Computer-aided Tools in Engineering ENG-A2001

Exercise/Demo session

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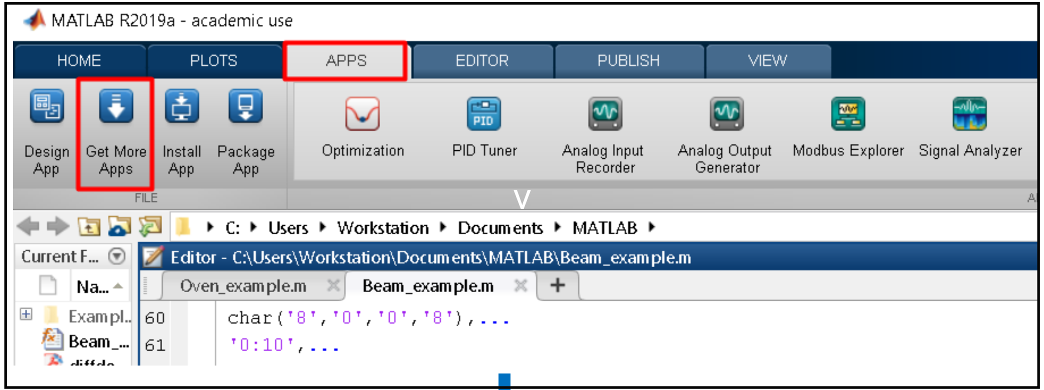
Software setup

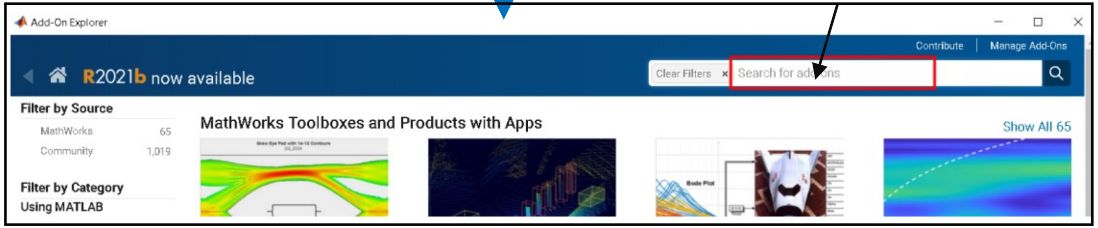
Install Matlab (https://download.aalto.fi/)

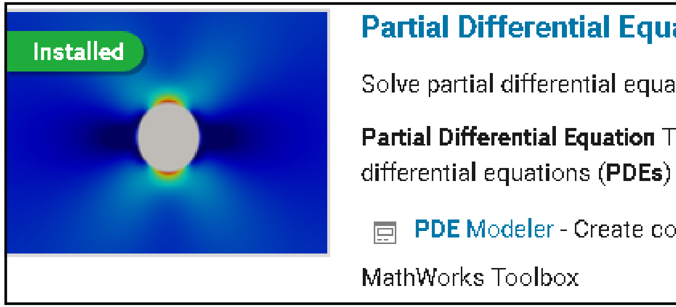
Install add-on PDE-Toolbox (or Modeler) on Matlab

Installing PDE Modeler



PDE





PDE problem forms

- Types of PDEs supported:

Today

Type	Description	Example Applications
Elliptic	$-\nabla \cdot (\mathbf{c} \otimes \nabla u) + \mathbf{a}u = \mathbf{f}$	electrostatic, magnetostatic, heat conduction, piezoelectric
Parabolic	$\mathbf{d}\frac{\partial u}{\partial t} - \nabla \cdot (\mathbf{c} \otimes \nabla u) + \mathbf{a}u = \mathbf{f}$	heat transfer (diffusion), reaction-diffusion
Hyperbolic	$\mathbf{d}\frac{\partial^2 u}{\partial^2 t} - \nabla \cdot (\mathbf{c} \otimes \nabla u) + \mathbf{a}u = \mathbf{f}$	wave, structural dynamics
Eigenvalue	$-\nabla \cdot (\mathbf{c} \otimes \nabla u) + \mathbf{a}u = \lambda \mathbf{d}u$	structural mode shapes

Computer exercise 3.1

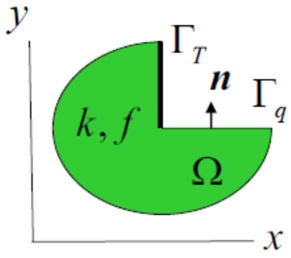
2D heat conduction/diffusion

Strong form. The *partial differential equation* and *boundary conditions* for *stationary isotropic heat diffusion/conduction* read as follows: Find $T = T(x,y)$ such that

$$(1) \quad -\nabla \cdot (k(x,y)\nabla T(x,y)) = f(x,y), \quad (x,y) \in \Omega$$

$$(2a) \quad T(x,y) = T_0(x,y), \quad (x,y) \in \Gamma_T$$

$$(2b) \quad \boldsymbol{q}(x,y) \cdot \boldsymbol{n} = q_0(x,y), \quad (x,y) \in \Gamma_q$$



T temperature (unknown function)

k thermal conductivity (given material data)

f heat supply (given loading data), Ω domain (given geometrical data)

Γ_T boundary part for given temperature (given boundary data)

Γ_q boundary part for given heat flux (given boundary data)

T_0 temperature on the boundary (given essential, Dirichlet, boundary data)

q_0 heat flux on the boundary (given natural, Neumann, boundary data).

Beneath the surface – Weak form

Weak form. Find $T = T(x, y)$ such that it satisfies $T_{|\Gamma_T} = T_0$,

$$\int_\Omega (k\nabla T)\cdot\nabla\hat{T}\,\mathrm{d}A = \int_\Omega f\,\hat{T}\,\mathrm{d}A - \int_{\Gamma_q} q_0\hat{T}\,\mathrm{ds}$$

for all $\hat{T} = \hat{T}(x, y)$ satisfying $\hat{T}_{|\Gamma_T} = 0.$

Computer exercise 3.1 – Matlab PDE Modeler

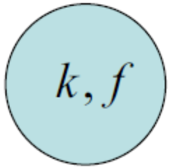
Let us consider isotropic and homogeneous heat diffusion in a circle with radius *R* and the following problem data:

$$-\nabla\cdot(k(x,y)\nabla T(x,y)) = f(x,y),$$

$$k = 0.1\ \mathrm{W/(m\,^\circ C)},\ R = 1\ \mathrm{m}$$

$$f = 10\ \mathrm{W/m^3},\ T_0 = 0\ ^\circ\mathrm{C}$$

$T = T_0 = 0$



(i) Solve the temperature distribution approximately via the finite element method by applying Matlab PDE Modeler with three different mesh sizes.

(ii) Plot the corresponding triangular meshes, temperature distributions (*T*) and heat flux fields (***q***) for each finite element solution.

(iii) List three, perhaps simplified, engineering problems which you can be seen as applications of this model problem.

Hint: Take part in the guided tour in the exercise session.

Computer exercise 3.1 – Matlab PDE Modeler

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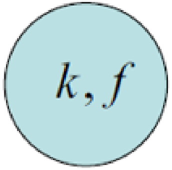


$$-\nabla \cdot (\mathrm{k}(\mathrm{x},\mathrm{y})\nabla T) = f(x,y) = \text{In Matlab} = Q(x,y) + h(x,y)(T_{ext}(x,y) - T)$$

Simplifies in this case: $-\mathrm{k}\nabla \cdot \nabla T = k\frac{\partial^2 T}{\partial x^2} + k\frac{\partial^2 T}{\partial y^2} = f \qquad , \qquad T = T(x,y)$

$k, f = Q$ are constants

+ Boundary conditions: $T = T_0$: at boundary

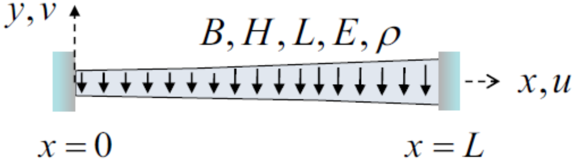
Computer exercise 3.2

Computer exercise 3.2 – Matlab PDE Modeler

Let us consider a 2D/3D *beam bending* problem as a 2D *plane stress elasticity* problem with displacements *u = u(x,y)* and *v = v(x,y)* in the *x*- and *y*-directions, correspondingly, as primary variables (i.e., displacement vector is $\mathbf{u} = (u, v)$) and self-weight as a body load $\mathbf{f} = (f_x, f_y)$:

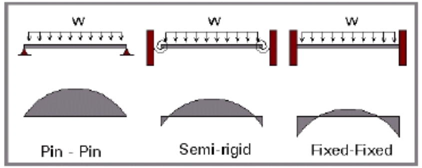
$$B = 4 \text{ cm}, H = 2 \text{ cm}, L = 1 \text{ m}$$

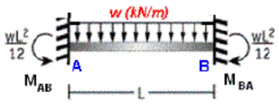
$$E = 200 \text{ GPa}, \rho = 8 \text{ kg/m}^3$$

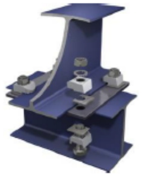


Solve the 2D deformation and stress state of the beam (meaning displacement components *u* and *v* and stress components by utilizing Matlab *PDE Modeler* relying on the finite element method (with triangular elements of linear basis functions).

Hint: Take part in the guided tour in the exercise session.







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Beneath the surface – Strong form

For given loadings $\boldsymbol{b}$ and $\boldsymbol{t}$ and displacement $\boldsymbol{u}_0$, find $\boldsymbol{u}$ such that

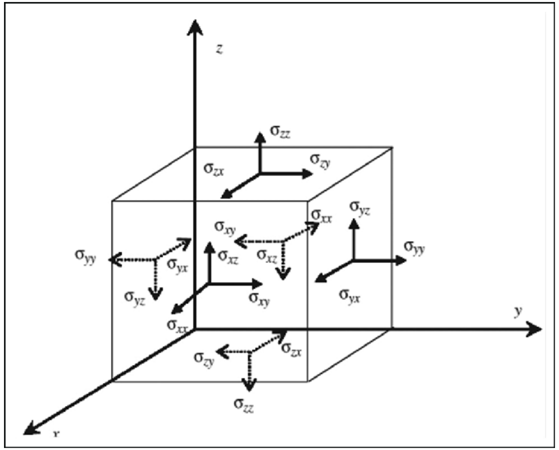
$$-\operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{b} \quad \text{in } \Omega,$$

$$\boldsymbol{u} = \boldsymbol{u}_0 \quad \text{on } S_u \subset \partial\Omega,$$

$$\boldsymbol{\sigma n} = \boldsymbol{t} \quad \text{on } S_t \subset \partial\Omega,$$

$$\varepsilon(\boldsymbol{u}) = \overleftrightarrow{\nabla} \boldsymbol{u} = (\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)/2,$$

$$\begin{aligned} \boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{D}\varepsilon(\boldsymbol{u}) &= 2\mu\, \varepsilon(\boldsymbol{u}) + \lambda \operatorname{tr}(\varepsilon(\boldsymbol{u}))\, \boldsymbol{I} \\ &= 2\mu\, \varepsilon(\boldsymbol{u}) + \lambda \operatorname{div} \boldsymbol{u}\, \boldsymbol{I} \end{aligned}$$



Beneath the surface – Weak form

Weak form of the linear elasticity problem: Let a three-dimensional body be under the loading $\boldsymbol{b} \in [L^2(\Omega)]^3$, $\Omega \subset R^3$. Find $\boldsymbol{u} \in \boldsymbol{U}$ such that

$$a(\boldsymbol{u}, \hat{\boldsymbol{u}}) = l(\hat{\boldsymbol{u}}) \quad \forall \hat{\boldsymbol{u}} \in \boldsymbol{U},$$

with the bilinear form, load functional and variational space

$$a(\boldsymbol{u}, \hat{\boldsymbol{u}}) = \int_\Omega (\boldsymbol{D}\varepsilon(\boldsymbol{u})) : \varepsilon(\hat{\boldsymbol{u}})\, d\Omega,$$

$$l(\hat{\boldsymbol{u}}) = \int_\Omega \boldsymbol{b} \cdot \hat{\boldsymbol{u}}\, d\Omega + \int_{S_t} \boldsymbol{t} \cdot \hat{\boldsymbol{u}}\, dS,$$

$$\boldsymbol{U} = \{\ \boldsymbol{v} \in [H^1(\Omega)]^3 \ \mid \ \boldsymbol{v}_{|S_u} = \boldsymbol{0}\ \}.$$

Remark. The bilinear form can be written in the form

$$a(\boldsymbol{u}, \hat{\boldsymbol{u}}) = \int_\Omega (2\mu\, \varepsilon(\boldsymbol{u}) : \varepsilon(\hat{\boldsymbol{u}}) + \lambda \operatorname{div} \boldsymbol{u} \ \operatorname{div} \hat{\boldsymbol{u}})\, d\Omega,$$

$$\boldsymbol{\tau} : \boldsymbol{\theta} = \sum_{i,j=1}^{n} \tau_{ij}\theta_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} \tau_{ij}\theta_{ij}.$$

Recommendations

- **Describe *shortly* each step**
  - Formulas if available
  - Making the domain = geometry
  - Boundary conditions
  - Mesh size (at least approximation)
  - Etc.
- **Label axes (x,y,z) and titles of figures**