Aalto CS-C3100 Computer Graphics

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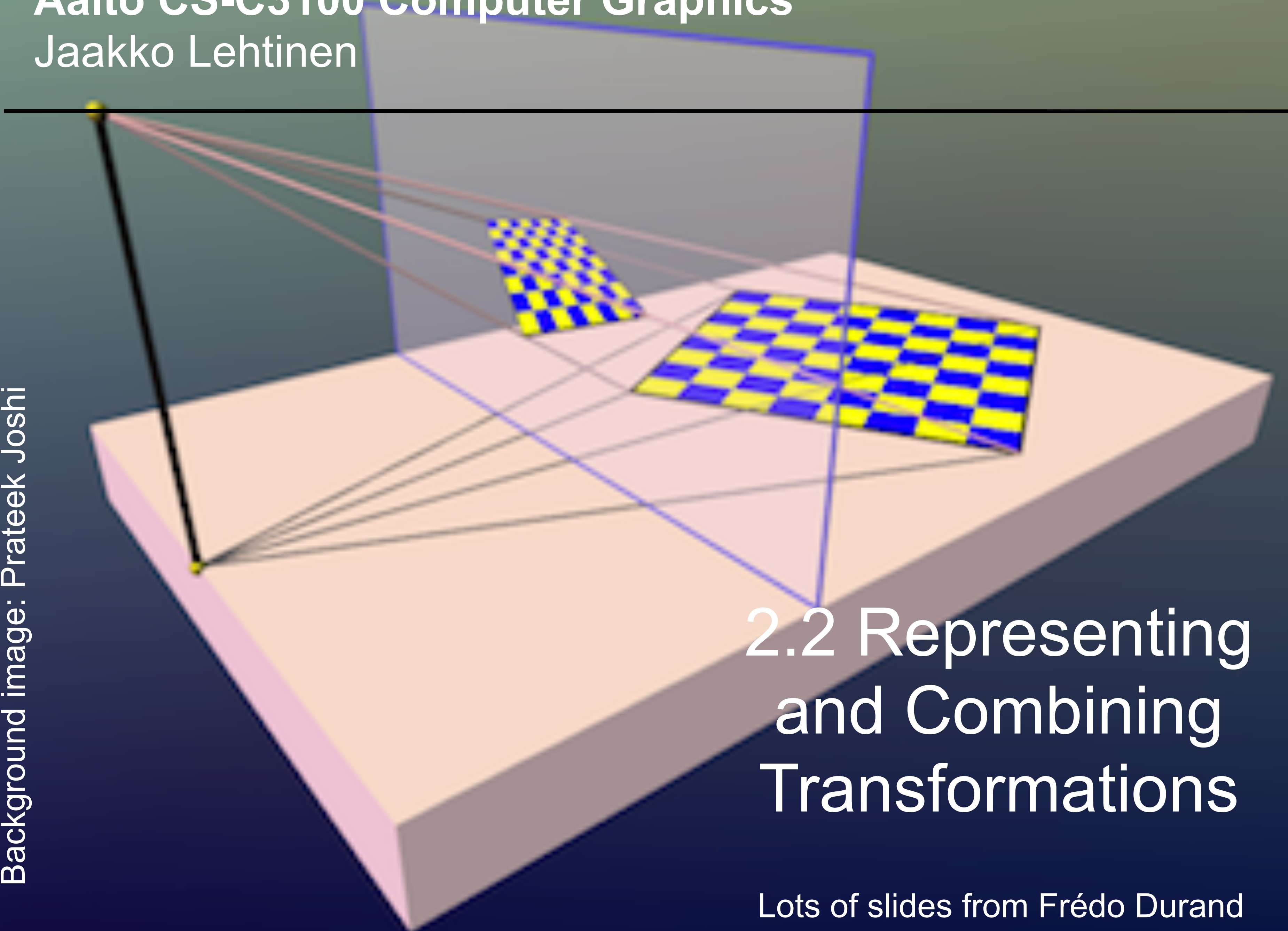


2.2 Representing and Combining Transformations

Lots of slides from Frédo Durand

Background image: Prateek Joshi

In This Video

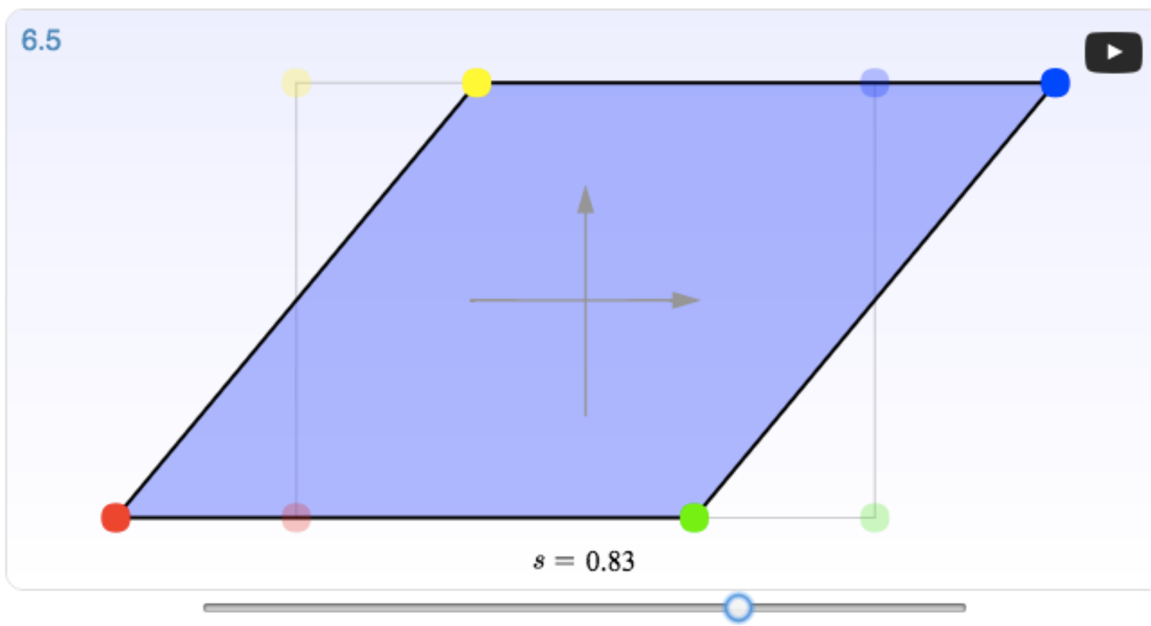
- Representing standard transformations using matrices and vectors
- Combining transformations

Interlude: ImmersiveMath.com

- If your matrices and vectors are a little rusty, and even if they aren't, read the first chapters from this really neat interactive linear algebra "book"!

Interactive Illustration 6.4: The two sliders above control the scaling factors f_x and f_y, in the x- and y-direction, respectively.

The effect of a shear matrix is best seen before it is described in detail, so we recommend that the reader explores Interactive Illustration 6.5 first, and then a formal definition will be provided.



Matrix representations for common transformations

How are Linear Transforms Represented?

Linear transformations include rotation, scaling, reflection, and shear

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$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$p' = M p$$

How are Linear Transforms Represented?

$$x' = ax + by$$
$$y' = dx + ey$$

Linear transformations include rotation, scaling, reflection, and shear

Note that the origin always stays fixed.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$p' = M p$$

How are Affine Transforms Represented?

Affine transformations include all linear transformations, plus translation

$$x' = ax + by + c$$
$$y' = dx + ey + f$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix}$$

$$p' = M p + t$$

Can we use matrices for affine transforms?

$$x' = ax + by + c$$
$$y' = dx + ey + f$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$p' = Mp \text{ ?}$$

The Homogeneous Coordinate Trick

This is what we want:

$$x' = ax + by + c$$
$$y' = dx + ey + f$$

Affine formulation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ f \end{bmatrix}$$

$$p' = M p + t$$

Homogeneous formulation

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

The Homogeneous Coordinate Trick

This is what we want:

$$x' = ax + by + c$$
$$y' = dx + ey + f$$

Affine for

Homogeneous formulation

See the next video for more details on homogeneous coordinates!

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} \; \\ y \end{bmatrix} + \begin{bmatrix} \; \\ f \end{bmatrix}$$

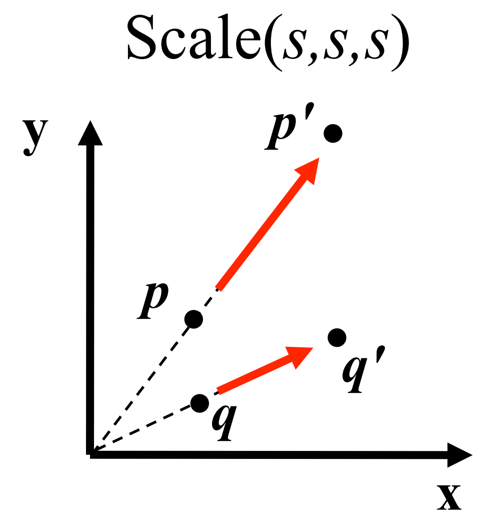
$$\begin{bmatrix} \; \\ \; \\ 1 \end{bmatrix} \quad \begin{bmatrix} & & c \\ & & f \\ & & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$p' = Mp + t$$

$$p' = Mp$$

Scale (s_x, s_y, s_z)

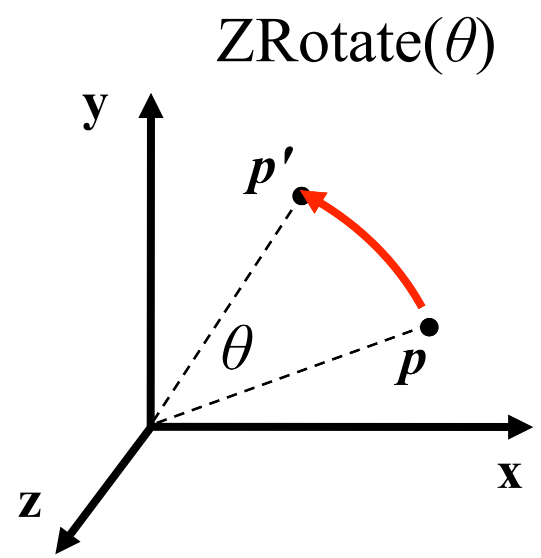
- Isotropic (uniform) scaling: $s_x = s_y = s_z$



$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Rotation

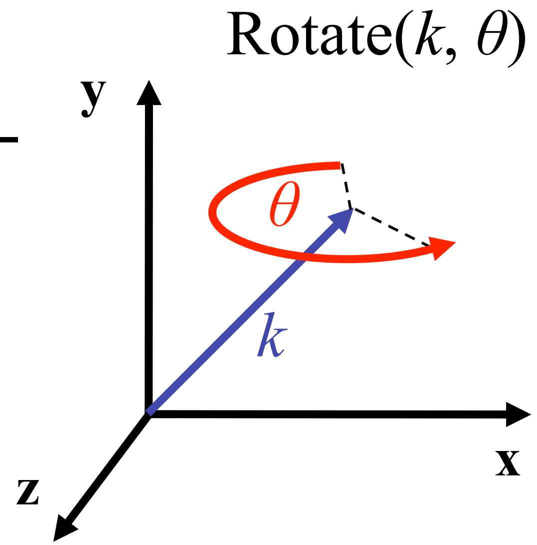
- About z axis



$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Rotation

- About (k_x, k_y, k_z), a unit vector on an arbitrary axis (Rodrigues Formula)



$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} k_x k_x(1-c)+c & k_x k_y(1-c)-k_z s & k_x k_z(1-c)+k_y s & 0 \\ k_y k_x(1-c)+k_z s & k_y k_y(1-c)+c & k_y k_z(1-c)-k_x s & 0 \\ k_z k_x(1-c)-k_y s & k_z k_y(1-c)+k_x s & k_z k_z(1-c)+c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

where $c = \cos\theta$ & $s = \sin\theta$

Rotations and Matrices

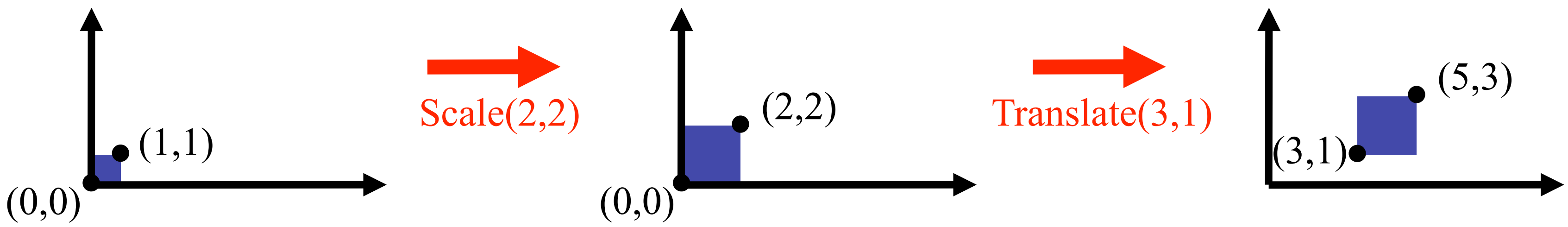
- Rotations are represented by orthonormal matrices, i.e., matrices such that $\mathbf{M}^T\mathbf{M} = \mathbf{I}$
  - This implies $\det(\mathbf{M}) = \pm 1$ (why?)
  - Furthermore, to rule out reflections, require $\det(\mathbf{M})=1$

- Rotations are a group of their own
  - **Can you prove this based on the above properties?**

Combining transformations

How are transforms combined?

Scale then Translate



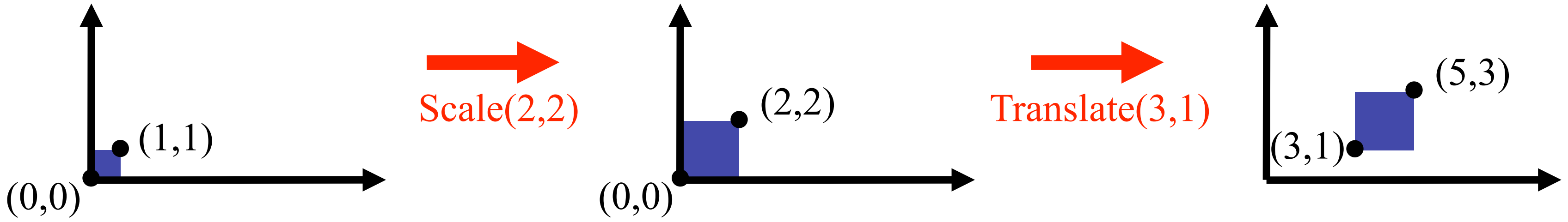
Use matrix multiplication: $p' = T(S\,p) = TS\,p$

$$TS = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

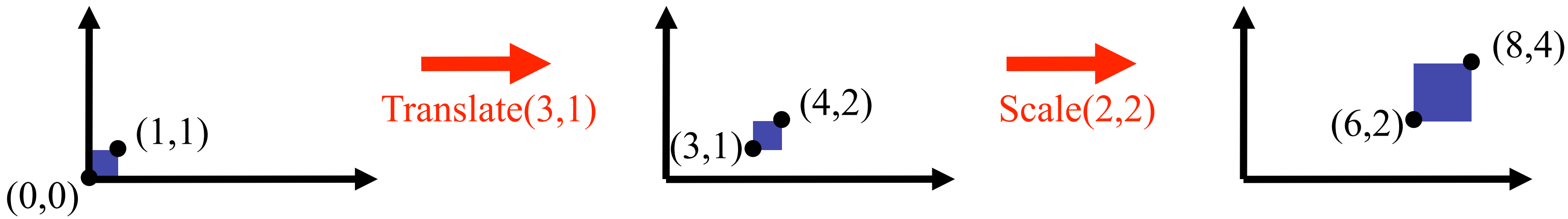
Caution: matrix multiplication is NOT commutative!

Non-commutative Composition

Scale then Translate: $p' = T(Sp) = TSp$



Translate then Scale: $p' = S(Tp) = STp$



Non-commutative Composition

Scale then Translate: $p' = T(S\,p) = TS\,p$

$$TS = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Translate then Scale: $p' = S(T\,p) = ST\,p$

$$ST = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Key Takeaways

- Matrices represent linear transformations
- Adding an extra dimension allow affine transformations to be represented by matrices
- Successive transformations=matrix multiplication
  - Order matters!