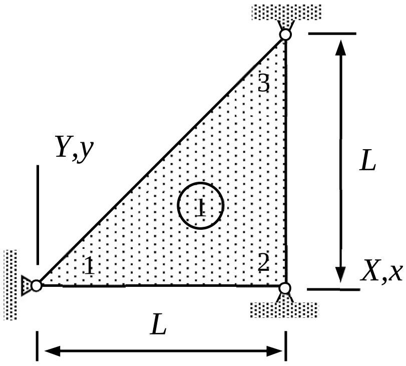
Name________________________ Student number________________

Assignment 5

Determine the angular speed of free vibrations for the thin triangular slab shown. Assume plane stress conditions. The material properties E, ν, ρ and thickness h of the slab are constants. Use the approximations $u=0$ and $v=(1-x/L)u_{Y1}$ in which the nodal value u_{Y1} is a function of time.



Solution

The virtual work densities of the internal and inertia forces for the thin slab model (plane stress conditions assumed) are given by

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta u/\partial y+\partial\delta v/\partial x \end{Bmatrix}^{\mathrm{T}} t[E]_{\sigma}\begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y+\partial v/\partial x \end{Bmatrix} \text{ and } \delta w_{\Omega}^{\text{ine}} = -\begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} t\rho\begin{Bmatrix} \ddot{u} \\ \ddot{v} \end{Bmatrix}$$

where the elasticity matrix of the plane stress

$$[E]_{\sigma} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}.$$

The approximations to the displacement components are given (the linear interpolants of the nodal values can also be deduced easily from the figure). Hence

$$u=0 \text{ and } v=(1-\frac{x}{L})u_{Y1} \quad \Rightarrow$$

$$\frac{\partial u}{\partial x}=0,\ \frac{\partial u}{\partial y}=0,\ \frac{\partial\delta u}{\partial x}=0,\ \frac{\partial\delta u}{\partial y}=0,\ \delta u=0,\ \ddot{u}=0$$

$$\frac{\partial v}{\partial x}=-\frac{1}{L}u_{Y1},\ \frac{\partial v}{\partial y}=0,\ \frac{\partial\delta v}{\partial x}=-\frac{1}{L}\delta u_{Y1},\ \frac{\partial\delta v}{\partial y}=0,\ \delta v=(1-\frac{x}{L})\delta u_{Y1},\ \ddot{v}=(1-\frac{x}{L})\ddot{u}_{Y1}$$

When the approximations are substituted there, virtual work density of the internal and inertia forces simplify to

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} 0 \\ 0 \\ -\delta u_{Y1}/L \end{Bmatrix}^{\mathrm{T}} \frac{hE}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ -u_{Y1}/L \end{Bmatrix} = -\delta u_{Y1}\frac{h}{L^2}Gu_{Y1},$$

$$\delta w_{\Omega}^{\text{ine}} = -\begin{Bmatrix} 0 \\ (1-\frac{x}{L})\delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} h\rho \begin{Bmatrix} 0 \\ (1-\frac{x}{L})\ddot{u}_{Y1} \end{Bmatrix} = -\delta u_{Y1} h\rho (1-\frac{x}{L})^2 \ddot{u}_{Y1} .$$

Integrations over the triangular domain of the element gives

$$\delta W^{\text{int}} = \int_A \delta w_{\Omega}^{\text{int}} \mathrm{d}A = \delta w_{\Omega}^{\text{int}} \frac{L^2}{2} = -\delta u_{Y1} \frac{h}{2} G u_{Y1} ,$$

$$\delta W^{\text{ext}} = \int_A \delta w_{\Omega}^{\text{ine}} \mathrm{d}A = \int_0^L (\int_0^x \delta w_{\Omega}^{\text{ine}} dy) dx \quad \Rightarrow$$

$$\delta W^{\text{ext}} = -\delta u_{Y1} h\rho \int_0^L (\int_0^x (1-\frac{x}{L})^2 dy) dx \, \ddot{u}_{Y1} = -\delta u_{Y1} h\rho \int_0^L x(1-\frac{x}{L})^2 dx \, \ddot{u}_{Y1} = -\delta u_{Y1} h\rho \frac{1}{12} L^2 \ddot{u}_{Y1} .$$

Virtual work expression of the structure takes the form

$$\delta W = -\delta u_{Y1} (\frac{t}{2} G u_{Y1} + h\rho \frac{1}{12} L^2 \ddot{u}_{Y1}) .$$

Principle of virtual work $\delta W = 0 \; \forall \delta \mathrm{a}$ and the fundamental lemma of variation calculus give

$$\frac{h}{2} G u_{Y1} + h\rho \frac{1}{12} L^2 \ddot{u}_{Y1} = 0 \quad \Leftrightarrow \quad \ddot{u}_{Y1} + 6 \frac{G}{\rho L^2} u_{Y1} = 0 .$$

As the ordinary differential equation is of the form $\ddot{u} + \omega^2 u = 0$, the angular speed of free vibrations is

$$\omega = \sqrt{6 \frac{G}{\rho L^2}} .$$