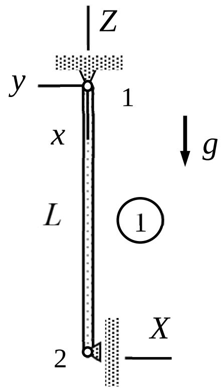
Name______________________________ Student number____________________

Assignment 3



A bar is loaded by its own weight as shown in the figure. Determine the equilibrium equation in terms of the dimensionless displacement $\mathrm{a} = u_{Z2}/L$ with the large deformation theory. Without external loading, area of the cross-section, length of the bar, and density of the material are A, L, and ρ, respectively. Young's modulus of the material is C. Also find the displacement according to the linear theory by simplifying the equilibrium equation with the assumption $|\mathrm{a}| \ll 1$.

Solution template

Virtual work densities of the non-linear bar model

$$\delta w_{\Omega^\circ}^{\text{int}} = -\left(\frac{d\delta u}{dx} + \frac{du}{dx}\frac{d\delta u}{dx} + \frac{dv}{dx}\frac{d\delta v}{dx} + \frac{dw}{dx}\frac{d\delta w}{dx}\right) C A^\circ \left[\frac{du}{dx} + \frac{1}{2}\left(\frac{du}{dx}\right)^2 + \frac{1}{2}\left(\frac{dv}{dx}\right)^2 + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right],$$

$$\delta w_{\Omega^\circ}^{\text{ext}} = A^\circ \rho^\circ (\delta u g_x + \delta v g_y + \delta w g_z)$$

are based on the Green-Lagrange strain definition, which works also when rotations/displacements are large. The expressions depend on all displacement components, material property is denoted by C (kind of Young's modulus), and the superscript in the cross-sectional area A° (and in other quantities) refers to the initial geometry where strain and stress vanish.

The non-zero displacement component of the structure is the vertical displacement of node 2. Linear approximations to the displacement components in terms of the displacement/rotation components of the structural system are

$$u = -\frac{x}{L} u_{Z2} \quad \text{and} \quad v = w = 0 \quad \Rightarrow \quad \frac{du}{dx} = -\frac{u_{Z2}}{L} \quad \text{and} \quad \frac{dv}{dx} = \frac{dw}{dx} = 0\,.$$

In terms of the dimensionless displacement $\mathrm{a} = u_{Z2}/L$ and its variation $\delta \mathrm{a} = \delta u_{Z2}/L$, virtual work densities simplify to

$$\delta w_{\Omega^\circ}^{\text{int}} = -\delta \mathrm{a}(-1+\mathrm{a}) C A (-\mathrm{a} + \frac{1}{2}\mathrm{a}^2),$$

$$\delta w_{\Omega^\circ}^{\text{ext}} = -\frac{x}{L} \delta u_{Z2} A \rho g = -\delta \mathrm{a} x A \rho g\,.$$

Virtual work expressions are integrals of the densities over the domain occupied by the element

$$\delta W = \int_0^L (\delta w_{\Omega^\circ}^{\text{int}} + \delta w_{\Omega^\circ}^{\text{ext}})dx = -\delta \mathrm{a}[(-1+\mathrm{a})CAL(-\mathrm{a}+\frac{1}{2}\mathrm{a}^2)+\frac{1}{2}L^2 A\rho g].$$

Principle of virtual work and the fundamental lemma of variation calculus imply that

$$C(-2\mathrm{a}+\mathrm{a}^2)(-1+\mathrm{a})+L\rho g = 0 \quad \text{in which } \mathrm{a} = \frac{u_{Z2}}{L}.$$

Assuming that $|\mathrm{a}| \ll 1$, only the linear part matters and the equilibrium equation simplifies to

$$CAL\mathrm{a}+\frac{1}{2}L^2 A\rho g = 0 \quad \Rightarrow \quad \mathrm{a} = -\frac{1}{2}\frac{L\rho g}{C}.$$