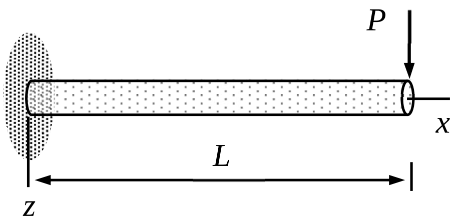
Name________________________ Student number________________

Assignment 3

Find the transverse displacement w and rotation θ of the xz – plane cantilever beam shown. Start with the Bernoulli beam model planar bending equations in their first order forms



$$\begin{Bmatrix} \dfrac{dQ}{dx} \\ \dfrac{dM}{dx} - Q \end{Bmatrix} = 0 , \quad \begin{Bmatrix} M \\ 0 \end{Bmatrix} = \begin{Bmatrix} EI\dfrac{d\theta}{dx} \\ \dfrac{dw}{dx} + \theta \end{Bmatrix} ,$$

where E and I are constants and Q, M are the shear force and bending moment, respectively,

Solution

There are two different ways to find the solution to rotation θ and the transverse displacement w. The first one is based on elimination of the variables from the first order equations above to get first an equation for the transverse displacement only. The second is based on integration of the first order equations directly.

Let us start with the elimination of the shear force, bending moment, and rotation. Considering also the conditions at the endpoints of the beam, the outcome is the boundary value problem: find $w(x)$ such that

$$-EI\frac{d^4w}{dx^4} = 0 \quad x \in (0,L) \ , \quad w = -\frac{dw}{dx} = 0 \quad x = 0 , \quad EI\frac{d^2w}{dx^2} = 0 \quad x = L , \text{ and } -EI\frac{d^3w}{dx^3} = P \quad x = L .$$

The fourth order beam equation is the usual form of textboooks. The generic solution to the differential equation

$$w(x) = ax^3 + bx^2 + cx + d$$

contains 4 parameters. When subtituted in the boundary conditions

$$d = -c = 0 , \ EI(6aL + 2b) = 0 , \text{ and } -EI(6a) = P \quad \Leftrightarrow \quad a = -\frac{P}{6EI} , \quad b = \frac{PL}{2EI} , \text{ and } d = c = 0 .$$

The final expressions for the transverse displacement and rotation take the forms

$$w(x) = \frac{P}{6EI}(3Lx^2 - x^3) \quad \text{and} \quad \theta(x) = -\frac{dw}{dx} = \frac{P}{2EI}(x^2 - 2Lx) . \ \leftarrow$$

The second method uses the first order equations one-by-one in certain order. Let us start with the equilibrium equations with boundary conditions at the free end

$$\frac{dQ}{dx} = 0 \quad x \in (0, L) \text{ and } Q = P \quad x = L \quad \Rightarrow \quad Q(x) = P$$

$$\frac{dM}{dx} = Q = P \quad x \in (0, L) \text{ and } M = 0 \quad x = L \quad \Rightarrow \quad M(x) = P(x - L).$$

Knowing the force resultants, constitutive equation and the Bernoulli constraint can be integrated for the rotation and the transverse displacement

$$\frac{d\theta}{dx} = \frac{M}{EI} = \frac{P}{EI}(x - L) \quad x \in (0, L) \text{ and } \theta = 0 \quad x = 0 \quad \Rightarrow \quad \theta(x) = \frac{P}{2EI}(x^2 - 2xL),$$

$$\frac{dw}{dx} = -\theta = -\frac{P}{EI}(\frac{1}{2}x^2 - xL) \quad x \in (0, L) \text{ and } w = 0 \quad x = 0 \quad \Rightarrow \quad w(x) = -\frac{P}{6EI}(x^3 - 3x^2 L).$$

The latter method is usually more straightforward but requires integration of the equations in the "right order".