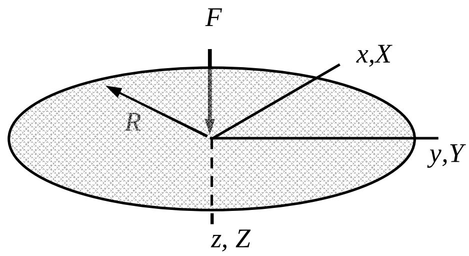
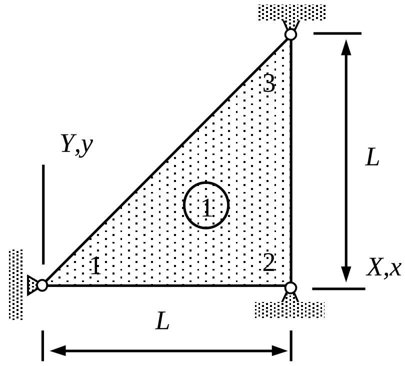
MEC-E8001 Finite Element Analysis, Onsite exam 22.02.2023

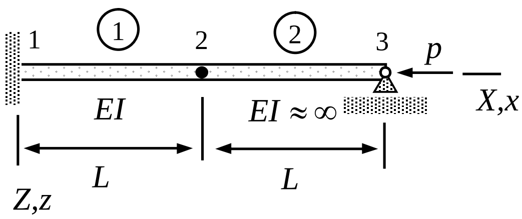
1. A circular plate of radius R, which is simply supported at the outer edge, is loaded by force F at the center point. Use the Kirchhoff plate model to find the transverse displacement at the center point. Use the approximation $w = a_0(x^2 + y^2 - R^2)$ for the transverse displacement. Material properties E, ν and thickness t are constants.



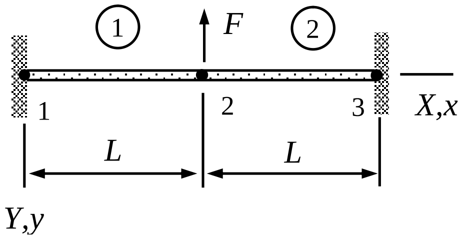
2. Determine the angular speed of free vibrations for the thin triangular slab shown. Assume plane stress conditions. The material properties E, ν, ρ and thickness h of the slab are constants. Use the approximations $u = 0$ and $v = (1 - x/L)u_{Y1}$ in which the nodal value u_{Y1} is a function of time.



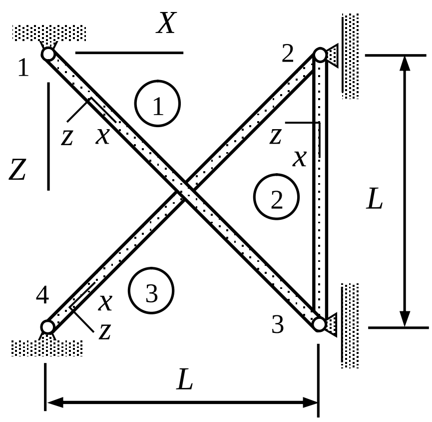
3. The structure shown consist of two beams, each of length L. As beam 2 is much stiffer than beam 1, the setting is simplified by considering beam 2 as a rigid body. Displacements are confined to the xz-plane. Cross-section properties of beam 1 are A and I and Young's modulus of the material is E. Determine the buckling load p_{cr}.



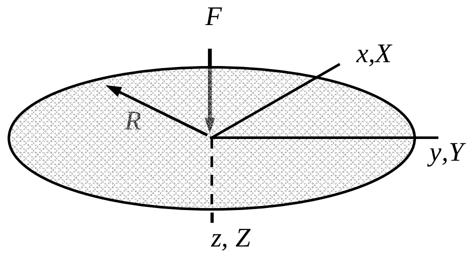
4. Determine the displacement at node 2 of the elastic bar shown by the large deformation theory. Take into account only the transverse displacement u_{Y2} ($u_{X2} = 0$). When $F = 0$, the cross-sectional area and length of the bar are A and L, respectively. Constitutive equation of the material is $S_{xx} = C\mathrm{E}_{xx}$, in which C is constant. Use two elements with linear shape functions.



5. Determine the static displacements $u_{Z2} = -u_{Z3}$ of nodes 2 and 3 due to the temperature increase $\Delta\vartheta$ at nodes 2 and 3 (actually in the wall). The material constants are E and α. The cross-sectional area of bar 1 and 3 is A and that of bar 2 is $\sqrt{2}A$. The initial temperature is $\vartheta°$.



A circular plate of radius R, which is simply supported at the outer edge, is loaded by force F at the center point. Use the Kirchhoff plate model to find the transverse displacement at the center point. Use the approximation $w = a_0(x^2 + y^2 - R^2)$ for the transverse displacement. Material properties E, ν and thickness t are constants.



Solution

Assuming that the material coordinate system is chosen so that the plate bending and thin slab modes decouple, it is enough to consider the virtual work densities of the bending mode only

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial^2 \delta w / \partial x^2 \\ \partial^2 \delta w / \partial y^2 \\ 2\partial^2 \delta w / \partial x \partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12} [E]_\sigma \begin{Bmatrix} \partial^2 w / \partial x^2 \\ \partial^2 w / \partial y^2 \\ 2\partial^2 w / \partial x \partial y \end{Bmatrix}, \quad \delta w_\Omega^{\text{ext}} = \delta w f_z \, .$$

in which the elasticity matrix of plane stress

$$[E]_\sigma = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}.$$

In the present case, distributed force vanishes i.e. $f_z = 0$ and the point force is taken into account by a point force element.

2p Approximation to the transverse displacement is given by (a_0 is not associated with any point but it just a parameter of the approximation)

$$w(x, y) = a_0(x^2 + y^2 - R^2) \quad \Rightarrow \quad \frac{\partial^2 w}{\partial x^2} = 2a_0 \, , \quad \frac{\partial^2 w}{\partial y^2} = 2a_0 \, , \text{ and } \quad \frac{\partial^2 w}{\partial x \partial y} = 0 \, .$$

When the approximation is substituted there, virtual work density of internal forces simplifies to

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} 2\delta a_0 \\ 2\delta a_0 \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{t^3 E}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} 2a_0 \\ 2a_0 \\ 0 \end{Bmatrix} = -\delta a_0 \frac{2}{3} \frac{t^3 E}{1-\nu} a_0 \, .$$

Virtual work expression of the plate bending element (element 1 here) is integral of the virtual work density over the domain occupied by the element. As the density expression is constant it is enough to multiply by the area of the domain

$$\delta W^1 = \int_\Omega \delta w_\Omega^{\text{int}} dA = \delta w_\Omega^{\text{int}} \pi R^2 = -\delta a_0 \frac{2\pi}{3} \frac{R^2 t^3 E}{1-\nu} a_0 \, .$$

2p Virtual work expression of the point force (element 2 here) follows from the definition of work (notice the use of virtual displacement at the point of action $x = y = 0$)

$$\delta W^2 = \delta w(0,0)F = -\delta a_0 R^2 F \ .$$

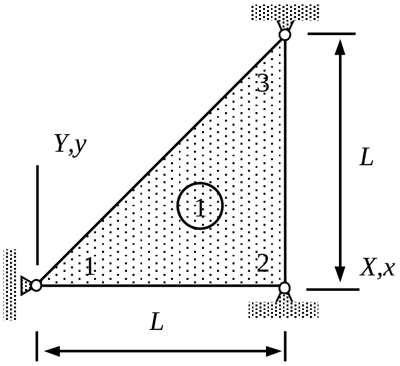
1p Principle of virtual work and the fundamental lemma of variation calculus give

$$\delta W = \delta W^1 + \delta W^2 = -\delta a_0 \left(\frac{2\pi}{3}\frac{R^2 t^3 E}{1-\nu} a_0 + R^2 F\right) = 0 \quad \Rightarrow \quad a_0 = -\frac{3}{2\pi}\frac{F}{t^3 E}(1-\nu) \ .$$

1p Displacement at the center point

$$w(0,0) = -a_0 R^2 = \frac{3}{2\pi}\frac{FR^2}{t^3 E}(1-\nu) \ .$$

Determine the angular speed of free vibrations for the thin triangular slab shown. Assume plane stress conditions. The material properties E, ν, ρ and thickness h of the slab are constants. Use the approximations $u=0$ and $v=(1-x/L)u_{Y1}$ in which the nodal value u_{Y1} is a function of time.



Solution

The virtual work densities of the internal and inertia forces for the thin slab model (plane stress conditions assumed) are given by

$$\delta w_{\Omega}^{\rm int}=-\begin{Bmatrix}\partial\delta u/\partial x\\ \partial\delta v/\partial y\\ \partial\delta u/\partial y+\partial\delta v/\partial x\end{Bmatrix}^{\rm T} t[E]_{\sigma}\begin{Bmatrix}\partial u/\partial x\\ \partial v/\partial y\\ \partial u/\partial y+\partial v/\partial x\end{Bmatrix} \text{ and } \delta w_{\Omega}^{\rm ine}=-\begin{Bmatrix}\delta u\\ \delta v\end{Bmatrix}^{\rm T} t\rho\begin{Bmatrix}\ddot{u}\\ \ddot{v}\end{Bmatrix}$$

where the elasticity matrix of the plane stress

$$[E]_{\sigma}=\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & (1-\nu)/2\end{bmatrix}.$$

1p The approximations to the displacement components are given (the linear interpolants of the nodal values can also be deduced easily from the figure). Hence

$$u=0 \text{ and } v=(1-\frac{x}{L})u_{Y1} \quad\Rightarrow$$

$$\frac{\partial u}{\partial x}=0,\ \frac{\partial u}{\partial y}=0,\ \frac{\partial\delta u}{\partial x}=0,\ \frac{\partial\delta u}{\partial y}=0,\ \delta u=0,\ \ddot{u}=0$$

$$\frac{\partial v}{\partial x}=-\frac{1}{L}u_{Y1},\ \frac{\partial v}{\partial y}=0,\ \frac{\partial\delta v}{\partial x}=-\frac{1}{L}\delta u_{Y1},\ \frac{\partial\delta v}{\partial y}=0,\ \delta v=(1-\frac{x}{L})\delta u_{Y1},\ \ddot{v}=(1-\frac{x}{L})\ddot{u}_{Y1}$$

4p When the approximations are substituted there, virtual work density of the internal and inertia forces simplify to

$$\delta w_{\Omega}^{\rm int}=-\begin{Bmatrix}0\\ 0\\ -\delta u_{Y1}/L\end{Bmatrix}^{\rm T}\frac{hE}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & (1-\nu)/2\end{bmatrix}\begin{Bmatrix}0\\ 0\\ -u_{Y1}/L\end{Bmatrix}=-\delta u_{Y1}\frac{h}{L^2}Gu_{Y1},$$

$$\delta w_{\Omega}^{\rm ine}=-\begin{Bmatrix}0\\ (1-\frac{x}{L})\delta u_{Y1}\end{Bmatrix}^{\rm T} h\rho\begin{Bmatrix}0\\ (1-\frac{x}{L})\ddot{u}_{Y1}\end{Bmatrix}=-\delta u_{Y1}h\rho(1-\frac{x}{L})^2\ddot{u}_{Y1}.$$

Integrations over the triangular domain of the element gives

$$\delta W^{\text{int}} = \int_A \delta w_\Omega^{\text{int}} \mathrm{d}A = \delta w_\Omega^{\text{int}} \frac{L^2}{2} = -\delta u_{Y1} \frac{h}{2} G u_{Y1} ,$$

$$\delta W^{\text{ine}} = \int_A \delta w_\Omega^{\text{ine}} \mathrm{d}A = \int_0^L (\int_0^x \delta w_\Omega^{\text{ine}} dy) dx \quad \Rightarrow$$

$$\delta W^{\text{ine}} = -\delta u_{Y1} h\rho \int_0^L (\int_0^x (1-\frac{x}{L})^2 dy) dx\, \ddot{u}_{Y1} = -\delta u_{Y1} h\rho \int_0^L x(1-\frac{x}{L})^2 dx\, \ddot{u}_{Y1} = -\delta u_{Y1} h\rho \frac{1}{12} L^2 \ddot{u}_{Y1} .$$

$^{\text{ine}}$ Virtual work expression of the structure takes the form

$$\delta W = -\delta u_{Y1} (\frac{t}{2} G u_{Y1} + h\rho \frac{1}{12} L^2 \ddot{u}_{Y1}) .$$

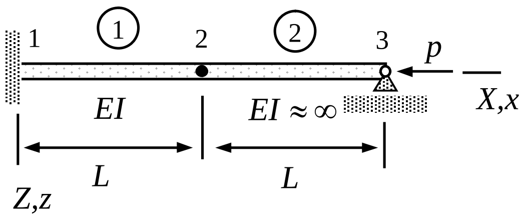
Principle of virtual work $\delta W = 0 \;\forall \delta \mathrm{a}$ and the fundamental lemma of variation calculus give

$$\frac{h}{2} G u_{Y1} + h\rho \frac{1}{12} L^2 \ddot{u}_{Y1} = 0 \quad \Leftrightarrow \quad \ddot{u}_{Y1} + 6 \frac{G}{\rho L^2} u_{Y1} = 0 .$$

1p As the ordinary differential equation is of the form $\ddot{u} + \omega^2 u = 0$, the angular speed of free vibrations is

$$\omega = \sqrt{6 \frac{G}{\rho L^2}} . \quad \Leftarrow$$

The structure shown consist of two beams, each of length L. As beam 2 is much stiffer than beam 1, the setting is simplified by considering beam 2 as a rigid body. Displacements are confined to the $xz-$plane. Cross-section properties of beam 1 are A and I and Young's modulus of the material is E. Determine the buckling load p_{cr}.



Solution

The axial forces in the beams follow directly from a free body diagram and it is enough to consider virtual work associated with the bending and interaction between bar and bending modes

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\text{T}} \frac{EI_{yy}}{h^3} \begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix},$$

$$\delta W^{\text{sta}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\text{T}} \frac{N}{30h} \begin{bmatrix} 36 & -3h & -36 & -3h \\ -3h & 4h^2 & 3h & -h^2 \\ -36 & 3h & 36 & 3h \\ -3h & -h^2 & 3h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix}.$$

3p Also, as beam 2 is considered as rigid, virtual work expression of internal forces vanish and modelling uses kinematical constraints $\vec{u}_{\text{B}} = \vec{u}_{\text{A}} + \vec{\theta}_{\text{A}} \times \vec{\rho}_{\text{AB}}$ and $\vec{\theta}_{\text{B}} = \vec{\theta}_{\text{A}}$. Let us choose A to be node 3 and B as node 2. Then

$u_{Z2} = \theta_{Y3} L$ and $\theta_{Y2} = \theta_{Y3}$.

Although axial displacement is non-zero, it is not needed as the axial force in the structure $N = -p$ (negative means compression) can be deduced without calculations on the axial displacement.

2p The internal force and coupling parts of beam 1 take the forms ($u_{z1} = 0$, $\theta_{y1} = 0$, $u_{z2} = u_{Z2} = \theta_{Y3} L$, $\theta_{y2} = \theta_{Y2} = \theta_{Y3}$)

$$\delta W^{\text{int}} = -\begin{Bmatrix} 0 \\ 0 \\ L\delta\theta_{Y3} \\ \delta\theta_{Y3} \end{Bmatrix}^{\text{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ L\theta_{Y3} \\ \theta_{Y3} \end{Bmatrix} = -\delta\theta_{Y3} 28 \frac{EI}{L} \theta_{Y3},$$

$$\delta W^{\text{sta}} = -\begin{Bmatrix} 0 \\ 0 \\ L\delta\theta_{Y3} \\ \delta\theta_{Y3} \end{Bmatrix}^{\text{T}} \frac{-p}{30L} \begin{bmatrix} 36 & -3L & -36 & -3L \\ -3L & 4L^2 & 3L & -L^2 \\ -36 & 3L & 36 & 3L \\ -3L & -L^2 & 3L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ L\theta_{Y3} \\ \theta_{Y3} \end{Bmatrix} = \delta\theta_{Y3} \frac{46}{30} pL\theta_{Y3}.$$

Virtual work expression of the structure is the sum of the internal and stability parts

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{sta}} = -\delta\theta_{Y3}(28\frac{EI}{L} - \frac{46}{30}pL)\theta_{Y3}.$$

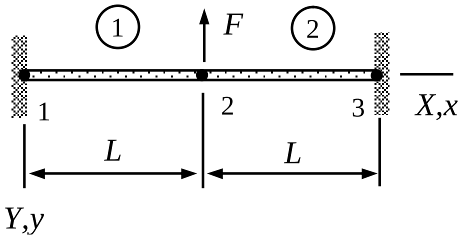
1p Principle of virtual work and the fundamental lemma of variation calculus imply that

$$(28\frac{EI}{L} - \frac{46}{30}pL)\theta_{Y3} = 0.$$

In stability analysis the aim is to find a non-zero equilibrium solution so $\theta_{Y3} \neq 0$ and the multiplier needs to vanish for a solution. Therefore, the buckling load is given by

$$p_{\text{cr}} = \frac{30}{46}28\frac{EI}{L^2} = \frac{420}{23}\frac{EI}{L^2}.$$

Determine the displacement at node 2 of the elastic bar shown by the large deformation theory. Take into account only the transverse displacement u_{Y2} ($u_{X2}=0$). When $F=0$, the cross-sectional area and length of the bar are A and L, respectively. Constitutive equation of the material is $S_{xx}=C\text{E}_{xx}$, in which C is constant. Use two elements with linear shape functions.



Solution

Virtual work density of the non-linear bar model

$$\delta w_{\Omega^\circ}^{\text{int}} = -\left(\frac{d\delta u}{dx}+\frac{du}{dx}\frac{d\delta u}{dx}+\frac{dv}{dx}\frac{d\delta v}{dx}+\frac{dw}{dx}\frac{d\delta w}{dx}\right)CA^\circ\left[\frac{du}{dx}+\frac{1}{2}\left(\frac{du}{dx}\right)^2+\frac{1}{2}\left(\frac{dv}{dx}\right)^2+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]$$

is based on the Green-Lagrange strain definition which is physically correct also when rotations/displacements are large. The expression depends on all displacement components, material property is denoted by C (constitutive equation $S_{xx}=C\text{E}_{xx}$), and the superscript in the cross-sectional area A° (and in other quantities) refers to the initial geometry (strain and stress vanishes). Otherwise, equilibrium equations follow in the same manner as in the linear case.

1p For element 1, the non-zero displacement components is $u_{y2}=u_{Y2}$. As the initial length of the element $h^\circ = L$, linear approximations to the displacement components

$$u=w=0 \text{ and } v=\frac{x}{L}u_{Y2} \quad\Rightarrow\quad \frac{du}{dx}=\frac{dw}{dx}=0 \text{ and } \frac{dv}{dx}=\frac{u_{Y2}}{L}.$$

3p When the approximation is substituted there, virtual work density of the internal forces and thereby the virtual work expression (density is constant) simplify to

$$\delta w_{\Omega^\circ}^{\text{int}} = -\frac{\delta u_{Y2}}{L}\frac{u_{Y2}}{L}\frac{CA}{2}\left(\frac{u_{Y2}}{L}\right)^2 \quad\Rightarrow\quad \delta W^1 = -\delta u_{Y2}\frac{CA}{2}\left(\frac{u_{Y2}}{L}\right)^3.$$

For element 2, the non-zero displacement component $u_{y2}=u_{Y2}$. As the initial length of the element $h^\circ = L$, linear approximations to the displacement components

$$u=w=0 \text{ and } v=\left(1-\frac{x}{L}\right)u_{Y2} \quad\Rightarrow\quad \frac{du}{dx}=\frac{dw}{dx}=0 \text{ and } \frac{dv}{dx}=-\frac{u_{Y2}}{L}.$$

When the approximation is substituted there, virtual work density of the internal forces and thereby the virtual work expression (density is constant) simplifies to

$$\delta w_{\Omega^\circ}^{\text{int}} = -\frac{\delta u_{Y2}}{L}\frac{u_{Y2}}{L}\frac{CA}{2}\left(\frac{u_{Y2}}{L}\right)^2 \quad\Rightarrow\quad \delta W^2 = -\delta u_{Y2}\frac{CA}{2}\left(\frac{u_{Y2}}{L}\right)^3.$$

Virtual work expression of the point force is

$$\delta W^3 = -F\delta u_{Y2}.$$

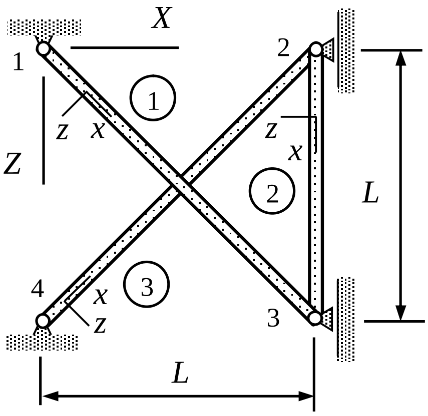
2p Virtual work expression of the structure is obtained as the sum of the element contributions

$$\delta W = -\delta u_{Y2}\left[\frac{CA}{2}\left(\frac{u_{Y2}}{L}\right)^3 + \frac{CA}{2}\left(\frac{u_{Y2}}{L}\right)^3 + F\right].$$

Principle of virtual work and the fundamental lemma of variation calculus imply that

$$\left(\frac{u_{Y2}}{L}\right)^3 + \frac{F}{CA} = 0 \quad \Rightarrow \quad u_{Y2} = -\left(\frac{FL^3}{CA}\right)^{1/3}. \quad \leftarrow$$

Determine the static displacements $u_{Z2} = -u_{Z3}$ of nodes 2 and 3 due to the temperature increase $\Delta\vartheta$ at nodes 2 and 3 (actually in the wall). The material constants are E and α. The cross-sectional area of bar 1 and 3 is A and that of bar 2 is $\sqrt{2}A$. The initial temperature is $\vartheta°$.



Solution

As temperature is known and the external distributed force vanishes, only the virtual work expressions of the internal and coupling parts

$$\delta W^{\text{int}} = -\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}} \frac{EA}{h}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix} \text{ and } \delta W^{\text{cpl}} = \begin{Bmatrix}\delta u_{x1}\\ \delta u_{x1}\end{Bmatrix}^{\mathrm{T}} \frac{\alpha EA}{2}\begin{bmatrix}-1 & -1\\ 1 & 1\end{bmatrix}\begin{Bmatrix}\Delta\vartheta_1\\ \Delta\vartheta_2\end{Bmatrix}$$

are needed in the calculations. Term $\Delta\vartheta = \vartheta - \vartheta°$ is the difference between temperature at the deformed and initial geometries.

5p The nodal displacements and temperatures of bar 1 $u_{x1} = 0$, $u_{x3} = u_{Z3}/\sqrt{2}$, $\Delta\vartheta_1 = \vartheta° - \vartheta° = 0$, and $\Delta\vartheta_3 = \Delta\vartheta$ give (notice that the variation of a given function is always zero)

$$\delta W^1 = -\begin{Bmatrix}0\\ \delta u_{Z3}/\sqrt{2}\end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{\sqrt{2}L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}0\\ u_{Z3}/\sqrt{2}\end{Bmatrix} - \frac{\alpha EA}{2}\begin{bmatrix}-1 & -1\\ 1 & 1\end{bmatrix}\begin{Bmatrix}0\\ \Delta\vartheta\end{Bmatrix}\right) \Leftrightarrow$$

$$\delta W^1 = -\delta u_{Z3}\left(\frac{EA}{2\sqrt{2}L}u_{Z3} - \frac{\alpha EA}{2\sqrt{2}}\Delta\vartheta\right).$$

The nodal displacements and temperatures of bar 2 $u_{x2} = u_{Z2} = -u_{Z3}$, $u_{x3} = u_{Z3}$, $\Delta\vartheta_2 = \Delta\vartheta$, and $\Delta\vartheta_3 = \Delta\vartheta$ give

$$\delta W^2 = -\begin{Bmatrix}-\delta u_{Z3}\\ \delta u_{Z3}\end{Bmatrix}^{\mathrm{T}} \left(\frac{E\sqrt{2}A}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}-u_{Z3}\\ u_{Z3}\end{Bmatrix} - \frac{\alpha E\sqrt{2}A}{2}\begin{bmatrix}-1 & -1\\ 1 & 1\end{bmatrix}\begin{Bmatrix}\Delta\vartheta\\ \Delta\vartheta\end{Bmatrix}\right) \Leftrightarrow$$

$$\delta W^2 = -\delta u_{Z3}\left(4\sqrt{2}\frac{EA}{L}u_{Z3} - 4\sqrt{2}\frac{\alpha EA}{2}\Delta\vartheta\right).$$

The nodal displacements and temperatures of bar 3 $u_{x4} = 0$, $u_{x2} = -u_{Z2}/\sqrt{2} = u_{Z3}/\sqrt{2}$, $\Delta\vartheta_1 = \vartheta° - \vartheta° = 0$, and $\Delta\vartheta_3 = \Delta\vartheta$ give

$$\delta W^3 = -\begin{Bmatrix}0\\ \delta u_{Z3}/\sqrt{2}\end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{\sqrt{2}L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}0\\ u_{Z3}/\sqrt{2}\end{Bmatrix} - \frac{\alpha EA}{2}\begin{bmatrix}-1 & -1\\ 1 & 1\end{bmatrix}\begin{Bmatrix}0\\ \Delta\vartheta\end{Bmatrix}\right) \Leftrightarrow$$

$$\delta W^3 = -\delta u_{Z3}\left(\frac{EA}{2\sqrt{2}L}u_{Z3} - \frac{\alpha EA}{2\sqrt{2}}\Delta\vartheta\right).$$

1p Virtual work expression of the structure is the sum of element contributions

$$\delta W = -\delta u_{Z3} 2(\frac{EA}{2\sqrt{2}L} u_{Z3} - \frac{\alpha EA}{2\sqrt{2}} \Delta\vartheta) - \delta u_{Z3}(4\sqrt{2}\frac{EA}{L} u_{Z3} - 4\sqrt{2}\frac{\alpha EA}{2}\Delta\vartheta) \quad \Leftrightarrow$$

$$\delta W = -\delta u_{Z3}(9\frac{EA}{\sqrt{2}L} u_{Z3} - 5\frac{\alpha EA}{\sqrt{2}}\Delta\vartheta) .$$

Principle of virtual work and the fundamental lemma of variation calculus imply

$$\frac{9}{\sqrt{2}L} EAu_{Z3} - \frac{5}{\sqrt{2}} EA\alpha\Delta\vartheta = 0 \quad \Leftrightarrow \quad u_{Z3} = \frac{5}{9}\alpha L\Delta\vartheta \ .$$