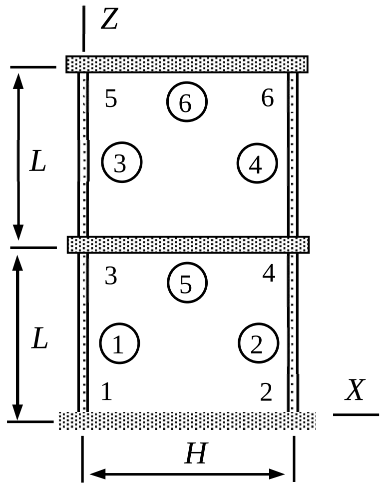
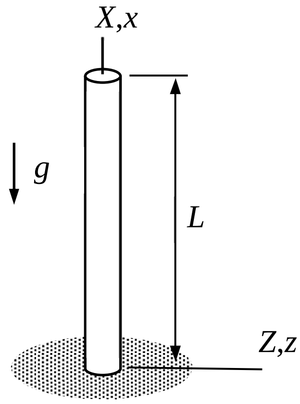
MEC-E8001 Finite Element Analysis, online exam 21.02.2024

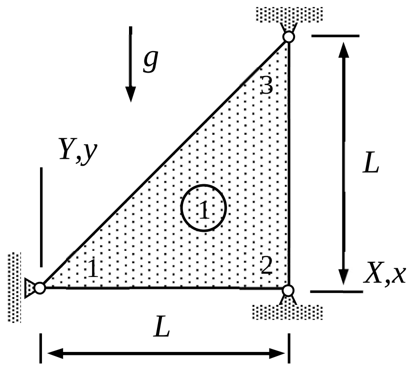
1. The XZ – plane model of a building shown consists of two rigid floors of mass m, each, and four columns modelled as massless bending beams. The structure is welded so the displacements and rotations of the floors and columns coincide at the contact points. Determine the angular speeds of the free horizontal vibrations of the structure. Young's modulus of the beam material and the second moment of the cross-section are E and I, respectively.



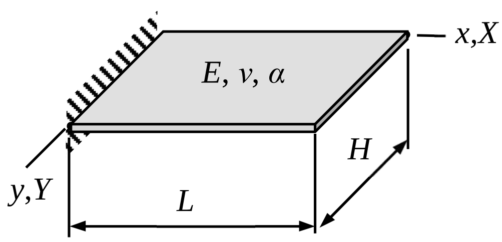
2. A building of height L is modelled as a beam loaded by its own weight. Cross-section of the building is idealized as a circle of radius r. Find the critical radius r_{cr} for the buckling to occur. Material properties ρ, E and acceleration by gravity g are constants. Use the approximations $u = a(x/L)$, $v = 0$ and $w = b(x/L)^2$, in which a and b are parameters of the approximations. Start with the virtual work densities for the beam model.



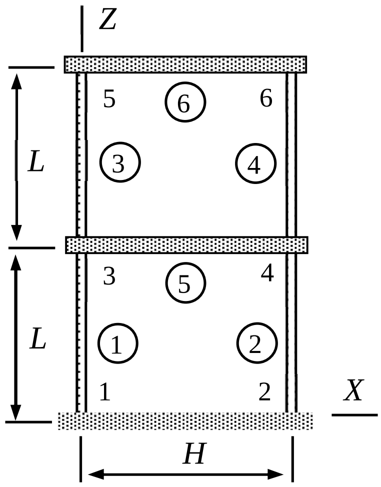
3. A thin triangular slab which is loaded by its own weight can move vertically at node 1 whereas nodes 2 and 3 are fixed. Assuming plane stress conditions, derive the equilibrium equation for the structure according to the large displacement theory. Material parameters C, ν, ρ and thickness t at the initial geometry of the slab are constants.



4. A bending plate of thickness t, which is clamped on one edge, is assembled at constant temperature $2\vartheta°$. Find the transverse displacement due to heating on the upper side $z = -t/2$ and cooling on the lower side $z = t/2$ resulting in surface temperatures $3\vartheta°$ and $\vartheta°$, respectively. Assume that temperature in plate is linear in z and does not depend on x or y. Use approximation $w = a_0 x^2$ in which a_0 is the parameter to be determined. Young's modulus E, Poisson's ratio ν, and coefficient of thermal expansion α are constants.



The XZ – plane model of a building shown consists of two rigid floors of mass m, each, and four columns modelled as massless bending beams. The structure is welded so the displacements and rotations of the floors and columns coincide at the contact points. Determine the angular speeds of the free horizontal vibrations of the structure. Young's modulus of the beam material and the second moment of the cross-section are E and I, respectively.



Solution

Beam bending and rigid body model virtual work expression of internal and inertia forces are available in the formulae collection. As beams are assumed to be massless, only the internal part is needed. For the floors, only the translation part applies as the floors translate in horizontal direction. Therefore

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI_{yy}}{h^3} \begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix}, \quad \delta W^{\text{ine}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{y1} \\ \delta u_{z1} \end{Bmatrix}^{\mathrm{T}} m \begin{Bmatrix} \ddot{u}_{x1} \\ \ddot{u}_{y1} \\ \ddot{u}_{z1} \end{Bmatrix}.$$

2p The non-zero displacement components of the structure are $u_{X4} = u_{X3}$ and $u_{X6} = u_{X5}$. Let us start with the element contributions of the beams. Since the beam are assumed to be massless, only the virtual work expressions of the internal forces (available in the formulae collection) are needed.

$$\delta W^1 = \delta W^2 = -\begin{Bmatrix} 0 \\ 0 \\ \delta u_{X3} \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_{X3} \\ 0 \end{Bmatrix} = -\begin{Bmatrix} \delta u_{X3} \\ \delta u_{X5} \end{Bmatrix}^{\mathrm{T}} 12\frac{EI}{L^3} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} u_{X3} \\ u_{X5} \end{Bmatrix}$$

$$\delta W^3 = \delta W^4 = -\begin{Bmatrix} \delta u_{X3} \\ 0 \\ \delta u_{X5} \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} u_{X3} \\ 0 \\ u_{X5} \\ 0 \end{Bmatrix} = -\begin{Bmatrix} \delta u_{X3} \\ \delta u_{X5} \end{Bmatrix}^{\mathrm{T}} 12\frac{EI}{L^3} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{X3} \\ u_{X5} \end{Bmatrix}$$

Then the particles. Element contribution of the rigid body (formulae collection) simplifies to

$$\delta W^5 = -\begin{Bmatrix} \delta u_{X3} \\ 0 \\ 0 \end{Bmatrix}^{\mathrm{T}} m \begin{Bmatrix} \ddot{u}_{X3} \\ 0 \\ 0 \end{Bmatrix} = -\begin{Bmatrix} \delta u_{X3} \\ \delta u_{X5} \end{Bmatrix}^{\mathrm{T}} m \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{u}_{X3} \\ \ddot{u}_{X5} \end{Bmatrix},$$

$$\delta W^6 = -\begin{Bmatrix} \delta u_{X5} \\ 0 \\ 0 \end{Bmatrix}^{\mathrm{T}} m \begin{Bmatrix} \ddot{u}_{X5} \\ 0 \\ 0 \end{Bmatrix} = -\begin{Bmatrix} \delta u_{X3} \\ \delta u_{X5} \end{Bmatrix}^{\mathrm{T}} m \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{u}_{X3} \\ \ddot{u}_{X5} \end{Bmatrix}.$$

2p Virtual work expression of structure is the sum of element contributions.

$$\delta W = \sum \delta W^i = -\begin{Bmatrix} \delta u_{X3} \\ \delta u_{X5} \end{Bmatrix}^{\mathrm{T}} (24\frac{EI}{L^3}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X3} \\ u_{X5} \end{Bmatrix} + m\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{u}_{X3} \\ \ddot{u}_{X5} \end{Bmatrix}).$$

Principle of virtual work and the fundamental lemma of variation calculus imply a set of ordinary differential equations:

$$24\frac{EI}{L^3}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X3} \\ u_{X5} \end{Bmatrix} + m\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{u}_{X3} \\ \ddot{u}_{X5} \end{Bmatrix} = 0 \quad \leftarrow$$

or written in the standard form

$$\begin{Bmatrix} \ddot{u}_{X3} \\ \ddot{u}_{X5} \end{Bmatrix} + \mathbf{\Omega}^2 \begin{Bmatrix} u_{X3} \\ u_{X5} \end{Bmatrix} = 0, \text{ where } \mathbf{\Omega}^2 = 24\frac{EI}{mL^3}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

2p The angular speeds of free vibrations are the eigenvalues of matrix $\mathbf{\Omega}$. The easiest way to find the eigenvalues uses the result that the eigenvalues of $\mathbf{\Omega}$ are square roots of those for $\mathbf{\Omega}^2$. Let us consider first the eigenvalues of $\mathbf{\Omega}^2$

$$\det(\mathbf{\Omega}^2 - \lambda\mathbf{I}) = \det(24\frac{EI}{mL^3}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = 0 \;\Rightarrow$$

$$\det(\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \gamma\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = (2-\gamma)(1-\gamma) - 1 = 0 \quad \text{where} \quad \gamma = \frac{1}{24}\frac{mL^3}{EI}\lambda.$$

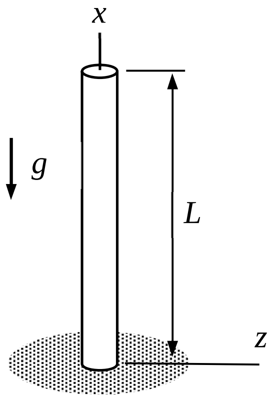
The roots are

$$\gamma = \frac{3 \pm \sqrt{5}}{2} \quad \text{so} \quad \lambda = 24\frac{EI}{mL^3}\gamma = 24\frac{EI}{mL^3}\frac{3 \pm \sqrt{5}}{2}.$$

Eigenvalues of $\mathbf{\Omega}$ are square roots of eigenvalues of $\mathbf{\Omega}^2$

$$\omega_1 = \sqrt{\lambda_1} = \sqrt{12\frac{EI}{mL^3}(3+\sqrt{5})} \quad \text{and} \quad \omega_2 = \sqrt{\lambda_2} = \sqrt{12\frac{EI}{mL^3}(3-\sqrt{5})}. \quad \leftarrow$$

A building of height L is modelled as a beam loaded by its own weight. Cross-section of the building is idealized as a circle of radius r. Find the critical radius r_{cr} for the buckling to occur. Material properties ρ, E and acceleration by gravity g are constants. Use the approximations $u = a(x/L)$, $v = 0$ and $w = b(x/L)^2$, in which a and b are parameters of the approximations. Start with the virtual work densities for the beam model.



Solution

1p Virtual work densities

$$\delta w_{\Omega}^{\text{int}} = -\frac{d\delta u}{dx} EA \frac{du}{dx}, \quad \delta w_{\Omega}^{\text{ext}} = \delta u f_x ,$$

$$\delta w_{\Omega}^{\text{int}} = -\frac{d^2\delta w}{dx^2} EI_{yy} \frac{d^2 w}{dx^2} , \quad \delta w_{\Omega}^{\text{sta}} = -\frac{d\delta w}{dx} N \frac{dw}{dx} \text{ , where } N = EA\frac{du}{dx}$$

take into account the bar and bending modes and their interaction.

2p As the connection in bar and bending modes is one-way, let us start with the bar mode where the external distributed force due to gravity $f_x = -\rho g A$ and $u = a(x/L)$. Virtual work expression

$$\delta W = \int_0^L (\delta w_{\Omega}^{\text{int}} + \delta w_{\Omega}^{\text{ext}}) dx = \int_0^L (-\frac{\delta a}{L} EA \frac{a}{L} - \delta a \frac{x}{L} \rho g A) dx = -\delta a (\frac{EA}{L} a + \frac{1}{2} \rho g L A)$$

and principle of virtual work $\delta W = 0 \quad \forall \delta a$ and the fundamental lemma of variational calculation give

$$a = -\frac{1}{2}\frac{\rho g L^2}{E} \quad \text{hence } \; u = a\frac{x}{L} = -\frac{1}{2}\frac{\rho g L}{E} x \; \text{ and } \; N = EA\frac{du}{dx} = -\frac{1}{2}\rho g L A \; \text{ (constant).}$$

3p The bending mode, composed of the internal and coupling parts, and approximation $w = b(x/L)^2$ give the virtual work expression

$$\delta W = \int_0^L (\delta w_{\Omega}^{\text{int}} + \delta w_{\Omega}^{\text{sta}}) dx = \int_0^L (-2\frac{\delta b}{L^2} EI_{yy} 2\frac{b}{L^2} - 2\frac{\delta b}{L^2} x N 2 \frac{b}{L^2} x) dx = -4\delta b (\frac{EI_{yy}}{L^3} - \frac{1}{6}\rho g A) b .$$

Principle of virtual work $\delta W = 0 \quad \forall \delta b$ and the fundamental lemma of variational calculation give the equilibrium equation

$$(\frac{EI_{yy}}{L^3} - \frac{1}{6}\rho g A) b = 0 .$$

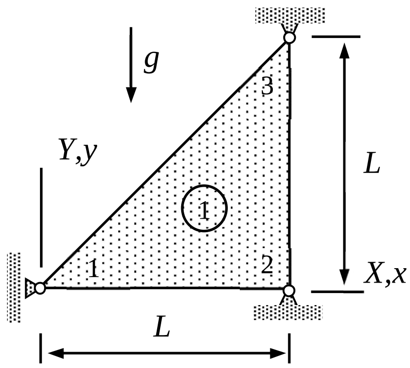
In stability analysis, the goal is to find the condition under which the solution becomes non-unique. Clearly, this is possibly only if

$$\frac{EI_{yy}}{L^3} - \frac{1}{6}\rho g A = 0 .$$

Finally let us substitute the cross-section moments $A = \pi r^2$, $I_{yy} = \pi r^4 / 4$, and solve for the critical value of radius in terms of the other parameters of the problem

$$r_{cr} = \sqrt{\frac{2}{3}\frac{\rho g L^3}{E}} .$$

A thin triangular slab which is loaded by its own weight can move vertically at node 1 whereas nodes 2 and 3 are fixed. Assuming plane stress conditions, derive the equilibrium equation for the structure according to the large displacement theory. Material parameters C, ν, ρ and thickness t at the initial geometry of the slab are constants.



Solution

According to the large displacement theory, virtual work densities of the thin slab model under plane strain conditions are

$$\delta w_{\Omega^\circ}^{\text{int}} = -\begin{Bmatrix} \delta E_{xx} \\ \delta E_{yy} \\ 2\delta E_{xy} \end{Bmatrix}^{\mathrm{T}} \frac{tC}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\begin{Bmatrix} E_{xx} \\ E_{yy} \\ 2E_{xy} \end{Bmatrix}, \quad \begin{Bmatrix} E_{xx} \\ E_{yy} \\ 2E_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x}+\dfrac{1}{2}(\dfrac{\partial u}{\partial x})^2+\dfrac{1}{2}(\dfrac{\partial v}{\partial x})^2 \\ \dfrac{\partial v}{\partial y}+\dfrac{1}{2}(\dfrac{\partial u}{\partial y})^2+\dfrac{1}{2}(\dfrac{\partial v}{\partial y})^2 \\ \dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x}+\dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x}\dfrac{\partial v}{\partial y} \end{Bmatrix}.$$

$$\delta w_{\Omega^\circ}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} t\rho^\circ \begin{Bmatrix} g_x \\ g_y \end{Bmatrix}$$

in which g_x and g_y are the components of acceleration by gravity and ρ° the density at the initial geometry. Above, constitutive equation is assumed to be of the same form as that for the linear theory with possibly different elasticity parameters C and ν.

1p Shape function $N_1 = 1 - x/L$ of node 1 can be deduced from the figure. Linear approximations to the displacement components and their derivatives are (with $a = u_{Y1}/L$)

$$u = 0 \text{ and } v = (1-\frac{x}{L})u_{Y1} = (L-x)a \quad \Rightarrow \quad \frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial x} = -a, \text{ and } \frac{\partial v}{\partial y} = 0.$$

2p When the approximation is substituted into the Green-Lagrange strain component vector and that is used in the virtual work densities

$$\delta w_{\Omega^\circ}^{\text{int}} = -\begin{Bmatrix} a\delta a \\ 0 \\ -\delta a \end{Bmatrix}^{\mathrm{T}} \frac{tC}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\begin{Bmatrix} a^2/2 \\ 0 \\ -a \end{Bmatrix} = -\delta a \frac{tC}{1-\nu^2}\frac{1}{2}[a^3 + (1-\nu)a],$$

$$\delta w_{\Omega^\circ}^{\text{ext}} = -t\rho g \delta v = -t\rho g (L-x)\delta a.$$

2p Integration over the domain occupied by the body at the initial geometry gives the virtual work expressions

$$\delta W^{\text{int}} = -\delta a \frac{tC}{1-\nu^2} \frac{L^2}{4} [a^3 + (1-\nu)a] ,$$

$$\delta W^{\text{ext}} = \int_0^L \left(\int_0^x \delta w_{\Omega^\circ}^{\text{ext}} dy \right) dx = \int_0^L \left(\int_0^x -t\rho g (L-x) \delta a dy \right) dx = -\delta a t \rho g \frac{1}{6} L^3 .$$

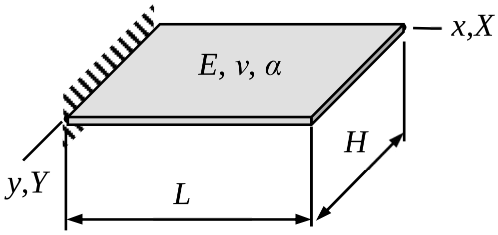
1p Virtual work expression in the sum of the internal and external parts. Written in the standard form

$$\delta W = -\delta a \left[\frac{tC}{1-\nu^2} \frac{L^2}{4} (a^3 + a - \nu a) + t\rho g \frac{1}{6} L^3 \right] .$$

Principle of virtual work and the fundamental lemma of variation calculus imply the equilibrium equation

$$\frac{tC}{1-\nu^2} \frac{L^2}{4} (a^3 + a - \nu a) + t\rho g \frac{1}{6} L^3 = 0 \quad \text{where} \quad a = \frac{u_{Y1}}{L} .$$

A bending plate of thickness t, which is clamped on one edge, is assembled at constant temperature $2\vartheta°$. Find the transverse displacement due to heating on the upper side $z=-t/2$ and cooling on the lower side $z=t/2$ resulting in surface temperatures $3\vartheta°$ and $\vartheta°$, respectively. Assume that temperature in plate is linear in z and does not depend on x or y. Use approximation $w=a_0 x^2$ in which a_0 is the parameter to be determined. Young's modulus E, Poisson's ratio ν, and coefficient of thermal expansion α are constants.



Solution

Assuming that the material coordinate system is chosen so that the plate bending and thin slab modes decouple, the plate model virtual work densities of internal force and coupling terms are given by

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial^2 \delta w / \partial x^2 \\ \partial^2 \delta w / \partial y^2 \\ 2\partial^2 \delta w / \partial x \partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12} \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{Bmatrix} \partial^2 w / \partial x^2 \\ \partial^2 w / \partial y^2 \\ 2\partial^2 w / \partial x \partial y \end{Bmatrix},$$

$$\delta w_{\Omega}^{\text{cpl}} = -\begin{Bmatrix} \partial^2 \delta w / \partial x^2 \\ \partial^2 \delta w / \partial y^2 \end{Bmatrix}^{\mathrm{T}} \int z \Delta \vartheta dz \frac{\alpha E}{1-\nu} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

3p The coupling term contains an integral of temperature change over the thickness of the plate. First, as temperature is linear

$$\vartheta_{\text{asm}}(z) = 2\vartheta° \quad \text{and} \quad \vartheta_{\text{fin}}(z) = 2\vartheta°(1-\frac{z}{t}) \quad \Rightarrow \quad \Delta\vartheta = \vartheta_{\text{fin}} - \vartheta_{\text{asm}} = -2\vartheta° \frac{z}{t}.$$

So the integral of the coupling term becomes

$$\int z \Delta \vartheta dz = -2\frac{\vartheta°}{t} \int_{-t/2}^{t/2} z^2 dz = -2\frac{\vartheta°}{t} \frac{1}{3}(\frac{t^3}{8} + \frac{t^3}{8}) = -\frac{1}{6} \vartheta° t^2.$$

and virtual work density for the coupling part

$$\delta w_{\Omega}^{\text{cpl}} = \begin{Bmatrix} 2\delta \mathrm{a} \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{1}{6} \vartheta° t^2 \frac{\alpha E}{1-\nu} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \delta \mathrm{a} \frac{1}{3} \vartheta° t^2 \frac{\alpha E}{1-\nu}.$$

Virtual work expression is integral of the density over the domain occupied by the plate/element. As integrand is constants, it is enough to multiply by the area of the mid-plane to get

$$\delta W^{\text{cpl}} = \int_\Omega \delta w_\Omega^{\text{cpl}} dA = LH\delta \text{a} \frac{1}{3} \vartheta^\circ t^2 \frac{\alpha E}{1-\nu}.$$

2p Approximation to the transverse displacement

$$w = \text{a}\, x^2 \quad \Rightarrow \quad \frac{\partial^2 w}{\partial x^2} = 2\text{a} \text{ and } \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial x \partial y} = 0.$$

When the approximation is substituted there, virtual work density of the internal forces simplifies to

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} 2\delta\text{a} \\ 0 \\ 0 \end{Bmatrix}^{\text{T}} \frac{t^3}{12} \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{Bmatrix} 2\text{a} \\ 0 \\ 0 \end{Bmatrix} = -\delta\text{a} \frac{t^3}{3} \frac{E}{1-\nu^2} \text{a}.$$

Virtual work expression is integral of the density over the domain occupied by the plate/element. As integrand is constants, it is enough to multiply by the area of the mid-plane to get

$$\delta W^{\text{int}} = \int_\Omega \delta w_\Omega^{\text{int}} dA = -LH\delta\text{a} \frac{t^3}{3} \frac{E}{1-\nu^2} \text{a}.$$

1p Virtual work expression is the sum of the parts

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{cpl}} = -\delta\text{a}(LH \frac{t^3}{3} \frac{E}{1-\nu^2} \text{a} - LH \frac{1}{3} \vartheta^\circ t^2 \frac{\alpha E}{1-\nu}).$$

Principle of virtual work $\delta W = 0 \; \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus give

$$\delta W = -\delta\text{a}(LH \frac{t^3}{3} \frac{E}{1-\nu^2} \text{a} - LH \frac{1}{3} \vartheta^\circ t^2 \frac{\alpha E}{1-\nu}) = 0 \quad \forall \delta\text{a} \quad \Leftrightarrow$$

$$LH \frac{t^3}{3} \frac{E}{1-\nu^2} \text{a} - LH \frac{1}{3} \vartheta^\circ t^2 \frac{\alpha E}{1-\nu} = 0 \quad \Rightarrow \quad \text{a} = (1+\nu) \frac{1}{t} \alpha \vartheta^\circ.$$

Transverse displacement $w = (1+\nu)\alpha\vartheta^\circ \frac{1}{t} x^2$. ←