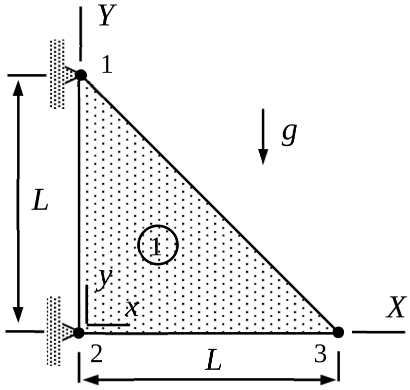
Name________________________ Student number________________

Assignment 5



A thin triangular slab of thickness t is loaded by its own weight. Derive the virtual work expression δW of the structure and solve for the nodal displacements u_{X3} and u_{Y3}. Approximation is linear and elasticity parameters E, ν and density ρ are constants. Assume plane stress conditions.

Solution

The virtual work densities (virtual works per unit area) of the thin slab model under the plane stress conditions

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta u/\partial y + \partial\delta v/\partial x \end{Bmatrix}^{\mathrm{T}} t[E]_{\sigma} \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix} \text{ and } \delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix} \text{ where}$$

$$[E]_{\sigma} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}$$

take into account the internal forces (stress), external forces acting on the element domain, and external forces acting on the edges. Notice that the components f_x and f_y are external forces per unit area.

Expressions of linear shape functions in material $xy-$coordinates can be deduced from the figure. Only the shape function $N_3 = x/L$ of node 3 is actually needed. Hence

$$u = \frac{x}{L}u_{X3} \quad \Rightarrow \quad \frac{\partial u}{\partial x} = \frac{1}{L}u_{X3} \quad \text{and} \quad \frac{\partial u}{\partial y} = 0\,,$$

$$v = \frac{x}{L}u_{Y3} \quad \Rightarrow \quad \frac{\partial v}{\partial x} = \frac{1}{L}u_{Y3} \quad \text{and} \quad \frac{\partial v}{\partial y} = 0\,.$$

When the approximation is substituted there, virtual work expression of internal forces per unit area simplifies to

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \delta u_{X3} \\ 0 \\ \delta u_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{1}{L}\frac{tE}{2(1-\nu^2)}\begin{bmatrix} 2 & 2\nu & 0 \\ 2\nu & 2 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix}\frac{1}{L}\begin{Bmatrix} u_{X3} \\ 0 \\ u_{Y3} \end{Bmatrix} \quad \Leftrightarrow$$

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \delta u_{X3} \\ \delta u_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{tE}{2L^2(1-\nu^2)} \begin{bmatrix} 2 & 0 \\ 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} u_{X3} \\ u_{Y3} \end{Bmatrix}.$$

As the integrand is constant, integration over the triangular domain gives

$$\delta W^{\text{int}} = \int_A \delta w_{\Omega}^{\text{int}} \mathrm{d}A = \delta w_{\Omega}^{\text{int}} \frac{L^2}{2} = -\begin{Bmatrix} \delta u_{X3} \\ \delta u_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{tE}{4(1-\nu^2)} \begin{bmatrix} 2 & 0 \\ 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} u_{X3} \\ u_{Y3} \end{Bmatrix}.$$

In the virtual work density of the external forces $f_x = 0$ and $f_y = -\rho g t$ so

$$\delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix} = -\rho g t \frac{x}{L} \delta u_{Y3}.$$

Integration over the domain occupied by the element gives

$$\delta W^{\text{ext}} = \int_A \delta w_{\Omega}^{\text{ext}} \mathrm{d}A = \int_0^L \left(\int_0^{L-x} -\rho g t \frac{x}{L} \delta u_{Y3} dy\right) dx = -\frac{\rho g t L^2}{6} \delta u_{Y3} = -\begin{Bmatrix} \delta u_{X3} \\ \delta u_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{\rho g t L^2}{6} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}.$$

Virtual work expression of the structure takes the form

$$\delta W = -\begin{Bmatrix} \delta u_{X3} \\ \delta u_{Y3} \end{Bmatrix}^{\mathrm{T}} \left(\frac{tE}{4(1-\nu^2)} \begin{bmatrix} 2 & 0 \\ 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} u_{X3} \\ u_{Y3} \end{Bmatrix} + \frac{\rho g t L^2}{6} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \right).$$

Principle of virtual work $\delta W = 0 \ \forall \delta \mathrm{a}$ and the fundamental lemma of variation calculus give

$$\frac{tE}{4(1-\nu^2)} \begin{bmatrix} 2 & 0 \\ 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} u_{X3} \\ u_{Y3} \end{Bmatrix} + \frac{\rho g t L^2}{6} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix} u_{X3} \\ u_{Y3} \end{Bmatrix} = -\frac{4}{6} \frac{\rho g L^2}{E} (1+\nu) \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}. \quad \Leftarrow$$