CS-C3100 Computer Graphics

Bézier Curves and Splines

3.2 Cubic Bézier Splines

Majority of slides from Frédo Durand

vectorportal.com

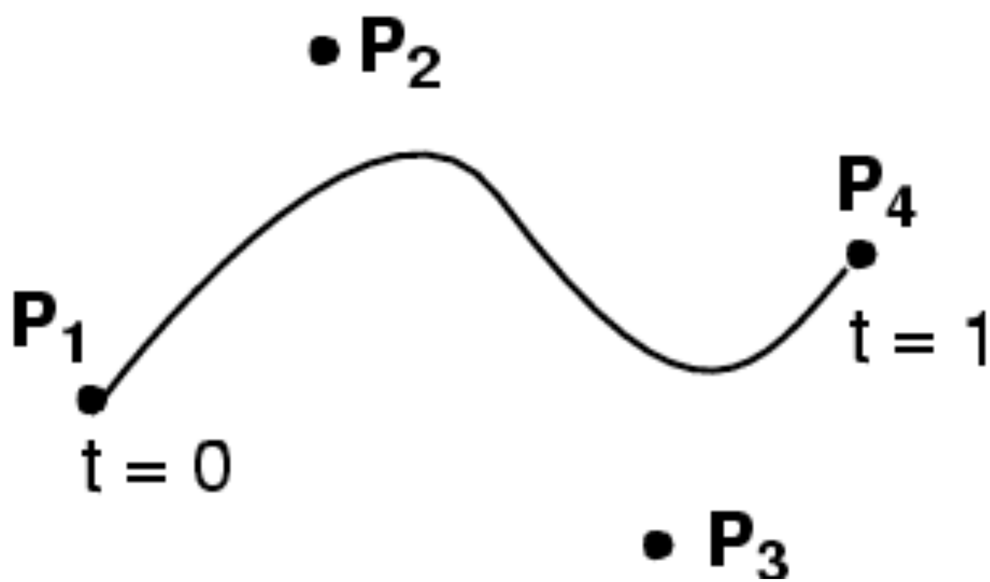
In This Video

- Cubic Bézier Curves: the prototype of a spline
- Manipulating polynomials with matrices
- A general formulation for polynomial splines
  – Not just Bézier: also Catmull-Rom, B-Splines, …

The Cubic Bézier Curve

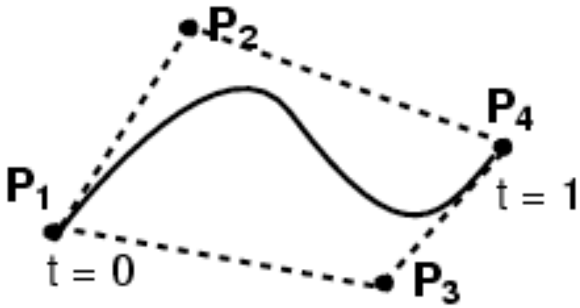
Cubic Bézier Curve

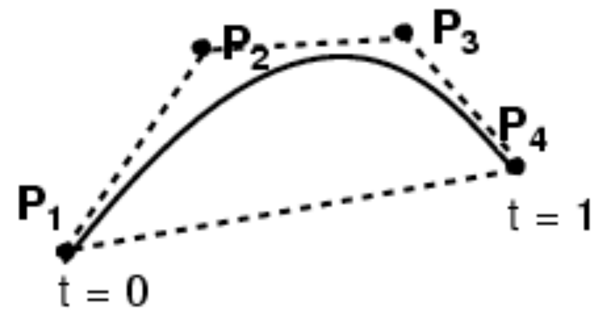
- User specifies 4 control points $\mathbf{P}_1 \ldots \mathbf{P}_4$
- Curve goes through (interpolates) the ends $\mathbf{P}_1$, $\mathbf{P}_4$
- Approximates the two other ones
- Cubic polynomial

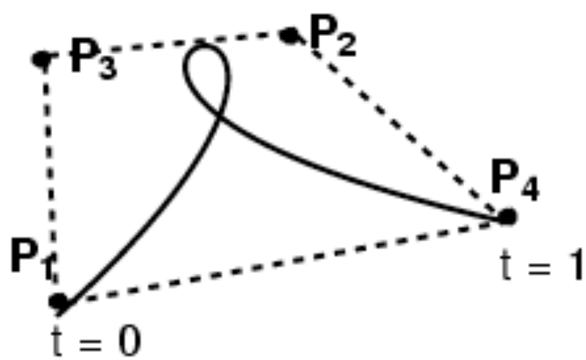


Cubic Bézier Curve

- 4 control points
- Curve passes through first & last control point
- Curve is tangent at $\mathbf{P_1}$ to $(\mathbf{P_1}-\mathbf{P_2})$ and at $\mathbf{P_4}$ to $(\mathbf{P_4}-\mathbf{P_3})$







A Bézier curve is bounded by the **convex hull** of its control points.

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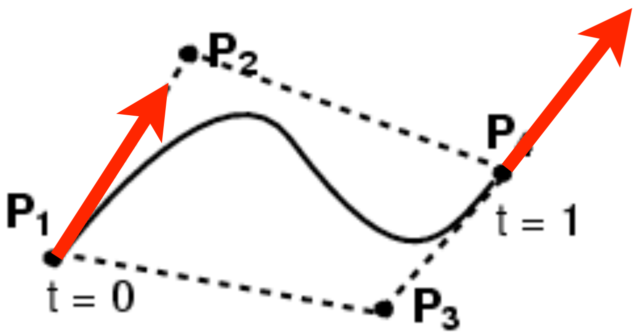
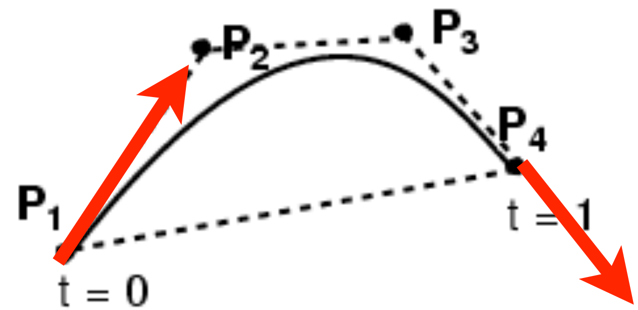
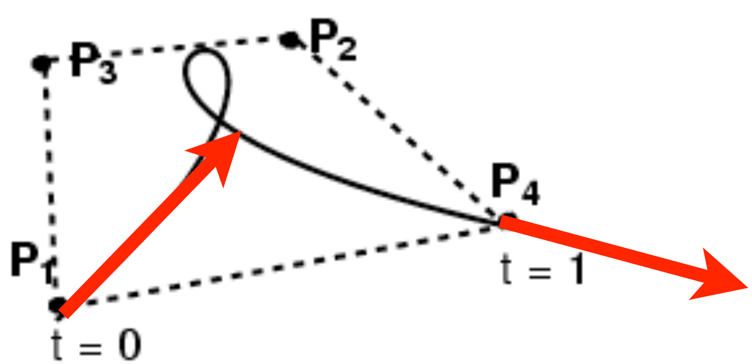


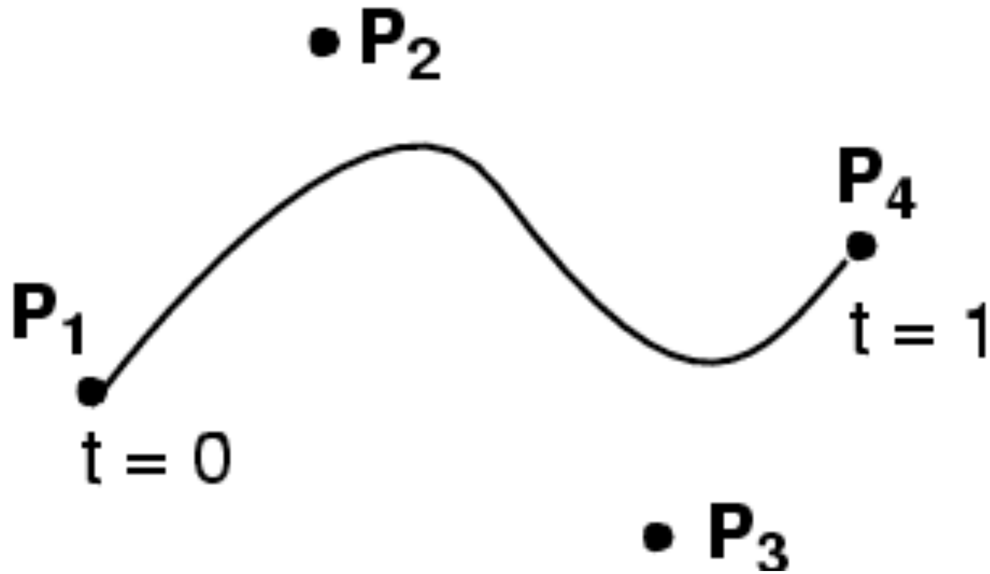




A Bézier curve is bounded by the **convex hull** of its control points.

Cubic Bézier Curve

- $\mathbf{P}(t) = (1-t)^3 \mathbf{P}_1 + 3t(1-t)^2 \mathbf{P}_2 + 3t^2(1-t) \mathbf{P}_3 + t^3 \mathbf{P}_4$



That is,

$$x(t) = (1-t)^3\ x_1 + 3t(1-t)^2\ x_2 + 3t^2(1-t)\ x_3 + t^3\ x_4$$

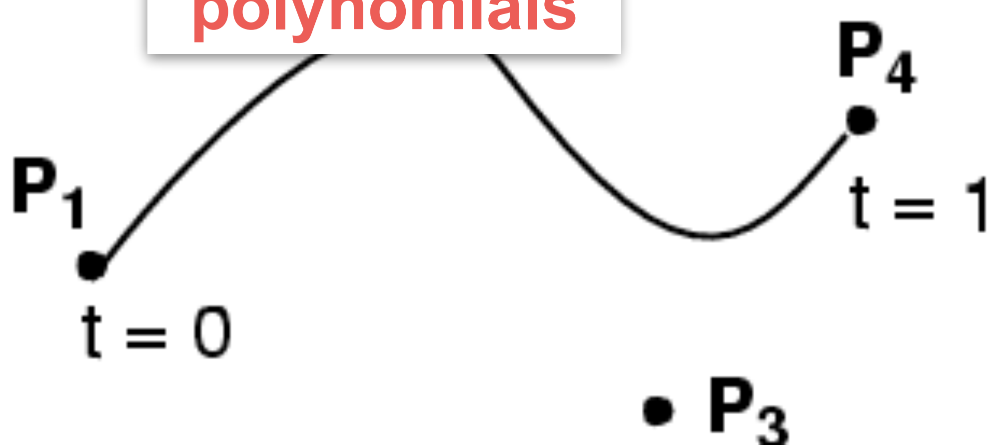
$$y(t) = (1-t)^3\ y_1 + 3t(1-t)^2\ y_2 + 3t^2(1-t)\ y_3 + t^3\ y_4$$

Cubic Bézier Curve

- $\mathbf{P}(t) = (1-t)^3 \; \mathbf{P}_1 + 3t(1-t)^2 \; \mathbf{P}_2 + 3t^2(1-t) \; \mathbf{P}_3 + t^3 \; \mathbf{P}_4$

Vectors with coordinates of control points

Scalar polynomials



That is,

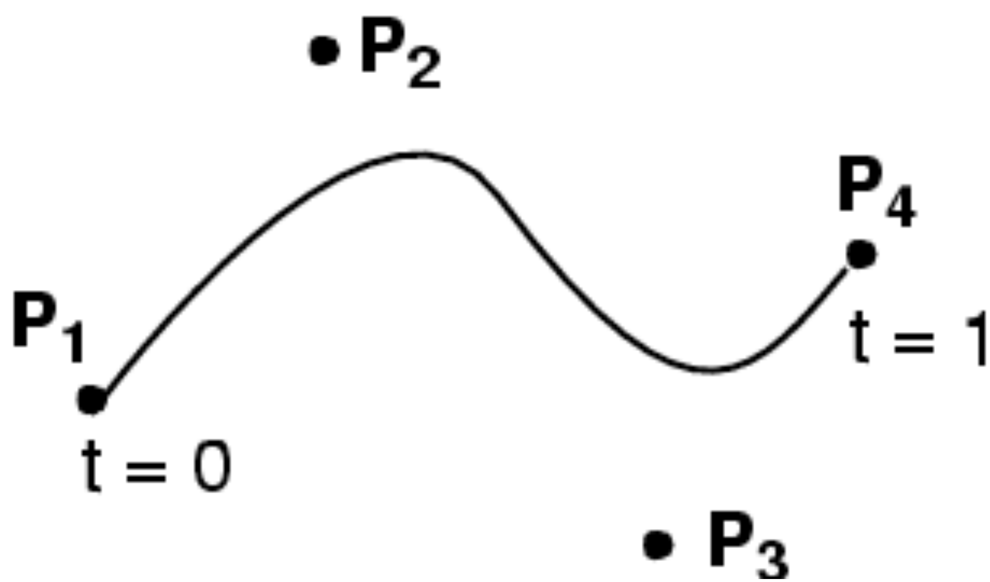
$$(1-t)^3\ x_1 +$$
$$t(1-t)^2\ x_2 +$$
$$3t^2(1-t)\ x_3 +$$
$$t^3\ x_4$$

$$y(t) = (1-t)^3\ y_1 +$$
$$3t(1-t)^2\ y_2 +$$
$$3t^2(1-t)\ y_3 +$$
$$t^3\ y_4$$

Cubic Bézier Curve

- $$\mathbf{P}(t) = (1-t)^3 \mathbf{P}_1 + 3t(1-t)^2 \mathbf{P}_2 + 3t^2(1-t) \mathbf{P}_3 + t^3 \mathbf{P}_4$$

Verify what happens for t=0 and t=1



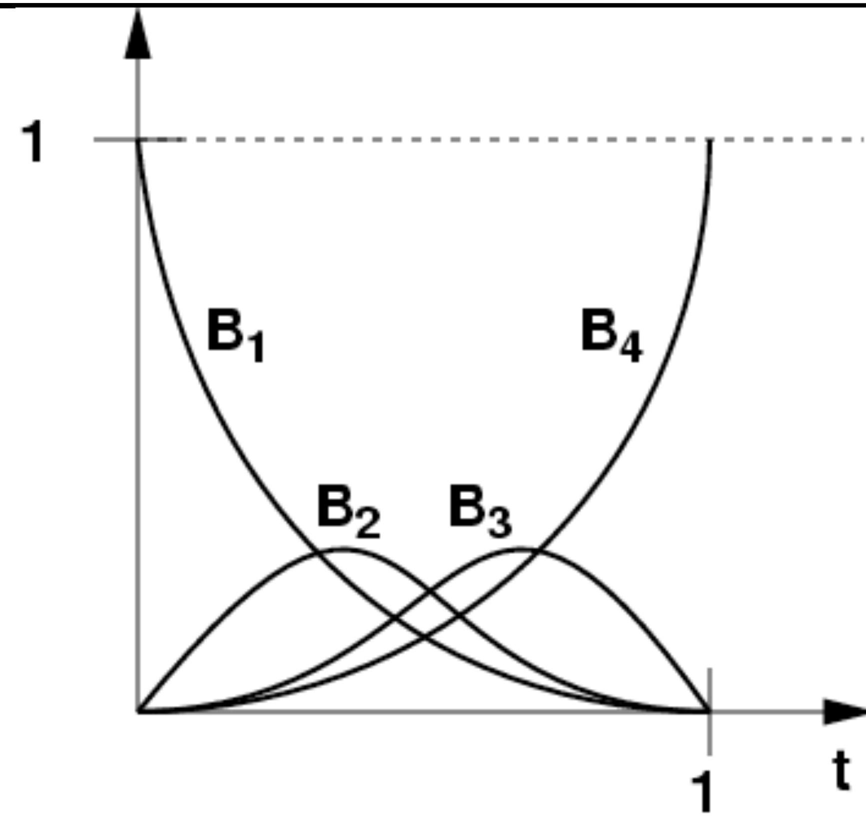
"Bernstein Polynomials"

For cubic:

- $B_1(t)=(1-t)^3$
- $B_2(t)=3t(1-t)^2$
- $B_3(t)=3t^2(1-t)$
- $B_4(t)=t^3$

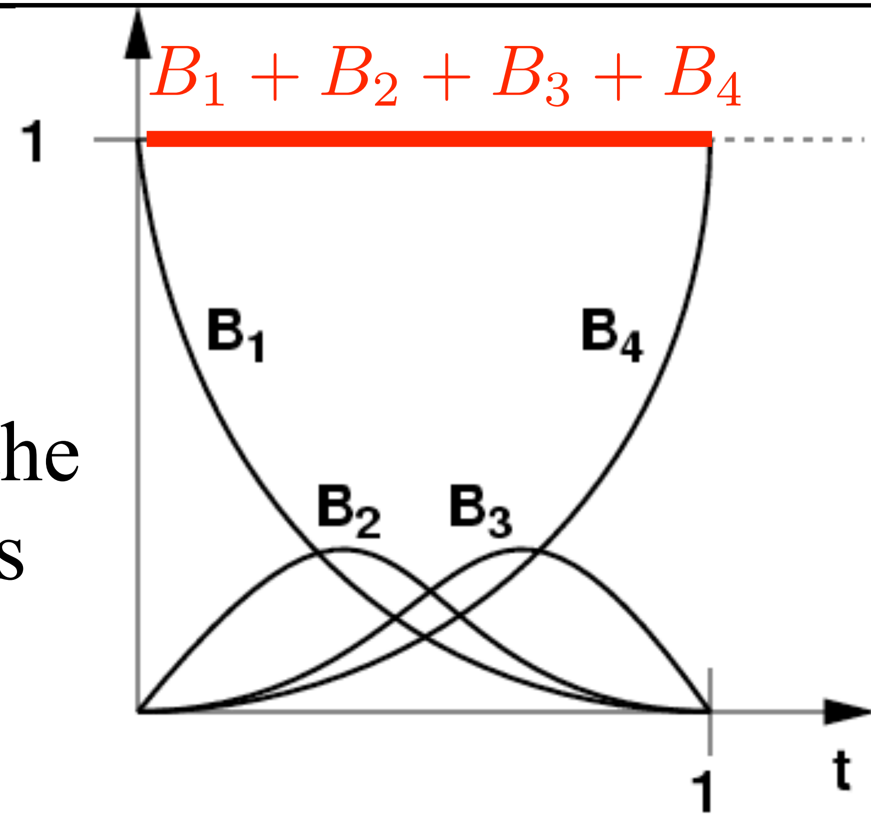
  – (careful with indices, many authors start at 0)

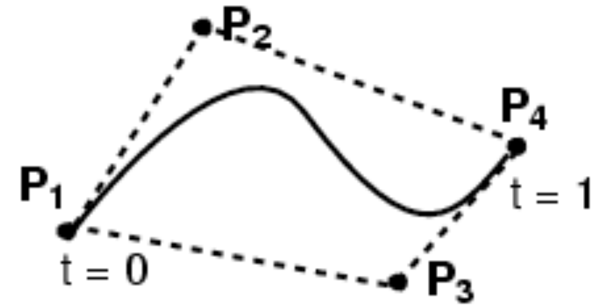
- But defined for any degree



Properties of Bernstein polynomials

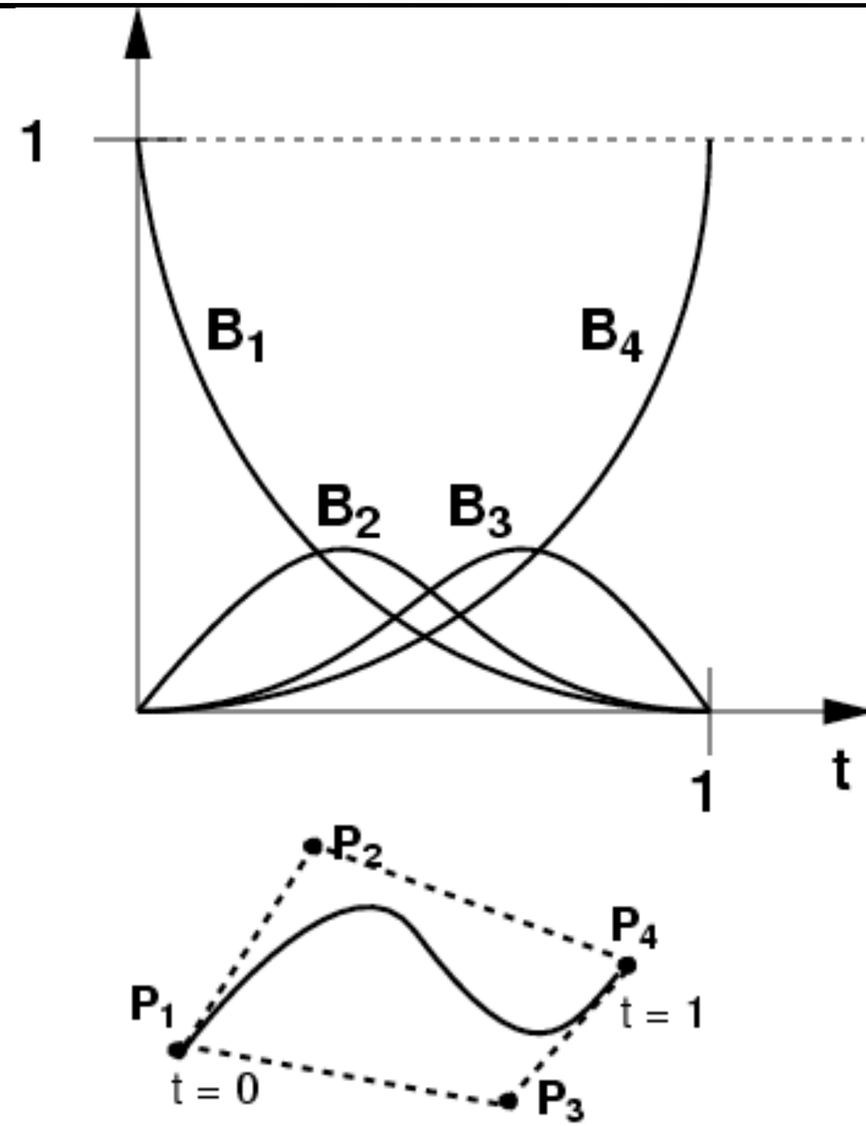
- ≥ 0 for all $0 \leq t \leq 1$
- Sum to 1 for every t
  - called *partition of unity*
- (These two together are the reason why Bézier curves lie within convex hull)
- Only B_1 is non-zero at 0
  - Bezier interpolates P_1
  - Same for B_4 and P_4 for t=1





Interpretation as "Influence"

- Each B_i specifies the influence of P_i
- First, P_1 is the most influential point, then P_2, P_3, and P_4
- P_2 and P_3 never have full influence
  - Not interpolated!



Bézier Curves, Concise Notation

- $\mathbf{P}(t) = \mathbf{P}_1 B_1(t) + \mathbf{P}_2 B_2(t) + \mathbf{P}_3 B_3(t) + \mathbf{P}_4 B_4(t)$
  - $\mathbf{P}_i$ are 2D control points (x_i, y_i)
  - For each t, the point $\mathbf{P}(t)$ on a Bézier curve is a linear combination of the control points with weights given by the Bernstein polynomials at t

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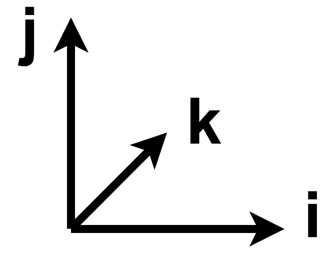
- Very nice, but only works for Bernstein polynomials.. there are other splines too!

Basis for Cubic Polynomials

$$\vec{v} = x\,\vec{\boldsymbol{i}} + y\,\vec{\boldsymbol{j}} + z\,\vec{\boldsymbol{k}}$$

In 3D

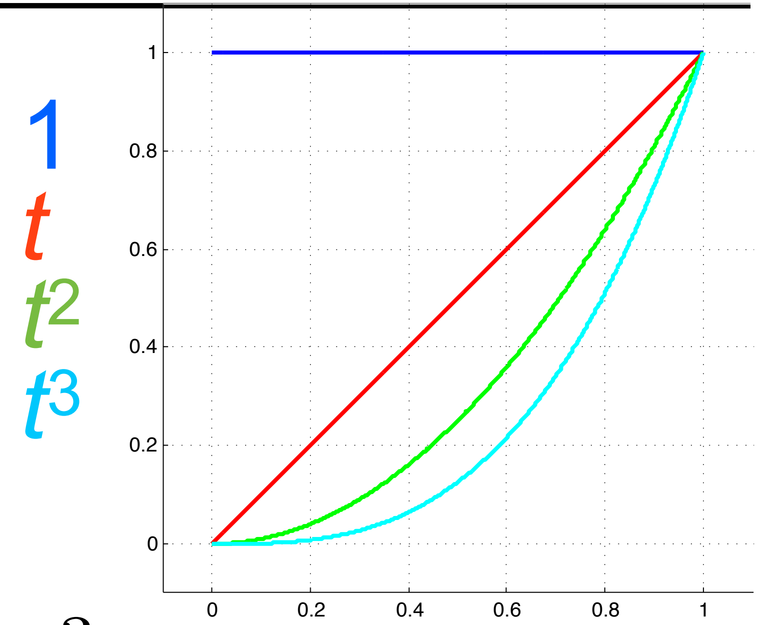


What's a basis?

- A set of "atomic" vectors
  - Called **basis vectors**
  - Linear combinations of basis vectors span the space
- Linearly independent
  - Means that no basis vector can be obtained from the others by linear combination
    - Example: **i**, **j**, **i**+**j** is not a basis (missing **k** direction!)

Canonical Basis for Cubics

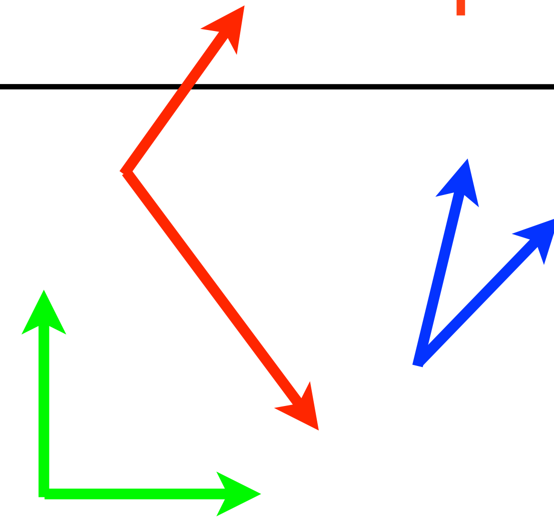
$$\{1, t, t^2, t^3\}$$



- *Any* cubic polynomial is a linear combination of these:
  $$a_0 * 1 + a_1\, t + a_2\, t^2 + a_3\, t^3$$
  - The as are the weights
- They are linearly independent
  - Means you can't write any of the four monomials as a a linear combination of the others. (You can try.)

Different basis

2D examples



- For example:
  - {1, 1+t, 1+2t+t^2, 1+t-t^2+6t^3}
  - {t^3, t^3+t^2, t^3+t, t^3+1}
- These can all be obtained from $1, t, t^2, t^3$ by linear combination
  - Just like all bases for Euclidean space can be obtained by linear combinations of the canonical **i**, **j**, …
- Infinite number of possibilities, just like you have an infinite number of bases to span R^2

Why we bother:

Matrix-Vector Notation For Polynomials

- For example:
  - 1, 1+t, 1+t+t^2, 1+t-t^2+t^3
  - t^3, t^3+t^2, t^3+t, t^3+1

Change-of-basis matrix

"Canonical" monomial basis

These relationships hold for each value of *t*

$$\begin{pmatrix} 1 \\ 1+t \\ 1+t+t^2 \\ 1+t-t^2+t^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

$$\begin{pmatrix} t^3 \\ t^3+t^2 \\ t^3+t \\ t^3+1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

Matrix-Vector Notation For Polynomials

- For example:
  - $1,\ 1+t,\ 1+t+t^2,\ 1+t-t^2+t^3$
  - $t^3,\ t^3+t^2,\ t^3+t,\ t^3+1$

Change-of-basis matrix

"Canonical" monomial basis

$$\begin{pmatrix} 1 \\ 1+t \\ 1+t+t^2 \\ 1+t-t^2+t^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

$$\begin{pmatrix} t^3 \\ t^3+t^2 \\ t^3+t \\ t^3+1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

Not any matrix will do! If it's singular, the basis set will be linearly dependent, i.e., redundant.

Matrix Form of Bernstein

Cubic Bernstein:

- $B_1(t)=(1-t)^3$
- $B_2(t)=3t(1-t)^2$
- $B_3(t)=3t^2(1-t)$
- $B_4(t)=t^3$

Expand these out and collect powers of *t*. The coefficients are the entries in the matrix B!

$$\begin{pmatrix} B_1(t) \\ B_2(t) \\ B_3(t) \\ B_4(t) \end{pmatrix} = \overbrace{\begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}}^{\boldsymbol{B}} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

Bézier in Matrix-Vector Notation

- Remember:
  $\mathbf{P}(t) = \mathbf{P}_1 \mathrm{B}_1(t) + \mathbf{P}_2 \mathrm{B}_2(t) + \mathbf{P}_3 \mathrm{B}_3(t) + \mathbf{P}_4 \mathrm{B}_4(t)$
  is a linear combination of control points

- or, in matrix-vector notation

Bernstein polynomials
(4x1 vector)

$$P(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix} \begin{pmatrix} B_1(t) \\ B_2(t) \\ B_3(t) \\ B_4(t) \end{pmatrix}$$

point on curve
(2x1 vector)

matrix of
control points (2 x 4)

Bézier in Matrix-Vector Notation

$$\begin{pmatrix} B_1(t) \\ B_2(t) \\ B_3(t) \\ B_4(t) \end{pmatrix} = \overbrace{\begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}}^{\boldsymbol{B}} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

<= Flasback from two slides ago, let's combine with below:

Bernstein polynomials (4x1 vector)

$$P(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix} \begin{pmatrix} B_1(t) \\ B_2(t) \\ B_3(t) \\ B_4(t) \end{pmatrix}$$

point on curve (2x1 vector)

matrix of control points (2 x 4)

Phase 3: Profit

- Combined, we get cubic Bézier in matrix notation

point on curve
(2x1 vector)

$$\mathrm{P}(\mathrm{t}) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} =$$

Canonical
monomial basis

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix} \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

"Geometry matrix"
of control points $P_1..P_4$
(2 x 4)

"Spline matrix"
(Bernstein)

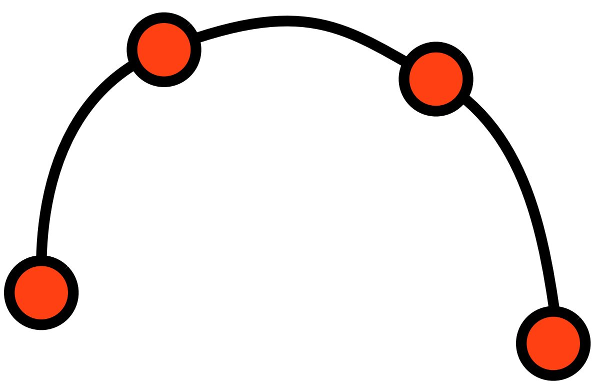
General Spline Formulation

$$Q(t) = \mathbf{GBT(t)} \;= \text{Geometry } \mathbf{G} \cdot \text{Spline Basis } \mathbf{B} \cdot \text{Power Basis } \mathbf{T(t)}$$

- Geometry: control points coordinates assembled into a matrix $\mathbf{G}=(P_1, P_2, \ldots, P_{n+1})$
- Spline matrix **B**: defines the type of spline
  - Bernstein for Bézier
- Power basis **T**: the monomials $(1, t, ..., t^n)^T$
- Advantage of general formulation
  - Compact expression
  - Easy to convert between types of splines
  - Dimensionality (plane or space) doesn't really matter

What can we do with this?

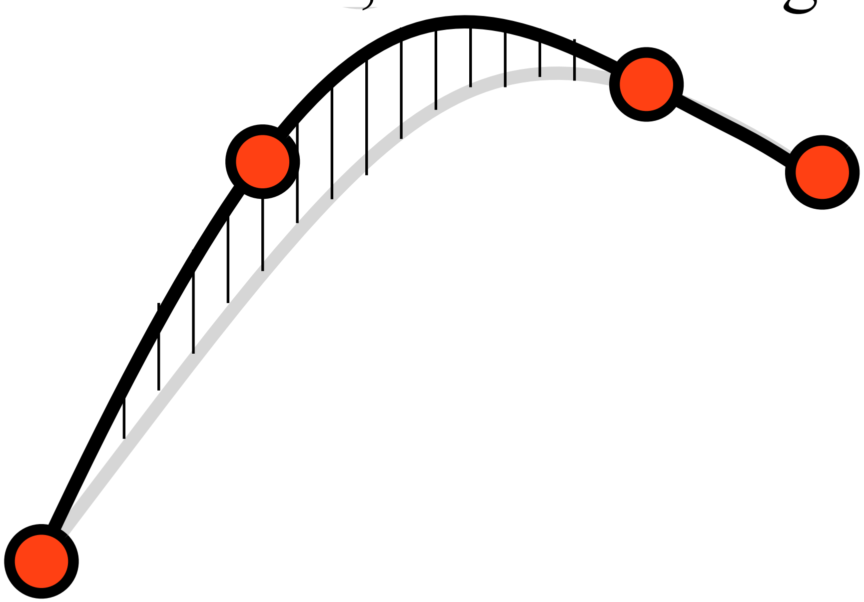
- Cubic polynomials form a vector space.
- Bernstein basis defines Bézier curves.
- Can do other bases as well: Catmull-Rom splines interpolate all the control points, for instance



$$\mathbf{B}_{\mathrm{CR}} = \frac{1}{2}\begin{pmatrix} 0 & -1 & 2 & -1 \\ 2 & 0 & -5 & 3 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

3D Spline has 3D control points

- The 2D control points can be replaced by 3D points – this yields space curves.
  - All the above math stays the same except with the addition of 3rd row to control point matrix (*z* coords)
  - In fact, can do homogeneous coordinates as well!



Linear Transformations & Cubics

- What if we want to transform each point on the curve with a linear transformation **M**?

$$P'(t) = \mathbf{M}\left[\begin{pmatrix} P_{1,x} & P_{2,x} & P_{3,x} & P_{4,x} \\ P_{1,y} & P_{2,y} & P_{3,y} & P_{4,y} \end{pmatrix}\begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}\right]$$

Linear Transformations & Cubics

- What if we want to transform each point on the curve with a linear transformation **M**?
  - Because everything is linear, it's the same as transforming the only the control points

$$P'(t) = \mathbf{M}\left[\begin{pmatrix} P_{1,x} & P_{2,x} & P_{3,x} & P_{4,x} \\ P_{1,y} & P_{2,y} & P_{3,y} & P_{4,y} \end{pmatrix} \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}\right]$$

$$= \left[\mathbf{M} \begin{pmatrix} P_{1,x} & P_{2,x} & P_{3,x} & P_{4,x} \\ P_{1,y} & P_{2,y} & P_{3,y} & P_{4,y} \end{pmatrix}\right] \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

Linear Transformations & Cubics

- Homogeneous coordinates also work
  - Means you can translate, rotate, shear, etc.
  - Also, changing *w* gives a "tension" parameter
    - Note though that you need to normalize P' by 1/*w*

$$P'(t) = \mathbf{M}\left[\begin{pmatrix} P_{1,x} & P_{2,x} & P_{3,x} & P_{4,x} \\ P_{1,y} & P_{2,y} & P_{3,y} & P_{4,y} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \end{pmatrix} \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}\right]$$

$$= \left[\mathbf{M} \begin{pmatrix} P_{1,x} & P_{2,x} & P_{3,x} & P_{4,x} \\ P_{1,y} & P_{2,y} & P_{3,y} & P_{4,y} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \end{pmatrix}\right] \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

Recap

- Splines turn control points into smooth curves
- Cubic polynomials form a vector space
- Bernstein basis gives rise to Bézier curves
  - Can be seen as influence function of data points
  - Or data points are coordinates of the curve in the Bernstein basis
- We can change between bases with matrices