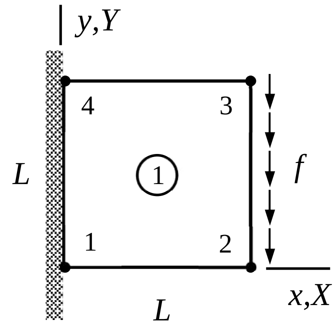
Name________________________ Student number________________

Assignment 4



A thin slab is loaded by distributed force on its outer edge as shown in the figure. Determine the vertical displacement of the outer edge 2-3 by using a bi-linear interpolation to the nodal values. Edge 1-4 is welded to a rigid wall so that the displacements vanish. Thickness of the slab *t*, Young's modulus *E*, and Poisson's ratio ν are constants. Assume plane stress conditions. Simplify the setting with conditions $u_{Y3} = u_{Y2}$ and $u_{X3} = u_{X2} = 0$.

Solution template

Under the plane stress conditions, the virtual work densities (virtual works per unit area) of the thin slab model are given by

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta u / \partial x \\ \partial\delta v / \partial y \\ \partial\delta u / \partial y + \partial\delta v / \partial x \end{Bmatrix}^{\mathrm{T}} t[E]_{\sigma} \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix} \text{ and } \delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix} \text{ where}$$

$$[E]_{\sigma} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}.$$

Expressions take into account the internal forces (stress) and the external area forces acting on the element domain. The external forces t_x and t_y (tractions per unit length in this case) acting on the element edges can be taken into account by a separate force element with the density expression (per unit length)

$$\delta w_{\partial\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} t_x \\ t_y \end{Bmatrix}.$$

The approximation on the boundary is just the restriction of the element approximation to the boundary (corresponds to a linear two-node element).

Only the shape functions associated with nodes 2 and 3 are needed as the other nodes are fixed (displacement vanishes). By deducing the expression, i.e., combining the linear shape functions in the $x-$directions and $y-$directions

$$N_2 = (1 - \frac{y}{L})\frac{x}{L} \quad \text{and} \quad N_3 = \frac{y}{L}\frac{x}{L}.$$

In terms of the vertical displacement component u_{Y2} of node 2, approximations to the displacement components and their derivatives are

$$u(x,y)=\mathbf{N}^{\mathrm{T}}\mathbf{a}=0 \;\Rightarrow\; \frac{\partial u}{\partial x}=0 \quad\text{and}\quad \frac{\partial u}{\partial y}=0\,,$$

$$v(x,y)=\mathbf{N}^{\mathrm{T}}\mathbf{a}=u_{Y2}\frac{x}{L} \;\Rightarrow\; \frac{\partial v}{\partial x}=u_{Y2}\frac{1}{L} \quad\text{and}\quad \frac{\partial v}{\partial y}=0\,.$$

Virtual work density of the internal forces simplifies to (when the approximations are substituted there)

$$\delta w_{\Omega}^{\text{int}}=-\begin{Bmatrix}0\\0\\\delta u_{Y2}/L\end{Bmatrix}^{\mathrm{T}}\frac{Et}{1-\nu^2}\begin{bmatrix}1&\nu&0\\\nu&1&0\\0&0&(1-\nu)/2\end{bmatrix}\begin{Bmatrix}0\\0\\u_{Y2}/L\end{Bmatrix}=-\delta u_{Y2}\frac{1}{L^2}\frac{Et}{2(1+\nu)}u_{Y2}\,.$$

Virtual work density is constant in this case. Integration over the element gives the virtual work expression of internal forces

$$\delta W^{\text{int}}=\int_0^L\int_0^L \delta w_{\Omega}^{\text{int}}dxdy=-\delta u_{Y2}\frac{Et}{2(1+\nu)}u_{Y2}\,.$$

Virtual work expression of external distributed force components $t_x=0$ and $t_y=-f$ is obtained as an integral over the edge defined by $x=L$. The restriction of approximation to $x=L$ is given by

$$u(L,y)=0 \quad\text{and}\quad v(L,y)=u_{Y2}$$

so the virtual work density expression simplifies to

$$\delta w_{\partial\Omega}^{\text{ext}}=\begin{Bmatrix}\delta u\\\delta v\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}t_x\\t_y\end{Bmatrix}=-\delta u_{Y2}f$$

giving the virtual work expression

$$\delta W^{\text{ext}}=\int_0^L w_{\partial\Omega}^{\text{ext}}dy=-\delta u_{Y2}Lf\,.$$

Virtual work expression is the sum of internal and external parts

$$\delta W=\delta W^{\text{int}}+\delta W^{\text{ext}}=-\delta u_{Y2}\left(\frac{Et}{2(1+\nu)}u_{Y2}+Lf\right).$$

Principle of virtual work $\delta W=0\;\forall\delta\mathbf{a}$ and the fundamental lemma of variation calculus in the form $\delta\mathbf{a}^{\mathrm{T}}\mathbf{R}=0\;\forall\delta\mathbf{a} \;\Leftrightarrow\; \mathbf{R}=0$ give

$$\frac{Et}{2(1+\nu)} u_{Y2} + Lf = 0 \quad \Leftrightarrow \quad u_{Y2} = -\frac{Lf\,2(1+\nu)}{Et} = -\frac{Lf}{tG}.$$