

**MEC-E1050**

**FINITE ELEMENT METHOD IN  
SOLIDS 2024**

# 3 BAR AND BEAM STRUCTURES

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## LEARNING OUTCOMES

Students can solve the lecture problems, home problems, and exercise problems on the topics of the week:

- Bar truss displacements. Element contribution of a bar in the structural coordinate system.
- Principle of virtual work and the fundamental lemma of variation calculus. Virtual work expressions of elements and structures.
- Beam element loading modes and the element.
- Kinematic constraints, kinematic links, and boundary conditions.

## 3.1 BAR TRUSS

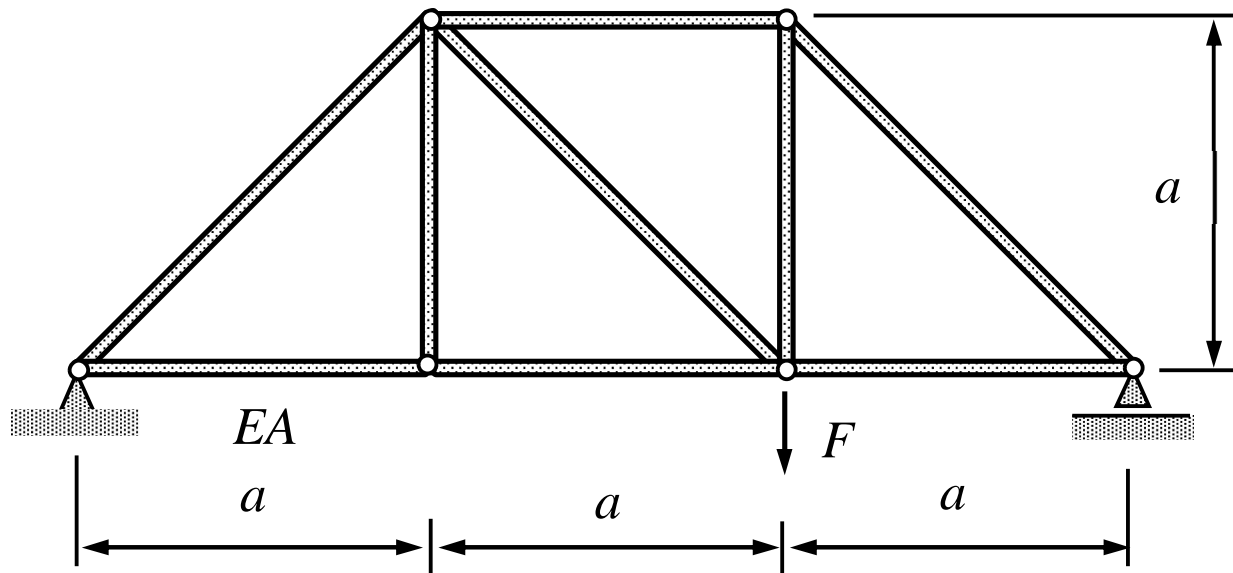
Slender structural parts of a truss may act as bars or beams depending on the loading and the type of joints. If internal forces are aligned with the axes of the parts, a simple bar model may give satisfactory results!



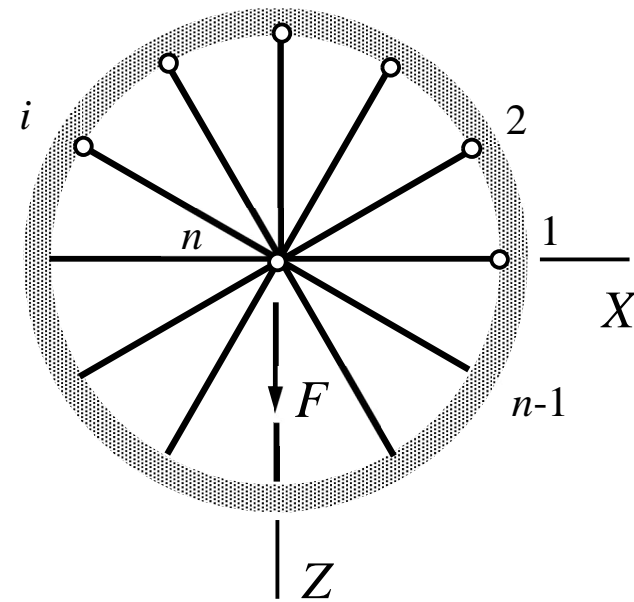
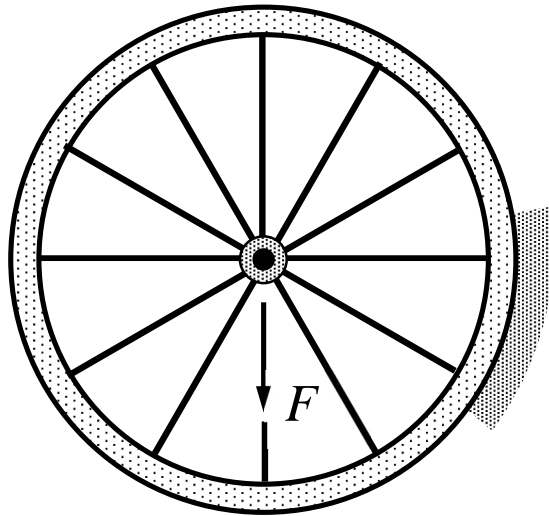
Typical applications are bridges, cranes, roof trusses etc.

## BAR TRUSS

Bar truss consists of straight slender structural parts connected by cylindrical or spherical joints so that internal forces are aligned with the axes of the structural parts (a straight line between the joints). The unknowns are the nodal displacements. Rotations do not matter as they do not appear in the element contributions.

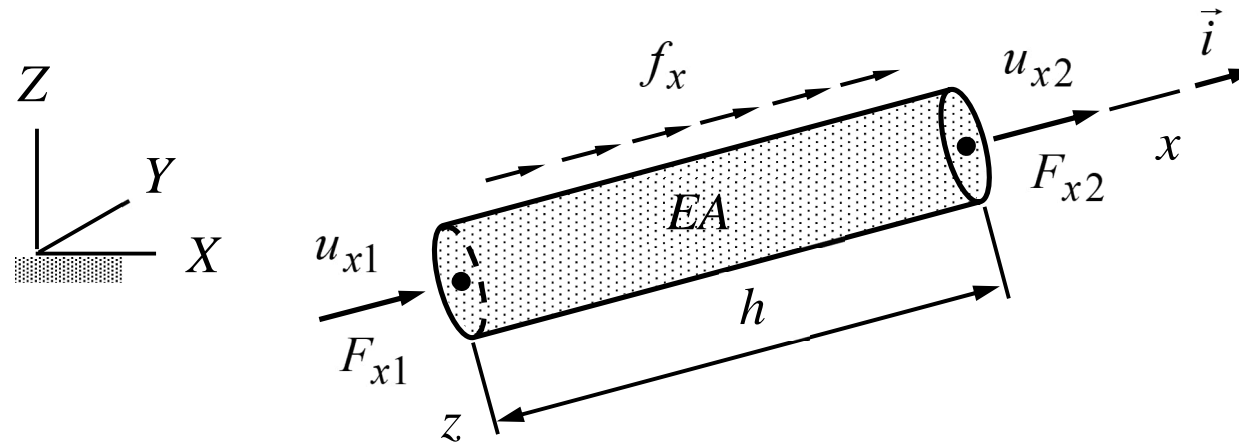


**EXAMPLE 3.1.** The outer rim and center of a wheel are assumed to be fully rigid. The center is fastened with 12 (diameter  $d = 1\text{mm}$ ) steel ( $E = 210\text{GN/m}^2$ ) rods (length  $L = 300\text{mm}$ ). Using bar elements, calculate the displacement of the center when a load of  $F = 1\text{kN}$  is placed (buckling does not occur)?



**Answer**  $U_{Zn} = \frac{FL}{6EA} (= \frac{1}{525\pi} [\text{m}])$ , when  $n = 13$

## BAR ELEMENT IN THE STRUCTURAL SYSTEM



$$\begin{Bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \end{Bmatrix} = \frac{EA}{h} \begin{bmatrix} \mathbf{i}\mathbf{i}^T & -\mathbf{i}\mathbf{i}^T \\ -\mathbf{i}\mathbf{i}^T & \mathbf{i}\mathbf{i}^T \end{bmatrix} \begin{Bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{Bmatrix} - \frac{f_x h}{2} \begin{Bmatrix} \mathbf{i} \\ \mathbf{i} \end{Bmatrix}, \text{ where } \mathbf{a} = \begin{Bmatrix} u_X \\ u_Y \\ u_Z \end{Bmatrix} \text{ and } \mathbf{R} = \begin{Bmatrix} F_X \\ F_Y \\ F_Z \end{Bmatrix}$$

The displacement components of the material system are expressed in terms of those in the structural system, which brings the orientation into the element contribution. Column matrix  $\mathbf{i}$  contains the components of the unit vector  $\vec{i}$  in the structural coordinate system!

- The starting point is the element contribution in terms of displacement and force components in the material system (the simplest representation)

$$\begin{Bmatrix} F_{x1} \\ F_{x2} \end{Bmatrix} = \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

- With notations  $\mathbf{a} = \{u_X \quad u_Y \quad u_Z\}^T$  and  $\mathbf{R} = \{F_X \quad F_Y \quad F_Z\}^T$  and taking into account that  $F_y = F_z = 0$  for a bar

$$u_x = \mathbf{i}^T \begin{Bmatrix} u_X \\ u_Y \\ u_Z \end{Bmatrix} = \mathbf{i}^T \mathbf{a} \quad \text{and} \quad \mathbf{R} = \begin{Bmatrix} F_X \\ F_Y \\ F_Z \end{Bmatrix} = \mathbf{i} F_x \quad \Rightarrow$$

$$\begin{Bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \end{Bmatrix} = \begin{bmatrix} \mathbf{i} & \mathbf{0} \\ \mathbf{0} & \mathbf{i} \end{bmatrix} \begin{Bmatrix} F_{x1} \\ F_{x2} \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} = \begin{bmatrix} \mathbf{i}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{i}^T \end{bmatrix} \begin{Bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{Bmatrix}.$$



- Therefore, element contribution in the structural system

$$\begin{bmatrix} \mathbf{i} & \mathbf{0} \\ \mathbf{0} & \mathbf{i} \end{bmatrix} \begin{Bmatrix} F_{x1} \\ F_{x2} \end{Bmatrix} = \frac{EA}{h} \begin{bmatrix} \mathbf{i} & \mathbf{0} \\ \mathbf{0} & \mathbf{i} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2} \begin{bmatrix} \mathbf{i} & \mathbf{0} \\ \mathbf{0} & \mathbf{i} \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \Rightarrow$$

$$\begin{Bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \end{Bmatrix} = \frac{EA}{h} \begin{bmatrix} \mathbf{i} & \mathbf{0} \\ \mathbf{0} & \mathbf{i} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{i}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{i}^T \end{bmatrix} \begin{Bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{Bmatrix} - \frac{f_x h}{2} \begin{bmatrix} \mathbf{i} & \mathbf{0} \\ \mathbf{0} & \mathbf{i} \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \Rightarrow$$

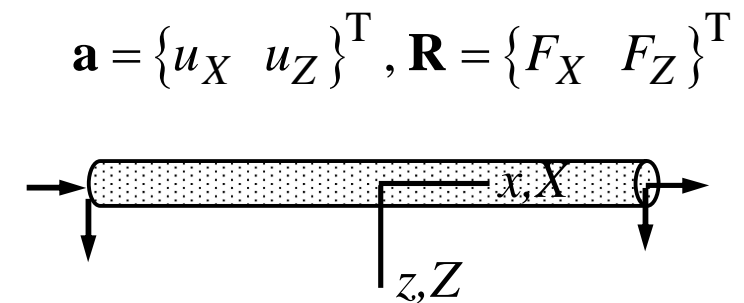
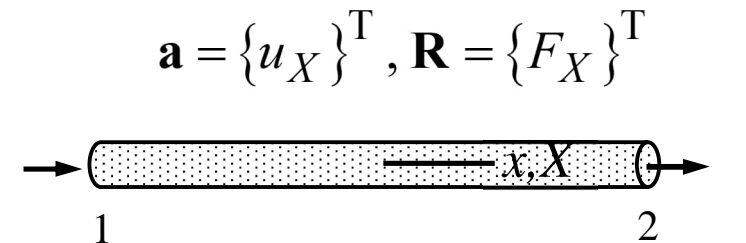
$$\begin{Bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \end{Bmatrix} = \frac{EA}{h} \begin{bmatrix} \mathbf{i}\mathbf{i}^T & -\mathbf{i}\mathbf{i}^T \\ -\mathbf{i}\mathbf{i}^T & \mathbf{i}\mathbf{i}^T \end{bmatrix} \begin{Bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{Bmatrix} - \frac{f_x h}{2} \begin{Bmatrix} \mathbf{i} \\ \mathbf{i} \end{Bmatrix} \quad \leftarrow$$

contains also the orientation  $\mathbf{i}$  of the bar.

- The actual size of the matrix etc. depends on the number of components in **i** (dimension of the problem). For example, assuming that the axes of the material and structural coordinate systems are aligned, the bar elements for the uni-axial (X-axis), and planar (XZ-plane) are

$$\begin{Bmatrix} F_{X1} \\ F_{X2} \end{Bmatrix} = \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{X2} \end{Bmatrix} - \frac{f_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix},$$

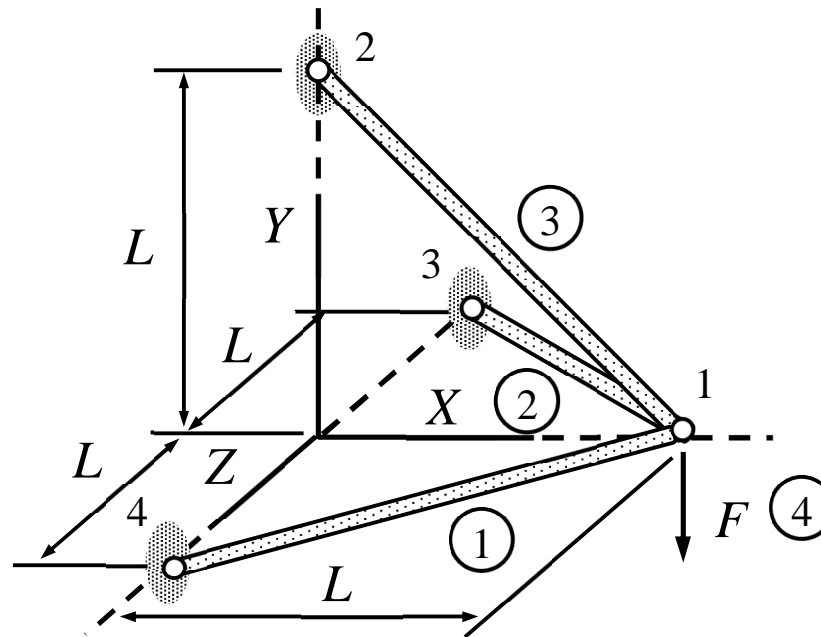
$$\begin{Bmatrix} F_{X1} \\ F_{Z1} \\ F_{X2} \\ F_{Z2} \end{Bmatrix} = \frac{EA}{h} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{Z1} \\ u_{X2} \\ u_{Z2} \end{Bmatrix} - \frac{f_x h}{2} \begin{Bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{Bmatrix}.$$



**EXAMPLE 3.2.** The nodes of a bar are at  $(0,0,0)$  (node 1) and  $(L,L,L)$  (node 2) in the structural coordinate system and the positive  $x$ -axis is directed from node 1 to 2. Determine the element contribution  $\mathbf{R} = \mathbf{K}\mathbf{a} - \mathbf{F}$  in the structural coordinate system if  $f_x$  and  $EA$  are constants.

$$\text{Answer } \left\{ \begin{array}{c} F_{X1} \\ F_{Y1} \\ F_{Z1} \\ \hline F_{X2} \\ F_{Y2} \\ F_{Z2} \end{array} \right\} = \frac{EA}{3\sqrt{3}L} \left[ \begin{array}{ccc|ccc} 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ \hline -1 & -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 \end{array} \right] \left\{ \begin{array}{c} u_{X1} \\ u_{Y1} \\ u_{Z1} \\ \hline u_{X2} \\ u_{Y2} \\ u_{Z2} \end{array} \right\} - \frac{f_x L}{2} \left\{ \begin{array}{c} 1 \\ 1 \\ 1 \\ \hline 1 \\ 1 \\ 1 \end{array} \right\}$$

**EXAMPLE 3.3.** If the space truss of the figure is loaded by a vertical force  $F$  acting on node 1, determine the displacement of node 1. Assume that the displacement in  $Z$ -direction vanishes due to the symmetry i.e.  $u_{Z1} = 0$ . Young's modulus of the material  $E$  and the cross-sectional area  $A$  are constants. Gravity is negligible.



**Answer**  $u_{X1} = -\sqrt{2} \frac{FL}{EA}$  and  $u_{Y1} = -3\sqrt{2} \frac{FL}{EA}$

- The bar element contribution in the structural coordinate system

$$\begin{Bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \end{Bmatrix} = \frac{EA}{h} \begin{bmatrix} \mathbf{i}\mathbf{i}^T & -\mathbf{i}\mathbf{i}^T \\ -\mathbf{i}\mathbf{i}^T & \mathbf{i}\mathbf{i}^T \end{bmatrix} \begin{Bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{Bmatrix} - \frac{f_x h}{2} \begin{Bmatrix} \mathbf{i} \\ \mathbf{i} \end{Bmatrix}, \quad \mathbf{R} = \begin{Bmatrix} F_X \\ F_Y \\ F_Z \end{Bmatrix}, \quad \text{and} \quad \mathbf{a} = \begin{Bmatrix} u_X \\ u_Y \\ u_Z \end{Bmatrix}$$

is useful in hand calculations. The elements of  $\mathbf{i}$  are the components of the unit vector  $\vec{i}$  in the structural coordinate system which can be deduced from the figure:

$$\mathbf{i}^1 = \frac{1}{\sqrt{2}} \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} \text{ and } h = \sqrt{2}L, \quad \mathbf{i}^2 = \frac{1}{\sqrt{2}} \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix} \text{ and } h = \sqrt{2}L, \quad \mathbf{i}^3 = \frac{1}{\sqrt{2}} \begin{Bmatrix} 1 \\ -1 \\ 0 \end{Bmatrix} \text{ and } h = \sqrt{2}L.$$

- The element contributions of the three bars and one point force are

$$\text{Bar 1: } \begin{Bmatrix} F_{X4}^1 \\ F_{Y4}^1 \\ F_{Z4}^1 \\ \hline F_{X1}^1 \\ F_{Y1}^1 \\ F_{Z1}^1 \end{Bmatrix} = \frac{EA}{\sqrt{8}L} \begin{bmatrix} 1 & 0 & -1 & | & -1 & 0 & 1 \\ 0 & 0 & 0 & | & 0 & 0 & 0 \\ -1 & 0 & 1 & | & 1 & 0 & -1 \\ \hline -1 & 0 & 1 & | & 1 & 0 & -1 \\ 0 & 0 & 0 & | & 0 & 0 & 0 \\ 1 & 0 & -1 & | & -1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \hline u_{X1} \\ u_{Y1} \\ 0 \end{Bmatrix}, \quad (\mathbf{i}^1 = \frac{1}{\sqrt{2}} \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}, \quad h = \sqrt{2}L)$$

$$\text{Bar 2: } \begin{Bmatrix} F_{X3}^2 \\ F_{Y3}^2 \\ F_{Z3}^2 \\ \hline F_{X1}^2 \\ F_{Y1}^2 \\ F_{Z1}^2 \end{Bmatrix} = \frac{EA}{\sqrt{8}L} \begin{bmatrix} 1 & 0 & 1 & | & -1 & 0 & -1 \\ 0 & 0 & 0 & | & 0 & 0 & 0 \\ 1 & 0 & 1 & | & -1 & 0 & -1 \\ \hline -1 & 0 & -1 & | & 1 & 0 & 1 \\ 0 & 0 & 0 & | & 0 & 0 & 0 \\ -1 & 0 & -1 & | & 1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \hline u_{X1} \\ u_{Y1} \\ 0 \end{Bmatrix}, \quad (\mathbf{i}^2 = \frac{1}{\sqrt{2}} \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix}, \quad h = \sqrt{2}L)$$

$$\text{Bar 3: } \begin{Bmatrix} F_{X2}^3 \\ F_{Y2}^3 \\ F_{Z2}^3 \\ \hline F_{X1}^3 \\ F_{Y1}^3 \\ F_{Z1}^3 \end{Bmatrix} = \frac{EA}{\sqrt{8}L} \begin{bmatrix} 1 & -1 & 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -1 & 1 & 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \hline u_{X1} \\ u_{Y1} \\ 0 \end{Bmatrix}, \quad (\mathbf{i}^3 = \frac{1}{\sqrt{2}} \begin{Bmatrix} 1 \\ -1 \\ 0 \end{Bmatrix}, h = \sqrt{2}L)$$

$$\text{Force 4: } \begin{Bmatrix} F_{X1}^4 \\ F_{Y1}^4 \\ F_{Z1}^4 \end{Bmatrix} = - \begin{Bmatrix} 0 \\ -F \\ 0 \end{Bmatrix}.$$

- In assembly, internal forces acting on the nodes are added to end up with the equilibrium equations for the nodes. To get the minimal system for the unknown displacement components, only the non-constrained directions are considered first (the remaining

equilibrium equation can be used to get the solution to the constraint forces and element contributions to solution for the internal forces)

$$\begin{Bmatrix} F_{X1}^1 \\ F_{Y1}^1 \end{Bmatrix} + \begin{Bmatrix} F_{X1}^2 \\ F_{Y1}^2 \end{Bmatrix} + \begin{Bmatrix} F_{X1}^3 \\ F_{Y1}^3 \end{Bmatrix} + \begin{Bmatrix} F_{X1}^4 \\ F_{Y1}^4 \end{Bmatrix} = \frac{EA}{\sqrt{8}L} \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} + F \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = 0.$$

- The unknown displacement components follow as the solution of the system equations

$$\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} = -\sqrt{8} \frac{FL}{EA} \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = -\frac{\sqrt{8}}{2} \frac{FL}{EA} \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = -\sqrt{2} \frac{FL}{EA} \begin{Bmatrix} 1 \\ 3 \end{Bmatrix} \Leftrightarrow$$

$$u_{X1} = -\sqrt{2} \frac{FL}{EA} \quad \text{and} \quad u_{Y1} = -3\sqrt{2} \frac{FL}{EA} . \quad \leftarrow$$



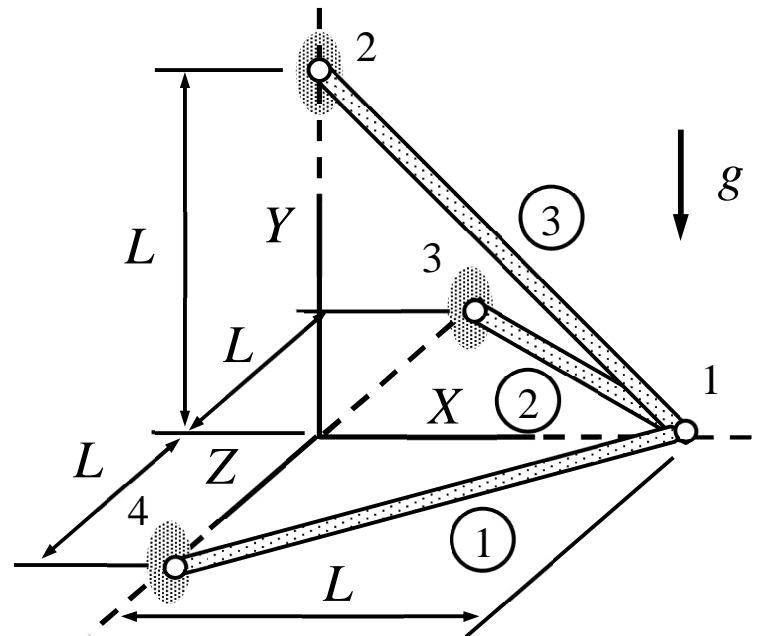
## EFFECT OF WEIGHT

Weight  $\vec{f} = \rho A \vec{g}$  acting on bars of a truss may not be aligned with the axes. Assuming that the joints are not capable for taking moments, the bar model may give a good picture about the internal forces if the weight is taken to act at the element nodes according to

$$\begin{Bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \end{Bmatrix} = \frac{EA}{h} \begin{bmatrix} \mathbf{i}\mathbf{i}^T & -\mathbf{i}\mathbf{i}^T \\ -\mathbf{i}\mathbf{i}^T & \mathbf{i}\mathbf{i}^T \end{bmatrix} \begin{Bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{Bmatrix} - \frac{h\rho A}{2} \begin{Bmatrix} \mathbf{g} \\ \mathbf{g} \end{Bmatrix}, \text{ where } \mathbf{a} = \begin{Bmatrix} u_X \\ u_Y \\ u_Z \end{Bmatrix}, \mathbf{R} = \begin{Bmatrix} F_X \\ F_Y \\ F_Z \end{Bmatrix} \text{ and } \mathbf{g} = \begin{Bmatrix} g_X \\ g_Y \\ g_Z \end{Bmatrix}.$$

Above,  $\mathbf{g}$  contains the components of acceleration by gravity in the structural coordinate system. The elements of  $\mathbf{i}$  are the components of the unit vector  $\vec{i}$  in the structural coordinate system.

**EXAMPLE 3.4.** A space truss is loaded by its own weight. If the joints do not take any moments, determine the displacement of node 1. Young's modulus  $E$ , density  $\rho$ , and cross-sectional area  $A$  are constants. Use symmetry.



**Answer**  $u_{X1} = -3 \frac{\rho g L^2}{E}$  and  $u_{Y1} = -9 \frac{\rho g L^2}{E}$

- The Length of all the bars  $h = \sqrt{2}L$ . The unit vectors to the directions of the  $x$ -axes and the components of the acceleration by gravity are (figure)

$$\mathbf{i}^1 = \frac{1}{\sqrt{2}} \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}, \quad \mathbf{i}^2 = \frac{1}{\sqrt{2}} \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix}, \quad \mathbf{i}^3 = \frac{1}{\sqrt{2}} \begin{Bmatrix} 1 \\ -1 \\ 0 \end{Bmatrix}, \quad \text{and } \mathbf{g} = \begin{Bmatrix} 0 \\ -g \\ 0 \end{Bmatrix}.$$

- The element contributions need to be expressed in the structural coordinate system:

$$\text{Bar 1: } \begin{Bmatrix} F_{X4}^1 \\ F_{Y4}^1 \\ F_{Z4}^1 \\ F_{X1}^1 \\ F_{Y1}^1 \\ F_{Z1}^1 \end{Bmatrix} = \frac{EA}{\sqrt{8}L} \begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & -1 \\ -1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ u_{X1} \\ u_{Y1} \\ 0 \end{Bmatrix} - \frac{\rho ALg}{\sqrt{2}} \begin{Bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ -1 \\ 0 \end{Bmatrix},$$

$$\text{Bar 2: } \begin{Bmatrix} F_{X3}^2 \\ F_{Y3}^2 \\ F_{Z3}^2 \\ F_{X1}^2 \\ F_{Y1}^2 \\ F_{Z1}^2 \end{Bmatrix} = \frac{EA}{\sqrt{8}L} \begin{bmatrix} 1 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 & -1 \\ \hline -1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ u_{X1} \\ u_{Y1} \\ 0 \end{Bmatrix} - \frac{\rho ALg}{\sqrt{2}} \begin{Bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ -1 \\ 0 \end{Bmatrix},$$

$$\text{Bar 3: } \begin{Bmatrix} F_{X2}^3 \\ F_{Y2}^3 \\ F_{Z2}^3 \\ F_{X1}^3 \\ F_{Y1}^3 \\ F_{Z1}^3 \end{Bmatrix} = \frac{EA}{\sqrt{8}L} \begin{bmatrix} 1 & -1 & 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -1 & 1 & 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ u_{X1} \\ u_{Y1} \\ 0 \end{Bmatrix} - \frac{\rho ALg}{\sqrt{2}} \begin{Bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ -1 \\ 0 \end{Bmatrix}.$$

- Sum of the forces acting on node 1 need to vanish for the equilibrium

$$0 = \begin{Bmatrix} F_{X1}^1 + F_{X1}^2 + F_{X1}^3 \\ F_{Y1}^1 + F_{Y1}^2 + F_{Y1}^3 \end{Bmatrix} = \frac{EA}{2\sqrt{2}L} \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} - \frac{\rho ALg}{\sqrt{2}} \begin{Bmatrix} 0 \\ -3 \end{Bmatrix}.$$

- The values of the unknown displacement components are obtained from the equation system

$$\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} = 2 \frac{\rho L^2 g}{E} \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ -3 \end{Bmatrix} = \frac{\rho g L^2}{E} \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{Bmatrix} 0 \\ -3 \end{Bmatrix} = \frac{\rho g L^2}{E} \begin{Bmatrix} -3 \\ -9 \end{Bmatrix} \Leftrightarrow$$

$$u_{X1} = -\frac{3\rho g L^2}{E} \text{ and } u_{Y1} = -\frac{9\rho g L^2}{E}. \quad \leftarrow$$

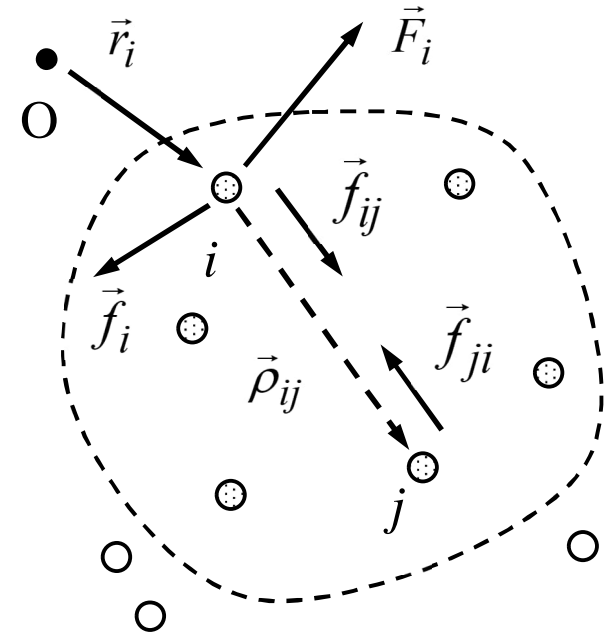
## 3.2 PRINCIPLE OF VIRTUAL WORK

Principle of virtual work is one of the equivalent forms of equilibrium equations (an important form).

**Virtual work**  $\delta W = \delta W^{\text{ext}} + \delta W^{\text{int}} = 0 \quad \forall \delta \vec{r}_i$

External forces  $\delta W^{\text{ext}} = \sum \delta \vec{r}_i \cdot \vec{F}_i$

Internal forces  $\delta W^{\text{int}} = \sum \delta \vec{r}_i \cdot \vec{f}_i = -\sum \delta \rho_{ij} f_{ij}$



The principle is very useful, for example, in connection with kinematical constraints. As an example,  $\delta W^{\text{int}} = 0$  for a rigid body, as the distances between particles are constants and therefore  $\delta \rho_{ij} = 0$ !

- The starting point is the equilibrium equations of a particle system

$$\sum_{i \in I} \delta \vec{r}_i \cdot (\vec{F}_i + \vec{f}_i) = \delta W^{\text{ext}} + \delta W^{\text{int}} = 0 \quad \forall \delta \vec{r}_i, \text{ where}$$

$$\delta W^{\text{ext}} = \sum_{i \in I} \delta \vec{r}_i \cdot \vec{F}_i \quad \text{and} \quad \delta W^{\text{int}} = \sum_{i \in I} \delta \vec{r}_i \cdot \vec{f}_i .$$

- The virtual work of internal forces can be written in a more concise form: Let us consider a typical pair  $(i, j)$  of particles:

$$W_{(i,j)}^{\text{int}} = \vec{f}_{ij} \cdot \delta \vec{r}_i + \vec{f}_{ji} \cdot \delta \vec{r}_j = \vec{f}_{ij} \cdot \delta(\vec{r}_i - \vec{r}_j) = -\vec{f}_{ij} \cdot \delta \vec{\rho}_{ij} = -f_{ij} \delta \rho_{ij},$$

where  $\vec{\rho}_{ij} = \vec{r}_j - \vec{r}_i$  is the position of particle  $j$  relative to particle  $i$ . The expression for a body (a closed system of particles) is obtained as a sum over the particle pairs.

## FUNDAMENTAL LEMMA OF VARIATION CALCULUS

The fundamental lemma of variation calculus in one form or another is an important tool in FEM. In MEC-E1050 the lemma tells how to deduce the equilibrium equations of a structure using a virtual work expression and the principle of virtual work:

$$\square \quad u, v \in \mathbb{R} \quad : \quad vu = 0 \quad \forall v \quad \Leftrightarrow \quad u = 0$$

$$\square \quad \mathbf{u}, \mathbf{v} \in \mathbb{R}^n \quad : \quad \mathbf{v}^T \mathbf{u} = 0 \quad \forall \mathbf{v} \quad \Leftrightarrow \quad \mathbf{u} = 0$$

$$\square \quad u, v \in C^0(\Omega) \quad : \quad \int_{\Omega} uv d\Omega = 0 \quad \forall v \quad \Leftrightarrow \quad u(x, y, \dots) = 0 \quad \text{in } \Omega$$

In mechanics of the materials, variable or function  $v$  is often chosen as the kinematically admissible variation of the displacement field  $\delta u$ .



## PRINCIPLE OF VIRTUAL WORK IN FEM

Principle of virtual work  $\delta W = 0 \quad \forall \delta \mathbf{a}$  is just one form of equilibrium equations. In connection with MEC-E1050, the principle is a representation of the equilibrium equations of *nodes*.

forces from  
elements

given forces  
acting on nodes

$$\delta W = - \sum_{e \in E} \sum_i (\delta \vec{u}_i \cdot \vec{F}_i^e + \delta \vec{\theta}_i \cdot \vec{M}_i^e) + \sum_{i \in I} (\delta \vec{u}_i \cdot \vec{F}_i + \delta \vec{\theta}_i \cdot \vec{M}_i) \Rightarrow$$

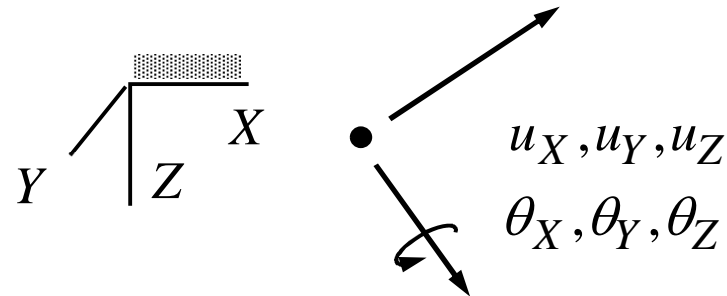
$$\delta W = \sum_{e \in E} \delta W^e = -\delta \mathbf{a}^T (\mathbf{K} \mathbf{a} - \mathbf{F}). \quad (\text{point forces treated as one-node elements})$$

The negative sign in the first term is due to the selection that the forces acting on the *elements* are positive in the direction of displacement. Therefore, according to Newton's 3rd law, forces acting on the nodes are negative.

## FORCE ELEMENT CONTRIBUTION

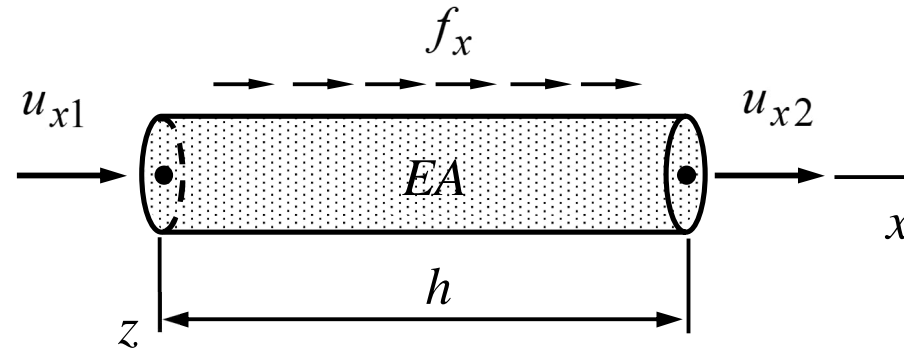
External point forces and moments are assumed to act on the joints. They are treated as elements associated with one node only. Virtual work expression is usually simplest in the structural coordinate system:

$$\delta W = \begin{Bmatrix} \delta u_X \\ \delta u_Y \\ \delta u_Z \end{Bmatrix}^T \begin{Bmatrix} F_X \\ F_Y \\ F_Z \end{Bmatrix} + \begin{Bmatrix} \delta \theta_X \\ \delta \theta_Y \\ \delta \theta_Z \end{Bmatrix}^T \begin{Bmatrix} M_X \\ M_Y \\ M_Z \end{Bmatrix}$$



Above,  $F_X, F_Y, F_Z$  and  $M_X, M_Y, M_Z$  are the given components. A rigid body can be modeled as a particle at the center of mass connected to the other joints of the body by rigid links!

## BAR ELEMENT CONTRIBUTION



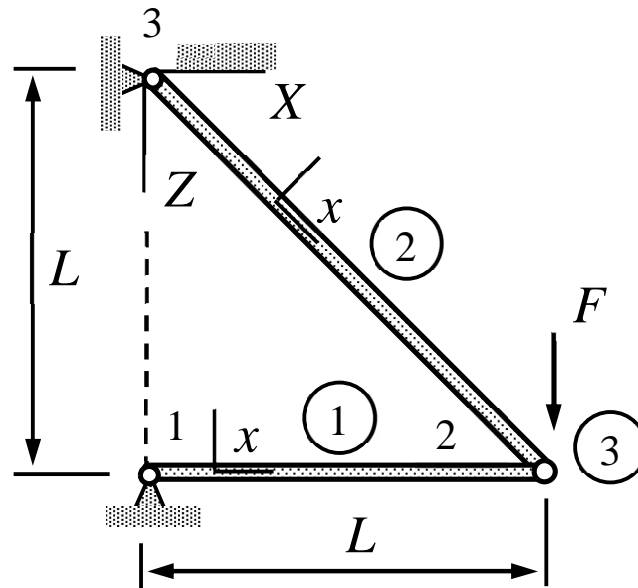
$$\delta W = - \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^T \left( \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \right), \text{ where } u_x = \mathbf{i}^T \begin{Bmatrix} u_X \\ u_Y \\ u_Z \end{Bmatrix}.$$

Element contribution in its variational form is a scalar which simplifies assembly considerably. Mathematica code of MEC-E1050 uses the variational form of the element contribution!

## DISPLACEMENT ANALYSIS; THE IMPROVED RECIPE

- Express the nodal displacements and rotations  $u_x, u_y, u_z, \theta_x, \theta_y, \theta_z$  of the material coordinate systems in terms of those in the structural coordinate system  $u_X, u_Y, u_Z, \theta_X, \theta_Y, \theta_Z$  ( $u_x = \{u_X \ u_Y \ u_Z\} \mathbf{i}$  etc.) and write down the element contributions  $\delta W = -\delta \mathbf{a}^T (\mathbf{K} \mathbf{a} - \mathbf{F})$ .
- Sum the element contributions to end up with the virtual work expression of the structure  $\delta W = \sum_{e \in E} \delta W^e$  (point forces can be considered as elements also). Re-structure to get the “standard” form  $\delta W = -\delta \mathbf{a}^T (\mathbf{K} \mathbf{a} - \mathbf{F})$ .
- Use the principle of virtual work  $\delta W = 0 \ \forall \delta \mathbf{a}$ , the fundamental lemma of variation calculus for  $\delta \mathbf{a} \in \mathbb{R}^n$ , and solve for the dofs from the system equations  $\mathbf{K} \mathbf{a} - \mathbf{F} = 0$ .

**EXAMPLE 3.5.** A *bar truss* is loaded by a point force having magnitude  $F$  as shown in the figure. Derive the equilibrium equations and determine the nodal displacements. The cross-sectional area of bar 1 is  $A$  and that for bar 2  $\sqrt{8}A$ . Young's modulus is  $E$  and weight is omitted. Use the principle of virtual work.



**Answer** 
$$\frac{EA}{L} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} u_{X2} \\ u_{Z2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ F \end{Bmatrix} = 0 \quad \Leftrightarrow \quad u_{X2} = -\frac{FL}{EA} \quad \text{and} \quad u_{Z2} = 2\frac{FL}{EA}.$$

- Element contributions  $\delta W = -\delta \mathbf{a}^T (\mathbf{K} \mathbf{a} - \mathbf{F})$  to the virtual work expression of the structure are

$$\text{Bar 1: } \delta W^1 = - \begin{Bmatrix} 0 \\ \delta u_{X2} \end{Bmatrix}^T \left( \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ u_{X2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \right) = -\frac{EA}{L} u_{X2} \delta u_{X2},$$

$$\text{Bar 2: } \delta W^2 = -\frac{1}{\sqrt{2}} \begin{Bmatrix} 0 \\ \delta u_{X2} + \delta u_{Z2} \end{Bmatrix}^T \left( \frac{E\sqrt{8}A}{\sqrt{2}L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{Bmatrix} 0 \\ u_{X2} + u_{Z2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \right) \Leftrightarrow$$

$$\delta W^2 = -\frac{EA}{L} (\delta u_{X2} + \delta u_{Z2})(u_{X2} + u_{Z2}),$$

$$\text{Force 3: } \delta W^3 = \delta u_{Z2} F.$$

- Virtual work expression is obtained as the sum of the element contributions

$$\delta W = -\frac{EA}{L} \delta u_{X2} u_{X2} - \frac{EA}{L} (\delta u_{X2} + \delta u_{Z2}) (u_{X2} + u_{Z2}) + \delta u_{Z2} F \quad \Leftrightarrow$$

$$\delta W = -\delta u_{X2} \left( 2\frac{EA}{L} u_{X2} + \frac{EA}{L} u_{Z2} \right) - \delta u_{Z2} \left( -F + \frac{EA}{L} u_{X2} + \frac{EA}{L} u_{Z2} \right) \quad \Leftrightarrow$$

$$\delta W = -\begin{Bmatrix} \delta u_{X2} \\ \delta u_{Z2} \end{Bmatrix}^T \left( \frac{EA}{L} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} u_{X2} \\ u_{Z2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ F \end{Bmatrix} \right). \quad \text{"standard" form}$$

- Principle of virtual work  $\delta W = 0 \quad \forall \delta \mathbf{a}$  and the fundamental lemma of variation calculus imply

$$\frac{EA}{L} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} u_{X2} \\ u_{Z2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ F \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix} u_{X2} \\ u_{Z2} \end{Bmatrix} = \frac{LF}{EA} \begin{Bmatrix} -1 \\ 2 \end{Bmatrix}. \quad \leftarrow$$

### 3.3 BEAM ELEMENT CONTRIBUTION

The beam element is obtained by combining the virtual work expressions of tension, torsion, and bending loading modes (*b*~bending, *t*~torsion, *s*~stretching)!

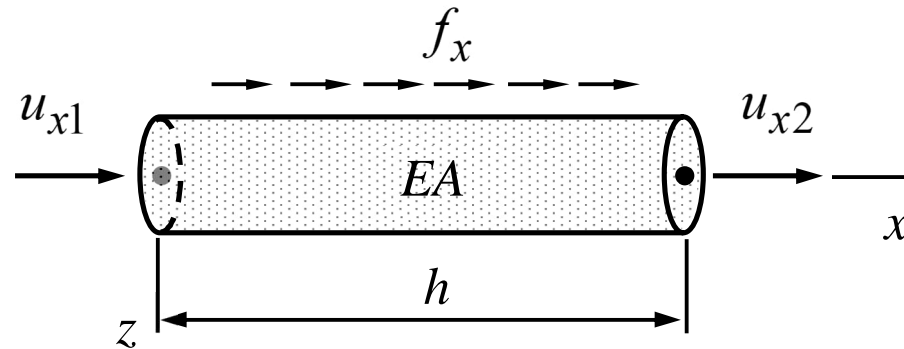
**Virtual work of a structure:**  $\delta W = \sum_{e \in E} \delta W^e$

**Virtual work of a beam:**  $\delta W^e = \sum_m \delta W_m^e = \delta W_{bxz}^e + \delta W_{bxy}^e + \delta W_{tx}^e + \delta W_{sx}^e$

In hand calculations, one starts with the expressions in the material coordinate system, expresses the nodal displacements and rotations in the structural coordinate system, and sums over the elements and loading modes. The remaining follows from the principle of virtual work  $\delta W = 0 \quad \forall \delta \mathbf{a}$ .



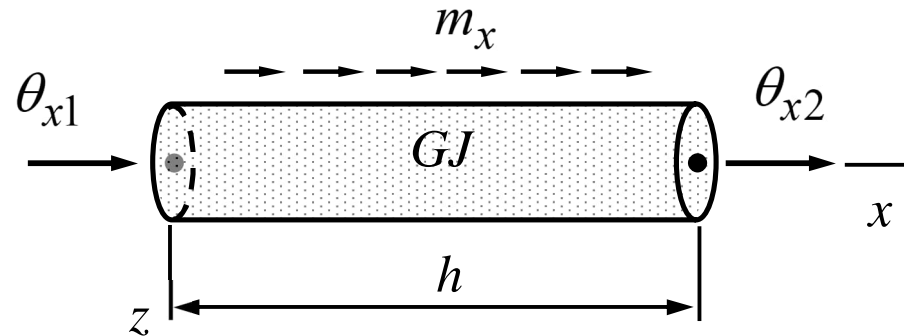
## BAR MODE



$$\delta W = - \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^T \left( \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \right), \quad u_x = \mathbf{i}^T \begin{Bmatrix} u_X \\ u_Y \\ u_Z \end{Bmatrix}, \quad \text{where } \mathbf{i} = \frac{1}{h} \begin{Bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{Bmatrix}.$$

Above,  $f_x$  and  $EA$  are assumed constants and the elements of the column matrix  $\mathbf{i}$  are the components of the unit vector  $\vec{i}$  in the structural coordinate system.

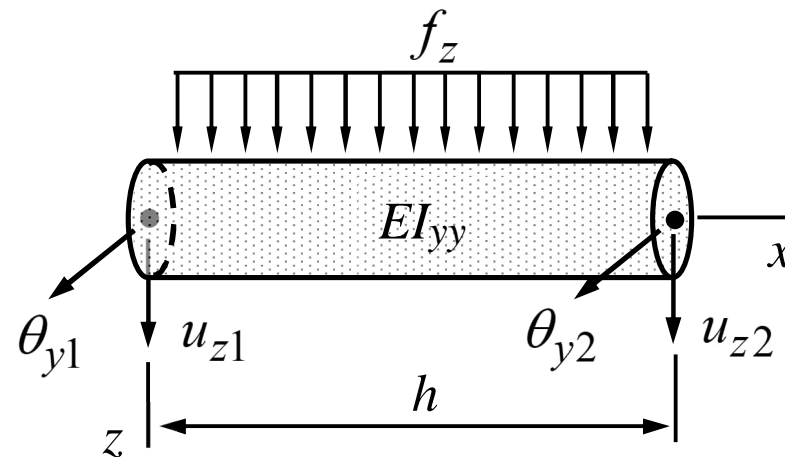
## TORSION MODE



$$\delta W = - \begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}^T \left( \frac{GJ}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix} - \frac{m_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \right), \quad \theta_x = \mathbf{i}^T \begin{Bmatrix} \theta_X \\ \theta_Y \\ \theta_Z \end{Bmatrix}, \quad \text{where } \mathbf{i} = \frac{1}{h} \begin{Bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{Bmatrix}.$$

Above,  $m_x$  and  $GJ$  are assumed constants and the elements of the column matrix  $\mathbf{i}$  are the components of the unit vector  $\vec{i}$  in the structural coordinate system.

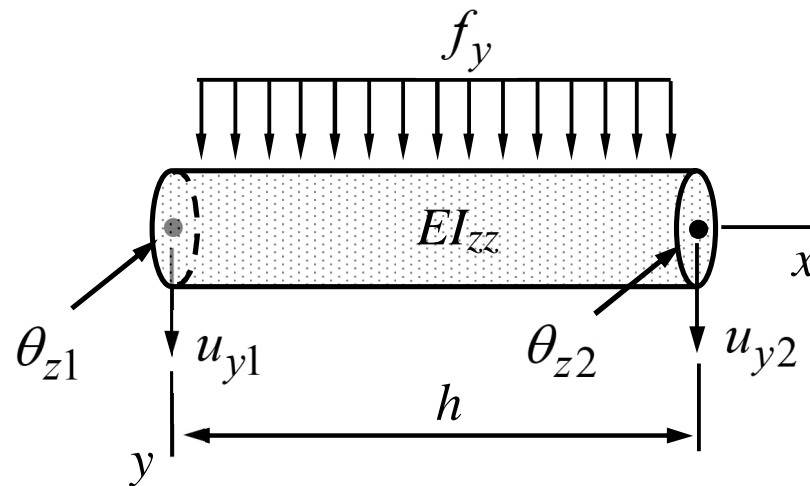
## **xz-PLANE BENDING MODE**



$$\delta W = - \begin{Bmatrix} \delta u_{z1} \\ \delta \theta_{y1} \\ \delta u_{z2} \\ \delta \theta_{y2} \end{Bmatrix}^T \left( \frac{EI_{yy}}{h^3} \begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix} - \frac{f_z h}{12} \begin{Bmatrix} 6 \\ -h \\ 6 \\ h \end{Bmatrix} \right), u_z = \mathbf{k}^T \begin{Bmatrix} u_X \\ u_Y \\ u_Z \end{Bmatrix} \text{ etc.}$$

Above,  $f_z$  and  $EI_{yy}$  are assumed to be constants and the elements of the column matrices  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{k}$  are the components of the unit vectors  $\vec{i}$ ,  $\vec{j}$  and  $\vec{k}$  in the structural coordinate system.

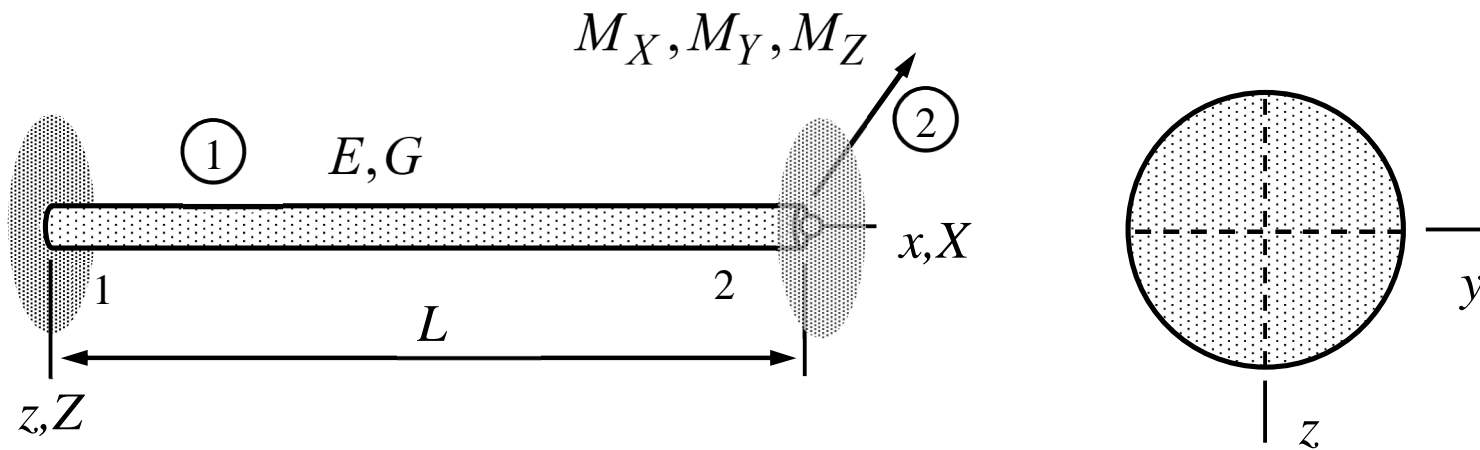
## xy-PLANE BENDING MODE



$$\delta W = - \begin{Bmatrix} \delta u_{y1} \\ \delta \theta_{z1} \\ \delta u_{y2} \\ \delta \theta_{z2} \end{Bmatrix}^T \left( \frac{EI_{zz}}{h^3} \begin{bmatrix} 12 & 6h & -12 & 6h \\ 6h & 4h^2 & -6h & 2h^2 \\ -12 & -6h & 12 & -6h \\ 6h & 2h^2 & -6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \\ u_{y2} \\ \theta_{z2} \end{Bmatrix} - \frac{f_y h}{12} \begin{Bmatrix} 6 \\ h \\ 6 \\ -h \end{Bmatrix} \right), \quad u_y = \mathbf{j}^T \begin{Bmatrix} u_X \\ u_Y \\ u_Z \end{Bmatrix} \text{ etc.}$$

Above,  $f_y$  and  $EI_{zz}$  are assumed to be constants and the column matrices  $\mathbf{I}$ ,  $\mathbf{j}$  and  $\mathbf{k}$  contain the components of the unit vectors  $\vec{i}$ ,  $\vec{j}$  and  $\vec{k}$  in the structural coordinate system.

**EXAMPLE 3.6.** Consider the beam of the figure and determine the rotation of point 2 by using a generic beam element. The  $x$ -axis coincides with the geometrical axis, the spherical joint at point 2 is frictionless, and the components of the external moment acting on point 2 are  $M_X, M_Y$ , and  $M_Z$ . The second moments of area are  $I_{yy} = I_{zz} = I$  and  $J = 2I$ .



**Answer**  $\theta_{X2} = \frac{1}{2} \frac{M_X L}{GI}$ ,  $\theta_{Y2} = \frac{1}{4} \frac{M_Y L}{EI}$ , and  $\theta_{Z2} = \frac{1}{4} \frac{M_Z L}{EI}$

- The element contribution consists of parts of the loading modes. The active degrees of freedom are rotations  $\theta_{X2}$ ,  $\theta_{Y2}$  and  $\theta_{Z2}$  (element and structural systems coincide here)

$$\delta W_{bxz}^1 = - \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta\theta_{Y2} \end{Bmatrix}^T \left( \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \theta_{Y2} \end{Bmatrix} \right) = -\delta\theta_{Y2} \frac{EI}{L^3} 4L^2 \theta_{Y2}$$

$$\delta W_{bxy}^1 = - \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta\theta_{Z2} \end{Bmatrix}^T \left( \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \theta_{Z2} \end{Bmatrix} \right) = -\delta\theta_{Z2} 4L^2 \frac{EI}{L^3} \theta_{Z2}$$

$$\delta W_{tx}^1 = - \begin{Bmatrix} 0 \\ \delta\theta_{X2} \end{Bmatrix}^T \left( \frac{2GI}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ \theta_{X2} \end{Bmatrix} \right) = -\delta\theta_{X2} \frac{G2I}{L} \theta_{X2} \quad (J = 2I)$$

$$\delta W^2 = \delta\theta_{X2}M_X + \delta\theta_{Y2}M_Y + \delta\theta_{Z2}M_Z.$$

- Virtual work expression of the structure is sum over the elements and their loading modes

$$\delta W = \sum_e \delta W^e = \delta W^1 + \delta W^2 = (\delta W_{bxz}^1 + \delta W_{bxy}^1 + \delta W_{tx}^1) + \delta W^2 \Rightarrow$$

$$\delta W = -\delta\theta_{Y2} \frac{EI}{L^3} 4L^2 \theta_{Y2} - \delta\theta_{Z2} 4L^2 \frac{EI}{L^3} \theta_{Z2} - \delta\theta_{X2} \frac{2GI}{L} \theta_{X2} + \delta\theta_{X2} M_X +$$

$$\delta\theta_{Y2} M_Y + \delta\theta_{Z2} M_Z \quad \Leftrightarrow$$

$$\delta W = -\delta\theta_{X2} \left( \frac{2GI}{L} \theta_{X2} - M_X \right) - \delta\theta_{Y2} \left( \frac{EI}{L^3} 4L^2 \theta_{Y2} - M_Y \right) - \delta\theta_{Z2} \left( 4L^2 \frac{EI}{L^3} \theta_{Z2} - M_Z \right)$$

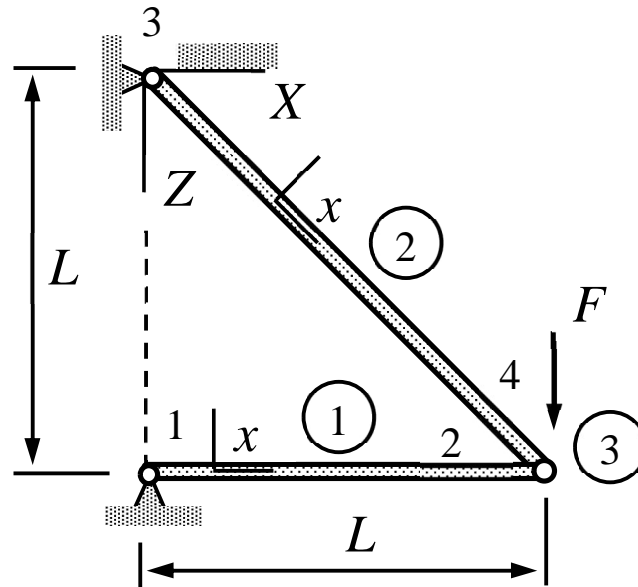
$$\delta W = - \begin{Bmatrix} \delta\theta_{X2} \\ \delta\theta_{Y2} \\ \delta\theta_{Z2} \end{Bmatrix}^T \left( \frac{I}{L} \begin{bmatrix} 2G & 0 & 0 \\ 0 & 4E & 0 \\ 0 & 0 & 4E \end{bmatrix} \begin{Bmatrix} \theta_{X2} \\ \theta_{Y2} \\ \theta_{Z2} \end{Bmatrix} - \begin{Bmatrix} M_X \\ M_Y \\ M_Z \end{Bmatrix} \right).$$

- Principle of virtual work  $\delta W = 0 \quad \forall \delta \mathbf{a}$  and the fundamental lemma of variation calculus in the form  $\delta \mathbf{a}^T \mathbf{R} = 0 \quad \forall \delta \mathbf{a} \Leftrightarrow \mathbf{R} = 0$  give

$$\frac{2I}{L} \begin{bmatrix} G & 0 & 0 \\ 0 & 2E & 0 \\ 0 & 0 & 2E \end{bmatrix} \begin{Bmatrix} \theta_{X2} \\ \theta_{Y2} \\ \theta_{Z2} \end{Bmatrix} - \begin{Bmatrix} M_X \\ M_Y \\ M_Z \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix} \theta_{X2} \\ \theta_{Y2} \\ \theta_{Z2} \end{Bmatrix} = \frac{L}{2I} \begin{Bmatrix} M_X G \\ M_Y / (2E) \\ M_Z / (2E) \end{Bmatrix}. \quad \leftarrow$$



**EXAMPLE 3.7.** A *beam truss* is loaded by a point force having magnitude  $F$  as shown in the figure. Determine the nodal displacements. The cross-sectional area of beam 1 is  $A$  and that for beam 2  $\sqrt{8}A$ . Young's modulus is  $E$  and weight is omitted. Use the principle of virtual work.



**Answer**  $u_{X2} = u_{X4} = -\frac{FL}{EA}$  and  $u_{Z2} = u_{Z4} = 2\frac{FL}{EA}$ .

- A joint is generated by using a duplicate node in the Mathematica code. The displacement components coincide at the nodes but the rotations may not:

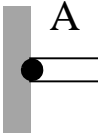

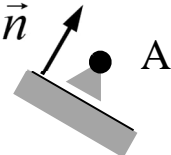
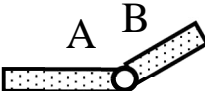
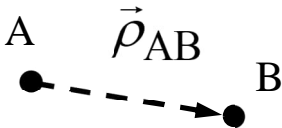
	model	properties	geometry
1	BEAM	$\{\{E, G\}, \{A, I, I\}\}$	Line[{1, 2}]
2	BEAM	$\{\{E, G\}, \{2\sqrt{2} A, I, I\}\}$	Line[{3, 4}]
3	FORCE	$\{0, 0, F\}$	Point[{4}]

	$\{X, Y, Z\}$	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	$\{0, 0, L\}$	$\{0, 0, 0\}$	$\{0, \theta_Y[1], 0\}$
2	$\{L, 0, L\}$	$\{u_X[2], 0, u_Z[2]\}$	$\{0, \theta_Y[2], 0\}$
3	$\{0, 0, 0\}$	$\{0, 0, 0\}$	$\{0, \theta_Y[3], 0\}$
4	$\{L, 0, L\}$	$\{u_X[2], 0, u_Z[2]\}$	$\{0, \theta_Y[4], 0\}$

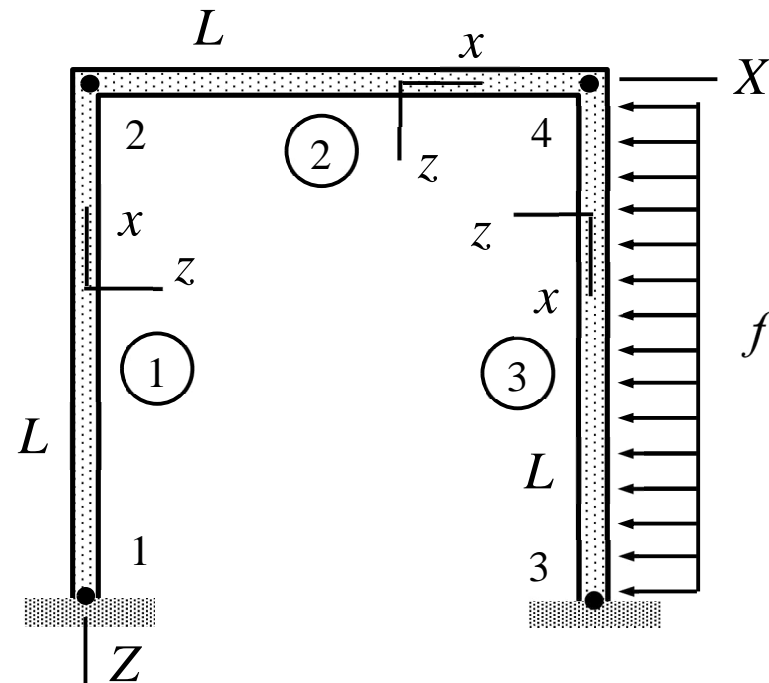
$$\left\{ u_X[2] \rightarrow -\frac{F L}{A E}, u_Z[2] \rightarrow \frac{2 F L}{A E}, \theta_Y[1] \rightarrow -\frac{2 F}{A E}, \right. \\ \left. \theta_Y[2] \rightarrow -\frac{2 F}{A E}, \theta_Y[3] \rightarrow -\frac{3 F}{2 A E}, \theta_Y[4] \rightarrow -\frac{3 F}{2 A E} \right\}$$

Solution to the displacements is the same as with the bar model!

### 3.4 CONSTRAINTS AND LINKS

name	symbol	equation
clamped		$\vec{u}_A = 0 \text{ and } \vec{\theta}_A = 0$
fixed		$\vec{u}_A = 0$
slide		$\vec{n} \cdot \vec{u}_A = 0$
joint		$\vec{u}_B = \vec{u}_A$
rigid		$\vec{u}_B = \vec{u}_A + \vec{\theta}_A \times \vec{\rho}_{AB} \text{ and } \vec{\theta}_B = \vec{\theta}_A$

**EXAMPLE 3.8.** Consider the beam truss of the figure. Determine the displacements and rotations of nodes 2 and 4. Assume that the beams are rigid in the axial directions so that the axial *strain* vanishes. Bending rigidity of the beams  $EI$  is constant.



**Answer**  $u_{X2} = u_{X4} = -\frac{3}{112} \frac{fL^4}{EI}$ ,  $\theta_{Y2} = \frac{19}{1008} \frac{fL^3}{EI}$ , and  $\theta_{Y4} = \frac{5}{1008} \frac{fL^3}{EI}$

- Only the bending in XZ-plane needs to be accounted for. The non-zero displacement and rotation components of the structure are  $u_{X2}$ ,  $\theta_{Y2}$ , and  $\theta_{Y4}$ . As the axial strain of beam 2 vanishes, axial displacements satisfy  $u_{X4} = u_{X2}$ .

$$\delta W_{bxz}^1 = - \begin{Bmatrix} 0 \\ 0 \\ \delta u_{X2} \\ \delta \theta_{Y2} \end{Bmatrix}^T \left( \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_{X2} \\ \theta_{Y2} \end{Bmatrix} \right) \quad (u_{z2} = u_{X2}, \theta_{y2} = \theta_{Y2})$$

$$\delta W_{bxz}^2 = - \begin{Bmatrix} 0 \\ \delta \theta_{Y2} \\ 0 \\ \delta \theta_{Y4} \end{Bmatrix}^T \left( \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ \theta_{Y2} \\ 0 \\ \theta_{Y4} \end{Bmatrix} \right) \quad (\theta_{y1} = \theta_{Y2}, \theta_{y2} = \theta_{Y4})$$

$$\delta W_{bxz}^2 = - \begin{Bmatrix} -\delta u_{X2} \\ \delta \theta_{Y4} \\ 0 \\ 0 \end{Bmatrix}^T \left( \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} -u_{X2} \\ \theta_{Y4} \\ 0 \\ 0 \end{Bmatrix} - \frac{fL}{12} \begin{Bmatrix} 6 \\ -L \\ 6 \\ L \end{Bmatrix} \right)$$

$(u_{z1} = -u_{X2})$

- Virtual work expression of the structure is

$$\delta W = \delta W_{bxz}^1 + \delta W_{bxz}^2 + \delta W_{bxz}^3 = - \begin{Bmatrix} \delta u_{X2} \\ \delta \theta_{Y2} \\ \delta \theta_{Y4} \end{Bmatrix}^T \left( \frac{EI}{L^3} \begin{bmatrix} 24 & 6L & 6L \\ 6L & 8L^2 & 2L^2 \\ 6L & 2L^2 & 8L^2 \end{bmatrix} \begin{Bmatrix} u_{X2} \\ \theta_{Y2} \\ \theta_{Y4} \end{Bmatrix} - \frac{fL}{12} \begin{Bmatrix} -6 \\ 0 \\ -L \end{Bmatrix} \right).$$

- Principle of virtual work  $\delta W = 0 \quad \forall \delta \mathbf{a}$  and the fundamental lemma of variation calculus give

$$\frac{EI}{L^3} \begin{bmatrix} 24 & 6L & 6L \\ 6L & 8L^2 & 2L^2 \\ 6L & 2L^2 & 8L^2 \end{bmatrix} \begin{Bmatrix} u_{X2} \\ \theta_{Y2} \\ \theta_{Y4} \end{Bmatrix} - \frac{fL}{12} \begin{Bmatrix} -6 \\ 0 \\ -L \end{Bmatrix} = 0 \Leftrightarrow \begin{Bmatrix} u_{X2} \\ \theta_{Y2} \\ \theta_{Y4} \end{Bmatrix} = \frac{fL^3}{1008EI} \begin{Bmatrix} -27L \\ 19 \\ 5 \end{Bmatrix}. \quad \leftarrow$$

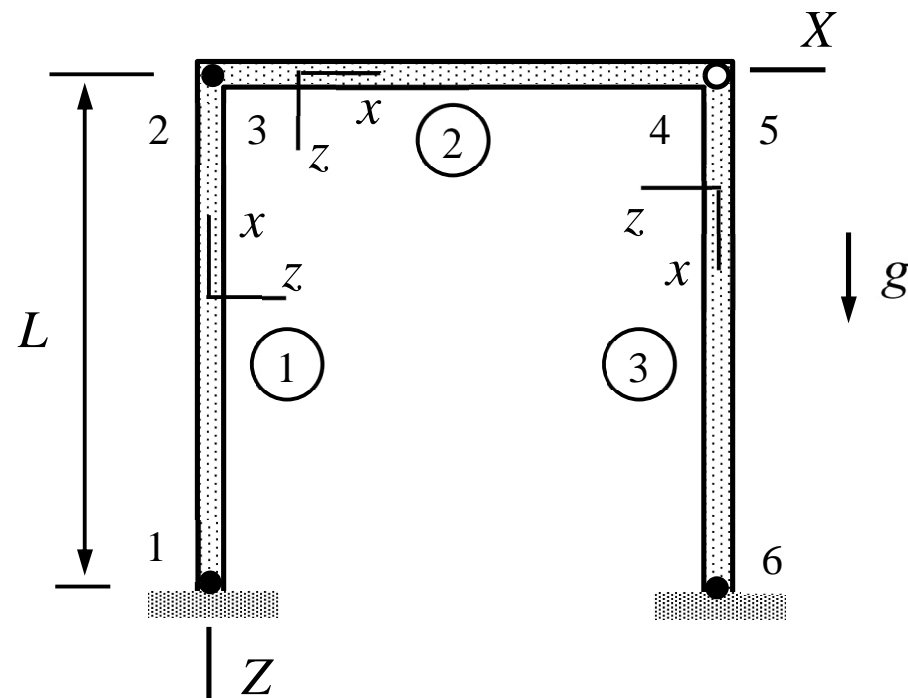
- In Mathematica code calculation, horizontal displacements of nodes 2 and 4 are forced to be same ( $u_{X4} = u_{X2}$ )

	model	properties	geometry
1	BEAM	$\{\{E, G\}, \{A, I, I\}\}$	Line[ {1, 2} ]
2	BEAM	$\{\{E, G\}, \{A, I, I\}\}$	Line[ {2, 4} ]
3	BEAM	$\{\{E, G\}, \{A, I, I\}, \{-f, 0, 0\}\}$	Line[ {4, 3} ]

	$\{X, Y, Z\}$	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	$\{0, 0, L\}$	$\{0, 0, 0\}$	$\{0, 0, 0\}$
2	$\{0, 0, 0\}$	$\{u_X[2], 0, 0\}$	$\{0, \theta_Y[2], 0\}$
3	$\{L, 0, L\}$	$\{0, 0, 0\}$	$\{0, 0, 0\}$
4	$\{L, 0, 0\}$	$\{u_X[2], 0, 0\}$	$\{0, \theta_Y[4], 0\}$

$$\left\{ u_X[2] \rightarrow -\frac{3 f L^4}{112 E I}, \theta_Y[2] \rightarrow \frac{19 f L^3}{1008 E I}, \theta_Y[4] \rightarrow \frac{5 f L^3}{1008 E I} \right\}$$

**EXAMPLE 3.9.** Consider the beam truss of the figure and displacements and rotations at nodes 2 (3) and 4 (5) modeled by using duplicate nodes. Write down the element tables by considering 4 (5) as a cylindrical frictionless joint.





- The structural parts can be joined by kinematical constraints. At nodes (black circle), displacement and rotation components coincide. At a joint (white circle), only displacement components need to coincide:

	model	properties	geometry
1	BEAM	$\{\{E, G\}, \{A, I, I\}, \{0, 0, A g \rho\}\}$	Line[ {1, 2} ]
2	BEAM	$\{\{E, G\}, \{A, I, I\}, \{0, 0, A g \rho\}\}$	Line[ {3, 4} ]
3	BEAM	$\{\{E, G\}, \{A, I, I\}, \{0, 0, A g \rho\}\}$	Line[ {5, 6} ]

	$\{X, Y, Z\}$	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	$\{0, 0, L\}$	$\{0, 0, 0\}$	$\{0, 0, 0\}$
2	$\{0, 0, 0\}$	$\{uX[2], 0, uZ[2]\}$	$\{0, \theta Y[2], 0\}$
3	$\{0, 0, 0\}$	$\{uX[2], 0, uZ[2]\}$	$\{0, \theta Y[2], 0\}$
4	$\{L, 0, 0\}$	$\{uX[4], 0, uZ[4]\}$	$\{0, \theta Y[4], 0\}$
5	$\{L, 0, 0\}$	$\{uX[4], 0, uZ[4]\}$	$\{0, \theta Y[5], 0\}$
6	$\{L, 0, L\}$	$\{0, 0, 0\}$	$\{0, 0, 0\}$

Solution to the problem is a bit lengthy so it is not given here (see the examples section of the Mathematica code).

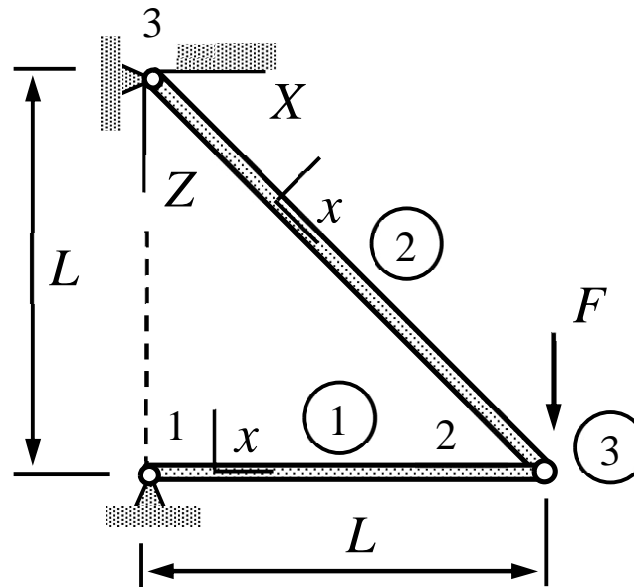
## POINT CONSTRAINT CONTRIBUTION

Displacement and rotation constraints can be enforced by using a given value in calculations. Alternatively, one may use a one-node constraint element:

$$\delta W = \begin{Bmatrix} \delta u_X \\ \delta u_Y \\ \delta u_Z \end{Bmatrix}^T \begin{Bmatrix} F_X \\ F_Y \\ F_Z \end{Bmatrix} + \begin{Bmatrix} u_X - \underline{u}_X \\ u_Y - \underline{u}_Y \\ u_Z - \underline{u}_Z \end{Bmatrix}^T \begin{Bmatrix} \delta F_X \\ \delta F_Y \\ \delta F_Z \end{Bmatrix} + \begin{Bmatrix} \delta \theta_X \\ \delta \theta_Y \\ \delta \theta_Z \end{Bmatrix}^T \begin{Bmatrix} M_X \\ M_Y \\ M_Z \end{Bmatrix} + \begin{Bmatrix} \theta_X - \underline{\theta}_X \\ \theta_Y - \underline{\theta}_Y \\ \theta_Z - \underline{\theta}_Z \end{Bmatrix}^T \begin{Bmatrix} \delta M_X \\ \delta M_Y \\ \delta M_Z \end{Bmatrix}$$

Above,  $F_X, F_Y, F_Z$  and  $M_X, M_Y, M_Z$  are considered as unknown constraint forces/moments whenever the corresponding displacement/rotation should be constrained to the value indicated by an underline. Notice that the variation of a given quantity vanishes. Explicit constraint in this form can be used to find some of the internal forces in calculations based on the virtual work expressions.

**EXAMPLE 3.10.** A *bar truss* is loaded by a point force having magnitude  $F$  as shown in the figure. The cross-sectional area of bar 1 is  $A$  and that for bar 2  $\sqrt{8}A$ . Young's modulus is  $E$  and weight is omitted. Determine the nodal displacements. Enforce the zero displacement conditions at nodes 1 and 3 by point constraints



**Answer**  $u_{X2} = -\frac{FL}{EA}$  and  $u_{Z2} = 2\frac{FL}{EA}$ .

- An alternative way to enforce displacement/rotation constraints uses a one-node constraint element:

	model	properties	geometry
1	BAR	$\{\{E\}, \{A\}\}$	Line $[\{1, 2\}]$
2	BAR	$\{\{E\}, \{2\sqrt{2} A\}\}$	Line $[\{3, 2\}]$
3	FORCE	$\{0, 0, F\}$	Point $[\{2\}]$
4	RIGID	$\{\{0, 0, 0\}, \{0, 0, 0\}\}$	Point $[\{1\}]$
5	RIGID	$\{\{0, 0, 0\}, \{0, 0, 0\}\}$	Point $[\{3\}]$

	$\{X, Y, Z\}$	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	$\{0, 0, L\}$	$\{uX[1], 0, uZ[1]\}$	$\{0, 0, 0\}$
2	$\{L, 0, L\}$	$\{uX[2], 0, uZ[2]\}$	$\{0, 0, 0\}$
3	$\{0, 0, 0\}$	$\{uX[3], 0, uZ[3]\}$	$\{0, 0, 0\}$

$$\left\{ \begin{aligned} &FX[1] \rightarrow F, FX[3] \rightarrow -F, FZ[1] \rightarrow 0, FZ[3] \rightarrow -F, uX[1] \rightarrow 0, \\ &uX[2] \rightarrow -\frac{FL}{AE}, uX[3] \rightarrow 0, uZ[1] \rightarrow 0, uZ[2] \rightarrow \frac{2FL}{AE}, uZ[3] \rightarrow 0 \end{aligned} \right\}$$