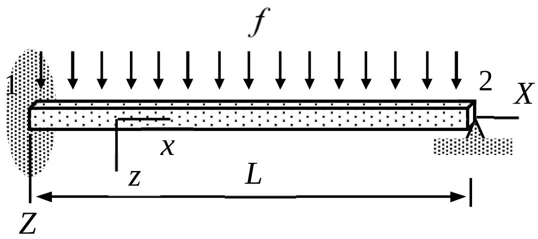
Name________________________ Student number________________

Assignment 4

Determine rotation θ_{Y2} of the bending beam shown at the support of the right end (use one element). The x-axis of the material coordinate system coincides with the neutral axis of the beam. Young's modulus E of the material and the second moment of cross-section $I_{yy} = I$ are constants. Use the virtual work density of the beam $xz-$plane bending mode and a cubic approximation to the transverse displacement.



Solution template

In the $xz-$plane problem bending problem, when x-axis is chosen to coincide with the neutral axis, virtual work densities of the Bernoulli beam model are

$$\delta w_{\Omega}^{\text{int}} = -\frac{d^2\delta w}{dx^2} EI_{yy} \frac{d^2 w}{dx^2} \quad \text{and} \quad \delta w_{\Omega}^{\text{ext}} = \delta w f_z .$$

Approximation is the first thing to be considered. The left end of the beam is clamped and the right end support allows only rotation. As only $\theta_{y2} = \theta_{Y2}$ is non-zero, approximation to w in terms of $\xi = x/L$ simplifies into the form (see the formulae collection for the cubic beam bending approximation)

$$w(\xi) = \begin{Bmatrix} (1-\xi)^2(1+2\xi) \\ L(1-\xi)^2\xi \\ (3-2\xi)\xi^2 \\ L\xi^2(\xi-1) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{z1} \\ -\theta_{y1} \\ u_{z2} \\ -\theta_{y2} \end{Bmatrix} = \begin{Bmatrix} (1-\xi)^2(1+2\xi) \\ L(1-\xi)^2\xi \\ (3-2\xi)\xi^2 \\ L\xi^2(\xi-1) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -\theta_{Y2} \end{Bmatrix} = L\xi^2(1-\xi)\theta_{Y2} \qquad \Rightarrow$$

$$w(x) = \frac{1}{L^2} x^2 (L-x)\theta_{Y2} \qquad \Rightarrow \qquad \frac{d^2 w}{dx^2} = \frac{1}{L^2}(2L-6x)\theta_{Y2} \quad \text{so}$$

$$\delta w(x) = \frac{1}{L^2} x^2 (L-x)\delta\theta_{Y2} \quad \text{and} \quad \frac{d^2\delta w}{dx^2} = \frac{1}{L^2}(2L-6x)\delta\theta_{Y2} .$$

When the approximation is substituted there, virtual work densities of the internal and external forces (external distributed force is constant) simplify to

$$\delta w_\Omega^{\text{int}} = -\frac{d^2\delta w}{dx^2} EI_{yy} \frac{d^2 w}{dx^2} = -\delta\theta_{Y2} \frac{EI}{L^4}(2L-6x)^2 \theta_{Y2},$$

$$\delta w_\Omega^{\text{ext}} = \delta w f_z = \delta\theta_{Y2} \frac{1}{L^2} x^2 (L-x) f \ .$$

Integration over the domain $\Omega =]0, L[$ gives the virtual work expressions of the internal and external forces

$$\delta W^{\text{int}} = \int_0^L \delta w_\Omega^{\text{int}} dx = -\delta\theta_{Y2} 4 \frac{EI}{L} \theta_{Y2},$$

$$\delta W^{\text{ext}} = \int_0^L \delta w_\Omega^{\text{ext}} dx = \delta\theta_{Y2} \frac{1}{12} L^2 f \ .$$

Principle of virtual work $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = 0 \quad \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus imply the solution

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = -\delta\theta_{Y2}(4\frac{EI}{L}\theta_{Y2} - \frac{1}{12}L^2 f) = 0 \quad \forall \delta\theta_{Y2} \quad \Leftrightarrow$$

$$4\frac{EI}{L}\theta_{Y2} - \frac{1}{12}L^2 f = 0 \quad \Leftrightarrow \quad \theta_{Y2} = \frac{1}{48}\frac{L^3 f}{EI} .$$