CS-E4850 Computer Vision, Answers to Exercise Round 6

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November 8, 2018

Exercise 1. Least-squares fitting for affine transformations.

a) Compute the gradient of the least squares error with respect to the parameters of the transformation (i.e. elements of matrix $\mathbf{M}$ and vector $\mathbf{t}$).

$$E = \sum_{i=1}^{n} ||x_i' - Mx_i - t||^2 = \sum_{i=1}^{n} || \begin{bmatrix} x_i' \\ y_i' \end{bmatrix} - \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix} ||^2$$

$$= \sum_{i=1}^{n} \left[(x_i' - M_1 x_i - M_2 y_i - t_1)^2 + (y_i' - M_i x_3 - M_4 y_i - t_2)^2 \right]$$

We then differentiate this with respect to each of the 6 parameters $(M_1, M_2, M_3, M_4, t_1, t_2)$

$$\frac{dE}{dM_1} = \sum_{i=1}^{n} 2(x_i' - M_1 x_i - M_2 y_i - t_1)(-x_i)$$

$$\frac{dE}{dM_2} = \sum_{i=1}^{n} 2(x_i' - M_1 x_i - M_2 y_i - t_1)(-y_i)$$

$$\frac{dE}{dM_3} = \sum_{i=1}^{n} 2(y_i' - M_3 x_i - M_4 y_i - t_2)(-x_i)$$

$$\frac{dE}{dM_4} = \sum_{i=1}^{n} 2(y_i' - M_3 x_i - M_4 y_i - t_2)(-y_i)$$

$$\frac{dE}{dt_1} = \sum_{i=1}^{n} 2(x_i' - M_1 x_i - M_2 y_i - t_1)(-1)$$

$$\frac{dE}{dt_2} = \sum_{i=1}^{n} 2(y_i' - M_3 x_i - M_4 y_i - t_2)(-1)$$

b) Show that by setting the aforementioned gradient to zero you will get an equation of the form $\mathbf{Sh} = \mathbf{u}$, where vector $\mathbf{h}$ contains the unknown parameters of the transformation, and 6×6 matrix $\mathbf{S}$ and 6×1 vector $\mathbf{u}$ depend on the coordinates of the point correspondences

$$\{x_i', x_i\}, i = 1, ..., n$$

. We let $\mathbf{S}$ be the matrix that contains the coefficients of parameters of the gradient.

$$S = \begin{pmatrix} \sum_{i=1}^n x_i^2 & \sum_{i=1}^n x_i y_i & 0 & 0 & \sum_{i=1}^n x_i & 0 \\ \sum_{i=1}^n x_i y_i & \sum_{i=1}^n y_i^2 & 0 & 0 & \sum_{i=1}^n y_i & 0 \\ 0 & 0 & \sum_{i=1}^n x_i^2 & \sum_{i=1}^n x_i y_i & 0 & \sum_{i=1}^n x_i \\ 0 & 0 & \sum_{i=1}^n x_i y_i & \sum_{i=1}^n y_i^2 & 0 & \sum_{i=1}^n y_i \\ \sum_{i=1}^n x_i & \sum_{i=1}^n y_i & 0 & 0 & n & 0 \\ 0 & 0 & \sum_{i=1}^n x_i & \sum_{i=1}^n y_i & 0 & n \end{pmatrix}$$

$$h = \begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ t_1 \\ t_2 \end{pmatrix}, \quad u = \begin{pmatrix} \sum_{i=1}^n x_i x' \\ \sum_{i=1}^n y_i x' \\ \sum_{i=1}^n x_i y' \\ \sum_{i=1}^n y_i y' \\ \sum_{i=1}^n x_i \\ \sum_{i=1}^n y_i \end{pmatrix}$$

c) Thus, one may solve the transformation by computing $h = S^{-1}u$. Compute the affine transformation from the following point correspondences $(0, 0) \to (1, 2)$, $(1, 0) \to (3, 2)$, and $(0, 1) \to (1, 4)$.

Exercise 2 Similarity transformation from two point correspondences. A similarity transformation consists of rotation, scaling and translation and is defined in two dimensions as follows:

$$\mathbf{x}' = s\mathbf{R}\mathbf{x} + \mathbf{t} \Leftrightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = s \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix}$$

Describe a method for solving the parameters s, θ, t_x, t_y of a similarity transformation from two point correspondences $\{x_1 \to x_1'\}$, $\{x_2 \to x_2'\}$ using the following stages:

a) Compute the vectors $v' = x_2' - x_1'$ and $v = x_2 - x_1$ and present a formula to recover the rotation angle θ from the corresponding unit vectors.

$$v = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}, \quad v' = \begin{pmatrix} x_2' - x_1' \\ y_2' - y_1' \end{pmatrix} = s \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$$

To recover the angle θ, we can use

$$\begin{aligned} \cos(\theta) &= \frac{v' \cdot v}{||v'|| \cdot ||v||} \\ &= \frac{(x_2' - x_1')(x_2 - x_1) + (y_2' - y_1')(y_2 - y_1)}{\sqrt{(x_2' - x_1')^2 + (y_2' - y_1')^2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} \end{aligned}$$

$$\theta = cos^-1(\frac{(x_2' - x_1')(x_2 - x_1) + (y_2' - y_1')(y_2 - y_1)}{\sqrt{(x_2' - x_1')^2 + (y_2' - y_1')^2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}})$$

b) Compute the scale factor s as the ratio of the norms of vectors v' and v.

$$s = \frac{||v'||}{||v||} = \frac{\sqrt{(x_2' - x_1')^2 + (y_2' - y_1')^2}}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

c) After solving s and θ compute t using equation (1) and either one of the two point correspondences.

Recall equation (1):

$$\mathbf{x}' = s\mathbf{R}\mathbf{x} + \mathbf{t} \Leftrightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = s \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix}$$

$$\begin{aligned} x' &= s\cos(\theta)x - s\sin(\theta)y + t_x \quad &\to \quad t_x = x' - s\cos(\theta)x + s\sin(\theta)y \\ y' &= s\sin(\theta)x + s\cos(\theta)y + t_y \quad &\to \quad t_y = y' - s\sin(\theta)x - s\cos(\theta)y \end{aligned}$$

d) Use the procedure to compute the transformation from the following point correspondences: $\{(\frac{1}{2}, 0) \rightarrow (0, 0)\}, \{(0, \frac{1}{2}) \rightarrow (-1, -1)\}$.

$$(x_1, y_1) = (\frac{1}{2}, 0), \quad (x_2, y_2) = (0, \frac{1}{2}), \quad (x_1', y_1') = (0, 0), \quad (x_2', y_2') = (-1, -1)$$

$$v = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix}, \quad v' = \begin{pmatrix} x_2' - x_1' \\ y_2' - y_1' \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$\begin{aligned}
\theta &= cos^{-1}(\frac{(x_2' - x_1')(x_2 - x_1) + (y_2' - y_1')(y_2 - y_1)}{\sqrt{(x_2' - x_1')^2 + (y_2' - y_1')^2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}) \\
&= \cos^{-1} \frac{(-1)(1/2) - (-1)(1/2)}{\sqrt{(x_2' - x_1')^2 + (y_2' - y_1')^2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} \\
&= \cos^{-1}(0) = \frac{\pi}{2}
\end{aligned}$$

$$\begin{aligned}
s &= \frac{||v'||}{||v||} = \frac{\sqrt{(x_2' - x_1')^2 + (y_2' - y_1')^2}}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} \\
&= \sqrt{\frac{(-1)^2 + (-1)^2}{(-1/2)^2 + (1/2)^2}} = \sqrt{\frac{2}{1/2}} = \sqrt{4} \\
&= 2
\end{aligned}$$

$$\begin{aligned}
t_x &= x' - s\cos(\theta)x + s\sin(\theta)y \\
&= 0 - 2(0)(\frac{1}{2}) + 2(0)(1) = 0
\end{aligned}$$

$$\begin{aligned}
t_y &= y' - s\sin(\theta)x - s\cos(\theta)y \\
&= 0 - 2(1)(\frac{1}{2}) + 2(0)(0) = -\frac{1}{2}
\end{aligned}$$

The equation for the transformation is as follows:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = 2\begin{pmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$