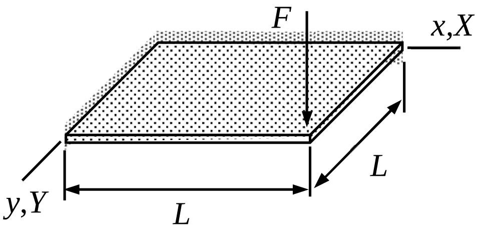
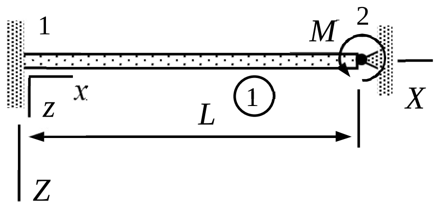
MEC-E8001 Finite Element Analysis, exam 05.06.2024

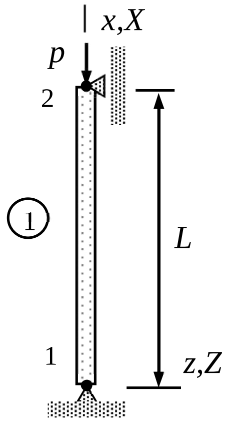
1. A plate, loaded by point force F acting at the free corner, is simply supported on two edges and free on the other two edges as shown in the figure. Determine the parameter a_0 of approximation $w(x,y) = a_0(x/L)(y/L)$ and displacement at the center point. Use the virtual work density of the plate bending mode with constant E, ν, ρ and t.



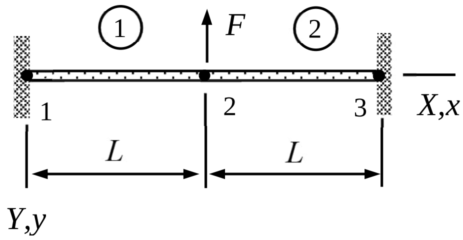
2. The beam of the figure is subjected to moment M when $t < 0$. At $t = 0$, the moment is suddenly removed and the beam starts to vibrate. Derive the initial value problem giving $\theta_{Y2}(t)$ for $t > 0$. The beam is thin so that the rotational part of the inertia term is negligible. The geometrical quantities of the cross-section are A, I and the material constants E and ρ.



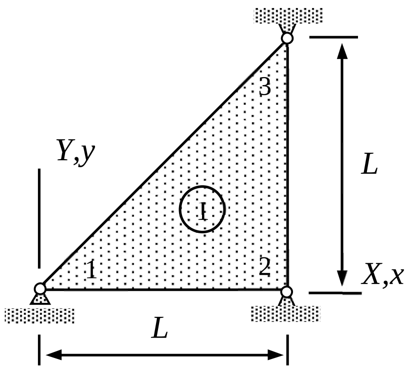
3. Determine the buckling force p_{cr} and the buckled shape of the structure shown by using one beam element. Displacements are confined to the xz – plane. Parameters E, A, and I are constants.



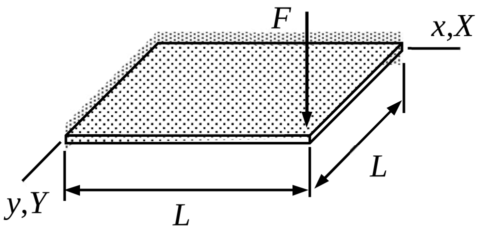
4. Determine the displacement at node 2 of the elastic bar shown by the large deformation theory. Consider only the transverse displacement u_{Y2} ($u_{X2} = 0$). When $F = 0$, the cross-sectional area and length of the bar are A and L, respectively. Constitutive equation of the material is $S_{xx} = C\mathrm{E}_{xx}$, in which C is constant. Use two elements with linear shape functions.



5. A thin triangular slab (plane stress conditions) is allowed to move horizontally at node 1 and nodes 2 and 3 are fixed. Stress is zero when temperature (assumed constant) is ϑ°. Determine the non-zero displacement component u_{X1}, if the temperature of slab is increased to $2\vartheta^{\circ}$.



A Kirchhoff plate, loaded by point force F acting at the free corner, is simply supported on two edges and free on the other two edges as shown in the figure. Determine the parameter a_0 of approximation $w(x,y)=a_0(x/L)(y/L)$ and displacement at the center point. Use the virtual work density of the Kirchhoff plate model with constant E, ν, ρ and t.



Solution

Assuming that the material coordinate system is chosen so that the plate bending and thin slab modes decouple, it is enough to consider the virtual work densities of the bending mode only

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial^2\delta w/\partial x^2 \\ \partial^2\delta w/\partial y^2 \\ 2\partial^2\delta w/\partial x\partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12}[E]_{\sigma} \begin{Bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ 2\partial^2 w/\partial x\partial y \end{Bmatrix},\ \delta w_{\Omega}^{\text{ext}} = \delta w f_z .$$

in which the elasticity matrix of plane stress

$$[E]_{\sigma} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}.$$

In the present case, distributed force vanishes i.e. $f_z = 0$ and the point force is taken into account by a point force element.

2p Approximation to the transverse displacement is chosen to be (a_0 is not associated with any point but it just a parameter of the approximation)

$$w(x,y) = a_0 \frac{x}{L}\frac{y}{L} \quad \Rightarrow \quad \frac{\partial^2 w}{\partial x^2} = 0,\ \frac{\partial^2 w}{\partial y^2} = 0,\ \text{and}\ \frac{\partial^2 w}{\partial x \partial y} = \frac{1}{L^2} a_0 .$$

When the approximation is substituted there, virtual work density of internal forces simplifies to

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial^2\delta w/\partial x^2 \\ \partial^2\delta w/\partial y^2 \\ 2\partial^2\delta w/\partial x\partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3 E}{12(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ 2\partial^2 w/\partial x\partial y \end{Bmatrix} = -\delta a_0 \frac{Et^3}{6(1+\nu)}\frac{1}{L^4} a_0 ,$$

2p Virtual work expression of the plate bending element (element 1 here) is integral of the virtual work density over the domain occupied by the element

$$\delta W^1 = \int_0^L \int_0^L \delta w_{\Omega}^{\text{int}} dxdy = -\delta a_0 \frac{Et^3}{6(1+\nu)}\frac{1}{L^2} a_0 .$$

Virtual work expression of the point force (element 2 here) follows from the definition of work (notice the use of virtual displacement of the point of action $x = y = L$)

$$\delta W^2 = \delta w(L,L)F = \delta a_0 F \ .$$

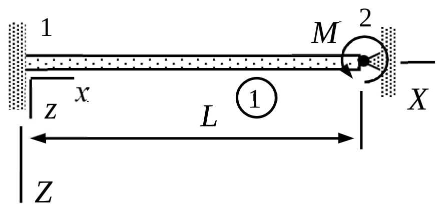
2p Principle of virtual work and the fundamental lemma of variation calculus give

$$\delta W = \delta W^1 + \delta W^2 = -\delta a_0 \left(\frac{Et^3}{6(1+\nu)} \frac{1}{L^2} a_0 - F \right) = 0 \quad \Rightarrow \quad a_0 = 6(1+\nu)\frac{FL^2}{Et^3} \ .$$

Displacement at the center point

$$w\left(\frac{L}{2},\frac{L}{2}\right) = a_0 \frac{1}{4} = \frac{3}{2}(1+\nu)\frac{FL^2}{Et^3} \ . \quad \leftarrow$$

The beam of the figure is subjected to moment M when $t<0$. At $t=0$, the moment is suddenly removed and the beam starts to vibrate. Derive the initial value problem giving $\theta_{Y2}(t)$ for $t>0$. The beam is thin so that the rotational part of the inertia term is negligible. The geometrical quantities of the cross-section are A, I and the material constants E and ρ.



Solution

2p Virtual work expression consists of parts coming from internal and inertial forces. Finding the equation of motion is the first thing to do. The beam element contributions needed in the problem are (the term having to do with rotational inertia is omitted)

$$\delta W^{\text{int}} = -\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \theta_{Y2} \end{Bmatrix} = -\delta\theta_{Y2} 4\frac{EI}{L}\theta_{Y2},$$

$$\delta W^{\text{ine}} = -\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{\rho AL}{420} \begin{bmatrix} 156 & -22L & 54 & 13L \\ -22L & 4L^2 & -13L & -3L^2 \\ 54 & -13L & 156 & 22L \\ 13L & -3L^2 & 22L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \ddot{\theta}_{Y2} \end{Bmatrix} = -\delta\theta_{Y2} \frac{\rho AL^3}{105}\ddot{\theta}_{Y2}$$

giving

$$\delta W^1 = -\delta\theta_{Y2}(4\frac{EI}{L}\theta_{Y2} + \frac{\rho AL^3}{105}\ddot{\theta}_{Y2}).$$

In terms of moment $P(t)$ (positive in the positive direction of Y-axis) which is piecewise constant in time so that $P(t)=M \quad t\le 0$ and $P(t)=0 \quad t>0$, the element contribution of the moment is

$$\delta W^2 = \delta\theta_{Y2}P.$$

2p Virtual work expression is the sum of element contributions:

$$\delta W = \delta W^1 + \delta W^2 = -\delta\theta_{Y2}(4\frac{EI}{L}\theta_{Y2} + \frac{\rho AL^3}{105}\ddot{\theta}_{Y2} - P) = 0.$$

Principle of virtual work and the fundamental lemma of variation calculus imply the ordinary differential equation

$$\delta W = -\delta\theta_{Y2}(4\frac{EI}{L}\theta_{Y2} + \frac{\rho AL^3}{105}\ddot{\theta}_{Y2} - P) = 0 \quad \forall \delta\theta_{Y2} \quad \Leftrightarrow$$

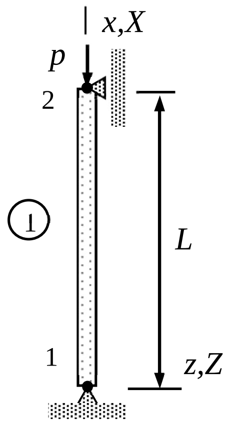
$$4\frac{EI}{L}\theta_{Y2} + \frac{\rho AL^3}{105}\ddot{\theta}_{Y2} - P = 0. \quad \Leftarrow$$

2p When $t \le 0$, external moment $P = M$ is acting on node 2 and the system is at rest. Therefore, the equation of motion becomes an equilibrium equation giving as its solution the initial rotation

$$\theta_{Y2} = \frac{1}{4}\frac{ML}{EI}.$$

When $t > 0$, external moment is zero and acceleration does not vanish. The initial value problem giving as its solution $\theta_{Y2}(t)$ for $t > 0$ takes the form

$$4\frac{EI}{L}\theta_{Y2} + \frac{\rho A L^3}{105}\ddot{\theta}_{Y2} = 0 \quad t > 0, \quad \theta_{Y2}(0) = \frac{1}{4}\frac{ML}{EI}, \quad \text{and} \quad \dot{\theta}_{Y2}(0) = 0 .$$



Determine the buckling force p_{cr} and the buckled shape of the structure shown by using one beam element. Displacements are confined to the xz – plane. Parameters E, A, and I are constants.

Solution

3p The non-zero displacement/rotation components of the structure are and $\theta_{y1} = \theta_{Y1}$, $\theta_{y2} = \theta_{Y2}$, and $u_{x2} = u_{X2}$. The normal force in the beam $N = -p$ can be deduced without calculations on the axial displacement. Therefore, it is enough to consider only the bending and coupling terms of the virtual work expression. As buckling is confined to the xz – plane

$$\delta W = -\begin{Bmatrix} 0 \\ \delta\theta_{Y1} \\ 0 \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} - \frac{p}{30L}\begin{bmatrix} 36 & -3L & -36 & -3L \\ -3L & 4L^2 & 3L & -L^2 \\ -36 & 3L & 36 & 3L \\ -3L & -L^2 & 3L & 4L^2 \end{bmatrix}\right)\begin{Bmatrix} 0 \\ \theta_{Y1} \\ 0 \\ \theta_{Y2} \end{Bmatrix} \Leftrightarrow$$

$$\delta W = -\begin{Bmatrix} \delta\theta_{Y1} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EI}{L}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} - \frac{pL}{30}\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}\right)\begin{Bmatrix} \theta_{Y1} \\ \theta_{Y2} \end{Bmatrix}.$$

According to the principle of virtual work

$$\left(\frac{EI}{L}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} - \frac{pL}{30}\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}\right)\begin{Bmatrix} \theta_{Y1} \\ \theta_{Y2} \end{Bmatrix} = 0.$$

2p A homogeneous equation system has a non-trivial solution only if the matrix is singular

$$\det\left(\begin{bmatrix} 4\frac{EI}{L} - 4\frac{pL}{30} & 2\frac{EI}{L} + \frac{pL}{30} \\ 2\frac{EI}{L} + \frac{pL}{30} & 4\frac{EI}{L} - 4\frac{pL}{30} \end{bmatrix}\right) = \left(4\frac{EI}{L} - 4\frac{pL}{30}\right)^2 - \left(2\frac{EI}{L} + \frac{pL}{30}\right)^2 = 0 \quad \Rightarrow \quad \frac{pL^2}{EI} \in \{12, 60\}.$$

The smallest eigenvalue gives the critical loading

$$p_{\mathrm{cr}} = 12\frac{EI}{L^2}. \quad \leftarrow$$

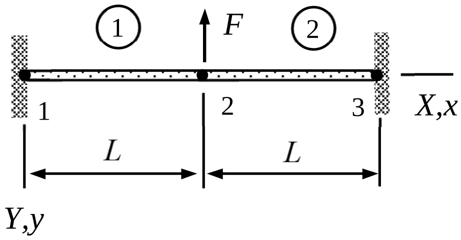
1p The corresponding eigenvector (mode) is given by

$$\begin{bmatrix} 4\frac{EI}{L} - 4\frac{p_{\mathrm{cr}}L}{30} & 2\frac{EI}{L} + \frac{p_{\mathrm{cr}}L}{30} \\ 2\frac{EI}{L} + \frac{p_{\mathrm{cr}}L}{30} & 4\frac{EI}{L} - 4\frac{p_{\mathrm{cr}}L}{30} \end{bmatrix}\begin{Bmatrix} \theta_{Y1} \\ \theta_{Y2} \end{Bmatrix} = \frac{72}{30}\frac{EI}{L}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} \theta_{Y1} \\ \theta_{Y2} \end{Bmatrix} = 0 \quad \Rightarrow \quad \begin{Bmatrix} \theta_{Y1} \\ \theta_{Y2} \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \text{ (say).}$$

Shape of the buckled beam follows from approximation when the mode is substituted there (see the formulae collection)

$$w(x)=\begin{Bmatrix}(1-\xi)^2(1+2\xi)\\ L(1-\xi)^2\xi\\ (3-2\xi)\xi^2\\ L\xi^2(\xi-1)\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}0\\ -1\\ 0\\ 1\end{Bmatrix}=-L(\xi-1)\xi \quad \text{where} \quad \xi=\frac{x}{L}\ .$$

Determine the displacement at node 2 of the elastic bar shown by the large deformation theory. Take into account only the transverse displacement u_{Y2} $(u_{X2}=0)$. When $F=0$, the cross-sectional area and length of the bar are A and L, respectively. Constitutive equation of the material is $S_{xx}=C\mathrm{E}_{xx}$, in which C is constant. Use two elements with linear shape functions.



Solution

Virtual work density of the non-linear bar model

$$\delta w_{\Omega^\circ}^{\text{int}}=-(\frac{d\delta u}{dx}+\frac{du}{dx}\frac{d\delta u}{dx}+\frac{dv}{dx}\frac{d\delta v}{dx}+\frac{dw}{dx}\frac{d\delta w}{dx})CA^\circ[\frac{du}{dx}+\frac{1}{2}(\frac{du}{dx})^2+\frac{1}{2}(\frac{dv}{dx})^2+\frac{1}{2}(\frac{dw}{dx})^2]$$

is based on the Green-Lagrange strain definition which is physically correct also when rotations/displacements are large. The expression depends on all displacement components, material property is denoted by C (constitutive equation $S_{xx}=C\mathrm{E}_{xx}$), and the superscript in the cross-sectional area A° (and in other quantities) refers to the initial geometry (strain and stress vanishes). Otherwise, equilibrium equations follow in the same manner as in the linear case.

4p For element 1, the non-zero displacement components is $u_{y2}=u_{Y2}$. As the initial length of the element $h^\circ=L$, linear approximations to the displacement components

$$u=w=0 \text{ and } v=\frac{x}{L}u_{Y2} \quad\Rightarrow\quad \frac{du}{dx}=\frac{dw}{dx}=0 \text{ and } \frac{dv}{dx}=\frac{u_{Y2}}{L}.$$

When the approximation is substituted there, virtual work density of the internal forces and thereby the virtual work expression (density is constant) simplify to

$$\delta w_{\Omega^\circ}^{\text{int}}=-\frac{\delta u_{Y2}}{L}\frac{u_{Y2}}{L}\frac{CA}{2}(\frac{u_{Y2}}{L})^2 \quad\Rightarrow\quad \delta W^1=-\delta u_{Y2}\frac{CA}{2}(\frac{u_{Y2}}{L})^3.$$

For element 2, the non-zero displacement component $u_{y2}=u_{Y2}$. As the initial length of the element $h^\circ=L$, linear approximations to the displacement components

$$u=w=0 \text{ and } v=(1-\frac{x}{L})u_{Y2} \quad\Rightarrow\quad \frac{du}{dx}=\frac{dw}{dx}=0 \text{ and } \frac{dv}{dx}=-\frac{u_{Y2}}{L}.$$

When the approximation is substituted there, virtual work density of the internal forces and thereby the virtual work expression (density is constant) simplifies to

$$\delta w_{\Omega^\circ}^{\text{int}}=-\frac{\delta u_{Y2}}{L}\frac{u_{Y2}}{L}\frac{CA}{2}(\frac{u_{Y2}}{L})^2 \quad\Rightarrow\quad \delta W^2=-\delta u_{Y2}\frac{CA}{2}(\frac{u_{Y2}}{L})^3.$$

Virtual work expression of the point force is

$$\delta W^3=-F\delta u_{Y2}.$$

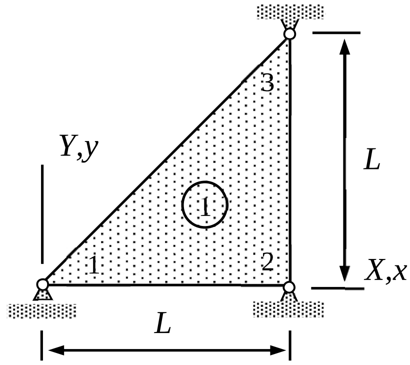
2p Virtual work expression of the structure is obtained as the sum of the element contributions

$$\delta W = -\delta u_{Y2}[\frac{CA}{2}(\frac{u_{Y2}}{L})^3 + \frac{CA}{2}(\frac{u_{Y2}}{L})^3 + F].$$

Principle of virtual work and the fundamental lemma of variation calculus imply that

$$(\frac{u_{Y2}}{L})^3 + \frac{F}{CA} = 0 \quad \Rightarrow \quad u_{Y2} = -(\frac{FL^3}{CA})^{1/3}. \quad \leftarrow$$

A thin triangular slab (plane stress conditions) is allowed to move horizontally at node 1 and nodes 2 and 3 are fixed. Stress is zero when temperature (assumed constant) is ϑ°. Determine the non-zero displacement component u_{X1}, if the temperature of slab is increased to $2\vartheta^{\circ}$.



Solution

As temperature is known and the external distributed force vanishes, the virtual work densities needed here are (formulae collection)

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta u/\partial y + \partial\delta v/\partial x \end{Bmatrix}^{\mathrm{T}} t[E]_{\sigma} \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix}, \quad \delta w_{\Omega}^{\text{cpl}} = \begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial x \end{Bmatrix}^{\mathrm{T}} \frac{E\alpha}{1-\nu} \int \Delta\vartheta dz \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

in which $\Delta\vartheta = \vartheta - \vartheta^{\circ}$ is the difference between temperature at the deformed and initial and deformed geometries. At the initial geometry stress is assumed to vanish and the integral

$$\int \Delta\vartheta dz = t\Delta\vartheta .$$

2p Approximation is the first thing to be considered. Linear shape functions can be deduced from the figure

$$N_1 = 1 - \frac{x}{L}, \quad N_3 = \frac{y}{L}, \quad \text{and} \quad N_2 = 1 - N_1 - N_3 = \frac{x-y}{L}.$$

Approximations to the displacement components and temperature difference are

$$u = (1 - \frac{x}{L}) u_{X1}, \quad v = 0, \quad \text{and} \quad \Delta\vartheta = \vartheta^{\circ}.$$

3p When the approximations are substituted there, virtual work densities take the forms

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} -\delta u_{X1}/L \\ 0 \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{Et}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} -u_{X1}/L \\ 0 \\ 0 \end{Bmatrix} = -\delta u_{X1} \frac{1}{L^2} \frac{Et}{1-\nu^2} u_{X1},$$

$$\delta w_{\Omega}^{\text{cpl}} = \begin{Bmatrix} -\delta u_{X1}/L \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{Et\alpha}{1-\nu} \vartheta^{\circ} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = -\delta u_{X1} \frac{1}{L} \frac{Et\alpha}{1-\nu} \vartheta^{\circ} \qquad \Rightarrow$$

$$\delta w_{\Omega} = \delta w_{\Omega}^{\text{int}} + \delta w_{\Omega}^{\text{cpl}} = -\frac{\delta u_{X1}}{L} \frac{Et}{1-\nu^2} \frac{u_{X1}}{L} - \frac{\delta u_{X1}}{L} \frac{Et}{1-\nu} \alpha\vartheta^{\circ}.$$

Virtual work expression is the integral of the density over the domain occupied by the element. Here, virtual work density is constant so that it is enough to multiply by the area:

$$\delta W = \delta w_\Omega \frac{L^2}{2} = -\delta u_{X1}\left(\frac{1}{2}\frac{Et}{1-\nu^2}u_{X1} + \frac{1}{2}\frac{Et}{1-\nu}L\alpha\vartheta^\circ\right).$$

1p Principle of virtual work and the fundamental lemma of variation calculus give

$$\frac{1}{2}\frac{Et}{1-\nu^2}u_{X1} + \frac{1}{2}\frac{Et}{1-\nu}L\alpha\vartheta^\circ = 0 \quad \Leftrightarrow \quad u_{X1} = -(1+\nu)\alpha L\vartheta^\circ. \quad \leftarrow$$