Exercise 1. Homogeneous coordinates.

a) The equation of a line in the plane is

$$ax + by + c = 0.$$

Show that by using homogeneous coordinates this can be written as

$$\mathbf{x}^\top \mathbf{l} = 0$$

where $\mathbf{l} = \begin{pmatrix} a & b & c \end{pmatrix}^\top$.

The homogeneous coordinates is $x^T = \begin{bmatrix} xw & yw & w \end{bmatrix}$ and $l = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

=> $x^T l = \begin{bmatrix} xw & yw & w \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} axw + byw + cw \end{bmatrix} = \begin{bmatrix} w(ax + by + c) \end{bmatrix} = \begin{bmatrix} w \times 0 \end{bmatrix} = 0$ (proven)

b) Show that the intersection of two lines $\mathbf{l}$ and $\mathbf{l}'$ is the point $\mathbf{x} = \mathbf{l} \times \mathbf{l}'$.

Let $l = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $l' = \begin{bmatrix} a' \\ b' \\ c' \end{bmatrix}$ => $l \times l' = \begin{bmatrix} bc' - b'c \\ ca' - c'a \\ ab' - a'b \end{bmatrix} = X$

Converting homogeneous coordinates to Euclidean coordinate:

$x = \begin{bmatrix} \frac{X_1}{X_3} \\ \frac{X_2}{X_3} \end{bmatrix} = \begin{bmatrix} \frac{bc' - b'c}{ab' - a'b} \\ \frac{ca' - c'a}{ab' - a'b} \end{bmatrix}$. Plugging x into both line equation of l and l', we have:

For l: $a\frac{bc' - b'c}{ab' - a'b} + b\frac{ca' - c'a}{ab' - a'b} + c = \frac{abc' - ab'c + a'cb - abc'}{ab' - a'b} + c = -\frac{c(ab' - a'b)}{ab' - a'b} + c = -c + c = 0$

For l': $a'\frac{bc' - b'c}{ab' - a'b} + b'\frac{ca' - c'a}{ab' - a'b} + c' = \frac{a'bc' - a'b'c + a'b'c - ab'c'}{ab' - a'b} + c' = -\frac{c'(ab' - a'b)}{ab' - a'b} + c' = -c' + c' = 0$

The point x lies on both lines l and l'. Therefore, the cross product of two lines is their intersection if the lines lie on the same plane.

c) Show that the line through two points $\mathbf{x}$ and $\mathbf{x}'$ is $\mathbf{l} = \mathbf{x} \times \mathbf{x}'$.

Let $x = \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}$ and $x' = \begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix} \Rightarrow x \times x' = \begin{bmatrix} y_1 - y_2 \\ x_2 - x_1 \\ x_1 y_2 - x_2 y_1 \end{bmatrix} = l$

The equation of l becomes: $(y_1 - y_2)x + (x_2 - x_1)y + x_1 y_2 - x_2 y_1 = 0$

Plugging x and x' into l, we have:

For x: $(y_1 - y_2)x_1 + (x_2 - x_1)y_1 + x_1 y_2 - x_2 y_1 = x_1 y_1 - x_1 y_2 + x_2 y_1 - x_1 y_1 + x_1 y_2 - x_2 y_1 = 0$

For x': $(y_1 - y_2)x_2 + (x_2 - x_1)y_2 + x_1 y_2 - x_2 y_1 = x_2 y_1 - x_2 y_2 + x_2 y_2 - x_1 y_2 + x_1 y_2 - x_2 y_1 = 0$

Therefore, the cross product of two points is a line that goes through them.

d) Show that for all $\alpha \in \mathbb{R}$ the point $\mathbf{y} = \alpha\mathbf{x} + (1-\alpha)\mathbf{x}'$ lies on the line through points $\mathbf{x}$ and $\mathbf{x}'$.
(Hint: In tasks *b*, *c*, *d* above, you can utilize the fact that for three-element vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ the scalar triple product, $(\mathbf{a} \times \mathbf{b})^\top \mathbf{c}$, is zero if any two of the vectors are parallel.)

From (c), the line that goes through points x and x' is $(y_1 - y_2)x + (x_2 - x_1)y + x_1 y_2 - x_2 y_1 = 0$

All the points lying on the line $y = \alpha x + (1-\alpha)x' = \alpha \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} + (1-\alpha) \begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha x_1 + (1-\alpha)x_2 \\ \alpha y_1 + (1-\alpha)y_2 \\ \alpha + (1-\alpha) \end{bmatrix}$

Converting the homogenous coordinate into Euclidean coordinate, we have:

$X = \begin{bmatrix} \alpha x_1 + (1-\alpha)x_2 \\ \alpha y_1 + (1-\alpha)y_2 \\ \alpha + (1-\alpha) \end{bmatrix} \Rightarrow x = \begin{bmatrix} \dfrac{\alpha x_1 + (1-\alpha)x_2}{\alpha + (1-\alpha)} \\ \dfrac{\alpha y_1 + (1-\alpha)y_2}{\alpha + (1-\alpha)} \end{bmatrix}$. Plugging into the line equation above:

$$(y_1 - y_2)\frac{\alpha x_1 + (1-\alpha)x_2}{\alpha + (1-\alpha)} + (x_2 - x_1)\frac{\alpha y_1 + (1-\alpha)y_2}{\alpha + (1-\alpha)} + x_1 y_2 - x_2 y_1$$

$$= \frac{(y_1 - y_2)(\alpha x_1 + (1-\alpha)x_2) + (x_2 - x_1)(\alpha y_1 + (1-\alpha)y_2)}{\alpha + (1-\alpha)} + x_1 y_2 - x_2 y_1$$

$$= \frac{\alpha x_1 y_1 + (1-\alpha)x_2 y_1 - \alpha x_1 y_2 - (1-\alpha)x_2 y_2 + \alpha x_2 y_1 + (1-\alpha)x_2 y_2 - \alpha x_1 y_1 - (1-\alpha)x_1 y_2}{\alpha + (1-\alpha)} + x_1 y_2 - x_2 y_1$$

$$= \frac{\alpha(x_1 y_1 - x_1 y_2 + x_2 y_1 - x_1 y_1) + (1-\alpha)(x_2 y_1 - x_2 y_2 + x_2 y_2 - x_1 y_2)}{\alpha + (1-\alpha)} + x_1 y_2 - x_2 y_1$$

$$= \frac{\alpha(x_2 y_1 - x_1 y_2) + (1-\alpha)(x_2 y_1 - x_1 y_2)}{\alpha + (1-\alpha)} + x_1 y_2 - x_2 y_1 = \frac{(\alpha + (1-\alpha))(x_2 y_1 - x_1 y_2)}{\alpha + (1-\alpha)} + x_1 y_2 - x_2 y_1$$

$$= x_2 y_1 - x_1 y_2 + x_1 y_2 - x_2 y_1 = 0 \;\; (proven)$$

Exercise 2. Transformations in 2D.
a) Use homogeneous coordinates and give the matrix representations of the following transformation groups: translation, Euclidean transformation (rotation+translation), similarity transformation (scaling+rotation+translation), affine transformation, projective transformation.

Given a homogeneous coordinate point: $X = \begin{bmatrix} x & y & 1 \end{bmatrix}^T$

- The translation matrix: $T = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$

- The rotation matrix: $R_{clockwise} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *and* $R_{anticlockwise} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- The scaling matrix: $S = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- The Euclidean transformation matrix:

$$E = TR_{clockwise} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

- The similarity transformation matrix:

$$Sim = TR_{clockwise}S = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} S_x\cos\theta & -S_y\sin\theta & t_x \\ S_x\sin\theta & S_y\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

- The affine transformation matrix incorporates both change of basis and origin:

$$A = Origin \cdot Basis = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

- The projective transformation matrix incorporates all types of transformations above:

$$P = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$

b) What is the number of degrees of freedom in these transformations?

Number of freedoms in
- Translation transformation: 2 Dof
- Euclidean transformation: 3 Dof
- Similarity transformation: 4 Dof
- Affine transformation: 6 Dof
- Projection transformation: 8 Dof

c) Why is the number of degrees of freedom in a projective transformation less than the number of elements in a 3×3 matrix?
(Hint: The answers to the first two sub-tasks are directly given in Table 2.1 in Hartley & Zisserman.)

The projection or homography transformation matrix is defined up to a scale. In other words, it can be multiplied by a non zero constant without any affect on projective transformation. Let this constant be the inverse of any element in the 3x3 projection matrix. Thus, multiply the homography matrix by this constant results in one of the elements equals to 1, while the other elements are still unknown. Therefore, the projection matrix has 8 degree of freedom even though it contains 9 elements (3x3 matrix) because the number of unknowns that need to be solved for the matrix is only 8.

Exercise 3. Planar projective transformation.
The equation of a line on a plane, $ax + by + c = 0$, can be written as $\mathbf{l}^\top \mathbf{x} = 0$, where $\mathbf{l} = [a\ b\ c]^\top$ and $\mathbf{x}$ are homogeneous coordinates for lines and points, respectively. Under a planar projective transformation, represented with an invertible 3×3 matrix $\mathbf{H}$, points transform as

$$\mathbf{x}' = \mathbf{H}\mathbf{x}.$$

a) Given the matrix $\mathbf{H}$ for transforming points, as defined above, define the line transformation (i.e. transformation that gives $\mathbf{l}'$ which is a transformed version of $\mathbf{l}$).

We have $l^T x = 0$, $l'^T x' = 0 \Rightarrow l'^T x' = l^T x$ and also $x' = Hx \Rightarrow x = H^{-1}x'$
$\Rightarrow l'^T x' = l^T H^{-1} x'$
$\Rightarrow l'^T = l^T H^{-1}$ or $\Rightarrow l' = H^{-T} l$ (answer)

b) A projective invariant is a quantity which does not change its value in the transformation. Using the transformation rules for points and lines, show that two lines, $\mathbf{l}_1, \mathbf{l}_2$, and two points, $\mathbf{x}_1, \mathbf{x}_2$, not lying on the lines have the following invariant under projective transformation:

$$I = \frac{(\mathbf{l}_1^\top \mathbf{x}_1)(\mathbf{l}_2^\top \mathbf{x}_2)}{(\mathbf{l}_1^\top \mathbf{x}_2)(\mathbf{l}_2^\top \mathbf{x}_1)}.$$

Why similar construction does not give projective invariants with fewer number of points or lines? (Hint: Projective invariants defined via homogeneous coordinates must be invariant also to arbitrary scaling of the homogeneous coordinate vectors with a non-zero scaling factor.)

Note: Exercise 3 above is from Chapter 2 of the book by Hartley and Zisserman and that chapter is helpful reference.

From (3), we have: $l^T x = l'^T x' = l'^T Hx \Rightarrow l^T = l'^T H$
Under the projective transformation, the identity is:

$$I = \frac{\left(l_1^T x_1\right)\left(l_2^T x_2\right)}{\left(l_1^T x_2\right)\left(l_2^T x_1\right)} = \frac{\left(l_2^T H H^{-1} x_2\right)\left(l_1^T H^{-1} H x_1\right)}{\left(l_2^T x_1\right)\left(l_1^T x_2\right)} = \frac{\left(l_2^T x_2\right)\left(l_1^T x_1\right)}{\left(l_2^T x_1\right)\left(l_1^T x_2\right)} = I$$

Therefore, this identity is invariant under projective transformation

Because projective invariants defined via homogeneous coordinates must be invariant also to arbitrary scaling of the homogeneous coordinate vectors with a non-zero scaling vector, we define a, b, c, d as the scaling factor for I identity

$$\frac{\left(al_1^T bx_1\right)\left(cl_2^T dx_2\right)}{\left(al_1^T dx_2\right)\left(cl_2^T bx_1\right)} = \frac{\left(al_2^T HH^{-1}bx_2\right)\left(cl_1^T H^{-1}Hdx_1\right)}{\left(al_2^T dx_1\right)\left(cl_1^T bx_2\right)} = \frac{ab\left(l_2^T x_2\right)cd\left(l_1^T x_1\right)}{ad\left(l_2^T x_1\right)cb\left(l_1^T x_2\right)} = I$$

Therefore, two lines and two points are required to cancel out the scaling factors from the denominator and nominator. Similar construction with fewer points or lines is not possible in projective transformation because the scaling factors are not totally cancelled out and thus the identity is variant to arbitrary scaling of the homogeneous coordinate vectors.