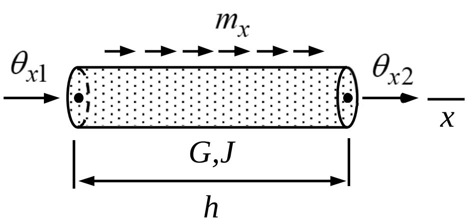
Name________________________ Student number________________

Assignment 1

Derive the virtual work expression $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}}$ of the bar element (length h) if the approximation is linear (a two-node element) and J, G and m_x are constants.



Solution template

Virtual work densities of the internal and external forces of the torsion bar model are

$$\delta w_\Omega^{\text{int}} = -\frac{d\delta\phi}{dx} GJ \frac{d\phi}{dx} \quad \text{and} \quad \delta w_\Omega^{\text{ext}} = \delta\phi m_x ,$$

in which J is the second moment of area with respect to x-axis, G is the shear modulus, and m_x is the external moment per unit length.

Let us start with the linear approximation to the rotation angle. The origin of the material coordinate system is at node 1 and the length of the bar is h.

$$\phi = \frac{1}{h}\begin{Bmatrix} h-x \\ x \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix}, \quad \frac{d\phi}{dx} = \frac{1}{h}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix},$$

$$\delta\phi = \frac{1}{h}\begin{Bmatrix} h-x \\ x \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}, \quad \frac{d\delta\phi}{dx} = \frac{1}{h}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}.$$

When the approximation is substituted there, virtual works of internal and external forces per unit take the forms

$$\delta w_\Omega^{\text{int}} = -\frac{d\delta\phi}{dx} GJ \frac{d\phi}{dx} = -\begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{GJ}{h^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix},$$

$$\delta w_\Omega^{\text{ext}} = \delta\phi m_x = \begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{m_x}{h}\begin{Bmatrix} h-x \\ x \end{Bmatrix}.$$

Virtual work expressions are integrals of the densities over the length. Then, virtual work expressions of the bar element are given by

$$\delta W^{\text{int}} = \int_0^h \delta w_\Omega^{\text{int}} dx = -\begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{GJ}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix}, \quad \leftarrow$$

$$\delta W^{\text{ext}} = \int_0^h \delta w_\Omega^{\text{ext}} dx = \begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{m_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$