MEC-E8001 Finite Element Analysis; Schedule 2023

Week	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Orientation							
2		**14:15-15:30** Lecture 1/2 (R008/216) **15:30-16:00** Assignment 1 (R008/216) **16:00-17:00** Calculation hours (Zoom) **23:55 DL** Assignment 1 (MyCourses)	**14:15-15:30** Lecture 2/2 (R008/213a) **15:30-16:00** Assignment 2 (R008/213a) **16:00-17:00** Calculation hours (Zoom) **23:55 DL** Assignment 2 (MyCourses)	**12:15-13:30** FEM solver (R001/U1) **13:30-14:00** Modelling assignment (R001/U1) **14:15-16:00** Calculation Hours (R001/U1)	Experiment 09:00-16:00 Puumiehenkuja 5L		**23:55 DL** Modelling assignment (3,4,5) (MyCourses)
Lectures and exercises							
3		**14:15-15:30** Lecture 1/2 (R008/216) **15:30-16:00** Assignment 1 (R008/216) **16:00-17:00** Calculation hours (Zoom) **23:55 DL** Assignment 1 (MyCourses)	**14:15-15:30** Lecture 2/2 (R008/213a) **15:30-16:00** Assignment 2 (R008/213a) **16:00-17:00** Calculation hours (Zoom) **23:55 DL** Assignment 2 (MyCourses)	**12:15-13:30** Calculation examples (R001/U1) **13:30-14:00** Assignments 3,4,5 (R001/U1) **14:15-16:00** Calculation Hours (R001/U1)			**23:55 DL** Assignments 3,4,5 (MyCourses)
4		**14:15-15:30** Lecture 1/2 (R008/216) **15:30-16:00** Assignment 1 (R008/216) **16:00-17:00** Calculation hours (Zoom) **23:55 DL** assignment 1 (MyCourses)	**14:15-15:30** Lecture 2/2 (R008/213a) **15:30-16:00** Assignment 2 (R008/213a) **16:00-17:00** Calculation hours (Zoom) **23:55 DL** assignment 2 (MyCourses)	**12:15-13:30** Calculation examples (R001/U1) **13:30-14:00** Assignments 3,4,5 (R001/U1) **14:15-16:00** Calculation Hours (R001/U1)			**23:55 DL** Assignments 3,4,5 (MyCourses)
5		**14:15-15:30** Lecture 1/2 (R008/216) **15:30-16:00** Assignment 1	**14:15-15:30** Lecture 2/2 (R008/213a) **15:30-16:00** Assignment 2	**12:15-13:30** Calculation examples (R001/U1) **13:30-14:00**			**23:55 DL** Assignments 3,4,5 (MyCourses)

		(R008/216) **16:00-17:00** Calculation hours (Zoom) **23:55 DL** Assignment 1 (MyCourses)	(R008/213a) **16:00-17:00** Calculation hours (Zoom) **23:55 DL** Assignment 2 (MyCourses)	Assignments 3,4,5 (R001/U1) **14:15-16:00** Calculation Hours (R001/U1)			
6		**14:15-15:30** Lecture 1/2 (R008/216) **15:30-16:00** Assignment 1 (R008/216) **16:00-17:00** Calculation hours (Zoom) **23:55 DL** Assignment 1 (MyCourses)	**14:15-15:30** Lecture 2/2 (R008/213a) **15:30-16:00** Assignment 2 (R008/213a) **16:00-17:00** Calculation hours (Zoom) **23:55 DL** Assignment 2 (MyCourses)	**12:15-13:30** Calculation examples (R001/U1) **13:30-14:00** Assignments 3,4,5 (R001/U1) **14:15-16:00** Calculation Hours (R001/U1)			**23:55 DL** Assignments 3,4,5 (MyCourses)
7		**14:15-15:30** Lecture 1/2 (R008/216) **15:30-16:00** Assignment 1 (R008/216) **16:00-17:00** Calculation hours (Zoom) **23:55 DL** Assignment 1 (MyCourses)	**14:15-15:30** Lecture 2/2 (R008/213a) **15:30-16:00** Assignment 2 (R008/213a) **16:00-17:00** Calculation hours (Zoom) **23:55 DL** Assignment 2 (MyCourses)	**12:15-13:30** Calculation examples (R001/U1) **13:30-14:00** Assignments 3,4,5 (R001/U1) **14:15-16:00** Calculation Hours (R001/U1)			**23:55 DL** Assignments 3,4,5 (MyCourses)
Exams							
8			**13:00-17:00** Final exam (R008/213a) (MyCourses)				

MEC-E8001 Finite Element Analysis; Formulae

GENERAL

Displacement: $\vec{u} = u_X \vec{I} + u_Y \vec{J} + u_Z \vec{K} = u_x \vec{i} + u_y \vec{j} + u_z \vec{k} = u\vec{i} + v\vec{j} + w\vec{k}$

Rotation (small): $\vec{\theta} = \theta_X \vec{I} + \theta_Y \vec{J} + \theta_Z \vec{K} = \theta_x \vec{i} + \theta_y \vec{j} + \theta_z \vec{k} = \phi\vec{i} + \theta\vec{j} + \psi\vec{k}$

Coordinate systems: $\begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = \begin{bmatrix} i_X & i_Y & i_Z \\ j_X & j_Y & j_Z \\ k_X & k_Y & k_Z \end{bmatrix} \begin{Bmatrix} \vec{I} \\ \vec{J} \\ \vec{K} \end{Bmatrix}$

Stress-strain: $\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} = [E] \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix}, \quad \begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = G \begin{Bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$

Elasticity matrices: $[C] \equiv [E], \; [C]_\sigma \equiv [E]_\sigma, \; G = \dfrac{E}{2(1+\nu)}$

$$[E] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix}, \quad [E]^{-1} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix}$$

$$[E]_\sigma = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}, \quad [E]_\varepsilon = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix}$$

Strain-displacement: $\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} = \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial w / \partial z \end{Bmatrix}, \quad \begin{Bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{Bmatrix} \partial u / \partial y + \partial v / \partial x \\ \partial v / \partial z + \partial w / \partial y \\ \partial w / \partial x + \partial u / \partial z \end{Bmatrix}$

$$\begin{Bmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ \mathrm{E}_{zz} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} + \frac{1}{2} \begin{Bmatrix} (\partial u / \partial x)^2 + (\partial v / \partial x)^2 + (\partial w / \partial x)^2 \\ (\partial u / \partial y)^2 + (\partial v / \partial y)^2 + (\partial w / \partial y)^2 \\ (\partial u / \partial z)^2 + (\partial v / \partial z)^2 + (\partial w / \partial z)^2 \end{Bmatrix}$$

$$\begin{Bmatrix} \mathrm{E}_{xy} \\ \mathrm{E}_{yz} \\ \mathrm{E}_{zx} \end{Bmatrix} = \frac{1}{2} \begin{Bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} + \frac{1}{2} \begin{Bmatrix} (\partial u / \partial x)(\partial u / \partial y) + (\partial v / \partial x)(\partial v / \partial y) + (\partial w / \partial x)(\partial w / \partial y) \\ (\partial u / \partial y)(\partial u / \partial z) + (\partial v / \partial y)(\partial v / \partial z) + (\partial w / \partial y)(\partial w / \partial z) \\ (\partial u / \partial z)(\partial u / \partial x) + (\partial v / \partial z)(\partial v / \partial x) + (\partial w / \partial z)(\partial w / \partial x) \end{Bmatrix}$$

PRINCIPLE OF VIRTUAL WORK

$$\delta W=\sum_{e\in E}\ \delta W^{e}=0\ \ \forall\delta\mathbf{a}\,,\ \delta W=\int_{\Omega}\ \delta w d\Omega$$

Bar (x): $\delta w_{\Omega}^{\text{int}}=-\frac{\partial\delta u}{\partial x}EA\frac{\partial u}{\partial x}\,,\ \delta w_{\Omega}^{\text{ext}}=\delta u f_{x}\,,\ \ \delta w_{\Omega}^{\text{ine}}=-\delta u\rho A\frac{\partial^{2}u}{\partial t^{2}}\,,\ \delta w_{\Omega}^{\text{cpl}}=\frac{d\delta u}{dx}EA\alpha\Delta\vartheta$

$$\delta w_{\Omega^{\circ}}^{\text{int}}=-\delta\mathrm{E}_{xx}A^{\circ}C\mathrm{E}_{xx}\,,\ \ \delta w_{\Omega^{\circ}}^{\text{ext}}=A^{\circ}\rho^{\circ}(\delta u g_{x}+\delta v g_{y}+\delta w g_{z})$$

$$\delta p_{\Omega}^{\text{int}}=-\frac{d\delta\vartheta}{dx}kA\frac{d\vartheta}{dx},\ \ \delta p_{\Omega}^{\text{ext}}=\delta\vartheta s$$

Torsion (x): $\delta w_{\Omega}^{\text{int}}=-\frac{\partial\delta\phi}{\partial x}GJ\frac{\partial\phi}{\partial x},\ \delta w_{\Omega}^{\text{ext}}=\delta\phi m_{x}\,,\ \delta w_{\Omega}^{\text{ine}}=-\delta\phi\rho J\frac{\partial^{2}\phi}{\partial t^{2}}$

Bending (xz): $\delta w_{\Omega}^{\text{int}}=-\frac{\partial^{2}\delta w}{\partial x^{2}}EI_{yy}\frac{\partial^{2}w}{\partial x^{2}},\ \delta w_{\Omega}^{\text{ext}}=\delta w f_{z}$

$$\delta w_{\Omega}^{\text{ine}}=-\delta w\rho A\frac{\partial^{2}w}{\partial t^{2}}-\frac{\partial\delta w}{\partial x}\rho I_{yy}\frac{\partial^{2}}{\partial t^{2}}\frac{\partial w}{\partial x},\ \delta w_{\Omega}^{\text{sta}}=-\frac{d\delta w}{dx}N\frac{dw}{dx}\ \text{ where }\ N=EA\frac{du}{dx}.$$

Bending (xy): $\delta w_{\Omega}^{\text{int}}=-\frac{\partial^{2}\delta v}{\partial x^{2}}EI_{zz}\frac{\partial^{2}v}{\partial x^{2}},\ \delta w_{\Omega}^{\text{ext}}=\delta v f_{y}\,,$

$$\delta w_{\Omega}^{\text{ine}}=-\delta v\rho A\frac{\partial^{2}v}{\partial t^{2}}-\frac{\partial\delta v}{\partial x}\rho I_{zz}\frac{\partial^{2}}{\partial t^{2}}\frac{\partial v}{\partial x},\ \delta w_{\Omega}^{\text{sta}}=-\frac{d\delta v}{dx}N\frac{dv}{dx}\ \text{ where }\ N=EA\frac{du}{dx}$$

Thin-slab (xy):

$$\delta w_{\Omega}^{\text{int}}=-\begin{Bmatrix}\partial\delta u/\partial x\\ \partial\delta v/\partial y\\ \partial\delta u/\partial y+\partial\delta v/\partial x\end{Bmatrix}^{\mathrm{T}}t[E]_{\sigma}\begin{Bmatrix}\partial u/\partial x\\ \partial v/\partial y\\ \partial u/\partial y+\partial v/\partial x\end{Bmatrix},\ \delta w_{\Omega}^{\text{ext}}=\begin{Bmatrix}\delta u\\ \delta v\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}f_{x}\\ f_{y}\end{Bmatrix}$$

$$\delta w_{\partial\Omega}^{\text{ext}}=\begin{Bmatrix}\delta u\\ \delta v\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}t_{x}\\ t_{y}\end{Bmatrix},\ \ \delta w_{\Omega}^{\text{ine}}=-\begin{Bmatrix}\delta u\\ \delta v\end{Bmatrix}^{\mathrm{T}}t\rho\frac{\partial^{2}}{\partial t^{2}}\begin{Bmatrix}u\\ v\end{Bmatrix},$$

$$\delta w_{\Omega^{\circ}}^{\text{int}}=-\begin{Bmatrix}\delta\mathrm{E}_{xx}\\ \delta\mathrm{E}_{yy}\\ 2\delta\mathrm{E}_{xy}\end{Bmatrix}^{\mathrm{T}}t[C]_{\sigma}\begin{Bmatrix}\mathrm{E}_{xx}\\ \mathrm{E}_{yy}\\ 2\mathrm{E}_{xy}\end{Bmatrix},\ \delta w_{\Omega^{\circ}}^{\text{ext}}=\begin{Bmatrix}\delta u\\ \delta v\end{Bmatrix}^{\mathrm{T}}\rho^{\circ}t^{\circ}\begin{Bmatrix}g_{x}\\ g_{y}\end{Bmatrix}$$

$$\delta w_{\Omega}^{\text{cpl}}=\begin{Bmatrix}\partial\delta u/\partial x\\ \partial\delta v/\partial x\end{Bmatrix}^{\mathrm{T}}\frac{E\alpha}{1-\nu}\int\ \Delta\vartheta dz\begin{Bmatrix}1\\ 1\end{Bmatrix},\ \ \delta p_{\Omega}^{\text{int}}=-\begin{Bmatrix}\partial\delta\vartheta/\partial x\\ \partial\delta\vartheta/\partial y\end{Bmatrix}^{\mathrm{T}}tk\begin{Bmatrix}\partial\vartheta/\partial x\\ \partial\vartheta/\partial y\end{Bmatrix},\ \ \delta p_{\Omega}^{\text{ext}}=\delta\vartheta s$$

Bending (xy):

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial^2\delta w/\partial x^2 \\ \partial^2\delta w/\partial y^2 \\ 2\partial^2\delta w/\partial x\partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12}[E]_{\sigma} \begin{Bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ 2\partial^2 w/\partial x\partial y \end{Bmatrix}, \ \delta w_{\Omega}^{\text{ext}} = \delta w f_z$$

$$\delta w_{\Omega}^{\text{ine}} = -\begin{Bmatrix} \partial\delta w/\partial x \\ \partial\delta w/\partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12}\rho\frac{\partial^2}{\partial t^2}\begin{Bmatrix} \partial w/\partial x \\ \partial w/\partial y \end{Bmatrix} - \delta w t\rho\frac{\partial^2 w}{\partial t^2}$$

$$\delta w_{\Omega}^{\text{sta}} = -\begin{Bmatrix} \partial\delta w/\partial x \\ \partial\delta w/\partial y \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} N_{xx} & N_{xy} \\ N_{xy} & N_{yy} \end{bmatrix}\begin{Bmatrix} \partial w/\partial x \\ \partial w/\partial y \end{Bmatrix} \text{ where } \begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = t[E]_{\sigma}\begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix}$$

$$\delta w_{\Omega}^{\text{cpl}} = -\begin{Bmatrix} \partial^2\delta w/\partial x^2 \\ \partial^2\delta w/\partial y^2 \end{Bmatrix}^{\mathrm{T}} \frac{\alpha E}{1-\nu}\int z\Delta\vartheta dz\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Solid:

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta w/\partial z \end{Bmatrix}^{\mathrm{T}} [E]\begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial w/\partial z \end{Bmatrix} - \begin{Bmatrix} \partial\delta u/\partial y + \partial\delta v/\partial x \\ \partial\delta v/\partial z + \partial\delta w/\partial y \\ \partial\delta w/\partial x + \partial\delta u/\partial z \end{Bmatrix}^{\mathrm{T}} G\begin{Bmatrix} \partial u/\partial y + \partial v/\partial x \\ \partial v/\partial z + \partial w/\partial y \\ \partial w/\partial x + \partial u/\partial z \end{Bmatrix}$$

$$\delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix}, \ \delta w_{\Omega^\circ}^{\text{int}} = -\begin{Bmatrix} \delta \mathrm{E}_{xx} \\ \delta \mathrm{E}_{yy} \\ \delta \mathrm{E}_{zz} \end{Bmatrix}^{\mathrm{T}} [C]\begin{Bmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ \mathrm{E}_{zz} \end{Bmatrix} - \begin{Bmatrix} \delta \mathrm{E}_{xy} \\ \delta \mathrm{E}_{yz} \\ \delta \mathrm{E}_{zx} \end{Bmatrix}^{\mathrm{T}} 4G\begin{Bmatrix} \mathrm{E}_{xy} \\ \mathrm{E}_{yz} \\ \mathrm{E}_{zx} \end{Bmatrix}$$

$$\delta w_{\Omega}^{\text{cpl}} = \begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta w/\partial z \end{Bmatrix}^{\mathrm{T}} \frac{E\alpha}{1-2\nu}\Delta\vartheta\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}, \ \delta p_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta\vartheta/\partial x \\ \partial\delta\vartheta/\partial y \\ \partial\delta\vartheta/\partial z \end{Bmatrix}^{\mathrm{T}} k\begin{Bmatrix} \partial\vartheta/\partial x \\ \partial\vartheta/\partial y \\ \partial\vartheta/\partial z \end{Bmatrix}, \ \delta p_{\Omega}^{\text{ext}} = \delta\vartheta s$$

APPROXIMATIONS (some) $u = \mathbf{N}^{\mathrm{T}}\mathbf{a}$, $\xi = \dfrac{x}{h}$

Quadratic: $\mathbf{N} = \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} = \begin{Bmatrix} 1-3\xi+2\xi^2 \\ 4\xi(1-\xi) \\ \xi(2\xi-1) \end{Bmatrix}$, $\mathbf{a} = \begin{Bmatrix} u_{x1} \\ u_{x2} \\ u_{x3} \end{Bmatrix}$ (bar)

Cubic: $$\mathbf{N}=\begin{Bmatrix} N_{10} \\ N_{11} \\ \hdashline N_{20} \\ N_{21} \end{Bmatrix}=\begin{Bmatrix} (1-\xi)^2(1+2\xi) \\ h(1-\xi)^2\xi \\ \hdashline (3-2\xi)\xi^2 \\ h\xi^2(\xi-1) \end{Bmatrix},\quad \mathbf{a}=\begin{Bmatrix} u_{10} \\ u_{11} \\ \hdashline u_{20} \\ u_{21} \end{Bmatrix}(=\begin{Bmatrix} u_{z1} \\ -\theta_{y1} \\ \hdashline u_{z2} \\ -\theta_{y2} \end{Bmatrix}\text{beam xz-plane bending})$$

Linear: $$\mathbf{N}=\begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix}^{-1}\begin{Bmatrix} 1 \\ x \end{Bmatrix},\ \mathbf{N}=\begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}^{-1}\begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix},\ \mathbf{N}=\begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix}^{-1}\begin{Bmatrix} 1 \\ x \\ y \\ z \end{Bmatrix}$$

VIRTUAL WORK EXPRESSIONS $\ddot{a}\equiv\frac{d^2}{dt^2}a$

Rigid body/point force: $$\delta W^{\text{ext}}=\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \\ \delta u_{Z1} \end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix} \underline{F}_X \\ \underline{F}_Y \\ \underline{F}_Z \end{Bmatrix}+\begin{Bmatrix} \delta\theta_{X1} \\ \delta\theta_{Y1} \\ \delta\theta_{Z1} \end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix} \underline{M}_X \\ \underline{M}_Y \\ \underline{M}_Z \end{Bmatrix}$$

$$\delta W^{\text{ine}}=-\begin{Bmatrix} \delta u_{x1} \\ \delta u_{y1} \\ \delta u_{z1} \end{Bmatrix}^{\mathrm{T}} m\begin{Bmatrix} \ddot{u}_{x1} \\ \ddot{u}_{y1} \\ \ddot{u}_{z1} \end{Bmatrix}-\begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{y1} \\ \delta\theta_{z1} \end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix} J_{xx}\ddot{\theta}_{x1} \\ J_{yy}\ddot{\theta}_{y1} \\ J_{zz}\ddot{\theta}_{z1} \end{Bmatrix}$$

Bar: $$\delta W^{\text{int}}=-\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}}\frac{EA}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix},\quad \delta W^{\text{ext}}=\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}}\frac{f_x h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\delta W^{\text{ine}}=-\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}}\frac{\rho Ah}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} \ddot{u}_{x1} \\ \ddot{u}_{x2} \end{Bmatrix},\quad \delta W^{\text{cpl}}=\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}}\frac{\alpha EA}{2}\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} \Delta\vartheta_1 \\ \Delta\vartheta_2 \end{Bmatrix}$$

$$\delta P^{\text{int}}=-\begin{Bmatrix} \delta\vartheta_1 \\ \delta\vartheta_2 \end{Bmatrix}^{\mathrm{T}}\frac{kA}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \vartheta_1 \\ \vartheta_2 \end{Bmatrix},\quad \delta P^{\text{ext}}=\begin{Bmatrix} \delta\vartheta_1 \\ \delta\vartheta_2 \end{Bmatrix}^{\mathrm{T}}\frac{sh}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\delta W^{\text{int}}=-\delta h\frac{h}{h^\circ}CA^\circ\frac{1}{2}[(\frac{h}{h^\circ})^2-1],\quad \delta W^{\text{ext}}=\begin{Bmatrix} g_x\delta u_{x1}+g_y\delta u_{y1}+g_z\delta u_{z1} \\ g_x\delta u_{x2}+g_y\delta u_{y2}+g_z\delta u_{z2} \end{Bmatrix}^{\mathrm{T}}\frac{\rho^\circ A^\circ h^\circ}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$h^2=(h^\circ+u_{x2}-u_{x1})^2+(u_{y2}-u_{y1})^2+(u_{z2}-u_{z1})^2$$

Torsion: $$\delta W^{\text{int}}=-\begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}^{\mathrm{T}}\frac{GJ}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix},\quad \delta W^{\text{ext}}=\begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}^{\mathrm{T}}\frac{m_x h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\delta W^{\mathrm{ine}}=-\begin{Bmatrix}\delta\theta_{x1}\\ \delta\theta_{x2}\end{Bmatrix}^{\mathrm{T}}\frac{\rho Jh}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{Bmatrix}\ddot{\theta}_{x1}\\ \ddot{\theta}_{x2}\end{Bmatrix}$$

Bending (xz):

$$\delta W^{\mathrm{int}}=-\begin{Bmatrix}\delta u_{z1}\\ \delta\theta_{y1}\\ \delta u_{z2}\\ \delta\theta_{y2}\end{Bmatrix}^{\mathrm{T}}\frac{EI_{yy}}{h^3}\begin{bmatrix}12 & -6h & -12 & -6h\\ -6h & 4h^2 & 6h & 2h^2\\ -12 & 6h & 12 & 6h\\ -6h & 2h^2 & 6h & 4h^2\end{bmatrix}\begin{Bmatrix}u_{z1}\\ \theta_{y1}\\ u_{z2}\\ \theta_{y2}\end{Bmatrix},\quad \delta W^{\mathrm{ext}}=\begin{Bmatrix}\delta u_{z1}\\ \delta\theta_{y1}\\ \delta u_{z2}\\ \delta\theta_{y2}\end{Bmatrix}^{\mathrm{T}}\frac{f_z\,h}{12}\begin{Bmatrix}6\\ -h\\ 6\\ h\end{Bmatrix}$$

$$\delta W^{\mathrm{ine}}=-\begin{Bmatrix}\delta u_{z1}\\ \delta\theta_{y1}\\ \delta u_{z2}\\ \delta\theta_{y2}\end{Bmatrix}^{\mathrm{T}}\left(\frac{\rho I_{yy}}{30h}\begin{bmatrix}36 & -3h & -36 & -3h\\ -3h & 4h^2 & 3h & -h^2\\ -36 & 3h & 36 & 3h\\ -3h & -h^2 & 3h & 4h^2\end{bmatrix}+\frac{\rho Ah}{420}\begin{bmatrix}156 & -22h & 54 & 13h\\ -22h & 4h^2 & -13h & -3h^2\\ 54 & -13h & 156 & 22h\\ 13h & -3h^2 & 22h & 4h^2\end{bmatrix}\right)\begin{Bmatrix}\ddot{u}_{z1}\\ \ddot{\theta}_{y1}\\ \ddot{u}_{z2}\\ \ddot{\theta}_{y2}\end{Bmatrix}$$

$$\delta W^{\mathrm{sta}}=-\begin{Bmatrix}\delta u_{z1}\\ \delta\theta_{y1}\\ \delta u_{z2}\\ \delta\theta_{y2}\end{Bmatrix}^{\mathrm{T}}\frac{N}{30h}\begin{bmatrix}36 & -3h & -36 & -3h\\ -3h & 4h^2 & 3h & -h^2\\ -36 & 3h & 36 & 3h\\ -3h & -h^2 & 3h & 4h^2\end{bmatrix}\begin{Bmatrix}u_{z1}\\ \theta_{y1}\\ u_{z2}\\ \theta_{y2}\end{Bmatrix}\ \text{where}\ N=EA\left(\frac{u_{x2}-u_{x1}}{h}\right)$$

Bending (xy):

$$\delta W^{\mathrm{int}}=-\begin{Bmatrix}\delta u_{y1}\\ \delta\theta_{z1}\\ \delta u_{y2}\\ \delta\theta_{z2}\end{Bmatrix}^{\mathrm{T}}\frac{EI_{zz}}{h^3}\begin{bmatrix}12 & 6h & -12 & 6h\\ 6h & 4h^2 & -6h & 2h^2\\ -12 & -6h & 12 & -6h\\ 6h & 2h^2 & -6h & 4h^2\end{bmatrix}\begin{Bmatrix}u_{y1}\\ \theta_{z1}\\ u_{y2}\\ \theta_{z2}\end{Bmatrix},\quad \delta W^{\mathrm{ext}}=\begin{Bmatrix}\delta u_{y1}\\ \delta\theta_{z1}\\ \delta u_{y2}\\ \delta\theta_{z2}\end{Bmatrix}^{\mathrm{T}}\frac{f_y\,h}{12}\begin{Bmatrix}6\\ h\\ 6\\ -h\end{Bmatrix}$$

$$\delta W^{\mathrm{ine}}=-\begin{Bmatrix}\delta u_{y1}\\ \delta\theta_{z1}\\ \delta u_{y2}\\ \delta\theta_{z2}\end{Bmatrix}^{\mathrm{T}}\left(\frac{\rho I_{zz}}{30h}\begin{bmatrix}36 & 3h & -36 & 3h\\ 3h & 4h^2 & -3h & -h^2\\ -36 & -3h & 36 & -3h\\ 3h & -h^2 & -3h & 4h^2\end{bmatrix}+\frac{\rho Ah}{420}\begin{bmatrix}156 & 22h & 54 & -13h\\ 22h & 4h^2 & 13h & -3h^2\\ 54 & 13h & 156 & -22h\\ -13h & -3h^2 & -22h & 4h^2\end{bmatrix}\right)\begin{Bmatrix}\ddot{u}_{y1}\\ \ddot{\theta}_{z1}\\ \ddot{u}_{y2}\\ \ddot{\theta}_{z2}\end{Bmatrix}$$

$$\delta W^{\mathrm{sta}}=-\begin{Bmatrix}\delta u_{y1}\\ \delta\theta_{z1}\\ \delta u_{y2}\\ \delta\theta_{z2}\end{Bmatrix}^{\mathrm{T}}\frac{N}{30h}\begin{bmatrix}36 & 3h & -36 & 3h\\ 3h & 4h^2 & -3h & -h^2\\ -36 & -3h & 36 & -3h\\ 3h & -h^2 & -3h & 4h^2\end{bmatrix}\begin{Bmatrix}u_{y1}\\ \theta_{z1}\\ u_{y2}\\ \theta_{z2}\end{Bmatrix}\ \text{where}\ N=EA\left(\frac{u_{x2}-u_{x1}}{h}\right)$$

CONSTRAINTS

Frictionless contact: $\vec{n}\cdot\vec{u}_{\mathrm{A}}=0$

Joint: $\vec{u}_{\mathrm{B}}=\vec{u}_{\mathrm{A}}$

Rigid body (link): $\vec{u}_{\mathrm{B}}=\vec{u}_{\mathrm{A}}+\vec{\theta}_{\mathrm{A}}\times\vec{\rho}_{\mathrm{AB}}$, $\vec{\theta}_{\mathrm{B}}=\vec{\theta}_{\mathrm{A}}$.

MATHEMATICS

Polar representation: $e^{i\alpha}=\cos\alpha+i\sin\alpha$, $\sin i\alpha=i\sinh\alpha$, $\cos i\alpha=\cosh\alpha$, $i^2=-1$

Eigenvalue decomposition: $\mathbf{A}=\mathbf{X}\boldsymbol{\lambda}\mathbf{X}^{-1}$

Matrix function: If $\mathbf{A}=\mathbf{X}\boldsymbol{\lambda}\mathbf{X}^{-1}$, then $f(\mathbf{A})=\mathbf{X}f(\boldsymbol{\lambda})\mathbf{X}^{-1}$

Newton's method: If $\mathbf{a}=\mathbf{a}-(\frac{\partial\mathbf{R}(\mathbf{a})}{\partial\mathbf{a}})^{-1}\mathbf{R}(\mathbf{a})\equiv\mathbf{G}(\mathbf{a})$, then $\mathbf{R}(\mathbf{a})=0$

Taylor series: $f(x+\mathrm{a})=\sum_{i=0}^{n}\frac{1}{i!}(\mathrm{a}\frac{d}{dx})^i f(x)+\frac{1}{(n+1)!}f^{(n+1)}(\underline{x})\mathrm{a}^{n+1}\quad \underline{x}\in[x,x+\mathrm{a}]$

TIME INTEGRATION (free vibrations)

Crank-Nicholson: $\begin{Bmatrix}\mathbf{a}_0\\ \mathbf{a}_1\Delta t\end{Bmatrix}^{(i+1)}=\begin{bmatrix}\mathbf{I} & -\mathbf{I}/2\\ \boldsymbol{\alpha}/2 & \mathbf{I}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{I} & \mathbf{I}/2\\ -\boldsymbol{\alpha}/2 & \mathbf{I}\end{bmatrix}\begin{Bmatrix}\mathbf{a}_0\\ \mathbf{a}_1\Delta t\end{Bmatrix}^{(i)}$, $\boldsymbol{\alpha}=\mathbf{M}^{-1}\mathbf{K}\Delta t^2$

Disc. Galerkin: $\begin{Bmatrix}\mathbf{a}_0\\ \mathbf{a}_1\Delta t\end{Bmatrix}^{(i+1)}=\begin{bmatrix}\boldsymbol{\alpha} & \mathbf{I}-\boldsymbol{\alpha}/2\\ -\mathbf{I}-\boldsymbol{\alpha}/2 & \boldsymbol{\alpha}/3\end{bmatrix}^{-1}\begin{bmatrix}0 & \mathbf{I}\\ -\mathbf{I} & -\mathbf{I}\end{bmatrix}\begin{Bmatrix}\mathbf{a}_0\\ \mathbf{a}_1\Delta t\end{Bmatrix}^{(i)}$, $\boldsymbol{\alpha}=\mathbf{M}^{-1}\mathbf{K}\Delta t^2$

MEC-E8001 Finite Element Analysis; Mathematica

"Structure is a collection of elements connected by nodes. Geometry, displacement, temperature etc. of the structure are defined by the nodal values of coordinates, translation, rotation, temperature etc. of which some are known and some unknown."

STRUCTURE

$prb = \{ele, fun\}$ where

$ele = \{prt_1, prt_2, \ldots\}$ .... elements

$fun = \{val_1, val_2, \ldots\}$ .... nodes

Element

$prt = \{typ, pro, geo\}$ where

$typ = \mathrm{BAR} \mid \mathrm{TORSION} \mid \mathrm{BEAM} \mid \mathrm{RIGID} \mid \ldots \mid$ .... model

$pro = \{p_1, p_2, \ldots, p_n\}$ ....properties

$geo = \mathrm{Point}[\{n_1\}] \mid \mathrm{Line}[\{n_1, n_2\}] \mid \mathrm{Polygon}[\{n_1, n_2, n_3\}] \mid \ldots \mid$ ....geometry

Nodes

$val = \{crd, trn, rot\} \mid \{crd, trn, rot, tmp\}$ where

$crd = \{X, Y, Z\}$ ....structural coordinates

$trn = \{u_X, u_Y, u_Z\}$ .... translation components

$rot = \{\theta_X, \theta_Y, \theta_Z\}$ ....rotation components

$tmp = \vartheta$ ....temperature

DISPLACEMENT ANALYSIS

Constraint

$\{\text{JOINT},\{\}\,|\,\{\{\underline{u}_X, \underline{u}_Y, \underline{u}_Z\}\}, \text{Point}[\{n_1\}]\}$ ....displacement constraint

$\{\text{JOINT},\{\}, \text{Line}[\{n_1, n_2\}]\}$ ....displacement constraint

$\{\text{RIGID},\{\}\,|\,\{\{\underline{u}_X, \underline{u}_Y, \underline{u}_Z\},\{\underline{\theta}_X, \underline{\theta}_Y, \underline{\theta}_Z\}\}, \text{Point}[\{n_1\}]\}$ ....displacement/rotation constraint

$\{\text{RIGID},\{\}, \text{Line}[\{n_1, n_2\}]\}$ ....rigid constraint

$\{\text{SLIDER},\{n_X, n_Y, n_Z\}, \text{Point}[\{n_1\}]\}$ .... slider constraint

Force

$\{\text{FORCE},\{F_X, F_Y, F_Z\}, \text{Point}[\{n_1\}]\}$ .... point force

$\{\text{FORCE},\{F_X, F_Y, F_Z, M_X, M_Y, M_Z\}, \text{Point}[\{n_1\}]\}$ ....point load

$\{\text{FORCE},\{f_X, f_Y, f_Z\}, \text{Line}[\{n_1, n_2\}]\}$ ....distributed force

$\{\text{FORCE},\{f_X, f_Y, f_Z\}, \text{Polygon}[\{n_1, n_2, n_3\}]\}$ ....distributed force

Beam model

$\{\text{BAR},\{\{E\},\{A\},\{f_X, f_Y, f_Z\}\}, \text{Line}[\{n_1, n_2\}]\}$ ....bar mode

$\{\text{TORSION},\{\{G\},\{J\},\{m_X, m_Y, m_Z\}\}, \text{Line}[\{n_1, n_2\}]\}$ .... torsion mode

$\{\text{BEAM},\{\{E,G\},\{A, I_{yy}, I_{zz}\},\{f_X, f_Y, f_Z\}\}, \text{Line}[\{n_1, n_2\}]\}$ ....beam

$\{\text{BEAM},\{\{E,G\},\{A, I_{yy}, I_{zz},\{j_X, j_Y, j_Z\}\},\{f_X, f_Y, f_Z\}\}, \text{Line}[\{n_1, n_2\}]\}$ ....beam

Plate model

$\{\text{PLANE},\{\{E,\nu\},\{t\},\{f_X, f_Y, f_Z\}\}, \text{Polygon}[\{n_1, n_2, n_3\}]\}$ .... thin slab mode

$\{\text{PLANE},\{\{E,\nu\},\{t\},\{f_X, f_Y, f_Z\}\}, \text{Polygon}[\{n_1, n_2, n_3, n_4\}]\}$ .... thin slab mode

$\{\text{PLATE},\{\{E,\nu\},\{t\},\{f_X, f_Y, f_Z\}\}, \text{Polygon}[\{n_1, n_2, n_3\}]\}$ .... bending mode

$\{\text{SHELL},\{\{E,\nu\},\{t\},\{f_X, f_Y, f_Z\}\}, \text{Polygon}[\{n_1, n_2, n_3\}]\}$ ....plate

Solid model

$\{\text{SOLID},\{\{E,\nu\},\{f_X, f_Y, f_Z\}\}, \text{Tetrahedron}[\{n_1, n_2, n_3, n_4\}]\}$ ....solid

$\{\text{SOLID},\{\{E,\nu\},\{f_X, f_Y, f_Z\}\}, \text{Hexahedron}[\{n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8\}]\}$ ....solid

$\{\text{SOLID},\{\{E,\nu\},\{f_X, f_Y, f_Z, m_X, m_Y, m_Z,\}\}, \text{Tetrahedron}[\{n_1, n_2, n_3, n_4\}]\}$ ....solid

OPERATIONS

$prb = \text{REFINE}[prb]$ .... refine structure representation

$\text{Out} = \text{FORMATTED}[prb]$ ....display problem definition

$\text{Out} = \text{STANDARDFORM}[\,prb\,]$ .... display virtual work expression

$\text{sol} = \text{SOLVE}[\{\text{DISP}\}, prb] \;\; | \;\; \text{SOLVE}[\,prb\,]$ .... solve the unknowns

VIBRATION ANALYSIS

Constraint

$\{\text{JOINT},\{\}\,|\,\{\{\underline{u}_X,\underline{u}_Y,\underline{u}_Z\}\},\text{Point}[\{n_1\}]\}$ .... displacement constraint

$\{\text{JOINT},\{\},\text{Line}[\{n_1,n_2\}]\}$ .... displacement constraint

$\{\text{RIGID},\{\}\,|\,\{\{\underline{u}_X,\underline{u}_Y,\underline{u}_Z\},\{\underline{\theta}_X,\underline{\theta}_Y,\underline{\theta}_Z\}\},\text{Point}[\{n_1\}]\}$ .... displacement/rotation constraint

$\{\text{RIGID},\{\},\text{Line}[\{n_1,n_2\}]\}$ .... rigid constraint

$\{\text{SLIDER},\{n_X,n_Y,n_Z\},\text{Point}[\{n_1\}]\}$ .... slider constraint

Force

$\{\text{FORCE},\{F_X,F_Y,F_Z\},\text{Point}[\{n_1\}]\}$ .... point force

$\{\text{FORCE},\{F_X,F_Y,F_Z,M_X,M_Y,M_Z\},\text{Point}[\{n_1\}]\}$ .... point load

$\{\text{FORCE},\{f_X,f_Y,f_Z\},\text{Line}[\{n_1,n_2\}]\}$ .... distributed force

$\{\text{FORCE},\{f_X,f_Y,f_Z\},\text{Polygon}[\{n_1,n_2,n_3\}]\}$ .... distributed force

$\{\text{FORCE},\{\{m\}\},\text{Point}[\{n_1\}]\}$ .... inertia effect

$\{\text{FORCE},\{\{m,J\}\},\text{Point}[\{n_1\}]\}$ .... inertia effect

$\{\text{FORCE},\{\{m,J\},\{\{i_X,i_Y,i_Z\},\{j_X,j_Y,j_Z\}\}\},\text{Point}[\{n_1\}]\}$ .... inertia effect

Beam model

$\{\text{BAR},\{\{E,\rho\},\{A\},\{f_X,f_Y,f_Z\}\},\text{Line}[\{n_1,n_2\}]\}$ .... bar mode

$\{\text{TORSION},\{\{G,\rho\},\{J\},\{m_X,m_Y,m_Z\}\},\text{Line}[\{n_1,n_2\}]\}$ .... torsion mode

$\{\text{BEAM},\{\{E,G,\rho\},\{A,I_{yy},I_{zz}\},\{f_X,f_Y,f_Z\}\},\text{Line}[\{n_1,n_2\}]\}$ .... beam

$\{\text{BEAM},\{\{E,G,\rho\},\{A,I_{yy},I_{zz},\{j_X,j_Y,j_Z\}\},\{f_X,f_Y,f_Z\}\},\text{Line}[\{n_1,n_2\}]\}$ .... beam

Plate model

$\{\text{PLANE},\{\{E,\nu,\rho\},\{t\},\{f_X,f_Y,f_Z\}\},\text{Polygon}[\{n_1,n_2,n_3\}]\}$ .... thin slab mode

$\{\text{PLANE},\{\{E,\nu,\rho\},\{t\},\{f_X,f_Y,f_Z\}\},\text{Polygon}[\{n_1,n_2,n_3,n_4\}]\}$ .... thin slab mode

$\{\text{PLATE},\{\{E,\nu,\rho\},\{t\},\{f_X,f_Y,f_Z\}\},\text{Polygon}[\{n_1,n_2,n_3\}]\}$ .... bending mode

$\{\text{SHELL},\{\{E,\nu,\rho\},\{t\},\{f_X,f_Y,f_Z\}\},\text{Polygon}[\{n_1,n_2,n_3\}]\}$ .... plate

Solid model

$\{\text{SOLID},\{\{E,\nu,\rho\},\{f_X,f_Y,f_Z\}\},\text{Tetrahedron}[\{n_1,n_2,n_3,n_4\}]\}$ .... solid

$\{\text{SOLID},\{\{E,\nu,\rho\},\{f_X,f_Y,f_Z\}\},\text{Hexahedron}[\{n_1,n_2,n_3,n_4,n_5,n_6,n_7,n_8\}]\}$ .... solid

$\{\text{SOLID},\{\{E,\nu,\rho\},\{f_X,f_Y,f_Z,m_X,m_Y,m_Z,\}\},\text{Tetrahedron}[\{n_1,n_2,n_3,n_4\}]\}$ .... solid

OPERATIONS

$prb = \text{REFINE}[prb]$ .... refine structure representation

$\text{Out} = \text{FORMATTED}[prb]$ ....display problem definition

$\text{Out} = \text{STANDARDFORM}[\{\text{VIBR}\}, prb]$ ....display virtual work expression

$\text{sol} = \text{SOLVE}[\{\text{VIBR}\}, prb]$ ....solve the eigenfrequencies and modes

$\text{sol} = \text{SOLVE}[\{\text{VIBR,ini}\}, prb]$ .... solve the unknowns (does not work with DAEs)

STABILITY ANALYSIS

Constraint

$\{\text{JOINT},\{\}|\{\{\underline{u}_X,\underline{u}_Y,\underline{u}_Z\}\},\text{Point}[\{n_1\}]\}$ ....displacement constraint

$\{\text{JOINT},\{\},\text{Line}[\{n_1,n_2\}]\}$ ....displacement constraint

$\{\text{RIGID},\{\}|\{\{\underline{u}_X,\underline{u}_Y,\underline{u}_Z\},\{\underline{\theta}_X,\underline{\theta}_Y,\underline{\theta}_Z\}\},\text{Point}[\{n_1\}]\}$ ....displacement/rotation constraint

$\{\text{RIGID},\{\},\text{Line}[\{n_1,n_2\}]\}$ ....rigid constraint

$\{\text{SLIDER},\{n_X,n_Y,n_Z\},\text{Point}[\{n_1\}]\}$ .... slider constraint

Force

$\{\text{FORCE},\{F_X,F_Y,F_Z\},\text{Point}[\{n_1\}]\}$ .... point force

$\{\text{FORCE},\{F_X,F_Y,F_Z,M_X,M_Y,M_Z\},\text{Point}[\{n_1\}]\}$ ....point load

$\{\text{FORCE},\{f_X,f_Y,f_Z\},\text{Line}[\{n_1,n_2\}]\}$ ....distributed force

$\{\text{FORCE},\{f_X,f_Y,f_Z\},\text{Polygon}[\{n_1,n_2,n_3\}]\}$ ....distributed force

Beam model

$\{\text{BAR},\{\{E\},\{A\},\{f_X,f_Y,f_Z\}\},\text{Line}[\{n_1,n_2\}]\}$ ....bar mode

$\{\text{TORSION},\{\{G\},\{J\},\{\{m_X,m_Y,m_Z\}\}\},\text{Line}[\{n_1,n_2\}]\}$ .... torsion mode

$\{\text{BENDING},\{\{E\},\{I_y,I_z\},\{f_X,f_Y,f_Z\}\},\text{Line}[\{n_1,n_2\}]\}$................................. bending mode

$\{\text{BEAM},\{\{E,G\},\{A,I_{yy},I_{zz}\},\{f_X,f_Y,f_Z\}\},\text{Line}[\{n_1,n_2\}]\}$ ....beam

$\{\text{BEAM},\{\{E,G\},\{A,I_{yy},I_{zz},\{j_X,j_Y,j_Z\}\},\{f_X,f_Y,f_Z\}\},\text{Line}[\{n_1,n_2\}]\}$.......................beam

Plate model

$\{\text{PLANE},\{\{E,\nu\},\{t\},\{f_X,f_Y,f_Z\}\},\text{Polygon}[\{n_1,n_2,n_3\}]\}$.............................. thin slab mode

$\{\text{PLANE},\{\{E,\nu\},\{t\},\{f_X,f_Y,f_Z\}\},\text{Polygon}[\{n_1,n_2,n_3,n_4\}]\}$......................... thin slab mode

$\{\text{PLATE},\{\{E,\nu\},\{t\},\{f_X,f_Y,f_Z\}\},\text{Polygon}[\{n_1,n_2,n_3\}]\}$ .... bending mode

$\{\text{SHELL},\{\{E,\nu\},\{t\},\{f_X,f_Y,f_Z\}\},\text{Polygon}[\{n_1,n_2,n_3\}]\}$ ....plate

Solid model

$\{\text{SOLID},\{\{E,\nu\},\{f_X,f_Y,f_Z\}\},\text{Tetrahedron}[\{n_1,n_2,n_3,n_4\}]\}$..solid

$\{\text{SOLID},\{\{E,\nu\},\{f_X,f_Y,f_Z\}\},\text{Hexahedron}[\{n_1,n_2,n_3,n_4,n_5,n_6,n_7,n_8\}]\}$ ....solid

$\{\text{SOLID},\{\{E,\nu\},\{f_X,f_Y,f_Z,m_X,m_Y,m_Z,\}\},\text{Tetrahedron}[\{n_1,n_2,n_3,n_4\}]\}$ ....solid

OPERATIONS

$prb = \text{REFINE}[prb]$ .... refine structure representation

$\text{Out} = \text{FORMATTED}[prb]$ ....display problem definition

$\text{Out} = \text{STANDARDFORM}[\{\text{STAB}\}, prb]$ ....display virtual work expression

$sol = \text{SOLVE}[\{\text{STAB}, p\}, prb]$ ....find the critical values of p and the modes

NONLINEAR ANALYSIS

Constraint

$\{\text{JOINT},\{\}|\{\{\underline{u}_X,\underline{u}_Y,\underline{u}_Z\}\},\text{Point}[\{n_1\}]\}$ ....displacement constraint

$\{\text{JOINT},\{\},\text{Line}[\{n_1,n_2\}]\}$ ....displacement constraint

$\{\text{RIGID},\{\}|\{\{\underline{u}_X,\underline{u}_Y,\underline{u}_Z\},\{\underline{\theta}_X,\underline{\theta}_Y,\underline{\theta}_Z\}\},\text{Point}[\{n_1\}]\}$ ....displacement/rotation constraint

$\{\text{RIGID},\{\},\text{Line}[\{n_1,n_2\}]\}$ ....rigid constraint

$\{\text{SLIDER},\{n_X,n_Y,n_Z\},\text{Point}[\{n_1\}]\}$ .... slider constraint

Force

$\{\text{FORCE},\{F_X,F_Y,F_Z\},\text{Point}[\{n_1\}]\}$ .... point force

$\{\text{FORCE},\{F_X,F_Y,F_Z,M_X,M_Y,M_Z\},\text{Point}[\{n_1\}]\}$ ....point load

$\{\text{FORCE},\{f_X,f_Y,f_Z\},\text{Line}[\{n_1,n_2\}]\}$ ....distributed force

$\{\text{FORCE},\{f_X,f_Y,f_Z\},\text{Polygon}[\{n_1,n_2,n_3\}]\}$ ....distributed force

Beam model

$\{\text{BAR},\{\{E\},\{A\},\{f_X,f_Y,f_Z\}\},\text{Line}[\{n_1,n_2\}]\}$ ....bar mode

Plate model

$\{\text{PLANE},\{\{E,\nu\},\{t\},\{f_X,f_Y,f_Z\}\},\text{Polygon}[\{n_1,n_2,n_3\}]\}$ .... thin slab mode

$\{\text{PLANE},\{\{E,\nu\},\{t\},\{f_X,f_Y,f_Z\}\},\text{Polygon}[\{n_1,n_2,n_3,n_4\}]\}$ .... thin slab mode

Solid model

$\{\text{SOLID},\{\{E,\nu\},\{f_X,f_Y,f_Z\}\},\text{Tetrahedron}[\{n_1,n_2,n_3,n_4\}]\}$ ....(nonlinear) solid

$\{\text{SOLID},\{\{E,\nu\},\{f_X,f_Y,f_Z\}\},\text{Hexahedron}[\{n_1,n_2,n_3,n_4,n_5,n_6,n_7,n_8\}]\}$ ....(nonlinear) solid

$\{\text{SOLID},\{\{E,\nu\},\{f_X,f_Y,f_Z,m_X,m_Y,m_Z,\}\},\text{Tetrahedron}[\{n_1,n_2,n_3,n_4\}]\}$ ...(nonlinear) solid

OPERATIONS

$prb = \text{REFINE}[prb]$ .... refine structure representation

$\text{Out} = \text{FORMATTED}[\{\text{NONL}\},prb]$ ....display problem definition

$\text{Out} = \text{STANDARDFORM}[\{\text{NONL}\},prb]$ ....display virtual work expression

$sol = \text{SOLVE}[\{\text{NONL}\},prb]$ ....find the likely numerical solution

$sol = \text{SOLVE}[\{\text{NONL,ALL}\},prb]$ .... find all solutions

THERMO-MECHANICAL ANALYSIS

Constraint

$\{\mathrm{JOINT},\{\}|\{\{\underline{u}_X,\underline{u}_Y,\underline{u}_Z\}\},\mathrm{Point}[\{n_1\}]\}$ ....displacement constraint

$\{\mathrm{JOINT},\{\},\mathrm{Line}[\{n_1,n_2\}]\}$ ....displacement constraint

$\{\mathrm{RIGID},\{\}|\{\{\underline{u}_X,\underline{u}_Y,\underline{u}_Z\},\{\underline{\theta}_X,\underline{\theta}_Y,\underline{\theta}_Z\}\},\mathrm{Point}[\{n_1\}]\}$ ....displacement/rotation constraint

$\{\mathrm{RIGID},\{\},\mathrm{Line}[\{n_1,n_2\}]\}$ ....rigid constraint

$\{\mathrm{SLIDER},\{n_X,n_Y,n_Z\},\mathrm{Point}[\{n_1\}]\}$ .... slider constraint

Force

$\{\mathrm{FORCE},\{F_X,F_Y,F_Z\},\mathrm{Point}[\{n_1\}]\}$ .... point force

$\{\mathrm{FORCE},\{F_X,F_Y,F_Z,M_X,M_Y,M_Z\},\mathrm{Point}[\{n_1\}]\}$ ....point load

$\{\mathrm{FORCE},\{f_X,f_Y,f_Z\},\mathrm{Line}[\{n_1,n_2\}]\}$ ....distributed force

$\{\mathrm{FORCE},\{f_X,f_Y,f_Z\},\mathrm{Polygon}[\{n_1,n_2,n_3\}]\}$ ....distributed force

Beam model

$\{\mathrm{BAR},\{\{E,\alpha,k\},\{A\},\{\{f_X,f_Y,f_Z\},\{s,\vartheta_0\}\}\},\mathrm{Line}[\{n_1,n_2\}]\}$ ....bar mode

Plate model

$\{\mathrm{PLANE},\{\{E,\nu,\alpha,k\},\{t\},\{\{f_X,f_Y,f_Z\},\{s,\vartheta_0\}\}\},\mathrm{Polygon}[\{n_1,n_2,n_3\}]\}$ .... thin slab mode

$\{\mathrm{PLANE},\{\{E,\nu,\alpha,k\},\{t\},\{\{f_X,f_Y,f_Z\},s,\vartheta_0\}\},\mathrm{Polygon}[\{n_1,n_2,n_3,n_4\}]\}$ ...... thin slab mode

Solid model

$\{\mathrm{SOLID},\{\{E,\nu,\alpha,k\},\{\{f_X,f_Y,f_Z\},\{s,\vartheta_0\}\}\},\mathrm{Tetrahedron}[\{n_1,n_2,n_3,n_4\}]\}$ ....solid

Functions

$prb = \mathrm{REFINE}[prb]$ .... refine structure representation

$\mathrm{Out} = \mathrm{FORMATTED}[prb]$ ....display problem definition

$\mathrm{Out} = \mathrm{STANDARDFORM}[\{\mathrm{TMEC}\},prb]$ ....display virtual work expression

$\mathrm{sol} = \mathrm{SOLVE}[\{\mathrm{TMEC}\},prb]$ ....solve the unknowns

MEC-E8001

Finite Element Analysis

2023

WEEK 2: INTRODUCTION

1 INTRODUCTION

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1.3 FINITE ELEMENT ANALYSIS 24

1.4 FE-CODE OF MEC-E8001 45

LEARNING OUTCOMES

Students get an overall picture about prerequisites of the course, the roles of engineering models in structure modelling, and finite element method in displacement analysis of structures. The topics of the week are

- Experiment vs. modelling
- Structure modelling
- Structure analysis
- Mathematica language and the finite element solver of MEC-E8001

EXPERIMENT VS. MODELLING

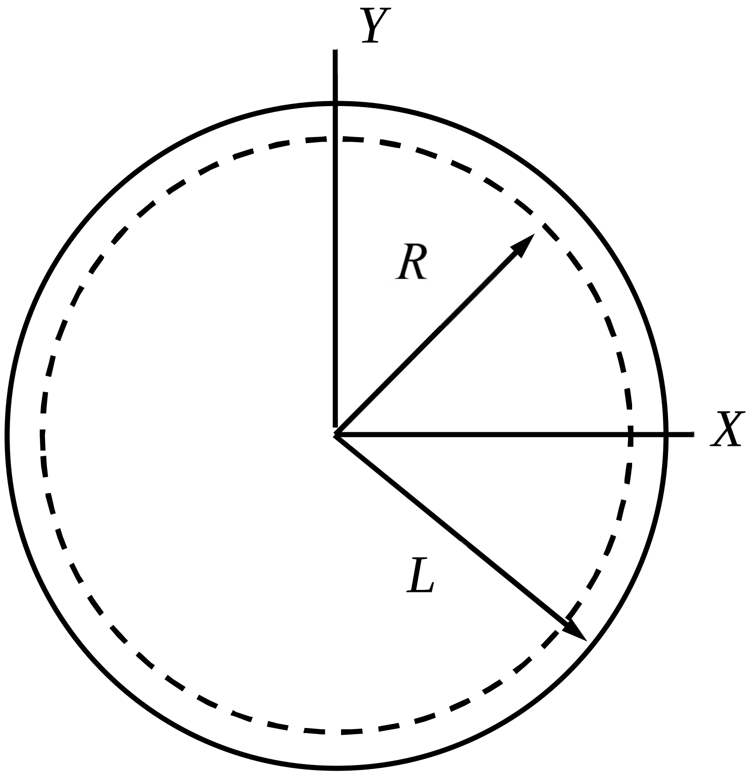
In design of a simple pendulum of a tall-case clock, the required information is the dependency of period T on mass m, initial angle ϕ_0 from the stable equilibrium position, acceleration by gravity g, and length L. The main options are

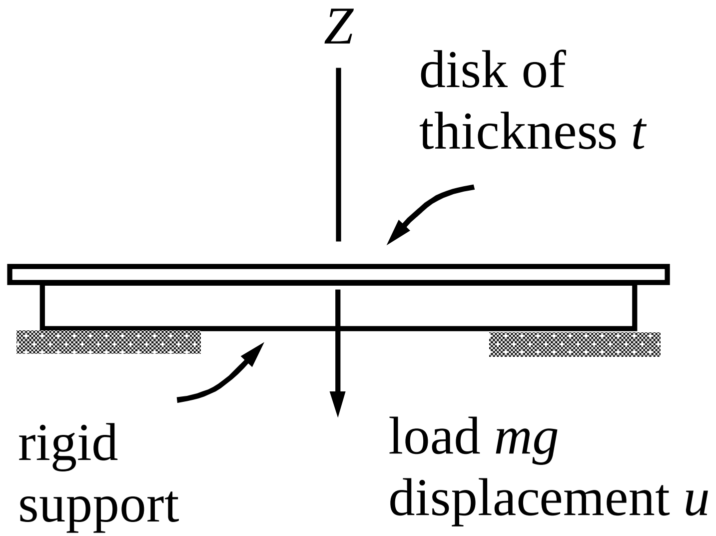
Straightforward experiment: Measurement of T on various physical structures (characterized by m and L), with various initial angles ϕ_0, and on various places on earth (characterized by g).

Dimension analysis: Application of generic principles of physics to get $T = \sqrt{L/g}\, f(\phi_0)$ and measurement of $T\sqrt{g/L}$ as the function of ϕ_0.

Mathematical modelling: Application of simplifying assumptions, the basic laws of mechanics, and rules of mathematics to get $T = 2\pi\sqrt{L/g}$.

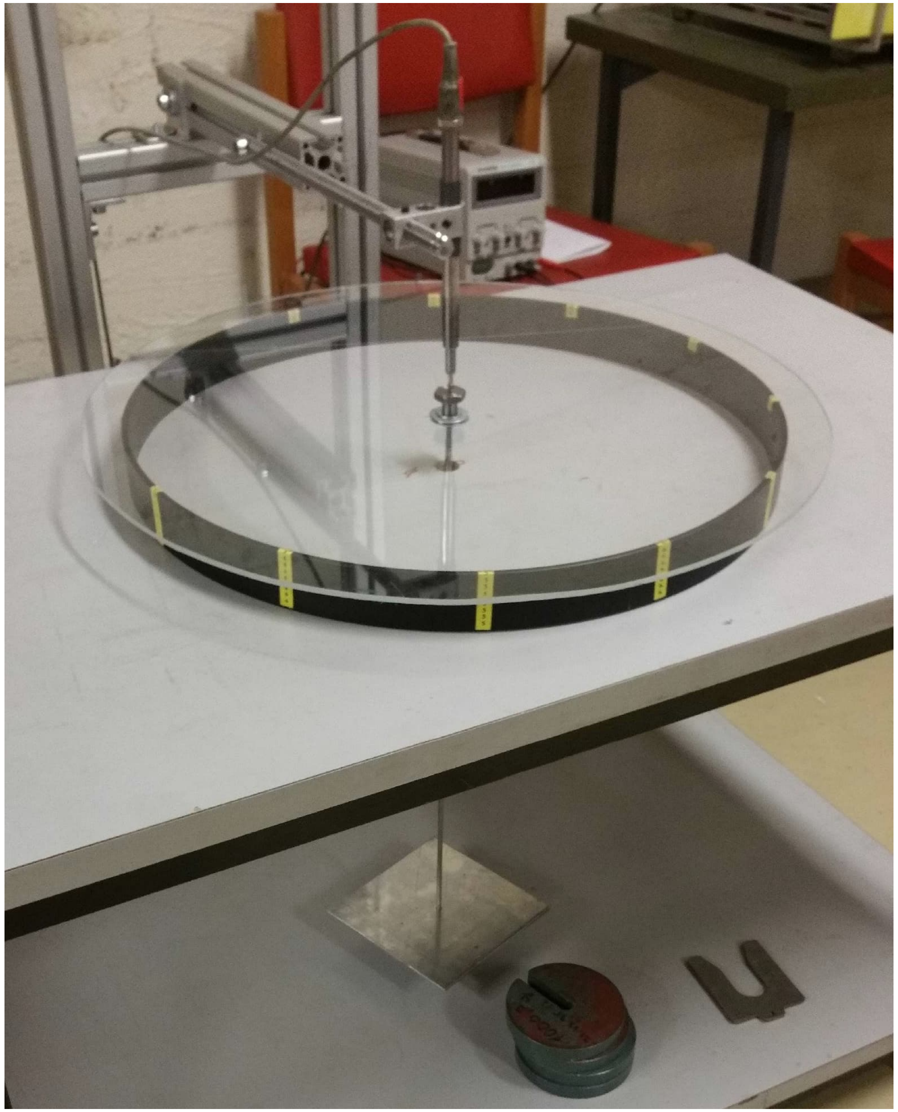
RIGIDITY OF ELASTIC DISK ?





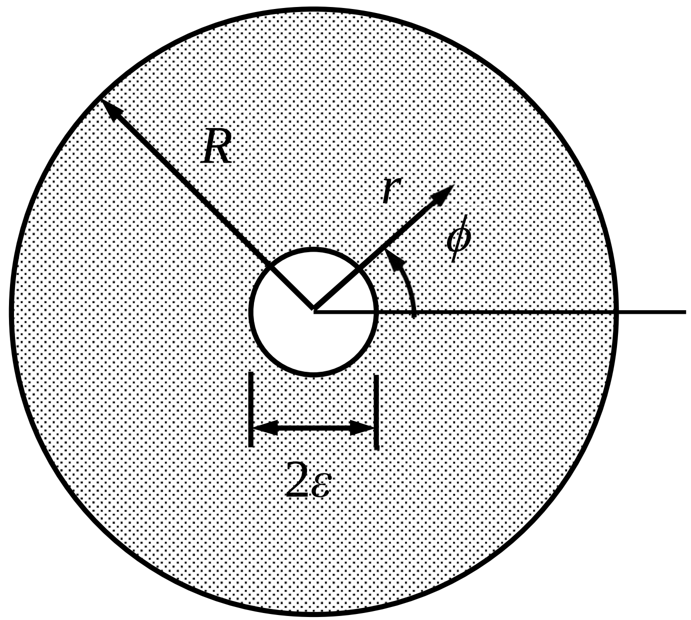
Dimension analysis with quantities E, ν, R, L, t, m, g, and u : $\quad \dfrac{mgR^2}{Et^4} = f\left(\dfrac{u}{t}, \dfrac{L}{R}, \nu\right)$

EXPERIMENT



ANALYTICAL METHOD

Kirchhoff plate model in polar coordinates with the assumption of rotation symmetry. Point force is modelled as distributed load acting on the inner boundary of radius $\varepsilon \to 0$ (finally). Boundary value problem for the transverse displacement *w*:



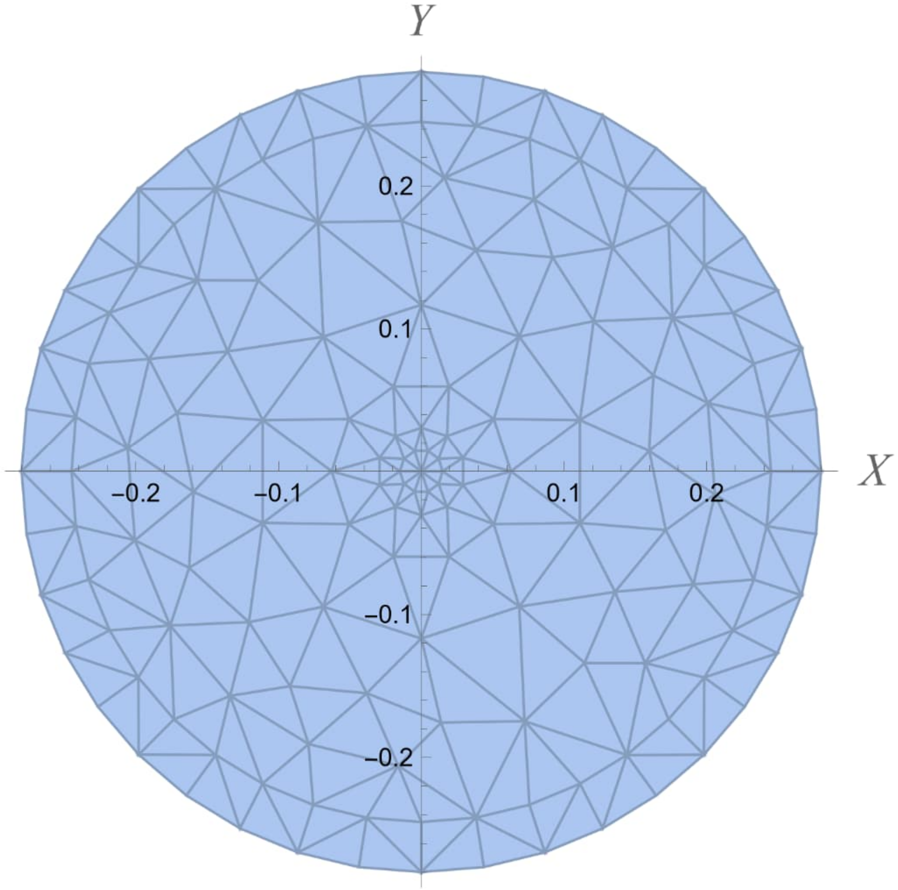
$$(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr})(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr})w = 0 \quad \text{in} \quad (\varepsilon, R),$$

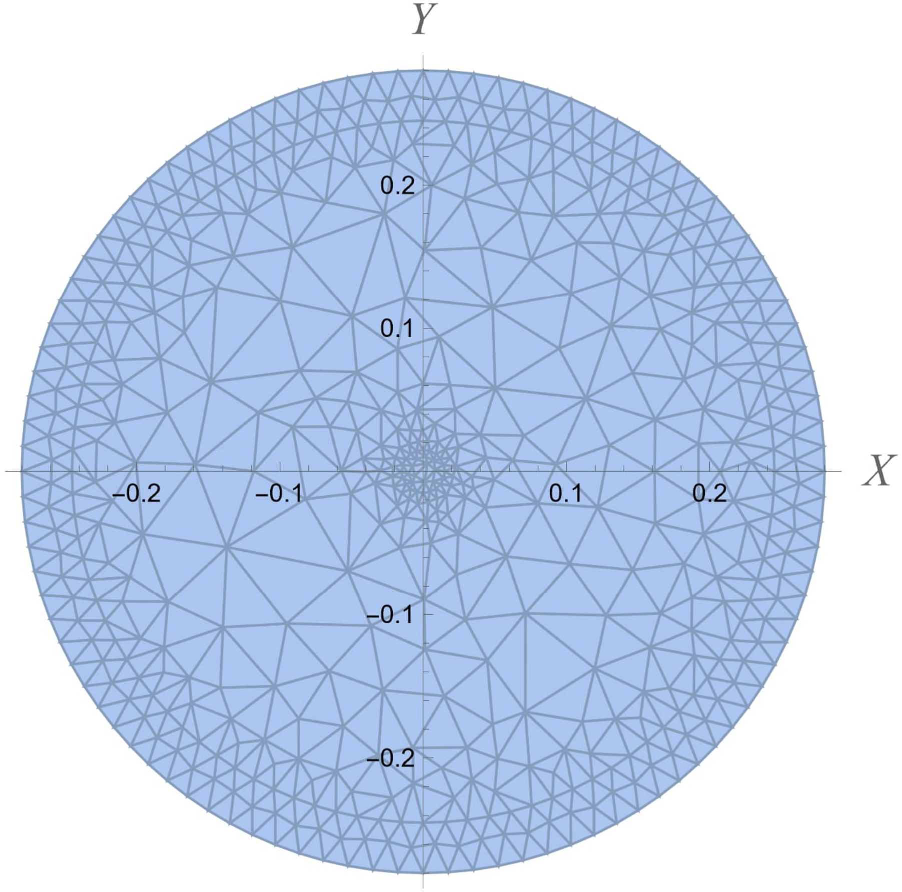
$$M_{rr}(R) = 0 \quad \text{and} \quad w(R) = 0,$$

$$Q_r(\varepsilon) + \frac{F}{2\pi\varepsilon} = 0 \quad \text{and} \quad M_{rr}(\varepsilon) = 0,$$

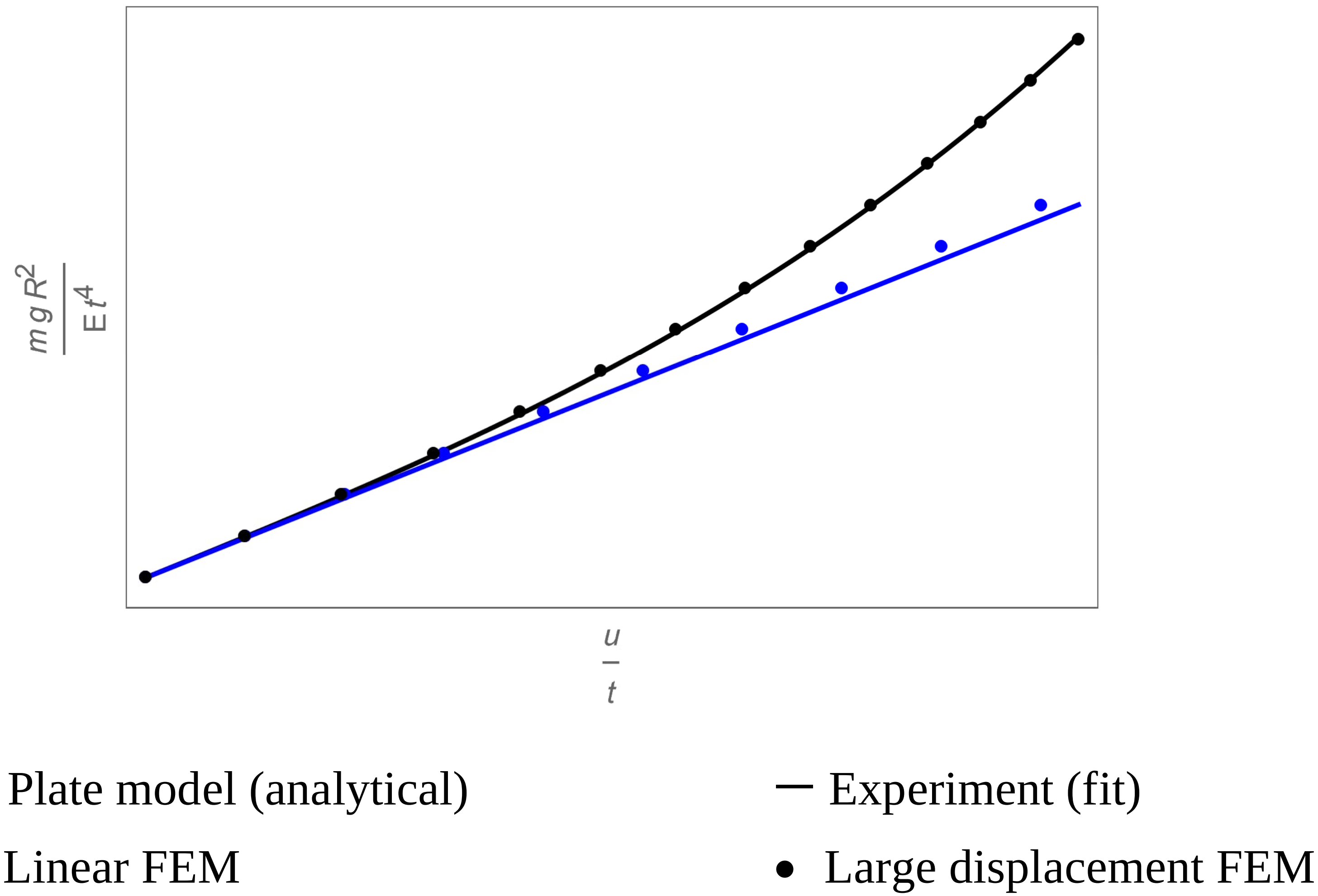
where $Q_r = -D(\frac{d^3w}{dr^3} + \frac{1}{r}\frac{d^2w}{dr^2} - \frac{1}{r^2}\frac{dw}{dr})$ and $M_{rr} = -D(\frac{d^2w}{dr^2} + \nu\frac{1}{r}\frac{dw}{dr})$, $D = \frac{t^3}{12}\frac{E}{1-\nu^2}$

FINITE ELEMENT METHOD





With a numerical solution method, restrictions to the model are less severe. In the present case, e.g., large displacement analysis with a non-linear model is needed unless $u \ll t$.



$\frac{mgR^2}{Et^4}$
$\frac{u}{t}$
Plate model (analytical)
Experiment (fit)
Linear FEM
Large displacement FEM

ERROR COMPONENTS

Measuring error $\|\bar{u}-u\|$: measured ($\bar{u}$) and exact (u)

Modeling error $\|u-\hat{u}\|$: exact (u) and exact to model ($\hat{u}$)

Numerical error $\|\hat{u}-\tilde{u}\|$: exact to model ($\hat{u}$) and numerical ($\tilde{u}$)

Error $\|\bar{u}-\tilde{u}\| = \|\bar{u}-u+u-\hat{u}+\hat{u}-\tilde{u}\| \leq \|\bar{u}-u\|+\|u-\hat{u}\|+\|\hat{u}-\tilde{u}\|$

An engineering model and, thereby, predictions by the model contain always error due to the simplifications made. Therefore, in practice, numerical error of the same order in finding the predictions (solution to the model) can be considered as acceptable.

1.1 STRUCTURE MODELLING

MODELLING STEPS

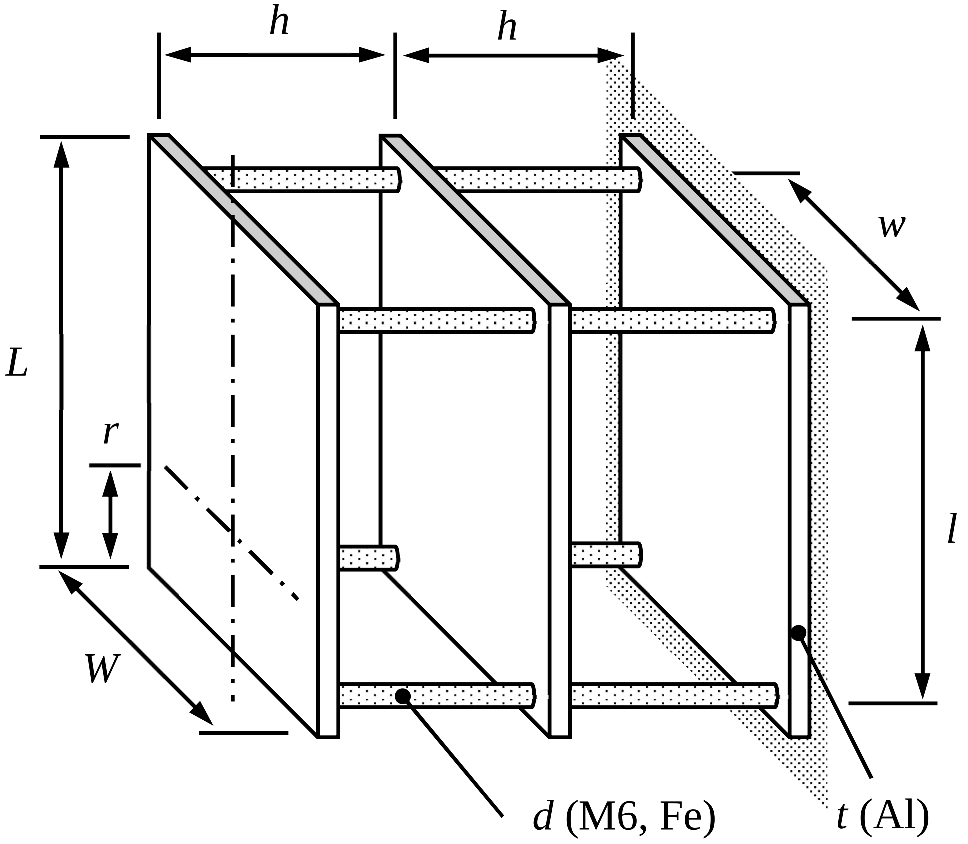
- **Crop**: Decide the boundary of structure. Interaction with surroundings need to be described in terms of known forces, moments, displacements, and rotations. All uncertainties with this respect bring uncertainty to the model too.

- **Idealize**: Simplify the geometry and decide the model. Ignoring the details, not likely to affect the outcome, may simplify analysis a lot.

- **Parameterize**: Assign symbols to geometric and material parameter of the idealized structure. Measure or find the values needed in numerical calculations.

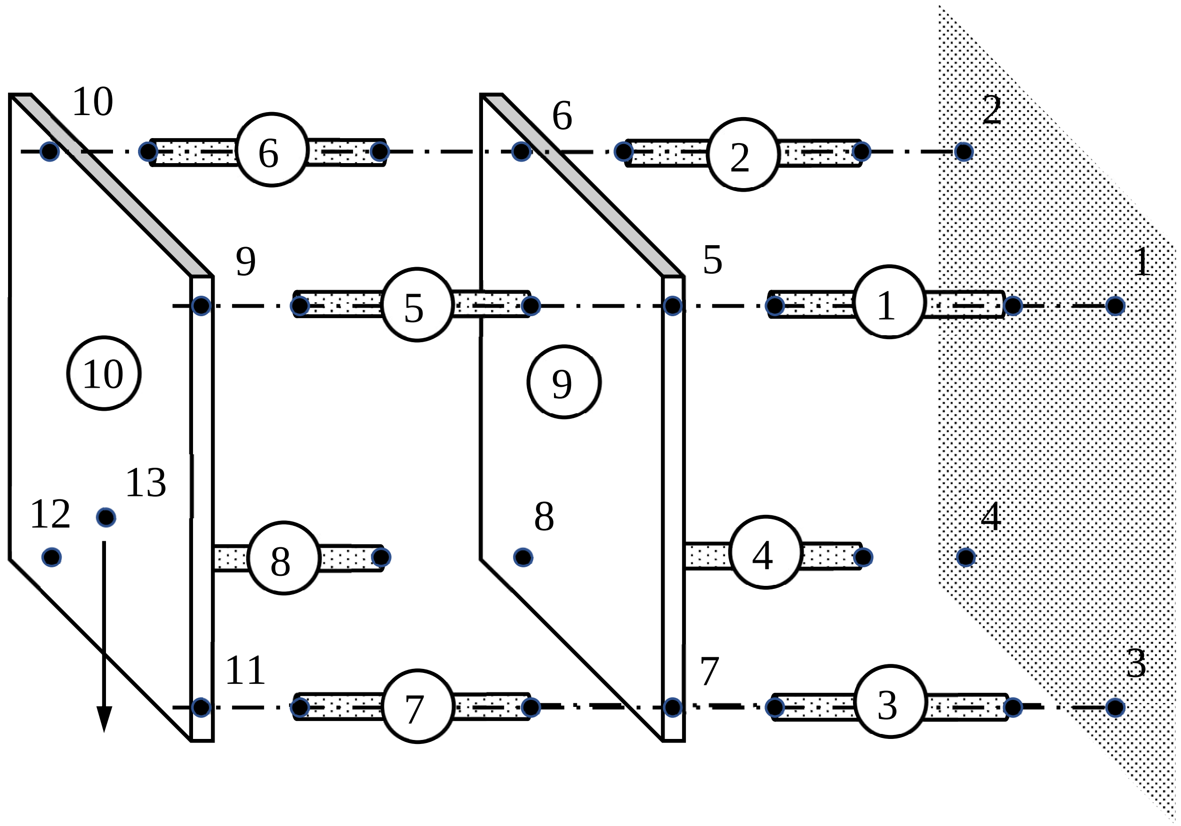
- **Divide-and-rule**: Represent a complex structure as a set of structural parts interacting with each other through connection points and surroundings with interaction models.

CROP-IDEALIZE-PARAMETERIZE



d	4.8 mm
h	0.156 m
l	0.4 m
w	0.243 m
L	0.44 m
W	0.295 m
t	1.5 cm
r	6.5 cm

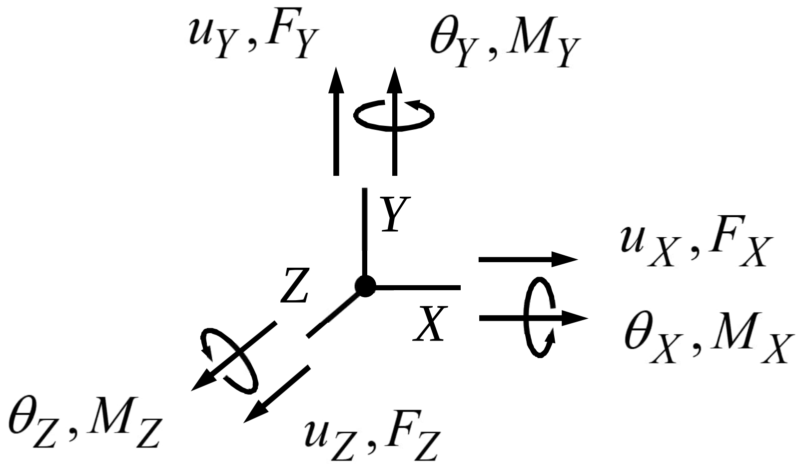
DIVIDE-AND-RULE



The book-keeping with unique identifiers for the structural parts and connection points allow a consistent naming of the kinematic and kinetic quantities of analysis.

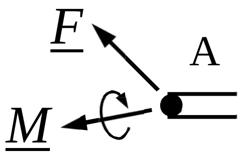
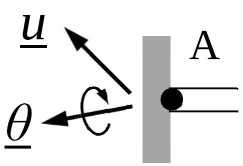
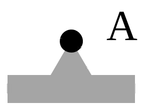
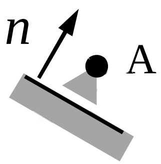
KINEMATIC AND KINETIC QUANTITIES

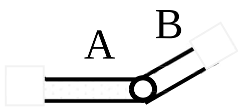
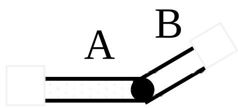
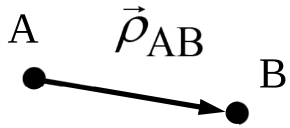
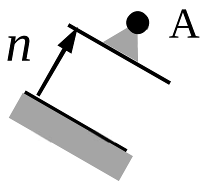
The primary quantities of analysis are displacements, rotations, forces and moments at the connection points of the structural parts. The components of the vector quantities (magnitude and direction) are taken to be positive in the directions of the coordinate axes.



Vector quantities are invariants in the sense $\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k} = a_X\vec{I} + a_Y\vec{J} + a_Z\vec{K}$, and can be transformed from one coordinate system to another using the property.

INTERACTION MODELS

name	symbol	equations
force		$\vec{F}_{\mathrm{A}} = \underline{\vec{F}},\ \vec{M}_{\mathrm{A}} = \underline{\vec{M}}$
fixed		$\vec{u}_{\mathrm{A}} = \underline{\vec{u}},\ \vec{\theta}_{\mathrm{A}} = \underline{\vec{\theta}}$
joint		$\vec{u}_{\mathrm{A}} = 0,\ \vec{M}_{\mathrm{A}} = 0$
slider		$\vec{n} \cdot \vec{u}_{\mathrm{A}} = 0,\ \vec{F}_{\mathrm{A}} - (\vec{F}_{\mathrm{A}} \cdot \vec{n})\vec{n} = 0,\ \vec{M}_{\mathrm{A}} = 0$

joint		$\vec{u}_{\mathrm{B}} = \vec{u}_{\mathrm{A}},\ \vec{M}_{\mathrm{A}} = 0,\ \vec{M}_{\mathrm{B}} = 0$
fixed		$\vec{u}_{\mathrm{B}} = \vec{u}_{\mathrm{A}},\ \vec{\theta}_{\mathrm{B}} = \vec{\theta}_{\mathrm{A}}$
rigid		$\vec{u}_{\mathrm{B}} = \vec{u}_{\mathrm{A}} + \vec{\theta}_{\mathrm{A}} \times \vec{\rho}_{\mathrm{AB}},\ \vec{\theta}_{\mathrm{B}} = \vec{\theta}_{\mathrm{A}}$
contact		$\vec{n} \cdot \vec{u}_{\mathrm{A}} \geq 0,\ \vec{n} \cdot \vec{F}_{\mathrm{A}} \geq 0,\ (\vec{n} \cdot \vec{u}_{\mathrm{A}})(\vec{n} \cdot \vec{F}_{\mathrm{A}}) = 0$

Interaction models define a kinematic quantity (displacements and rotations) or its work conjugate (forces and moments)!

NEWTON's LAWS OF MOTION

I In an inertial frame of reference, an object either remains at rest or continues to move at a constant velocity, unless acted upon by a force.

II The vector sum of the forces on an object is equal to the mass of that object multiplied by the acceleration of the object (assuming that the mass is constant).

III When one body exerts a force on a second body, the second body simultaneously exerts a force equal in magnitude and opposite in direction on the first body.

Newton's laws in their original forms apply to particles only. The formulation for rigid bodies and deformable bodies requires slight modifications.

MODELS FOR STRUCTURAL PARTS

Structural part model is a relationship between the displacements, rotations, forces, moments at the connection points and external given forces (like weight) acting on the structural parts. The relationship is. One may consider the relationship as the generalization of the simple spring model affected by the assumptions used (beam, plate, solid model), material model, the number of connection and additional points, and the shape of the structural parts. Assuming a linear and stationary case

nodal displacements   distributed forces

$$\{F\} = [K]\{u\} - \{f\}$$

nodal forces   stiffness

1.2 STRUCTURE ANALYSIS

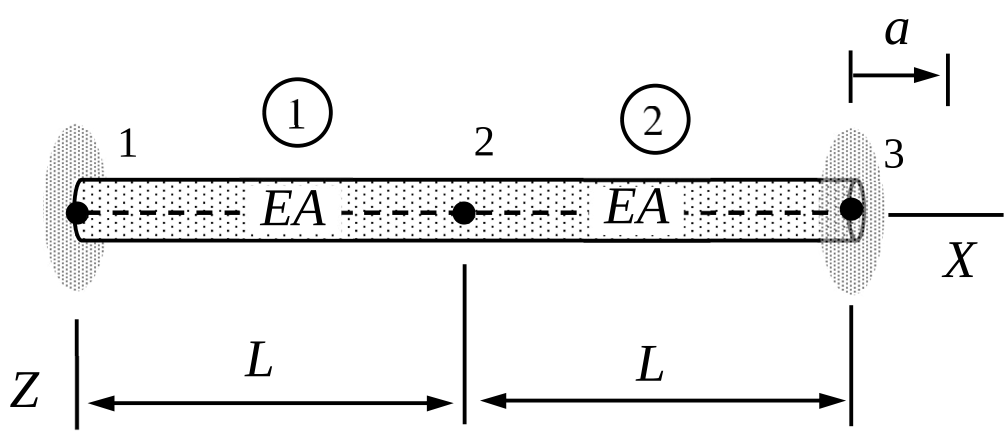
- Idealize a complex structure as a set of structural parts, whose behavior can be approximated by using the usual engineering models (bar, beam, plate, rigid body etc.).

- Write down the equilibrium equations at the connections (Newton **III**), the force-displacement relationships of the structural parts, and constraints concerning the nodal displacements (displacements and rotations should match).

- Solve the nodal displacements and rotations and the forces and moments acting on the structural parts from the equation system.

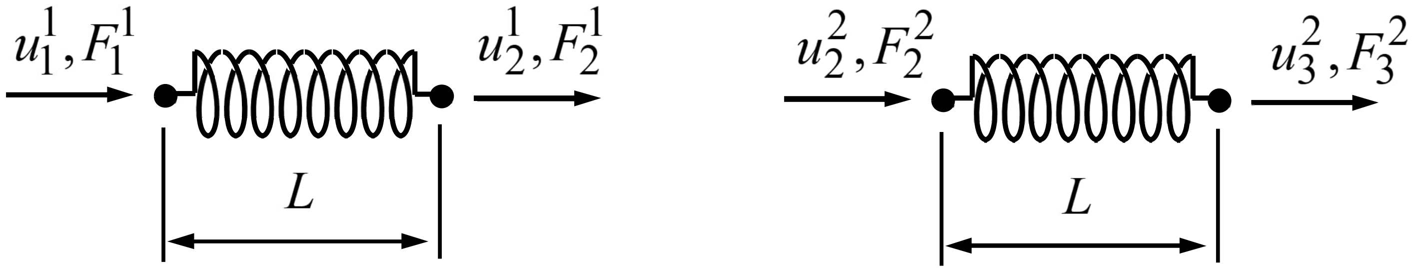
- Determine the stress in the structural parts one-by-one according to the engineering model used (optional step).

EXAMPLE 1.1 A connector bar is welded at its ends to rigid walls. If the right end wall displacement is a, determine the displacements of connection points 1, 2, and 3 and the forces acting on structural parts 1 and 2. Cross sectional area A and Young's modulus of the material E are constants and the displacement force relationship of a bar is the same as that of a spring with coefficient $k = EA/L$.



Answer $u_1^1 = 0$, $u_2^1 = u_2^2 = \frac{1}{2}a$, $u_3^2 = a$, $F_1^1 = F_2^2 = -\frac{1}{2}ka$, $F_2^1 = F_3^2 = \frac{1}{2}ka$.

- Let us omit the index for the component (x,y,z as unnecessary in this simplistic case) and use the two-index notation u_i^e , F_i^e for the displacements and forces acting on the structure. Superscript denotes the structural element and subscript the connection point. The exploded structure with the displacements and forces is given by



- Interaction model between the bars and with the surroundings is of type "fixed" so $u_1^1 = 0$, $u_2^1 = u_2^2$, and $u_3^2 = a$ (left edge welding, integrity of structure at the connection, and displacement of the right end wall). The force constraints are due to Newton **III** which requires that F_2^1 and F_2^2 are equal in magnitude and opposite in signs i.e. $F_2^1 + F_2^2 = 0$.

- As the structural parts can be considered as springs of coefficient $k = EA/L$, $F_1^1 = k(u_1^1 - u_2^1)$, $F_1^2 = k(u_2^1 - u_1^1)$, $F_2^2 = k(u_2^2 - u_3^2)$, and $F_3^2 = k(u_3^2 - u_2^2)$.

- Altogether, the 8 equations determining the 4 displacement components u_1^1, u_2^1, u_2^2, u_3^2 and the 4 force components F_1^1, F_2^1, F_2^2, F_3^2 are given by

$$F_1^1 = k(u_1^1 - u_2^1),\ F_1^2 = k(u_2^1 - u_1^1),\ F_2^2 = k(u_2^2 - u_3^2),\ F_3^2 = k(u_3^2 - u_2^2).$$

$$u_1^1 = 0,\ u_2^1 = u_2^2,\ u_3^2 = a,$$

$$F_2^1 + F_2^2 = 0.$$

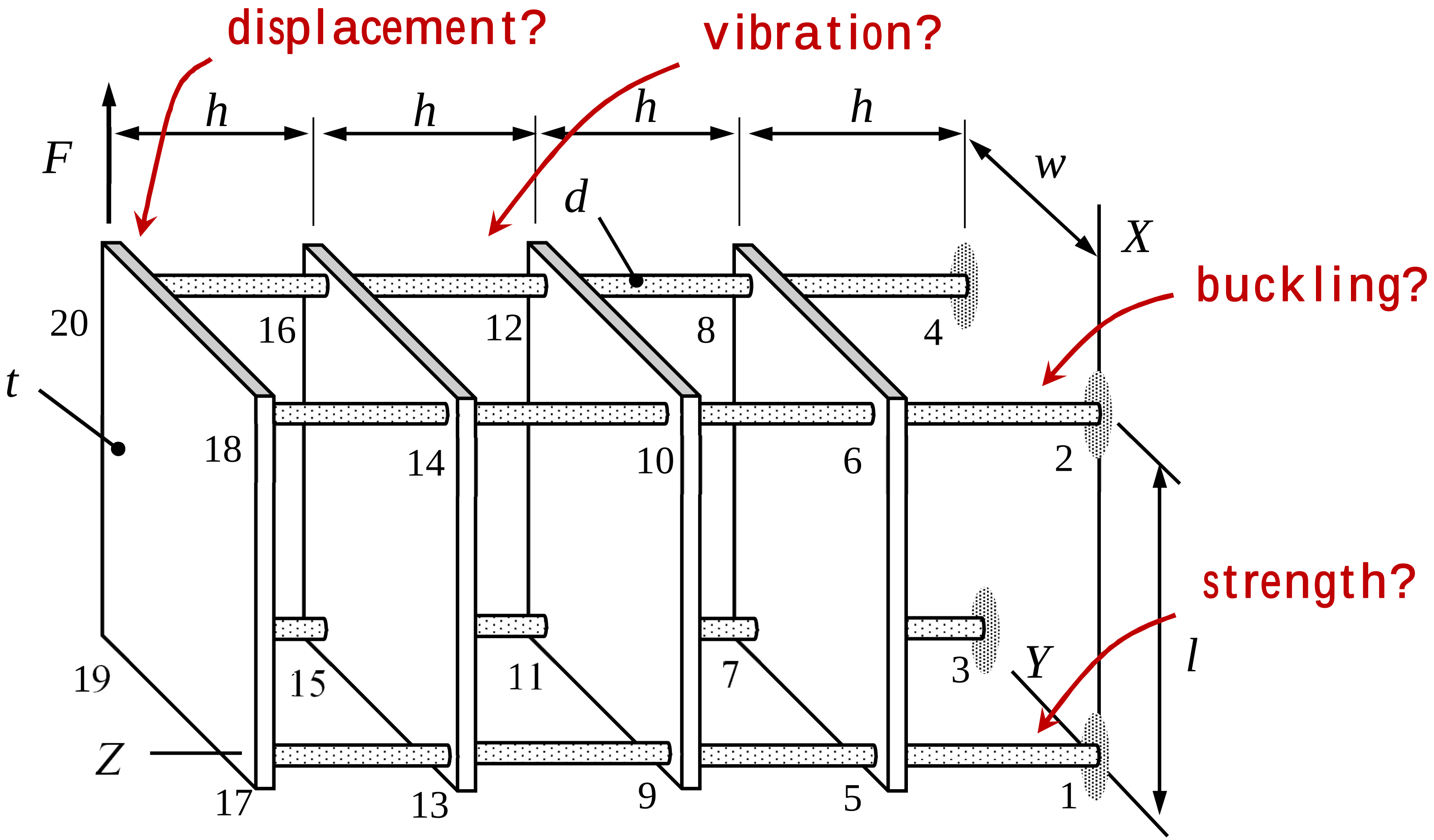
- The linear equation system can be solved, e.g., by considering the equations in a proper order (to be discussed later in more detail), by Gauss elimination, by Mathematica, …

$$u_1^1 = 0,\ u_2^1 = \frac{1}{2}a,\ u_2^2 = \frac{1}{2}a,\ u_3^2 = a,$$

$$F_1^1 = -\frac{1}{2}ka,\ F_2^1 = \frac{1}{2}ka,\ F_2^2 = -\frac{1}{2}ka,\ F_3^2 = \frac{1}{2}ka.$$

The example and exercise problems of MEC-E8001 can be solved either by hand calculation (above) or by representing the problem in the two-table form used by the FE-code of the course.

1.3 FINITE ELEMENT ANALYSIS



Displacement and stress analysis according to the linear elasticity theory may not entirely explain the behavior of a structure!

WHY FINITE ELEMENTS ?

Design of machines and structures: Solution to stress or displacement by analytical method is often impossible due to complex geometry, heterogeneous material etc. Lack of the "exact solution" to an "approximate problem" is not an issue in engineering work.

Finite element method is the standard of solid mechanics: Commercial codes in common use are based on the finite element method. A graphical user interface may make living easier, but a user should always understand what the problem is and in what sense it is solved!

Finite element method has a strong theory: Although approximate solution is acceptable, knowing nothing about the numerical error is not acceptable.

NUMERICAL ERROR

Numerical method replaces the original problem (solution $\hat{u}$**)** by a *numerically convenient problem* (solution $\tilde{u}$). In FEM the error $e = \|\hat{u} - \tilde{u}\|$ can be made as small as wanted by increasing the numerical work.

Error is usually of the form $e = \|\hat{u} - \tilde{u}\| \leq C_e n^{-\alpha}$, in which n characterizes the size of *numerically convenient problem* (typically the number of linear equations) and C_e and α are positive constants.

Numerical work (number of arithmetic operations needed) depends on the details of the recipe, but it grows typically at a polynomial rate $C_w n^{\beta}$ (hence $e \sim 1/w^{(\alpha/\beta)}$)

DISPLACEMENT FEA

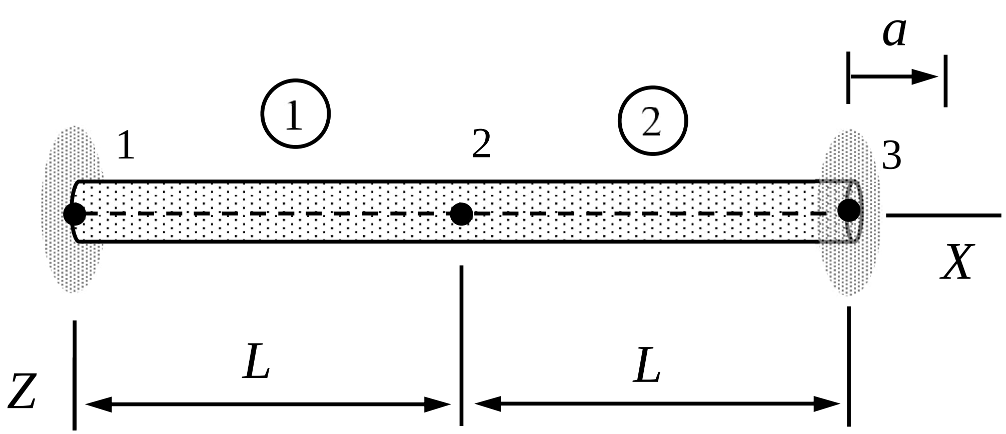
- Model the structure as a collection of elements (solid, plate, beam). Derive the element contributions $\delta W^e = \delta W^{\text{int}} + \delta W^{\text{ext}}$ in terms of the nodal displacement and rotation components of the structural coordinate system.

- Sum the element contributions to end up with the virtual work expression of the structure $\delta W = \sum_{e \in E} \delta W^e$. Re-arrange to get the "standard" form $\delta W = -\delta \mathbf{a}^{\mathrm{T}}(\mathbf{Ka} - \mathbf{F}) = 0$.

- Use the principle of virtual work $\delta W = 0 \ \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus for $\delta \mathbf{a} \in \mathbb{R}^n$ to deduce the linear equations $\mathbf{Ka} - \mathbf{F} = 0$.

- Solve the equations for displacements and rotations $\mathbf{a}$.

EXAMPLE 1.2 A connector bar is welded at its ends to rigid walls. Assuming linearly elastic behavior and the right end wall displacement a, determine the displacements of nodes 1, 2 and 3. Model the structure as a collection of two bar elements of cross-sectional area A and Young's modulus E.



Answer $u_{X2} = \frac{1}{2}a$

In finite element analysis, the conditions related with the integrity of the structure are (usually) satisfied ‘a priori’ to eliminate the internal forces automatically. Structure needs to modelled as a set of elements but exploded structure, with all kinematic and kinetic quantities in it, is not needed. Let us follow the recipe:

- First, virtual work expressions of the elements in terms of the displacement components in the structural coordinate system

$$\delta W^1 = -\begin{Bmatrix} 0 \\ \delta u_{X2} \end{Bmatrix}^{\mathrm{T}} \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ u_{X2} \end{Bmatrix} = -\delta u_{X2} \frac{EA}{L} u_{X2},$$

$$\delta W^2 = -\begin{Bmatrix} \delta u_{X2} \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{X2} \\ a \end{Bmatrix} = -\delta u_{X2} \frac{EA}{L} (u_{X2} - a).$$

- Second, virtual work expression of the structure in its "standard" form

$$\delta W=\delta W^{1}+\delta W^{2}=-\delta u_{X2}\frac{EA}{L}u_{X2}-\delta u_{X2}\frac{EA}{L}(u_{X2}-a)=-\delta u_{X2}\frac{EA}{L}(2u_{X2}-a).$$

- Third, principle of virtual work $\delta W=0\ \forall\delta\mathbf{a}$ and the fundamental lemma of variation calculus for $\delta\mathbf{a}\in\mathbb{R}^{n}$ imply a linear equation to the unknown u_{X2} and, thereby, the solution

$$\frac{EA}{L}(2u_{X2}-a)=0\quad\Leftrightarrow\quad u_{X2}=\frac{1}{2}a.$$

The recipe works no matter the complexity of the structure, analysis type, and method of calculation (hand or FE-code) with slight modifications depending mainly on the analysis type.

VIBRATION FEA

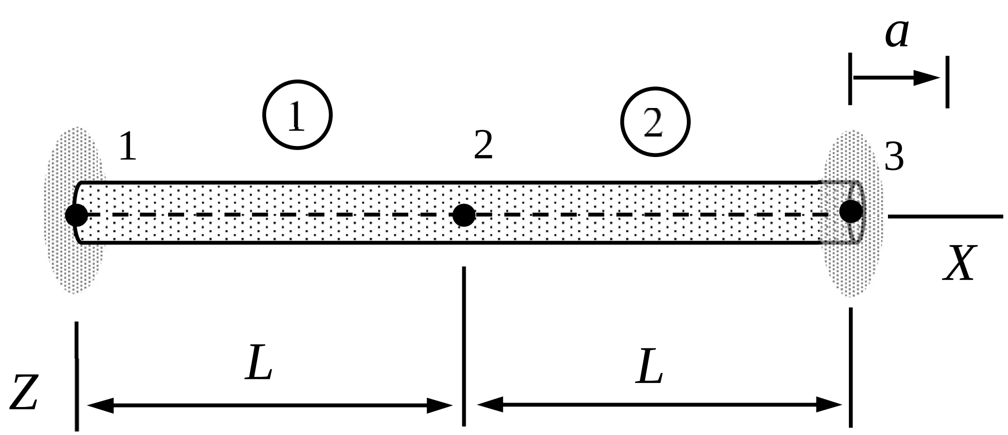
- Model the structure as a collection of elements (solid, plate, beam). Derive the element contributions $\delta W^{e}=\delta W^{\text{int}}+\delta W^{\text{ext}}+\delta W^{\text{ine}}$ in terms of the nodal displacement and rotation components of the structural coordinate system.

- Sum the element contributions to end up with the virtual work expression of the structure $\delta W=\sum_{e\in E}\ \delta W^{e}$. Re-arrange to get $\delta W=-\delta\mathbf{a}^{\mathrm{T}}(\mathbf{M}\ddot{\mathbf{a}}+\mathbf{K}\mathbf{a}-\mathbf{F})$.

- Use the principle of virtual work $\delta W=0\ \forall\delta\mathbf{a}$ and the fundamental lemma of variation calculus for $\delta\mathbf{a}\in\mathbb{R}^{n}$ to deduce the ordinary differential equations $\mathbf{M}\ddot{\mathbf{a}}+\mathbf{K}\mathbf{a}-\mathbf{F}=0$.

- Solve the equations for the natural angular speeds of vibrations and the corresponding modes $(\omega,\mathbf{a})_{i}$ or for displacements and rotations as the functions of time $\mathbf{a}(t)$.

EXAMPLE 1.3 A connector bar is welded at its ends to rigid walls. If the welding fails at (time) $t = 0$ when the right end wall displacement is a, determine the displacement of the midpoint 2 as the function of time. Model the structure as a collection of two bar elements of cross-sectional area A, Young's modulus E, and density ρ.



Answer $u_{X2}(t) = \frac{1}{4}a(1+\sqrt{2})\cos(t\sqrt{\frac{6}{7}(5-3\sqrt{2})\frac{E}{\rho L^2}}) - \frac{1}{4}a(\sqrt{2}-1)\cos(t\sqrt{\frac{6}{7}(5+3\sqrt{2})\frac{\mathrm{E}}{\rho L^2}})$

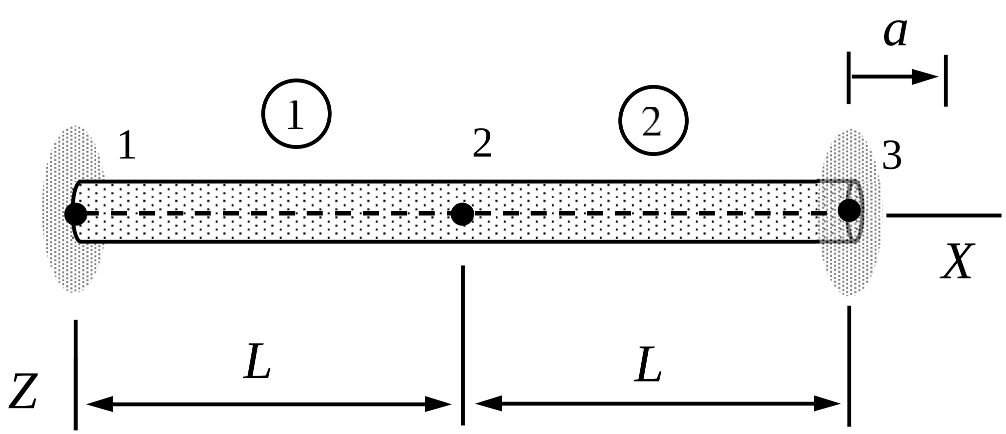
STABILITY FEA

- Model the structure as a collection of beam, plate, etc. elements. Derive the element contributions $\delta W^e = \delta W^{\text{int}} + \delta W^{\text{ext}} + \delta W^{\text{sta}}$ in terms of the nodal displacement and rotation components of the structural coordinate system.

- Sum the element contributions to end up with the virtual work expression of the structure $\delta W = \sum_{e \in E} \delta W^e$. Re-arrange to get $\delta W = -\delta \mathbf{a}^{\mathrm{T}} \mathbf{R}(\mathbf{a})$ and use the principle of virtual work $\delta W = 0 \quad \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus for $\delta \mathbf{a} \in \mathbb{R}^n$ to deduce the (non-linear) equilibrium equations $\mathbf{R}(\mathbf{a}) = 0$.

- Find the values of the control parameter values and the corresponding modes $(p, \mathbf{a})_i$ for non-unique solutions of the equilibrium equations. The smallest of the control parameter values is the critical one.

EXAMPLE 1.4 A connector bar is welded at its ends to rigid walls. Determine the displacement a at which the buckling of structure occurs. Model the structure as a collection of two beams of cross-section moments A, I, Young's modulus E and shear modulus G.



Answer $a = -20\dfrac{I}{AL}$

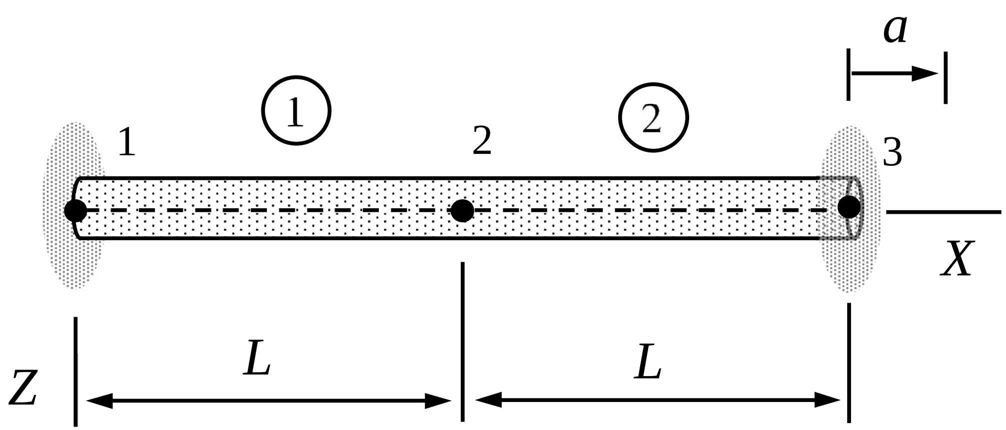
NON-LINEAR FEA

- Model the structure as a collection of beam, plate, etc. elements by considering the initial geometry. Derive the element contributions $\delta W^e = \delta W^{\mathrm{int}} + \delta W^{\mathrm{ext}}$ in terms of the nodal displacement and rotation components of the structural coordinate system.

- Sum the element contributions to end up with the virtual work expression of the structure $\delta W = \sum_{e \in E} \delta W^e$. Re-arrange to get $\delta W = -\delta \mathbf{a}^{\mathrm{T}} \mathbf{R}(\mathbf{a})$ and use the principle of virtual work $\delta W = 0 \ \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus for $\delta \mathbf{a} \in \mathbb{R}^n$ to deduce the equilibrium equations $\mathbf{R}(\mathbf{a}) = 0$.

- Find a physically meaningful solution $\mathbf{a}$ by using any of the standard numerical methods for non-linear algebraic equation systems.

EXAMPLE 1.5 A connector bar is welded at its ends to rigid walls. Determine the axial displacement of midpoint 2 according to the large displacement theory when the right end wall displacement is a. Model the structure as a collection of two bar elements of cross-section area A and Young's modulus E. Use the problem parameter values $L = 1\text{m}$, $A = 0.01\text{m}^2$, $E = 100\,\text{N/m}^2$, and $a = -L/10$.



Answer $u_{X2} = -0.05$ ($u_{X2} = -0.05 - 1.31i$, $u_{X2} = -0.05 + 1.31i$)

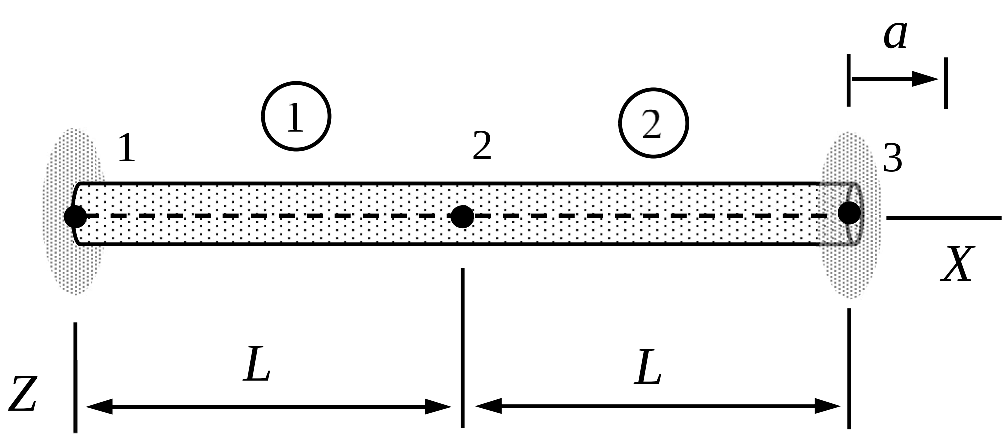
THERMO-MECHANICAL (MULTI-PHYSICS) FEA

- Model the structure as a collection of beam, plate, etc. elements. Derive the element contributions δW^e and δP^e in terms of nodal displacements/rotation components of the structural coordinate system and temperature.

- Sum the element contributions to end up with the variational expression for the structure. Re-arrange to get $\delta W + \tau\delta P = -\delta\mathbf{a}^{\mathrm{T}}\mathbf{R}(\mathbf{a},\mathbf{b}) - \tau\delta\mathbf{b}^{\mathrm{T}}\mathbf{R}(\mathbf{b})$ (τ is a dimensionally correct but otherwise arbitrary constant). Use the principle $\delta W + \tau\delta P = 0 \quad \forall\delta\mathbf{a},\delta\mathbf{b}$ and the fundamental lemma of variation calculus to deduce $\mathbf{R}(\mathbf{a},\mathbf{b}) = 0$ and $\mathbf{R}(\mathbf{b}) = 0$.

- Solve the linear algebraic equations for the nodal displacements, rotations, and temperatures (due to the one-sided coupling of the stationary problem, solving the temperature first is always possible).

EXAMPLE 1.6 A connector bar is welded at its ends to rigid walls. Determine the stationary displacement u_{X2} and temperature ϑ_2 at node 2, when the temperature of the right end is increased to $2\vartheta^{\circ}$ and the right end wall displacement is a, Model the structure as a collection of two bar elements of cross-section area A, Young's modulus E, thermal conductivity k, and thermal expansion coefficient α. Stress in the bar vanishes, when the temperature in the wall and bar is ϑ° and $a = 0$.



Answer $u_{X2} = \frac{1}{4}(4a - L\alpha\vartheta^{\circ})$, $\vartheta_2 = \frac{3}{2}\vartheta^{\circ}$

MATHEMATICAL REPRESENTATIONS

- **Small displacement analysis** $\mathbf{R}(\mathbf{a}) = \mathbf{Ka} - \mathbf{F} = 0$
- **Vibration analysis** $\mathbf{R}(\mathbf{a}) = \mathbf{M}\ddot{\mathbf{a}} + \mathbf{Ka} - \mathbf{F} = 0 \; t > t_0$, $\dot{\mathbf{a}} = \dot{\mathbf{a}}_0 \; t = t_0$, $\mathbf{a} = \mathbf{a}_0 \; t = t_0$
- **Eigenfrequency analysis** $\mathbf{R}(\mathbf{a}, \omega) = (-\mathbf{M}\omega^2 + \mathbf{K})\mathbf{a} = 0$
- **Stability analysis** $\mathbf{R}(\mathbf{a}, p) = (-p\mathbf{F} + \mathbf{K})\mathbf{a} = 0$
- **Large displacement analysis** $\mathbf{R}(\mathbf{a}) = 0$
- **Thermo-mechanical analysis** $\mathbf{R}(\mathbf{a}, \mathbf{b}) = 0$

PREREQUISITE; MATRIX ALGEBRA I

Addition	$\mathbf{C} = \mathbf{A} + \mathbf{B}$	$C_{ij} = A_{ij} + B_{ij}$
Multiplication (scalar)	$\mathbf{C} = \alpha\mathbf{A}$	$C_{ij} = \alpha A_{ij}$
Multiplication (matrix)	$\mathbf{C} = \mathbf{AB}$	$C_{ij} = \sum_{k\in\{1\ldots n\}} A_{ik} B_{kj}$

Unit matrix	$\mathbf{I}$	$\delta_{ij} = 1 \quad i = j, \quad \delta_{ij} = 0 \quad i \neq j$
Symmetric matrix	$\mathbf{A} = \mathbf{A}^{\mathrm{T}}$	$A_{ij} = A_{ji}$
Skew symmetric matrix	$\mathbf{A} = -\mathbf{A}^{\mathrm{T}}$	$A_{ij} = -A_{ji}$
Positive definite matrix	$\mathbf{x}^{\mathrm{T}}\mathbf{Ax} > 0 \quad \forall \mathbf{x} \neq 0$	

PREREQUISITE; MATRIX ALGEBRA II

Transpose	$\mathbf{A}^{\mathrm{T}}$	$A_{ij}^{\mathrm{T}} = A_{ji}$
Inverse	$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$	$\sum_{k\in\{1\ldots n\}} A_{ik} A_{kj}^{-1} = \delta_{ij}$
Derivative	$\dot{\mathbf{x}}$	$\dot{x}_i = dx_i / dt$

Linear equation system	Find $\mathbf{x}$ such that $\mathbf{A}\mathbf{x} = \mathbf{b}$
Eigenvalue problem	Find all $(\lambda, \mathbf{x})$ such that $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$
Eigenvalue composition	$\mathbf{A} = \mathbf{X}\boldsymbol{\lambda}\mathbf{X}^{-1}$, where $\mathbf{X} = [\mathbf{x}_1 \ldots \mathbf{x}_n]$ and $\boldsymbol{\lambda} = diag[\lambda_1 \ldots \lambda_n]$
Matrix function	If $\mathbf{A} = \mathbf{X}\boldsymbol{\lambda}\mathbf{X}^{-1}$, then $f(\mathbf{A}) = \mathbf{X}f(\boldsymbol{\lambda})\mathbf{X}^{-1}$

EXAMPLE 1.7 Determine the eigenvalues λ_1, λ_2 and the corresponding eigenvectors $\mathbf{x}_1$, $\mathbf{x}_2$ of the 2×2 matrix $\mathbf{A}$. Write down also the eigenvalue decomposition $\mathbf{A} = \mathbf{X}\lambda\mathbf{X}^{-1}$ when

$$\mathbf{A} = \begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix}.$$

Answer $\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix}$

- In an eigenvalue problem of matrix $\mathbf{A}$, the goal is to find all pairs $(\lambda, \mathbf{x})$ such that $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$. As the equation is homogeneous, a non-zero solution to $\mathbf{x}$ requires that the matrix is singular, i.e., $\det(\mathbf{A}-\lambda\mathbf{I})=0$. Hence

$$\det\begin{bmatrix} 3-\lambda & 0 \\ -2 & 1-\lambda \end{bmatrix} = (3-\lambda)(1-\lambda)=0 \;\Rightarrow\; \lambda_1 = 3 \quad \text{or} \quad \lambda_2 = 1.$$

- After finding the possible values of λ, the corresponding vectors (eigenvectors) are given by the original equation $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$. Solution to $\mathbf{x}$ is not unique and any *non-zero* solution suffices (in practice one may choose one or more components of $\mathbf{x}$ and solve the equation for the remaining)

$$\lambda_1: \quad \begin{bmatrix} 3-3 & 0 \\ -2 & 1-3 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0 \;\Rightarrow\; \mathbf{x}_1 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix},$$

$$\lambda_2:\begin{bmatrix}3-1 & 0\\ -2 & 1-1\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\end{Bmatrix}=0 \quad\Rightarrow\quad \mathbf{x}_2=\begin{Bmatrix}x_1\\ x_2\end{Bmatrix}=\begin{Bmatrix}0\\ 1\end{Bmatrix}.$$

- Matrix of eigenvalues $\boldsymbol{\lambda}$, matrix of eigenvectors $\mathbf{X}$ and its inverse $\mathbf{X}^{-1}$ are now

$$\boldsymbol{\lambda}=\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}=\begin{bmatrix}3 & 0\\ 0 & 1\end{bmatrix},\quad \mathbf{X}=\left[\mathbf{x}_1\ \mathbf{x}_2\right]=\left[\begin{array}{c:c}-1 & 0\\ 1 & 1\end{array}\right] \text{ and } \mathbf{X}^{-1}=\begin{bmatrix}-1 & 0\\ 1 & 1\end{bmatrix}.$$

- Eigenvalue decomposition $\mathbf{A}=\mathbf{X}\boldsymbol{\lambda}\mathbf{X}^{-1}$ is a very useful representation of the original matrix (for example $f(\mathbf{A})=\mathbf{X}diag[f(\lambda_1)\ldots f(\lambda_n)]\mathbf{X}^{-1}$)

$$\mathbf{A}=\begin{bmatrix}-1 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}3 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}-1 & 0\\ 1 & 1\end{bmatrix}=\begin{bmatrix}3 & 0\\ -2 & 1\end{bmatrix}.$$

1.4 FE-CODE OF MEC-E8001

"Structure is a collection of elements connected by nodes. Geometry, displacement, temperature etc. of the structure are defined by the nodal values of coordinates, translation, rotation, temperature etc. of which some are known and some unknown."

DATA STRUCTURE

$prb = \{ele, fun\}$ where

$ele = \{prt_1, prt_2, \ldots\}$ .... elements

$fun = \{val_1, val_2, \ldots\}$ .... nodes

Element

$prt = \{typ, pro, geo\}$ where

$typ = \text{BAR} \mid \text{TORSION} \mid \text{BEAM} \mid \text{RIGID} \mid \ldots \mid$ .... model

$pro = \{p_1, p_2, \ldots, p_n\}$ ....properties

$geo = \text{Point}[\{n_1\}] \mid \text{Line}[\{n_1, n_2\}] \mid \text{Polygon}[\{n_1, n_2, n_3\}] \mid \ldots \mid$.............................. geometry

Nodes

$val = \{crd, trn, rot\} \mid \{crd, trn, rot, tmp\}$ where

$crd = \{X, Y, Z\}$.. structural coordinates

$trn = \{u_X, u_Y, u_Z\}$.. translation components

$rot = \{\theta_X, \theta_Y, \theta_Z\}$ ....rotation components

$tmp = \vartheta$ .... temperature

DISPLACEMENT ANALYSIS

Constraint

$\{\text{JOINT},\{\}\,|\,\{\{\underline{u}_X, \underline{u}_Y, \underline{u}_Z\}\}, \text{Point}[\{n_1\}]\}$ ....displacement constraint

$\{\text{JOINT},\{\}, \text{Line}[\{n_1, n_2\}]\}$ ....displacement constraint

$\{\text{RIGID},\{\}\,|\,\{\{\underline{u}_X, \underline{u}_Y, \underline{u}_Z\},\{\underline{\theta}_X, \underline{\theta}_Y, \underline{\theta}_Z\}\}, \text{Point}[\{n_1\}]\}$ ... displacement/rotation constraint

$\{\text{RIGID},\{\}, \text{Line}[\{n_1, n_2\}]\}$ .... rigid constraint

$\{\text{SLIDER},\{n_X, n_Y, n_Z\}, \text{Point}[\{n_1\}]\}$ ....slider constraint

Force

$\{\text{FORCE},\{F_X, F_Y, F_Z\}, \text{Point}[\{n_1\}]\}$ .... point force

$\{\text{FORCE},\{F_X, F_Y, F_Z, M_X, M_Y, M_Z\}, \text{Point}[\{n_1\}]\}$ .... point load

$\{\text{FORCE},\{f_X, f_Y, f_Z\}, \text{Line}[\{n_1, n_2\}]\}$ .... distributed force

$\{\text{FORCE},\{f_X, f_Y, f_Z\}, \text{Polygon}[\{n_1, n_2, n_3\}]\}$ .... distributed force

Beam model

$\{\text{BAR},\{\{E\},\{A\},\{f_X, f_Y, f_Z\}\},\text{Line}[\{n_1,n_2\}]\}$ ....bar mode

$\{\text{TORSION},\{\{G\},\{J\},\{\{m_X, m_Y, m_Z\}\}\},\text{Line}[\{n_1,n_2\}]\}$ .... torsion mode

$\{\text{BEAM},\{\{E,G\},\{A,I_{yy},I_{zz}\},\{f_X, f_Y, f_Z\}\},\text{Line}[\{n_1,n_2\}]\}$ ....beam

$\{\text{BEAM},\{\{E,G\},\{A,I_{yy},I_{zz},\{j_X, j_Y, j_Z\}\},\{f_X, f_Y, f_Z\}\},\text{Line}[\{n_1,n_2\}]\}$ ....beam

Plate model

$\{\text{PLANE},\{\{E,\nu\},\{t\},\{f_X, f_Y, f_Z\}\},\text{Polygon}[\{n_1,n_2,n_3\}]\}$........................thin slab mode

$\{\text{PLANE},\{\{E,\nu\},\{t\},\{f_X, f_Y, f_Z\}\},\text{Polygon}[\{n_1,n_2,n_3,n_4\}]\}$..................thin slab mode

$\{\text{PLATE},\{\{E,\nu\},\{t\},\{f_X, f_Y, f_Z\}\},\text{Polygon}[\{n_1,n_2,n_3\}]\}$ ....bending mode

$\{\text{SHELL},\{\{E,\nu\},\{t\},\{f_X, f_Y, f_Z\}\},\text{Polygon}[\{n_1,n_2,n_3\}]\}$..plate

Solid model

$\{\text{SOLID},\{\{E,\nu\},\{f_X, f_Y, f_Z\}\},\text{Tetrahedron}[\{n_1,n_2,n_3,n_4\}]\}$ ....solid

$\{\mathrm{SOLID},\{\{E,\nu\},\{f_X, f_Y, f_Z\}\},\mathrm{Hexahedron}[\{n_1,n_2,n_3,n_4,n_5,n_6,n_7,n_8\}]\}$ ....solid

$\{\mathrm{SOLID},\{\{E,\nu\},\{f_X, f_Y, f_Z, m_X, m_Y, m_Z\}\},\mathrm{Tetrahedron}[\{n_1,n_2,n_3,n_4\}]\}$ ....solid

Operations

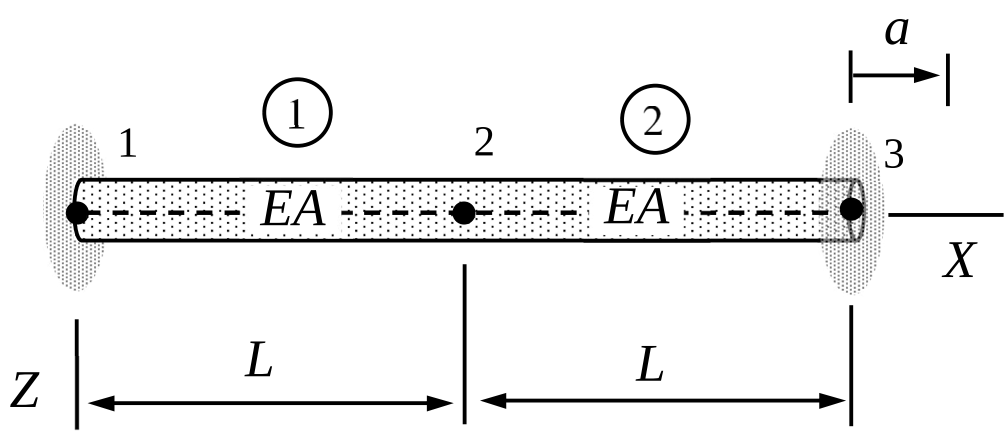
$prb = \mathrm{REFINE}[prb]$ ....refine structure representation

$\mathrm{Out} = \mathrm{FORMATTED}[prb]$ .... display problem definition

$\mathrm{Out} = \mathrm{STANDARDFORM}[prb]$..................................... display virtual work expression

$\mathrm{sol} = \mathrm{SOLVE}[\{\mathrm{DISP}\}, prb] \;|\; \mathrm{SOLVE}[prb]$ ....solve the unknowns

EXAMPLE 1.1 A connector bar is welded at its ends to rigid walls. If the right end wall displacement is a, determine the displacements of connection points 1, 2, and 3 and the forces acting on structural parts 1 and 2. Cross sectional area A and Young's modulus of the material E are constants and the displacement force relationship of a bar is the same as that of a spring with coefficient $k = EA / L$.



Answer $u_1^1 = 0$, $u_2^1 = u_2^2 = \frac{1}{2}a$, $u_3^2 = a$, $F_1^1 = F_2^2 = -\frac{1}{2}ka$, $F_2^1 = F_3^2 = \frac{1}{2}ka$.

- Problem description for the FE-solver uses duplicate node at the center point. Solution to the problem uses the replacement rule concept of Mathematica.

	model	properties	geometry
1	BAR	{{E}, {A}}	Line[{1, 2}]
2	BAR	{{E}, {A}}	Line[{3, 4}]
3	JOINT	{}	Point[{1}]
4	JOINT	{}	Line[{2, 3}]
5	JOINT	{a, 0, 0}	Point[{4}]

	$\{X,Y,Z\}$	$\{u_X,u_Y,u_Z\}$	$\{\theta_X,\theta_Y,\theta_Z\}$
1	{0, 0, 0}	{uX[1], 0, 0}	{0, 0, 0}
2	{L, 0, 0}	{uX[2], 0, 0}	{0, 0, 0}
3	{L, 0, 0}	{uX[3], 0, 0}	{0, 0, 0}
4	{2 L, 0, 0}	{uX[4], 0, 0}	{0, 0, 0}

$$\left\{\mathrm{FX}[1]\to-\frac{a\,A\,E}{2\,L},\ \mathrm{FX}[2]\to\frac{a\,A\,E}{2\,L},\ \mathrm{FX}[4]\to\frac{a\,A\,E}{2\,L},\ \mathrm{uX}[1]\to0,\ \mathrm{uX}[2]\to\frac{a}{2},\ \mathrm{uX}[3]\to\frac{a}{2},\ \mathrm{uX}[4]\to a\right\}$$

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Assignment 1

Determine the displacements w_i $i \in \{1,2,3\}$, if the vector of displacements $\mathbf{a}$, stiffness matrix $\mathbf{K}$, and the loading vector $\mathbf{F}$ of the equilibrium equations $-\mathbf{Ka}+\mathbf{F}=0$ are given by

$$\mathbf{a}=\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}, \quad \mathbf{K}=k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \text{ and } \mathbf{F}=P\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}.$$

Solution template

With linear equation systems $-\mathbf{Ka}+\mathbf{F}=0$ of more than two unknows, using the matrix inverse to get $\mathbf{a}=\mathbf{K}^{-1}\mathbf{F}$ is not practical in hand calculation. Gauss elimination is based on row operations aiming at an upper diagonal matrix. After that, solution for the unknowns is obtained step-by-step starting from the last equation. In standard form, the equation system is given by

$$k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}=P\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}.$$

Let us multiply the 1:st equation by 1/2 and add it to the 2:nd equation to get

$$k\begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}=P\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}.$$

Let us multiply the 2:nd equation by 2/3 and add to it the 3:rd equation to get the upper triangular matrix

$$k\begin{bmatrix} 2 & -1 & 0 \\ 0 & 3/2 & -1 \\ 0 & 0 & 1/3 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}=P\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}.$$

Solution to the modified equation system coincides with that of the original system as the equations are just linear combinations of the original ones. However, with the modified form solution to the unknowns is obtained step-by-step starting from the last equation:

$$w_3=3\frac{P}{k}, \qquad w_2=\frac{2}{3}w_3=2\frac{P}{k}, \quad \text{and} \quad w_1=\frac{1}{2}w_2=\frac{P}{k}.$$

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Assignment 2

Determine the eigenvalues λ_1, λ_2 and the corresponding eigenvectors $\mathbf{a}_1$, $\mathbf{a}_2$ of the 2×2 matrix $\mathbf{A}$. Consider the possible $(\lambda, \mathbf{a})$ pairs giving solutions to linear equation system $(\mathbf{A}-\lambda\mathbf{I})\mathbf{a}=\mathbf{0}$ with

$$\mathbf{A}=\begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix},\ \mathbf{I}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ and } \mathbf{a}=\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}.$$

Solution template

As the matrix needs to be singular for a non-zero solution to $\mathbf{a}$, the possible values of λ follow from the characteristic equation $\det(\mathbf{A}-\lambda\mathbf{I})=0$

$$\det\begin{bmatrix} 1-\lambda & 0 \\ -3 & 2-\lambda \end{bmatrix}=(1-\lambda)(2-\lambda)=0 \quad\Rightarrow\quad \lambda_1=1 \quad\text{or}\quad \lambda_2=2.$$

Eigenvector $\mathbf{a}$ (non-zero) corresponding to a possible value of λ follows from $(\mathbf{A}-\lambda\mathbf{I})\mathbf{a}=0$ when the value of λ is substituted there:

$$\lambda_1=1:\quad \begin{bmatrix} 1-1 & 0 \\ -3 & 2-1 \end{bmatrix}\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}=0 \quad\Rightarrow\quad \mathbf{a}_1=\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}=\begin{Bmatrix} 1 \\ 3 \end{Bmatrix}$$

$$\lambda_2=2:\quad \begin{bmatrix} 1-2 & 0 \\ -3 & 2-2 \end{bmatrix}\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}=0 \quad\Rightarrow\quad \mathbf{a}_2=\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

Hence, the eigenvalue-eigenvector pairs of $\mathbf{A}$ are given by

$$(\lambda,\mathbf{a})_1=(1,\begin{Bmatrix} 1 \\ 3 \end{Bmatrix}) \quad\text{and}\quad (\lambda,\mathbf{a})_2=(2,\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}).$$

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Assignment 3

Consider the disk rigidity problem on page 1-4 of the lecture notes. Simplify the structure by omitting the disk part outside the support. Use literature to find the analytical transverse displacement solution to a circular elastic plate loaded at the center point. Use the expression to deduce the coefficient a of (predicted by dimension analysis)

$$\frac{mgR^2}{Et^4} = f\left(\frac{u}{t}, \nu\right) = a\frac{u}{t},$$

where m is the mass used for loading, g is the acceleration by gravity, R is the disk radius, t is the disk thickness, E is the Young's modulus of the disk material, ν its Poisson's ratio, and u the transverse displacement at the center point. The latter form assumes linearity and vanishing displacement without external loading, i.e., $u = 0$ when $m = 0$.

Solution

The small displacement solution to simply supported circular plate loaded by a point force is well-known (see, for example, the lecture notes of MEC-E8003 Beam, Plate, and Shell Models about the plate model in Mycourses)

$$u = w(0) = \frac{1}{16\pi}\frac{FR^2}{D}\frac{3+\nu}{1+\nu} \quad \text{where} \quad D = \frac{t^3}{12}\frac{E}{1-\nu^2}.$$

When written in terms of the dimensionless groups, the mass-displacement relationship takes the form

$$\frac{mgR^2}{Et^4} = \frac{4\pi}{3}\frac{1}{(3+\nu)(1-\nu)}\frac{u}{t} = a\frac{u}{t}.$$

Therefore $a = \dfrac{4\pi}{3}\dfrac{1}{(3+\nu)(1-\nu)}$. ←

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Assignment 4

Consider the disk rigidity problem on page 1-4 of the lecture notes and the mass-displacement relationship given by dimension analysis

$$\frac{mgR^2}{Et^4} = f(\frac{u}{t}, \frac{L}{R}, \nu) \approx a(\frac{u}{t}) + b(\frac{u}{t})^3,$$

where the latter form uses the first two odd order terms of Taylor expansion of f with respect to u/t and, therefore, coefficients a and b may depend of L/R and ν. Instead of (expensive) physical experiments, one may use simulation by a model for finding, e.g., the dependency of the coefficients on L/R and ν. Use the mass-displacement table below, given by the course software with a large displacement plate model, to determine a and b when $E = 4.22\,\text{GPa}$, $R = 0.245\,\text{m}$, $L = 0.280\,\text{m}$, $t = 4.1\,\text{mm}$, and $g = 9.81\,\text{m/s}^2$. Also, use the outcome to estimate the values of the parameters for $\nu = 0.32$.

m [kg]	u [mm] ($\nu = 0.1$)	u [mm] ($\nu = 0.4$)
0	0.00	0.00
1	1.26	0.94
2	2.34	1.81
3	3.21	2.59
4	3.94	3.28
5	4.56	3.89
6	5.10	4.44
7	5.58	4.93

Solution

Let us use notations $\underline{m} = mgR^2/(Et^4)$ and $\underline{u} = u/t$ for the dimensionless mass and displacement, respectively. To find the coefficients a and b of relationship, one may use the least-squares method giving the values of a and b as the minimizers of function

$$\prod(a,b) = \frac{1}{2}\sum (a\underline{u}_i + b\underline{u}_i^3 - \underline{m}_i)^2,$$

where the sum is over all the measured mass-displacement values. The method looks for parameter values giving as good as possible overall match to the data. For a perfect fit with the best values of a and b function $\prod = 0$ so the value of $\prod$ at the minimum point is a measure of the quality of the fit.

At the minimum point, partial derivatives of $\prod(a,b)$ with respect to a and b should vanish. Therefore, one obtains

$$\frac{\partial\prod(a,b)}{\partial a}=\sum \underline{u}_i(a\underline{u}_i+b\underline{u}_i^3-\underline{m}_i)=0 \quad\text{and}\quad \frac{\partial\prod(a,b)}{\partial b}=\sum \underline{u}_i^3(a\underline{u}_i+b\underline{u}_i^3-\underline{m}_i)=0$$

or written in a more convenient form

$$(\sum\begin{bmatrix}\underline{u}_i^2 & \underline{u}_i^4\\ \underline{u}_i^4 & \underline{u}_i^6\end{bmatrix})\begin{Bmatrix}a\\ b\end{Bmatrix}-\sum\begin{Bmatrix}\underline{u}_i\underline{m}_i\\ \underline{u}_i^3\underline{m}_i\end{Bmatrix}=0 \quad\Leftrightarrow\quad \begin{Bmatrix}a\\ b\end{Bmatrix}=(\sum\begin{bmatrix}\underline{u}_i^2 & \underline{u}_i^4\\ \underline{u}_i^4 & \underline{u}_i^6\end{bmatrix})^{-1}(\sum\begin{Bmatrix}\underline{u}_i\underline{m}_i\\ \underline{u}_i^3\underline{m}_i\end{Bmatrix}).$$

From this point on, it is convenient to use Mathematica, Matlab, Excel or some other computational tool. The first step is to transform the measured data into dimensionless form using the definitions and the given values $E=4.22\,\text{GPa}$, $\nu=0.32$, $R=0.245\,\text{m}$, $t=4.1\,\text{mm}$, and $g=9.81\,\text{m}/\text{s}^2$

m [kg]	$mgR^2/(Et^4)$	u [mm] ($\nu=0.1$)	u/t ($\nu=0.1$)	u [mm] ($\nu=0.4$)	u/t ($\nu=0.4$)
0	0.00	0.00	0.00	0.00	0.00
1	0.49	1.26	0.31	0.94	0.23
2	0.99	2.34	0.57	1.81	0.44
3	1.48	3.21	0.78	2.59	0.63
4	1.98	3.94	0.96	3.28	0.80
5	2.47	4.56	1.11	3.89	0.95
6	2.96	5.10	1.24	4.44	1.08
7	3.46	5.58	1.36	4.93	1.20

With the data in the table for $\nu=0.1$, the linear equation system to coefficients a and b and the solution to the values become

$$\begin{bmatrix}6.59 & 8.70\\ 8.70 & 13.01\end{bmatrix}\begin{Bmatrix}a\\ b\end{Bmatrix}-\begin{Bmatrix}14.91\\ 20.47\end{Bmatrix}=0 \quad\Leftrightarrow\quad \begin{Bmatrix}a\\ b\end{Bmatrix}=\begin{bmatrix}6.59 & 8.70\\ 8.70 & 13.01\end{bmatrix}^{-1}\begin{Bmatrix}14.91\\ 20.47\end{Bmatrix}=\begin{Bmatrix}1.57\\ 0.53\end{Bmatrix}.$$

With the data in the table for $\nu=0.4$, the linear equation system to coefficients a and b and the solution to the values become

$$\begin{bmatrix}4.81 & 4.89\\ 4.89 & 5.70\end{bmatrix}\begin{Bmatrix}a\\ b\end{Bmatrix}-\begin{Bmatrix}12.77\\ 13.36\end{Bmatrix}=0 \quad\Leftrightarrow\quad \begin{Bmatrix}a\\ b\end{Bmatrix}=\begin{bmatrix}4.81 & 4.89\\ 4.89 & 5.70\end{bmatrix}^{-1}\begin{Bmatrix}12.77\\ 13.36\end{Bmatrix}=\begin{Bmatrix}2.14\\ 0.51\end{Bmatrix}.$$

These values correspond to $\nu=0.1$ and $\nu=0.4$ when $L/R=1.14$. Interpolation gives the prediction for the coefficients when $\nu=0.32$, when $L/R=1.14$ (one may use, e.g., linear shape functions)

$$\begin{Bmatrix}a\\ b\end{Bmatrix}_{\nu}=\frac{\nu-0.4}{0.1-0.4}\begin{Bmatrix}a\\ b\end{Bmatrix}_{0.1}+\frac{\nu-0.1}{0.4-0.1}\begin{Bmatrix}a\\ b\end{Bmatrix}_{0.4} \quad\Rightarrow\quad \begin{Bmatrix}a\\ b\end{Bmatrix}_{0.32}=\begin{Bmatrix}1.99\\ 0.51\end{Bmatrix}.$$

Simulation with Poisson's ratio $\nu = 0.32$ gives $\begin{Bmatrix} a \\ b \end{Bmatrix}_{0.32} = \begin{Bmatrix} 1.93 \\ 0.51 \end{Bmatrix}$.

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Assignment 5

Consider the disk rigidity problem on page 1-4 of the lecture notes. First, measure the displacement of disks center point u as the function of mass m used as loading (page 1-5 of the lecture notes). Thereafter, use the mass-displacement data to find the coefficients a and b of relationship

$$\frac{mgR^2}{Et^4} = f(\frac{u}{t}, \frac{L}{R}, \nu) \approx a(\frac{u}{t}) + b(\frac{u}{t})^3 .$$

The latter form uses the first two odd order terms of Taylor expansion of f with respect to u/t. The values of the geometrical and material parameters are $E = 4.22\,\mathrm{GPa}$, $\nu = 0.32$, $R = 0.245\,\mathrm{m}$, $L = 0.280\,\mathrm{m}$, $t = 4.1\,\mathrm{mm}$, and $g = 9.81\,\mathrm{m/s^2}$.

Experiment: The set-up is located in Puumiehenkuja 5L (Konemiehentie side of the building). The hall is open during the office hours (9:00-16:00) on Fri 13.01.2023. Place a mass on the loading tray and record the displacement shown on the laptop display. Disk material is not purely elastic so wait for the displacement reading to settle (almost). Gather enough mass-displacement data for finding the coefficients a and b reliably. For example, you may repeat a measurement with certain loading several times to reduce the effect of random error by averaging etc. You may also consider different loading sequences (like increasing and decreasing the mass) to minimize the effect of the viscous part of material response.

Solution

Let us use notations $\underline{m} = mgR^2/(Et^4)$ and $\underline{u} = u/t$ for the dimensionless mass and displacement, respectively. To find the coefficients a and b of relationship, one may use the least-squares method giving the values of a and b as the minimizers of function

$$\prod(a,b) = \frac{1}{2}\sum (a\underline{u}_i + b\underline{u}_i^3 - \underline{m}_i)^2 ,$$

where the sum is over all the measured mass-displacement values. The method looks for parameter values giving as good as possible overall match to the data. For a perfect fit with the best values of a and b function $\prod = 0$ so the value of $\prod$ at the minimum point is a measure of the quality of the fit.

At the minimum point, partial derivatives of $\prod(a,b)$ with respect to a and b should vanish. Therefore, one obtains

$$\frac{\partial \prod(a,b)}{\partial a} = \sum \underline{u}_i(a\underline{u}_i + b\underline{u}_i^3 - \underline{m}_i) = 0 \quad \text{and} \quad \frac{\partial \prod(a,b)}{\partial b} = \sum \underline{u}_i^3(a\underline{u}_i + b\underline{u}_i^3 - \underline{m}_i) = 0$$

or written in a more convenient form

$$(\sum \begin{bmatrix} \underline{u}_i^2 & \underline{u}_i^4 \\ \underline{u}_i^4 & \underline{u}_i^6 \end{bmatrix}) \begin{Bmatrix} a \\ b \end{Bmatrix} - \sum \begin{Bmatrix} \underline{u}_i \underline{m}_i \\ \underline{u}_i^3 \underline{m}_i \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix} a \\ b \end{Bmatrix} = (\sum \begin{bmatrix} \underline{u}_i^2 & \underline{u}_i^4 \\ \underline{u}_i^4 & \underline{u}_i^6 \end{bmatrix})^{-1} (\sum \begin{Bmatrix} \underline{u}_i \underline{m}_i \\ \underline{u}_i^3 \underline{m}_i \end{Bmatrix}).$$

From this point on, it is convenient to use Mathematica, Matlab, Excel or some other computational tool. The first step is to transform the measured data into dimensionless form using the definitions and the given values $E = 4.22\,\mathrm{GPa}$, $\nu = 0.32$, $R = 0.245\,\mathrm{m}$, $t = 4.1\,\mathrm{mm}$, and $g = 9.81\,\mathrm{m/s^2}$

m [kg]	u [mm]	$mgR^2/(Et^4)$	u/t
0	0.00	0.00	0.00
1	1.15	0.49	0.28
2	2.09	0.99	0.51
3	2.89	1.48	0.70
4	3.61	1.98	0.88
6.5	4.96	3.21	1.21
2.5	2.37	1.23	0.58
5	3.89	2.47	0.95

With the data in the table, the linear equation system to coefficients a and b and the solution to the values become

$$\begin{bmatrix} 4.31 & 3.95 \\ 3.95 & 4.51 \end{bmatrix} \begin{Bmatrix} a \\ b \end{Bmatrix} - \begin{Bmatrix} 10.36 \\ 10.04 \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix} a \\ b \end{Bmatrix} = \begin{bmatrix} 4.31 & 3.95 \\ 3.95 & 4.51 \end{bmatrix}^{-1} \begin{Bmatrix} 10.36 \\ 10.04 \end{Bmatrix} = \begin{Bmatrix} 1.90 \\ 0.55 \end{Bmatrix}. \quad \leftarrow$$

These values correspond to $\nu = 0.32$ and $L/R = 1.14$. For a more precise picture about the effects of ν and L/R, experiment needs to be repeated with varying values of these parameters.

MEC-E8001

Finite Element Analysis

2023

WEEK 3: DISPLACEMENT ANALYSIS

2 DISPLACEMENT ANALYSIS

2.1 LINEAR ELASTICITY 5

2.2 DISPLACEMENT FEA 9

2.3 ELEMENT CONTRIBUTIONS.. 28

LEARNING OUTCOMES

Students are able to solve the weekly lecture problems, home problems, and exercise problems related to displacement FEA:

- Engineering paradigm in FEM, elements and nodes, nodal quantities and sign conventions.

- Displacement analysis of simple structures by using the virtual work expressions of the elements.

- Calculations of the element contributions of force, solid, beam, and plate elements out of virtual work density of the model and element approximation.

BALANCE LAWS OF MECHANICS

Balance of mass (def. of a body or a material volume) Mass of a body is constant

Balance of linear momentum (Newton 2) The rate of change of linear momentum within a material volume equals the external force resultant acting on the material volume. 🡐

Balance of angular momentum (Cor. of Newton 2) The rate of change of angular momentum within a material volume equals the external moment resultant acting on the material volume. 🡐

Balance of energy (Thermodynamics 1)

Entropy growth (Thermodynamics 2)

PREREQUISITE: FUNDAMENTAL LEMMA OF VARIATION CALCULUS

The fundamental lemma of variation calculus in one form or another is an important tool in FEM. The lemma tells how to deduce the equilibrium equations of a structure using a virtual work expression and the principle of virtual work:

- ☐ $u,v \in \mathbb{R} \quad : \quad vu = 0 \;\; \forall v \quad \Leftrightarrow \quad u = 0$

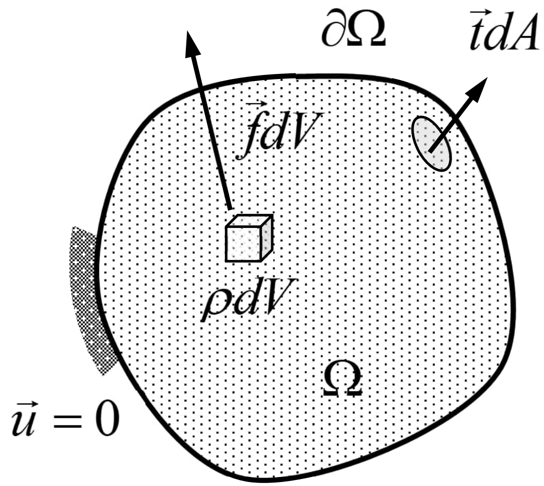
- ☐ $\mathbf{u},\mathbf{v} \in \mathbb{R}^n \quad : \quad \mathbf{v}^{\mathrm{T}}\mathbf{u} = 0 \;\; \forall \mathbf{v} \quad \Leftrightarrow \quad \mathbf{u} = 0$ 🡐

- ☐ $u,v \in C^0(\Omega) \quad : \quad \int_\Omega uv d\Omega = 0 \;\; \forall v \quad \Leftrightarrow \quad u(x,y,\ldots) = 0 \;\; \text{in} \;\; \Omega$

In mechanics of the materials, variable or function v is (usually) chosen as the kinematically admissible variation of displacement δu.

2.1 LINEAR ELASTICITY

Assuming equilibrium of a solid body (a set of particles) inside domain Ω, the aim is to find displacement $\vec{u}$ of the particles, when external forces or boundary conditions are changed in some manner:



Equilibrium equations $\nabla \cdot \vec{\vec{\sigma}} + \vec{f} = 0 \quad \text{in} \quad \Omega,$

Hooke's law $\vec{\vec{\sigma}} = \dfrac{E}{1+\nu}\left(\dfrac{\nu}{1-2\nu}\vec{\vec{I}}\nabla \cdot \vec{u} + \vec{\vec{\varepsilon}}\right) \quad \text{in} \quad \Omega,$

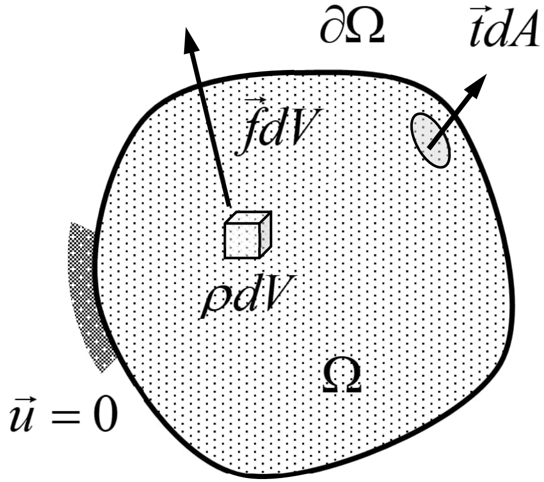
Boundary conditions $\vec{n} \cdot \vec{\vec{\sigma}} = \vec{t} \quad \text{or} \quad \vec{u} = \vec{g} \quad \text{on} \quad \partial\Omega,$

The balance law of angular momentum is satisfied ‘a priori’ by the form of Hooke’s law.

PRINCIPLE OF VIRTUAL WORK

Principle of virtual work $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = 0 \ \ \forall \delta\vec{u}$ is a concise representation of the boundary value problem. In terms of virtual work densities δw^{int} , δw_V^{ext} , and δw_A^{ext}

Internal forces: $\delta W^{\text{int}} = \int_{\Omega} \delta w_V^{\text{int}} dV$



External forces: $\delta W^{\text{ext}} = \int_{\Omega} \delta w_V^{\text{ext}} dV + \int_{\partial\Omega} \delta w_A^{\text{ext}} dA$

Although the two representations are equivalent, principle of virtual work combines the equations in a way which is the key for multiple important applications in mechanics. Finite element method is just one of them.

DENSITY EXPRESSIONS

Virtual work densities (virtual work per unit volume or area) of the internal forces, external volume forces, and external surface forces are

$$\delta w_V^{\mathrm{int}} = -\begin{Bmatrix} \delta\varepsilon_{xx} \\ \delta\varepsilon_{yy} \\ \delta\varepsilon_{zz} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} - \begin{Bmatrix} \delta\gamma_{xy} \\ \delta\gamma_{yz} \\ \delta\gamma_{zx} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix},$$

$$\delta w_V^{\mathrm{ext}} = \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} \text{ and } \delta w_A^{\mathrm{ext}} = \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix}.$$

The terms of the expressions consist of work conjugate pairs of kinematic and kinetic quantities.

GENERALIZED HOOKE'S LAW

The model $g(\vec{\sigma},\vec{u})=0$ for isotropic homogeneous material can be expressed, e.g., in its compliance form as

Strain-stress: $$\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\end{Bmatrix}=\frac{1}{E}\begin{bmatrix}1 & -\nu & -\nu\\ -\nu & 1 & -\nu\\ -\nu & -\nu & 1\end{bmatrix}=[E]^{-1}\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\end{Bmatrix} \text{ and } \begin{Bmatrix}\gamma_{xy}\\ \gamma_{yz}\\ \gamma_{zx}\end{Bmatrix}=\frac{1}{G}\begin{Bmatrix}\sigma_{xy}\\ \sigma_{yz}\\ \sigma_{zx}\end{Bmatrix}$$

Strain-displacement: $$\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\end{Bmatrix}=\begin{Bmatrix}\partial u_x/\partial x\\ \partial u_y/\partial y\\ \partial u_z/\partial z\end{Bmatrix} \text{ and } \begin{Bmatrix}\gamma_{xy}\\ \gamma_{yz}\\ \gamma_{zx}\end{Bmatrix}=\begin{Bmatrix}\partial u_x/\partial y+\partial u_y/\partial x\\ \partial u_y/\partial z+\partial u_z/\partial y\\ \partial u_z/\partial x+\partial u_x/\partial z\end{Bmatrix}$$

Above, E is the Young's modulus, ν the Poisson's ratio, and $G=E/(2+2\nu)$ the shear modulus. Strain and stress are symmetric (the matrix of components is symmetric).

2.2 DISPLACEMENT ANALYSIS

- Model the structure as a collection of elements (solid, plate, beam). Derive the element contributions $\delta W^e = \delta W^{\text{int}} + \delta W^{\text{ext}}$ in terms of the nodal displacement and rotation components of the structural coordinate system.

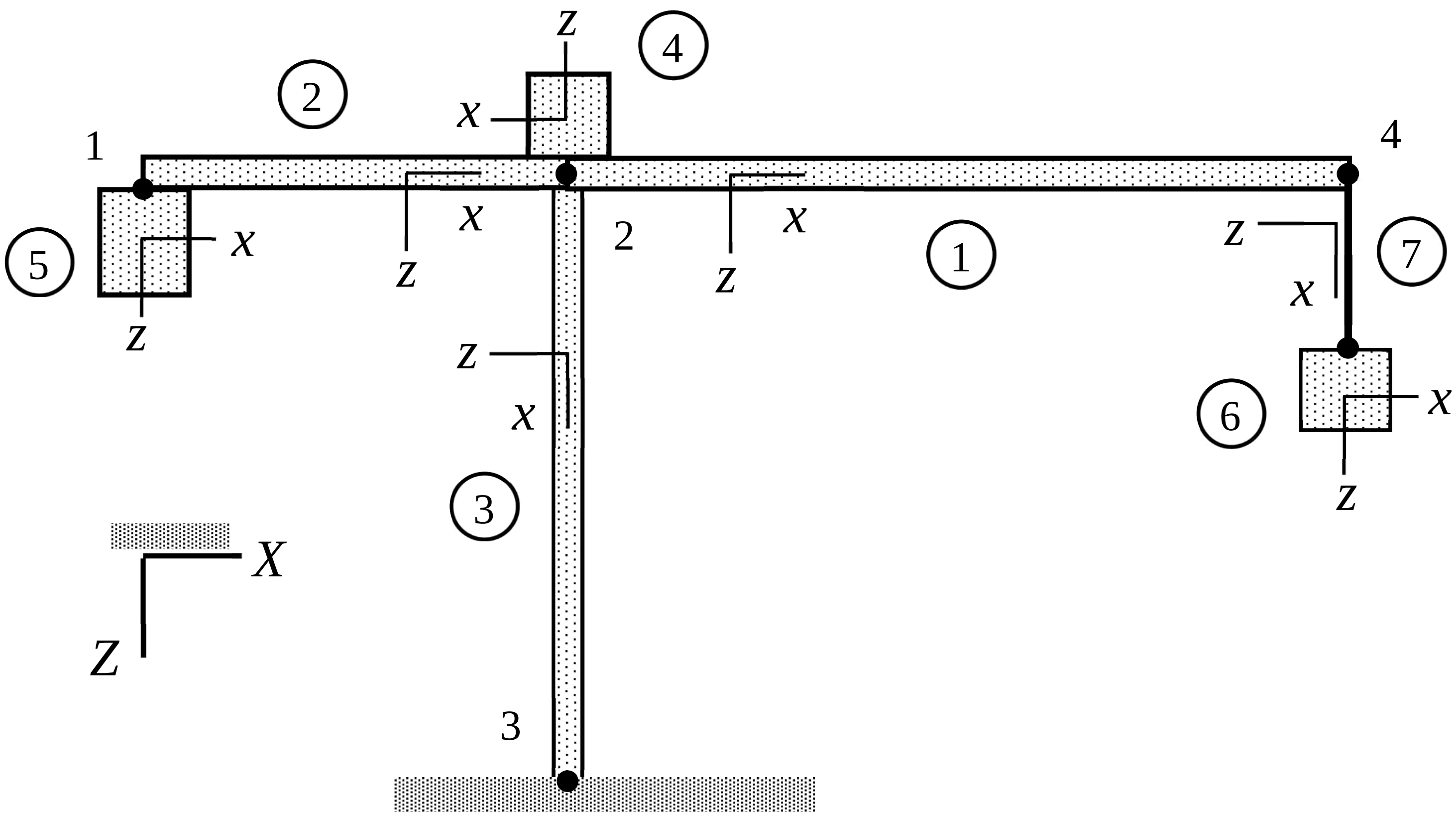
- Sum the element contributions to end up with the virtual work expression of the structure $\delta W = \sum_{e \in E} \delta W^e$. Re-arrange to get the "standard" form $\delta W = -\delta \mathbf{a}^{\mathrm{T}}(\mathbf{Ka} - \mathbf{F}) = 0$.

- Use the principle of virtual work $\delta W = 0 \ \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus for $\delta \mathbf{a} \in \mathbb{R}^n$ to deduce the linear equation system $\mathbf{Ka} - \mathbf{F} = 0$.

- Solve the equations for displacements and rotations $\mathbf{a}$.

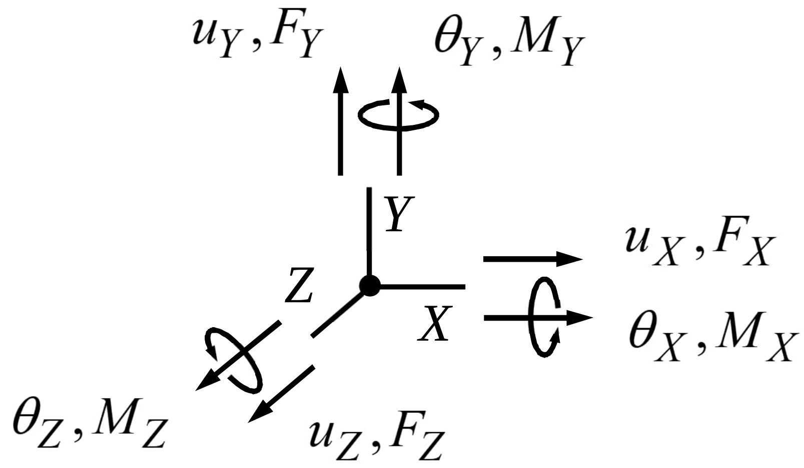
FINITE ELEMENT ANALYSIS

A complex structure is modelled as a collection of structural parts (or elements) modelled as rigid bodies, beams, plates, or solids. Elements are connected by nodes.



KINEMATIC AND KINETIC QUANTITIES

The primary quantities of analysis are displacements, rotations, forces and moments at the connection points of the structural parts. The components of the vector quantities (magnitude and direction) are taken to be positive in the directions of the coordinate axes.



Vector quantities are invariants in the sense $\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k} = a_X\vec{I} + a_Y\vec{J} + a_Z\vec{K}$, and can be transformed from one coordinate system to another using the property.

SIGN CONVENTIONS AND NOTATIONS

Displacements, rotations, forces and moments are vector quantities whose components are positive in the directions of the chosen coordinate axes. The convention may differ from that used in mechanics of materials courses (be careful with that).

	Displacement	**Force**	**Rotation**	**Moment**
Material	u_x, u_y, u_z	F_x, F_y, F_z	$\theta_x, \theta_y, \theta_z$	M_x, M_y, M_z
Structural	u_X, u_Y, u_Z	F_X, F_Y, F_Z	$\theta_X, \theta_Y, \theta_Z$	M_X, M_Y, M_Z

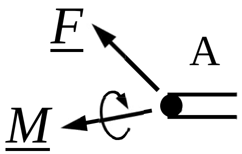
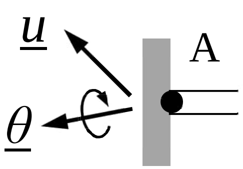
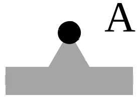
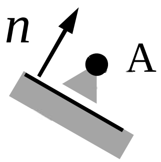
The basis vectors of the material and structural systems are $(\vec{i}, \vec{j}, \vec{k})$ and $(\vec{I}, \vec{J}, \vec{K})$, respectively!

- In FE calculations, one needs to express displacement and rotation components in the material coordinate system in terms of those in the structural coordinate system. Expressing $\vec{i}, \vec{j}, \vec{k}$ (basis vectors of the material coordinate system) in terms of $\vec{I}, \vec{J}, \vec{K}$ (basis vectors of the structural coordinate system) and coordinate system invariance in form $\vec{a} = a_x \vec{i} + a_y \vec{j} + a_z \vec{k} = a_X \vec{I} + a_Y \vec{J} + a_Z \vec{K}$, one obtains

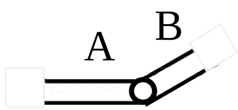
$$\begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = \begin{bmatrix} i_X & i_Y & i_Z \\ j_X & j_Y & j_Z \\ k_X & k_Y & k_Z \end{bmatrix} \begin{Bmatrix} \vec{I} \\ \vec{J} \\ \vec{K} \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = \begin{Bmatrix} a_X \\ a_Y \\ a_Z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vec{I} \\ \vec{J} \\ \vec{K} \end{Bmatrix} \Rightarrow$$

$$\begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix} = \begin{bmatrix} i_X & i_Y & i_Z \\ j_X & j_Y & j_Z \\ k_X & k_Y & k_Z \end{bmatrix} \begin{Bmatrix} a_X \\ a_Y \\ a_Z \end{Bmatrix} \text{ or } \begin{Bmatrix} a_X \\ a_Y \\ a_Z \end{Bmatrix} = \begin{bmatrix} i_X & i_Y & i_Z \\ j_X & j_Y & j_Z \\ k_X & k_Y & k_Z \end{bmatrix}^{\mathrm{T}} \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix}. \quad \Leftarrow$$

INTERACTION MODELS

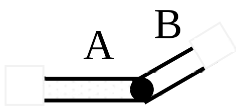
name	symbol	equations
force		$\vec{F}_{\mathrm{A}} = \underline{F},\ \vec{M}_{\mathrm{A}} = \underline{M}$
fixed		$\vec{u}_{\mathrm{A}} = \underline{u},\ \vec{\theta}_{\mathrm{A}} = \underline{\theta}$
joint		$\vec{u}_{\mathrm{A}} = 0,\ \vec{M}_{\mathrm{A}} = 0$
slider		$\vec{n} \cdot \vec{u}_{\mathrm{A}} = 0,\ \vec{F}_{\mathrm{A}} - (\vec{F}_{\mathrm{A}} \cdot \vec{n})\vec{n} = 0,\ \vec{M}_{\mathrm{A}} = 0$

joint



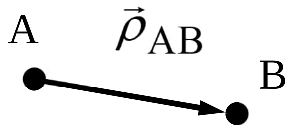
$$\vec{u}_B = \vec{u}_A,\ \vec{M}_A = 0,\ \vec{M}_B = 0$$

fixed



$$\vec{u}_B = \vec{u}_A,\ \vec{\theta}_B = \vec{\theta}_A$$

rigid



$$\vec{u}_B = \vec{u}_A + \vec{\theta}_A \times \vec{\rho}_{AB},\ \vec{\theta}_B = \vec{\theta}_A$$

Interaction models define a kinematic quantity (displacements and rotations) or its work conjugate (forces and moments). In practice, only the kinematic conditions need to be imposed explicitly.

BEAMS, PLATES AND SOLIDS

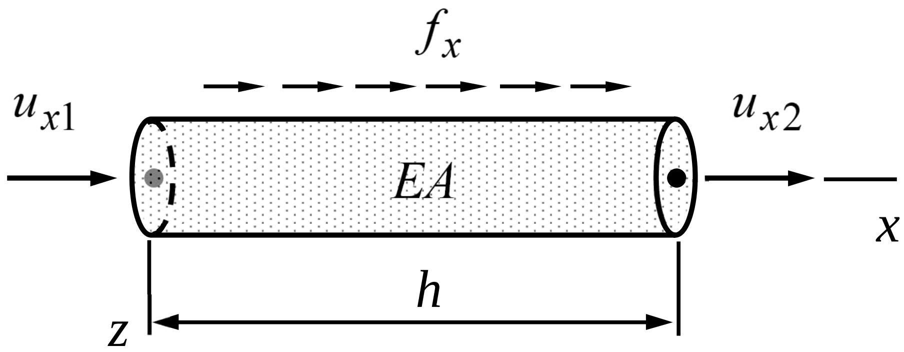
Elements of the structure may be modelled as rigid bodies, beams, plates, or solids or their simplified versions considering only the active loading modes, i.e., *bar*, *torsion*, and *bending* modes for the beam model and *thin slab* and *bending* modes for the plate model:

Beam: $\delta W = \delta W_{\text{bar}} + \delta W_{\text{tor}} + \delta W_{\text{xz-bnd}} + \delta W_{\text{xy-bnd}}$

Plate: $\delta W = \delta W_{\text{slb}} + \delta W_{\text{bnd}}$

The simple expressions above assume a clever positioning of material coordinate system and, thereby, uncoupling of the loading modes. Then one may treat the modes in the same manner as the elements of the structure (virtual work expression is obtained as the sum over the elements and the loading modes of them).

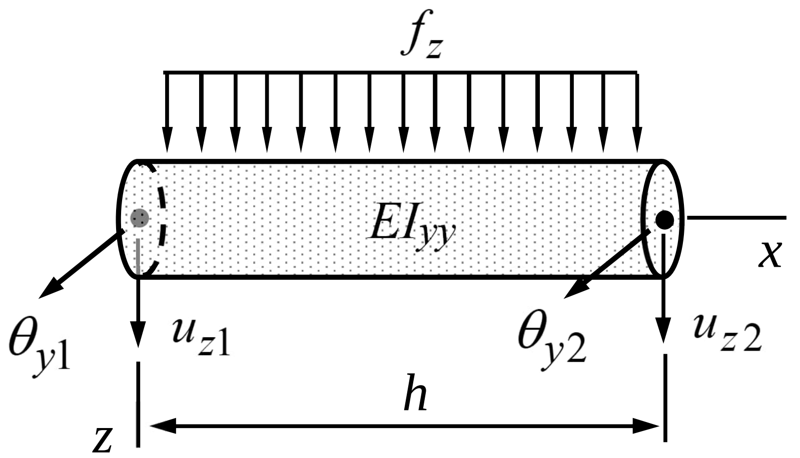
BAR MODE



$$\delta W = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \right)$$

Above, f_x, E, and A are assumed to be constants. In terms of the unit vector in the direction of the $x-$axis $u_x = \vec{i} \cdot \vec{u} = i_X u_X + i_Y u_Y + i_Z u_Z$ and $\delta u_x = \vec{i} \cdot \delta\vec{u} = i_X \delta u_X + i_Y \delta u_Y + i_Z \delta u_Z$.

BENDING MODE



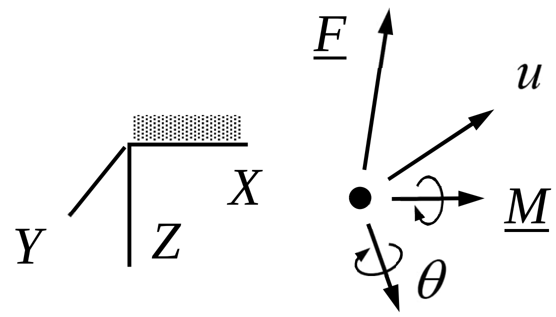
$$\delta W = -\left\{\begin{array}{c}\delta u_{z1}\\ \delta\theta_{y1}\\ \hline \delta u_{z2}\\ \delta\theta_{y2}\end{array}\right\}^{\mathrm{T}} \left(\frac{EI_{yy}}{h^3}\left[\begin{array}{cc|cc} 12 & -6h & -12 & -6h\\ -6h & 4h^2 & 6h & 2h^2\\ \hline -12 & 6h & 12 & 6h\\ -6h & 2h^2 & 6h & 4h^2\end{array}\right]\left\{\begin{array}{c}u_{z1}\\ \theta_{y1}\\ \hline u_{z2}\\ \theta_{y2}\end{array}\right\} - \frac{f_z h}{12}\left\{\begin{array}{c}6\\ -h\\ \hline 6\\ h\end{array}\right\}\right)$$

Above, f_z , I_{yy} and E are assumed to be constants. In terms of the basis vectors of the $xyz-$ system $u_z = \vec{k}\cdot\vec{u}$, $\delta u_z = \vec{k}\cdot\delta\vec{u}$, $\theta_y = \vec{j}\cdot\vec{\theta}$, and $\delta\theta_y = \vec{j}\cdot\delta\vec{\theta}$.

FORCE ELEMENT

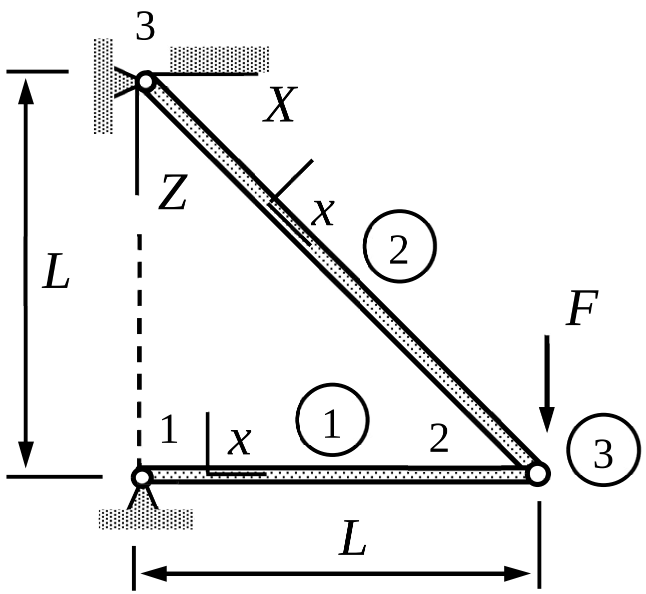
External point forces and moments are assumed to act on the joints. They are treated as elements associated with one node only. Virtual work expression is usually simplest in the structural coordinate system:

$$\delta W = \begin{Bmatrix} \delta u_X \\ \delta u_Y \\ \delta u_Z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \underline{F}_X \\ \underline{F}_Y \\ \underline{F}_Z \end{Bmatrix} + \begin{Bmatrix} \delta \theta_X \\ \delta \theta_Y \\ \delta \theta_Z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \underline{M}_X \\ \underline{M}_Y \\ \underline{M}_Z \end{Bmatrix}$$



Above, $\underline{F}_X$, $\underline{F}_Y$, $\underline{F}_Z$ and $\underline{M}_X$, $\underline{M}_Y$, $\underline{M}_Z$ are the given external force and moment components. A rigid body can be modeled as a particle at the center of mass connected to the other joints of the body by rigid links!

EXAMPLE 2.1 A bar truss is loaded by a point force having magnitude F as shown in the figure. Determine the nodal displacements. Cross-sectional area of bar 1-2 is A and that for bar 3-2 $\sqrt{8}A$. Young's modulus is E and weight is omitted.



Answer $\begin{Bmatrix} u_{X1} \\ u_{Z1} \end{Bmatrix} = \frac{LF}{EA}\begin{Bmatrix} -1 \\ 2 \end{Bmatrix}$

- For element 1, the relationships between the nodal displacement components in the material and structural systems are $u_{x1} = 0$ and $u_{x2} = u_{X2}$. Element contribution δW^1 to the virtual work expression of the structure is

$$\delta W^1 = -\begin{Bmatrix} 0 \\ \delta u_{X2} \end{Bmatrix}^{\mathrm{T}} (\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_{X2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}) = -\frac{EA}{L} u_{X2} \delta u_{X2}.$$

- For element 2, $u_{x3} = 0$ and $u_{x2} = (u_{X2} + u_{Z2}) / \sqrt{2}$. Element contribution takes the form

$$\delta W^2 = -\frac{1}{\sqrt{2}}\begin{Bmatrix} 0 \\ \delta u_{X2} + \delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} (\frac{E\sqrt{8}A}{\sqrt{2}L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{Bmatrix} 0 \\ u_{X2} + u_{Z2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}) \quad \Leftrightarrow$$

$$\delta W^2 = -\frac{EA}{L}(\delta u_{X2} + \delta u_{Z2})(u_{X2} + u_{Z2}).$$

- Virtual work expression of the point force follows from the definition of work

$$\delta W^3 = \delta u_{Z2} F \,.$$

- Virtual work expression of the structure is obtained as the sum of the element contributions. Then

$$\delta W = -\frac{EA}{L}\delta u_{X2} u_{X2} - \frac{EA}{L}(\delta u_{X2} + \delta u_{Z2})(u_{X2} + u_{Z2}) + \delta u_{Z2} F \quad \Leftrightarrow$$

$$\delta W = -\begin{Bmatrix} \delta u_{X2} \\ \delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{L}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{Z2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ F \end{Bmatrix} \right). \qquad \text{"standard" form}$$

- Using the principle of virtual work $\delta W = 0 \;\; \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus

$$\frac{EA}{L}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{Z2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ F \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix} u_{X2} \\ u_{Z2} \end{Bmatrix} = \frac{LF}{EA}\begin{Bmatrix} -1 \\ 2 \end{Bmatrix}.$$

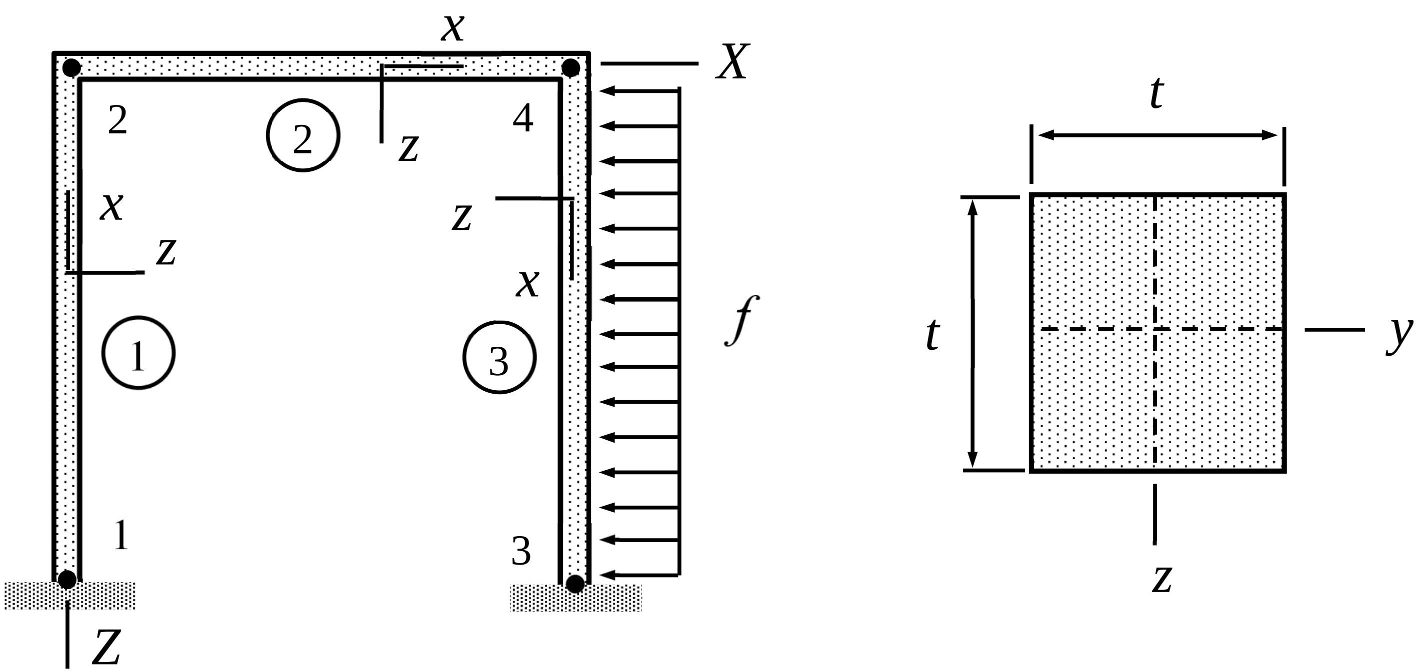
- The Mathematica description of the problem and solution are given by

	model	properties	geometry
1	BAR	{{E}, {A}}	Line[{1, 2}]
2	BAR	$\{\{E\}, \{2\sqrt{2}\ A\}\}$	Line[{3, 2}]
3	FORCE	{0, 0, F}	Point[{2}]

	{X,Y,Z}	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	{0, 0, L}	{0, 0, 0}	{0, 0, 0}
2	{L, 0, L}	{uX[2], 0, uZ[2]}	{0, 0, 0}
3	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}

$$\left\{uX[2] \to -\frac{F\,L}{A\,E},\ uZ[2] \to \frac{2\,F\,L}{A\,E}\right\}$$

EXAMPLE 2.2 Consider the beam truss shown. Determine the displacements and rotations of nodes 2 and 4. Assume that the beams are rigid in the axial directions so that the axial *strain* vanishes. Cross-sections and lengths are the same and Young's modulus E is constant.



Answer $u_{X2} = u_{X4} = -\frac{3}{112}\frac{fL^4}{EI}$, $\theta_{Y2} = \frac{19}{1008}\frac{fL^3}{EI}$, and $\theta_{Y4} = \frac{5}{1008}\frac{fL^3}{EI}$

- Only the bending in *XZ*-plane needs to be accounted for. The displacement and rotation components of the structure are u_{X2}, θ_{Y2}, and θ_{Y4}. As the axial strain of beam 2 vanishes, axial displacements satisfy $u_{X4} = u_{X2}$.

$$\delta W^1 = -\left\{\begin{array}{c} 0 \\ 0 \\ \hdashline \delta u_{X2} \\ \delta\theta_{Y2} \end{array}\right\}^{\mathrm{T}} \left(\frac{EI}{L^3}\left[\begin{array}{cc:cc} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ \hdashline -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{array}\right]\left\{\begin{array}{c} 0 \\ 0 \\ \hdashline u_{X2} \\ \theta_{Y2} \end{array}\right\}\right) \quad (u_{z2} = u_{X2}, \theta_{y2} = \theta_{Y2})$$

$$\delta W^2 = -\left\{\begin{array}{c} 0 \\ \delta\theta_{Y2} \\ \hdashline 0 \\ \delta\theta_{Y4} \end{array}\right\}^{\mathrm{T}} \left(\frac{EI}{L^3}\left[\begin{array}{cc:cc} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ \hdashline -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{array}\right]\left\{\begin{array}{c} 0 \\ \theta_{Y2} \\ \hdashline 0 \\ \theta_{Y4} \end{array}\right\}\right) \quad (\theta_{y2} = \theta_{Y2}, \theta_{y4} = \theta_{Y4})$$

$$\delta W^3 = -\left\{\begin{array}{c} -\delta u_{X2} \\ \delta\theta_{Y4} \\ \hdashline 0 \\ 0 \end{array}\right\}^{\mathrm{T}} \left(\frac{EI}{L^3}\left[\begin{array}{cc:cc} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ \hdashline -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{array}\right]\left\{\begin{array}{c} -u_{X2} \\ \theta_{Y4} \\ \hdashline 0 \\ 0 \end{array}\right\} - \frac{fL}{12}\left\{\begin{array}{c} 6 \\ -L \\ \hdashline 6 \\ L \end{array}\right\}\right) \quad (u_{z4} = -u_{X2})$$

- Virtual work expression of the structure is

$$\delta W = \delta W^1 + \delta W^2 + \delta W^3 = -\begin{Bmatrix} \delta u_{X2} \\ \delta\theta_{Y2} \\ \delta\theta_{Y4} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EI}{L^3}\begin{bmatrix} 24 & 6L & 6L \\ 6L & 8L^2 & 2L^2 \\ 6L & 2L^2 & 8L^2 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ \theta_{Y2} \\ \theta_{Y4} \end{Bmatrix} - \frac{fL}{12}\begin{Bmatrix} -6 \\ 0 \\ -L \end{Bmatrix}\right).$$

- Principle of virtual work $\delta W = 0 \;\forall \delta\mathbf{a}$ and the fundamental lemma of variation calculus give

$$\frac{EI}{L^3}\begin{bmatrix} 24 & 6L & 6L \\ 6L & 8L^2 & 2L^2 \\ 6L & 2L^2 & 8L^2 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ \theta_{Y2} \\ \theta_{Y4} \end{Bmatrix} - \frac{fL}{12}\begin{Bmatrix} -6 \\ 0 \\ -L \end{Bmatrix} = 0 \Leftrightarrow \begin{Bmatrix} u_{X2} \\ \theta_{Y2} \\ \theta_{Y4} \end{Bmatrix} = \frac{fL^3}{1008EI}\begin{Bmatrix} -27L \\ 19 \\ 5 \end{Bmatrix}. \leftarrow$$

- In the Mathematica code calculation, horizontal displacements of nodes 2 and 4 are forced to be same ($u_{X4} = u_{X2}$)

	model	properties	geometry
1	BEAM	{{E, G}, {A, I, I}}	Line[{1, 2}]
2	BEAM	{{E, G}, {A, I, I}}	Line[{2, 4}]
3	BEAM	{{E, G}, {A, I, I}, {-f, 0, 0}}	Line[{4, 3}]

	{X,Y,Z}	$\{u_X,u_Y,u_Z\}$	$\{\theta_X,\theta_Y,\theta_Z\}$
1	{0, 0, L}	{0, 0, 0}	{0, 0, 0}
2	{0, 0, 0}	{uX[2], 0, 0}	{0, θY[2], 0}
3	{L, 0, L}	{0, 0, 0}	{0, 0, 0}
4	{L, 0, 0}	{uX[2], 0, 0}	{0, θY[4], 0}

$$\left\{uX[2] \to -\frac{3\,f\,L^4}{112\,E\,I},\ \theta Y[2] \to \frac{19\,f\,L^3}{1008\,E\,I},\ \theta Y[4] \to \frac{5\,f\,L^3}{1008\,E\,I}\right\}$$

2.3 ELEMENT CONTRIBUTIONS

Virtual work expressions for the solid, beam, plate elements combine virtual work densities representing the model and a case dependent approximation. To derive the expression for an element:

- Start with the virtual work densities $\delta w_{\Omega}^{\text{int}}$ and $\delta w_{\Omega}^{\text{ext}}$ of the formulae collection (if not available there, derive the expression in the manner discussed in MEC-E1050).

- Represent the unknown functions by interpolation of the nodal displacement and rotations (see formulae collection). Substitute the approximations into the density expressions.

- Integrate the virtual work density over the domain occupied by the element to get δW.

ELEMENT APPROXIMATION

Approximation of a function is a polynomial interpolant of the nodal displacement and rotations in terms of shape functions. In displacement analysis, shape functions depend on (x, y, z) and the nodal values are parameters to be evaluated by FEM.

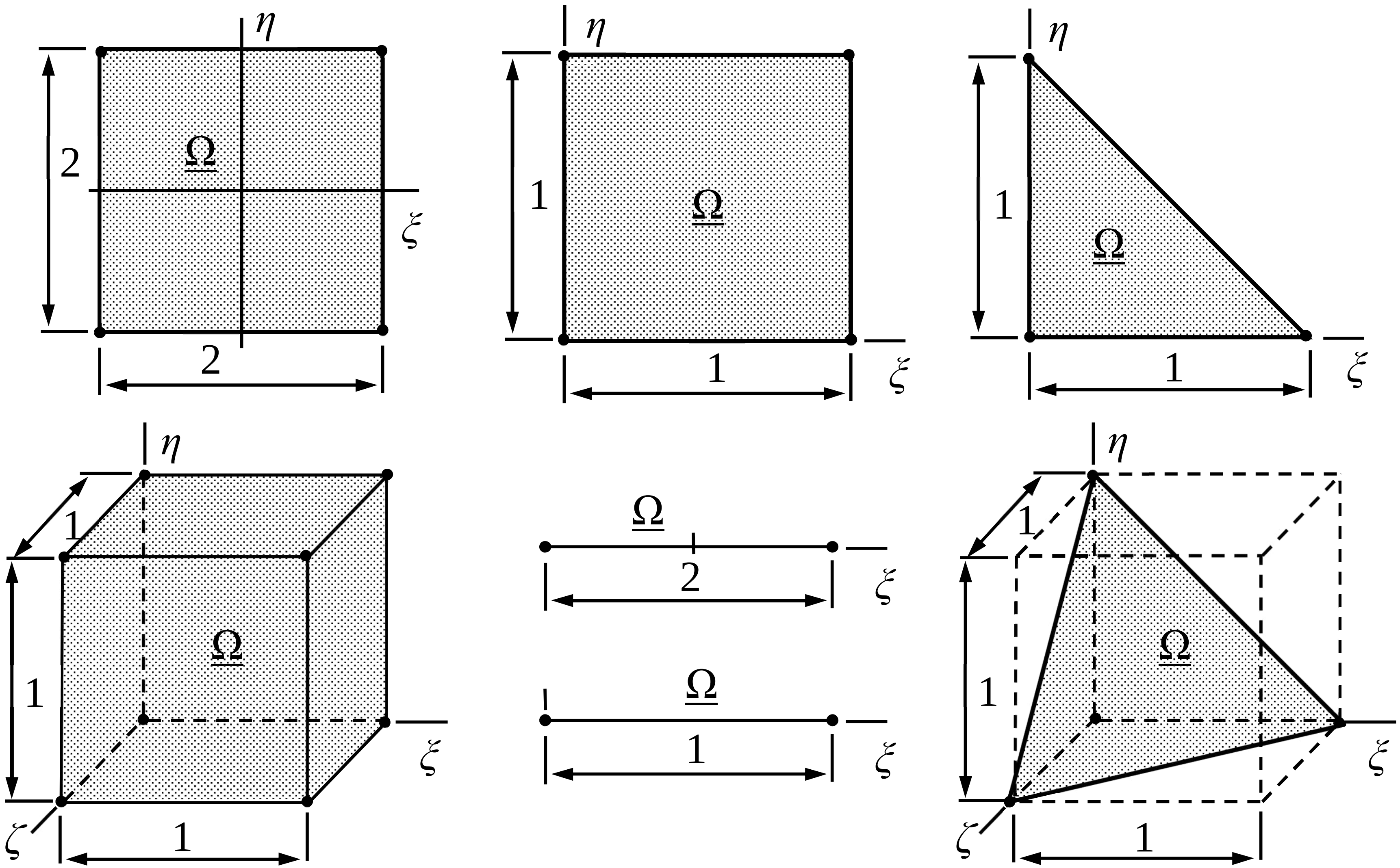
Approximation $\mathbf{u} = \mathbf{N}^{\mathrm{T}}\mathbf{a}$ always of the same form!

Shape functions $\mathbf{N} = \{N_1(x, y, z) \quad N_2(x, y, z) \quad \ldots \quad N_n(x, y, z)\}^{\mathrm{T}}$

Parameters $\mathbf{a} = \{\mathrm{a}_1 \quad \mathrm{a}_2 \quad \ldots \quad \mathrm{a}_n\}^{\mathrm{T}}$

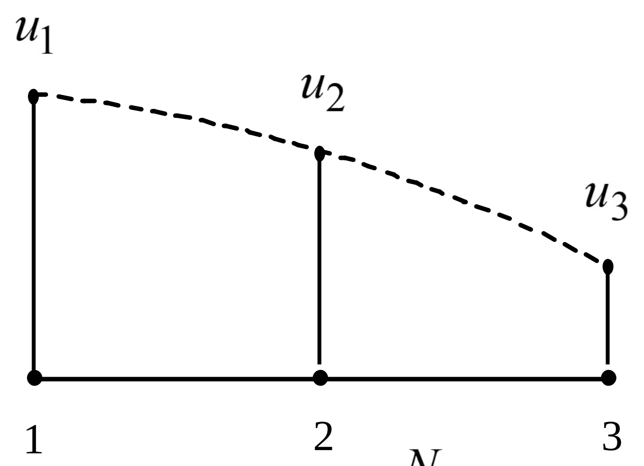
Nodal parameters $\mathrm{a} \in \{u_x, u_y, u_z, \theta_x, \theta_y, \theta_z\}$ may be just displacement or rotation components or a mixture of them (as with the Bernoulli beam model).

ELEMENT GEOMETRY



QUADRATIC SHAPE FUNCTIONS

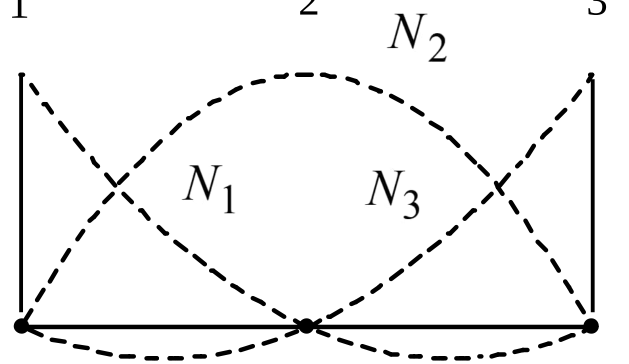
Piecewise quadratic approximation is continuous in Ω and second order polynomial inside the elements. In a typical element Ω^e



Approximation: $u = \mathbf{N}^{\mathrm{T}}\mathbf{a}$

Nodal values: $\mathbf{a} = \{u_1 \quad u_2 \quad u_3\}^{\mathrm{T}}$

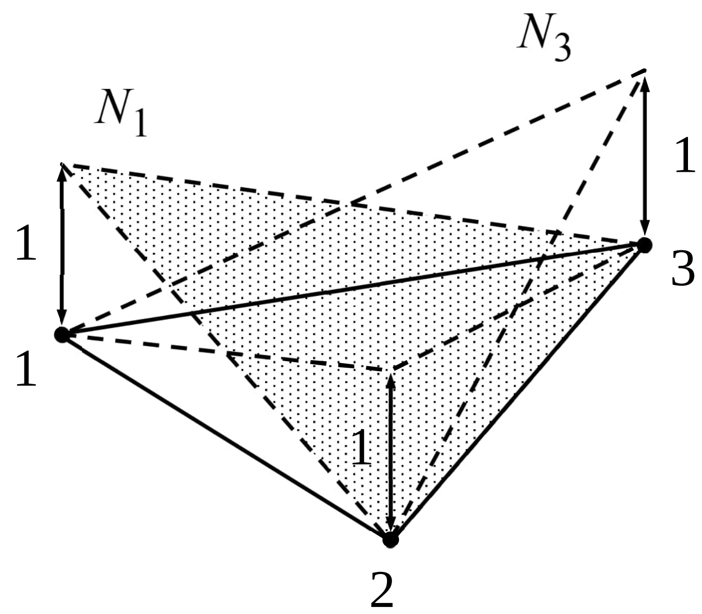
Shape functions: $\mathbf{N} = \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} = \begin{Bmatrix} 1-3\xi+2\xi^2 \\ 4\xi(1-\xi) \\ \xi(2\xi-1) \end{Bmatrix}, \quad \xi = \dfrac{x}{h}$



More nodes can be used to generate higher order approximations!

LINEAR SHAPE FUNCTIONS

A piecewise linear approximation is continuous in Ω and linear inside each element of triangle shape. In a typical element



Approximation: $u = \mathbf{N}^{\mathrm{T}}\mathbf{a}$

Nodal values: $\mathbf{a} = \{u_1 \quad u_2 \quad u_3\}^{\mathrm{T}}$

Shape functions: $\mathbf{N} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}$

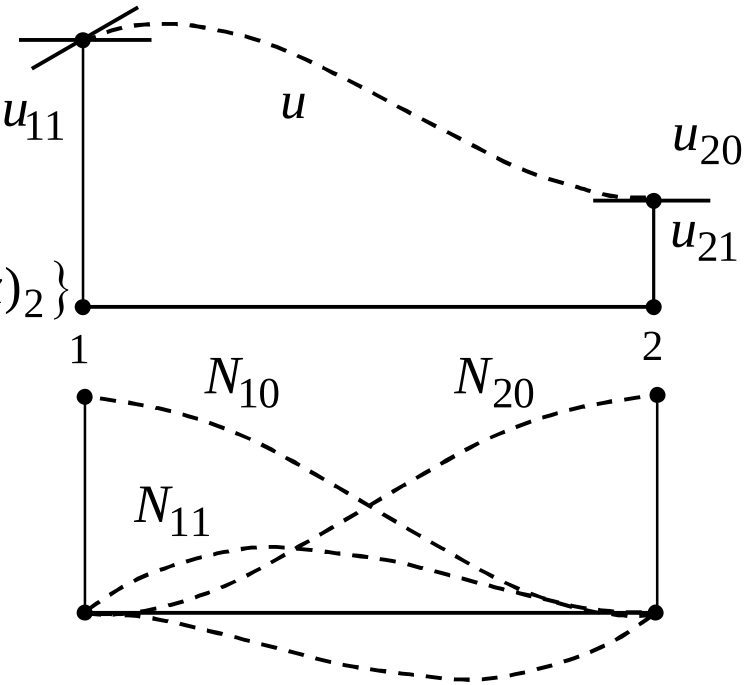
Triangle element is the simplest element in two dimensions. Division of any 2D domain into triangles is always possible, which makes the element quite useful.

CUBIC SHAPE FUNCTIONS

Piecewise cubic approximation has continuous derivatives up to the first order in Ω and is a third order polynomial inside the elements.

Approximation: $u = \mathbf{N}^{\mathrm{T}}\mathbf{a}$

Nodal values: $\mathbf{a} = \{u_1 \quad (du/dx)_1 \mid u_2 \quad (du/dx)_2\}$



Shape functions: $\mathbf{N} = \begin{Bmatrix} N_{10} \\ N_{11} \\ \hline N_{20} \\ N_{21} \end{Bmatrix} = \begin{Bmatrix} (1-\xi)^2(1+2\xi) \\ h(1-\xi)^2\xi \\ \hline (3-2\xi)\xi^2 \\ h\xi^2(\xi-1) \end{Bmatrix}$

In $xz-$plane bending $u = u_z,\ du/dx = -\theta_y$ and in $xy-$plane bending $u = u_y,\ du/dx = \theta_z$.

SOLID MODEL

The model does not contain assumptions in addition to those of linear elasticity theory.

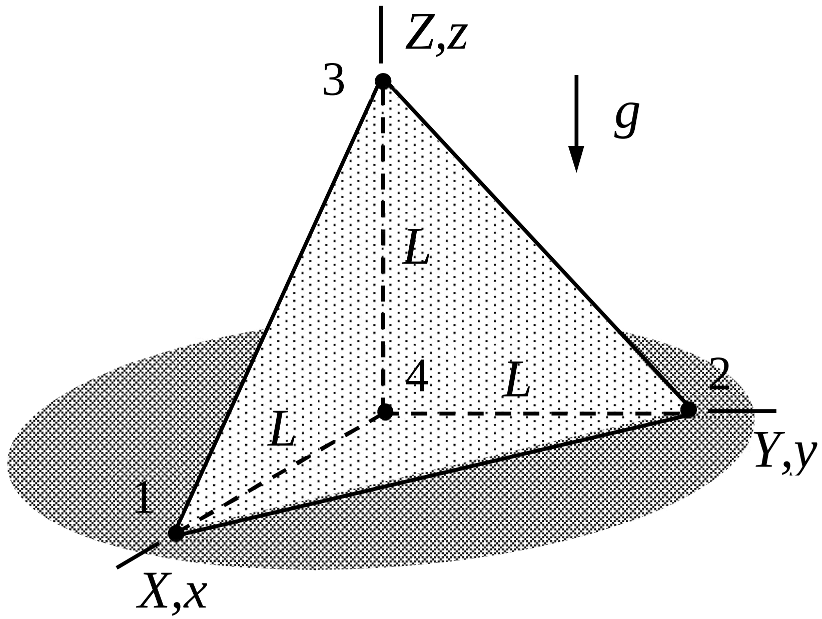
$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta w/\partial z \end{Bmatrix}^{\mathrm{T}} [E] \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial w/\partial z \end{Bmatrix} - \begin{Bmatrix} \partial\delta u/\partial y + \partial\delta v/\partial x \\ \partial\delta v/\partial z + \partial\delta w/\partial y \\ \partial\delta w/\partial x + \partial\delta u/\partial z \end{Bmatrix}^{\mathrm{T}} G \begin{Bmatrix} \partial u/\partial y + \partial v/\partial x \\ \partial v/\partial z + \partial w/\partial y \\ \partial w/\partial x + \partial u/\partial z \end{Bmatrix},$$

$$\delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} \text{ and } \delta w_{\partial\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix} \text{ in which } [E] = E\begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix}^{-1}.$$

The solution domain can be represented, e.g, by tetrahedron elements with linear interpolation of the displacement components $u(x,y,z)$, $v(x,y,z)$, and $w(x,y,z)$

EXAMPLE 2.3 A tetrahedron of edge length L, density ρ, and elastic properties E and ν is subjected to its own weight on a horizontal floor. Calculate the displacement u_{Z3} of node 3 with one tetrahedron element and linear approximation. Assume that $u_{X3} = u_{Y3} = 0$, and that the bottom surface is fixed.

Answer: $u_{Z3} = -\dfrac{1}{4}\dfrac{\rho g L^2}{E}\dfrac{1-\nu-2\nu^2}{1-\nu}$



- Linear shape functions can be deduced directly from the figure $N_1 = x/L$, $N_2 = y/L$, $N_3 = z/L$, and $N_4 = 1 - x/L - y/L - z/L$. However, only the shape function of node 3 is needed as the other nodes are fixed. Approximations to the displacement components are

$$u = 0,\ \ v = 0,\ \text{ and }\ w = \frac{z}{L} u_{Z3},\ \text{ giving }\ \frac{\partial w}{\partial x} = \frac{\partial w}{\partial y} = 0\ \text{ and }\ \frac{\partial w}{\partial z} = \frac{1}{L} u_{Z3}.$$

- When the approximation is substituted there, the virtual work densities of the internal and external forces simplify to

$$\delta w_V^{\text{int}} = -\begin{Bmatrix} 0 \\ 0 \\ \delta_{Z3} \end{Bmatrix}^{\mathrm{T}} \frac{E}{L^2(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_{Z3} \end{Bmatrix} = \frac{-E(1-\nu)}{(1+\nu)(1-2\nu)} \frac{u_{Z3}\delta u_{Z3}}{L^2}$$

$$\delta w_V^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \delta u_{Z3} \end{Bmatrix}^{\mathrm{T}} \frac{z}{L} \begin{Bmatrix} 0 \\ 0 \\ -\rho g \end{Bmatrix} = -\frac{z}{L} \rho g \delta u_{Z3}.$$

- Virtual work expressions are obtained as integrals of densities over the volume:

$$\delta W^{\text{int}} = \int_{\Omega} \delta w_{\Omega}^{\text{int}} dV = \delta w_{\Omega}^{\text{int}} \frac{L^3}{6} = -\frac{1}{6} \frac{1-\nu}{(1+\nu)(1-2\nu)} ELu_{Z3} \delta u_{Z3},$$

$$\delta W^{\text{ext}} = \int_{\Omega} \delta w_{\Omega}^{\text{ext}} dV = -\frac{L^3}{24} \rho g \delta u_{Z3}.$$

- Finally, principle of virtual work $\delta W = 0 \quad \forall \delta \mathbf{a}$ with $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}}$ and the fundamental lemma of variation calculus imply

$$u_{Z3} = -\frac{1}{4}\frac{\rho g L^2}{E}\frac{1-\nu-2\nu^2}{1-\nu}.$$

- For the Mathematica code of the course, the problem description is given by

	model	properties	geometry
1	SOLID	{{E, ν}, {0, 0, -g ρ}}	Tetrahedron[{1, 2, 3, 4}]

	{X,Y,Z}	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	{L, 0, 0}	{0, 0, 0}	{0, 0, 0}
2	{0, L, 0}	{0, 0, 0}	{0, 0, 0}
3	{0, 0, L}	{uX[3], uY[3], uZ[3]}	{0, 0, 0}
4	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}

$$\left\{uX[3] \to 0,\ uY[3] \to 0,\ uZ[3] \to -\frac{g\, L^2 \left(-1+\nu+2\,\nu^2\right)\rho}{4\,E\,(-1+\nu)}\right\}$$

BEAM MODEL

In the beam model, the displacement and rotation components to be interpolated on a line segment of $x-$axis are $u(x)$, $v(x)$, $w(x)$, and $\phi(x)$. Virtual work densities are given by

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} d\delta u / dx \\ d^2\delta v / dx^2 \\ d^2\delta w / dx^2 \end{Bmatrix}^{\mathrm{T}} E \begin{bmatrix} A & -S_z & -S_y \\ -S_z & I_{zz} & I_{zy} \\ -S_y & I_{yz} & I_{yy} \end{bmatrix} \begin{Bmatrix} du / dx \\ d^2 v / dx^2 \\ d^2 w / dx^2 \end{Bmatrix} - \frac{d\delta\phi}{dx} GJ \frac{d\phi}{dx},$$

$$\delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} + \begin{Bmatrix} \delta\phi \\ -d\delta w / dx \\ d\delta v / dx \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} m_x \\ m_y \\ m_z \end{Bmatrix}.$$

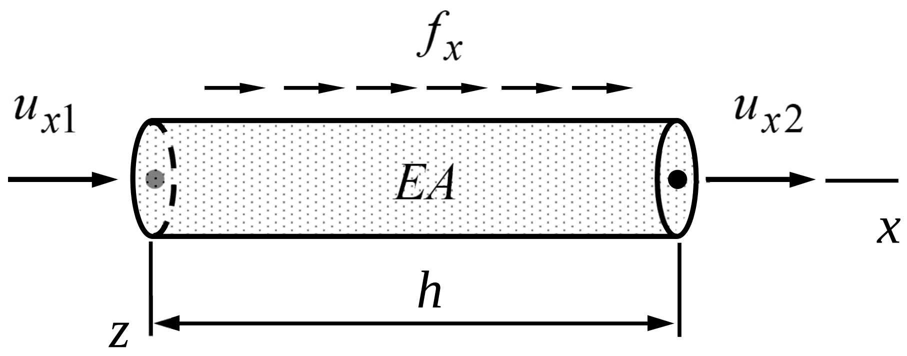
In what follows, the first and cross moments of the cross-section are assumed to vanish to disconnect the bar, torsion, and bending modes of the beam ($S_z = S_y = I_{yz} = 0$).

BAR MODE

Assuming a linear interpolation to $u(x)$ in terms of the end point displacements u_{x1}, u_{x2}, virtual work expressions of the internal and external forces take the forms

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix},$$

$$\delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{f_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$



Above, f_x, E, and A are assumed to be constants. The relationship between the axial displacement component and the displacement components in the structural coordinate system is $u_x = \vec{i} \cdot \vec{u} = i_X u_X + i_Y u_Y + i_Z u_Z$.

- First, element interpolant $u=\mathbf{N}^{\mathrm{T}}\mathbf{a}$ and its variation $\delta u=\mathbf{N}^{\mathrm{T}}\delta\mathbf{a}=\delta\mathbf{a}^{\mathrm{T}}\mathbf{N}$ are substituted into the virtual work expression to get (here $\Omega=]0,h[$ and $d\Omega=dx$)

$$\delta W=\int_0^h (-\frac{d\delta u}{dx}EA\frac{du}{dx}+\delta u f_x)dx \qquad \Rightarrow$$

$$\delta W=-\int_0^h \delta\mathbf{a}^{\mathrm{T}}\frac{d\mathbf{N}}{dx}EA\frac{d\mathbf{N}^{\mathrm{T}}}{dx}\mathbf{a}dx+\int_0^h \delta\mathbf{a}^{\mathrm{T}}\mathbf{N}f_x dx \qquad \Leftrightarrow$$

$$\delta W=-\delta\mathbf{a}^{\mathrm{T}}(\int_0^h \frac{d\mathbf{N}}{dx}EA\frac{d\mathbf{N}^{\mathrm{T}}}{dx}dx\mathbf{a}-\int_0^h \mathbf{N}f_x dx). \qquad \Leftarrow$$

- If the interpolant is taken to be linear, shape functions and the nodal values are given by

$$\mathbf{N}=\frac{1}{h}\begin{Bmatrix}h-x\\ x\end{Bmatrix},\quad \frac{d}{dx}\mathbf{N}=\frac{1}{h}\begin{Bmatrix}-1\\ 1\end{Bmatrix},\quad \mathbf{a}=\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix},\text{ and } \delta\mathbf{a}=\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}$$

- If Young's modulus E, cross-sectional area A, and the distributed force f_x are constants, integration over the element domain gives (the expressions of the shape functions need to be substituted now)

$$\delta W = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \left(\int_0^h \frac{1}{h} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} EA \frac{1}{h} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}^{\mathrm{T}} dx \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \int_0^h \frac{1}{h} \begin{Bmatrix} h - x \\ x \end{Bmatrix} f_x dx \right) \quad \Leftrightarrow$$

$$\delta W = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \right). \quad \Leftarrow$$

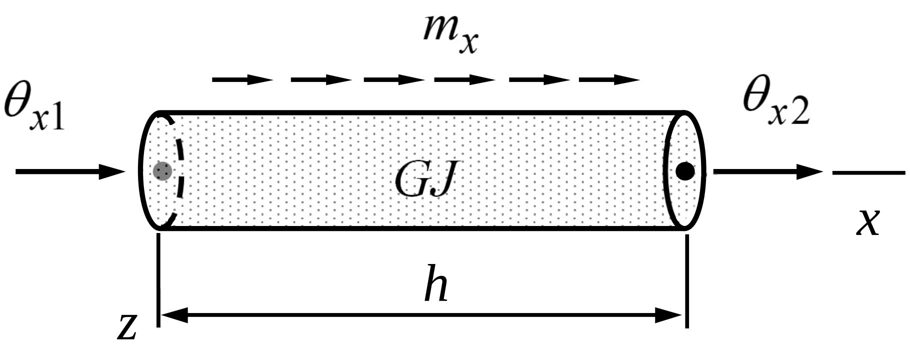
Derivation out of virtual work densities works also when Young's modulus E, cross-sectional area A, and the distributed force f_x are not constants. Also, approximation to axial displacement $u(x)$ may be chosen in various ways.

TORSION MODE

Assuming a linear interpolation to $\phi(x)$ in terms of the end point rotations θ_{x1} and θ_{x2}, virtual work expressions of the internal and external forces take the forms

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{GJ}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix},$$

$$\delta W^{\text{ext}} = \begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{m_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$



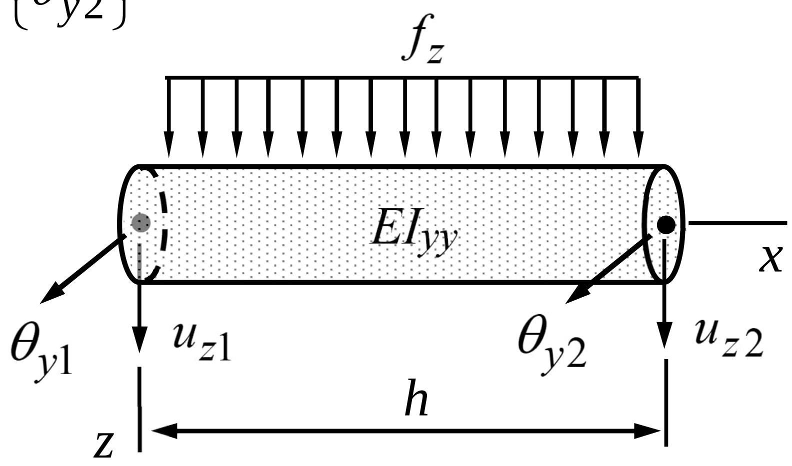
Above, m_x, E, and J are assumed to be constants. The relationship between the axial rotation component and the rotation components in the structural coordinate system is $\theta_x = \vec{i} \cdot \vec{u} = i_X \theta_X + i_Y \theta_Y + i_Z \theta_Z$.

BENDING MODE (*xz*-plane)

Assuming a cubic approximation to $w(x)$ in terms of the end point displacements u_{z1}, u_{z2} and rotations θ_{y1} and θ_{y2}, virtual work expressions of the internal and external forces

$$\delta W^{\mathrm{int}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI_{yy}}{h^3} \begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix}$$

$$\delta W^{\mathrm{ext}} = \begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{f_z h}{12} \begin{Bmatrix} 6 \\ -h \\ 6 \\ h \end{Bmatrix}$$



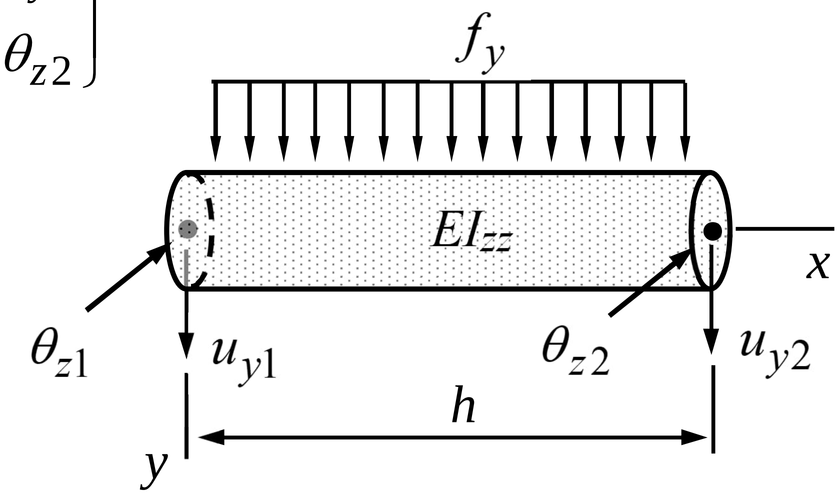
Above, f_z, I_{yy} and E are assumed to be constants.

BENDING MODE (*xy*-plane)

Assuming a cubic approximation to $v(x)$ in terms of point displacements u_{y1}, u_{y2} and rotations θ_{z1} and θ_{z2}, virtual work expressions of the internal and external forces

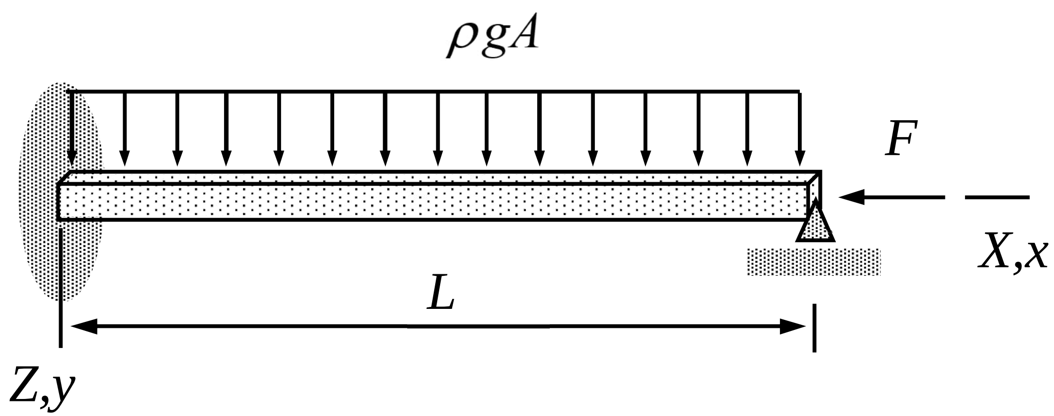
$$\delta W^{\text{int}} = \begin{Bmatrix} \delta u_{y1} \\ \delta\theta_{z1} \\ \delta u_{y2} \\ \delta\theta_{z2} \end{Bmatrix}^{\mathrm{T}} \frac{EI_{zz}}{h^3} \begin{bmatrix} 12 & 6h & -12 & 6h \\ 6h & 4h^2 & -6h & 2h^2 \\ -12 & -6h & 12 & -6h \\ 6h & 2h^2 & -6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \\ u_{y2} \\ \theta_{z2} \end{Bmatrix},$$

$$\delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{y1} \\ \delta\theta_{z1} \\ \delta u_{y2} \\ \delta\theta_{z2} \end{Bmatrix} \frac{f_y h}{12} \begin{Bmatrix} 6 \\ h \\ 6 \\ -h \end{Bmatrix}.$$



Above, f_y , I_{zz} and E are assumed to be constants.

EXAMPLE 2.4 The Bernoulli beam of the figure is loaded by its own weight $f=\rho g A$ and a point force F acting on the right end. Determine the displacement and rotation of the right end with the Mathematica code of MEC-E8001. The x-axis of the material coordinate system is placed at the geometric centroid of the rectangle cross-section. Beam cross-section properties A, I_{yy}, I_{zz}, and material properties E, ρ are constants.



Answer: $u_{X2}=\dfrac{FL}{EA}$ and $\theta_{Y2}=\dfrac{1}{48}\dfrac{\rho g A L^3}{EI_{zz}}$

- Bernoulli beam element of the Mathematica code requires the orientation of the y –axis unless y –axis and Y –axis are aligned. Orientation is given by additional parameter defining the components of $\vec{j}$ in the structural coordinate system:

	model	properties	geometry
1	BEAM	{{E, G}, {A, Iyy, Izz, {0, 0, 1}}, {0, f, 0}}	Line[{1, 2}]
2	FORCE	{-F, 0, 0}	Point[{2}]

	{X,Y,Z}	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}
2	{L, 0, 0}	{uX[2], 0, 0}	{0, θY[2], 0}

$$\left\{uX[2] \to -\frac{F\,L}{A\,E},\ \theta Y[2] \to \frac{f\,L^3}{48\,E\,Izz}\right\}$$

PLATE MODEL

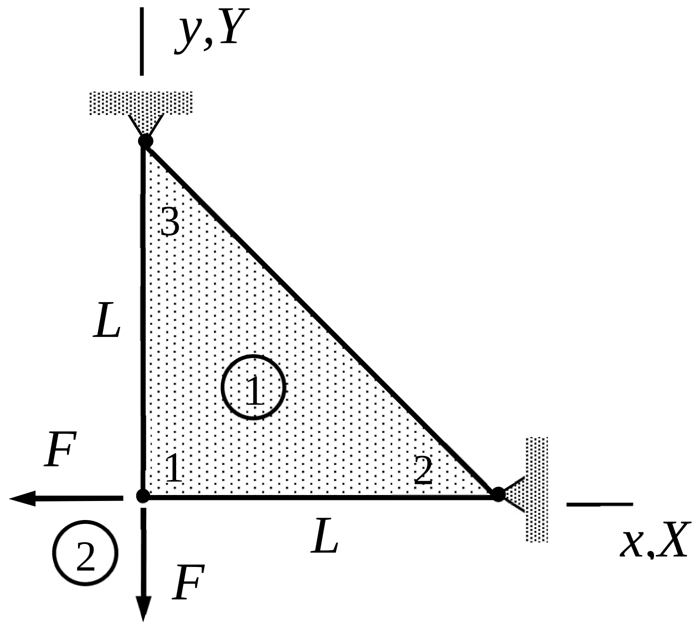
Virtual work densities combine the plane-stress and plate bending modes. Assuming that the material coordinate system is placed at the geometric centroid

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta u / \partial x \\ \partial\delta v / \partial y \\ \partial\delta u / \partial y + \partial\delta v / \partial x \end{Bmatrix}^{\mathrm{T}} t[E]_{\sigma} \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix} - \begin{Bmatrix} \partial^2\delta w / \partial x^2 \\ \partial^2\delta w / \partial y^2 \\ 2\partial^2\delta w / \partial x\partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12}[E]_{\sigma} \times$$

$$\times \begin{Bmatrix} \partial^2 w / \partial x^2 \\ \partial^2 w / \partial y^2 \\ 2\partial^2 w / \partial x\partial y \end{Bmatrix}, \quad \delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix}, \text{ and } \delta w_{\partial\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix}.$$

Approximation to the displacement components $u(x, y)$, $v(x, y)$, $w(x, y)$ should be continuous and $w(x, y)$ should also have continuous derivatives at the element interfaces.

EXAMPLE 2.5. Consider the thin triangular structure shown. Young's modulus E, Poisson's ratio ν, and thickness t are constants. Distributed external force vanishes. Assume plane-stress conditions, $XY-$plane deformation and determine the displacement of node 1 when the force components acting on the node are as shown in the figure.



Answer: $\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} = -\frac{F}{Et}\frac{(1+\nu)(1-2\nu)}{1-\nu}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

- Nodes 2 are 3 are fixed and the non-zero displacement components are u_{X1} and u_{Y1}. Linear shape functions $N_1 = (L - x - y)/L$, $N_2 = x/L$ and $N_3 = y/L$ are easy to deduce from the figure. Therefore

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \frac{L - x - y}{L} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} \quad \Rightarrow \quad \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial x \end{Bmatrix} = -\frac{1}{L} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} \text{ and } \begin{Bmatrix} \partial u / \partial y \\ \partial v / \partial y \end{Bmatrix} = -\frac{1}{L} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix}.$$

- Virtual work density of internal forces is given by

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \\ \delta u_{X1} + \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \frac{1}{L^2} \frac{tE}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{Y1} \\ u_{X1} + u_{Y1} \end{Bmatrix}.$$

- Integration over the triangular domain gives (integrand is constant)

$$\delta W^1 = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \\ \delta u_{X1}+\delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \frac{1}{2}\frac{tE}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{Y1} \\ u_{X1}+u_{Y1} \end{Bmatrix} \quad \Leftrightarrow$$

$$\delta W^1 = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \frac{1}{4}\frac{tE}{1-\nu^2}\begin{bmatrix} 3-\nu & 1+\nu \\ 1+\nu & 3-\nu \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix}.$$

- Virtual work expression for the point forces follows from the definition of work

$$\delta W^2 = \begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} -F \\ -F \end{Bmatrix} .$$

- Principle of virtual work in the form $\delta W = \delta W^1 + \delta W^2 = 0 \ \ \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus give

$$\delta W = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \left(\frac{1}{4}\frac{tE}{1-\nu^2}\begin{bmatrix} 3-\nu & 1+\nu \\ 1+\nu & 3-\nu \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} + F\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}\right) = 0 \quad \forall \begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix} \quad \Rightarrow$$

$$\frac{1}{4}\frac{tE}{1-\nu^2}\begin{bmatrix} 3-\nu & 1+\nu \\ 1+\nu & 3-\nu \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} + F\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = 0 \quad \Leftrightarrow$$

$$\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} = -\frac{F}{tE}(1-\nu^2)\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

The point forces acting on a thin slab should be considered as "equivalent nodal forces" i.e. just representations of tractions acting on some part of the boundary. Under the action of an actual point force, displacement becomes non-bounded. In practice, numerical solution to the displacement at the point of action increases when the mesh is refined.

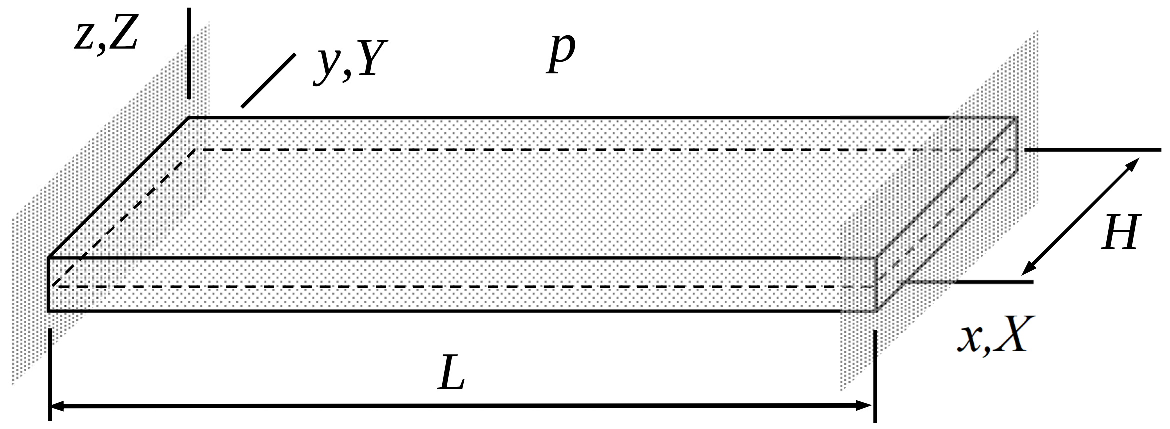
- In Mathematica code of the course, the problem description is given by

	model	properties	geometry
1	PLANE	{{E, ν}, {t}}	Polygon[{1, 2, 3}]
2	FORCE	{-F, -F, 0}	Point[{1}]

	{X,Y,Z}	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	{0, 0, 0}	{uX[1], uY[1], 0}	{0, 0, 0}
2	{L, 0, 0}	{0, 0, 0}	{0, 0, 0}
3	{0, L, 0}	{0, 0, 0}	{0, 0, 0}

$$\left\{uX[1] \rightarrow \frac{F\,(-1+\nu)\,(1+\nu)}{t\,E},\ uY[1] \rightarrow -\frac{F - F\,\nu^2}{t\,E}\right\}$$

EXAMPLE 2.6 Consider a plate strip loaded by pressure p acting on the upper surface. Determine the deflection w at the center point according to the Kirchhoff model. Thickness, length and width of the plate are t, L, and H, respectively. Young's modulus E, and Poisson's ratio ν are constants. Use the one parameter approximation $w(x) = a_0(1 - x/L)^2(x/L)^2$.



Answer: $w = -\dfrac{1}{32}(\dfrac{L}{t})^3 \dfrac{Lp}{E}(1-\nu^2)$

- Approximation satisfies the displacement boundary conditions ‘a priori’ and contains a free parameter a_0 (not associated with any node) to be solved by using the principle of virtual work:

$$w = a_0(1-\frac{x}{L})^2(\frac{x}{L})^2 \quad \Rightarrow \quad \frac{\partial^2 w}{\partial x^2} = a_0 \frac{2}{L^2}[1-6\frac{x}{L}+6(\frac{x}{L})^2] \text{ and } \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial x \partial y} = 0.$$

- When the approximation is substituted there, virtual work densities (formulae collection) simplify to

$$\delta w_{\Omega}^{\text{int}} = -a_0 \delta a_0 \frac{Et^3}{3(1-\nu^2)} \frac{1}{L^4}[1-6\frac{x}{L}+6(\frac{x}{L})^2]^2,$$

$$\delta w_{\Omega}^{\text{ext}} = -\delta a_0 (1-\frac{x}{L})^2(\frac{x}{L})^2 p.$$

- Integrations over the domain $\Omega =]0,L[\times]0,H[$ give the virtual works of internal and external forces

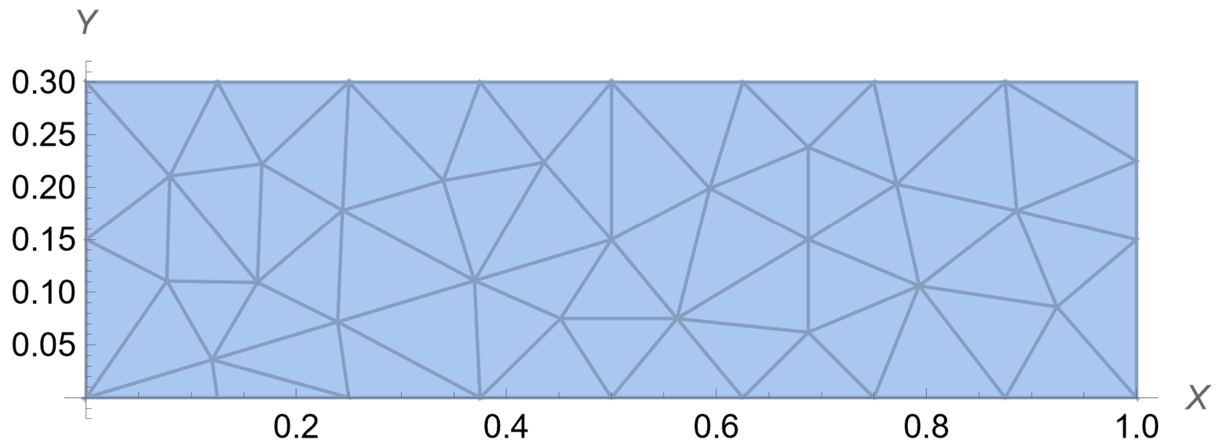
$$\delta W^{\mathrm{int}} = \int_\Omega \delta w_\Omega^{\mathrm{int}} d\Omega = -a_0 \delta a_0 \frac{1}{15}\frac{HEt^3}{L^3(1-\nu^2)},$$

$$\delta W^{\mathrm{ext}} = \int_\Omega \delta w_\Omega^{\mathrm{ext}} d\Omega = -\delta a_0 \frac{1}{30} pLH .$$

- Principle of virtual work $\delta W = \delta W^{\mathrm{int}} + \delta W^{\mathrm{ext}} = 0 \;\; \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus give finally $\forall \delta a_0$

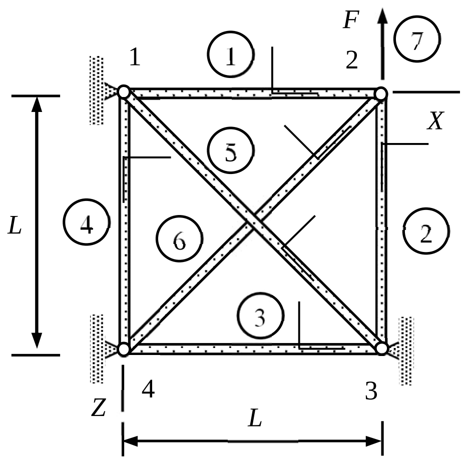
$$\delta W = -\delta a_0 \left(\frac{1}{15}\frac{HEt^3}{L^3(1-\nu^2)} a_0 + \frac{1}{30} pLH\right) = 0 \quad \Leftrightarrow \quad a_0 = -\frac{1}{2}\frac{pL^4}{Et^3}(1-\nu^2).$$

- The problem can be solved numerically also by using the Reissner-Mindlin plate model and plate bending element of the Mathematica code. For example, assuming parameter values $p(L/t)^3/E=10$, $\nu=0.33$, $H/L=0.3$, and $t/L=0.01$ (a thin plate), the one parameter approximation to displacement gives $w/L=-0.278$ at the centerpoint whereas the solution on a regular (rough) mesh of about 300 unknown displacement/rotation components gives $w/L=-0.278$ (a fine mesh gives $w/L=-0.289$)



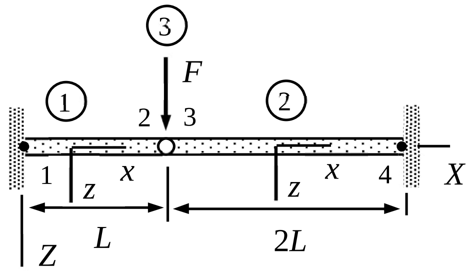
MEC-E8001 Finite Element Analysis, week 3/2023

1. Determine the nodal displacements when force F is acting on the structure as shown. The cross-sectional area of bars 1,2,3 and 4 is A and the cross-sectional area of bars 5 and 6 is $2\sqrt{2}A$. Young's modulus of the material is E. Use the principle of virtual work.



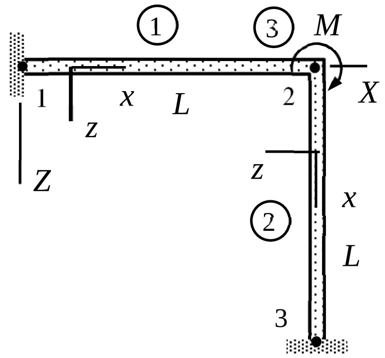
Answer $u_{X2} = -\dfrac{1}{3}\dfrac{FL}{EA}$, $u_{Z2} = -\dfrac{2}{3}\dfrac{FL}{EA}$

2. Joint between the beams of the figure is frictionless. Force F acting on the joint and displacement of the beam are restricted to the XZ-plane. Determine the rotations and displacement at the joint. Use two beam elements. The second moment of area I and Young's modulus of the material E are constants.



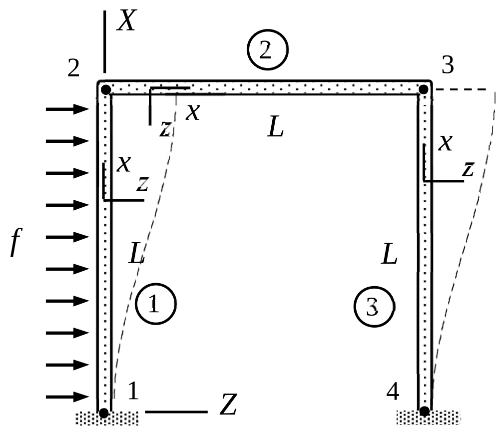
Answer $u_{Z2} = \dfrac{8}{27}\dfrac{FL^3}{EI}$, $\theta_{Y2} = -\dfrac{4}{9}\dfrac{FL^2}{EI}$ (short), and $\theta_{Y3} = \dfrac{2}{9}\dfrac{FL^2}{EI}$ (long).

3. Determine the rotation θ_{Y2} at node 2 of the structure loaded by a point moment (magnitude M) acting on node 2. Use beam elements (1) and (2) of equal length and a point moment element (3). Assume that the beams are inextensible in the axial directions. Young's modulus of material E and the second moment of area I are constants.



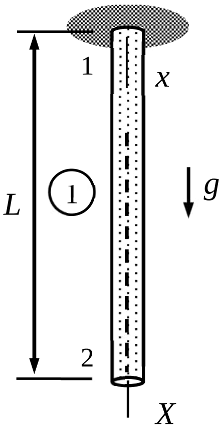
Answer $\theta_{Y2} = -\dfrac{1}{8}\dfrac{LM}{EI}$

4. Frame of the figure consists of a rigid body (2) and beam elements (1) and (3). Determine the non-zero displacements and rotations. The beams are identical and can be assumed rigid in the axial directions. Displacements are confined to the XZ-plane. Young's modulus E, second moment of area I, and distributed force f acting on element 1 are constants.



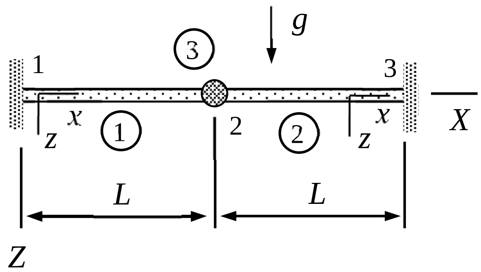
Answer $u_{Z2} = \dfrac{1}{48}\dfrac{fL^4}{EI}$

5. Consider a bar of length L loaded by its own weight (figure). Determine the displacement u_{X2} at the free end. Start with the virtual work density expression $\delta w_{\Omega} = -(d\delta u / dx)EA(du / dx) + \delta u f_x$ and approximation $u = (1 - x / L)u_{x1} + (x / L)u_{x2}$. Cross-sectional area A, acceleration by gravity g, and material properties E and ρ are constants.



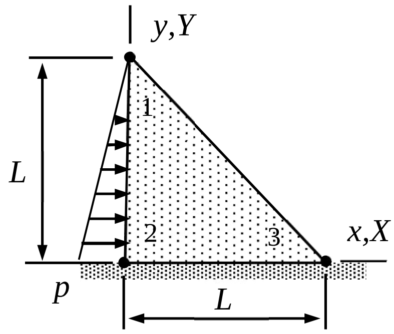
Answer $u_{X2} = \dfrac{\rho g L^2}{2E}$

6. The XZ-plane structure shown consists of two *massless* beams and a homogeneous disk considered as a rigid body. Determine the displacement u_{Z2} and rotation θ_{Y2} at node 2. Young's modulus E of the beam material and the second moment of area I are constants.



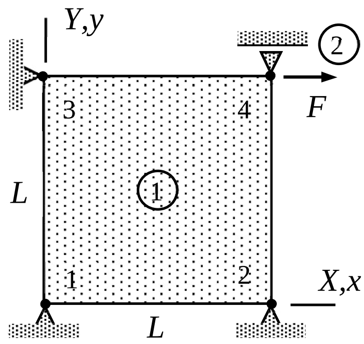
Answer $u_{Z2} = \dfrac{1}{24}\dfrac{mgL^3}{EI}$ and $\theta_{Y2} = 0$

7. A long dam of homogeneous, isotropic, linearly elastic material, is subjected to water pressure on one side. Material properties E and ν are constants. Determine the displacement components u_{X1} and u_{Y1} of node 1. Nodes 2 and 3 are fixed. Use a three-node element and assume plane strain conditions. Consider a slab of thickness t in calculations. The peak value of the linearly varying pressure is p.



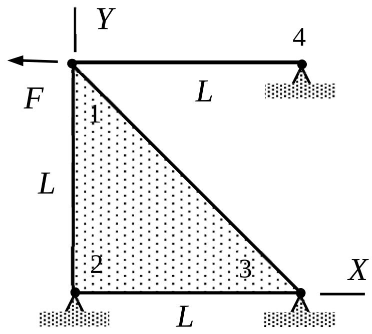
Answer $u_{X1} = \dfrac{2}{3}\dfrac{pL}{E}(1+\nu)$, $u_{Y1} = 0$

8. A thin slab (1) of square shape is loaded by a point force (2) as shown in the figure. Derive the relationship between the force F and the displacement u_{X4} of its point of action. Young's modulus E, Poisson's ratio ν, and thickness of the slab t are constants. The external distributed forces are zeros. Assume plane-stress conditions and use bilinear approximation.



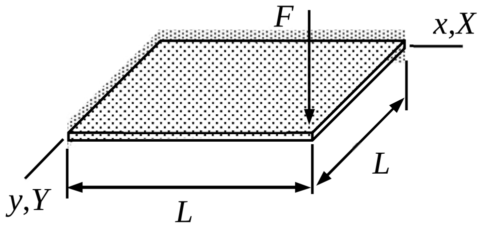
Answer $u_{X4} = \dfrac{6F}{Et}\dfrac{1-\nu^2}{3-\nu}$

9. A structure, consisting of a thin slab and a bar, is loaded by a horizontal force F acting on node 1. Material properties are E and ν, thickness of the slab is t and the cross-sectional area of the bar is A. Determine displacement of node 1 u_{X1} and u_{Y1} by using a linear bar element and a linear plane-stress element.



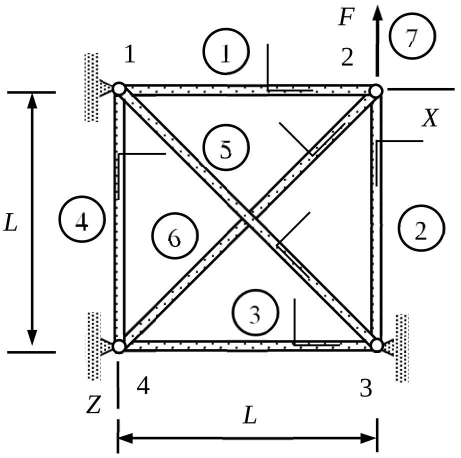
Answer $u_{X1} = -4\dfrac{L(1+\nu)}{Lt+4A(1+\nu)}\dfrac{F}{E}$ and $u_{Y1} = 0$

10. A plate, loaded by point force F acting at the free corner, is simply supported on two edges and free on the other two edges as shown in the figure. Determine the parameter a_0 of approximation $w(x,y) = a_0(x/L)(y/L)$ and displacement at the center point. Use the virtual work density of the plate bending mode with constant E, ν, ρ and t.



Answer $a_0 = 6\dfrac{FL^2}{Et^3}(1+\nu)$, $w(\dfrac{L}{2},\dfrac{L}{2}) = \dfrac{3}{2}\dfrac{FL^2}{Et^3}(1+\nu)$

Determine the nodal displacements when force F is acting on the structure as shown. The cross-sectional area of bars 1,2,3 and 4 is A and the cross-sectional area of bars 5 and 6 is $2\sqrt{2}A$. Young's modulus of the material is E. Use the principle of virtual work.



Solution

Element and node tables contain the information needed in displacement and stress analysis of the structure. In hand calculations, it is often enough to complete the figure by the material coordinate systems and express the nodal displacements/rotations in terms symbols for the nodal displacements and rotations and/or values known a priori. The components in the material coordinate systems can also be deduced directly form the figure (in simple cases). Virtual work expression of the bar element is given by

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \right).$$

Nodal displacements/rotations of the structure are zeros except u_{X2} and u_{Z2}. Element contributions in their virtual work forms are (nodal displacements of the material coordinate system need to be expressed in terms of the structural system components)

Bar 1: $u_{x1} = 0$, $u_{x2} = u_{X2}$: $\quad \delta W^1 = -\delta u_{X2} \dfrac{EA}{L} u_{X2}$,

Bar 2: $u_{x2} = u_{Z2}$, $u_{x3} = 0$: $\quad \delta W^2 = -\delta u_{Z2} \dfrac{EA}{L} u_{Z2}$,

Bar 3: $u_{x4} = 0$ and $u_{x3} = 0$: $\quad \delta W^3 = 0$,

Bar 4: $u_{x1} = 0$ and $u_{x4} = 0$: $\quad \delta W^4 = 0$,

Bar 5: $u_{x1} = 0$ and $u_{x3} = 0$: $\quad \delta W^5 = 0$,

Bar 6: $u_{x4} = 0$, $u_{x2} = \dfrac{1}{\sqrt{2}}(u_{X2} - u_{Z2})$: $\quad \delta W^6 = -(\delta u_{X2} - \delta u_{Z2}) \dfrac{EA}{L} (u_{X2} - u_{Z2})$

Force 7: $\quad \delta W^7 = -\delta u_{Z2} F$.

Virtual work expression of the structure is sum of the element contributions

$$\delta W = \sum_e \delta W^e = \delta W^1 + \delta W^2 + \delta W^3 + \delta W^4 + \delta W^5 + \delta W^6 + \delta W^7 \Rightarrow$$

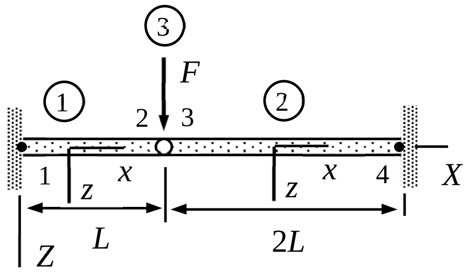
$$\delta W = -\delta u_{X2}\frac{EA}{L}u_{X2} - \delta u_{Z2}\frac{EA}{L}\mathrm{a}_2 + 0 + 0 + 0 - (\delta u_{X2} - \delta u_{Z2})\frac{EA}{L}(u_{X2} - u_{Z2}) - \delta u_{Z2}F \quad \Leftrightarrow$$

$$\delta W = -\begin{Bmatrix} \delta u_{X2} \\ \delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} (\frac{EA}{L}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{Z2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ -F \end{Bmatrix}).$$

Principle of virtual work $\delta W = 0 \, \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus in the form $\delta \mathbf{a}^{\mathrm{T}} \mathbf{R} = 0 \;\; \forall \delta \mathbf{a} \;\Leftrightarrow\; \mathbf{R} = 0$ imply

$$\frac{EA}{L}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{Z2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ -F \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix} u_{X2} \\ u_{Z2} \end{Bmatrix} = \frac{FL}{EA}\frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} 0 \\ -1 \end{Bmatrix} = -\frac{FL}{EA}\begin{Bmatrix} 1/3 \\ 2/3 \end{Bmatrix}.$$

Joint between the beams of the figure is frictionless. Force F acting on the joint and displacement of the beam are restricted to the XZ – plane. Determine the rotations and displacement at the joint. Use two beam elements. The second moment of area I and Young's modulus of the material E are constants.



Solution

Only the displacement in Z – direction and rotation in Y – direction matter in the planar beam bending problem. Rotation may not be continuous at the joint and, therefore, a double node with labels 2 and 3 are introduced there. At the joint, displacement is continuous and therefore $u_{Z3} = u_{Z2}$.

For element 1, the non-zero displacement/rotation components of the material coordinate system are $u_{z2} = u_{Z2}$ and $\theta_{y2} = \theta_{Y2}$. The element contribution of a xz – plane beam in bending (formulae collection) takes the form (the

$$\delta W^1 = -\begin{Bmatrix} 0 \\ 0 \\ \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_{Z2} \\ \theta_{Y2} \end{Bmatrix} = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{2L^3} \begin{bmatrix} 24 & 12L & 0 \\ 12L & 8L^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \\ \theta_{Y3} \end{Bmatrix}.$$

For element 2, the non-zero displacement/rotation components of the material coordinate system are $u_{z2} = u_{Z3} = u_{Z2}$ and $\theta_{y2} = \theta_{Y3}$. The element contribution is

$$\delta W^2 = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y3} \\ 0 \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{EI}{2L^3} \begin{bmatrix} 3 & -3L & -3 & -3L \\ -3L & 4L^2 & 3 & 2L^2 \\ -3 & 3L & 3 & 3L \\ -3L & 2L^2 & 3L & 4L^2 \end{bmatrix} \begin{Bmatrix} u_{Z2} \\ \theta_{Y3} \\ 0 \\ 0 \end{Bmatrix} = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{2L^3} \begin{bmatrix} 3 & 0 & -3L \\ 0 & 0 & 0 \\ -3L & 0 & 4L^2 \end{bmatrix} \begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \\ \theta_{Y3} \end{Bmatrix}.$$

Element 3 is a point force whose virtual work expression follows from the definition of work

$$\delta W^3 = F\delta u_{Z2} = \begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} F \\ 0 \\ 0 \end{Bmatrix}.$$

Virtual work expression of a structure is the sum of the element contributions

$$\delta W = \delta W^1 + \delta W^2 + \delta W^3 = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EI}{2L^3} \begin{bmatrix} 27 & 12L & -3L \\ 12L & 8L^2 & 0 \\ -3L & 0 & 4L^2 \end{bmatrix} \begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \\ \theta_{Y3} \end{Bmatrix} - \begin{Bmatrix} F \\ 0 \\ 0 \end{Bmatrix}\right).$$

Principle of virtual work $\delta W = 0 \quad \forall \delta\mathbf{a}$ and the fundamental lemma of variation calculus imply the linear equation system

$$\frac{EI}{2L^3}\begin{bmatrix} 27 & 12L & -3L \\ 12L & 8L^2 & 0 \\ -3L & 0 & 4L^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \\ \theta_{Y3} \end{Bmatrix}-\begin{Bmatrix} F \\ 0 \\ 0 \end{Bmatrix}=0.$$

Solving a system of linear equations is one of the basic tasks in FEM (reduction to a triangular system by row operations works well in hand calculations). Multiply the first row by 4 and the third row by 3/L to get

$$\frac{EI}{2L^3}\begin{bmatrix} 108 & 48L & -12L \\ 12L & 8L^2 & 0 \\ -9 & 0 & 12L \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \\ \theta_{Y3} \end{Bmatrix}-\begin{Bmatrix} 4F \\ 0 \\ 0 \end{Bmatrix}=0.$$

Add the last row to the first row to get

$$\frac{EI}{2L^3}\begin{bmatrix} 99 & 48L & 0 \\ 12L & 8L^2 & 0 \\ -9 & 0 & 12L \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \\ \theta_{Y3} \end{Bmatrix}-\begin{Bmatrix} 4F \\ 0 \\ 0 \end{Bmatrix}=0.$$

Then multiply the second row by $-6/L$

$$\frac{EI}{2L^3}\begin{bmatrix} 99 & 48L & 0 \\ -72 & -48L & 0 \\ -9 & 0 & 12L \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \\ \theta_{Y3} \end{Bmatrix}-\begin{Bmatrix} 4F \\ 0 \\ 0 \end{Bmatrix}=0$$

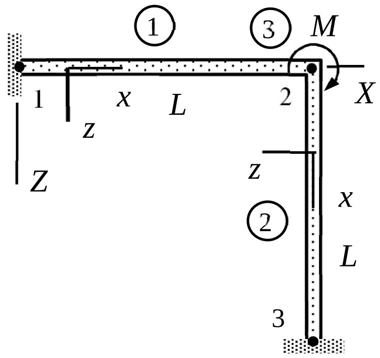
and add the second row to the first row to get

$$\frac{EI}{2L^3}\begin{bmatrix} 27 & 0 & 0 \\ -72 & -48L & 0 \\ -9 & 0 & 12L \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \\ \theta_{Y3} \end{Bmatrix}-\begin{Bmatrix} 4F \\ 0 \\ 0 \end{Bmatrix}=0.$$

After these steps, the matrix is a lower diagonal one, and solution follows by considering the equations in a proper order one at a time:

$$u_{Z2}=\frac{8FL^3}{27EI},\quad \theta_{Y2}=-\frac{72}{48}\frac{1}{L}u_{Z2}=-\frac{4}{9}\frac{FL^2}{EI}\quad \text{and}\quad \theta_{Y3}=\frac{9}{12L}u_{Z2}=\frac{2}{9}\frac{FL^2}{EI}.\ \leftarrow$$

Determine the rotation θ_{Y2} at node 2 of the structure loaded by a point moment (magnitude M) acting on node 2. Use beam elements (1) and (2) of equal length and a point moment element (3). Assume that the beams are inextensible in the axial directions. Young's modulus of material E and the second moment of area I are constants.



Solution

In a planar problem, torsion and out-plane bending deformation modes can be omitted. As beams are assumed to be inextensible in the axial direction and there are no axial distributed forces, the bar mode virtual work expression vanishes. Virtual work expressions of the beam xz – plane bending element and point force/moment elements are given by

$$\delta W = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} (\frac{EI_{yy}}{h^3}\begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix}\begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix} - \frac{f_z h}{12}\begin{Bmatrix} 6 \\ -h \\ 6 \\ h \end{Bmatrix}),$$

$$\delta W = \begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \\ \delta u_{Z1} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} F_{X1} \\ F_{Y1} \\ F_{Z1} \end{Bmatrix} + \begin{Bmatrix} \delta\theta_{X1} \\ \delta\theta_{Y1} \\ \delta\theta_{Z1} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} M_{X1} \\ M_{Y1} \\ M_{Z1} \end{Bmatrix}.$$

Nodal displacements/rotations of the structure are clearly zeros except those for node 2. Displacement at node 2 vanishes also as both beams are inextensible in the axial directions. Therefore, the only non-zero displacement/rotation component of the structure is θ_{Y2}.

Beam 1: $u_{z1} = 0$, $\theta_{y1} = 0$, $u_{z2} = 0$, and $\theta_{y2} = \theta_{Y2}$

$$\delta W^1 = -\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \theta_{Y2} \end{Bmatrix} = -\delta\theta_{Y2} 4\frac{EI}{L}\theta_{Y2}.$$

Beam 2: $u_{z1} = 0$, $\theta_{y1} = \theta_{Y2}$, $u_{z2} = 0$, and $\theta_{y2} = 0$

$$\delta W^2 = -\begin{Bmatrix} 0 \\ \delta\theta_{Y2} \\ 0 \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ \theta_{Y2} \\ 0 \\ 0 \end{Bmatrix} = -\delta\theta_{Y2} 4\frac{EI}{L}\theta_{Y2}.$$

Point moment 3:

$$\delta W^3 = \begin{Bmatrix} \delta u_{X2} \\ \delta u_{Y2} \\ \delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} F_{X2} \\ F_{Y2} \\ F_{Z2} \end{Bmatrix} + \begin{Bmatrix} \delta\theta_{X2} \\ \delta\theta_{Y2} \\ \delta\theta_{Z2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} M_{X2} \\ M_{Y2} \\ M_{Z2} \end{Bmatrix} = -\delta\theta_{Y2} M \,.$$

Virtual work expression of the structure is sum of the element contributions

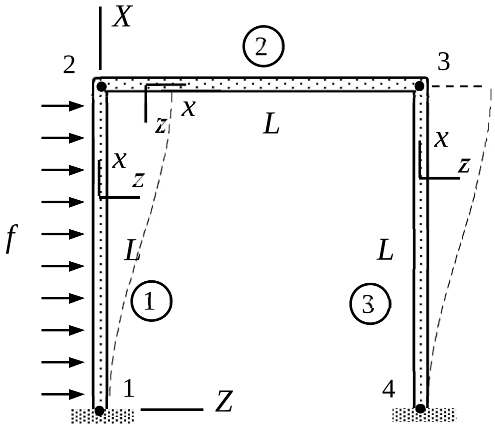
$$\delta W = \sum\nolimits_e \delta W^e = \delta W^1 + \delta W^2 + \delta W^3 \Rightarrow$$

$$\delta W = -\delta\theta_{Y2} 4\frac{EI}{L}\theta_{Y2} - \delta\theta_{Y2} 4\frac{EI}{L}\theta_{Y2} + 0 - \delta\theta_{Y2} M = -\delta\theta_{Y2}(8\frac{EI}{L}\theta_{Y2} + M)\,.$$

Principle of virtual work $\delta W = 0 \; \forall \delta\mathbf{a}$ and the fundamental lemma of variation calculus in the form $\delta\mathbf{a}^{\mathrm{T}}\mathbf{R} = 0 \;\; \forall \delta\mathbf{a} \;\Leftrightarrow\; \mathbf{R} = 0$ imply

$$8\frac{EI}{L}\theta_{Y2} + M = 0 \quad \Leftrightarrow \quad \theta_{Y2} = -\frac{1}{8}\frac{ML}{EI}\,. \;\leftarrow$$

The frame of the figure consists of a rigid body (2) and beam elements (1) and (3). Determine the non-zero displacements and rotations. The beams are identical and can be assumed rigid in the axial directions. Displacements are confined to the XZ – plane. Young's modulus E, second moment of area I, and distributed force f acting on element 1 are constants.



Solution

As element 2 is a rigid body and the other beam are rigid in the axial directions, only the horizontal displacement components $u_{Z3} = u_{Z2}$ are non-zeros. Element contributions to the virtual work expression are

$$\delta W^1 = -\begin{Bmatrix} 0 \\ 0 \\ \delta u_{Z2} \\ 0 \end{Bmatrix}^{\mathrm{T}} \left(\frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ u_{Z2} \\ 0 \end{Bmatrix} - \frac{fL}{12}\begin{Bmatrix} 6 \\ -L \\ 6 \\ L \end{Bmatrix}\right) = -\delta u_{Z2}\left(12\frac{EI}{L^3}u_{Z2} - \frac{fL}{2}\right),$$

$$\delta W^2 = 0,$$

$$\delta W^3 = -\begin{Bmatrix} 0 \\ 0 \\ \delta u_{Z2} \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ u_{Z2} \\ 0 \end{Bmatrix} = -\delta u_{Z2} 12\frac{EI}{L^3}u_{Z2}.$$

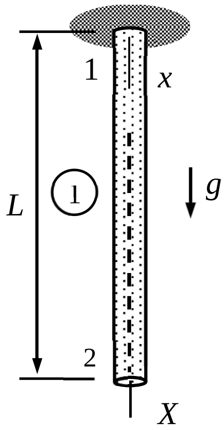
Virtual work expression of the structure is the sum of element contributions

$$\delta W = \delta W^1 + \delta W^2 + \delta W^3 = -\delta u_{Z2}\left(24\frac{EI}{L^3}u_{Z2} - \frac{fL}{2}\right).$$

Principle of virtual work $\delta W = 0 \ \forall \delta \mathrm{a}$ and the fundamental lemma of variation calculus imply

$$u_{Z2} = \frac{1}{48}\frac{fL^4}{EI}.$$

Consider a bar of length L loaded by its own weight (figure). Determine the displacement u_{X2} at the free end. Start with the virtual work density expression $\delta w_\Omega = -(d\delta u/dx)EA(du/dx)+\delta u f_x$ and approximation $u=(1-x/L)u_{x1}+(x/L)u_{x2}$. Cross-sectional area A, acceleration by gravity g and material properties E and ρ are constants.



Solution

The concise representation of the element contribution consists of a virtual work density expression and approximations to the displacement and rotation components. Approximations are just substituted into the density expression followed by integration over the domain occupied by the element (line segment, triangle etc.). Here the two building blocks are

$$\delta w_\Omega = -\frac{d\delta u}{dx}EA\frac{du}{dx}+\delta u f_x \quad \text{and} \quad u=(1-\frac{x}{L})u_{x1}+\frac{x}{L}u_{x2}.$$

The quantities needed in the virtual work density are the axial displacement, variation of the axial displacement, and variation of the derivative of the axial displacement

$$u=\begin{Bmatrix}1-x/L\\ x/L\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix} \quad\Rightarrow\quad \delta u=\begin{Bmatrix}1-x/L\\ x/L\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}=\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}1-x/L\\ x/L\end{Bmatrix},$$

$$\frac{du}{dx}=\frac{1}{L}\begin{Bmatrix}-1\\ 1\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix} \quad\Rightarrow\quad \frac{d\delta u}{dx}=\frac{1}{L}\begin{Bmatrix}-1\\ 1\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}=\frac{1}{L}\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}-1\\ 1\end{Bmatrix}.$$

When the approximation is substituted there, virtual work density expression of the bar model takes the form

$$\delta w_\Omega = -\frac{d\delta u}{dx}EA\frac{du}{dx}+\delta u f_x = -\frac{1}{L}\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}-1\\ 1\end{Bmatrix}EA\frac{1}{L}\begin{Bmatrix}-1\\ 1\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix}+\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}1-x/L\\ x/L\end{Bmatrix}f \quad\Leftrightarrow$$

$$\delta w_\Omega = -\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}(\begin{Bmatrix}-1\\ 1\end{Bmatrix}\frac{1}{L}EA\frac{1}{L}\begin{Bmatrix}-1\\ 1\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix}-\begin{Bmatrix}1-x/L\\ x/L\end{Bmatrix}f) \quad\Leftrightarrow$$

$$\delta w_\Omega = -\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}(\frac{EA}{L^2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix}-\begin{Bmatrix}1-x/L\\ x/L\end{Bmatrix}f)\ .$$

Finally, integration over the element gives the virtual work expression of the bar element

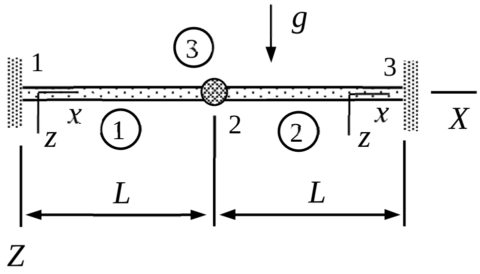
$$\delta W=\int_0^L \delta w_\Omega dx = -\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}(\frac{EA}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix}-\frac{fL}{2}\begin{Bmatrix}1\\ 1\end{Bmatrix}).$$

Finding the displacement of the free end follows the usual lines. Here, $f_x = \rho g A$, $u_{x1} = u_{X1} = 0$, and $u_{x2} = u_{X2}$

$$\delta W = -\begin{Bmatrix} 0 \\ \delta u_{X2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_{X2} \end{Bmatrix} - \frac{\rho g A L}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}\right) = -\delta u_{X2}\left(\frac{EA}{L}u_{X2} - \frac{\rho g A L}{2}\right) = 0 \quad \forall \delta u_{X2} \quad \Leftrightarrow$$

$$\frac{EA}{L}u_{X2} - \frac{\rho g A L}{2} = 0 \quad \Leftrightarrow \quad u_{X2} = \frac{\rho g L^2}{2E}.$$

The $XZ-$plane structure shown consists of two *massless* beams and a homogeneous disk of mass *m* considered as a rigid body. Determine the displacement u_{Z2} and rotation θ_{Y2} at node 2. Young's modulus *E* of the beam material and the second moment of area I are constants.



Solution

Only the displacement in the $Z-$direction and rotation in the $Y-$direction matter in the planar beam bending problem. From the figure, the non-zero displacement and rotation components are u_{Z2} and θ_{Y2}. For element 1, the non-zero displacement/rotation components of the material coordinate system are $u_{z2}=u_{Z2}$ and $\theta_{y2}=\theta_{Y2}$. The element contribution of a plane beam in bending (formulae collection) is

$$\delta W^1 = -\begin{Bmatrix} 0 \\ 0 \\ \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ u_{Z2} \\ \theta_{Y2} \end{Bmatrix} = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3}\begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix}.$$

For element 2, the non-zero displacement/rotation components of the material coordinate system are $u_{z2}=u_{Z2}$ and $\theta_{y2}=\theta_{Y2}$. The element contribution of a *xz*-plane beam in bending is

$$\delta W^2 = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \\ 0 \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \\ 0 \\ 0 \end{Bmatrix} = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3}\begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix}.$$

Element 3 is a rigid body. In static displacement analysis, only the weight acting at the mass centroid matters. Virtual work expression of the point force of magnitude mg follows from the definition of work

$$\delta W^3 = mg\delta u_{Z2} = \begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix} mg \\ 0 \end{Bmatrix}.$$

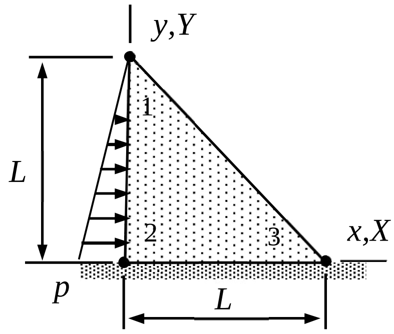
Virtual work expression of a structure is the sum of the element contributions

$$\delta W = \delta W^1 + \delta W^2 + \delta W^3 = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EI}{L^3}\begin{bmatrix} 24 & 0 \\ 0 & 8L^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix} - \begin{Bmatrix} mg \\ 0 \end{Bmatrix}\right).$$

Principle of virtual work $\delta W = 0 \;\; \forall \delta\mathbf{a}$ and the fundamental lemma of variation calculus imply the linear equation system

$$\frac{EI}{L^3}\begin{bmatrix} 24 & 0 \\ 0 & 8L^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix} - \begin{Bmatrix} mg \\ 0 \end{Bmatrix} = 0 \quad \Leftrightarrow \quad u_{Z2} = \frac{1}{24}\frac{mgL^3}{EI} \quad \text{and} \quad \theta_{Y2} = 0 .$$

A long dam of homogeneous, isotropic, linearly elastic material, is subjected to water pressure on one side. Material properties E and ν are constants. Determine the displacement components u_{X1} and u_{Y1} of node 1. Nodes 2 and 3 are fixed. Use a three-node element and assume plane strain conditions. Consider a slab of thickness t in calculations. The peak value of the linearly varying pressure is p.



Solution

Under the plane strain conditions, the virtual work densities of thin slab are

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta u/\partial y + \partial\delta v/\partial x \end{Bmatrix}^{\mathrm{T}} t[E]_{\varepsilon} \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix} \text{ and } \delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix} \text{ where}$$

$$[E]_{\varepsilon} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix}.$$

The external forces t_x and t_y (force per unit length in this case) acting on the element edges can be taken into account by a separate force element with the density expression (per unit length)

$$\delta w_{\partial\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} t_x \\ t_y \end{Bmatrix}$$

although the expression is actually part of the thin slab model. The approximation on the boundary is just the restriction of the element approximation to the boundary.

Only the shape function for node 1 is needed as the other nodes are fixed (displacement vanishes). In terms of the displacement components u_{X1} and u_{Y1} of node 1, element approximations of the displacement components and their derivatives are

$$u = \frac{y}{L} u_{X1} \quad \Rightarrow \quad \frac{\partial u}{\partial x} = 0 \text{ and } \frac{\partial u}{\partial y} = \frac{1}{L} u_{X1},$$

$$v = \frac{y}{L} u_{Y1} \quad \Rightarrow \quad \frac{\partial v}{\partial x} = 0 \text{ and } \frac{\partial v}{\partial y} = \frac{1}{L} u_{Y1}.$$

When the approximation is substituted there, the virtual work densities simplify to

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} 0 \\ \delta u_{Y1}/L \\ \delta u_{X1}/L \end{Bmatrix}^{\mathrm{T}} \frac{Et}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix} \begin{Bmatrix} 0 \\ u_{Y1}/L \\ u_{X1}/L \end{Bmatrix} \Rightarrow$$

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \dfrac{Et}{2(1+\nu)L^2} & 0 \\ 0 & \dfrac{Et(1-\nu)}{(1+\nu)(1-2\nu)L^2} \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix},$$

$$\delta w_{\partial\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} t_x \\ t_y \end{Bmatrix} = \begin{Bmatrix} \delta u_{X1} y / L \\ \delta u_{Y1} y / L \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} pt(1-y/L) \\ 0 \end{Bmatrix} = \begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} pt(1-y/L)y/L \\ 0 \end{Bmatrix}.$$

Integrations over the element and edge 2-1 give the virtual work expressions (notice that the virtual work density of internal forces is constant)

$$\delta W^{\text{int}} = \int_{\Omega} \delta w_{\Omega}^{\text{int}} d\Omega = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \dfrac{Et}{4(1+\nu)} & 0 \\ 0 & \dfrac{Et(1-\nu)}{2(1+\nu)(1-2\nu)} \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix},$$

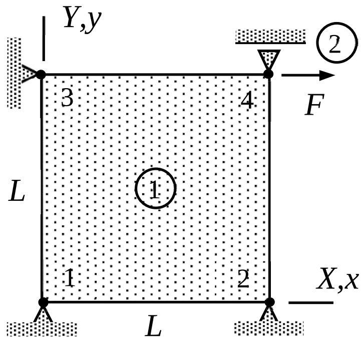
$$\delta W^{\text{ext}} = \int_0^L \delta w_{\partial\Omega}^{\text{ext}} dy = \begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} ptL/6 \\ 0 \end{Bmatrix}.$$

Principle of virtual work $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = 0 \quad \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus give

$$\delta W = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} (\begin{bmatrix} \dfrac{Et}{4(1+\nu)} & 0 \\ 0 & \dfrac{Et(1-\nu)}{2(1+\nu)(1-2\nu)} \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} - \begin{Bmatrix} ptL/6 \\ 0 \end{Bmatrix}) = 0 \qquad \Rightarrow$$

$$\begin{bmatrix} \dfrac{Et}{4(1+\nu)} & 0 \\ 0 & \dfrac{Et(1-\nu)}{2(1+\nu)(1-2\nu)} \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} - \begin{Bmatrix} ptL/6 \\ 0 \end{Bmatrix} = 0 \qquad \Leftrightarrow$$

$u_{X1} = \dfrac{2}{3} \dfrac{pL}{E}(1+\nu)$ and $u_{Y1} = 0$.



A thin slab (1) of square shape is loaded by a point force (2) as shown in the figure. Derive the relationship between the force F and the displacement u_{X4} of its point of action. Young's modulus E, Poisson's ratio ν, and thickness of the slab t are constants. The external distributed forces are zeros. Assume plane stress conditions and use bilinear approximation.

Solution

Let us start with the shape functions of element 1 and approximations. As nodes 1, 2, and 3 are fixed, it is enough to deduce the shape function of node 4

$$N_4 = \frac{xy}{L^2} .$$

Approximations to the displacement components and their derivatives with respect to x and y are

$$u = \frac{xy}{L^2} u_{X4}, \quad \frac{\partial u}{\partial x} = \frac{y}{L^2} u_{X4}, \text{ and } \frac{\partial u}{\partial y} = \frac{x}{L^2} u_{X4}$$

$$v = 0, \quad \frac{\partial v}{\partial x} = 0, \text{ and } \frac{\partial v}{\partial y} = 0 .$$

When the approximations are substituted there, the virtual work density of thin slab model simplifies to (plane stress conditions, only the internal part is needed)

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta u / \partial x \\ \partial\delta v / \partial y \\ \partial\delta u / \partial y + \partial\delta v / \partial x \end{Bmatrix}^{\mathrm{T}} \frac{tE}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix} \quad \Rightarrow$$

$$\delta w_{\Omega}^{\text{int}} = -\delta u_{X4} \frac{tE}{1-\nu^2} \frac{1}{L^4} (y^2 + \frac{1-\nu}{2} x^2) u_{X4} .$$

Integration over the domain occupied by the element gives the element contribution

$$\delta W^1 = \int_0^L \int_0^L \delta w_{\Omega}^{\text{int}} dxdy = -\delta u_{X4} \frac{Et}{6} \frac{3-\nu}{1-\nu^2} u_{X4} .$$

Virtual work expression of the point force (element 2) follows from the definition of work

$$\delta W^2 = \delta u_{X4} F .$$

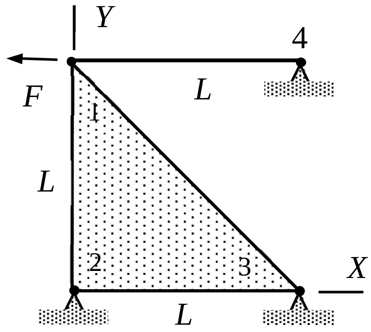
Virtual work expression of a structure is the sum of element contributions

$$\delta W = \delta W^1 + \delta W^2 = \delta u_{X4} (-\frac{Et}{6} \frac{3-\nu}{1-\nu^2} u_{X4} + F) .$$

Finally, principle of virtual work in the form $\delta W = 0 \quad \forall \delta \mathrm{a}$ and the fundamental lemma of variation calculus imply that

$$u_{X4} = \frac{6F}{Et}\frac{1-\nu^2}{3-\nu}.$$

A structure, consisting of a thin slab and a bar, is loaded by a horizontal force F acting on node 1. Material properties are E and ν, thickness of the slab is t, and the cross-sectional area of the bar A are constants. Determine displacement components u_{X1} and u_{Y1} of node 1 by using a linear bar element and a linear plane-stress element.



Solution

Under the plane stress conditions, the virtual work densities (virtual works per unit area) of the thin slab model

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta u/\partial y + \partial\delta v/\partial x \end{Bmatrix}^{\mathrm{T}} t[E]_\sigma \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix} \text{ and } \delta w_\Omega^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix} \text{where}$$

$$[E]_\sigma = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}$$

take into account the internal forces (stress) and external forces acting on the element domain. Notice that the components f_x and f_y are external forces per unit area. Forces acting on the element edges can be taken into account by separate force elements.

Element contribution for the thin slab needs to be derived from approximation and virtual work densities. Approximations to the displacement components depend only on the shape function associated with node 1 as the other nodes are fixed (displacement vanishes). In terms of the displacement components u_{X1} and u_{Y1}

$$u = u_{X1}\frac{y}{L} \quad \Rightarrow \quad \frac{\partial u}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} = u_{X1}\frac{1}{L},$$

$$v = u_{Y1}\frac{y}{L} \quad \Rightarrow \quad \frac{\partial v}{\partial x} = 0 \quad \text{and} \quad \frac{\partial v}{\partial y} = u_{Y1}\frac{1}{L}.$$

Virtual work density of the internal forces simplifies to (when the approximations are substituted there)

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} 0 \\ \delta u_{Y1} \\ \delta u_{X1} \end{Bmatrix}^{\mathrm{T}} \frac{1}{L^2}\frac{Et}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\begin{Bmatrix} 0 \\ u_{Y1} \\ u_{X1} \end{Bmatrix} = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \frac{1}{L^2}\frac{Et}{1-\nu^2}\begin{bmatrix} (1-\nu)/2 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix}$$

Virtual work expression is the integral of density over the domain occupied by the element (note that the virtual work density is constant in this case). Therefore

$$\delta W^1 = \int_\Omega \delta w_\Omega^{\text{int}} d\Omega = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \frac{1}{2}\frac{Et}{1-\nu^2}\begin{bmatrix} (1-\nu)/2 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix}.$$

Virtual work expression of the bar element is given in the formula collection with $u_{x1} = u_{X1}$ and $u_{x2} = 0$

$$\delta W^2 = -\begin{Bmatrix} \delta u_{X1} \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X1} \\ 0 \end{Bmatrix} = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \frac{EA}{L}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix}.$$

Virtual work expression of the point force follows e.g. directly from the definition (force multiplied by the virtual displacement in its direction)

$$\delta W^3 = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} F \\ 0 \end{Bmatrix}.$$

Virtual work expression of the structure is the sum of element contributions $\delta W = \delta W^1 + \delta W^2 + \delta W^3$

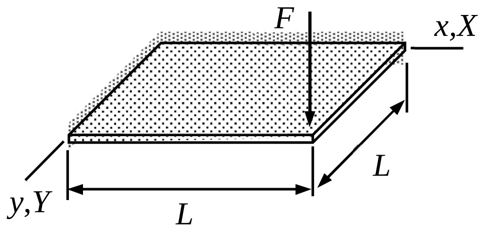
$$\delta W = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} (\frac{1}{2}\frac{Et}{1-\nu^2}\begin{bmatrix} (1-\nu)/2 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} + \frac{EA}{L}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} + \begin{Bmatrix} F \\ 0 \end{Bmatrix}) \quad \Leftrightarrow$$

$$\delta W = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} (\begin{bmatrix} \frac{1}{4}\frac{Et}{1+\nu} + \frac{EA}{L} & 0 \\ 0 & \frac{1}{2}\frac{Et}{1-\nu^2} \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} + \begin{Bmatrix} F \\ 0 \end{Bmatrix}).$$

Principle of virtual work $\delta W = 0 \quad \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus give

$$\begin{bmatrix} \frac{1}{4}\frac{Et}{1+\nu} + \frac{EA}{L} & 0 \\ 0 & \frac{1}{2}\frac{Et}{1-\nu^2} \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} + \begin{Bmatrix} F \\ 0 \end{Bmatrix} = 0 \quad \Leftrightarrow \quad u_{X1} = -\frac{4(1+\nu)L}{tL + 4(1+\nu)A}\frac{F}{E} \text{ and } u_{Y1} = 0.$$

A Kirchhoff plate, loaded by point force F acting at the free corner, is simply supported on two edges and free on the other two edges as shown in the figure. Determine the parameter a_0 of approximation $w(x,y)=a_0(x/L)(y/L)$ and displacement at the center point. Use the virtual work density of the Kirchhoff plate model with constant E, ν, ρ and t.



Solution

Assuming that the material coordinate system is chosen so that the plate bending and thin slab modes decouple, it is enough to consider the virtual work densities of the bending mode only

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial^2\delta w/\partial x^2 \\ \partial^2\delta w/\partial y^2 \\ 2\partial^2\delta w/\partial x\partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12}[E]_\sigma \begin{Bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ 2\partial^2 w/\partial x\partial y \end{Bmatrix}, \quad \delta w_\Omega^{\text{ext}} = \delta w f_z .$$

in which the elasticity matrix of plane stress

$$[E]_\sigma = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}.$$

In the present case, distributed force vanishes i.e. $f_z = 0$ and the point force is taken into account by a point force element.

Approximation to the transverse displacement is chosen to be (a_0 is not associated with any point but it just a parameter of the approximation)

$$w(x,y) = a_0 \frac{x}{L}\frac{y}{L} \quad \Rightarrow \quad \frac{\partial^2 w}{\partial x^2} = 0, \ \frac{\partial^2 w}{\partial y^2} = 0, \text{ and } \frac{\partial^2 w}{\partial x\partial y} = \frac{1}{L^2}a_0 .$$

When the approximation is substituted there, virtual work density of internal forces simplifies to

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial^2\delta w/\partial x^2 \\ \partial^2\delta w/\partial y^2 \\ 2\partial^2\delta w/\partial x\partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3 E}{12(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\begin{Bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ 2\partial^2 w/\partial x\partial y \end{Bmatrix} = -\delta a_0 \frac{Et^3}{6(1+\nu)}\frac{1}{L^4}a_0 ,$$

Virtual work expression of the plate bending element (element 1 here) is integral of the virtual work density over the domain occupied by the element

$$\delta W^1 = \int_0^L \int_0^L \delta w_\Omega^{\text{int}} dxdy = -\delta a_0 \frac{Et^3}{6(1+\nu)}\frac{1}{L^2}a_0 .$$

Virtual work expression of the point force (element 2 here) follows from the definition of work (notice the use of virtual displacement of the point of action $x = y = L$)

$$\delta W^2 = \delta w(L,L)F = \delta a_0 F \ .$$

Principle of virtual work and the fundamental lemma of variation calculus give

$$\delta W = \delta W^1 + \delta W^2 = -\delta a_0 \left(\frac{Et^3}{6(1+\nu)}\frac{1}{L^2}a_0 - F\right) = 0 \quad \Rightarrow \quad a_0 = 6(1+\nu)\frac{FL^2}{Et^3} \ .$$

Displacement at the center point

$$w\left(\frac{L}{2},\frac{L}{2}\right) = a_0\frac{1}{4} = \frac{3}{2}(1+\nu)\frac{FL^2}{Et^3} \ . \quad \leftarrow$$

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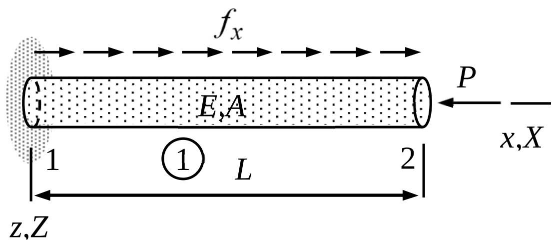
Assignment 1

Find the displacement u_{X2} of the bar shown. Left end of the bar (node 1) is fixed and the given external force P is acting on node 2. Young's modulus E and cross-sectional area A are constants and distributed force $f_x = 3P/L$.



Solution template

The bar element contribution is

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \right),$$

in which A is the cross-sectional area, E is the Young's modulus, and f_x is the external distributed force in $x-$direction. The point force/moment element contribution is given by

$$\delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \\ \delta u_{Z1} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \underline{F}_X \\ \underline{F}_Y \\ \underline{F}_Z \end{Bmatrix} + \begin{Bmatrix} \delta \theta_{X1} \\ \delta \theta_{Y1} \\ \delta \theta_{Z1} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \underline{M}_X \\ \underline{M}_Y \\ \underline{M}_Z \end{Bmatrix}.$$

When the known nodal displacement of node 1 and the relationship $u_{x2} = u_{X2}$ are used there, the bar element contribution (element 1 here) simplifies to

$$\delta W^1 = \delta W^1 = -\begin{Bmatrix} 0 \\ \delta u_{X2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ u_{X2} \end{Bmatrix} - \begin{Bmatrix} 3P/2 \\ 3P/2 \end{Bmatrix} \right) = -\delta u_{X2} \left(\frac{EA}{L} u_{X2} - \frac{3}{2} P \right).$$

The force element contribution (element 2 here) simplifies to

$$\delta W^2 = -\delta u_{X2} P.$$

Virtual work expression of a structure is the sum of the element contributions

$$\delta W = \delta W^1 + \delta W^2 = -\delta u_{X2} \left(\frac{EA}{L} u_{X2} - \frac{1}{2} P \right).$$

Principle of virtual work and the fundamental lemma of variation calculus imply the equilibrium equation

$$\frac{EA}{L}u_{X2} - \frac{1}{2}P = 0\,.$$

Solution to the nodal displacement is given by

$$u_{X2} = \frac{1}{2}\frac{PL}{EA}\,.$$

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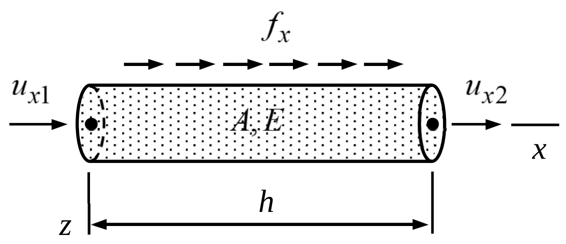
Assignment 2

Consider a bar element when A and E and distributed force f_x is the linear distributed force. Derive the virtual work expression of *internal* forces starting with the approximation $u=(1-x/h)u_{x1}+(x/h)u_{x2}$ and the virtual work density expressions $\delta w_{\Omega}^{\text{int}}=-(d\delta u/dx)EA(du/dx)$ and $\delta w_{\Omega}^{\text{ext}}=\delta u f_x$ of the bar mode.



Solution template

Displacement quantities in the virtual work density:

$$u=\begin{Bmatrix}1-x/h\\ x/h\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix}\quad\Rightarrow\quad\frac{du}{dx}=\begin{Bmatrix}-1/h\\ 1/h\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix},$$

$$\delta u=\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}1-x/h\\ x/h\end{Bmatrix}\quad\Rightarrow\quad\frac{d\delta u}{dx}=\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}-1/h\\ 1/h\end{Bmatrix}.$$

When the approximation is substituted there, virtual work density of internal forces $\delta w_{\Omega}^{\text{int}}$ becomes

$$\delta w_{\Omega}^{\text{int}}=-\frac{d\delta u}{dx}EA\frac{du}{dx}=-\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}\dfrac{EA}{h^2} & -\dfrac{EA}{h^2}\\ -\dfrac{EA}{h^2} & \dfrac{EA}{h^2}\end{bmatrix}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix}$$

Virtual work of internal forces δW^{int} is the integral of $\delta w_{\Omega}^{\text{int}}$ over the domain occupied by the element

$$\delta W^{\text{int}}=\int_0^h \delta w_{\Omega}^{\text{int}}dx=-\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}\dfrac{EA}{h} & -\dfrac{EA}{h}\\ -\dfrac{EA}{h} & \dfrac{EA}{h}\end{bmatrix}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix}.$$

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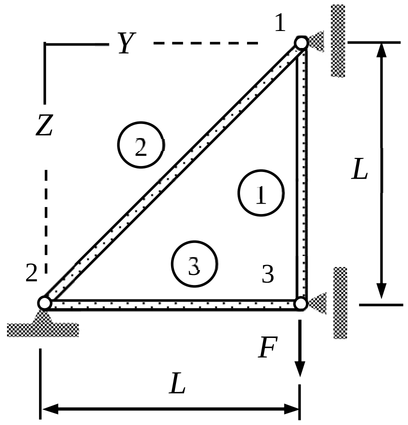
Assignment 3



The structure shown consists of three elastic bars connected by joints and a point force acting on node 3. Young's modulus of the material is E. The cross-sectional area of bars 1 and 3 is A and that for bar 2 $\sqrt{2}A$. Determine the displacement components u_{Z1} and u_{Z3}.

Solution template

Virtual work expression of the bar element is of the form

$$\delta W = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \right).$$

In the problem, the structure is loaded only by the point force so $f_x = 0$. To express the axial components (in the virtual work expression above) in terms of those in the structural coordinate system, one has to assign a material coordinate system to each bar element.

For element 1, let the $x-$axis be aligned from node 1 to 3. In terms of displacement components in the structural system, the displacement components in the direction of the bar axis are

$u_{x1} = u_{Z1}$ and $u_{x3} = u_{Z3}$.

In terms of displacement components in the structural system, element 1 contribution takes the form

$$\delta W^1 = -\begin{Bmatrix} \delta u_{Z1} \\ \delta u_{Z3} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix} \begin{Bmatrix} u_{Z1} \\ u_{Z3} \end{Bmatrix}.$$

For element 2, let the $x-$axis be aligned from node 2 to 1. In terms of displacement components in the structural system, the displacement components in the direction of the bar axis are

$u_{x2} = 0$ and $u_{x1} = -\frac{1}{\sqrt{2}} u_{Z1}$.

In terms of displacement components in the structural system, element 2 contribution takes the form

$$\delta W^2 = -\begin{Bmatrix} 0 \\ -\dfrac{\delta u_{Z1}}{\sqrt{2}} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \dfrac{EA}{L} & -\dfrac{EA}{L} \\ -\dfrac{EA}{L} & \dfrac{EA}{L} \end{bmatrix} \begin{Bmatrix} 0 \\ -\dfrac{u_{Z1}}{\sqrt{2}} \end{Bmatrix} = -\delta u_{Z1} \frac{EA}{2L} u_{Z1}.$$

For element 3, let the $x-$axis be aligned from node 2 to 3. In terms of displacement components in the structural system, the displacement components in the direction of the bar axis are

$u_{x2} = 0$ and $u_{x3} = 0$.

In terms of displacement components in the structural system, element 3 contribution takes the form

$$\delta W^3 = -\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \dfrac{EA}{L} & -\dfrac{EA}{L} \\ -\dfrac{EA}{L} & \dfrac{EA}{L} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = 0.$$

Virtual work expression of a point force (taken as element 4) follows from the definition of work

$$\delta W^4 = \delta u_{Z3} F .$$

Virtual work expression of the structure is sum of the element contributions

$$\delta W = \delta W^1 + \delta W^2 + \delta W^3 + \delta W^4 = -\begin{Bmatrix} \delta u_{Z1} \\ \delta u_{Z3} \end{Bmatrix}^{\mathrm{T}} \left(\begin{bmatrix} \dfrac{3}{2}\dfrac{EA}{L} & -\dfrac{EA}{L} \\ -\dfrac{EA}{L} & \dfrac{EA}{L} \end{bmatrix} \begin{Bmatrix} u_{Z1} \\ u_{Z3} \end{Bmatrix} + \begin{Bmatrix} 0 \\ -F \end{Bmatrix} \right).$$

Principle of virtual work $\delta W = 0 \; \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus in the form $\delta \mathbf{a}^{\mathrm{T}} \mathbf{R} = 0 \;\; \forall \delta \mathbf{a} \;\Leftrightarrow\; \mathbf{R} = 0$ imply

$$\begin{bmatrix} \dfrac{3}{2}\dfrac{EA}{L} & -\dfrac{EA}{L} \\ -\dfrac{EA}{L} & \dfrac{EA}{L} \end{bmatrix} \begin{Bmatrix} u_{Z1} \\ u_{Z3} \end{Bmatrix} + \begin{Bmatrix} 0 \\ -F \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix} u_{Z1} \\ u_{Z3} \end{Bmatrix} = \begin{Bmatrix} 2\dfrac{FL}{EA} \\ 3\dfrac{FL}{EA} \end{Bmatrix}.$$

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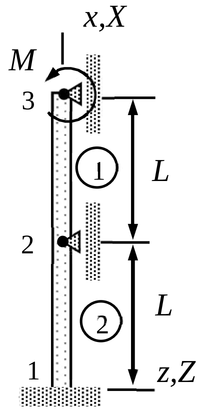
Assignment 4



Beam structure of the figure is loaded by a point moment acting on node 3. Determine the rotations θ_{Y2} and θ_{Y3} by using two beam bending elements. Displacements are confined to the XZ-plane. The cross-section properties of the beam A, I and Young's modulus of the material E are constants.

Solution template

Virtual work expression for the displacement analysis consists of parts coming from internal and external forces δW^{int} and δW^{ext}. For the beam bending mode in xz-plane, the virtual work expressions are

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\text{T}} \frac{EI_{yy}}{h^3} \begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix}, \quad \delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\text{T}} \frac{f_z h}{12} \begin{Bmatrix} 6 \\ -h \\ 6 \\ h \end{Bmatrix},$$

The element contribution of the point force/moment follows from the definition or work and is given by

$$\delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \\ \delta u_{Z1} \end{Bmatrix}^{\text{T}} \begin{Bmatrix} \underline{F}_X \\ \underline{F}_Y \\ \underline{F}_Z \end{Bmatrix} + \begin{Bmatrix} \delta\theta_{X1} \\ \delta\theta_{Y1} \\ \delta\theta_{Z1} \end{Bmatrix}^{\text{T}} \begin{Bmatrix} \underline{M}_X \\ \underline{M}_Y \\ \underline{M}_Z \end{Bmatrix}.$$

Distributed force $f_z = 0$ and $I_{yy} = I$ in the present problem. In the first step of analysis, the virtual work expressions (given in material coordinate systems of the element) are written in terms of the nodal displacements and rotation components in the structural coordinate system. As the coordinate axes of the two systems are aligned, transformation is simple. Virtual work expression of beam 1

$$\delta W^1 = -\begin{Bmatrix} 0 \\ \delta\theta_{Y2} \\ 0 \\ \delta\theta_{Y3} \end{Bmatrix}^{\text{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ \theta_{Y2} \\ 0 \\ \theta_{Y3} \end{Bmatrix} = -\begin{Bmatrix} \delta\theta_{Y2} \\ \delta\theta_{Y3} \end{Bmatrix}^{\text{T}} \begin{bmatrix} 4\frac{EI}{L} & 2\frac{EI}{L} \\ 2\frac{EI}{L} & 4\frac{EI}{L} \end{bmatrix} \begin{Bmatrix} \theta_{Y2} \\ \theta_{Y3} \end{Bmatrix},$$

When written in the standard form having the $\delta -$ quantity vector as the multiplier, virtual work expression for beam 2 takes the form

$$\delta W^2 = -\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \theta_{Y2} \end{Bmatrix} = -\begin{Bmatrix} \delta\theta_{Y2} \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 4\frac{EI}{L} & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \theta_{Y2} \\ \theta_{Y3} \end{Bmatrix}.$$

Virtual work expression of the point moment (also written in the 'standard' form having the $\delta-$quantity vector as the multiplier)

$$\delta W^3 = -\begin{Bmatrix} \delta\theta_{Y2} \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ -M \end{Bmatrix}.$$

Virtual work expression of structure is sum of the element contributions, i.e.,

$$\delta W = \delta W^1 + \delta W^2 + \delta W^3 = -\begin{Bmatrix} \delta\theta_{Y2} \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \left(\begin{bmatrix} 8\frac{EI}{L} & 2\frac{EI}{L} \\ 2\frac{EI}{L} & 4\frac{EI}{L} \end{bmatrix} \begin{Bmatrix} \theta_{Y2} \\ \theta_{Y3} \end{Bmatrix} + \begin{Bmatrix} 0 \\ -M \end{Bmatrix} \right).$$

Principle of virtual work and the fundamental lemma of variation calculus imply that

$$\begin{bmatrix} 8\frac{EI}{L} & 2\frac{EI}{L} \\ 2\frac{EI}{L} & 4\frac{EI}{L} \end{bmatrix} \begin{Bmatrix} \theta_{Y2} \\ \theta_{Y3} \end{Bmatrix} + \begin{Bmatrix} 0 \\ -M \end{Bmatrix} = 0.$$

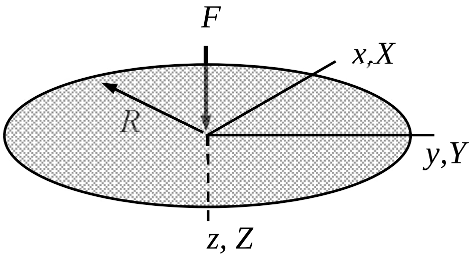
Solution to the linear equations system is given by

$$\theta_{Y2} = -\frac{1}{14}\frac{ML}{EI} \quad \text{and} \quad \theta_{Y3} = \frac{2}{7}\frac{ML}{EI}.$$

Name_________________________ Student number________________

Assignment 5

A circular plate of radius R, which is simply supported at the outer edge, is loaded by force F at the center point. Use the Kirchhoff plate model to find the transverse displacement at the center point. Use the approximation $w = a_0(x^2 + y^2 - R^2)$ for the transverse displacement. Material properties E, ν and thickness t are constants.



Solution

Assuming that the material coordinate system is chosen so that the plate bending and thin slab modes decouple, it is enough to consider the virtual work densities of the bending mode only

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial^2 \delta w / \partial x^2 \\ \partial^2 \delta w / \partial y^2 \\ 2\partial^2 \delta w / \partial x \partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12} [E]_\sigma \begin{Bmatrix} \partial^2 w / \partial x^2 \\ \partial^2 w / \partial y^2 \\ 2\partial^2 w / \partial x \partial y \end{Bmatrix}, \; \delta w_\Omega^{\text{ext}} = \delta w f_z .$$

in which the elasticity matrix of plane stress

$$[E]_\sigma = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}.$$

In the present case, distributed force vanishes i.e. $f_z = 0$ and the point force is taken into account by a point force element.

Approximation to the transverse displacement is given by (a_0 is not associated with any point but it just a parameter of the approximation)

$$w(x,y) = a_0(x^2 + y^2 - R^2) \quad \Rightarrow \quad \frac{\partial^2 w}{\partial x^2} = 2a_0, \; \frac{\partial^2 w}{\partial y^2} = 2a_0, \text{ and } \frac{\partial^2 w}{\partial x \partial y} = 0 .$$

When the approximation is substituted there, virtual work density of internal forces simplifies to

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} 2\delta a_0 \\ 2\delta a_0 \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{t^3 E}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} 2a_0 \\ 2a_0 \\ 0 \end{Bmatrix} = -\delta a_0 \frac{2}{3} \frac{t^3 E}{1-\nu} a_0 .$$

Virtual work expression of the plate bending element (element 1 here) is integral of the virtual work density over the domain occupied by the element. As the density expression is constant it is enough to multiply by the area of the domain

$$\delta W^1 = \int_\Omega \delta w_\Omega^{\text{int}} dA = \delta w_\Omega^{\text{int}} \pi R^2 = -\delta a_0 \frac{2\pi}{3} \frac{R^2 t^3 E}{1-\nu} a_0 .$$

Virtual work expression of the point force (element 2 here) follows from the definition of work (notice the use of virtual displacement at the point of action $x = y = 0$)

$$\delta W^2 = \delta w(0,0) F = -\delta a_0 R^2 F .$$

Principle of virtual work and the fundamental lemma of variation calculus give

$$\delta W = \delta W^1 + \delta W^2 = -\delta a_0 \left(\frac{2\pi}{3} \frac{R^2 t^3 E}{1-\nu} a_0 + R^2 F\right) = 0 \quad \Rightarrow \quad a_0 = -\frac{3}{2\pi} \frac{F}{t^3 E} (1-\nu) .$$

Displacement at the center point

$$w(0,0) = -a_0 R^2 = \frac{3}{2\pi} \frac{F R^2}{t^3 E} (1-\nu) .$$

MEC-E8001

Finite Element Analysis

2023

WEEK 4: VIBRATION ANALYSIS

3 VIBRATION ANALYSIS

3.1 LINEAR NON-STATIONARY ELASTICITY 6

3.2 PERIODIC MOTION AND VIBRATION 9

3.3 VIBRATION FEA 19

3.4 ELEMENT CONTRIBUTIONS 30

LEARNING OUTCOMES

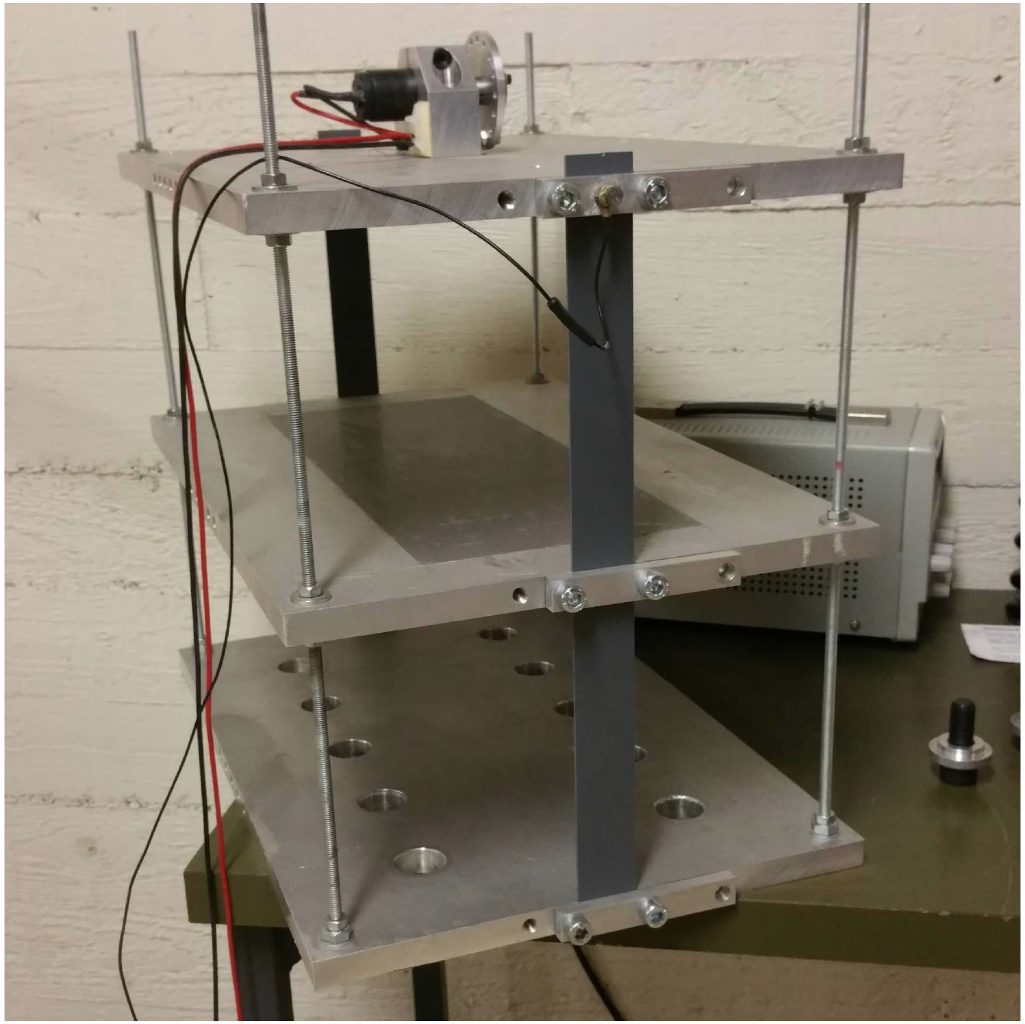
Students are able to solve the weekly lecture problems, home problems, and exercise problems related to vibration FEA:

- Vibration problem, natural frequencies and modes of vibration, solution to vibration problem as the function of time.

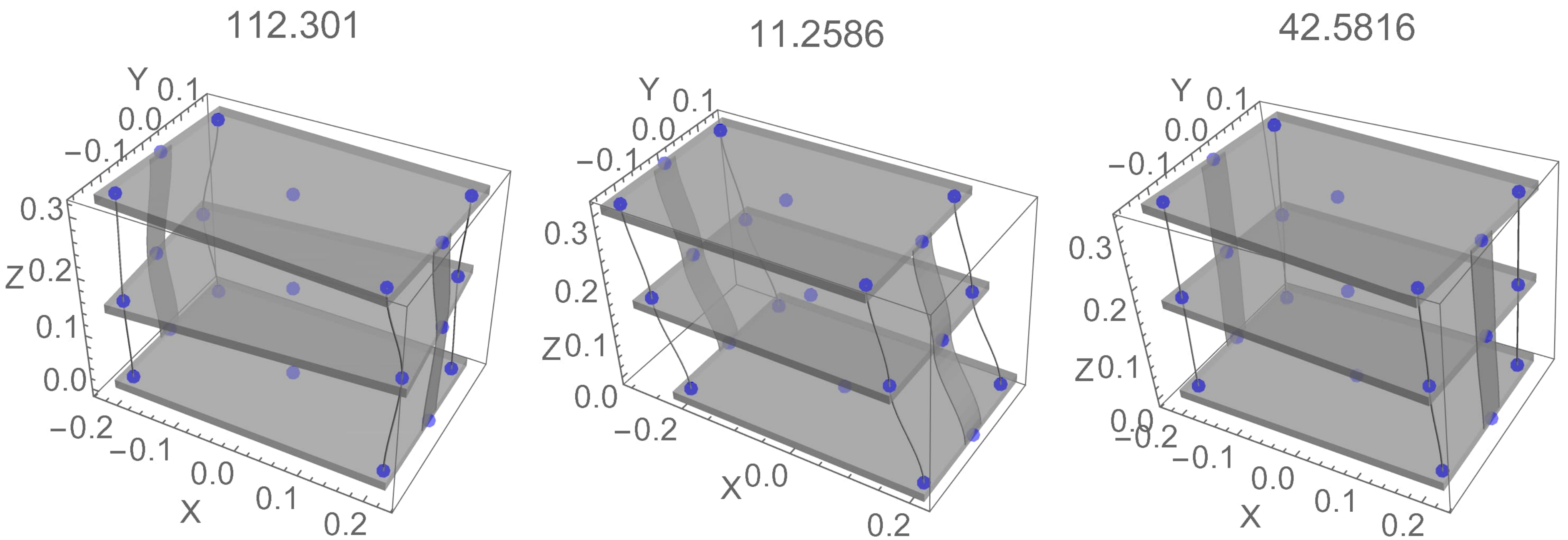
- Time dependent linear elasticity problem, principle of virtual work in a time-dependent case and vibration analysis by FEM.

- Inertia term element contributions for the solid-, beam-, plate-, and rigid body elements.

VIBRATION EXPERIMENT



TWO SMALLEST EIGENFREQUENCIES



Experiment	11.4 1/s	29.6 1/s
FEA	11.3 1/s	29.6 1/s

BALANCE LAWS OF MECHANICS

Balance of mass (def. of a body or a material volume) Mass of a body is constant

Balance of linear momentum (Newton 2) The rate of change of linear momentum within a material volume equals the external force resultant acting on the material volume. ←

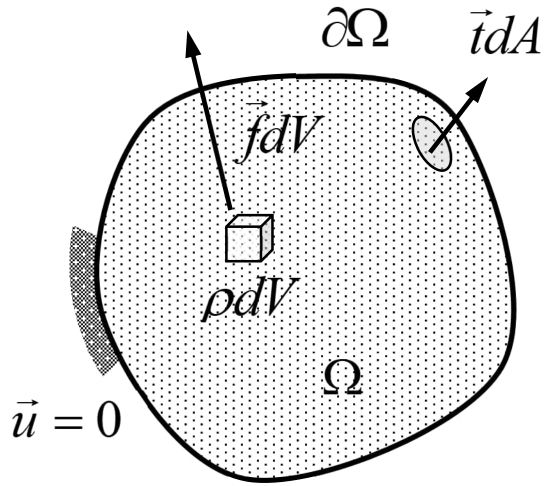
Balance of angular momentum (Cor. of Newton 2) The rate of change of angular momentum within a material volume equals the external moment resultant acting on the material volume. ←

Balance of energy (Thermodynamics 1)

Entropy growth (Thermodynamics 2)

3.1 LINEAR NON-STATIONARY ELASTICITY

Assuming equilibrium of a solid body (a set of particles) inside domain Ω, the aim is to find displacement $\vec{u}$ of the particles as functions of time, when external forces or boundary conditions are changed in some manner:



Equilibrium equations $\nabla \cdot \vec{\vec{\sigma}} + \vec{f} = \rho \ddot{\vec{u}}$ in Ω, $t > 0$

Hooke's law $\vec{\vec{\sigma}} = \frac{E}{1+\nu}\left(\frac{\nu}{1-2\nu}\vec{\vec{I}}\nabla \cdot \vec{u} + \vec{\vec{\varepsilon}}\right)$ in Ω, $t > 0$

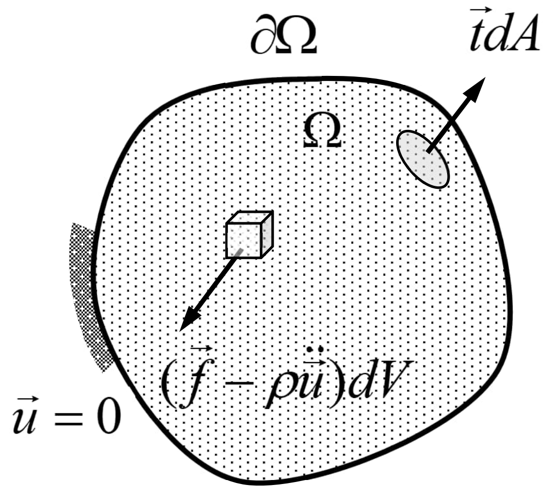
Boundary conditions $\vec{n} \cdot \vec{\vec{\sigma}} = \vec{t}$ or $\vec{u} = \vec{g}$ on $\partial\Omega$, $t > 0$

Initial conditions $\vec{u} = \vec{u}_0$ and $\dot{\vec{u}} = \dot{\vec{u}}_0$ in Ω at $t = 0$

The balance law of angular momentum is satisfied 'a priori' by the form of Hooke's law.

PRINCIPLE OF VIRTUAL WORK

Principle of virtual work $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} + \delta W^{\text{ine}} = 0 \;\; \forall \delta\vec{u}$ is just one form of the equations of motion, where



Internal forces: $\delta W^{\text{int}} = \int_{\Omega} \delta w_V^{\text{int}} dV$

External forces: $\delta W^{\text{ext}} = \int_{\Omega} \delta w_V^{\text{ext}} dV + \int_{\partial\Omega} \delta w_A^{\text{ext}} dA$

Inertia forces: $\delta W^{\text{ine}} = \int_{\Omega} \delta w_V^{\text{ine}} dV$.

In connection with the principle, time is considered as a parameter and inertia term is treated as a part of the volume force although it is not a force (it does not have a counterpart which is opposite in direction and equal in magnitude).

VIRTUAL WORK DENSITIES

Virtual work densities of the internal forces, inertia forces, external volume forces, and external surface forces are

$$\delta w_V^{\text{int}} = -\begin{Bmatrix} \delta\varepsilon_{xx} \\ \delta\varepsilon_{yy} \\ \delta\varepsilon_{zz} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} - \begin{Bmatrix} \delta\gamma_{xy} \\ \delta\gamma_{yz} \\ \delta\gamma_{zx} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix},$$

Partial derivatives
with respect to time

$$\delta w_V^{\text{ext}} = \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix}, \quad \delta w_A^{\text{ext}} = \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix}, \text{ and } \quad \delta w_V^{\text{ine}} = -\begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix}^{\mathrm{T}} \rho \begin{Bmatrix} \ddot{u}_x \\ \ddot{u}_y \\ \ddot{u}_z \end{Bmatrix}.$$

Virtual work densities consist of terms containing kinematic quantities and their “work conjugates” !

3.2 PERIODIC MOTION AND VIBRATION

Constrained motion: there exists c and C such that $c < x(t) < C$

Periodic motion: there exists T such that $x(t+T) = x(t)$ for any t

Vibration: "periodic motion near static equilibrium"

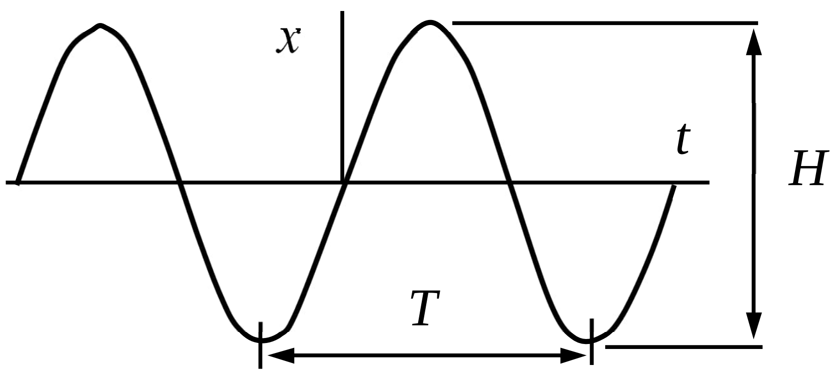
Harmonic vibration: $x(t) = X \sin \omega t$

Period T

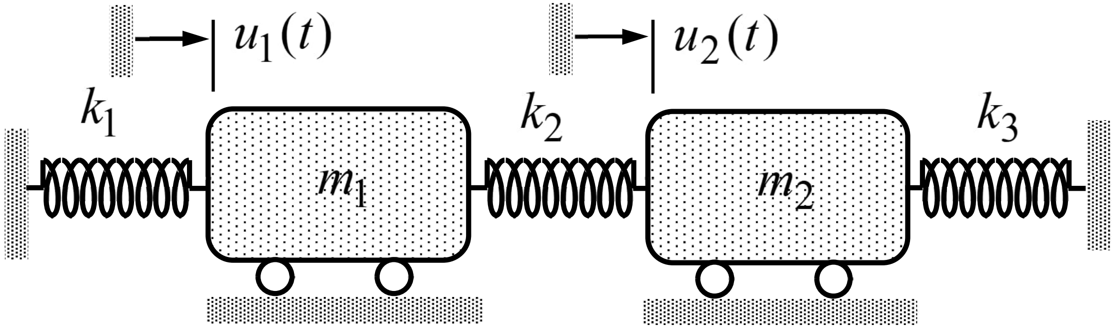
Frequency $f = 1/T$

Angular speed $\omega = 2\pi f$

Amplitude $X = H/2$



FREE UNDAMPED VIBRATION



Initial value problem $\mathbf{M}\ddot{\mathbf{a}}+\mathbf{K}\mathbf{a}=0 \quad t>0$, $\quad \dot{\mathbf{a}}=\dot{\mathbf{a}}_0 \quad t=0$, and $\mathbf{a}=\mathbf{a}_0 \quad t=0$

Solution $\quad \mathbf{a}(t)=\cos(\mathbf{\Omega}t)\mathbf{a}_0+\sin(\mathbf{\Omega}t)\mathbf{\Omega}^{-1}\dot{\mathbf{a}}_0$

Problem parameter $\mathbf{\Omega}=(\mathbf{M}^{-1}\mathbf{K})^{1/2}=\mathbf{X}\boldsymbol{\omega}\mathbf{X}^{-1}$

In practice, the main task is to find the eigenvalue decomposition $\mathbf{\Omega}=\mathbf{X}\boldsymbol{\omega}\mathbf{X}^{-1}$ or it's form $\mathbf{\Omega}^2=\mathbf{X}\boldsymbol{\omega}^2\mathbf{X}^{-1}$ (see: the definition of matrix function)!

EXAMPLE 3.1 Determine the angular speeds and modes of free vibrations, when the differential equations in their standard form are given by

$$\begin{Bmatrix} \ddot{a}_1 \\ \ddot{a}_2 \end{Bmatrix} + \mathbf{\Omega}^2 \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = 0 \quad \text{in which} \quad \mathbf{\Omega}^2 = \begin{bmatrix} 3 & -1/3 \\ -3 & 3 \end{bmatrix}.$$

Answer $\omega_1 = \sqrt{2}$, $\mathbf{x}_1 = \begin{Bmatrix} 1 \\ 3 \end{Bmatrix}$ and $\omega_2 = 2$, $\mathbf{x}_2 = \begin{Bmatrix} 1 \\ -3 \end{Bmatrix}$

- Characteristic equation $\det(\mathbf{\Omega}^2 - \lambda\mathbf{I}) = \det\begin{bmatrix} 3-\lambda & -1/3 \\ -3 & 3-\lambda \end{bmatrix} = (3-\lambda)^2 - 1 = 0$

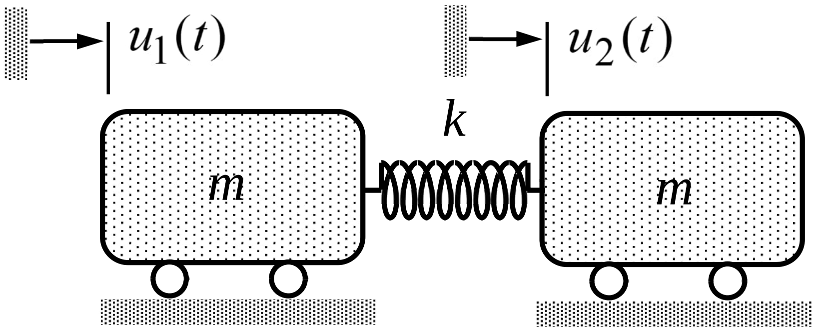
  Mode $\lambda_1 = 2$: $\begin{bmatrix} 3-2 & -1/3 \\ -3 & 3-2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0 \quad \Rightarrow \quad \mathbf{x}_1 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 3 \end{Bmatrix}$

  Mode $\lambda_2 = 4$: $\begin{bmatrix} 3-4 & -1/3 \\ -3 & 3-4 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0 \quad \Rightarrow \quad \mathbf{x}_2 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -3 \end{Bmatrix}$

- Eigenvalue decomposition $\mathbf{\Omega}^2 = \mathbf{X}\boldsymbol{\omega}^2\mathbf{X}^{-1} = \begin{bmatrix} 1 & 1 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 3 & -3 \end{bmatrix}^{-1}$

- Positive square root $\mathbf{\Omega} = \mathbf{X}\boldsymbol{\omega}\mathbf{X}^{-1} = \begin{bmatrix} 1 & 1 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} \sqrt{2} & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 3 & -3 \end{bmatrix}^{-1}$ ←

EXAMPLE 3.2 Write down the equations of motion for the system shown consisting of two particles and a spring. After that, determine the angular speeds and modes of free vibrations.



Answer $\omega_1 = 0$ and $\mathbf{x}_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$ (translation mode),

$\omega_2 = \sqrt{2\dfrac{k}{m}}$ and $\mathbf{x}_2 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$ (vibration mode).

- Matrices $\mathbf{M}=\begin{bmatrix}0 & m\\ m & 0\end{bmatrix}$ and $\mathbf{K}=\begin{bmatrix}-k & k\\ k & -k\end{bmatrix}$ give

$$\mathbf{\Omega}^2=\mathbf{M}^{-1}\mathbf{K}=\begin{bmatrix}0 & m\\ m & 0\end{bmatrix}^{-1}\begin{bmatrix}-k & k\\ k & -k\end{bmatrix}=\frac{k}{m}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

- Angular speeds of the free vibration modes are the eigenvalues of $\mathbf{\Omega}$. Let us calculate first the eigenvalues of $\mathbf{\Omega}^2$ and the corresponding modes

$$\det(\mathbf{\Omega}^2-\lambda\mathbf{I})=\det(\begin{bmatrix}\frac{k}{m}-\lambda & -\frac{k}{m}\\ -\frac{k}{m} & \frac{k}{m}-\lambda\end{bmatrix})=(\frac{k}{m}-\lambda)^2-(\frac{k}{m})^2=0 \quad\Rightarrow\quad \lambda\in\{0,2\frac{k}{m}\}.$$

$$\lambda_1 = 0 : \quad \begin{bmatrix} \frac{k}{m} - 0 & -\frac{k}{m} \\ -\frac{k}{m} & \frac{k}{m} - 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0 \quad \Rightarrow \quad \mathbf{x}_1 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix},$$

$$\lambda_2 = 2\frac{k}{m} : \quad \begin{bmatrix} \frac{k}{m} - 2\frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{k}{m} - 2\frac{k}{m} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0 \quad \Rightarrow \quad \mathbf{x}_2 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}.$$

- Angular speeds of the free vibrations and the corresponding modes are

$$\omega_1 = \sqrt{\lambda_1} = 0, \ \mathbf{x}_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \quad \text{and} \quad \omega_2 = \sqrt{\lambda_2} = \sqrt{2\frac{k}{m}}, \ \mathbf{x}_2 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}. \qquad \Leftarrow$$

In one-step methods for second order initial value problems, temporal domain is divided into sub-domains $t \in [t^{(i-1)}, t^{(i)}]$ $i \in \{1 \ldots n\}$. Differential equations are replaced by difference equations:

initial conditions

$$\begin{Bmatrix} \mathbf{a}_0 \\ \hline \mathbf{a}_1 \Delta t \end{Bmatrix}^{(i+1)} = \left[\begin{array}{c|c} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \hline \mathbf{A}_{21} & \mathbf{A}_{22} \end{array} \right] \begin{Bmatrix} \mathbf{a}_0 \\ \hline \mathbf{a}_1 \Delta t \end{Bmatrix}^{(i)} + \begin{Bmatrix} \mathbf{B}_1 \\ \hline \mathbf{B}_2 \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \mathbf{a}_0 \\ \hline \mathbf{a}_1 \Delta t \end{Bmatrix}^{(0)} = \begin{Bmatrix} \underline{\mathbf{a}}_0 \\ \hline \underline{\mathbf{a}}_1 \Delta t \end{Bmatrix}$$

Iteration on the difference equations gives values of the unknowns $\mathbf{a}_0^{(i)}$ and their first time-derivatives $\mathbf{a}_1^{(i)}$ at $t^{(i)}$ $i \in \{0 \ldots n\}$. Iteration matrix $\mathbf{A}$ depends on the mass matrix $\mathbf{M}$, stiffness matrix $\mathbf{K}$, and the step size Δt. Vector $\mathbf{B}$ depends also on the external forces.

ONE-STEP INTEGRATION METHODS

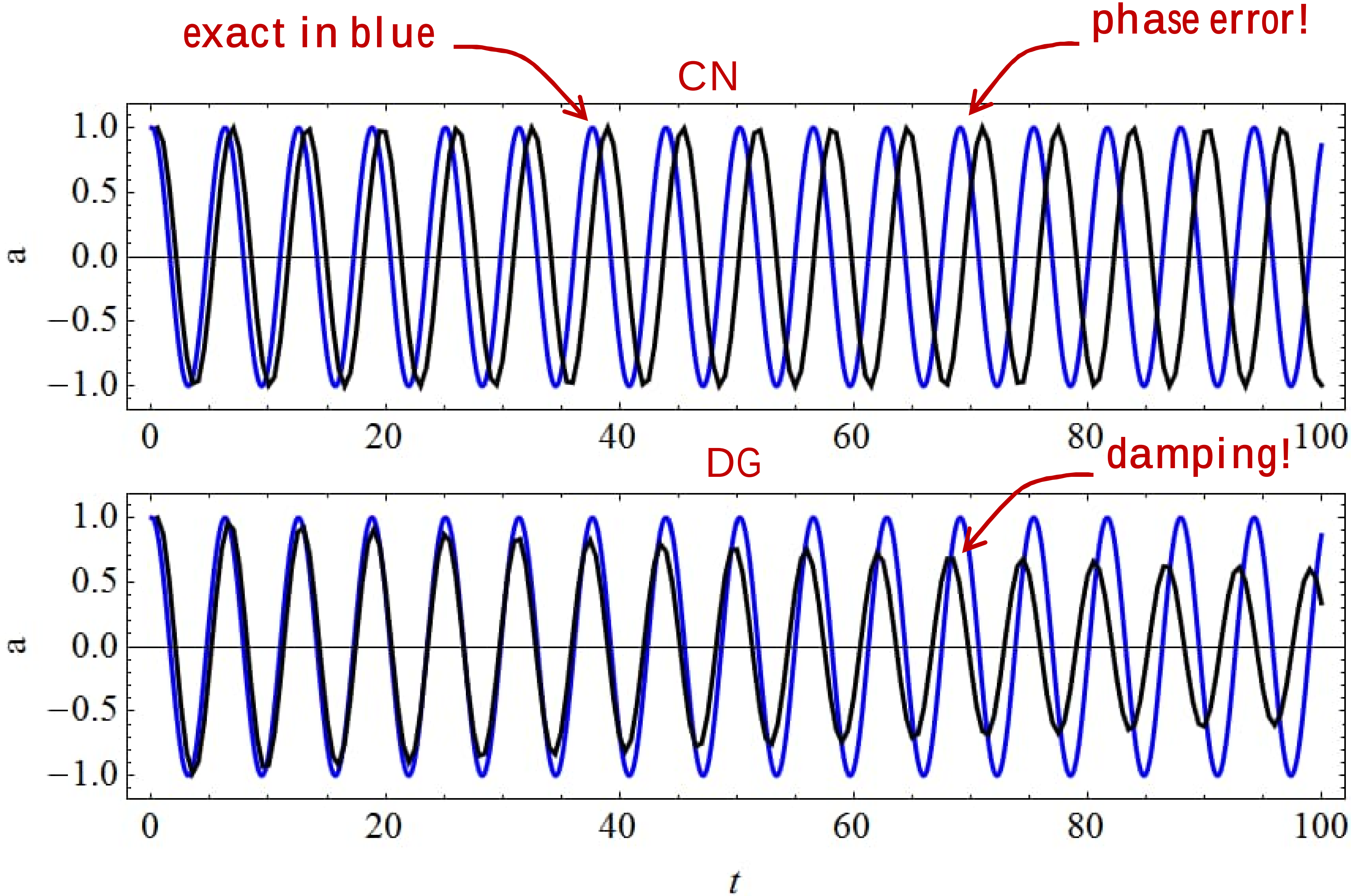
The recipes for a single equation and an equation systems are the same. For problem $\mathbf{M}\ddot{\mathbf{a}}+\mathbf{K}\mathbf{a}=0$

Crank-Nicholson: $$\begin{Bmatrix}\mathbf{a}_0\\ \mathbf{a}_1\Delta t\end{Bmatrix}^{(i+1)}=\begin{bmatrix}\mathbf{I} & -\mathbf{I}/2\\ \alpha/2 & \mathbf{I}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{I} & \mathbf{I}/2\\ -\alpha/2 & \mathbf{I}\end{bmatrix}\begin{Bmatrix}\mathbf{a}_0\\ \mathbf{a}_1\Delta t\end{Bmatrix}^{(i)}$$

Disc. Galerkin: $$\begin{Bmatrix}\mathbf{a}_0\\ \mathbf{a}_1\Delta t\end{Bmatrix}^{(i+1)}=\begin{bmatrix}\alpha & \mathbf{I}-\alpha/2\\ -\mathbf{I}-\alpha/2 & \alpha/3\end{bmatrix}^{-1}\begin{bmatrix}0 & \mathbf{I}\\ -\mathbf{I} & -\mathbf{I}\end{bmatrix}\begin{Bmatrix}\mathbf{a}_0\\ \mathbf{a}_1\Delta t\end{Bmatrix}^{(i)}$$

The proper step-size of depends on the largest eigenvalue of parameter $\alpha=\mathbf{M}^{-1}\mathbf{K}\Delta t^2$. A small amount of numerical damping is advantageous, if the step-size, according to the largest eigenvalue, becomes impractically small.

ACCURACY AND STABILITY $\Delta t = 1/2$ & $\alpha = 1/4$



3.3 VIBRATION FEA

- Model the structure as a collection of beam, plate, etc. elements. Derive the element contributions δW^e and express the nodal displacement and rotation components of the material coordinate system in terms of those in the structural coordinate system.

- Sum the element contributions to end up with the virtual work expression of the structure $\delta W = \sum_{e \in E} \delta W^e$. Re-arrange to get $\delta W = -\delta \mathbf{a}^{\mathrm{T}}(\mathbf{M}\ddot{\mathbf{a}} + \mathbf{K}\mathbf{a} - \mathbf{F})$.

- Use the principle of virtual work $\delta W = 0 \;\; \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus for $\delta \mathbf{a} \in \mathbb{R}^n$ to deduce the equations of motion $\mathbf{M}\ddot{\mathbf{a}} + \mathbf{K}\mathbf{a} - \mathbf{F} = 0$.

- Solve the equations for the natural angular speeds of vibrations and the corresponding modes or solve for the displacements and rotations as functions of time.

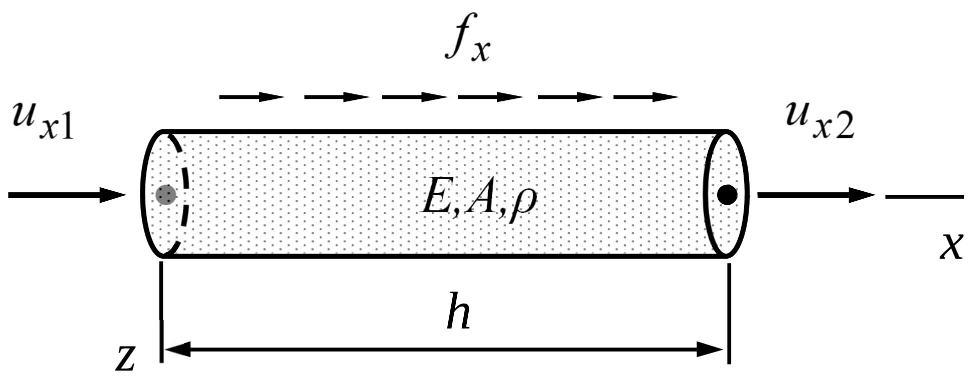
BAR MODE

Assuming a linear approximation to the axial displacement $u(x,t)$ with respect to x, virtual work expressions of the internal, external, and inertia forces take the forms

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix},$$

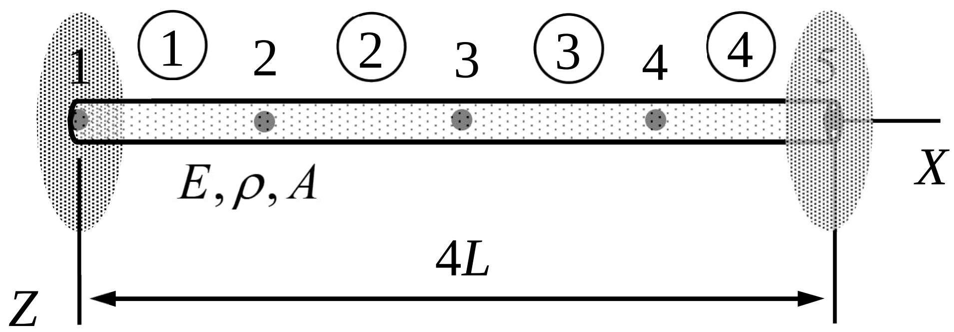
$$\delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{f_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix},$$

$$\delta W^{\text{ine}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{\rho A h}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_{x1} \\ \ddot{u}_{x2} \end{Bmatrix}.$$



Above, f_x and E, A, ρ are taken as constants.

EXAMPLE 3.3 Consider the free vibrations of the bar shown, when material properties E, ρ and cross-sectional area A are constants. Determine the set of ordinary differential equations giving as their solution the nodal displacements (assuming that initial displacement and velocity are known). Use four elements of equal size.



Answer $$\frac{EA}{L}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{X3} \\ u_{X4} \end{Bmatrix} + \frac{\rho AL}{6}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix}\begin{Bmatrix} \ddot{u}_{X2} \\ \ddot{u}_{X3} \\ \ddot{u}_{X4} \end{Bmatrix} = 0$$

- Let us assume that the structural and material coordinate systems coincide (for simplicity). Virtual work expressions of the elements taking into account the internal and inertia parts are

$$\delta W^1 = -\begin{Bmatrix} 0 \\ \delta u_{X2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_{X2} \end{Bmatrix} + \frac{\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} 0 \\ \ddot{u}_{X2} \end{Bmatrix}\right),$$

$$\delta W^2 = -\begin{Bmatrix} \delta u_{X2} \\ \delta u_{X3} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} + \frac{\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} \ddot{u}_{X2} \\ \ddot{u}_{X3} \end{Bmatrix}\right),$$

$$\delta W^3 = -\begin{Bmatrix} \delta u_{X3} \\ \delta u_{X4} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X3} \\ u_{X4} \end{Bmatrix} + \frac{\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} \ddot{u}_{X3} \\ \ddot{u}_{X4} \end{Bmatrix}\right),$$

$$\delta W^4 = -\begin{Bmatrix} \delta u_{X4} \\ 0 \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X4} \\ 0 \end{Bmatrix} + \frac{\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} \ddot{u}_{X4} \\ 0 \end{Bmatrix}\right).$$

- Virtual work expression of the structure is the sum of the element contributions $\delta W = \sum \delta W^e$. When expressed in the "standard form"

$$\delta W = -\begin{Bmatrix} \delta u_{X2} \\ \delta u_{X3} \\ \delta u_{X4} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{L} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{Bmatrix} u_{X2} \\ u_{X3} \\ u_{X4} \end{Bmatrix} + \frac{\rho AL}{6} \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} \begin{Bmatrix} \ddot{u}_{X2} \\ \ddot{u}_{X3} \\ \ddot{u}_{X4} \end{Bmatrix} \right).$$

- Principle of virtual work $\delta W = 0 \ \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus for $\delta \mathbf{a}$ give

$$\frac{EA}{L} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{Bmatrix} u_{X2} \\ u_{X3} \\ u_{X4} \end{Bmatrix} + \frac{\rho AL}{6} \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} \begin{Bmatrix} \ddot{u}_{X2} \\ \ddot{u}_{X3} \\ \ddot{u}_{X4} \end{Bmatrix} = 0. \quad \Leftarrow$$

- Mathematica code of the course can be used to build the set of ordinary differential equations and check the outcome of hand calculations (details in the notebook).

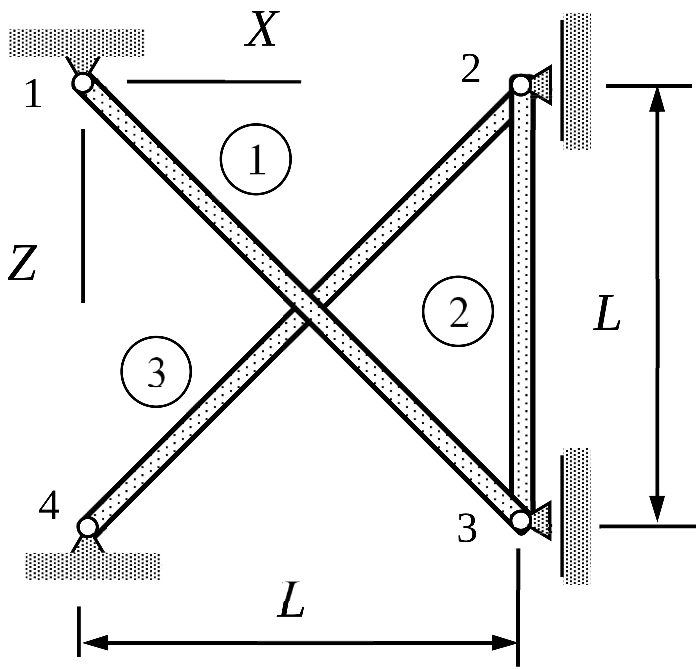
	model	properties	geometry
1	BAR	{{E, ρ}, {A}}	Line[{1, 2}]
2	BAR	{{E, ρ}, {A}}	Line[{2, 3}]
3	BAR	{{E, ρ}, {A}}	Line[{3, 4}]
4	BAR	{{E, ρ}, {A}}	Line[{4, 5}]

	{X,Y,Z}	$\{u_X,u_Y,u_Z\}$	$\{\theta_X,\theta_Y,\theta_Z\}$
1	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}
2	{L, 0, 0}	{uX[1], 0, 0}	{0, 0, 0}
3	{2 L, 0, 0}	{uX[2], 0, 0}	{0, 0, 0}
4	{3 L, 0, 0}	{uX[3], 0, 0}	{0, 0, 0}
5	{4 L, 0, 0}	{0, 0, 0}	{0, 0, 0}

In Mathematica representation, derivatives with respect to time are indicated by indices. Therefore, e.g., $u_{Xn} \sim uX[n,0]$, $\ddot{u}_{Xn} \sim uX[n,2]$ (zero order derivative means function itself).

EXAMPLE 3.4 Consider free vibrations of a truss of three bar elements of which bar 2 is inextensible and bars 1 and 3 massless. Determine the displacement of node 2 as function of time. Initially, displacements are zeros and velocity of nodes 2 and 3 are $\dot{U}$ downwards. Use linear bar elements. Cross-sectional areas of bars 1 and 3 are A and that of bar 2 $\sqrt{8}A$.

Answer $u_{Z2}(t) = 2\dot{U}\sqrt{\frac{L^2\rho}{E}}\sin(\frac{1}{2}t\sqrt{\frac{E}{L^2\rho}})$



- Only the displacements of nodes 2 and 3 in the $Z-$direction matter. As bar 2 is known to be rigid, vertical displacements of nodes 2 and 3 coincide i.e. $u_{Z2} = u_{Z3}$. From the figure, the nodal displacement and length of bar 1 are $u_{x1} = 0$, $u_{x2} = u_{Z2} / \sqrt{2}$ and $h = \sqrt{2}L$. As the bar is assumed to be massless, inertia term vanishes and

$$\delta W^1 = -\begin{Bmatrix} 0 \\ \delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} \frac{EA}{\sqrt{8}L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ u_{Z2} \end{Bmatrix} = -\delta u_{Z2} \frac{EA}{\sqrt{8}L} u_{Z2}.$$

- The relationships for bar 2 are $u_{x1} = u_{Z2}$, $u_{x2} = u_{Z2}$ and $h = L$. The cross-sectional area is $\sqrt{8}A$. As the axial displacements coincide, internal part vanishes and

$$\delta W^2 = -\begin{Bmatrix} \delta u_{Z2} \\ \delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} \frac{\rho\sqrt{8}AL}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_{Z2} \\ \ddot{u}_{Z2} \end{Bmatrix} = -\rho\sqrt{8}AL\delta u_{Z2}\ddot{u}_{Z2}.$$

- The relationships for bar 3 are $u_{x1}=0$, $u_{x2}=-u_{Z2}/\sqrt{2}$ and $h=\sqrt{2}L$. As the bar is assumed to be massless

$$\delta W^3=-\begin{Bmatrix}0\\-\delta u_{Z2}\end{Bmatrix}^{\mathrm{T}}\frac{EA}{\sqrt{8}L}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{Bmatrix}0\\-u_{Z2}\end{Bmatrix}=-\delta u_{Z2}\frac{EA}{\sqrt{8}L}u_{Z2}.$$

- Virtual work expression of the structure is the sum of element contributions

$$\delta W=\delta W^1+\delta W^2+\delta W^3=-\delta u_{Z2}(\frac{EA}{\sqrt{2}L}u_{Z2}+\rho\sqrt{8}AL\ddot{u}_{Z2}).$$

- Principle of virtual work $\delta W=0\quad\forall\delta\mathbf{a}$ and the fundamental lemma of variation calculus $\delta\mathbf{a}^{\mathrm{T}}\mathbf{F}=0\;\forall\delta\mathbf{a}\quad\Leftrightarrow\quad\mathbf{F}=0$ imply that

$$\frac{EA}{\sqrt{2}L}u_{Z2}+\rho\sqrt{8}AL\ddot{u}_{Z2}=0.$$

- What remains, is solving for the displacement as function of time with the additional information of the problem description. The initial value problem consists of the differential equation and two initial conditions:

$$\ddot{u}_{Z2} + \frac{1}{4}\frac{E}{\rho L^2} u_{Z2} = 0 \quad t > 0, \quad u_{Z2}(0) = 0 \quad \text{and} \quad \dot{u}_{Z2}(0) = \dot{U} .$$

- Solution to the equations is given by

$$u_{Z2}(t) = 2\dot{U}\sqrt{\frac{L^2\rho}{E}} \sin\left(\frac{1}{2}t\sqrt{\frac{E}{L^2\rho}}\right) \quad t > 0. \quad \Leftarrow$$

- Mathematica code of the course can be used to solve the set of ordinary differential equations for the nodal displacements and rotations in simple cases and check the outcome of the hand calculations:

	model	properties	geometry
1	BAR	{{E, 0}, {A}}	Line[{1, 3}]
2	BAR	$\{\{E, \rho\}, \{2\sqrt{2}\,A\}\}$	Line[{2, 3}]
3	BAR	{{E, 0}, {A}}	Line[{4, 2}]

	{X,Y,Z}	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}
2	{L, 0, 0}	{0, 0, uZ[2]}	{0, 0, 0}
3	{L, 0, L}	{0, 0, uZ[2]}	{0, 0, 0}
4	{0, 0, L}	{0, 0, 0}	{0, 0, 0}

$$\left\{uZ[2] \to \frac{2\,\dot{U}\,\mathrm{Sin}\left[\frac{1}{2}\,t\,\sqrt{\frac{E}{L^2\,\rho}}\right]}{\sqrt{\frac{E}{L^2\,\rho}}}\right\}$$

3.4 ELEMENT CONTRIBUTIONS

Virtual work expressions for solid, beam, plate elements combine virtual work densities representing the model and the element shape and type dependent approximation. To derive the expression for an element:

- □ Start with the virtual work densities $\delta w_{\Omega}^{\text{int}}$, $\delta w_{\Omega}^{\text{ine}}$, and $\delta w_{\Omega}^{\text{ext}}$ of the formulae collection.

- □ Represent the unknown functions by spatial interpolation of the nodal displacement and rotations (see formulae collection). Substitute the approximations into the density expressions.

- □ Integrate the virtual work density over the domain occupied by the element to get δW.

ELEMENT APPROXIMATION

In MEC-E8001 element approximation is a polynomial interpolant of the nodal displacement and rotations in terms of shape functions. In vibration analysis, shape functions depend on x, y, z and the nodal values on time t. Time is treated as just a parameter of the problem.

Approximation $\mathbf{u} = \mathbf{N}^{\mathrm{T}}\mathbf{a}$ always of the same form!

Shape functions $\mathbf{N} = \{N_1(x, y, z) \quad N_2(x, y, z) \quad \ldots \quad N_n(x, y, z)\}^{\mathrm{T}}$

Parameters $\mathbf{a} = \{\mathrm{a}_1(t) \quad \mathrm{a}_2(t) \quad \ldots \quad \mathrm{a}_n(t)\}^{\mathrm{T}}$

Nodal parameters $\mathrm{a} \in \{u_x, u_y, u_z, \theta_x, \theta_y, \theta_z\}$ may be only displacement or rotation components or a mixture of them (as with the beam model).

SOLID MODEL

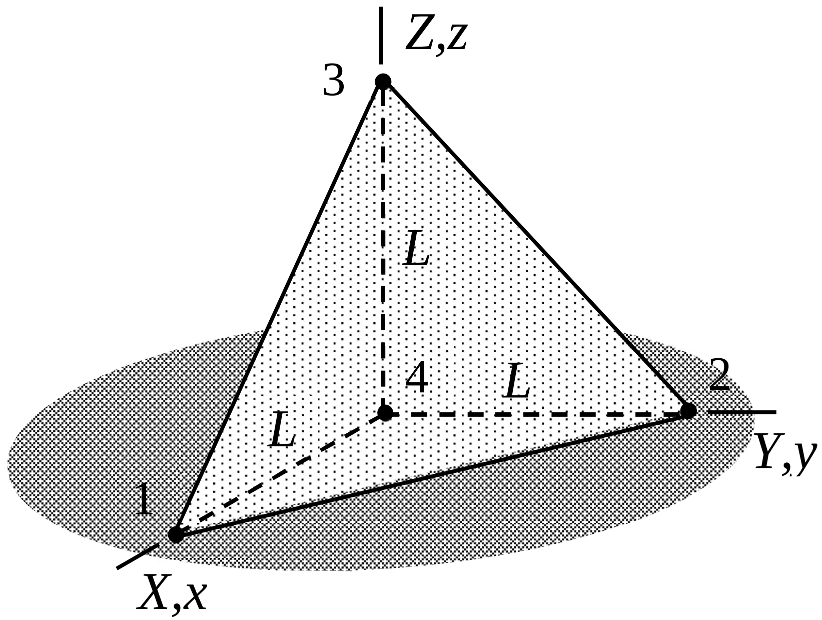
The model does not contain any assumptions in addition to those of the linear elasticity theory.

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta w/\partial z \end{Bmatrix}^{\mathrm{T}} [E] \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial w/\partial z \end{Bmatrix} - \begin{Bmatrix} \partial\delta u/\partial y + \partial\delta v/\partial x \\ \partial\delta v/\partial z + \partial\delta w/\partial y \\ \partial\delta w/\partial x + \partial\delta u/\partial z \end{Bmatrix}^{\mathrm{T}} G \begin{Bmatrix} \partial u/\partial y + \partial v/\partial x \\ \partial v/\partial z + \partial w/\partial y \\ \partial w/\partial x + \partial u/\partial z \end{Bmatrix},$$

$$\delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix}, \quad \delta w_{\partial\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix}, \text{ and } \delta w_{\Omega}^{\text{ine}} = -\begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \rho \begin{Bmatrix} \ddot{u} \\ \ddot{v} \\ \ddot{w} \end{Bmatrix}.$$

The solution domain can be represented, e.g., by tetrahedron elements with linear interpolation of $u(x,y,z,t)$, $v(x,y,z,t)$, and $w(x,y,z,t)$ in spatial coordinates.

EXAMPLE 3.5 Consider a tetrahedron of edge length L, density ρ, and elastic properties E and $\nu = 0$ on a horizontal floor. Calculate the displacement $u_{Z3}(t)$ of node 3 with one tetrahedron element and linear approximation. Assume that $u_{X3} = u_{Y3} = 0$, the bottom surface is fixed, and $u_{Z3} = U$ and $\dot{u}_{Z3} = 0$ at $t = 0$. Stress vanishes at the initial geometry when $u_{Z3} = 0$.



Answer: $u_{Z3}(t) = U\cos(t\sqrt{10\frac{E}{\rho L^2}})$

- Linear shape functions can be deduced directly from the figure $N_1 = x/L$, $N_2 = y/L$, $N_3 = z/L$, and $N_4 = 1 - x/L - y/L - z/L$. However, only the shape function of node 3 is needed as the other nodes are fixed. Approximations to the displacement components are

$$u = 0,\ v = 0, \text{ and } w = \frac{z}{L}u_{Z3},\ \text{ giving } \frac{\partial w}{\partial x} = \frac{\partial w}{\partial y} = 0,\ \frac{\partial w}{\partial z} = \frac{1}{L}u_{Z3}, \text{ and } \ddot{w} = \frac{z}{L}\ddot{u}_{Z3}.$$

- When the approximation is substituted there, the virtual work densities of the internal external, and inertia forces simplify to (here $\nu = 0$)

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} 0 \\ 0 \\ \partial\delta w/\partial z \end{Bmatrix}^{\mathrm{T}} \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \partial w/\partial z \end{Bmatrix} = -\frac{E}{L^2}u_{Z3}\delta u_{Z3},$$

$$\delta w_{\Omega}^{\text{ine}} = -\begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \rho \begin{Bmatrix} \ddot{u} \\ \ddot{v} \\ \ddot{w} \end{Bmatrix} = -\begin{Bmatrix} 0 \\ 0 \\ z/L\delta u_{Z3} \end{Bmatrix}^{\mathrm{T}} \rho \begin{Bmatrix} 0 \\ 0 \\ z/L\ddot{u}_{Z3} \end{Bmatrix} = -\rho(\frac{z}{L})^2 \delta u_{Z3} \ddot{u}_{Z3}.$$

- Virtual work expressions are obtained as integrals of densities over the volume:

$$\delta W^{\text{int}} = \int_{\Omega} \delta w_{\Omega}^{\text{int}} dV = \int_0^L \int_0^{L-z} \int_0^{L-z-y} \delta w_{\Omega}^{\text{int}} dxdydz = -\frac{1}{6} EL\delta u_{Z3} u_{Z3},$$

$$\delta W^{\text{ine}} = \int_{\Omega} \delta w_{\Omega}^{\text{ine}} dV = \int_0^L \int_0^{L-z} \int_0^{L-z-y} \delta w_{\Omega}^{\text{ine}} dxdydz = -\frac{L^3 \rho}{60} \delta u_{Z3} \ddot{u}_{Z3}.$$

- Finally, principle of virtual work $\delta W = 0$ with $\delta W = \delta W^{\text{int}} + \delta W^{\text{ine}}$ implies that

$$\frac{1}{6} ELu_{Z3} + \frac{L^3 \rho}{60} \ddot{u}_{Z3} = 0 \quad \Leftrightarrow \quad \ddot{u}_{Z3} + 10 \frac{E}{\rho L^2} u_{Z3} = 0.$$ **The standard form!**

- Solution to the ordinary differential equations with the initial conditions $u_{Z3} = U$ and $\dot{u}_{Z3} = 0$ at $t = 0$ is given by (

$$u_{Z3}(t) = U\cos(\sqrt{10\frac{E}{\rho L^2}}t).$$

Virtual work density of the inertia term is of the same form as that of the external force, if the distributed external force is replaced by the "inertia force" (not a true force actually). Virtual work density of the inertia forces of the beam is given by

$$\delta w_{\Omega}^{\mathrm{ine}} = -\begin{Bmatrix} \delta u \\ \delta\psi \\ -\delta\theta \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} A & -S_z & -S_y \\ -S_z & I_{zz} & I_{yz} \\ -S_y & I_{zy} & I_{yy} \end{bmatrix} \rho \begin{Bmatrix} \ddot{u} \\ \ddot{\psi} \\ -\ddot{\theta} \end{Bmatrix} - \begin{Bmatrix} \delta v \\ \delta w \\ \delta\phi \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} A & 0 & -S_y \\ 0 & A & S_z \\ -S_y & S_z & J \end{bmatrix} \rho \begin{Bmatrix} \ddot{v} \\ \ddot{w} \\ \ddot{\phi} \end{Bmatrix},$$

in which (Bernoulli constraints) $\psi = \partial v / \partial x$, $\theta = -\partial w / \partial x$, $\ddot{\psi} = \partial\ddot{v} / \partial x$, and $\ddot{\theta} = -\partial\ddot{w} / \partial x$. The terms for the bar, torsion bar, and the two ending modes follow from the generic expression above. Often, the rotation terms in bending are omitted as negligible.

- Let us consider the inertia term per unit length under the kinematic assumptions of the Timoshenko beam model $u_x = u + z\theta - y\psi$, $u_y = v - z\phi$, and $u_z = w + y\phi$

$$\delta w_\Omega^{\text{ine}} = -\int_A (\rho\ddot{\vec{u}} \cdot \delta\vec{u})dA = (\delta w_\Omega^{\text{ine}})_x + (\delta w_\Omega^{\text{ine}})_y + (\delta w_\Omega^{\text{ine}})_z, \quad \text{where}$$

$$(\delta w_\Omega^{\text{ine}})_x = -\int_A \delta u_x \rho \ddot{u}_x dA = -\begin{Bmatrix} \delta u \\ \delta\psi \\ -\delta\theta \end{Bmatrix}^{\mathrm{T}} \int_A \begin{bmatrix} 1 & -y & -z \\ -y & y^2 & yz \\ -z & zy & z^2 \end{bmatrix} \rho dA \begin{Bmatrix} \ddot{u} \\ \ddot{\psi} \\ -\ddot{\theta} \end{Bmatrix},$$

$$(\delta w_\Omega^{\text{ine}})_y = -\int_A \delta u_y \rho \ddot{u}_y dA = -\begin{Bmatrix} \delta v \\ \delta\phi \end{Bmatrix}^{\mathrm{T}} \int_A \begin{bmatrix} 1 & -z \\ -z & z^2 \end{bmatrix} \rho dA \begin{Bmatrix} \ddot{v} \\ \ddot{\phi} \end{Bmatrix},$$

$$(\delta w_\Omega^{\text{ine}})_z = -\int_A \delta u_z \rho \ddot{u}_z dA = -\begin{Bmatrix} \delta w \\ \delta\phi \end{Bmatrix}^{\mathrm{T}} \int_A \begin{bmatrix} 1 & y \\ y & y^2 \end{bmatrix} \rho dA \begin{Bmatrix} \ddot{w} \\ \ddot{\phi} \end{Bmatrix}.$$

- Assuming that cross-section geometry and density are constants, integration over the area gives with the assumptions $S_y = S_z = 0$ and $I_{yz} = I_{zy} = 0$

$$(\delta w_{\Omega}^{\text{ine}})_x = -\begin{Bmatrix} \delta u \\ \delta\psi \\ -\delta\theta \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} A & -S_z & -S_y \\ -S_z & I_{zz} & I_{yz} \\ -S_y & I_{zy} & I_{yy} \end{bmatrix} \rho \begin{Bmatrix} \ddot{u} \\ \ddot{\psi} \\ -\ddot{\theta} \end{Bmatrix},$$

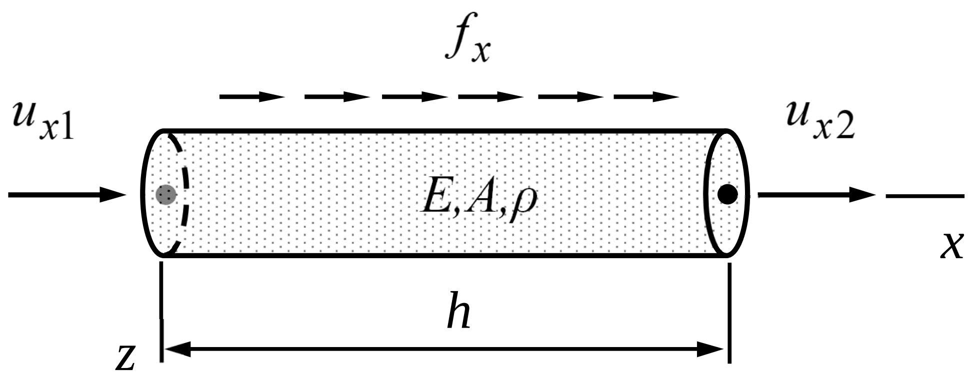
$$(\delta w_{\Omega}^{\text{ine}})_y + (\delta w_{\Omega}^{\text{ine}})_z = -\begin{Bmatrix} \delta v \\ \delta v \\ \delta\phi \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} A & 0 & -S_y \\ 0 & A & S_z \\ -S_y & S_z & J \end{bmatrix} \rho \begin{Bmatrix} \ddot{v} \\ \ddot{w} \\ \ddot{\phi} \end{Bmatrix},$$

in which $J = I_{yy} + I_{zz}$, $\psi = dv/dx$ and $\theta = -dw/dx$ (Bernoulli constraints).

BAR MODE

Bar mode element contribution follows with the assumptions $v=0$, $w=0$, $\phi=0$, and a linear approximation to $u(x)$

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix},$$



$$\delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{f_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix},$$

$$\delta W^{\text{ine}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{\rho A h}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_{x1} \\ \ddot{u}_{x2} \end{Bmatrix}.$$

Above, f_x and E, A, ρ are taken as constants.

- Virtual work density of the inertia term is of the same form as the terms coming from the external distributed forces with $f_x = -\rho A\ddot{u}$ (inertia force per unit length). Hence virtual work densities are

$$\delta w_\Omega^{\text{int}} = -\frac{\partial \delta u}{\partial x} EA \frac{\partial u}{\partial x}, \quad \delta w_\Omega^{\text{ext}} = \delta u f_x, \text{ and } \delta w_\Omega^{\text{ine}} = -\delta u \rho A \ddot{u}.$$

Cross-sectional area A, Young's modulus E, density ρ, and external force per unit length f_x may depend on x and time t.

- Element approximation of the bar model with semi-discretization $u(x,t) = \mathbf{N}(x)^{\mathrm{T}} \mathbf{a}(t)$

$$u = \frac{1}{h}\{h - x \quad x\}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} \quad \Rightarrow \quad \frac{\partial u}{\partial x} = \frac{1}{h}\{-1 \quad 1\}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} \quad \text{and} \quad \ddot{u} = \frac{1}{h}\{h - x \quad x\}\begin{Bmatrix} \ddot{u}_{x1} \\ \ddot{u}_{x2} \end{Bmatrix}$$

- Virtual work density of the inertia force (expressions for the internal and external forces have been discussed in MEC-E1050) is given by

$$\delta w_{\Omega}^{\text{ine}} = -\delta u \rho A \ddot{u} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{\rho A}{h^2} \begin{bmatrix} (h-x)^2 & x(h-x) \\ x(h-x) & x^2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_{x1} \\ \ddot{u}_{x2} \end{Bmatrix}$$

- Assuming that A, ρ are constants, integration over the length gives

$$\delta W^{\text{ine}} = \int_0^h \delta w_{\Omega}^{\text{ine}} dx = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{\rho A h}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_{x1} \\ \ddot{u}_{x2} \end{Bmatrix}. \Leftarrow$$

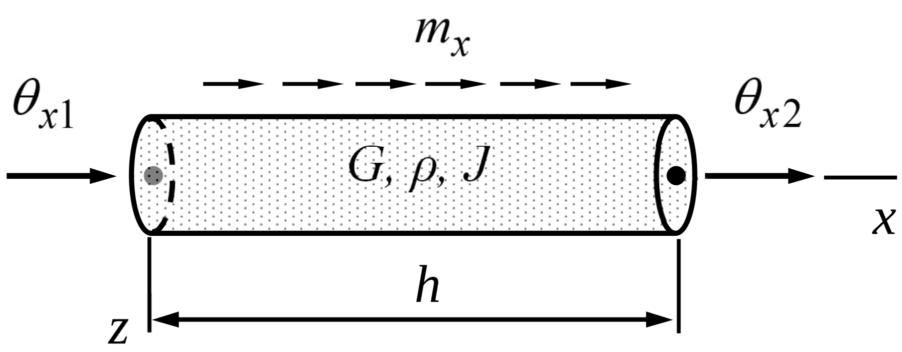
TORSION MODE

Torsion mode element contribution follows with the assumptions $u=0,\ v=0,\ w=0$, and a linear approximation to $\phi(x)$

$$\delta W^{\text{int}}=-\begin{Bmatrix}\delta\theta_{x1}\\ \delta\theta_{x2}\end{Bmatrix}^{\mathrm{T}}\frac{GJ}{h}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}\theta_{x1}\\ \theta_{x2}\end{Bmatrix},$$

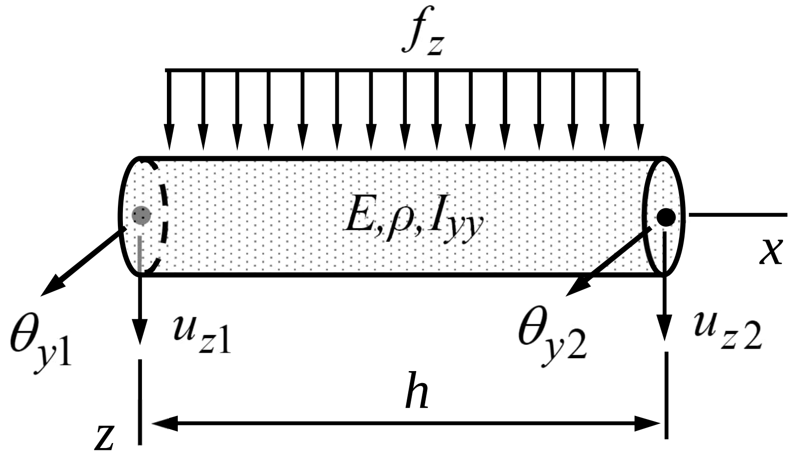
$$\delta W^{\text{ext}}=\begin{Bmatrix}\delta\theta_{x1}\\ \delta\theta_{x2}\end{Bmatrix}^{\mathrm{T}}\frac{m_x h}{2}\begin{Bmatrix}1\\ 1\end{Bmatrix},$$

$$\delta W^{\text{ine}}=-\begin{Bmatrix}\delta\theta_{x1}\\ \delta\theta_{x2}\end{Bmatrix}^{\mathrm{T}}\frac{\rho Jh}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{Bmatrix}\ddot{\theta}_{x1}\\ \ddot{\theta}_{x2}\end{Bmatrix}.$$



Above, m_x, G, ρ and J are assumed to be constants.

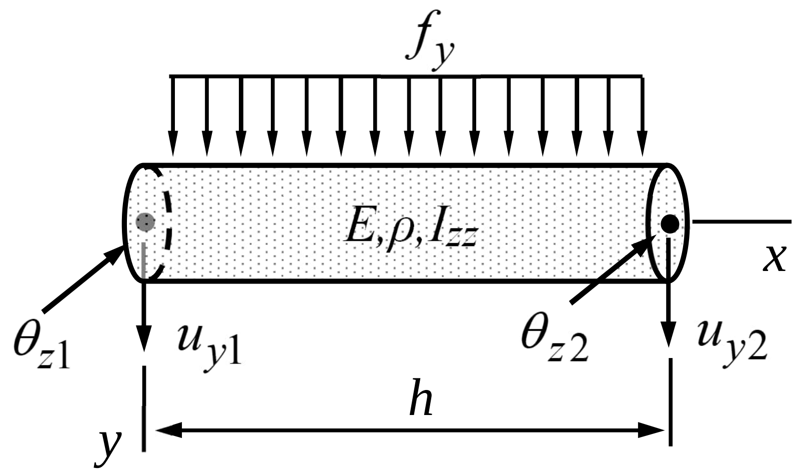
BENDING MODE (*xz*-plane)



$$\delta W^{\text{ine}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{\rho I_{yy}}{30h} \begin{bmatrix} 36 & -3h & -36 & -3h \\ -3h & 4h^2 & 3h & -h^2 \\ -36 & 3h & 36 & 3h \\ -3h & -h^2 & 3h & 4h^2 \end{bmatrix} + \frac{\rho A h}{420} \begin{bmatrix} 156 & -22h & 54 & 13h \\ -22h & 4h^2 & -13h & -3h^2 \\ 54 & -13h & 156 & 22h \\ 13h & -3h^2 & 22h & 4h^2 \end{bmatrix} \right) \begin{Bmatrix} \ddot{u}_{z1} \\ \ddot{\theta}_{y1} \\ \ddot{u}_{z2} \\ \ddot{\theta}_{y2} \end{Bmatrix}$$

The first term is negligible whenever a beam element is thin in the sense $\alpha = t / h << 1$!

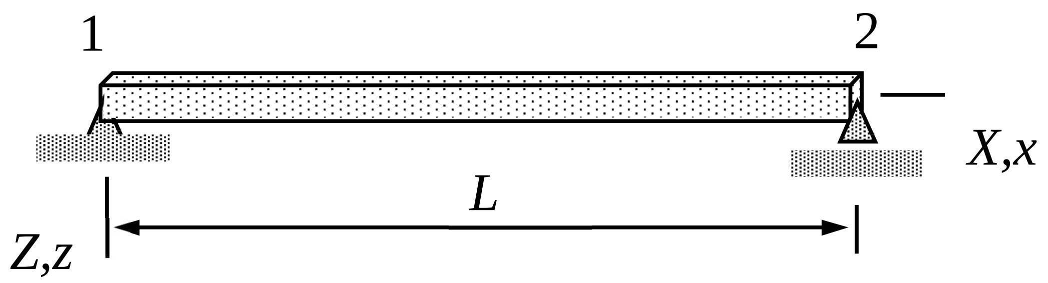
BENDING MODE (*xy*-plane)



$$\delta W^{\text{ine}} = -\begin{Bmatrix} \delta u_{y1} \\ \delta\theta_{z1} \\ \delta u_{y2} \\ \delta\theta_{z2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{\rho I_{zz}}{30h} \begin{bmatrix} 36 & 3h & -36 & 3h \\ 3h & 4h^2 & -3h & -h^2 \\ -36 & -3h & 36 & -3h \\ 3h & -h^2 & -3h & 4h^2 \end{bmatrix} + \frac{\rho A h}{420} \begin{bmatrix} 156 & 22h & 54 & -13h \\ 22h & 4h^2 & 13h & -3h^2 \\ 54 & 13h & 156 & -22h \\ -13h & -3h^2 & -22h & 4h^2 \end{bmatrix} \right) \begin{Bmatrix} \ddot{u}_{y1} \\ \ddot{\theta}_{z1} \\ \ddot{u}_{y2} \\ \ddot{\theta}_{z2} \end{Bmatrix}$$

The first term is negligible whenever a beam element is thin in the sense $\alpha = t / h << 1$!

EXAMPLE 3.6 Consider bending of a simply supported beam of length L in $XZ-$plane. Determine the ordinary differential equations giving as their solution the rotation components of the end nodes as functions of time. Determine also the natural angular speeds of free vibrations and the corresponding modes. Cross-section properties A, I and material properties E, ρ are constants.



Answer $\omega_1 = \sqrt{2520\dfrac{EI}{\rho AL^4}}$, $\begin{Bmatrix}\theta_{Y1}\\ \theta_{Y2}\end{Bmatrix}_1 = \begin{Bmatrix}1\\ 1\end{Bmatrix}$ and $\omega_2 = \sqrt{120\dfrac{EI}{\rho AL^4}}$, $\begin{Bmatrix}\theta_{Y1}\\ \theta_{Y2}\end{Bmatrix}_2 = \begin{Bmatrix}-1\\ 1\end{Bmatrix}$

- As the material and structural coordinate systems coincide here, virtual work expression considering the internal and inertia forces simplifies to (the second bending term is omitted in the inertia part)

$$\delta W = -\begin{Bmatrix} \delta\theta_{Y1} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EI}{L^3}\begin{bmatrix} 4L^2 & 2L^2 \\ 2L^2 & 4L^2 \end{bmatrix}\begin{Bmatrix} \theta_{Y1} \\ \theta_{Y2} \end{Bmatrix} + \frac{\rho AL}{420}\begin{bmatrix} 4L^2 & -3L^2 \\ -3L^2 & 4L^2 \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_{Y1} \\ \ddot{\theta}_{Y2} \end{Bmatrix}\right) = 0.$$

- Principle of virtual work and the fundamental lemma of variation calculus give the ordinary differential equations

$$\frac{EI}{L}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}\begin{Bmatrix} \theta_{Y1} \\ \theta_{Y2} \end{Bmatrix} + \frac{\rho AL^3}{420}\begin{bmatrix} 4 & -3 \\ -3 & 4 \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_{Y1} \\ \ddot{\theta}_{Y2} \end{Bmatrix} = 0.$$

- Angular speeds of free vibrations $\boldsymbol{\omega}$ are the eigenvalues of $\boldsymbol{\Omega}$ which is related with the matrices of the differential equations by $\boldsymbol{\Omega}^2 = \mathbf{M}^{-1}\mathbf{K} = \mathbf{X}\boldsymbol{\omega}^2\mathbf{X}^{-1}$

$$\boldsymbol{\Omega}^2 = 120\frac{EI}{\rho AL^4}\begin{bmatrix} 11 & 10 \\ 10 & 11 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\frac{EI}{\rho AL^4}\begin{bmatrix} 2520 & 0 \\ 0 & 120 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1}.$$

- The latter form of $\boldsymbol{\Omega}^2$ (eigenvalue decomposition) gives

$$\omega_1 = \sqrt{2520\frac{EI}{\rho AL^4}},\ \ \mathbf{x}_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \ \text{ and } \ \omega_2 = \sqrt{120\frac{EI}{\rho AL^4}},\ \ \mathbf{x}_2 = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}.$$

- Mathematica code takes into account both inertia terms

	model	properties	geometry
1	BEAM	{{E, G, ρ}, {A, I, I}}	Line[{1, 2}]

	{X,Y,Z}	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	{0, 0, 0}	{0, 0, 0}	{0, θY[1], 0}
2	{L, 0, 0}	{0, 0, 0}	{0, θY[2], 0}

$$\left\{\left\{\omega[1] \to 6\sqrt{70}\sqrt{\frac{E\,I\,(A\,L^2 + 10\,I)}{(A^2\,L^6 + 52\,A\,L^4\,I + 420\,L^2\,I^2)\,\rho}},\ \{\theta Y[1] \to 1,\ \theta Y[2] \to 1\}\right\},\right.$$

$$\left.\left\{\omega[2] \to 2\sqrt{30}\sqrt{\frac{E\,I\,(A\,L^2 + 42\,I)}{(A^2\,L^6 + 52\,A\,L^4\,I + 420\,L^2\,I^2)\,\rho}},\ \{\theta Y[1] \to -1,\ \theta Y[2] \to 1\}\right\}\right\}$$

PLATE MODEL

The generic element contribution of plate is obtained by combining the virtual work expressions of thin slab and plate bending modes. Assuming that the origin of the material coordinate system is placed at the mid-plane and material properties are constants through the thickness, virtual work density is given by

$$\delta w_{\Omega}^{\text{ine}} = -\begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} t\rho \begin{Bmatrix} \ddot{u} \\ \ddot{v} \\ \ddot{w} \end{Bmatrix} - \begin{Bmatrix} \partial \delta w / \partial x \\ \partial \delta w / \partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3 \rho}{12} \begin{Bmatrix} \partial \ddot{w} / \partial x \\ \partial \ddot{w} / \partial y \end{Bmatrix}.$$

The planar solution domain (reference-plane) can be represented by triangular or rectangular elements. Interpolation of displacement components should be continuous and $w(x, y)$ should have also continuous derivatives at the element interfaces.

- Let us consider (first) the virtual work density of the inertia forces under the kinematic assumptions of the Reissner-Mindlin model $u_x = u + \theta z$, $u_y = v - \phi z$, and $u_z = w$.

$$\delta w_\Omega^{\text{ine}} = -\int_t (\rho \ddot{\vec{u}} \cdot \delta \vec{u}) dz = (\delta w_\Omega^{\text{ine}})_x + (\delta w_\Omega^{\text{ine}})_y + (\delta w_\Omega^{\text{ine}})_z, \quad \text{where}$$

$$(\delta w_\Omega^{\text{ine}})_x = -\int \delta u_x \rho \ddot{u}_x dz = -\begin{Bmatrix} \delta u \\ -\delta\theta \end{Bmatrix}^{\mathrm{T}} \int \begin{bmatrix} 1 & -z \\ -z & z^2 \end{bmatrix} \rho dz \begin{Bmatrix} \ddot{u} \\ -\ddot{\theta} \end{Bmatrix},$$

$$(\delta w_\Omega^{\text{ine}})_y = -\int \delta u_y \rho \ddot{u}_y dz = -\begin{Bmatrix} \delta v \\ \delta\phi \end{Bmatrix}^{\mathrm{T}} \int \begin{bmatrix} 1 & -z \\ -z & z^2 \end{bmatrix} \rho dz \begin{Bmatrix} \ddot{v} \\ \ddot{\phi} \end{Bmatrix},$$

$$(\delta w_\Omega^{\text{ine}})_z = -\int \delta u_z \rho \ddot{u}_z dz = -\delta w \int \rho dz \ddot{w}.$$

- Assuming that thickness and density are constants, and the origin of the z –axis is placed at the geometric centroid, integration over the thickness $z \in [-t/2, t/2]$ gives

$$(\delta w_{\Omega}^{\text{ine}})_x = -\begin{Bmatrix} \delta u \\ -\delta\theta \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} t & 0 \\ 0 & t^3/12 \end{bmatrix} \rho \begin{Bmatrix} \ddot{u} \\ -\ddot{\theta} \end{Bmatrix},$$

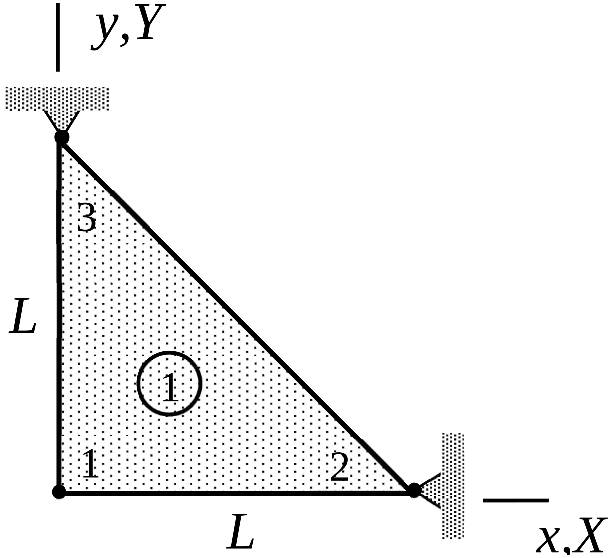
$$(\delta w_{\Omega}^{\text{ine}})_y = -\begin{Bmatrix} \delta v \\ \delta\phi \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} t & 0 \\ 0 & t^3/12 \end{bmatrix} \rho \begin{Bmatrix} \ddot{v} \\ \ddot{\phi} \end{Bmatrix},$$

$$(\delta w_{\Omega}^{\text{ine}})_z = -\delta w t \rho \ddot{w}.$$

- Summing up the terms with the Kirchhoff constraints $\phi = \partial w / \partial y$ and $\theta = -\partial w / \partial x$ (to end up with the Kirchhoff model expressions) gives the final form:

$$\delta w_{\Omega}^{\text{ine}} = -\begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} t\rho \begin{Bmatrix} \ddot{u} \\ \ddot{v} \\ \ddot{w} \end{Bmatrix} - \begin{Bmatrix} \partial \delta w / \partial x \\ \partial \delta w / \partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3 \rho}{12} \begin{Bmatrix} \partial \ddot{w} / \partial x \\ \partial \ddot{w} / \partial y \end{Bmatrix}. \quad \Leftarrow$$

EXAMPLE 3.7 Consider the thin triangular structure shown. Young's modulus *E*, Poisson's ratio ν, and thickness *h* are constants. Assume plane-stress conditions and derive the ordinary differential equations giving as their solutions the free vibrations of the structure.



Answer: $$\frac{1}{4}\frac{hE}{1-\nu^2}\begin{bmatrix} 3-\nu & 1+\nu \\ 1+\nu & 3-\nu \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} + \frac{L^2}{12}h\rho\begin{Bmatrix} \ddot{u}_{X1} \\ \ddot{u}_{Y1} \end{Bmatrix} = 0$$

- Nodes 2 are 3 are fixed and the non-zero displacements/rotations are u_{X1} and u_{Y1}. Linear shape functions $N_1 = (L-x-y)/L$, $N_2 = x/L$ and $N_3 = y/L$ are easy to deduce from the figure. Therefore

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \frac{L-x-y}{L} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} \quad \Rightarrow$$

$$\begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial x \end{Bmatrix} = -\frac{1}{L} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix}, \ \begin{Bmatrix} \partial u/\partial y \\ \partial v/\partial y \end{Bmatrix} = -\frac{1}{L} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix}, \text{ and } \begin{Bmatrix} \ddot{u} \\ \ddot{v} \end{Bmatrix} = \frac{L-x-y}{L} \begin{Bmatrix} \ddot{u}_{X1} \\ \ddot{u}_{Y1} \end{Bmatrix}.$$

- Virtual work densities of internal and inertia forces are given by

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \\ \delta u_{X1} + \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \frac{1}{L^2} \frac{hE}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{Y1} \\ u_{X1} + u_{Y1} \end{Bmatrix},$$

$$\delta w_{\Omega}^{\text{ine}} = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} h\rho\left(\frac{L-x-y}{L}\right)^2 \begin{Bmatrix} \ddot{u}_{X1} \\ \ddot{u}_{Y1} \end{Bmatrix}.$$

- Integration over the triangular domain gives (integrand of the internal part is constant)

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \\ \delta u_{X1} + \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \frac{1}{2}\frac{hE}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{Y1} \\ u_{X1} + u_{Y1} \end{Bmatrix} \Leftrightarrow$$

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \frac{1}{4}\frac{hE}{1-\nu^2} \begin{bmatrix} 3-\nu & 1+\nu \\ 1+\nu & 3-\nu \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix},$$

$$\delta W^{\text{ine}} = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} h\rho \begin{Bmatrix} \ddot{u}_{X1} \\ \ddot{u}_{Y1} \end{Bmatrix} \int_0^L \int_0^{L-x} \int \left(\frac{L-x-y}{L}\right)^2 dydx \;\Rightarrow$$

$$\delta W^{\text{ine}} = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \frac{L^2}{12} h\rho \begin{Bmatrix} \ddot{u}_{X1} \\ \ddot{u}_{Y1} \end{Bmatrix}.$$

- Principle of virtual work in the form $\delta W = \delta W^{\text{int}} + \delta W^{\text{ine}} = 0 \;\; \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus give

$$\delta W = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} (\frac{1}{4}\frac{hE}{1-\nu^2}\begin{bmatrix} 3-\nu & 1+\nu \\ 1+\nu & 3-\nu \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} + \frac{L^2}{12} h\rho \begin{Bmatrix} \ddot{u}_{X1} \\ \ddot{u}_{Y1} \end{Bmatrix}) = 0 \quad \forall \begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix} \quad \Rightarrow$$

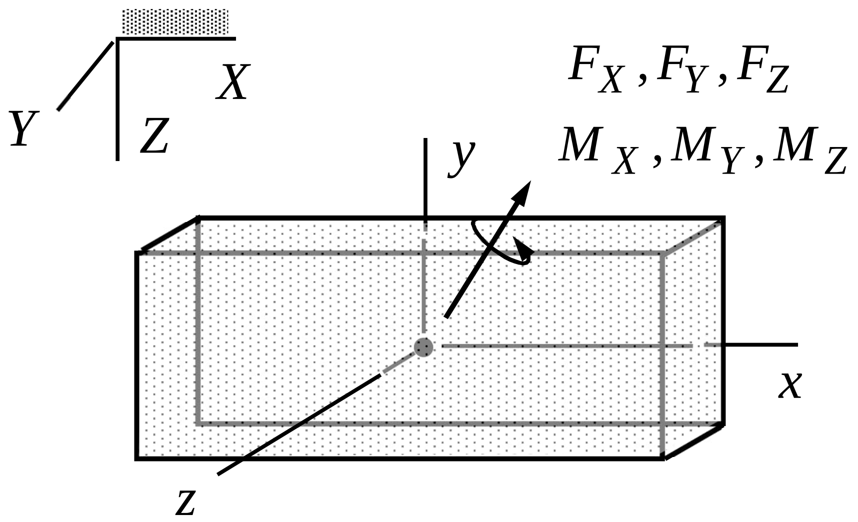
$$\frac{1}{4}\frac{hE}{1-\nu^2}\begin{bmatrix} 3-\nu & 1+\nu \\ 1+\nu & 3-\nu \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} + \frac{L^2}{12} h\rho \begin{Bmatrix} \ddot{u}_{X1} \\ \ddot{u}_{Y1} \end{Bmatrix} = 0 \;. \quad \Leftarrow$$

RIGID BODY

Inertia term takes into account translation and rotation parts which depend on the mass m and the 3×3 inertia matrix $\mathbf{J}$. For a sphere $\mathbf{J} = \tfrac{2}{5} mR^2 \mathbf{I}$ ($\mathbf{I}$ is the 3×3 unit matrix).

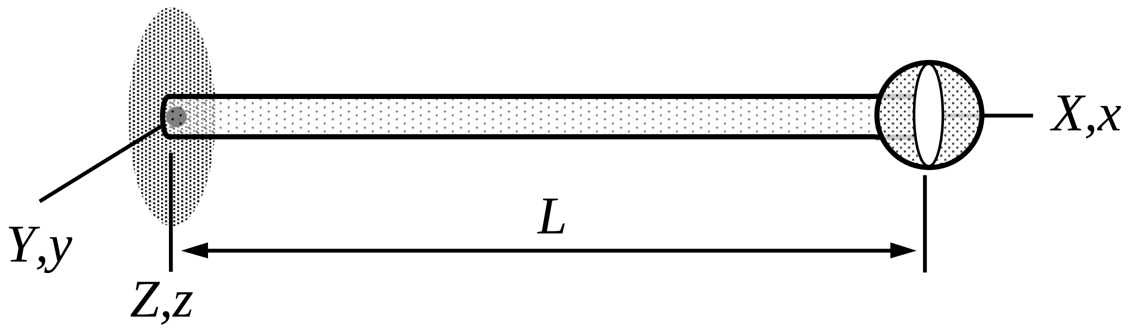
$$\delta W^{\text{ext}} = \begin{Bmatrix} \delta u_X \\ \delta u_Y \\ \delta u_Z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} F_X \\ F_Y \\ F_Z \end{Bmatrix} + \begin{Bmatrix} \delta\theta_X \\ \delta\theta_Y \\ \delta\theta_Z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} M_X \\ M_Y \\ M_Z \end{Bmatrix},$$

$$\delta W^{\text{ine}} = -\begin{Bmatrix} \delta u_X \\ \delta u_Y \\ \delta u_Z \end{Bmatrix}^{\mathrm{T}} m \begin{Bmatrix} \ddot{u}_X \\ \ddot{u}_Y \\ \ddot{u}_Z \end{Bmatrix} - \begin{Bmatrix} \delta\theta_x \\ \delta\theta_y \\ \delta\theta_z \end{Bmatrix}^{\mathrm{T}} \mathbf{J} \begin{Bmatrix} \ddot{\theta}_x \\ \ddot{\theta}_y \\ \ddot{\theta}_z \end{Bmatrix}.$$



The form above assumes that the first moments of mass and the off-diagonal terms of the second moments of mass vanish (origin of the material coordinate system at the center of mass etc.). Expressions for large rotations are more complex.

EXAMPLE 3.8 The mass of a cantilever (circular cross section) is negligible compared to the mass of a rigid spherical body welded to the free end. Determine the angular speeds and modes of the free vibrations. The mass of the sphere is m and the moment of inertia $\mathbf{J} = \frac{1}{5} mL^2 \mathbf{I}$.



Answer $\omega_3 = \omega_4 = \sqrt{2\dfrac{EI}{mL^3}}$, $\omega_5 = \omega_6 = \sqrt{30\dfrac{EI}{mL^3}}$, $\omega_2 = \sqrt{10\dfrac{GI}{mL^3}}$, and $\omega_1 = \sqrt{\dfrac{EA}{mL}}$

- Frequency analysis by the Mathematica code gives

$$\omega[1] \to \sqrt{\frac{A\,E}{L\,m}} \qquad \{uX[2] \to 1,\ uY[2] \to 0,\ uZ[2] \to 0,\ \theta X[2] \to 0,\ \theta Y[2] \to 0,\ \theta Z[2] \to 0\}$$

$$\omega[2] \to \sqrt{10}\ \sqrt{\frac{G\,I}{L^3\,m}} \qquad \{uX[2] \to 0,\ uY[2] \to 0,\ uZ[2] \to 0,\ \theta X[2] \to 1,\ \theta Y[2] \to 0,\ \theta Z[2] \to 0\}$$

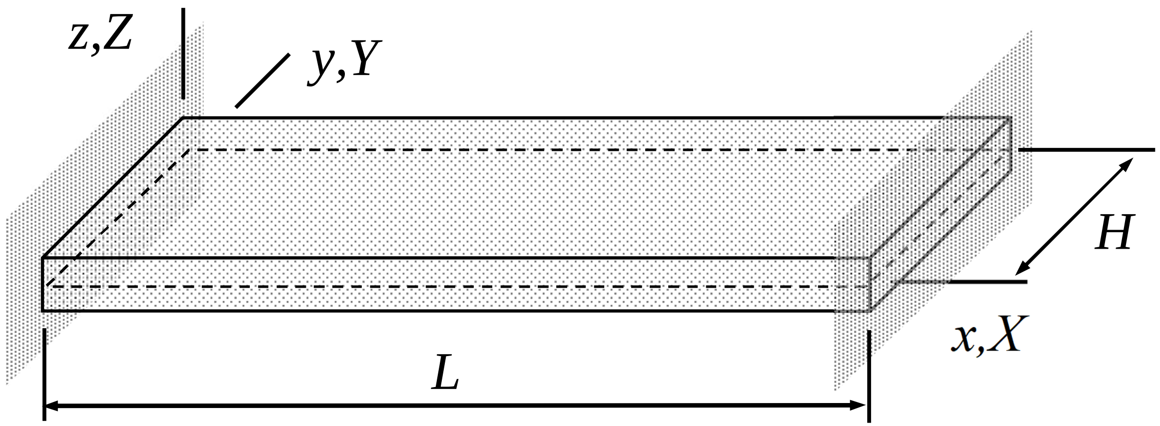
$$\omega[3] \to \sqrt{2}\ \sqrt{\frac{E\,I}{L^3\,m}} \qquad \left\{uX[2] \to 0,\ uY[2] \to \frac{3\,L}{5},\ uZ[2] \to 0,\ \theta X[2] \to 0,\ \theta Y[2] \to 0,\ \theta Z[2] \to 1\right\}$$

$$\omega[4] \to \sqrt{2}\ \sqrt{\frac{E\,I}{L^3\,m}} \qquad \left\{uX[2] \to 0,\ uY[2] \to 0,\ uZ[2] \to -\frac{3\,L}{5},\ \theta X[2] \to 0,\ \theta Y[2] \to 1,\ \theta Z[2] \to 0\right\}$$

$$\omega[5] \to \sqrt{2}\ \sqrt{\frac{E\,I}{L^3\,m}} \qquad \left\{uX[2] \to 0,\ uY[2] \to \frac{3\,L}{5},\ uZ[2] \to 0,\ \theta X[2] \to 0,\ \theta Y[2] \to 0,\ \theta Z[2] \to 1\right\}$$

$$\omega[6] \to \sqrt{2}\ \sqrt{\frac{E\,I}{L^3\,m}} \qquad \left\{uX[2] \to 0,\ uY[2] \to 0,\ uZ[2] \to -\frac{3\,L}{5},\ \theta X[2] \to 0,\ \theta Y[2] \to 1,\ \theta Z[2] \to 0\right\}$$

EXAMPLE 3.9 Find the frequency of the free transverse vibrations of a plate strip using the one parameter approximation $w(x,t) = a(t)(1 - x/L)^2 (x/L)^2$ and the virtual work densities for Kirchhoff model bending mode. Thickness, length, and width of the plate are t, L, and H, respectively. Young's modulus E, and Poisson's ratio ν, and density ρ are constants.



Answer: $f = \dfrac{1}{2\pi}\dfrac{t}{L^2}\sqrt{42\dfrac{E}{\rho(1-\nu^2)}}$

- Approximation satisfies the displacement boundary conditions ‘a priori’ and contains a unknown function $a(t)$ to be determined by using the principle of virtual work (the outcome is an ordinary differential equation). The non-zero derivatives in the virtual work densities are given by

$$w(x,t)=a(t)\frac{1}{L^4}(L^2x^2-2Lx^3+x^4) \quad \Rightarrow \quad \frac{\partial w}{\partial x}=w(x,t)=a(t)\frac{1}{L^4}(2L^2x-6Lx^2+4x^3),$$

$$\frac{\partial^2 w}{\partial x^2}=a(t)\frac{2}{L^4}(L^2-6Lx+6x^2),\ \frac{\partial^2 w}{\partial t^2}=\ddot{a}(t)\frac{1}{L^4}(L^2x^2-2Lx^3+x^4).$$

- When the approximation is substituted there, virtual work densities simplify to (omitting the rotation term of the inertia part as negligible)

$$\delta w_{\Omega}^{\text{int}}=-a\delta a 4\frac{D}{L^8}(L^2-6Lx+6x^2)^2,$$

$$\delta w_{\Omega}^{\text{ine}} = -t\rho\ddot{a}\frac{1}{L^8}(L^2x^2 - 2Lx^3 + x^4)^2\delta a.$$

- Integrations over the domain $\Omega =]0, L[\times]0, H[$ give the virtual work expression of the internal and inertia forces

$$\delta W^{\text{int}} = \int_{\Omega} \delta w_{\Omega}^{\text{int}} d\Omega = -a\delta a\frac{1}{15}\frac{HEt^3}{L^3(1-\nu^2)},$$

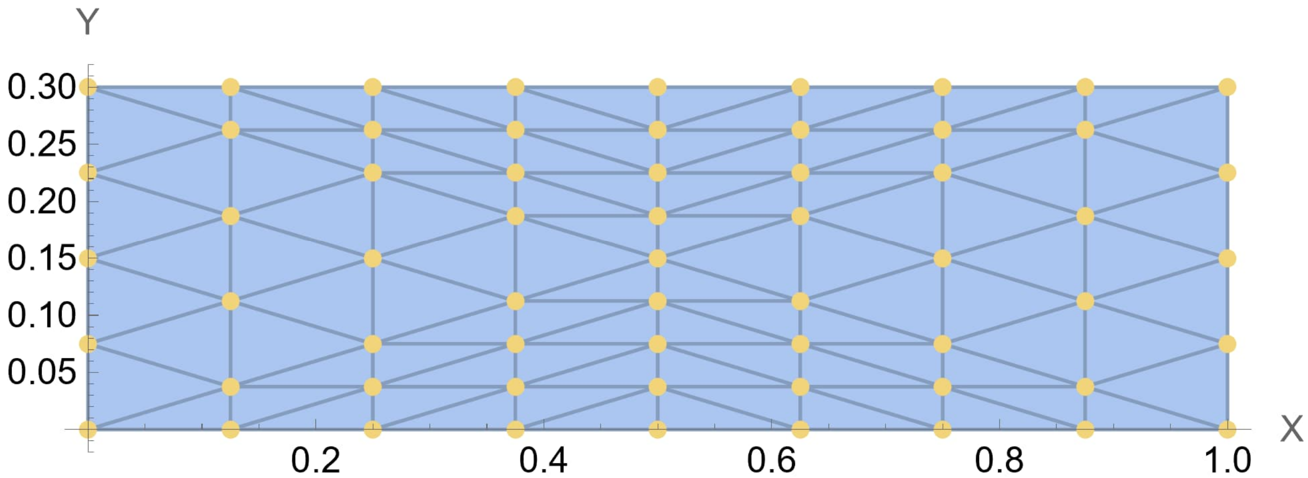
$$\delta W^{\text{ine}} = \int_{\Omega} \delta w_{\Omega}^{\text{ine}} d\Omega = -\delta a\frac{1}{630}tLH\rho\ddot{a}.$$

- Principle of virtual work $\delta W = \delta W^{\text{int}} + \delta W^{\text{ine}} = 0 \quad \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus give finally

$$\delta W = -\delta a\left(\frac{1}{15}\frac{HEt^3}{L^3(1-\nu^2)}a + \frac{1}{630}tLH\rho\ddot{a}\right) = 0 \quad \forall \delta a \quad \Leftrightarrow \quad \frac{1}{15}\frac{HEt^3}{L^3(1-\nu^2)}a + \frac{1}{630}tLH\rho\ddot{a} = 0$$

$$\ddot{a} + \frac{630}{15} \frac{Et^2}{L^4 \rho(1-\nu^2)} a = 0 \quad \text{so} \quad f = \frac{1}{2\pi} \frac{t}{L^2} \sqrt{42 \frac{E}{\rho(1-\nu^2)}} \, .$$

- The problem can be solved numerically by using the Reissner-Mindlin plate model and plate bending element of the Mathematica code. For example, assuming parameter values $p(L/t)^3/E = 10$, $\nu = 0.33$, $H/L = 0.3$, and $t/L = 0.01$ (thin plate), the one parameter approximation gives $f = 0.345\,\text{Hz}$ whereas the solution on the mesh shown gives $f = 0.349\,\text{Hz}$.



MEC-E8001 Finite Element Analysis, week 4/2023

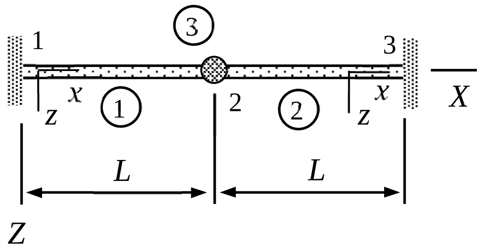
1. Determine the eigenvalue decomposition $\mathbf{A}=\mathbf{X}\boldsymbol{\lambda}\mathbf{X}^{-1}$ and $\sqrt{\mathbf{A}}$ when $\mathbf{A}=\begin{bmatrix} 4 & 0 \\ -1 & 1 \end{bmatrix}$.

Answer $\mathbf{A}=\mathbf{X}\boldsymbol{\lambda}\mathbf{X}^{-1}=\begin{bmatrix} -3 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1/3 & 0 \\ 1/3 & 1 \end{bmatrix}$ and $\sqrt{\mathbf{A}}=\pm\begin{bmatrix} 2 & 0 \\ -1/3 & 1 \end{bmatrix}$

2. Derive the consistent mass matrix $\mathbf{M}$ of a two-node beam element (bending in *xz*-plane). Assume that density is constant, cross section is constant, and the beam element is thin in the sense $t/h \ll 1$, so that $\delta w_{\Omega}^{\text{ine}}=-\delta w \rho A \ddot{w}$.

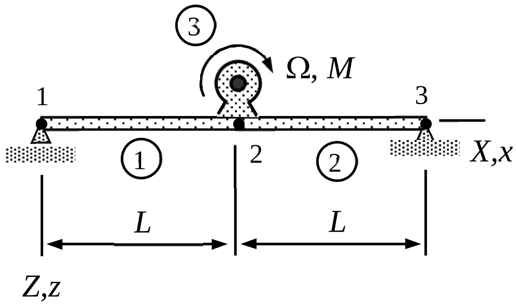
Answer $\mathbf{M}=\dfrac{\rho A h}{420}\left[\begin{array}{cc:cc} 156 & -22h & 54 & 13h \\ -22h & 4h^2 & -13h & -3h^2 \\ \hdashline 54 & -13h & 156 & 22h \\ 13h & -3h^2 & 22h & 4h^2 \end{array}\right]$

3. The *XZ*-plane structure shown consists of two *massless* beams and a homogeneous disk considered as a rigid body. Derive the equations of motion in terms of displacements u_{Z2} and θ_{Y2}. Young's modulus of the beam material and the second moment of area are E and I, and the mass and moment of inertia of the disk are m and J, respectively.



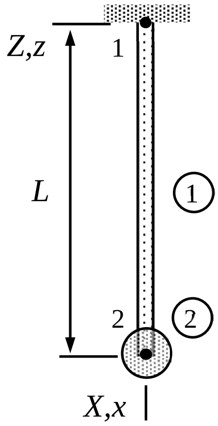
Answer $\dfrac{EI}{L^3}\begin{bmatrix} 24 & 0 \\ 0 & 8L^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix}+\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{Bmatrix} \ddot{u}_{Z2} \\ \ddot{\theta}_{Y2} \end{Bmatrix}=0$

4. The rotor of the machine shown rotates with angular speed Ω. Determine the bending stiffness EI so that the angular speed (free vibrations) of the foundation-machine system coincides with Ω. The foundation is modeled as two *massless* beams and the machine as a particle of mass *M*. Assume that $\theta_{Y1}=-\theta_{Y3}$ and $\theta_{Y2}=0$.



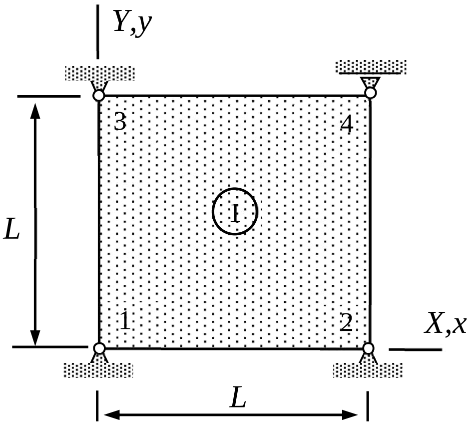
Answer $EI=\dfrac{1}{6}mL^3\Omega^2$

5. XZ-plane structure shown consists of a beam and a homogeneous disk considered as a rigid body. Derive the equations of motion in terms of u_{Z2}, θ_{Y2} and determine the angular speeds of free vibrations. Assume that mass of the beam is negligible compared to that of the disk and that the beam is inextensible in the axial direction. Young's modulus E of the beam material and the second moment of area I are constants. Mass and moment of inertia of the disk are m and $\mathbf{J} = \frac{1}{5} mL^2 \mathbf{I}$, respectively.



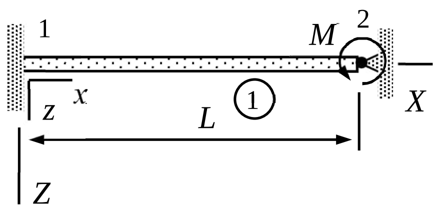
Answer $\omega_1 = \sqrt{\lambda_1} = \sqrt{2}\sqrt{\frac{EI}{mL^3}}$, $\omega_2 = \sqrt{\lambda_2} = \sqrt{30}\sqrt{\frac{EI}{mL^3}}$

6. Node 4 of a thin rectangular slab (assume plane stress conditions) is allowed to move horizontally and nodes 1, 2, and 3 are fixed. Derive the initial value problem giving as its solution the horizontal displacement $u_{X4}(t)$ of node 4 as function of time, if $u_{X4}(0) = U$ and $\dot{u}_{X4}(0) = 0$. Use just one bilinear element. Material parameters E, $\nu = 0$, ρ and thickness h of the slab are constants.



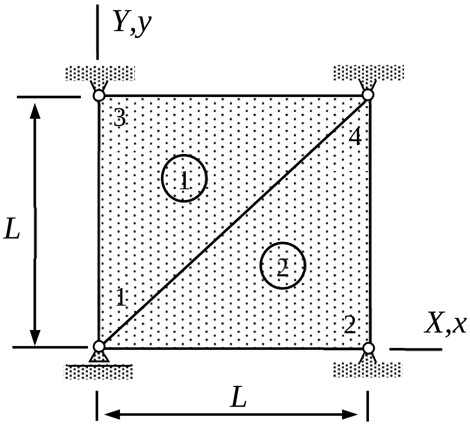
Answer $\ddot{u}_{X4} + \frac{9}{2}\frac{E}{L^2\rho} u_{X4} = 0 \quad t > 0$, $u_{X4}(0) = U$, $\dot{u}_{X4}(0) = 0$

7. The beam of the figure is subjected to moment M when $t < 0$. At $t = 0$, the moment is suddenly removed and the beam starts to vibrate. Derive the initial value problem giving $\theta_{Y2}(t)$ for $t > 0$. The beam is thin so that the rotational part of the inertia term is negligible. The geometrical quantities of the cross-section are A, I and the material constants E and ρ.



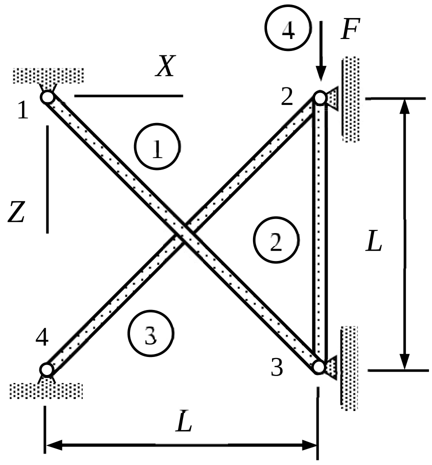
Answer $4\frac{EI}{L}\theta_{Y2} + \frac{\rho A L^3}{105}\ddot{\theta}_{Y2} = 0 \quad t > 0$, $\theta_{Y2}(0) = \frac{1}{4}\frac{ML}{EI}$, $\dot{\theta}_{Y2}(0) = 0$

8. Node 1 of a thin rectangular slab (assume plane stress conditions) is allowed to move horizontally at node 1 whereas nodes 2, 3 and 4 are fixed. Derive the expression of horizontal displacement $u_{X1}(t)$ of node 1 as function of time, if $u_{X1}(0) = U$ and $\dot{u}_{X1}(0) = 0$. Use two linear triangle elements. Material parameters E, ν, ρ, and thickness h of the slab are constants.



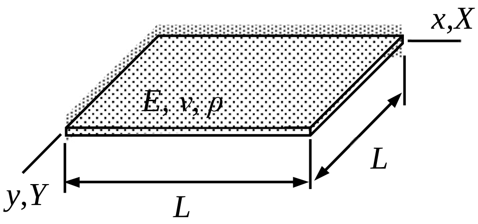
Answer $u_{X1}(t) = U\cos\left(t\sqrt{\frac{3}{2}\frac{3-\nu}{1-\nu^2}\frac{E}{\rho L^2}}\right) \quad t > 0$

9. Bars 1 and 3 of the structure shown are massless and bar 2 is rigid. Force F is acting on node 2. Determine the displacement $u_{Z2}(t)$ of node 2 for $t>0$, if the force is removed at $t=0$. Young's modulus of bars 1 and 3 is E and density of bar 2 is ρ. Cross-sectional area is constant A.



Answer $u_{Z2}(t)=F\dfrac{\sqrt{2}L}{EA}\cos\left(\dfrac{t}{L}\sqrt{\dfrac{E}{\sqrt{2}\rho}}\right)\quad t>0$

10. A plate is simply supported on two edges and free on the other two edges as shown. Use the approximation $w(x,y,t)=a(t)xy/L^2$ to determine the transverse displacement as function of time $t>0$. Material properties E, ν, and ρ are constants and thickness of the plate is h. At $t=0$, initial conditions are $\dot{w}(x,y,0)=0$ and $w(x,y,0)=Uxy/L^2$. Assume that the plate is thin so that the rotation part of the inertia term is negligible.



Answer $w(x,y,t)=U\cos\left(\sqrt{3\dfrac{G}{\rho}}\dfrac{h}{L^2}t\right)\dfrac{xy}{L^2}\quad t>0$

Determine the eigenvalue decomposition $\mathbf{A} = \mathbf{X}\boldsymbol{\lambda}\mathbf{X}^{-1}$ and $\sqrt{\mathbf{A}}$ when $\mathbf{A} = \begin{bmatrix} 4 & 0 \\ -1 & 1 \end{bmatrix}$.

Solution

Let us solve for the eigenvalues first from $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

$$\det\begin{bmatrix} 4-\lambda & 0 \\ -1 & 1-\lambda \end{bmatrix} = (4-\lambda)(1-\lambda) = 0 \quad \Leftrightarrow \quad \lambda = 1 \text{ or } \lambda = 4.$$

The corresponding eigenvectors $\mathbf{x}$ follow from $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$ when the eigenvalues are substituted there

$$\lambda_1 = 1: \begin{bmatrix} 4-1 & 0 \\ -1 & 1-1 \end{bmatrix}\begin{Bmatrix} a \\ 1 \end{Bmatrix} = 0 \quad \Rightarrow \quad a = 0 \quad \Rightarrow \quad \mathbf{x}_1 = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix},$$

$$\lambda_2 = 4: \begin{bmatrix} 4-4 & 0 \\ -1 & 1-4 \end{bmatrix}\begin{Bmatrix} a \\ 1 \end{Bmatrix} = 0 \quad \Rightarrow \quad a = -3 \quad \Rightarrow \quad \mathbf{x}_2 = \begin{Bmatrix} -3 \\ 1 \end{Bmatrix}.$$

Therefore

$$\mathbf{A} = \mathbf{X}\boldsymbol{\lambda}\mathbf{X}^{-1} = \begin{bmatrix} 0 & -3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1/3 & 1 \\ -1/3 & 0 \end{bmatrix}.$$

Let us use the definition: if $\mathbf{A} = \mathbf{X}\boldsymbol{\lambda}\mathbf{X}^{-1}$ then $f(\mathbf{A}) = \mathbf{X}f(\boldsymbol{\lambda})\mathbf{X}^{-1}$. When applied to the present case of a square root

$$\sqrt{\mathbf{A}} = \mathbf{X}\left(\pm\sqrt{\boldsymbol{\lambda}}\right)\mathbf{X}^{-1} = \begin{bmatrix} 0 & -3 \\ 1 & 1 \end{bmatrix}\left(\pm\begin{bmatrix} \sqrt{1} & 0 \\ 0 & \sqrt{4} \end{bmatrix}\right)\begin{bmatrix} 1/3 & 1 \\ -1/3 & 0 \end{bmatrix} = \pm\begin{bmatrix} 2 & 0 \\ -1/3 & 1 \end{bmatrix}.$$

Derive the consistent mass matrix $\mathbf{M}$ of a two-node beam element (bending in *xz*-plane). Assume that density is constant, cross section is constant, and the beam element is thin in the sense $t/h \ll 1$, so that $\delta w_{\Omega}^{\text{ine}} = -\delta w \rho A \ddot{w}$.

Solution

The starting is the virtual work density of inertia forces and the element approximation of the beam model (see the formulae collection)

$$w(x,t) = \begin{Bmatrix} (1-\xi)^2(1+2\xi) \\ h(1-\xi)^2\xi \\ (3-2\xi)\xi^2 \\ h\xi^2(\xi-1) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{z1}(t) \\ -\theta_{y1}(t) \\ u_{z2}(t) \\ -\theta_{y2}(t) \end{Bmatrix} \;\Rightarrow\; \delta w(x,t) = \begin{Bmatrix} \delta u_{z1}(t) \\ \delta\theta_{y1}(t) \\ \delta u_{z2}(t) \\ \delta\theta_{y2}(t) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} (1-\xi)^2(1+2\xi) \\ -h(1-\xi)^2\xi \\ (3-2\xi)\xi^2 \\ -h\xi^2(\xi-1) \end{Bmatrix},$$

$$\ddot{w}(x,t) = \begin{Bmatrix} (1-\xi)^2(1+2\xi) \\ -h\xi(1-\xi)^2 \\ (3-2\xi)\xi^2 \\ -h\xi^2(\xi-1) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \ddot{u}_{z1}(t) \\ \ddot{\theta}_{y1}(t) \\ \ddot{u}_{z2}(t) \\ \ddot{\theta}_{y2}(t) \end{Bmatrix} \quad (\text{here } \xi = \frac{x}{h}).$$

Virtual work expression of the inertia forces consists of terms taking into account translation and rotation of the cross-section. Here, rotation part is assumed to be negligible so that

$$\delta w_{\Omega}^{\text{ine}} = -\delta w \rho A \ddot{w}.$$

When the approximation is substituted there, virtual work density takes the form

$$\delta w_{\Omega}^{\text{ine}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \rho A \left\{ \begin{Bmatrix} (1-\xi)^2(1+2\xi) \\ -h\xi(1-\xi)^2 \\ (3-2\xi)\xi^2 \\ -h\xi^2(\xi-1) \end{Bmatrix} \begin{Bmatrix} (1-\xi)^2(1+2\xi) \\ -h\xi(1-\xi)^2 \\ (3-2\xi)\xi^2 \\ -h\xi^2(\xi-1) \end{Bmatrix}^{\mathrm{T}} \right\} \begin{Bmatrix} \ddot{u}_{z1} \\ \ddot{\theta}_{y1} \\ \ddot{u}_{z2} \\ \ddot{\theta}_{y2} \end{Bmatrix}.$$

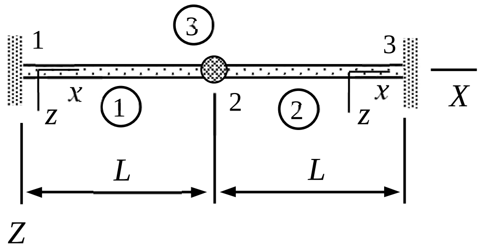
Integration over the spatial domain gives (use Mathematica in this step)

$$\delta W^{\text{ine}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{\rho A h}{420} \begin{bmatrix} 156 & -22h & 54 & 13h \\ -22h & 4h^2 & -13h & -3h^2 \\ 54 & -13h & 156 & 22h \\ 13h & -3h^2 & 22h & 4h^2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_{z1} \\ \ddot{\theta}_{y1} \\ \ddot{u}_{z2} \\ \ddot{\theta}_{y2} \end{Bmatrix}.$$

Therefore, the mass matrix

$$\mathbf{M} = \frac{\rho A h}{420}\begin{bmatrix} 156 & -22h & 54 & 13h \\ -22h & 4h^2 & -13h & -3h^2 \\ 54 & -13h & 156 & 22h \\ 13h & -3h^2 & 22h & 4h^2 \end{bmatrix}.$$

The XZ-plane structure shown consists of two *massless* beams and a homogeneous disk considered as a rigid body. Derive the equations of motion in terms of displacements u_{Z2} and θ_{Y2}. Young's modulus of the beam material and the second moment of area are E and I, and the mass and moment of inertia of the disk are m and J, respectively.



Solution

The non-zero displacement/rotation components of the structure are u_{Z2} and θ_{Y2}. Let us start with the element contributions. Since the beam is assumed to be massless, only the virtual work expressions of the internal forces (available in the formulae collection) is needed.

$$\delta W^1 = -\begin{Bmatrix} 0 \\ 0 \\ \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ u_{Z2} \\ \theta_{Y2} \end{Bmatrix} = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3}\begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix},$$

$$\delta W^2 = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \\ 0 \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \\ 0 \\ 0 \end{Bmatrix} = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3}\begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix}.$$

Element contribution of the rigid body (formulae collection) simplifies to

$$\delta W^3 = -\begin{Bmatrix} 0 \\ 0 \\ \delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} m\begin{Bmatrix} 0 \\ 0 \\ \ddot{u}_{Z2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ \delta\theta_{Y2} \\ 0 \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ J\ddot{\theta}_{Y2} \\ 0 \end{Bmatrix} = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{Bmatrix} \ddot{u}_{Z2} \\ \ddot{\theta}_{Y2} \end{Bmatrix}.$$

Virtual work expression of structure is the sum of element contributions.

$$\delta W = \delta W^1 + \delta W^2 + \delta W^3 = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EI}{L^3}\begin{bmatrix} 24 & 0 \\ 0 & 8L^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix} + \begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{Bmatrix} \ddot{u}_{Z2} \\ \ddot{\theta}_{Y2} \end{Bmatrix}\right).$$

Finally, principle of virtual work and the fundamental lemma of variation calculus imply a set of ordinary differential equations:

$$\frac{EI}{L^3}\begin{bmatrix} 24 & 0 \\ 0 & 8L^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix} + \begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{Bmatrix} \ddot{u}_{Z2} \\ \ddot{\theta}_{Y2} \end{Bmatrix} = 0. \quad \leftarrow$$

The angular speeds of free vibrations can be deduced from the stiffness and mass matrix of the equation system

$$\mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix} \text{ and } \mathbf{K} = \frac{EI}{L^3}\begin{bmatrix} 24 & 0 \\ 0 & 8L^2 \end{bmatrix} \Rightarrow \mathbf{\Omega}^2 = \mathbf{M}^{-1}\mathbf{K} = \frac{EI}{L^3}\begin{bmatrix} 24/m & 0 \\ 0 & 8L^2/J \end{bmatrix}.$$

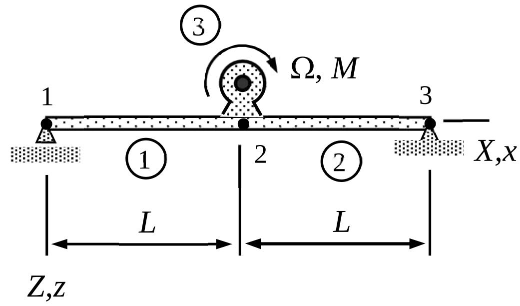
The angular speeds of free vibrations are the eigenvalues of $\mathbf{\Omega}$. Let us start with the eigenvalues of $\mathbf{\Omega}^2 = \mathbf{M}^{-1}\mathbf{K}$

$$\det(\frac{EI}{L^3}\begin{bmatrix} 24/m & 0 \\ 0 & 8L^2/J \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = (24\frac{EI}{mL^3} - \lambda)(8\frac{EI}{JL} - \lambda) = 0 \quad \Rightarrow \quad \lambda \in \{24\frac{EI}{mL^3}, 8\frac{EI}{JL}\}.$$

Eigenvalues of $\mathbf{\Omega}$ are square roots of eigenvalues of $\mathbf{\Omega}^2$

$$\omega_1 = \sqrt{\lambda_1} = \sqrt{24\frac{EI}{mL^3}} \quad \text{and} \quad \omega_2 = \sqrt{\lambda_2} = \sqrt{8\frac{EI}{JL}}.$$

The rotor of the machine shown rotates with angular speed Ω. Determine the bending stiffness EI so that the (smallest) angular speed of free vibrations of the foundation-machine system coincides with Ω. The foundation is modeled as two *massless* beams and the machine as particle of mass *M*. Assume that $\theta_{Y1} = -\theta_{Y3}$ and $\theta_{Y2} = 0$.



Solution

The non-zero displacement/rotation components of the structure are u_{Z2}, θ_{Y1}, and $\theta_{Y3} = -\theta_{Y1}$. Let us start with the element contributions. Since the beam is assumed to be massless, only the virtual work expressions of the internal forces (available in formulae collection) is needed.

$$\delta W^1 = -\begin{Bmatrix} 0 \\ \delta\theta_{Y1} \\ \delta u_{Z2} \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ \theta_{Y1} \\ u_{Z2} \\ 0 \end{Bmatrix} = -\begin{Bmatrix} \delta\theta_{Y1} \\ \delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3}\begin{bmatrix} 4L^2 & 6L \\ 6L & 12 \end{bmatrix}\begin{Bmatrix} \theta_{Y1} \\ u_{Z2} \end{Bmatrix},$$

$$\delta W^2 = -\begin{Bmatrix} \delta u_{Z2} \\ 0 \\ 0 \\ -\delta\theta_{Y1} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ 0 \\ 0 \\ -\theta_{Y1} \end{Bmatrix} = -\begin{Bmatrix} \delta\theta_{Y1} \\ \delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3}\begin{bmatrix} 4L^2 & 6L \\ 6L & 12 \end{bmatrix}\begin{Bmatrix} \theta_{Y1} \\ u_{Z2} \end{Bmatrix}.$$

Element contribution of the rigid body (formulae collection) simplifies to

$$\delta W^3 = -\begin{Bmatrix} 0 \\ 0 \\ \delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} m\begin{Bmatrix} 0 \\ 0 \\ \ddot{u}_{Z2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = -\begin{Bmatrix} \delta\theta_{Y1} \\ \delta u_{Z2} \end{Bmatrix}^{\mathrm{T}}\begin{bmatrix} 0 & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_{Y1} \\ \ddot{u}_{Z2} \end{Bmatrix}.$$

Virtual work expression of structure is the sum of element contributions.

$$\delta W = \delta W^1 + \delta W^2 + \delta W^3 = -\begin{Bmatrix} \delta\theta_{Y1} \\ \delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} (\frac{EI}{L^3}\begin{bmatrix} 8L^2 & 12L \\ 12L & 24 \end{bmatrix}\begin{Bmatrix} \theta_{Y1} \\ u_{Z2} \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_{Y1} \\ \ddot{u}_{Z2} \end{Bmatrix}) \ .$$

Finally, principle of virtual work and the fundamental lemma of variation calculus imply a differential algebraic system (DAE):

$$(\frac{EI}{L^3}\begin{bmatrix} 8L^2 & 12L \\ 12L & 24 \end{bmatrix}\begin{Bmatrix} \theta_{Y1} \\ u_{Z2} \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_{Y1} \\ \ddot{u}_{Z2} \end{Bmatrix} = 0 .$$

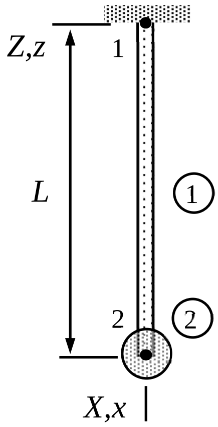
Let us eliminate the rotation from the differential equation by using the algebraic equation $8L^2\theta_{Y1} + 12Lu_{Z2} = 0 \quad \Leftrightarrow \quad \theta_{Y1} = -u_{Z2}3/(2L)$. Therefore

$$\frac{EI}{L^3}(12L\theta_{Y1} + 24u_{Z2}) + m\ddot{u}_{Z2} = 0 \quad \Leftrightarrow \quad \frac{EI}{L^3}6u_{Z2} + m\ddot{u}_{Z2} = 0 \quad \text{or} \quad \ddot{u}_{Z2} + 6\frac{EI}{mL^3}u_{Z2} = 0 .$$

The angular speed of free vibrations should match the angular speed of rotor (the condition of resonance and increasing amplitude in vibrations)

$$\omega = \sqrt{6\frac{EI}{mL^3}} = \Omega \quad \Rightarrow \quad EI = \frac{1}{6} mL^3 \Omega^2 .$$

The XZ-plane structure shown consists of a beam and a homogeneous disk considered as a rigid body. Derive the equations of motion in terms of u_{Z2}, θ_{Y2} and determine the angular speeds of free vibrations. Assume that mass of the beam is negligible compared to that of the disk and that the beam is inextensible in the axial direction. Young's modulus E of the beam material and the second moment of area I are constants. Mass and moment of inertia of the disk are m and $\mathbf{J} = \frac{1}{5} mL^2 \mathbf{I}$, respectively.



Solution

Virtual work expressions of the beam and rigid body elements are given by (inertia contribution is omitted from the beam contribution and rigid body has only the inertia part)

$$\delta W^1 = -\begin{Bmatrix} 0 \\ 0 \\ \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_{Z2} \\ \theta_{Y2} \end{Bmatrix} = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 12\dfrac{EI}{L^3} & 6\dfrac{EI}{L^2} \\ 6\dfrac{EI}{L^2} & 4\dfrac{EI}{L} \end{bmatrix} \begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix},$$

$$\delta W^2 = -\begin{Bmatrix} 0 \\ 0 \\ \delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} m \begin{Bmatrix} 0 \\ 0 \\ \ddot{u}_{Z2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ \delta\theta_{Y2} \\ 0 \end{Bmatrix}^{\mathrm{T}} \mathbf{J} \begin{Bmatrix} 0 \\ \ddot{\theta}_{Y2} \\ 0 \end{Bmatrix} = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} m & 0 \\ 0 & \dfrac{mL^2}{5} \end{bmatrix} \begin{Bmatrix} \ddot{u}_{Z2} \\ \ddot{\theta}_{Y2} \end{Bmatrix}.$$

Principle of virtual work $\delta W = \delta W^1 + \delta W^2 = 0 \quad \forall \delta \mathrm{a}$ gives

$$\delta W = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \left(\begin{bmatrix} 12\dfrac{EI}{L^3} & 6\dfrac{EI}{L^2} \\ 6\dfrac{EI}{L^2} & 4\dfrac{EI}{L} \end{bmatrix} \begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix} + \begin{bmatrix} m & 0 \\ 0 & mL^2/5 \end{bmatrix} \begin{Bmatrix} \ddot{u}_{Z2} \\ \ddot{\theta}_{Y2} \end{Bmatrix} \right) = 0 \quad \Rightarrow$$

$$\begin{bmatrix} 12\dfrac{EI}{L^3} & 6\dfrac{EI}{L^2} \\ 6\dfrac{EI}{L^2} & 4\dfrac{EI}{L} \end{bmatrix} \begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix} + \begin{bmatrix} m & 0 \\ 0 & \dfrac{mL^2}{5} \end{bmatrix} \begin{Bmatrix} \ddot{u}_{Z2} \\ \ddot{\theta}_{Y2} \end{Bmatrix} = 0.$$

The angular speeds ω_1 and ω_2 of free vibrations can be obtained (as square roots of the eigenvalues) from the eigenvalue decomposition $\mathbf{\Omega}^2 = \mathbf{M}^{-1}\mathbf{K} = \mathbf{X}\boldsymbol{\omega}^2\mathbf{X}^{-1}$. Let us start with

$$\mathbf{\Omega}^2 = \mathbf{M}^{-1}\mathbf{K} = \begin{bmatrix} m & 0 \\ 0 & \dfrac{mL^2}{5} \end{bmatrix}^{-1} \begin{bmatrix} 12\dfrac{EI}{L^3} & 6\dfrac{EI}{L^2} \\ 6\dfrac{EI}{L^2} & 4\dfrac{EI}{L} \end{bmatrix} = \begin{bmatrix} 12\dfrac{EI}{mL^3} & 6\dfrac{EI}{mL^2} \\ 30\dfrac{EI}{mL^4} & 20\dfrac{EI}{mL^3} \end{bmatrix}.$$

and continue with the characteristic equation

$$\det(\mathbf{\Omega}^2 - \lambda\mathbf{I}) = (12\frac{EI}{mL^3} - \lambda)(20\frac{EI}{mL^3} - \lambda) - 180\frac{EI}{mL^2}\frac{EI}{mL^4} = 0$$

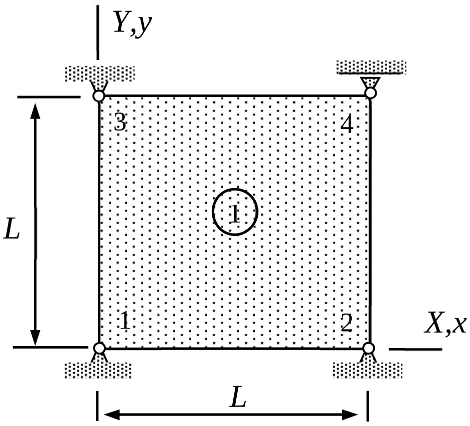
giving the eigenvalue solutions

$$\lambda_1 = 2\frac{EI}{mL^3} \quad \text{and} \quad \lambda_2 = 30\frac{EI}{mL^3} .$$

Finally, the angular speeds are square roots of the eigenvalues

$$\omega_1 = \sqrt{\lambda_1} = \sqrt{2}\sqrt{\frac{EI}{mL^3}} \quad \text{and} \quad \omega_2 = \sqrt{\lambda_2} = \sqrt{30}\sqrt{\frac{EI}{mL^3}} .$$

Node 4 of a thin rectangular slab (assume plane stress conditions) is allowed to move horizontally and nodes 1, 2, and 3 are fixed. Derive the initial value problem giving as its solution the horizontal displacement $u_{X4}(t)$ of node 4 as function of time, if $u_{X4}(0)=U$ and $\dot{u}_{X4}(0)=0$. Use just one bilinear element. Material parameters E, $\nu=0$, ρ and thickness h of the slab are constants.



Solution

Let us use the xy – coordinate system of the figure as the material coordinate system for the thin slab element 1. Only the shape function of node 4 is needed in the approximations:

$$u=\frac{x}{L}\frac{y}{L}u_{X4},\quad \frac{\partial u}{\partial x}=\frac{1}{L}\frac{y}{L}u_{X4},\quad \frac{\partial u}{\partial y}=\frac{x}{L}\frac{1}{L}u_{X4},\quad \ddot{u}=\frac{x}{L}\frac{y}{L}\ddot{u}_{X4}\quad \text{and}\quad v=0.$$

When the approximations are substituted there, virtual work densities of internal and inertia forces simplify to (here $\nu=0$)

$$\delta w_{\Omega}^{\text{int}}=-\begin{Bmatrix}\frac{y}{L^2}\delta u_{X4}\\ 0\\ \frac{x}{L^2}\delta u_{X4}\end{Bmatrix}^{\mathrm{T}}\frac{hE}{2}\begin{bmatrix}2&0&0\\0&2&0\\0&0&1\end{bmatrix}\begin{Bmatrix}\frac{y}{L^2}u_{X4}\\ 0\\ \frac{x}{L^2}u_{X4}\end{Bmatrix}=-\delta u_{X4}u_{X4}\frac{hE}{L^4}(y^2+\frac{1}{2}x^2),$$

$$\delta w_{\Omega}^{\text{ine}}=-\begin{Bmatrix}\delta u\\ \delta v\end{Bmatrix}^{\mathrm{T}}h\rho\begin{Bmatrix}\ddot{u}\\ \ddot{v}\end{Bmatrix}=-\begin{Bmatrix}\delta u_{X4}\frac{x}{L}\frac{y}{L}\\ 0\end{Bmatrix}^{\mathrm{T}}h\rho\begin{Bmatrix}\ddot{u}_{X4}\frac{x}{L}\frac{y}{L}\\ 0\end{Bmatrix}=-\delta u_{X4}\ddot{u}_{X4}\frac{h\rho}{L^4}x^2y^2.$$

Virtual work expressions are obtained by integrating the densities over the domain occupied by the element

$$\delta W^{\text{int}}=\int_0^L\int_0^L \delta w_{\Omega}^{\text{int}}dydx=-\delta u_{X4}u_{X4}\frac{hE}{2},$$

$$\delta W^{\text{ine}}=\int_0^L\int_0^L \delta w_{\Omega}^{\text{ine}}dxdy=-\delta u_{X4}\ddot{u}_{X4}\frac{1}{9}h\rho L^2.$$

Virtual work expression is the sum of the terms

$$\delta W=\delta W^{\text{int}}+\delta W^{\text{ine}}=-\delta u_{X4}(\frac{hE}{2}u_{X4}+\frac{1}{9}h\rho L^2\ddot{u}_{X4}).$$

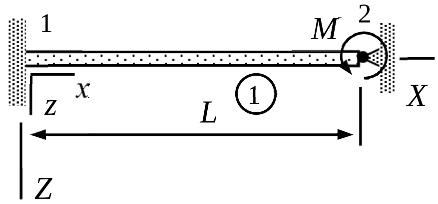
Principle of virtual work $\delta W=0\ \ \forall\delta\mathbf{a}$ and the fundamental lemma of variation calculus $\delta\mathbf{a}^{\mathrm{T}}\mathbf{F}=0\ \forall\delta\mathbf{a}\ \Leftrightarrow\ \mathbf{F}=0$ imply the ordinary differential equation

$$\frac{hE}{2}u_{X4}+\frac{1}{9}h\rho L^2\ddot{u}_{X4}=0\,.$$

Initial value problem consists of the second order ordinary differential equation above and additional conditions at $t=0$

$$\ddot{u}_{X4}+\frac{9}{2}\frac{E}{L^2\rho}u_{X4}=0 \quad t>0 \quad \text{and} \quad u_{X4}=U\,,\ \dot{u}_{X4}=0 \quad \text{at} \quad t=0\,.$$

The beam of the figure is subjected to moment M when $t<0$. At $t=0$, the moment is suddenly removed and the beam starts to vibrate. Derive the initial value problem giving $\theta_{Y2}(t)$ for $t>0$. The beam is thin so that the rotational part of the inertia term is negligible. The geometrical quantities of the cross-section are A, I and the material constants E and ρ.



Solution

Virtual work expression consists of parts coming from internal and inertial forces. Finding the equation of motion is the first thing to do. The beam element contributions needed in the problem are (the term having to do with rotational inertia is omitted)

$$\delta W^{\text{int}} = -\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \theta_{Y2} \end{Bmatrix} = -\delta\theta_{Y2} 4\frac{EI}{L}\theta_{Y2},$$

$$\delta W^{\text{ine}} = -\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{\rho AL}{420} \begin{bmatrix} 156 & -22L & 54 & 13L \\ -22L & 4L^2 & -13L & -3L^2 \\ 54 & -13L & 156 & 22L \\ 13L & -3L^2 & 22L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \ddot{\theta}_{Y2} \end{Bmatrix} = -\delta\theta_{Y2} \frac{\rho AL^3}{105}\ddot{\theta}_{Y2}$$

giving

$$\delta W^1 = -\delta\theta_{Y2}(4\frac{EI}{L}\theta_{Y2} + \frac{\rho AL^3}{105}\ddot{\theta}_{Y2}).$$

In terms of moment $P(t)$ (positive in the positive direction of Y-axis) which is piecewise constant in time so that $P(t)=M \quad t\le 0$ and $P(t)=0 \quad t>0$, the element contribution of the moment is

$$\delta W^2 = \delta\theta_{Y2}P.$$

Virtual work expression is the sum of element contributions:

$$\delta W = \delta W^1 + \delta W^2 = -\delta\theta_{Y2}(4\frac{EI}{L}\theta_{Y2} + \frac{\rho AL^3}{105}\ddot{\theta}_{Y2} - P) = 0.$$

Principle of virtual work and the fundamental lemma of variation calculus imply the ordinary differential equation

$$\delta W = -\delta\theta_{Y2}(4\frac{EI}{L}\theta_{Y2} + \frac{\rho AL^3}{105}\ddot{\theta}_{Y2} - P) = 0 \quad \forall \delta\theta_{Y2} \quad \Leftrightarrow$$

$$4\frac{EI}{L}\theta_{Y2} + \frac{\rho AL^3}{105}\ddot{\theta}_{Y2} - P = 0.$$

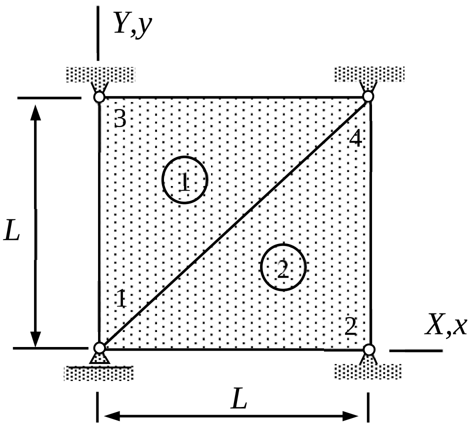
When $t \le 0$, external moment $P = M$ is acting on node 2 and the system is at rest. Therefore, the equation of motion becomes an equilibrium equation giving as its solution the initial rotation

$$\theta_{Y2} = \frac{1}{4}\frac{ML}{EI}.$$

When $t > 0$, external moment is zero and acceleration does not vanish. The initial value problem giving as its solution $\theta_{Y2}(t)$ for $t > 0$ takes the form

$$4\frac{EI}{L}\theta_{Y2} + \frac{\rho A L^3}{105}\ddot{\theta}_{Y2} = 0 \quad t > 0, \quad \theta_{Y2}(0) = \frac{1}{4}\frac{ML}{EI}, \quad \text{and} \quad \dot{\theta}_{Y2}(0) = 0 .$$

Node 1 of a thin rectangular slab (assume plane stress conditions) is allowed to move horizontally at node 1 whereas nodes 2, 3 and 4 are fixed. Derive the expression of horizontal displacement $u_{X1}(t)$ of node 1 as function of time, if $u_{X1}(0)=U$ and $\dot{u}_{X1}(0)=0$. Use two linear triangle elements. Material parameters E, ν, ρ and thickness h of the slab are constants.



Solution

Let us use the xy – coordinate system of the figure as the material coordinate system for the thin slab elements 1 and 2. Only the displacement $u_{X1}(t)$ of node 1 in the X – direction matters.

Shape functions of element 1 can be deduced from the figure. However, only the shape function $N_1 = 1 - y/L$ is needed as the other nodes are fixed. Approximations to the in-plane displacement components are $v \equiv 0$ and

$$u = (1-\frac{y}{L})u_{X1} \;\Rightarrow\; \frac{\partial u}{\partial x} = 0\,,\; \frac{\partial u}{\partial y} = -\frac{1}{L}u_{X1}\,, \text{ and } \ddot{u} = (1-\frac{y}{L})\ddot{u}_{X1}.$$

When the approximations above are substituted there, virtual work densities of internal and inertia forces simplify to

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} 0 \\ 0 \\ -\delta u_{X1}/L \end{Bmatrix}^{\mathrm{T}} \frac{hE}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ -u_{X1}/L \end{Bmatrix} = -\delta u_{X1}\frac{hE}{2L^2(1+\nu)}u_{X1},$$

$$\delta w_{\Omega}^{\text{ine}} = -\begin{Bmatrix} \delta u_{X1}(1-y/L) \\ 0 \end{Bmatrix}^{\mathrm{T}} h\rho \begin{Bmatrix} \ddot{u}_{X1}(1-y/L) \\ 0 \end{Bmatrix} = -\delta u_{X1}(1-\frac{y}{L})^2 h\rho \ddot{u}_{X1}.$$

Integration over the domain occupied by the element gives the virtual work expression. The limits of the double integral over a triangle are not constants (equation of the tilted edge is $y = x$)

$$\delta W^1 = \int_0^L \int_x^L (\delta w_{\Omega}^{\text{int}} + \delta w_{\Omega}^{\text{ine}})dydx \quad \Rightarrow$$

$$\delta W^1 = \int_0^L [-\delta u_{X1}\frac{hE}{2L^2(1+\nu)}u_{X1}(L-x) - \frac{L}{3}\delta u_{X1}(1-\frac{x}{L})^3 h\rho\ddot{u}_{X1}]dx \quad \Rightarrow$$

$$\delta W^1 = -\delta u_{X1}\frac{h}{12}(3\frac{E}{1+\nu}u_{X1} + \rho L^2 \ddot{u}_{X1})\,.$$

In the same manner, shape functions of element 2 can be deduced from the figure. Only $N_1 = 1 - x/L$ is needed as the other nodes are fixed. Approximations to the in-plane displacement components are $v \equiv 0$ and

$$u=(1-\frac{x}{L})u_{X1} \quad \Rightarrow \quad \frac{\partial u}{\partial y}=0\,,\ \ \frac{\partial u}{\partial x}=-\frac{1}{L}u_{X1}\,,\ \ \text{and}\ \ \ddot{u}=(1-\frac{x}{L})\ddot{u}_{X1}.$$

When the approximations are substituted there, virtual work densities of internal and inertia forces simplify to

$$\delta w_{\Omega}^{\text{int}}=-\begin{Bmatrix}-\delta u_{X1}/L\\0\\0\end{Bmatrix}^{\text{T}}\frac{hE}{1-\nu^2}\begin{bmatrix}1&\nu&0\\\nu&1&0\\0&0&(1-\nu)/2\end{bmatrix}\begin{Bmatrix}-u_{X1}/L\\0\\0\end{Bmatrix}=-\delta u_{X1}\frac{hE}{L^2(1-\nu^2)}u_{X1},$$

$$\delta w_{\Omega}^{\text{ine}}=-\begin{Bmatrix}(1-x/L)\delta u_{X1}\\0\end{Bmatrix}^{\text{T}}h\rho\begin{Bmatrix}(1-x/L)\ddot{u}_{X1}\\0\end{Bmatrix}=-\delta u_{X1}(1-\frac{x}{L})^2h\rho\ddot{u}_{X1}.$$

Integration over the domain occupied by the element gives the virtual work expression (notice the limits of the double integral and the order of the integrations)

$$\delta W^2=\int_0^L\int_0^x(\delta w_{\Omega}^{\text{int}}+\delta w_{\Omega}^{\text{ine}})dydx \quad \Rightarrow$$

$$\delta W^2=\int_0^L[-\delta u_{X1}\frac{hE}{L^2(1-\nu^2)}u_{X1}x-\frac{1}{L}\delta u_{X1}(1-\frac{x}{L})^2h\rho\ddot{u}_{X1}x]dx \quad \Rightarrow$$

$$\delta W^2=-\delta u_{X1}\frac{h}{12}(6\frac{E}{1-\nu^2}u_{X1}+\rho L^2\ddot{u}_{X1}).$$

Virtual work expression of a structure is sum over the element contributions

$$\delta W=\delta W^1+\delta W^2=-\delta u_{X1}\frac{h}{12}(3\frac{E}{1+\nu}u_{X1}+\rho L^2\ddot{u}_{X1})-\delta u_{X1}\frac{h}{12}(6\frac{E}{1-\nu^2}u_{X1}+\rho L^2\ddot{u}_{X1}) \quad \Leftrightarrow$$

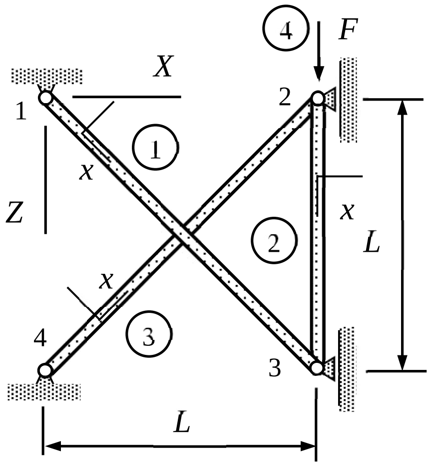
$$\delta W=-\delta u_{X1}\frac{h}{12}(\frac{3-\nu}{1-\nu^2}3Eu_{X1}+2\rho L^2\ddot{u}_{X1}).$$

Principle of virtual work $\delta W=0\ \ \forall\delta\mathbf{a}$ and the fundamental lemma of variation calculus $\delta\mathbf{a}^{\text{T}}\mathbf{F}=0\ \forall\delta\mathbf{a} \ \Leftrightarrow\ \mathbf{F}=0$ imply

$$\frac{3-\nu}{1-\nu^2}3Eu_{X1}+2\rho L^2\ddot{u}_{X1}=0 \quad \text{or} \quad \ddot{u}_{X1}+\Omega^2u_{X1}=0 \ \ \text{in which}\ \ \Omega^2=\frac{3}{2}\frac{3-\nu}{1-\nu^2}\frac{E}{\rho L^2}.$$

What remains, is solving for the displacement from the ordinary differential equation above for $t>0$ and the initial conditions $u_{X1}(0)=U$ and $\dot{u}_{X1}(0)=0$. Solution to equations is (this can be shown, e.g., by substituting the solution in the equations above)

$$u_{X1}(t)=U\cos(\sqrt{\frac{3}{2}\frac{3-\nu}{1-\nu^2}\frac{E}{\rho L^2}}t) \quad t>0. \ \leftarrow$$



Bars 1 and 3 of the structure shown are massless and bar 2 is rigid. Force F is acting on node 2. Determine the displacement $u_{Z2}(t)$ of node 2 for $t>0$, if the force is removed at $t=0$. Young's modulus of bars 1 and 3 is E and density of bar 2 is ρ. Cross-sectional area is constant A.

Solution

Only the displacement of nodes 2 and 3 in the Z-direction matter. As bar 2 is known to be rigid, vertical displacements of nodes 2 and 3 coincide i.e. $u_{Z2}=u_{Z3}$. Bar element contributions of the formulae collection are

$$\delta W^{\text{int}}=-\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\frac{EA}{h}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix} \text{ and } \delta W^{\text{ine}}=-\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\frac{\rho Ah}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{Bmatrix}\ddot{u}_{x1}\\ \ddot{u}_{x2}\end{Bmatrix}.$$

From the figure, the nodal displacement and length of bar 1 are $u_{x1}=0$, $u_{x2}=u_{Z2}/\sqrt{2}$ and $h=\sqrt{2}L$. As the bar is assumed to be massless, inertia term vanishes and

$$\delta W^{1}=-\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\frac{EA}{h}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix}=-\begin{Bmatrix}0\\ \delta u_{Z2}\end{Bmatrix}^{\mathrm{T}}\frac{EA}{\sqrt{8}L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}0\\ u_{Z2}\end{Bmatrix}=-\delta u_{Z2}\frac{EA}{\sqrt{8}L}u_{Z2}.$$

The relationships for bar 2 are $u_{x1}=u_{Z2}$, $u_{x2}=u_{Z2}$ and $h=L$. As the axial displacements coincide, internal part vanishes and

$$\delta W^{2}=-\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\frac{\rho Ah}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{Bmatrix}\ddot{u}_{x1}\\ \ddot{u}_{x2}\end{Bmatrix}=-\begin{Bmatrix}\delta u_{Z2}\\ \delta u_{Z2}\end{Bmatrix}^{\mathrm{T}}\frac{\rho AL}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{Bmatrix}\ddot{u}_{Z2}\\ \ddot{u}_{Z2}\end{Bmatrix}=-\rho AL\delta u_{Z2}\ddot{u}_{Z2}.$$

The relationships for bar 3 are $u_{x1}=0$, $u_{x2}=-u_{Z2}/\sqrt{2}$ and $h=\sqrt{2}L$. As the bar is assumed to be massless

$$\delta W^{3}=-\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\frac{EA}{h}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix}=-\begin{Bmatrix}0\\ -\delta u_{Z2}\end{Bmatrix}^{\mathrm{T}}\frac{EA}{\sqrt{8}L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}0\\ -u_{Z2}\end{Bmatrix}=-\delta u_{Z2}\frac{EA}{\sqrt{8}L}u_{Z2}.$$

Point force $P(t)$ acting on node 2 is piecewise constant in time so that $P(t)=F\quad t\le 0$ and $P(t)=0\quad t>0$. Virtual work expression is

$$\delta W^{4}=\delta u_{Z2}P.$$

Virtual work expression of the structure is the sum of element contributions

$$\delta W=\delta W^{1}+\delta W^{2}+\delta W^{3}+\delta W^{4}=-\delta u_{Z2}\left(\frac{EA}{\sqrt{2}L}u_{Z2}+\rho AL\ddot{u}_{Z2}-P\right).$$

Principle of virtual work $\delta W = 0 \quad \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus $\delta \mathbf{a}^{\mathrm{T}} \mathbf{F} = 0 \ \forall \delta \mathbf{a} \quad \Leftrightarrow \quad \mathbf{F} = 0$ imply that

$$\frac{EA}{\sqrt{2}L} u_{Z2} + \rho A L \ddot{u}_{Z2} - P = 0 .$$

When $t \leq 0$, u_{Z2} does not depend on time and therefore $\ddot{u}_{Z2} = \dot{u}_{Z2} = 0$. As the second derivative vanishes and $P = F$, the ordinary differential equation simplifies to an algebraic one giving

$$\frac{EA}{\sqrt{2}L} u_{Z2} - F = 0 \quad \Leftrightarrow \quad u_{Z2} = \frac{\sqrt{2}L}{EA} F \quad \text{when} \quad t \leq 0 .$$

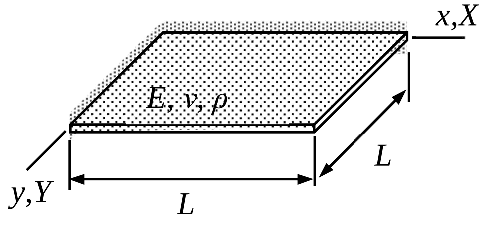
When $t > 0$, $P = 0$ and the initial value problem for the displacement becomes (notice that the initial conditions are taken from the solution for $t \leq 0$)

$$\frac{EA}{\sqrt{2}L} u_{Z2} + \rho A L \ddot{u}_{Z2} = 0 \quad t > 0 , \quad u_{Z2}(0) = \frac{\sqrt{2}L}{EA} F \quad \text{and} \quad \dot{u}_{Z2}(0) = 0 .$$

Solution to the equations is given by

$$u_{Z2}(t) = F \frac{\sqrt{2}L}{EA} \cos\left(\frac{t}{L} \sqrt{\frac{E}{\sqrt{2}\rho}}\right) \quad t > 0 . \quad \leftarrow$$

A plate is simply supported on two edges and free on the other two edges as shown. Use the approximation $w(x,y,t)=a(t)xy/L^2$ to determine the transverse displacement as function of time $t>0$. Material properties E, ν, and ρ are constants and thickness of the plate is h. At $t=0$, initial conditions are $\dot{w}(x,y,0)=0$ and $w(x,y,0)=Uxy/L^2$. Assume that the plate is thin so that the rotation part of the inertia term is negligible.



Solution

Only the bending mode of the plate matters. When the approximation $w=a(t)xy/L^2$ is substituted there, virtual work densities of internal and inertia forces (without the rotation part) of the plate simplify to (shear modulus $G=E/(2+2\nu)$)

$$\delta w_{\Omega}^{\text{int}}=-\begin{Bmatrix}\partial^2\delta w/\partial x^2\\ \partial^2\delta w/\partial y^2\\ 2\partial^2\delta w/\partial x\partial y\end{Bmatrix}^{\mathrm{T}}\frac{h^3}{12}\frac{E}{1-\nu^2}\begin{bmatrix}1&\nu&0\\ \nu&1&0\\ 0&0&(1-\nu)/2\end{bmatrix}\begin{Bmatrix}\partial^2 w/\partial x^2\\ \partial^2 w/\partial y^2\\ 2\partial^2 w/\partial x\partial y\end{Bmatrix}=-\delta a\frac{1}{L^4}\frac{h^3}{3}Ga\,,$$

$$\delta w_{\Omega}^{\text{ine}}=-\begin{Bmatrix}\partial\delta w/\partial x\\ \partial\delta w/\partial y\end{Bmatrix}^{\mathrm{T}}\frac{t^3}{12}\rho\begin{Bmatrix}\partial\ddot{w}/\partial x\\ \partial\ddot{w}/\partial y\end{Bmatrix}-\delta wt\rho\ddot{w}=-\delta a(\frac{x}{L})^2(\frac{y}{L})^2h\rho\ddot{a}$$

in which h is thickness of the plate. Integration over the domain occupied by the element gives the virtual work expressions

$$\delta W^{\text{int}}=\int_0^L\int_0^L\delta w_{\Omega}^{\text{int}}dydx=\int_0^L\int_0^L-\delta a\frac{1}{L^4}\frac{h^3}{3}Gadydx=-\delta a\frac{1}{L^2}\frac{h^3}{3}Ga\,,$$

$$\delta W^{\text{ine}}=\int_0^L\int_0^L\delta w_{\Omega}^{\text{ine}}dydx=\int_0^L\int_0^L-\delta a(\frac{x}{L})^2(\frac{y}{L})^2h\rho\ddot{a}dxdy=-\delta a\frac{L^2}{9}h\rho\ddot{a}\,.$$

Virtual work expression of the structure consists of the internal and inertia parts

$$\delta W=\delta W^{\text{int}}+\delta W^{\text{ext}}=-\delta a(\frac{1}{L^2}\frac{h^3}{3}Ga+\frac{L^2}{9}h\rho\ddot{a})\,.$$

Principle of virtual work $\delta W=0\quad\forall\delta\mathbf{a}$ and the fundamental lemma of variation calculus $\delta\mathbf{a}^{\mathrm{T}}\mathbf{F}=0\ \forall\delta\mathbf{a}\quad\Leftrightarrow\quad\mathbf{F}=0$ imply

$$\frac{1}{L^2}\frac{h^3}{3}Ga+\frac{L^2}{9}h\rho\ddot{a}=0\,.$$

What remains, is solving for the displacement from the initial value problem

$$\ddot{a}+3\frac{Gh^2}{\rho L^4}a=0\quad t>0\,,\quad a(0)=U\,,\quad \dot{a}(0)=0\,.$$

Solution to equations is (this can be shown e.g. by substituting the solution in the equations above)

$$a(t) = U\cos\left(\sqrt{3\frac{G}{\rho}}\frac{h}{L^2}t\right) \quad t > 0\,.$$

Finally, substituting the solution to parameter $a(t)$ into the approximation gives

$$w(x, y, t) = U\cos\left(\sqrt{3\frac{G}{\rho}}\frac{h}{L^2}t\right)\frac{xy}{L^2}\,.$$

Name________________________ Student number________________

Assignment 1

Determine the eigenvalues λ_1, λ_2 and the corresponding eigenvectors $\mathbf{x}_1$, $\mathbf{x}_2$ of the 2×2 matrix $\mathbf{A}$. Write down also the eigenvalue decomposition $\mathbf{A} = \mathbf{X}\boldsymbol{\lambda}\mathbf{X}^{-1}$. Consider

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Solution template

Eigenvalues given by the characteristic equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

$$\det\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 - 1 = 0 \quad \Rightarrow \quad \lambda_1 = 1 \text{ or } \lambda_2 = 3$$

Non-zero eigenvectors given by equations $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$

$$\lambda_1 : \quad \begin{bmatrix} 2-1 & 1 \\ 1 & 2-1 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0 \quad \Rightarrow \quad \mathbf{x}_1 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

$$\lambda_2 : \quad \begin{bmatrix} 2-3 & 1 \\ 1 & 2-3 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0 \quad \Rightarrow \quad \mathbf{x}_2 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Matrix of eigenvalues $\boldsymbol{\lambda}$, matrix of eigenvectors $\mathbf{X} = [\mathbf{x}_1 \quad \mathbf{x}_2]$ and its inverse $\mathbf{X}^{-1}$

$$\boldsymbol{\lambda} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{X}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Eigenvalue decomposition $\mathbf{A} = \mathbf{X}\boldsymbol{\lambda}\mathbf{X}^{-1}$

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

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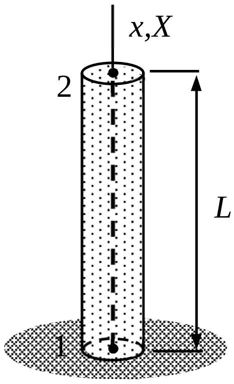

Assignment 2

Assuming that node 1 of the bar shown is fixed, derive the expression of the axial displacement $u_{X2}(t)$ at the free end for $t>0$. The initial conditions at $t=0$ are $u_{X2}(0)=0$ and $\dot{u}_{X2}(0)=V$.

Solution template

Virtual work expression of internal and inertia forces of the bar model is given by

$$\delta W=-\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}(\frac{EA}{h}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix}+\frac{\rho Ah}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{Bmatrix}\ddot{u}_{x1}\\ \ddot{u}_{x2}\end{Bmatrix})$$

in which A is the cross-sectional area, E is the Young's modulus, and ρ is the density of the material. In terms of the displacement components of the structural coordinate system

$$\delta W=-\begin{Bmatrix}0\\ \delta u_{X2}\end{Bmatrix}^{\mathrm{T}}(\frac{EA}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}0\\ u_{X2}\end{Bmatrix}+\frac{\rho AL}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{Bmatrix}0\\ \ddot{u}_{X2}\end{Bmatrix}).$$

$$\delta W=-\delta u_{X2}(\frac{EA}{L}u_{X2}+\frac{\rho AL}{3}\ddot{u}_{X2})$$

Initial value problem, consisting of an ordinary differential equation (implied by the virtual work expression) and initial conditions, is given by

$$\ddot{u}_{X2}+3\frac{E}{\rho L^2}u_{X2}=0 \qquad t>0,$$

$$u_{X2}=0 \quad \text{and} \quad \dot{u}_{X2}=V \qquad t=0.$$

Expression $u_{X2}(t)=A\sin(\omega t)$, which describes a harmonic periodic motion, satisfies all the equations with selections

$$\omega=\sqrt{3\frac{E}{\rho L^2}} \quad \text{and} \quad A=\frac{V}{\omega}=V/\sqrt{3\frac{E}{\rho L^2}}.$$

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Assignment 3

Consider the equations of motion

$$\frac{GJ}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}\theta_{X1}\\ \theta_{X2}\end{Bmatrix}+\frac{\rho LJ}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix}\ddot{\theta}_{X1}\\ \ddot{\theta}_{X2}\end{Bmatrix}=0$$

for the end point rotations of a certain torsion bar of length L. Above, J is the second moment of area with respect to the axis of the bar (polar moment), G is the shear modulus, and ρ is the density of material. Derive the angular speeds and the corresponding modes of the free vibrations.

Solution template

The set of ordinary differential equations as given by the principle of virtual work $\mathbf{M}\ddot{\mathbf{a}}+\mathbf{K}\mathbf{a}=0$ consists of the inertia and stiffness parts. The symmetric mass matrix $\mathbf{M}$ and the stiffness matrix $\mathbf{K}$ depend on the structure. Angular speeds of the free vibrations are the eigenvalues of $\mathbf{\Omega}=\sqrt{\mathbf{M}^{-1}\mathbf{K}}$. In practice, it is easier to calculate first the eigenvalues $\mathbf{\Omega}^2=\mathbf{M}^{-1}\mathbf{K}$ as the eigenvalues of $\mathbf{\Omega}$ are the square roots of those for $\mathbf{\Omega}^2$ and the eigenvectors coincide.

In the present case, the matrices are

$$\mathbf{K}=\frac{GJ}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} \text{ and } \mathbf{M}=\frac{\rho JL}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

Therefore

$$\mathbf{\Omega}^2=\mathbf{M}^{-1}\mathbf{K}=\begin{bmatrix}\frac{2G}{\rho L^2} & -\frac{2G}{\rho L^2}\\ -\frac{2G}{\rho L^2} & \frac{2G}{\rho L^2}\end{bmatrix}.$$

In the eigenvalue problem of matrix $\mathbf{A}$, the goal is to find all pairs $(\lambda,\mathbf{x})$ such that $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0$. The linear homogeneous equation system can have a non-zero solution only if $\det(\mathbf{A}-\lambda\mathbf{I})=0$. The eigenvalues are obtained as solutions to this characteristic equation. The characteristic equation for the eigenvalues of $\mathbf{\Omega}^2$ is

$$\det(\mathbf{\Omega}^2-\lambda\mathbf{I})=(\frac{2G}{\rho L^2}-\lambda)^2-(\frac{2G}{\rho L^2})^2=0.$$

The two solutions for the eigenvalues are ($(a-\lambda)^2-b^2=0 \Leftrightarrow \lambda=a\pm b$)

$$\lambda_1 = \frac{4G}{\rho L^2} \quad \text{and} \quad \lambda_2 = 0\,.$$

The corresponding eigenvectors are obtained as solutions to $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$. The eigenvectors are not unique and it is enough to find some of them. However, the eigenvectors should be linearly independent so that, e.g., the zero vector is not a valid choice.

$$\lambda_1: \qquad \begin{bmatrix} \frac{2G}{\rho L^2} - \frac{4G}{\rho L^2} & -\frac{2G}{\rho L^2} \\ -\frac{2G}{\rho L^2} & \frac{2G}{\rho L^2} - \frac{4G}{\rho L^2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0 \quad \Rightarrow \quad \mathbf{x}_1 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix},$$

$$\lambda_2: \qquad \begin{bmatrix} \frac{2G}{\rho L^2} - 0 & -\frac{2G}{\rho L^2} \\ -\frac{2G}{\rho L^2} & \frac{2G}{\rho L^2} - 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0 \quad \Rightarrow \quad \mathbf{x}_2 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

The representation of the matrix in terms of its eigenvalues and eigenvectors $\mathbf{\Omega}^2 = \mathbf{X}\boldsymbol{\lambda}\mathbf{X}^{-1}$ implies that $\mathbf{\Omega} = \mathbf{X}\sqrt{\boldsymbol{\lambda}}\mathbf{X}^{-1}$. As taking a square root of the diagonal matrix means just taking the square roots of the diagonal terms, the angular speeds of the free vibrations

$$(\omega_1, \mathbf{x}_1) = (\sqrt{\lambda_1}, \mathbf{x}_1) = (\frac{2}{L}\sqrt{\frac{G}{\rho}}, \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}) \quad \text{and} \quad (\omega_2, \mathbf{x}_2) = (\sqrt{\lambda_2}, \mathbf{x}_2) = (0, \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}). \quad \leftarrow$$

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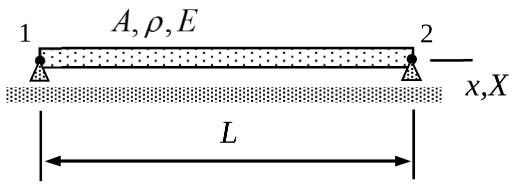
Assignment 4

A bar is free to move in the horizontal direction as shown. Determine the angular velocities of the free vibrations and the corresponding modes. Use one bar element of nodes 1 and 2. Cross-sectional are A, density ρ of the material, and Young's modulus E of the material are constants.



Solution template

The non-zero displacement/rotation components of the structure are u_{X1} and u_{X2}. Let us start with the element contributions for the internal and inertia parts

$$\delta W^{\text{int}} = -\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}} \frac{EA}{h}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix},\quad \delta W^{\text{ine}} = -\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}} \frac{\rho Ah}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{Bmatrix}\ddot{u}_{x1}\\ \ddot{u}_{x2}\end{Bmatrix}.$$

As the axes of the material and structural coordinate systems coincide, virtual work expression of the structure takes the form

$$\delta W = -\begin{Bmatrix}\delta u_{X1}\\ \delta u_{X2}\end{Bmatrix}^{\mathrm{T}} (\begin{bmatrix}\frac{EA}{L} & -\frac{EA}{L}\\ -\frac{EA}{L} & \frac{EA}{L}\end{bmatrix}\begin{Bmatrix}u_{X1}\\ u_{X2}\end{Bmatrix} + \begin{bmatrix}\frac{\rho AL}{3} & \frac{\rho AL}{6}\\ \frac{\rho AL}{6} & \frac{\rho AL}{3}\end{bmatrix}\begin{Bmatrix}\ddot{u}_{X1}\\ \ddot{u}_{X2}\end{Bmatrix}).$$

Principle of virtual work and fundamental lemma of variation calculus imply the set of ordinary differential equations

$$\begin{bmatrix}\frac{EA}{L} & -\frac{EA}{L}\\ -\frac{EA}{L} & \frac{EA}{L}\end{bmatrix}\begin{Bmatrix}u_{X1}\\ u_{X2}\end{Bmatrix} + \begin{bmatrix}\frac{\rho AL}{3} & \frac{\rho AL}{6}\\ \frac{\rho AL}{6} & \frac{\rho AL}{3}\end{bmatrix}\begin{Bmatrix}\ddot{u}_{X1}\\ \ddot{u}_{X2}\end{Bmatrix} = 0. \leftarrow$$

The angular speeds of free vibrations can be deduced from the stiffness and mass matrix of the equation system

$$\mathbf{M} = \begin{bmatrix}\frac{\rho AL}{3} & \frac{\rho AL}{6}\\ \frac{\rho AL}{6} & \frac{\rho AL}{3}\end{bmatrix} \text{ and } \mathbf{K} = \begin{bmatrix}\frac{EA}{L} & -\frac{EA}{L}\\ -\frac{EA}{L} & \frac{EA}{L}\end{bmatrix} \Rightarrow \boldsymbol{\Omega}^2 = \mathbf{M}^{-1}\mathbf{K} = \begin{bmatrix}\frac{6E}{\rho L^2} & -\frac{6E}{\rho L^2}\\ -\frac{6E}{\rho L^2} & \frac{6E}{\rho L^2}\end{bmatrix}.$$

The angular speeds of free vibrations are the eigenvalues of $\mathbf{\Omega}$. Let us start with the eigenvalues $\lambda=\omega^2$ of $\mathbf{\Omega}^2=\mathbf{M}^{-1}\mathbf{K}$ to get

$$\det(\begin{bmatrix} \dfrac{6E}{\rho L^2} & -\dfrac{6E}{\rho L^2} \\ -\dfrac{6E}{\rho L^2} & \dfrac{6E}{\rho L^2} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = \det(\begin{bmatrix} \dfrac{6E}{\rho L^2}-\lambda & -\dfrac{6E}{\rho L^2} \\ -\dfrac{6E}{\rho L^2} & \dfrac{6E}{\rho L^2}-\lambda \end{bmatrix}) = (\frac{6E}{\rho L^2}-\lambda)^2-(\frac{6E}{\rho L^2})^2=0 \quad \Rightarrow$$

$$\lambda_1=0 \quad \text{and} \quad \lambda_2=\frac{12E}{\rho L^2}.$$

The corresponding modes follow from the linear equation system of the eigenvalue problem when the eigenvalues are substituted there (one by one)

$$\lambda_1=0 \quad \text{and} \quad \begin{bmatrix} \dfrac{6E}{\rho L^2} & -\dfrac{6E}{\rho L^2} \\ -\dfrac{6E}{\rho L^2} & \dfrac{6E}{\rho L^2} \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{X2} \end{Bmatrix} = 0 \quad \Rightarrow \quad \mathbf{x}_1 = \begin{Bmatrix} u_{X1} \\ u_{X2} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix},$$

$$\lambda_2=\frac{12E}{\rho L^2} \quad \text{and} \quad \begin{bmatrix} -\dfrac{6E}{\rho L^2} & -\dfrac{6E}{\rho L^2} \\ -\dfrac{6E}{\rho L^2} & -\dfrac{6E}{\rho L^2} \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{X2} \end{Bmatrix} = 0 \quad \Rightarrow \quad \mathbf{x}_2 = \begin{Bmatrix} u_{X1} \\ u_{X2} \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}.$$

As $\omega=\sqrt{\lambda}$, the angular velocities of the free vibrations and the associated modes are

$$(\omega_1,\mathbf{x}_1)=(0,\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}) \quad \text{and} \quad (\omega_2,\mathbf{x}_2)=(\sqrt{\frac{12E}{\rho L^2}},\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}). \quad \leftarrow$$

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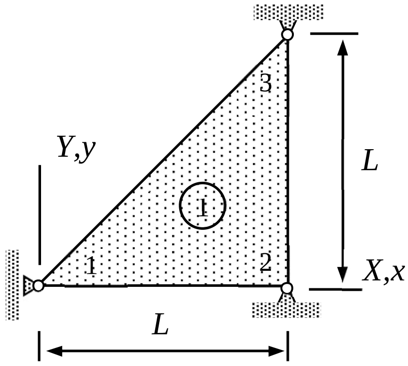
Assignment 5



Determine the angular speed of free vibrations for the thin triangular slab shown. Assume plane stress conditions. The material properties E, ν, ρ and thickness h of the slab are constants. Use the approximations $u=0$ and $v=(1-x/L)u_{Y1}$ in which the nodal value u_{Y1} is a function of time.

Solution

The virtual work densities of the internal and inertia forces for the thin slab model (plane stress conditions assumed) are given by

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta u/\partial y+\partial\delta v/\partial x \end{Bmatrix}^{\mathrm{T}} t[E]_{\sigma} \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y+\partial v/\partial x \end{Bmatrix} \text{ and } \delta w_{\Omega}^{\text{ine}} = -\begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} t\rho \begin{Bmatrix} \ddot{u} \\ \ddot{v} \end{Bmatrix}$$

where the elasticity matrix of the plane stress

$$[E]_{\sigma} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}.$$

The approximations to the displacement components are given (the linear interpolants of the nodal values can also be deduced easily from the figure). Hence

$$u=0 \text{ and } v=(1-\frac{x}{L})u_{Y1} \quad \Rightarrow$$

$$\frac{\partial u}{\partial x}=0\,,\ \frac{\partial u}{\partial y}=0\,,\ \frac{\partial\delta u}{\partial x}=0\,,\ \frac{\partial\delta u}{\partial y}=0\,,\ \delta u=0\,,\ \ddot{u}=0$$

$$\frac{\partial v}{\partial x}=-\frac{1}{L}u_{Y1},\ \frac{\partial v}{\partial y}=0\,,\ \frac{\partial\delta v}{\partial x}=-\frac{1}{L}\delta u_{Y1},\ \frac{\partial\delta v}{\partial y}=0\,,\ \delta v=(1-\frac{x}{L})\delta u_{Y1},\ \ddot{v}=(1-\frac{x}{L})\ddot{u}_{Y1}$$

When the approximations are substituted there, virtual work density of the internal and inertia forces simplify to

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} 0 \\ 0 \\ -\delta u_{Y1}/L \end{Bmatrix}^{\mathrm{T}} \frac{hE}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ -u_{Y1}/L \end{Bmatrix} = -\delta u_{Y1}\frac{h}{L^2}Gu_{Y1},$$

$$\delta w_{\Omega}^{\text{ine}} = -\begin{Bmatrix} 0 \\ (1-\frac{x}{L})\delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} h\rho \begin{Bmatrix} 0 \\ (1-\frac{x}{L})\ddot{u}_{Y1} \end{Bmatrix} = -\delta u_{Y1} h\rho (1-\frac{x}{L})^2 \ddot{u}_{Y1} .$$

Integrations over the triangular domain of the element gives

$$\delta W^{\text{int}} = \int_A \delta w_{\Omega}^{\text{int}} \mathrm{d}A = \delta w_{\Omega}^{\text{int}} \frac{L^2}{2} = -\delta u_{Y1} \frac{h}{2} G u_{Y1} ,$$

$$\delta W^{\text{ext}} = \int_A \delta w_{\Omega}^{\text{ine}} \mathrm{d}A = \int_0^L (\int_0^x \delta w_{\Omega}^{\text{ine}} dy) dx \quad \Rightarrow$$

$$\delta W^{\text{ext}} = -\delta u_{Y1} h\rho \int_0^L (\int_0^x (1-\frac{x}{L})^2 dy) dx \, \ddot{u}_{Y1} = -\delta u_{Y1} h\rho \int_0^L x(1-\frac{x}{L})^2 dx \, \ddot{u}_{Y1} = -\delta u_{Y1} h\rho \frac{1}{12} L^2 \ddot{u}_{Y1} .$$

Virtual work expression of the structure takes the form

$$\delta W = -\delta u_{Y1} (\frac{t}{2} G u_{Y1} + h\rho \frac{1}{12} L^2 \ddot{u}_{Y1}) .$$

Principle of virtual work $\delta W = 0 \ \forall \delta \mathrm{a}$ and the fundamental lemma of variation calculus give

$$\frac{h}{2} G u_{Y1} + h\rho \frac{1}{12} L^2 \ddot{u}_{Y1} = 0 \quad \Leftrightarrow \quad \ddot{u}_{Y1} + 6\frac{G}{\rho L^2} u_{Y1} = 0 .$$

As the ordinary differential equation is of the form $\ddot{u} + \omega^2 u = 0$, the angular speed of free vibrations is

$$\omega = \sqrt{6\frac{G}{\rho L^2}} . \quad \Leftarrow$$

MEC-E8001

Finite Element Analysis

2023

WEEK 5: STABILITY ANALYSIS

4 STABILITY ANALYSIS

4.1 STRAIN MEASURES 7

4.2 BUCKLING OF BEAMS AND PLATES 14

4.3 STABILITY FEA 22

4.4 ELEMENT CONTRIBUTIONS 36

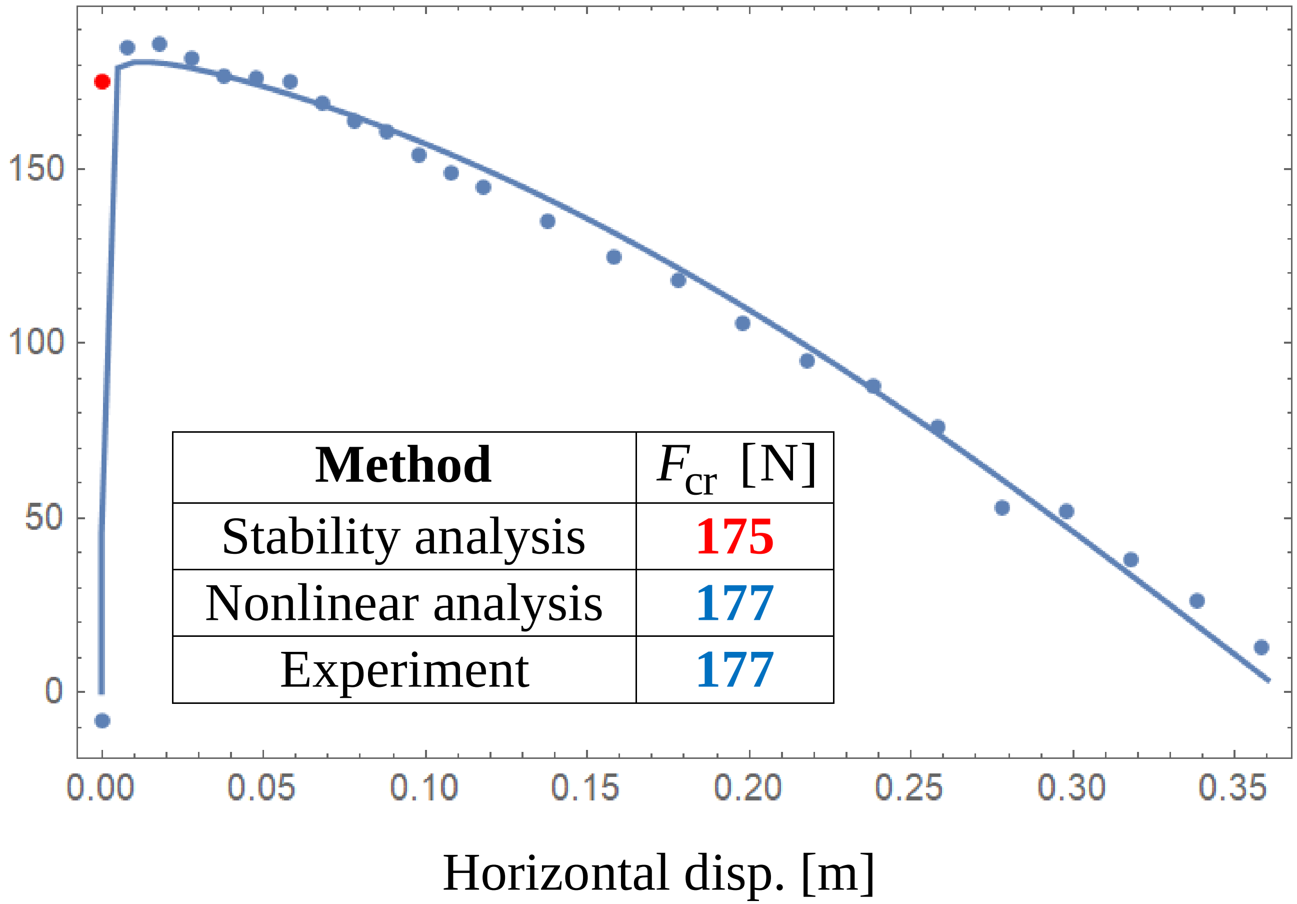
LEARNING OUTCOMES

Students are able to solve the weekly lecture problems, home problems, and exercise problems about stability FEA:

- ☐ Stability of structures and principle of virtual work for large displacements
- ☐ Aim of stability analysis and stability FEA
- ☐ Beam and plate element contributions for stability analysis

BUCKLING EXPERIMENT

EXPERIMENT VS. MODEL



Method	F_{cr} [N]
Stability analysis	**175**
Nonlinear analysis	**177**
Experiment	**177**

BALANCE LAWS OF MECHANICS

Balance of mass (def. of a body or a material volume) Mass of a body is constant

Balance of linear momentum (Newton 2) The rate of change of linear momentum within a material volume equals the external force resultant acting on the material volume. 🡐

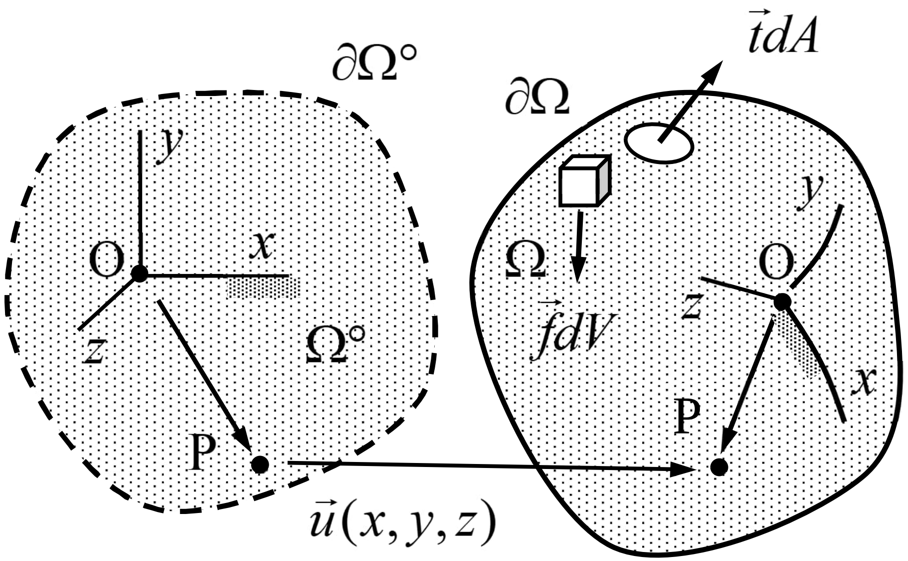
Balance of angular momentum (Cor. of Newton 2) The rate of change of angular momentum within a material volume equals the external moment resultant acting on the material volume. 🡐

Balance of energy (Thermodynamics 1)

Entropy growth (Thermodynamics 2)

INITIAL AND DEFORMED DOMAINS

Assuming equilibrium on the initial domain Ω°, the aim is to find a new equilibrium on the deformed domain Ω, when e.g., external forces acting on the structure are changed.



The local forms of the balance laws are concerned with the deformed domain which depends on the displacement! Precise treatment of large displacements requires modifications in stress and strain concepts of linear theory.

4.1 STRAIN MEASURES

A rigid body motion should not induce strains! A proper strain measure with this respect is always non-linear in displacement components (small strain $|h-h^{\circ}|\ll h^{\circ}$)

Linear strain $\quad \varepsilon=\dfrac{h}{h^{\circ}}-1 \quad \Rightarrow \quad 2\vec{\vec{\varepsilon}}=\nabla\vec{u}+(\nabla\vec{u})_{\mathrm{c}}$ $\quad$ epsilon

Green-Lagrange $\quad \mathrm{E}=\dfrac{1}{2}[(\dfrac{h}{h^{\circ}})^{2}-1] \quad \Rightarrow \quad 2\vec{\vec{\mathrm{E}}}=\nabla\vec{u}+(\nabla\vec{u})_{\mathrm{c}}+\nabla\vec{u}\cdot(\nabla\vec{u})_{\mathrm{c}}$ $\quad$ capital epsilon

Superscript ° refers to the initial geometry and subscript c denotes conjugate tensor. At the initial geometry, material coordinate system is usually assumed to be Cartesian so that $\nabla=\vec{i}\,\partial/\partial x+\vec{j}\,\partial/\partial y+\vec{k}\,\partial/\partial z$.

GENERALIZED HOOKE'S LAW

Under small displacement assumption, the model for an isotropic homogeneous material can be expressed as

Strain-stress: $$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} = [E]^{-1}\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} \text{ and } \begin{Bmatrix} \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{Bmatrix} = \frac{1}{2G}\begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix}$$

Strain-displacement: $$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} = \begin{Bmatrix} \partial u_x / \partial x \\ \partial u_y / \partial y \\ \partial u_z / \partial z \end{Bmatrix} \text{ and } \begin{Bmatrix} \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{Bmatrix} = \frac{1}{2}\begin{Bmatrix} \partial u_x / \partial y + \partial u_y / \partial x \\ \partial u_y / \partial z + \partial u_z / \partial y \\ \partial u_z / \partial x + \partial u_x / \partial z \end{Bmatrix}$$

Above, E is the Young's modulus, ν the Poisson's ratio, and $G = E/(2+2\nu)$ the shear modulus. Strain and stress are assumed to be symmetric.

GREEN-LAGRANGE STRAIN

A rigid body motion should not induce strains! The proper strain measures with this respect are non-linear in displacement components

$$\begin{Bmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ \mathrm{E}_{zz} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} + \frac{1}{2} \begin{Bmatrix} (\partial u_x / \partial x)^2 + (\partial u_y / \partial x)^2 + (\partial u_z / \partial x)^2 \\ (\partial u_x / \partial y)^2 + (\partial u_y / \partial y)^2 + (\partial u_z / \partial y)^2 \\ (\partial u_x / \partial z)^2 + (\partial u_y / \partial z)^2 + (\partial u_z / \partial z)^2 \end{Bmatrix},$$

$$\begin{Bmatrix} \mathrm{E}_{xy} \\ \mathrm{E}_{yz} \\ \mathrm{E}_{zx} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{Bmatrix} + \frac{1}{2} \begin{Bmatrix} (\partial u_x / \partial x)(\partial u_x / \partial y) + (\partial u_y / \partial x)(\partial u_y / \partial y) + (\partial u_z / \partial x)(\partial u_z / \partial y) \\ (\partial u_x / \partial y)(\partial u_x / \partial z) + (\partial u_y / \partial y)(\partial u_y / \partial z) + (\partial u_z / \partial y)(\partial u_z / \partial z) \\ (\partial u_x / \partial z)(\partial u_x / \partial x) + (\partial u_y / \partial z)(\partial u_y / \partial x) + (\partial u_z / \partial z)(\partial u_z / \partial x) \end{Bmatrix}.$$

All measures boil down to the definition of linear displacement analysis when strains and rotations of material elements are small!

EXAMPLE. Consider a bar whose left end is simply supported (joint) and right end is free to move. Displacement of the typical particle (x, y) of the bar

$$\begin{Bmatrix} u_x \\ u_y \end{Bmatrix} = \begin{bmatrix} (1+\varepsilon)\cos\alpha - 1 & -\sin\alpha \\ (1+\varepsilon)\sin\alpha & \cos\alpha - 1 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix}$$

describes rotation with angle α and length increase $\Delta h = \varepsilon h$. Determine the linear strain component ε_{xx} and the Green-Lagrange strain component E_{xx}.

Answer $\mathrm{E}_{xx} = \varepsilon + \frac{1}{2}\varepsilon^2 \approx \varepsilon$ when $|\varepsilon| \ll 1$ and $\varepsilon_{xx} = (1+\varepsilon)\cos\alpha - 1 \approx \varepsilon$ when $|\alpha| \ll 1$

- Partial derivatives of the displacement components are

$$\begin{Bmatrix} \partial u_x / \partial x \\ \partial u_y / \partial x \end{Bmatrix} = \begin{Bmatrix} (1+\varepsilon)\cos\alpha - 1 \\ (1+\varepsilon)\sin\alpha \end{Bmatrix} \text{ and } \begin{Bmatrix} \partial u_x / \partial y \\ \partial u_y / \partial y \end{Bmatrix} = \begin{Bmatrix} -\sin\alpha \\ \cos\alpha - 1 \end{Bmatrix}.$$

- Linear and Green-Lagrange axial strain components

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = (1+\varepsilon)\cos\alpha - 1 \text{ and } \mathrm{E}_{xx} = \frac{\partial u_x}{\partial x} + \frac{1}{2}(\frac{\partial u_x}{\partial x})^2 + \frac{1}{2}(\frac{\partial u_y}{\partial x})^2 = \varepsilon + \frac{1}{2}\varepsilon^2. \quad \Leftarrow$$

The former depends strongly on the rotation angle even when ε is small although pure rotation should not cause any strains. The latter does not depend on the rotation at all. Also, for small length changes, the Green-Lagrange strain is close to the relative change of length $\varepsilon = \Delta h / h^\circ$.

ELASTIC MATERIAL

Under the assumption of large displacements and small strains the Green-Lagrange strain measure does not differ much from the linear setting with small displacements and small strains. Constitutive equations

$$\begin{Bmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ \mathrm{E}_{zz} \end{Bmatrix} = \frac{1}{C} \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} \begin{Bmatrix} S_{xx} \\ S_{yy} \\ S_{zz} \end{Bmatrix} \text{ and } \begin{Bmatrix} \mathrm{E}_{xy} \\ \mathrm{E}_{yz} \\ \mathrm{E}_{zx} \end{Bmatrix} = \frac{1}{2G} \begin{Bmatrix} S_{xy} \\ S_{yz} \\ S_{zx} \end{Bmatrix},$$

with material parameters C (which replaces E), ν, and $G = C/(2+2\nu)$ are same as those of the linear case, are assumed to simplify the setting. Also, the uni-axial and two-axial (plane) stress and strain relationships follows just by using strains instead of engineering strains and C instead of E.

STRAIN COMPONENTS FOR BUCKLING ANALYSIS

In buckling analysis of beams and plates, the setting is simplified by using the displacement assumptions of the small displacement theory and only the most significant terms of the Green-Lagrange axial strain expressions:

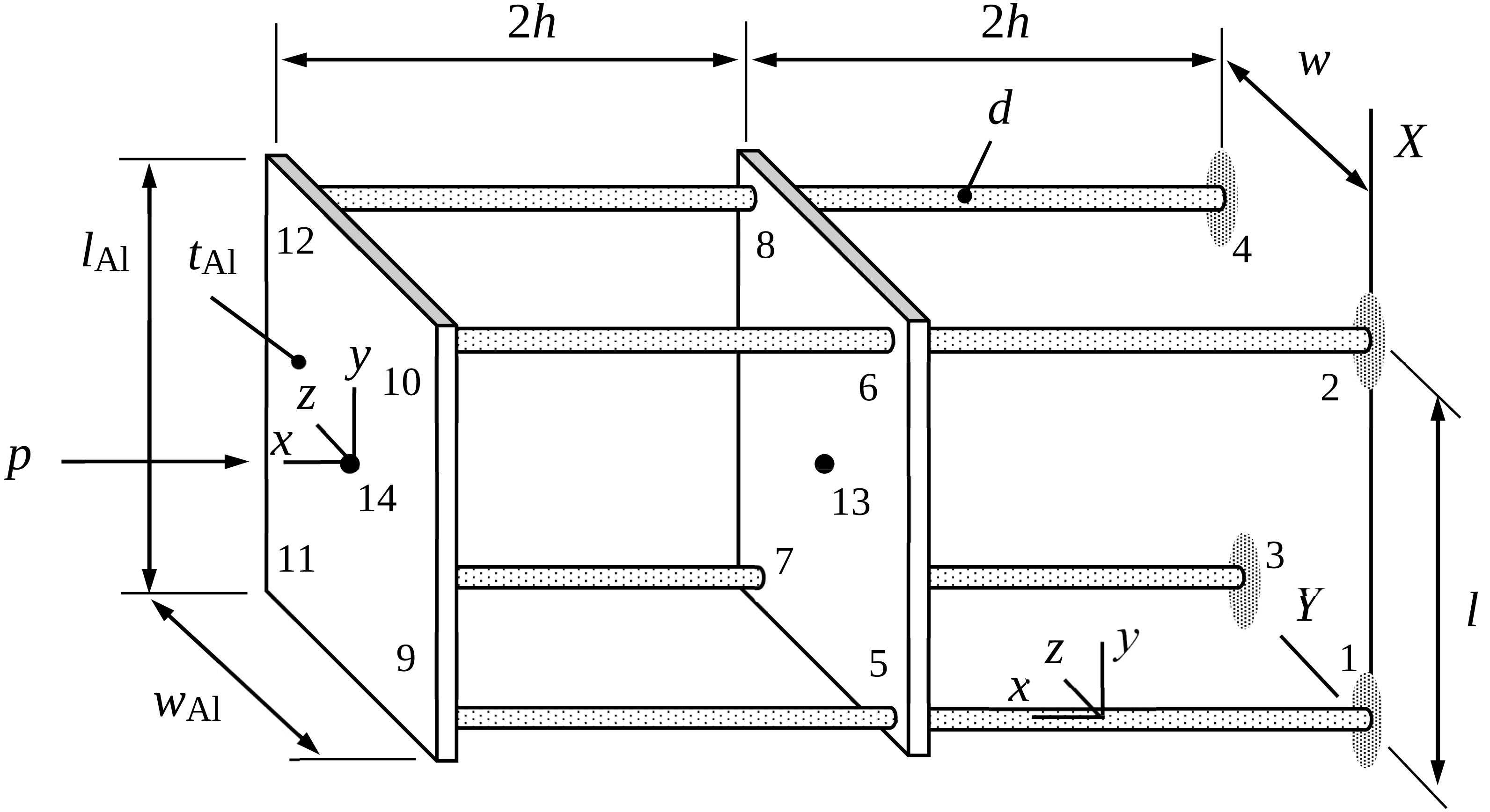
Beam: $\mathrm{E}_{xx} \approx \varepsilon_{xx} + \frac{1}{2}(\frac{dv}{dx})^2 + \frac{1}{2}(\frac{dw}{dx})^2$ and $S_{xx} = C\mathrm{E}_{xx}$,

Plate: $\begin{Bmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ 2\mathrm{E}_{xy} \end{Bmatrix} \approx \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} + \frac{1}{2} \begin{Bmatrix} (\partial w / \partial x)^2 \\ (\partial w / \partial y)^2 \\ 2(\partial w / \partial x)(\partial w / \partial y) \end{Bmatrix}$ and $\begin{Bmatrix} S_{xx} \\ S_{yy} \\ S_{xy} \end{Bmatrix} = [E]_{\sigma} \begin{Bmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ 2\mathrm{E}_{xy} \end{Bmatrix}$.

In large displacement theory, also the displacement assumptions need to be modified to keep the idea of rigid body motion of cross-sections (beams) or line segments (plates).

4.2 BUCKLING OF BEAMS AND PLATES

In stability analysis, the goal is to find the critical value p_{cr} of parameter p (force, load, displacement etc.) so that the zero and non-zero bending solutions may co-exist.



NON-LINEAR COUPLING OF THE MODES

Buckling analysis considers the coupling of the bar/ thin-slab and bending modes. There, the bending mode is affected by the bar/thin slab mode but not the other way round. Equilibrium equations for the Bernoulli beam model and Kirchhoff plate model bending modes change to

$$EI\frac{d^4w}{dx^4}-N\frac{d^2w}{dx^2}=0 \quad x\in\Omega,$$

Non-linear coupling of the thin slab and plate bending modes

$$D\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)-N_{xx}\frac{\partial^2 w}{\partial x^2}-2N_{xy}\frac{\partial^2 w}{\partial x\partial y}-N_{yy}\frac{\partial^2 w}{\partial y^2}=0 \quad (x,y)\in\Omega,$$

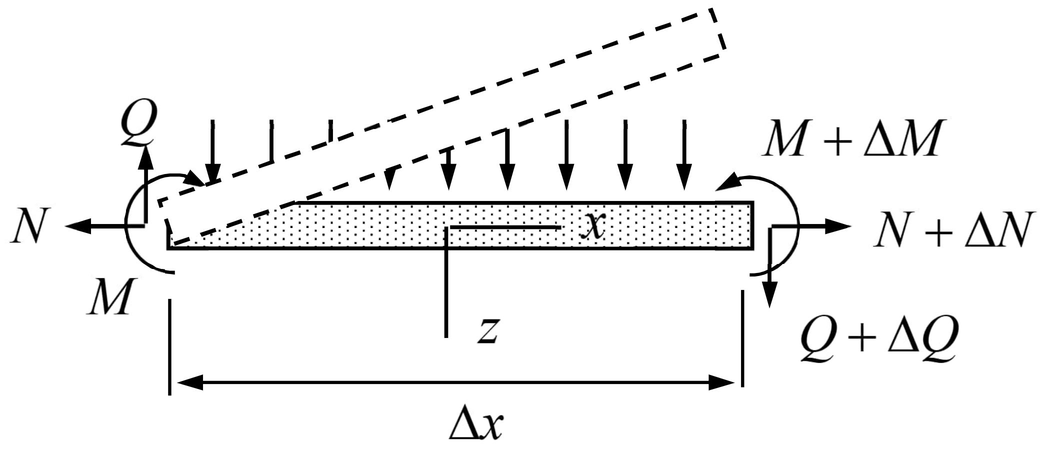
assuming that the axial or in-plane stress resultants of the bar mode or thin-slab mode are constants (as one of the assumptions).

- The simplified buckling analysis also considers the effect of the normal force on bending. By considering the equilibrium of a beam element in $xz-$plane

$$\frac{dN}{dx}=0 \quad x\in]0,L[,$$

$$\frac{dM}{dx}-Q+N\frac{dw}{dx}=0 \quad x\in]0,L[,$$

$$\frac{dQ}{dx}+f_z=0 \quad x\in]0,L[,$$



where $M=-EId^2w/dx^2$ and $N=EAdu/dx$. The more precise equilibrium equations couple the bar and bending modes (bending mode is affected by the bar mode but not the other way around).

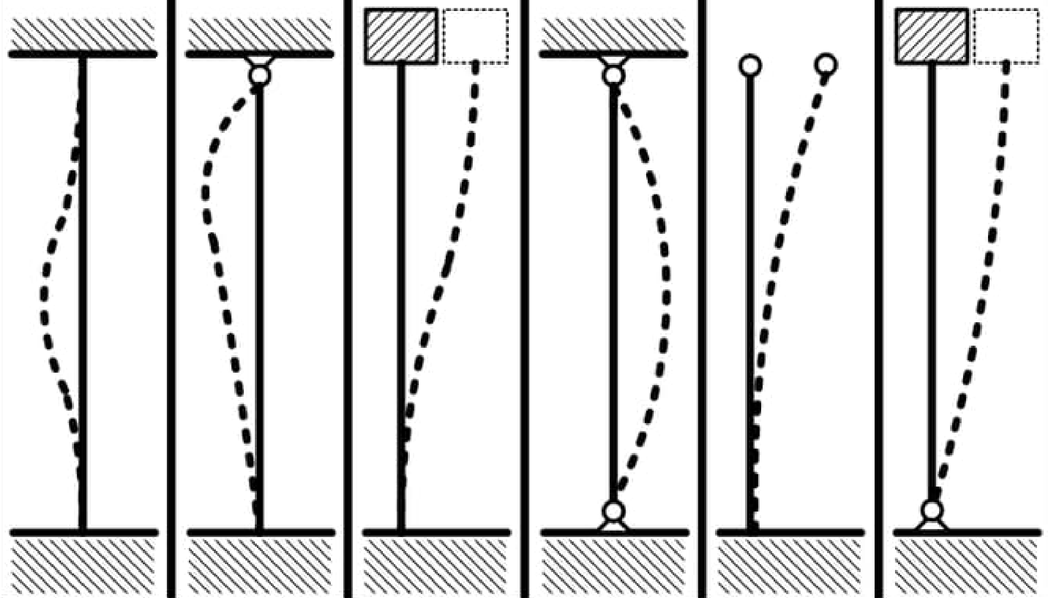
- The table by George William Herbert - *Own work, after Table C.1.8.1 in Steel Construction Manual, 8th edition, 2nd revised printing, American Institute of Steel Construction, 1987, CC BY-SA 2.5*, is based on the equilibrium equation

$$-EI\frac{d^4w}{dx^4}+N\frac{d^2w}{dx^2}=0 \quad x\in]0,L[\,,$$

for the $xz-$plane bending with a compressive $N=-p$. The different values in the table are due to different boundary and symmetry conditions imposed on the generic solution

$$w=a+bx+c\sin(\sqrt{\frac{p}{EI}}x)+d\cos(\sqrt{\frac{p}{EI}}x)\,.$$

BUCKLING LOAD OF BEAM $p_{cr} = \pi^2 \dfrac{EI}{(KL)^2}$

Buckled shape of column shown by dashed line						
Theoretical K value	0.5	0.7	1.0	1.0	2.0	2.0
Recommended design value K	0.65	0.80	1.2	1.0	2.10	2.0

VIRTUAL WORK DENSITIES

The refined virtual work densities contain also the work done by the axial force in bending. The simplified forms of Green-Lagrange strains in derivation of virtual work densities give additional contributions (coupling terms)

Beam: $\delta w_{\Omega}^{\text{sta}} = -\begin{Bmatrix} d\delta v / dx \\ d\delta w / dx \end{Bmatrix}^{\mathrm{T}} N \begin{Bmatrix} dv / dx \\ dw / dx \end{Bmatrix}$ where $N = EA\dfrac{du}{dx}$,

Plate: $\delta w_{\Omega}^{\text{sta}} = -\begin{Bmatrix} \partial\delta w / \partial x \\ \partial\delta w / \partial y \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} N_{xx} & N_{xy} \\ N_{xy} & N_{yy} \end{bmatrix} \begin{Bmatrix} \partial w / \partial x \\ \partial w / \partial y \end{Bmatrix}$, $\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = t[E]_{\sigma} \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix}$.

Coupling affects the bending mode only as the variations are concerned with the transverse displacements of the bending modes.

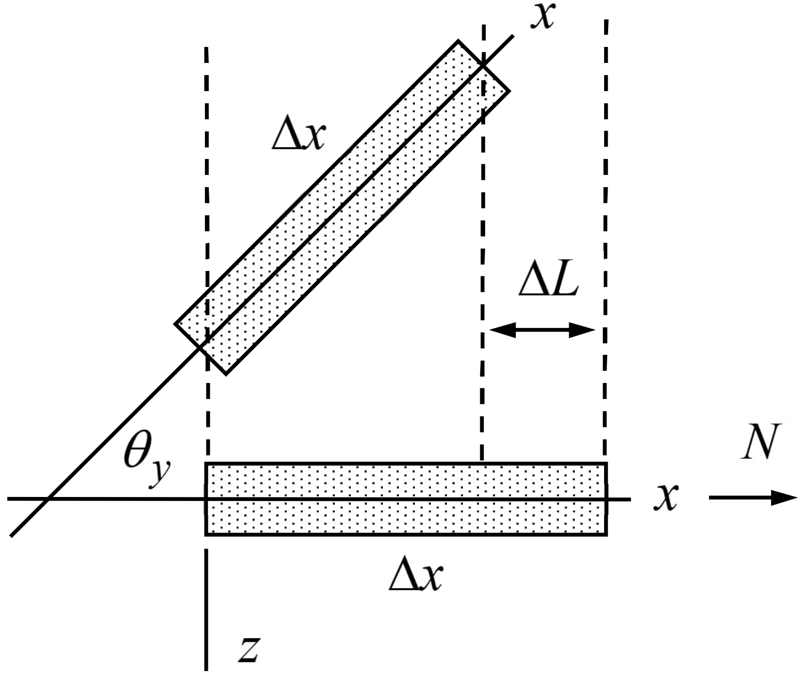
- Derivation based on the virtual work of the external axial force, is also possible. The axial displacement of the free end of a cantilever *due to the bending only* can be obtained by considering an inextensible material element of length Δx. The length change in the direction of the force is given by (Taylor series $\cos(x) = 1 - x^2/2 + \ldots$)

$$\Delta L = \Delta x - \Delta x \cos\theta_y \quad \Rightarrow$$

$$\frac{dL}{dx} = 1 - \cos\theta_y \approx \frac{1}{2}\theta_y^2 = \frac{1}{2}(-\frac{dw}{dx})^2 \quad \Rightarrow$$

$$u(L) = -\int_0^L \frac{1}{2}(\frac{dw}{dx})^2 dx \quad \Rightarrow$$

$$\delta u(L) = -\int_0^L \frac{d\delta w}{dx}\frac{dw}{dx}dx.$$



- Virtual work of the external force due to the bending effect is therefore given by

$$\delta W^{\text{sta}} = N\delta u(L) = N\int_0^L \frac{d\delta w}{dx}\frac{dw}{dx}dx.$$

- In the simultaneous bending in both directions, the length change of an inextensible material element Δx in the axial direction is given by

$$\Delta L = \Delta x - \Delta x \cos\theta_y \cos\theta_z \approx \Delta x - \Delta x(1-\frac{1}{2}\theta_y^2)(1-\frac{1}{2}\theta_z^2) \approx \Delta x \frac{1}{2}(\theta_y^2 + \theta_z^2) \Rightarrow$$

$$\Delta L \approx \Delta x \frac{1}{2}(\frac{dw}{dx}\frac{dw}{dx} + \frac{dv}{dx}\frac{dv}{dx}) \Rightarrow \delta u(L) = -\int_0^L (\frac{d\delta w}{dx}\frac{dw}{dx} + \frac{d\delta v}{dx}\frac{dv}{dx})dx$$

Hence, the coupling term is the sum of coupling terms of the planar problems!

4.3 STABILITY FEA

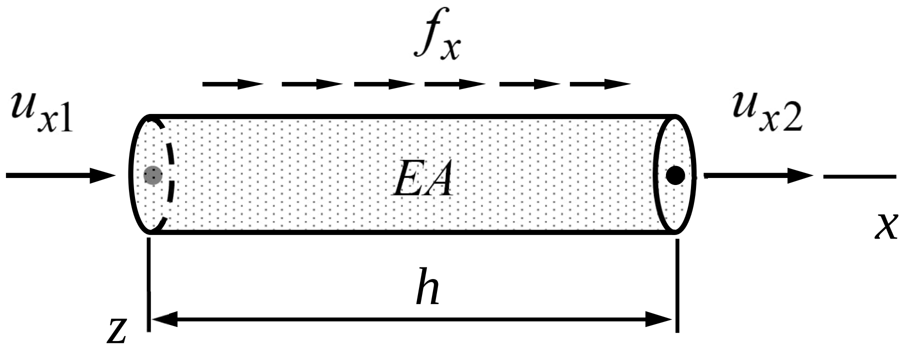
- Model a structure as a collection of beam, plate, etc. elements. Derive the element contributions $\delta W^e = \delta W^{\text{int}} + \delta W^{\text{ext}} + \delta W^{\text{sta}}$ in terms of the nodal displacement and rotation components of the structural coordinate system.

- Sum the element contributions to end up with the virtual work expression of the structure $\delta W = \sum_{e \in E} \delta W^e$. Re-arrange to get $\delta W = -\delta \mathbf{a}^{\mathrm{T}} \mathbf{R}(\mathbf{a})$.

- Use the principle of virtual work $\delta W = 0 \;\forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus for $\delta \mathbf{a} \in \mathbb{R}^n$ to deduce the equilibrium equations $\mathbf{R}(\mathbf{a}) = 0$. Finally, find the values of the loading parameter p making the solution non-unique. In practice, solve for the bar/thin slab modes from the linear part and use the solution to express the axial and in-plane stress resultants of the non-linear terms in terms of p.

BAR MODE

In terms of the nodal axial forces N_{x1}, N_{x2} and nodal displacements u_{x1}, u_{x2} virtual work expressions of the internal and external forces take the forms

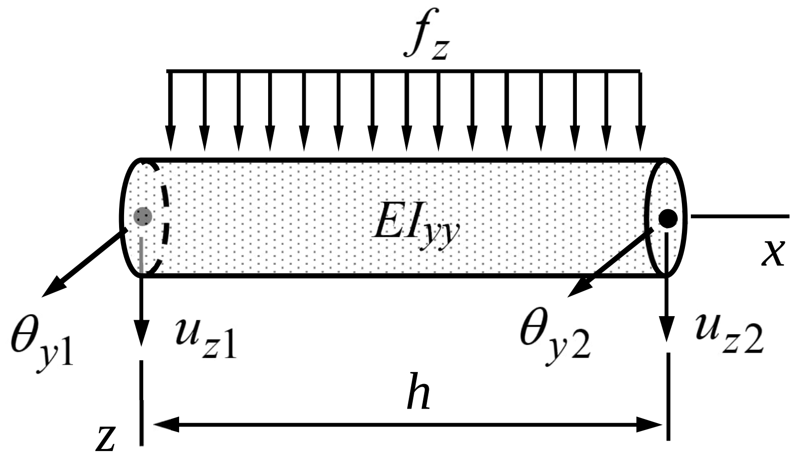


$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} N_{x1} \\ N_{x2} \end{Bmatrix} \quad \text{where} \quad \begin{Bmatrix} N_{x1} \\ N_{x2} \end{Bmatrix} = \frac{EA}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix},$$

$$\delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{f_x h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

BENDING MODE (*xz*-plane)

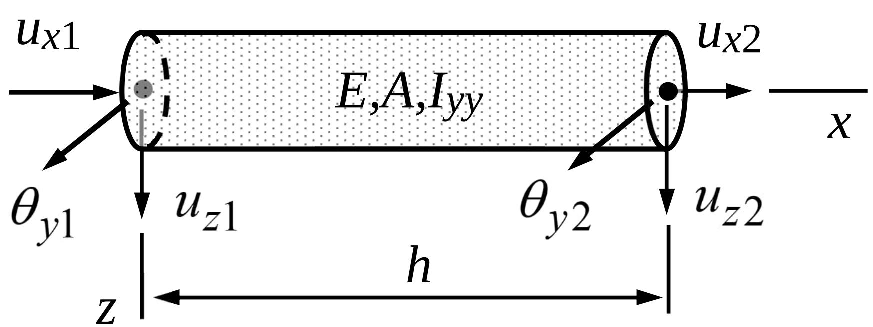
In terms of the shear forces Q_{z1}, Q_{z2}, bending moments M_{y1}, M_{y2}, displacements u_{x1}, transverse displacements u_{z1}, u_{z2}, and rotations θ_{y1},θ_{y2}, virtual work expression of internal forces



$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} Q_{z1} \\ M_{y1} \\ Q_{z2} \\ M_{y2} \end{Bmatrix} \quad \text{where} \quad \begin{Bmatrix} Q_{z1} \\ M_{y1} \\ Q_{z2} \\ M_{y2} \end{Bmatrix} = \frac{EI_{yy}}{h^3} \begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix}.$$

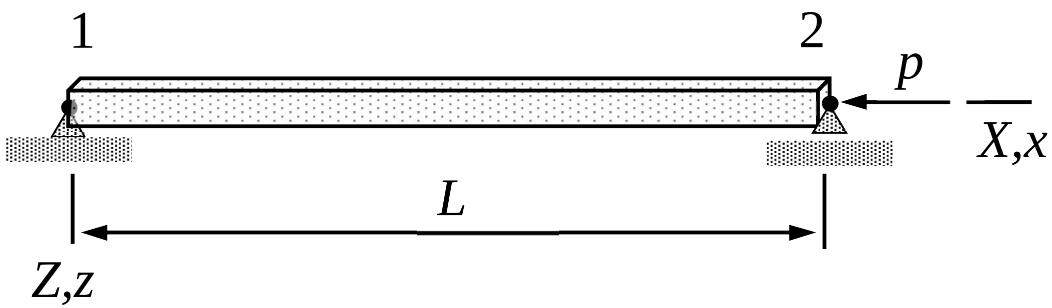
BENDING-BAR COUPLING (*xz*-plane)

Assuming a cubic approximation to $w(x)$ of nodal displacements/rotations u_{z1}, u_{z2}, θ_{y1}, and θ_{y2}, and a linear approximation to $u(x)$ of the nodal displacements u_{x1}, u_{x2}



$$\delta W^{\text{sta}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{N}{30h} \begin{bmatrix} 36 & -3h & -36 & -3h \\ -3h & 4h^2 & 3h & -h^2 \\ -36 & 3h & 36 & 3h \\ -3h & -h^2 & 3h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix}, \text{ where } N = EA\frac{u_{x2} - u_{x1}}{h}$$

EXAMPLE 4.1 Consider a simply supported beam loaded by a compressive axial force p acting on the right end. Assuming that displacement is confined to the $xz-$plane, use a single beam element to determine the buckling force p_{cr}. Cross-section properties A, I and Young's modulus E are constants.



Answer $p_{\mathrm{cr}} = 12\dfrac{EI}{L^2}$ (exact to the model $p_{\mathrm{cr}} = \pi^2 \dfrac{EI}{L^2}$)

- The non-zero nodal displacements/rotations are θ_{Y1}, θ_{Y2}, and u_{X2}. Virtual work expression for the beam $\delta W^1 = \delta W^{\text{int}} + \delta W^{\text{sta}}$ and the point force δW^2 are (here $N = EA(u_{x2} - u_{x1})/h = EAu_{X2}/L$)

$$\delta W^1 = -\delta u_{X2} \frac{EA}{L} u_{X2} - \begin{Bmatrix} \delta\theta_{Y1} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L} \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \begin{Bmatrix} \theta_{Y1} \\ \theta_{Y2} \end{Bmatrix} - \begin{Bmatrix} \delta\theta_{Y1} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{NL}{30} \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix} \begin{Bmatrix} \theta_{Y1} \\ \theta_{Y2} \end{Bmatrix},$$

$$\delta W^2 = -p\delta u_{X2}.$$

- Virtual work expression is sum of the element contributions

$$\delta W = -\begin{Bmatrix} \delta u_{X2} \\ \delta\theta_{Y1} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} [\frac{1}{L} \begin{bmatrix} EA & 0 & 0 \\ 0 & 4EI & 2EI \\ 0 & 2EI & 4EI \end{bmatrix} + \frac{EAu_{X2}}{30} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}) \begin{Bmatrix} u_{X2} \\ \theta_{Y1} \\ \theta_{Y2} \end{Bmatrix} + \begin{Bmatrix} p \\ 0 \\ 0 \end{Bmatrix}].$$

- Principle of virtual work and the fundamental lemma of variation calculus imply that

$$
\left(\frac{1}{L}\begin{bmatrix} EA & 0 & 0 \\ 0 & 4EI & 2EI \\ 0 & 2EI & 4EI \end{bmatrix} + \frac{EAu_{X2}}{30}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}\right)\begin{Bmatrix} u_{X2} \\ \theta_{Y1} \\ \theta_{Y2} \end{Bmatrix} + \begin{Bmatrix} p \\ 0 \\ 0 \end{Bmatrix} = 0.
$$

- The remaining task is to solve the (non-linear) equations for the values of the loading parameter p and the corresponding modes. Solving for the axial displacements (and thereby the axial forces) of the beams allowed to buckle as functions of the loading parameters is always the first step. The first equation gives

$$
\frac{1}{L}EAu_{X2} + p = 0 \quad \Leftrightarrow \quad u_{X2} = -\frac{pL}{EA}.
$$

- When the solution is substituted there, the remaining equations simplify to the homogeneous form

$$
\left(\frac{EI}{L}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} - \frac{pL}{30}\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}\right)\begin{Bmatrix} \theta_{Y1} \\ \theta_{Y2} \end{Bmatrix} = 0.
$$

- A non-trivial solution (zero rotations satisfy the equations always) is possible only if the matrix in parenthesis is singular

$$
\det\left(\frac{EI}{L}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} - \frac{pL}{30}\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}\right) = \left(4\frac{EI}{L} - 4\frac{pL}{30}\right)^2 - \left(2\frac{EI}{L} + \frac{pL}{30}\right)^2 = 0 \quad \Rightarrow
$$

$$
\frac{pL^2}{EI} \in \{12, 60\}.
$$

- The smallest of the values is the critical one

$$
p_{\mathrm{cr}} = 12\frac{EI}{L^2}.
$$

- Stability analysis by the Mathematica code gives

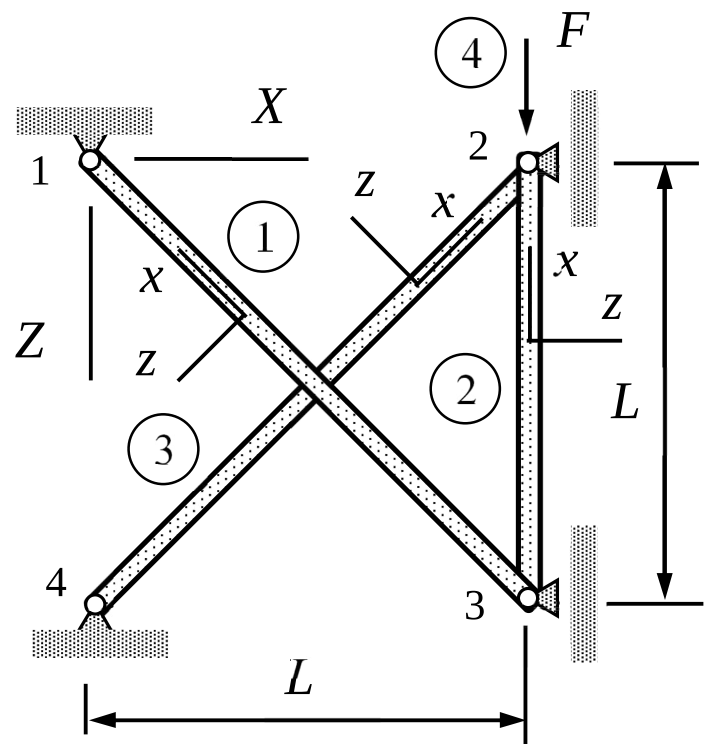
	model	properties	geometry
1	BEAM	{{E, G}, {A, I, I}}	Line[{1, 2}]
2	FORCE	{-p, 0, 0}	Point[{2}]

	{X,Y,Z}	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	{0, 0, 0}	{0, 0, 0}	{0, θY[1], 0}
2	{L, 0, 0}	{uX[2], 0, 0}	{0, θY[2], 0}

$$p[1] \to \frac{60\,E\,I}{L^2} \quad \{uX[2] \to 0,\ \theta Y[1] \to 1,\ \theta Y[2] \to 1\}$$

$$p[2] \to \frac{12\,E\,I}{L^2} \quad \{uX[2] \to 0,\ \theta Y[1] \to -1,\ \theta Y[2] \to 1\}$$

EXAMPLE 4.2 Consider the truss shown in which elements 1 and 3 are modelled as bars and element 2 as a beam. Determine the critical value of force F for buckling of the beam element. Cross-sectional area of element 1 and 3 are $\sqrt{8}A$. Cross sectional area of element 2 is A and the second moment of area I. Young's modulus of the material is E. Assume that $\theta_{Y3} = -\theta_{Y2}$.



Answer $F_{\mathrm{cr}} = 36\dfrac{EI}{L^2}$

- The non-zero nodal displacements/rotations are θ_{Y2}, $\theta_{Y3} = -\theta_{Y2}$, u_{Z2}, and u_{Z3}. Virtual work expressions of the elements are (here the axial force is given by $N = EA(u_{x2} - u_{x3})/L = EA(u_{Z3} - u_{Z2})/L$)

$$\delta W^1 = -\begin{Bmatrix} -\delta u_{Z3} \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{E\sqrt{8}A}{\sqrt{8}L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} -u_{Z3} \\ 0 \end{Bmatrix} = \begin{Bmatrix} \delta u_{Z3} \\ \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EA}{L}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} u_{Z3} \\ u_{Z2} \\ \theta_{Y2} \end{Bmatrix},$$

$$\delta W^2 = -\begin{Bmatrix} \delta u_{Z3} \\ \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} (\frac{1}{L}\begin{bmatrix} EA & -EA & 0 \\ -EA & EA & 0 \\ 0 & 0 & 4EI + NL^2/3 \end{bmatrix}\begin{Bmatrix} u_{Z3} \\ u_{Z2} \\ \theta_{Y2} \end{Bmatrix},$$

$$\delta W^3 = -\begin{Bmatrix} 0 \\ -\delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} \frac{E\sqrt{8}A}{\sqrt{8}L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ -u_{Z2} \end{Bmatrix} = \begin{Bmatrix} \delta u_{Z3} \\ \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} u_{Z3} \\ u_{Z2} \\ \theta_{Y2} \end{Bmatrix},$$

$$\delta W^4 = \delta u_{Z2} F = \begin{Bmatrix} \delta u_{Z3} \\ \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ F \\ 0 \end{Bmatrix}.$$

- Virtual work expression is the sum of element contributions

$$\delta W = -\begin{Bmatrix} \delta u_{Z3} \\ \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{1}{L} \begin{bmatrix} 2EA & -EA & 0 \\ -EA & 2EA & 0 \\ 0 & 0 & 4EI + NL^2/3 \end{bmatrix} \begin{Bmatrix} u_{Z3} \\ u_{Z2} \\ \theta_{Y2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ F \\ 0 \end{Bmatrix} \right).$$

- Principle of virtual work and the fundamental lemma of variation calculus imply that

$$\frac{1}{L} \begin{bmatrix} 2EA & -EA & 0 \\ -EA & 2EA & 0 \\ 0 & 0 & 4EI + NL^2/3 \end{bmatrix} \begin{Bmatrix} u_{Z3} \\ u_{Z2} \\ \theta_{Y2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ F \\ 0 \end{Bmatrix} = 0 \quad \text{where} \quad N = \frac{EA}{L}(u_{Z3} - u_{Z2}).$$

- The remaining task is to solve the (non-linear) equations for the values of the loading parameter F making the solution non-unique (the corresponding modes might be of some interest also). The first two equations give

$$\frac{EA}{L}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} u_{Z3} \\ u_{Z2} \end{Bmatrix}-\begin{Bmatrix} 0 \\ F \end{Bmatrix}=0 \quad \Leftrightarrow \quad \begin{Bmatrix} u_{Z3} \\ u_{Z2} \end{Bmatrix}=\frac{FL}{3EA}\begin{Bmatrix} 1 \\ 2 \end{Bmatrix}.$$

- When the solution is substituted there, the axial force expression and the remaining third equation give

$$N=\frac{EA}{L}(u_{Z3}-u_{Z2})=-\frac{F}{3} \quad \Rightarrow \quad (4EI-\frac{1}{3}\frac{FL^2}{3})\theta_{Y2}=0 \ .$$

- A non-trivial solution $\theta_{Y2} \neq 0$ is possible only if

$$4EI-\frac{FL^2}{9}=0 \quad \Leftrightarrow \quad F_{\rm cr}=36\frac{EI}{L^2}\,.$$

- Stability analysis by the Mathematica code gives

	model	properties	geometry
1	BAR	$\{\{E\},\{2\sqrt{2}A\}\}$	Line[{1, 3}]
2	BEAM	{{E, G}, {A, I, I}}	Line[{2, 3}]
3	BAR	$\{\{E\},\{2\sqrt{2}A\}\}$	Line[{4, 2}]
4	FORCE	{0, 0, F}	Point[{2}]

	{X,Y,Z}	$\{u_X,u_Y,u_Z\}$	$\{\theta_X,\theta_Y,\theta_Z\}$
1	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}
2	{L, 0, 0}	{0, 0, uZ[2]}	{0, θY[2], 0}
3	{L, 0, L}	{0, 0, uZ[3]}	{0, -θY[2], 0}
4	{0, 0, L}	{0, 0, 0}	{0, 0, 0}

$F[1]\to\frac{36EI}{L^2}\quad \{uZ[2]\to 0,\ uZ[3]\to 0,\ \theta Y[2]\to 1\}$

4.4 ELEMENT CONTRIBUTIONS

Virtual work expressions for the beam and plate elements combine virtual work densities of the model and approximation depending on the element shape and type. To derive the expression:

- Start with the virtual work densities $\delta w_{\Omega}^{\text{int}}$, $\delta w_{\Omega}^{\text{sta}}$, and $\delta w_{\Omega}^{\text{ext}}$ of the formulae collection.

- Represent the unknown functions by interpolation of the nodal displacement and rotations (see formulae collection). Substitute the approximations into the density expressions.

- Integrate the virtual work density over the domain occupied by the element to get δW .

ELEMENT APPROXIMATION

In MEC-E8001 element approximation is a polynomial interpolant of the nodal displacement and rotations in terms of shape functions. In stability analysis, shape functions depend on *x*, *y*, and *z*.

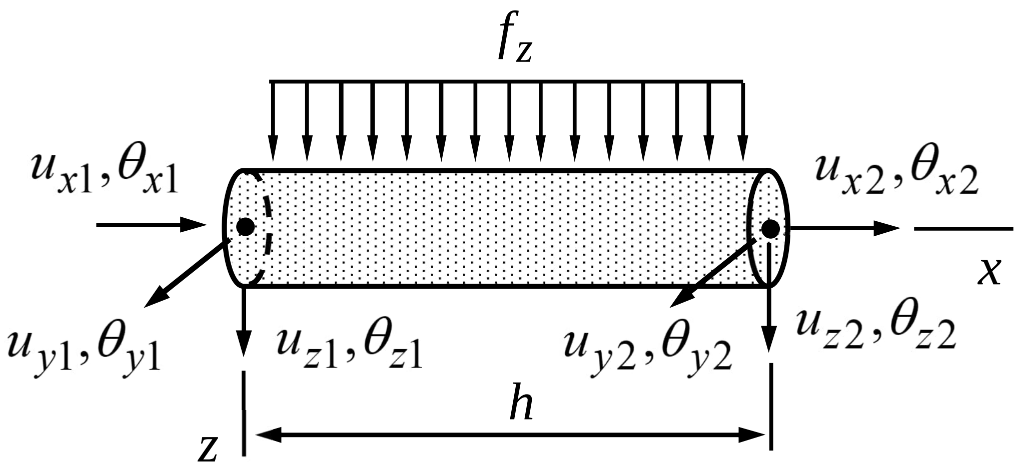
Approximation $\mathbf{u}=\mathbf{N}^{\mathrm{T}}\mathbf{a}$ always of the same form!

Shape functions $\mathbf{N}=\{N_1(x,y,z) \quad N_2(x,y,z) \quad \ldots \quad N_n(x,y,z)\}^{\mathrm{T}}$

Parameters $\mathbf{a}=\{\mathrm{a}_1 \quad \mathrm{a}_2 \quad \ldots \quad \mathrm{a}_n\}^{\mathrm{T}}$

Nodal parameters $\mathrm{a}\in\{u_x,u_y,u_z,\theta_x,\theta_y,\theta_z\}$ may be just displacement or rotation components or a mixture of them (as with the beam model).

BEAM MODEL



Coupling term: $\delta w_{\Omega}^{\text{sta}} = -\frac{d\delta v}{dx} N \frac{dv}{dx} - \frac{d\delta w}{dx} N \frac{dw}{dx}$, where $N = EA \frac{du}{dx}$.

The additional coupling term is part of the virtual work density of internal forces $\delta w_{\Omega} = (\delta w_{\Omega}^{\text{int}} + \delta w_{\Omega}^{\text{sta}}) + \delta w_{\Omega}^{\text{ext}}$ and assumes that $S_y = S_z = I_{yz} = 0$. The coupling of the bar and bending modes is the most significant non-linear term.

- The coupling terms of the bending and bar modes follow from the large displacement virtual work expression and displacement assumptions. For the beam model $u_x = u - zdw/dx - ydv/dx$, $u_y = v(x)$, and $u_z = w(x)$. Considering only the most significant terms of the Green-Lagrange axial strain expression

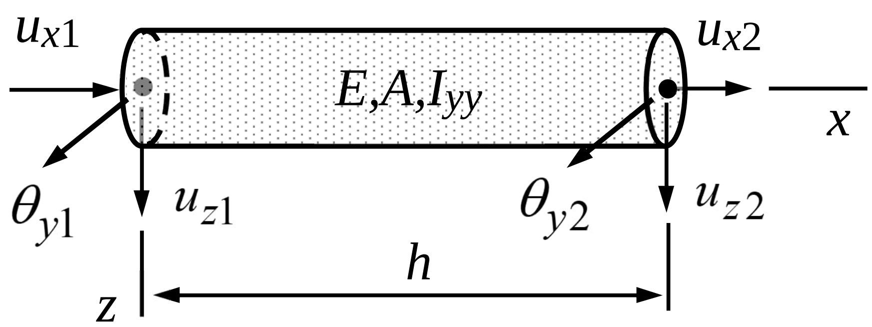
$$E_{xx} = \frac{du}{dx} - z\frac{d^2w}{dx^2} - y\frac{d^2v}{dx^2} + \frac{1}{2}(\frac{dv}{dx})^2 + \frac{1}{2}(\frac{dw}{dx})^2 \text{ and } S_{xx} = CE_{xx},$$

- Integration of $\delta w^{\text{int}} = -\delta E_{xx}S_{xx}$ over the cross-section gives the virtual work densities of the bar mode, bending modes, and the additional coupling term. Assuming that $S_y = S_z = I_{yz} = 0$, the additional coupling term takes the form

$$\delta w_{\Omega}^{\text{sta}} = -\frac{d\delta v}{dx} N \frac{dv}{dx} - \frac{d\delta w}{dx} N \frac{dw}{dx}, \text{ where } N = EA\frac{du}{dx}.$$

BENDING-BAR COUPLING (*xz*-plane)

Assuming that $v = 0$, $\phi = 0$, a cubic approximation to $w(x)$ in terms of nodal displacements/rotations u_{z1}, u_{z2}, θ_{y1}, and θ_{y2}, and a linear approximation to $u(x)$ in terms of the nodal displacements u_{x1}, u_{x2},



$$\delta W^{\text{sta}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{N}{30h} \begin{bmatrix} 36 & -3h & -36 & -3h \\ -3h & 4h^2 & 3h & -h^2 \\ -36 & 3h & 36 & 3h \\ -3h & -h^2 & 3h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix}, \text{ where } N = EA\frac{u_{x2} - u_{x1}}{h}.$$

- Virtual work density of the bending-bar mode coupling term in the $xz-$plane is given by

$$\delta w_{\Omega}^{\mathrm{sta}} = -N\frac{d\delta w}{dx}\frac{dw}{dx} \quad \text{where } N = EA\frac{du}{dx}$$

and the cross-sectional area A and Young's modulus E may depend on x. Element approximations (simplest possible) are $du/dx = (u_{x2} - u_{x1})/h$ and

$$w = \frac{1}{h^3}\begin{Bmatrix} (h-x)^2(h+2x) \\ -h(h-x)^2 x \\ (3h-2x)x^2 \\ (h-x)x^2 \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix} \Rightarrow \frac{dw}{dx} = \frac{1}{h^3}\begin{Bmatrix} -6(h-x)x \\ -h(h-3x)(h-x) \\ 6(h-x)x \\ h(2h-3x)x \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix}.$$

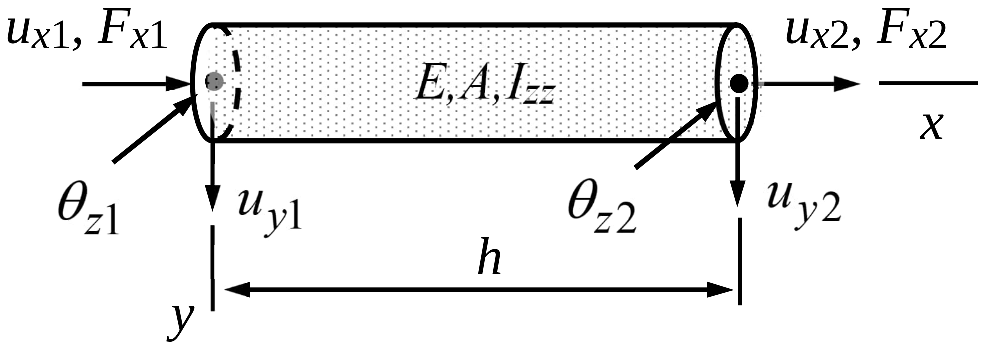
- Integration over the domain occupied by the element gives

$$\delta W^{\text{sta}} = \int_0^h \delta w_\Omega^{\text{sta}} dx = -N \int_0^h \frac{d\delta w}{dx}\frac{dw}{dx} dx \quad \left(N = EA\frac{du}{dx} \text{ is constant here} \right) \quad \Rightarrow$$

$$\delta W^{\text{sta}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{N}{30h} \begin{bmatrix} 36 & -3h & -36 & -3h \\ -3h & 4h^2 & 3h & -h^2 \\ -36 & 3h & 36 & 3h \\ -3h & -h^2 & 3h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix}, \text{ where } N = EA\frac{u_{x2} - u_{x1}}{h}. \;\Leftarrow$$

BENDING-BAR COUPLING (*xy*-plane)

Assuming a cubic approximation to $v(x)$ in terms of nodal displacements/rotations u_{y1}, u_{y2}, θ_{z1}, and θ_{z2}, and linear approximation to $u(x)$ in terms of nodal displacements u_{x1}, u_{x2},



$$\delta W^{\text{sta}} = -\begin{Bmatrix} \delta u_{y1} \\ \delta\theta_{z1} \\ \delta u_{y2} \\ \delta\theta_{z2} \end{Bmatrix}^{\mathrm{T}} \frac{N}{30h} \begin{bmatrix} 36 & 3h & -36 & 3h \\ 3h & 4h^2 & -3h & -h^2 \\ -36 & -3h & 36 & -3h \\ 3h & -h^2 & -3h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \\ u_{y2} \\ \theta_{z2} \end{Bmatrix}, \quad \text{where } N = EA\frac{u_{x2} - u_{x1}}{h}.$$

PLATE MODEL

Virtual work density combines the thin-slab and plate bending modes. Assuming that the material coordinate system is placed at the geometric mid-plane, bending mode is affected by the thin slab mode but not vice versa. The additional coupling term for stability analysis

Coupling: $\delta w_{\Omega}^{\mathrm{sta}} = -\begin{Bmatrix} \partial\delta w / \partial x \\ \partial\delta w / \partial y \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} N_{xx} & N_{xy} \\ N_{yx} & N_{yy} \end{bmatrix} \begin{Bmatrix} \partial w / \partial x \\ \partial w / \partial y \end{Bmatrix}$, where $\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = t[E]_{\sigma} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}$

depends on the in-plane stress resultants N_{xx}, N_{yy}, and $N_{xy} = N_{yx}$ of the thin-slab mode. The additional coupling term is part of the virtual work density of internal forces $\delta w_{\Omega} = \delta w_{\Omega}^{\mathrm{int}} + \delta w_{\Omega}^{\mathrm{sta}} + \delta w_{\Omega}^{\mathrm{ext}}$. As stability term affects only the bending mode, dependence of the stress resultants on the loading parameter can be obtained from a thin-slab problem.

- The coupling term of the plate bending and thin-slab loading modes follows from the generic non-linear virtual work density of the internal forces and the kinematic assumptions of the Kirchhoff plate model $u_x = u - z\partial w / \partial x$, $u_y = v - z\partial w / \partial y$, and $u_z = w(x, y)$. If only the most significant terms are accounted for, Green-Lagrange strain and the corresponding second Piola-Kirchhoff stress components

$$\begin{Bmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ 2\mathrm{E}_{xy} \end{Bmatrix} \approx \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix} - z \begin{Bmatrix} \partial^2 w / \partial x^2 \\ \partial^2 w / \partial y^2 \\ 2\partial^2 w / \partial x \partial y \end{Bmatrix} + \frac{1}{2} \begin{Bmatrix} (\partial w / \partial x)^2 \\ (\partial w / \partial y)^2 \\ 2(\partial w / \partial x)(\partial w / \partial y) \end{Bmatrix},$$

$$\begin{Bmatrix} S_{xx} \\ S_{yy} \\ S_{xy} \end{Bmatrix} = [E]_\sigma \begin{Bmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ 2\mathrm{E}_{xy} \end{Bmatrix}.$$

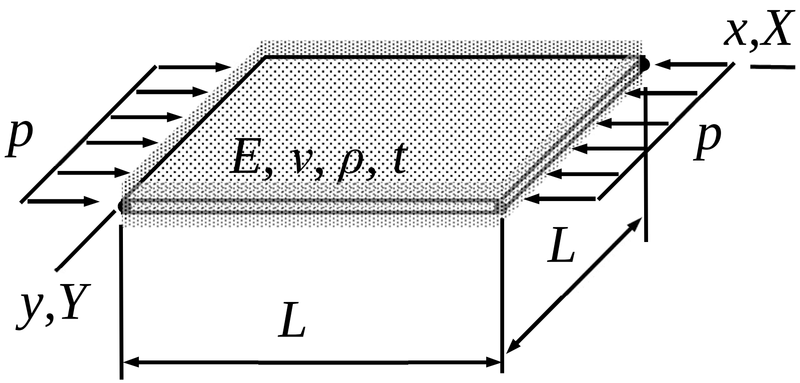
- Assuming that the material coordinate system is placed at the geometric mid-plane, integration of the virtual work density gives the virtual work density of the thin-slab mode, virtual work density of plate bending mode, and the coupling term (considering only the most significant terms)

$$\delta w_{\Omega}^{\mathrm{sta}} = -\begin{Bmatrix} \partial\delta w / \partial x \\ \partial\delta w / \partial y \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} N_{xx} & N_{xy} \\ N_{yx} & N_{yy} \end{bmatrix} \begin{Bmatrix} \partial w / \partial x \\ \partial w / \partial y \end{Bmatrix},$$

where the in-plane stress resultants

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = t\left[E\right]_{\sigma} \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix}.$$

EXAMPLE 4.3 Determine the critical value of the in-plane loading $p_{\rm cr}$ making the plate of the figure to buckle. Use the approximation $w(x,y)=a_0(xy/L^2)(1-x/L)(1-y/L)$. Assume that the edge conditions are such that solution to the in-plane stress resultants is given by $N_{xx}=-p$ and $N_{yy}=N_{xy}=0$ (solution to the thin-slab problem).



Answer $p_{\rm cr}=\dfrac{11}{3}\dfrac{Et^3}{L^2(1-\nu^2)}$ (exact $p_{\rm cr}=\dfrac{\pi^2}{3}\dfrac{Et^3}{L^2(1-\nu^2)}$).

- Assuming that the material coordinate system is chosen so that the linear plate bending and thin slab modes decouple, the plate model virtual work densities of the bending mode and the coupling term are given by ($N_{xx} = -p$ and $N_{yy} = N_{xy} = 0$)

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial^2 \delta w / \partial x^2 \\ \partial^2 \delta w / \partial y^2 \\ 2\partial^2 \delta w / \partial x \partial y \end{Bmatrix}^{\mathrm{T}} D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \partial^2 w / \partial x^2 \\ \partial^2 w / \partial y^2 \\ 2\partial^2 w / \partial x \partial y \end{Bmatrix} \text{ where } D = \frac{t^3}{12} \frac{E}{1-\nu^2},$$

$$\delta w_\Omega^{\text{sta}} = -\begin{Bmatrix} \partial \delta w / \partial x \\ \partial \delta w / \partial y \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} N_{xx} & N_{xy} \\ N_{yx} & N_{yy} \end{bmatrix} \begin{Bmatrix} \partial w / \partial x \\ \partial w / \partial y \end{Bmatrix} = \frac{\partial \delta w}{\partial x} p \frac{\partial w}{\partial x}.$$

- When the approximation is substituted there, virtual work expressions of the plate bending mode and that of the coupling between the thin-slab and bending modes simplify to

$$\delta W^{\text{int}} = \int_0^L \int_0^L \delta w_\Omega^{\text{int}} dxdy = -\delta a_0 \frac{22}{45}\frac{D}{L^2} a_0,$$

$$\delta W^{\text{sta}} = \int_0^L \int_0^L \delta w_\Omega^{\text{sta}} dxdy = \delta a_0 \frac{1}{90} p a_0 .$$

- Virtual work expression is the sum of the two parts

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{sta}} = -\delta a_0 (\frac{22}{45}\frac{D}{L^2} - \frac{1}{90} p) a_0 .$$

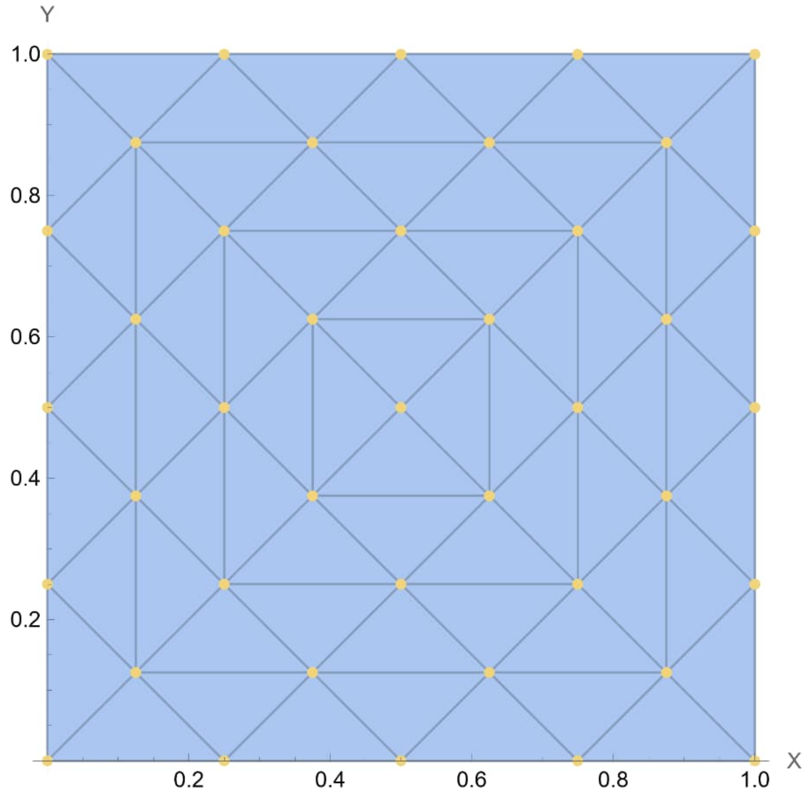
- Principle of virtual work $\delta W = 0 \, \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus give

$$\delta W = -\delta a_0 (\frac{22}{45}\frac{D}{L^2} - \frac{1}{90} p) a_0 = 0 \quad \forall \delta a_0 \quad \Rightarrow \quad (\frac{22}{45}\frac{D}{L^2} - \frac{1}{90} p) a_0 = 0 .$$

For a non-trivial solution $a_0 \neq 0$, the loading parameter needs to take the value

$$p_{\mathrm{cr}} = 44\frac{D}{L^2} = \frac{11}{3}\frac{Et^3}{L^2(1-\nu^2)} \,.$$

- The problem can be solved numerically by using the Reissner-Mindlin plate model and the Mathematica code. Assuming parameter values $E = 210\,\mathrm{GPa}$, $\nu = 0.33$, $L = 1\,\mathrm{m}$, and $t = 1\,\mathrm{mm}$, the one parameter approximation gives $p_{\mathrm{cr}} = 0.86\,\mathrm{Nm}^{-1}$ whereas the solution on the mesh shown gives $p_{\mathrm{cr}} = 0.78\,\mathrm{Nm}^{-1}$.



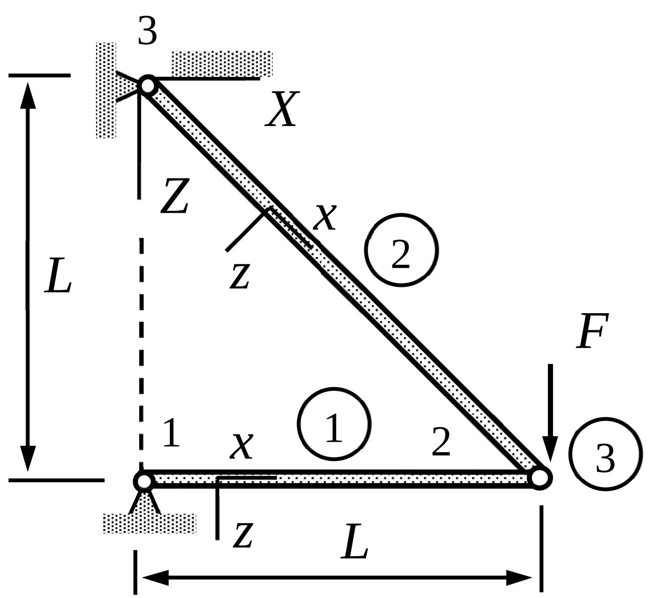
STABILITY ANALYSIS OF TRUSS SIMPLIFIED

In hand calculations, one may use the fact that the bar model predicts the axial forces correctly when beams of a truss are connected with joints. Then, the first step is a linear displacement analysis for finding the displacements of the nodes and thereby the axial forces $N(p)$ as functions of the loading parameter. After that, the buckling loads of each beam under compression follows from the buckling criterion (N is negative in compression)

$$-N(p)=\pi^2\frac{EI}{L^2}$$

for a simply supported beam. The first beam to buckle (or the smallest p given by the conditions above) defines the critical load p_{cr}.

EXAMPLE 4.4 A beam truss is loaded by a vertical point force having magnitude F and acting in the positive or negative direction of the Z-axis. Determine the critical load magnitude F_{cr} for buckling of beam 1 or 2 of the truss. Cross-sectional area of element 1 is A and that for element 2 $\sqrt{8}A$, Young's modulus E is constant, and the second moment of area is I for both beams. The beams are connected by frictionless joints.



Answer $F_{\mathrm{cr}} = \dfrac{\pi^2}{\sqrt{8}} \dfrac{EI}{L^2}$ when $F < 0$.

- The relationships between the nodal displacement components in the material and structural systems are $u_{x1}=0$ and $u_{x2}=u_{X2}$. Element contribution δW^1 to the virtual work expression of the structure is

$$\delta W^1=-\begin{Bmatrix}0\\ \delta u_{X2}\end{Bmatrix}^{\mathrm{T}}(\frac{EA}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}0\\ u_{X2}\end{Bmatrix}-\begin{Bmatrix}0\\ 0\end{Bmatrix})=-\frac{EA}{L}u_{X2}\delta u_{X2}.$$

- For element 2, $u_{x3}=0$ and $u_{x2}=(u_{X2}+u_{Z2})/\sqrt{2}$. Element contribution takes the form

$$\delta W^2=-\frac{1}{\sqrt{2}}\begin{Bmatrix}0\\ \delta u_{X2}+\delta u_{Z2}\end{Bmatrix}^{\mathrm{T}}(\frac{E\sqrt{8}A}{\sqrt{2}L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{Bmatrix}0\\ u_{X2}+u_{Z2}\end{Bmatrix}-\begin{Bmatrix}0\\ 0\end{Bmatrix}) \quad \Leftrightarrow$$

$$\delta W^2=-\frac{EA}{L}(\delta u_{X2}+\delta u_{Z2})(u_{X2}+u_{Z2}).$$

- Virtual work expression of the point force follows from the definition of work. The direction may be up or down and hence F may also be negative (which means up)

$$\delta W^3 = \delta u_{Z2} F .$$

- Virtual work expression of a structure is obtained as the sum of the element contributions

$$\delta W = -\frac{EA}{L}\delta u_{X2} u_{X2} - \frac{EA}{L}(\delta u_{X2} + \delta u_{Z2})(u_{X2} + u_{Z2}) + \delta u_{Z2} F \quad \Leftrightarrow$$

$$\delta W = -\begin{Bmatrix} \delta u_{X2} \\ \delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} (\frac{EA}{L}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{Z2} \end{Bmatrix} + \begin{Bmatrix} 0 \\ F \end{Bmatrix}) .$$

- Using the principle of virtual work $\delta W = 0 \ \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus

$$\frac{EA}{L}\begin{bmatrix}2 & 1\\1 & 1\end{bmatrix}\begin{Bmatrix}u_{X2}\\u_{Z2}\end{Bmatrix}-\begin{Bmatrix}0\\F\end{Bmatrix}=0 \quad\Leftrightarrow\quad \begin{Bmatrix}u_{X2}\\u_{Z2}\end{Bmatrix}=\frac{LF}{EA}\begin{Bmatrix}-1\\2\end{Bmatrix}.$$

- For buckling of beam 1, the axial force should be compression (negative) and therefore the external force should be acting downwards.

$$N=\frac{EA}{L}(u_{x2}-u_{x1})=\frac{EA}{L}u_{X2}=-F \quad\Rightarrow\quad F_{\mathrm{cr}}=\pi^2\frac{EI}{L^2} \ \text{ when } \ F>0.$$

- For buckling of beam 2, the axial force should be compression (negative) and therefore the external force should be acting upwards. When $F<0$

$$N=\frac{E\sqrt{8}A}{\sqrt{2}L}(u_{x2}-u_{x3})=\sqrt{2}\frac{EA}{L}(u_{X2}+u_{Z2})=-\sqrt{2}F \quad\Rightarrow\quad F_{\mathrm{cr}}=\frac{\pi^2}{\sqrt{8}}\frac{EI}{L^2}\,.$$

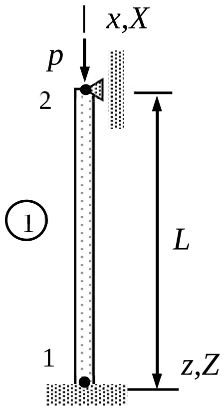
MEC-E8001 Finite Element Analysis, week 5/2023

1. Virtual work expression of a beam, which takes into account the bar and bending modes and the coupling between them, gives a non-linear equation system for the axial and transverse displacement. Determine the critical load p_{cr} causing the beam to buckle if the equation system is given by

$$-\frac{1}{5L^3}\begin{Bmatrix} 5L^3 p + 5L^2 EAu_{X2} \\ 60EIu_{Z2} + 6LEAu_{X2}u_{Z2} \end{Bmatrix} = 0 \ .$$

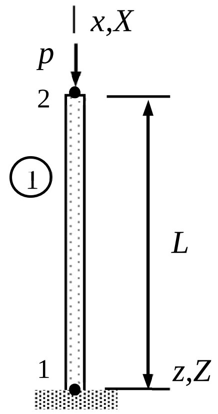
Answer $p_{\text{cr}} = 10\dfrac{EI}{L^2}$

2. Determine the buckling force p_{cr} and the buckled shape of the structure shown by using one beam element. Displacements are confined to the $xz-$plane. Parameters E, A, and I are constants.



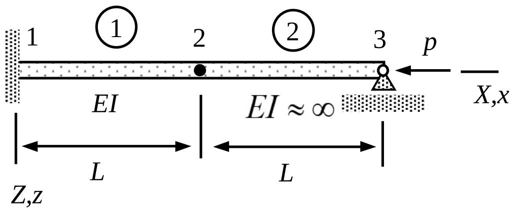
Answer $p_{\text{cr}} = 30\dfrac{EI}{L^2}$, $w = k(\dfrac{x}{L})^2(1-\dfrac{x}{L})$ (k is arbitrary)

3. Determine the buckling force p_{cr} of the beam shown by using one beam element. Displacements are confined to the $xz-$plane. The cross-section and material properties A, I, and E are constants.



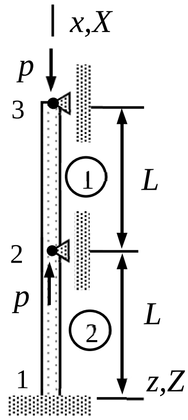
Answer $p_{\text{cr}} = \dfrac{EI}{L^2}\dfrac{4}{3}\left(13 - 2\sqrt{31}\right)$

4. The structure shown consist of two beams, each of length L. As beam 2 is much stiffer than beam 1, the setting is simplified by considering beam 2 as a rigid body. Displacements are confined to the $xz-$plane. Cross-section properties of beam 1 are A and I and Young's modulus of the material is E. Determine the buckling load p_{cr}.



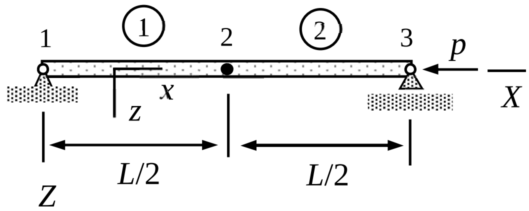
Answer $p_{\text{cr}} = \dfrac{420}{23}\dfrac{EI}{L^2}$

5. Beam structure of the figure is loaded by opposite forces of magnitude p acting on nodes 2 and 3. Determine the buckling force p_{cr} of the structure using two beam elements. Displacements are confined to the xz–plane. The cross-section properties of the beam A, I and Young's modulus of the material E are constants.



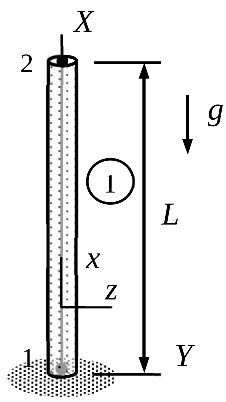
Answer $p_{cr} = 20\dfrac{EI}{L^2}$

6. The simply supported uniform beam shown is divided into two identical beam elements, each of length $L/2$. Displacements are confined to the xz–plane. Cross-section properties are A and I and Young's modulus of the material E. Determine the buckling load p_{cr}. Assume that rotation angles satisfy $\theta_{Y1} = -\theta_{Y3}$ and $\theta_{Y2} = 0$.



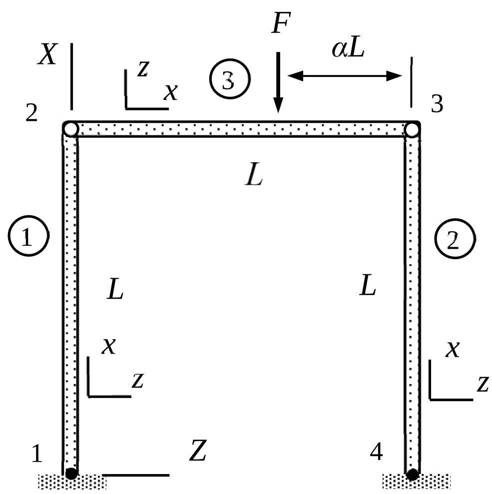
Answer $p_{cr} = \dfrac{240}{13+2\sqrt{31}}\dfrac{EI}{L^2}$

7. Find the density ρ_{cr} causing the beam of the figure to buckle in xz–plane. Start with the virtual work density taking into account the interaction of the bar and bending modes. Choose first $\delta w = 0$ in the virtual work density to solve for the axial displacement and the axial force N. After that, choose $\delta u = 0$ to find the virtual work expression taking into account the internal and coupling parts.



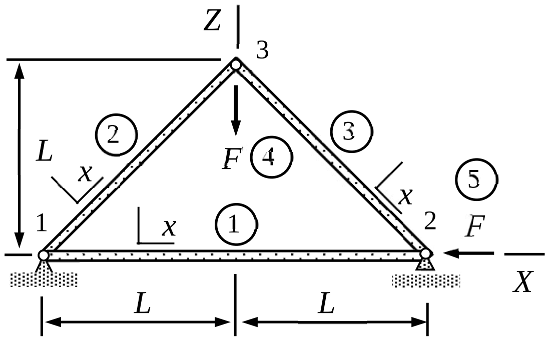
Answer $\rho_{cr} = \dfrac{120}{13+2\sqrt{31}}\dfrac{EI}{AgL^3}$

8. The plane frame of the figure consists of a rigid body 3 and beam elements 1 and 2. The joints at nodes 2 and 3 are frictionless. Determine the critical value of the force F acting on the rigid body at distance αL $\alpha \in [0,1]$ from node 3 making the frame to buckle laterally. The cross-section properties A and I and Young's modulus of the material E are constants.



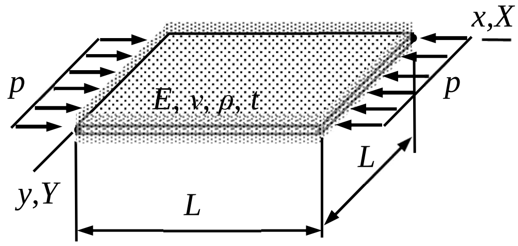
Answer $F = \dfrac{EI}{L^2}\dfrac{8}{3}(13-2\sqrt{31}) \approx 4.97\dfrac{EI}{L^2}$

9. Determine the critical value of force F causing some beam of the truss shown to buckle. First, use the bar model to solve for the nodal displacements and thereby the axial forces as functions of the loading F (assumed to be positive). After that, use criterion $N(F)=\pi^2 EI/h^2$ to find the first beam to buckle and the critical value F_{cr}. Cross-sectional areas of beams 2 and 3 are $\sqrt{8}A$ and that of beam 1 $2A$. The second moments of cross-sections I and the Young's modulus E of the material are constants.



Answer $F_{cr}=\frac{1}{2}\pi^2\frac{EI}{L^2}$ (beam 1 buckles)

10. Determine the critical value of the in-plane loading p_{cr} making the plate shown to buckle. Use $w(x,y)=a_0\sin(\pi x/L)\sin(\pi y/L)$ as the approximation and assume that $N_{xx}=-p$, $N_{yy}=0$, and $N_{xy}=0$. Problem parameters E, ν, ρ and t are constants. Integrals of sin and cos satisfy



$$\int_0^L \sin(i\pi\frac{x}{L})\sin(j\pi\frac{x}{L})dx=\frac{L}{2}\delta_{ij} \text{ and } \int_0^L \cos(i\pi\frac{x}{L})\cos(j\pi\frac{x}{L})dx=\frac{L}{2}\delta_{ij}.$$

Answer $p_{cr}=4\frac{t^3}{12}\frac{E}{1-\nu^2}(\frac{\pi}{L})^2$

Virtual work expression of a beam, which takes into account the bar, bending, and the coupling of the modes, gives a non-linear equation system for the axial and transverse displacement. Determine the critical load p_{cr} causing the beam to buckle if the equation system is given by

$$-\frac{1}{5L^3}\begin{Bmatrix} 5L^3 p + 5L^2 EAu_{X2} \\ 60EIu_{Z2} + 6LEAu_{X2}u_{Z2} \end{Bmatrix} = 0 .$$

Solution

Although the equation system is non-linear, it can be solved in two steps for the critical load. Finding the normal forces (or axial displacements) as function of the load parameter is the first step. The first equation gives

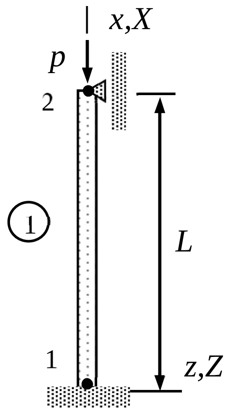
$$5L^3 p + 5L^2 EAu_{X2} = 0 \quad \Leftrightarrow \quad \frac{EA}{L}u_{X2} + p = 0 \quad \Leftrightarrow \quad u_{X2} = -\frac{pL}{EA} .$$

With this expression, the second equation simplifies to

$$60EIu_{Z2} + 6LEAu_{X2}u_{Z2} = 0 \;\Rightarrow\; (60EI - 6LEA\frac{pL}{EA})u_{Z2} = 0 \quad \Leftrightarrow \quad (60EI - 6L^2 p)u_{Z2} = 0$$

A non-trivial solution $u_{Z2} \neq 0$ is possible only if

$$60EI - 6L^2 p = 0 \quad \Leftrightarrow \quad p = 10\frac{EI}{L^2} .$$



Determine the buckling force p_{cr} and the buckled shape of the structure shown by using one beam element. Displacements are confined to the $xz-$plane. Parameters *E*, *A*, and *I* are constants.

Solution

The non-zero displacement/rotation components of the structure are $\theta_{y2} = \theta_{Y2}$ and $u_{x2} = u_{X2}$. In this case, the normal force in the beam $N = -p$ can be deduced without calculations on the axial displacement and it is enough to consider only the bending and coupling terms of the virtual work expression. As buckling is confined to the $xz-$plane

$$\delta W = -\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} - \frac{p}{30L}\begin{bmatrix} 36 & -3L & -36 & -3L \\ -3L & 4L^2 & 3L & -L^2 \\ -36 & 3L & 36 & 3L \\ -3L & -L^2 & 3L & 4L^2 \end{bmatrix}\right)\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \theta_{Y2} \end{Bmatrix} \Leftrightarrow$$

$$\delta W = -\delta\theta_{Y2}\left(\frac{EI}{L^3}4L^2 - \frac{p}{30L}4L^2\right)\theta_{Y2}.$$

Principle of virtual work and the fundamental lemma of variation calculus imply

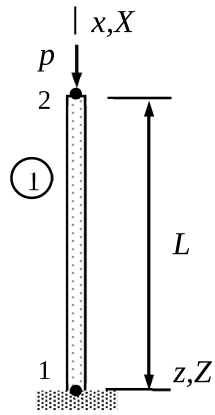
$$\left(\frac{EI}{L^3}4L^2 - \frac{p}{30L}4L^2\right)\theta_{Y2} = 0.$$

A non-trivial solution $\theta_{Y2} \neq 0$ is obtained only if

$$\frac{EI}{L^3}4L^2 - \frac{p}{30L}4L^2 = 0 \quad \Rightarrow \quad p_{cr} = 30\frac{EI}{L^2}.$$

The shape function associated with θ_{Y2} is $N = -x^2/L + x^3/L^2$. Therefore, the buckled shape is given by (save an arbitrary multiplier)

$$w = -\frac{x^2}{L} + \frac{x^3}{L^2}.$$



Determine the buckling force p_{cr} of the beam shown by using one beam element. Displacements are confined to the xz-plane. The cross-section and material properties A, I, and E are constants.

Solution

The non-zero displacement/rotation components of the structure are $\theta_{y2}=\theta_{Y2}$, $u_{z2}=u_{Z2}$ and $u_{x2}=u_{X2}$. As the normal force in the beam

$$N=-p$$

can be deduced directly, it is enough to consider only the bending and coupling terms of the virtual work expression. For buckling in the xz-plane

$$\delta W=-\begin{Bmatrix}0\\0\\\delta u_{Z2}\\\delta\theta_{Y2}\end{Bmatrix}^{\mathrm{T}}(\frac{EI}{L^3}\begin{bmatrix}12&-6L&-12&-6L\\-6L&4L^2&6L&2L^2\\-12&6L&12&6L\\-6L&2L^2&6L&4L^2\end{bmatrix}-\frac{p}{30L}\begin{bmatrix}36&-3L&-36&-3L\\-3L&4L^2&3L&-L^2\\-36&3L&36&3L\\-3L&-L^2&3L&4L^2\end{bmatrix})\begin{Bmatrix}0\\0\\u_{Z2}\\\theta_{Y2}\end{Bmatrix}\Leftrightarrow$$

$$\delta W=-\begin{Bmatrix}\delta u_{Z2}\\\delta\theta_{Y2}\end{Bmatrix}^{\mathrm{T}}(\frac{EI}{L^3}\begin{bmatrix}12&6L\\6L&4L^2\end{bmatrix}-\frac{p}{30L}\begin{bmatrix}36&3L\\3L&4L^2\end{bmatrix})\begin{Bmatrix}u_{Z2}\\\theta_{Y2}\end{Bmatrix}.$$

Principle of virtual work and the fundamental lemma of variation calculus imply (notice the scaling of the rotation and the force which make the two matrices dimensionless and simplify the eigenvalue calculations)

$$(\begin{bmatrix}12&6\\6&4\end{bmatrix}-\lambda\begin{bmatrix}36&3\\3&4\end{bmatrix})\begin{Bmatrix}u_{Z2}\\L\theta_{Y2}\end{Bmatrix}=0 \text{ where } \lambda=\frac{pL^2}{30EI}.$$

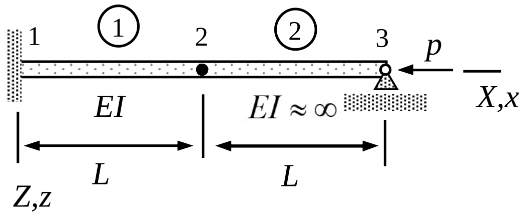
A non-trivial solution is obtained only if

$$\det(\begin{bmatrix}12&6\\6&4\end{bmatrix}-\lambda\begin{bmatrix}36&3\\3&4\end{bmatrix})=(12-36\lambda)(4-4\lambda)-(6-3\lambda)^2=0 \quad\Rightarrow\quad \lambda=\frac{2}{45}\left(13\pm2\sqrt{31}\right).$$

The smallest of the values is the critical one

$$p_{\mathrm{cr}}=\frac{EI}{L^2}\frac{60}{45}(13-2\sqrt{31})\approx2.48\frac{EI}{L^2}.$$

The structure shown consist of two beams, each of length L. As beam 2 is much stiffer than beam 1, the setting is simplified by considering beam 2 as a rigid body. Displacements are confined to the xz – plane. Cross-section properties of beam 1 are A and I and Young's modulus of the material is E. Determine the buckling load p_{cr}.



Solution

The axial forces in the beams follow directly from a free body diagram and it is enough to consider virtual work associated with the bending and interaction between bar and bending modes

$$\delta W^{\mathrm{int}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI_{yy}}{h^3} \begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix},$$

$$\delta W^{\mathrm{sta}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{N}{30h} \begin{bmatrix} 36 & -3h & -36 & -3h \\ -3h & 4h^2 & 3h & -h^2 \\ -36 & 3h & 36 & 3h \\ -3h & -h^2 & 3h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix}.$$

Also, as beam 2 is considered as rigid, virtual work expression of internal forces vanish and modelling uses a kinematical constraints $\vec{u}_{\mathrm{B}} = \vec{u}_{\mathrm{A}} + \vec{\theta}_{\mathrm{A}} \times \vec{\rho}_{\mathrm{AB}}$ and $\vec{\theta}_{\mathrm{B}} = \vec{\theta}_{\mathrm{A}}$. Let us choose A to be node 3 and B as node 2. Then

$u_{Z2} = \theta_{Y3}L$ and $\theta_{Y2} = \theta_{Y3}$.

Although axial displacement is non-zero, it is not needed as the axial force in the structure $N = -p$ (negative means compression) can be deduced without calculations on the axial displacement.

The internal force and coupling parts of beam 1 take the forms ($u_{z1} = 0$, $\theta_{y1} = 0$, $u_{z2} = u_{Z2} = \theta_{Y3}L$, $\theta_{y2} = \theta_{Y2} = \theta_{Y3}$)

$$\delta W^{\mathrm{int}} = -\begin{Bmatrix} 0 \\ 0 \\ L\delta\theta_{Y3} \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ L\theta_{Y3} \\ \theta_{Y3} \end{Bmatrix} = -\delta\theta_{Y3} 28 \frac{EI}{L} \theta_{Y3},$$

$$\delta W^{\mathrm{sta}} = -\begin{Bmatrix} 0 \\ 0 \\ L\delta\theta_{Y3} \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{-p}{30L} \begin{bmatrix} 36 & -3L & -36 & -3L \\ -3L & 4L^2 & 3L & -L^2 \\ -36 & 3L & 36 & 3L \\ -3L & -L^2 & 3L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ L\theta_{Y3} \\ \theta_{Y3} \end{Bmatrix} = \delta\theta_{Y3} \frac{46}{30} pL\theta_{Y3}.$$

Virtual work expression of the structure is the sum of the internal and stability parts

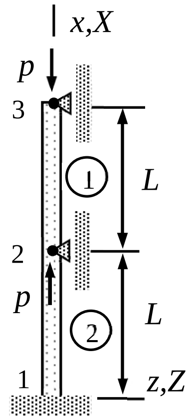
$$\delta W = \delta W^{\text{int}} + \delta W^{\text{sta}} = -\delta\theta_{Y3}(28\frac{EI}{L} - \frac{46}{30}pL)\theta_{Y3}.$$

Principle of virtual work and the fundamental lemma of variation calculus imply that

$$(28\frac{EI}{L} - \frac{46}{30}pL)\theta_{Y3} = 0.$$

In stability analysis the aim is to find a non-zero equilibrium solution so $\theta_{Y3} \neq 0$ and the multiplier needs to vanish for a solution. Therefore, the buckling load is given by

$$p_{\text{cr}} = \frac{30}{46}28\frac{EI}{L^2} = \frac{420}{23}\frac{EI}{L^2}.$$



Beam structure of the figure is loaded by opposite forces of magnitude p acting on nodes 2 and 3. Determine the buckling force p_{cr} of the structure using two beam elements. Displacements are confined to the $xz-$ plane. The cross-section properties of the beam A, I and Young's modulus of the material E are constants.

Solution

The axial forces in the beams follow directly from a free body diagram and it is enough to consider virtual work associated with the bending and interaction modes. The non-zero displacement/rotation components of the structure are θ_{Y2} and θ_{Y3}.

For beam 1, $\theta_{y2}=\theta_{Y2}$ and $\theta_{y3}=\theta_{Y3}$. The axial force acting on the element $N=-p$ (negative means compression) follows from a free-body diagram. Therefore

$$\delta W^{\text{int}}=-\begin{Bmatrix}0\\ \delta\theta_{Y2}\\ 0\\ \delta\theta_{Y3}\end{Bmatrix}^{\mathrm{T}}\frac{EI}{L^3}\begin{bmatrix}12 & -6L & -12 & -6L\\ -6L & 4L^2 & 6L & 2L^2\\ -12 & 6L & 12 & 6L\\ -6L & 2L^2 & 6L & 4L^2\end{bmatrix}\begin{Bmatrix}0\\ \theta_{Y2}\\ 0\\ \theta_{Y3}\end{Bmatrix}=-\begin{Bmatrix}\delta\theta_{Y2}\\ \delta\theta_{Y3}\end{Bmatrix}^{\mathrm{T}}\frac{EI}{L}\begin{bmatrix}4 & 2\\ 2 & 4\end{bmatrix}\begin{Bmatrix}\theta_{Y2}\\ \theta_{Y3}\end{Bmatrix},$$

$$\delta W^{\text{sta}}=-\begin{Bmatrix}0\\ \delta\theta_{Y2}\\ 0\\ \delta\theta_{Y3}\end{Bmatrix}^{\mathrm{T}}\frac{N}{30L}\begin{bmatrix}36 & -3L & -36 & -3L\\ -3L & 4L^2 & 3L & -L^2\\ -36 & 3L & 36 & 3L\\ -3L & -L^2 & 3L & 4L^2\end{bmatrix}\begin{Bmatrix}0\\ \theta_{Y2}\\ 0\\ \theta_{Y3}\end{Bmatrix}=\begin{Bmatrix}\delta\theta_{Y2}\\ \delta\theta_{Y3}\end{Bmatrix}^{\mathrm{T}}\frac{pL}{30}\begin{bmatrix}4 & -1\\ -1 & 4\end{bmatrix}\begin{Bmatrix}\theta_{Y2}\\ \theta_{Y3}\end{Bmatrix}$$

giving

$$\delta W^1=\delta W^{\text{int}}+\delta W^{\text{sta}}=-\begin{Bmatrix}\delta\theta_{Y2}\\ \delta\theta_{Y3}\end{Bmatrix}^{\mathrm{T}}\left(\frac{EI}{L}\begin{bmatrix}4 & 2\\ 2 & 4\end{bmatrix}-\frac{pL}{30}\begin{bmatrix}4 & -1\\ -1 & 4\end{bmatrix}\right)\begin{Bmatrix}\theta_{Y2}\\ \theta_{Y3}\end{Bmatrix}.$$

For beam 2, $\theta_{y2}=\theta_{Y2}$ and the axial force $N=0$. Therefore $\delta W^{\text{sta}}=0$ and

$$\delta W^2=-\begin{Bmatrix}0\\ 0\\ 0\\ \delta\theta_{Y2}\end{Bmatrix}^{\mathrm{T}}\frac{EI}{L^3}\begin{bmatrix}12 & -6L & -12 & -6L\\ -6L & 4L^2 & 6L & 2L^2\\ -12 & 6L & 12 & 6L\\ -6L & 2L^2 & 6L & 4L^2\end{bmatrix}\begin{Bmatrix}0\\ 0\\ 0\\ \theta_{Y2}\end{Bmatrix}=-\begin{Bmatrix}\delta\theta_{Y2}\\ \delta\theta_{Y3}\end{Bmatrix}^{\mathrm{T}}\frac{EI}{L}\begin{bmatrix}4 & 0\\ 0 & 0\end{bmatrix}\begin{Bmatrix}\theta_{Y2}\\ \theta_{Y3}\end{Bmatrix}.$$

Virtual work expression of structure is the sum of element contributions

$$\delta W=\delta W^1+\delta W^2=-\begin{Bmatrix}\delta\theta_{Y2}\\ \delta\theta_{Y3}\end{Bmatrix}^{\mathrm{T}}\left(\frac{EI}{L}\begin{bmatrix}8 & 2\\ 2 & 4\end{bmatrix}-\frac{pL}{30}\begin{bmatrix}4 & -1\\ -1 & 4\end{bmatrix}\right)\begin{Bmatrix}\theta_{Y2}\\ \theta_{Y3}\end{Bmatrix}.$$

Principle of virtual work and the fundamental lemma of variation calculus imply that

$$\left(\frac{EI}{L}\begin{bmatrix} 8 & 2 \\ 2 & 4 \end{bmatrix} - \frac{pL}{30}\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}\right)\begin{Bmatrix} \theta_{Y2} \\ \theta_{Y3} \end{Bmatrix} = 0 .$$

In stability analysis the aim is to find the critical values (smallest of them typically) of the load parameter p such that the solution becomes non-unique. As the equilibrium equations are homogeneous, non-zero solution is obtained only if the matrix (above) is singular:

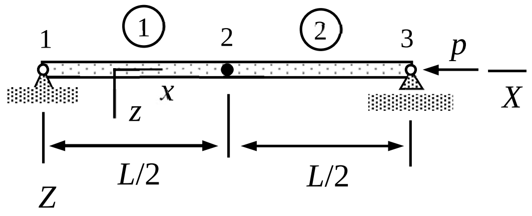
$$\det\left(\frac{EI}{L}\begin{bmatrix} 8 & 2 \\ 2 & 4 \end{bmatrix} - \frac{pL}{30}\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}\right) = \left(8\frac{EI}{L} - 4\frac{pL}{30}\right)\left(4\frac{EI}{L} - 4\frac{pL}{30}\right) - \left(2\frac{EI}{L} + \frac{pL}{30}\right)^2 = 0 \quad \Rightarrow$$

$$\frac{pL^2}{EI} \in \{20, 84\} .$$

The smallest of the values is the critical one

$$p_{\mathrm{cr}} = 20\frac{EI}{L^2} .$$

The simply supported uniform beam shown is divided into two identical beam elements, each of length $L/2$. Displacements are confined to the xz – plane. Cross-section properties are A and I and Young's modulus of the material E. Determine the buckling load $p_{\rm cr}$. Assume that rotation angles satisfy $\theta_{Y1}=-\theta_{Y3}$ and $\theta_{Y2}=0$.



Solution

The axial force in the beams follows directly from a free body diagram and it is enough to consider the virtual work associated with bending and interaction modes. The non-zero displacements and rotations of the structure are θ_{Y1}, u_{Z2} and $\theta_{Y3}=-\theta_{Y1}$.

For beam 1, $\theta_{y1}=\theta_{Y1}$ and $u_{z2}=u_{Z2}$. The axial force $N=-p$ (negative means compression) follows from a free-body diagram. Here $h=L/2$

$$\delta W^{\rm int}=-\begin{Bmatrix}0\\ \delta\theta_{Y1}\\ \delta u_{Z2}\\ 0\end{Bmatrix}^{\rm T}\frac{EI}{h^3}\begin{bmatrix}12 & -6h & -12 & -6h\\ -6h & 4h^2 & 6h & 2h^2\\ -12 & 6h & 12 & 6h\\ -6h & 2h^2 & 6h & 4h^2\end{bmatrix}\begin{Bmatrix}0\\ \theta_{Y1}\\ u_{Z2}\\ 0\end{Bmatrix}=-\begin{Bmatrix}\delta\theta_{Y1}\\ \delta u_{Z2}\end{Bmatrix}^{\rm T}8\frac{EI}{L^3}\begin{bmatrix}L^2 & 3L\\ 3L & 12\end{bmatrix}\begin{Bmatrix}\theta_{Y1}\\ u_{Z2}\end{Bmatrix},$$

$$\delta W^{\rm sta}=-\begin{Bmatrix}0\\ \delta\theta_{Y1}\\ \delta u_{Z2}\\ 0\end{Bmatrix}^{\rm T}\frac{N}{30h}\begin{bmatrix}36 & -3h & -36 & -3h\\ -3h & 4h^2 & 3h & -h^2\\ -36 & 3h & 36 & 3h\\ -3h & -h^2 & 3h & 4h^2\end{bmatrix}\begin{Bmatrix}0\\ \theta_{Y1}\\ u_{Z2}\\ 0\end{Bmatrix}=\begin{Bmatrix}\delta\theta_{Y1}\\ \delta u_{Z2}\end{Bmatrix}^{\rm T}\frac{p}{15L}\begin{bmatrix}L^2 & 3L/2\\ 3L/2 & 36\end{bmatrix}\begin{Bmatrix}\theta_{Y1}\\ u_{Z2}\end{Bmatrix}$$

giving

$$\delta W^1=\delta W^{\rm int}+\delta W^{\rm sta}=-\begin{Bmatrix}\delta\theta_{Y1}\\ \delta u_{Z2}\end{Bmatrix}^{\rm T}(8\frac{EI}{L^3}\begin{bmatrix}L^2 & 3L\\ 3L & 12\end{bmatrix}-\frac{p}{15L}\begin{bmatrix}L^2 & 3L/2\\ 3L/2 & 36\end{bmatrix})\begin{Bmatrix}\theta_{Y1}\\ u_{Z2}\end{Bmatrix}.$$

For beam 2, $u_{z2}=u_{Z2}$, $\theta_{y3}=\theta_{Y3}=-\theta_{Y1}$ and the axial force $N=-p$. As $h=L/2$

$$\delta W^{\rm int}=-\begin{Bmatrix}\delta u_{Z2}\\ 0\\ 0\\ -\delta\theta_{Y1}\end{Bmatrix}^{\rm T}\frac{EI}{h^3}\begin{bmatrix}12 & -6h & -12 & -6h\\ -6h & 4h^2 & 6h & 2h^2\\ -12 & 6h & 12 & 6h\\ -6h & 2h^2 & 6h & 4h^2\end{bmatrix}\begin{Bmatrix}u_{Z2}\\ 0\\ 0\\ -\theta_{Y1}\end{Bmatrix}=-\begin{Bmatrix}\delta\theta_{Y1}\\ \delta u_{Z2}\end{Bmatrix}^{\rm T}8\frac{EI}{L^3}\begin{bmatrix}L^2 & 3L\\ 3L & 12\end{bmatrix}\begin{Bmatrix}\theta_{Y1}\\ u_{Z2}\end{Bmatrix},$$

$$\delta W^{\rm sta}=-\begin{Bmatrix}\delta u_{Z2}\\ 0\\ 0\\ -\delta\theta_{Y1}\end{Bmatrix}^{\rm T}\frac{N}{30h}\begin{bmatrix}36 & -3h & -36 & -3h\\ -3h & 4h^2 & 3h & -h^2\\ -36 & 3h & 36 & 3h\\ -3h & -h^2 & 3h & 4h^2\end{bmatrix}\begin{Bmatrix}u_{Z2}\\ 0\\ 0\\ -\theta_{Y1}\end{Bmatrix}=\begin{Bmatrix}\delta\theta_{Y1}\\ \delta u_{Z2}\end{Bmatrix}^{\rm T}\frac{p}{15L}\begin{bmatrix}L^2 & 3L/2\\ 3L/2 & 36\end{bmatrix}\begin{Bmatrix}\theta_{Y1}\\ u_{Z2}\end{Bmatrix},$$

so that

$$\delta W^2 = \delta W^{\text{int}} + \delta W^{\text{sta}} = -\begin{Bmatrix} \delta\theta_{Y1} \\ \delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} (8\frac{EI}{L^3}\begin{bmatrix} L^2 & 3L \\ 3L & 12 \end{bmatrix} - \frac{p}{15L}\begin{bmatrix} L^2 & 3L/2 \\ 3L/2 & 36 \end{bmatrix})\begin{Bmatrix} \theta_{Y1} \\ u_{Z2} \end{Bmatrix}.$$

Virtual work expression of structure is sum of the element contributions

$$\delta W = \delta W^1 + \delta W^2 = -\begin{Bmatrix} \delta\theta_{Y1} \\ \delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} (16\frac{EI}{L^3}\begin{bmatrix} L^2 & 3L \\ 3L & 12 \end{bmatrix} - \frac{2p}{15L}\begin{bmatrix} L^2 & 3L/2 \\ 3L/2 & 36 \end{bmatrix})\begin{Bmatrix} \theta_{Y1} \\ u_{Z2} \end{Bmatrix}.$$

Principle of virtual work and the fundamental lemma of variation calculus imply that

$$(16\frac{EI}{L^3}\begin{bmatrix} L^2 & 3L \\ 3L & 12 \end{bmatrix} - \frac{2p}{15L}\begin{bmatrix} L^2 & 3L/2 \\ 3L/2 & 36 \end{bmatrix})\begin{Bmatrix} \theta_{Y1} \\ u_{Z2} \end{Bmatrix} = 0.$$

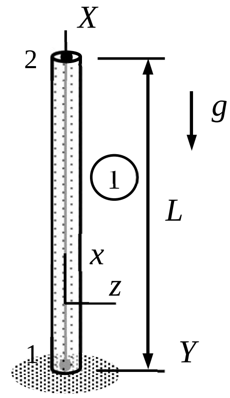
As the equilibrium equations are homogeneous, non-zero solution is obtained only if the matrix (above) is singular:

$$\det(16\frac{EI}{L^3}\begin{bmatrix} L^2 & 3L \\ 3L & 12 \end{bmatrix} - \frac{2p}{15L}\begin{bmatrix} L^2 & 3L/2 \\ 3L/2 & 36 \end{bmatrix}) = 0 \quad \Rightarrow \quad \frac{pL^2}{EI} = \frac{16}{3}\left(13 \pm 2\sqrt{31}\right).$$

The smallest of the values is the critical one

$$p_{\text{cr}} = \frac{16}{3}\left(13 - 2\sqrt{31}\right)\frac{EI}{L^2} = \frac{240}{13 + 2\sqrt{31}}\frac{EI}{L^2} \approx 9.94\frac{EI}{L^2}.$$

Find the density ρ_{cr} causing the beam of the figure to buckle in $xz-$plane. Start with the virtual work density taking into account the interaction of the bar and bending modes. Choose first $\delta w=0$ in the virtual work density to solve for the axial displacement and the axial force N. After that, choose $\delta u=0$ to find the virtual work expression taking into account the internal and coupling parts.



Solution

The displacements/rotations of the structure are $u_{x2}=u_{X2}$, $u_{z2}=u_{Y2}$ and $\theta_{y2}=-\theta_{Z2}$. The starting point is the virtual work density

$$\delta w_{\Omega}=-\frac{d\delta u}{dx}EA\frac{du}{dx}-\frac{d^2\delta w}{dx^2}EI_{zz}\frac{d^2 w}{dx^2}-N\frac{d\delta w}{dx}\frac{dw}{dx}+\delta u f_x \quad \text{where} \quad N=EA\frac{du}{dx}$$

which takes into account the bar and bending modes and their interaction. Approximations to axial displacement u and transverse displacement w ($\xi=x/h$ and $h=L$) are

$$u=\frac{x}{L}u_{X2} \quad \Rightarrow \quad \frac{du}{dx}=\frac{1}{L}u_{X2},$$

$$w=\begin{Bmatrix}(3-2\frac{x}{L})(\frac{x}{L})^2 \\ L(\frac{x}{L})^2(\frac{x}{L}-1)\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}u_{Y2} \\ \theta_{Z2}\end{Bmatrix} \quad \Rightarrow \quad \frac{dw}{dx}=\begin{Bmatrix}\frac{6x}{L^2}-\frac{6x^2}{L^3} \\ \frac{3x^2}{L^2}-2\frac{x}{L}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}u_{Y2} \\ \theta_{Z2}\end{Bmatrix} \quad \Rightarrow \quad \frac{d^2 w}{dx^2}=\begin{Bmatrix}\frac{6}{L^2}-\frac{12x}{L^3} \\ \frac{6x}{L^2}-\frac{2}{L}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}u_{Y2} \\ \theta_{Z2}\end{Bmatrix}.$$

In the first step $\delta w=0$. When the approximation to u is substituted there, virtual work density simplifies to

$$\delta w_{\Omega}=\delta w_{\Omega}=-\frac{d\delta u}{dx}EA\frac{du}{dx}+\delta u f_x=-\frac{\delta u_{X2}}{L}EA\frac{u_{X2}}{L}-\delta u_{X2}\frac{x}{L}\rho g A \quad \Rightarrow$$

$$\delta W=\int_0^L \delta w_{\Omega}dx=-\delta u_{X2}(\frac{EA}{L}u_{X2}+\frac{L}{2}\rho g A).$$

Principle of virtual work and the fundamental lemma of variation calculus imply that (notice that the actual axial force is linear)

$$u_{X2}=-\frac{L^2}{2}\frac{\rho g}{E} \quad \text{giving as the axial force} \quad N=EA\frac{du}{dx}=\frac{EA}{L}u_{X2}=-\frac{L}{2}\rho g A.$$

In the second step $\delta u=0$. When the approximation to w is substituted there, the virtual work density becomes (virtual work expression is available also in the formulae collection)

$$\delta w_{\Omega} = -\frac{d^2\delta w}{dx^2} EI_{zz} \frac{d^2 w}{dx^2} - N \frac{d\delta w}{dx}\frac{dw}{dx} \quad \Rightarrow$$

$$\delta w_{\Omega} = -\begin{Bmatrix} \delta u_{Y2} \\ \delta\theta_{Z2} \end{Bmatrix}^{\mathrm{T}} \left(\begin{Bmatrix} \dfrac{6}{L^2} - \dfrac{12x}{L^3} \\ \dfrac{6x}{L^2} - \dfrac{2}{L} \end{Bmatrix} EI \begin{Bmatrix} \dfrac{6}{L^2} - \dfrac{12x}{L^3} \\ \dfrac{6x}{L^2} - \dfrac{2}{L} \end{Bmatrix}^{\mathrm{T}} + \begin{Bmatrix} \dfrac{6x}{L^2} - \dfrac{6x^2}{L^3} \\ \dfrac{3x^2}{L^2} - 2\dfrac{x}{L} \end{Bmatrix} N \begin{Bmatrix} \dfrac{6x}{L^2} - \dfrac{6x^2}{L^3} \\ \dfrac{3x^2}{L^2} - 2\dfrac{x}{L} \end{Bmatrix}^{\mathrm{T}} \right) \begin{Bmatrix} u_{Y2} \\ \theta_{Z2} \end{Bmatrix} \Rightarrow$$

$$\delta W = \int_0^L \delta w_{\Omega} dx = -\begin{Bmatrix} \delta u_{Y2} \\ \delta\theta_{Z2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EI}{L^3} \begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix} + \frac{N}{30L} \begin{bmatrix} 36 & -3L \\ -3L & 4L^2 \end{bmatrix} \right) \begin{Bmatrix} u_{Y2} \\ \theta_{Z2} \end{Bmatrix}.$$

Principle of virtual work and the fundamental lemma of variation calculus imply (with $N = -L\rho g A / 2$)

$$\left(\frac{EI}{L^3} \begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix} - \frac{\rho g A}{60} \begin{bmatrix} 36 & -3L \\ -3L & 4L^2 \end{bmatrix} \right) \begin{Bmatrix} u_{Y2} \\ \theta_{Z2} \end{Bmatrix} = 0.$$

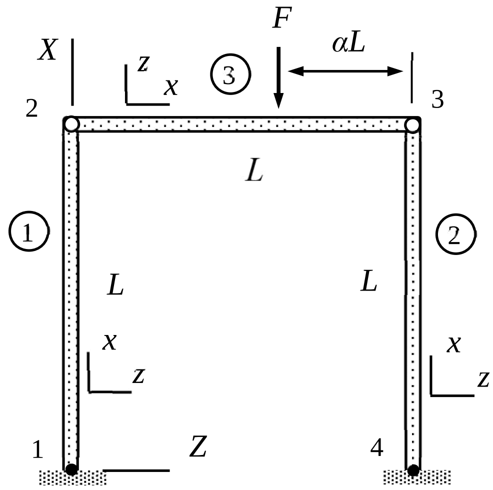
In stability analysis, the goal is to find the value of the loading parameter such that the solution is not unique. This is possibly only if

$$\det\left(\frac{EI}{L^3} \begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix} - \frac{\rho g A}{60} \begin{bmatrix} 36 & -3L \\ -3L & 4L^2 \end{bmatrix} \right) = 0 \quad \Rightarrow \quad 12\left(\frac{EI}{L^2 Ag}\right)^2 - \frac{13}{5}\left(\frac{EI}{L^2 Ag}\right)(L\rho) + \frac{3}{80}(L\rho)^2 = 0$$

giving (the smallest ρ matters)

$$\rho = \frac{8}{3}(13 \pm 2\sqrt{31})\frac{EI}{AgL^3} \quad \Rightarrow \quad \rho_{\mathrm{cr}} = \frac{8}{3}(13 - 2\sqrt{31})\frac{EI}{AgL^3} = \frac{120}{13 + 2\sqrt{31}}\frac{EI}{AgL^3}.$$

The plane frame of the figure consists of a rigid body 3 and beam elements 1 and 2. The joints at nodes 2 and 3 are frictionless. Determine the critical value of the force F acting on the rigid body at distance αL $\alpha \in [0,1]$ from node 3 making the frame to buckle laterally. The cross-section properties A, I and Young's modulus of the material E are constants.



Solution

The non-zero displacement and rotation components are $u_{Z2} = u_{Z3}$, $\theta_{Y2} = \theta_{Y3}$, u_{X2}, and u_{X3}. The vertical contact forces between the beams at nodes 2 and 3 follow from the equilibrium equations of the rigid body 3. Therefore, the axial force in beam 1 is $N = -\alpha F$ and that in beam 2 $N = -(1-\alpha)F$ (both compression) and it is enough to consider only the bending of beams 1 and 2.

For beam 1, the non-zero displacement/rotation components are (omitting the axial one as only the bending mode is considered) $u_{z2} = u_{Z2}$ and $\theta_{y2} = \theta_{Y2}$

$$\delta W^{\text{int}} = -\begin{Bmatrix} 0 \\ 0 \\ \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_{Z2} \\ \theta_{Y2} \end{Bmatrix} = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix},$$

$$\delta W^{\text{sta}} = -\begin{Bmatrix} 0 \\ 0 \\ \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{N}{30L} \begin{bmatrix} 36 & -3L & -36 & -3L \\ -3L & 4L^2 & 3L & -L^2 \\ -36 & 3L & 36 & 3L \\ -3L & -L^2 & 3L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_{Z2} \\ \theta_{Y2} \end{Bmatrix} = \begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{\alpha F}{30L} \begin{bmatrix} 36 & 3L \\ 3L & 4L^2 \end{bmatrix} \begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix}$$

therefore

$$\delta W^1 = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EI}{L^3} \begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix} - \frac{\alpha F}{30L} \begin{bmatrix} 36 & 3L \\ 3L & 4L^2 \end{bmatrix} \right) \begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix}.$$

For beam 2, the non-zero displacement/rotation components are (again omitting the axial one as only the bending mode is considered) $u_{z3} = u_{Z2} = u_{Z3}$ and $\theta_{y3} = \theta_{Y3} = \theta_{Y2}$

$$\delta W^{\text{int}} = -\begin{Bmatrix} 0 \\ 0 \\ \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_{Z2} \\ \theta_{Y2} \end{Bmatrix} = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix},$$

$$\delta W^{\text{sta}} = -\begin{Bmatrix} 0 \\ 0 \\ \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{N}{30L}\begin{bmatrix} 36 & -3L & -36 & -3L \\ -3L & 4L^2 & 3L & -L^2 \\ -36 & 3L & 36 & 3L \\ -3L & -L^2 & 3L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ u_{Z2} \\ \theta_{Y2} \end{Bmatrix} = \begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{(1-\alpha)F}{30L}\begin{bmatrix} 36 & 3L \\ 3L & 4L^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix}$$

therefore

$$\delta W^2 = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} (\frac{EI}{L^3}\begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix} - \frac{(1-\alpha)F}{30L}\begin{bmatrix} 36 & 3L \\ 3L & 4L^2 \end{bmatrix})\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix}.$$

Virtual work expression of the structure is the sum of element contributions i.e

$$\delta W = \delta W^1 + \delta W^2 = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} (2\frac{EI}{L^3}\begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix} - \frac{F}{30L}\begin{bmatrix} 36 & 3L \\ 3L & 4L^2 \end{bmatrix})\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix}.$$

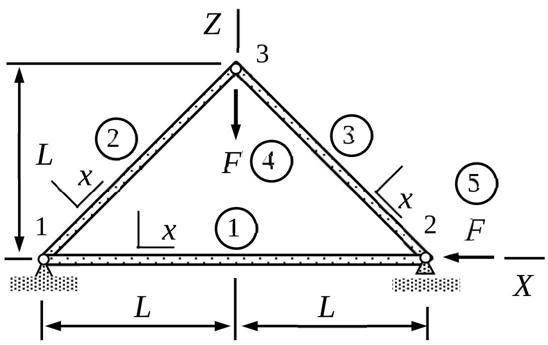
A non-trivial solution is possible only if the matrix inside parenthesis is singular

$$\det(\begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix} - \frac{FL^2}{60EI}\begin{bmatrix} 36 & 3L \\ 3L & 4L^2 \end{bmatrix}) = 0 \quad \Rightarrow \quad \frac{FL^2}{EI} = \frac{8}{3}(13 \pm 2\sqrt{31}).$$

Critical value is the smallest of the solutions

$$F = \frac{EI}{L^2}\frac{8}{3}(13 - 2\sqrt{31}) \approx 4.97\frac{EI}{L^2}\ .$$

Determine the critical value of force F causing some beam of the truss shown to buckle. First, use the bar model to solve for the nodal displacements and thereby the axial forces as functions of the loading F (assumed to be positive). After that, use criterion $N(F)=\pi^2 EI/h^2$ to find the first beam to buckle and the critical value F_{cr}. Cross-sectional areas of beams 2 and 3 are $\sqrt{8}A$ and that of beam 1 $2A$. The second moments of cross-sections I and the Young's modulus E of the material are constants.



Solution

In the first step, the structure is considered as bar structure to find the nodal displacements as functions of the loading. Virtual work expression of the bar element needed in the displacement analysis

$$\delta W^{\mathrm{int}}=-\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\frac{EA}{h}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix} \text{ and } \delta W^{\mathrm{ext}}=\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\frac{f_x h}{2}\begin{Bmatrix}1\\ 1\end{Bmatrix}$$

depend on the cross-sectional area A, Young's modulus E, bar length h, force per unit length of the bar f_x in the direction of the $x-$axis. The non-zero displacement/rotation components of the structure are u_{X2}, u_{X3}, and u_{Z3}. Virtual work expression of the elements are (no distributed forces)

Bar 1: $u_{x1}=0$, $u_{x2}=u_{X2}$,

$$\delta W^1=-\begin{Bmatrix}0\\ \delta u_{X2}\end{Bmatrix}^{\mathrm{T}}\frac{E2A}{2L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}0\\ u_{X2}\end{Bmatrix}=-\begin{Bmatrix}\delta u_{X2}\\ \delta u_{X3}\\ \delta u_{Z3}\end{Bmatrix}^{\mathrm{T}}\frac{EA}{L}\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}u_{X2}\\ u_{X3}\\ u_{Z3}\end{Bmatrix},$$

Bar 2: $u_{x1}=0$, $u_{x3}=\dfrac{1}{\sqrt{2}}(u_{X3}+u_{Z3})$

$$\delta W^2=-\begin{Bmatrix}0\\ \delta u_{X3}+\delta u_{Z3}\end{Bmatrix}^{\mathrm{T}}\frac{E\sqrt{8}A}{\sqrt{8}L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}0\\ u_{X3}+u_{Z3}\end{Bmatrix}=-\begin{Bmatrix}\delta u_{X2}\\ \delta u_{X3}\\ \delta u_{Z3}\end{Bmatrix}^{\mathrm{T}}\frac{EA}{L}\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 1\\ 0 & 1 & 1\end{bmatrix}\begin{Bmatrix}u_{X2}\\ u_{X3}\\ u_{Z3}\end{Bmatrix}$$

Bar 3: $u_{x3}=\dfrac{1}{\sqrt{2}}(u_{X3}-u_{Z3})$, $u_{x2}=\dfrac{1}{\sqrt{2}}u_{X2}$

$$\delta W^3=-\begin{Bmatrix}\delta u_{X3}-\delta u_{Z3}\\ \delta u_{X2}\end{Bmatrix}^{\mathrm{T}}\frac{E\sqrt{8}A}{\sqrt{8}L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_{X3}-u_{Z3}\\ u_{X2}\end{Bmatrix}=-\begin{Bmatrix}\delta u_{X2}\\ \delta u_{X3}\\ \delta u_{Z3}\end{Bmatrix}^{\mathrm{T}}\frac{EA}{L}\begin{bmatrix}1 & -1 & 1\\ -1 & 1 & -1\\ 1 & -1 & 1\end{bmatrix}\begin{Bmatrix}u_{X2}\\ u_{X3}\\ u_{Z3}\end{Bmatrix}$$

Force 4: $\delta W^4=-\begin{Bmatrix}\delta u_{X2}\\ \delta u_{X3}\\ \delta u_{Z3}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}0\\ 0\\ F\end{Bmatrix}$

Force 5: $\delta W^5 = -\begin{Bmatrix} \delta u_{X2} \\ \delta u_{X3} \\ \delta u_{Z3} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} F \\ 0 \\ 0 \end{Bmatrix}$

Virtual work expression of a structure is sum of the element contributions

$$\delta W = -\begin{Bmatrix} \delta u_{X2} \\ \delta u_{X3} \\ \delta u_{Z3} \end{Bmatrix}^{\mathrm{T}} (\frac{EA}{L}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{X3} \\ u_{Z3} \end{Bmatrix} + \begin{Bmatrix} F \\ 0 \\ F \end{Bmatrix}).$$

Principle of virtual work and the fundamental lemma of variation calculus imply the linear equation system and thereby the solution to nodal displacements

$$\frac{EA}{L}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{X3} \\ u_{Z3} \end{Bmatrix} + F\begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix} u_{X2} \\ u_{X3} \\ u_{Z3} \end{Bmatrix} = \frac{FL}{EA}\begin{Bmatrix} -1/2 \\ -1/4 \\ -1/4 \end{Bmatrix}.$$

The axial forces of the beams become (notice that the expression depends on the displacement components in the material coordinate systems of the beams)

Beam 1: $N = \frac{EA}{L}u_{X2} = \frac{EA}{L}(-\frac{1}{2}\frac{FL}{EA}) = -\frac{1}{2}F$,

Beam 2: $N = \frac{E\sqrt{8}A}{\sqrt{2}L}\frac{1}{\sqrt{2}}(u_{X3} + u_{Z3}) = -\frac{1}{\sqrt{2}}F$,

Beam 3: $N = \frac{E\sqrt{8}A}{\sqrt{2}L}\frac{1}{\sqrt{2}}(u_{X2} - u_{X3} + u_{Z3}) = -\frac{1}{\sqrt{2}}F$.

The critical loading of the truss as predicted by criterion $N = \pi^2 EI/h^2$ in which N is the magnitude of the compressive axial force

Beam 1: $F = \frac{1}{2}\pi^2\frac{EI}{L^2} \approx 4.93\frac{EI}{L^2}$,

Beam 2: $F = \frac{1}{\sqrt{2}}\pi^2\frac{EI}{L^2} \approx 6.98\frac{EI}{L^2}$,

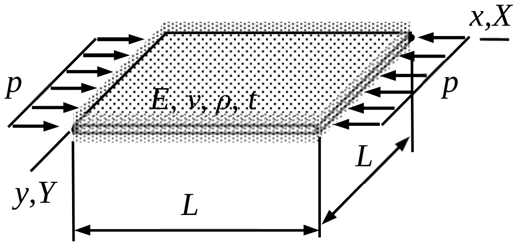
Beam 3: $F = \frac{1}{\sqrt{2}}\pi^2\frac{EI}{L^2} \approx 6.98\frac{EI}{L^2}$.

The critical load of the truss is the smallest of the critical loads calculated for the beams

$$F_{\mathrm{cr}} = \frac{1}{2}\pi^2\frac{EI}{L^2} \approx 4.93\frac{EI}{L^2}.$$

Beam 1 is likely to buckle first.

Determine the critical value of the in-plane loading p_{cr} making the plate shown to buckle. Use $w(x,y)=a_0\sin(\pi x/L)\sin(\pi y/L)$ as the approximation and assume that $N_{xx}=-p$, $N_{yy}=0$, and $N_{xy}=0$. Problem parameters E, ν, ρ and t are constants. Integrals of sin and cos satisfy, e.g.,



$$\int_0^L \sin(i\pi\frac{x}{L})\sin(j\pi\frac{x}{L})dx=\frac{L}{2}\delta_{ij} \text{ and } \int_0^L \cos(i\pi\frac{x}{L})\cos(j\pi\frac{x}{L})dx=\frac{L}{2}\delta_{ij}.$$

Solution

Assuming that the material coordinate system is chosen so that the plate bending and thin slab modes decouple in the linear analysis and that the in-plane stress resultants are known (from linear displacement analysis, say), it is enough to consider the virtual work densities of plate bending mode and the coupling of the bending and thin-slab modes

$$\delta w_\Omega^{\text{int}}=-\begin{Bmatrix}\partial^2\delta w/\partial x^2\\ \partial^2\delta w/\partial y^2\\ 2\partial^2\delta w/\partial x\partial y\end{Bmatrix}^{\mathrm{T}}\frac{t^3}{12}[E]_\sigma\begin{Bmatrix}\partial^2 w/\partial x^2\\ \partial^2 w/\partial y^2\\ 2\partial^2 w/\partial x\partial y\end{Bmatrix},\quad \delta w_\Omega^{\text{sta}}=-\begin{Bmatrix}\partial\delta w/\partial x\\ \partial\delta w/\partial y\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}N_{xx} & N_{xy}\\ N_{yx} & N_{yy}\end{bmatrix}\begin{Bmatrix}\partial w/\partial x\\ \partial w/\partial y\end{Bmatrix}$$

where the elasticity matrix of plane stress

$$[E]_\sigma=\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & (1-\nu)/2\end{bmatrix}.$$

Approximation to the transverse displacement and its derivatives are

$$w(x,y)=a_0\sin(\pi\frac{x}{L})\sin(\pi\frac{y}{L}) \quad\Rightarrow$$

$$\frac{\partial w}{\partial x}=a_0(\frac{\pi}{L})\cos(\pi\frac{x}{L})\sin(\pi\frac{y}{L}),\quad \frac{\partial w}{\partial y}=a_0(\frac{\pi}{L})\sin(\pi\frac{x}{L})\cos(\pi\frac{y}{L}),$$

$$\frac{\partial^2 w}{\partial x^2}=\frac{\partial^2 w}{\partial y^2}=-a_0(\frac{\pi}{L})^2\sin(\pi\frac{x}{L})\sin(\pi\frac{y}{L}),\quad \frac{\partial^2 w}{\partial x\partial y}=a_0(\frac{\pi}{L})^2\cos(\pi\frac{x}{L})\cos(\pi\frac{y}{L}).$$

When the approximation is substituted there, virtual work densities of the internal and forces and that of the coupling simplify to ($N_{xx}=-p$ and $N_{yy}=N_{xy}=0$)

$$\delta w_\Omega^{\text{int}}=-\delta a_0\frac{t^3E}{12(1-\nu^2)}(\frac{\pi}{L})^4 2[\sin^2(\frac{\pi x}{L})\sin^2(\frac{\pi y}{L})(1+\nu)+(1-\nu)\cos^2(\frac{\pi x}{L})\cos^2(\frac{\pi y}{L})]a_0,$$

$$\delta w_\Omega^{\text{sta}}=\delta a_0 p(\frac{\pi}{L})^2\cos^2(\pi\frac{x}{L})\sin^2(\pi\frac{y}{L})a_0.$$

Virtual work expressions are integrals of the densities over the domain occupied by the plate/element

$$\delta W^{\text{int}} = \int_0^L \int_0^L \delta w_\Omega^{\text{int}} dxdy = -\delta a_0 \frac{t^3 E}{3(1-\nu^2)} (\frac{\pi}{L})^4 (\frac{L}{2})^2 a_0 \,,$$

$$\delta W^{\text{sta}} = \int_0^L \int_0^L \delta w_\Omega^{\text{sta}} dxdy = \delta a_0 \, p(\frac{\pi}{L})^2 (\frac{L}{2})^2 a_0 \,.$$

Virtual work expression

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{sta}} = -\delta a_0 [\frac{t^3 E}{3(1-\nu^2)} (\frac{\pi}{L})^4 (\frac{L}{2})^2 - p(\frac{\pi}{L})^2 (\frac{L}{2})^2] a_0 \,.$$

Principle of virtual work $\delta W = 0 \; \forall \delta a_0$ and the fundamental lemma of variation calculus give

$$\delta W = -\delta a_0 [\frac{t^3 E}{3(1-\nu^2)} (\frac{\pi}{L})^4 (\frac{L}{2})^2 - p(\frac{\pi}{L})^2 (\frac{L}{2})^2] a_0 = 0 \quad \forall \delta a_0 \quad \Leftrightarrow$$

$$[\frac{t^3 E}{3(1-\nu^2)} (\frac{\pi}{L})^4 (\frac{L}{2})^2 - p(\frac{\pi}{L})^2 (\frac{L}{2})^2] a_0 = 0 \,.$$

For a non-trivial solution $a_0 \neq 0$, the loading parameter needs to take the value

$$p_{\text{cr}} = \frac{1}{3} \frac{t^3 E}{1-\nu^2} (\frac{\pi}{L})^2 \,. \quad \leftarrow$$

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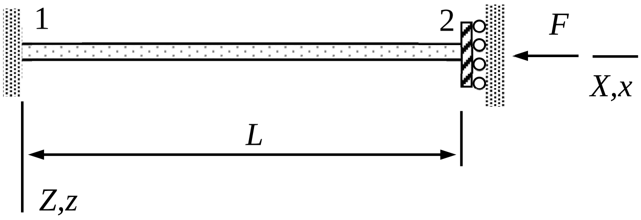
Assignment 1

Determine the buckling force F_{cr} of the beam shown by using one element. Second moment of area I and Young's modulus E are constants.



Solution template

Linear and non-linear parts of virtual work expression of internal forces of a beam element (displacements in xz-plane) are given by

$$\delta W^{\mathrm{int}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI_{yy}}{h^3} \begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix},$$

$$\delta W^{\mathrm{sta}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{N}{30h} \begin{bmatrix} 36 & -3h & -36 & -3h \\ -3h & 4h^2 & 3h & -h^2 \\ -36 & 3h & 36 & 3h \\ -3h & -h^2 & 3h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix}$$

in which I_{yy} is the second moment of area, E is the Young's modulus, and N is the axial force in the beam. The axial stress resultant N of the beam in terms of the loading parameter F (use the figure to deduce the relationship)

$$N = -F\,.$$

Linear and non-linear parts of virtual work expression of internal forces of the beam (substitute also the expression for the axial stress resultant N) are

$$\delta W^{\mathrm{int}} = -\begin{Bmatrix} 0 \\ 0 \\ \delta u_{Z2} \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_{Z2} \\ 0 \end{Bmatrix} = -\delta u_{Z2} \frac{12EI}{L^3} u_{Z2},$$

$$\delta W^{\text{sta}} = -\begin{Bmatrix} 0 \\ 0 \\ \delta u_{Z2} \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{N}{30L}\begin{bmatrix} 36 & -3L & -36 & -3L \\ -3L & 4L^2 & 3L & -L^2 \\ -36 & 3L & 36 & 3L \\ -3L & -L^2 & 3L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ u_{Z2} \\ 0 \end{Bmatrix} = \delta u_{Z2}\frac{6}{5}\frac{F}{L}u_{Z2}.$$

Principle of virtual work $\delta W = \delta W^{\text{int}} + \delta W^{\text{sta}} = 0 \quad \forall \delta u_{Z2}$ implies (assuming that $u_{Z2} \neq 0$)

$$\left(\frac{12EI}{L^3} - \frac{6}{5}\frac{F}{L}\right)u_{Z2} = 0 \quad \Rightarrow \quad F_{\text{cr}} = 10\frac{EI}{L^2}.$$

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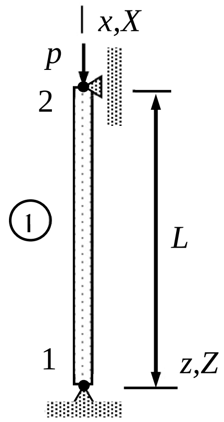

Assignment 2

Determine the buckling force p_{cr} of the structure shown by using one beam element. Displacements are confined to the xz – plane. Parameters E, A, and I are constants.

Solution template

The normal force in the beam

$$N = -p$$

can be deduced without calculations on the axial displacement. Therefore, it is enough to consider only the bending and coupling terms of the virtual work expression. As displacement is confined to the xz – plane and the beam is simply supported, virtual work expression

$$\delta W = -\begin{Bmatrix} 0 \\ \delta\theta_{Y1} \\ 0 \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} + \frac{N}{30L}\begin{bmatrix} 36 & -3L & -36 & -3L \\ -3L & 4L^2 & 3L & -L^2 \\ -36 & 3L & 36 & 3L \\ -3L & -L^2 & 3L & 4L^2 \end{bmatrix} \right) \begin{Bmatrix} 0 \\ \theta_{Y1} \\ 0 \\ \theta_{Y2} \end{Bmatrix}$$

simplifies to

$$\delta W = -\begin{Bmatrix} \delta\theta_{Y1} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 4\frac{EI}{L} - 4\frac{pL}{30} & 2\frac{EI}{L} + \frac{pL}{30} \\ 2\frac{EI}{L} + \frac{pL}{30} & 4\frac{EI}{L} - 4\frac{pL}{30} \end{bmatrix} \begin{Bmatrix} \theta_{Y1} \\ \theta_{Y2} \end{Bmatrix}.$$

According to the principle of virtual work and the fundamental lemma of variational calculation

$$\begin{bmatrix} 4\frac{EI}{L} - 4\frac{pL}{30} & 2\frac{EI}{L} + \frac{pL}{30} \\ 2\frac{EI}{L} + \frac{pL}{30} & 4\frac{EI}{L} - 4\frac{pL}{30} \end{bmatrix} \begin{Bmatrix} \theta_{Y1} \\ \theta_{Y2} \end{Bmatrix} = 0.$$

A homogeneous linear equation system has a non-trivial solution only if the matrix is singular (notice that equation $a^2 - b^2 = 0$ implies $a = \pm b$)

$$\det\left(\begin{bmatrix} 4\frac{EI}{L}-4\frac{pL}{30} & 2\frac{EI}{L}+\frac{pL}{30} \\ 2\frac{EI}{L}+\frac{pL}{30} & 4\frac{EI}{L}-4\frac{pL}{30} \end{bmatrix}\right) = (4\frac{EI}{L}-4\frac{pL}{30})^2 - (2\frac{EI}{L}+\frac{pL}{30})^2 = 0 \quad \Leftrightarrow$$

$$p = 12\frac{EI}{L^2} \quad \text{or} \quad p = 60\frac{EI}{L^2}.$$

The critical loading is given by the smallest of the buckling forces

$$p_{\text{cr}} = 12\frac{EI}{L^2}. \quad \leftarrow$$

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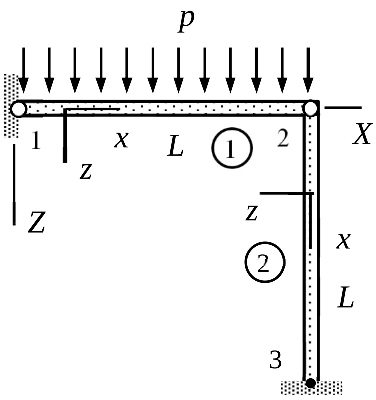
Assignment 3



Beam structure of the figure is loaded by distributed force p acting on beam 1. Determine the critical value p_{cr} causing beam 2 to buckle. Assume that beam 1 is inextensible in the axial direction. Displacements are confined to the $XZ-$plane. Cross-sectional properties A and I of the beam structure and Young's modulus E of the material are constants.

Solution template

The aim of the stability analysis is to find the condition for a non-zero transverse displacement solution for beam 2. Solving for the axial displacement of beam 2 is not necessary as the axial force in terms of the loading parameter p follows from the (moment) equilibrium of beam 1

$$N=-\frac{pL}{2}.$$

As beam 1 is inextensible in the axial direction The non-zero displacement/rotation component for beam 2 is θ_{Y2}. Element contribution, taking into account the beam bending mode and the interaction of the bar and beam bending modes, are given by

$$\delta W^{\mathrm{int}}=-\begin{Bmatrix}0\\ \delta\theta_{Y2}\\ 0\\ 0\end{Bmatrix}^{\mathrm{T}}\frac{EI}{L^3}\begin{bmatrix}12 & -6L & -12 & -6L\\ -6L & 4L^2 & 6L & 2L^2\\ -12 & 6L & 12 & 6L\\ -6L & 2L^2 & 6L & 4L^2\end{bmatrix}\begin{Bmatrix}0\\ \theta_{Y2}\\ 0\\ 0\end{Bmatrix}=-\delta\theta_{Y2}4\frac{EI}{L}\theta_{Y2},$$

$$\delta W^{\mathrm{sta}}=-\begin{Bmatrix}0\\ \delta\theta_{Y2}\\ 0\\ 0\end{Bmatrix}^{\mathrm{T}}\frac{N}{30L}\begin{bmatrix}36 & -3L & -36 & -3L\\ -3L & 4L^2 & 3L & -L^2\\ -36 & 3L & 36 & 3L\\ -3L & -L^2 & 3L & 4L^2\end{bmatrix}\begin{Bmatrix}0\\ \theta_{Y2}\\ 0\\ 0\end{Bmatrix}=\delta\theta_{Y2}\frac{1}{15}pL^2\theta_{Y2}.$$

Virtual work expression is sum of the internal and stability parts

$$\delta W=\delta W^{\mathrm{int}}+\delta W^{\mathrm{sta}}=-\delta\theta_{Y2}\left(4\frac{EI}{L}-\frac{1}{15}pL^2\right)\theta_{Y2}.$$

Principle of virtual work and the fundamental lemma of variation calculus imply the equilibrium equation

$$(4\frac{EI}{L}-\frac{1}{15}pL^2)\theta_{Y2}=0\,.$$

A non-trivial solution is possible (something that is non-zero) only if the expression in parenthesis vanishes. Therefore, the critical value of the loading parameter p, making the solution non-unique, is

$$4\frac{EI}{L}-\frac{1}{15}pL^2=0 \quad \Leftrightarrow \quad p_{\mathrm{cr}}=60\frac{EI}{L^3}\,.$$

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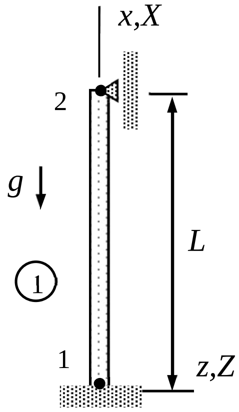
Assignment 4



A beam is loaded by its own weight as shown in the figure. Assume that displacement is confined to the $XZ-$plane. Derive the equilibrium equations for buckling analysis giving the axial displacement and the critical density ρ_{cr} of the material. Start with the virtual work density and approximations to the axial and transverse displacements. The cross-section properties A, I and material properties E, ρ are constants.

Solution template

Virtual work expressions for the buckling analysis of a beam in $xz-$plane consist of the internal parts for the bar and bending modes, coupling (stability expression) between them, and virtual work of the external point force. Altogether ($f_x = -\rho A g$)

$$\delta w_\Omega = -\frac{d^2\delta w}{dx^2} EI_{yy} \frac{d^2 w}{dx^2} - \frac{d\delta u}{dx} EA \frac{du}{dx} - \frac{d\delta w}{dx} N \frac{dw}{dx} + \delta u f_x, \text{ where } N = EA\frac{du}{dx}.$$

In terms of the non-zero displacement/rotation components of the structural system, approximations to the axial displacement u, transverse displacement w, and the axial force N simplify to

$$u(x) = \begin{Bmatrix} 1-x/L \\ x/L \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ u_{X2} \end{Bmatrix} = \frac{x}{L} u_{X2},$$

$$w(x) = \begin{Bmatrix} (1-x/L)^2(1+2x/L) \\ L(1-x/L)^2 x/L \\ (3-2x/L)(x/L)^2 \\ L(x/L)^2(x/L-1) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -\theta_{Y2} \end{Bmatrix} = \frac{1}{L^2}(Lx^2 - x^3)\theta_{Y2},$$

$$N = EA\frac{du}{dx} = \frac{EA}{L} u_{X2}.$$

When the approximations are substituted there, virtual work density simplifies to (substitute the expression for the axial force N and distributed force f_x)

$$\delta w_\Omega = -\delta\theta_{Y2}\frac{EI}{L^4}(2L-6x)^2\theta_{Y2} - \delta u_{X2}\frac{EA}{L^2}u_{X2} - \delta\theta_{Y2}(2Lx-3x^2)^2 u_{X2}\frac{EA}{L^5}\theta_{Y2} - \delta u_{X2}\frac{x}{L}\rho A g.$$

Integration over the length of the beam gives

$$\delta W = \int_0^L \delta w_\Omega dx = -\delta\theta_{Y2} 4\frac{EI}{L}\theta_{Y2} - \delta u_{X2}\frac{EA}{L}u_{X2} - \delta\theta_{Y2}\frac{2}{15}EAu_{X2}\theta_{Y2} - \delta u_{X2}\frac{1}{2}L\rho Ag \quad \Leftrightarrow$$

$$\delta W = -\begin{Bmatrix} \delta u_{X2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \dfrac{EA}{L}u_{X2} + \dfrac{1}{2}L\rho Ag \\ (4\dfrac{EI}{L} + \dfrac{2}{15}EAu_{X2})\theta_{Y2} \end{Bmatrix}.$$

Principle of virtual work and the fundamental lemma of variation calculus imply equilibrium equations

$$\begin{Bmatrix} \dfrac{EA}{L}u_{X2} + \dfrac{1}{2}L\rho Ag \\ (4\dfrac{EI}{L} + \dfrac{2}{15}EAu_{X2})\theta_{Y2} \end{Bmatrix} = 0.$$

The first equation is linear and can be solved for the axial displacement

$$\frac{EA}{L}u_{X2} + \frac{1}{2}L\rho Ag = 0 \quad \Leftrightarrow \quad u_{X2} = -\frac{L^2}{2}\frac{\rho g}{E}.$$

When the solution to the axial displacement is substituted there, the second (non-linear) equation simplifies to

$$(4\frac{EI}{L} - \frac{1}{15}L^2 A\rho g)\theta_{Y2} = 0.$$

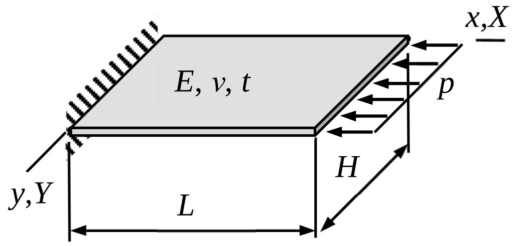
The remaining task is to deduce the possible solutions: If the expression in parenthesis is non-zero, the equation implies that $\theta_{Y2} = 0$. If the expression in parenthesis is zero, the equation is satisfied no matter the non-zero value of θ_{Y2}. Therefore, buckling may occur when (here density ρ stands for the loading parameter)

$$4\frac{EI}{L} - \frac{1}{15}L^2 A\rho g = 0 \quad \Leftrightarrow \quad \rho_{\mathrm{cr}} = 60\frac{EI}{AgL^3}.$$

Name________________________ Student number________________

Assignment 5

The clamping of the plate shown allows displacement in y – direction. At the free edge, the plate is loaded by distributed force p. Determine the critical value p_{cr} of the distributed force making the plate to buckle. Use the approximation $w(x,y)=a_0(x/L)^2$ and assume that $N_{xx}=-p$ and $N_{yy}=N_{xy}=0$. Material parameters E, ν and thickness of the plate t are constants.



Solution

Assuming that the material coordinate system is chosen so that the plate bending and thin slab modes decouple in the linear analysis and that the in-plane stress resultants are known (from linear displacement analysis, say), it is enough to consider the virtual work densities of plate bending mode and the coupling of the bending and thin-slab modes

$$\delta w_{\Omega}^{\text{int}}=-\begin{Bmatrix}\partial^2\delta w/\partial x^2\\ \partial^2\delta w/\partial y^2\\ 2\partial^2\delta w/\partial x\partial y\end{Bmatrix}^{\mathrm{T}}\frac{t^3}{12}[E]_\sigma\begin{Bmatrix}\partial^2 w/\partial x^2\\ \partial^2 w/\partial y^2\\ 2\partial^2 w/\partial x\partial y\end{Bmatrix},\quad \delta w_{\Omega}^{\text{sta}}=-\begin{Bmatrix}\partial\delta w/\partial x\\ \partial\delta w/\partial y\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}N_{xx} & N_{xy}\\ N_{yx} & N_{yy}\end{bmatrix}\begin{Bmatrix}\partial w/\partial x\\ \partial w/\partial y\end{Bmatrix}$$

where the elasticity matrix of plane stress

$$[E]_\sigma=\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & (1-\nu)/2\end{bmatrix}.$$

As the support at the clamped edge allows displacement in the y – direction, solution to the in-plane stress resultants $N_{xx}=-p$ and $N_{yy}=N_{xy}=0$ can be deduced without calculations. Approximation to the transverse displacement and its non-zero derivatives are given by

$$w(x,y)=a_0\left(\frac{x}{L}\right)^2 \quad\Rightarrow\quad \frac{\partial w}{\partial x}=2a_0\frac{x}{L^2} \text{ and } \frac{\partial^2 w}{\partial x^2}=2\frac{a_0}{L^2}.$$

When the approximation is substituted there, virtual work density of the internal forces and that of the coupling simplify to

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} 2\delta a_0 / L^2 \\ 0 \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12} \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} 2a_0 / L^2 \\ 0 \\ 0 \end{Bmatrix} = -\delta a_0 \frac{1}{3} \frac{t^3}{L^4} \frac{E}{1-\nu^2} a_0 \,,$$

$$\delta w_{\Omega}^{\text{sta}} = -\begin{Bmatrix} 2\delta a_0 x / L^2 \\ 0 \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} N_{xx} & N_{xy} \\ N_{yx} & N_{yy} \end{bmatrix} \begin{Bmatrix} 2a_0 x / L^2 \\ 0 \end{Bmatrix} = \delta a_0 4x^2 \frac{p}{L^4} a_0 \,.$$

Virtual work expressions are integrals of the densities over the domain occupied by the plate

$$\delta W^{\text{int}} = \int_0^H \int_0^L \delta w_{\Omega}^{\text{int}} dxdy = -\delta a_0 \frac{1}{3} \frac{t^3}{L^3} H \frac{E}{1-\nu^2} a_0 \,,$$

$$\delta W^{\text{sta}} = \int_0^H \int_0^L \delta w_{\Omega}^{\text{sta}} dxdy = \delta a_0 \frac{4}{3} \frac{H}{L} p a_0 \,.$$

Virtual work expression

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{sta}} = -\delta a_0 \left(\frac{1}{3} \frac{t^3}{L^3} \frac{H}{L} \frac{E}{1-\nu^2} - \frac{4}{3} \frac{H}{L} p\right) a_0 \,,$$

principle of virtual work $\delta W = 0 \ \forall \delta a_0$, and the fundamental lemma of variation calculus give

$$\left(\frac{1}{3} \frac{t^3}{L^3} \frac{H}{L} \frac{E}{1-\nu^2} - \frac{4}{3} \frac{H}{L} p\right) a_0 = 0 \,.$$

For a non-trivial solution $a_0 \neq 0$, the loading parameter needs to take the value

$$p_{\text{cr}} = \frac{1}{4} \frac{E}{1-\nu^2} \frac{t^3}{L^2} \,. \quad \leftarrow$$

MEC-E8001

Finite Element Analysis

2023

WEEK 6: NONLINEAR ANALYSIS

5 NONLINEAR ANALYSIS

5.1 LARGE DISPLACEMENT ELASTICITY.. 12

5.2 LARGE DISPLACEMENT FEA 22

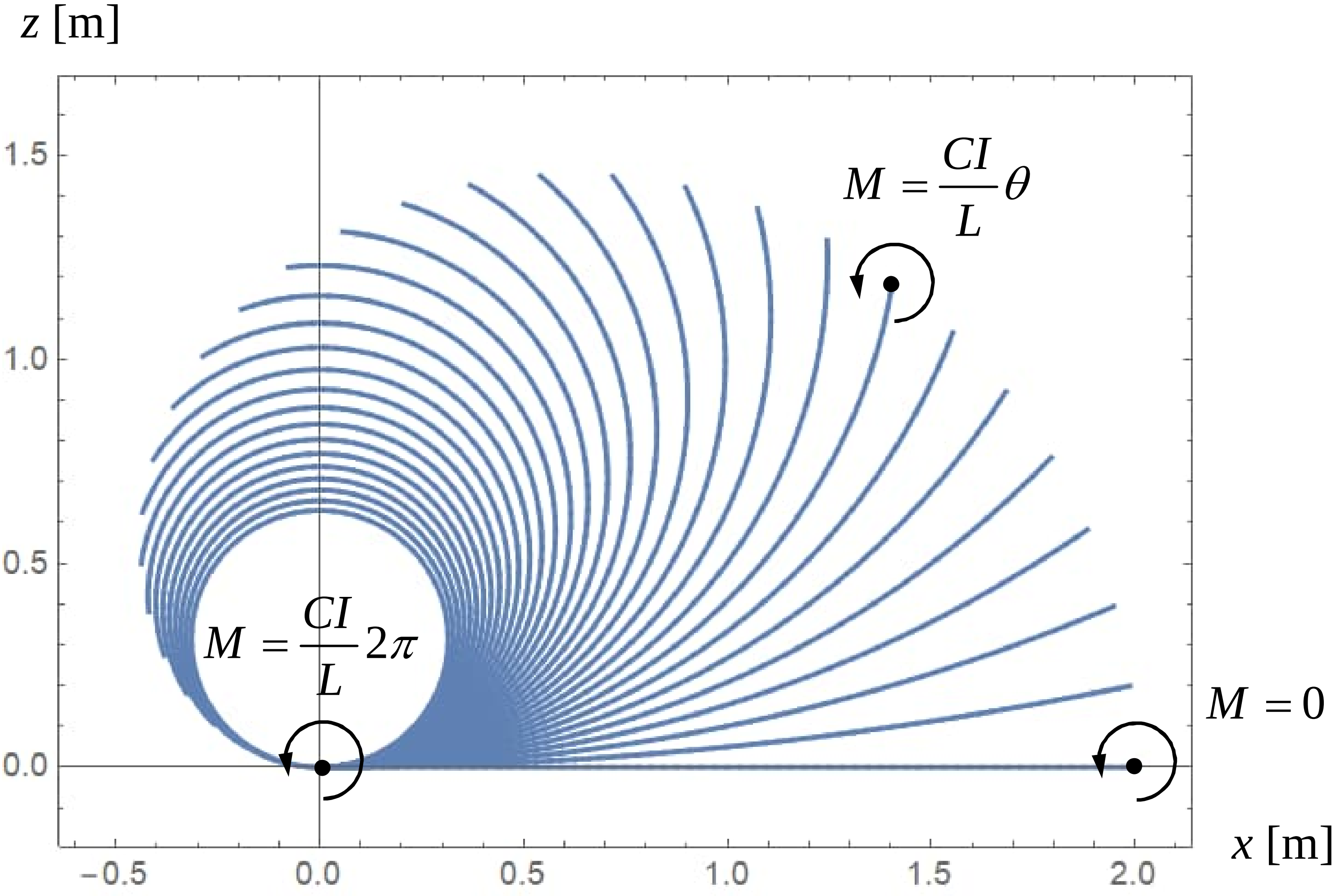
5.3 ELEMENT CONTRIBUTIONS.. 30

LEARNING OUTCOMES

Students are able to solve the weekly lecture problems, home problems, and exercise problems about large displacement FEA:

- ☐ Large displacement elasticity theory, principle of virtual work
- ☐ Large displacement FEA for solid, thin slab, and bar models
- ☐ Non-linear element contributions of solid, thin slab, and bar models

BENDING OF BEAM



BALANCE LAWS OF MECHANICS

Balance of mass (def. of a body or a material volume) Mass of a body is constant

Balance of linear momentum (Newton 2) The rate of change of linear momentum within a material volume equals the external force resultant acting on the material volume.

Balance of angular momentum (Cor. of Newton 2) The rate of change of angular momentum within a material volume equals the external moment resultant acting on the material volume.

Balance of energy (Thermodynamics 1)

Entropy growth (Thermodynamics 2)

SOURCES OF NON-LINEARITY

- **Geometry:** Equilibrium equations should be satisfied in deformed geometry depending on displacement. Strain measures of large displacements are always non-linear.

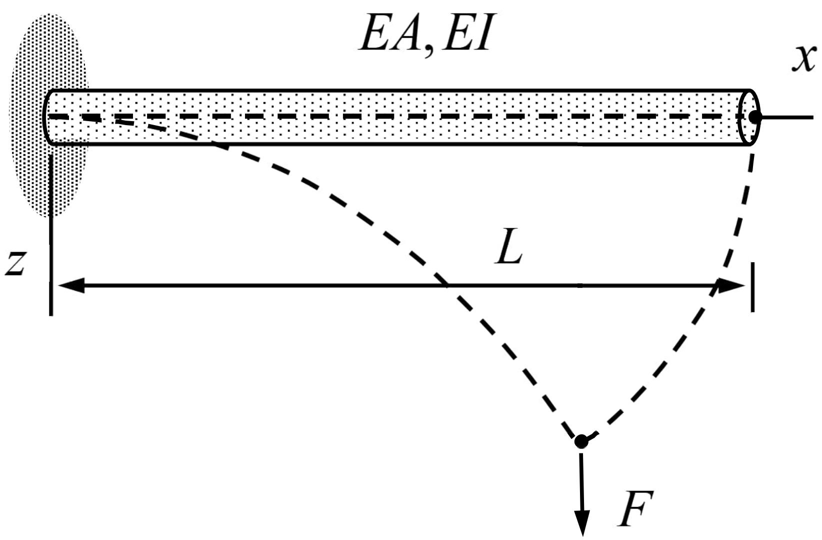
- **Material:** Constitutive equation $g(\sigma,u)=0$ may be non-linear. Near reference geometry, truncated Taylor series $g^{\circ}+(\partial g/\partial\sigma)^{\circ}\Delta\sigma+(\partial g/\partial u)^{\circ}\Delta u=0$ gives a useful approximation.

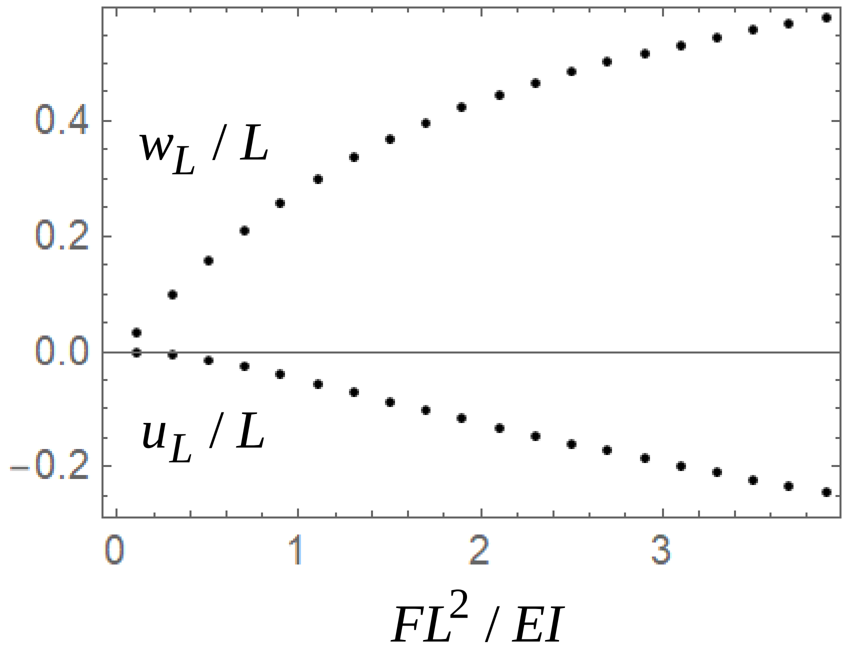
- **External forces:** External forces may be non-linear. Even the simplest contact conditions containing inequalities are always non-linear.

In non-linear mechanics $g(\sigma,\varepsilon)=0$ and $f(\varepsilon,u)=0$ the effect of material and geometry cannot be separated in the same manner as !

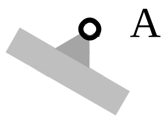
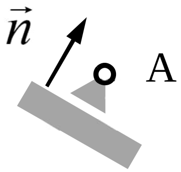
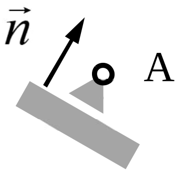
EFFECT OF GEOMETRY

Displacement at the free end (u_L, w_L) caused by force F in bending of a cantilever. Axial stiffness EA is assumed to be much larger than the bending stiffness EI $(I \ll AL^2)$. Then, length of the axis is (almost) constant L no matter the deformation.

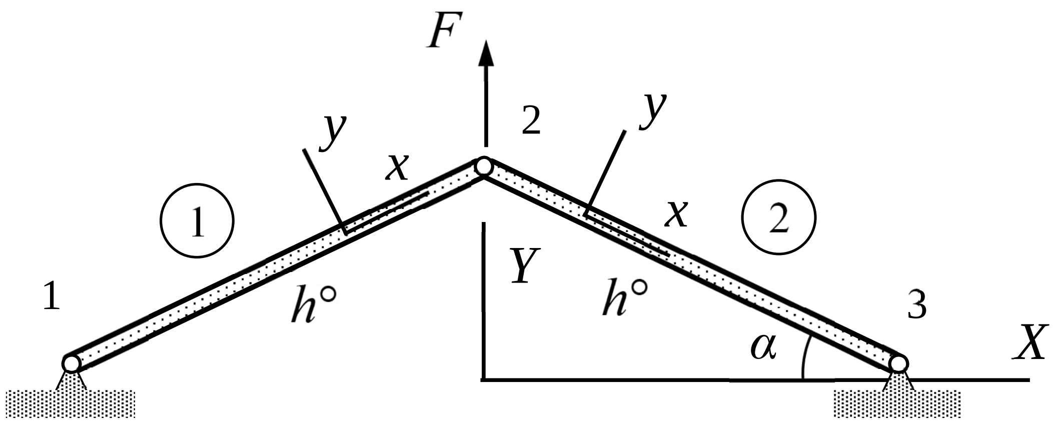




BOUNDARY CONDITIONS

name	symbol	equation
joint		$\vec{u}_{\mathrm{A}} = 0$
slider		$u_n = \vec{n} \cdot \vec{u}_{\mathrm{A}} = 0$
contact		One-sided (non-linear) boundary condition! $u_{\mathrm{A}} = \vec{n} \cdot \vec{u}_{\mathrm{A}} \geq 0 \ , \ F_{\mathrm{A}} = \vec{n} \cdot \vec{F}_{\mathrm{A}} \geq 0 \ , \ u_{\mathrm{A}} F_{\mathrm{A}} = 0$

EXAMPLE. Determine the relationship between the vertical displacement of node 2 (positive upwards) and force F acting on node 2 for the structure shown. Assume that the force-length relationship is given by $N = EAe$ and $e = h / h^{\circ} - 1$ in which EA is constant, h° is the length when $N = 0$, and h is the length at the deformed geometry (takes into account the displacement).



Answer $\dfrac{F}{EA} - 2(\sin\alpha + \mathrm{a})\dfrac{\sqrt{1+2\mathrm{a}\sin\alpha+\mathrm{a}^2}-1}{\sqrt{1+2\mathrm{a}\sin\alpha+\mathrm{a}^2}} = 0$, where $\mathrm{a} = \dfrac{u_{Y2}}{h^{\circ}}$

- Strain definition should not induce stress under rigid body motion of motion of a bar. Strain measure $e = h/h^\circ - 1$, based on the relative length change, satisfies the criterion. At the deformed geometry, when displacement is u_{Y2},

$$h = \left| h^\circ \cos\alpha \vec{I} + (h^\circ \sin\alpha + u_{Y2})\vec{J} \right| = h^\circ \sqrt{1 + 2\mathrm{a}\sin\alpha + \mathrm{a}^2} \quad \Rightarrow$$

$$\delta h = \frac{\partial h}{\partial u_{Y2}} \delta u_{Y2} = \frac{\sin\alpha + \mathrm{a}}{\sqrt{1 + 2\mathrm{a}\sin\alpha + \mathrm{a}^2}} \delta u_{Y2} \Rightarrow$$

$$N = EA\left(\frac{h}{h^\circ} - 1\right) = EA\left(\sqrt{1 + 2\mathrm{a}\sin\alpha + \mathrm{a}^2} - 1\right), \text{ where } \mathrm{a} = \frac{u_{Y2}}{h^\circ}.$$

- Virtual work expressions of external and internal forces for one bar element, written at the deformed geometry with length h, are $\delta W^{\mathrm{ext}} = F\delta u_{Y2}$ and $\delta W^{\mathrm{int}} = -N\delta h$. As the structure consists of two bars (internal parts of the bars are the same by symmetry), virtual work expression of the structure

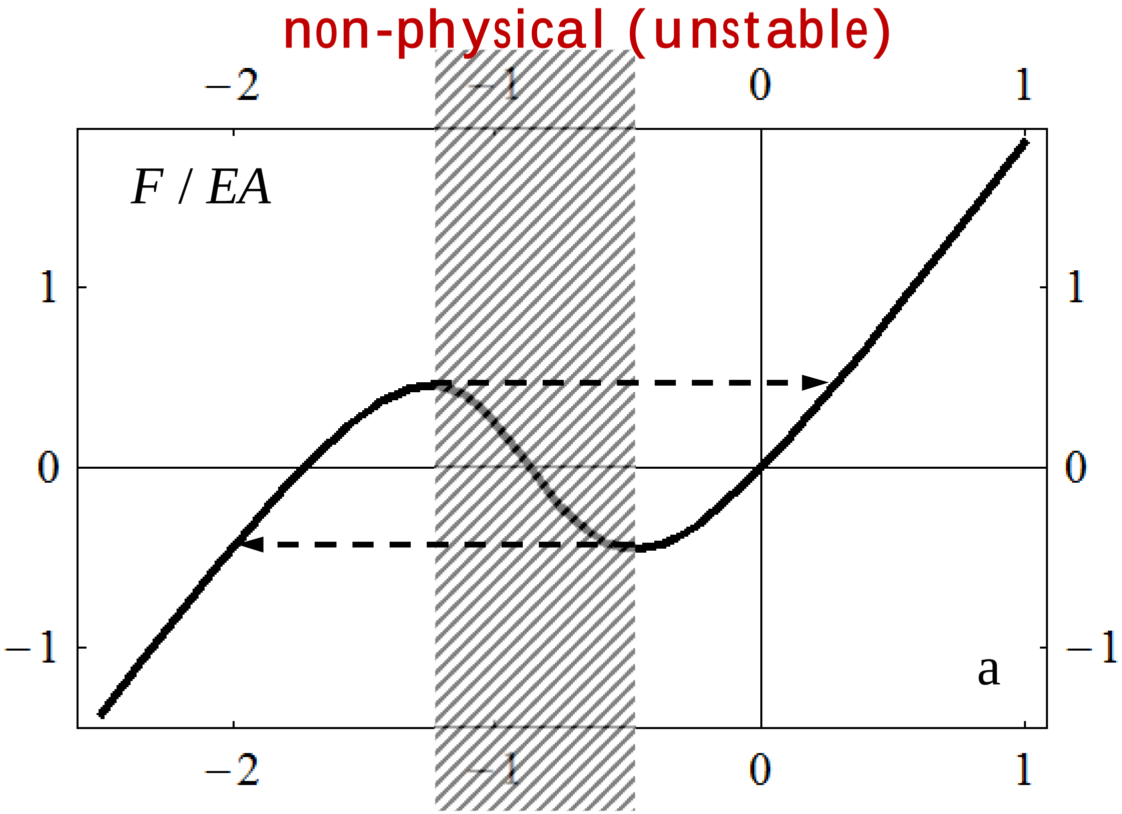
$$\delta W = [F - 2EA(\sin\alpha + \mathrm{a})\frac{\sqrt{1+2\mathrm{a}\sin\alpha + \mathrm{a}^2} - 1}{\sqrt{1+2\mathrm{a}\sin\alpha + \mathrm{a}^2}}]\delta u_{Y2}.$$

- Principle of virtual work and the fundamental lemma of variation calculus are valid also in large displacement analysis

$$F - 2EA(\sin\alpha + \mathrm{a})\frac{\sqrt{1+2\mathrm{a}\sin\alpha + \mathrm{a}^2} - 1}{\sqrt{1+2\mathrm{a}\sin\alpha + \mathrm{a}^2}} = 0. \quad \Leftarrow$$

The remaining –mathematical problem– is to find a solution or solutions to the non-linear algebraic equilibrium equation.

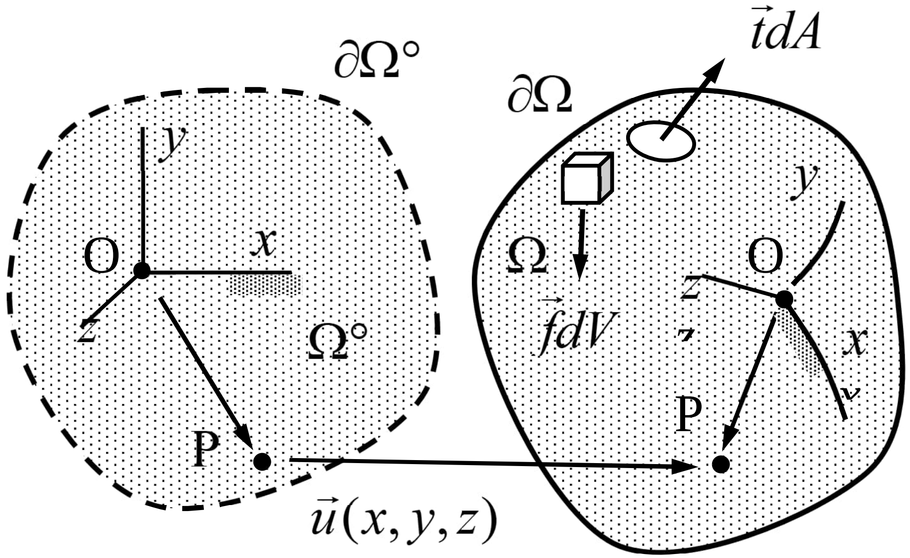
FORCE-DISPLACEMENT RELATIONSHIP $\alpha = \pi / 3$



Finding the solution by a numerical method can be tricky as a mathematically correct solution may not be physically feasible, displacement (solution) may not depend continuously on the force (data), solution depends on the loading path, etc.

5.1 LARGE DISPLACEMENT ELASTICITY

Assuming equilibrium on the initial domain Ω°, the aim is to find a new equilibrium on the deformed domain Ω, when, e.g., external forces acting on the structure are changed.



The local forms of the balance laws are concerned with the deformed domain which depends on the displacement! Precise treatment of large displacements requires modifications in stress and strain concepts of linear theory.

KINEMATICS OF LARGE DISPLACEMENTS

Displacement .... $\vec{r} = \vec{r}^\circ + \vec{u}(x^\circ, y^\circ, z^\circ)$

Deformation gradient .... $\overleftrightarrow{F}_c = \overleftrightarrow{I} + \nabla^\circ \vec{u}$

Green-Lagrange .... $2\overleftrightarrow{E} = \overleftrightarrow{F}_c \cdot \overleftrightarrow{F} - \overleftrightarrow{I} = \nabla^\circ \vec{u} + (\nabla^\circ \vec{u})_c + (\nabla^\circ \vec{u}) \cdot (\nabla^\circ \vec{u})_c$

Variation .... $\delta \overleftrightarrow{E} = \overleftrightarrow{F}_c \cdot \delta \overleftrightarrow{\varepsilon} \cdot \overleftrightarrow{F}$ where $2\overleftrightarrow{\varepsilon} = \nabla \vec{u} + (\nabla \vec{u})_c$

Domain element .... $dV = J dV^\circ$

Jacobian .... $J = |\det[F]|$

Nanson .... $\vec{n} dA = J \overleftrightarrow{F}_c^{-1} \cdot \vec{n}^\circ dA^\circ$ or $d\vec{A} = J \overleftrightarrow{F}_c^{-1} \cdot d\vec{A}^\circ$

KINETICS OF LARGE DISPLACEMENTS

Piola-Kirchhoff 1............... $J\vec{\vec{\sigma}} = \vec{\vec{P}} \cdot \vec{\vec{F}}_{\mathrm{c}}$

Piola-Kirchhoff 2............... $J\vec{\vec{\sigma}} = \vec{\vec{F}} \cdot \vec{\vec{S}} \cdot \vec{\vec{F}}_{\mathrm{c}} \qquad (\vec{\vec{F}} \cdot \vec{\vec{S}} = \vec{\vec{P}})$

Force element .... $d\vec{F} = \vec{t}dA = \vec{n} \cdot \vec{\vec{\sigma}} dA = \vec{\vec{\sigma}}_{\mathrm{c}} \cdot \vec{n} dA = \vec{\vec{P}} \cdot \vec{n}^{\circ} dA^{\circ}$

Virtual work density .... $\delta w_{V^{\circ}}^{\mathrm{int}} = -\vec{\vec{S}} : \delta \vec{\vec{E}}_{\mathrm{c}} = -\vec{\vec{\sigma}} : \delta \vec{\vec{\varepsilon}}_{\mathrm{c}} J$

Elastic material.................. $\vec{\vec{S}} = \lambda \mathrm{tr}(\vec{\vec{E}}) \vec{\vec{I}} + 2\mu \vec{\vec{E}}$

Analysis uses the PK2 stress concept. Cauchy (true) stress follows from the relationship between the quantities. In practice, the simple constitutive equation applies to isotropic material subjected to small strains (displacements may be large).

GREEN-LAGRANGE STRAIN

A rigid body motion should not induce strains! The proper strain measures with this respect are non-linear in displacement components

$$\begin{Bmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ \mathrm{E}_{zz} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} + \frac{1}{2} \begin{Bmatrix} (\partial u_x / \partial x)^2 + (\partial u_y / \partial x)^2 + (\partial u_z / \partial x)^2 \\ (\partial u_x / \partial y)^2 + (\partial u_y / \partial y)^2 + (\partial u_z / \partial y)^2 \\ (\partial u_x / \partial z)^2 + (\partial u_y / \partial z)^2 + (\partial u_z / \partial z)^2 \end{Bmatrix},$$

$$\begin{Bmatrix} \mathrm{E}_{xy} \\ \mathrm{E}_{yz} \\ \mathrm{E}_{zx} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{Bmatrix} + \frac{1}{2} \begin{Bmatrix} (\partial u_x / \partial x)(\partial u_x / \partial y) + (\partial u_y / \partial x)(\partial u_y / \partial y) + (\partial u_z / \partial x)(\partial u_z / \partial y) \\ (\partial u_x / \partial y)(\partial u_x / \partial z) + (\partial u_y / \partial y)(\partial u_y / \partial z) + (\partial u_z / \partial y)(\partial u_z / \partial z) \\ (\partial u_x / \partial z)(\partial u_x / \partial x) + (\partial u_y / \partial z)(\partial u_y / \partial x) + (\partial u_z / \partial z)(\partial u_z / \partial x) \end{Bmatrix}.$$

All measures boil down to the definitions of linear displacement analysis when strains and rotations of material elements are small!

ELASTIC MATERIAL

Under the assumption of large displacement and small strains, the Green-Lagrange strain measure does not differ much from the linear setting with small displacements and small strains. Constitutive equations

$$\begin{Bmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ \mathrm{E}_{zz} \end{Bmatrix} = \frac{1}{C}\begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix}\begin{Bmatrix} S_{xx} \\ S_{yy} \\ S_{zz} \end{Bmatrix} \text{ and } \begin{Bmatrix} 2\mathrm{E}_{xy} \\ 2\mathrm{E}_{yz} \\ 2\mathrm{E}_{zx} \end{Bmatrix} = \frac{1}{G}\begin{Bmatrix} S_{xy} \\ S_{yz} \\ S_{zx} \end{Bmatrix},$$

with material parameters C (which replaces E), ν, and $G = C/(2+2\nu)$ are same as those of the linear case, are assumed to simplify the setting. Also, the uni-axial and two-axial (plane) stress and strain relationships follow just by using Green-Lagrange strains instead of linear strains and C instead of E.

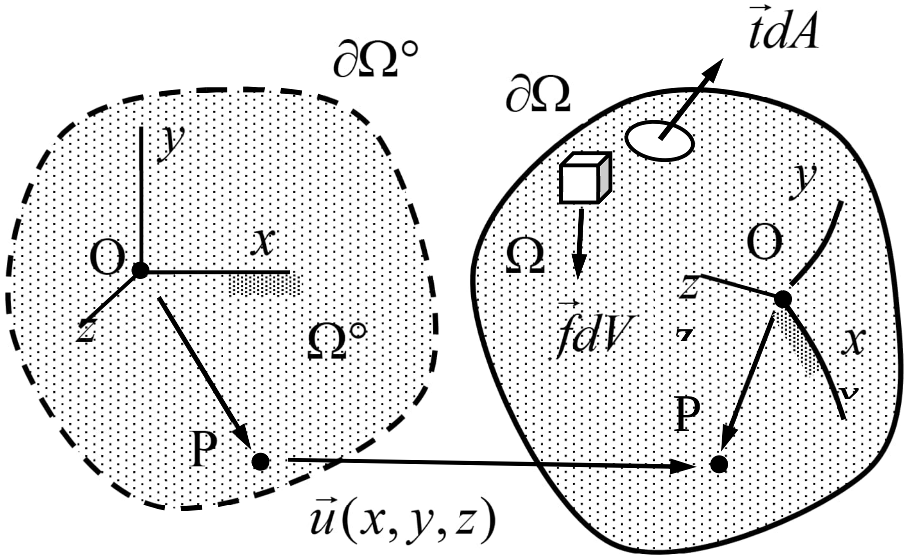
PRINCIPLE OF VIRTUAL WORK

Principle of virtual work $\delta W^{\text{int}} + \delta W^{\text{ext}} = 0 \;\; \forall \delta\vec{u}$ is concerned with the deformed domain Ω. In large displacement theory, all quantities are expressed in the Cartesian $xyz-$system of the initial geometry

$$\delta W^{\text{int}} = \int_{\Omega} \delta w_V^{\text{int}} dV = \int_{\Omega^\circ} \delta w_{V^\circ}^{\text{int}} dV^\circ,$$

$$\delta W^{\text{ext}} = \int_{\Omega} \delta w_V^{\text{ext}} dV + \int_{\partial\Omega} \delta w_A^{\text{ext}} dA$$

$$= \int_{\Omega^\circ} \delta w_{V^\circ}^{\text{ext}} dV^\circ + \int_{\partial\Omega^\circ} \delta w_{A^\circ}^{\text{ext}} dA^\circ.$$



Physics is related with domain Ω occupied by the deformed body but mathematics with the initial domain Ω° of fixed geometry.

- Principle of virtual work $\delta W^{\text{int}} + \delta W^{\text{ext}} = 0 \;\forall \delta\vec{u}$ holds at the equilibrium and therefore at the deformed geometry. In non-linear analysis, virtual work density of internal forces is expressed in terms of Green-Lagrange strain measure and PK2 stress with $\delta\vec{\vec{E}} = \vec{\vec{F}}_{\text{c}} \cdot \delta\vec{\vec{\varepsilon}} \cdot \vec{\vec{F}}$ and $dV = JdV^{\circ}$ (tensor identity $\vec{\vec{a}} : (\vec{\vec{b}}_{\text{c}} \cdot \vec{\vec{c}} \cdot \vec{\vec{b}}) = (\vec{\vec{b}} \cdot \vec{\vec{a}} \cdot \vec{\vec{b}}_{\text{c}}) : \vec{\vec{c}}$)

$$\delta W^{\text{int}} = -\int_{\Omega} (\vec{\vec{\sigma}} : \delta\vec{\vec{\varepsilon}}_{\text{c}}) dV = -\int_{\Omega^{\circ}} \vec{\vec{\sigma}} : (\vec{\vec{F}}_{\text{c}}^{-1} \cdot \delta\vec{\vec{E}}_{\text{c}} \cdot \vec{\vec{F}}^{-1}) JdV^{\circ} \quad \Rightarrow$$

$$\delta W^{\text{int}} = -\int_{\Omega^{\circ}} (\vec{\vec{F}}^{-1} \cdot \vec{\vec{\sigma}} \cdot \vec{\vec{F}}_{\text{c}}^{-1} J) : \delta\vec{\vec{E}}_{\text{c}}) dV^{\circ} = -\int_{\Omega^{\circ}} (\vec{\vec{S}} : \delta\vec{\vec{E}}_{\text{c}}) dV^{\circ}. \quad \Leftarrow$$

$$\delta W^{\text{ext}} = \int_{\Omega} (\rho\vec{g} \cdot \delta\vec{u}) dV + ... = \int_{\Omega^{\circ}} (\rho^{\circ}\vec{g} \cdot \delta\vec{u}) dV^{\circ} + ... \quad \Leftarrow$$

The virtual work density due to gravity uses the balance law of mass in its local form $\rho dV = \rho^{\circ} dV^{\circ}$ or $\rho J = \rho^{\circ}$ (also $\vec{t} dA = \vec{t}^{\circ} dA^{\circ}$).

VIRTUAL WORK DENSITIES

Virtual work densities of the internal forces, inertia forces, external volume forces due to gravity are

$$\delta w_{V^\circ}^{\text{int}} = -\begin{Bmatrix} \delta \mathrm{E}_{xx} \\ \delta \mathrm{E}_{yy} \\ \delta \mathrm{E}_{zz} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} S_{xx} \\ S_{yy} \\ S_{zz} \end{Bmatrix} - \begin{Bmatrix} 2\delta \mathrm{E}_{xy} \\ 2\delta \mathrm{E}_{yz} \\ 2\delta \mathrm{E}_{zx} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} S_{xy} \\ S_{yz} \\ S_{zx} \end{Bmatrix},$$

$$\delta w_{V^\circ}^{\text{ext}} = \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix}^{\mathrm{T}} \rho^\circ \begin{Bmatrix} g_x \\ g_y \\ g_z \end{Bmatrix}.$$

External distributed force due to gravity

Virtual work densities consist of terms containing kinematic quantities and their “work conjugates” !

DENSITY EXPRESSIONS FOR BEAMS AND PLATES

In large displacement theory, the displacement assumptions need to be modified to keep the idea of rigid body motion of cross-sections (beams) or line segments (plates). In terms of strain measures, the virtual work densities of internal forces

Beam: $\delta w_{\Omega^\circ}^{\text{int}} = -\delta \frac{1}{2}(CA\mathrm{E}^2 + CI\,\kappa^2 + GJ\,\tau^2)$,

Plate: $\delta w_{\Omega^\circ}^{\text{int}} = -\delta \frac{1}{2}(\begin{Bmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ 2\mathrm{E}_{xy} \end{Bmatrix}^{\mathrm{T}} t[C]_\sigma \begin{Bmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ 2\mathrm{E}_{xy} \end{Bmatrix} + \begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ 2\kappa_{xy} \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12}[C]_\sigma \begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ 2\kappa_{xy} \end{Bmatrix})$.

The strain measures of the bar, bending and torsion modes of the beam expression depend on Green-Lagrange axial strain E, curvature κ, and torsion τ of the mid-curve.

- Finally, the strain measures need to be expressed in terms of displacement components. For example, in a $xz-$plane beam problem

$$\mathrm{E} = \frac{du}{dx} + \frac{1}{2}(\frac{du}{dx})^2 + \frac{1}{2}(\frac{dw}{dx})^2,$$

$$\kappa = [\frac{dw}{dx}\frac{d^2u}{dx^2} - (1+\frac{du}{dx})\frac{d^2w}{dx^2}]/[(1+\frac{du}{dx})^2 + (\frac{dw}{dx})^2]^{3/2}.$$

These virtual work densities and the strain measure expressions assume. e.g., a stress-free flat initial geometry, retain only the most significant terms etc. The generic expressions in terms of the three displacement components are lengthy.

5.2 LARGE DISPLACEMENT FEA

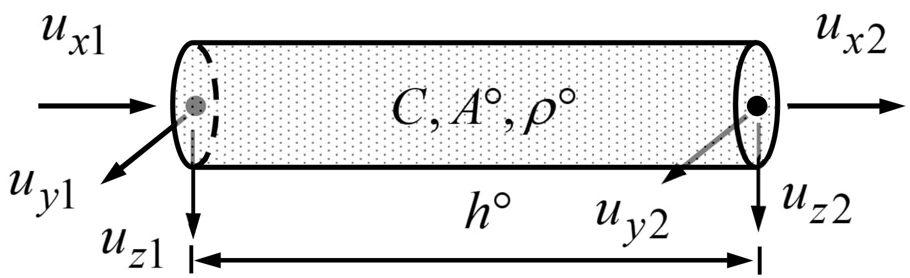
- Model a structure as a collection of beam, plate, etc. elements by considering the initial geometry. Derive the element contributions $\delta W^e = \delta W^{\text{int}} + \delta W^{\text{ext}}$ in terms of the nodal displacement and rotation components of the structural coordinate system.

- Sum the element contributions to end up with the virtual work expression of the structure $\delta W = \sum_{e \in E} \delta W^e$. Re-arrange to get $\delta W = -\delta \mathbf{a}^{\mathrm{T}} \mathbf{R}(\mathbf{a})$

- Use the principle of virtual work $\delta W = 0 \;\; \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus for $\delta \mathbf{a} \in \mathbb{R}^n$ to deduce the system equations $\mathbf{R}(\mathbf{a}) = 0$. Find a physically meaningful solution by any of the standard numerical methods for non-linear algebraic equation systems.

BAR MODE

Virtual work expression can be expressed in a concise form in terms of initial and deformed lengths of a bar element

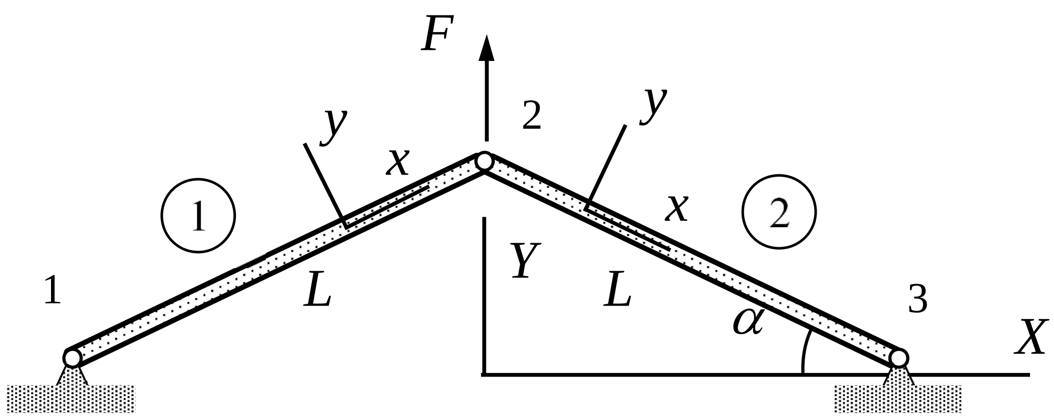


$$\delta W^{\text{int}} = -\delta \mathrm{E}_{xx} C A° \mathrm{E}_{xx},$$

$$\delta W^{\text{ext}} = \begin{Bmatrix} \delta\vec{u}_1 \cdot \vec{g} \\ \delta\vec{u}_2 \cdot \vec{g} \end{Bmatrix}^{\mathrm{T}} \frac{\rho° A° h°}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix},$$

where $\mathrm{E}_{xx} = [(h / h°)^2 - 1] / 2$ and $h^2 = (h° + u_{x2} - u_{x1})^2 + (u_{y2} - u_{y1})^2 + (u_{z2} - u_{z1})^2$ of the deformed element depends also on the nodal displaments in the $y-$ and $z-$directions. Transformation into the components of the structural system follows the lines of the linear displacement analysis.

EXAMPLE 5.1 Consider the bar structure shown subjected to large displacements. Determine the relationship between the vertical displacement of node 2 (positive upwards) and force F acting on node 2. Use the principle of virtual work and assume the constitutive equation $S_{xx} = C\mathrm{E}_{xx}$, in which Green-Lagrange strain $\mathrm{E}_{xx} = [(h/h°)^2 - 1]/2$ and C is constant. Cross-sectional area of the initial geometry is $A°$.



Answer $\dfrac{F}{CA°} - 2(\sin\alpha + \mathrm{a})(\mathrm{a}\sin\alpha + \dfrac{1}{2}\mathrm{a}^2) = 0$ where $\mathrm{a} = \dfrac{u_{Y2}}{L}$

- In (geometrically) non-linear analysis, equilibrium equations are satisfied at the deformed geometry, although the mathematics is related with the initial geometry. Virtual work expressions of internal forces of the bar element and the point force are

$$\delta W^{\text{int}} = -\delta \mathrm{E}_{xx} C A^{\circ} \mathrm{E}_{xx} = -\delta h \frac{h}{h^{\circ}} C A^{\circ} \frac{1}{2}[(\frac{h}{h^{\circ}})^2 - 1] \quad \text{and} \quad \delta W^{\text{ext}} = F \delta u_{Y2}.$$

- For element 1, the relationship between the displacement components in the material coordinate system are $u_{x2} = u_{Y2} \sin\alpha$ and $u_{y2} = u_{Y2} \cos\alpha$ giving $(\mathrm{a} = u_{Y2} / L)$

$$h^2 = (L + u_{Y2} \sin\alpha)^2 + (u_{Y2} \cos\alpha)^2 = L^2(1+2\mathrm{a}\sin\alpha + \mathrm{a}^2) \;\Rightarrow$$

$$h\delta h = \delta u_{Y2} 2(L\sin\alpha + u_{Y2}) = \delta \mathrm{a} L^2 (\sin\alpha + \mathrm{a}).$$

- For element 1, the virtual work expression of internal forces takes the form

$$\delta W^{\text{int}} = -\delta h \frac{h}{h^\circ} CA^\circ \frac{1}{2}[(\frac{h}{h^\circ})^2 - 1] = -\delta \mathrm{a} L(\sin\alpha + \mathrm{a}) CA^\circ \frac{1}{2}(2\mathrm{a}\sin\alpha + \mathrm{a}^2).$$

- Virtual work expression of the structure becomes (the internal contribution for bar 2 is the same due to the symmetry). Hence

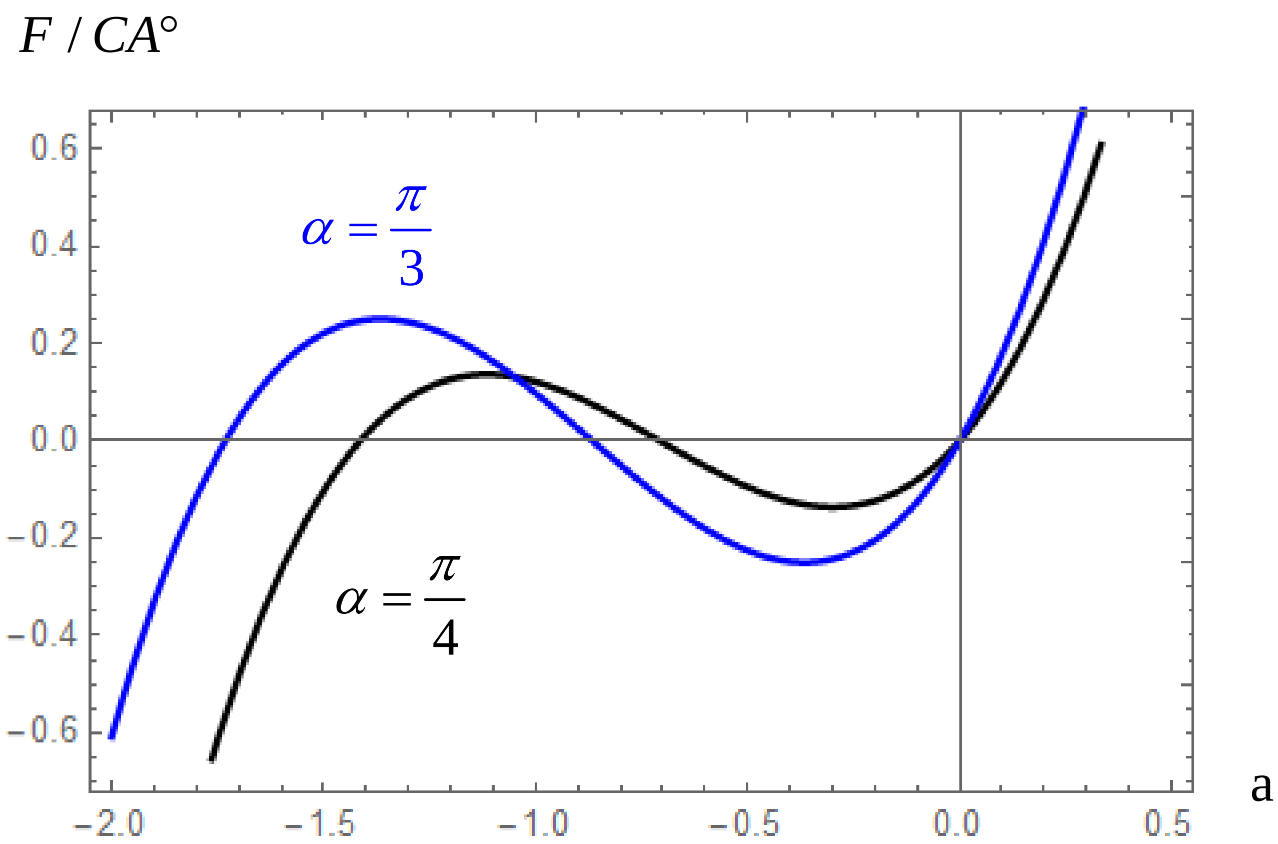
$$\delta W = 2\delta W^{\text{int}} + \delta W^{\text{ext}} = -\delta \mathrm{a} L(\sin\alpha + \mathrm{a}) CA^\circ (2\mathrm{a}\sin\alpha + \mathrm{a}^2) + FL\delta \mathrm{a}.$$

- Principle of virtual work and the fundamental lemma of variation calculus give

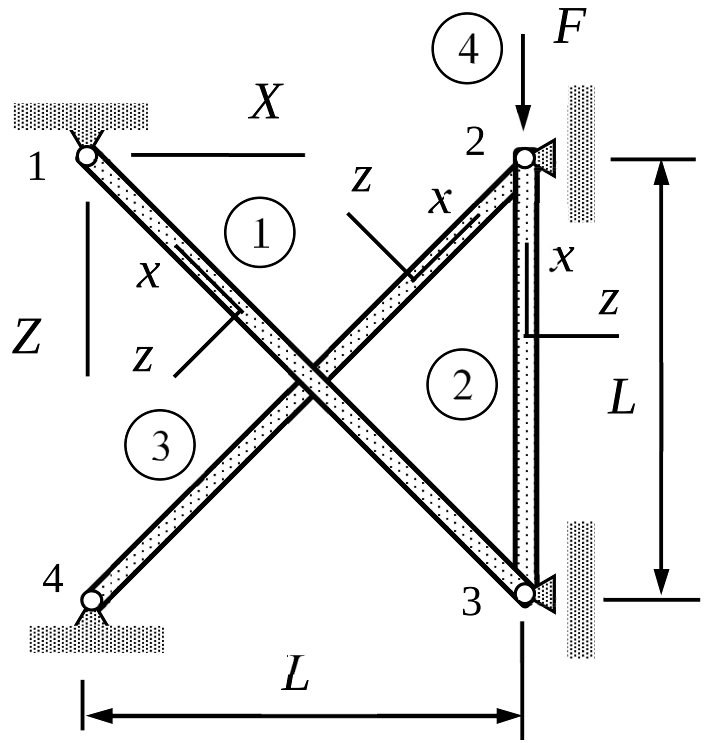
$$-\delta \mathrm{a}[L(\sin\alpha + \mathrm{a}) CA^\circ (2\mathrm{a}\sin\alpha + \mathrm{a}^2) - FL] = 0 \quad \forall \delta \mathrm{a} \quad \Leftrightarrow$$

$$(\sin\alpha + \mathrm{a})\mathrm{a}(2\sin\alpha + \mathrm{a}) - \frac{F}{CA^\circ} = 0. \quad \Leftarrow$$

FORCE-DISPLACEMENT RELATIONSHIP



EXAMPLE 5.2 Determine the nodal displacement u_{Z2} and u_{Z3} of the bar structure shown. Use non-linear bar elements and linear approximations. Cross-sectional areas and length of the initial geometry are $A = 0.01\text{m}^2$ and $L = 1\text{m}$. Elasticity parameter $C = 100\,\text{Nm}^{-2}$ and external force $F = 0.05\,\text{N}$.



Answer $u_{Z2} \approx 0.085\text{m}$ and $u_{Z3} \approx 0.061\text{m}$

- The physically correct solution is just one of the mathematically correct solutions to the nodal displacements (in this case the number of solutions is 6). The solver for non-linear analysis returns a real valued solution with the minimal norm. In the example, when $L = 1\text{m}$, $A = 0.01\text{m}^2$, $C = E = 100\,\text{Nm}^{-2}$, and $F = 1/20\,\text{N}$:

	model	properties	geometry
1	BAR	{{E}, {A}}	Line[{3, 1}]
2	BAR	{{E}, {A}}	Line[{3, 2}]
3	BAR	{{E}, {A}}	Line[{4, 2}]
4	FORCE	{0, 0, F}	Point[{2}]

	{X,Y,Z}	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}
2	{L, 0, 0}	{0, 0, uZ[2]}	{0, 0, 0}
3	{L, 0, L}	{0, 0, uZ[3]}	{0, 0, 0}
4	{0, 0, L}	{0, 0, 0}	{0, 0, 0}

{uZ[2] → 0.0854082, uZ[3] → 0.0609567}

5.3 ELEMENT CONTRIBUTIONS

Virtual work expressions for the elements combine virtual work densities of the model and an approximation depending on the element shape and type. To derive the expression for an element:

- Start with the large displacement versions of the virtual work densities $\delta w_{\Omega^\circ}^{\text{int}}$ and $\delta w_{\Omega^\circ}^{\text{ext}}$ of the formulae collection.

- Represent the unknown functions by interpolation of the nodal displacement and rotations (see formulae collection). Substitute the approximations into the density expressions.

- Integrate the virtual work density over the domain occupied by the element at the initial geometry to get δW .

ELEMENT APPROXIMATION

In MEC-E8001 element approximation is a polynomial interpolant of the nodal displacements and rotations in terms of shape functions. In non-linear analysis, approximations, shape functions etc. are written for the initial geometry.

Approximation $\mathbf{u}=\mathbf{N}^{\mathrm{T}}\mathbf{a}$ always of the same form!

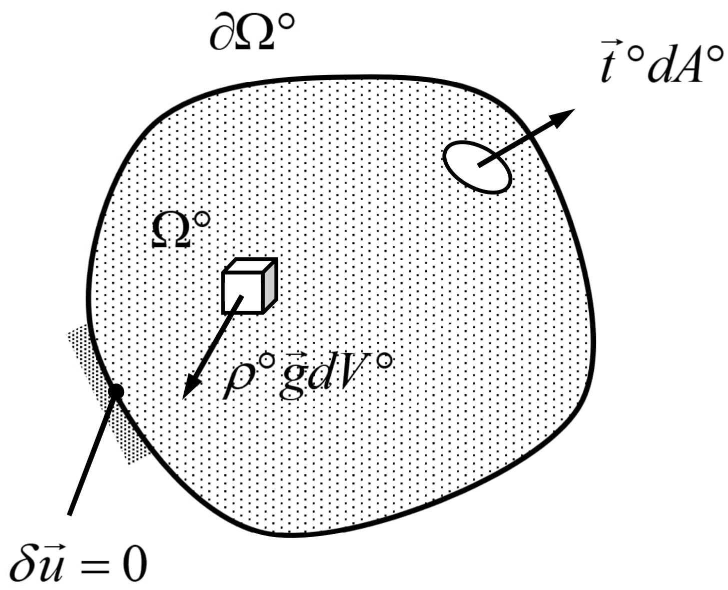
Shape functions $\mathbf{N}=\{N_1(x,y,z) \quad N_2(x,y,z) \quad \dots \quad N_n(x,y,z)\}^{\mathrm{T}}$

Parameters $\mathbf{a}=\{\mathrm{a}_1 \quad \mathrm{a}_2 \quad \dots \quad \mathrm{a}_n\}^{\mathrm{T}}$

Nodal parameters $\mathrm{a}\in\{u_x,u_y,u_z,\theta_x,\theta_y,\theta_z\}$ may be just displacement or rotation components or a mixture of them (as with the beam model).

SOLID MODEL

The model does not contain kinetic or kinematic assumptions in addition to those of non-linear elasticity theory. Virtual work density expression of the internal and external forces for the initial geometry are given by

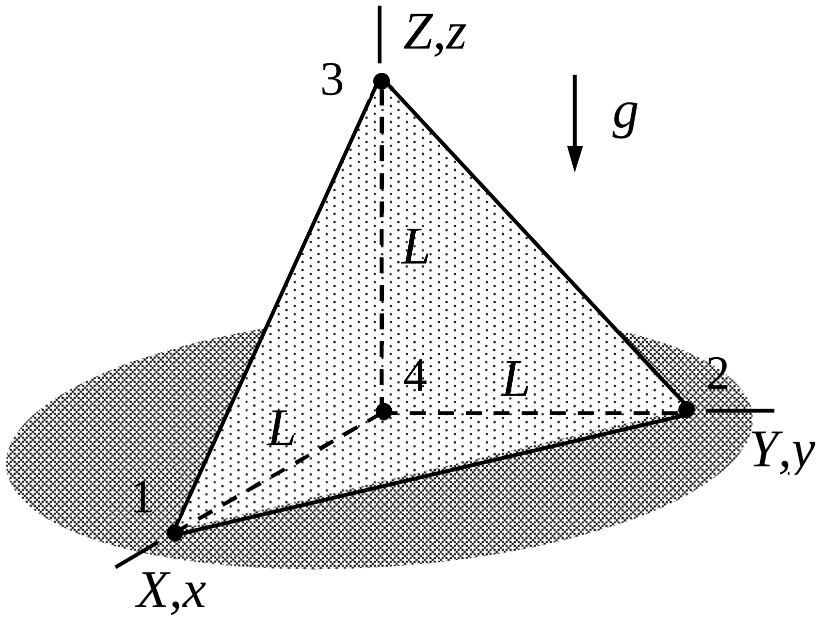


$$\delta w_{V^\circ}^{\text{int}} = -\begin{Bmatrix} \delta \mathrm{E}_{xx} \\ \delta \mathrm{E}_{yy} \\ \delta \mathrm{E}_{zz} \end{Bmatrix}^{\mathrm{T}} [C] \begin{Bmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ \mathrm{E}_{zz} \end{Bmatrix} - \begin{Bmatrix} 2\delta \mathrm{E}_{xy} \\ 2\delta \mathrm{E}_{yz} \\ 2\delta \mathrm{E}_{zx} \end{Bmatrix}^{\mathrm{T}} G \begin{Bmatrix} 2\mathrm{E}_{xy} \\ 2\mathrm{E}_{yz} \\ 2\mathrm{E}_{zx} \end{Bmatrix},$$

$$\delta w_{V^\circ}^{\text{ext}} = \delta\vec{u} \cdot \vec{g}\rho^\circ \quad \text{and} \quad \delta w_{A^\circ}^{\text{ext}} = \delta\vec{u} \cdot \vec{t}^\circ.$$

The solution domain can be represented, e.g., by tetrahedron elements with linear interpolation of the displacement components $u(x, y, z)$, $v(x, y, z)$, and $w(x, y, z)$

EXAMPLE 5.3 A tetrahedron of edge length L, density ρ, and elastic properties C and ν is subjected to its own weight on a horizontal floor. Determine the equilibrium equation for the displacement u_{Z3} of node 3 with one tetrahedron element and linear approximation. Assume that $u_{X3} = u_{Y3} = 0$ and that the bottom surface is fixed and that the geometry and density described is concerned with the initial geometry (gravity omitted).



Answer: $(1+\mathrm{a})\mathrm{a}(1+\frac{1}{2}\mathrm{a}) + F = 0$ where

$$F = \frac{1}{4}\frac{1-\nu-2\nu^2}{1-\nu}\frac{\rho g L^3}{C} \quad \text{and} \quad \mathrm{a} = \frac{u_{Z3}}{L} .$$

- Linear shape functions can be deduced directly from the figure $N_1 = x/L$, $N_2 = y/L$, $N_3 = z/L$, and $N_4 = 1 - x/L - y/L - z/L$. Only the shape function of node 3 is actually needed as the other nodes are fixed. Approximations to the displacement components are

$$u_x = u_y = 0 \quad \text{and} \quad u_z = \frac{z}{L} u_{Z3}, \quad \text{giving} \quad \frac{\partial u_z}{\partial x} = \frac{\partial u_z}{\partial y} = 0 \quad \text{and} \quad \frac{\partial u_z}{\partial z} = \frac{1}{L} u_{Z3}.$$

- When the approximation is substituted there, the non-zero Green-Lagrange strain component takes the form

$$\mathrm{E}_{zz} = \frac{1}{L} u_{Z3} + \frac{1}{2L^2} u_{Z3}^2 \quad \Rightarrow \quad \delta \mathrm{E}_{zz} = \frac{1}{L} \delta u_{Z3} + \frac{1}{L^2} u_{Z3} \delta u_{Z3}.$$

- Virtual work densities of the internal and external forces simplify to (we assume that the material is described by the constitutive equation of linear elasticity theory in which the Young's modulus E is replaced by elasticity parameter C)

$$\delta w_{V^\circ}^{\text{int}} = -\delta \mathrm{E}_{zz} S_{zz} = \frac{-C(1-\nu)}{(1+\nu)(1-2\nu)} \delta u_{Z3} \left(\frac{1}{L} + \frac{1}{L^2} u_{Z3}\right)\left(\frac{1}{L} u_{Z3} + \frac{1}{2L^2} u_{Z3}^2\right),$$

$$\delta w_{V^\circ}^{\text{ext}} = -\delta u_z \rho g = -\frac{z}{L} \rho g \delta u_{Z3}.$$

- Virtual work expressions are obtained as integrals of densities over the volume occupied by the body at the initial geometry. With $\mathrm{a} = u_{Z3} / L$

$$\delta W^{\text{int}} = \int_{\Omega^\circ} \delta w_{V^\circ}^{\text{int}} dV = \delta w_{V^\circ}^{\text{int}} \frac{L^3}{6} = -\frac{L^2}{6} \frac{1-\nu}{(1+\nu)(1-2\nu)} C \delta u_{Z3} (1+\mathrm{a})(\mathrm{a} + \frac{1}{2}\mathrm{a}^2),$$

$$\delta W^{\text{ext}} = \int_{\Omega^\circ} \delta w_{V^\circ}^{\text{ext}} dV = -\frac{L^3}{24}\rho g \delta u_{Z3}.$$

- Finally, principle of virtual work $\delta W = 0$ with $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}}$ implies the equilibrium equation

$$\frac{L^2}{6}\frac{C(1-\nu)}{(1+\nu)(1-2\nu)}(1+\mathrm{a})(\mathrm{a}+\frac{1}{2}\mathrm{a}^2) + \frac{L^3}{24}\rho g = 0.$$

- In terms of $F = \frac{1}{4}\frac{1-\nu-2\nu^2}{1-\nu}\frac{\rho g L}{C}$ the physically meaningful solution is given by

$$\mathrm{a} = \frac{1}{3^{1/3}\alpha} + \frac{\alpha}{3^{2/3}} - 1 \quad \text{where} \quad \alpha = (-9F + \sqrt{3}\sqrt{-1+27F^2})^{1/3}.$$

THIN SLAB MODE

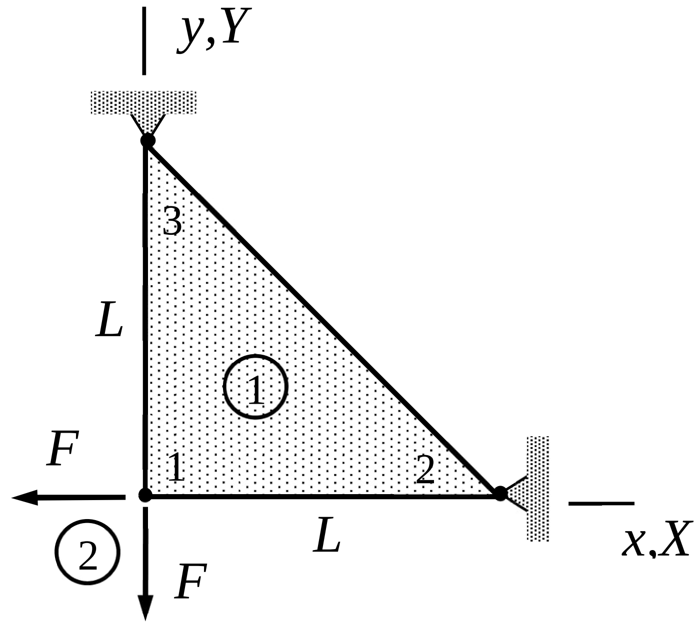
Virtual work densities of plate combine the thin-slab and plate bending modes. Assuming that the two modes de-couple and the bending mode can be omitted

$$\delta w_{\Omega^\circ}^{\text{int}} = -\begin{Bmatrix} \delta \mathrm{E}_{xx} \\ \delta \mathrm{E}_{yy} \\ 2\delta \mathrm{E}_{xy} \end{Bmatrix}^{\mathrm{T}} t^\circ [C]_\sigma \begin{Bmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ 2\mathrm{E}_{xy} \end{Bmatrix}, \quad \delta w_{\Omega^\circ}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \rho^\circ t^\circ \begin{Bmatrix} g_x \\ g_y \end{Bmatrix} \quad \text{where}$$

$$\begin{Bmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ 2\mathrm{E}_{xy} \end{Bmatrix} = \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix} + \begin{Bmatrix} (\partial u / \partial x)^2 / 2 + (\partial v / \partial x)^2 / 2 \\ (\partial u / \partial y)^2 / 2 + (\partial v / \partial y)^2 / 2 \\ (\partial u / \partial x)(\partial u / \partial y) + (\partial v / \partial x)(\partial v / \partial y) \end{Bmatrix}.$$

The planar solution domain Ω° (reference-plane of the initial geometry) can be represented by triangular or rectangular elements.

EXAMPLE 5.4 Consider the thin triangular structure shown. Assuming plane-stress conditions and $xy-$plane deformation, determine the equation for the displacement $u_{X1} = \mathrm{a}L$ and $u_{Y1} = \mathrm{a}L$ of node 1 according to the large displacement theory. Young's modulus E, Poisson's ratio ν, and thickness t are constants and distributed external force vanishes.



Answer: $(-1+2\mathrm{a})L\dfrac{tE}{1-\nu^2}\mathrm{a}(-1+\mathrm{a})-F=0$

- Nodes 2 are 3 are fixed and the non-zero displacement/rotation components are $u_{X1} = \text{a}L$ and $u_{Y1} = \text{a}L$. Linear shape functions $N_1 = (L - x - y)/L$, $N_2 = x/L$ and $N_3 = y/L$ are easy to deduce from the figure. Therefore $u = v = (L - x - y)\text{a}$ and

$$\begin{Bmatrix} \text{E}_{xx} \\ \text{E}_{yy} \\ 2\text{E}_{xy} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \\ 2 \end{Bmatrix} (-\text{a} + \text{a}^2) \quad \Rightarrow \quad \begin{Bmatrix} \delta\text{E}_{xx} \\ \delta\text{E}_{yy} \\ 2\delta\text{E}_{xy} \end{Bmatrix} = \delta\text{a}(-1 + 2\text{a}) \begin{Bmatrix} 1 \\ 1 \\ 2 \end{Bmatrix}.$$

- Virtual work density of internal forces simplifies to

$$\delta w_{\Omega^\circ}^{\text{int}} = -\begin{Bmatrix} \delta\text{E}_{xx} \\ \delta\text{E}_{yy} \\ 2\delta\text{E}_{xy} \end{Bmatrix}^{\text{T}} t[C]_\sigma \begin{Bmatrix} \text{E}_{xx} \\ \text{E}_{yy} \\ 2\text{E}_{xy} \end{Bmatrix} = -\delta\text{a}(-1 + 2\text{a}) \frac{4tE}{1 - \nu^2} (-\text{a} + \text{a}^2).$$

- Integration over the triangular domain gives (integrand is constant)

$$\delta W^1 = -\delta \mathrm{a}(-1+2\mathrm{a})L^2 \frac{2tE}{1-\nu^2}(-\mathrm{a}+\mathrm{a}^2).$$

- Virtual work expression for the point forces follows from the definition of work

$$\delta W^2 = -2\delta \mathrm{a} LF \ .$$

- Principle of virtual work in the form $\delta W = \delta W^1 + \delta W^2 = 0 \ \ \forall \delta \mathrm{a}$ and the fundamental lemma of variation calculus give

$$\delta W = -\delta \mathrm{a} L[(-1+2\mathrm{a})L \frac{2tE}{1-\nu^2}(-\mathrm{a}+\mathrm{a}^2) - 2F] = 0 \quad \forall \delta \mathrm{a} \quad \Leftrightarrow$$

$$(-1+2\mathrm{a})L \frac{2tE}{1-\nu^2}(-\mathrm{a}+\mathrm{a}^2) - 2F = 0. \quad \Leftarrow$$

The point forces acting on a thin slab should be considered as “equivalent nodal forces” i.e. just representations of tractions acting on some part of the boundary. Under the action of an actual point force, displacement becomes non-bounded. In practice, numerical solution to the displacement at the point of action increases when the mesh is refined.

- In the Mathematica code of the course, the problem description is given by

	model	properties	geometry
1	PLANE	{{E, ν}, {t}}	Polygon[{1, 2, 3}]
2	FORCE	{-F, -F, 0}	Point[{1}]

	{X,Y,Z}	$\{u_X,u_Y,u_Z\}$	$\{\theta_X,\theta_Y,\theta_Z\}$
1	{0, 0, 0}	{L a[1], L a[1], 0}	{0, 0, 0}
2	{L, 0, 0}	{0, 0, 0}	{0, 0, 0}
3	{0, L, 0}	{0, 0, 0}	{0, 0, 0}

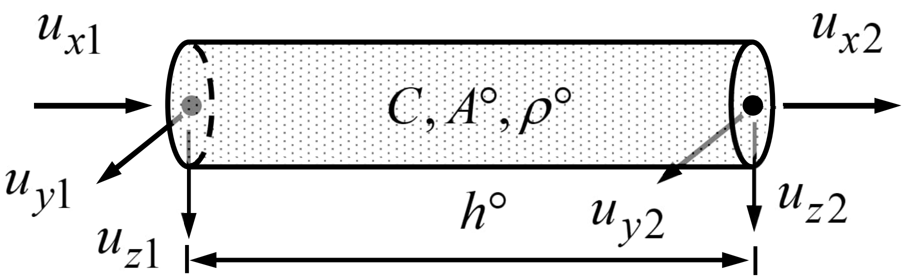
$$\delta W = -(\delta a[1])^{T}\left(-\frac{2L\left(F-F\nu^{2}+L\,t\,E\,a[1]\left(1-3a[1]+2a[1]^{2}\right)\right)}{-1+\nu^{2}}\right)$$

BAR MODE

With the assumptions of the bar model $\vec{u} = u(x)\vec{i} + v(x)\vec{j} + w(x)\vec{k}$, $\overleftrightarrow{S} = S_{xx}\vec{i}\vec{i}$ etc. in the generic expressions for large displacement analysis for the solid model simplify to

$$\delta w_{\Omega^\circ}^{\text{int}} = -\delta \mathrm{E}_{xx} A^\circ C \mathrm{E}_{xx},$$

$$\delta w_{\Omega^\circ}^{\text{ext}} = A^\circ \rho^\circ \delta\vec{u} \cdot \vec{g} \,,$$



where $\mathrm{E}_{xx} = \dfrac{du}{dx} + \dfrac{1}{2}(\dfrac{du}{dx})^2 + \dfrac{1}{2}(\dfrac{dv}{dx})^2 + \dfrac{1}{2}(\dfrac{dw}{dx})^2$.

In FEA, the solution domain (a line segment) is represented by line elements and the displacement components $u(x)$, $v(x)$, $w(x)$ by their interpolants.

- Let us start with the kinematical assumption $\vec{u} = u(x)\vec{i} + v(x)\vec{j} + w(x)\vec{k}$. The kinetic assumption is $\overleftrightarrow{S} = S_{xx}\vec{i}\vec{i}$. Green-Lagrange strain and its variation are

$$\mathrm{E}_{xx} = \frac{du}{dx} + \frac{1}{2}(\frac{du}{dx})^2 + \frac{1}{2}(\frac{dv}{dx})^2 + \frac{1}{2}(\frac{dw}{dx})^2, \quad \delta \mathrm{E}_{xx} = \frac{d\delta u}{dx} + \frac{d\delta u}{dx}\frac{du}{dx} + \frac{d\delta v}{dx}\frac{dv}{dx} + \frac{d\delta w}{dx}\frac{dw}{dx}.$$

- Assuming the constitutive equation $S_{xx} = C\mathrm{E}_{xx}$, virtual work densities of the internal and external forces per unit length of the initial domain become (expression is integrated over the cross section of the initial geometry)

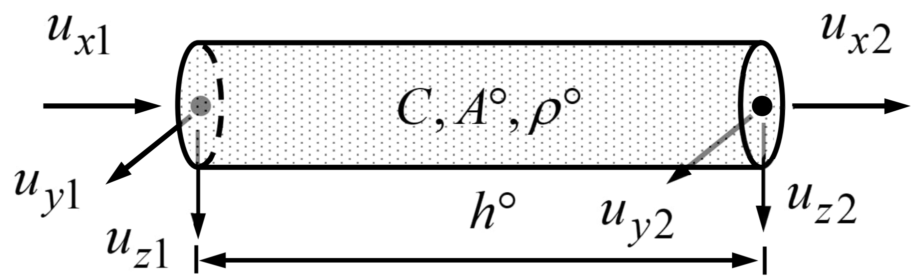
$$\delta w_{\Omega^\circ}^{\mathrm{int}} = -\delta \mathrm{E}_{xx} A^\circ C \mathrm{E}_{xx} \quad \text{and} \quad \delta w_{\Omega^\circ}^{\mathrm{ext}} = A^\circ \rho^\circ (\delta u g_x + \delta v g_y + \delta w g_z). \quad \Leftarrow$$

BAR MODE

Virtual work expression can be expressed in a concise form in terms of initial and deformed lengths of a bar element

$$\delta W^{\text{int}} = -\delta \mathrm{E}_{xx} C A^{\circ} \mathrm{E}_{xx},$$

$$\delta W^{\text{ext}} = \begin{Bmatrix} \delta \vec{u}_1 \cdot \vec{g} \\ \delta \vec{u}_2 \cdot \vec{g} \end{Bmatrix}^{\mathrm{T}} \frac{\rho^{\circ} A^{\circ} h^{\circ}}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix},$$



where $\mathrm{E}_{xx} = [(h / h^{\circ})^2 - 1] / 2$ and $h^2 = (h^{\circ} + u_{x2} - u_{x1})^2 + (u_{y2} - u_{y1})^2 + (u_{z2} - u_{z1})^2$ of the deformed element depends also on the nodal displaments in the $y-$ and $z-$directions. Transformation into the components of the structural system follows the lines of the linear displacement analysis.

- Linear approximations to the displacement components give constant values to the derivatives du/dx, dv/dx, and dw/dx and the Green-Lagrange strain component E_{xx} is simply the relative difference in the squares of lengths:

$$\mathrm{E}_{xx}=\frac{1}{2}\frac{h^2-(h^\circ)^2}{(h^\circ)^2}=\frac{1}{2}[(\frac{h}{h^\circ})^2-1] \quad \text{and} \quad \delta\mathrm{E}_{xx}=\frac{\delta h}{h^\circ}\frac{h}{h^\circ}.$$

- As virtual work density of internal forces is constant and the approximation linear Virtual works of internal and external forces become

$$\delta W^{\text{int}}=\delta w_{\Omega^\circ}^{\text{int}}h^\circ=-\delta h\frac{h}{h^\circ}CA^\circ\frac{1}{2}[(\frac{h}{h^\circ})^2-1], \quad \Leftarrow$$

$$\delta W^{\text{ext}}=\begin{Bmatrix} g_x\delta u_{x1}+g_y\delta u_{y1}+g_z\delta u_{z1} \\ g_x\delta u_{x2}+g_y\delta u_{y2}+g_z\delta u_{z2} \end{Bmatrix}^{\mathrm{T}}\frac{1}{2}\rho^\circ h^\circ A^\circ\begin{Bmatrix}1\\1\end{Bmatrix}. \quad \Leftarrow$$

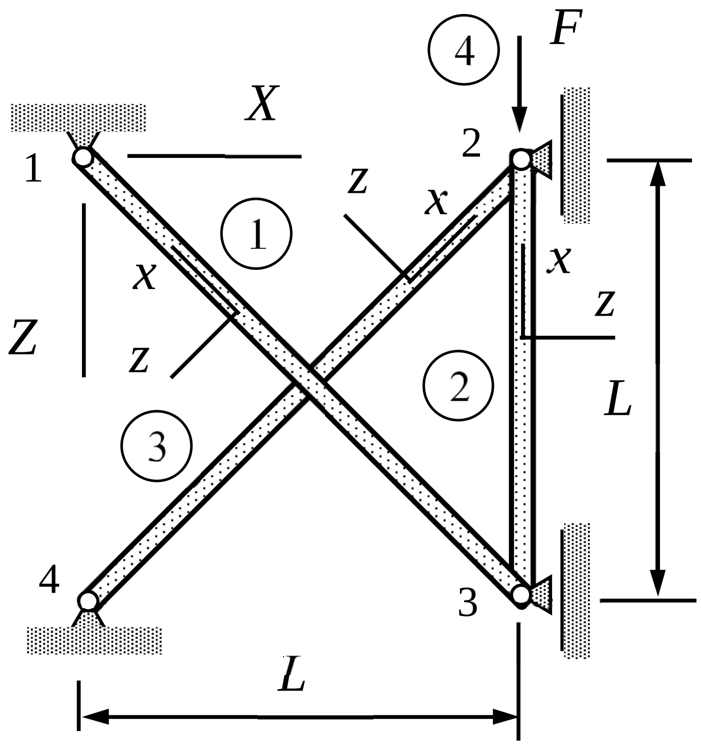
- It is noteworthy that PK2 does not represent the true stress in bar. The constitutive equation for the (true) axial force in terms of Green-Lagrange strain follows from the relationship between the Cauchy stress and PK2 stress. Here the relationship $J\vec{\sigma} = \vec{F} \cdot \vec{S} \cdot \vec{F}_{\mathrm{c}}$ simplifies to $J\sigma = FSF$ in which $J = V / V^{\circ} = hA / h^{\circ}A^{\circ}$ and $F = h / h^{\circ}$ giving

$$S = \frac{h^{\circ}}{h}\sigma\frac{h^{\circ}}{h}\frac{hA}{h^{\circ}A^{\circ}} = \frac{h^{\circ}}{hA^{\circ}}(\sigma A) = \frac{h^{\circ}}{hA^{\circ}}N \quad \Rightarrow \quad N = \frac{h}{h^{\circ}}A^{\circ}S = \frac{h}{h^{\circ}}A^{\circ}C\mathrm{E}.$$

- Using the axial force N and the variation δh (at deformed geometry)

$$\delta W^{\mathrm{int}} = -N\delta h = -\delta h\frac{h}{h^{\circ}}CA^{\circ}\frac{1}{2}[(\frac{h}{h^{\circ}})^{2} - 1] \quad \text{(same as earlier).}$$

EXAMPLE 5.5 Write the virtual work expression of the structure shown in terms of the nodal displacement u_{Z2} and u_{Z3}. Use non-linear bar elements and linear approximations. Solve for the nodal displacement when the cross-sectional areas and material properties are $L = 1\,\mathrm{m}$, $A^{\circ} = 1/100\,\mathrm{m}^2$, $C = 100\,\mathrm{Nm}^{-2}$ and $F = 1/20\,\mathrm{N}$.



Answer $u_{Z2} = 0.085\mathrm{m}$ and $u_{Z3} = 0.061\mathrm{m}$

- For bar 1, the nodal displacement components of material coordinate system are $u_{x1} = u_{z1} = 0$, $u_{x3} = -u_{Z3} / \sqrt{2}$, and $u_{x3} = u_{Z3} / \sqrt{2}$. As approximations are linear, derivatives are

$$\frac{du}{dx} = (0 + \frac{u_{Z3}}{\sqrt{2}})\frac{1}{\sqrt{2}L} = \frac{u_{Z3}}{2L}, \; \frac{dv}{dx} = 0, \quad \frac{dw}{dx} = (0 - \frac{u_{Z3}}{\sqrt{2}})\frac{1}{\sqrt{2}L} = -\frac{u_{Z3}}{2L}.$$

$$\mathrm{E}_{xx} = \frac{1}{2}\frac{u_{Z3}}{L}(1 + \frac{1}{2}\frac{u_{Z3}}{L}) \quad \Rightarrow \quad \delta\mathrm{E}_{xx} = \frac{1}{2}\frac{\delta u_{Z3}}{L}(1 + \frac{u_{Z3}}{L}).$$

When the approximations are substituted there, virtual work expression of internal forces simplifies to (density is constant)

$$\delta W^1 = -\delta u_{Z3}(1 + \frac{u_{Z3}}{L})\frac{CA^\circ}{4\sqrt{2}}\frac{u_{Z3}}{L}(2 + \frac{u_{Z3}}{L}).$$

- For bar 2, the nodal displacement components are $u_{x3}=-u_{Z3}$, $u_{x2}=-u_{Z2}$ and $u_{z2}=u_{z3}=0$. As approximations are linear, derivatives and the Green-Lagrange strains take the forms

$$\frac{du}{dx}=\frac{u_{x2}-u_{x3}}{L}=\frac{u_{Z3}-u_{Z2}}{L},\ \frac{dv}{dx}=0,\text{ and }\frac{dw}{dx}=0.$$

$$\mathrm{E}_{xx}=\frac{u_{Z3}-u_{Z2}}{L}(1+\frac{1}{2}\frac{u_{Z3}-u_{Z2}}{L})\quad\Rightarrow\quad\delta\mathrm{E}_{xx}=\frac{\delta u_{Z3}-\delta u_{Z2}}{L}(1+\frac{u_{Z3}-u_{Z2}}{L}).$$

When the approximations are substituted there, virtual work expression of internal forces simplifies to

$$\delta W^{2}=-(\delta u_{Z3}-\delta u_{Z2})(1+\frac{u_{Z3}-u_{Z2}}{L})CA^{\circ}\frac{u_{Z3}-u_{Z2}}{L}(1+\frac{1}{2}\frac{u_{Z3}-u_{Z2}}{L}).$$

- For bar 3, the nodal displacement components are $u_{x4} = u_{z4} = 0$, $u_{x2} = -u_{Z2} / \sqrt{2}$, and $u_{z2} = -u_{Z2} / \sqrt{2}$. As approximations are linear, derivatives and the Green-Lagrange strain take the forms

$$\frac{du}{dx} = (-\frac{u_{Z2}}{\sqrt{2}})\frac{1}{\sqrt{2}L} = -\frac{u_{Z2}}{2L}, \quad \frac{dv}{dx} = 0, \text{ and } \frac{dw}{dx} = (-\frac{u_{Z2}}{\sqrt{2}})\frac{1}{\sqrt{2}L} = -\frac{u_{Z2}}{2L}.$$

$$\mathrm{E}_{xx} = \frac{1}{2}\frac{u_{Z2}}{L}(-1 + \frac{1}{2}\frac{u_{Z2}}{L}) \quad \Rightarrow \quad \delta\mathrm{E}_{xx} = \frac{1}{2}\frac{\delta u_{Z2}}{L}(-1 + \frac{u_{Z2}}{L}).$$

When the approximations are substituted there, virtual work expression of internal forces simplifies to (density is constant)

$$\delta W^{3} = -\frac{CA^{\circ}}{4\sqrt{2}}\delta u_{Z2}(-1 + \frac{u_{Z2}}{L})\frac{u_{Z2}}{L}(-2 + \frac{u_{Z2}}{L}).$$

- Virtual work expression is sum of the element contributions. By taking into account also the point force contribution $\delta W^4 = \delta u_{Z2} F$

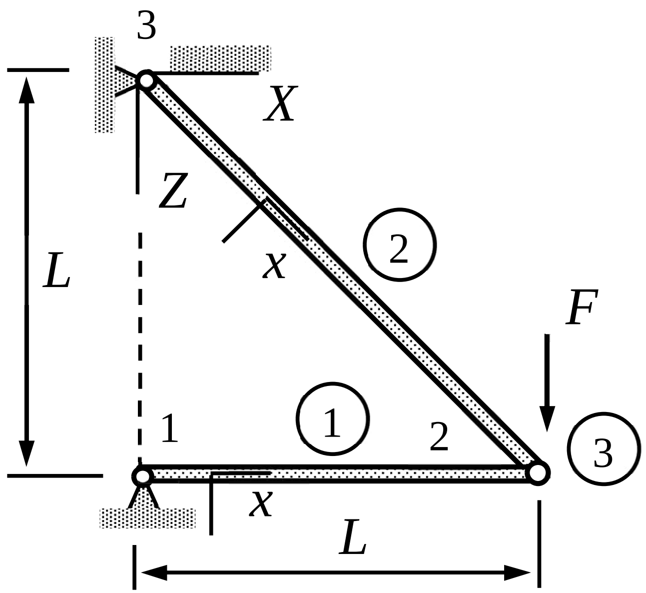
$$\delta W = -\delta u_{Z3}(1+\frac{u_{Z3}}{L})\frac{CA^\circ}{4\sqrt{2}}\frac{u_{Z3}}{L}(2+\frac{u_{Z3}}{L}) - (\delta u_{Z3} - \delta u_{Z2})(1+\frac{u_{Z3}-u_{Z2}}{L})\times$$

$$CA^\circ \frac{u_{Z3}-u_{Z2}}{L}(1+\frac{1}{2}\frac{u_{Z3}-u_{Z2}}{L}) - \delta u_{Z2}(-1+\frac{u_{Z2}}{L})\frac{CA^\circ}{4\sqrt{2}}\frac{u_{Z2}}{L}(-2+\frac{u_{Z2}}{L}) + \delta u_{Z2} F \,.$$

- Principle of virtual work and the fundamental lemma of variation calculus give a non-linear algebraic equation system for the non-zero displacement components u_{Z2} and u_{Z3}. In most cases, finding an analytical solution in terms of the parameters of the problem is not possible.

EXAMPLE 5.6 A bar truss is loaded by a point force having magnitude F as shown in the figure. Determine the equilibrium equations according to the large displacement theory. At the initial (non-loaded) geometry, cross-sectional area of bar 1 is $A°$ and that for bar 2 $A°/\sqrt{2}$. Also, find the solution for $L=1\text{m}$, $A°=1/100\,\text{m}^2$, $C=100\,\text{Nm}^{-2}$ and $F=1/20\,\text{N}$.

Answer $u_{X2}=-0.085\text{m}$ and $u_{Z2}=0.25\text{m}$



- For bar 1, the nodal displacement components of material coordinate system are $u_{x1} = u_{z1} = 0$, $u_{x2} = u_{X2}$, and $u_{z2} = u_{Z2}$. As the approximations are linear

$$\frac{du}{dx} = \frac{u_{X2}}{L}, \ \frac{dv}{dx} = 0, \ \text{and} \ \frac{dw}{dx} = \frac{u_{Z2}}{L}$$

and the virtual work expression (density is constant) of internal forces simplifies to

$$\delta W^1 = -(\delta u_{X2} + \delta u_{X2}\frac{u_{X2}}{L} + \delta u_{Z2}\frac{u_{Z2}}{L})CA^{\circ}[\frac{u_{X2}}{L} + \frac{1}{2}(\frac{u_{X2}}{L})^2 + \frac{1}{2}(\frac{u_{Z2}}{L})^2].$$

- For bar 2, the nodal displacement components of material coordinate system are $u_{x3} = u_{z3} = 0$, $u_{x2} = (u_{X2} + u_{Z2})/\sqrt{2}$ and $u_{z2} = (-u_{X2} + u_{Z2})/\sqrt{2}$ (notice the use of initial geometry). As the approximations are linear

$$\frac{du}{dx} = \frac{u_{X2} + u_{Z2}}{L}, \quad \frac{dw}{dx} = \frac{u_{Z2} - u_{X2}}{L}$$

and the virtual work expression (density is constant) of internal forces simplifies to

$$\delta W^2 = -[\delta u_{X2} + \delta u_{Z2} + (\delta u_{X2} + \delta u_{Z2})(\frac{u_{X2} + u_{Z2}}{L}) + (\delta u_{Z2} - \delta u_{X2})(\frac{u_{Z2} - u_{X2}}{L})]\times$$

$$CA^{\circ}[(\frac{u_{X2} + u_{Z2}}{L}) + \frac{1}{2}(\frac{u_{X2} + u_{Z2}}{L})^2 + \frac{1}{2}(\frac{u_{Z2} - u_{X2}}{L})^2].$$

- Virtual work expression of the point follows from definition of work

$$\delta W^3 = F\delta u_{Z2}.$$

- Virtual work expression is sum of the element contributions. After a considerable amount of manipulations, the standard form with notations $a_1 = u_{X2} / L$ and $a_2 = u_{Z2} / L$

$$\delta W=-\frac{EA}{8}\begin{Bmatrix}\delta u_{X2}\\ \delta u_{Z2}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}-(1+a_1)(10a_1+5a_1^2+2a_2+5a_2^2)\\ 8\dfrac{F}{EA}-[2a_1(1+5a_2)+a_1^2(1+5a_2)+a_2(2+3a_2+5a_2^2)]\end{Bmatrix}=0.$$

- Principle of virtual work and the fundamental lemma of variation calculus give a non-linear algebraic equation system $(a_1=u_{Z3}/L$ and $a_2=u_{Z2}/L)$

$$\begin{Bmatrix}-(1+a_1)(10a_1+5a_1^2+2a_2+5a_2^2)\\ 8\dfrac{F}{EA}-[2a_1(1+5a_2)+a_1^2(1+5a_2)+a_2(2+3a_2+5a_2^2)]\end{Bmatrix}=0. \quad \Leftarrow$$

- It is obvious that finding an analytical solution in terms of the parameters of the problem becomes impossible even when the truss is very simple if the number of non-zero displacement components exceeds one. Mathematica code of the course gives the real valued solution with the minimal norm (that is likely to be the physically meaningful solution when the initial displacement is zero) ($L = 1\,\mathrm{m}$, $A^{\circ} = 1/100\,\mathrm{m}^2$, $E = C = 100\,\mathrm{N m}^{-2}$ and $F = 1/20\,\mathrm{N}$.

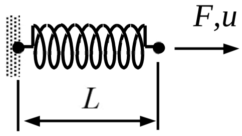
	model	properties	geometry
1	BAR	{{E}, {A}}	Line[{1, 2}]
2	BAR	$\left\{\{E\}, \left\{\frac{A}{\sqrt{2}}\right\}\right\}$	Line[{3, 2}]
3	FORCE	{0, 0, F}	Point[{2}]

	{X,Y,Z}	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	{0, 0, L}	{0, 0, 0}	{0, 0, 0}
2	{L, 0, L}	{uX[2], 0, uZ[2]}	{0, 0, 0}
3	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}

{uX[2] → −0.0848497, uZ[2] → 0.25}

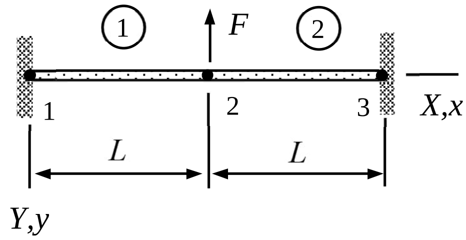
MEC-E8001 Finite Element Analysis, week 6/2023

1. The spring force of non-linear spring depends on the dimensionless displacement $a = u/L$ according to $F = k(a - a^2 + a^3/3)$. Determine the dimensionless displacement $a = u/L$ if force $F = k/4$.



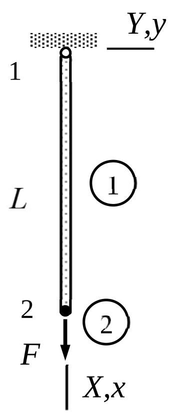
Answer $a = \dfrac{u}{L} \approx 0.370$

2. Determine the displacement at node 2 of the elastic bar shown by the large deformation theory. Take into account only the transverse displacement u_{Y2} ($u_{X2} = 0$). When $F = 0$, the cross-sectional area and length of the bar are A and L, respectively. Constitutive equation of the material is $S_{xx} = C E_{xx}$, in which C is constant. Use two elements with linear shape functions.



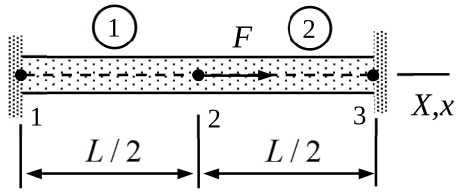
Answer $u_{Y2} = -(\dfrac{FL^3}{AC})^{1/3}$

3. Consider the bar shown loaded by a point force. Determine the equilibrium equations in terms of the dimensionless displacement components $a_1 = u_{X2}/L$ and $a_2 = u_{Y2}/L$ according to the large displacement bar theory. Assume that displacement component $w = 0$ and use linear approximation to the non-zero components u and v. Without loading, the area of cross-section and the length of bar are $A°$ and $L°$, respectively. Constitutive equation of the material is $S_{xx} = C E_{xx}$, in which C is constant.



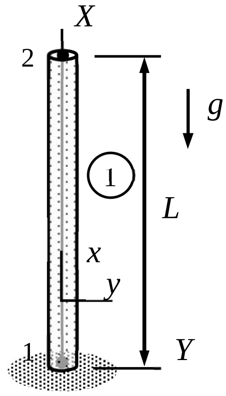
Answer $(1 + a_1)(a_1 + \dfrac{1}{2}a_1^2 + \dfrac{1}{2}a_2^2) - \dfrac{F}{A°C} = 0$ and $a_2(2a_1 + a_1^2 + a_2^2) = 0$

4. Determine the equilibrium equation of the elastic bar of the figure with the large deformation theory. The active degree of freedom is u_{X2} and the cross-sectional area and length of the bar are A and L without the point force F acting on node 2. Constitutive equation of the material is $S_{xx} = C E_{xx}$, in which C is constant. Use two elements with linear shape functions.



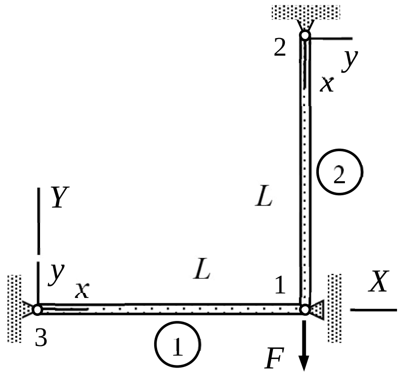
Answer $a(1 + 2a^2) - \dfrac{1}{4}\dfrac{F}{AC} = 0$ where $a = \dfrac{u_{X2}}{L}$

5. Consider the structure shown loaded by its own weight. Determine the equations giving the displacement u_{X2} of the free end according to large displacement bar theory. Without gravity, cross-sectional area, length, and density of the bar are A, L, and ρ, respectively. Constitutive equation of the material is $S_{xx} = C E_{xx}$, in which C is constant. Use a linear approximation.



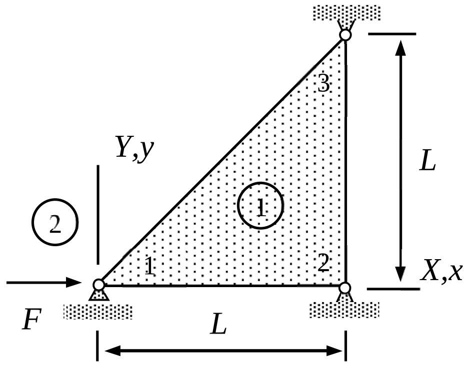
Answer $(1+\frac{u_{X2}}{L})\frac{u_{X2}}{L}(2+\frac{u_{X2}}{L})+\frac{L\rho g}{C}=0$

6. Derive the equilibrium equation of the elastic truss shown with the large deformation theory. The cross-sectional areas and length of the bars are A and L when $F=0$. Constitutive equation of the material is $S_{xx} = C E_{xx}$, in which C is constant. Assume a planar problem of two elements.



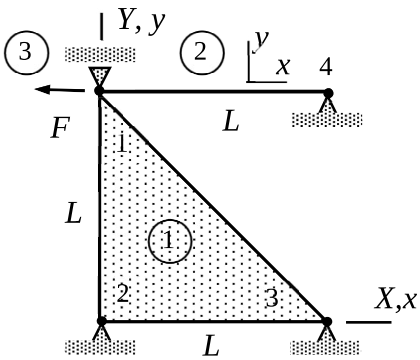
Answer $\frac{u_{Y1}}{L}\frac{CA}{2}[2(\frac{u_{Y1}}{L})^2-3\frac{u_{Y1}}{L}+2]+F=0$

7. A thin triangular slab (assume plane stress conditions) loaded by a horizontal force can move horizontally at node 1 and nodes 2 and 3 are fixed. Derive the equilibrium equation for the structure according to the large displacement theory. Material parameters C, ν and thickness t at the initial geometry of the slab are constants.



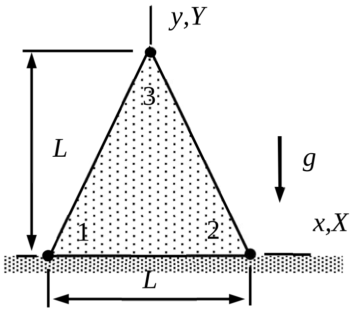
Answer $\frac{1}{2}\frac{tLC}{1-\nu^2}\mathrm{a}(-1+\mathrm{a})(-1+\frac{1}{2}\mathrm{a})-F=0$ where $\mathrm{a}=\frac{u_{X1}}{L}$

8. A structure, consisting of a thin slab under the plane stress conditions and a bar, is loaded by a horizontal force F acting on node 1. Material properties are C and ν, thickness of the slab is t, and the cross-sectional area of the bar A at the initial unloaded geometry. Determine the equilibrium equation giving as its solution the displacement component u_{X1} of node 1 according to the large displacement theory.



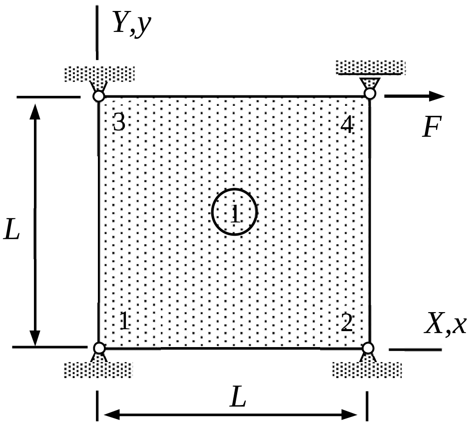
Answer $\frac{L}{4}\frac{tC}{1-\nu^2}\mathrm{a}(\mathrm{a}^2+1-\nu)+CA(-1+\mathrm{a})\mathrm{a}(-\mathrm{a}+\frac{1}{2}\mathrm{a})+F=0$ where $\mathrm{a}=\frac{u_{X1}}{L}$

9. A long wall having triangular cross-section, and made of homogeneous, isotropic, linearly elastic material, is subjected to its own weight. Determine the equilibrium equation giving as its solution displacement components u_{Y3} according to the large displacement theory. Nodes 1 and 2 are fixed. Use a three-node element and assume plane stress conditions and symmetry $u_{X3} = 0$. Material properties C, ν and the density ρ of the initial geometry are constants.



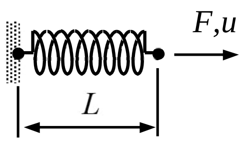
Answer $(1+\mathrm{a})\mathrm{a}(1+\frac{1}{2}\mathrm{a})+\frac{1}{3}(1-\nu^2)\frac{L\rho g}{E}=0$ where $\mathrm{a}=\frac{u_{Y3}}{L}$.

10. Node 4 of a thin rectangular slab, loaded by force F, can move horizontally and nodes 1, 2, and 3 are fixed. Assume plane stress conditions and derive the equilibrium equation of the structure according to the large deformation theory. Use just one bilinear element. Material parameters C and $\nu = 0$. Thickness of the slab at the initial geometry is t.



Answer $\frac{1}{2}\mathrm{a}+\frac{5}{8}\mathrm{a}^2+\frac{14}{45}\mathrm{a}^3-\frac{F}{tLC}=0$ where $\mathrm{a}=\frac{u_{X4}}{L}$.

The spring force of non-linear spring depends on the dimensionless displacement $a = u/L$ according to $F = k(a - a^2 + a^3/3)$. Determine the dimensionless displacement $a = u/L$ if force $F = k/4$.



Solution

As the equilibrium equation is non-linear, finding the displacement as function of the force by hand calculations is difficult (but possible for a third order polynomial). Mathematica gives three mathematically correct solution

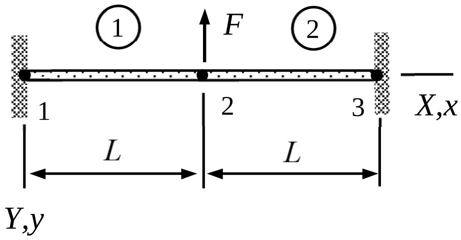
$$\left\{\left\{a \to 1 - \frac{1}{2^{2/3}}\right\}, \left\{a \to 1 + \frac{1 - i\sqrt{3}}{2 \times 2^{2/3}}\right\}, \left\{a \to 1 + \frac{1 + i\sqrt{3}}{2 \times 2^{2/3}}\right\}\right\}$$

of which the real valued is obviously the physically correct one. A simple graphical method for finding one solution to

$$R(a) = F - k(a - a^2 + \frac{1}{3}a^3)$$

in a given range $a \in [a_{min}, a_{max}]$ uses an iterative refinement of the range so that the sign change of $R(a)$ is bracketed inside a smaller and smaller range.

Determine the displacement at node 2 of the elastic bar shown by the large deformation theory. Take into account only the transverse displacement u_{Y2} ($u_{X2}=0$). When $F=0$, the cross-sectional area and length of the bar are A and L, respectively. Constitutive equation of the material is $S_{xx}=C\mathrm{E}_{xx}$, in which C is constant. Use two elements with linear shape functions.



Solution

Virtual work density of the non-linear bar model

$$\delta w_{\Omega^\circ}^{\text{int}}=-(\frac{d\delta u}{dx}+\frac{du}{dx}\frac{d\delta u}{dx}+\frac{dv}{dx}\frac{d\delta v}{dx}+\frac{dw}{dx}\frac{d\delta w}{dx})CA^\circ[\frac{du}{dx}+\frac{1}{2}(\frac{du}{dx})^2+\frac{1}{2}(\frac{dv}{dx})^2+\frac{1}{2}(\frac{dw}{dx})^2]$$

is based on the Green-Lagrange strain definition which is physically correct also when rotations/displacements are large. The expression depends on all displacement components, material property is denoted by C (constitutive equation $S_{xx}=C\mathrm{E}_{xx}$), and the superscript in the cross-sectional area A° (and in other quantities) refers to the initial geometry (strain and stress vanishes). Otherwise, equilibrium equations follow in the same manner as in the linear case.

For element 1, the non-zero displacement components is $u_{y2}=u_{Y2}$. As the initial length of the element $h^\circ=L$, linear approximations to the displacement components

$$u=w=0 \text{ and } v=\frac{x}{L}u_{Y2} \quad\Rightarrow\quad \frac{du}{dx}=\frac{dw}{dx}=0 \text{ and } \frac{dv}{dx}=\frac{u_{Y2}}{L}.$$

When the approximation is substituted there, virtual work density of the internal forces and thereby the virtual work expression (density is constant) simplify to

$$\delta w_{\Omega^\circ}^{\text{int}}=-\frac{\delta u_{Y2}}{L}\frac{u_{Y2}}{L}\frac{CA}{2}(\frac{u_{Y2}}{L})^2 \quad\Rightarrow\quad \delta W^1=-\delta u_{Y2}\frac{CA}{2}(\frac{u_{Y2}}{L})^3.$$

For element 2, the non-zero displacement component $u_{y2}=u_{Y2}$. As the initial length of the element $h^\circ=L$, linear approximations to the displacement components

$$u=w=0 \text{ and } v=(1-\frac{x}{L})u_{Y2} \quad\Rightarrow\quad \frac{du}{dx}=\frac{dw}{dx}=0 \text{ and } \frac{dv}{dx}=-\frac{u_{Y2}}{L}.$$

When the approximation is substituted there, virtual work density of the internal forces and thereby the virtual work expression (density is constant) simplifies to

$$\delta w_{\Omega^\circ}^{\text{int}}=-\frac{\delta u_{Y2}}{L}\frac{u_{Y2}}{L}\frac{CA}{2}(\frac{u_{Y2}}{L})^2 \quad\Rightarrow\quad \delta W^2=-\delta u_{Y2}\frac{CA}{2}(\frac{u_{Y2}}{L})^3.$$

Virtual work expression of the point force is

$$\delta W^3=-F\delta u_{Y2}.$$

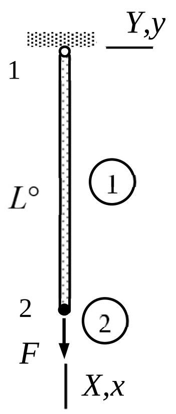
Virtual work expression of the structure is obtained as the sum of the element contributions

$$\delta W = -\delta u_{Y2}[\frac{CA}{2}(\frac{u_{Y2}}{L})^3 + \frac{CA}{2}(\frac{u_{Y2}}{L})^3 + F].$$

Principle of virtual work and the fundamental lemma of variation calculus imply that

$$(\frac{u_{Y2}}{L})^3 + \frac{F}{CA} = 0 \quad \Rightarrow \quad u_{Y2} = -(\frac{FL^3}{CA})^{1/3}.$$

Consider the bar shown loaded by a point force. Determine the equilibrium equations in terms of the dimensionless displacement components $a_1 = u_{X2}/L$ and $a_2 = u_{Y2}/L$ according to the large displacement bar theory. Assume that displacement component $w = 0$ and use linear approximation to the non-zero components u and v. Without loading, the area of cross-section and the length of bar are A° and L°, respectively. Constitutive equation of the material is $S_{xx} = C E_{xx}$, in which C is constant.



Solution

Virtual work density of internal forces is

$$\delta w_{\Omega^\circ}^{\text{int}} = -\left(\frac{d\delta u}{dx} + \frac{du}{dx}\frac{d\delta u}{dx} + \frac{dv}{dx}\frac{d\delta v}{dx} + \frac{dw}{dx}\frac{d\delta w}{dx}\right) CA^\circ \left[\frac{du}{dx} + \frac{1}{2}\left(\frac{du}{dx}\right)^2 + \frac{1}{2}\left(\frac{dv}{dx}\right)^2 + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right].$$

Assuming a linear approximation to displacement components with $u_{x2} = u_{X2}$ and $u_{y2} = u_{Y2}$

$$u = \frac{x}{L^\circ} u_{X2}, \quad v = \frac{x}{L^\circ} u_{Y2}, \text{ and } \quad w = 0 \quad \Rightarrow \quad \frac{du}{dx} = \frac{u_{X2}}{L^\circ}, \quad \frac{dv}{dx} = \frac{u_{Y2}}{L^\circ}, \text{ and } \frac{dw}{dx} = 0.$$

Virtual work expression is obtained as integral of the density over the domain occupied by the body (notice that the virtual work density is constant when the approximations are substituted there):

$$\delta W^1 = -\left(\frac{\delta u_{X2}}{L^\circ} + \frac{u_{X2}}{L^\circ}\frac{\delta u_{X2}}{L^\circ} + \frac{u_{Y2}}{L^\circ}\frac{\delta u_{Y2}}{L^\circ}\right) L^\circ CA^\circ \left[\frac{u_{X2}}{L^\circ} + \frac{1}{2}\left(\frac{u_{X2}}{L^\circ}\right)^2 + \frac{1}{2}\left(\frac{u_{Y2}}{L^\circ}\right)^2\right],$$

$$\delta W^2 = F \delta u_{X2}.$$

Virtual work expression of the structure is $\delta W = \delta W^1 + \delta W^2$. In terms of dimensionless displacements $a_1 = u_{X2}/L^\circ$ and $a_2 = u_{Y2}/L^\circ$ (introduced just to simplify the expressions)

$$\delta W = -(\delta a_1 + a_1 \delta a_1 + a_2 \delta a_2) L^\circ CA^\circ \left(a_1 + \frac{1}{2}a_1^2 + \frac{1}{2}a_2^2\right) + FL^\circ \delta a_1 \quad \Leftrightarrow$$

$$\delta W = -CA^\circ \begin{Bmatrix} \delta a_1 \\ \delta a_2 \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} (1+a_1)\left(a_1 + \frac{1}{2}a_1^2 + \frac{1}{2}a_2^2\right) - \dfrac{F}{CA^\circ} \\ a_2\left(a_1 + \frac{1}{2}a_1^2 + \frac{1}{2}a_2^2\right) \end{Bmatrix}.$$

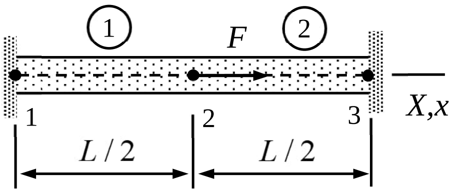
principle of virtual work and the fundamental lemma of variation calculus imply that

$$(1+a_1)\left(a_1 + \frac{1}{2}a_1^2 + \frac{1}{2}a_2^2\right) - \frac{F}{CA^\circ} = 0 \quad \text{and} \quad a_2\left(a_1 + \frac{1}{2}a_1^2 + \frac{1}{2}a_2^2\right) = 0. \quad \leftarrow$$

In this case, the solution can be deduced without numerical calculations: the latter equation implies that $a_2 = 0$ as the other option $a_1 + a_1^2/2 + a_2^2/2 = 0$ would mean an inconsistency with the first equation. Knowing this (the real valued solution)

$$a_1 = \frac{1}{3}\left(-3 - \frac{3^{2/3}}{\alpha} - 3^{1/3}\alpha\right) \quad \text{where} \quad \alpha = \left(-9f + \sqrt{-3 + 81f^2}\right)^{1/3} \quad \text{and} \quad f = \frac{F}{CA^\circ} \ .$$

Derive the equilibrium equation of the elastic bar of the figure with the large deformation theory. The non-zero displacement component is u_{X2} and the cross-sectional area and length of the bar are A and L, when the point force F acting on node 2 is zero. Constitutive equation of the material is $S = C\mathrm{E}$, in which C is constant. Use two elements with linear shape functions.



Solution

Virtual work density of the non-linear bar model

$$\delta w_{\Omega^\circ}^{\text{int}} = -(\frac{d\delta u}{dx} + \frac{du}{dx}\frac{d\delta u}{dx} + \frac{dv}{dx}\frac{d\delta v}{dx} + \frac{dw}{dx}\frac{d\delta w}{dx})CA^\circ[\frac{du}{dx} + \frac{1}{2}(\frac{du}{dx})^2 + \frac{1}{2}(\frac{dv}{dx})^2 + \frac{1}{2}(\frac{dw}{dx})^2]$$

is based on the Green-Lagrange strain definition which works also when rotations/displacements are large. The expression depends on all displacement components, material property is denoted by C (constitutive equation $S_{xx} = C\mathrm{E}_{xx}$), and the superscript in the cross-sectional area A° (and in other quantities) refers to the initial geometry (strain and stress vanishes). Otherwise, equilibrium equations follow in the same manner as in the linear case.

For element 1, $u_{x2} = u_{X2}$. As the initial length of the element $h^\circ = L/2$, linear approximations to the displacement components

$$v = w = 0 \text{ and } u = 2\frac{x}{L}u_{X2} \quad \Rightarrow \quad \frac{du}{dx} = 2\frac{u_{X2}}{L}.$$

When the approximation is substituted there, virtual work density of the internal forces and thereby the virtual work expression (density is constant) simplify to

$$\delta w_{\Omega^\circ}^{\text{int}} = -2\frac{\delta u_{X2}}{L}(1 + 2\frac{u_{X2}}{L})CA2\frac{u_{X2}}{L}(1 + \frac{1}{2}2\frac{u_{X2}}{L}) \quad \Rightarrow$$

$$\delta W^1 = -\delta u_{X2}(1 + 2\frac{u_{X2}}{L})2CA\frac{u_{X2}}{L}(1 + \frac{u_{X2}}{L}).$$

For element 2, $u_{x2} = u_{X2}$. As the initial length of the element $h^\circ = L/2$, linear approximations to the displacement components

$$v = w = 0 \text{ and } u = (1 - 2\frac{x}{L})u_{X2} \quad \Rightarrow \quad \frac{du}{dx} = -2\frac{u_{X2}}{L}.$$

When the approximation is substituted there, virtual work density of the internal forces and thereby the virtual work expression (density is constant) simplify to

$$\delta w_{\Omega^\circ}^{\text{int}} = -2(-\frac{\delta u_{X2}}{L})(1 - 2\frac{u_{X2}}{L})2CA(-\frac{u_{X2}}{L})(1 - \frac{u_{X2}}{L}) \quad \Rightarrow$$

$$\delta W^2 = -\delta u_{X2}(1-2\frac{u_{X2}}{L})2CA\frac{u_{X2}}{L}(1-\frac{u_{X2}}{L}) .$$

Virtual work expression of the force is

$$\delta W^3 = F\delta u_{X2} .$$

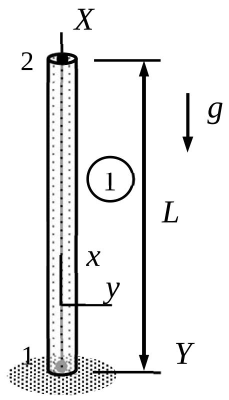
Virtual work expression of the structure is obtained as sum over the element contributions

$$\delta W = -\delta u_{X2}[(1+2\frac{u_{X2}}{L})2CA\frac{u_{X2}}{L}(1+\frac{u_{X2}}{L})+(1-2\frac{u_{X2}}{L})2CA\frac{u_{X2}}{L}(1-\frac{u_{X2}}{L})-F] .$$

Principle of virtual work and the fundamental lemma of variation calculus imply that

$$\frac{u_{X2}}{L}[(1+2\frac{u_{X2}}{L})(1+\frac{u_{X2}}{L})+(1-2\frac{u_{X2}}{L})(1-\frac{u_{X2}}{L})]-\frac{F}{2CA}=0 \quad \Rightarrow$$

$$\mathrm{a}(1+2\mathrm{a}^2)-\frac{F}{4CA}=0 \quad \text{in which} \quad \mathrm{a}=\frac{u_{X2}}{L} .$$



Consider the structure shown loaded by its own weight. Determine the equations giving the displacement u_{X2} of the free end according to large displacement bar theory. Without gravity, cross-sectional area, length, and density of the bar are A, L, and ρ, respectively. Constitutive equation of the material is $S_{xx} = C\mathrm{E}_{xx}$, in which C is constant. Use a linear approximation.

Solution

As $v = w = 0$, virtual work densities of internal and external distributed forces of the non-linear bar model simplify to

$$\delta w_{\Omega^\circ}^{\text{int}} = -(\frac{d\delta u}{dx} + \frac{du}{dx}\frac{d\delta u}{dx})CA^\circ[\frac{du}{dx} + \frac{1}{2}(\frac{du}{dx})^2] \quad \text{and} \quad \delta w_{\Omega^\circ}^{\text{ext}} = -\delta u \rho g A$$

the negative sign of the external part takes into account the direction of gravity with respect to the x-axis. The non-zero displacement component of the structure is the vertical displacement of node 2 i.e. $u_{x2} = u_{X2}$. Linear approximation (two-node element) is

$$u = \frac{x}{L} u_{X2} \quad \Rightarrow \quad \frac{du}{dx} = \frac{u_{X2}}{L}.$$

When the approximation is substituted there, virtual work densities simplify to

$$\delta w_{\Omega^\circ}^{\text{int}} = -(\frac{\delta u_{X2}}{L})(1 + \frac{u_{X2}}{L})\frac{CA}{2}(\frac{u_{X2}}{L})(2 + \frac{u_{X2}}{L}) \quad \text{and} \quad \delta w_{\Omega^\circ}^{\text{ext}} = -\frac{x}{L}\delta u_{X2}\rho g A.$$

Virtual work expression is integral of the virtual work density over the domain occupied by the element at the initial geometry:

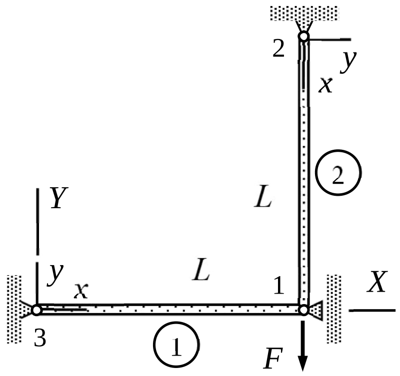
$$\delta W^{\text{int}} = \int_0^L \delta w_{\Omega^\circ}^{\text{int}} dx = -\delta u_{X2}(1 + \frac{u_{X2}}{L})\frac{CA}{2}(\frac{u_{X2}}{L})(2 + \frac{u_{X2}}{L}),$$

$$\delta W^{\text{ext}} = \int_0^L \delta w_{\Omega^\circ}^{\text{ext}} dx = -\frac{1}{2} L \delta u_{X2} \rho g A.$$

Principle of virtual work with $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}}$ and the fundamental lemma of variation calculus imply that

$$(1 + \frac{u_{X2}}{L})\frac{CA}{2}(\frac{u_{X2}}{L})(2 + \frac{u_{X2}}{L}) + \frac{1}{2} L \rho g A = 0 \quad \Rightarrow \quad (1 + \mathrm{a})\mathrm{a}(2 + \mathrm{a}) + \frac{L\rho g}{C} = 0, \ \mathrm{a} = \frac{u_{X2}}{L}. \quad \leftarrow$$

Derive the equilibrium equation of the elastic truss shown with the large deformation theory. The cross-sectional areas and length of the bars are A and L when $F=0$. Constitutive equation of the material is $S_{xx}=C\text{E}_{xx}$, in which C is constant. Assume a planar problem of two elements.



Solution

As $w=0$ and cross-sectional area of the initial geometry is A, virtual work density of internal forces of the large displacement bar model simplifies to

$$\delta w_{\Omega^\circ}^{\text{int}}=-(\frac{d\delta u}{dx}+\frac{du}{dx}\frac{d\delta u}{dx}+\frac{dv}{dx}\frac{d\delta v}{dx})CA^\circ[\frac{du}{dx}+\frac{1}{2}(\frac{du}{dx})^2+\frac{1}{2}(\frac{dv}{dx})^2].$$

In element 1, linear approximations to the displacement components expressed in terms of u_{Y1} are

$$u=0 \text{ and } v=\frac{x}{L}u_{Y1} \quad \Rightarrow \quad \frac{du}{dx}=0 \text{ and } \frac{dv}{dx}=\frac{u_{Y1}}{L}.$$

When the approximation is substituted there, virtual work density of internal forces and the virtual work expression take the forms

$$\delta w_{\Omega^\circ}^{\text{int}}=-\frac{u_{Y1}}{L}\frac{\delta u_{Y1}}{L})CA\frac{1}{2}(\frac{u_{Y1}}{L})^2,$$

$$\delta W^1=\int_0^L \delta w_{\Omega^\circ}^{\text{int}}dx=-\delta u_{Y1}CA\frac{1}{2}(\frac{u_{Y1}}{L})^3.$$

In element 2, linear approximations to the displacement components expressed in terms of u_{Y1} are

$$u=-\frac{x}{L}u_{Y1} \text{ and } v=0 \quad \Rightarrow \quad \frac{du}{dx}=-\frac{u_{Y1}}{L} \text{ and } \frac{dv}{dx}=0.$$

When the approximation is substituted there, virtual work density of internal forces and thereby the virtual work expression take the forms

$$\delta w_{\Omega^\circ}^{\text{int}}=-(\frac{\delta u_{Y1}}{L})(1-\frac{u_{Y1}}{L})CA(\frac{u_{Y1}}{L})(1-\frac{1}{2}\frac{u_{Y1}}{L}),$$

$$\delta W^2=\int_0^L \delta w_{\Omega^\circ}^{\text{int}}dx=-\delta u_{Y1}(1-\frac{u_{Y1}}{L})CA(\frac{u_{Y1}}{L})(1-\frac{1}{2}\frac{u_{Y1}}{L}).$$

Element 3 contribution (point force)

$$\delta W^3=-F\delta u_{Y1}.$$

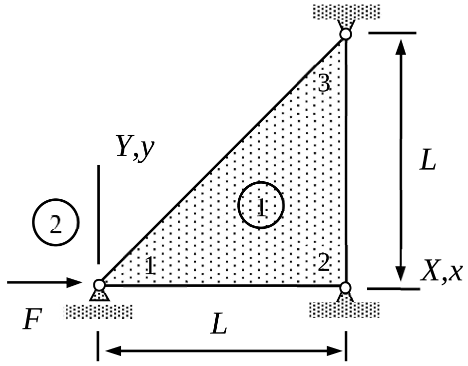
Virtual work expression of the structure is sum over the element contributions. In the standard form

$$\delta W = -\delta u_{Y1}[\frac{u_{Y1}}{L} CA \frac{1}{2} (\frac{u_{Y1}}{L})^2 + (1 - \frac{u_{Y1}}{L}) CA (\frac{u_{Y1}}{L})(1 - \frac{1}{2} \frac{u_{Y1}}{L}) + F].$$

Principle of virtual work and the fundamental lemma of variation calculus imply the equilibrium equation

$$\frac{u_{Y1}}{L} \frac{CA}{2} [2(\frac{u_{Y1}}{L})^2 - 3\frac{u_{Y1}}{L} + 2] + F = 0.$$

A thin triangular slab (assume plane stress conditions) loaded by a horizontal force can move horizontally at node 1 and nodes 2 and 3 are fixed. Derive the equilibrium equation for the structure according to the large displacement theory. Material parameters C, ν and thickness t at the initial geometry of the slab are constants.



Solution

Virtual work density of internal force, when modified for large displacement analysis with the same constitutive equation as in the linear case of plane stress, is given by

$$\delta w_{\Omega^\circ}^{\text{int}} = -\begin{Bmatrix} \delta \mathrm{E}_{xx} \\ \delta \mathrm{E}_{yy} \\ 2\delta \mathrm{E}_{xy} \end{Bmatrix}^{\mathrm{T}} \frac{tC}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ 2\mathrm{E}_{xy} \end{Bmatrix}, \quad \begin{Bmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ 2\mathrm{E}_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} + \dfrac{1}{2}(\dfrac{\partial u}{\partial x})^2 + \dfrac{1}{2}(\dfrac{\partial v}{\partial x})^2 \\ \dfrac{\partial v}{\partial y} + \dfrac{1}{2}(\dfrac{\partial u}{\partial y})^2 + \dfrac{1}{2}(\dfrac{\partial v}{\partial y})^2 \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\dfrac{\partial v}{\partial y} \end{Bmatrix}.$$

Let us start with the approximations and the corresponding components of the Green-Lagrange strain. Linear shape functions can be deduced from the figure. Only the shape function $N_1 = (1 - x/L)$ of node 1 is needed. Displacement components $v = w = 0$ and

$$u = (1 - \frac{x}{L})u_{X1} \;\Rightarrow\; \frac{\partial u}{\partial x} = -\frac{u_{X1}}{L}, \;\; \frac{\partial u}{\partial y} = 0, \;\; \mathrm{E}_{yy} = \mathrm{E}_{xy} = 0 \;\text{ and }\; \mathrm{E}_{xx} = -\frac{u_{X1}}{L} + \frac{1}{2}(-\frac{u_{X1}}{L})^2.$$

When the strain component expression are substituted there, virtual work density simplifies to

$$\delta w_{\Omega^\circ}^{\text{int}} = -\delta \mathrm{E}_{xx} \frac{tC}{1-\nu^2} \mathrm{E}_{xx} = -\frac{\delta u_{X1}}{L}(-1 + \frac{u_{X1}}{L}) \frac{tC}{1-\nu^2} \frac{u_{X1}}{L}(-1 + \frac{1}{2}\frac{u_{X1}}{L}).$$

Integration over the (initial) domain gives the virtual work expression. As the integrand is constant

$$\delta W^1 = -\frac{L^2}{2} \frac{\delta u_{X1}}{L}(-1 + \frac{u_{X1}}{L}) \frac{tC}{1-\nu^2} \frac{u_{X1}}{L}(-1 + \frac{1}{2}\frac{u_{X1}}{L})$$

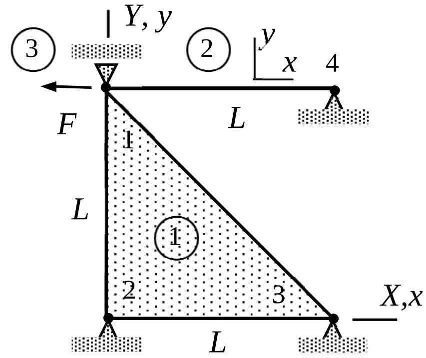
Virtual work expression of the point force follows from the definition of work

$$\delta W^2 = \delta u_{X1} F = \frac{\delta u_{X1}}{L} LF.$$

Virtual work expression of the structure is obtained as sum over the element contributions. In terms of the dimensionless displacement $\mathrm{a} = u_{X1}/L$

$$\delta W = -\frac{L^2}{2} \delta \mathrm{a}(-1 + \mathrm{a}) \frac{tC}{1-\nu^2} \mathrm{a}(-1 + \frac{1}{2}\mathrm{a}) + \delta \mathrm{a} LF \;\;\Rightarrow\;\; \frac{L}{2}(-1 + \mathrm{a}) \frac{tC}{1-\nu^2}(-\mathrm{a} + \frac{1}{2}\mathrm{a}^2) - F = 0.$$

A structure, consisting of a thin slab under the plane stress conditions and a bar, is loaded by a horizontal force F acting on node 1. Material properties are C and ν, thickness of the slab is t, and the cross-sectional area of the bar A at the initial unloaded geometry. Determine the equilibrium equation giving as its solution the displacement component u_{X1} of node 1 according to the large displacement theory.



Solution

Virtual work densities of the thin slab and bar models, when modified for large displacement analysis with the same constitutive equation as in the linear case, are given by

$$\delta w_{\Omega^\circ}^{\text{int}} = -\begin{Bmatrix} \delta \mathrm{E}_{xx} \\ \delta \mathrm{E}_{yy} \\ 2\delta \mathrm{E}_{xy} \end{Bmatrix}^{\mathrm{T}} \frac{tC}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ 2\mathrm{E}_{xy} \end{Bmatrix}, \quad \begin{Bmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ 2\mathrm{E}_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} + \dfrac{1}{2}(\dfrac{\partial u}{\partial x})^2 + \dfrac{1}{2}(\dfrac{\partial v}{\partial x})^2 \\ \dfrac{\partial v}{\partial y} + \dfrac{1}{2}(\dfrac{\partial u}{\partial y})^2 + \dfrac{1}{2}(\dfrac{\partial v}{\partial y})^2 \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\dfrac{\partial v}{\partial y} \end{Bmatrix}.$$

$$\delta w_{\Omega^\circ}^{\text{int}} = -\delta \mathrm{E}_{xx} C A^\circ \mathrm{E}_{xx}, \quad \mathrm{E}_{xx} = \frac{du}{dx} + \frac{1}{2}(\frac{du}{dx})^2 + \frac{1}{2}(\frac{dv}{dx})^2 + \frac{1}{2}(\frac{dw}{dx})^2 .$$

Element contributions need to be derived from approximations and virtual work densities. Approximations to the displacement components depend only on the shape function associated with node 1 as the other nodes are fixed (displacement vanishes).

Let us start with the thin slab element. In terms of the displacement component u_{X1}

$$u = \frac{y}{L} u_{X1} \text{ and } v = 0 \quad \Rightarrow \quad \frac{\partial u}{\partial x} = 0, \ \frac{\partial u}{\partial y} = \frac{u_{X1}}{L}, \text{ and } \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0,$$

giving

$$\begin{Bmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ 2\mathrm{E}_{xy} \end{Bmatrix} = \frac{1}{2}\mathrm{a} \begin{Bmatrix} 0 \\ \mathrm{a} \\ 2 \end{Bmatrix} \text{ and } \begin{Bmatrix} \delta \mathrm{E}_{xx} \\ \delta \mathrm{E}_{yy} \\ 2\delta \mathrm{E}_{xy} \end{Bmatrix} = \delta \mathrm{a} \begin{Bmatrix} 0 \\ \mathrm{a} \\ 1 \end{Bmatrix} \text{ where } \mathrm{a} = \frac{u_{X1}}{L} \text{ and } \delta \mathrm{a} = \frac{\delta u_{X1}}{L}.$$

Virtual work density of the internal forces simplifies to (when the approximations are substituted there)

$$\delta w_{\Omega^\circ}^{\text{int}} = -\frac{tC}{1-\nu^2} \delta \mathrm{a} \begin{Bmatrix} 0 \\ \mathrm{a} \\ 1 \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \frac{1}{2}\mathrm{a} \begin{Bmatrix} 0 \\ \mathrm{a} \\ 2 \end{Bmatrix} = -\delta \mathrm{a} \frac{1}{2} \mathrm{a} \frac{tC}{1-\nu^2} (\mathrm{a}^2 + 1 - \nu).$$

Virtual work expression is the integral of density over the domain occupied by the element (note that the virtual work density is constant in this case). Therefore

$$\delta W^1 = \delta w_{\Omega^\circ}^{\text{int}} \frac{L^2}{2} = -\delta \mathrm{a} \frac{1}{2} \mathrm{a} \frac{L^2}{2} \frac{tC}{1-\nu^2} (\mathrm{a}^2 + 1 - \nu) .$$

The linear approximations to the displacement of the bar element are $w = v = 0$ and

$$u = (1 - \frac{x}{L}) u_{X1} \quad \Rightarrow \quad \frac{du}{dx} = -\frac{u_{X1}}{L} = -\mathrm{a} \;, \quad \text{and} \quad E_{xx} = -\frac{u_{X1}}{L} + \frac{1}{2}(-\frac{u_{X1}}{L})^2 = -\mathrm{a} + \frac{1}{2}\mathrm{a}^2 .$$

For the bar element, virtual work density of the internal forces and thereby the virtual work expression (density is constant) simplifies to

$$\delta W^2 = -\delta \mathrm{a}(-1 + \mathrm{a}) LCA \mathrm{a}(-\mathrm{a} + \frac{1}{2}\mathrm{a}) .$$

Virtual work expression of the point force follows, e.g., directly from the definition (force multiplied by the virtual displacement in its direction)

$$\delta W^3 = -\delta u_{X1} F = -\delta \mathrm{a} LF .$$

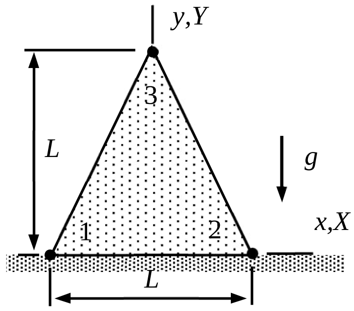
Virtual work expression of a structure is the sum of element contributions

$$\delta W = -\delta \mathrm{a}[\frac{1}{2} \mathrm{a} \frac{L^2}{2} \frac{tC}{1-\nu^2} (\mathrm{a}^2 + 1 - \nu) + (-1 + \mathrm{a}) LCA \mathrm{a}(-\mathrm{a} + \frac{1}{2}\mathrm{a}) + LF] .$$

Principle of virtual work and the fundamental lemma of variation calculus give

$$\frac{L}{4} \frac{tC}{1-\nu^2} \mathrm{a}(\mathrm{a}^2 + 1 - \nu) + CA(-1 + \mathrm{a})\mathrm{a}(-\mathrm{a} + \frac{1}{2}\mathrm{a}) + F = 0 .$$

A long wall having triangular cross-section, and made of homogeneous, isotropic, linearly elastic material, is subjected to its own weight. Determine the equilibrium equation giving as its solution displacement components u_{Y3} according to the large displacement theory. Nodes 1 and 2 are fixed. Use a three-node element and assume plane stress conditions and symmetry $u_{X3} = 0$. Material properties C, ν and the density ρ of the initial geometry are constants.



Solution

According to the large displacement theory, virtual work densities of the thin slab model under plane strain conditions are

$$\delta w_{\Omega^\circ}^{\text{int}} = -\begin{Bmatrix} \delta \mathrm{E}_{xx} \\ \delta \mathrm{E}_{yy} \\ 2\delta \mathrm{E}_{xy} \end{Bmatrix}^{\mathrm{T}} \frac{tC}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ 2\mathrm{E}_{xy} \end{Bmatrix}, \quad \begin{Bmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ 2\mathrm{E}_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} + \dfrac{1}{2}(\dfrac{\partial u}{\partial x})^2 + \dfrac{1}{2}(\dfrac{\partial v}{\partial x})^2 \\ \dfrac{\partial v}{\partial y} + \dfrac{1}{2}(\dfrac{\partial u}{\partial y})^2 + \dfrac{1}{2}(\dfrac{\partial v}{\partial y})^2 \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\dfrac{\partial v}{\partial y} \end{Bmatrix}.$$

$$\delta w_{\Omega^\circ}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} t\rho^\circ \begin{Bmatrix} g_x \\ g_y \end{Bmatrix}$$

in which g_x and g_y are the components of acceleration by gravity and ρ° the density at the initial geometry. Above, constitutive equation is assumed to be of the same form as that for the linear theory with possibly different elasticity parameters C and ν.

Shape function $N_3 = y/L$ of node 3 can be deduced from the figure. Linear approximations to the displacement components and their derivatives are

$$u = 0 \text{ and } v = \frac{y}{L}u_{Y3} \quad \Rightarrow \quad \frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial x} = 0, \text{ and } \frac{\partial v}{\partial y} = \frac{u_{Y3}}{L}.$$

When the approximation is substituted there, the non-zero Green-Lagrange strain component and its variation take the forms

$$\mathrm{E}_{yy} = \frac{u_{Y3}}{L} + \frac{1}{2}(\frac{u_{Y3}}{L})^2 \text{ and } \delta \mathrm{E}_{yy} = \frac{\delta u_{Y3}}{L} + \frac{\delta u_{Y3}}{L}\frac{u_{Y3}}{L}.$$

Virtual work densities simplify to

$$\delta w_{\Omega^\circ}^{\text{int}} = -\frac{\delta u_{Y3}}{L}(1 + \frac{u_{Y3}}{L})\frac{tE}{1-\nu^2}\frac{u_{Y3}}{L}(1 + \frac{1}{2}\frac{u_{Y3}}{L}),$$

$$\delta w_{\Omega^\circ}^{\text{ext}} = -\delta u_{Y3}\frac{y}{L}t\rho g.$$

Integration over the domain occupied by the body at the initial geometry gives the virtual work expressions

$$\delta W^{\text{int}} = -\frac{\delta u_{Y3}}{L}(1+\frac{u_{Y3}}{L})\frac{L^2}{2}\frac{tE}{1-\nu^2}\frac{u_{Y3}}{L}(1+\frac{1}{2}\frac{u_{Y3}}{L}),$$

$$\delta W^{\text{ext}} = \int_0^L (\int_{(y-L)/2}^{(L-y)/2} \delta w_{\Omega^\circ}^{\text{ext}} dx)dy = -\frac{\delta u_{Y3}}{L}\frac{L^3 t\rho g}{6}.$$

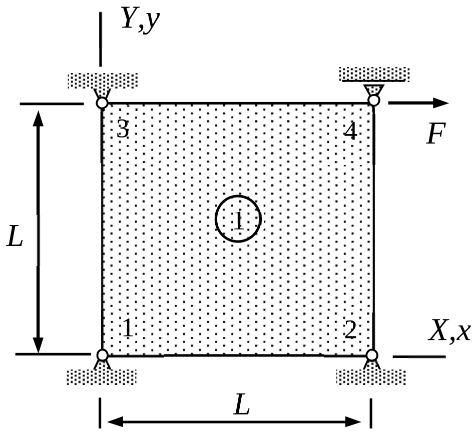
Virtual work expression in the sum of the internal and external parts. Written in the standard form

$$\delta W = -\frac{\delta u_{Y3}}{L}[(1+\frac{u_{Y3}}{L})\frac{L^2}{2}\frac{tE}{1-\nu^2}\frac{u_{Y3}}{L}(1+\frac{1}{2}\frac{u_{Y3}}{L})+\frac{L^3 t\rho g}{6}].$$

Principle of virtual work and the fundamental lemma of variation calculus imply the equilibrium equations

$$(1+\mathrm{a})\mathrm{a}(1+\frac{1}{2}\mathrm{a})+\frac{1}{3}(1-\nu^2)\frac{L\rho g}{E} = 0 \quad \text{where} \quad \mathrm{a} = \frac{u_{Y3}}{L}.$$

Node 4 of a thin rectangular slab, loaded by force F, can move horizontally and nodes 1, 2, and 3 are fixed. Assume plane stress conditions and derive the equilibrium equation of the structure according to the large deformation theory. Use just one bilinear element. Material parameters C and $\nu = 0$. Thickness of the slab at the initial geometry is t.



Solution

According to the large displacement theory, virtual work density of the thin slab model (plane stress condition) is

$$\delta w_{\Omega^\circ}^{\text{int}} = -\begin{Bmatrix} \delta \mathrm{E}_{xx} \\ \delta \mathrm{E}_{yy} \\ 2\delta \mathrm{E}_{xy} \end{Bmatrix}^{\mathrm{T}} \frac{tC}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\begin{Bmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ 2\mathrm{E}_{xy} \end{Bmatrix}, \quad \begin{Bmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ 2\mathrm{E}_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} + \dfrac{1}{2}(\dfrac{\partial u}{\partial x})^2 + \dfrac{1}{2}(\dfrac{\partial v}{\partial x})^2 \\ \dfrac{\partial v}{\partial y} + \dfrac{1}{2}(\dfrac{\partial u}{\partial y})^2 + \dfrac{1}{2}(\dfrac{\partial v}{\partial y})^2 \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\dfrac{\partial v}{\partial y} \end{Bmatrix}.$$

Only the displacement of node 4 in the $X-$direction matters. Shape function $N_4 = xy/L^2$ gives

$$v = 0 \;\text{ and }\; u = xy\frac{u_{X4}}{L^2} \;\Rightarrow\; \frac{\partial u}{\partial x} = y\frac{u_{X4}}{L^2} \;\text{ and }\; \frac{\partial u}{\partial y} = x\frac{u_{X4}}{L^2}.$$

When the approximations are substituted there, the Green-Lagrange strain components and their variations simplify to

$$\begin{Bmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ 2\mathrm{E}_{xy} \end{Bmatrix} = \frac{u_{X4}}{L^2}\begin{Bmatrix} y \\ 0 \\ x \end{Bmatrix} + \frac{1}{2}(\frac{u_{X4}}{L^2})^2\begin{Bmatrix} y^2 \\ x^2 \\ 2xy \end{Bmatrix} \;\text{ and }\; \begin{Bmatrix} \delta\mathrm{E}_{xx} \\ \delta\mathrm{E}_{yy} \\ 2\delta\mathrm{E}_{xy} \end{Bmatrix} = \frac{\delta u_{X4}}{L^2}(\begin{Bmatrix} y \\ 0 \\ x \end{Bmatrix} + \frac{u_{X4}}{L^2}\begin{Bmatrix} y^2 \\ x^2 \\ 2xy \end{Bmatrix}).$$

Virtual work density of the internal forces according to the large displacement theory simplify to (with the Poisson's ratio $\nu = 0$)

$$\delta w_{\Omega^\circ}^{\text{int}} = -\frac{\delta u_{X4}}{L^2}[\begin{Bmatrix} y \\ 0 \\ x \end{Bmatrix} + \frac{u_{X4}}{L^2}\begin{Bmatrix} y^2 \\ x^2 \\ 2xy \end{Bmatrix}]^{\mathrm{T}} tC\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}[\frac{u_{X4}}{L^2}\begin{Bmatrix} y \\ 0 \\ x \end{Bmatrix} + \frac{1}{2}(\frac{u_{X4}}{L^2})^2\begin{Bmatrix} y^2 \\ x^2 \\ 2xy \end{Bmatrix}],$$

The four terms of the virtual work density

$$(\delta w_{\Omega^\circ}^{\text{int}})_1 = -\frac{\delta u_{X4}}{L}\frac{tC}{L^2}(y^2 + \frac{1}{2}x^2)\frac{u_{X4}}{L},$$

$$(\delta w_{\Omega^\circ}^{\text{int}})_2 = -\frac{\delta u_{X4}}{L}\frac{tC}{L}(y^3 + x^2 y)\frac{1}{2}(\frac{u_{X4}}{L^2})^2,$$

$$(\delta w_{\Omega^\circ}^{\text{int}})_3 = -\frac{\delta u_{X4}}{L}\frac{u_{X4}}{L}\frac{tC}{L^3}(y^3 + x^2 y)\frac{u_{X4}}{L},$$

$$(\delta w_{\Omega^\circ}^{\text{int}})_4 = -\frac{\delta u_{X4}}{L}\frac{u_{X4}}{L}\frac{tC}{L^2}(y^4 + x^4 + 2x^2 y^2)\frac{1}{2}(\frac{u_{X4}}{L^2})^2.$$

Virtual work expressions are obtained by integrating the densities over the domain occupied by the element

$$\delta W_1^{\text{int}} = \int_0^L \int_0^L (\delta w_{\Omega^\circ}^{\text{int}})_1 dydx = -\frac{\delta u_{X4}}{L}\frac{1}{2}L^2 tC\frac{u_{X4}}{L},$$

$$\delta W_2^{\text{int}} = \int_0^L \int_0^L (\delta w_{\Omega^\circ}^{\text{int}})_2 dydx = -\frac{\delta u_{X4}}{L}tCL^2\frac{5}{24}(\frac{u_{X4}}{L})^2,$$

$$\delta W_3^{\text{int}} = \int_0^L \int_0^L (\delta w_{\Omega^\circ}^{\text{int}})_3 dydx = -\frac{\delta u_{X4}}{L}\frac{u_{X4}}{L}tCL^2\frac{5}{12}\frac{u_{X4}}{L},$$

$$\delta W_4^{\text{int}} = \int_0^L \int_0^L (\delta w_{\Omega^\circ}^{\text{int}})_4 dydx = -\frac{\delta u_{X4}}{L}\frac{u_{X4}}{L}tCL^2\frac{14}{45}(\frac{u_{X4}}{L})^2.$$

Virtual work expression of the point force

$$\delta W^{\text{ext}} = FL\frac{\delta u_{X4}}{L}.$$

Virtual work expression is the sum of the terms. In terms of the dimensionless displacement $a = u_{X4}/L$

$$\delta W = -tCL^2\delta a(\frac{1}{2}a + \frac{5}{8}a^2 + \frac{14}{45}a^3 - \frac{F}{tLC}).$$

Principle of virtual work and the fundamental lemma of variation calculus imply the equilibrium equation

$$\frac{1}{2}a + \frac{5}{8}a^2 + \frac{14}{45}a^3 - \frac{F}{tLC} = 0.$$

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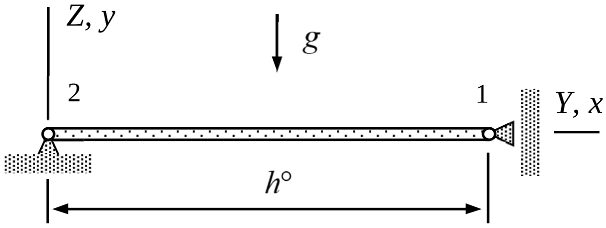
Assignment 1

Derive the virtual work expression for the bar element (planar problem) shown in terms of nodal displacement components of the structural system.



Solution template

Virtual work expressions of the bar model according to the large displacement theory are

$$\delta W^{\text{int}} = -\delta \mathrm{E} C A^{\circ} h^{\circ} \mathrm{E} \quad \text{in which} \quad \mathrm{E} = \frac{1}{2}[(\frac{h}{h^{\circ}})^2 - 1],$$

$$\delta W^{\text{ext}} = \begin{Bmatrix} g_x \delta u_{x1} + g_y \delta u_{y1} + g_z \delta u_{z1} \\ g_x \delta u_{x2} + g_y \delta u_{y2} + g_z \delta u_{z2} \end{Bmatrix}^{\mathrm{T}} \frac{\rho^{\circ} A^{\circ} h^{\circ}}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

In the expressions, A° and h° are the cross-sectional area and length of bar at the initial geometry, g_x, g_y, g_z are the components of the distributed body force (force per unit volume), and ρ°, C are the density and elasticity parameter of the material. The squared length of the deformed bar

$$h^2 = (h^{\circ} + u_{x2} - u_{x1})^2 + (u_{y2} - u_{y1})^2 + (u_{z2} - u_{z1})^2$$

depends on the nodal displacements.

Let us start with the displacement components of the material coordinate system in terms of those of the structural system and the body force components

$u_{x1} = 0$, $u_{y1} = 0$, $u_{z1} = 0$,

$u_{x2} = 0$, $u_{y2} = u_{Z1}$, $u_{z2} = 0$,

$g_x = 0$, $g_y = -g$, $g_z = 0$.

Length of the deformed bar squared is given by

$$h^2 = (h^{\circ} + u_{x2} - u_{x1})^2 + (u_{y2} - u_{y1})^2 + (u_{z2} - u_{z1})^2 = h^{\circ 2} + u_{Z1}^2.$$

Therefore, the Green-Lagrange strain measure and its variation take the forms

$$\mathrm{E} = \frac{1}{2}[(\frac{h}{h^{\circ}})^2 - 1] = \frac{1}{2}(\frac{u_{Z1}}{h^{\circ}})^2, \quad \delta \mathrm{E} = \frac{u_{Z1}}{h^{\circ}} \frac{\delta u_{Z1}}{h^{\circ}}.$$

Using the quantities above, virtual work expressions of the internal and external forces simplify to

$$\delta W^{\text{int}} = -\delta u_{Z1} C A^{\circ} \frac{1}{2} (\frac{u_{Z1}}{h^{\circ}})^3 \ ,$$

$$\delta W^{\text{ext}} = \delta u_{Z1} \frac{\rho^{\circ} A^{\circ} h^{\circ}}{2} g \ .$$

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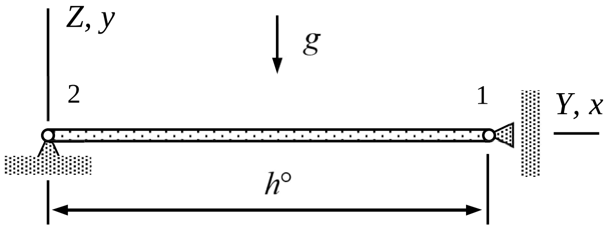
Assignment 2

Derive the virtual work expressions for the element shown in terms of the nodal displacement components of the structural system. Use linear approximations to the displacement components. Cross-sectional area and density of the initial geometry are A° and ρ°, respectively, and elasticity parameter C.



Solution template

Virtual work densities of the bar model according to the large displacement theory are given by

$$\delta w_{\Omega^\circ}^{\text{int}} = -\delta \mathrm{E}_{xx} C A^\circ \mathrm{E}_{xx}, \quad \delta w_{\Omega^\circ}^{\text{ext}} = \rho^\circ A^\circ (\delta u g_x + \delta v g_y + \delta w g_z)$$

in which the Green-Lagrange strain measure and its variation

$$\mathrm{E}_{xx} = \frac{du}{dx} + \frac{1}{2}\left(\frac{du}{dx}\right)^2 + \frac{1}{2}\left(\frac{dv}{dx}\right)^2 + \frac{1}{2}\left(\frac{dw}{dx}\right)^2, \ \delta \mathrm{E}_{xx} = \frac{d\delta u}{dx} + \frac{d\delta u}{dx}\frac{du}{dx} + \frac{d\delta v}{dx}\frac{dv}{dx} + \frac{d\delta w}{dx}\frac{dw}{dx}.$$

Linear approximations to displacement components in terms of nodal displacement components of the structural system and the body force components are given by

$$u = 0, \qquad v = \frac{x}{h^\circ} u_{Z1}, \qquad w = 0,$$

$$g_x = 0, \qquad g_y = -g, \qquad g_z = 0.$$

Green-Lagrange strain measure and its variation in terms of displacement components of the structural system are

$$\mathrm{E}_{xx} = \frac{1}{2}\left(\frac{u_{Z1}}{h^\circ}\right)^2, \quad \delta \mathrm{E}_{xx} = \frac{u_{Z1}}{h^\circ}\frac{\delta u_{Z1}}{h^\circ}.$$

Virtual work densities of internal and external distributed forces

$$\delta w_{\Omega^\circ}^{\text{int}} = -\frac{u_{Z1}}{h^\circ}\frac{\delta u_{Z1}}{h^\circ} C A^\circ \frac{1}{2}\left(\frac{u_{Z1}}{h^\circ}\right)^2,$$

$$\delta w_{\Omega^\circ}^{\text{ext}} = -\frac{x}{h^\circ}\delta u_{Z1} \rho^\circ A^\circ g .$$

Finally, virtual work expressions are integrals over the initial domain

$$\delta W^{\text{int}} = \int_0^{h^\circ} \delta w_{\Omega^\circ}^{\text{int}} dx = -\delta u_{Z1} C A^\circ \frac{1}{2} \left(\frac{u_{Z1}}{h^\circ}\right)^3 ,$$

$$\delta W^{\text{ext}} = \int_0^{h^\circ} \delta w_{\Omega^\circ}^{\text{ext}} dx = -\delta u_{Z1} \frac{\rho^\circ A^\circ h^\circ}{2} g \, .$$

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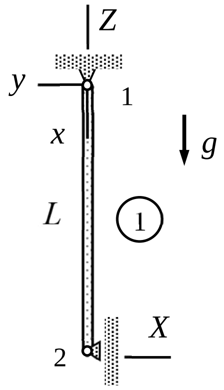
Assignment 3



A bar is loaded by its own weight as shown in the figure. Determine the equilibrium equation in terms of the dimensionless displacement $a = u_{Z2} / L$ with the large deformation theory. Without external loading, area of the cross-section, length of the bar, and density of the material are *A*, *L*, and ρ, respectively. Young's modulus of the material is *C*. Also find the displacement according to the linear theory by simplifying the equilibrium equation with the assumption $|a| \ll 1$.

Solution template

Virtual work densities of the non-linear bar model

$$\delta w_{\Omega^\circ}^{\text{int}} = -(\frac{d\delta u}{dx} + \frac{du}{dx}\frac{d\delta u}{dx} + \frac{dv}{dx}\frac{d\delta v}{dx} + \frac{dw}{dx}\frac{d\delta w}{dx})CA^\circ[\frac{du}{dx} + \frac{1}{2}(\frac{du}{dx})^2 + \frac{1}{2}(\frac{dv}{dx})^2 + \frac{1}{2}(\frac{dw}{dx})^2],$$

$$\delta w_{\Omega^\circ}^{\text{ext}} = A^\circ \rho^\circ (\delta u g_x + \delta v g_y + \delta w g_z)$$

are based on the Green-Lagrange strain definition, which works also when rotations/displacements are large. The expressions depend on all displacement components, material property is denoted by C (kind of Young's modulus), and the superscript in the cross-sectional area A° (and in other quantities) refers to the initial geometry where strain and stress vanish.

The non-zero displacement component of the structure is the vertical displacement of node 2. Linear approximations to the displacement components in terms of the displacement/rotation components of the structural system are

$$u = -\frac{x}{L}u_{Z2} \text{ and } v = w = 0 \quad \Rightarrow \quad \frac{du}{dx} = -\frac{u_{Z2}}{L} \text{ and } \frac{dv}{dx} = \frac{dw}{dx} = 0.$$

In terms of the dimensionless displacement $a = u_{Z2} / L$ and its variation $\delta a = \delta u_{Z2} / L$, virtual work densities simplify to

$$\delta w_{\Omega^\circ}^{\text{int}} = -\delta a(-1 + a)CA(-a + \frac{1}{2}a^2),$$

$$\delta w_{\Omega^\circ}^{\text{ext}} = -\frac{x}{L}\delta u_{Z2} A\rho g = -\delta a x A\rho g.$$

Virtual work expressions are integrals of the densities over the domain occupied by the element

$$\delta W = \int_0^L (\delta w_{\Omega^\circ}^{\text{int}} + \delta w_{\Omega^\circ}^{\text{ext}}) dx = -\delta \mathrm{a}[(-1+\mathrm{a})CAL(-\mathrm{a}+\frac{1}{2}\mathrm{a}^2)+\frac{1}{2}L^2 A\rho g].$$

Principle of virtual work and the fundamental lemma of variation calculus imply that

$$C(-2\mathrm{a}+\mathrm{a}^2)(-1+\mathrm{a})+L\rho g = 0 \quad \text{in which } \mathrm{a} = \frac{u_{Z2}}{L}. \leftarrow$$

Assuming that $|\mathrm{a}| \ll 1$, only the linear part matters and the equilibrium equation simplifies to

$$CAL\mathrm{a} + \frac{1}{2}L^2 A\rho g = 0 \quad \Rightarrow \quad \mathrm{a} = -\frac{1}{2}\frac{L\rho g}{C}. \leftarrow$$

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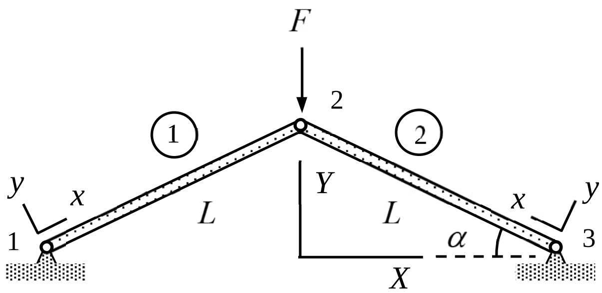
Assignment 4

Consider the bar structure subjected to large displacements in the figure. Determine the relationship between the displacement of node 2 and force F. Start with the virtual work density $\delta w_{\Omega^\circ}^{\text{int}}$ of the non-linear bar model, linear approximations to displacement components, and assume that $u_{X2}=0$ (due to symmetry). Cross-sectional area of the initial geometry is A° and Young's modulus of the material is C.



Solution template

Virtual work density of the non-linear bar model

$$\delta w_{\Omega^\circ}^{\text{int}} = -(\frac{d\delta u}{dx}+\frac{du}{dx}\frac{d\delta u}{dx}+\frac{dv}{dx}\frac{d\delta v}{dx}+\frac{dw}{dx}\frac{d\delta w}{dx})CA^\circ[\frac{du}{dx}+\frac{1}{2}(\frac{du}{dx})^2+\frac{1}{2}(\frac{dv}{dx})^2+\frac{1}{2}(\frac{dw}{dx})^2]$$

is based on the Green-Lagrange strain definition which is physically correct also when rotations/displacements are large. The expression depends on all displacement components, material property is denoted by C (constitutive equation $S_{xx}=CE_{xx}$), and the superscript in the cross-sectional area A° (and in other quantities) refers to the initial geometry (strain and stress vanishes). Otherwise, equilibrium equations follow in the same manner as in the linear case.

For element 1, the non-zero displacement components are $u_{y2}=u_{Y2}\cos\alpha$ and $u_{x2}=u_{Y2}\sin\alpha$. As the initial length of the element $h^\circ = L$, linear approximations to the displacement components are

$$u=\frac{x}{L}u_{Y2}\sin\alpha,\quad v=\frac{x}{L}u_{Y2}\cos\alpha,\quad \text{and}\quad w=0 \quad\Rightarrow$$

$$\frac{du}{dx}=\frac{u_{Y2}}{L}\sin\alpha,\quad \frac{dv}{dx}=\frac{u_{Y2}}{L}\cos\alpha,\ \text{and}\quad \frac{dw}{dx}=0.$$

When the approximations are substituted there, virtual work density of the internal forces and thereby the virtual work expression (density is constant) in terms of the dimensionless displacement $a=u_{Y2}/L$ simplify to

$$\delta w_{\Omega^\circ}^{\text{int}} = -(\delta a\sin\alpha + a\delta a)CA^\circ(a\sin\alpha+\frac{1}{2}a^2) \quad\Rightarrow$$

$$\delta W^1 = -(\delta a\sin\alpha + a\delta a)CA^\circ L(a\sin\alpha+\frac{1}{2}a^2).$$

For element 2, the non-zero displacement components are $u_{y2}=u_{Y2}\cos\alpha$ and $u_{x2}=u_{Y2}\sin\alpha$. As the initial length of the element $h^\circ=L$, linear approximations to the displacement components are

$$u=\frac{x}{L}u_{Y2}\sin\alpha\,,\quad v=\frac{x}{L}u_{Y2}\cos\alpha\,,\quad \text{and}\quad w=0 \quad\Rightarrow$$

$$\frac{du}{dx}=\frac{u_{Y2}}{L}\sin\alpha\,,\quad \frac{dv}{dx}=\frac{u_{Y2}}{L}\cos\alpha\,,\ \text{and}\quad \frac{dw}{dx}=0\,.$$

When the approximations are substituted there, virtual work density of the internal forces and thereby the virtual work expression (density is constant) in terms of the dimensionless displacement $\mathrm{a}=u_{Y2}/L$ simplify to

$$\delta w_{\Omega^\circ}^{\text{int}}=-(\delta\mathrm{a}\sin\alpha+\mathrm{a}\delta\mathrm{a})CA^\circ(\mathrm{a}\sin\alpha+\frac{1}{2}\mathrm{a}^2)\quad\Rightarrow$$

$$\delta W^2=-(\delta\mathrm{a}\sin\alpha+\mathrm{a}\delta\mathrm{a})CA^\circ L(\mathrm{a}\sin\alpha+\frac{1}{2}\mathrm{a}^2)\,.$$

Virtual work expression of the point force, in terms of the dimensionless displacement $\mathrm{a}=u_{Y2}/L$, is

$$\delta W^3=-F\delta u_{Y2}=-FL\delta\mathrm{a}\,.$$

Virtual work expression of the structure is obtained as the sum of the element contributions

$$\delta W=-\delta\mathrm{a}((\sin\alpha+\mathrm{a})2CA^\circ L(\mathrm{a}\,\sin\alpha+\frac{1}{2}\mathrm{a}^2)+FL)\,.$$

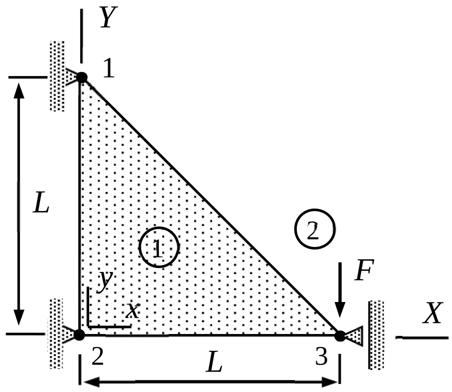
Principle of virtual work and the fundamental lemma of variation calculus imply that

$$CA^\circ\mathrm{a}(\sin\alpha+\mathrm{a})(2\sin\alpha+\mathrm{a})+F=0 \quad\text{in which}\quad \mathrm{a}=\frac{u_{Y2}}{L}\,.$$

Name________________________ Student number________________

Assignment 5

A thin triangular slab is loaded by a point force at node 3. Nodes 1 and 2 are fixed and node 3 moves only in the vertical direction. Derive the equilibrium equation of the structure according to the large displacement theory in terms of the dimensionless displacement component $\mathrm{a} = u_{Y3}/L$. Approximation is linear and material parameters C and ν are constants. Assume plane-stress conditions. When $F = 0$, side length and thickness of the slab are L and t, respectively. Also find the solution to a small displacement problem by simplifying the equilibrium equations with the assumption $|\mathrm{a}| \ll 1$.



Solution

Virtual work density of internal forces, when modified for large displacement analysis with the same constitutive equation as in the linear case of plane stress, is given by

$$\delta w_{\Omega^\circ}^{\mathrm{int}} = -\begin{Bmatrix} \delta \mathrm{E}_{xx} \\ \delta \mathrm{E}_{yy} \\ 2\delta \mathrm{E}_{xy} \end{Bmatrix}^{\mathrm{T}} \frac{tC}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ 2\mathrm{E}_{xy} \end{Bmatrix}, \quad \begin{Bmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ 2\mathrm{E}_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} + \dfrac{1}{2}(\dfrac{\partial u}{\partial x})^2 + \dfrac{1}{2}(\dfrac{\partial v}{\partial x})^2 \\ \dfrac{\partial v}{\partial y} + \dfrac{1}{2}(\dfrac{\partial u}{\partial y})^2 + \dfrac{1}{2}(\dfrac{\partial v}{\partial y})^2 \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\dfrac{\partial v}{\partial y} \end{Bmatrix}.$$

Let us start with the approximations and the corresponding components of the Green-Lagrange strain. Linear shape functions can be deduced from the figure. Only the shape function $N_3 = x/L$ of node 3 is needed. Displacement components and their non-zero derivatives are

$$u = 0 \text{ and } v = \frac{x}{L}u_{Y3} \;\Rightarrow\; \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0 \text{ and } \frac{\partial v}{\partial x} = \frac{u_{Y3}}{L} = \mathrm{a}, \; \frac{\partial v}{\partial y} = 0.$$

Green-Lagrange strain measures and their variations

$$\begin{Bmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ 2\mathrm{E}_{xy} \end{Bmatrix} = \begin{Bmatrix} \mathrm{a}^2/2 \\ 0 \\ \mathrm{a} \end{Bmatrix} \;\Rightarrow\; \begin{Bmatrix} \delta\mathrm{E}_{xx} \\ \delta\mathrm{E}_{yy} \\ 2\delta\mathrm{E}_{xy} \end{Bmatrix} = \begin{Bmatrix} \mathrm{a}\delta\mathrm{a} \\ 0 \\ \delta\mathrm{a} \end{Bmatrix}.$$

When the strain component expressions are substituted there, virtual work density simplifies to

$$\delta w_{\Omega^\circ}^{\text{int}} = -\frac{tC}{1-\nu^2}\begin{Bmatrix} \mathrm{a}\delta\mathrm{a} \\ 0 \\ \delta\mathrm{a} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \mathrm{a}^2/2 \\ 0 \\ \mathrm{a} \end{Bmatrix} \Rightarrow$$

$$\delta w_{\Omega^\circ}^{\text{int}} = -\frac{tC}{1-\nu^2}\delta\mathrm{a}(\frac{1}{2}\mathrm{a}^3 + \frac{1-\nu}{2}\mathrm{a})$$

Integration over the (initial) domain gives the virtual work expression. As the integrand is constant

$$\delta W^1 = \frac{L^2}{2}\delta w_{\Omega^\circ}^{\text{int}} = -\frac{L^2}{2}\frac{tC}{1-\nu^2}\delta\mathrm{a}(\frac{1}{2}\mathrm{a}^3 + \frac{1-\nu}{2}\mathrm{a}).$$

Virtual work expression of the external point force components

$$\delta W^2 = -F\delta u_{Y3} = -FL\delta\mathrm{a}.$$

Virtual work expression of the structure is obtained as sum over the element contributions. In terms of the dimensionless displacement

$$\delta W = -\frac{L^2}{2}\frac{tC}{1-\nu^2}\delta\mathrm{a}(\frac{1}{2}\mathrm{a}^3 + \frac{1-\nu}{2}\mathrm{a}) - FL\delta\mathrm{a}$$

or, when written in the standard form,

$$\delta W = -\delta\mathrm{a}[\frac{L^2}{2}\frac{tC}{1-\nu^2}(\frac{1}{2}\mathrm{a}^3 + \frac{1-\nu}{2}\mathrm{a}) + FL].$$

Principle of virtual work and the basic lemma of variation calculus imply the equilibrium equation

$$(\mathrm{a}^2 + 1 - \nu)\mathrm{a} + 4(1-\nu^2)\frac{F}{tLC} = 0. \leftarrow$$

Assuming that $|\mathrm{a}| \ll 1$ the equilibrium equation simplifies to

$$(1-\nu)\mathrm{a} + 4(1-\nu^2)\frac{F}{tLC} = 0 \quad \Rightarrow \quad \mathrm{a} = -4\frac{F}{tCL}(1+\nu). \leftarrow$$

MEC-E8001

Finite Element Analysis

2023

WEEK 7: THERMO-MECHANICAL ANALYSIS

6 THERMO-MECHANICAL ANALYSIS

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LEARNING OUTCOMES

Students are able to solve the weekly lecture problems, home problems, and exercise problems on thermo-mechanical FEA:

- ☐ Balance laws and constitutive equations of isotropic thermo-mechanics

- ☐ Stationary thermo-mechanical FEA with solid, plate, and beam elements

- ☐ Virtual work densities of solid, plate, and beam models

MULTIPHYSICS FEA

Multiphysics simulation employs temperature, water contents, etc. with additional balance laws and constitutive equations to predict displacement, temperature, concentration etc. under complex interactions. *A thermo-mechanical model* considers the effect of temperature on mechanical behavior:

- As an unwanted mechanical effect, pipelines and continuous welded rails may bend or buckle in a hot summer.

- Press fit take advantage of thermal expansion and contraction: enveloping parts are assembled into position while hot, then allowed to cool and contract back to their former size. Loosening of a jar lid under heating is based on the opposite mechanism.

- Temperature changes may induce very large stresses.

BALANCE LAWS OF MECHANICS

Balance of mass (def. of a body or a material volume) Mass of a body is constant ←

Balance of linear momentum (Newton 2) The rate of change of linear momentum within a material volume equals the external force resultant acting on the material volume. ←

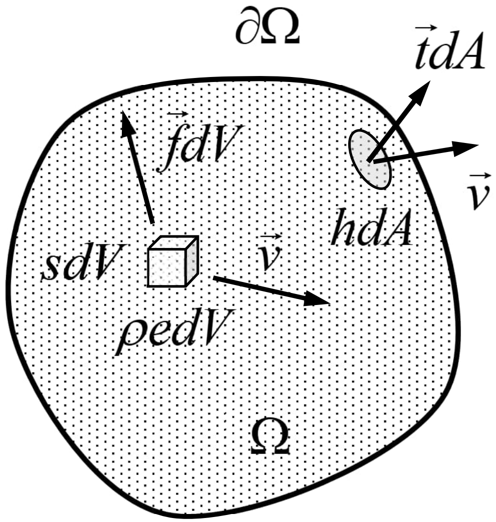
Balance of angular momentum (Cor. of Newton 2) The rate of change of angular momentum within a material volume equals the external moment resultant acting on the material volume. ←

Balance of energy (Thermodynamics 1) ←

Entropy growth (Thermodynamics 2)

BALANCE OF ENERGY

The rate of change of kinetic and internal energies equals the powers of external forces and added heat, i.e., $\dot{U} + \dot{T} = P_W + P_Q$ where



Internal energy $U = \int_{\Omega} \rho e dV$

Kinetic energy $T = \int_{\Omega} \frac{1}{2} \rho \vec{v} \cdot \vec{v} dV$

Power of forces $P_W = \int_{\Omega} \vec{f} \cdot \vec{v} dV + \int_{\partial\Omega} \vec{t} \cdot \vec{v} dA$

Power of heat $P_Q = \int_{\Omega} s dV + \int_{\partial\Omega} h dA$

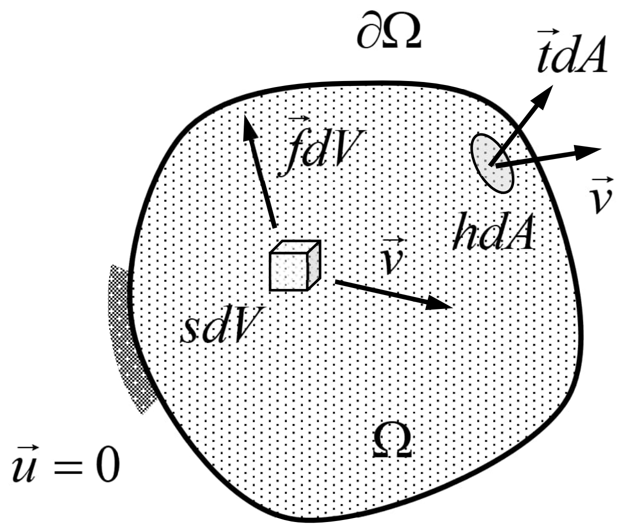
Temperature ϑ, heat Q, and internal energy U are concepts of continuum mechanics that do not have direct counterparts in particle mechanics (force and displacements have).

6.1 LINEAR THERMO-MECHANICS

Balance law	**Local form in Ω**	**Local form on $\partial\Omega$**
$\dfrac{Dm}{Dt} = 0$	$\rho^{\circ} = J\rho$	–
$\dfrac{D\vec{p}}{Dt} = \vec{F}$	$\rho^{\circ}\dfrac{\partial^2\vec{u}}{\partial t^2} = \nabla\cdot\overleftrightarrow{\sigma} + \vec{f}$	$\vec{n}\cdot\overleftrightarrow{\sigma} = \vec{t}$
$\dfrac{D\vec{L}}{Dt} = \vec{M}$	$\overleftrightarrow{\sigma} = \overleftrightarrow{\sigma}_{\mathrm{c}}$	–
$\dfrac{D(U+K)}{Dt} = P_W + P_Q$	$\rho^{\circ}\dfrac{\partial e}{\partial t} = \overleftrightarrow{\sigma} : \overleftrightarrow{d}_{\mathrm{c}} + s - \nabla\cdot\vec{q}$	$\vec{n}\cdot\vec{q} = h$

BOUNDARY VALUE PROBLEM

Given the initial stationary equilibrium temperature and displacement on Ω, the aim is to find new stationary equilibrium temperature and displacement, when external forces, heating etc. are changed in some manner.



Balance of momentun $\nabla\cdot\vec{\vec{\sigma}}+\vec{f}=0$ in Ω,

Balance of energy $-\nabla\cdot\vec{q}+s=0$ in Ω,

Displacement BC:s $\vec{n}\cdot\vec{\vec{\sigma}}=\vec{t}$ or $\vec{u}=\vec{g}$ on $\partial\Omega$,

Temperature BC:s $\vec{n}\cdot\vec{q}=h$ or $\vartheta=\underline{\vartheta}$ on $\partial\Omega$.

Constitutive equations of the form $\vec{q}(\vartheta)$ (heat flux) and $\vec{\vec{\sigma}}(\vec{u},\vartheta)$ (stress) are needed for a closed equation system in terms of displacement and temperature.

GENERALIZED HOOKE'S LAW

The generalized Hooke's law, also considering the change of temperature $\Delta\vartheta = \vartheta - \vartheta^{\circ}$, is given by ($\vec{\sigma} = 0$ and $\Delta\vartheta = 0$ at the initial geometry)

Strain-stress: $$\begin{Bmatrix} \varepsilon_{xx} - \alpha\Delta\vartheta \\ \varepsilon_{yy} - \alpha\Delta\vartheta \\ \varepsilon_{zz} - \alpha\Delta\vartheta \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix}\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} \text{ and } \begin{Bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \frac{1}{G}\begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix}$$

Strain-displacement: $$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} = \begin{Bmatrix} \partial u_x / \partial x \\ \partial u_y / \partial y \\ \partial u_z / \partial z \end{Bmatrix} \text{ and } \begin{Bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{Bmatrix} \partial u_x / \partial y + \partial u_y / \partial x \\ \partial u_y / \partial z + \partial u_z / \partial y \\ \partial u_z / \partial x + \partial u_x / \partial z \end{Bmatrix}$$

Above, E is the Young's modulus, ν the Poisson's ratio, $G = E/(2+2\nu)$ the shear modulus, and α the thermal expansion coefficient. Strain and stress are symmetric.

FOURIER LAW OF HEAT CONDUCTION

When bodies at different temperatures are in contact, heat flows toward the cooler body until temperatures are the same. The Fourier law of heat conduction for an isotropic homogeneous material are (stress is assumed to vanish at the initial geometry) is given by

Heat-temperature: $\begin{Bmatrix} q_x \\ q_y \\ q_z \end{Bmatrix} = -\begin{bmatrix} k_{xx} & k_{xy} & k_{xz} \\ k_{xy} & k_{yy} & k_{yz} \\ k_{xz} & k_{yz} & k_{zz} \end{bmatrix} \begin{Bmatrix} \partial \vartheta / \partial x \\ \partial \vartheta / \partial y \\ \partial \vartheta / \partial z \end{Bmatrix} = -k \begin{Bmatrix} \partial \vartheta / \partial x \\ \partial \vartheta / \partial y \\ \partial \vartheta / \partial z \end{Bmatrix}$ Isotropic material

Thermal conductivity $k([\mathrm{N}/(\mathrm{Ks})]$ or $[\mathrm{W/(Km)}])$ depends on the material. The forms for the uni-axial and planar problems can be deduced from the generic form in the same manner as those for the stress-strain relationship.

EXAMPLE. Derive the stress-strain-temperature relationship of isotropic homogeneous material under (a) the $xy-$plane stress and (b) uni-axial stress conditions. Start with the generic strain-stress-temperature relationship.

Answer $\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = [E]_\sigma \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} - \alpha\Delta\vartheta \frac{E}{1-\nu} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}$ and $\sigma_{xx} = E(\varepsilon_{xx} - \alpha\Delta\vartheta)$

- Under the plane stress assumption, only σ_{xx}, σ_{yy}, and σ_{xy} are non-zeros. The relationship for the in-plane normal stress resultants follows from the generic strain-temperature-stress relationship modified according to the kinetic assumption:

$$\begin{Bmatrix} \varepsilon_{xx}-\alpha\Delta\vartheta \\ \varepsilon_{yy}-\alpha\Delta\vartheta \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix}\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} \Leftrightarrow \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = [E]_{\sigma}\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} - \alpha\Delta\vartheta\frac{E}{1-\nu}\begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}.$$

- Under the uni-axial stress assumption, only σ_{xx} is non-zero. The relationship follows directly from the generic strain-stress-temperature relationship. Inversion gives the stress-strain-temperature relationship for the uni-axial case

$$\varepsilon_{xx}-\alpha\Delta\vartheta = \frac{1}{E}\sigma_{xx} \quad \Leftrightarrow \quad \sigma_{xx} = E(\varepsilon_{xx}-\alpha\Delta\vartheta).$$

MATERIAL PARAMETERS

Material	ρ **[** $\mathrm{kg/m^3}$ **]**	E [GPa]	ν **[** 1 **]**
Steel	7800	210	0.3
Aluminum	2700	70	0.33
Copper	8900	120	0.34
Glass	2500	60	0.23
Granite	2700	65	0.23
Birch	600	16	-
Rubber	900	10^{-2}	0.5
Concrete	2300	25	0.1

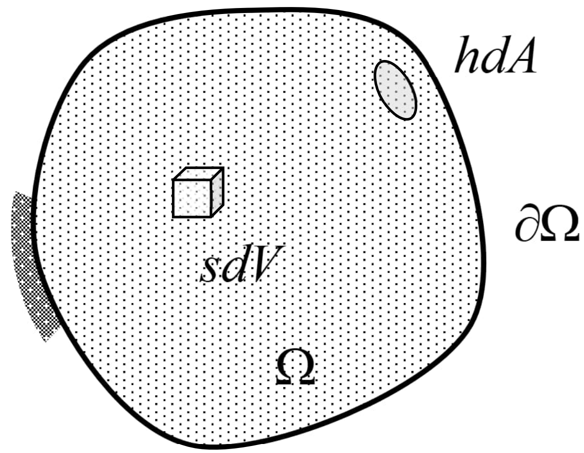
MATERIAL PARAMETERS

Material	k [W / (Km)]	α [μm / mK]	c [J / kgK]
Steel	45…50	12…13	520
Aluminum	205…240	23…24	900
Copper	385…400	17	
Glass, ordinary	0.8…1	8…9	800
Granite	0.7…0.9		
Wood	0.1…0.2	30	1300
Rubber	0.2	0.1	
Concrete	1	12	850

VARIATIONAL REPRESENTATION

The variational form $\delta P = \delta P^{\text{int}} + \delta P^{\text{ext}} = 0 \quad \forall \delta\vartheta$ is the concise representation of the stationary heat conduction boundary value problem. In terms of density expressions $\delta p_{\Omega}^{\text{int}}$, $\delta p_{\Omega}^{\text{ext}}$, and $\delta p_{\partial\Omega}^{\text{ext}}$

Internal part: $\delta P^{\text{int}} = \int_{\Omega} \delta p_{\Omega}^{\text{int}} dV$,



External part: $\delta P^{\text{ext}} = \int_{\Omega} \delta p_{\Omega}^{\text{ext}} dV + \int_{\partial\Omega} \delta p_{\partial\Omega}^{\text{ext}} dA$.

The variational form lacks a clear physical interpretation although the meaning is clear from the mathematical viewpoint. The physical dimensions of δP $[\text{WK}]$ and δW $[\text{J}]$ differ, the former being power and the latter work.

- In derivation, the local form of energy balance is multiplied by $\delta\vartheta$, integrated over the domain followed by integration by parts in the heat flux term. Manipulations give the equivalent representations

$$-\nabla\cdot\vec{q}+s=0 \quad \text{in} \quad \Omega \quad \Leftrightarrow$$

$$\int_{\Omega}\delta\vartheta(-\nabla\cdot\vec{q}+s)dV=\int_{\Omega}(\nabla\delta\vartheta\cdot\vec{q}+\delta\vartheta s)dV-\int_{\partial\Omega}\delta\vartheta\vec{n}\cdot\vec{q}dA=0 \quad \forall\delta\vartheta.$$

- Assumption $\delta\vartheta=0$ (temperature specified) or $\vec{n}\cdot\vec{q}+h=0$ (heat flux specified) on $\partial\Omega$ gives the final form

$$\delta P=0 \quad \forall\delta\vartheta \quad \text{where} \quad \delta P=\int_{\Omega}\nabla\delta\vartheta\cdot\vec{q}dV+\int_{\Omega}\delta\vartheta s dV+\int_{\partial\Omega}\delta\vartheta h dA\,. \quad \Leftarrow$$

DENSITY EXPRESSIONS

The integrands of the variational form represent the model in the same manner as the virtual work densities in principle of virtual work:

Internal part: $\delta p_{\Omega}^{\text{int}} = \begin{Bmatrix} \partial\delta\vartheta / \partial x \\ \partial\delta\vartheta / \partial y \\ \partial\delta\vartheta / \partial z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} q_x \\ q_y \\ q_z \end{Bmatrix} = -\begin{Bmatrix} \partial\delta\vartheta / \partial x \\ \partial\delta\vartheta / \partial y \\ \partial\delta\vartheta / \partial z \end{Bmatrix}^{\mathrm{T}} k \begin{Bmatrix} \partial\vartheta / \partial x \\ \partial\vartheta / \partial y \\ \partial\vartheta / \partial z \end{Bmatrix},$ Isotropic material

External parts: $\delta p_{\Omega}^{\text{ext}} = \delta\vartheta s$ and $\delta p_{\partial\Omega}^{\text{ext}} = \delta\vartheta h$.

Thermal conductivity k $[\mathrm{W/(Km)}]$, power of heat per unit volume s $[\mathrm{W/m^3}]$, and power of heat per unit area h $[\mathrm{W/m^2}]$ may depend on position. For non-isotropic materials thermal conductivity is a (positive definite) matrix.

6.2 THERMO-MECHANICAL FEA

- Model the structure as a collection of beam, plate, etc. elements. Derive the element contributions $\delta W^e = \delta W^{\text{int}} + \delta W^{\text{ext}} + \delta W^{\text{cpl}}$ and $\delta P^e = \delta P^{\text{int}} + \delta P^{\text{ext}}$ in terms of nodal displacements/rotation components of the structural coordinate system and temperature.

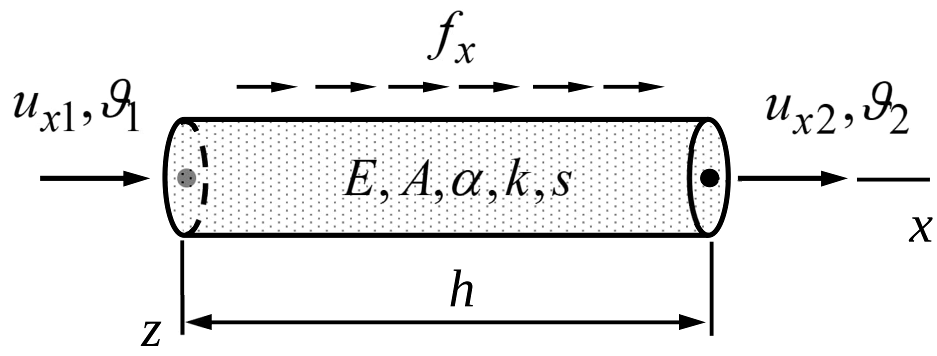
- Sum the element contributions to end up with the variational expression for the structure. Re-arrange to get $\delta W + \tau\delta P = -\delta\mathbf{a}^{\mathrm{T}}\mathbf{R}(\mathbf{a},\mathbf{b}) - \tau\delta\mathbf{b}^{\mathrm{T}}\mathbf{R}(\mathbf{b})$ (τ is a dimensionally correct but otherwise arbitrary constant).

- Use the principle $\delta W + \tau\delta P = 0 \quad \forall \delta\mathbf{a}, \delta\mathbf{b}$ and the fundamental lemma of variation calculus to deduce $\mathbf{R}(\mathbf{a},\mathbf{b}) = 0$ and $\mathbf{R}(\mathbf{b}) = 0$. Solve the linear algebraic equations for the nodal displacements, rotations, and temperatures (due to the one-sided coupling of the stationary problem, solving the temperature first is always possible).

BAR MODE

Assuming that $v=0,\ w=0,\ \phi=0$ and a linear interpolation to the axial displacement $u(x)$ and temperature $\vartheta(x)$

$$\delta P^{\text{int}} = -\begin{Bmatrix}\delta\vartheta_1\\ \delta\vartheta_2\end{Bmatrix}^{\mathrm{T}} \frac{kA}{h}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}\vartheta_1\\ \vartheta_2\end{Bmatrix},$$

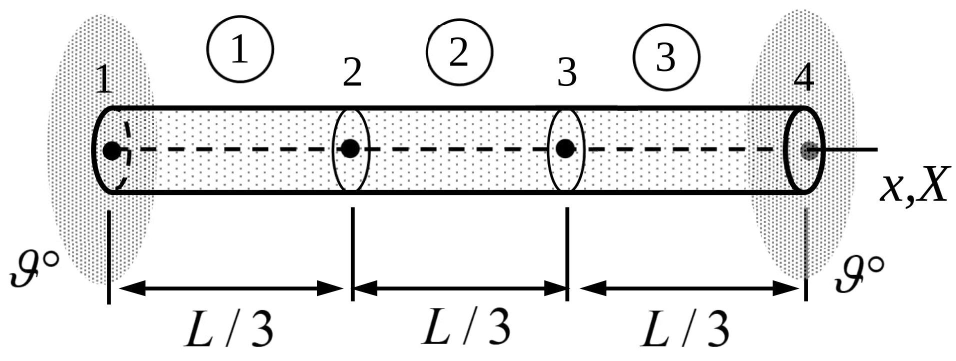


$$\delta W^{\text{cpl}} = \begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}} \frac{\alpha EA}{2}\begin{bmatrix}-1 & -1\\ 1 & 1\end{bmatrix}\begin{Bmatrix}\Delta\vartheta_1\\ \Delta\vartheta_2\end{Bmatrix},$$

$$\delta P^{\text{ext}} = \begin{Bmatrix}\delta\vartheta_1\\ \delta\vartheta_2\end{Bmatrix}^{\mathrm{T}} \frac{Ash}{2}\begin{Bmatrix}1\\ 1\end{Bmatrix}.$$

Heat flux through the end-planes is treated by point elements in the same manner as traction on the end-plates by point forces and moments.

EXAMPLE 6.1 The bar of the figure consists of three linear elements of identical lengths. Determine the stationary temperatures ϑ_2 at node 2 and ϑ_3 at node 3 when the end temperature is ϑ° and heat generation s per unit volume are constants. Take only the heat conduction along the bar axis into account. Problem parameters E, A, and k are constants.



Answer $\begin{Bmatrix} \vartheta_2 \\ \vartheta_3 \end{Bmatrix} = (\vartheta^\circ + \frac{1}{9}\frac{sL^2}{k})\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

- Element contributions for the temperature distribution problem are (temperature is not affected by displacement)

$$\delta P^{\text{int}} = -\begin{Bmatrix} \delta\vartheta_1 \\ \delta\vartheta_2 \end{Bmatrix}^{\mathrm{T}} \frac{kA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \vartheta_1 \\ \vartheta_2 \end{Bmatrix}, \quad \delta P^{\text{ext}} = \begin{Bmatrix} \delta\vartheta_1 \\ \delta\vartheta_2 \end{Bmatrix}^{\mathrm{T}} \frac{Ash}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

- When the actual nodal values are substituted there, element contributions simplify to

$$\delta P^{1} = -\begin{Bmatrix} 0 \\ \delta\vartheta_2 \end{Bmatrix}^{\mathrm{T}} \left(\frac{3kA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \vartheta^{\circ} \\ \vartheta_2 \end{Bmatrix} - \frac{AsL}{6} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}\right),$$

$$\delta P^{2} = -\begin{Bmatrix} \delta\vartheta_2 \\ \delta\vartheta_3 \end{Bmatrix}^{\mathrm{T}} \left(\frac{3kA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \vartheta_2 \\ \vartheta_3 \end{Bmatrix} - \frac{AsL}{6} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}\right),$$

$$\delta P^3 = -\begin{Bmatrix} \delta\vartheta_3 \\ 0 \end{Bmatrix}^{\mathrm{T}} (\frac{3kA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \vartheta_3 \\ \vartheta^\circ \end{Bmatrix} - \frac{AsL}{6}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}).$$

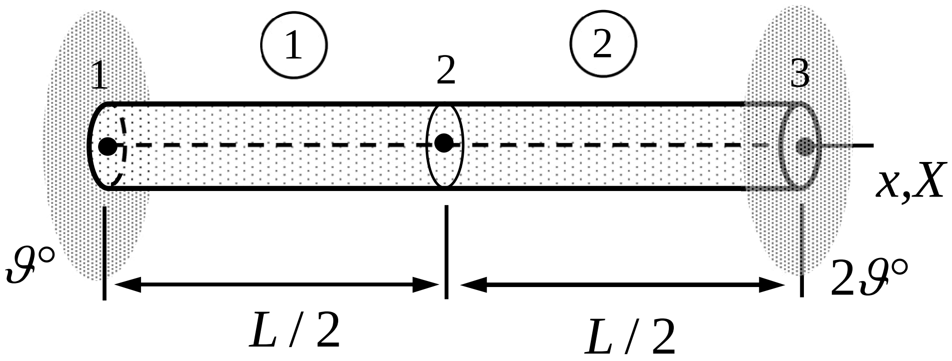
- Variational expression for a structure is the sum of the element contributions

$$\delta P = -\begin{Bmatrix} \delta\vartheta_2 \\ \delta\vartheta_3 \end{Bmatrix}^{\mathrm{T}} (\frac{3kA}{L}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} \vartheta_2 \\ \vartheta_3 \end{Bmatrix} - \frac{3kA}{L}\begin{Bmatrix} \vartheta^\circ \\ \vartheta^\circ \end{Bmatrix} - \frac{AsL}{6}\begin{Bmatrix} 2 \\ 2 \end{Bmatrix}).$$

- Variational principle $\delta P = 0 \quad \forall \delta\mathbf{a}$ and the fundamental lemma of variation calculus imply a linear equation system and thereby the solution

$$\frac{3kA}{L}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} \vartheta_2 \\ \vartheta_3 \end{Bmatrix} - \frac{3kA}{L}\begin{Bmatrix} \vartheta^\circ \\ \vartheta^\circ \end{Bmatrix} - \frac{AsL}{6}\begin{Bmatrix} 2 \\ 2 \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix} \vartheta_2 \\ \vartheta_3 \end{Bmatrix} = (\vartheta^\circ + \frac{1}{9}\frac{sL^2}{k})\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}. \Leftarrow$$

EXAMPLE 6.2 The bar of the figure consists of two elements having the same material properties. Stress is zero, when the temperature in the wall and bar is ϑ°. Determine the stationary displacement u_{X2} and temperature ϑ_2 at node 2, when the temperature of the right end is increased to $2\vartheta^\circ$. Take only the heat conduction along the bar axis into account. Use two linear elements. Problem parameters E, A, k, and α are constants.



Answer $u_{X2} = -\dfrac{1}{8} L\alpha\vartheta^\circ$ and $\vartheta_2 = \dfrac{3}{2}\vartheta^\circ$

- Element contributions for the thermo-mechanical problem needed in this case are (no heat production, nor external distributed forces, and $\Delta\vartheta = \vartheta - \vartheta^{\circ}$).

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix}, \quad \delta W^{\text{cpl}} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{\alpha EA}{2} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ \vartheta_2 - \vartheta^{\circ} \end{Bmatrix},$$

$$\delta P^{\text{int}} = -\begin{Bmatrix} \delta\vartheta_1 \\ \delta\vartheta_2 \end{Bmatrix}^{\mathrm{T}} \frac{kA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \vartheta_1 \\ \vartheta_2 \end{Bmatrix}.$$

- As the nodal values for bar 1 are $u_{x1} = 0$, $u_{x2} = u_{X2}$, $\Delta\vartheta_1 = 0$, and $\Delta\vartheta_2 = \vartheta_2 - \vartheta^{\circ}$, the element contributions $\delta W^{\text{int}} + \delta W^{\text{cpl}}$ and δP^{int} simplify to

$$\delta W^1 = -\begin{Bmatrix} 0 \\ \delta u_{X2} \end{Bmatrix}^{\mathrm{T}} \frac{2EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ u_{X2} \end{Bmatrix} + \begin{Bmatrix} 0 \\ \delta u_{X2} \end{Bmatrix}^{\mathrm{T}} \frac{\alpha EA}{2} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ \vartheta_2 - \vartheta^{\circ} \end{Bmatrix} \Leftrightarrow$$

$$\delta W^1 = -\delta u_{X2} \frac{2EA}{L} u_{X2} + \delta u_{X2} \frac{\alpha EA}{L} (\vartheta_2 - \vartheta^\circ),$$

$$\delta P^1 = -\begin{Bmatrix} 0 \\ \delta\vartheta_2 \end{Bmatrix}^{\mathrm{T}} \frac{2kA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \vartheta^\circ \\ \vartheta_2 \end{Bmatrix} = -\delta\vartheta_2 \frac{2kA}{L} (\vartheta_2 - \vartheta^\circ).$$

- As the nodal values for bar 2 are $u_{x3} = 0$, $u_{x2} = u_{X2}$, $\Delta\vartheta_3 = 2\vartheta^\circ - \vartheta^\circ = \vartheta^\circ$, and $\Delta\vartheta_2 = \vartheta_2 - \vartheta^\circ$, the element contributions $\delta W^{\mathrm{int}} + \delta W^{\mathrm{cpl}}$ and δP^{int} simplify to

$$\delta W^2 = -\begin{Bmatrix} \delta u_{X2} \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{2EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{X2} \\ 0 \end{Bmatrix} + \begin{Bmatrix} \delta u_{X2} \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{\alpha EA}{2} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \vartheta_2 - \vartheta^\circ \\ \vartheta^\circ \end{Bmatrix} \Leftrightarrow$$

$$\delta W^2 = -\delta u_{X2} \frac{2EA}{L} u_{X2} - \delta u_{X2} \frac{\alpha EA}{2} \vartheta_2,$$

$$\delta P^2 = -\begin{Bmatrix} \delta\vartheta_2 \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{2kA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \vartheta_2 \\ 2\vartheta^\circ \end{Bmatrix} = -\delta\vartheta_2 \frac{2kA}{L} (\vartheta_2 - 2\vartheta^\circ).$$

- Variational expressions for the mechanical and thermal parts are sums of the element contributions

$$\delta W = \delta W^1 + \delta W^2 = -\delta u_{X2}(\frac{4EA}{L}u_{X2} + \frac{\alpha EA}{2}\vartheta^\circ),$$

$$\delta P = \delta P^1 + \delta P^2 = -\delta\vartheta_2 \frac{2kA}{L}(2\vartheta_2 - 3\vartheta^\circ).$$

- Variational principle $\delta W + \tau\delta P = 0 \quad \forall\delta\mathbf{a}$ and the fundamental lemma of variation calculus imply the equations

$$\frac{4EA}{L}u_{X2} + \frac{\alpha EA}{2}\vartheta^\circ = 0 \text{ and } \frac{2kA}{L}(2\vartheta_2 - 3\vartheta^\circ) = 0 \quad \Leftrightarrow$$

$$\vartheta_2 = \frac{3}{2}\vartheta^\circ \text{ and } u_{X2} = -\frac{\alpha L}{8}\vartheta^\circ.$$

- In Mathematica notation, the problem description is given by

	model	properties	geometry
1	BAR	$\{\{E, \alpha, k\}, \{A\}, \{\{0, \vartheta0\}\}\}$	Line[{1, 2}]
2	BAR	$\{\{E, \alpha, k\}, \{A\}, \{\{0, \vartheta0\}\}\}$	Line[{2, 3}]

	$\{X,Y,Z\}$	$\{u_X,u_Y,u_Z\}$	$\{\Theta_X,\Theta_Y,\Theta_Z\}$	ϑ
1	$\{0, 0, 0\}$	$\{0, 0, 0\}$	$\{0, 0, 0\}$	$\vartheta0$
2	$\{\frac{L}{2}, 0, 0\}$	$\{uX[2], 0, 0\}$	$\{0, 0, 0\}$	$\vartheta[2]$
3	$\{L, 0, 0\}$	$\{0, 0, 0\}$	$\{0, 0, 0\}$	$2\,\vartheta0$

$$\left\{uX[2] \to -\frac{1}{8}\, L\, \alpha\, \vartheta0,\ \vartheta[2] \to \frac{3\,\vartheta0}{2}\right\}$$

6.3 ELEMENT CONTRIBUTIONS

Variational expressions for the elements combine the density expressions of a model and approximations depending on the element shape and type. To derive the expression for an element:

- Start with the densities $\delta w_{\Omega}^{\text{int}}$, $\delta w_{\Omega}^{\text{ext}}$, $\delta w_{\Omega}^{\text{cpl}}$, $\delta p_{\Omega}^{\text{int}}$, and $\delta p_{\Omega}^{\text{ext}}$ of the model. If not given in the formulae collection, derive the expressions starting from the 3D versions.

- Represent the unknown functions by interpolation of the nodal displacements, rotations, and temperatures. Substitute the approximations into the density expressions.

- Integrate the densities over the domain occupied by the element to end up with $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} + \delta W^{\text{cpl}}$ and $\delta P = \delta P^{\text{int}} + \delta P^{\text{ext}}$

ELEMENT APPROXIMATION

In MEC-E8001 element approximation is a polynomial interpolant of the nodal displacements and rotations in terms of shape functions. In thermo-mechanical analysis, temperature is represented in the same manner by using nodal temperatures.

Approximation $u = \mathbf{N}^{\mathrm{T}}\mathbf{a},\ v = \mathbf{N}^{\mathrm{T}}\mathbf{a},\ \ldots.,\ \vartheta = \mathbf{N}^{\mathrm{T}}\mathbf{a}$ always of the same form!

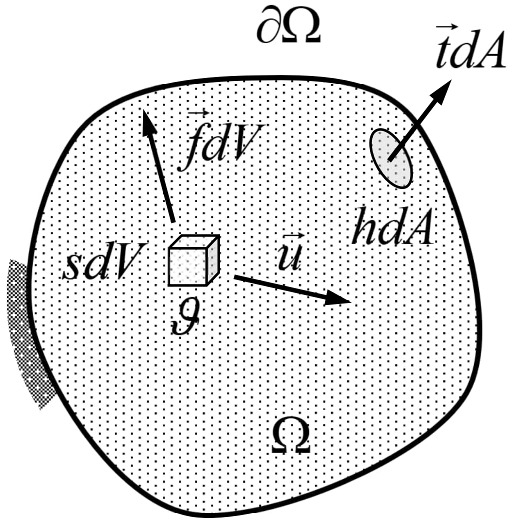
Shape functions $\mathbf{N} = \{N_1(x,y,z) \quad N_2(x,y,z) \quad \ldots \quad N_n(x,y,z)\}^{\mathrm{T}}$

Parameters $\mathbf{a} = \{\mathrm{a}_1 \quad \mathrm{a}_2 \quad \ldots \quad \mathrm{a}_n\}^{\mathrm{T}}$

Nodal parameters $\mathrm{a} \in \{u_x, u_y, u_z, \theta_x, \theta_y, \theta_z, \vartheta\}$ may be just displacement or rotation components or a mixture of them (as with the Bernoulli beam model). Nodal parameters may also represent temperature.

SOLID MODEL

The model does not contain any kinetic or kinematic assumptions. Virtual work densities of the internal and external distributed forces $\delta w_{\Omega}^{\text{int}}$ and $\delta w_{\Omega}^{\text{ext}}$ are the same as in linear displacement analysis. The additional terms are

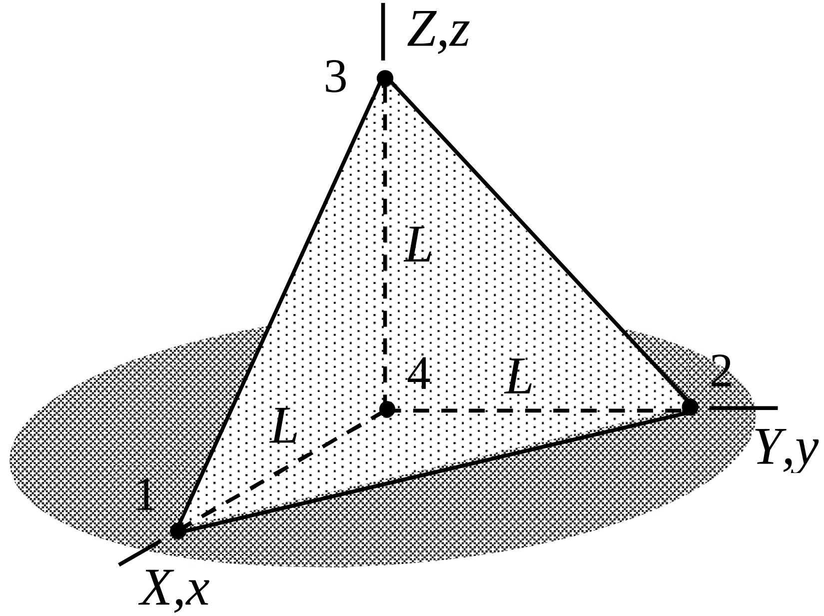


$$\delta w_{\Omega}^{\text{cpl}} = \begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta w/\partial z \end{Bmatrix}^{\text{T}} \frac{E\alpha\Delta\vartheta}{1-2\nu} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix},$$

$$\delta p_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta\vartheta/\partial x \\ \partial\delta\vartheta/\partial y \\ \partial\delta\vartheta/\partial z \end{Bmatrix}^{\text{T}} k \begin{Bmatrix} \partial\vartheta/\partial x \\ \partial\vartheta/\partial y \\ \partial\vartheta/\partial z \end{Bmatrix}, \quad \delta p_{\Omega}^{\text{ext}} = \delta\vartheta s.$$

The solution domain can be represented, e.g., by tetrahedron elements with linear interpolation of $u(x,y,z)$, $v(x,y,z)$, $w(x,y,z)$ and $\vartheta(x,y,z)$.

EXAMPLE 6.3 Consider a tetrahedron of edge length L on a horizontal floor. Determine displacement u_{Z3} when temperature is increased by constant $\Delta\vartheta$ and before that stress vanishes. Assume that $u_{X3} = u_{Y3} = 0$ and that the bottom surface is fixed. Stress vanishes at the initial geometry when $u_{Z3} = 0$. Material parameters E, $\nu = 0$, and α are constants.



Answer: $u_{Z3} = L\alpha\Delta\vartheta$

- Only the shape function $N_3 = z/L$ of node 3 is needed as the other nodes are fixed. Approximations to the displacement components are

$$u = 0,\ v = 0, \text{ and } w = \frac{z}{L}u_{Z3}, \text{ giving } \frac{\partial w}{\partial x} = \frac{\partial w}{\partial y} = 0, \text{ and } \frac{\partial w}{\partial z} = \frac{1}{L}u_{Z3}.$$

- As temperature is known, it is enough to consider the displacement problem. With the approximation, the internal and coupling densities simplify to ($\nu = 0$)

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} 0 \\ 0 \\ \delta u_{Z3}/L \end{Bmatrix}^{\mathrm{T}} \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_{Z3}/L \end{Bmatrix} = -\frac{E}{L^2}u_{Z3}\delta u_{Z3},$$

$$\delta w_{\Omega}^{\text{cpl}} = \begin{Bmatrix} 0 \\ 0 \\ \delta u_{Z3}/L \end{Bmatrix}^{\mathrm{T}} \frac{E\alpha\Delta\vartheta}{1-2\nu} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \frac{\delta u_{Z3}}{L} E\alpha\Delta\vartheta.$$

- Virtual work expressions are integrals of the densities over the volume. Here, the densities are constants, and it is enough to multiply by the volume $L^3 / 6$

$$\delta W^{\text{int}} = \int_{\Omega} \delta w_{\Omega}^{\text{int}} dV = -\delta u_{Z3} \frac{1}{6} E L u_{Z3},$$

$$\delta W^{\text{cpl}} = \int_{\Omega} \delta w_{\Omega}^{\text{cpl}} dV = \delta u_{Z3} \frac{1}{6} L^2 E \alpha \Delta \vartheta .$$

- Variational principle (here principle of virtual work) $\delta W = \delta W^{\text{int}} + \delta W^{\text{cpl}} = 0$ implies that

$$-\frac{1}{6} E L u_{Z3} + \frac{1}{6} L^2 E \alpha \Delta \vartheta = 0 \quad \Leftrightarrow \quad u_{Z3} = L \alpha \Delta \vartheta . \quad \leftarrow$$

PLATE MODEL

Virtual work densities combine the plane-stress and plate bending modes. Assuming that the material coordinate system is placed at the mid-plane, and material properties do not depend on the transverse coordinate,

$$\delta w_{\Omega}^{\mathrm{cpl}} = \begin{Bmatrix} \partial\delta u / \partial x \\ \partial\delta v / \partial y \end{Bmatrix}^{\mathrm{T}} \int \Delta\vartheta dz \frac{\alpha E}{1-\nu} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} - \begin{Bmatrix} \partial^2\delta w / \partial x^2 \\ \partial^2\delta w / \partial y^2 \end{Bmatrix}^{\mathrm{T}} \int z\Delta\vartheta dz \frac{\alpha E}{1-\nu} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix},$$

$$\delta p_{\Omega}^{\mathrm{int}} = -\begin{Bmatrix} \partial\delta\vartheta / \partial x \\ \partial\delta\vartheta / \partial y \\ \partial\delta\vartheta / \partial z \end{Bmatrix}^{\mathrm{T}} k \begin{Bmatrix} \partial\vartheta / \partial x \\ \partial\vartheta / \partial y \\ \partial\vartheta / \partial z \end{Bmatrix}, \quad \delta p_{\Omega}^{\mathrm{ext}} = \delta\vartheta s \quad \text{and} \quad \delta p_{\partial\Omega}^{\mathrm{ext}} = \delta\vartheta h.$$

Approximation to the transverse displacement depends only on the planar coordinates but temperature and its approximation may depend on all the coordinates.

- The constitutive equations of a linearly elastic isotropic material and kinetic assumption $\sigma_{zz} = 0$ give the non-zero stress components

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = [E]_\sigma \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} - \alpha \Delta \vartheta \frac{E}{1-\nu} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \text{with} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix} - z \begin{Bmatrix} \partial^2 w / \partial x^2 \\ \partial^2 w / \partial y^2 \\ 2\partial^2 w / \partial x \partial y \end{Bmatrix}.$$

- The generic expression of $\delta w_\Omega^{\text{int}}$ simplifies to a sum of thin slab, bending and interaction parts. Assuming that material properties do not depend on z, and that the origin of the material coordinate system is placed at the mid-plane, virtual work density of internal forces consists of the internal parts of the plate thin-slab and bending modes $\delta w_\Omega^{\text{int}}$ and the coupling parts for the thin-slab and bending modes (the integral is over the thickness)

$$\delta w_{\Omega}^{\mathrm{cpl}} = \begin{Bmatrix} \partial\delta u / \partial x \\ \partial\delta v / \partial y \end{Bmatrix}^{\mathrm{T}} \int \Delta\vartheta dz \frac{\alpha E}{1-\nu} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix},$$

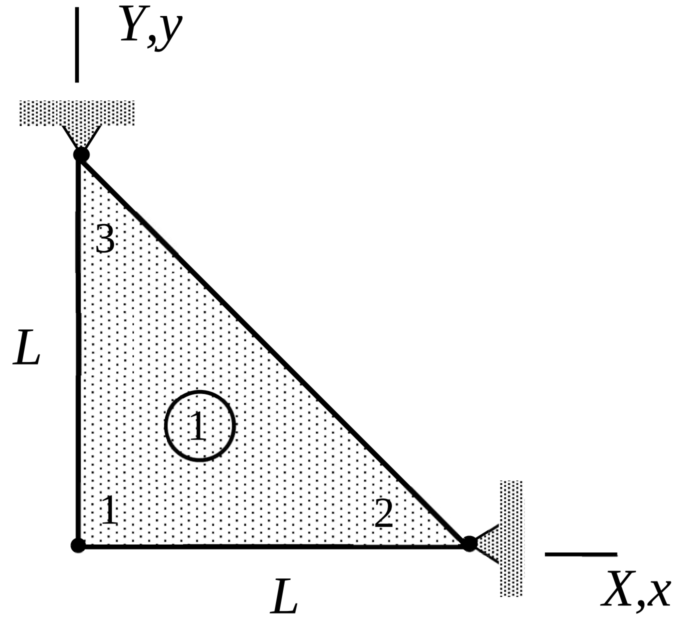
$$\delta w_{\Omega}^{\mathrm{cpl}} = -\begin{Bmatrix} \partial^2\delta w / \partial x^2 \\ \partial^2\delta w / \partial y^2 \end{Bmatrix}^{\mathrm{T}} \int z\Delta\vartheta dz \frac{\alpha E}{1-\nu} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}. \quad \Leftarrow$$

- As temperature is not assumed to be constant in the thickness direction, variational expression for the temperature calculation is based on the generic expressions. Therefore, also the approximation, e.g., of the type

$$\vartheta(x, y, z) = \mathbf{N}^{\mathrm{T}}(x, y)\mathbf{a}(z) \quad \text{where} \quad \mathbf{a}(z) = \mathbf{a}_0 + \mathbf{a}_z z$$

is used for the actual domain of the plate.

EXAMPLE 6.4 Consider the triangular thin slab shown. Determine displacements u_{X1} and u_{Y1}, when temperature is increased by constant $\Delta\vartheta$ and before that stress vanishes. Use a linear approximation and assume plane stress conditions. Thickness of the slab is t and material parameters E, ν, and α are constants.



Answer $$\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} = -\frac{1+\nu}{2} L\alpha\Delta\vartheta \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

- The non-zero displacement components are $u_{x1} = u_{X1}$ and $u_{y1} = u_{Y1}$. The linear shape functions $N_1 = (L - x - y)/L$, $N_2 = x/L$ and $N_3 = y/L$ can be deduced from the figure. Therefore, approximations are

$$u = N_1 u_{x1} = \frac{1}{L}(L - x - y)u_{X1} \;\text{ and }\; v = N_1 u_{y1} = \frac{1}{L}(L - x - y)u_{Y1} \;\Rightarrow$$

$$\frac{\partial u}{\partial x} = -\frac{u_{X1}}{L},\; \frac{\partial u}{\partial y} = -\frac{u_{X1}}{L},\; \frac{\partial v}{\partial x} = -\frac{u_{Y1}}{L} \;\text{ and }\; \frac{\partial v}{\partial y} = -\frac{u_{Y1}}{L}.$$

- Densities of internal and coupling terms simplify to

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} -\delta u_{X1} \\ -\delta u_{Y1} \\ -\delta u_{X1} - \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \frac{1}{L^2}\frac{Et}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\begin{Bmatrix} -u_{X1} \\ -u_{Y1} \\ -u_{X1} - u_{Y1} \end{Bmatrix} \Leftrightarrow$$

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \left(\frac{Et}{1-\nu^2} \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix} + \frac{Et}{2(1+\nu)} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right) \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} \frac{1}{L^2},$$

$$\delta w_{\Omega}^{\text{cpl}} = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \frac{1}{L} \frac{E\alpha t}{1-\nu} \Delta\vartheta \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

- Integration over the element gives (densities are constants)

$$\delta W^{\text{int}} = \int_{\Omega} \delta w_{\Omega}^{\text{int}} dA = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \left(\frac{Et}{2(1-\nu^2)} \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix} + \frac{Et}{4(1+\nu)} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right) \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix},$$

$$\delta W^{\text{cpl}} = \int_{\Omega} \delta w_{\Omega}^{\text{cpl}} dA = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \frac{L}{2} \frac{E\alpha t}{1-\nu} \Delta\vartheta \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

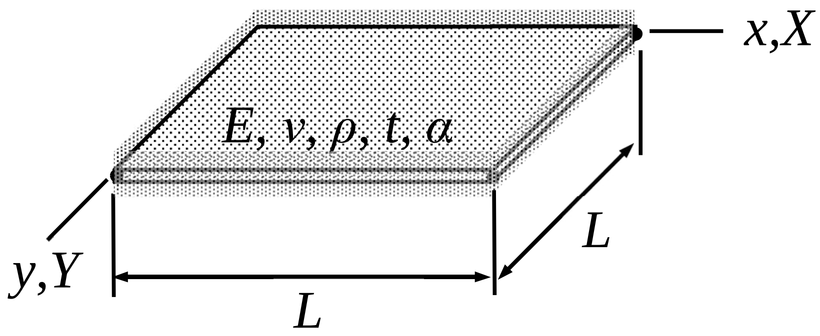
- Variation principle $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = 0 \ \forall \delta \mathbf{a}$ and fundamental lemma of variation calculus imply the equilibrium equations

$$\left(\frac{Et}{2(1-\nu^2)}\begin{bmatrix}1 & \nu\\ \nu & 1\end{bmatrix}+\frac{Et}{4(1+\nu)}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\right)\begin{Bmatrix}u_{X1}\\ u_{Y1}\end{Bmatrix}+\frac{L}{2}\frac{E\alpha t}{1-\nu}\Delta\vartheta\begin{Bmatrix}1\\ 1\end{Bmatrix}=0 \quad \Leftrightarrow$$

$$\begin{Bmatrix}u_{X1}\\ u_{Y1}\end{Bmatrix}=-\begin{bmatrix}1/(1-\nu)+1/2 & \nu/(1-\nu)+1/2\\ \nu/(1-\nu)+1/2 & 1/(1-\nu)+1/2\end{bmatrix}^{-1}\frac{1+\nu}{1-\nu}L\alpha\Delta\vartheta\begin{Bmatrix}1\\ 1\end{Bmatrix} \quad \Leftrightarrow$$

$$\begin{Bmatrix}u_{X1}\\ u_{Y1}\end{Bmatrix}=-\frac{1+\nu}{2}L\alpha\Delta\vartheta\begin{Bmatrix}1\\ 1\end{Bmatrix}.$$

EXAMPLE 6.5 Simply supported plate of the figure is assembled at constant temperature $3\vartheta^\circ$. Find the transverse displacement when the upper side temperature is $4\vartheta^\circ$ and that of the lower side $2\vartheta^\circ$. Assume that temperature in plate is linear in z. Use the polynomial approximation $w(x,y) = \mathrm{a}(xy/L^2)(1-x/L)(1-y/L)$. Problem parameters E, ν, ρ, α and t are constants.



Answer $w(x,y) = -\dfrac{30}{11}\alpha\vartheta^\circ(1+\nu)\dfrac{L^2}{t}\dfrac{xy}{L^2}(1-\dfrac{x}{L})(1-\dfrac{y}{L})$

- Assuming that the material coordinate system is chosen so that the plate bending and thin slab modes decouple, the bending mode virtual work densities of the internal and coupling parts are given by

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial^2 \delta w / \partial x^2 \\ \partial^2 \delta w / \partial y^2 \\ 2\partial^2 \delta w / \partial x \partial y \end{Bmatrix}^{\mathrm{T}} D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \partial^2 w / \partial x^2 \\ \partial^2 w / \partial y^2 \\ 2\partial^2 w / \partial x \partial y \end{Bmatrix} \text{ where } D = \frac{t^3}{12} \frac{E}{1-\nu^2},$$

$$\delta w_{\Omega}^{\text{cpl}} = -\begin{Bmatrix} \partial^2 \delta w / \partial x^2 \\ \partial^2 \delta w / \partial y^2 \end{Bmatrix}^{\mathrm{T}} \int z \Delta \vartheta dz \frac{\alpha E}{1-\nu} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

- Approximation to the transverse displacement and its derivatives are

$$w(x,y) = \mathrm{a} \frac{xy}{L^2} (1-\frac{x}{L})(1-\frac{y}{L}) \quad \Rightarrow$$

$$\frac{\partial^2 w}{\partial x^2} = -2a\frac{y}{L^3}(1-\frac{y}{L}),\ \frac{\partial^2 w}{\partial y^2} = -2a\frac{x}{L^3}(1-\frac{x}{L}),\ \frac{\partial^2 w}{\partial x \partial y} = a\frac{1}{L^2}(1-2\frac{x}{L})(1-2\frac{y}{L}).$$

- Temperature difference and its weighted integral over the thickness (integral of the coupling term)

$$\Delta\vartheta = \vartheta(z) - 3\vartheta^\circ = (\frac{1}{2}+\frac{z}{t})2\vartheta^\circ + (\frac{1}{2}-\frac{z}{t})4\vartheta^\circ - 3\vartheta^\circ = -\frac{z}{t}2\vartheta^\circ \quad \Rightarrow$$

$$\int z\Delta\vartheta dz = -\int_{-t/2}^{t/2} z\frac{z}{t}2\vartheta^\circ dz = -\frac{1}{6}\vartheta^\circ t^2 \ .$$

- When the approximation is substituted there, virtual work expressions of the internal and coupling terms simplify to

$$\delta W^{\text{int}} = \int_0^L \int_0^L \delta w_\Omega^{\text{int}} dxdy = -\delta \mathrm{a} \frac{22}{45} \frac{1}{L^2} \frac{t^3}{12} \frac{E}{1-\nu^2} \mathrm{a},$$

$$\delta W^{\text{cpl}} = \int_0^L \int_0^L \delta w_\Omega^{\text{cpl}} dxdy = -\delta \mathrm{a} \frac{1}{9} \frac{\alpha E}{1-\nu} \vartheta^\circ t^2.$$

- Virtual work expression is the sum of the internal and coupling parts

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{cpl}} = -\delta \mathrm{a} \left(\frac{22}{45} \frac{1}{L^2} \frac{t^3}{12} \frac{E}{1-\nu^2} \mathrm{a} + \frac{1}{9} \frac{\alpha E}{1-\nu} \vartheta^\circ t^2 \right).$$

- Principle of virtual work $\delta W = 0 \, \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus give

$$\mathrm{a} = -\frac{30}{11} \alpha \vartheta^\circ (1+\nu) \frac{L^2}{t} \quad \Rightarrow \quad w(x,y) = -\frac{30}{11} \alpha \vartheta^\circ (1+\nu) \frac{L^2}{t} \frac{xy}{L^2} (1-\frac{x}{L})(1-\frac{y}{L}) \, .$$

BEAM MODEL

Virtual work densities combine the bar, bending, and torsion modes. Assuming that material properties are constants, and the material coordinate system is placed so that the first and the cross moments of the cross section vanish

$$\delta w_{\Omega}^{\mathrm{cpl}} = E\alpha(\begin{Bmatrix} d\delta u / dx \\ d^2\delta v / dx^2 \\ d^2\delta w / dx^2 \end{Bmatrix}^{\mathrm{T}} \int \Delta\vartheta \begin{Bmatrix} 1 \\ -y \\ -z \end{Bmatrix} dA, \quad \delta p_{\Omega}^{\mathrm{int}} = -\begin{Bmatrix} \partial\delta\vartheta / \partial x \\ \partial\delta\vartheta / \partial y \\ \partial\delta\vartheta / \partial z \end{Bmatrix}^{\mathrm{T}} k \begin{Bmatrix} \partial\vartheta / \partial x \\ \partial\vartheta / \partial y \\ \partial\vartheta / \partial z \end{Bmatrix}, \quad \text{and}$$

$$\delta p_{\Omega}^{\mathrm{ext}} = \delta\vartheta\, s \quad \text{and} \quad \delta p_{\partial\Omega}^{\mathrm{ext}} = \delta\vartheta\, h.$$

Approximation to the transverse displacement depends only on the axial coordinate but temperature and its approximation may depend on all the coordinates in the expressions.

- The displacement components of the Bernoulli beam model are $u_x = u - (dw/dx)z - (dv/dx)y$, $u_y = v - \phi z$ and $u_z = w + \phi y$. With the kinetic assumption $\sigma_{zz} = \sigma_{yy} = 0$, stress and strain components take the forms

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{xy} \\ \sigma_{xz} \end{Bmatrix} = \begin{bmatrix} E & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \gamma_{xy} \\ \gamma_{xz} \end{Bmatrix} - E\alpha\Delta\vartheta \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \text{ where } \begin{Bmatrix} \varepsilon_{xx} \\ \gamma_{xy} \\ \gamma_{xz} \end{Bmatrix} = \begin{Bmatrix} \dfrac{du}{dx} - \dfrac{d^2w}{dx^2}z - \dfrac{d^2v}{dx^2}y \\ -z\dfrac{d\phi}{dx} \\ y\dfrac{d\phi}{dx} \end{Bmatrix}.$$

- Assuming that material properties are constants, and the material coordinate system is placed so that the first and the cross moments of the cross section vanish, the virtual work density of the coupling term simplifies to (after integration over the cross section)

$$\delta w_{\Omega}^{\mathrm{cpl}} = E\alpha(\frac{d\delta u}{dx}\int \Delta\vartheta dA - \frac{d^2\delta w}{dx^2}\int z\Delta\vartheta dA - \frac{d^2\delta v}{dx^2}\int y\Delta\vartheta dA). \quad \Leftarrow$$

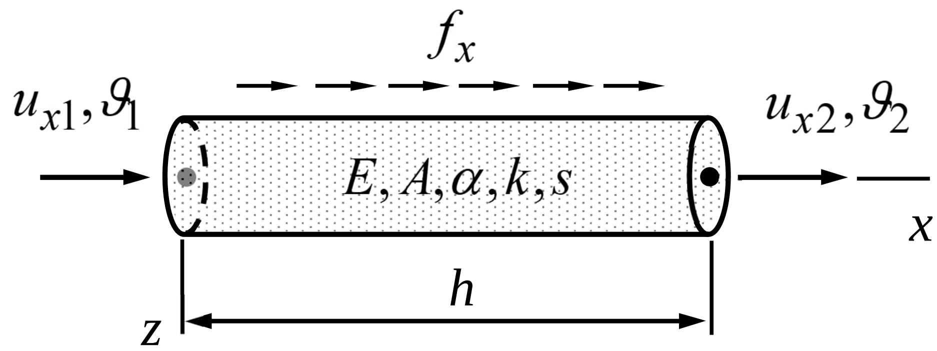
- As temperature is not assumed to be constant in the thickness direction, variational expression for the temperature calculation is based on the generic expressions. Accordingly, the approximation depends on all the coordinates. Approximation of the type

$$\vartheta(x,y,z) = \mathbf{N}^{\mathrm{T}}(x)\mathbf{a}(y,z) \quad \text{where} \quad \mathbf{a}(y,z) = \mathbf{a}_0 + \mathbf{a}_y y + \mathbf{a}_z z$$

is one of the possibilities.

BAR MODE

Assuming that $v = 0,\ w = 0,\ \phi = 0$ and a linear interpolation to the axial displacement $u(x)$ and temperature $\vartheta(x)$



$$\delta P^{\text{int}} = -\begin{Bmatrix} \delta\vartheta_1 \\ \delta\vartheta_2 \end{Bmatrix}^{\mathrm{T}} \frac{kA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \vartheta_1 \\ \vartheta_2 \end{Bmatrix},$$

$$\delta W^{\text{cpl}} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{\alpha EA}{2} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \Delta\vartheta_1 \\ \Delta\vartheta_2 \end{Bmatrix},$$

$$\delta P^{\text{ext}} = \begin{Bmatrix} \delta\vartheta_1 \\ \delta\vartheta_2 \end{Bmatrix}^{\mathrm{T}} \frac{Ash}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

Heat flux through the end-planes is treated by point elements in the same manner as traction on the end-plates by point forces and moments.

- Bar model assumes that $v(x) = w(x) = 0$ or that coupling between the bar and bending modes vanish. After integration over the cross section, the generic expressions for the 3D case simplify to

$$\delta w_\Omega^{\text{int}} = -\frac{d\delta u}{dx} EA \frac{du}{dx}\ , \ \delta w_\Omega^{\text{ext}} = \delta u f_x,\ \ \delta w_\Omega^{\text{cpl}} = \frac{d\delta u}{dx} EA\alpha\Delta\vartheta,$$

$$\delta p_\Omega^{\text{int}} = -\frac{d\delta\vartheta}{dx} kA \frac{d\vartheta}{dx},\ \ \delta p_\Omega^{\text{ext}} = \delta\vartheta s,$$

in which cross-sectional area A, Young's modulus E, external force per unit length f_x, thermal conductivity k, coefficient of thermal expansion α, and heat production rate per unit length s may depend on x.

- Linear interpolants to the axial displacement and temperature are

$$u=\frac{1}{h}\{h-x \quad x\}\begin{Bmatrix}u_{x1}\\u_{x2}\end{Bmatrix},\ \vartheta=\frac{1}{h}\{h-x \quad x\}\begin{Bmatrix}\vartheta_1\\\vartheta_2\end{Bmatrix},\text{ and }\ \Delta\vartheta=\frac{1}{h}\{h-x \quad x\}\begin{Bmatrix}\Delta\vartheta_1\\\Delta\vartheta_2\end{Bmatrix}.$$

- After substituting the approximations into the densities and integration over the domain occupied by the element with the assumedly constant material properties

$$\delta W^{\text{cpl}}=\begin{Bmatrix}\delta u_{x1}\\\delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\frac{\alpha EA}{2}\begin{bmatrix}-1 & -1\\1 & 1\end{bmatrix}\begin{Bmatrix}\Delta\vartheta_1\\\Delta\vartheta_2\end{Bmatrix},$$

$$\delta P^{\text{int}}=-\begin{Bmatrix}\delta\vartheta_1\\\delta\vartheta_2\end{Bmatrix}^{\mathrm{T}}\frac{kA}{h}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{Bmatrix}\vartheta_1\\\vartheta_2\end{Bmatrix},\ \delta P^{\text{ext}}=\begin{Bmatrix}\delta\vartheta_1\\\delta\vartheta_2\end{Bmatrix}^{\mathrm{T}}\frac{sAh}{2}\begin{Bmatrix}1\\1\end{Bmatrix}. \quad \Leftarrow$$

Assuming a cubic interpolation to $w(x)$ and $v(x)$ and linear interpolation to the "coefficients" of the representation $\Delta\vartheta(x,z)=\Delta\vartheta_0(x)+\Delta\vartheta_y(x)y+\Delta\vartheta_z(x)z$, the coupling term

$$\delta W^{\mathrm{cpl}}=-\begin{Bmatrix}\delta u_{y1}\\ \delta\theta_{z1}\\ \delta u_{y2}\\ \delta\theta_{z2}\end{Bmatrix}^{\mathrm{T}}\frac{EI_{zz}\alpha}{h^2}\begin{bmatrix}-\frac{1}{h} & \frac{1}{h}\\ -1 & 0\\ \frac{1}{h} & -\frac{1}{h}\\ 0 & 1\end{bmatrix}\begin{Bmatrix}\Delta\vartheta_{y1}\\ \Delta\vartheta_{y2}\end{Bmatrix}-\begin{Bmatrix}\delta u_{z1}\\ \delta\theta_{y1}\\ \delta u_{z2}\\ \delta\theta_{y2}\end{Bmatrix}^{\mathrm{T}}\frac{EI_{yy}\alpha}{h^2}\begin{bmatrix}-\frac{1}{h} & \frac{1}{h}\\ 1 & 0\\ \frac{1}{h} & -\frac{1}{h}\\ 0 & -1\end{bmatrix}\begin{Bmatrix}\Delta\vartheta_{z1}\\ \Delta\vartheta_{z2}\end{Bmatrix}$$

Under the assumptions used, the displacement-temperature coupling of the bar and the bending modes can be treated by adding a coupling term for each mode.

- Cubic interpolants to the transverse displacements and the "Taylor series" type linear approximation to the temperature difference are

$$v = \begin{Bmatrix} (1-\xi)^2(1+2\xi) \\ h(1-\xi)^2\xi \\ (3-2\xi)\xi^2 \\ h\xi^2(\xi-1) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \\ u_{y2} \\ \theta_{z2} \end{Bmatrix}, \quad w = \begin{Bmatrix} (1-\xi)^2(1+2\xi) \\ -h(1-\xi)^2\xi \\ (3-2\xi)\xi^2 \\ -h\xi^2(\xi-1) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix},$$

$$\Delta\vartheta = \begin{Bmatrix} 1-\xi \\ \xi \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \Delta\vartheta_1 \\ \Delta\vartheta_2 \end{Bmatrix} + y \begin{Bmatrix} 1-\xi \\ \xi \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \Delta\vartheta_{y1} \\ \Delta\vartheta_{y2} \end{Bmatrix} + z \begin{Bmatrix} 1-\xi \\ \xi \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \Delta\vartheta_{z1} \\ \Delta\vartheta_{z2} \end{Bmatrix} \text{ where } \xi = \frac{x}{h}.$$

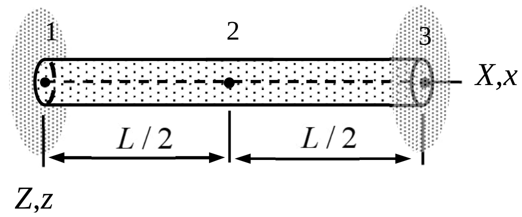
- When the approximation is substituted there, integration of the density over the cross sections gives the coupling expression (notice that the first term of the temperature approximation contributes to the bar mode only).

MEC-E8001 Finite Element Analysis, week 7/2023

1. The variational densities (correspond to virtual work densities of a displacement problem) of a heat conduction problem in a bar are given by $\delta p_\Omega^{\text{int}} = -(d\delta\vartheta / dx)kA(d\vartheta / dx)$ and $\delta p_\Omega^{\text{ext}} = \delta\vartheta s$ in which ϑ is the temperature, A is the cross-sectional area, k is the thermal conductivity, and s is the rate of heat production per unit length. Determine the element contributions δP^{int} and δP^{ext} if the approximation to temperature is linear, length of the element is h, and the given functions of the density expression are constants.

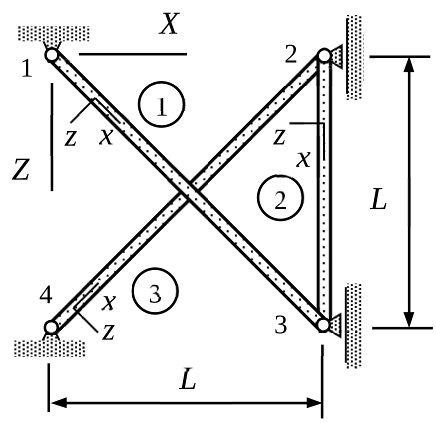
Answer $\delta P^{\text{int}} = -\begin{Bmatrix} \delta\vartheta_1 \\ \delta\vartheta_2 \end{Bmatrix}^{\text{T}} \dfrac{kA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \vartheta_1 \\ \vartheta_2 \end{Bmatrix}$, $\delta P^{\text{ext}} = \begin{Bmatrix} \delta\vartheta_1 \\ \delta\vartheta_2 \end{Bmatrix}^{\text{T}} \dfrac{sh}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$.

2. Determine the stationary displacement u_{X2} and temperature ϑ_2 at node 2, when the temperature of the left and right ends are $\vartheta°$ and $2\vartheta°$, respectively. Use just one three node quadratic element. Stress is zero initially when the temperature in the wall and bar is $\vartheta°$. Problem parameters E, A, k, and α are constants.



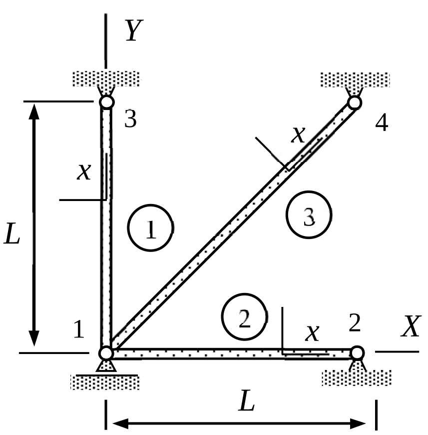
Answer $u_{X2} = -\dfrac{1}{8} L\alpha\vartheta°$, $\vartheta_2 = \dfrac{3}{2}\vartheta°$

3. Determine the static displacements $u_{Z2} = -u_{Z3}$ of nodes 2 and 3 due to the temperature increase $\Delta\vartheta$ at nodes 2 and 3 (actually in the wall). The material constants are E and α. The cross-sectional area of bar 1 and 3 is A and that of bar 2 is $\sqrt{2}A$. The initial temperature is $\vartheta°$.



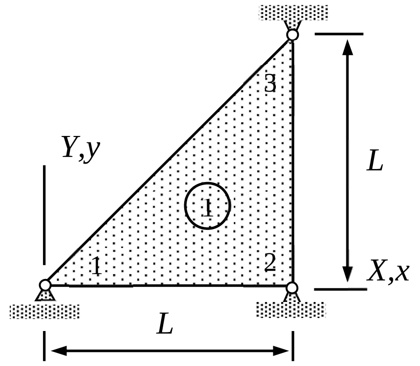
Answer $u_{Z2} = -u_{Z3} = -\dfrac{5}{9} L\alpha\Delta\vartheta$

4. The truss shown consists of bars having the same cross-sectional area A, Young's modulus E, coefficient of thermal expansion α, and thermal conductivity k. The truss is stress-free when the initial temperature of all the nodes is $\vartheta°$. Determine the stationary displacement u_{X1} of node 1, when the temperature of node 2 is changed to $2\vartheta°$ and nodes 1, 3 and 4 are in temperature $\vartheta°$.



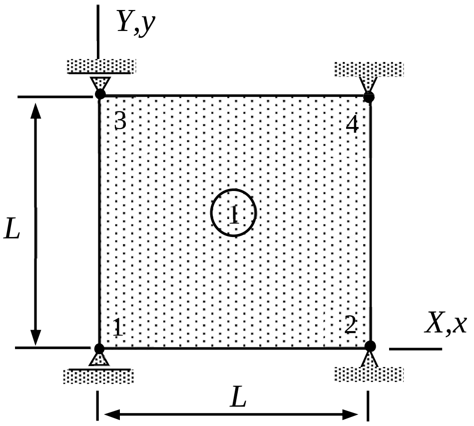
Answer $u_{X1} = -\dfrac{2}{4+\sqrt{2}} L\alpha\vartheta°$

5. A thin triangular slab (plane stress conditions) is allowed to move horizontally at node 1 and nodes 2 and 3 are fixed. Stress is zero when temperature (assumed constant) is ϑ°. Determine the non-zero displacement component u_{X1}, if the temperature of slab is increased to $2\vartheta^\circ$.



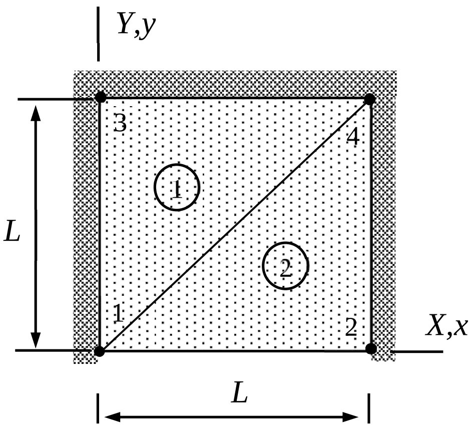
Answer $u_{X1} = -(1+\nu)\alpha L\vartheta^\circ$

6. Nodes 1 and 3 of a thin rectangular slab (assume plane stress conditions) shown are allowed to move horizontally and nodes 2 and 4 are fixed. Stress is zero when temperature is ϑ°. Determine the displacement components $u_{X1} = u_{X3}$ if the temperature of slab is increased to $2\vartheta^\circ$. Also, determine the strain and stress in the slab. Material parameters and thickness are E, ν, α and t, respectively.



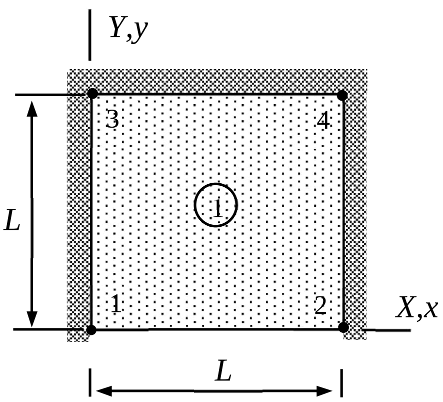
Answer $u_{X1} = -L\alpha\vartheta^\circ(1+\nu)$, $\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \alpha\vartheta^\circ(1+\nu)\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$, $\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = -E\alpha\vartheta^\circ\begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$

7. Determine the stationary temperature distribution in a thin slab shown. Edge 1-2 is at constant temperature ϑ° and heat flux through the other edges vanishes. Use a two-triangle mesh with ϑ_3 and $\vartheta_4 = \vartheta_3$ as the unknown node temperatures and consider $\vartheta_1 = \vartheta_2 = \vartheta^\circ$ as known. Thickness t, thermal conductivity k, and heat production rate per unit area s are constants.



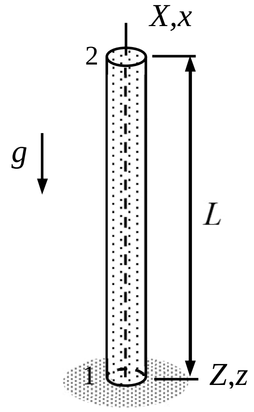
Answer $\vartheta_3 = \vartheta^\circ + \frac{1}{2}\frac{sL^2}{tk}$

8. Determine the stationary temperature distribution in a thin slab shown. Edge 1-2 is at constant temperature ϑ° and heat flux through the other edges vanishes. Use a rectangle element with bilinear approximation and consider $\vartheta_1 = \vartheta_2 = \vartheta^\circ$ as known and $\vartheta_4 = \vartheta_3$ as the unknown nodal temperatures. Thickness t, thermal conductivity k, and heat production rate per unit area s are constants.



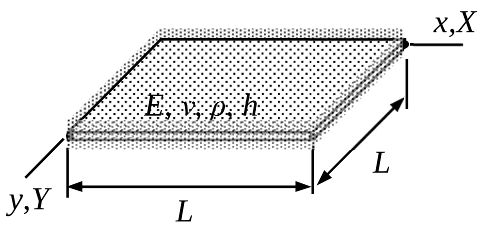
Answer $\vartheta_3 = \vartheta^\circ + \frac{1}{2}\frac{sL^2}{tk}$

9. At the initial temperature $\vartheta°$ and without external forces, the length of the bar shown is L. Calculate the displacement of node 2 due to the combined effect of gravity and change of temperature with the nodal values $\vartheta_1 = 2\vartheta°$ and $\vartheta_2 = \vartheta°$. Cross sectional area A, coefficient of thermal expansion α, and density ρ are considered as constants. Use linear interpolation to displacement and temperature and start with the virtual work density expressions.



Answer $u_{X2} = \dfrac{\alpha}{2} L\vartheta° - \dfrac{\rho g}{2E} L^2$

10. The simply supported plate shown is assembled at constant temperature $3\vartheta°$. Find the transverse displacement when the upper side temperature is $4\vartheta°$ and that of the lower side $2\vartheta°$. Assume that temperature in plate is linear in z and does not depend on x or y. Use $w(x,y) = a\sin(\pi x/L)\sin(\pi y/L)$ as the approximation. Problem parameters E, ν, ρ, α and t are constants. Integrals of sin and cos functions satisfy



$$\int_0^L \sin^2(\pi \frac{x}{L})dx = \int_0^L \cos^2(\pi \frac{x}{L})dx = \frac{L}{2} \text{ and } \int_0^L \sin(\pi \frac{x}{L})dx = \frac{2L}{\pi}.$$

Answer $w(x,y) = -\dfrac{16}{\pi^4} \dfrac{\alpha\vartheta° L^2}{t}(1+\nu)\sin(\pi \dfrac{x}{L})\sin(\pi \dfrac{y}{L})$

The variational densities (correspond to virtual work densities of a displacement problem) of a heat conduction problem in a bar are given by $\delta p_{\Omega}^{\text{int}} = -(d\delta\vartheta / dx)kA(d\vartheta / dx)$ and $\delta p_{\Omega}^{\text{ext}} = \delta\vartheta s$ in which ϑ is the temperature, A is the cross-sectional area, k is the thermal conductivity, and s is the rate of heat production per unit length. Determine the element contributions δP^{int} and δP^{ext} if the approximation to temperature is linear, length of the element is h, and the given functions of the density expression are constants.

Solution

In a pure heat conduction problem, density expressions of the bar model are given by

$$\delta p_{\Omega}^{\text{int}} = -\frac{d\delta\vartheta}{dx} kA \frac{d\vartheta}{dx} \text{ and } \delta p_{\Omega}^{\text{ext}} = \delta\vartheta s$$

in which ϑ is the temperature, k the thermal conductivity, and s the rate of heat production (per unit length). Although the physical meanings of the quantities differ from those of the displacement problem, finite element method works in the same manner. In particularly, the element contributions are derived in the same manner.

Assuming an element of size h and nodal values ϑ_1 and ϑ_2, the linear approximation to temperature, its variation, and their derivatives become

$$\vartheta = \frac{1}{h}\begin{Bmatrix} h-x \\ x \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vartheta_1 \\ \vartheta_2 \end{Bmatrix} \quad \Rightarrow \quad \frac{d\vartheta}{dx} = \frac{1}{h}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vartheta_1 \\ \vartheta_2 \end{Bmatrix} \text{ and}$$

$$\delta\vartheta = \frac{1}{h}\begin{Bmatrix} h-x \\ x \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \delta\vartheta_1 \\ \delta\vartheta_2 \end{Bmatrix} = \begin{Bmatrix} \delta\vartheta_1 \\ \delta\vartheta_2 \end{Bmatrix}^{\mathrm{T}} \frac{1}{h}\begin{Bmatrix} h-x \\ x \end{Bmatrix} \quad \Rightarrow \quad \frac{d\delta\vartheta}{dx} = \frac{1}{h}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \delta\vartheta_1 \\ \delta\vartheta_2 \end{Bmatrix} = \begin{Bmatrix} \delta\vartheta_1 \\ \delta\vartheta_2 \end{Bmatrix}^{\mathrm{T}} \frac{1}{h}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}.$$

When the approximations are substituted there, the variational density expressions take the forms

$$\delta p_{\Omega}^{\text{int}} = -\begin{Bmatrix} \delta\vartheta_1 \\ \delta\vartheta_2 \end{Bmatrix}^{\mathrm{T}} \frac{1}{h}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix} kA \frac{1}{h}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vartheta_1 \\ \vartheta_2 \end{Bmatrix} = -\begin{Bmatrix} \delta\vartheta_1 \\ \delta\vartheta_2 \end{Bmatrix}^{\mathrm{T}} \frac{kA}{h^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \vartheta_1 \\ \vartheta_2 \end{Bmatrix} \text{ and}$$

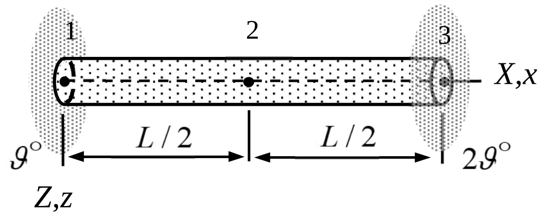
$$\delta p_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta\vartheta_1 \\ \delta\vartheta_2 \end{Bmatrix}^{\mathrm{T}} \frac{1}{h}\begin{Bmatrix} h-x \\ x \end{Bmatrix} s.$$

Element contributions are obtained as integrals over the domain occupied by the element

$$\delta P^{\text{int}} = \int_0^h \delta p_{\Omega}^{\text{int}} dx = -\begin{Bmatrix} \delta\vartheta_1 \\ \delta\vartheta_2 \end{Bmatrix}^{\mathrm{T}} \frac{kA}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \vartheta_1 \\ \vartheta_2 \end{Bmatrix}, \leftarrow$$

$$\delta P^{\text{ext}} = \int_0^h \delta p_{\Omega}^{\text{ext}} dx = \begin{Bmatrix} \delta\vartheta_1 \\ \delta\vartheta_2 \end{Bmatrix}^{\mathrm{T}} \frac{sh}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}. \leftarrow$$

Determine the stationary displacement u_{X2} and temperature ϑ_2 at node 2, when the temperature of the left and right ends are ϑ° and $2\vartheta^\circ$, respectively. Use just one three node quadratic element. Stress is zero initially when the temperature in the wall and bar is ϑ°. Problem parameters E, A, k, and α are constants.



Solution

In a temperature dependent case, variational density expressions of the bar model are

$$\delta w_\Omega^{\rm int} = -\frac{d\delta u}{dx} EA \frac{du}{dx}, \quad \delta w_\Omega^{\rm cpl} = \frac{d\delta u}{dx} EA\alpha\Delta\vartheta, \quad \text{and} \quad \delta p_\Omega^{\rm int} = -\frac{d\delta\vartheta}{dx} kA \frac{d\vartheta}{dx}.$$

In the second expression, $\Delta\vartheta = \vartheta - \vartheta^\circ$ is the temperature difference between the deformed and initial geometries (same material point). Variational expression is of the form $\delta W + \tau\delta P$ in which τ is an arbitrary but dimensionally correct multiplier (expression should be dimensionally homogeneous). The coupling in the stationary thermo-mechanical problem is one-sided so that it is possible to solve for the temperature first.

Approximation with the three-node element is quadratic. The shape functions can be deduced from the figure $N_1 = (1-\xi)(1-2\xi)$, $N_2 = 4(1-\xi)\xi$ and $N_3 = \xi(2\xi-1)$ in which $\xi = x/L$. The non-zero nodal displacements and temperatures are $u_{x2} = u_{X2}$ and ϑ_2 (material and structural coordinate systems coincide here). Therefore

$$u = \begin{Bmatrix} (1-\xi)(1-2\xi) \\ 4(1-\xi)\xi \\ \xi(2\xi-1) \end{Bmatrix}^{\rm T} \begin{Bmatrix} 0 \\ u_{x2} \\ 0 \end{Bmatrix} = 4(1-\frac{x}{L})\frac{x}{L}u_{X2} \quad \Rightarrow \quad \frac{du}{dx} = 4\frac{1}{L}(1-2\frac{x}{L})u_{X2},$$

$$\vartheta = \begin{Bmatrix} (1-\xi)(1-2\xi) \\ 4(1-\xi)\xi \\ \xi(2\xi-1) \end{Bmatrix}^{\rm T} \begin{Bmatrix} \vartheta^\circ \\ \vartheta_2 \\ 2\vartheta^\circ \end{Bmatrix} = [1-5\frac{x}{L}+6(\frac{x}{L})^2]\vartheta^\circ + 4\frac{x}{L}(1-\frac{x}{L})\vartheta_2 \quad \Rightarrow$$

$$\frac{d\vartheta}{dx} = \frac{1}{L}(-5+12\frac{x}{L})\vartheta^\circ + 4\frac{1}{L}(1-2\frac{x}{L})\vartheta_2.$$

Temperature difference between the deformed and initial geometries is

$$\Delta\vartheta = \vartheta - \vartheta^\circ = \frac{x}{L}(6\frac{x}{L}-5)\vartheta^\circ + 4\frac{x}{L}(1-\frac{x}{L})\vartheta_2.$$

When the approximations are substituted here, density expressions $\delta w_\Omega = \delta w_\Omega^{\rm int} + \delta w_\Omega^{\rm cpl}$ and $\delta p_\Omega = \delta p_\Omega^{\rm int}$ simplify to

$$\delta w_\Omega = -\delta u_{X2}\frac{EA}{L^2}[4(1-2\frac{x}{L})]^2 u_{X2} + 4(1-2\frac{x}{L})\delta u_{X2}\frac{EA}{L}\alpha[\frac{x}{L}(6\frac{x}{L}-5)\vartheta^\circ + 4\frac{x}{L}(1-\frac{x}{L})\vartheta_2],$$

$$\delta p_{\Omega} = -\delta\vartheta_2 kA[4\frac{1}{L}(1-2\frac{x}{L})]^2\vartheta_2 .$$

Element contributions are integrals of the densities over the element domain

$$\delta W = \int_0^L \delta w_{\Omega} dx = -\delta u_{X2}(\frac{16}{3}\frac{AE}{L}u_{X2} + \frac{2}{3}AE\alpha\vartheta^{\circ}) ,$$

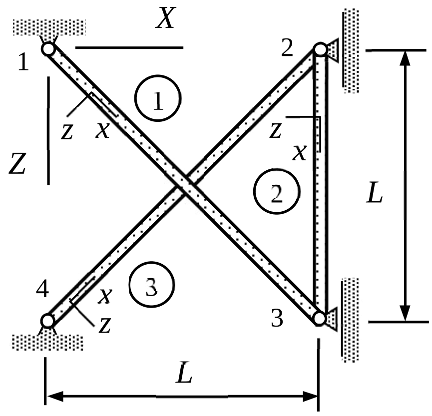
$$\delta P = \int_0^L \delta p_{\Omega} dx = -\delta\vartheta_2(\frac{16}{3}\frac{Ak}{L}\vartheta_2 - 8\frac{Ak}{L}\vartheta^{\circ}) .$$

Variation principle and the fundamental lemma of variation calculus give the equations

$$\frac{16}{3}\frac{AE}{L}u_{X2} + \frac{2}{3}AE\alpha\vartheta^{\circ} = 0 \quad \text{and} \quad \frac{16}{3}\frac{Ak}{L}\vartheta_2 - 8\frac{Ak}{L}\vartheta^{\circ} = 0 \quad \Leftrightarrow$$

$$u_{X2} = -\frac{1}{8}L\alpha\vartheta^{\circ} \quad \text{and} \quad \vartheta_2 = \frac{3}{2}\vartheta^{\circ} .$$

Determine the static displacements $u_{Z2} = -u_{Z3}$ of nodes 2 and 3 due to the temperature increase $\Delta\vartheta$ at nodes 2 and 3 (actually in the wall). The material constants are E and α. The cross-sectional area of bar 1 and 3 is A and that of bar 2 is $\sqrt{2}A$. The initial temperature is ϑ°.



Solution

As temperature is known and the external distributed force vanishes, only the virtual work expressions of the internal and coupling parts

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{EA}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} \text{ and } \delta W^{\text{cpl}} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x1} \end{Bmatrix}^{\mathrm{T}} \frac{\alpha EA}{2}\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} \Delta\vartheta_1 \\ \Delta\vartheta_2 \end{Bmatrix}$$

are needed in the calculations. Term $\Delta\vartheta = \vartheta - \vartheta^\circ$ is the difference between temperature at the deformed and initial geometries.

The nodal displacements and temperatures of bar 1 $u_{x1} = 0$, $u_{x3} = u_{Z3}/\sqrt{2}$, $\Delta\vartheta_1 = \vartheta^\circ - \vartheta^\circ = 0$, and $\Delta\vartheta_3 = \Delta\vartheta$ give (notice that the variation of a given function is always zero)

$$\delta W^1 = -\begin{Bmatrix} 0 \\ \delta u_{Z3}/\sqrt{2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{\sqrt{2}L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_{Z3}/\sqrt{2} \end{Bmatrix} - \frac{\alpha EA}{2}\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ \Delta\vartheta \end{Bmatrix}\right) \quad \Leftrightarrow$$

$$\delta W^1 = -\delta u_{Z3}\left(\frac{EA}{2\sqrt{2}L}u_{Z3} - \frac{\alpha EA}{2\sqrt{2}}\Delta\vartheta\right).$$

The nodal displacements and temperatures of bar 2 $u_{x2} = u_{Z2} = -u_{Z3}$, $u_{x3} = u_{Z3}$, $\Delta\vartheta_2 = \Delta\vartheta$, and $\Delta\vartheta_3 = \Delta\vartheta$ give

$$\delta W^2 = -\begin{Bmatrix} -\delta u_{Z3} \\ \delta u_{Z3} \end{Bmatrix}^{\mathrm{T}} \left(\frac{E\sqrt{2}A}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} -u_{Z3} \\ u_{Z3} \end{Bmatrix} - \frac{\alpha E\sqrt{2}A}{2}\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} \Delta\vartheta \\ \Delta\vartheta \end{Bmatrix}\right) \quad \Leftrightarrow$$

$$\delta W^2 = -\delta u_{Z3}\left(4\sqrt{2}\frac{EA}{L}u_{Z3} - 4\sqrt{2}\frac{\alpha EA}{2}\Delta\vartheta\right).$$

The nodal displacements and temperatures of bar 3 $u_{x4} = 0$, $u_{x2} = -u_{Z2}/\sqrt{2} = u_{Z3}/\sqrt{2}$, $\Delta\vartheta_1 = \vartheta^\circ - \vartheta^\circ = 0$, and $\Delta\vartheta_3 = \Delta\vartheta$ give

$$\delta W^3 = -\begin{Bmatrix} 0 \\ \delta u_{Z3}/\sqrt{2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{\sqrt{2}L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_{Z3}/\sqrt{2} \end{Bmatrix} - \frac{\alpha EA}{2}\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ \Delta\vartheta \end{Bmatrix}\right) \quad \Leftrightarrow$$

$$\delta W^3 = -\delta u_{Z3}\left(\frac{EA}{2\sqrt{2}L}u_{Z3} - \frac{\alpha EA}{2\sqrt{2}}\Delta\vartheta\right).$$

Virtual work expression of the structure is the sum of element contributions

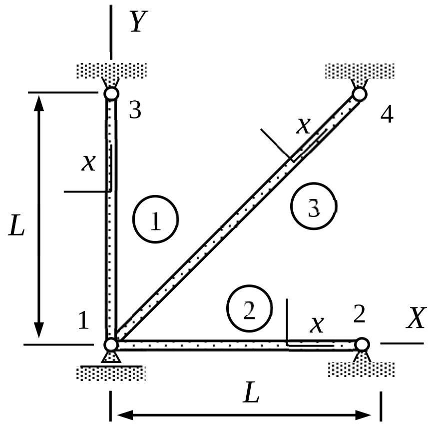
$$\delta W = -\delta u_{Z3} 2(\frac{EA}{2\sqrt{2}L} u_{Z3} - \frac{\alpha EA}{2\sqrt{2}} \Delta\vartheta) - \delta u_{Z3}(4\sqrt{2}\frac{EA}{L} u_{Z3} - 4\sqrt{2}\frac{\alpha EA}{2}\Delta\vartheta) \quad \Leftrightarrow$$

$$\delta W = -\delta u_{Z3}(9\frac{EA}{\sqrt{2}L} u_{Z3} - 5\frac{\alpha EA}{\sqrt{2}}\Delta\vartheta) .$$

Principle of virtual work and the fundamental lemma of variation calculus imply

$$\frac{9}{\sqrt{2}L} EAu_{Z3} - \frac{5}{\sqrt{2}} EA\alpha\Delta\vartheta = 0 \quad \Leftrightarrow \quad u_{Z3} = \frac{5}{9}\alpha L\Delta\vartheta \ .$$

The truss shown consists of bars having the same cross-sectional area A, Young's modulus E, coefficient of thermal expansion α, and thermal conductivity k. The truss is stress-free when the initial temperature of all the joints is ϑ°. Determine the stationary displacement u_{X1} of node 1, when the temperature of node 2 is changed to $2\vartheta^\circ$ and nodes 1, 3 and 4 are in temperature ϑ°.



Solution

Let us start with the virtual work density although also the virtual work expressions are available in the formulae collection. As temperature is known and external distributed force vanishes, the virtual work density simplifies to

$$\delta w_\Omega = \delta w_\Omega^{\text{int}} + \delta w_\Omega^{\text{cpl}} = -\frac{d\delta u}{dx} EA(\frac{du}{dx} - \alpha\Delta\vartheta) .$$

The nodal displacements and temperatures of bar 1 are $u_{x1} = u_{x3} = 0$, and $\Delta\vartheta_1 = \Delta\vartheta_3 = 0$. Using linear approximations to the axial displacement and temperature

$$u = 0 \text{ and } \Delta\vartheta = 0 \quad \Rightarrow \quad \delta w_\Omega = -\frac{d\delta u}{dx} EA(\frac{du}{dx} - \alpha\Delta\vartheta) = 0 \quad \Rightarrow \quad \delta W^1 = \int_0^L \delta w_\Omega dx = 0 .$$

The nodal displacements and temperatures of bar 2 are $u_{x1} = u_{X1}$, $u_{x2} = 0$, $\Delta\vartheta_1 = 0$, and $\Delta\vartheta_2 = 2\vartheta^\circ - \vartheta^\circ = \vartheta^\circ$. With the linear approximations to axial displacement and temperature

$$u = (1 - \frac{x}{L})u_{X1} \text{ and } \Delta\vartheta = \frac{x}{L}\vartheta^\circ \quad \Rightarrow \quad \delta w_\Omega = -(-\frac{\delta u_{X1}}{L})EA(-\frac{u_{X1}}{L} - \alpha\frac{x}{L}\vartheta^\circ) \quad \Rightarrow$$

$$\delta W^2 = \int_0^L \delta w_\Omega dx = -\delta u_{X1} EA(\frac{u_{X1}}{L} + \alpha\frac{L}{2}\vartheta^\circ) .$$

The nodal displacements and temperatures of bar 3 are $u_{x1} = u_{X1}/\sqrt{2}$, $u_{x4} = 0$, $\Delta\vartheta_1 = 0$, and $\Delta\vartheta_4 = 0$. With the linear approximations to axial displacement and temperature

$$u = (1 - \frac{x}{\sqrt{2}L})\frac{u_{X1}}{\sqrt{2}} \text{ and } \Delta\vartheta = 0 \quad \Rightarrow \quad \delta w_\Omega = -(-\frac{\delta u_{X1}}{2L})EA(-\frac{u_{X1}}{2L}) \quad \Rightarrow$$

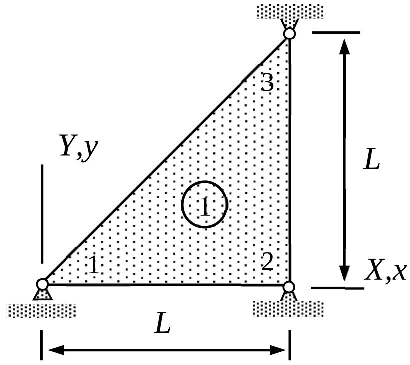
$$\delta W^3 = \int_0^{\sqrt{2}L} \delta w_\Omega dx = -\delta u_{X1} \frac{EA}{2\sqrt{2}L} u_{X1} .$$

Virtual work expression of a structure is the sum of element contributions

$$\delta W = \delta W^1 + \delta W^2 + \delta W^3 = -\delta u_{X1} EA[(1 + \frac{1}{2\sqrt{2}})\frac{u_{X1}}{L} + \alpha\vartheta^\circ\frac{1}{2}] .$$

Principle of virtual work $\delta W = 0 \; \forall \delta \mathrm{a}$ and the fundamental lemma of variation calculus imply

$$(1+\frac{1}{2\sqrt{2}})\frac{u_{X1}}{L}+\alpha\vartheta^{\circ}\frac{1}{2}=0 \quad \Leftrightarrow \quad u_{X1}=-\alpha L\vartheta^{\circ}\frac{2}{4+\sqrt{2}}.$$



A thin triangular slab (plane stress conditions) is allowed to move horizontally at node 1 and nodes 2 and 3 are fixed. Stress is zero when temperature (assumed constant) is ϑ°. Determine the non-zero displacement component u_{X1}, if the temperature of slab is increased to $2\vartheta^{\circ}$.

Solution

As temperature is known and the external distributed force vanishes, the virtual work densities needed here are (formulae collection)

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta u/\partial y + \partial\delta v/\partial x \end{Bmatrix}^{\mathrm{T}} t[E]_{\sigma} \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix}, \quad \delta w_{\Omega}^{\text{cpl}} = \begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial x \end{Bmatrix}^{\mathrm{T}} \frac{E\alpha t}{1-\nu}\Delta\vartheta \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

in which $\Delta\vartheta = \vartheta - \vartheta^{\circ}$ is the difference between temperature at the deformed and initial and deformed geometries. At the initial geometry stress is assumed to vanish.

Approximation is the first thing to be considered. Linear shape functions can be deduced from the figure

$$N_1 = 1 - \frac{x}{L}, \quad N_3 = \frac{y}{L}, \quad \text{and} \quad N_2 = 1 - N_1 - N_3 = \frac{x-y}{L}.$$

Approximations to the displacement components and temperature difference are

$$u = (1 - \frac{x}{L})u_{X1}, \quad v = 0, \quad \text{and} \quad \Delta\vartheta = \vartheta^{\circ}.$$

When the approximations are substituted there, virtual work densities take the forms

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} -\delta u_{X1}/L \\ 0 \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{Et}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} -u_{X1}/L \\ 0 \\ 0 \end{Bmatrix} = -\delta u_{X1} \frac{1}{L^2} \frac{Et}{1-\nu^2} u_{X1},$$

$$\delta w_{\Omega}^{\text{cpl}} = \begin{Bmatrix} -\delta u_{X1}/L \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{Et\alpha}{1-\nu}\vartheta^{\circ} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = -\delta u_{X1} \frac{1}{L} \frac{Et\alpha}{1-\nu}\vartheta^{\circ} \qquad \Rightarrow$$

$$\delta w_{\Omega} = \delta w_{\Omega}^{\text{int}} + \delta w_{\Omega}^{\text{cpl}} = -\frac{\delta u_{X1}}{L} \frac{Et}{1-\nu^2} \frac{u_{X1}}{L} - \frac{\delta u_{X1}}{L} \frac{Et}{1-\nu} \alpha\vartheta^{\circ}.$$

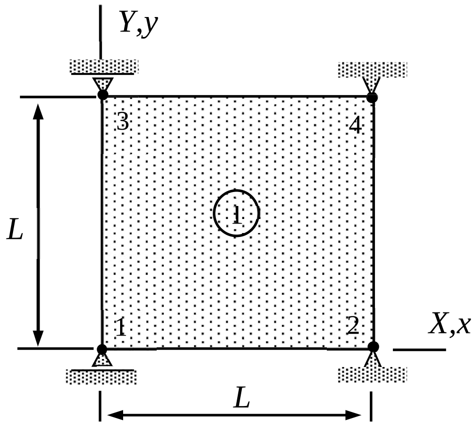
Virtual work expression is the integral of the density over the domain occupied by the element. Here, virtual work density is constant so that it is enough to multiply by the area:

$$\delta W = \delta w_{\Omega} \frac{L^2}{2} = -\delta u_{X1} (\frac{1}{2} \frac{Et}{1-\nu^2} u_{X1} + \frac{1}{2} \frac{Et}{1-\nu} L\alpha\vartheta^{\circ}).$$

Principle of virtual work and the fundamental lemma of variation calculus give

$$\frac{1}{2}\frac{Et}{1-\nu^2}u_{X1}+\frac{1}{2}\frac{Et}{1-\nu}L\alpha\vartheta^\circ=0 \quad\Leftrightarrow\quad u_{X1}=-(1+\nu)\alpha L\vartheta^\circ .$$

Nodes 1 and 3 of a thin rectangular slab (assume plane stress conditions) shown are allowed to move horizontally and nodes 2 and 4 are fixed. Stress is zero when temperature is ϑ°. Determine the displacement components $u_{X1} = u_{X3}$ if the temperature of slab is increased to $2\vartheta^{\circ}$. Also, determine the strain and stress in the slab. Material parameters and thickness are E, ν, α and t, respectively.



Solution

As temperature is known and the external distributed force vanishes, virtual work densities needed here are (formulae collection)

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta u / \partial x \\ \partial\delta v / \partial y \\ \partial\delta u / \partial y + \partial\delta v / \partial x \end{Bmatrix}^{\mathrm{T}} t[E]_{\sigma} \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix}, \quad \delta w_{\Omega}^{\text{cpl}} = \begin{Bmatrix} \partial\delta u / \partial x \\ \partial\delta v / \partial x \end{Bmatrix}^{\mathrm{T}} \frac{E\alpha t}{1-\nu} \Delta\vartheta \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

in which $\Delta\vartheta = \vartheta - \vartheta^{\circ}$ is the difference between temperature at the deformed and initial geometries.

Approximations are the first thing to be considered. As the origin of the material xy – coordinate system is placed at node 1 and the axes are aligned with the axes of the structural XY – coordinate system

$$u = \begin{Bmatrix} (1-\xi)(1-\eta) \\ \xi(1-\eta) \\ (1-\xi)\eta \\ \xi\eta \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{X1} \\ 0 \\ u_{X1} \\ 0 \end{Bmatrix} = (1-\frac{x}{L})u_{X1}, \quad v = 0, \text{ and } \Delta\vartheta = \vartheta^{\circ} \text{ (constant).}$$

When the approximations are substituted there, virtual work density simplifies to

$$\delta w_{\Omega} = -\begin{Bmatrix} -\delta u_{X1} / L \\ 0 \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{Et}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} -u_{X1} / L \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} -\delta u_{X1} / L \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{Et}{1-\nu} \alpha\vartheta^{\circ} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \quad \Leftrightarrow$$

$$\delta w_{\Omega} = -\frac{\delta u_{X1}}{L} \frac{Et}{1-\nu^2} \frac{u_{X1}}{L} - \frac{\delta u_{X1}}{L} \frac{Et}{1-\nu} \alpha\vartheta^{\circ}.$$

Virtual work expression is integral of the density over the domain occupied by the element. Here, virtual work density is constant so that it is enough to multiply by the area:

$$\delta W = \delta w_{\Omega} L^2 = -\delta u_{X1} \frac{Et}{1-\nu^2} u_{X1} - \delta u_{X1} \frac{Et}{1-\nu} L\alpha\vartheta^{\circ}.$$

Principle of virtual work and the fundamental lemma of variation calculus give

$$\frac{Et}{1-\nu^2}u_{X1}+\frac{Et}{1-\nu}L\alpha\vartheta^{\circ}=0 \quad \Leftrightarrow \quad u_{X1}=-(1+\nu)\alpha L\vartheta^{\circ}.$$

Strain components can be obtained from derivatives of the displacement components u and v

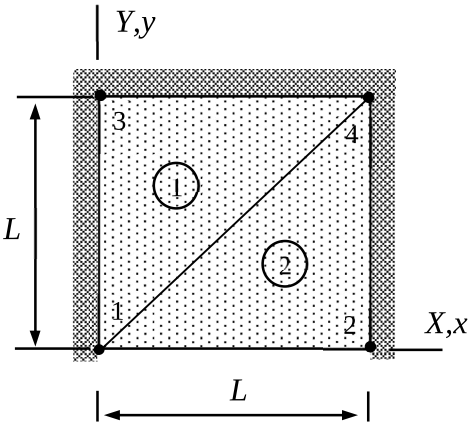
$$\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{Bmatrix}=\begin{Bmatrix}\partial u/\partial x\\ \partial v/\partial y\\ \partial u/\partial y+\partial v/\partial x\end{Bmatrix}=(1+\nu)\alpha\vartheta^{\circ}\begin{Bmatrix}1\\0\\0\end{Bmatrix}.$$

Cauchy stress components can be calculated from the stress-strain relationship of plane-stress case of the thin slab model taking into account the temperature change (see the lecture notes)

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{Bmatrix}=\frac{E}{1-\nu^2}(\begin{bmatrix}1&\nu&0\\ \nu&1&0\\ 0&0&(1-\nu)/2\end{bmatrix}\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{Bmatrix}-(1+\nu)\alpha\Delta\vartheta\begin{Bmatrix}1\\1\\0\end{Bmatrix}) \quad \Leftrightarrow$$

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{Bmatrix}=\frac{E}{1-\nu^2}(\begin{bmatrix}1&\nu&0\\ \nu&1&0\\ 0&0&(1-\nu)/2\end{bmatrix}\alpha\vartheta^{\circ}(1+\nu)\begin{Bmatrix}1\\0\\0\end{Bmatrix}-(1+\nu)\alpha\vartheta^{\circ}\begin{Bmatrix}1\\1\\0\end{Bmatrix})=-E\alpha\vartheta^{\circ}\begin{Bmatrix}0\\1\\0\end{Bmatrix}.$$

Determine the stationary temperature distribution in a thin slab shown. Edge 1-2 is at constant temperature ϑ° and heat flux through the other edges vanishes. Use a two-triangle mesh with ϑ_3 and $\vartheta_4 = \vartheta_3$ as the unknown node temperatures and consider $\vartheta_1 = \vartheta_2 = \vartheta^\circ$ as known. Thickness t, thermal conductivity k, and heat production rate per unit area s are constants.



Solution

The density expressions associated with the pure heat conduction problem in a thin slab are

$$\delta p_\Omega^{\text{int}} = -\begin{Bmatrix} \partial\delta\vartheta / \partial x \\ \partial\delta\vartheta / \partial y \end{Bmatrix}^{\mathrm{T}} tk \begin{Bmatrix} \partial\vartheta / \partial x \\ \partial\vartheta / \partial y \end{Bmatrix} \text{ and } \delta p_\Omega^{\text{ext}} = \delta\vartheta s\,.$$

in which ϑ is the temperature, k the thermal conductivity, and s the rate of heat production (per unit area). For a thin-slab element, element contributions need to be calculated from scratch starting with the densities and approximations.

The shape functions of element 1 (deduced from the figure) $N_1 = 1 - y/L$, $N_4 = x/L$, and $N_3 = 1 - N_1 - N_4 = (y - x)/L$ give approximations

$$\vartheta = \begin{Bmatrix} N_1 \\ N_4 \\ N_3 \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vartheta^\circ \\ \vartheta_3 \\ \vartheta_3 \end{Bmatrix} = (1 - \frac{y}{L})\vartheta^\circ + \frac{y}{L}\vartheta_3\,,\ \ \frac{\partial\vartheta}{\partial x} = 0\,,\ \ \frac{\partial\vartheta}{\partial y} = \frac{\vartheta_3 - \vartheta^\circ}{L} \text{ and}$$

$$\delta\vartheta = \frac{y}{L}\delta\vartheta_3\,,\ \ \frac{\partial\delta\vartheta}{\partial x} = 0\,,\ \ \frac{\partial\delta\vartheta}{\partial y} = \frac{\delta\vartheta_3}{L} \text{ (variation of } \vartheta^\circ \text{ vanishes).}$$

When the approximation is substituted there, density expression simplifies to

$$\delta p_\Omega = \delta p_\Omega^{\text{int}} + \delta p_\Omega^{\text{ext}} = -\begin{Bmatrix} \partial\delta\vartheta / \partial x \\ \partial\delta\vartheta / \partial y \end{Bmatrix}^{\mathrm{T}} tk \begin{Bmatrix} \partial\vartheta / \partial x \\ \partial\vartheta / \partial y \end{Bmatrix} + \delta\vartheta s = -\frac{\delta\vartheta_3}{L} tk \frac{\vartheta_3 - \vartheta^\circ}{L} + \frac{y}{L}\delta\vartheta_3 s\,.$$

Element contribution is the integral of the density expression over the domain occupied by the element:

$$\delta P^1 = -\delta\vartheta_3 (tk \frac{\vartheta_3 - \vartheta^\circ}{2} - \frac{L^2}{3}s)\,.$$

The shape functions of element 2 (deduced from the figure) $N_1 = 1 - x/L$, $N_4 = y/L$, and $N_2 = 1 - N_1 - N_4 = (x - y)/L$ give approximations

$$\vartheta = \begin{Bmatrix} N_1 \\ N_2 \\ N_4 \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vartheta^\circ \\ \vartheta^\circ \\ \vartheta_3 \end{Bmatrix} = (1-\frac{y}{L})\vartheta^\circ + \frac{y}{L}\vartheta_3\,, \quad \frac{\partial \vartheta}{\partial x} = 0\,, \quad \frac{\partial \vartheta}{\partial y} = \frac{\vartheta_3 - \vartheta^\circ}{L}\,, \quad \text{and}$$

$$\delta\vartheta = \frac{y}{L}\delta\vartheta_3\,, \quad \frac{\partial \delta\vartheta}{\partial x} = 0\,, \quad \frac{\partial \delta\vartheta}{\partial y} = \frac{\delta\vartheta_3}{L} \quad \text{(variation of } \vartheta^\circ \text{ vanishes).}$$

When the approximation is substituted there, density simplifies to

$$\delta p_\Omega = \delta p_\Omega^{\mathrm{int}} + \delta p_\Omega^{\mathrm{ext}} = -\frac{\delta\vartheta_3}{L} tk \frac{\vartheta_3 - \vartheta^\circ}{L} + \frac{y}{L}\delta\vartheta_3 s\,.$$

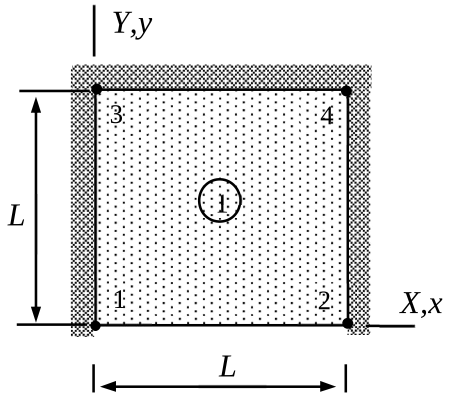
Element contribution is the integral of the density expression over the domain occupied by the element, so

$$\delta P^2 = -\delta\vartheta_3 (tk \frac{\vartheta_3 - \vartheta^\circ}{2} - \frac{L^2}{6} s)\,.$$

Variation principle $\delta P = \delta P^1 + \delta P^2 = 0 \quad \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus imply that

$$tk(\vartheta_3 - \vartheta^\circ) - \frac{L^2}{2} s = 0 \quad \Leftrightarrow \quad \vartheta_3 = \vartheta^\circ + \frac{sL^2}{2tk}\,. \quad \leftarrow$$

Determine the stationary temperature distribution in a thin slab shown. Edge 1-2 is at constant temperature ϑ° and heat flux through the other edges vanishes. Use a rectangle element with bilinear approximation and consider $\vartheta_1 = \vartheta_2 = \vartheta^\circ$ as known and $\vartheta_4 = \vartheta_3$ as the unknown nodal temperatures. Thickness t, thermal conductivity k, and heat production rate per unit area s are constants.



Solution

The density expressions associated with the pure heat conduction problem in a thin slab are

$$\delta p_\Omega^{\text{int}} = -\begin{Bmatrix} \partial\delta\vartheta/\partial x \\ \partial\delta\vartheta/\partial y \end{Bmatrix}^{\mathrm{T}} tk \begin{Bmatrix} \partial\vartheta/\partial x \\ \partial\vartheta/\partial y \end{Bmatrix} \text{ and } \delta p_\Omega^{\text{ext}} = \delta\vartheta s\,.$$

in which ϑ is the temperature, k the thermal conductivity, and s the rate of heat production (per unit area). For a thin-slab element, element contributions need to be calculated from scratch starting with the densities and approximation.

The shape functions can be deduced from the figure. Approximation

$$\vartheta = \begin{Bmatrix} (1-x/L)(1-y/L) \\ (x/L)(1-y/L) \\ (1-x/L)(y/L) \\ (x/L)(y/L) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vartheta^\circ \\ \vartheta^\circ \\ \vartheta_3 \\ \vartheta_3 \end{Bmatrix} = (1-\frac{y}{L})\vartheta^\circ + \frac{y}{L}\vartheta_3\,,\quad \frac{\partial\vartheta}{\partial x} = 0\,,\quad \frac{\partial\vartheta}{\partial y} = \frac{\vartheta_3 - \vartheta^\circ}{L} \quad \text{and}$$

$$\delta\vartheta = \frac{y}{L}\delta\vartheta_3\,,\quad \frac{\partial\delta\vartheta}{\partial x} = 0\,,\quad \frac{\partial\delta\vartheta}{\partial y} = \frac{\delta\vartheta_3}{L} \quad \text{(variation of } \vartheta^\circ \text{ vanishes).}$$

When the approximation is substituted there, density simplifies to

$$\delta p_\Omega = \delta p_\Omega^{\text{int}} + \delta p_\Omega^{\text{ext}} = -\frac{\delta\vartheta_3}{L} tk \frac{\vartheta_3 - \vartheta^\circ}{L} + \frac{y}{L}\delta\vartheta_3 s\,.$$

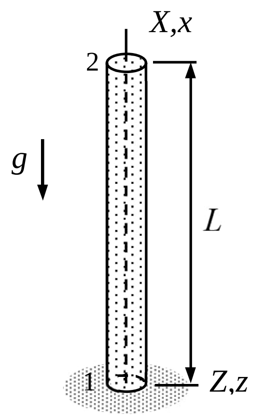
Element contribution is the integral of the density expression over the domain occupied by the element:

$$\delta P^1 = \int_0^L \int_0^L \delta p_\Omega dxdy = -\delta\vartheta_3[tk(\vartheta_3 - \vartheta^\circ) - \frac{L^2}{2}s]\,.$$

Variation principle $\delta P = 0 \quad \forall \delta\mathbf{a}$ and the fundamental lemma of variation calculus imply that

$$tk(\vartheta_3 - \vartheta^\circ) - \frac{L^2}{2}s = 0 \quad \Leftrightarrow \quad \vartheta_3 = \vartheta^\circ + \frac{sL^2}{2kt}\,.$$

At the initial temperature $\vartheta°$ and without external forces, the length of the bar shown is L. Calculate the displacement of node 2 due to the combined effect of gravity and change of temperature with the nodal values $\vartheta_1 = 2\vartheta°$ and $\vartheta_2 = \vartheta°$. Cross sectional area A, coefficient of thermal expansion α, and density ρ are considered as constants. Use linear interpolation to displacement and temperature and start with the virtual work density expressions.



Solution

Here temperature is given and the aim is to find the deformation implied by the temperature change. Virtual work density expressions of the bar model needed in the calculation are

$$\delta w_\Omega^{\text{int}} = -\frac{d\delta u}{dx} EA \frac{du}{dx}, \quad \delta w_\Omega^{\text{cpl}} = \frac{d\delta u}{dx} EA\alpha\Delta\vartheta \quad \text{and} \quad \delta w_\Omega^{\text{ext}} = \delta u \, f_x .$$

in which $\Delta\vartheta$ is the temperature change, α coefficient of thermal expansion, and f_x the distributed force per unit length.

The given nodal temperatures are $\vartheta_1 = 2\vartheta°$ and $\vartheta_2 = \vartheta°$. As the initial temperature is $\vartheta°$, the changes of the nodal values are $\Delta\vartheta_1 = \vartheta°$ and $\Delta\vartheta_2 = 0$. Linear interpolations to displacement and temperature in terms of the nodal values are

$$u = \begin{Bmatrix} 1 - x/L \\ x/L \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ u_{X2} \end{Bmatrix} = \frac{x}{L} u_{X2}, \quad \frac{du}{dx} = \frac{1}{L} u_{X2}, \quad \text{and} \quad \frac{d\delta u}{dx} = \frac{1}{L} \delta u_{X2},$$

$$\Delta\vartheta = \begin{Bmatrix} 1 - x/L \\ x/L \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vartheta° \\ 0 \end{Bmatrix} = (1 - \frac{x}{L})\vartheta° .$$

When the approximation is substituted there, density expression $\delta w_\Omega = \delta w_\Omega^{\text{int}} + \delta w_\Omega^{\text{cpl}} + \delta w_\Omega^{\text{ext}}$ simplifies to

$$\delta w_\Omega = -\delta u_{X2} \frac{EA}{L^2} u_{X2} + \delta u_{X2} \frac{EA\alpha}{L} (1 - \frac{x}{L})\vartheta° - \delta u_{X2} \frac{x}{L} \rho g A .$$

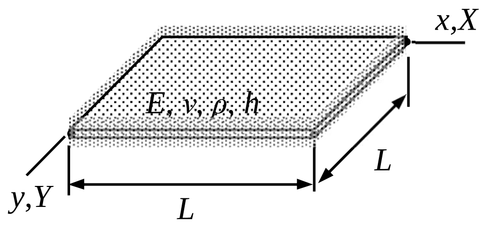
Virtual work expression is the integral of the density over the element domain

$$\delta W = -\delta u_{X2} \frac{EA}{L} u_{X2} + \delta u_{X2} \frac{EA\alpha}{2} \vartheta° - \delta u_{X2} \frac{\rho g A L}{2} .$$

Principle of virtual work $\delta W = 0 \;\; \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus give

$$-\frac{EA}{L} u_{X2} + \frac{EA\alpha}{2} \vartheta° - \frac{\rho g A L}{2} = 0 \quad \Leftrightarrow \quad u_{X2} = \frac{\alpha}{2} L \vartheta° - \frac{\rho g}{2E} L^2 .$$

The simply supported plate shown is assembled at constant temperature $3\vartheta°$. Find the transverse displacement when the upper side temperature is $4\vartheta°$ and that of the lower side $2\vartheta°$. Assume that temperature in plate is linear in z and does not depend on x or y. Use $w(x,y) = \mathrm{a}\sin(\pi x/L)\sin(\pi y/L)$ as the approximation. Problem parameters E, ν, ρ, α and t are constants. Integrals of sin and cos functions satisfy



$$\int_0^L \sin^2(\pi\frac{x}{L})dx = \int_0^L \cos^2(\pi\frac{x}{L})dx = \frac{L}{2} \text{ and } \int_0^L \sin(\pi\frac{x}{L})dx = \frac{2L}{\pi}.$$

Solution

Assuming that the material coordinate system is chosen so that the plate bending and thin slab modes decouple, the plate model virtual work densities of internal force and coupling terms are given by

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial^2\delta w/\partial x^2 \\ \partial^2\delta w/\partial y^2 \\ 2\partial^2\delta w/\partial x\partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12}[E]_\sigma \begin{Bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ 2\partial^2 w/\partial x\partial y \end{Bmatrix}, \quad \delta w_\Omega^{\text{cpl}} = -\begin{Bmatrix} \partial^2\delta w/\partial x^2 \\ \partial^2\delta w/\partial y^2 \end{Bmatrix}^{\mathrm{T}} \int z\Delta\vartheta dz \frac{\alpha E}{1-\nu}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

The coupling term contains an integral of temperature over the thickness of the plate. Approximation to the transverse displacement and its derivatives are

$$w(x,y) = \mathrm{a}\sin(\pi\frac{x}{L})\sin(\pi\frac{y}{L}) \quad \Rightarrow$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial y^2} = -\mathrm{a}\,(\frac{\pi}{L})^2\sin(\pi\frac{x}{L})\sin(\pi\frac{y}{L}), \quad \frac{\partial^2 w}{\partial x\partial y} = \mathrm{a}\,(\frac{\pi}{L})^2\cos(\pi\frac{x}{L})\cos(\pi\frac{y}{L}).$$

Temperature difference and its weighted integral over the thickness (integral of the coupling term)

$$\Delta\vartheta = \vartheta(z) - 3\vartheta° = (\frac{1}{2}+\frac{z}{t})2\vartheta° + (\frac{1}{2}-\frac{z}{t})4\vartheta° - 3\vartheta° = -\frac{z}{t}2\vartheta° \quad \Rightarrow$$

$$\int z\Delta\vartheta dz = -\int_{-t/2}^{t/2} z\frac{z}{t}2\vartheta° dz = -\frac{1}{6}\vartheta° t^2.$$

When the approximation to the transverse displacement is substituted there, virtual work densities of the internal and the coupling parts simplify to

$$\delta w_\Omega^{\text{int}} = -\delta\mathrm{a}\frac{t^3 E}{12(1-\nu^2)}(\frac{\pi}{L})^4 2[\sin^2(\frac{\pi x}{L})\sin^2(\frac{\pi y}{L})(1+\nu) + (1-\nu)\cos^2(\frac{\pi x}{L})\cos^2(\frac{\pi y}{L})]\mathrm{a},$$

$$\delta w_\Omega^{\text{cpl}} = -\delta\mathrm{a}(\frac{\pi}{L})^2\sin(\pi\frac{x}{L})\sin(\pi\frac{y}{L})\frac{1}{3}\vartheta° t^2\frac{\alpha E}{1-\nu}.$$

Virtual work expressions are integrals of the densities over the domain occupied by the plate/element

$$\delta W^{\text{int}} = \int_0^L \int_0^L \delta w_\Omega^{\text{int}} dxdy = -\delta \text{a} 4 \frac{t^3 E}{12(1-\nu^2)} (\frac{\pi}{L})^4 (\frac{L}{2})^2 \text{a} \,,$$

$$\delta W^{\text{cpl}} = \int_0^L \int_0^L \delta w_\Omega^{\text{cpl}} dxdy = -\delta \text{a} \frac{4}{3} \vartheta^\circ \frac{\alpha E t^2}{1-\nu} \,.$$

Virtual work expression is the sum of the parts

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{cpl}} = -\delta \text{a}[4 \frac{t^3 E}{12(1-\nu^2)} (\frac{\pi}{L})^4 (\frac{L}{2})^2 \text{a} + \frac{4}{3} \vartheta^\circ \frac{\alpha E t^2}{1-\nu}] \,.$$

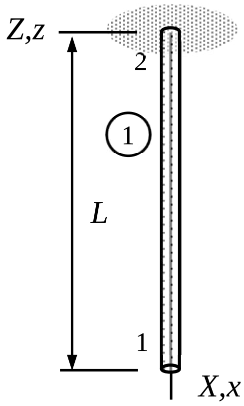
Principle of virtual work $\delta W = 0 \ \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus give

$$\delta W = -\delta \text{a}[4 \frac{t^3 E}{12(1-\nu^2)} (\frac{\pi}{L})^4 (\frac{L}{2})^2 \text{a} + \frac{4}{3} \vartheta^\circ \frac{\alpha E t^2}{1-\nu}] = 0 \quad \forall \delta \text{a} \quad \Leftrightarrow$$

$$4 \frac{t^3 E}{12(1-\nu^2)} (\frac{\pi}{L})^4 (\frac{L}{2})^2 \text{a} + \frac{4}{3} \vartheta^\circ \frac{\alpha E t^2}{1-\nu} = 0 \quad \Rightarrow \quad \text{a} = -\frac{16}{\pi^4} \frac{\alpha \vartheta^\circ L^2}{t} (1+\nu) \,. \leftarrow$$

Name________________________________ Student number_____________________

Assignment 1



Determine the displacement of node 1 of the bar structure shown at the constant temperature ϑ°. Use a linear approximation and assume that parameters E, A and α are constants. At the initial temperature $2\vartheta^\circ$, length of the bar is L and stress in the bar vanishes.

Solution template

In stationary thermo-elasticity without external forces, the virtual work density of the bar model is given by

$$\delta w_\Omega = -\frac{d\delta u}{dx} EA \frac{du}{dx} + \frac{d\delta u}{dx} EA\alpha\Delta\vartheta .$$

Linear interpolants to axial displacement $u(x)$ and temperature change $\Delta\vartheta(x)$ are

$$u(x) = \frac{x}{L} u_{X1} ,$$

$$\Delta\vartheta(x) = \vartheta^\circ - 2\vartheta^\circ = -\vartheta^\circ .$$

When $u(x)$ and $\Delta\vartheta(x)$ are substituted there, virtual work density simplifies to

$$\delta w_\Omega = -\delta u_{X1} \frac{1}{L} EA \frac{1}{L} u_{X1} - \delta u_{X1} \frac{1}{L} EA\alpha\vartheta^\circ .$$

Integration over the element gives

$$\delta W = -\delta u_{X1} \left(\frac{EA}{L} u_{X1} + EA\alpha\vartheta^\circ\right) .$$

Principle of virtual work $\delta W = 0 \quad \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus imply the nodal displacement

$$u_{X1} = -L\alpha\vartheta^\circ .$$

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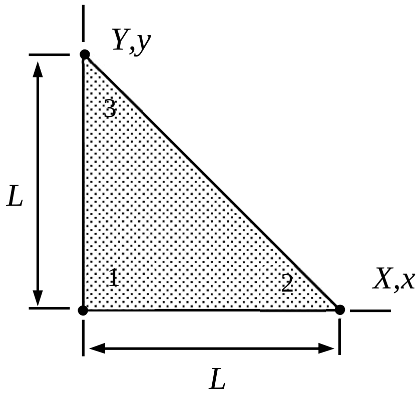
Assignment 2



Consider the temperature distribution in the structure shown which is composed of one triangle element. Assuming that the thermal conductivity k and thickness t of the element are constants, derive the element contribution δP^{int}. Temperature at nodes 1 and 2 is known to be ϑ° and the unknown nodal temperature is ϑ_3.

Solution template

In stationary thermo-elasticity, the variational densities of the thin slab mode of the plate model

$$\delta p_\Omega^{\text{int}} = -\begin{Bmatrix} \partial\delta\vartheta / \partial x \\ \partial\delta\vartheta / \partial y \end{Bmatrix}^{\text{T}} kt \begin{Bmatrix} \partial\vartheta / \partial x \\ \partial\vartheta / \partial y \end{Bmatrix}, \; \delta p_\Omega^{\text{ext}} = \delta\vartheta s$$

represent the energy balance. Linear shape functions of the temperature approximation can be deduced from the figure

$$N_2 = \frac{x}{L}, \quad N_3 = \frac{y}{L}, \quad N_1 = 1 - N_2 - N_3 = 1 - \frac{x}{L} - \frac{y}{L}.$$

Approximation to $\vartheta(x,y)$ and its variation $\delta\vartheta(x,y)$ (notice that the variation of a given quantity vanishes)

$$\vartheta = (1 - \frac{y}{L})\vartheta^\circ + \frac{y}{L}\vartheta_3, \quad \frac{\partial\vartheta}{\partial x} = 0, \quad \frac{\partial\vartheta}{\partial y} = \frac{1}{L}(\vartheta_3 - \vartheta^\circ),$$

$$\delta\vartheta = \frac{y}{L}\delta\vartheta_3, \quad \frac{\partial\delta\vartheta}{\partial x} = 0, \quad \frac{\partial\delta\vartheta}{\partial y} = \frac{1}{L}\delta\vartheta_3.$$

When the approximation is substituted there, the variational density simplifies to

$$\delta p_\Omega^{\text{int}} = -\frac{\delta\vartheta_3}{L} kt \frac{1}{L}(\vartheta_3 - \vartheta^\circ).$$

Integration over the element gives

$$\delta P^{\text{int}} = \int_\Omega \delta p_\Omega^{\text{int}} dA = -\delta\vartheta_3 \frac{kt}{2}(\vartheta_3 - \vartheta^\circ). \quad \leftarrow$$

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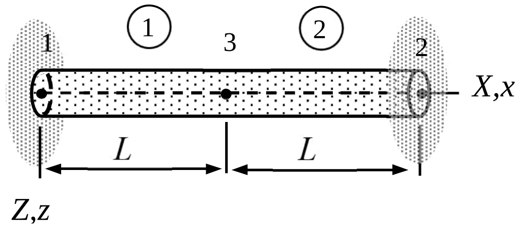
Assignment 3

Electric current causes heat generation in the bar shown. Calculate the temperature at the centre if the wall temperature (nodes 1 and 2) is ϑ°. Cross sectional area A, thermal conductivity k, and heat production rate per unit length s are constants.



Solution template

In a pure heat conduction problem, density expressions of the bar model are given by

$$\delta p_\Omega^{\text{int}} = -\frac{d\delta\vartheta}{dx} kA \frac{d\vartheta}{dx} \text{ and } \delta p_\Omega^{\text{ext}} = \delta\vartheta s$$

in which ϑ is the temperature, k the thermal conductivity, and s the rate of heat production (per unit length).

For bar 1, the nodal temperatures are $\vartheta_1 = \vartheta^\circ$ and ϑ_3 of which the latter is unknown. With a linear interpolation to temperature (notice that variation of ϑ° vanishes)

$$\vartheta = \begin{Bmatrix} 1 - x/L \\ x/L \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vartheta^\circ \\ \vartheta_3 \end{Bmatrix} = (1-\frac{x}{L})\vartheta^\circ + \frac{x}{L}\vartheta_3 \quad \Rightarrow \quad \frac{d\vartheta}{dx} = \frac{\vartheta_3 - \vartheta^\circ}{L},$$

$$\delta\vartheta = \frac{x}{L}\delta\vartheta_3 \quad \Rightarrow \quad \frac{d\delta\vartheta}{dx} = \frac{\delta\vartheta_3}{L}.$$

When the approximation is substituted there, density expression $\delta p_\Omega = \delta p_\Omega^{\text{int}} + \delta p_\Omega^{\text{ext}}$ simplifies to

$$\delta p_\Omega = -\frac{\delta\vartheta_3}{L} kA \frac{\vartheta_3 - \vartheta^\circ}{L} + \frac{x}{L}\delta\vartheta_3 s,$$

Virtual work expression is the integral of the density over the element domain

$$\delta P^1 = \int_0^L \delta p_\Omega dx = -\delta\vartheta_3 (kA\frac{\vartheta_3 - \vartheta^\circ}{L} - \frac{1}{2}Ls).$$

The nodal temperatures of bar 2 are ϑ_3 and $\vartheta_2 = \vartheta^\circ$. Linear interpolation gives (variations of the given quantities like ϑ° vanish)

$$\vartheta = \begin{Bmatrix} 1 - x/L \\ x/L \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vartheta_3 \\ \vartheta^\circ \end{Bmatrix} = (1-\frac{x}{L})\vartheta_3 + \frac{x}{L}\vartheta^\circ \quad \Rightarrow \quad \frac{d\vartheta}{dx} = \frac{\vartheta^\circ - \vartheta_3}{L},$$

$$\delta\vartheta = (1-\frac{x}{L})\delta\vartheta_3 \quad \Rightarrow \quad \frac{d\delta\vartheta}{dx} = -\frac{\delta\vartheta_3}{L}.$$

When the approximation is substituted there, density expression $\delta p_\Omega = \delta p_\Omega^{\text{int}} + \delta p_\Omega^{\text{ext}}$ simplifies to

$$\delta p_\Omega = -(-\frac{\delta\vartheta_3}{L})kA\frac{\vartheta^\circ - \vartheta_3}{L} + (1-\frac{x}{L})\delta\vartheta_3 s.$$

Element contribution to the variational expressions is the integral of density over the element domain

$$\delta P^2 = \int_0^L \delta p_\Omega dx = -\delta\vartheta_3(kA\frac{\vartheta_3 - \vartheta^\circ}{L} - \frac{L}{2}s).$$

Variational expression is sum of the element contributions

$$\delta P = \delta P^1 + \delta P^2 = -\delta\vartheta_3(2kA\frac{\vartheta_3 - \vartheta^\circ}{L} - Ls).$$

Variation principle $\delta P = 0 \quad \forall \delta\mathbf{a}$ and the fundamental lemma of variation calculus give

$$2\frac{kA}{L}(\vartheta_3 - \vartheta^\circ) - Ls = 0 \quad \Leftrightarrow \quad \vartheta_3 = \vartheta^\circ + \frac{1}{2}\frac{L^2 s}{kA}.$$

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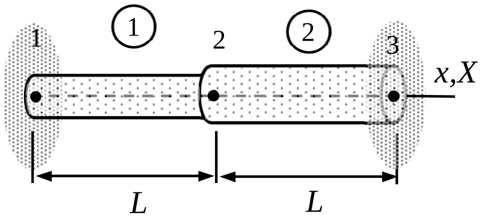
Assignment 4

The bar shown consists of two elements having different cross-sectional areas $A_1 = A$, $A_2 = 4A$. Material properties E, k, and α are the same. Determine the stationary displacement u_{X2} and temperature ϑ_2 at node 2, when the temperature at the left wall (node 1) is $2\vartheta^\circ$ and that of the right wall is ϑ° (node 3). Stress vanishes, when the temperature in the wall and bar is ϑ°.



Solution template

Element contribution of a bar needed in this case are

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix}, \quad \delta W^{\text{cpl}} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{\alpha EA}{2} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \Delta\vartheta_1 \\ \Delta\vartheta_2 \end{Bmatrix},$$

$$\delta P^{\text{int}} = -\begin{Bmatrix} \delta\vartheta_1 \\ \delta\vartheta_2 \end{Bmatrix}^{\mathrm{T}} \frac{kA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \vartheta_1 \\ \vartheta_2 \end{Bmatrix}.$$

The expressions assume linear approximations and constant material properties. The temperature relative to the initial temperature without stress is denoted by $\Delta\vartheta = \vartheta - \vartheta^\circ$. The unknown nodal displacement and temperature are u_{X2} and ϑ_2.

When the nodal displacements and temperatures are substituted there, the element contributions of bar 1 take the forms

$$\delta W^1 = -\begin{Bmatrix} 0 \\ \delta u_{X2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ u_{X2} \end{Bmatrix} - \frac{\alpha EA}{2} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \vartheta^\circ \\ \vartheta_2 - \vartheta^\circ \end{Bmatrix}\right)$$

$$= -\delta u_{X2}\left(\frac{EA}{L} u_{X2} - \frac{\alpha EA}{2} \vartheta_2\right),$$

$$\delta P^1 = -\begin{Bmatrix} 0 \\ \delta\vartheta_2 \end{Bmatrix}^{\mathrm{T}} \frac{kA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 2\vartheta^\circ \\ \vartheta_2 \end{Bmatrix} = -\delta\vartheta_2\left(\frac{kA}{L}\vartheta_2 - 2\frac{kA}{L}\vartheta^\circ\right).$$

When the displacements and temperatures are substituted there, the element contributions of bar 2 take the forms

$$\delta W^2 = -\begin{Bmatrix} \delta u_{X2} \\ 0 \end{Bmatrix}^{\mathrm{T}} (\frac{4EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ 0 \end{Bmatrix} - 2\alpha EA\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} \vartheta_2 - \vartheta^\circ \\ 0 \end{Bmatrix})$$

$$= -\delta u_{X2}(\frac{4EA}{L}u_{X2} + 2\alpha EA\vartheta_2 - 2\alpha EA\vartheta^\circ)$$

$$\delta P^2 = -\begin{Bmatrix} \delta\vartheta_2 \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{4kA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \vartheta_2 \\ \vartheta^\circ \end{Bmatrix} = -\delta\vartheta_2(4\frac{kA}{L}\vartheta_2 - 4\frac{kA}{L}\vartheta^\circ).$$

Virtual work expression is the sum of element contributions

$$\delta W = -\delta u_{X2}(5\frac{EA}{L}u_{X2} + \frac{3}{2}\alpha EA\vartheta_2 - 2\alpha EA\vartheta^\circ),$$

$$\delta P = -\delta\vartheta_2(5\frac{kA}{L}\vartheta_2 - 6\frac{kA}{L}\vartheta^\circ).$$

Variational principle $\delta P = 0$ and $\delta W = 0$ $\forall \mathbf{a}$ gives a linear equation system

$$\begin{bmatrix} 5\frac{EA}{L} & \frac{3}{2}EA\alpha \\ 0 & 5\frac{kA}{L} \end{bmatrix}\begin{Bmatrix} u_{X2} \\ \vartheta_2 \end{Bmatrix} - \begin{Bmatrix} 2\alpha EA\vartheta^\circ \\ 6\frac{kA}{L}\vartheta^\circ \end{Bmatrix} = 0 \quad \Leftrightarrow$$

$$\vartheta_2 = \frac{6}{5}\vartheta^\circ \quad \text{and} \quad u_{X2} = \frac{1}{25}\alpha L\vartheta^\circ.$$

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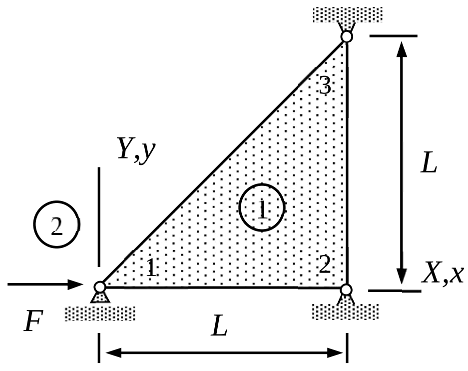
Assignment 5



A thin triangular slab (assume plane stress conditions) loaded by a horizontal force is allowed to move horizontally at node 1 and nodes 2 and 3 are fixed. At the constant initial temperature ϑ° and loading $F=0$, stress vanishes. If the slab is heated to the constant temperature $2\vartheta^\circ$, what is the required force F to have $u_{X1}=0$? Material properties E, ν, α and thickness t of the slab are constants.

Solution

As temperature is known and the external distributed force vanishes, virtual work densities needed are (formulae collection)

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \delta\varepsilon_{xx} \\ \delta\varepsilon_{yy} \\ \delta\gamma_{xy} \end{Bmatrix}^{\mathrm{T}} t[E]_\sigma \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \text{ where } \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix},$$

$$\delta w_\Omega^{\text{cpl}} = \begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial x \end{Bmatrix}^{\mathrm{T}} \frac{E\alpha t}{1-\nu}\Delta\vartheta \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

in which $\Delta\vartheta = \vartheta - \vartheta^\circ$ is the difference between temperature at the deformed and initial geometries.

Approximation is the first thing to be considered. As the origin of the material $xy-$coordinate system is placed at node 1 and the axes are aligned with the axes of the structural $XY-$coordinate system

$$u = (1-\frac{x}{L})u_{X1},\ \ v=0,\ \text{ and } \Delta\vartheta = \vartheta^\circ \text{ (constant).}$$

When the approximations are substituted there, virtual work density (composed of the internal and coupling parts) simplifies to

$$\delta w_\Omega = -\begin{Bmatrix} -\delta u_{X1}/L \\ 0 \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{Et}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\begin{Bmatrix} -u_{X1}/L \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} -\delta u_{X1}/L \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{Et}{1-\nu}\alpha\vartheta^\circ \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \quad \Leftrightarrow$$

$$\delta w_{\Omega}=-\frac{\delta u_{X1}}{L}\frac{Et}{1-\nu^2}\frac{u_{X1}}{L}-\frac{\delta u_{X1}}{L}\frac{Et}{1-\nu}\alpha\vartheta^{\circ}.$$

Virtual work expression is integral of the density expression over the domain occupied by the element. Here, virtual work density is constant so that it is enough to multiply by the area. Virtual work expressions of element 1 and 2 (point force) become

$$\delta W^1=\delta w_{\Omega}\frac{L^2}{2}=-\delta u_{X1}(\frac{1}{2}\frac{Et}{1-\nu^2}u_{X1}+\frac{1}{2}\frac{Et}{1-\nu}L\alpha\vartheta^{\circ}),$$

$$\delta W^2=\delta u_{X1}F.$$

Virtual work expression of the structure $\delta W=\delta W^1+\delta W^2$, principle of virtual work, and the fundamental lemma of variation calculus imply the equilibrium equation

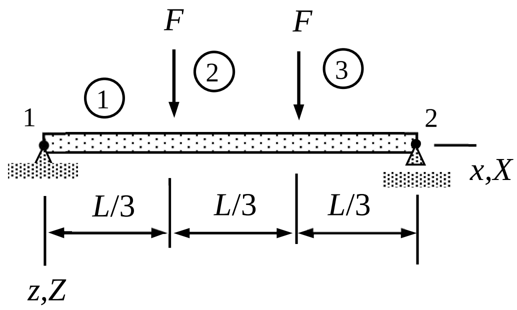
$$\frac{1}{2}\frac{Et}{1-\nu^2}u_{X1}+\frac{1}{2}\frac{Et}{1-\nu}L\alpha\vartheta^{\circ}-F=0 \quad\Leftrightarrow\quad u_{X1}=\frac{1-\nu^2}{Et}(2F-\frac{Et}{1-\nu}L\alpha\vartheta^{\circ}).$$

Displacement vanishes with the force (this is also the horizontal constraint force when the node is fixed)

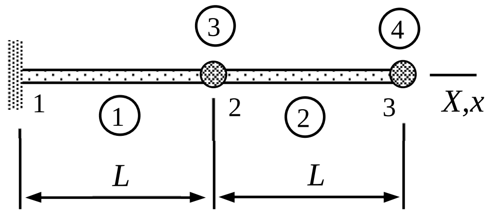
$$F=\frac{1}{2}\frac{Et}{1-\nu}L\alpha\vartheta^{\circ}.$$

MEC-E8001 Finite Element Analysis, Online exam 22.02.2023

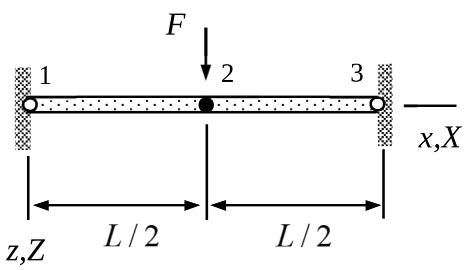
1. Find the transverse displacement $w(x)$ of the structure consisting of *one* beam element of cubic approximation and point forces 2 and 3. The rotations of the endpoints are assumed to be equal in magnitudes but opposite in directions, i.e., $\theta_{Y2} = -\theta_{Y1}$. Problem parameters E and $I_{yy} = I$ are constants.



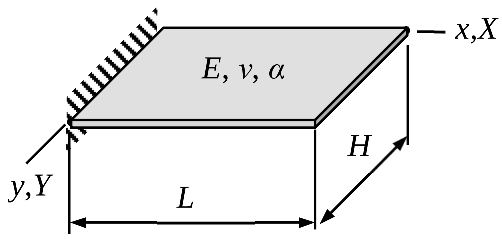
2. The bar structure shown consists of two *massless* bars and two particles of mass m, each. Find the angular speeds of the free axial vibrations of the structure. Young's modulus of the material and the cross-sectional area of the bars are E and A, respectively.



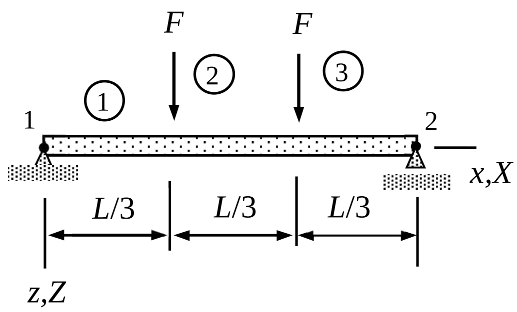
3. Find the relationship between the force F acting on the beam and the displacement u_{Z2} of its centerpoint. Combine the virtual work density expressions for the beam bending mode of *small (linear) displacement analysis* and that for the bar mode of *large displacement analysis.* Assume $u = v = 0$ and use the approximation $w = u_{Z2} 4(x/L)(1-x/L)$ for the transverse displacement. Assume that material parameter $C = E$ and cross-sectional area $A° = A$.



4. A bending plate of thickness t, which is clamped on one edge, is assembled at constant temperature $2\vartheta°$. Find the transverse displacement due to heating on the upper side $z = -t/2$ and cooling on the lower side $z = t/2$ resulting in surface temperatures $3\vartheta°$ and $\vartheta°$, respectively. Assume that temperature in plate is linear in z and does not depend on x or y. Use approximation $w = a_0 x^2$ in which a_0 is the parameter to be determined. Young's modulus E, Poisson's ratio ν, and coefficient of thermal expansion α are constants.



Find the transverse displacement $w(x)$ of the structure consisting of *one* beam element of cubic approximation and point forces 2 and 3. The rotations of the endpoints are assumed to be equal in magnitudes but opposite in directions, i.e., $\theta_{Y2}=-\theta_{Y1}$. Problem parameters E and $I_{yy}=I$ are constants.



Solution

Virtual work expression of the internal forces

$$\delta W^{\text{int}}=-\begin{Bmatrix}\delta u_{z1}\\ \delta\theta_{y1}\\ \delta u_{z2}\\ \delta\theta_{y2}\end{Bmatrix}^{\mathrm{T}}\frac{EI_{yy}}{h^3}\begin{bmatrix}12 & -6h & -12 & -6h\\ -6h & 4h^2 & 6h & 2h^2\\ -12 & 6h & 12 & 6h\\ -6h & 2h^2 & 6h & 4h^2\end{bmatrix}\begin{Bmatrix}u_{z1}\\ \theta_{y1}\\ u_{z2}\\ \theta_{y2}\end{Bmatrix}$$

available in the formulae collection applies here. The contribution of the point forces (acting inside the element) follows from the definition of work as usually. However, the virtual displacement needs to be expressed in terms of the displacement and rotation of nodes by using the cubic approximation for bending.

3p Transverse displacements vanish at the nodes and rotations satisfy $\theta_{Y2}=-\theta_{Y1}$ so ($\xi=x/L$)

$$w=\begin{Bmatrix}(1-\xi)^2(1+2\xi)\\ L(1-\xi)^2\xi\\ (3-2\xi)\xi^2\\ L\xi^2(\xi-1)\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}0\\ -\theta_{Y1}\\ 0\\ \theta_{Y1}\end{Bmatrix}=x(\frac{x}{L}-1)\theta_{Y1}\ \text{ and }\ \delta w=x(\frac{x}{L}-1)\delta\theta_{Y1}.$$

At the points of action of the forces 2 and 3, the virtual displacements are

$$\delta w(\frac{L}{3})=\frac{L}{3}(\frac{1}{3}-1)\delta\theta_{Y1}=-\frac{2}{9}L\delta\theta_{Y1}\quad\text{and}\quad\delta w(\frac{2L}{3})=\frac{2L}{3}(\frac{2}{3}-1)\delta\theta_{Y1}=-\frac{2}{9}L\delta\theta_{Y1}.$$

Therefore, the virtual work expression of the point forces

$$\delta W^{\text{ext}}=-\frac{2}{9}L\delta\theta_{Y1}F-\frac{2}{9}L\delta\theta_{Y1}F=-\delta\theta_{Y1}\frac{4}{9}LF\,.$$

2p Virtual work expression of the internal forces simplifies to

$$\delta W^{\text{int}}=-\begin{Bmatrix}0\\ \delta\theta_{Y1}\\ 0\\ -\delta\theta_{Y1}\end{Bmatrix}^{\mathrm{T}}\frac{EI}{L^3}\begin{bmatrix}12 & -6L & -12 & -6L\\ -6L & 4L^2 & 6L & 2L^2\\ -12 & 6L & 12 & 6L\\ -6L & 2L^2 & 6L & 4L^2\end{bmatrix}\begin{Bmatrix}0\\ \theta_{Y1}\\ 0\\ -\theta_{Y1}\end{Bmatrix}=-\delta\theta_{Y1}4\frac{EI}{L}\theta_{Y1}.$$

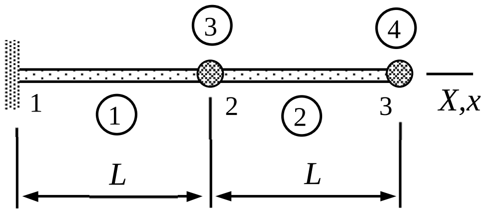
1p Principle of virtual work and the fundamental lemma of variation calculus imply

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = -\delta\theta_{Y1}(4\frac{EI}{L}\theta_{Y1} + \frac{4}{9}LF) = 0 \quad \Rightarrow \quad \theta_{Y1} = -\frac{1}{9}\frac{FL^2}{EI}.$$

Therefore, transverse displacement

$$w = -\frac{1}{9}\frac{FL^2}{EI}x(\frac{x}{L}-1). \leftarrow$$

The bar structure shown consists of two *massless* bars and two particles of mass m, each. Find the angular speeds of the free axial vibrations of the structure. Young's modulus of the material and the cross-sectional area of the bars are E and A, respectively.



Solution

Bar and rigid body model virtual work expression of internal and inertia forces are available in the formulae collection. As bars are assumed to be massless, only the internal part is needed. For a particle, only translation part applies (moments of inertia are zeros). Therefore

$$\delta W^{\text{int}} = -\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}} \frac{EA}{h}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix}, \quad \delta W^{\text{ine}} = -\begin{Bmatrix}\delta u_{x1}\\ \delta u_{y1}\\ \delta u_{z1}\end{Bmatrix}^{\mathrm{T}} m\begin{Bmatrix}\ddot{u}_{x1}\\ \ddot{u}_{y1}\\ \ddot{u}_{z1}\end{Bmatrix}.$$

2p The non-zero displacement components of the structure are u_{X2} and u_{X3}. Let us start with the element contributions of the bars. Since the bars are assumed to be massless, only the virtual work expressions of the internal forces (available in the formulae collection) are needed.

$$\delta W^1 = -\begin{Bmatrix}0\\ \delta u_{X2}\end{Bmatrix}^{\mathrm{T}} \frac{EA}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}0\\ u_{X2}\end{Bmatrix} = -\delta u_{X2}\frac{EA}{L}u_{X2} = -\begin{Bmatrix}\delta u_{X2}\\ \delta u_{X3}\end{Bmatrix}^{\mathrm{T}} \frac{EA}{L}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{Bmatrix}u_{X2}\\ u_{X3}\end{Bmatrix},$$

$$\delta W^2 = -\begin{Bmatrix}\delta u_{X2}\\ \delta u_{X3}\end{Bmatrix}^{\mathrm{T}} \frac{EA}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_{X2}\\ u_{X3}\end{Bmatrix}.$$

Then the two particle elements. Element contribution of the rigid body (formula collection) simplifies to

$$\delta W^3 = -\begin{Bmatrix}\delta u_{X2}\\ 0\\ 0\end{Bmatrix}^{\mathrm{T}} m\begin{Bmatrix}\ddot{u}_{X2}\\ 0\\ 0\end{Bmatrix} = -\delta u_{X2} m\ddot{u}_{X2} = -\begin{Bmatrix}\delta u_{X2}\\ \delta u_{X3}\end{Bmatrix}^{\mathrm{T}} m\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{Bmatrix}\ddot{u}_{X2}\\ \ddot{u}_{X3}\end{Bmatrix},$$

$$\delta W^4 = -\begin{Bmatrix}\delta u_{X3}\\ 0\\ 0\end{Bmatrix}^{\mathrm{T}} m\begin{Bmatrix}\ddot{u}_{X3}\\ 0\\ 0\end{Bmatrix} = -\delta u_{X3} m\ddot{u}_{X3} = -\begin{Bmatrix}\delta u_{X2}\\ \delta u_{X3}\end{Bmatrix}^{\mathrm{T}} m\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix}\ddot{u}_{X2}\\ \ddot{u}_{X3}\end{Bmatrix}$$

2p Virtual work expression of structure is the sum of element contributions.

$$\delta W = \delta W^1 + \delta W^2 + \delta W^3 + \delta W^4 = -\begin{Bmatrix}\delta u_{X2}\\ \delta u_{X3}\end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{L}\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_{X2}\\ u_{X3}\end{Bmatrix} + m\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix}\ddot{u}_{X2}\\ \ddot{u}_{X3}\end{Bmatrix}\right).$$

Principle of virtual work and the fundamental lemma of variation calculus imply a set of ordinary differential equations:

$$\frac{EA}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} + m\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{u}_{X2} \\ \ddot{u}_{X3} \end{Bmatrix} = 0 \quad \Leftarrow$$

or written in the standard form

$$\begin{Bmatrix} \ddot{u}_{X2} \\ \ddot{u}_{X3} \end{Bmatrix} + \mathbf{\Omega}^2 \begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} = 0, \text{ where } \mathbf{\Omega}^2 = \frac{EA}{mL}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

2p The angular speeds of free vibrations are the eigenvalues of matrix $\mathbf{\Omega}$. The easiest way to find the eigenvalues uses the result that the eigenvalues of $\mathbf{\Omega}$ are square roots of those for $\mathbf{\Omega}^2$. Let us consider first the eigenvalues of $\mathbf{\Omega}^2$

$$\det(\mathbf{\Omega}^2 - \lambda \mathbf{I}) = \det(\frac{EA}{mL}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = 0 \Rightarrow$$

$$\det(\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \gamma \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = (2-\gamma)(1-\gamma) - 1 = 0 \quad \text{where} \quad \gamma = \frac{mL}{EA}\lambda .$$

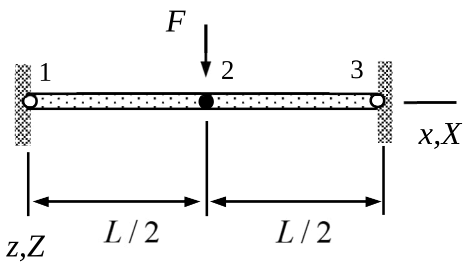
The roots are

$$\gamma = \frac{3 \pm \sqrt{5}}{2} \quad \text{so} \quad \lambda = \frac{EA}{mL}\gamma = \frac{EA}{mL}\frac{3 \pm \sqrt{5}}{2}.$$

Eigenvalues of $\mathbf{\Omega}$ are square roots of the eigenvalues of $\mathbf{\Omega}^2$

$$\omega_1 = \sqrt{\lambda_1} = \sqrt{\frac{EA}{mL}\frac{3+\sqrt{5}}{2}} \quad \text{and} \quad \omega_2 = \sqrt{\lambda_2} = \sqrt{\frac{EA}{mL}\frac{3-\sqrt{5}}{2}}. \quad \Leftarrow$$

Find the relationship between the force F acting on the beam and the displacement u_{Z2} of its centerpoint. Combine the virtual work density expressions for the beam bending mode of *small (linear) displacement analysis* and that for the bar mode of *large displacement analysis*. Assume $u = v = 0$ and use the approximation $w = u_{Z2} 4(x/L)(1 - x/L)$ for the transverse displacement. Assume that material parameter $C = E$ and cross-sectional area $A^\circ = A$.



Solution

Virtual work density expressions for the beam bending mode for the small displacement analysis and that for the bar mode for large displacement analysis are given by

$$\delta w_{\Omega}^{\text{int}} = -\frac{d^2\delta w}{dx^2} EI_{yy} \frac{d^2 w}{dx^2},$$

$$\delta w_{\Omega^\circ}^{\text{int}} = -\left(\frac{d\delta u}{dx} + \frac{du}{dx}\frac{d\delta u}{dx} + \frac{dv}{dx}\frac{d\delta v}{dx} + \frac{dw}{dx}\frac{d\delta w}{dx}\right) CA^\circ \left[\frac{du}{dx} + \frac{1}{2}\left(\frac{du}{dx}\right)^2 + \frac{1}{2}\left(\frac{dv}{dx}\right)^2 + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right].$$

The latter is based on the Green-Lagrange strain definition which is physically correct also when rotations/displacements are large and considers also the off-axis displacement components.

1p The quadratic approximation to the transverse displacement gives

$$w = u_{Z2} 4\frac{x}{L}\left(1 - \frac{x}{L}\right) = \frac{u_{Z2}}{L^2} 4(Lx - x^2) \quad \Rightarrow \quad \frac{dw}{dx} = \frac{u_{Z2}}{L^2} 4(L - 2x) \quad \Rightarrow \quad \frac{d^2 w}{dx^2} = -8\frac{u_{Z2}}{L^2} \quad \text{and}$$

$$\delta w = \frac{\delta u_{Z2}}{L^2} 4(Lx - x^2) \quad \Rightarrow \quad \frac{d\delta w}{dx} = \frac{\delta u_{Z2}}{L^2} 4(L - 2x) \quad \Rightarrow \quad \frac{d^2\delta w}{dx^2} = -8\frac{\delta u_{Z2}}{L^2}.$$

3p With the assumptions $u = v = 0$, $C = E$ and $A^\circ = A$, virtual work density of the internal forces simplifies to

$$\delta w^{\text{int}} = -\frac{d^2\delta w}{dx^2} EI \frac{d^2 w}{dx^2} - \frac{d\delta w}{dx}\frac{1}{2} EA\left(\frac{dw}{dx}\right)^3.$$

Substituting into the density expression and integration over the element gives the virtual work expression for internal forces

$$\delta w^{\text{int}} = -\frac{\delta u_{Z2}}{L^2} 64EI \frac{u_{Z2}}{L^2} - \frac{\delta u_{Z2}}{L^2} 128EA\left(\frac{u_{Z2}}{L^2}\right)^3 (L - 2x)^4 \quad \Rightarrow$$

$$\delta W^{\text{int}} = \int_0^L \delta w^{\text{int}} dx = -\frac{\delta u_{Z2}}{L} 64\left[\frac{EI}{L}\left(\frac{u_{Z2}}{L}\right) + \frac{2}{5} EAL\left(\frac{u_{Z2}}{L}\right)^3\right]. \qquad \left(\int_0^L (L - 2x)^4 dx = \frac{L^5}{5}\right)$$

1p Virtual work for the external force

$$\delta W^{\text{ext}} = \delta u_{Z2} F = (\frac{\delta u_{Z2}}{L}) 64 \frac{LF}{64} \,.$$

1p Virtual work expression

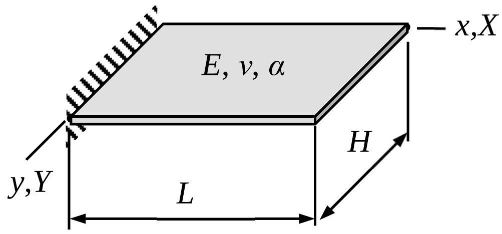
$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = -\frac{\delta u_{Z2}}{L} 64 [\frac{EI}{L}(\frac{u_{Z2}}{L}) + \frac{2}{5} EAL (\frac{u_{Z2}}{L})^3 - \frac{LF}{64}] \,.$$

Principle of virtual work and the fundamental lemma of variational calculus imply the equilibrium equations

$$\frac{EI}{L}(\frac{u_{Z2}}{L}) + \frac{2}{5} EAL (\frac{u_{Z2}}{L})^3 - \frac{LF}{64} = 0 \quad \text{or} \quad (\text{with} \;\; a = \frac{u_{Z2}}{L})$$

$$\frac{EI}{L} a + \frac{2}{5} EALa^3 - \frac{LF}{64} = 0 \quad \Leftrightarrow \quad a + \frac{2}{5} \frac{AL^2}{I} a^3 = \frac{1}{64} \frac{FL^2}{EI} \,. \quad \leftarrow$$

A bending plate of thickness t, which is clamped on one edge, is assembled at constant temperature $2\vartheta^\circ$. Find the transverse displacement due to heating on the upper side $z=-t/2$ and cooling on the lower side $z=t/2$ resulting in surface temperatures $3\vartheta^\circ$ and ϑ°, respectively. Assume that temperature in plate is linear in z and does not depend on x or y. Use approximation $w=a_0x^2$ in which a_0 is the parameter to be determined. Young's modulus E, Poisson's ratio ν, and coefficient of thermal expansion α are constants.



Solution

Assuming that the material coordinate system is chosen so that the plate bending and thin slab modes decouple, the plate model virtual work densities of internal force and coupling terms are given by

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial^2\delta w/\partial x^2 \\ \partial^2\delta w/\partial y^2 \\ 2\partial^2\delta w/\partial x\partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12}\frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}\begin{Bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ 2\partial^2 w/\partial x\partial y \end{Bmatrix},$$

$$\delta w_\Omega^{\text{cpl}} = -\begin{Bmatrix} \partial^2\delta w/\partial x^2 \\ \partial^2\delta w/\partial y^2 \end{Bmatrix}^{\mathrm{T}} \int z\Delta\vartheta dz \frac{\alpha E}{1-\nu}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

3p The coupling term contains an integral of temperature change over the thickness of the plate. First as temperature is linear

$$\vartheta_{\text{asm}}(z) = 2\vartheta^\circ \quad \text{and} \quad \vartheta_{\text{fin}}(z) = 2\vartheta^\circ(1-\frac{z}{t}) \quad \Rightarrow \quad \Delta\vartheta = \vartheta_{\text{fin}} - \vartheta_{\text{asm}} = -2\vartheta^\circ\frac{z}{t}.$$

So the integral of the coupling term becomes

$$\int z\Delta\vartheta dz = -2\frac{\vartheta^\circ}{t}\int_{-t/2}^{t/2} z^2 dz = -2\frac{\vartheta^\circ}{t}\frac{1}{3}(\frac{t^3}{8}+\frac{t^3}{8}) = -\frac{1}{6}\vartheta^\circ t^2.$$

and virtual work density for the coupling part

$$\delta w_\Omega^{\text{cpl}} = \begin{Bmatrix} 2\delta \mathrm{a} \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{1}{6}\vartheta^\circ t^2 \frac{\alpha E}{1-\nu}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \delta \mathrm{a}\frac{1}{3}\vartheta^\circ t^2 \frac{\alpha E}{1-\nu}.$$

Virtual work expression is integral of the density over the domain occupied by the plate/element. As integrand is constants, it is enough to multiply by the area of the mid-plane to get

$$\delta W^{\text{cpl}} = \int_\Omega \delta w_\Omega^{\text{cpl}} dA = LH\delta \mathrm{a}\frac{1}{3}\vartheta^\circ t^2 \frac{\alpha E}{1-\nu}.$$

2p Approximation to the transverse displacement

$$w = \mathrm{a}\,x^2 \quad \Rightarrow \quad \frac{\partial^2 w}{\partial x^2} = 2\mathrm{a} \;\text{ and }\; \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial x \partial y} = 0\,.$$

When the approximation is substituted there, virtual work density of the internal forces simplifies to

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} 2\delta\mathrm{a} \\ 0 \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12}\frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}\begin{Bmatrix} 2\mathrm{a} \\ 0 \\ 0 \end{Bmatrix} = -\delta\mathrm{a}\frac{t^3}{3}\frac{E}{1-\nu^2}\mathrm{a}\,.$$

Virtual work expression is integral of the density over the domain occupied by the plate/element. As integrand is constants, it is enough to multiply by the area of the mid-plane to get

$$\delta W^{\text{int}} = \int_{\Omega} \delta w_{\Omega}^{\text{int}} dA = -LH\delta\mathrm{a}\frac{t^3}{3}\frac{E}{1-\nu^2}\mathrm{a}\,.$$

1p Virtual work expression is the sum of the parts

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{cpl}} = -\delta\mathrm{a}\left(LH\frac{t^3}{3}\frac{E}{1-\nu^2}\mathrm{a} - LH\frac{1}{3}\vartheta^{\circ}t^2\frac{\alpha E}{1-\nu}\right).$$

Principle of virtual work $\delta W = 0\ \forall \delta\mathbf{a}$ and the fundamental lemma of variation calculus give

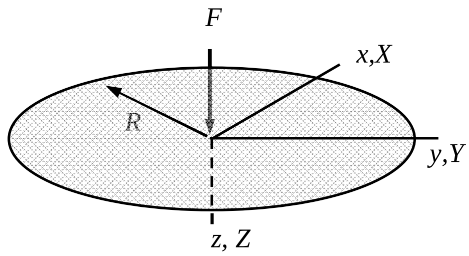
$$\delta W = -\delta\mathrm{a}\left(LH\frac{t^3}{3}\frac{E}{1-\nu^2}\mathrm{a} - LH\frac{1}{3}\vartheta^{\circ}t^2\frac{\alpha E}{1-\nu}\right) = 0 \quad \forall\delta\mathrm{a} \quad \Leftrightarrow$$

$$LH\frac{t^3}{3}\frac{E}{1-\nu^2}\mathrm{a} - LH\frac{1}{3}\vartheta^{\circ}t^2\frac{\alpha E}{1-\nu} = 0 \quad \Rightarrow \quad \mathrm{a} = (1+\nu)\frac{1}{t}\alpha\vartheta^{\circ}\,.$$

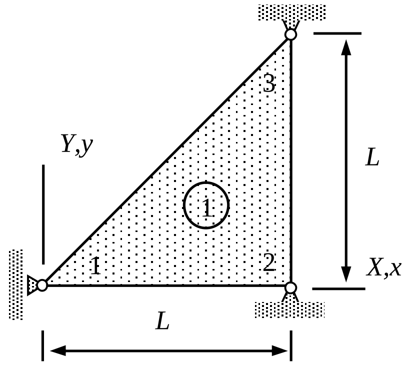
Transverse displacement $w = (1+\nu)\alpha\vartheta^{\circ}\frac{1}{t}x^2$. ←

MEC-E8001 Finite Element Analysis, Onsite exam 22.02.2023

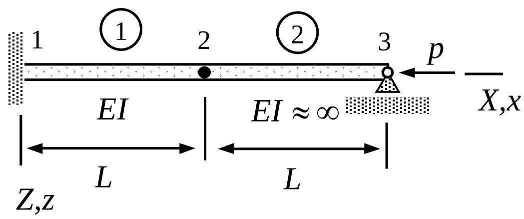
1. A circular plate of radius R, which is simply supported at the outer edge, is loaded by force F at the center point. Use the Kirchhoff plate model to find the transverse displacement at the center point. Use the approximation $w = a_0(x^2 + y^2 - R^2)$ for the transverse displacement. Material properties E, ν and thickness t are constants.



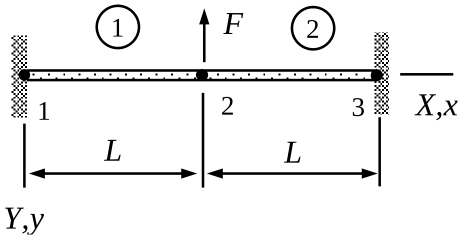
2. Determine the angular speed of free vibrations for the thin triangular slab shown. Assume plane stress conditions. The material properties E, ν, ρ and thickness h of the slab are constants. Use the approximations $u = 0$ and $v = (1 - x/L)u_{Y1}$ in which the nodal value u_{Y1} is a function of time.



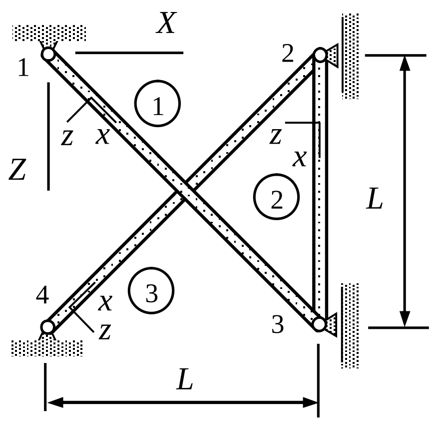
3. The structure shown consist of two beams, each of length L. As beam 2 is much stiffer than beam 1, the setting is simplified by considering beam 2 as a rigid body. Displacements are confined to the xz-plane. Cross-section properties of beam 1 are A and I and Young's modulus of the material is E. Determine the buckling load p_{cr}.



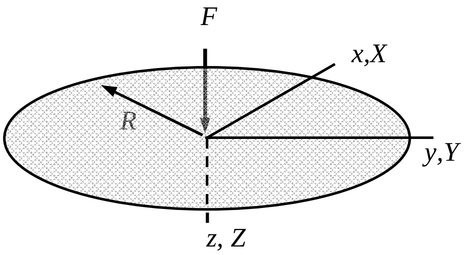
4. Determine the displacement at node 2 of the elastic bar shown by the large deformation theory. Take into account only the transverse displacement u_{Y2} ($u_{X2} = 0$). When $F = 0$, the cross-sectional area and length of the bar are A and L, respectively. Constitutive equation of the material is $S_{xx} = C\text{E}_{xx}$, in which C is constant. Use two elements with linear shape functions.



5. Determine the static displacements $u_{Z2} = -u_{Z3}$ of nodes 2 and 3 due to the temperature increase $\Delta\vartheta$ at nodes 2 and 3 (actually in the wall). The material constants are E and α. The cross-sectional area of bar 1 and 3 is A and that of bar 2 is $\sqrt{2}A$. The initial temperature is $\vartheta°$.



A circular plate of radius R, which is simply supported at the outer edge, is loaded by force F at the center point. Use the Kirchhoff plate model to find the transverse displacement at the center point. Use the approximation $w = a_0(x^2 + y^2 - R^2)$ for the transverse displacement. Material properties E, ν and thickness t are constants.



Solution

Assuming that the material coordinate system is chosen so that the plate bending and thin slab modes decouple, it is enough to consider the virtual work densities of the bending mode only

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial^2 \delta w / \partial x^2 \\ \partial^2 \delta w / \partial y^2 \\ 2\partial^2 \delta w / \partial x \partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12} [E]_\sigma \begin{Bmatrix} \partial^2 w / \partial x^2 \\ \partial^2 w / \partial y^2 \\ 2\partial^2 w / \partial x \partial y \end{Bmatrix}, \ \delta w_\Omega^{\text{ext}} = \delta w f_z .$$

in which the elasticity matrix of plane stress

$$[E]_\sigma = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}.$$

In the present case, distributed force vanishes i.e. $f_z = 0$ and the point force is taken into account by a point force element.

2p Approximation to the transverse displacement is given by (a_0 is not associated with any point but it just a parameter of the approximation)

$$w(x,y) = a_0(x^2 + y^2 - R^2) \quad \Rightarrow \quad \frac{\partial^2 w}{\partial x^2} = 2a_0, \ \frac{\partial^2 w}{\partial y^2} = 2a_0, \text{ and } \frac{\partial^2 w}{\partial x \partial y} = 0 .$$

When the approximation is substituted there, virtual work density of internal forces simplifies to

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} 2\delta a_0 \\ 2\delta a_0 \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{t^3 E}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} 2a_0 \\ 2a_0 \\ 0 \end{Bmatrix} = -\delta a_0 \frac{2}{3} \frac{t^3 E}{1-\nu} a_0 .$$

Virtual work expression of the plate bending element (element 1 here) is integral of the virtual work density over the domain occupied by the element. As the density expression is constant it is enough to multiply by the area of the domain

$$\delta W^1 = \int_\Omega \delta w_\Omega^{\text{int}} dA = \delta w_\Omega^{\text{int}} \pi R^2 = -\delta a_0 \frac{2\pi}{3} \frac{R^2 t^3 E}{1-\nu} a_0 .$$

2p Virtual work expression of the point force (element 2 here) follows from the definition of work (notice the use of virtual displacement at the point of action $x = y = 0$)

$$\delta W^2 = \delta w(0,0) F = -\delta a_0 R^2 F \ .$$

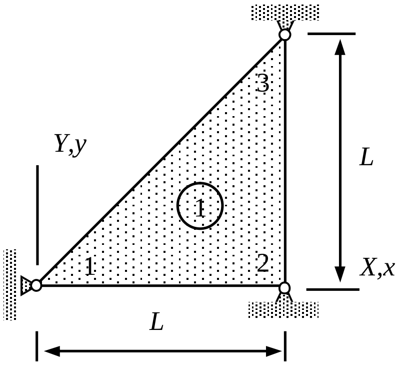
1p Principle of virtual work and the fundamental lemma of variation calculus give

$$\delta W = \delta W^1 + \delta W^2 = -\delta a_0 \left(\frac{2\pi}{3} \frac{R^2 t^3 E}{1-\nu} a_0 + R^2 F\right) = 0 \quad \Rightarrow \quad a_0 = -\frac{3}{2\pi} \frac{F}{t^3 E}(1-\nu) \ .$$

1p Displacement at the center point

$$w(0,0) = -a_0 R^2 = \frac{3}{2\pi} \frac{FR^2}{t^3 E}(1-\nu) \ . \quad \leftarrow$$

Determine the angular speed of free vibrations for the thin triangular slab shown. Assume plane stress conditions. The material properties E, ν, ρ and thickness h of the slab are constants. Use the approximations $u=0$ and $v=(1-x/L)u_{Y1}$ in which the nodal value u_{Y1} is a function of time.



Solution

The virtual work densities of the internal and inertia forces for the thin slab model (plane stress conditions assumed) are given by

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta u/\partial y + \partial\delta v/\partial x \end{Bmatrix}^{\mathrm{T}} t[E]_{\sigma} \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix} \text{ and } \delta w_{\Omega}^{\text{ine}} = -\begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} t\rho \begin{Bmatrix} \ddot{u} \\ \ddot{v} \end{Bmatrix}$$

where the elasticity matrix of the plane stress

$$[E]_{\sigma} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}.$$

1p The approximations to the displacement components are given (the linear interpolants of the nodal values can also be deduced easily from the figure). Hence

$$u = 0 \text{ and } v = (1-\frac{x}{L})u_{Y1} \quad \Rightarrow$$

$$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial \delta u}{\partial x} = 0, \quad \frac{\partial \delta u}{\partial y} = 0, \quad \delta u = 0, \quad \ddot{u} = 0$$

$$\frac{\partial v}{\partial x} = -\frac{1}{L}u_{Y1}, \quad \frac{\partial v}{\partial y} = 0, \quad \frac{\partial \delta v}{\partial x} = -\frac{1}{L}\delta u_{Y1}, \quad \frac{\partial \delta v}{\partial y} = 0, \quad \delta v = (1-\frac{x}{L})\delta u_{Y1}, \quad \ddot{v} = (1-\frac{x}{L})\ddot{u}_{Y1}$$

4p When the approximations are substituted there, virtual work density of the internal and inertia forces simplify to

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} 0 \\ 0 \\ -\delta u_{Y1}/L \end{Bmatrix}^{\mathrm{T}} \frac{hE}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ -u_{Y1}/L \end{Bmatrix} = -\delta u_{Y1}\frac{h}{L^2}Gu_{Y1},$$

$$\delta w_{\Omega}^{\text{ine}} = -\begin{Bmatrix} 0 \\ (1-\frac{x}{L})\delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} h\rho \begin{Bmatrix} 0 \\ (1-\frac{x}{L})\ddot{u}_{Y1} \end{Bmatrix} = -\delta u_{Y1} h\rho (1-\frac{x}{L})^2 \ddot{u}_{Y1}.$$

Integrations over the triangular domain of the element gives

$$\delta W^{\text{int}} = \int_A \delta w_\Omega^{\text{int}} \mathrm{d}A = \delta w_\Omega^{\text{int}} \frac{L^2}{2} = -\delta u_{Y1} \frac{h}{2} G u_{Y1} ,$$

$$\delta W^{\text{ine}} = \int_A \delta w_\Omega^{\text{ine}} \mathrm{d}A = \int_0^L (\int_0^x \delta w_\Omega^{\text{ine}} dy) dx \quad \Rightarrow$$

$$\delta W^{\text{ine}} = -\delta u_{Y1} h\rho \int_0^L (\int_0^x (1-\frac{x}{L})^2 dy) dx\, \ddot{u}_{Y1} = -\delta u_{Y1} h\rho \int_0^L x(1-\frac{x}{L})^2 dx\, \ddot{u}_{Y1} = -\delta u_{Y1} h\rho \frac{1}{12} L^2\, \ddot{u}_{Y1} .$$

$^{\text{ine}}$ Virtual work expression of the structure takes the form

$$\delta W = -\delta u_{Y1} (\frac{t}{2} G u_{Y1} + h\rho \frac{1}{12} L^2\, \ddot{u}_{Y1}) .$$

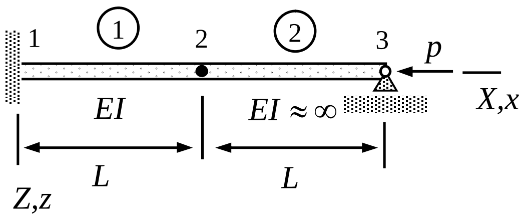
Principle of virtual work $\delta W = 0\ \forall \delta \mathrm{a}$ and the fundamental lemma of variation calculus give

$$\frac{h}{2} G u_{Y1} + h\rho \frac{1}{12} L^2\, \ddot{u}_{Y1} = 0 \quad \Leftrightarrow \quad \ddot{u}_{Y1} + 6 \frac{G}{\rho L^2} u_{Y1} = 0 .$$

1p As the ordinary differential equation is of the form $\ddot{u} + \omega^2 u = 0$, the angular speed of free vibrations is

$$\omega = \sqrt{6 \frac{G}{\rho L^2}} . \quad \Leftarrow$$

The structure shown consist of two beams, each of length L. As beam 2 is much stiffer than beam 1, the setting is simplified by considering beam 2 as a rigid body. Displacements are confined to the xz–plane. Cross-section properties of beam 1 are A and I and Young's modulus of the material is E. Determine the buckling load p_{cr}.



Solution

The axial forces in the beams follow directly from a free body diagram and it is enough to consider virtual work associated with the bending and interaction between bar and bending modes

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI_{yy}}{h^3} \begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix},$$

$$\delta W^{\text{sta}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{N}{30h} \begin{bmatrix} 36 & -3h & -36 & -3h \\ -3h & 4h^2 & 3h & -h^2 \\ -36 & 3h & 36 & 3h \\ -3h & -h^2 & 3h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix}.$$

3p Also, as beam 2 is considered as rigid, virtual work expression of internal forces vanish and modelling uses kinematical constraints $\vec{u}_{\text{B}} = \vec{u}_{\text{A}} + \vec{\theta}_{\text{A}} \times \vec{\rho}_{\text{AB}}$ and $\vec{\theta}_{\text{B}} = \vec{\theta}_{\text{A}}$. Let us choose A to be node 3 and B as node 2. Then

$u_{Z2} = \theta_{Y3} L$ and $\theta_{Y2} = \theta_{Y3}$.

Although axial displacement is non-zero, it is not needed as the axial force in the structure $N = -p$ (negative means compression) can be deduced without calculations on the axial displacement.

2p The internal force and coupling parts of beam 1 take the forms ($u_{z1} = 0$, $\theta_{y1} = 0$, $u_{z2} = u_{Z2} = \theta_{Y3} L$, $\theta_{y2} = \theta_{Y2} = \theta_{Y3}$)

$$\delta W^{\text{int}} = -\begin{Bmatrix} 0 \\ 0 \\ L\delta\theta_{Y3} \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ L\theta_{Y3} \\ \theta_{Y3} \end{Bmatrix} = -\delta\theta_{Y3} 28 \frac{EI}{L} \theta_{Y3},$$

$$\delta W^{\text{sta}} = -\begin{Bmatrix} 0 \\ 0 \\ L\delta\theta_{Y3} \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{-p}{30L} \begin{bmatrix} 36 & -3L & -36 & -3L \\ -3L & 4L^2 & 3L & -L^2 \\ -36 & 3L & 36 & 3L \\ -3L & -L^2 & 3L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ L\theta_{Y3} \\ \theta_{Y3} \end{Bmatrix} = \delta\theta_{Y3} \frac{46}{30} pL\theta_{Y3}.$$

Virtual work expression of the structure is the sum of the internal and stability parts

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{sta}} = -\delta\theta_{Y3}(28\frac{EI}{L} - \frac{46}{30}pL)\theta_{Y3}.$$

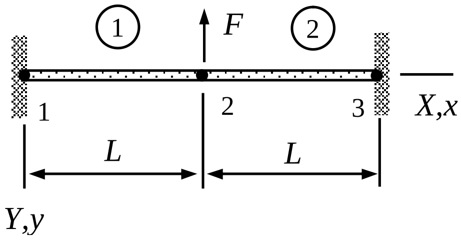
1p Principle of virtual work and the fundamental lemma of variation calculus imply that

$$(28\frac{EI}{L} - \frac{46}{30}pL)\theta_{Y3} = 0.$$

In stability analysis the aim is to find a non-zero equilibrium solution so $\theta_{Y3} \neq 0$ and the multiplier needs to vanish for a solution. Therefore, the buckling load is given by

$$p_{\text{cr}} = \frac{30}{46}28\frac{EI}{L^2} = \frac{420}{23}\frac{EI}{L^2}. \quad \leftarrow$$

Determine the displacement at node 2 of the elastic bar shown by the large deformation theory. Take into account only the transverse displacement u_{Y2} ($u_{X2}=0$). When $F=0$, the cross-sectional area and length of the bar are A and L, respectively. Constitutive equation of the material is $S_{xx}=C\mathrm{E}_{xx}$, in which C is constant. Use two elements with linear shape functions.



Solution

Virtual work density of the non-linear bar model

$$\delta w_{\Omega^\circ}^{\text{int}}=-(\frac{d\delta u}{dx}+\frac{du}{dx}\frac{d\delta u}{dx}+\frac{dv}{dx}\frac{d\delta v}{dx}+\frac{dw}{dx}\frac{d\delta w}{dx})CA^\circ[\frac{du}{dx}+\frac{1}{2}(\frac{du}{dx})^2+\frac{1}{2}(\frac{dv}{dx})^2+\frac{1}{2}(\frac{dw}{dx})^2]$$

is based on the Green-Lagrange strain definition which is physically correct also when rotations/displacements are large. The expression depends on all displacement components, material property is denoted by C (constitutive equation $S_{xx}=C\mathrm{E}_{xx}$), and the superscript in the cross-sectional area A° (and in other quantities) refers to the initial geometry (strain and stress vanishes). Otherwise, equilibrium equations follow in the same manner as in the linear case.

1p For element 1, the non-zero displacement components is $u_{y2}=u_{Y2}$. As the initial length of the element $h^\circ=L$, linear approximations to the displacement components

$$u=w=0 \text{ and } v=\frac{x}{L}u_{Y2} \quad\Rightarrow\quad \frac{du}{dx}=\frac{dw}{dx}=0 \text{ and } \frac{dv}{dx}=\frac{u_{Y2}}{L}.$$

3p When the approximation is substituted there, virtual work density of the internal forces and thereby the virtual work expression (density is constant) simplify to

$$\delta w_{\Omega^\circ}^{\text{int}}=-\frac{\delta u_{Y2}}{L}\frac{u_{Y2}}{L}\frac{CA}{2}(\frac{u_{Y2}}{L})^2 \quad\Rightarrow\quad \delta W^1=-\delta u_{Y2}\frac{CA}{2}(\frac{u_{Y2}}{L})^3.$$

For element 2, the non-zero displacement component $u_{y2}=u_{Y2}$. As the initial length of the element $h^\circ=L$, linear approximations to the displacement components

$$u=w=0 \text{ and } v=(1-\frac{x}{L})u_{Y2} \quad\Rightarrow\quad \frac{du}{dx}=\frac{dw}{dx}=0 \text{ and } \frac{dv}{dx}=-\frac{u_{Y2}}{L}.$$

When the approximation is substituted there, virtual work density of the internal forces and thereby the virtual work expression (density is constant) simplifies to

$$\delta w_{\Omega^\circ}^{\text{int}}=-\frac{\delta u_{Y2}}{L}\frac{u_{Y2}}{L}\frac{CA}{2}(\frac{u_{Y2}}{L})^2 \quad\Rightarrow\quad \delta W^2=-\delta u_{Y2}\frac{CA}{2}(\frac{u_{Y2}}{L})^3.$$

Virtual work expression of the point force is

$$\delta W^3=-F\delta u_{Y2}.$$

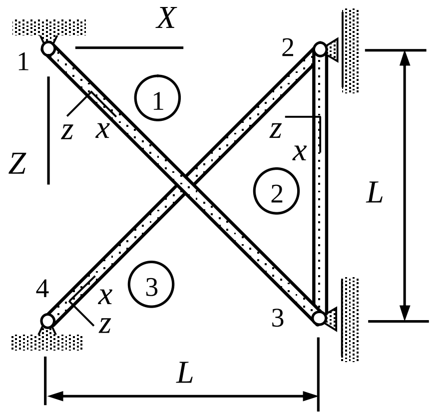
2p Virtual work expression of the structure is obtained as the sum of the element contributions

$$\delta W = -\delta u_{Y2}[\frac{CA}{2}(\frac{u_{Y2}}{L})^3 + \frac{CA}{2}(\frac{u_{Y2}}{L})^3 + F].$$

Principle of virtual work and the fundamental lemma of variation calculus imply that

$$(\frac{u_{Y2}}{L})^3 + \frac{F}{CA} = 0 \quad \Rightarrow \quad u_{Y2} = -(\frac{FL^3}{CA})^{1/3}.$$

Determine the static displacements $u_{Z2} = -u_{Z3}$ of nodes 2 and 3 due to the temperature increase $\Delta\vartheta$ at nodes 2 and 3 (actually in the wall). The material constants are E and α. The cross-sectional area of bar 1 and 3 is A and that of bar 2 is $\sqrt{2}A$. The initial temperature is ϑ°.



Solution

As temperature is known and the external distributed force vanishes, only the virtual work expressions of the internal and coupling parts

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} \text{ and } \delta W^{\text{cpl}} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x1} \end{Bmatrix}^{\mathrm{T}} \frac{\alpha EA}{2} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \Delta\vartheta_1 \\ \Delta\vartheta_2 \end{Bmatrix}$$

are needed in the calculations. Term $\Delta\vartheta = \vartheta - \vartheta^\circ$ is the difference between temperature at the deformed and initial geometries.

5p The nodal displacements and temperatures of bar 1 $u_{x1} = 0$, $u_{x3} = u_{Z3} / \sqrt{2}$, $\Delta\vartheta_1 = \vartheta^\circ - \vartheta^\circ = 0$, and $\Delta\vartheta_3 = \Delta\vartheta$ give (notice that the variation of a given function is always zero)

$$\delta W^1 = -\begin{Bmatrix} 0 \\ \delta u_{Z3} / \sqrt{2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{\sqrt{2}L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ u_{Z3} / \sqrt{2} \end{Bmatrix} - \frac{\alpha EA}{2} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ \Delta\vartheta \end{Bmatrix} \right) \quad \Leftrightarrow$$

$$\delta W^1 = -\delta u_{Z3} \left(\frac{EA}{2\sqrt{2}L} u_{Z3} - \frac{\alpha EA}{2\sqrt{2}} \Delta\vartheta \right).$$

The nodal displacements and temperatures of bar 2 $u_{x2} = u_{Z2} = -u_{Z3}$, $u_{x3} = u_{Z3}$, $\Delta\vartheta_2 = \Delta\vartheta$, and $\Delta\vartheta_3 = \Delta\vartheta$ give

$$\delta W^2 = -\begin{Bmatrix} -\delta u_{Z3} \\ \delta u_{Z3} \end{Bmatrix}^{\mathrm{T}} \left(\frac{E\sqrt{2}A}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} -u_{Z3} \\ u_{Z3} \end{Bmatrix} - \frac{\alpha E\sqrt{2}A}{2} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \Delta\vartheta \\ \Delta\vartheta \end{Bmatrix} \right) \quad \Leftrightarrow$$

$$\delta W^2 = -\delta u_{Z3} \left(4\sqrt{2} \frac{EA}{L} u_{Z3} - 4\sqrt{2} \frac{\alpha EA}{2} \Delta\vartheta \right).$$

The nodal displacements and temperatures of bar 3 $u_{x4} = 0$, $u_{x2} = -u_{Z2} / \sqrt{2} = u_{Z3} / \sqrt{2}$, $\Delta\vartheta_1 = \vartheta^\circ - \vartheta^\circ = 0$, and $\Delta\vartheta_3 = \Delta\vartheta$ give

$$\delta W^3 = -\begin{Bmatrix} 0 \\ \delta u_{Z3} / \sqrt{2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{\sqrt{2}L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ u_{Z3} / \sqrt{2} \end{Bmatrix} - \frac{\alpha EA}{2} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ \Delta\vartheta \end{Bmatrix} \right) \quad \Leftrightarrow$$

$$\delta W^3 = -\delta u_{Z3} \left(\frac{EA}{2\sqrt{2}L} u_{Z3} - \frac{\alpha EA}{2\sqrt{2}} \Delta\vartheta \right).$$

1p Virtual work expression of the structure is the sum of element contributions

$$\delta W = -\delta u_{Z3} 2(\frac{EA}{2\sqrt{2}L} u_{Z3} - \frac{\alpha EA}{2\sqrt{2}} \Delta\vartheta) - \delta u_{Z3}(4\sqrt{2}\frac{EA}{L} u_{Z3} - 4\sqrt{2}\frac{\alpha EA}{2}\Delta\vartheta) \quad \Leftrightarrow$$

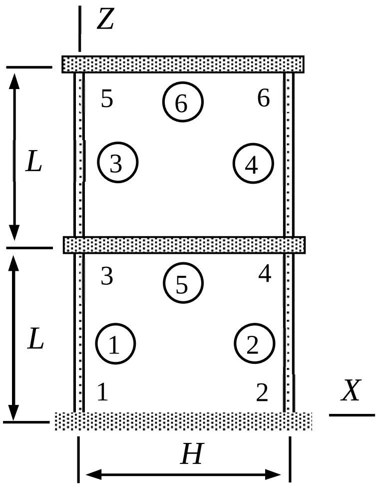
$$\delta W = -\delta u_{Z3}(9\frac{EA}{\sqrt{2}L} u_{Z3} - 5\frac{\alpha EA}{\sqrt{2}}\Delta\vartheta) .$$

Principle of virtual work and the fundamental lemma of variation calculus imply

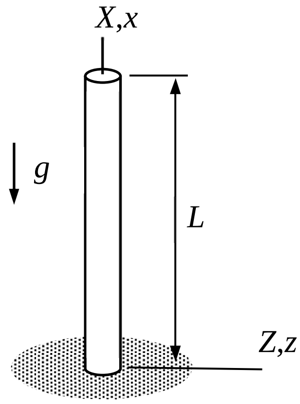
$$\frac{9}{\sqrt{2}L} EAu_{Z3} - \frac{5}{\sqrt{2}} EA\alpha\Delta\vartheta = 0 \quad \Leftrightarrow \quad u_{Z3} = \frac{5}{9}\alpha L\Delta\vartheta \ . \quad \leftarrow$$

MEC-E8001 Finite Element Analysis, online exam 21.02.2024

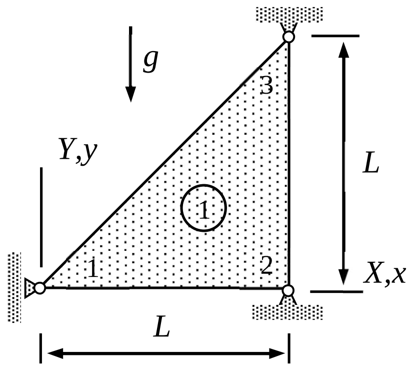
1. The XZ – plane model of a building shown consists of two rigid floors of mass m, each, and four columns modelled as massless bending beams. The structure is welded so the displacements and rotations of the floors and columns coincide at the contact points. Determine the angular speeds of the free horizontal vibrations of the structure. Young's modulus of the beam material and the second moment of the cross-section are E and I, respectively.



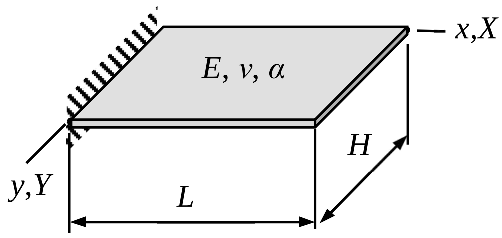
2. A building of height L is modelled as a beam loaded by its own weight. Cross-section of the building is idealized as a circle of radius r. Find the critical radius r_{cr} for the buckling to occur. Material properties ρ, E and acceleration by gravity g are constants. Use the approximations $u = a(x/L)$, $v = 0$ and $w = b(x/L)^2$, in which a and b are parameters of the approximations. Start with the virtual work densities for the beam model.



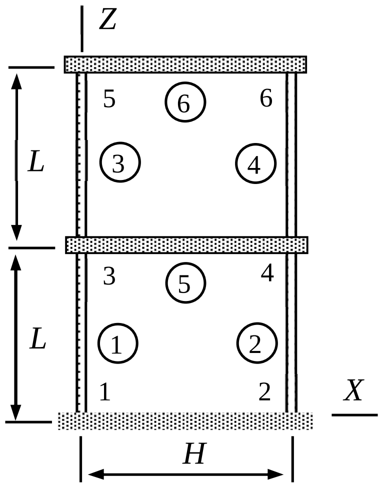
3. A thin triangular slab which is loaded by its own weight can move vertically at node 1 whereas nodes 2 and 3 are fixed. Assuming plane stress conditions, derive the equilibrium equation for the structure according to the large displacement theory. Material parameters C, ν, ρ and thickness t at the initial geometry of the slab are constants.



4. A bending plate of thickness t, which is clamped on one edge, is assembled at constant temperature $2\vartheta°$. Find the transverse displacement due to heating on the upper side $z = -t/2$ and cooling on the lower side $z = t/2$ resulting in surface temperatures $3\vartheta°$ and $\vartheta°$, respectively. Assume that temperature in plate is linear in z and does not depend on x or y. Use approximation $w = a_0 x^2$ in which a_0 is the parameter to be determined. Young's modulus E, Poisson's ratio ν, and coefficient of thermal expansion α are constants.



The XZ – plane model of a building shown consists of two rigid floors of mass m, each, and four columns modelled as massless bending beams. The structure is welded so the displacements and rotations of the floors and columns coincide at the contact points. Determine the angular speeds of the free horizontal vibrations of the structure. Young's modulus of the beam material and the second moment of the cross-section are E and I, respectively.



Solution

Beam bending and rigid body model virtual work expression of internal and inertia forces are available in the formulae collection. As beams are assumed to be massless, only the internal part is needed. For the floors, only the translation part applies as the floors translate in horizontal direction. Therefore

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI_{yy}}{h^3} \begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix}, \quad \delta W^{\text{ine}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{y1} \\ \delta u_{z1} \end{Bmatrix}^{\mathrm{T}} m \begin{Bmatrix} \ddot{u}_{x1} \\ \ddot{u}_{y1} \\ \ddot{u}_{z1} \end{Bmatrix}.$$

2p The non-zero displacement components of the structure are $u_{X4} = u_{X3}$ and $u_{X6} = u_{X5}$. Let us start with the element contributions of the beams. Since the beam are assumed to be massless, only the virtual work expressions of the internal forces (available in the formulae collection) are needed.

$$\delta W^1 = \delta W^2 = -\begin{Bmatrix} 0 \\ 0 \\ \delta u_{X3} \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_{X3} \\ 0 \end{Bmatrix} = -\begin{Bmatrix} \delta u_{X3} \\ \delta u_{X5} \end{Bmatrix}^{\mathrm{T}} 12\frac{EI}{L^3} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} u_{X3} \\ u_{X5} \end{Bmatrix}$$

$$\delta W^3 = \delta W^4 = -\begin{Bmatrix} \delta u_{X3} \\ 0 \\ \delta u_{X5} \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} u_{X3} \\ 0 \\ u_{X5} \\ 0 \end{Bmatrix} = -\begin{Bmatrix} \delta u_{X3} \\ \delta u_{X5} \end{Bmatrix}^{\mathrm{T}} 12\frac{EI}{L^3} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{X3} \\ u_{X5} \end{Bmatrix}$$

Then the particles. Element contribution of the rigid body (formulae collection) simplifies to

$$\delta W^5 = -\begin{Bmatrix} \delta u_{X3} \\ 0 \\ 0 \end{Bmatrix}^{\mathrm{T}} m \begin{Bmatrix} \ddot{u}_{X3} \\ 0 \\ 0 \end{Bmatrix} = -\begin{Bmatrix} \delta u_{X3} \\ \delta u_{X5} \end{Bmatrix}^{\mathrm{T}} m \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{u}_{X3} \\ \ddot{u}_{X5} \end{Bmatrix},$$

$$\delta W^6 = -\begin{Bmatrix} \delta u_{X5} \\ 0 \\ 0 \end{Bmatrix}^{\mathrm{T}} m \begin{Bmatrix} \ddot{u}_{X5} \\ 0 \\ 0 \end{Bmatrix} = -\begin{Bmatrix} \delta u_{X3} \\ \delta u_{X5} \end{Bmatrix}^{\mathrm{T}} m \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{u}_{X3} \\ \ddot{u}_{X5} \end{Bmatrix}.$$

2p Virtual work expression of structure is the sum of element contributions.

$$\delta W = \sum \delta W^i = -\begin{Bmatrix} \delta u_{X3} \\ \delta u_{X5} \end{Bmatrix}^{\mathrm{T}} (24\frac{EI}{L^3}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X3} \\ u_{X5} \end{Bmatrix} + m\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{u}_{X3} \\ \ddot{u}_{X5} \end{Bmatrix}).$$

Principle of virtual work and the fundamental lemma of variation calculus imply a set of ordinary differential equations:

$$24\frac{EI}{L^3}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X3} \\ u_{X5} \end{Bmatrix} + m\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{u}_{X3} \\ \ddot{u}_{X5} \end{Bmatrix} = 0 \quad \leftarrow$$

or written in the standard form

$$\begin{Bmatrix} \ddot{u}_{X3} \\ \ddot{u}_{X5} \end{Bmatrix} + \mathbf{\Omega}^2 \begin{Bmatrix} u_{X3} \\ u_{X5} \end{Bmatrix} = 0, \text{ where } \mathbf{\Omega}^2 = 24\frac{EI}{mL^3}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

2p The angular speeds of free vibrations are the eigenvalues of matrix $\mathbf{\Omega}$. The easiest way to find the eigenvalues uses the result that the eigenvalues of $\mathbf{\Omega}$ are square roots of those for $\mathbf{\Omega}^2$. Let us consider first the eigenvalues of $\mathbf{\Omega}^2$

$$\det(\mathbf{\Omega}^2 - \lambda\mathbf{I}) = \det(24\frac{EI}{mL^3}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = 0 \Rightarrow$$

$$\det(\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \gamma\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = (2-\gamma)(1-\gamma) - 1 = 0 \quad \text{where } \gamma = \frac{1}{24}\frac{mL^3}{EI}\lambda.$$

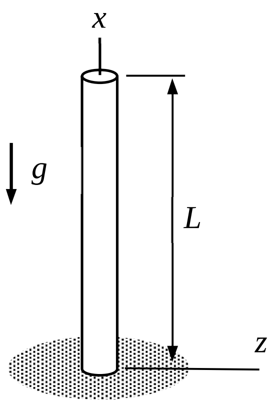
The roots are

$$\gamma = \frac{3 \pm \sqrt{5}}{2} \quad \text{so} \quad \lambda = 24\frac{EI}{mL^3}\gamma = 24\frac{EI}{mL^3}\frac{3 \pm \sqrt{5}}{2}.$$

Eigenvalues of $\mathbf{\Omega}$ are square roots of eigenvalues of $\mathbf{\Omega}^2$

$$\omega_1 = \sqrt{\lambda_1} = \sqrt{12\frac{EI}{mL^3}(3+\sqrt{5})} \quad \text{and} \quad \omega_2 = \sqrt{\lambda_2} = \sqrt{12\frac{EI}{mL^3}(3-\sqrt{5})}. \quad \leftarrow$$

A building of height L is modelled as a beam loaded by its own weight. Cross-section of the building is idealized as a circle of radius r. Find the critical radius r_{cr} for the buckling to occur. Material properties ρ, E and acceleration by gravity g are constants. Use the approximations $u = a(x/L)$, $v = 0$ and $w = b(x/L)^2$, in which a and b are parameters of the approximations. Start with the virtual work densities for the beam model.



Solution

1p Virtual work densities

$$\delta w_\Omega^{\text{int}} = -\frac{d\delta u}{dx} EA \frac{du}{dx}, \quad \delta w_\Omega^{\text{ext}} = \delta u f_x,$$

$$\delta w_\Omega^{\text{int}} = -\frac{d^2\delta w}{dx^2} EI_{yy} \frac{d^2 w}{dx^2}, \quad \delta w_\Omega^{\text{sta}} = -\frac{d\delta w}{dx} N \frac{dw}{dx}, \text{ where } N = EA\frac{du}{dx}$$

take into account the bar and bending modes and their interaction.

2p As the connection in bar and bending modes is one-way, let us start with the bar mode where the external distributed force due to gravity $f_x = -\rho g A$ and $u = a(x/L)$. Virtual work expression

$$\delta W = \int_0^L (\delta w_\Omega^{\text{int}} + \delta w_\Omega^{\text{ext}}) dx = \int_0^L (-\frac{\delta a}{L} EA \frac{a}{L} - \delta a \frac{x}{L} \rho g A) dx = -\delta a (\frac{EA}{L} a + \frac{1}{2} \rho g L A)$$

and principle of virtual work $\delta W = 0 \quad \forall \delta a$ and the fundamental lemma of variational calculation give

$$a = -\frac{1}{2}\frac{\rho g L^2}{E} \quad \text{hence} \quad u = a\frac{x}{L} = -\frac{1}{2}\frac{\rho g L}{E} x \quad \text{and} \quad N = EA\frac{du}{dx} = -\frac{1}{2}\rho g L A \quad \text{(constant).}$$

3p The bending mode, composed of the internal and coupling parts, and approximation $w = b(x/L)^2$ give the virtual work expression

$$\delta W = \int_0^L (\delta w_\Omega^{\text{int}} + \delta w_\Omega^{\text{sta}}) dx = \int_0^L (-2\frac{\delta b}{L^2} EI_{yy} 2\frac{b}{L^2} - 2\frac{\delta b}{L^2} x N 2 \frac{b}{L^2} x) dx = -4\delta b (\frac{EI_{yy}}{L^3} - \frac{1}{6}\rho g A) b.$$

Principle of virtual work $\delta W = 0 \quad \forall \delta b$ and the fundamental lemma of variational calculation give the equilibrium equation

$$(\frac{EI_{yy}}{L^3} - \frac{1}{6}\rho g A) b = 0.$$

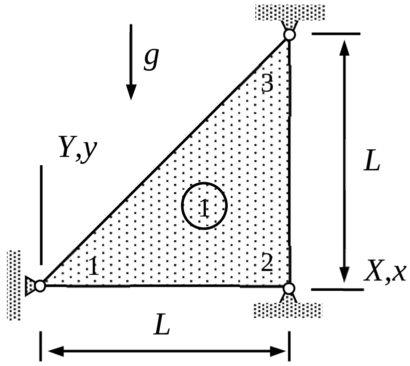
In stability analysis, the goal is to find the condition under which the solution becomes non-unique. Clearly, this is possibly only if

$$\frac{EI_{yy}}{L^3} - \frac{1}{6}\rho g A = 0\,.$$

Finally let us substitute the cross-section moments $A = \pi r^2$, $I_{yy} = \pi r^4 / 4$, and solve for the critical value of radius in terms of the other parameters of the problem

$$r_{cr} = \sqrt{\frac{2}{3}\frac{\rho g L^3}{E}}\,. \leftarrow$$

A thin triangular slab which is loaded by its own weight can move vertically at node 1 whereas nodes 2 and 3 are fixed. Assuming plane stress conditions, derive the equilibrium equation for the structure according to the large displacement theory. Material parameters C, ν, ρ and thickness t at the initial geometry of the slab are constants.



Solution

According to the large displacement theory, virtual work densities of the thin slab model under plane strain conditions are

$$\delta w_{\Omega^\circ}^{\text{int}} = -\begin{Bmatrix} \delta \mathrm{E}_{xx} \\ \delta \mathrm{E}_{yy} \\ 2\delta \mathrm{E}_{xy} \end{Bmatrix}^{\mathrm{T}} \frac{tC}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\begin{Bmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ 2\mathrm{E}_{xy} \end{Bmatrix}, \quad \begin{Bmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ 2\mathrm{E}_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} + \frac{1}{2}(\frac{\partial u}{\partial x})^2 + \frac{1}{2}(\frac{\partial v}{\partial x})^2 \\ \frac{\partial v}{\partial y} + \frac{1}{2}(\frac{\partial u}{\partial y})^2 + \frac{1}{2}(\frac{\partial v}{\partial y})^2 \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} \end{Bmatrix}.$$

$$\delta w_{\Omega^\circ}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} t\rho^\circ \begin{Bmatrix} g_x \\ g_y \end{Bmatrix}$$

in which g_x and g_y are the components of acceleration by gravity and ρ° the density at the initial geometry. Above, constitutive equation is assumed to be of the same form as that for the linear theory with possibly different elasticity parameters C and ν.

1p Shape function $N_1 = 1 - x/L$ of node 1 can be deduced from the figure. Linear approximations to the displacement components and their derivatives are (with $a = u_{Y1}/L$)

$$u = 0 \text{ and } v = (1-\frac{x}{L})u_{Y1} = (L-x)a \quad \Rightarrow \quad \frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial x} = -a, \text{ and } \frac{\partial v}{\partial y} = 0.$$

2p When the approximation is substituted into the Green-Lagrange strain component vector and that is used in the virtual work densities

$$\delta w_{\Omega^\circ}^{\text{int}} = -\begin{Bmatrix} a\delta a \\ 0 \\ -\delta a \end{Bmatrix}^{\mathrm{T}} \frac{tC}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\begin{Bmatrix} a^2/2 \\ 0 \\ -a \end{Bmatrix} = -\delta a \frac{tC}{1-\nu^2}\frac{1}{2}[a^3 + (1-\nu)a],$$

$$\delta w_{\Omega^\circ}^{\text{ext}} = -t\rho g \delta v = -t\rho g(L-x)\delta a.$$

2p Integration over the domain occupied by the body at the initial geometry gives the virtual work expressions

$$\delta W^{\text{int}} = -\delta a \frac{tC}{1-\nu^2}\frac{L^2}{4}[a^3 + (1-\nu)a],$$

$$\delta W^{\text{ext}} = \int_0^L \left(\int_0^x \delta w_{\Omega^\circ}^{\text{ext}} dy\right)dx = \int_0^L \left(\int_0^x -t\rho g(L-x)\delta a dy\right)dx = -\delta a t\rho g \frac{1}{6}L^3 .$$

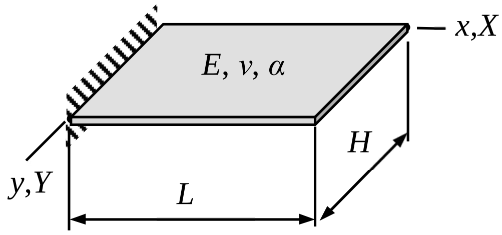
1p Virtual work expression in the sum of the internal and external parts. Written in the standard form

$$\delta W = -\delta a\left[\frac{tC}{1-\nu^2}\frac{L^2}{4}(a^3 + a - \nu a) + t\rho g \frac{1}{6}L^3\right].$$

Principle of virtual work and the fundamental lemma of variation calculus imply the equilibrium equation

$$\frac{tC}{1-\nu^2}\frac{L^2}{4}(a^3 + a - \nu a) + t\rho g \frac{1}{6}L^3 = 0 \quad \text{where} \quad a = \frac{u_{Y1}}{L}.$$

A bending plate of thickness t, which is clamped on one edge, is assembled at constant temperature $2\vartheta^\circ$. Find the transverse displacement due to heating on the upper side $z=-t/2$ and cooling on the lower side $z=t/2$ resulting in surface temperatures $3\vartheta^\circ$ and ϑ°, respectively. Assume that temperature in plate is linear in z and does not depend on x or y. Use approximation $w=a_0x^2$ in which a_0 is the parameter to be determined. Young's modulus E, Poisson's ratio ν, and coefficient of thermal expansion α are constants.



Solution

Assuming that the material coordinate system is chosen so that the plate bending and thin slab modes decouple, the plate model virtual work densities of internal force and coupling terms are given by

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial^2\delta w/\partial x^2 \\ \partial^2\delta w/\partial y^2 \\ 2\partial^2\delta w/\partial x\partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12}\frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}\begin{Bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ 2\partial^2 w/\partial x\partial y \end{Bmatrix},$$

$$\delta w_\Omega^{\text{cpl}} = -\begin{Bmatrix} \partial^2\delta w/\partial x^2 \\ \partial^2\delta w/\partial y^2 \end{Bmatrix}^{\mathrm{T}} \int z\Delta\vartheta dz \frac{\alpha E}{1-\nu}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

3p The coupling term contains an integral of temperature change over the thickness of the plate. First, as temperature is linear

$$\vartheta_{\text{asm}}(z) = 2\vartheta^\circ \quad \text{and} \quad \vartheta_{\text{fin}}(z) = 2\vartheta^\circ(1-\frac{z}{t}) \quad \Rightarrow \quad \Delta\vartheta = \vartheta_{\text{fin}} - \vartheta_{\text{asm}} = -2\vartheta^\circ\frac{z}{t}.$$

So the integral of the coupling term becomes

$$\int z\Delta\vartheta dz = -2\frac{\vartheta^\circ}{t}\int_{-t/2}^{t/2} z^2 dz = -2\frac{\vartheta^\circ}{t}\frac{1}{3}(\frac{t^3}{8}+\frac{t^3}{8}) = -\frac{1}{6}\vartheta^\circ t^2.$$

and virtual work density for the coupling part

$$\delta w_\Omega^{\text{cpl}} = \begin{Bmatrix} 2\delta a \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{1}{6}\vartheta^\circ t^2 \frac{\alpha E}{1-\nu}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \delta a\frac{1}{3}\vartheta^\circ t^2\frac{\alpha E}{1-\nu}.$$

Virtual work expression is integral of the density over the domain occupied by the plate/element. As integrand is constants, it is enough to multiply by the area of the mid-plane to get

$$\delta W^{\text{cpl}} = \int_\Omega \delta w_\Omega^{\text{cpl}} dA = LH\delta \mathrm{a}\frac{1}{3}\vartheta^\circ t^2 \frac{\alpha E}{1-\nu}.$$

2p Approximation to the transverse displacement

$$w = \mathrm{a}\, x^2 \quad \Rightarrow \quad \frac{\partial^2 w}{\partial x^2} = 2\mathrm{a} \text{ and } \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial x \partial y} = 0.$$

When the approximation is substituted there, virtual work density of the internal forces simplifies to

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} 2\delta\mathrm{a} \\ 0 \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12}\frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}\begin{Bmatrix} 2\mathrm{a} \\ 0 \\ 0 \end{Bmatrix} = -\delta\mathrm{a}\frac{t^3}{3}\frac{E}{1-\nu^2}\mathrm{a}.$$

Virtual work expression is integral of the density over the domain occupied by the plate/element. As integrand is constants, it is enough to multiply by the area of the mid-plane to get

$$\delta W^{\text{int}} = \int_\Omega \delta w_\Omega^{\text{int}} dA = -LH\delta\mathrm{a}\frac{t^3}{3}\frac{E}{1-\nu^2}\mathrm{a}.$$

1p Virtual work expression is the sum of the parts

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{cpl}} = -\delta\mathrm{a}(LH\frac{t^3}{3}\frac{E}{1-\nu^2}\mathrm{a} - LH\frac{1}{3}\vartheta^\circ t^2\frac{\alpha E}{1-\nu}).$$

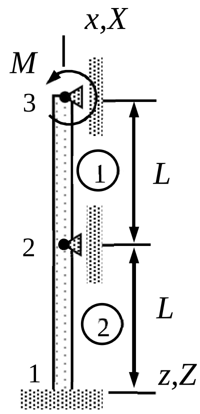
Principle of virtual work $\delta W = 0 \ \forall \delta\mathbf{a}$ and the fundamental lemma of variation calculus give

$$\delta W = -\delta\mathrm{a}(LH\frac{t^3}{3}\frac{E}{1-\nu^2}\mathrm{a} - LH\frac{1}{3}\vartheta^\circ t^2\frac{\alpha E}{1-\nu}) = 0 \quad \forall\delta\mathrm{a} \quad \Leftrightarrow$$

$$LH\frac{t^3}{3}\frac{E}{1-\nu^2}\mathrm{a} - LH\frac{1}{3}\vartheta^\circ t^2\frac{\alpha E}{1-\nu} = 0 \quad \Rightarrow \quad \mathrm{a} = (1+\nu)\frac{1}{t}\alpha\vartheta^\circ.$$

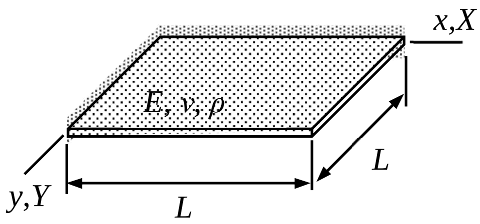
Transverse displacement $w = (1+\nu)\alpha\vartheta^\circ\frac{1}{t}x^2$. 🡐

MEC-E8001 Finite Element Analysis, onsite exam 21.02.2024

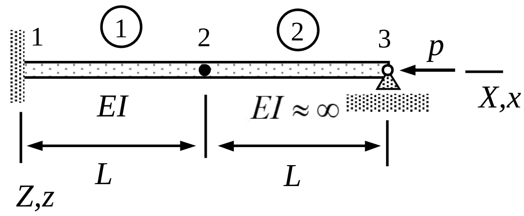


1. Beam structure of the figure is loaded by a point moment acting on node 3. Determine the rotations θ_{Y2} and θ_{Y3} by using two beam bending elements. Displacements are confined to the XZ-plane. The cross-section properties of the beam A, I and Young's modulus of the material E are constants.

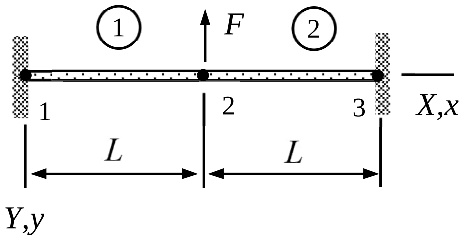
2. A plate is simply supported on two edges and free on the other two edges as shown. Use the approximation $w(x,y,t) = a(t)xy/L^2$ to determine the transverse displacement as function of time $t > 0$. Material properties E, ν, and ρ are constants and thickness of the plate is h. At $t = 0$, initial conditions are $\dot{w}(x,y,0) = 0$ and $w(x,y,0) = Uxy/L^2$. Assume that the plate is thin so that the rotation part of the inertia term is negligible.



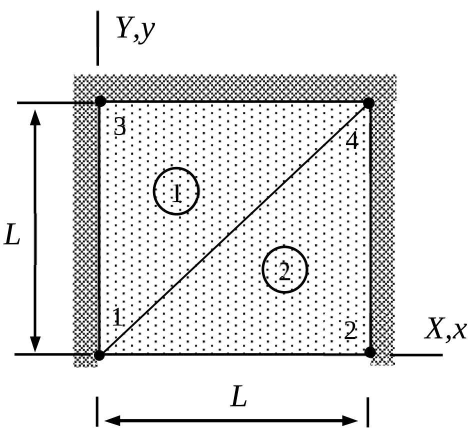
3. The structure shown consists of two beams, each of length L. As beam 2 is much stiffer than beam 1, the setting is simplified by considering beam 2 as a rigid body. Displacements are confined to the $xz-$plane. Cross-section properties of beam 1 are A and I and Young's modulus of the material is E. Determine the buckling load p_{cr}.



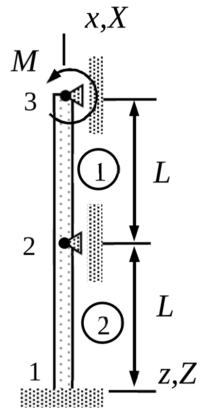
4. Determine the displacement at node 2 of the elastic bar shown by the large deformation theory. Take into account only the transverse displacement u_{Y2} ($u_{X2} = 0$). When $F = 0$, the cross-sectional area and length of the bar are A and L, respectively. Constitutive equation of the material is $S_{xx} = C\mathrm{E}_{xx}$, in which C is constant. Use two elements with linear shape functions.



5. Determine the stationary temperature distribution in a thin slab shown. Edge 1-2 is at constant temperature ϑ° and heat flux through the other edges vanishes. Use a two-triangle mesh with ϑ_3 and $\vartheta_4 = \vartheta_3$ as the unknown node temperatures and consider $\vartheta_1 = \vartheta_2 = \vartheta^\circ$ as known. Thickness t, thermal conductivity k, and heat production rate per unit area s are constants.



Beam structure of the figure is loaded by a point moment acting on node 3. Determine the rotations θ_{Y2} and θ_{Y3} by using two beam bending elements. Displacements are confined to the XZ-plane. The cross-section properties of the beam A, I and Young's modulus of the material E are constants.



Solution

Virtual work expression for the displacement analysis consists of parts coming from internal and external forces δW^{int} and δW^{ext}. For the beam bending mode in xz-plane, the virtual work expressions are

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI_{yy}}{h^3} \begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix}, \quad \delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{f_z h}{12} \begin{Bmatrix} 6 \\ -h \\ 6 \\ h \end{Bmatrix},$$

The element contribution of the point force/moment follows from the definition or work and is given by

$$\delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \\ \delta u_{Z1} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \underline{F}_X \\ \underline{F}_Y \\ \underline{F}_Z \end{Bmatrix} + \begin{Bmatrix} \delta\theta_{X1} \\ \delta\theta_{Y1} \\ \delta\theta_{Z1} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \underline{M}_X \\ \underline{M}_Y \\ \underline{M}_Z \end{Bmatrix}.$$

2p Distributed force $f_z = 0$ and $I_{yy} = I$ in the present problem. In the first step of analysis, the virtual work expressions (given in material coordinate systems of the element) are written in terms of the nodal displacements and rotation components in the structural coordinate system. As the coordinate axes of the two systems are aligned, transformation is simple. Virtual work expression of beam 1

$$\delta W^1 = -\begin{Bmatrix} 0 \\ \delta\theta_{Y2} \\ 0 \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ \theta_{Y2} \\ 0 \\ \theta_{Y3} \end{Bmatrix} = -\begin{Bmatrix} \delta\theta_{Y2} \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 4\dfrac{EI}{L} & 2\dfrac{EI}{L} \\ 2\dfrac{EI}{L} & 4\dfrac{EI}{L} \end{bmatrix} \begin{Bmatrix} \theta_{Y2} \\ \theta_{Y3} \end{Bmatrix},$$

2p When written in the standard form having the $\delta-$ quantity vector as the multiplier, virtual work expression for beam 2 takes the form

$$\delta W^2 = -\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \theta_{Y2} \end{Bmatrix} = -\begin{Bmatrix} \delta\theta_{Y2} \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 4\dfrac{EI}{L} & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \theta_{Y2} \\ \theta_{Y3} \end{Bmatrix}.$$

1p Virtual work expression of the point moment (also written in the 'standard' form having the $\delta -$ quantity vector as the multiplier)

$$\delta W^3 = -\begin{Bmatrix} \delta\theta_{Y2} \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ -M \end{Bmatrix}.$$

1p Virtual work expression of structure is sum of the element contributions, i.e.,

$$\delta W = \delta W^1 + \delta W^2 + \delta W^3 = -\begin{Bmatrix} \delta\theta_{Y2} \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} (\begin{bmatrix} 8\dfrac{EI}{L} & 2\dfrac{EI}{L} \\ 2\dfrac{EI}{L} & 4\dfrac{EI}{L} \end{bmatrix} \begin{Bmatrix} \theta_{Y2} \\ \theta_{Y3} \end{Bmatrix} + \begin{Bmatrix} 0 \\ -M \end{Bmatrix}).$$

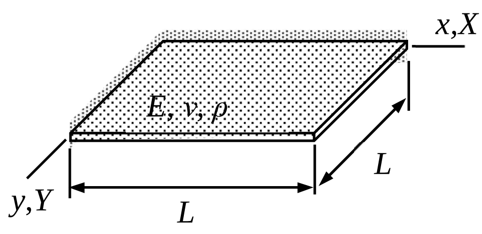
Principle of virtual work and the fundamental lemma of variation calculus imply that

$$\begin{bmatrix} 8\dfrac{EI}{L} & 2\dfrac{EI}{L} \\ 2\dfrac{EI}{L} & 4\dfrac{EI}{L} \end{bmatrix} \begin{Bmatrix} \theta_{Y2} \\ \theta_{Y3} \end{Bmatrix} + \begin{Bmatrix} 0 \\ -M \end{Bmatrix} = 0.$$

Solution to the linear equations system is given by

$$\theta_{Y2} = -\frac{1}{14}\frac{ML}{EI} \quad \text{and} \quad \theta_{Y3} = \frac{2}{7}\frac{ML}{EI}. \quad \leftarrow$$

A plate is simply supported on two edges and free on the other two edges as shown. Use the approximation $w(x,y,t) = a(t)xy/L^2$ to determine the transverse displacement as function of time $t > 0$. Material properties E, ν, and ρ are constants and thickness of the plate is h. At $t = 0$, initial conditions are $\dot{w}(x,y,0) = 0$ and $w(x,y,0) = Uxy/L^2$. Assume that the plate is thin so that the rotation part of the inertia term is negligible.



Solution

4p Only the bending mode of the plate matters. When the approximation $w = a(t)xy/L^2$ is substituted there, virtual work densities of internal and inertia forces (without the rotation part) of the plate simplify to (shear modulus $G = E/(2+2\nu)$)

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial^2\delta w/\partial x^2 \\ \partial^2\delta w/\partial y^2 \\ 2\partial^2\delta w/\partial x\partial y \end{Bmatrix}^{\mathrm{T}} \frac{h^3}{12}\frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\begin{Bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ 2\partial^2 w/\partial x\partial y \end{Bmatrix} = -\delta a\frac{1}{L^4}\frac{h^3}{3}Ga\,,$$

$$\delta w_{\Omega}^{\text{ine}} = -\begin{Bmatrix} \partial\delta w/\partial x \\ \partial\delta w/\partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12}\rho\begin{Bmatrix} \partial\ddot{w}/\partial x \\ \partial\ddot{w}/\partial y \end{Bmatrix} - \delta w t\rho\ddot{w} = -\delta a(\frac{x}{L})^2(\frac{y}{L})^2 h\rho\ddot{a}$$

in which h is thickness of the plate. Integration over the domain occupied by the element gives the virtual work expressions

$$\delta W^{\text{int}} = \int_0^L \int_0^L \delta w_{\Omega}^{\text{int}} dydx = \int_0^L \int_0^L -\delta a\frac{1}{L^4}\frac{h^3}{3}Gadydx = -\delta a\frac{1}{L^2}\frac{h^3}{3}Ga\,,$$

$$\delta W^{\text{ine}} = \int_0^L \int_0^L \delta w_{\Omega}^{\text{ine}} dydx = \int_0^L \int_0^L -\delta a(\frac{x}{L})^2(\frac{y}{L})^2 h\rho\ddot{a}dxdy = -\delta a\frac{L^2}{9}h\rho\ddot{a}\,.$$

Virtual work expression of the structure consists of the internal and inertia parts

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = -\delta a(\frac{1}{L^2}\frac{h^3}{3}Ga + \frac{L^2}{9}h\rho\ddot{a})\,.$$

Principle of virtual work $\delta W = 0 \quad \forall\delta\mathbf{a}$ and the fundamental lemma of variation calculus $\delta\mathbf{a}^{\mathrm{T}}\mathbf{F} = 0 \ \forall\delta\mathbf{a} \quad\Leftrightarrow\quad \mathbf{F} = 0$ imply

$$\frac{1}{L^2}\frac{h^3}{3}Ga + \frac{L^2}{9}h\rho\ddot{a} = 0\,.$$

2p What remains, is solving for the displacement from the initial value problem

$$\ddot{a} + 3\frac{Gh^2}{\rho L^4}a = 0 \quad t > 0\,,\ \ a(0) = U\,,\ \ \dot{a}(0) = 0\,.$$

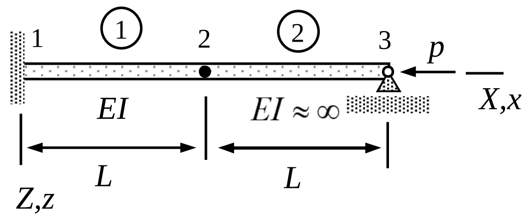
Solution to equations is (this can be shown e.g. by substituting the solution in the equations above)

$$a(t) = U\cos(\sqrt{3\frac{G}{\rho}\frac{h}{L^2}}t) \quad t > 0\,.$$

Finally, substituting the solution to parameter $a(t)$ into the approximation gives

$$w(x, y, t) = U\cos(\sqrt{3\frac{G}{\rho}\frac{h}{L^2}}t)\frac{xy}{L^2}\,.$$

The structure shown consists of two beams, each of length L. As beam 2 is much stiffer than beam 1, the setting is simplified by considering beam 2 as a rigid body. Displacements are confined to the xz – plane. Cross-section properties of beam 1 are A and I and Young's modulus of the material is E. Determine the buckling load p_{cr}.



Solution

The axial forces in the beams follow directly from a free body diagram and it is enough to consider virtual work associated with the bending and interaction between bar and bending modes

$$\delta W^{\mathrm{int}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI_{yy}}{h^3} \begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix},$$

$$\delta W^{\mathrm{sta}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{N}{30h} \begin{bmatrix} 36 & -3h & -36 & -3h \\ -3h & 4h^2 & 3h & -h^2 \\ -36 & 3h & 36 & 3h \\ -3h & -h^2 & 3h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix}.$$

2p Also, as beam 2 is considered as rigid, virtual work expression of internal forces vanish and modelling uses kinematical constraints $\vec{u}_{\mathrm{B}} = \vec{u}_{\mathrm{A}} + \vec{\theta}_{\mathrm{A}} \times \vec{\rho}_{\mathrm{AB}}$ and $\vec{\theta}_{\mathrm{B}} = \vec{\theta}_{\mathrm{A}}$. Let us choose A to be node 3 and B as node 2. Then

$$u_{Z2} = \theta_{Y3}L \quad \text{and} \quad \theta_{Y2} = \theta_{Y3}.$$

Although axial displacement is non-zero, it is not needed as the axial force in the structure $N = -p$ (negative means compression) can be deduced without calculations on the axial displacement.

4p The internal force and coupling parts of beam 1 take the forms ($u_{z1} = 0$, $\theta_{y1} = 0$, $u_{z2} = u_{Z2} = \theta_{Y3}L$, $\theta_{y2} = \theta_{Y2} = \theta_{Y3}$)

$$\delta W^{\mathrm{int}} = -\begin{Bmatrix} 0 \\ 0 \\ L\delta\theta_{Y3} \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ L\theta_{Y3} \\ \theta_{Y3} \end{Bmatrix} = -\delta\theta_{Y3} 28 \frac{EI}{L} \theta_{Y3},$$

$$\delta W^{\mathrm{sta}} = -\begin{Bmatrix} 0 \\ 0 \\ L\delta\theta_{Y3} \\ \delta\theta_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{-p}{30L} \begin{bmatrix} 36 & -3L & -36 & -3L \\ -3L & 4L^2 & 3L & -L^2 \\ -36 & 3L & 36 & 3L \\ -3L & -L^2 & 3L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ L\theta_{Y3} \\ \theta_{Y3} \end{Bmatrix} = \delta\theta_{Y3} \frac{46}{30} pL\theta_{Y3}.$$

Virtual work expression of the structure is the sum of the internal and stability parts

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{sta}} = -\delta\theta_{Y3}(28\frac{EI}{L} - \frac{46}{30}pL)\theta_{Y3}.$$

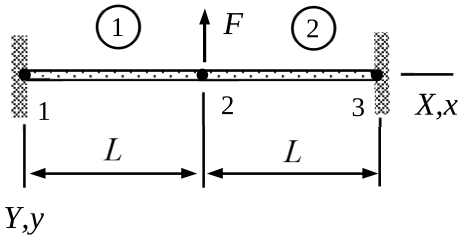
Principle of virtual work and the fundamental lemma of variation calculus imply that

$$(28\frac{EI}{L} - \frac{46}{30}pL)\theta_{Y3} = 0.$$

In stability analysis the aim is to find a non-zero equilibrium solution so $\theta_{Y3} \neq 0$ and the multiplier needs to vanish for a solution. Therefore, the buckling load is given by

$$p_{\text{cr}} = \frac{30}{46}28\frac{EI}{L^2} = \frac{420}{23}\frac{EI}{L^2}.$$

Determine the displacement at node 2 of the elastic bar shown by the large deformation theory. Take into account only the transverse displacement u_{Y2} ($u_{X2}=0$). When $F=0$, the cross-sectional area and length of the bar are A and L, respectively. Constitutive equation of the material is $S_{xx}=C\mathrm{E}_{xx}$, in which C is constant. Use two elements with linear shape functions.



Solution

Virtual work density of the non-linear bar model

$$\delta w_{\Omega^\circ}^{\text{int}}=-(\frac{d\delta u}{dx}+\frac{du}{dx}\frac{d\delta u}{dx}+\frac{dv}{dx}\frac{d\delta v}{dx}+\frac{dw}{dx}\frac{d\delta w}{dx})CA^\circ[\frac{du}{dx}+\frac{1}{2}(\frac{du}{dx})^2+\frac{1}{2}(\frac{dv}{dx})^2+\frac{1}{2}(\frac{dw}{dx})^2]$$

is based on the Green-Lagrange strain definition which is physically correct also when rotations/displacements are large. The expression depends on all displacement components, material property is denoted by C (constitutive equation $S_{xx}=C\mathrm{E}_{xx}$), and the superscript in the cross-sectional area A° (and in other quantities) refers to the initial geometry (strain and stress vanishes). Otherwise, equilibrium equations follow in the same manner as in the linear case.

2p For element 1, the non-zero displacement components is $u_{y2}=u_{Y2}$. As the initial length of the element $h^\circ=L$, linear approximations to the displacement components

$$u=w=0 \text{ and } v=\frac{x}{L}u_{Y2} \quad\Rightarrow\quad \frac{du}{dx}=\frac{dw}{dx}=0 \text{ and } \frac{dv}{dx}=\frac{u_{Y2}}{L}.$$

When the approximation is substituted there, virtual work density of the internal forces and thereby the virtual work expression (density is constant) simplify to

$$\delta w_{\Omega^\circ}^{\text{int}}=-\frac{\delta u_{Y2}}{L}\frac{u_{Y2}}{L}\frac{CA}{2}(\frac{u_{Y2}}{L})^2 \quad\Rightarrow\quad \delta W^1=-\delta u_{Y2}\frac{CA}{2}(\frac{u_{Y2}}{L})^3.$$

2p For element 2, the non-zero displacement component $u_{y2}=u_{Y2}$. As the initial length of the element $h^\circ=L$, linear approximations to the displacement components

$$u=w=0 \text{ and } v=(1-\frac{x}{L})u_{Y2} \quad\Rightarrow\quad \frac{du}{dx}=\frac{dw}{dx}=0 \text{ and } \frac{dv}{dx}=-\frac{u_{Y2}}{L}.$$

When the approximation is substituted there, virtual work density of the internal forces and thereby the virtual work expression (density is constant) simplifies to

$$\delta w_{\Omega^\circ}^{\text{int}}=-\frac{\delta u_{Y2}}{L}\frac{u_{Y2}}{L}\frac{CA}{2}(\frac{u_{Y2}}{L})^2 \quad\Rightarrow\quad \delta W^2=-\delta u_{Y2}\frac{CA}{2}(\frac{u_{Y2}}{L})^3.$$

2p Virtual work expression of the point force is

$$\delta W^3=-F\delta u_{Y2}.$$

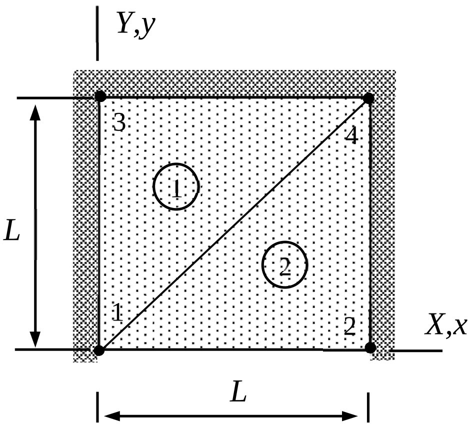
Virtual work expression of the structure is obtained as the sum of the element contributions

$$\delta W = -\delta u_{Y2}[\frac{CA}{2}(\frac{u_{Y2}}{L})^3 + \frac{CA}{2}(\frac{u_{Y2}}{L})^3 + F].$$

Principle of virtual work and the fundamental lemma of variation calculus imply that

$$(\frac{u_{Y2}}{L})^3 + \frac{F}{CA} = 0 \quad \Rightarrow \quad u_{Y2} = -(\frac{FL^3}{CA})^{1/3}.$$

Determine the stationary temperature distribution in a thin slab shown. Edge 1-2 is at constant temperature ϑ° and heat flux through the other edges vanishes. Use a two-triangle mesh with ϑ_3 and $\vartheta_4 = \vartheta_3$ as the unknown node temperatures and consider $\vartheta_1 = \vartheta_2 = \vartheta^\circ$ as known. Thickness t, thermal conductivity k, and heat production rate per unit area s are constants.



Solution

The density expressions associated with the pure heat conduction problem in a thin slab are

$$\delta p_\Omega^{\text{int}} = -\begin{Bmatrix} \partial\delta\vartheta/\partial x \\ \partial\delta\vartheta/\partial y \end{Bmatrix}^{\mathrm{T}} tk \begin{Bmatrix} \partial\vartheta/\partial x \\ \partial\vartheta/\partial y \end{Bmatrix} \text{ and } \delta p_\Omega^{\text{ext}} = \delta\vartheta s \,.$$

in which ϑ is the temperature, k the thermal conductivity, and s the rate of heat production (per unit area). For a thin-slab element, element contributions need to be calculated from scratch starting with the densities and approximations.

3p The shape functions of element 1 (deduced from the figure) $N_1 = 1 - y/L$, $N_4 = x/L$, and $N_3 = 1 - N_1 - N_4 = (y-x)/L$ give approximations

$$\vartheta = \begin{Bmatrix} N_1 \\ N_4 \\ N_3 \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vartheta^\circ \\ \vartheta_3 \\ \vartheta_3 \end{Bmatrix} = (1 - \frac{y}{L})\vartheta^\circ + \frac{y}{L}\vartheta_3 \,, \quad \frac{\partial\vartheta}{\partial x} = 0 \,, \quad \frac{\partial\vartheta}{\partial y} = \frac{\vartheta_3 - \vartheta^\circ}{L} \quad \text{and}$$

$$\delta\vartheta = \frac{y}{L}\delta\vartheta_3 \,, \quad \frac{\partial\delta\vartheta}{\partial x} = 0 \,, \quad \frac{\partial\delta\vartheta}{\partial y} = \frac{\delta\vartheta_3}{L} \quad \text{(variation of } \vartheta^\circ \text{ vanishes).}$$

When the approximation is substituted there, density expression simplifies to

$$\delta p_\Omega = \delta p_\Omega^{\text{int}} + \delta p_\Omega^{\text{ext}} = -\begin{Bmatrix} \partial\delta\vartheta/\partial x \\ \partial\delta\vartheta/\partial y \end{Bmatrix}^{\mathrm{T}} tk \begin{Bmatrix} \partial\vartheta/\partial x \\ \partial\vartheta/\partial y \end{Bmatrix} + \delta\vartheta s = -\frac{\delta\vartheta_3}{L} tk \frac{\vartheta_3 - \vartheta^\circ}{L} + \frac{y}{L}\delta\vartheta_3 s \,.$$

Element contribution is the integral of the density expression over the domain occupied by the element:

$$\delta P^1 = -\delta\vartheta_3 (tk \frac{\vartheta_3 - \vartheta^\circ}{2} - \frac{L^2}{3} s) \,.$$

3p The shape functions of element 2 (deduced from the figure) $N_1 = 1 - x/L$, $N_4 = y/L$, and $N_2 = 1 - N_1 - N_4 = (x-y)/L$ give approximations

$$\vartheta=\begin{Bmatrix}N_1\\N_2\\N_4\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\vartheta^\circ\\\vartheta^\circ\\\vartheta_3\end{Bmatrix}=(1-\frac{y}{L})\vartheta^\circ+\frac{y}{L}\vartheta_3\,,\quad\frac{\partial\vartheta}{\partial x}=0\,,\quad\frac{\partial\vartheta}{\partial y}=\frac{\vartheta_3-\vartheta^\circ}{L}\,,\quad\text{and}$$

$$\delta\vartheta=\frac{y}{L}\delta\vartheta_3\,,\quad\frac{\partial\delta\vartheta}{\partial x}=0\,,\quad\frac{\partial\delta\vartheta}{\partial y}=\frac{\delta\vartheta_3}{L}\quad\text{(variation of }\vartheta^\circ\text{ vanishes).}$$

When the approximation is substituted there, density simplifies to

$$\delta p_\Omega=\delta p_\Omega^{\mathrm{int}}+\delta p_\Omega^{\mathrm{ext}}=-\frac{\delta\vartheta_3}{L}tk\frac{\vartheta_3-\vartheta^\circ}{L}+\frac{y}{L}\delta\vartheta_3 s\,.$$

Element contribution is the integral of the density expression over the domain occupied by the element, so

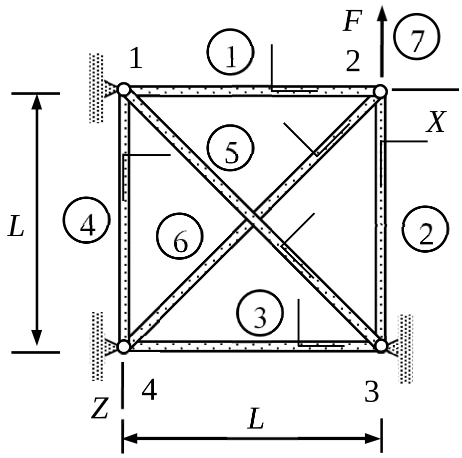
$$\delta P^2=-\delta\vartheta_3(tk\frac{\vartheta_3-\vartheta^\circ}{2}-\frac{L^2}{6}s)\,.$$

Variation principle $\delta P=\delta P^1+\delta P^2=0\ \ \forall\delta\mathbf{a}$ and the fundamental lemma of variation calculus imply that

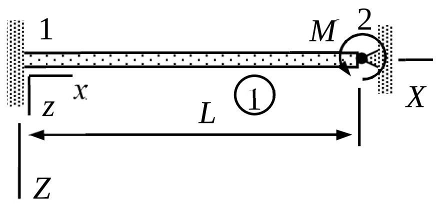
$$tk(\vartheta_3-\vartheta^\circ)-\frac{L^2}{2}s=0\quad\Leftrightarrow\quad\vartheta_3=\vartheta^\circ+\frac{sL^2}{2tk}\,.$$

MEC-E8001 Finite Element Analysis, exam 17.04.2024

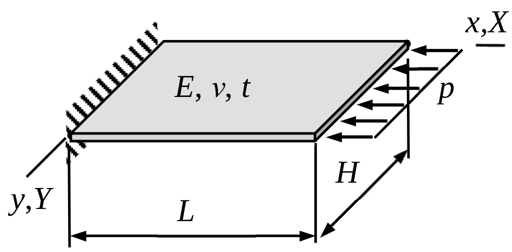
1. Determine the nodal displacements when force F is acting on the structure as shown. The cross-sectional area of bars 1,2,3 and 4 is A and the cross-sectional area of bars 5 and 6 is $2\sqrt{2}A$. Young's modulus of the material is E. Use the principle of virtual work.



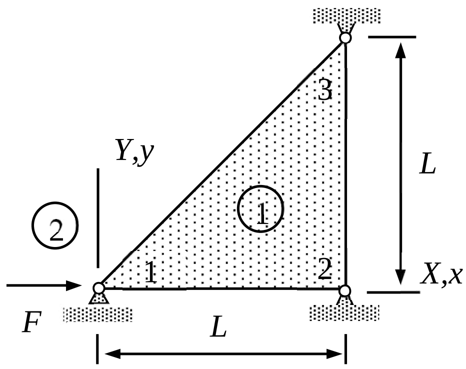
2. The beam of the figure is subjected to moment M when $t<0$. At $t=0$, the moment is suddenly removed and the beam starts to vibrate. Derive the initial value problem giving $\theta_{Y2}(t)$ for $t>0$. The beam is thin so that the rotational part of the inertia term is negligible. The geometrical quantities of the cross-section are A, I and the material constants E and ρ.



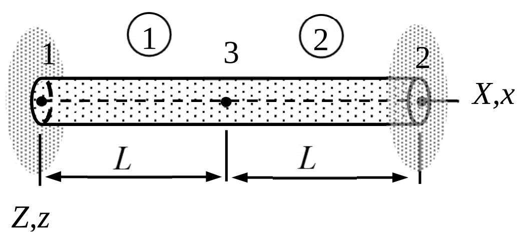
3. The clamping of the plate shown allows displacement in $y-$direction. At the free edge, the plate is loaded by distributed force p. Determine the critical value p_{cr} of the distributed force making the plate to buckle. Use the approximation $w(x,y)=a_0(x/L)^2$ and assume that $N_{xx}=-p$ and $N_{yy}=N_{xy}=0$. Material parameters E, ν and thickness of the plate t are constants.



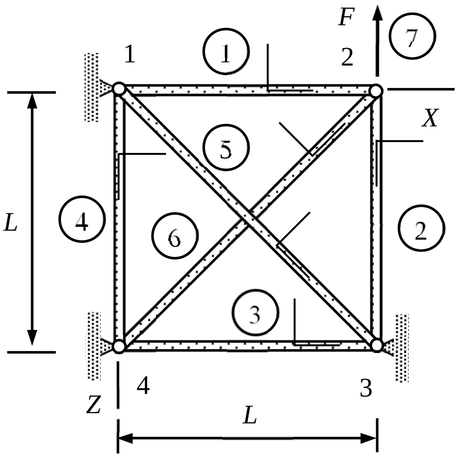
4. A thin triangular slab (assume plane stress conditions) loaded by a horizontal force can move horizontally at node 1 and nodes 2 and 3 are fixed. Derive the equilibrium equation for the structure according to the large displacement theory. Material parameters C, ν and thickness t at the initial geometry of the slab are constants.



5. Electric current causes heat generation in the bar shown. Calculate the temperature at the centre if the wall temperature (nodes 1 and 2) is $\vartheta°$. Cross sectional area A, thermal conductivity k, and heat production rate per unit length s are constants.



Determine the nodal displacements when force F is acting on the structure as shown. The cross-sectional area of bars 1,2,3 and 4 is A and the cross-sectional area of bars 5 and 6 is $2\sqrt{2}A$. Young's modulus of the material is E. Use the principle of virtual work.



Solution

Element and node tables contain the information needed in displacement and stress analysis of the structure. In hand calculations, it is often enough to complete the figure by the material coordinate systems and express the nodal displacements/rotations in terms symbols for the nodal displacements and rotations and/or values known a priori. The components in the material coordinate systems can also be deduced directly form the figure (in simple cases). Virtual work expression of the bar element is given by

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} (\frac{EA}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}).$$

5p Nodal displacements/rotations of the structure are zeros except u_{X2} and u_{Z2}. Element contributions in their virtual work forms are (nodal displacements of the material coordinate system need to be expressed in terms of the structural system components)

Bar 1: $u_{x1} = 0$, $u_{x2} = u_{X2}$: $\quad \delta W^1 = -\delta u_{X2}\frac{EA}{L}u_{X2}$,

Bar 2: $u_{x2} = u_{Z2}$, $u_{x3} = 0$: $\quad \delta W^2 = -\delta u_{Z2}\frac{EA}{L}u_{Z2}$,

Bar 3: $u_{x4} = 0$ and $u_{x3} = 0$: $\quad \delta W^3 = 0$,

Bar 4: $u_{x1} = 0$ and $u_{x4} = 0$: $\quad \delta W^4 = 0$,

Bar 5: $u_{x1} = 0$ and $u_{x3} = 0$: $\quad \delta W^5 = 0$,

Bar 6: $u_{x4} = 0$, $u_{x2} = \frac{1}{\sqrt{2}}(u_{X2} - u_{Z2})$: $\quad \delta W^6 = -(\delta u_{X2} - \delta u_{Z2})\frac{EA}{L}(u_{X2} - u_{Z2})$

Force 7: $\quad \delta W^7 = -\delta u_{Z2} F$.

Virtual work expression of the structure is sum of the element contributions

$$\delta W = \sum_e \delta W^e = \delta W^1 + \delta W^2 + \delta W^3 + \delta W^4 + \delta W^5 + \delta W^6 + \delta W^7 \Rightarrow$$

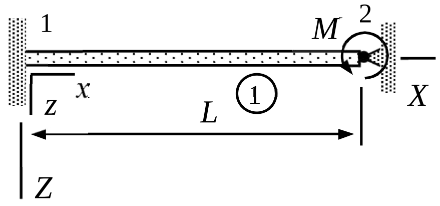
$$\delta W = -\delta u_{X2}\frac{EA}{L}u_{X2} - \delta u_{Z2}\frac{EA}{L}\mathrm{a}_2 + 0 + 0 + 0 - (\delta u_{X2} - \delta u_{Z2})\frac{EA}{L}(u_{X2} - u_{Z2}) - \delta u_{Z2}F \quad \Leftrightarrow$$

$$\delta W = -\begin{Bmatrix} \delta u_{X2} \\ \delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{L}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{Z2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ -F \end{Bmatrix}\right).$$

1p Principle of virtual work $\delta W = 0\ \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus in the form $\delta \mathbf{a}^{\mathrm{T}}\mathbf{R} = 0\ \ \forall \delta \mathbf{a} \ \Leftrightarrow\ \mathbf{R} = 0$ imply

$$\frac{EA}{L}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{Z2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ -F \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix} u_{X2} \\ u_{Z2} \end{Bmatrix} = \frac{FL}{EA}\frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} 0 \\ -1 \end{Bmatrix} = -\frac{FL}{EA}\begin{Bmatrix} 1/3 \\ 2/3 \end{Bmatrix}. \ \leftarrow$$

The beam of the figure is subjected to moment M when $t<0$. At $t=0$, the moment is suddenly removed and the beam starts to vibrate. Derive the initial value problem giving $\theta_{Y2}(t)$ for $t>0$. The beam is thin so that the rotational part of the inertia term is negligible. The geometrical quantities of the cross-section are A, I and the material constants E and ρ.



Solution

4p Virtual work expression consists of parts coming from internal and inertial forces. Finding the equation of motion is the first thing to do. The beam element contributions needed in the problem are (the term having to do with rotational inertia is omitted)

$$\delta W^{\text{int}}=-\begin{Bmatrix}0\\0\\0\\\delta\theta_{Y2}\end{Bmatrix}^{\mathrm{T}}\frac{EI}{L^3}\begin{bmatrix}12 & -6L & -12 & -6L\\-6L & 4L^2 & 6L & 2L^2\\-12 & 6L & 12 & 6L\\-6L & 2L^2 & 6L & 4L^2\end{bmatrix}\begin{Bmatrix}0\\0\\0\\\theta_{Y2}\end{Bmatrix}=-\delta\theta_{Y2}4\frac{EI}{L}\theta_{Y2},$$

$$\delta W^{\text{ine}}=-\begin{Bmatrix}0\\0\\0\\\delta\theta_{Y2}\end{Bmatrix}^{\mathrm{T}}\frac{\rho AL}{420}\begin{bmatrix}156 & -22L & 54 & 13L\\-22L & 4L^2 & -13L & -3L^2\\54 & -13L & 156 & 22L\\13L & -3L^2 & 22L & 4L^2\end{bmatrix}\begin{Bmatrix}0\\0\\0\\\ddot{\theta}_{Y2}\end{Bmatrix}=-\delta\theta_{Y2}\frac{\rho AL^3}{105}\ddot{\theta}_{Y2}$$

giving

$$\delta W^1=-\delta\theta_{Y2}(4\frac{EI}{L}\theta_{Y2}+\frac{\rho AL^3}{105}\ddot{\theta}_{Y2}).$$

In terms of moment $P(t)$ (positive in the positive direction of Y-axis) which is piecewise constant in time so that $P(t)=M\ \ t\le 0$ and $P(t)=0\ \ t>0$, the element contribution of the moment is

$$\delta W^2=\delta\theta_{Y2}P.$$

Virtual work expression is the sum of element contributions:

$$\delta W=\delta W^1+\delta W^2=-\delta\theta_{Y2}(4\frac{EI}{L}\theta_{Y2}+\frac{\rho AL^3}{105}\ddot{\theta}_{Y2}-P)=0.$$

Principle of virtual work and the fundamental lemma of variation calculus imply the ordinary differential equation

$$\delta W=-\delta\theta_{Y2}(4\frac{EI}{L}\theta_{Y2}+\frac{\rho AL^3}{105}\ddot{\theta}_{Y2}-P)=0\quad\forall\delta\theta_{Y2}\quad\Leftrightarrow$$

$$4\frac{EI}{L}\theta_{Y2}+\frac{\rho AL^3}{105}\ddot{\theta}_{Y2}-P=0.$$

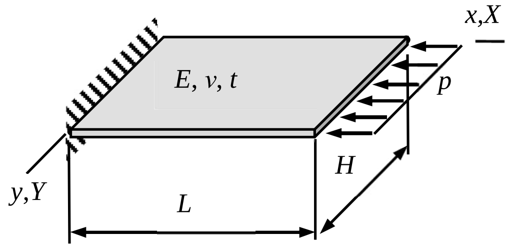
2p When $t \le 0$, external moment $P = M$ is acting on node 2 and the system is at rest. Therefore, the equation of motion becomes an equilibrium equation giving as its solution the initial rotation

$$\theta_{Y2} = \frac{1}{4}\frac{ML}{EI}.$$

When $t > 0$, external moment is zero and acceleration does not vanish. The initial value problem giving as its solution $\theta_{Y2}(t)$ for $t > 0$ takes the form

$$4\frac{EI}{L}\theta_{Y2} + \frac{\rho A L^3}{105}\ddot{\theta}_{Y2} = 0 \quad t > 0, \quad \theta_{Y2}(0) = \frac{1}{4}\frac{ML}{EI}, \quad \text{and} \quad \dot{\theta}_{Y2}(0) = 0 .$$

The clamping of the plate shown allows displacement in $y-$direction. At the free edge, the plate is loaded by distributed force p. Determine the critical value $p_{\rm cr}$ of the distributed force making the plate to buckle. Use the approximation $w(x,y)=a_0(x/L)^2$ and assume that $N_{xx}=-p$ and $N_{yy}=N_{xy}=0$. Material parameters E, ν and thickness of the plate t are constants.



Solution

Assuming that the material coordinate system is chosen so that the plate bending and thin slab modes decouple in the linear analysis and that the in-plane stress resultants are known (from linear displacement analysis, say), it is enough to consider the virtual work densities of plate bending mode and the coupling of the bending and thin-slab modes

$$\delta w_\Omega^{\rm int}=-\begin{Bmatrix}\partial^2\delta w/\partial x^2\\ \partial^2\delta w/\partial y^2\\ 2\partial^2\delta w/\partial x\partial y\end{Bmatrix}^{\rm T}\frac{t^3}{12}[E]_\sigma\begin{Bmatrix}\partial^2 w/\partial x^2\\ \partial^2 w/\partial y^2\\ 2\partial^2 w/\partial x\partial y\end{Bmatrix},\ \delta w_\Omega^{\rm sta}=-\begin{Bmatrix}\partial\delta w/\partial x\\ \partial\delta w/\partial y\end{Bmatrix}^{\rm T}\begin{bmatrix}N_{xx} & N_{xy}\\ N_{yx} & N_{yy}\end{bmatrix}\begin{Bmatrix}\partial w/\partial x\\ \partial w/\partial y\end{Bmatrix}$$

where the elasticity matrix of plane stress

$$[E]_\sigma=\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & (1-\nu)/2\end{bmatrix}.$$

4p As the support at the clamped edge allows displacement in the $y-$direction, solution to the in-plane stress resultants $N_{xx}=-p$ and $N_{yy}=N_{xy}=0$ can be deduced without calculations. Approximation to the transverse displacement and its non-zero derivatives are given by

$$w(x,y)=a_0(\frac{x}{L})^2\quad\Rightarrow\quad\frac{\partial w}{\partial x}=2a_0\frac{x}{L^2}\ \text{and}\ \frac{\partial^2 w}{\partial x^2}=2\frac{a_0}{L^2}.$$

When the approximation is substituted there, virtual work density of the internal forces and that of the coupling simplify to

$$\delta w_\Omega^{\rm int}=-\begin{Bmatrix}2\delta a_0/L^2\\ 0\\ 0\end{Bmatrix}^{\rm T}\frac{t^3}{12}\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & (1-\nu)/2\end{bmatrix}\begin{Bmatrix}2a_0/L^2\\ 0\\ 0\end{Bmatrix}=-\delta a_0\frac{1}{3}\frac{t^3}{L^4}\frac{E}{1-\nu^2}a_0,$$

$$\delta w_\Omega^{\rm sta}=-\begin{Bmatrix}2\delta a_0 x/L^2\\ 0\end{Bmatrix}^{\rm T}\begin{bmatrix}N_{xx} & N_{xy}\\ N_{yx} & N_{yy}\end{bmatrix}\begin{Bmatrix}2a_0 x/L^2\\ 0\end{Bmatrix}=\delta a_0 4x^2\frac{p}{L^4}a_0.$$

Virtual work expressions are integrals of the densities over the domain occupied by the plate

$$\delta W^{\mathrm{int}} = \int_0^H \int_0^L \delta w_\Omega^{\mathrm{int}} dxdy = -\delta a_0 \frac{1}{3}\frac{t^3}{L^3} H \frac{E}{1-\nu^2} a_0 ,$$

$$\delta W^{\mathrm{sta}} = \int_0^H \int_0^L \delta w_\Omega^{\mathrm{sta}} dxdy = \delta a_0 \frac{4}{3}\frac{H}{L} p a_0 .$$

1p Virtual work expression

$$\delta W = \delta W^{\mathrm{int}} + \delta W^{\mathrm{sta}} = -\delta a_0 \left(\frac{1}{3}\frac{t^3}{L^3}\frac{H}{L}\frac{E}{1-\nu^2} - \frac{4}{3}\frac{H}{L} p\right) a_0 ,$$

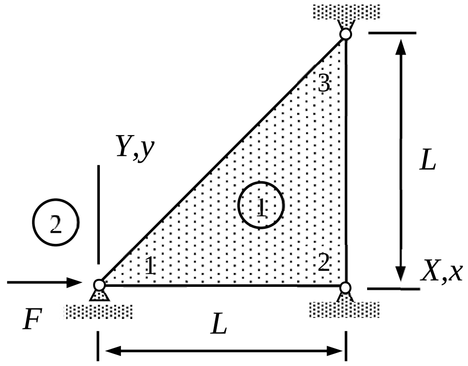
principle of virtual work $\delta W = 0 \ \forall \delta a_0$, and the fundamental lemma of variation calculus give

$$\left(\frac{1}{3}\frac{t^3}{L^3}\frac{H}{L}\frac{E}{1-\nu^2} - \frac{4}{3}\frac{H}{L} p\right) a_0 = 0 .$$

1p For a non-trivial solution $a_0 \neq 0$, the loading parameter needs to take the value

$$p_{\mathrm{cr}} = \frac{1}{4}\frac{E}{1-\nu^2}\frac{t^3}{L^2} . \quad \Leftarrow$$

A thin triangular slab (assume plane stress conditions) loaded by a horizontal force can move horizontally at node 1 and nodes 2 and 3 are fixed. Derive the equilibrium equation for the structure according to the large displacement theory. Material parameters C, ν and thickness t at the initial geometry of the slab are constants.



Solution

Virtual work density of internal force, when modified for large displacement analysis with the same constitutive equation as in the linear case of plane stress, is given by

$$\delta w_{\Omega^\circ}^{\text{int}} = -\begin{Bmatrix} \delta \mathrm{E}_{xx} \\ \delta \mathrm{E}_{yy} \\ 2\delta \mathrm{E}_{xy} \end{Bmatrix}^{\mathrm{T}} \frac{tC}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ 2\mathrm{E}_{xy} \end{Bmatrix}, \quad \begin{Bmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ 2\mathrm{E}_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} + \dfrac{1}{2}(\dfrac{\partial u}{\partial x})^2 + \dfrac{1}{2}(\dfrac{\partial v}{\partial x})^2 \\ \dfrac{\partial v}{\partial y} + \dfrac{1}{2}(\dfrac{\partial u}{\partial y})^2 + \dfrac{1}{2}(\dfrac{\partial v}{\partial y})^2 \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\dfrac{\partial v}{\partial y} \end{Bmatrix}.$$

2p Let us start with the approximations and the corresponding components of the Green-Lagrange strain. Linear shape functions can be deduced from the figure. Only the shape function $N_1 = (1 - x/L)$ of node 1 is needed. Displacement components $v = w = 0$ and

$$u = (1-\frac{x}{L})u_{X1} \quad \Rightarrow \quad \frac{\partial u}{\partial x} = -\frac{u_{X1}}{L}, \quad \frac{\partial u}{\partial y} = 0, \quad \mathrm{E}_{yy} = \mathrm{E}_{xy} = 0 \quad \text{and} \quad \mathrm{E}_{xx} = -\frac{u_{X1}}{L} + \frac{1}{2}(-\frac{u_{X1}}{L})^2.$$

2p When the strain component expression are substituted there, virtual work density simplifies to

$$\delta w_{\Omega^\circ}^{\text{int}} = -\delta \mathrm{E}_{xx} \frac{tC}{1-\nu^2} \mathrm{E}_{xx} = -\frac{\delta u_{X1}}{L}(-1+\frac{u_{X1}}{L})\frac{tC}{1-\nu^2}\frac{u_{X1}}{L}(-1+\frac{1}{2}\frac{u_{X1}}{L}).$$

Integration over the (initial) domain gives the virtual work expression. As the integrand is constant

$$\delta W^1 = -\frac{L^2}{2}\frac{\delta u_{X1}}{L}(-1+\frac{u_{X1}}{L})\frac{tC}{1-\nu^2}\frac{u_{X1}}{L}(-1+\frac{1}{2}\frac{u_{X1}}{L})$$

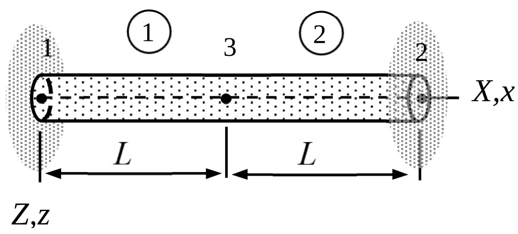
2p Virtual work expression of the point force follows from the definition of work

$$\delta W^2 = \delta u_{X1} F = \frac{\delta u_{X1}}{L} LF.$$

Virtual work expression of the structure is obtained as sum over the element contributions. In terms of the dimensionless displacement $\mathrm{a} = u_{X1}/L$

$$\delta W = -\frac{L^2}{2}\delta \mathrm{a}(-1+\mathrm{a})\frac{tC}{1-\nu^2}\mathrm{a}(-1+\frac{1}{2}\mathrm{a}) + \delta \mathrm{a} LF \quad \Rightarrow \quad \frac{L}{2}(-1+\mathrm{a})\frac{tC}{1-\nu^2}(-\mathrm{a}+\frac{1}{2}\mathrm{a}^2) - F = 0.$$

Electric current causes heat generation in the bar shown. Calculate the temperature at the centre if the wall temperature (nodes 1 and 2) is ϑ°. Cross sectional area A, thermal conductivity k, and heat production rate per unit length s are constants.



Solution

In a pure heat conduction problem, density expressions of the bar model are given by

$$\delta p_\Omega^{\text{int}} = -\frac{d\delta\vartheta}{dx} kA \frac{d\vartheta}{dx} \text{ and } \delta p_\Omega^{\text{ext}} = \delta\vartheta s$$

in which ϑ is the temperature, k the thermal conductivity, and s the rate of heat production (per unit length).

2p For bar 1, the nodal temperatures are $\vartheta_1 = \vartheta^\circ$ and ϑ_3 of which the latter is unknown. With a linear interpolation to temperature (notice that variation of ϑ° vanishes)

$$\vartheta = \begin{Bmatrix} 1-x/L \\ x/L \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vartheta^\circ \\ \vartheta_3 \end{Bmatrix} = (1-\frac{x}{L})\vartheta^\circ + \frac{x}{L}\vartheta_3 \quad \Rightarrow \quad \frac{d\vartheta}{dx} = \frac{\vartheta_3 - \vartheta^\circ}{L},$$

$$\delta\vartheta = \frac{x}{L}\delta\vartheta_3 \quad \Rightarrow \quad \frac{d\delta\vartheta}{dx} = \frac{\delta\vartheta_3}{L}.$$

When the approximation is substituted there, density expression $\delta p_\Omega = \delta p_\Omega^{\text{int}} + \delta p_\Omega^{\text{ext}}$ simplifies to

$$\delta p_\Omega = -\frac{\delta\vartheta_3}{L} kA \frac{\vartheta_3 - \vartheta^\circ}{L} + \frac{x}{L}\delta\vartheta_3 s,$$

Virtual work expression is the integral of the density over the element domain

$$\delta P^1 = \int_0^L \delta p_\Omega dx = -\delta\vartheta_3 (kA\frac{\vartheta_3 - \vartheta^\circ}{L} - \frac{1}{2}Ls).$$

2p The nodal temperatures of bar 2 are ϑ_3 and $\vartheta_2 = \vartheta^\circ$. Linear interpolation gives (variations of the given quantities like ϑ° vanish)

$$\vartheta = \begin{Bmatrix} 1-x/L \\ x/L \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \vartheta_3 \\ \vartheta^\circ \end{Bmatrix} = (1-\frac{x}{L})\vartheta_3 + \frac{x}{L}\vartheta^\circ \quad \Rightarrow \quad \frac{d\vartheta}{dx} = \frac{\vartheta^\circ - \vartheta_3}{L},$$

$$\delta\vartheta = (1-\frac{x}{L})\delta\vartheta_3 \quad \Rightarrow \quad \frac{d\delta\vartheta}{dx} = -\frac{\delta\vartheta_3}{L}.$$

When the approximation is substituted there, density expression $\delta p_\Omega = \delta p_\Omega^{\text{int}} + \delta p_\Omega^{\text{ext}}$ simplifies to

$$\delta p_\Omega = -(-\frac{\delta\vartheta_3}{L}) kA \frac{\vartheta^\circ - \vartheta_3}{L} + (1-\frac{x}{L})\delta\vartheta_3 s.$$

Element contribution to the variational expressions is the integral of density over the element domain

$$\delta P^2 = \int_0^L \delta p_\Omega dx = -\delta\vartheta_3 (kA\frac{\vartheta_3 - \vartheta^\circ}{L} - \frac{L}{2}s) .$$

2p Variational expression is sum of the element contributions

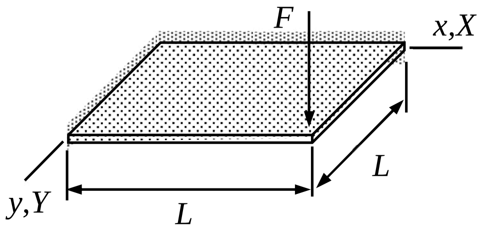
$$\delta P = \delta P^1 + \delta P^2 = -\delta\vartheta_3 (2kA\frac{\vartheta_3 - \vartheta^\circ}{L} - Ls) .$$

Variation principle $\delta P = 0 \quad \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus give

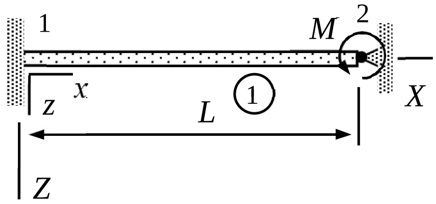
$$2\frac{kA}{L}(\vartheta_3 - \vartheta^\circ) - Ls = 0 \quad \Leftrightarrow \quad \vartheta_3 = \vartheta^\circ + \frac{1}{2}\frac{L^2 s}{kA} .$$

MEC-E8001 Finite Element Analysis, exam 05.06.2024

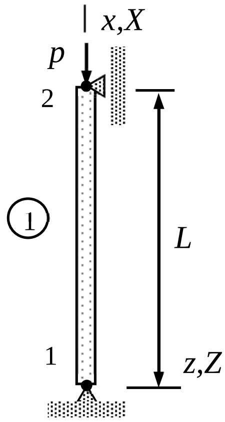
1. A plate, loaded by point force F acting at the free corner, is simply supported on two edges and free on the other two edges as shown in the figure. Determine the parameter a_0 of approximation $w(x,y) = a_0(x/L)(y/L)$ and displacement at the center point. Use the virtual work density of the plate bending mode with constant E, ν, ρ and t.



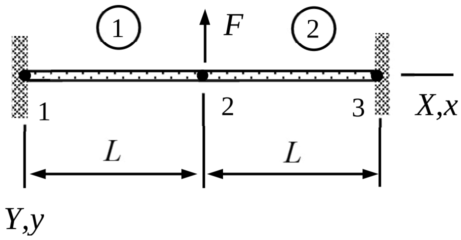
2. The beam of the figure is subjected to moment M when $t<0$. At $t=0$, the moment is suddenly removed and the beam starts to vibrate. Derive the initial value problem giving $\theta_{Y2}(t)$ for $t>0$. The beam is thin so that the rotational part of the inertia term is negligible. The geometrical quantities of the cross-section are A, I and the material constants E and ρ.



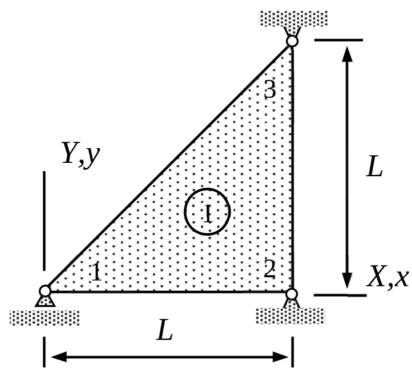
3. Determine the buckling force p_{cr} and the buckled shape of the structure shown by using one beam element. Displacements are confined to the xz – plane. Parameters E, A, and I are constants.



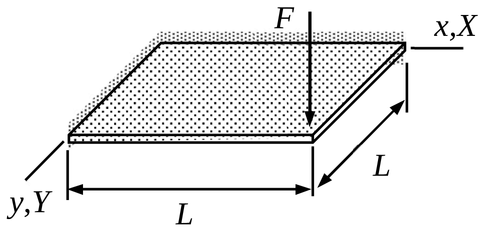
4. Determine the displacement at node 2 of the elastic bar shown by the large deformation theory. Consider only the transverse displacement u_{Y2} ($u_{X2}=0$). When $F=0$, the cross-sectional area and length of the bar are A and L, respectively. Constitutive equation of the material is $S_{xx} = C\mathrm{E}_{xx}$, in which C is constant. Use two elements with linear shape functions.



5. A thin triangular slab (plane stress conditions) is allowed to move horizontally at node 1 and nodes 2 and 3 are fixed. Stress is zero when temperature (assumed constant) is ϑ°. Determine the non-zero displacement component u_{X1}, if the temperature of slab is increased to $2\vartheta^\circ$.



A Kirchhoff plate, loaded by point force F acting at the free corner, is simply supported on two edges and free on the other two edges as shown in the figure. Determine the parameter a_0 of approximation $w(x,y) = a_0(x/L)(y/L)$ and displacement at the center point. Use the virtual work density of the Kirchhoff plate model with constant E, ν, ρ and t.



Solution

Assuming that the material coordinate system is chosen so that the plate bending and thin slab modes decouple, it is enough to consider the virtual work densities of the bending mode only

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial^2 \delta w / \partial x^2 \\ \partial^2 \delta w / \partial y^2 \\ 2\partial^2 \delta w / \partial x \partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12} [E]_\sigma \begin{Bmatrix} \partial^2 w / \partial x^2 \\ \partial^2 w / \partial y^2 \\ 2\partial^2 w / \partial x \partial y \end{Bmatrix}, \; \delta w_\Omega^{\text{ext}} = \delta w f_z .$$

in which the elasticity matrix of plane stress

$$[E]_\sigma = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}.$$

In the present case, distributed force vanishes i.e. $f_z = 0$ and the point force is taken into account by a point force element.

2p Approximation to the transverse displacement is chosen to be (a_0 is not associated with any point but it just a parameter of the approximation)

$$w(x,y) = a_0 \frac{x}{L}\frac{y}{L} \quad \Rightarrow \quad \frac{\partial^2 w}{\partial x^2} = 0, \; \frac{\partial^2 w}{\partial y^2} = 0, \text{ and } \frac{\partial^2 w}{\partial x \partial y} = \frac{1}{L^2} a_0 .$$

When the approximation is substituted there, virtual work density of internal forces simplifies to

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial^2 \delta w / \partial x^2 \\ \partial^2 \delta w / \partial y^2 \\ 2\partial^2 \delta w / \partial x \partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3 E}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \partial^2 w / \partial x^2 \\ \partial^2 w / \partial y^2 \\ 2\partial^2 w / \partial x \partial y \end{Bmatrix} = -\delta a_0 \frac{Et^3}{6(1+\nu)} \frac{1}{L^4} a_0 ,$$

2p Virtual work expression of the plate bending element (element 1 here) is integral of the virtual work density over the domain occupied by the element

$$\delta W^1 = \int_0^L \int_0^L \delta w_\Omega^{\text{int}} dx dy = -\delta a_0 \frac{Et^3}{6(1+\nu)} \frac{1}{L^2} a_0 .$$

Virtual work expression of the point force (element 2 here) follows from the definition of work (notice the use of virtual displacement of the point of action $x = y = L$)

$$\delta W^2 = \delta w(L,L)F = \delta a_0 F \,.$$

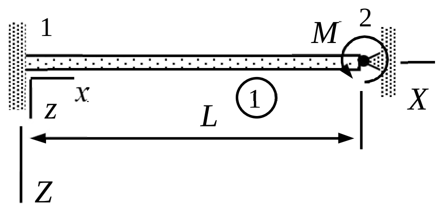
2p Principle of virtual work and the fundamental lemma of variation calculus give

$$\delta W = \delta W^1 + \delta W^2 = -\delta a_0 \left(\frac{Et^3}{6(1+\nu)} \frac{1}{L^2} a_0 - F \right) = 0 \quad \Rightarrow \quad a_0 = 6(1+\nu)\frac{FL^2}{Et^3} \,.$$

Displacement at the center point

$$w(\frac{L}{2},\frac{L}{2}) = a_0 \frac{1}{4} = \frac{3}{2}(1+\nu)\frac{FL^2}{Et^3} \,.$$

The beam of the figure is subjected to moment M when $t<0$. At $t=0$, the moment is suddenly removed and the beam starts to vibrate. Derive the initial value problem giving $\theta_{Y2}(t)$ for $t>0$. The beam is thin so that the rotational part of the inertia term is negligible. The geometrical quantities of the cross-section are A, I and the material constants E and ρ.



Solution

2p Virtual work expression consists of parts coming from internal and inertial forces. Finding the equation of motion is the first thing to do. The beam element contributions needed in the problem are (the term having to do with rotational inertia is omitted)

$$\delta W^{\text{int}} = -\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \theta_{Y2} \end{Bmatrix} = -\delta\theta_{Y2}4\frac{EI}{L}\theta_{Y2},$$

$$\delta W^{\text{ine}} = -\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{\rho AL}{420}\begin{bmatrix} 156 & -22L & 54 & 13L \\ -22L & 4L^2 & -13L & -3L^2 \\ 54 & -13L & 156 & 22L \\ 13L & -3L^2 & 22L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \ddot{\theta}_{Y2} \end{Bmatrix} = -\delta\theta_{Y2}\frac{\rho AL^3}{105}\ddot{\theta}_{Y2}$$

giving

$$\delta W^1 = -\delta\theta_{Y2}(4\frac{EI}{L}\theta_{Y2} + \frac{\rho AL^3}{105}\ddot{\theta}_{Y2}).$$

In terms of moment $P(t)$ (positive in the positive direction of Y-axis) which is piecewise constant in time so that $P(t)=M \quad t\le 0$ and $P(t)=0 \quad t>0$, the element contribution of the moment is

$$\delta W^2 = \delta\theta_{Y2}P.$$

2p Virtual work expression is the sum of element contributions:

$$\delta W = \delta W^1 + \delta W^2 = -\delta\theta_{Y2}(4\frac{EI}{L}\theta_{Y2} + \frac{\rho AL^3}{105}\ddot{\theta}_{Y2} - P) = 0.$$

Principle of virtual work and the fundamental lemma of variation calculus imply the ordinary differential equation

$$\delta W = -\delta\theta_{Y2}(4\frac{EI}{L}\theta_{Y2} + \frac{\rho AL^3}{105}\ddot{\theta}_{Y2} - P) = 0 \quad \forall\delta\theta_{Y2} \quad \Leftrightarrow$$

$$4\frac{EI}{L}\theta_{Y2} + \frac{\rho AL^3}{105}\ddot{\theta}_{Y2} - P = 0. \leftarrow$$

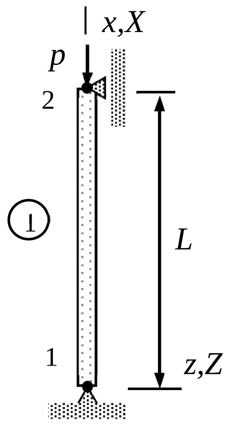
2p When $t \le 0$, external moment $P = M$ is acting on node 2 and the system is at rest. Therefore, the equation of motion becomes an equilibrium equation giving as its solution the initial rotation

$$\theta_{Y2} = \frac{1}{4}\frac{ML}{EI}.$$

When $t > 0$, external moment is zero and acceleration does not vanish. The initial value problem giving as its solution $\theta_{Y2}(t)$ for $t > 0$ takes the form

$$4\frac{EI}{L}\theta_{Y2} + \frac{\rho A L^3}{105}\ddot{\theta}_{Y2} = 0 \quad t > 0, \quad \theta_{Y2}(0) = \frac{1}{4}\frac{ML}{EI}, \quad \text{and} \quad \dot{\theta}_{Y2}(0) = 0 .$$

Determine the buckling force p_{cr} and the buckled shape of the structure shown by using one beam element. Displacements are confined to the xz – plane. Parameters *E*, *A*, and *I* are constants.



Solution

3p The non-zero displacement/rotation components of the structure are and $\theta_{y1}=\theta_{Y1}$, $\theta_{y2}=\theta_{Y2}$, and $u_{x2}=u_{X2}$. The normal force in the beam $N=-p$ can be deduced without calculations on the axial displacement. Therefore, it is enough to consider only the bending and coupling terms of the virtual work expression. As buckling is confined to the xz – plane

$$\delta W=-\begin{Bmatrix}0\\ \delta\theta_{Y1}\\ 0\\ \delta\theta_{Y2}\end{Bmatrix}^{\mathrm{T}}\left(\frac{EI}{L^3}\begin{bmatrix}12 & -6L & -12 & -6L\\ -6L & 4L^2 & 6L & 2L^2\\ -12 & 6L & 12 & 6L\\ -6L & 2L^2 & 6L & 4L^2\end{bmatrix}-\frac{p}{30L}\begin{bmatrix}36 & -3L & -36 & -3L\\ -3L & 4L^2 & 3L & -L^2\\ -36 & 3L & 36 & 3L\\ -3L & -L^2 & 3L & 4L^2\end{bmatrix}\right)\begin{Bmatrix}0\\ \theta_{Y1}\\ 0\\ \theta_{Y2}\end{Bmatrix}\Leftrightarrow$$

$$\delta W=-\begin{Bmatrix}\delta\theta_{Y1}\\ \delta\theta_{Y2}\end{Bmatrix}^{\mathrm{T}}\left(\frac{EI}{L}\begin{bmatrix}4 & 2\\ 2 & 4\end{bmatrix}-\frac{pL}{30}\begin{bmatrix}4 & -1\\ -1 & 4\end{bmatrix}\right)\begin{Bmatrix}\theta_{Y1}\\ \theta_{Y2}\end{Bmatrix}.$$

According to the principle of virtual work

$$\left(\frac{EI}{L}\begin{bmatrix}4 & 2\\ 2 & 4\end{bmatrix}-\frac{pL}{30}\begin{bmatrix}4 & -1\\ -1 & 4\end{bmatrix}\right)\begin{Bmatrix}\theta_{Y1}\\ \theta_{Y2}\end{Bmatrix}=0.$$

2p A homogeneous equation system has a non-trivial solution only if the matrix is singular

$$\det\left(\begin{bmatrix}4\frac{EI}{L}-4\frac{pL}{30} & 2\frac{EI}{L}+\frac{pL}{30}\\ 2\frac{EI}{L}+\frac{pL}{30} & 4\frac{EI}{L}-4\frac{pL}{30}\end{bmatrix}\right)=\left(4\frac{EI}{L}-4\frac{pL}{30}\right)^2-\left(2\frac{EI}{L}+\frac{pL}{30}\right)^2=0\quad\Rightarrow\quad\frac{pL^2}{EI}\in\{12,60\}.$$

The smallest eigenvalue gives the critical loading

$$p_{\text{cr}}=12\frac{EI}{L^2}.$$

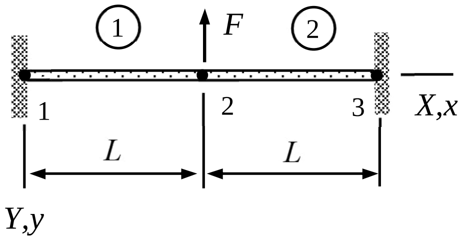
1p The corresponding eigenvector (mode) is given by

$$\begin{bmatrix}4\frac{EI}{L}-4\frac{p_{\text{cr}}L}{30} & 2\frac{EI}{L}+\frac{p_{\text{cr}}L}{30}\\ 2\frac{EI}{L}+\frac{p_{\text{cr}}L}{30} & 4\frac{EI}{L}-4\frac{p_{\text{cr}}L}{30}\end{bmatrix}\begin{Bmatrix}\theta_{Y1}\\ \theta_{Y2}\end{Bmatrix}=\frac{72}{30}\frac{EI}{L}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{Bmatrix}\theta_{Y1}\\ \theta_{Y2}\end{Bmatrix}=0\quad\Rightarrow\quad\begin{Bmatrix}\theta_{Y1}\\ \theta_{Y2}\end{Bmatrix}=\begin{Bmatrix}1\\ -1\end{Bmatrix}\text{ (say).}$$

Shape of the buckled beam follows from approximation when the mode is substituted there (see the formulae collection)

$$w(x)=\begin{Bmatrix}(1-\xi)^2(1+2\xi)\\ L(1-\xi)^2\xi\\ (3-2\xi)\xi^2\\ L\xi^2(\xi-1)\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}0\\ -1\\ 0\\ 1\end{Bmatrix}=-L(\xi-1)\xi \quad \text{where} \quad \xi=\frac{x}{L}\,.$$

Determine the displacement at node 2 of the elastic bar shown by the large deformation theory. Take into account only the transverse displacement u_{Y2} ($u_{X2}=0$). When $F=0$, the cross-sectional area and length of the bar are A and L, respectively. Constitutive equation of the material is $S_{xx}=C\mathrm{E}_{xx}$, in which C is constant. Use two elements with linear shape functions.



Solution

Virtual work density of the non-linear bar model

$$\delta w_{\Omega^\circ}^{\text{int}}=-(\frac{d\delta u}{dx}+\frac{du}{dx}\frac{d\delta u}{dx}+\frac{dv}{dx}\frac{d\delta v}{dx}+\frac{dw}{dx}\frac{d\delta w}{dx})CA^\circ[\frac{du}{dx}+\frac{1}{2}(\frac{du}{dx})^2+\frac{1}{2}(\frac{dv}{dx})^2+\frac{1}{2}(\frac{dw}{dx})^2]$$

is based on the Green-Lagrange strain definition which is physically correct also when rotations/displacements are large. The expression depends on all displacement components, material property is denoted by C (constitutive equation $S_{xx}=C\mathrm{E}_{xx}$), and the superscript in the cross-sectional area A° (and in other quantities) refers to the initial geometry (strain and stress vanishes). Otherwise, equilibrium equations follow in the same manner as in the linear case.

4p For element 1, the non-zero displacement components is $u_{y2}=u_{Y2}$. As the initial length of the element $h^\circ=L$, linear approximations to the displacement components

$$u=w=0 \text{ and } v=\frac{x}{L}u_{Y2} \quad\Rightarrow\quad \frac{du}{dx}=\frac{dw}{dx}=0 \text{ and } \frac{dv}{dx}=\frac{u_{Y2}}{L}.$$

When the approximation is substituted there, virtual work density of the internal forces and thereby the virtual work expression (density is constant) simplify to

$$\delta w_{\Omega^\circ}^{\text{int}}=-\frac{\delta u_{Y2}}{L}\frac{u_{Y2}}{L}\frac{CA}{2}(\frac{u_{Y2}}{L})^2 \quad\Rightarrow\quad \delta W^1=-\delta u_{Y2}\frac{CA}{2}(\frac{u_{Y2}}{L})^3.$$

For element 2, the non-zero displacement component $u_{y2}=u_{Y2}$. As the initial length of the element $h^\circ=L$, linear approximations to the displacement components

$$u=w=0 \text{ and } v=(1-\frac{x}{L})u_{Y2} \quad\Rightarrow\quad \frac{du}{dx}=\frac{dw}{dx}=0 \text{ and } \frac{dv}{dx}=-\frac{u_{Y2}}{L}.$$

When the approximation is substituted there, virtual work density of the internal forces and thereby the virtual work expression (density is constant) simplifies to

$$\delta w_{\Omega^\circ}^{\text{int}}=-\frac{\delta u_{Y2}}{L}\frac{u_{Y2}}{L}\frac{CA}{2}(\frac{u_{Y2}}{L})^2 \quad\Rightarrow\quad \delta W^2=-\delta u_{Y2}\frac{CA}{2}(\frac{u_{Y2}}{L})^3.$$

Virtual work expression of the point force is

$$\delta W^3=-F\delta u_{Y2}.$$

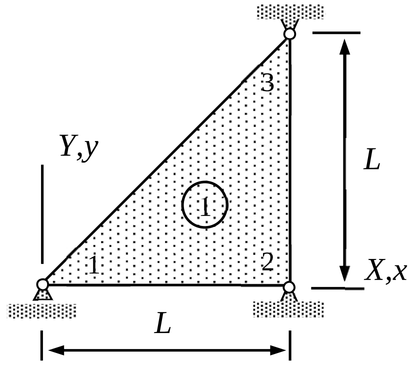
2p Virtual work expression of the structure is obtained as the sum of the element contributions

$$\delta W = -\delta u_{Y2}[\frac{CA}{2}(\frac{u_{Y2}}{L})^3 + \frac{CA}{2}(\frac{u_{Y2}}{L})^3 + F].$$

Principle of virtual work and the fundamental lemma of variation calculus imply that

$$(\frac{u_{Y2}}{L})^3 + \frac{F}{CA} = 0 \quad \Rightarrow \quad u_{Y2} = -(\frac{FL^3}{CA})^{1/3}. \quad \leftarrow$$

A thin triangular slab (plane stress conditions) is allowed to move horizontally at node 1 and nodes 2 and 3 are fixed. Stress is zero when temperature (assumed constant) is ϑ°. Determine the non-zero displacement component u_{X1}, if the temperature of slab is increased to $2\vartheta^{\circ}$.



Solution

As temperature is known and the external distributed force vanishes, the virtual work densities needed here are (formulae collection)

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta u/\partial y + \partial\delta v/\partial x \end{Bmatrix}^{\mathrm{T}} t[E]_{\sigma} \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix}, \quad \delta w_{\Omega}^{\text{cpl}} = \begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial x \end{Bmatrix}^{\mathrm{T}} \frac{E\alpha}{1-\nu} \int \Delta\vartheta dz \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

in which $\Delta\vartheta = \vartheta - \vartheta^{\circ}$ is the difference between temperature at the deformed and initial and deformed geometries. At the initial geometry stress is assumed to vanish and the integral

$$\int \Delta\vartheta dz = t\Delta\vartheta .$$

2p Approximation is the first thing to be considered. Linear shape functions can be deduced from the figure

$$N_1 = 1 - \frac{x}{L}, \quad N_3 = \frac{y}{L}, \quad \text{and} \quad N_2 = 1 - N_1 - N_3 = \frac{x-y}{L}.$$

Approximations to the displacement components and temperature difference are

$$u = (1 - \frac{x}{L})u_{X1}, \quad v = 0, \quad \text{and} \quad \Delta\vartheta = \vartheta^{\circ}.$$

3p When the approximations are substituted there, virtual work densities take the forms

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} -\delta u_{X1}/L \\ 0 \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{Et}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} -u_{X1}/L \\ 0 \\ 0 \end{Bmatrix} = -\delta u_{X1} \frac{1}{L^2} \frac{Et}{1-\nu^2} u_{X1},$$

$$\delta w_{\Omega}^{\text{cpl}} = \begin{Bmatrix} -\delta u_{X1}/L \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{Et\alpha}{1-\nu} \vartheta^{\circ} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = -\delta u_{X1} \frac{1}{L} \frac{Et\alpha}{1-\nu} \vartheta^{\circ} \qquad \Rightarrow$$

$$\delta w_{\Omega} = \delta w_{\Omega}^{\text{int}} + \delta w_{\Omega}^{\text{cpl}} = -\frac{\delta u_{X1}}{L} \frac{Et}{1-\nu^2} \frac{u_{X1}}{L} - \frac{\delta u_{X1}}{L} \frac{Et}{1-\nu} \alpha\vartheta^{\circ}.$$

Virtual work expression is the integral of the density over the domain occupied by the element. Here, virtual work density is constant so that it is enough to multiply by the area:

$$\delta W = \delta w_\Omega \frac{L^2}{2} = -\delta u_{X1}\left(\frac{1}{2}\frac{Et}{1-\nu^2}u_{X1} + \frac{1}{2}\frac{Et}{1-\nu}L\alpha\vartheta^\circ\right).$$

1p Principle of virtual work and the fundamental lemma of variation calculus give

$$\frac{1}{2}\frac{Et}{1-\nu^2}u_{X1} + \frac{1}{2}\frac{Et}{1-\nu}L\alpha\vartheta^\circ = 0 \quad \Leftrightarrow \quad u_{X1} = -(1+\nu)\alpha L\vartheta^\circ. \quad \leftarrow$$