4. (a) $\mathbf{E}(\mathbf{R},t) = \mathbf{E}(x,t) = \mathbf{a}_y E_0 \cos(\omega t - \kappa x)$

i. We can derive the magnetic field using Faraday's law:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}$$

The curl of the electric field is:

$$\nabla \times \mathbf{E}(\mathbf{R},t) = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_0\cos(\omega t - \kappa x) & 0 \end{vmatrix} = -\frac{\partial}{\partial z}(E_0 \cos(\omega t - \kappa x))\,\mathbf{a}_x + \frac{\partial}{\partial x}(E_0 \cos(\omega t - \kappa x))\,\mathbf{a}_z$$

$$= E_0 \kappa \sin(\omega t - \kappa x)\mathbf{a}_z$$

Thus, we get the magnetic field:

$$\mathbf{H}(\mathbf{R},t) = -\int \frac{1}{\mu_0} \nabla \times \mathbf{E}(\mathbf{R},t)\,\mathrm{d}t = \frac{E_0\kappa}{\omega\mu_0}\cos(\omega t - \kappa x)\mathbf{a}_z$$

ii. Faraday's law is obviously satisfied. Let us check Ampere's law:

$$\nabla \times \mathbf{H} = \underbrace{\mathbf{J}}_{=0} + \frac{\partial \mathbf{D}}{\partial t} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

The curl of the magnetic field is:

$$\nabla \times \mathbf{H}(\mathbf{R},t) = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & \frac{E_0\kappa}{\omega\mu_0}\cos(\omega t - \kappa x) \end{vmatrix} = \frac{\partial}{\partial y}\left(\frac{E_0\kappa}{\omega\mu_0}\cos(\omega t - \kappa x)\right)\mathbf{a}_x - \frac{\partial}{\partial x}\left(\frac{E_0\kappa}{\omega\mu_0}\cos(\omega t - \kappa x)\right)\mathbf{a}_y$$

$$= -\frac{E_0\kappa^2}{\omega\mu_0}\sin(\omega t - \kappa x)\mathbf{a}_y$$

Thus, we get the electric field:

$$\mathbf{E}(\mathbf{R},t) = \int \frac{1}{\epsilon_0}\nabla \times \mathbf{H}(\mathbf{R},t)\,\mathrm{d}t = \frac{E_0\kappa^2}{\omega^2\mu_0\epsilon_0}\cos(\omega t - \kappa x)\mathbf{a}_y$$

This gives us the condition from the original electric field expression:

$$\frac{\kappa^2}{\omega^2\mu_0\epsilon_0} = 1 \quad \Rightarrow \quad \kappa = \omega\sqrt{\mu_0\epsilon_0}$$

Which is the expression for the wave number in free space. Thus we can conclude that Ampere's law is also satisfied. Now, as for the divergence equations we have:

$$\nabla \cdot \mathbf{D} = \underbrace{\rho_v}_{=0} = 0$$

$$\nabla \cdot \mathbf{D} = \epsilon_0 \nabla \cdot \mathbf{E} = \epsilon_0 \left[\frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(E_0\cos(\omega t - \kappa x)) + \frac{\partial}{\partial z}(0)\right] = 0$$

and

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \cdot \mathbf{B} = \mu_0 \nabla \cdot \mathbf{H} = \mu_0 \left[\frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}\left(\frac{E_0\kappa}{\omega\mu_0}\cos(\omega t - \kappa x)\right)\right] = 0$$

Thus, the Maxwell's equations are satisfied for $\mathbf{J} = 0$ and $\rho_v = 0$, if the constant $\kappa = \omega\sqrt{\mu_0\epsilon_0}$ is the wave number of the electromagnetic field.

(b) i. $\mathrm{j}^{-3} = \frac{1}{\mathrm{j}^3} = \frac{\mathrm{j}}{\mathrm{j}^4} = \frac{\mathrm{j}}{(-1)^2} = \mathrm{j}$

ii. $\sqrt{\mathrm{j}} = a + \mathrm{j}b, \qquad a, b \in \mathbb{R}$

$\Rightarrow \quad \mathrm{j} = (a+\mathrm{j}b)^2 = a^2 + \mathrm{j}2ab - b^2$

$$\Rightarrow \begin{cases} 2ab = 1 \\ a^2 - b^2 = 0 \end{cases} \Leftrightarrow \begin{cases} a = \frac{1}{2b} \\ \frac{1}{4b^2} - b^2 = 0 \end{cases} \Leftrightarrow \begin{cases} a = \pm\frac{1}{\sqrt{2}} \\ b = \pm\frac{1}{\sqrt{2}} \end{cases}$$

$\Rightarrow \sqrt{\mathrm{j}} = \pm\left(\frac{1}{\sqrt{2}} + \mathrm{j}\frac{1}{\sqrt{2}}\right)$

iii. $\sqrt{2+\mathrm{j}2} = a + \mathrm{j}b, \qquad a, b \in \mathbb{R}$

$\Rightarrow \quad 2 + \mathrm{j}2 = (a+\mathrm{j}b)^2 = a^2 + \mathrm{j}2ab - b^2$

$$\Rightarrow \begin{cases} 2ab = 2 \\ a^2 - b^2 = 2 \end{cases} \Rightarrow \begin{cases} a = \frac{1}{b} \\ \frac{1}{b^2} - b^2 = 2 \end{cases} \Rightarrow \begin{cases} a = \frac{1}{b} \\ b^4 + 2b^2 - 1 = 0 \end{cases} \Rightarrow \begin{cases} a = \pm\frac{1}{\sqrt{\sqrt{2}-1}} = \pm\sqrt{\sqrt{2}+1} \\ b = \pm\sqrt{\sqrt{2}-1} \end{cases}$$

$\Rightarrow \sqrt{2+\mathrm{j}2} = \pm\left(\sqrt{\sqrt{2}+1} + \mathrm{j}\sqrt{\sqrt{2}-1}\right)$

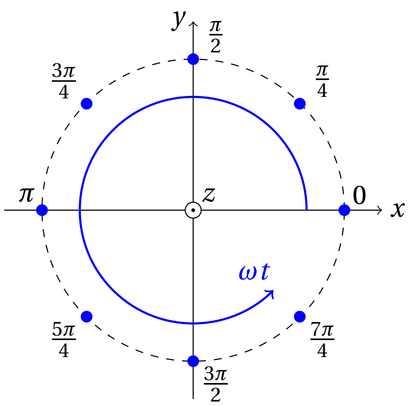
iv. $\frac{3+\mathrm{j}}{2-\mathrm{j}} = \frac{(3+\mathrm{j})(2+\mathrm{j})}{(2-\mathrm{j})(2+\mathrm{j})} = \frac{3\cdot 2 + \mathrm{j}3 + \mathrm{j}2 + \mathrm{j}^2}{2^2+1^2} = \frac{5+\mathrm{j}5}{5} = 1 + \mathrm{j}$

(c) i. The electric field $\mathbf{a}_z E_1 \mathrm{e}^{\mathrm{j}kx}$ only has a z-component, and thus the plane wave in this case is clearly linearly polarized.

ii. The electric field $(\mathbf{a}_x - \mathrm{j}\mathbf{a}_y)E_2 \mathrm{e}^{\mathrm{j}kz}$ propagates in the -z-direction, and has a x-directed and a y-directed component. Since the two components are equal in length and have a phase difference of 90°, the plane wave is circularly polarized. To determine the handedness of the plane wave, we can examine the instantaneous expression:

$$\begin{aligned}\mathbf{E}(z,t) &= \Re\left\{\left[E_2\mathrm{e}^{\mathrm{j}kz}\mathbf{a}_x - \mathrm{j}E_2\mathrm{e}^{\mathrm{j}kz}\mathbf{a}_y\right]\mathrm{e}^{\mathrm{j}\omega t}\right\} \\ &= \Re\left\{E_2\mathrm{e}^{\mathrm{j}(\omega t+kz)}\mathbf{a}_x + E_2\mathrm{e}^{\mathrm{j}(\omega t+kz-\frac{\pi}{2})}\mathbf{a}_y\right\} \\ &= E_2\cos(\omega t+kz)\mathbf{a}_x + E_2\cos\left(\omega t+kz-\frac{\pi}{2}\right)\mathbf{a}_y\end{aligned}$$

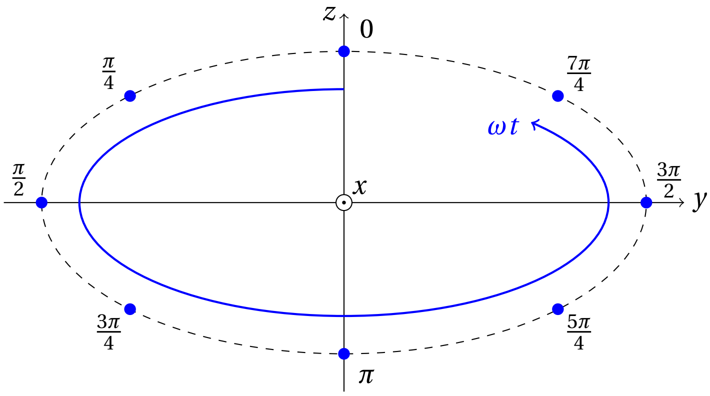
For convenience, let us set $z = 0$ as we examine how the field changes direction as ωt increases:
$\mathbf{E}(0,t) = E_2\cos(\omega t)\mathbf{a}_x + E_2\sin(\omega t)\mathbf{a}_y$



We can see that the field rotates in the counterclockwise direction as ωt increases. Since the wave propagates in the -z-direction, placing our thumb in the propagation direction and our fingers in the rotation direction gives us a left-hand circularly polarized wave.

iii. The electric field $(\mathbf{a}_z + 2\mathrm{j}\mathbf{a}_y)E_3\,\mathrm{e}^{\mathrm{j}kx}$ propagates in the -x-direction, and similarly to part ii, there is a 90° phase shift between the y-directed and z-directed component. In this case however, the lengths of the components are not equal, and thus we have an elliptical polarization. We can determine the handedness in a similar fashion:

$\mathbf{E}(0,t) = E_3\cos(\omega t)\mathbf{a}_z - 2E_3\sin(\omega t)\mathbf{a}_y$



As we can see, the field rotates in the counterclockwise direction as ωt increases. Since the wave propagates in the -x-direction, we have a left-hand elliptically polarized wave.

iv. The electric field $(\mathbf{a}_z + \mathbf{a}_x)E_4\,\mathrm{e}^{\mathrm{j}ky}$ has two components in the z-direction and in the x-direction. Since the two components are in the same phase, the wave is linearly polarized, however the polarization is tilted 45° in the xz-plane.