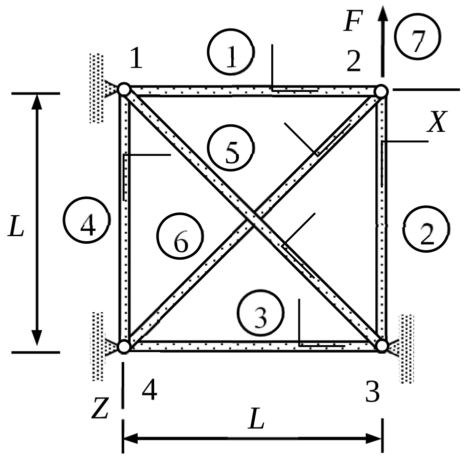
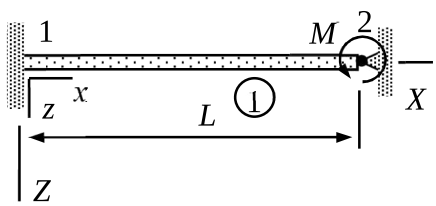
MEC-E8001 Finite Element Analysis, exam 17.04.2024

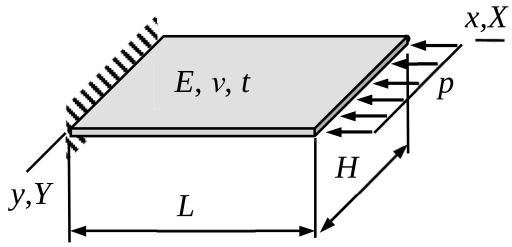
1. Determine the nodal displacements when force F is acting on the structure as shown. The cross-sectional area of bars 1,2,3 and 4 is A and the cross-sectional area of bars 5 and 6 is $2\sqrt{2}A$. Young's modulus of the material is E. Use the principle of virtual work.



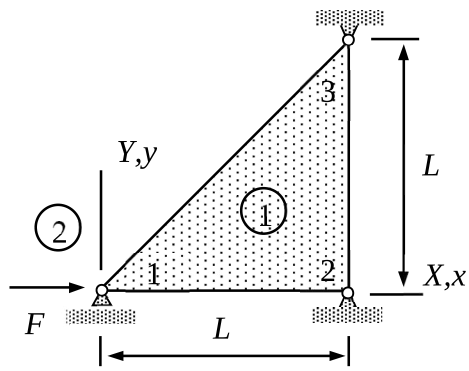
2. The beam of the figure is subjected to moment M when $t<0$. At $t=0$, the moment is suddenly removed and the beam starts to vibrate. Derive the initial value problem giving $\theta_{Y2}(t)$ for $t>0$. The beam is thin so that the rotational part of the inertia term is negligible. The geometrical quantities of the cross-section are A, I and the material constants E and ρ.



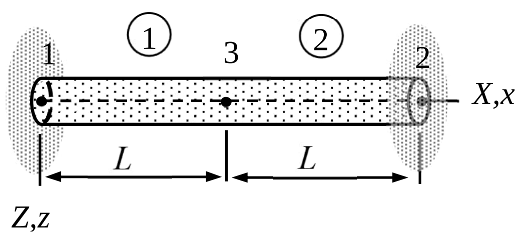
3. The clamping of the plate shown allows displacement in $y-$direction. At the free edge, the plate is loaded by distributed force p. Determine the critical value p_{cr} of the distributed force making the plate to buckle. Use the approximation $w(x,y)=a_0(x/L)^2$ and assume that $N_{xx}=-p$ and $N_{yy}=N_{xy}=0$. Material parameters E, ν and thickness of the plate t are constants.



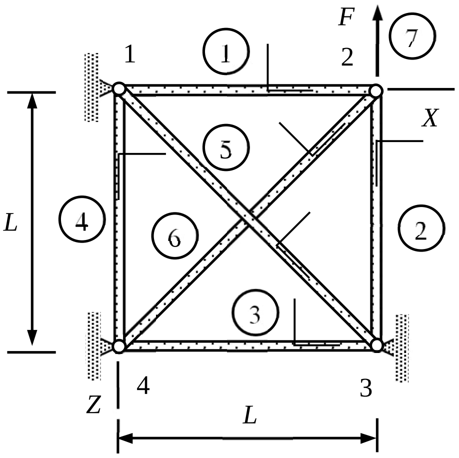
4. A thin triangular slab (assume plane stress conditions) loaded by a horizontal force can move horizontally at node 1 and nodes 2 and 3 are fixed. Derive the equilibrium equation for the structure according to the large displacement theory. Material parameters C, ν and thickness t at the initial geometry of the slab are constants.



5. Electric current causes heat generation in the bar shown. Calculate the temperature at the centre if the wall temperature (nodes 1 and 2) is $\vartheta°$. Cross sectional area A, thermal conductivity k, and heat production rate per unit length s are constants.



Determine the nodal displacements when force F is acting on the structure as shown. The cross-sectional area of bars 1,2,3 and 4 is A and the cross-sectional area of bars 5 and 6 is $2\sqrt{2}A$. Young's modulus of the material is E. Use the principle of virtual work.



Solution

Element and node tables contain the information needed in displacement and stress analysis of the structure. In hand calculations, it is often enough to complete the figure by the material coordinate systems and express the nodal displacements/rotations in terms symbols for the nodal displacements and rotations and/or values known a priori. The components in the material coordinate systems can also be deduced directly form the figure (in simple cases). Virtual work expression of the bar element is given by

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}\right).$$

5p Nodal displacements/rotations of the structure are zeros except u_{X2} and u_{Z2}. Element contributions in their virtual work forms are (nodal displacements of the material coordinate system need to be expressed in terms of the structural system components)

Bar 1: $u_{x1} = 0$, $u_{x2} = u_{X2}$: $\quad \delta W^1 = -\delta u_{X2} \dfrac{EA}{L} u_{X2}$,

Bar 2: $u_{x2} = u_{Z2}$, $u_{x3} = 0$: $\quad \delta W^2 = -\delta u_{Z2} \dfrac{EA}{L} u_{Z2}$,

Bar 3: $u_{x4} = 0$ and $u_{x3} = 0$: $\quad \delta W^3 = 0$,

Bar 4: $u_{x1} = 0$ and $u_{x4} = 0$: $\quad \delta W^4 = 0$,

Bar 5: $u_{x1} = 0$ and $u_{x3} = 0$: $\quad \delta W^5 = 0$,

Bar 6: $u_{x4} = 0$, $u_{x2} = \dfrac{1}{\sqrt{2}}(u_{X2} - u_{Z2})$: $\quad \delta W^6 = -(\delta u_{X2} - \delta u_{Z2})\dfrac{EA}{L}(u_{X2} - u_{Z2})$

Force 7: $\quad \delta W^7 = -\delta u_{Z2} F$.

Virtual work expression of the structure is sum of the element contributions

$$\delta W = \sum\nolimits_e \delta W^e = \delta W^1 + \delta W^2 + \delta W^3 + \delta W^4 + \delta W^5 + \delta W^6 + \delta W^7 \;\Rightarrow$$

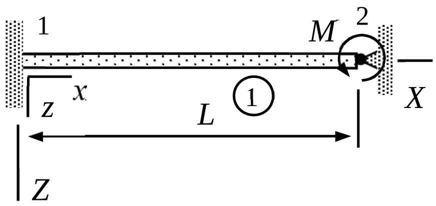
$$\delta W=-\delta u_{X2}\frac{EA}{L}u_{X2}-\delta u_{Z2}\frac{EA}{L}\mathrm{a}_2+0+0+0-(\delta u_{X2}-\delta u_{Z2})\frac{EA}{L}(u_{X2}-u_{Z2})-\delta u_{Z2}F \quad\Leftrightarrow$$

$$\delta W=-\begin{Bmatrix}\delta u_{X2}\\ \delta u_{Z2}\end{Bmatrix}^{\mathrm{T}}(\frac{EA}{L}\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}\begin{Bmatrix}u_{X2}\\ u_{Z2}\end{Bmatrix}-\begin{Bmatrix}0\\ -F\end{Bmatrix}).$$

1p Principle of virtual work $\delta W=0\ \forall\delta\mathbf{a}$ and the fundamental lemma of variation calculus in the form $\delta\mathbf{a}^{\mathrm{T}}\mathbf{R}=0\ \ \forall\delta\mathbf{a}\ \Leftrightarrow\ \mathbf{R}=0$ imply

$$\frac{EA}{L}\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}\begin{Bmatrix}u_{X2}\\ u_{Z2}\end{Bmatrix}-\begin{Bmatrix}0\\ -F\end{Bmatrix}=0 \quad\Leftrightarrow\quad \begin{Bmatrix}u_{X2}\\ u_{Z2}\end{Bmatrix}=\frac{FL}{EA}\frac{1}{3}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{Bmatrix}0\\ -1\end{Bmatrix}=-\frac{FL}{EA}\begin{Bmatrix}1/3\\ 2/3\end{Bmatrix}.$$

The beam of the figure is subjected to moment M when $t<0$. At $t=0$, the moment is suddenly removed and the beam starts to vibrate. Derive the initial value problem giving $\theta_{Y2}(t)$ for $t>0$. The beam is thin so that the rotational part of the inertia term is negligible. The geometrical quantities of the cross-section are A, I and the material constants E and ρ.



Solution

4p Virtual work expression consists of parts coming from internal and inertial forces. Finding the equation of motion is the first thing to do. The beam element contributions needed in the problem are (the term having to do with rotational inertia is omitted)

$$\delta W^{\text{int}} = -\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta\theta_{Y2} \end{Bmatrix}^{\text{T}} \frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \theta_{Y2} \end{Bmatrix} = -\delta\theta_{Y2} 4\frac{EI}{L}\theta_{Y2},$$

$$\delta W^{\text{ine}} = -\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta\theta_{Y2} \end{Bmatrix}^{\text{T}} \frac{\rho AL}{420}\begin{bmatrix} 156 & -22L & 54 & 13L \\ -22L & 4L^2 & -13L & -3L^2 \\ 54 & -13L & 156 & 22L \\ 13L & -3L^2 & 22L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \ddot{\theta}_{Y2} \end{Bmatrix} = -\delta\theta_{Y2}\frac{\rho AL^3}{105}\ddot{\theta}_{Y2}$$

giving

$$\delta W^1 = -\delta\theta_{Y2}(4\frac{EI}{L}\theta_{Y2} + \frac{\rho AL^3}{105}\ddot{\theta}_{Y2}).$$

In terms of moment $P(t)$ (positive in the positive direction of Y-axis) which is piecewise constant in time so that $P(t)=M \quad t\le 0$ and $P(t)=0 \quad t>0$, the element contribution of the moment is

$$\delta W^2 = \delta\theta_{Y2}P.$$

Virtual work expression is the sum of element contributions:

$$\delta W = \delta W^1 + \delta W^2 = -\delta\theta_{Y2}(4\frac{EI}{L}\theta_{Y2} + \frac{\rho AL^3}{105}\ddot{\theta}_{Y2} - P) = 0.$$

Principle of virtual work and the fundamental lemma of variation calculus imply the ordinary differential equation

$$\delta W = -\delta\theta_{Y2}(4\frac{EI}{L}\theta_{Y2} + \frac{\rho AL^3}{105}\ddot{\theta}_{Y2} - P) = 0 \quad \forall\delta\theta_{Y2} \quad \Leftrightarrow$$

$$4\frac{EI}{L}\theta_{Y2} + \frac{\rho AL^3}{105}\ddot{\theta}_{Y2} - P = 0. \quad \Leftarrow$$

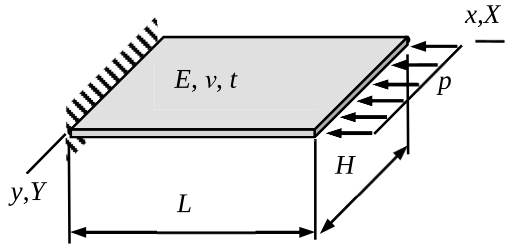
2p When $t \leq 0$, external moment $P = M$ is acting on node 2 and the system is at rest. Therefore, the equation of motion becomes an equilibrium equation giving as its solution the initial rotation

$$\theta_{Y2} = \frac{1}{4}\frac{ML}{EI}.$$

When $t > 0$, external moment is zero and acceleration does not vanish. The initial value problem giving as its solution $\theta_{Y2}(t)$ for $t > 0$ takes the form

$$4\frac{EI}{L}\theta_{Y2} + \frac{\rho A L^3}{105}\ddot{\theta}_{Y2} = 0 \quad t > 0, \quad \theta_{Y2}(0) = \frac{1}{4}\frac{ML}{EI}, \quad \text{and} \quad \dot{\theta}_{Y2}(0) = 0 \ .$$ ←

The clamping of the plate shown allows displacement in $y-$direction. At the free edge, the plate is loaded by distributed force p. Determine the critical value p_{cr} of the distributed force making the plate to buckle. Use the approximation $w(x,y)=a_0(x/L)^2$ and assume that $N_{xx}=-p$ and $N_{yy}=N_{xy}=0$. Material parameters E, ν and thickness of the plate t are constants.



Solution

Assuming that the material coordinate system is chosen so that the plate bending and thin slab modes decouple in the linear analysis and that the in-plane stress resultants are known (from linear displacement analysis, say), it is enough to consider the virtual work densities of plate bending mode and the coupling of the bending and thin-slab modes

$$\delta w_\Omega^{\mathrm{int}}=-\begin{Bmatrix}\partial^2\delta w/\partial x^2\\ \partial^2\delta w/\partial y^2\\ 2\partial^2\delta w/\partial x\partial y\end{Bmatrix}^{\mathrm{T}}\frac{t^3}{12}[E]_\sigma\begin{Bmatrix}\partial^2 w/\partial x^2\\ \partial^2 w/\partial y^2\\ 2\partial^2 w/\partial x\partial y\end{Bmatrix},\quad \delta w_\Omega^{\mathrm{sta}}=-\begin{Bmatrix}\partial\delta w/\partial x\\ \partial\delta w/\partial y\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}N_{xx} & N_{xy}\\ N_{yx} & N_{yy}\end{bmatrix}\begin{Bmatrix}\partial w/\partial x\\ \partial w/\partial y\end{Bmatrix}$$

where the elasticity matrix of plane stress

$$[E]_\sigma=\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & (1-\nu)/2\end{bmatrix}.$$

4p As the support at the clamped edge allows displacement in the $y-$direction, solution to the in-plane stress resultants $N_{xx}=-p$ and $N_{yy}=N_{xy}=0$ can be deduced without calculations. Approximation to the transverse displacement and its non-zero derivatives are given by

$$w(x,y)=a_0(\frac{x}{L})^2\quad\Rightarrow\quad\frac{\partial w}{\partial x}=2a_0\frac{x}{L^2}\text{ and }\frac{\partial^2 w}{\partial x^2}=2\frac{a_0}{L^2}.$$

When the approximation is substituted there, virtual work density of the internal forces and that of the coupling simplify to

$$\delta w_\Omega^{\mathrm{int}}=-\begin{Bmatrix}2\delta a_0/L^2\\ 0\\ 0\end{Bmatrix}^{\mathrm{T}}\frac{t^3}{12}\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & (1-\nu)/2\end{bmatrix}\begin{Bmatrix}2a_0/L^2\\ 0\\ 0\end{Bmatrix}=-\delta a_0\frac{1}{3}\frac{t^3}{L^4}\frac{E}{1-\nu^2}a_0,$$

$$\delta w_\Omega^{\mathrm{sta}}=-\begin{Bmatrix}2\delta a_0 x/L^2\\ 0\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}N_{xx} & N_{xy}\\ N_{yx} & N_{yy}\end{bmatrix}\begin{Bmatrix}2a_0 x/L^2\\ 0\end{Bmatrix}=\delta a_0 4x^2\frac{p}{L^4}a_0.$$

Virtual work expressions are integrals of the densities over the domain occupied by the plate

$$\delta W^{\text{int}} = \int_0^H \int_0^L \delta w_\Omega^{\text{int}} dxdy = -\delta a_0 \frac{1}{3}\frac{t^3}{L^3} H \frac{E}{1-\nu^2} a_0 ,$$

$$\delta W^{\text{sta}} = \int_0^H \int_0^L \delta w_\Omega^{\text{sta}} dxdy = \delta a_0 \frac{4}{3}\frac{H}{L} p a_0 .$$

1p Virtual work expression

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{sta}} = -\delta a_0 (\frac{1}{3}\frac{t^3}{L^3}\frac{H}{L}\frac{E}{1-\nu^2} - \frac{4}{3}\frac{H}{L} p) a_0 ,$$

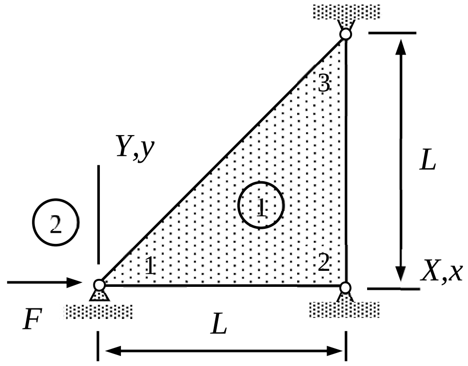
principle of virtual work $\delta W = 0 \ \forall \delta a_0$, and the fundamental lemma of variation calculus give

$$(\frac{1}{3}\frac{t^3}{L^3}\frac{H}{L}\frac{E}{1-\nu^2} - \frac{4}{3}\frac{H}{L} p) a_0 = 0 .$$

1p For a non-trivial solution $a_0 \neq 0$, the loading parameter needs to take the value

$$p_{\text{cr}} = \frac{1}{4}\frac{E}{1-\nu^2}\frac{t^3}{L^2} . \quad \leftarrow$$

A thin triangular slab (assume plane stress conditions) loaded by a horizontal force can move horizontally at node 1 and nodes 2 and 3 are fixed. Derive the equilibrium equation for the structure according to the large displacement theory. Material parameters C, ν and thickness t at the initial geometry of the slab are constants.



Solution

Virtual work density of internal force, when modified for large displacement analysis with the same constitutive equation as in the linear case of plane stress, is given by

$$\delta w_{\Omega^\circ}^{\text{int}} = -\begin{Bmatrix} \delta \mathrm{E}_{xx} \\ \delta \mathrm{E}_{yy} \\ 2\delta \mathrm{E}_{xy} \end{Bmatrix}^{\mathrm{T}} \frac{tC}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ 2\mathrm{E}_{xy} \end{Bmatrix}, \quad \begin{Bmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ 2\mathrm{E}_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} + \dfrac{1}{2}(\dfrac{\partial u}{\partial x})^2 + \dfrac{1}{2}(\dfrac{\partial v}{\partial x})^2 \\ \dfrac{\partial v}{\partial y} + \dfrac{1}{2}(\dfrac{\partial u}{\partial y})^2 + \dfrac{1}{2}(\dfrac{\partial v}{\partial y})^2 \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\dfrac{\partial v}{\partial y} \end{Bmatrix}.$$

2p Let us start with the approximations and the corresponding components of the Green-Lagrange strain. Linear shape functions can be deduced from the figure. Only the shape function $N_1 = (1 - x/L)$ of node 1 is needed. Displacement components $v = w = 0$ and

$$u = (1 - \frac{x}{L})u_{X1} \quad \Rightarrow \quad \frac{\partial u}{\partial x} = -\frac{u_{X1}}{L}, \quad \frac{\partial u}{\partial y} = 0, \quad \mathrm{E}_{yy} = \mathrm{E}_{xy} = 0 \quad \text{and} \quad \mathrm{E}_{xx} = -\frac{u_{X1}}{L} + \frac{1}{2}(-\frac{u_{X1}}{L})^2.$$

2p When the strain component expression are substituted there, virtual work density simplifies to

$$\delta w_{\Omega^\circ}^{\text{int}} = -\delta \mathrm{E}_{xx} \frac{tC}{1-\nu^2} \mathrm{E}_{xx} = -\frac{\delta u_{X1}}{L}(-1 + \frac{u_{X1}}{L}) \frac{tC}{1-\nu^2} \frac{u_{X1}}{L}(-1 + \frac{1}{2}\frac{u_{X1}}{L}).$$

Integration over the (initial) domain gives the virtual work expression. As the integrand is constant

$$\delta W^1 = -\frac{L^2}{2} \frac{\delta u_{X1}}{L}(-1 + \frac{u_{X1}}{L}) \frac{tC}{1-\nu^2} \frac{u_{X1}}{L}(-1 + \frac{1}{2}\frac{u_{X1}}{L})$$

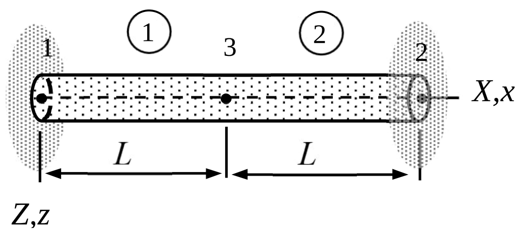
2p Virtual work expression of the point force follows from the definition of work

$$\delta W^2 = \delta u_{X1} F = \frac{\delta u_{X1}}{L} LF.$$

Virtual work expression of the structure is obtained as sum over the element contributions. In terms of the dimensionless displacement $\mathrm{a} = u_{X1}/L$

$$\delta W = -\frac{L^2}{2}\delta \mathrm{a}(-1+\mathrm{a})\frac{tC}{1-\nu^2}\mathrm{a}(-1+\frac{1}{2}\mathrm{a}) + \delta \mathrm{a} LF \quad \Rightarrow \quad \frac{L}{2}(-1+\mathrm{a})\frac{tC}{1-\nu^2}(-\mathrm{a}+\frac{1}{2}\mathrm{a}^2) - F = 0. \leftarrow$$

Electric current causes heat generation in the bar shown. Calculate the temperature at the centre if the wall temperature (nodes 1 and 2) is ϑ°. Cross sectional area A, thermal conductivity k, and heat production rate per unit length s are constants.



Solution

In a pure heat conduction problem, density expressions of the bar model are given by

$$\delta p_\Omega^{\text{int}} = -\frac{d\delta\vartheta}{dx} kA \frac{d\vartheta}{dx} \text{ and } \delta p_\Omega^{\text{ext}} = \delta\vartheta s$$

in which ϑ is the temperature, k the thermal conductivity, and s the rate of heat production (per unit length).

2p For bar 1, the nodal temperatures are $\vartheta_1 = \vartheta^\circ$ and ϑ_3 of which the latter is unknown. With a linear interpolation to temperature (notice that variation of ϑ° vanishes)

$$\vartheta = \begin{Bmatrix} 1 - x/L \\ x/L \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vartheta^\circ \\ \vartheta_3 \end{Bmatrix} = (1 - \frac{x}{L})\vartheta^\circ + \frac{x}{L}\vartheta_3 \quad \Rightarrow \quad \frac{d\vartheta}{dx} = \frac{\vartheta_3 - \vartheta^\circ}{L},$$

$$\delta\vartheta = \frac{x}{L}\delta\vartheta_3 \quad \Rightarrow \quad \frac{d\delta\vartheta}{dx} = \frac{\delta\vartheta_3}{L}.$$

When the approximation is substituted there, density expression $\delta p_\Omega = \delta p_\Omega^{\text{int}} + \delta p_\Omega^{\text{ext}}$ simplifies to

$$\delta p_\Omega = -\frac{\delta\vartheta_3}{L} kA \frac{\vartheta_3 - \vartheta^\circ}{L} + \frac{x}{L}\delta\vartheta_3 s,$$

Virtual work expression is the integral of the density over the element domain

$$\delta P^1 = \int_0^L \delta p_\Omega dx = -\delta\vartheta_3 (kA \frac{\vartheta_3 - \vartheta^\circ}{L} - \frac{1}{2} Ls).$$

2p The nodal temperatures of bar 2 are ϑ_3 and $\vartheta_2 = \vartheta^\circ$. Linear interpolation gives (variations of the given quantities like ϑ° vanish)

$$\vartheta = \begin{Bmatrix} 1 - x/L \\ x/L \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vartheta_3 \\ \vartheta^\circ \end{Bmatrix} = (1 - \frac{x}{L})\vartheta_3 + \frac{x}{L}\vartheta^\circ \quad \Rightarrow \quad \frac{d\vartheta}{dx} = \frac{\vartheta^\circ - \vartheta_3}{L},$$

$$\delta\vartheta = (1 - \frac{x}{L})\delta\vartheta_3 \quad \Rightarrow \quad \frac{d\delta\vartheta}{dx} = -\frac{\delta\vartheta_3}{L}.$$

When the approximation is substituted there, density expression $\delta p_\Omega = \delta p_\Omega^{\text{int}} + \delta p_\Omega^{\text{ext}}$ simplifies to

$$\delta p_\Omega = -(-\frac{\delta\vartheta_3}{L}) kA \frac{\vartheta^\circ - \vartheta_3}{L} + (1 - \frac{x}{L})\delta\vartheta_3 s.$$

Element contribution to the variational expressions is the integral of density over the element domain

$$\delta P^2 = \int_0^L \delta p_\Omega dx = -\delta\vartheta_3 (kA\frac{\vartheta_3 - \vartheta^\circ}{L} - \frac{L}{2} s) .$$

2p Variational expression is sum of the element contributions

$$\delta P = \delta P^1 + \delta P^2 = -\delta\vartheta_3 (2kA\frac{\vartheta_3 - \vartheta^\circ}{L} - Ls) .$$

Variation principle $\delta P = 0 \quad \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus give

$$2\frac{kA}{L}(\vartheta_3 - \vartheta^\circ) - Ls = 0 \quad \Leftrightarrow \quad \vartheta_3 = \vartheta^\circ + \frac{1}{2}\frac{L^2 s}{kA} .$$