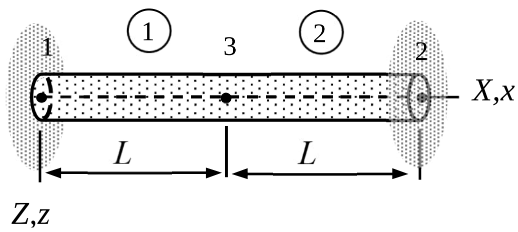
Name________________________ Student number________________

Assignment 3

Electric current causes heat generation in the bar shown. Calculate the temperature at the centre if the wall temperature (nodes 1 and 2) is ϑ°. Cross sectional area A, thermal conductivity k, and heat production rate per unit length s are constants.



Solution template

In a pure heat conduction problem, density expressions of the bar model are given by

$$\delta p_\Omega^{\text{int}} = -\frac{d\delta\vartheta}{dx} kA \frac{d\vartheta}{dx} \text{ and } \delta p_\Omega^{\text{ext}} = \delta\vartheta s$$

in which ϑ is the temperature, k the thermal conductivity, and s the rate of heat production (per unit length).

For bar 1, the nodal temperatures are $\vartheta_1 = \vartheta^\circ$ and ϑ_3 of which the latter is unknown. With a linear interpolation to temperature (notice that variation of ϑ° vanishes)

$$\vartheta = \begin{Bmatrix} 1-x/L \\ x/L \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vartheta^\circ \\ \vartheta_3 \end{Bmatrix} = (1-\frac{x}{L})\vartheta^\circ + \frac{x}{L}\vartheta_3 \quad \Rightarrow \quad \frac{d\vartheta}{dx} = \frac{\vartheta_3 - \vartheta^\circ}{L},$$

$$\delta\vartheta = \frac{x}{L}\delta\vartheta_3 \quad \Rightarrow \quad \frac{d\delta\vartheta}{dx} = \frac{\delta\vartheta_3}{L}.$$

When the approximation is substituted there, density expression $\delta p_\Omega = \delta p_\Omega^{\text{int}} + \delta p_\Omega^{\text{ext}}$ simplifies to

$$\delta p_\Omega = -\frac{\delta\vartheta_3}{L} kA \frac{\vartheta_3 - \vartheta^\circ}{L} + \frac{x}{L}\delta\vartheta_3 s,$$

Virtual work expression is the integral of the density over the element domain

$$\delta P^1 = \int_0^L \delta p_\Omega dx = -\delta\vartheta_3 (kA\frac{\vartheta_3 - \vartheta^\circ}{L} - \frac{1}{2}Ls).$$

The nodal temperatures of bar 2 are ϑ_3 and $\vartheta_2 = \vartheta^\circ$. Linear interpolation gives (variations of the given quantities like ϑ° vanish)

$$\vartheta = \begin{Bmatrix} 1-x/L \\ x/L \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vartheta_3 \\ \vartheta^\circ \end{Bmatrix} = (1-\frac{x}{L})\vartheta_3 + \frac{x}{L}\vartheta^\circ \quad \Rightarrow \quad \frac{d\vartheta}{dx} = \frac{\vartheta^\circ - \vartheta_3}{L},$$

$$\delta\vartheta = (1-\frac{x}{L})\delta\vartheta_3 \quad \Rightarrow \quad \frac{d\delta\vartheta}{dx} = -\frac{\delta\vartheta_3}{L}.$$

When the approximation is substituted there, density expression $\delta p_\Omega = \delta p_\Omega^{\text{int}} + \delta p_\Omega^{\text{ext}}$ simplifies to

$$\delta p_\Omega = -(-\frac{\delta\vartheta_3}{L})kA\frac{\vartheta^\circ - \vartheta_3}{L} + (1-\frac{x}{L})\delta\vartheta_3 s.$$

Element contribution to the variational expressions is the integral of density over the element domain

$$\delta P^2 = \int_0^L \delta p_\Omega dx = -\delta\vartheta_3(kA\frac{\vartheta_3 - \vartheta^\circ}{L} - \frac{L}{2}s).$$

Variational expression is sum of the element contributions

$$\delta P = \delta P^1 + \delta P^2 = -\delta\vartheta_3(2kA\frac{\vartheta_3 - \vartheta^\circ}{L} - Ls).$$

Variation principle $\delta P = 0 \quad \forall \delta\mathbf{a}$ and the fundamental lemma of variation calculus give

$$2\frac{kA}{L}(\vartheta_3 - \vartheta^\circ) - Ls = 0 \quad \Leftrightarrow \quad \vartheta_3 = \vartheta^\circ + \frac{1}{2}\frac{L^2 s}{kA}.$$