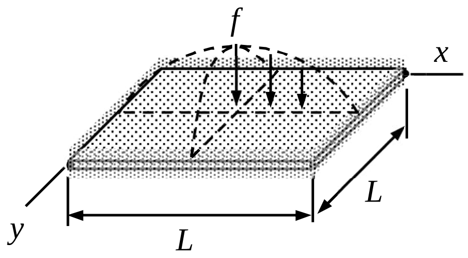
Name__________________________ Student number__________________

Assignment 4

A simply supported plate is loaded by distributed force $f_z = f\sin(\pi x/L)\sin(\pi y/L)$ as shown in the figure. Determine the displacement $w(x,y)$ by using the principle of virtual work. Consider the plate bending mode only and use approximation $w = a_0\sin(\pi x/L)\sin(\pi y/L)$ in which a_0 is a parameter. Material properties E, ν, ρ and thickness t are constants. The shape functions of the approximation satisfy, e.g.,



$$\int_0^L \sin(i\pi\frac{x}{L})\sin(j\pi\frac{x}{L})dx = \frac{L}{2}\delta_{ij} .$$

Solution template

Assuming that the material coordinate system is chosen so that the plate bending and thin slab modes decouple, virtual work densities of the Kirchhoff plate model are given by

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial^2\delta w/\partial x^2 \\ \partial^2\delta w/\partial y^2 \\ 2\partial^2\delta w/\partial x\partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12}[E]_\sigma \begin{Bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ 2\partial^2 w/\partial x\partial y \end{Bmatrix} \quad \text{and} \quad \delta w_\Omega^{\text{ext}} = \delta w f_z .$$

in which the elasticity matrix of plane stress

$$[E]_\sigma = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}.$$

Approximation to the transverse displacement and its derivatives

$$w = a_0\sin(\pi\frac{x}{L})\sin(\pi\frac{y}{L}) \quad \Rightarrow$$

$$\frac{\partial^2 w}{\partial x^2} = -a_0(\frac{\pi}{L})^2\sin(\pi\frac{x}{L})\sin(\pi\frac{y}{L}),$$

$$\frac{\partial^2 w}{\partial y^2} = -a_0(\frac{\pi}{L})^2\sin(\pi\frac{x}{L})\sin(\pi\frac{y}{L}),$$

$$\frac{\partial^2 w}{\partial x \partial y} = a_0 (\frac{\pi}{L})^2 \cos(\pi \frac{x}{L}) \cos(\pi \frac{y}{L}) .$$

When the approximation and the expression for the distributed force are substituted there, virtual work densities simplify to

$$\delta w_\Omega^{\text{int}} = -\delta a_0 \frac{t^3 E}{12(1-\nu^2)} (\frac{\pi}{L})^4 2[\sin^2(\frac{\pi x}{L}) \sin^2(\frac{\pi y}{L})(1+\nu) + (1-\nu)\cos^2(\frac{\pi x}{L})\cos^2(\frac{\pi y}{L})] a_0 ,$$

$$\delta w_\Omega^{\text{ext}} = \delta a_0 f \sin^2(\pi \frac{x}{L}) \sin^2(\pi \frac{y}{L}) .$$

Virtual work expressions are integrals of the virtual work densities over the domain occupied by the element

$$\delta W^{\text{int}} = \int_0^L \int_0^L \delta w_\Omega^{\text{int}} dxdy = -\delta a_0 4 \frac{t^3 E}{12(1-\nu^2)} (\frac{\pi}{L})^4 (\frac{L}{2})^2 a_0 ,$$

$$\delta W^{\text{ext}} = \int_0^L \int_0^L \delta w_\Omega^{\text{ext}} dxdy = \delta a_0 f (\frac{L}{2})^2 .$$

Principle of virtual work and the fundamental lemma of variation calculus give

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = -\delta a_0 [4 \frac{t^3 E}{12(1-\nu^2)} (\frac{\pi}{L})^4 (\frac{L}{2})^2 a_0 - f (\frac{L}{2})^2] = 0 \quad \forall \delta a_0 \quad \Leftrightarrow$$

$$a_0 = 3\pi^4 \frac{f L^4}{t^3 E} (1-\nu^2) .$$

Displacement

$$w(x,y) = \frac{3}{\pi^4} \frac{f L^4}{t^3 E} (1-\nu^2) \sin(\pi \frac{x}{L}) \sin(\pi \frac{y}{L}) . \quad \leftarrow$$