Chapter 6: Decision making under certainty and with multiple objectives: multiattribute value functions

6.0 Summary

1. In many situations, multiple objectives are relevant when making a decision. They require the evaluation of alternatives by means of a multiattribute value function.
2. The simplest and most important multiattribute value function is the additive one. Using this value function, the (total) value of an alternative is computed as a weighted sum of (individual) values per attribute.
3. The additive model can only be employed if certain independence conditions between the considered attributes are fulfilled.
4. The weights of the attributes can reasonably be determined using the *trade-off* method or the *swing* method. In the case of the trade-off method, objective weights are derived from the trade-off relationship between two attributes. In the case of the swing method, objective weights are computed from point valuations of different alternatives.
5. The widely used *direct-ratio* method is fraught with problems.
6. Attribute weights can only be reasonably interpreted with respect to the ranges of the attribute levels. There is no "per se" importance of particular attributes.
7. If the decision maker is incapable of specifying exact weights, the optimal alternative might also be determined on the basis of imperfect information (e.g. the specification of an interval) or at least dominant alternatives can be eliminated.
8. Sensitivity analyses can provide important information on acceptable changes of the objective weights without changing the optimal alternative.
9. Individuals make systematic mistakes when trying to determine their attribute weights. In particular, they do not sufficiently take into account the attribute ranges (see 6. above) and overestimate the weight of an objective if it is split into multiple subordinate objectives.

6.1 Value functions for multiple attributes

Decision making with multiple conflicting objectives is a key problem in many areas of application. For that reason, the literature on this topic is particularly multifaceted and extensive (for a recent survey, see Wallenius et al. 2008). We will focus on the multiattribute valuation concept in this chapter. Building on the ideas in Chapter 5, in which we analyzed value functions for only one objective, we will now consider value functions for multiple attributes. A multiattribute value func-

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tion assigns a particular value to every alternative, depending on the respective levels of the individual attributes. We assume certainty concerning the levels.

Let us return to the example from Chapter 5 and consider the recent graduate who faces several employment opportunities. They no longer differ only in their respective annual starting salary but now also in their average weekly working time: occupation in a management consultancy, as a teaching assistant at a university, or as a sailing instructor (Table 6-1).

Table 6-1: Three employment opportunities with two attributes

Alternative	Salary	Working hours
(*a*) Consultancy	€80,000	60 hours
(*b*) University	€50,000	40 hours
(*c*) Sailing instructor	€30,000	20 hours

Using a *multiattribute value function* v, we want to try to model his preferences according to his multiple objectives (i.e. salary and working hours) in order to facilitate the complex decision. (Having acquired the relevant knowledge, he will be able to do so himself.) Similarly to the value functions for a single attribute, the value function v shall express the strength of preference towards the different alternatives. Among two alternatives, the one with the higher (preference) value is chosen. We are hence looking for a function v with

$$a \succ b \Leftrightarrow v(a) > v(b).$$

But how do we obtain such a function? First off, there needs to be a guarantee that such a function exists in the first place. This does not mean that the decision maker needs to know it; the "existence" of such a function rather means that the decision maker is capable of making statements that allow the construction of the function.

Let us assume that the value function v exists. What could it look like? Obviously, a particularly simple form, typical of the additive model, is desirable. We will thus restrict our attention to this model, especially as it has the highest relevance in practice. In Section 6.3, we will discuss what conditions are necessary for the existence of an additive multiattribute value function.

6.2 The additive model

The alternative $a \in A$ is characterized by the vector $a = (a_1, \ldots, a_m)$. The a_r indicate the levels of the attribute X_r for the alternative a. The decision maker has a value function $v_r(x_r)$ (one-dimensional attribute value function, individual value function) over every attribute X_r. The individual value functions v_r are normalized on the interval [0, 1]. This means that all relevant levels lie between x_r^- (worst level) and x_r^+ (best level) and that

$$v_r(x_r^-) = 0 \quad \text{and} \quad v_r(x_r^+) = 1 \tag{6.1}$$

hold.

The boundaries x_r^- and x_r^+ have to enclose the attribute levels of all alternatives that are being evaluated but they can be larger than necessary (e.g. to avoid re-normalizations in the case of the occurrence of new alternatives). However, the interval should generally be as narrow as possible as this leads to more accuracy in the subsequent evaluations. If, for instance, your job offers yield salaries of €55,000 and €68,000, it is possible to determine the value function over this interval or over a slightly larger one, say from €50,000 to €80,000; however, it would not be sensible to use an interval from €0 to €1,000,000.

The additive model determines the value of an alternative a as

$$v(a) = \sum_{r=1}^{m} w_r v_r(a_r) \tag{6.2}$$

where $w_r > 0$ and

$$\sum_{r=1}^{m} w_r = 1. \tag{6.3}$$

As stated above, a_r indicates the level of the attribute X_r for the alternative a, and $v_r(a_r)$ indicates the respective value of the attribute value function v_r. The w_r are objective weights or attribute weights. Later on, we will explain that this term is misleading because attributes actually do not have any weights per se. However, we will use this term because it is both convenient and widely used, although it would be more accurate to speak of scaling constants. The weights relate the valuations of the different attributes to each other. Essentially, only the relative size of the weights is important, and not their absolute size. The normalization that is postulated by condition (6.3) has the single purpose of choosing one specific combination out of the set of many equivalent weight combinations allowing us to speak of *unique weights*. In multiattribute value theory, it is common to normalize by fixing the sum of weights at 1 (and this is actually in line with our standard comprehension of "weighting"). However, we could as well postulate that w_1 always has to be 1 (a normalization that you will observe later on in Chapter 11, when we discuss intertemporal decision making and interpret it as a special case of a multiattribute decision problem).

If you consider just two attributes X and Y, the additive model can be illustrated graphically. Figure 6-1 shows how the total value is aggregated from the attribute value functions v_X and v_Y and the weighting factors w_X and $w_Y = 1 - w_X$. x^- indicates the worst, x^+ the best level of the attribute X; the same applies to Y. We obtain the left (total) value function if attribute X is weighted heavily and the right one if a lot of weight is assigned to attribute Y. The meaning of the weighting factors can easily be seen: the weighting factor w_X indicates the increase in value when attribute X is changed from its lowest level (individual value = 0) to the highest level (individual value = 1), leaving all other attributes equal. In general, the value of an alternative with the maximum level in one attribute w_r and the minimum level in all other attributes is equal to the weight of this attribute:

$$\begin{aligned} & v(x_1^-, x_2^-, \ldots, x_r^+, x_{r+1}^-, \ldots, x_m^-) \\ &= w_1 v_1(x_1^-) + w_2 v_2(x_2^-) + \ldots + w_r v_r(x_r^+) + w_{r+1} v_{r+1}(x_{r+1}^-) + \ldots + w_m v_m(x_m^-) \\ &= w_1 \cdot 0 + w_2 \cdot 0 + \ldots + w_r \cdot 1 + w_{r+1} \cdot 0 + \ldots + w_m \cdot 0 \\ &= w_r. \end{aligned}$$

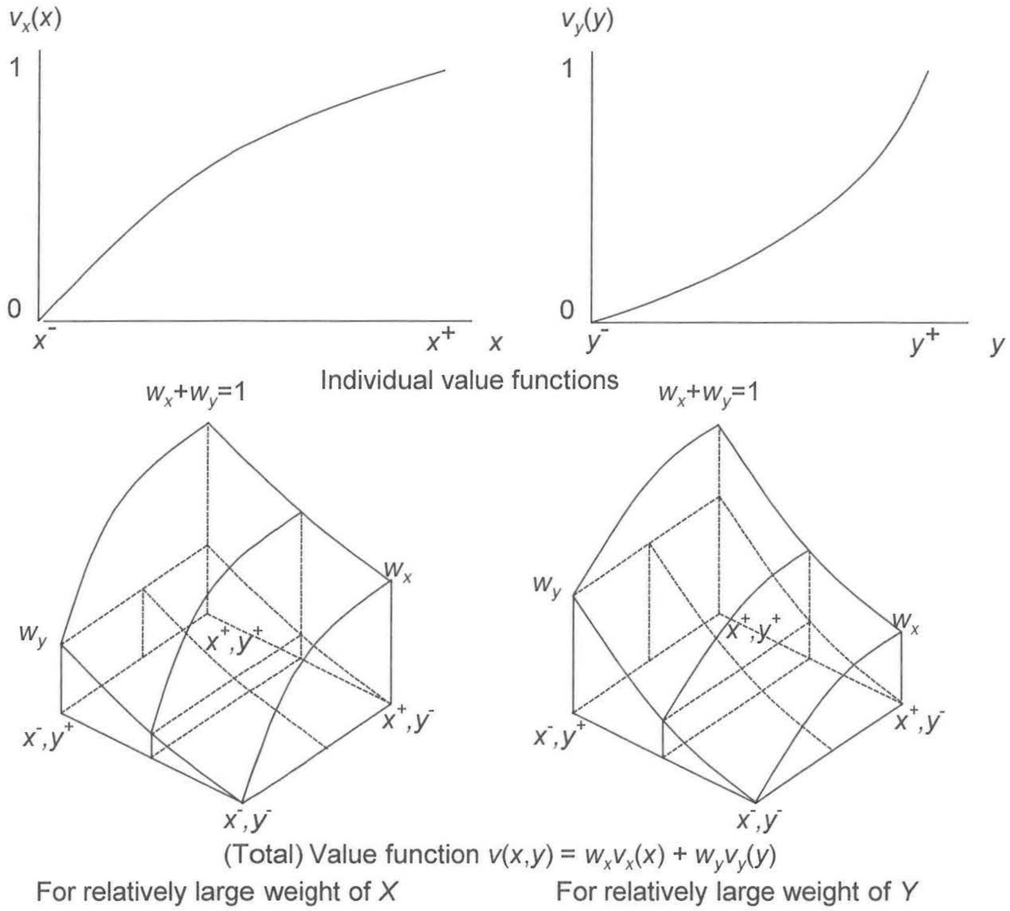

Figure 6-1: Graphical illustration of the additive model for two attributes

Let us assume we had identified measurable attribute value functions v_1 and v_2 for our job example (using the techniques described in Chapter 5). Let us further assume that the value functions in this case are monotone. This implies that the decision maker prefers more money to less money and less work to more work at all times (this is a quite plausible assumption but need not invariably be the case).

The value functions in Table 6-2 were determined over the attribute intervals €30,000 to €80,000 annual salary and 20 to 60 weekly working hours, respectively.

Table 6-2: Three job offers with two attributes and their respective values

Alternative	Salary x_1	Value of salary $v_1(x_1)$	Working hours x_2	Value of working hours $v_2(x_2)$
(*a*) Consultancy	€80,000	1.0	60 hours	0.0
(*b*) University	€50,000	0.6	40 hours	0.5
(*c*) Sailing instructor	€30,000	0.0	20 hours	1.0

With the help of the value function v, the total value can now be derived from the individual values. Let us assume that we know both objective weights w_1 and w_2. This allows us to evaluate the three alternatives. From Table 6-3, we see that for $w_1 = 0.6$ and $w_2 = 0.4$, the job as a consultant yields the highest total value.

Table 6-3: Evaluation of the three job offers using attribute weights of 0.6 for the salary and 0.4 for working hours

Alternative	Value of salary $v_1(x_1)$	Weighted value of salary $w_1v_1(x_1)$	Value of working hours $v_2(x_2)$	Weighted value of working hours $w_2v_2(x_2)$	Total value $w_1v_1(x_1) + w_2v_2(x_2)$
(*a*) Consultancy	1.0	0.60	0.0	0.00	0.60
(*b*) University	0.6	0.36	0.5	0.20	0.56
(*c*) Sailing instr.	0.0	0.00	1.0	0.40	0.40

6.3 Requirements for the applicability of the additive model

Due to the simplicity and elegance of the additive value function, the model is (often hastily) used in a variety of applications. For example, it is known by the terms "scoring model", "point valuation method", or "cost utility analysis". In these methods, scores between 0 and 10 or 0 and 100 are attached to the attribute levels of each alternative. The importance of an attribute is reflected by a weight expressed as a percentage. The percentages add up to 100% for all attributes. The percentages are used to weight the scores, summing up to the total value of an alternative. Examples can be found in a variety of domains, e.g. in product testing, methods of analytical job performance evaluation, the evaluation of alternative technical systems, performance evaluations in sports, or even at school.

In order to be able to rationally justify the use of an additive value model, certain conditions concerning the independence of the attribute evaluation have to be fulfilled. A measurable value function

$$v(x_1, x_2, \ldots, x_m) = w_1v_1(x_1) + w_2v_2(x_2) + \ldots + w_mv_m(x_m)$$

obviously expresses the concept that a certain increase in one attribute causes a change in total value completely independent of the level of all other attributes. For example:

- In a decathlon, the improvement from 11.5 to 11.0 seconds in the 100-meter sprint yields an additional score that is independent of the performance in the long jump or shot put event.
- In building projects, short construction time, a high quality standard, and low construction costs are desirable. The reduction of the construction time from 10 to 9 months generates the same value for you, no matter whether costs and quality are high or low.

We have already discussed this property in Chapter 3, where we called it preferential independence. Let us now define it formally:

Definition 6.1 (Simple preferential independence)
Let

$$a = (a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_m)$$
$$b = (a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_m)$$

be two alternatives that only differ in the i-th attribute, and

$$a' = (a'_1, \ldots, a'_{i-1}, a_i, a'_{i+1}, \ldots, a'_m)$$
$$b' = (a'_1, \ldots, a'_{i-1}, b_i, a'_{i+1}, \ldots, a'_m)$$

be two other alternatives that also only differ in the i-th attribute, but for this feature the same levels as a and b. The attribute X_i is then called *(simply) preferentially independent* of the remaining attributes if it holds for any a, b, a', b' defined as above that:

$$a \succ b \Leftrightarrow a' \succ b' \quad \text{(analogous for } \sim, \prec\text{).}$$

To clarify: you want to buy a new car and choose among various models. For you, one of the most important attributes is the color and you are only interested in a black or a white car. For every model, both colors are available. If you prefer black on *any* car model, the attribute "color" is simply preferentially independent of all other attributes. However, if you preferred an Opel in black and a VW in white, the attribute color would not be independent of the other attributes.

Extending the definition of preferential independence to the case of multiple attributes does not pose a problem. A subset of the attribute set is called "preferentially independent of the remaining attributes" if the preference of the decision maker with respect to the attribute levels on this subset is not influenced by the levels of the other attributes (as long as they are the same for all alternatives). We refrain here from presenting a formal definition of this property (it would be a little technical and rather confusing) and explain the extended concept by means of an example:

A professor is looking for an undergraduate research assistant and a number of students have applied for the job. After a first screening, two of them remain on the short list. The relevant objectives are grade point average (GPA), length of study, knowledge of foreign languages, computer skills, and maximum workload in hours per week. An assistant is asked to support the preparation of the final decision by summarizing the relevant attribute levels for both candidates. He constrains his summary to the two attributes of grade point average (3.5 vs 3.0) and maximum workload (3 hrs/w vs. 8 hrs/w) and justifies his approach with the argument that "the candidates do not differ on the other attributes anyway". However, in doing so he has implicitly assumed that the attributes GPA and maximum workload are preferentially independent of the other attributes and that the explicit levels of the other attributes do not even need to be mentioned. Whether this is a sensible assumption is very doubtful. It may well be that the difference in maximum workload is much more important (relative to the GPA) if both candidates have extraordinary computer skills, while it is less relevant if both candidates are rather weak in this respect.

We want to stress once more the fundamental difference between preferential independence and statistical independence. Preferential independence has nothing to do with the empirical observation that students with better computer skills have also better GPAs and less time available, for instance (the authors have not checked whether this is indeed an empirical fact; it is only an example), it is instead concerned with the feature that the decision maker has a preference with respect to different GPA/workload combinations that is not influenced by other attributes like computer skills.

A particularly comfortable decision situation is given if the above defined condition of preferential independence is given for any subset of the attribute set; we will call such a scenario "mutual preferential independence".

Definition 6.2 (Mutual preferential independence)
The attributes $X_1, ..., X_m$ are *mutually preferentially independent* if every subset of these attributes is preferentially independent of its complementary set of attributes (R. L. Keeney and Raiffa 1976, p. 111).

Let us look at an example: if you want to buy a new car and consider the four criteria (attributes) "engine power", "trunk size", "consumption", and "price" for your evaluation, the condition implies that the subset {engine power, trunk size, consumption} should be preferentially independent of the complementary set {price}. At the same time, the subset {engine power, price} has to be preferentially independent of the complementary set {trunk size, consumption} and so on (for each possible subset of attributes).

If mutual preferential independence is satisfied, the preference can be represented by an additive multiattribute value function. However, mutual preferential independence is only sufficient for non-measurable value functions. These are value functions that are able to arrange the alternatives in the right order, but whose values cannot be used to measure value differences ("strength of preference"). Since we would like to obtain measurable value functions (which means that we want to be able to interpret value differences), preferences have to meet even stricter requirements.

Definition 6.3 (Difference independence)
Let

$$a = (a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_m)$$
$$b = (a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_m)$$

be two alternatives that only differ in the i-th attribute, and

$$a' = (a'_1, \ldots, a'_{i-1}, a_i, a'_{i+1}, \ldots, a'_m)$$
$$b' = (a'_1, \ldots, a'_{i-1}, b_i, a'_{i+1}, \ldots, a'_m)$$

be two more alternatives that also only differ in the i-th attribute, but for this feature the same levels as a and b. The attribute X_i is then called *difference independent* of the remaining attributes if for any a, b, a', b' defined as above it holds that

$$(a \to b) \sim (a' \to b').$$

Let us look at an example: the attribute "maximum speed" is difference independent of the attributes "price" and "consumption", if the additional value that you attach to a particular increase in maximum speed is independent of the vehicle being a €50,000 car with an average consumption of 20 liters/100km or a €20,000 car with an average fuel consumption of 10 liters/100km.

An additive measurable value function requires that additive difference independence holds for every attribute; it is easy to verify that this also implies mutual preferential independence.

Hence, our earlier statement that "preferential independence" is the necessary condition for the additive model was very sloppy. More exactly, we have to require mutual preferential independence and, if we ask for a measurable value function (as we usually do), we also need difference independence.

If it is impossible for a decision maker to define the attribute value function of an attribute without knowing the level of another attribute, preferential independence is obviously not given. Let us assume that, in a decision between two jobs, both the annual salary and the annual number of days off are important besides other attributes. The decision maker now wants to construct his attribute value function for the annual number of days off and makes use of the bisection method. He hence has to determine the number of days off that is right in the middle of 20 and 40 vacation days from a value perspective. He should now ask himself: is it possible to determine this without knowing my salary? If I have a very low salary (€35,000 gross), I have to spend a large part of my net income on daily life goods so there will not remain much money for holidays. Therefore, holiday time exceeding three weeks is of very little value. Even though I can read or take long walks, I will probably soon miss the stimulating atmosphere at work. The midpoint level from a value perspective might be about 24 days. However, if my salary is close to the upper limit of the considered alternatives (€60,000 gross), additional holiday time is of much greater value (luxurious long-distance journeys). The midpoint level from a value perspective might then be about 29 days.

The difference independence condition will tend to be fulfilled (approximately) more easily if the ranges between the upper and lower limits of the respective attributes are small. If the minimum salary is €50,000 and the maximum salary is €60,000, it can be expected that the value function over the days off is less dependent on the salary in comparison to a range from €30,000 to €60,000.

Figure 6-2 shows two *non-additive* value functions over two attributes X and Y. In the left example, the increase in value when switching from x^- to x^+ is quite modest at a low level of Y. The higher Y becomes, the more value appreciation is produced by the increase in the level of X; there is a complementary relationship between the attributes. Consider a medical decision and let X denominate the life duration and Y the quality of life. If we (plausibly) assume that the decision maker values an additional year of life more if he enjoys a higher quality of life, we have complementarity. The reverse case is displayed in the right part of Figure 6-2: Here, we observe a substitute relationship between the attributes. The higher the level of one attribute, the less additional value is produced by an improvement of the other attribute.

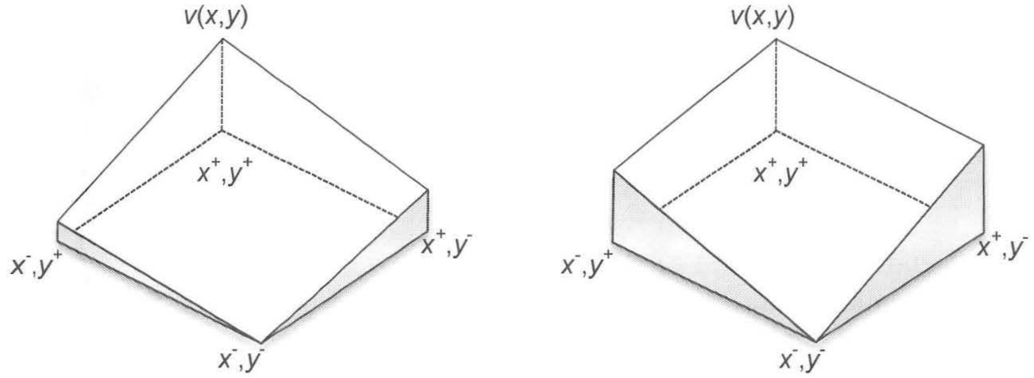

Figure 6-2: Non-additive value models with two attributes

If the conditions for the additive model are not fulfilled, you should – as already mentioned in Chapter 3 – try to "produce" independence via a different, better formulation of the objectives. For example, you could split the attribute "number of days off" into three sub-attributes:

- Number of days off for long-distance journeys,
- Number of days off for trips within Europe,
- Number of days off spent at home.

For each of these attributes, you could generate a value function. In addition, you would have to decide how you plan to assign your days off to the three categories, depending on your annual salary.

Another example: a manager who is in charge of a profit center tries to plan the development of his unit over the next five years. For him, not only the expected total profit over this time span is important but also the development of the profits over time. He prefers a continuous upward trend to a volatile up-and-down movement that comes, in the worst case, with a sharp decline in the final year. If the profit in each of the five years was introduced as a separate attribute, there would be no difference independence between them. However, it might be possible to model the manager's preferences through the two attributes, say "total profit over five years" and "average annual profit growth", which fulfill the independence condition.

Since an additive value model strongly simplifies the decision calculus, it is worthwhile to carefully think about possibilities to redefine attributes in order to eliminate existing dependencies. Von Winterfeldt and Edwards, who are not only scientists but also experienced decision analysis practitioners, are convinced that this should be possible in virtually every case of application ({{324 von Winterfeldt,Detlof. 1986/y;}},1986, p. 309).

6.4 Determination of the weights

6.4.1 The attribute value functions in the example "Choosing a job"

From now on, we will always assume that preferences can be properly modeled by a measurable, additive value function. There are a number of possible ways to determine the attribute weights w_r. We will outline three of these approaches and illustrate them by means of examples. In order to do so, we extend the above-mentioned job choice problem and introduce a third objective, which we will call "career perspectives". The respective attribute can assume the levels "excellent", "good", and "bad". We do not want to go into further detail regarding the measurement and interpretation of these terms. However, it is important to note that the decision maker himself has to have a clear understanding of their meaning; otherwise he will not be able to determine a proper attribute value function and an appropriate weighting of this attribute.

Table 6-4 displays the extended alternatives. The value function of the decision maker for the starting salary, v_1, is assumed to have a shape as presented in Figure 6-3. The elicited points were approximated by the continuous function

$$v_1(x_1) = 1.225 \cdot (1 - e^{-1.695(x_1 - 30{,}000)/(80{,}000-30{,}000)}).$$

Table 6-4: The three available jobs with three attributes

Alternative	Salary	Working hours	Career perspectives
(*a*) Consultancy	€80,000	60 hours	good
(*b*) University	€50,000	40 hours	excellent
(*c*) Sailing instructor	€30,000	20 hours	bad

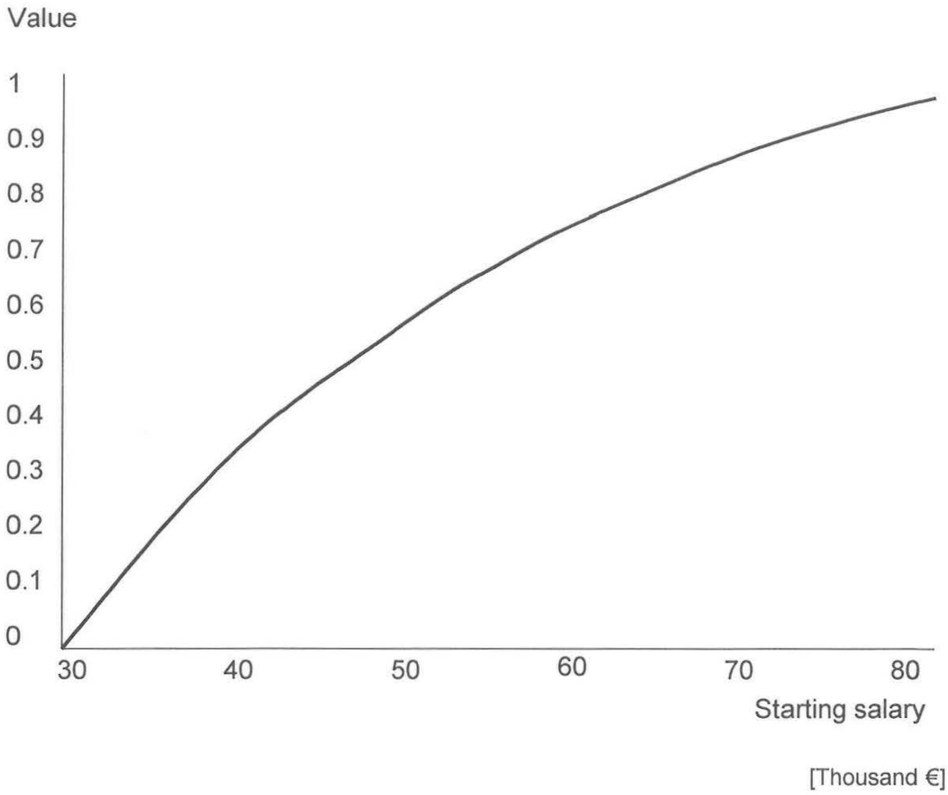

Figure 6-3: Value function for an annual salary in a range between €30,000 and €80,000

For the attribute "working hours", the decision maker is assumed to have a linear value function in a range between 20 and 60 hours; it thus holds that $v_2(x_2) = (60 - x_2) / 40$. For the attribute "career perspectives", the attribute levels are evaluated according to the last column of Table 6-5.

Table 6-5: Individual values of the alternatives in the respective attributes

Alternative	Salary	Working hours	Career perspectives
(a) Consultancy	1	0	0.7
(b) University	0.6	0.5	1
(c) Sailing instructor	0	1	0

6.4.2 Determination of the weights by use of the trade-off method

A *trade-off* is essentially an exchange relationship. Determining the weights according to the trade-off method means that you ask for the exchange relationship of two objective variables for which the decision maker is indifferent. The value functions have to be known. You proceed as follows: you need pairs of alternatives that differ only in two attributes and are considered equally attractive by the decision maker. An indifference statement, for instance, between the alternatives

$$f = (f_1, \ldots, f_{i-1}, f_i, f_{i+1}, \ldots, f_{j-1}, f_j, f_{j+1}, \ldots, f_m)$$

and

$$g = (f_1, \ldots, f_{i-1}, g_i, f_{i+1}, \ldots, f_{j-1}, g_j, f_{j+1}, \ldots, f_m)$$

can be used to conclude how heavily the decision maker weights the attributes X_i and X_j. Due to the additivity, the equation $v(f) = v(g)$ can be reduced to

$$w_i v_i(f_i) + w_j v_j(f_j) = w_i v_i(g_i) + w_j v_j(g_j).$$

A set of $m - 1$ of such equations in combination with the normalization condition $\Sigma w_r = 1$ yields a system of m equations in m variables. This system has a unique solution if there are no redundancies between the equations. Redundancies occur if you can derive an equation of the system from two or more other equations within the system.

Let us begin by comparing the attributes "annual salary" and "working hours". Since it is often very difficult for a decision maker to immediately quantify precisely an exchange relationship, you would prefer to draw nearer step-by-step, starting with the extreme values. At first, the decision maker is asked if he prefers an alternative f with the characteristics

$$f = (€80{,}000, 60 \text{ hours}, *)$$

which is the combination of highest possible salary and worst outcome in terms of working hours, to an alternative g with

$$g = (€30{,}000, 20 \text{ hours}, *)$$

which is the combination with the worst possible salary and the best working hours. The attribute level for career perspectives is omitted here, because it is irrelevant for the preference determination as long as the level is the same for both alternatives, due to the assumed preferential independence; that is why an asterisk has been put in the third place. It is sufficient for the decision maker to focus solely on the first two attributes when determining the *trade-off*.

If the decision maker states, for instance, that he prefers alternative f to g, then the high salary of €80,000 can be lowered to €60,000, whereupon the decision maker has to decide between f' and g with

$$f' = (€60{,}000, 60 \text{ hours}, *)$$
$$g = (€30{,}000, 20 \text{ hours}, *)$$

If he still prefers f', the salary of €60,000 can be reduced further. If his preference reverses to g, the salary of €60,000 will be increased again. This process is repeated a couple of times until a sum of €x with $f'' = (€x, 60 \text{ hours}, *)$ is found that makes f'' and g equally attractive. Let us assume €55,000 is the amount we are looking for. Hence it holds

$$(€55{,}000, 60 \text{ hours}, *) \sim (€30{,}000, 20 \text{ hours}, *)$$

meaning that the decision maker is willing to trade an additional salary of €25,000 (on top of the €30,000) against 40 hours of weekly working time (on top of the 20 hours); this is his trade-off.

It holds that

$$w_1v_1(€55{,}000) + w_2v_2(60 \text{ hours}) + w_3v_3(*) =$$
$$w_1v_1(€30{,}000) + w_2v_2(20 \text{ hours}) + w_3v_3(*).$$

Since * is equal in both cases, the term can be canceled out on both sides. Solving for w_1 yields

$$w_1 = \frac{v_2(20) - v_2(60)}{v_1(55{,}000) - v_1(30{,}000)} \cdot w_2 = \frac{1}{0.7} \cdot w_2 = 1.429 \cdot w_2. \tag{6.4}$$

The approximate value for the salary $v_1(55{,}000) = 0.7$ can be taken from Figure 6-3. The exact value is derived by inserting the argument into the given exponential function.

We now compare the attributes “career perspectives” and “annual salary”. Let us assume the decision maker arrives (after an appropriate convergence procedure) at the following equally attractive alternatives:

$$(€70{,}000, *, \text{bad}) \sim (€30{,}000, *, \text{excellent})$$

This would lead us to the following equation:

$$w_1 = \frac{v_3(\text{excellent}) - v_3(\text{bad})}{v_1(70{,}000) - v_1(30{,}000)} \cdot w_3 = \frac{1}{0.91} \cdot w_3 = 1.099 \cdot w_3. \tag{6.5}$$

Adding the normalization condition $\Sigma w_r = 1$, we can easily derive the respective w_r from the equations. We find $w_1 = 0.38$, $w_2 = 0.27$, and $w_3 = 0.35$.

The decision maker can use these weighting factors to obtain a ranking of his preferred alternatives:

$$v(a) = 0.38 \cdot 1 + 0.27 \cdot 0 + 0.35 \cdot 0.7 = 0.63$$
$$v(b) = 0.38 \cdot 0.6 + 0.27 \cdot 0.5 + 0.35 \cdot 1 = 0.71$$
$$v(c) = 0.38 \cdot 0 + 0.27 \cdot 1 + 0.35 \cdot 0 = 0.27.$$

As we can see, the occupation as a teaching assistant at the university is the optimal alternative, whereas the job as a sailing instructor ends up far behind the two other alternatives.

One potential problem with the determination of trade-offs can be caused by attributes that assume only a few possible levels. There is no continuous value function. This can be problematic for the reason that the level of a particular attribute can only be adjusted discontinuously. For example, the attribute “location” could play an important role in choosing a job. The head office of the consultancy is in Frankfurt, the university is located in Paderborn, and the sailing academy in Kiel. The decision maker has assigned a value of 1 to his preferred location (e.g. Kiel), a value of 0 to his least preferred location (e.g. Paderborn), and an intermediate value to the third location (Frankfurt). When comparing the location with a continuous attribute like the starting salary, it is no problem to find equally attractive alternatives because it is possible to vary the level of the continuous attribute. If, for instance, the decision maker states (Paderborn, €80,000) $\succ$ (Kiel, €30,000), then you can increase the salary that is associated with the location Kiel until the decision maker is indifferent between the alternatives. However, you should avoid a comparison of two discrete attributes because it would only be by mere chance that you would find two alternatives with the same attractiveness.

Moreover, it may happen that there is such a large preference jump between two adjacent levels of the discrete attribute that it cannot be compensated for by any adjustment of the continuous attribute within its predefined range. In our case, e.g., the number of annual holidays (as another attribute) may play a role as well. The number of holidays has a very narrow range of variation; let us say 20 to 24 days. Beginning with a combination (Kiel, 20 days), it might not be possible to compensate the transition to a less preferred location by increasing the number of holidays. Hence, there are no two combinations within the given range of alternatives that are equally attractive. However, since we are hardly restricted in our choice of picking attribute pairs to construct the trade-offs (only keeping the problem of redundancies in mind), we should not encounter a problem here.

In our sample calculations, trade-offs were always determined on the basis of alternatives that featured only minimal or maximal levels of an attribute. It is problematic that the decision maker is thereby confronted with rather unrealistic alternatives. He has to be imaginative and will have a harder time making reliable indifference statements.

You should therefore try to use realistic comparisons when identifying trade-offs. In a comparison between two different combinations of salary and working hours, the decision maker may well be able to make the following statement:

$$(€80{,}000, 60\text{ hours}, *) \sim (€65{,}000, 52\text{ hours}, *)$$

The left alternative is equivalent to the alternative "consultancy" in both considered attributes. It should be fairly easy for the decision maker to imagine this combination. From the indifference statement we can derive the following relation:

$$w_1 = \frac{v_2(52) - v_2(60)}{v_1(80{,}000) - v_1(65{,}000)} \cdot w_2 = \frac{0.2}{0.15} \cdot w_2 = 1.333 \cdot w_2.$$

You can see that this weight relation does not exactly match the relation ($w_1 = 1.429\ w_2$) that was determined before. However, if the additive model was valid (as we presume) and if the attribute value functions were properly elicited beforehand, we should have obtained the same ratio in both cases. Differences of such kind are inevitable due to people's limited cognitive abilities. For that reason, it is sensible not to limit ourselves to the determination of $m - 1$ trade-offs to generate a consistent system of equations but to deliberately create more than the minimal required number of trade-off measurements in order to check for consistency of the statements. In Section 6.5, we will investigate ways and means of dealing with the unavoidable fuzziness of preference measurements.

6.4.3 Determination of the weights by use of the swing method

Unlike the trade-off method, the swing method does not require that the attribute value functions are known. The decision maker imagines that he has at hand the worst defined alternative

$$a^- = (x_1^-, x_2^-, \ldots, x_m^-).$$

Assume that he has the choice of increasing one of the attributes to its highest level, leaving all other attributes at their lowest levels. He ranks the attributes according to his preference for increasing their level to the maximum. Let us call these artificial alternatives

$$b^r = (x_1^-, \ldots, x_{r-1}^-, x_r^+, x_{r+1}^-, \ldots, x_m^-)$$

with $r = 1, \ldots, m$ and remind ourselves that $v(b^r) = w_r$.

Through the ranking of the b^r, we have already structured the weights in descending order; it now remains to quantify them. Let us assign a value of zero to a^- and an arbitrary value of 100 points to the most preferred b^r. Subsequently, the decision maker assigns points to the remaining b^r in such a way that the value differences between them are reflected. The final step is the normalization of the weights to 1. Let t_i be the number of points, it then holds that

$$w_r = \frac{t_r}{\sum_{i=1}^{m} t_i} \tag{6.6}$$

In summary, the procedure works as follows:

1. Specification of a ranking of the artificial alternatives b^r,
2. Base allocation of points: 0 for a^-, 100 for the best b^r,
3. Evaluation of the remaining b^r in such a way that the value differences between them are reflected,
4. Determination of the attribute weights by normalization of the evaluations.

The approach can be clarified by means of our example. At first, the decision maker considers the extreme alternative

$$a^- = (€30{,}000, 60\text{ hours, bad}).$$

He is then asked for which attribute a change from the lowest to the highest level would be most attractive for him. He hence has to decide which of the alternatives

$$b^1 = (€80{,}000, 60\text{ hours, bad})$$
$$b^2 = (€30{,}000, 20\text{ hours, bad})$$
$$b^3 = (€30{,}000, 60\text{ hours, excellent})$$

he prefers. Let us assume he chooses b^1 as his preferred alternative, followed by b^3 and finally b^2.

Let us further assume that the decision maker arrives at an evaluation as depicted in Table 6-6.

Table 6-6: Evaluation of the three artificial alternatives b^r

Rank	Alternative b^r	Points
1	b^1	100
2	b^3	70
3	b^2	60

We then derive the weights:

$$w_1 = 100 / (100 + 70 + 60) = 0.44$$
$$w_2 = 60 / (100 + 70 + 60) = 0.26$$
$$w_3 = 70 / (100 + 70 + 60) = 0.30.$$

The calculation of the weights from the preference statements by the swing method is fairly easy. However, it should be noted that – in the same spirit as in the comparison of the value function elicitation methods in Section 5.2.5 –the allegedly simpler swing method actually requires much more cognitive work on part of the decision maker than does the trade-off method. While the trade-off method asks for a large number of very simple preference statements (i.e., "I like this combination better than that one"), the "point assignment by preference strength" (step 3) of the swing method is a very complex valuation problem that additionally leaves a large margin of interpretation (what exactly does "assigning points by preference strength" even mean?). Therefore, the major advantage of the swing method should not be seen in the property that the weights are generated in such an easy way (with respect to quality and reliability of the elicited weights this should actually be seen as detrimental) but rather in the fact that the detailed shapes of the attribute value functions are not needed in the process (as only extreme levels are compared).

6.4.4 Determination of the weights by use of the direct-ratio method

In practice, the direct-ratio method is widely used (even though it is often referred to by another name). Strictly speaking, it should not even be presented in a textbook because it is quite unreliable. However, we must outline this method for two reasons: first, it is very widely used in practice and second, we want to clarify its logical defect.

When using this method, you first have to rank the attributes according to their "importance". This does not cause too much trouble for most people, although a question concerning the importance of an attribute per se is pointless. The objective variable itself cannot be important, only the difference between its levels. For example, it is pointless to say that salary is more important than holidays. The following statement would be more sensible: an increase of annual salary from €50,000 to €53,000 is more important to me than an increase of annual holiday time from 25 to 30 days. We will come back to this issue in Section 6.6.

Let us assume that you still have the feeling that the annual salary is more important to you than your career perspectives and that your career perspectives are more important than the working hours. You now compare two attributes at a time; let us start with the less important ones. The question is: "how much more important to you are your career perspectives than the working hours?" Your answer might be "just a bit". This answer is of little use and you are therefore asked to attach a concrete figure to it. The wording of the question is now as follows: "If the level of importance that is attached to the attribute "working hours" equals 1, how important is the attribute "career perspectives"?" Your answer might be "1.2". You proceed accordingly when comparing the attributes "annual salary" and "working hours". "If the level of importance that is attached to the attribute "working hours" equals 1, how important is the attribute "annual salary"?" Your answer: 2.

From these statements you can derive objective weights. It holds that

$$w_1 / w_2 = 2$$

and

$$w_3 / w_2 = 1.2.$$

It follows for the weights that

$$\begin{aligned} w_1 &= 2/(1+1.2+2) = 0.48 \\ w_2 &= 1/(1+1.2+2) = 0.24 \\ w_3 &= 1{,}2/(1+1.2+2) = 0.29. \end{aligned}$$

Obviously, it is again advisable to check for consistency by comparing the importance of the attributes salary and career perspectives. In this comparison, the decision maker would be consistent if he quoted a ratio of $w_1/w_3 = 2/1.2 = 1.7$.

6.4.5 Application of multiple methods and alternative procedures

As has been repeatedly emphasized, it is advisable to attempt to validate the statements that lead to the determination of weights. This could happen within a single method, but multiple methods could be used as well. For example, you could identify the weights using the trade-off method and test these weights with the swing method, or vice versa.

An alternative method that has some appeal for practical applications of multiattribute decision making was developed by Hammond et al. (1998). It is called the *even-swap* method. The procedure differs fundamentally from the methods discussed so far as it does not determine attribute value functions and attribute weights in isolation; however, the even-swap method builds on the same conceptual ideas as the trade-off method, in particular on the assumption that mutual preferential independence allows the comparison of changes on some attribute levels without concern for the levels of the other attributes. More explicitly, the even-swap method at each step derives changes in exactly two attributes (one improving and one worsening) that leave the decision maker's preference unchanged. In contrast to the trade-off method, however, these even swaps are not used to derive attribute weights but to transform a given alternative into an equally attractive alternative. This is repeated again and again with other even swaps until obvious dominance relations can be spotted (see the sample case at the end of Chapter 6). In this process, the (hypothetical) alternatives that are successively generated by the even swaps are only considered for easy comparison. After the optimal alternative in the set of modified alternatives has been identified, we carry over the preference ordering to the original alternatives.

The even-swap method has the advantage that it does not need terms like attribute value functions and attribute weights even though it is based on exactly these concepts. Furthermore, the method is easy to explain and to understand; in practical applications, this can be an important advantage. And, if supported by appropriate software, even the search for dominance relations can be arranged in a very efficient way (Mustajoki and Hämäläinen 2005). For a deeper analysis of complex problems, the classical approach of determining value functions and attribute weights can hardly be circumvented.

6.5 Incomplete information about the weights

6.5.1 Handling of inconsistent or incomplete information

Usually, the generation of redundant information will lead to inconclusive weights due to inconsistencies in the statements of the decision maker. He should become aware of these inconsistencies and reassess and revise his statements until they are consistent. However, it is also possible that the decision maker is not willing or capable of correcting the inconsistencies in his statements. It is surely true that one should think carefully and invest significant effort into the preparation of an important decision. Nonetheless, there is no point in "helping" people in their decision making through unnerving questions with the result that they actually lose all confidence in their own answers. Individuals will have much greater confidence in the validity of their statements if they are allowed to make statements about value intervals instead of exact point values. For clarification, compare the following two statements:

1. The combination of an annual salary of €100,000 and 25 days off is equal to an annual salary of €90,000 and 35 days off.
2. I prefer the combination of an annual salary of €100,000 and 25 days off to an annual salary of €90,000 and 30 days off, but do not prefer it to an annual salary of €90,000 and 40 days off.