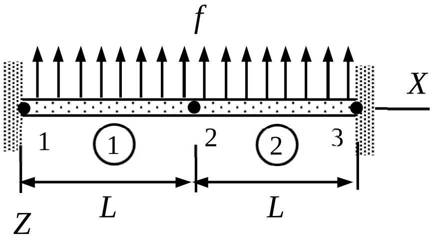
Name________________________ Student number________________

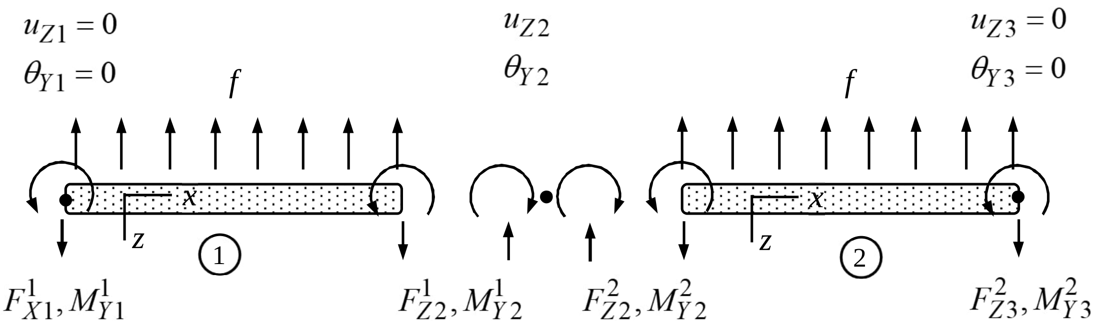
Assignment 5

External load f acting on the beam shown is constant. Determine displacement u_{Z2} and rotation θ_{Y2} of the mid-point (node 2). Young's modulus of the material and the second moments of area are E and I, respectively. Use two beam elements of equal length.



Solution

Only the displacement in the $Z-$direction and rotation in the $Y-$direction matter in the planar beam bending problem. From the figure, the non-zero displacement/rotation components are u_{Z2} and θ_{Y2}. Free body diagrams of the two bending beam elements and node 2 are (nodes 1 and 3 are constrained)



Element contributions of the two $xz-$plane bending beams (formulae collection) and the equilibrium equations of node 2 are (notice that the distributed force in the element contribution is the transverse component in the material system associated with beam)

Beam 1: $$\begin{Bmatrix} F_{Z1}^1 \\ M_{Y1}^1 \\ F_{Z2}^1 \\ M_{Y2}^1 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ u_{Z2} \\ \theta_{Y2} \end{Bmatrix} + \frac{fL}{12}\begin{Bmatrix} 6 \\ -L \\ 6 \\ L \end{Bmatrix}, \quad (f_z = -f)$$

Beam 2: $$\begin{Bmatrix} F_{Z2}^2 \\ M_{Y2}^2 \\ F_{Z3}^2 \\ M_{Y3}^2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \\ 0 \\ 0 \end{Bmatrix} + \frac{fL}{12}\begin{Bmatrix} 6 \\ -L \\ 6 \\ L \end{Bmatrix}, \quad (f_z = -f)$$

Node 2: $-F_{Z2}^1 - F_{Z2}^2 = 0$ and $-M_{Y2}^1 - M_{Y2}^2 = 0$.

Elimination of the internal forces from the two equilibrium equations of node 2 using the element contributions gives the forms

Node 2: $$-[\frac{EI}{L^3}(12u_{Z2}+6L\theta_{Y2})+6\frac{fL}{12}]-[\frac{EI}{L^3}(12u_{Z2}-6L\theta_{Y2})+6\frac{fL}{12}]=0 \quad \text{and}$$

$$-[\frac{EI}{L^3}(6Lu_{Z2}+4L^2\theta_{Y2})+L\frac{fL}{12}]-[\frac{EI}{L^3}(-6Lu_{Z2}+4L^2\theta_{Y2})-L\frac{fL}{12}]=0.$$

Matrix representation of the two equilibrium equations, containing u_{Z2} and θ_{Y2} as the unknowns, is

$$\frac{EI}{L^3}\begin{bmatrix}24 & 0\\ 0 & 8L^2\end{bmatrix}\begin{Bmatrix}u_{Z2}\\ \theta_{Y2}\end{Bmatrix}+fL\begin{Bmatrix}1\\ 0\end{Bmatrix}=0 \quad \Leftrightarrow$$

$$\begin{Bmatrix}u_{Z2}\\ \theta_{Y2}\end{Bmatrix}=-\frac{fL^4}{EI}\begin{bmatrix}24 & 0\\ 0 & 8L^2\end{bmatrix}^{-1}\begin{Bmatrix}1\\ 0\end{Bmatrix}=-\frac{fL^4}{EI}\begin{bmatrix}1/24 & 0\\ 0 & 1/(8L^2)\end{bmatrix}\begin{Bmatrix}1\\ 0\end{Bmatrix}=-\frac{1}{24}\frac{fL^4}{EI}\begin{Bmatrix}1\\ 0\end{Bmatrix} \quad \Leftrightarrow$$

$$u_{Z2}=-\frac{1}{24}\frac{fL^4}{EI} \quad \text{and} \quad \theta_{Y2}=0.$$