**5.** (a) Since $\frac{\sigma}{\omega\epsilon} = \frac{3}{2\pi 1\cdot 10^6(\frac{1}{36\pi}\cdot 10^{-7})80} = 675 \gg 1$ we can use the formulas for good conductors:

The attenuation constant:

$\alpha = \sqrt{\pi f \mu\sigma} = \sqrt{\pi 10^6 4\pi\cdot 10^{-7} 3} \approx 3.44\ \mathrm{Np/m}$

The phase constant:

$\beta = \sqrt{\pi f \mu\sigma} \approx 3.44\ \mathrm{rad/m}$

The intrinsic impedance:

$$\eta_c = (1+\mathrm{j})\sqrt{\frac{\pi f\mu}{\sigma}} = (1+\mathrm{j})\sqrt{\frac{\pi 10^6 4\pi\cdot 10^{-7}}{3}} \approx (1+\mathrm{j})1.15\,\Omega$$

The average power density of the wave is given by the time-average Poynting vector:

$$\begin{aligned}\mathscr{P}_{\mathrm{av}}(z) &= \tfrac{1}{2}\Re\left\{\mathbf{E}\times\mathbf{H}^*\right\} = \tfrac{1}{2}\Re\left\{\left(E_0 e^{-\alpha z}e^{-\mathrm{j}\beta z}\mathbf{a}_x\right)\times\left(\tfrac{E_0}{\eta_c}e^{-\alpha z}e^{-\mathrm{j}\beta z}\mathbf{a}_y\right)^*\right\}\\ &= \tfrac{1}{2}\Re\left\{\left(E_0 e^{-\alpha z}e^{-\mathrm{j}\beta z}\mathbf{a}_x\right)\times\left(\tfrac{E_0^*}{\eta_c^*}e^{-\alpha z}e^{+\mathrm{j}\beta z}\mathbf{a}_y\right)\right\}\\ &= \tfrac{1}{2}\Re\left\{\tfrac{|E_0|^2}{\eta_c^*}e^{-2\alpha z}\mathbf{a}_x\times\mathbf{a}_y\right\} = \Re\left\{\tfrac{1}{\eta_c^*}\right\}\tfrac{|E_0|^2}{2}e^{-2\alpha z}\mathbf{a}_z\end{aligned}$$

i. The average power density at the surface is:

$\mathscr{P}_{\mathrm{av}}(0) = \Re\left\{\frac{1}{\eta_c^*}\right\}\frac{|E_0|^2}{2}e^{-2\alpha\cdot 0} = 0.435\cdot\frac{|10|^2}{2} \approx 22\ \mathrm{W/m^2}$

ii. What is the average power density of the wave (in $\mathrm{W/m^2}$) two meters below the surface ($z = 2$ m)?

$\mathscr{P}_{\mathrm{av}}(2) = \Re\left\{\frac{1}{\eta_c^*}\right\}\frac{|E_0|^2}{2}e^{-2\alpha\cdot 2} = 0.435\cdot\frac{|10|^2}{2}e^{-2\cdot 3.44\cdot 2} \approx 23\ \mu\mathrm{W/m^2}$

iii. We can get the attenuation in decibels directly from the attenuation constant if we remember that $1\mathrm{Np} \approx 8.69\mathrm{dB}$:

$\alpha = 3.44\cdot 8.69 \approx 29.9\mathrm{dB/m}$

Thus the wave attenuates roughly 60dB over the 2m distance.

(b) Let's assume an incident electric field of $\mathbf{E}_i(z) = E_0 e^{-\mathrm{j}kz}\mathbf{a}_x$ which has the corresponding incident magnetic field of $\mathbf{H}_i(z) = \frac{E_0}{\eta_0}e^{-\mathrm{j}kz}\mathbf{a}_y$. The plane wave has a normal incidence $\theta_1 = 0°$.

i. The complex Poynting vector is:

$$\mathbf{S}_i = \tfrac{1}{2}\mathbf{E}\times\mathbf{H}^* = \tfrac{1}{2}\left(E_0 e^{-\mathrm{j}kz}\mathbf{a}_x\right)\times\left(\tfrac{E_0}{\eta_0}e^{-\mathrm{j}kz}\mathbf{a}_y\right)^* = \tfrac{1}{2}\left(E_0 e^{-\mathrm{j}kz}\mathbf{a}_x\right)\times\left(\tfrac{E_0^*}{\eta_0}e^{+\mathrm{j}kz}\mathbf{a}_y\right) = \tfrac{|E_0|^2}{2\eta_0}\underbrace{e^{-\mathrm{j}kz}e^{+\mathrm{j}kz}}_{=1}\overbrace{\mathbf{a}_x\times\mathbf{a}_y}^{=\mathbf{a}_z}$$

$= \frac{|E_0|^2}{2\eta_0}\mathbf{a}_z$

Thus the amplitude is $S_i = \frac{|E_0|^2}{2\eta_0}$.

ii. The reflection coefficient is:

$$\Gamma = \frac{\eta_2-\eta_1}{\eta_2+\eta_1} = \frac{\frac{1}{\sqrt{4}}\eta_0-\eta_0}{\frac{1}{\sqrt{4}}\eta_0+\eta_0} = \frac{\frac{1}{2}-1}{\frac{1}{2}+1} = -\frac{1}{3}$$

Since we are dealing with power density, the reflection coefficient has to be squared. Thus we get the Poynting vector for the reflected wave:

$\mathbf{S}_r = -\left(-\frac{1}{3}\right)^2\frac{|E_0|^2}{2\eta_0}\mathbf{a}_z$

Which gives us the amplitude of $S_r = \frac{1}{9}\frac{|E_0|^2}{2\eta_0}$

iii. The transmission coefficient is:

$$\tau = \frac{2\eta_2}{\eta_2+\eta_1} = \frac{\frac{2}{\sqrt{4}}\eta_0}{\frac{1}{\sqrt{4}}\eta_0+\eta_0} = \frac{1}{\frac{1}{2}+1} = \frac{2}{3}$$

Again, since we are dealing with power density, the transmission coefficient has to be squared. Thus we get the Poynting vector for the transmitted wave:

$$\mathbf{S}_t = \left(\frac{2}{3}\right)^2 \frac{|E_0|^2}{2\eta_2}\mathbf{a}_z$$

Which gives us the amplitude of $S_t = \frac{4}{9}\frac{|E_0|^2}{2\eta_2}$

iv. The net power flow should be the same on both sides of the boundary:

$$\mathbf{S}_i + \mathbf{S}_r = \frac{|E_0|^2}{2\eta_0}\mathbf{a}_z - \frac{1}{9}\frac{|E_0|^2}{2\eta_0}\mathbf{a}_z = \left(1-\frac{1}{9}\right)\frac{|E_0|^2}{2\eta_0}\mathbf{a}_z = \frac{4\cdot 2}{9}\frac{|E_0|^2}{2\eta_0}\mathbf{a}_z = \frac{4}{9}\frac{|E_0|^2}{2\eta_2}\mathbf{a}_z = \mathbf{S}_t$$

Thus, the energy balance is satisfied.

(c) Snell's law of refraction gives us the relation:

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{\eta_2}{\eta_1}$$

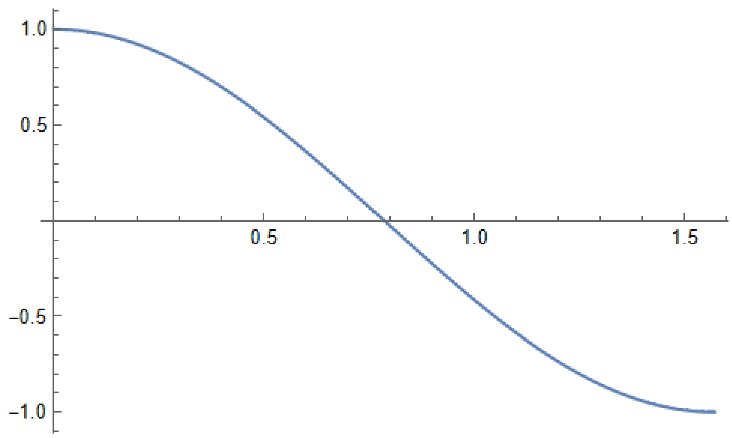
Let us substitute this into the equation for the reflection coefficient for perpendicular polarization:

$$\Gamma_\perp = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t} = \frac{\frac{\sin\theta_t}{\sin\theta_i}\eta_1\cos\theta_i - \eta_1\cos\theta_t}{\frac{\sin\theta_t}{\sin\theta_i}\eta_1\cos\theta_i + \eta_1\cos\theta_t} = \frac{\sin\theta_t\cos\theta_i - \sin\theta_i\cos\theta_t}{\sin\theta_t\cos\theta_i + \sin\theta_i\cos\theta_t} = -\frac{\sin(\theta_i-\theta_t)}{\sin(\theta_i+\theta_t)}$$

At Brewster's angle, we know that $\theta_i + \theta_t = 90°$. Thus, we get:

$$\Gamma_\perp = -\frac{\sin[\theta_i - (90° - \theta_i)]}{\sin[\theta_i + (90° - \theta_i)]} = -\frac{\sin(2\theta_i - 90°)}{\sin(90°)} = -\sin(2\theta_i)\cos(90°) + \sin(90°)\cos(2\theta_i) = \cos(2\theta_i)$$

Where $\theta_i = \theta_{Br}$. Now we can plot $\Gamma_\perp$ as a function of $\theta_{Br}$ for all possible incidence angles $0° \cdots 90°$:




$\Gamma_\perp = \cos(2\theta_{Br})$