COE-C3005 Finite Element and Finite Difference methods 2021

Week	Mon	Tue	Wed	Thu	Fri	Sun
Orientation						
16		**09:15-10:30** Course introduction and modelling assignment	**09:15-10:30** Particle surrogate modelling	**09:15-11:00** Mathematical methods and Mathematica tools	**09:15-11:00** Calculation hours	**23:55 DL** Modelling assignment
Lectures and exercises						
17		**09:15-10:30** Lecture 1/2 **10:35-11:30** Assignment 1	**09:15-10:30** Lecture 2/2 **10:35-11:30** Assignment 2	**09:15-11:00** Calculation examples	**09:15-11:00** Calculation hours	**23:55 DL** Assignments 3,4,5
18		**09:15-10:30** Lecture 1/2 **10:35-11:30** Assignment 1	**09:15-10:30** Lecture 2/2 **10:35-11:30** Assignment 2	**09:15-11:00** Calculation Examples	**09:15-11:00** Calculation hours	**23:55 DL** Assignments 3,4,5
19		**09:15-10:30** Lecture 1/2 **10:35-11:30** Assignment 1	**09:15-10:30** Lecture 2/2 **10:35-11:30** Assignment 2		**09:15-11:00** Calculation hours	**23:55 DL** Assignments 3,4,5
20		**09:15-10:30** Lecture 1/2 **10:35-11:30** Assignment 1	**09:15-10:30** Lecture 2/2 **10:35-11:30** Assignment 2	**09:15-11:00** Calculation Examples	**09:15-11:00** Calculation hours	**23:55 DL** Assignments 3,4,5
21		**09:15-10:30** Lecture 1/2 **10:35-11:30** Assignment 1	**09:15-10:30** Lecture 2/2 **10:35-11:30** Assignment 2	**09:15-11:00** Calculation Examples	**09:15-11:00** Calculation hours	**23:55 DL** Assignments 3,4,5
Exams						
22					**09:00-13:00** Final exam	

COE-C3005 Finite Element and Finite difference methods

BAR-STRING MODELS

$$k\frac{\partial^2 a}{\partial x^2}+f'=m'\frac{\partial^2 a}{\partial t^2} \quad x\in\Omega\setminus I \quad \text{and} \quad \left[\!\left[k\frac{\partial a}{\partial x}\right]\!\right]+F=0 \quad x\in I \quad t>0$$

$$a=\underline{a} \quad \text{or} \quad -n_x(k\frac{\partial a}{\partial x})+F=0 \quad x\in\partial\Omega \quad t>0,$$

$$a=g \quad \text{and} \quad \frac{\partial a}{\partial t}=h \quad x\in\Omega \quad t=0$$

PARTICLE SURROGATE METHOD

$$\frac{k}{\Delta x}(a_{i-1}-2a_i+a_{i+1})+F_i=m_i\ddot{a}_i \quad t>0$$

$$a_0=\underline{a}_0 \quad \text{or} \quad \frac{k}{\Delta x}(a_1-a_0)+F_0=m_0\ddot{a}_0 \quad \text{and} \quad a_n=\underline{a}_n \quad \text{or} \quad \frac{k}{\Delta x}(a_{n-1}-a_n)+F_n=m_n\ddot{a}_n \quad t>0$$

$$a_i=g_i \quad \text{and} \quad \dot{a}_i=h_i \quad t=0$$

FINITE DIFFERENCE METHOD

$$\frac{k}{\Delta x^2}(a_{i-1}-2a_i+a_{i+1})+f'=m'\ddot{a}_i \quad \text{or} \quad \frac{k}{\Delta x}(a_{i-1}-2a_i+a_{i+1})+F=0 \quad t>0$$

$$a_0=\underline{a}_0 \quad \text{or} \quad \frac{k}{\Delta x}(a_1-a_0)+F_0=0 \quad \text{and} \quad a_n=\underline{a}_n \quad \text{or} \quad \frac{k}{\Delta x}(a_{n-1}-a_n)+F_n=0 \quad t>0$$

$$a_i=g_i \quad \text{and} \quad \dot{a}_i=h_i \quad t=0$$

FINITE ELEMENT METHOD

$$\frac{k}{\Delta x}(a_{i-1}-2a_i+a_{i+1})+F_i+f'\Delta x=m'\frac{\Delta x}{6}(\ddot{a}_{i-1}+4\ddot{a}_i+\ddot{a}_{i+1}) \quad i\in\{1,2,\ldots,n-1\}$$

$$\frac{k}{\Delta x}(a_1-a_0)+F_0+f'\frac{\Delta x}{2}-m'\frac{\Delta x}{6}(2\ddot{a}_0+\ddot{a}_1)=0 \quad \text{or} \quad a_0=\underline{a}_0,$$

$$\frac{k}{\Delta x}(a_{n-1}-a_n)+F_n+f'\frac{\Delta x}{2}-m'\frac{\Delta x}{6}(2\ddot{a}_n+\ddot{a}_{n-1})=0 \quad \text{or} \quad a_n=\underline{a}_n,$$

$$a_i-g_i=0 \quad \text{and} \quad \dot{a}_i-h_i=0.$$

MEMBRANE MODEL

$$S'(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2})+f'=m'\frac{\partial^2 w}{\partial t^2} \quad (x,y)\in\Omega \quad t>0$$

$$w=\underline{w} \quad \text{or} \quad S'(n_x\frac{\partial w}{\partial x}+n_y\frac{\partial w}{\partial y})=F' \quad (x,y)\in\partial\Omega \quad t>0$$

$$w=g \quad \text{and} \quad \frac{\partial w}{\partial t}=h \quad (x,y)\in\Omega \quad t=0$$

FINITE DIFFERENCE METHOD

$$\frac{S'}{h^2}[w_{(i-1,j)}+w_{(i,j-1)}-4w_{(i,j)}+w_{(i+1,j)}+w_{(i,j+1)}]+f'=m'\ddot{w}_{(i,j)} \quad (i,j)\in I \quad t>0$$

$$w_{(i,j)}=0 \quad (i,j)\in\partial I \quad t>0$$

$$w_{(i,j)}-g_{(i,j)}=0 \quad \text{and} \quad \dot{w}_{(i,j)}-h_{(i,j)}=0 \quad (i,j)\in I \quad t=0$$

FINITE ELEMENT METHOD

$$S'[w_{(i-1,j)}+w_{(i,j-1)}-4w_{(i,j)}+w_{(i+1,j)}+w_{(i,j+1)}]+h^2 f'=$$

$$m'h^2\frac{1}{12}[\ddot{w}_{(i-1,j-1)}+\ddot{w}_{(i-1,j)}+\ddot{w}_{(i,j-1)}+6\ddot{w}_{(i,j)}+\ddot{w}_{(i+1,j)}+\ddot{w}_{(i,j+1)}+\ddot{w}_{(i+1,j+1)}] \quad (i,j)\in I \quad t>0$$

$$w_{(i,j)}=0 \quad (i,j)\in\partial I \quad t>0$$

$$w_{(i,j)}-g_{(i,j)}=0 \quad \text{and} \quad \dot{w}_{(i,j)}-h_{(i,j)}=0 \quad (i,j)\in I \quad t=0$$

SOLUTION METHODS

FOURIER SINE SERIES

$$\int_0^L \sin(k\pi\frac{x}{L})\sin(l\pi\frac{x}{L})dx=\frac{L}{2}\delta_{kl}$$

$$\alpha_k=\frac{2}{L}\int_0^L \sin(k\pi\frac{x}{L})a(x)dx \quad k\in\{1,2,\ldots\} \quad \Leftrightarrow \quad a(x)=\sum_{k\in\{1,2,\ldots\}}\alpha_k\sin(k\pi\frac{x}{L})$$

MODAL ANALYSIS AND MODE SUPERPOSITION

$$A(x)=\delta\sin(\lambda x)+\gamma\cos(\lambda x)\,, \quad \lambda=\omega\sqrt{\frac{m'}{k'}}$$

$$a(x,t) = \sum \quad A_j[\frac{1}{\omega_j}\alpha_j \sin(\omega_j t) + \beta_j \cos(\omega_j t)]$$

$$\alpha_j = \frac{1}{\mathrm{A}_j^2}\int_\Omega \; A_j(x)hdx\,, \quad \beta_j = \frac{1}{\mathrm{A}_j^2}\int_\Omega \; A_j(x)gdx \text{ and } \mathrm{A}_j^2 = \int_\Omega \; A_j(x)A_j(x)dx\,.$$

MODAL ANALYSIS AND MODE SUPERPOSITION

$$(\mathbf{K} - \omega^2\mathbf{M})\mathbf{A} = 0$$

$$\mathbf{a}(t) = \sum \; \mathbf{A}_j[\frac{1}{\omega_j}\alpha_j \sin(\omega_j t) + \beta_j \cos(\omega_j t)]$$

$$\alpha_j = \frac{1}{\mathrm{A}_j^2}\mathbf{A}_j^{\mathrm{T}}\mathbf{h}\,, \quad \beta_j = \frac{1}{\mathrm{A}_j^2}\mathbf{A}_j^{\mathrm{T}}\mathbf{g}\,, \quad \mathrm{A}_j^2 = \mathbf{A}_j^{\mathrm{T}}\mathbf{A}_j$$

CRANK-NICOLSON

$$\begin{Bmatrix} a \\ \dot{a}\Delta t \end{Bmatrix}_i = \frac{1}{4+\alpha^2}\begin{bmatrix} 4-\alpha^2 & 4 \\ -4\alpha^2 & 4-\alpha^2 \end{bmatrix}\begin{Bmatrix} a \\ \dot{a}\Delta t \end{Bmatrix}_{i-1}, \quad \begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_0 = \begin{Bmatrix} g \\ \Delta th \end{Bmatrix} \text{ where } \alpha = \sqrt{\frac{k}{m}}\Delta t$$

$$\begin{bmatrix} \mathbf{I} & -\frac{1}{2}\mathbf{I} \\ \frac{\Delta t}{2}\mathbf{K} & \Delta t\mathbf{M} \end{bmatrix}\begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}}\Delta t \end{Bmatrix}_i = \begin{bmatrix} \mathbf{I} & \frac{1}{2}\mathbf{I} \\ -\frac{\Delta t}{2}\mathbf{K} & \mathbf{M} \end{bmatrix}\begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}}\Delta t \end{Bmatrix}_{i-1}, \quad \begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}}\Delta t \end{Bmatrix}_0 = \begin{Bmatrix} \mathbf{g} \\ \mathbf{h}\Delta t \end{Bmatrix}$$

DISCONTINUOUS-GALERKIN

$$\begin{Bmatrix} a \\ \dot{a}\Delta t \end{Bmatrix}_i = \frac{2}{12+\alpha^4}\begin{bmatrix} 6-3\alpha^2 & 6-\alpha^2 \\ -6\alpha^2 & 6-3\alpha^2 \end{bmatrix}\begin{Bmatrix} a \\ \dot{a}\Delta t \end{Bmatrix}_{i-1}, \quad \begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_0 = \begin{Bmatrix} g \\ \Delta th \end{Bmatrix} \text{ where } \alpha = \sqrt{\frac{k}{m}}\Delta t$$

$$\begin{bmatrix} \Delta t^2\mathbf{K} & -\frac{1}{2}\Delta t^2\mathbf{K} + \mathbf{M} \\ \frac{1}{2}\Delta t^2\mathbf{K} - \mathbf{M} & \mathbf{M} - \frac{1}{6}\Delta t^2\mathbf{K} \end{bmatrix}\begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}}\Delta t \end{Bmatrix}_i = \begin{bmatrix} \mathbf{0} & \mathbf{M} \\ -\mathbf{M} & \mathbf{0} \end{bmatrix}\begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}}\Delta t \end{Bmatrix}_{i-1}, \quad \begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}}\Delta t \end{Bmatrix}_0 = \begin{Bmatrix} \mathbf{g} \\ \mathbf{h}\Delta t \end{Bmatrix}$$

COE-C3005
Finite Element and Finite Difference Methods 2021

WEEK 16: INTRODUCTION

Fri 09:15-11:00 Calculation hours (JF & MÅ)

CONTENTS

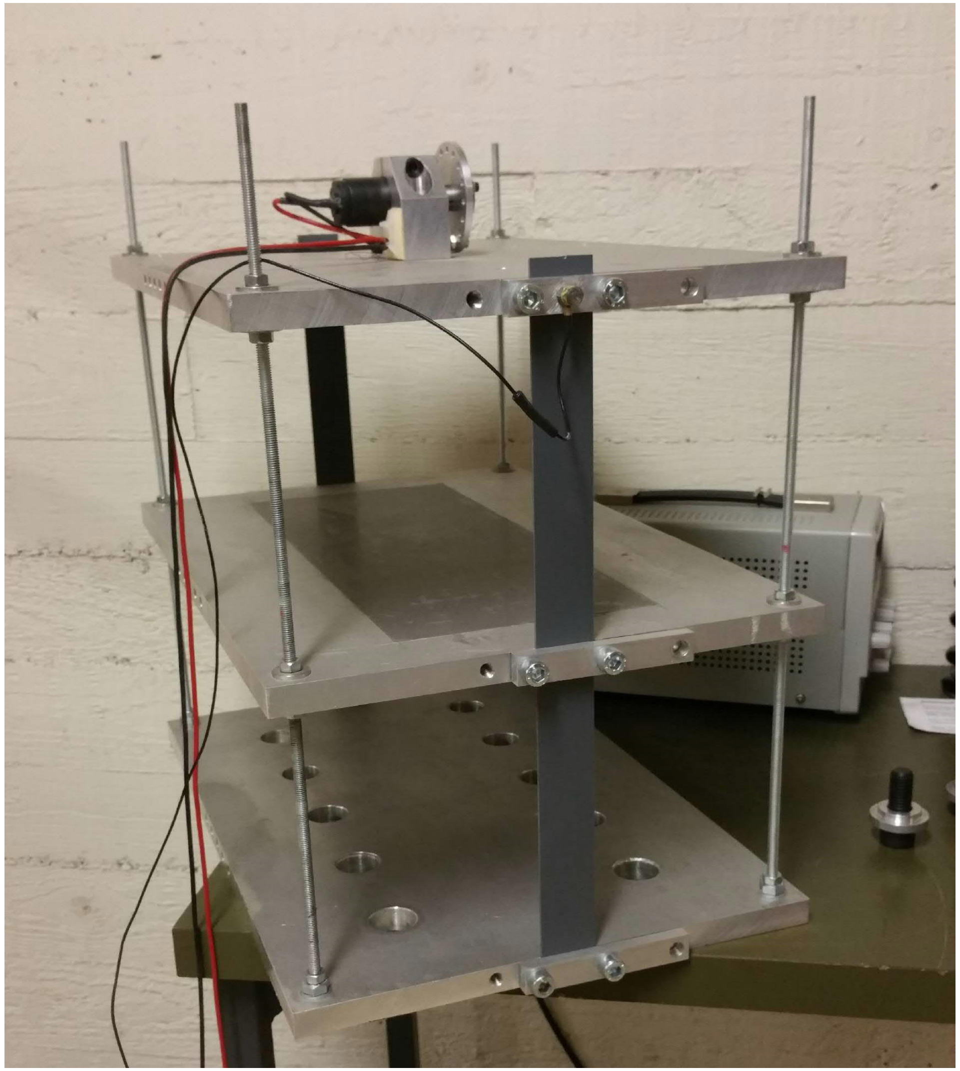
LEARNING OUTCOMES

Introduction to approximate methods for *initial* and *boundary* value problems in solid mechanics. After the course, student understands the physical background of the bar and string model problems, knows the basic ideas of (1) particle surrogate, (2) finite difference, (3) finite element methods, is able to apply the methods to the model problems (1D), and knows the extensions to the thin slab and membrane models of solid mechanics (2D).

Prerequisites: Linear algebra, ordinary differential equations

1 INTRODUCTION

VIBRATION OF 3-STORY BUILDING



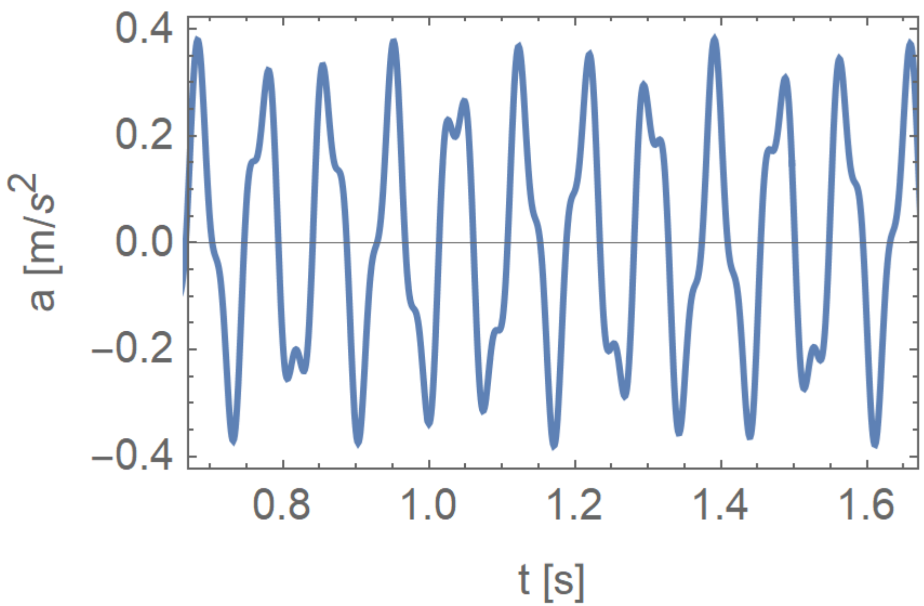
MODELLING ASSIGNMENT

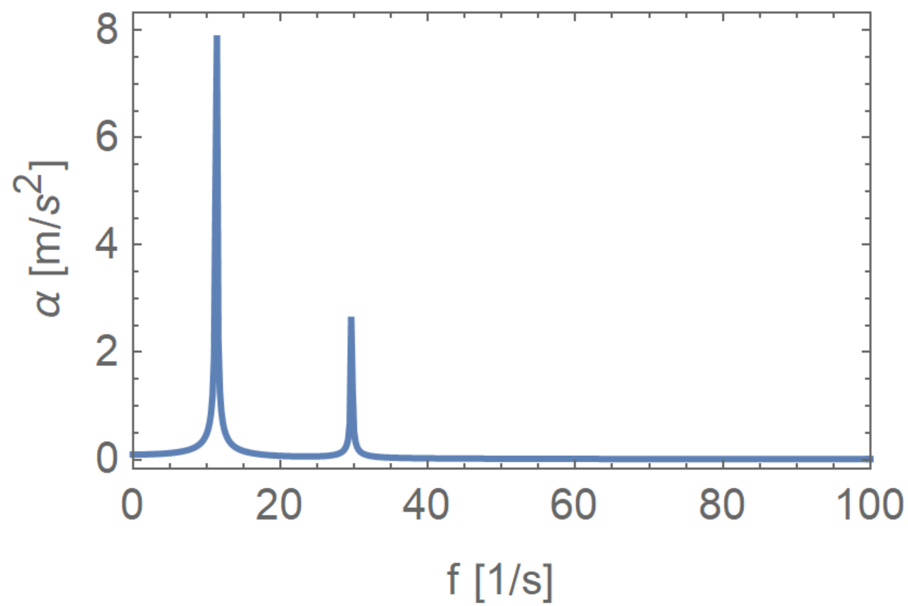
In the modelling assignment, you will determine the two first frequencies of the free vibrations of the 3-story building using a model and

1. Particle Surrogate Method (PSM)
2. Finite Difference Method (FDM)
3. Finite Element Method (FDM)

To report the outcome, supplement the assignment paper with experimental results and the outcome of calculations (table for results in light blue shading). Return your report (in PDF) on Sun 25.04.2021 23:55 at the latest (MyCourses).

VIBRATION EXPERIMENT





Experimental data consists of the acceleration time-series measured by the accelerometer at one point. In processing of data, the time-acceleration representation is transformed to frequency-mode magnitude form by Discrete Fourier Transform (DFT).

MODELLING STEPS

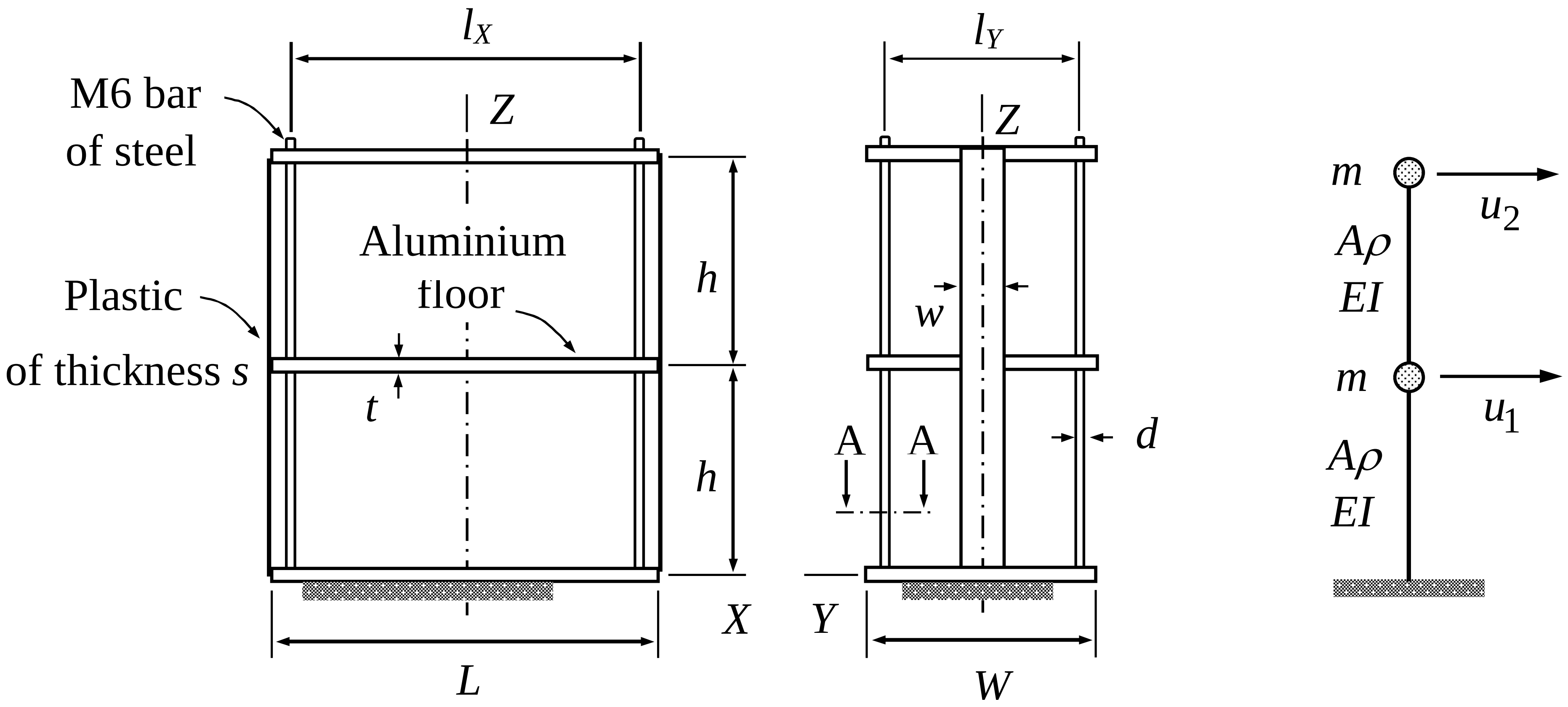
- **Crop:** Decide the boundary of a structure. Interaction with surroundings need to be described in terms of known forces, moments, displacements, and rotations. All uncertainties with this respect bring uncertainty to the model too.

- **Idealize and parameterize:** Simplify the geometry. Ignoring the details not likely to affect the outcome may simplify the analysis a lot. Assign symbols to geometric and material parameter of the idealized structure.

- **Model:** Write the equilibrium equations, constitutive equations, and boundary conditions of the structure.

- **Solve:** Use an analytical or approximate method and hand calculation or a code to find the solution.

STRUCTURE IDEALIZATION



The simplified model considers the columns as bending beams, floors as rigid bodies, omits the plastic strips, and assumes that the floors move horizontally in the XZ – plane. The horizontal displacements of the floors are denoted by $u_1(t)$ and $u_2(t)$.

APPROXIMATE METHODS

The simplest approximate equations of motion by Particle Surrogate Method, Finite Difference Method, and Finite Element Method, contain only the horizontal displacements of the first and second floors:

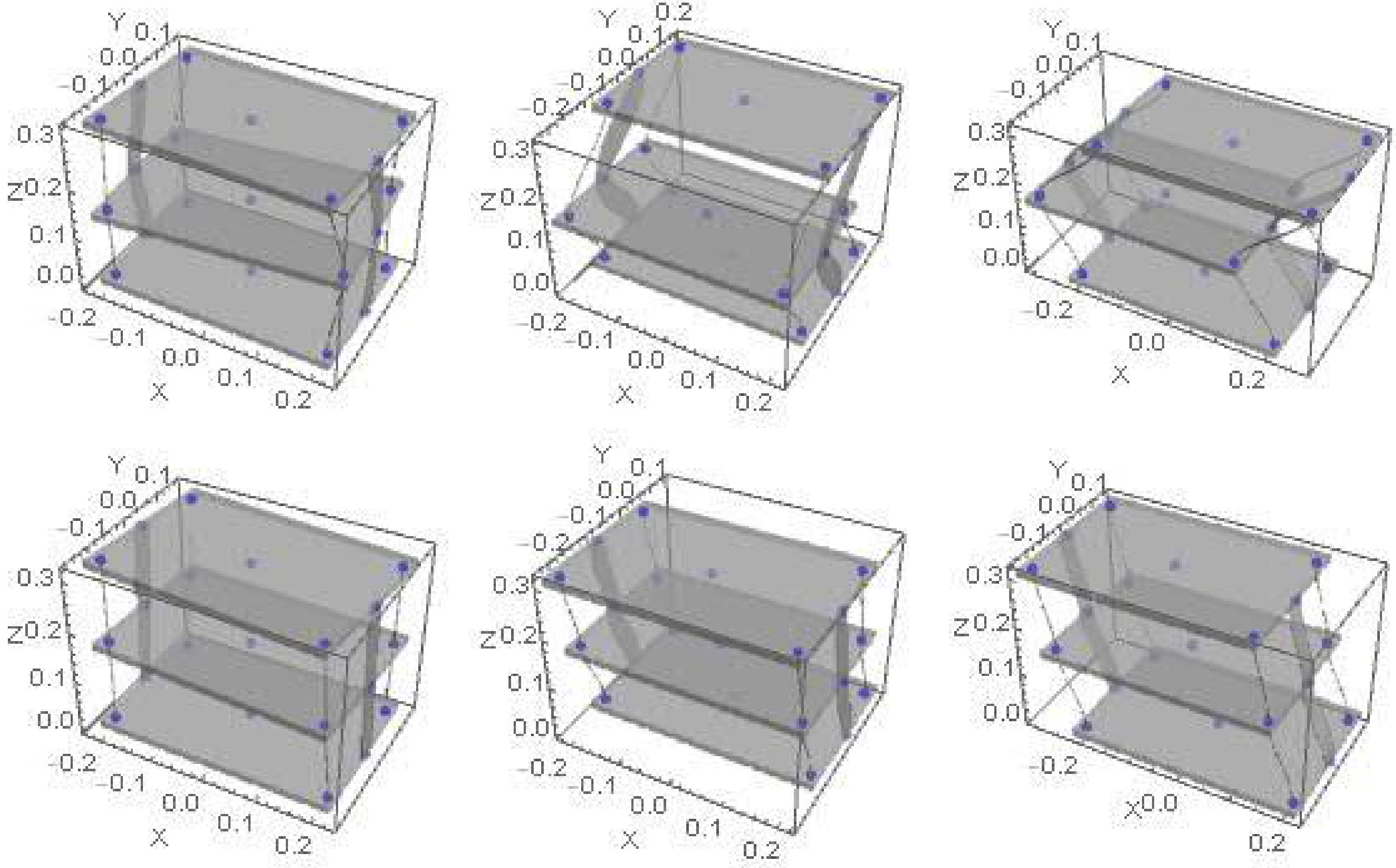
$$\text{PSM:}\quad \left(m\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + 4\times\frac{1}{2}\rho A h\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\right)\frac{d^2}{dt^2}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} + 4\times 12\frac{EI}{h^3}\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} = 0$$

$$\text{FDM:}\quad \left(\frac{m}{h}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + 4\times\rho A\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\frac{d^2}{dt^2}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} + 4\times 12\frac{EI}{h^4}\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} = 0$$

$$\text{FEM:}\quad \left(m\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + 4\times\frac{\rho A h}{6}\begin{bmatrix}4 & 1\\ 1 & 2\end{bmatrix}\right)\frac{d^2}{dt^2}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} + 4\times 12\frac{EI}{h^3}\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} = 0$$

Mode analysis for the frequencies assumes solution of the form $\mathbf{a} = \mathbf{A}\exp(\mathrm{i}\omega t)$ where $\omega = 2\pi f$, $\mathrm{i}^2 = -1$, $\mathbf{a}(t) = \{u_1 \quad u_2\}^{\mathrm{T}}$, and $\mathbf{A} = \{A_1 \quad A_2\}^{\mathrm{T}}$ (some constants).

FEM SIMULATION WITH A DETAILED MODEL



1.1 PARTICLE SURROGATE MODELLING (PSM)

In a particle model surrogates, discretization replaces a continuum model by a particle model on a grid of spatial resolution *h*. There

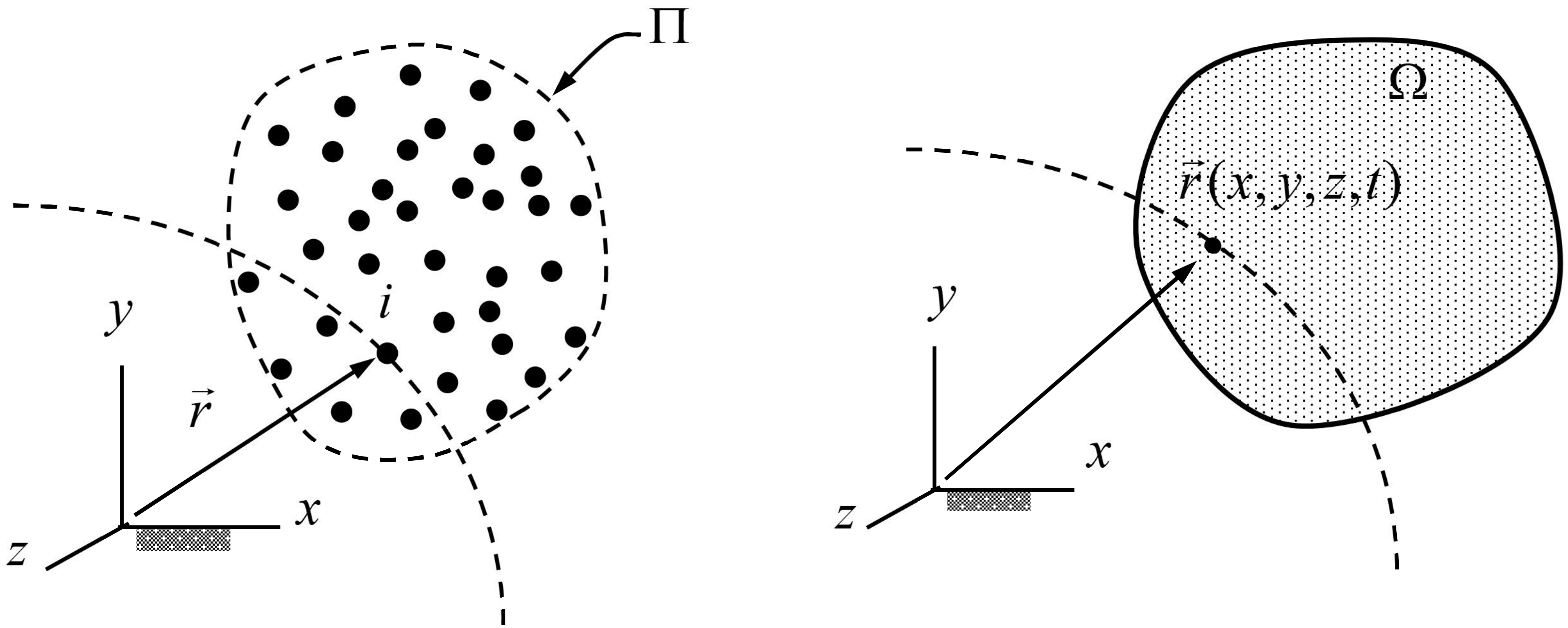
- Inertia forces (actually mass) and external forces are lumped to the grid points. Elastic properties of the material are used to deduce an interaction model of particles at the grid points, i.e., internal forces of the particle system.

- The main unknowns are the displacement of the particles. The model consists of equations of motion of the particles and possible initial conditions (if known).

Discretization replaces the original problem with computable problem whose complexity depends on resolution *h*. Particle discretization introduces modelling error compared with the continuous model which should reduce in *h* (by design of the method) and vanish in the limit $h \to 0$.

PARTICLE AND CONTINUUM MODELS

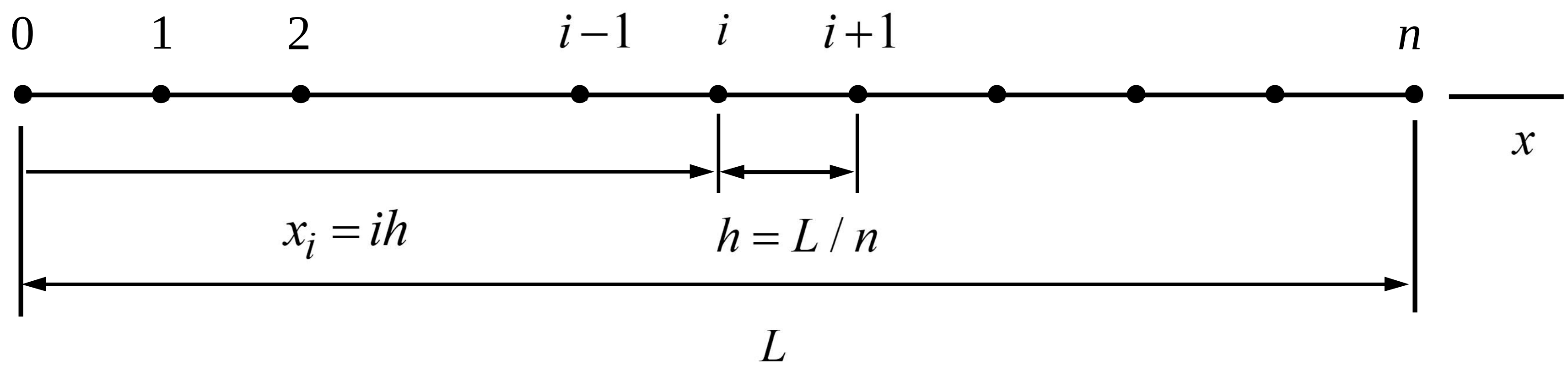
Particle surrogate model replaces the continuum model of solid mechanics by a particle model by reverting the reasoning used commonly in derivation of the continuum models.



In particle models, index $i \in \Pi \subset \mathbb{N}$ is used for labelling. In continuum models, material coordinates $(x, y, z) \in \Omega \subset \mathbb{R}^3$ are used for the purposes.

REGULAR GRID IN 1D

On a regular grid, the grid points are distributed evenly. Here, grid point numbering starts from 0 and increases without gaps in the direction of the $x-$axis the total number of grid points being $n+1$. The line segments of numbered from 1 in the same manner.



The numbering convention above fits well hand calculations in 1D case but it will be refined later for an unified geometrical description including, e.g., non-regular grids in several physical dimensions.

NEWTON’S LAWS OF MOTION

I In an inertial frame of reference, an object either remains at rest or continues to move at a constant velocity, unless acted upon by a force.

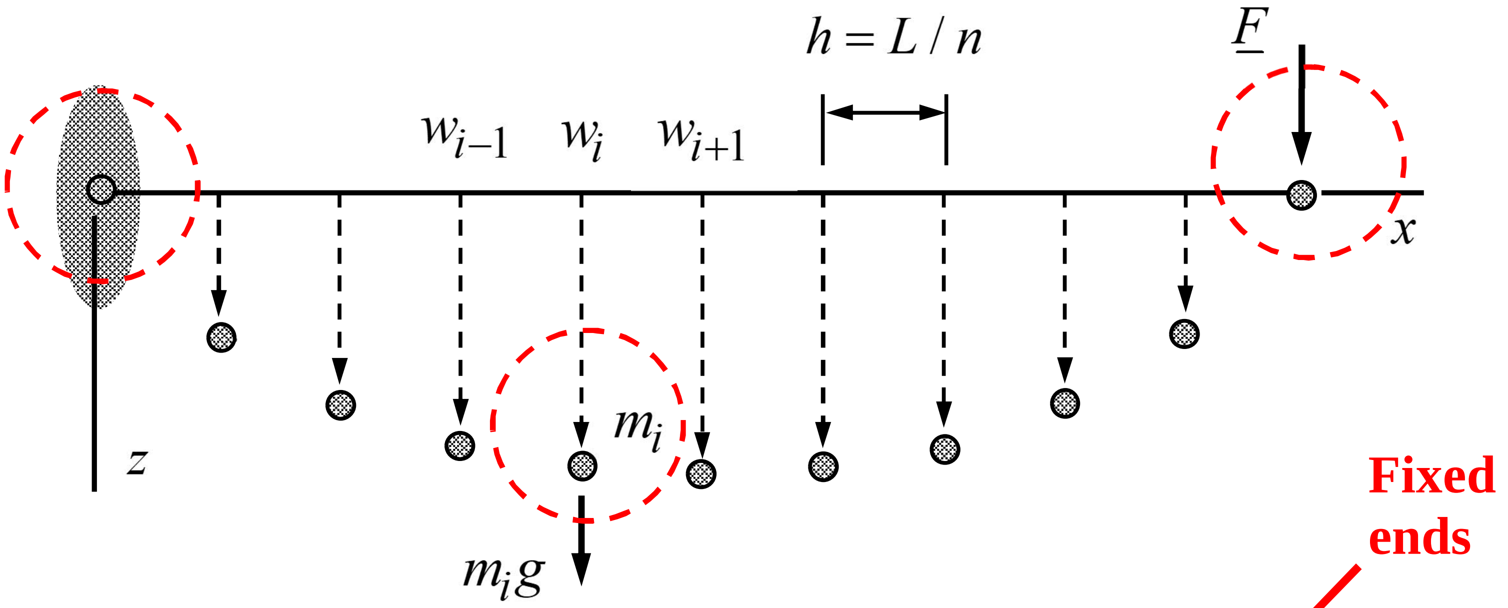
II The vector sum of the forces on an object is equal to the mass of that object multiplied by the acceleration of the object (assuming that the mass is constant).

III When one body exerts a force on a second body, the second body simultaneously exerts a force equal in magnitude and opposite in direction on the first body.

Newton’s laws in their original forms above apply to each particle separately. The formulation for average behavior of particle systems, rigid bodies, deformable bodies, open system of particles etc. require slight modifications.

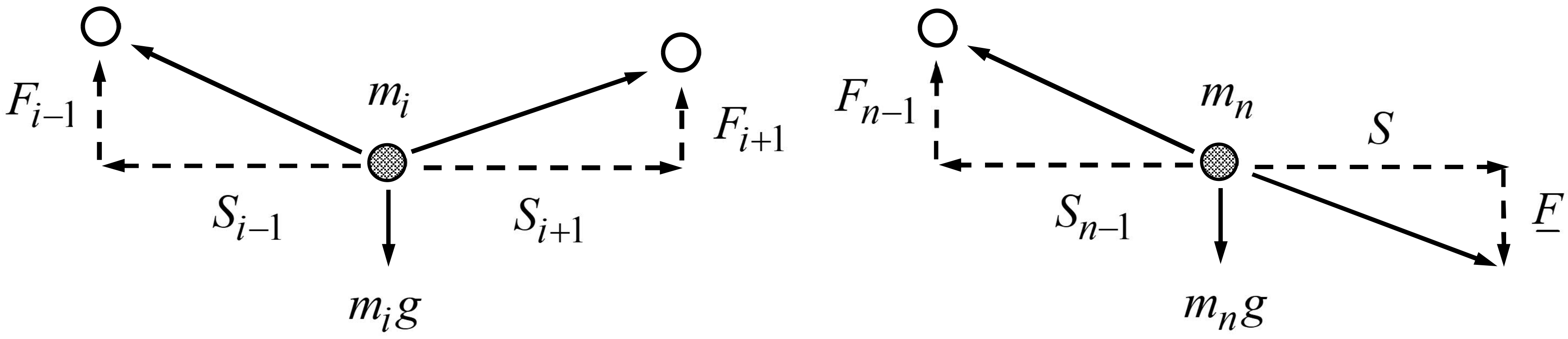
PARTICLE MODEL OF STRING

In the horizontal string model, particles move in vertical direction and forces between the particles are aligned with the string. In PSM, mass of the string and distributed transverse force are lumped as particles and point forces on a regular grid.



$$\frac{S}{h}(w_{i-1} - 2w_i + w_{i+1}) + m_i g = m_i \frac{d^2 w_i}{dt^2} \quad i \in \{1, 2, \ldots, n-1\}, \quad w_i = 0 \quad i \in \{0, n\}$$

Free body diagrams for the typical particle and the particle at the end of the string



As particles move in the vertical direction, horizontal tightening must be constant S. Assuming also constant density of the material and constant cross-sectional area, external distributed force due to gravity $m_i g$ is the same for all particles. Using the free body diagrams and the geometry of the figures, where $m_i = \rho h A$, equations of motion in the transverse direction become

$$-F_{i-1} - F_{i+1} + m_i g = m_i \frac{d^2 w_i}{dt^2} \quad \text{where} \quad \frac{F_{i-1}}{S} = \frac{w_i - w_{i-1}}{h} \quad \text{and} \quad \frac{F_{i+1}}{S} = \frac{w_i - w_{i+1}}{h},$$

$$-F_{n-1} + \underline{F} + m_n g = m_n \frac{d^2 w_n}{dt^2} \quad \text{where} \quad \frac{F_{n-1}}{S} = \frac{w_n - w_{n-1}}{h}.$$

At the fixed boundary transverse displacement vanishes or coincides with that of the surroundings $w_0 = \underline{w}$ so the equations describing the displacement of the particles are

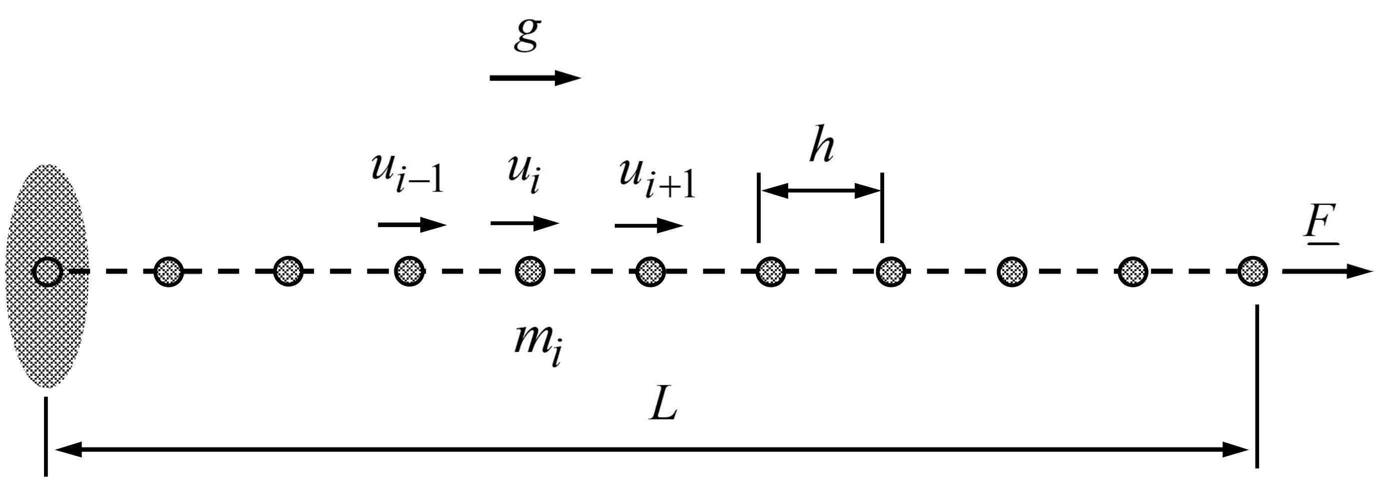
$$\frac{S}{h}(w_{i-1} - 2w_i + w_{i+1}) + m_i g = m_i \frac{d^2 w_i}{dt^2} \quad i \in \{1, 2, \ldots, n-1\}$$

$$w_0 = \underline{w} \quad \text{and} \quad -S\left(\frac{w_n - w_{n-1}}{h}\right) + \underline{F} + m_n g = m_n \frac{d^2 w_n}{dt^2}.$$

For a unique solution, the second order ordinary differential equations in time require initial conditions specifying the positions and velocities of the particles at the initial time $t = 0$.

PARTICLE MODEL OF BAR

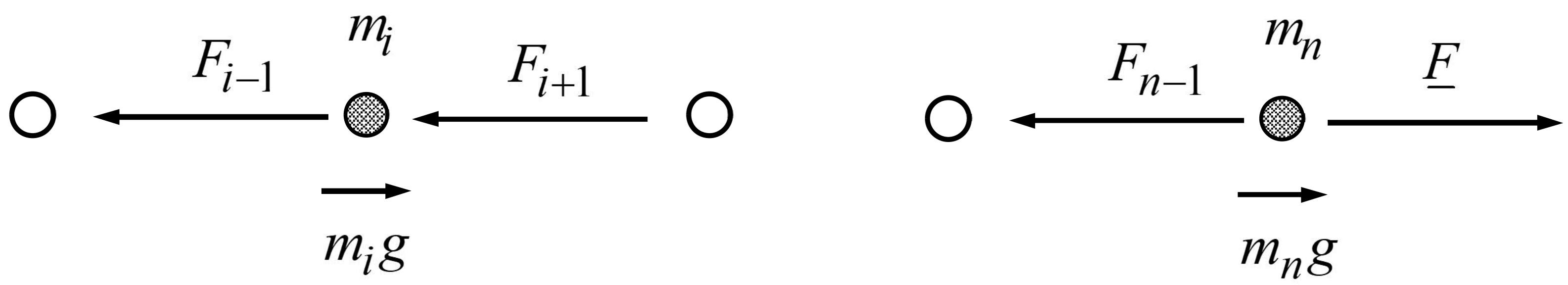
In the horizontal bar model, particles move in the horizontal direction and forces between the particles are aligned with the bar. In PSM, mass of the bar and the distributed horizontal force are lumped as particles and point forces on a regular grid.



Fixed ends

$$\frac{EA}{h}(u_{i-1} - 2u_i + u_{i+1}) + m_i g = m_i \frac{d^2 u_i}{dt^2} \quad i \in \{1, 2, \ldots, n-1\} \quad \text{and} \quad u_i = 0 \quad i \in \{0, n\}$$

In the particle surrogate model of an elastic bar, interaction of the particles are modelled by using elastic springs of spring constant $k = EA/h$, where A is the cross-sectional area and E the Young's modulus of the material. The external forces acting on particles are due to gravity.



Using the free body diagrams for the typical particle and the particle at the free end of the string

$$-F_{i-1} - F_{i+1} + m_i g = m_i \frac{d^2 u_i}{dt^2} \quad \text{where} \quad F_{i-1} = \frac{EA}{h}(u_i - u_{i-1}) \quad \text{and} \quad F_{i+1} = \frac{EA}{h}(u_i - u_{i+1}),$$

$$-F_{n-1}+\underline{F}+m_n g=m_n\frac{d^2u_n}{dt^2} \quad \text{where} \quad F_{n-1}=\frac{EA}{h}(u_n-u_{n-1}).$$

At the fixed boundary transverse displacement vanishes $u_0=0$ or coincides with that of the surroundings $u_0=\underline{u}$ so the equations describing the displacement of the particles are

$$\frac{EA}{h}(u_{i-1}-2u_i+u_{i+1})+m_i g=m_i\frac{d^2u_i}{dt^2} \quad i\in\{1,2,\ldots,n-1\},$$

$$u_0=\underline{u} \quad \text{and} \quad -\frac{EA}{h}(u_n-u_{n-1})+\underline{F}+m_n g=m_n\frac{d^2u_n}{dt^2}.$$

For a unique solution, the second order ordinary differential equations in time require initial conditions specifying the positions and velocities of the particles at the initial time $t=0$.

DIFFERENCE AND MATRIX REPRESENTATIONS

In their mathematical forms, the particle models for string and bar coincide. Assuming fixed ends, both can be considered as particular cases of a bit more generic set of ordinary second order difference-differential equations (notation $\dot{a} \equiv da/dt$, $\ddot{a} \equiv d^2a/dt^2$)

$$k_i(a_{i-1} - 2a_i + a_{i+1}) + F_i = m_i\ddot{a}_i \quad i \in \{1,2,\ldots,n-1\} \text{ and } a_i = 0 \quad i \in \{0,n\} \quad t > 0$$

$$a_i = g_i \text{ and } \dot{a}_i = h_i \quad i \in \{1,2,\ldots,n-1\} \quad t = 0$$

or, using the more concise matrix representation, as a set of ordinary second order differential equations

$$-\mathbf{Ka} + \mathbf{F} = \mathbf{M}\ddot{\mathbf{a}} \quad t > 0, \quad \mathbf{a} = \mathbf{g} \text{ and } \dot{\mathbf{a}} = \mathbf{h} \quad t = 0.$$

The two (mathematically equivalent) representations are the starting points for difference equation and matrix based solution methods for displacement and vibration analyses.

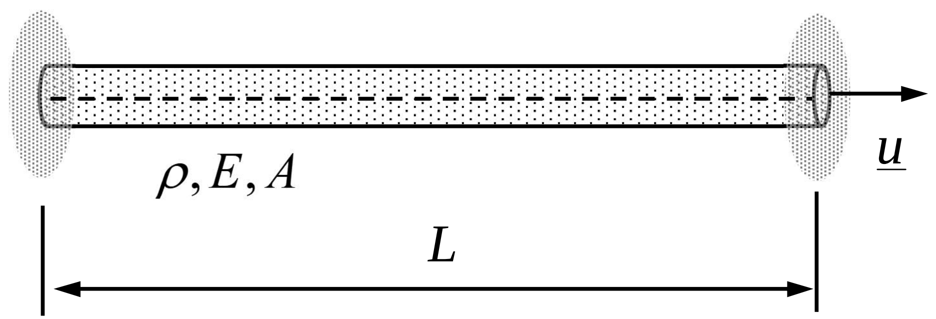
The $(n-1)\times(n-1)$ stiffness and mass matrices of the matrix representation are given by

$$\mathbf{K}=\begin{bmatrix} 2k_1 & -k_1 & & \\ -k_2 & 2k_2 & -k_2 & \\ & & \ddots & \\ & & -k_{n-1} & 2k_{n-1} \end{bmatrix},\quad \mathbf{M}=\begin{bmatrix} m_1 & & & \\ & m_2 & & \\ & & \ddots & \\ & & & m_{n-1} \end{bmatrix}$$

and the $(n-1)$ column matrices for the displacement of free particles, external forces, initial displacements and velocities

$$\mathbf{a}=\begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{Bmatrix},\quad \mathbf{F}=\begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_{n-1} \end{Bmatrix},\quad \mathbf{g}=\begin{Bmatrix} g_1 \\ g_2 \\ \vdots \\ g_{n-1} \end{Bmatrix},\quad \mathbf{h}=\begin{Bmatrix} h_1 \\ h_2 \\ \vdots \\ h_{n-1} \end{Bmatrix}.$$

EXAMPLE A connector bar is welded at its ends to rigid walls. If the right end wall displacement is $\underline{u}$, write the matrix representation for the stationary particle surrogate model. Cross sectional area A and Young's modulus of the material E are constants Use a regular grid of points $i \in \{0,1,2,3\}$. Also, find the solution to the axial displacements.



Answer $u_0 = 0$, $u_1 = \frac{1}{3}\underline{u}$, $u_2 = \frac{2}{3}\underline{u}$, and $u_3 = \underline{u}$

In time independent problem without external distributed forces, the two difference equations for the free interior particles and the conditions for the boundary particles simplify to

$$u_0 = 0, \quad \frac{EA}{h}(u_0 - 2u_1 + u_2) = 0, \quad \frac{EA}{h}(u_1 - 2u_2 + u_3) = 0, \text{ and } \; u_3 = \underline{u}.$$

In matrix representation, one considers the equations for the free particles and uses the known displacements of the fixed particles in their expressions to get

$$-\frac{EA}{h}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \frac{EA}{h}\begin{Bmatrix} 0 \\ \underline{u} \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}^{-1}\begin{Bmatrix} 0 \\ \underline{u} \end{Bmatrix} = \underline{u}\begin{Bmatrix} 1/3 \\ 2/3 \end{Bmatrix}.$$

1.2 DISPLACEMENT ANALYSIS

A stationary surrogate particle model is composed of equilibrium equations for the free particles and equations defining the displacement of fixed particles (displacement boundary conditions). The equilibrium equations for the free interior end boundary particles differ (see the derivation)

Free interior $k(a_{i-1} - 2a_i + a_{i+1}) + F_i = 0 \quad i \in \{1, 2, \ldots, n-1\}$

Free boundary $k(a_0 - a_1) = \underline{F}_0$ or $k(a_n - a_{n-1}) = \underline{F}_n$

Fixed $a_i = \underline{a}_i \quad i \in \{0, n\}$

For a boundary particle, one may give the force acting on a particle or displacement of the particle but not both. Also, displacement condition should be specified for one particle to make the solution unique (otherwise rigid body motion is not constrained).

MATRIX REPRESENTATION

Representing the displacement of the free particles by column matrix $\mathbf{a}$, the coefficient of internal force, and the external force terms by square stiffness matrix $\mathbf{K}$ and column matrix $\mathbf{F}$, respectively, gives the set of algebraic equations in their "standard" forms

$$-\mathbf{K}\mathbf{a}+\mathbf{F}=\mathbf{0} \quad \text{where} \quad \mathbf{K}=k\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & & \ddots & & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}, \quad \mathbf{a}=\begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ \\ a_{n-1} \end{Bmatrix}, \text{ and } \mathbf{F}=\begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ \\ F_{n-1} \end{Bmatrix}.$$

Matrix representation allows difference equations of various forms in the same model. Solution to the displacement follows with the standard methods for linear equations system. Therefore, the computational work in calculations is some power (1…3) of n and $n \sim 1/h$ (depends on the method used).

DIFFERENCE EQUATION REPRESENTATION

Difference equation is a mathematical equality involving the differences between successive values of a function of a discrete variable, typically values of a function at discrete spatial or temporal domain. Assuming constant coefficients α, β, γ and a polynomial δ_i

Difference equation $\alpha a_{i-1} + \beta a_i + \gamma a_{i+1} + \delta_i = 0 \quad \Rightarrow$

Generic solution $a_i = A r_1^i + B r_2^i$ or $a_i = r_1^i (A + Bi)$ where $\alpha + \beta r + \gamma r^2 = 0$

Particular solution $a_i = C + Di + Ei^2 + \ldots$

Difference equations of discrete variables correspond to differential equations of continuous variables: Solution is composed of the generic solution to homogeneous equations and a particular solution. Uniqueness of the solution require additional (boundary) conditions of number indicated by the order.

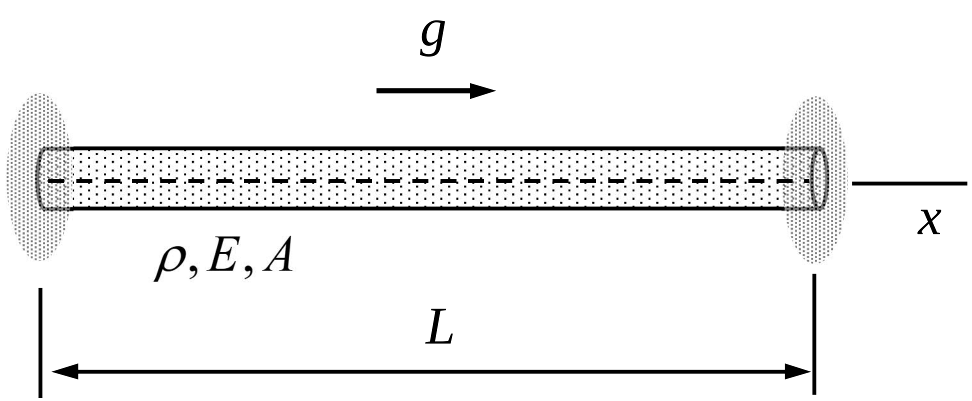
The well-known methods for 2nd order ordinary differential equations work with slight modifications. Solution to the homogeneous equation $\delta_i = 0$ is obtained with $a_i = Ar^i$:

$$\alpha Ar^{i-1} + \beta Ar^i + \gamma Ar^{i+1} = Ar^{i-1}(\alpha + \beta r + \gamma r^2) = 0 \;\Rightarrow\; \alpha + \beta r + \gamma r^2 = 0.$$

Separate roots imply the generic solution of the form $a_i = Ar_1^i + Br_2^i$. A double root implies the generic solution of the form $a_i = (A + Bi)r_1^i$. Assuming a polynomial δ_i, particular solution follows with a higher order polynomial trial etc. Finally, the two parameters A and B of the solution follow from the equations for the boundary particles $i \in \{0, n\}$.

Representation by difference equation is particularly useful on regular grids and a simple δ_i as the difference equation can be solved analytically in the same manner as the underlying ordinary differential equation. Therefore, effort for solving the problem does not depend on n at all.

EXAMPLE A connector bar, which is loaded by its own weight, is welded at its ends to rigid walls. Use a particle surrogate model on a regular grid of points $i \in \{0,1,2,3,4\}$ to find the displacements at the grid points. Use first the difference equation method to determine displacements on the generic grid and apply that to get the solution for the case $n=4$. Cross sectional area *A*, density of the material ρ, and Young's modulus of the material *E* are constants.



Answer $u_i = \dfrac{\rho L^2 g}{E}\dfrac{i(n-i)}{2n^2}$ and $u_i = \dfrac{\rho L^2 g}{E}\dfrac{i(4-i)}{32}$ when $n=4$

Let us use the difference equation method for generic n

$$k(u_{i-1}-2u_i+u_{i+1})+F=0 \quad i\in\{1,2,\ldots,n-1\} \quad \text{and} \quad u_i=0 \quad i\in\{0,n\},$$

where $k=EA/h$, $m=\rho Ah$, $F=mg$, and $h=L/n$. Solution to the homogeneous equation follows with $a_i=Ar^i$ giving, when substituted to the difference equation,

$$r^2-2r+1=(r-1)^2=0.$$

The double root $r=1$ implies the generic solution $u_i=A+Bi$. As loading F is constant, i.e., a zero-order polynomial in i, a second order polynomial $u_i=Ci^2$ might work as the particular solution. Substitution into the difference equation gives

$$k[C(i-1)^2-2Ci^2+C(i+1)^2]+F=0 \;\Rightarrow\; C=-\frac{F}{2k}.$$

When the solution is substituted there, the two displacement conditions $u_0 = u_n = 0$ give equations for the two parameters of the generic solution part

$$u_0 = A = 0 \quad \text{and} \quad u_n = A + Bn - \frac{F}{2k}n^2 = 0 \quad \Leftrightarrow \quad A = 0 \quad \text{and} \quad B = \frac{F}{2k}n.$$

Therefore, the analytic solution to the bar problem by PSM becomes

$$u_i = \frac{F}{2k} i(n-i) \quad \text{where} \quad \frac{F}{k} = \frac{\rho L^2 g}{En^2}.$$

NOTICE: In terms of the coordinates of the grid points $x_i = hi$ and $L = hn$

$$u_i = \frac{\rho L^2 g}{E} \frac{i(n-i)}{2n^2} = \frac{\rho g}{E} \frac{x_i(L-x_i)}{2} \quad \text{(continuous model } u(x) = \frac{\rho g}{E} \frac{x(L-x)}{2}\text{)}.$$

1.3 VIBRATION ANALYSIS

In time dependent case, the model is composed of equations for the interior and boundary particles, and two initial conditions for the free particles. Considering displacement component $a(t)$ on a regular grid

Particles $i \in \{1,2,\ldots,n-1\}$**:** $k(a_{i-1} - 2a_i + a_{i+1}) + F_i = m_i \ddot{a}_i \quad i \in \{1,2,\ldots,n-1\} \quad t > 0$

Particle 0**:** $a_0 = \underline{a}_0$ or $-k(a_0 - a_1) + \underline{F}_0 + m_0 g = m_0 \ddot{w}_0 \quad t > 0$

Particle n**:** $a_n = \underline{a}_n$ or $-k(a_n - a_{n-1}) + \underline{F}_n + m_n g = m_n \ddot{w}_n \quad t > 0$

Initial conditions: $a_i = g_i$ and $\dot{a}_i = h_i \quad i \in \{1,2,\ldots,n-1\} \quad t = 0$

In solid mechanics, one may give the force acting on a particle or displacement of the particle as the boundary condition but not both.

MATRIX REPRESENTATION

Representing the displacement of the free particles by column matrix $\mathbf{a}(t)$, coefficients of $\ddot{\mathbf{a}}(t)$ by square mass matrix $\mathbf{M}$, coefficients of $\mathbf{a}(t)$ by square stiffness matrix $\mathbf{K}$, and the external force terms by column matrix $\mathbf{F}$ in the difference equations of the free particles gives the second order initial value problem of ordinary differential equations

$$-\mathbf{K}\mathbf{a} + \mathbf{F} = \mathbf{M}\ddot{\mathbf{a}} \quad t > 0 \qquad \mathbf{a} = \mathbf{g} \text{ and } \dot{\mathbf{a}} = \mathbf{h} \quad t = 0.$$

The column matrices $\mathbf{g}$ and $\mathbf{h}$ represent the initial positions and velocities of the free particles. Matrix representation is the concise starting point for

(1) mode analysis for frequencies and modes of free vibrations

(2) displacement solutions based on the frequencies and modes

(3) step-by-step time integration methods on temporal grid of time instants

The $n-1$ by $n-1$ matrices and the $n-1$ column matrix corresponding to a problem of free particles $i \in \{1, 2, \ldots, n-1\}$, conditions $a_0 = a_n = 0$, and $k_i = k$, $m_i = m$, $F_i = F$

$$\mathbf{K} = k\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}, \quad \mathbf{M} = m\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{and } \mathbf{F} = F\begin{Bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{Bmatrix}.$$

The coefficients for the bar of length $L = nh$ are $k = EA/h$, $m = \rho Ah$, and $F = g\rho Ah$ (gravity in the direction of the axis) and the coefficients for the string of length $L = nh$ are $k = S/h$, $m = \rho Ah$, and $F = g\rho Ah$ (gravity in the transverse direction).

MODAL ANALYSIS

For constant $\mathbf{M}$, $\mathbf{K}$ and $\mathbf{F}=\mathbf{0}$, displacement can be considered as the sum of harmonic components. In mode analysis, a harmonic trial solution is used to transform the ordinary differential equations into algebraic one for the angular velocity and mode pairs $(\omega_j, \mathbf{A}_j)$:

$$\mathbf{M}\ddot{\mathbf{a}}+\mathbf{K}\mathbf{a}=\mathbf{0} \text{ and } \mathbf{a}(t)=\mathbf{A}e^{\mathrm{i}\omega t} \quad \Rightarrow \quad (-\mathbf{M}\omega^2+\mathbf{K})\mathbf{A}=0.$$

The necessary condition for a non-zero solution to $\mathbf{A}$ is $\det(-\mathbf{M}\omega^2+\mathbf{K})=0$. The algebraic polynomial equation gives ω_j $j\in\{1,2,\ldots\}$ of number of the free particles as its solutions (positive square roots of ω^2). After that, the modes follow from

$$(-\mathbf{M}\omega_j^2+\mathbf{K})\mathbf{A}_j=0$$

up to an arbitrary multiplier. The angular velocity ω and frequency f are related by $\omega=2\pi f$.

MODE SUPERPOSITION

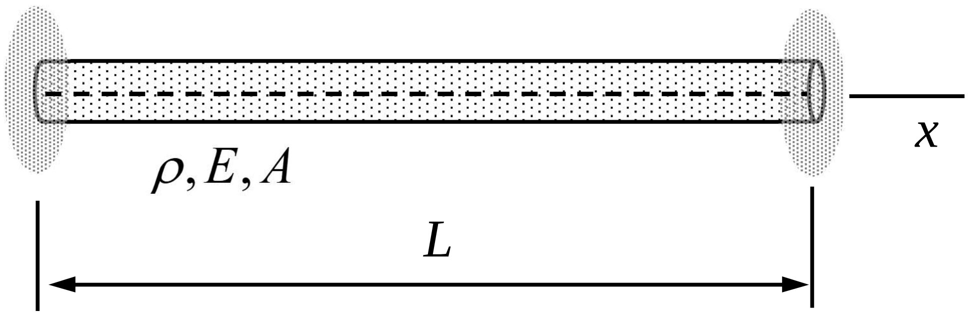
If the initial conditions concerning position and displacement of the particles are known (quite exceptional case), the outcome of the modal analysis $(\omega_j, \mathbf{A}_j)$ $j \in \{1,2,\ldots\}$ can be used to construct a displacement solution for the given initial data. The combination of the modes for $\mathbf{a} = \mathbf{g}$ and $\dot{\mathbf{a}} = \mathbf{h}$ is given by

$$\mathbf{a}(t) = \sum_{j\in\{1,2,\ldots\}} \mathbf{A}_j[\frac{1}{\omega_j}\alpha_j \sin(\omega_j t) + \beta_j \cos(\omega_j t)] \text{ where}$$

$$\alpha_j = \frac{\mathbf{A}_j^{\mathrm{T}}\mathbf{h}}{\mathbf{A}_j^{\mathrm{T}}\mathbf{A}_j} \text{ and } \beta_j = \frac{\mathbf{A}_j^{\mathrm{T}}\mathbf{g}}{\mathbf{A}_j^{\mathrm{T}}\mathbf{A}_j}.$$

As the first term contains division by ω_k, one should use $\lim_{\omega\to 0}\sin(\omega t)/\omega = t$ if $\omega_k = 0$. The simple formula relies on orthogonality of the modes $\mathbf{A}_j^{\mathrm{T}}\mathbf{A}_l = 0$ whenever $j \neq l$. One may think that the coefficients α_j and β_j are given by discrete Fourier series.

EXAMPLE A connector bar is welded at its ends to rigid walls. Use a particle surrogate model on a regular grid of points $i \in \{0,1,\ldots,n\}$ to find displacements at the grid points as functions of time for the initial data $g_i = U$ and $h_i = 0$, respectively. Cross sectional area A, density of the material ρ, and Young's modulus of the material E are constants. Use the matrix method and consider the case $n = 3$



Answer $\mathbf{a}(t) = U \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \cos(\sqrt{\frac{k}{m}} t) = U \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \cos(\frac{3}{L}\sqrt{\frac{E}{\rho}} t)$

Let us start with modal analysis with the stiffness and mass matrix of the example problem where $k = EA/h$, $m = \rho Ah$, and $h = L/3$. The number of free particles is 2 so

$$(m\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\omega^2 + k\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix})\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = k(\begin{bmatrix} \lambda - 2 & 1 \\ 1 & \lambda - 2 \end{bmatrix})\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0 \quad (\text{denote } \lambda = \frac{m\omega^2}{k})$$

The homogeneous linear equation system can yield a non-zero solution to the mode only if the matrix in parenthesis is singular, i.e., its determinant vanishes

$$\det\begin{bmatrix} \lambda - 2 & 1 \\ 1 & \lambda - 2 \end{bmatrix} = (\lambda - 2)^2 - 1 = 0 \text{ so } \lambda_1 = 1 \text{ or } \lambda_2 = 3.$$

Knowing the possible angular velocities, solution to the modes are given by the linear equation systems:

$$\lambda_1 = 1: \quad \omega_1 = \sqrt{\lambda \frac{k}{m}} \text{ and } \begin{bmatrix} 1-2 & 1 \\ 1 & 1-2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0 \quad \text{so} \quad (\omega_1, \mathbf{A}_1) = (\sqrt{\frac{k}{m}}, \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}).$$

$$\lambda_2 = 3: \quad \omega_2 = \sqrt{3 \frac{k}{m}} \text{ and } \begin{bmatrix} 3-2 & 1 \\ 1 & 3-2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0 \quad \text{so} \quad (\omega_2, \mathbf{A}_2) = (\sqrt{3 \frac{k}{m}}, \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}).$$

The coefficients of the series solution are $\alpha_1 = \alpha_2 = 0$ and

$$\beta_1 = \frac{\mathbf{A}_1^{\mathrm{T}} \mathbf{g}}{\mathbf{A}_1^{\mathrm{T}} \mathbf{A}_1} = \frac{\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} U \\ U \end{Bmatrix}}{\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}} = U, \quad \beta_2 = \frac{\mathbf{A}_2^{\mathrm{T}} g}{\mathbf{A}_2^{\mathrm{T}} \mathbf{A}_2} = \frac{\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} U \\ U \end{Bmatrix}}{\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}}) = 0 \quad \Rightarrow$$

$$\mathbf{a}(t) = \sum_{k \in \{1,2\}} \mathbf{A}_j [\frac{1}{\omega_j} \alpha_j \sin(\omega_j t) + \beta_j \cos(\omega_j t)] = U \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \cos(\sqrt{\frac{k}{m}} t).$$

DIFFERENTIAL-DIFFERENCE EQUATION METHOD

The analytical solution method for differential-difference equations on a regular grid uses a trial solution mode which gives an algebraic equation for the corresponding angular velocity. Then the outcome of modal analysis takes the form

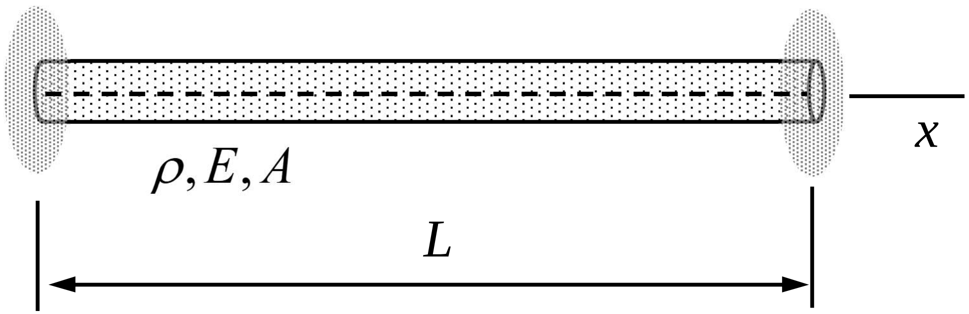
$$(\omega_j, \mathbf{A}_j) \quad \text{where} \quad \omega_j = \sqrt{2\frac{k}{m}[1-\cos(\frac{j\pi}{n})]} \quad \text{and} \quad (\mathbf{A}_j)_i = \alpha\cos(\pi j\frac{i}{n}) + \beta\sin(\pi j\frac{i}{n}),$$

where α and β are determined by the (homogeneous) boundary conditions. The combination of the modes for $\mathbf{a} = \mathbf{g}$ and $\dot{\mathbf{a}} = \mathbf{h}$ at $t = 0$

$$\mathbf{a}(t) = \sum_{j\in\{1,2,\ldots\}} \mathbf{A}_j[\frac{1}{\omega_j}\alpha_j\sin(\omega_j t) + \beta_j\cos(\omega_j t)], \quad \alpha_j = \frac{\mathbf{A}_j^{\mathrm{T}}\mathbf{h}}{\mathbf{A}_j^{\mathrm{T}}\mathbf{A}_j} \quad \text{and} \quad \beta_k = \frac{\mathbf{A}_j^{\mathrm{T}}\mathbf{g}}{\mathbf{A}_j^{\mathrm{T}}\mathbf{A}_j}$$

is the same as with the matrix formulation.

EXAMPLE A connector bar is welded at its ends to rigid walls. Use a particle surrogate model on a regular grid of points with $n = 3$ to find displacements at the grid points as functions of time for the initial data $g_i = U$ and $h_i = 0$, respectively. Cross sectional area A, density of the material ρ, and Young's modulus of the material E are constants. Use the differential-difference equation method.



Answer $\mathbf{a}(t) = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \cos(\frac{3}{L}\sqrt{\frac{E}{\rho}}t)$

The method is based on the closed form solution to the modal analysis

$$(\omega_j, \mathbf{A}_j) \quad \text{where} \quad \omega_j = \sqrt{2\frac{k}{m}[1-\cos(\frac{j\pi}{n})]} \quad \text{and} \quad (\mathbf{A}_j)_i = \gamma\cos(\pi j\frac{i}{n}) + \delta\sin(\pi j\frac{i}{n})$$

As both ends are fixed, the parameters of the modes are chosen to be $\gamma = 0$ and $\delta = 1$ (say). As $n = 3$ $j \in \{1, 2\}$ so

$$\omega_1 = \sqrt{2\frac{k}{m}[1-\cos(\pi\frac{1}{3})]} = \sqrt{\frac{k}{m}} \quad \text{and} \quad \mathbf{A}_1 = \begin{Bmatrix} \sin(\pi/3) \\ \sin(2\pi/3) \end{Bmatrix} = \frac{\sqrt{3}}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix},$$

$$\omega_2 = \sqrt{2\frac{k}{m}[1-\cos(2\pi\frac{1}{3})]} = \sqrt{3\frac{k}{m}} \quad \text{and} \quad \mathbf{A}_2 = \begin{Bmatrix} \sin(2\pi/3) \\ \sin(4\pi/3) \end{Bmatrix} = \frac{\sqrt{3}}{2}\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

Notice that the modes differ only in scaling from the ones of the matrix method. Therefore, the solution by the two methods coincide as should be the case.

DISCRETE SINE SERIES

The discrete Fourier series (various forms exist) can be used to represent a list as the sum of lists of harmonic terms. For example, the sine-transformation pair for a list a_i $i \in \{1, 2, \ldots, n-1\}$ is given by

$$\alpha_j = \frac{2}{n} \sum_{i \in \{1,2,\ldots,n-1\}} \sin(j\pi \frac{i}{n}) a_i \quad j \in \{1, 2, \ldots, n-1\}$$

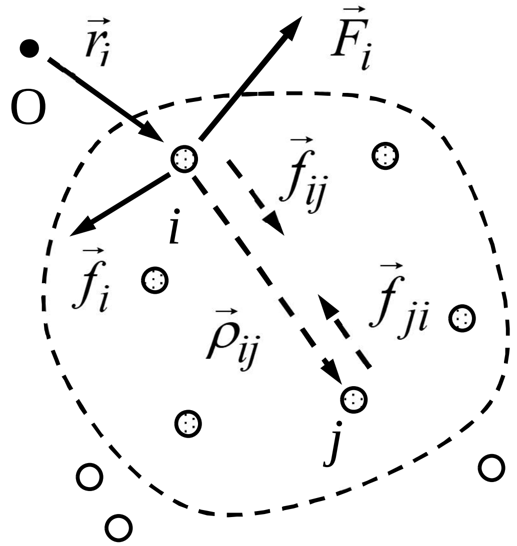
$$a_i = \sum_{j \in \{1,2,\ldots,n-1\}} \alpha_j \sin(j\pi \frac{i}{n}) \quad i \in \{1, 2, \ldots, n-1\}$$

The transformation pair is based on the orthogonality of the modes (Cronecker delta $\delta_{jl} = 1$ if $j = l$ and $\delta_{jl} = 0$ if $j \neq l$)

$$\sum_{j \in \{1,2,\ldots,n-1\}} \sin(j\pi \frac{i}{n}) \sin(l\pi \frac{i}{n}) = \delta_{jl} \frac{n}{2}.$$

1.4 PRINCIPLE OF VIRTUAL WORK

Principle of virtual work is one of the equivalent forms of equilibrium equations and equations of motion (an important one). According to the principle, work of forces acting on the particles vanishes in all virtual displacements of the particles. In short $\delta W = \delta W^{\text{ext}} + \delta W^{\text{int}} + \delta W^{\text{ine}} = 0 \quad \forall \delta\vec{r}_i$.



External $\qquad \delta W^{\text{ext}} = \sum \vec{F}_i \cdot \delta\vec{r}_i$

Internal $\qquad \delta W^{\text{int}} = \sum \vec{f}_i \cdot \delta\vec{r}_i = -\sum f_p \delta\rho_p$

Inertia $\qquad \delta W^{\text{ine}} = -\sum m_i \vec{a}_i \cdot \delta\vec{r}_i$

Principle of virtual work has a physical interpretation which is useful in connection with the variety of particle and continuum models and numerical methods in solid mechanics.

Let us consider the set of particles $i \in I$ and the set of interacting pairs $p \in P \subset I \times I$. The two equivalent representations for the equations of motion are

$$\vec{F}_i + \vec{f}_i = m_i \vec{a}_i \quad \forall i \in I \qquad \Leftrightarrow \qquad \sum_{i \in I} \delta \vec{r}_i \cdot (\vec{F}_i + \vec{f}_i - m_i \vec{a}_i) = 0 \quad \forall \delta \vec{r}_i,$$

where $\delta \vec{r}_i$ is the virtual displacement (a virtual offset) of particle *i*. The two-ways implication follows from the fundamental lemma of variation calculus. Works of the forces in the virtual displacement (forces are not affected by the virtual offset)

$$\delta W^{\text{ext}} = \sum_{i \in I} \delta \vec{r}_i \cdot \vec{F}_i \,, \quad \delta W^{\text{int}} = \sum_{i \in I} \delta \vec{r}_i \cdot \vec{f}_i \,, \text{ and } \quad \delta W^{\text{ine}} = -\sum_{i \in I} \delta \vec{r}_i \cdot m_i \vec{a}_i \,.$$

Above, $\vec{f}_i$ denotes the sum of the internal forces acting on particle i. Let us consider a typical pair $p = (i, j)$ of particles and interaction $\vec{f}_{ij} = -\vec{f}_{ji}$. The contribution to δW^{int} can be written in a more concise form

$$\delta W_p^{\text{int}} = \vec{f}_{ij} \cdot \delta\vec{r}_i + \vec{f}_{ji} \cdot \delta\vec{r}_j = \vec{f}_{ij} \cdot \delta(\vec{r}_i - \vec{r}_j) = -\vec{f}_{ij} \cdot \delta\vec{\rho}_{ij} = -f_p \delta\rho_p,$$

where $\vec{r}_j = \vec{r}_i + \vec{\rho}_{ij}$ (assuming a simple force interaction). The overall work of the internal forces is obtained as the sum over all the interacting pairs, i.e,

$$\delta W^{\text{int}} = \sum_{p \in P} \delta W_p^{\text{int}} = -\sum_{p \in P} f_p \delta\rho_p .$$

VIRTUAL WORK EXPRESSIONS FOR STRING

Let us consider particles $i \in I = \{0,1,\ldots,n\}$, interacting particle pairs $p \in P \subset I \times I$, choose $\delta w_i = 0$ whenever $w_i = \underline{w}_i$ (known), and denote $\Delta w_p = w_i - w_j$ when $p = (i, j)$

Internal forces: $\delta W^{\mathrm{int}} = -\sum_{p \in P} \delta\Delta w_p \dfrac{S}{h} \Delta w_p$

External forces: $\delta W^{\mathrm{ext}} = \sum_{i \in I} \delta w_i F_i$

Inertia forces: $\delta W^{\mathrm{ine}} = -\sum_{i \in I} \delta w_i \rho A h \ddot{w}_i$

Principle of virtual work and the virtual work expressions give a concise reprentation of the string and bar equations of PSM. Various different boundary conditions can be included by modification of the expression using the physical work interpretation. The representation is almost indispensible with membrane and thin slab models of the course and irregular grids on generic solution domains.

The external and inertia parts are obvious. In the string model, particle i interacts with the neighbors $i-1$ and $i+1$ only. Therefore, virtual work of the internal forces (all particles accounted for)

$$-\delta W^{\text{int}} = \delta w_0 F_1 + \delta w_1 (F_0 + F_3) + \delta w_2 (F_1 + F_3) + \ldots + \delta w_n F_{n-1}.$$

Substituting expressions $F_{i-1} = S(w_i - w_{i-1})/h$ and $F_{i+1} = S(w_i - w_{i+1})/h$ for the left and right neighbour interactions and rearranging

$$-\delta W^{\text{int}} \frac{h}{S} = (\delta w_1 - \delta w_0)(w_1 - w_0) + (\delta w_2 - \delta w_1)(w_2 - w_1) + \ldots + (\delta w_n - \delta w_{n-1})(w_n - w_{n-1})$$

and, finally, using the concise sum notation

$$\delta W^{\text{int}} = -\sum_{i \in \{1,2,\ldots,n\}} (\delta w_i - \delta w_{i-1}) \frac{S}{h} (w_i - w_{i-1}) = -\sum_{p \in P} \delta \Delta w_p \frac{S}{h} \Delta w_p . \quad \Leftarrow$$

VIRTUAL WORK EXPRESSIONS

Principle of virtual work is just a concise representation of the equations-of-motion (or equilibrium equations) and boundary conditions of a particle surrogate model. Virtual work expression depends on the problem, but the principle does not.

Virtual work	Bar	String
δW^{int}	$-\sum_{p\in P} \delta\Delta u_p \frac{EA}{h}\Delta u_p$	$-\sum_{p\in P} \delta\Delta w_p \frac{S}{h}\Delta w_p$
δW^{ext}	$\sum_{i\in I} \delta u_i F_i$	$\sum_{i\in I} \delta w_i F_i$
δW^{ine}	$-\sum_{i\in I} \delta u_i \rho A h \ddot{u}_i$	$-\sum_{i\in I} \delta w_i \rho A h \ddot{w}_i$

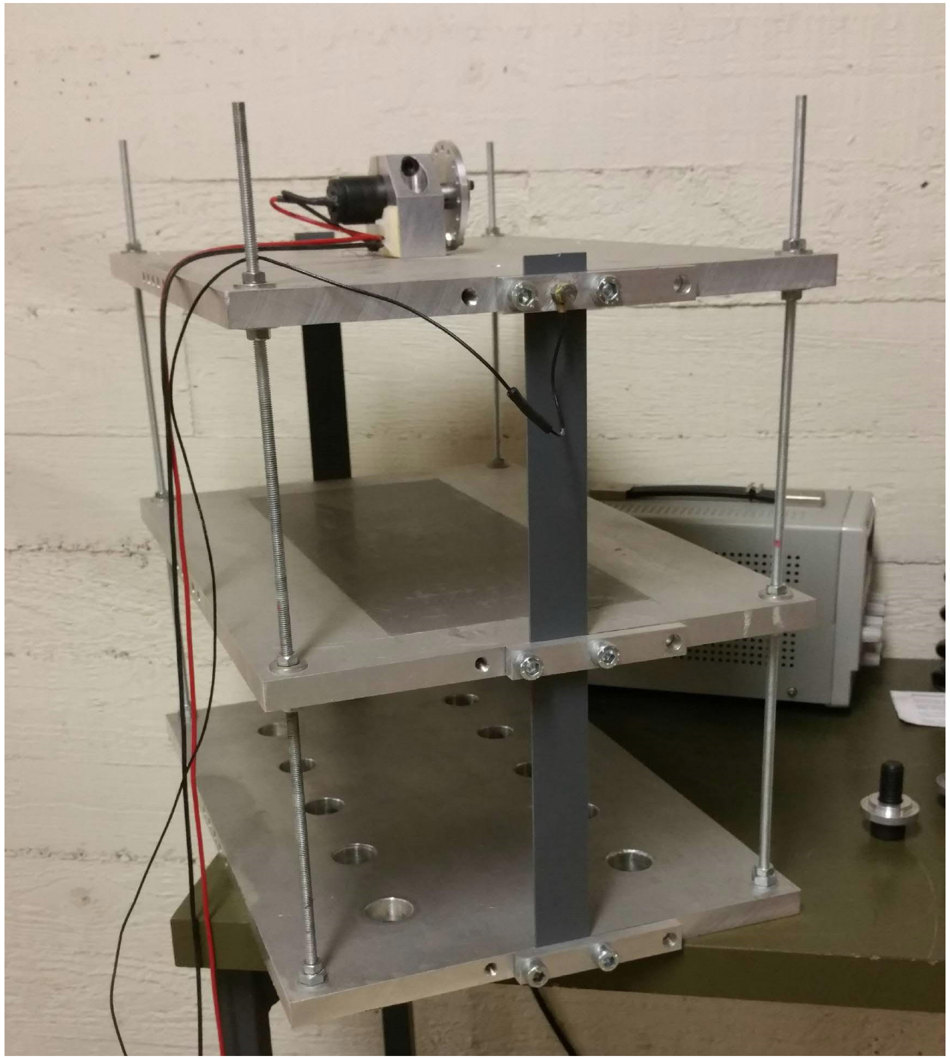
COE-C3005

Finite Element and Finite Difference Methods 2021

WEEK 16: INTRODUCTION

Fri 09:15-11:00 Calculation hours (JF & MÅ)

VIBRATION EXPERIMENT



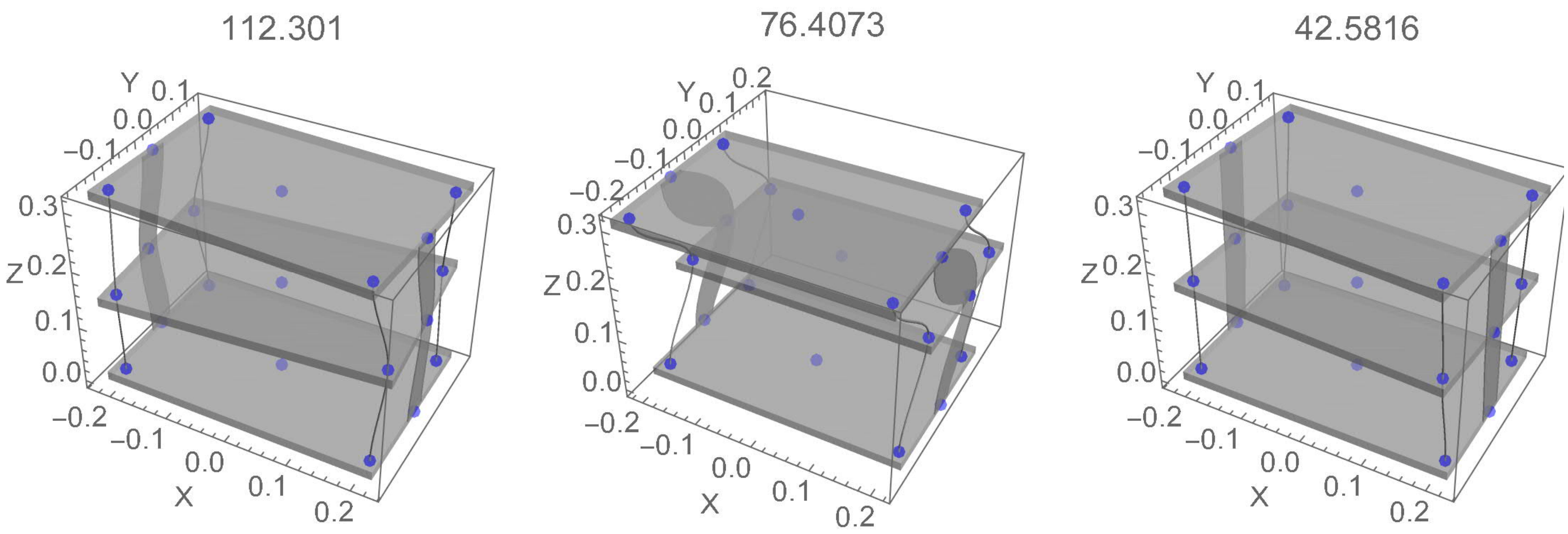
MODELLING ASSIGNMENT

In the modelling assignment, you will determine the two first frequencies of the free vibrations of the 3-story building using a model and

1. Particle Surrogate Method (PSM)
2. Finite Difference Method (FDM)
3. Finite Element Method (FDM)

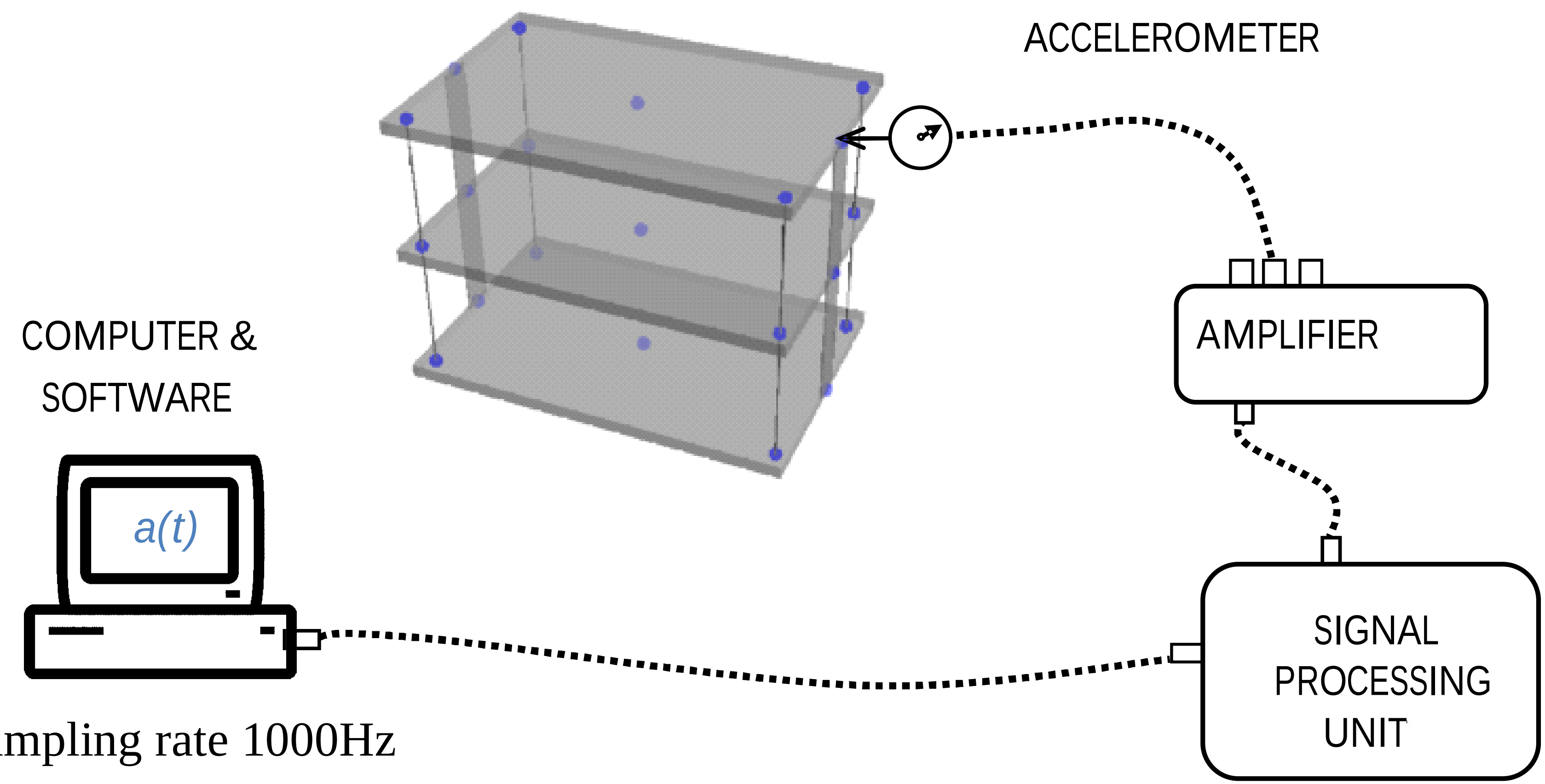
To report the outcome, supplement the assignment paper with experimental results and the outcome of calculations (table for results in light blue shading). Return your report (in PDF) on Sun 25.04.2021 23:55 at the latest (MyCourses).

FREE VIBRATIONS OF STRUCTURE



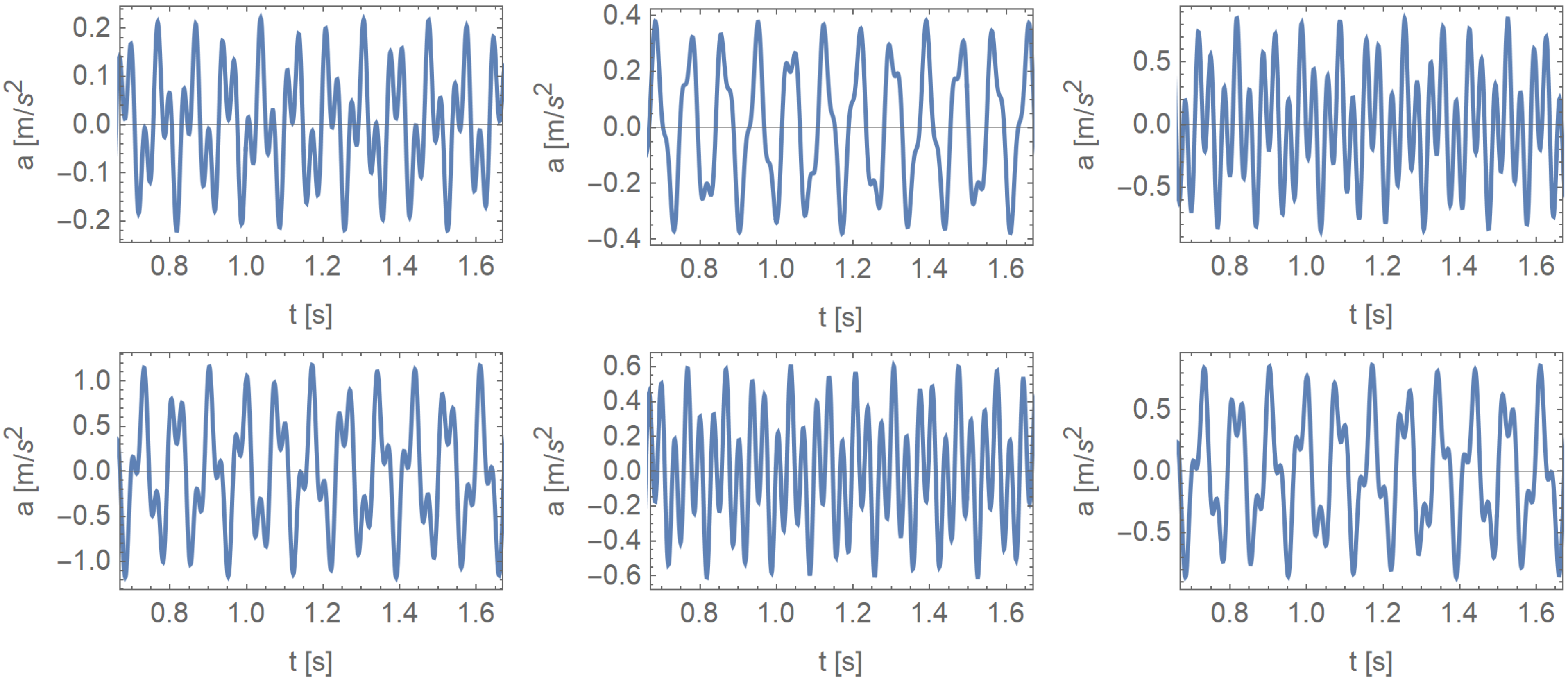
Assuming linearly elastic material, displacements $\mathbf{a}(t)$ at the grid points (in blue) can be represented as the sum $\mathbf{a}(t) = \sum \mathbf{A}_j[\alpha_j \sin(\omega_j t)/\omega_j + \beta_j \cos(\omega_j t)]$, where $\mathbf{A}_j$ are the modes (deformation patterns of the figure) and $\omega_j = 2\pi f_j$ the angular velocities associated with the modes.

VIBRATION EXPERIMENT



Sampling rate 1000Hz

ACCELERATION TIME-SERIES



Experimental data consists of the acceleration time-series measured by the accelerometer at one point. Experiment is repeated 6 times.

DISCRETE SINE SERIES

The discrete Fourier series (various forms exist) can be used to represent a list as the sum of lists of harmonic terms. For example, the sine-transformation pair for a list a_i $i \in \{1,2,\ldots,n-1\}$ is given by

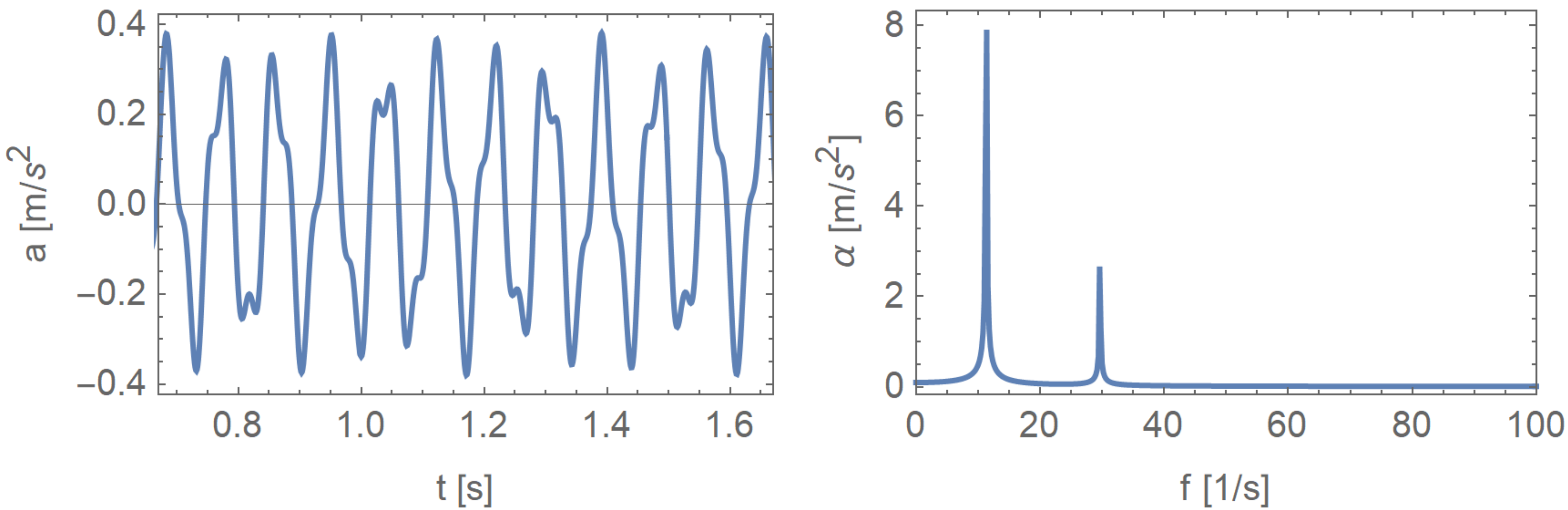
$$\alpha_j = \frac{2}{n}\sum_{i\in\{1,2,\ldots,n-1\}} \sin(j\pi\frac{i}{n})a_i \quad j \in \{1,2,\ldots,n-1\}$$

$$a_i = \sum_{j\in\{1,2,\ldots,n-1\}} \alpha_j \sin(j\pi\frac{i}{n}) \quad i \in \{1,2,\ldots,n-1\}$$

The transformation pair is based on the orthogonality of the modes (Cronecker delta $\delta_{jl} = 1$ if $j = l$ and $\delta_{jl} = 0$ if $j \neq l$)

$$\sum_{j\in\{1,2,\ldots,n-1\}} \sin(j\pi\frac{i}{n})\sin(l\pi\frac{i}{n}) = \delta_{jl}\frac{n}{2}.$$

PROCESSING OF DATA



Experimental data consists of the acceleration time-series measured by the accelerometer at one point. In processing of data, the time-acceleration representation is transformed to frequency-mode magnitude form by Discrete Fourier Transform (DFT).

MODELLING STEPS

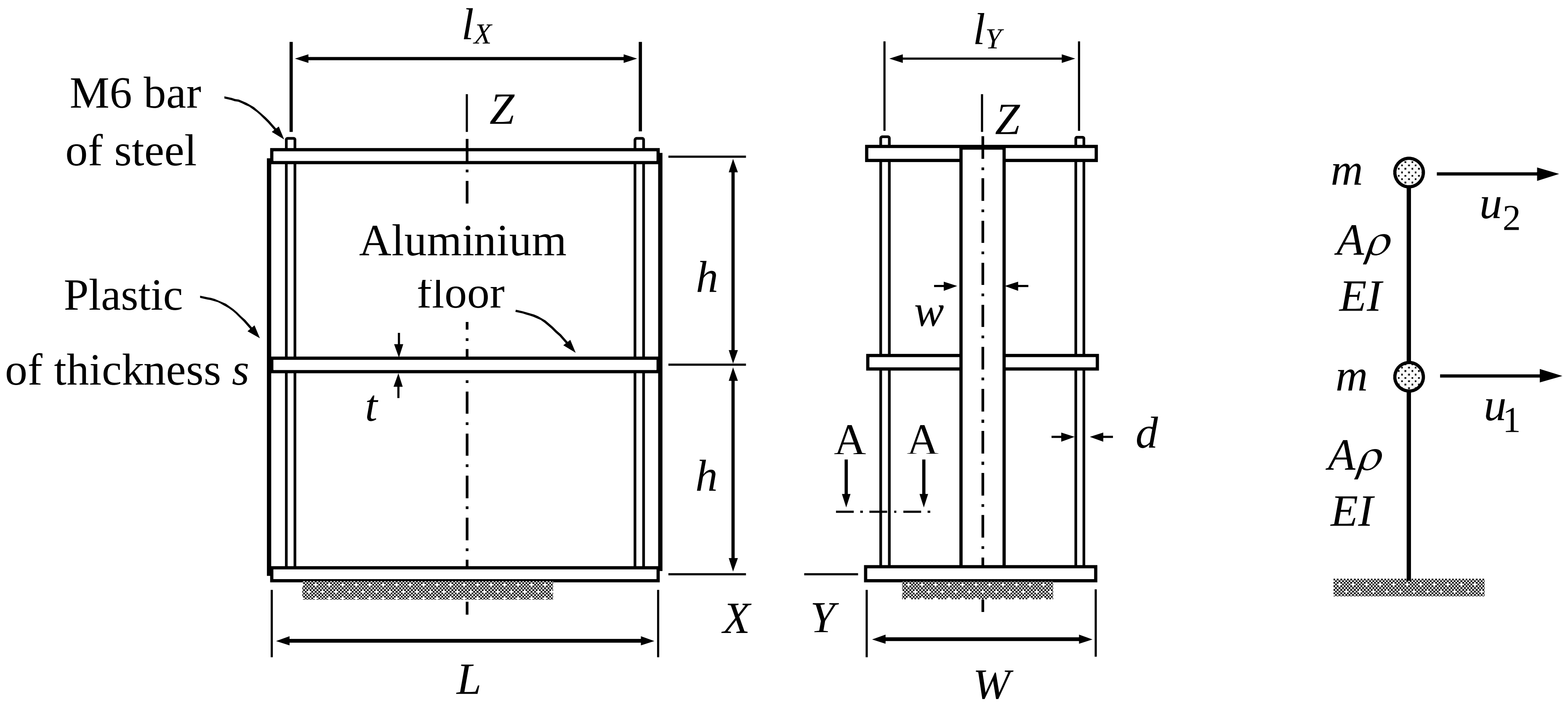
- **Crop:** Decide the boundary of a structure. Interaction with surroundings need to be described in terms of known forces, moments, displacements, and rotations. All uncertainties with this respect bring uncertainty to the model too.

- **Idealize and parameterize:** Simplify the geometry. Ignoring the details not likely to affect the outcome may simplify the analysis a lot. Assign symbols to geometric and material parameter of the idealized structure.

- **Model:** Write the equilibrium equations, constitutive equations, and boundary conditions of the structure.

- **Solve:** Use an analytical or approximate method and hand calculation or a code to find the solution.

STRUCTURE IDEALIZATION



The simplified model considers the columns as bending beams, floors as rigid bodies, omits the plastic strips, and assumes that the floors move horizontally in the $XZ-$plane. The horizontal displacements of the floors are denoted by $u_1(t)$ and $u_2(t)$.

PARAMETERIZATION

Parameter	symbol	value
Column thickness	d	0.0048 m
Room height	h	0.156 m
Column distance (x)	l_x	0.4 m
Column distance (y)	l_y	0.243 m
Floor length	L	0.44 m
Floor width	W	0.295 m
Floor thickness	t	0.015 m
Strip width	w	0.04 m
Strip thickness	s	0.002 m

APPROXIMATE METHODS

The simplest approximate equations of motion by Particle Surrogate Method, Finite Difference Method, and Finite Element Method, contain only the horizontal displacements of the first and second floors:

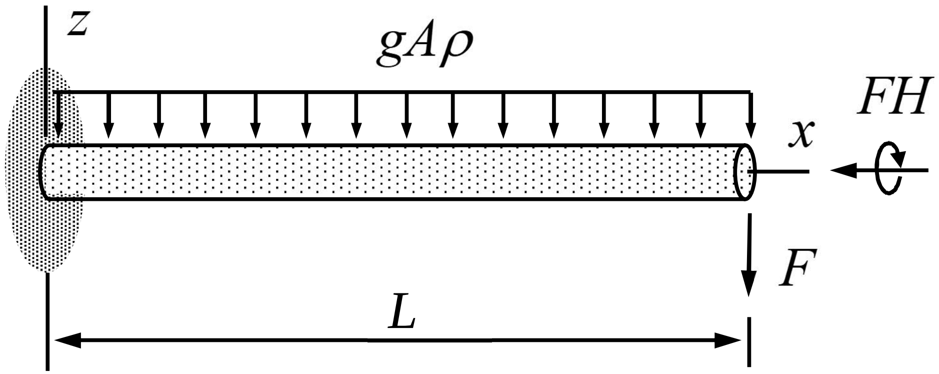
$$\text{PSM:} \quad \left(m\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + 4\times\frac{1}{2}\rho_s Ah\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\right)\frac{d^2}{dt^2}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} + 4\times 12\frac{E_s I}{h^3}\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} = 0$$

$$\text{FDM:} \quad \left(\frac{m}{h}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + 4\times\rho_s A\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\frac{d^2}{dt^2}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} + 4\times 12\frac{EI}{h^4}\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} = 0$$

$$\text{FEM:} \quad \left(m\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + 4\times\frac{\rho_s Ah}{6}\begin{bmatrix}4 & 1\\ 1 & 2\end{bmatrix}\right)\frac{d^2}{dt^2}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} + 4\times 12\frac{EI}{h^3}\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} = 0$$

Mode analysis for the frequencies assumes solution of the form $\mathbf{a} = \mathbf{A}\exp(\mathrm{i}\omega t)$ where $\omega = 2\pi f$, $\mathrm{i}^2 = -1$, $\mathbf{a}(t) = \{u_1 \quad u_2\}^{\mathrm{T}}$, and $\mathbf{A} = \{A_1 \quad A_2\}^{\mathrm{T}}$ (some constants).

BEAM THEORY



Domain: $-EI_{yy}\dfrac{d^4w}{dx^4}+b_z=0,\;\; GJ\dfrac{d^2\phi}{dx^2}=0$ in $(0,L)$

Free end: $-EI_{yy}\dfrac{d^2w}{dx^2}=\underline{M}_y,\;\; -EI_{yy}\dfrac{d^3w}{dx^3}=\underline{F}_z\,,\;\; GJ\dfrac{d\phi}{dx}=\underline{M}_x$ at $x=L$

Clamped end: $w=0\,,\;\; \theta=-\dfrac{dw}{dx}=0,\;\phi=0$ at $x=0$

MOMENTS OF AREA

Zero moment: $A = \int dA$

First moments: $S_z = \int ydA$ and $S_y = \int zdA$

Second moments: $I_{zz} = \int y^2 dA,\ I_{yy} = \int z^2 dA,$ and $I_{zy} = I_{yz} = \int yzdA$

Polar moment: $J_B = I_{rr} = \int y^2 + z^2 dA = I_{zz} + I_{yy}$

The polar moment according to the standard model is usually (way) too large for profiles that do not actually remain planar in deformation.

FIRST TWO EIGENFREQUENCIES AND MODES

method	f_1 [Hz]	A_1 [–]	A_2 [–]	f_2 [Hz]	A_1 [–]	A_2 [–]
EXP		-	-		-	-
PSM						
FDM						
FEM						

2 BAR AND STRING MODELS

MATHEMATICAL PREREQUISITES

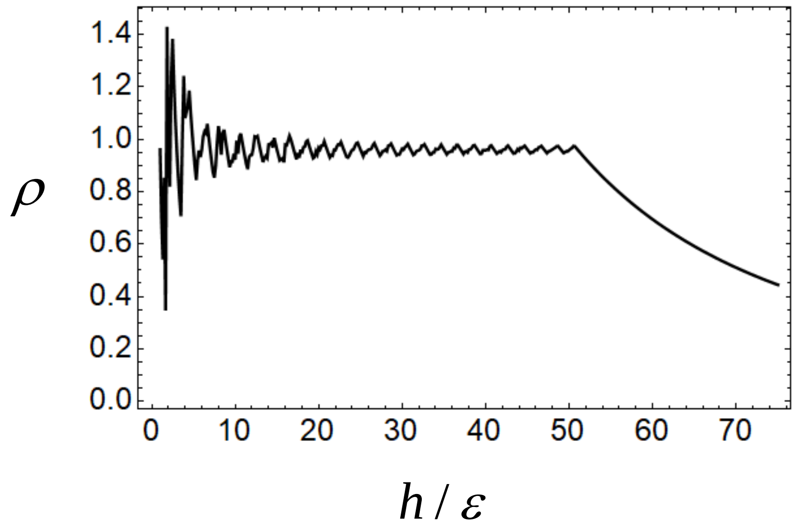
In analytical solution method, solution trial is used to transform a partial differential differentia equation into an ordinary differential equation, another solution trial is used to transform the ordinary differential equation into an algebraic equation etc.

Equation	Solution trial	Outcome
$k'\dfrac{\partial^2 a}{\partial x^2} - m'\dfrac{\partial^2 a}{\partial t^2} = 0$	$a(x,t) = A(x)e^{\mathrm{i}\omega t}$	$k'\dfrac{d^2 A}{dx^2} + m'\omega^2 A = 0$
$k'\dfrac{d^2 A}{dx^2} + m'\omega^2 A = 0$	$A(x) = ae^{\mathrm{i}\lambda x}$	$-\lambda^2 k' + m'\omega^2 = 0$

$$a(x,t) = \sum (\alpha \sin\omega_j t + \beta\cos\omega_j t)(\delta\sin\lambda_j x + \gamma\cos\lambda_j x) \quad \text{where} \quad \lambda_j = \omega_j\sqrt{\frac{m'}{k'}}$$

CONTINUUM MODEL

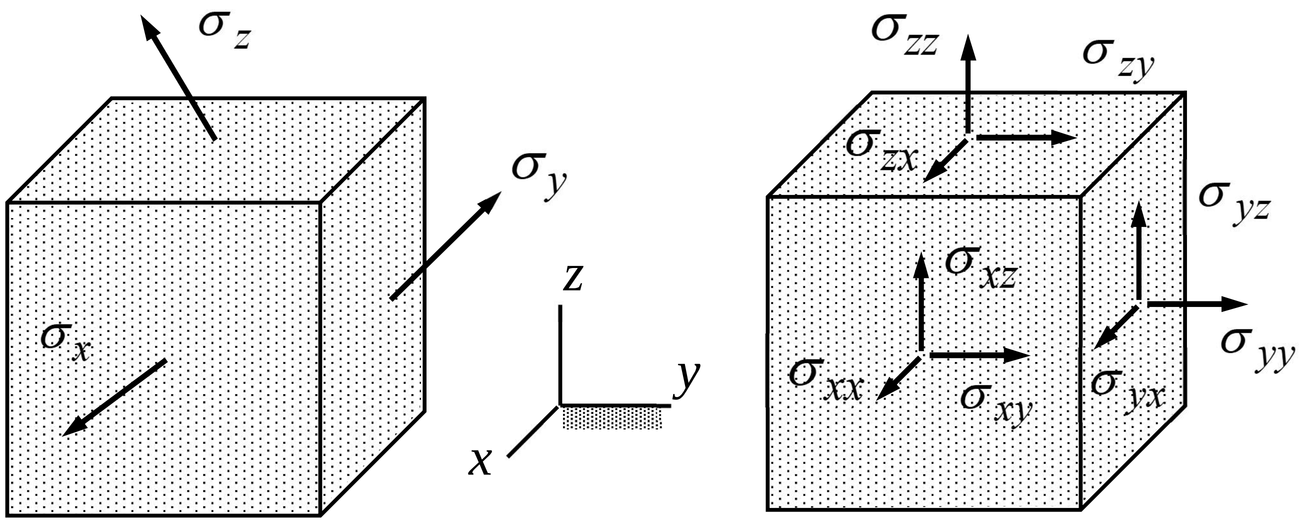
Continuum model is concerned with the average behavior of particles in a material element. The element is assumed to be small compared with the scale L of the body and large compared with the scale ε of the microstructure of the material.



Continuum model compromises modelling error with simplicity by assuming a scale $\varepsilon \ll h \ll L$ on which, e.g., material properties can be described by densities like mass per unit volume $\rho = \Delta m / \Delta V$ (mass density).

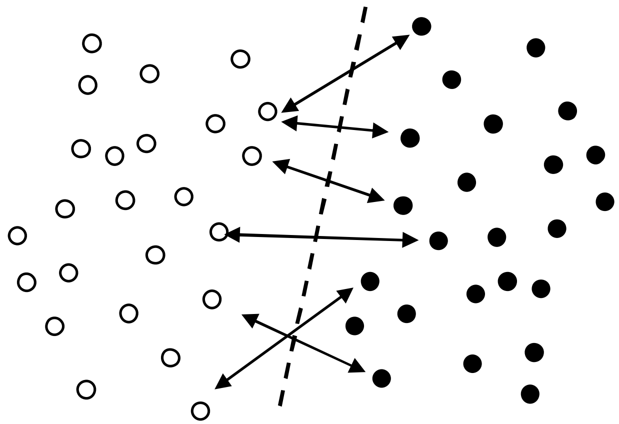
TRACTION AND STRESS

Traction $\vec{\sigma} = \Delta\vec{F} / \Delta A$ (a vector) describes the surface force between material elements of a body. Stress $\vec{\vec{\sigma}}$ describes the surface forces acting on all edges of a material element. Traction and stress are related by $\vec{\sigma} = \vec{n} \cdot \vec{\vec{\sigma}}$ in which $\vec{n}$ is the outward unit normal vector to face of the material element.



The first index of a stress component refers to the direction of the surface normal and the second that of the force component. On opposite edges components directions are opposite.

Traction and stress describe the internal forces. i.e., the interaction of the neighboring material elements. The white particles, belonging to certain material element, impose forces of resultant $\Delta\vec{F}$ to black particles of the neighboring element through area element ΔA .



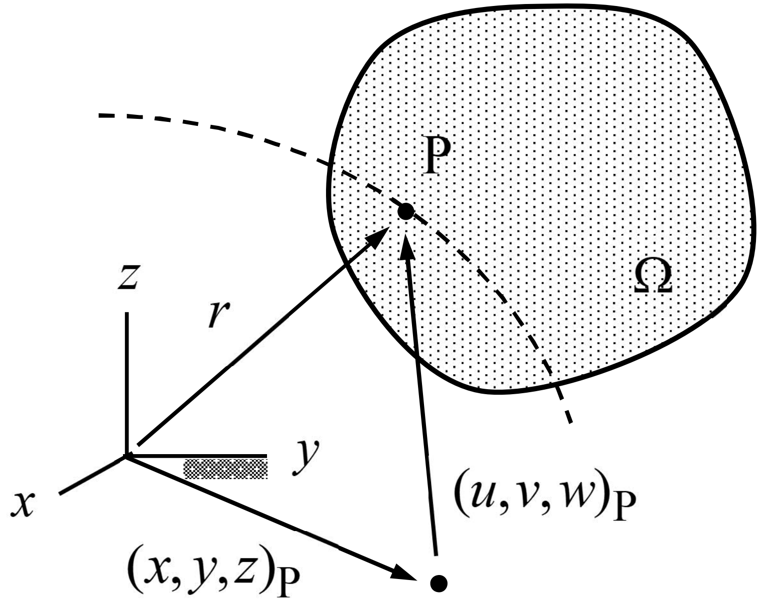
The ratio of $\Delta\vec{F}$ to area ΔA is assumed to be constant $\vec{\sigma}$, if the material element is small compared with the scale L of the body and large compared with the scale ε of the microstructure of the material so $\Delta\vec{F} = \vec{\sigma}\Delta A$. Theory assumes that the relationship holds also in form $d\vec{F} = \vec{\sigma} dA$ no matter the scale.

DISPLACEMENT

In continuum mechanics with solids, the motion of particle (x, y, z) is described by displacement components $u(x, y, z, t)$, $v(x, y, z, t)$, $w(x, y, z, t)$ in the directions of the coordinate axes relative to the initial position (x, y, z) at $t = 0$.

Non-stationary $$\begin{Bmatrix} r_x \\ r_y \\ r_z \end{Bmatrix} = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} + \begin{Bmatrix} u(x, y, z, t) \\ v(x, y, z, t) \\ w(x, y, z, t) \end{Bmatrix}$$

Stationary $$\begin{Bmatrix} r_x \\ r_y \\ r_z \end{Bmatrix} = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} + \begin{Bmatrix} u(x, y, z) \\ v(x, y, z) \\ w(x, y, z) \end{Bmatrix}$$



In stationary case, one considers only the initial and the final positions of the particles. In non-stationary case, the final position depends on parameter t (time).

FIRST PRINCIPLES

I Balance of mass Mass of a body is constant.

II Balance of linear momentum The rate of change of linear momentum within a material volume equals the external force resultant acting on the material volume. ←

III Balance of angular momentum The rate of change of angular momentum within a material volume equals the external moment resultant acting on the material volume. ←

IV Balance of energy (Thermodynamics 1)

V Entropy growth (Thermodynamics 2)

The first principles apply in these simple forms to a closed set of particles, i.e., a set which consist of the same particles all the time.

2.1 ENGINEERINGS MODELS

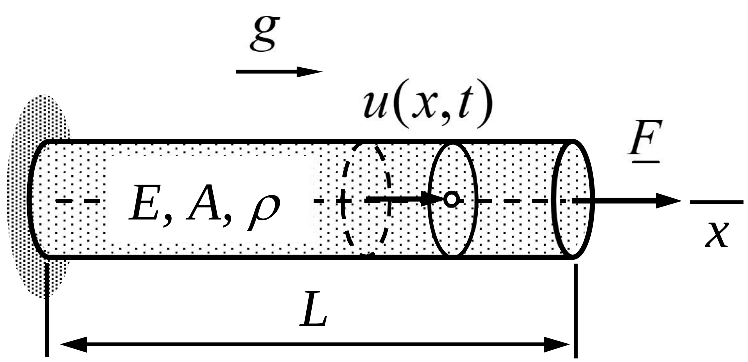
BAR is a body which is thin in two dimensions and has straight initial geometry. Displacement has only the axial component. Internal force is aligned with the axis.

STRING is a body which is thin in two dimensions and has straight initial geometry. Displacement has only the transverse component. Internal force is aligned with the axis (initial and deformed geometries).

THIN SLAB is a body which is thin in one dimension and has planar initial geometry. Displacement has only the mid-plane components. Internal force does not have transverse component.

MEMBRANE is a body which is very thin in one dimension and has planar initial geometry. Displacement has only the transverse component. Internal force does not have transverse component (initial and deformed geometries).

BAR MODEL

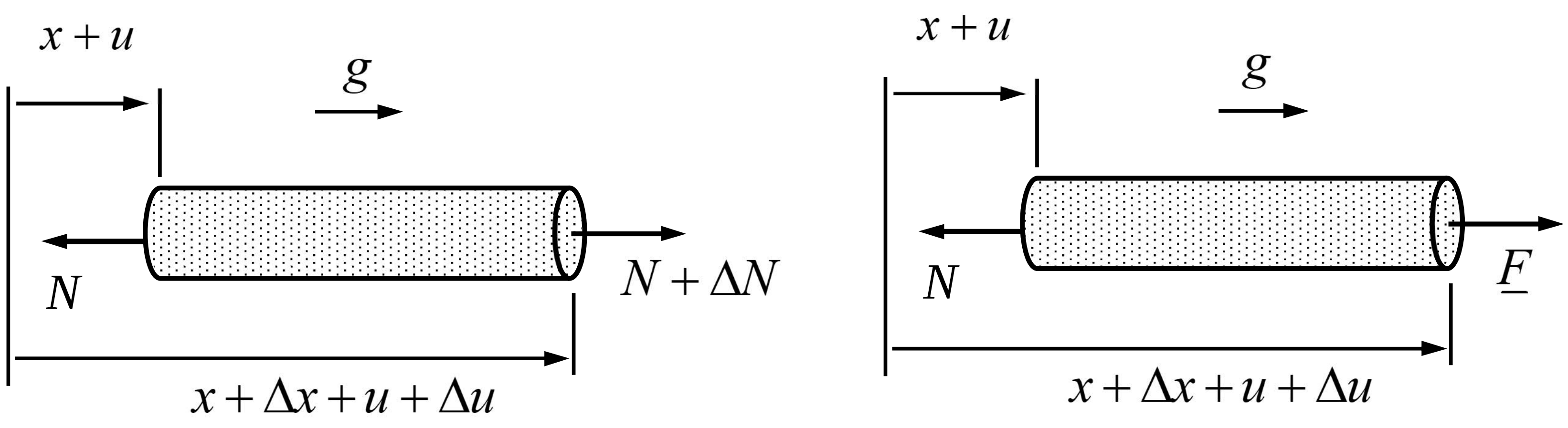


Equation of motion $$EA\frac{\partial^2 u}{\partial x^2} + \rho A g = \rho A \frac{\partial^2 u}{\partial t^2} \qquad x \in]0, L[\quad t > 0,$$

Boundary conditions $$u = \underline{u} \quad \text{or} \quad EA\frac{\partial u}{\partial x} = \underline{F} \qquad x \in \{0, L\} \quad t > 0,$$

Initial conditions $$u = g(x) \quad \text{and} \quad \frac{\partial u}{\partial t} = h(x) \qquad x \in [0, L] \quad t = 0.$$

Let us apply the first principles to a material element of initial length Δx at the initial and final geometries.



The cases where the material element is inside the bar and located at the free boundary differ.

Mass balance: $\Delta m = (\rho A)^{\circ}\Delta x = (\rho A)(\Delta x + \Delta u)$

Momentum balance $\rightarrow$: $N + \Delta N - N + g\Delta m = \Delta m \dfrac{\partial^2 u}{\partial t^2}$

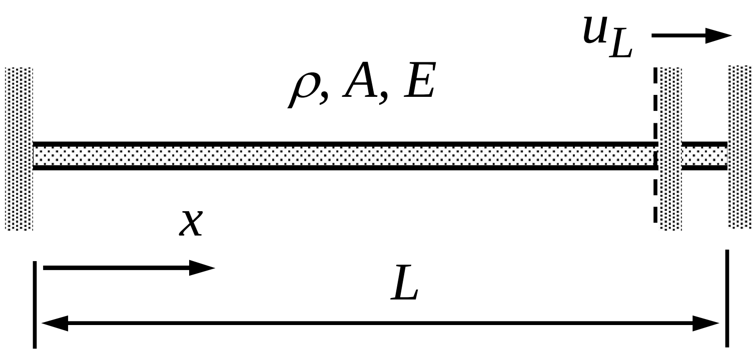
Momentum balance $\rightarrow$: $F - N + g\Delta m = \Delta m \dfrac{\partial^2 u}{\partial t^2}$

The limit model assumes that $\Delta N / \Delta x$ exists also when $\Delta x \rightarrow 0$. In case of a discontinuity, like a point force P at x_0, one obtains the "jump" condition $[\![N]\!] + P = 0$ where $[\![a]\!] = \lim_{\varepsilon \rightarrow 0}[a(x_0 + \varepsilon) - a(x_0 - \varepsilon)]$.

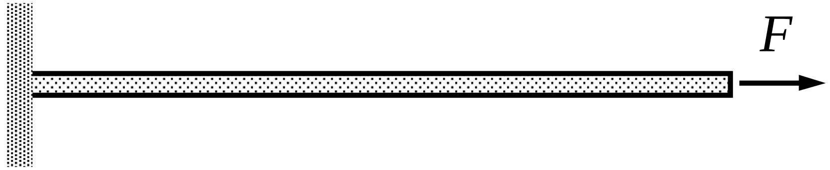
Hooke's law: $\sigma = E \dfrac{\Delta u}{\Delta x} \;\Rightarrow\; N = EA \dfrac{\Delta u}{\Delta x}$

relates the length change and force acting on a material element. Material model is the weakest element of the model as it lacks the generality of the first principles and brings the major portion of the modelling error!

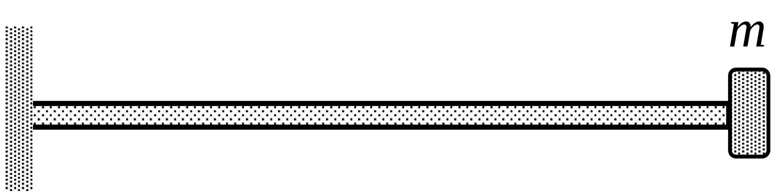
BOUNDARY CONDITIONS



$$u(0,t) = u(L,t) - u_L(t) = 0$$



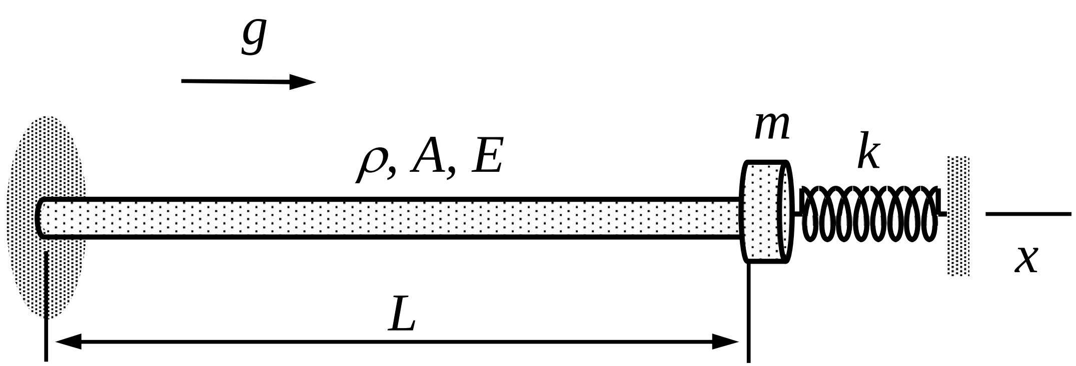
$$u(0,t) = EA\frac{\partial}{\partial x}u(L,t) - F(t) = 0$$



$$u(0,t) = EA\frac{\partial}{\partial x}u(L,t) + m\frac{\partial^2}{\partial t^2}u(L,t) = 0$$

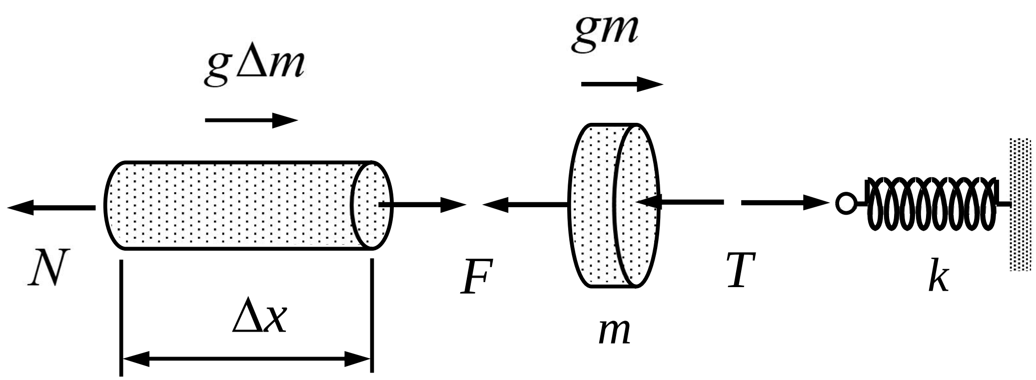
$$u(0,t) = EA\frac{\partial}{\partial x}u(L,t) + ku(L,t) = 0$$

EXAMPLE Derive the boundary condition for the case shown starting from the first principles. The particle at the right end is connected to rigid wall with a spring. Spring force vanishes when displacement at the end $u(L,t)=0$.



Answer $EA\frac{\partial u}{\partial x}+ku-mg+m\frac{\partial^2 u}{\partial t^2}=0 \quad x=L \quad t>0$

Let us apply the balance of momentum to a material element using the free body diagrams of the bar element and particle

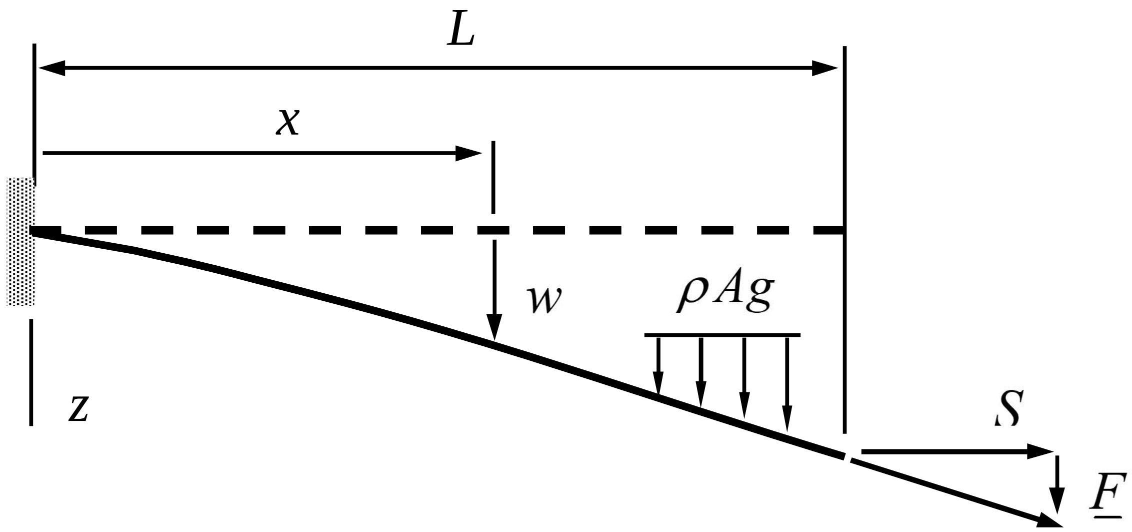


$$-N + \rho A g \Delta x + F = \rho A \Delta x \frac{\partial^2 u}{\partial t^2}\ , \qquad -F - T + mg = m\frac{\partial^2 u}{\partial t^2}$$

Eliminating the internal force F, using $N = EA\partial u/\partial x$ and $T = ku$, and considering ja the limit $\Delta x \to 0$

$$-EA\frac{\partial u}{\partial x} - ku + mg - m\frac{\partial^2 u}{\partial t^2} = 0 \quad x = L \quad t > 0.$$

STRING MODEL

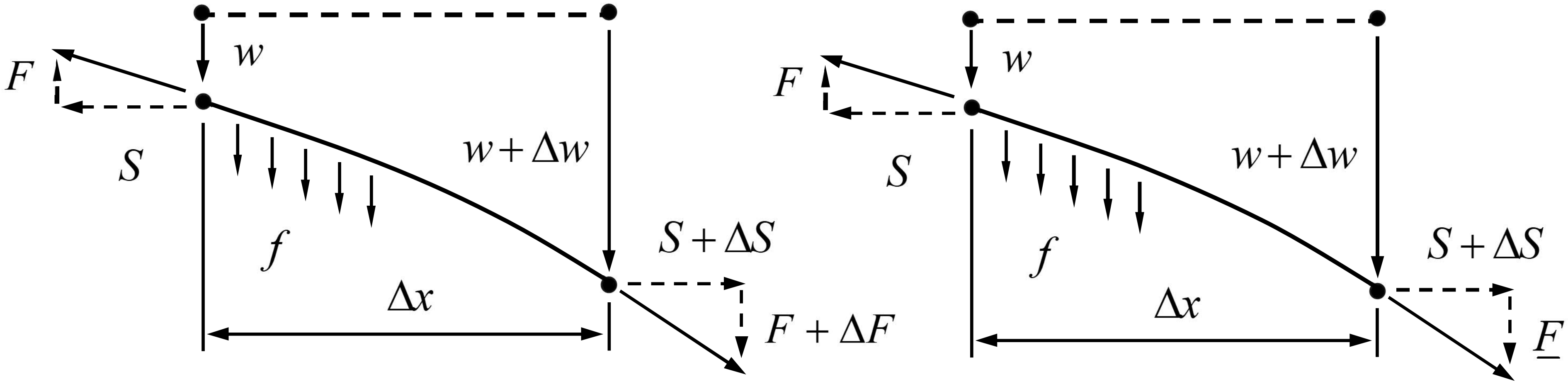


Equation of motion $S\dfrac{\partial^2 w}{\partial x^2} + \rho A g = \rho A \dfrac{\partial^2 w}{\partial t^2} \quad x \in]0, L[\quad t > 0$

Boundary conditions $w = \underline{w}$ or $S\dfrac{\partial w}{\partial x} - \underline{F} = 0 \quad x \in \{0, L\} \quad t > 0$

Initial conditions $w = g(x)$ and $\dfrac{\partial w}{\partial t} = h(x) \quad x \in]0, L[\quad t = 0$

Let us apply principles **I** and **II** to an originally straight material element $[x, x+\Delta x]$ of initial length Δx by assuming a planar problem and only the transverse displacement $w(x,t)$.



Mass balance: $\Delta m = \Delta x(\rho A)^{\circ} = (\rho A)\Delta x\sqrt{1+(\Delta w/\Delta x)^2}$

Momentum balance $\rightarrow$: $S+\Delta S - S = 0$

Momentum balance $\downarrow$: $F + \Delta F - F + f^\circ \Delta x = (\rho A)^\circ \Delta x \dfrac{\partial^2 w}{\partial t^2}$

Momentum balance $\downarrow$: $-F - \underline{F} + f^\circ \Delta x = (\rho A)^\circ \Delta x \dfrac{\partial^2 w}{\partial t^2}$

The constitutive equations is also required. The momentum balance in the horizontal direction implies that tightening S of string is constant. Therefore, according to the figure

$$\frac{F}{S} = \frac{\partial w}{\partial x} \;\Leftrightarrow\; F = S\frac{\partial w}{\partial x} \quad \text{and} \quad \frac{F + \Delta F}{S} = \frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2}\Delta x \;\Leftrightarrow\; F + \Delta F = S\left(\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2}\Delta x\right)$$

The second order limit problem $\Delta x \to 0$ assumes continuous derivatives at all the points inside the domain. In case of a vertical point force P acting on the string, the limit equation for the point of action becomes $[\![F]\!] + P = 0$.

BOUNDARY-INITIAL VALUE PROBLEM

In their mathematical forms, the continuum models for string and bar coincide. Denoting the displacement by $a(x,t)$, the boundary-initial value problems are given by equations

Differential equation: $k'\dfrac{\partial^2 a}{\partial x^2}+f'=m'\dfrac{\partial^2 a}{\partial t^2} \quad x\in\Omega \quad t>0$

Boundary conditions: $a=\underline{a}$ or $n(k'\dfrac{\partial a}{\partial x})-\underline{F}=0 \quad x\in\partial\Omega \quad t>0$

Initial conditions: $a=g(x)$ and $\dfrac{\partial a}{\partial t}=h(x) \quad x\in\Omega \quad t=0$

In a boundary condition, one may specify displacement or force but not both simultaneously. For a unique solution in stationary case, the displacement needs to be specified at least one point. In the force boundary condition, $n=\pm 1$ depending on the boundary point.

2.2 DISPLACEMENT ANALYSIS

In the displacement problem, one assumed that all quantities are independent in time so derivatives with respect to time vanish and partial derivatives with respct to the spatiol coordinate become ordinary one. Assuming that point forces, if any, act on I

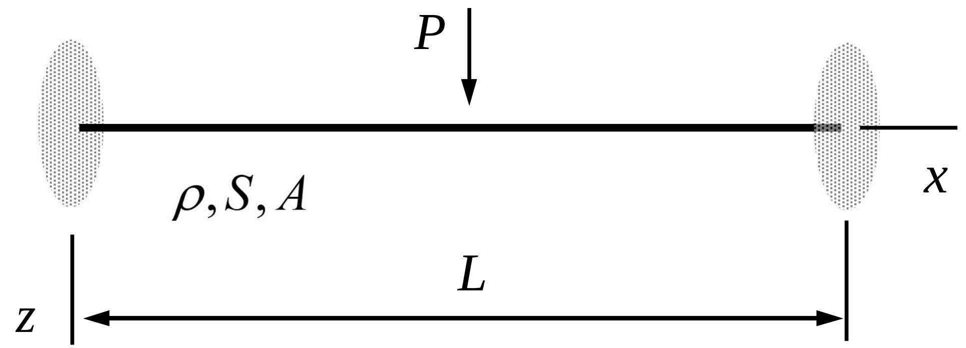
Differential equation $k'\dfrac{d^2a}{dx^2}+f'=0 \quad x\in\Omega\setminus I$

Continuity conditions $k'\left[\!\left[\dfrac{da}{dx}\right]\!\right]+P=0$ and $[\![a]\!]=0 \quad x\in I$

Boundary conditions $a=\underline{a}$ or $n(k'\dfrac{da}{dx})-\underline{F}=0 \quad x\in\partial\Omega$

At the boundary, one may specify the force or displacement but not both. Also, displacement should be specified at one point for an unique solution.

EXAMPLE A string of length L, tightening S, cross-sectional area A, and density ρ, is loaded by a point force P at its center point. If the ands are fixed and the initial geometry without loading is straight, find the solution to the transverse displacement as function of x using the continuum model.



Answer $$w(x)=\begin{cases}\dfrac{P}{2S}x & 0\le x<\dfrac{L}{2}\\[2mm] \dfrac{P}{2S}(L-x) & \dfrac{L}{2}<x\le L\end{cases}$$

The boundary value problem is given by equilibrium equations for the interior points, continuity conditions at the center point, and boundary conditions for the end points

$$S\frac{d^2w}{dx^2}=0 \quad x\in]0,\frac{L}{2}[\quad \text{or} \quad x\in]\frac{L}{2},L[,$$

$$S\left[\!\!\left[\frac{dw}{dx}\right]\!\!\right]+P=0\ ,\ \ [\![w]\!]=0 \quad x=\frac{L}{2}, \qquad \text{and} \qquad w(x)=0 \quad x\in\{0,L\}.$$

The generic solution to the differential equation is the same form $w=a+bx$ in both parts of the domain with different integration constants. Using the remaining conditions

$$w(x)=\begin{cases} a_1+b_1x & 0<x<\dfrac{L}{2} \\[2ex] a_2+b_2x & \dfrac{L}{2}<x<L \end{cases} \quad \Rightarrow \quad w(x)=\begin{cases} \dfrac{P}{2S}x & 0\le x<\dfrac{L}{2} \\[2ex] \dfrac{P}{2S}(L-x) & \dfrac{L}{2}<x\le L \end{cases}.$$

DISPLACEMENT SOLUTION

Displacement calculation with the bar or string model means solving a boundary value problem of ordinary linear second order differential equation. In the present context, coefficient are constants so finding the displacement is straightforward.

(a) Find the generic solution to the differential equation by repetitive integrations on both sides. Each integration creates a new integration constant. In case of point forces etc. apply the integration separately in subdomains.

(b) Use the boundary conditions conerning given displacement and forces on the boundaries to find the values of two integration constants. In case of point forces etc. glue the solutions to the subdomains together with continuity of displacement and the condition implied by momentum equilibrium written for, e.g., the point of action of a point force.

2.3 VIBRATION ANALYSIS

The equations describing vibration consist of equation of motion, boundary conditions for displacement or force, and initial conditions for the displacement and velocity

Equation of motion $k'\dfrac{\partial^2 a}{\partial x^2} + f' = m'\dfrac{\partial^2 a}{\partial t^2} \quad x \in \Omega \quad t > 0$

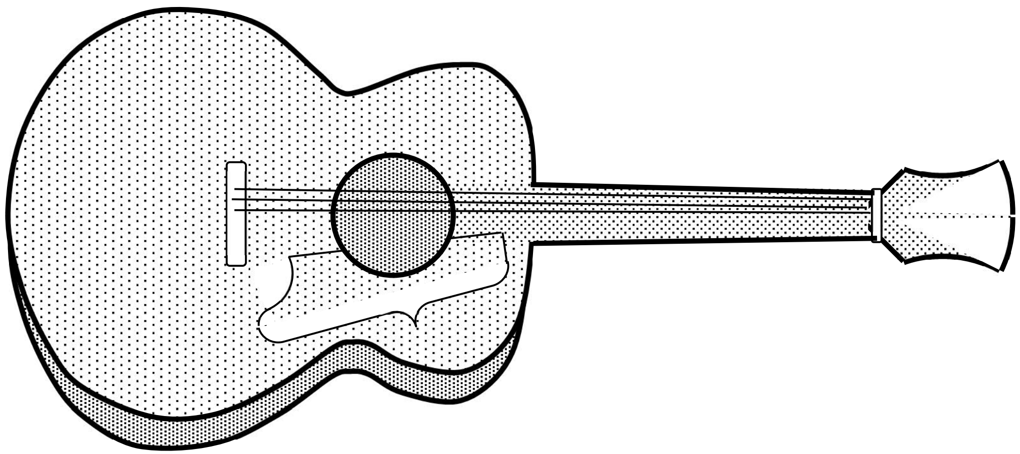
Boundary conditions $a = \underline{a}$ or $n(k'\dfrac{\partial a}{\partial x}) - \underline{F} = 0 \;\; x \in \partial\Omega \quad t > 0$

Initial conditions $a = g(x)$ and $\dfrac{\partial a}{\partial t} = h(x) \quad x \in \Omega \quad t = 0$

Initial and boundary conditions for displacement should match for a regular solution of a vibration problem (but not always, like when a bar moving with a constant velocity hits a rigid wall). In vibration problems, the boundary conditions are homogeneous so $\underline{a} = \underline{F} = 0$.

PYTHAGORAS' DISCOVERY

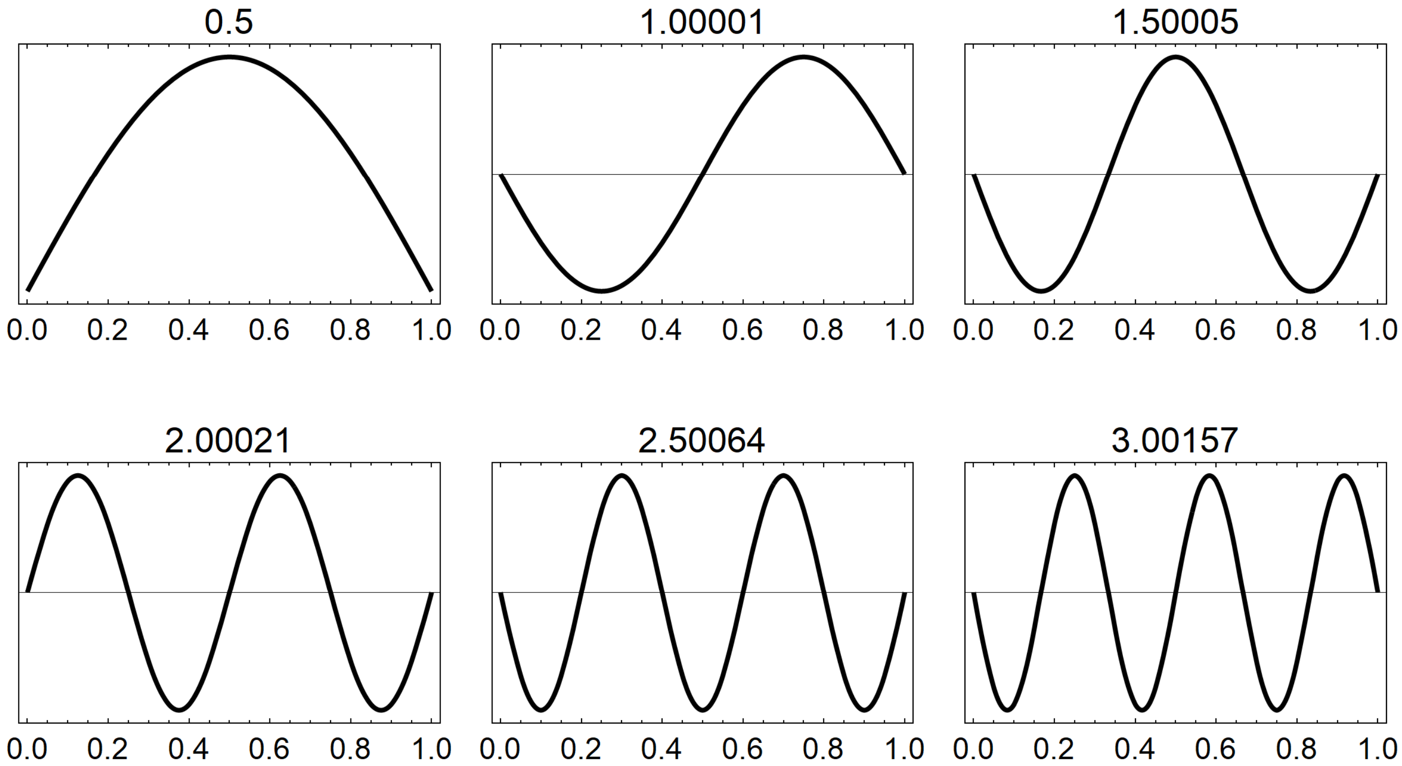
According to the experiments by Pythagoras, pitch of a note f depends on the length of string L, stress σ and density ρ of material!



String model $f_{\min} = \dfrac{1}{2L}\sqrt{\dfrac{S}{\rho A}} = \dfrac{1}{2L}\sqrt{\dfrac{\sigma}{\rho}}$

VIBRATION MODES OF STRING

Displacement of string in vibration can be represented as sum of harmonic modes each vibrating in its own frequency f [1/s]. For a string of fixed ends, the frequency-mode pairs are (dimensionless value $Lf / \sqrt{S / \rho A}$)



MODAL ANALYSIS

In vibration analysis of structures, one is usually interested in the frequencies f ($\omega = 2\pi f$) or angular velocity-mode pairs (ω, A) for the free vibrations. Using the generic form valid for the bar and string models, the modes and angular velocities are related by

(a) $A(x) = \delta \sin(\lambda x) + \gamma \cos(\lambda x)$ and $\lambda = \omega \sqrt{\dfrac{m'}{k'}}$

The (assumedly homogeneous) boundary conditions at the ends

(b) $A(x) = 0$ or $n(k' \dfrac{dA}{dx}) = 0 \quad x \in \partial\Omega$

give 2 conditions for the 3 parameters δ, γ, λ and, thereby, the possible pairs $(\omega, A)_j$. Notice that the modes are determined up to an arbitrary scaling only (3 parameters and 2 equations).

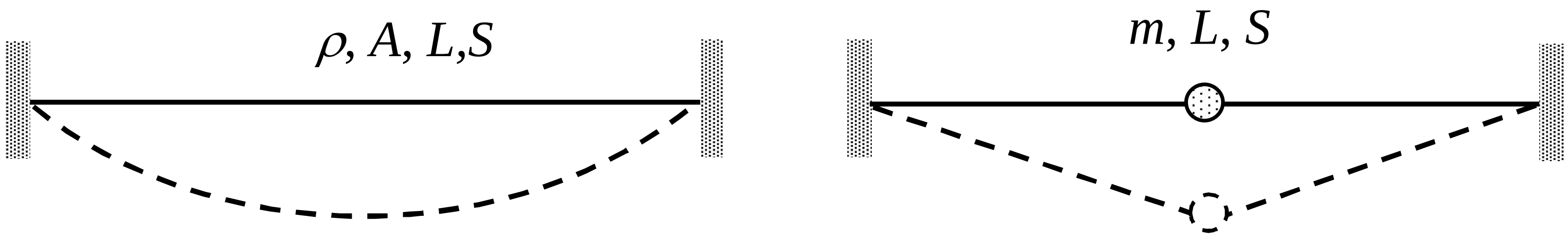
In mode detail, the possible angular velocity-mode pairs (ω, A) follow from displacement assumption $a(x,t) = A(x)e^{\mathrm{i}\omega t}$ which transforms the partial differential equation into an ordinary one. Using the generic form for both model problems, the steps to the generic form of the modes

$$k' \frac{\partial^2 a}{\partial x^2} = m' \frac{\partial^2 a}{\partial t^2} \quad \Rightarrow \quad \frac{d^2 A}{dx^2} + \omega^2 \frac{k'}{m'} A = 0.$$

The generic solution to the angular velocity-mode pairs (ω, A) is therefore given by

$$A(x) = \delta \sin(\lambda x) + \gamma \cos(\lambda x) \text{ where } \lambda = \omega \sqrt{\frac{k'}{m'}}. \quad \Leftarrow$$

EXAMPLE What is relative error if the smallest frequency of the free vibrations $f_{\min}$ is calculated with a particle surrogate model of the figure ($\bar{f}_{\min}$), where the mass of the particle is half of that of the string?



Answer $e = \dfrac{\bar{f}_{\min} - f_{\min}}{f_{\min}} \times 100\% = (\dfrac{2\sqrt{2}}{\pi} - 1) \times 100\% \approx -10\%$

Let us start with the continuum model considered as the precise model. In string equations, $k' = S$, $m' = \rho A$, and $a = w(x,t)$. External force and displacement at the boundaries vanish and $\Omega =]0, L[$. The possible angular velocity-mode pairs follow from the generic solution

$$A(x) = \delta \sin(\lambda x) + \gamma \cos(\lambda x) \quad \text{and} \quad \lambda = \omega \sqrt{\rho A / S}.$$

Boundary conditions $A(0) = A(L) = 0$ give 2 conditions for the 3 parameters γ, δ, λ

$$A(0) = \gamma = 0 \quad \text{and} \quad A(L) = \delta \sin(\lambda L).$$

A non-zero solution is possible only if $\sin(\lambda L) = 0$ so $\lambda L = \pi j \quad j \in \{1, 2, \ldots\}$. The smallest angular velocity, and therefore the frequency, follows with selection $j = 1$ so

$$f_j = \frac{\omega_j}{2\pi} = \frac{1}{L}\sqrt{\frac{S}{\rho A}}\frac{1}{2} j \quad \Rightarrow \quad f_{\min} = \frac{1}{L}\sqrt{\frac{S}{\rho A}}\frac{1}{2}. \quad \leftarrow$$

Particle surrogate model for string with $i = \{0,1,2\}$ and fixed particles at the end points gives the equation of motion:

$$2\frac{S}{h}w_1 + \rho A h \frac{d^2 w_1}{dt^2} = 0 \quad \Leftrightarrow \quad \frac{d^2 w_1}{dt^2} + \omega_e^2 w_1 = 0 \quad \text{where} \quad \omega = 2\pi f = \sqrt{2\frac{S}{\rho A h^2}}\,.$$

As $h = L/2$, the prediction by PSM $\bar{f}_{\min} = \frac{\sqrt{2}}{\pi L}\sqrt{\frac{S}{\rho A}}$. ←

MODE SUPERPOSITION

If the initial conditions concerning position and displacement of the particles are known (quite exceptional case in practice), the outcome of the modal analysis $(\omega, A)_j$ can be used to construct a displacement solution for the given initial data starting with the series

(a) $$a(x,t) = \sum A_j(x)[\alpha_j \frac{1}{\omega_j}\sin(\omega_j t) + \beta_j \cos(\omega_j t)].$$

The combination of the modes giving $a = g(x)$ and $\partial a / \partial t = h(x)$ at $t = 0$ follow with the coefficient expression

(b) $$\alpha_j = \frac{1}{\mathrm{A}_j^2}\int_\Omega A_j(x) h dx \text{ and } \beta_j = \frac{1}{\mathrm{A}_j^2}\int_\Omega A_j(x) g dx \text{ where } \mathrm{A}_j^2 = \int_\Omega A_j(x) A_j(x) dx.$$

The coefficients correspond to the spatial Fourier series of the initial data obtained with the orthogonal harmonic modes from the modal analysis.

The series representation follows directly from the displacement assumption written with the polar representation $e^{\mathrm{i}\alpha} = \cos\alpha + \mathrm{i}\cos\alpha$ and real valued coefficients

$$a(x,t) = \sum A_j(x) e^{\mathrm{i}\omega_j t} = \sum A_j(x)[\alpha_j \frac{1}{\omega_j}\sin(\omega_j t) + \beta_j \cos(\omega_j t)].$$

Initial conditions imply that $\sum A_j(x)\beta_j = g(x)$ and $\sum A_j(x)\alpha_j = g(x)$. Multiplying both sides with $A_l(x)$, integrating over the domain, and using the orthogonality gives

$$\alpha_j = \frac{1}{\mathrm{A}_j^2}\int_\Omega A_j(x)h(x)dx, \quad \beta_j = \frac{1}{\mathrm{A}_j^2}\int_\Omega A_j(x)g(x)dx, \quad \mathrm{A}_j^2 = \int_\Omega A_j(x)A_j(x)dx. \leftarrow$$

FOURIER SERIES

The Fourier series (various forms exist) can be used to represent a function as the sum of harmonic terms. For example, the sine-transformation pair for a function $a(x)$ $x \in [0,L]$ with vanishing values at the end points is given by

$$\alpha_j = \frac{2}{L}\int_0^L \sin(j\pi\frac{x}{L})a(x)dx \quad j \in \{1,2,\ldots\} \quad \Leftrightarrow \quad a(x) = \sum_{j\in\{1,2,\ldots\}} \alpha_j \sin(j\pi\frac{x}{L}).$$

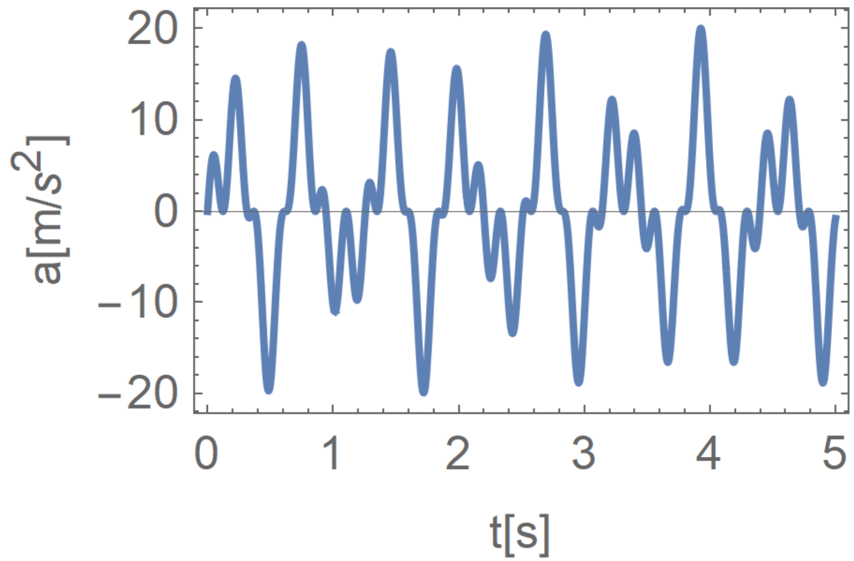
The transformation pair is based on the orthogonality of the modes

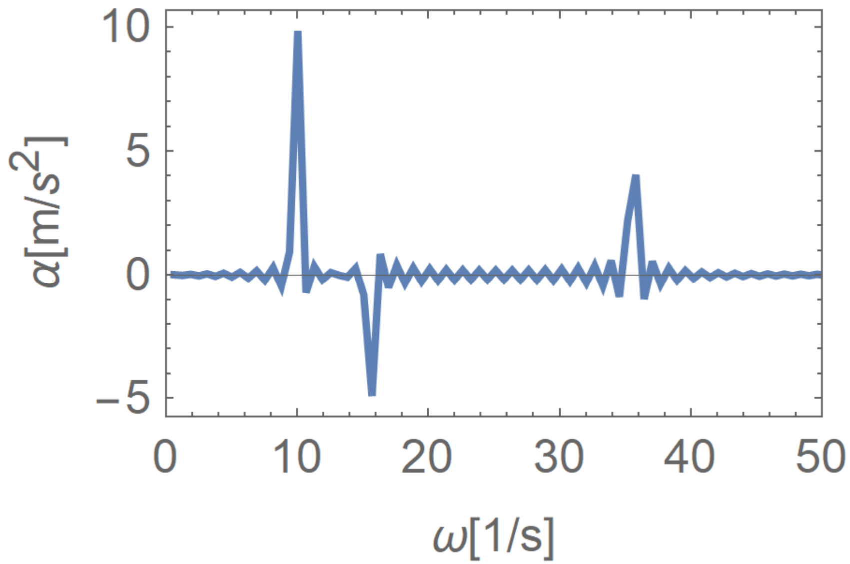
$$\int_0^L \sin(j\pi\frac{x}{L})\sin(l\pi\frac{x}{L})dx = \frac{L}{2}\delta_{jl} \text{ (Kronecker delta).}$$

The transformation (with respect to time) can be used to analyze frequency contents of data, filtering, to find the combination of the terms of the generic series solution for bar and string models satisfying the initial conditions, etc.

FREQUENCY CONTENTS OF DATA

Fourier transform (with respect to time) can be used, e.g., to analyze frequency contents of data, filtering of data etc. As an example, transform (right) of the measured acceleration (left), imply $a(t) = 10\sin(10t) - 5\sin(15.6t) + 5\sin(35.6t)$ (in appropriate units)





In filtering, one may just omit, e.g., components having frequencies over some value or maybe components of amplitudes of small values depending on the application.

EXAMPLE Find the transverse displacement of the string shown as function of time. The initial displacement is sinusoidal $g = W\sin(\pi x / L)$ (W is constant) and the initial speed vanishes, i.e., $h = 0$. Assume that the string is tightened with S in the horizontal direction and mass per unit length ρA is constant. Use the outcome of the modal analysis for a string fixed at both ends $A_j(x) = \sin(\lambda_j x)$ and $\omega_j = \lambda_j \sqrt{S/(\rho A)}$, where $\lambda_j = \pi j / L \quad j \in \{1, 2, \ldots\}$

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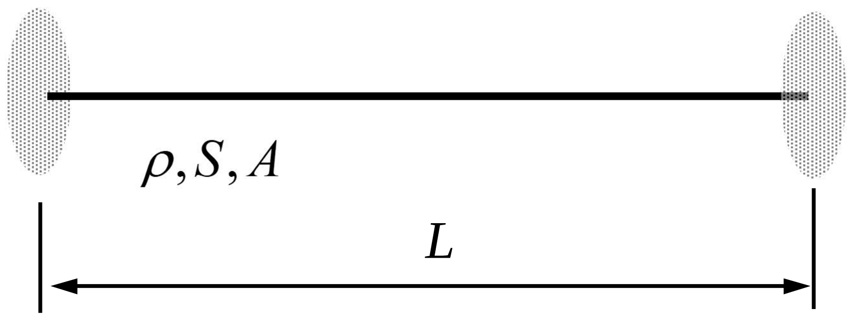


Answer $w(x,t) = W\cos(\frac{\pi}{L}\sqrt{\frac{S}{\rho A}}t)\sin(\frac{\pi}{L}x)$

After modal analysis, one may superpose the modes to find a particular combination satisfying the initial conditions. According to the recipe

$$w(x,t) = A_j(x)[\alpha_j \frac{1}{\omega_j}\sin(\omega_j t) + \beta_j \cos(\omega_j t)]$$

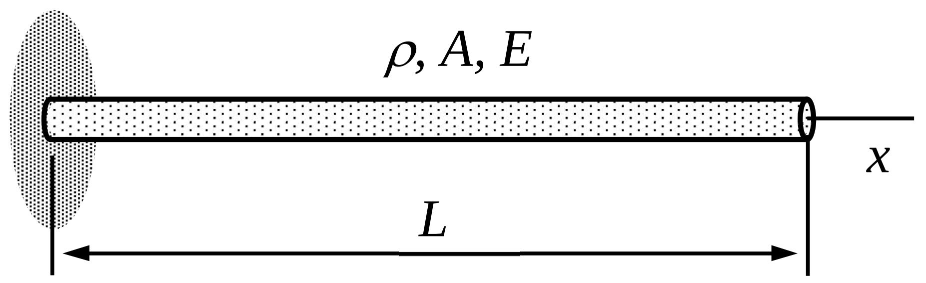
where

$$\alpha_j = \frac{1}{\mathrm{A}_j^2}\int_\Omega A_j(x) h dx \quad \text{and} \quad \beta_j = \frac{1}{\mathrm{A}_j^2}\int_\Omega A_j(x) g dx \ \text{where} \ \mathrm{A}_j^2 = \int_\Omega A_j(x) A_j(x) dx$$

The initial data $g(x) = W\sin(\pi x / L)$, $h(x) = 0$ and modes $A_j(x) = \sin(j\pi x / L)$ give $\mathrm{A}_j^2 = L/2$, $\alpha_j = 0$, and $\beta_j = W\delta_{1j}$ $j \in \{1, 2, \ldots\}$ and, thereby, the solution

$$w(x,t) = W\sin(\pi \frac{x}{L})\cos(\frac{\pi}{L}\sqrt{\frac{S}{\rho A}}t).$$

EXAMPLE Find the axial displacement of the bar shown as the functions of time t and position x, if the initial displacement and velocity at $t = 0$ are given by $g(x) = U\sin(\pi x/2L)$ and $h(x) = 0$, respectively.



Answer $u(x,t) = U\cos(\sqrt{\frac{E}{\rho}}\frac{\pi}{2L}t)\sin\frac{\pi x}{2L}$

Let us start with the modal analysis. According to the recipe, the modes are given by

$$A(x) = \delta \sin(\lambda x) + \gamma \cos(\lambda x),$$

where the proper combination of parameters δ, γ, λ follow from the boundary conditions. In the present case

$$A(0) = \gamma = 0 \quad \text{and} \quad (\frac{dA}{dx})_{x=L} = \delta\lambda \cos(\lambda L) - \gamma\lambda \sin(\lambda L) = 0.$$

The angular velocity-mode pairs (bar model $m' = \rho A$ and $k' = EA$) implied by the boundary conditions ($\cos(\lambda L) = 0$) are

$$A_j(x) = \sin(\lambda_j x) \quad \text{and} \quad \omega_j = \lambda_j \sqrt{\frac{E}{\rho}} \quad \text{where} \quad \lambda_j L = \frac{\pi}{2}(1 + 2j) \quad j \in \{0,1,2,\ldots\}.$$

Knowing the angular velocity-mode pairs, the particular combination satisfying the initial conditions follows from

$$u(x,t)=\sum_{j\in\{0,1,2,\ldots\}} A_j(x)[\alpha_j \frac{1}{\omega_j}\sin(\omega_j t)+\beta_j\cos(\omega_j t)]$$

where

$$\alpha_j=\frac{1}{\mathrm{A}_j^2}\int_\Omega A_j(x)h dx,\ \ \beta_j=\frac{1}{\mathrm{A}_j^2}\int_\Omega A_j(x)g dx \ \text{ and } \ \mathrm{A}_j^2=\int_\Omega A_j(x)A_j(x)dx.$$

The initial data $g(x)=U\sin(\pi x/2L)$, $h(x)=0$ give the coefficients $\alpha_j=0$ and $\beta_j=W\delta_{0j}$ $\quad j\in\{0,1,2,\ldots\}$ and, thereby, the solution

$$u(x,t)=U\cos(\sqrt{\frac{E}{\rho}}\frac{\pi}{2L}t)\sin\frac{\pi x}{2L}.$$ ←

2.4 PRINCIPLE OF VIRTUAL WORK

Principle of virtual work for particle and continuum models are just concise representations of equilibrium equations or equations-of-motion and boundary conditions of the particle and continuum models.

Virtual work	**Particle**	**Continuum**
δW^{int}	$-\sum_{e\in P} \left(\frac{\Delta\delta u_e}{\Delta x} EA \frac{\Delta u_e}{\Delta x}\right)h$	$-\int_\Omega \left(\frac{\partial\delta u}{\partial x} EA \frac{\partial u}{\partial x}\right)dx$
δW^{ext}	$\sum_{i\in I} (\delta u_i f)h$	$\int_\Omega (\delta u f)dx$
δW^{ine}	$-\sum_{i\in I} \left(\delta u_i \rho A \frac{\partial^2 u_i}{\partial t^2}\right)h$	$-\int_\Omega \left(\delta u \rho A \frac{\partial^2 u}{\partial t^2}\right)dx$

The expressions for the continuum can be considered as the limit cases of the particle model ones when $n \to \infty$ and $h = L / n$. The limit expression for the bar, string, thin slab membrane etc. models of solid mechanics play an important role in the Finite Element Method to be discussed later.

COE-C3005 Finite Element and Finite difference methods

1. Determine the displacements w_i $i \in \{1,2,3\}$, if the vector of displacements $\mathbf{a}$, stiffness matrix $\mathbf{K}$, and the loading vector $\mathbf{F}$ of the equilibrium equations $-\mathbf{Ka}+\mathbf{F}=0$ are given by

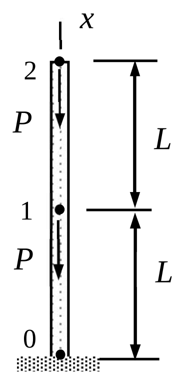
$$\mathbf{a}=\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix},\ \mathbf{K}=k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix},\ \text{and } \mathbf{F}=P\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}.$$

Answer $w_1=\dfrac{P}{k}$, $w_2=2\dfrac{P}{k}$, $w_3=3\dfrac{P}{k}$

2. Determine the angular velocities and modes $(\omega,\mathbf{A})$ of free vibrations, if the vector of unknowns $\mathbf{a}(t)$, stiffness matrix $\mathbf{K}$, and the mass matrix $\mathbf{M}$ of equations of motion $\mathbf{Ka}+\mathbf{M\ddot{a}}=0$ are given by

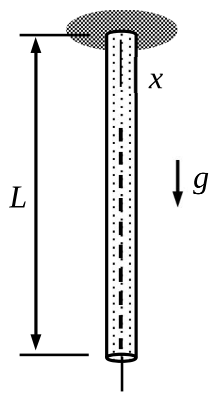
$$\mathbf{a}=\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix},\ \mathbf{K}=k\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix},\ \text{and } \mathbf{M}=m\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}.$$

Answer $(\omega,\mathbf{A})_1=(\sqrt{\dfrac{k}{m}},\begin{Bmatrix} 1 \\ -1 \end{Bmatrix})$, $(\omega,\mathbf{A})_2=(\sqrt{\dfrac{1}{5}\dfrac{k}{m}},\begin{Bmatrix} 1 \\ 1 \end{Bmatrix})$



3. The bar shown is loaded by point forces of equal magnitudes P but opposite directions acting on points 1 and 2. Use the Particle Surrogate Method on the regular grid shown to write the equilibrium equations of points 1 and 2. After that, solve the equations for the axial displacements u_1 and u_2. Cross-sectional area A and Young's modulus E of the material are constants.

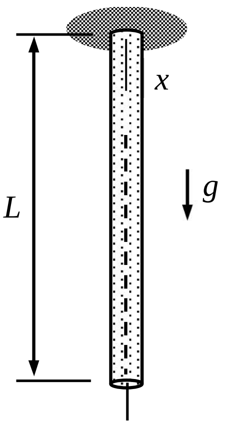
Answer $u_1=-2\dfrac{PL}{EA}$, $u_2=-3\dfrac{PL}{EA}$



4. Consider a bar of length L loaded by its own weight. Determine the displacements at the mid-point and at the free end using the continuum model. Cross-sectional area A, acceleration by gravity g, and material properties E and ρ are constants.

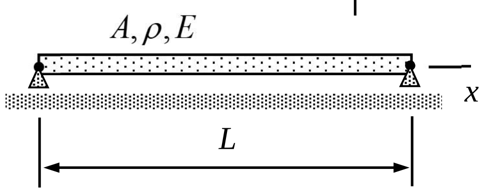
Answer $u(L/2)=\dfrac{3}{8}\dfrac{\rho g L^2}{E}$, $u(L)=\dfrac{1}{2}\dfrac{\rho g L^2}{E}$

5. Consider a bar of length L loaded by its own weight. Determine the displacements at the mid-point and at the free end using the Particle Surrogate Method of the bar with a regular grid $i \in \{0,1,2\}$. Cross-sectional area A, acceleration by gravity g, and material properties E and ρ are constants.



Answer $u_1 = \dfrac{3}{8}\dfrac{\rho g L^2}{E}$, $u_2 = \dfrac{1}{2}\dfrac{\rho g L^2}{E}$

6. A bar is free to move in the horizontal direction as shown. Determine the angular velocities and modes $(\omega, \mathbf{A})$ of free vibrations using the Particle Surrogate Method and a regular grid $i \in \{0,1\}$. Cross-sectional are A, density ρ of the material, and Young's modulus E of the material are constants.



Answer $(\omega, \mathbf{A})_1 = (0, \begin{Bmatrix} 1 \\ 1 \end{Bmatrix})$ $(\omega, \mathbf{A})_2 = (\dfrac{2}{L}\sqrt{\dfrac{E}{\rho}}, \begin{Bmatrix} 1 \\ -1 \end{Bmatrix})$

Determine the displacements w_i $i \in \{1,2,3\}$, if the vector of displacements $\mathbf{a}$, stiffness matrix $\mathbf{K}$, and the loading vector $\mathbf{F}$ of the equilibrium equations $-\mathbf{Ka}+\mathbf{F}=0$ are given by

$$\mathbf{a}=\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}, \quad \mathbf{K}=k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \text{ and } \mathbf{F}=P\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}.$$

Solution

With linear equation systems $-\mathbf{Ka}+\mathbf{F}=0$ of more than two unknows, using the matrix inverse to get $\mathbf{a}=\mathbf{K}^{-1}\mathbf{F}$ is not practical in hand calculation. Gauss elimination is based on row operations aiming at an upper diagonal matrix. After that, solution for the unknowns is obtained step-by-step starting from the last equation. In standard form, the equation system is given by

$$k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}=P\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}.$$

Let us multiply the 2:nd equation by 2 and add to it equation 1 to get

$$k\begin{bmatrix} 2 & -1 & 0 \\ -2 & 4 & -2 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}=P\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \quad \Rightarrow \quad k\begin{bmatrix} 2 & -1 & 0 \\ 0 & 3 & -2 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}=P\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}.$$

Let us multiply 3:rd equation by 3 and add to it the 2:nd equation to get the upper triangular matrix.

$$k\begin{bmatrix} 2 & -1 & 0 \\ 0 & 3 & -2 \\ 0 & -3 & 3 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}=P\begin{Bmatrix} 0 \\ 0 \\ 3 \end{Bmatrix} \quad \Rightarrow \quad k\begin{bmatrix} 2 & -1 & 0 \\ 0 & 3 & -2 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}=P\begin{Bmatrix} 0 \\ 0 \\ 3 \end{Bmatrix}.$$

Solution to the obtained equation system coincides with that of the original system as the equations are just linear combinations of the original one. After that, solution for the unknowns is obtained step-by-step starting from the last equation:

$$kw_3=3P \quad \Rightarrow \quad w_3=3\frac{P}{k}, \quad \leftarrow$$

$$k(3w_2-2w_3)=0 \quad \Rightarrow \quad w_2=\frac{2}{3}w_3=2\frac{P}{k}, \quad \leftarrow$$

$$k(2w_1-w_2)=0 \quad \Rightarrow \quad w_1=\frac{1}{2}w_2=\frac{P}{k}. \quad \leftarrow$$

Determine the angular velocities and modes $(\omega,\mathbf{A})$ of free vibrations, if the vector of unknowns $\mathbf{a}(t)$, stiffness matrix $\mathbf{K}$, and the mass matrix $\mathbf{M}$ of equations of motion $\mathbf{Ka}+\mathbf{M}\ddot{\mathbf{a}}=0$ are given by

$$\mathbf{a}=\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}, \quad \mathbf{K}=k\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \text{ and } \mathbf{M}=m\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}.$$

Solution

The solution method for the second order ordinary differential equations is based on a trial solution on the form $\mathbf{a}=\mathbf{A}\exp(\mathrm{i}\omega t)$ in which $\mathbf{A}$ is the mode and ω the angular velocity associated with it. The goal of the modal analysis is to find all the possible pairs $(\omega,\mathbf{A})$ in the solution trial. Solution trial transforms the second order ordinary differential equation set into an algebraic one:

$$k\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}+m\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix}=0 \text{ and } \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}=\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}e^{\mathrm{i}\omega t} \quad \Rightarrow$$

$$(\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}-\lambda\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix})\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}=0 \quad \text{where } \lambda=\frac{m\omega^2}{k} .$$

A homogeneous linear equation system can yield a non-zero solution only if the matrix is singular, i.e., its determinant vanishes. The condition can be used to find the possible values of λ

$$\det(\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}-\lambda\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix})=\det(\begin{bmatrix} 2-4\lambda & -1-\lambda \\ -1-\lambda & 2-4\lambda \end{bmatrix})=(2-4\lambda)^2-(1+\lambda)^2=0 \quad \Rightarrow$$

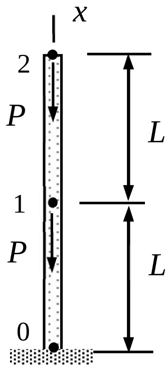
$$(2-4\lambda)=(1+\lambda) \quad \text{or} \quad (2-4\lambda)=-(1+\lambda) \quad \Rightarrow \quad \lambda=1/5 \quad \text{or} \quad \lambda=1 .$$

Knowing the possible values for a non-zero solution, the modes follow from the linear equation system when the values of λ:s are substituted there (one at the time):

$$\lambda=1 : \; \omega=\sqrt{\frac{k}{m}} \quad \text{and} \quad \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix}\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}=0 \quad \text{so} \quad (\omega,\mathbf{A})_1=(\sqrt{\frac{k}{m}},\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}),$$

$$\lambda=\frac{1}{5} : \; \omega=\sqrt{\frac{1}{5}\frac{k}{m}} \quad \text{and} \quad \frac{1}{5}\begin{bmatrix} 6 & -6 \\ -6 & 6 \end{bmatrix}\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}=0 \quad \text{so} \quad (\omega,\mathbf{A})_2=(\sqrt{\frac{1}{5}\frac{k}{m}},\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}).$$

The bar shown is loaded by point forces of equal magnitudes P but opposite directions acting on points 1 and 2. Use the Particle Surrogate Method on the regular grid shown to write the equilibrium equations of points 1 and 2. After that, solve the equations for the axial displacements u_1 and u_2. Cross-sectional area A and Young's modulus E of the material are constants.



Solution

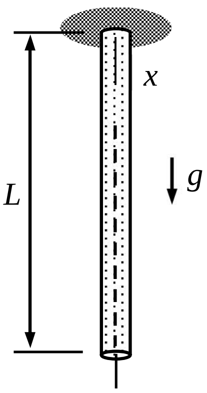
The equilibrium equations of the two free particles and one fixed for the bar model according to the particle surrogate method are given by (formulae collection)

$$u_0 = 0\,,\quad \frac{EA}{h}(u_0 - 2u_1 + u_2) - P = 0\,,\quad \frac{EA}{h}(u_1 - u_2) - P = 0$$

where $h = L$. Notice that the given point force needs to be taken into account in the sum of forces on the left-hand side of equation of motion for particle 1 (in the formulae collection, only the effect of gravity is considered). The matrix notation uses only the equations of the free particles and the boundary condition given by particle 0 to eliminate u_0 from the equilibrium equations $-\mathbf{Ka}+\mathbf{F}=\mathbf{0}$

$$-\frac{EA}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + P\begin{Bmatrix} -1 \\ -1 \end{Bmatrix} = 0 \quad \Leftrightarrow$$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{PL}{EA}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}^{-1}\begin{Bmatrix} -1 \\ -1 \end{Bmatrix} = \frac{PL}{EA}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} -1 \\ -1 \end{Bmatrix} = -\frac{PL}{EA}\begin{Bmatrix} 2 \\ 3 \end{Bmatrix}. \quad \leftarrow$$



Consider a bar of length L loaded by its own weight. Determine the displacements at the mid-point and at the free end using the continuum model. Cross-sectional area A, acceleration by gravity g, and material properties E and ρ are constants.

Solution

In stationary case, the continuum model for the problem is given by equations

$$EA\frac{d^2u}{dx^2}+\rho Ag=0 \quad x\in]0,L[\;, \quad u=0 \quad x=0\,, \text{ and } \; EA\frac{du}{dx}=0 \quad x=L\,.$$

Repetitive integrations in the differential equation give the generic solution containing two integration constants:

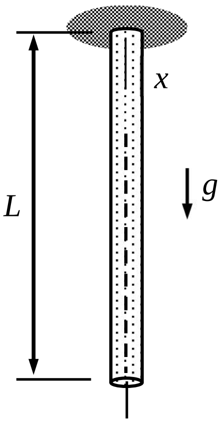
$$\frac{d^2u}{dx^2}=-\frac{\rho g}{E} \;\Rightarrow\; \frac{du}{dx}=-\frac{\rho g}{E}x+a \;\Rightarrow\; u=-\frac{\rho g}{E}\frac{1}{2}x^2+ax+b\,.$$

Then, substituting the generic solution into the boundary conditions

$$u(0)=b=0 \text{ and } \frac{du}{dx}(L)=-\frac{\rho g}{E}L+a=0 \quad\Leftrightarrow\quad a=\frac{\rho g}{E}L \text{ and } b=0\,.$$

Solution to the problem for all points give also the values at the center and end points

$$u(x)=\frac{\rho g}{E}x(L-\frac{1}{2}x) \quad\Rightarrow\quad u(\frac{L}{2})=\frac{3}{8}\frac{\rho gL^2}{E} \text{ and } u(L)=\frac{\rho gL^2}{2E}\,.$$



Consider a bar of length L loaded by its own weight. Determine the displacements at the mid-point and at the free end using the Particle Surrogate Method of the bar with a regular grid $i \in \{0,1,2\}$. Cross-sectional area A, acceleration by gravity g, and material properties E and ρ are constants.

Solution

In Particle Surrogate Method, mass is lumped at the grid points and external forces act on the particles. Interaction of the particles corresponds to a spring of coefficient $k = EA/h$ where $h = L/2$ with the present grid. Equations for particles $i \in \{0,1,2\}$ are

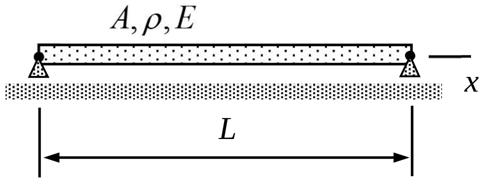
$$u_0 = 0, \quad 2\frac{EA}{L}(u_0 - 2u_1 + u_2) + \frac{L}{2}A\rho g = 0, \text{ and } \quad 2\frac{EA}{L}(u_1 - u_2) + \frac{L}{4}A\rho g = 0.$$

In matrix form for the last two equations, in which the first equations is used to eliminate u_0,

$$-2\frac{EA}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \frac{L}{4}A\rho g\begin{Bmatrix} 2 \\ 1 \end{Bmatrix} = 0 \quad \Leftrightarrow \quad 2\frac{EA}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{L}{4}A\rho g\begin{Bmatrix} 2 \\ 1 \end{Bmatrix} \quad \Leftrightarrow$$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{L^2}{8}\frac{\rho g}{E}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}^{-1}\begin{Bmatrix} 2 \\ 1 \end{Bmatrix} = \frac{L^2}{8}\frac{\rho g}{E}\frac{1}{1}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} 2 \\ 1 \end{Bmatrix} = \frac{L^2}{8}\frac{\rho g}{E}\begin{Bmatrix} 3 \\ 4 \end{Bmatrix}.$$

A bar is free to move in the horizontal direction as shown. Determine the angular velocities and modes $(\omega,\mathbf{A})$ of free vibrations using the Particle Surrogate Method and a regular grid $i \in \{0,1\}$. Cross-sectional are A, density ρ of the material, and Young's modulus E of the material are constants.



Solution

In Particle Surrogate Method, mass is lumped at the grid points and external and internal forces act on the particles. Interaction of the particles corresponds to a spring of coefficient $k = EA/h$ where the spacing $h = L$ in the present case. The equations for particles $i \in \{0,1\}$ take the forms

$$\frac{EA}{h}(u_1 - u_0) = \frac{h}{2}\rho A h \ddot{u}_0 \text{ and } \frac{EA}{h}(u_0 - u_1) = \frac{h}{2}\rho A h \ddot{u}_1$$

or using the matrix notation

$$\frac{EA}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_0 \\ u_1 \end{Bmatrix} + \frac{h}{2}\rho A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{u}_0 \\ \ddot{u}_1 \end{Bmatrix} = 0 .$$

The solution method for the second order ordinary differential equations is based on a trial solution of the form $\mathbf{a} = \mathbf{A}\exp(\mathrm{i}\omega t)$ in which $\mathbf{A}$ is the mode and ω the angular velocity associated with it. The goal of the modal analysis is to find all the possible pairs $(\omega,\mathbf{A})$. When substituted there, solution trial transforms the second order ordinary differential equation set into an algebraic one:

$$(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix})\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0 \text{ where } \lambda = \frac{1}{2}\omega^2 \frac{h^2 \rho}{E} = \frac{1}{2}\omega^2 \frac{L^2 \rho}{E} .$$

A homogeneous linear equation system can give a non-zero solution only if the matrix is singular, i.e., its determinant vanishes. The condition can be used to find the possible values of λ

$$\det(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = \det(\begin{bmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{bmatrix}) = (1-\lambda)^2 - 1 = 0 \quad \Rightarrow \quad \lambda = 0 \text{ or } \lambda = 2 .$$

Knowing the possible values of λ for a non-zero solution, the modes follow from the linear equation system when the values of λ :s are substituted there (one at the time):

$$\lambda = 0 : \ \omega = 0 \text{ and } \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0 \quad \text{so} \quad (\omega,\mathbf{A})_1 = (0, \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}) ,$$

$$\lambda = 2 : \ \omega = \frac{2}{L}\sqrt{\frac{E}{\rho}} \text{ and } \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0 \text{ so } (\omega,\mathbf{A})_2 = (\frac{2}{L}\sqrt{\frac{E}{\rho}}, \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}) .$$

LECTURE ASSIGNMENT 1

Determine the eigenvalues λ_1, λ_2 and the corresponding eigenvectors $\mathbf{a}_1$, $\mathbf{a}_2$ of the $2{\times}2$ matrix $\mathbf{A}$. Consider the possible $(\lambda, \mathbf{a})$ pairs giving solutions to linear equation system $(\mathbf{A} - \lambda \mathbf{I})\mathbf{a} = \mathbf{0}$ with

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}, \quad \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ and } \mathbf{a} = \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}.$$

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As the matrix needs to be singular for a non-zero solution to $\mathbf{a}$, the possible values of λ follow from the characteristic equation $\det(\mathbf{A}-\lambda\mathbf{I})=0$

$$\det\begin{bmatrix} 1-\lambda & 0 \\ -3 & 2-\lambda \end{bmatrix} = (1-\lambda)(2-\lambda)=0 \quad \Rightarrow \lambda_1 = 1 \quad \text{or} \quad \lambda_2 = 2 .$$

Eigenvector $\mathbf{a}$ (non-zero) corresponding to a possible value of λ follows from $(\mathbf{A}-\lambda\mathbf{I})\mathbf{a}=0$ when the value of λ is substituted there:

$$\lambda_1 = 1 : \quad \begin{bmatrix} 1-1 & 0 \\ -3 & 2-1 \end{bmatrix}\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = 0 \quad \Rightarrow \quad \mathbf{a}_1 = \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 3 \end{Bmatrix}$$

$$\lambda_2 = 2 : \quad \begin{bmatrix} 1-2 & 0 \\ -3 & 2-2 \end{bmatrix}\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = 0 \quad \Rightarrow \quad \mathbf{a}_2 = \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

Hence, the eigenvalue-eigenvector pairs of $\mathbf{A}$ are given by

$$(\lambda, \mathbf{a})_1 = (1, \begin{Bmatrix} 1 \\ 3 \end{Bmatrix}) \quad \text{and} \quad (\lambda, \mathbf{a})_2 = (2, \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}) .$$

LECTURE ASSIGNMENT 2

Find the displacement $u(x)$ of a bar of length L using the boundary value problem

$$EA\frac{d^2u}{dx^2}+\rho Ag=0 \quad x\in]0,L[\,,\quad u(0)=u(L)=0$$

given by the continuum model. Assume that the cross-sectional area A, Young's modulus E of the material, density ρ of the material, and acceleration by gravity g are constants.

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First, repeated integrations with the differential equation are used to find the generic solution. Let the integration constants be a and b:

$$\frac{d^2u}{dx^2} = -\frac{\rho Ag}{EA} \quad \Rightarrow \quad \frac{du}{dx} = -\frac{\rho Ag}{EA}x + a \quad \Rightarrow \quad u(x) = -\frac{\rho Ag}{EA}\frac{1}{2}x^2 + ax + b\,.$$

Second, boundary conditions are used to find the values of the integration constants a and b:

$$u(0) = b = 0 \quad \text{and} \quad u(L) = -\frac{\rho Ag}{EA}\frac{1}{2}L^2 + aL + b = 0 \quad \Rightarrow \quad b = 0 \quad \text{and} \quad a = \frac{\rho Ag}{EA}\frac{1}{2}L$$

Finally, the values of the integration constants are substituted into the generic solution to get the solution:

$$u(x) = \frac{\rho Ag}{EA}\frac{1}{2}x(L - x)\,. \quad \Leftarrow$$

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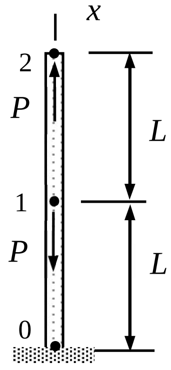
Home assignment 1



The bar shown is loaded by point forces of equal magnitudes P but opposite directions acting on points 1 and 2. Use the particle surrogate method (PSM) on the regular grid shown to write the equilibrium equations of points 1 and 2. After that, solve the equations for the axial displacements u_1 and u_2. Cross-sectional area A and Young's modulus E of the material are constants.

Solution

The equilibrium equations of the two free particles and one fixed for the bar model according to the particle surrogate method are given by (formulae collection)

$$u_0 = 0\,,\quad \frac{EA}{h}(u_0 - 2u_1 + u_2) - P = 0\,,\quad \frac{EA}{h}(u_1 - u_2) + P = 0$$

where $h = L$. Notice that the given point force needs to be taken into account in the sum of forces on the left hand side of equation of motion for particle 1 (in the formulae collection, only the effect of gravity is considered). The matrix notation uses only the equations of the free particles and the boundary condition given by particle 0 to eliminate u_0 from the equilibrium equations $-\mathbf{Ka} + \mathbf{F} = \mathbf{0}$

$$-\frac{EA}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + P\begin{Bmatrix} -1 \\ 1 \end{Bmatrix} = 0 \quad \Leftrightarrow$$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{PL}{EA}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}^{-1}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix} = \frac{PL}{EA}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix} = \frac{PL}{EA}\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}.$$

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Home assignment 2

On grid $i \in \{0,1,2,3\}$, particle surrogate method (PSM) gives the second order ordinary differential equations

$$\frac{S}{h}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} + \rho A h \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{w}_1 \\ \ddot{w}_2 \end{Bmatrix} = 0$$

for a vibration problem of a string of length L. Assuming that the horizontal tightening S, cross-sectional area A, density of material ρ, and spacing h of the grid points are constants, derive the angular speeds and the corresponding modes of the free vibrations.

Solution

Calculation of the angular velocity-mode pairs $(\omega, \mathbf{A})$ is based on the use of trial solution $\mathbf{a}(t) = \mathbf{A}\exp(\mathrm{i}\omega t)$ which transform the set of ordinary differential equations into algebraic ones. In the present case and

$$\frac{S}{h}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} + \rho A h \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{w}_1 \\ \ddot{w}_2 \end{Bmatrix} = 0 \;\Rightarrow\; (\frac{S}{h}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \rho A h \omega^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix})\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0 .$$

where $h = L/3$. To simplify the calculations, let us write the equations first with notation $\lambda = \omega^2 \rho A h^2 / S$

$$(\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix})\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix}\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0 .$$

The homogeneous linear equation system can yield a non-zero solution to the mode only if the matrix in parenthesis is singular, i.e., its determinant vanishes. The condition can be used to find the possible values of λ

$$\det\begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 - 1 = 0 \quad \text{so} \quad \lambda_1 = 1 \quad \text{or} \quad \lambda_2 = 3 .$$

Knowing the possible values for a non-zero solution, the modes follow from the algebraic equation when the values of parameter λ are substituted there (one at the time):

$$\lambda_1 = 1: \quad \omega_1 = \frac{3}{L}\sqrt{\frac{S}{\rho A}} \quad \text{and} \quad \begin{bmatrix} 2-1 & -1 \\ -1 & 2-1 \end{bmatrix}\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0 \quad \text{so} \quad (\omega_1, \mathbf{A}_1) = (\frac{3}{L}\sqrt{\frac{S}{\rho A}}, \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}) ,$$

$$\lambda_2 = 3: \;\; \omega_2 = \frac{3}{L}\sqrt{3\frac{S}{\rho A}} \;\; \text{and} \;\; \begin{bmatrix} 2-3 & -1 \\ -1 & 2-3 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0 \;\; \text{so} \;\; (\omega_2, \mathbf{A}_2) = \left(\frac{3}{L}\sqrt{3\frac{S}{\rho A}}, \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}\right).$$

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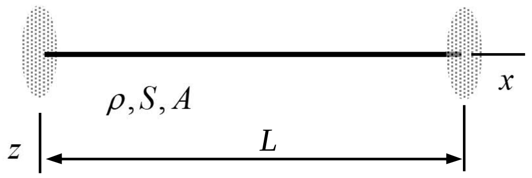
Home assignment 3

Find the transverse displacement $w(x,t)$ of the string shown if the initial velocity and displacement at $t=0$ are $\partial w/\partial t=\dot{W}\sin(\pi x/L)$ ($\dot{W}$ is constant) and $w=0$, respectively. Use the continuum model and assume that the string tightening S and the mass per unit length ρA are constants.



Solution

Vibration problem is solved in two steps. First, modal analysis is used to find the generic solution to the equation of motion. The step is based on a trial solution converting the partial differential equation into an ordinary one. Boundary conditions are used find the combinations of the parameters in the solution related with the problem. The outcome are the angular velocity-mode pairs. Second, the series composed of the trial solutions is tuned to match the initial conditions in the mode superposition step.

Let us start with the modal analysis. According to the recipe, the modes are of the form

$$A(x)=\delta\sin(\lambda x)+\gamma\cos(\lambda x)\,,\quad \lambda=\omega\sqrt{\frac{m'}{k'}}$$

where the proper combination of the three parameters δ, γ, λ follow from the boundary conditions. In the present case, both ends are fixed so

$$A(0)=\gamma=0 \quad\text{and}\quad A(L)=\delta\sin(\lambda L)+\gamma\cos(\lambda L)=0\,.$$

A non-zero mode is obtained by selection $\sin(\lambda L)=0$ so $\lambda L=j\pi \quad j\in\{1,2,\ldots\}$. The mode angular velocity pairs (string model $m'=\rho A$ and $k'=S$)

$$A_j(x)=\sin(\lambda_j x) \quad\text{and}\quad \omega_j=\lambda_j\sqrt{\frac{S}{\rho A}} \quad\text{where}\quad \lambda_j L=j\pi \quad j\in\{1,2,\ldots\}\,.$$

Knowing the modes, the particular combination satisfying the initial conditions follows with mode superposition according to

$$u(x,t)=\sum\nolimits_{j\in\{1,2,\ldots\}} A_j(x)[\alpha_j\frac{1}{\omega_j}\sin(\omega_j t)+\beta_j\cos(\omega_j t)]$$

where $\alpha_j = \dfrac{1}{\mathrm{A}_j^2}\int_0^L A_j(x)h dx$, $\beta_j = \dfrac{1}{\mathrm{A}_j^2}\int_0^L A_j(x)g dx$ where $\mathrm{A}_j^2 = \int_0^L A_j(x)A_j(x)dx$

Using the initial condition data $g(x)=0$, $h(x)=\dot{W}\sin(\pi x/L)$ and the modes (orthogonal as can be verified, e.g., graphically, by hand calculation, or with Mathematica)

$$A_1(x)=\sin(\pi\frac{x}{L}),\quad A_1(x)=\sin(2\pi\frac{x}{L}),\ \cdots\ ,A_j(x)=\sin(j\pi\frac{x}{L})$$

of which the first is of the same form as the given initial velocity, one obtains that $\alpha_1 \neq 0$ the remaining coefficients being zeros. The value of the coefficient

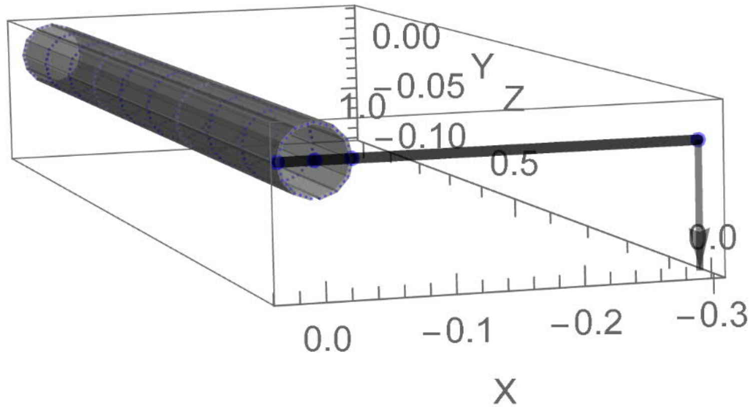
$$\alpha_1 = \frac{\int_0^L A_1(x)h dx}{\int_0^L A_1(x)A_1(x)dx} = \frac{\int_0^L \sin(\pi\frac{x}{L})\dot{W}\sin(\pi\frac{x}{L})dx}{\int_0^L \sin(\pi\frac{x}{L})\sin(\pi\frac{x}{L})dx} = \dot{W}\ .$$

Using the series with only the non-zero term and the relationship $\lambda_1 = \pi/L = \omega_1\sqrt{\rho A/S}$ gives

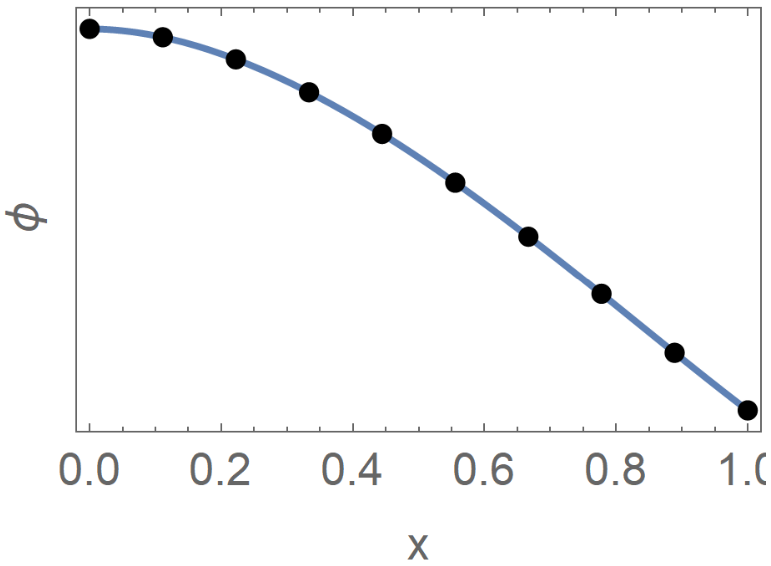
$$w(x,t) = A_1(x)\alpha_1\frac{1}{\omega_1}\sin(\omega_1 t) = \dot{W}\sin(\pi\frac{x}{L})\frac{L}{\pi}\sqrt{\frac{\rho A}{S}}\sin(\frac{\pi}{L}\sqrt{\frac{S}{\rho A}}t)\ .$$

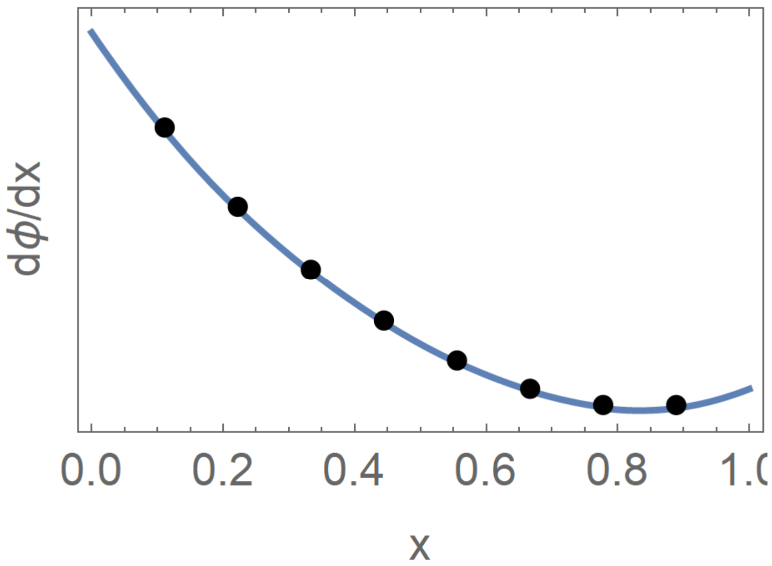
3 FINITE DIFFERENCE METHOD

SIMULATION EXPERIMENT



The outcome of the simulation experiment is the dataset $\{(x_0,\phi_0),(x_1,\phi_1),\ldots,(x_n,\phi_n)\}$ consisting of axial rotation angles on a regular grid on the axis. Processing of data is required to find the derivative of the rotation with respect to the axial coordinate.





Interpolation of the dataset gives a continuous representation (in blue on the left) of continuous derivative (in blue on the right). The dataset (in black on the left) can also be used directly to find the dataset for derivates (in black on the righ). The outcomes differ but not much.

PROCESSING OF DATA

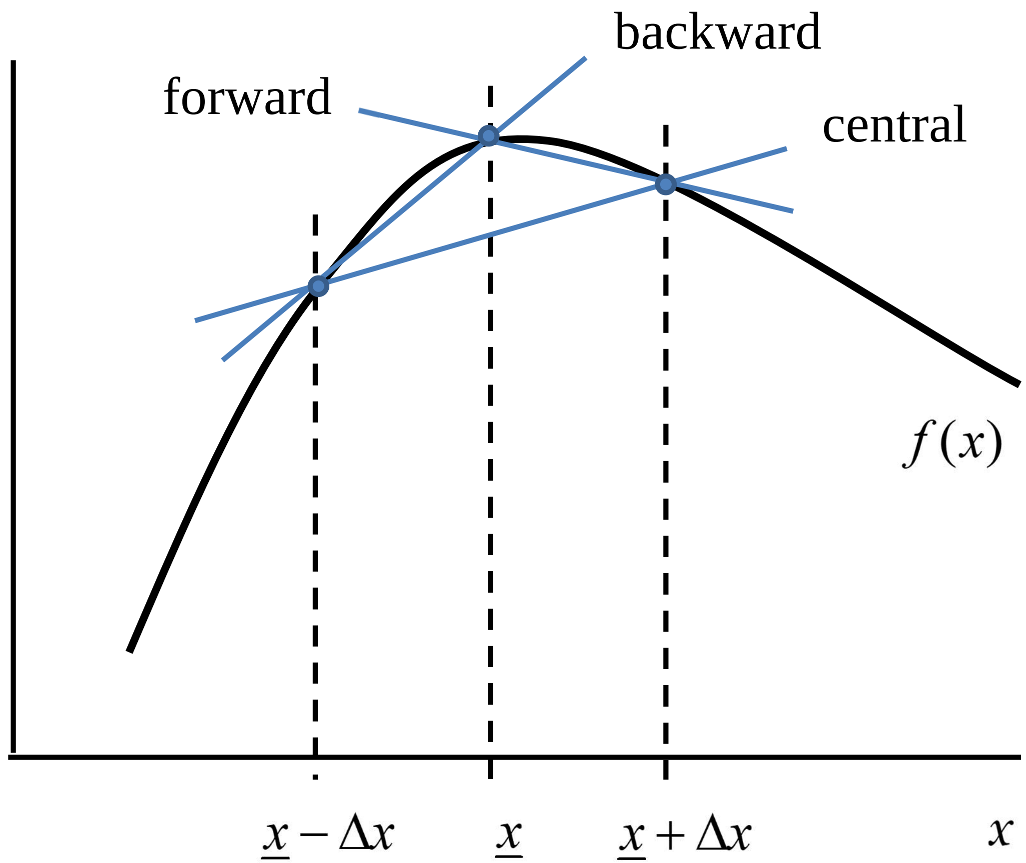
In a typical design, dataset of an experiment $\{\ldots,(x_i, f_i),(x_{i+1}, f_{i+1}),\ldots\}$ is considered as sampling of the underlying continuous *dependent quantity* $f(x)$ at values $\{\ldots, x_i, x_{i+1},\ldots\}$ of the *independent quantity* x. In further processing of data, one may

- use the dataset to find a continuous approximation $g(x)$ to $f(x)$. Thereafter finding the value at any point, calculation of derivatives, integration etc. with *generic methods* is possible.

- use the dataset directly to find, e.g., derivatives at the sampling points, integrals, etc. using *dedicated methods* like difference approximations and quadratures (numerical integration).

Although the details of the methods differ, the results at the sampling points may not differ too much from the engineering viewpoint.

3.1 APPROXIMATION TO DERIVATIVES



Judging from the figure, central difference $f'(\underline{x})=[-f(\underline{x}-\Delta x)+f(\underline{x}+\Delta x)]/(2\Delta x)$ gives the best approximation to the first derivative at $\underline{x}$.

TAYLOR'S THEOREM

Taylor's series with the remainder term is an important tool in numerical methods. Theorem tells how to approximate a function in some neighborhood of a point by a polynomial (below $f^{(i)}(x)$ denotes the i:th derivative of $f(x)$)

1D: $$f(x+\Delta x)= f(x)+\frac{1}{1!}f^{(1)}(x)\Delta x+\ldots+\frac{1}{(n-1)!}f^{(n-1)}(x)\Delta x^{n-1}+\frac{1}{n!}f^{(n)}(\xi)\Delta x^{n}$$

nD: $$f(x+\mathrm{a})=\sum_{i=0}^{n-1}\frac{1}{i!}(\Delta\vec{x}\cdot\nabla)^{i}f(\vec{x})+\frac{1}{n!}[(\Delta\vec{x}\cdot\nabla)^{n}f(\vec{x})]_{\vec{x}=\vec{\xi}}$$

Theorem assumes existence of the n:th derivative. In the remainder term, ξ is some point to the interval which is different in each occurrence. In the finite difference method, approximations to derivatives in terms function values at certain points are often derived with the aid of the theorem.

DIFFERENCE APPROXIMATIONS TO $f'(x)$

For function $f(x)$ values on a regularly spaced grid of resolution Δx, an order k difference approximation to $f'(x)$ on the grid points has a remainder term proportional to Δx^k. The approximation is exact to a polynomial $f(x)$ of degree k.

Type	$f'(x)$	Order
Backward	$\dfrac{f(x)-f(x-\Delta x)}{\Delta x}$	1
Forward	$\dfrac{-f(x)+f(x+\Delta x)}{\Delta x}$	1
Central	$\dfrac{-f(x-\Delta x)+f(x+\Delta x)}{2\Delta x}$	2
Central	$\dfrac{f(x-2\Delta x)-8f(x-\Delta x)+8f(x+\Delta x)-f(x+2\Delta x)}{12\Delta x}$	4

Difference approximations follow from the Taylor's representation truncated at certain term and written for points $x \pm k\Delta x$. The central differences can be derived using the versions

$$f(x+\Delta x) = f(x) + f'(x)\Delta x + \frac{1}{2} f''(x)\Delta x^2 + \frac{1}{6} f^{(3)}(\xi_1)\Delta x^3,$$

$$f(x-\Delta x) = f(x) - f'(x)\Delta x + \frac{1}{2} f''(x)\Delta x^2 - \frac{1}{6} f^{(3)}(\xi_2)\Delta x^3.$$

Adding and subtracting on both sides, rearranging, and dividing with an appropriate power of Δx

$$\frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} = f'(x) + \frac{1}{12}[f^{(3)}(\xi_1) - f^{(3)}(\xi_2)]\Delta x^2,$$

$$\frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{\Delta x^2} = f''(x) + \frac{1}{6}[f^{(3)}(\xi_1) - f^{(3)}(\xi_2)]\Delta x.$$

DIFFERENCE APPROXIMATIONS TO $f''(x)$

Type	$f''(x)$	Order
Backward	$\dfrac{f(x-2\Delta x)-2f(x-\Delta x)+f(x)}{\Delta x^2}$	1
Forward	$\dfrac{f(x)-2f(x+\Delta x)+f(x+2\Delta x)}{\Delta x^2}$	1
Central	$\dfrac{f(x-\Delta x)-2f(x)+f(x+\Delta x)}{\Delta x^2}$	2
Central	$\dfrac{-f(x-2\Delta x)+16f(x-\Delta x)-30f(x)+16f(x+\Delta x)-f(x+2\Delta x)}{12\Delta x}$	4

Difference approximations follow from the Taylor's representation truncated at certain term and written for points $x \pm k\Delta x$. Backward differences can be obtained by using the versions

$$2f(x-\Delta x) = 2f(x) - 2f'(x)\Delta x + f''(x)\Delta x^2 - \frac{1}{3} f^{(3)}(\xi_1)\Delta x^3,$$

$$-f(x-2\Delta x) = -f(x) + 2f'(x)\Delta x - 2f''(x)\Delta x^2 + \frac{4}{3} f^{(3)}(\xi_2)\Delta x^3.$$

Adding and subtracting on both sides, rearranging, and dividing with an appropriate power of Δx

$$2f(x-\Delta x) - f(x-2\Delta x) = f(x) - f''(x)\Delta x^2 - \frac{1}{3} f^{(3)}(\xi_1)\Delta x^3 + \frac{4}{3} f^{(3)}(\xi_2)\Delta x^3,$$

$$2f(x-\Delta x) - f(x-2\Delta x) = f(x) - f''(x)\Delta x^2 - \frac{1}{3} f^{(3)}(\xi_1)\Delta x^3 + \frac{4}{3} f^{(3)}(\xi_2)\Delta x^3$$

EXAMPLE A straightforward way to construct difference formulas for derivatives uses a polynomial interpolant to dataset $\{\ldots,(x_i, f_i),(x_{i+1}, f_{i+1}),\ldots\}$ and derivatives of the interpolant at the grid points. As an example, let us consider the interpolant $p(x)$ to $\{(x_{i-1}, f_{i-1}),(x_i, f_i),(x_{i+1}, f_{i+1})\}$ to find the difference approximations to the first and second derivatives at x_i by calculating the derivatives of the interpolant at that point. Assume a regular grid of points of spacing Δx.

Answer $f_i' = p'(x_i) = \dfrac{-f_{i-1} + f_{i+1}}{2\Delta x}$ and $f_i'' = p''(x_i) = \dfrac{f_{i-1} - 2f_i + f_{i+1}}{\Delta x^2}$

The well-known Lagrange interpolation polynomial $p_n(x)$ of degree n and its error formula are for dataset $\{(x_0, f_0),(x_1, f_1),\ldots(x_n, f_n)\}$

$$p_n(x) = \sum_{i\in\{0,1,\ldots,n\}} f_i \Pi_{j\in\{0,1,\ldots i-1,i+1,\ldots,n\}} \frac{x - x_j}{x_i - x_j},$$

$$f(x) - p_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi)\Pi_{i\in\{0,1,\ldots,n\}}(x - x_i) \ .$$

Notice the removal of index i in the product term inside the sum of the interpolation formula. With dataset $\{(x_{i-1}, f_{i-1}),(x_i, f_i),(x_{i+1}, f_{i+1})\}$

$$p(x) = f_{i-1}\frac{x - x_i}{x_{i-1} - x_i}\frac{x - x_{i+1}}{x_{i-1} - x_{i+1}} + f_i\frac{x - x_{i-1}}{x_i - x_{i-1}}\frac{x - x_{i+1}}{x_i - x_{i+1}} + f_{i+1}\frac{x - x_{i-1}}{x_{i+1} - x_{i-1}}\frac{x - x_i}{x_{i+1} - x_i}.$$

Selection $x_i = i\Delta x$ and representation with monomials of increasing powers, which is more convenient in calculation of derivatives, gives the (equivalent) form

$$p(x) = f_i - \frac{x}{2\Delta x}(-f_{i-1} + f_{i+1}) + \frac{x^2}{2\Delta x^2}(f_{i-1} - 2f_i + f_{i+1}).$$

Therefore, the calculation with the interpolant to the dataset implies the well-known 2:nd order accurate difference approximations to the first and second derivatives

$$f_i' = p'(0) = \frac{-f_{i-1} + f_{i+1}}{2\Delta x} \quad \text{and} \quad f_i'' = p''(0) = \frac{f_{i-1} - 2f_i + f_{i+1}}{\Delta x^2}.$$

The power Δx^2 in the remainder term can be verified by a direct calculation with the remainder expression.

DIFFERENCE STENCILS

Derivative	Accuracy	−5	−4	−3	−2	−1	0	1	2	3	4	5
1	2					−1/2	0	1/2				
	4				1/12	−2/3	0	2/3	−1/12			
	6			−1/60	3/20	−3/4	0	3/4	−3/20	1/60		
	8		1/280	−4/105	1/5	−4/5	0	4/5	−1/5	4/105	−1/280	
2	2					1	−2	1				
	4				−1/12	4/3	−5/2	4/3	−1/12			
	6			1/90	−3/20	3/2	−49/18	3/2	−3/20	1/90		
	8		−1/560	8/315	−1/5	8/5	−205/72	8/5	−1/5	8/315	−1/560	
3	2				−1/2	1	0	−1	1/2			
	4			1/8	−1	13/8	0	−13/8	1	−1/8		
	6		−7/240	3/10	−169/120	61/30	0	−61/30	169/120	−3/10	7/240	
4	2				1	−4	6	−4	1			
	4			−1/6	2	−13/2	28/3	−13/2	2	−1/6		
	6		7/240	−2/5	169/60	−122/15	91/8	−122/15	169/60	−2/5	7/240	
5	2			−1/2	2	−5/2	0	5/2	−2	1/2		
	4		1/6	−3/2	13/3	−29/6	0	29/6	−13/3	3/2	−1/6	
	6	−13/288	19/36	−87/32	13/2	−323/48	0	323/48	−13/2	87/32	−19/36	13/288

https://en.wikipedia.org/wiki/Finite_difference_coefficient

3.2 FINITE DIFFERENCE METHOD

Finite Difference Method is a numerical technique for solving ordinary and partial differential equations by approximating derivatives with finite difference formulas. If applied with a regular grid to the string and bar models, the discrete equations by the FDM

Interior $\dfrac{k'}{h^2}(a_{i-1} - 2a_i + a_{i+1}) + f' = m'\ddot{a}_i \quad i \in \{1, 2, \ldots, n-1\}$

Boundary $a_0 = \underline{a}_0$ or $-\dfrac{k'}{h}(a_1 - a_0) = \underline{F}_0$ and $a_n = \underline{a}_n$ or $\dfrac{k'}{h}(a_n - a_{n-1}) = \underline{F}_n$

Initial conditions $a_i - g_i = 0$ and $\dot{a}_i - h_i = 0 \quad i \in \{1, 2, \ldots, n-1\}$

Then, the outcome is a set of Ordinary Differential Equations of the same type as by the Particle Surrogate Method. Therefore, the matrix and difference equation techniques for PSM also apply to FDM.

Continuum model of bar or string of known displacement or loading at the end points and known initial position and velocity is given by equations

$$k'\frac{\partial^2 a}{\partial x^2}+f'=m'\frac{\partial^2 a}{\partial t^2} \quad x\in\Omega \quad t>0$$

$$a=\underline{a} \;\text{ or }\; n_x(k'\frac{\partial a}{\partial x})=\underline{F} \quad x\in\partial\Omega \quad t>0,$$

$$a=g \quad\text{and}\quad \frac{\partial a}{\partial t}=h \quad x\in\Omega \quad t=0.$$

By using the 2:nd order accurate central difference approximation for the second partial derivative in the equation of the motion and 1:st order accurate difference approximation for the derivative in the boundary condition, one obtains

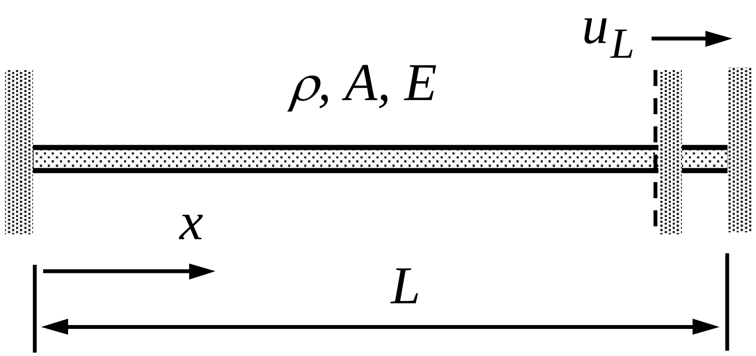
$$\frac{k'}{h^2}(a_{i-1}-2a_i+a_{i+1})+f'=m'\ddot{a}_i \quad i\in\{1,2,\ldots,n-1\}$$

$$a_0 = \underline{a}_0 \text{ or } -\frac{k'}{h}(a_1 - a_0) = \underline{F}_0 \text{ and } a_n = \underline{a}_n \text{ or } \frac{k'}{h}(a_n - a_{n-1}) = \underline{F}_n$$

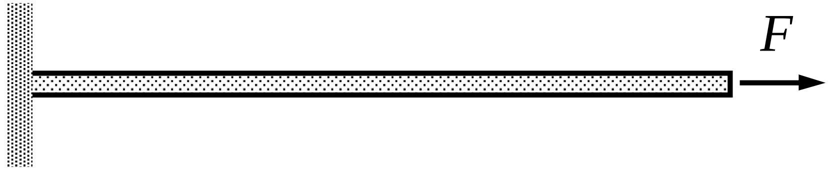
$$w_i - g_i = 0 \quad \text{and} \quad \dot{w}_i - h_i = 0 \quad i \in \{1, 2, \ldots, n-1\} \quad t = 0.$$

Multiplication of both sides of the equation of motion by h (here spacing of the grip points) gives the final form which differs only in the equations for the boundary points from those for the Particle Surrogate Method. As the starting point of FDM are the differential equations of the continuum model, e.g., point forces, point masses etc. not located at the boundaries need to be represented correctly in the continuum model before the use of FDM. At non-regular points, the differential equation should be replaced by jump conditions implied by the first principles of mechanics. Also, the grid should be adjusted to have a grid point at positions of point forces and masses.

1st ORDER BOUNDARY CONDITIONS FOR BAR



$$u_0 = 0, \quad u_n - \underline{u} = 0$$

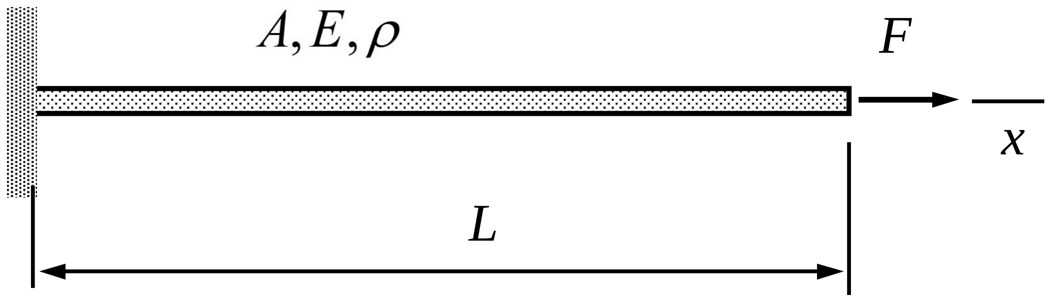


$$u_0 = 0, \quad EA\frac{u_n - u_{n-1}}{h} - F_n = 0$$

$$u_0 = 0, \quad EA\frac{u_n - u_{n-1}}{h} + m\ddot{u}_n = 0$$

$$u_0 = 0, \quad EA\frac{u_n - u_{n-1}}{h} + ku_n = 0$$

EXAMPLE The elastic bar shown is loaded by a point force at the right end. The left end is fixed. Determine the stationary solution displacement using the Finite Difference Method on a regular grid $i \in \{0,1,\ldots,n\}$ on the solution domain of length L. Material properties and cross-sectional area are constants. What is limit solution when $n \to \infty$ and $h = L/n$?



Answer $u_i = ih\dfrac{F}{EA} = x_i\dfrac{F}{EA}$, limit solution $u(x) = x\dfrac{F}{EA}$

Using the 2:nd order accurate central difference approximation to the second derivative in the equilibrium equation for typical grid point $i \in \{1,2,\dots,n-1\}$ and first order accurate approximation to the first derivative in the boundary condition at the loaded end

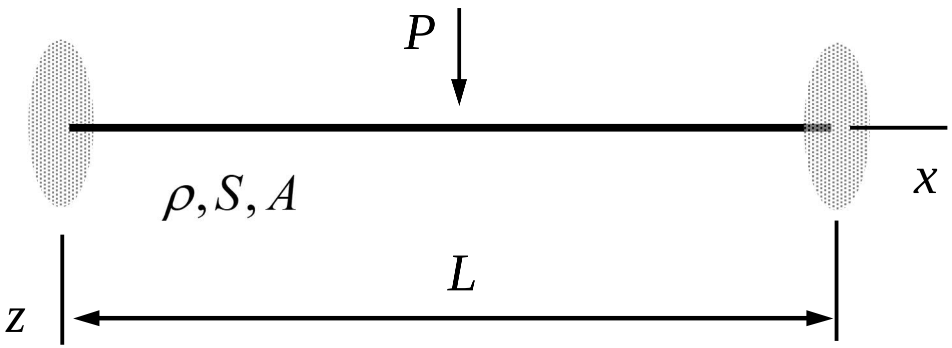
$$EA\left(\frac{u_{i-1}-2u_i+u_{i+1}}{h^2}\right)=0 \quad i \in \{1,2,\dots,n-1\}, \; u_0=0 \; , \text{ and } \; EA\frac{u_n-u_{n-1}}{h}=F \,.$$

When substituted into the difference equation, the solution trial $u_i = ar^i$ implies the condition $1-2r+r^2=(1-r)^2=0$ so the generic solution is $u_i = a+bi$ (double root $r=1$) . The two constants follow from equations for the boundary points

$$u_0 = a = 0 \quad \text{and} \quad EA\frac{a+bn-a-b(n-1)}{h}=F \quad \Leftrightarrow \quad a=0 \; \text{ and } \; b=\frac{hF}{EA}.$$

Hence $u_i = ih\dfrac{F}{EA}=x_i\dfrac{F}{EA}$ (in the limit $u(x)=x\dfrac{F}{EA}$). ←

EXAMPLE A string of length L, tightening S, cross-sectional area A, and density ρ, is loaded by a point force P at its center point. If the ends are fixed and the initial geometry without loading is straight, find the solution to the transverse displacement as function of x using the finite difference method on a regular grid of three points $i \in \{0,1,2\}$.



Answer $w_1 = \dfrac{PL}{4S}$

The boundary value problem is given by equilibrium equations for the regular interior points, jump conditions at the center point (non-regular point due to the point force), and boundary conditions for the end points

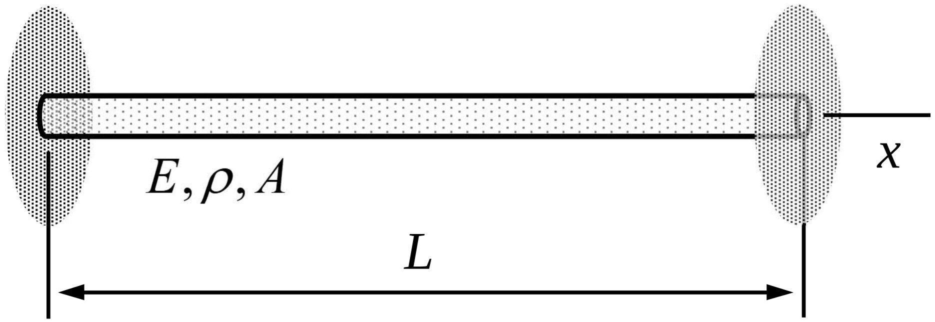
$$S\frac{d^2w}{dx^2}=0 \quad x\in]0,\frac{L}{2}[\quad \text{or} \quad x\in]\frac{L}{2},L[,$$

$$S\left[\!\left[\frac{dw}{dx}\right]\!\right]+P=0\ , \ [\![w]\!]=0 \quad x=\frac{L}{2}, \ \text{and } w(x)=0 \quad x\in\{0,L\}.$$

As the end points are fixed and there is a discontinuity at the midpoint, only the jump condition applies. Let us use the first order accurate backward and forward two-point difference approximations to the left and right derivatives, to get ($w_0=w_2=0$ and $\Delta x=L/2$):

$$S(\frac{w_2-w_1}{\Delta x}-\frac{w_1-w_0}{\Delta x})+P=0 \quad \Rightarrow \quad w_1=\frac{PL}{4S}.$$

EXAMPLE Write the equations of motion for the free vibrations of the bar shown by using FDM. Use the matrix formulation on a regular grid with $i \in \{0,1,2,3\}$. Material properties E, ρ and the cross-sectional area A are constants. Also, determine the two lowest angular velocities and the corresponding modes of the free vibrations.



Answer $\frac{EA}{h^2}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \rho A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ u_2 \end{Bmatrix} = 0$ where $h = \frac{L}{3}$

The equations for the points inside the domain, as given by the 2:nd order accurate central difference approximation to the second derivative (with respect to x), are

$$u_0 = 0\ ,\ \frac{EA}{h^2}(u_0 - 2u_1 + u_2) = \rho A \ddot{u}_1\ ,\ \frac{EA}{h^2}(u_1 - 2u_2 + u_3) = \rho A \ddot{u}_2, \text{ and } u_3 = 0.$$

In matrix notation and $k = EA/h$, $m = \rho Ah$, and $h = L/3$, the equations for points 1 and 2 are (when the known displacements at the boundary points are used there)

$$k\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + m\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ u_2 \end{Bmatrix} = 0. \quad \Leftarrow$$

Frequencies and modes of the free vibrations follow with the trial solution $\mathbf{u} = \mathbf{A}\exp(\mathrm{i}\omega t)$. Using the notation $\lambda = \omega^2 m/k$

$$(\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix})\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0\ .$$

The homogeneous linear equation system can yield a non-zero solution to the mode only if the matrix in parenthesis is singular so its determinant needs to vanish

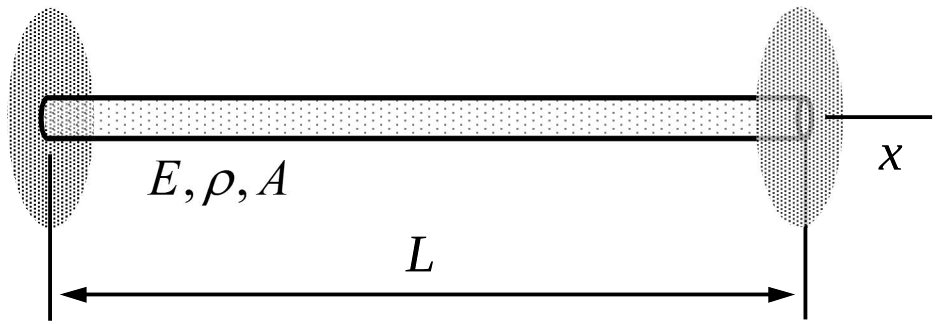
$$\det\begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 - 1 = 0 \quad \Rightarrow \quad \lambda_1 = 1 \text{ or } \lambda_2 = 3.$$

Knowing the possible λ:s and also the angular velocities from $\lambda = \omega^2 m / k$, solution to the modes are given by the linear equation systems:

$$\lambda_1 = 1: \quad \omega_1 = \sqrt{\lambda \frac{k}{m}} \quad \text{and} \quad \begin{bmatrix} 2-1 & -1 \\ -1 & 2-1 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0 \quad \text{so} \quad (\omega_1, \mathbf{A}_1) = (\sqrt{\frac{k}{m}}, \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}).$$

$$\lambda_2 = 3: \quad \omega_2 = \sqrt{3 \frac{k}{m}} \quad \text{and} \quad \begin{bmatrix} 2-3 & -1 \\ -1 & 2-3 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0 \quad \text{so} \quad (\omega_2, \mathbf{A}_2) = (\sqrt{3 \frac{k}{m}}, \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}).$$

EXAMPLE Consider the free vibrations of the bar shown, when material properties E, ρ and cross-sectional area A are constants. Use finite difference method with a second order accurate central difference approximation to find the displacement as the function time on a regular grid $i \in \{0,1,\ldots,n\}$ when the initial displacement and velocity are $u_i = \sum_{k\in\{1,2,\ldots,n-1\}} \alpha_k \sin(k\pi i / n)$ and $\dot{u}_i = 0$ $i \in \{0,1,\ldots,n\}$ respectively. Hint. Use the displacement assumption $u_i = \sum_{k\in\{1,2,\ldots,n-1\}} a_k(t) \sin(k\pi i / n)$



Answer $u_i(t) = \sum_{k\in\{1,2,\ldots,n-1\}} \alpha_k \cos(\omega_k t) \sin(k\pi \frac{i}{n})$ and $\omega_k = \sqrt{\frac{2E}{\rho L^2} n^2 [1 - \cos(\pi \frac{k}{n})]}$

Lets start with the difference equations

$$\frac{EA}{h^2}(u_{i-1} - 2u_i + u_{i+1}) = \rho A\ddot{u}_i \quad \text{and} \quad u_i = 0 \quad i \in \{0, n\}$$

and use the solution assumption motivated by the form of the initial condition (can be considered as the discrete fourier series of some continuous initial displacement at the grid points)

$$u_i(t) = \sum_{k\in\{1,2,\ldots,n-1\}} a_k(t)\sin(k\pi\frac{i}{n}).$$

Notice that the number terms correspond to the number of interior grid points) $i \in \{1, 2, \ldots, n-1\}$. As the differential equation is linear it is enough to consider a typical term k giving when substituted into the difference expression and second time derivative

$$u_{i-1} - 2u_i + u_{i+1} = 2a_k(t)[\cos(\pi\frac{k}{n}) - 1]\sin(k\pi\frac{i}{n}) \quad \text{and} \quad \ddot{u}_i = \sin(k\pi\frac{i}{n})\ddot{a}_k.$$

Consequently, the difference-differential equation and the initial conditions associated with the k:th term boil down to an initial value problem for the unknown $a_k(t)$ of the displacement assumption

$$\ddot{a}_k + \omega_k^2 a_k = 0 \quad t > 0 \,, \quad a_k(0) = \alpha_k \text{ and } \dot{a}_k(0) = 0 \quad \text{where} \quad \omega_k = \sqrt{\frac{2E}{\rho h^2}[1 - \cos(\pi \frac{k}{n})]}$$

whose solution is $a_k(t) = \alpha_k \cos(\omega_k t)$. Putting everything together, the solution to the vibration problem with initial displacement in terms of the discrete Fourier sine series becomes

$$u_i(t) = \sum\nolimits_{k \in \{1,2,\ldots,n-1\}} \alpha_k \cos(\omega_k t) \sin(k\pi \frac{i}{n}) \quad \text{where} \quad \omega_k = \sqrt{\frac{2E}{\rho L^2} n^2 [1 - \cos(\pi \frac{k}{n})]} \,.$$ ←

DISCRETE SINE SERIES

The discrete Fourier series (various forms exist) can be used to represent a list as the sum of lists of harmonic terms. For example, the sine-transformation pair for a list a_i $i \in \{1,2,\ldots,n-1\}$ is given by

$$\alpha_j = \frac{2}{n}\sum_{i\in\{1,2,\ldots,n-1\}} \sin(j\pi\frac{i}{n})a_i \quad j \in \{1,2,\ldots,n-1\}$$

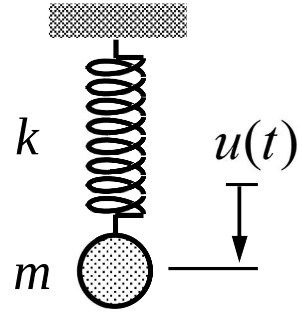
$$a_i = \sum_{j\in\{1,2,\ldots,n-1\}} \alpha_j \sin(j\pi\frac{i}{n}) \quad i \in \{1,2,\ldots,n-1\}$$

The transformation pair is based on the orthogonality of the modes (Cronecker delta $\delta_{jl} = 1$ if $j = l$ and $\delta_{jl} = 0$ if $j \neq l$)

$$\sum_{j\in\{1,2,\ldots,n-1\}} \sin(j\pi\frac{i}{n})\sin(l\pi\frac{i}{n}) = \delta_{jl}\frac{n}{2}.$$

3.3 TIME INTEGRATION

In time integration, the solution is sought step-by-step using a regular grid on the temporal domain $t_i = i\Delta t \quad i \in \{0,1,\ldots\}$, where Δt is the step size. The exact one particle vibration solution to displacement and velocity at the grid points represents the generic idea of a recursive one-step time-integration method:



$$\begin{Bmatrix} u \\ \dot{u} \end{Bmatrix} = \begin{bmatrix} \cos(\omega t) & \omega^{-1}\sin(\omega t) \\ -\omega\sin(\omega t) & \cos(\omega t) \end{bmatrix} \begin{Bmatrix} g \\ h \end{Bmatrix} \text{ where } \omega = \sqrt{\frac{k}{m}} \Rightarrow$$

$$\begin{Bmatrix} u \\ \Delta t\dot{u} \end{Bmatrix}_i = \begin{bmatrix} \cos(\alpha i) & \alpha^{-1}\sin(\alpha i) \\ -\alpha\sin(\alpha i) & \cos(\alpha i) \end{bmatrix} \begin{Bmatrix} g \\ \Delta t h \end{Bmatrix} \text{ where } \alpha = \Delta t\omega = \Delta t\sqrt{\frac{k}{m}} \Rightarrow$$

$$\begin{Bmatrix} u \\ \Delta t\dot{u} \end{Bmatrix}_i = \begin{bmatrix} \cos\alpha & \alpha^{-1}\sin\alpha \\ -\alpha\sin\alpha & \cos\alpha \end{bmatrix} \begin{Bmatrix} u \\ \Delta t\dot{u} \end{Bmatrix}_{i-1} \quad i \in \{1,2,\ldots\} \text{ and } \begin{Bmatrix} u \\ \Delta t\dot{u} \end{Bmatrix}_0 = \begin{Bmatrix} g \\ \Delta t h \end{Bmatrix}.$$

Let us consider free vibration of one particle with known position and velocity at the initial time described by the initial value problem

$$m\ddot{u} + ku = 0 \;\; t > 0, \;\; \dot{u}(0) = h, \quad u(0) = g\,.$$

The exact solutions to displacement and velocity can be expressed in the form

$$\begin{Bmatrix} u \\ \dot{u} \end{Bmatrix} = \begin{bmatrix} \cos(\omega t) & \omega^{-1}\sin(\omega t) \\ -\omega\sin(\omega t) & \cos(\omega t) \end{bmatrix} \begin{Bmatrix} g \\ h \end{Bmatrix} \text{ where } \; \omega = \sqrt{\frac{k}{m}}\,.$$

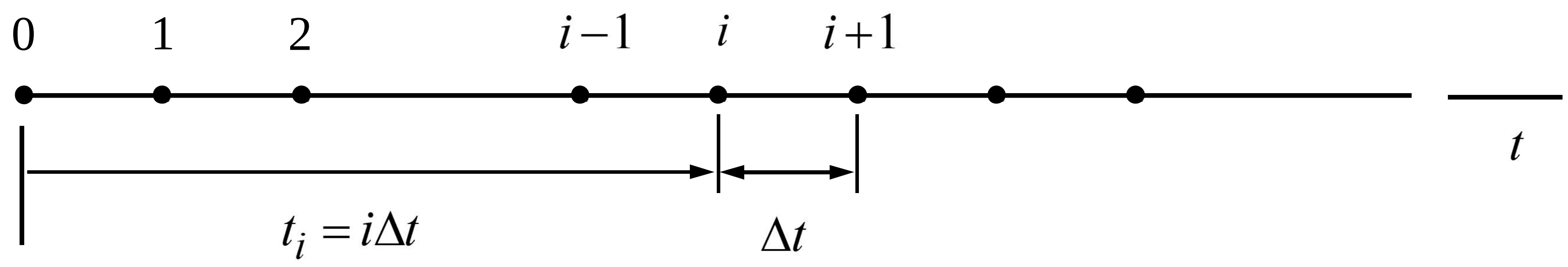
Solution at point $t_i = i\Delta t$ of the regular temporal grid

$$\begin{Bmatrix} u \\ \Delta t\dot{u} \end{Bmatrix}_i = \begin{bmatrix} \cos(\alpha i) & \alpha^{-1}\sin(\alpha i) \\ -\alpha\sin(\alpha i) & \cos(\alpha i) \end{bmatrix} \begin{Bmatrix} g \\ \Delta t h \end{Bmatrix} \text{ where } \alpha = \Delta t\omega = \Delta t\sqrt{\frac{k}{m}}\,.$$

In recursive form, initial conditions give the solution at the end of the first time-interval to be treated as the initial conditions for the next time-interval and so on.

REGULAR TEMPORAL GRID

On a regular grid, the grid points are distributed evenly. The number of the point at the origin is 0 and the numbering increases in the direction of the $t-$axis without gaps. The time intervals are referenced by their end point indices (i.e., interval i is between grid points $i-1$ and i.



As the temporal domain for an initial value problem does not have an upper bound (strictly speaking), the length of the intervals can (and often are) chosen to match the behavior of the solution (small steps for the rapid changes).

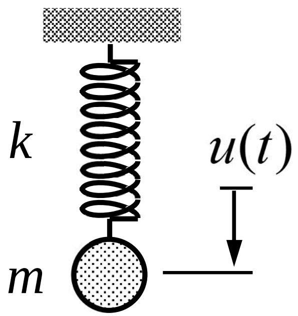
TIME INTEGRATION

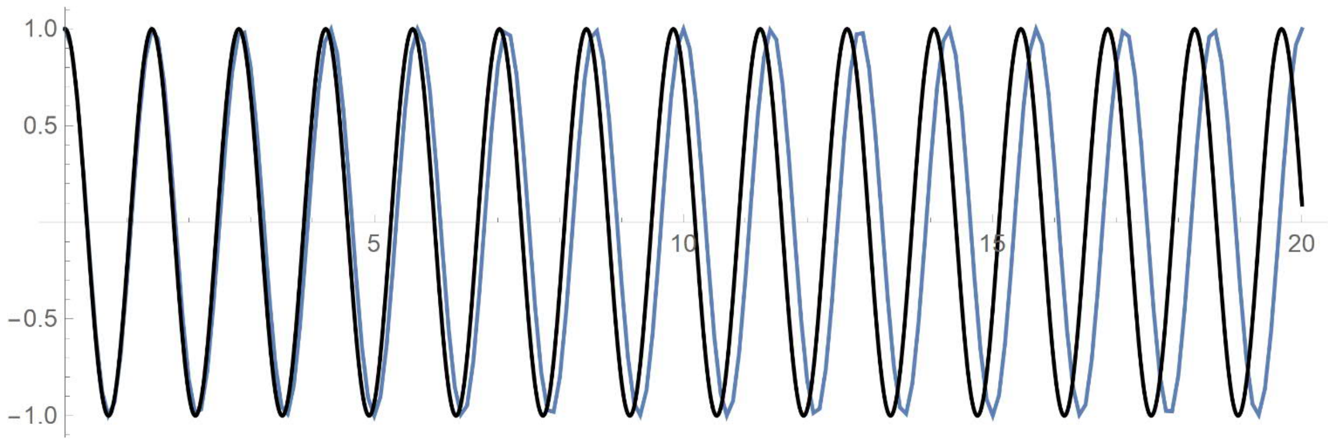
Method	Iteration $i \in \{1,2,\ldots\}$	Initial $i=0$
EX	$\begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_i = \begin{bmatrix} \cos\alpha & \alpha^{-1}\sin\alpha \\ -\alpha\sin\alpha & \cos\alpha \end{bmatrix} \begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_{i-1}$	$\begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_0 = \begin{Bmatrix} g \\ \Delta t h \end{Bmatrix}$
CN	$\begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_i = \dfrac{1}{4+\alpha^2} \begin{bmatrix} 4-\alpha^2 & 4 \\ -4\alpha^2 & 4-\alpha^2 \end{bmatrix} \begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_{i-1}$	$\begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_0 = \begin{Bmatrix} g \\ \Delta t h \end{Bmatrix}$
DG	$\begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_i = \dfrac{2}{12+\alpha^4} \begin{bmatrix} 6-3\alpha^2 & 6-\alpha^2 \\ -6\alpha^2 & 6-3\alpha^2 \end{bmatrix} \begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_{i-1}$	$\begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_0 = \begin{Bmatrix} g \\ \Delta t h \end{Bmatrix}$

The methods coincide at the limit of vanishing step-size when $\alpha = \sqrt{\dfrac{k}{m}}\Delta t \to 0$.

ACCURACY AND STABILITY

Numerical integration involves discretization error in each step and error accumulation may spoil the solution after certain number of steps. Crank-Nicolson does not reduce the amplitude but the phase error is clear from comparison of the exact and numerical solutions to a vibrating particle problem.





With multiple particles and various time-scales of vibrations, certain amount of numerical dumping is actually a desirable property of a numerical integration method!

TIME INTEGRATION

Method	Iteration $i \in \{1,2,\ldots\}$	Initial $i=0$
DG	$\begin{bmatrix} \Delta t\mathbf{K} & -\frac{1}{2}\Delta t^2\mathbf{K}+\mathbf{M} \\ \frac{1}{2}\Delta t^2\mathbf{K}-\mathbf{M} & \Delta t\mathbf{M}-\frac{1}{6}\Delta t^3\mathbf{K} \end{bmatrix}\begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}} \end{Bmatrix}_i = \begin{bmatrix} \mathbf{0} & \mathbf{M} \\ -\mathbf{M} & \mathbf{0} \end{bmatrix}\begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}} \end{Bmatrix}_{i-1}$	$\begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}} \end{Bmatrix}_0 = \begin{Bmatrix} \mathbf{g} \\ \mathbf{h} \end{Bmatrix}$
CN	$\begin{bmatrix} \mathbf{I} & -\frac{\Delta t}{2}\mathbf{I} \\ \frac{\Delta t}{2}\mathbf{K} & \Delta t\mathbf{M} \end{bmatrix}\begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}} \end{Bmatrix}_i = \begin{bmatrix} \mathbf{I} & \frac{\Delta t}{2}\mathbf{I} \\ -\frac{\Delta t}{2}\mathbf{K} & \Delta t\mathbf{M} \end{bmatrix}\begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}} \end{Bmatrix}_{i-1}$	$\begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}} \end{Bmatrix}_0 = \begin{Bmatrix} \mathbf{g} \\ \mathbf{h} \end{Bmatrix}$

The proper step-size Δt depends on the largest eigenvalue of parameter $\boldsymbol{\alpha} = \mathbf{M}^{-1}\mathbf{K}\Delta t^2$. The numerical damping of DG exceeds that of CN whereas the phase error of CN exceeds that of the DG method.

Derivation of the Crank-Nicolson method uses Taylor series with respect to time for displacement and velocity with the mean value approximation to the remainder

$$\mathbf{a}_i = \mathbf{a}_{i-1} + \Delta t \frac{\dot{\mathbf{a}}_{i-1} + \dot{\mathbf{a}}_i}{2}\ , \quad \dot{\mathbf{a}}_i = \dot{\mathbf{a}}_{i-1} + \Delta t \frac{\ddot{\mathbf{a}}_{i-1} + \ddot{\mathbf{a}}_i}{2}$$

and the differential equation written at the ends of the time interval

$$\mathbf{M}\ddot{\mathbf{a}}_{i-1} + \mathbf{K}\mathbf{a}_{i-1} - \mathbf{F}_{i-1} = 0\ , \ \ \mathbf{M}\ddot{\mathbf{a}}_i + \mathbf{K}\mathbf{a}_i - \mathbf{F}_i = 0.$$

Solving for $\mathbf{a}_i$ and $\dot{\mathbf{a}}_i$ in terms of $\mathbf{a}_{i-1}$ and $\dot{\mathbf{a}}_{i-1}$ from the equations

$$\begin{bmatrix} \mathbf{I} & -\frac{\Delta t}{2}\mathbf{I} \\ \frac{\Delta t}{2}\mathbf{K} & \Delta t\mathbf{M} \end{bmatrix} \begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}} \end{Bmatrix}_i = \begin{bmatrix} \mathbf{I} & \frac{\Delta t}{2}\mathbf{I} \\ -\frac{\Delta t}{2}\mathbf{K} & \Delta t\mathbf{M} \end{bmatrix} \begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}} \end{Bmatrix}_{i-1} + \frac{\Delta t}{2}\left(\begin{Bmatrix} \mathbf{0} \\ \mathbf{F} \end{Bmatrix}_{i-1} + \begin{Bmatrix} \mathbf{0} \\ \mathbf{F} \end{Bmatrix}_i\right).$$

ONE-STEP METHOD

Taylor series with respect to time and the mean value approximation to the remainder (the number of terms or the approximation may differ from those below) give, e.g.,

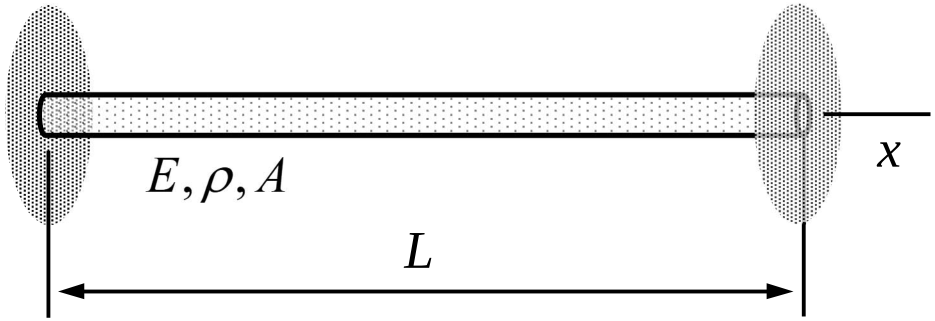
$$\mathbf{a}_i = \mathbf{a}_{i-1} + \Delta t \frac{\dot{\mathbf{a}}_{i-1} + \dot{\mathbf{a}}_i}{2} \quad \text{and} \quad \dot{\mathbf{a}}_i = \dot{\mathbf{a}}_{i-1} + \Delta t \frac{\ddot{\mathbf{a}}_{i-1} + \ddot{\mathbf{a}}_i}{2}$$

Differential equations written at the end points of the time interval contain the derivatives of the remainder terms (the equation may be differentiated more with respect to time

$$\mathbf{M}\ddot{\mathbf{a}}_{i-1} + \mathbf{K}\mathbf{a}_{i-1} - \mathbf{F}_{i-1} = 0 \quad \text{and} \quad \mathbf{M}\ddot{\mathbf{a}}_i + \mathbf{K}\mathbf{a}_i - \mathbf{F}_i = 0.$$

Solving the equations for $\mathbf{a}_i$ and $\dot{\mathbf{a}}_i$ in terms of $\mathbf{a}_{i-1}$ and $\dot{\mathbf{a}}_{i-1}$ gives the well-known Crank-Nicolson method. The same recipe applies with more terms in the Taylor series approximations to displacement and velocity.

EXAMPLE Finite Difference Method is applied to the bar problem shown using a regular grid with $i \in \{0,1,2,3\}$. Thereafter, Crank-Nicolson methods is applied to find the solution at the temporal grid $t_j = j\Delta t \quad j \in \{0,1,\ldots\}$. Derive the iteration formula giving the displacements and velocities of points of the spatial discretization for any initial displacement and velocities. Material properties E,ρ and cross-sectional area A are constants.



Answer $\frac{EA}{h^2}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \rho A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ u_2 \end{Bmatrix} = 0$ where $h = \frac{L}{3}$

Use of the Finite Difference Method and 2:nd order central difference approximation on a regular grid with $i \in \{0,1,2,3\}$ gives the ordinary differential equations

$$\frac{EA}{h^2}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \rho A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ u_2 \end{Bmatrix} = 0 \quad \text{where} \quad h = \frac{L}{3}$$

for the interior points 1 and 2. With notation

$$\mathbf{M} = \rho A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \mathbf{K} = \frac{EA}{h^2}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix},\ \mathbf{a} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix},\ \dot{\mathbf{a}} = \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix},\ \mathbf{g} = \begin{Bmatrix} g_1 \\ g_2 \end{Bmatrix},\ \text{and}\ \ \mathbf{h} = \begin{Bmatrix} h_1 \\ h_2 \end{Bmatrix}$$

the time-integration according to the Crank-Nicolson method follows from

$$\begin{bmatrix} \mathbf{I} & -\frac{\Delta t}{2}\mathbf{I} \\ \frac{\Delta t}{2}\mathbf{K} & \Delta t\mathbf{M} \end{bmatrix}\begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}} \end{Bmatrix}_i = \begin{bmatrix} \mathbf{I} & \frac{\Delta t}{2}\mathbf{I} \\ -\frac{\Delta t}{2}\mathbf{K} & \Delta t\mathbf{M} \end{bmatrix}\begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}} \end{Bmatrix}_{i-1} \quad \text{and} \quad \begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}} \end{Bmatrix}_0 = \begin{Bmatrix} \mathbf{g} \\ \mathbf{h} \end{Bmatrix}.$$

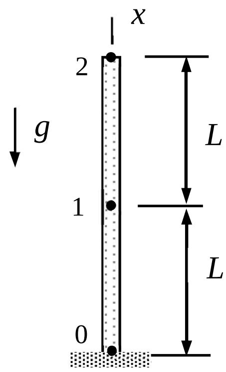
COE-C3005 Finite Element and Finite difference methods

1. Find the five point difference approximation to $f_i^{(4)}$ (fourth derivative) using the dataset $\{(-2\Delta x, f_{i-2}),(-\Delta x, f_{i-1}),(0, f_i),(\Delta x, f_{i+1}),(2\Delta x, f_{i+2})\}$. Use the Lagrange interpolation polynomial $p(x)$ to the dataset and calculate the derivative approximation using $f_i^{(4)} = p^{(4)}(0)$ (Mathematica may be useful).

   Answer $f_i^{(4)} = \dfrac{1}{\Delta x^4}(f_{i-2} - 4f_{i-1} + 6f_i - 4f_{i+1} + f_{i+2})$

2. Derive the Crank-Nicolson time integration iteration using (1) Taylor series of displacement $a(t)$ and velocity $\dot{a}(t)$ with respect to time and the mean value approximation to the remainder containing the second time derivative, and (2) differential equation $m\ddot{a} + ka = 0$ written at the end points of the time interval of length Δt.

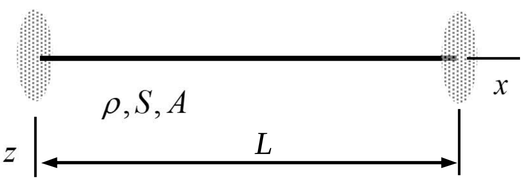
   Answer $\begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_i = \dfrac{1}{4+\alpha^2}\begin{bmatrix} 4-\alpha^2 & 4 \\ -4\alpha^2 & 4-\alpha^2 \end{bmatrix}\begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_{i-1}$ where $\alpha = \sqrt{\dfrac{k}{m}}\Delta t$



3. The bar shown is loaded by its own weight. Determine the displacements at the grid points 1 and 2 using the Finite Difference Method. Cross-sectional area A, Young's modulus E, and density ρ of the material are constants.

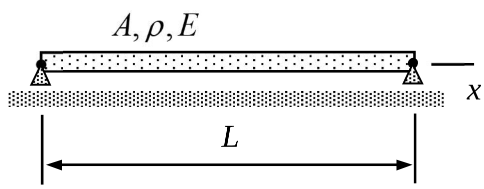
   Answer $u_0 = 0$, $u_1 = u_2 = -\dfrac{\rho g L^2}{E}$

4. Consider the string of tightening S and mass per unit length ρA shown. Use the Finite Difference Method with second order accurate central differences on a regular grid $i \in \{0,1,\ldots,n\}$ to find the angular velocities ω_k of the free vibrations using the solution trial $w_i = a(t)\sin(k\pi i / n)$ $k \in \{1,2\ldots,n-1\}$.



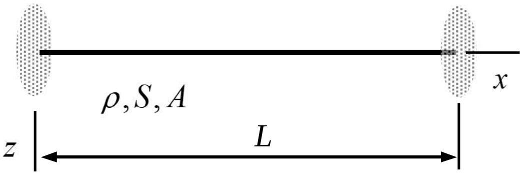
   Answer $\omega_k = \dfrac{2}{L}\sqrt{\dfrac{S}{A\rho}}n\sin(\dfrac{k\pi}{2n})$ $k \in \{1,2\ldots,n-1\}$ (exact $\omega_k = \dfrac{k\pi}{L}\sqrt{\dfrac{S}{A\rho}}$)

5. A bar is free to move in the horizontal direction as shown. Write the equation system $\mathbf{Ka} + \mathbf{M\ddot{a}} = 0$ given by the Finite Difference Method on a regular grid with $i \in \{0,1,2\}$. Also, determine the angular velocities and modes of the free vibrations. Cross-sectional area A, density ρ of the material, and Young's modulus E of the material are constants.



Answer $2\dfrac{EA}{L^2}\begin{bmatrix} L & -L & 0 \\ -2 & 4 & -2 \\ 0 & -L & L \end{bmatrix}\begin{Bmatrix} u_0 \\ u_1 \\ u_2 \end{Bmatrix} + \rho A \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \ddot{u}_0 \\ \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} = 0$ and $(\omega, \mathbf{A})_1 = (0, \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix})$

6. Consider the string of tightening S and mass per unit length ρA shown. First, use the Finite Difference Method with the second order accurate central differences on a regular grid $i \in \{0,1,2\}$ to find the equations of motion of the form $ka + m\ddot{a} = 0$. Second, write the iteration equation for a typical time-step of size Δt according to Crank-Nicolson method giving the values of displacement and velocity on the temporal grid.



Answer $8\dfrac{EA}{L^2}u_1 + \rho A \ddot{u}_1 = 0$, $\begin{Bmatrix} u \\ \Delta t \dot{u} \end{Bmatrix}_i = \dfrac{1}{4+\alpha^2}\begin{bmatrix} 4-\alpha^2 & 4 \\ -4\alpha^2 & 4-\alpha^2 \end{bmatrix}\begin{Bmatrix} u \\ \Delta t \dot{u} \end{Bmatrix}_{i-1}$ where $\alpha = \sqrt{8\dfrac{E}{\rho}}\dfrac{\Delta t}{L}$

Find the five point difference approximation to $f_i^{(4)}$ (fourth derivative) using the dataset $\{(-2\Delta x, f_{i-2}), (-\Delta x, f_{i-1}), (0, f_i), (\Delta x, f_{i+1}), (2\Delta x, f_{i+2})\}$. Use the Lagrange interpolation polynomial $p(x)$ to the dataset and calculate the derivative approximation using $f_i^{(4)} = p^{(4)}(0)$ (Mathematica may be useful).

Solution

Let us start with the Lagrange interpolation polynomials taking the value one at grid points and vanishing at all the other grid points of the dataset. The fourth derivatives of the fourth order polynomial are given by denominators multiplied by $4\times 3\times 2\times 1 = 24$ (fourth derivative of the nominator):

$$p_{i-2}(x) = \frac{(x+\Delta x)(x)(x-\Delta x)(x-2\Delta x)}{(-2\Delta x+\Delta x)(-2\Delta x)(-2\Delta x-\Delta x)(-2\Delta x-2\Delta x)} \quad \Rightarrow \quad p_{i-2}^{(4)}(0) = \frac{1}{\Delta x^4},$$

$$p_{i-1}(x) = \frac{(x+2\Delta x)(x)(x-\Delta x)(x-2\Delta x)}{(-\Delta x+2\Delta x)(-\Delta x)(-\Delta x-\Delta x)(-\Delta x-2\Delta x)} \quad \Rightarrow \quad p_{i-1}^{(4)}(0) = -\frac{4}{\Delta x^4},$$

$$p_i(x) = \frac{(x+2\Delta x)(x+\Delta x)(x-\Delta x)(x-2\Delta x)}{(0+2\Delta x)(0+\Delta x)(0-\Delta x)(0-2\Delta x)} \quad \Rightarrow \quad p_i^{(4)}(0) = \frac{6}{\Delta x^4},$$

$$p_{i+1}(x) = \frac{(x+2\Delta x)(x+\Delta x)(x)(x-2\Delta x)}{(\Delta x+2\Delta x)(\Delta x+\Delta x)(\Delta x)(\Delta x-2\Delta x)} \quad \Rightarrow \quad p_{i+1}^{(4)}(0) = -\frac{4}{\Delta x^4},$$

$$p_{i+2}(x) = \frac{(x+2\Delta x)(x+\Delta x)(x)(x-\Delta x)}{(2\Delta x+2\Delta x)(2\Delta x+\Delta x)(2\Delta x)(2\Delta x-\Delta x)} \quad \Rightarrow \quad p_{i+2}^{(4)}(0) = \frac{1}{\Delta x^4}.$$

The fourth derivative of the Lagrange interpolation polynomial $p(x)$ gives an approximation to derivative of the assumed continuous $f(x)$ (evaluated at the grid points to get the dataset) at $x_i = 0$

$$f_i^{(4)} = p^{(4)}(0) = p_{i-2}^{(4)}(0)\, f_{i-2} + p_{i-1}^{(4)}(0)\, f_{i-1} + p_i^{(4)}(0)\, f_i + p_{i+1}^{(4)}(0)\, f_{i+1} + p_{i+2}^{(4)}(0)\, f_{i+2} \quad \Rightarrow$$

$$f_i^{(4)} = \frac{1}{\Delta x^4}(f_{i-2} - 4 f_{i-1} + 6 f_i - 4 f_{i+1} + f_{i+2}).$$

Derive the Crank-Nicolson time integration iteration using (1) Taylor series of displacement $a(t)$ and velocity $\dot{a}(t)$ with respect to time and the mean value approximation to the remainder containing the second time derivative, and (2) differential equation $m\ddot{a}+ka=0$ written at the end points of the time interval of length Δt.

Solution

Taylor series of displacement $a(t)$ and velocity $\dot{a}(t)$ with respect to time with remainders containing the second time derivative are

$$a(t+\Delta t)=a(t)+\dot{a}(t)\Delta t+\frac{1}{2}\ddot{a}(\xi)\Delta t^2 \quad \text{and} \quad \dot{a}(t+\Delta t)=\dot{a}(t)+\ddot{a}(\xi)\Delta t\,,$$

where $\xi\in[t,t+\Delta t]$ is different in all its occurrences. Denoting the end points of the time interval $[t,t+\Delta t]$ and values of $a(t)$ by indices by $i-1$ and i and the mean value approximations to the second derivatives

$$a_i=a_{i-1}+\dot{a}_{i-1}\Delta t+\frac{1}{4}(\ddot{a}_i+\ddot{a}_{i-1})\Delta t^2 \quad \text{and} \quad \dot{a}_i=\dot{a}_{i-1}+\frac{1}{2}(\ddot{a}_i+\ddot{a}_{i-1})\Delta t\,.$$

Differential equation $m\ddot{a}+ka=0$ written at the end points of the time interval and index notation for the end points give

$$m\ddot{a}_{i-1}+ka_{i-1}=0 \quad \text{and} \quad m\ddot{a}_i+ka_i=0\,.$$

Elimination of second derivatives in the Taylor's series using the differential equations gives the forms

$$ma_i=ma_{i-1}+m\dot{a}_{i-1}\Delta t+\frac{1}{4}(m\ddot{a}_i+m\ddot{a}_{i-1})\Delta t^2=ma_{i-1}+m\dot{a}_{i-1}\Delta t-\frac{1}{4}(ka_i+ka_{i-1})\Delta t^2\,,$$

$$m\dot{a}_i=m\dot{a}_{i-1}+\frac{1}{2}(m\ddot{a}_i+m\ddot{a}_{i-1})\Delta t=m\dot{a}_{i-1}-\frac{1}{2}(ka_i+ka_{i-1})\Delta t\,.$$

In matrix form, containing a_i and $\dot{a}_i$ on the left-hand side and a_{i-1} and $\dot{a}_{i-1}$ on the right-hand side, the equations are

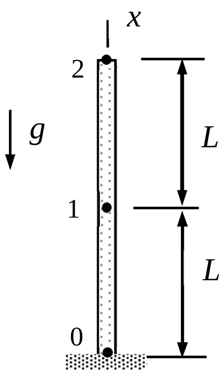
$$\begin{bmatrix} m+\frac{1}{4}k\Delta t^2 & 0 \\ \frac{1}{2}k\Delta t & \frac{1}{\Delta t}m \end{bmatrix}\begin{Bmatrix} a_i \\ \Delta t\dot{a}_i \end{Bmatrix}=\begin{bmatrix} m-\frac{1}{4}k\Delta t^2 & m \\ -\frac{1}{2}k\Delta t & \frac{1}{\Delta t}m \end{bmatrix}\begin{Bmatrix} a_{i-1} \\ \Delta t\dot{a}_{i-1} \end{Bmatrix}, \quad \alpha=\sqrt{\frac{k}{m}}\Delta t$$

or after left multiplication by the inverse of the matrix on the left-hand side

$$\begin{Bmatrix} a_i \\ \Delta t\dot{a}_i \end{Bmatrix}=\begin{bmatrix} m+\frac{1}{4}k\Delta t^2 & 0 \\ \frac{1}{2}k\Delta t & m/\Delta t \end{bmatrix}^{-1}\begin{bmatrix} m-\frac{1}{4}k\Delta t^2 & m \\ -\frac{1}{2}k\Delta t & m/\Delta t \end{bmatrix}\begin{Bmatrix} a_{i-1} \\ \Delta t\dot{a}_{i-1} \end{Bmatrix}, \quad \alpha=\sqrt{\frac{k}{m}}\Delta t$$

which gives finally the iteration

$$\begin{Bmatrix} a \\ \Delta t \dot{a} \end{Bmatrix}_i = \frac{1}{4+\alpha^2} \begin{bmatrix} 4-\alpha^2 & 4 \\ -4\alpha^2 & 4-\alpha^2 \end{bmatrix} \begin{Bmatrix} a \\ \Delta t \dot{a} \end{Bmatrix}_{i-1}, \quad \alpha = \sqrt{\frac{k}{m}} \Delta t .$$



The bar shown is loaded by its own weight. Determine the displacements at the grid points 1 and 2 using the Finite Difference Method. Cross-sectional area A, Young's modulus E, and density ρ of the material are constants.

Solution

The discrete equations for the Finite Difference Method on a regular grid and stationary case are given

$$\frac{k}{\Delta x^2}(a_{i-1} - 2a_i + a_{i+1}) + f' = 0 \quad \text{or} \quad \frac{k}{\Delta x}(a_{i-1} - 2a_i + a_{i+1}) + F = 0,$$

$$a_0 = \underline{a}_0 \quad \text{or} \quad \frac{k}{\Delta x}(a_1 - a_0) + \underline{F}_0 = 0 \quad \text{and} \quad a_n = \underline{a}_n \quad \text{or} \quad \frac{k}{\Delta x}(a_{n-1} - a_n) + F_n = 0.$$

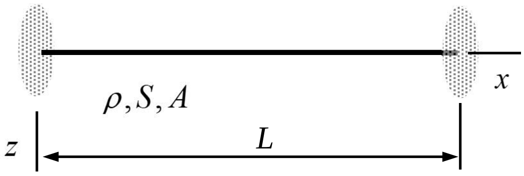
In the present case of a bar and regular grid with $i \in \{0,1,2\}$, $a = u$, $k = EA$, $f' = -\rho A g$, and $\Delta x = L$, the equations for the grid points are

$$u_0 = 0, \quad \frac{EA}{\Delta x^2}(u_0 - 2u_1 + u_2) - A\rho g = 0, \quad \text{and} \quad \frac{EA}{\Delta x}(u_1 - u_2) = 0.$$

Solution to the displacements are ($\Delta x = L$)

$$u_0 = 0, \; u_1 = u_2 = -\frac{\rho g L^2}{E}.$$

Consider the string of tightening S and mass per unit length ρA shown. Use the Finite Difference Method with second order accurate central differences on a regular grid $i \in \{0,1,\ldots,n\}$ to find the angular velocities ω_k of the free vibrations using the solution trial $w_i = a(t)\sin(k\pi i / n) \;\; k \in \{1,2\ldots,n-1\}$.



Solution

The trial solution is chosen in such manner that the zero displacement conditions at the end points are satisfied 'a priori'. Let us use the difference equation for the generic point inside the domain $i \in \{1,2\ldots,n-1\}$ (no external forces)

$$\frac{S}{\Delta x^2}(w_{i-1} - 2w_i + w_{i+1}) = \rho A \ddot{w}_i$$

to deduce the expression for $a(t)$ which is the unknown of the solution trial. Substituting the trial solution and using the trigonometric identity $\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$ (or letting Mathematica to do the manipulation)

$$w_{i-1} - 2w_i + w_{i+1} = a(t)\sin(k\pi\frac{i-1}{n}) - 2a(t)\sin(k\pi\frac{i}{n}) + a(t)\sin(k\pi\frac{i+1}{n}) \quad \Leftrightarrow$$

$$w_{i-1} - 2w_i + w_{i+1} = a(t)\sin(k\pi\frac{i}{n})2[-1+\cos(\frac{k\pi}{n})],$$

$$\ddot{w}_i = \ddot{a}(t)\sin(k\pi\frac{i}{n}).$$

Therefore, the difference equation for the generic point takes the form

$$[\ddot{a}(t) + \omega_k^2 a(t)]\sin(k\pi\frac{i}{n}) = 0 \quad \text{where} \quad \omega_k^2 = \frac{S}{\rho A \Delta x^2}2[1-\cos(\frac{k\pi}{n})].$$

The equation to hold, $a(t)$ should be the solution to $\ddot{a}(t) + \omega_k^2 a(t) = 0$ which is a linear combination of $\sin(\omega_k t)$ and $\cos(\omega_k t)$. As a conclusion, the angular velocity of the free vibration

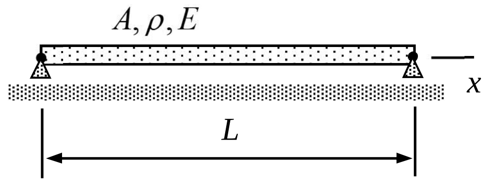
$$\omega_k = \frac{1}{L}n\sqrt{2\frac{S}{\rho A}[1-\cos(\frac{k\pi}{n})]} = \frac{2}{L}\sqrt{\frac{S}{A\rho}}n\sin(\frac{k\pi}{2n}). \quad \Leftarrow$$

Let us note that, in the limit $n \to \infty$,

$$\omega_k = \frac{k\pi}{L}\sqrt{\frac{S}{A\rho}},$$

which is the angular velocity given by the continuum model.

A bar is free to move in the horizontal direction as shown. Write the equation system $\mathbf{Ka}+\mathbf{M\ddot{a}}=0$ given by the Finite Difference Method on a regular grid with $i \in \{0,1,2\}$. Also, determine the angular velocities and modes of the free vibrations. Cross-sectional area A, density ρ of the material, and Young's modulus E of the material are constants.



Solution

The generic equation set for the model problems and the Finite Difference Method on a regular grid with the simplest possible difference approximations to the derivatives is given by

$$\frac{k}{\Delta x^2}(a_{i-1}-2a_i+a_{i+1})+f'=m'\ddot{a}_i \quad \text{or} \quad \frac{k}{\Delta x}(a_{i-1}-2a_i+a_{i+1})+F=0 \qquad t>0$$

$$a_0=\underline{a}_0 \;\text{ or }\; \frac{k}{\Delta x}(a_1-a_0)+F_0=0 \;\text{ and }\; a_n=\underline{a}_n \;\text{ or }\; \frac{k}{\Delta x}(a_{n-1}-a_n)+F_n=0 \qquad t>0$$

$$a_i=g_i \;\text{ and }\; \dot{a}_i=h_i \qquad t=0$$

In the bar application external forces vanish, $k=EA$, $m'=\rho A$, and $\Delta x = L/2$. Initial conditions do not matter in modal analysis. Equations for $i \in \{0,1,2\}$ simplify to

$$u_1-u_0=0\,, \quad \frac{EA}{\Delta x^2}(u_0-2u_1+u_2)=\rho A\ddot{u}_1\,, \quad a_1-a_2=0$$

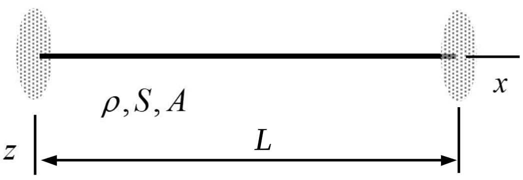
In solution methods for time dependent problem, algebraic equations are used to eliminate the displacements of the boundary points from the differential equation, which simplifies to

$$\ddot{u}_1=0\,.$$

Using the solution trial of the modal analysis $u_1 = Ae^{\mathrm{i}\omega t}$ gives the angular velocity value $\omega=0$ the corresponding mode being $A=1$ (say), so trial gives $u_1=1$ (not the solution to the problem but solution to the mode) and then with use of the algebraic equations $u_0=u_1=u_2=1$ so

$$(\omega, A)=(0,\begin{Bmatrix}1\\1\\1\end{Bmatrix})\,.$$

Consider the string of tightening S and mass per unit length ρA shown. First, use the Finite Difference Method with the second order accurate central differences on a regular grid $i \in \{0,1,2\}$ to find the equations of motion of the form $ka + m\ddot{a} = 0$. Second, write the iteration equation for a typical time-step of size Δt according to Crank-Nicolson method giving the values of displacement and velocity on the temporal grid.



Solution

The generic equation set for the model problems and the Finite Difference Method on a regular grid with the simplest possible difference approximations to the derivatives is given by

$$\frac{k}{\Delta x^2}(a_{i-1} - 2a_i + a_{i+1}) + f' = m'\ddot{a}_i \quad \text{or} \quad \frac{k}{\Delta x}(a_{i-1} - 2a_i + a_{i+1}) + F = 0 \quad t > 0$$

$$a_0 = \underline{a}_0 \quad \text{or} \quad \frac{k}{\Delta x}(a_1 - a_0) + F_0 = 0 \quad \text{and} \quad a_n = \underline{a}_n \quad \text{or} \quad \frac{k}{\Delta x}(a_{n-1} - a_n) + F_n = 0 \quad t > 0$$

$$a_i = g_i \quad \text{and} \quad \dot{a}_i = h_i \quad t = 0$$

In the string application external forces vanish, $k = S$, $m' = \rho A$, and $\Delta x = L/2$. Initial conditions do not matter in modal analysis. Equations for $i \in \{0,1,2\}$ simplify to

$$w_0 = 0, \quad \frac{S}{\Delta x^2}(w_0 - 2w_1 + w_2) = \rho A \ddot{w}_1, \text{ and } w_2 = 0 \quad t > 0$$

$$w_1 = g_1 \quad \text{and} \quad \dot{w}_1 = h_1 \quad t = 0$$

In solution methods for time dependent problem, algebraic equations are used to eliminate the displacements of the boundary points from the differential equation, so the initial value problem simplifies to

$$2\frac{S}{\Delta x^2} w_1 + \rho A \ddot{w}_1 = 0 \quad t > 0, \; w_1 = g \text{ and } \dot{w}_1 = h \quad t = 0.$$

With definition $w_1 = a$, time integration by Crank-Nicolson method is given by iteration

$$\begin{Bmatrix} a \\ \Delta t \dot{a} \end{Bmatrix}_i = \frac{1}{4+\alpha^2}\begin{bmatrix} 4-\alpha^2 & 4 \\ -4\alpha^2 & 4-\alpha^2 \end{bmatrix}\begin{Bmatrix} a \\ \Delta t \dot{a} \end{Bmatrix}_{i-1}, \quad \begin{Bmatrix} a \\ \Delta t \dot{a} \end{Bmatrix}_0 = \begin{Bmatrix} g \\ \Delta t h \end{Bmatrix}_0 \quad \text{where } \alpha = 2\sqrt{2\frac{S}{\rho A}}\frac{\Delta t}{L}.$$

LECTURE ASSIGNMENT 1

Polynomial interpolant $p(x)$ to dataset $\{\ldots,(x_{i-1}, f_{i-1}),(x_i, f_i),\ldots\}$ is the simplest possible continuous polynomial giving the prescribed values at the grid points, i.e., $p(x_i) = f_i$ for all indices. Use the Lagrange interpolation polynomial on a regular grid of spacing Δx to find the three point backward, central, and forward difference formulas for the first and second derivatives at point i.

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Backward difference approximations use dataset $\{(-2\Delta x, f_{i-2}),(-\Delta x, f_{i-1}),(0, f_i)\}$. Central difference approximations use dataset $\{(-\Delta x, f_{i-1}),(0, f_i),(\Delta x, f_{i+1})\}$. Forward difference approximations use dataset $\{(0, f_i),(\Delta x, f_{i+1}),(2\Delta x, f_{i+2})\}$. Interpolants to the datasets follow straightforwardly by using the Lagrange interpolation polynomial:

$$p_{\mathrm{b}}(x) = f_i + \frac{x}{2\Delta x}(f_{i-2} - 4f_{i-1} + 3f_i) + \frac{x^2}{2\Delta x^2}(f_{i-2} - 2f_{i-1} + f_i),$$

$$p_{\mathrm{c}}(x) = f_i + \frac{x}{2\Delta x}(-f_{i-1} + f_{i+1}) + \frac{x^2}{2\Delta x^2}(f_{i-1} - 2f_i + f_{i+1}),$$

$$p_{\mathrm{f}}(x) = f_i - \frac{x}{2\Delta x}(3f_i - 4f_{i+1} + f_{i+2}) + \frac{x^2}{2\Delta x^2}(f_i - 2f_{i+1} + f_{i+2}).$$

Thereafter, using the definitions $f_i' = p_{\mathrm{b}}'(0)$, $f_i'' = p_{\mathrm{b}}''(0)$, $f_i' = p_{\mathrm{c}}'(0)$, $f_i'' = p_{\mathrm{c}}''(0)$, $f_i'' = p_{\mathrm{f}}''(0)$, and $f_i'' = p_{\mathrm{f}}''(0)$ (origin is placed at point i):

	Backward	Central	Forward
f_i'	$\dfrac{f_{i-2} - 4f_{i-1} + 3f_i}{2\Delta x}$	$\dfrac{-f_{i-1} + f_{i+1}}{2\Delta x}$	$-\dfrac{3f_i - 4f_{i+1} + f_{i+2}}{2\Delta x}$
f_i''	$\dfrac{f_{i-2} - 2f_{i-1} + f_i}{\Delta x^2}$	$\dfrac{f_{i-1} - 2f_i + f_{i+1}}{\Delta x^2}$	$\dfrac{f_i - 2f_{i+1} + f_{i+2}}{\Delta x^2}$

LECTURE ASSIGNMENT 2

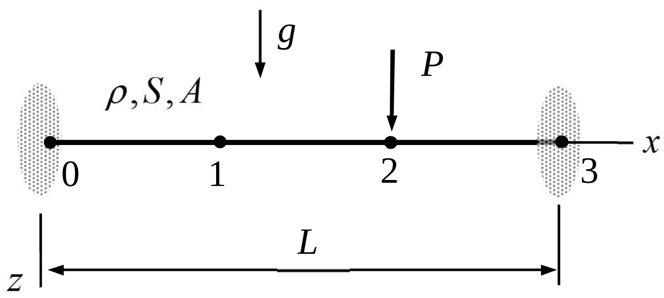
The continuum model for the string shown is given by equations

$$S\frac{d^2w}{dx^2}+\rho Ag=0 \quad x\in]0,\frac{2}{3}L] \ \text{ or } \ x\in]\frac{2}{3}L,L]$$

$$[\![S\frac{dw}{dx}]\!] + P = 0 \quad x=\frac{2}{3}L\,, \quad w=0 \quad x=0\,, \ \text{ and } \ w=0 \quad x=L\,.$$

Write the equations according to the Finite Difference Method using a regular grid $i\in\{0,1,2,3\}$, if the backward and forward difference approximations to the first derivative and the central difference approximation to the second derivative are given by

$$w_i'=\frac{1}{\Delta x}(w_i-w_{i-1})\,, \quad w_i'=\frac{1}{\Delta x}(w_{i+1}-w_i)\,, \ \text{ and } \ w_i''=\frac{1}{\Delta x^2}(w_{i-1}-2w_i+w_{i+1})\,.$$



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Finite difference Method uses the continuum model and difference approximations to derivatives at the grid points to discretize with respect to the spatial coordinate. Proper outcome requires a correct representation of the continuum model (obviously). Let us write the equations by considering the points one-by-one:

At point $i=0$, one uses the boundary condition

$$w=0: \quad w_0=0$$

At the regular point $i=1$, one uses the differential equation

$$S\frac{d^2w}{dx^2}+\rho Ag=0: \quad 9\frac{S}{L^2}(w_0-2w_1+w_2)+\rho Ag=0$$

At the non-regular point $i=2$ of the point force, one uses the jump condition

$$\left[\!\left[S\frac{dw}{dx}\right]\!\right]+P=0: \quad 3\frac{S}{L}(w_3-2w_2+w_1)+P=0$$

At point $i=3$, one uses the boundary condition

$$w=0: \quad w_3=0$$

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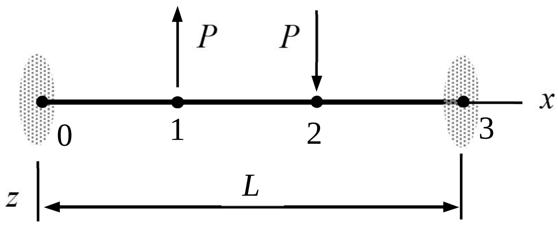
Home assignment 1

A string of length L and tightening S is loaded by point forces of magnitudes P as shown. If the ends are fixed and the initial geometry without loading is straight, find the transverse displacements at the grid points using the finite difference method (FDM) on a regular grid $i \in \{0,1,2,3\}$.



Solution

The boundary value problem is given by equilibrium equations for the regular interior points, jump conditions at the center point (non-regular point due to the point force), and boundary conditions for the end points

$$S\frac{d^2w}{dx^2}=0 \quad x\in]0,\frac{1}{3}L[\quad \text{or} \quad x\in]\frac{1}{3}L,\frac{2}{3}L[\quad \text{or} \quad x\in]\frac{2}{3}L,L[\,,$$

$$S\left[\!\left[\frac{dw}{dx}\right]\!\right]+P=0\,,\ [\![w]\!]=0 \quad x\in\{\frac{1}{3}L,\frac{2}{3}L\}\,, \text{ and } w(x)=0 \quad x\in\{0,L\}\,.$$

As the grid points inside the domain are at the locations of the point forces, the equations by the Finite Difference Method consist of displacement boundary conditions and jump conditions. Let us use the first order accurate backward and forward two-point difference approximations to the left and right derivatives, to get ($\Delta x = L/3$)

$$w_0=0\,,\ S(\frac{w_2-w_1}{\Delta x}-\frac{w_1-w_0}{\Delta x})-P=0\,,\ S(\frac{w_3-w_2}{\Delta x}-\frac{w_2-w_1}{\Delta x})+P=0\,, \text{and } w_3=0\,.$$

Using only the jump conditions with the known values of the boundary displacements

$$-\frac{S}{\Delta x}\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}\begin{Bmatrix}w_1\\ w_2\end{Bmatrix}+P\begin{Bmatrix}-1\\ 1\end{Bmatrix}=0 \quad \Leftrightarrow \quad \begin{Bmatrix}w_1\\ w_2\end{Bmatrix}=\frac{P\Delta x}{S}\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}^{-1}\begin{Bmatrix}-1\\ 1\end{Bmatrix}=\frac{1}{9}\frac{PL}{S}\begin{Bmatrix}-1\\ 1\end{Bmatrix}.$$

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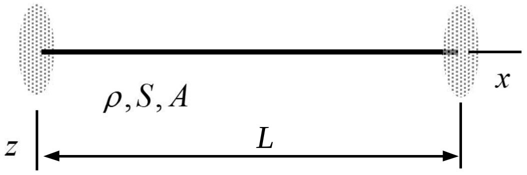
Home assignment 2

What is relative (numerical) error in the smallest frequency of the free vibrations $f_{\min}$ of the string shown if predicted by the Finite Difference Method and a regular spatial grid $i \in \{0,1,2,3\}$? Cross-sectional area A, density of the material ρ, and horizontal tightening S are constants. Use the three-point central difference approximation to the second derivative with respect to x. The smallest frequency given by the continuum model $f_{\min} = \sqrt{S/(\rho A)}/(2L)$.



Solution

The equations for the boundary points and for the two points inside the domain, as given by the 2:nd order accurate central difference approximation to the second derivative (with respect to *x*), are

$$w_0 = 0 \ , \ \frac{S}{\Delta x^2}(w_0 - 2w_1 + w_2) = \rho A \ddot{w}_1 \ , \ \frac{S}{\Delta x^2}(w_1 - 2w_2 + w_3) = \rho A \ddot{w}_2 \text{, and } \ w_3 = 0 \ .$$

In matrix notation and $\Delta x = L/3$, the equations for points 1 and 2 are (when the known displacements at the boundary points are used there)

$$\frac{S}{\Delta x^2}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} + \rho A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{w}_1 \\ \ddot{w}_2 \end{Bmatrix} = 0 \ . \quad \leftarrow$$

Frequencies and modes of the free vibrations follow with the trial solution $\mathbf{a} = \mathbf{A}\exp(\mathrm{i}\omega t)$. Using the notation $\lambda = \omega^2 \Delta x^2 \rho A / S$, the conditions for the possible angular velocity ω and mode $\mathbf{A}$ pairs takes the form

$$(\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix})\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0 \ .$$

A homogeneous linear equation system can yield a non-zero solution only if the matrix is singular. The condition implies that

$$\det\begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 - 1 = 0 \quad \Rightarrow \quad \lambda_1 = 1 \ \text{ or } \ \lambda_2 = 3 .$$

The angular velocities follow from the known relationship $\lambda = \omega^2 \Delta x^2 \rho A / S$ and frequency from $\omega = 2\pi f$. The smallest frequency is given by $\lambda_1 = 1$:

$\underline{f}_{\min} = \frac{3}{2\pi}\frac{1}{L}\sqrt{\frac{S}{\rho A}}$ the exact value being $f_{\min} = \frac{1}{2}\frac{1}{L}\sqrt{\frac{S}{\rho A}}$.

The relative error in the smallest value $\frac{\underline{f}_{\min} - f_{\min}}{f_{\min}} 100\% = (\frac{3}{\pi} - 1)100\% \approx -5\%$.

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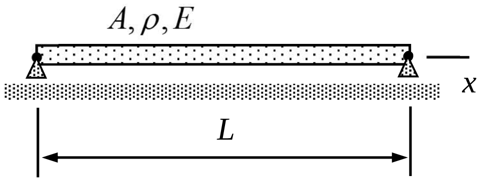
Home assignment 3

A bar is free to move in the horizontal direction as shown. At $t=0$, the bar moves with constant velocity $\dot{U}$ to the direction of the $x-$axis displacements being zeros. Use the Finite Difference Method on a regular grid with $i \in \{0,1,2\}$ and the Crank-Nicolson method with step size Δt to find the displacements and velocities at $t=\Delta t$. Cross-sectional area A, density ρ of the material, and Young's modulus E of the material are constants.



Solution

The continuum model for the problem is given by equations

$$EA\frac{\partial^2 u}{\partial x^2}=\rho A\frac{\partial^2 u}{\partial t^2} \quad x\in]0,L[\quad \text{and} \quad n_x EA\frac{\partial u}{\partial x}=0 \quad x\in\{0,L\} \quad t>0,$$

$$\frac{\partial u}{\partial t}=\dot{U} \text{ and } u=0 \quad x\in]0,L[\quad t=0.$$

In the Finite Difference Method, derivatives with respect to the spatial coordinates are replaced by difference approximations. Assuming the central three-point approximation to the second derivative and two point backward and forward approximations to the derivatives in the boundary conditions (difference approximations cannot use points outside the region), the equations for the points $i\in\{0,1,2\}$ become

$$\frac{EA}{\Delta x}(u_0-u_1)=0, \quad \frac{EA}{\Delta x^2}(u_0-2u_1+u_2)=\rho A\ddot{u}_1, \quad \frac{EA}{\Delta x}(u_2-u_1)=0 \quad t>0,$$

$$\dot{u}_1=\dot{U} \text{ and } u_1=0 \quad t=0$$

of which the first and the last are algebraic equations and do not require discretization in the temporal domain. Also, the initial conditions apply only to the points inside the domain, i.e., to point 1 whose equation is an ordinary second order differential equation in time. The algebraic equations can be used to eliminate u_2 and u_0 from the differential equation to get the initial value problem

$$\rho A\ddot{u}_1=0 \quad t>0, \quad \dot{u}_1=\dot{U} \quad t=0, \text{ and } u_1=0 \quad t=0$$

having the exact solution $u_1(t)=\dot{U}t$ and using the algebraic equations of the boundary conditions $u_0(t)=u_2(t)=u_1(t)=\dot{U}t$. Let us apply the Crank-Nicolson method to see how well it predicts the displacements:

$$\begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_i = \frac{1}{4+\alpha^2}\begin{bmatrix} 4-\alpha^2 & 4 \\ -4\alpha^2 & 4-\alpha^2 \end{bmatrix}\begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_{i-1} \quad \text{and} \quad \begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_0 = \begin{Bmatrix} g \\ \Delta t h \end{Bmatrix}.$$

With the present problem $a=u_1$ and $\alpha=\sqrt{k/m}\Delta t=0$ so the iteration simplifies to

$$\begin{Bmatrix} u_1 \\ \Delta t\dot{u}_1 \end{Bmatrix}_i = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ \Delta t\dot{u}_1 \end{Bmatrix}_{i-1} \quad \text{and} \quad \begin{Bmatrix} u_1 \\ \Delta t\dot{u}_1 \end{Bmatrix}_0 = \begin{Bmatrix} 0 \\ \Delta t\dot{U} \end{Bmatrix}$$

which gives after one step

$$\begin{Bmatrix} u_1 \\ \Delta t\dot{u}_1 \end{Bmatrix}_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ \Delta t\dot{u}_1 \end{Bmatrix}_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ \Delta t\dot{U} \end{Bmatrix} = \begin{Bmatrix} \Delta t\dot{U} \\ \Delta t\dot{U} \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} u_1 \\ \dot{u}_1 \end{Bmatrix}_1 = \begin{Bmatrix} \Delta t\dot{U} \\ \dot{U} \end{Bmatrix}.$$

The displacements and velocities of the boundary points follow from the two algebraic equations given by the boundary conditions in the same manner as with the exact solution with respect to time:

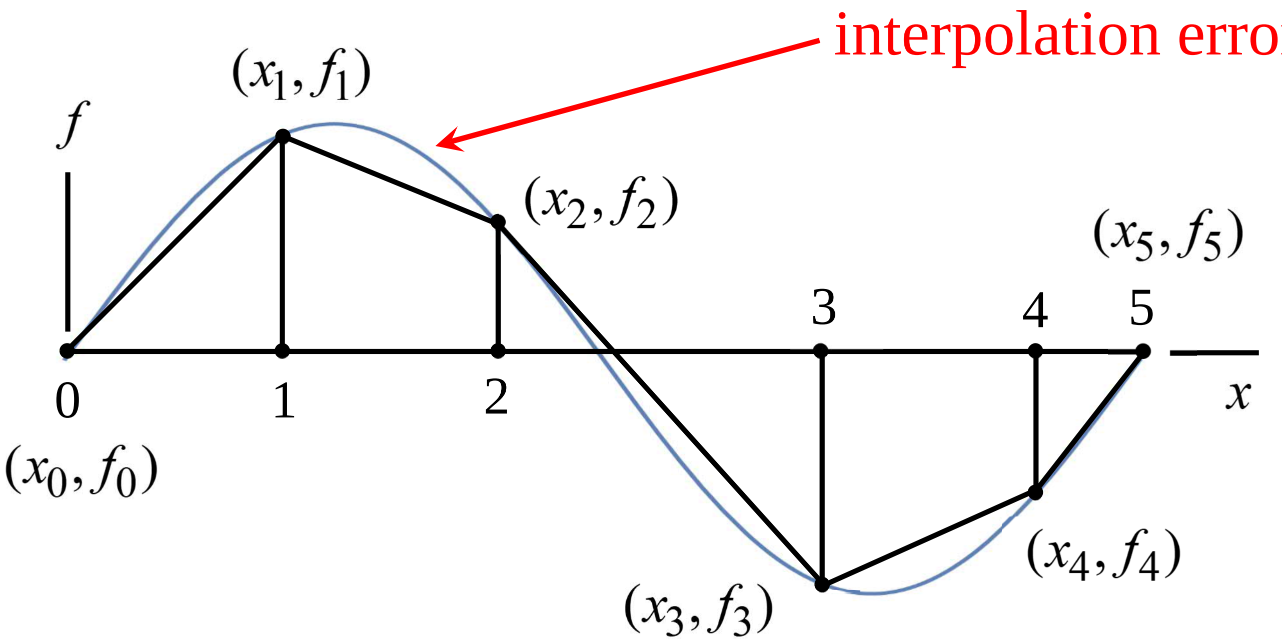
$$\begin{Bmatrix} u_0 \\ u_1 \\ u_2 \end{Bmatrix}_1 = \dot{U}\Delta t\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \dot{u}_0 \\ \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix}_1 = \dot{U}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}. \leftarrow$$

In this particular case, the outcome by the numerical time integration method coincides with the exact solution.

4 FINITE ELEMENT METHOD

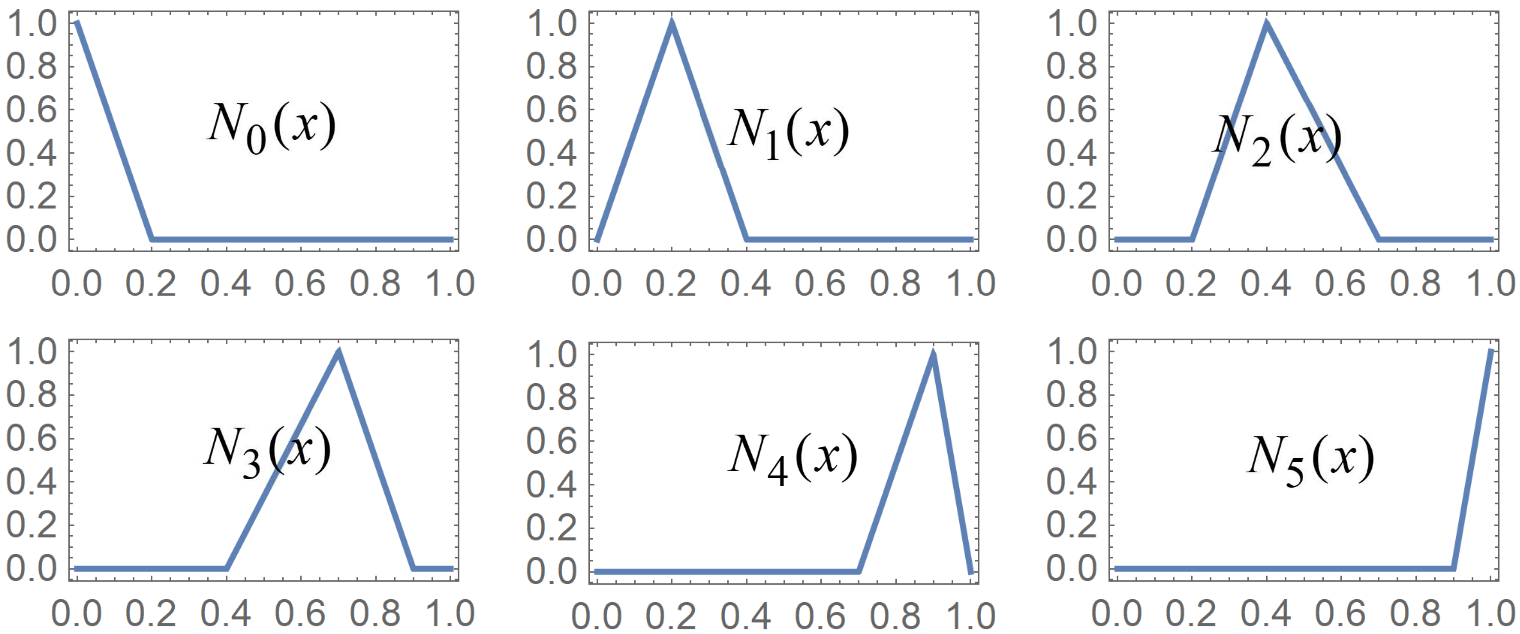
4.1 INTERPOLANT AND APPROXIMATION

Piecewise linear interpolant $p(x)$ to dataset $\{(x_0, f_0),(x_1, f_1),\ldots,(x_n, f_n)\}$ gives the simplest continuous polynomial representation. Assuming that the dataset is sampling of a continuous $f(x)$, $p(x)$ can also be considered as an approximation to $f(x)$.



SHAPE FUNCTIONS

In the Finite Element Method, linear interpolants to datasets $\{(x_0,\delta_{0i}),(x_1,\delta_{1i}),\ldots,(x_n,\delta_{ni})\}$, where $i \in \{0,1,\ldots,n\}$ and δ_{ji} is the Kronecker delta defined as $\delta_{ji}=1$ when $i=j$ and $\delta_{ji}=0 \quad i \neq j$, are called as the linear shape functions $N_i(x) \quad i \in \{0,1,\ldots,n\}$. As an example, the shape functions on the irregular grid $\{0,3,4,7,9,10\}/10$ of 6 nodes on $\Omega=[0,1]$ are:



The datasets $\{(x_0,\delta_{0i}),(x_1,\delta_{1i}),\ldots,(x_n,\delta_{ni})\}$ $i\in\{0,1,\ldots,n\}$ correspond to value 1 for grid point i the remaining being zeros. The $n+1$ interpolants to the $n+1$ dataset are denoted $N_i(x)$ and called as the shape functions. With this concept the linear interpolant to dataset $\{(x_0, f_0),(x_1, f_1),\ldots,(x_n, f_n)\}$ is given by

$$p(x)=\sum_{i\in\{0,1,\ldots,n\}} f_i N_i(x)$$

where f_i $i\in\{0,1,\ldots,n\}$ are the values at the grid points. In a typical line segment $[x_i,x_j]$, only the shape functions of grid points i and j are non-zeros the shape function and the interpolant expressions being ($x\in[x_i,x_j]$)

$$\begin{Bmatrix} N_i(x) \\ N_j(x) \end{Bmatrix}=\begin{bmatrix} 1 & 1 \\ x_i & x_j \end{bmatrix}^{-1}\begin{Bmatrix} 1 \\ x \end{Bmatrix} \text{ and } p(x)=\begin{Bmatrix} f_i \\ f_j \end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix} N_i(x) \\ N_j(x) \end{Bmatrix}=\begin{Bmatrix} f_i \\ f_j \end{Bmatrix}^{\mathrm{T}}\begin{bmatrix} 1 & 1 \\ x_i & x_j \end{bmatrix}^{-1}\begin{Bmatrix} 1 \\ x \end{Bmatrix}.$$

4.2 WEIGHTED RESIDUAL APPROXIMATION

Finding an approximation $g(x)$ to function $f(x)$ is one the basic tasks in numerical mathematics. In the Least Squares Method and Weighted Residual Methods, the grid point values g_i of approximation $g(x) = \sum g_i N_i(x) = \mathbf{N}^\mathbf{T}\mathbf{g}$ follow from the steps

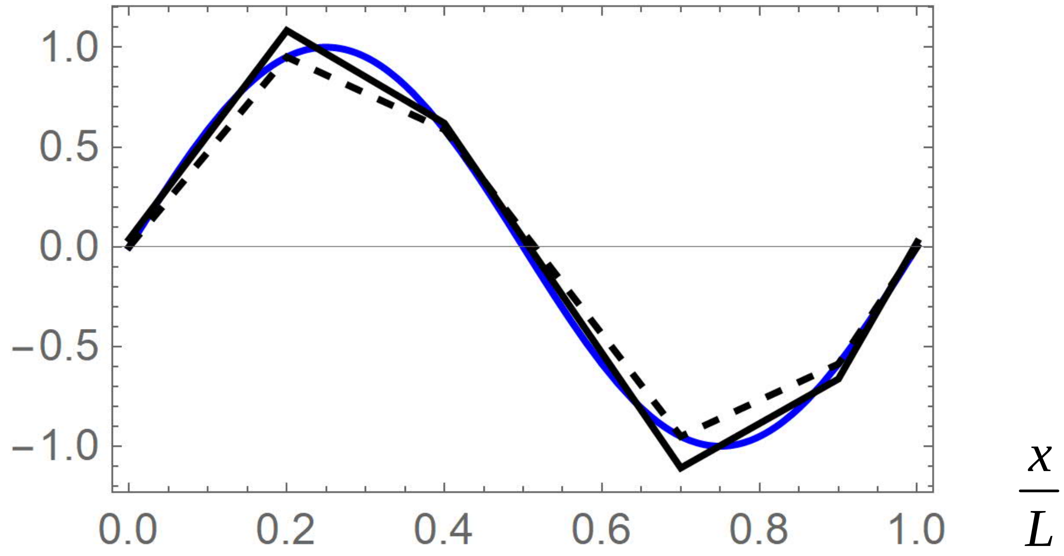
Distance: $\Pi(\mathbf{g}) = \dfrac{1}{2}\int_0^L (g - f)^2 dx = \dfrac{1}{2}\int_0^L (\mathbf{N}^\mathrm{T}\mathbf{g} - f)^2 dx$,

Minimizer: $\mathbf{Kg} - \mathbf{F} = \mathbf{0}$ where $\mathbf{K} = \int_0^L \mathbf{N}\mathbf{N}^\mathrm{T} dx$ and $\mathbf{F} = \int_0^L \mathbf{N} f dx$,

Nodal values: $\mathbf{g} = \mathbf{K}^{-1}\mathbf{F}$.

In practice, the nodal values $\mathbf{g}$ are solved from the linear equation system without matrix inversion (to avoid excess computational work). The method works in the same manner irrespective of the series approximation used.

Interpolant $p(x)$ to function $f(x)$ is accurate on the grid points but the interpolation error at the other points is not under control. Least squares method considers all points of the domain and control the error everywhere as well as possible.



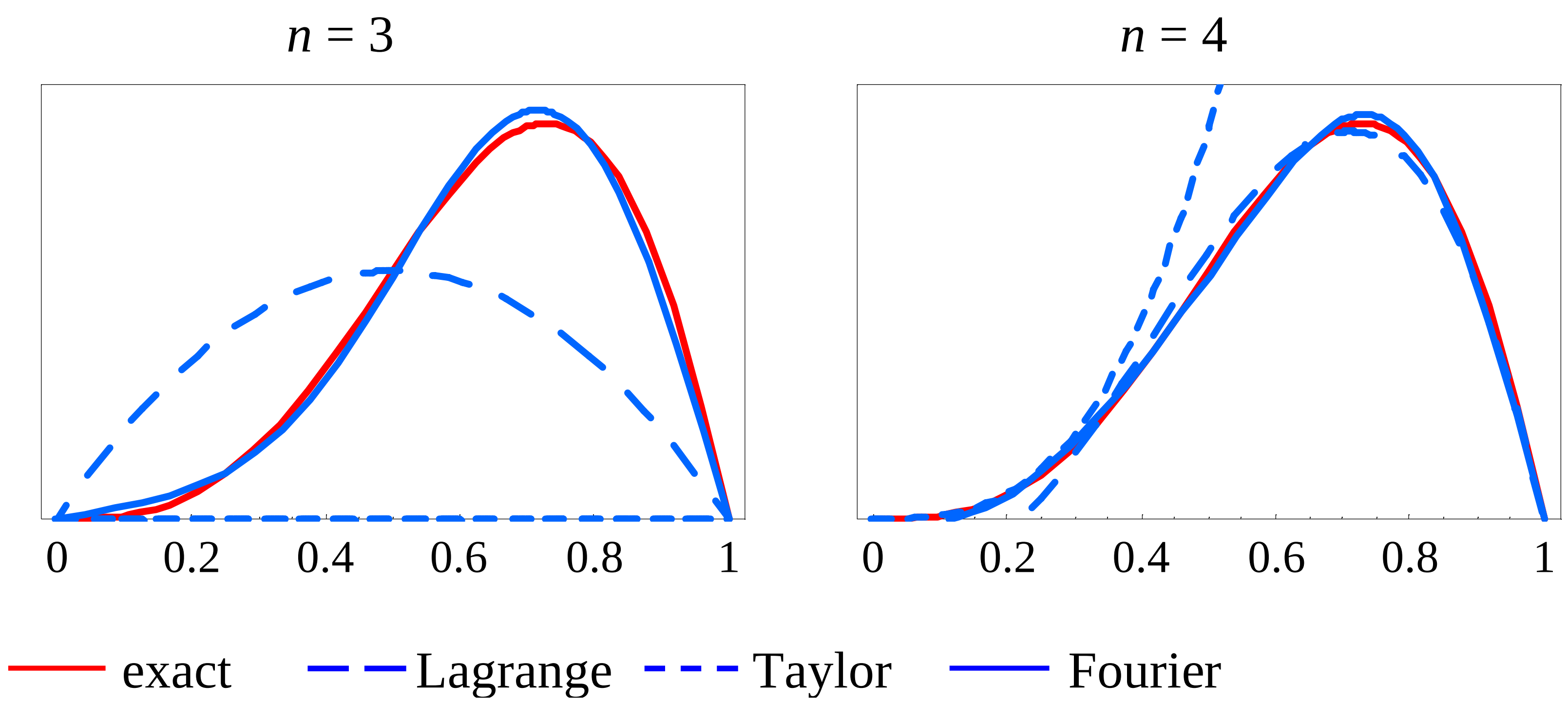
The figure above compares the interpolation (broken-black) and approximation (solid-black) of $f(x) = \sin(2\pi x / L)$ (solid-blue) on irregular grid $x / L \in \{0, 2, 4, 7, 9, 10\} / 10$ of 6 nodes on $\Omega = [0, L]$. Judging from the figure, approximation $g(x)$ gives a better fit to $f(x)$ than interpolant $p(x)$.

Least Squares Method is useful in various tasks in numerical mathematics. One of the applications is related with the condition for the minimum of Π, which can be written in the form

$$\int_0^L N_i R dx = 0 \quad i \in \{0,1,\ldots\}$$

where $R = g(x) - f(x)$ is called as the residual. In the weighted residual interpretation of the method, linear equations giving the values of the approximation are obtained as the weighted residuals with the shape functions. The idea extends to residuals of differential equations and is one of the starting points for the Finite Element Method for bar and string model problems.

EXAMPLE Find an approximation to $f(x) = 5x^2 \sin(\pi x)$ $x \in [0,1]$ by using: (a) Lagrange interpolation polynomial (n evenly spaced points), (b) Taylor series (at $x = 0$ and n terms), and (c) Fourier sine-series (n terms). Consider the cases $n = 3$ and $n = 4$.



Answer $f(x)$ and the approximations are shown in the figure.

Lagrange interpolation polynomial is continuous and it coincides with a function on a given set of points

recipe!

$$p = \mathbf{N}^{\mathrm{T}}\mathbf{p} \quad \text{where} \quad N_i = \Pi_{j\in\{1\ldots n\}\setminus i} \frac{x_j - x}{x_j - x_i} \quad \text{and} \quad p_i = f(x_i)$$

For the given set of points $\mathbf{x} \in \{0,1,2\}/2$:

$$\mathbf{p} = \{0 \;\; 5/4 \;\; 0\}^{\mathrm{T}} \text{ and } \mathbf{N}^{\mathrm{T}} = \{\frac{(1/2-x)(1-x)}{(1/2-0)(1-0)} \quad \frac{(0-x)(1-x)}{(0-1/2)(1-1/2)} \quad \frac{(0-x)(1/2-x)}{(0-1)(1/2-1)}\}$$

$$p(x) = \mathbf{N}^{\mathrm{T}}\mathbf{p} = 0 + \frac{5}{4}\frac{(0-x)(1-x)}{(0-1/2)(1-1/2)} + 0 = 5(x - x^2).$$

The given (shape) functions can be taken as monomials like $N_i = x^{(i-1)}$ without affecting the approximation, but then the recipe is worse from the numerical viewpoint.

A truncated Taylor series is a continuous polynomial whose derivatives coincide with those of the given function up to a point.

recipe!

$$t(x) = \mathbf{N}^{\mathrm{T}}\mathbf{t} \quad \text{where} \quad N_i = \frac{1}{i!}(x - x_0)^{(i)} \quad \text{and} \quad t_i = \left(\frac{d^i f}{dx^i}\right)_{x=x_0}$$

If the number of terms is chosen to be 3 and $x_0 = 0$,

$$\mathbf{t} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{and} \quad \mathbf{N} = \begin{Bmatrix} 1 \\ x \\ x^2/2 \end{Bmatrix} \quad \text{so} \quad t(x) = \mathbf{N}^{\mathrm{T}}\mathbf{t} = 0 + 0 + 0 = 0.$$

The given functions can be chosen in some other way without affecting the approximation, but then the recipe is not as good.

Sine-series is a linear combination of sine-functions where the coefficient are given by the Least-Squares Method (or Weighted Residuals)

recipe!

$$s(x) = \mathbf{N}^{\mathrm{T}}\mathbf{s} \quad \text{where} \quad N_i = \sqrt{\frac{2}{L}}\sin(i\pi\frac{x}{L}) \quad \text{and} \quad \mathrm{s}_i = \int_0^L f N_i dx$$

If the number of terms is chosen as 3, the series becomes

$$\mathbf{s} = \frac{\sqrt{2}}{\pi^2}\{5(\frac{\pi^2}{2} - \frac{1}{4}) \quad -\frac{40}{9} \quad \frac{15}{16}\}^{\mathrm{T}} \quad \text{and} \quad \mathbf{N}^{\mathrm{T}} = \sqrt{2}\{\sin(\pi x) \quad \sin(2\pi x) \quad \sin(3\pi x)\}$$

$$s(x) = \mathbf{N}^{\mathrm{T}}\mathbf{s} = \frac{2}{\pi^2}[5(\frac{\pi^2}{2} - \frac{1}{4})\sin(\pi x) - \frac{40}{9}\sin(2\pi x) + \frac{15}{16}\sin(3\pi x)].$$

The criterion for the parameters produces a good approximation. Clearly, considering all points of the domain in the recipe may be a good idea!

APPROXIMATIONS TO DERIVATIVES

Linear interpolation and shape functions give an alternative way to find difference stencils for derivatives at the interior points $i \in \{1, 2, \ldots, n-1\}$. The weighted average of a derivative using N_i is just interpreted as an approximation to derivative at point i (multiplied by Δx)

Term	Weighted residual	Stencil
a	$\int_0^L N_i a dx$	$\frac{\Delta x}{6}(a_{i-1} + 4a_i + a_{i+1})$
$\frac{\partial a}{\partial x}$	$\int_0^L N_i \frac{\partial a}{\partial x} dx$	$\frac{1}{2}(a_{i+1} - a_{i-1})$
$\frac{\partial^2 a}{\partial x^2}$	$-\int_0^L \frac{\partial N_i}{\partial x} \frac{\partial a}{\partial x} dx$	$\frac{1}{\Delta x}(a_{i-1} - 2a_i + a_{i+1})$

The stencils by the weighted residuals with N_i and linear interpolant to a on the regular grid coincide with the 2:nd order accurate central finite differences for the first and second derivatives. For the 0:th derivative (function itself)

$$\int_0^L N_0 a dx = \int_{x_0}^{x_0+\Delta x} N_0 a dx = \frac{\Delta x}{6}(2a_0 + a_1).$$

$$\int_0^L N_i a dx = \int_{x_i-\Delta x}^{x_i+\Delta x} N_i a dx = \frac{\Delta x}{6}(a_{i-1} + 4a_i + a_{i+1}) \quad i \in \{1, 2, \ldots, n-1\},$$

$$\int_0^L N_n a dx = \int_{x_0}^{x_0+\Delta x} N_n a dx = \frac{\Delta x}{6}(a_{n-1} + 2a_n).$$

For the first derivative

$$\int_0^L N_0 \frac{\partial a}{\partial x} dx = \int_{x_0}^{x_0+\Delta x} N_0 \frac{\partial a}{\partial x} dx = \frac{1}{2}(a_0 - a_1)$$

$$\int_0^L N_i \frac{\partial a}{\partial x} dx = \int_{x_i - \Delta x}^{x_i + \Delta x} N_i \frac{\partial a}{\partial x} dx = \frac{1}{2}(a_{i+1} - a_{i-1}) \quad i \in \{1, 2, \ldots, n-1\},$$

$$\int_0^L N_n \frac{\partial a}{\partial x} dx = \frac{1}{2}(a_n - a_{n-1}).$$

Weighted residual approximation to the second derivative uses an integral identity for a piecewise linear $a(x)$

$$-\int_0^L \frac{\partial N_i}{\partial x}\frac{\partial a}{\partial x} dx = -\int_{x_i - \Delta x}^{x_i + \Delta x} (\frac{\partial N_i}{\partial x}\frac{\partial a}{\partial x}) dx = \frac{1}{\Delta x}(a_{i-1} - 2a_i + a_{i+1}) \quad i \in \{1, 2, \ldots, n-1\}.$$

The derivation of the bar and string models indicate that at points where derivative is not continuous, the second derivative in the equation of motion needs to be replaced by a jump condition. Integral identity

$$\sum_{i \in \{1,2,\ldots,n-1\}} N_i [[\frac{\partial a}{\partial x}]] + N_0 (\frac{\partial a}{\partial x})_0 - N_n (\frac{\partial a}{\partial x})_n = -\int_0^L \frac{\partial N_i}{\partial x}\frac{\partial a}{\partial x} dx$$

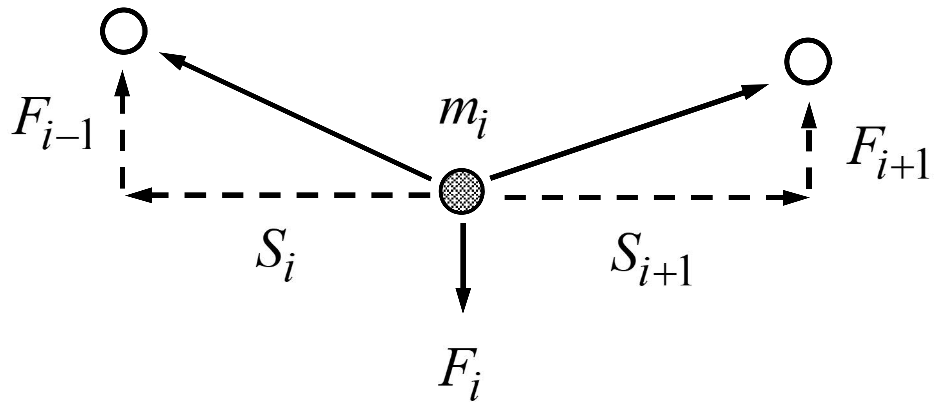
indicates that the jump conditions can be calculated as a weighted average using the derivative of the shape function as the weight. Jump bracket is a shorthand notation for the difference of the limit values $[\![a(x_i)]\!] = \lim_{\varepsilon\to 0}[a(x_i+\varepsilon) - a(x_i-\varepsilon)]$.

4.3 PRINCIPLE OF VIRTUAL WORK

Principle of virtual work $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} + \delta W^{\text{ine}} = 0 \quad \forall \delta a$ is an alternative representation of the equations of Particle Surrogate Method using the concept of work of forces acting on a particle. The principle holds also at the limit $n \to \infty$ and $\Delta x = L/n$:

Virtual work	**Particle**	**Continuum**
δW^{int}	$-\sum_{e \in P} \left(\frac{\Delta \delta a_e}{\Delta x} k' \frac{\Delta a_e}{\Delta x} \right) \Delta x$	$-\int_{\Omega} \left(\frac{\partial \delta a}{\partial x} k' \frac{\partial a}{\partial x} \right) dx$
δW^{ext}	$\sum_{i \in I} \left(\delta u_i \frac{F_i}{\Delta x} \right) \Delta x$	$\int_{\Omega} (\delta a f') dx + \sum_{I} \delta a F$
δW^{ine}	$-\sum_{i \in I} \left(\delta a_i \frac{m}{\Delta x} \ddot{a}_i \right) \Delta x$	$-\int_{\Omega} \left(\delta a m' \frac{\partial^2 a}{\partial t^2} \right) dx$

Let us start with the Particle Surrogate Method for a string with $i \in I = \{0,1,\ldots,n\}$, denote particle spacing by Δx, interacting particle pairs $p \in P \subset I \times I$, and choose $\delta w_i = 0$ whenever $w_i = \underline{w}_i$ (known), and denote $\Delta w_p = w_i - w_j$ when $p = (i, j)$.



The sum of works done by the internal, external and inertia forces on the displacement δw_i (fixed particles cannot move so $\delta w_i = 0$)

External forces: $\delta W^{\text{ext}} = \sum_{i \in I} \delta w_i (F_i + \rho A \Delta x g)$

Inertia forces: $\delta W^{\text{ine}} = -\sum_{i \in I} \delta w_i m_i \ddot{w}_i$

Internal forces: $\delta W^{\text{int}} = -\sum_{p\in P} \delta\Delta w_p \frac{S}{\Delta x} \Delta w_p$

External part is obvious as the sum of works of external forces acting on the particles. Inertia part uses the inertia force interpretation of acceleration term which is moved to the left hand side of (formally) equilibrium equations of particles. The internal part follows with some manipulations: particle i interacts with the neighbors $i-1$ and $i+1$ only. Therefore, virtual work of the internal forces (all particles accounted for as possible conditions $\delta w_i = 0$ can be applied after manipulations)

$$-\delta W^{\text{int}} = \delta w_0 F_1 + \delta w_1 (F_0 + F_3) + \delta w_2 (F_1 + F_3) + \ldots + \delta w_n F_{n-1}.$$

Substituting expressions $F_{i-1} = S(w_i - w_{i-1}) / \Delta x$ and $F_{i+1} = S(w_i - w_{i+1}) / \Delta x$ for the left and right neighbour interactions and rearranging

$$-\delta W^{\text{int}} \frac{\Delta x}{S} = (\delta w_1 - \delta w_0)(w_1 - w_0) + \ldots + (\delta w_n - \delta w_{n-1})(w_n - w_{n-1})$$

and, finally, using the notation $\Delta w_p = w_i - w_j$

$$\delta W^{\text{int}} = -\sum_{p\in P} \delta\Delta w_p \frac{S}{\Delta x} \Delta w_p. \quad \Leftarrow$$

Principle of virtual work and the virtual work expressions give a concise reprentation of the string and bar equations of the Particle Surrogate Method. Various different boundary conditions can be included by modification of the expression using the physical work interpretation.

Finite Element Method uses the linit expressions $n \to \infty$ and $\Delta x = L / n$ corresponding to the continuum model. There, linear shape functions are used for a piecewise linear approximation and the weight function is chosen to be the shape functions of the free points get the equation of the motion. At the fixed points, equation $w_i - \underline{w}_i = 0$ is used instead.

4.4 FINITE ELEMENT METHOD

Finite Element Method (FEM) is a numerical technique for solving differential equations. If applied to the model problems on a regular grid of points on the spatial domain

Interior $\frac{k}{\Delta x}(a_{i-1} - 2a_i + a_{i+1}) + F_i + f'\Delta x = m'\frac{\Delta x}{6}(\ddot{a}_{i-1} + 4\ddot{a}_i + \ddot{a}_{i+1}) \quad i \in \{1, 2, \ldots, n-1\}$

Boundary $\frac{k}{\Delta x}(a_1 - a_0) + F_0 + \frac{\Delta x}{2}f' - m'\frac{\Delta x}{6}(2\ddot{a}_0 + \ddot{a}_1) = 0$ or $a_0 = \underline{a}_0$,

Boundary $\frac{k}{\Delta x}(a_{n-1} - a_n) + F_n + \frac{\Delta x}{2}f' - m'\frac{\Delta x}{6}(2\ddot{a}_n + \ddot{a}_{n-1}) = 0$ or $a_n = \underline{a}_n$,

Initial $a_i - g_i = 0$ and $\dot{a}_i - h_i = 0$ (for ODE:s of the set).

Then, the outcome is a set of ordinary differential equations of the same type as by the Particle Surrogate Method.

In FEM, the starting point is the weighted residual expression implied by the principle of virtual work. Using a linear piecewise approximation to the transverse displacement in spatial grid, selecting $\delta a = N_i$, considering the displacement values $w_i(t)$ as functions of time, and assuming constant properties, for $i \in \{0,1,\ldots,n\}$

$$-\int_{\Omega} (\frac{\partial N_i}{\partial x} S \frac{\partial w}{\partial x}) dx + \sum_I N_i F - \int_{\Omega} N_i \rho A (g - \frac{\partial^2 w}{\partial t^2}) dx = 0 \quad \text{or} \quad w_i - \underline{w} = 0 \quad t > 0,$$

$$\int_{\Omega} N_i (w - g) dx = 0 \quad \text{and} \quad \int_{\Omega} N_i (\frac{\partial w}{\partial t} - h) dx = 0 \quad t = 0.$$

The integral equation and the corresponding initial conditions are used at all points where displacement is not known. Considering a regular grid, a piecewise linear approximation to w in terms of the nodal displacements, and assuming that g and h of initial conditions are of the same form as the approximation (tacitly accepting the possible interpolation error due to the simplification).

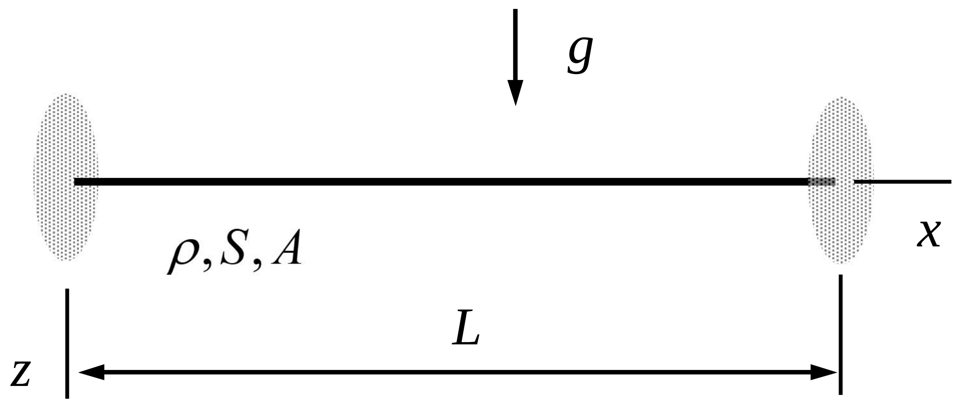
$$\frac{S}{\Delta x}(w_{i-1} - 2w_i + w_{i+1}) + \rho A g \Delta x = \rho A \frac{\Delta x}{6}(\ddot{w}_{i-1} + 4\ddot{w}_i + \ddot{w}_{i+1}) \quad i \in \{1, 2, \ldots, n-1\},$$

$$S(\frac{w_1 - w_0}{\Delta x}) + F_0 + \frac{\Delta x}{2}\rho A g - \rho A \frac{\Delta x}{6}(2\ddot{w}_0 + \ddot{w}_1) = 0 \quad \text{or} \quad w_0 = \underline{w}_0,$$

$$S(\frac{w_{n-1} - w_n}{\Delta x}) + F_n + \frac{\Delta x}{2}\rho A g - \rho A \frac{\Delta x}{6}(2\ddot{w}_n + \ddot{w}_{n-1}) = 0 \quad \text{or} \quad w_n = \underline{w}_n,$$

$w_i - g_i = 0$ and $\dot{w}_i - h_i = 0$ (for Ordinary Differential Equations).

EXAMPLE A string of length L, tightening S, cross section area A, and density ρ, and is loaded by its own weigh. If the ends are fixed and the initial geometry without loading is straight, find the solution to the transverse displacement by using the Finite Element Method and a regular grid of points $i \in \{0,1,\ldots,n\}$.



Answer $w_i = \frac{\rho A g L^2}{2S}\frac{i(n-i)}{n^2}$ (limit solution $w(x) = \frac{\rho A g}{2S}x(L-x)$)

The algebraic equations according to the Finite Element Method, n elements of the same size $\Delta x = L/n$, and regular node numbering $i = \{0,1,\ldots,n\}$ are given by

$$\frac{S}{\Delta x}(w_{i-1} - 2w_i + w_{i+1}) + \rho Ag\Delta x = 0 \quad i = \{1,2\ldots,n-1\} \quad \text{and} \quad w_i = 0 \quad i = \{0,n\}.$$

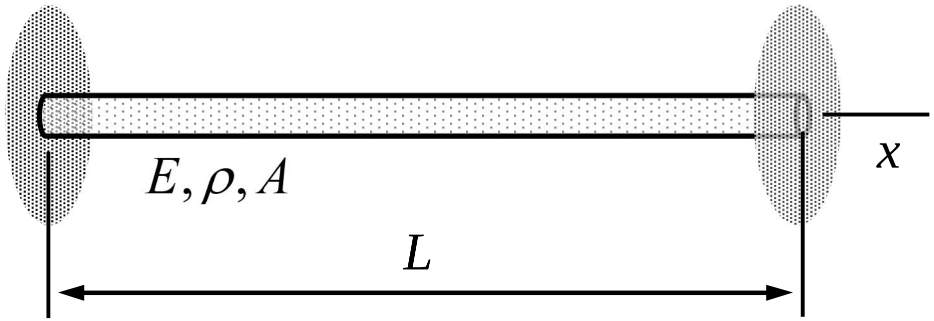
The generic solution to the difference equations consists of the generic solution to the homogeneous equation (trial solution $w_i = ar^i$) and a particular solution (trial solution $w_i = ai^2$):

$$w_i = a + bi - \frac{\rho Ag\Delta x^2}{2S} i^2.$$

Using the equations of the boundary points $w_0 = w_n = 0$ for the constants a and b

$$w_i = \frac{\rho AgL^2}{2S} \frac{i(n-i)}{n^2}.$$

EXAMPLE Write the equations of motion for the free vibrations of the bar shown by using the Finite Element Method and determine the frequencies and the corresponding modes of free vibrations. Use the matrix formulation on a regular grid $i \in \{0,1,2,3\}$. Material properties E, ρ and cross-sectional area A are constants.



Answer $(f_1, \mathbf{A}_1) = (\frac{1}{2\pi}\sqrt{\frac{6}{5}\frac{k}{m}}, \begin{Bmatrix} 1 \\ 1 \end{Bmatrix})$ and $(f_2, \mathbf{A}_2) = (\frac{1}{2\pi}\sqrt{\frac{k}{m}}, \begin{Bmatrix} 1 \\ -1 \end{Bmatrix})$

In bar problem $k = EA$, $m' = \rho A$, and external forces vanish. Equations for the points $i \in \{0,1,2,3\}$ are $u_0 = 0$ and $u_3 = 0$

$$\frac{EA}{\Delta x}(u_0 - 2u_1 + u_2) - \rho A \Delta x \frac{1}{6}(4\ddot{u}_0 + 4\ddot{u}_1 + \ddot{u}_2) = 0 \ ,$$

$$\frac{EA}{\Delta x}(u_1 - 2u_2 + u_3) - \rho A \Delta x \frac{1}{6}(4\ddot{u}_1 + 4\ddot{u}_2 + \ddot{u}_3) = 0$$

In matrix notation, when the known displacements at the boundary points $i \in \{0,3\}$ are used to simplify equations of points $i \in \{1,2\}$

$$\frac{EA}{\Delta x}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \rho A \Delta x \frac{1}{6}\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} = 0 . \quad \Leftarrow$$

With the trial solution $\mathbf{u} = \mathbf{A}e^{\mathrm{i}\omega t}$, the algebraic equations for the angular velocities ω and the corresponding modes $\mathbf{A}$ takes the form

$$\left(\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\right)\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0 \quad \text{where} \quad \lambda = \omega^2 \frac{\rho \Delta x^2}{6E}.$$

The possible values of λ are given by

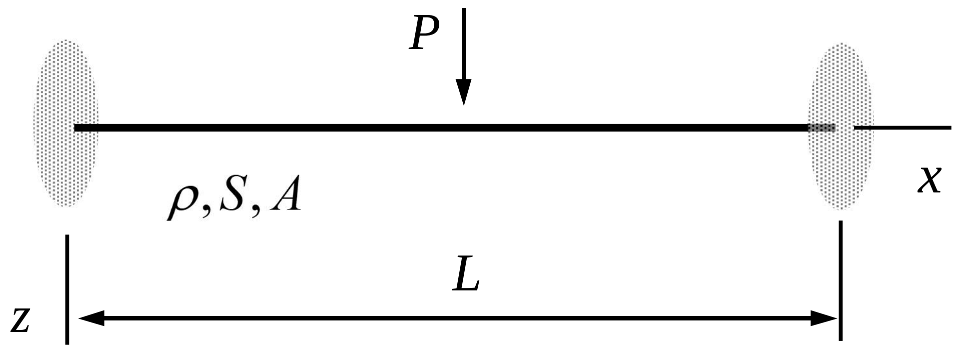
$$\det\begin{bmatrix} 2-4\lambda & -1-\lambda \\ -1-\lambda & 2-4\lambda \end{bmatrix} = (2-4\lambda)^2 - (1+\lambda)^2 = 0 \quad \Leftrightarrow \lambda = \frac{1}{5} \quad \text{or} \quad \lambda = 1.$$

The corresponding modes follow from the linear equations when the values of λ are substituted there one at a time

$$\lambda_1 = \frac{1}{5}: \; \left(\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \frac{1}{5}\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\right)\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \frac{6}{5}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0 \;\Rightarrow\; \mathbf{A}_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix},$$

$$\lambda_2 = 1: \; \left(\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - 1\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\right)\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 2\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0 \;\Rightarrow\; \mathbf{A}_2 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}.$$

EXAMPLE A string of length L, tightening S, cross-sectional area A, and density ρ, is loaded by a point force P at its center point. If the ends are fixed and the initial geometry without loading is straight, find the solution to the transverse displacement as function of x using the Finite Element Method on a regular grid of three points $i \in \{0,1,2\}$.



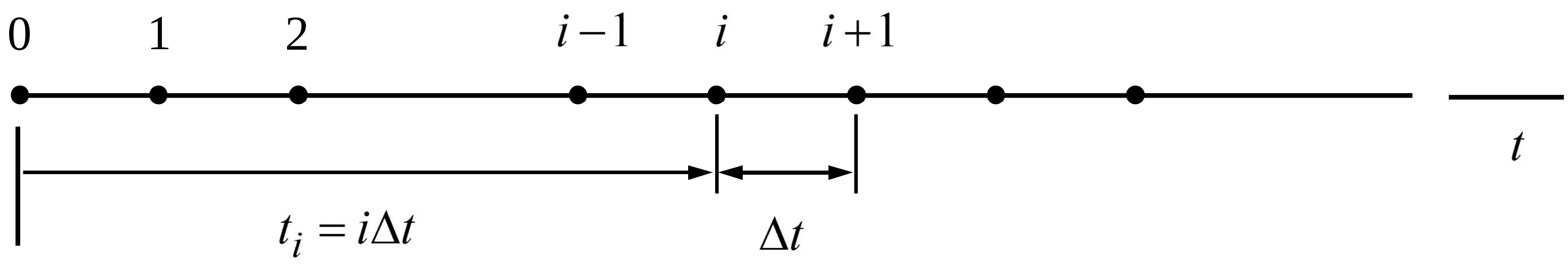
Answer $w_1 = \dfrac{PL}{4S}$

In the string problem $a_i = w_i$, $k = S$, $F_1 = P$, and $\Delta x = L/2$. The equations by the Finite Element Method considers also the possibility of points forces. Therefore

$$\frac{S}{\Delta x}(w_0 - 2w_1 + w_2) + P = 0, \quad w_0 = 0, \text{ and } w_2 = 0 \quad \Rightarrow \quad w_1 = \frac{PL}{4S}.$$

4.5 TIME INTEGRATION (DG)

In the one-step DG (Discontinuous Galerkin) method, the solution is sought step-by-step in the same manner as with the CN (Crank-Nicolson) method. Derivation of the method is, however, based on a polynomial approximation and the weighted residuals for the differential equations.



As the temporal domain for an initial value problem does not have an upper bound (strictly speaking). Also, the length of the intervals can be chosen to match the behavior of the solution (small steps for the rapid changes).

TIME INTEGRATION

Method	Iteration $i \in \{1, 2, \ldots\}$	Initial $i = 0$
EX	$\begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_i = \begin{bmatrix} \cos\alpha & \alpha^{-1}\sin\alpha \\ -\alpha\sin\alpha & \cos\alpha \end{bmatrix} \begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_{i-1}$	$\begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_0 = \begin{Bmatrix} g \\ \Delta th \end{Bmatrix}$
CN	$\begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_i = \dfrac{1}{4+\alpha^2} \begin{bmatrix} 4-\alpha^2 & 4 \\ -4\alpha^2 & 4-\alpha^2 \end{bmatrix} \begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_{i-1}$	$\begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_0 = \begin{Bmatrix} g \\ \Delta th \end{Bmatrix}$
DG	$\begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_i = \dfrac{2}{12+\alpha^4} \begin{bmatrix} 6-3\alpha^2 & 6-\alpha^2 \\ -6\alpha^2 & 6-3\alpha^2 \end{bmatrix} \begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_{i-1}$	$\begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_0 = \begin{Bmatrix} g \\ \Delta th \end{Bmatrix}$

The methods coincide at the limit of vanishing step-size when $\alpha = \sqrt{\dfrac{k}{m}}\Delta t \to 0$.

MATRIX REPRESENTATION

Representing the unknown displacements by column matrix $\mathbf{a}(t)$, coefficients of $\ddot{\mathbf{a}}(t)$ by square mass matrix $\mathbf{M}$, coefficients of $\mathbf{a}(t)$ by square stiffness matrix $\mathbf{K}$, and the external loading by column matrix $\mathbf{F}$, the second order initial value problems by Particle Surrogate Method, Finite Difference Method, and the Finite Element Method are given by

$$-\mathbf{K}\mathbf{a} + \mathbf{F} = \mathbf{M}\ddot{\mathbf{a}} \quad t > 0 \qquad \mathbf{a} = \mathbf{g} \;\text{ and }\; \dot{\mathbf{a}} = \mathbf{h} \quad t = 0 .$$

The column matrices $\mathbf{g}$ and $\mathbf{h}$ represent the initial positions and velocities of the free particles. Matrix representation is the concise starting point for

(1) mode analysis for frequencies and modes of free vibrations

(2) displacement solutions based on the frequencies and modes

(3) step-by-step time integration methods on temporal grid of time instants 🡐

DERIVATION OF THE METHOD

Polynomial approximation $\mathbf{a}(t) = \boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1 t + \ldots$ for the typical time-interval $t \in [t_{i-1}, t_i]$ where $t_i = t_{i-1} + \Delta t$. The lowest order method is given by a linear approximation

$$\mathbf{a}(t) = \boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1 t \,.$$

Weighted residual method for the second order ordinary differential equations with the shape functions $p = 1$ and $p = t$ of the approximation as the weights in

$$\int_{t_{i-1}}^{t_i} p(\mathbf{M}\ddot{\mathbf{a}} + \mathbf{K}\mathbf{a} - \mathbf{F})dt + [p\mathbf{M}(\dot{\mathbf{a}} - \mathbf{h})]_{i-1} - [\dot{p}\mathbf{M}(\mathbf{a} - \mathbf{g})]_{i-1} = 0 \,.$$

Solving the equations for $\boldsymbol{\alpha}_0$, $\boldsymbol{\alpha}_1$ and use of $\mathbf{a}_i = \boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1 \Delta t$ and $\dot{\mathbf{a}}_{i+1} = \dot{\mathbf{a}}_i$ gives the typical step of the Discontinuous Galerkin method for a second order initial values problem. The same recipe applies with more terms.

According to the recipe for the simplest DG time integration step to get $\{\mathbf{a},\dot{\mathbf{a}}\}_i$ in terms of solution to the previous step evaluated at its end point $\{\mathbf{a},\dot{\mathbf{a}}\}_{i-1} = \{\mathbf{g},\mathbf{h}\}$

$$\int_{t_{i-1}}^{t_i} (\mathbf{Ka}-\mathbf{F})dt + \mathbf{M}(\dot{\mathbf{a}}_{i-1}-\mathbf{h}) = \mathbf{K}(\boldsymbol{\alpha}_0\Delta t + \boldsymbol{\alpha}_1\frac{1}{2}\Delta t^2) - \mathbf{F}\Delta t + \mathbf{M}(\boldsymbol{\alpha}_1-\mathbf{h}) = 0,$$

$$\int_{t_{i-1}}^{t_i} t(\mathbf{Ka}-\mathbf{F})dt - \mathbf{M}(\mathbf{a}_{i-1}-\mathbf{g}) = \mathbf{K}(\frac{1}{2}\Delta t^2\boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1\frac{1}{3}\Delta t^3) - \mathbf{F}\frac{1}{2}\Delta t^2 - \mathbf{M}(\boldsymbol{\alpha}_0-\mathbf{g}) = 0.$$

Rearranging the equations

$$\Delta t\mathbf{K}\boldsymbol{\alpha}_0 + (\frac{1}{2}\Delta t^2\mathbf{K}+\mathbf{M})\boldsymbol{\alpha}_1 = \mathbf{Mh}+\mathbf{F}\Delta t,$$

$$(\frac{1}{2}\Delta t^2\mathbf{K}-\mathbf{M})\boldsymbol{\alpha}_0 + \frac{1}{3}\Delta t^3\mathbf{K}\boldsymbol{\alpha}_1 = -\mathbf{Mg}+\mathbf{F}\frac{1}{2}\Delta t^2$$

and using the writing the equations in the recursive form of iteration with

$$\begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}} \end{Bmatrix}_i = \begin{bmatrix} \mathbf{I} & \mathbf{I}\Delta t \\ 0 & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\alpha}_0 \\ \boldsymbol{\alpha}_1 \end{Bmatrix} \quad \Leftrightarrow \quad \begin{Bmatrix} \boldsymbol{\alpha}_0 \\ \boldsymbol{\alpha}_1 \end{Bmatrix} = \begin{bmatrix} \mathbf{I} & -\mathbf{I}\Delta t \\ 0 & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}} \end{Bmatrix}_i$$

gives

$$\begin{bmatrix} \Delta t\mathbf{K} & \frac{1}{2}\Delta t^2\mathbf{K} + \mathbf{M} \\ \frac{1}{2}\Delta t^2\mathbf{K} - \mathbf{M} & \frac{1}{3}\Delta t^3\mathbf{K} \end{bmatrix} \begin{bmatrix} \mathbf{I} & -\mathbf{I}\Delta t \\ 0 & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}} \end{Bmatrix}_{i+1} = \begin{bmatrix} 0 & \mathbf{M} \\ -\mathbf{M} & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}} \end{Bmatrix}_i + \mathbf{F} \begin{Bmatrix} \Delta t \\ \frac{1}{2}\Delta t^2 \end{Bmatrix}$$

and finally

$$\begin{bmatrix} \Delta t\mathbf{K} & -\frac{1}{2}\Delta t^2\mathbf{K} + \mathbf{M} \\ \frac{1}{2}\Delta t^2\mathbf{K} - \mathbf{M} & \Delta t\mathbf{M} - \frac{1}{6}\Delta t^3\mathbf{K} \end{bmatrix} \begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}} \end{Bmatrix}_i = \begin{bmatrix} 0 & \mathbf{M} \\ -\mathbf{M} & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}} \end{Bmatrix}_{i-1} + \mathbf{F} \begin{Bmatrix} \Delta t \\ \frac{1}{2}\Delta t^2 \end{Bmatrix}.$$

ONE-STEP METHODS FOR EQUATION SYSTEM

DG:

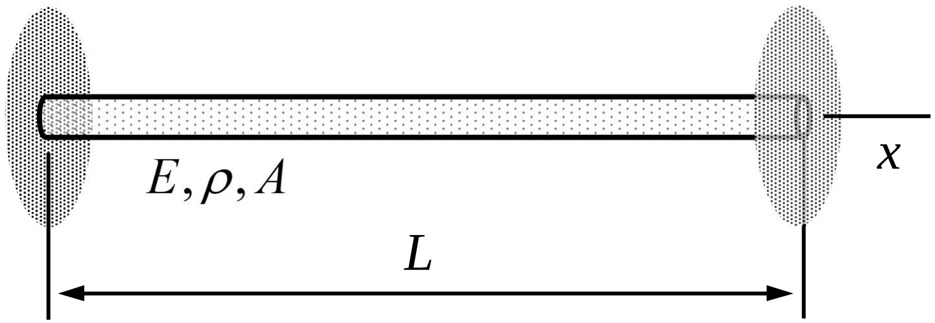
$$\begin{bmatrix} \Delta t^2\mathbf{K} & -\frac{1}{2}\Delta t^2\mathbf{K}+\mathbf{M} \\ \frac{1}{2}\Delta t^2\mathbf{K}-\mathbf{M} & \mathbf{M}-\frac{1}{6}\Delta t^2\mathbf{K} \end{bmatrix} \begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}}\Delta t \end{Bmatrix}_i = \begin{bmatrix} \mathbf{0} & \mathbf{M} \\ -\mathbf{M} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}}\Delta t \end{Bmatrix}_{i-1}, \quad \begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}}\Delta t \end{Bmatrix}_0 = \begin{Bmatrix} \mathbf{g} \\ \mathbf{h}\Delta t \end{Bmatrix}$$

CN:

$$\begin{bmatrix} \mathbf{I} & -\frac{1}{2}\mathbf{I} \\ \frac{\Delta t}{2}\mathbf{K} & \mathbf{M} \end{bmatrix} \begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}}\Delta t \end{Bmatrix}_i = \begin{bmatrix} \mathbf{I} & \frac{1}{2}\mathbf{I} \\ -\frac{\Delta t}{2}\mathbf{K} & \mathbf{M} \end{bmatrix} \begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}}\Delta t \end{Bmatrix}_{i-1}, \quad \begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}}\Delta t \end{Bmatrix}_0 = \begin{Bmatrix} \mathbf{g} \\ \mathbf{h}\Delta t \end{Bmatrix}$$

The proper step-size Δt depends on the largest eigenvalue of parameter $\boldsymbol{\alpha} = \mathbf{M}^{-1}\mathbf{K}\Delta t^2$. The numerical damping of DG exceeds that of CN whereas the phase error of CN exceeds that of the DG method.

EXAMPLE Finite Element Method is applied to the bar problem shown using a regular grid with $i \in \{0,1,2,3\}$. Thereafter, Discontinuous-Galerkin method is applied to find the solution at the temporal grid $t_j = j\Delta t \;\; j \in \{0,1,\ldots\}$. Derive the iteration formula giving the displacements and velocities of points of the spatial discretization for initial displacement and velocities given by $\mathbf{g}$ and $\mathbf{h}$. Material properties E and ρ and cross-sectional area A are constants.



Answer $3\frac{EA}{L}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \frac{L}{3}\rho A \frac{1}{6}\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ u_2 \end{Bmatrix} = 0$

Using the Finite Element Method on a regular grid of points $i \in \{0,1,2,3\}$ gives the equations set: $u_0 = 0$ and $u_3 = 0$

$$\frac{EA}{\Delta x}(u_0 - 2u_1 + u_2) - \rho A \Delta x \frac{1}{6}(4\ddot{u}_0 + 4\ddot{u}_1 + \ddot{u}_2) = 0,$$

$$\frac{EA}{\Delta x}(u_1 - 2u_2 + u_3) - \rho A \Delta x \frac{1}{6}(4\ddot{u}_1 + 4\ddot{u}_2 + \ddot{u}_3) = 0.$$

Or written in the matrix form by taking into account only the points of unknown displacements and $\Delta x = L/3$

$$3\frac{EA}{L}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \frac{L}{3}\rho A \frac{1}{6}\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ u_2 \end{Bmatrix} = 0.$$

With notation

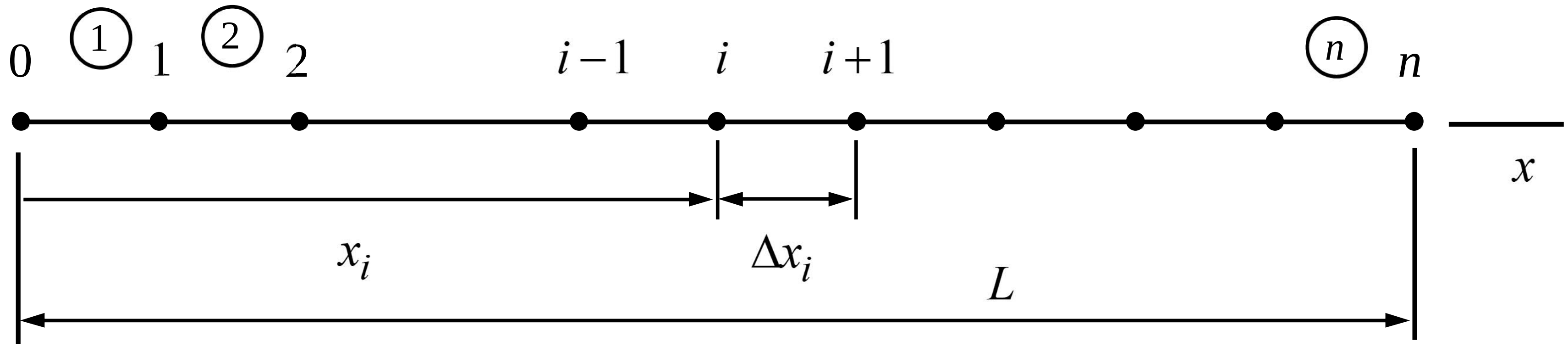
$$\mathbf{M} = \frac{L}{3}\rho A \frac{1}{6}\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix},\ \mathbf{K} = 3\frac{EA}{L}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix},\ \mathbf{a} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix},\ \dot{\mathbf{a}} = \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix}$$

the time-integration according to the Discontinuous-Galerkin method is given by

$$\begin{bmatrix} \Delta t^2 \mathbf{K} & -\frac{1}{2}\Delta t^2 \mathbf{K} + \mathbf{M} \\ \frac{1}{2}\Delta t^2 \mathbf{K} - \mathbf{M} & \mathbf{M} - \frac{1}{6}\Delta t^2 \mathbf{K} \end{bmatrix} \begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}}\Delta t \end{Bmatrix}_i = \begin{bmatrix} \mathbf{0} & \mathbf{M} \\ -\mathbf{M} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}}\Delta t \end{Bmatrix}_{i-1}, \quad \begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}}\Delta t \end{Bmatrix}_0 = \begin{Bmatrix} \mathbf{g} \\ \mathbf{h}\Delta t \end{Bmatrix}.$$

4.5 ELEMENTS AND NODES

In Finite Element Method, grid points and the line segments between them are called as the nodes and elements, respectively. The representation of geometry or dataset by elements and separate lists of coordinates and function values contains the regular grid representations used in Particle Surrogate Method and Finite Difference Method as a particular case.



The element-node representation is more flexible than the regular grid and fits particularly well in the Finite Element Method also with several spatial dimensions.

Example of the representation of dataset $\{(x_0, f_0),(x_1, f_1),\ldots,(x_n, f_n)\}$ with the element concept is

$$\{(0,1),(1,2),\ldots,(n-1,n)\},\ \{x_0, x_1,\ldots, x_n\},\ \{f_0, f_1,\ldots, f_n\}$$

The element description consist of the indices of the end points and coordinates and function values are given in separate lists. In implementation of the numerical method, the numbering of the nodes starts from 1 due to the usual referencing convention to elements of the lists and tables so the representation of the dataset would likely be

$$\{(1,2),(2,3),\ldots,(n,n+1)\},\ \{x_0, x_1,\ldots, x_n\},\ \{f_0, f_1,\ldots, f_n\}$$

or even more likely

$$\{(1,2),(2,3),\ldots,(n,n+1)\},\ \{x_1, x_2,\ldots, x_{n+1}\},\ \{f_1, f_2,\ldots, f_{n+1}\}.$$

In hand calculations, the convention used does not matter.

COE-C3005 Finite Element and Finite difference methods

1. Consider a piecewise linear approximation $g(x)$ to $f(x)=x^2$ $x\in[0,L]$ on the regular grid of 3 points. Use the weighted residual method

$$\int_0^L \mathbf{N}(\mathbf{N}^{\mathrm{T}}\mathbf{g}-f)dx=0,$$

where $\mathbf{N}^{\mathrm{T}}=\{N_0 \quad N_1 \quad N_2\}$ to find the values $\mathbf{g}^{\mathrm{T}}=\{g_0 \quad g_1 \quad g_2\}$ of the approximation at the grid points.

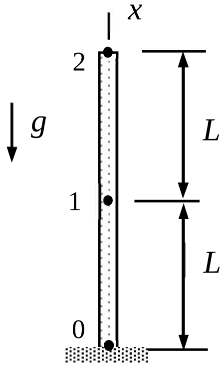
Answer $\mathbf{g}^{\mathrm{T}}=\dfrac{L^2}{24}\{-1 \quad 5 \quad 23\}$

2. Derive the iteration according to the Discontinuous Galerkin Method using (1) a polynomial approximation $a(t)=\alpha_0+\alpha_1 t$ to displacement in a typical time interval of length Δt, and (2) weighted residual expression

$$\int_{t_{i-1}}^{t_i} p(m\ddot{a}+ka)dt+[pm(\dot{a}-h)]_{i-1}-[\dot{p}m(a-g)]_{i-1}=0 \quad \text{where } p\in\{1,t\}.$$

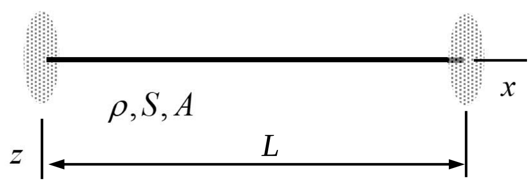
Answer $\begin{Bmatrix} a \\ \dot{a}\Delta t \end{Bmatrix}_i=\dfrac{2}{12+\alpha^4}\begin{bmatrix} 6-3\alpha^2 & 6-\alpha^2 \\ -6\alpha^2 & 6-3\alpha^2 \end{bmatrix}\begin{Bmatrix} a \\ \dot{a}\Delta t \end{Bmatrix}_{i-1}$ where $\alpha=\sqrt{\dfrac{k}{m}}\Delta t$

3. The bar shown is loaded by its own weight. Determine the displacements at the grid points 1 and 2 using the Finite Element Method. Cross-sectional area A, Young's modulus E, and density ρ of the material are constants.



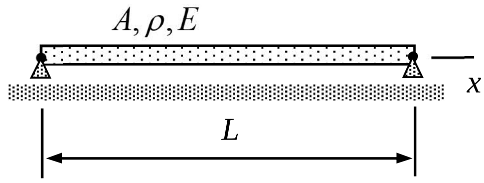
Answer $u_0=0$, $u_1=-\dfrac{3}{2}\dfrac{\rho g L^2}{E}$, $u_2=-2\dfrac{\rho g L^2}{E}$

4. Consider the string of tightening S and mass per unit length ρA shown. Use the Finite Element Method on a regular grid $i\in\{0,1,\ldots,n\}$ to find the angular velocities ω_k of the free vibrations using the solution trial $w_i=a_k(t)\sin(k\pi i/n)$ $k\in\{1,2\ldots,n-1\}$.



Answer $\omega_k=\dfrac{12}{L}\sqrt{\dfrac{S}{A\rho}}n\sin(\dfrac{k\pi}{2n})/\sqrt{2+\cos(\dfrac{k\pi}{n})}$ $k\in\{1,2\ldots,n-1\}$ (exact $\omega_k=\dfrac{k\pi}{L}\sqrt{\dfrac{S}{A\rho}}$)

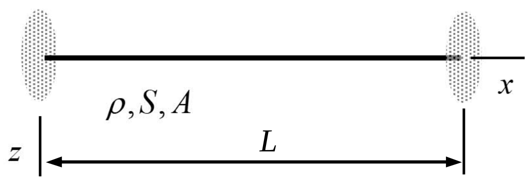
5. A bar is free to move in the horizontal direction as shown. Write the equation system $\mathbf{Ka}+\mathbf{M\ddot{a}}=0$ given by the Finite Element Method on a regular grid with $i\in\{0,1,2\}$. Also, determine the angular velocities and modes of the free vibrations. Cross-sectional area A, density ρ of the material, and Young's modulus E of the material are constants.



Answer $2\dfrac{EA}{L}\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix}\begin{Bmatrix}u_0\\ u_1\\ u_2\end{Bmatrix}+\rho A\dfrac{L}{2}\begin{bmatrix}2 & 1 & 0\\ 1 & 4 & 1\\ 0 & 1 & 2\end{bmatrix}\begin{Bmatrix}\ddot{u}_0\\ \ddot{u}_1\\ \ddot{u}_2\end{Bmatrix}=0$ and $(\omega,\mathbf{A})_1=(0,\begin{Bmatrix}1\\ 1\\ 1\end{Bmatrix})$,

$(\omega,\mathbf{A})_2=(\dfrac{2}{L}\sqrt{3\dfrac{E}{\rho}},\begin{Bmatrix}-1\\ 0\\ 1\end{Bmatrix})$, $(\omega,\mathbf{A})_3=(\dfrac{4}{L}\sqrt{3\dfrac{E}{\rho}},\begin{Bmatrix}1\\ -1\\ 1\end{Bmatrix})$

6. Consider the string of tightening S and mass per unit length ρA shown. First, use the Finite Element Method with the second order accurate central differences on a regular grid $i\in\{0,1,2\}$ to find the equation of motion of the form $ka+m\ddot{a}=0$. Second, write the iteration equation for a typical time-step of size Δt according to Discontinuous Galerkin Method giving the values of displacement and velocity on the temporal grid.



Answer $4\dfrac{EA}{L}u_1+\rho AL\dfrac{1}{3}\ddot{u}_1=0$, $\begin{Bmatrix}a\\ \dot{a}\Delta t\end{Bmatrix}_i=\dfrac{2}{12+\alpha^4}\begin{bmatrix}6-3\alpha^2 & 6-\alpha^2\\ -6\alpha^2 & 6-3\alpha^2\end{bmatrix}\begin{Bmatrix}a\\ \dot{a}\Delta t\end{Bmatrix}_{i-1}$, $\alpha=\sqrt{12\dfrac{E}{\rho}}\dfrac{\Delta t}{L}$

Consider a piecewise linear approximation $g(x)$ to $f(x) = x^2 \quad x \in [0, L]$ on the regular grid of 3 points. Use the weighted residual method

$$\int_0^L \mathbf{N}(\mathbf{N}^{\mathrm{T}}\mathbf{g} - f)dx = 0,$$

where $\mathbf{N}^{\mathrm{T}} = \{N_0 \quad N_1 \quad N_2\}$ to find the values $\mathbf{g}^{\mathrm{T}} = \{g_0 \quad g_1 \quad g_2\}$ of the approximation at the grid points.

Solution

In the weighted residual method with piecewise linear approximation, the aim is to find the approximation $g = \mathbf{N}^{\mathrm{T}}\mathbf{g}$ so that

$$\int_0^L \mathbf{N}(g - f)dx = 0.$$

As the approximation is piecewise continuous, the integral needs to be calculated as the sum of integrals over the line segments between the grid points. The equations given by the three shape functions take the form

$$\int_0^{L/2} N_0(N_0 g_0 + N_1 g_1 - f)dx = 0,$$

$$\int_0^L N_1(g - f)dx = \int_0^{L/2} N_1(N_0 g_0 + N_1 g_1 - f)dx + \int_{L/2}^L N_1(N_1 g_1 + N_2 g_2 - f)dx = 0,$$

$$\int_0^L N_2(N_1 g_1 + N_2 g_2 - f)dx = 0.$$

Using $f(x) = x^2$ and the shape function expression

$$N_0 = 1 - 2\frac{x}{L} \quad x < \frac{L}{2} \quad \text{and} \quad N_0 = 0 \quad x > \frac{L}{2},$$

$$N_1 = 2\frac{x}{L} \quad x < \frac{L}{2} \quad \text{and} \quad N_1 = 2(1 - \frac{x}{L}) \quad x > \frac{L}{2},$$

$$N_2 = 2\frac{x}{L} - 1 \quad x < \frac{L}{2} \quad \text{and} \quad N_1 = 2(1 - \frac{x}{L}) \quad x > \frac{L}{2},$$

The three equations become

$$\int_0^L N_0(N_0 g_0 + N_1 g_1 - f)dx = g_0 \frac{L}{6} + g_1 \frac{L}{12} - \frac{L^3}{96} = 0,$$

$$\int_0^L N_1(g - f)dx = \frac{L}{12} g_0 + \frac{L}{3} g_1 + \frac{L}{12} g_2 - \frac{7L^3}{48} = 0,$$

$$\int_0^L N_2(N_1 g_1 + N_2 g_2 - f)dx = \frac{L}{12} g_1 + \frac{L}{6} g_2 - \frac{17L^3}{96} = 0\,.$$

Or when written in matrix form and simplified somewhat

$$\begin{bmatrix} 16 & 8 & 0 \\ 8 & 32 & 8 \\ 0 & 8 & 16 \end{bmatrix} \begin{Bmatrix} g_0 \\ g_1 \\ g_2 \end{Bmatrix} - L^2 \begin{Bmatrix} 1 \\ 14 \\ 17 \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix} g_0 \\ g_1 \\ g_2 \end{Bmatrix} = \frac{L^2}{24} \begin{Bmatrix} -1 \\ 5 \\ 23 \end{Bmatrix}.$$

Derive the iteration according to the Discontinuous Galerkin Method using (1) a polynomial approximation $a(t)=\alpha_0+\alpha_1 t$ to displacement in a typical time interval of length Δt, and (2) weighted residual expression

$$\int_{t_{i-1}}^{t_i} p(m\ddot{a}+ka)dt+[pm(\dot{a}-h)]_{i-1}-[\dot{p}m(a-g)]_{i-1}=0 \quad \text{where } p\in\{1,t\}.$$

Solution

The simplest time discontinuous Galerkin method for an initial value problems of a second order ordinary differential equations uses the polynomial approximations $a(t)=\alpha_0+\alpha_1 t$ and weighted residual expression of the equations

$$\int_{t_{i-1}}^{t_i} p(m\ddot{a}+ka)dt+[pm(\dot{a}-h)]_{i-1}-[\dot{p}m(a-g)]_{i-1}=0,$$

where the weighting $p\in\{1,t\}$. Considering the time interval $[0,\Delta t]$ so $t_{i-1}=0$ and $t_i=\Delta t$ (just to simplify the manipulations) substituting the approximation $a(t)=\alpha_0+\alpha_1 t$, and writing the weighted residual expression with the two selections of the weighting function gives

$$k(\alpha_0\Delta t+\frac{1}{2}\alpha_1\Delta t^2)+[m(\alpha_1-h)]=0 \quad \text{and} \quad k(\alpha_0\frac{1}{2}\Delta t^2+\alpha_1\frac{1}{6}\Delta t^3)-m(\alpha_0-g)=0.$$

Or, when written in the matrix form

$$\begin{bmatrix} k\frac{1}{2}\Delta t^2-m & k\frac{1}{6}\Delta t^3 \\ k\Delta t & \frac{1}{2}\Delta t^2+m \end{bmatrix}\begin{Bmatrix}\alpha_0\\ \alpha_1\end{Bmatrix}=m\begin{Bmatrix}-g\\ h\end{Bmatrix}.$$

Assuming that the initial conditions are given by the solution to the previous time interval $h=\dot{a}_{i-1}$ and $g=a_{i-1}$ and writing the approximation $a(t)=\alpha_0+\alpha_1 t$ at the end point of the time interval $a_i=\alpha_0+\alpha_1\Delta t$ and $\dot{a}_i=\alpha_1$:

$$\begin{Bmatrix}a\\ \dot{a}\end{Bmatrix}_i=\begin{bmatrix}1 & \Delta t\\ 0 & 1\end{bmatrix}\begin{Bmatrix}\alpha_0\\ \alpha_1\end{Bmatrix} \quad \Leftrightarrow \quad \begin{Bmatrix}\alpha_0\\ \alpha_1\end{Bmatrix}=\begin{bmatrix}1 & -\Delta t\\ 0 & 1\end{bmatrix}\begin{Bmatrix}a\\ \dot{a}\end{Bmatrix}_i \quad \text{and} \quad \begin{Bmatrix}-g\\ h\end{Bmatrix}=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix}a\\ \dot{a}\end{Bmatrix}_{i-1}.$$

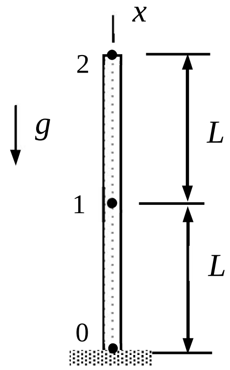
Substituting into to the linear equations given by the weighted residual method

$$\begin{bmatrix} k\frac{1}{2}\Delta t^2-m & k\frac{1}{6}\Delta t^3 \\ k\Delta t & \frac{1}{2}\Delta t^2+m \end{bmatrix}\begin{bmatrix}1 & -\Delta t\\ 0 & 1\end{bmatrix}\begin{Bmatrix}a\\ \dot{a}\end{Bmatrix}_i=m\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix}a\\ \dot{a}\end{Bmatrix}_{i-1}$$

which gives after simplification the iteration

$$\begin{Bmatrix} a \\ \dot{a}\Delta t \end{Bmatrix}_i = \frac{2}{12+\alpha^4}\begin{bmatrix} 6-3\alpha^2 & 6-\alpha^2 \\ -6\alpha^2 & 6-3\alpha^2 \end{bmatrix}\begin{Bmatrix} a \\ \dot{a}\Delta t \end{Bmatrix}_{i-1}, \quad \alpha = \sqrt{\frac{k}{m}}\Delta t .$$

The bar shown is loaded by its own weight. Determine the displacements at the grid points 1 and 2 using the Finite Element Method. Cross-sectional area A, Young's modulus E, and density ρ of the material are constants.



Solution

The difference equations to the bar and string problems according to the Finite Element Method are given by (omitting the time derivatives and initial conditions)

$$\frac{k}{\Delta x}(a_{i-1}-2a_i+a_{i+1})+F_i+f'\Delta x=0 \quad i\in\{1,2,\ldots,n-1\}$$

$$\frac{k}{\Delta x}(a_1-a_0)+F_0+f'\frac{\Delta x}{2}=0 \quad \text{or} \quad a_0=\underline{a}_0,$$

$$\frac{k}{\Delta x}(a_{n-1}-a_n)+F_n+f'\frac{\Delta x}{2}=0 \quad \text{or} \quad a_n=\underline{a}_n,$$

where in this case of a bar problem $k=EA$, $f'=-\rho Ag$, $\Delta x=L$, and $a=u$. Equations for the three grid points are

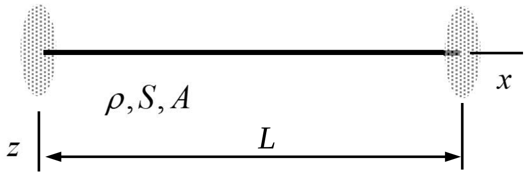
$$u_0=0, \; \frac{EA}{L}(u_0-2u_1+u_2)-\rho AgL=0, \text{ and } \frac{EA}{L}(u_1-u_2)-\rho Ag\frac{L}{2}=0.$$

In matrix notation

$$-\frac{EA}{L}\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}+\frac{1}{2}\rho AgL\begin{Bmatrix}-2\\ -1\end{Bmatrix}=0 \quad \Leftrightarrow$$

$$\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}=\frac{1}{2}\frac{\rho gL^2}{E}\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}^{-1}\begin{Bmatrix}-2\\ -1\end{Bmatrix}=\frac{1}{2}\frac{\rho gL^2}{E}\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}\begin{Bmatrix}-2\\ -1\end{Bmatrix}=-\frac{1}{2}\frac{\rho gL^2}{E}\begin{Bmatrix}3\\ 4\end{Bmatrix}. \quad \leftarrow$$

Consider the string of tightening S and mass per unit length ρA shown. Use the Finite Element Method on a regular grid $i \in \{0,1,\ldots,n\}$ to find the angular velocities ω_k of the free vibrations using the solution trial $w_i = a_k(t)\sin(k\pi i/n)$ $k \in \{1,2\ldots,n-1\}$.



Solution

The difference equations to the bar and string problems according to the Finite Element Method are given by (omitting the initial conditions as they are needed in modal analysis)

$$\frac{k}{\Delta x}(a_{i-1}-2a_i+a_{i+1})+F_i+f'\Delta x = m'\frac{\Delta x}{6}(\ddot{a}_{i-1}+4\ddot{a}_i+\ddot{a}_{i+1}) \quad i \in \{1,2,\ldots,n-1\}$$

$$\frac{k}{\Delta x}(a_1-a_0)+F_0+f'\frac{\Delta x}{2}-m'\frac{\Delta x}{6}(2\ddot{a}_0+\ddot{a}_1)=0 \quad \text{or} \quad a_0=\underline{a}_0,$$

$$\frac{k}{\Delta x}(a_{n-1}-a_n)+F_n+f'\frac{\Delta x}{2}-m'\frac{\Delta x}{6}(2\ddot{a}_n+\ddot{a}_{n-1})=0 \quad \text{or} \quad a_n=\underline{a}_n.$$

Where in this case of a bar problem $k=S$, $m'=\rho A$, $\Delta x = L/n$, $a=w$ and external forces vanish. As the trial solution satisfies the fixed end conditions, it is enough to consider a typical equation inside the domain. Substituting the trial solution and using the trigonometric identity $\sin(\alpha\pm\beta)=\sin(\alpha)\cos(\beta)\pm\cos(\alpha)\sin(\beta)$

$$w_{i-1}-2w_i+w_{i+1}=-2(1-\cos\frac{k\pi}{n})a_k(t)\sin(k\pi\frac{i}{n}),$$

$$\ddot{w}_{i-1}+4\ddot{w}_i+\ddot{w}_{i+1}=2(2+\cos\frac{k\pi}{n})\ddot{a}_k(t)\sin(k\pi\frac{i}{n})$$

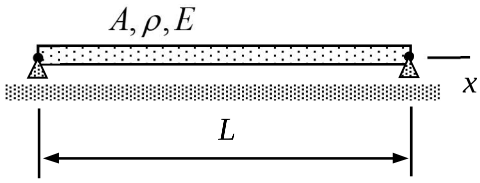
so the equation of motion simplifies to

$$\ddot{a}_k+\omega_k^2 a_k=0 \quad \text{where} \quad \omega_k=\frac{1}{L}\sqrt{6\frac{S}{\rho A}n^2\frac{1-\cos k\pi/n}{2+\cos k\pi/n}}.$$

In the limit, when $n\to\infty$ the angular velocity coincides with that of the continuum model

$$\omega_k = k\frac{\pi}{L}\sqrt{\frac{S}{\rho A}}.$$

A bar is free to move in the horizontal direction as shown. Write the equation system $\mathbf{Ka}+\mathbf{M\ddot{a}}=0$ given by the Finite Element Method on a regular grid with $i \in \{0,1,2\}$. Also, determine the angular velocities and modes of the free vibrations. Cross-sectional area A, density ρ of the material, and Young's modulus E of the material are constants.



Solution

The difference equations to the bar and string problems according to the Finite Element Method are given by (omitting the initial conditions as they are needed in modal analysis)

$$\frac{k}{\Delta x}(a_{i-1}-2a_i+a_{i+1})+F_i+f'\Delta x=m'\frac{\Delta x}{6}(\ddot{a}_{i-1}+4\ddot{a}_i+\ddot{a}_{i+1}) \qquad i\in\{1,2,\ldots,n-1\}$$

$$\frac{k}{\Delta x}(a_1-a_0)+F_0+f'\frac{\Delta x}{2}-m'\frac{\Delta x}{6}(2\ddot{a}_0+\ddot{a}_1)=0 \quad \text{or} \quad a_0=\underline{a}_0,$$

$$\frac{k}{\Delta x}(a_{n-1}-a_n)+F_n+f'\frac{\Delta x}{2}-m'\frac{\Delta x}{6}(2\ddot{a}_n+\ddot{a}_{n-1})=0 \quad \text{or} \quad a_n=\underline{a}_n,$$

Where in this case of a bar problem $k=EA$, $m'=\rho A, \Delta x=L$, $a=u$ and external forces vanish. Equations for the three grid points are

$$\frac{EA}{L}(u_1-u_0)-\rho A\frac{L}{6}(2\ddot{u}_0+\ddot{u}_1)=0$$

$$\frac{AE}{L}(u_0-2u_1+u_2)-\rho A\frac{L}{6}(\ddot{u}_0+4\ddot{u}_1+\ddot{u}_2)=0$$

$$\frac{EA}{L}(u_1-u_2)-\rho A\frac{L}{6}(2\ddot{u}_2+\ddot{u}_1)=0$$

In matrix notation, the equation are

$$\frac{EA}{L}\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix}\begin{Bmatrix}u_0\\ u_1\\ u_2\end{Bmatrix}+\rho A\frac{L}{6}\begin{bmatrix}2 & 1 & 0\\ 1 & 4 & 1\\ 0 & 1 & 2\end{bmatrix}\begin{Bmatrix}\ddot{u}_0\\ \ddot{u}_1\\ \ddot{u}_2\end{Bmatrix}=0$$

Modal analysis uses the trial solution $\mathbf{u}=\mathbf{A}e^{\mathrm{i}\omega t}$ in which $\mathbf{A}$ represent mode and ω the corresponding angular velocity. Substitution into the set of differential equations gives a set of algebraic equations for the possible combinations $(\omega,\mathbf{A})$:

$$(\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix}-\lambda\begin{bmatrix}2 & 1 & 0\\ 1 & 4 & 1\\ 0 & 1 & 2\end{bmatrix})\begin{Bmatrix}A_0\\ A_1\\ A_2\end{Bmatrix}=0 \quad \text{where} \quad \lambda=\frac{\rho}{E}\frac{L^2}{6}\omega^2 \quad \Leftrightarrow \quad \omega=\frac{1}{L}\sqrt{6\lambda\frac{E}{\rho}}\,.$$

First, the possible λ values:

$$\det\left(\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}\right) = 0 \quad \Rightarrow \quad \lambda = 0 \quad \text{or} \quad \lambda = \frac{1}{2} \quad \text{or} \quad \lambda = 2\,.$$

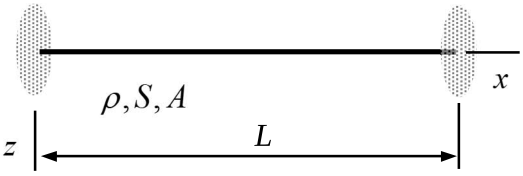
Then the corresponding modes one-by-one

$$\lambda_1 = 0: \quad \omega_1 = 0, \quad \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} A_0 \\ A_1 \\ A_2 \end{Bmatrix} = 0 \quad \Rightarrow \mathbf{A}_1 = \begin{Bmatrix} A_0 \\ A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix},$$

$$\lambda_2 = \frac{1}{2}: \quad \omega_2 = \frac{2}{L}\sqrt{3\frac{E}{\rho}}, \quad \begin{bmatrix} 0 & -3 & 0 \\ -3 & 0 & -3 \\ 0 & -3 & 0 \end{bmatrix} \begin{Bmatrix} A_0 \\ A_1 \\ A_2 \end{Bmatrix} = 0 \quad \Rightarrow \mathbf{A}_2 = \begin{Bmatrix} A_0 \\ A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix},$$

$$\lambda_3 = 2: \quad \omega_3 = \frac{4}{L}\sqrt{3\frac{E}{\rho}}, \quad \begin{bmatrix} -3 & -3 & 0 \\ -3 & -6 & -3 \\ 0 & -3 & -3 \end{bmatrix} \begin{Bmatrix} A_0 \\ A_1 \\ A_2 \end{Bmatrix} = 0 \quad \Rightarrow \mathbf{A}_3 = \begin{Bmatrix} A_0 \\ A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}.$$

Consider the string of tightening S and mass per unit length ρA shown. First, use the Finite Element Method with the second order accurate central differences on a regular grid $i \in \{0,1,2\}$ to find the equation of motion of the form $ka + m\ddot{a} = 0$. Second, write the iteration equation for a typical time-step of size Δt according to Discontinuous Galerkin Method giving the values of displacement and velocity on the temporal grid.



Solution

The generic equation set for the model problems and the Finite Element Method on a regular grid is given by

$$\frac{k}{\Delta x}(a_{i-1} - 2a_i + a_{i+1}) + F_i + f'\Delta x = m'\frac{\Delta x}{6}(\ddot{a}_{i-1} + 4\ddot{a}_i + \ddot{a}_{i+1}) \qquad i \in \{1,2,\ldots,n-1\}$$

$$\frac{k}{\Delta x}(a_1 - a_0) + F_0 + f'\frac{\Delta x}{2} - m'\frac{\Delta x}{6}(2\ddot{a}_0 + \ddot{a}_1) = 0 \quad \text{or} \quad a_0 = \underline{a}_0 ,$$

$$\frac{k}{\Delta x}(a_{n-1} - a_n) + F_n + f'\frac{\Delta x}{2} - m'\frac{\Delta x}{6}(2\ddot{a}_n + \ddot{a}_{n-1}) = 0 \quad \text{or} \quad a_n = \underline{a}_n ,$$

$$a_i - g_i = 0 \quad \text{and} \quad \dot{a}_i - h_i = 0 .$$

In the string application of the problem, external forces vanish, $k = S$, $m' = \rho A$, $a = w$, $\Delta x = L/2$., and external forces vanish. Equations for $i \in \{0,1,2\}$ simplify to

$$w_0 = 0 , \quad 2\frac{S}{L}(w_0 - 2w_1 + w_2) = \rho A\frac{L}{12}(\ddot{w}_0 + 4\ddot{w}_1 + \ddot{w}_2) , \text{ and } w_2 = 0 \quad t > 0 ,$$

$$w_1 = g_1 \quad \text{and} \quad \dot{w}_1 = h_1 \quad t = 0 .$$

In solution methods for time dependent problem, algebraic equations are used to eliminate the displacements of the boundary points from the differential equation, so the initial value problem simplifies to

$$4\frac{S}{L}w_1 + \rho A\frac{L}{12}4\ddot{w}_1 = 0 \quad t > 0 , \; w_1 = g \text{ and } \dot{w}_1 = h \quad t = 0 .$$

With definition $w_1 = a$, time integration by Discontinuous-Galerkin method is given by iteration

$$\begin{Bmatrix} a \\ \dot{a}\Delta t \end{Bmatrix}_i = \frac{2}{12 + \alpha^4}\begin{bmatrix} 6 - 3\alpha^2 & 6 - \alpha^2 \\ -6\alpha^2 & 6 - 3\alpha^2 \end{bmatrix}\begin{Bmatrix} a \\ \dot{a}\Delta t \end{Bmatrix}_{i-1} , \quad \begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_0 = \begin{Bmatrix} g \\ \Delta t h \end{Bmatrix}_0 \quad \text{where } \alpha = \sqrt{12\frac{S}{\rho A}}\frac{\Delta t}{L} .$$

LECTURE ASSIGNMENT 1

Verify by direct calculation the weighted residual expression for the first derivative

$$\int_0^L N_i \frac{\partial a}{\partial x} dx = \frac{1}{2}(a_{i+1} - a_{i-1})$$

for a regular grid of spacing Δx. In a line segment of end points x_i and x_j, the non-zero linear shape functions and the interpolant $a(x)$ are given by

$$\begin{Bmatrix} N_i \\ N_j \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ x_i & x_j \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \end{Bmatrix} \quad \text{and} \quad a(x) = \begin{Bmatrix} a_i \\ a_j \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} N_i \\ N_j \end{Bmatrix}.$$

Place the origin of the $x-$coordinate system at point i.

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As the shape function N_i is non-zero only in the line segments having the point i in common and origin is placed at point i, it is enough to consider $x \in [-\Delta x, \Delta x]$. Using the expressions of the shape functions in line segments $(i-1,i)$ and $(i,i+1)$

$$x \in [-\Delta x, 0]: \begin{Bmatrix} N_{i-1} \\ N_i \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ -\Delta x & 0 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \end{Bmatrix} = \begin{Bmatrix} -x/\Delta x \\ x/\Delta x + 1 \end{Bmatrix} \text{ and } a(x) = \begin{Bmatrix} a_{i-1} \\ a_i \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} N_{i-1} \\ N_i \end{Bmatrix} \Rightarrow$$

$$N_i = 1 + \frac{x}{\Delta x}, \quad a(x) = -a_{i-1}\frac{x}{\Delta x} + a_i \frac{x+\Delta x}{\Delta x}, \text{ and } \frac{\partial a}{\partial x} = \frac{a_i - a_{i-1}}{\Delta x}$$

$$x \in [0, \Delta x]: \begin{Bmatrix} N_i \\ N_{i+1} \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & \Delta x \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \end{Bmatrix} = \begin{Bmatrix} 1 - x/\Delta x \\ x/\Delta x \end{Bmatrix} \text{ and } a(x) = \begin{Bmatrix} a_i \\ a_{i+1} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} N_i \\ N_{i+1} \end{Bmatrix} \Rightarrow$$

$$N_i = 1 - \frac{x}{\Delta x}, \quad a(x) = a_i(1 - \frac{x}{\Delta x}) + a_{i+1}\frac{x}{\Delta x}, \text{ and } \frac{\partial a}{\partial x} = \frac{a_{i+1} - a_i}{\Delta x}$$

Integral over the domain is the sum of the integrals over the line segments. Direct calculation gives first

$$\int_{-\Delta x}^{0} N_i \frac{\partial a}{\partial x} dx = \frac{a_i - a_{i-1}}{2} \quad \text{and} \quad \int_{0}^{\Delta x} N_i \frac{\partial a}{\partial x} dx = \frac{a_{i+1} - a_i}{2}$$

and, after that, combining the integrals over the line segment having the point i in common (N_i vanishes elsewhere):

$$\int_0^L N_i \frac{\partial a}{\partial x} dx = \int_{-\Delta x}^{0} N_i \frac{\partial a}{\partial x} dx + \int_{0}^{\Delta x} N_i \frac{\partial a}{\partial x} dx = \frac{1}{2}(a_{i+1} - a_{i-1}) .$$

LECTURE ASSIGNMENT 2

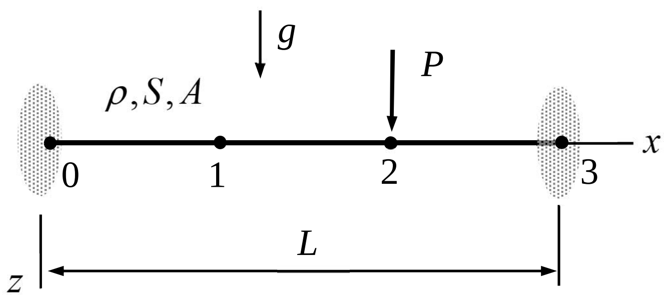
The equations for stationary string and bar problems given by the Finite Element Method on a regular spatial are

$$\frac{k}{\Delta x}(a_{i-1} - 2a_i + a_{i+1}) + F_i + f'\Delta x = 0 \quad i \in \{1, 2, \ldots, n-1\},$$

$$\frac{k}{\Delta x}(a_1 - a_0) + F_0 + f'\frac{\Delta x}{2} = 0 \quad \text{or} \quad a_0 = \underline{a}_0,$$

$$\frac{k}{\Delta x}(a_{n-1} - a_n) + F_n + f'\frac{\Delta x}{2} = 0 \quad \text{or} \quad a_n = \underline{a}_n.$$

Write the equations for the stationary string problem of grid points $i \in \{0,1,2,3\}$ shown in the figure. Tightening S, cross-sectional area A, and density of the material ρ are constants.



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At point $i=0$, the displacement boundary condition applies

$$w_0 = 0$$

At point $i=1$, the equilibrium equation applies

$$3\frac{S}{L}(w_0 - 2w_1 + w_2) + \rho A g \frac{L}{3} = 0$$

At point $i=2$, the equilibrium equation applies

$$3\frac{S}{L}(w_1 - 2w_2 + w_3) + P + \rho A g \frac{L}{3} = 0$$

At point $i=3$, the displacement boundary condition applies

$$w_3 = 0$$

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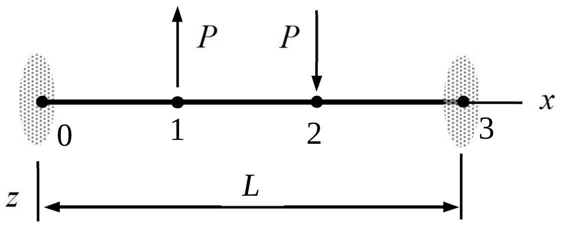
Home assignment 1

A string of length L and tightening S is loaded by point forces of magnitudes P as shown. If the ends are fixed and the initial geometry without loading is straight, find the transverse displacements at the grid points $i \in \{0,1,2,3\}$ using the Finite Element Method



Solution

The generic equation set for the string and bar models is given the Finite Element Method on a regular grid is

$$\frac{k}{\Delta x}(a_{i-1}-2a_i+a_{i+1})+F_i+f'\Delta x=m'\frac{\Delta x}{6}(\ddot{a}_{i-1}+4\ddot{a}_i+\ddot{a}_{i+1}) \qquad i\in\{1,2,\ldots,n-1\}$$

$$\frac{k}{\Delta x}(w_1-w_0)+F_0+\frac{\Delta x}{2}f'-m'\frac{\Delta x}{6}(2\ddot{a}_0+\ddot{a}_1)=0 \quad \text{or} \quad a_0=\underline{a}_0,$$

$$\frac{k}{\Delta x}(a_{n-1}-a_n)+F_n+\frac{\Delta x}{2}f'-m'\frac{\Delta x}{6}(2\ddot{a}_n+\ddot{a}_{n-1})=0 \quad \text{or} \quad a_n=\underline{a}_n,$$

$$a_i-g_i=0 \quad \text{and} \quad \dot{a}_i-h_i=0.$$

In the stationary case all time derivatives vanish and initial conditions are not used. With $a_i=w_i$, $k=S$, $m'=\rho A$, and $\Delta x=L/3$, the equations for the grid points $i\in\{0,1,2,3\}$ simplify to

$$w_0=0,\; 3\frac{S}{L}(w_0-2w_1+w_1)-P=0,\; 3\frac{S}{L}(w_1-2w_2+w_3)+P=0, \text{ and } w_3=0.$$

Using only the equations for points $i\in\{1,2\}$ simplified with the known values of the boundary displacements at points $i\in\{0,3\}$

$$-3\frac{S}{L}\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}\begin{Bmatrix}w_1\\ w_2\end{Bmatrix}+P\begin{Bmatrix}-1\\ 1\end{Bmatrix}=0 \quad\Leftrightarrow\quad \begin{Bmatrix}w_1\\ w_2\end{Bmatrix}=\frac{PL}{3S}\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}^{-1}\begin{Bmatrix}-1\\ 1\end{Bmatrix}=\frac{1}{9}\frac{PL}{S}\begin{Bmatrix}-1\\ 1\end{Bmatrix}. \leftarrow$$

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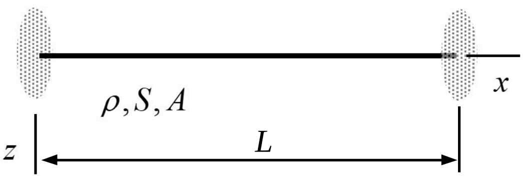
Home assignment 2

What is relative (numerical) error in the smallest frequency of the free vibrations $\underline{f}_{\min}$ of the string shown obtained with the Finite Element Method and a regular spatial grid $i \in \{0,1,2,3\}$? Cross-sectional area A, density of the material ρ, and horizontal tightening S are constants. The smallest frequency given by the continuum model $f_{\min} = \sqrt{S/(\rho A)}/(2L)$.



Solution

The generic equation set for the string and bar models is given the Finite Element Method on a regular grid is

$$\frac{k}{\Delta x}(a_{i-1} - 2a_i + a_{i+1}) + F_i + f'\Delta x = m'\frac{\Delta x}{6}(\ddot{a}_{i-1} + 4\ddot{a}_i + \ddot{a}_{i+1}) \qquad i \in \{1,2,\ldots,n-1\}$$

$$\frac{k}{\Delta x}(w_1 - w_0) + F_0 + \frac{\Delta x}{2} f' - m'\frac{\Delta x}{6}(2\ddot{a}_0 + \ddot{a}_1) = 0 \quad \text{or} \quad a_0 = \underline{a}_0 ,$$

$$\frac{k}{\Delta x}(a_{n-1} - a_n) + F_n + \frac{\Delta x}{2} f' - m'\frac{\Delta x}{6}(2\ddot{a}_n + \ddot{a}_{n-1}) = 0 \quad \text{or} \quad a_n = \underline{a}_n ,$$

$$a_i - g_i = 0 \quad \text{and} \quad \dot{a}_i - h_i = 0 .$$

In modal analysis, initial conditions are not needed. With $a_i = w_i$, $k = S$, and $m' = \rho A$, the equations for the grid points $i \in \{0,1,2,3\}$ simplify to

$$w_0 = 0 , \quad \frac{S}{\Delta x}(w_0 - 2w_1 + w_2) - \rho A\frac{\Delta x}{6}(\ddot{w}_0 + 4\ddot{w}_1 + \ddot{w}_2) = 0 ,$$

$$\frac{S}{\Delta x}(w_1 - 2w_2 + w_3) - \rho A\frac{\Delta x}{6}(\ddot{w}_1 + 4\ddot{w}_2 + \ddot{w}_3) = 0 , \quad w_3 = 0$$

In matrix notation, the equations for points 1 and 2 are (when the known displacements at the boundary points are used there)

$$\frac{S}{\Delta x}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} + \rho A \Delta x \frac{1}{6}\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\begin{Bmatrix} \ddot{w}_1 \\ \ddot{w}_2 \end{Bmatrix} = 0 .$$

With the trial solution $\mathbf{a} = \mathbf{A}\exp(\mathrm{i}\omega t)$, the condition for the possible angular velocity ω and mode $\mathbf{A}$ pairs takes the form

$$(\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix})\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0 \quad \text{where} \quad \lambda = \omega^2 \frac{1}{6}\frac{\rho A}{S}\Delta x^2 ,$$

A homogeneous linear equation system can give a non-zero solution only if the matrix is singular so

$$\det\begin{bmatrix} 2-4\lambda & -1-\lambda \\ -1-\lambda & 2-4\lambda \end{bmatrix} = (2-4\lambda)^2 - (1+\lambda)^2 = 0 \quad \Rightarrow \quad \lambda_1 = \frac{1}{5} \quad \text{or} \quad \lambda_2 = 1 .$$

The angular velocities follow from the known relationships

$$\lambda = \omega^2 \frac{1}{6}\frac{\rho A}{S}\Delta x^2 \quad \text{and} \quad \omega = 2\pi f \; .$$

The smallest frequency corresponds $\lambda_{\min} = 1/5$

$$\underline{f}_{\min} = \frac{3}{2\pi}\sqrt{\frac{6}{5}}\frac{1}{L}\sqrt{\frac{S}{\rho A}} \quad \text{the exact being} \quad f_{\min} = \frac{1}{2}\frac{1}{L}\sqrt{\frac{S}{\rho A}} \; .$$

The relative error in the smallest value

$$\frac{\underline{f}_{\min} - f_{\min}}{f_{\min}} 100\% = (\frac{3}{\pi}\sqrt{\frac{6}{5}} - 1)100\% \approx 5\% \; .$$

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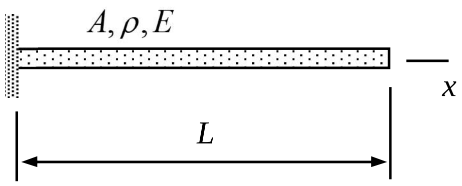
Home assignment 3

A bar is free to move in the horizontal direction as shown. At $t=0$, displacement of the free end is U and velocity vanishes. Use the Finite Element Method on a regular spatial grid with $i \in \{0,1\}$ and the Discontinuous-Galerkin method with step size Δt to find the displacement and velocity of the free end at $t=\Delta t$. Cross-sectional area A, density ρ of the material, and Young's modulus E of the material are constants.



Solution

On a regular grid of points of spacing Δx, the generic equations for the string and bar problems according to the Finite Element Method are given by

$$\frac{k}{\Delta x}(a_{i-1}-2a_i+a_{i+1})+F_i+f'\Delta x=m'\frac{\Delta x}{6}(\ddot{a}_{i-1}+4\ddot{a}_i+\ddot{a}_{i+1}) \quad i\in\{1,2,\ldots,n-1\},$$

$$\frac{k}{\Delta x}(a_1-a_0)+F_0+f'\frac{\Delta x}{2}-m'\frac{\Delta x}{6}(2\ddot{a}_0+\ddot{a}_1)=0 \quad \text{or} \quad a_0=\underline{a}_0,$$

$$\frac{k}{\Delta x}(a_{n-1}-a_n)+F_n+f'\frac{\Delta x}{2}-m'\frac{\Delta x}{6}(2\ddot{a}_n+\ddot{a}_{n-1})=0 \quad \text{or} \quad a_n=\underline{a}_n,$$

$$a_i-g_i=0 \quad \text{and} \quad \dot{a}_i-h_i=0.$$

In the bar problem show, $k=EA$, $m'=\rho A$ and the equations for points $i\in\{0,1\}$ become

$$u_0=0 \quad \text{and} \quad \frac{EA}{\Delta x}(u_0-u_1)-\rho A\frac{\Delta x}{6}(2\ddot{u}_1+\ddot{u}_0)=0 \quad t>0,$$

$$u_1=U \quad \text{and} \quad \dot{u}_1=0 \quad t=0.$$

The algebraic equation for point 0 can be used to eliminate u_0 from the differential equation for point 1. The outcome is the initial value problem

$$\frac{EA}{\Delta x}u_1+\rho A\frac{\Delta x}{3}\ddot{u}_1=0 \quad t>0, \quad u_1=U \quad t=0, \text{ and } \dot{u}_1=0 \quad t=0.$$

After discretization with respect to the spatial coordinate, one may apply analytical solution method or discretize with respect to the temporal coordinate to get am algebraic equation system for typical time-step. Let us choose the latter approach and apply the Discontinuous-Galerkin method for a single equation

$$\begin{Bmatrix} a \\ \dot{a}\Delta t \end{Bmatrix}_i = \frac{2}{12+\alpha^4}\begin{bmatrix} 6-3\alpha^2 & 6-\alpha^2 \\ -6\alpha^2 & 6-3\alpha^2 \end{bmatrix}\begin{Bmatrix} a \\ \dot{a}\Delta t \end{Bmatrix}_{i-1}, \quad \begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_0 = \begin{Bmatrix} g \\ \Delta t h \end{Bmatrix} \text{ where } \alpha = \sqrt{\frac{k}{m}}\Delta t\,.$$

With the present problem $a = u_1 = u$ so the iteration gives for the first step

$$\begin{Bmatrix} u \\ \dot{u}\Delta t \end{Bmatrix}_1 = \frac{2}{12+\alpha^4}\begin{bmatrix} 6-3\alpha^2 & 6-\alpha^2 \\ -6\alpha^2 & 6-3\alpha^2 \end{bmatrix}\begin{Bmatrix} U \\ 0 \end{Bmatrix} = \frac{2U}{12+\alpha^4}\begin{Bmatrix} 6-3\alpha^2 \\ -6\alpha^2 \end{Bmatrix} \text{ where } \alpha = \sqrt{3\frac{E}{\rho}}\frac{\Delta t}{\Delta x}\,. \quad \leftarrow$$

5 FDM FOR MEMBRANE MODEL

ENGINEERINGS MODELS

BAR is a body which is thin in two dimensions and has straight initial geometry. Displacement has only the axial component. Internal force is aligned with the axis.

STRING is a body which is thin in two dimensions and has straight initial geometry. Displacement has only the transverse component. Internal force is aligned with the tangent of the mid-curve at the initial and deformed geometries.

THIN SLAB is a body which is thin in one dimension and has planar initial geometry. Displacement has only the mid-plane components. Internal force does not have transverse component.

MEMBRANE is a body which is very thin in one dimension and has planar initial geometry. Displacement has only the transverse component. Internal force is aligned with the tangent of the mid-plane at the initial and deformed geometries.

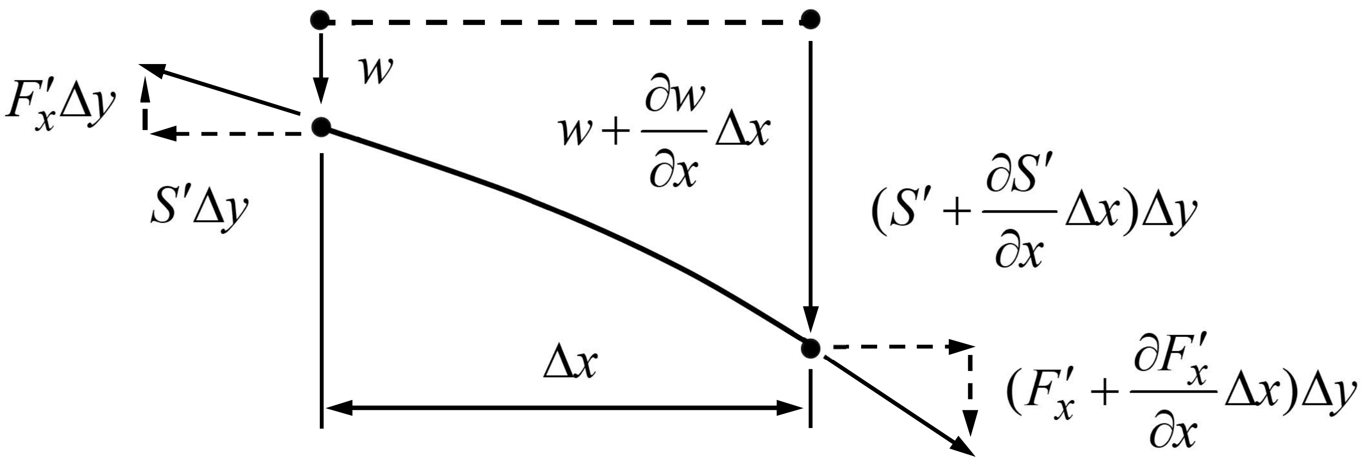
MATHEMATICAL PREREQUISITES

In an analytical solution method, solution trial is used to transform a partial differential differentia equation into an ordinary differential equation, another solution trial is used to transform the ordinary differential equation into an algebraic equation etc.

Equation	Solution trial
$k'(\frac{\partial^2 a}{\partial x^2}+\frac{\partial^2 a}{\partial y^2})-m'\frac{\partial^2 a}{\partial t^2}=0$	$a(x,y,t)=A(t)e^{\mathrm{i}(\lambda_x x+\lambda_y y)}$
$A(t)k'(\lambda_x^2+\lambda_y^2)+m'\ddot{A}(t)=0$	$A(t)=\alpha e^{\mathrm{i}\omega t}$

$$a(x,y,t)=\sum(\alpha_t\sin\omega t+\beta_t\cos\omega t)(\alpha_x\sin\lambda_x x+\beta_x\cos\lambda_x x)(\alpha_y\sin\lambda_y y+\beta_y\cos\lambda_y y)$$

5.1 MEMBRANE MODEL

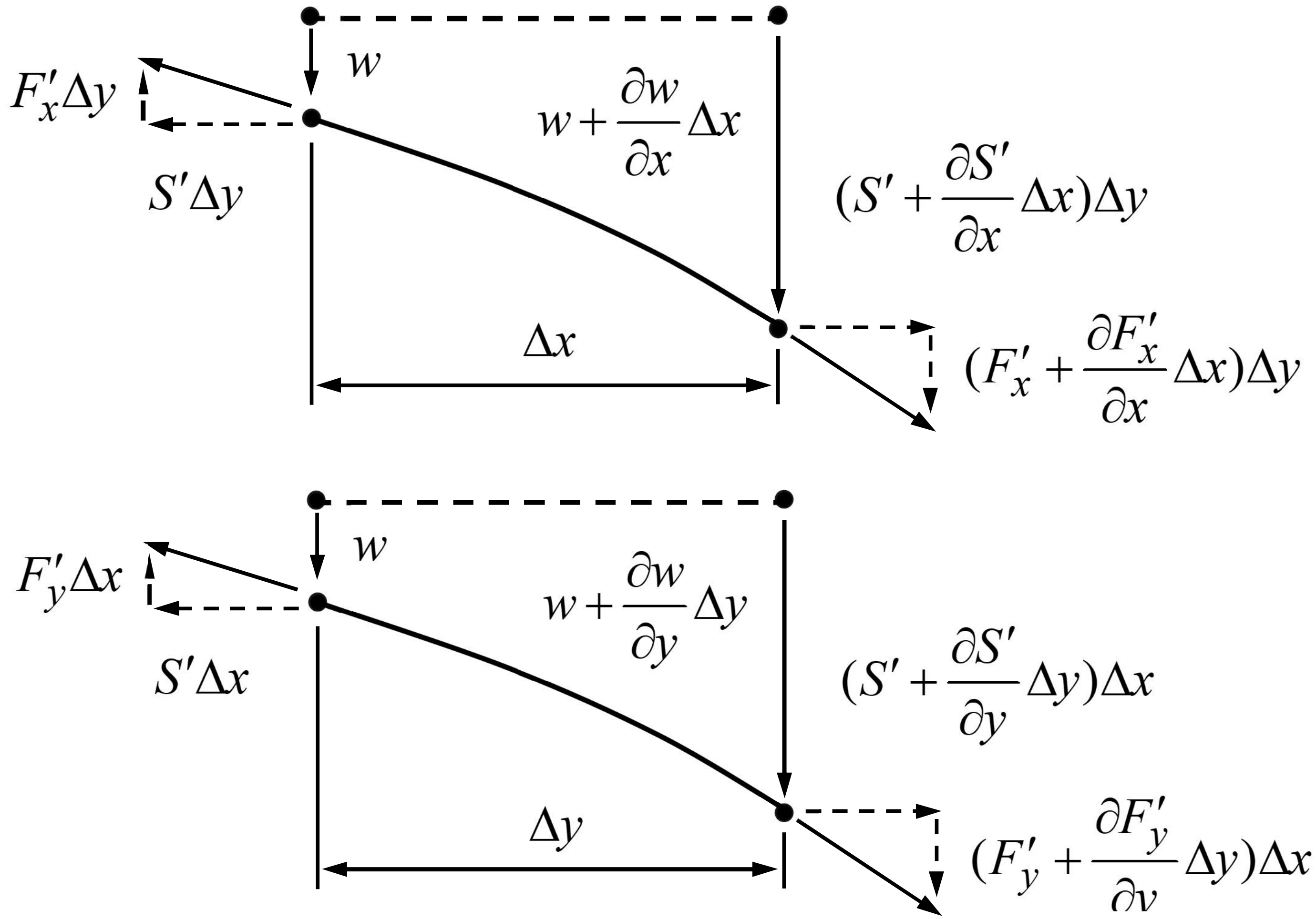


Equation of motion $S'(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2})+f'=m'\frac{\partial^2 w}{\partial t^2} \quad (x,y)\in\Omega \quad t>0,$

Boundary conditions $w=\underline{w}$ or $S'(n_x\frac{\partial w}{\partial x}+n_y\frac{\partial w}{\partial y})=F' \quad (x,y)\in\partial\Omega \ t>0,$

Initial conditions $w=g$ and $\frac{\partial w}{\partial t}=h \quad (x,y)\in\Omega \quad t=0.$

Derivation starting from the first principles goes in the same manner as for the string model but uses a rectangular material element of side lengths Δx and Δy:



The tightening S'_x and S'_y (force per unit length) in the $x-$ and $y-$directions may differ, in principle. Let us consider the isotropic case $S'_x = S'_y = S'$. As material elements are assumed to move only in the transverse direction, equations of motions in the horizontal directions become

Momentum balance (x) : $(S' + \frac{\partial S'}{\partial x}\Delta x)\Delta y - S'\Delta y = 0 \quad \Rightarrow \quad \frac{\partial S'}{\partial x} = 0$,

Momentum balance (y) : $(S' + \frac{\partial S'}{\partial y}\Delta y)\Delta x - S'\Delta x = 0 \quad \Rightarrow \quad \frac{\partial S'}{\partial y} = 0$.

Hence, S' (force per unit length) needs to be constant. Using the deformed geometry of the material element and the assumption that the internal forces are aligned with the tangent of the mid-plane, gives the representations

$$F'_x = S'\frac{\partial w}{\partial x} \quad \text{and} \quad F'_y = S'\frac{\partial w}{\partial y}$$

so the equation of motion in the transverse direction takes the form

$$\frac{\partial F_x'}{\partial x}+\frac{\partial F_y'}{\partial y}+f'=S'(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2})+f'=m'\frac{\partial^2 w}{\partial t^2}. \quad \Leftarrow$$

where $m'=\rho t$ is the mass per unit area.

NOTICE: The string and membrane equations as defined in this course do not follow, e.g., from the principle of virtual work for linear elasticity theory in the same manner as, e.g., the well-known beam, plate etc. models but requires the use of large-displacement theory with the kinetic assumption that tightening of the initial flat geometry is constant and not affected by the transverse displacement.

FOURIER SERIES

The Fourier series (various forms exist) can be used to represent a function as the sum of harmonic terms. For example, the sine-transformation pair for a function $a(x)$ $x \in [0, L]$ with vanishing values at the end points is given by

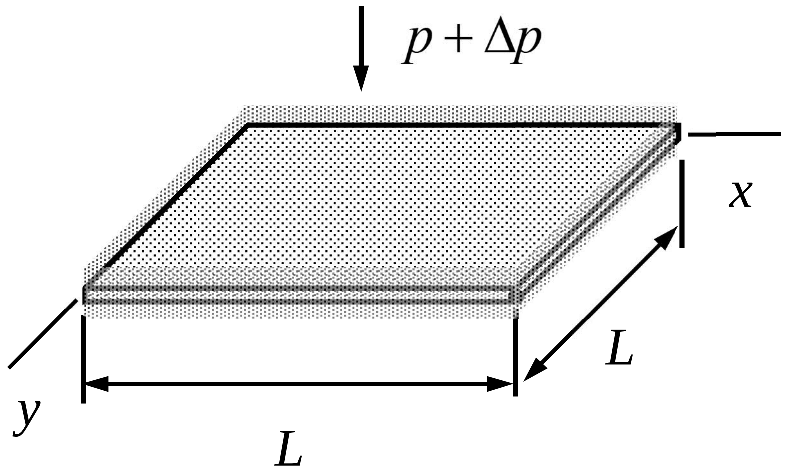
$$\alpha_j = \frac{2}{L}\int_0^L \sin(j\pi\frac{x}{L})a(x)dx \quad j \in \{1,2,\ldots\} \quad \Leftrightarrow \quad a(x) = \sum_{j\in\{1,2,\ldots\}} \alpha_j \sin(j\pi\frac{x}{L}).$$

The transformation pair is based on the orthogonality of the modes

$$\int_0^L \sin(j\pi\frac{x}{L})\sin(l\pi\frac{x}{L})dx = \frac{L}{2}\delta_{jl} \text{ (Kronecker delta).}$$

The transformation (with respect to time) can be used to analyze frequency contents of data, filtering, to find the combination of the terms of the generic series solution for bar, string and membrane models satisfying the initial conditions, etc.

EXAMPLE A rectangular membrane of fixed edges and constant tightening S' (force per unit length) is loaded by pressures $p+\Delta p$ acting on the upper surface and p acting on the lower surface. Find the transverse displacement by using the double sine series trial solution and double sine series representation of the loading.



Answer $w(x,y)=\sum_{k\in\{1,3,5\ldots\}}\sum_{l\in\{1,3,5\ldots\}}\frac{16}{\pi^4}\frac{\Delta pL^2}{S'}\frac{1}{kl(k^2+l^2)}\sin(k\pi\frac{x}{L})\sin(l\pi\frac{y}{L})$

According to the problem decription, displacement and the constant loading due to excess pressure on the upper surface, should be presented by double sine series ($k,l \in \{1,2,\ldots\}$)

$$w(x,y)=\sum\sum w_{kl}\sin(k\pi\frac{x}{L})\sin(l\pi\frac{y}{L}), \quad f(x,y)=\sum\sum f_{kl}\sin(k\pi\frac{x}{L})\sin(l\pi\frac{y}{L}).$$

As the boundary conditions are satisfied by the trial solution, it is enough to concentrate on the differential equation. Substituting the typical terms of series representations, gives a condition for the multipliers of the sine terms:

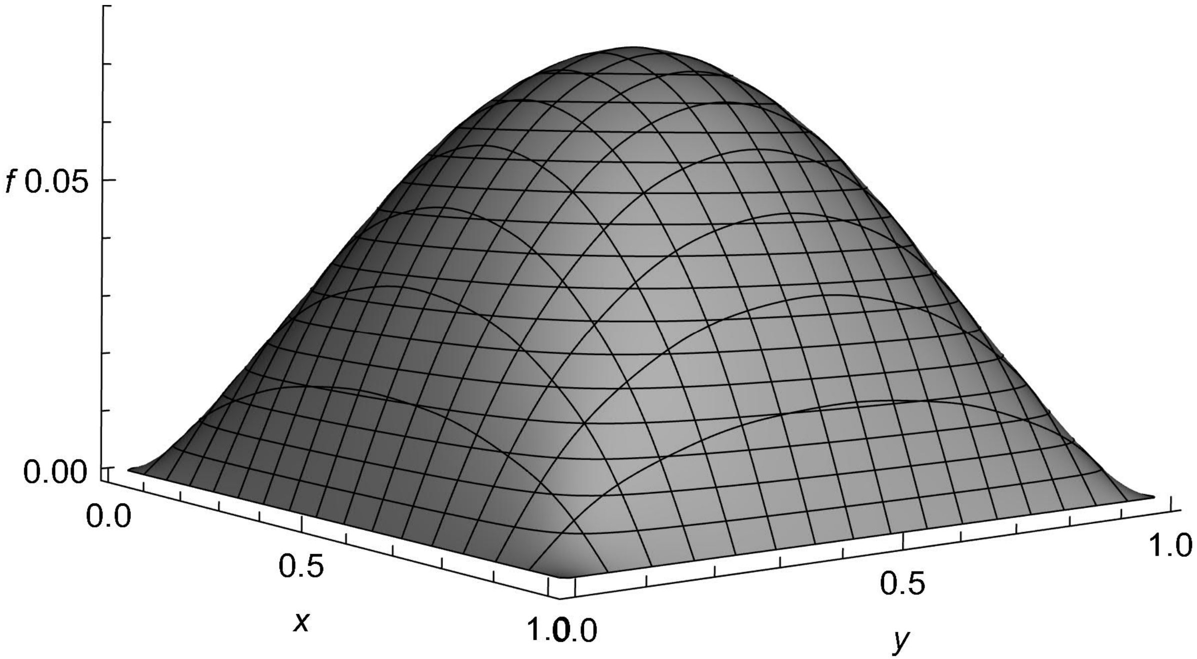
$$-w_{kl}S'[(\frac{k\pi}{L})^2+(\frac{l\pi}{L})^2]+f_{kl}=0 \quad \Leftrightarrow \quad w_{kl}=\frac{L^2}{S'\pi^2}\frac{1}{k^2+l^2}f_{kl}.$$

What remains, is finding the coefficients of the double sine representation of the loading. Using the orthogonality of sines in both coordinate directions

$$\Delta p = \sum \sum f_{kl} \sin(k\pi \frac{x}{L}) \sin(l\pi \frac{y}{L}) \iff f_{kl} = \frac{16}{\pi^2}\frac{1}{kl}\Delta p \text{ where } k,l \in \{1,3,5,\ldots\}$$

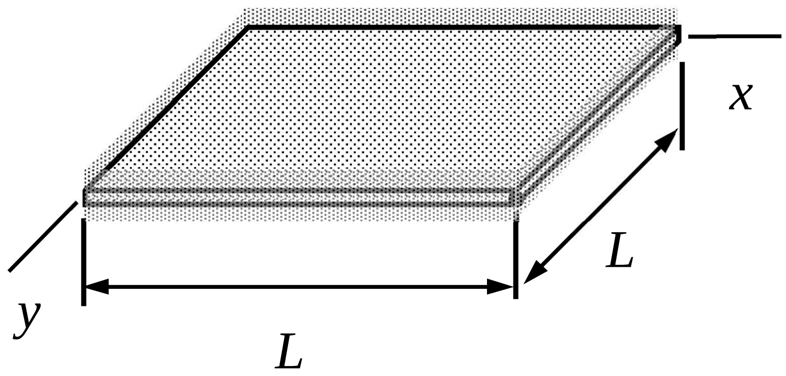
the other coefficients being zeros. Therefore, the series solution takes the form

$$w(x,y) = \sum_{k\in\{1,3,5\ldots\}} \sum_{l\in\{1,3,5\ldots\}} \frac{16}{\pi^4}\frac{\Delta p L^2}{S'}\frac{1}{kl(k^2+l^2)} \sin(k\pi \frac{x}{L}) \sin(l\pi \frac{y}{L}).$$



EXAMPLE Consider a rectangular drumhead of fixed edges, constant tightening S' (force per unit length) and density ρt (per unit area). Find the frequencies of the free vibrations by using the double sine series trial solution

$$w(x,y,t) = \sum \sum w_{kl}(t) \sin(k\pi \frac{x}{L}) \sin(l\pi \frac{y}{L}).$$



Answer $f_{kl} = \dfrac{\pi}{2L}\sqrt{(k^2 + l^2)\dfrac{S'}{\rho t}}$

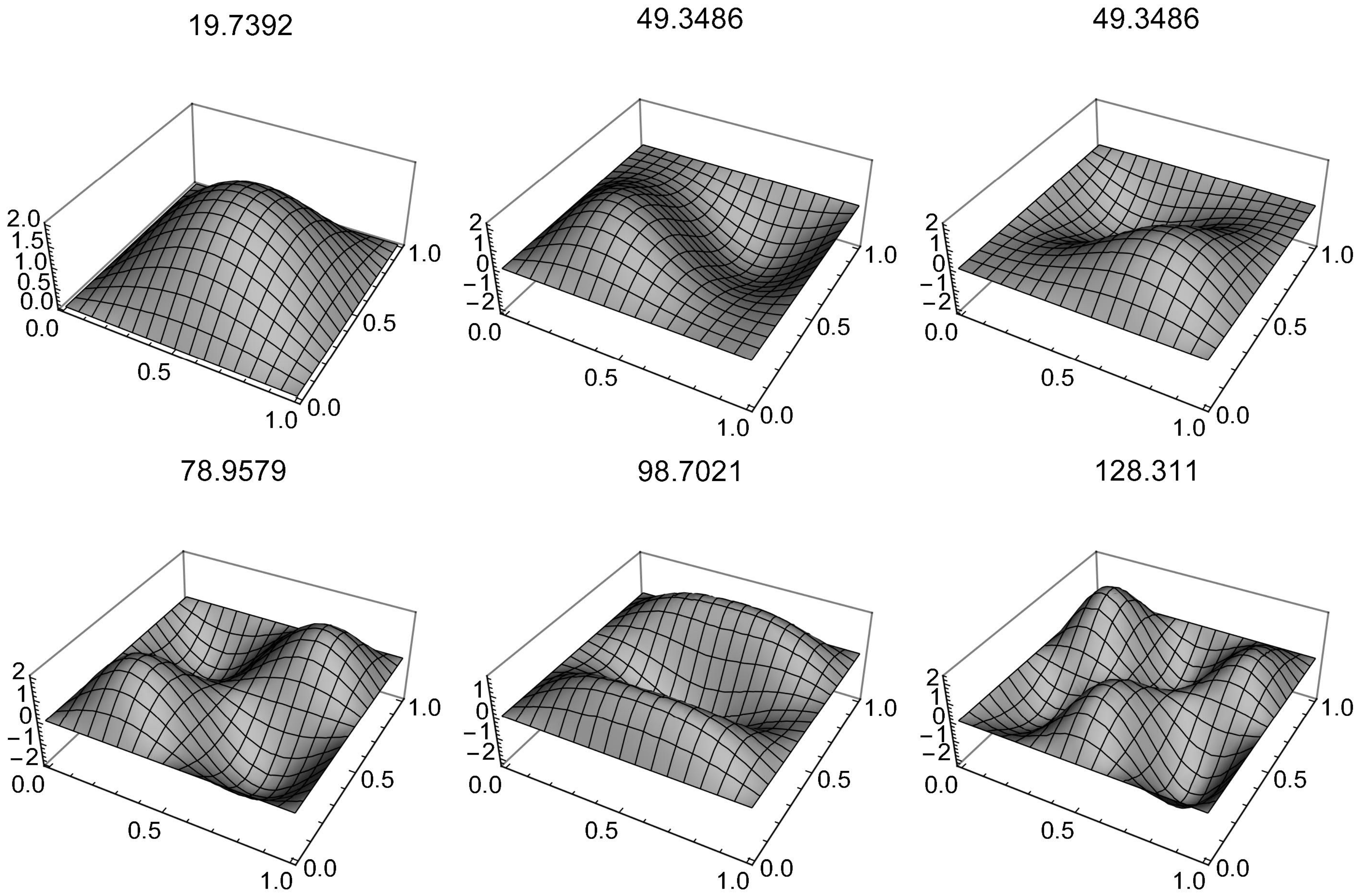
The solution trial, composed of the sines in both directions, and amplitudes depending on time is given by $k, l \in \{1, 2, \ldots\}$)

$$w(x, y, t) = \sum \sum w_{kl}(t) \sin(k\pi \frac{x}{L}) \sin(l\pi \frac{y}{L}).$$

As the boundary conditions are satisfied by the trial solution, it is enough to concentrate on the differential equation. Substitution of the typical term, gives the ordinary differential equation

$$\ddot{w}_{kl} + \omega_{kl}^2 w_{kl} = 0 \quad \text{where} \quad \omega_{kl} = 2\pi f_{kl} = \frac{\pi}{L} \sqrt{(k^2 + l^2) \frac{S'}{\rho t}}$$

the corresponding modes being the double sine terms of the trial solution. The smallest frequency is given by selection $k = l = 1$: $f_{11} = \sqrt{S' / \rho t} / \sqrt{2} L$.


19.7392
49.3486
49.3486
78.9579
98.7021
128.311

MODE SUPERPOSITION

If the initial conditions concerning position and displacement of the particles are known (quite exceptional case in practice), the outcome of the modal analysis $(\omega, A)_j$ can be used to construct a displacement solution for the given initial data starting with the series

(a) $$a(x,y,t) = \sum \sum A_{kl}(x,y)[\alpha_{kl} \frac{1}{\omega_{kl}} \sin(\omega_{kl} t) + \beta_{kl} \cos(\omega_{kl} t)].$$

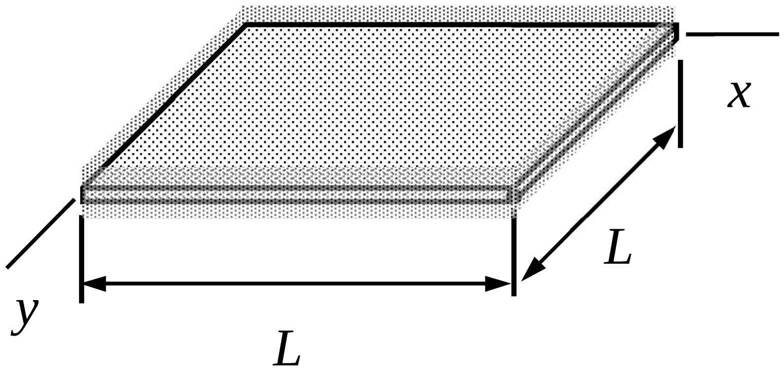
The combination of the modes giving $a = g(x,y)$ and $\partial a / \partial t = h(x,y)$ at $t = 0$ follow with the expressions

(b) $$\alpha_{kl} = \frac{1}{\mathrm{A}_{kl}^2} \int_\Omega A_{kl}(x,y) h dA, \;\; \beta_{kl} = \frac{1}{\mathrm{A}_{kl}^2} \int_\Omega A_{kl}(x,y) g dA, \;\; \mathrm{A}_{kl}^2 = \int_\Omega A_{kl}(x,y) A_{kl}(x,y) dA.$$

The coefficients correspond to the spatial Fourier series of the initial data obtained with the orthogonal harmonic modes from the modal analysis.

EXAMPLE Consider a rectangular drumhead of fixed edges, constant tightening S' (force per unit length) and density ρt (per unit area). Find the solution to the transverse displacement if the initial displacement $g(x,y) = W\sin(k\pi x/L)\sin(l\pi y/L)$ and initial velocity vanishes. The outcome of the modal analysis is the angular velocity-mode pairs

$$\omega_{kl} = \frac{\pi}{L}\sqrt{(k^2+l^2)\frac{S'}{\rho t}} \quad \text{and} \quad A_{kl}(x,y) = \sin(k\pi\frac{x}{L})\sin(l\pi\frac{y}{L}).$$



Answer $w(x,y,t) = W\sin(k\pi\frac{x}{L})\sin(l\pi\frac{y}{L})\cos(\frac{\pi}{L}\sqrt{(k^2+l^2)\frac{S'}{\rho t}}t)$

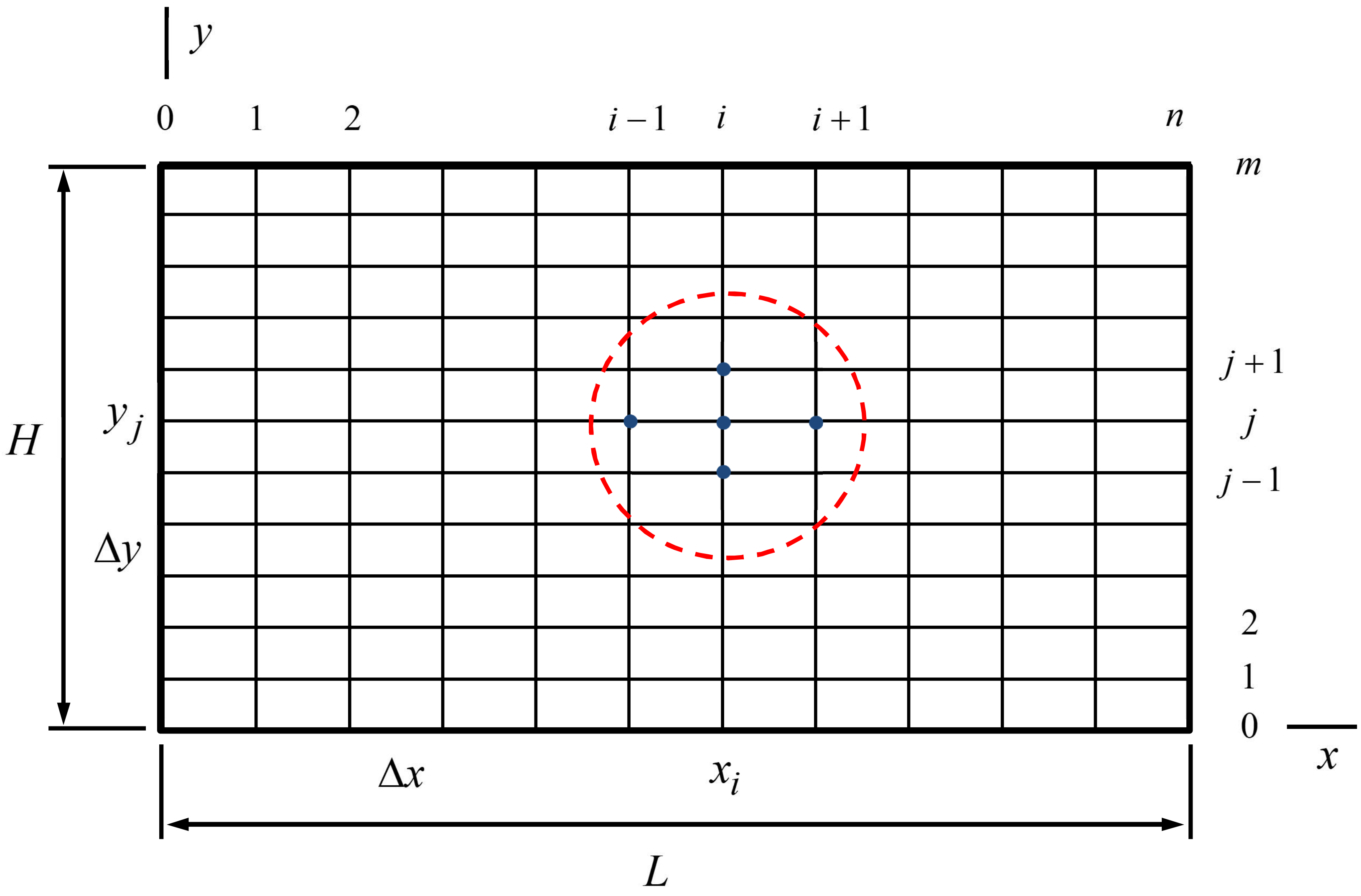
In the same manner as with the string problem, solution as the function of spatial coordinates and time is obtained by mode superposition with

$$a(x,y,t) = \sum \sum A_{kl}(x,y)[\alpha_{kl} \frac{1}{\omega_{kl}} \sin(\omega_{kl} t) + \beta_{kl} \cos(\omega_{kl} t)].$$

As initial velocity vanishes $\alpha_{kl} = 0$. The initial displacement is one of the modes (some fixed k and l, so $\beta_{kl} = W$ the remaining being zeros (no summing now)

$$w(x,y,t) = W \sin(k\pi \frac{x}{L}) \sin(l\pi \frac{y}{L}) \cos(\frac{\pi}{L} \sqrt{(k^2 + l^2) \frac{S'}{\rho t}} t).$$ ←

5.2 APPROXIMATION TO DERIVATIVES



TAYLOR'S THEOREM

Taylor series with the remainder term is an important tool in numerics, e.g., in the finite difference method. Theorem tells how to approximate a function in some neighborhood of a point by a polynomial.

1D: $$f(x+\Delta x) = \sum_{i=0}^{n} \frac{1}{i!}(\Delta x \frac{d}{dx})^{i} f(x) + [\frac{1}{(n+1)!}(\Delta x \frac{d}{dx})^{n+1} f(x)]_{\xi}$$

nD: $$f(\mathbf{x}+\Delta \mathbf{x}) = \sum_{i=0}^{n} \frac{1}{i!}(\Delta \mathbf{x}^{\mathrm{T}} \nabla)^{i} f(\mathbf{x}) + [\frac{1}{n+1!}(\Delta \mathbf{x}^{\mathrm{T}} \nabla)^{n+1} f(\mathbf{x})]_{\boldsymbol{\xi}}$$

Theorem assumes existence of the *n*:th derivative. In the remainder term, ξ is some point to the interval and is different in each occurrence). For example, finite difference approximations to derivatives in terms of values of pointwise values of a function follow from the theorem.

The generic form simplifies to

$$f(x+\Delta x, y+\Delta y) = \sum_{i=0}^{n} \frac{1}{i!}(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y})^{i} f(x,y) + R(\xi,\eta)$$

when expressions $\Delta \mathbf{x}^{\mathrm{T}} = \{\Delta x \quad \Delta y\}$, $\nabla^{\mathrm{T}} = \{\partial / \partial x \quad \partial / \partial y\}$ and $\boldsymbol{\xi} = (\xi,\eta)$ are used there. The remainder term is given by

$$R(\xi,\eta) = [\frac{1}{n+1!}(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y})^{n+1} f(x,y)]_{(\xi,\eta)},$$

where $\xi \in [x, x+\Delta x]$ and $\eta \in [y, y+\Delta y]$.

DIFFERENCE APPROXIMATIONS

Derivative	Central difference	Order
$\left(\dfrac{\partial^2 f}{\partial x^2}\right)_{(i,j)}$	$\dfrac{f_{(i-1,j)} - 2f_{(i,j)} + f_{(i+1,j)}}{\Delta x^2}$	2
$\left(\dfrac{\partial^2 f}{\partial y^2}\right)_{(i,j)}$	$\dfrac{f_{(i,j-1)} - 2f_{(i,j)} + f_{(i,j+1)}}{\Delta y^2}$	2
$\left(\dfrac{\partial^2 f}{\partial x \partial y}\right)_{(i,j)}$	$\dfrac{f_{(i+1,j+1)} - f_{(i+1,j-1)} - f_{(i-1,j+1)} - f_{(i-1,j-1)}}{4\Delta x \Delta y}$	2

Although the expressions follow in the same manner as in the one-dimensional case, hand calculations are a bit tedious with the method based on Taylor series.

The method using the interpolation of a dataset work also in two-dimensions although Mathematica may prove to be necessary in manipulations. Let us consider the stencil $\{i-1,i,i+1\}\times\{j-1,j,j+1\}$ of constant spacing Δx and Δy and interpolation $p(x,y)=\mathbf{N}^{\mathrm{T}}\mathbf{f}$ with (9) shape functions $\mathbf{N}(x,y)=\mathbf{N}(y)\times\mathbf{N}(x)$ and function values $\mathbf{f}$ where

$$\mathbf{N}(\xi)=\{\frac{(\xi)(\xi-\Delta\xi)}{(-\Delta\xi)(-\Delta\xi-\Delta\xi)},\frac{(\xi+\Delta\xi)(\xi-\Delta\xi)}{(0+\Delta\xi)(0-\Delta\xi)},\frac{(\xi+\Delta\xi)(\xi)}{(\Delta\xi+\Delta\xi)(\Delta\xi)}\}\quad \xi\in\{x,y\},$$

$$\mathbf{f}=\{f_{(i-1,j-1)},f_{(i,j-1)},f_{(i+1,j-1)},f_{(i-1,j)},f_{(i,j)},f_{(i+1,j)},f_{(i-1,j+1)},f_{(i,j+1)},f_{(i+1,j+1)}\}.$$

Difference approximations follow also from the Taylor's representation truncated at certain term and written for $\{i-1,i,i+1\}\times\{j-1,j,j+1\}$, adding and subtracting on both sides, rearranging, and dividing with an appropriate power of Δx. However, the proper combination depends on the derivative which makes the Taylor series method a bit tedious in several physical dimensions.

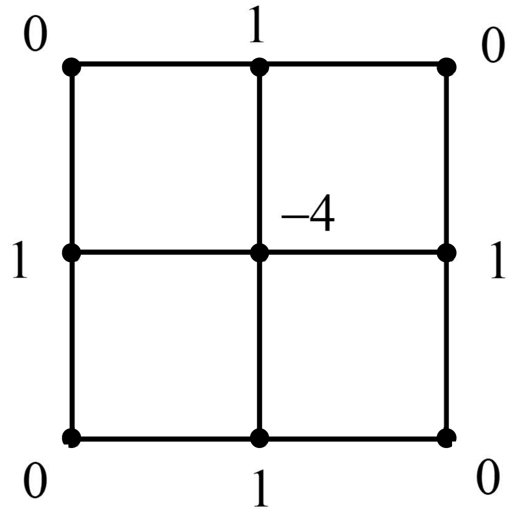
DIFFERENCE APPROXIMATIONS

Derivative	Central difference	Order
$(\frac{\partial f}{\partial x})_{(i,j)}$	$\frac{-f_{(i-1,j)} + f_{(i+1,j)}}{2\Delta x}$	2
$(\frac{\partial f}{\partial y})_{(i,j)}$	$\frac{-f_{(i,j-1)} + f_{(i,j+1)}}{2\Delta y}$	2
$(\frac{\partial f}{\partial x})_{(i,j)}$	$\frac{-f_{(i,j)} + f_{(i,j+1)}}{\Delta y}$	1
$(\frac{\partial f}{\partial y})_{(i,j)}$	$\frac{-f_{(i,j-1)} + f_{(i,j)}}{\Delta y}$	1

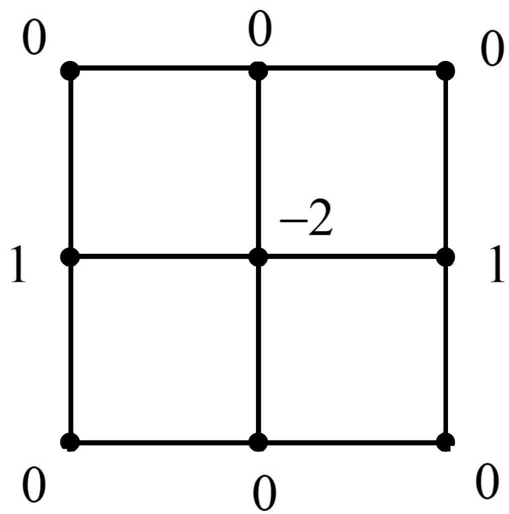
In two-dimensions, various stencils can be used in the difference approximation to derivative at point (i, j).

STENCIL

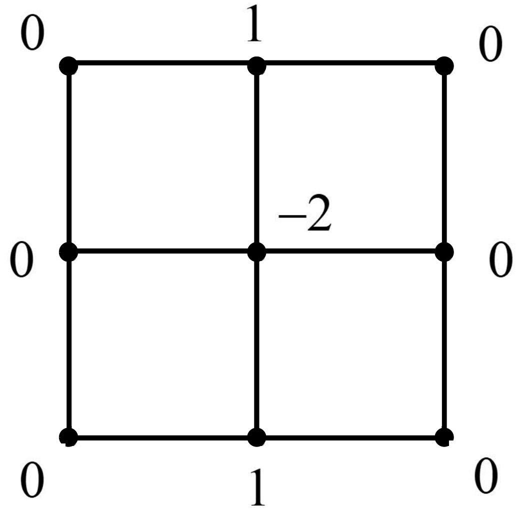
Stencil is used as a concise way to represent the difference approximations with a geometric pattern on the grid with the associated multipliers of the function values on the grid. For a regular grid $\Delta x = \Delta y$



$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$



$$\frac{\partial^2}{\partial x^2}$$



$$\frac{\partial^2}{\partial y^2}$$

5.3 FINITE DIFFERENCE METHOD

Finite Difference Method is a numerical technique for solving ordinary and partial differential equations by approximating derivatives with finite difference formulas. On a regular grid $\Delta x = \Delta y = h$, the membrane model with zero displacement boundary conditions

Interior $\dfrac{S'}{h^2}[w_{(i-1,j)} + w_{(i,j-1)} - 4w_{(i,j)} + w_{(i+1,j)} + w_{(i,j+1)}] + f_i' = m_i' \ddot{w}_{(i,j)} \quad (i,j) \in I$

Boundary $w_{(i,j)} = 0 \quad (i,j) \in \partial I$

Initial conditions $w_{(i,j)} - g_{(i,j)} = 0$ and $\dot{w}_{(i,j)} - h_{(i,j)} = 0 \quad (i,j) \in I$

Where the interior grid point are denoted by I and the boundary grid points ∂I. Then, the outcome is a set of Ordinary Differential Equations which can be solved with the matrix and difference equation methods used for the bar and string models.

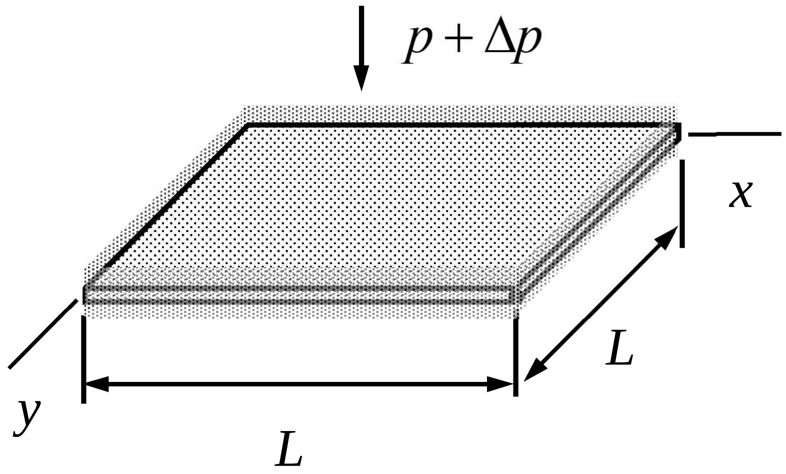
In the finite difference method, derivatives in the continuum model with respect to spatial coordinates are replaced by difference approximations, to get a set of ordinary differential equatios or a set of algebraic equations. The aim is replace a problem, which may be difficult solve as it stands, by a mathematically simpler problem. The prize one has to pay comes from the discretization error. For a membrane problem of fixed boundaries, the continuum model is given by

$$S'(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2})+f'=m'\frac{\partial^2 w}{\partial t^2} \quad (x,y)\in\Omega, \qquad w=0 \quad (x,y)\in\partial\Omega \quad t>0,$$

$$w=g \quad \text{and} \quad \frac{\partial w}{\partial t}=h \quad (x,y)\in\Omega \quad t=0.$$

Using the central difference approximation to the two partial derivatives and denoting the interior grid point by I and the boundary grid points ∂I, the equation system transforms to ordinary differential equations.

EXAMPLE A rectangular membrane of fixed edges and constant tightening s (force per unit length) is loaded by pressures $p+\Delta p$ acting on the upper surface and p acting on the lower surface. Find the solution to the transverse displacement by using the Finite Difference Method and a regular grid $(i, j) \in \{0,1,2\} \times \{0,1,2\}$.



Answer $w_{(1,1)} = \dfrac{1}{16}\dfrac{\Delta p L^2}{S'} \approx 0.0625 \dfrac{\Delta p L^2}{S'}$ (exact to the model $0.0737 \dfrac{\Delta p L^2}{S'}$)

In the finite difference method, derivatives in the continuum model with respect to spatial coordinates are replaced by difference approximations, to get a set of algebraic equations. In the present problem the interior and boundary grid points and the equations for the grid points are

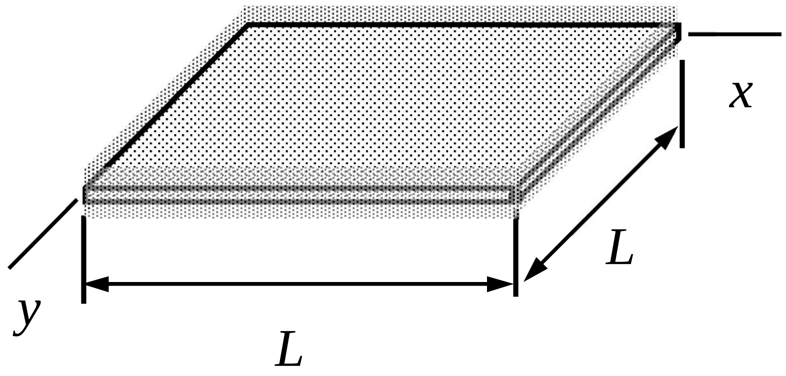
$$\frac{S'}{h^2}[w_{(i-1,j)} + w_{(i,j-1)} - 4w_{(i,j)} + w_{(i+1,j)} + w_{(i,j+1)}] + \Delta p = 0 \quad (i,j) \in I = \{(1,1)\}$$

$$w_{(i,j)} = 0 \quad (i,j) \in \partial I = \{0,1,2\} \times \{0,1,2\} \setminus \{(1,1)\} \quad \text{(interior point excluded)}$$

where $\Delta x = \Delta x = h = L/2$. Eliminating the displacements of the boundary points from the equation for the interior point

$$4\frac{S'}{L^2}[-4w_{(1,1)}] + \Delta p = 0 \quad \Rightarrow \quad w_{(1,1)} = \frac{1}{16}\frac{\Delta p L^2}{S'}.$$

EXAMPLE Consider a rectangular drumhead of fixed edges, constant tightening S' (force per unit length) and density ρt (per unit area). Find the frequency of the free vibrations by using the Finite Difference Method and a regular grid $(i, j) \in \{0,1,2\} \times \{0,1,2\}$.



Answer $f = \frac{4}{2\pi} \frac{1}{L} \sqrt{\frac{S'}{\rho t}} \approx 0.64 \frac{1}{L} \sqrt{\frac{S'}{\rho t}}$ (exact to the model $\approx 0.71 \frac{1}{L} \sqrt{\frac{S'}{\rho t}}$)

In the finite difference method, derivatives in the continuum model with respect to spatial coordinates are replaced by difference approximations to get a set of ordinary differential equations. In the present problem the interior and boundary grid points and the equations for the grid points are

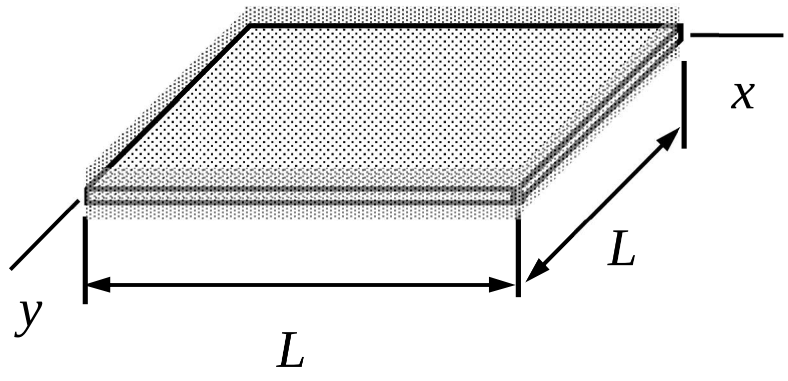
$$\frac{S'}{h^2}[w_{(i-1,j)}+w_{(i,j-1)}-4w_{(i,j)}+w_{(i+1,j)}+w_{(i,j+1)}]=\rho t\ddot{w}_{(i,j)} \quad (i,j)\in I=\{(1,1)\}$$

$$w_{(i,j)}=0 \quad (i,j)\in\partial I=\{0,1,2\}\times\{0,1,2\}\setminus\{(1,1)\} \quad \text{(interior point excluded)}$$

where $\Delta x=\Delta x=h=L/2$. Eliminating the displacements of the boundary points from the equation for the interior point

$$\ddot{w}_{(1,1)}+\omega^2 w_{(1,1)}=0 \quad \text{where} \quad \omega=2\pi f=\frac{4}{L}\sqrt{\frac{S'}{\rho t}} \quad \text{so} \quad f=\frac{4}{2\pi}\frac{1}{L}\sqrt{\frac{S'}{\rho t}}\,.$$

EXAMPLE Consider a rectangular drumhead of fixed edges, constant tightening S' (force per unit length) and density ρt (per unit area). Find the angular velocities of the free vibrations and the corresponding modes as predicted by the Finite Difference Method and a regular grid $(i, j) \in \{0,1,2,\ldots,n\} \times \{0,1,2,\ldots,n\}$. Use the trial solution for the typical mode $w_{(i,j)}(t) = a(t)\sin(k\pi i / n)\sin(l\pi j / n)$.



Answer $\omega_{kl} = \dfrac{1}{L}\sqrt{\dfrac{S'}{\rho t}\dfrac{2}{n^2}(2 - \cos\dfrac{k\pi}{n} - \cos\dfrac{l\pi}{n})}$

The equations for the boundary points are satisfied by the trial solution so it is enough to consider the ordinary differential equations for the interior points

$$\frac{S'}{h^2}[w_{(i-1,j)} + w_{(i,j-1)} - 4w_{(i,j)} + w_{(i+1,j)} + w_{(i,j+1)}] = \rho t \ddot{w}_{(i,j)}$$

When substituted into the difference expression on the left-hand side, the trial solution $w_{(i,j)} = a(t)\sin(k\pi i/n)\sin(l\pi j/n)$ and identity $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$ give expression

$$w_{(i-1,j)} + w_{(i,j-1)} - 4w_{(i,j)} + w_{(i+1,j)} + w_{(i,j+1)} = -2(2 - \cos\frac{k\pi}{n} - \cos\frac{l\pi}{n})w_{(i,j)}$$

So the partial-difference and ordinary-differential equation simplifies to an ordinary-differential equation

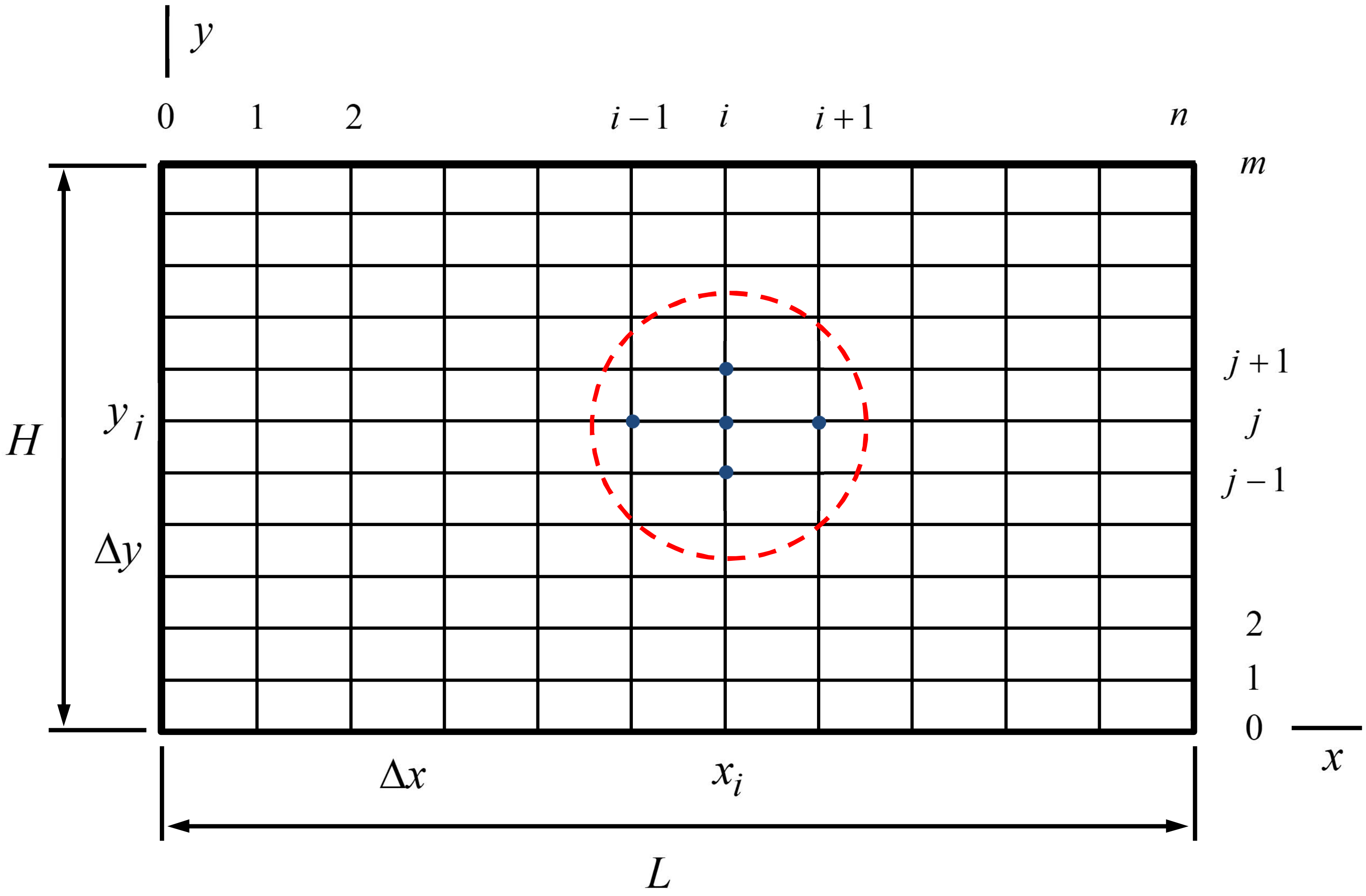
$$\ddot{w}_{(i,j)} + \omega_{kl}^2 w_{(i,j)} = 0 \quad \text{where} \quad \omega_{kl} = \frac{1}{L}\sqrt{\frac{S'}{\rho t}\frac{2}{n^2}\left(2 - \cos\frac{k\pi}{n} - \cos\frac{l\pi}{n}\right)} \; .$$

In verification that the solution to the limit problem $n \to \infty$ coincides with the exact solution (a desirable property of a numerical method)

$$\omega_{kl} = \frac{\pi}{L}\sqrt{\frac{S'}{\rho t}(k^2 + l^2)}$$

one may assume that k, l are bounded and use $\cos\alpha \approx 1 - \alpha^2 / 2$ when $\alpha \ll 1$.

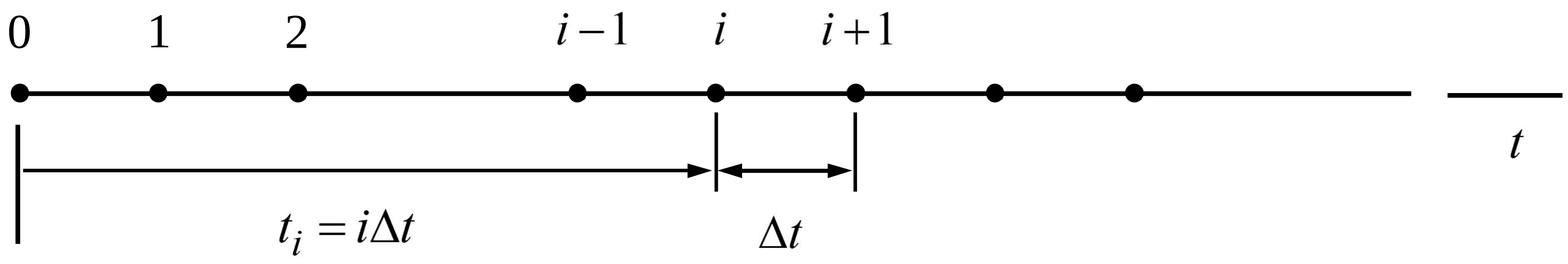
ONE-INDEX NUMBERING



In practice, a regular two-index numbering is used mainly with rectangular solution domains. The one index labelling of the grid points in any order works well in calculations with the matrix representation of the equilibrium equations and equations of motion as the order of the equations or labelling does not matter (if the number of algebraic operations needed to find the solution is not considered).

5.5 TIME INTEGRATION

The one-step DG (Discontinuous Galerkin) and CN (Crank-Nicolson) methods can be applied to the membrane problem in the same manner as for the bar, and string problems to find the solution on a grid of the temporal domain.



As the temporal domain for an initial value problem does not have an upper bound (strictly speaking). Also, the length of the intervals can be chosen to match the behavior of the solution (small steps for the rapid changes).

TIME INTEGRATION

Method	Iteration $i \in \{1, 2, \ldots\}$	Initial $i = 0$
EX	$\begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_i = \begin{bmatrix} \cos\alpha & \alpha^{-1}\sin\alpha \\ -\alpha\sin\alpha & \cos\alpha \end{bmatrix} \begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_{i-1}$	$\begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_0 = \begin{Bmatrix} g \\ \Delta t h \end{Bmatrix}$
CN	$\begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_i = \dfrac{1}{4+\alpha^2} \begin{bmatrix} 4-\alpha^2 & 4 \\ -4\alpha^2 & 4-\alpha^2 \end{bmatrix} \begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_{i-1}$	$\begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_0 = \begin{Bmatrix} g \\ \Delta t h \end{Bmatrix}$
DG	$\begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_i = \dfrac{2}{12+\alpha^4} \begin{bmatrix} 6-3\alpha^2 & 6-\alpha^2 \\ -6\alpha^2 & 6-3\alpha^2 \end{bmatrix} \begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_{i-1}$	$\begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_0 = \begin{Bmatrix} g \\ \Delta t h \end{Bmatrix}$

The methods coincide at the limit of vanishing step-size when $\alpha = \sqrt{\dfrac{k}{m}}\Delta t \to 0$.

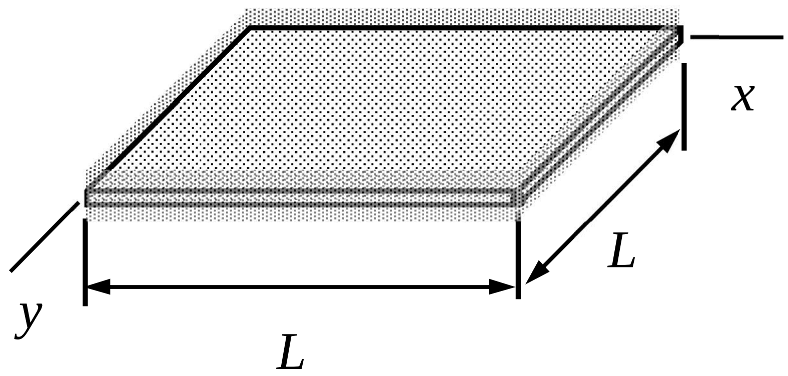
ONE-STEP METHODS FOR EQUATION SYSTEM

DG: $$\begin{bmatrix} \Delta t^2\mathbf{K} & -\frac{1}{2}\Delta t^2\mathbf{K}+\mathbf{M} \\ \frac{1}{2}\Delta t^2\mathbf{K}-\mathbf{M} & \mathbf{M}-\frac{1}{6}\Delta t^2\mathbf{K} \end{bmatrix} \begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}}\Delta t \end{Bmatrix}_i = \begin{bmatrix} \mathbf{0} & \mathbf{M} \\ -\mathbf{M} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}}\Delta t \end{Bmatrix}_{i-1}, \quad \begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}}\Delta t \end{Bmatrix}_0 = \begin{Bmatrix} \mathbf{g} \\ \mathbf{h}\Delta t \end{Bmatrix}$$

CN: $$\begin{bmatrix} \mathbf{I} & -\frac{1}{2}\mathbf{I} \\ \frac{\Delta t}{2}\mathbf{K} & \mathbf{M} \end{bmatrix} \begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}}\Delta t \end{Bmatrix}_i = \begin{bmatrix} \mathbf{I} & \frac{1}{2}\mathbf{I} \\ -\frac{\Delta t}{2}\mathbf{K} & \mathbf{M} \end{bmatrix} \begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}}\Delta t \end{Bmatrix}_{i-1}, \quad \begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}}\Delta t \end{Bmatrix}_0 = \begin{Bmatrix} \mathbf{g} \\ \mathbf{h}\Delta t \end{Bmatrix}$$

The proper step-size Δt depends on the largest eigenvalue of parameter $\boldsymbol{\alpha} = \mathbf{M}^{-1}\mathbf{K}\Delta t^2$. The numerical damping of DG exceeds that of CN whereas the phase error of CN exceeds that of the DG method.

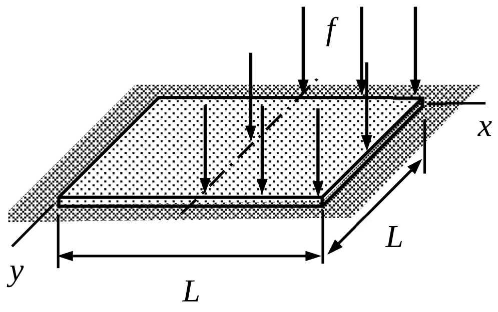
EXAMPLE Finite Difference Method using a regular grid $(i, j) \in \{0,1,2\} \times \{0,1,2\}$ is applied to discretize the equations for the rectangular drum head shown. Thereafter, Crank-Nicolson method is applied to find the solution at the temporal grid $t_k = k\Delta t \;\; k \in \{0,1,\ldots\}$. Derive the iteration formula giving the displacements and velocities of points of the spatial discretization starting from the known initial displacement and velocities. Membrane tightening S' and density per unit area ρt are constants.



Answer Discussed during the lectures of week 20

COE-C3005 Finite Element and Finite difference methods

1. A rectangular membrane of side length L, density ρ, thickness t, and tightening S' is loaded by a constant distributed force f acting on half of the membrane as shown. If the edges are fixed, find the transverse displacement by using the continuum model and double sine series representation of the displacement and force.



Answer $w(x,y)=\sum\sum \frac{f}{S'}(\frac{L}{\pi})^4\frac{1}{kl}\frac{1}{k^2+l^2}[\cos(k\frac{\pi}{2})-\cos(k\pi)][1-\cos(l\pi)]\sin(k\pi\frac{x}{L})\sin(l\pi\frac{y}{L})$

2. Derive the partial difference equation to the membrane model according to the Finite Difference Method and a regular grid of different spacings Δx and Δy in the coordinate directions. Consider a generic interior point (i,j). Start with the equation of motion for the continuum model

$$S'(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2})+f'=m'\frac{\partial^2 w}{\partial t^2} \quad (x,y)\in\Omega \quad t>0$$

where tightening per unit length S', distributed force f', and mass per unit area m' are constants.

Answer $\frac{S'}{h^2}[\alpha^2 w_{(i-1,j)}+\alpha^2 w_{(i+1,j)}-2(1+\alpha^2)w_{(i,j)}+w_{(i,j-1)}+w_{(i,j+1)}]+f'=m'\ddot{w}_{(i,j)}$

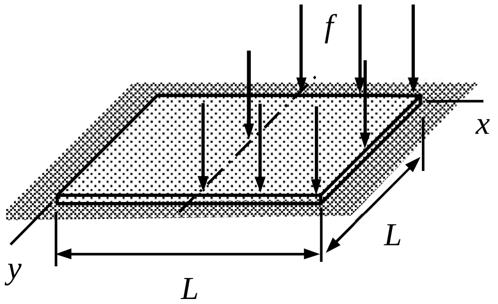
3. Derive the partial difference equation to the membrane model according to the Finite Difference Method and a regular grid of different spacings Δx and Δy in the coordinate directions. Consider a generic boundary point (i,j). Start with the equilibrium equation for the continuum model

$$S'(n_x\frac{\partial w}{\partial x}+n_y\frac{\partial w}{\partial y})=F' \quad (x,y)\in\partial\Omega \quad t>0,$$

where tightening per unit length S' and transverse external force per unit length F' are constants. Above, n_x and n_y are the components of the unit outward normal to the boundary.

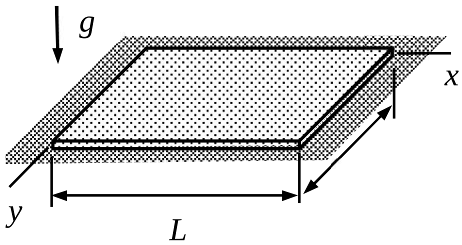
Answer When $n_x=1$ and $n_y=0$: $\frac{S'}{\Delta x}(-w_{(i-1,j)}+w_{(i,j)})=F'$ (for example)

4. A rectangular membrane of side length L and tightening S' is loaded by a constant distributed force f acting on half of the membrane as shown. If the edges are fixed, find the transverse displacement using the Finite Difference Method on a regular grid $(i,j)\in\{0,1,2,3\}\times\{0,1,2,3\}$.



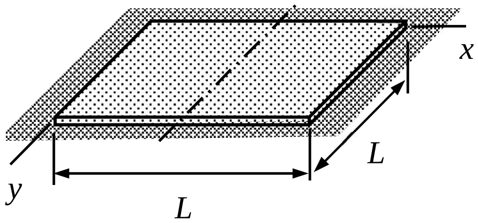
Answer $\begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} = \dfrac{fL^2}{S'}\dfrac{1}{72}\begin{Bmatrix} 1 \\ 3 \end{Bmatrix}$

5. A rectangular membrane of side length L, density ρ, thickness t, and tightening S' is loaded by its own weight as shown. If the edges are fixed, find the transverse displacements using the Finite Difference Method on a regular grid $(i,j)\in\{0,1,2,3,4\}\times\{0,1,2\}$.



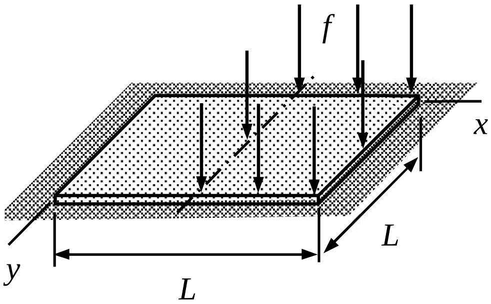
Answer $\begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} = \dfrac{\rho tgL^2}{S'}\dfrac{1}{136}\begin{Bmatrix} 7 \\ 9 \end{Bmatrix}$

6. Consider a rectangular membrane of side length L, density ρ, thickness t, and tightening S'. If the edges are fixed, find angular velocities of the modes, that are reflection symmetric with respect to the line through the center point shown using the Finite Difference Method. Use a regular grid $(i,j)\in\{0,1,2,3,4\}\times\{0,1,2\}$ of different spacings in the coordinate directions.



Answer $\omega = \dfrac{2}{L}\sqrt{(10\pm 4\sqrt{2})\dfrac{S'}{\rho t}}$

A rectangular membrane of side length L, density ρ, thickness t, and tightening S' is loaded by a constant distributed force f acting on half of the membrane as shown. If the edges are fixed, find the transverse displacement by using the continuum model and double sine series representation of the displacement and force.



Solution

According to the problem, transverse displacement and force are considered to be given by double sine series

$$w(x,y)=\sum\sum w_{kl}\sin(k\pi\frac{x}{L})\sin(l\pi\frac{y}{L}) \text{ and } f(x,y)=\sum\sum f_{kl}\sin(k\pi\frac{x}{L})\sin(l\pi\frac{y}{L}),$$

where the sums are over the sets $k\in\{1,2,\ldots\}$, $l\in\{1,2,\ldots\}$ and w_{kl}, f_{kl} should be determined by using equilibrium equation and the known distribution of the external force. Both expressions vanish on the boundaries no matter the multipliers. Let us substitute first into the equilibrium equation

$$S'(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2})+f'=0 \quad\Rightarrow\quad \sum\sum \sin(k\pi\frac{x}{L})\sin(l\pi\frac{y}{L})\{[-S'w_{kl}(\frac{k\pi}{L})^2+(\frac{l\pi}{L})^2]+f_{kl}\}=0$$

which implies the relationship

$$w_{kl}=\frac{f_{kl}}{S'}(\frac{L}{\pi})^2\frac{1}{k^2+l^2}.$$

Orthogonality of sines according to

$$\int_0^L \sin(k\pi\frac{x}{L})\sin(i\pi\frac{x}{L})dx=\frac{L}{2}\delta_{ik} \text{ and } \int_0^L \sin(l\pi\frac{y}{L})\sin(j\pi\frac{y}{L})dx=\frac{L}{2}\delta_{jl}$$

gives the expression

$$f_{ij}=(\frac{2}{L})^2\int_0^L\int_0^L \sin(i\pi\frac{x}{L})\sin(j\pi\frac{y}{L})f(x,y)dxdy \quad\Rightarrow$$

$$f_{ij}=f(\frac{2}{L})^2(\int_{L/2}^L \sin(i\pi\frac{x}{L})dx)(\int_0^L \sin(j\pi\frac{y}{L})dy) \quad\Rightarrow$$

$$f_{ij}=f(\frac{2}{\pi})^2\frac{1}{ij}[\cos(i\frac{\pi}{2})-\cos(i\pi)][1-\cos(j\pi)].$$

Combining the results

$$w(x,y)=\sum\sum \frac{f}{S'}(\frac{L}{\pi})^4\frac{1}{kl}\frac{1}{k^2+l^2}[\cos(k\frac{\pi}{2})-\cos(k\pi)][1-\cos(l\pi)]\sin(k\pi\frac{x}{L})\sin(l\pi\frac{y}{L}).$$

Derive the partial difference equation to the membrane model according to the Finite Difference Method and a regular grid of different spacings Δx and Δy in the coordinate directions. Consider a generic interior point (i, j). Start with the equation of motion for the continuum model

$$S'(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2})+f'=m'\frac{\partial^2 w}{\partial t^2} \quad (x,y)\in\Omega \quad t>0$$

where tightening per unit length S', distributed force f', and mass per unit area m' are constants.

Solution

In the Finite Difference Method, the derivatives of the differential equation with respect to the spatial coordinates are replaced by difference approximations. Using the second order accurate central difference approximations for derivatives with respect to x and y at point (i, j) and evaluating the second time derivative at that point:

$$\frac{\partial^2 w}{\partial x^2}=\frac{1}{\Delta x^2}(w_{(i-1,j)}-2w_{(i,j)}+w_{(i+1,j)}) \text{ and } \frac{\partial^2 w}{\partial y^2}=\frac{1}{\Delta y^2}(w_{(i,j-1)}-2w_{(i,j)}+w_{(i,j+1)}),$$

$$\frac{\partial^2 w}{\partial t^2}=\ddot{w}_{(i,j)}.$$

Differential equation

$$S'(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2})+f'=m'\frac{\partial^2 w}{\partial t^2}$$

gives the difference-differential equation

$$S'\frac{1}{\Delta x^2}[w_{(i-1,j)}-2w_{(i,j)}+w_{(i+1,j)}]+S'\frac{1}{\Delta y^2}[w_{(i,j-1)}-2w_{(i,j)}+w_{(i,j+1)}]+f'=m'\ddot{w}_{(i,j)} \quad \Leftrightarrow$$

$$\frac{1}{\Delta y^2}\frac{1}{\Delta x^2}[\alpha^2 w_{(i-1,j)}-2\alpha^2 w_{(i,j)}+\alpha^2 w_{(i+1,j)}]+[w_{(i,j-1)}-2w_{(i,j)}+w_{(i,j+1)}]+f'=m'\ddot{w}_{(i,j)} \quad \Leftrightarrow$$

$$\frac{S'}{h^2}[\alpha^2 w_{(i-1,j)}+\alpha^2 w_{(i+1,j)}-2(1+\alpha^2)w_{(i,j)}+w_{(i,j-1)}+w_{(i,j+1)}]+f'=m'\ddot{w}_{(i,j)},$$

where $\Delta y=h$ and $\alpha=\Delta y/\Delta x$.

Derive the partial difference equation to the membrane model according to the Finite Difference Method and a regular grid of different spacings Δx and Δy in the coordinate directions. Consider a generic boundary point (i, j). Start with the equilibrium equation for the continuum model

$$S'(n_x \frac{\partial w}{\partial x} + n_y \frac{\partial w}{\partial y}) = F' \quad (x, y) \in \partial\Omega \quad t > 0,$$

where tightening per unit length S' and transverse external force per unit length F' are constants. Above, n_x and n_y are the components of the unit outward normal to the boundary.

Solution

In the Finite Difference Method, the derivatives of the differential equation with respect to the spatial coordinates are replaced by difference approximations. The equation used and the type of difference approximation depend on the location of the point. The equilibrium equation (in stationary and non-stationary cases) for boundary point (i, j)

$$S'[n_x (\frac{\partial w}{\partial x})_{(i,j)} + n_y (\frac{\partial w}{\partial y})_{(i,j)}] = F'$$

depends on the components n_x and n_y of the unit outward normal vector to the domain. Let us consider a rectangle domain, assume that the edges are aligned with the coordinate axes, omit the corner points, and use first order accurate approximations at point (i, j) for derivatives with respect to x and y. Then, $n_x = \pm 1$ and $n_y = 0$ or $n_y = \pm 1$ and $n_x = 0$ and the backward and forward difference approximations to be used depend on n_x and n_y:

$$n_x = 1: \quad (\frac{\partial w}{\partial x})_{(i,j)} = \frac{1}{\Delta x}(-w_{(i-1,j)} + w_{(i,j)}),$$

$$n_x = -1: \quad (\frac{\partial w}{\partial x})_{(i,j)} = \frac{1}{\Delta x}(w_{(i+1,j)} - w_{(i,j)}),$$

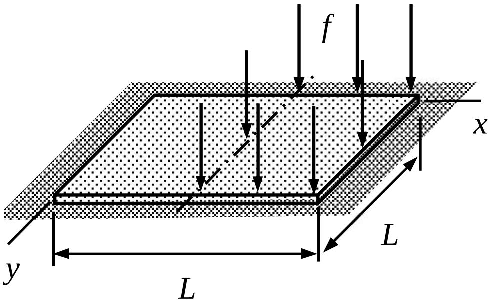
$$n_y = 1: \quad (\frac{\partial w}{\partial y})_{(i,j)} = \frac{1}{\Delta y}(-w_{(i,j-1)} + w_{(i,j)}),$$

$$n_y = -1: \quad (\frac{\partial w}{\partial y})_{(i,j)} = \frac{1}{\Delta y}(w_{(i,j+1)} - w_{(i,j)}).$$

For example, when the boundary is defined by $n_x = 1$ and $n_y = 0$:

$$\frac{S'}{\Delta x}(-w_{(i-1,j)} + w_{(i,j)}) = F'. \quad \leftarrow$$

A rectangular membrane of side length L and tightening S' is loaded by a constant distributed force f acting on half of the membrane as shown. If the edges are fixed, find the transverse displacement using the Finite Difference Method on a regular grid $(i,j)\in\{0,1,2,3\}\times\{0,1,2,3\}$.



Solution

The generic equations for the membrane model with fixed boundaries, as given by the Finite Difference Method on a regular grid, are

$$\frac{S'}{h^2}[w_{(i-1,j)}+w_{(i,j-1)}-4w_{(i,j)}+w_{(i+1,j)}+w_{(i,j+1)}]+f'=m'\ddot{w}_{(i,j)}\quad (i,j)\in I\,,$$

$$w_{(i,j)}=0\quad (i,j)\in\partial I\,,$$

$$w_{(i,j)}-g_{(i,j)}=0\quad\text{and}\quad \dot{w}_{(i,j)}-h_{(i,j)}=0\qquad (i,j)\in I\,.$$

In the present problem, time derivatives vanish, initial conditions are not needed, and solution is reflection symmetric with respect to lines through the center point and aligned with the coordinate axes. Therefore, transverse displacements at the grid points satisfy

$$w_{(1,1)}=w_{(1,2)}=w_1\,,$$

$$w_{(2,1)}=w_{(2,2)}=w_2\,.$$

As the equations by the Finite Difference Method for points $(1,1)$, $(1,2)$ and $(2,1)$, $(2,2)$ do not differ, it is enough consider $(1,1)$ and $(1,2)$ (say) with the displacement constraints. Here $h=L/3$:

$$9\frac{S'}{L^2}[w_{(0,1)}+w_{(1,0)}-4w_{(1,1)}+w_{(2,1)}+w_{(1,2)}]=0\quad\Rightarrow\quad 9\frac{S'}{L^2}(-3w_1+w_2)=0\,,$$

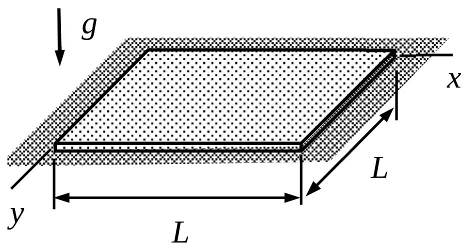
$$9\frac{S'}{L^2}[w_{(1,1)}+w_{(2,0)}-4w_{(2,1)}+w_{(3,1)}+w_{(2,2)}]+\rho tg=0\quad\Rightarrow\quad 9\frac{S'}{L^2}(w_1-3w_2)+f=0\,.$$

Using the matrix represntation

$$-9\frac{S'}{L^2}\begin{bmatrix}3 & -1\\ -1 & 3\end{bmatrix}\begin{Bmatrix}w_1\\ w_2\end{Bmatrix}+f\begin{Bmatrix}0\\ 1\end{Bmatrix}=0\quad\Leftrightarrow$$

$$\begin{Bmatrix}w_1\\ w_2\end{Bmatrix}=\frac{1}{9}\frac{fL^2}{S'}\begin{bmatrix}3 & -1\\ -1 & 3\end{bmatrix}^{-1}\begin{Bmatrix}0\\ 1\end{Bmatrix}=\frac{1}{9}\frac{fL^2}{S'}\frac{1}{8}\begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix}\begin{Bmatrix}0\\ 1\end{Bmatrix}=\frac{fL^2}{S'}\frac{1}{72}\begin{Bmatrix}1\\ 3\end{Bmatrix}.$$

A rectangular membrane of side length L, density ρ, thickness t, and tightening S' is loaded by its own weight as shown. If the edges are fixed, find the transverse displacements using the Finite Difference Method on a regular grid $(i,j)\in\{0,1,2,3,4\}\times\{0,1,2\}$.



Solution

The difference equations for regular grid but different spacings of the grid points in the coordinate directions follow from the continuum model boundary value problem when the difference approximations are substituted there. Equations

$$S'(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2})+f'=0 \quad (x,y)\in\Omega \quad \text{and} \quad w=0 \quad (x,y)\in\partial\Omega\,,$$

$$(\frac{\partial^2 w}{\partial x^2})_{(i,j)}=\frac{1}{\Delta x^2}(w_{(i-1,j)}-2w_{(i,j)}+w_{(i+1,j)}),\quad (\frac{\partial^2 w}{\partial y^2})_{(i,j)}=\frac{1}{\Delta y^2}(w_{(i,j-1)}-2w_{(i,j)}+w_{(i,j+1)})$$

give

$$\frac{S'}{h^2}[\alpha^2 w_{(i-1,j)}+\alpha^2 w_{(i+1,j)}-2(1+\alpha^2)w_{(i,j)}+w_{(i,j-1)}+w_{(i,j+1)}]+f'=0 \quad (i,j)\in I\,,$$

$$w_{(i,j)}=0 \quad (i,j)\in\partial I\,,$$

where $h=\Delta y=L/2$, $\alpha=\Delta y/\Delta x=2$, I is the index set for the interior points, and ∂I that for the boundary points. In the present problem, $I=\{(1,1),(2,1),(3,1)\}$ and one may use symmetry by defining

$$w_{(1,1)}=w_{(3,1)}=w_1 \quad \text{and} \quad w_{(2,1)}=w_2\,.$$

As the equations given by the (present) Finite Difference Method for the constrained points do not differ, it is enough to write the equations for $(i,j)=(1,1)$ and $(i,j)=(2,1)$

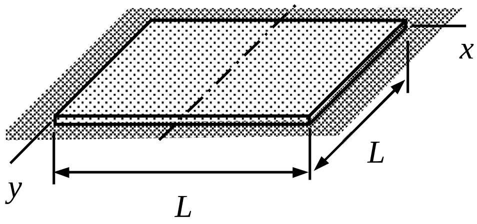
$$4\frac{S'}{L^2}[4w_{(0,1)}+4w_{(2,1)}-10w_{(1,1)}+w_{(1,0)}+w_{(1,2)}]+\rho t g=0 \quad \Rightarrow \quad 4\frac{S'}{L^2}(4w_2-10w_1)+\rho t g=0\,,$$

$$4\frac{S'}{L^2}[4w_{(1,1)}+4w_{(3,1)}-10w_{(2,1)}+w_{(2,0)}+w_{(2,2)}]+\rho g t=0 \quad \Rightarrow \quad 4\frac{S'}{L^2}(8w_1-10w_2)+\rho t g=0\,.$$

In matrix notation, the equations are

$$-4\frac{S'}{L^2}\begin{bmatrix}10 & -4\\ -8 & 10\end{bmatrix}\begin{Bmatrix}w_1\\ w_2\end{Bmatrix}+\rho t g\begin{Bmatrix}1\\ 1\end{Bmatrix}=0 \quad \Leftrightarrow \quad \begin{Bmatrix}w_1\\ w_2\end{Bmatrix}=\rho t g\frac{h^2}{S'}\begin{bmatrix}10 & -4\\ -8 & 10\end{bmatrix}^{-1}\begin{Bmatrix}1\\ 1\end{Bmatrix}=\frac{\rho t g L^2}{S'}\frac{1}{136}\begin{Bmatrix}7\\ 9\end{Bmatrix}.$$

Consider a rectangular membrane of side length L, density ρ, thickness t, and tightening S'. If the edges are fixed, find angular velocities of the modes, that are reflection symmetric with respect to the line through the center point shown using the Finite Difference Method. Use a regular grid $(i,j)\in\{0,1,2,3,4\}\times\{0,1,2\}$ of different spacings in the coordinate directions.



Solution

The difference equations for a regular grid but different point spacing in the coordinate directions follow from the continuum model when the difference approximations are substituted there. Equations

$$S'(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2})+f'=m'\frac{\partial^2 w}{\partial t^2} \quad (x,y)\in\Omega \quad \text{and} \quad w=0 \quad (x,y)\in\partial\Omega ,$$

$$(\frac{\partial^2 w}{\partial x^2})_{(i,j)}=\frac{1}{\Delta x^2}(w_{(i-1,j)}-2w_{(i,j)}+w_{(i+1,j)}), \quad (\frac{\partial^2 w}{\partial y^2})_{(i,j)}=\frac{1}{\Delta y^2}(w_{(i,j-1)}-2w_{(i,j)}+w_{(i,j+1)})$$

give (initial conditions do not matter in modal analysis)

$$\frac{S'}{h^2}[\alpha^2 w_{(i-1,j)}+\alpha^2 w_{(i+1,j)}-2(1+\alpha^2)w_{(i,j)}+w_{(i,j-1)}+w_{(i,j+1)}]+f'=m'\ddot{w}_{(i,j)} \quad (i,j)\in I$$

$$w_{(i,j)}=0 \quad (i,j)\in\partial I$$

where $h=\Delta y=L/2$, $\alpha=\Delta y/\Delta x=2$, I is the index set for the interior points, and ∂I that for the boundary points. In the present problem, $I=\{(1,1),(2,1),(3,1)\}$ and one may use symmetry by defining

$$w_{(1,1)}=w_{(3,1)}=w_1 \quad \text{and} \quad w_{(2,1)}=w_2 \, .$$

As the equations given by the (present) Finite Difference Method for the constrained points do not differ, it is enough to write the equations for $(i,j)=(1,1)$ and $(i,j)=(2,1)$ to get

$$4\frac{S'}{L^2}(4w_2-10w_1)=\rho t\ddot{w}_1 \text{ and } 4\frac{S'}{L^2}(8w_1-10w_2)=\rho t\ddot{w}_2 .$$

In matrix notation, the equations can be written as

$$\begin{bmatrix} 10 & -4 \\ -8 & 10 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix}+\frac{\rho t L^2}{4S'}\begin{Bmatrix} \ddot{w}_1 \\ \ddot{w}_2 \end{Bmatrix}=0 .$$

Solution trial $\mathbf{w} = \mathbf{A}e^{\mathrm{i}\omega t}$ gives an algebraic equation system for the mode $\mathbf{A}$ and the corresponding angular velocity ω

$$(\begin{bmatrix} 10 & -4 \\ -8 & 10 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0 \text{, where } \lambda = \frac{\rho t L^2}{4S'} \omega^2 \quad \text{or} \quad \omega = \frac{2}{L}\sqrt{\lambda \frac{S'}{\rho t}} \text{ .}$$

Solution to the possible angular velocities follow from condition

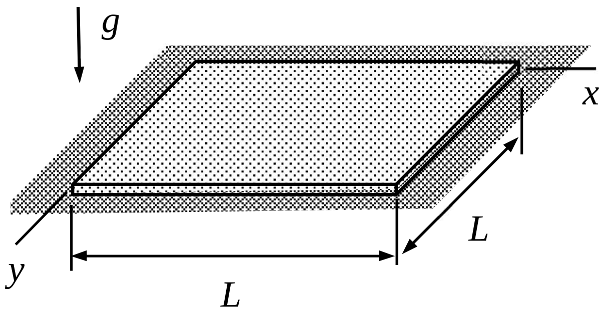
$$\det(\begin{bmatrix} 10-\lambda & -4 \\ -8 & 10-\lambda \end{bmatrix}) = (10-\lambda)^2 - 32 = 0 \quad \Rightarrow \quad \lambda = 10 \pm 4\sqrt{2} \text{ .}$$

The corresponding angular velocities follow from the relationship between ω and λ

$$\omega_1 = \frac{2}{L}\sqrt{\lambda_1 \frac{S'}{\rho t}} = \frac{2}{L}\sqrt{(10+4\sqrt{2})\frac{S'}{\rho t}} \text{ and } \omega_2 = \frac{2}{L}\sqrt{\lambda_2 \frac{S'}{\rho t}} = \frac{2}{L}\sqrt{(10-4\sqrt{2})\frac{S'}{\rho t}} \text{ .} \quad \leftarrow$$

LECTURE ASSIGNMENT 1

A rectangular membrane of side length L, density ρ, thickness t, and tightening S' (force per unit length) is loaded by its own weight as shown. If the edges are fixed, find the transverse displacements at the grid points $(i, j) \in \{0,1,2,3\} \times \{0,1,2,3\}$ of a regular grid using the Finite Difference Method. Use symmetry to reduce the number of non-zero independent displacements to one.



Name______________________________ Student number______________

In a stationary problem, the discrete equations given by the Finite Difference Method on regular grid of spacing h are

$$\frac{S'}{h^2}[w_{(i-1,j)} + w_{(i,j-1)} - 4w_{(i,j)} + w_{(i+1,j)} + w_{(i,j+1)}] + f_i' = 0 \quad (i,j) \in I \,,$$

$$w_{(i,j)} = 0 \quad (i,j) \in \partial I \,.$$

In the present problem, the set of interior points is given by

$$I = \{(1,1),(1,2),(2,1),(2,2)\}$$

the remaining of $(i,j) \in \{0,1,2,3\} \times \{0,1,2,3\}$ being boundary points ∂I of vanishing displacements. Due to symmetry, displacements at the interior points should be equal. Denoting the value by w_1, all equations for the interior point I boil down to

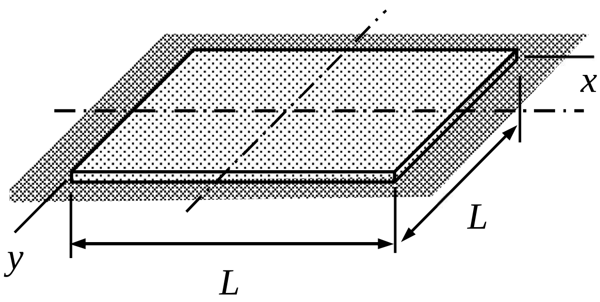
$$9\frac{S'}{L^2}(-4w_1 + w_1 + w_1) + \rho g t = 0$$

giving as the displacement at the interior points

$$w_1 = \frac{1}{18}\frac{\rho g t L^2}{S'} \,.$$

LECTURE ASSIGNMENT 2

Consider vibration of a rectangular membrane of side length L, density ρ, thickness t, and tightening S' (force per unit length). If the edges are fixed, find the angular velocity of the free vibrations using the Finite Difference Method on a regular grid of points $(i,j) \in \{0,1,2,3\} \times \{0,1,2,3\}$. Consider the mode, which is reflection symmetric with respect to the lines through the center point (figure).



Name______________________________ Student number______________

In a time-dependent membrane problem, the equations given by the Finite Difference Method on regular grid of spacing h are

$$\frac{S'}{h^2}[w_{(i-1,j)} + w_{(i,j-1)} - 4w_{(i,j)} + w_{(i+1,j)} + w_{(i,j+1)}] + f_i' = \rho t \ddot{w}_{(i,j)} \quad (i,j) \in I \,,$$

$$w_{(i,j)} = 0 \quad (i,j) \in \partial I \,.$$

Initial conditions are not needed in modal analysis of the problem. In the present problem, the set of interior points is given by

$$I = \{(1,1),(1,2),(2,1),(2,2)\}$$

the remaining of $(i,j) \in \{0,1,2,3\} \times \{0,1,2,3\}$ being boundary points ∂I of vanishing displacements. Due to the reflection symmetry, displacements at the interior points are equal, w_1 say. Consequently, all equations for the interior points $(i,j) \in I$ boil down to

$$\ddot{w}_1 + \omega^2 w_1 = 0 \quad \text{where} \quad \omega = \frac{3}{L}\sqrt{2\frac{S'}{\rho t}} \,.$$

Therefore, the frequency of the assumed mode shape

$$f = \frac{3}{2\pi L}\sqrt{2\frac{S'}{\rho t}} \,.$$

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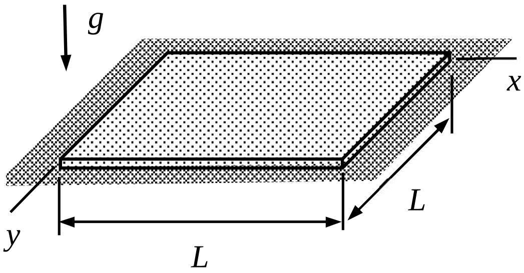
Home assignment 1

A rectangular membrane of side length L, density ρ, thickness t, and tightening S' (force per unit length) is loaded by its own weigh as shown. If the edges are fixed, find the transverse displacements at the grid points $(i,j)\in\{0,1,2,3,4\}\times\{0,1,2,3,4\}$ of a regular grid using the Finite Difference Method. Use symmetry to reduce the number of non-zero independent displacements to three.



Solution

The generic equations for the membrane model with fixed boundaries as given by the Finite Difference Method on a regular grid are

$$\frac{S'}{h^2}[w_{(i-1,j)}+w_{(i,j-1)}-4w_{(i,j)}+w_{(i+1,j)}+w_{(i,j+1)}]+f_i'=m_i'\ddot{w}_{(i,j)}\quad (i,j)\in I\,,$$

$$w_{(i,j)}=0\quad (i,j)\in\partial I\,,$$

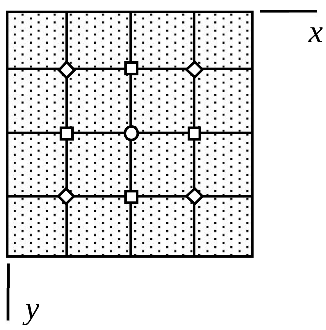
$$w_{(i,j)}-g_{(i,j)}=0\quad\text{and}\quad \dot{w}_{(i,j)}-h_{(i,j)}=0\qquad (i,j)\in I\,.$$

In the present problem, time derivatives vanish, initial conditions are not needed, and solution is reflection symmetric with respect to lines through the center point and aligned with the coordinate axes. Therefore, transverse displacements at the grid points satisfy

$$w_{(1,1)}=w_{(1,3)}=w_{(3,1)}=w_{(3,3)}=w_1\,,$$

$$w_{(1,2)}=w_{(2,1)}=w_{(3,2)}=w_{(2,3)}=w_2\,,$$

$$w_{(2,2)}=w_3\,,$$



and the number of independent equilibrium equations is 3. Considering only the independent equations with $f'=\rho t g$ and $h=L/4$

$$16\frac{S'}{L^2}[w_{(0,1)}+w_{(1,0)}-4w_{(1,1)}+w_{(2,1)}+w_{(1,2)}]+\rho tg=16\frac{S'}{L^2}(-4w_1+2w_2)+\rho tg=0\,,$$

$$16\frac{S'}{L^2}[w_{(0,1)}+w_{(1,1)}-4w_{(1,2)}+w_{(2,2)}+w_{(1,3)}]+\rho tg=16\frac{S'}{L^2}(-4w_2+w_3+2w_1)+\rho tg=0\,,$$

$$16\frac{S'}{L^2}[w_{(1,2)}+w_{(2,1)}-4w_{(2,2)}+w_{(3,2)}+w_{(2,3)}]+\rho tg=16\frac{S'}{L^2}(-4w_3+4w_2)+\rho tg=0$$

or using the matrix notation

$$\begin{bmatrix}4 & -2 & 0\\ -2 & 4 & -1\\ 0 & -4 & 4\end{bmatrix}\begin{Bmatrix}w_1\\ w_2\\ w_3\end{Bmatrix}-\frac{1}{16}\frac{\rho gtL^2}{S'}\begin{Bmatrix}1\\ 1\\ 1\end{Bmatrix}=0\,.$$

Then, using row operations to get an equivalent upper triangular matrix representation

$$\begin{bmatrix}2 & -1 & 0\\ -2 & 4 & -1\\ 0 & -4 & 4\end{bmatrix}\begin{Bmatrix}w_1\\ w_2\\ w_3\end{Bmatrix}-\frac{1}{16}\frac{\rho gtL^2}{S'}\begin{Bmatrix}1/2\\ 1\\ 1\end{Bmatrix}=0 \quad\Leftrightarrow\quad \begin{bmatrix}2 & -1 & 0\\ 0 & 3 & -1\\ 0 & -4 & 4\end{bmatrix}\begin{Bmatrix}w_1\\ w_2\\ w_3\end{Bmatrix}-\frac{1}{16}\frac{\rho gtL^2}{S'}\begin{Bmatrix}1/2\\ 3/2\\ 1\end{Bmatrix}=0 \quad\Leftrightarrow$$

$$\begin{bmatrix}2 & -1 & 0\\ 0 & 12 & -4\\ 0 & -12 & 12\end{bmatrix}\begin{Bmatrix}w_1\\ w_2\\ w_3\end{Bmatrix}-\frac{1}{16}\frac{\rho gtL^2}{S'}\begin{Bmatrix}1/2\\ 6\\ 3\end{Bmatrix}=0 \quad\Leftrightarrow\quad \begin{bmatrix}2 & -1 & 0\\ 0 & 12 & -4\\ 0 & 0 & 8\end{bmatrix}\begin{Bmatrix}w_1\\ w_2\\ w_3\end{Bmatrix}-\frac{1}{16}\frac{\rho gtL^2}{S'}\begin{Bmatrix}1/2\\ 6\\ 9\end{Bmatrix}=0\,.$$

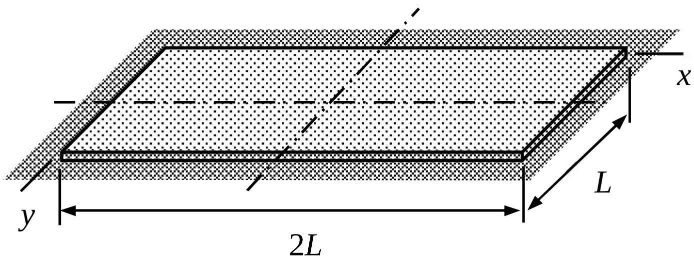
Then, using the equations starting from the last one

$$w_3=\frac{9}{128}\frac{\rho gtL^2}{S'}\,,\quad w_2=\frac{7}{128}\frac{\rho gtL^2}{S'}\,,\ \text{and}\ w_1=\frac{11}{256}\frac{\rho gtL^2}{S'}\,.$$ ⟸

Name________________________ Student number________________

Home assignment 2

Consider vibration of a rectangular membrane of side lengths $2L$ and L, density ρ, thickness t, and tightening S' (force per unit length). If the edges are fixed, find the angular velocity of the free vibrations using the Finite Difference Method on a regular grid of points $(i, j) \in \{0,1,2,3,4\} \times \{0,1,2\}$. Consider the modes, that are reflection symmetric with respect to the lines through the center point (figure).



Solution

The generic equations for the membrane model with fixed boundaries as given by the Finite Difference Method on a regular grid are

$$\frac{S'}{h^2}[w_{(i-1,j)} + w_{(i,j-1)} - 4w_{(i,j)} + w_{(i+1,j)} + w_{(i,j+1)}] + f' = m'\ddot{w}_{(i,j)} \quad (i, j) \in I,$$

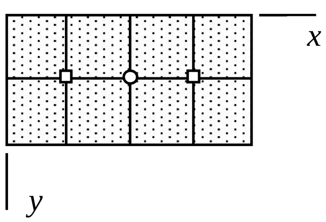
$$w_{(i,j)} = 0 \quad (i, j) \in \partial I,$$

$$w_{(i,j)} - g_{(i,j)} = 0 \quad \text{and} \quad \dot{w}_{(i,j)} - h_{(i,j)} = 0 \quad (i, j) \in I.$$

In modal analysis, initial conditions are not needed. As the mode is assumed to be reflection symmetric with respect to lines through the center point and aligned with the coordinate axes, transverse displacements at the grid points satisfy

$$w_{(1,1)} = w_{(3,1)} = w_1,$$

$$w_{(2,1)} = w_2,$$



the remaining displacements at the boundary points being zeros. Considering only the independent equations with $f' = 0$, $m' = \rho t$, and $h = L/2$

$$4\frac{S'}{L^2}[w_{(0,1)} + w_{(1,0)} - 4w_{(1,1)} + w_{(2,1)} + w_{(1,2)}] = 4\frac{S'}{L^2}(-4w_1 + w_2) = \rho t \ddot{w}_1,$$

$$4\frac{S'}{L^2}[w_{(1,1)}+w_{(2,0)}-4w_{(2,1)}+w_{(3,1)}+w_{(2,2)}]=4\frac{S'}{L^2}(-4w_2+2w_1)=\rho t\ddot{w}_2$$

or using the matrix notation

$$\begin{bmatrix} 4 & -1 \\ -2 & 4 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix}+\frac{1}{4}\frac{\rho t L^2}{S'}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{w}_1 \\ \ddot{w}_2 \end{Bmatrix}=0 .$$

Solution to the angular velocities and the corresponding modes follow with the trial solution $\mathbf{a}=\mathbf{A}e^{\mathrm{i}\omega t}$:

$$(\begin{bmatrix} 4 & -1 \\ -2 & 4 \end{bmatrix}-\lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix})\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}=0 \text{ where } \lambda=\omega^2\frac{1}{4}\frac{\rho t L^2}{S'} \quad \Leftrightarrow \quad \omega=\frac{2}{L}\sqrt{\lambda\frac{S'}{\rho t}} .$$

A homogeneous linear equation system can yield a non-zero solution to the mode only if the matrix is singular, i.e., its determinant vanishes. The condition can be used to find the possible values of λ

$$\det\begin{bmatrix} 4-\lambda & -1 \\ -2 & 4-\lambda \end{bmatrix}=(4-\lambda)^2-2=0 \quad \text{so} \quad \lambda_1=4-\sqrt{2} \quad \text{or} \quad \lambda_2=4+\sqrt{2} .$$

Knowing the possible values for a non-zero solution, the modes follow from the algebraic equation when the values of parameter λ are substituted there (one at a time):

$$\lambda_1=4-\sqrt{2}: \ \omega_1=\frac{2}{L}\sqrt{(4-\sqrt{2})\frac{S'}{\rho t}} \quad \text{and} \quad \begin{bmatrix} \sqrt{2} & -1 \\ -2 & \sqrt{2} \end{bmatrix}\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}=0 \ \text{ so}$$

$$(\omega_1,\mathbf{A}_1)=(\frac{2}{L}\sqrt{(4-\sqrt{2})\frac{S'}{\rho t}},\begin{Bmatrix} 1 \\ \sqrt{2} \end{Bmatrix}) .$$

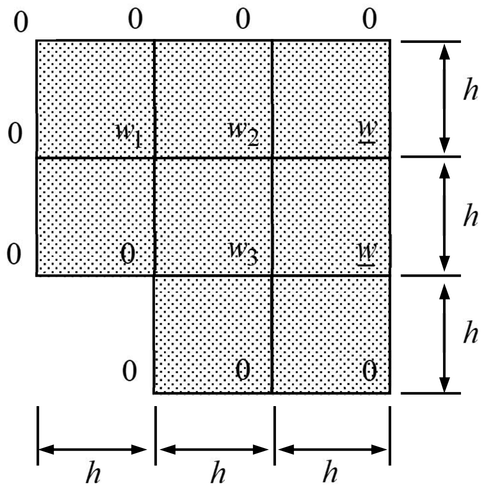
$$\lambda_2=4+\sqrt{2}: \ \omega_2=\frac{2}{L}\sqrt{(4+\sqrt{2})\frac{S'}{\rho t}} \quad \text{and} \quad \begin{bmatrix} -\sqrt{2} & -1 \\ -2 & -\sqrt{2} \end{bmatrix}\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}=0 \ \text{ so}$$

$$(\omega_2,\mathbf{A}_2)=(\frac{2}{L}\sqrt{(4+\sqrt{2})\frac{S'}{\rho t}},\begin{Bmatrix} 1 \\ -\sqrt{2} \end{Bmatrix}) .$$

Name________________________ Student number__________________

Home assignment 3

Consider the membrane of polygonal shape of the figure for which density ρ, thickness t, tightening S' (force per unit length) are constants. Using the given values at the grid points on the boundary, determine the unknown transverse displacement values w_1, w_2 and w_3 using the Finite Difference Method.



Solution

The generic equations for the membrane model with fixed boundaries as given by the Finite Difference Method on a regular grid are

$$\frac{S'}{h^2}[w_{(i-1,j)} + w_{(i,j-1)} - 4w_{(i,j)} + w_{(i+1,j)} + w_{(i,j+1)}] + f_i' = m_i'\ddot{w}_{(i,j)} \quad (i,j) \in I\,,$$

$$w_{(i,j)} = 0 \quad (i,j) \in \partial I\,,$$

$$w_{(i,j)} - g_{(i,j)} = 0 \quad \text{and} \quad \dot{w}_{(i,j)} - h_{(i,j)} = 0 \qquad (i,j) \in I\,.$$

In the present problem, time derivatives vanish and initial conditions are not needed. The equilibrium equations for the interior points are

$$\frac{S'}{h^2}(-4w_1 + w_2) = 0\,,\ \frac{S'}{h^2}(-4w_2 + w_1 + w_3 + \underline{w}) = 0\,,\ \text{and}\ \frac{S'}{h^2}(-4w_3 + w_2 + \underline{w}) = 0\,.$$

Using the matrix notation

$$\begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} - \underline{w} \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} = 0\,.$$

Then, using row operations to get an equivalent upper triangular matrix representation

$$\begin{bmatrix} 4 & -1 & 0 \\ -4 & 16 & -4 \\ 0 & -1 & 4 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} - \underline{w} \begin{Bmatrix} 0 \\ 4 \\ 1 \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{bmatrix} 4 & -1 & 0 \\ 0 & 15 & -4 \\ 0 & -1 & 4 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} - \underline{w} \begin{Bmatrix} 0 \\ 4 \\ 1 \end{Bmatrix} = 0 \quad \Leftrightarrow$$

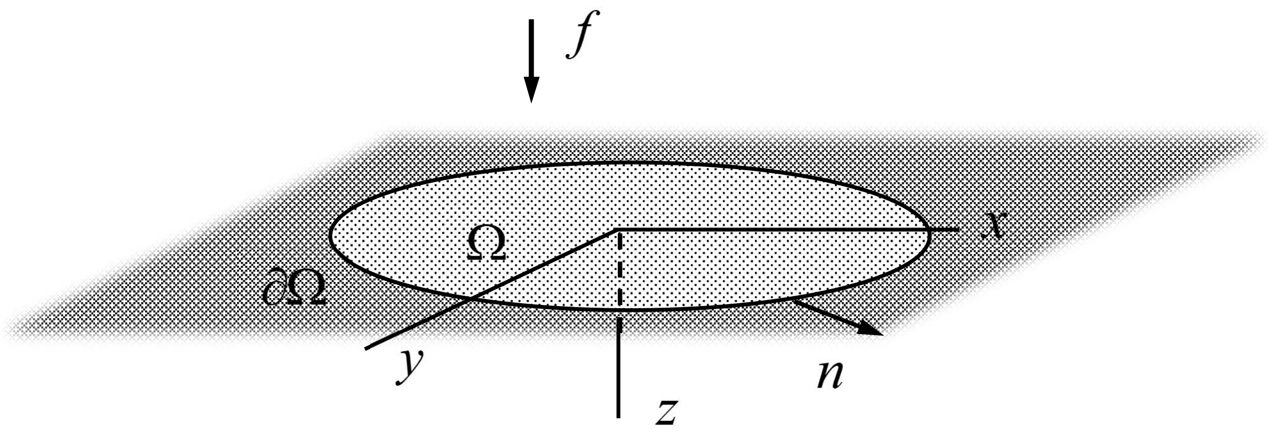
$$\begin{bmatrix} 4 & -1 & 0 \\ 0 & 15 & -4 \\ 0 & -15 & 60 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} - \underline{w} \begin{Bmatrix} 0 \\ 4 \\ 15 \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{bmatrix} 4 & -1 & 0 \\ 0 & 15 & -4 \\ 0 & 0 & 56 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} - \underline{w} \begin{Bmatrix} 0 \\ 4 \\ 19 \end{Bmatrix} = 0 .$$

After that, equations can be solved one-by-one for the displacement starting from the last one

$$w_3 = \frac{19}{56} \underline{w}, \quad w_2 = \frac{5}{14} \underline{w}, \text{ and } w_1 = \frac{5}{56} \underline{w} .$$

6 FEM FOR MEMBRANE MODEL

MEMBRANE EQUATIONS

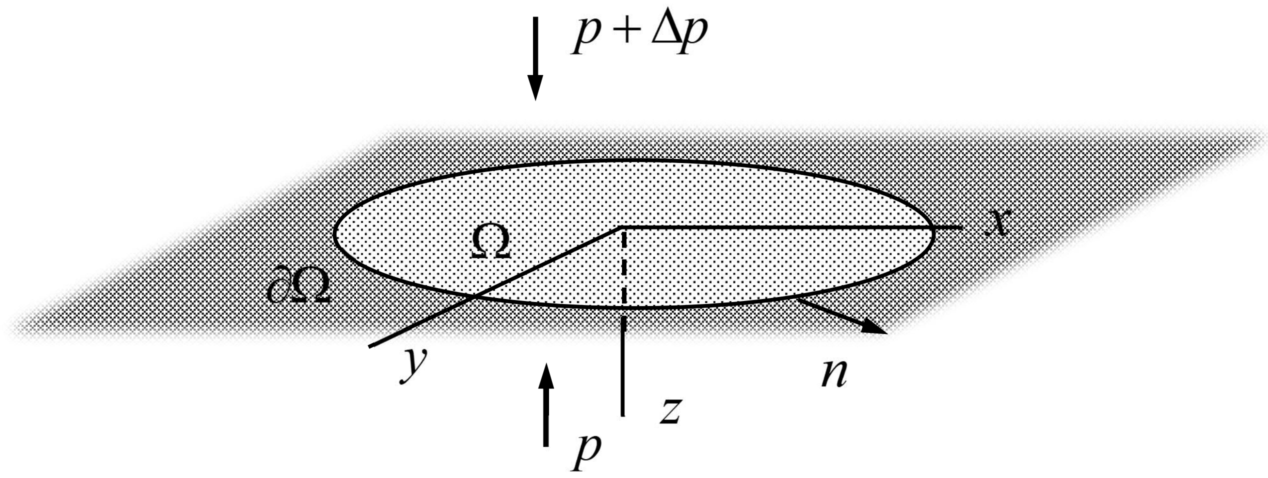


Equation of motion $S'(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}) + f' = m' \frac{\partial^2 w}{\partial t^2} \quad (x, y) \in \Omega \quad t > 0,$

Boundary conditions $w = \underline{w}$ or $S'(n_x \frac{\partial w}{\partial x} + n_y \frac{\partial w}{\partial y}) = F' \quad (x, y) \in \partial\Omega \;\; t > 0,$

Initial conditions $w = g$ and $\frac{\partial w}{\partial t} = h \quad (x, y) \in \Omega \quad t = 0.$

EXAMPLE A circular membrane of radius R, fixed edges, and constant tightening S' (force per unit length) is loaded by pressure $p+\Delta p$ acting on the upper surface and p acting on the lower surface. Find the transverse displacement assuming that the solution is rotation symmetric.



Answer $w(r)=\dfrac{\Delta p}{S'}\dfrac{1}{4}(R^2-r^2)$

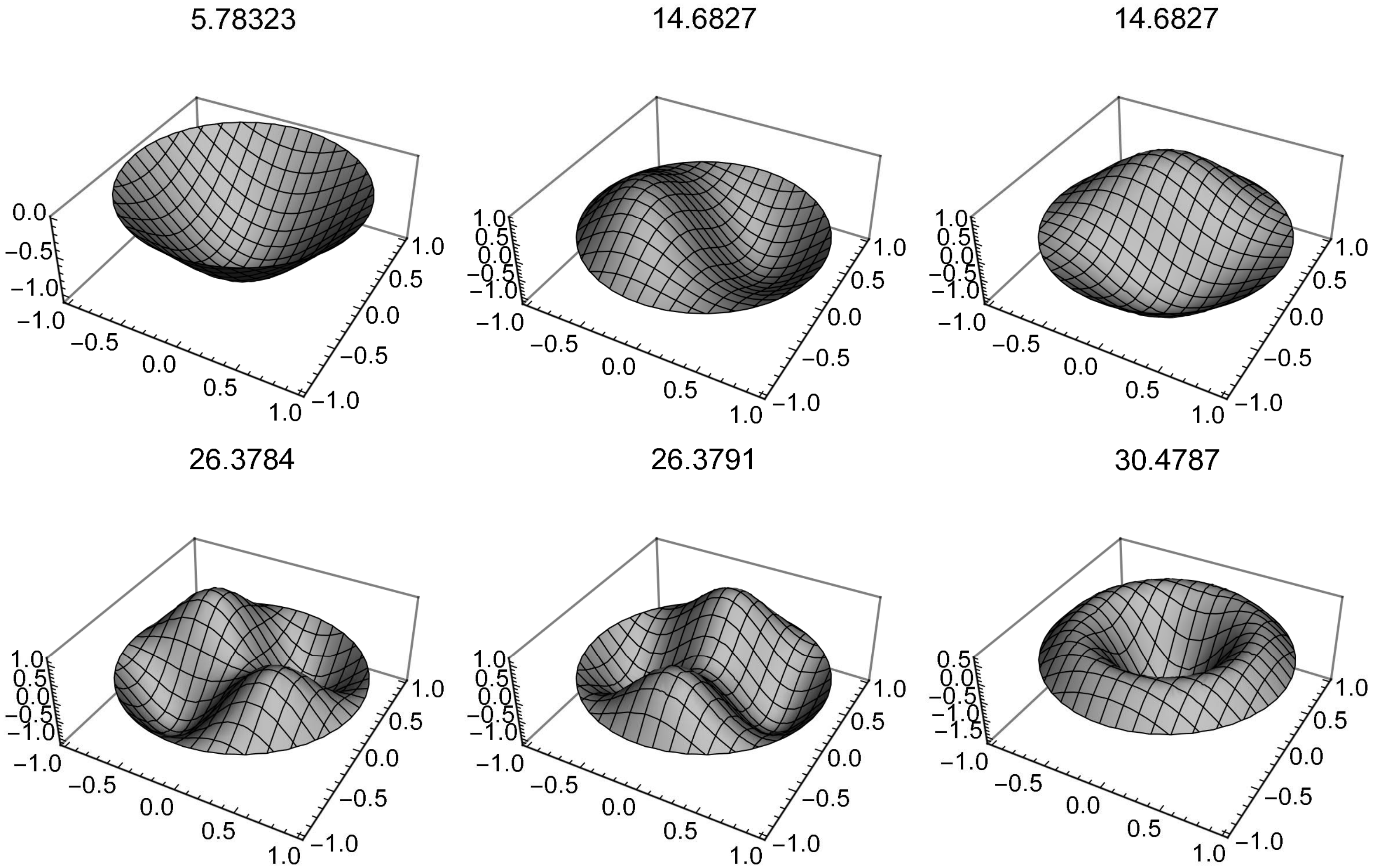
According to the problem description, solution is rotational symmetric so it can depend only on the distance from the centerpoint $r = \sqrt{x^2 + y^2}$. Then, membrane equilibrium equation simplifies to the ordinary differential equation (see: Laplace operator in polar coordinate system)

$$S'\frac{1}{r}\frac{d}{dr}(r\frac{dw}{dr}) + \Delta p = 0 \quad \Leftrightarrow \quad w(r) = -\frac{\Delta p}{S'}\frac{1}{4}r^2 + a\ln r + b.$$

Solution should be bounded at the origin so $a = 0$. The value of the second parameter follows from the boundary condition $w(R) = 0$. Therefore

$$w(r) = \frac{\Delta p}{S'}\frac{1}{4}(R^2 - r^2) \ .$$

VIBRATION PATTERNS OF DRUMHEAD

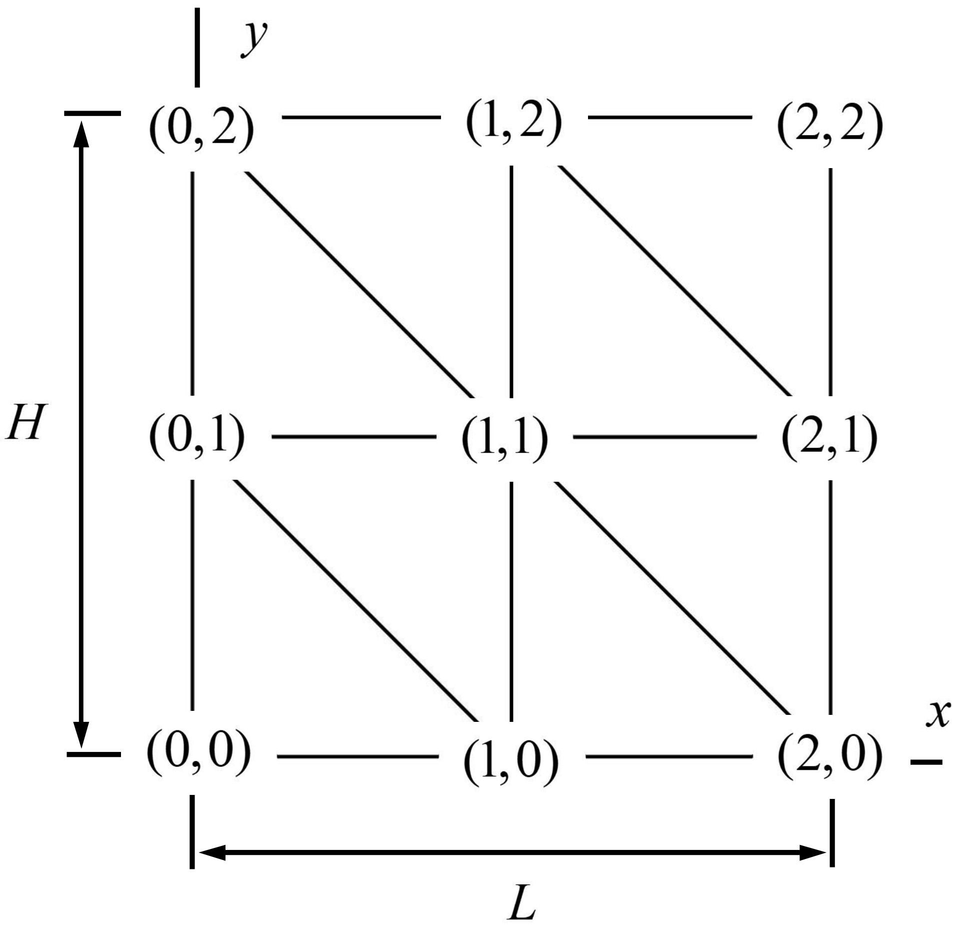


PRINCIPLE OF VIRTUAL WORK

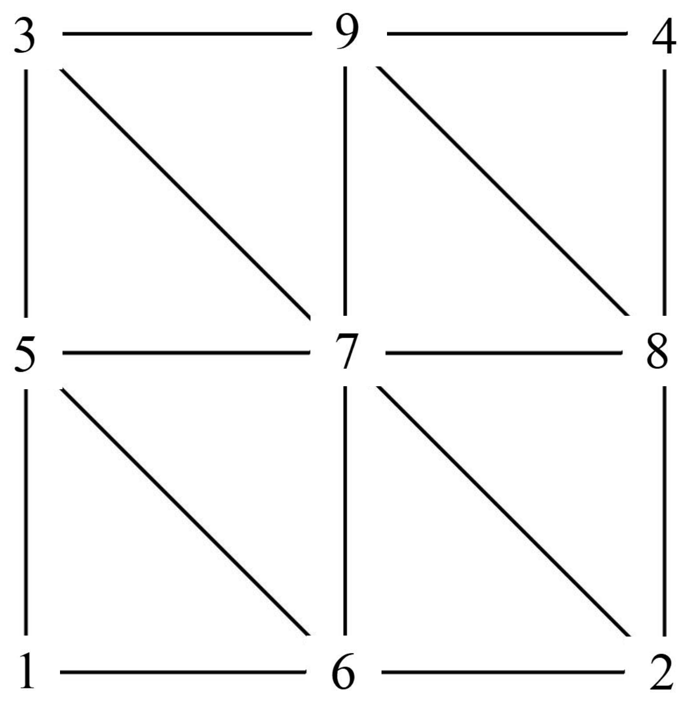
Principle of virtual work for particle and continuum models is just a concise representations of equations-of-motion and boundary conditions of the models.

Virtual work	String	Membrane
δW^{int}	$-\int_{\Omega} S(\frac{\partial \delta w}{\partial x}\frac{\partial w}{\partial x})dx$	$-\int_{\Omega} S'(\frac{\partial \delta w}{\partial x}\frac{\partial w}{\partial x}+\frac{\partial \delta w}{\partial y}\frac{\partial w}{\partial y})dA$
δW^{ext}	$\int_{\Omega} (\delta w f)dx$	$\int_{\Omega} (\delta w f')dA$
δW^{ine}	$-\int_{\Omega} (\delta w \rho A\frac{\partial^2 w}{\partial t^2})dx$	$-\int_{\Omega} (\delta w \rho t\frac{\partial^2 w}{\partial t^2})dA$

6.1 INTERPOLANT AND APPROXIMATION



2-index labeling



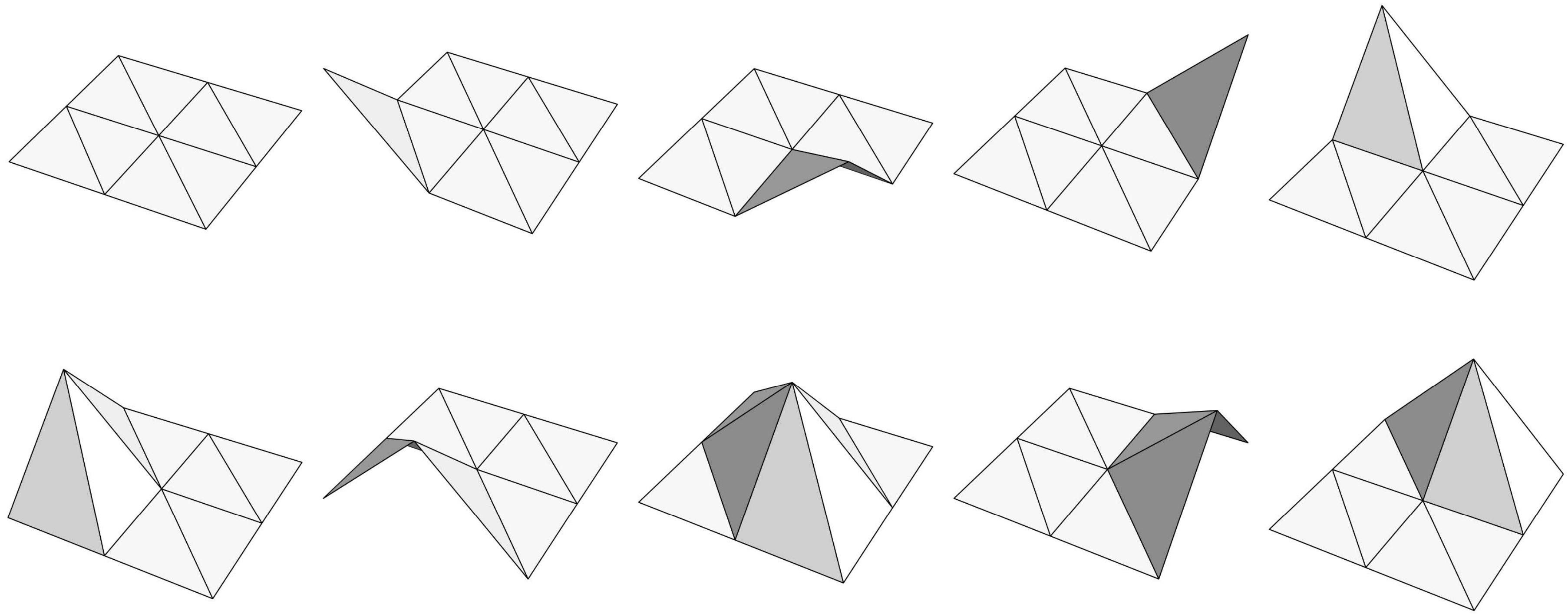
1-index labeling

In Finite Element Method, grid points and triangles, having the grid points as the vertices, are called as the nodes and elements, respectively. The representation of geometry or dataset uses a list of triangle vertex labels and separate lists of coordinates and function values. The regular grid representations used in Particle Surrogate Method and Finite Difference Method are particular cases of the more generic representation.

Piecewise linear interpolant $p(x, y)$ to the dataset $\{\ldots,(x_i, y_{i,} f_i),\ldots\}$ (one-index labeling) uses a triangle representation having the grid points as vertices. Regular triangle representation on a regular grid of the dataset repeats the same triangle element pattern for all interior points. Assuming that the dataset is sampling of function $f(x, y)$ at the grid points, $p(x, y)$ can also be considered as an approximation to $f(x, y)$. Piecewise linear interpolation with the triangle division works also without a regular grid.

SHAPE FUNCTIONS

Piecewise linear interpolants to datasets $\{\ldots,(x_i, y_i, f_i),\ldots\}$, where f_i is chosen to be one at one of the grid points the remaining being zeros, are called as the piecewise linear shape functions $N_i(x, y)$:



With the shape function concept, the linear interpolant to dataset $\{\ldots,(x_i, y_i, f_i),\ldots\}$ (x_i, y_i, f_i) can be represented in the same form as in the one-dimensional case

$$p(x,y)=\sum_{i\in I} f_i N_i(x,y)$$

in which f_i $i\in I$ are the nodal values and I is the labelling set. In a typical triangle element of the vertex nodes (i,j,k), only the shape functions of nodes i, j, and k are non-zeros. The expression of the shape functions and the interpolant are

$$\begin{Bmatrix} N_i(x,y) \\ N_j(x,y) \\ N_k(x,y) \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_i & x_j & x_k \\ y_i & y_j & y_k \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} \text{ and } p(x,y)=\begin{Bmatrix} f_i \\ f_j \\ f_k \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} N_i(x,y) \\ N_j(x,y) \\ N_k(x,y) \end{Bmatrix},$$

respectively.

6.2 WEIGHTED RESIDUAL APPROXIMATION

Finding an approximation $g(x,y)$ to function $f(x,y)$ is one the basic tasks in numerical mathematics. In the Least Squares Method and Weighted Residual Methods, the nodal values g_i of approximation $g(x) = \sum g_i N_i(x) = \mathbf{N}^{\mathbf{T}}\mathbf{g}$ follow from the steps

Distance: $\Pi(\mathbf{g}) = \frac{1}{2}\int_{\Omega} (g - f)^2 dA = \frac{1}{2}\int_{\Omega} (\mathbf{N}^{\mathrm{T}}\mathbf{g} - f)^2 dA,$

Minimizer: $\mathbf{Kg} - \mathbf{F} = \mathbf{0}$ where $\mathbf{K} = \int_{\Omega} \mathbf{N}\mathbf{N}^{\mathrm{T}} dA$ and $\mathbf{F} = \int_{\Omega} \mathbf{N}f \, dA,$

Nodal values: $\mathbf{g} = \mathbf{K}^{-1}\mathbf{F}.$

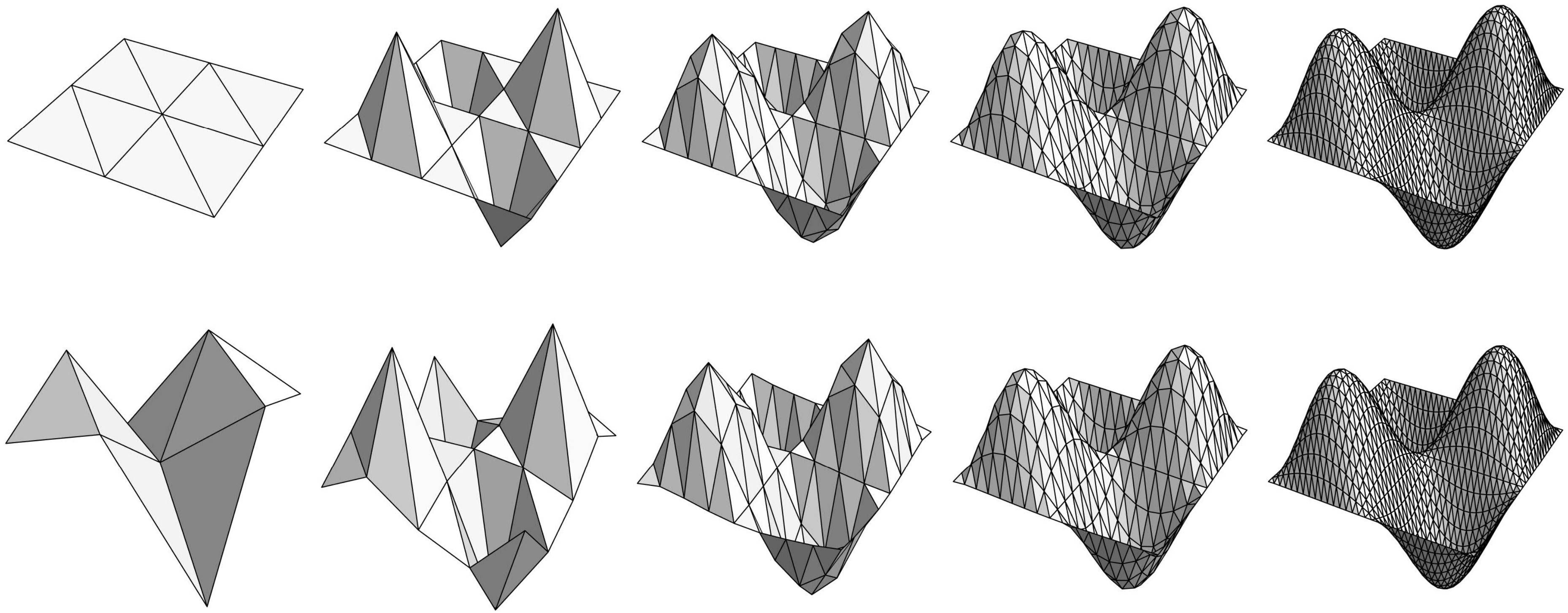
In practice, the nodal values $\mathbf{g}$ are solved from the linear equation system without matrix inversion (to avoid excess computational work). The method works in the same manner irrespective of the series approximation used.

Least Squares Method is useful in various tasks in numerical mathematics. One of the applications is related with the condition for the minimum of Π, which can be written in the form

$$\int_{\Omega} N_i R dA = 0 \quad i \in \{0,1,\ldots\}$$

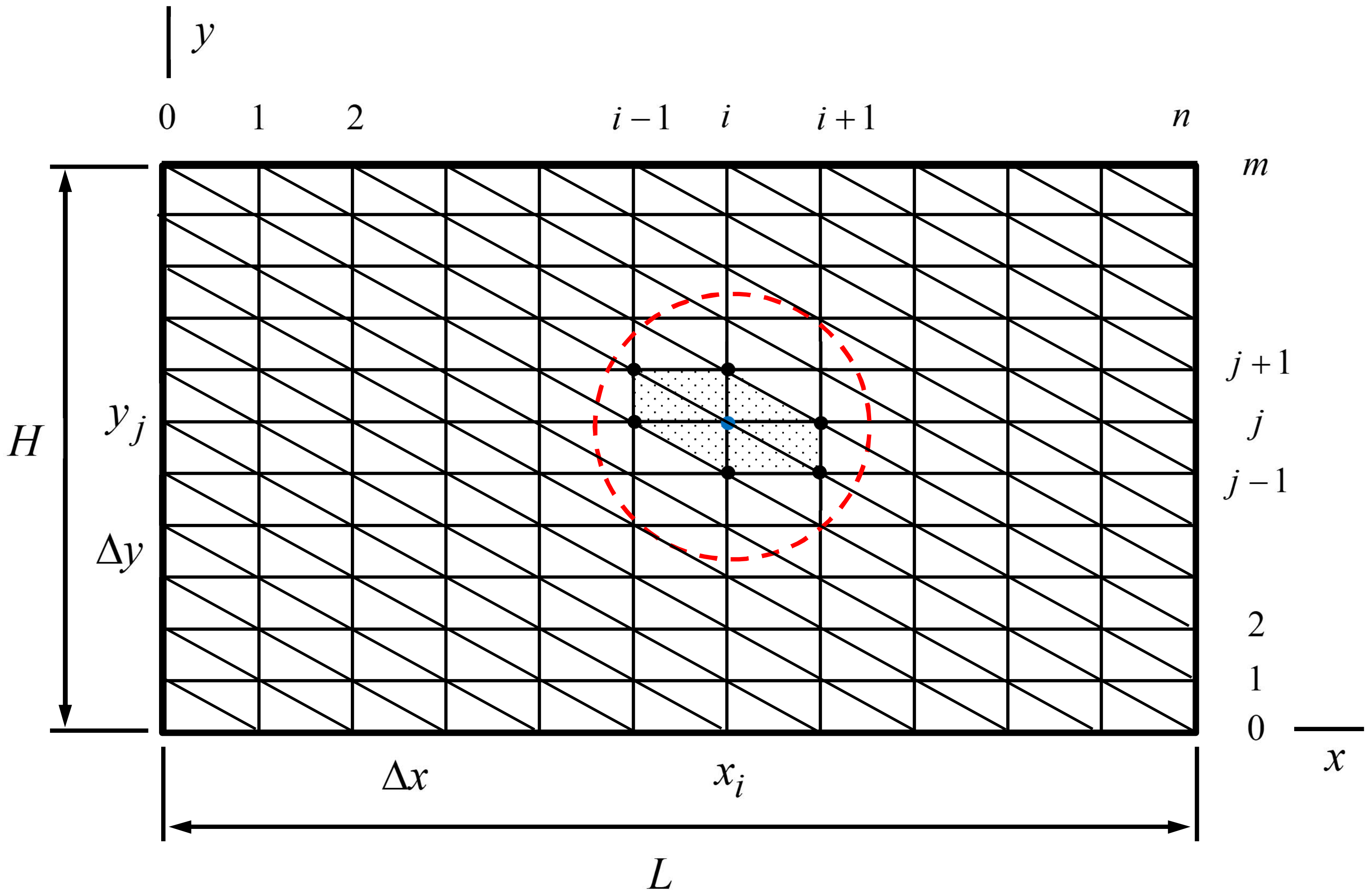
where $R = g(x,y) - f(x,y)$ is called as the residual. In the weighted residual interpretation of the method, linear equations giving the values of the approximation are obtained as the weighted residuals with the shape functions. The idea extends to residuals of differential equations and is one of the starting points for the Finite Element Method.

EXAMPLE Let us consider function $f(x, y) = \sin(2\pi x / L)\sin(\pi y / L)/2$ on the square domain $(x, y) \in [0, L] \times [0, L]$. Using regular triangle elements on a regular grid of points (nodes), piecewise linear interpolant $p(x, y)$, and the least-squares approximation $g(x, y)$ to $f(x, y)$ are:



Interpolant is accurate on the grid points but the interpolation error at the other points is not controlled. As nodal values on the rough 3×3 point grid vanish, also the interpolant is identically zero. Least squares method considers all points of the domain and control the error everywhere. On the rough grid, the piecewise linear approximation is not particularly accurate but better than that given by the interpolant. When the number of elements is increase, both approximations converge to $f(x,y)$.

REGULAR GRID OF TRIANGLES



APPROXIMATION TO DERIVATIVES

The weighted average, using $N_{(i,j)}$, of a derivative of the piecewise linear interpolant $a = \sum_{(i,j)\in I\times I} N_{(i,j)} a_{(i,j)}$ where $I = \{0,1,\ldots,n\}$ is interpreted as an approximation to derivative at the interior grid points (multiplied by ΔA due to the integration).

Term	Weighted residual
$a(x,y)$	$\int_\Omega N_{(i,j)} a\,dA$
$\dfrac{\partial a}{\partial x}$	$\int_\Omega N_{(i,j)} \dfrac{\partial a}{\partial x} dA$
$\dfrac{\partial^2 a}{\partial x^2} + \dfrac{\partial^2 a}{\partial y^2}$	$-\int_A \left(\dfrac{\partial N_{(i,j)}}{\partial x}\dfrac{\partial a}{\partial x} + \dfrac{\partial N_{(i,j)}}{\partial y}\dfrac{\partial a}{\partial y}\right) dA$

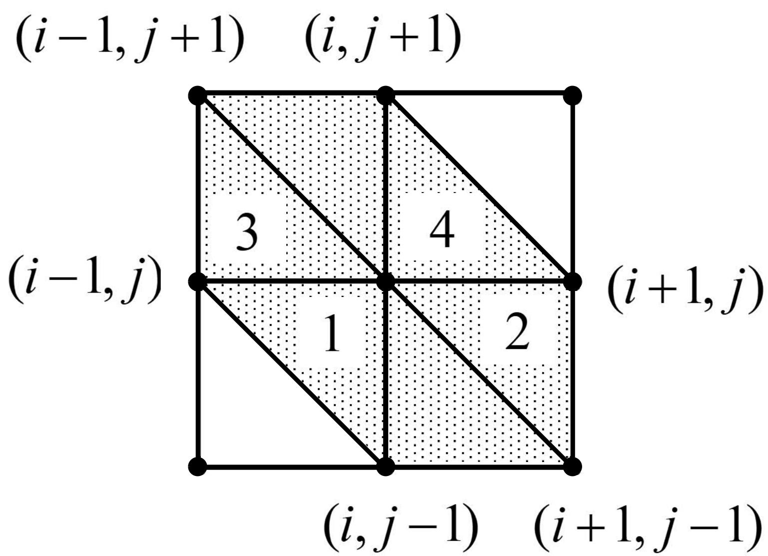
As an example, let us consider the approximation

$$\Delta A\left(\frac{\partial^2 a}{\partial x^2}\right)_{(i,j)} \approx -\int_{\Omega} \left(\frac{\partial N_{(i,j)}}{\partial x}\frac{\partial a}{\partial x}\right) dA = -\sum_e \int_{\Omega^e} \left(\frac{\partial N_{(i,j)}}{\partial x}\frac{\partial a}{\partial x}\right) dA,$$

where the sum is over the elements having the grid point (i, j) in common ($N_{(i,j)}$ vanishes elsewhere). Considering the 6 elements separately and using the fact the derivatives of the shape functions of a piecewise linear interpolation are piecewise constants, and placing the origin of the coordinate system at (i, j):

1 and 3 : $\dfrac{\partial a}{\partial x} = \dfrac{a_{(i,j)} - a_{(i-1,j)}}{\Delta x}, \quad \dfrac{\partial N_{(i,j)}}{\partial x} = \dfrac{1}{\Delta x}$

2 and 4: $\dfrac{\partial a}{\partial x} = \dfrac{a_{(i+1,j)} - a_{(i,j)}}{\Delta x}, \quad \dfrac{\partial N_{(i,j)}}{\partial x} = -\dfrac{1}{\Delta x}$



Therefore, the outcome

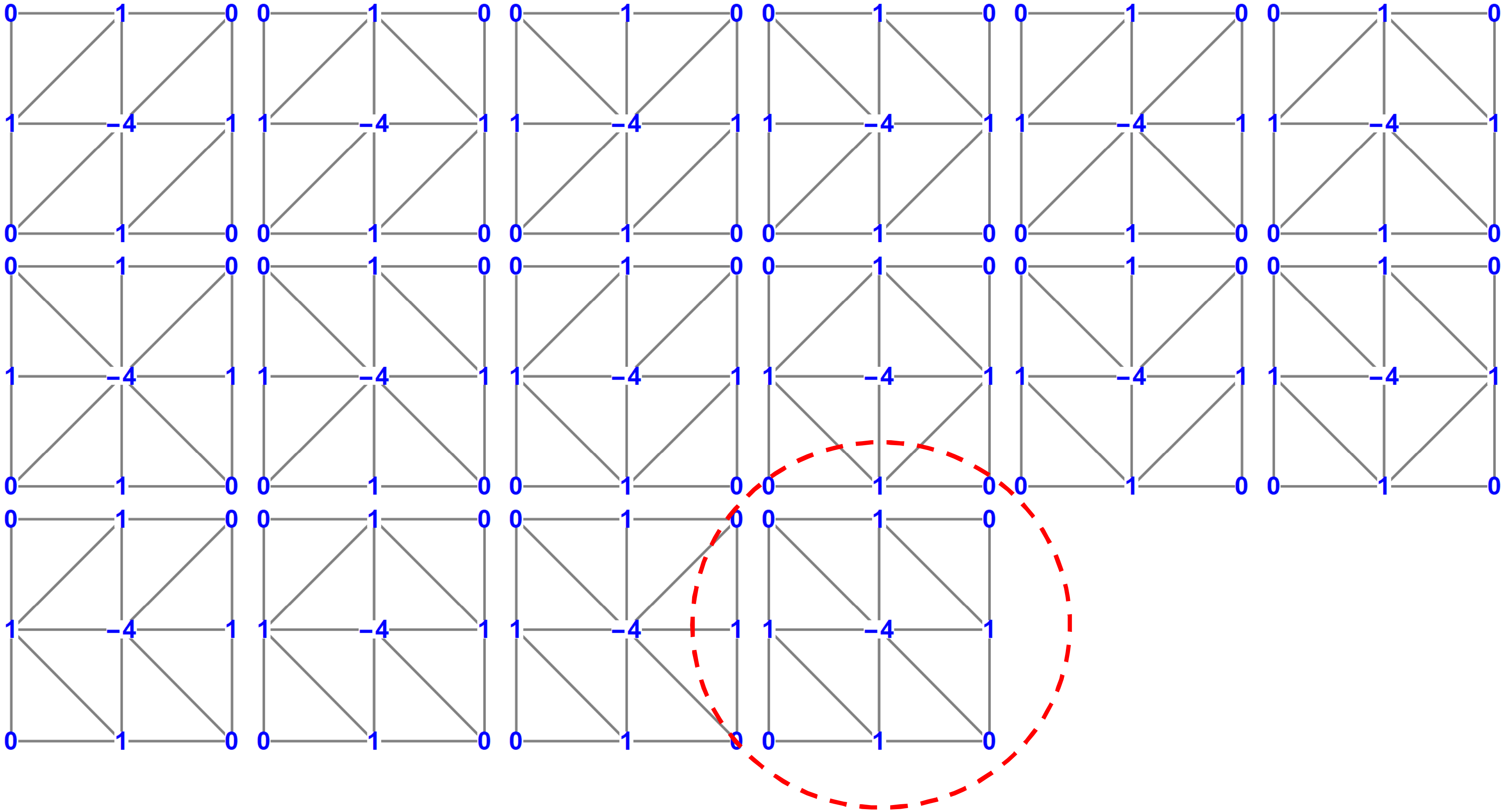
$$\Delta A(\frac{\partial^2 a}{\partial x^2})_{(i,j)} \approx -\sum_e \int_{A^e} (\frac{\partial N_{(i,j)}}{\partial x}\frac{\partial a}{\partial x})dA \approx \frac{\Delta y}{\Delta x}[a_{(i-1,j)} - 2a_{(i,j)} + a_{(i+1,j)}])$$

differs from the expression by the Finite Difference Method by multiplier

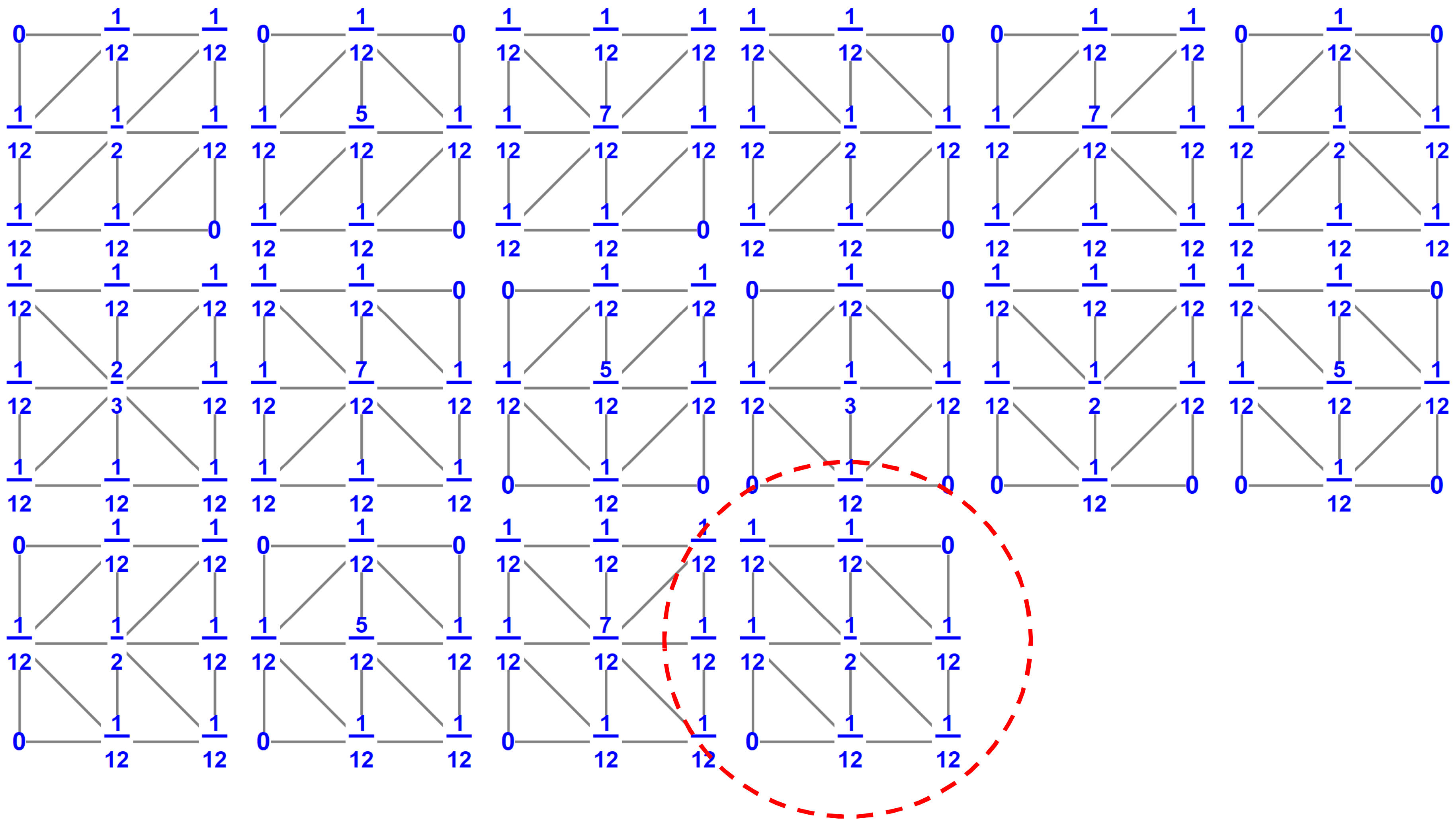
$$\Delta A \approx \int_{\Omega} N_{(i,j)} dA = \Delta x \Delta y$$

so the weighted residual approximation can be considered as the shape function weighted average. The rule for finding the difference approximation by the weighted residual method is the same for any division of the domain into triangle elements but the outcomes may differ. For the regular triangle division used in the example (geometry is the same for all interior grid points) the outcome can be described by a stencil of constant weights.

STENCIL OF LAPLACIAN

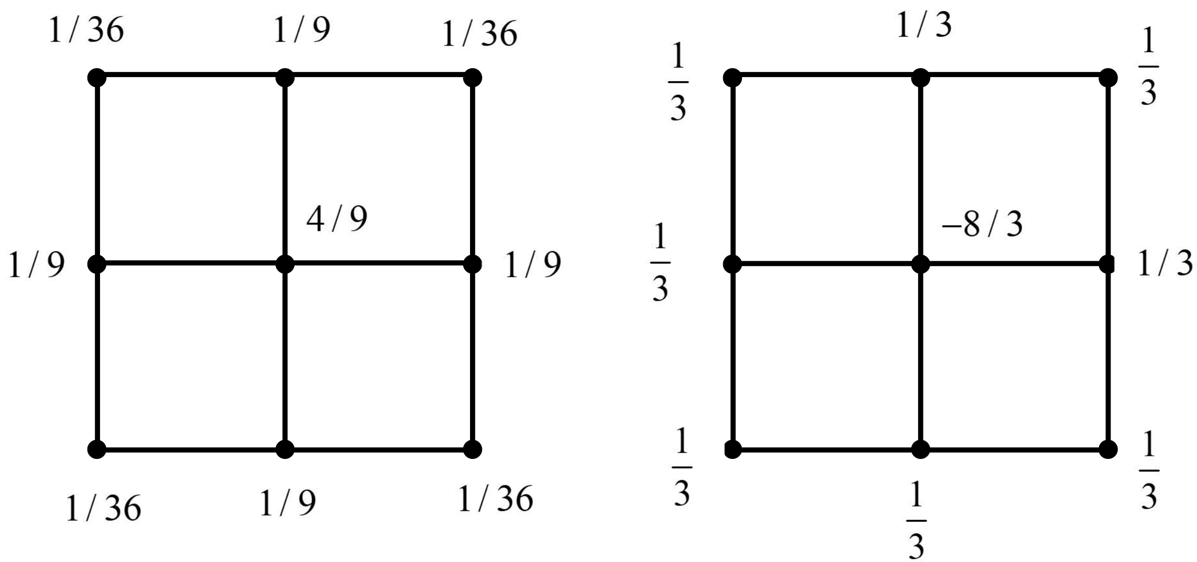


STENCIL OF IDENTITY MAPPING



OTHER STENCILS

Even on a regular grid, derivative approximations based on the weighted residual expression depend on the interpolation used. For a bi-linear interpolation (another common interpolation type in the Finite Element Method) based on four vertex points of rectangular elements, the stencils for the identity and Laplacian operators take the forms:



In each element, bi-linear approximation is combination of 1, x, y, and xy. The shape functions take the value one at one grid point and vanishes at all the other grid points.

WEIGHTED RESIDUAL EXPRESSIONS FOR MEMBRANE

Difference equation approximation to the membrane model follow from the principle of virtual work, triangle representation of the domain, piecewise linear approximation to the transverse displacement, and using the shape functions as the weights:

Virtual work	**String**	**Membrane**
δW^{int}	$-\int_\Omega S(\frac{\partial N_i}{\partial x}\frac{\partial w}{\partial x})dx$	$-\int_\Omega S'(\frac{\partial N_{(i,j)}}{\partial x}\frac{\partial w}{\partial x}+\frac{\partial N_{(i,j)}}{\partial y}\frac{\partial w}{\partial y})dA$
δW^{ext}	$\int_\Omega (N_i f)dx$	$\int_\Omega (N_{(i,j)} f')dA$
δW^{ine}	$-\int_\Omega (N_i \rho A\frac{\partial^2 w}{\partial t^2})dx$	$-\int_\Omega (N_{(i,j)}\rho t\frac{\partial^2 w}{\partial t^2})dA$

6.3 FINITE ELEMENT METHOD

Considering a regular grid of points of constant spacing $\Delta x = \Delta y = h$ and linear interpolation using a regular triangle division, the weighted residual expression for the membrane problem give the difference equations (2-index notation)

$$S'[w_{(i-1,j)} + w_{(i,j-1)} - 4w_{(i,j)} + w_{(i+1,j)} + w_{(i,j+1)}] + h^2 f' =$$

$$\frac{\rho t h^2}{12}[\ddot{w}_{(i-1,j-1)} + \ddot{w}_{(i-1,j)} + \ddot{w}_{(i,j-1)} + 6\ddot{w}_{(i,j)} + \ddot{w}_{(i+1,j)} + \ddot{w}_{(i,j+1)} + \ddot{w}_{(i+1,j+1)}] \quad (i,j) \in I,$$

$$w_{(i,j)} = 0 \quad (i,j) \in \partial I,$$

$$w_{(i,j)} - g_{(i,j)} = 0 \quad \text{and} \quad \dot{w}_{(i,j)} - h_{(i,j)} = 0 \quad (i,j) \in I,$$

where the interior grid point are denoted by I and the boundary grid points by ∂I.

Finite Element Method replaces the differential equations by difference equations with a compact and generic recipe for stencils of Identity, Laplacian etc. operators on regular and irregular grid of points for all dimensions. The starting point is the weighted residual expression implied by the principle of virtual work.

Using a regular triangle element representation of the solution domain, piecewise linear interpolation to the transverse displacements on the spatial grid, considering the displacement values $w_i(t)$ as functions of time, and assuming constant properties, for $(i, j) \in \{0,1,\ldots,n\} \times \{0,1,\ldots,n\}$, the weighted residual expression

$$-\int_{\Omega} S'\left(\frac{\partial N_{(i,j)}}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial N_{(i,j)}}{\partial y}\frac{\partial w}{\partial y}\right)dA + \int_{\Omega} N_{(i,j)} f' dA = \int_{\Omega} N_{(i,j)} \rho t \frac{\partial^2 w}{\partial t^2} dA$$

gives the difference equation for the interior grid points $I = \{1,2\ldots,n-1\} \times \{1,2\ldots,n-1\}$ (nodes)

$$S'[w_{(i-1,j)} + w_{(i,j-1)} - 4w_{(i,j)} + w_{(i+1,j)} + w_{(i,j+1)}] + h^2 f' =$$

$$\frac{\rho t h^2}{12}[\ddot{w}_{(i-1,j-1)} + \ddot{w}_{(i-1,j)} + \ddot{w}_{(i,j-1)} + 6\ddot{w}_{(i,j)} + \ddot{w}_{(i+1,j)} + \ddot{w}_{(i,j+1)} + \ddot{w}_{(i+1,j+1)}]. \quad t > 0$$

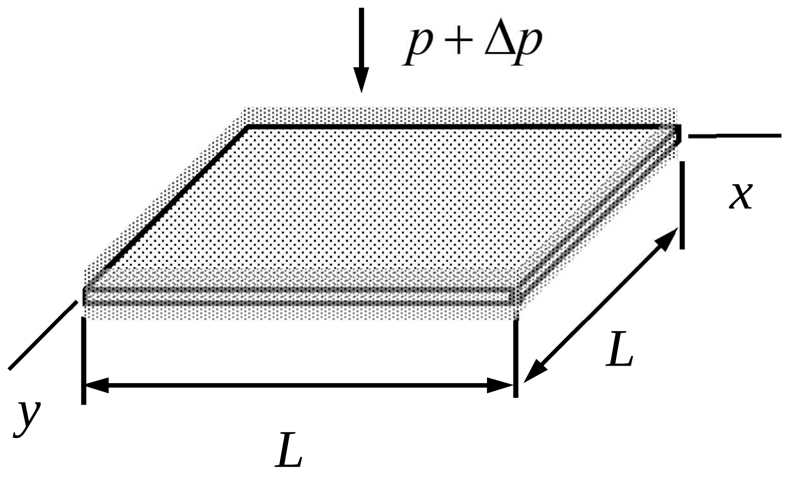
On the boundary points ∂I, displacement vanishes (the second option, force loading on the boundary, is not considered in this simplified setting) so

$$w_{(i,j)} = 0 \ \ t > 0.$$

Assuming that g and h of initial conditions are of the same form as the approximation, the initial conditions

$$w_{(i,j)} - g_{(i,j)} = 0 \quad \text{and} \quad \dot{w}_{(i,j)} - h_{(i,j)} = 0 \quad (i,j) \in I.$$

EXAMPLE A rectangular membrane of fixed edges and constant tightening s (force per unit length) is loaded by pressures $p+\Delta p$ acting on the upper surface and p acting on the lower surface. Find the solution to the transverse displacement using the Finite Element Method, regular triangle division of the domain with the regular grid $(i, j) \in \{0,1,2\} \times \{0,1,2\}$, and a piecewise linear approximation to the transverse displacement.



Answer $w_{(1,1)} = \dfrac{1}{16}\dfrac{\Delta p L^2}{S'} \approx 0.0625 \dfrac{\Delta p L^2}{S'}$ (exact to the model $0.0737 \dfrac{\Delta p L^2}{S'}$)

Finite Element Method is based on the principle of virtual work, element representation of the domain, intrpolation of the grid values inside the elements, and using the shape functions of the interpolation as the weights. In the present problem the interior and boundary grid points and the equations for the grid points are

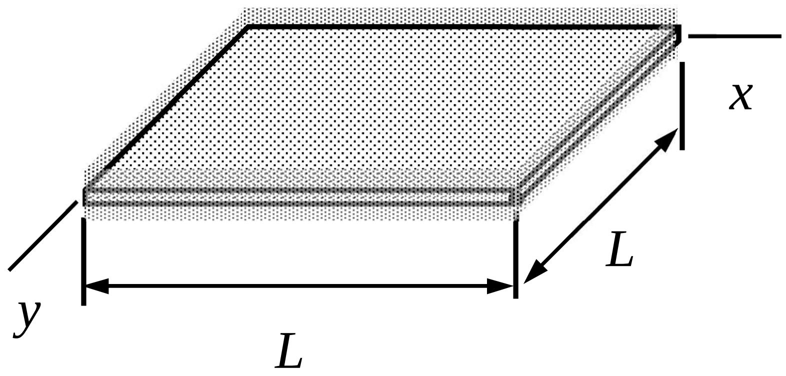
$$S'[w_{(i-1,j)} + w_{(i,j-1)} - 4w_{(i,j)} + w_{(i+1,j)} + w_{(i,j+1)}] + \Delta p h^2 = 0 \quad (i,j) \in I = \{(1,1)\}$$

$$w_{(i,j)} = 0 \quad (i,j) \in \partial I = \{0,1,2\} \times \{0,1,2\} \setminus \{(1,1)\} \quad \text{(interior point excluded)}$$

where $\Delta x = \Delta x = h = L/2$. Eliminating the displacements of the boundary points from the equation for the interior point

$$S'[-4w_{(1,1)}] + \Delta p \frac{L^2}{4} = 0 \quad \Rightarrow \quad w_{(1,1)} = \frac{1}{16} \frac{\Delta p L^2}{S'}.$$

EXAMPLE Consider a rectangular (side length L) drumhead of fixed edges, constant tightening S' (force per unit length) and density ρt (per unit area). Find the frequencies of the free vibrations by using the Finite Element Method on a regular grid $I = \{0,1,2\} \times \{0,1,2\}$ with piecewise linear approximation to the transverse displacement.



Answer $f = \dfrac{2}{\pi L}\sqrt{2\dfrac{S'}{\rho t}} \approx 0.90\dfrac{1}{L}\sqrt{\dfrac{S'}{\rho t}}$ (exact to the model $\approx 0.71\dfrac{1}{L}\sqrt{\dfrac{S'}{\rho t}}$)

Finite Element Method uses the principle of virtual work, element representation of the domain, interpolation of the grid point values inside the elements, and the shape functions of the approximation as the weights. In the present problem the interior and boundary grid points and the equations for the grid points are

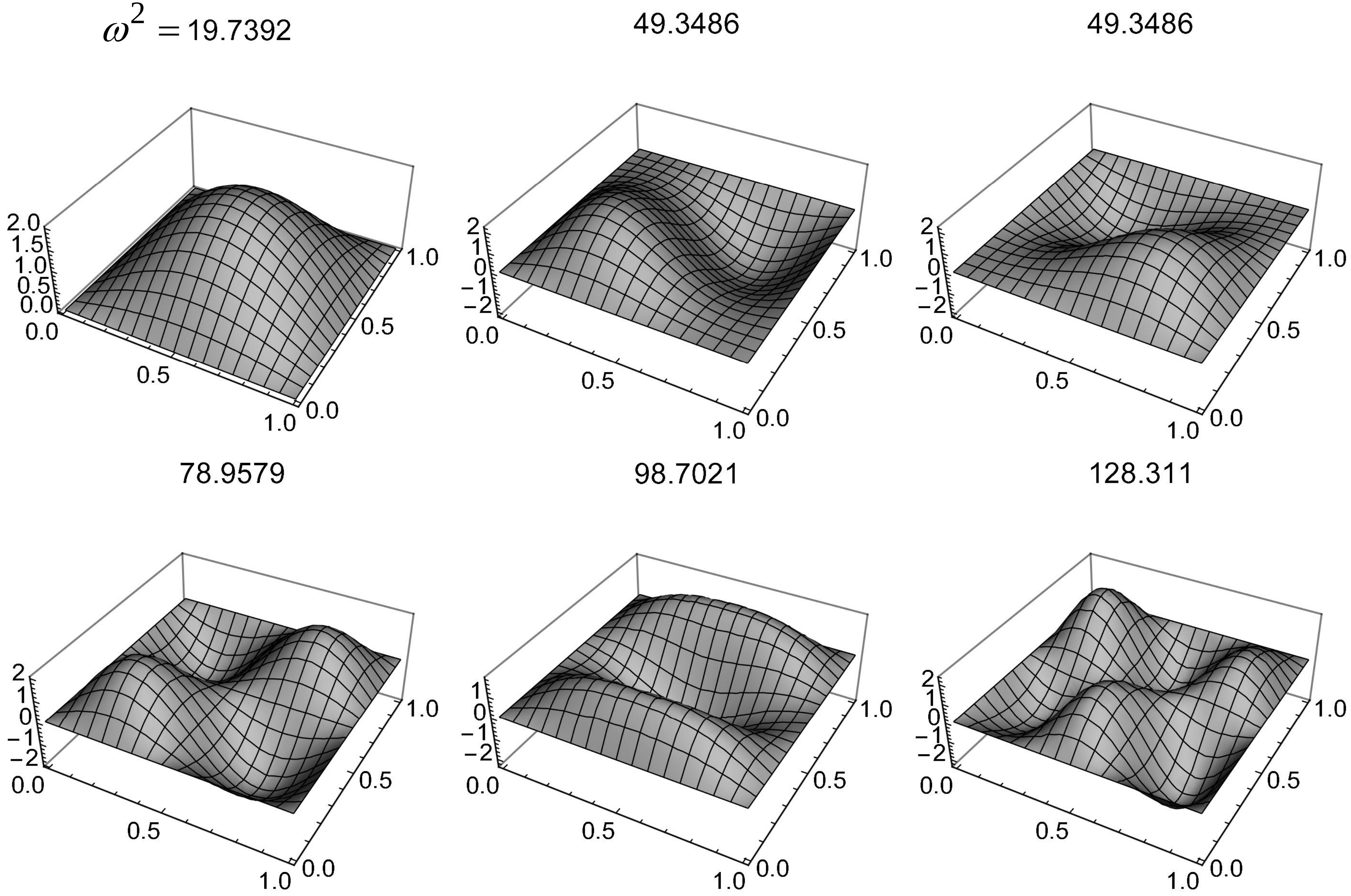
$$S'[w_{(i-1,j)} + w_{(i,j-1)} - 4w_{(i,j)} + w_{(i+1,j)} + w_{(i,j+1)}] =$$

$$\frac{\rho t h^2}{12}[\ddot{w}_{(i-1,j-1)} + \ddot{w}_{(i-1,j)} + \ddot{w}_{(i,j-1)} + 6\ddot{w}_{(i,j)} + \ddot{w}_{(i+1,j)} + \ddot{w}_{(i,j+1)} + \ddot{w}_{(i+1,j+1)}] \quad (i,j) = (1,1),$$

$$w_{(i,j)} = 0 \quad (i,j) \in \{0,1,2\} \times \{0,1,2\} \setminus \{(1,1)\} \quad \text{(interior point excluded)}$$

where $\Delta x = \Delta x = h = L/2$. Eliminating the displacements of the boundary points from the equation for the interior point

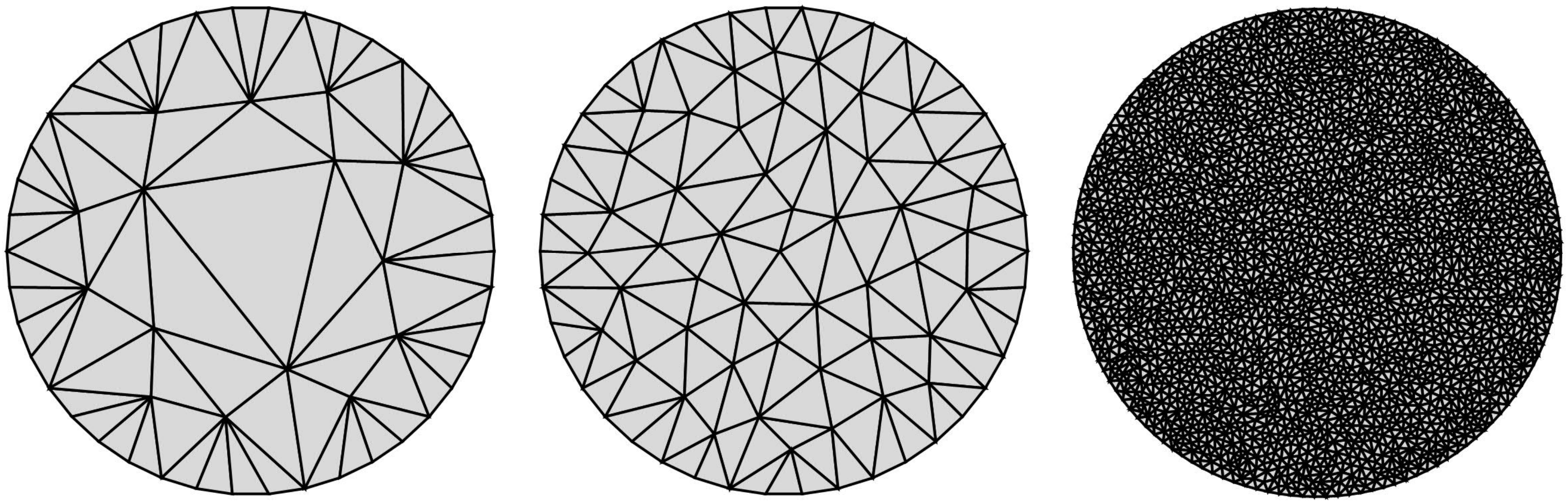
$$\ddot{w}_{(1,1)} + \omega^2 w_{(1,1)} = 0 \quad \text{where} \quad \omega = 2\pi f = \frac{4}{L}\sqrt{2\frac{S'}{\rho t}} \quad \text{so} \quad f = \frac{2}{\pi L}\sqrt{2\frac{S'}{\rho t}} \;.$$



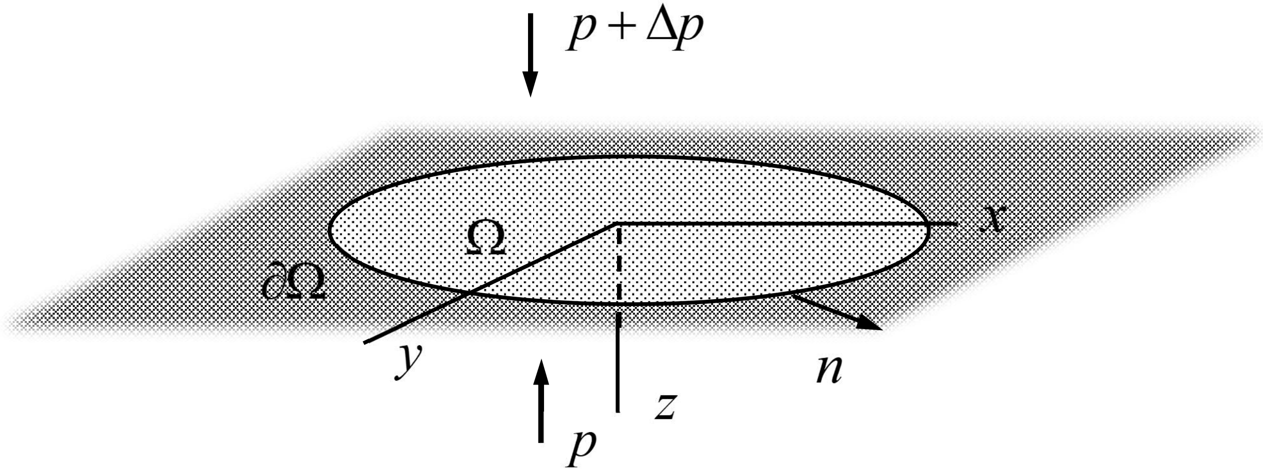
6-30

6.4 NON-REGULAR GRIDS

Finite Element Method does not impose restrictions on the solution domain geometry or require regularity of the grid or elements having the grid points as vertices. No matter the case, function values at the grid points are interpolated inside the elements and the integrals are calculated elementwise. The outcome is a set algebraic equations or ordinary differential equations that can be solved with matrix methods.



EXAMPLE A circular membrane of radius R, fixed edges, and constant tightening S' (force per unit length) is loaded by pressure $p+\Delta p$ acting on the upper surface and p acting on the lower surface. Find the transverse displacement at the centerpoint by using the Finite Element Method and piecewise linear approximations on triangle elements.



Answer (Exact $w_{\max} = \dfrac{1}{4}\dfrac{\Delta p R^2}{S'}$)

In numerical calculations, the problem parameters need to be given values unless the number of grid points is small and the code used allows symbolic calculations (e.g., Mathematica does). Let us consider a combination for which the dimensionless group $\Delta p R^2 / S' = 4$ and let Mathematica to do the triangulation and find the interior and boundary points

```
Ω = Disk[{0, 0}];
ℛ = DiscretizeRegion[Ω, MaxCellMeasure → 0.05]
bℛ = RegionBoundary[ℛ];
bp = MeshCells[bℛ, 0] /. Point[any_] → any;
ip = Complement[Table[i, {i, 1, MeshCellCount[ℛ, 0]}], bp];
```

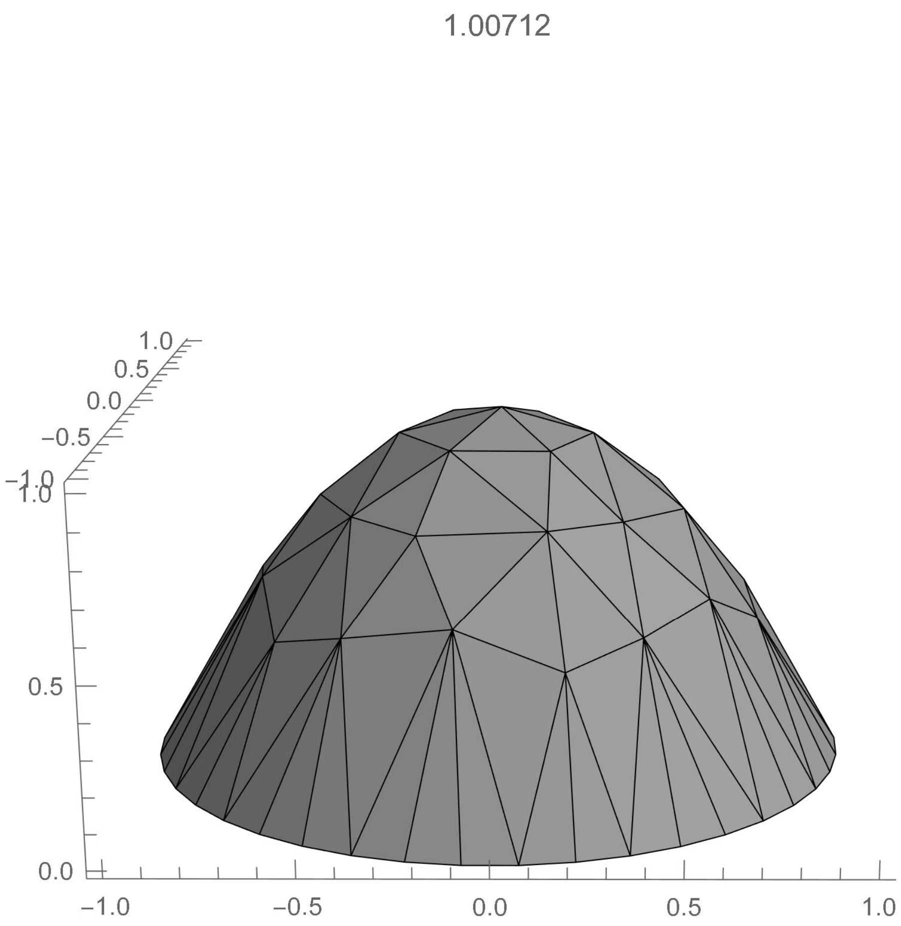
After that, use the stencils of Laplacian and loading at the interior points (given by the weighted residual expression and piecewise linear approximation to the transverse displacement on the triangle representation) to find the matrix representation of the equilibrium equations and solve for the displacements

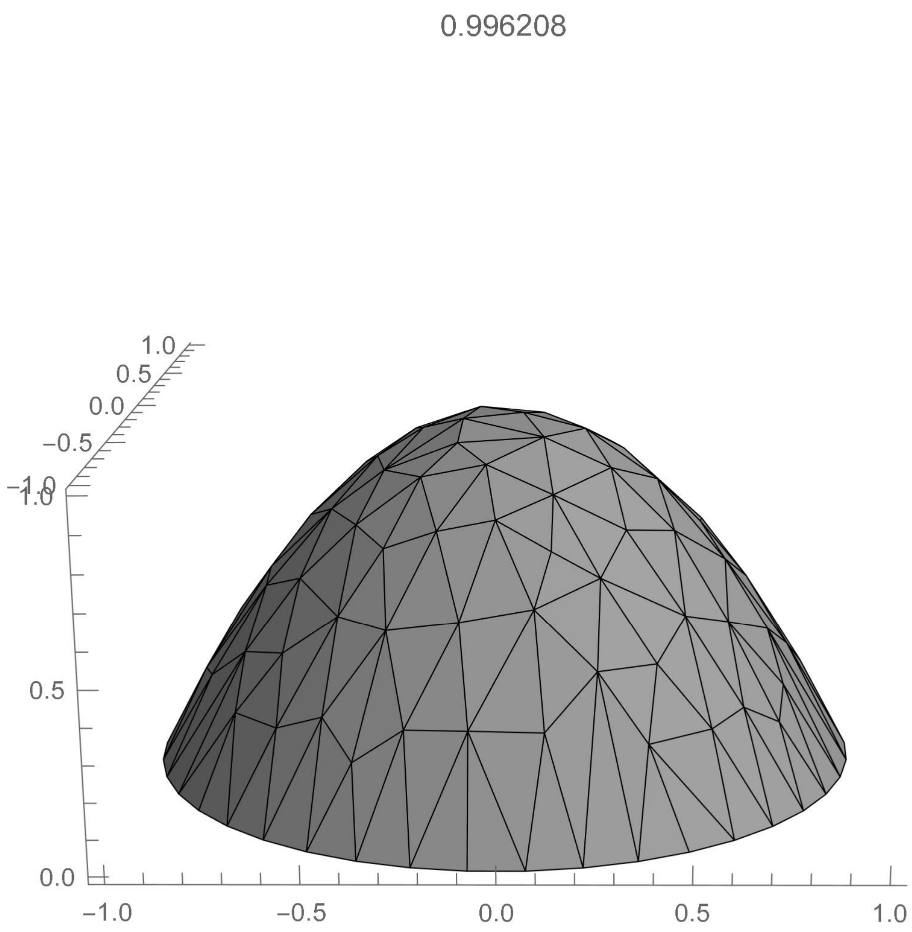
```
FF = LOAD[ℛ, 4];
KK = LAPLACIAN[ℛ];
iw = LinearSolve[KK[[ip, ip]], -FF[[ip]]]
ww = Table[0, {i, 1, MeshCellCount[ℛ, 0]}];
ww[[ip]] = iw;
```

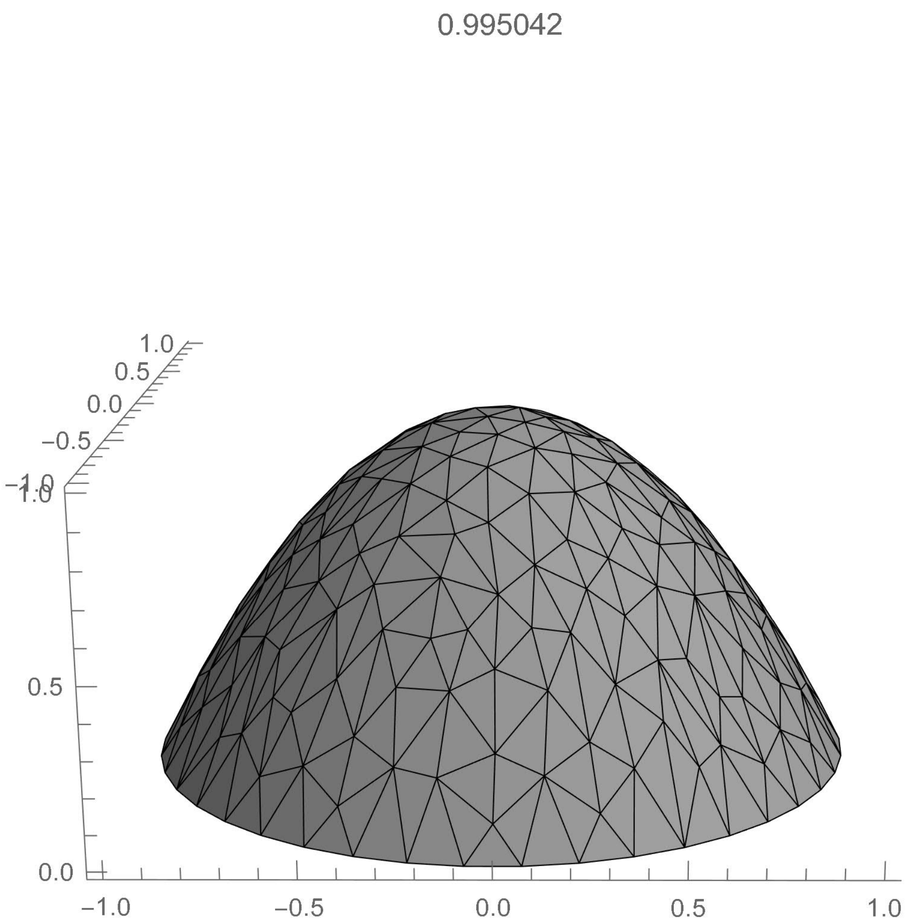
Finally, some post-processing to check the outcome and find the maximal value of the displacement

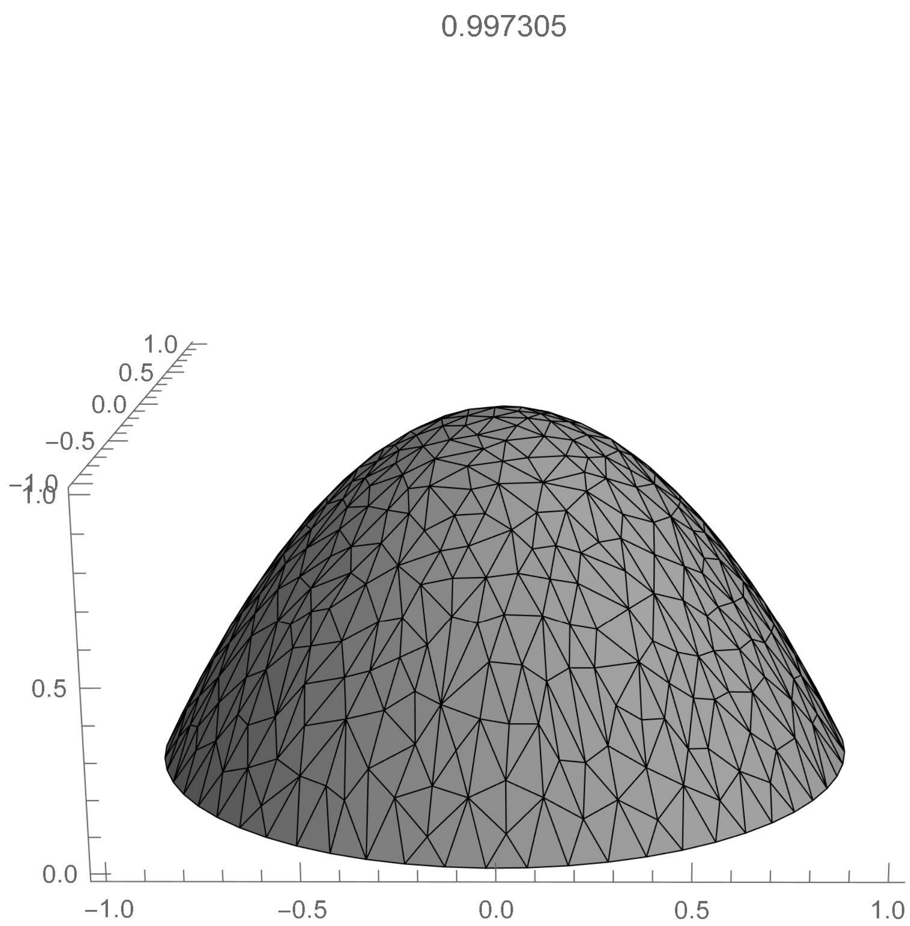
```
xyw = Transpose[Append[Transpose[MeshCoordinates[ℛ]], ww]];
wℛ = MeshRegion[xyw, MeshCells[ℛ, 2]];
Show[HighlightMesh[wℛ, {Style[{2}, Gray], Style[{1}, Black]}], Axes → True, PlotLabel -> Max[ww]]
```

For a picture about the discretization error, problem can be solved a few times by reducing the size of the elements (reducing the parameter MaxCellMeasure in the beginning of the Mathematica code):


1.00712
1.0
0.5
0.0
−0.5
−1.0
0.5
0.0
−1.0
−0.5
0.0
0.5
1.0


0.996208
1.0
0.5
0.0
−0.5
−1.0
0.5
0.0
−1.0
−0.5
0.0
0.5
1.0


0.995042
1.0
0.5
0.0
−0.5
−1.0
1.0
0.5
0.0
−1.0
−0.5
0.0
0.5
1.0

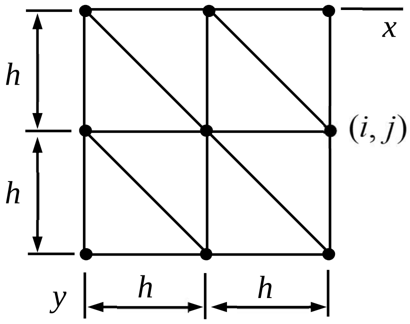

0.997305
1.0
0.5
0.0
−0.5
−1.0
1.0
0.5
0.0
−1.0
−0.5
0.0
0.5
1.0

COE-C3005 Finite Element and Finite difference methods

1. Derive the linear interpolant expression for line and triangle elements

$$w(x)=\begin{Bmatrix} w_i \\ w_j \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & 1 \\ x_i & x_j \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \end{Bmatrix} \text{ and } w(x,y)=\begin{Bmatrix} w_i \\ w_j \\ w_k \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & 1 & 1 \\ x_i & x_j & x_k \\ y_i & y_j & y_k \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}.$$

2. Derive the difference equation for boundary point (i, j) of a regular grid shown. Use piecewise linear approximation w and the weighted residual expression $R_{(i,j)}^{\text{int}} + R_{(i,j)}^{\text{ext}} = 0$, where ($S'$ and f' are constants)

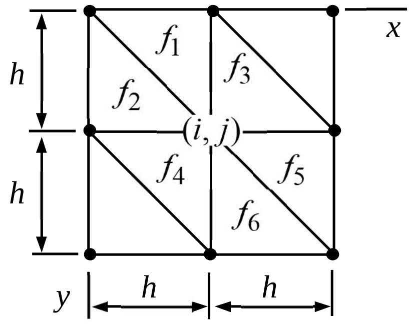


$$R_{(i,j)}^{\text{int}} = -\int_{\Omega} S'(\frac{\partial N_{(i,j)}}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial N_{(i,j)}}{\partial y}\frac{\partial w}{\partial y})dA,$$

$$R_{(i,j)}^{\text{ext}} = \int_{\Omega} (N_{(i,j)} f')dA.$$

Answer $\frac{1}{2}S[2w_{(i-1,j)} + w_{(i,j-1)} - 4w_{(i,j)} + w_{(i,j+1)}] + f'\frac{h^2}{2} = 0$

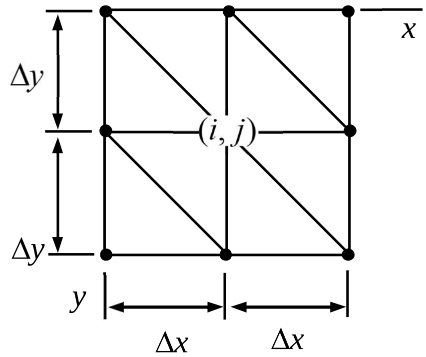
3. Find the weighted residual expression $R_{(i,j)}^{\text{ext}}$ for an external distributed force f' which is piecewise constant. Use the notation in the figure for the constant values in the triangle elements and expression



$$R_{(i,j)}^{\text{ext}} = \int_{\Omega} (N_{(i,j)} f')dA.$$

Answer $R_{(i,j)}^{\text{ext}} = \frac{h^2}{6}(f_1 + f_2 + \ldots + f_6)$

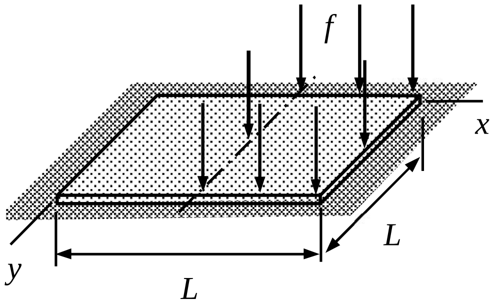
4. Find the weighted residual expression $R_{(i,j)}^{\text{int}}$ for internal forces when the spacing of regular grid differs in the $x-$ and $y-$directions. Use the expression



$$R_{(i,j)}^{\text{int}} = -\int_{\Omega} S'(\frac{\partial N_{(i,j)}}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial N_{(i,j)}}{\partial y}\frac{\partial w}{\partial y})dA.$$

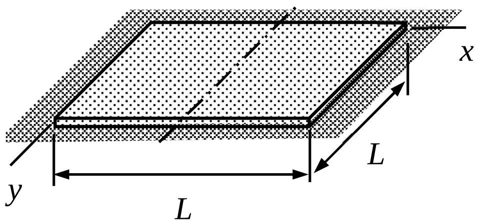
Answer $R_{(i,j)}^{\text{int}} = -S'\frac{\Delta x}{\Delta y}[-w_{(i,j-1)} + 2w_{(i,j)} - w_{(i,j+1)}] - S'\frac{\Delta y}{\Delta x}[-w_{(i-1,j)} + 2w_{(i,j)} - w_{(i+1,j)}]$

5. A rectangular membrane of side length L and tightening S' is loaded by a constant distributed force f acting on half of the membrane as shown. If the edges are fixed, find the transverse displacement using the Finite Element Method on a regular grid $(i,j) \in \{0,1,2\} \times \{0,1,2\}$.



Answer $w_{(1,1)} = \dfrac{h^2 f}{8S'}$

6. Consider a rectangular membrane of side length L, ρ, thickness t, and tightening S'. If the edges are fixed, find angular velocities of the modes, that are reflection symmetric with respect to the line through the center point shown using the Finite Element Method. Use a regular grid $(i,j) \in \{0,1,2,3,4\} \times \{0,1,2\}$ of different spacings in the coordinate directions.



Answer $\omega = \dfrac{4}{L}\sqrt{3(2 \pm \sqrt{2})\dfrac{S'}{\rho t}}$

Derive the linear interpolant expression for line and triangle elements

$$w(x)=\begin{Bmatrix} w_i \\ w_j \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & 1 \\ x_i & x_j \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \end{Bmatrix} \quad \text{and} \quad w(x,y)=\begin{Bmatrix} w_i \\ w_j \\ w_k \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & 1 & 1 \\ x_i & x_j & x_k \\ y_i & y_j & y_k \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}.$$

Solution

Let us start with the line element. By definition, interpolant $w(x)$ is linear in x

$$w(x)=a+bx=\begin{Bmatrix} a \\ b \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 1 \\ x \end{Bmatrix}$$

and takes the prescribed values at the end points so $w(x_i)=a+bx_i=w_i$ and $w(x_j)=a+bx_j=w_j$. Using the matrix notation

$$\begin{bmatrix} 1 & x_i \\ 1 & x_j \end{bmatrix} \begin{Bmatrix} a \\ b \end{Bmatrix} = \begin{Bmatrix} w_i \\ w_j \end{Bmatrix} \quad \Leftrightarrow \quad \begin{Bmatrix} a \\ b \end{Bmatrix} = \begin{bmatrix} 1 & x_i \\ 1 & x_j \end{bmatrix}^{-1} \begin{Bmatrix} w_i \\ w_j \end{Bmatrix}.$$

Therefore, in terms of the end point values

$$w(x)=\begin{Bmatrix} a \\ b \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 1 \\ x \end{Bmatrix} = \begin{Bmatrix} w_i \\ w_j \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & 1 \\ x_i & x_j \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \end{Bmatrix}.$$

Then using the same steps in case of a triangle element: By definition, interpolant $w(x,y)$ is linear in x and y

$$w(x,y)=a+bx+cy=\begin{Bmatrix} a \\ b \\ c \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}$$

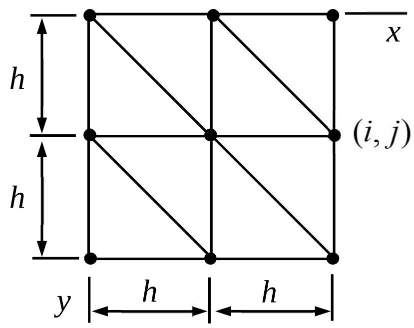
and takes the prescribed values at the end points so $w(x_\alpha,y_\alpha)=a+bx_\alpha+cy_\alpha=w_\alpha$ where $\alpha\in\{i,j,k\}$. Using the matrix notation

$$\begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \end{Bmatrix} = \begin{Bmatrix} w_i \\ w_j \\ w_k \end{Bmatrix} \quad \Leftrightarrow \quad \begin{Bmatrix} a \\ b \\ c \end{Bmatrix} = \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix}^{-1} \begin{Bmatrix} w_i \\ w_j \\ w_k \end{Bmatrix}.$$

Therefore, in terms of the end point values

$$w(x,y)=a+bx+cy=\begin{Bmatrix} w_i \\ w_j \\ w_k \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & 1 & 1 \\ x_i & x_j & x_k \\ y_i & y_j & y_k \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}.$$

Derive the difference equation for boundary point (i, j) of a regular grid shown. Use piecewise linear approximation w and the weighted residual expression $R^{\text{int}}_{(i,j)} + R^{\text{ext}}_{(i,j)} = 0$, where ($S'$ and f' are constants)



$$R^{\text{int}}_{(i,j)} = -\int_{\Omega} S'(\frac{\partial N_{(i,j)}}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial N_{(i,j)}}{\partial y}\frac{\partial w}{\partial y})dA,$$

$$R^{\text{ext}}_{(i,j)} = \int_{\Omega} (N_{(i,j)} f')dA.$$

Solution

The integrals of the residuals are calculated element-by-element using the linear expression inside the elements using Mathematica or deducing the expression based on the geometrical picture about plane defined by the values of w at the vertex points and the piecewise linear shape function taking the value one at point (i, j) and vanishing at all other grid points. It is enough to consider only the elements having (i, j) as one of the vertex point as $N_{(i,j)}$ vanishes elsewhere. Let us start with $R^{\text{int}}_{(i,j)}$

Triangle Ω_1 of vertices (i, j), $(i, j-1)$, and $(i-1, j-1)$ (counterclockwise):

$$\frac{\partial N_{(i,j)}}{\partial x} = 0, \quad \frac{\partial N_{(i,j)}}{\partial y} = \frac{1}{h}, \quad \frac{\partial w}{\partial x} = \frac{w_{(i,j-1)} - w_{(i-1,j-1)}}{h}, \quad \frac{\partial w}{\partial y} = \frac{w_{(i,j)} - w_{(i,j-1)}}{h} \quad \Rightarrow$$

$$-\int_{\Omega_1} S'(\frac{\partial N_{(i,j)}}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial N_{(i,j)}}{\partial y}\frac{\partial w}{\partial y})dA = -S'\frac{1}{2}(w_{(i,j)} - w_{(i,j-1)}).$$

Triangle Ω_2 of vertices (i, j), $(i-1, j-1)$, and $(i-1, j)$ (counterclockwise):

$$\frac{\partial N_{(i,j)}}{\partial x} = \frac{1}{h}, \quad \frac{\partial N_{(i,j)}}{\partial y} = 0, \quad \frac{\partial w}{\partial x} = \frac{w_{(i,j)} - w_{(i-1,j)}}{h}, \quad \frac{\partial w}{\partial y} = \frac{w_{(i-1,j)} - w_{(i-1,j-1)}}{h} \quad \Rightarrow$$

$$-\int_{\Omega_2} S'(\frac{\partial N_{(i,j)}}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial N_{(i,j)}}{\partial y}\frac{\partial w}{\partial y})dA = -S'\frac{1}{2}(w_{(i,j)} - w_{(i-1,j)})$$

Triangle Ω_3 of vertices (i, j), $(i-1, j)$, and $(i, j+1)$ (counterclockwise):

$$\frac{\partial N_{(i,j)}}{\partial x} = \frac{1}{h}, \quad \frac{\partial N_{(i,j)}}{\partial y} = -\frac{1}{h}, \quad \frac{\partial w}{\partial x} = \frac{w_{(i,j)} - w_{(i-1,j)}}{h}, \quad \frac{\partial w}{\partial y} = \frac{w_{(i,j+1)} - w_{(i,j)}}{h} \quad \Rightarrow$$

$$-\int_{\Omega_3} S'(\frac{\partial N_{(i,j)}}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial N_{(i,j)}}{\partial y}\frac{\partial w}{\partial y})dA = -S'\frac{1}{2}(-w_{(i-1,j)} + 2w_{(i,j)} - w_{(i,j+1)})$$

Therefore

$$R^{\text{int}}_{(i,j)} = -\int_{\Omega} S'\left(\frac{\partial N_{(i,j)}}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial N_{(i,j)}}{\partial y}\frac{\partial w}{\partial y}\right)dA = \frac{1}{2}S(2w_{(i-1,j)} + w_{(i,j-1)} - 4w_{(i,j)} + w_{(i,j+1)})\,. \leftarrow$$

Let us consider then $R^{\text{ext}}_{(i,j)}$ in the same manner. As distributed force is constant it can be taken outside the integral. The remaining task is to find the integral of $N_{(i,j)}$ over the elements. Using the generic expression for an interpolant taking the value 1 at

Triangle Ω_1 of vertices (i,j), $(i,j-1)$, and $(i-1,j-1)$.

$$N_{(i,j)} = \begin{Bmatrix}1\\0\\0\end{Bmatrix}^{\mathrm{T}} \begin{bmatrix}1 & 1 & 1\\ 2h & 2h & h\\ h & 0 & 0\end{bmatrix}^{-1} \begin{Bmatrix}1\\x\\y\end{Bmatrix} = \frac{y}{h} \quad\Rightarrow\quad \int_h^{2h}\int_0^{x-h}\frac{y}{h}\,dydx = \int_h^{2h}\frac{1}{2}\frac{(x-h)^2}{h}\,dx = \frac{h^2}{6}$$

Triangle Ω_2 of vertices (i,j), $(i-1,j-1)$, and $(i-1,j)$:

$$N_{(i,j)} = \begin{Bmatrix}1\\0\\0\end{Bmatrix}^{\mathrm{T}} \begin{bmatrix}1 & 1 & 1\\ 2h & h & h\\ h & 0 & h\end{bmatrix}^{-1} \begin{Bmatrix}1\\x\\y\end{Bmatrix} = \frac{x}{h} - 1 \quad\Rightarrow\quad \int_h^{2h}\int_{x-h}^{h}\left(\frac{x}{h} - 1\right)dydx = \frac{h^2}{6}$$

Triangle Ω_3 of vertices (i,j), $(i-1,j)$, and $(i,j+1)$:

$$N_{(i,j)} = \begin{Bmatrix}1\\0\\0\end{Bmatrix}^{\mathrm{T}} \begin{bmatrix}1 & 1 & 1\\ 2h & h & 2h\\ h & h & 2h\end{bmatrix}^{-1} \begin{Bmatrix}1\\x\\y\end{Bmatrix} = \frac{x}{h} - \frac{y}{h} \quad\Rightarrow\quad \int_h^{2h}\int_h^{x}\left(\frac{x}{h} - \frac{y}{h}\right)dydx = \frac{h^2}{6}$$

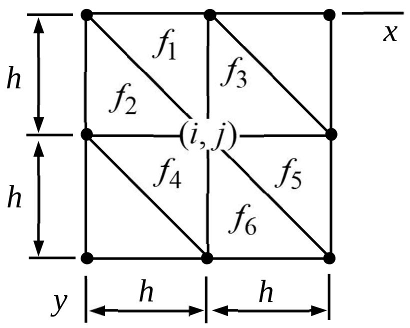
Therefore

$$R^{\text{ext}}_{(i,j)} = f'\left(\frac{h^2}{6} + \frac{h^2}{6} + \frac{h^2}{6}\right) = f'\frac{h^2}{2}\,. \leftarrow$$

And the difference equation for boundary point (i,j)

$$R^{\text{int}}_{(i,j)} + R^{\text{ext}}_{(i,j)} = \frac{1}{2}S(2w_{(i-1,j)} + w_{(i,j-1)} - 4w_{(i,j)} + w_{(i,j+1)}) + f'\frac{h^2}{2} = 0\,. \leftarrow$$

Find the weighted residual expression $R^{\text{ext}}_{(i,j)}$ for an external distributed force f' which is piecewise constant. Use the notation in the figure for the constant values in the triangle elements and expression



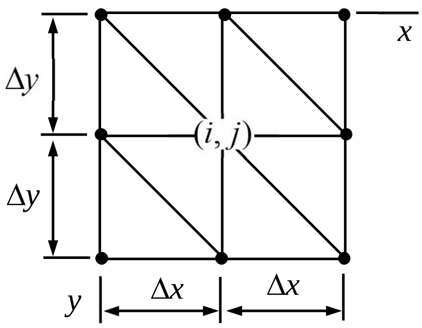
$$R^{\text{ext}}_{(i,j)} = \int_{\Omega} (N_{(i,j)} f')dA \,.$$

Solution

According to problem 2, for a regular grid of equal spacing h in the coordinate directions, integral of the shape function of point (i, j) over a tringle having (i, j) as one of its vertex points is always $h^2 / 6$. As distributed force f' constant in each element

$$R^{\text{ext}}_{(i,j)} = \int_{\Omega} (N_{(i,j)} f')dA = \sum_{e\in\{1,\ldots,6\}} f_e \int_{\Omega_e} N_{(i,j)})dA = \frac{h^2}{6} \sum_{e\in\{1,\ldots,6\}} f_e \,.$$

Find the weighted residual expression $R^{\text{int}}_{(i,j)}$ for internal forces when the spacing of regular grid differs in the $x-$ and $y-$ directions. Use the expression



$$R^{\text{int}}_{(i,j)} = -\int_{\Omega} S'(\frac{\partial N_{(i,j)}}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial N_{(i,j)}}{\partial y}\frac{\partial w}{\partial y})dA .$$

Solution

The integrals of the residuals are calculated element-by-element using the linear expression inside the elements using Mathematica or deducing the expression based on the geometrical picture about a plane defined by the values of w at the vertex points and the piecewise linear shape function taking the value one at point (i, j) and vanishing at all other grid points. It is enough to consider only the elements having (i, j) as one of the vertex point as $N_{(i,j)}$ vanishes elsewhere.

Triangle Ω_1 of vertices (i, j), $(i, j-1)$, and $(i-1, j-1)$:

$$\frac{\partial N_{(i,j)}}{\partial x} = 0 , \quad \frac{\partial N_{(i,j)}}{\partial y} = \frac{1}{\Delta y}, \quad \frac{\partial w}{\partial x} = \frac{w_{(i,j-1)} - w_{(i-1,j-1)}}{\Delta x}, \quad \frac{\partial w}{\partial y} = \frac{w_{(i,j)} - w_{(i,j-1)}}{\Delta y} \quad \Rightarrow$$

$$-\int_{\Omega_1} S'(\frac{\partial N_{(i,j)}}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial N_{(i,j)}}{\partial y}\frac{\partial w}{\partial y})dA = -S'\frac{1}{2}\frac{\Delta x}{\Delta y}[w_{(i,j)} - w_{(i,j-1)}] .$$

Triangle Ω_2 of vertices (i, j), $(i-1, j-1)$, and $(i-1, j)$:

$$\frac{\partial N_{(i,j)}}{\partial x} = \frac{1}{\Delta x}, \quad \frac{\partial N_{(i,j)}}{\partial y} = 0 , \quad \frac{\partial w}{\partial x} = \frac{w_{(i,j)} - w_{(i-1,j)}}{\Delta x}, \quad \frac{\partial w}{\partial y} = \frac{w_{(i-1,j)} - w_{(i-1,j-1)}}{\Delta y} \quad \Rightarrow$$

$$-\int_{\Omega_2} S'(\frac{\partial N_{(i,j)}}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial N_{(i,j)}}{\partial y}\frac{\partial w}{\partial y})dA = -S'\frac{1}{2}\frac{\Delta y}{\Delta x}[w_{(i,j)} - w_{(i-1,j)}] .$$

Triangle Ω_3 of vertices (i, j), $(i-1, j)$, and $(i, j+1)$:

$$\frac{\partial N_{(i,j)}}{\partial x} = \frac{1}{\Delta x}, \quad \frac{\partial N_{(i,j)}}{\partial y} = -\frac{1}{\Delta y}, \quad \frac{\partial w}{\partial x} = \frac{w_{(i,j)} - w_{(i-1,j)}}{\Delta x}, \quad \frac{\partial w}{\partial y} = \frac{w_{(i,j+1)} - w_{(i,j)}}{\Delta y} \quad \Rightarrow$$

$$-\int_{\Omega_3} S'(\frac{\partial N_{(i,j)}}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial N_{(i,j)}}{\partial y}\frac{\partial w}{\partial y})dA = -S'\frac{1}{2}\frac{\Delta y}{\Delta x}[w_{(i,j)} - w_{(i-1,j)}] + S'\frac{1}{2}\frac{\Delta x}{\Delta y}[w_{(i,j+1)} - w_{(i,j)}] .$$

Triangle Ω_4 of vertices (i, j), $(i, j+1)$, and $(i+1, j+1)$:

$$\frac{\partial N_{(i,j)}}{\partial x} = 0 , \quad \frac{\partial N_{(i,j)}}{\partial y} = -\frac{1}{\Delta y}, \quad \frac{\partial w}{\partial x} = \frac{w_{(i+1,j+1)} - w_{(i,j+1)}}{\Delta x}, \quad \frac{\partial w}{\partial y} = \frac{w_{(i,j+1)} - w_{(i,j)}}{\Delta y} \quad \Rightarrow$$

$$-\int_{\Omega_4} S'\left(\frac{\partial N_{(i,j)}}{\partial x}\frac{\partial w}{\partial x}+\frac{\partial N_{(i,j)}}{\partial y}\frac{\partial w}{\partial y}\right)dA = S'\frac{1}{2}\frac{\Delta x}{\Delta y}[w_{(i,j+1)}-w_{(i,j)}].$$

Triangle Ω_5 of vertices (i, j), $(i+1, j+1)$, and $(i+1, j)$:

$$\frac{\partial N_{(i,j)}}{\partial x}=-\frac{1}{\Delta x}, \quad \frac{\partial N_{(i,j)}}{\partial y}=0, \quad \frac{\partial w}{\partial x}=\frac{w_{(i+1,j)}-w_{(i,j)}}{\Delta x}, \quad \frac{\partial w}{\partial y}=\frac{w_{(i+1,j+1)}-w_{(i+1,j)}}{\Delta y} \quad \Rightarrow$$

$$-\int_{\Omega_5} S'\left(\frac{\partial N_{(i,j)}}{\partial x}\frac{\partial w}{\partial x}+\frac{\partial N_{(i,j)}}{\partial y}\frac{\partial w}{\partial y}\right)dA = S'\frac{1}{2}\frac{\Delta y}{\Delta x}[w_{(i+1,j)}-w_{(i,j)}].$$

Triangle Ω_6 of vertices (i, j), $(i+1, j+1)$, and $(i+1, j)$:

$$\frac{\partial N_{(i,j)}}{\partial x}=-\frac{1}{\Delta x}, \quad \frac{\partial N_{(i,j)}}{\partial y}=\frac{1}{\Delta y}, \quad \frac{\partial w}{\partial x}=\frac{w_{(i+1,j)}-w_{(i,j)}}{\Delta x}, \quad \frac{\partial w}{\partial y}=\frac{w_{(i,j)}-w_{(i,j-1)}}{\Delta y} \quad \Rightarrow$$

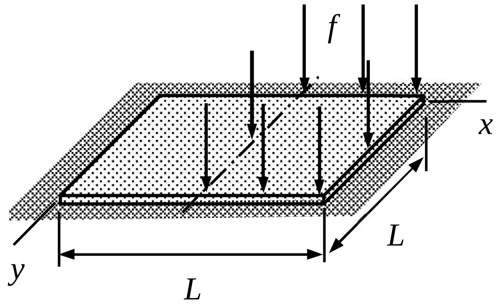
$$-\int_{\Omega_6} S'\left(\frac{\partial N_{(i,j)}}{\partial x}\frac{\partial w}{\partial x}+\frac{\partial N_{(i,j)}}{\partial y}\frac{\partial w}{\partial y}\right)dA = S'\frac{1}{2}\frac{\Delta y}{\Delta x}[w_{(i+1,j)}-w_{(i,j)}]-S'\frac{1}{2}\frac{\Delta x}{\Delta y}[w_{(i,j)}-w_{(i,j-1)}].$$

Therefore

$$R^{\text{int}}_{(i,j)} = -\int_{\Omega} S'\left(\frac{\partial N_{(i,j)}}{\partial x}\frac{\partial w}{\partial x}+\frac{\partial N_{(i,j)}}{\partial y}\frac{\partial w}{\partial y}\right)dA \quad \Rightarrow$$

$$R^{\text{int}}_{(i,j)} = -S'\frac{\Delta x}{\Delta y}[-w_{(i,j-1)}+2w_{(i,j)}-w_{(i,j+1)}]-S'\frac{\Delta y}{\Delta x}[-w_{(i-1,j)}+2w_{(i,j)}-w_{(i+1,j)}].$$

A rectangular membrane of side length L and tightening S' is loaded by a constant distributed force f acting on half of the membrane as shown. If the edges are fixed, find the transverse displacement using the Finite Element Method on a regular grid $(i,j)\in\{0,1,2\}\times\{0,1,2\}$.



Solution

In stationary problem, the generic equations for the membrane model with fixed boundaries as given by the Finite Element Method on a regular grid are

$$S'[w_{(i-1,j)}+w_{(i,j-1)}-4w_{(i,j)}+w_{(i+1,j)}+w_{(i,j+1)}]+\frac{h^2}{6}\sum f=0 \quad (i,j)\in I\,,$$

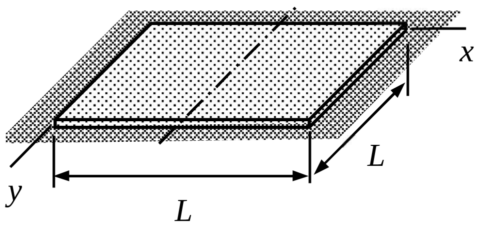
$$w_{(i,j)}=0 \quad (i,j)\in\partial I\,,$$

where the distributed loading is assumed to be piecewise constant (constant in each element) and the sum is over the constant loads in the elements having the point (i,j) as one of the vertices. In the present problem, time derivatives vanish, initial conditions are not needed. Equilibrium equation of the only interior point $(1,1)$

$$S'[w_{(0,1)}+w_{(1,0)}-4w_{(1,1)}+w_{(2,1)}+w_{(1,2)}]+\frac{h^2}{6}(f+f+f)=0 \quad \Rightarrow$$

$$S'[-4w_{(1,1)}]+\frac{h^2 f}{2}=0 \quad \Leftrightarrow \quad w_{(1,1)}=\frac{h^2 f}{8S'}\,.$$

Consider a rectangular membrane of side length L, density ρ, thickness t, and tightening S'. If the edges are fixed, find angular velocities of the modes, that are reflection symmetric with respect to the line through the center point shown using the Finite Element Method. Use a regular grid $(i,j)\in\{0,1,2,3,4\}\times\{0,1,2\}$ of different spacings in the coordinate directions.



Solution

Assuming regular grid of points and regular triangle division, the generic equations for the membrane model with fixed boundaries as given by the Finite Element Method are (the left hand side is given by the solution to problem 4, the right hand side can be obtained just by replacing h^2 in the expression of the formulae collection by $\Delta x\Delta y$)

$$-S'\frac{\Delta x}{\Delta y}[-w_{(i,j-1)}+2w_{(i,j)}-w_{(i,j+1)}]-S'\frac{\Delta y}{\Delta x}[-w_{(i-1,j)}+2w_{(i,j)}-w_{(i+1,j)}]=$$

$$m'\Delta x\Delta y\frac{1}{12}[\ddot{w}_{(i-1,j-1)}+\ddot{w}_{(i-1,j)}+\ddot{w}_{(i,j-1)}+6\ddot{w}_{(i,j)}+\ddot{w}_{(i+1,j)}+\ddot{w}_{(i,j+1)}+\ddot{w}_{(i+1,j+1)}]\quad (i,j)\in I\quad t>0$$

$$w_{(i,j)}=0\quad (i,j)\in\partial I\quad t>0\,,$$

As the mode is assumed to be reflection symmetric with respect to lines through the center point and aligned with the coordinate axes, transverse displacements at the grid points satisfy

$$w_{(1,1)}=w_{(3,1)}=w_1\quad\text{and}\quad w_{(2,1)}=w_2\,,$$

the remaining displacements at the boundary points being zeros. Using $m'=\rho t$, $\Delta x=L/4$, $\Delta y=L/2$ in equations for points $(1,1)$, $(2,1)$, and $(3,1)$

$$-S'\frac{1}{2}[-w_{(1,0)}+2w_{(1,1)}-w_{(1,2)}]-S'\frac{2}{1}[-w_{(0,1)}+2w_{(1,1)}-w_{(2,1)}]=$$

$$m'\frac{L^2}{8}\frac{1}{12}[\ddot{w}_{(0,0)}+\ddot{w}_{(0,1)}+\ddot{w}_{(1,0)}+6\ddot{w}_{(1,1)}+\ddot{w}_{(2,1)}+\ddot{w}_{(1,2)}+\ddot{w}_{(2,2)}]\quad\Rightarrow$$

$$-S'(10w_1-4w_2)=\frac{\rho tL^2}{48}(6\ddot{w}_1+\ddot{w}_2)\,.$$

$$-S'\frac{1}{2}[-w_{(2,0)}+2w_{(2,1)}-w_{(2,2)}]-S'\frac{2}{1}[-w_{(1,1)}+2w_{(2,1)}-w_{(3,1)}]=$$

$$m'\frac{L^2}{8}\frac{1}{12}[\ddot{w}_{(1,0)}+\ddot{w}_{(1,1)}+\ddot{w}_{(2,0)}+6\ddot{w}_{(2,1)}+\ddot{w}_{(3,1)}+\ddot{w}_{(2,2)}+\ddot{w}_{(3,2)}]\quad\Rightarrow$$

$$-S'(-8w_1+10w_2)=\rho t\frac{L^2}{48}(2\ddot{w}_1+6\ddot{w}_2)\,.$$

$$-S'\frac{1}{2}[-w_{(3,0)}+2w_{(3,1)}-w_{(3,2)}]-S'\frac{2}{1}[-w_{(2,1)}+2w_{(3,1)}-w_{(4,1)}]=$$

$$m'\frac{L^2}{8}\frac{1}{12}[\ddot{w}_{(2,0)}+\ddot{w}_{(2,1)}+\ddot{w}_{(3,0)}+6\ddot{w}_{(3,1)}+\ddot{w}_{(4,1)}+\ddot{w}_{(3,2)}+\ddot{w}_{(4,2)}] \quad \Rightarrow$$

$$-S'(-4w_2+10w_1)=\rho t\frac{L^2}{48}(\ddot{w}_2+6\ddot{w}_1)\,.$$

According to the principle of virtual work, the equation for a constrained displacement is the sum of equations for the constrained points. Here the equations coincide so one may use. e.g., equation for point $(1,1)$. In matrix notation

$$\begin{bmatrix} 10 & -4 \\ -8 & 10 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix}+\frac{1}{48}\frac{\rho t L^2}{S'}\begin{bmatrix} 6 & 1 \\ 2 & 6 \end{bmatrix}\begin{Bmatrix} \ddot{w}_1 \\ \ddot{w}_2 \end{Bmatrix}=0\,.$$

Solution to the angular velocities and the corresponding modes follow with the trial solution $\mathbf{a}=\mathbf{A}e^{\mathrm{i}\omega t}$:

$$(\begin{bmatrix} 10 & -4 \\ -8 & 10 \end{bmatrix}-\lambda\begin{bmatrix} 6 & 1 \\ 2 & 6 \end{bmatrix})\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}=0 \text{ where } \lambda=\omega^2\frac{1}{48}\frac{\rho t L^2}{S'} \quad \Leftrightarrow \quad \omega=\frac{4}{L}\sqrt{3\lambda\frac{S'}{\rho t}}\,.$$

A homogeneous linear equation system can yield a non-zero solution to the mode only if the matrix is singular, i.e., its determinant vanishes. The condition can be used to find the possible values of λ

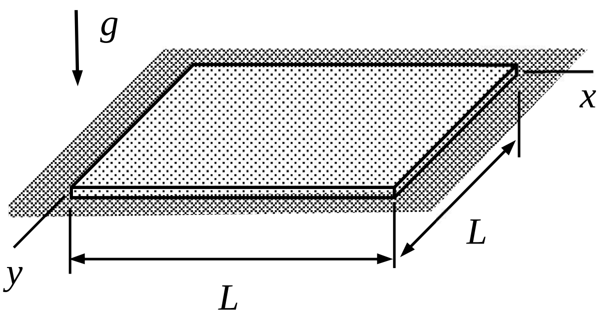
$$\det\begin{bmatrix} 10-6\lambda & -4-\lambda \\ -8-2\lambda & 10-6\lambda \end{bmatrix}=(10-6\lambda)^2-2(4+\lambda)^2=0 \quad \Rightarrow \quad \lambda=2\pm\sqrt{2}\,.$$

Knowing the possible values of parameter λ, the angular velocities follow from the relationship between λ and ω:

$$\omega=\frac{4}{L}\sqrt{3(2\pm\sqrt{2})\frac{S'}{\rho t}}\,.$$

LECTURE ASSIGNMENT 1

A rectangular membrane of side length L, density ρ, thickness t, and tightening S' (force per unit length) is loaded by its own weight as shown. If the edges are fixed, find the transverse displacements at the grid points $(i, j) \in \{0,1,2,3\} \times \{0,1,2,3\}$ of a regular grid using the Finite Element Method. Use symmetry to reduce the number of non-zero independent displacements to one.



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In a stationary problem, the discrete equations given by the Finite Element Method on regular grid of spacing h and piecewise linear approximation on triangle elements

$$S'[w_{(i-1,j)} + w_{(i,j-1)} - 4w_{(i,j)} + w_{(i+1,j)} + w_{(i,j+1)}] + h^2 f_i' = 0 \quad (i,j) \in I \,,$$

$$w_{(i,j)} = 0 \quad (i,j) \in \partial I \,.$$

Displacement vanishes at the boundary points and, due to the symmetry, displacements at the interior points should be equal. Denoting the common value by

$$w_{(1,1)} = w_{(1,2)} = w_{(2,1)} = w_{(2,2)} = w_1$$

all equations for the interior points boil down to

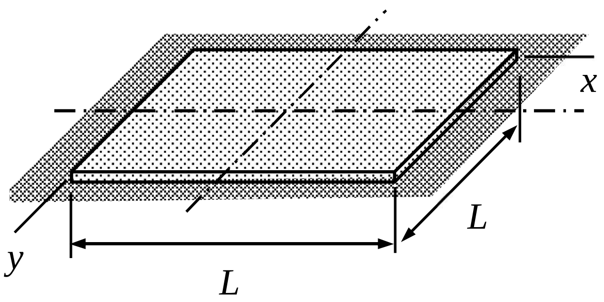
$$-S'2w_1 + \frac{L^2}{9}\rho g t = 0$$

giving as the displacement at the interior points

$$w_1 = \frac{1}{18}\frac{\rho g t L^2}{S'} \,.$$

LECTURE ASSIGNMENT 2

Consider vibration of a rectangular membrane of side length L, density ρ, thickness t, and tightening S' (force per unit length). If the edges are fixed, find the angular velocity of the free vibrations using the Finite Element Method on a regular grid of points $(i,j) \in \{0,1,2,3\} \times \{0,1,2,3\}$. Consider the mode, which is reflection symmetric with respect to the lines through the center point (figure).



Notice: According to the principle of virtual work, the equation for a constrained displacement is the sum of equations for the constrained points.

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In a time-dependent membrane problem without external forces, the equations given by the Finite Element Method on regular grid of spacing h and piecewise linear approximation on a regular triangle division are

$$S'[w_{(i-1,j)}+w_{(i,j-1)}-4w_{(i,j)}+w_{(i+1,j)}+w_{(i,j+1)}]=$$

$$m'h^2\frac{1}{12}[\ddot{w}_{(i-1,j-1)}+\ddot{w}_{(i-1,j)}+\ddot{w}_{(i,j-1)}+6\ddot{w}_{(i,j)}+\ddot{w}_{(i+1,j)}+\ddot{w}_{(i,j+1)}+\ddot{w}_{(i+1,j+1)}],$$

$$w_{(i,j)}=0.$$

Initial conditions are not needed in modal analysis. In the present problem, displacement vanishes at the boundary points and, due to the symmetry, displacements at the interior points should be equal. Denoting the common value by

$$w_{(1,1)}=w_{(1,2)}=w_{(2,1)}=w_{(2,2)}=w_1.$$

equation for point $(1,1)$ simplifies to (equations for $(1,2)$, $(2,1)$, and $(2,2)$ may differ but that is omitted in the assignment)

$$\ddot{w}_1+\omega^2 w_1=0 \quad \text{where} \quad \omega=\frac{2}{L}\sqrt{6\frac{S'}{\rho t}}.$$

Therefore, the frequency of the assumed mode shape

$$f=\frac{1}{\pi L}\sqrt{6\frac{S'}{\rho t}}.$$

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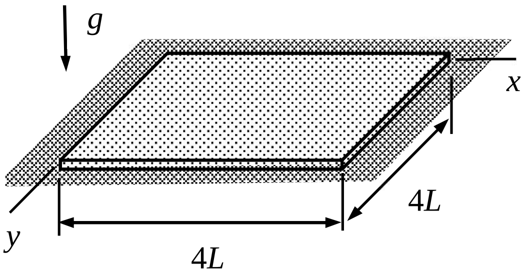
Home assignment 1

A rectangular membrane of side length $4L$, density ρ, thickness t, and tightening S' (force per unit length) is loaded by its own weigh as shown. If the edges are fixed, find the transverse displacements at the grid points $(i,j) \in \{0,1,2,3,4\} \times \{0,1,2,3,4\}$ of constant spacing. Use the Finite Element Method with a piecewise linear approximation on regular triangle elements. Use symmetry to reduce the number of non-zero independent displacements to three.



Solution

In stationary problem, the generic equations for the membrane model with fixed boundaries as given by the Finite Element Method on a regular grid are

$$S'[w_{(i-1,j)} + w_{(i,j-1)} - 4w_{(i,j)} + w_{(i+1,j)} + w_{(i,j+1)}] + h^2 f' = 0 \quad (i,j) \in I,$$

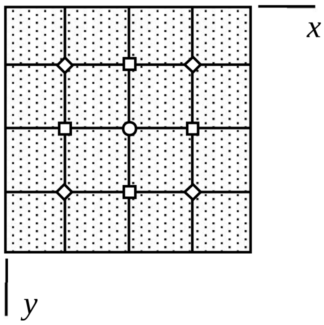
$$w_{(i,j)} = 0 \quad (i,j) \in \partial I,$$

In the present problem, time derivatives vanish, initial conditions are not needed, and solution is reflection symmetric with respect to lines through the center point and aligned with the coordinate axes. Therefore, transverse displacements at the grid points satisfy

$$w_{(1,1)} = w_{(1,3)} = w_{(3,1)} = w_{(3,3)} = w_1,$$

$$w_{(1,2)} = w_{(2,1)} = w_{(3,2)} = w_{(2,3)} = w_2,$$

$$w_{(2,2)} = w_3,$$



and the number of independent equilibrium equations is 3. According to the principle of virtual work, the equation for a constrained displacement is the sum of equations for the constrained points. In a stationary problem, it is enough to consider just one of them as the equations coincide. Considering only the equations for points $(1,1)$, $(1,2)$, and $(2,2)$ with $f' = \rho t g$ and $h = L$

$$S'[w_{(0,1)} + w_{(1,0)} - 4w_{(1,1)} + w_{(2,1)} + w_{(1,2)}] + L^2\rho tg = S'(-4w_1 + 2w_2) + L^2\rho tg = 0\,,$$

$$S'[w_{(0,1)} + w_{(1,1)} - 4w_{(1,2)} + w_{(2,2)} + w_{(1,3)}] + L^2\rho tg = S'(-4w_2 + w_3 + 2w_1) + L^2\rho tg = 0\,,$$

$$S'[w_{(1,2)} + w_{(2,1)} - 4w_{(2,2)} + w_{(3,2)} + w_{(2,3)}] + L^2\rho tg = S'(-4w_3 + 4w_2) + L^2\rho tg = 0$$

or using the matrix notation

$$\begin{bmatrix} 4 & -2 & 0 \\ -2 & 4 & -1 \\ 0 & -4 & 4 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} - \frac{\rho gtL^2}{S'} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = 0\,.$$

Then, using row operations to get an equivalent upper triangular matrix representation

$$\begin{bmatrix} 2 & -1 & 0 \\ -2 & 4 & -1 \\ 0 & -4 & 4 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} - \frac{\rho gtL^2}{S'} \begin{Bmatrix} 1/2 \\ 1 \\ 1 \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3 & -1 \\ 0 & -4 & 4 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} - \frac{\rho gtL^2}{S'} \begin{Bmatrix} 1/2 \\ 3/2 \\ 1 \end{Bmatrix} = 0 \quad \Leftrightarrow$$

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & 12 & -4 \\ 0 & -12 & 12 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} - \frac{\rho gtL^2}{S'} \begin{Bmatrix} 1/2 \\ 6 \\ 3 \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{bmatrix} 2 & -1 & 0 \\ 0 & 12 & -4 \\ 0 & 0 & 8 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} - \frac{\rho gtL^2}{S'} \begin{Bmatrix} 1/2 \\ 6 \\ 9 \end{Bmatrix} = 0\,.$$

Then, using the equations starting from the last one

$$w_3 = \frac{9}{8}\frac{\rho gtL^2}{S'}, \quad w_2 = \frac{7}{8}\frac{\rho gtL^2}{S'}, \text{ and } w_1 = \frac{11}{16}\frac{\rho gtL^2}{S'}\,.$$

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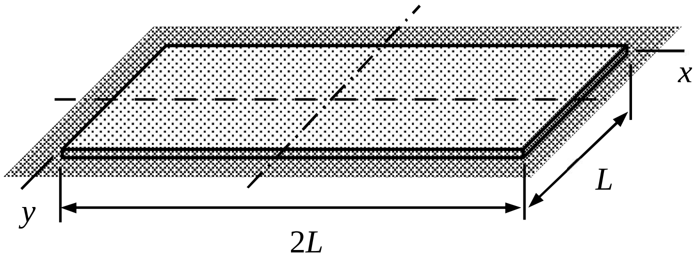
Home assignment 2

Consider free vibration of a rectangular membrane of side lengths $2L$ and L, density ρ, thickness t, and tightening S'. If the edges are fixed, find the angular velocities of the free vibrations using a regular grid $(i,j)\in\{0,1,2,3,4\}\times\{0,1,2\}$ of constant spacing and the Finite Element Method with a piecewise linear approximation on regular triangle elements. Consider the modes, that are reflection symmetric with respect to the lines through the center point (figure).



Solution

Assuming regular grid of points and regular triangle division, the generic equations for the membrane model with fixed boundaries as given by the Finite Element Method are

$$S'[w_{(i-1,j)}+w_{(i,j-1)}-4w_{(i,j)}+w_{(i+1,j)}+w_{(i,j+1)}]+h^2 f'=$$

$$m'h^2\frac{1}{12}[\ddot{w}_{(i-1,j-1)}+\ddot{w}_{(i-1,j)}+\ddot{w}_{(i,j-1)}+6\ddot{w}_{(i,j)}+\ddot{w}_{(i+1,j)}+\ddot{w}_{(i,j+1)}+\ddot{w}_{(i+1,j+1)}]\quad (i,j)\in I\quad t>0$$

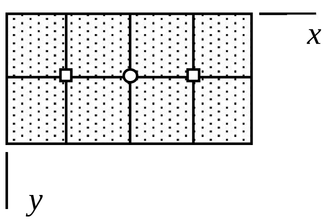
$$w_{(i,j)}=0\quad (i,j)\in\partial I\quad t>0,$$

$$w_{(i,j)}-g_{(i,j)}=0\quad\text{and}\quad \dot{w}_{(i,j)}-h_{(i,j)}=0\quad (i,j)\in I\quad t=0.$$

In modal analysis, initial conditions are not needed. As the mode is assumed to be reflection symmetric with respect to lines through the center point and aligned with the coordinate axes, transverse displacements at the grid points satisfy

$$w_{(1,1)}=w_{(3,1)}=w_1,$$

$$w_{(2,1)}=w_2,$$



the remaining displacements at the boundary points being zeros. According to the principle of virtual work, the equation for a constrained displacement is the sum of equations for the constrained points. In the present problem, it is enough to consider just one of them as the equations of points $(1,1)$ and

(3,1) coincide. Considering only the equations of points (1,1) and (2,1) with $f'=0$, $m'=\rho t$, and $h=L/2$

$$S'(-4w_1+w_2)=\rho tL^2\frac{1}{48}(6\ddot{w}_1+\ddot{w}_2),$$

$$S'(-4w_2+2w_1)=\rho tL^2\frac{1}{48}(6\ddot{w}_2+2\ddot{w}_1)$$

or using the matrix notation

$$\begin{bmatrix}4 & -1\\ -2 & 4\end{bmatrix}\begin{Bmatrix}w_1\\ w_2\end{Bmatrix}+\frac{1}{48}\frac{\rho tL^2}{S'}\begin{bmatrix}6 & 1\\ 2 & 6\end{bmatrix}\begin{Bmatrix}\ddot{w}_1\\ \ddot{w}_2\end{Bmatrix}=0.$$

Solution to the angular velocities and the corresponding modes follow with the trial solution $\mathbf{a}=\mathbf{A}e^{\mathrm{i}\omega t}$:

$$(\begin{bmatrix}4 & -1\\ -2 & 4\end{bmatrix}-\lambda\begin{bmatrix}6 & 1\\ 2 & 6\end{bmatrix})\begin{Bmatrix}A_1\\ A_2\end{Bmatrix}=0 \text{ where } \lambda=\omega^2\frac{1}{48}\frac{\rho tL^2}{S'} \quad\Leftrightarrow\quad \omega=\frac{4}{L}\sqrt{3\lambda\frac{S'}{\rho t}}.$$

A homogeneous linear equation system can yield a non-zero solution to the mode only if the matrix is singular, i.e., its determinant vanishes. The condition can be used to find the possible values of λ

$$\det\begin{bmatrix}4-6\lambda & -1-\lambda\\ -2-2\lambda & 4-6\lambda\end{bmatrix}=(4-6\lambda)^2-2(1+\lambda)^2=14-52\lambda+34\lambda^2=0 \quad\Rightarrow$$

$$\lambda_1=\frac{1}{17}(13-5\sqrt{2}) \text{ or } \lambda_2=\frac{1}{17}(13+5\sqrt{2}).$$

Knowing the possible values of parameter λ, the angular velocities follow from the relationship between λ and ω:

$$\lambda_1=\frac{1}{17}(13-5\sqrt{2}):\ \omega_1=\frac{4}{L}\sqrt{\frac{3}{17}(13-5\sqrt{2})\frac{S'}{\rho t}}, \leftarrow$$

$$\lambda_2=\frac{1}{17}(13+5\sqrt{2}):\ \omega_2=\frac{4}{L}\sqrt{\frac{3}{17}(13+5\sqrt{2})\frac{S'}{\rho t}}. \leftarrow$$

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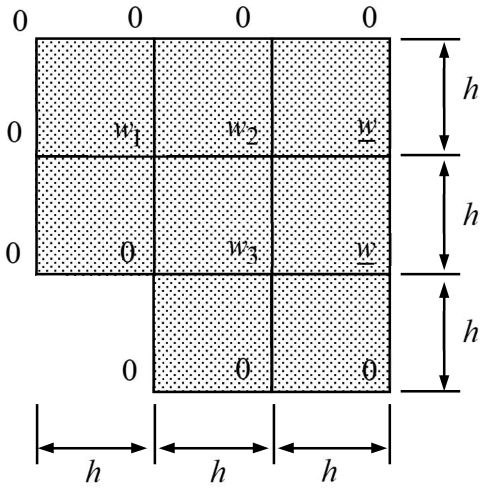
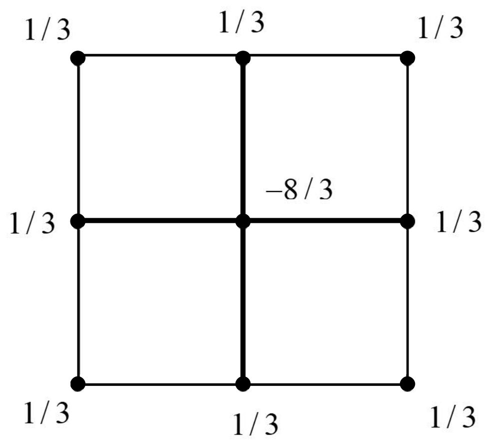
Home assignment 3

Consider the membrane of polygonal shape of the figure for which density ρ, thickness t, tightening S' (force per unit length) are constants. Using the given displacement values at the grid points on the boundary, determine the unknown transverse displacement values w_1, w_2 and w_3 using the given Laplacian stencil for the interior points given by the Finite Element Method and bi-linear approximation on rectangle elements.





Solution

The equations for the interior points can be obtained by centering the stencil at the grip one at a time and using the multipliers as the weighting of the function values. The outcomes for the interior points 1, 2, and 3 are (notice that the values at the boundary points are known)

$$\frac{S'}{3}(-8w_1 + w_2 + w_3) = 0\,, \quad \frac{S'}{3}(-8w_2 + w_1 + w_3 + 2\underline{w}) = 0\,, \text{ and } \frac{S'}{3}(-8w_3 + w_1 + w_2 + 2\underline{w}) = 0$$

Using the matrix notation

$$\begin{bmatrix} 8 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & 8 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} - 2\underline{w} \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} = 0\,.$$

Then, using row operations to get an equivalent upper triangular matrix representation

$$\begin{bmatrix} 8 & -1 & -1 \\ -8 & 64 & -8 \\ -8 & -8 & 64 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} - 2\underline{w} \begin{Bmatrix} 0 \\ 8 \\ 8 \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{bmatrix} 8 & -1 & -1 \\ 0 & 63 & -9 \\ 0 & -9 & 63 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} - 2\underline{w} \begin{Bmatrix} 0 \\ 8 \\ 8 \end{Bmatrix} = 0 \quad \Leftrightarrow$$

$$\begin{bmatrix} 8 & -1 & -1 \\ 0 & 63 & -9 \\ 0 & -63 & 441 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} - 2\underline{w} \begin{Bmatrix} 0 \\ 8 \\ 56 \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{bmatrix} 8 & -1 & -1 \\ 0 & 63 & -9 \\ 0 & 0 & 432 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} - 2\underline{w} \begin{Bmatrix} 0 \\ 8 \\ 64 \end{Bmatrix} = 0 .$$

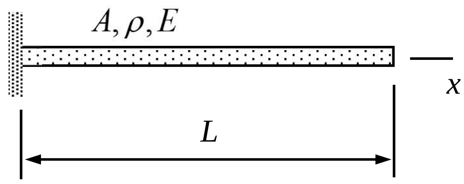
After that, equations can be solved one-by-one for the displacement starting from the last one

$$w_3 = \frac{8}{27}\underline{w}, \quad w_2 = \frac{8}{27}\underline{w}, \text{ and } w_1 = \frac{2}{27}\underline{w}.$$

COE-C3005 Finite Element and Finite difference methods, Remote exam 04.06.2021

Problem 1

Determine the angular velocities of the free vibrations of the bar shown by using the Finite Element Method on a regular grid with $i \in \{0,1,2\}$. Cross-sectional area A, density ρ of the material, and Young's modulus E of the material are constants.



Solution

The difference equations to the bar and string problems according to the Finite Element Method are given by (omitting the initial conditions as they are needed in modal analysis)

$$\frac{k}{\Delta x}(a_{i-1} - 2a_i + a_{i+1}) + F_i + f'\Delta x = m'\frac{\Delta x}{6}(\ddot{a}_{i-1} + 4\ddot{a}_i + \ddot{a}_{i+1}) \qquad i \in \{1,2,\ldots,n-1\}$$

$$\frac{k}{\Delta x}(a_1 - a_0) + F_0 + f'\frac{\Delta x}{2} - m'\frac{\Delta x}{6}(2\ddot{a}_0 + \ddot{a}_1) = 0 \quad \text{or} \quad a_0 = \underline{a}_0,$$

$$\frac{k}{\Delta x}(a_{n-1} - a_n) + F_n + f'\frac{\Delta x}{2} - m'\frac{\Delta x}{6}(2\ddot{a}_n + \ddot{a}_{n-1}) = 0 \quad \text{or} \quad a_n = \underline{a}_n.$$

Where in this case of a bar problem $k = EA$, $m' = \rho A$, $\Delta x = L/2$, $a = u$ and external forces vanish. Equations for the three grid points are

$$u_0 = 0,$$

$$2\frac{AE}{L}(u_0 - 2u_1 + u_2) - \rho A\frac{L}{12}(\ddot{u}_0 + 4\ddot{u}_1 + \ddot{u}_2) = 0,$$

$$2\frac{EA}{L}(u_1 - u_2) - \rho A\frac{L}{12}(2\ddot{u}_2 + \ddot{u}_1) = 0.$$

After using the first algebraic equation to eliminate u_0 from the ordinary differential equations, the matrix representation, required by the modal analysis, takes the form

$$2\frac{EA}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \rho A\frac{L}{12}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} = 0 .$$

Modal analysis uses the trial solution $\mathbf{u} = \mathbf{A}e^{\mathrm{i}\omega t}$ in which $\mathbf{A}$ represent mode and ω the corresponding angular velocity. Substitution into the set of differential equations gives a set of algebraic equations for the possible combinations $(\omega, \mathbf{A})$:

$$(\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \lambda\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix})\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0 \quad \text{where} \quad \lambda = \frac{1}{24}\frac{\rho L^2}{E}\omega^2 \quad \Leftrightarrow \quad \omega = \frac{2}{L}\sqrt{6\lambda\frac{E}{\rho}} .$$

First, the possible λ values:

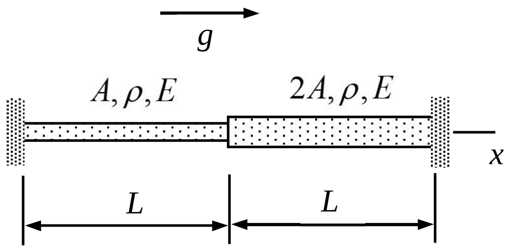
$$\det(\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \lambda\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}) = 2(1-2\lambda)^2 - (1+\lambda)^2 = 0 \quad \Rightarrow \quad \lambda = \frac{1}{7}(5 \pm 3\sqrt{2}) .$$

Therefore, the angular velocities of the free vibrations

$$\omega = \frac{2}{L}\sqrt{\frac{6}{7}(5 \pm 3\sqrt{2})\frac{E}{\rho}} . \quad \leftarrow$$

Problem 2

The stationary boundary value problem for the bar shown (jump in the cross-sectional area at $x = L$) consists of the equilibrium equations for the regular interior points, jump condition at $x = L$, and displacement boundary conditions for the end points. Write the equation system $-\mathbf{Ku} + \mathbf{F} = 0$ according to the Finite Difference Method on a regular grid with $i \in \{0,1,2,3,4\}$ and solve for the displacements. Use the proper forward/backward difference approximation to the first derivative in the jump condition. Young's modulus E and density ρ of the material are constants.



Solution

The zero displacement boundary conditions for points 0 and 4 are obvious, The difference equations for the regular interior points 2 and 3, follow when the second order derivative in the stationary differential equation is replaced by central difference approximation and the distributed load due to gravity is evaluated at those points. In the jump condition for the cemterpoint, the first derivatives are replaced by backward and forward approximations:

$$u_0 = 0 \,,$$

$$\frac{EA}{\Delta x^2}(u_0 - 2u_1 + u_2) + \rho g A = 0 \,,$$

$$\frac{E2A}{\Delta x}(u_3 - u_2) - \frac{EA}{\Delta x}(u_2 - u_1) = 0 \,,$$

$$\frac{E2A}{\Delta x^2}(u_2 - 2u_3 + u_4) + \rho g 2A = 0 \,,$$

$$u_4 = 0 \,.$$

Next, using the boundary conditions to eliminate u_0 and u_4, the remaining equations can be written in the matrix form

$$-\frac{EA}{L^2}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 4 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} + \rho A g \begin{Bmatrix} 1 \\ 0 \\ 2 \end{Bmatrix} = 0 \,.$$

Solution can be obtained by (Gauss) elimination (or using Mathematica). First, row operations to get an equivalent upper diagonal form:

$$\begin{bmatrix} 2 & -1 & 0 \\ -2 & 6 & -4 \\ 0 & -2 & 4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{\rho g L^2}{E} \begin{Bmatrix} 1 \\ 0 \\ 2 \end{Bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 2 & -1 & 0 \\ 0 & 5 & -4 \\ 0 & -2 & 4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{\rho g L^2}{E} \begin{Bmatrix} 1 \\ 1 \\ 2 \end{Bmatrix},$$

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & 5 & -4 \\ 0 & -2 & 4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{\rho g L^2}{E} \begin{Bmatrix} 1 \\ 1 \\ 2 \end{Bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 2 & -1 & 0 \\ 0 & 5 & -4 \\ 0 & 3 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{\rho g L^2}{E} \begin{Bmatrix} 1 \\ 1 \\ 3 \end{Bmatrix}.$$

Then, considering the equations one-by-one starting from the last one

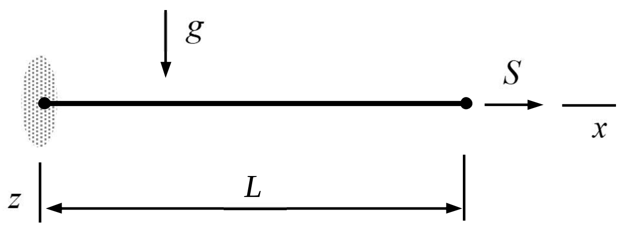
$$u_2 = \frac{\rho g L^2}{E} \quad \Rightarrow \quad u_3 = -\frac{1}{4}\left(\frac{\rho g L^2}{E} - 5u_2\right) = \frac{\rho g L^2}{E} \quad \Rightarrow \quad u_1 = \frac{1}{2}\left(\frac{\rho g L^2}{E} + u_2\right) = \frac{\rho g L^2}{E}.$$

Altogether

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{\rho g L^2}{E} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}.$$

Problem 3

A string of length L, density ρ, cross-sectional area A, and tightening S is loaded by its own weight as shown. If the left end is fixed, the right end is free, and the initial geometry without loading is straight, find the stationary transverse displacement according to the Finite Element Method with regular grid $i \in \{0,1,\ldots,n\}$. What is the limit solution when $n \to \infty$?



Solution

In the stationary case all time derivatives vanish and initial conditions are not needed. The generic equation set for the string and bar models is given the Finite Element Method on a regular grid simplifies to

$$\frac{S}{\Delta x}(w_{i-1} - 2w_i + w_{i+1}) + \rho A g \Delta x = 0 \quad i \in \{1,2,\ldots,n-1\},$$

$$w_0 = 0 \quad \text{and} \quad \frac{S}{\Delta x}(w_{n-1} - w_n) + \frac{\Delta x}{2}\rho g A = 0,$$

where $\Delta x = L/n$. Let us find first the generic solution to the difference equation for the interior points using the same approach as with differential equations. The generic solution is composed of the generic solution to the homogeneous equation of the form $w_h = Ar^i$ to the homogeneous equation (loading omitted) and particular solution $w_p = Ci^2$. Substituting the trial solution w_h

$$Ar^{i-1} - 2Ar^i + Ar^{i+1} = (1 - 2r + r^2)Ar^{i-1} = (1-r)^2 Ar^{i-1} = 0.$$

Due to the double root $r = 1$, one obtains $w_h = A + Bi$. The particular solution follows with the solution trial $w_p = Ci^2$. Substitution into the (full) equilibrium equation gives

$$\frac{S}{\Delta x}[C(i-1)^2 - 2Ci^2 + C(i+1)^2] + \rho A g \Delta x = 0 \quad \Rightarrow \quad C = -\frac{\rho A g}{2S}\Delta x^2.$$

The generic solution to the difference equation for the interior points

$$w_i = w_h + w_p = A + Bi - \frac{\rho A g \Delta x^2}{2S} i^2$$

contains parameters A and B to be determined from the equations for the boundary points.

$$w_0 = A = 0 \quad \text{and} \quad B = \frac{\rho A g \Delta x^2}{S} n .$$

Therefore, finally

$$w_i = \frac{\rho A g \Delta x^2}{2S} i(2n - i) . \quad \leftarrow$$

The limit solution can be obtained by writing the solution first into the form with notation $x_i = \Delta x i$

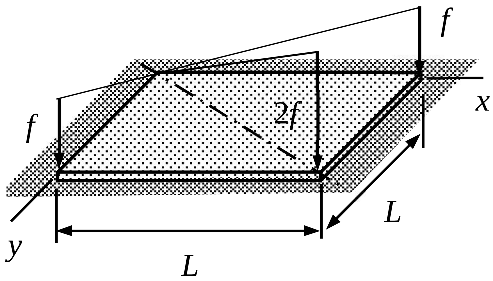
$$w_i = \frac{\rho A g}{2S} x_i (2L - x_i)$$

and considering $n \to \infty$ meaning that $\Delta x = L / n \to 0$. Then labelling the points with indices is not possible and material coordinate x needs to be used for particle identification:

$$w(x) = \frac{\rho A g}{2S} x(2L - x) . \quad \leftarrow$$

Problem 4

A rectangular membrane of side length L and tightening S' is loaded by a non-constant distributed force $f'(x,y)=(x+y)f/L$ in which f is constant. If the edges are fixed, find the transverse displacement using the Finite Difference Method on a regular grid $(i,j)\in\{0,1,2,3\}\times\{0,1,2,3\}$ and reflection symmetry with respect to the line shown in figure.



Solution

The generic equations for the membrane model with fixed boundaries, as given by the Finite Difference Method on a regular grid, are

$$\frac{S'}{h^2}[w_{(i-1,j)}+w_{(i,j-1)}-4w_{(i,j)}+w_{(i+1,j)}+w_{(i,j+1)}]+f'=m'\ddot{w}_{(i,j)} \quad (i,j)\in I\,,$$

$$w_{(i,j)}=0 \quad (i,j)\in\partial I\,,$$

$$w_{(i,j)}-g_{(i,j)}=0 \quad \text{and} \quad \dot{w}_{(i,j)}-h_{(i,j)}=0 \qquad (i,j)\in I\,.$$

In the present problem, time derivatives vanish, initial conditions are not needed, and solution is reflection symmetric with respect to line shown in the figure. Therefore, transverse displacements at the interior grid points satisfy

$$w_{(1,1)}=w_1\,,\; w_{(1,2)}=w_{(2,1)}=w_2\,, \text{ and } \; w_{(2,2)}=w_3\,.$$

Let us write the equilibrium equations for the interior points one-by-one with the displacement constraints. In the Finite Difference Method, external distributed force is evaluated at the grid points. Here $h=L/3$:

$$9\frac{S'}{L^2}[w_{(0,1)}+w_{(1,0)}-4w_{(1,1)}+w_{(2,1)}+w_{(1,2)}]+\frac{2}{3}f=0 \quad\Rightarrow\quad 9\frac{S'}{L^2}(-4w_1+2w_2)+\frac{2}{3}f=0\,,$$

$$9\frac{S'}{L^2}[w_{(0,2)}+w_{(1,1)}-4w_{(1,2)}+w_{(2,2)}+w_{(1,3)}]+f=0 \quad\Rightarrow\quad 9\frac{S'}{L^2}(w_1-4w_2+w_3)+f=0\,,$$

$$9\frac{S'}{L^2}[w_{(1,1)}+w_{(2,0)}-4w_{(2,1)}+w_{(3,1)}+w_{(2,2)}]+f=0 \quad\Rightarrow\quad 9\frac{S'}{L^2}(w_1-4w_2+w_3)+f=0\,,$$

$$9\frac{S'}{L^2}[w_{(1,2)}+w_{(2,1)}-4w_{(2,2)}+w_{(3,2)}+w_{(2,3)}]+\frac{4}{3}f=0 \quad \Rightarrow \quad 9\frac{S'}{L^2}(2w_2-4w_3)+\frac{4}{3}f=0.$$

As the equations by the Finite Difference Method for the symmetry points $(1,2)$ and $(2,1)$ do not differ, it is enough to consider equations for $(1,1)$, $(1,2)$, and $(2,2)$, say. Using the matrix representation

$$-9\frac{S'}{L^2}\begin{bmatrix} 4 & -2 & 0 \\ -1 & 4 & -1 \\ 0 & -2 & 4 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}+\frac{f}{3}\begin{Bmatrix} 2 \\ 3 \\ 4 \end{Bmatrix}=0.$$

The solution can be obtained by Mathematica or (Gauss) elimination. First row operation to get an upper triangular matrix:

$$\begin{bmatrix} 4 & -2 & 0 \\ -4 & 16 & -4 \\ 0 & -2 & 4 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}-\frac{fL^2}{27S'}\begin{Bmatrix} 2 \\ 12 \\ 4 \end{Bmatrix}=0 \quad \Rightarrow \quad \begin{bmatrix} 4 & -2 & 0 \\ 0 & 14 & -4 \\ 0 & -2 & 4 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}-\frac{fL^2}{27S'}\begin{Bmatrix} 2 \\ 14 \\ 4 \end{Bmatrix}=0,$$

$$\begin{bmatrix} 4 & -2 & 0 \\ 0 & 14 & -4 \\ 0 & -14 & 28 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}-\frac{fL^2}{27S'}\begin{Bmatrix} 2 \\ 14 \\ 28 \end{Bmatrix}=0 \quad \Rightarrow \quad \begin{bmatrix} 4 & -2 & 0 \\ 0 & 14 & -4 \\ 0 & 0 & 24 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}-\frac{fL^2}{27S'}\begin{Bmatrix} 2 \\ 14 \\ 42 \end{Bmatrix}=0.$$

Then using the equations in reverse order (starting from the last one)

$$w_3=\frac{7}{108}\frac{fL^2}{S'} \quad \Rightarrow \quad w_2=\frac{fL^2}{27S'}+\frac{4}{14}w_3=\frac{6}{108}\frac{fL^2}{S'} \quad \Rightarrow \quad w_1=\frac{1}{2}(\frac{fL^2}{27S'}+w_2)=\frac{5}{108}\frac{fL^2}{S'}.$$

Altogether

$$\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}=\frac{fL^2}{S'}\frac{1}{108}\begin{Bmatrix} 5 \\ 6 \\ 7 \end{Bmatrix}.$$