CS-E4950 Computer Vision
Exercise Round 2

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1 Pinhole Camera

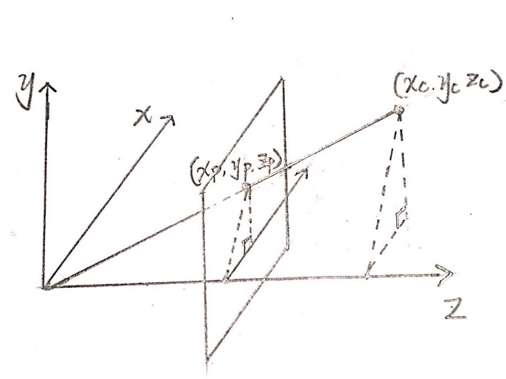

Figure 1: Perspective Projection

According the similar triangles shown in the figure above, the image plane is parallel to x_c, y_c, z_c plane. focal length $z_p = f$. We could get the equations below:

$$\begin{aligned} \frac{x_c}{z_c} &= \frac{x_p}{f}, x_p = f\frac{x_c}{z_c} \\ \frac{y_c}{z_c} &= \frac{y_p}{f}, y_p = f\frac{y_c}{z_c} \end{aligned} \tag{1}$$

2 Pixel Coordinate Frame

a)

Given u and v axis are parallel to x and y axis respectively. Transform image coordinate to pixel coordinate:

$$\begin{bmatrix} m_u & 0 & 0 \\ 0 & m_v & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f\frac{x_c}{z_c} + p_x \\ f\frac{y_c}{z_c} + p_y \\ 1 \end{bmatrix} = \begin{bmatrix} m_u f\frac{x_c}{z_c} + m_u p_x \\ m_v f\frac{y_c}{z_c} + m_v p_y \\ 1 \end{bmatrix} = \begin{bmatrix} m_u x_p + u_0 \\ m_v y_p + v_o \\ 1 \end{bmatrix} \tag{2}$$

The formula to transform point x_p to pixel coordinate is:

$$u = m_u x_p + u_0, \ v = m_v y_p + v_o \tag{3}$$

b)

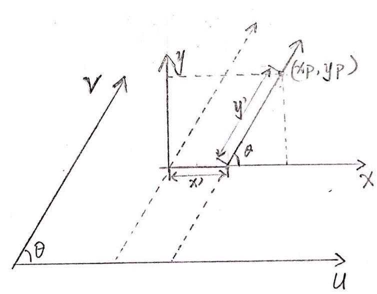

Figure 2: Pixel Coordinate Frame

Given u axis is parallel to x axis and the angle between u and v axis is θ. The figure is given above, where:

$$\begin{aligned} \sin\theta = \frac{y_p}{y'}, y' = \frac{y_p}{\sin\theta} \\ \tan\theta = \frac{y_p}{x_p - x'}, x' = x_p - \frac{y_p}{\tan\theta} \end{aligned} \tag{4}$$

The formula to transform point x_p to pixel coordinate is:

$$\begin{aligned} u = m_u x' + u_0, v = m_v y' + v_o \\ u = m_u x_p - \frac{m_u}{\tan\theta} y_p + u_0, v = m_v \frac{y_p}{\sin\theta} + v_0 \end{aligned} \tag{5}$$

3 Intrinsic camera calibration matrix

Use homogeneous coordinates to represent case (2.b) above with the camera's intrinsic calibration matrix.

$$\mathbf{K_{x\times3}} \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \tag{6}$$

According to equation (5) and (6):

$$\begin{aligned} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} &= \begin{bmatrix} m_u & -\frac{m_u}{\tan\theta} & u_0 \\ 0 & \frac{m_v}{\sin\theta} & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_p \\ y_p \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} m_u & -\frac{m_u}{\tan\theta} & u_0 \\ 0 & \frac{m_v}{\sin\theta} & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} fx_c \\ fy_c \\ fz_c \end{bmatrix} \end{aligned} \tag{7}$$

4 Camera projection matrix

From real world to pixel camera matrix:

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \mathbf{Kx_c} = \mathbf{K}(\mathbf{Rx_w} + \mathbf{t}) = \mathbf{K}\begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix} \begin{bmatrix} \mathbf{x_w} \\ 1 \end{bmatrix} = \mathbf{P} \begin{bmatrix} \mathbf{x_w} \\ 1 \end{bmatrix} \tag{8}$$

5 Rotation matrix

a)

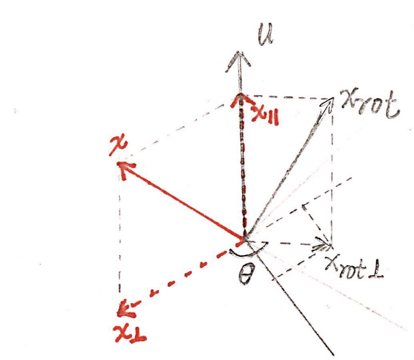

Figure 3: Rodrigues Formula

Given arbitrary vector $\mathbf{x}$ rotates about rotation axis $\mathbf{u}$ by arbitrary angel θ, vector $\mathbf{x}$ could be decomposed into parallel and perpendicular to axix $\mathbf{u}$, $\mathbf{x} = \mathbf{x}_{\parallel} + \mathbf{x}_{\perp}$, where:

$$\begin{aligned} \mathbf{x}_{\parallel} &= (\mathbf{x} \cdot \mathbf{u})\mathbf{u} \\ \mathbf{x}_{\perp} &= \mathbf{x} - \mathbf{x}_{\parallel} = -\mathbf{u} \times (\mathbf{u} \times \mathbf{x}) \end{aligned} \tag{9}$$

Similarly, $\mathbf{x_{rot}}$ is the vector after rotation. From the figure 3, we could get:

$$\begin{aligned} \mathbf{x_{rot}} &= \mathbf{x_{rot_{\perp}}} + \mathbf{x}_{\parallel} \\ &= (\sin\theta \cdot \mathbf{u} \times \mathbf{x} + \cos\theta \cdot \mathbf{x}_{\perp}) + (\mathbf{x} \cdot \mathbf{u})\mathbf{u} \\ &= \sin\theta \cdot \mathbf{u} \times \mathbf{x} + \cos\theta \cdot (\mathbf{x} - \mathbf{x}_{\parallel}) + (\mathbf{x} \cdot \mathbf{u})\mathbf{u} \\ &= \sin\theta \cdot \mathbf{u} \times \mathbf{x} + \cos\theta \cdot [\mathbf{x} - (\mathbf{x} \cdot \mathbf{u})\mathbf{u}] + (\mathbf{x} \cdot \mathbf{u})\mathbf{u} \\ &= \sin\theta \cdot \mathbf{u} \times \mathbf{x} + \cos\theta \cdot \mathbf{x} + (1 - \cos\theta) \cdot (\mathbf{x} \cdot \mathbf{u})\mathbf{u} \end{aligned} \tag{10}$$

Given the notation in this exercise, rotate a vector$\mathbf{x}$ to $\mathbf{x}' = \mathbf{R}\mathbf{x} + \mathbf{t}$ by the angle θ about the axis $\mathbf{u}$, the Rodrigues Formula can be rewritten as:

$$\mathbf{R}\mathbf{x} = \sin\theta \cdot \mathbf{u} \times \mathbf{x} + \cos\theta \cdot \mathbf{x} + (1 - \cos\theta) \cdot (\mathbf{x} \cdot \mathbf{u})\mathbf{u} \tag{11}$$

b)

Represent $\mathbf{x}$ and $\mathbf{u} \times \mathbf{x}$ as column matrices:

$$\mathbf{u} \times \mathbf{x} = \begin{bmatrix} u_2x_3 - u_3x_2 \\ u_3x_1 - u_1x_3 \\ u_1x_2 - u_2x_1 \end{bmatrix} = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{12}$$

Use $\mathbf{U}$ to denote cross-product matrix:

$$\begin{aligned} \mathbf{U} &= \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \\ \mathbf{U}\mathbf{x} &= \mathbf{u} \times \mathbf{x} \end{aligned} \tag{13}$$

Hence, Rodrigues Formula derived by $\mathbf{u}$ and θ can be written as:

$$\mathbf{U} = \mathbf{I} + \sin\theta \cdot \mathbf{U} + (1 - \cos\theta)\mathbf{U}^2 \tag{14}$$