CS-E4850 Computer Vision, Answers to Exercise Round 5

Adam Ilyas 725819

October 12, 2018

Exercise 1. Total least squares line fitting.
An overview of least squares line fitting is presented on the slide 13 of Lecture 4. Study it in detail and present the derivation with the following stages:

1) Given a line $ax + by - d = 0$, where the coefficients are normalized so that $a^2 + b^2 = 1$, show that the distance between a point (x_i, y_i) and the line is $|ax_i + by_i - d|$. **Solution**

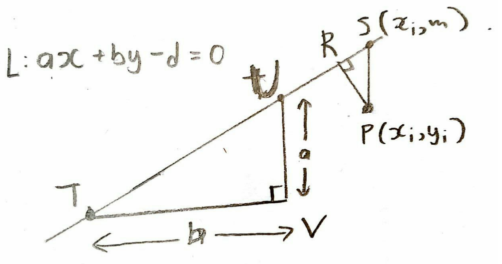


Figure 1: Similar triangles

ΔPRS and ΔTVU are Similar triangles Hence $\frac{|PR|}{|PS|} = \frac{|TV|}{|TU|}$

$Distance = |PR| = \frac{|PS| * |TV|}{|TU|} = \frac{|y_i - m| * |b|}{\sqrt{a^2+b^2}}$

We let S to be (x_i, m) Since S is on line L, it follows $ax_i + bm - d = 0$

$m = \frac{-ax_i+d}{b}$,

$$distance = |yi - \frac{-ax_i+d}{b}| \times |b|/\sqrt{a^2+b^2} = |\frac{byi+ax_i-d}{b}| \times b/\sqrt{a^2+b^2}$$
$$= |\frac{ax_i+byi-d}{\sqrt{a^2+b^2}}|$$

Since $a^2+b^2=1$, $distance = |ax_i + byi - d|$

2) Thus, given n points $(x_i, y_i), i = 1, ..., n$, the sum of squared distances between the points and the line is $E = {}^{n}_{i=1}(ax_i + by_i - d)^2$. In order to find the minimum of E, compute the partial derivative $\delta E/\delta d$, set it to zero, and solve d in terms of a and b.

Partial Derivative $\delta E/\delta d = \sum_{i=1}^{n} -2(ax_i + by_i - d) = 0$

$$a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} y_i - \sum_{i=1}^{n} d) = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} y_i - nd)$$

$$nd = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} y_i$$
$$d = \frac{a}{n}\sum_{i=1}^{n} x_i + \frac{b}{n}\sum_{i=1}^{n} y_i = a\bar{x} + b\bar{y}$$

3) Substitute the expression obtained for d to the formula of E, and show that then $E = (a\ b)U^\intercal U(a\ b)^\intercal$, where matrix U depends on the point coordinates (x_i, y_i).

Substitute $E = \sum_{i=1}^{n}(ax_i + by_i - a\bar{x} - b\bar{y})^2$

$$E = \sum_{i=1}^{n}(a(x_i - \bar{x}) + b(y_i - b\bar{y}))^2$$
$$= \left\| \begin{bmatrix} x_1 - \bar{x} & y_1 - \bar{y} \\ x_2 - \bar{x} & y_2 - \bar{y} \\ \vdots & \vdots \\ x_n - \bar{x} & y_n - \bar{y} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \right\|^2 = \|UN\|^2 = (UN)^\intercal(UN)$$
$$= N^\intercal U^\intercal UN$$

4) Thus, the task is to minimize $||U(a\ b)^\intercal||$ under the constraint $a^2 + b^2 = 1$. The solution for $(a\ b)^\intercal$ is the eigenvector of $U^\intercal U$ corresponding to the smallest eigenvalue, and d can be solved thereafter using the expression obtained above in the stage two.

$$U = \begin{bmatrix} x_1 - \bar{x} & y_1 - \bar{y} \\ \vdots & \vdots \\ x_n - \bar{x} & y_n - \bar{y} \end{bmatrix}$$

$$U^\intercal U = \begin{bmatrix} \sum_{i=1}^{n}(x_i - \bar{x})^2 & \sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) \\ \sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) & \sum_{i=1}^{n}(y_i - \bar{y})^2 \end{bmatrix}$$

Thus, if we use this second moment matrix $U^\intercal U$, we can get the following:

$$\begin{aligned} E &= (UN)^\intercal(UN) \\ &= N^\intercal U^\intercal U N \\ &= \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} \sum_{i=1}^{n}(x_i - \bar{x})^2 & \sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) \\ \sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) & \sum_{i=1}^{n}(y_i - \bar{y})^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \end{aligned}$$

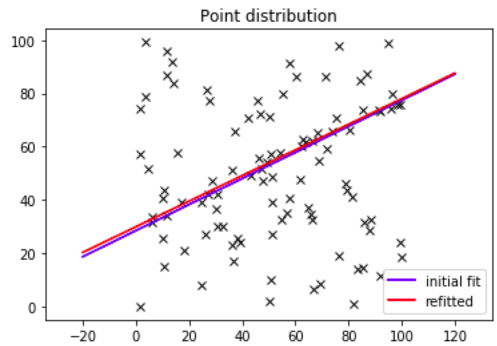
We can solve for min E:

$$\frac{dE}{dN} = 2(U^\intercal U)N = 0$$

Solve for

$$(U^\intercal U)N = 0$$

Robust line fitting using RANSAC.




```
In [9]: print(f'Estimated line equation: {m_new}x + {b_new}')

        Estimated line equation: 0.48057749432376784x + 29.918244462622305
```

Figure 2: Line Fitting