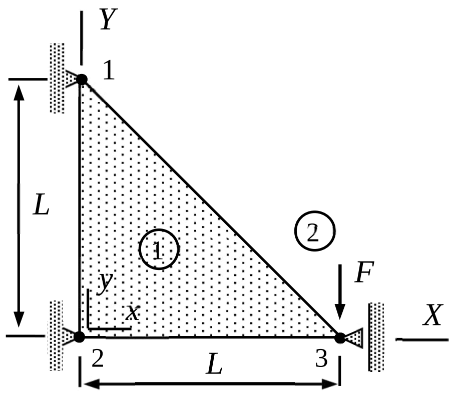
Name___________________________ Student number__________________

Assignment 5

A thin triangular slab is loaded by a point force at node 3. Nodes 1 and 2 are fixed and node 3 moves only in the vertical direction. Derive the equilibrium equation of the structure according to the large displacement theory in terms of the dimensionless displacement component $\mathrm{a} = u_{Y3}/L$. Approximation is linear and material parameters C and ν are constants. Assume plane-stress conditions. When $F=0$, side length and thickness of the slab are L and t, respectively. Also find the solution to a small displacement problem by simplifying the equilibrium equations with the assumption $|\mathrm{a}| \ll 1$.



Solution

Virtual work density of internal forces, when modified for large displacement analysis with the same constitutive equation as in the linear case of plane stress, is given by

$$\delta w_{\Omega^\circ}^{\text{int}} = -\begin{Bmatrix} \delta \mathrm{E}_{xx} \\ \delta \mathrm{E}_{yy} \\ 2\delta \mathrm{E}_{xy} \end{Bmatrix}^{\mathrm{T}} \frac{tC}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\begin{Bmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ 2\mathrm{E}_{xy} \end{Bmatrix}, \quad \begin{Bmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ 2\mathrm{E}_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} + \dfrac{1}{2}(\dfrac{\partial u}{\partial x})^2 + \dfrac{1}{2}(\dfrac{\partial v}{\partial x})^2 \\ \dfrac{\partial v}{\partial y} + \dfrac{1}{2}(\dfrac{\partial u}{\partial y})^2 + \dfrac{1}{2}(\dfrac{\partial v}{\partial y})^2 \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\dfrac{\partial v}{\partial y} \end{Bmatrix}.$$

Let us start with the approximations and the corresponding components of the Green-Lagrange strain. Linear shape functions can be deduced from the figure. Only the shape function $N_3 = x/L$ of node 3 is needed. Displacement components and their non-zero derivatives are

$$u = 0 \text{ and } v = \frac{x}{L}u_{Y3} \;\Rightarrow\; \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0 \text{ and } \frac{\partial v}{\partial x} = \frac{u_{Y3}}{L} = \mathrm{a}, \; \frac{\partial v}{\partial y} = 0.$$

Green-Lagrange strain measures and their variations

$$\begin{Bmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ 2\mathrm{E}_{xy} \end{Bmatrix} = \begin{Bmatrix} \mathrm{a}^2/2 \\ 0 \\ \mathrm{a} \end{Bmatrix} \;\Rightarrow\; \begin{Bmatrix} \delta \mathrm{E}_{xx} \\ \delta \mathrm{E}_{yy} \\ 2\delta \mathrm{E}_{xy} \end{Bmatrix} = \begin{Bmatrix} \mathrm{a}\delta \mathrm{a} \\ 0 \\ \delta \mathrm{a} \end{Bmatrix}.$$

When the strain component expressions are substituted there, virtual work density simplifies to

$$\delta w_{\Omega^\circ}^{\text{int}} = -\frac{tC}{1-\nu^2}\begin{Bmatrix} \mathrm{a}\delta\mathrm{a} \\ 0 \\ \delta\mathrm{a} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \mathrm{a}^2/2 \\ 0 \\ \mathrm{a} \end{Bmatrix} \Rightarrow$$

$$\delta w_{\Omega^\circ}^{\text{int}} = -\frac{tC}{1-\nu^2}\delta\mathrm{a}(\frac{1}{2}\mathrm{a}^3 + \frac{1-\nu}{2}\mathrm{a})$$

Integration over the (initial) domain gives the virtual work expression. As the integrand is constant

$$\delta W^1 = \frac{L^2}{2}\delta w_{\Omega^\circ}^{\text{int}} = -\frac{L^2}{2}\frac{tC}{1-\nu^2}\delta\mathrm{a}(\frac{1}{2}\mathrm{a}^3 + \frac{1-\nu}{2}\mathrm{a}).$$

Virtual work expression of the external point force components

$$\delta W^2 = -F\delta u_{Y3} = -FL\delta\mathrm{a}.$$

Virtual work expression of the structure is obtained as sum over the element contributions. In terms of the dimensionless displacement

$$\delta W = -\frac{L^2}{2}\frac{tC}{1-\nu^2}\delta\mathrm{a}(\frac{1}{2}\mathrm{a}^3 + \frac{1-\nu}{2}\mathrm{a}) - FL\delta\mathrm{a}$$

or, when written in the standard form,

$$\delta W = -\delta\mathrm{a}[\frac{L^2}{2}\frac{tC}{1-\nu^2}(\frac{1}{2}\mathrm{a}^3 + \frac{1-\nu}{2}\mathrm{a}) + FL].$$

Principle of virtual work and the basic lemma of variation calculus imply the equilibrium equation

$$(\mathrm{a}^2 + 1 - \nu)\mathrm{a} + 4(1-\nu^2)\frac{F}{tLC} = 0. \leftarrow$$

Assuming that $|\mathrm{a}| \ll 1$ the equilibrium equation simplifies to

$$(1-\nu)\mathrm{a} + 4(1-\nu^2)\frac{F}{tLC} = 0 \quad \Rightarrow \quad \mathrm{a} = -4\frac{F}{tCL}(1+\nu). \leftarrow$$