Name________________________ Student number________________

Assignment 4

Consider the disk rigidity problem on page 1-4 of the lecture notes and the mass-displacement relationship given by dimension analysis

$$\frac{mgR^2}{Et^4} = f(\frac{u}{t}, \frac{L}{R}, \nu) \approx a(\frac{u}{t}) + b(\frac{u}{t})^3,$$

where the latter form uses the first two odd order terms of Taylor expansion of f with respect to u/t and, therefore, coefficients a and b may depend of L/R and ν. Instead of (expensive) physical experiments, one may use simulation by a model for finding, e.g., the dependency of the coefficients on L/R and ν. Use the mass-displacement table below, given by the course software with a large displacement plate model, to determine a and b when $E = 4.22\,\mathrm{GPa}$, $R = 0.245\,\mathrm{m}$, $L = 0.280\,\mathrm{m}$, $t = 4.1\mathrm{mm}$, and $g = 9.81\mathrm{m}/\mathrm{s}^2$. Also, use the outcome to estimate the values of the parameters for $\nu = 0.32$.

m [kg]	u [mm] ($\nu = 0.1$)	u [mm] ($\nu = 0.4$)
0	0.00	0.00
1	1.26	0.94
2	2.34	1.81
3	3.21	2.59
4	3.94	3.28
5	4.56	3.89
6	5.10	4.44
7	5.58	4.93

Solution

Let us use notations $\underline{m} = mgR^2/(Et^4)$ and $\underline{u} = u/t$ for the dimensionless mass and displacement, respectively. To find the coefficients a and b of relationship, one may use the least-squares method giving the values of a and b as the minimizers of function

$$\Pi(a,b) = \frac{1}{2}\sum (a\underline{u}_i + b\underline{u}_i^3 - \underline{m}_i)^2,$$

where the sum is over all the measured mass-displacement values. The method looks for parameter values giving as good as possible overall match to the data. For a perfect fit with the best values of a and b function $\Pi = 0$ so the value of Π at the minimum point is a measure of the quality of the fit.

At the minimum point, partial derivatives of $\prod(a,b)$ with respect to a and b should vanish. Therefore, one obtains

$$\frac{\partial\prod(a,b)}{\partial a}=\sum \underline{u}_i(a\underline{u}_i+b\underline{u}_i^3-\underline{m}_i)=0 \quad \text{and} \quad \frac{\partial\prod(a,b)}{\partial b}=\sum \underline{u}_i^3(a\underline{u}_i+b\underline{u}_i^3-\underline{m}_i)=0$$

or written in a more convenient form

$$(\sum \begin{bmatrix} \underline{u}_i^2 & \underline{u}_i^4 \\ \underline{u}_i^4 & \underline{u}_i^6 \end{bmatrix})\begin{Bmatrix} a \\ b \end{Bmatrix}-\sum \begin{Bmatrix} \underline{u}_i\underline{m}_i \\ \underline{u}_i^3\underline{m}_i \end{Bmatrix}=0 \quad \Leftrightarrow \quad \begin{Bmatrix} a \\ b \end{Bmatrix}=(\sum \begin{bmatrix} \underline{u}_i^2 & \underline{u}_i^4 \\ \underline{u}_i^4 & \underline{u}_i^6 \end{bmatrix})^{-1}(\sum \begin{Bmatrix} \underline{u}_i\underline{m}_i \\ \underline{u}_i^3\underline{m}_i \end{Bmatrix}).$$

From this point on, it is convenient to use Mathematica, Matlab, Excel or some other computational tool. The first step is to transform the measured data into dimensionless form using the definitions and the given values $E=4.22\,\text{GPa}$, $\nu=0.32$, $R=0.245\,\text{m}$, $t=4.1\text{mm}$, and $g=9.81\,\text{m/s}^2$

m [kg]	$mgR^2/(Et^4)$	u [mm] ($\nu=0.1$)	u/t ($\nu=0.1$)	u [mm] ($\nu=0.4$)	u/t ($\nu=0.4$)
0	0.00	0.00	0.00	0.00	0.00
1	0.49	1.26	0.31	0.94	0.23
2	0.99	2.34	0.57	1.81	0.44
3	1.48	3.21	0.78	2.59	0.63
4	1.98	3.94	0.96	3.28	0.80
5	2.47	4.56	1.11	3.89	0.95
6	2.96	5.10	1.24	4.44	1.08
7	3.46	5.58	1.36	4.93	1.20

With the data in the table for $\nu=0.1$, the linear equation system to coefficients a and b and the solution to the values become

$$\begin{bmatrix} 6.59 & 8.70 \\ 8.70 & 13.01 \end{bmatrix}\begin{Bmatrix} a \\ b \end{Bmatrix}-\begin{Bmatrix} 14.91 \\ 20.47 \end{Bmatrix}=0 \quad \Leftrightarrow \quad \begin{Bmatrix} a \\ b \end{Bmatrix}=\begin{bmatrix} 6.59 & 8.70 \\ 8.70 & 13.01 \end{bmatrix}^{-1}\begin{Bmatrix} 14.91 \\ 20.47 \end{Bmatrix}=\begin{Bmatrix} 1.57 \\ 0.53 \end{Bmatrix}. \leftarrow$$

With the data in the table for $\nu=0.4$, the linear equation system to coefficients a and b and the solution to the values become

$$\begin{bmatrix} 4.81 & 4.89 \\ 4.89 & 5.70 \end{bmatrix}\begin{Bmatrix} a \\ b \end{Bmatrix}-\begin{Bmatrix} 12.77 \\ 13.36 \end{Bmatrix}=0 \quad \Leftrightarrow \quad \begin{Bmatrix} a \\ b \end{Bmatrix}=\begin{bmatrix} 4.81 & 4.89 \\ 4.89 & 5.70 \end{bmatrix}^{-1}\begin{Bmatrix} 12.77 \\ 13.36 \end{Bmatrix}=\begin{Bmatrix} 2.14 \\ 0.51 \end{Bmatrix}. \leftarrow$$

These values correspond to $\nu=0.1$ and $\nu=0.4$ when $L/R=1.14$. Interpolation gives the prediction for the coefficients when $\nu=0.32$, when $L/R=1.14$ (one may use, e.g., linear shape functions)

$$\begin{Bmatrix} a \\ b \end{Bmatrix}_{\nu}=\frac{\nu-0.4}{0.1-0.4}\begin{Bmatrix} a \\ b \end{Bmatrix}_{0.1}+\frac{\nu-0.1}{0.4-0.1}\begin{Bmatrix} a \\ b \end{Bmatrix}_{0.4} \quad \Rightarrow \quad \begin{Bmatrix} a \\ b \end{Bmatrix}_{0.32}=\begin{Bmatrix} 1.99 \\ 0.51 \end{Bmatrix}. \leftarrow$$

Simulation with Poisson's ratio $\nu = 0.32$ gives $\begin{Bmatrix} a \\ b \end{Bmatrix}_{0.32} = \begin{Bmatrix} 1.93 \\ 0.51 \end{Bmatrix}$.