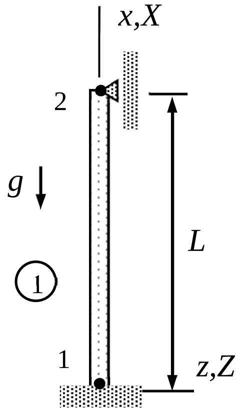
Name________________________ Student number________________

Assignment 4



A beam is loaded by its own weight as shown in the figure. Assume that displacement is confined to the $XZ-$plane. Derive the equilibrium equations for buckling analysis giving the axial displacement and the critical density ρ_{cr} of the material. Start with the virtual work density and approximations to the axial and transverse displacements. The cross-section properties A, I and material properties E, ρ are constants.

Solution template

Virtual work expressions for the buckling analysis of a beam in $xz-$plane consist of the internal parts for the bar and bending modes, coupling (stability expression) between them, and virtual work of the external point force. Altogether ($f_x=-\rho Ag$)

$$\delta w_\Omega=-\frac{d^2\delta w}{dx^2}EI_{yy}\frac{d^2w}{dx^2}-\frac{d\delta u}{dx}EA\frac{du}{dx}-\frac{d\delta w}{dx}N\frac{dw}{dx}+\delta u f_x,\ \text{where}\ N=EA\frac{du}{dx}.$$

In terms of the non-zero displacement/rotation components of the structural system, approximations to the axial displacement u, transverse displacement w, and the axial force N simplify to

$$u(x)=\begin{Bmatrix}1-x/L\\ x/L\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\_\_\_\_\\ \_\_\_\_\end{Bmatrix}=\_\_\_\_\_\_\_\_\_\_\_\_,$$

$$w(x)=\begin{Bmatrix}(1-x/L)^2(1+2x/L)\\ L(1-x/L)^2x/L\\ (3-2x/L)(x/L)^2\\ L(x/L)^2(x/L-1)\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\_\_\_\_\\ \_\_\_\_\\ \_\_\_\_\\ \_\_\_\_\end{Bmatrix}=\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_,$$

$$N=EA\frac{du}{dx}=\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_.$$

When the approximations are substituted there, virtual work density simplifies to (substitute the expression for the axial force N and distributed force f_x)

$$\delta w_\Omega=\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

$$\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Integration over the length of the beam gives

$$\delta W = \int_0^L \delta w_\Omega dx = \underline{\hspace{10cm}}$$

$$\underline{\hspace{10cm}} \quad \Leftrightarrow$$

$$\delta W = -\begin{Bmatrix} \delta u_{X2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \underline{\hspace{6cm}} \\ \underline{\hspace{6cm}} \end{Bmatrix}.$$

Principle of virtual work and the fundamental lemma of variation calculus imply equilibrium equations

$$\begin{Bmatrix} \underline{\hspace{6cm}} \\ \underline{\hspace{6cm}} \end{Bmatrix} = 0.$$

The first equation is linear and can be solved for the axial displacement

$$\underline{\hspace{5cm}} = 0 \quad \Leftrightarrow \quad u_{X2} = \underline{\hspace{4cm}}.$$

When the solution to the axial displacement is substituted there, the second (non-linear) equation simplifies to

$$(\underline{\hspace{5cm}})\theta_{Y2} = 0.$$

The remaining task is to deduce the possible solutions: If the expression in parenthesis is non-zero, the equation implies that $\theta_{Y2} = 0$. If the expression in parenthesis is zero, the equation is satisfied no matter the non-zero value of θ_{Y2}. Therefore, buckling may occur when (here density ρ stands for the loading parameter)

$$\underline{\hspace{5cm}} = 0 \quad \Leftrightarrow \quad \rho_{\mathrm{cr}} = \underline{\hspace{5cm}}.$$