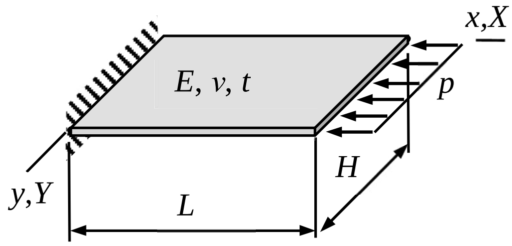
Name__________________________ Student number__________________

Assignment 5

The clamping of the plate shown allows displacement in $y-$direction. At the free edge, the plate is loaded by distributed force p. Determine the critical value p_{cr} of the distributed force making the plate to buckle. Use the approximation $w(x,y)=a_0(x/L)^2$ and assume that $N_{xx}=-p$ and $N_{yy}=N_{xy}=0$. Material parameters E, ν and thickness of the plate t are constants.



Solution

Assuming that the material coordinate system is chosen so that the plate bending and thin slab modes decouple in the linear analysis and that the in-plane stress resultants are known (from linear displacement analysis, say), it is enough to consider the virtual work densities of plate bending mode and the coupling of the bending and thin-slab modes

$$\delta w_{\Omega}^{\mathrm{int}}=-\begin{Bmatrix}\partial^2\delta w/\partial x^2\\ \partial^2\delta w/\partial y^2\\ 2\partial^2\delta w/\partial x\partial y\end{Bmatrix}^{\mathrm{T}}\frac{t^3}{12}[E]_{\sigma}\begin{Bmatrix}\partial^2 w/\partial x^2\\ \partial^2 w/\partial y^2\\ 2\partial^2 w/\partial x\partial y\end{Bmatrix},\ \delta w_{\Omega}^{\mathrm{sta}}=-\begin{Bmatrix}\partial\delta w/\partial x\\ \partial\delta w/\partial y\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}N_{xx} & N_{xy}\\ N_{yx} & N_{yy}\end{bmatrix}\begin{Bmatrix}\partial w/\partial x\\ \partial w/\partial y\end{Bmatrix}$$

where the elasticity matrix of plane stress

$$[E]_{\sigma}=\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & (1-\nu)/2\end{bmatrix}.$$

As the support at the clamped edge allows displacement in the $y-$direction, solution to the in-plane stress resultants $N_{xx}=-p$ and $N_{yy}=N_{xy}=0$ can be deduced without calculations. Approximation to the transverse displacement and its non-zero derivatives are given by

$$w(x,y)=a_0(\frac{x}{L})^2 \quad\Rightarrow\quad \frac{\partial w}{\partial x}=2a_0\frac{x}{L^2} \text{ and } \frac{\partial^2 w}{\partial x^2}=2\frac{a_0}{L^2}.$$

When the approximation is substituted there, virtual work density of the internal forces and that of the coupling simplify to

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} 2\delta a_0 / L^2 \\ 0 \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12} \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} 2a_0 / L^2 \\ 0 \\ 0 \end{Bmatrix} = -\delta a_0 \frac{1}{3} \frac{t^3}{L^4} \frac{E}{1-\nu^2} a_0 ,$$

$$\delta w_{\Omega}^{\text{sta}} = -\begin{Bmatrix} 2\delta a_0 x / L^2 \\ 0 \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} N_{xx} & N_{xy} \\ N_{yx} & N_{yy} \end{bmatrix} \begin{Bmatrix} 2a_0 x / L^2 \\ 0 \end{Bmatrix} = \delta a_0 4x^2 \frac{p}{L^4} a_0 .$$

Virtual work expressions are integrals of the densities over the domain occupied by the plate

$$\delta W^{\text{int}} = \int_0^H \int_0^L \delta w_{\Omega}^{\text{int}} dxdy = -\delta a_0 \frac{1}{3} \frac{t^3}{L^3} H \frac{E}{1-\nu^2} a_0 ,$$

$$\delta W^{\text{sta}} = \int_0^H \int_0^L \delta w_{\Omega}^{\text{sta}} dxdy = \delta a_0 \frac{4}{3} \frac{H}{L} p a_0 .$$

Virtual work expression

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{sta}} = -\delta a_0 \left(\frac{1}{3} \frac{t^3}{L^3} \frac{H}{L} \frac{E}{1-\nu^2} - \frac{4}{3} \frac{H}{L} p\right) a_0 ,$$

principle of virtual work $\delta W = 0 \; \forall \delta a_0$, and the fundamental lemma of variation calculus give

$$\left(\frac{1}{3} \frac{t^3}{L^3} \frac{H}{L} \frac{E}{1-\nu^2} - \frac{4}{3} \frac{H}{L} p\right) a_0 = 0 .$$

For a non-trivial solution $a_0 \neq 0$, the loading parameter needs to take the value

$$p_{\text{cr}} = \frac{1}{4} \frac{E}{1-\nu^2} \frac{t^3}{L^2} . \quad \leftarrow$$