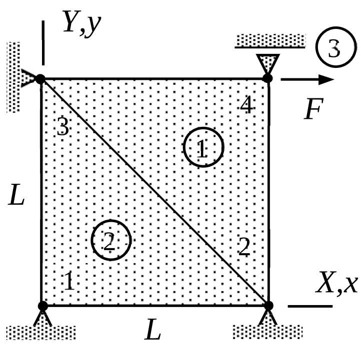
Name________________________ Student number________________

Assignment 3



A thin slab of square shape is loaded by a point force as shown. Derive the relationship between the force F and the displacement u_{X4} of its point of action. Young's modulus E, Poisson's ratio ν, and thickness of the slab t are constants. External distributed forces vanish. Assume plane stress conditions and use two linear triangle elements.

Solution template

Under the plane stress conditions, the virtual work densities (virtual works per unit area) of the thin slab model are given by

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta u/\partial y + \partial\delta v/\partial x \end{Bmatrix}^{\mathrm{T}} t[E]_{\sigma} \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix} \text{ and } \delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix} \text{ where}$$

$$[E]_{\sigma} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}.$$

All nodes of element 2 are fixed so its element contribution vanishes and it is enough to consider only element 1 and 3. Let us start with the shape functions of element 1. As two of the edges are aligned with the coordinate axes, deducing the shape function expressions is not too difficult

$$N_3 = 1 - \frac{x}{L}, \quad N_2 = 1 - \frac{y}{L}, \text{ and } \quad N_4 = 1 - N_2 - N_3 = \frac{x}{L} + \frac{y}{L} - 1.$$

Approximations to the displacement components and their derivatives with respect to x and y are

$$u(x,y) = \mathbf{N}^{\mathrm{T}}\mathbf{a} = (\frac{x}{L} + \frac{y}{L} - 1)u_{X4}, \quad \frac{\partial u}{\partial x} = \frac{1}{L}u_{X4}, \text{ and } \frac{\partial u}{\partial y} = \frac{1}{L}u_{X4},$$

$$v(x,y) = \mathbf{N}^{\mathrm{T}}\mathbf{a} = 0, \quad \frac{\partial v}{\partial x} = 0, \text{ and } \frac{\partial v}{\partial y} = 0.$$

When the approximations are substituted there, the virtual work density of thin slab model simplifies to (plane stress conditions, only the internal part is needed)

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \delta u_{X4}/L \\ 0 \\ \delta u_{X4}/L \end{Bmatrix}^{\mathrm{T}} \frac{tE}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\begin{Bmatrix} u_{X4}/L \\ 0 \\ u_{X4}/L \end{Bmatrix} = -\delta u_{X4}\frac{tE}{1-\nu^2}\frac{1}{L^2}\frac{3-\nu}{2}u_{X4} \ .$$

Integration over the domain occupied by the element gives the element contribution (notice that the integrand is constant so it is enough to multiply by the area of the triangle)

$$\delta W^1 = \int_{\Omega} \delta w_{\Omega}^{\text{int}} dA = -\delta u_{X4}\frac{1}{4}tE\frac{3-\nu}{1-\nu^2}u_{X4} .$$

Virtual work expression of the point force (element 3) follows from the definition of work

$$\delta W^3 = \delta u_{X4} F \ .$$

Virtual work expression of a structure is the sum of element contributions

$$\delta W = \delta W^1 + \delta W^3 = -\delta u_{X4}\left(\frac{1}{4}tE\frac{3-\nu}{1-\nu^2}u_{X4} - F\right).$$

Principle of virtual work $\delta W = 0 \ \ \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus in the form $\delta \mathbf{a}^{\mathrm{T}}\mathbf{R} = 0 \ \ \forall \delta \mathbf{a} \ \Leftrightarrow \ \mathbf{R} = 0$ give

$$\frac{1}{4}tE\frac{3-\nu}{1-\nu^2}u_{X4} - F = 0 \qquad \Leftrightarrow \qquad u_{X4} = \frac{4F}{Et}\frac{1-\nu^2}{3-\nu} .$$