CS-E4950 Computer Vision
Exercise Round 1

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Exercise 1 Solution

a)
Any point in the projective plane is represented by a triple (X, Y, Z), which is called the homogeneous coordinates or projective coordinates of the point, where $X, Y, Z \neq 0$.

In Euclidean plane, homogeneous coordinates can be represented as $(\frac{X}{Z}, \frac{Y}{Z})$

The point represented by a given set of homogeneous coordinates is unchanged if the coordinates are multiplied by a common factor. Conversely, two sets of homogeneous coordinates represent the same point if and only if one is obtained from the other by multiplying all the coordinates by the same non-zero constant.

The equation of a line in the plane could be re-written as:

$$ax + by + c = \begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{x}^\top \mathbf{l} = 0 \tag{1}$$

, where $\mathbf{l} = \begin{pmatrix} a & b & c \end{pmatrix}^\top$. Like the discussion above, if both sides multiply with arbitrary nonzero scalar n, all 3-vectors $n\mathbf{l} = n \begin{pmatrix} a & b & c \end{pmatrix}^\top$ still represent the same line. In terms of homogeneous coordinates, points $\begin{pmatrix} x & y \end{pmatrix}$ on the line l if and only if $\mathbf{x}^\top \mathbf{l} = 0$.

b)
Given point $\mathbf{x}$ is the intersection of two lines $\mathbf{l}$ and $\mathbf{l}'$, thus $\mathbf{x}$ fulfills both $\mathbf{x}^\top \mathbf{l} = 0$ and $\mathbf{x}^\top \mathbf{l}' = 0$. Considering triple scalar product identity $\mathbf{l} \dot{(} \mathbf{l} \times \mathbf{l}') = \mathbf{l}' \dot{(} \mathbf{l} \times \mathbf{l}') = 0$, we could see that $x = \mathbf{l} \times \mathbf{l}'$. Hence, the intersection of two lines $\mathbf{l}$ and $\mathbf{l}'$ is the point $x = \mathbf{l} \times \mathbf{l}'$.

c)
Similar to b), given both points $\mathbf{x}, \mathbf{x}'$ lie on the line $\mathbf{l}$, thus $\mathbf{x}^\top \mathbf{l} = 0$ and $\mathbf{x}'^\top \mathbf{l} = 0$

Considering triple scalar product identity $\mathbf{x}\dot{(}\mathbf{x} \times \mathbf{x}') = \mathbf{x}'\dot{(}\mathbf{x} \times \mathbf{x}')$ we could see that $\mathbf{l} = \mathbf{x} \times \mathbf{x}'$. Hence, the line goes through the two points.

d)
According to c) the line through points $\mathbf{x}, \mathbf{x}'$ is $\mathbf{l} = \mathbf{x} \times \mathbf{x}'$. If point $\mathbf{y}$ lies on the line $\mathbf{l}$, then

$$\begin{aligned} \mathbf{y}^\top \mathbf{l} &= 0 \\ \mathbf{y}^\top (\mathbf{x} \times \mathbf{x}') &= 0 \\ (\alpha \mathbf{x} + (1-\alpha)\mathbf{x}')^\top (\mathbf{x} \times \mathbf{x}') &= 0 \\ \alpha \mathbf{x}^\top (\mathbf{x} \times \mathbf{x}') + (1-\alpha)\mathbf{x}'^\top (\mathbf{x} \times \mathbf{x}') &= 0 \end{aligned} \tag{2}$$

According to triple scalar product identity, $\mathbf{x}\dot{(}\mathbf{x} \times \mathbf{x}') = \mathbf{x}'\dot{(}\mathbf{x} \times \mathbf{x}')$, we could say $\mathbf{y}$ lies on the line $\mathbf{l}$.

Exercise 2 Solution

a) & b)

Translation:

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \tag{3}$$

2 degrees of freedom (t_x, t_y)

Euclidean transformation(rotation + translation):

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \tag{4}$$

3 degrees of freedom (t_x, t_y, θ)

Similarity transformation(scaling+rotation+translation):

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \tag{5}$$

4 degrees of freedom (t_x, t_y, θ, s)

Affine translation:

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \tag{6}$$

6 degrees of freedom $(t_x, t_y, a_{11}, a_{12}, a_{21}, a_{22})$

Projective translation:

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \tag{7}$$

8 degrees of freedom among 9 elements in a 3×3 matrix(the scale of the elements doesn't count according to homogeneous coordinate)

c)
We can scale our matrix by arbitrary scalar value and it won't change the interpretation of the homogeneous transformation. For example, dividing the matrix by the h_{33} term, we get the following expression:

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} \frac{h_{11}}{h_{33}} & \frac{h_{12}}{h_{33}} & \frac{h_{13}}{h_{33}} \\ \frac{h_{21}}{h_{33}} & \frac{h_{22}}{h_{33}} & \frac{h_{23}}{h_{33}} \\ \frac{h_{31}}{h_{33}} & \frac{h_{32}}{h_{33}} & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \tag{8}$$

With k representing an arbitrary scaling factor and new notations for the matrix terms based on these eight ratios, we rewrite the equation:

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = k \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \tag{9}$$

It is the RATIOS of the nine values that actually matter, there are eight independent ratios among nine values. Hence, a projective transformation has 8 degrees of freedom, not 9.

Exercise 3 Solution

a)
Given the matrix $\mathbf{H}$ for transformaing points, then $\tilde{\mathbf{l}}^\top \mathbf{H}^{-1} \mathbf{H} \tilde{\mathbf{x}} = 0$
All points $\mathbf{H}\tilde{\mathbf{x}}$ lie on the line $\tilde{\mathbf{l}}^\top \mathbf{H}^- 1$. Thus, the line transformation is $\tilde{\mathbf{l}}^\top \mathbf{H}^- 1$.
a)
we construct the straight line passing through the two points of the configuration. This line intersects the two straight lines of the configuration in two points, which can be at finite or infinite distance. We thus have four points on the same straight line: the two points of the configuration, and the two intersection points; these points define a cross ratio which is an invariant for the problem. The set of planar collineations which are defined by a 3 x 3 matrix up to a scale factor, is a manifold of dimension 8. The constraint (determinant not equal to zero) removes a submanifold of transformations, but leaves the dimension unchanged. A straight line or a point are defined by 3 parameters, but up to a scale factor: thus it belongs to a manifold of dimension 2. The configurations contain two lines and two points and also form a manifold of dimension 8.