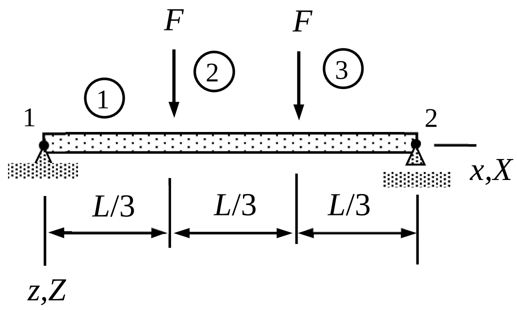
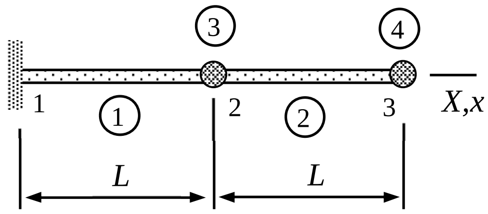
MEC-E8001 Finite Element Analysis, Online exam 22.02.2023

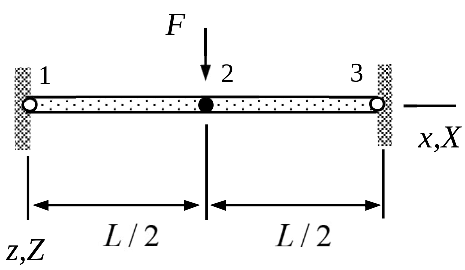
1. Find the transverse displacement $w(x)$ of the structure consisting of *one* beam element of cubic approximation and point forces 2 and 3. The rotations of the endpoints are assumed to be equal in magnitudes but opposite in directions, i.e., $\theta_{Y2} = -\theta_{Y1}$. Problem parameters E and $I_{yy} = I$ are constants.



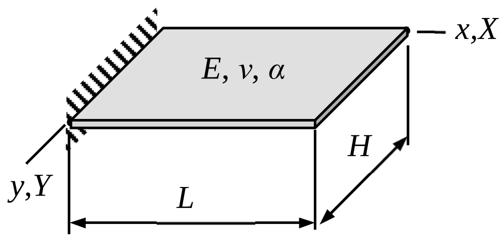
2. The bar structure shown consists of two *massless* bars and two particles of mass m, each. Find the angular speeds of the free axial vibrations of the structure. Young's modulus of the material and the cross-sectional area of the bars are E and A, respectively.



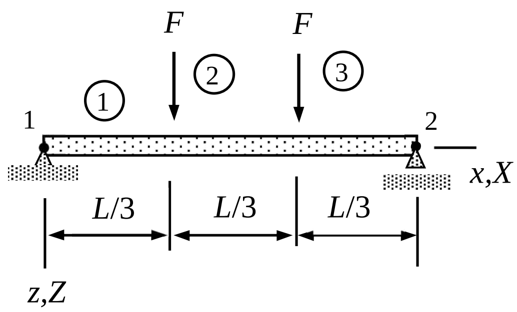
3. Find the relationship between the force F acting on the beam and the displacement u_{Z2} of its centerpoint. Combine the virtual work density expressions for the beam bending mode of *small (linear) displacement analysis* and that for the bar mode of *large displacement analysis.* Assume $u = v = 0$ and use the approximation $w = u_{Z2} 4(x/L)(1-x/L)$ for the transverse displacement. Assume that material parameter $C = E$ and cross-sectional area $A° = A$.



4. A bending plate of thickness t, which is clamped on one edge, is assembled at constant temperature $2\vartheta°$. Find the transverse displacement due to heating on the upper side $z = -t/2$ and cooling on the lower side $z = t/2$ resulting in surface temperatures $3\vartheta°$ and $\vartheta°$, respectively. Assume that temperature in plate is linear in z and does not depend on x or y. Use approximation $w = a_0 x^2$ in which a_0 is the parameter to be determined. Young's modulus E, Poisson's ratio ν, and coefficient of thermal expansion α are constants.



Find the transverse displacement $w(x)$ of the structure consisting of *one* beam element of cubic approximation and point forces 2 and 3. The rotations of the endpoints are assumed to be equal in magnitudes but opposite in directions, i.e., $\theta_{Y2} = -\theta_{Y1}$. Problem parameters E and $I_{yy} = I$ are constants.



Solution

Virtual work expression of the internal forces

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI_{yy}}{h^3} \begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix}$$

available in the formulae collection applies here. The contribution of the point forces (acting inside the element) follows from the definition of work as usually. However, the virtual displacement needs to be expressed in terms of the displacement and rotation of nodes by using the cubic approximation for bending.

3p Transverse displacements vanish at the nodes and rotations satisfy $\theta_{Y2} = -\theta_{Y1}$ so ($\xi = x/L$)

$$w = \begin{Bmatrix} (1-\xi)^2(1+2\xi) \\ L(1-\xi)^2\xi \\ (3-2\xi)\xi^2 \\ L\xi^2(\xi-1) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ -\theta_{Y1} \\ 0 \\ \theta_{Y1} \end{Bmatrix} = x(\frac{x}{L}-1)\theta_{Y1} \text{ and } \delta w = x(\frac{x}{L}-1)\delta\theta_{Y1}.$$

At the points of action of the forces 2 and 3, the virtual displacements are

$$\delta w(\frac{L}{3}) = \frac{L}{3}(\frac{1}{3}-1)\delta\theta_{Y1} = -\frac{2}{9}L\delta\theta_{Y1} \quad \text{and} \quad \delta w(\frac{2L}{3}) = \frac{2L}{3}(\frac{2}{3}-1)\delta\theta_{Y1} = -\frac{2}{9}L\delta\theta_{Y1}.$$

Therefore, the virtual work expression of the point forces

$$\delta W^{\text{ext}} = -\frac{2}{9}L\delta\theta_{Y1}F - \frac{2}{9}L\delta\theta_{Y1}F = -\delta\theta_{Y1}\frac{4}{9}LF .$$

2p Virtual work expression of the internal forces simplifies to

$$\delta W^{\text{int}} = -\begin{Bmatrix} 0 \\ \delta\theta_{Y1} \\ 0 \\ -\delta\theta_{Y1} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ \theta_{Y1} \\ 0 \\ -\theta_{Y1} \end{Bmatrix} = -\delta\theta_{Y1}4\frac{EI}{L}\theta_{Y1}.$$

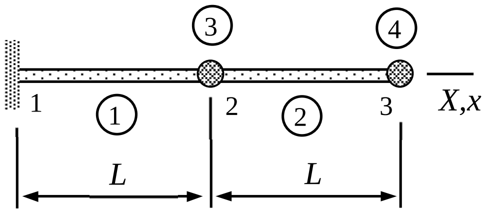
1p Principle of virtual work and the fundamental lemma of variation calculus imply

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = -\delta\theta_{Y1}(4\frac{EI}{L}\theta_{Y1} + \frac{4}{9}LF) = 0 \quad \Rightarrow \quad \theta_{Y1} = -\frac{1}{9}\frac{FL^2}{EI}.$$

Therefore, transverse displacement

$$w = -\frac{1}{9}\frac{FL^2}{EI}x(\frac{x}{L}-1). \quad \leftarrow$$

The bar structure shown consists of two *massless* bars and two particles of mass m, each. Find the angular speeds of the free axial vibrations of the structure. Young's modulus of the material and the cross-sectional area of the bars are E and A, respectively.



Solution

Bar and rigid body model virtual work expression of internal and inertia forces are available in the formulae collection. As bars are assumed to be massless, only the internal part is needed. For a particle, only translation part applies (moments of inertia are zeros). Therefore

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix}, \quad \delta W^{\text{ine}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{y1} \\ \delta u_{z1} \end{Bmatrix}^{\mathrm{T}} m \begin{Bmatrix} \ddot{u}_{x1} \\ \ddot{u}_{y1} \\ \ddot{u}_{z1} \end{Bmatrix}.$$

2p The non-zero displacement components of the structure are u_{X2} and u_{X3}. Let us start with the element contributions of the bars. Since the bars are assumed to be massless, only the virtual work expressions of the internal forces (available in the formulae collection) are needed.

$$\delta W^1 = -\begin{Bmatrix} 0 \\ \delta u_{X2} \end{Bmatrix}^{\mathrm{T}} \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ u_{X2} \end{Bmatrix} = -\delta u_{X2} \frac{EA}{L} u_{X2} = -\begin{Bmatrix} \delta u_{X2} \\ \delta u_{X3} \end{Bmatrix}^{\mathrm{T}} \frac{EA}{L} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix},$$

$$\delta W^2 = -\begin{Bmatrix} \delta u_{X2} \\ \delta u_{X3} \end{Bmatrix}^{\mathrm{T}} \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix}.$$

Then the two particle elements. Element contribution of the rigid body (formula collection) simplifies to

$$\delta W^3 = -\begin{Bmatrix} \delta u_{X2} \\ 0 \\ 0 \end{Bmatrix}^{\mathrm{T}} m \begin{Bmatrix} \ddot{u}_{X2} \\ 0 \\ 0 \end{Bmatrix} = -\delta u_{X2} m \ddot{u}_{X2} = -\begin{Bmatrix} \delta u_{X2} \\ \delta u_{X3} \end{Bmatrix}^{\mathrm{T}} m \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{u}_{X2} \\ \ddot{u}_{X3} \end{Bmatrix},$$

$$\delta W^4 = -\begin{Bmatrix} \delta u_{X3} \\ 0 \\ 0 \end{Bmatrix}^{\mathrm{T}} m \begin{Bmatrix} \ddot{u}_{X3} \\ 0 \\ 0 \end{Bmatrix} = -\delta u_{X3} m \ddot{u}_{X3} = -\begin{Bmatrix} \delta u_{X2} \\ \delta u_{X3} \end{Bmatrix}^{\mathrm{T}} m \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{u}_{X2} \\ \ddot{u}_{X3} \end{Bmatrix}$$

2p Virtual work expression of structure is the sum of element contributions.

$$\delta W = \delta W^1 + \delta W^2 + \delta W^3 + \delta W^4 = -\begin{Bmatrix} \delta u_{X2} \\ \delta u_{X3} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{L} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} + m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{u}_{X2} \\ \ddot{u}_{X3} \end{Bmatrix} \right).$$

Principle of virtual work and the fundamental lemma of variation calculus imply a set of ordinary differential equations:

$$\frac{EA}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} + m\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{u}_{X2} \\ \ddot{u}_{X3} \end{Bmatrix} = 0 \quad \Leftarrow$$

or written in the standard form

$$\begin{Bmatrix} \ddot{u}_{X2} \\ \ddot{u}_{X3} \end{Bmatrix} + \mathbf{\Omega}^2 \begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} = 0, \text{ where } \mathbf{\Omega}^2 = \frac{EA}{mL}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

2p The angular speeds of free vibrations are the eigenvalues of matrix $\mathbf{\Omega}$. The easiest way to find the eigenvalues uses the result that the eigenvalues of $\mathbf{\Omega}$ are square roots of those for $\mathbf{\Omega}^2$. Let us consider first the eigenvalues of $\mathbf{\Omega}^2$

$$\det(\mathbf{\Omega}^2 - \lambda \mathbf{I}) = \det(\frac{EA}{mL}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = 0 \Rightarrow$$

$$\det(\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \gamma \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = (2-\gamma)(1-\gamma) - 1 = 0 \quad \text{where} \quad \gamma = \frac{mL}{EA}\lambda .$$

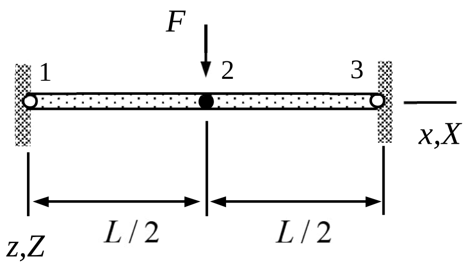
The roots are

$$\gamma = \frac{3 \pm \sqrt{5}}{2} \quad \text{so} \quad \lambda = \frac{EA}{mL}\gamma = \frac{EA}{mL}\frac{3 \pm \sqrt{5}}{2}.$$

Eigenvalues of $\mathbf{\Omega}$ are square roots of the eigenvalues of $\mathbf{\Omega}^2$

$$\omega_1 = \sqrt{\lambda_1} = \sqrt{\frac{EA}{mL}\frac{3+\sqrt{5}}{2}} \quad \text{and} \quad \omega_2 = \sqrt{\lambda_2} = \sqrt{\frac{EA}{mL}\frac{3-\sqrt{5}}{2}}. \quad \Leftarrow$$

Find the relationship between the force F acting on the beam and the displacement u_{Z2} of its centerpoint. Combine the virtual work density expressions for the beam bending mode of *small (linear) displacement analysis* and that for the bar mode of *large displacement analysis*. Assume $u = v = 0$ and use the approximation $w = u_{Z2} 4(x/L)(1-x/L)$ for the transverse displacement. Assume that material parameter $C = E$ and cross-sectional area $A° = A$.



Solution

Virtual work density expressions for the beam bending mode for the small displacement analysis and that for the bar mode for large displacement analysis are given by

$$\delta w_{\Omega}^{\text{int}} = -\frac{d^2\delta w}{dx^2} EI_{yy} \frac{d^2 w}{dx^2},$$

$$\delta w_{\Omega°}^{\text{int}} = -(\frac{d\delta u}{dx} + \frac{du}{dx}\frac{d\delta u}{dx} + \frac{dv}{dx}\frac{d\delta v}{dx} + \frac{dw}{dx}\frac{d\delta w}{dx}) CA°[\frac{du}{dx} + \frac{1}{2}(\frac{du}{dx})^2 + \frac{1}{2}(\frac{dv}{dx})^2 + \frac{1}{2}(\frac{dw}{dx})^2].$$

The latter is based on the Green-Lagrange strain definition which is physically correct also when rotations/displacements are large and considers also the off-axis displacement components.

1p The quadratic approximation to the transverse displacement gives

$$w = u_{Z2} 4\frac{x}{L}(1-\frac{x}{L}) = \frac{u_{Z2}}{L^2} 4(Lx - x^2) \;\Rightarrow\; \frac{dw}{dx} = \frac{u_{Z2}}{L^2} 4(L-2x) \;\Rightarrow\; \frac{d^2 w}{dx^2} = -8\frac{u_{Z2}}{L^2} \quad \text{and}$$

$$\delta w = \frac{\delta u_{Z2}}{L^2} 4(Lx - x^2) \;\Rightarrow\; \frac{d\delta w}{dx} = \frac{\delta u_{Z2}}{L^2} 4(L-2x) \;\Rightarrow\; \frac{d^2\delta w}{dx^2} = -8\frac{\delta u_{Z2}}{L^2}.$$

3p With the assumptions $u = v = 0$, $C = E$ and $A° = A$, virtual work density of the internal forces simplifies to

$$\delta w^{\text{int}} = -\frac{d^2\delta w}{dx^2} EI \frac{d^2 w}{dx^2} - \frac{d\delta w}{dx}\frac{1}{2} EA(\frac{dw}{dx})^3.$$

Substituting into the density expression and integration over the element gives the virtual work expression for internal forces

$$\delta w^{\text{int}} = -\frac{\delta u_{Z2}}{L^2} 64EI \frac{u_{Z2}}{L^2} - \frac{\delta u_{Z2}}{L^2} 128EA(\frac{u_{Z2}}{L^2})^3 (L-2x)^4 \quad \Rightarrow$$

$$\delta W^{\text{int}} = \int_0^L \delta w^{\text{int}} dx = -\frac{\delta u_{Z2}}{L} 64[\frac{EI}{L}(\frac{u_{Z2}}{L}) + \frac{2}{5} EAL(\frac{u_{Z2}}{L})^3]. \quad (\int_0^L (L-2x)^4 dx = \frac{L^5}{5})$$

1p Virtual work for the external force

$$\delta W^{\text{ext}} = \delta u_{Z2} F = (\frac{\delta u_{Z2}}{L})64\frac{LF}{64} .$$

1p Virtual work expression

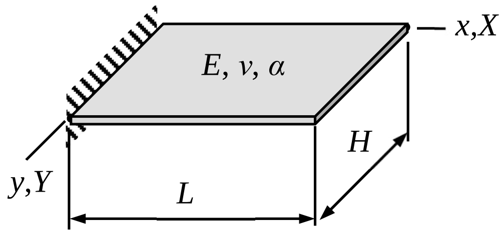
$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = -\frac{\delta u_{Z2}}{L} 64[\frac{EI}{L}(\frac{u_{Z2}}{L}) + \frac{2}{5} EAL(\frac{u_{Z2}}{L})^3 - \frac{LF}{64}] .$$

Principle of virtual work and the fundamental lemma of variational calculus imply the equilibrium equations

$$\frac{EI}{L}(\frac{u_{Z2}}{L}) + \frac{2}{5} EAL(\frac{u_{Z2}}{L})^3 - \frac{LF}{64} = 0 \quad \text{or} \quad (\text{with} \quad a = \frac{u_{Z2}}{L})$$

$$\frac{EI}{L} a + \frac{2}{5} EALa^3 - \frac{LF}{64} = 0 \quad \Leftrightarrow \quad a + \frac{2}{5}\frac{AL^2}{I} a^3 = \frac{1}{64}\frac{FL^2}{EI} .$$

A bending plate of thickness t, which is clamped on one edge, is assembled at constant temperature $2\vartheta°$. Find the transverse displacement due to heating on the upper side $z=-t/2$ and cooling on the lower side $z=t/2$ resulting in surface temperatures $3\vartheta°$ and $\vartheta°$, respectively. Assume that temperature in plate is linear in z and does not depend on x or y. Use approximation $w=a_0x^2$ in which a_0 is the parameter to be determined. Young's modulus E, Poisson's ratio ν, and coefficient of thermal expansion α are constants.



Solution

Assuming that the material coordinate system is chosen so that the plate bending and thin slab modes decouple, the plate model virtual work densities of internal force and coupling terms are given by

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial^2\delta w/\partial x^2 \\ \partial^2\delta w/\partial y^2 \\ 2\partial^2\delta w/\partial x\partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12}\frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}\begin{Bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ 2\partial^2 w/\partial x\partial y \end{Bmatrix},$$

$$\delta w_{\Omega}^{\text{cpl}} = -\begin{Bmatrix} \partial^2\delta w/\partial x^2 \\ \partial^2\delta w/\partial y^2 \end{Bmatrix}^{\mathrm{T}} \int z\Delta\vartheta dz \frac{\alpha E}{1-\nu}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

3p The coupling term contains an integral of temperature change over the thickness of the plate. First as temperature is linear

$$\vartheta_{\text{asm}}(z) = 2\vartheta° \quad \text{and} \quad \vartheta_{\text{fin}}(z) = 2\vartheta°(1-\frac{z}{t}) \quad \Rightarrow \quad \Delta\vartheta = \vartheta_{\text{fin}} - \vartheta_{\text{asm}} = -2\vartheta°\frac{z}{t}.$$

So the integral of the coupling term becomes

$$\int z\Delta\vartheta dz = -2\frac{\vartheta°}{t}\int_{-t/2}^{t/2} z^2 dz = -2\frac{\vartheta°}{t}\frac{1}{3}(\frac{t^3}{8}+\frac{t^3}{8}) = -\frac{1}{6}\vartheta° t^2.$$

and virtual work density for the coupling part

$$\delta w_{\Omega}^{\text{cpl}} = \begin{Bmatrix} 2\delta a \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{1}{6}\vartheta° t^2 \frac{\alpha E}{1-\nu}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \delta a\frac{1}{3}\vartheta° t^2\frac{\alpha E}{1-\nu}.$$

Virtual work expression is integral of the density over the domain occupied by the plate/element. As integrand is constants, it is enough to multiply by the area of the mid-plane to get

$$\delta W^{\text{cpl}} = \int_{\Omega} \delta w_{\Omega}^{\text{cpl}} dA = LH\delta a\frac{1}{3}\vartheta° t^2\frac{\alpha E}{1-\nu}.$$

2p Approximation to the transverse displacement

$$w = \mathrm{a}\, x^2 \quad \Rightarrow \quad \frac{\partial^2 w}{\partial x^2} = 2\mathrm{a} \ \text{ and } \ \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial x \partial y} = 0 \,.$$

When the approximation is substituted there, virtual work density of the internal forces simplifies to

$$\delta w_{\Omega}^{\mathrm{int}} = -\begin{Bmatrix} 2\delta\mathrm{a} \\ 0 \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12} \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{Bmatrix} 2\mathrm{a} \\ 0 \\ 0 \end{Bmatrix} = -\delta\mathrm{a} \frac{t^3}{3} \frac{E}{1-\nu^2} \mathrm{a} \,.$$

Virtual work expression is integral of the density over the domain occupied by the plate/element. As integrand is constants, it is enough to multiply by the area of the mid-plane to get

$$\delta W^{\mathrm{int}} = \int_{\Omega} \delta w_{\Omega}^{\mathrm{int}} dA = -LH \delta\mathrm{a} \frac{t^3}{3} \frac{E}{1-\nu^2} \mathrm{a} \,.$$

1p Virtual work expression is the sum of the parts

$$\delta W = \delta W^{\mathrm{int}} + \delta W^{\mathrm{cpl}} = -\delta\mathrm{a}\Big(LH \frac{t^3}{3} \frac{E}{1-\nu^2} \mathrm{a} - LH \frac{1}{3} \vartheta^{\circ} t^2 \frac{\alpha E}{1-\nu}\Big) \,.$$

Principle of virtual work $\delta W = 0 \ \forall \delta\mathbf{a}$ and the fundamental lemma of variation calculus give

$$\delta W = -\delta\mathrm{a}\Big(LH \frac{t^3}{3} \frac{E}{1-\nu^2} \mathrm{a} - LH \frac{1}{3} \vartheta^{\circ} t^2 \frac{\alpha E}{1-\nu}\Big) = 0 \quad \forall \delta\mathrm{a} \quad \Leftrightarrow$$

$$LH \frac{t^3}{3} \frac{E}{1-\nu^2} \mathrm{a} - LH \frac{1}{3} \vartheta^{\circ} t^2 \frac{\alpha E}{1-\nu} = 0 \quad \Rightarrow \quad \mathrm{a} = (1+\nu) \frac{1}{t} \alpha \vartheta^{\circ} \,.$$

Transverse displacement $\quad w = (1+\nu)\alpha\vartheta^{\circ} \frac{1}{t} x^2 \,.$ ←