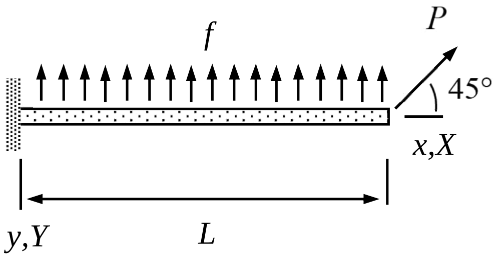
Name________________________ Student number________________

Assignment 3

Consider a cantilever in *xy*-plane loaded by distributed force f and point force $P=\sqrt{2}fL$. Determine the displacement and rotation of the free end. Young's modulus E and Poisson's ratio ν are constants. Cross-section is a rectangle of side length *t*. Assume that the neutral axis coincides with the *x*-axis of the material coordinate system. Use the bar and bending modes of the beam model.



Solution template

Assuming that the material coordinate system is chosen so that the bending and stretching modes decouple, the two modes can be taken into account as if they were separate elements. Therefore, one may use the virtual work expressions for the beam *xy*-plane bending and bar modes of the formulae collection

$$\delta W^{\text{int}} = -\begin{Bmatrix}\delta u_{y1}\\ \delta\theta_{z1}\\ \delta u_{y2}\\ \delta\theta_{z2}\end{Bmatrix}^{\mathrm{T}} \frac{EI_{zz}}{h^3}\begin{bmatrix}12 & 6h & -12 & 6h\\ 6h & 4h^2 & -6h & 2h^2\\ -12 & -6h & 12 & -6h\\ 6h & 2h^2 & -6h & 4h^2\end{bmatrix}\begin{Bmatrix}u_{y1}\\ \theta_{z1}\\ u_{y2}\\ \theta_{z2}\end{Bmatrix},\quad \delta W^{\text{ext}} = \begin{Bmatrix}\delta u_{y1}\\ \delta\theta_{z1}\\ \delta u_{y2}\\ \delta\theta_{z2}\end{Bmatrix}^{\mathrm{T}} \frac{f_y h}{12}\begin{Bmatrix}6\\ h\\ 6\\ -h\end{Bmatrix},$$

$$\delta W^{\text{int}} = -\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}} \frac{EA}{h}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix},\quad \delta W^{\text{ext}} = \begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}} \frac{f_x h}{2}\begin{Bmatrix}1\\ 1\end{Bmatrix}.$$

The nodal displacements and rotations of the material coordinate systems need to be expressed in terms of those of the structural coordinate system. By using the figure

$u_{x1}=0$, $\quad u_{y1}=0$, $\quad \theta_{z1}=0$,

$u_{x2}=u_{X2}$, $\quad u_{y2}=u_{Y2}$, $\quad \theta_{z2}=\theta_{Z2}$.

The cross-section properties and the distributed force (per unit length) components in the material coordinate system are

$$A=t^2,\quad I_{zz}=\frac{1}{12}t^4,\quad f_x=0,\quad f_y=-f\,.$$

When the relationships are used in the element contribution of the beam bending mode the generic expressions simplify to

$$\delta W^1 = -\begin{Bmatrix} \delta u_{Y2} \\ \delta\theta_{Z2} \end{Bmatrix}^{\mathrm{T}} \left(\begin{bmatrix} \frac{Et^4}{L^3} & -\frac{Et^4}{2L^2} \\ -\frac{Et^4}{2L^2} & \frac{Et^4}{3L} \end{bmatrix} \begin{Bmatrix} u_{Y2} \\ \theta_{Z2} \end{Bmatrix} - \begin{Bmatrix} -\frac{fL}{2} \\ \frac{fL^2}{12} \end{Bmatrix} \right).$$

The bar mode expression takes the form

$$\delta W^2 = -\delta u_{X2} \frac{Et^2}{L} u_{X2}.$$

Virtual work expression of the point force

$$\delta W^3 = \begin{Bmatrix} \delta u_{X2} \\ \delta u_{Y2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} fL \\ -fL \end{Bmatrix}.$$

Virtual work expression is the sum of the element/mode contributions

$$\delta W = -\begin{Bmatrix} \delta u_{X2} \\ \delta u_{Y2} \\ \delta\theta_{Z2} \end{Bmatrix}^{\mathrm{T}} \left(\begin{bmatrix} \frac{Et^2}{L} & 0 & 0 \\ 0 & \frac{Et^4}{L^3} & -\frac{Et^4}{2L^2} \\ 0 & -\frac{Et^4}{2L^2} & \frac{Et^4}{3L} \end{bmatrix} \begin{Bmatrix} u_{X2} \\ u_{Y2} \\ \theta_{Z2} \end{Bmatrix} - \begin{Bmatrix} fL \\ -\frac{3}{2} fL \\ \frac{1}{12} fL^2 \end{Bmatrix} \right).$$

Principle of virtual work and the fundamental lemma of variational calculus imply the linear equations system

$$\begin{bmatrix} \frac{Et^2}{L} & 0 & 0 \\ 0 & \frac{Et^4}{L^3} & -\frac{Et^4}{2L^2} \\ 0 & -\frac{Et^4}{2L^2} & \frac{Et^4}{3L} \end{bmatrix} \begin{Bmatrix} u_{X2} \\ u_{Y2} \\ \theta_{Z2} \end{Bmatrix} - \begin{Bmatrix} fL \\ -\frac{3}{2} fL \\ \frac{1}{12} fL^2 \end{Bmatrix} = 0.$$

The first equation is not connected to the second and third. Therefore, the solution can be found without inverting the 3-by-3 matrix (the bending mode equations are connected so a 2-by-2 matrix needs to be inverted)

$$u_{X2} = \frac{f}{E}\left(\frac{L}{t}\right)^2,$$

$$\begin{Bmatrix} u_{Y2} \\ \theta_{Z2} \end{Bmatrix} = \begin{bmatrix} \dfrac{Et^4}{L^3} & -\dfrac{Et^4}{2L^2} \\ -\dfrac{Et^4}{2L^2} & \dfrac{Et^4}{3L} \end{bmatrix}^{-1} \begin{Bmatrix} -\dfrac{3}{2} f L \\ \dfrac{1}{12} f L^2 \end{Bmatrix} = \begin{bmatrix} 4\dfrac{L^3}{t^4 E} & 6\dfrac{L^2}{t^4 E} \\ 6\dfrac{L^2}{t^4 E} & 12\dfrac{L}{t^4 E} \end{bmatrix} \begin{Bmatrix} -\dfrac{3}{2} f L \\ \dfrac{1}{12} f L^2 \end{Bmatrix} = \begin{Bmatrix} -\dfrac{11}{2}\dfrac{f}{E}(\dfrac{L}{t})^4 \\ -8\dfrac{f}{Et}(\dfrac{L}{t})^3 \end{Bmatrix}.$$