Convex Optimization: Homework 2

Due to Oct. 29, 2023

Problem 1: For each of the following functions determine whether it is convex, concave, quasiconvex, or quasiconcave?

1. $f(x) = e^x - 1$ on $\mathbf{R}$.
2. $f(x_1, x_2) = x_1 x_2$ on $\mathbf{R}^2_+$.
3. $f(x_1, x_2) = 1/(x_1 x_2)$ on $\mathbf{R}^2_+$.
4. $f(x_1, x_2) = x_1/x_2$ on $\mathbf{R}^2_+$.
5. $f(x_1, x_2) = x_1^2/x_2$ on $\mathbf{R} \times \mathbf{R}_+$.
6. $f(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$, where $0 \le \alpha \le 1$ on $\mathbf{R}^2_+$.

Problem 2: Let x be a real-valued random variable which takes values in $\{a_1, \ldots, a_n\}$ where $a_1 < a_2 < \ldots < a_n$, with $\mathbf{prob}(x = a_i) = p_i$, $i = 1, \ldots, n$. For each of the following functions of p (on the probability simplex $\{\mathbf{p} \in \mathbf{R}^n_+ | \mathbf{1}^T\mathbf{p} = 1\}$), determine if the function is convex, concave, quasiconvex, or quasiconcave.

1. $\mathbf{E}\ x$
2. $\mathbf{prob}(x \ge \alpha)$
3. $\mathbf{prob}(\alpha \le x \le \beta)$
4. $\sum_{i=1}^n p_i \log p_i$, the negative entropy of the distribution.
5. $\mathbf{var} x = \mathbf{E}(x - \mathbf{E}\ x)^2$
6. $\mathbf{quartile}(x) = \inf\{\beta |\ \mathbf{prob}(x \le \beta) \ge 0.25\}$.

7. The cardinality of the smallest set $A \subseteq \{a_1, \ldots, a_n\}$ with probability $\geq 90\%$. (By cardinality we mean the number of elements in A.
8. The minimum width interval that contains 90% of the probability, i.e.,

$$\inf\{\beta - \alpha | \ \mathbf{prob}(\alpha \leq x \leq \beta) \geq 0.9\}.$$

Problem 3: Prove that for strictly positive definite symmetric matrices, the following holds

- $f(\mathbf{X}) = \mathrm{tr}\{\mathbf{X}^{-1}\}$ is convex.
- $f(\mathbf{X}) = (\det \mathbf{X})^{1/n}$ is concave.

Problem 4: Prove that if $h(\mathbf{x})$ is twice continuously differentiable, then $h(\mathbf{x})$ is convex in R^n is equivalent to

$$\nabla^2 h(\mathbf{x}) \succeq 0 \qquad \forall \mathbf{x} \in R^n.$$

Problem 5: First, show that the sum of r largest components of a vector $\mathbf{x} \in \mathbf{R}^n$ is a convex function.

Then let the eigenvalues of symmetric $n \times n$ matrix $\mathbf{X}$ are denoted and ordered as

$$\lambda_1(\mathbf{X}) \geq \lambda_2(\mathbf{X}) \geq \cdots \geq \lambda_n(\mathbf{X}).$$

Take integer k as $1 \leq k \leq n$, and consider the function defined as

$$f(\mathbf{X}) = \sum_{i=1}^{k} \lambda_k(\mathbf{X}).$$

For $k = n$, this function is called the nuclear norm of a matrix, and for $k < n$ it is just a partial sum of eigenvalues.

Prove that the above defined function $f(\mathbf{X})$ is a convex function.

Hint: Show that

$$f(\mathbf{X}) = \sup\left\{\mathrm{tr}(\mathbf{U}^T\mathbf{X}\mathbf{U}) \mid \mathbf{U} \in R^{n \times k}, \ \mathbf{U}^T\mathbf{U} = I_k\right\}.$$

Problem 6: *A quadratic-over-linear composition:* Suppose that $f : \mathbf{R}^n \to \mathbf{R}$ is nonnegative and convex, and $g : \mathbf{R}^n \to \mathbf{R}$ is positive and concave.

Show that the function f^2/g, with domain $\mathrm{dom} f \cap \mathrm{dom} g$, is convex.

Problem 7: Let D_{KL} be the Kullback-Leibler divergence (if you do not know what it is, see equation (3.17) in the textbook). Prove the information theoretic inequality:

$$D_{\mathrm{KL}}(\mathbf{u}, \mathbf{v}) \geq 0$$

for all $\mathbf{u}$, $\mathbf{v}$.

Also show that

$$D_{\mathrm{KL}}(\mathbf{u}, \mathbf{v}) = 0$$

if and only if $\mathbf{u} = \mathbf{v}$.

It is useful to know that the Kullback-Leibler divergence can be expressed as

$$D_{\mathrm{KL}} = f(\mathbf{u}) - f(\mathbf{v}) - \bigtriangledown f^T(\mathbf{v})(\mathbf{u} - \mathbf{v})$$

where $f(\mathbf{v}) = \sum_{i=1}^{n} v_i \log v_i$ is the negative entropy of $\mathbf{v}$.