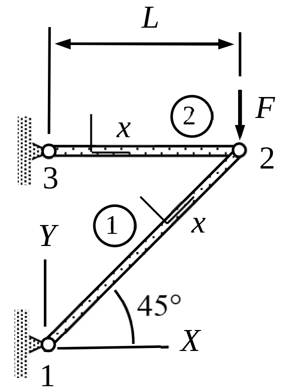
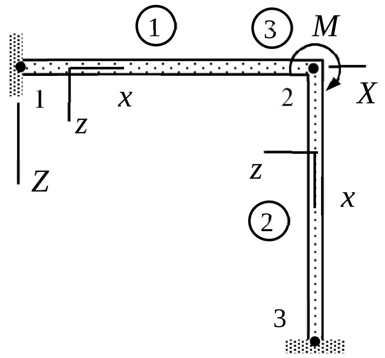
MEC-E1050 Finite Element Method in Solids, onsite exam 04.12.2023

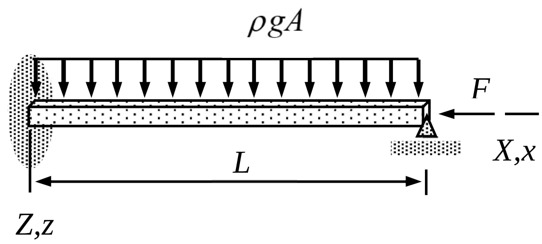
1. Determine the horizontal and vertical displacements of node 2. Cross-sectional area of bar 1 is $\sqrt{2}A$ and that of bar 2 is A. Young's modulus of the material is E.



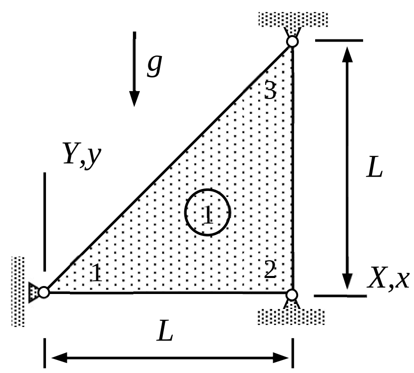
2. Determine the rotation θ_{Y2} at node 2 of the structure loaded by a point moment (magnitude M) acting on node 2. Use beam elements (1) and (2) of length L and a point moment element (3). Assume that the beams are inextensible in the axial directions. Young's modulus E and the second moment of area I are constants. Use the principle of virtual work.



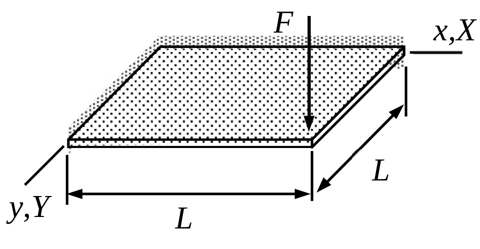
3. The beam of the figure is loaded by its own weight and a point force acting on the right end. Determine the displacement and rotation of the right end starting with the virtual density of the Bernoulli beam model. The x-axis of the material coordinate system is placed at the geometric centroid of the rectangle cross-section. Beam properties A, I_{yy}, I_{zz} and E of the planar problem are constants.



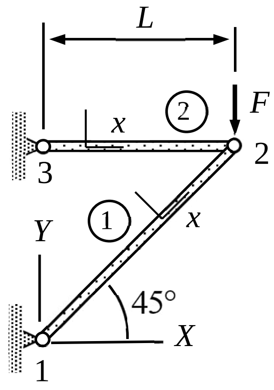
4. A thin triangular slab (assume plane stress conditions) loaded by its own weight is allowed to move vertically at node 1 and nodes 2 and 3 are fixed. Find the displacement u_{Y1}. Material parameters E, ν, ρ and thickness t of the slab are constants.



5. A plate, loaded by point force F acting at the free corner, is simply supported on two edges and free on the other two edges as shown in the figure. Determine the parameter a_0 of approximation $w(x,y) = a_0(x/L)(y/L)$ and displacement at the center point. Use the virtual work density of the plate bending mode with constant E, ν, ρ and t.



Determine the horizontal and vertical displacements of node 2. Cross-sectional area of bar 1 is $\sqrt{2}A$ and that of bar 2 is A. Young's modulus of the material is E.



Solution (option 1)

Element contribution, written in terms of displacement components of the structural coordinate system,

$$\begin{Bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \end{Bmatrix} = \frac{EA}{h}\begin{bmatrix} \mathbf{i}\mathbf{i}^{\mathrm{T}} & -\mathbf{i}\mathbf{i}^{\mathrm{T}} \\ -\mathbf{i}\mathbf{i}^{\mathrm{T}} & \mathbf{i}\mathbf{i}^{\mathrm{T}} \end{bmatrix}\begin{Bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{Bmatrix} - \frac{f_x h}{2}\begin{Bmatrix} \mathbf{i} \\ \mathbf{i} \end{Bmatrix}, \text{ where } \mathbf{i} = \frac{1}{h}\begin{Bmatrix} \Delta X \\ \Delta Y \end{Bmatrix}$$

depends on the cross-sectional area A, Young's modulus E, bar length h, force per unit length f_x and the components of the basis vector $\vec{i}$ in the structural coordinate system (the components define the orientation).

4p Element contributions are first written in terms of the nodal displacements of the structural coordinate system (notice that the point force is treated as a one-node element)

$$\text{Bar 1: } h = \sqrt{2}L\,,\ \mathbf{i} = \begin{Bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{Bmatrix},\quad \begin{Bmatrix} F_{X1}^1 \\ F_{Y1}^1 \\ F_{X2}^1 \\ F_{Y2}^1 \end{Bmatrix} = \frac{EA}{2L}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ u_{X2} \\ u_{Y2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix},$$

$$\text{Bar 2: } h = L\,,\ \mathbf{i} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix},\quad \begin{Bmatrix} F_{X3}^2 \\ F_{Y3}^2 \\ F_{X2}^2 \\ F_{Y2}^2 \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ u_{X2} \\ u_{Y2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix},$$

$$\text{Force 3: } \begin{Bmatrix} F_{X2}^3 \\ F_{Y2}^3 \end{Bmatrix} = -\begin{Bmatrix} 0 \\ -F \end{Bmatrix}.$$

2p In assembly, the internal forces acting on the non-constrained directions are added to get the equilibrium equations of the structure. The unknown displacement components follow as the solution to the equilibrium equations:

$$\sum \begin{Bmatrix} F_{X2}^e \\ F_{Y2}^e \end{Bmatrix} = \begin{Bmatrix} F_{X2}^1 \\ F_{Y2}^1 \end{Bmatrix} + \begin{Bmatrix} F_{X2}^2 \\ F_{Y2}^2 \end{Bmatrix} + \begin{Bmatrix} F_{X2}^3 \\ F_{Y2}^3 \end{Bmatrix} = \frac{EA}{2L}\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{Y2} \end{Bmatrix} + \begin{Bmatrix} 0 \\ F \end{Bmatrix} = 0 \quad \Leftrightarrow$$

$$u_{X2}=\frac{FL}{EA} \quad \text{and} \quad u_{Y2}=-3\frac{FL}{EA}.$$

Solution (option 2)

In the material coordinate system, virtual work expression of the bar model is given by

$$\delta W=-\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\left(\frac{EA}{h}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix}-\frac{f_x h}{2}\begin{Bmatrix}1\\ 1\end{Bmatrix}\right).$$

4p In terms of the nodal displacement components of the structural system, axial displacements of bar 1 are $u_{x1}=0$ and $u_{x2}=(u_{X2}+u_{Y2})/\sqrt{2}$. Length of the bar $h=\sqrt{2}L$, cross-sectional area is $\sqrt{2}A$, and the external distributed force $f_x=0$. Therefore

$$\delta W^1=-\begin{Bmatrix}0\\ (\delta u_{X2}+\delta u_{Y2})/\sqrt{2}\end{Bmatrix}^{\mathrm{T}}\frac{E\sqrt{2}A}{\sqrt{2}L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}0\\ (u_{X2}+u_{Y2})/\sqrt{2}\end{Bmatrix}=-\begin{Bmatrix}\delta u_{X2}\\ \delta u_{Y2}\end{Bmatrix}^{\mathrm{T}}\frac{EA}{2L}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{Bmatrix}u_{X2}\\ u_{Y2}\end{Bmatrix}$$

Axial displacements of bar 2 are $u_{x3}=0$ and $u_{x2}=u_{X2}$. Length of the bar $h=L$, cross-sectional area is A, and the external distributed force $f_x=0$. Hence

$$\delta W^2=-\begin{Bmatrix}0\\ \delta u_{X2}\end{Bmatrix}^{\mathrm{T}}\frac{EA}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}0\\ u_{X2}\end{Bmatrix}=-\begin{Bmatrix}\delta u_{X2}\\ \delta u_{Y2}\end{Bmatrix}^{\mathrm{T}}\frac{EA}{L}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{Bmatrix}u_{X2}\\ u_{Y2}\end{Bmatrix}.$$

Virtual work expression of the external force follows, for example, from the definition of work

$$\delta W^3=-\delta u_{Y2}F=-\begin{Bmatrix}\delta u_{X2}\\ \delta u_{Y2}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}0\\ F\end{Bmatrix}.$$

2p Virtual work expression of the structure is the sum of element contributions

$$\delta W=-\begin{Bmatrix}\delta u_{X2}\\ \delta u_{Y2}\end{Bmatrix}^{\mathrm{T}}\frac{EA}{2L}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{Bmatrix}u_{X2}\\ u_{Y2}\end{Bmatrix}-\begin{Bmatrix}\delta u_{X2}\\ \delta u_{Y2}\end{Bmatrix}^{\mathrm{T}}\frac{EA}{2L}\begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix}\begin{Bmatrix}u_{X2}\\ u_{Y2}\end{Bmatrix}-\begin{Bmatrix}\delta u_{X2}\\ \delta u_{Y2}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}0\\ F\end{Bmatrix}$$

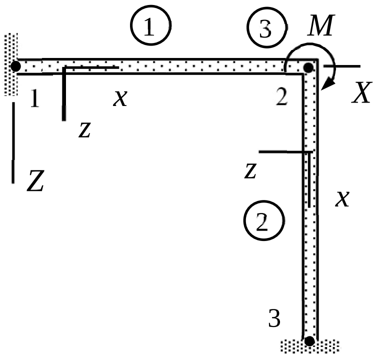
or when written in the standard form

$$\delta W=-\begin{Bmatrix}\delta u_{X2}\\ \delta u_{Y2}\end{Bmatrix}^{\mathrm{T}}\left(\frac{EA}{2L}\begin{bmatrix}3 & 1\\ 1 & 1\end{bmatrix}\begin{Bmatrix}u_{X2}\\ u_{Y2}\end{Bmatrix}+\begin{Bmatrix}0\\ F\end{Bmatrix}\right).$$

Principle of virtual work and the fundamental lemma of variation calculus imply

$$\frac{EA}{2L}\begin{bmatrix}3 & 1\\ 1 & 1\end{bmatrix}\begin{Bmatrix}u_{X2}\\ u_{Y2}\end{Bmatrix}+\begin{Bmatrix}0\\ F\end{Bmatrix}=0 \quad\Leftrightarrow\quad \begin{Bmatrix}u_{X2}\\ u_{Y2}\end{Bmatrix}=-2\frac{FL}{EA}\frac{1}{2}\begin{bmatrix}1 & -1\\ -1 & 3\end{bmatrix}\begin{Bmatrix}0\\ 1\end{Bmatrix}=-\frac{FL}{EA}\begin{Bmatrix}-1\\ 3\end{Bmatrix}.$$

Determine the rotation θ_{Y2} at node 2 of the structure loaded by a point moment (magnitude M) acting on node 2. Use beam elements (1) and (2) of length L and a point moment element (3). Assume that the beams are inextensible in the axial directions. Young's modulus E and the second moment of area I are constants. Use the principle of virtual work.



Solution

In a planar problem, torsion and out-plane bending deformation modes can be omitted. As beams are assumed to be inextensible in the axial direction and there are no axial distributed forces, the bar mode virtual work expression vanishes. Virtual work expressions of the beam xz-plane bending element and point force/moment elements are given by

$$\delta W=-\begin{Bmatrix}\delta u_{z1}\\ \delta\theta_{y1}\\ \delta u_{z2}\\ \delta\theta_{y2}\end{Bmatrix}^{\mathrm{T}}\left(\frac{EI_{yy}}{h^3}\begin{bmatrix}12 & -6h & -12 & -6h\\ -6h & 4h^2 & 6h & 2h^2\\ -12 & 6h & 12 & 6h\\ -6h & 2h^2 & 6h & 4h^2\end{bmatrix}\begin{Bmatrix}u_{z1}\\ \theta_{y1}\\ u_{z2}\\ \theta_{y2}\end{Bmatrix}-\frac{f_z h}{12}\begin{Bmatrix}6\\ -h\\ 6\\ h\end{Bmatrix}\right),$$

$$\delta W=\begin{Bmatrix}\delta u_{X1}\\ \delta u_{Y1}\\ \delta u_{Z1}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}F_{X1}\\ F_{Y1}\\ F_{Z1}\end{Bmatrix}+\begin{Bmatrix}\delta\theta_{X1}\\ \delta\theta_{Y1}\\ \delta\theta_{Z1}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}M_{X1}\\ M_{Y1}\\ M_{Z1}\end{Bmatrix}.$$

4p Nodal displacements/rotations of the structure are clearly zeros except those of node 2. Displacement of node 2 vanishes as both beams are inextensible in the axial directions. Therefore, the only non-zero displacement/rotation component of the structure is θ_{Y2}.

Beam 1: $u_{z1}=0$, $\theta_{y1}=0$, $u_{z2}=0$, and $\theta_{y2}=\theta_{Y2}$

$$\delta W^1=-\begin{Bmatrix}0\\ 0\\ 0\\ \delta\theta_{Y2}\end{Bmatrix}^{\mathrm{T}}\frac{EI}{L^3}\begin{bmatrix}12 & -6L & -12 & -6L\\ -6L & 4L^2 & 6L & 2L^2\\ -12 & 6L & 12 & 6L\\ -6L & 2L^2 & 6L & 4L^2\end{bmatrix}\begin{Bmatrix}0\\ 0\\ 0\\ \theta_{Y2}\end{Bmatrix}=-\delta\theta_{Y2}4\frac{EI}{L}\theta_{Y2}.$$

Beam 2: $u_{z2}=0$, $\theta_{y2}=\theta_{Y2}$, $u_{z3}=0$, and $\theta_{y3}=0$

$$\delta W^2=-\begin{Bmatrix}0\\ \delta\theta_{Y2}\\ 0\\ 0\end{Bmatrix}^{\mathrm{T}}\frac{EI}{L^3}\begin{bmatrix}12 & -6L & -12 & -6L\\ -6L & 4L^2 & 6L & 2L^2\\ -12 & 6L & 12 & 6L\\ -6L & 2L^2 & 6L & 4L^2\end{bmatrix}\begin{Bmatrix}0\\ \theta_{Y2}\\ 0\\ 0\end{Bmatrix}=-\delta\theta_{Y2}4\frac{EI}{L}\theta_{Y2}.$$

Point moment 3:

$$\delta W^3 = \begin{Bmatrix} \delta u_{X2} \\ \delta u_{Y2} \\ \delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} F_{X2} \\ F_{Y2} \\ F_{Z2} \end{Bmatrix} + \begin{Bmatrix} \delta\theta_{X2} \\ \delta\theta_{Y2} \\ \delta\theta_{Z2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} M_{X2} \\ M_{Y2} \\ M_{Z2} \end{Bmatrix} = -\delta\theta_{Y2} M \,.$$

2p Virtual work expression of the structure is sum of the element contributions

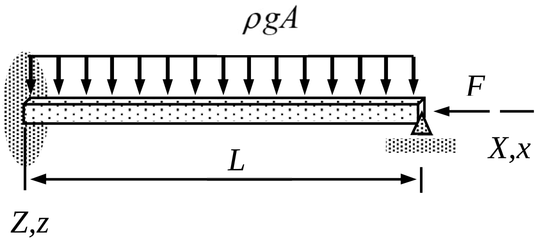
$$\delta W = \sum_e \delta W^e = \delta W^1 + \delta W^2 + \delta W^3 \Rightarrow$$

$$\delta W = -\delta\theta_{Y2} 4\frac{EI}{L}\theta_{Y2} - \delta\theta_{Y2} 4\frac{EI}{L}\theta_{Y2} + 0 - \delta\theta_{Y2} M = -\delta\theta_{Y2}(8\frac{EI}{L}\theta_{Y2} + M) \,.$$

Principle of virtual work $\delta W = 0 \; \forall \delta\mathbf{a}$ and the fundamental lemma of variation calculus in the form $\delta\mathbf{a}^{\mathrm{T}}\mathbf{R} = 0 \;\; \forall \delta\mathbf{a} \;\Leftrightarrow\; \mathbf{R} = 0$ imply

$$8\frac{EI}{L}\theta_{Y2} + M = 0 \quad \Leftrightarrow \quad \theta_{Y2} = -\frac{1}{8}\frac{ML}{EI} \,. \;\leftarrow$$

The beam of the figure is loaded by its own weight and a point force acting on the right end. Determine the displacement and rotation of the right end starting with the virtual density of the Bernoulli beam model. The x-axis of the material coordinate system is placed at the geometric centroid of the rectangle cross-section. Beam properties A, I_{yy}, I_{zz} and E of the planar problem are constants.



Solution

2p The left end of the beam is clamped and the right end simply supported. As the material and structural coordinate systems coincide $u_{x2} = u_{X2}$ and $\theta_{y2} = \theta_{Y2}$, the approximations of u and w simplify to

$$\begin{Bmatrix} u \\ w \end{Bmatrix} = \begin{Bmatrix} x/L u_{X2} \\ L(x/L)^2(1-x/L)\theta_{Y2} \end{Bmatrix} \quad \Rightarrow \quad \begin{Bmatrix} du/dx \\ d^2w/dx^2 \end{Bmatrix} = \frac{1}{L}\begin{Bmatrix} u_{X2} \\ (2-6x/L)\theta_{Y2} \end{Bmatrix}.$$

3p The moments of cross-section $S_y = S_z = 0$, I_{yy}, I_{zz} and $I_{yz} = 0$. As here $v = \phi = 0$, $f_x = f_y = 0$ and $m_x = m_y = m_z = 0$, virtual work densities take the forms

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} d\delta u/dx \\ d^2\delta w/dx^2 \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} EA & 0 \\ 0 & EI_{yy} \end{bmatrix} \begin{Bmatrix} du/dx \\ d^2w/dx^2 \end{Bmatrix} \text{ and } \delta w_\Omega^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ \rho g A \end{Bmatrix} \quad \Rightarrow$$

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \delta u_{X2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{E}{L^2} \begin{bmatrix} A & 0 \\ 0 & I_{yy}(2-6x/L)^2 \end{bmatrix} \begin{Bmatrix} u_{X2} \\ \theta_{Y2} \end{Bmatrix},$$

$$\delta w_\Omega^{\text{ext}} = \begin{Bmatrix} \delta u_{X2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ L(x/L)^2(1-x/L)\rho g A \end{Bmatrix}.$$

Integrations over the domain $\Omega =]0, L[$ give the virtual work expressions

$$\delta W^{\text{int}} = \int_\Omega \delta w_\Omega^{\text{int}} d\Omega = -\begin{Bmatrix} \delta u_{X2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{E}{L} \begin{bmatrix} A & 0 \\ 0 & 4I_{yy} \end{bmatrix} \begin{Bmatrix} u_{X2} \\ \theta_{Y2} \end{Bmatrix},$$

$$\delta W^{\text{ext}} = \int_\Omega \delta w_\Omega^{\text{ext}} d\Omega = \begin{Bmatrix} \delta u_{X2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{\rho g A L^2}{12} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \quad \Rightarrow$$

$$\delta W^1 = \delta W^{\text{int}} + \delta W^{\text{ext}} = -\begin{Bmatrix} \delta u_{X2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{E}{L} \begin{bmatrix} A & 0 \\ 0 & 4I_{yy} \end{bmatrix} \begin{Bmatrix} u_{X2} \\ \theta_{Y2} \end{Bmatrix} - \frac{\rho g A L^2}{12} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \right).$$

Virtual work expression of the point force follows from definition of work (or from the expression of formulae collection)

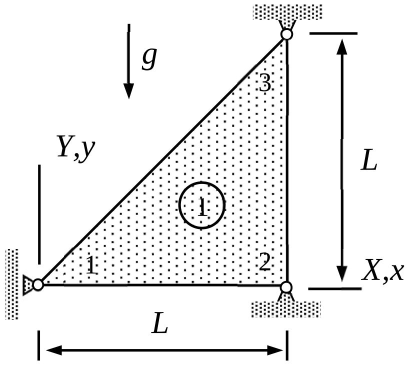
$$\delta W^2 = \begin{Bmatrix} \delta u_{X2} \\ \delta \theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} -F \\ 0 \end{Bmatrix}.$$

1p Principle of virtual work $\delta W = \delta W^1 + \delta W^2 = 0 \quad \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus give

$$\delta W = -\begin{Bmatrix} \delta u_{X2} \\ \delta \theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{E}{L} \begin{bmatrix} A & 0 \\ 0 & 4I_{yy} \end{bmatrix} \begin{Bmatrix} u_{X2} \\ \theta_{Y2} \end{Bmatrix} - \begin{Bmatrix} -F \\ \rho g A L^2 / 12 \end{Bmatrix} \right) \quad \forall \begin{Bmatrix} \delta u_{X2} \\ \delta \theta_{Y2} \end{Bmatrix} \quad \Leftrightarrow$$

$$\frac{E}{L} \begin{bmatrix} A & 0 \\ 0 & 4I \end{bmatrix} \begin{Bmatrix} u_{X2} \\ \theta_{Y2} \end{Bmatrix} - \begin{Bmatrix} -F \\ \rho g A L^2 / 12 \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix} u_{X2} \\ \theta_{Y2} \end{Bmatrix} = \begin{Bmatrix} -LF / EA \\ \rho g A L^3 / (48 E I_{yy}) \end{Bmatrix}.$$

A thin triangular slab (assume plane stress conditions) loaded by its own weight is allowed to move vertically at node 1 and nodes 2 and 3 are fixed. Find the displacement u_{Y1}. Material parameters E, ν, ρ and thickness t of the slab are constants.



Solution

For the plane stress conditions and the thin-slab model, virtual work density of internal and external volume forces are given by

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta u / \partial x \\ \partial\delta v / \partial y \\ \partial\delta u / \partial y + \partial\delta v / \partial x \end{Bmatrix}^{\mathrm{T}} t[E]_{\sigma} \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix}, \text{ where } [E]_{\sigma} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix},$$

$$\delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix}.$$

2p Let us start with the approximations. Only the shape function of node 1 is needed as the other nodes are fixed. By using linearity and conditions $N_1(0,0) = 1$, $N_1(L,0) = N_1(0,L) = 0$

$$N_1(x,y) = 1 - \frac{x}{L}.$$

Displacement components simplify to

$$u = 0 \quad \Rightarrow \quad \frac{\partial u}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} = 0,$$

$$v = (1-\frac{x}{L})u_{Y1} \quad \Rightarrow \quad \frac{\partial v}{\partial x} = -\frac{u_{Y1}}{L} \quad \text{and} \quad \frac{\partial v}{\partial y} = 0.$$

4p When approximations are substituted there, virtual work density simplifies to

$$\delta w_{\Omega}^{\text{int}} = -\delta u_{Y1} \frac{1}{L^2} \frac{tE}{2+2\nu} u_{Y1}.$$

Integration over the domain gives the virtual work expression. As the integrand is constant

$$\delta W^{\text{int}} = \int_{\Omega} \delta w_{\Omega}^{\text{int}} dA = \frac{L^2}{2} \delta w_{\Omega}^{\text{int}} = -\delta u_{Y1} \frac{1}{4} \frac{tE}{1+\nu} u_{Y1}.$$

Virtual work expression of the external volume force due to gravity takes the form

$$\delta W^{\text{ext}} = \int_{\Omega} \delta w_{\Omega}^{\text{ext}} dA = -\int_0^L \int_0^x (1-\frac{x}{L}) \delta u_{Y1} t\rho g dy dx = -\delta u_{Y1} \frac{1}{6} t\rho g L^2.$$

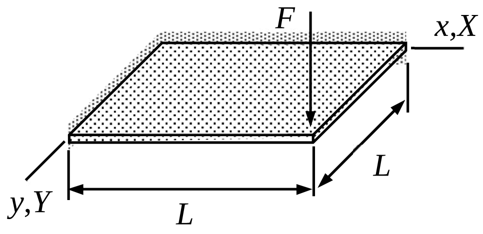
Virtual work expression of the thin slab is sum of the internal and external parts

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = -\delta u_{Y1}\left(\frac{1}{4}\frac{tE}{1+\nu}u_{Y1} + \frac{1}{6}t\rho g L^2\right).$$

Principle of virtual work $\delta W = 0 \;\forall \delta \mathrm{a}$ and the fundamental lemma of variation calculus give

$$\frac{1}{4}\frac{tE}{1+\nu}u_{Y1} + \frac{1}{6}t\rho g L^2 = 0 \quad \Leftrightarrow \quad u_{Y1} = -\frac{2}{3}(1+\nu)\frac{\rho g L^2}{E}.$$

A Kirchhoff plate, loaded by point force F acting at the free corner, is simply supported on two edges and free on the other two edges as shown in the figure. Determine the parameter a_0 of approximation $w(x,y)=a_0(x/L)(y/L)$ and displacement at the center point. Use the virtual work density of the Kirchhoff plate model with constant E, ν, ρ and t.



Solution

Assuming that the material coordinate system is chosen so that the plate bending and thin slab modes decouple, it is enough to consider the virtual work densities of the bending mode only

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial^2\delta w/\partial x^2 \\ \partial^2\delta w/\partial y^2 \\ 2\partial^2\delta w/\partial x\partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12}[E]_\sigma \begin{Bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ 2\partial^2 w/\partial x\partial y \end{Bmatrix}, \ \delta w_\Omega^{\text{ext}} = \delta w f_z .$$

in which the elasticity matrix of plane stress

$$[E]_\sigma = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}.$$

In the present case, distributed force vanishes i.e. $f_z = 0$ and the point force is taken into account by a point force element.

4p Approximation to the transverse displacement is chosen to be (a_0 is not associated with any point but it just a parameter of the approximation)

$$w(x,y) = a_0 \frac{x}{L}\frac{y}{L} \quad \Rightarrow \quad \frac{\partial^2 w}{\partial x^2} = 0, \ \frac{\partial^2 w}{\partial y^2} = 0, \text{ and } \frac{\partial^2 w}{\partial x \partial y} = \frac{1}{L^2}a_0 .$$

When the approximation is substituted there, virtual work density of internal forces simplifies to

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial^2\delta w/\partial x^2 \\ \partial^2\delta w/\partial y^2 \\ 2\partial^2\delta w/\partial x\partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3 E}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ 2\partial^2 w/\partial x\partial y \end{Bmatrix} = -\delta a_0 \frac{Et^3}{6(1+\nu)}\frac{1}{L^4}a_0 ,$$

Virtual work expression of the plate bending element (element 1 here) is integral of the virtual work density over the domain occupied by the element

$$\delta W^1 = \int_0^L \int_0^L \delta w_\Omega^{\text{int}} dxdy = -\delta a_0 \frac{Et^3}{6(1+\nu)}\frac{1}{L^2}a_0 .$$

Virtual work expression of the point force (element 2 here) follows from the definition of work (notice the use of virtual displacement of the point of action $x = y = L$)

$$\delta W^2 = \delta w(L,L)F = \delta a_0 F \,.$$

2p Principle of virtual work and the fundamental lemma of variation calculus give

$$\delta W = \delta W^1 + \delta W^2 = -\delta a_0 \left(\frac{Et^3}{6(1+\nu)}\frac{1}{L^2}a_0 - F\right) = 0 \quad \Rightarrow \quad a_0 = 6(1+\nu)\frac{FL^2}{Et^3} \,.$$

Displacement at the center point

$$w\left(\frac{L}{2},\frac{L}{2}\right) = a_0 \frac{1}{4} = \frac{3}{2}(1+\nu)\frac{FL^2}{Et^3} \,. \quad \Leftarrow$$