1. (a) To calculate the surface distance between Otaniemi and Paris, we start by defining two vectors from the center of the Earth to the coordinates of the cities. Let's use a spherical coordinate system, where the θ-angle is given by the latitudinal coordinates, and the ϕ-angle is given by the longitudinal coordinates. Since one arc minute is $1' = 1°/60$, we get the coordinates:

Otaniemi:	$\theta_O = 90° - 60°11' \approx 29.817°$	$\phi_O = 24°50' \approx 24.833°$
Paris:	$\theta_P = 90° - 48°52' \approx 41.133°$	$\phi_P = 2°21' \approx 2.35°$

Next, we can calculate the cartesian unit vectors from the center of the Earth to the respective locations:

$$\mathbf{u} = \sin\theta\cos\phi\,\mathbf{u}_x + \sin\theta\sin\phi\,\mathbf{u}_y + \cos\theta\,\mathbf{u}_z$$

Which gives us:

$$\mathbf{u}_O = 0.4512\mathbf{u}_x + 0.2088\mathbf{u}_y + 0.8676\mathbf{u}_z$$
$$\mathbf{u}_P = 0.6573\mathbf{u}_x + 0.0270\mathbf{u}_y + 0.7532\mathbf{u}_z$$

Since both of these vectors have a unitary length, we get the angle between them from the dot product:

$$\begin{aligned}\alpha &= \arccos(\mathbf{u}_O \cdot \mathbf{u}_P)\\ &= 0.2988 \qquad (\approx 17.12°)\end{aligned}$$

Since we know the radius of the Earth (R=6370km), we can now use the angle α in radians to calculate the distance by using the length of an arc:

$$\begin{aligned}l &= R\alpha\\ &\approx 1900\text{km}\end{aligned}$$

(b) In this exercise the vector formula sheet in MyCourses might come in handy.

i. From the vector formula sheet, we get that $\nabla \times \mathbf{a}_\phi$ can be calculated in cylindrical coordinates as:

$$\begin{aligned}\nabla \times \mathbf{a}_\phi &= \frac{1}{r}\begin{vmatrix} \mathbf{a}_r & r\mathbf{a}_\phi & \mathbf{a}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & r & 0 \end{vmatrix}\\ &= \frac{1}{r}\left(0 - \frac{\partial}{\partial z} r\mathbf{a}_r - 0 + 0 + \frac{\partial}{\partial r} r\mathbf{a}_z - 0\right)\\ &= \frac{1}{r}\mathbf{a}_z\end{aligned}$$

ii. First we need to convert the unit vector $\mathbf{a}_\phi$ to cartesian coordinates using the formula sheet:

$$\begin{aligned}\mathbf{F} &= -\sin\phi\,\mathbf{a}_x + \cos\phi\,\mathbf{a}_y\\ &= -\frac{y}{r}\mathbf{a}_x + \frac{x}{r}\mathbf{a}_y\\ &= -\frac{y}{\sqrt{x^2+y^2}}\mathbf{a}_x + \frac{x}{\sqrt{x^2+y^2}}\mathbf{a}_y\end{aligned}$$

Now we can use either the formula sheet or Equation (2-95) in the textbook to calculate the curl:

$$\begin{aligned}\nabla \times \mathbf{F} &= \mathbf{a}_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \mathbf{a}_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \mathbf{a}_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\\ &= \mathbf{a}_x\left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right) + \mathbf{a}_y\left(\frac{\partial}{\partial z}\left(-\frac{y}{\sqrt{x^2+y^2}}\right) - \frac{\partial}{\partial x}(0)\right)\\ &\quad + \mathbf{a}_z\left(\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right) - \frac{\partial}{\partial y}\left(-\frac{y}{\sqrt{x^2+y^2}}\right)\right)\\ &= \mathbf{a}_z\left(\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right) - \frac{\partial}{\partial y}\left(-\frac{y}{\sqrt{x^2+y^2}}\right)\right)\end{aligned}$$

Now, since:

$$\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right) = \frac{1}{\sqrt{x^2+y^2}} - \frac{x^2}{\sqrt{x^2+y^2}^3} = \frac{\sqrt{x^2+y^2}^2 - x^2}{\sqrt{x^2+y^2}^3} = \frac{y^2}{\sqrt{x^2+y^2}^3}$$

and

$$\frac{\partial}{\partial y}\left(-\frac{y}{\sqrt{x^2+y^2}}\right) = -\frac{1}{\sqrt{x^2+y^2}} + \frac{y^2}{\sqrt{x^2+y^2}^3} = \frac{y^2 - \sqrt{x^2+y^2}^2}{\sqrt{x^2+y^2}^3} = -\frac{x^2}{\sqrt{x^2+y^2}^3}$$

Finally, we get $\nabla \times \mathbf{F}$ by inserting the results into the expression:

$$\begin{aligned}\nabla \times \mathbf{F} &= \left(\frac{y^2}{\sqrt{x^2+y^2}^3} - \left(-\frac{x^2}{\sqrt{x^2+y^2}^3}\right)\right)\mathbf{a}_z = \frac{x^2+y^2}{\sqrt{x^2+y^2}^3}\mathbf{a}_z \\ &= \frac{1}{\sqrt{x^2+y^2}}\mathbf{a}_z\end{aligned}$$

iii. If we take a look at the expression we got in part ii, we can see that if we convert the expression back to cylindrical coordinates, we get the same solution as in part i. This is naturally the case, since it does not matter what coordinate system we use in our calculations. However, choosing the "right" coordinate system might make calculations a lot easier. Hence, the calculations in part i required much less work. We can also see that the curl vector points in the z-direction when the field points in the ϕ-direction, in accordance with the right-hand rule.