CS-C3100 Computer Graphics
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Picture: A. Selle et al.

10.1 Higher order ODE solvers (and why we need them)

Lots of slides from Frédo Durand

In This Video

- Why Euler’s method is not great
- More accurate ODE solvers

Euler's method: Not Always Stable

- "Test equation" $f(x,t) = -kx$

- Since $f(x,t)$ is supposed to give the derivative of $x(t)$, we're asking for a function $x(t)$ that, when differentiated w.r.t. t, gives back itself times $-k$

Euler's method: Not Always Stable

- "Test equation" $f(x,t) = -kx$

- Exact solution is a decaying exponential:

$$x(t) = x_0 e^{-kt}$$

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$$\frac{\mathrm{d}}{\mathrm{d}t} x(t)$$

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How so?

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(x_0\, e^{-kt}\right) =$$

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$x(t) = x_0 e^{-kt}$ **How so?**

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = \frac{\mathrm{d}}{\mathrm{d}t} (x_0 \, e^{-kt}) =$$

$$x_0 \frac{\mathrm{d}}{\mathrm{d}t} e^{-kt}$$

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$$x_0 \frac{\mathrm{d}}{\mathrm{d}t} e^{-kt} = -k x_0 e^{-kt}$$

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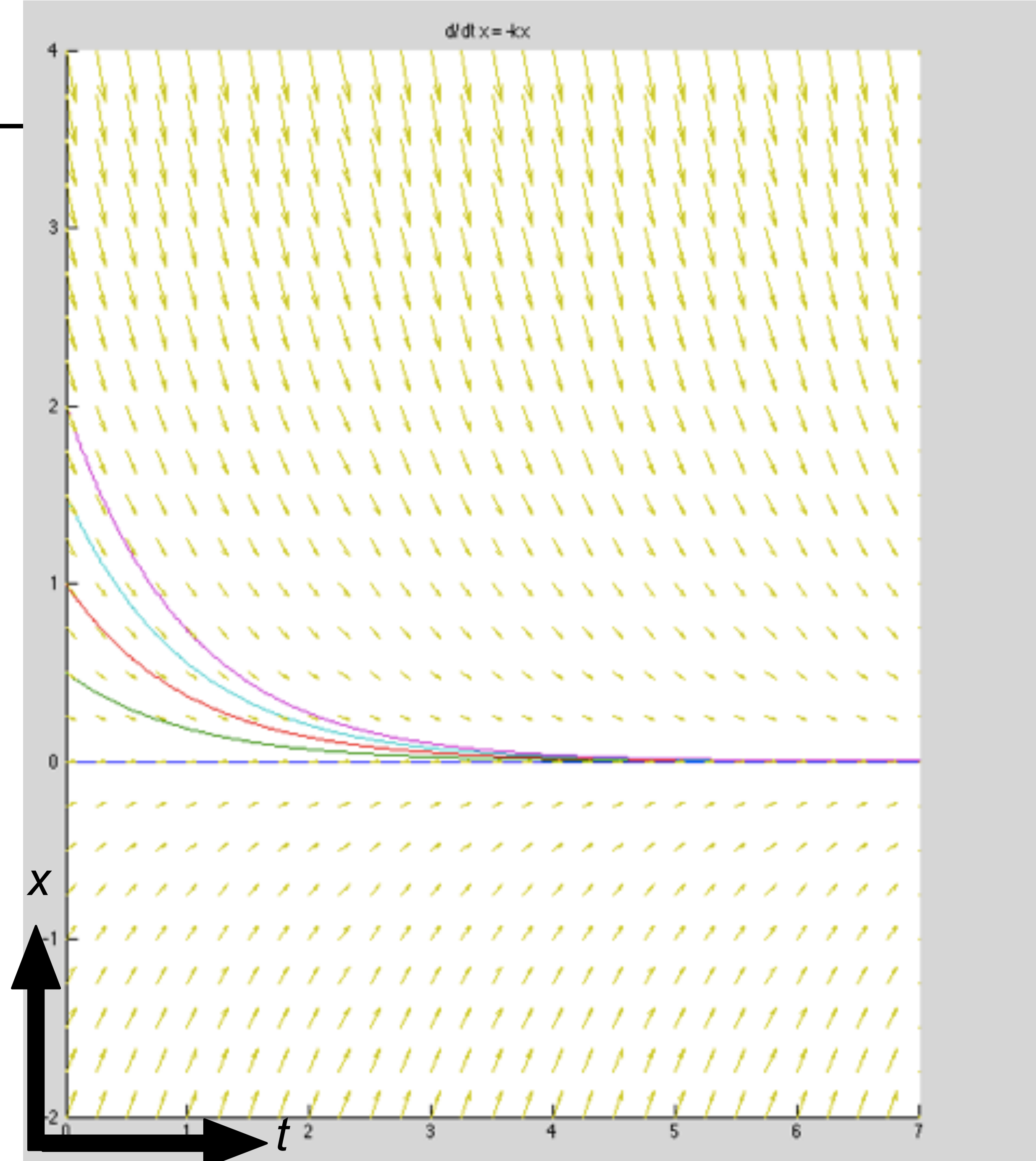
How so?

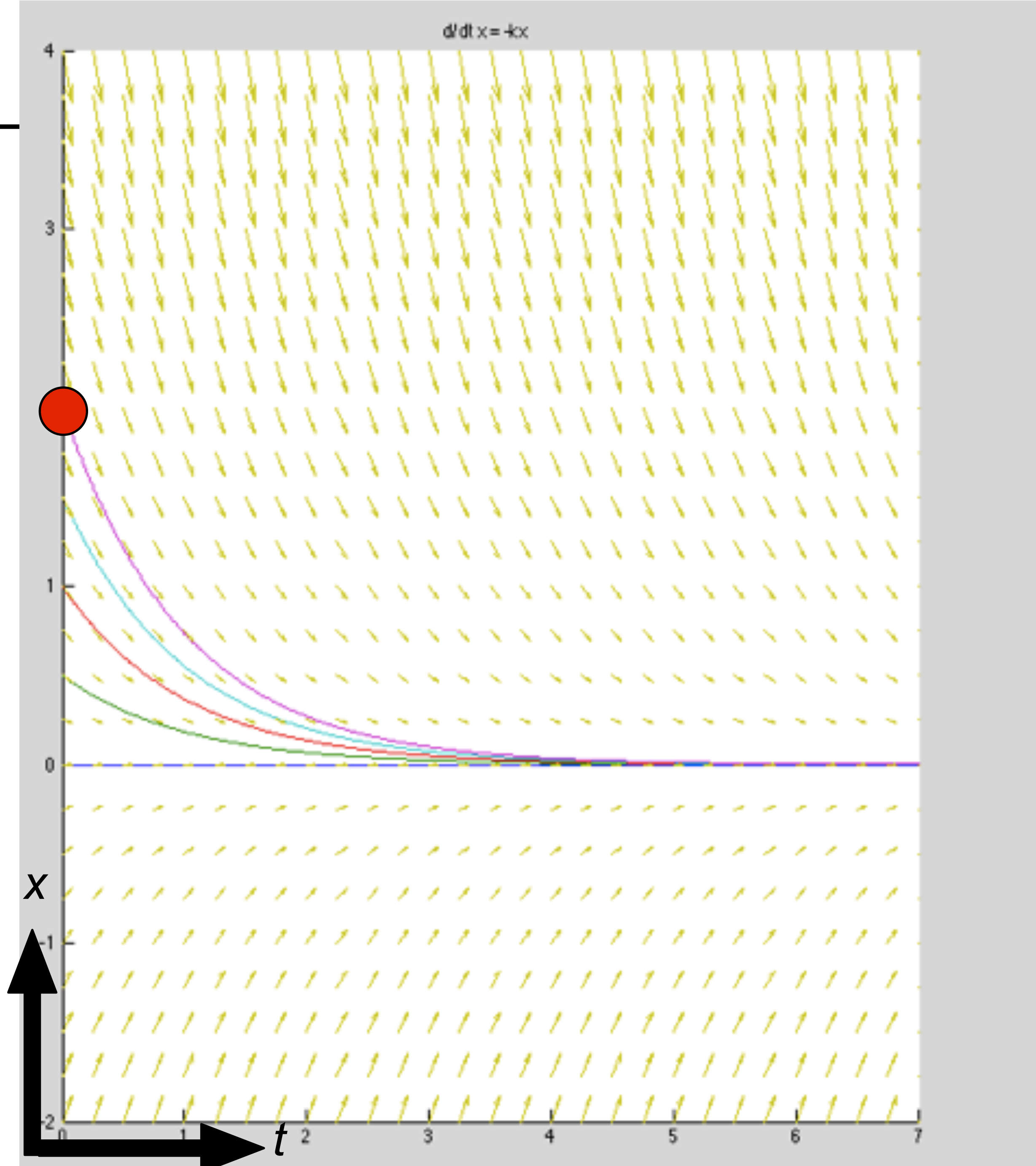
$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(x_0\, e^{-kt}\right) =$$

$$x_0 \frac{\mathrm{d}}{\mathrm{d}t} e^{-kt} = -k x_0 e^{-kt} = -k x(t) \quad ✔$$

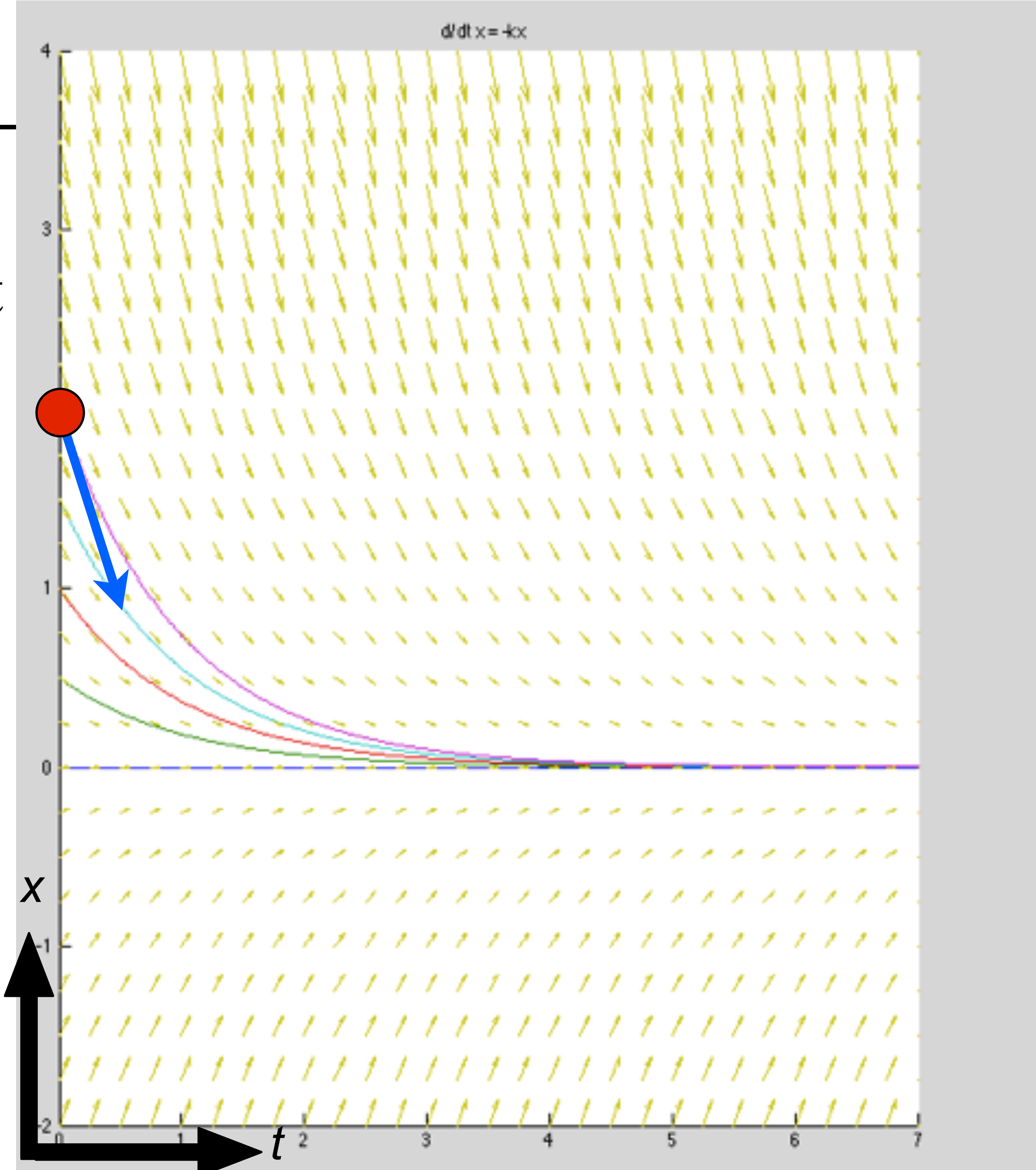
- f(x,t)=-kx
- Arrows show the derivatives that we're supposed to follow
- Solid curves are the analytic solutions for different starting points



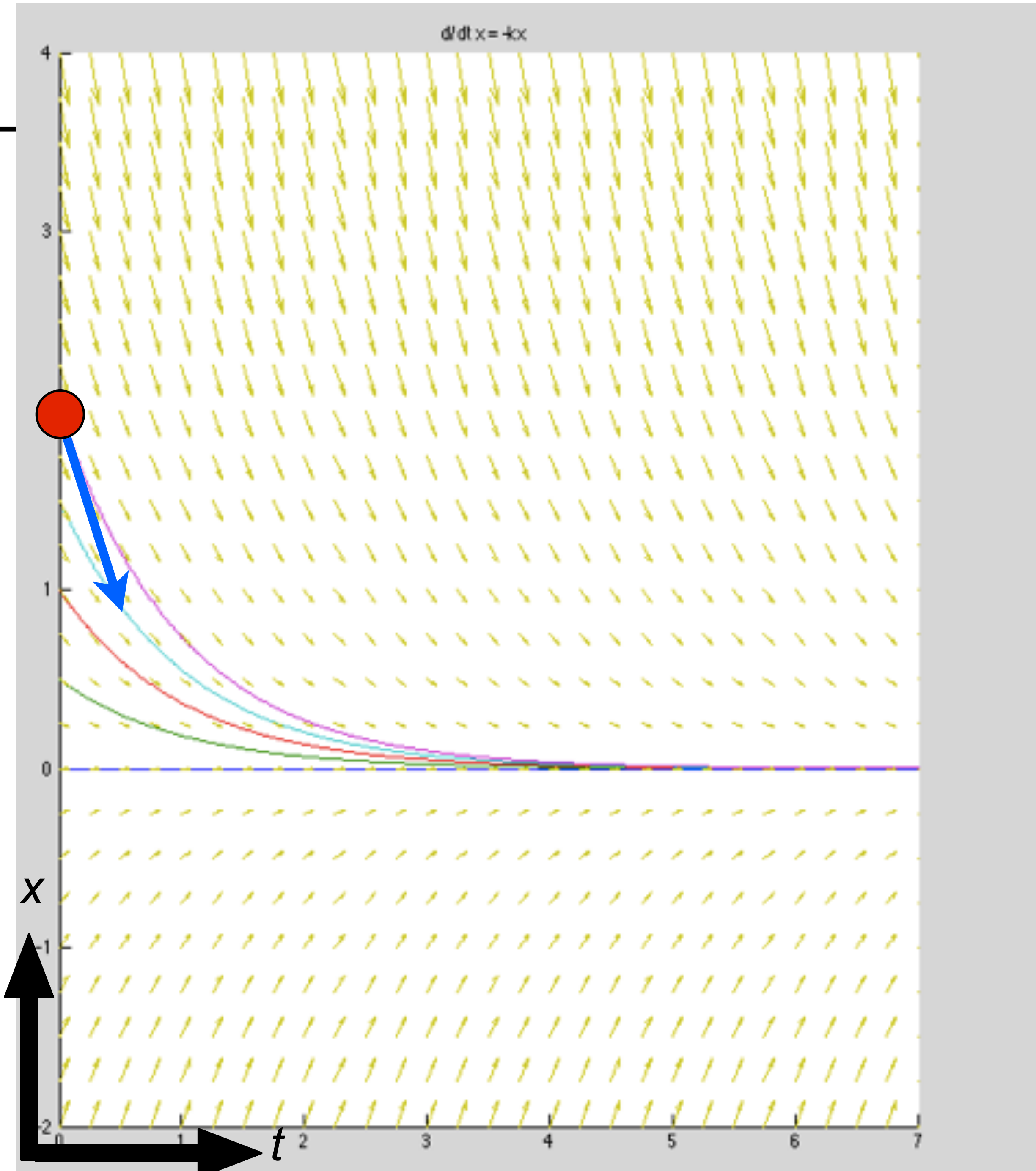
- Let's apply Euler starting from the red dot



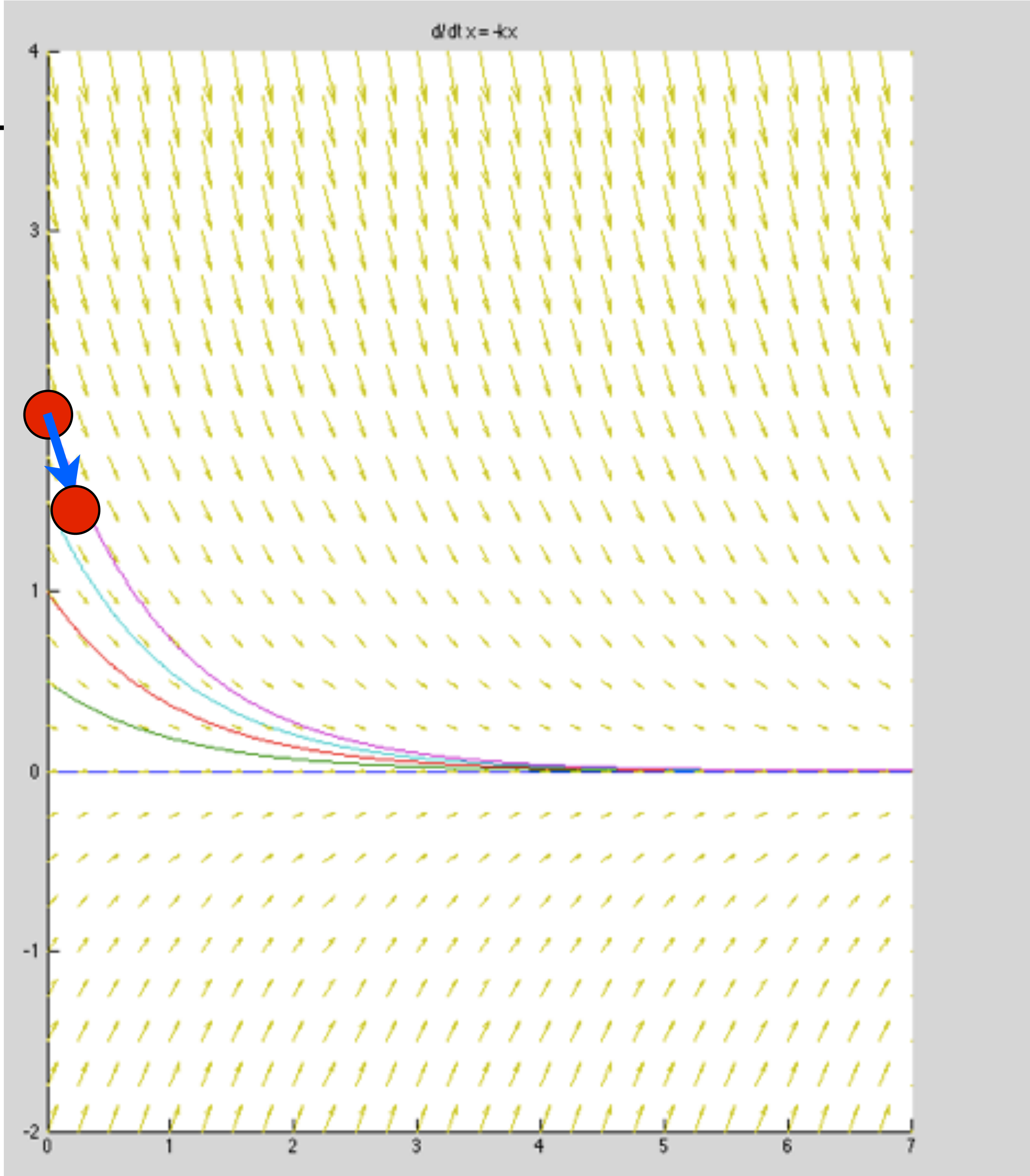
- Evaluating f(x,t) at this point gives the derivative



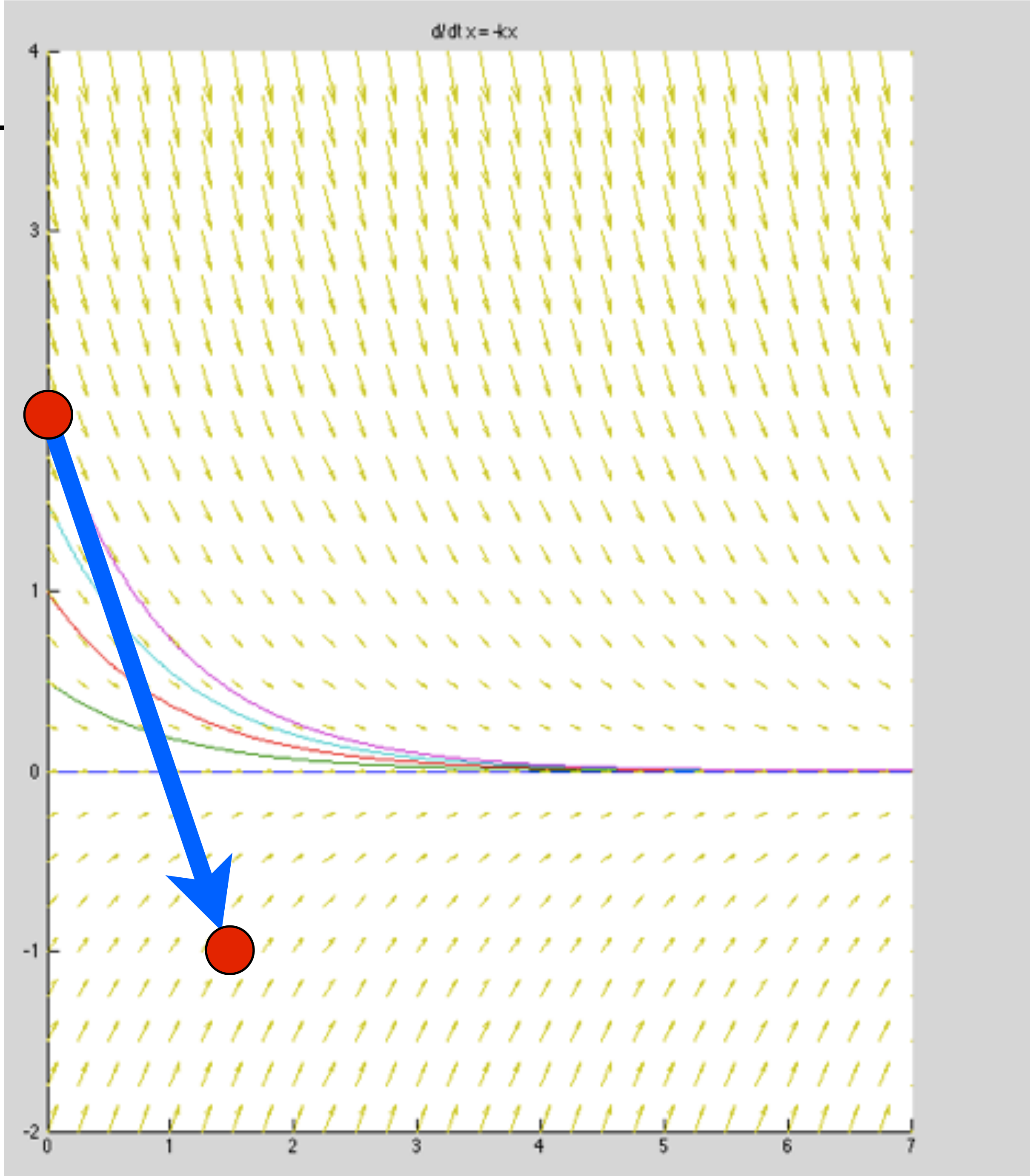
- Euler says: staying on the line defined by the derivative, let's step forward in time by step size *h*



- If we take a small enough step we keep close to the actual solution curve



- But if we take a large step, we overshoot badly!



Euler's method: Not Always Stable

- "Test equation" $f(x,t) = -kx$

- Exact solution is a decaying exponential:

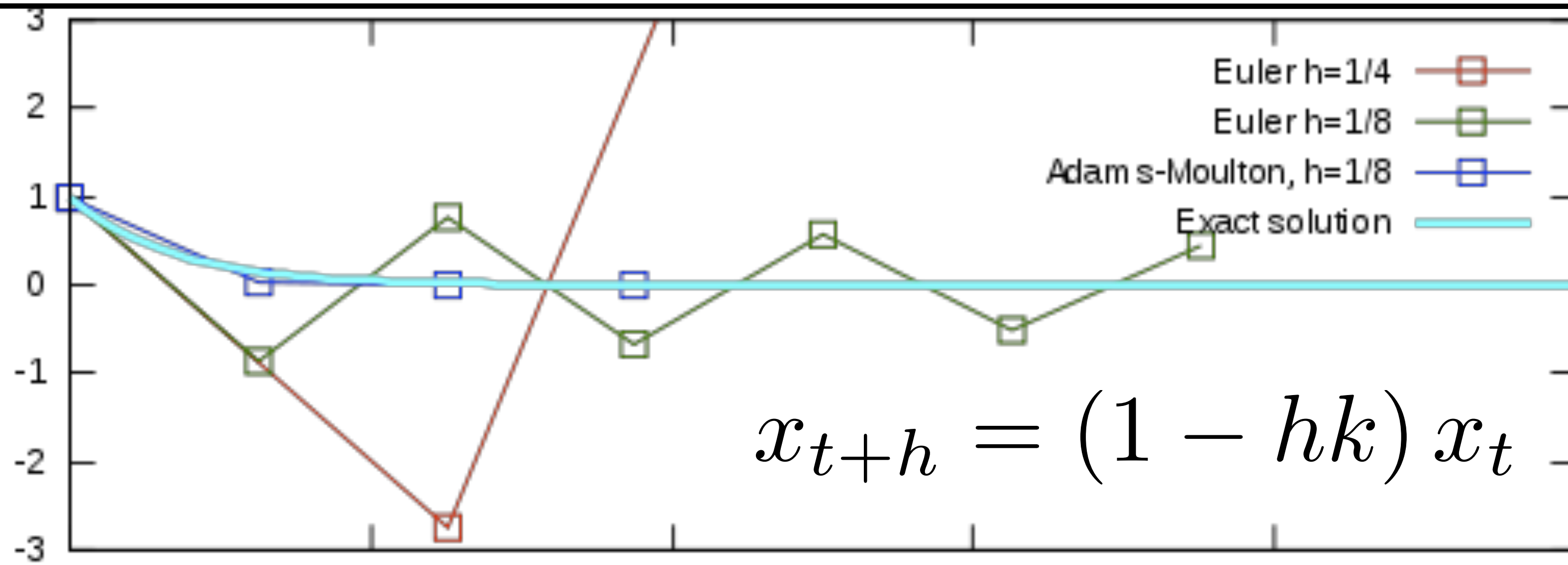
$$x(t) = x_0 e^{-kt}$$

- Let's apply Euler's method:

$$\begin{aligned} x_{t+h} &= x_t + h\,f(x_t, t) \\ &= x_t - hkx_t \\ &= \boxed{(1 - hk)\,x_t} \end{aligned}$$

Euler’s method: Not Always Stable

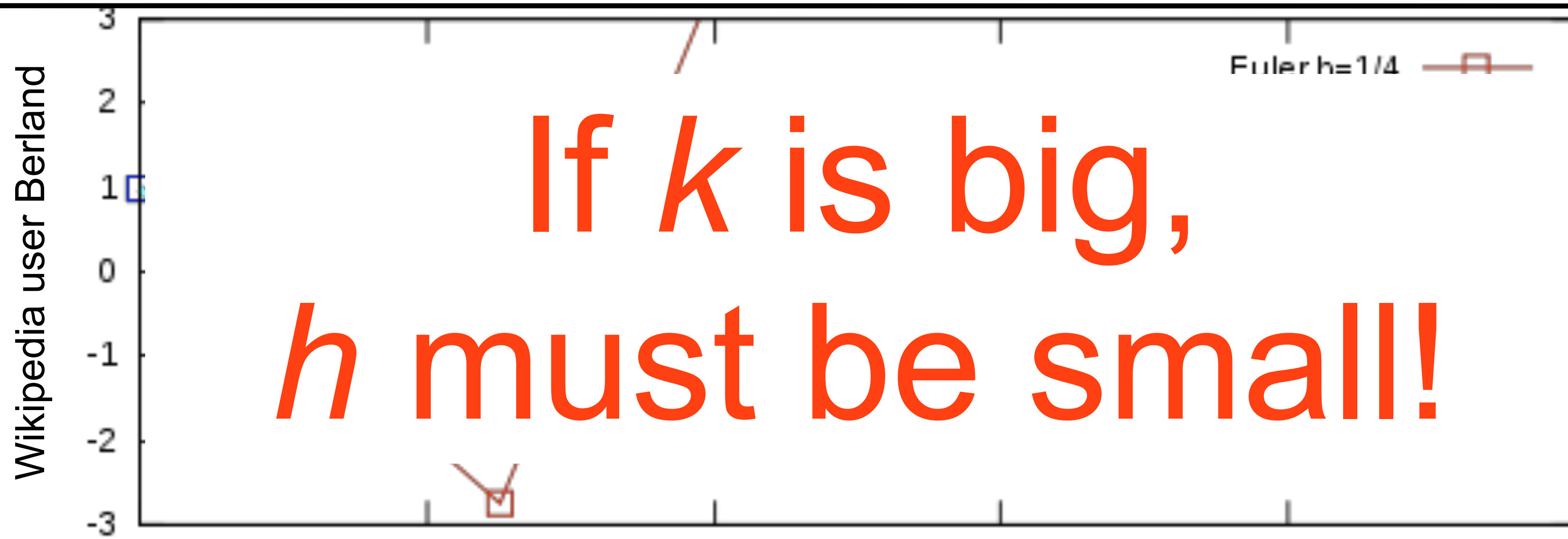
Wikipedia user Berland



$$x_{t+h} = (1 - hk)\, x_t$$

- Limited step size!
  - When $0 \leq (1 - hk) < 1 \Leftrightarrow h < 1/k$ things are fine, the solution decays
  - When $-1 \leq (1 - hk) \leq 0 \Leftrightarrow 1/k \leq h \leq 2/k$ we get oscillation
  - When $(1 - hk) < -1 \Leftrightarrow h > 2/k$ things explode!

Euler’s method: Not Always Stable



If k is big,
h must be small!

- Limited step size!
  - When $0 \leq (1 - hk) < 1 \Leftrightarrow h < 1/k$ things are fine, the solution decays
  - When $-1 \leq (1 - hk) \leq 0 \Leftrightarrow 1/k \leq h \leq 2/k$ we get oscillation
  - When $(1 - hk) < -1 \Leftrightarrow h > 2/k$ things explode!

Analysis: Taylor series

- Expand exact solution $\mathbf{X}(t)$

$$\mathbf{X}(t_0+h)=\mathbf{X}(t_0)+h\left(\tfrac{d}{dt}\mathbf{X}(t)\right)\Big|_{t_0}+\tfrac{h^2}{2!}\left(\tfrac{d^2}{dt^2}\mathbf{X}(t)\right)\Big|_{t_0}+\tfrac{h^3}{3!}\left(\cdots\right)+\cdots$$

- Euler's method approximates:

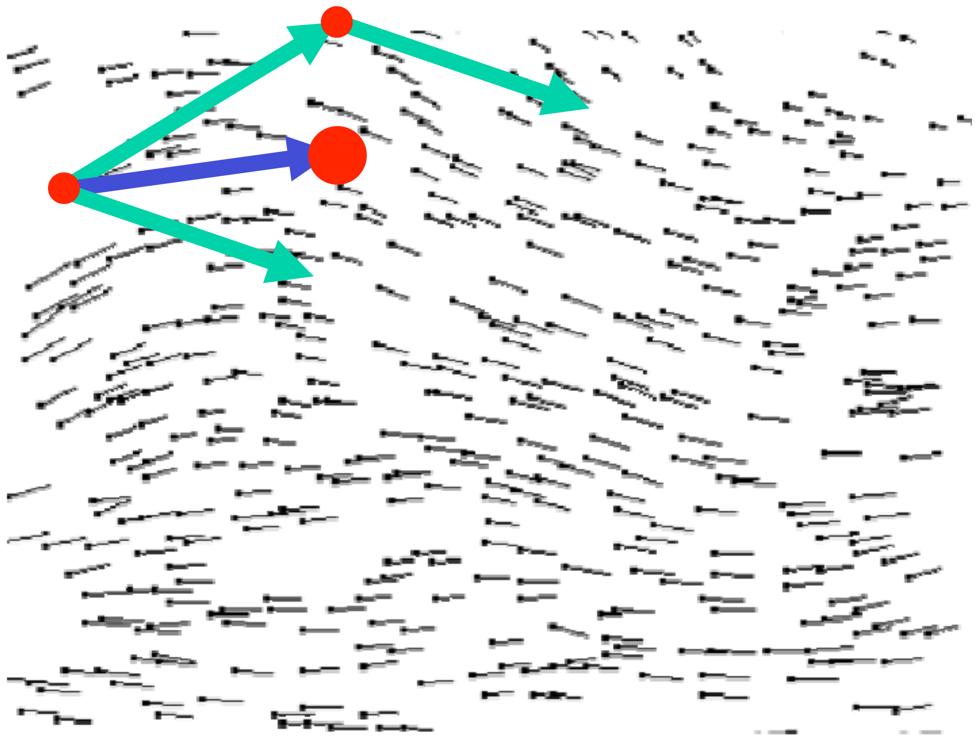
$$\mathbf{X}(t_0+h)=\mathbf{X}_0+h\,f(\mathbf{X}_0,t_0) \qquad \ldots+O(h^2)\,\text{error}$$

$$\begin{aligned} h\to h/2 \;\Rightarrow\; & \mathit{error}\to \mathit{error}/4\,\text{per step}\times\text{twice as many steps} \\ & \to \mathit{error}/2 \end{aligned}$$

- First-order method: Accuracy varies with h
- To get 100x better accuracy need 100x more steps

Can we do better?

- Problem: *f* varies along our Euler step
- Idea 1: look at *f* at the arrival of the step and compensate for variation



2nd Order Methods

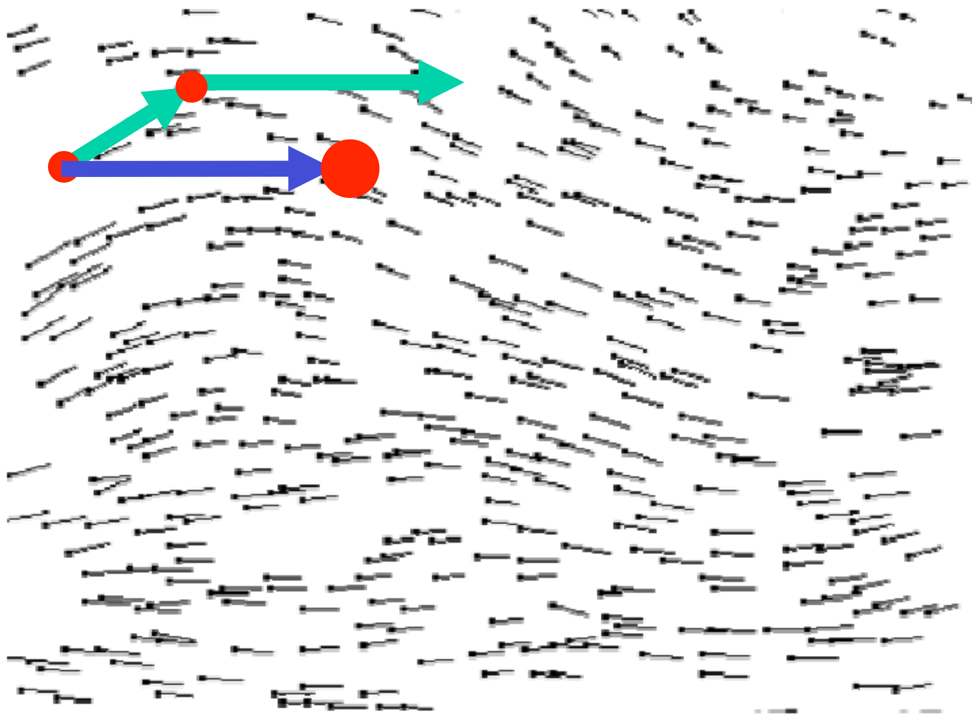
- Let $$\boxed{\begin{aligned} f_0 &= f(\mathbf{X}_0, t_0) \\ f_1 &= f(\mathbf{X}_0 + h f_0, t_0 + h) \end{aligned}}$$

- Then $$\boxed{\mathbf{X}(t_0 + h) = \mathbf{X}_0 + \tfrac{h}{2}(f_0 + f_1)} + O(h^3)$$

- This is the *trapezoid method*
  - Analysis omitted
- Note! "2nd order method" means that the error goes down with h^2, not h – the *equation* is still 1st order!

Can we do better, another try

- Problem: *f* has varied along our Euler step
- Idea 2: look at *f* after a smaller step, use that value for a full step from initial position



2nd Order Methods cont'd

- This translates to...

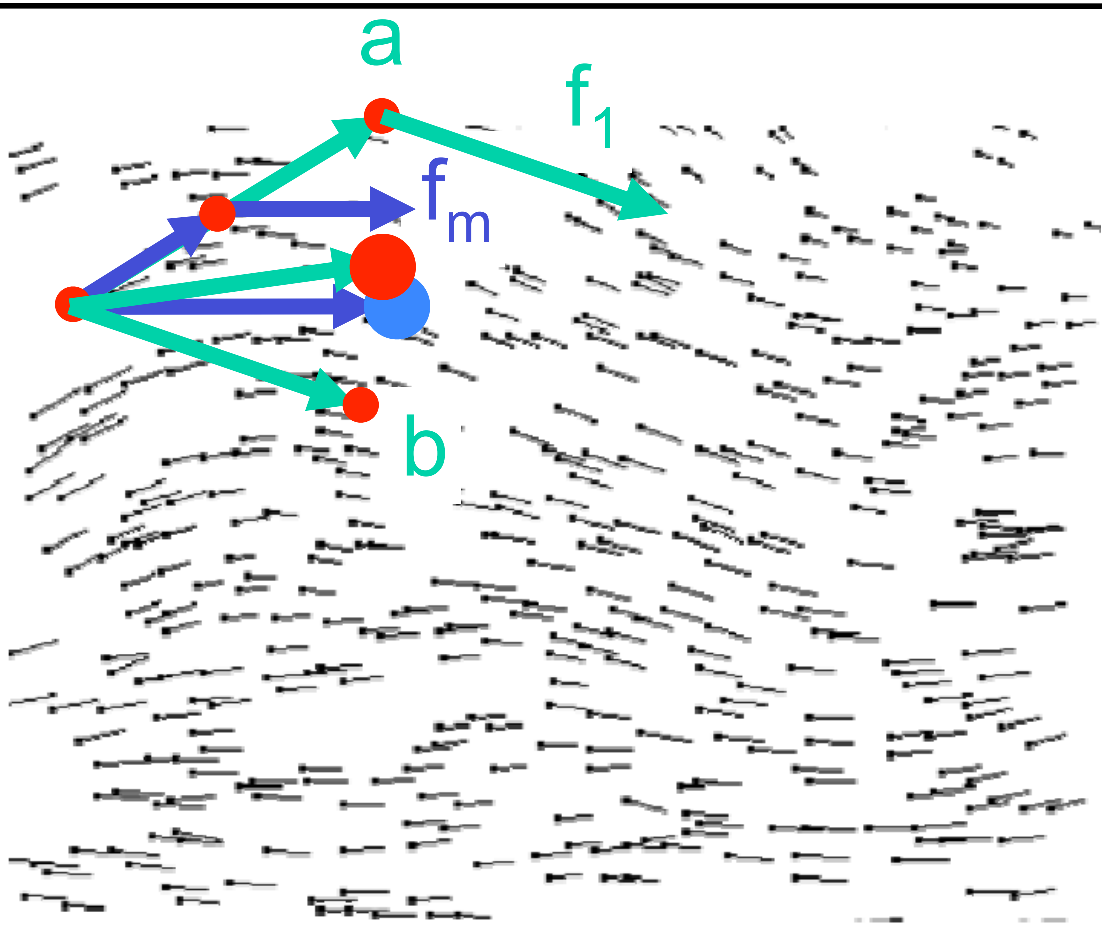
$$\boxed{\begin{aligned} f_0 &= f(\mathbf{X}_0, t_0) \\ f_m &= f(\mathbf{X}_0 + \tfrac{h}{2} f_0, t_0 + \tfrac{h}{2}) \end{aligned}}$$

- and we get $\boxed{\mathbf{X}(t_0 + h) = \mathbf{X}_0 + h\, f_m} + O(h^3)$

- This is the *midpoint method*
  - Analysis omitted again,
    but it's not very complicated, see here.

Comparison

- Midpoint:
  - ½ Euler step
  - evaluate f_m
  - full step using f_m
- Trapezoid:
  - Euler step (a)
  - evaluate f_1
  - full step using f_1 (b)
  - average (a) and (b)
- Not exactly same result, but same order of accuracy

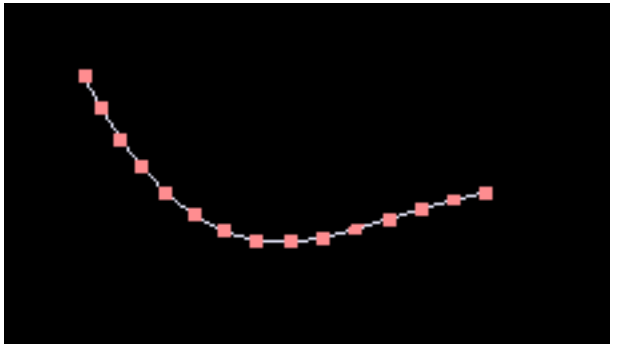


Can we do even better?

- You bet!
- You will implement Runge-Kutta for Assignment 4

- Again, see Witkin, Baraff, Kass: Physically-based Modeling Course Notes, SIGGRAPH 2001

- Comparison Demo a little later



That’s It..

- Next: Mass-spring modelling