Name________________________ Student number________________

Assignment 3

Consider the equations of motion

$$\frac{GJ}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \theta_{X1} \\ \theta_{X2} \end{Bmatrix} + \frac{\rho LJ}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_{X1} \\ \ddot{\theta}_{X2} \end{Bmatrix} = 0$$

for the end point rotations of a certain torsion bar of length L. Above, J is the second moment of area with respect to the axis of the bar (polar moment), G is the shear modulus, and ρ is the density of material. Derive the angular speeds and the corresponding modes of the free vibrations.

Solution template

The set of ordinary differential equations as given by the principle of virtual work $\mathbf{M}\ddot{\mathbf{a}} + \mathbf{K}\mathbf{a} = 0$ consists of the inertia and stiffness parts. The symmetric mass matrix $\mathbf{M}$ and the stiffness matrix $\mathbf{K}$ depend on the structure. Angular speeds of the free vibrations are the eigenvalues of $\mathbf{\Omega} = \sqrt{\mathbf{M}^{-1}\mathbf{K}}$. In practice, it is easier to calculate first the eigenvalues $\mathbf{\Omega}^2 = \mathbf{M}^{-1}\mathbf{K}$ as the eigenvalues of $\mathbf{\Omega}$ are the square roots of those for $\mathbf{\Omega}^2$ and the eigenvectors coincide.

In the present case, the matrices are

$$\mathbf{K} = \frac{GJ}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ and } \mathbf{M} = \frac{\rho JL}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore

$$\mathbf{\Omega}^2 = \mathbf{M}^{-1}\mathbf{K} = \begin{bmatrix} \_\_\_\_\_\_\_\_ & \_\_\_\_\_\_\_\_ \\ \_\_\_\_\_\_\_\_ & \_\_\_\_\_\_\_\_ \end{bmatrix}.$$

In the eigenvalue problem of matrix $\mathbf{A}$, the goal is to find all pairs $(\lambda, \mathbf{x})$ such that $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$. The linear homogeneous equation system can have a non-zero solution only if $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$. The eigenvalues are obtained as solutions to this characteristic equation. The characteristic equation for the eigenvalues of $\mathbf{\Omega}^2$ is

$$\det(\mathbf{\Omega}^2 - \lambda\mathbf{I}) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ = 0.$$

The two solutions for the eigenvalues are ($(a-\lambda)^2 - b^2 = 0 \iff \lambda = a \pm b$)

$\lambda_1 =$ ________ and $\lambda_2 =$ ________ .

The corresponding eigenvectors are obtained as solutions to $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$. The eigenvectors are not unique and it is enough to find some of them. However, the eigenvectors should be linearly independent so that, e.g., the zero vector is not a valid choice.

$$\lambda_1: \qquad \begin{bmatrix} \underline{\qquad\qquad} & \underline{\qquad\qquad} \\ \underline{\qquad\qquad} & \underline{\qquad\qquad} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0 \quad \Rightarrow \quad \mathbf{x}_1 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} \underline{\qquad\qquad} \\ \underline{\qquad\qquad} \end{Bmatrix},$$

$$\lambda_2: \qquad \begin{bmatrix} \underline{\qquad\qquad} & \underline{\qquad\qquad} \\ \underline{\qquad\qquad} & \underline{\qquad\qquad} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0 \quad \Rightarrow \quad \mathbf{x}_2 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} \underline{\qquad\qquad} \\ \underline{\qquad\qquad} \end{Bmatrix}.$$

The representation of the matrix in terms of its eigenvalues and eigenvectors $\mathbf{\Omega}^2 = \mathbf{X}\boldsymbol{\lambda}\mathbf{X}^{-1}$ implies that $\mathbf{\Omega} = \mathbf{X}\sqrt{\boldsymbol{\lambda}}\mathbf{X}^{-1}$. As taking a square root of the diagonal matrix means just taking the square roots of the diagonal terms, the angular speeds of the free vibrations

$$(\omega_1, \mathbf{x}_1) = (\sqrt{\lambda_1}, \mathbf{x}_1) = (\underline{\qquad\quad}, \begin{Bmatrix} \underline{\qquad\quad} \\ \underline{\qquad\quad} \end{Bmatrix}) \quad \text{and} \quad (\omega_2, \mathbf{x}_2) = (\sqrt{\lambda_2}, \mathbf{x}_2) = (\underline{\qquad\quad}, \begin{Bmatrix} \underline{\qquad\quad} \\ \underline{\qquad\quad} \end{Bmatrix}).$$