Aalto University
School of Business

Linear programming (LP) – basic concepts

- *LP: A definition*
- *Graphical solution procedure for LP*
- *Binding and redundant constraints*
- *Special cases: Infeasible and unbounded problems; alternate optima*
- *Computer solution of LPs*
- *Application Examples*

Mathematical programming (optimization) problem

- A math. programming/optimization problem is defined by
  1. decision variables ($x = (x_1, \dots, x_m)$)
  2. objective function (min $f(x)$ or max $f(x)$)
  3. constraints (equalities $h(x) = c$, or inequalities $g(x) \leq c$)

- A solution = some values for the decision variables
  - A <u>feasible solution</u> satisfies all the problem's constraints
    - The set of all feasible solution is called the <u>feasible region</u>
  - An <u>optimal solution</u> is a feasible solution that results in the best possible value for the objective function
    - The best value =
      - The lowest in minimization problems
      - The highest in maximization problems

Linear Programming (LP) Problem

- If both the objective function and the constraints are linear, the problem is referred to as a <u>linear programming (LP) problem</u>

- <u>Linear functions</u> are functions in which each (decision) variable appears in a separate term raised to the first power and is multiplied by a constant (which could be zero).

- <u>Linear constraints</u> are linear functions that are restricted to be "*less than or equal to*“ (≤), "*equal to*“ (=), or "*greater than or equal to*“ (≥) a constant

- Question: Why do you think strict inequalities (>, <) are not that relevant for LP, or optimization in general? Consider e.g., the problem

$$\max 2x$$
$$x < 1$$

Thus, on this course we only have ≤, ≥, and = constraints

Example: Iron Works, Inc.

- Iron Works, Inc. manufactures two grades of steel and receives *19* kilotons of iron ore per day
  - It takes *2* kt of ore to make one kt of grade 1 steel
  - It takes *3* kt of ore to make one kt of grade 2 steel
  - Facilities allow to produce at most *8* kt of steel daily
  - At most 6 kt of grade 1 steel can be produced daily due to labor restrictions
  - Revenue from one kt of steel is 50000 and 70000 euros for grades 1 and 2, respectively.
- Question:
  - Provide an interpretation of the decision variables (x_1, x_2), constraints and objective function.

LP Formulation

$$\text{Max} \quad 5x_1 + 7x_2$$

Obj. func.

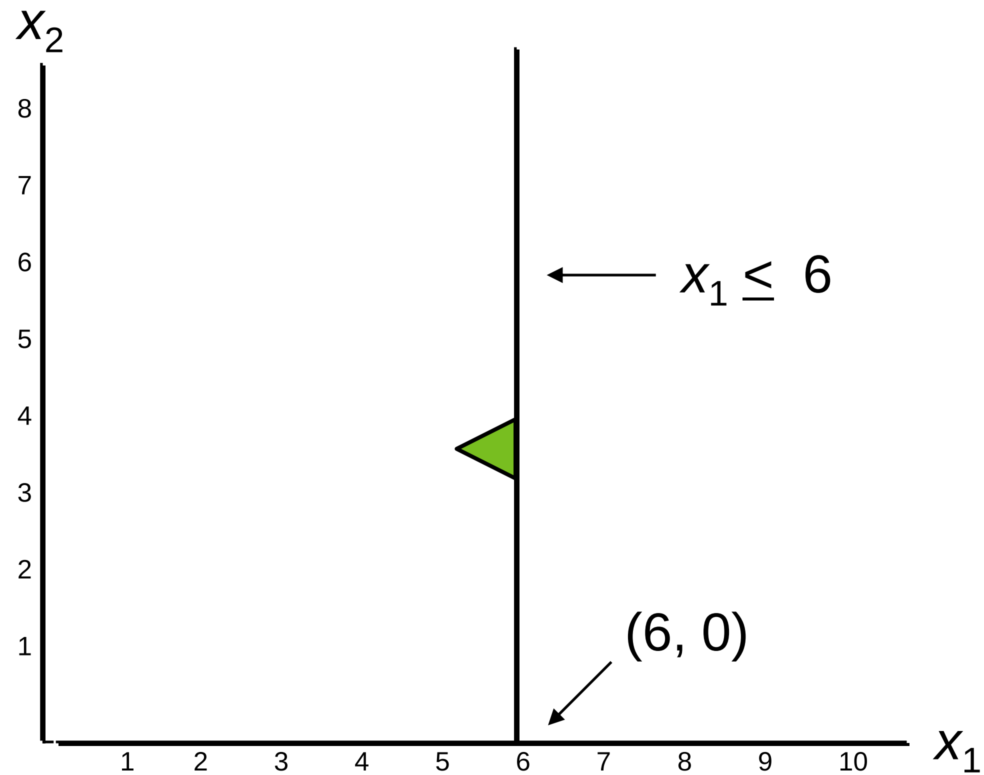
$$\begin{aligned} x_1 &\le 6 \\ 2x_1 + 3x_2 &\le 19 \\ x_1 + x_2 &\le 8 \\ x_1, x_2 &\ge 0 \end{aligned}$$

constraints

Example 1: Graphical Solution

At most 6 tons of grade 1 steel can be produced

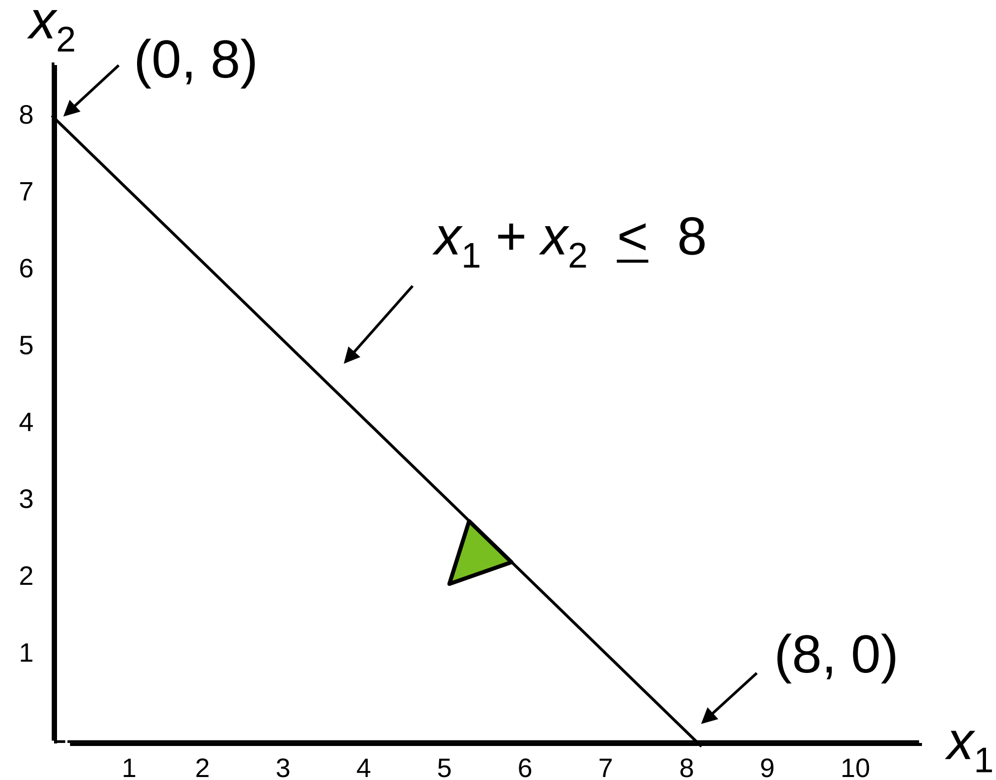
- x_1 = tons of grade 1 steel produced
- x_2 = tons of grade 2 steel produced



Example: Graphical Solution

At most 8 tons of steel can be produced

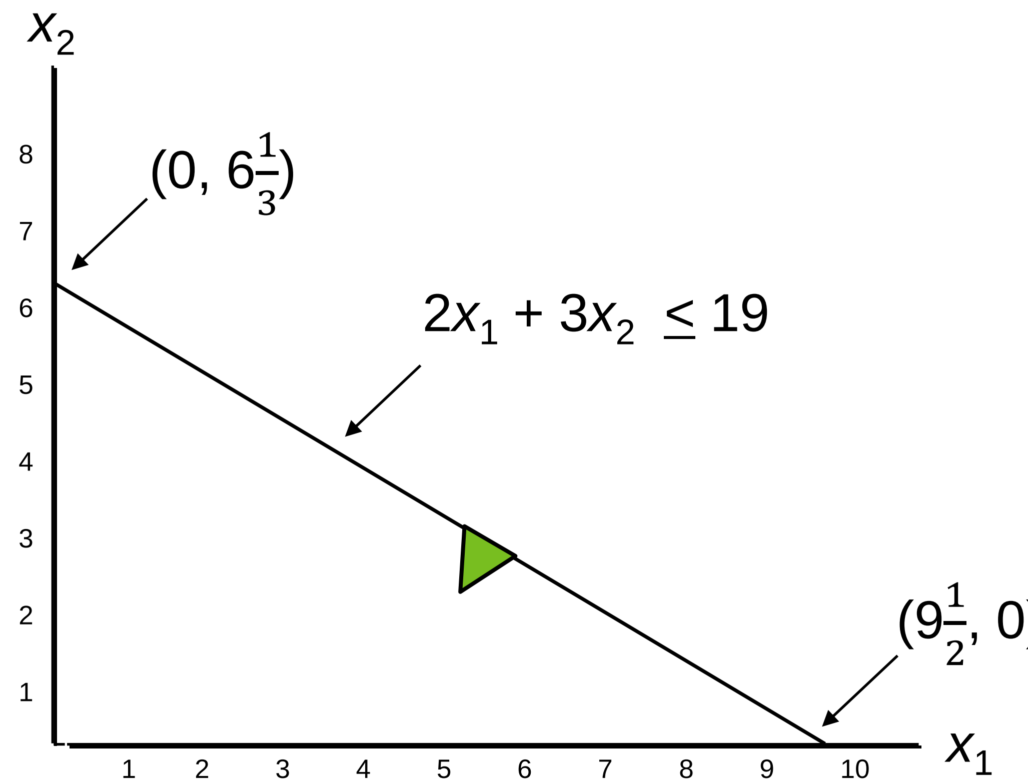
- x_1 = tons of grade 1 steel produced
- x_2 = tons of grade 2 steel produced



Example: Graphical Solution

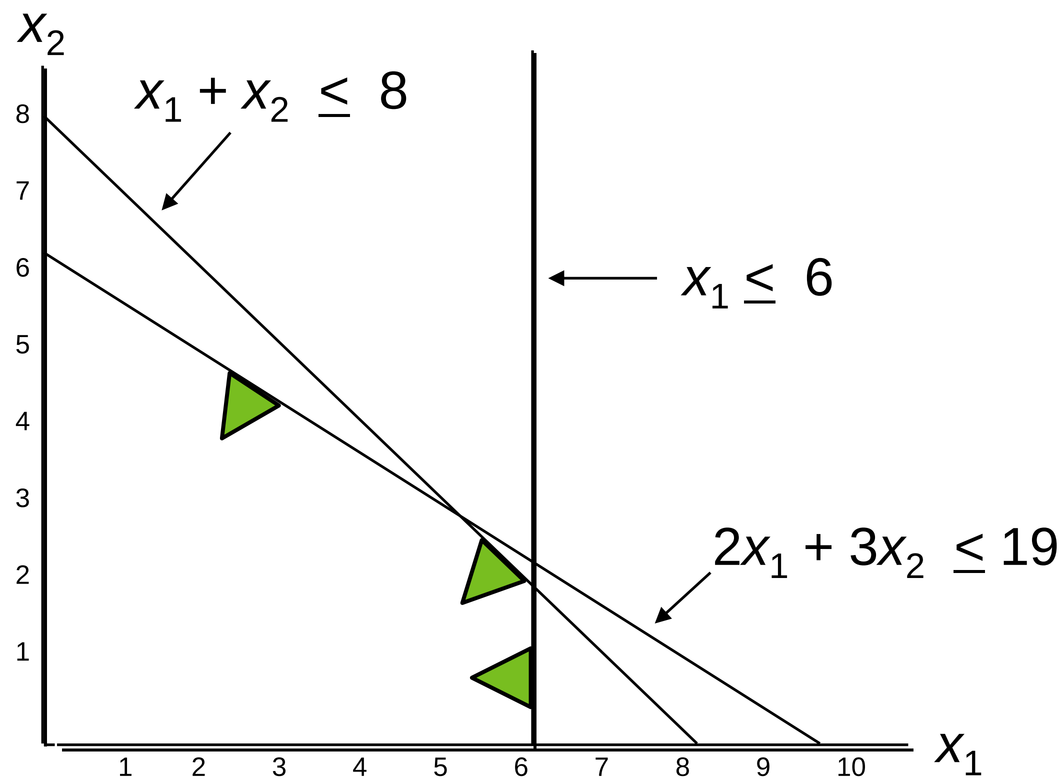
19 tons of iron ore available

- x_1 = tons of grade 1 steel produced
- x_2 = tons of grade 2 steel produced



Example: Graphical Solution

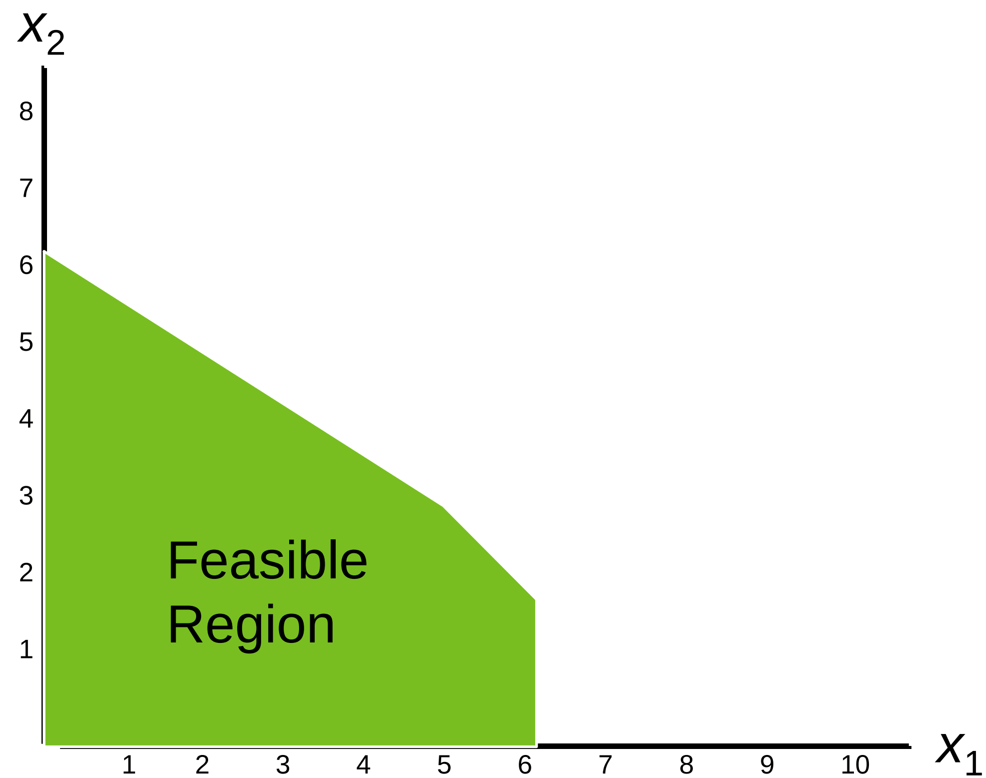
Combined-Constraint Graph



Example: Graphical Solution

feasible region

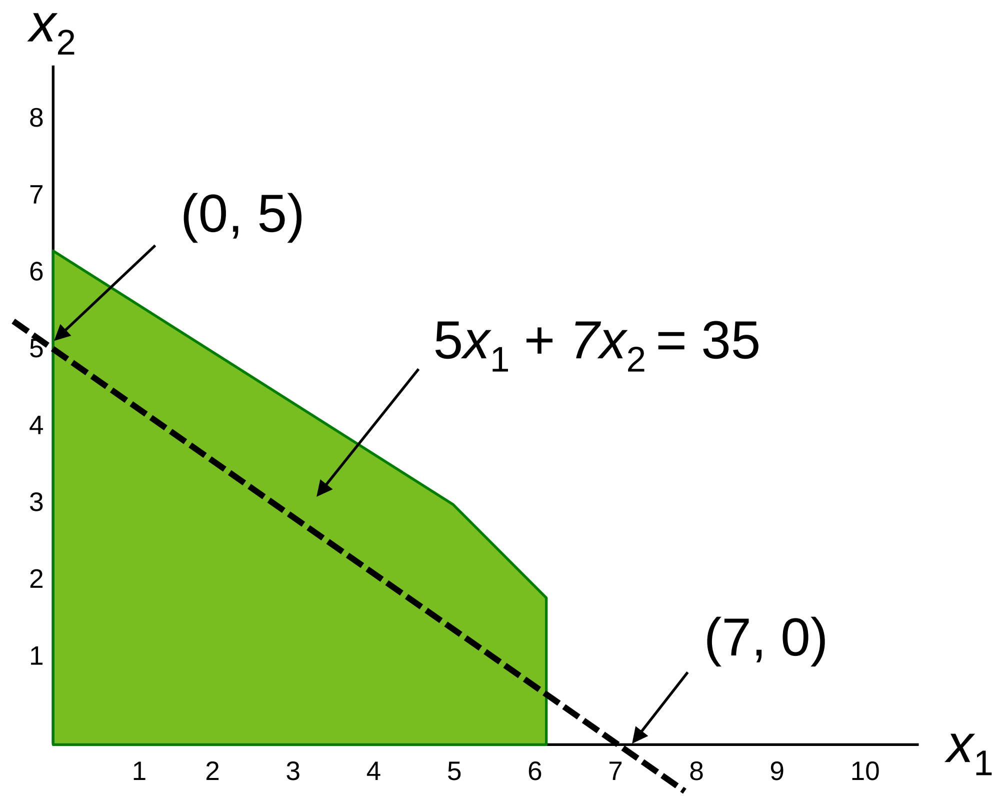
(=the set of feasible solutions)



Example: Graphical Solution

Objective Function Line

- Cf. revenue
  - Grade 1: 50k euros
  - Grade 2: 70k euros



Example: Graphical Solution

<u>Optimal Solution:</u>

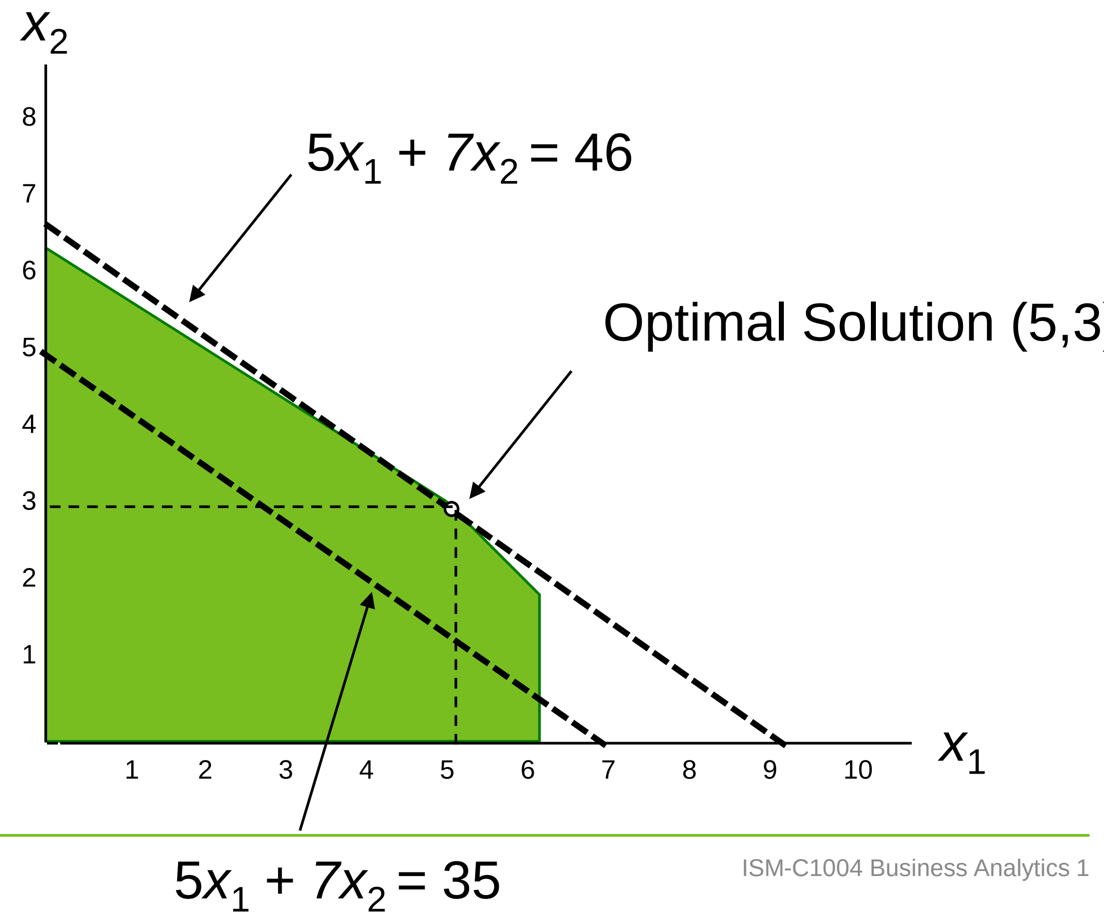
- x_1 =5 tons of grade 1 steel produced
- x_2 = 3 tons of grade 2 steel produced

<u>Optimal objective function value:</u>

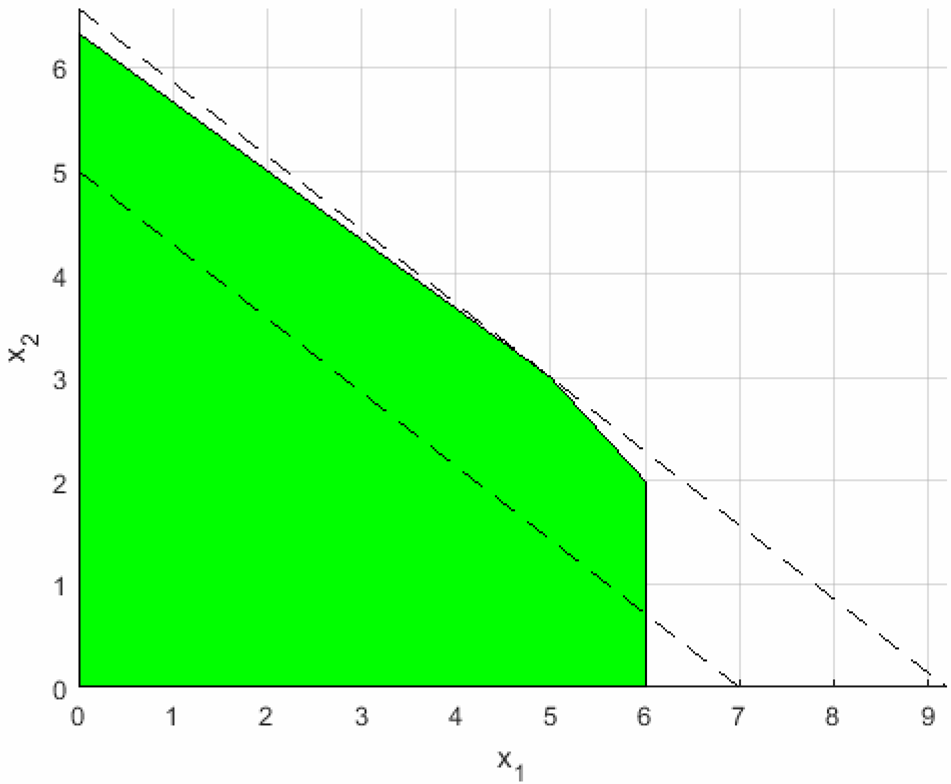
- 460000 euros

<u>Binding constraints</u> at optimum:

- $2x_1 + 3x_2 \leq 19$
- $x_1 + x_2 \leq 8$



Example: Graphical Solution



LP Formulation

$$\text{Max} \quad 5x_1 + 7x_2$$

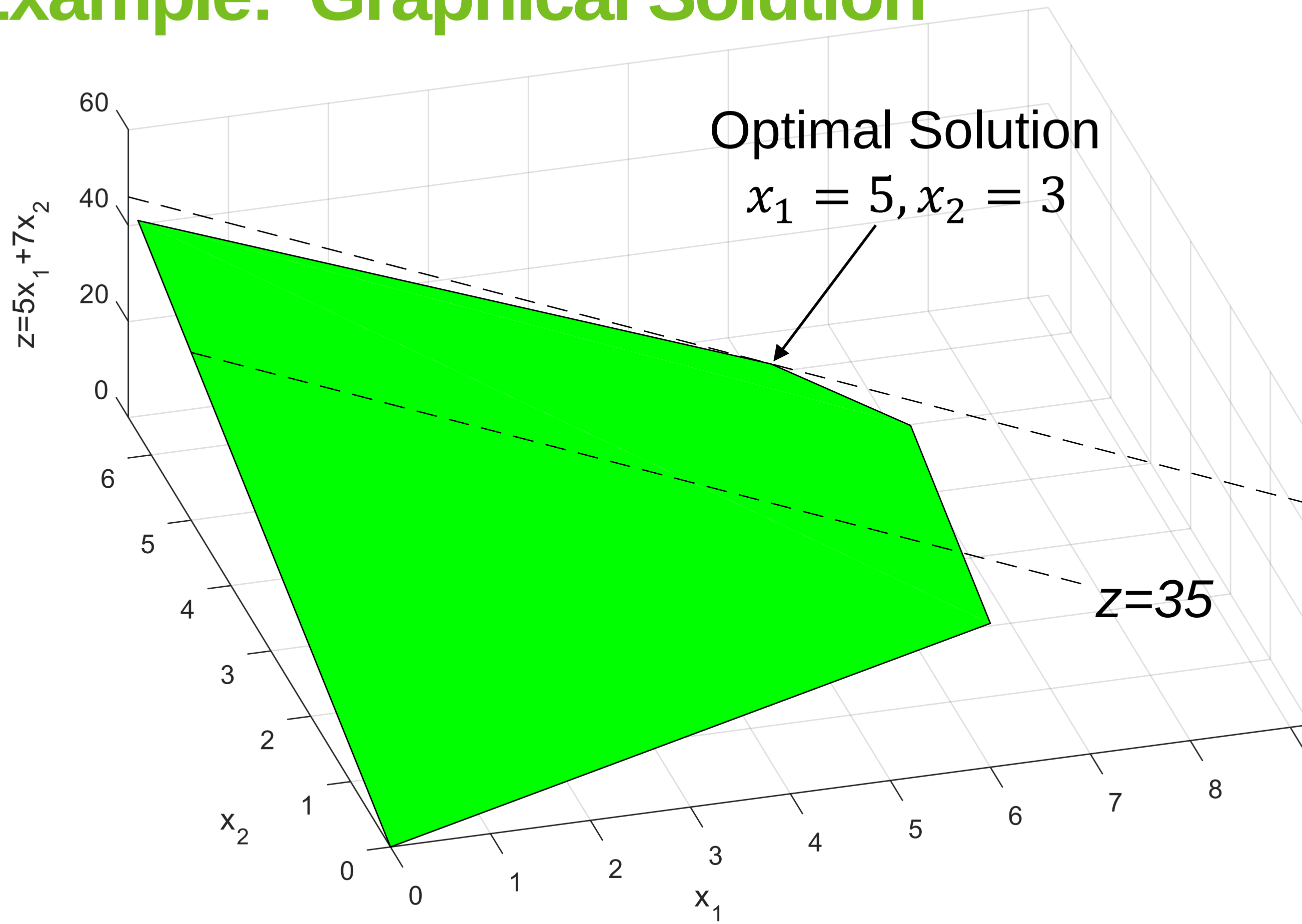
$$x_1 \leq 6$$

$$2x_1 + 3x_2 \leq 19$$

$$x_1 + x_2 \leq 8$$

$$x_1, x_2 \geq 0$$

Example: Graphical Solution



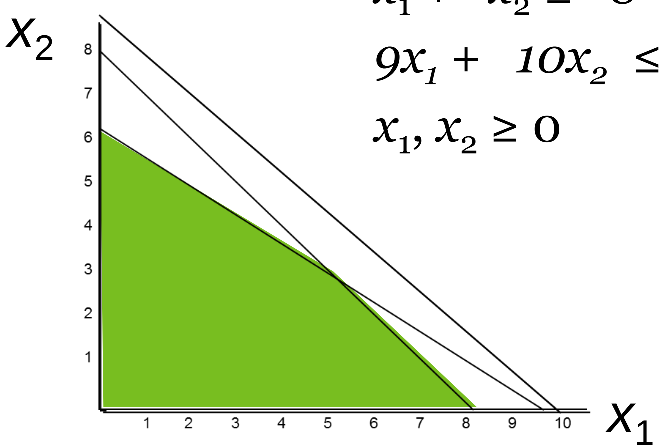
LP Formulation

$$\begin{aligned} \text{Max} \quad & 5x_1 + 7x_2 \\ x_1 \quad & \leq \ 6 \\ 2x_1 + 3x_2 & \leq \ 19 \\ x_1 + \ x_2 & \leq \ 8 \\ x_1, x_2 & \geq 0 \end{aligned}$$

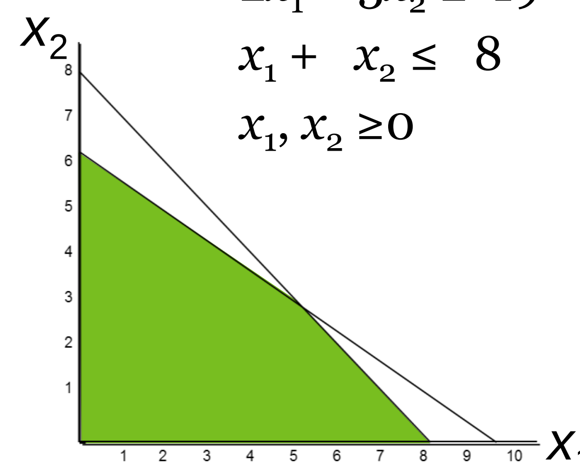
Redundant constraints

- A constraint is **redundant** if removing it does not change the feasible region
- Example:

$$2x_1 + 3x_2 \leq 19$$
$$x_1 + x_2 \leq 8$$
$$9x_1 + 10x_2 \leq 90$$
$$x_1, x_2 \geq 0$$



$$2x_1 + 3x_2 \leq 19$$
$$x_1 + x_2 \leq 8$$
$$x_1, x_2 \geq 0$$



→ Constraint $9x_1 + 10x_2 < 90$ is redundant!

Example: A Minimization Problem

LP Formulation

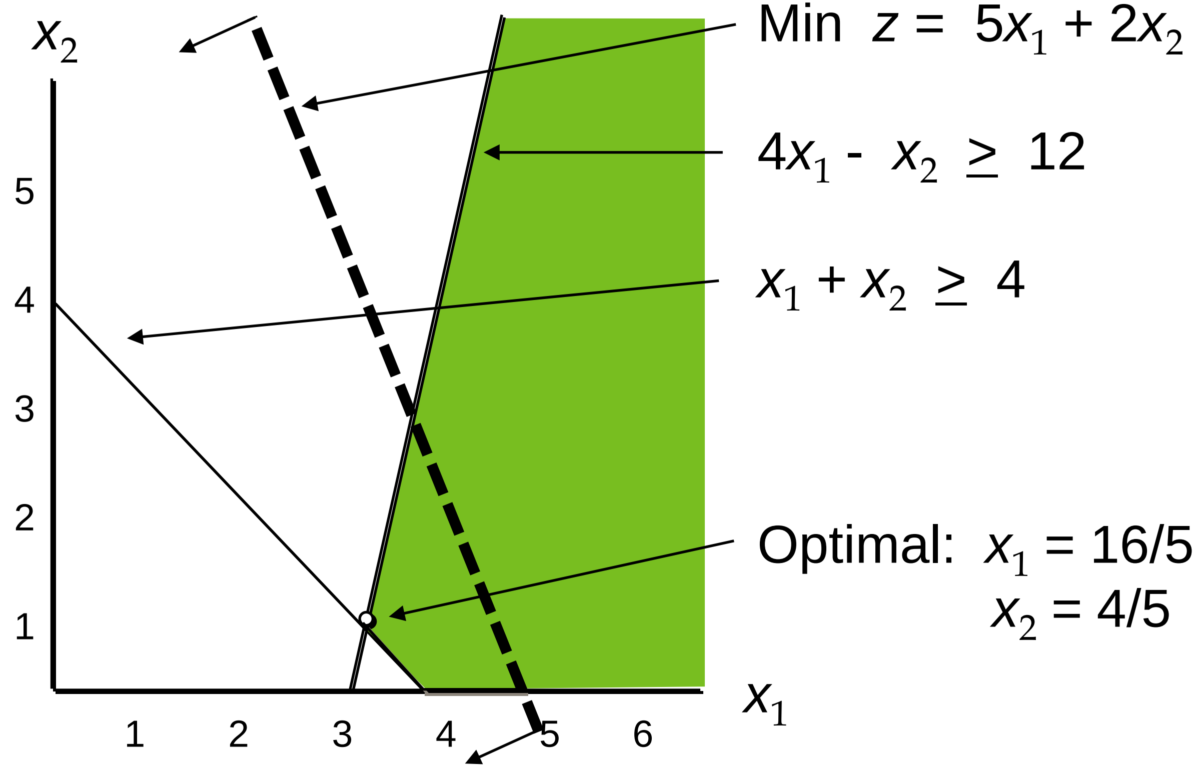
Min $z = 5x_1 + 2x_2$

$4x_1 - x_2 \geq 12$ (1)

$x_1 + x_2 \geq 4$ (2)

$x_1 \geq 0$ (3)

$x_2 \geq 0$ (4)



Question: Which constraints are binding?

Summary of the Graphical Solution Procedure

- Prepare a graph of the feasible solutions for each of the constraints
- Determine the feasible region
  - The set of those solutions that satisfy all the constraints
- Draw an objective function line
  - Move parallel objective function lines toward improved objective function values without entirely leaving the feasible region
  - Any feasible solution on the objective function line with the largest (smallest) value is an optimal solution

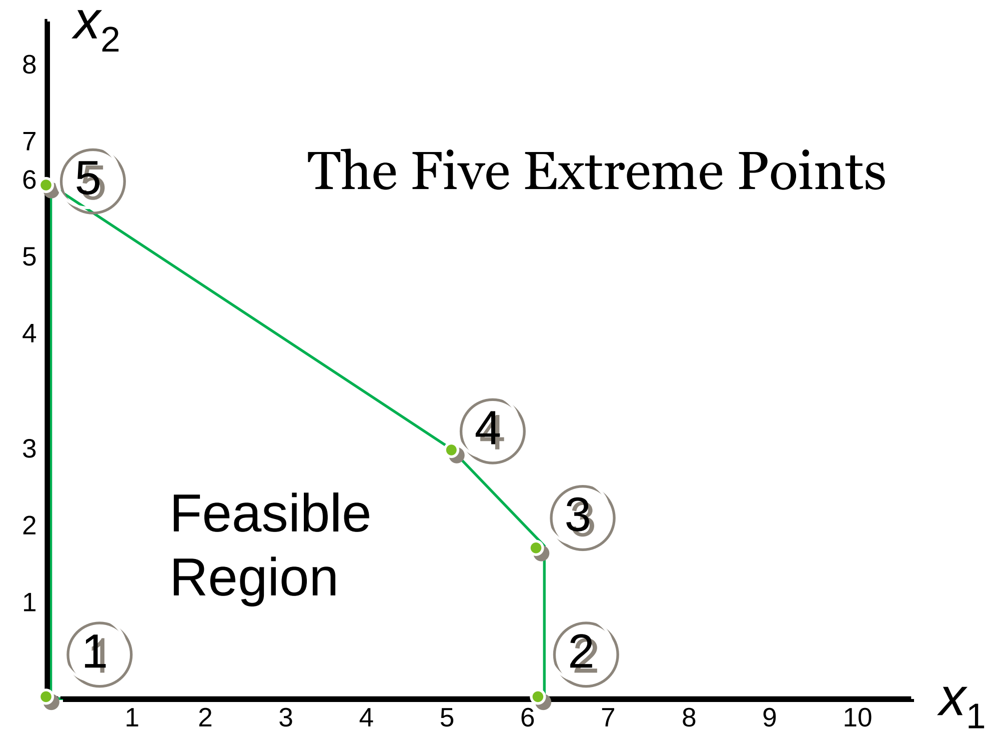
- Another example on the graphical solution procedure can be found at the end of this slide set

Extreme Points and the Optimal Solution

- The corners of the feasible region are referred to as the extreme points.
- At least one of the extreme points is an optimal solution*

→An alternative (graphical) solution method:

- Compute the objective function value in each extreme point
- The extreme point with the highest objective function value is an optimal solution



*) as long as the problem has an optimal solution (cf. unbounded and infeasible problems introduced soon)

LP special cases

- Each LP problem falls into one of the three categories:

1. The problem has one or more optimal solutions
   - Several alternative optimal solutions exists if all points of a line segment between two extreme points yield the optimal objective function value

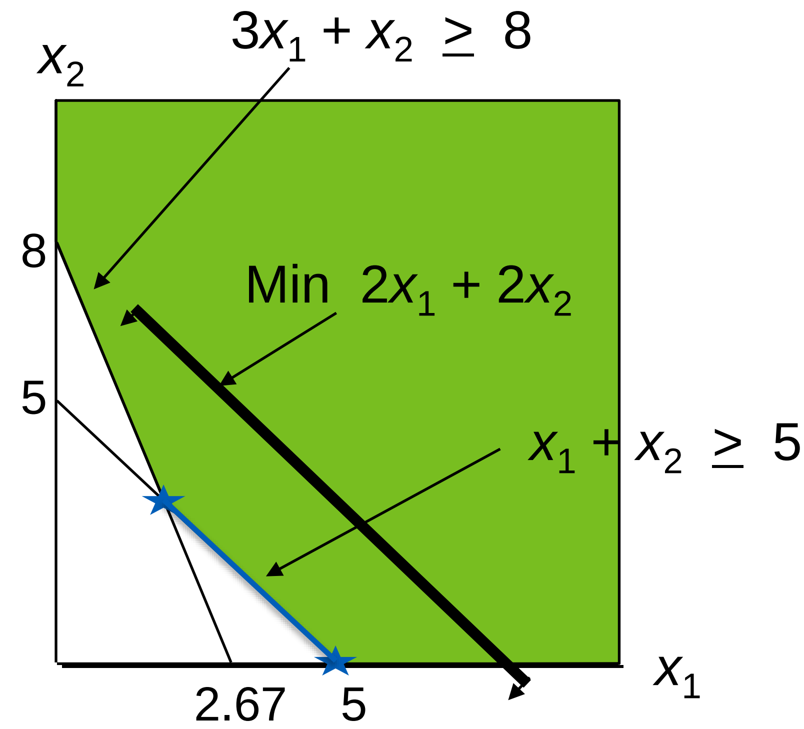
2. The problem is infeasible
   - An over constrained LP with no point that satisfies all the constraints (i.e., the feasible region is empty)

3. The problem is unbounded
   - The objective function value can be improved without a bound in the feasible region

Example: Alternate optimal solutions

$$\text{Min} \quad z = 2x_1 + 2x_2$$
$$x_1 + x_2 \geq 5$$
$$3x_1 + x_2 \geq 8$$
$$x_1, x_2 \geq 0$$

- The objective function line is parallel to a boundary constraint in the direction of optimization
- The points (5,0) and (1.5 , 3.5) and all points on the line segment in between are optimal
  - Objective function value is 10



Example: An Infeasible Problem

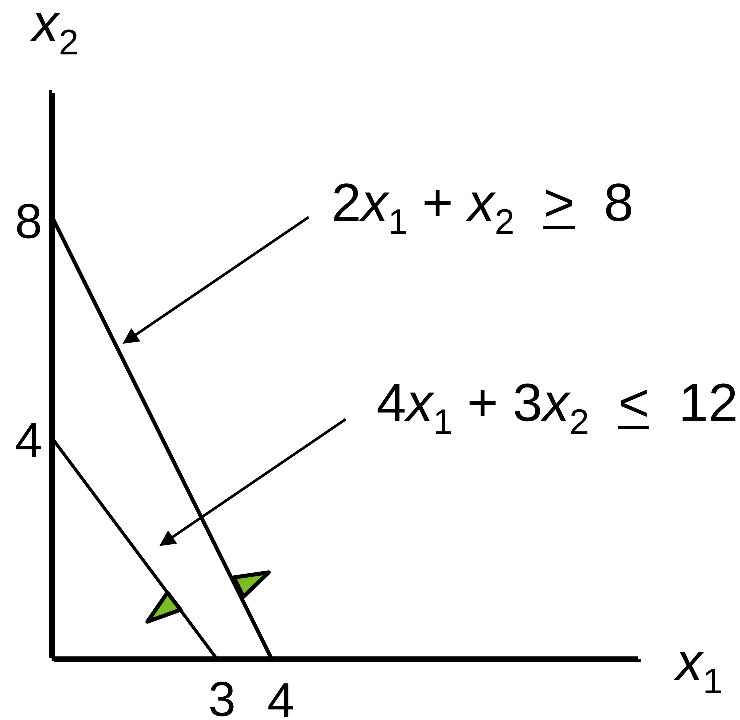
$$\text{Max} \quad z = 2x_1 + 6x_2$$

$$4x_1 + 3x_2 \leq 12$$

$$2x_1 + x_2 \geq 8$$

$$x_1, x_2 \geq 0$$

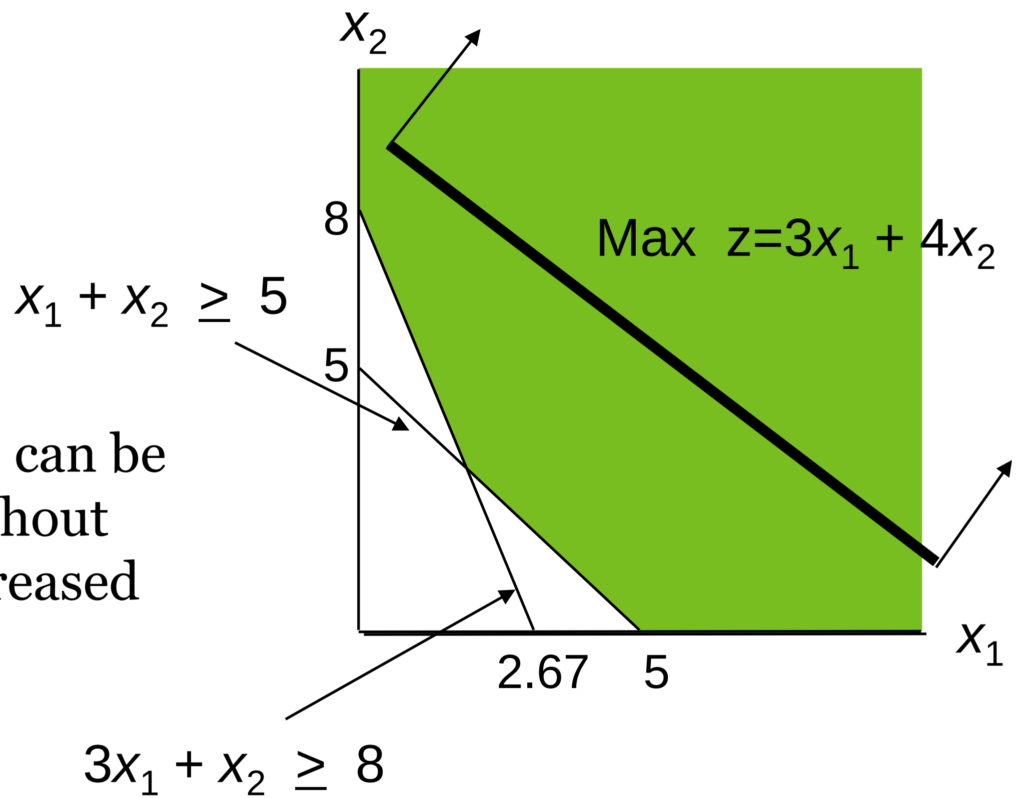
- There are no points that satisfy all constraints
- Hence, this problem has
  - An empty feasible region
  - No feasible solution
  - No optimal solution



Example: An Unbounded Problem

$$\text{Max} \quad z = 3x_1 + 4x_2$$
$$x_1 + x_2 \geq 5$$
$$3x_1 + x_2 \geq 8$$
$$x_1, x_2 \geq 0$$

- The objective function line can be moved parallel to itself without bound so that z can be increased infinitely



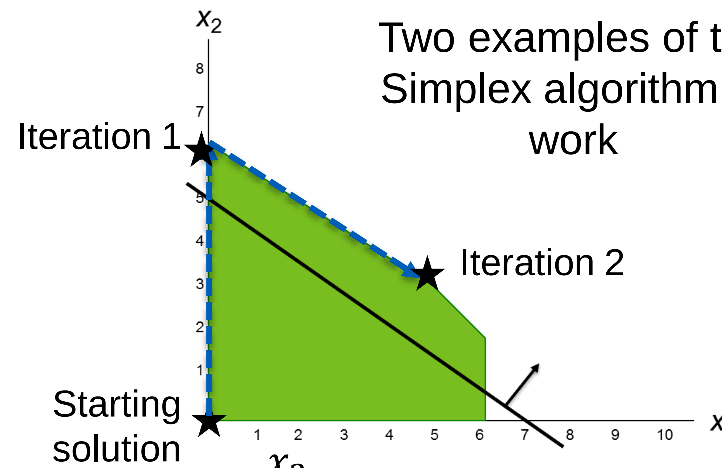
Note: Changing max to min would make the problem bounded (see two slides back).
This shows that in general, even if the feasible region contains solutions with infinitely large/small decision variable values, it does not necessarily mean that the LP problem is unbounded.

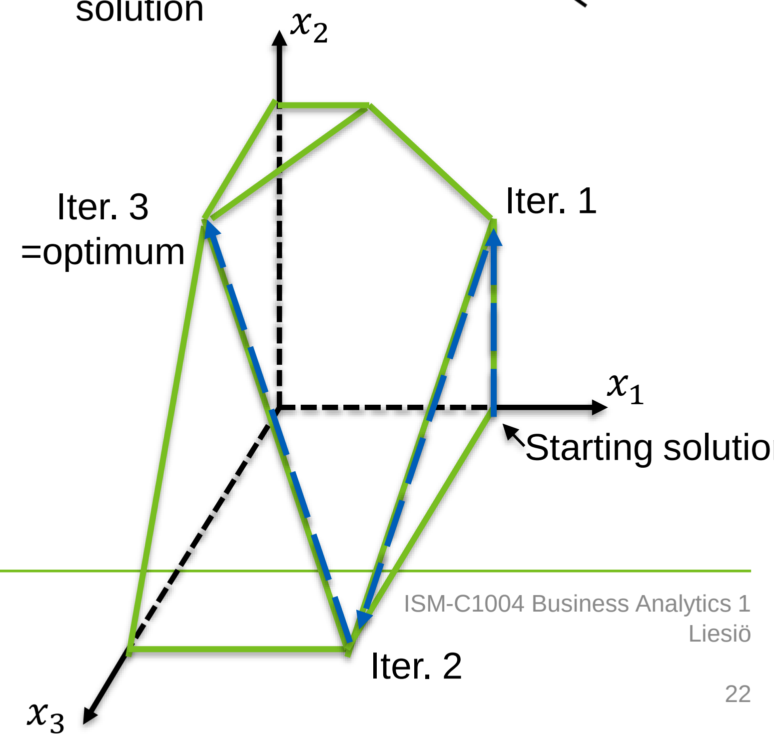
The Simplex Algorithm

- Solution Algorithms for LP
  - **Simplex** (G.B. Dantzig, 1947)
  - Interior point methods (Karmarkar, 1984)

- Simplex (of its modification) is implemented in many software
  - **Excel Solver**
  - lp_solve (open source)
  - CPLEX
  - Gurobi
  - XPress

Two examples of the Simplex algorithm at work





Excel solver

D4 | =SUMPRODUCT(B$2:C$2;B4:C4)

	A	B	C			
1		x1	x2			
2	Decision variable	5	3			
3						
4	Objective f. coefficients	5	7	46		
5						
6						
7	Constraint #1	1	0	5	<=	6
8	Constraint #2	2	3	19	<=	19
9	Constraint #3	1	1	8	<=	8
10						

LP Formulation

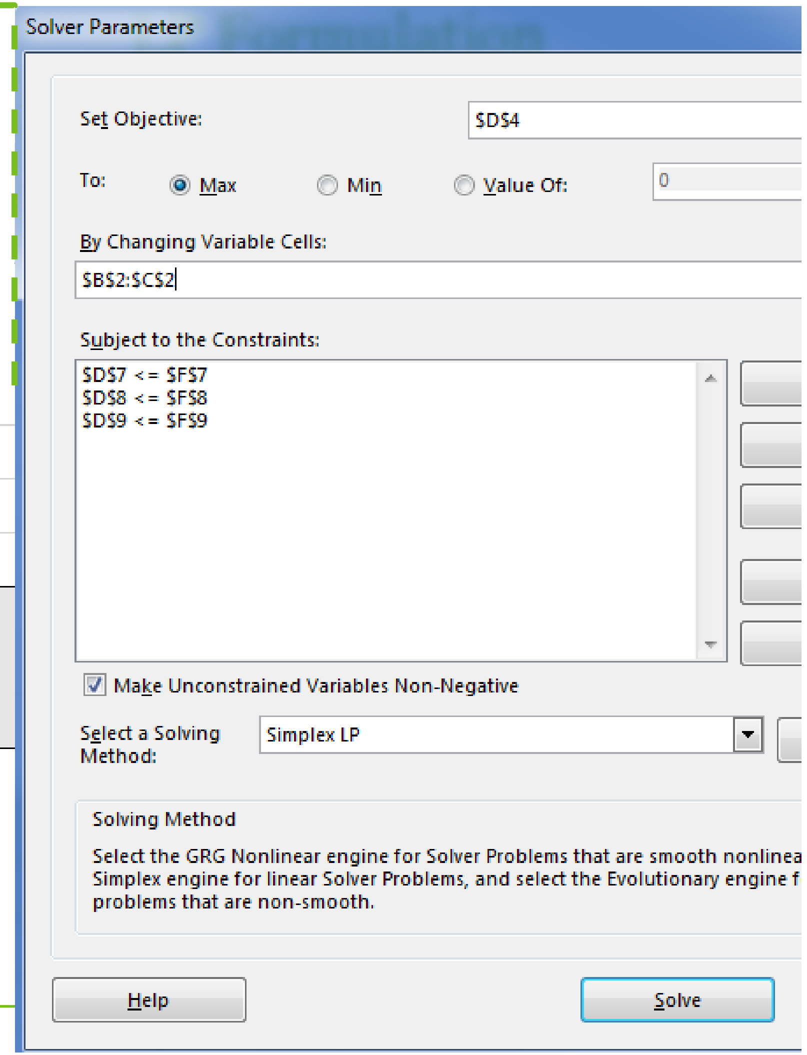
$$\text{Max} \quad 5x_1 + 7x_2$$

$$x_1 \leq 6$$

$$2x_1 + 3x_2 \leq 19$$

$$x_1 + x_2 \leq 8$$

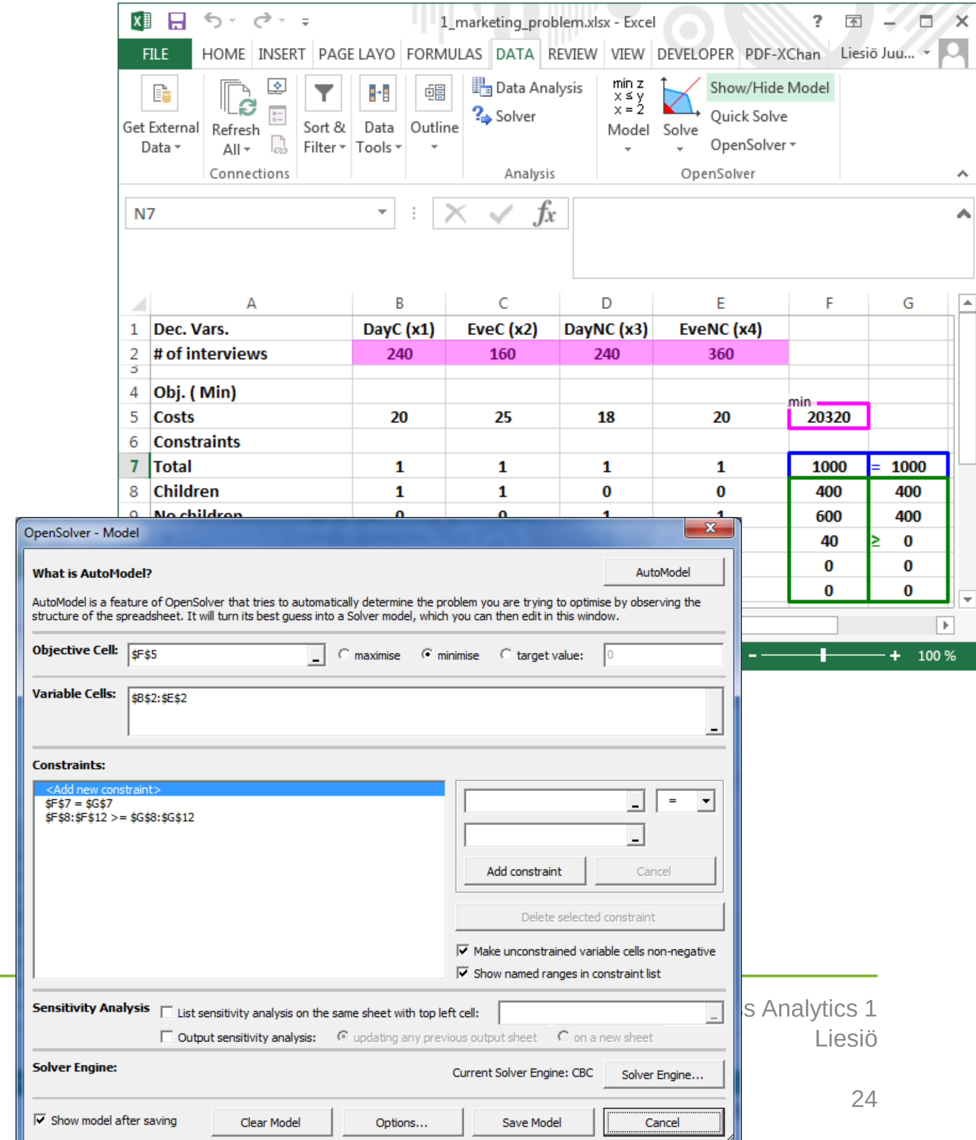
$$x_1, x_2 \geq 0$$



- Details of building and solving the model can be found at the end of this slide set

Open solver

- Excel Solver is not particularly powerful
  - Difficult to solve
    - large problems (i.e., any constrains/ decision variables)
    - integer LP problems (covered later)
  - Mac versions have bugs
- Open solver offers a free alternative for Win and Mac
  - Download: http://opensolver.org/
  - More powerful solution algorithm
  - Better user interface
  - This also has some bugs…



Guidelines for Building Spreadsheet Models

- All data should be visible and labeled
  - Organize and clearly identify the data
    - Use Borders, shading, and/or colors to distinguish cell types (data/parameters, decision variables, formulas, the objective function)
  - Enter each piece of data into one cell only
    - This makes the model much easier to modify later

- Show entire model on the spreadsheet
  - Avoid putting numbers directly in formulas
  - Break out complicated formulas into subtotals
  - All constraints should be on the spreadsheet (not buried in Solver)

- Try to enter the formula just once, and then use Excel's 'fill' capability.
  - This makes the model easer to build and reduces typos

Formulas in Excel that result in a LP model

Problem parameters (data) located in cells D1:D6

Decision variables located in cells C1:C6.

SUMPRODUCT(D4:D6, C4:C6)
[(D1 + D2) / D3] * C4
SUM(D4:D6)
2*C1 + 3*C4 + C6
C1 + C2 + C3

→ **LP model**

SUMPRODUCT(C4:C6, C1:C3)
[(C1 + C2) / C3] * D4
ABS(C1)
SQRT(C1)
C1 * C2
C1 / C2
C1 ^2

→ **NOT a LP model**

A function that is linear in the decision variables x_i: $f(x) = \sum_{i=1}^{n} a_i x_i + b$

Example: Marketing Research

- Market Survey Inc. has been asked to conduct 1000 interviews to find out how consumers react to a new household product
- The client has also given the following guidelines:
  - Interview at least 400 households with children
  - Interview at least 400 households with no children
  - At least as many interviews in the evening as during the day
  - Conduct at least 40% of the children household interviews in the evening
  - Conduct at least 60% of the no children household interviews in the evening

Household	Interview Cost	
	Day	*Evening*
Children	\$20	\$25
No Children	\$18	\$20

Example: Marketing Research (Cont'd)

Household	Interview Cost	
	Day	*Evening*
Children	\$20	\$25
No Children	\$18	\$20

- Interview at least 400 households with children
- Interview at least 400 households with no children
- At least as many interviews in the evening as during the day
- Conduct at least 40% of the children household interviews in the evening
- Conduct at least 60% of the no children household interviews in the evening

Question:

- A summer trainee developed an LP model to optimize the interview plan. Help the management to understand the model:
  - What do the decision variables represent?
  - What does the objective function measure?
  - What requirements do the constraint capture?

$$\min\ 20x_1 + 25x_2 + 18x_3 + 20x_4$$

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 1000 \\ x_1 + x_2 &\geq 400 \\ x_3 + x_4 &\geq 400 \\ x_2 + x_4 &\geq x_1 + x_3 \\ x_2 &\geq 0.4(x_1 + x_2) \\ x_4 &\geq 0.6(x_3 + x_4) \end{aligned}$$

Example: Marketing Research (Cont'd)

- Simplified LP model and its spreadsheet implementation

$$\min 20x_1 + 25x_2 + 18x_3 + 20x_4$$

$$x_1 + x_2 + x_3 + x_4 = 1000$$

$$x_1 + x_2 \geq 400$$
$$x_3 + x_4 \geq 400$$

$$x_2 + x_4 - x_1 - x_3 \geq 0$$

$$-0.4x_1 + 0.6x_2 \geq 0$$

$$-0.6x_3 + 0.4x_4 \geq 0$$

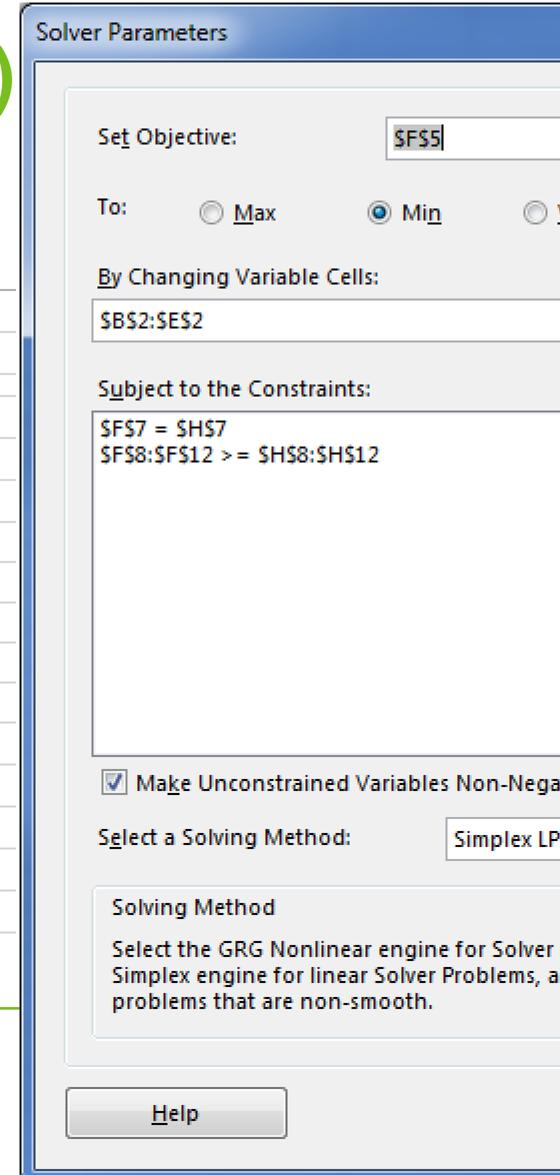
F7 =SUMPRODUCT(B7:E7;$B$2:$E$2)

	A	B	C	D	E	F	G	H
1	Dec. Vars.	DayC (x1)	EveC (x2)	DayNC (x3)	EveNC (x4)			
2	# of interviews	700	0	0	300			
3								
4	Obj. (Min)							
5	Costs	20	25	18	20	20000		
6	Constraints							
7	Total	1	1	1	1	1000	=	1000
8	Children	1	1	0	0	700	>=	400
9	No children	0	0	1	1	300	>=	400
10	Evening	-1	1	-1	1	-400	>=	0
11	Children in evening	-0.4	0.6	0	0	-280	>=	0
12	No chidren in evening	0	0	-0.6	0.4	120	>=	0
13								
14	Decision variables							
15	Parameters							
16	Formulas							

Example: Marketing Research (Cont'd)

- Solver settings and optimal solution

	A	B	C	D	E	F	G	H
1	Dec. Vars.	DayC (x1)	EveC (x2)	DayNC (x3)	EveNC (x4)			
2	# of interviews	240	160	240	360			
3								
4	Obj. (Min)							
5	Costs	20	25	18	20	20320		
6	Constraints							
7	Total	1	1	1	1	1000	=	1000
8	Children	1	1	0	0	400	>=	400
9	No children	0	0	1	1	600	>=	400
10	Evening	-1	1	-1	1	40	>=	0
11	Children in evening	-0.4	0.6	0	0	0	>=	0
12	No chidren in evening	0	0	-0.6	0.4	0	>=	0
13								
14	Decision variables							
15	Parameters							
16	Formulas							



Other properties of LP problems

- Optimal solution will not change if
  - A constant is added to the objective function
  - The objective function is multiplied with a positive constant
  - The objective function (i) is multiplied with a negative constant **and** (ii) min (max) is changed to max (min)
- Any problem can be formulated such that it only has one of the constraint types =, ≥ and ≤
- Adding a constraint can never improve the objective function value

$$\begin{aligned} \min \quad & z = -5x_1 - 7x_2 \\ & 2x_1 + 3x_2 = 19 \\ & x_1 + x_2 \le 8 \end{aligned}$$

same optimal solution (x_1, x_2)

$$\begin{aligned} \max \quad & y = 50x_1 + 70x_2 - 500 \\ & -2x_1 - 3x_2 \le -19 \\ & 2x_1 + 3x_2 \le 19 \\ & x_1 + x_2 \le 8 \end{aligned}$$

optimal w ≥ optimal y

$$\begin{aligned} \max \quad & w = 50x_1 + 70x_2 - 500 \\ & 2x_1 + 3x_2 \le 19 \\ & x_1 + x_2 \le 8 \end{aligned}$$

Slack and Surplus Variables

- A linear program in which all the variables are non-negative and all the constraints are equalities is said to be in <u>standard form</u>
  - Standard form can be obtained by adding <u>slack variables</u> to ≤-constraints, and by subtracting <u>surplus variables</u> from ≥-constraints
  - Slack/surplus variables
    - Represent the difference between the left and right sides of the constraints.
    - Have objective function coefficients equal to zero

- An LP Formulation

$$\begin{aligned} \text{Max} \quad & z = 5x_1 + 7x_2 \\ \text{s.t.} \quad & x_1 \le 6 \\ & 2x_1 + 3x_2 \le 19 \\ & x_1 + x_2 \le 8 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Formulation in Standard Form

$$\begin{aligned} \text{Max} \quad & z = 5x_1 + 7x_2 + 0s_1 + 0s_2 + 0s_3 \\ \text{s.t.} \quad & x_1 + s_1 = 6 \\ & 2x_1 + 3x_2 + s_2 = 19 \\ & x_1 + x_2 + s_3 = 8 \\ & x_1, x_2, s_1, s_2, s_3 \ge 0 \end{aligned}$$

Uses of additional decision variables

- Decision variables often have an explicit connection to the real-life decisions, e.g.,
  - x_1: number of houses with children interviewed during day
  - x_2: kilotons of steel to produce
- Sometimes the link is implicit, e.g.,
  - Value of decision variable s_1 depends on how much we decide to produce grade 1 steal, i.e., the value of x_1
- Use of such auxiliary decision variables can make it easier to apply LP in more complex problems (cf. example on next slide)

- An LP Formulation

$$
\begin{aligned}
\text{Max} \quad & z = 5x_1 + 7x_2 \\
\text{s.t.} \quad & x_1 \leq 6 \\
& 2x_1 + 3x_2 \leq 19 \\
& x_1 + x_2 \leq 8 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

Formulation in Standard Form

$$
\begin{aligned}
\text{Max} \quad & z = 5x_1 + 7x_2 + 0s_1 + 0s_2 + 0s_3 \\
\text{s.t.} \quad & x_1 + s_1 = 6 \\
& 2x_1 + 3x_2 + s_2 = 19 \\
& x_1 + x_2 + s_3 = 8 \\
& x_1, x_2, s_1, s_2, s_3 \geq 0
\end{aligned}
$$

Application in production scheduling

- A company produces computers
  - **R**egular production capacity is 160 computers per week
    - Production costs 190€ per computer
  - Additional 50 computers per week can be produced with **o**vertime (260€ / cmp.)
  - Cost of holding a computer in **i**nventory to satisfy future demand is 10€ / cmp.
  - Demand for the upcoming four weeks (105, 170, 230, 180 computers) needs to be satisfied

$$\min 190r_1 + 260o_1 + 10i_1 + 190r_2 + 260o_2 + 10i_2 + 190r_3 + 260o_3 + 10i_3 + 190r_4 + 260o_4$$

$$r_1 + o_1 - i_1 = 105$$
$$r_2 + o_2 + i_1 - i_2 = 170$$
$$r_3 + o_3 + i_2 - i_3 = 230$$
$$r_4 + o_4 + i_3 = 180$$

$$0 \le r_j \le 160, j = 1, \dots, 4$$
$$0 \le o_j \le 50, j = 1, \dots, 4$$
$$i_j \ge 0, j = 1, \dots, 3$$

Question:

- Help the management understand this LP model for production planning:
  - What does decision variable i_2 represent?
  - What is the interpretation of constraint $r_2 + o_2 + i_1 - i_2 = 170$?

Auxiliary decision variables in production scheduling

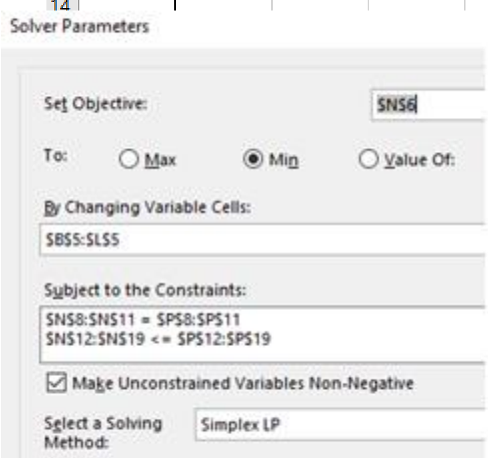
N9 =SUMPRODUCT(B$5:L$5;B9:L9)

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
1																
2		Week 1			Week 2			Week 3			Week 4					
3		Regular prod.	Overtime prod.	Inventory carried over to next	Regular prod.	Overtime prod.	inventory to nect week	Regular prod.	Overtime prod.	inventory to nect week	Regular prod.	Overtime prod.				
4		r_1	o_1	i_1	r_2	o_2	i_2	r_3	o_3	i_3	r_4	o_4				
5	Units	160	0	55	160	0	45	160	25	0	160	20		*Tot. cost*		
6	Unit costs	190	260	10	190	260	10	190	260	10	190	260		134300		
7																
8		1	1	-1										105	=	105
9				1	1	1	-1							170	=	170
10							1	1	1	-1				230	=	230
11										1	1	1		180	=	180
12		1												160	<=	160
13			1											0	<=	50
14					1									160	<=	160
						1								0	<=	50
								1						160	<=	160
									1					25	<=	50
											1			160	<=	160
												1		20	<=	50

$$\begin{aligned}\min\ & 190r_1 + 260o_1 + 10i_1 \\ &+ 190r_2 + 260o_2 + 10i_2 \\ &+ 190r_3 + 260o_3 + 10i_3 \\ &+ 190r_4 + 260o_4\end{aligned}$$

$$\begin{aligned} r_1 + o_1 - i_1 &= 105 \\ r_2 + o_2 + i_1 - i_2 &= 170 \\ r_3 + o_3 + i_2 - i_3 &= 230 \\ r_4 + o_4 + i_3 &= 180 \end{aligned}$$

$$r_j \leq 160, j = 1, \ldots, 4$$
$$o_j \leq 50, j = 1, \ldots, 4$$



General matrix formulation of an LP problem

- Since each constraint can be presented with ≤-constraints and the objective function can be multiplied by -1 to change "min" to "max," any LP problem can be written in the form

$$\max \sum_{i=1}^{n} c_i x_i$$

$$\sum_{i=1}^{n} a_{ji} x_i \le b_j, j = 1, \dots, m$$

$$\Leftrightarrow$$

$$\max \boldsymbol{c}^T \boldsymbol{x}$$

$$\boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b}$$

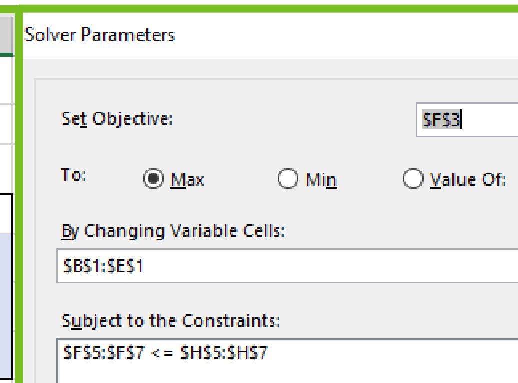
$$\Leftrightarrow$$

$$\max [c_1 \quad \dots \quad c_n] \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \le \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$$\Updownarrow$$

	A	B	C	D	E	F	G	H
1	$\boldsymbol{x}$	x_1	x_2	x_3	x_4			
2								
3	$\boldsymbol{c}^T$	c_1	c_2	c_3	c_4	`=SUMPRODUCT($B$1:$E$1;B3:E3)`		
4								$\boldsymbol{b}$
5	$\boldsymbol{A}$	$a_{1,1}$	$a_{1,2}$	$a_{1,3}$	$a_{1,4}$	`=SUMPRODUCT($B$1:$E$1;B5:E5)`	≤	b_1
6		$a_{2,1}$	$a_{2,2}$	$a_{2,3}$	$a_{2,4}$	`=SUMPRODUCT($B$1:$E$1;B6:E6)`	≤	b_2
7		$a_{3,1}$	$a_{3,2}$	$a_{3,3}$	$a_{3,4}$	`=SUMPRODUCT($B$1:$E$1;B7:E7)`	≤	b_3



Additional examples

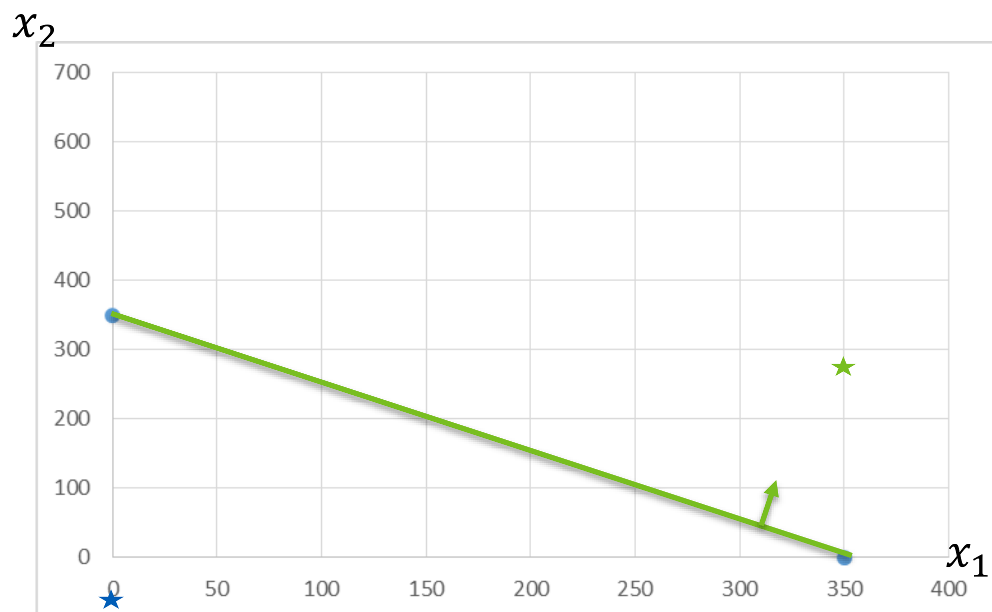
Example 1:

Graphical solution approach

Example 1: Solving LPs graphically

- Constraint (1)
  - Draw line $x_1 + x_2 = 350$
    - If x_1=0 then x_2=350
    - If x_2=0 then x_1=350
  - Hence the line goes through points (0,350) and (350,0)
  - Which side of the line is feasible?
    - (0,0) gives $0 + 0 \geq 350 \rightarrow$ constraint not satisfied
    - (350,350) gives $350 + 350 \geq 350 \rightarrow$ constraint satisfied
  - Thus, points located up and right of the line are feasible

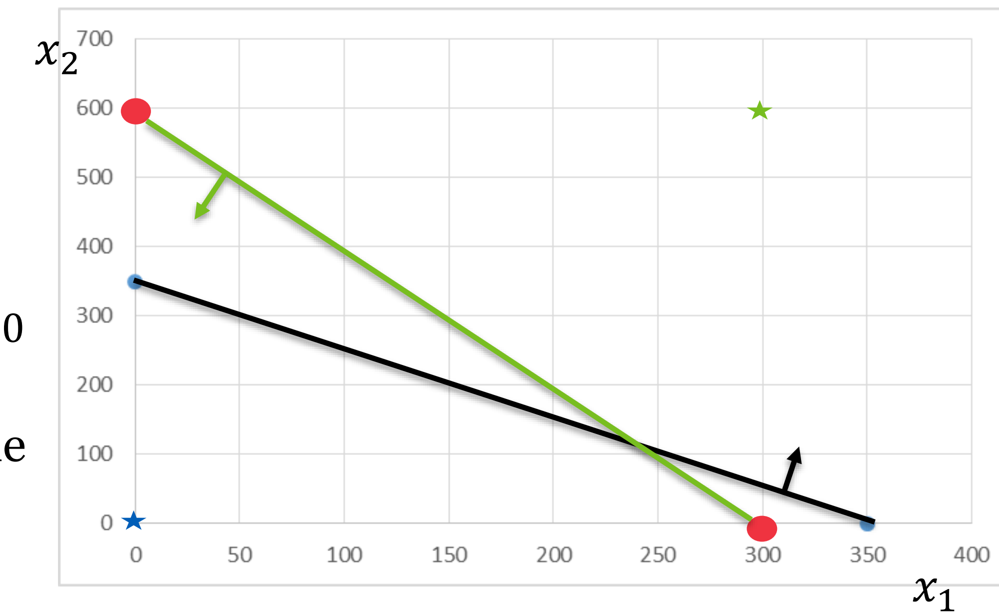
$$\min 2x_1 + 3x_2$$
$$x_1 + x_2 \geq 350 \quad (1)$$
$$2x_1 + x_2 \leq 600 \quad (2)$$
$$x_1, x_2 \geq 0$$



Example 1: Solving LPs graphically

$$\min 2x_1 + 3x_2$$
$$x_1 + x_2 \geq 350 \quad (1)$$
$$2x_1 + x_2 \leq 600 \quad (2)$$
$$x_1, x_2 \geq 0$$

- Constraint (2)
  - Draw line $2x_1 + x_2 = 600$
    - If x_1=0 then x_2=600
    - If x_2=0 then x_1=300
  - Hence the line goes through points (0,600) and (300,0)
  - Which side of the line is feasible?
    - ★ (0,0) gives $0 \leq$600 → constraint satisfied
    - ★ (300,600) gives $2 * 300 + 600 \leq 600$ → constraint not satisfied
  - Points down left and right of the line are feasible

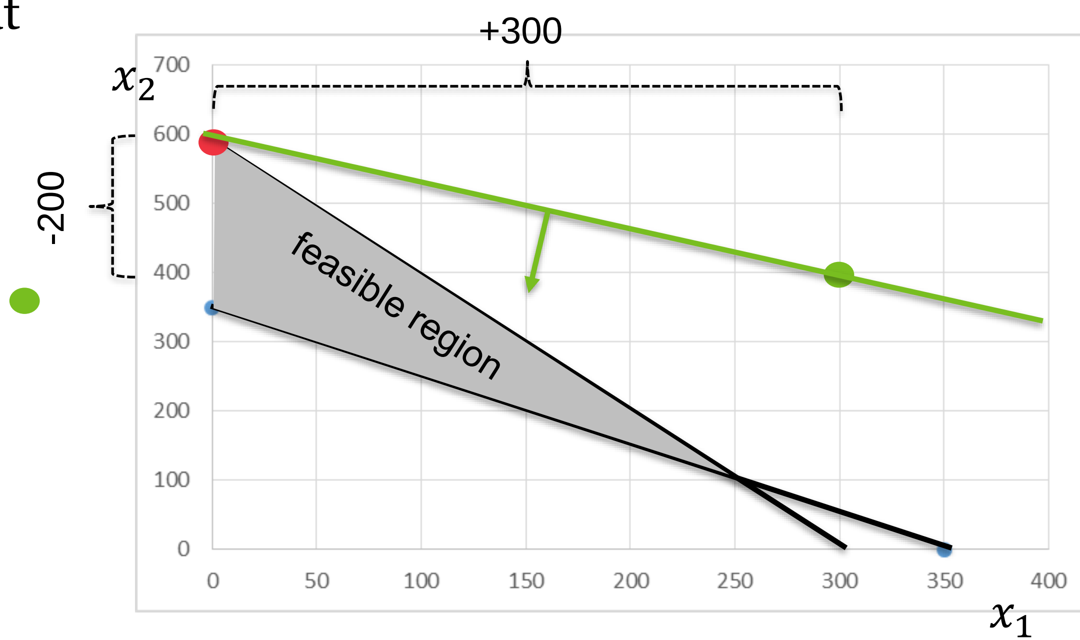


Example 1: Solving LPs graphically

- Objective function line
  - Evaluate objective function at point (0,600)
    $$z = 2 * 0 + 3 * 600 = 1800$$
  - Find point $(300, x_2)$ such that z=1800
    $$z = 2 * 300 + 3 * x_2 = 1800$$
    $$\Rightarrow x_2 = 400$$
  - Objective function value is 1800 also at point (300,400)

- The contours of a linear function are straight lines

$$\min z = 2x_1 + 3x_2$$
$$x_1 + x_2 \geq 350 \quad (1)$$
$$2x_1 + x_2 \leq 600 \quad (2)$$
$$x_1, x_2 \geq 0$$



$$2 * (+300) + 3 * (-200) = 600 - 600 = 0$$

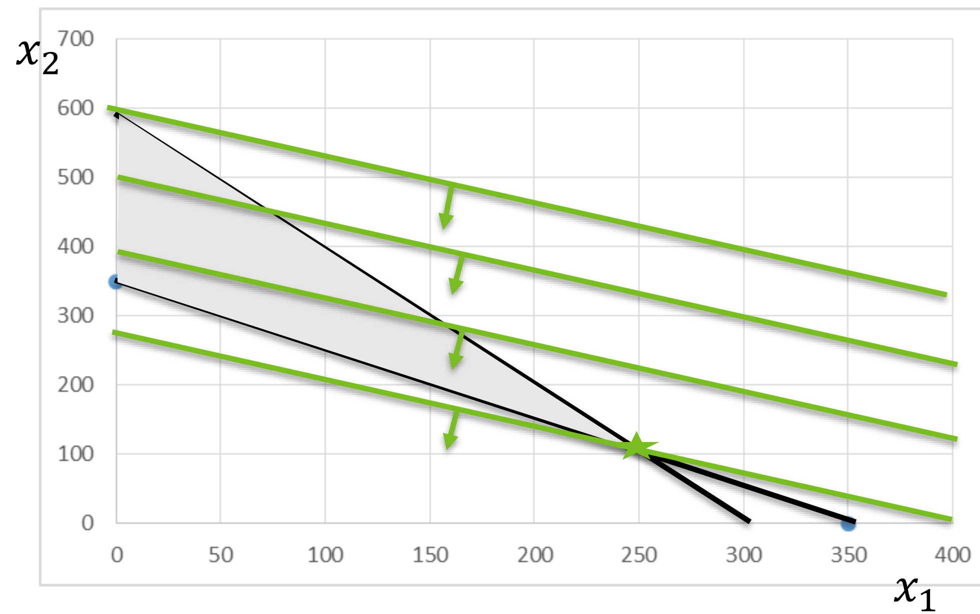
Example 1: Solving LPs graphically

$$\min z = 2x_1 + 3x_2$$
$$x_1 + x_2 \geq 350 \quad (1)$$
$$2x_1 + x_2 \leq 600 \quad (2)$$
$$x_1, x_2 \geq 0$$

- Move objective function line
  - Clearly at (0,0) z=0 → better solutions are found by when both decision variable decrease
- Optimum is at the extreme point defined by constraints (1) and (2)
  - Solve (x_1, x_2) such that $x_1 + x_2 = 350$ and $2x_1 + x_2 = 600$

$$x_2 = 350 - x_1$$
$$2x_1 + (350 - x_1) = 600$$
$$\Rightarrow x_1 = 250$$
$$\Rightarrow x_2 = 350 - 250 = 100$$
$$\Rightarrow (x_1, x_2) = (250,100) \bigstar$$

- Optimal objective function value: z=2*250+3*100=800



Example 1: Binding vs. Redundant constraints

- Constraints (1) and (2) are **binding** since at optimum we have

$$x_1 + x_2 = 350$$
$$2x_1 + x_2 = 600$$

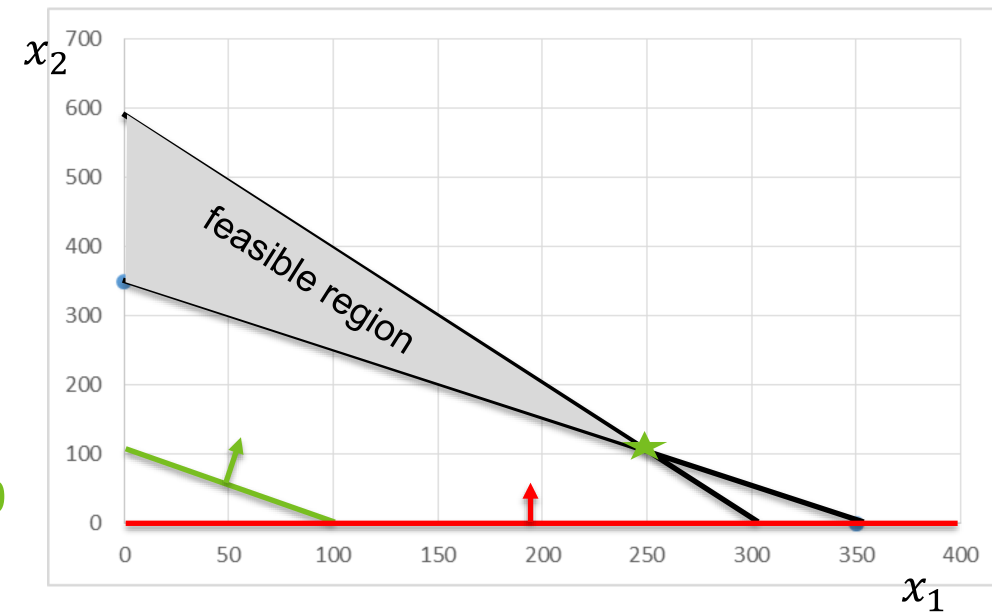
  - $x_1 \geq 0, x_2 \geq 0$ are not binding

- A constraint is **redundant** if removing it would not change the feasible region

  $\rightarrow$ The constraint $x_2 \geq 0$ is redundant

  - Also the constraint $x_1 + x_2 \geq 100$ would be redundant

$$\min z = 2x_1 + 3x_2$$
$$x_1 + x_2 \geq 350 \quad (1)$$
$$2x_1 + x_2 \leq 600 \quad (2)$$
$$x_1, x_2 \geq 0$$



Example 2:

Optimization with Excel Solver

Example 2: Spreadsheet formulation

LP Formulation

$$\begin{aligned}
\text{Max} \quad & z = 5x_1 + 7x_2 \\
\text{s.t.} \quad & x_1 \le 6 \\
& 2x_1 + 3x_2 \le 19 \\
& x_1 + x_2 \le 8 \\
& x_1, x_2 \ge 0
\end{aligned}$$

1. Enter the problem data.

- Objective coefficients are in cells B4 and C4.
- Constraint coefficients are in cells B7, C7, B8, C8, B9, C9.

2. Specify cell locations for all decision variables.

- Cell B2 is reserved for x1, cell C2 is reserved for x2.

3. Select a cell and enter a formula for computing the value of the objective function.

- Cell D4 contains formula for computing value of the obj. function.

	A	B	C	D	E	F
1		x1	x2			
2	Decision Variables					
3						
4	Objective Coefficients	5	7	=SUMPRODUCT(B$2:C$2,B4:C4)		
5						
6						
7	Constraint #1	1	0		<=	6
8	Constraint #2	2	3		<=	19
9	Constraint #3	1	1		<=	8

Example 2: Spreadsheet formulation

4. Select a cell and enter a formula for computing the left-hand-side of each constraint.

- Cells D7, D8, D9 contain formulas for computing the LHSs of the constraints.

5. Select a cell and enter the value of the RHS of each constraint

- Cells F7, F8, F9 contain the RHS values.

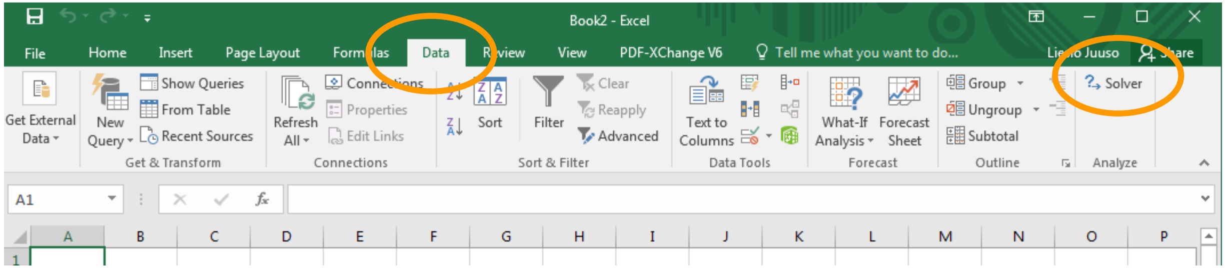
LP Formulation

$$\text{Max} \quad z = 5x_1 + 7x_2$$
$$\text{s.t.} \quad x_1 \leq 6$$
$$2x_1 + 3x_2 \leq 19$$
$$x_1 + x_2 \leq 8$$
$$x_1, x_2 \geq 0$$

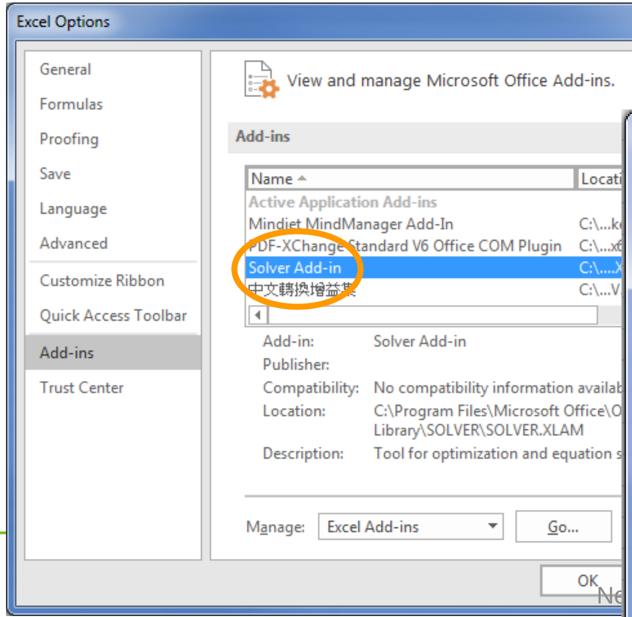
	A	B	C	D	E	F
1		**x1**	**x2**			
2	**Decision Variables**					
3						
4	**Objective Coefficients**	5	7	=SUMPRODUCT(B$2:C$2,B4:C4)		
5						
6						
7	**Constraint #1**	1	0	=SUMPRODUCT(B$2:C$2,B7:C7)	<=	6
8	**Constraint #2**	2	3	=SUMPRODUCT(B$2:C$2,B8:C8)	<=	19
9	**Constraint #3**	1	1	=SUMPRODUCT(B$2:C$2,B9:C9)	<=	8

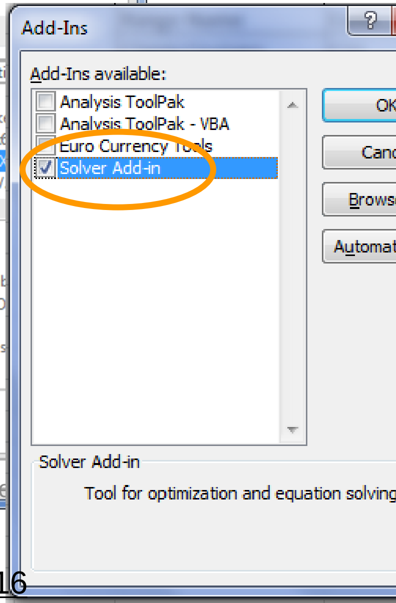
Example 2: Installing and opening the solver

- Select the **Solver** option from **Data** tab



- If the solver is not visible you have to install it first:
  - File -> Options -> Add-ins
  - In the Add-ins box, select Solver Add-In and click Go.
  - In the Add-Ins available box, check the Solver Add-in and then OK.

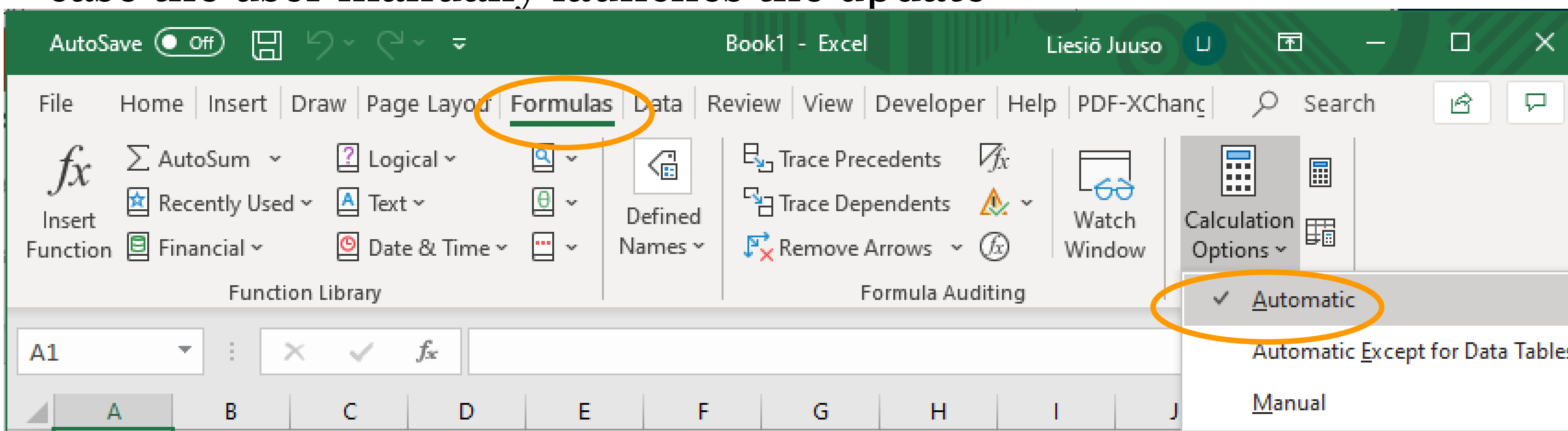




MacOS: https://support.ivey.ca/hc/en-us/articles/209172446-Solver-and-Data-Analysis-Add-ins-for-Excel-for-Mac-2016

Example 2: Set formula calculation to ”Automatic” – otherwise solver will not work!

- Solver needs to know how the changes it makes to the decision variable values affects the values of the objective function and the constraints
- This is not possible if the values of these cells are calculated only in the case the user manually launches the update

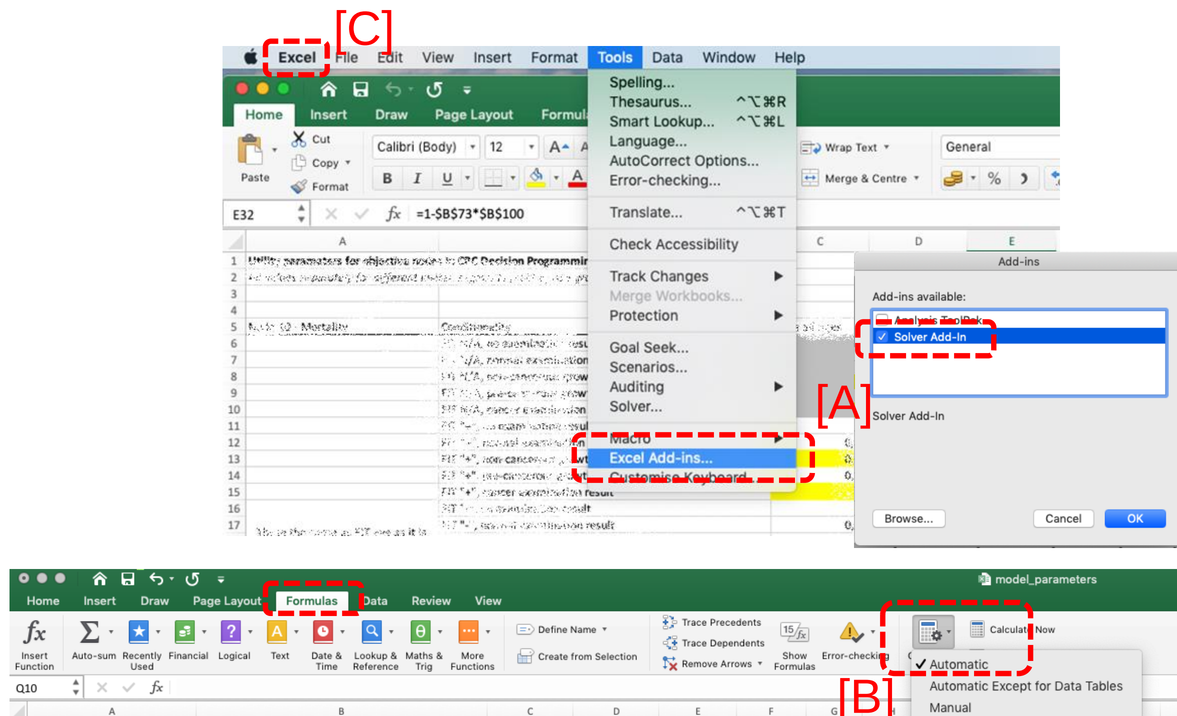


Example 2: Using Excel solver on Mac

- Solver is on the ‘**Data**’-tab if it is installed

- To install Solver [A]
  - **Tools → Excel Add-ins → Solver**

- Enable automatic update of call values from `**Formulas**’-tab [B]
  - Verify that Calculation is set to Automatic in **Excel-**menu → **Preferences → Calculation** [C]



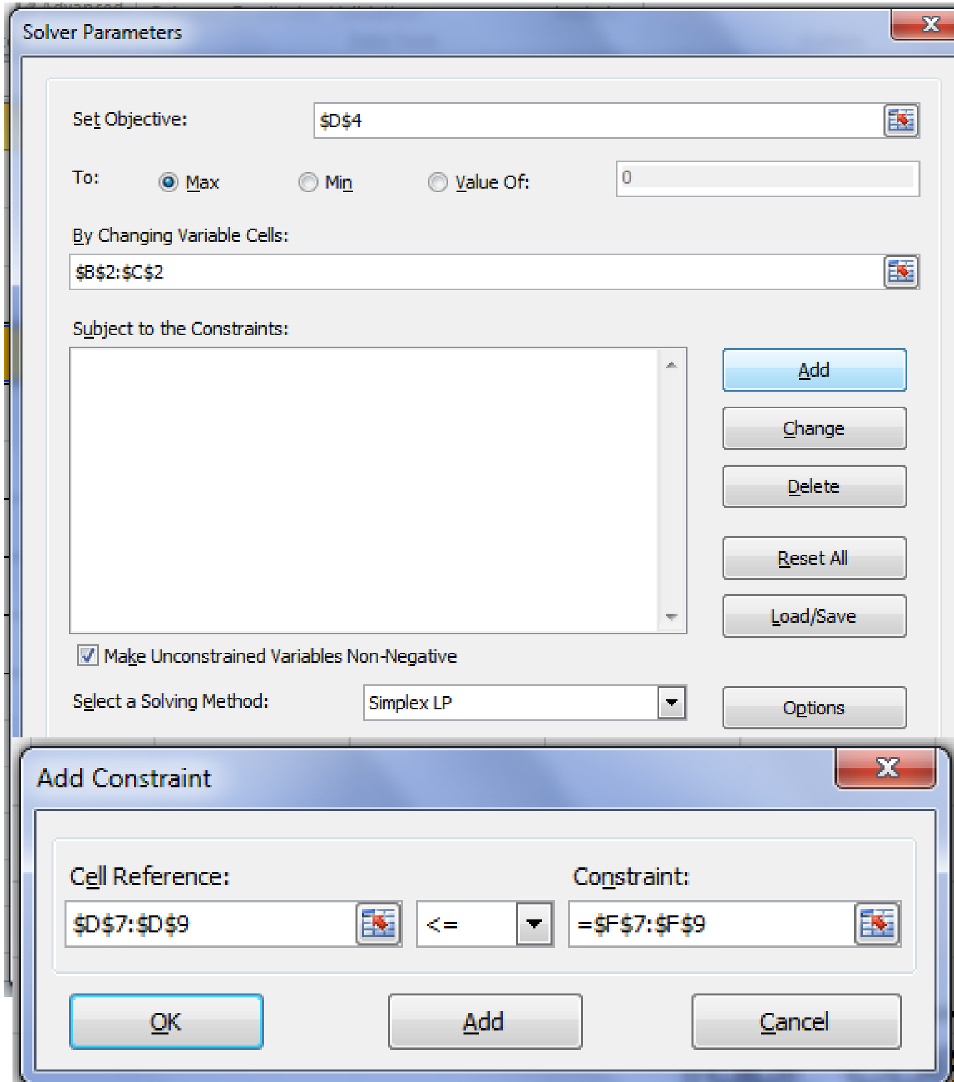
Example 2: Using the Excel Solver

In **solver parameters** dialog box:

- Enter D4 into the **Set Target Cell** box
- Select the **Max** option
- Enter B2:C2 into the **By Changing Cells** box
- Choose **Add**

In **Add Constraint** dialog box:

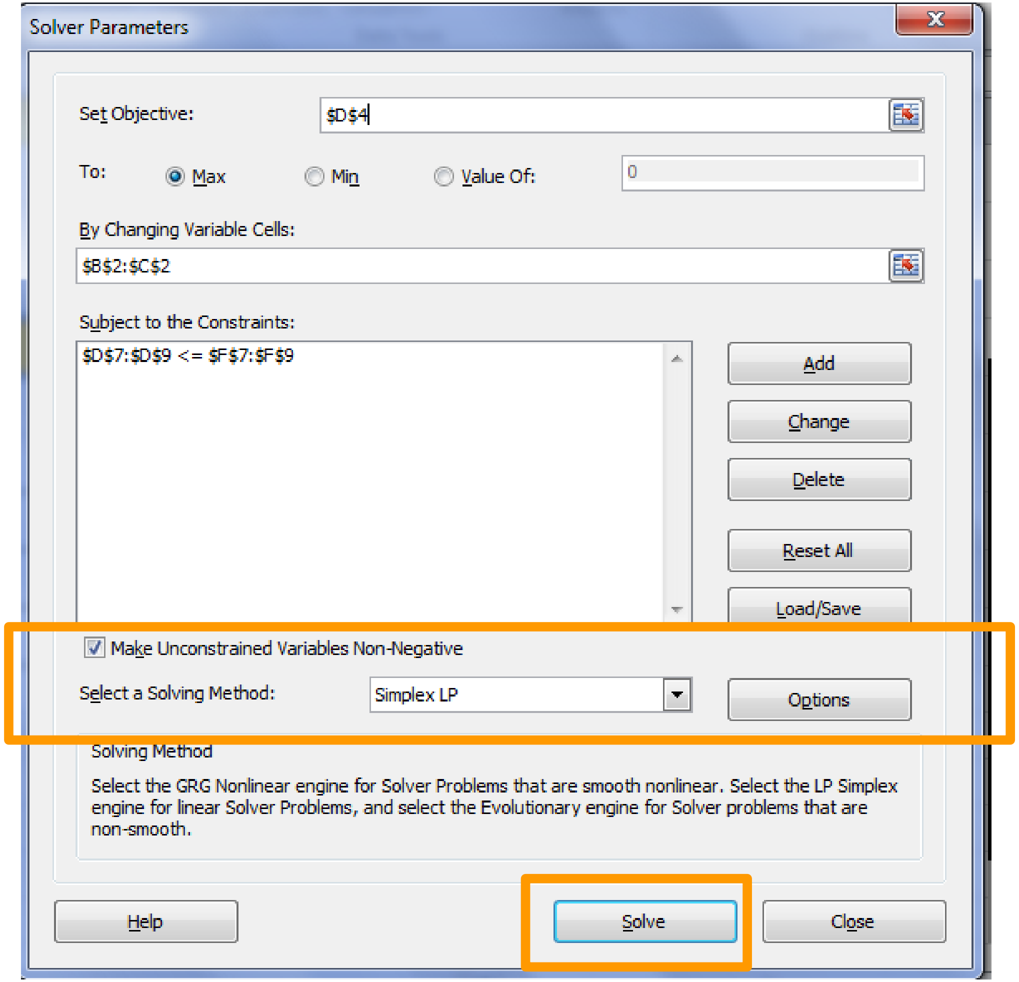
- Enter D7:D9 in the **Cell Reference** box
- Select <=
- Enter F7:F9 into the **Constraint** box
- Choose **OK**



Example 2: Using the Excel Solver

In the **solver parameters** dialog box:

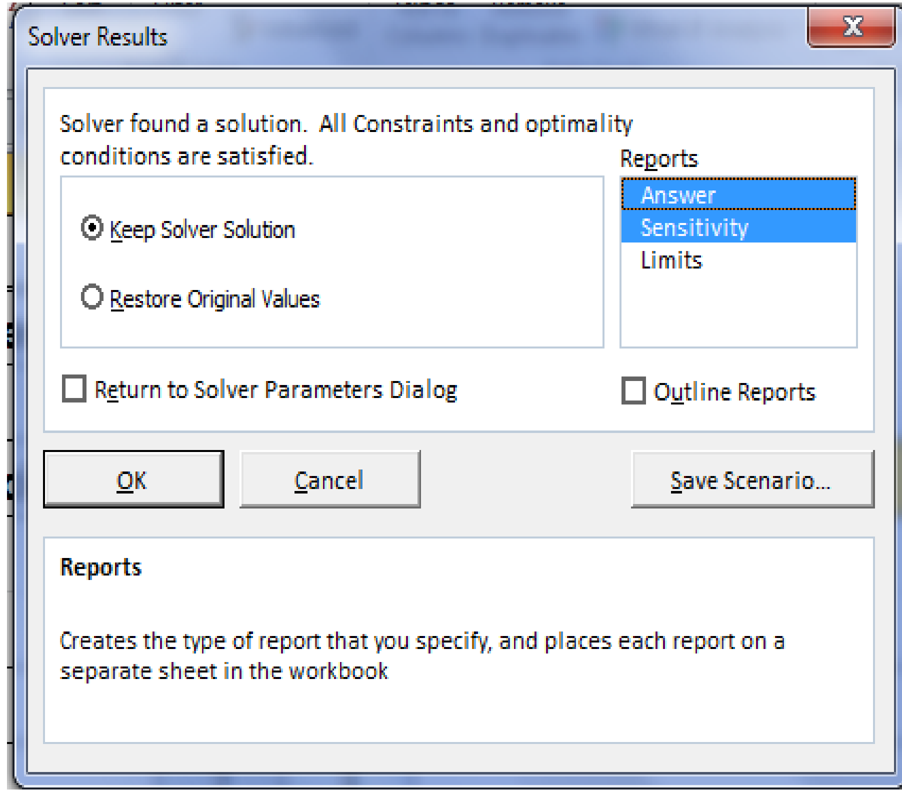
- Select **Simplex LP** as a Solving Method
- Make unconstrained Variables **Non-negative**
- Choose **Solve**



Example 2: Using the Excel Solver

When the **Solver Results** dialog box appears:

- Select **Keep Solver Solution**
- Select **Answer** and **Sensitivity** reports
- Choose **OK** to produce the optimal solution output.



Example 2: Solver solution

	A	B	C	D	E	F
1		x1	x2			
2	Decision Variables	5	3			
3						
4	Objective Coefficients	5	7	46		
5						
6						
7	Constraint #1	1	0	5	<=	6
8	Constraint #2	2	3	19	<=	19
9	Constraint #3	1	1	8	<=	8

Example 2: Answer report

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$D$4	Objective Coefficients	0	46

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$2	Decision Variables x1	0	5	Contin
$C$2	Decision Variables x2	0	3	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$D$7	Constraint #1	5	$D$7<=$F$7	Not Binding	1
$D$8	Constraint #2	19	$D$8<=$F$8	Binding	0
$D$9	Constraint #3	8	$D$9<=$F$9	Binding	0

Example 2: Sensitivity report

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$2	Decision Variables x1	5	0	5	2	0.333333333
$C$2	Decision Variables x2	3	0	7	0.5	2

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$D$7	Constraint #1	5	0	6	1E+30	1
$D$8	Constraint #2	19	2	19	5	1
$D$9	Constraint #3	8	1	8	0.333333333	1.666666667