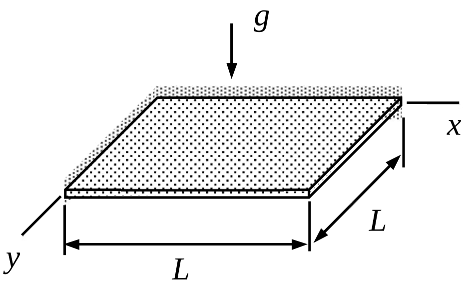
Name_________________________ Student number_________________

Assignment 5

A Kirchhoff plate of thickness t is loaded by its own weight. The plate is clamped on the edges where $x=0$ or $y=0$ and free on the other two edges. Material parameters E, $\nu=0$, and ρ are constants. Determine the parameter a_0 of the approximation $w(x,y)=a_0x^2y^2$. Use the principle of virtual work in form $\delta W=0\quad \forall \delta a_0 \in \mathbb{R}$.



Solution

Virtual work densities of the plate bending mode are given by

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial^2 \delta w / \partial x^2 \\ \partial^2 \delta w / \partial y^2 \\ 2\partial^2 \delta w / (\partial x \partial y) \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12}[E]_\sigma \begin{Bmatrix} \partial^2 w / \partial x^2 \\ \partial^2 w / \partial y^2 \\ 2\partial^2 w / (\partial x \partial y) \end{Bmatrix} \text{ and } \delta w^{\text{ext}} = \delta w f_z ,$$

in which f_z is the $z-$component of the distributed force per unit area and the elasticity matrix of plane stress

$$[E]_\sigma = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}.$$

Approximation to the transverse displacement and its derivatives are

$$w(x,y) = a_0 x^2 y^2 \quad \Rightarrow \quad \frac{\partial^2 w}{\partial x^2} = 2a_0 y^2 , \quad \frac{\partial^2 w}{\partial y^2} = 2a_0 x^2 , \quad \text{and} \quad \frac{\partial^2 w}{\partial x \partial y} = 4a_0 xy .$$

When the approximation is substituted there, virtual work densities simplify to

$$\delta w_\Omega^{\text{int}} = -2\delta a_0 \begin{Bmatrix} y^2 \\ x^2 \\ 4xy \end{Bmatrix}^{\mathrm{T}} \frac{t^3 E}{12} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{Bmatrix} y^2 \\ x^2 \\ 4xy \end{Bmatrix} 2a_0 = -\delta a_0 \frac{t^3 E}{3}(y^4 + x^4 + 8x^2y^2)a_0 ,$$

$$\delta w_\Omega^{\text{ext}} = \delta w f_z = \delta a_0 x^2 y^2 \rho g t .$$

Virtual work expressions are integrals of the virtual work densities over the mathematical solution domain (here mid-plane)

$$\delta W^{\text{int}} = \int_0^L \int_0^L \delta w_\Omega^{\text{int}} dxdy = -\delta a_0 \frac{t^3 E}{3} L^6 \frac{58}{45} a_0 \,,$$

$$\delta W^{\text{ext}} = \int_0^L \int_0^L \delta w_\Omega^{\text{ext}} dxdy = \delta a_0 \frac{1}{9} L^6 \rho g t \,.$$

Principle of virtual work $\delta W = 0 \ \forall \delta \mathrm{a}$ and the fundamental lemma of variation calculus give

$$\delta W = -\delta a_0 \left(\frac{t^3 E}{3} L^6 \frac{58}{45} a_0 - \frac{1}{9} L^6 \rho g t\right) = 0 \quad \forall \delta a_0 \quad \Leftrightarrow \quad \frac{t^3 E}{3} L^6 \frac{58}{45} a_0 - \frac{1}{9} L^6 \rho g t = 0 \quad \Leftrightarrow$$

$$a_0 = \frac{15}{58} \frac{\rho g}{t^2 E} \,.$$