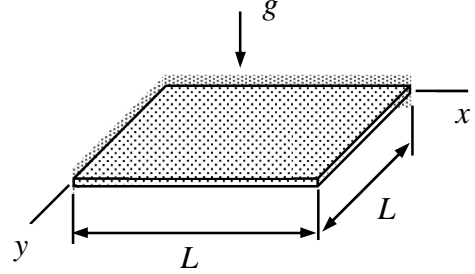


Name _____ Student number _____

Assignment 5

A Kirchhoff plate of thickness t is loaded by its own weight. The plate is clamped on the edges where $x=0$ or $y=0$ and free on the other two edges. Material parameters E , $\nu=0$, and ρ are constants. Determine the parameter a_0 of the approximation $w(x, y) = a_0 x^2 y^2$. Use the principle of virtual work in form $\delta W = 0 \quad \forall \delta a_0 \in \mathbb{R}$.



Solution

Virtual work densities of the plate bending mode are given by

$$\delta w_{\Omega}^{\text{int}} = - \left\{ \begin{array}{c} \partial^2 \delta w / \partial x^2 \\ \partial^2 \delta w / \partial y^2 \\ 2 \partial^2 \delta w / (\partial x \partial y) \end{array} \right\}^T \frac{t^3}{12} [E]_{\sigma} \left\{ \begin{array}{c} \partial^2 w / \partial x^2 \\ \partial^2 w / \partial y^2 \\ 2 \partial^2 w / (\partial x \partial y) \end{array} \right\} \quad \text{and} \quad \delta w^{\text{ext}} = \delta w f_z,$$

in which f_z is the z -component of the distributed force per unit area and the elasticity matrix of plane stress

$$[E]_{\sigma} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}.$$

Approximation to the transverse displacement and its derivatives are

$$w(x, y) = a_0 x^2 y^2 \quad \Rightarrow \quad \frac{\partial^2 w}{\partial x^2} = 2a_0 y^2, \quad \frac{\partial^2 w}{\partial y^2} = 2a_0 x^2, \quad \text{and} \quad \frac{\partial^2 w}{\partial x \partial y} = 4a_0 xy.$$

When the approximation is substituted there, virtual work densities simplify to

$$\delta w_{\Omega}^{\text{int}} = -2\delta a_0 \left\{ \begin{array}{c} y^2 \\ x^2 \\ 4xy \end{array} \right\}^T \frac{t^3 E}{12} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \left\{ \begin{array}{c} y^2 \\ x^2 \\ 4xy \end{array} \right\} 2a_0 = -\delta a_0 \frac{t^3 E}{3} (y^4 + x^4 + 8x^2 y^2) a_0,$$

$$\delta w_{\Omega}^{\text{ext}} = \delta w f_z = \delta a_0 x^2 y^2 \rho g t.$$

Virtual work expressions are integrals of the virtual work densities over the mathematical solution domain (here mid-plane)

$$\delta W^{\text{int}} = \int_0^L \int_0^L \delta w_{\Omega}^{\text{int}} dx dy = -\delta a_0 \frac{t^3 E}{3} L^6 \frac{58}{45} a_0 ,$$

$$\delta W^{\text{ext}} = \int_0^L \int_0^L \delta w_{\Omega}^{\text{ext}} dx dy = \delta a_0 \frac{1}{9} L^6 \rho g t .$$

Principle of virtual work $\delta W = 0 \forall \delta a$ and the fundamental lemma of variation calculus give

$$\delta W = -\delta a_0 \left(\frac{t^3 E}{3} L^6 \frac{58}{45} a_0 - \frac{1}{9} L^6 \rho g t \right) = 0 \quad \forall \delta a_0 \quad \Leftrightarrow \quad \frac{t^3 E}{3} L^6 \frac{58}{45} a_0 - \frac{1}{9} L^6 \rho g t = 0 \quad \Leftrightarrow$$

$$a_0 = \frac{15}{58} \frac{\rho g}{t^2 E} . \quad \leftarrow$$