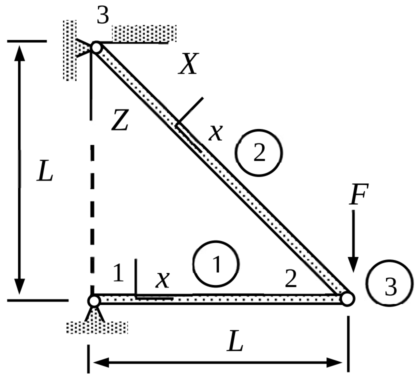
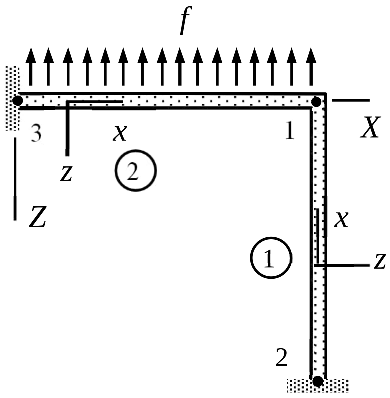
MEC-E1050 Finite Element Method in Solids, onsite exam 02.12.2024

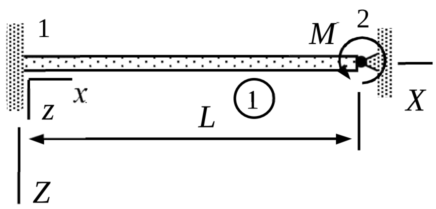
1. A *bar truss* is loaded by a point force having magnitude F as shown in the figure. Derive the equilibrium equations and determine the nodal displacements. The cross-sectional area of bar 1 is A and that for bar 2 $\sqrt{8}A$. Young's modulus is E and weight is omitted. Use the principle of virtual work.



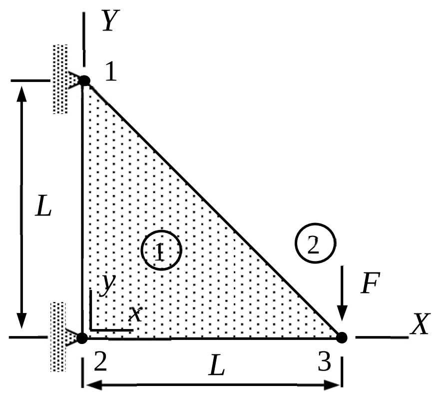
2. Determine the rotation θ_{Y1} at node 1 of the structure shown. Use two beam elements of length L. Assume that the beams are inextensible in the axial directions. Young's modulus of the material E and the second moment of area I are constants. Use the principle of virtual work.



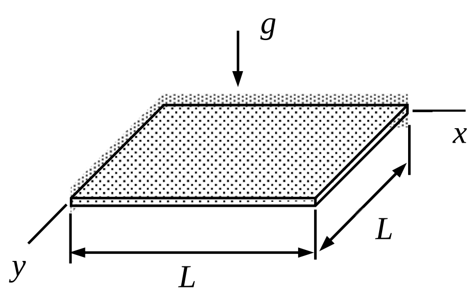
3. Determine the rotation θ_{Y2} of the beam shown at the support of the right end which allows rotation but not transverse displacement. Young's modulus E of the material and second moment of cross-section $I_{yy} = I$ are constants. Use the virtual work density of beam bending mode $\delta w_{\Omega} = -(d^2\delta w / dx^2)EI_{yy}(d^2 w / dx^2) + \delta w f_z$ and cubic approximation to the transverse displacement.



4. A thin triangular slab of thickness t is loaded by a point force at node 3. Nodes 1 and 2 are fixed. Derive the virtual work expression δW of the structure in terms of u_{X3} and u_{Y3}, and solve for the nodal displacements. Approximation is linear and material parameters E and ν are constants. Assume plane-stress conditions.



5. A Kirchhoff plate of thickness t is loaded by its own weight. The plate is clamped on the edges where $x = 0$ or $y = 0$ and free on the other two edges. Material parameters E, $\nu = 0$, and ρ are constants. Determine the parameter a_0 of the approximation $w(x, y) = a_0 x^2 y^2$. Use the principle of virtual work in form $\delta W = 0 \quad \forall \delta a_0 \in \mathbb{R}$.



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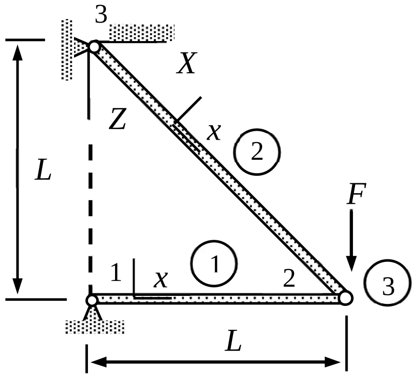


Solution

4p Element contributions $\delta W = -\delta \mathbf{a}^{\mathrm{T}}(\mathbf{K}\mathbf{a}-\mathbf{F})$ to the virtual work expression of the structure are

Bar 1: $\delta W^1 = -\begin{Bmatrix} 0 \\ \delta u_{X2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_{X2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}\right) = -\frac{EA}{L} u_{X2}\delta u_{X2}$,

Bar 2: $\delta W^2 = -\frac{1}{\sqrt{2}}\begin{Bmatrix} 0 \\ \delta u_{X2}+\delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{E\sqrt{8}A}{\sqrt{2}L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{Bmatrix} 0 \\ u_{X2}+u_{Z2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}\right) \Leftrightarrow$

$$\delta W^2 = -\frac{EA}{L}(\delta u_{X2}+\delta u_{Z2})(u_{X2}+u_{Z2}),$$

Force 3: $\delta W^3 = \delta u_{Z2} F$.

2p Virtual work expression is the sum of the element contributions

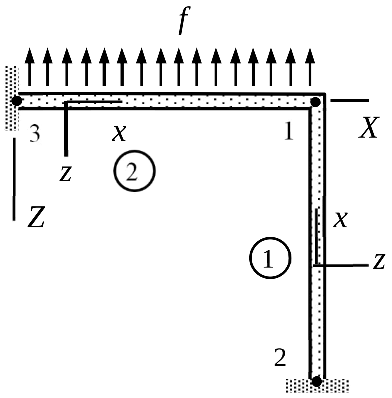
$$\delta W = -\frac{EA}{L}\delta u_{X2} u_{X2} - \frac{EA}{L}(\delta u_{X2}+\delta u_{Z2})(u_{X2}+u_{Z2}) + \delta u_{Z2} F \quad \Leftrightarrow$$

$$\delta W = -\delta u_{X2}\left(2\frac{EA}{L}u_{X2} + \frac{EA}{L}u_{Z2}\right) - \delta u_{Z2}\left(-F + \frac{EA}{L}u_{X2} + \frac{EA}{L}u_{Z2}\right) \quad \Leftrightarrow$$

$$\delta W = -\begin{Bmatrix} \delta u_{X2} \\ \delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{L}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{Z2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ F \end{Bmatrix}\right).$$

Principle of virtual work $\delta W = 0 \quad \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus imply

$$\frac{EA}{L}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{Z2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ F \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix} u_{X2} \\ u_{Z2} \end{Bmatrix} = \frac{LF}{EA}\begin{Bmatrix} -1 \\ 2 \end{Bmatrix}.$$



Determine the rotation θ_{Y1} at node 1 of the structure shown. Use two beam elements of length L. Assume that the beams are inextensible in the axial directions. Young's modulus of the material E and the second moment of area I are constants. Use the principle of virtual work.

Solution

Virtual work expression of the xz-plane bending beam element

$$\delta W = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EI_{yy}}{h^3}\begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix}\begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix} - \frac{f_z h}{12}\begin{Bmatrix} 6 \\ -h \\ 6 \\ h \end{Bmatrix}\right)$$

depends on the second moment of cross-section area I_{yy}, Young's modulus E, beam length h, and force per unit length f_z.

The displacement and rotation components of the material coordinate system need to be expressed in terms of the components of the structural system. As beams are inextensible in the axial directions, the structure has only the rotation degree of freedom θ_{Y1} and it is enough to consider bending in the $xz-$plane only.

2p Beam 1: $u_{z2} = 0$, $\theta_{y2} = 0$, $u_{z1} = 0$, $\theta_{y1} = \theta_{Y1}$, and $f_z = 0$

$$\delta W^1 = -\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta\theta_{Y1} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \theta_{Y1} \end{Bmatrix} - \frac{0\,L}{12}\begin{Bmatrix} 6 \\ -L \\ 6 \\ L \end{Bmatrix}\right) = -\delta\theta_{Y1}\frac{EI}{L^3}4L^2\theta_{Y1}$$

2p Beam 2: $u_{z3} = 0$, $\theta_{y3} = 0$, $u_{z1} = 0$, $\theta_{y1} = \theta_{Y1}$, and $f_z = -f$

$$\delta W^2 = -\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta\theta_{Y1} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \theta_{Y1} \end{Bmatrix} - \frac{-fL}{12}\begin{Bmatrix} 6 \\ -L \\ 6 \\ L \end{Bmatrix}\right) = -\delta\theta_{Y1}\left(\frac{EI}{L^3}4L^2\theta_{Y1} + \frac{fL^2}{12}\right)$$

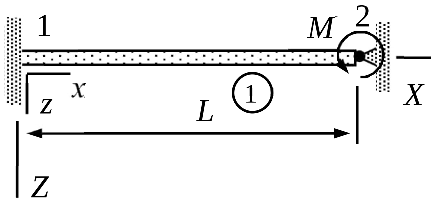
2p Virtual work expression of the structure is obtained by summing the element contributions. After that, the virtual work expression is rearranged into the standard form (similar to the virtual work expression of an element):

$$\delta W = \delta W^1 + \delta W^2 = -\delta\theta_{Y1}\frac{EI}{L^3}4L^2\theta_{Y1} - \delta\theta_{Y1}\left(\frac{EI}{L^3}4L^2\theta_{Y1} + \frac{fL^2}{12}\right) = -\delta\theta_{Y1}\left(8\frac{EI}{L}\theta_{Y1} + \frac{fL^2}{12}\right).$$

Principle of virtual work $\delta W = 0 \quad \forall \delta\mathbf{a}$ and the fundamental lemma of variation calculus imply the equilibrium equation and thereby solution to θ_{Y1}

$$8\frac{EI}{L}\theta_{Y1} + \frac{f\,L^2}{12} = 0 \quad \Leftrightarrow \quad \theta_{Y1} = -\frac{1}{96}\frac{f\,L^3}{EI}.$$

Determine the rotation θ_{Y2} of the beam shown at the support of the right end which allows rotation but not transverse displacement. Young's modulus E of the material and second moment of cross-section $I_{yy} = I$ are constants. Use the virtual work density of beam bending mode $\delta w_\Omega = -\delta(d^2w/dx^2)EI_{yy}(d^2w/dx^2) + \delta w f_z$ and cubic approximation to the transverse displacement.



Solution

In the $xz-$plane problem bending problem, when *x*-axis is chosen to coincide with the neutral axis, virtual work densities of the beam bending mode are

$$\delta w_\Omega^{\text{int}} = -\frac{d^2\delta w}{dx^2} EI_{yy} \frac{d^2 w}{dx^2} \quad \text{and} \quad \delta w_\Omega^{\text{ext}} = \delta w f_z .$$

2p Approximation is the first thing to be considered. The left end of the beam is clamped and the right end support does not allow transverse displacement. As only $\theta_{y2} = \theta_{Y2}$ is non-zero, approximation to w simplifies into the form (see the formulae collection for the cubic beam bending approximation)

$$w = \begin{Bmatrix} (1-\xi)^2(1+2\xi) \\ L(1-\xi)^2\xi \\ (3-2\xi)\xi^2 \\ L\xi^2(\xi-1) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -\theta_{Y2} \end{Bmatrix} = L(\frac{x}{L})^2(1-\frac{x}{L})\theta_{Y2} \quad \Rightarrow \quad \frac{d^2w}{dx^2} = \frac{1}{L}(2-6\frac{x}{L})\theta_{Y2} \text{ and}$$

$$\delta w = L(\frac{x}{L})^2(1-\frac{x}{L})\delta\theta_{Y2} \quad \Rightarrow \quad \frac{d^2\delta w}{dx^2} = \frac{1}{L}(2-6\frac{x}{L})\delta\theta_{Y2} .$$

3p When the approximation is substituted there, virtual work density takes the form (external distributed force vanishes)

$$\delta w_\Omega = -\frac{d^2\delta w}{dx^2} EI_{yy} \frac{d^2w}{dx^2} = -\delta\theta_{Y2}\frac{EI}{L^4}\theta_{Y2}(2L-6x)^2 .$$

Integration over the domain $\Omega =]0, L[$ gives the virtual work expressions (beam is considered as element 1)

$$\delta W^1 = \int_0^L \delta w_\Omega dx = -\delta\theta_{Y2} 4\frac{EI}{L}\theta_{Y2} .$$

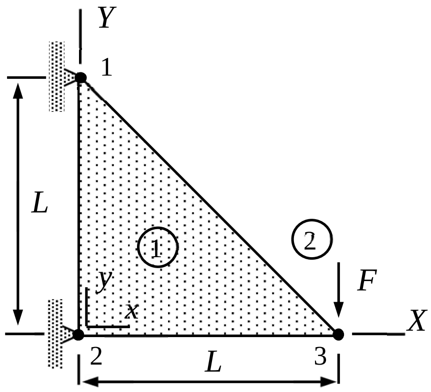
The external moment gives the contribution (element 2)

$$\delta W^2 = \delta\theta_{Y2} M .$$

1p Principle of virtual work $\delta W = \delta W^1 + \delta W^2 = 0 \quad \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus imply the solution

$$\delta W = -\delta\theta_{Y2}\left(4\frac{EI}{L}\theta_{Y2} - M\right) = 0 \quad \forall\,\delta u_{Z2} \quad \Leftrightarrow \quad 4\frac{EI}{L}\theta_{Y2} - M = 0 \quad \Leftrightarrow \quad \theta_{Y2} = \frac{1}{4}\frac{ML}{EI} \;.$$

A thin triangular slab of thickness t is loaded by a point force at node 3. Nodes 1 and 2 are fixed. Derive the virtual work expression δW of the structure in terms of u_{X3} and u_{Y3}, and solve for the nodal displacements. Approximation is linear and material parameters E and ν are constants. Assume plane stress conditions.



Solution

The virtual work densities (virtual works per unit area) of the thin slab model under the plane stress conditions

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta u/\partial y + \partial\delta v/\partial x \end{Bmatrix}^{\mathrm{T}} t[E]_\sigma \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix} \text{ and } \delta w_\Omega^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix} \text{where}$$

$$[E]_\sigma = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}$$

take into account the internal forces (stress), external forces acting on the element domain, and external forces acting on the edges. Notice that the components f_x and f_y are external forces per unit area. The forces acting on the element edges are taken into account by separate force elements.

2p Expressions of linear shape functions in material $xy-$coordinates can be deduced from the figure. Only the shape function of node 3 is actually needed:

$$N_3 = \frac{x}{L},\quad N_1 = \frac{y}{L}\text{, and } N_2 = 1 - N_1 - N_3 = 1 - \frac{x}{L} - \frac{y}{L} \quad \Rightarrow$$

$$u = N_1 0 + N_2 0 + N_3 u_{X3} = \frac{x}{L}u_{X3} \quad \Rightarrow \quad \frac{\partial u}{\partial x} = \frac{1}{L}u_{X3} \quad \text{and} \quad \frac{\partial u}{\partial y} = 0,$$

$$v = N_1 0 + N_2 0 + N_3 u_{Y3} = \frac{x}{L}u_{Y3} \quad \Rightarrow \quad \frac{\partial v}{\partial x} = \frac{1}{L}u_{Y3} \quad \text{and} \quad \frac{\partial v}{\partial y} = 0.$$

2p When the approximation is substituted there, virtual work expression of internal forces per unit area simplifies to

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \delta u_{X3} \\ 0 \\ \delta u_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{1}{L}\frac{tE}{2(1-\nu^2)}\begin{bmatrix} 2 & 2\nu & 0 \\ 2\nu & 2 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix}\frac{1}{L}\begin{Bmatrix} u_{X3} \\ 0 \\ u_{Y3} \end{Bmatrix} = -\begin{Bmatrix} \delta u_{X3} \\ \delta u_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{tE}{2L^2(1-\nu^2)}\begin{bmatrix} 2 & 0 \\ 0 & 1-\nu \end{bmatrix}\begin{Bmatrix} u_{X3} \\ u_{Y3} \end{Bmatrix}$$

As the integrand is constant, integration over the triangular domain gives

$$\delta W^{\text{int}} = \int_A \delta w_\Omega^{\text{int}} \mathrm{d}A = \delta w_\Omega^{\text{int}} \frac{L^2}{2} = -\begin{Bmatrix} \delta u_{X3} \\ \delta u_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{tE}{4(1-\nu^2)} \begin{bmatrix} 2 & 0 \\ 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} u_{X3} \\ u_{Y3} \end{Bmatrix}.$$

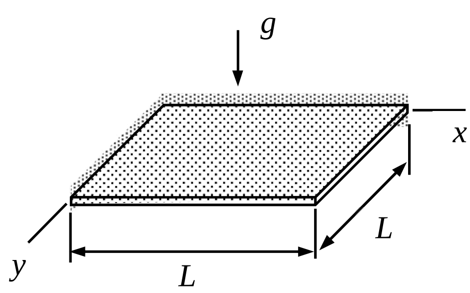
2p If also the point force is accounted for, the virtual work expression of the structure takes the form

$$\delta W = -\begin{Bmatrix} \delta u_{X3} \\ \delta u_{Y3} \end{Bmatrix}^{\mathrm{T}} \left(\frac{tE}{4(1-\nu^2)} \begin{bmatrix} 2 & 0 \\ 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} u_{X3} \\ u_{Y3} \end{Bmatrix} + \begin{Bmatrix} 0 \\ F \end{Bmatrix} \right). \leftarrow$$

Principle of virtual work $\delta W = 0 \; \forall \delta \mathrm{a}$ and the fundamental lemma of variation calculus give

$$\frac{tE}{4(1-\nu^2)} \begin{bmatrix} 2 & 0 \\ 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} u_{X3} \\ u_{Y3} \end{Bmatrix} + \begin{Bmatrix} 0 \\ F \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix} u_{X3} \\ u_{Y3} \end{Bmatrix} = -4(1+\nu) \frac{F}{tE} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}. \quad \leftarrow$$

A Kirchhoff plate of thickness t is loaded by its own weight. The plate is clamped on the edges where $x=0$ or $y=0$ and free on the other two edges. Material parameters E, $\nu=0$, and ρ are constants. Determine the parameter a_0 of the approximation $w(x,y)=a_0x^2y^2$. Use the principle of virtual work in form $\delta W=0\quad\forall\delta a_0\in\mathbb{R}$.



Solution

Virtual work densities of the plate bending mode are given by

$$\delta w_\Omega^{\text{int}}=-\begin{Bmatrix}\partial^2\delta w/\partial x^2\\ \partial^2\delta w/\partial y^2\\ 2\partial^2\delta w/(\partial x\partial y)\end{Bmatrix}^{\mathrm{T}}\frac{t^3}{12}[E]_\sigma\begin{Bmatrix}\partial^2 w/\partial x^2\\ \partial^2 w/\partial y^2\\ 2\partial^2 w/(\partial x\partial y)\end{Bmatrix}\text{ and }\ \delta w^{\text{ext}}=\delta w f_z,$$

in which f_z is the $z-$component of the distributed force per unit area and the elasticity matrix of plane stress

$$[E]_\sigma=\frac{E}{1-\nu^2}\begin{bmatrix}1&\nu&0\\ \nu&1&0\\ 0&0&(1-\nu)/2\end{bmatrix}.$$

2p Approximation to the transverse displacement and its derivatives are

$$w(x,y)=a_0x^2y^2\quad\Rightarrow\quad\frac{\partial^2 w}{\partial x^2}=2a_0y^2,\quad\frac{\partial^2 w}{\partial y^2}=2a_0x^2,\quad\text{and}\quad\frac{\partial^2 w}{\partial x\partial y}=4a_0xy.$$

When the approximation is substituted there, virtual work densities simplify to

$$\delta w_\Omega^{\text{int}}=-2\delta a_0\begin{Bmatrix}y^2\\ x^2\\ 4xy\end{Bmatrix}^{\mathrm{T}}\frac{t^3E}{12}\begin{bmatrix}1&0&0\\ 0&1&0\\ 0&0&1/2\end{bmatrix}\begin{Bmatrix}y^2\\ x^2\\ 4xy\end{Bmatrix}2a_0=-\delta a_0\frac{t^3E}{3}(y^4+x^4+8x^2y^2)a_0,$$

$$\delta w_\Omega^{\text{ext}}=\delta w f_z=\delta a_0x^2y^2\rho gt.$$

2p Virtual work expressions are integrals of the virtual work densities over the mathematical solution domain (here mid-plane)

$$\delta W^{\text{int}}=\int_0^L\int_0^L\delta w_\Omega^{\text{int}}dxdy=-\delta a_0\frac{t^3E}{3}L^6\frac{58}{45}a_0,$$

$$\delta W^{\text{ext}}=\int_0^L\int_0^L\delta w_\Omega^{\text{ext}}dxdy=\delta a_0\frac{1}{9}L^6\rho gt.$$

2p Principle of virtual work $\delta W=0\ \forall\delta a$ and the fundamental lemma of variation calculus give

$$\delta W = -\delta a_0\left(\frac{t^3 E}{3} L^6 \frac{58}{45} a_0 - \frac{1}{9} L^6 \rho g t\right) = 0 \quad \forall \delta a_0 \quad \Leftrightarrow \quad \frac{t^3 E}{3} L^6 \frac{58}{45} a_0 - \frac{1}{9} L^6 \rho g t = 0 \quad \Leftrightarrow$$

$$a_0 = \frac{15}{58} \frac{\rho g}{t^2 E} \ .$$