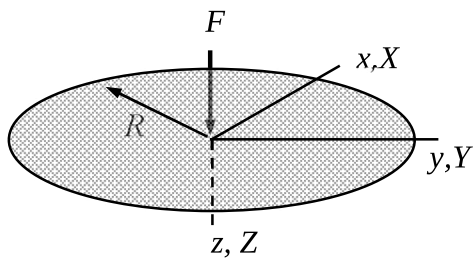
Name______________________________ Student number____________________

Assignment 5

A circular plate of radius R, which is simply supported at the outer edge, is loaded by force F at the center point. Use the Kirchhoff plate model to find the transverse displacement at the center point. Use the approximation $w = a_0(x^2 + y^2 - R^2)$ for the transverse displacement. Material properties E, ν and thickness t are constants.



Solution

Assuming that the material coordinate system is chosen so that the plate bending and thin slab modes decouple, it is enough to consider the virtual work densities of the bending mode only

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial^2 \delta w / \partial x^2 \\ \partial^2 \delta w / \partial y^2 \\ 2\partial^2 \delta w / \partial x \partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12} [E]_{\sigma} \begin{Bmatrix} \partial^2 w / \partial x^2 \\ \partial^2 w / \partial y^2 \\ 2\partial^2 w / \partial x \partial y \end{Bmatrix}, \quad \delta w_{\Omega}^{\text{ext}} = \delta w f_z .$$

in which the elasticity matrix of plane stress

$$[E]_{\sigma} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}.$$

In the present case, distributed force vanishes i.e. $f_z = 0$ and the point force is taken into account by a point force element.

Approximation to the transverse displacement is given by (a_0 is not associated with any point but it just a parameter of the approximation)

$$w(x, y) = a_0(x^2 + y^2 - R^2) \quad \Rightarrow \quad \frac{\partial^2 w}{\partial x^2} = 2a_0, \; \frac{\partial^2 w}{\partial y^2} = 2a_0, \text{ and } \frac{\partial^2 w}{\partial x \partial y} = 0 .$$

When the approximation is substituted there, virtual work density of internal forces simplifies to

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} 2\delta a_0 \\ 2\delta a_0 \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{t^3 E}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} 2a_0 \\ 2a_0 \\ 0 \end{Bmatrix} = -\delta a_0 \frac{2}{3} \frac{t^3 E}{1-\nu} a_0 .$$

Virtual work expression of the plate bending element (element 1 here) is integral of the virtual work density over the domain occupied by the element. As the density expression is constant it is enough to multiply by the area of the domain

$$\delta W^1 = \int_\Omega \delta w_\Omega^{\mathrm{int}} dA = \delta w_\Omega^{\mathrm{int}} \pi R^2 = -\delta a_0 \frac{2\pi}{3}\frac{R^2 t^3 E}{1-\nu} a_0 .$$

Virtual work expression of the point force (element 2 here) follows from the definition of work (notice the use of virtual displacement at the point of action $x = y = 0$)

$$\delta W^2 = \delta w(0,0) F = -\delta a_0 R^2 F .$$

Principle of virtual work and the fundamental lemma of variation calculus give

$$\delta W = \delta W^1 + \delta W^2 = -\delta a_0 \left(\frac{2\pi}{3}\frac{R^2 t^3 E}{1-\nu} a_0 + R^2 F\right) = 0 \quad \Rightarrow \quad a_0 = -\frac{3}{2\pi}\frac{F}{t^3 E}(1-\nu) .$$

Displacement at the center point

$$w(0,0) = -a_0 R^2 = \frac{3}{2\pi}\frac{FR^2}{t^3 E}(1-\nu) .$$