Aalto University
School of Engineering

Kul-49.4350 Fatigue of Structures

Lecture 11: Multiaxial fatigue

Course contents

Week		Description
43	**Lecture 1-2**	**Fatigue phenomenon and fatigue design principles**
	Assignment 1	Fatigue Damage process, design principle and Rainflow counting – dl after week 43
44	**Lecture 3-4**	**Stress-based fatigue assessment**
	Assignment 2	Fatigue life estimation using stress-based approach – dl after week 44
45	**Lecture 5-6**	**Strain-based fatigue assessment**
	Assignment 3	Fatigue crack initiation life by strain-based approach – dl after week 46
46	**Lectures 7-8**	**Fracture mechanics -based assessment**
	Assignment 4	Fatigue crack propagation life by fracture mechanics – dl after week 46
47	**Lectures 9-10**	**Fatigue assessment of welded structures and residual stress effect**
	Assignment 5	Fatigue life estimation of welded joint – dl after week 48
48	**Lecture 11-12**	**Multiaxial fatigue and statistic of fatigue testing**
	Assignment 6	Fatigue life estimation for multiaxial loading and statistical analysis – dl after week 48
49	**Exam**	**Course exam**
	Project work	Delivery of final project (optional) – dl on week 50

Assignment are delivered in pdf format using MyCourse and the following naming ***AssignmentNumber_FirstName_Surname.pdf.***

Learning outcomes

After the lecture, you

- *understand* multiaxial fatigue phenomena in materials and structures
- *understand* the non-proportional loading and stressing of structures
- *can apply* the critical plane and equivalent stress methods for fatigue strength assessment

Contents

- Motivation
- Stresses and critical planes
- Non-proportional loading and stressing
- Equivalent stress methods
- General rules

Motivation

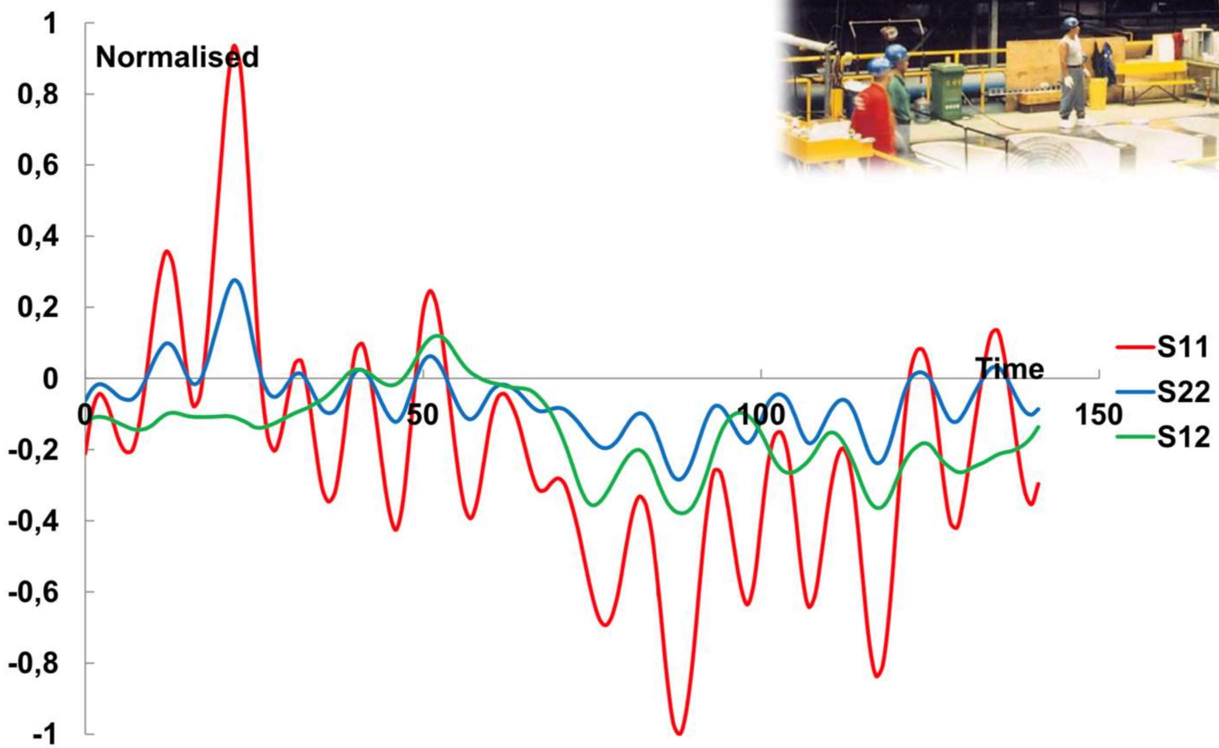
Multiaxial states of stress are very common and multiaxial strain is rarely avoided.

For instance:

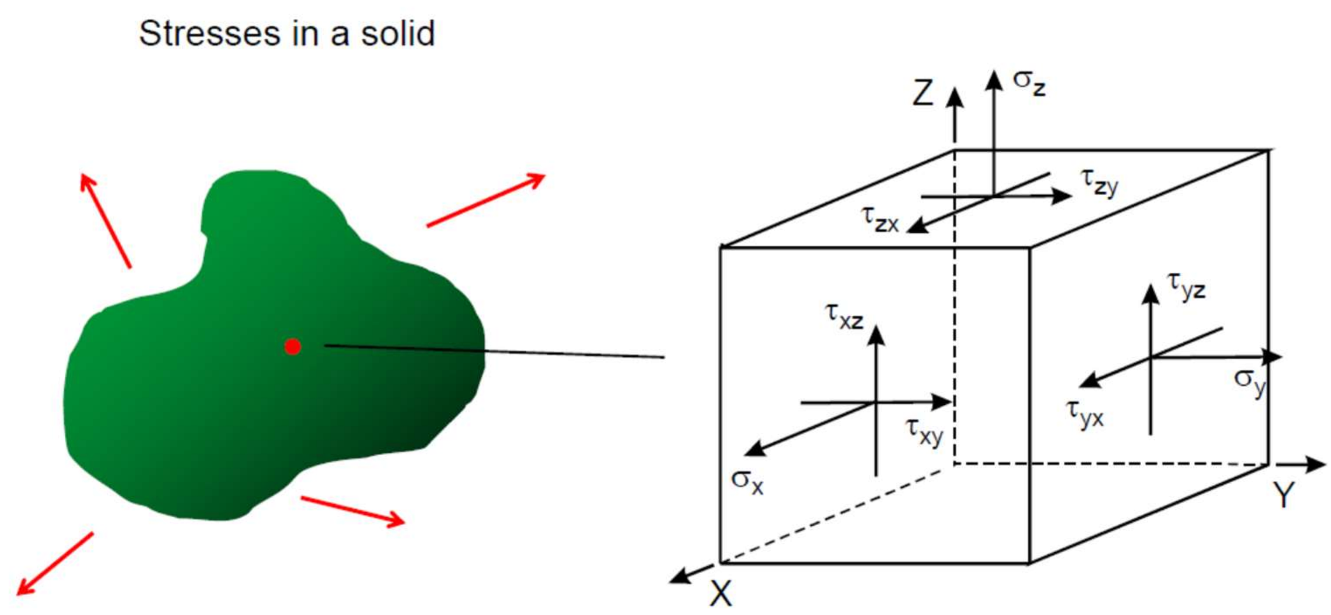
- shaft with biaxial stress state
- thin-walled pressure vessel under cyclic pressure with biaxial stress state
- tensile bar with triaxial strain state

Motivation

Normalized stress component histories of crankshaft hot-spot



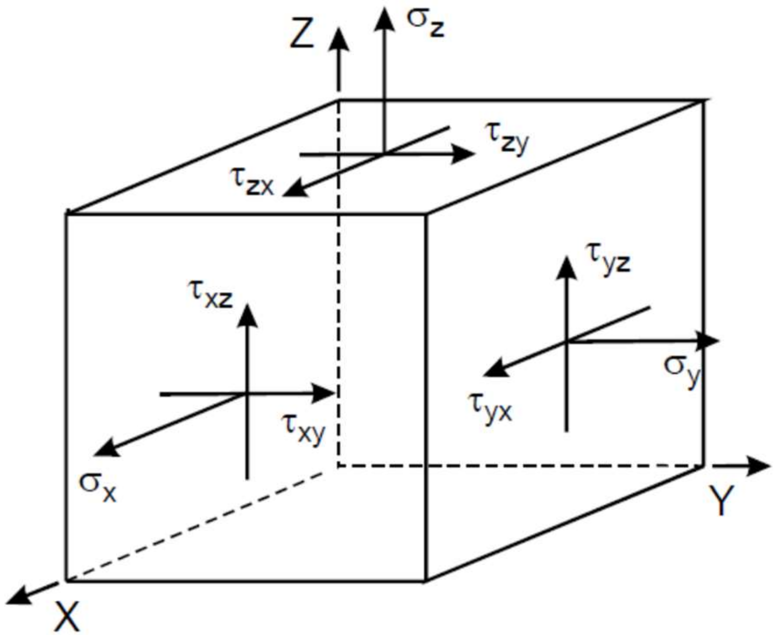
Stresses and Critical Planes



Stresses and Critical Planes

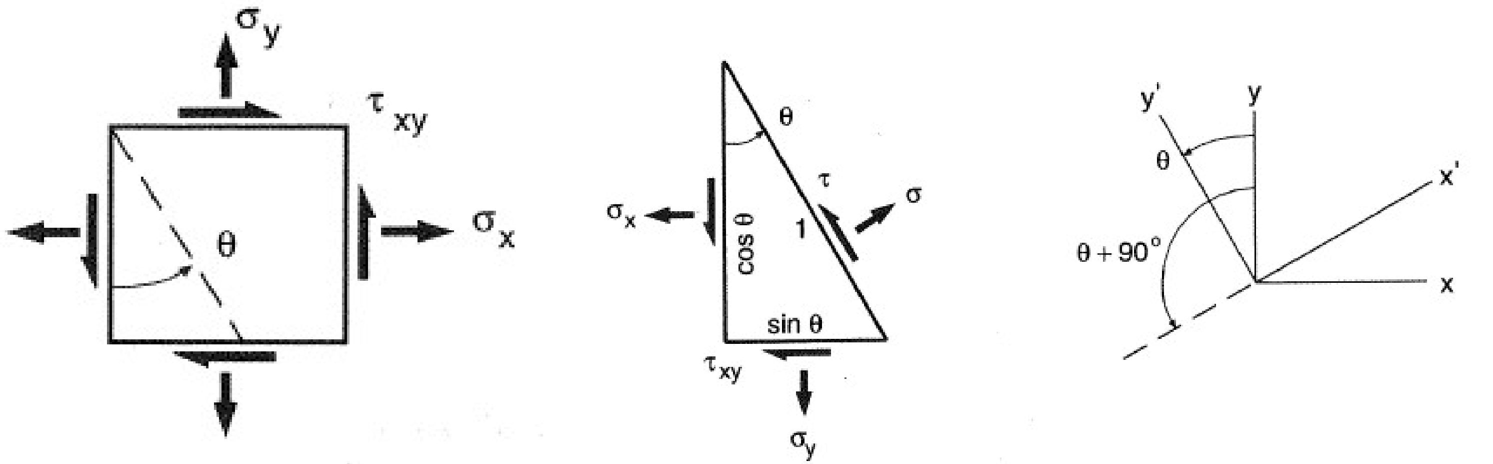
Stresses in a solid

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$



Principal stress - 2D (plane stress)

e.g. free surface of a body



Transformation equations -normal and shear stress at θ plane.

- They reflect the force balance at different orientations

$$\begin{cases} \sigma = \dfrac{\sigma_x + \sigma_y}{2} + \dfrac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \\ \tau = -\dfrac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta \end{cases}$$

NOTE: parametric equations of a circle

$$\left(\sigma - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$$

Principal stresses

2D: Principal stresses: corresponding normal stresses σ at an angle θ_n at which the shear stress τ is zero.

Orientation

$$\boxed{\tan 2\theta_n = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}} \longleftarrow \boldsymbol{\tau=0} \longleftarrow \tau = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \quad \text{or} \quad \frac{d\sigma}{d\theta} = 0$$

Magnitude

$$\boxed{\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}} \longleftarrow \sigma = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

θ_n

or

Diagonalization of stress tensor

$$\begin{pmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_y & 0 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

2 planes, corresponding to MAX and MIN principal stresses (σ_1 and σ_2), separated by 90º in θ.

Maximum shear stress

2D: Another important parameter is the maximum shear stress.

Orientation

$$\boxed{\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}}$$

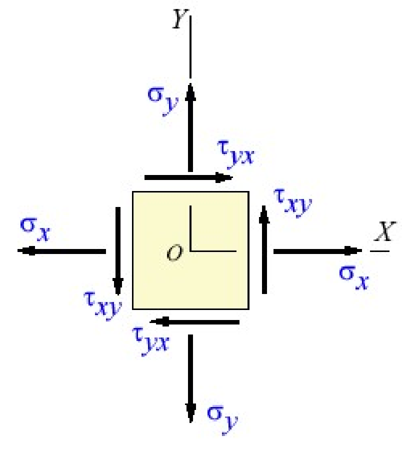
$$\frac{d\tau}{d\theta} = 0 \longleftarrow \text{Max of} \quad \tau = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

Magnitude

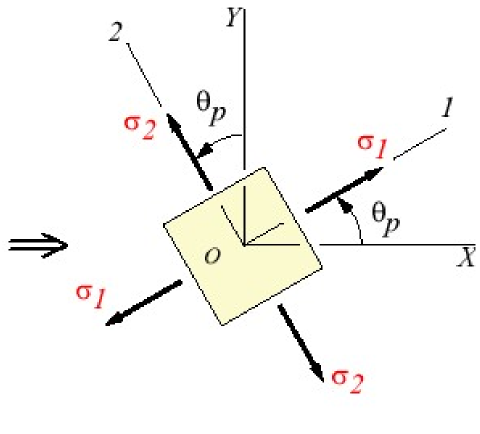
$$\boxed{\tau_3 = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}} \longleftarrow \tau_3 = \frac{\sigma_1 - \sigma_2}{2}$$

NOTE: $\theta_s = \theta_n \pm 45^{\circ}$

Examples



Stresses in given coordinate system



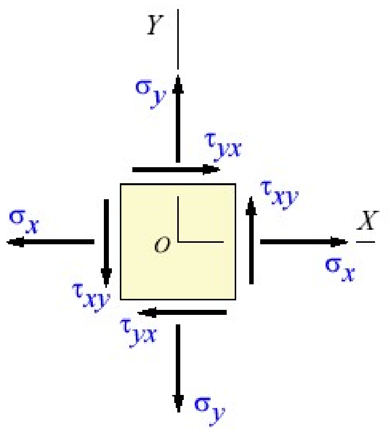
Principal stresses

Principal stresses

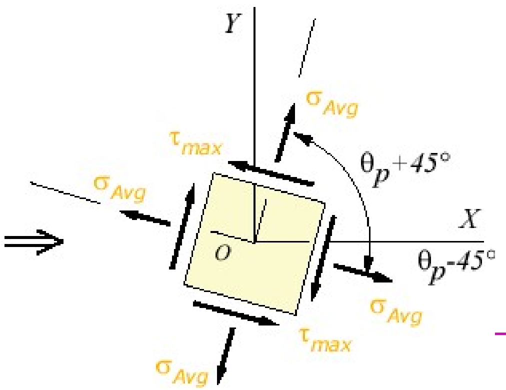
Note: $\boldsymbol{\theta_n}$ is the $\boldsymbol{\theta_p}$ here

Max shear stress

$$\boldsymbol{\sigma_{Avg}} = \frac{\boldsymbol{\sigma_1 + \sigma_2}}{\mathbf{2}} = \frac{\boldsymbol{\sigma_x + \sigma_y}}{\mathbf{2}}$$



Stresses in given coordinate system



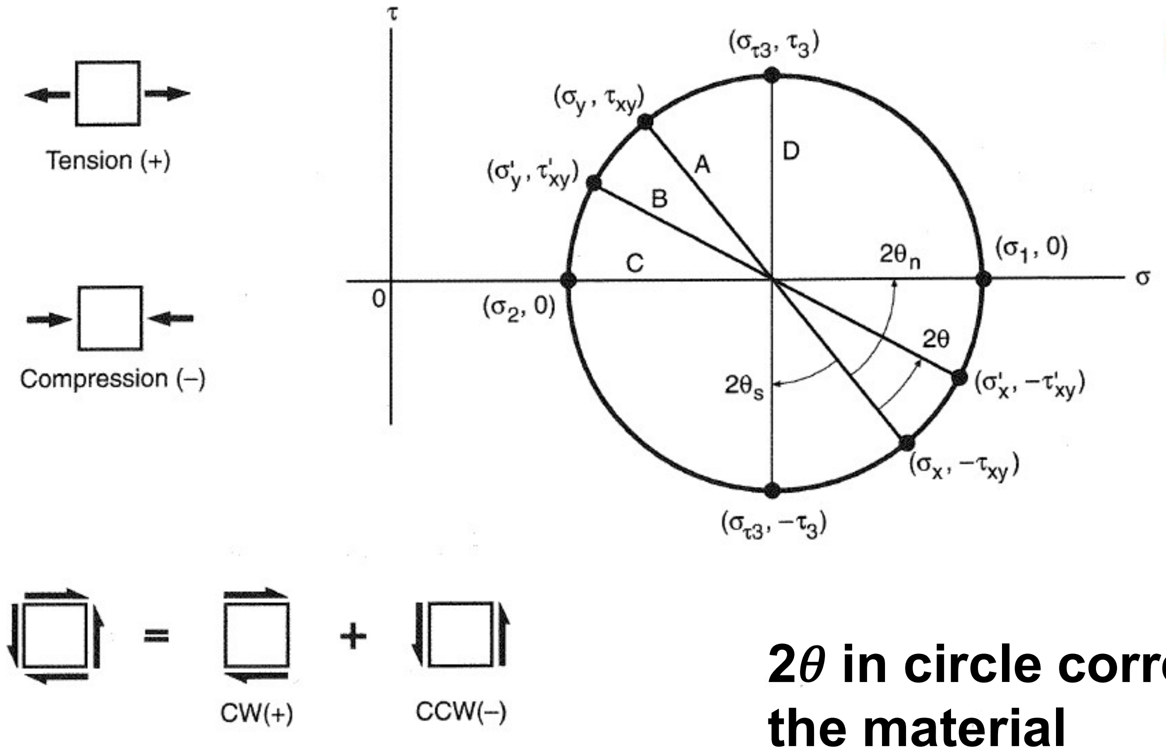
Maximum shear stress

Practical example:

- Shaft in torsion or torsion+bending. How do you chose strain gage positioning for steel?

Mohr's circle (1880s)

Alternative "graphical" way to deal with principal stresses.
Based on the same equations seen before. Why CIRCLE?
Recall: transformations equations were parametric of a circle.

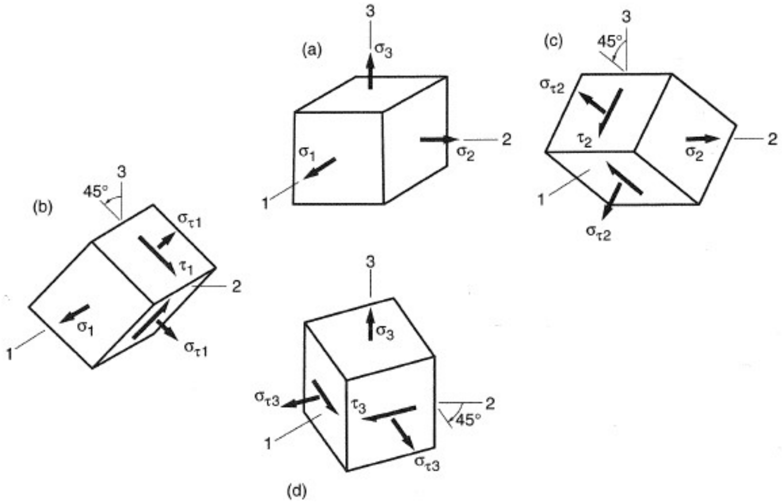


$$\left(\sigma - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$$

$$(\sigma - a)^2 + (\tau - b)^2 = r^2$$

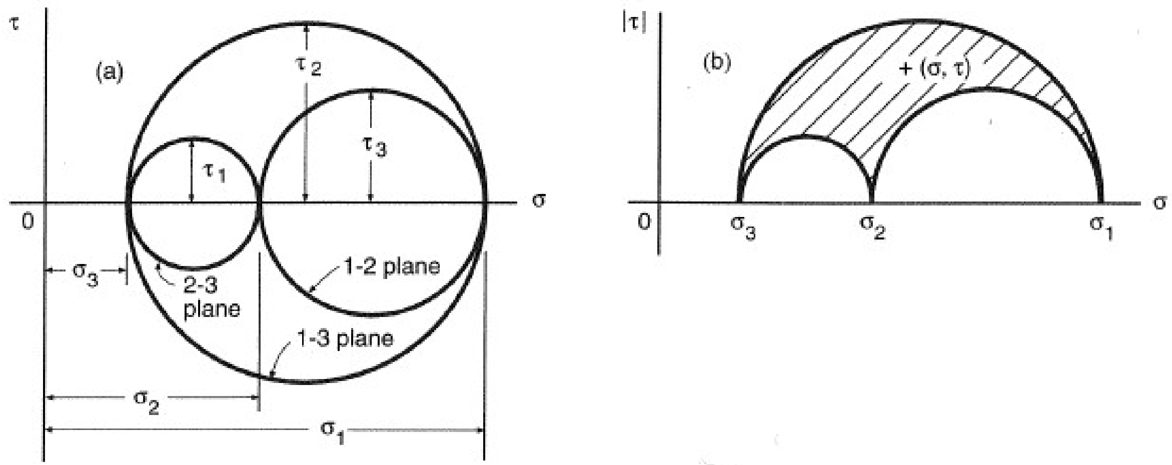
2θ in circle corresponds to θ in the material

3D case



$$\tau_1 = \frac{|\sigma_2 - \sigma_3|}{2}, \qquad \tau_2 = \frac{|\sigma_1 - \sigma_3|}{2}, \qquad \tau_3 = \frac{|\sigma_1 - \sigma_2|}{2}$$

$$\sigma_{\tau 1} = \frac{\sigma_2 + \sigma_3}{2}, \qquad \sigma_{\tau 2} = \frac{\sigma_1 + \sigma_3}{2}, \qquad \sigma_{\tau 3} = \frac{\sigma_1 + \sigma_2}{2}$$

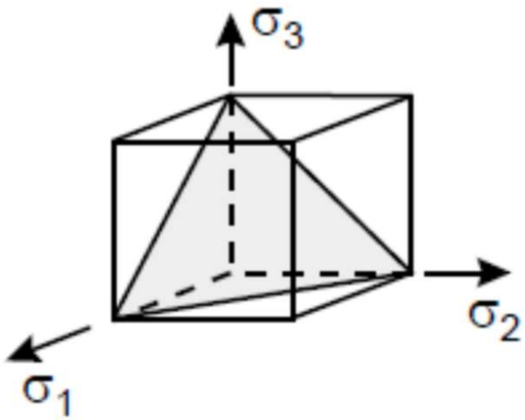


Max shear becomes the max of the principal shear stresses.

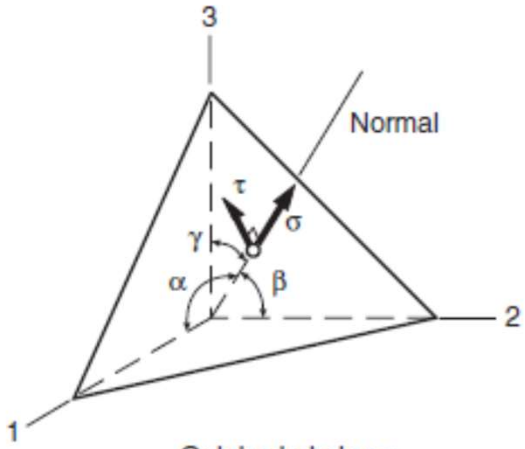
Octahedral planes

- Octahedral planes are of importance in yielding prediction and fatigue analysis, since we can derive stress invariant quantities.
- 8 octahedral planes making equal angles with the three principal stress directions.

Octahedral plane



All octahedral planes have the same shear stress - *Invariant*



Octahedral plane:
$\alpha = \beta = \gamma$
$\sigma = \sigma_h$
$\tau = \tau_h$

Octahedral planes

Octahedral normal stress or the hydrostatic stress

$$\boldsymbol{Normal\ stress\ \sigma_{oct} = \sigma_h = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)}$$

Octahedral shear stress or von Mises equivalent stress

$$\tau_h = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

$$\tau_h = \frac{1}{3}\sqrt{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)}$$

Stresses and Critical Planes

Hydrostatic stress is the average of the three normal stress components of any stress tensor.

<u>Hydrostatic</u> stress is related to <u>volume change</u>. Scalar.

$$\sigma_h = \frac{\sigma_x + \sigma_y + \sigma_z}{3}$$

Proportional and non-proportional loading

PROPORTIONAL LOADING:
Orientation of the principal axes with respect the loading axes is fixed.
Size of the Mohr's circle varies during cycle loading but angle does not.

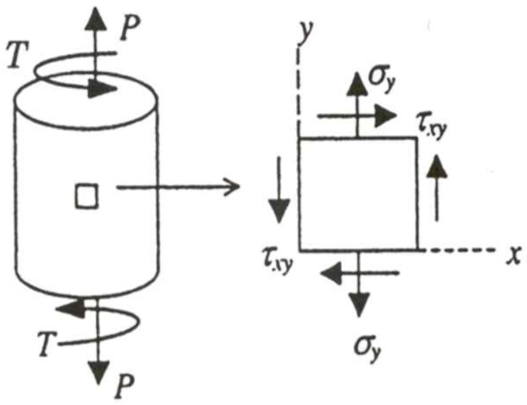
NON-PROPORTIONAL LOADING:
The principal directions are not fixed, but change orientation during cycle loading.

Out-of-phase: cycle max and min do not happen at the same time;
In-phase: cycle max and min happen at the same time.

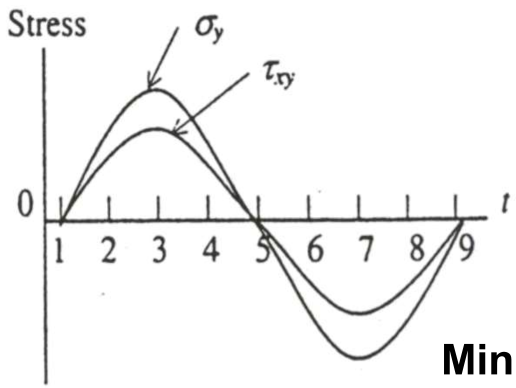
Biaxial (or triaxial) loading in the absence of applied shear strains will always be proportional

Examples: Proportional, in-phase

Max in 3

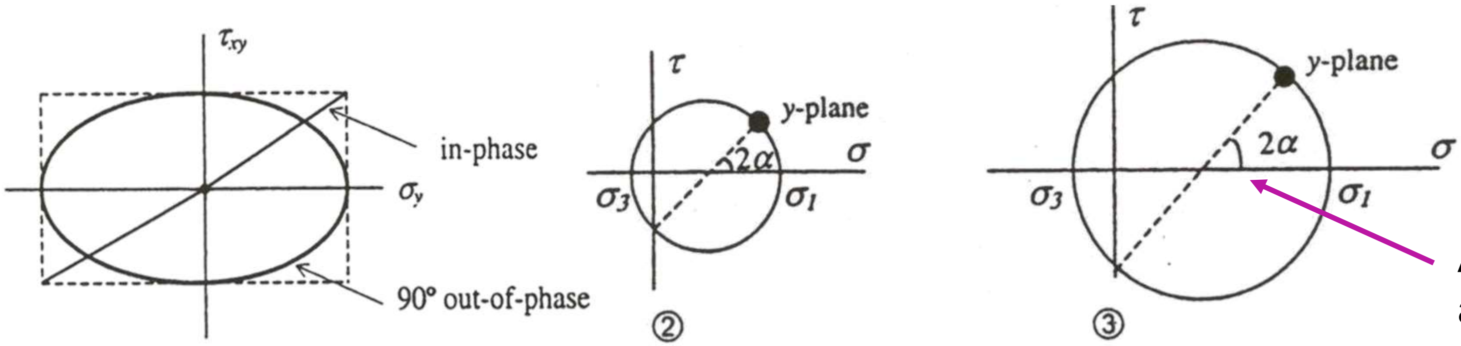


(a)



Min in 7

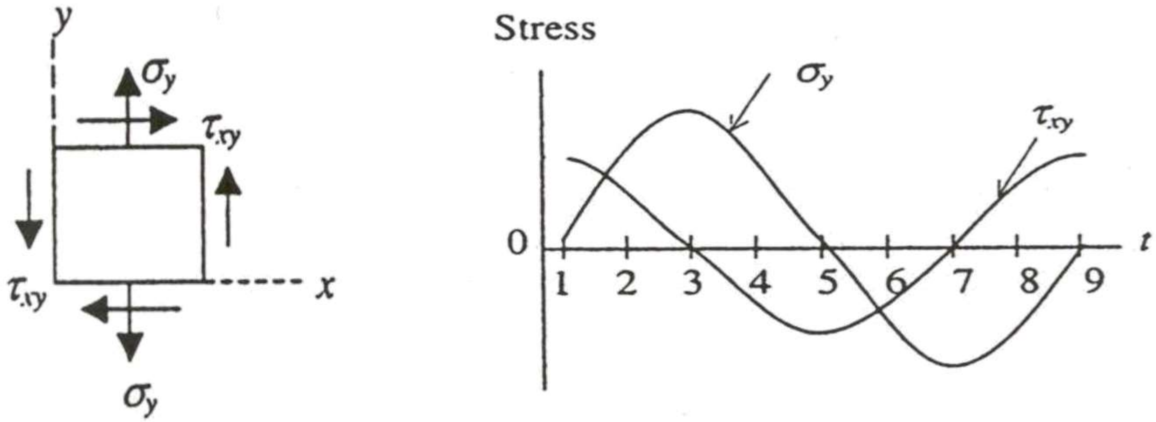
(b)



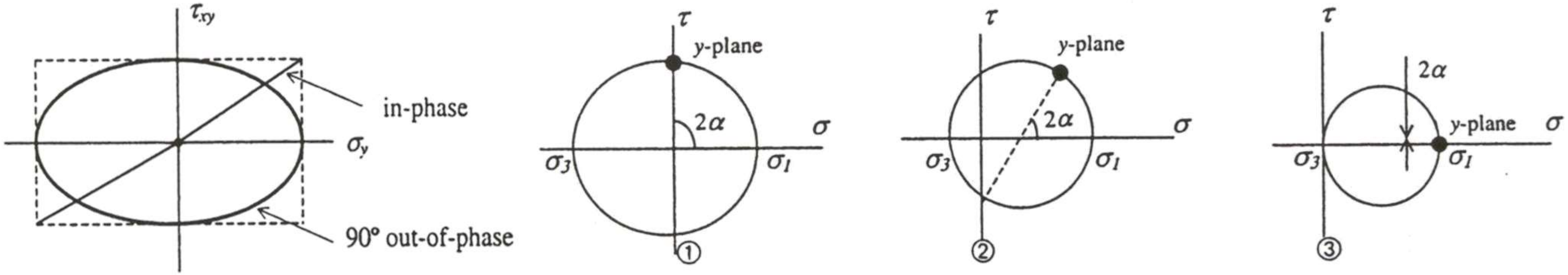
(e)

Angle is always the same

Examples: Non-proportional, out-of-phase

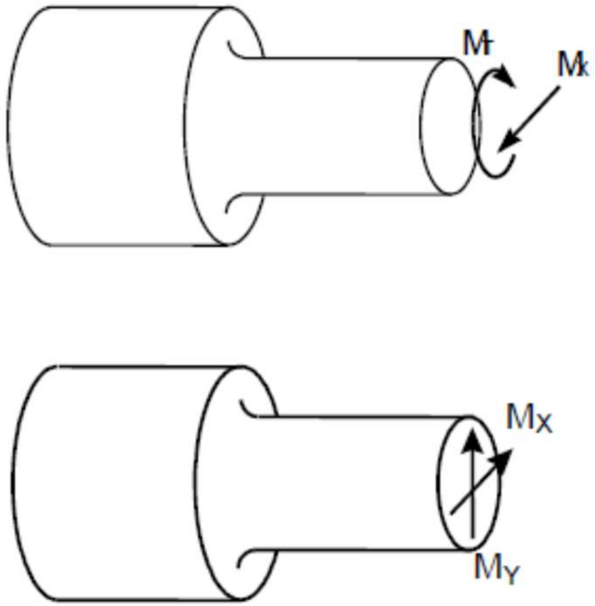


(d)



(f)

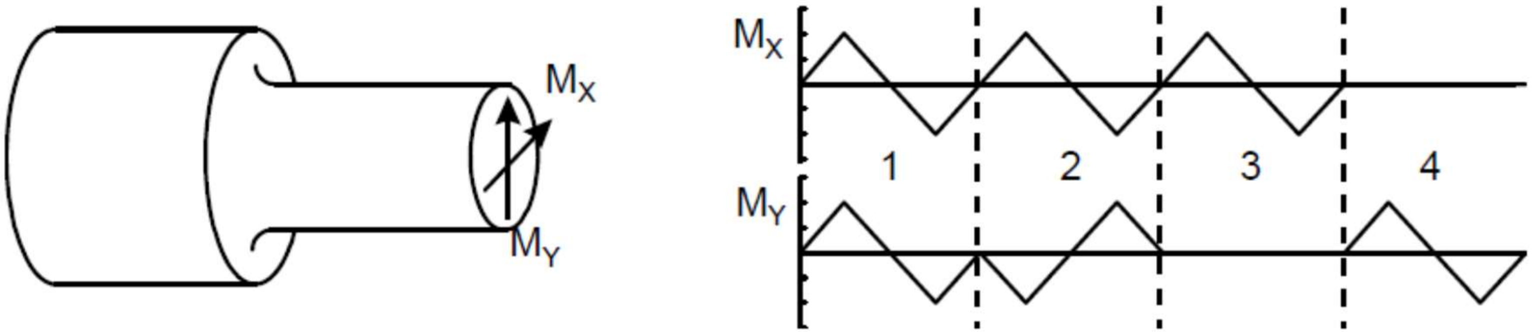
Non-proportional loading and non-proportional stressing



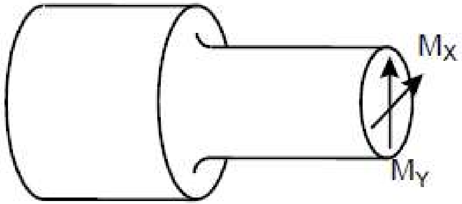
Proportional or nonproportional loading describes the loads acting on a structure while *proportional and nonproportional stressing* describes the resulting local stresses on the material.

Nonproportional loading may result in proportional stressing.

Non-proportional loading and non-proportional stressing

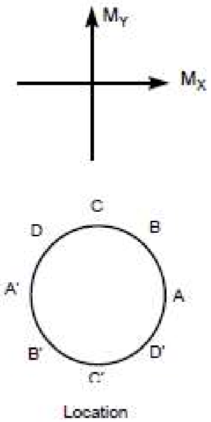


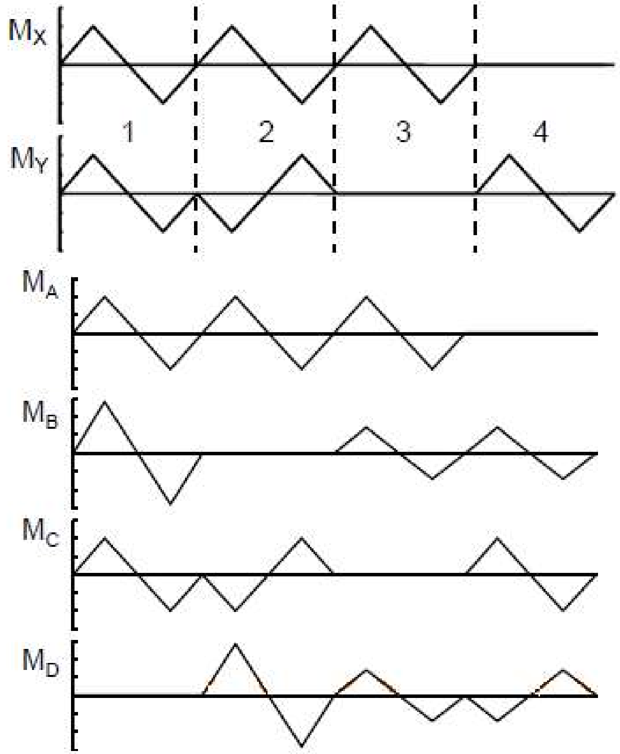
Non-proportional loading and non-proportional stressing



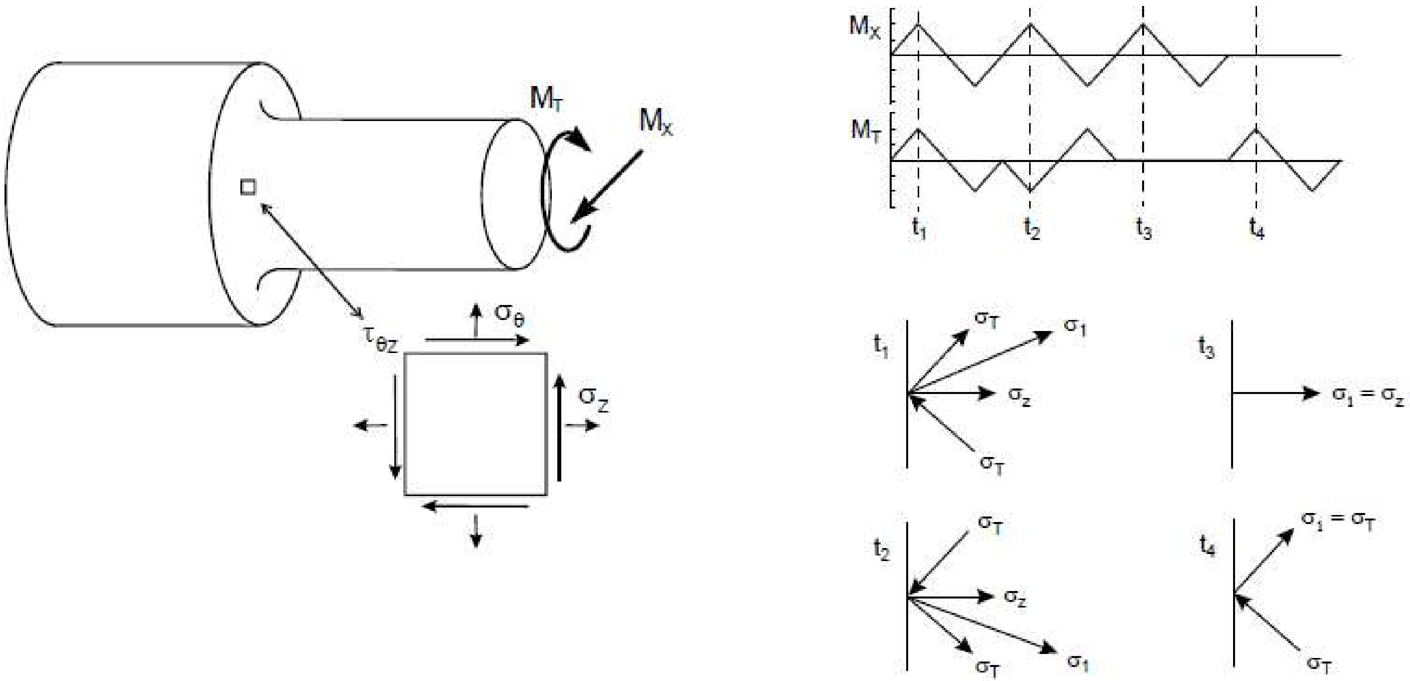
the state of stress remains uniaxial

$M_Z(\theta) = M_X \sin\theta + M_Y \cos\theta$

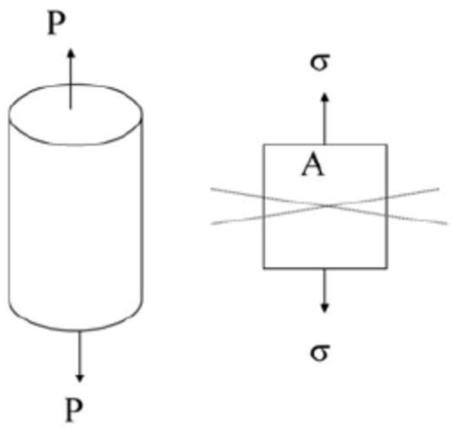


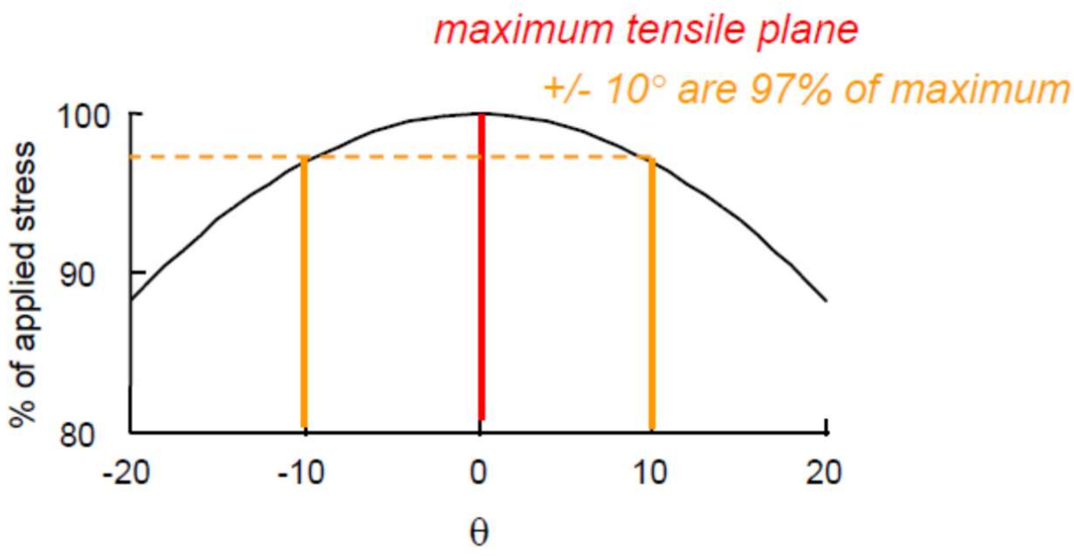


Non-proportional loading and non-proportional stressing



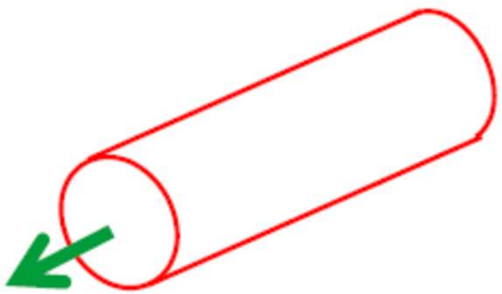
Stresses and Critical Planes



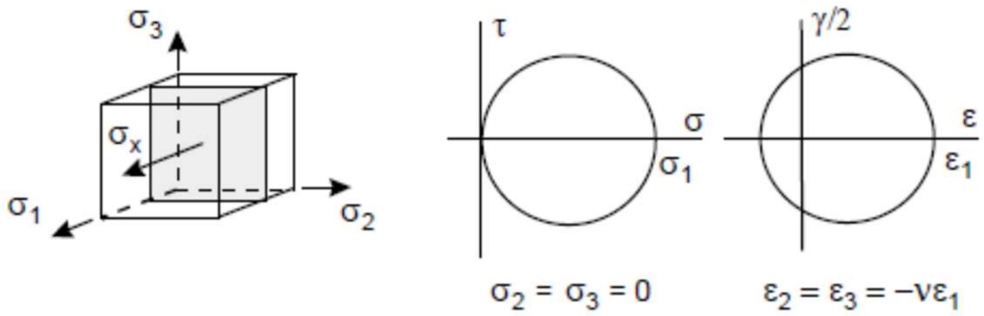


Mathematically – one plane is maximum
Engineering – near planes are nearly the same

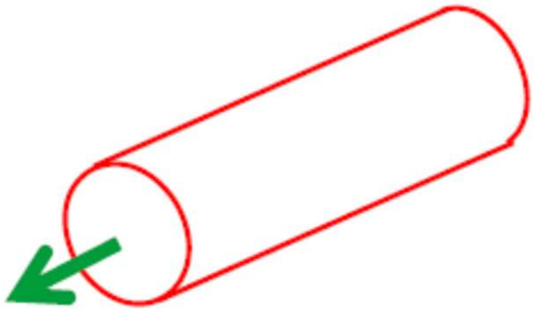
Stresses and Critical Planes



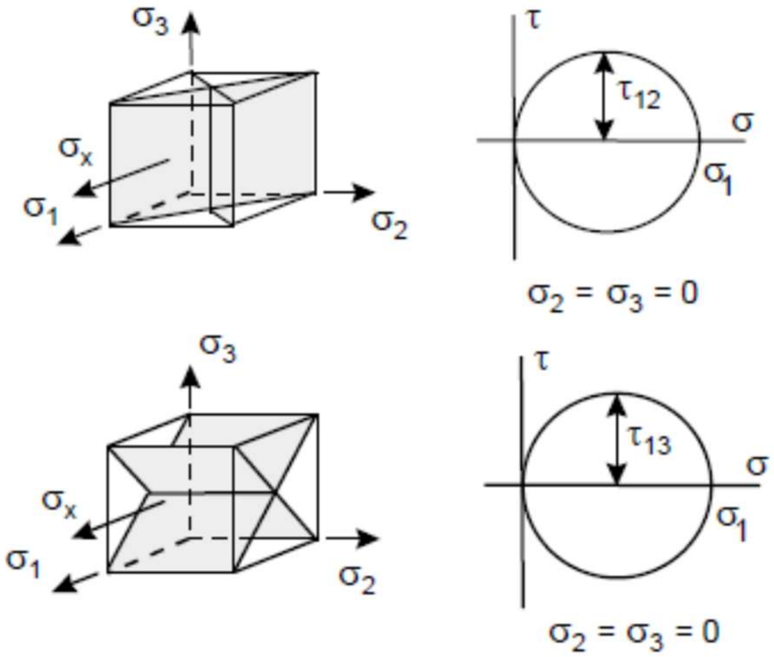
principal normal stress plane



Stresses and Critical Planes

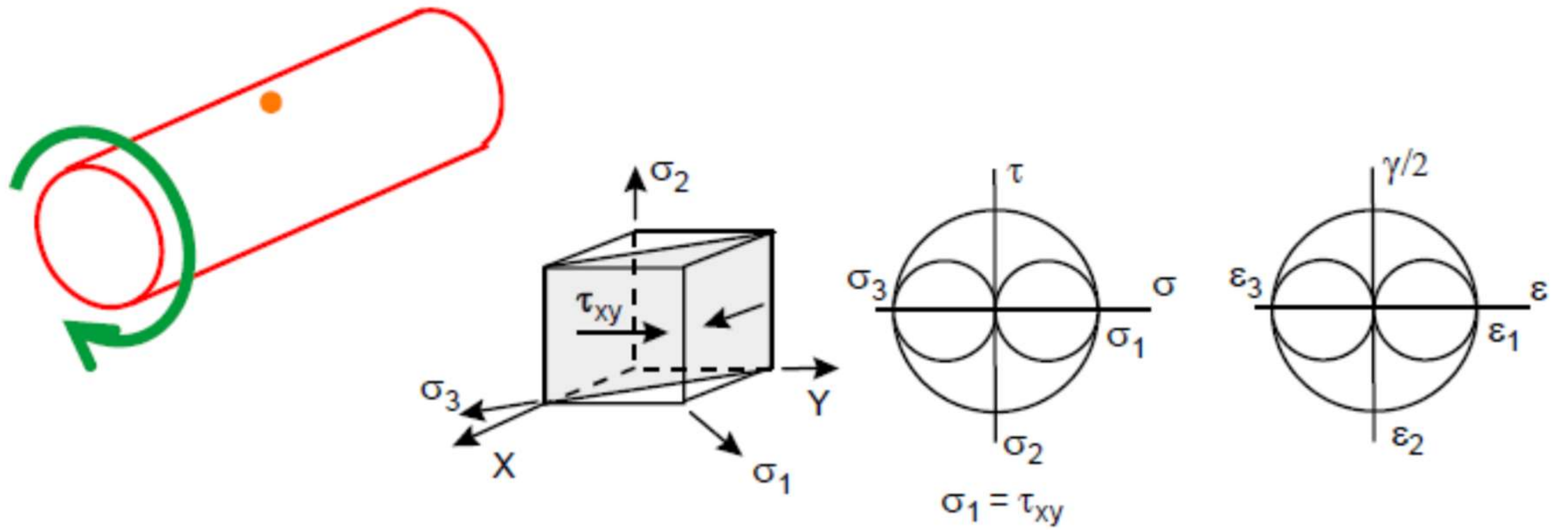


Four planes of maximum shear stress

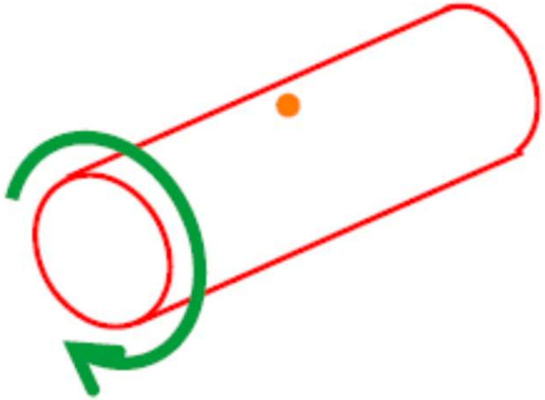


Stresses and Critical Planes

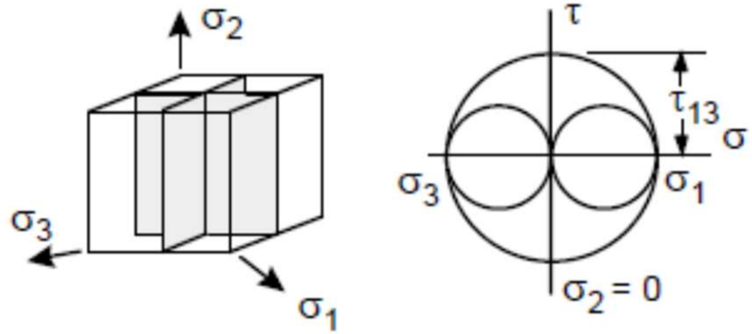
principal normal stress plane for torsion



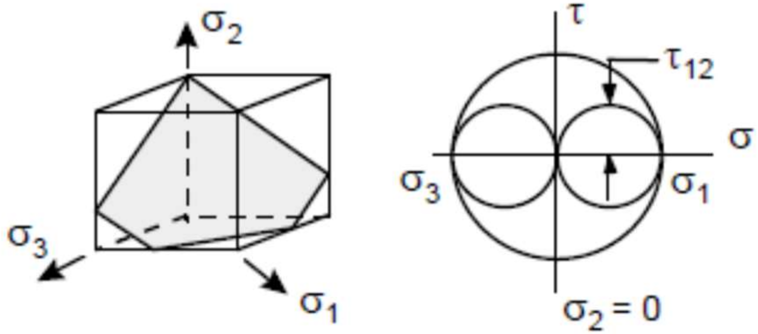
Stresses and Critical Planes



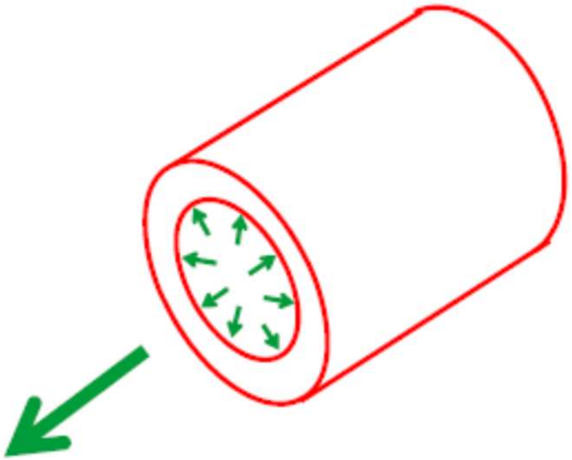
Two planes of maximum shear stress



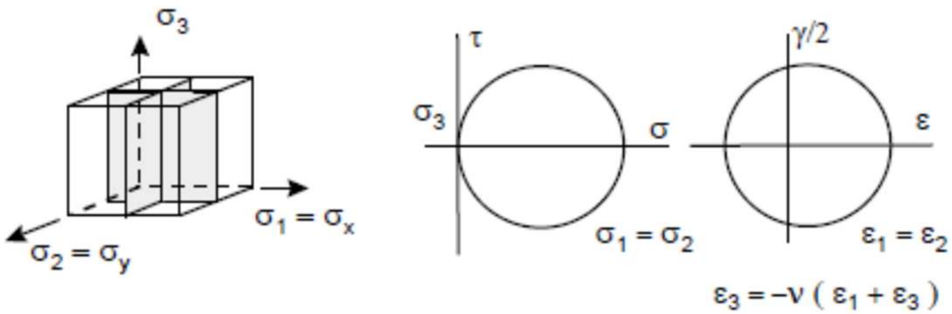
Four secondary shear stress planes



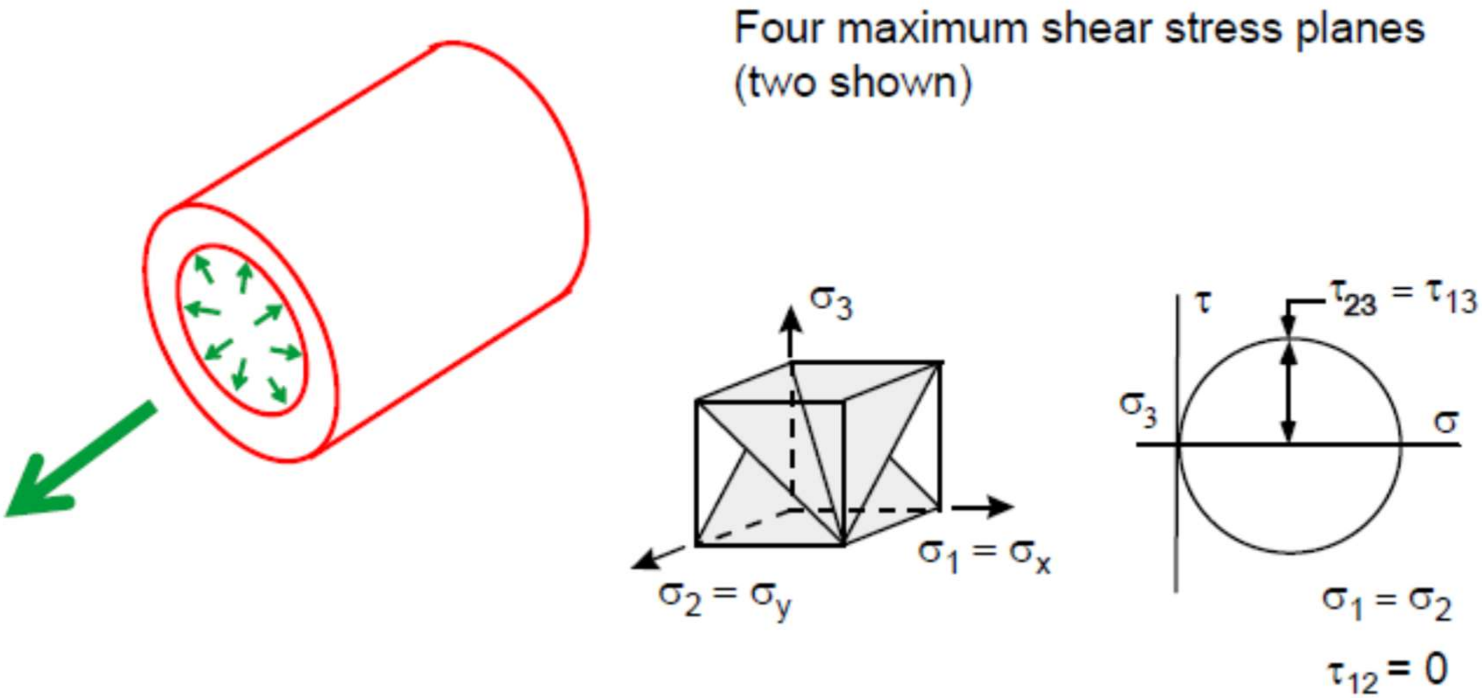
Stresses and Critical Planes



Two maximum stress planes for equi-biaxial tension



Stresses and Critical Planes



Stresses and Critical Planes

Common parameters for fatigue life Prediction:

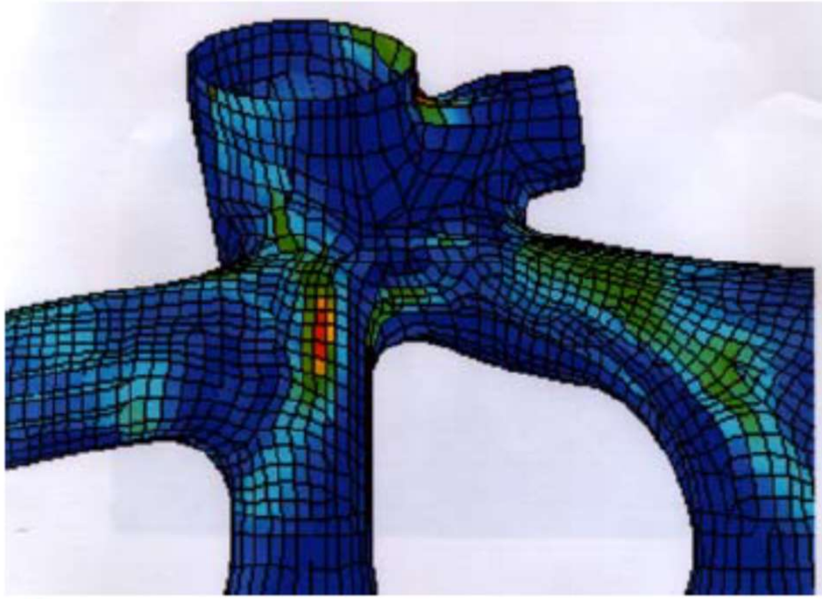
$\Delta\sigma$	normal stress range
$\Delta\tau$	shear stress range
$\sigma_{n,\ max}$	maximum normal stress
τ_{max}	maximum shear stress
$\Delta\varepsilon$	normal strain range
$\Delta\varepsilon^p$	normal plastic strain range
$\Delta\gamma$	shear strain range
$\varepsilon_{n,\ max}$	maximum normal strain

Multiaxial fatigue assessment

Can we apply our knowledge and data from uniaxial behaviour and tests to multiaxial situations?

- **Stress-based approaches (equivalent stress approaches, Sines method)**
- Strain-based
- Energy-based approaches
- Critical plane models
- Fracture Mech for crack growth

Equivalent stress approaches



$$\Delta\sigma_{eq} = \frac{1}{\sqrt{2}}\left[\left(\Delta\sigma_1 - \Delta\sigma_2\right)^2 + \left(\Delta\sigma_2 - \Delta\sigma_3\right)^2 + \left(\Delta\sigma_3 - \Delta\sigma_1\right)^2\right]^{\frac{1}{2}}$$

FEM → ”equivalent” stress → uniaxial life prediction

Equivalent stress approaches

Reduce multiaxial stress state to uniaxial stress state.

Maximum principal stress theory: $S_{qa} = S_{a1}$

Better for brittle material since they do not fail in shear.

Maximum shear stress theory: $S_{qa} = S_{a1} - S_{a3}$

Octahedral shear stress theory:

$$S_{qa} = \frac{1}{\sqrt{2}} \sqrt{(S_{a1} - S_{a2})^2 + (S_{a2} - S_{a3})^2 + (S_{a3} - S_{a1})^2}$$

Alternating stresses
Most used for ductile material.

(remind: von Mises stress)

$$\sigma_e = \frac{1}{\sqrt{2}} \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

Principal stresses

Equivalent stress approaches

If mean stress is present, additional term is needed

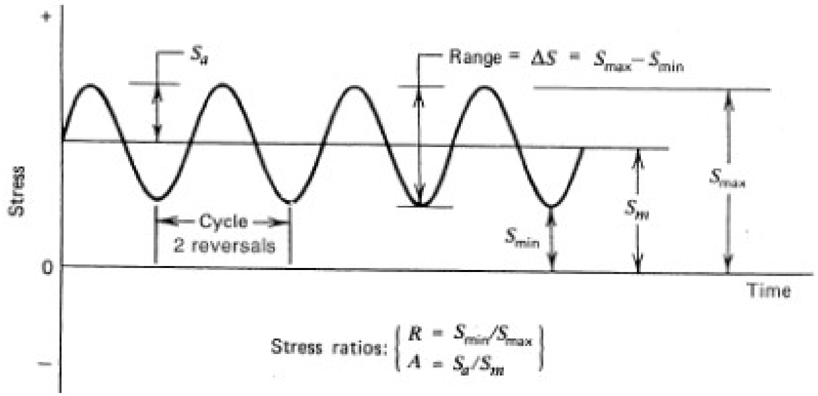
Option 1:

$$S_{qm} = \frac{1}{\sqrt{2}}\sqrt{(S_{m1} - S_{m2})^2 + (S_{m2} - S_{m3})^2 + (S_{m3} - S_{m1})^2}$$

Option 2:

$$S_{qm} = S_{m1} + S_{m2} + S_{m3} = S_{mx} + S_{my} + S_{mz}$$

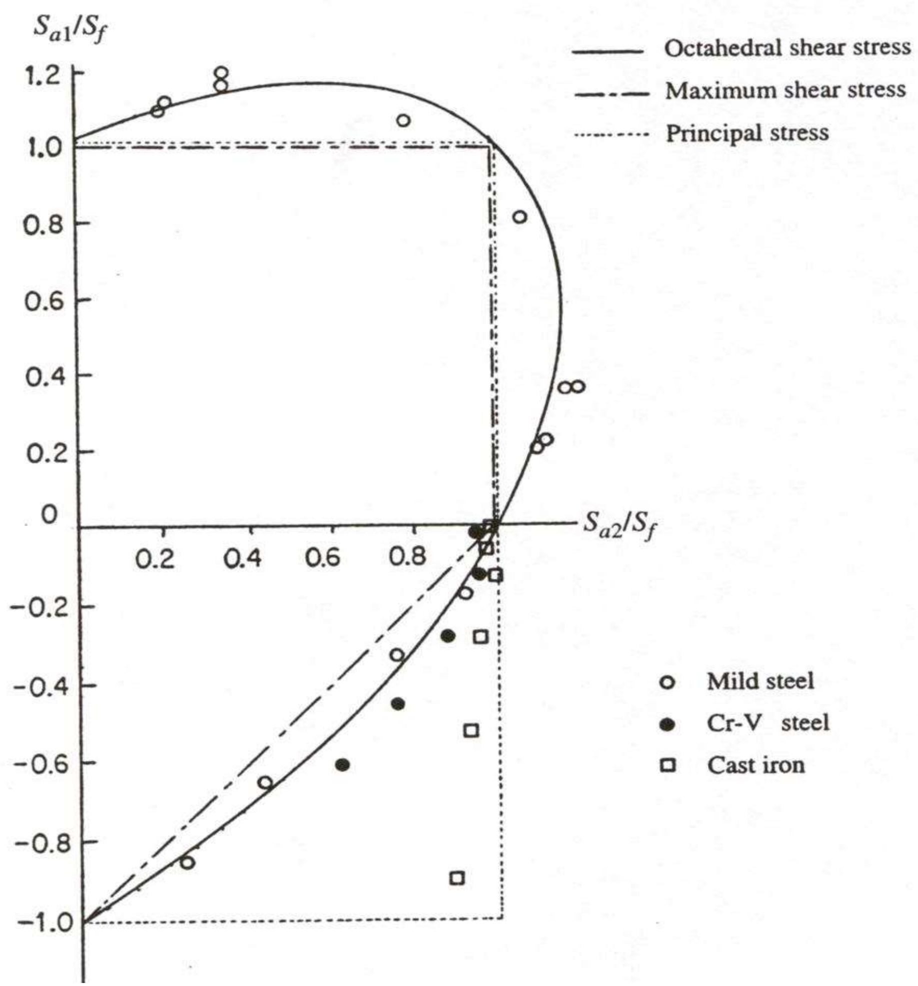
Principal mean nominal stresses



Equivalent stress methods

Example

- biaxial stress state (S3=0)
- mild steel and Cr-V agree well with octahedral shear stress criterion (ductile)
- cast iron (brittle), agrees better with the maximum principal stress criterion



Sines method

Alternating octahedral shear stress for cyclic stresses and hydrostatic stress for mean stresses.

$$\sqrt{(S_{ax}-S_{ay})^2+(S_{ay}-S_{az})^2+(S_{az}-S_{ax})^2+6(\tau^2_{axy}+\tau^2_{ayz}+\tau^2_{azx})}+m(S_{mx}+S_{my}+S_{mz})=\sqrt{2}\,S_{Nf}$$

- S_{Nf} is the uniaxial fully reversed fatigue strength that is expected to give the same fatigue life on uniaxial smooth specimens as the multiaxial stress state.
- ***m*** is the coefficient of mean stress influence.
  - It can be determined experimentally by obtaining a fatigue strength with a nonzero mean stress level (i.e. for example, uniaxial fatigue strength for $R = 0$ condition where $S_m = S_a$).
  - The value of m is on the order of 0.5

Note: Sines method is ok for proportional loading. Very good for long life fatigue and can be extended to strain-controlled low cycle fatigue. However, as all the stress-based approaches, it is suitable for long life fatigue situations (elastic strains mainly).

Example

Example 1 A closed-end, thin-walled tube made of 1020 sheet steel, with inside diameter $d = 100$ mm (4 in.) and wall thickness $t = 3$ mm (0.12 in.), is subject to internal pressure, p, which fluctuates from 0 to 15 MPa (2.18 ksi). What is the expected fatigue life?

Stress analysis shows a longitudinal stress varying from 0 minimum to $pd/4t = (15 \times 100)/12 = 125$ MPa maximum and a circumferential stress varying from 0 to $pd/2t = 250$ MPa maximum, which is in-phase with, or proportional to, the longitudinal stress. These stresses are also the principal stresses such that $S_1 = 250$ MPa and $S_2 = 125$ MPa. The radial stress in a thin-walled tube is small compared to the longitudinal and circumferential stresses such that $S_3 \approx 0$.

For fatigue analysis we separate the stresses into alternating and mean components:

$$S_{a1} = S_{m1} = 125 \text{ MPa} \quad \text{and} \quad S_{a2} = S_{m2} = 62.5 \text{ MPa}$$

We then form "equivalent" alternating and mean stresses. They are equivalent because we expect their joint effect to give the same life in uniaxial tests that we expect from the multiaxial situation. The equivalent alternating stress is calculated from Eq. 10.9:

$$S_{qa} = \frac{1}{\sqrt{2}}\sqrt{(125 - 62.5)^2 + (62.5 - 0)^2 + (0 - 125)^2} = 108 \text{ MPa}$$

The equivalent mean stress from Eq. 10.11 is simply the sum of the mean normal stresses in three mutually perpendicular directions, $S_{qm} = 125 + 62.5 = 187.5$ MPa. With S_{qa} and S_{qm} values known, we can use the modified Goodman equation (Eq. 4.8) to obtain the uniaxial, fully reversed fatigue strength, S_{Nf}. From Table A.2 for 1020 HR sheet steel, $S_u = 441$ MPa. Then

Example

$$\frac{S_{qa}}{S_{Nf}} + \frac{S_{qm}}{S_u} = \frac{108}{S_{Nf}} + \frac{187.5}{441} = 1 \quad \text{or} \quad S_{Nf} = 188 \text{ MPa}$$

Now the fatigue life can be calculated using Basquin's S–N equation with cyclic properties of the material from Table A.2

$$S_{Nf} = \sigma_f'(2N_f)^b = 1384\ (2N_f)^{-0.156}$$

Substituting $S_{Nf} = 188$ MPa results in $N_f = 180\,000$ cycles. If we use the Sines method with $m = 0.5$, Eq. 10.12 results in

$$\sqrt{(125 - 62.5)^2 + (62.5 - 0)^2 + (0 - 125)^2} + 0.5\ (125 + 62.5 + 0) = \sqrt{2}\, S_{Nf}$$

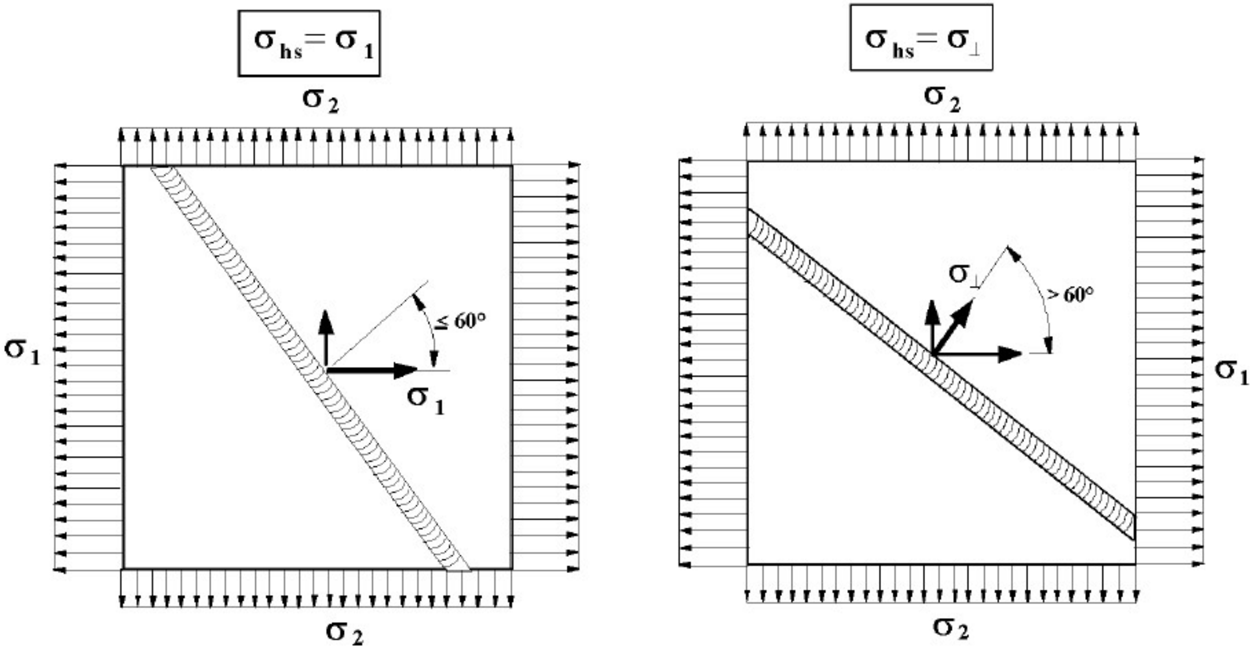
from which we obtain $S_{Nf} = 175$ MPa, resulting in $N_f = 290\,000$ cycles. The difference between the two results in less than a factor of 2.

Metal fatigue in engineering: Chapter 10

Maximum principal stress criteria

Biaxial stresses at weld toe (IIW recommendation example)

- use the principal stress, which is approximately in line with the perpendicular to the weld toe, i.e. within a deviation of ±60°



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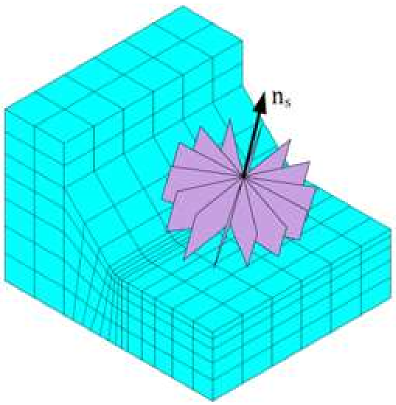
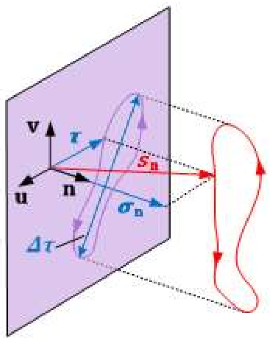
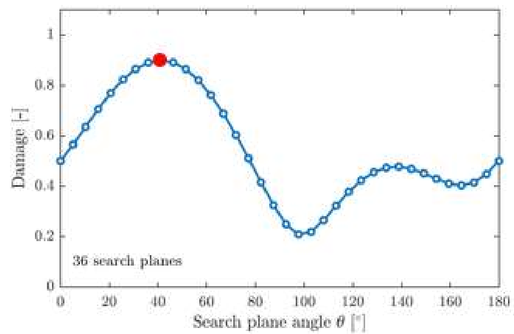
Critical plane approaches

- Allows the analysis of non-proportional load history, in where direction of maximum principal stress varies
- Stresses and strains during cyclic loading are determined for various orientations (planes)
- Stresses and strains acting on the most severely loaded plane are used to predict fatigue failure



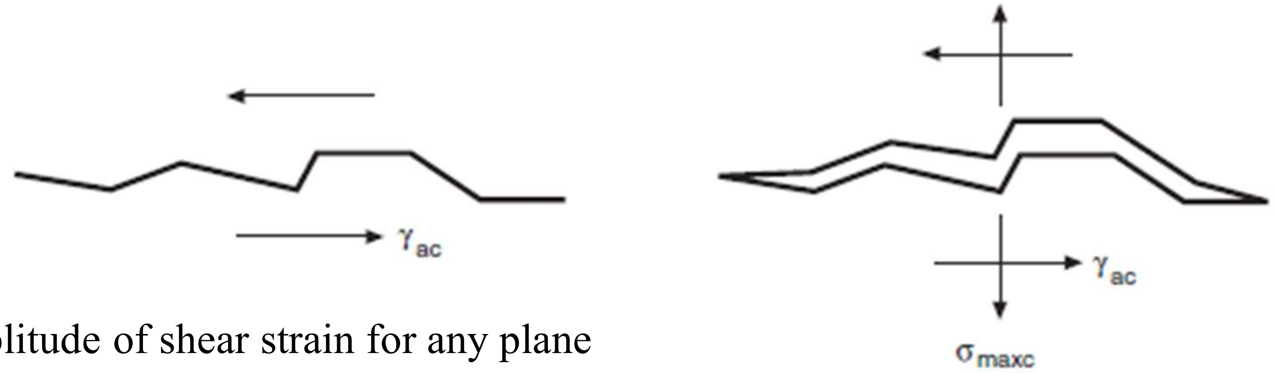




Critical plane approaches

Fatemi and Socie

$$\gamma_{ac}\left(1+\frac{\alpha\sigma_{\max c}}{\sigma_o'}\right)=\frac{\tau_f'}{G}(2N_f)^b+\gamma_f'(2N_f)^c$$



- γ_{ac} the largest amplitude of shear strain for any plane
- $\sigma_{\max c}$ the peak tensile stress normal to the plane of γ_{ac}
- α an empirical constant, α = 0.6 to 1.0
- σ_o the yield strength for the cyclic stress–strain curve
- Constants τ_f,b, γ_f, and c *describe* the strain–life curve from tests in pure shear

General remarks

- Don’t ignore multiaxial stress state.
- Do check whether the alternating stresses or strains have fixed principal directions. If so, the loading is proportional and fairly simple methods for life estimation can be used.
- Don't ignore the effects of nonproportional cyclic loading since it can produce additional cyclic hardening and often results in a shorter fatigue life compared to proportional loading.
- Damage mechanism dominated by shear or tensile cracking? Different fatigue damage models apply to each case.

Summary

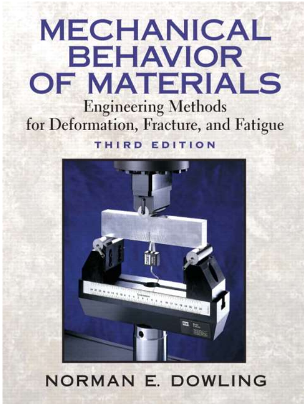
Load and
stress history

Stress and
strain state

Analysis methods for
multiaxial fatigue

Readings – Course material

Course book



Mechanical Behavior of Materials Engineering Methods for Deformation, Fracture, and Fatigue, Norman E. Dowling

- Section 6.1-6.3

Additional papers and reports given in MyCourses webpages

- Metal Fatigue in Engineering Book: Chapter 10 – Multiaxial stresses