Exercise 1. Least-squares fitting for affine transformations. (pen & paper problem) A brief overview of affine transformation estimation is presented on slides 17-19 of Lecture 5. Present a derivation and compute an example by performing the following stages:

a) Compute the gradient of the least squares error $E = \sum_{i=1}^{n} ||\mathbf{x}_i' - \mathbf{M}\mathbf{x}_i - \mathbf{t}||^2$ with respect to the parameters of the transformation (i.e. elements of matrix $\mathbf{M}$ and vector $\mathbf{t}$).

Expanding the least squares error by applying the affine transformation definition:

$$E = \sum_{i=1}^{n} \left\| x_i' - Mx_i - t \right\|^2 = \sum_{i=1}^{n} \left\| \begin{bmatrix} x_i' \\ y_i' \end{bmatrix} - \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix} - \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} \right\|^2 = \begin{bmatrix} \sum_{i=1}^{n} \left(x_i' - m_1 x_i - m_2 y_i - t_1 \right)^2 \\ \sum_{i=1}^{n} \left(y_i' - m_3 x_i - m_4 y_i - t_2 \right)^2 \end{bmatrix}$$

The gradient of the least squares error with respect to each parameters of the transformation is:

$$\Rightarrow \frac{\partial E}{\partial m_1} = \begin{bmatrix} \sum_{i=1}^{n} -2x_i \left(x_i' - m_1 x_i - m_2 y_i - t_1 \right) \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{\partial E}{\partial m_2} = \begin{bmatrix} \sum_{i=1}^{n} -2y_i \left(x_i' - m_1 x_i - m_2 y_i - t_1 \right) \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{\partial E}{\partial m_3} = \begin{bmatrix} 0 \\ \sum_{i=1}^{n} -2x_i \left(y_i' - m_3 x_i - m_4 y_i - t_2 \right) \end{bmatrix}$$

$$\Rightarrow \frac{\partial E}{\partial m_4} = \begin{bmatrix} 0 \\ \sum_{i=1}^{n} -2y_i \left(y_i' - m_3 x_i - m_4 y_i - t_2 \right) \end{bmatrix}$$

$$\Rightarrow \frac{\partial E}{\partial t_1} = \begin{bmatrix} \sum_{i=1}^{n} -2 \left(x_i' - m_1 x_i - m_2 y_i - t_1 \right) \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{\partial E}{\partial t_2} = \begin{bmatrix} 0 \\ \sum_{i=1}^{n} -2 \left(y_i' - m_3 x_i - m_4 y_i - t_2 \right) \end{bmatrix}$$

b) Show that by setting the aforementioned gradient to zero you will get an equation of the form $\mathbf{S}\mathbf{h} = \mathbf{u}$, where vector $\mathbf{h}$ contains the unknown parameters of the transformation, and 6×6 matrix $\mathbf{S}$ and 6×1 vector $\mathbf{u}$ depend on the coordinates of the point correspondences $\{\mathbf{x}_i', \mathbf{x}_i\}$, $i = 1, \ldots, n$.

Setting all the partial derivatives in 1a to zero, we get a system of 6 equations:

(1) $\sum_{i=1}^{n} -2x_i \left(x_i' - m_1 x_i - m_2 y_i - t_1 \right) = 0 \Rightarrow \sum_{i=1}^{n} x_i x_i' = m_1 \sum_{i=1}^{n} x_i^2 + m_2 \sum_{i=1}^{n} x_i y_i + t_1 \sum_{i=1}^{n} x_i$

(2) $\sum_{i=1}^{n} -2y_i \left(x_i' - m_1 x_i - m_2 y_i - t_1 \right) = 0 \Rightarrow \sum_{i=1}^{n} y_i x_i' = m_1 \sum_{i=1}^{n} x_i y_i + m_2 \sum_{i=1}^{n} y_i^2 + t_1 \sum_{i=1}^{n} y_i$

(3) $\sum_{i=1}^{n} -2x_i \left(y_i' - m_3 x_i - m_4 y_i - t_2 \right) = 0 \Rightarrow \sum_{i=1}^{n} x_i y_i' = m_3 \sum_{i=1}^{n} x_i^2 + m_4 \sum_{i=1}^{n} x_i y_i + t_2 \sum_{i=1}^{n} x_i$

(4) $\sum_{i=1}^{n} -2y_i \left(y_i' - m_3 x_i - m_4 y_i - t_2 \right) = 0 \Rightarrow \sum_{i=1}^{n} y_i y_i' = m_3 \sum_{i=1}^{n} x_i y_i + m_4 \sum_{i=1}^{n} y_i^2 + t_2 \sum_{i=1}^{n} y_i$

(5) $\sum_{i=1}^{n} -2\left(x_i' - m_1 x_i - m_2 y_i - t_1\right) = 0 \Rightarrow \sum_{i=1}^{n} x_i' = m_1 \sum_{i=1}^{n} x_i + m_2 \sum_{i=1}^{n} y_i + nt_1$

(6) $\sum_{i=1}^{n} -2\left(y_i' - m_3 x_i - m_4 y_i - t_2\right) = 0 \Rightarrow \sum_{i=1}^{n} y_i' = m_3 \sum_{i=1}^{n} x_i + m_4 \sum_{i=1}^{n} y_i + nt_2$

The matrix form of the equation **Sh = u** becomes:

$$\begin{bmatrix} \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i y_i & 0 & 0 & \sum_{i=1}^{n} x_i & 0 \\ \sum_{i=1}^{n} x_i y_i & \sum_{i=1}^{n} y_i^2 & 0 & 0 & \sum_{i=1}^{n} y_i & 0 \\ 0 & 0 & \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i y_i & 0 & \sum_{i=1}^{n} x_i \\ 0 & 0 & \sum_{i=1}^{n} x_i y_i & \sum_{i=1}^{n} y_i^2 & 0 & \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} y_i & 0 & 0 & n & 0 \\ 0 & 0 & \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} y_i & 0 & n \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} x_i x_i' \\ \sum_{i=1}^{n} y_i x_i' \\ \sum_{i=1}^{n} x_i y_i' \\ \sum_{i=1}^{n} y_i y_i' \\ \sum_{i=1}^{n} x_i' \\ \sum_{i=1}^{n} y_i' \end{bmatrix}$$

c) Thus, one may solve the transformation by computing $\mathbf{h} = \mathbf{S}^{-1}\mathbf{u}$. Compute the affine transformation from the following point correspondences $\{(0,0) \rightarrow (1,2)\}$, $\{(1,0) \rightarrow (3,2)\}$, and $\{(0,1) \rightarrow (1,4)\}$.
(Hint: This calculation can be done with pen and paper but you may check the correct answer by running the function `affinefit` given for the third exercise round in `Exercise03.zip`. Another way to check the answer is to draw the point correspondences on paper and visually determine the correct solution.)

The identities in the S matrix can be calculated as follows:

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i y_i & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i y_i & \sum_{i=1}^{n} y_i^2 & \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} y_i & n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

Likewise, the identities in the u vector can be calculated as:

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1' & y_1' & 1 \\ x_2' & y_2' & 1 \\ x_3' & y_3' & 1 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} x_i x_i' & \sum_{i=1}^{n} x_i y_i' & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} y_i x_i' & \sum_{i=1}^{n} y_i y' & \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i' & \sum_{i=1}^{n} y_i' & n \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 4 & 1 \\ 5 & 8 & 3 \end{bmatrix}$$

Finally, the optimal parameters of the affine transformation are:

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 4 \\ 5 \\ 8 \end{bmatrix} \Rightarrow \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 1 \\ 2 \\ 4 \\ 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \\ 1 \\ 2 \end{bmatrix} \text{(answer)}$$

Exercise 2. Similarity transformation from two point correspondences. (pen & paper) A similarity transformation consists of rotation, scaling and translation and is defined in two dimensions as follows:

$$\mathbf{x}' = s\mathbf{R}\mathbf{x} + \mathbf{t} \quad \Leftrightarrow \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = s \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix} \tag{1}$$

Describe a method for solving the parameters s, θ, t_x, t_y of a similarity transformation from two point correspondences $\{\mathbf{x}_1 \to \mathbf{x}'_1\}$, $\{\mathbf{x}_2 \to \mathbf{x}'_2\}$ using the following stages:

a) Compute the vectors $\mathbf{v}' = \mathbf{x}'_2 - \mathbf{x}'_1$ and $\mathbf{v} = \mathbf{x}_2 - \mathbf{x}_1$ and present a formula to recover the rotation angle θ from the corresponding unit vectors.

Let denote the vectors as follows:

$$v = \mathbf{x_2} - \mathbf{x_1} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix},\ v' = \mathbf{x'_2} - \mathbf{x'_1} = \begin{bmatrix} x'_2 - x'_1 \\ y'_2 - y'_1 \end{bmatrix} = s \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}$$

Then, the formula to recover the rotation angle from the corresponding unit vectors is:

$$\theta = \arccos\left(\frac{v'v}{\|v'\|\|v\|}\right) = \arccos\left(\frac{(x'_2 - x'_1)(x_2 - x_1) + (y'_2 - y'_1)(y_2 - y_1)}{\sqrt{(x'_2 - x'_1)^2 + (y'_2 - y'_1)^2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}\right) \text{(answer)}$$

b) Compute the scale factor s as the ratio of the norms of vectors $\mathbf{v}'$ and $\mathbf{v}$.

The scale factor as the ratio of the norms is:

$$s = \frac{\|v'\|}{\|v\|} = \frac{\sqrt{(x'_2 - x'_1)^2 + (y'_2 - y'_1)^2}}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} \text{(answer)}$$

c) After solving s and θ compute $\mathbf{t}$ using equation (1) and either one of the two point correspondences.

The equation (1) is:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = s \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix} \Rightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} sx\cos\theta - sy\sin\theta + t_x \\ sx\sin\theta + sy\cos\theta + t_y \end{bmatrix}$$

Then we derive the translation => $\begin{bmatrix} t_x \\ t_y \end{bmatrix} = \begin{bmatrix} x' - sx\cos\theta + sy\sin\theta \\ y' - sx\sin\theta - sy\cos\theta \end{bmatrix}$ (answer)

d) Use the procedure to compute the transformation from the following point correspondences: $\{(\frac{1}{2},0) \rightarrow (0,0)\}$, $\{(0,\frac{1}{2}) \rightarrow (-1,-1)\}$.
(Hint: Drawing the point correspondences on a grid paper may help you to check your answer.)

From the points, the vectors are constructed as:

$$v = \begin{bmatrix} 0-1/2 \\ 1/2-0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}, \; v' = \begin{bmatrix} -1-0 \\ -1-0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

Calculating the angle:

$$\theta = \arccos\left(\frac{(-1-0)(0-1/2)+(-1-0)(1/2-0)}{\sqrt{(-1-0)^2+(-1-0)^2}\sqrt{(0-1/2)^2+(1/2-0)^2}} \right) = \arccos(0) \Rightarrow \theta = \frac{\pi}{2}$$

Calculating the scale:

$$s = \frac{\sqrt{(-1-0)^2+(-1-0)^2}}{\sqrt{(0-1/2)^2+(1/2-0)^2}} = 2$$

Pick a random endpoint, which is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$

Calculare the translation:

$$\begin{bmatrix} t_x \\ t_y \end{bmatrix} = \begin{bmatrix} 0-2\times\frac{1}{2}\times\cos\frac{\pi}{2}+2\times 0\times\sin\frac{\pi}{2} \\ 0-2\times\frac{1}{2}\times\sin\frac{\pi}{2}-2\times 0\times\cos\frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Finally, the transformation becomes:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = 2\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$ (answer)