CS-E4950 Computer Vision
Exercise Round 5

Rui Qu
rui.qu@aalto.fi

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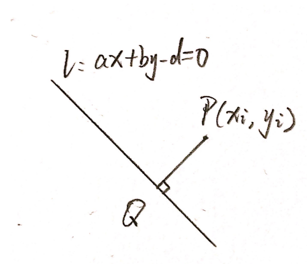
Exercise 1 Solution

1.1



Given a line $l:\ ax + by - d = 0$ and a point $P(x_i, y_i)$, line PQ is perpendicular to line l, hence:

$$k_{PQ} \cdot k_l = -1,\ k_{PQ} = \frac{b}{a} \tag{1}$$

The equation of line PQ is:

$$y - y_i = \frac{b}{a}(x - x_i) \tag{2}$$

The intersection of line l and PQ is:

$$Q(\frac{b^2 x_i - aby_o + ad}{a^2 + b^2},\ \frac{a^2 y_i - abx_i + bd}{a^2 + b^2}) \tag{3}$$

Hence, the distance between point P and line l is:

$$\begin{aligned} |PQ| &= \sqrt{(\frac{b^2 x_i - aby_i + ad}{a^2 + b^2} - x_i)^2 + (\frac{a^2 y_i - abx_i + bd}{a^2 + b^2} - y_i)^2} \\ &= \frac{|ax_i + by_i - d|}{\sqrt{a^2 + b^2}} \end{aligned} \tag{4}$$

Since $a^2 + b^2 = 1$, the distance is $|ax_i + by_i - d|$.

1.2

Compute the partial derivative:

$$\frac{\partial E}{\partial d} = \sum_{i=1}^{n} -2(ax_i + by_i - d) \tag{5}$$

Set it to zero and solve d in terms of a and b:

$$\begin{aligned} a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} y_i - d\sum_{i=1}^{n} = 0 \\ nd = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} y_i \\ d = \frac{a}{n}\sum_{i=1}^{n} x_i + \frac{b}{n}\sum_{i=1}^{n} y_i = a\overline{x} + b\overline{y} \end{aligned} \tag{6}$$

1.3

Substitute d to the formula of E:

$$\begin{aligned} E &= \sum_{i=1}^{n}(ax_i + by_i - a\overline{x} - b\overline{y})^2 \\ &= \left\| \begin{bmatrix} x_1 - \overline{x} & y_1 - \overline{y} \\ \vdots & \vdots \\ x_n - \overline{x} & y_n - \overline{y} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \right\|^2 \\ &= (UN)^T(UN) \\ &= N^T U^T U N \end{aligned} \tag{7}$$

1.4

Continue the derivation in 1.3:

$$\begin{aligned} E &= N^T U^T U N \\ &= \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} \sum_{i=1}^{n}(x_i - \overline{x})^2 & \sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y}) \\ \sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y}) & \sum_{i=1}^{n}(y_i - \overline{y})^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \end{aligned} \tag{8}$$

Solution to $2(U^TU)N = 0$, subject to $||N||^2 = 1$, eigenvector of U^TU

$$\begin{aligned} \frac{\mathrm{d}E}{\mathrm{d}N} = 2(U^TU)N = 0 \\ (U^TU)N = 0 \end{aligned} \tag{9}$$