MEC-E1050 Finite Element Method in Solids; Formulae

LINEAR ELASTICITY

Coordinate systems: $\begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = \begin{bmatrix} i_X & i_Y & i_Z \\ \hdashline j_X & j_Y & j_Z \\ \hdashline k_X & k_Y & k_Z \end{bmatrix} \begin{Bmatrix} \vec{I} \\ \vec{J} \\ \vec{K} \end{Bmatrix} = \{\mathbf{i} \quad \mathbf{j} \quad \mathbf{k}\}^{\mathrm{T}} \begin{Bmatrix} \vec{I} \\ \vec{J} \\ \vec{K} \end{Bmatrix}$

Strain-stress: $\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix}$, $\begin{Bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \frac{1}{G} \begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix}$, $G = \frac{E}{2(1+\nu)}$ or

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} \equiv [E] \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix}, \quad \begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = G \begin{Bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

$$[E]_\sigma = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}, \quad [E]_\varepsilon = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix}$$

Strain-displacement: $\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} = \begin{Bmatrix} \partial u_x / \partial x \\ \partial u_y / \partial y \\ \partial u_z / \partial z \end{Bmatrix}$, $\begin{Bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{Bmatrix} \partial u_x / \partial y + \partial u_y / \partial x \\ \partial u_y / \partial z + \partial u_z / \partial y \\ \partial u_z / \partial x + \partial u_x / \partial z \end{Bmatrix}$

ELEMENT CONTRIBUTION (constant load)

Bar (axial): $\begin{Bmatrix} F_{x1} \\ F_{x2} \end{Bmatrix} = \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

$\begin{Bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \end{Bmatrix} = \frac{EA}{h} \begin{bmatrix} \mathbf{i}\mathbf{i}^{\mathrm{T}} & -\mathbf{i}\mathbf{i}^{\mathrm{T}} \\ -\mathbf{i}\mathbf{i}^{\mathrm{T}} & \mathbf{i}\mathbf{i}^{\mathrm{T}} \end{bmatrix} \begin{Bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{Bmatrix} - \frac{f_x h}{2} \begin{Bmatrix} \mathbf{i} \\ \mathbf{i} \end{Bmatrix}$, in which $\mathbf{i} = \frac{1}{h} \begin{Bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{Bmatrix}$

Bar (torsion): $\begin{Bmatrix} M_{x1} \\ M_{x2} \end{Bmatrix} = \frac{GI_{rr}}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix} - \frac{m_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

Beam (*xz*): $\begin{Bmatrix} F_{z1} \\ M_{y1} \\ F_{z2} \\ M_{y2} \end{Bmatrix} = \frac{EI_{yy}}{h^3} \begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix} - \frac{f_z h}{12} \begin{Bmatrix} 6 \\ -h \\ 6 \\ h \end{Bmatrix}$

Point loads: $\begin{Bmatrix} F_{X1} \\ F_{Y1} \\ F_{Z1} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{Y1} \\ u_{Z1} \end{Bmatrix} - \begin{Bmatrix} F_X \\ F_Y \\ F_Z \end{Bmatrix}$, $\begin{Bmatrix} M_{X1} \\ M_{Y1} \\ M_{Z1} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \theta_{X1} \\ \theta_{Y1} \\ \theta_{Z1} \end{Bmatrix} - \begin{Bmatrix} M_X \\ M_Y \\ M_Z \end{Bmatrix}$

PRINCIPLE OF VIRTUAL WORK

$\delta W = \delta W^{\text{ext}} + \delta W^{\text{int}}$, $\delta W = \sum_{e \in E} \delta W^e = 0 \quad \forall \delta \mathbf{a}$, $\delta W = \int_\Omega \delta w d\Omega$

Bar: $\delta w^{\text{int}} = -\dfrac{d\delta u}{dx} EA \dfrac{du}{dx}$, $\delta w^{\text{ext}} = \delta u f_x$

Torsion: $\delta w^{\text{int}} = -\dfrac{d\delta\phi}{dx} GI_{rr} \dfrac{d\phi}{dx}$, $\delta w^{\text{ext}} = \delta\phi m_x$

Beam bending (*xz*-plane): $\delta w^{\text{int}} = -\dfrac{d^2\delta w}{dx^2} EI_{yy} \dfrac{d^2 w}{dx^2}$, $\delta w^{\text{ext}} = \delta w f_z$

Beam bending (*xy*-plane): $\delta w^{\text{int}} = -\dfrac{d^2\delta v}{dx^2} EI_{zz} \dfrac{d^2 v}{dx^2}$, $\delta w^{\text{ext}} = \delta v f_y$

Beam (Bernoulli):

$$\delta w^{\text{int}} = -\begin{Bmatrix} d\delta u / dx \\ d^2\delta v / dx^2 \\ d^2\delta w / dx^2 \end{Bmatrix}^{\mathrm{T}} E \begin{bmatrix} A & -S_z & -S_y \\ -S_z & I_{zz} & I_{zy} \\ -S_y & I_{yz} & I_{yy} \end{bmatrix} \begin{Bmatrix} du / dx \\ d^2 v / dx^2 \\ d^2 w / dx^2 \end{Bmatrix} - \frac{d\delta\phi}{dx} GI_{rr} \frac{d\phi}{dx},$$

$$\delta w^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} + \begin{Bmatrix} \delta\phi \\ -d\delta w / dx \\ d\delta v / dx \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} -S_y f_y + S_z f_z \\ S_y f_x \\ -S_z f_x \end{Bmatrix}$$

Thin slab (plane-stress):

$$\delta w^{\text{int}} = -\begin{Bmatrix} \partial\delta u / \partial x \\ \partial\delta v / \partial y \\ \partial\delta u / \partial y + \partial\delta v / \partial x \end{Bmatrix}^{\mathrm{T}} t[E]_\sigma \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix}, \quad \delta w^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix}$$

Thin slab (plane-strain):

$$\delta w^{\text{int}} = -\begin{Bmatrix} \partial\delta u / \partial x \\ \partial\delta v / \partial y \\ \partial\delta u / \partial y + \partial\delta v / \partial x \end{Bmatrix}^{\mathrm{T}} t[E]_\varepsilon \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix}, \quad \delta w^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix}$$

Kirchhoff plate:

$$\delta w^{\rm int} = -\begin{Bmatrix} \partial^2\delta w/\partial x^2 \\ \partial^2\delta w/\partial y^2 \\ 2\partial^2\delta w/(\partial x\partial y) \end{Bmatrix}^{\rm T} \frac{t^3}{12}[E]_\sigma \begin{Bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ 2\partial^2 w/(\partial x\partial y) \end{Bmatrix},\ \delta w^{\rm ext} = \delta w f_z$$

Reissner-Mindlin plate:

$$\delta w^{\rm int} = -\begin{Bmatrix} -\partial\delta\theta/\partial x \\ \partial\delta\phi/\partial y \\ \partial\delta\phi/\partial x - \partial\delta\theta/\partial y \end{Bmatrix}^{\rm T} \frac{t^3}{12}[E]_\sigma \begin{Bmatrix} -\partial\theta/\partial x \\ \partial\phi/\partial y \\ \partial\phi/\partial x - \partial\theta/\partial y \end{Bmatrix} - \begin{Bmatrix} \partial\delta w/\partial y - \delta\phi \\ \partial\delta w/\partial x + \delta\theta \end{Bmatrix}^{\rm T} tG \begin{Bmatrix} \partial w/\partial y - \phi \\ \partial w/\partial x + \theta \end{Bmatrix},$$

$$\delta w^{\rm ext} = \delta w f_z$$

Body: $\delta w^{\rm int} = -\begin{Bmatrix} \delta\varepsilon_{xx} \\ \delta\varepsilon_{yy} \\ \delta\varepsilon_{zz} \end{Bmatrix}^{\rm T} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} - \begin{Bmatrix} \delta\gamma_{xy} \\ \delta\gamma_{yz} \\ \delta\gamma_{zx} \end{Bmatrix}^{\rm T} \begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix},\ \delta w^{\rm ext} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\rm T} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix}$ or

$$\delta w^{\rm int} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta w/\partial z \end{Bmatrix}^{\rm T} [E] \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial w/\partial z \end{Bmatrix} - \begin{Bmatrix} \partial\delta u/\partial y + \partial\delta v/\partial x \\ \partial\delta v/\partial z + \partial\delta w/\partial y \\ \partial\delta w/\partial x + \partial\delta u/\partial z \end{Bmatrix}^{\rm T} G \begin{Bmatrix} \partial u/\partial y + \partial v/\partial x \\ \partial v/\partial z + \partial w/\partial y \\ \partial w/\partial x + \partial u/\partial z \end{Bmatrix}$$

APPROXIMATIONS (some) $u = \mathbf{N}^{\rm T}\mathbf{a}$, $\xi = \dfrac{x}{h}$

Quadratic line: $\mathbf{N} = \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} = \begin{Bmatrix} 1-3\xi+2\xi^2 \\ 4\xi(1-\xi) \\ \xi(2\xi-1) \end{Bmatrix},\ \mathbf{a} = \begin{Bmatrix} u_{x1} \\ u_{x2} \\ u_{x3} \end{Bmatrix}$ (bar)

Cubic line: $\mathbf{N} = \begin{Bmatrix} N_{10} \\ N_{11} \\ \hdashline N_{20} \\ N_{21} \end{Bmatrix} = \begin{Bmatrix} (1-\xi)^2(1+2\xi) \\ h(1-\xi)^2\xi \\ \hdashline (3-2\xi)\xi^2 \\ h\xi^2(\xi-1) \end{Bmatrix},\ \mathbf{a} = \begin{Bmatrix} u_{10} \\ u_{11} \\ \hdashline u_{20} \\ u_{21} \end{Bmatrix} (= \begin{Bmatrix} u_{z1} \\ -\theta_{y1} \\ \hdashline u_{z2} \\ -\theta_{y2} \end{Bmatrix})$ (beam bending)

Linear: $\mathbf{N} = \begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \end{Bmatrix},\ \mathbf{N} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix},\ \mathbf{N} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \\ y \\ z \end{Bmatrix}$

VIRTUAL WORK EXPRESSIONS

Force: $\delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{Xi} \\ \delta u_{Yi} \\ \delta u_{Zi} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} F_{Xi} \\ F_{Yi} \\ F_{Zi} \end{Bmatrix} + \begin{Bmatrix} \delta\theta_{Xi} \\ \delta\theta_{Yi} \\ \delta\theta_{Zi} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} M_{Xi} \\ M_{Yi} \\ M_{Zi} \end{Bmatrix}$

Bar: $\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix}$, $\delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{f_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

Torsion: $\delta W^{\text{int}} = -\begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{GI_{rr}}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix}$, $\delta W^{\text{ext}} = \begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{m_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$)

Beam bending (*xz*-plane):

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI_{yy}}{h^3} \begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix}, \quad \delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{f_z h}{12} \begin{Bmatrix} 6 \\ -h \\ 6 \\ h \end{Bmatrix}$$

Beam bending (*xy*-plane):

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{y1} \\ \delta\theta_{z1} \\ \delta u_{y2} \\ \delta\theta_{z2} \end{Bmatrix}^{\mathrm{T}} \frac{EI_{zz}}{h^3} \begin{bmatrix} 12 & 6h & -12 & 6h \\ 6h & 4h^2 & -6h & 2h^2 \\ -12 & -6h & 12 & -6h \\ 6h & 2h^2 & -6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \\ u_{y2} \\ \theta_{z2} \end{Bmatrix}, \quad \delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{y1} \\ \delta\theta_{z1} \\ \delta u_{y2} \\ \delta\theta_{z2} \end{Bmatrix}^{\mathrm{T}} \frac{f_y h}{12} \begin{Bmatrix} 6 \\ h \\ 6 \\ -h \end{Bmatrix})$$

CONSTRAINTS

Frictionless contact: $\vec{n} \cdot \vec{u}_{\mathrm{A}} = 0$

Joint: $\vec{u}_{\mathrm{B}} = \vec{u}_{\mathrm{A}}$

Rigid body (link): $\vec{u}_{\mathrm{B}} = \vec{u}_{\mathrm{A}} + \vec{\theta}_{\mathrm{A}} \times \vec{\rho}_{\mathrm{AB}}$, $\vec{\theta}_{\mathrm{B}} = \vec{\theta}_{\mathrm{A}}$.