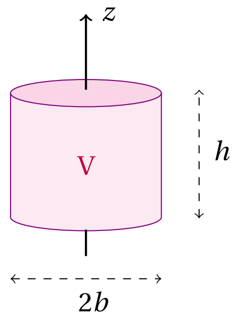
2. (a) When calculating the outward flux through a surface, one has to keep in mind that $\oint_S \mathbf{D}\cdot d\mathbf{S} = \oint_S \mathbf{D}\cdot\mathbf{n}dS$, where $\mathbf{n}$ is the outward normal of the surface. It is natural for us to do our calculations in a cylindrical coordinate system and convert the Cartesian coordinates when necessary. More on the divergence theorem can be found on page 48 in the course book.



i. First, we calculate the divergence $\nabla\cdot\mathbf{D}$, and then we integrate over the volume. The divergence is:

$$\nabla\cdot\mathbf{D} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1 = 3$$

Thus, we get:

$$\int_V \nabla\cdot\mathbf{D}\,dV = \int_V 3\,dV = 3\int_{-h/2}^{h/2}\int_0^{2\pi}\int_0^b r\,dr\,d\phi\,dz = \frac{3b^2}{2}\int_{-h/2}^{h/2}\int_0^{2\pi} d\phi\,dz = 3\pi b^2\int_{-h/2}^{h/2} dz = 3\pi b^2 h$$

ii. When calculating the top surface, z=h/2 is constant. The unit vector of the outward surface normal is $\mathbf{a}_z$:

$$\int_{S_1}\mathbf{D}\cdot d\mathbf{S} = \iint_{z=h/2}\left(x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z\right)\cdot\mathbf{a}_z r\,dr\,d\phi = \iint_{z=h/2}\underbrace{z}_{=h/2} r\,dr\,d\phi$$

$$= \frac{h}{2}\int_0^b r\,dr\int_0^{2\pi} d\phi = \frac{\pi b^2 h}{2}$$

iii. When calculating the top surface, z=-h/2 is constant. The unit vector of the outward surface normal is $-\mathbf{a}_z$:

$$\int_{S_2}\mathbf{D}\cdot d\mathbf{S} = \iint_{z=-h/2}\left(x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z\right)\cdot(-\mathbf{a}_z)\,r\,dr\,d\phi = \iint_{z=-h/2}\underbrace{-z}_{=h/2} r\,dr\,d\phi = \frac{\pi b^2 h}{2}$$

iv. When calculating the side surface, the radius is constant r=b. The unit vector of the outward surface normal is $\mathbf{a}_r$:

$$\int_{S_3}\mathbf{D}\cdot d\mathbf{S} = \iint_{r=b}\left(x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z\right)\cdot\mathbf{a}_r\,r\,d\phi\,dz$$

$$= \iint_{r=b}\left(\underbrace{r\cos\phi}_{=x}\mathbf{a}_x + \underbrace{r\sin\phi}_{=y}\mathbf{a}_y + z\mathbf{a}_z\right)\cdot\underbrace{\left(\cos\phi\mathbf{a}_x + \sin\phi\mathbf{a}_y\right)}_{=\mathbf{a}_r} r\,d\phi\,dz$$

$$= \iint_{r=b} r\cos^2\phi + r\sin^2\phi\, r\,d\phi\,dz = \iint_{r=b}\underbrace{r^2}_{=b^2}\underbrace{\left(\cos^2\phi + \sin^2\phi\right)}_{=1} d\phi\,dz$$

$$= b^2\int_0^{2\pi} d\phi\int_{-h/2}^{h/2} dz = 2\pi b^2 h$$

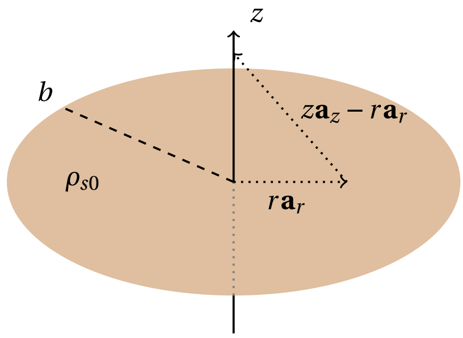
v. The outward flux through the total surface is equal to the sum of the flux through the individual surfaces, which we have already calculated:

$$\oint_S \mathbf{D}\cdot d\mathbf{S} = \frac{\pi b^2 h}{2} + \frac{\pi b^2 h}{2} + 2\pi b^2 h = 3\pi b^2 h$$

vi. The divergence theorem states that the volume integral of the divergence of a vector field equals the total outward flux of the vector through the surface that bounds the volume $\int_V \nabla\cdot\mathbf{D}\,dV = \oint_S \mathbf{D}\cdot d\mathbf{S}$.

The results in part i and v seem to agree with the theorem.

(b) A circular planar surface charge floats in free space (permittivity ε_0). The surface charge density is constant over the disk ρ_{s0} (with units $\mathrm{As/m^2}$). Let's fix the coordinate system such that the disk is in the xy plane ($z=0$) and its center in the origin. Compute the electric field caused by this source at the symmetry axis (z).



i. The distance D from any charge point on the surface is the length of the vector $z\mathbf{a}_z - r\mathbf{a}_r$:

$$D = |z\mathbf{a}_z - r\mathbf{a}_r| = \sqrt{z^2+r^2}$$

Thus, the electric scalar potential $V(z)$ at the z axis is:

$$V(z) = \int_S \frac{\rho_{s0}}{4\pi\varepsilon_0(z^2+r^2)^{1/2}}\mathrm{d}S = \frac{\rho_{s0}}{4\pi\varepsilon_0}\int_0^{2\pi}\mathrm{d}\phi\int_0^b \frac{r}{(z^2+r^2)^{1/2}}\mathrm{d}r = \frac{\rho_{s0}}{2\varepsilon_0}\Big|_0^b(\sqrt{z^2+r^2})$$
$$= \frac{\rho_{s0}}{2\varepsilon_0}(\sqrt{z^2+b^2}-|z|)$$

ii. We can write the electric field using the negative gradient of the potential $\mathbf{E} = -\nabla V$:

$$\mathbf{E}(z) = -\nabla V = -\left(\frac{\partial}{\partial r}(V)\mathbf{a}_r + \frac{1}{r}\frac{\partial}{\partial\phi}(V)\mathbf{a}_\phi + \frac{\partial}{\partial z}(V)\mathbf{a}_z\right) = -\frac{\rho_{s0}}{2\varepsilon_0}\frac{\partial}{\partial z}(\sqrt{z^2+b^2}-|z|)\mathbf{a}_z$$
$$= -\frac{\rho_{s0}}{2\varepsilon_0}\left(\frac{z}{\sqrt{z^2+b^2}}-1\right)\mathbf{a}_z$$

iii. Far away from the source, $|z| \gg b$ and we can simplify the expression using Taylor series approximation:

$$\frac{z}{\sqrt{z^2+b^2}} = \frac{1}{\sqrt{1+(b/z)^2}} = (1+(b/z)^2)^{-1/2} \approx 1-\frac{b^2}{2z^2}$$

Which gives us the expression:

$$\mathbf{E}(z) = -\frac{\rho_{s0}}{2\varepsilon_0}\left(\frac{z}{\sqrt{z^2+b^2}}-1\right)\mathbf{a}_z \approx \frac{\rho_{s0}b^2}{4\varepsilon_0 z^2}\mathbf{a}_z$$

What we might notice is that this is essentially the electric field from Coulombs law for a point charge:

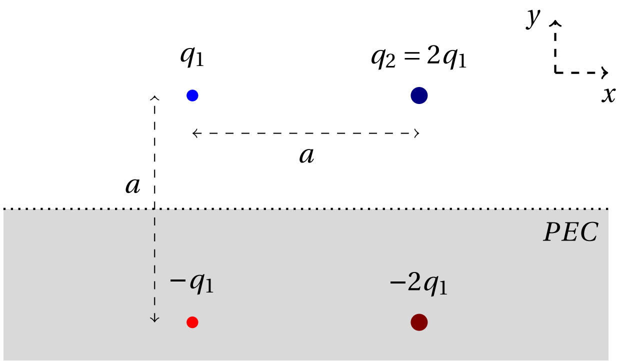
$$\mathbf{E}(z) \approx \frac{1}{4\pi\varepsilon_0}\frac{\overbrace{\rho_{s0}\pi b^2}^{\text{total charge}}}{z^2}\mathbf{a}_z$$

iv. Close to the disk, when $|z| \ll b$, the z-dependent expression becomes very small and we get the approximation:

$$\mathbf{E}(z) = -\frac{\rho_{s0}}{2\varepsilon_0}\left(\frac{z}{\sqrt{z^2+b^2}}-1\right)\mathbf{a}_z \approx \frac{\rho_{s0}}{2\varepsilon_0}\mathbf{a}_z$$

Which is the electric field intensity of an infinite planar charge with a uniform surface charge density ρ_{s0} (on the positive z-axis).

(c) Due to the conducting plane, the point charges will be mirrored at the surface, equal in magnitude with the opposite sign. Each charge will experience a force that is the sum of the forces due to each of the three other charges.



The Coulomb force experienced by a charge q_A due to the electric field $\mathbf{E}_A B$ due to the charges q_A and q_B is:

$$\mathbf{F}_{AB} = q_A \mathbf{E}_{AB} = \frac{q_A q_B}{4\pi\varepsilon_0 R_{AB}^2}\mathbf{a}_{AB} \quad [\mathrm{N}]$$

Are these forces equally strong? If not, which one is larger, and how many percent larger?

Thus, the force experienced by charge q_1 is:

$$\begin{aligned} F_1 &= \frac{q_1}{4\pi\varepsilon_0}\left(\frac{q_2}{a^2}(-\mathbf{a}_x) + \frac{-q_2}{(\sqrt{2}a)^2}\frac{-\mathbf{a}_x + \mathbf{a}_y}{\sqrt{2}} + \frac{-q_1}{a^2}\mathbf{a}_y\right) \\ &= \frac{q_1^2}{4\pi\varepsilon_0 a^2}\left[\left(-2 + \frac{1}{\sqrt{2}}\right)\mathbf{a}_x + \left(-1 - \frac{1}{\sqrt{2}}\right)\mathbf{a}_y\right] \\ &\approx \frac{q_1^2}{4\pi\varepsilon_0 a^2}\left(-1.293\mathbf{a}_x - 1.707\mathbf{a}_y\right) \end{aligned}$$

Similarly, the force experienced by charge q_2 is:

$$\begin{aligned} F_2 &= \frac{q_2}{4\pi\varepsilon_0}\left(\frac{q_1}{a^2}\mathbf{a}_x + \frac{-q_1}{(\sqrt{2}a)^2}\frac{\mathbf{a}_x + \mathbf{a}_y}{\sqrt{2}} + \frac{-q_2}{a^2}\mathbf{a}_y\right) \\ &= \frac{q_1^2}{4\pi\varepsilon_0 a^2}\left[\left(2 - \frac{1}{\sqrt{2}}\right)\mathbf{a}_x + \left(-4 - \frac{1}{\sqrt{2}}\right)\mathbf{a}_y\right] \\ &\approx \frac{q_1^2}{4\pi\varepsilon_0 a^2}\left(1.293\mathbf{a}_x - 4.707\mathbf{a}_y\right) \end{aligned}$$

As we can see the forces are not equally strong. The forces are equally strong in the horizontal direction, however the ground plane has a stronger attraction to the larger charge. The magnitude of the vector part of the force acting on the first charge is ≈ 2.14 and for the second charge the magnitude of the vector is ≈ 4.88. The constants in front of the expression are the same, and thus the force acting on the second charge is $\approx 130\%$ larger.