
  - RoundUP(2.5, 1.5)=(3,2)✖

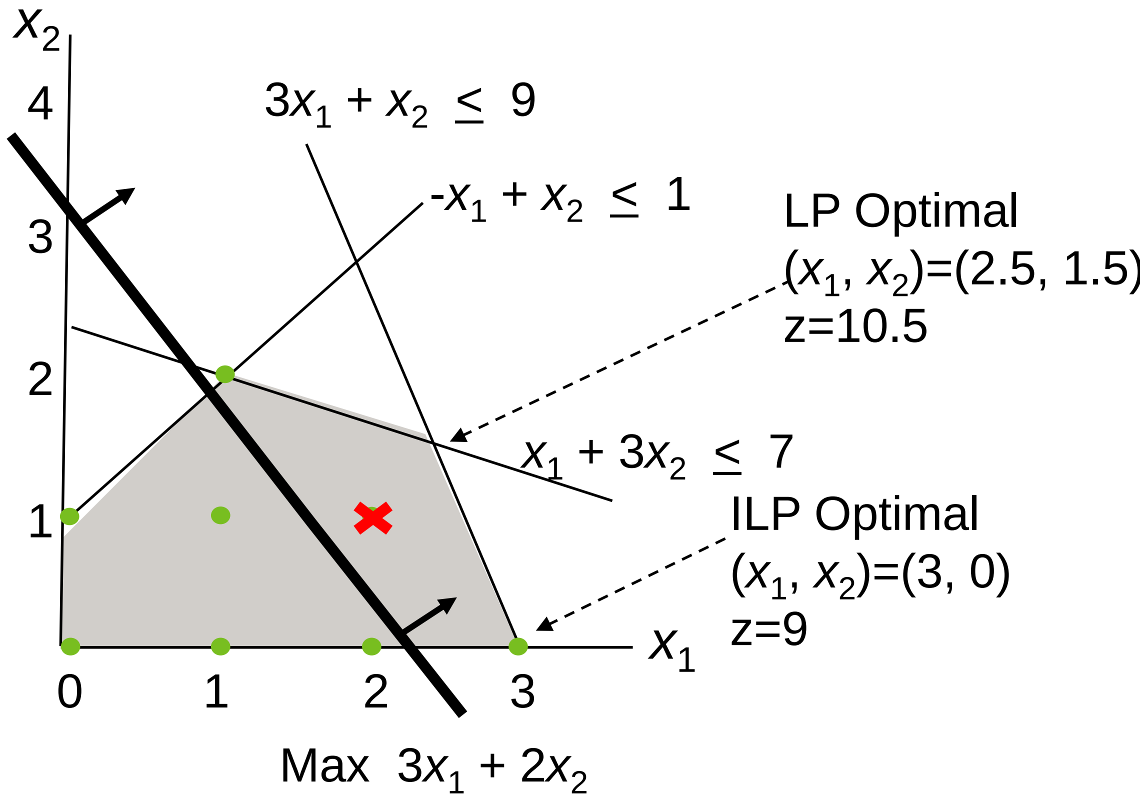


Cautionary note about solving (M)ILP problems by "rounding"

- Trying to solve the problem by first solving the LP relaxation and then rounding-down gives an sub-optimal solution:

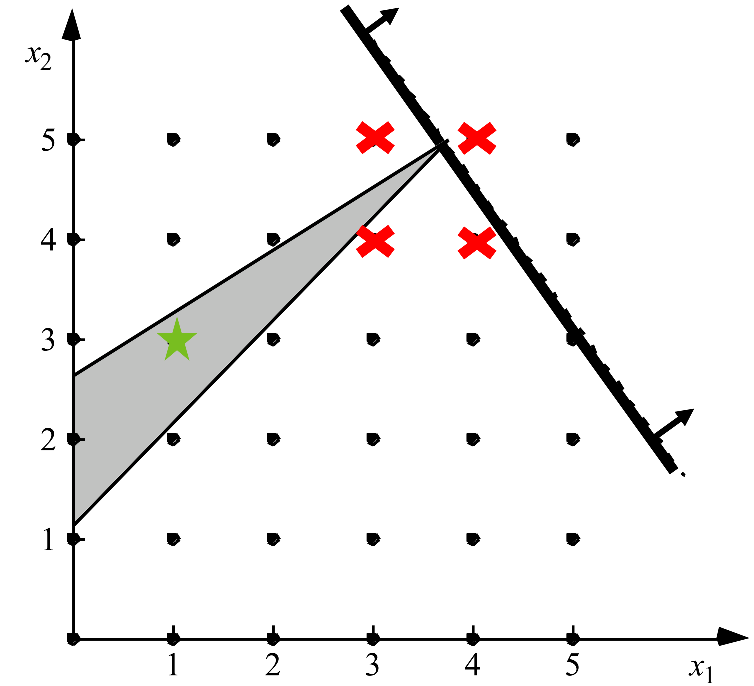
RoundDW(2.5, 1.5)=(2,1) ✖

z'=3(2) + 2(1)=8 < 9



Cautionary note about solving (M)ILP problems by "rounding"

- Rounded solution may not be feasible.
- Rounded solution may not be close to optimal.
- There can be *many* rounded solutions.
  - Example: Consider a problem with 30 variables that have non-integer values in the LP-solution. How many possible rounded solutions are there?



✖ Solutions obtained by rounding optimal solutions of the LP relaxation (are infeasible)

★ ILP optimal solution

Cautionary note about solving (M)ILP problems by "rounding"

When are "non-integer" solutions okay?

- Solution is naturally divisible (e.g., $, pounds, hours)
- Solution represents a rate (e.g., units per week)

When is rounding okay?

- When numbers are large
  - e.g., rounding 114.286 to 114 is *probably* okay.

When is rounding not okay?

- When numbers are small
  - e.g., rounding 2.6 to 2 or 3 may be a problem.
- Binary variables
  - yes-or-no decisions

Cautionary note about sensitivity analysis in (M)ILP problems

- A B&B algorithm usually does not provide information on the solution sensitivity (cf. Simplex algorithm)
  →No "Sensitivity report" for (M)ILP problems
- Yet, analyzing sensitivity of (M)ILP problems is important
  - Maybe more important than for LP problems
- Sensitivity analysis requires re-optimizing the problem

$$\max z = 40x_1 + 60x_2 + 70x_3 + 160x_4$$
$$16x_1 + 35x_2 + 45x_3 + 85x_4 \leq \mathbf{100}$$
$$x_1, x_2, x_3, x_4 \in \{0,1\}$$

→Optimum $x = (1,1,1,0), z = 170$

$$\max z = 40x_1 + 60x_2 + 70x_3 + 160x_4$$
$$16x_1 + 35x_2 + 45x_3 + 85x_4 \leq \mathbf{101}$$
$$x_1, x_2, x_3, x_4 \in \{0,1\}$$

→Optimum $x = (1,0,0,1), z = 200$

MILP Example: Fixed-Charge Problem

- A product can be assembled on any of the five assembly lines.
  - For each line, the table below gives the cost of assembling a product, the assembly time required per product, the start-up cost, and the maximum number of hours the line can be operated during the next month.
- At least 350 units of product must be assembled next month.

Line	Start-up Cost	Prod. Cost/unit	Prod.Time (hrs./unit)	Maximum Prod.hrs.
A	\$ 6,000	\$ 80	5	510
B	\$10,000	\$ 60	6	480
C	\$ 2,000	\$110	10	600
D	\$ 7,500	\$ 75	4	440
E	\$15,000	\$ 40	3	480

MILP Example: Fixed-Charge Problem (cont'd)

$$\min \quad 80x_A + \ldots + 40x_E + 6000y_A + \ldots + 15000y_E$$

s.t.

1. $x_A + x_B + \ldots + x_E \geq 350$
2. $5x_A \leq 510\, y_A$
3. $6x_B \leq 480\, y_B$
4. $10x_C \leq 600\, y_C$
5. $4x_D \leq 440\, y_D$
6. $3x_E \leq 480\, y_E$

$y_A, \ldots, y_E \in \{0,1\}$

$x_A, \ldots, x_E \geq 0$

Line	Start-up Cost	Prod. Cost/unit	Prod.Time (hrs./unit)	Maximum Prod.hrs.
A	\$ 6,000	\$ 80	5	510
B	\$10,000	\$ 60	6	480
C	\$ 2,000	\$110	10	600
D	\$ 7,500	\$ 75	4	440
E	\$15,000	\$ 40	3	480

Question

- Interpret the decision variables, the objective function and the constraints

MILP Example: Fixed-Charge Problem (cont'd)

	A	B	C	D	E	F	G	H	I	J
1										
2				Production Lines				MINIMIZE		
3		A	B	C	D	E		TOTAL		
4	Start-up cost	6000	10000	2000	7500	15000		COST		
5	Cost/unit	80	60	110	75	40		49150		
6										
7	Start? (y)	1	0	0	1	1		Units produced total		minimum
8	Units produced (x)	70	1.42E-14	0	110	170		350	>=	350
9										
10	Prod time in hours / unit	5	6	10	4	3				
11	Hours total	350	8.53E-14	0	440	510				
12		<=	<=	<=	<=	<=				
13	Hours available	510	0	0	440	510				
14	Hours max	510	480	600	440	510				
15										

Solver Parameters

Set Objective: $H$5

To: ( ) Max (•) Min ( ) Value Of: 0

By Changing Variable Cells:
$B$7:$F$8

Subject to the Constraints:
$B$11:$F$11 <= $B$13:$F$13
$B$7:$F$7 = binary
$H$8 >= $J$8

[x] Make Unconstrained Variables Non-Negative

Select a Solving Method: Simplex LP

MILP Example: Fixed-Charge Problem (cont'd)

Question: Help the management to formulate the additional restrictions:

1. If line E is operated, then line B must also be operated

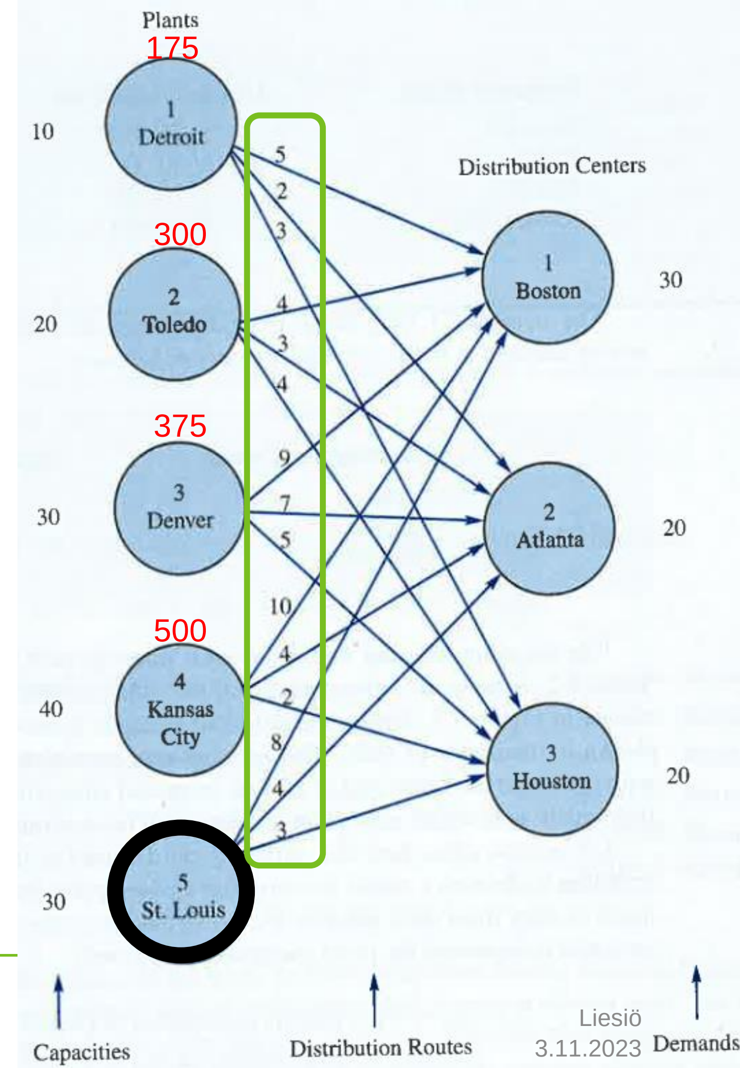
2. If line A is operated, then lines D and E may not be operated

3. If line B is operated, then at least 50 units must be produced on that line

4. If line C is operated, then no more than 150 units may be produced on lines B and D combined

MILP Example: Distribution System Design

- Company operates a plant in St. Louis
  - Annual capacity of 30,000 units
- Products shipped to 3 distribution centers in Boston, Atlanta and Houston
  - Different demands
- Possible locations for new plants: Detroit, Toledo, Denver, Kansas City
  - Differ in terms of annual fixed operating costs and cost of shipment to distribution centers
- In which cities should new plants be built?
  - Satisfy demands with minimal cost

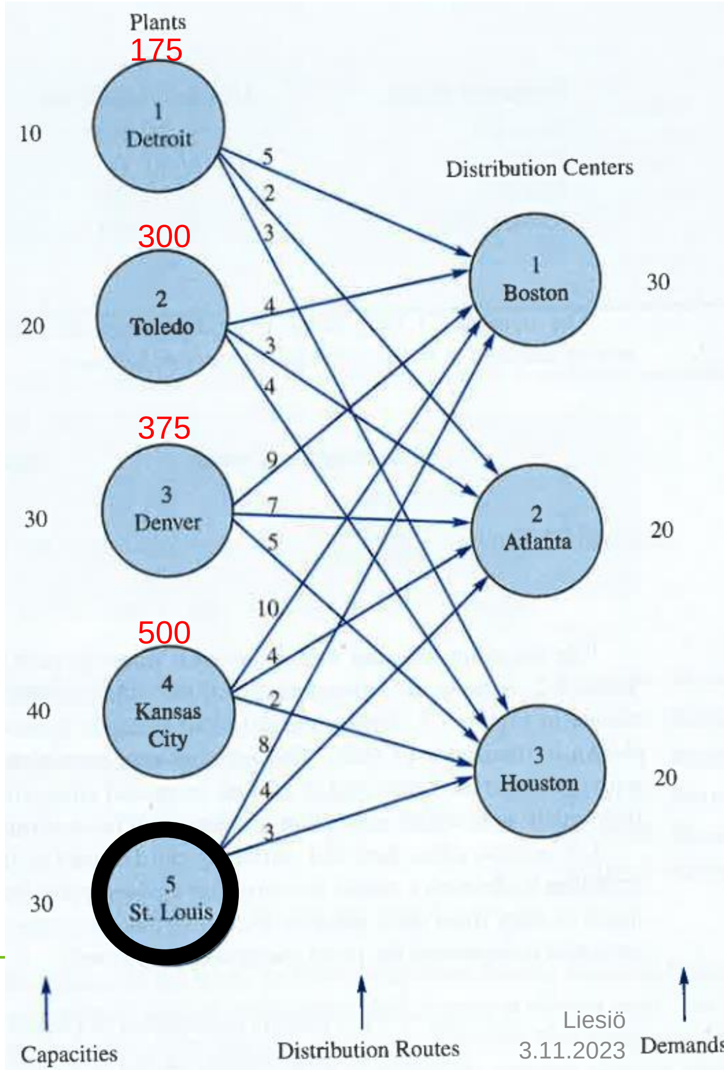


MILP Example: Distribution System Design (Cont'd)

$$\min 5x_{11} + 2x_{12} + 3x_{13} + \cdots + 4x_{52} + 3x_{53}$$
$$+175y_1 + 300y_2 + 375y_3 + 500y_4$$
$$x_{11} + x_{12} + x_{13} \le 10y_1$$
$$x_{21} + x_{22} + x_{23} \le 20y_2$$
$$x_{31} + x_{32} + x_{33} \le 30y_3$$
$$x_{41} + x_{42} + x_{43} \le 40y_4$$
$$x_{51} + x_{52} + x_{53} \le 30$$
$$x_{11} + x_{21} + x_{31} + x_{41} + x_{51} = 30$$
$$x_{12} + x_{22} + x_{32} + x_{42} + x_{52} = 20$$
$$x_{13} + x_{23} + x_{33} + x_{43} + x_{53} = 20$$
$$x_{ij} \ge 0, y_k \in \{0,1\},$$

Question:

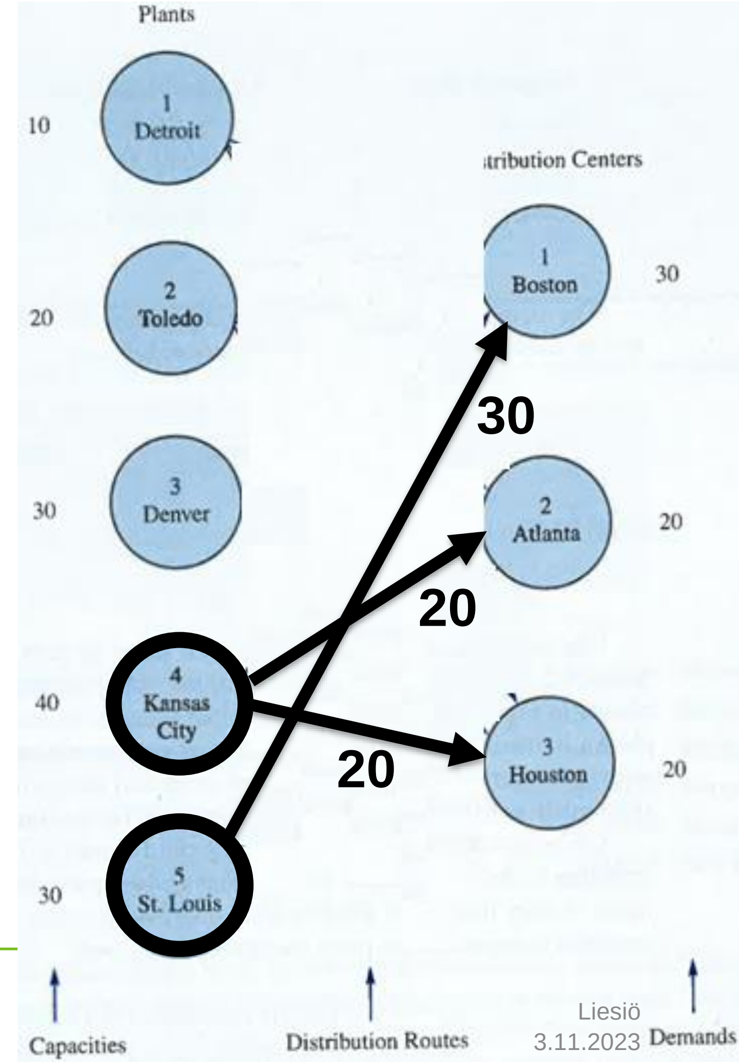
- Interpret the model, i.e., decision variables, objective function, constraints



MILP Example: Distribution System Design (Cont'd)

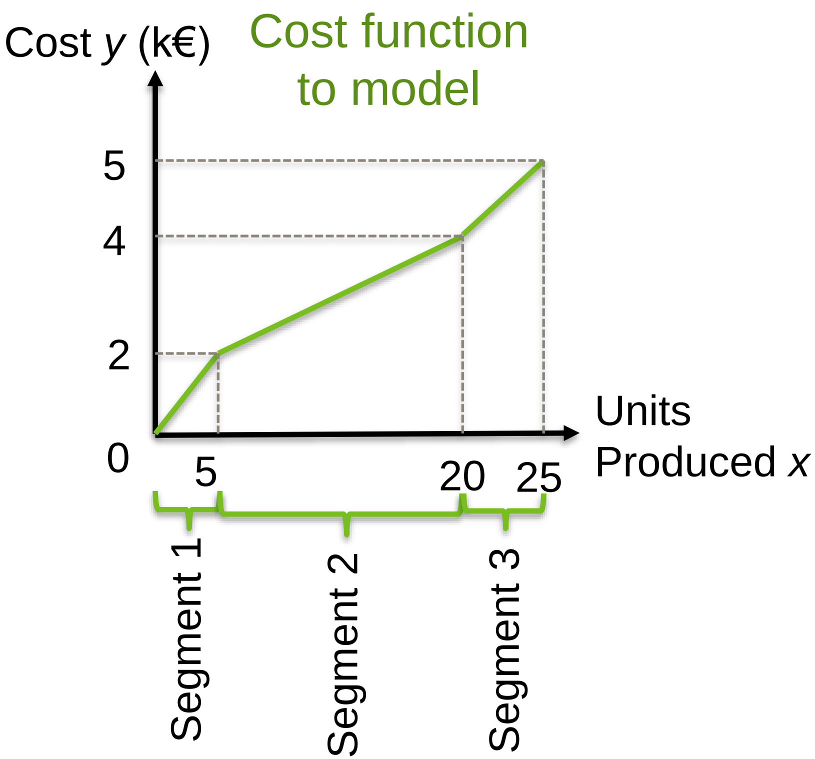
$$\min 5x_{11} + 2x_{12} + 3x_{13} + \cdots + 4x_{52} + 3x_{53}$$
$$+175y_1 + 300y_2 + 375y_3 + 500y_4$$
$$x_{11} + x_{12} + x_{13} \leq 10y_1$$
$$x_{21} + x_{22} + x_{23} \leq 20y_2$$
$$x_{31} + x_{32} + x_{33} \leq 30y_3$$
$$x_{41} + x_{42} + x_{43} \leq 40y_4$$
$$x_{51} + x_{52} + x_{53} \leq 30$$
$$x_{11} + x_{21} + x_{31} + x_{41} + x_{51} = 30$$
$$x_{12} + x_{22} + x_{32} + x_{42} + x_{52} = 20$$
$$x_{13} + x_{23} + x_{33} + x_{43} + x_{53} = 20$$
$$x_{ij} \geq 0, y_k \in \{0,1\},$$

- Optimal solution (non-zero dec. var.):
  $$x_{42} = 20, x_{43} = 20, x_{51} = 30, y_4 = 1$$



Modelling non-constant marginal costs with MILP

- MILP can capture arbitrary piecewise linear functions in the objective function or in the constraints



MILP formulation

- Decision variables:
  - y: cost
  - x: number of units produced
  - $z_1, z_2, z_3 \in \{0,1\}$: x located on segment with z_i=1
  - $c_1, c_2, c_3, c_4 \in [0,1]$: 'Weights for segment borders'
- Constraints:

$$0c_1 + 5c_2 + 20c_3 + 25c_4 = x$$
$$c_1 + c_2 + c_3 + c_4 = 1$$
$$z_1 + z_2 + z_3 = 1$$
$$c_1 \leq z_1$$
$$c_2 \leq z_1 + z_2$$
$$c_3 \leq z_2 + z_3$$
$$c_4 \leq z_3$$
$$y = 0c_1 + 2c_2 + 4c_3 + 5c_4$$

Modelling variable marginal costs with MILP (cont'd)

- Consider x=10 units are produced
- Then MILP has to find $c_i \in [0,1]$ such that

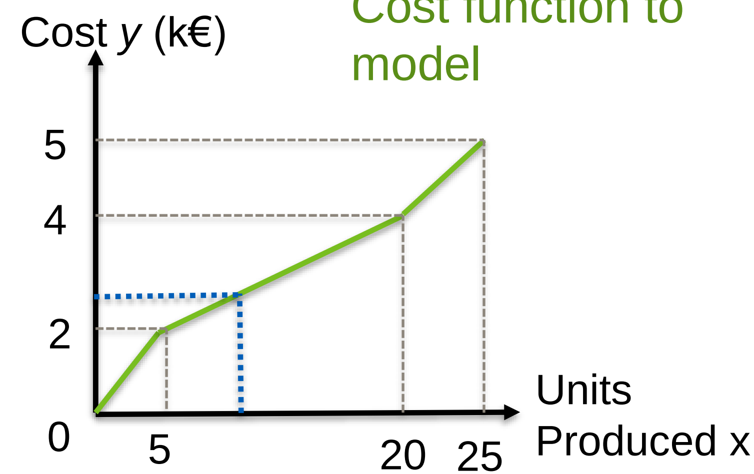
$$0c_1 + 5c_2 + 20c_3 + 25c_4 = 10$$

- However, only two consecutive c_i:s can have a non-zero value due to z_i:s:
  - z_1=1 and $0c_1 + 5c_2 = 10$ ➔ infeasible
  - z_2=1 and $5c_2 + 20c_3 = 10$ ➔ c_2=2/3, c_3=1/3
  - z_3=1 and $20c_3 + 25c_4 = 10$ ➔ infeasible
- Hence, the cost is equal to

$$y = 0(0) + 2\left(\frac{2}{3}\right) + 4\left(\frac{1}{3}\right) + 5(0) = \frac{8}{3} = 2\,\frac{2}{3},$$

which is inline with the original cost function!

Cost function to model



$$0c_1 + 5c_2 + 20c_3 + 25c_4 = x$$
$$c_1 + c_2 + c_3 + c_4 = 1$$
$$z_1 + z_2 + z_3 = 1$$
$$c_1 \le z_1$$
$$c_2 \le z_1 + z_2$$
$$c_3 \le z_2 + z_3$$
$$c_4 \le z_3$$
$$y = 0c_1 + 2c_2 + 4c_3 + 5c_4$$

Integer Linear Programming - Summary

- Different types: MILP, pure ILP, BLP,....

- Compared to LP models, MILP models can capture
  - Yes/no decisions
  - Logical dependencies among decisions
  - Fixed-charges (e.g. start-up costs)
  - Piecewise linear functions (e.g. non-constant marginal production costs)

- Solving of ILPs is fundamentally different from solving LPs
  - Solution time may increase exponentially as a function of the problem size