Convex Optimization: HW 1 (Convex Sets)

Due to Oct. 8, 2023

You may also need to check Chapter 2 in

Textbook: https://web.stanford.edu/~boyd/cvxbook/bv_cvxbook.pdf

Problem 1: Express the convex set

$$\mathcal{S} = \left\{ [x_1\, x_2]^T \in R_+^2 \;\middle|\; x_1 x_2 \geq 1 \right\}$$

as an intersection of halfspaces.

One possible Hint. If a and b are non-negative and $0 \leq \theta \leq 1$, then $a^\theta b^{1-\theta} \leq \theta a + (1-\theta)b$.

Using MATLAB or Python, visualize this set as an intersection of halfspaces.

Problem 2: Is the set

$$\mathcal{S} = \{\mathbf{a} \in R^{k+1} \mid p(0) = 1,\ |p(t)| \leq 1 \text{ for } \alpha \leq t \leq \beta\},$$

where

$$p(t) = a_0 + a_1 t + \cdots + a_k t^k,$$

convex?

Using MATLAB or Python, visualize the set for $k = 2$ as an intersection of slabs.

Note that this problem is similar to the one in Lec02 'Convex Sets', slide 7, where we visualized the set of Fourier coefficients for $m = 2$.

Problem 3: *Representations of ellipsoid.* Show analytically (by derivations or formal arguments) the equivalence between the following three representations of ellipsoid:

$$\mathcal{E} = \left\{ x \mid (x - x_\mathrm{c})^T A^{-1} (x - x_\mathrm{c}) \leq 1 \right\}$$

where A is a symmetric positive definite matrix and $x_c \in R^n$ is the center of ellipsoid.

$$\mathcal{E} = \{Bu + x_c \mid \|u\|_2 \leq 1\}$$

where $\|u\|_2$ is the Euclidean norm of u, and

$$\mathcal{E} = \{x \mid f(x) \leq 0\}$$

where $f(x) = x^T Cx + 2d^T x + e$, C is a symmetric positive definite matrix, and $e - d^T C^{-1} d < 0$.

See also Lec02 'Convex Sets', slide 9.

Problem 4: *Conic hull of outer products.* Consider the set of rank-k outer products, defined as $\{\mathbf{X}\mathbf{X}^T \mid \mathbf{X} \in R^{n\times k},\ \operatorname{rank}(\mathbf{X}) = k\}$.

Describe its conic hull in simple terms.

Problem 5: *Generalized inequalities.* To better understand how the generalized inequities works prove the properties of (nonstrict and strict) generalized inequalities for cones as they are defined in Lec02 'Convex Sets', slide 18 (closed, non-empty interior, pointed).

- $\preceq_K$ is preserved under addition: if $x \preceq_K y$ and $u \preceq_K v$, then $x + u \preceq_K y + v$.
- $\preceq_K$ is transitive: if $x \preceq_K y$ and $y \preceq_K z$, then $x \preceq_K z$.
- $\preceq_K$ is preserved under nonnegative scaling: if $x \preceq_K y$ and $\alpha \geq 0$, then $\alpha x \preceq_K \alpha y$.
- $\preceq_K$ is reflexive: $x \preceq_K x$.
- $\preceq_K$ is antisymmetric: if $x \preceq_K y$ and $y \preceq_K x$, then $x = y$.
- $\preceq_K$ is preserved under limits: if $x_i \preceq_K y_i$ for $i = 1, 2, \cdots$, $x_i \to x$ and $y_i \to y$ as $i \to \infty$, then $x \preceq_K y$.
- if $x \prec_K y$ then $x \preceq_K y$.
- if $x \prec_K y$ and $u \preceq_K v$, then $x + u \prec_K y + v$.

- if $x \prec_K y$ and $\alpha > 0$, then $\alpha x \prec_K \alpha y$.
- $x \prec_K x$ is not true.
- if $x \prec_K y$, then for u and v small enough, $x + u \prec_K y + v$.

Problem 6: *Dual cones in R^2.* Describe the dual cone for each of the following cones.

- $K = \{0\}$.
- $K = R^2$.
- $K = \{(x_1, x_2) \mid |x_1| \leq x_2\}$.
- $K = \{(x_1, x_2) \mid x_1 + x_2 = 0\}$.

Problem 7: *Self-duality of SOC.* Prove that Second-Order Cone (SOC)

$$\mathcal{K} = \{(x, t) \mid \|x\|_2 \leq t\}$$

is self-dual.