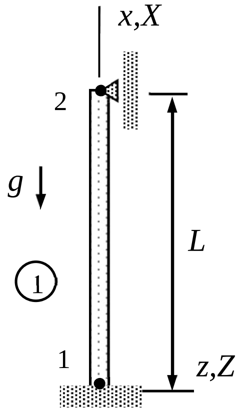
Name________________________ Student number________________

Assignment 4



A beam is loaded by its own weight as shown in the figure. Assume that displacement is confined to the $XZ-$plane. Derive the equilibrium equations for buckling analysis giving the axial displacement and the critical density ρ_{cr} of the material. Start with the virtual work density and approximations to the axial and transverse displacements. The cross-section properties A, I and material properties E, ρ are constants.

Solution template

Virtual work expressions for the buckling analysis of a beam in $xz-$plane consist of the internal parts for the bar and bending modes, coupling (stability expression) between them, and virtual work of the external point force. Altogether ($f_x = -\rho A g$)

$$\delta w_\Omega = -\frac{d^2\delta w}{dx^2} EI_{yy} \frac{d^2 w}{dx^2} - \frac{d\delta u}{dx} EA \frac{du}{dx} - \frac{d\delta w}{dx} N \frac{dw}{dx} + \delta u f_x, \text{ where } N = EA\frac{du}{dx}.$$

In terms of the non-zero displacement/rotation components of the structural system, approximations to the axial displacement u, transverse displacement w, and the axial force N simplify to

$$u(x) = \begin{Bmatrix} 1-x/L \\ x/L \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ u_{X2} \end{Bmatrix} = \frac{x}{L} u_{X2},$$

$$w(x) = \begin{Bmatrix} (1-x/L)^2(1+2x/L) \\ L(1-x/L)^2 x/L \\ (3-2x/L)(x/L)^2 \\ L(x/L)^2(x/L-1) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -\theta_{Y2} \end{Bmatrix} = \frac{1}{L^2}(Lx^2 - x^3)\theta_{Y2},$$

$$N = EA\frac{du}{dx} = \frac{EA}{L} u_{X2}.$$

When the approximations are substituted there, virtual work density simplifies to (substitute the expression for the axial force N and distributed force f_x)

$$\delta w_\Omega = -\delta\theta_{Y2} \frac{EI}{L^4}(2L-6x)^2 \theta_{Y2} - \delta u_{X2} \frac{EA}{L^2} u_{X2} - \delta\theta_{Y2}(2Lx - 3x^2)^2 u_{X2} \frac{EA}{L^5} \theta_{Y2} - \delta u_{X2} \frac{x}{L} \rho A g.$$

Integration over the length of the beam gives

$$\delta W = \int_0^L \delta w_\Omega dx = -\delta\theta_{Y2} 4\frac{EI}{L}\theta_{Y2} - \delta u_{X2}\frac{EA}{L}u_{X2} - \delta\theta_{Y2}\frac{2}{15}EAu_{X2}\theta_{Y2} - \delta u_{X2}\frac{1}{2}L\rho Ag \quad \Leftrightarrow$$

$$\delta W = -\begin{Bmatrix} \delta u_{X2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \frac{EA}{L}u_{X2} + \frac{1}{2}L\rho Ag \\ (4\frac{EI}{L} + \frac{2}{15}EAu_{X2})\theta_{Y2} \end{Bmatrix}.$$

Principle of virtual work and the fundamental lemma of variation calculus imply equilibrium equations

$$\begin{Bmatrix} \frac{EA}{L}u_{X2} + \frac{1}{2}L\rho Ag \\ (4\frac{EI}{L} + \frac{2}{15}EAu_{X2})\theta_{Y2} \end{Bmatrix} = 0.$$

The first equation is linear and can be solved for the axial displacement

$$\frac{EA}{L}u_{X2} + \frac{1}{2}L\rho Ag = 0 \quad \Leftrightarrow \quad u_{X2} = -\frac{L^2}{2}\frac{\rho g}{E}.$$

When the solution to the axial displacement is substituted there, the second (non-linear) equation simplifies to

$$(4\frac{EI}{L} - \frac{1}{15}L^2 A\rho g)\theta_{Y2} = 0.$$

The remaining task is to deduce the possible solutions: If the expression in parenthesis is non-zero, the equation implies that $\theta_{Y2} = 0$. If the expression in parenthesis is zero, the equation is satisfied no matter the non-zero value of θ_{Y2}. Therefore, buckling may occur when (here density ρ stands for the loading parameter)

$$4\frac{EI}{L} - \frac{1}{15}L^2 A\rho g = 0 \quad \Leftrightarrow \quad \rho_{\mathrm{cr}} = 60\frac{EI}{AgL^3}.$$