MEC-E1050

FINITE ELEMENT METHOD IN SOLIDS 2024

4 ELEMENT CONTRIBUTIONS

LEARNING OUTCOMES

Students are able to solve the lecture problems, home problems, and exercise problems on the topics of the week:

☐ The basic building blocks of element contributions: virtual work density and element interpolant (to the nodal values).

☐ Derivation of the beam element contribution starting with the basic building blocks

☐ Element interpolant and shape functions

EXAMPLE 4.1. Consider the beam truss of the figure. Determine the displacements and rotations of nodes 2 and 4. Assume that the beams are rigid in the axial directions. Cross-sections and lengths are the same and Young's modulus E is constant.
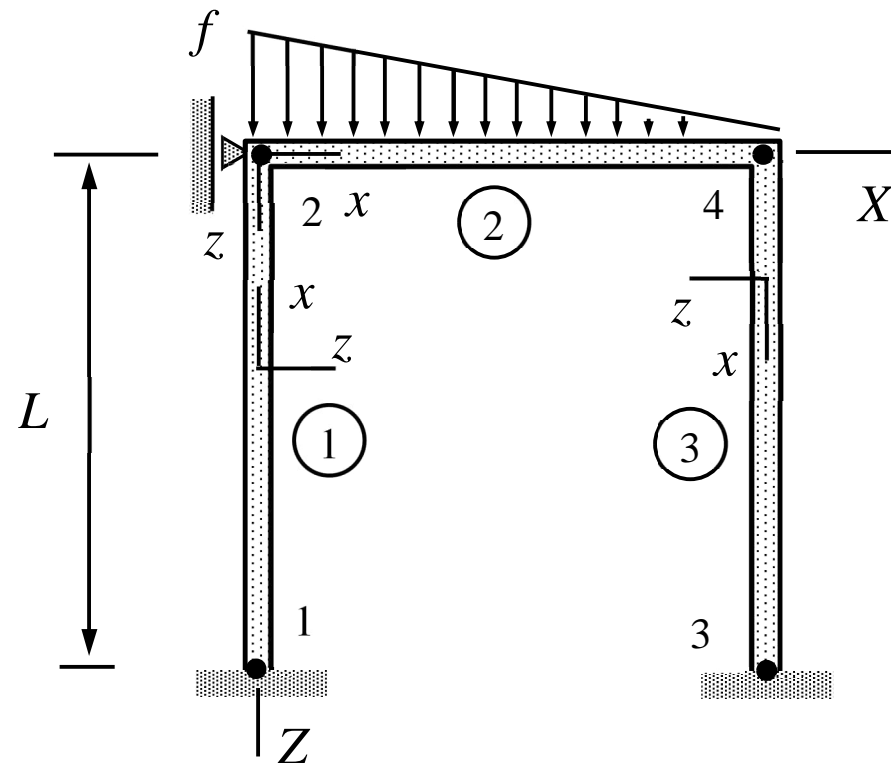
Answer $\theta_{Y2} = -\dfrac{7}{900}\dfrac{fL^3}{EI}$ and $\theta_{Y4} = \dfrac{11}{1800}\dfrac{fL^3}{EI}$

- The Mathematica code solution is given by (f_z is specified by its nodal values)

	model	properties	geometry
1	BEAM	{{E, G}, {A, I, I}}	Line[{1, 2}]
2	BEAM	{{E, G}, {A, I, I}, {0, 0, {f, 0}}}	Line[{2, 4}]
3	BEAM	{{E, G}, {A, I, I}}	Line[{4, 3}]

	{X,Y,Z}	{u$_X$,u$_Y$,u$_Z$}	{θ_X,θ_Y,θ_Z}
1	{0, 0, L}	{0, 0, 0}	{0, 0, 0}
2	{0, 0, 0}	{0, 0, 0}	{0, θY[2], 0}
3	{L, 0, L}	{0, 0, 0}	{0, 0, 0}
4	{L, 0, 0}	{0, 0, 0}	{0, θY[4], 0}

$$\left\{ \theta Y[2] \to -\frac{7\,f\,L^3}{900\,E\,I}, \; \theta Y[4] \to \frac{11\,f\,L^3}{1800\,E\,I} \right\}$$

Parameters of the problem can be functions of x. Then, derivation of the element contribution by using the exact solution may not be practical and, with 2D/3D elements for plates etc., impossible.

Week 46-4

4.1 VIRTUAL WORK EXPRESSION

To find the virtual work expression of an element without recourse to the exact solution of a boundary value problem (which may not be available due to non-constant material properties, distributed forces etc.)

☐ Start with the basic building blocks: virtual work density for the model and a polynomial interpolant to nodal displacements and rotations.

☐ Substitute the interpolant to the virtual work density expression and integrate the density over the mathematical domain occupied by the element (the density represents virtual work per unit length, area etc.).

☐ Rearrange to get the standard form $\delta W = -\delta \mathbf{a}^{\mathrm{T}}(\mathbf{Ka} - \mathbf{F})$.
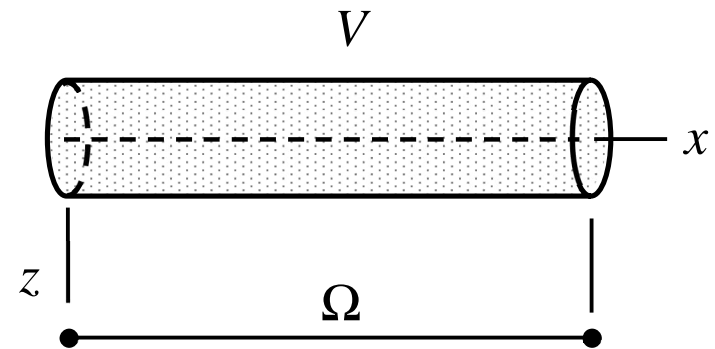
VIRTUAL WORK DENSITY

Virtual work densities are concise representations of engineering models (bar, beam, plate, shell, etc.). For the four loading modes of the beam model virtual (density = virtual work per unit length)

Bar: $\delta w_\Omega = -\dfrac{d\delta u}{dx} EA \dfrac{du}{dx} + \delta u f_x$

Torsion: $\delta w_\Omega = -\dfrac{d\delta\phi}{dx} GJ \dfrac{d\phi}{dx} + \delta\phi m_x$

Bending (xz): $\delta w_\Omega = -\dfrac{d^2\delta w}{dx^2} EI_{yy} \dfrac{d^2\delta w}{dx^2} + \delta w f_z$

Bending (xy): $\delta w_\Omega = -\dfrac{d^2\delta v}{dx^2} EI_{zz} \dfrac{d^2 v}{dx^2} + \delta v f_y$

STRUCTURE ANALYSIS; IMPROVED RECIPE

☐ Derive the element contributions δW^e from *virtual work density of the model* and *polynomial interpolation* of the nodal displacements and rotations in the material coordinate system. a new step

☐ Express the nodal displacements and rotations of the material coordinate system in terms of those in the structural coordinate system.

☐ Sum the element contributions over the elements and their loading modes to end up with the virtual work expression $\delta W = \sum_{e \in E} \delta W^e = \sum_{e \in E} (\sum_{m} \delta W_m^e)$ of structure. Restructure to get the form $\delta W = -\delta \mathbf{a}^{\mathrm{T}}(\mathbf{Ka} - \mathbf{F})$

☐ Use the principle of virtual work $\delta W = 0 \ \forall \delta \mathbf{a}$, fundamental lemma of variation calculus for $\delta \mathbf{a} \in \mathbb{R}^n$, and solve the dofs from $\mathbf{Ka} - \mathbf{F} = 0$.

4.2 BAR MODE
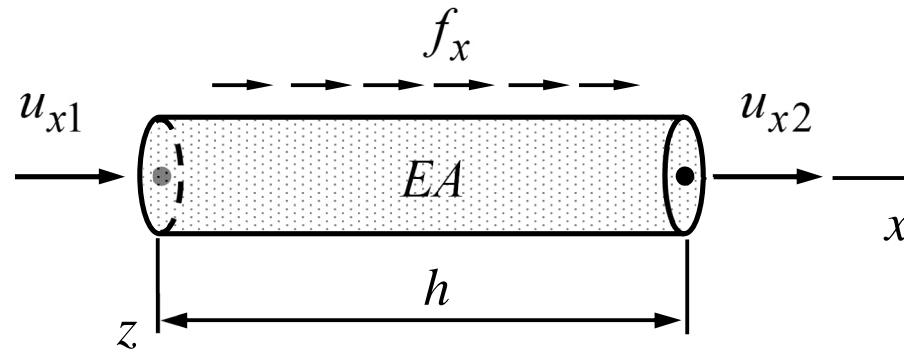
**Virtual work density:** $\delta w_\Omega = \delta w_\Omega^{\text{int}} + \delta w_\Omega^{\text{ext}} = -\dfrac{d\delta u}{dx} EA \dfrac{du}{dx} + \delta u f_x$

Linear interpolant: $u(x) = \mathbf{N}^{\text{T}}\mathbf{a} \equiv \dfrac{1}{h}\begin{Bmatrix} h-x \\ x \end{Bmatrix}^{\text{T}} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix}$

Cross-sectional area A, Young's modulus E, and force per unit length f_x (acting on the x-axis) may depend on position. Virtual work density depends only on the model but the interpolant (or approximation) can be chosen in various ways!

BAR ELEMENT CONTRIBUTION
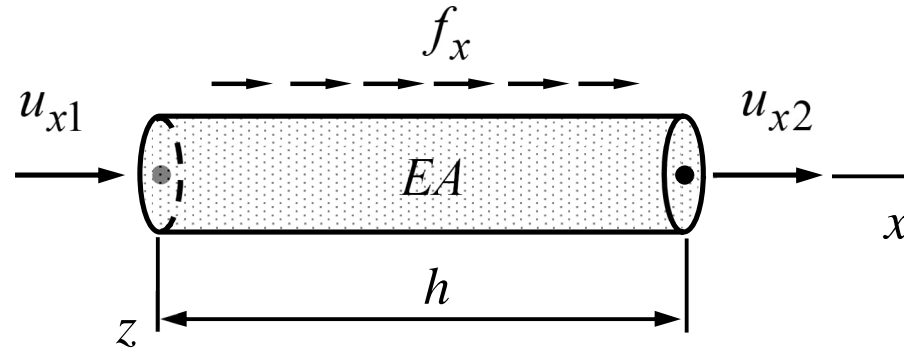
$$\delta W = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\text{T}} (\frac{EA}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}), \ u_x = \mathbf{i}^{\text{T}}\begin{Bmatrix} u_X \\ u_Y \\ u_Z \end{Bmatrix}, \text{ where } \mathbf{i} = \frac{1}{h}\begin{Bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{Bmatrix}$$

Above, f_x and EA are assumed constants and the elements of matrix $\mathbf{i}$ (1×1, 2×1, 3×1) are the components of the unit vector $\vec{i}$ in the structural coordinate system. The algorithm of Mathematica code is based on element contributions in its variational form!

- First, element interpolant $u = \mathbf{N}^{\mathrm{T}}\mathbf{a}$ and its variation $\delta u = \mathbf{N}^{\mathrm{T}}\delta\mathbf{a} = \delta\mathbf{a}^{\mathrm{T}}\mathbf{N}$ are substituted into the virtual work expression to get (here $\Omega =]0, h[$ and $d\Omega = dx$)

$$\delta W = \int_0^h (-\frac{d\delta u}{dx} EA \frac{du}{dx} + \delta u f_x) dx \qquad \Rightarrow$$

$$\delta W = -\int_0^h \delta\mathbf{a}^{\mathrm{T}} \frac{d\mathbf{N}}{dx} EA \frac{d\mathbf{N}^{\mathrm{T}}}{dx} \mathbf{a} dx + \int_0^h \delta\mathbf{a}^{\mathrm{T}} \mathbf{N} f_x dx \qquad \Leftrightarrow$$

$$\delta W = -\delta\mathbf{a}^{\mathrm{T}} (\int_0^h \frac{d\mathbf{N}}{dx} EA \frac{d\mathbf{N}^{\mathrm{T}}}{dx} dx \mathbf{a} - \int_0^h \mathbf{N} f_x dx). \qquad \leftarrow$$

- If the interpolant is taken to be linear, shape functions and the nodal values are given by

$$\mathbf{N} = \frac{1}{h} \begin{Bmatrix} h - x \\ x \end{Bmatrix}, \quad \frac{d}{dx}\mathbf{N} = \frac{1}{h} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}, \quad \mathbf{a} = \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix}, \text{ and } \delta\mathbf{a} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}$$

- Assuming that Young's modulus E, cross-sectional area A, and the distributed force f_x are constants, integration over the element domain gives (the expressions of the shape functions need to be substituted now)

$$\delta W = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^T (\int_0^h \frac{1}{h}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix} EA \frac{1}{h}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}^T dx \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \int_0^h \frac{1}{h}\begin{Bmatrix} h-x \\ x \end{Bmatrix} f_x dx) \quad \Leftrightarrow$$

$$\delta W = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^T (\frac{EA}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}). \quad \Leftarrow$$

Derivation out of virtual work densities works also when Young's modulus E, cross-sectional area A, and the distributed force f_x are not constants!

EXAMPLE 4.2 Consider the bar model and a piecewise linear interpolant of the nodal values. Determine the *equivalent nodal forces* $\mathbf{F}$ of the element contribution $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}}$, in which $\delta W^{\text{int}} = -\delta \mathbf{a}^{\text{T}} \mathbf{K} \mathbf{a}$ and $\delta W^{\text{ext}} = \delta \mathbf{a}^{\text{T}} \mathbf{F}$, when length of the element is h and

(a) f_x is constant,

(b) f_x is piecewise linear $f_x = \mathbf{N}^{\text{T}} \mathbf{f}$, where the nodal values are $\mathbf{f}^{\text{T}} = \{ f_{x1} \quad f_{x2} \}$,

(c) $f_x = F_x \delta(\dfrac{x}{h} - \dfrac{1}{2})$, where δ is the Dirac-delta and F_x is a point force.

Answer (a) $\mathbf{F} = \dfrac{f_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$ (b) $\mathbf{F} = \dfrac{h}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} f_{x1} \\ f_{x2} \end{Bmatrix}$ (c) $\mathbf{F} = \dfrac{F_x}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

- The equivalent nodal forces are obtained by using $u = \mathbf{N}^{\mathrm{T}}\mathbf{a}$ in the virtual work expression of the external forces

$$\Leftarrow \quad \delta W^{\mathrm{ext}} = \int_0^h \delta u f_x dx, \quad \delta u = \delta \mathbf{a}^{\mathrm{T}}\mathbf{N} \text{ and } \mathbf{N} = \frac{1}{h}\begin{Bmatrix} h-x \\ x \end{Bmatrix} \Rightarrow \mathbf{F} = \int_0^h \frac{1}{h}\begin{Bmatrix} h-x \\ x \end{Bmatrix} f_x dx \quad \Leftarrow$$

- With the constant, linear and Dirac delta distributions

$$\mathbf{F} = \int_0^h \frac{1}{h}\begin{Bmatrix} h-x \\ x \end{Bmatrix} f_x dx = f_x \int_0^h \frac{1}{h}\begin{Bmatrix} h-x \\ x \end{Bmatrix} dx = \frac{f_x h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \quad \Leftarrow$$

$$\mathbf{F} = \int_0^h \frac{1}{h}\begin{Bmatrix} h-x \\ x \end{Bmatrix} f_x dx = \int_0^h \frac{1}{h}\begin{Bmatrix} h-x \\ x \end{Bmatrix}\begin{Bmatrix} h-x \\ x \end{Bmatrix}^{\mathrm{T}} dx \begin{Bmatrix} f_{x1} \\ f_{x2} \end{Bmatrix} = \frac{h}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} f_{x1} \\ f_{x2} \end{Bmatrix} \quad \Leftarrow$$

$$\mathbf{F} = \int_0^h \frac{1}{h}\begin{Bmatrix} h-x \\ x \end{Bmatrix} F_x \delta(x - \frac{h}{2}) dx = \frac{F_x}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \quad \Leftarrow$$

EXAMPLE 4.3 The bar of the figure (*EA* is constant) is loaded by a quadratic distributed force $f_x = f\xi(2-\xi)$ where $\xi = x/L$. Determine the displacement at the free end by the finite element method. Use one, two, and four elements of equal lengths.
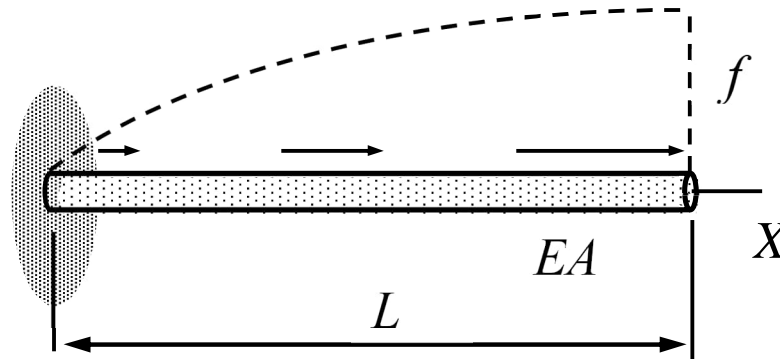
Answer $u_{X2} = \dfrac{5}{12}\dfrac{fL^2}{EA}$ no matter the number of elements (exact $u(L) = \dfrac{5}{12}\dfrac{fL^2}{EA}$)

- Distributed force f_x, Young's modulus E, and the cross-sectional area A may depend on x. In Mathematica code a quadratic distributed force f_x is defined by its values on the nodes and at the midpoint.

	model	properties	geometry
1	BAR	$\{\{E\}, \{A\}, \{\{0, \frac{3\,f}{4}, f\}, 0, 0\}\}$	Line[{1, 2}]

	{X,Y,Z}	{u_X,u_Y,u_Z}	{Θ_X,Θ_Y,Θ_Z}
1	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}
2	{L, 0, 0}	{uX[2], 0, 0}	{0, 0, 0}

$$\left\{ \frac{5\,f\,L^2}{12\,A\,E}, \frac{5\,f\,L^2}{12\,A\,E}, \frac{5\,f\,L^2}{12\,A\,E} \right\}$$

Above, the problem has been solved three times with 1, 2, and 4 elements and displacements at the free end are given as a list (see the Mathematica notebook for the details).

EXAMPLE 4.4 The cross-sectional area of a bar is given by $A / A_0 = 1 - x / (2L)$. Assuming that the approximation of displacement u is (piecewise) linear, Young's modulus E and density ρ of the material are constants and distributed loading f_x is due to the gravity, determine the displacement at the free end of the bar. Use two elements of equal length.
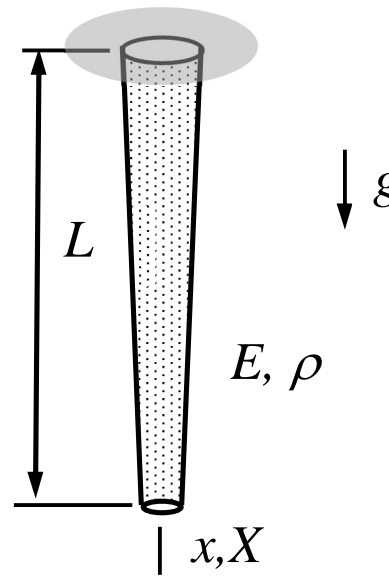
Answer $u(L) = \dfrac{29}{70} \dfrac{g \rho L^2}{E}$ ... (exact $u(L) = \dfrac{3 - \log 4}{4} \dfrac{g \rho L^2}{E}$, error 2.7%)

- Element interpolants of displacement, cross-sectional area (in terms of its nodal values), and weight per unit length are here

$$u(x) = \frac{1}{h}\{h - x \quad x\}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} \implies \frac{du}{dx} = \frac{1}{h}\{-1 \quad 1\}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} \text{ and } \frac{d\delta u}{dx} = \frac{1}{h}\{-1 \quad 1\}\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix},$$

$$A(x) = \frac{1}{h}\{h - x \quad x\}\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} \implies f_x = \rho g A = \frac{\rho g}{h}\{h - x \quad x\}\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}.$$

- Virtual works of internal and external forces per unit length of a bar are given by

$$\delta w_\Omega^{int} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{EA}{h^2}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}\{-1 \quad 1\}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{EA}{h^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} \implies$$

$$\delta w_\Omega^{int} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{E}{h^3}\{h - x \quad x\}\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix},$$

$$\delta w_\Omega^{\text{ext}} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\text{T}} \frac{1}{h^2} \begin{Bmatrix} h-x \\ x \end{Bmatrix} \{ h-x \quad x \} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} \rho g \ .$$

- Element contribution of a typical element is obtained as integral over the domain occupied by the element

$$\delta W^{\text{int}} = \int_0^h \delta w_\Omega^{\text{int}} dx = - \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\text{T}} \frac{E(A_1 + A_2)}{2h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix},$$

$$\delta W^{\text{ext}} = \int_0^h \delta w_\Omega^{\text{ext}} dx = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\text{T}} \frac{\rho g h}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} \quad \Rightarrow$$

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = - \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\text{T}} (\frac{E}{h} \frac{A_1 + A_2}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{\rho g h}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix})$$

- Element contributions of the two elements

$$\delta W^1 = -\left\{\begin{matrix} 0 \\ \delta u_{X2} \end{matrix}\right\}^{\mathrm{T}} (\frac{7EA_0}{4L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\left\{\begin{matrix} 0 \\ u_{X2} \end{matrix}\right\} - \frac{\rho g L A_0}{48}\left\{\begin{matrix} 11 \\ 10 \end{matrix}\right\}) \quad (A_1 = A_0, \ A_2 = \frac{3A_0}{4}),$$

$$\delta W^2 = -\left\{\begin{matrix} \delta u_{X2} \\ \delta u_{X3} \end{matrix}\right\}^{\mathrm{T}} (\frac{5EA_0}{4L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\left\{\begin{matrix} u_{X2} \\ u_{X3} \end{matrix}\right\} - \frac{\rho g L A_0}{48}\left\{\begin{matrix} 8 \\ 7 \end{matrix}\right\}) \quad (A_1 = \frac{3A_0}{4}, \ A_2 = \frac{A_0}{2}).$$

- Virtual work of the structure is the sum over elements $\delta W = \sum \delta W^e = \delta W^1 + \delta W^2$

$$\delta W = -\delta u_{X2}(\frac{7EA_0}{4L}u_{X2} - \frac{A_0\rho g L}{48}10) - \left\{\begin{matrix} \delta u_{X2} \\ \delta u_{X3} \end{matrix}\right\}^{\mathrm{T}} (\frac{A_0}{4L}\begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix}\left\{\begin{matrix} u_{X2} \\ u_{X3} \end{matrix}\right\} - \frac{\rho g L A_0}{48}\left\{\begin{matrix} 8 \\ 7 \end{matrix}\right\})$$

$$\delta W = -\left\{\begin{matrix} \delta u_{X2} \\ \delta u_{X3} \end{matrix}\right\}^{\mathrm{T}} (\frac{EA_0}{4L}\begin{bmatrix} 12 & -5 \\ -5 & 5 \end{bmatrix}\left\{\begin{matrix} u_{X2} \\ u_{X3} \end{matrix}\right\} - \frac{\rho g L A_0}{48}\left\{\begin{matrix} 18 \\ 7 \end{matrix}\right\})$$

- Principle of virtual work and the fundamental lemma of variation calculus give

$$\delta W = -\begin{Bmatrix} \delta u_{X2} \\ \delta u_{X3} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA_0}{4L} \begin{bmatrix} 12 & -5 \\ -5 & 5 \end{bmatrix} \begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} - \frac{\rho g L A_0}{48} \begin{Bmatrix} 18 \\ 7 \end{Bmatrix} \right) = 0 \quad \forall \begin{Bmatrix} \delta u_{X2} \\ \delta u_{X3} \end{Bmatrix} \quad \Leftrightarrow$$

$$\frac{EA_0}{4L} \begin{bmatrix} 12 & -5 \\ -5 & 5 \end{bmatrix} \begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} - \frac{\rho g L A_0}{48} \begin{Bmatrix} 18 \\ 7 \end{Bmatrix} = 0 \quad \Leftrightarrow$$

$$\begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} = \frac{\rho g L^2}{12E} \begin{bmatrix} 12 & -5 \\ -5 & 5 \end{bmatrix}^{-1} \begin{Bmatrix} 18 \\ 7 \end{Bmatrix} = \frac{\rho g L^2}{12 \cdot 35 E} \begin{bmatrix} 5 & 5 \\ 5 & 12 \end{bmatrix} \begin{Bmatrix} 18 \\ 7 \end{Bmatrix} = \frac{\rho g L^2}{E} \begin{Bmatrix} 25/84 \\ 29/70 \end{Bmatrix}. \quad \Leftarrow$$
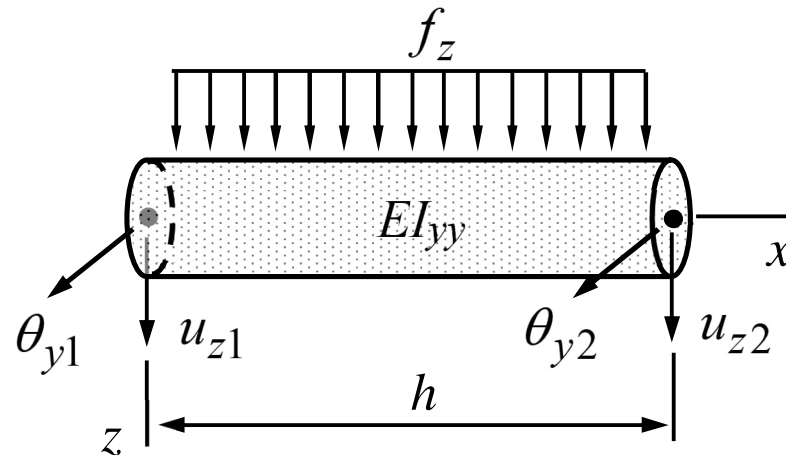
- In the Mathematica code of the course, the given quantities may vary linearly

	model	properties	geometry
1	BAR	$\left\{\{E\}, \left\{\left\{A0, \frac{3\,A0}{4}\right\}\right\}, \left\{\left\{A0\,g\,\rho, \frac{3\,A0\,g\,\rho}{4}\right\}, 0, 0\right\}\right\}$	Line[{1, 2}]
2	BAR	$\left\{\{E\}, \left\{\left\{\frac{3\,A0}{4}, \frac{A0}{2}\right\}\right\}, \left\{\left\{\frac{3\,A0\,g\,\rho}{4}, \frac{A0\,g\,\rho}{2}\right\}, 0, 0\right\}\right\}$	Line[{2, 3}]

	{X,Y,Z}	{u_X,u_Y,u_Z}	{Θ_X,Θ_Y,Θ_Z}
1	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}
2	$\left\{\frac{L}{2}, 0, 0\right\}$	{uX[2], 0, 0}	{0, 0, 0}
3	{L, 0, 0}	{uX[3], 0, 0}	{0, 0, 0}

$$\left\{uX[2] \to \frac{25\,g\,L^2\,\rho}{84\,E}, \; uX[3] \to \frac{29\,g\,L^2\,\rho}{70\,E}\right\}$$
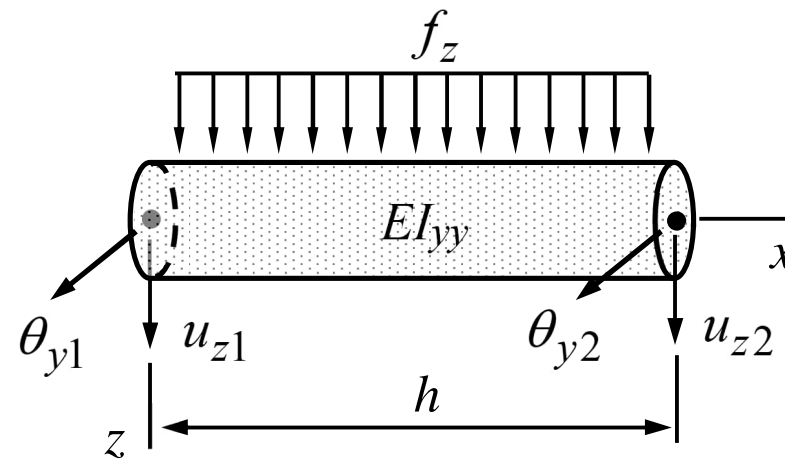
4.3 BENDING MODE

Virtual work density: $\quad \delta w_\Omega = \delta w_\Omega^{\text{int}} + \delta w_\Omega^{\text{ext}} = -\dfrac{d^2 \delta w}{dx^2} EI_{yy} \dfrac{d^2 w}{dx^2} + \delta w f_z$

Cubic interpolant: $\quad w(x) = \left\{ \begin{array}{c} (1-\xi)^2 (1+2\xi) \\ -h(1-\xi)^2 \xi \\ \overline{} \\ (3-2\xi)\xi^2 \\ -h\xi^2(\xi-1) \end{array} \right\}^{\mathrm{T}} \left\{ \begin{array}{c} u_{z1} \\ \theta_{y1} \\ \overline{} \\ u_{z2} \\ \theta_{y2} \end{array} \right\}$ where $\xi = \dfrac{x}{h}$.

BEAM BENDING ELEMENT

$$\delta W = -\left\{\begin{array}{c} \delta u_{z1} \\ \delta \theta_{y1} \\ \hdashline \delta u_{z2} \\ \delta \theta_{y2} \end{array}\right\}^{\text{T}} \left(\frac{EI_{yy}}{h^3} \left[\begin{array}{cc:cc} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ \hdashline -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{array}\right] \left\{\begin{array}{c} u_{z1} \\ \theta_{y1} \\ \hdashline u_{z2} \\ \theta_{y2} \end{array}\right\} - \frac{f_z h}{12} \left\{\begin{array}{c} 6 \\ -h \\ \hdashline 6 \\ h \end{array}\right\}\right), \quad u_z = \mathbf{k}^{\text{T}} \left\{\begin{array}{c} u_X \\ u_Y \\ u_Z \end{array}\right\} \text{ etc.}$$

Above, f_z and EI_{yy} are assumed to be constants and the elements of matrices $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ (1×1, 2×1, 3×1) are the components of the unit vectors $\vec{i}, \vec{j}$ and $\vec{k}$ in the structural coordinate system.

- First, element interpolant (approximation) $w = \mathbf{N}^{\mathrm{T}}\mathbf{a}$ and its variation $\delta w = \delta\mathbf{a}^{\mathrm{T}}\mathbf{N}$ are substituted into the virtual work expression to get

$$\delta W = \int_0^h \left(-\frac{d^2\delta w}{dx^2}EI_{yy}\frac{d^2 w}{dx^2} + \delta w f_z\right)dx \quad \Rightarrow$$

$$\delta W = -\int_0^h \delta\mathbf{a}^{\mathrm{T}}\frac{d^2\mathbf{N}}{dx^2}EI_{yy}\frac{d^2\mathbf{N}^{\mathrm{T}}}{dx^2}\mathbf{a}\,dx + \int_0^h \delta\mathbf{a}^{\mathrm{T}}\mathbf{N}f_z dx \quad \Leftrightarrow$$

$$\delta W = -\delta\mathbf{a}^{\mathrm{T}}\left(\int_0^h \frac{d^2\mathbf{N}}{dx^2}EI_{yy}\frac{d^2\mathbf{N}^{\mathrm{T}}}{dx^2}dx\,\mathbf{a} - \int_0^h \mathbf{N}f_z dx\right). \quad \leftarrow$$

- The shape function expressions and their second derivatives are (Mathematica is useful in the calculations)

$$\mathbf{N} = \left\{ \begin{array}{c} (1 - x/h)^2 (1 + 2x/h) \\[2mm] \dfrac{-h(1 - x/h)^2 x/h}{(3 - 2x/h)(x/h)^2} \\[4mm] -h(x/h)^2 (x/h - 1) \end{array} \right\} \quad \text{and} \quad \frac{d^2 \mathbf{N}}{dx^2} = \frac{1}{h^2} \left\{ \begin{array}{c} 6(2x/h - 1) \\[2mm] \dfrac{-2(3x/h - 2)h}{6(1 - 2x/h)} \\[4mm] -(3x/h - 1)h \end{array} \right\}.$$

- In the next step, the shape function expressions are substituted into the virtual work expression. Integration over the domain occupied by the element gives the element contribution. A derivation along these lines is valid also when the given functions are not constants!

EXAMPLE 4.5 The integral representation of *equivalent nodal forces* is $\mathbf{F} = \int_{\Omega^e} \mathbf{N} f_z \, d\Omega$.

Determine the equivalent nodal forces of a beam element for (a) $f_z = \text{const.}$, (b) $f_z = f_{z1}(1-\xi) + \xi f_{z2}$, and (c) $f_z = F_z \delta(x - h/2)$ (Dirac delta at the midpoint), when

$$\mathbf{N} = \left\{ (1-\xi)^2(1+2\xi) \quad -h(1-\xi)^2\xi \quad (3-2\xi)\xi^2 \quad -h\xi^2(\xi-1) \right\}^{\mathrm{T}}$$

Answer
$$\mathbf{F} = \frac{hf_z}{12} \left\{ \begin{array}{c} 6 \\ -h \\ 6 \\ h \end{array} \right\}, \quad \mathbf{F} = \frac{hf_{z1}}{60} \left\{ \begin{array}{c} 21 \\ -3h \\ 9 \\ 2h \end{array} \right\} + \frac{hf_{z2}}{60} \left\{ \begin{array}{c} 9 \\ -2h \\ 21 \\ 3h \end{array} \right\}, \quad \text{and } \mathbf{F} = \frac{F_z}{8} \left\{ \begin{array}{c} 4 \\ -h \\ 4 \\ h \end{array} \right\}$$

$$\mathbf{F} = \int_0^h \left\{ \begin{array}{c} (1-x/h)^2(1+2x/h) \\ -h(1-x/h)^2 x/h \\ \hline (3-2x/h)(x/h)^2 \\ -h(x/h)^2(x/h-1) \end{array} \right\} f_z \, dx = \frac{hf_z}{12} \left\{ \begin{array}{c} 6 \\ -h \\ \hline 6 \\ h \end{array} \right\} \quad \textcolor{red}{\Leftarrow}$$

$$\mathbf{F} = \int_0^h \left\{ \begin{array}{c} (1-x/h)^2(1+2x/h) \\ -h(1-x/h)^2 x/h \\ \hline (3-2x/h)(x/h)^2 \\ -h(x/h)^2(x/h-1) \end{array} \right\} [(1-\frac{x}{h})f_{z1} + \frac{x}{h}f_{z2}] \, dx = \frac{hf_{z1}}{60} \left\{ \begin{array}{c} 21 \\ -3h \\ 9 \\ 2h \end{array} \right\} + \frac{hf_{z2}}{60} \left\{ \begin{array}{c} 9 \\ -2h \\ 21 \\ 3h \end{array} \right\} \quad \textcolor{red}{\Leftarrow}$$

$$\mathbf{F} = \int_0^h \left\{ \begin{array}{c} (1-x/h)^2(1+2x/h) \\ -h(1-x/h)^2 x/h \\ \hline (3-2x/h)(x/h)^2 \\ -h(x/h)^2(x/h-1) \end{array} \right\} F_z \delta(x-\frac{h}{2}) \, dx = \frac{F_z}{8} \left\{ \begin{array}{c} 4 \\ -h \\ 4 \\ h \end{array} \right\} \quad \textcolor{red}{\Leftarrow}$$

Week 46-27

4.4 INTERPOLATION

Piecewise linear interpolant to nodal values $\{(x_0, f_0), (x_1, f_1), \ldots, (x_n, f_n)\}$ gives the simplest continuous polynomial approximation to $f(x)$.
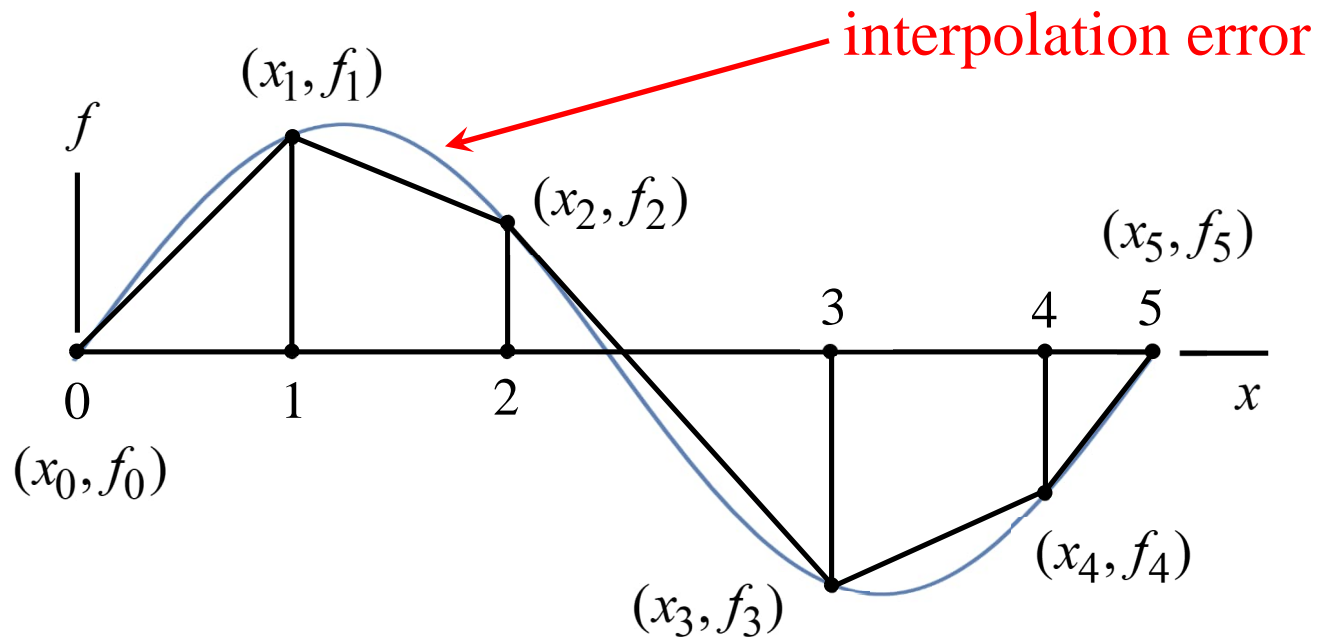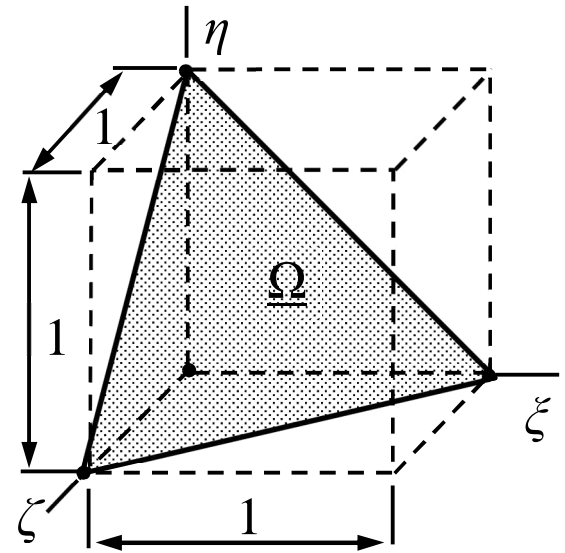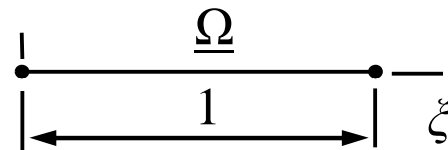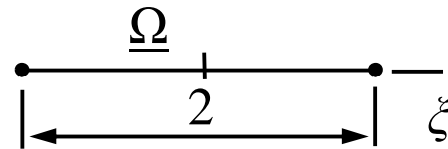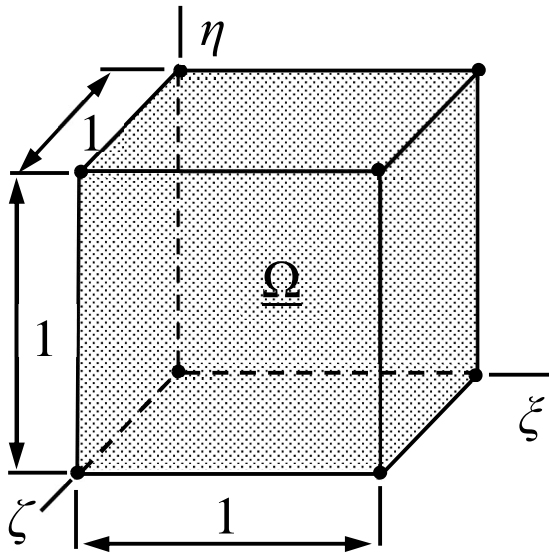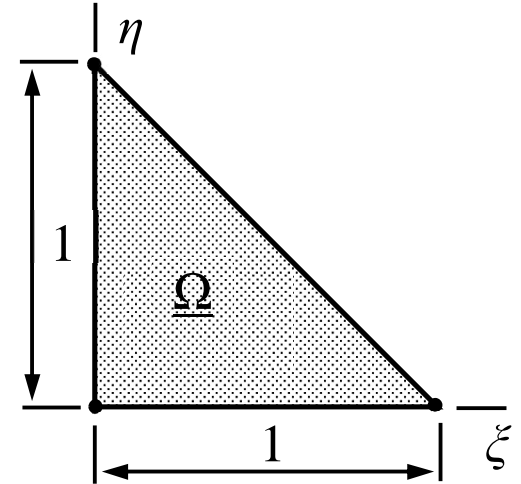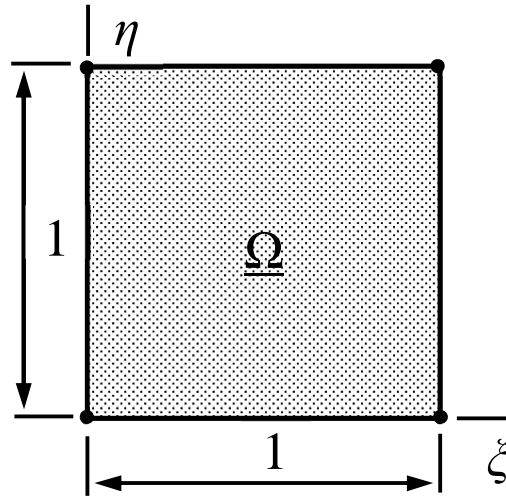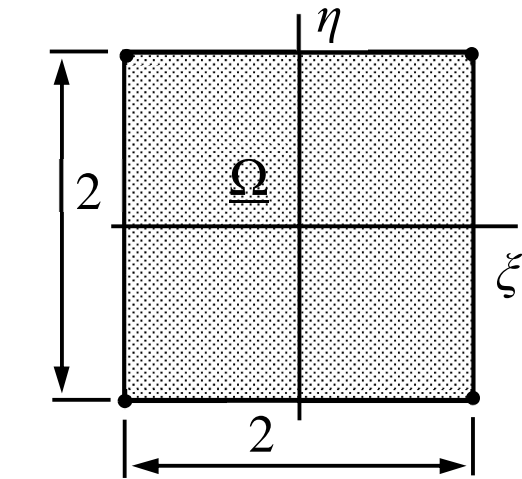

Interpolation with piecewise linear polynomials extends straightforwardly to more dimensions, higher order polynomials, and divisions of the domain into elements.

EXAMPLE 4.6 Interpolants of $f(x,y)/F = \sin(2\pi x/L)\sin(\pi y/L)/4$ on square domain $(x,y)/L \in [0,1] \times [0,1]$ with triangle and rectangle elements of increasing number.
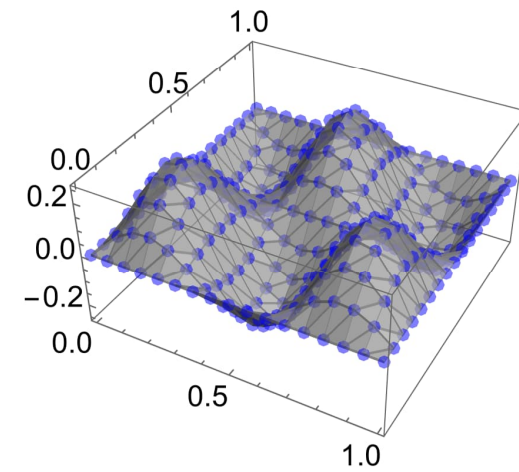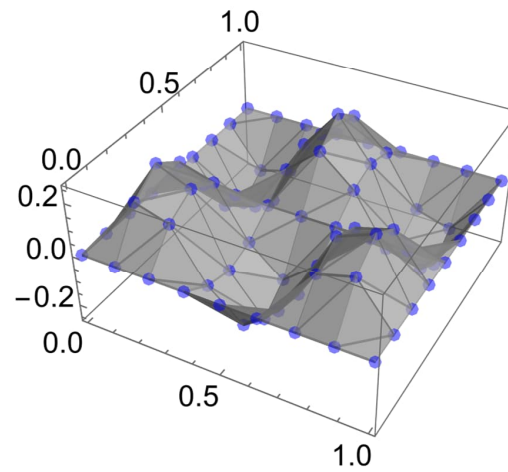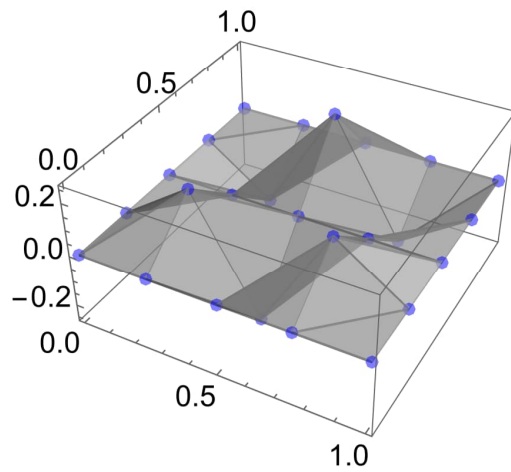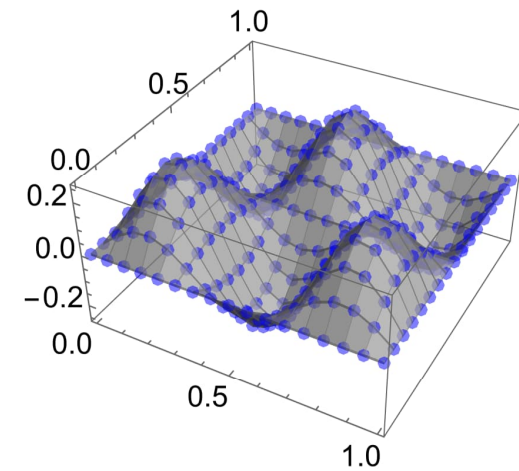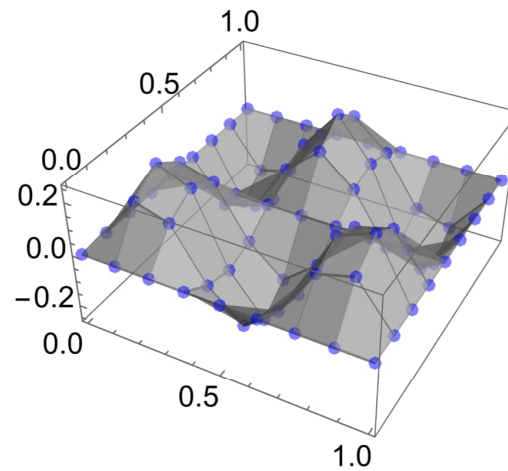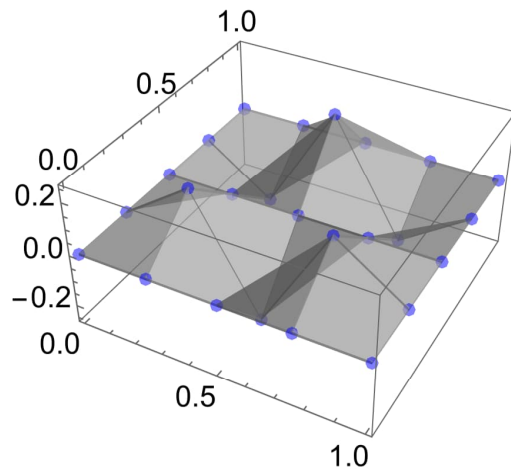
ELEMENTS

SHAPE FUNCTIONS

Shape functions are used to interpolate the nodal values inside the elements. The shape function N_i of node i in element Ω^e

☐ is the lowest order polynomial taking the value 1 at node i and the value 0 at all the other nodes of the element.

☐ shape functions should satisfy the previous condition on each edge (as an example, shape function should be linear on an edge of two nodes)

☐ Sum of the shape functions of an element should be 1.

The shape functions can often be deduced directly by using the conditions above or/and by using the Lagrange interpolation polynomials.

Lagrange interpolation polynomial $p_n(x)$ of degree n and its error formula are for dataset $\{(x_0, f_0), (x_1, f_1), \ldots (x_n, f_n)\}$

$$p_n(x) = \sum_{i \in \{0,1,\ldots,n\}} f_i \prod_{j \in \{0,1,\ldots i-1, i+1, \ldots, n\}} \frac{x - x_j}{x_i - x_j},$$

$$f(x) - p_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{i \in \{0,1,\ldots,n\}} (x - x_i).$$

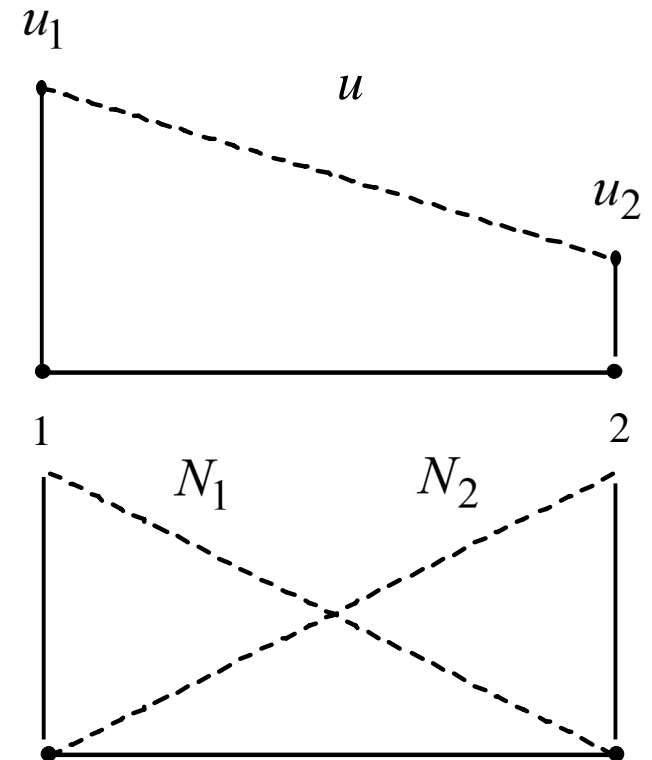Notice the removal of index i in the product term inside the sum of the interpolation formula.

LINEAR SHAPE FUNCTIONS

Piecewise linear approximation in one dimension is continuous in Ω and a first order polynomial inside the elements. In element Ω^e

Approximation: $u = \mathbf{N}^T \mathbf{a}$

Nodal values: $\mathbf{a} = \{u_1 \ u_2\}^T$

Shape functions: $\mathbf{N} = \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} = \begin{Bmatrix} 1-\xi \\ \xi \end{Bmatrix}$ where $\xi = \dfrac{x}{h}$

Piecewise linear approximation is the simplest choice e.g. for the bar model.

- The method based on combining given polynomials gives (use of the scaled coordinate ξ simplifies the expressions)

$$\mathbf{N} = \begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \end{Bmatrix} = \frac{1}{h}\begin{bmatrix} x_2 & -1 \\ -x_1 & 1 \end{bmatrix}\begin{Bmatrix} 1 \\ x \end{Bmatrix} = \frac{1}{h}\begin{Bmatrix} x_2 - x \\ x - x_1 \end{Bmatrix} = \begin{Bmatrix} 1-\xi \\ \xi \end{Bmatrix}, \quad \text{where } \xi = \frac{x}{h}$$

- The virtual work expression (e.g. of the bar model) contains integrals of the shape functions in certain combinations. The most common are (here $\Omega^e =]x_1, x_2[$, $d\Omega = dx$, and $h = |x_2 - x_1|$)

$$\int_{\Omega^e} \mathbf{N}\,d\Omega = \frac{h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \quad \int_{\Omega^e} \mathbf{N}\mathbf{N}^{\mathrm{T}}\,d\Omega = \frac{h}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \text{and} \quad \int_{\Omega^e} \frac{d\mathbf{N}}{dx}\frac{d\mathbf{N}^{\mathrm{T}}}{dx}\,d\Omega = \frac{1}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$
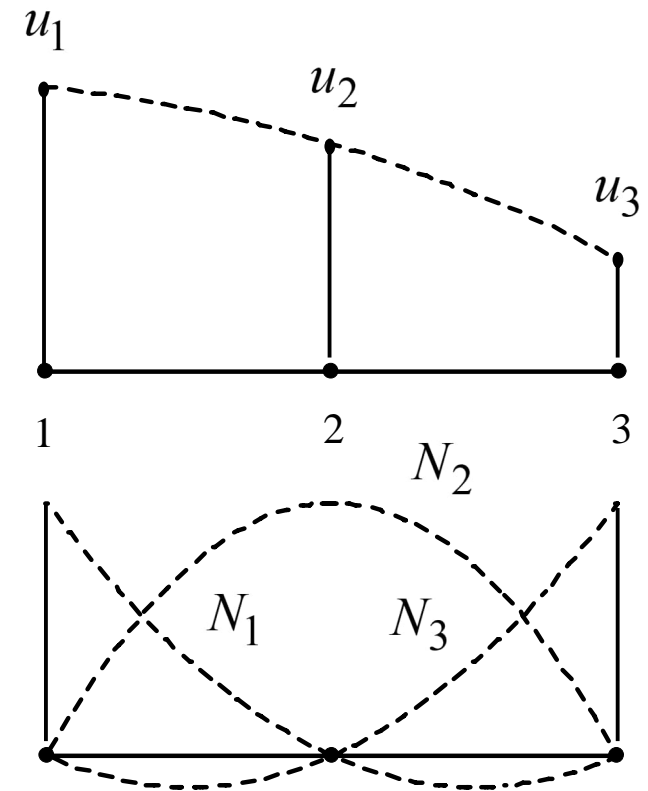
QUADRATIC SHAPE FUNCTIONS

Piecewise quadratic approximation in one dimension is continuous in Ω and a second order polynomial inside the elements. In element Ω^e

Approximation: $u = \mathbf{N}^T \mathbf{a}$

Nodal values: $\mathbf{a} = \{u_1 \quad u_2 \quad u_3\}^T$

Shape functions: $\mathbf{N} = \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} = \begin{Bmatrix} 1 - 3\xi + 2\xi^2 \\ 4\xi(1-\xi) \\ \xi(2\xi - 1) \end{Bmatrix}, \quad \xi = \dfrac{x}{h}$

More nodes can be used to generate higher order approximations!

- Derivation based on the Lagrange interpolation polynomials is convenient in the one-dimensional case. The idea is to write a polynomial vanishing on some set of points and scale the expression to take the value one at a certain point. In terms of $\xi = x/h$

$$N_1 = \frac{(\xi - 1/2)(\xi - 1)}{(0 - 1/2)(0 - 1)} = (2\xi - 1)(\xi - 1) \text{ and } N_2 = \frac{(\xi - 0)(\xi - 1)}{(1/2 - 0)(1/2 - 1)} = 4\xi(1 - \xi) \text{ etc.}$$

- Some integrals of the virtual work expression are given by

$$\int_0^h \mathbf{N} dx = \frac{h}{6} \begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix}, \quad \int_0^h \mathbf{N}\mathbf{N}^{\mathrm{T}} dx = \frac{h}{30} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}, \quad \int_0^h \frac{d\mathbf{N}}{dx}\frac{d\mathbf{N}}{dx}^{\mathrm{T}} dx = \frac{1}{3h} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}.$$

EXAMPLE 4.7 Find the virtual work expression $\delta W = \int_{\Omega} \delta w d\Omega$ of a bar element, when $\delta w_{\Omega} = -(d\delta u / dx)EA(du / dx) + \delta u f_x$, the shape functions are quadratic (a three-node element) and the force per unit length is (a) $f_x = $ constant (b) $f_x = F_x \delta(\xi - 1/2)$. The length of the element is h.

Answer:

(a) $\delta W = -\left\{ \begin{matrix} \delta u_{x1} \\ \delta u_{x2} \\ \delta u_{x3} \end{matrix} \right\}^{\mathrm{T}} \left(\dfrac{EA}{3h} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} \left\{ \begin{matrix} u_{x1} \\ u_{x2} \\ u_{x3} \end{matrix} \right\} - \dfrac{f_x h}{6} \left\{ \begin{matrix} 1 \\ 4 \\ 1 \end{matrix} \right\} \right)$

(b) $\delta W = -\left\{ \begin{matrix} \delta u_{x1} \\ \delta u_{x2} \\ \delta u_{x3} \end{matrix} \right\}^{\mathrm{T}} \left(\dfrac{EA}{3h} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} \left\{ \begin{matrix} u_{x1} \\ u_{x2} \\ u_{x3} \end{matrix} \right\} - F_x \left\{ \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} \right\} \right)$

- The quadratic shape functions of a three-node element can be obtained e.g. by using the Lagrange interpolation polynomials ($\xi = x/h$)

$$\mathbf{N} = \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} = \begin{Bmatrix} 1 - 3x/h + 2(x/h)^2 \\ 4x/h - 4(x/h)^2 \\ 2(x/h)^2 - x/h \end{Bmatrix} \quad \Rightarrow \quad \frac{d\mathbf{N}}{dx} = \begin{Bmatrix} dN_1/dx \\ dN_2/dx \\ dN_3/dx \end{Bmatrix} = \frac{1}{h} \begin{Bmatrix} -3 + 4(x/h) \\ 4 - 8(x/h) \\ 4(x/h) - 1 \end{Bmatrix}.$$

- Approximation, its derivative and variations needed in the virtual work density are

$$\frac{du}{dx} = \begin{Bmatrix} dN_1/dx \\ dN_2/dx \\ dN_3/dx \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{x1} \\ u_{x2} \\ u_{x3} \end{Bmatrix}, \quad \frac{d\delta u}{dx} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \\ \delta u_{x3} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} dN_1/dx \\ dN_2/dx \\ dN_3/dx \end{Bmatrix}, \quad \text{and} \quad \delta u = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \\ \delta u_{x3} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix}.$$

- When the approximation is substituted there, virtual work density takes the form

$$\delta w = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \\ \delta u_{x3} \end{Bmatrix}^{\mathrm{T}} (EA \begin{bmatrix} \dfrac{dN_1}{dx}\dfrac{dN_1}{dx} & \dfrac{dN_1}{dx}\dfrac{dN_2}{dx} & \dfrac{dN_1}{dx}\dfrac{dN_3}{dx} \\[2em] \dfrac{dN_2}{dx}\dfrac{dN_1}{dx} & \dfrac{dN_2}{dx}\dfrac{dN_2}{dx} & \dfrac{dN_2}{dx}\dfrac{dN_3}{dx} \\[2em] \dfrac{dN_3}{dx}\dfrac{dN_1}{dx} & \dfrac{dN_3}{dx}\dfrac{dN_2}{dx} & \dfrac{dN_3}{dx}\dfrac{dN_3}{dx} \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \\ u_{x3} \end{Bmatrix} - \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} f_x)$$

- Virtual work of the external volume force is given by integral $\delta W = \int_0^h \delta w\, dx$. If $f_x = \text{constant}$ or $f_x = F_x \delta(\xi - 1/2)$, the outcome is

$$\delta W = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \\ \delta u_{x3} \end{Bmatrix}^{\mathrm{T}} (\frac{EA}{3h} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \\ u_{x3} \end{Bmatrix} - \frac{f_x h}{6} \begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix}) \quad \Leftarrow$$

$$\delta W = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \\ \delta u_{x3} \end{Bmatrix}^{\mathrm{T}} (\frac{EA}{3h} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \\ u_{x3} \end{Bmatrix} - F_x \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}) \quad \Leftarrow$$
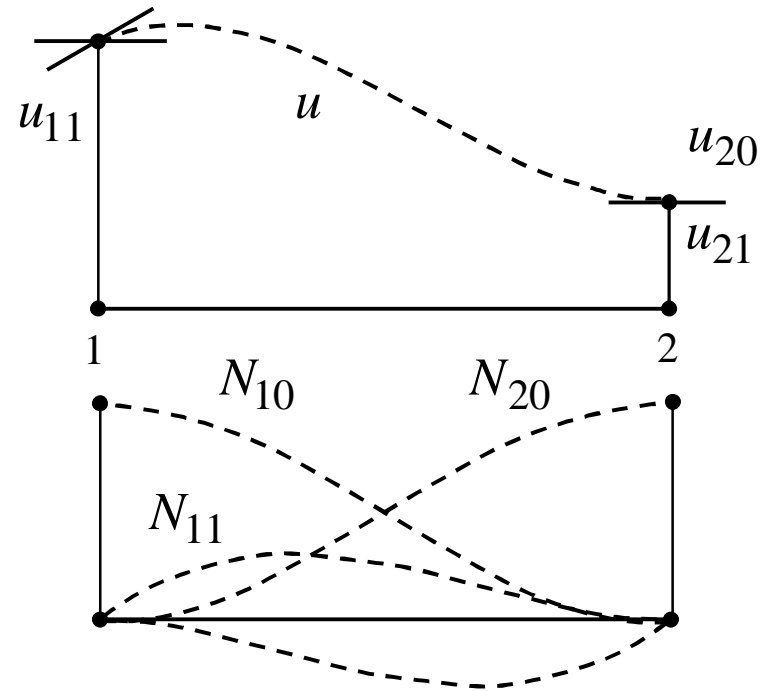
CUBIC SHAPE FUNCTIONS

Piecewise cubic approximation has continuous derivatives up to the first order in Ω and is a third order polynomial inside the elements.

Approximation: $u = \mathbf{N}^{\mathrm{T}}\mathbf{a}$

Nodal values: $\mathbf{a} = \{u_1 \quad du_1/dx \mid u_2 \quad du_2/dx\}$

Shape functions: $\mathbf{N} = \left\{ \begin{array}{c} N_{10} \\ N_{11} \\ \hline N_{20} \\ N_{21} \end{array} \right\} = \left\{ \begin{array}{c} (1-\xi)^2(1+2\xi) \\ h(1-\xi)^2\xi \\ \hline (3-2\xi)\xi^2 \\ h\xi^2(\xi-1) \end{array} \right\}$

In xz-plane bending $u = u_z$ and $du/dx = -\theta_y$, in xy-plane $u = u_y$ and $du/dx = \theta_z$.

- In one-dimensional case, the brute force approach works. Let us collect the coefficients of the monomials of the shape functions into a matrix, and use the definition of the shape functions

$$
\begin{Bmatrix} N_{10} \\ N_{11} \\ N_{20} \\ N_{21} \end{Bmatrix} = [A] \begin{Bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{Bmatrix} \quad \Rightarrow \quad
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [A]
\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & h & 1 \\ 0 & 0 & h^2 & 2h \\ 0 & 0 & h^3 & 3h^2 \end{bmatrix} \quad \Rightarrow
$$

$$
\begin{Bmatrix} N_{10} \\ N_{11} \\ N_{20} \\ N_{21} \end{Bmatrix} =
\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & h & 1 \\ 0 & 0 & h^2 & 2h \\ 0 & 0 & h^3 & 3h^2 \end{bmatrix}^{-1}
\begin{Bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{Bmatrix} =
\begin{Bmatrix} (-1+\xi)^2 (1+2\xi) \\ h(-1+\xi)^2 \xi \\ (3-2\xi)\xi^2 \\ h(-1+\xi)\xi^2 \end{Bmatrix}, \quad \text{where} \quad \xi = \frac{x}{h}.
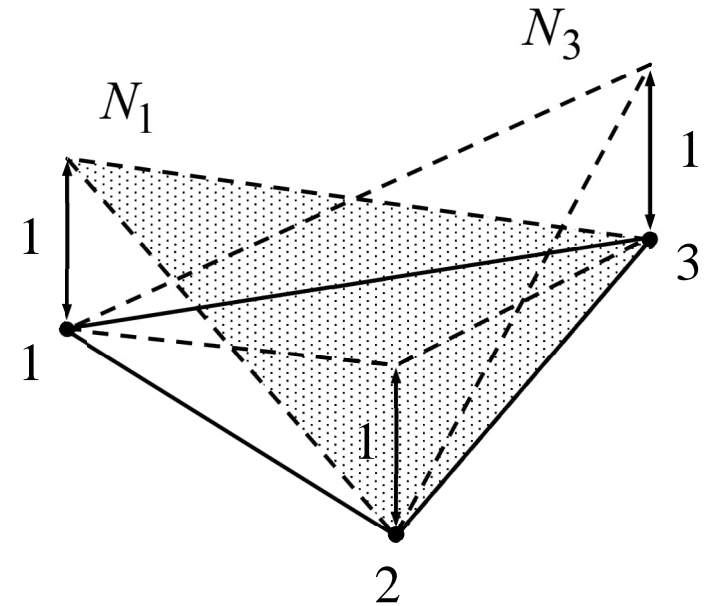$$

LINEAR SHAPE FUNCTIONS

Piecewise linear approximation in two-dimension is continuous in Ω and linear inside the elements of triangle shape. In element Ω^e

Approximation: $u = \mathbf{N}^{\mathrm{T}} \mathbf{a}$

Nodal values: $\mathbf{a} = \{u_1 \quad u_2 \quad u_3\}^{\mathrm{T}}$

Shape functions: $\mathbf{N} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}$

Triangle element is the simplest element in two dimensions. Division of any 2D domain into triangles is always possible, which makes the element quite useful.

- Let $\mathbf{N} = \{N_1 \ N_2 \ N_3\}^{\mathrm{T}}$ be the shape functions taking the value one at the vertices $\mathbf{x}_1 = (x_1, y_1)$, $\mathbf{x}_2 = (x_2, y_2)$ and $\mathbf{x}_3 = (x_3, y_3)$, respectively. Then

$$
\mathbf{N} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} = \begin{Bmatrix} \dfrac{x_2 y - x_3 y - x y_2 + x_3 y_2 + x y_3 - x_2 y_3}{x_2 y_1 - x_3 y_1 - x_1 y_2 + x_3 y_2 + x_1 y_3 - x_2 y_3} \\[2mm] \dfrac{-x_1 y + x_3 y + x y_1 - x_3 y_1 - x y_3 + x_1 y_3}{x_2 y_1 - x_3 y_1 - x_1 y_2 + x_3 y_2 + x_1 y_3 - x_2 y_3} \\[2mm] \dfrac{x_1 y - x_2 y - x y_1 + x_2 y_1 + x y_2 - x_1 y_2}{x_2 y_1 - x_3 y_1 - x_1 y_2 + x_3 y_2 + x_1 y_3 - x_2 y_3} \end{Bmatrix}. \ \leftarrow
$$

- Some integrals needed, e.g., in the virtual work expression of the thin slab model, are

$$
\int_{\Omega^e} \mathbf{N} \, d\Omega = \frac{A}{3} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}, \quad \int_{\Omega^e} \mathbf{N} \mathbf{N}^{\mathrm{T}} d\Omega = \frac{A}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.
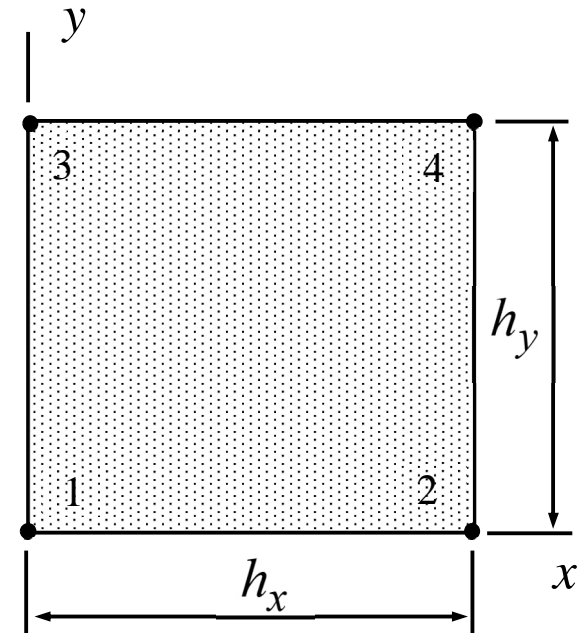$$

BI-LINEAR SHAPE FUNCTIONS

Bilinear approximation in two dimensions is continuous on Ω and linear with respect to both coordinates inside the elements of rectangular shape. In element Ω^e and notation $\xi = x / h_x$, $\eta = y / h_y$

Approximation: $u = \mathbf{N}^{\mathrm{T}} \mathbf{a}$

Nodal values: $\mathbf{a} = \{u_1 \quad u_2 \quad u_3 \quad u_4\}^{\mathrm{T}}$

Shape functions: $\mathbf{N} = \begin{Bmatrix} (1-\xi)(1-\eta) \\ \xi(1-\eta) \\ (1-\xi)\eta \\ \xi\eta \end{Bmatrix}$
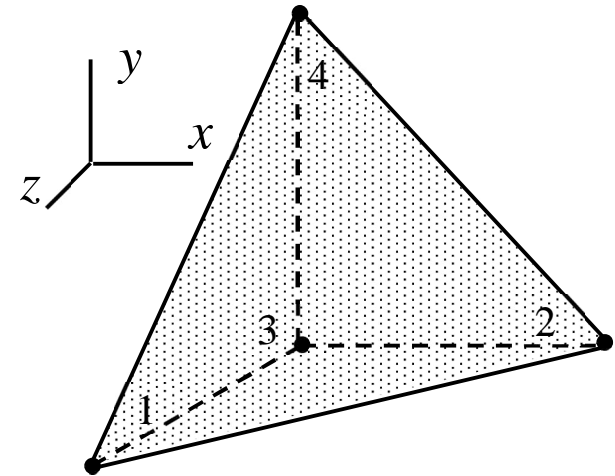
The ordering of the node numbers varies in literature.

LINEAR SHAPE FUNCTIONS

Piecewise linear approximation in three dimensions is continuous in Ω and a linear polynomial inside the tetrahedron elements. In a typical element Ω^e

Shape functions: $\quad \mathbf{N} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \\ y \\ z \end{Bmatrix}$
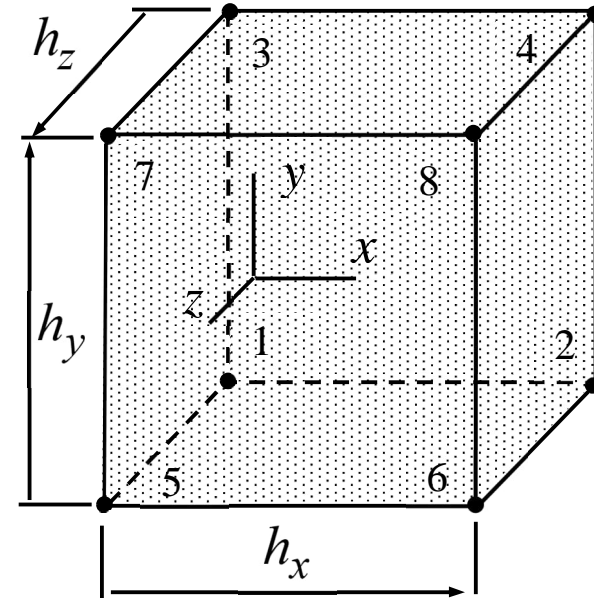
Tetrahedron is the simplest element in three dimensions. Division of any 3D domain into tetrahedrons is always possible, which makes also this element quite useful in practice.
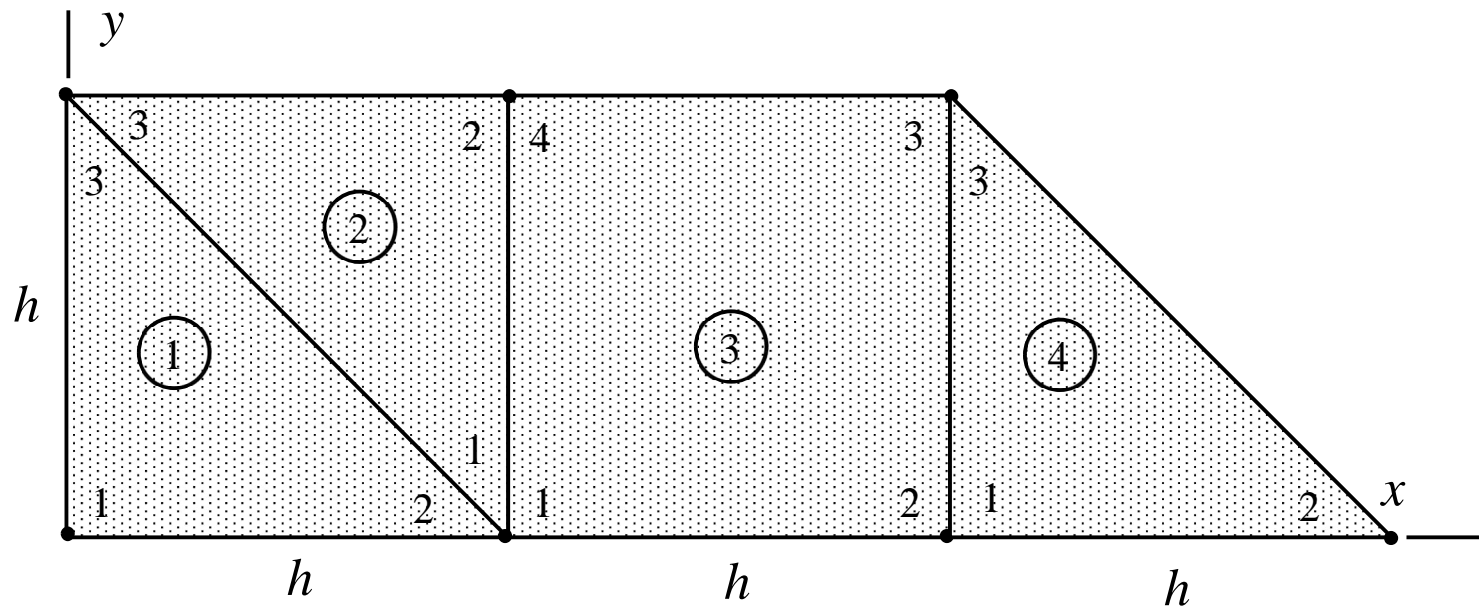
TRI-LINEAR SHAPE FUNCTIONS FOR A BODY

Approximation is continuous on Ω and tri-linear inside an element. In a typical element and with notations $\xi = (x - x_1)/h_x$, $\eta = (y - y_1)/h_y$, and $\zeta = (z - z_1)/h_z$,

$$\mathbf{N} = \begin{Bmatrix} (1-\xi)(1-\eta)(1-\zeta) \\ \xi(1-\eta)(1-\zeta) \\ (1-\xi)\eta(1-\zeta) \\ \xi\eta(1-\zeta) \\ (1-\xi)(1-\eta)\zeta \\ \xi(1-\eta)\zeta \\ (1-\xi)\eta\zeta \\ \xi\eta\zeta \end{Bmatrix}$$

Bi-linear and tri-linear shape functions of 2D and 3D cases are products of the linear shape functions of the 1D case.

EXAMPLE Consider the structure of the figure consisting of triangle and quadrilateral elements. Write down the shape functions of the elements in the $xy-$coordinates (the sum of the shape functions of an element is always 1).

Answer $N^1 = \dfrac{1}{h}\left\{\begin{array}{c} h - x - y \\ x \\ y \end{array}\right\}$, $N^2 = \dfrac{1}{h}\left\{\begin{array}{c} h - y \\ x + y - h \\ h - x \end{array}\right\}$, $N^3 = \dfrac{1}{h^2}\left\{\begin{array}{c} (2h - x)(h - y) \\ (x - h)(h - y) \\ (x - h)y \\ (2h - x)y \end{array}\right\}$, $N^4 = ?$