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Geometrical effects in fatigue: a unifying theoretical model

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Abstract

This paper presents a theory of fatigue behaviour in materials which encompasses two areas of the subject—the behaviour of cracks and the behaviour of notches—and which also accounts for size effects in these two types of geometrical feature. The basis of this theory is well established, and has been used by others to explain notch behaviour; it involves consideration of the elastic stresses over a critical distance ahead of the feature. In this paper it is shown that, by applying the same approach to cracked bodies, one can find the critical distance for a given material, avoiding the use of an empirically-derived value. This approach correctly predicts the fatigue limits of bodies containing both cracks and notches and also predicts the experimentally-observed effects of feature size on fatigue limit. It may be extended to consider differences between tensile and bending test results and the differing effects of surface treatments. The implication of this work is that there is no fundamental difference between the fatigue limit of an uncracked body—for which crack initiation is necessary—and that of a body which already contains a crack. The results are of practical value because the approach can easily be extended to cover geometrical features of any shape and size, such as occur in engineering components. © 1999 Elsevier Science Ltd. All rights reserved.

Keywords: Linear elastic fracture mechanics; Stress concentrations; Fatigue limit

1. Introduction

This paper is concerned with the prediction of the fatigue limit, or high-cycle endurance limit, $\Delta\sigma_o$. This parameter, when measured from plain specimens in tension, is of limited practical use because most engineering components and structures fail from points of stress concentration: geometric features which cause local increases in stress. In what follows the aim is to develop a theory which will be capable of predicting the fatigue limit of bodies containing such features, irrespective of shape and size. For convenience we will begin with two particular shapes: a blunt notch, represented in this case by a circular hole of radius a_n, and a crack of length $2a_c$ (Fig. 1). Also for simplicity's sake we will define the problem in two dimensions, thus the crack and hole are assumed to penetrate the thickness of a uniform plate whose dimensions are much larger than the feature size (a_n or a_c).

If a body contains a large blunt notch, then its nominal fatigue limit, $\Delta\sigma_{on}$ may be reduced by as much as K_t, the elastic stress concentration factor; thus:

$$\Delta\sigma_{on}=\Delta\sigma_o/K_t \tag{1}$$

and:

$$\Delta\sigma_{\max}=\Delta\sigma_o \tag{2}$$

where $\Delta\sigma_{\max}$ is the stress at the root of the notch, often called the 'hot-spot stress'. However, in many cases the notched fatigue limit is higher than this value, implying that stresses above $\Delta\sigma_o$ can exist at the notch root without causing failure. This effect is related both to material type and feature size. As a_n decreases, $\Delta\sigma_{on}$ increases, even for constant K_t, and must approach $\Delta\sigma_o$ as a_n approaches zero. This gives rise to complex behaviour for small features (see Fig. 2).

A number of workers have attempted to predict this behaviour by assuming that the apparent increase in strength for small notches is due to the fact that the stress concentration occurs over only a small volume of material. This work has been reviewed recently by Sheppard [1] and by Taylor and O'Donnell [2]. The underlying assumption is that, in order for failure to occur, the average stress must exceed $\Delta\sigma_o$ over some critical vol-

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Nomenclature

a_c	crack length (half length of centre-crack)
a_n	notch depth (and radius of circular hole)
a_o	ElHaddad constant
K_f	fatigue strength reduction factor
K_t	elastic stress concentration factor
ΔK	stress intensity factor range for a crack
ΔK_{th}	threshold value of ΔK for fatigue crack growth
r	distance measured from crack tip (or notch root)
R	distance measured from centre of hole
β	normalised stress gradient
ρ	notch root radius
$\Delta\sigma$	cyclic stress range (peak-to-peak)
$\Delta\sigma(r)$	$\Delta\sigma$ as a function of distance, r.
$\Delta\sigma_{\max}$	cyclic stress at notch root ('hot-spot' stress)
$\Delta\sigma_o$	plain-specimen fatigue limit
$\Delta\sigma_{oc}$	fatigue limit for a body containing a crack (nominal stress)
$\Delta\sigma_{on}$	fatigue limit for a body containing a notch (nominal stress)
$\Delta\sigma\|_{(r=a_o/2)}$	$\Delta\sigma$ at a point $r=a_o/2$
$\Delta\sigma_{av}\|_{(r=0-2a_o)}$	$\Delta\sigma$ averaged along a line from r=0 to $2a_o$
$\Delta\sigma_{av}\|_{(Area,r=a_o)}$	$\Delta\sigma$ averaged over a semi-circular area of radius $r=a_o$
θ	angle at which distance r is measured
Θ	angle at which distance R is measured

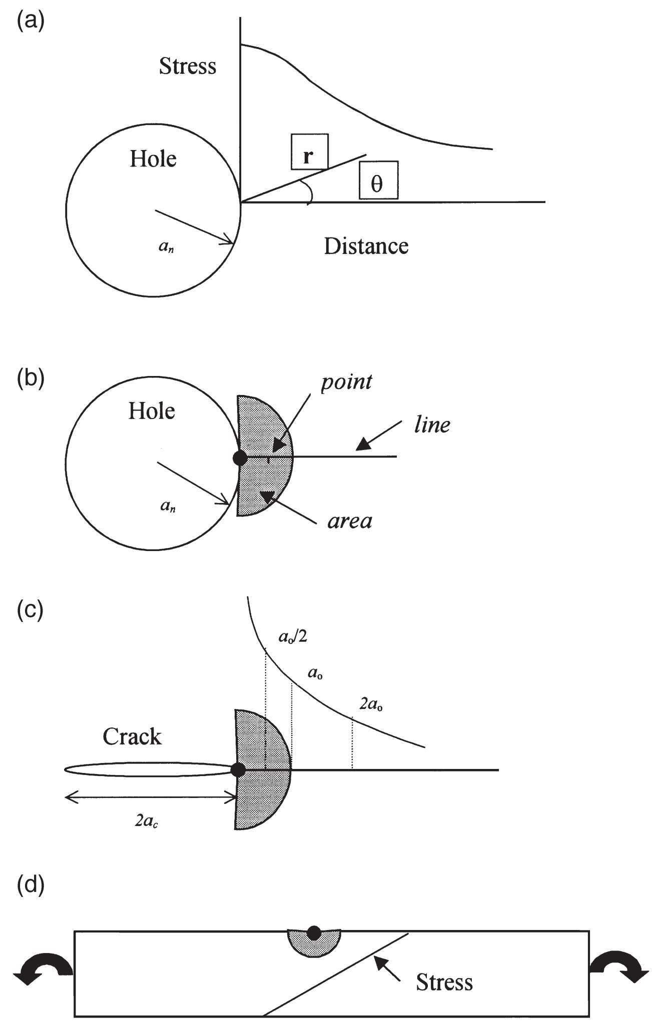

Fig. 1. Definition of geometry: (a) circular hole showing local stress field at θ=0 due to applied uniform tension in a vertical direction; (b) hole showing definition of *point*, *line* and *area*; (c) crack showing stress field and distance values; (d) specimen loaded in bend, showing stress variation and chosen area.

ume surrounding the hot-spot. For simplicity of calculation, this approach is often reduced to a consideration of the stress at a single point, or averaged over a given distance or area, as illustrated in Fig. 1. These various approximations will be referred to as *point*, *line*, *area* and *volume* methods, noting that in a 2D problem the *area* method is equivalent to the *volume* method.

Peterson [3] derived a relationship to predict the effect of notch size, based on a *point* method, simplified further by assuming a linear variation of stress with distance, r, from the hot-spot. Defining K_f as the experimentally-measured reduction in strength (thus $K_f = \Delta\sigma_o/\Delta\sigma_{on}$), he obtained:

$$K_f=1+(K_t-1)(1+\alpha/\rho)^{-1} \tag{3}$$

In this equation ρ is the root radius of the notch and α is a material constant—the critical distance—whose value can only be obtained experimentally. It was found that this method gave good approximations for various types of notches, and that the value of α varied greatly from material to material. Similar approaches, resulting in slightly different equations, were used by Neuber [4], and Siebel and Stieler [5]. For example, Neuber's equation takes the form:

$$K_f=1+(K_t-1)[(1+\rho'/\rho)^{0.5}]^{-1} \tag{4}$$

Peterson's equation has been tested against many different types of steel, and an empirical relationship has been found between α and the material's tensile strength [6], which is commonly used by industrial designers. Recently Sheppard [1] has obtained similar predictions

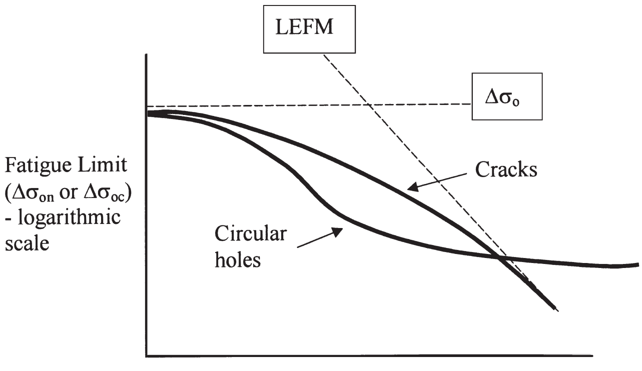

Fig. 2. Schematic showing the effect of feature size on fatigue limit. Large cracks conform to LEFM predictions (Eq. (6)) and large circular holes reduce the fatigue limit by K_t=3. Small features, however, show complex behaviour with the fatigue limit tending to $\Delta\sigma_o$ as feature size tends to zero.

by analysing the notch stress field in detail using finite element analysis.

Traditionally, a different approach has been used for analysing bodies containing cracks; these are treated using the methods of linear elastic fracture mechanics (LEFM). For a straight, through-crack in an infinite plate under uniform tension:

$$\Delta K=\Delta\sigma(\pi a_c)^{1/2} \tag{5}$$

Crack growth ceases below the threshold, ΔKth, so the fatigue limit of a cracked specimen, $\Delta\sigma_{oc}$, is:

$$\Delta\sigma_{oc}=\Delta K_{th}/(\pi a_c)^{1/2} \tag{6}$$

The fatigue limit behaviour of cracks also displays a size effect (Fig. 2) which has been extensively studied, in a manner quite different from the study of notch size effects. For example, ElHaddad et al [7] proposed an empirical equation which is a modification of Eq. (6), introducing a material constant a_o, thus:

$$\Delta\sigma_{oc}=\Delta K_{th}/(\pi[a_c+a_o])^{1/2} \tag{7}$$

and

$$a_o=(1/\pi).(\Delta K_{th}/\Delta\sigma_o)^2 \tag{8}$$

There has been considerable debate about the use of these equations and about the physical significance of a_o. However, Taylor and O'Donnell [2] showed that Eq. (7) provided a good approximation to the available experimental data on short cracks.

Developments over the last few decades have proceeded on parallel tracks, there being little overlap between the study of notches and the study of cracks, with two notable exceptions: Smith and Miller [8] and Pluvinage and co-workers [9,10]. Smith and Miller showed that, under certain circumstances, a sharp notch could be analysed as if it were a crack. These workers studied large features for which either Eq. (1) or Eq. (6) applied, and showed that a critical K_t value existed above which Eq. (6) could be used to predict notch behaviour, simply replacing a_c by a_n. The critical value of K_t depended both on material type and on a_n. This implies that certain notches are crack-like as regards their fatigue limits. Pluvinage and co-workers developed a 'notch stress intensity factor' which is a modified value of K which they obtained from examination of the stress field surrounding a notch. They also introduced a critical-distance concept [9] which will be discussed below. Neither of these researchers considered small notches or cracks.

The state of the art as regards the prediction of fatigue limit can be summarised as follows. Firstly, for large features there are two approaches, depending on whether the feature is a crack or a notch, with certain sharp notches being reclassified as cracks. Secondly, and probably more importantly, size effects modify the fatigue limits of both types of features; these size effects are considerable, they vary greatly depending on the type of material involved and currently they can only be predicted using empirical laws. However, from the above description of these laws it is clear that they have a common feature: both Peterson's equation (Eq. (3)) and ElHaddad's (Eq. (7)) rely on a material constant, with units of length: α and a_o respectively. In practice these constants have values which are the same order of magnitude. In what follows it will be shown that the behaviour of cracks can also be predicted using a critical-distance model, and that the same model accounts for size effects in both cracks and notches.

2. The theoretical model

Consider a through-crack in an infinite plate loaded by a uniform tensile stress range $\Delta\sigma$, applied normal to the crack. The elastic stress range (in the loading direction) as a function of distance from the crack tip [11], $\Delta\sigma(r)$ is:

$$\Delta\sigma(r)=\Delta\sigma/[1-(a_c/(a_c+r))^2]^{1/2} \tag{9}$$

This equation can be simplified when $r\ll a_c$ to give a result which is useful for large cracks:

$$\Delta\sigma(r)=\Delta\sigma(a_c/2r)^{1/2} \tag{10}$$

Here r represents distance measured in the crack plane, horizontally in Fig. 1(c), but Eq. (10) can be extended to include stress values at any angle, θ (Fig. 1):

$$\Delta\sigma(r)= \Delta\sigma(a_c/2r)^{1/2}.\cos(\theta/2).\{1+\sin(\theta/2).\sin(3\theta/2)\} \tag{11}$$

Combining the above equations, three interesting and novel identities can be obtained regarding the fatigue limit of long cracks. Firstly, using Eqs. (8) and (10), we find that, when $\Delta K=\Delta K_{th}$ the stress range at $r=a_o/2$ is equal to $\Delta\sigma_o$:

$$\Delta\sigma|_{(r=a_o/2)}=\Delta\sigma_o \tag{12}$$

This is equivalent to the *point* method mentioned above, and implies that the critical distance approach can be used for cracks in the same way as for notches. The criterion for crack propagation, and therefore for the fatigue limit, is that the local stress at a distance $r=a_o/2$ must exceed $\Delta\sigma_o$—the plain-specimen fatigue limit. Secondly, it can be shown that the average stress range, $\Delta\sigma_{av}$, evaluated over the distance from the crack tip to $r=2a_o$ (at $\theta=0$) is also equal to the plain fatigue limit when $\Delta K=\Delta K_{th}$. Using Eqs. (8) and (10):

$$\Delta\sigma_{av}|_{(r=0-2a_o)}=\frac{1}{2a_o}\int_0^{2a_o}\Delta\sigma\sqrt{\frac{a_c}{2r}}dr=\Delta\sigma_o \tag{13}$$

This is equivalent to the *line* method mentioned above. Thirdly, there is a similar identity related to the *area* method. If the average stress is evaluated over a semicircular area ahead of the crack tip, of radius a_o, as shown in Fig. 1, then the threshold for crack propagation is characterised by an average stress which is slightly larger than $\Delta\sigma_o$, thus using Eqs. (8) and (11):

$$\Delta\sigma_{av}|_{(Area,r=a_o)}$$

$$=\frac{4}{\pi a_o^2}\int_0^{\pi/2}\int_0^{a_o}\Delta\sigma\sqrt{\frac{a_c}{2r}}\left(\cos\frac{\theta}{2}\right)\left(1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)rdrd\theta$$

$$=1.1\Delta\sigma_o \tag{14}$$

These relationships imply that the critical-distance concept can be used in conjunction with the elastic stress distribution ahead of a crack, to predict whether or not the crack will propagate. In this case the relevant distance is simply related to a_o.

For the *point* and *line* methods the correspondence is exact: for the *area* method, using a distance of a_o, the resulting prediction of the fatigue limit based on $\Delta\sigma_o$ (omitting the factor 1.1) would be a slightly conservative estimate.

3. Application of the model to short cracks

The next step is to extend this concept to consider short cracks. In order to do this, the exact result for the stress distribution has to be used (Eq. (9) instead of Eq. (10)) to allow consideration of cases where the crack length is comparable in size to a_o. Fig. 3 shows predictions of the fatigue limit ($\Delta\sigma_{oc}$) as a function of crack length. The material constants chosen are those for a typical medium-carbon steel [12], at $R=-1$. The figure also shows the ElHaddad empirical law (Eq. (7)). Using the stress distribution of Eq. (10) results in the prediction labelled LEFM which is valid only for long cracks. Use of Eq. (9) causes a reduction in fatigue limit which is more pronounced as a_c decreases, tending to $\Delta\sigma_o$ at zero crack length. Predictions for the *point* method are achieved simply by rearranging Eq. (9) with $r=a_o/2$, thus:

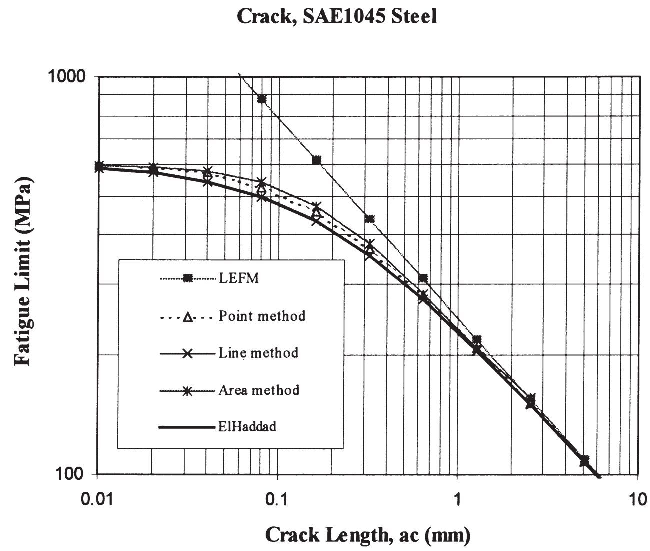

Fig. 3. Predictions of the fatigue limit as a function of crack size, a_c, using the *point*, *line*, and *area* methods. The line labelled LEFM refers to predictions using Eq. (6) which are valid only for long cracks. The ElHaddad prediction (Eq. (7)) coincides exactly with the predictions of the *line* method.

$$\Delta\sigma_{oc}=\Delta\sigma_o[1-(a_c/(a_c+a_o/2))^2]^{1/2} \tag{15}$$

The *line* method is used as follows: the average stress over the distance $0-2a_o$ using Eq. (9) is:

$$\Delta\sigma_{av}|_{(r=0-2a_o)}=\frac{1}{2a_o}\int_0^{2a_o}\Delta\sigma\frac{1}{\sqrt{1-\left(\frac{a_c}{a_c+r}\right)^2}}dr \tag{16}$$

$$=\Delta\sigma\sqrt{\frac{a_c+a_o}{a_o}}$$

The fatigue limit of the cracked body, $\Delta\sigma_{oc}$, is found by setting this average stress to $\Delta\sigma_o$, thus:

$$\Delta\sigma_{oc}=\Delta\sigma_o\sqrt{\frac{a_o}{a_c+a_o}} \tag{17}$$

By comparing this result with Eqs. (7) and (8), one can see that it is the same prediction as arises when using ElHaddad's law. This is an important result because it provides a rational explanation for the empirical law of ElHaddad and thus gives a physical significance to the length parameter a_o. Finally the *area* method can be used by finding the average stress in a manner similar to Eq. (14) but using the Westergaard form for the stress distribution. In this case the integral equation was solved numerically, as no simple analytical solution could be found. All three methods (*point*, *line* and *area*) follow the general form of the ElHaddad line, indicating that they would also give reasonable approximations to the experimental data.

4. Application of the model to notches

The same approach can be applied to the consideration of notches. The stresses in the region of a circular hole in an infinite plate with remote tensile stress σ are [13]:

$$\sigma_{\theta\theta}=(\sigma/2)[1+(a_n/R)^2+\cos 2\Theta(1+3(a_n/R)^4)] \tag{18a}$$

$$\sigma_{rr}=(\sigma/2)[1-(a_n/R)^2-cos2\Theta(1-4(a_n/R)^2+3(a_n/R)^4)] \tag{18b}$$

$$\sigma_{r\theta}=(\sigma/2)[1+2(a_n/R)^2-3(a_n/R)^4]\sin 2\Theta \tag{18c}$$

Eqs. (18a–18c) describe the tangential, radial and shear stress respectively, referred to polar coordinates (R,Θ) whose origin is at the hole centre. These can be used to find the stress in the vicinity of the hot-spot, as a function of coordinates (r,θ) with origin at the hot-spot (Fig. 1). In this way the stress field surrounding the hole can be described in the same terms as the stress field of a crack. Fig. 4 shows the results of the *point*, *line*, and *area* predictions for circular holes in the same medium carbon steel as mentioned above, using the same critical distance values. The *point* and *line* predictions are made simply using Eq. (18a) with Θ=0, giving, for the *point* method:

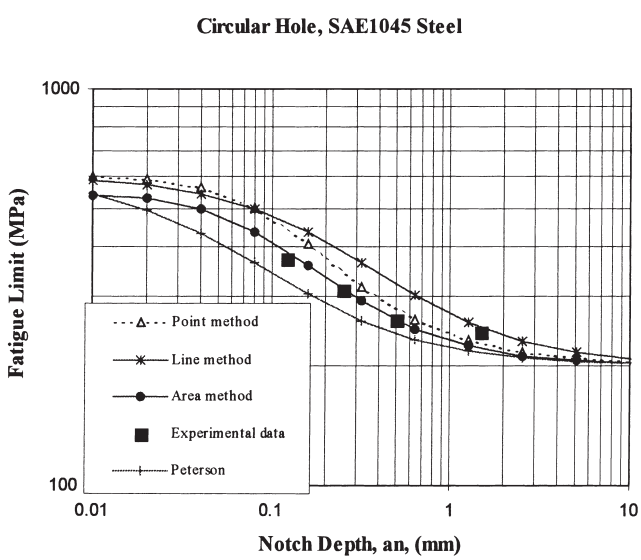

Fig. 4. Predictions of the fatigue limit of circular holes as a function of size, a_n, using the *point*, *line*, and *area* methods, compared with experimental data (DuQuesnay et al. [12]). The Peterson prediction (Eq. (3)) is also shown.

$$\Delta\sigma_{on}=2\Delta\sigma_o/[2+(a_n/(a_n+a_o/2))^2+3(a_n/(a_n+a_o/2))^4] \tag{19}$$

and for the *line* method:

$$\Delta\sigma_{on}=2a_o\Delta\sigma_o/[2a_o+a_n-0.5a_n^2/(a_n+2a_o)-0.5a_n^4/(a_n+2a_o)^3] \tag{20}$$

Predictions using the *area* method require the use of Eqs. (18a–18c) in order to find the maximum principal stress at all points in the chosen semi-circular area (Fig. 1(b)). The average stress over this area was found numerically, as no simple analytical solution could be identified.

Fig. 4 shows some experimental data [12] for four different hole diameters at R=−1 and also the Peterson prediction (Eq. (3)). All prediction lines show the same general features, tending to the plain fatigue limit as the notch size tends to zero, and tending to a value of $\Delta\sigma_o/K_t$ (K_t=3) for large notches. There is somewhat more variation between the three different methods than was found for cracks (Fig. 3) but all three give reasonable predictions of the data. The *area* method gives the best estimate of the experimental results, with an average error of 3.6% which is less that the error involved in measuring these fatigue limits experimentally.

5. Discussion

It has been shown that a single theory can predict the fatigue limits of both cracks and notches and can also account for the size effects in both of these features. This is a useful result because it brings together two areas of the subject which have traditionally been treated quite separately. The present approach also has the advantage that it removes the empiricism which was inherent in earlier treatments of the size effect. This has been done by identifying the size of the process zone, showing that it can be calculated knowing two other experimental quantities: the plain-specimen fatigue limit ($\Delta\sigma_o$) and the crack propagation threshold (ΔK_{th}). Since the threshold value is relatively difficult to measure, a simpler approach is to deduce ΔK_{th} from the fatigue limit of specimens containing sharp notches. In principle it is possible to find all three quantities ($\Delta\sigma_o$, ΔK_{th} and a_o) from fatigue limit data measured from any two different geometric features, e.g. notches of two different shapes or cracks of two different sizes.

Two previous approaches to this problem can usefully be discussed. The first is the work of Klesnil and Lucas [14] who predicted the fatigue limit for a notch by assuming that the critical stage in the process was the growth of a crack of length l_c, a length which is almost the same as a_o. This method has the advantage that it models the known physical process of fatigue: the growth of a crack. Disadvantages are the simplifying assumptions that must be made concerning the stress intensity of this crack/notch combination in the elastic/plastic stress field and, more importantly, uncertainty about the growth rate of the small crack, which will be subject to the size effect discussed above. Also it is difficult to imagine how this method could be applied to cracks. Another approach of interest is that of Boukharouba et al. [10] who extended their concept of a notch stress intensity factor to include the idea of a critical distance, Xe. Their approach is similar to the *point* method used in the present work, except that Xe is related to the elastic/plastic notch stress distribution, being an estimate of the distance over which the stress remains relatively high compared to the hot-spot stress. They define this as the region over which their notch stress intensity factor is unable to predict the stress level. Material properties are introduced because the stress distribution is modified in an elastic-plastic analysis which takes account of the material's cyclic stress-strain behaviour. Grain size is also used, though it is not clear how this is taken into account. Overall the model is similar to the present one but the definition of the critical distance is complex and liable to extrapolation errors. It requires specialised material information—the cyclic stress-strain curve—which is relatively difficult to obtain. The method does not seem to be able to predict the size effect, or the behaviour of cracks.

Four different approaches have been mentioned here: the *point*, *line*, *area* and *volume* methods. It has been shown formally that the *point*, *line* and *area* methods can give solutions which are exact for long cracks. For short cracks, the solutions differ from the ElHaddad law and from each other by less than 10%. For the case of circular holes the differences were larger, with the *area* method being the most successful, but care should be exercised here because the type of experimental data used has an uncertainty of at least 10%, so a large amount of data would have to be analysed in order to distinguish between the various methods. It has also been shown that the value of the distance to be used varies with the three methods, but these three distances are all simply related to the ratio ($\Delta K_{th}/\Delta\sigma_o$) and thus to a_o.

It is generally acknowledged by proponents of critical distance methods that it would be most appropriate to find the average stress over some volume in the vicinity of the hot-spot; thus the *point*, *line* and *area* methods are seen as simplifications of the *volume* method. The underlying assumption is that fatigue will occur, i.e. that a crack will initiate and grow within the chosen volume, if the average elastic stress is high enough. We know that the fatigue process always involves plastic deformation, so in fact the stresses which actually occur in this volume will not be correctly predicted by a purely elastic analysis. This is especially true for sharp stress concentrations such as cracks but will also apply to any location at which fatigue damage is occurring. We assume that the average elastic stress is critical because it is related to some other quantity, such as the traction or energy available in this volume, a minimum amount of which is needed in order to cause the required damage to the material. Similar assumptions have been recognised for many years in the field of linear elastic fracture mechanics, where parameters based on elastic analysis are used to predict the essentially plastic process of crack propagation. The size of the critical volume can be seen to be related to material strength, being larger in low-strength materials. This is presumably because low-strength materials tend to have coarse microstructures (e.g. a large grain size) and relatively large manufacturing defects (e.g. casting porosity). The critical distance may be related to the minimum volume which is representative of the material (e.g. a volume containing several grains, defects etc.): a crack may be able to initiate in a smaller volume but will fail to propagate because it will encounter some obstacle (e.g. a grain boundary). Thus we assume that the most correct approach is one which defines a critical volume, though in 2D problems this reduces to an area. Here a semicircular area was used, but this choice was arbitrary and should be investigated further. The reason why this volume is the same size as ElHaddad's constant, a_o, is not clear at present; it is relevant to point out that the size constants used by previous workers such as Peterson [3], Neuber [4] and Siebel and Stieler [5], all have similar magnitudes. On a more fundamental level they all appear to be reflecting the size of the representative volume discussed above.

An interesting finding of the present work is that there appears to be no fundamental difference between the behaviour of cracks, the behaviour of blunter stress-concentrations such as holes or indeed the behaviour of plain, unnotched specimens. The fatigue limit of all three categories of geometrical feature can be predicted using the same general law. Now it is recognised that fatigue failure involves two quite different processes: the initiation of a crack and its subsequent propagation through the body. But the present work suggests that, at least as far as the fatigue limit is concerned, these processes are indistinguishable, so that the presence of a pre-existing crack does not alter the mechanism by which fatigue proceeds to failure. One way to envisage this in terms of the physics of the process, is that crack propagation near the threshold level of stress intensity in fact involves a continual re-initiation of the crack. Some support for this idea is found from the appearance of fracture surfaces which are similar for crack initiation (so-called 'Stage 1' behaviour) and for long cracks growing near-threshold (so-called 'structure-sensitive' growth).

The theory can also be extended to cover a number of other problems in fatigue. For the sake of brevity, three cases will be mentioned briefly here without attempting a rigorous analysis. The first case is that of stress concentrations which are not notches. Engineering components frequently fail from corners, bends, keyways and other geometrically-complex features. Current methods (e.g. Smith and Miller [8] and Peterson [3]) are unsuitable in these cases because they rely on parameters such as notch depth and root radius which may not be well defined in component features. Taylor and co-workers [15,16] have developed a method by which the Smith and Miller approach can be adapted to such features, but this approach applied only to large features, i.e. it did not include the size effect. The present approach, on the other hand, can easily be extended to consider features of any shape for which the elastic stress field has been found, for example by finite element analysis.

The second case is a well-known problem in fatigue: the difference between fatigue limits obtained by testing specimens in uniaxial tension and in bending. It is found that the measured fatigue limit is higher in bending beams (the value of $\Delta\sigma$ being defined as the stress on the surface of the beam), and that this effect is greater in low-strength materials such as cast irons, negligible in high-strength materials. It seems likely that this effect is also explicable through the critical-distance approach: in a bending beam the average stress measured over any finite area or volume extending into the beam will be lower than the surface stress. A full analysis of this problem is not attempted here, but a simple example will serve to illustrate the use of the method. A grey cast iron material was tested [17] in simple uniaxial tension, using specimens of diameter 7 mm and in bend, using beams of square section, 10 mm × 10 mm, at the same R ratio (0.1). The fatigue limits were found to be 110 MPa in tension and 165 MPa in bend. The fatigue limit for the bend test can be predicted using the *area* method, taking a semi-circular area centred on a point on the beam's surface (see Fig. 1). If the normalised stress gradient is defined as β where:

$$\beta=\frac{1}{\sigma_{\max}}\frac{\mathrm{d}\sigma}{\mathrm{d}r} \tag{21}$$

where $\sigma_{\max}$ is the stress at the beam's surface and σ is the stress at a depth r, then the average stress range over this area is found to be:

$$\Delta\sigma_{av}|_{(Area,r=a_o)}=\frac{\Delta\sigma_{\max}}{\left(1-\frac{4}{3\pi}a_o\beta\right)} \tag{22}$$

For this material ΔK_{th} was measured [18] to be 11.2 $\mathrm{MPa(m)}^{1/2}$, giving a_o=3.3 mm, which is a particularly high value. Eq. (22) was used to predict the fatigue limit of the bend specimen: the result was 151.2 MPa, which is within 8% of the correct result. Further analysis is needed to confirm that the method is appropriate for a range of materials and specimens.

Finally, an interesting case for which this method may be useful is that of surface treatments. These treatments can alter the microstructure and strength of a surface layer, up to some depth d, which may vary from a few microns to several millimetres. Often the treatment will induce residual stresses in the surface layer; surface roughness may also be affected. One phenomenon which has been difficult to explain is the contribution of the layer depth, d. For example, fatigue-improving treatments such as shot-peening and roller burnishing have a strong beneficial effect due to compressive residual stresses, but a simple machining process can be shown to induce stresses of similar magnitude without changing $\Delta\sigma_o$. The difference is that d is much smaller for the machining process. Another problem is that the effect of surface roughness can be very significant in high strength materials, negligible in low-strength materials.

These phenomena can be explained, at least qualitatively, by reference to the present theory. If stresses are averaged over an area of radius a_o, then a residual stress will only be effective if d is similar to, or greater than, a_o. Surface roughness, which can be thought of as a series of small notches or cracks, will be insignificant if it is much less than a_o. Since a_o varies considerably from one material to another, from 3 mm in some cast irons to less than 0.01 mm in high-strength alloys, the material dependence of a given surface treatment will be considerable. Quantitative treatment of these problems is beyond the scope of this paper.

6. Conclusions

1. The fatigue limit of a body containing a crack can be predicted using a critical distance concept of the type which is already in use for the prediction of the behaviour of bodies containing notches. This analysis allows a value for the critical distance to be obtained, which is found to be related to the a_o parameter of ElHaddad et al. [7].
2. This same critical distance value can be used to predict the behaviour of notched bodies and also to predict the effect of size for both cracks and notches. The method gives results which are similar to those from current empirical laws (ElHaddad [7] and Peterson [3]) and also similar to experimental data. The difference between results from tensile and bending specimens can also be predicted and the approach may be useful in understanding the effects of surface treatments.
3. This approach removes the empiricism of previous methods, because the critical distance can be calculated from other material parameters. It can be easily extended to consider notches of any arbitrary geometry, including stress concentrations on components.

4. The method unifies the analysis of notches and cracks and thus bridges a gap between two areas of the subject which have traditionally used different approaches. Results imply that there is no fundamental difference in the mechanism of fatigue between those cases where crack initiation is required and those where a crack already exists.

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