Name______________________________ Student number____________________

# Assignment 3

Consider the equations of motion

$$\frac{GJ}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}\theta_{X1}\\ \theta_{X2}\end{Bmatrix}+\frac{\rho LJ}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix}\ddot{\theta}_{X1}\\ \ddot{\theta}_{X2}\end{Bmatrix}=0$$

for the end point rotations of a certain torsion bar of length $L$. Above, $J$ is the second moment of area with respect to the axis of the bar (polar moment), $G$ is the shear modulus, and $\rho$ is the density of material. Derive the angular speeds and the corresponding modes of the free vibrations.

**Solution template**

The set of ordinary differential equations as given by the principle of virtual work $\mathbf{M}\ddot{\mathbf{a}}+\mathbf{K}\mathbf{a}=0$ consists of the inertia and stiffness parts. The symmetric mass matrix $\mathbf{M}$ and the stiffness matrix $\mathbf{K}$ depend on the structure. Angular speeds of the free vibrations are the eigenvalues of $\mathbf{\Omega}=\sqrt{\mathbf{M}^{-1}\mathbf{K}}$. In practice, it is easier to calculate first the eigenvalues $\mathbf{\Omega}^2=\mathbf{M}^{-1}\mathbf{K}$ as the eigenvalues of $\mathbf{\Omega}$ are the square roots of those for $\mathbf{\Omega}^2$ and the eigenvectors coincide.

In the present case, the matrices are

$$\mathbf{K}=\frac{GJ}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} \text{ and } \mathbf{M}=\frac{\rho JL}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

Therefore

$$\mathbf{\Omega}^2=\mathbf{M}^{-1}\mathbf{K}=\begin{bmatrix}\_\_\_\_\_\_\_\_ & \_\_\_\_\_\_\_\_\\ \_\_\_\_\_\_\_\_ & \_\_\_\_\_\_\_\_\end{bmatrix}.$$

In the eigenvalue problem of matrix $\mathbf{A}$, the goal is to find all pairs $(\lambda,\mathbf{x})$ such that $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$. The linear homogeneous equation system can have a non-zero solution only if $\det(\mathbf{A}-\lambda\mathbf{I})=0$. The eigenvalues are obtained as solutions to this characteristic equation. The characteristic equation for the eigenvalues of $\mathbf{\Omega}^2$ is

$$\det(\mathbf{\Omega}^2-\lambda\mathbf{I})=\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_=0.$$

The two solutions for the eigenvalues are ($(a-\lambda)^2-b^2=0 \Leftrightarrow \lambda=a\pm b$)

$\lambda_1=$ ________ and $\lambda_2=$ ________ .

The corresponding eigenvectors are obtained as solutions to $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$. The eigenvectors are not unique and it is enough to find some of them. However, the eigenvectors should be linearly independent so that, e.g., the zero vector is not a valid choice.

$$\lambda_1: \qquad \begin{bmatrix} \underline{\qquad\qquad} & \underline{\qquad\qquad} \\ \underline{\qquad\qquad} & \underline{\qquad\qquad} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0 \quad \Rightarrow \quad \mathbf{x}_1 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} \underline{\qquad\qquad} \\ \underline{\qquad\qquad} \end{Bmatrix},$$

$$\lambda_2: \qquad \begin{bmatrix} \underline{\qquad\qquad} & \underline{\qquad\qquad} \\ \underline{\qquad\qquad} & \underline{\qquad\qquad} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0 \quad \Rightarrow \quad \mathbf{x}_2 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} \underline{\qquad\qquad} \\ \underline{\qquad\qquad} \end{Bmatrix}.$$

The representation of the matrix in terms of its eigenvalues and eigenvectors $\mathbf{\Omega}^2 = \mathbf{X}\boldsymbol{\lambda}\mathbf{X}^{-1}$ implies that $\mathbf{\Omega} = \mathbf{X}\sqrt{\boldsymbol{\lambda}}\mathbf{X}^{-1}$. As taking a square root of the diagonal matrix means just taking the square roots of the diagonal terms, the angular speeds of the free vibrations

$$(\omega_1, \mathbf{x}_1) = (\sqrt{\lambda_1}, \mathbf{x}_1) = (\underline{\qquad\quad}, \begin{Bmatrix} \underline{\qquad\quad} \\ \underline{\qquad\quad} \end{Bmatrix}) \quad \text{and} \quad (\omega_2, \mathbf{x}_2) = (\sqrt{\lambda_2}, \mathbf{x}_2) = (\underline{\qquad\quad}, \begin{Bmatrix} \underline{\qquad\quad} \\ \underline{\qquad\quad} \end{Bmatrix}).$$