Convex Optimization: Homework 3

Due to Nov. 12, 2023

Problem 1: Consider the following simple optimization problem

$$\min f_0(x_1, x_2) \quad \text{s.t.} \quad 2x_1 + x_2 \geq 1,\ x_1 + 3x_2 \geq 1,\ x_1 \geq 0,\ x_2 \geq 0.$$

Write it in standard form and SDP form. Make a sketch of the feasible set. For each of the following objective functions, give the optimal set and the optimal value.

- $f_0(x_1, x_2) = x_1 + x_2$.
- $f_0(x_1, x_2) = -x_1 - x_2$.
- $f_0(x_1, x_2) = x_1$.
- $f_0(x_1, x_2) = \max\{x_1, x_2\}$.
- $f_0(x_1, x_2) = x_1^2 + 9x_2^2$.

Problem 2: In Lecture 11 (Nov. 1, 2023), you practiced the use of CVX for solving the Thruster problem described in Lec05, slides 4.3, 4.4. As a set up of the problem, the shape of a rigid body as well as locations of thrusters and desired orientation of the rigid body in terms of desired resulting horizontal and vertical forces and torque can be generated arbitrarily as you like for your own experimentation, but so that the problem would make sense. Using the code that you wrote in class complete the following tasks:

- Adopt the code for the problem formulation in slide 4.4.
- Adopt the code for the problem formulation in the middle of slide 4.8 as well.
- Compare the results for these two optimization problems that solve the same practical problem is different ways and draw your conclusions.

Problem 3: Implement using MATLAB or Python and CVX toolbox the Aalto light design problem (Lec00, slides 7, 8). You may not use log function is the objective as in slide 7, but may just use the absolute value of the difference between the actual and desired intensities. Use 11 lamps of maximum power of 100 W each and 7 patches. Generate the patches orientation in space and lamp positions arbitrarily as you like, but so that the problem would make sense. For example, all lamps should be on one side. Set the desired intensity in the patches so that the problem would be feasible.

- Program the minimax problem that we formulated in class and show your code and simulation results.
- Use unconstrained LS (solved by matrix inversion) as a solution approach. In this way, you can not consider constraints to the lamp maximum power and power non-negativity.
- Use LS with constraints to the lamp maximum power and power non-negativity. It is then not solved via matrix inversion and CVX toolbox has to be used.
- Compare the results by solving the same problem in 3 aforementioned ways and make conclusions.

Problem 4: Formulate the following problems as LPs. Explain in detail the relation between the optimal solution of each problem and the solution of its equivalent LP. As a hint, you can think about all of these problems as about different ways to solve a system of linear equations each promoting different features in the final solution. Then the choice of problem formulation can be connected to the prior knowledge that we have about the features of the solution that we are looking for.

- $\min \|\mathbf{Ax} - \mathbf{b}\|_\infty$.
- $\min \|\mathbf{Ax} - \mathbf{b}\|_1$.
- $\min \|\mathbf{Ax} - \mathbf{b}\|_1 \quad \text{subject to} \quad \|\mathbf{x}\|_\infty \leq 1$.
- $\min \|\mathbf{x}\|_1 \quad \text{subject to} \quad \|\mathbf{Ax} - \mathbf{b}\|_\infty \leq 1$.

- $\min \|\mathbf{Ax} - \mathbf{b}\|_1 + \|x\|_\infty$.

In each problem, $\mathbf{A} \in R^{m\times n}$ and $\mathbf{b} \in R^m$ are given.

Implement using MATLAB or Python and CVX toolbox all these optimization problems. Generate the data sets (matrix $\mathbf{A}$, vector $\mathbf{b}$) arbitrarily. Draw conclusions by comparing solutions to these problems to each other.

Problem 5: Implement using MATLAB or Python and CVX toolbox the Beamforming problem as it is formulated in Lec05, slides 4.22-4.24. Select the target direction as well as antenna geometry (the locations of antenna elements) arbitrarily, but take into account physical constraints. For example, the space between adjacent antenna elements should be less than the half of your source signal bandwidth to avoid ambiguity problem. It is recommended to select the number of antenna elements as 10, and the geometry of the array as uniform linear array, i.e., all antenna elements are located on a line and the distances between any adjacent elements are equal.

- Program the LS formulation in slide 4.23.
- Program the Chebyshev (minimax) design formulation in slide 4.24.
- Compare the results and draw conclusions.

Problem 6: The problem that is related to the beamforming problem (if beamforming problem is formulated as a variance minimization) is the *investment portfolio selection problem.* From the past, you know that the sum of first k largest components of vector $\mathbf{x} \in R^n$ ($k < n$) is a convex function. Denote this function as $f(\mathbf{x})$. Formulate the portfolio selection (portfolio-forming) problem using such function $f(\mathbf{x})$. Specifically, we wish to select from a total of n assets to form a portfolio (no short-selling is allowed). Asset i has an expected rate of return $\mu_i > 0$, and the covariance matrix is $\mathbf{\Sigma}$. We wish to minimize the variance of the portfolio while requiring that the expected rate of return to the portfolio is at least μ. Moreover, the weight of the first k largest components of investment should not exceed half of the total investment.