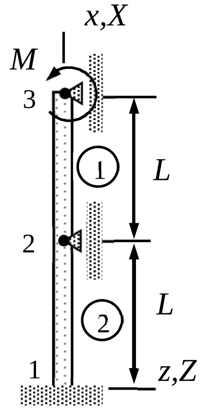
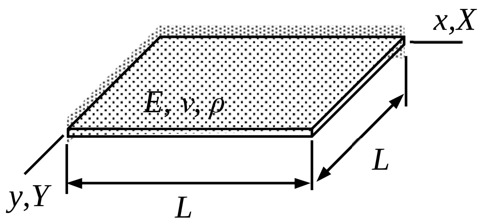
MEC-E8001 Finite Element Analysis, onsite exam 21.02.2024

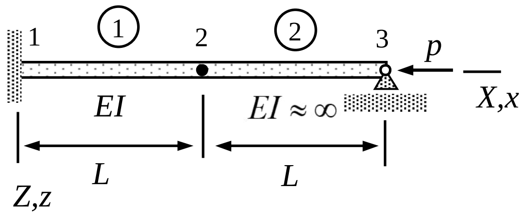
1. Beam structure of the figure is loaded by a point moment acting on node 3. Determine the rotations θ_{Y2} and θ_{Y3} by using two beam bending elements. Displacements are confined to the XZ-plane. The cross-section properties of the beam A, I and Young's modulus of the material E are constants.



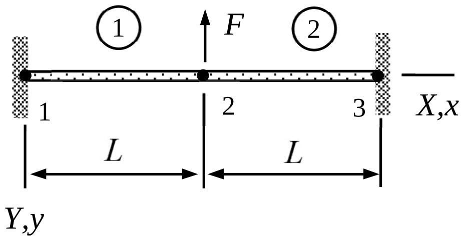
2. A plate is simply supported on two edges and free on the other two edges as shown. Use the approximation $w(x,y,t)=a(t)xy/L^2$ to determine the transverse displacement as function of time $t>0$. Material properties E, ν, and ρ are constants and thickness of the plate is h. At $t=0$, initial conditions are $\dot{w}(x,y,0)=0$ and $w(x,y,0)=Uxy/L^2$. Assume that the plate is thin so that the rotation part of the inertia term is negligible.



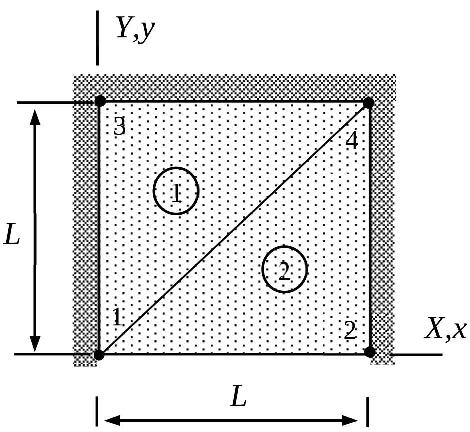
3. The structure shown consists of two beams, each of length L. As beam 2 is much stiffer than beam 1, the setting is simplified by considering beam 2 as a rigid body. Displacements are confined to the $xz-$plane. Cross-section properties of beam 1 are A and I and Young's modulus of the material is E. Determine the buckling load p_{cr}.



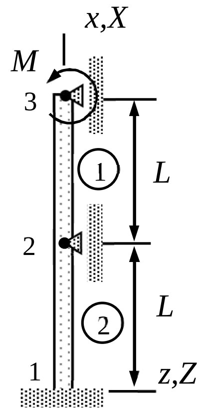
4. Determine the displacement at node 2 of the elastic bar shown by the large deformation theory. Take into account only the transverse displacement u_{Y2} ($u_{X2}=0$). When $F=0$, the cross-sectional area and length of the bar are A and L, respectively. Constitutive equation of the material is $S_{xx}=C\text{E}_{xx}$, in which C is constant. Use two elements with linear shape functions.



5. Determine the stationary temperature distribution in a thin slab shown. Edge 1-2 is at constant temperature ϑ° and heat flux through the other edges vanishes. Use a two-triangle mesh with ϑ_3 and $\vartheta_4=\vartheta_3$ as the unknown node temperatures and consider $\vartheta_1=\vartheta_2=\vartheta^\circ$ as known. Thickness t, thermal conductivity k, and heat production rate per unit area s are constants.



Beam structure of the figure is loaded by a point moment acting on node 3. Determine the rotations θ_{Y2} and θ_{Y3} by using two beam bending elements. Displacements are confined to the XZ-plane. The cross-section properties of the beam A, I and Young's modulus of the material E are constants.



Solution

Virtual work expression for the displacement analysis consists of parts coming from internal and external forces δW^{int} and δW^{ext}. For the beam bending mode in xz-plane, the virtual work expressions are

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI_{yy}}{h^3} \begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix}, \quad \delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{f_z h}{12} \begin{Bmatrix} 6 \\ -h \\ 6 \\ h \end{Bmatrix},$$

The element contribution of the point force/moment follows from the definition or work and is given by

$$\delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \\ \delta u_{Z1} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \underline{F}_X \\ \underline{F}_Y \\ \underline{F}_Z \end{Bmatrix} + \begin{Bmatrix} \delta\theta_{X1} \\ \delta\theta_{Y1} \\ \delta\theta_{Z1} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \underline{M}_X \\ \underline{M}_Y \\ \underline{M}_Z \end{Bmatrix}.$$

2p Distributed force $f_z = 0$ and $I_{yy} = I$ in the present problem. In the first step of analysis, the virtual work expressions (given in material coordinate systems of the element) are written in terms of the nodal displacements and rotation components in the structural coordinate system. As the coordinate axes of the two systems are aligned, transformation is simple. Virtual work expression of beam 1

$$\delta W^1 = -\begin{Bmatrix} 0 \\ \delta\theta_{Y2} \\ 0 \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ \theta_{Y2} \\ 0 \\ \theta_{Y3} \end{Bmatrix} = -\begin{Bmatrix} \delta\theta_{Y2} \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 4\dfrac{EI}{L} & 2\dfrac{EI}{L} \\ 2\dfrac{EI}{L} & 4\dfrac{EI}{L} \end{bmatrix} \begin{Bmatrix} \theta_{Y2} \\ \theta_{Y3} \end{Bmatrix},$$

2p When written in the standard form having the $\delta-$ quantity vector as the multiplier, virtual work expression for beam 2 takes the form

$$\delta W^2 = -\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \theta_{Y2} \end{Bmatrix} = -\begin{Bmatrix} \delta\theta_{Y2} \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 4\frac{EI}{L} & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \theta_{Y2} \\ \theta_{Y3} \end{Bmatrix}.$$

1p Virtual work expression of the point moment (also written in the 'standard' form having the $\delta-$ quantity vector as the multiplier)

$$\delta W^3 = -\begin{Bmatrix} \delta\theta_{Y2} \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ -M \end{Bmatrix}.$$

1p Virtual work expression of structure is sum of the element contributions, i.e.,

$$\delta W = \delta W^1 + \delta W^2 + \delta W^3 = -\begin{Bmatrix} \delta\theta_{Y2} \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} (\begin{bmatrix} 8\frac{EI}{L} & 2\frac{EI}{L} \\ 2\frac{EI}{L} & 4\frac{EI}{L} \end{bmatrix} \begin{Bmatrix} \theta_{Y2} \\ \theta_{Y3} \end{Bmatrix} + \begin{Bmatrix} 0 \\ -M \end{Bmatrix}).$$

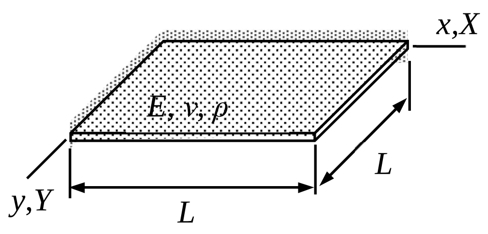
Principle of virtual work and the fundamental lemma of variation calculus imply that

$$\begin{bmatrix} 8\frac{EI}{L} & 2\frac{EI}{L} \\ 2\frac{EI}{L} & 4\frac{EI}{L} \end{bmatrix} \begin{Bmatrix} \theta_{Y2} \\ \theta_{Y3} \end{Bmatrix} + \begin{Bmatrix} 0 \\ -M \end{Bmatrix} = 0.$$

Solution to the linear equations system is given by

$$\theta_{Y2} = -\frac{1}{14}\frac{ML}{EI} \quad \text{and} \quad \theta_{Y3} = \frac{2}{7}\frac{ML}{EI}. \quad \leftarrow$$

A plate is simply supported on two edges and free on the other two edges as shown. Use the approximation $w(x,y,t)=a(t)xy/L^2$ to determine the transverse displacement as function of time $t>0$. Material properties E, ν, and ρ are constants and thickness of the plate is h. At $t=0$, initial conditions are $\dot{w}(x,y,0)=0$ and $w(x,y,0)=Uxy/L^2$. Assume that the plate is thin so that the rotation part of the inertia term is negligible.



Solution

4p Only the bending mode of the plate matters. When the approximation $w=a(t)xy/L^2$ is substituted there, virtual work densities of internal and inertia forces (without the rotation part) of the plate simplify to (shear modulus $G=E/(2+2\nu)$)

$$\delta w_{\Omega}^{\text{int}}=-\begin{Bmatrix}\partial^2\delta w/\partial x^2\\ \partial^2\delta w/\partial y^2\\ 2\partial^2\delta w/\partial x\partial y\end{Bmatrix}^{\mathrm{T}}\frac{h^3}{12}\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & (1-\nu)/2\end{bmatrix}\begin{Bmatrix}\partial^2 w/\partial x^2\\ \partial^2 w/\partial y^2\\ 2\partial^2 w/\partial x\partial y\end{Bmatrix}=-\delta a\frac{1}{L^4}\frac{h^3}{3}Ga,$$

$$\delta w_{\Omega}^{\text{ine}}=-\begin{Bmatrix}\partial\delta w/\partial x\\ \partial\delta w/\partial y\end{Bmatrix}^{\mathrm{T}}\frac{t^3}{12}\rho\begin{Bmatrix}\partial\ddot{w}/\partial x\\ \partial\ddot{w}/\partial y\end{Bmatrix}-\delta wt\rho\ddot{w}=-\delta a(\frac{x}{L})^2(\frac{y}{L})^2 h\rho\ddot{a}$$

in which h is thickness of the plate. Integration over the domain occupied by the element gives the virtual work expressions

$$\delta W^{\text{int}}=\int_0^L\int_0^L \delta w_{\Omega}^{\text{int}}dydx=\int_0^L\int_0^L -\delta a\frac{1}{L^4}\frac{h^3}{3}Gadydx=-\delta a\frac{1}{L^2}\frac{h^3}{3}Ga,$$

$$\delta W^{\text{ine}}=\int_0^L\int_0^L \delta w_{\Omega}^{\text{ine}}dydx=\int_0^L\int_0^L -\delta a(\frac{x}{L})^2(\frac{y}{L})^2 h\rho\ddot{a}dxdy=-\delta a\frac{L^2}{9}h\rho\ddot{a}.$$

Virtual work expression of the structure consists of the internal and inertia parts

$$\delta W=\delta W^{\text{int}}+\delta W^{\text{ext}}=-\delta a(\frac{1}{L^2}\frac{h^3}{3}Ga+\frac{L^2}{9}h\rho\ddot{a}).$$

Principle of virtual work $\delta W=0\quad\forall\delta\mathbf{a}$ and the fundamental lemma of variation calculus $\delta\mathbf{a}^{\mathrm{T}}\mathbf{F}=0\ \forall\delta\mathbf{a}\quad\Leftrightarrow\quad\mathbf{F}=0$ imply

$$\frac{1}{L^2}\frac{h^3}{3}Ga+\frac{L^2}{9}h\rho\ddot{a}=0.$$

2p What remains, is solving for the displacement from the initial value problem

$$\ddot{a}+3\frac{Gh^2}{\rho L^4}a=0\quad t>0,\quad a(0)=U,\quad \dot{a}(0)=0.$$

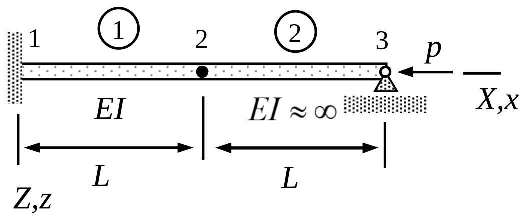
Solution to equations is (this can be shown e.g. by substituting the solution in the equations above)

$$a(t) = U\cos\left(\sqrt{3\frac{G}{\rho}\frac{h}{L^2}}\,t\right) \quad t > 0\,.$$

Finally, substituting the solution to parameter $a(t)$ into the approximation gives

$$w(x,y,t) = U\cos\left(\sqrt{3\frac{G}{\rho}\frac{h}{L^2}}\,t\right)\frac{xy}{L^2}\,.$$

The structure shown consists of two beams, each of length L. As beam 2 is much stiffer than beam 1, the setting is simplified by considering beam 2 as a rigid body. Displacements are confined to the xz – plane. Cross-section properties of beam 1 are A and I and Young's modulus of the material is E. Determine the buckling load p_{cr}.



Solution

The axial forces in the beams follow directly from a free body diagram and it is enough to consider virtual work associated with the bending and interaction between bar and bending modes

$$\delta W^{\mathrm{int}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI_{yy}}{h^3} \begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix},$$

$$\delta W^{\mathrm{sta}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{N}{30h} \begin{bmatrix} 36 & -3h & -36 & -3h \\ -3h & 4h^2 & 3h & -h^2 \\ -36 & 3h & 36 & 3h \\ -3h & -h^2 & 3h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix}.$$

2p Also, as beam 2 is considered as rigid, virtual work expression of internal forces vanish and modelling uses kinematical constraints $\vec{u}_{\mathrm{B}} = \vec{u}_{\mathrm{A}} + \vec{\theta}_{\mathrm{A}} \times \vec{\rho}_{\mathrm{AB}}$ and $\vec{\theta}_{\mathrm{B}} = \vec{\theta}_{\mathrm{A}}$. Let us choose A to be node 3 and B as node 2. Then

$$u_{Z2} = \theta_{Y3} L \quad \text{and} \quad \theta_{Y2} = \theta_{Y3}.$$

Although axial displacement is non-zero, it is not needed as the axial force in the structure $N = -p$ (negative means compression) can be deduced without calculations on the axial displacement.

4p The internal force and coupling parts of beam 1 take the forms ($u_{z1} = 0$, $\theta_{y1} = 0$, $u_{z2} = u_{Z2} = \theta_{Y3} L$, $\theta_{y2} = \theta_{Y2} = \theta_{Y3}$)

$$\delta W^{\mathrm{int}} = -\begin{Bmatrix} 0 \\ 0 \\ L\delta\theta_{Y3} \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ L\theta_{Y3} \\ \theta_{Y3} \end{Bmatrix} = -\delta\theta_{Y3} 28 \frac{EI}{L} \theta_{Y3},$$

$$\delta W^{\mathrm{sta}} = -\begin{Bmatrix} 0 \\ 0 \\ L\delta\theta_{Y3} \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{-p}{30L} \begin{bmatrix} 36 & -3L & -36 & -3L \\ -3L & 4L^2 & 3L & -L^2 \\ -36 & 3L & 36 & 3L \\ -3L & -L^2 & 3L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ L\theta_{Y3} \\ \theta_{Y3} \end{Bmatrix} = \delta\theta_{Y3} \frac{46}{30} pL\theta_{Y3}.$$

Virtual work expression of the structure is the sum of the internal and stability parts

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{sta}} = -\delta\theta_{Y3}(28\frac{EI}{L} - \frac{46}{30}pL)\theta_{Y3}.$$

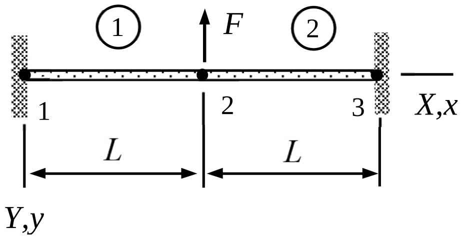
Principle of virtual work and the fundamental lemma of variation calculus imply that

$$(28\frac{EI}{L} - \frac{46}{30}pL)\theta_{Y3} = 0.$$

In stability analysis the aim is to find a non-zero equilibrium solution so $\theta_{Y3} \neq 0$ and the multiplier needs to vanish for a solution. Therefore, the buckling load is given by

$$p_{\text{cr}} = \frac{30}{46}28\frac{EI}{L^2} = \frac{420}{23}\frac{EI}{L^2}.$$

Determine the displacement at node 2 of the elastic bar shown by the large deformation theory. Take into account only the transverse displacement u_{Y2} ($u_{X2}=0$). When $F=0$, the cross-sectional area and length of the bar are A and L, respectively. Constitutive equation of the material is $S_{xx}=C\mathrm{E}_{xx}$, in which C is constant. Use two elements with linear shape functions.



Solution

Virtual work density of the non-linear bar model

$$\delta w_{\Omega^\circ}^{\text{int}}=-(\frac{d\delta u}{dx}+\frac{du}{dx}\frac{d\delta u}{dx}+\frac{dv}{dx}\frac{d\delta v}{dx}+\frac{dw}{dx}\frac{d\delta w}{dx})CA^\circ[\frac{du}{dx}+\frac{1}{2}(\frac{du}{dx})^2+\frac{1}{2}(\frac{dv}{dx})^2+\frac{1}{2}(\frac{dw}{dx})^2]$$

is based on the Green-Lagrange strain definition which is physically correct also when rotations/displacements are large. The expression depends on all displacement components, material property is denoted by C (constitutive equation $S_{xx}=C\mathrm{E}_{xx}$), and the superscript in the cross-sectional area A° (and in other quantities) refers to the initial geometry (strain and stress vanishes). Otherwise, equilibrium equations follow in the same manner as in the linear case.

2p For element 1, the non-zero displacement components is $u_{y2}=u_{Y2}$. As the initial length of the element $h^\circ=L$, linear approximations to the displacement components

$$u=w=0 \text{ and } v=\frac{x}{L}u_{Y2} \quad\Rightarrow\quad \frac{du}{dx}=\frac{dw}{dx}=0 \text{ and } \frac{dv}{dx}=\frac{u_{Y2}}{L}.$$

When the approximation is substituted there, virtual work density of the internal forces and thereby the virtual work expression (density is constant) simplify to

$$\delta w_{\Omega^\circ}^{\text{int}}=-\frac{\delta u_{Y2}}{L}\frac{u_{Y2}}{L}\frac{CA}{2}(\frac{u_{Y2}}{L})^2 \quad\Rightarrow\quad \delta W^1=-\delta u_{Y2}\frac{CA}{2}(\frac{u_{Y2}}{L})^3.$$

2p For element 2, the non-zero displacement component $u_{y2}=u_{Y2}$. As the initial length of the element $h^\circ=L$, linear approximations to the displacement components

$$u=w=0 \text{ and } v=(1-\frac{x}{L})u_{Y2} \quad\Rightarrow\quad \frac{du}{dx}=\frac{dw}{dx}=0 \text{ and } \frac{dv}{dx}=-\frac{u_{Y2}}{L}.$$

When the approximation is substituted there, virtual work density of the internal forces and thereby the virtual work expression (density is constant) simplifies to

$$\delta w_{\Omega^\circ}^{\text{int}}=-\frac{\delta u_{Y2}}{L}\frac{u_{Y2}}{L}\frac{CA}{2}(\frac{u_{Y2}}{L})^2 \quad\Rightarrow\quad \delta W^2=-\delta u_{Y2}\frac{CA}{2}(\frac{u_{Y2}}{L})^3.$$

2p Virtual work expression of the point force is

$$\delta W^3=-F\delta u_{Y2}.$$

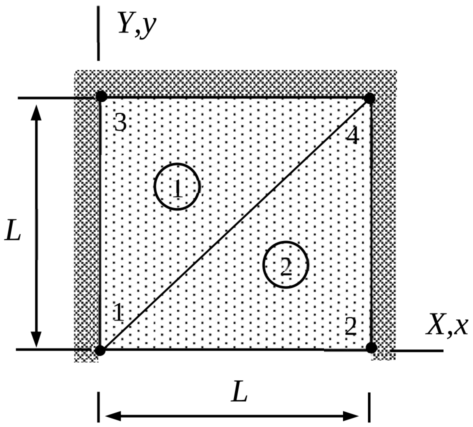
Virtual work expression of the structure is obtained as the sum of the element contributions

$$\delta W = -\delta u_{Y2}[\frac{CA}{2}(\frac{u_{Y2}}{L})^3 + \frac{CA}{2}(\frac{u_{Y2}}{L})^3 + F].$$

Principle of virtual work and the fundamental lemma of variation calculus imply that

$$(\frac{u_{Y2}}{L})^3 + \frac{F}{CA} = 0 \quad \Rightarrow \quad u_{Y2} = -(\frac{FL^3}{CA})^{1/3}.$$

Determine the stationary temperature distribution in a thin slab shown. Edge 1-2 is at constant temperature ϑ° and heat flux through the other edges vanishes. Use a two-triangle mesh with ϑ_3 and $\vartheta_4 = \vartheta_3$ as the unknown node temperatures and consider $\vartheta_1 = \vartheta_2 = \vartheta^{\circ}$ as known. Thickness t, thermal conductivity k, and heat production rate per unit area s are constants.



Solution

The density expressions associated with the pure heat conduction problem in a thin slab are

$$\delta p_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta\vartheta/\partial x \\ \partial\delta\vartheta/\partial y \end{Bmatrix}^{\mathrm{T}} tk \begin{Bmatrix} \partial\vartheta/\partial x \\ \partial\vartheta/\partial y \end{Bmatrix} \text{ and } \delta p_{\Omega}^{\text{ext}} = \delta\vartheta s\,.$$

in which ϑ is the temperature, k the thermal conductivity, and s the rate of heat production (per unit area). For a thin-slab element, element contributions need to be calculated from scratch starting with the densities and approximations.

3p The shape functions of element 1 (deduced from the figure) $N_1 = 1 - y/L$, $N_4 = x/L$, and $N_3 = 1 - N_1 - N_4 = (y-x)/L$ give approximations

$$\vartheta = \begin{Bmatrix} N_1 \\ N_4 \\ N_3 \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vartheta^{\circ} \\ \vartheta_3 \\ \vartheta_3 \end{Bmatrix} = (1-\frac{y}{L})\vartheta^{\circ} + \frac{y}{L}\vartheta_3\,,\quad \frac{\partial\vartheta}{\partial x} = 0\,,\quad \frac{\partial\vartheta}{\partial y} = \frac{\vartheta_3 - \vartheta^{\circ}}{L} \quad \text{and}$$

$$\delta\vartheta = \frac{y}{L}\delta\vartheta_3\,,\quad \frac{\partial\delta\vartheta}{\partial x} = 0\,,\quad \frac{\partial\delta\vartheta}{\partial y} = \frac{\delta\vartheta_3}{L} \quad \text{(variation of } \vartheta^{\circ} \text{ vanishes).}$$

When the approximation is substituted there, density expression simplifies to

$$\delta p_{\Omega} = \delta p_{\Omega}^{\text{int}} + \delta p_{\Omega}^{\text{ext}} = -\begin{Bmatrix} \partial\delta\vartheta/\partial x \\ \partial\delta\vartheta/\partial y \end{Bmatrix}^{\mathrm{T}} tk \begin{Bmatrix} \partial\vartheta/\partial x \\ \partial\vartheta/\partial y \end{Bmatrix} + \delta\vartheta s = -\frac{\delta\vartheta_3}{L} tk \frac{\vartheta_3 - \vartheta^{\circ}}{L} + \frac{y}{L}\delta\vartheta_3 s\,.$$

Element contribution is the integral of the density expression over the domain occupied by the element:

$$\delta P^1 = -\delta\vartheta_3 (tk\frac{\vartheta_3 - \vartheta^{\circ}}{2} - \frac{L^2}{3}s)\,.$$

3p The shape functions of element 2 (deduced from the figure) $N_1 = 1 - x/L$, $N_4 = y/L$, and $N_2 = 1 - N_1 - N_4 = (x-y)/L$ give approximations

$$\vartheta=\begin{Bmatrix}N_1\\N_2\\N_4\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\vartheta^\circ\\\vartheta^\circ\\\vartheta_3\end{Bmatrix}=(1-\frac{y}{L})\vartheta^\circ+\frac{y}{L}\vartheta_3\,,\quad\frac{\partial\vartheta}{\partial x}=0\,,\;\frac{\partial\vartheta}{\partial y}=\frac{\vartheta_3-\vartheta^\circ}{L}\,,\quad\text{and}$$

$$\delta\vartheta=\frac{y}{L}\delta\vartheta_3\,,\;\frac{\partial\delta\vartheta}{\partial x}=0\,,\;\frac{\partial\delta\vartheta}{\partial y}=\frac{\delta\vartheta_3}{L}\;\text{(variation of }\vartheta^\circ\text{ vanishes).}$$

When the approximation is substituted there, density simplifies to

$$\delta p_\Omega=\delta p_\Omega^{\mathrm{int}}+\delta p_\Omega^{\mathrm{ext}}=-\frac{\delta\vartheta_3}{L}tk\frac{\vartheta_3-\vartheta^\circ}{L}+\frac{y}{L}\delta\vartheta_3 s\,.$$

Element contribution is the integral of the density expression over the domain occupied by the element, so

$$\delta P^2=-\delta\vartheta_3(tk\frac{\vartheta_3-\vartheta^\circ}{2}-\frac{L^2}{6}s)\,.$$

Variation principle $\delta P=\delta P^1+\delta P^2=0\;\;\forall\delta\mathbf{a}$ and the fundamental lemma of variation calculus imply that

$$tk(\vartheta_3-\vartheta^\circ)-\frac{L^2}{2}s=0\quad\Leftrightarrow\quad\vartheta_3=\vartheta^\circ+\frac{sL^2}{2tk}\,.$$