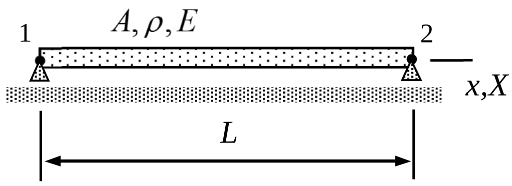
Name___________________________ Student number___________________

Assignment 4

A bar is free to move in the horizontal direction as shown. Determine the angular velocities of the free vibrations and the corresponding modes. Use one bar element of nodes 1 and 2. Cross-sectional are A, density ρ of the material, and Young's modulus E of the material are constants.



Solution template

The non-zero displacement/rotation components of the structure are u_{X1} and u_{X2}. Let us start with the element contributions for the internal and inertia parts

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{EA}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix}, \quad \delta W^{\text{ine}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{\rho A h}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} \ddot{u}_{x1} \\ \ddot{u}_{x2} \end{Bmatrix}.$$

As the axes of the material and structural coordinate systems coincide, virtual work expression of the structure takes the form

$$\delta W = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{X2} \end{Bmatrix}^{\mathrm{T}} \left(\begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{X2} \end{Bmatrix} + \begin{bmatrix} \frac{\rho AL}{3} & \frac{\rho AL}{6} \\ \frac{\rho AL}{6} & \frac{\rho AL}{3} \end{bmatrix}\begin{Bmatrix} \ddot{u}_{X1} \\ \ddot{u}_{X2} \end{Bmatrix} \right).$$

Principle of virtual work and fundamental lemma of variation calculus imply the set of ordinary differential equations

$$\begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{X2} \end{Bmatrix} + \begin{bmatrix} \frac{\rho AL}{3} & \frac{\rho AL}{6} \\ \frac{\rho AL}{6} & \frac{\rho AL}{3} \end{bmatrix}\begin{Bmatrix} \ddot{u}_{X1} \\ \ddot{u}_{X2} \end{Bmatrix} = 0.$$

The angular speeds of free vibrations can be deduced from the stiffness and mass matrix of the equation system

$$\mathbf{M} = \begin{bmatrix} \frac{\rho AL}{3} & \frac{\rho AL}{6} \\ \frac{\rho AL}{6} & \frac{\rho AL}{3} \end{bmatrix} \text{ and } \mathbf{K} = \begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix} \Rightarrow \mathbf{\Omega}^2 = \mathbf{M}^{-1}\mathbf{K} = \begin{bmatrix} \frac{6E}{\rho L^2} & -\frac{6E}{\rho L^2} \\ -\frac{6E}{\rho L^2} & \frac{6E}{\rho L^2} \end{bmatrix}.$$

The angular speeds of free vibrations are the eigenvalues of $\mathbf{\Omega}$. Let us start with the eigenvalues $\lambda=\omega^2$ of $\mathbf{\Omega}^2=\mathbf{M}^{-1}\mathbf{K}$ to get

$$\det\left(\begin{bmatrix} \frac{6E}{\rho L^2} & -\frac{6E}{\rho L^2} \\ -\frac{6E}{\rho L^2} & \frac{6E}{\rho L^2} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} \frac{6E}{\rho L^2}-\lambda & -\frac{6E}{\rho L^2} \\ -\frac{6E}{\rho L^2} & \frac{6E}{\rho L^2}-\lambda \end{bmatrix}\right) = \left(\frac{6E}{\rho L^2}-\lambda\right)^2 - \left(\frac{6E}{\rho L^2}\right)^2 = 0 \quad \Rightarrow$$

$\lambda_1 = 0$ and $\lambda_2 = \dfrac{12E}{\rho L^2}$.

The corresponding modes follow from the linear equation system of the eigenvalue problem when the eigenvalues are substituted there (one by one)

$$\lambda_1 = 0 \quad \text{and} \quad \begin{bmatrix} \frac{6E}{\rho L^2} & -\frac{6E}{\rho L^2} \\ -\frac{6E}{\rho L^2} & \frac{6E}{\rho L^2} \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{X2} \end{Bmatrix} = 0 \quad \Rightarrow \quad \mathbf{x}_1 = \begin{Bmatrix} u_{X1} \\ u_{X2} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix},$$

$$\lambda_2 = \frac{12E}{\rho L^2} \quad \text{and} \quad \begin{bmatrix} -\frac{6E}{\rho L^2} & -\frac{6E}{\rho L^2} \\ -\frac{6E}{\rho L^2} & -\frac{6E}{\rho L^2} \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{X2} \end{Bmatrix} = 0 \quad \Rightarrow \quad \mathbf{x}_2 = \begin{Bmatrix} u_{X1} \\ u_{X2} \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}.$$

As $\omega = \sqrt{\lambda}$, the angular velocities of the free vibrations and the associated modes are

$$(\omega_1, \mathbf{x}_1) = \left(0, \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}\right) \quad \text{and} \quad (\omega_2, \mathbf{x}_2) = \left(\sqrt{\frac{12E}{\rho L^2}}, \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}\right). \quad \leftarrow$$