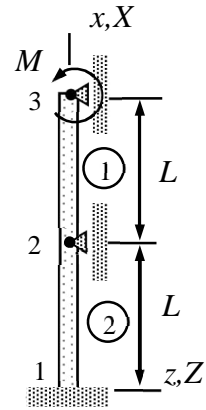


Name _____ Student number _____

Assignment 4

Beam structure of the figure is loaded by a point moment acting on node 3. Determine the rotations θ_{Y2} and θ_{Y3} by using two beam bending elements. Displacements are confined to the XZ-plane. The cross-section properties of the beam A , I and Young's modulus of the material E are constants.



Solution template

Virtual work expression for the displacement analysis consists of parts coming from internal and external forces δW^{int} and δW^{ext} . For the beam bending mode in xz -plane, the virtual work expressions are

$$\delta W^{\text{int}} = - \begin{Bmatrix} \delta u_{z1} \\ \delta \theta_{y1} \\ \delta u_{z2} \\ \delta \theta_{y2} \end{Bmatrix}^T \frac{EI_{yy}}{h^3} \begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix}, \quad \delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{z1} \\ \delta \theta_{y1} \\ \delta u_{z2} \\ \delta \theta_{y2} \end{Bmatrix}^T \frac{f_z h}{12} \begin{Bmatrix} 6 \\ -h \\ 6 \\ h \end{Bmatrix},$$

The element contribution of the point force/moment follows from the definition of work and is given by

$$\delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \\ \delta u_{Z1} \end{Bmatrix}^T \begin{Bmatrix} \underline{F}_X \\ \underline{F}_Y \\ \underline{F}_Z \end{Bmatrix} + \begin{Bmatrix} \delta \theta_{X1} \\ \delta \theta_{Y1} \\ \delta \theta_{Z1} \end{Bmatrix}^T \begin{Bmatrix} \underline{M}_X \\ \underline{M}_Y \\ \underline{M}_Z \end{Bmatrix}.$$

Distributed force $f_z = 0$ and $I_{yy} = I$ in the present problem. In the first step of analysis, the virtual work expressions (given in material coordinate systems of the element) are written in terms of the nodal displacements and rotation components in the structural coordinate system. As the coordinate axes of the two systems are aligned, transformation is simple. Virtual work expression of beam 1

$$\delta W^1 = - \begin{Bmatrix} 0 \\ \delta \theta_{Y2} \\ 0 \\ \delta \theta_{Y3} \end{Bmatrix}^T \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ \theta_{Y2} \\ 0 \\ \theta_{Y3} \end{Bmatrix} = - \begin{Bmatrix} \delta \theta_{Y2} \\ \delta \theta_{Y3} \end{Bmatrix}^T \begin{bmatrix} 4\frac{EI}{L} & 2\frac{EI}{L} \\ 2\frac{EI}{L} & 4\frac{EI}{L} \end{bmatrix} \begin{Bmatrix} \theta_{Y2} \\ \theta_{Y3} \end{Bmatrix},$$

When written in the standard form having the δ – quantity vector as the multiplier, virtual work expression for beam 2 takes the form

$$\delta W^2 = - \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta\theta_{Y2} \end{Bmatrix}^T \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \theta_{Y2} \end{Bmatrix} = - \begin{Bmatrix} \delta\theta_{Y2} \\ \delta\theta_{Y3} \end{Bmatrix}^T \begin{bmatrix} 4\frac{EI}{L} & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \theta_{Y2} \\ \theta_{Y3} \end{Bmatrix}.$$

Virtual work expression of the point moment (also written in the ‘standard’ form having the δ – quantity vector as the multiplier)

$$\delta W^3 = - \begin{Bmatrix} \delta\theta_{Y2} \\ \delta\theta_{Y3} \end{Bmatrix}^T \begin{Bmatrix} 0 \\ -M \end{Bmatrix}.$$

Virtual work expression of structure is sum of the element contributions, i.e.,

$$\delta W = \delta W^1 + \delta W^2 + \delta W^3 = - \begin{Bmatrix} \delta\theta_{Y2} \\ \delta\theta_{Y3} \end{Bmatrix}^T \left(\begin{bmatrix} 8\frac{EI}{L} & 2\frac{EI}{L} \\ 2\frac{EI}{L} & 4\frac{EI}{L} \end{bmatrix} \begin{Bmatrix} \theta_{Y2} \\ \theta_{Y3} \end{Bmatrix} + \begin{Bmatrix} 0 \\ -M \end{Bmatrix} \right).$$

Principle of virtual work and the fundamental lemma of variation calculus imply that

$$\begin{bmatrix} 8\frac{EI}{L} & 2\frac{EI}{L} \\ 2\frac{EI}{L} & 4\frac{EI}{L} \end{bmatrix} \begin{Bmatrix} \theta_{Y2} \\ \theta_{Y3} \end{Bmatrix} + \begin{Bmatrix} 0 \\ -M \end{Bmatrix} = 0.$$

Solution to the linear equations system is given by

$$\theta_{Y2} = -\frac{1}{14} \frac{ML}{EI} \quad \text{and} \quad \theta_{Y3} = \frac{2}{7} \frac{ML}{EI}. \quad \leftarrow$$