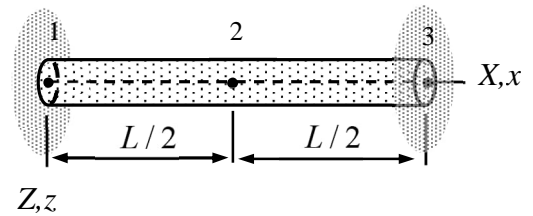# MEC-E8001 Finite Element Analysis, week 7/2024

1. The variational densities (correspond to virtual work densities of a displacement problem) of a heat conduction problem in a bar are given by $\delta p_\Omega^{int} = -(d\delta\vartheta/dx)kA(d\vartheta/dx)$ and $\delta p_\Omega^{ext} = \delta\vartheta s$ in which ϑ is the temperature, A is the cross-sectional area, k is the thermal conductivity, and s is the rate of heat production per unit length. Determine the element contributions δP^{int} and δP^{ext} if the approximation to temperature is linear, length of the element is h, and the given functions of the density expression are constants.

   Answer $\delta P^{int} = -\begin{Bmatrix} \delta\vartheta_1 \\ \delta\vartheta_2 \end{Bmatrix}^T \dfrac{kA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \vartheta_1 \\ \vartheta_2 \end{Bmatrix}$, $\delta P^{ext} = \begin{Bmatrix} \delta\vartheta_1 \\ \delta\vartheta_2 \end{Bmatrix}^T \dfrac{sh}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$.

2. Determine the stationary displacement u_{X2} and temperature ϑ_2 at node 2, when the temperature of the left and right ends are ϑ° and $2\vartheta^\circ$, respectively. Use just one three node quadratic element. Stress is zero initially when the temperature in the wall and bar is ϑ°. Problem parameters E, A, k, and α are constants.

   Answer $u_{X2} = -\dfrac{1}{8}L\alpha\vartheta^\circ$, $\vartheta_2 = \dfrac{3}{2}\vartheta^\circ$

3. Determine the static displacements $u_{Z2} = -u_{Z3}$ of nodes 2 and 3 due to the temperature increase $\Delta\vartheta$ at nodes 2 and 3 (actually in the wall). The material constants are E and α. The cross-sectional area of bar 1 and 3 is A and that of bar 2 is $\sqrt{2}A$. The initial temperature is ϑ°.
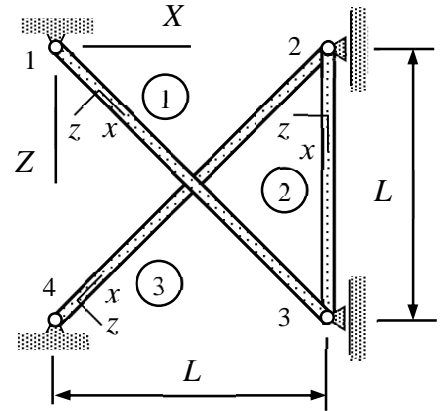
   **Answer** $u_{Z2} = -u_{Z3} = -\dfrac{5}{9}L\alpha\Delta\vartheta$

4. The truss shown consists of bars having the same cross-sectional area A, Young's modulus E, coefficient of thermal expansion α, and thermal conductivity k. The truss is stress-free when the initial temperature of all the nodes is ϑ°. Determine the stationary displacement u_{X1} of node 1, when the temperature of node 2 is changed to $2\vartheta^\circ$ and nodes 1, 3 and 4 are in temperature ϑ°.
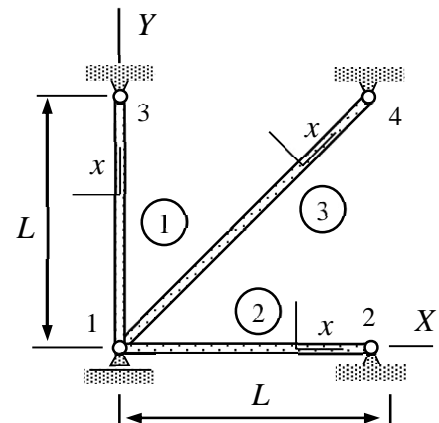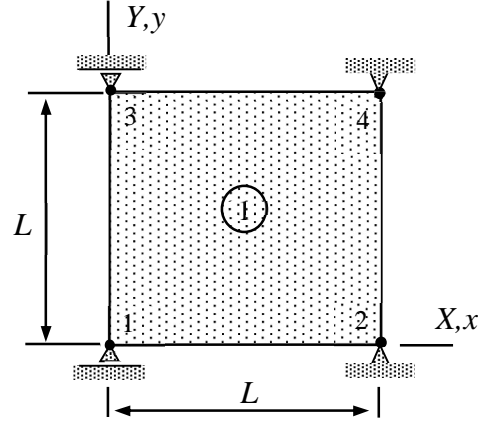
   Answer $u_{X1} = -\dfrac{2}{4+\sqrt{2}}L\alpha\vartheta^\circ$

5. A thin triangular slab (plane stress conditions) is allowed to move horizontally at node 1 and nodes 2 and 3 are fixed. Stress is zero when temperature (assumed constant) is ϑ°. Determine the non-zero displacement component u_{X1}, if the temperature of slab is increased to $2\vartheta^\circ$.

**Answer** $u_{X1} = -(1+\nu)\alpha L\vartheta^\circ$
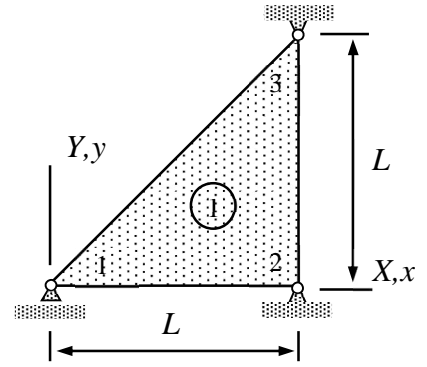
6. Nodes 1 and 3 of a thin rectangular slab (assume plane stress conditions) shown are allowed to move horizontally and nodes 2 and 4 are fixed. Stress is zero when temperature is ϑ°. Determine the displacement components $u_{X1} = u_{X3}$ if the temperature of slab is increased to $2\vartheta^\circ$. Also, determine the strain and stress in the slab. Material parameters and thickness are E, ν, α and t, respectively.

Answer $u_{X1} = -L\alpha\vartheta^\circ(1+\nu)$, $\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \alpha\vartheta^\circ(1+\nu)\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$, $\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = -E\alpha\vartheta^\circ\begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$

7. Determine the stationary temperature distribution in a thin slab shown. Edge 1-2 is at constant temperature ϑ° and heat flux through the other edges vanishes. Use a two-triangle mesh with ϑ_3 and $\vartheta_4 = \vartheta_3$ as the unknown node temperatures and consider $\vartheta_1 = \vartheta_2 = \vartheta^\circ$ as known. Thickness t, thermal conductivity k, and heat production rate per unit area s are constants.

Answer $\vartheta_3 = \vartheta^\circ + \dfrac{1}{2}\dfrac{sL^2}{tk}$
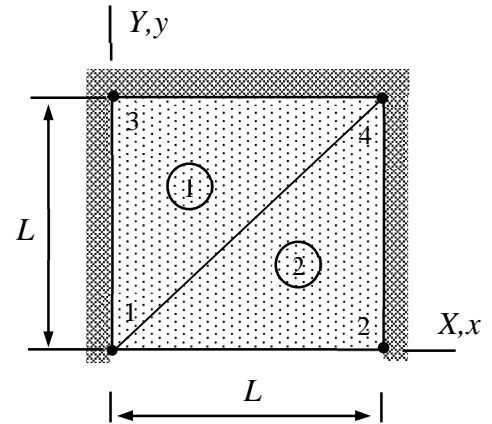
8. Determine the stationary temperature distribution in a thin slab shown. Edge 1-2 is at constant temperature ϑ° and heat flux through the other edges vanishes. Use a rectangle element with bilinear approximation and consider $\vartheta_1 = \vartheta_2 = \vartheta^\circ$ as known and $\vartheta_4 = \vartheta_3$ as the unknown nodal temperatures. Thickness t, thermal conductivity k, and heat production rate per unit area s are constants.

**Answer** $\vartheta_3 = \vartheta^\circ + \dfrac{1}{2}\dfrac{sL^2}{tk}$
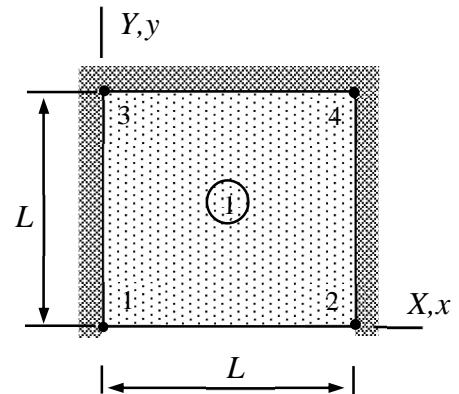
9. At the initial temperature ϑ° and without external forces, the length of the bar shown is L. Calculate the displacement of node 2 due to the combined effect of gravity and change of temperature with the nodal values $\vartheta_1 = 2\vartheta^\circ$ and $\vartheta_2 = \vartheta^\circ$. Cross sectional area A, coefficient of thermal expansion α, and density ρ are considered as constants. Use linear interpolation to displacement and temperature and start with the virtual work density expressions.
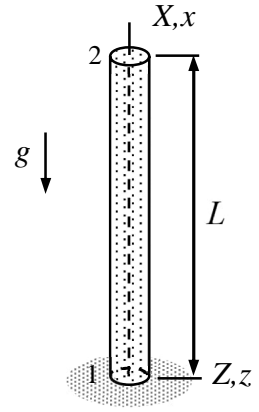
Answer $u_{X2} = \dfrac{\alpha}{2}L\vartheta^\circ - \dfrac{\rho g}{2E}L^2$

10. The simply supported plate shown is assembled at constant temperature $3\vartheta^\circ$. Find the transverse displacement when the upper side temperature is $4\vartheta^\circ$ and that of the lower side $2\vartheta^\circ$. Assume that temperature in plate is linear in z and does not depend on x or y. Use $w(x, y) = a\sin(\pi x / L)\sin(\pi y / L)$ as the approximation. Problem parameters E, v, ρ, α and t are constants. Integrals of sin and cos functions satisfy
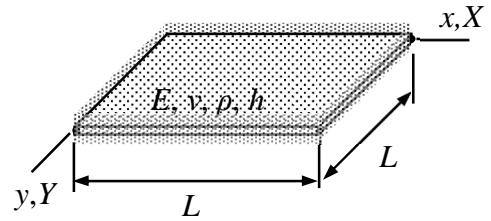
$$\int_0^L \sin^2(\pi \tfrac{x}{L})dx = \int_0^L \cos^2(\pi \tfrac{x}{L})dx = \frac{L}{2} \quad \text{and} \quad \int_0^L \sin(\pi \tfrac{x}{L})dx = \frac{2L}{\pi}.$$

Answer $w(x, y) = -\dfrac{16}{\pi^4}\dfrac{\alpha\vartheta^\circ L^2}{t}(1+v)\sin(\pi \tfrac{x}{L})\sin(\pi \tfrac{y}{L})$

The variational densities (correspond to virtual work densities of a displacement problem) of a heat conduction problem in a bar are given by $\delta p_\Omega^{\text{int}} = -(d\delta\vartheta/dx)kA(d\vartheta/dx)$ and $\delta p_\Omega^{\text{ext}} = \delta\vartheta s$ in which ϑ is the temperature, A is the cross-sectional area, k is the thermal conductivity, and s is the rate of heat production per unit length. Determine the element contributions δP^{int} and δP^{ext} if the approximation to temperature is linear, length of the element is h, and the given functions of the density expression are constants.

Solution

In a pure heat conduction problem, density expressions of the bar model are given by

$$\delta p_\Omega^{\text{int}} = -\frac{d\delta\vartheta}{dx}kA\frac{d\vartheta}{dx} \quad \text{and} \quad \delta p_\Omega^{\text{ext}} = \delta\vartheta s$$

in which ϑ is the temperature, k the thermal conductivity, and s the rate of heat production (per unit length). Although the physical meanings of the quantities differ from those of the displacement problem, finite element method works in the same manner. In particularly, the element contributions are derived in the same manner.

Assuming an element of size h and nodal values ϑ_1 and ϑ_2, the linear approximation to temperature, its variation, and their derivatives become

$$\vartheta = \frac{1}{h}\left\{\begin{matrix} h-x \\ x \end{matrix}\right\}^{\text{T}}\left\{\begin{matrix} \vartheta_1 \\ \vartheta_2 \end{matrix}\right\} \quad \Rightarrow \quad \frac{d\vartheta}{dx} = \frac{1}{h}\left\{\begin{matrix} -1 \\ 1 \end{matrix}\right\}^{\text{T}}\left\{\begin{matrix} \vartheta_1 \\ \vartheta_2 \end{matrix}\right\} \quad \text{and}$$

$$\delta\vartheta = \frac{1}{h}\left\{\begin{matrix} h-x \\ x \end{matrix}\right\}^{\text{T}}\left\{\begin{matrix} \delta\vartheta_1 \\ \delta\vartheta_2 \end{matrix}\right\} = \left\{\begin{matrix} \delta\vartheta_1 \\ \delta\vartheta_2 \end{matrix}\right\}^{\text{T}}\frac{1}{h}\left\{\begin{matrix} h-x \\ x \end{matrix}\right\} \quad \Rightarrow \quad \frac{d\delta\vartheta}{dx} = \frac{1}{h}\left\{\begin{matrix} -1 \\ 1 \end{matrix}\right\}^{\text{T}}\left\{\begin{matrix} \delta\vartheta_1 \\ \delta\vartheta_2 \end{matrix}\right\} = \left\{\begin{matrix} \delta\vartheta_1 \\ \delta\vartheta_2 \end{matrix}\right\}^{\text{T}}\frac{1}{h}\left\{\begin{matrix} -1 \\ 1 \end{matrix}\right\}.$$

When the approximations are substituted there, the variational density expressions take the forms

$$\delta p_\Omega^{\text{int}} = -\left\{\begin{matrix} \delta\vartheta_1 \\ \delta\vartheta_2 \end{matrix}\right\}^{\text{T}}\frac{1}{h}\left\{\begin{matrix} -1 \\ 1 \end{matrix}\right\}kA\frac{1}{h}\left\{\begin{matrix} -1 \\ 1 \end{matrix}\right\}^{\text{T}}\left\{\begin{matrix} \vartheta_1 \\ \vartheta_2 \end{matrix}\right\} = -\left\{\begin{matrix} \delta\vartheta_1 \\ \delta\vartheta_2 \end{matrix}\right\}^{\text{T}}\frac{kA}{h^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\left\{\begin{matrix} \vartheta_1 \\ \vartheta_2 \end{matrix}\right\} \quad \text{and}$$

$$\delta p_\Omega^{\text{ext}} = \left\{\begin{matrix} \delta\vartheta_1 \\ \delta\vartheta_2 \end{matrix}\right\}^{\text{T}}\frac{1}{h}\left\{\begin{matrix} h-x \\ x \end{matrix}\right\}s.$$

Element contributions are obtained as integrals over the domain occupied by the element

$$\delta P^{\text{int}} = \int_0^h \delta p_\Omega^{\text{int}}dx = -\left\{\begin{matrix} \delta\vartheta_1 \\ \delta\vartheta_2 \end{matrix}\right\}^{\text{T}}\frac{kA}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\left\{\begin{matrix} \vartheta_1 \\ \vartheta_2 \end{matrix}\right\}, \quad \Leftarrow$$

$$\delta P^{\text{ext}} = \int_0^h \delta p_\Omega^{\text{ext}}dx = \left\{\begin{matrix} \delta\vartheta_1 \\ \delta\vartheta_2 \end{matrix}\right\}^{\text{T}}\frac{sh}{2}\left\{\begin{matrix} 1 \\ 1 \end{matrix}\right\}. \quad \Leftarrow$$

Determine the stationary displacement u_{X2} and temperature ϑ_2 at node 2, when the temperature of the left and right ends are ϑ° and $2\vartheta^\circ$, respectively. Use just one three node quadratic element. Stress is zero initially when the temperature in the wall and bar is ϑ°. Problem parameters E, A, k, and α are constants.

Solution

In a temperature dependent case, variational density expressions of the bar model are

$$\delta w_\Omega^{\text{int}} = -\frac{d\delta u}{dx} EA \frac{du}{dx}, \quad \delta w_\Omega^{\text{cpl}} = \frac{d\delta u}{dx} EA\alpha\Delta\vartheta, \quad \text{and} \quad \delta p_\Omega^{\text{int}} = -\frac{d\delta\vartheta}{dx} kA \frac{d\vartheta}{dx}.$$

In the second expression, $\Delta\vartheta = \vartheta - \vartheta^\circ$ is the temperature difference between the deformed and initial geometries (same material point). Variational expression is of the form $\delta W + \tau\delta P$ in which τ is an arbitrary but dimensionally correct multiplier (expression should be dimensionally homogeneous). The coupling in the stationary thermo-mechanical problem is one-sided so that it is possible to solve for the temperature first.

Approximation with the three-node element is quadratic. The shape functions can be deduced from the figure $N_1 = (1-\xi)(1-2\xi)$, $N_2 = 4(1-\xi)\xi$ and $N_3 = \xi(2\xi-1)$ in which $\xi = x/L$. The non-zero nodal displacements and temperatures are $u_{x2} = u_{X2}$ and ϑ_2 (material and structural coordinate systems coincide here). Therefore

$$u = \left\{ \begin{array}{c} (1-\xi)(1-2\xi) \\ 4(1-\xi)\xi \\ \xi(2\xi-1) \end{array} \right\}^{\text{T}} \left\{ \begin{array}{c} 0 \\ u_{x2} \\ 0 \end{array} \right\} = 4(1-\frac{x}{L})\frac{x}{L}u_{X2} \quad \Rightarrow \quad \frac{du}{dx} = 4\frac{1}{L}(1-2\frac{x}{L})u_{X2},$$

$$\vartheta = \left\{ \begin{array}{c} (1-\xi)(1-2\xi) \\ 4(1-\xi)\xi \\ \xi(2\xi-1) \end{array} \right\}^{\text{T}} \left\{ \begin{array}{c} \vartheta^\circ \\ \vartheta_2 \\ 2\vartheta^\circ \end{array} \right\} = [1-5\frac{x}{L}+6(\frac{x}{L})^2]\vartheta^\circ + 4\frac{x}{L}(1-\frac{x}{L})\vartheta_2 \quad \Rightarrow$$

$$\frac{d\vartheta}{dx} = \frac{1}{L}(-5+12\frac{x}{L})\vartheta^\circ + 4\frac{1}{L}(1-2\frac{x}{L})\vartheta_2.$$

Temperature difference between the deformed and initial geometries is

$$\Delta\vartheta = \vartheta - \vartheta^\circ = \frac{x}{L}(6\frac{x}{L}-5)\vartheta^\circ + 4\frac{x}{L}(1-\frac{x}{L})\vartheta_2.$$

When the approximations are substituted here, density expressions $\delta w_\Omega = \delta w_\Omega^{\text{int}} + \delta w_\Omega^{\text{cpl}}$ and $\delta p_\Omega = \delta p_\Omega^{\text{int}}$ simplify to

$$\delta w_\Omega = -\delta u_{X2}\frac{EA}{L^2}[4(1-2\frac{x}{L})]^2 u_{X2} + 4(1-2\frac{x}{L})\delta u_{X2}\frac{EA}{L}\alpha[\frac{x}{L}(6\frac{x}{L}-5)\vartheta^\circ + 4\frac{x}{L}(1-\frac{x}{L})\vartheta_2],$$

$$\delta p_\Omega = -\delta \vartheta_2 kA[4\frac{1}{L}(1-2\frac{x}{L})]^2 \vartheta_2 .$$

Element contributions are integrals of the densities over the element domain

$$\delta W = \int_0^L \delta w_\Omega dx = -\delta u_{X2}(\frac{16}{3}\frac{AE}{L}u_{X2} + \frac{2}{3}AE\alpha\vartheta^\circ),$$

$$\delta P = \int_0^L \delta p_\Omega dx = -\delta\vartheta_2(\frac{16}{3}\frac{Ak}{L}\vartheta_2 - 8\frac{Ak}{L}\vartheta^\circ).$$

Variation principle and the fundamental lemma of variation calculus give the equations

$$\frac{16}{3}\frac{AE}{L}u_{X2} + \frac{2}{3}AE\alpha\vartheta^\circ = 0 \quad \text{and} \quad \frac{16}{3}\frac{Ak}{L}\vartheta_2 - 8\frac{Ak}{L}\vartheta^\circ = 0 \quad \Leftrightarrow$$

$$u_{X2} = -\frac{1}{8}L\alpha\vartheta^\circ \quad \text{and} \quad \vartheta_2 = \frac{3}{2}\vartheta^\circ . \quad \leftarrow$$

Determine the static displacements $u_{Z2} = -u_{Z3}$ of nodes 2 and 3 due to the temperature increase $\Delta\vartheta$ at nodes 2 and 3 (actually in the wall). The material constants are E and α. The cross-sectional area of bar 1 and 3 is A and that of bar 2 is $\sqrt{2}A$. The initial temperature is $\vartheta°$.

Solution

As temperature is known and the external distributed force vanishes, only the virtual work expressions of the internal and coupling parts

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\text{T}} \frac{EA}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} \quad \text{and} \quad \delta W^{\text{cpl}} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x1} \end{Bmatrix}^{\text{T}} \frac{\alpha EA}{2}\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} \Delta\vartheta_1 \\ \Delta\vartheta_2 \end{Bmatrix}$$

are needed in the calculations. Term $\Delta\vartheta = \vartheta - \vartheta°$ is the difference between temperature at the deformed and initial geometries.

The nodal displacements and temperatures of bar 1 $u_{x1} = 0$, $u_{x3} = u_{Z3}/\sqrt{2}$, $\Delta\vartheta_1 = \vartheta° - \vartheta° = 0$, and $\Delta\vartheta_3 = \Delta\vartheta$ give (notice that the variation of a given function is always zero)

$$\delta W^1 = -\begin{Bmatrix} 0 \\ \delta u_{Z3}/\sqrt{2} \end{Bmatrix}^{\text{T}}\left(\frac{EA}{\sqrt{2}L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_{Z3}/\sqrt{2} \end{Bmatrix} - \frac{\alpha EA}{2}\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ \Delta\vartheta \end{Bmatrix}\right) \quad \Leftrightarrow$$

$$\delta W^1 = -\delta u_{Z3}\left(\frac{EA}{2\sqrt{2}L}u_{Z3} - \frac{\alpha EA}{2\sqrt{2}}\Delta\vartheta\right).$$

The nodal displacements and temperatures of bar 2 $u_{x2} = u_{Z2} = -u_{Z3}$, $u_{x3} = u_{Z3}$, $\Delta\vartheta_2 = \Delta\vartheta$, and $\Delta\vartheta_3 = \Delta\vartheta$ give

$$\delta W^2 = -\begin{Bmatrix} -\delta u_{Z3} \\ \delta u_{Z3} \end{Bmatrix}^{\text{T}}\left(\frac{E\sqrt{2}A}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} -u_{Z3} \\ u_{Z3} \end{Bmatrix} - \frac{\alpha E\sqrt{2}A}{2}\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} \Delta\vartheta \\ \Delta\vartheta \end{Bmatrix}\right) \quad \Leftrightarrow$$

$$\delta W^2 = -\delta u_{Z3}\left(4\sqrt{2}\frac{EA}{L}u_{Z3} - 4\sqrt{2}\frac{\alpha EA}{2}\Delta\vartheta\right).$$

The nodal displacements and temperatures of bar 3 $u_{x4} = 0$, $u_{x2} = -u_{Z2}/\sqrt{2} = u_{Z3}/\sqrt{2}$, $\Delta\vartheta_1 = \vartheta° - \vartheta° = 0$, and $\Delta\vartheta_3 = \Delta\vartheta$ give

$$\delta W^3 = -\begin{Bmatrix} 0 \\ \delta u_{Z3}/\sqrt{2} \end{Bmatrix}^{\text{T}}\left(\frac{EA}{\sqrt{2}L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_{Z3}/\sqrt{2} \end{Bmatrix} - \frac{\alpha EA}{2}\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ \Delta\vartheta \end{Bmatrix}\right) \quad \Leftrightarrow$$

$$\delta W^3 = -\delta u_{Z3}\left(\frac{EA}{2\sqrt{2}L}u_{Z3} - \frac{\alpha EA}{2\sqrt{2}}\Delta\vartheta\right).$$

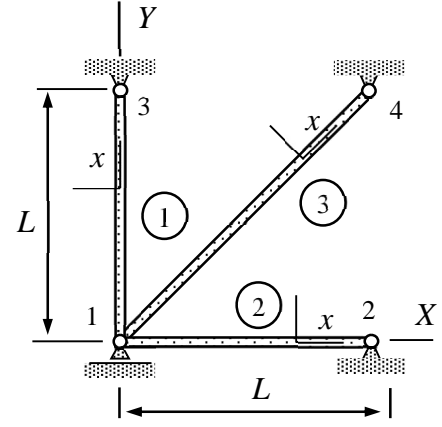Virtual work expression of the structure is the sum of element contributions

$$\delta W = -\delta u_{Z3} 2(\frac{EA}{2\sqrt{2}L} u_{Z3} - \frac{\alpha EA}{2\sqrt{2}} \Delta\vartheta) - \delta u_{Z3}(4\sqrt{2}\frac{EA}{L} u_{Z3} - 4\sqrt{2}\frac{\alpha EA}{2} \Delta\vartheta) \quad \Leftrightarrow$$

$$\delta W = -\delta u_{Z3}(9\frac{EA}{\sqrt{2}L} u_{Z3} - 5\frac{\alpha EA}{\sqrt{2}} \Delta\vartheta).$$

Principle of virtual work and the fundamental lemma of variation calculus imply

$$\frac{9}{\sqrt{2}L} EA u_{Z3} - \frac{5}{\sqrt{2}} EA\alpha\Delta\vartheta = 0 \quad \Leftrightarrow \quad u_{Z3} = \frac{5}{9}\alpha L\Delta\vartheta. \qquad \textcolor{red}{\leftarrow}$$

The truss shown consists of bars having the same cross-sectional area A, Young's modulus E, coefficient of thermal expansion α, and thermal conductivity k. The truss is stress-free when the initial temperature of all the joints is $\vartheta°$. Determine the stationary displacement u_{X1} of node 1, when the temperature of node 2 is changed to $2\vartheta°$ and nodes 1, 3 and 4 are in temperature $\vartheta°$.

Solution

Let us start with the virtual work density although also the virtual work expressions are available in the formulae collection. As temperature is known and external distributed force vanishes, the virtual work density simplifies to

$$\delta w_\Omega = \delta w_\Omega^{int} + \delta w_\Omega^{cpl} = -\frac{d\delta u}{dx} EA(\frac{du}{dx} - \alpha\Delta\vartheta).$$

The nodal displacements and temperatures of bar 1 are $u_{x1} = u_{x3} = 0$, and $\Delta\vartheta_1 = \Delta\vartheta_3 = 0$. Using linear approximations to the axial displacement and temperature

$$u = 0 \text{ and } \Delta\vartheta = 0 \quad \Rightarrow \quad \delta w_\Omega = -\frac{d\delta u}{dx} EA(\frac{du}{dx} - \alpha\Delta\vartheta) = 0 \quad \Rightarrow \quad \delta W^1 = \int_0^L \delta w_\Omega dx = 0.$$

The nodal displacements and temperatures of bar 2 are $u_{x1} = u_{X1}$, $u_{x2} = 0$, $\Delta\vartheta_1 = 0$, and $\Delta\vartheta_2 = 2\vartheta° - \vartheta° = \vartheta°$. With the linear approximations to axial displacement and temperature

$$u = (1 - \frac{x}{L})u_{X1} \text{ and } \Delta\vartheta = \frac{x}{L}\vartheta° \quad \Rightarrow \quad \delta w_\Omega = -(-\frac{\delta u_{X1}}{L})EA(-\frac{u_{X1}}{L} - \alpha\frac{x}{L}\vartheta°) \quad \Rightarrow$$

$$\delta W^2 = \int_0^L \delta w_\Omega dx = -\delta u_{X1} EA(\frac{u_{X1}}{L} + \alpha\frac{L}{2}\vartheta°).$$

The nodal displacements and temperatures of bar 3 are $u_{x1} = u_{X1}/\sqrt{2}$, $u_{x4} = 0$, $\Delta\vartheta_1 = 0$, and $\Delta\vartheta_4 = 0$. With the linear approximations to axial displacement and temperature

$$u = (1 - \frac{x}{\sqrt{2}L})\frac{u_{X1}}{\sqrt{2}} \text{ and } \Delta\vartheta = 0 \quad \Rightarrow \quad \delta w_\Omega = -(-\frac{\delta u_{X1}}{2L})EA(-\frac{u_{X1}}{2L}) \quad \Rightarrow$$

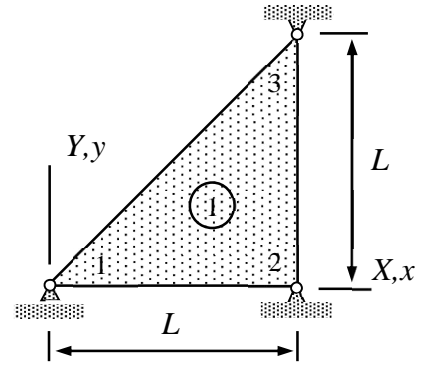$$\delta W^3 = \int_0^{\sqrt{2}L} \delta w_\Omega dx = -\delta u_{X1}\frac{EA}{2\sqrt{2}L}u_{X1}.$$

Virtual work expression of a structure is the sum of element contributions

$$\delta W = \delta W^1 + \delta W^2 + \delta W^3 = -\delta u_{X1}EA[(1 + \frac{1}{2\sqrt{2}})\frac{u_{X1}}{L} + \alpha\vartheta°\frac{1}{2}].$$

Principle of virtual work $\delta W = 0 \;\forall\delta a$ and the fundamental lemma of variation calculus imply

$$(1+\frac{1}{2\sqrt{2}})\frac{u_{X1}}{L}+\alpha\vartheta°\frac{1}{2}=0 \quad \Leftrightarrow \quad u_{X1}=-\alpha L\vartheta°\frac{2}{4+\sqrt{2}}. \quad \textcolor{red}{\leftarrow}$$

A thin triangular slab (plane stress conditions) is allowed to move horizontally at node 1 and nodes 2 and 3 are fixed. Stress is zero when temperature (assumed constant) is ϑ°. Determine the non-zero displacement component u_{X1}, if the temperature of slab is increased to $2\vartheta^\circ$.

Solution

As temperature is known and the external distributed force vanishes, the virtual work densities needed here are (formulae collection)

$$\delta w_\Omega^{\text{int}} = -\left\{\begin{array}{c} \partial \delta u/\partial x \\ \partial \delta v/\partial y \\ \partial \delta u/\partial y + \partial \delta v/\partial x \end{array}\right\}^{\mathrm{T}} t[E]_\sigma \left\{\begin{array}{c} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{array}\right\}, \quad \delta w_\Omega^{\text{cpl}} = \left\{\begin{array}{c} \partial \delta u/\partial x \\ \partial \delta v/\partial x \end{array}\right\}^{\mathrm{T}} \frac{E\alpha t}{1-v}\Delta\vartheta\left\{\begin{array}{c} 1 \\ 1 \end{array}\right\}$$

in which $\Delta\vartheta = \vartheta - \vartheta^\circ$ is the difference between temperature at the deformed and initial and deformed geometries. At the initial geometry stress is assumed to vanish.

Approximation is the first thing to be considered. Linear shape functions can be deduced from the figure

$$N_1 = 1 - \frac{x}{L}, \quad N_3 = \frac{y}{L}, \quad \text{and} \quad N_2 = 1 - N_1 - N_3 = \frac{x-y}{L}.$$

Approximations to the displacement components and temperature difference are

$$u = (1 - \frac{x}{L})u_{X1}, \quad v = 0, \quad \text{and} \quad \Delta\vartheta = \vartheta^\circ.$$

When the approximations are substituted there, virtual work densities take the forms

$$\delta w_\Omega^{\text{int}} = -\left\{\begin{array}{c} -\delta u_{X1}/L \\ 0 \\ 0 \end{array}\right\}^{\mathrm{T}} \frac{Et}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & (1-v)/2 \end{bmatrix}\left\{\begin{array}{c} -u_{X1}/L \\ 0 \\ 0 \end{array}\right\} = -\delta u_{X1}\frac{1}{L^2}\frac{Et}{1-v^2}u_{X1},$$

$$\delta w_\Omega^{\text{cpl}} = \left\{\begin{array}{c} -\delta u_{X1}/L \\ 0 \end{array}\right\}^{\mathrm{T}} \frac{Et\alpha}{1-v}\vartheta^\circ\left\{\begin{array}{c} 1 \\ 1 \end{array}\right\} = -\delta u_{X1}\frac{1}{L}\frac{Et\alpha}{1-v}\vartheta^\circ \qquad \Rightarrow$$

$$\delta w_\Omega = \delta w_\Omega^{\text{int}} + \delta w_\Omega^{\text{cpl}} = -\frac{\delta u_{X1}}{L}\frac{Et}{1-v^2}\frac{u_{X1}}{L} - \frac{\delta u_{X1}}{L}\frac{Et}{1-v}\alpha\vartheta^\circ.$$

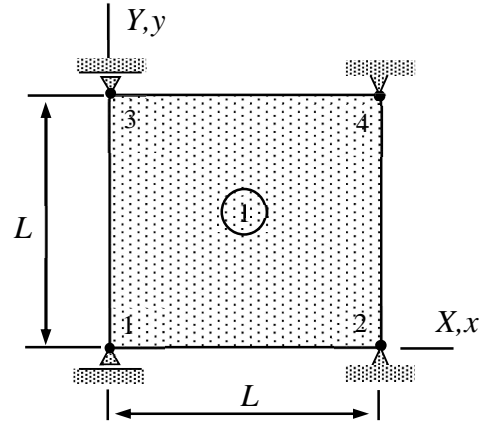Virtual work expression is the integral of the density over the domain occupied by the element. Here, virtual work density is constant so that it is enough to multiply by the area:

$$\delta W = \delta w_\Omega \frac{L^2}{2} = -\delta u_{X1}(\frac{1}{2}\frac{Et}{1-v^2}u_{X1} + \frac{1}{2}\frac{Et}{1-v}L\alpha\vartheta^\circ).$$

Principle of virtual work and the fundamental lemma of variation calculus give

$$\frac{1}{2}\frac{Et}{1-v^2}u_{X1}+\frac{1}{2}\frac{Et}{1-v}L\alpha\vartheta^\circ=0 \quad\Leftrightarrow\quad u_{X1}=-(1+v)\alpha L\vartheta^\circ. \quad\textcolor{red}{\leftarrow}$$

Nodes 1 and 3 of a thin rectangular slab (assume plane stress conditions) shown are allowed to move horizontally and nodes 2 and 4 are fixed. Stress is zero when temperature is ϑ°. Determine the displacement components $u_{X1} = u_{X3}$ if the temperature of slab is increased to $2\vartheta^\circ$. Also, determine the strain and stress in the slab. Material parameters and thickness are E, ν, α and t, respectively.

Solution

As temperature is known and the external distributed force vanishes, virtual work densities needed here are (formulae collection)

$$\delta w_\Omega^{int} = -\left\{ \begin{array}{c} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta u/\partial y + \partial\delta v/\partial x \end{array} \right\}^T t[E]_\sigma \left\{ \begin{array}{c} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{array} \right\}, \quad \delta w_\Omega^{cpl} = \left\{ \begin{array}{c} \partial\delta u/\partial x \\ \partial\delta v/\partial x \end{array} \right\}^T \frac{E\alpha t}{1-\nu} \Delta\vartheta \left\{ \begin{array}{c} 1 \\ 1 \end{array} \right\}$$

in which $\Delta\vartheta = \vartheta - \vartheta^\circ$ is the difference between temperature at the deformed and initial geometries.

Approximations are the first thing to be considered. As the origin of the material xy – coordinate system is placed at node 1 and the axes are aligned with the axes of the structural XY – coordinate system

$$u = \left\{ \begin{array}{c} (1-\xi)(1-\eta) \\ \xi(1-\eta) \\ (1-\xi)\eta \\ \xi\eta \end{array} \right\}^T \left\{ \begin{array}{c} u_{X1} \\ 0 \\ u_{X1} \\ 0 \end{array} \right\} = (1-\frac{x}{L})u_{X1}, \quad v = 0, \quad \text{and} \quad \Delta\vartheta = \vartheta^\circ \text{ (constant)}.$$

When the approximations are substituted there, virtual work density simplifies to

$$\delta w_\Omega = -\left\{ \begin{array}{c} -\delta u_{X1}/L \\ 0 \\ 0 \end{array} \right\}^T \frac{Et}{1-\nu^2} \left[\begin{array}{ccc} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{array} \right] \left\{ \begin{array}{c} -u_{X1}/L \\ 0 \\ 0 \end{array} \right\} + \left\{ \begin{array}{c} -\delta u_{X1}/L \\ 0 \end{array} \right\}^T \frac{Et}{1-\nu} \alpha\vartheta^\circ \left\{ \begin{array}{c} 1 \\ 1 \end{array} \right\} \quad \Leftrightarrow$$

$$\delta w_\Omega = -\frac{\delta u_{X1}}{L}\frac{Et}{1-\nu^2}\frac{u_{X1}}{L} - \frac{\delta u_{X1}}{L}\frac{Et}{1-\nu}\alpha\vartheta^\circ.$$

Virtual work expression is integral of the density over the domain occupied by the element. Here, virtual work density is constant so that it is enough to multiply by the area:

$$\delta W = \delta w_\Omega L^2 = -\delta u_{X1}\frac{Et}{1-\nu^2}u_{X1} - \delta u_{X1}\frac{Et}{1-\nu}L\alpha\vartheta^\circ.$$

Principle of virtual work and the fundamental lemma of variation calculus give

$$\frac{Et}{1-v^2}u_{X1} + \frac{Et}{1-v}L\alpha\vartheta^\circ = 0 \quad \Leftrightarrow \quad u_{X1} = -(1+v)\alpha L\vartheta^\circ. \quad \leftarrow$$

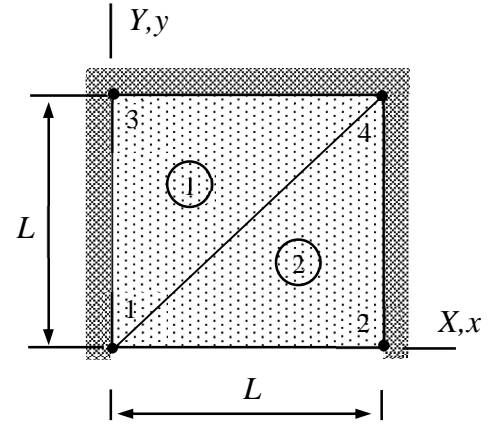Strain components can be obtained from derivatives of the displacement components u and v

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix} = (1+v)\alpha\vartheta^\circ \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}. \quad \leftarrow$$

Cauchy stress components can be calculated from the stress-strain relationship of plane-stress case of the thin slab model taking into account the temperature change (see the lecture notes)

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{1-v^2}(\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & (1-v)/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} - (1+v)\alpha\Delta\vartheta \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}) \quad \Leftrightarrow$$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{1-v^2}(\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & (1-v)/2 \end{bmatrix} \alpha\vartheta^\circ(1+v) \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} - (1+v)\alpha\vartheta^\circ \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}) = -E\alpha\vartheta^\circ \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}. \quad \leftarrow$$

Determine the stationary temperature distribution in a thin slab shown. Edge 1-2 is at constant temperature ϑ° and heat flux through the other edges vanishes. Use a two-triangle mesh with ϑ_3 and $\vartheta_4 = \vartheta_3$ as the unknown node temperatures and consider $\vartheta_1 = \vartheta_2 = \vartheta^\circ$ as known. Thickness t, thermal conductivity k, and heat production rate per unit area s are constants.

Solution

The density expressions associated with the pure heat conduction problem in a thin slab are

$$\delta p_\Omega^{int} = -\begin{Bmatrix} \partial \delta\vartheta / \partial x \\ \partial \delta\vartheta / \partial y \end{Bmatrix}^T tk \begin{Bmatrix} \partial \vartheta / \partial x \\ \partial \vartheta / \partial y \end{Bmatrix} \text{ and } \delta p_\Omega^{ext} = \delta\vartheta s \, .$$

in which ϑ is the temperature, k the thermal conductivity, and s the rate of heat production (per unit area). For a thin-slab element, element contributions need to be calculated from scratch starting with the densities and approximations.

The shape functions of element 1 (deduced from the figure) $N_1 = 1 - y/L$, $N_4 = x/L$, and $N_3 = 1 - N_1 - N_4 = (y - x)/L$ give approximations

$$\vartheta = \begin{Bmatrix} N_1 \\ N_4 \\ N_3 \end{Bmatrix}^T \begin{Bmatrix} \vartheta^\circ \\ \vartheta_3 \\ \vartheta_3 \end{Bmatrix} = (1 - \frac{y}{L})\vartheta^\circ + \frac{y}{L}\vartheta_3, \quad \frac{\partial \vartheta}{\partial x} = 0, \quad \frac{\partial \vartheta}{\partial y} = \frac{\vartheta_3 - \vartheta^\circ}{L} \quad \text{and}$$

$$\delta\vartheta = \frac{y}{L}\delta\vartheta_3, \quad \frac{\partial \delta\vartheta}{\partial x} = 0, \quad \frac{\partial \delta\vartheta}{\partial y} = \frac{\delta\vartheta_3}{L} \quad \text{(variation of } \vartheta^\circ \text{ vanishes).}$$

When the approximation is substituted there, density expression simplifies to

$$\delta p_\Omega = \delta p_\Omega^{int} + \delta p_\Omega^{ext} = -\begin{Bmatrix} \partial \delta\vartheta / \partial x \\ \partial \delta\vartheta / \partial y \end{Bmatrix}^T tk \begin{Bmatrix} \partial \vartheta / \partial x \\ \partial \vartheta / \partial y \end{Bmatrix} + \delta\vartheta s = -\frac{\delta\vartheta_3}{L} tk \frac{\vartheta_3 - \vartheta^\circ}{L} + \frac{y}{L}\delta\vartheta_3 s \, .$$
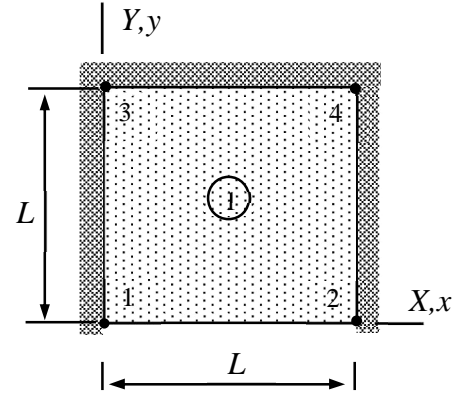
Element contribution is the integral of the density expression over the domain occupied by the element:

$$\delta P^1 = -\delta\vartheta_3 (tk \frac{\vartheta_3 - \vartheta^\circ}{2} - \frac{L^2}{3} s) \, .$$

The shape functions of element 2 (deduced from the figure) $N_1 = 1 - x/L$, $N_4 = y/L$, and $N_2 = 1 - N_1 - N_4 = (x - y)/L$ give approximations

$$\vartheta = \begin{Bmatrix} N_1 \\ N_2 \\ N_4 \end{Bmatrix}^T \begin{Bmatrix} \vartheta^\circ \\ \vartheta^\circ \\ \vartheta_3 \end{Bmatrix} = (1-\frac{y}{L})\vartheta^\circ + \frac{y}{L}\vartheta_3, \quad \frac{\partial \vartheta}{\partial x} = 0, \quad \frac{\partial \vartheta}{\partial y} = \frac{\vartheta_3 - \vartheta^\circ}{L}, \quad \text{and}$$

$$\delta\vartheta = \frac{y}{L}\delta\vartheta_3, \quad \frac{\partial \delta\vartheta}{\partial x} = 0, \quad \frac{\partial \delta\vartheta}{\partial y} = \frac{\delta\vartheta_3}{L} \quad \text{(variation of } \vartheta^\circ \text{ vanishes)}.$$

When the approximation is substituted there, density simplifies to

$$\delta p_\Omega = \delta p_\Omega^{\text{int}} + \delta p_\Omega^{\text{ext}} = -\frac{\delta\vartheta_3}{L}tk\frac{\vartheta_3 - \vartheta^\circ}{L} + \frac{y}{L}\delta\vartheta_3 s.$$

Element contribution is the integral of the density expression over the domain occupied by the element, so

$$\delta P^2 = -\delta\vartheta_3(tk\frac{\vartheta_3 - \vartheta^\circ}{2} - \frac{L^2}{6}s).$$

Variation principle $\delta P = \delta P^1 + \delta P^2 = 0 \ \forall \delta\mathbf{a}$ and the fundamental lemma of variation calculus imply that

$$tk(\vartheta_3 - \vartheta^\circ) - \frac{L^2}{2}s = 0 \quad \Leftrightarrow \quad \vartheta_3 = \vartheta^\circ + \frac{sL^2}{2tk}. \quad \leftarrow$$

Determine the stationary temperature distribution in a thin slab shown. Edge 1-2 is at constant temperature ϑ° and heat flux through the other edges vanishes. Use a rectangle element with bilinear approximation and consider $\vartheta_1 = \vartheta_2 = \vartheta^\circ$ as known and $\vartheta_4 = \vartheta_3$ as the unknown nodal temperatures. Thickness t, thermal conductivity k, and heat production rate per unit area s are constants.

Solution

The density expressions associated with the pure heat conduction problem in a thin slab are

$$\delta p_\Omega^{int} = -\begin{Bmatrix} \partial \delta\vartheta / \partial x \\ \partial \delta\vartheta / \partial y \end{Bmatrix}^T tk \begin{Bmatrix} \partial \vartheta / \partial x \\ \partial \vartheta / \partial y \end{Bmatrix} \quad \text{and} \quad \delta p_\Omega^{ext} = \delta\vartheta s .$$

in which ϑ is the temperature, k the thermal conductivity, and s the rate of heat production (per unit area). For a thin-slab element, element contributions need to be calculated from scratch starting with the densities and approximation.

The shape functions can be deduced from the figure. Approximation

$$\vartheta = \begin{Bmatrix} (1-x/L)(1-y/L) \\ (x/L)(1-y/L) \\ (1-x/L)(y/L) \\ (x/L)(y/L) \end{Bmatrix}^T \begin{Bmatrix} \vartheta^\circ \\ \vartheta^\circ \\ \vartheta_3 \\ \vartheta_3 \end{Bmatrix} = (1-\frac{y}{L})\vartheta^\circ + \frac{y}{L}\vartheta_3, \quad \frac{\partial\vartheta}{\partial x} = 0, \quad \frac{\partial\vartheta}{\partial y} = \frac{\vartheta_3 - \vartheta^\circ}{L} \quad \text{and}$$

$$\delta\vartheta = \frac{y}{L}\delta\vartheta_3, \quad \frac{\partial\delta\vartheta}{\partial x} = 0, \quad \frac{\partial\delta\vartheta}{\partial y} = \frac{\delta\vartheta_3}{L} \quad \text{(variation of } \vartheta^\circ \text{ vanishes).}$$

When the approximation is substituted there, density simplifies to

$$\delta p_\Omega = \delta p_\Omega^{int} + \delta p_\Omega^{ext} = -\frac{\delta\vartheta_3}{L} tk \frac{\vartheta_3 - \vartheta^\circ}{L} + \frac{y}{L}\delta\vartheta_3 s .$$

Element contribution is the integral of the density expression over the domain occupied by the element:

$$\delta P^1 = \int_0^L \int_0^L \delta p_\Omega dx dy = -\delta\vartheta_3 [tk(\vartheta_3 - \vartheta^\circ) - \frac{L^2}{2}s] .$$

Variation principle $\delta P = 0 \quad \forall \delta\mathbf{a}$ and the fundamental lemma of variation calculus imply that

$$tk(\vartheta_3 - \vartheta^\circ) - \frac{L^2}{2}s = 0 \quad \Leftrightarrow \quad \vartheta_3 = \vartheta^\circ + \frac{sL^2}{2kt} . \quad \leftarrow$$

At the initial temperature ϑ° and without external forces, the length of the bar shown is L. Calculate the displacement of node 2 due to the combined effect of gravity and change of temperature with the nodal values $\vartheta_1 = 2\vartheta^\circ$ and $\vartheta_2 = \vartheta^\circ$. Cross sectional area A, coefficient of thermal expansion α, and density ρ are considered as constants. Use linear interpolation to displacement and temperature and start with the virtual work density expressions.
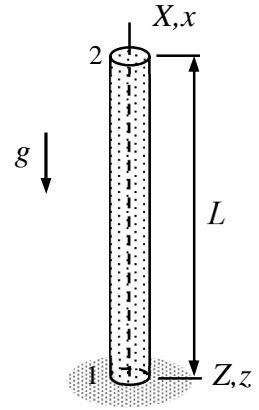
Solution

Here temperature is given and the aim is to find the deformation implied by the temperature change. Virtual work density expressions of the bar model needed in the calculation are

$$\delta w_\Omega^{\text{int}} = -\frac{d\delta u}{dx} EA \frac{du}{dx}, \quad \delta w_\Omega^{\text{cpl}} = \frac{d\delta u}{dx} EA\alpha\Delta\vartheta \quad \text{and} \quad \delta w_\Omega^{\text{ext}} = \delta u\, f_x.$$

in which $\Delta\vartheta$ is the temperature change, α coefficient of thermal expansion, and f_x the distributed force per unit length.

The given nodal temperatures are $\vartheta_1 = 2\vartheta^\circ$ and $\vartheta_2 = \vartheta^\circ$. As the initial temperature is ϑ°, the changes of the nodal values are $\Delta\vartheta_1 = \vartheta^\circ$ and $\Delta\vartheta_2 = 0$. Linear interpolations to displacement and temperature in terms of the nodal values are

$$u = \begin{Bmatrix} 1-x/L \\ x/L \end{Bmatrix}^{\text{T}} \begin{Bmatrix} 0 \\ u_{X2} \end{Bmatrix} = \frac{x}{L} u_{X2}, \quad \frac{du}{dx} = \frac{1}{L} u_{X2}, \quad \text{and} \quad \frac{d\delta u}{dx} = \frac{1}{L}\delta u_{X2},$$

$$\Delta\vartheta = \begin{Bmatrix} 1-x/L \\ x/L \end{Bmatrix}^{\text{T}} \begin{Bmatrix} \vartheta^\circ \\ 0 \end{Bmatrix} = (1-\frac{x}{L})\vartheta^\circ.$$

When the approximation is substituted there, density expression $\delta w_\Omega = \delta w_\Omega^{\text{int}} + \delta w_\Omega^{\text{cpl}} + \delta w_\Omega^{\text{ext}}$ simplifies to

$$\delta w_\Omega = -\delta u_{X2} \frac{EA}{L^2} u_{X2} + \delta u_{X2} \frac{EA\alpha}{L}(1-\frac{x}{L})\vartheta^\circ - \delta u_{X2} \frac{x}{L}\rho g A.$$

Virtual work expression is the integral of the density over the element domain

$$\delta W = -\delta u_{X2} \frac{EA}{L} u_{X2} + \delta u_{X2} \frac{EA\alpha}{2}\vartheta^\circ - \delta u_{X2} \frac{\rho g AL}{2}.$$

Principle of virtual work $\delta W = 0 \ \forall \delta\mathbf{a}$ and the fundamental lemma of variation calculus give

$$-\frac{EA}{L} u_{X2} + \frac{EA\alpha}{2}\vartheta^\circ - \frac{\rho g AL}{2} = 0 \quad \Leftrightarrow \quad u_{X2} = \frac{\alpha}{2} L\vartheta^\circ - \frac{\rho g}{2E} L^2. \quad \text{←}$$

The simply supported plate shown is assembled at constant temperature $3\vartheta°$. Find the transverse displacement when the upper side temperature is $4\vartheta°$ and that of the lower side $2\vartheta°$. Assume that temperature in plate is linear in z and does not depend on x or y. Use $w(x, y) = a\sin(\pi x / L)\sin(\pi y / L)$ as the approximation. Problem parameters E, v, ρ, α and t are constants. Integrals of sin and cos functions satisfy
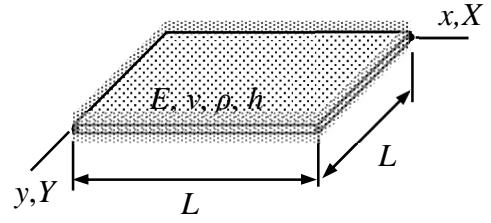
$$\int_0^L \sin^2(\pi\frac{x}{L})dx = \int_0^L \cos^2(\pi\frac{x}{L})dx = \frac{L}{2} \quad \text{and} \quad \int_0^L \sin(\pi\frac{x}{L})dx = \frac{2L}{\pi}.$$

Solution

Assuming that the material coordinate system is chosen so that the plate bending and thin slab modes decouple, the plate model virtual work densities of internal force and coupling terms are given by

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial^2\delta w/\partial x^2 \\ \partial^2\delta w/\partial y^2 \\ 2\partial^2\delta w/\partial x\partial y \end{Bmatrix}^{\text{T}} \frac{t^3}{12}[E]_\sigma \begin{Bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ 2\partial^2 w/\partial x\partial y \end{Bmatrix}, \quad \delta w_\Omega^{\text{cpl}} = -\begin{Bmatrix} \partial^2\delta w/\partial x^2 \\ \partial^2\delta w/\partial y^2 \end{Bmatrix}^{\text{T}} \int z\Delta\vartheta dz \frac{\alpha E}{1-v}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

The coupling term contains an integral of temperature over the thickness of the plate. Approximation to the transverse displacement and its derivatives are

$$w(x, y) = a\sin(\pi\frac{x}{L})\sin(\pi\frac{y}{L}) \quad \Rightarrow$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial y^2} = -a(\frac{\pi}{L})^2\sin(\pi\frac{x}{L})\sin(\pi\frac{y}{L}), \quad \frac{\partial^2 w}{\partial x\partial y} = a(\frac{\pi}{L})^2\cos(\pi\frac{x}{L})\cos(\pi\frac{y}{L}).$$

Temperature difference and its weighted integral over the thickness (integral of the coupling term)

$$\Delta\vartheta = \vartheta(z) - 3\vartheta° = (\frac{1}{2}+\frac{z}{t})2\vartheta° + (\frac{1}{2}-\frac{z}{t})4\vartheta° - 3\vartheta° = -\frac{z}{t}2\vartheta° \quad \Rightarrow$$

$$\int z\Delta\vartheta dz = -\int_{-t/2}^{t/2} z\frac{z}{t}2\vartheta° dz = -\frac{1}{6}\vartheta° t^2.$$

When the approximation to the transverse displacement is substituted there, virtual work densities of the internal and the coupling parts simplify to

$$\delta w_\Omega^{\text{int}} = -\delta a\frac{t^3 E}{12(1-v^2)}(\frac{\pi}{L})^4 2[\sin^2(\frac{\pi x}{L})\sin^2(\frac{\pi y}{L})(1+v)+(1-v)\cos^2(\frac{\pi x}{L})\cos^2(\frac{\pi y}{L})]a,$$

$$\delta w_\Omega^{\text{cpl}} = -\delta a(\frac{\pi}{L})^2\sin(\pi\frac{x}{L})\sin(\pi\frac{y}{L})\frac{1}{3}\vartheta° t^2\frac{\alpha E}{1-v}.$$

Virtual work expressions are integrals of the densities over the domain occupied by the plate/element

$$\delta W^{\text{int}} = \int_0^L \int_0^L \delta w_\Omega^{\text{int}} \, dx dy = -\delta a 4 \frac{t^3 E}{12(1-v^2)} (\frac{\pi}{L})^4 (\frac{L}{2})^2 a \,,$$

$$\delta W^{\text{cpl}} = \int_0^L \int_0^L \delta w_\Omega^{\text{cpl}} \, dx dy = -\delta a \frac{4}{3} \vartheta^\circ \frac{\alpha E t^2}{1-v} \,.$$

Virtual work expression is the sum of the parts

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{cpl}} = -\delta a [4 \frac{t^3 E}{12(1-v^2)} (\frac{\pi}{L})^4 (\frac{L}{2})^2 a + \frac{4}{3} \vartheta^\circ \frac{\alpha E t^2}{1-v}] \,.$$

Principle of virtual work $\delta W = 0 \ \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus give

$$\delta W = -\delta a [4 \frac{t^3 E}{12(1-v^2)} (\frac{\pi}{L})^4 (\frac{L}{2})^2 a + \frac{4}{3} \vartheta^\circ \frac{\alpha E t^2}{1-v}] = 0 \quad \forall \delta a \quad \Leftrightarrow$$

$$4 \frac{t^3 E}{12(1-v^2)} (\frac{\pi}{L})^4 (\frac{L}{2})^2 a + \frac{4}{3} \vartheta^\circ \frac{\alpha E t^2}{1-v} = 0 \quad \Rightarrow \quad a = -\frac{16}{\pi^4} \frac{\alpha \vartheta^\circ L^2}{t} (1+v) \,. \quad \textcolor{red}{\leftarrow}$$