MEC-E1050 Finite Element Method in Solids, Schedule 2024

Week	Mon	Tue	Wed	Thu	Fri	Sun
Orientation						
43	**12:15-13:30** Introduction 1 (R008/213a) **13:30-14:00** Lecture assignment 1 (R008/213a) **16:00-17:00** Calculation hours (Zoom) **23:55 DL** (MyCourses)	**14:15-15:30** Introduction 2 (R008/216) **15:30-16:00** Lecture assignment 2 (R008/216) **16:00-17:00** Calculation hours (Zoom) **23:55 DL** (MyCourses)	Experiment 9-12, 13-16 Puumiehenkuja 5L	**14:15-14:45** Core of load carrying / Konecranes Jari Kaiturinmäki (R008/216) **15:00-15:45** Mathematica FE-code (R008/216)	**9:15-11:00** Calculation hours (R008/216)	**23:55 DL** Modelling assignments 3,4,5 (MyCourses)
Lectures & exercises						
44	**12:15-13:30** Displacement analysis 1 (R008/213a) **13:30-14:00** Lecture assignment 1 (R008/213a) **16:00-17:00** Calculation hours (Zoom) **23:55 DL** (MyCourses)	**14:15-15:30** Displacement analysis 2 (R008/216) **15:30-16:00** Lecture assignment 2 (R008/216) **16:00-17:00** Calculation hours (Zoom) **23:55 DL** (MyCourses)		**14:15-15:30** Calculation examples (R008/216)	**9:15-11:00** Calculation hours (R008/216)	**23:55 DL** Home assignments 3,4,5 (MyCourses)
45	**12:15-13:30** Bar and beam structures 1 (R008/213a) **13:30-14:00** Lecture assignment 1 (R008/213a) **16:00-17:00** Calculation hours (Zoom) **23:55 DL** (MyCourses)	**14:15-15:30** Bar and beam structures 2 (R008/216) **15:30-16:00** Lecture assignment 2 (R008/216) **16:00-17:00** Calculation hours (Zoom) **23:55 DL** (MyCourses)		**14:15-15:30** Calculation examples (R008/216)	**9:15-11:00** Calculation hours (R008/216)	**23:55 DL** Home assignments 3,4,5 (MyCourses)

46	**12:15-13:30** Element contributions 1 (R008/213a) **13:30-14:00** Lecture assignment 1 (R008/213a) **16:00-17:00** Calculation hours (Zoom) **23:55 DL** (MyCourses)	**14:15-15:30** Element contributions 2 (R008/216) **15:30-16:00** Lecture assignment 2 (R008/216) **16:00-17:00** Calculation hours (Zoom) **23:55 DL** (MyCourses)		**08:15-9:30** Calculation examples (R008/216)	**9:15-11:00** Calculation hours (R008/216)	**23:55 DL** Assignments 3,4,5 (MyCourses)
47	**12:15-13:30** Virtual work density 1 (R008/213a) **13:30-14:00** Lecture assignment 1 (R008/213a) **16:00-17:00** Calculation hours (Zoom) **23:55 DL** (MyCourses)	**14:15-15:30** Virtual work density 2 (R008/216) **15:30-16:00** Lecture assignment 2 (R008/216) **16:00-17:00** Calculation hours (Zoom) **23:55 DL** (MyCourses)		**14:15-15:30** Calculation examples (R008/216)	**9:15-11:00** Calculation hours (R008/216)	**23:55 DL** Assignments 3,4,5 (MyCourses)
48	**12:15-13:30** Beam and plate models 1 (R008/213a) **13:30-14:00** Lecture assignment 1 (R008/213a) **16:00-17:00** Calculation hours (Zoom) **23:55 DL** (MyCourses)	**14:15-15:30** Beam and plate models 2 (R008/216) **15:30-16:00** Lecture assignment 2 (R008/216) **16:00-17:00** Calculation hours (Zoom) **23:55 DL** (MyCourses)		**14:15-15:30** Calculation examples (R008/216)	**9:15-11:00** Calculation hours (R008/216)	**23:55 DL** Assignments 3,4,5 (MyCourses)
Exams						
49	**13:00-17:00** Final exam (R008/215) (MyCourses)					

MEC-E1050 Finite Element Method in Solids; Formulae

LINEAR ELASTICITY

Coordinate systems: $\begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} = \begin{bmatrix} i_X & i_Y & i_Z \\ \hdashline j_X & j_Y & j_Z \\ \hdashline k_X & k_Y & k_Z \end{bmatrix} \begin{Bmatrix} \vec{I} \\ \vec{J} \\ \vec{K} \end{Bmatrix} = \{\mathbf{i} \quad \mathbf{j} \quad \mathbf{k}\}^{\mathrm{T}} \begin{Bmatrix} \vec{I} \\ \vec{J} \\ \vec{K} \end{Bmatrix}$

Strain-stress: $\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix}, \quad \begin{Bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \frac{1}{G} \begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix}, \quad G = \frac{E}{2(1+\nu)}$ or

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} \equiv [E] \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix}, \quad \begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = G \begin{Bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

$$[E]_\sigma = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}, \quad [E]_\varepsilon = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix}$$

Strain-displacement: $\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} = \begin{Bmatrix} \partial u_x / \partial x \\ \partial u_y / \partial y \\ \partial u_z / \partial z \end{Bmatrix}, \quad \begin{Bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{Bmatrix} \partial u_x / \partial y + \partial u_y / \partial x \\ \partial u_y / \partial z + \partial u_z / \partial y \\ \partial u_z / \partial x + \partial u_x / \partial z \end{Bmatrix}$

ELEMENT CONTRIBUTION (constant load)

Bar (axial): $\begin{Bmatrix} F_{x1} \\ F_{x2} \end{Bmatrix} = \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

$$\begin{Bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \end{Bmatrix} = \frac{EA}{h} \begin{bmatrix} \mathbf{i}\,\mathbf{i}^{\mathrm{T}} & -\mathbf{i}\,\mathbf{i}^{\mathrm{T}} \\ -\mathbf{i}\,\mathbf{i}^{\mathrm{T}} & \mathbf{i}\,\mathbf{i}^{\mathrm{T}} \end{bmatrix} \begin{Bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{Bmatrix} - \frac{f_x h}{2} \begin{Bmatrix} \mathbf{i} \\ \mathbf{i} \end{Bmatrix}, \text{ in which } \mathbf{i} = \frac{1}{h} \begin{Bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{Bmatrix}$$

Bar (torsion): $\begin{Bmatrix} M_{x1} \\ M_{x2} \end{Bmatrix} = \frac{GI_{rr}}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix} - \frac{m_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

Beam (*xz*): $\begin{Bmatrix} F_{z1} \\ M_{y1} \\ F_{z2} \\ M_{y2} \end{Bmatrix} = \frac{EI_{yy}}{h^3} \begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix} - \frac{f_z h}{12} \begin{Bmatrix} 6 \\ -h \\ 6 \\ h \end{Bmatrix}$

Point loads: $\begin{Bmatrix} F_{X1} \\ F_{Y1} \\ F_{Z1} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{Y1} \\ u_{Z1} \end{Bmatrix} - \begin{Bmatrix} F_X \\ F_Y \\ F_Z \end{Bmatrix}, \quad \begin{Bmatrix} M_{X1} \\ M_{Y1} \\ M_{Z1} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \theta_{X1} \\ \theta_{Y1} \\ \theta_{Z1} \end{Bmatrix} - \begin{Bmatrix} M_X \\ M_Y \\ M_Z \end{Bmatrix}$

PRINCIPLE OF VIRTUAL WORK

$$\delta W = \delta W^{\text{ext}} + \delta W^{\text{int}}, \ \delta W = \sum\nolimits_{e \in E} \ \delta W^e = 0 \ \ \forall \delta \mathbf{a}, \ \delta W = \int_\Omega \ \delta w d\Omega$$

Bar: $\delta w^{\text{int}} = -\dfrac{d\delta u}{dx} EA \dfrac{du}{dx}, \ \delta w^{\text{ext}} = \delta u f_x$

Torsion: $\delta w^{\text{int}} = -\dfrac{d\delta\phi}{dx} GI_{rr} \dfrac{d\phi}{dx}, \ \delta w^{\text{ext}} = \delta\phi m_x$

Beam bending (*xz*-plane): $\delta w^{\text{int}} = -\dfrac{d^2\delta w}{dx^2} EI_{yy} \dfrac{d^2 w}{dx^2}, \ \delta w^{\text{ext}} = \delta w f_z$

Beam bending (*xy*-plane): $\delta w^{\text{int}} = -\dfrac{d^2\delta v}{dx^2} EI_{zz} \dfrac{d^2 v}{dx^2}, \ \delta w^{\text{ext}} = \delta v f_y$

Beam (Bernoulli):

$$\delta w^{\text{int}} = -\begin{Bmatrix} d\delta u / dx \\ d^2 \delta v / dx^2 \\ d^2 \delta w / dx^2 \end{Bmatrix}^{\mathrm{T}} E \begin{bmatrix} A & -S_z & -S_y \\ -S_z & I_{zz} & I_{zy} \\ -S_y & I_{yz} & I_{yy} \end{bmatrix} \begin{Bmatrix} du / dx \\ d^2 v / dx^2 \\ d^2 w / dx^2 \end{Bmatrix} - \frac{d\delta\phi}{dx} GI_{rr} \frac{d\phi}{dx},$$

$$\delta w^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} + \begin{Bmatrix} \delta\phi \\ -d\delta w / dx \\ d\delta v / dx \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} -S_y f_y + S_z f_z \\ S_y f_x \\ -S_z f_z \end{Bmatrix}$$

Thin slab (plane-stress):

$$\delta w^{\text{int}} = -\begin{Bmatrix} \partial\delta u / \partial x \\ \partial\delta v / \partial y \\ \partial\delta u / \partial y + \partial\delta v / \partial x \end{Bmatrix}^{\mathrm{T}} t[E]_\sigma \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix}, \ \delta w^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix}$$

Thin slab (plane-strain):

$$\delta w^{\text{int}} = -\begin{Bmatrix} \partial\delta u / \partial x \\ \partial\delta v / \partial y \\ \partial\delta u / \partial y + \partial\delta v / \partial x \end{Bmatrix}^{\mathrm{T}} t[E]_\varepsilon \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix}, \ \delta w^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix}$$

Kirchhoff plate:

$$\delta w^{\mathrm{int}}=-\begin{Bmatrix}\partial^2\delta w/\partial x^2\\ \partial^2\delta w/\partial y^2\\ 2\partial^2\delta w/(\partial x\partial y)\end{Bmatrix}^{\mathrm{T}}\frac{t^3}{12}[E]_\sigma\begin{Bmatrix}\partial^2 w/\partial x^2\\ \partial^2 w/\partial y^2\\ 2\partial^2 w/(\partial x\partial y)\end{Bmatrix},\ \delta w^{\mathrm{ext}}=\delta w f_z$$

Reissner-Mindlin plate:

$$\delta w^{\mathrm{int}}=-\begin{Bmatrix}-\partial\delta\theta/\partial x\\ \partial\delta\phi/\partial y\\ \partial\delta\phi/\partial x-\partial\delta\theta/\partial y\end{Bmatrix}^{\mathrm{T}}\frac{t^3}{12}[E]_\sigma\begin{Bmatrix}-\partial\theta/\partial x\\ \partial\phi/\partial y\\ \partial\phi/\partial x-\partial\theta/\partial y\end{Bmatrix}-\begin{Bmatrix}\partial\delta w/\partial y-\delta\phi\\ \partial\delta w/\partial x+\delta\theta\end{Bmatrix}^{\mathrm{T}}tG\begin{Bmatrix}\partial w/\partial y-\phi\\ \partial w/\partial x+\theta\end{Bmatrix},$$

$$\delta w^{\mathrm{ext}}=\delta w f_z$$

Body: $\delta w^{\mathrm{int}}=-\begin{Bmatrix}\delta\varepsilon_{xx}\\ \delta\varepsilon_{yy}\\ \delta\varepsilon_{zz}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\end{Bmatrix}-\begin{Bmatrix}\delta\gamma_{xy}\\ \delta\gamma_{yz}\\ \delta\gamma_{zx}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\sigma_{xy}\\ \sigma_{yz}\\ \sigma_{zx}\end{Bmatrix},\ \delta w^{\mathrm{ext}}=\begin{Bmatrix}\delta u\\ \delta v\\ \delta w\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}f_x\\ f_y\\ f_z\end{Bmatrix}$ or

$$\delta w^{\mathrm{int}}=-\begin{Bmatrix}\partial\delta u/\partial x\\ \partial\delta v/\partial y\\ \partial\delta w/\partial z\end{Bmatrix}^{\mathrm{T}}[E]\begin{Bmatrix}\partial u/\partial x\\ \partial v/\partial y\\ \partial w/\partial z\end{Bmatrix}-\begin{Bmatrix}\partial\delta u/\partial y+\partial\delta v/\partial x\\ \partial\delta v/\partial z+\partial\delta w/\partial y\\ \partial\delta w/\partial x+\partial\delta u/\partial z\end{Bmatrix}^{\mathrm{T}}G\begin{Bmatrix}\partial u/\partial y+\partial v/\partial x\\ \partial v/\partial z+\partial w/\partial y\\ \partial w/\partial x+\partial u/\partial z\end{Bmatrix}$$

APPROXIMATIONS (some) $u=\mathbf{N}^{\mathrm{T}}\mathbf{a}$, $\xi=\dfrac{x}{h}$

Quadratic line: $\mathbf{N}=\begin{Bmatrix}N_1\\ N_2\\ N_3\end{Bmatrix}=\begin{Bmatrix}1-3\xi+2\xi^2\\ 4\xi(1-\xi)\\ \xi(2\xi-1)\end{Bmatrix}$, $\mathbf{a}=\begin{Bmatrix}u_{x1}\\ u_{x2}\\ u_{x3}\end{Bmatrix}$ (bar)

Cubic line: $\mathbf{N}=\begin{Bmatrix}N_{10}\\ N_{11}\\ \hdashline N_{20}\\ N_{21}\end{Bmatrix}=\begin{Bmatrix}(1-\xi)^2(1+2\xi)\\ h(1-\xi)^2\xi\\ \hdashline (3-2\xi)\xi^2\\ h\xi^2(\xi-1)\end{Bmatrix}$, $\mathbf{a}=\begin{Bmatrix}u_{10}\\ u_{11}\\ \hdashline u_{20}\\ u_{21}\end{Bmatrix}(=\begin{Bmatrix}u_{z1}\\ -\theta_{y1}\\ \hdashline u_{z2}\\ -\theta_{y2}\end{Bmatrix})$ (beam bending)

Linear: $\mathbf{N}=\begin{bmatrix}1 & 1\\ x_1 & x_2\end{bmatrix}^{-1}\begin{Bmatrix}1\\ x\end{Bmatrix}$, $\mathbf{N}=\begin{bmatrix}1 & 1 & 1\\ x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\end{bmatrix}^{-1}\begin{Bmatrix}1\\ x\\ y\end{Bmatrix}$, $\mathbf{N}=\begin{bmatrix}1 & 1 & 1 & 1\\ x_1 & x_2 & x_3 & x_4\\ y_1 & y_2 & y_3 & y_4\\ z_1 & z_2 & z_3 & z_4\end{bmatrix}^{-1}\begin{Bmatrix}1\\ x\\ y\\ z\end{Bmatrix}$

VIRTUAL WORK EXPRESSIONS

Force: $\delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{Xi} \\ \delta u_{Yi} \\ \delta u_{Zi} \end{Bmatrix}^{\text{T}} \begin{Bmatrix} F_{Xi} \\ F_{Yi} \\ F_{Zi} \end{Bmatrix} + \begin{Bmatrix} \delta\theta_{Xi} \\ \delta\theta_{Yi} \\ \delta\theta_{Zi} \end{Bmatrix}^{\text{T}} \begin{Bmatrix} M_{Xi} \\ M_{Yi} \\ M_{Zi} \end{Bmatrix}$

Bar: $\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\text{T}} \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix}$, $\delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\text{T}} \frac{f_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

Torsion: $\delta W^{\text{int}} = -\begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}^{\text{T}} \frac{GI_{rr}}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix}$, $\delta W^{\text{ext}} = \begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}^{\text{T}} \frac{m_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$)

Beam bending (*xz*-plane):

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\text{T}} \frac{EI_{yy}}{h^3} \begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix}, \quad \delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\text{T}} \frac{f_z h}{12} \begin{Bmatrix} 6 \\ -h \\ 6 \\ h \end{Bmatrix}$$

Beam bending (*xy*-plane):

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{y1} \\ \delta\theta_{z1} \\ \delta u_{y2} \\ \delta\theta_{z2} \end{Bmatrix}^{\text{T}} \frac{EI_{zz}}{h^3} \begin{bmatrix} 12 & 6h & -12 & 6h \\ 6h & 4h^2 & -6h & 2h^2 \\ -12 & -6h & 12 & -6h \\ 6h & 2h^2 & -6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \\ u_{y2} \\ \theta_{z2} \end{Bmatrix}, \quad \delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{y1} \\ \delta\theta_{z1} \\ \delta u_{y2} \\ \delta\theta_{z2} \end{Bmatrix}^{\text{T}} \frac{f_y h}{12} \begin{Bmatrix} 6 \\ h \\ 6 \\ -h \end{Bmatrix})$$

CONSTRAINTS

Frictionless contact: $\vec{n} \cdot \vec{u}_{\text{A}} = 0$

Joint: $\vec{u}_{\text{B}} = \vec{u}_{\text{A}}$

Rigid body (link): $\vec{u}_{\text{B}} = \vec{u}_{\text{A}} + \vec{\theta}_{\text{A}} \times \vec{\rho}_{\text{AB}}$, $\vec{\theta}_{\text{B}} = \vec{\theta}_{\text{A}}$.

MEC-E1050 Finite Element Method in Solids; Mathematica

“Structure is a collection of *elements* (earlier structural parts) connected by *nodes* (earlier connection points)“. Displacement of the structure is defined by nodal translations and rotations of which some are known and some unknown.”

Structure

$prb = \{ele, fun\}$ where

$ele = \{prt_1, prt_2, \ldots\}$ .... elements

$fun = \{val_1, val_2, \ldots\}$ .... nodes

Elements

$prt = \{typ, pro, geo\}$ where

$typ =$ BEAM | PLATE | SOLID | RIGID |…| .... model

$pro = \{p_1, p_2, \ldots, p_n\}$ ....properties

$geo =$ Point[$\{n_1\}$] | Line[$\{n_1, n_2\}$] | Polygon[$\{n_1, n_2, n_3\}$] | … | ....geometry

Nodes

$val = \{crd, tra, rot\}$ where

$crd = \{X, Y, Z\}$ ....structural coordinates

$tra = \{u_X, u_Y, u_Z\}$ .... translation components

$rot = \{\theta_X, \theta_Y, \theta_Z\}$ ....rotation components

Elements

Constraint

{JOINT,{}|{{$\underline{u}_X, \underline{u}_Y, \underline{u}_Z$}}, Point[{$n_1$}]} ....displacement constraint

{JOINT,{}, Line[{n_1, n_2}]} ....displacement constraint

{RIGID,{}|{{$\underline{u}_X, \underline{u}_Y, \underline{u}_Z$},{$\underline{\theta}_X, \underline{\theta}_Y, \underline{\theta}_Z$}}, Point[{$n_1$}]} ....displacement/rotation constraint

{RIGID,{}, Line[{n_1, n_2}]} ....rigid constraint

{SLIDER,{n_X, n_Y, n_Z}, Point[{n_1}]} .... slider constraint

Force

{FORCE,{F_X, F_Y, F_Z}, Point[{n_1}]} .... point force

{FORCE,{$F_X, F_Y, F_Z, M_X, M_Y, M_Z$}, Point[{$n_1$}]} ....point load

{FORCE,{f_X, f_Y, f_Z}, Line[{n_1, n_2}]} ....distributed force

$\{\mathrm{FORCE},\{f_X, f_Y, f_Z\},\mathrm{Polygon}[\{n_1,n_2,n_3\}]\}$ ....distributed force

Beam model

$\{\mathrm{BAR},\{\{E\},\{A\},\{f_X, f_Y, f_Z\}\},\mathrm{Line}[\{n_1,n_2\}]\}$ ....bar mode

$\{\mathrm{TORSION},\{\{G\},\{J\},\{\{m_X,m_Y,m_Z\}\}\},\mathrm{Line}[\{n_1,n_2\}]\}$ .... torsion mode

$\{\mathrm{BEAM},\{\{E,G\},\{A,I_{yy},I_{zz}\},\{f_X, f_Y, f_Z\}\},\mathrm{Line}[\{n_1,n_2\}]\}$ ....beam

$\{\mathrm{BEAM},\{\{E,G\},\{A,I_{yy},I_{zz},\{j_X, j_Y, j_Z\}\},\{f_X, f_Y, f_Z\}\},\mathrm{Line}[\{n_1,n_2\}]\}$ ....beam

Plate model

$\{\mathrm{PLANE},\{\{E,\nu\},\{t\},\{f_X, f_Y, f_Z\}\},\mathrm{Polygon}[\{n_1,n_2,n_3\}]\}$ .... thin slab mode

$\{\mathrm{PLANE},\{\{E,\nu\},\{t\},\{f_X, f_Y, f_Z\}\},\mathrm{Polygon}[\{n_1,n_2,n_3,n_4\}]\}$ .... thin slab mode

$\{\mathrm{PLATE},\{\{E,\nu\},\{t\},\{f_X, f_Y, f_Z\}\},\mathrm{Polygon}[\{n_1,n_2,n_3\}]\}$ ....plate

Solid model

$\{\mathrm{SOLID},\{\{E,\nu\},\{f_X, f_Y, f_Z\}\},\mathrm{Tetrahedron}[\{n_1,n_2,n_3,n_4\}]\}$ ....solid

$\{\mathrm{SOLID},\{\{E,\nu\},\{f_X, f_Y, f_Z\}\},\mathrm{Hexahedron}[\{n_1,n_2,n_3,n_4,n_5,n_6,n_7,n_8\}]\}$ ....solid

$\{\mathrm{SOLID},\{\{E,\nu\},\{f_X, f_Y, f_Z,m_X,m_Y,m_Z,\}\},\mathrm{Tetrahedron}[\{n_1,n_2,n_3,n_4\}]\}$ ....solid

Operations

$prb = \mathrm{REFINE}[prb]$ .... refine structure representation

$\mathrm{Out} = \mathrm{FORMATTED}[prb]$ ....display problem definition

$\mathrm{Out} = \mathrm{STANDARDFORM}[prb]$ ....display virtual work expression

$\mathrm{sol} = \mathrm{SOLVE}[prb]$ ....solve the unknowns

MEC-E1050

FINITE ELEMENT METHOD IN SOLIDS 2024

Week 43-0

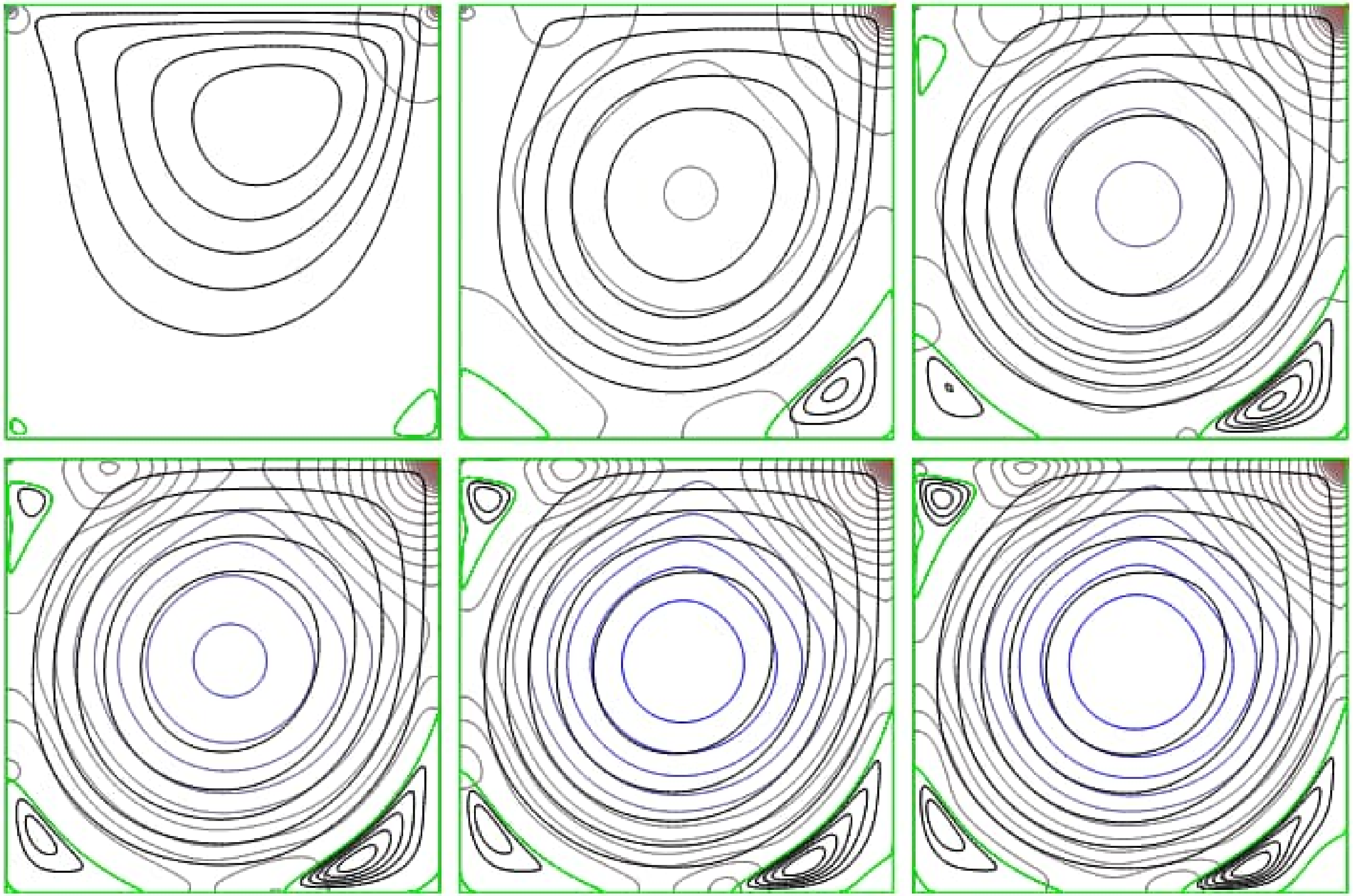
WHY FINITE ELEMENTS AND ITS THEORY?

Design of machines and structures: Solution to stress or displacement by analytical method is often impossible due to complex geometry, heterogeneous material etc. Lack of the “exact solution” to an “approximate problem” is not an issue in engineering work.

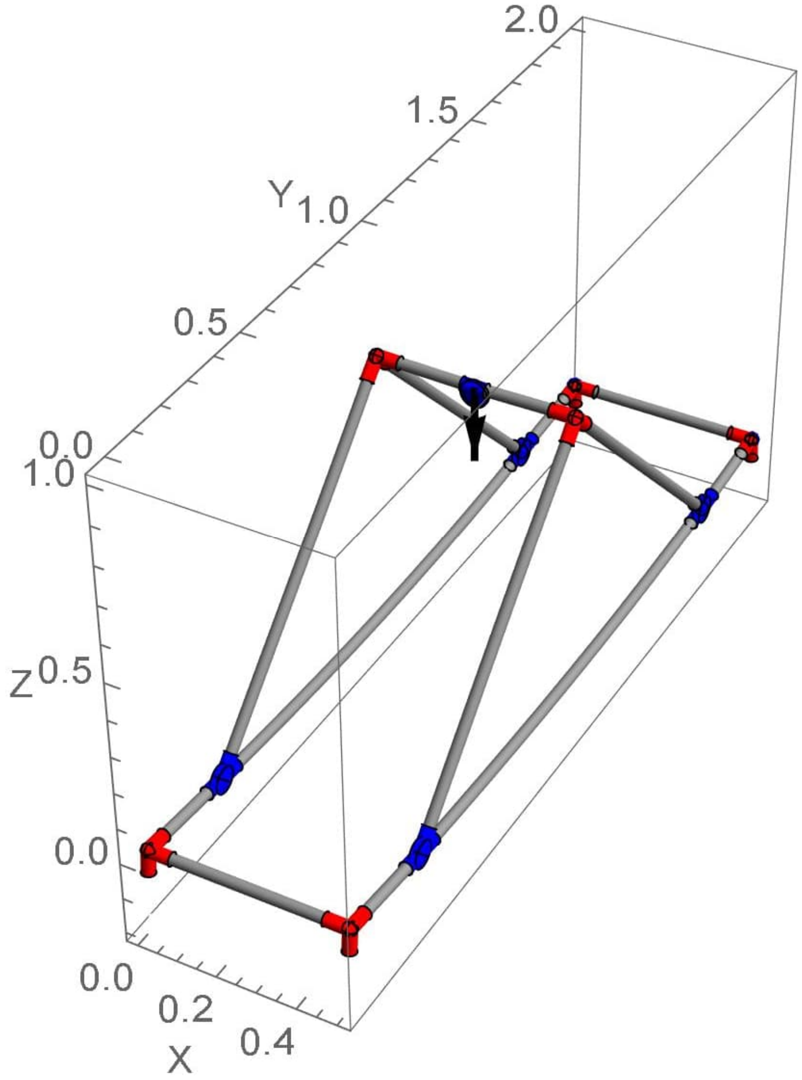
Finite element method is the standard of solid mechanics: Commercial codes in common use are based on the finite element method. A graphical user interface may make living easier, but a user should always understand what the problem is and in what sense it is solved!

Finite element method has a strong theory: Although approximate solution is acceptable, knowing nothing about the error is not acceptable.

FLUID MECHANICS APPLICATION



SOLID MECHANICS APPLICATION



Week 43-3

1 INTRODUCTION

LEARNING OUTCOMES

Students get an overall picture about prerequisites of the course, the roles of engineering models in structure modelling, and finite element method in displacement analysis of structures. The topics of the week are

- ☐ Structure modelling
- ☐ Engineering models
- ☐ Mathematica language and the finite element solver of MEC-E1050
- ☐ Prerequisites of MEC-E1050

1.1 STRUCTURE MODELLING

- **Crop**: Decide the boundary of a structure. Interaction with surroundings need to be described in terms of known forces, moments, displacements, and rotations. All uncertainties with this respect bring uncertainty to the model too.

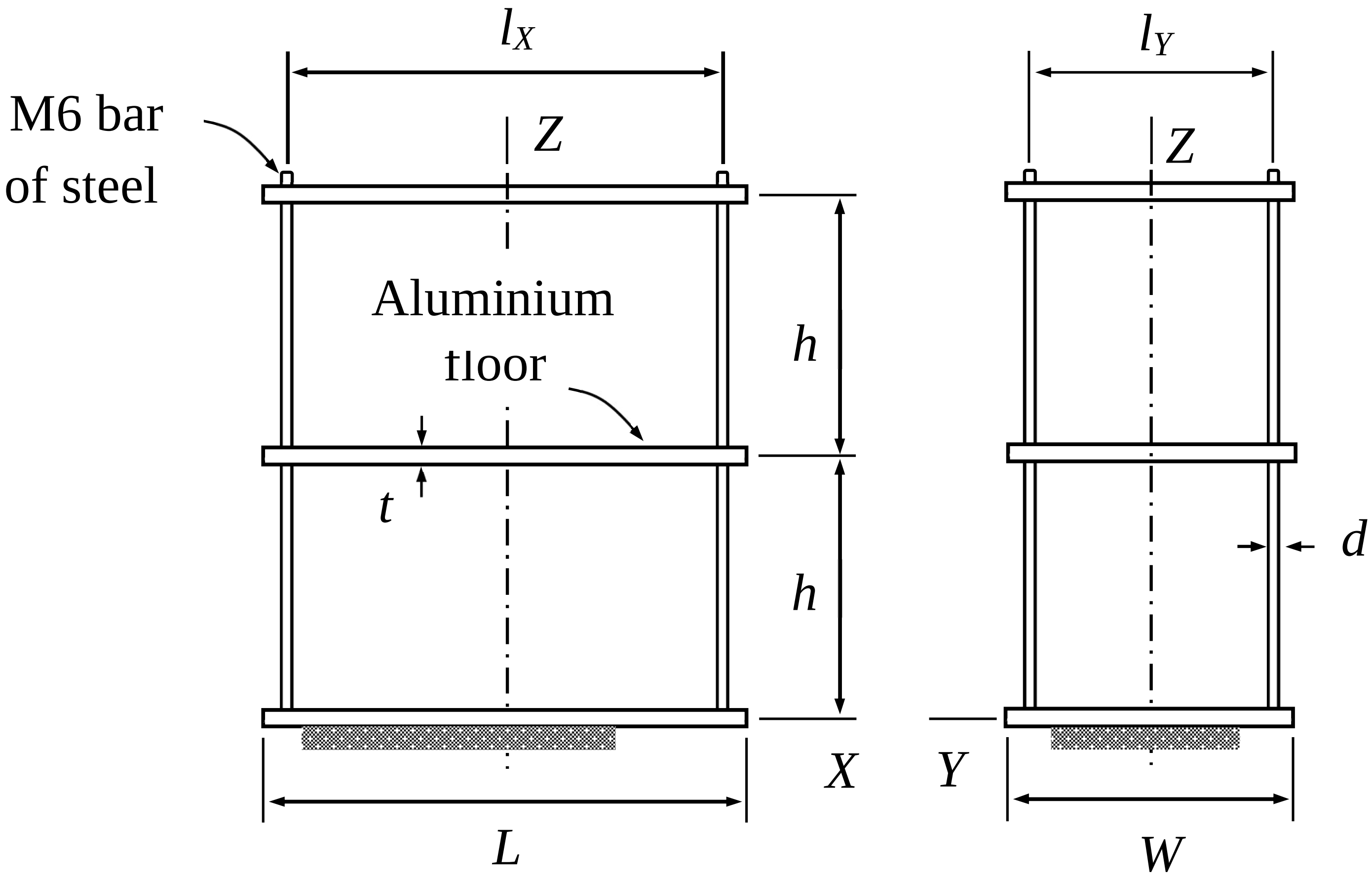
- **Idealize**: Simplify the geometry. Ignoring the details not likely to affect the outcome may simplify the analysis a lot.

- **Parametrize**: Assign symbols to geometric and material parameters of the idealized structure. Measure or find the values needed in numerical calculations.

- **Divide-and-rule**: Consider a complex structure as a set of structural parts interacting through connection points and find solutions one-by-one. Combine the solutions using the force and displacement conditions at the connection points.

MODEL OF A THREE FLOOR BUILDING

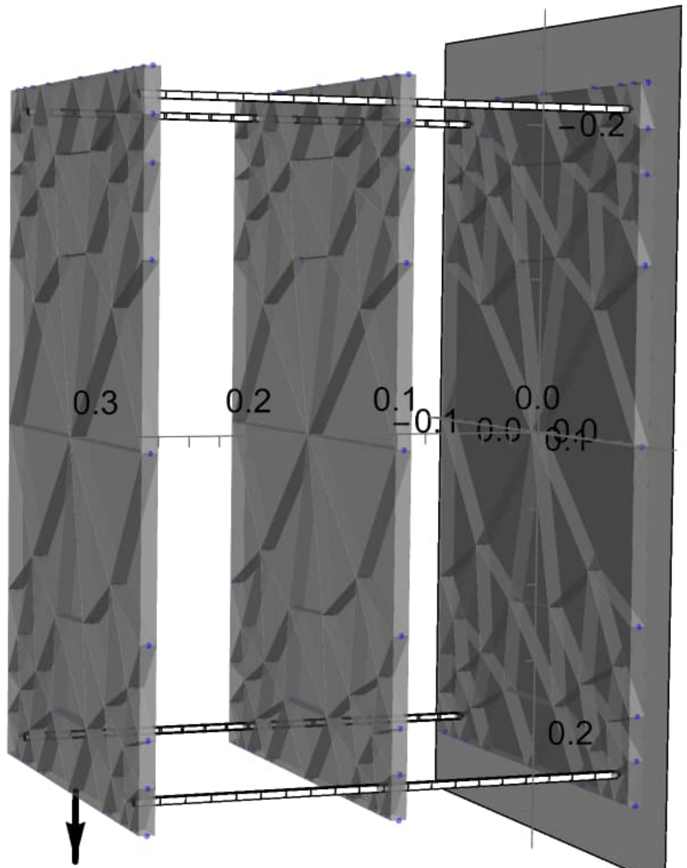
STRUCTURE IDEALIZATION AND …



… PARAMETRIZATION

Parameter	**symbol**	**value**
Effective column diameter	d	0.0048 m
Room height	h	0.156 m
Column spacing (x)	l_X	0.4 m
Column spacing (y)	l_Y	0.243 m
Floor length	L	0.44 m
Floor width	W	0.295 m
Floor thickness	t	0.015 m

FINITE ELEMENT SIMULATION



1.2 ENGINEERING MODELS

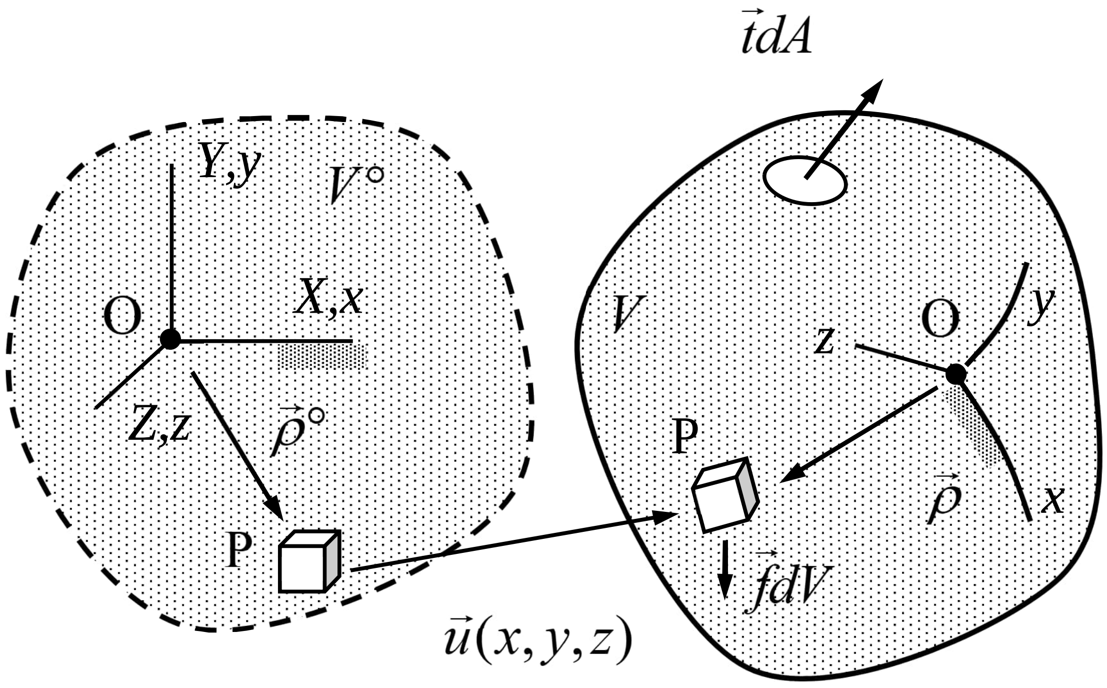
- **Solid model**: A linearly elastic body represents the “precise model” of the course. The generic solid model is used when more efficient engineering models do not apply.

- **Plate model**: Structural part of one small dimension (thin body). Deformation is described by thin slab and plate bending modes. In curved geometry, the model is called as the shell model.

- **Beam model**: Structural part of two small dimensions (slender body). Deformation is described by a bar mode, two bending modes, and a torsion mode.

- **Rigid body**: Structural part of negligible deformation. The idealization is useful when rigidities of structural parts differ significantly.

SOLID MODEL



The primary unknowns are $u(x,y,z)$, $v(x,y,z)$, $w(x,y,z)$, ($\phi(x,y,z)$, $\theta(x,y,z)$, $\psi(x,y,z)$). Material elements may translate, rotate, and deform. In short, for points P and Q of an element $\vec{u}_{\mathrm{Q}} = \vec{u}_{\mathrm{P}} + \vec{\theta}_{\mathrm{P}} \times \vec{\rho}_{\mathrm{PQ}} + \vec{\rho}_{\mathrm{PQ}} \cdot \vec{\vec{\varepsilon}}_{\mathrm{P}}$. Displacement follows from stress-strain relationship (generalized Hooke’s law) and equilibrium of material elements.

- Let us consider the displacement of a small material element centered at point P. As the material element is assumed to be small, first two terms of the Taylor series represent the displacement inside the material element

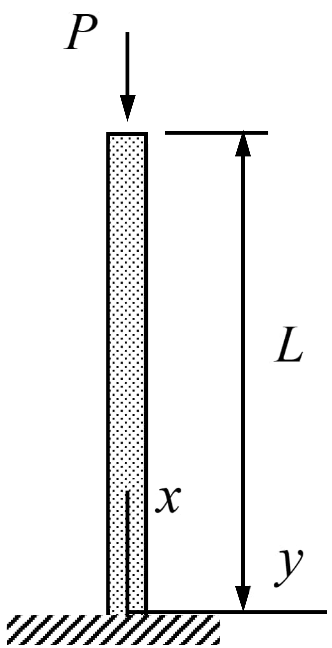
  $$\vec{u}_{\mathrm{Q}} = \vec{u}_{\mathrm{P}} + \vec{\rho}_{\mathrm{PQ}} \cdot (\nabla \vec{u})_{\mathrm{P}},$$

  where the relative position vector $\vec{\rho}_{\mathrm{PQ}} = \vec{r}_{\mathrm{Q}} - \vec{r}_{\mathrm{P}}$. Division of the displacement gradient into its anti-symmetric and symmetric parts $(\nabla \vec{u})_{\mathrm{P}} = \overleftrightarrow{\theta}_{\mathrm{P}} + \overleftrightarrow{\varepsilon}_{\mathrm{P}}$ and using the concept of an associated vector $\vec{\theta}$ to an antisymmetric tensor $\overleftrightarrow{\theta}$, gives

  $$\vec{u}_{\mathrm{Q}} = \vec{u}_{\mathrm{P}} + \vec{\theta}_{\mathrm{P}} \times \vec{\rho}_{\mathrm{PQ}} + \vec{\rho}_{\mathrm{PQ}} \cdot \overleftrightarrow{\varepsilon}_{\mathrm{P}}.$$

  The terms describe effects of translation, small rigid body rotation, and deformation (shape distortion) when the rotation part is small. Stress acting on the material element depends only on strain $\overleftrightarrow{\varepsilon}_{\mathrm{P}}$.

EXAMPLE The cross section of a cylindrical body is square of side length h. Density ρ, Young's modulus E, and Poisson's ratio ν of the linearly elastic isotropic and homogeneous material are constants. The body is loaded by a constant traction of magnitude P/h^2 at its free end. Determine stress $\vec{\vec{\sigma}}$ and displacement $\vec{u}$ using the solid model. Assume that the transverse (to the axis) displacement is not constrained by the support.



Answer $u = -\dfrac{P}{Eh^2}x,\ \ v = \nu\dfrac{P}{Eh^2}y\ ,\ w = \nu\dfrac{P}{Eh^2}z$

$$\sigma_{xx} = -\frac{P}{h^2}\ ,\ \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0$$

- The component forms of the equilibrium equations and constitutive equations of a linearly elastic isotropic material in a Cartesian $(x, y, z)-$coordinate system are

$$\begin{Bmatrix} \partial\sigma_{xx}/\partial x+\partial\sigma_{yx}/\partial y+\partial\sigma_{zx}/\partial z+f_x \\ \partial\sigma_{xy}/\partial x+\partial\sigma_{yy}/\partial y+\partial\sigma_{zy}/\partial z+f_y \\ \partial\sigma_{xz}/\partial x+\partial\sigma_{yz}/\partial y+\partial\sigma_{zz}/\partial z+f_z \end{Bmatrix}=0,$$

$$\begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial w/\partial z \end{Bmatrix}=\frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix}\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix}, \text{ and } \begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix}=\begin{Bmatrix} \sigma_{yx} \\ \sigma_{zy} \\ \sigma_{xz} \end{Bmatrix}=G\begin{Bmatrix} \partial u/\partial y+\partial v/\partial x \\ \partial v/\partial z+\partial w/\partial y \\ \partial w/\partial x+\partial u/\partial z \end{Bmatrix}.$$

- Let us assume that the non-zero stress and displacement components are $\sigma_{xx}(x)$, $u(x)$, $v(y)$ and $w(z)$. The axial stress follows from the equilibrium equation and the known traction at the free end $x=L$:

$$\frac{d\sigma_{xx}}{dx}=0 \quad 0<x<L \quad \text{and} \quad \sigma_{xx}(L)=-\frac{P}{h^2} \quad \Rightarrow \quad \sigma_{xx}(x)=-\frac{P}{h^2}.$$

- Generalized Hooke's law written for the uniaxial stress implies that

$$\frac{du}{dx}=\frac{\sigma_{xx}}{E}=-\frac{P}{Eh^2}, \quad \frac{dv}{dy}=-\frac{\nu}{E}\sigma_{xx}=\nu\frac{P}{Eh^2}, \quad \frac{dw}{dz}=-\frac{\nu}{E}\sigma_{xx}=\nu\frac{P}{Eh^2}.$$

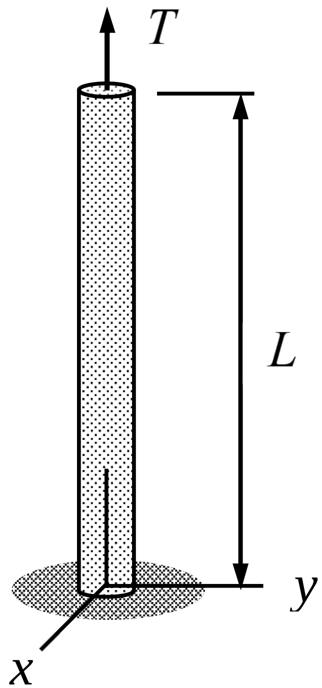
Axial displacement vanishes at the support and the transverse displacement at the axis:

$$\frac{du}{dx}=-\frac{P}{Eh^2} \quad 0<x<L \quad \text{and} \quad u(0)=0 \quad \Rightarrow \quad u(x)=-\frac{P}{Eh^2}x,$$

$$\frac{dv}{dy}=\nu\frac{P}{Eh^2} \quad -\frac{1}{2}h<y<\frac{1}{2}h \quad \text{and} \quad v(0)=0 \quad \Rightarrow \quad v(y)=\nu\frac{P}{Eh^2}y,$$

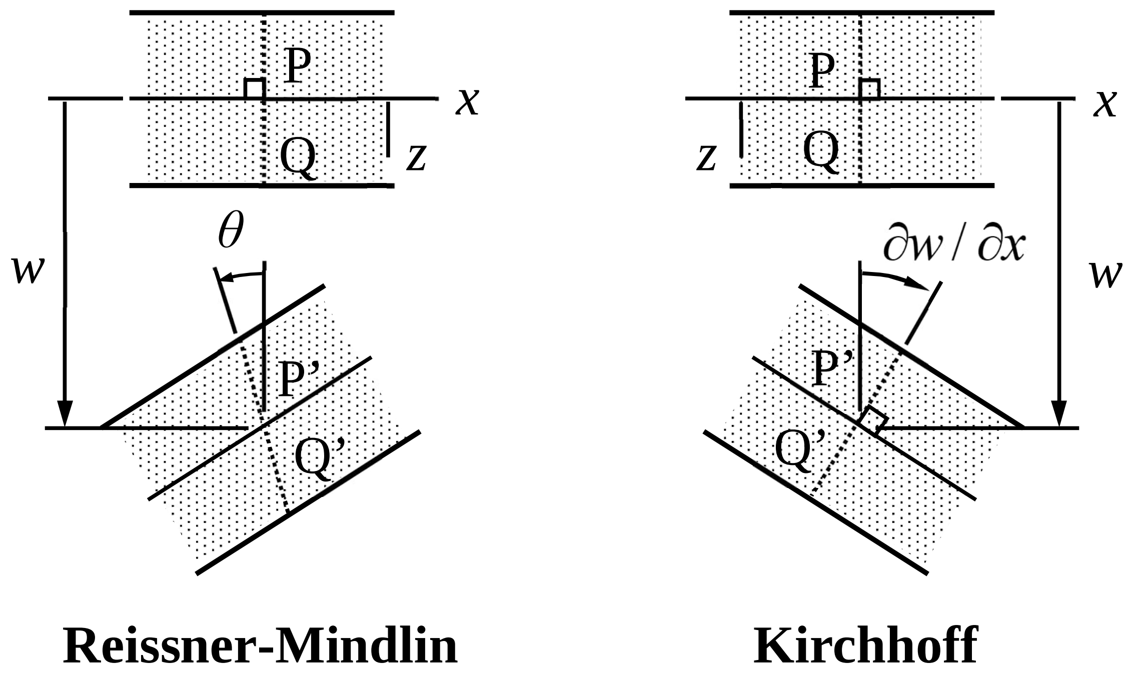
$$\frac{dw}{dz}=-\nu\frac{P}{Eh^2} \quad -\frac{1}{2}h<z<\frac{1}{2}h \quad \text{and} \quad w(0)=0 \quad \Rightarrow \quad w(z)=\nu\frac{P}{Eh^2}z.$$

EXAMPLE Consider a torsion of a cylindrical body of length L and circular cross-section of radius R. Shear modulus G of the material is constant. If one end is fixed and the other end is free to rotate, determine the relationship between torque T and rotation angle ψ at the free end. Assume that $u = -\psi(z)y$, $v = \psi(z)x$, and $w = 0$.



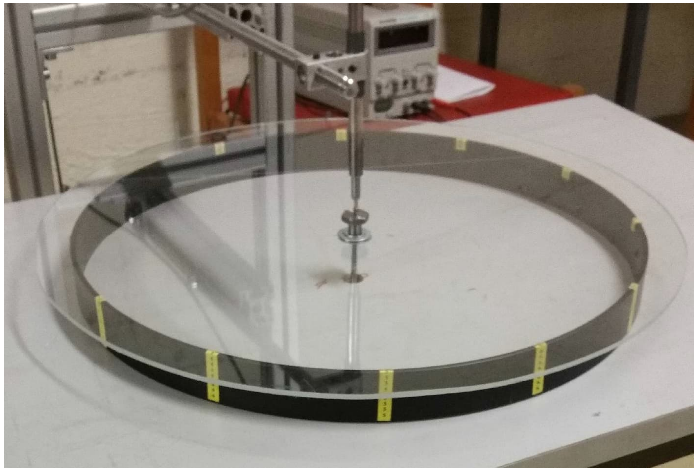
Answer $T = \dfrac{I_{rr}G}{L}\psi$ where $I_{rr} = \dfrac{\pi}{2}R^4$.

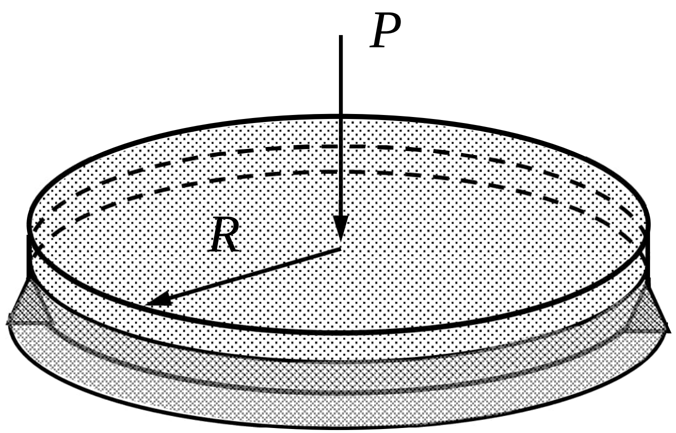
PLATE MODEL



The primary unknowns are $u(x,y)$, $v(x,y)$, $w(x,y)$, $\phi(x,y)$, $\theta(x,y)$, $\psi(x,y)$. Line segments perpendicular to the mid/reference-plane remain straight in deformation (Reissner-Mindlin) and perpendicular to the mid-plane (Kirchhoff). Mathematically $\vec{u}_{\mathrm{Q}} = \vec{u}_{\mathrm{P}} + \vec{\theta}_{\mathrm{P}} \times \vec{\rho}_{\mathrm{PQ}}$. Normal stress σ_{zz} is negligible.

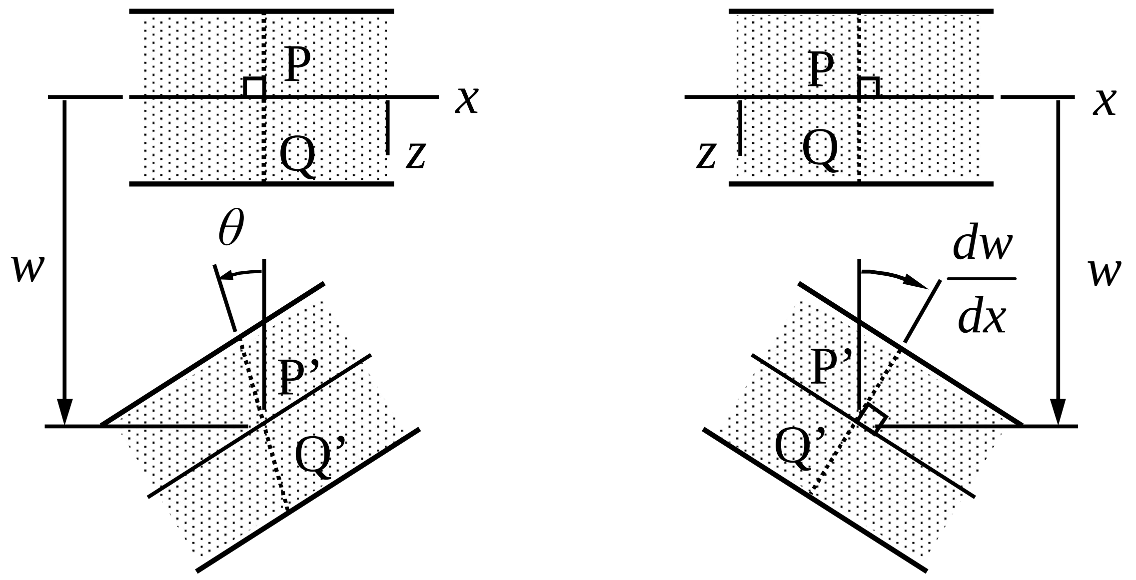
EXAMPLE A simply supported circular body of radius R and thickness t is loaded by a point force P acting at the midpoint as shown in the figure. Determine the transverse displacement w at the midpoint by using the plate model. Young's modulus E and Poisson's ratio ν of the isotropic material are constants. Assume that displacement depends on the radial coordinate only.





Answer: $$w(0) = -\frac{1}{16\pi}\frac{PR^2}{D}\frac{3+\nu}{1+\nu} = -\frac{3}{4\pi}\frac{PR^2}{Et^3}(3+\nu)(1-\nu)$$

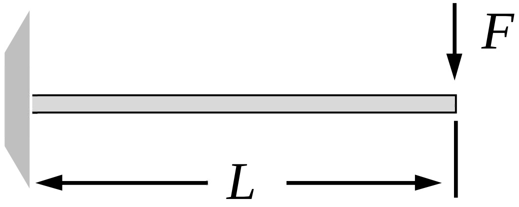
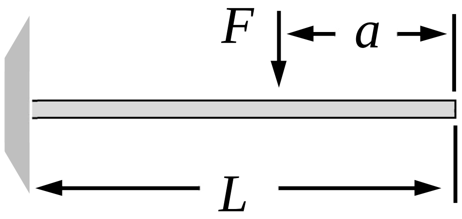
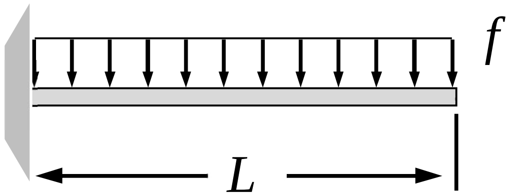
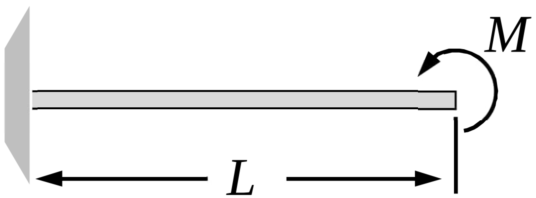
BEAM MODEL



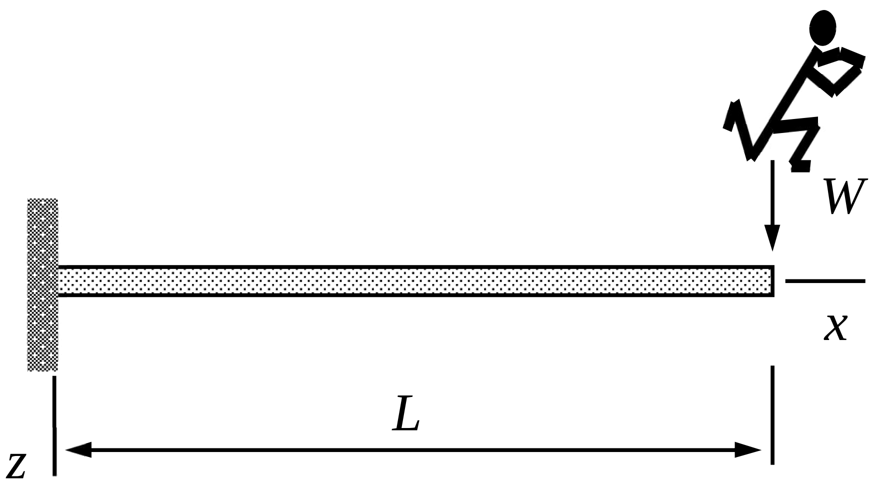
Timoshenko **Bernoulli**

The primary unknowns are $u(x)$, $v(x)$, $w(x)$, $\phi(x)$, $\theta(x)$, $\psi(x)$. Normal planes to the (material) axis of beam remain planes (Timoshenko) and normal to the axis (Bernoulli) in deformation. Mathematically $\vec{u}_{\mathrm{Q}} = \vec{u}_{\mathrm{P}} + \vec{\theta}_{\mathrm{P}} \times \vec{\rho}_{\mathrm{PQ}}$. Transverse normal stress is negligible, i.e., $\sigma_{yy} = \sigma_{zz} \ll \sigma_{xx}$.

BEAM BENDING

Loading case	Deflection (tip)	Rotation (tip)
	$w = \dfrac{FL^3}{3EI}$	$\theta = -\dfrac{dw}{dx} = -\dfrac{FL^2}{2EI}$
	$w = \dfrac{F(a-L)^2(a+2L)}{6EI}$	$\theta = -\dfrac{dw}{dx} = -\dfrac{F(L-a)^2}{2EI}$
	$w = \dfrac{fL^4}{8EI}$	$\theta = -\dfrac{dw}{dx} = -\dfrac{fL^3}{6EI}$
	$w = -\dfrac{L^2M}{2EI}$	$\theta = -\dfrac{dw}{dx} = \dfrac{LM}{EI}$

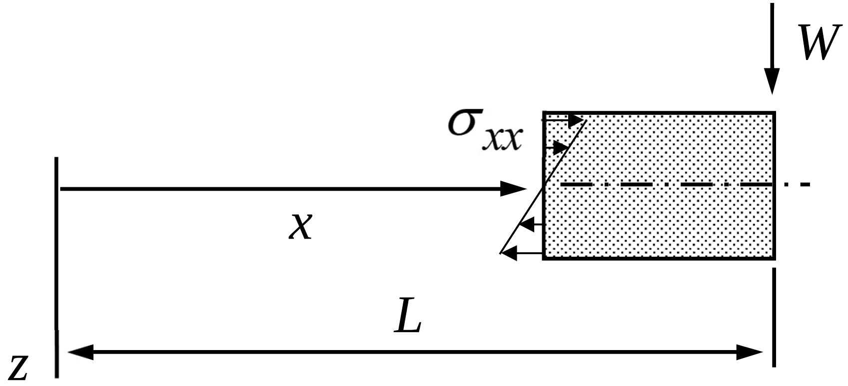
EXAMPLE A rigidly supported springboard of length L and cross-sectional area $A = bh$ is levelled without loading. Under which conditions displacement at the free end and stress at the support do not exceed the limit values δ and σ_{cr}, respectively, if a person of weight W is standing at the free end? Use the beam model and assume that the stress and the transverse displacement are related by $\sigma_{xx} = -Ezd^2w/dx^2$.



Answer $4\dfrac{WL^3}{Ebh^3} \le \delta$ and $6\dfrac{WL}{bh^2} \le \sigma_{\mathrm{cr}}$

- The relationship between the axial stress and transverse displacement follows from Hooke's law and assumptions of the beam model. Moment equilibrium gives

$$-\int_{-h/2}^{h/2} z\sigma_{xx} b\,dz - W(L-x) = 0 \quad \Rightarrow$$



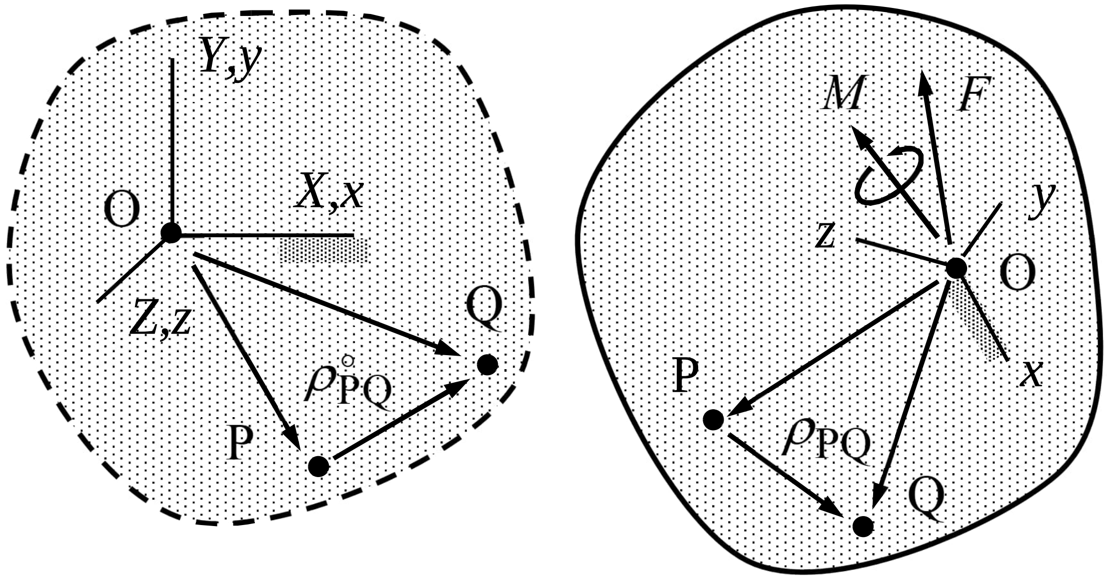
$$E\frac{bh^3}{12}\frac{d^2w}{dx^2} - W(L-x) = 0$$

- Transverse displacement and its derivative vanish at the support. Hence

$$\frac{d^2w}{dx^2} = 12\frac{W}{Ebh^3}(L-x) \quad \Rightarrow \quad w(x) = 2\frac{W}{Ebh^3}x^2(3L-x).$$

- Therefore $w(L) = 4\frac{WL^3}{Ebh^3} \le \delta$ and $\sigma_{\max}(0) = 6\frac{WL}{bh^2} \le \sigma_{\text{cr}}$.

RIGID BODY



The primary unknowns are u, v, w, ϕ, θ, ψ of the translation point. Body may translate and rotate but distance between any two points P and Q is constant. Mathematically, e.g., $\vec{u}_{Q} = \vec{u}_{P} + \vec{\theta}_{P} \times \vec{\rho}_{PQ}$. Forces acting on the body can be represented by a force-moment pair.

1.3 DISPLACEMENT ANALYSIS

- Idealize a complex structure as a set of structural parts, whose behavior can be approximated by using the engineering models (bar, beam, plate, rigid body etc.).

- Write the equilibrium equations for the connection points (Newton **III**), the force-displacement relationships of the structural parts, and constraints concerning the nodal displacements (displacements and rotations should match at the connection points).

- Solve the nodal displacements and rotations and the forces and moments acting on the structural parts (elements in FEM) from the equation system.

- Determine the stress in the structural parts one-by-one according to the engineering model used (optional step).

NEWTON's LAWS OF MOTION

I In an inertial frame of reference, an object either remains at rest or continues to move at a constant velocity, unless acted upon by a force.

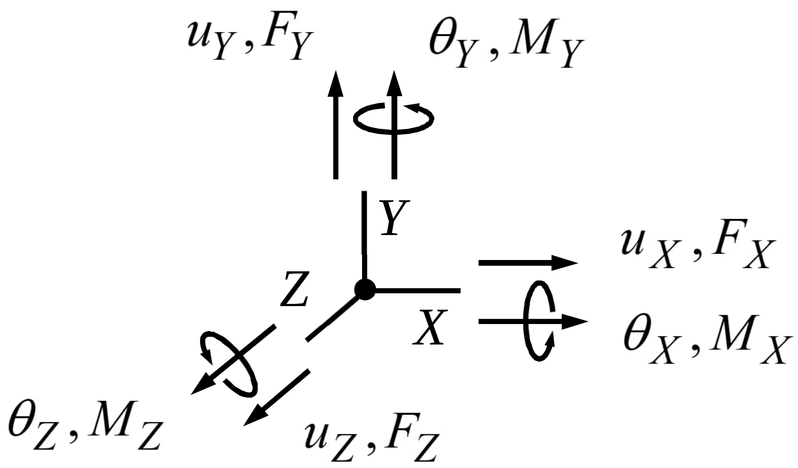
II The vector sum of the forces on an object is equal to the mass of that object multiplied by the acceleration of the object (assuming that the mass is constant).

III When one body exerts a force on a second body, the second body simultaneously exerts a force equal in magnitude and opposite in direction on the first body.

Newton's laws in their original forms apply to particles only. The formulation for rigid bodies and deformable bodies requires slight modifications.

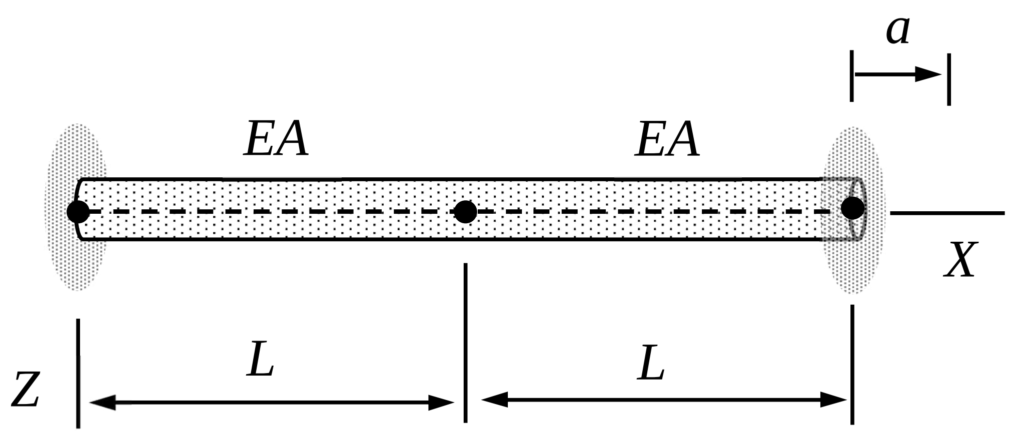
QUANTITIES OF ANALYSIS

The primary aim is to find displacements, rotations, forces and moments at the connection points of the structural parts. The components of the vector quantities (magnitude and direction) are taken to be positive in the directions of the coordinate axes.



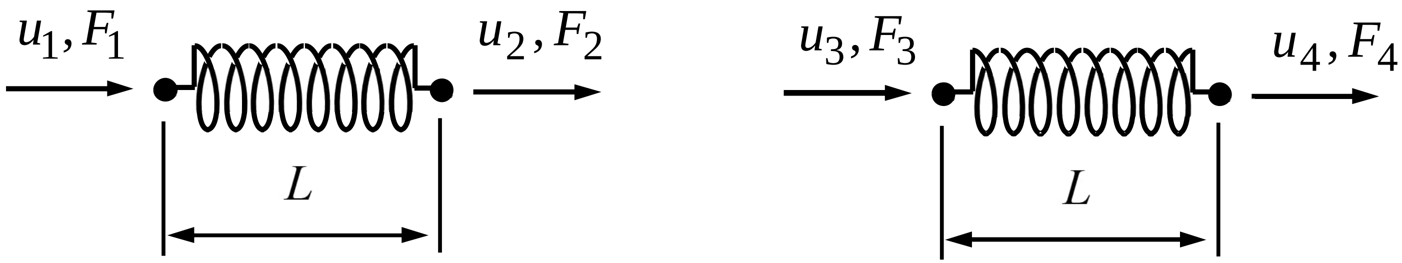
Vector quantities are invariants in the sense $\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k} = a_X\vec{I} + a_Y\vec{J} + a_Z\vec{K}$, and can be transformed from one coordinate system to another using the property.

EXAMPLE A connector bar is welded at its ends to rigid walls. If the right end wall displacement is a, determine the displacements of connection points 1, 2, and 3 and the forces acting on structural parts. Cross sectional area A and Young's modulus of the material E are constants, and the displacement force relationship of a bar is the same as that of a spring with coefficient $k = EA / L$. Model the structure as a collection of two bars (1 and 2).



Answer $u_1 = 0$, $u_2 = u_3 = \frac{1}{2}a$, $u_4 = a$, $F_1 = F_3 = -\frac{1}{2}ka$, $F_2 = F_4 = \frac{1}{2}ka$.

- As the structural parts can be considered as springs of coefficient $k = EA/L$, $F_1 = k(u_1 - u_2)$, $F_2 = k(u_2 - u_1)$, $F_3 = EA(u_3 - u_4)/L$, $F_4 = EA(u_4 - u_3)/L$



- The displacement constraints due to the left edge welding, displacement of the right end wall, and integrity of structure at the connection of the structural parts are $u_1 = 0$, $u_4 = a$, and $u_2 = u_3$.

- The force constraints are due to Newton **III** which requires that F_2 and F_3 are equal in magnitude and opposite in signs or $F_2 + F_3 = 0$.

- Altogether, the 8 equations determining the 4 displacement components u_1, u_2, u_3, u_4 and the 4 force components F_1, F_2, F_3, F_4 are given by

$$F_1 = k(u_1 - u_2),\ F_2 = k(u_2 - u_1),\ F_3 = k(u_3 - u_4),\ F_4 = k(u_4 - u_3),$$

$$u_1 = 0,\ \ u_4 = a,\ \ u_2 = u_3,\ \ F_2 + F_3 = 0.$$

- The linear equation system can be solved, e.g., by considering the equations in a proper order (to be discussed later in more detail), by Gauss elimination, by Mathematica, …

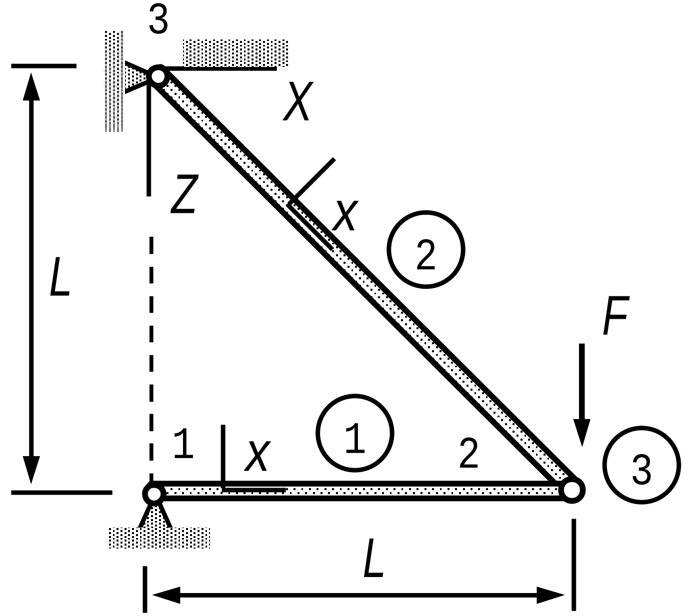
$$u_1 = 0,\ \ u_2 = \frac{1}{2}a,\ u_3 = \frac{1}{2}a,\ \ u_4 = a, \qquad \Leftarrow$$

$$F_1 = -\frac{1}{2}ka,\ \ F_2 = \frac{1}{2}ka,\ \ F_3 = -\frac{1}{2}ka,\ F_4 = \frac{1}{2}ka. \qquad \Leftarrow$$

1.4 FE-CODE OF MEC-E1050

	model	properties	geometry
1	BAR	{{$\mathbb{E}$}, {A}}	Line[{1, 2}]
2	BAR	{{$\mathbb{E}$}, {$2\sqrt{2}$ A}}	Line[{3, 2}]
3	FORCE	{0, 0, F}	Point[{2}]

	{X,Y,Z}	{u_X,u_Y,u_Z}	{θ_X,θ_Y,θ_Z}
1	{0, 0, L}	{0, 0, 0}	{0, 0, 0}
2	{L, 0, L}	{uX[2], 0, uZ[2]}	{0, 0, 0}
3	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}



STRUCTURE

"Structure is a collection of *elements* (earlier structural parts) connected by *nodes* (earlier connection points). Displacement of the structure is defined by nodal translations and rotations of which some are known and some unknown."

$prb = \{ele, fun\}$ where

$ele = \{prt_1, prt_2, \ldots\}$ .... elements

$fun = \{val_1, val_2, \ldots\}$ .... nodes

Elements

$prt = \{typ, pro, geo\}$ where

$typ = \mathrm{BAR} \,|\, \mathrm{TORSION} \,|\, \mathrm{BEAM} \,|\, \mathrm{RIGID} |\ldots|$ .... model

$pro = \{p_1, p_2, \ldots, p_n\}$ ....properties

$geo = \text{Point}[\{n_1\}]\,|\,\text{Line}[\{n_1, n_2\}]\,|\,\text{Triangle}[\{n_1, n_2, n_3\}]\,|\,\ldots\,|$ .... geometry

Nodes

$val = \{crd, tra, rot\}$ where

$crd = \{X, Y, Z\}$ .... structural coordinates

$tra = \{u_X, u_Y, u_Z\}$ .... translation components

$rot = \{\theta_X, \theta_Y, \theta_Z\}$ ....rotation components

ELEMENTS

Elements represent the structural parts modelled as solids, plates, beams, or rigid bodies or their simplified versions, external point and boundary forces and moments.

Constraint

$\{\text{JOINT},\{\}|\{\{\underline{u}_X,\underline{u}_Y,\underline{u}_Z\}\},\text{Point}[\{n_1\}]\}$ ....displacement constraint

$\{\text{JOINT},\{\},\text{Line}[\{n_1,n_2\}]\}$ ....displacement constraint

$\{\text{RIGID},\{\}|\{\{\underline{u}_X,\underline{u}_Y,\underline{u}_Z\},\{\underline{\theta}_X,\underline{\theta}_Y,\underline{\theta}_Z\}\},\text{Point}[\{n_1\}]\}$ ... displacement/rotation constraint

$\{\text{RIGID},\{\},\text{Line}[\{n_1,n_2\}]\}$ .... rigid constraint

$\{\text{SLIDER},\{n_X,n_Y,n_Z\},\text{Point}[\{n_1\}]\}$ ....slider constraint

Force

$\{\text{FORCE},\{F_X,F_Y,F_Z\},\text{Point}[\{n_1\}]\}$ .... point force

$\{\text{FORCE},\{F_X,F_Y,F_Z,M_X,M_Y,M_Z\},\text{Point}[\{n_1\}]\}$ .... point load

$\{\text{FORCE},\{f_X, f_Y, f_Z\},\text{Line}[\{n_1,n_2\}]\}$ .... distributed force

$\{\text{FORCE},\{f_X, f_Y, f_Z\},\text{Polygon}[\{n_1,n_2,n_3\}]\}$ .... distributed force

Beam model

$\{\text{BAR},\{\{E\},\{A\},\{f_X, f_Y, f_Z\}\},\text{Line}[\{n_1,n_2\}]\}$ ....bar mode

$\{\text{TORSION},\{\{G\},\{J\},\{\{m_X, m_Y, m_Z\}\}\},\text{Line}[\{n_1,n_2\}]\}$ .... torsion mode

$\{\text{BEAM},\{\{E,G\},\{A,I_{yy},I_{zz}\},\{f_X, f_Y, f_Z\}\},\text{Line}[\{n_1,n_2\}]\}$ ....beam

$\{\text{BEAM},\{\{E,G\},\{A,I_{yy},I_{zz},\{j_X, j_Y, j_Z\}\},\{f_X, f_Y, f_Z\}\},\text{Line}[\{n_1,n_2\}]\}$ ....beam

Plate model

$\{\text{PLANE},\{\{E,\nu\},\{t\},\{f_X, f_Y, f_Z\}\},\text{Polygon}[\{n_1,n_2,n_3\}]\}$........................thin slab mode

$\{\text{PLANE},\{\{E,\nu\},\{t\},\{f_X, f_Y, f_Z\}\},\text{Polygon}[\{n_1,n_2,n_3,n_4\}]\}$..................thin slab mode

$\{\text{PLATE},\{\{E,\nu\},\{t\},\{f_X, f_Y, f_Z\}\},\text{Polygon}[\{n_1,n_2,n_3\}]\}$..plate

Solid model

$\{\mathrm{SOLID},\{\{E,\nu\},\{f_X, f_Y, f_Z\}\},\mathrm{Tetrahedron}[\{n_1,n_2,n_3,n_4\}]\}$ ....solid

$\{\mathrm{SOLID},\{\{E,\nu\},\{f_X, f_Y, f_Z\}\},\mathrm{Hexahedron}[\{n_1,n_2,n_3,n_4,n_5,n_6,n_7,n_8\}]\}$ ....solid

$\{\mathrm{SOLID},\{\{E,\nu\},\{f_X, f_Y, f_Z, m_X, m_Y, m_Z,\}\},\mathrm{Tetrahedron}[\{n_1,n_2,n_3,n_4\}]\}$ ....solid

OPERATIONS

Operations act on structure as defined by prb. The main operations of MEC-E1050 are solving the unknowns in displacement analysis and displaying the problem definition in a formatted form.

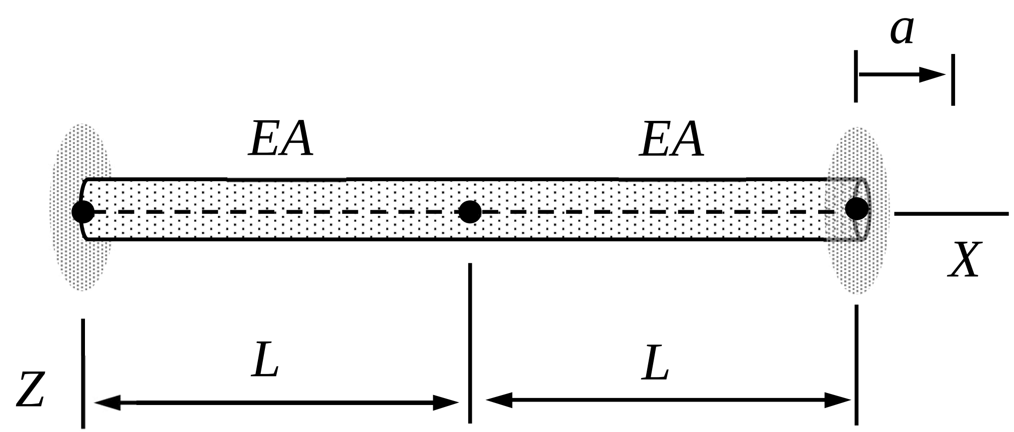
$prb = \text{REFINE}[prb]$ ....refine structure representation

$\text{Out} = \text{FORMATTED}[prb]$ .... display problem definition

$\text{Out} = \text{STANDARDFORM}[prb]$ .... display virtual work expression

$sol = \text{SOLVE}[prb]$ ....solve the unknowns

EXAMPLE 1.1 A connector bar is welded at its ends to rigid walls. If the right end wall displacement is a, determine the displacements of connection points 1, 2, and 3 and the forces acting on structural parts. Cross sectional area A and Young's modulus of the material E are constants and the displacement force relationship of a bar is the same as that of a spring with coefficient $k = EA/L$. Model the structure as a collection of two bars (1 and 2).



Answer $u_1 = 0,\ u_2 = u_3 = \frac{1}{2}a,\ u_4 = a,\ F_1 = F_3 = -\frac{1}{2}ka,\ F_2 = F_4 = \frac{1}{2}ka.$

- Problem description by $prb = \{ele, fun\}$ and two operations acting on it

```
ele = {
   {RIGID, {{0, 0, 0}, {0, 0, 0}}, Point[{1}]},
   {BAR, {{E}, {A}}, Line[{1, 2}]},
   {RIGID, {}, Line[{2, 3}]},
   {BAR, {{E}, {A}}, Line[{3, 4}]},
   {RIGID, {{a, 0, 0}, {0, 0, 0}}, Point[{4}]}};

fun = {
   {{0, 0, 0}, {uX[1], 0, 0}, {0, 0, 0}},
   {{L, 0, 0}, {uX[2], 0, 0}, {0, 0, 0}},
   {{L, 0, 0}, {uX[3], 0, 0}, {0, 0, 0}},
   {{2 L, 0, 0}, {uX[4], 0, 0}, {0, 0, 0}}};

FORMATTED[{ele, fun}]

SOLVE[{ele, fun}]
```

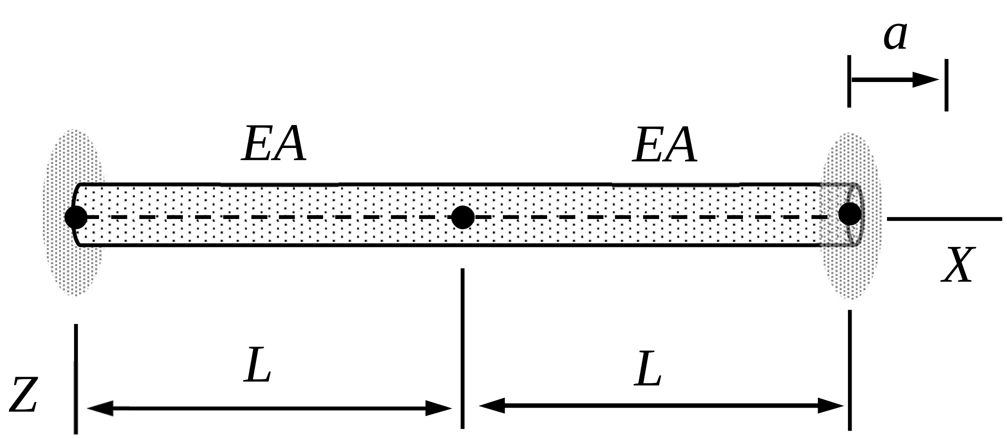
- Outcome of the operations is the structure description in table format and solution to the unknowns of the displacement problem (in format of a rule)

	model	properties	geometry
1	RIGID	{{0, 0, 0}, {0, 0, 0}}	Point[{1}]
2	BAR	{{E}, {A}}	Line[{1, 2}]
3	RIGID	{}	Line[{2, 3}]
4	BAR	{{E}, {A}}	Line[{3, 4}]
5	RIGID	{{a, 0, 0}, {0, 0, 0}}	Point[{4}]

	{X,Y,Z}	$\{u_X,u_Y,u_Z\}$	$\{\theta_X,\theta_Y,\theta_Z\}$
1	{0, 0, 0}	{uX[1], 0, 0}	{0, 0, 0}
2	{L, 0, 0}	{uX[2], 0, 0}	{0, 0, 0}
3	{L, 0, 0}	{uX[3], 0, 0}	{0, 0, 0}
4	{2 L, 0, 0}	{uX[4], 0, 0}	{0, 0, 0}

$$\left\{FX[1] \to -\frac{a\,A\,E}{2\,L},\ FX[4] \to \frac{a\,A\,E}{2\,L},\ FX[\{3,2\}] \to -\frac{a\,A\,E}{2\,L},\ uX[1] \to 0,\ uX[2] \to \frac{a}{2},\ uX[3] \to \frac{a}{2},\ uX[4] \to a\right\}$$

EXAMPLE 1.2. A connector bar is welded at its ends to rigid walls. If the right end wall displacement is a, determine the displacement of point 2. Cross sectional area A and Young's modulus of the material E are constants. Model the structure as a collection of two bars (1 and 2).



Answer $u_2 = \frac{1}{2}a$ (Mathematica notebook)

PREREQUISITE: MATRIX ALGEBRA I

Addition	$\mathbf{C} = \mathbf{A} + \mathbf{B}$	$C_{ij} = A_{ij} + B_{ij}$
Multiplication (scalar)	$\mathbf{C} = \alpha\mathbf{A}$	$C_{ij} = \alpha A_{ij}$
Multiplication (matrix)	$\mathbf{C} = \mathbf{AB}$	$C_{ij} = \sum_{k\in\{1\ldots n\}} A_{ik} B_{kj}$

Unit matrix	$\mathbf{I}$	$\delta_{ij} = 1 \quad i = j, \quad \delta_{ij} = 0 \quad i \neq j$
Symmetric matrix	$\mathbf{A} = \mathbf{A}^{\mathrm{T}}$	$A_{ij} = A_{ji}$
Skew symmetric matrix	$\mathbf{A} = -\mathbf{A}^{\mathrm{T}}$	$A_{ij} = -A_{ji}$
Positive definite matrix	$\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} > 0 \quad \forall \mathbf{x} \neq 0$	

PREREQUISITE: MATRIX ALGEBRA II

Transpose	$\mathbf{A}^{\mathrm{T}}$	$A_{ij}^{\mathrm{T}} = A_{ji}$
Inverse	$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$	$\sum_{k \in \{1 \ldots n\}} A_{ik} A_{kj}^{-1} = \delta_{ij}$
Derivative	$\dot{\mathbf{x}}$	$\dot{x}_i = dx_i / dt$

Linear equation system	Find $\mathbf{x}$ such that $\mathbf{A}\mathbf{x} = \mathbf{b}$
Eigenvalue problem	Find all $(\lambda, \mathbf{x})$ such that $(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = 0$
Eigenvalue composition	$\mathbf{A} = \mathbf{X}\boldsymbol{\lambda}\mathbf{X}^{-1}$, where $\mathbf{X} = [\mathbf{x}_1 \ldots \mathbf{x}_n]$ and $\boldsymbol{\lambda} = diag[\lambda_1 \quad \ldots \quad \lambda_n]$
Matrix function	If $\mathbf{A} = \mathbf{X}\boldsymbol{\lambda}\mathbf{X}^{-1}$, then $f(\mathbf{A}) = \mathbf{X} f(\boldsymbol{\lambda}) \mathbf{X}^{-1}$

EXAMPLE. Determine the square $\mathbf{A}^2$ and inverse $\mathbf{A}^{-1}$ of $\mathbf{A}$ if

$$\mathbf{A} = \begin{bmatrix} 5 & 2 \\ -1 & 3 \end{bmatrix} \quad \left(\text{note: } \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \quad \mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right)$$

Matrix squared $\quad \mathbf{A}^2 = \begin{bmatrix} 5 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 23 & 16 \\ -8 & 7 \end{bmatrix} \Leftarrow$

Inverse matrix $\quad \mathbf{A}^{-1} = \begin{bmatrix} 5 & 2 \\ -1 & 3 \end{bmatrix}^{-1} = \frac{1}{17}\begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix} \Leftarrow$

From the viewpoint of computational complexity, solving a system of linear equations $\mathbf{Ax} = \mathbf{b}$ by Gauss elimination makes more sense than using the matrix inverse with $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$!

PREREQUISITE: MATRIX ALGEBRA III

Partitioned matrix $\mathbf{A}=\begin{bmatrix}\mathbf{a}_{11} & \mathbf{a}_{12}\\ \mathbf{a}_{21} & \mathbf{a}_{22}\end{bmatrix}$ & $\mathbf{B}=\begin{Bmatrix}\mathbf{b}_1\\ \mathbf{b}_2\end{Bmatrix}$

block or sub-matrix

Transpose $\mathbf{A}^{\mathrm{T}}=\begin{bmatrix}\mathbf{a}_{11}^{\mathrm{T}} & \mathbf{a}_{21}^{\mathrm{T}}\\ \mathbf{a}_{12}^{\mathrm{T}} & \mathbf{a}_{22}^{\mathrm{T}}\end{bmatrix}$

Multiplication $\mathbf{AB}=\begin{bmatrix}\mathbf{a}_{11} & \mathbf{a}_{12}\\ \mathbf{a}_{21} & \mathbf{a}_{22}\end{bmatrix}\begin{Bmatrix}\mathbf{b}_1\\ \mathbf{b}_2\end{Bmatrix}=\begin{Bmatrix}\mathbf{a}_{11}\mathbf{b}_1+\mathbf{a}_{12}\mathbf{b}_2\\ \mathbf{a}_{21}\mathbf{b}_1+\mathbf{a}_{22}\mathbf{b}_2\end{Bmatrix}$

The rules are the same as with the ordinary matrices. The sizes of the blocks need to be consistent in operations like transposing and multiplication!

EXAMPLE Determine the displacements w_i, if $i \in \{1,2,3\}$ the vector of displacements $\mathbf{a}$, stiffness matrix $\mathbf{K}$, and the loading vector $\mathbf{F}$ of the equilibrium equations $\mathbf{Ka} - \mathbf{F} = 0$ are given by

$$\mathbf{a} = \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}, \quad \mathbf{K} = k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \text{ and } \mathbf{F} = P\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}.$$

Answer $w_1 = \dfrac{P}{k}$, $w_2 = 2\dfrac{P}{k}$, and $w_3 = 3\dfrac{P}{k}$.

- With linear equation systems of more than two unknows, using a matrix inverse is not efficient. Gauss elimination is based on row operations aiming at an upper diagonal matrix. After that, solution for the unknowns is obtained step-by-step starting from the last equation. In the present problem

$$k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} = P\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}.$$

- Let us multiply the 2:nd equation by 2 and add to it equation 1 to get

$$k\begin{bmatrix} 2 & -1 & 0 \\ -2 & 4 & -2 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} = P\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \quad \Rightarrow \quad k\begin{bmatrix} 2 & -1 & 0 \\ 0 & 3 & -2 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} = P\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}.$$

- Let us multiply 3:rd equation by 3 and add to it the 2:nd equation to get the upper triangular matrix.

$$k\begin{bmatrix} 2 & -1 & 0 \\ 0 & 3 & -2 \\ 0 & -3 & 3 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} = P\begin{Bmatrix} 0 \\ 0 \\ 3 \end{Bmatrix} \Rightarrow k\begin{bmatrix} 2 & -1 & 0 \\ 0 & 3 & -2 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} = P\begin{Bmatrix} 0 \\ 0 \\ 3 \end{Bmatrix}.$$

- After these steps, solution is obtained step-by-step starting from the last equation:

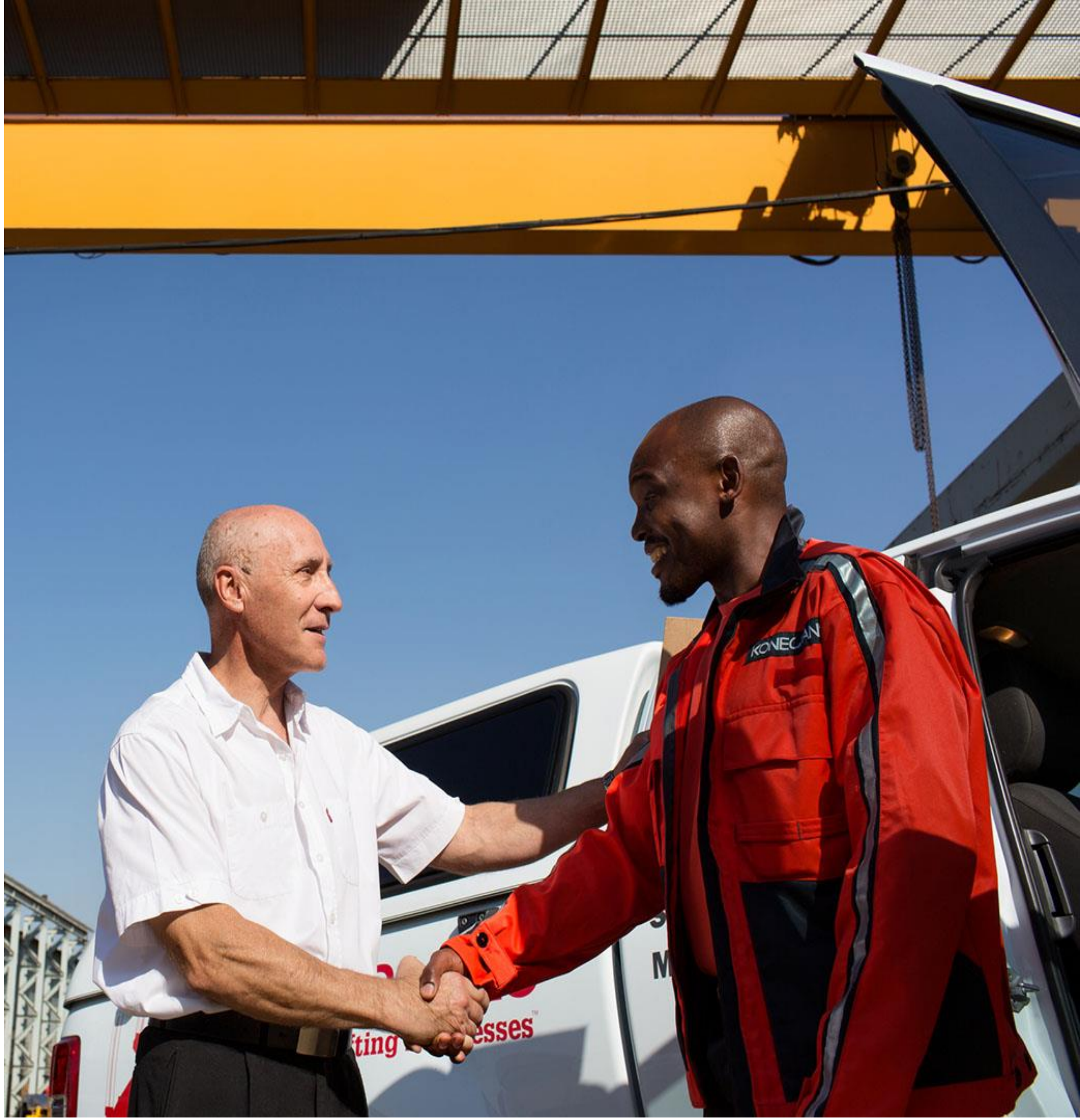
$$kw_3 = 3P \quad \Leftrightarrow \quad w_3 = 3\frac{P}{k}, \quad \leftarrow$$

$$k(3w_2 - 2w_3) = 0 \quad \Rightarrow \quad w_2 = \frac{2}{3}w_3 = 2\frac{P}{k}, \quad \leftarrow$$

$$k(2w_1 - w_2) = 0 \quad \Rightarrow \quad w_1 = \frac{1}{2}w_2 = \frac{P}{k}. \quad \leftarrow$$

Aalto 24.10.2024 MEC-E1050 - Finite Element Method in Solids

1. My journey at KC
2. KC intro
3. KC calculations tools – process cranes point of view – **Core of load carrying**
4. Some real cases
5. Q / A



KONECRANES

 |

This is Konecranes

Jari Kaiturinmäki

Aalto – KC 22.10.2024

Konecranes in numbers in 2023

ACTIVE IN ~50 COUNTRIES	EQUIPMENT IS ~60% OF TOTAL SALES	SERVICE IS ~40% OF TOTAL SALES
11.4% COMPARABLE EBITA MARGIN	€4.0 BILLION IN SALES	~16,600 EMPLOYEES

KONECRANES

A family of leading brands, globally and locally

END USERS

SINGLE GLOBAL BRAND

KONECRANES

END USERS

CRANE BUILDERS, DISTRIBUTORS, COMPONENT INTEGRATORS

MARKET-SPECIFIC BRANDS

DEMAG · SWF KRANTECHNIK · R&M MATERIALS HANDLING · VERLINDE LIFTING EQUIPMENT · donati

Our customer offering

An unmatched offering across our **three business segments**

SERVICE

Industry-leading lifecycle services for all types and makes of industrial cranes and hoists. **Unparalleled global service** network.

INDUSTRIAL EQUIPMENT

Extensive range of industrial cranes, from components and light duty applications to demanding process solutions. **Technology leadership** and **leading market position.**

PORT SOLUTIONS

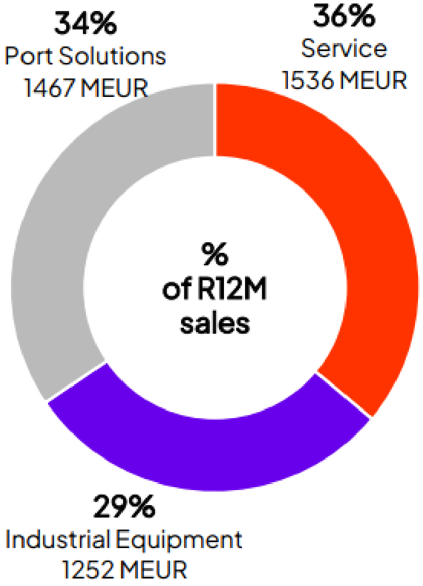
One of the leading global suppliers of **equipment**, **software**, **services** and **automated solutions** for the container handling industry and ports.

KONECRANES

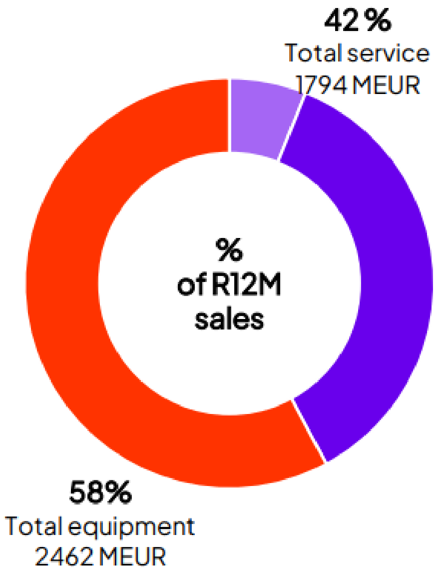
Q2 – 2024 Half-year-financial report

Group R12M sales split

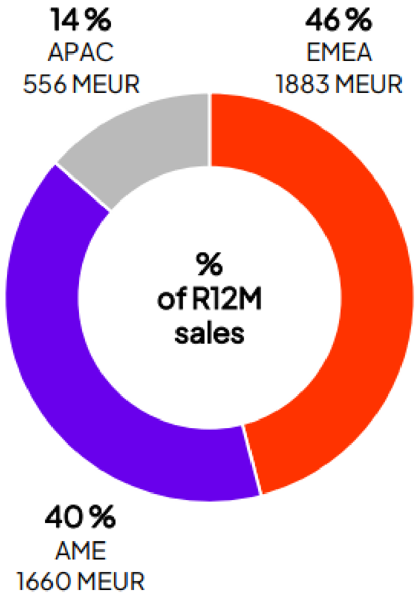
Group R12M sales by segment



Group R12M sales by offering type[1]



Group R12M sales by region



Note (1): Total service includes Service and Port Solutions' service sales, Total equipment includes Industrial Equipment and Port Solutions excluding Port Solutions' service sales

 |

KONECRANES

Industrial Equipment: Fulfilling every global need

Global leader in sustainable lifting solutions covering a full range of industrial applications

DUAL CHANNELS TO MARKET

Direct to end users

KONECRANES

Indirect distribution

R&M Materials Handling, DEMAG, SWF Krantechnik, donati, VERLINDE Lifting Equipment

INDUSTRY EXPERTISE/EMBEDDED SUSTAINABILITY

100%
Renewable electricity in manufacturing sites

Eco-optimized product features

COMPREHENSIVE OFFERING/ECONOMIES OF SCALE

Standard equipment


electric chain hoists


light crane systems


jib cranes


wire rope hoists


standard cranes

Process crane


winches


cranes

 |

Thank you!

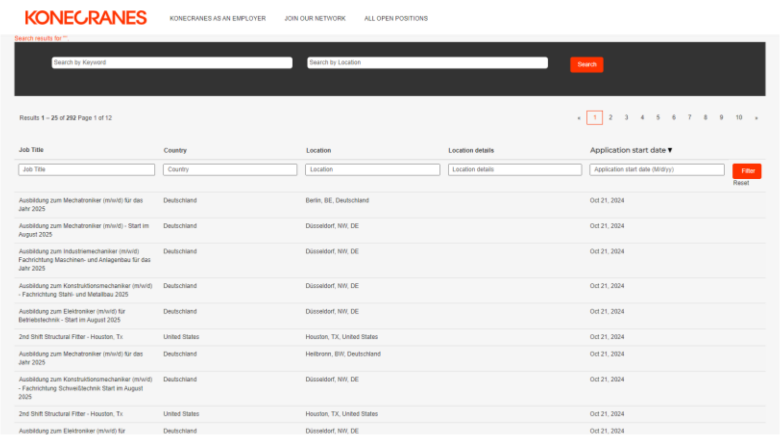
Jari Kaiturinmäki

Technical Product Manager - Process Cranes - Konecranes Finland Oy

jari.kaiturinmaki@konecranes.com

+358 40 573 21 41

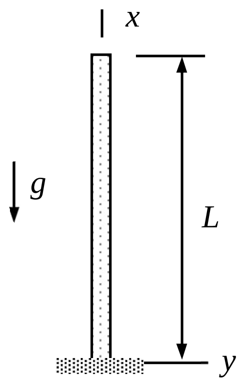
Konecranes Jobs



KONECRANES

Name________________________ Student number________________

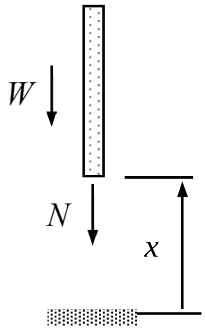
Assignment 1



The column of the figure is loaded by its own weight. Determine stress σ_{xx}, strain ε_{xx} and displacement u_x as functions of x. Cross-sectional area A and density ρ of the material are constants. Assume that stress and strain are related by Hooke's law $\sigma_{xx} = E\varepsilon_{xx}$.

Solution template

Let us start with the axial force N by considering the equilibrium of the column part shown



Weight of the column part $W = \rho Ag(L - x)$

Equilibrium equation $N + W = 0$

Axial force $N = \rho Ag(x - L)$

Stress at x follows from definition "force divided by the area" as directed area and force are aligned in the present problem.

Stress $\sigma_{xx} = \rho g(x - L)$.

Strain at x follows from the stress-strain relationship $\sigma_{xx} = E\varepsilon_{xx}$.

Strain $\varepsilon_{xx} = \dfrac{\rho g}{E}(x - L)$.

Displacement of the column at x follows from the definition of strain (strain-displacement relationship) $\varepsilon_{xx} = du_x / dx$ to be considered as an ordinary first order differential equation to displacement u_x. Let the integration constant be C.

Generic solution to displacement $u_x = \dfrac{\rho g}{E}(\dfrac{1}{2}x^2 - Lx) + C$

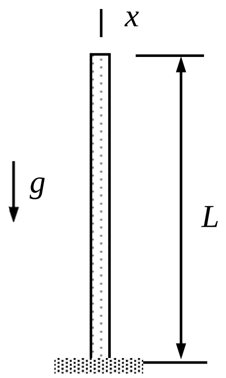
Displacement is known to vanish at $x = 0$. Elimination the integration constant by using the boundary condition $u_x(0) = 0$ gives the displacement for the problem.

Displacement $u_x = \dfrac{\rho g}{E}(\dfrac{1}{2}x^2 - Lx)$.

Name______________________ Student number________________

Assignment 2

Find the displacement $u(x)$ of the column shown by using the boundary value problem



$$EA\frac{d^2u}{dx^2}-\rho Ag=0 \quad x\in]0,L[\,,\quad u=0 \text{ when } x=0\,,\quad EA\frac{du}{dx}=0 \text{ when } x=L\,.$$

Assume that the cross-sectional area A, Young's modulus E of the material, density ρ of the material, and acceleration by gravity g are constants.

Solution template

First, repeated integrations with the differential equation are used to find the generic solution. Let the integration constants be a and b:

$$\frac{d^2u}{dx^2}=\frac{\rho g}{E} \quad\Rightarrow\quad \frac{du}{dx}=\frac{\rho g}{E}x+a \quad\Rightarrow\quad u(x)=\frac{\rho g}{E}\frac{1}{2}x^2+ax+b\,.$$

Second, boundary conditions are used to find the values of the integration constants a and b:

$$u(0)=b=0 \quad\text{and}\quad EA\frac{du}{dx}(L)=\frac{\rho g}{E}L+a=0 \quad\Rightarrow\quad b=0 \quad\text{and}\quad a=-\frac{\rho g}{E}L\,.$$

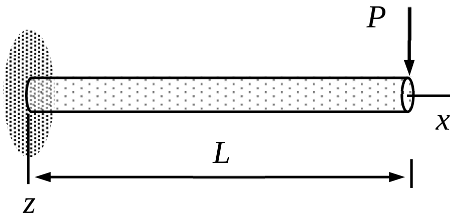
Finally, the values of the integration constants are substituted into the generic solution to get the displacement solution:

$$u(x)=\frac{\rho g}{E}(\frac{1}{2}x^2-xL)\,.$$

Name________________________ Student number________________

Assignment 3

Find the transverse displacement w and rotation θ of the $xz-$plane cantilever beam shown. Start with the Bernoulli beam model planar bending equations in their first order forms



$$\begin{Bmatrix} \dfrac{dQ}{dx} \\ \dfrac{dM}{dx}-Q \end{Bmatrix}=0\,,\quad \begin{Bmatrix} M \\ 0 \end{Bmatrix}=\begin{Bmatrix} EI\dfrac{d\theta}{dx} \\ \dfrac{dw}{dx}+\theta \end{Bmatrix},$$

where E and I are constants and Q, M are the shear force and bending moment, respectively,

Solution

There are two different ways to find the solution to rotation θ and the transverse displacement w. The first one is based on elimination of the variables from the first order equations above to get first an equation for the transverse displacement only. The second is based on integration of the first order equations directly.

Let us start with the elimination of the shear force, bending moment, and rotation. Considering also the conditions at the endpoints of the beam, the outcome is the boundary value problem: find $w(x)$ such that

$$-EI\frac{d^4w}{dx^4}=0 \quad x\in(0,L)\ , \quad w=-\frac{dw}{dx}=0 \quad x=0\,, \quad EI\frac{d^2w}{dx^2}=0 \quad x=L\,, \text{ and } -EI\frac{d^3w}{dx^3}=P \quad x=L\,.$$

The fourth order beam equation is the usual form of textboooks. The generic solution to the differential equation

$$w(x)=ax^3+bx^2+cx+d$$

contains 4 parameters. When subtituted in the boundary conditions

$$d=-c=0\,,\ EI(6aL+2b)=0\,, \text{ and } -EI(6a)=P \quad \Leftrightarrow \quad a=-\frac{P}{6EI}\,,\ \ b=\frac{PL}{2EI}\,, \text{ and } d=c=0\,.$$

The final expressions for the transverse displacement and rotation take the forms

$$w(x)=\frac{P}{6EI}(3Lx^2-x^3) \quad \text{and} \quad \theta(x)=-\frac{dw}{dx}=\frac{P}{2EI}(x^2-2Lx)\,.$$

The second method uses the first order equations one-by-one in certain order. Let us start with the equilibrium equations with boundary conditions at the free end

$$\frac{dQ}{dx}=0 \quad x\in(0,L) \text{ and } Q=P \quad x=L \quad \Rightarrow \quad Q(x)=P$$

$$\frac{dM}{dx}=Q=P \quad x\in(0,L) \text{ and } M=0 \quad x=L \quad \Rightarrow \quad M(x)=P(x-L)\,.$$

Knowing the force resultants, constitutive equation and the Bernoulli constraint can be integrated for the rotation and the transverse displacement

$$\frac{d\theta}{dx}=\frac{M}{EI}=\frac{P}{EI}(x-L) \quad x\in(0,L) \text{ and } \theta=0 \quad x=0 \quad \Rightarrow \quad \theta(x)=\frac{P}{2EI}(x^2-2xL)\,, \leftarrow$$

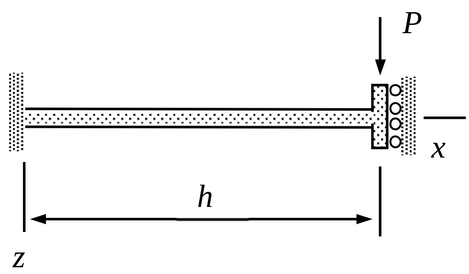
$$\frac{dw}{dx}=-\theta=-\frac{P}{EI}(\frac{1}{2}x^2-xL) \quad x\in(0,L) \text{ and } w=0 \quad x=0 \quad \Rightarrow \quad w(x)=-\frac{P}{6EI}(x^3-3x^2L)\,. \leftarrow$$

The latter method is usually more straightforward but requires integration of the equations in the “right order”.

Name________________________ Student number________________

Assignment 4

Consider building model on pages 7-9 of the lecture notes. Model the columns as massless bending beams, floors as massless rigid bodies, and assume that the floors move vertically in the XZ – plane. Find the vertical displacement u of the loading point as function of the weight F on the loading tray and thereby the effective stiffness (spring coefficient) k of the structure defined by $F = ku$. Use the displacement-force relationship for a typical column shown to deduce the displacement of the second floor.



Solution

As the floors are rigid bodies and the columns are rigidly connected to the floors, displacements of the first floor and the column end connected to the that are the same and rotations vanish. Let us start with the displacement-force relationship for the typical column. The Bernoulli beam model planar bending equations in their first order forms are given by

$$\begin{Bmatrix} \dfrac{dQ}{dx} \\ \dfrac{dM}{dx} - Q \end{Bmatrix} = 0, \quad \begin{Bmatrix} M \\ 0 \end{Bmatrix} = \begin{Bmatrix} EI\dfrac{d\theta}{dx} \\ \dfrac{dw}{dx} + \theta \end{Bmatrix},$$

where E and I are constants, and the boundary conditions require that rotation vanishes at both ends. Displacement at the left end vanishes and shear force is given at the right end. Elimination gives the fourth order boundary value problem to the transverse displacement

$$-EI\frac{d^4 w}{dx^4} = 0 \quad x \in (0,h), \quad \frac{dw}{dx} = 0 \quad x \in \{0,h\}, \quad w = 0 \quad x = h, \text{ and } -EI\frac{d^3 w}{dx^3} = P \quad x = h.$$

When the integration constants of the generic solution to the differential equation $w(x) = ax^3 + bx^2 + cx + d$ are chosen to satisfy the boundary conditions, the outcome is

$$w(x) = \frac{P}{12EI}(3hx^2 - 2x^3) \quad \Rightarrow \quad w(h) = \frac{Ph^3}{12EI}.$$

Now, there are four columns supporting a floor, each taking one fourth of the total force (the same force acts on the first and second floor columns) so $P = F/4$. In addition, the displacement of the second floor is twice that of the first floor so $u = 2w(h)$. Therefore

$$u = 2\frac{F}{4}h^3/(12EI) = \frac{Fh^3}{24EI}.$$

As the second moments of area is given by $I = \pi d^4/64$, the expression for the rigidity (spring coefficient) takes the form

$$k = \frac{3d^4 E\pi}{8h^3}.$$

When the values of the parameters are substituted there $k = 34.6\frac{\text{N}}{\text{mm}}$.

Name________________________ Student number________________

Assignment 5

Measure the effective rigidity of the structure model on pages 7-9 of the lecture notes. After finding enough loading-displacement pairs, process the data to find the experimental value of rigidity k ($F = ku$).

The set-up is located in Puumiehenkuja 5L (Konemiehentie side of the building). The hall is open during the office hours (9-12 and 13-16) on Wed 23.10.2024. Place a mass on the loading tray and record the reading of the displacement transducer. Gather enough loading-displacement data for finding the rigidity reliably.

Solution

Table below shows the loading-displacement pairs (F, u) given by the experiment. Displacement of the loading point is obtained as the mean value u of displacements u_1 and u_2 given by transducers. In the rigidity estimation, one assumes that displacement and loading are related by $F = ku$, experiment gives a sample of that (with possibly random error due to various sources), and the task is to find the k for the best fit of data.

m	u_1	u_2	F	u
kg	mm	mm	N	mm
0	0	0	0	0
1.824	0.505	0.491	17.89	0.498
3.672	1.034	1.006	26.02	1.020
4.668	1.312	1.277	45.79	1.294
7.171	3.40	2.021	1.967	1.994

To find the best fit of the form $F = ku$ one may use, e.g., the least-squares method which gives the value of k as the minimizer of function

$$\prod(k) = \frac{1}{2}\sum (ku_i - F_i)^2 ,$$

where the sum is over all the measured value pairs. The method looks for a k which gives as good as possible overall match to the data. At the minimum point, derivative of $\prod(k)$ with respect to k vanishes, so

$$\frac{d\prod(k)}{dk} = \sum u_i(ku_i - F_i) = 0$$

or when solved for rigidity

$$k = \frac{\sum u_i F_i}{\sum u_i u_i} \quad .$$

Finally, substituting the values in the table

$$k \approx 35 \frac{\text{N}}{\text{mm}} .$$

MEC-E1050

FINITE ELEMENT METHOD IN SOLIDS 2024

Week 44-0

2 DISPLACEMENT ANALYSIS

LEARNING OUTCOMES

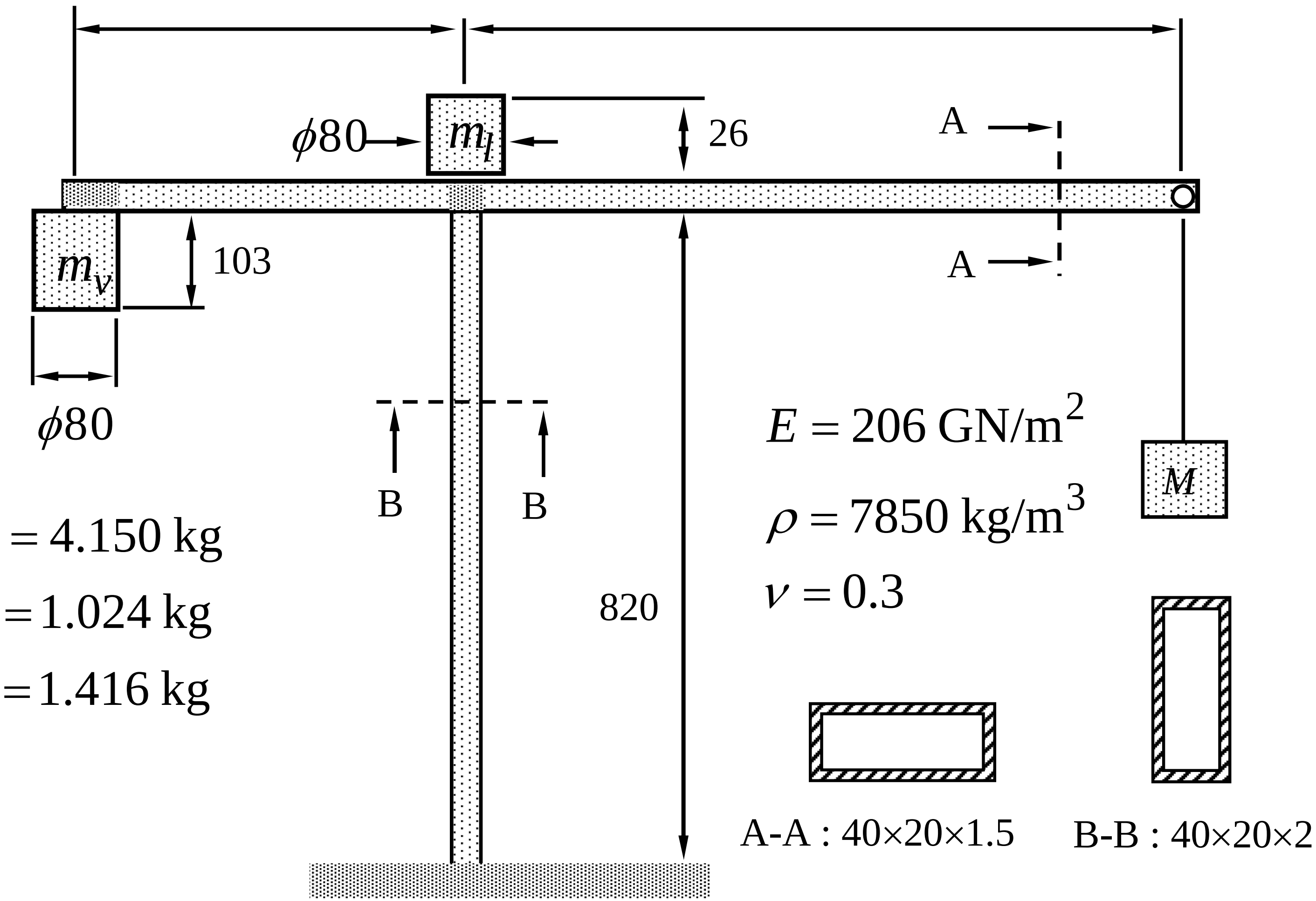
Students are able to solve the weekly lecture problems, home problems, and exercise problems on topics of the week:

- □ Engineering paradigm, elements and nodes, structural and material coordinate systems, displacements and rotations

- □ Equilibrium equations of nodes and element contributions (force-displacement relationships)

- □ Derivation of tension bar, torsion bar, and bending beam element contributions from the exact solutions to the corresponding boundary value problems.

2.1 STRUCTURE ANALYSIS



$m_v = 4.150$ kg

$m_l = 1.024$ kg

$M = 1.416$ kg

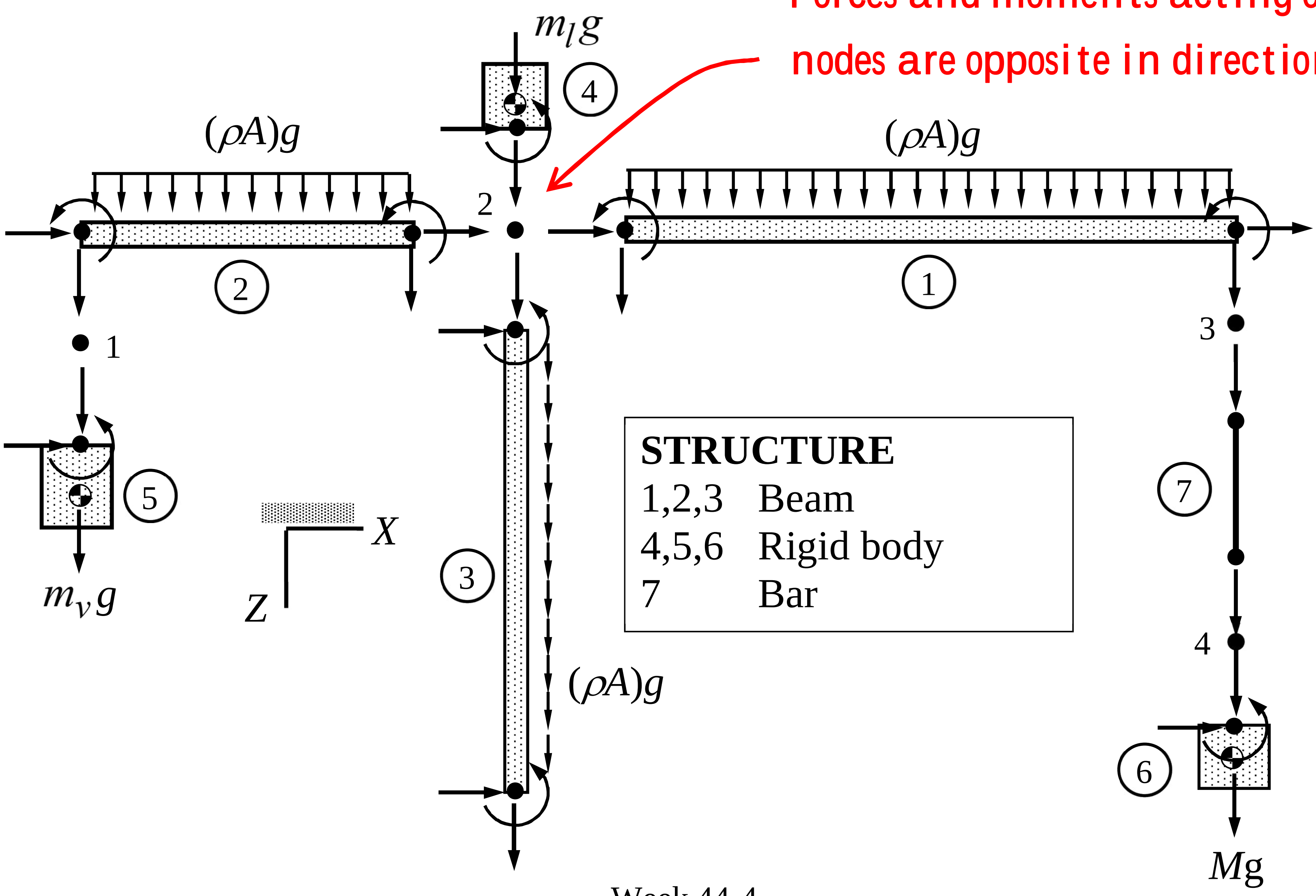
$E = 206\ \mathrm{GN/m^2}$

$\rho = 7850\ \mathrm{kg/m^3}$

$\nu = 0.3$

ELEMENTS AND NODES

Forces and moments acting on the nodes are opposite in directions!



NEWTON's LAWS OF MOTION

I In an inertial frame of reference, an object either remains at rest or continues to move at a constant velocity, unless acted upon by a force.

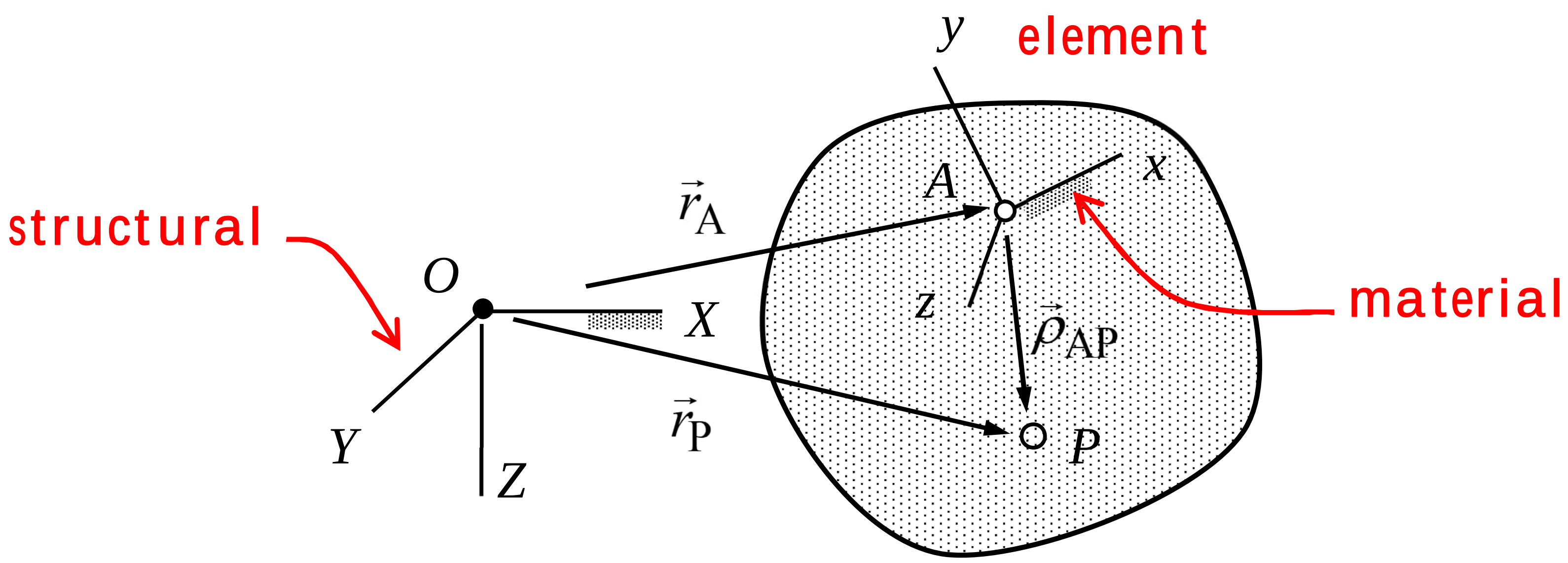
II The vector sum of the forces on an object is equal to the mass of that object multiplied by the acceleration of the object (assuming that the mass is constant).

III When one body exerts a force on a second body, the second body simultaneously exerts a force equal in magnitude and opposite in direction on the first body.

Newton's laws in the original form apply to particles only. The formulation for rigid bodies and deformable bodies require slight modifications.

COORDINATE SYSTEMS

The particles of elements are identified by coordinates (x, y, z) of the *material coordinate system* which moves and deforms with the body (in principle). The unique *structural coordinate system* (X, Y, Z) is needed, e.g., in description of geometry.



The basis vectors of the material and structural systems are denoted by $\vec{i}, \vec{j}, \vec{k}$ and $\vec{I}, \vec{J}, \vec{K}$, respectively!

SIGN CONVENTIONS AND NOTATIONS

Displacements, rotations, forces and moments are vector quantities (magnitude and direction) so the components are taken to be positive in the directions of the chosen coordinate axes. The convention may differ from that used in mechanics of materials courses (be careful).

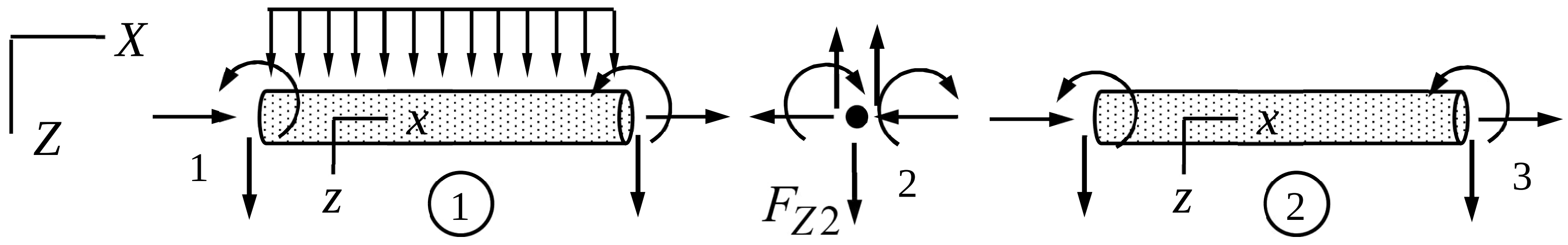
	Displacement	**Force**	**Rotation**	**Moment**
Material	u_x, u_y, u_z	F_x, F_y, F_z	$\theta_x, \theta_y, \theta_z$	M_x, M_y, M_z
Structural	u_X, u_Y, u_Z	F_X, F_Y, F_Z	$\theta_X, \theta_Y, \theta_Z$	M_X, M_Y, M_Z

Representation in one system can be transformed into another assuming that the relative orientations of the axes are known (example).

FREE BODY DIAGRAMS

Index e refers to an *element* (in a figure ○) and i to a *node* (in a figure **•**):

$$F_{X1}^{1}, F_{Z1}^{1}, M_{Y1}^{1} \qquad F_{X2}^{1}, F_{Z2}^{1}, M_{Y2}^{1} \qquad F_{X2}^{2}, F_{Z2}^{2}, M_{Y2}^{2} \qquad F_{X3}^{2}, F_{Z3}^{2}, M_{Y3}^{2}$$



External known forces and moments are acting on the nodes. Internal forces between the elements satisfy the law of action and reaction (Newton III) and act through the nodes. Components acting on the elements are considered as positive (with respect to the material coordinate system of the element).

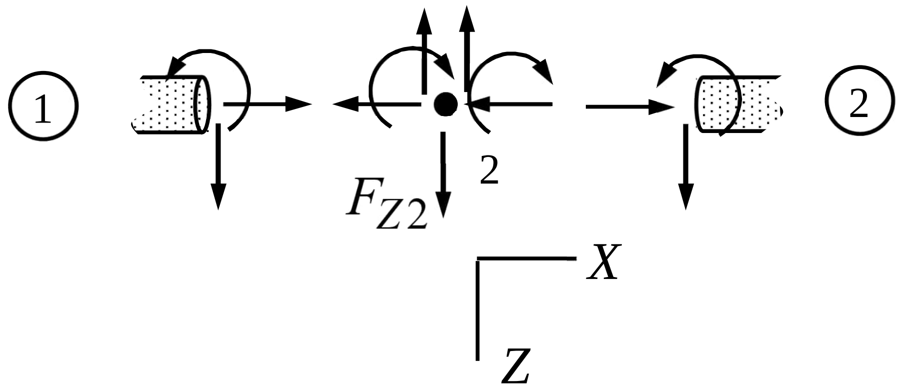
EQUILIBRIUM EQUATIONS

Principles of momentum and moment of momentum are applied to the nodes of structure: As nodes are massless particles (or control points), the sums of the forces and moments acting on the nodes $i \in I$ must vanish

$$-\sum_{e\in E} F_{Xi}^{e} + F_{Xi} = 0, \quad -\sum_{e\in E} M_{Xi}^{e} + M_{Xi} = 0$$

$$-\sum_{e\in E} F_{Yi}^{e} + F_{Yi} = 0, \quad -\sum_{e\in E} M_{Yi}^{e} + M_{Yi} = 0$$

$$-\sum_{e\in E} F_{Zi}^{e} + F_{Zi} = 0, \quad -\sum_{e\in E} M_{Zi}^{e} + M_{Zi} = 0$$



The sums extend over the elements connected to node i. Equilibrium equations for the constrained displacements and rotations may contain constraint forces and moments treated as unknown external forces in MEC-E1050.

2.2 DISPLACEMENT ANALYSIS

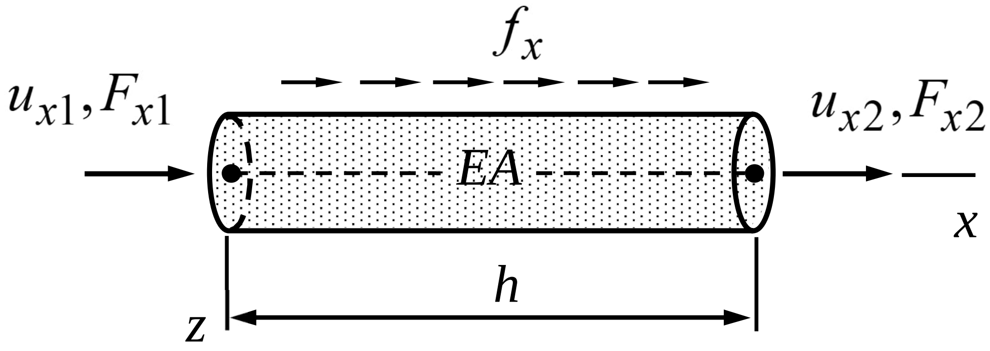
- Idealize a complex structure as a set of elements, whose behavior can be approximated by using the usual engineering models (bar, beam, plate, rigid body etc.).

- Write down the equilibrium equations of the nodes, the force-displacement relationships of the elements (element contributions), and constraints concerning the nodal displacements.

- Solve the nodal displacements and rotations and the internal forces and moments acting on the elements from the equation system.

- Determine the stress in elements one-by-one (optional step).

BAR ELEMENT

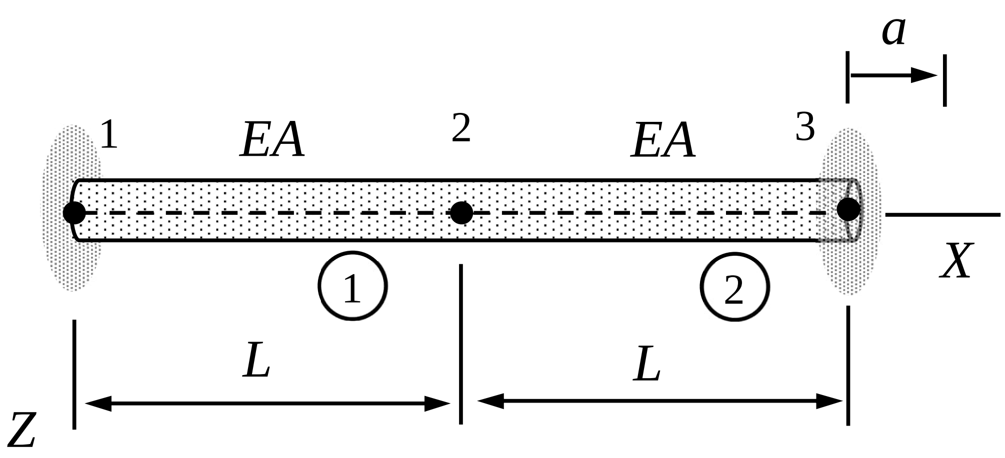


$$\begin{Bmatrix} F_{x1} \\ F_{x2} \end{Bmatrix} = \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

force-displacement relationship of a bar element!

The force-displacement relationships of elements are always expressed in material coordinate systems. In calculations, the displacement, rotation, force, and moment components of material coordinate systems need to be expressed in terms of those of the structural system.

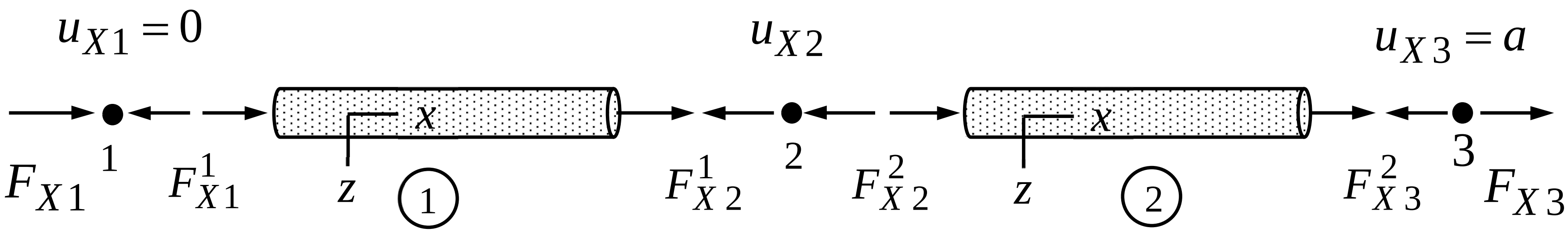
EXAMPLE 2.1 Determine the nodal displacements and the forces acting on the elements 1 and 2 of the figure. The displacement of node 3 is known to be a, EA is constant and the structure consists of two bars.



Answer $u_{X2} = \dfrac{a}{2}$, $F^1_{X1} = -\dfrac{EA}{2L}a$, $F^1_{X2} = \dfrac{EA}{2L}a$, $F^2_{X2} = -\dfrac{EA}{2L}a$, $F^2_{X3} = \dfrac{EA}{2L}a$,

$$F_{X1} = -\frac{EA}{2L}a,\ \ F_{X3} = \frac{EA}{2L}a.$$

- Free body diagram shows all the forces acting on the two bar elements and three nodes. External constraint forces F_{X1} and F_{X3} acting on points 1 and 3 due to the walls are unknown quantities of the problem, whereas displacements of points 1 and 3 are known ($u_{X1}=0$, $u_{X3}=a$).



- As the axes of the structural and material coordinate systems coincide in this case so for element 1 $F^1_{x1}=F^1_{X1}$, $F^1_{x2}=F^1_{X2}$, $u^1_{x1}=u_{X1}=0$, $u^1_{x2}=u_{X2}$ and for element 2 $F^2_{x2}=F^2_{X2}$, $F^2_{x3}=F^2_{X3}$, $u^2_{x2}=u_{X2}$, and $u^2_{x3}=u_{X3}=a$. In terms of the force and displacement components in the structural system, bar element contributions and the equilibrium equations of nodes are

$$\begin{Bmatrix} F_{X1}^{1} \\ F_{X2}^{1} \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_{X2} \end{Bmatrix}, \quad \begin{Bmatrix} F_{X2}^{2} \\ F_{X3}^{2} \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ a \end{Bmatrix},$$

$$F_{X1} - F_{X1}^{1} = 0, \quad -F_{X2}^{1} - F_{X2}^{2} = 0, \quad -F_{X3}^{2} + F_{X3} = 0$$

- The seven unknowns $F_{X1}, F_{X1}^{1}, F_{X2}^{1}, F_{X2}^{2}, F_{X3}^{2}, F_{X3}, u_{X2}$ can be solved from the system of seven equations above. The unknown displacement follows from the equilibrium equation of node 2 after elimination of the internal forces:

$$F_{X2}^{1} + F_{X2}^{2} = \frac{EA}{L}u_{X2} + \frac{EA}{L}u_{X2} - \frac{EA}{L}a = 0 \quad \Leftrightarrow \quad u_{X2} = \frac{a}{2}.$$

- After that, internal forces follow from the element contributions (the components in the material coordinate system are more useful, e.g., in stress calculations)

$$\begin{Bmatrix} F_{x1}^{1} \\ F_{x2}^{1} \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ a/2 \end{Bmatrix} = \frac{EA}{L}\frac{a}{2}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix},$$

$$\begin{Bmatrix} F_{x2}^{2} \\ F_{x3}^{2} \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} a/2 \\ a \end{Bmatrix} = \frac{EA}{L}\frac{a}{2}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}.$$

- Finally, the constraint forces (boundary reactions) follow from the remaining two equilibrium equations of the boundary nodes 1 and 2.

$$F_{X1} = F_{X1}^{1} = -\frac{EA}{2L}a, \quad F_{X3} = F_{X3}^{2} = \frac{EA}{2L}a.$$

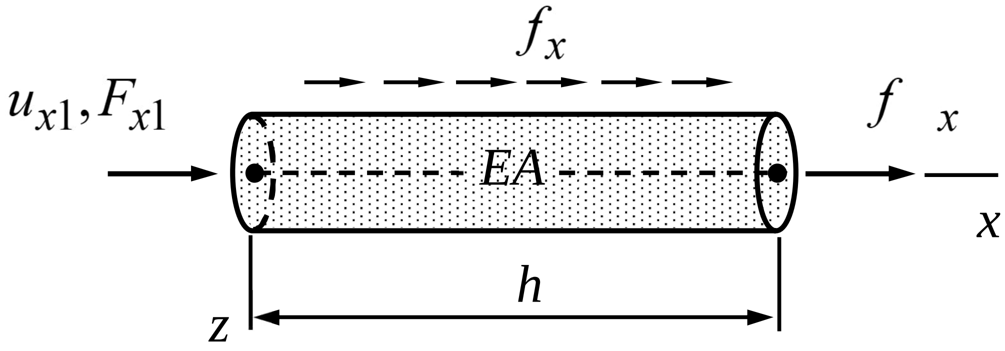
2.3 ELEMENT CONTRIBUTION

- Find the generic solution to the differential equation of the model. Assume that the external distributed forces are simple (for example constant or linear).

- Express the integration constants of the generic solution in terms of the nodal displacement and rotations. The number of integration constants and the number of nodal displacement and rotations should naturally match.

- Substitute the displacement back into the force-displacement relationship of the model and rearrange to get a matrix representation of the form $\mathbf{R} = \mathbf{K}\mathbf{a} - \mathbf{F}$ (to be called as the element contribution) in which $\mathbf{R}$ contains the nodal forces and moments and $\mathbf{a}$ the nodal displacements and rotations.

BAR ELEMENT



$$\begin{Bmatrix} F_{x1} \\ F_{x2} \end{Bmatrix} = \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

the force-displacement relationship of bar element!

Element contributions are always expressed in material coordinate systems. However, in calculations, the displacement, rotation, force, and moment components of material coordinate systems need to be expressed in terms of those of the structural system.

- Boundary value problem for a bar element of length h

$$EA\frac{d^2u}{dx^2}+f_x=0 \quad x\in]0,h[\qquad \text{(equilibrium equation)}$$

$$u(0)=u_{x1} \ \text{ and } \ u(h)=u_{x2} \quad \text{(given nodal displacements)}$$

$$EA\frac{du}{dx}(0)=-F_{x1} \ \text{ and } \ EA\frac{du}{dx}(h)=F_{x2} \ \text{(force-displacement relationship)}$$

- The generic solution to the equilibrium equation (f_x and EA are constants) is given by

$$u=a+bx-\frac{f_x}{2EA}x^2=\{1 \quad x\}\begin{Bmatrix} a \\ b \end{Bmatrix}-\frac{f_x}{2EA}x^2.$$

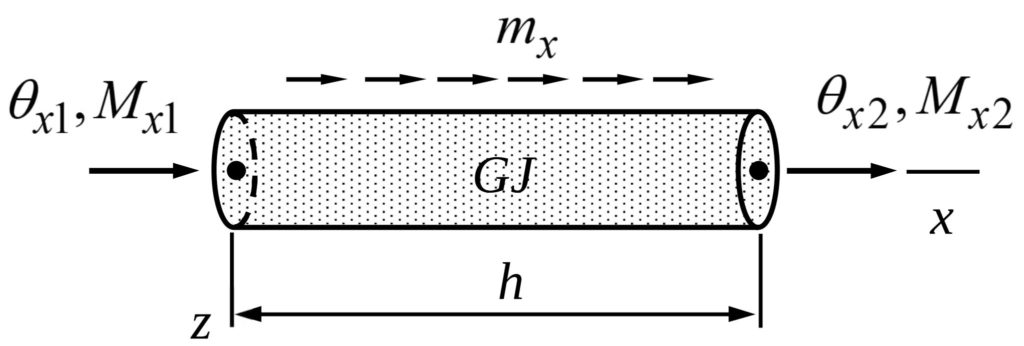
- Integration constants a and b need to be expressed in terms of the nodal displacements u_{x1} and u_{x2}

$$\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} = \begin{Bmatrix} u(0) \\ u(h) \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & h \end{bmatrix} \begin{Bmatrix} a \\ b \end{Bmatrix} - \frac{f_x}{2EA} \begin{Bmatrix} 0 \\ h^2 \end{Bmatrix} \Leftrightarrow \quad \begin{Bmatrix} a \\ b \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & h \end{bmatrix}^{-1} \left(\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} + \frac{f_x}{2EA} \begin{Bmatrix} 0 \\ h^2 \end{Bmatrix} \right)$$

- The relationship $\mathbf{R} = \mathbf{Ka} - \mathbf{F}$ between the nodal forces and displacement (element contribution) follows from the force-displacement relationship of the model

$$\begin{Bmatrix} F_{x1} \\ F_{x2} \end{Bmatrix} = EA \begin{Bmatrix} -\dfrac{du}{dx}(0) \\ \dfrac{du}{dx}(h) \end{Bmatrix} = \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}. \quad \Leftarrow$$

TORSION ELEMENT

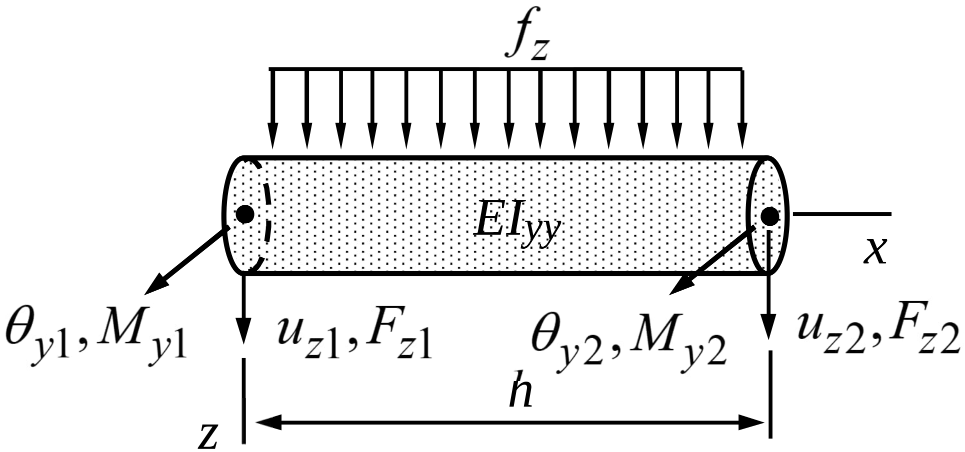


$$\begin{Bmatrix} M_{x1} \\ M_{x2} \end{Bmatrix} = \frac{GJ}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix} - \frac{m_x h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

the force-displacement relationship of torsion bar element!

The force-displacement relationships of elements are always expressed in material coordinate systems. However, in calculations, the displacement, rotation, force, and moment components of material coordinate systems need to be expressed in terms of those of the structural system.

BENDING BEAM ELEMENT



$$\begin{Bmatrix} F_{z1} \\ M_{y1} \\ \hline F_{z2} \\ M_{y2} \end{Bmatrix} = \frac{EI_{yy}}{h^3} \left[\begin{array}{cc:cc} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ \hdashline -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{array}\right] \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ \hline u_{z2} \\ \theta_{y2} \end{Bmatrix} - \frac{f_z h}{12} \begin{Bmatrix} 6 \\ -h \\ \hline 6 \\ h \end{Bmatrix}$$

the force-displacement relationship of bending beam element!

Notice that the displacements, rotations, forces, and moments are components of vectors of the material coordinate system!

- Boundary value problem for a bending beam (Bernoulli) element of length h

$$\frac{d^2}{dx^2}(EI_{yy}\frac{d^2w}{dx^2}) - f_z = 0 \quad x \in]0,h[\qquad \text{(equilibrium equation)}$$

$$\frac{dw}{dx}(0) = -\theta_{y1}, \quad \frac{dw}{dx}(h) = -\theta_{y2}, \quad w(0) = u_{z1}, \text{ and } \quad w(h) = u_{z2} \qquad \text{(boundary conditions)}$$

$$\frac{d^2w}{dx^2}(0) = \frac{M_{y1}}{EI_{yy}}, \quad \frac{d^2w}{dx^2}(h) = -\frac{M_{y2}}{EI_{yy}}, \quad \frac{d^3w}{dx^3}(0) = \frac{F_{z1}}{EI_{yy}}, \text{ and } \quad \frac{d^3w}{dx^3}(h) = -\frac{F_{z2}}{EI_{yy}}.$$

- First, the generic solution to the equilibrium equation is given by (f_z and EI_{yy} constants)

$$w(x) = a + bx + cx^2 + dx^3 + \frac{f_z}{24EI_{yy}}x^4 \quad (a,b,c,d \text{ are integration constants})$$

- Second, the integration constants a,b,c,d are expressed in terms of displacements u_{z1}, u_{z2} and rotations θ_{y1}, θ_{y2} by using conditions

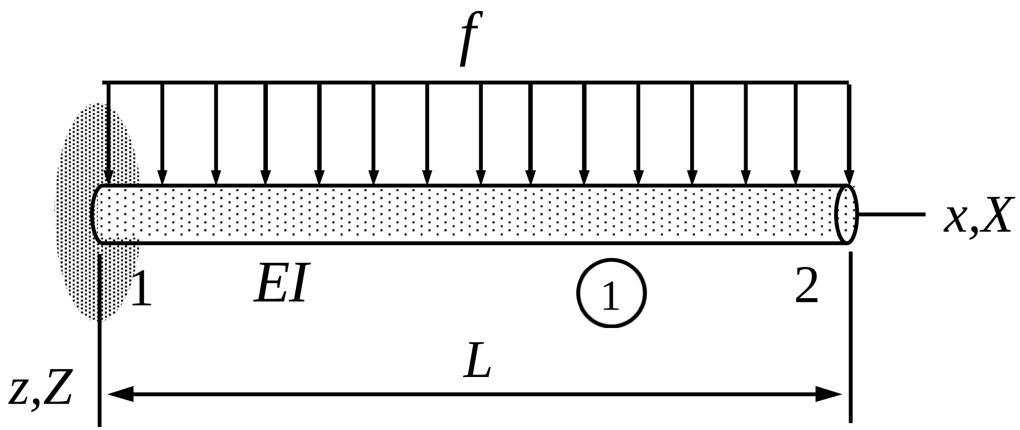
$$w(0)=u_{z1},\ w(h)=u_{z2},\ \frac{dw}{dx}(0)=-\theta_{y1},\ \text{and}\ \frac{dw}{dx}(h)=-\theta_{y2}$$

  Notice that derivatives and rotations are positive in opposite directions.

- Third, by using the force/moment-displacement/rotation relationship of the model

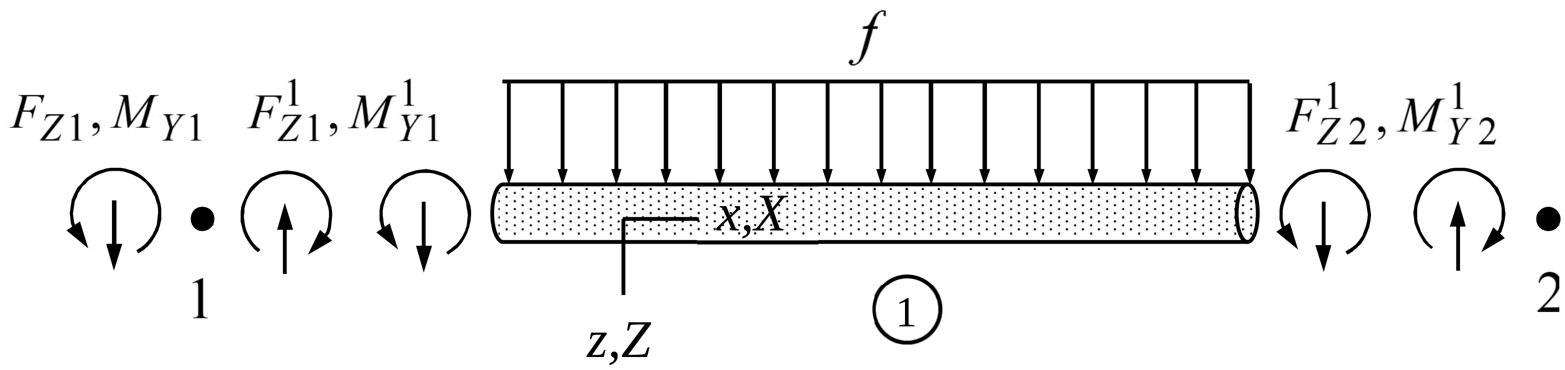
$$\begin{Bmatrix} F_{z1} \\ M_{y1} \\ \hdashline F_{z2} \\ M_{y2} \end{Bmatrix} = \frac{EI_{yy}}{h^3} \left[\begin{array}{cc:cc} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ \hdashline -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{array}\right] \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ \hdashline u_{z2} \\ \theta_{y2} \end{Bmatrix} - \frac{f_z h}{12} \begin{Bmatrix} 6 \\ -h \\ \hdashline 6 \\ h \end{Bmatrix}. \Leftarrow$$

EXAMPLE 2.2. Consider a bending beam loaded by its own weight and clamped on its left end (figure). Determine the displacement and rotation at the right-end by using one beam element. Bending rigidity of the beam EI is constant.



Answer $u_{Z2} = \frac{1}{8}\frac{fL^4}{EI}$ and $\theta_{Y2} = -\frac{1}{6}\frac{fL^3}{EI}$

- Free body diagram shows all the forces acting on the beam element and the two nodes. External constraint force and moment F_{Z1} and M_{Y1} acting on node 1 are unknown quantities of the problem, whereas displacement and rotation at the wall are known ($u_{Z1}=0,\ \theta_{Y1}=0$). At node 2, external forces are known (zeros), whereas the displacement are rotation u_{Z2} and θ_{Y2} are unknown.



- Element contribution, when the known displacement and rotation at the left end $u_{Z1} = \theta_{Y1} = 0$ are substituted there, and the equilibrium equations of nodes 1 and 2:

$$\begin{Bmatrix} F_{Z1}^1 \\ M_{Y1}^1 \\ F_{Z2}^1 \\ M_{Y2}^1 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_{Z2} \\ \theta_{Y2} \end{Bmatrix} - \frac{fL}{12} \begin{Bmatrix} 6 \\ -L \\ 6 \\ L \end{Bmatrix}.$$

Node 1: $F_{Z1} - F_{Z1}^1 = 0$ and $M_{Y1} - M_{Y1}^1 = 0$

Node 2: $-F_{Z2}^1 = 0$ and $-M_{Y2}^1 = 0$

- By eliminating the internal forces from the equilibrium equations which do NOT contain constraint forces (node 2) with the expressions of the element contribution

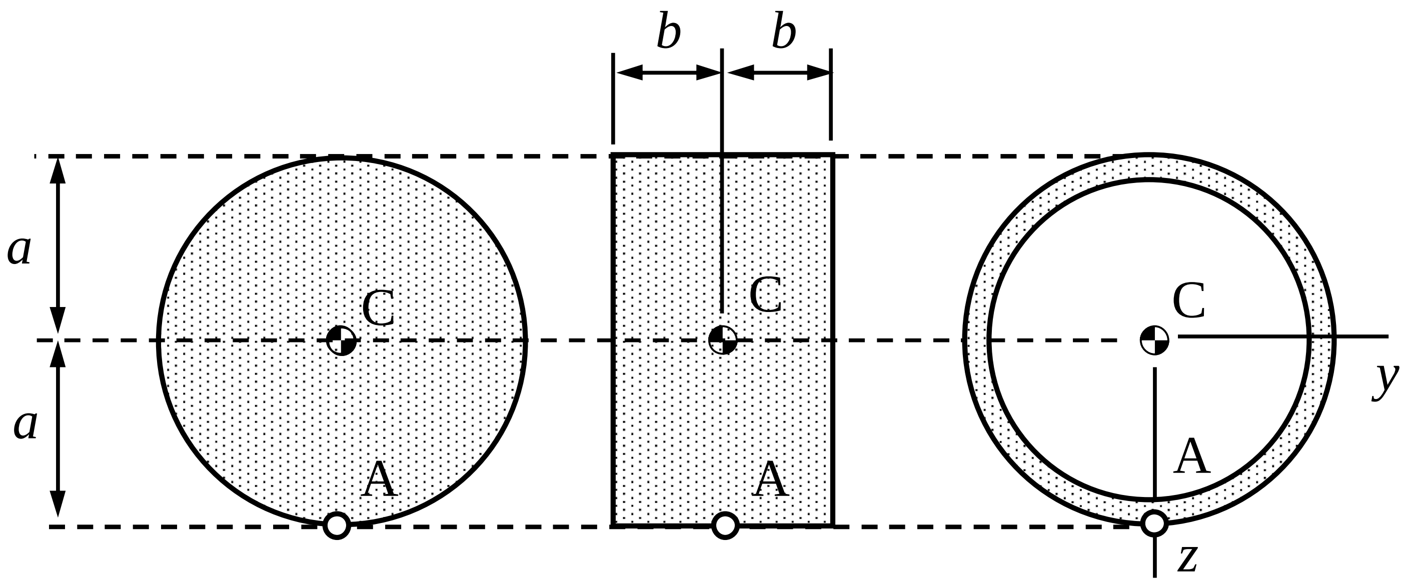
$$-\begin{Bmatrix} F_{Z2}^{1} \\ M_{Y2}^{1} \end{Bmatrix} = -\left(\frac{EI}{L^3}\begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix} - \frac{fL}{12}\begin{Bmatrix} 6 \\ L \end{Bmatrix}\right) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

- The equilibrium equations in this form give the solution to the displacement and rotation at the right-end

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix} - \frac{fL}{12}\begin{Bmatrix} 6 \\ L \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \Leftrightarrow \quad \begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix} = \frac{fL^4}{12EI}\begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix}^{-1}\begin{Bmatrix} 6 \\ L \end{Bmatrix} \quad \Leftrightarrow$$

$$\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix} = \frac{fL^4}{12EI}\begin{Bmatrix} 3/2 \\ -2/L \end{Bmatrix}. \leftarrow$$

SECOND MOMENTS OF CROSS-SECTION



$$I_{zz} = \frac{\pi}{4} a^4 \qquad I_{zz} = \frac{4}{3} ab^3 \qquad I_{zz} = \frac{\pi}{4}[a^4 - (a - t)^4]$$

MATERIAL PARAMETERS

Material	ρ **[**kg / m^3**]**	E [GN / m^2]	ν **[** 1 **]**
Steel	7800	210	0.3
Aluminum	2700	70	0.33
Copper	8900	120	0.34
Glass	2500	60	0.23
Granite	2700	65	0.23
Birch	600	16	-
Rubber	900	10^{-2}	0.5
Concrete	2300	25	0.1

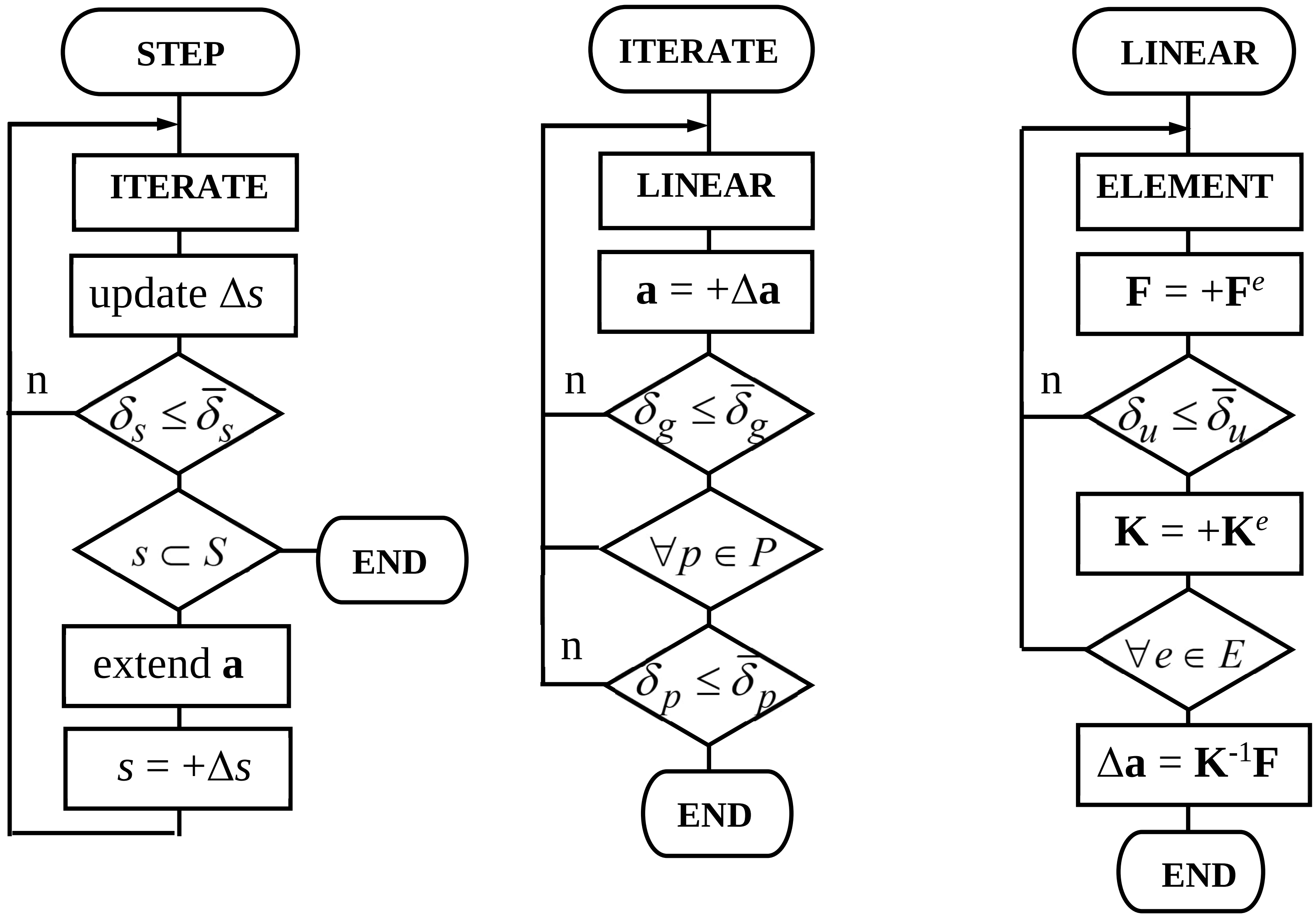
FORCE ELEMENT

In MEC-E1050, point forces and moments are taken into account by using a one-node force-moment element. The element contribution is given by

$$\begin{Bmatrix} F_{Xi} \\ F_{Yi} \\ F_{Zi} \end{Bmatrix} = -\begin{Bmatrix} F_X \\ F_Y \\ F_Z \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} M_{Xi} \\ M_{Yi} \\ M_{Zi} \end{Bmatrix} = -\begin{Bmatrix} M_X \\ M_Y \\ M_Z \end{Bmatrix}.$$

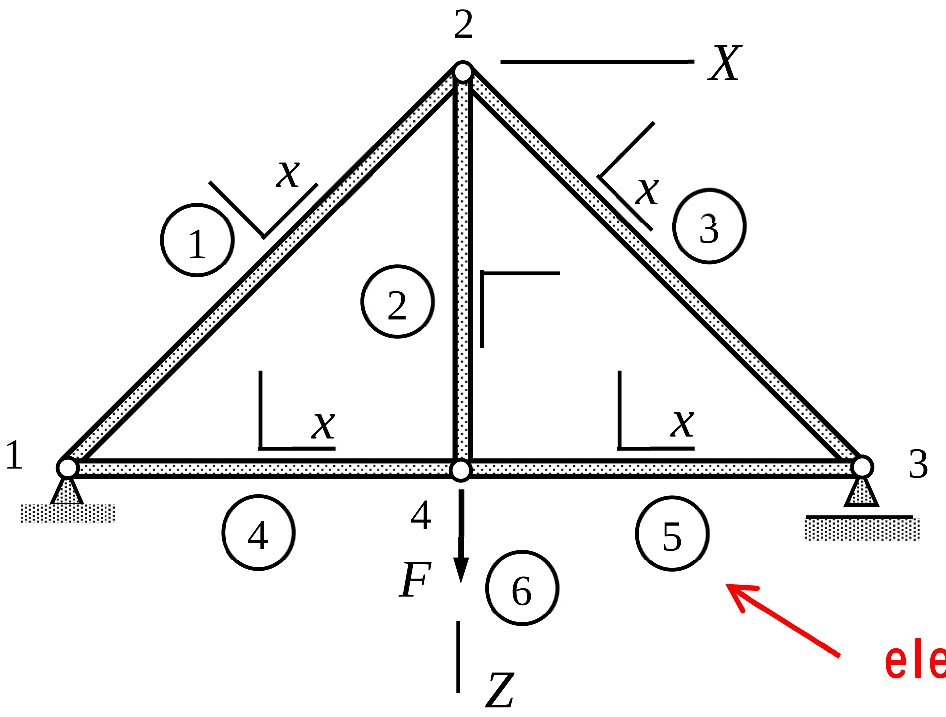
The quantities on the right-hand side are known whereas those on the left-hand side represent internal forces acting on the nodes. These “element contributions” are written directly in the structural system as that is needed finally.

2.4 ALGORITHM AND DATA STRUCTURE OF FEA



ELEMENT TABLE

Element table contains the quantities associated with elements. The table indicates also the topology of the structure (how elements are connected).

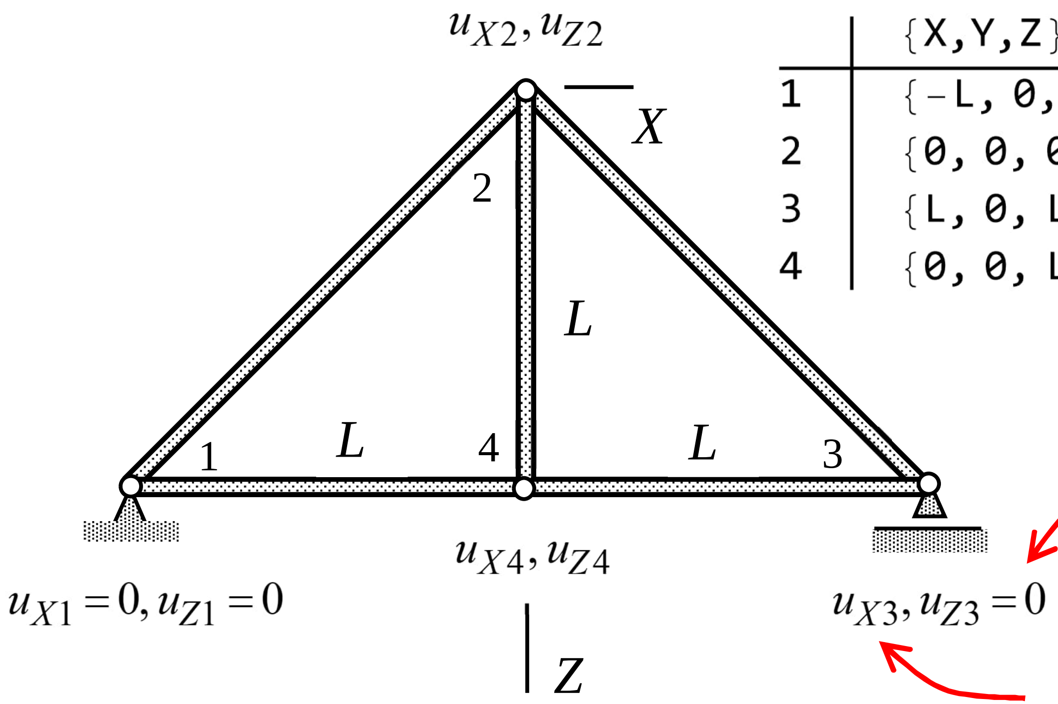


	model	properties	geometry
1	BAR	{{E}, {A}}	Line[{1, 2}]
2	BAR	{{E}, {A}}	Line[{2, 4}]
3	BAR	{{E}, {A}}	Line[{2, 3}]
4	BAR	{{E}, {A}}	Line[{1, 4}]
5	BAR	{{E}, {A}}	Line[{4, 3}]
6	FORCE	{0, 0, F}	Point[{4}]

The order of the nodal numbers in the table depends on the orientation of the elementwise material coordinate systems.

NODE TABLE

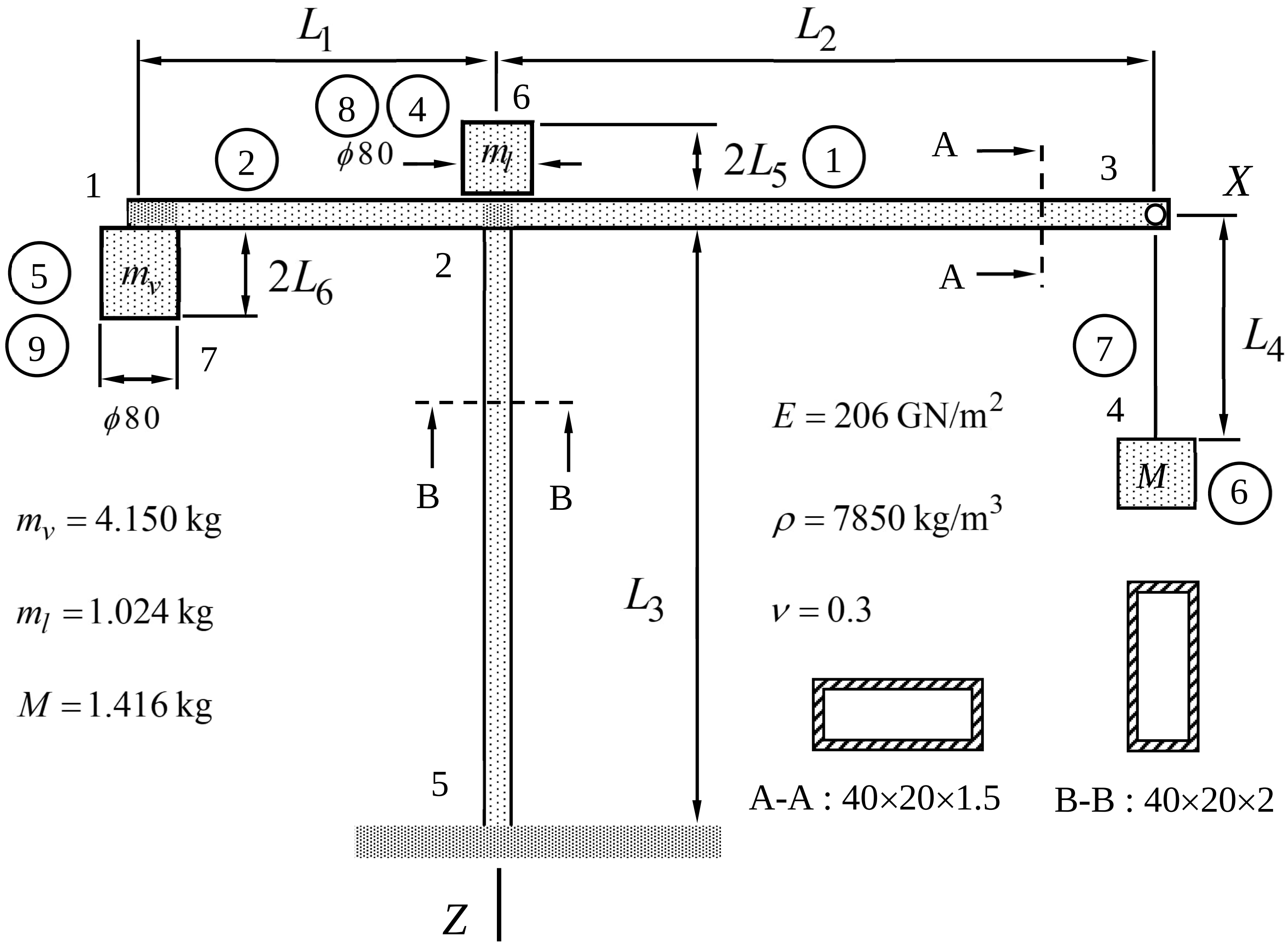
Node table contains the quantities associated with nodes. Nodal coordinates define the actual geometry. Nodal displacements and rotations represent the unknowns of the problem.



	{X,Y,Z}	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	{-L, 0, L}	{0, 0, 0}	{0, 0, 0}
2	{0, 0, 0}	{uX[2], 0, uZ[2]}	{0, 0, 0}
3	{L, 0, L}	{uX[3], 0, 0}	{0, 0, 0}
4	{0, 0, L}	{uX[4], 0, uZ[4]}	{0, 0, 0}

If the value of a nodal displacement or rotation is known, the value is used instead of a symbol in the table!

MINIATURE MODEL OF A CRANE



- Problem description consists of the element and node tables:

	model	properties	geometry
1	BEAM	{{E}, {A, I, I}, {0, 0, A g ρ}}	Line[{2, 3}]
2	BEAM	{{E}, {A, I, I}, {0, 0, A g ρ}}	Line[{1, 2}]
3	BEAM	{{E}, {A, I, I}, {0, 0, A g ρ}}	Line[{2, 5}]
4	FORCE	{0, 0, g ml}	Point[{6}]
5	FORCE	{0, 0, g mv}	Point[{7}]
6	FORCE	{0, 0, g M}	Point[{4}]
7	BAR	{{E}, {a}}	Line[{3, 4}]
8	RIGID	{}	Line[{6, 2}]
9	RIGID	{}	Line[{7, 1}]

	{X,Y,Z}	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	{-L1, 0, 0}	{uX[1], 0, uZ[1]}	{0, θY[1], 0}
2	{0, 0, 0}	{uX[2], 0, uZ[2]}	{0, θY[2], 0}
3	{L2, 0, 0}	{uX[3], 0, uZ[3]}	{0, θY[3], 0}
4	{L2, 0, L4}	{uX[3], 0, uZ[4]}	{0, 0, 0}
5	{0, 0, L3}	{0, 0, 0}	{0, 0, 0}
6	{0, 0, -L5}	{uX[6], 0, uZ[6]}	{0, θY[6], 0}
7	{-L1, 0, L6}	{uX[7], 0, uZ[7]}	{0, θY[7], 0}

ASSEMBLY OF SYSTEM EQUATIONS

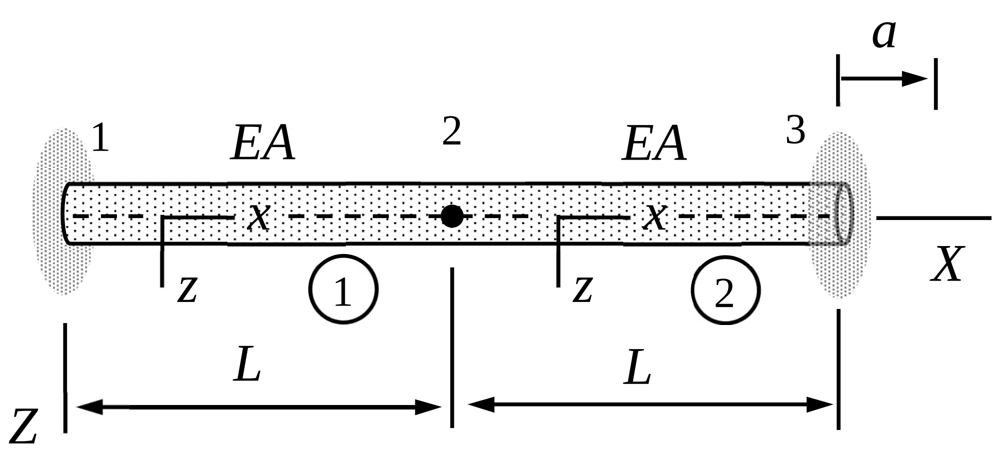
- ☐ Number the elements and nodes of the structure and express the problem data in the form of element and node tables (unless the structure is very simple),

- ☐ Write the element contributions $\mathbf{R}^e = \mathbf{Ka} - \mathbf{F}$ in terms of the displacement and rotation components of the structural coordinate system.

- ☐ Assemble the system equations $\mathbf{R} = \sum_{e \in E} \mathbf{R}^e = 0$ by summing the internal forces acting on the nodes in directions where displacements and rotations are not constrained.

- ☐ Solve the unknown displacements and rotations from the system equations $\mathbf{Ka} - \mathbf{F} = 0$ ($\mathbf{a} = \mathbf{K}^{-1}\mathbf{F}$).

EXAMPLE 2.3. Solve the nodal displacement u_{X2} of the bar shown, if the displacement of node 3 is a and rigidity EA is constant.



Answer $u_{X2} = \dfrac{a}{2}$

- Problem description consists of the element and node tables. In this case, the number of unknown displacements is one as the displacement of the right end is known.

	model	properties	geometry
1	BAR	{{E}, {A}}	Line[{1, 2}]
2	BAR	{{E}, {A}}	Line[{2, 3}]

	{X,Y,Z}	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}
2	{L, 0, 0}	{uX[2], 0, 0}	{0, 0, 0}
3	{2 L, 0, 0}	{a, 0, 0}	{0, 0, 0}

- Element contributions need to be written in terms of displacement and force components of the structural coordinate system. As the material and structural systems coincide and $f_x = 0$

$$\text{Bar 1}: \begin{Bmatrix} F_{X1}^{1} \\ F_{X2}^{1} \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ u_{X2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix},$$

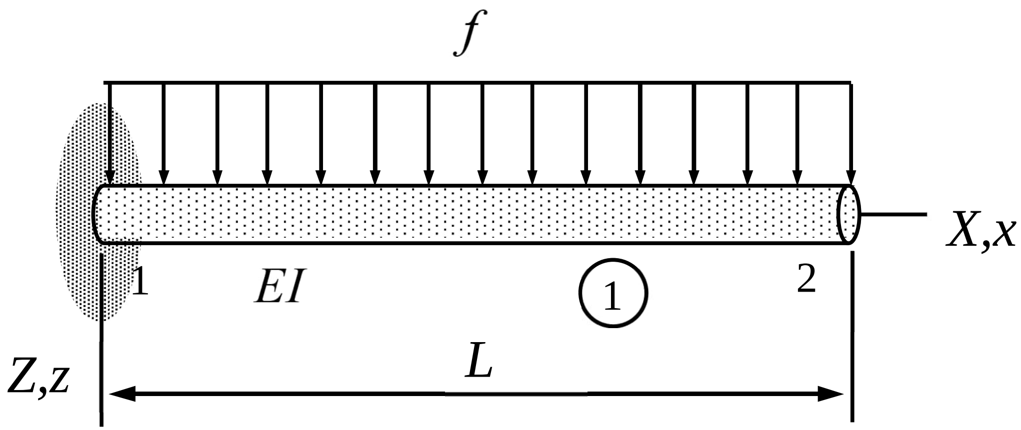
$$\text{Bar 2}: \begin{Bmatrix} F_{X2}^{2} \\ F_{X3}^{2} \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{X2} \\ a \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

- Sum of the internal forces, external and constraint forces acting on the nodes should vanish for the equilibrium. According to the algorithm, it is enough to consider the non-constrained directions for displacements and rotations. Hence

$$F_{X2}^{1} + F_{X2}^{2} = \frac{EA}{L} u_{X2} + \frac{EA}{L}(u_{X2} - a) = 0 \quad \Leftrightarrow \quad u_{X2} = \frac{a}{2}. \quad \Leftarrow$$

The problem can also be solved by the Mathematica code of the course. The code takes the problem description tables as input and returns the solution.

EXAMPLE 2.4. The beam of the figure is loaded by its own weight f (per unit length). Determine the end displacement and rotation by using a two-node beam element. Bending rigidity of the beam EI is constant.



Answer $u_{Z2} = \dfrac{1}{8}\dfrac{fL^4}{EI}$ and $\theta_{Y2} = -\dfrac{1}{6}\dfrac{fL^3}{EI}$

- Problem description consists of the element and node tables. In this case, the number of unknown displacements and rotations is 2.

	type	properties	geometry
1	BEAM	{{E, G}, {A, I, I}, {0, 0, f}}	Line[{1, 2}]

	{X,Y,Z}	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}
2	{L, 0, 0}	{0, 0, uZ[2]}	{0, θY[2], 0}

- Element contributions need to be written in the structural coordinate system

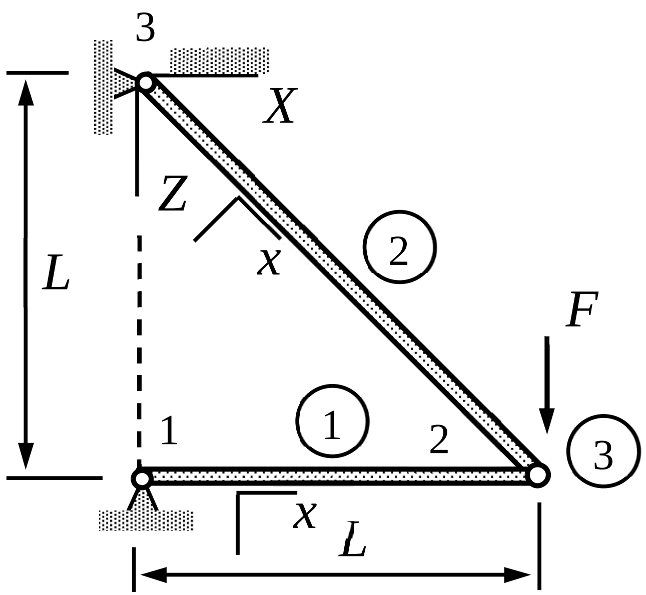
$$\text{Beam 1:}\quad \begin{Bmatrix} F_{Z1}^1 \\ M_{Y1}^1 \\ \hdashline F_{Z2}^1 \\ M_{Y2}^1 \end{Bmatrix} = \frac{EI}{L^3}\left[\begin{array}{cc:cc} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ \hdashline -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{array}\right]\begin{Bmatrix} 0 \\ 0 \\ \hdashline u_{Z2} \\ \theta_{Y2} \end{Bmatrix} - \frac{f_Z L}{12}\begin{Bmatrix} 6 \\ -L \\ \hdashline 6 \\ L \end{Bmatrix} \quad (h = L)$$

- Sums of the internal forces, external and constraint forces acting on the nodes should vanish for the equilibrium. According to the algorithm, it is enough to consider the non-constrained directions for displacements and rotations (now displacement and rotation at the right end)

$$\begin{Bmatrix} F_{Z2}^{1} \\ M_{Y2}^{1} \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix} - \frac{fL}{12}\begin{Bmatrix} 6 \\ L \end{Bmatrix} = 0 \qquad \Leftrightarrow$$

$$\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix} = \frac{L^3}{EI}\frac{f_z L}{12}\begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix}^{-1}\begin{Bmatrix} 6 \\ L \end{Bmatrix} = \frac{fL^4}{12EI}\begin{Bmatrix} 3/2 \\ -2/L \end{Bmatrix}. \qquad \Leftarrow$$

EXAMPLE 2.5. A bar structure is loaded by a point force having magnitude F as shown in the figure. Determine the nodal displacements of the bars. Cross-sectional area of bar 1 is A and that for bar 2 $\sqrt{8}A$. Young's modulus is E and weight is omitted.



Answer $u_{X2} = -\dfrac{FL}{EA}$ and $u_{Z2} = 2\dfrac{FL}{EA}$

- Problem description consists of the element and node tables

	model	properties	geometry
1	BAR	{{E}, {A}}	Line[{1, 2}]
2	BAR	$\{\{E\}, \{2\sqrt{2}\, A\}\}$	Line[{3, 2}]
3	FORCE	{0, 0, F}	Point[{2}]

	{X,Y,Z}	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	{0, 0, L}	{0, 0, 0}	{0, 0, 0}
2	{L, 0, L}	{uX[2], 0, uZ[2]}	{0, 0, 0}
3	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}

- The nodal displacements of the material and structural coordinate systems are related by (orientation angle β of the material coordinate is rotation in the positive direction along the Y – axis)

$$\begin{Bmatrix} u_x \\ u_z \end{Bmatrix} = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \begin{Bmatrix} u_X \\ u_Z \end{Bmatrix} \text{ and } \begin{Bmatrix} F_X \\ F_Z \end{Bmatrix} = \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix} \begin{Bmatrix} F_x \\ F_z \end{Bmatrix}.$$

- For bar 1, the relationships between the displacement and force components of the material and structural system and the bar element contribution are as $\beta = 0$ (notice that a bar element takes forces only in its direction and therefore $F_{z2} = 0$)

$$\begin{Bmatrix} F_{x1} \\ F_{x2} \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_{x2} \end{Bmatrix}, \text{ where } u_{x2} = u_{X2} \text{ and } \begin{Bmatrix} F^1_{X2} \\ F^1_{Z2} \end{Bmatrix} = \begin{Bmatrix} F_{x2} \\ 0 \end{Bmatrix} \Rightarrow$$

$$\begin{Bmatrix} F^1_{X2} \\ F^1_{Z2} \end{Bmatrix} = \frac{EA}{L}\begin{Bmatrix} u_{X2} \\ 0 \end{Bmatrix}.$$

- For bar 2, the relationships between the displacement and force components of the material and structural system and the element contribution are ($\beta = -45°$)

$$\begin{Bmatrix} F_{x3} \\ F_{x2} \end{Bmatrix} = \frac{E\sqrt{8}A}{\sqrt{2}L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_{x2} \end{Bmatrix} \text{ where } u_{x2} = \frac{u_{X2} + u_{Z2}}{\sqrt{2}}, \begin{Bmatrix} F^2_{X2} \\ F^2_{Z2} \end{Bmatrix} = \frac{1}{\sqrt{2}}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} F_{x2} \Rightarrow$$

$$\begin{Bmatrix} F_{X2}^{2} \\ F_{Z2}^{2} \end{Bmatrix} = \frac{1}{\sqrt{2}} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \frac{2EA}{L} \frac{1}{\sqrt{2}} \begin{Bmatrix} 1 & 1 \end{Bmatrix} \begin{Bmatrix} u_{X2} \\ u_{Z2} \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} u_{X2} \\ u_{Z2} \end{Bmatrix}.$$

- Element contribution of the point force is

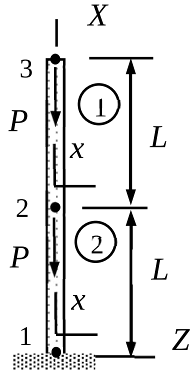
$$\begin{Bmatrix} F_{X}^{3} \\ F_{Z}^{3} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -F \end{Bmatrix}.$$

- Equilibrium requires that the sum of the forces acting on the non-constrained node 2 vanish:

$$\begin{Bmatrix} F_{X}^{1} \\ F_{Z}^{1} \end{Bmatrix} + \begin{Bmatrix} F_{X}^{2} \\ F_{Z}^{2} \end{Bmatrix} + \begin{Bmatrix} F_{X}^{3} \\ F_{Z}^{3} \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} u_{X2} \\ u_{Z2} \end{Bmatrix} + \begin{Bmatrix} 0 \\ -F \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix} u_{X2} \\ u_{Z2} \end{Bmatrix} = \frac{FL}{EA} \begin{Bmatrix} -1 \\ 2 \end{Bmatrix}. \leftarrow$$

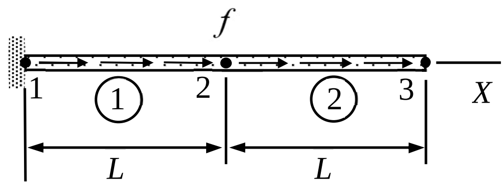
MEC-E1050 Finite Element Method in Solids, week 44/2024

1. The bar structure shown is loaded by point forces of equal magnitude P. Determine the nodal displacements u_{X2} and u_{X3}. Cross-sectional area A and Young's modulus E are constants. Use bar elements as indicated in the figure.



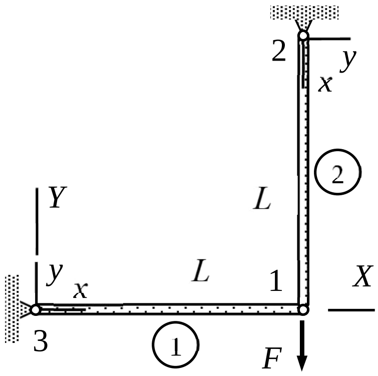
Answer $u_{X2} = -2\dfrac{PL}{EA}$, $u_{X3} = -3\dfrac{PL}{EA}$

2. Determine displacements u_{X2} and u_{X3} of nodes 2 and 3 of the bar shown. The external force per unit length is constant f and axial rigidity of the bar is EA. Use two bar elements of equal length and the bar element contribution given in the formulae collection.



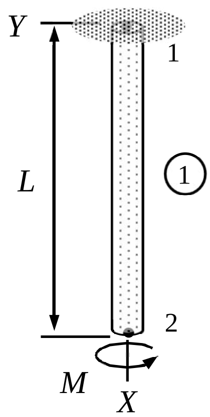
Answer $u_{X2} = \dfrac{3}{2}\dfrac{fL^2}{EA}$, $u_{X3} = 2\dfrac{fL^2}{EA}$

3. The bar structure shown is loaded by a point force at node 1. Draw the free body diagrams of the three nodes and two bars. Write down the equilibrium equations of the nodes, force-displacement relationships of the elements, and constraints on the displacements imposed by supports. Solve the nodal displacements from the equation system.



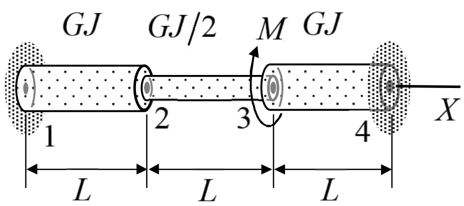
Answer $u_{X1} = 0$, $u_{Y1} = -\dfrac{FL}{EA}$

4. Consider the torsion bar of the figure loaded by torque M acting on the free end. Determine the rotation θ_{X2} at the free end if the polar moment of the cross-section J and shear modulus G are constants.



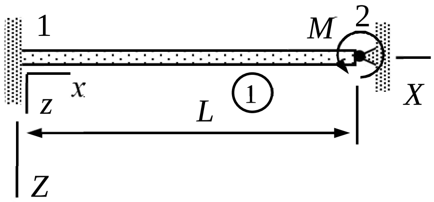
Answer $\theta_{X2} = -\dfrac{ML}{GJ}$

5. Torque M is acting in the direction of negative X-axis at node 3 of a torsion bar. Determine rotations θ_{X2} and θ_{X3} of nodes 2 and 3. Shear modulus G is constant and the polar moment of area J is piecewise constant. Use three elements of equal length.



Answer $\theta_{X2} = -\dfrac{1}{4}\dfrac{ML}{GJ}$, $\theta_{X3} = -\dfrac{3}{4}\dfrac{ML}{GJ}$

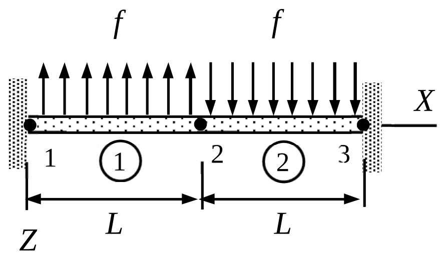
6. Determine rotation of the bending beam shown at node 2, internal forces and moments acting between the nodes and the beam element, and the constraint forces at the supports. The beam is clamped at the left end and simply supported at the right end. Young's modulus of the material E and the second moment of the cross-section $I_{yy} = I$ are constants. External distributed force $f_z = 0$.



Answer $\theta_{Y2} = \frac{1}{4}\frac{ML}{EI}$, $F_{Z1}^{1} = -\frac{3}{2}\frac{M}{L}$, $M_{Y1}^{1} = \frac{1}{2}M$, $F_{Z2}^{1} = \frac{3}{2}\frac{M}{L}$, $F_{Z1} = -\frac{3}{2}\frac{M}{L}$,

$M_{Y1} = \frac{1}{2}M$, $F_{Z2} = \frac{3}{2}\frac{M}{L}$.

7. External load acting on the beam shown consists of piecewise constant parts having equal magnitudes but opposite signs. Determine displacement u_{Z2} and rotation θ_{Y2} of the mid-point (point 2). Young's modulus of the material and the second moments of area are E and I, respectively. Use two beam elements of equal length.



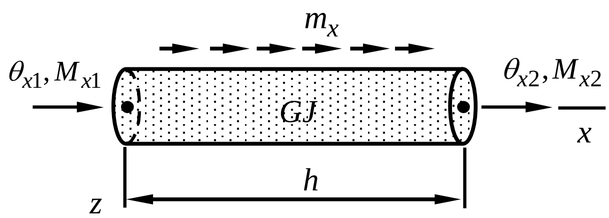
Answer $u_{Z2} = 0$, $\theta_{Y2} = -\frac{fL^3}{48EI}$

8. The boundary value problem defining the element contribution of a torsion bar consist of

$$GJ\frac{d^2\phi}{dx^2} + m_x = 0 \quad x \in]0, h[,$$

$$\phi(0) = \theta_{x1} \quad \text{and} \quad \phi(h) = \theta_{x2},$$

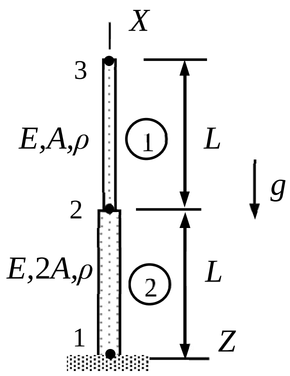
$$GJ\frac{d\phi}{dx}(0) = -M_{x1} \quad \text{and} \quad GJ\frac{d\phi}{dx}(h) = M_{x2},$$



in which the shear modulus G, cross-sectional area of the bar A, and external distributed moment per unit length m_x are constants. Derive the element contribution of a torsion bar element with the aid of the boundary value problem.

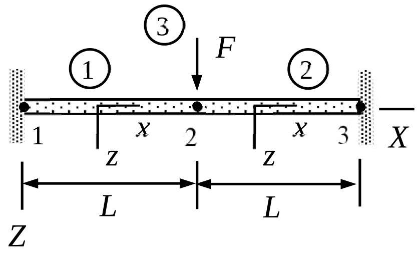
Answer $\begin{Bmatrix} M_{x1} \\ M_{x2} \end{Bmatrix} = \frac{GJ}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix} - \frac{m_x h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

9. Consider the bar structure of the figure loaded by its own weight. Determine the displacements u_{X2} and u_{X3} by using two bar elements. Acceleration by gravity g and material properties E and ρ are constants.

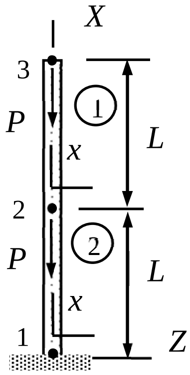


Answer $u_{X2} = -\frac{g\rho L^2}{E}$, $u_{X3} = -\frac{3}{2}\frac{g\rho L^2}{E}$

10. Determine displacement u_{Z2} at node 2 of the beam structure shown. Use two beam elements of equal length. Assume that rotation $\theta_{Y2} = 0$. Point force of magnitude F is acting on node 2. Young's modulus of the material E and the second moment of area I are constants.



Answer $u_{Z2} = \frac{1}{24}\frac{FL^3}{EI}$



The bar structure shown is loaded by two point forces of equal magnitude P. Determine the nodal displacements u_{X2} and u_{X3}. Cross-sectional area A and Young's modulus E are constants. Use two bar elements as indicated in the figure.

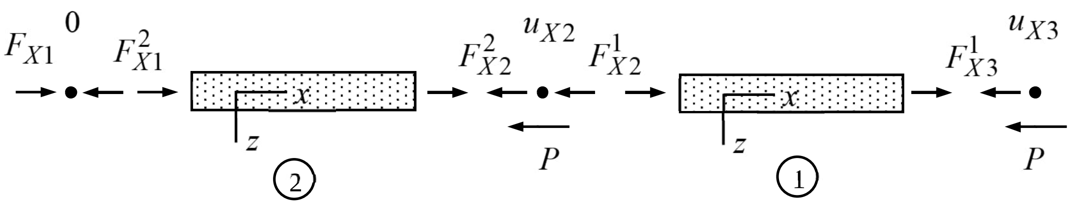
Solution

The generic force-displacement relationship of a bar element

$$\begin{Bmatrix} F_{x1} \\ F_{x2} \end{Bmatrix} = \frac{EA}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

depends on the cross-sectional area A, Young's modulus E, bar length h, and force per unit length of the bar f_x in the direction of the $x-$axis.

Let us start with the free body diagram of the structure consisting of two bar elements (the structure is rotated clockwise just to save space).



Element contributions (notice that $f_x = 0$ and the force components of the material and structural systems coincide here) are:

$$\text{bar 1}: \begin{Bmatrix} F^1_{X2} \\ F^1_{X3} \end{Bmatrix} = \begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \begin{matrix} \text{eq.1} \\ \text{eq.2} \end{matrix}$$

$$\text{bar 2}: \begin{Bmatrix} F^2_{X1} \\ F^2_{X2} \end{Bmatrix} = \begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix}\begin{Bmatrix} 0 \\ u_{X2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \begin{matrix} \text{eq.3} \\ \text{eq.4} \end{matrix}$$

Equilibrium equations of the nodes are:

node 1: $\sum F_X = F_{X1} - F^2_{X1} = 0$ eq.5

node 2: $\sum F_X = -F^2_{X2} - F^1_{X2} - P = 0$ eq.6

node 3: $\sum F_X = -F_{X3}^1 - P = 0$ eq.7

The outcome is 7 linear equations for the 2 displacements, 4 internal forces, and 1 constraint force. As the first step toward the solution (always), the internal forces are replaced in eq.6 and eq.7 (non-constrained nodes) by their expression given by eq.1, eq.2 and eq.4, to get the equilibrium equations of the nodes in terms of displacements:

node 2: $-(\frac{EA}{L}u_{X2}) - (\frac{EA}{L}u_{X2} - \frac{EA}{L}u_{X3}) - P = 0$

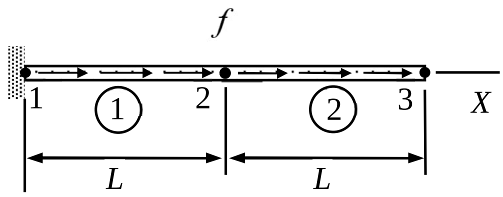
node 3: $-(-\frac{EA}{L}u_{X2} + \frac{EA}{L}u_{X3}) - P = 0$

After that, the unknown displacements follow from the system of linear equations for node 2 and 3. In matrix form

$$\begin{bmatrix} -2\frac{EA}{L} & \frac{EA}{L} \\ \frac{EA}{L} & -\frac{EA}{L} \end{bmatrix} \begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} - \begin{Bmatrix} P \\ P \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} = \begin{Bmatrix} -2\frac{PL}{EA} \\ -3\frac{PL}{EA} \end{Bmatrix}.$$

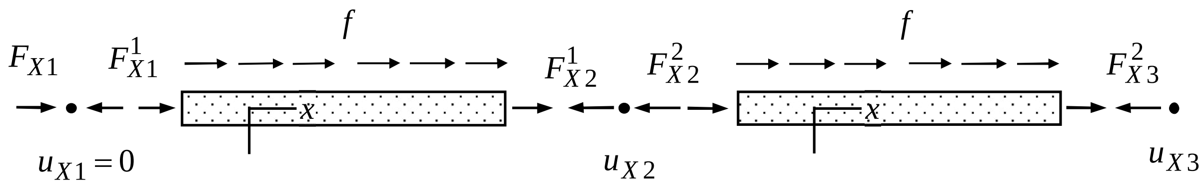
Use the code of MEC-E1050 to check the solution!

Determine displacements u_{X2} and u_{X3} of nodes 2 and 3 of the bar shown. The external force per unit length is constant f and axial rigidity of the bar is EA. Use two bar elements of equal length and the bar element contribution given in the formulae collection.



Solution

Only the displacement in the direction of the bar matters. From the figure, the non-zero displacement components are u_{X2} and u_{X3}. Free body diagram of the two bar elements and nodes 1, 2 and 3 is



Element contributions of the bar elements 1 and 2 (formulae collection) and the equilibrium equations of nodes 1, 2 and 3 are (written in terms of the force and displacement components of the structural system)

Bar 1: $\begin{Bmatrix} F^1_{X1} \\ F^1_{X2} \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_{X2} \end{Bmatrix} - \frac{fL}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$,

Bar 2: $\begin{Bmatrix} F^2_{X2} \\ F^2_{X3} \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} - \frac{fL}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$,

Node 1: $\sum F_X = F_{X1} - F^1_{x1} = 0$,

Node 2: $\sum F_X = -F^1_{X2} - F^2_{X2} = 0$,

Node 3: $\sum F_X = -F^2_{X3} = 0$.

Elimination of the internal forces from the two equilibrium equations of the non-constrained nodes 2 and 3 using the element contributions gives

Node 2: $-(\frac{EA}{L}u_{X2} - \frac{fL}{2}) - (\frac{EA}{L}u_{X2} - \frac{EA}{L}u_{X3} - \frac{fL}{2}) = 0$,

Node 3: $-(-\frac{EA}{L}u_{X2} + \frac{EA}{L}u_{X3} - \frac{fL}{2}) = 0$.

When the equilibrium equations are written in the "standard" matrix form

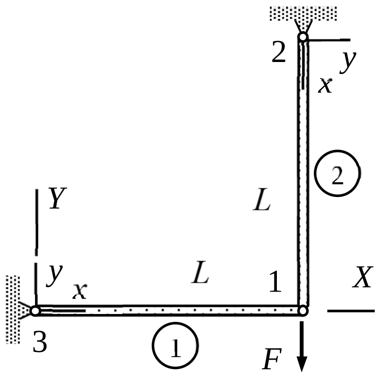
$$\frac{EA}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} - fL\begin{Bmatrix} 1 \\ 1/2 \end{Bmatrix} = 0 \quad \Leftrightarrow \quad (\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix})$$

$$\begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} = \frac{fL^2}{EA} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ 1/2 \end{Bmatrix} = \frac{fL^2}{EA} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} 1 \\ 1/2 \end{Bmatrix} = \frac{fL^2}{EA} \begin{Bmatrix} 3/2 \\ 2 \end{Bmatrix} \quad \Leftrightarrow$$

$u_{X2} = \dfrac{3}{2}\dfrac{fL^2}{EA}$ and $u_{X3} = 2\dfrac{fL^2}{EA}$. 🡐

Use the code of MEC-E1050 to check the solution!

The bar structure shown is loaded by a point force at node 1. Draw the free body diagrams of the three nodes and two bars. Write down the equilibrium equations of the nodes, force-displacement relationships of the elements, and constraints on the displacements imposed by supports. Solve the nodal displacements from the equation system.



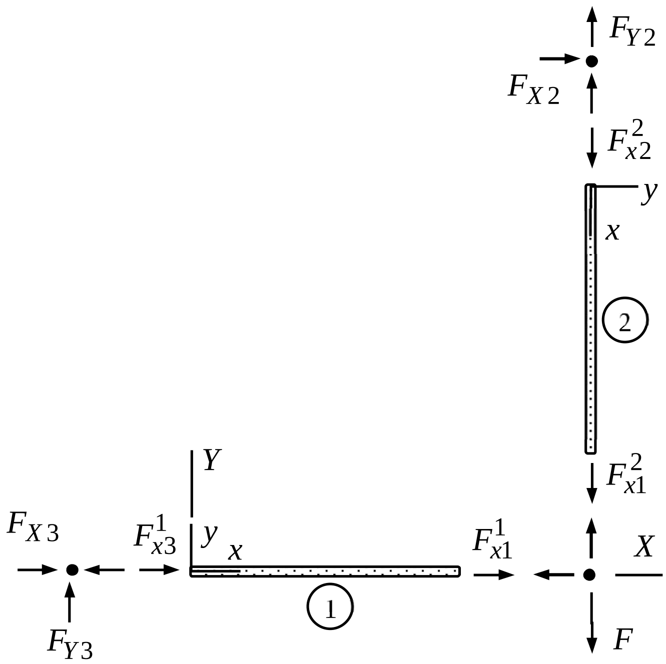
Solution

The generic force-displacement relationship of a bar element

$$\begin{Bmatrix} F_{x1} \\ F_{x2} \end{Bmatrix} = \frac{EA}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

depends on the cross-sectional area A, Young's modulus E, bar length h, and force per unit length of the bar f_x in the direction of the $x-$axis. In the present case, the distributed force $f_x = 0$.

Let us start with the free body diagram of the structure consisting of two bar elements. A bar takes only forces acting in its direction. The external point force acts on node 1. Supports are replaced by reaction forces which they impose on the structure.



Element contributions are written in terms of the force and displacement components in the structural system. All the components of the elementwise material coordinate systems need to be expressed in terms of those of the structural system before writing the equilibrium equations. In this case, the relationships between the material and structural system can easily be seen from the free body diagram ($F_{x1}^1 = F_{X1}^1$, $F_{x1}^2 = -F_{Y1}^2$, etc.)

bar 1 : $\begin{Bmatrix} F_{X3}^1 \\ F_{X1}^1 \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_{X1} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ eq.1 eq.2

bar 2 : $\begin{Bmatrix} -F_{Y2}^{2} \\ -F_{Y1}^{2} \end{Bmatrix} = \dfrac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ -u_{Y1} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ eq.3 eq.4

Force equilibrium equations of the nodes in the X- and Y- directions are:

node 1: $\sum \begin{Bmatrix} F_X \\ F_Y \end{Bmatrix} = \begin{Bmatrix} -F_{X1}^{1} \\ -F_{Y1}^{2} - F \end{Bmatrix} = 0$, eq.5 eq.6

node 2: $\sum \begin{Bmatrix} F_X \\ F_Y \end{Bmatrix} = \begin{Bmatrix} F_{X2} \\ F_{Y2}^{2} + F_{Y2} \end{Bmatrix} = 0$, eq.7 eq.8

node 3: $\sum \begin{Bmatrix} F_X \\ F_Y \end{Bmatrix} = \begin{Bmatrix} F_{X3} - F_{X3}^{1} \\ F_{Y3} \end{Bmatrix} = 0$. eq.9 eq.10

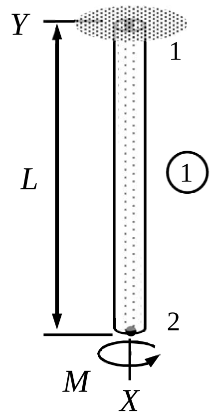
The outcome is a set of 10 linear equations for 2 displacement components, 4 internal force components, and 4 constraint force components. As the first step toward the solution (always), the internal forces are replaced in eq.5 and eq.6 (non-constrained directions) by their expression given in eq.2 and eq.4, to get the equilibrium equations of the nodes in terms of displacements:

node 1: $\sum \begin{Bmatrix} F_X \\ F_Y \end{Bmatrix} = \begin{Bmatrix} -\dfrac{EA}{L} u_{X1} \\ -\dfrac{EA}{L} u_{Y1} - F \end{Bmatrix} = \begin{bmatrix} -\dfrac{EA}{L} & 0 \\ 0 & -\dfrac{EA}{L} \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} - \begin{Bmatrix} 0 \\ F \end{Bmatrix} = 0$.

After that, the unknown displacements are solved from the system of linear equations

$$\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -\dfrac{FL}{EA} \end{Bmatrix}.$$

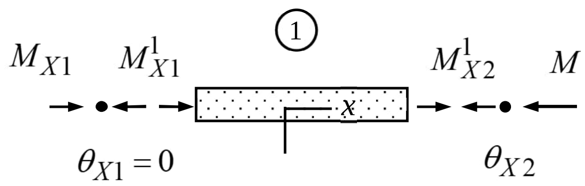
Use the code of MEC-E1050 to check the solution!



Consider the torsion bar of the figure loaded by torque M acting on the free end. Determine the rotation θ_{X2} at the free end if the polar moment of the cross-section J and shear modulus G are constants.

Solution

Only the rotation in the direction of the bar matters. From the figure, only the rotation component θ_{X2} may not be zero. Free body diagrams of the torsion bar and nodes 1 and 2 are (the structure is rotated just to save space)



Element contribution of the torsion bar and the equilibrium equations of nodes 1 and 2 are (the distributed moment vanishes here) written in terms of the rotation and moment components of the structural system:

Bar 1: $\begin{Bmatrix} M_{X1}^1 \\ M_{X2}^1 \end{Bmatrix} = \dfrac{GJ}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ \theta_{X2} \end{Bmatrix}$, eq.1 eq.2

Node 1: $\sum M_X = M_{X1} - M_{X1}^1 = 0$, eq.3

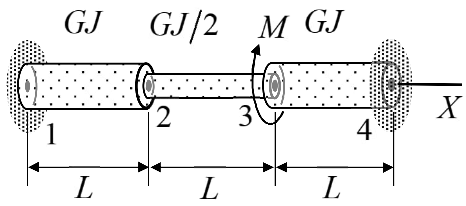
Node 2: $\sum M_X = -M_{X2}^1 - M = 0$. eq.4

Elimination of the internal forces from the equilibrium eq.4 for node 2 using the element contribution eq.2 gives

$$-\frac{GJ}{L}\theta_{X2} - M = 0 \quad \Leftrightarrow \quad \theta_{X2} = -\frac{ML}{GJ}.$$

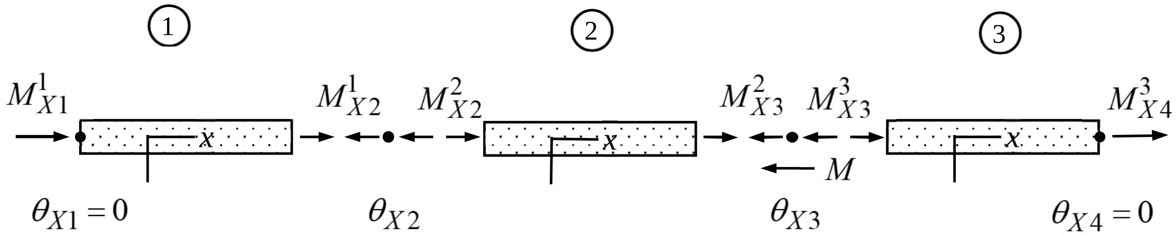
Solution to the unknown rotation was obtained from the equilibrium equation of a non-constrained node 2. The equilibrium equation of the constrained node 1 contains the constraint moment and is useful if that is needed too.

Torque M is acting in the direction of negative X-axis at node 3 of a torsion bar. Determine rotations θ_{X2} and θ_{X3} of nodes 2 and 3. Shear modulus G is constant and the polar moment of area J is piecewise constant. Use three elements of equal length.



Solution

Only the rotation in the direction of the bar matters. From the figure, the non-zero rotation components are θ_{X2} and θ_{X3}. Free body diagrams of the three torsion bar elements and nodes 2 and 3 are (nodes 1 and 4 are constrained and do not contribute to the system equations)



Element contributions of the torsion bar elements 1, 2 and 3 (formulae collection) and the equilibrium equations of nodes 2 and 3 are (notice that the distributed moment vanishes here)

Bar 1: $\begin{Bmatrix} M^1_{X1} \\ M^1_{X2} \end{Bmatrix} = \frac{GJ}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ \theta_{X2} \end{Bmatrix}$,

Bar 2: $\begin{Bmatrix} M^2_{X2} \\ M^2_{X3} \end{Bmatrix} = \frac{GJ}{2L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \theta_{X2} \\ \theta_{X3} \end{Bmatrix}$,

Bar 3: $\begin{Bmatrix} M^3_{X3} \\ M^3_{X4} \end{Bmatrix} = \frac{GJ}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \theta_{X3} \\ 0 \end{Bmatrix}$,

Node 2: $\sum M_X = -M^1_{X2} - M^2_{X2} = 0$,

Node 3: $\sum M_X = -M^2_{X3} - M^3_{X3} - M = 0$.

Elimination of the internal forces from the two equilibrium equations of the nodes using the element contributions gives the forms

Node 2: $-(\frac{GJ}{L}\theta_{X2}) - (\frac{GJ}{2L}\theta_{X2} - \frac{GJ}{2L}\theta_{X3}) = 0$,

Node 3: $-(-\frac{GJ}{2L}\theta_{X2} + \frac{GJ}{2L}\theta_{X3}) - (\frac{GJ}{L}\theta_{X3}) - M = 0$.

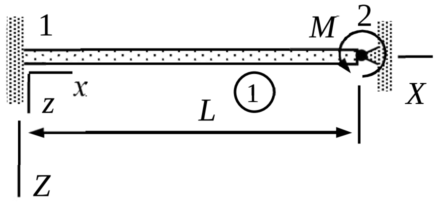
Matrix representation of the two equilibrium equations, containing the rotations of nodes 2 and 3 as unknowns, is

$$\frac{GJ}{2L}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}\begin{Bmatrix} \theta_{X2} \\ \theta_{X3} \end{Bmatrix} - M\begin{Bmatrix} 0 \\ -1 \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \left(\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}^{-1} = \frac{1}{8}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\right)$$

$$\begin{Bmatrix} \theta_{X2} \\ \theta_{X3} \end{Bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}^{-1} 2\frac{ML}{GJ}\begin{Bmatrix} 0 \\ -1 \end{Bmatrix} = \frac{1}{4}\frac{ML}{GJ}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\begin{Bmatrix} 0 \\ -1 \end{Bmatrix} = -\frac{1}{4}\frac{ML}{GJ}\begin{Bmatrix} 1 \\ 3 \end{Bmatrix} \quad \Leftrightarrow$$

$$\theta_{X2} = -\frac{1}{4}\frac{ML}{GJ} \quad \text{and} \quad \theta_{X3} = -\frac{3}{4}\frac{ML}{GJ}.$$

Determine rotation of the bending beam shown at node 2, internal forces and moments acting between the nodes and the beam element, and the constraint forces at the supports. The beam is clamped at the left end and simply supported at the right end. Young's modulus of the material E and the second moment of the cross-section $I_{yy} = I$ are constants. External distributed force $f_z = 0$.

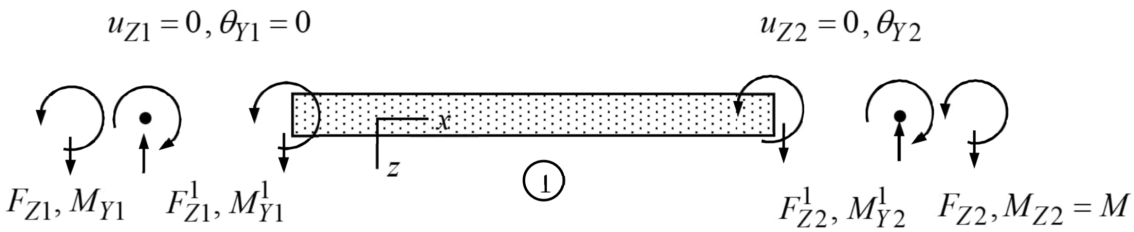


Solution

The generic force-displacement relationship of a bending beam element

$$\begin{Bmatrix} F_{z1} \\ M_{y1} \\ F_{z2} \\ M_{y2} \end{Bmatrix} = \frac{EI_{yy}}{h^3} \begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix} - \frac{f_z h}{12} \begin{Bmatrix} 6 \\ -h \\ 6 \\ h \end{Bmatrix}$$

depends on the second moment of area I_{yy}, Young's modulus E, beam length h, and force per unit length f_z in the direction of the $z-$axis. Let us start with the free body diagram of the beam and the two nodes. As the axis of the material and structural system coincide, the displacement, rotation, force, and moments components of the two systems are the same



When written in terms of displacement, rotation, force, and moment components in the structural system, the beam element contribution becomes (as the orientation of the material and structural coordinate system is the same, the components are the same)

Beam: $$\begin{Bmatrix} F_{Z1}^1 \\ M_{Y1}^1 \\ F_{Z2}^1 \\ M_{Y2}^1 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6h & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \theta_{Y2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$ eq. 1 eq. 2 eq. 3 eq. 4

Equilibrium equations of the nodes are

Node 1 : $\sum F_Z = F_{Z1} - F_{Z1}^1 = 0$ eq. 5

$\sum M_Y = M_{Y1} - M_{Y1}^1 = 0$ eq. 6

Node 2 : $\sum F_Z = F_{Z2} - F_{Z2}^1 = 0$ eq. 7

$\sum M_Y = M - M_{Y2}^1 = 0$ eq. 8

The outcome is a set of 8 linear equations for 1 rotation, 4 internal forces, and 3 constraint forces/moments. As the first step toward the solution (always), the internal forces in the node equilibrium equations are replaced by their expressions given by eq.1, eq.2 , eq.3 and eq.4. After that, the unknown displacements and rotations follow from the corresponding equilibrium equations. Eq.4 and eq.8 imply first

$$\sum M_Y = M - \frac{EI}{L^3} 4L^2 \theta_{Y2} = 0 \quad \Leftrightarrow \quad \theta_{Y2} = \frac{1}{4}\frac{ML}{EI}.$$

Use the code of MEC-E1050 to check the solution! Knowing the rotation angle, the remaining eq.1, eq.2 , and eq.3 of the beam element contribution give the internal forces

$$F_{Z1}^1 = -6L\frac{EI}{L^3}\frac{1}{4}\frac{ML}{EI} = -\frac{3}{2}\frac{M}{L},$$

$$M_{Y1}^1 = \frac{EI}{L^3} 2L^2 \frac{1}{4}\frac{ML}{EI} = \frac{1}{2}M,$$

$$F_{Z2}^1 = \frac{EI}{L^3} 6L \frac{1}{4}\frac{ML}{EI} = \frac{3}{2}\frac{M}{L}.$$

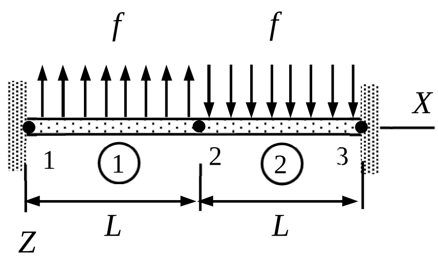
Constraint forces, due to the clamping at node 1 and simple support at node 2, follow from the remaining equilibrium eq.5, eq.6, and eq.7 and the solution to the internal forces

$$\sum F_Z = F_{Z1} + \frac{3}{2}\frac{M}{L} = 0 \quad \Leftrightarrow \quad F_{Z1} = -\frac{3}{2}\frac{M}{L},$$

$$\sum M_Y = M_{Y1} - \frac{1}{2}M = 0 \quad \Leftrightarrow \quad M_{Y1} = \frac{1}{2}M,$$

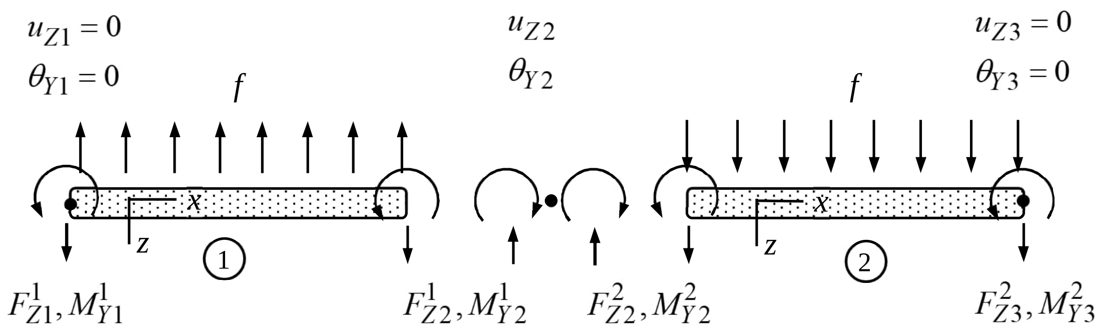
$$\sum F_Z = F_{Z2} - \frac{3}{2}\frac{M}{L} = 0 \quad \Leftrightarrow \quad F_{Z2} = \frac{3}{2}\frac{M}{L}.$$

External load acting on the beam shown consists of piecewise constant parts having equal magnitudes but opposite signs. Determine displacement u_{Z2} and rotation θ_{Y2} of the mid-point (point 2). Young's modulus of the material and the second moments of area are E and I , respectively. Use two beam elements of equal length.



Solution

Only the displacement in the Z – direction and rotation in the Y – direction matter in the planar beam bending problem. From the figure, the non-zero displacement/rotation components are u_{Z2} and θ_{Y2} . Free body diagrams of the two bending beam elements and node 2 are (nodes 1 and 3 are constrained)



Element contributions of the two xz – plane bending beams (formulae collection) and the equilibrium equations of node 2 are (notice that the distributed force in the element contribution is the transverse component in the material system associated with beam)

Beam 1: $$\begin{Bmatrix} F_{Z1}^1 \\ M_{Y1}^1 \\ F_{Z2}^1 \\ M_{Y2}^1 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ u_{Z2} \\ \theta_{Y2} \end{Bmatrix} + \frac{fL}{12}\begin{Bmatrix} 6 \\ -L \\ 6 \\ L \end{Bmatrix}, \quad (f_z = -f)$$

Beam 2: $$\begin{Bmatrix} F_{Z2}^2 \\ M_{Y2}^2 \\ F_{Z3}^2 \\ M_{Y3}^2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \\ 0 \\ 0 \end{Bmatrix} - \frac{fL}{12}\begin{Bmatrix} 6 \\ -L \\ 6 \\ L \end{Bmatrix}, \quad (f_z = f)$$

Node 2: $-F_{Z2}^1 - F_{Z2}^2 = 0$ and $-M_{Y2}^1 - M_{Y2}^2 = 0$.

Elimination of the internal forces from the two equilibrium equations of node 2 using the element contributions gives the forms

Node 2: $$-[\frac{EI}{L^3}(12u_{Z2}+6L\theta_{Y2})+6\frac{fL}{12}]-[\frac{EI}{L^3}(12u_{Z2}-6L\theta_{Y2})-6\frac{fL}{12}]=0 \qquad \text{and}$$

$$-[\frac{EI}{L^3}(6Lu_{Z2}+4L^2\theta_{Y2})+L\frac{fL}{12}]-[\frac{EI}{L^3}(-6Lu_{Z2}+4L^2\theta_{Y2})+L\frac{fL}{12}]=0\,.$$

Matrix representation of the two equilibrium equations, containing u_{Z2} and θ_{Y2} as the unknowns, is

$$\frac{EI}{L^3}\begin{bmatrix}24 & 0\\ 0 & 8L^2\end{bmatrix}\begin{Bmatrix}u_{Z2}\\ \theta_{Y2}\end{Bmatrix}-fL^2\begin{Bmatrix}0\\ -1/6\end{Bmatrix}=0 \quad \Leftrightarrow$$

$$\begin{Bmatrix}u_{Z2}\\ \theta_{Y2}\end{Bmatrix}=\frac{fL^5}{EI}\begin{bmatrix}24 & 0\\ 0 & 8L^2\end{bmatrix}^{-1}\begin{Bmatrix}0\\ -1/6\end{Bmatrix}=\frac{fL^5}{EI}\begin{bmatrix}1/24 & 0\\ 0 & 1/(8L^2)\end{bmatrix}\begin{Bmatrix}0\\ -1/6\end{Bmatrix}=\frac{1}{48}\frac{fL^3}{EI}\begin{Bmatrix}0\\ -1\end{Bmatrix} \quad \Leftrightarrow$$

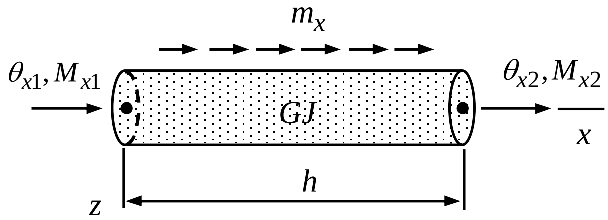
$u_{Z2}=0$ and $\theta_{Y2}=-\frac{1}{48}\frac{fL^3}{EI}$. ←

The boundary value problem defining the element contribution of a torsion bar consist of

$$GJ\frac{d^2\phi}{dx^2}+m_x=0 \quad x\in]0,h[\,,$$

$$\phi(0)=\theta_{x1} \quad \text{and} \quad \phi(h)=\theta_{x2}\,,$$

$$GJ\frac{d\phi}{dx}(0)=-M_{x1} \quad \text{and} \quad GJ\frac{d\phi}{dx}(h)=M_{x2}\,,$$



in which the shear modulus G, cross-sectional area of the bar A, and external distributed moment per unit length m_x are constants. Derive the element contribution of a torsion bar element with the aid of the boundary value problem.

Solution

The equations defining the element contribution of a torsion bar consist of the equilibrium equation, and boundary conditions for rotations and moments at the nodes. As the number of boundary conditions is four, existence of the solution is possible only under certain condition on ”data” GJ, m_x, h, θ_{x1}, θ_{x2}, M_{x1}, M_{x2}. The condition for the data is the torsion bar element contribution.

First, integration of the equilibrium twice is used to find the generic solution (any method to find the solution goes)

$$\phi=a+bx-\frac{m_x}{2GJ}x^2=\{1 \quad x\}\begin{Bmatrix}a\\b\end{Bmatrix}-\frac{m_x}{2GJ}x^2\,.$$

After that, the rotation boundary conditions are used to express the integration constants a and b in terms of the nodal rotations

$$\begin{Bmatrix}\theta_{x1}\\\theta_{x2}\end{Bmatrix}=\begin{Bmatrix}\phi(0)\\\phi(h)\end{Bmatrix}=\begin{bmatrix}1 & 0\\1 & h\end{bmatrix}\begin{Bmatrix}a\\b\end{Bmatrix}-\frac{m_x}{2GJ}\begin{Bmatrix}0\\h^2\end{Bmatrix} \quad\Leftrightarrow\quad \begin{Bmatrix}a\\b\end{Bmatrix}=\frac{1}{h}\begin{bmatrix}h & 0\\-1 & 1\end{bmatrix}(\begin{Bmatrix}\theta_{x1}\\\theta_{x2}\end{Bmatrix}+\frac{m_x}{2GJ}\begin{Bmatrix}0\\h^2\end{Bmatrix})$$

to get

$$\phi=\{1 \quad x\}\frac{1}{h}\begin{bmatrix}h & 0\\-1 & 1\end{bmatrix}(\begin{Bmatrix}\theta_{x1}\\\theta_{x2}\end{Bmatrix}+\frac{m_x}{2GJ}\begin{Bmatrix}0\\h^2\end{Bmatrix})-\frac{m_x}{2GJ}x^2\,.$$

Finally, the moment boundary conditions and the rotation solution give

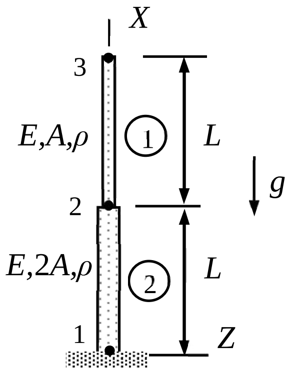
$$M_{x1}=-GJ\frac{d\phi}{dx}(0)=\frac{GJ}{h}(\theta_{x1}-\theta_{x2})-\frac{m_xh}{2}\,,$$

$$M_{x2}=GJ\frac{d\phi}{dx}(h)=\frac{GJ}{h}(\theta_{x2}-\theta_{x1})-\frac{m_xh}{2}$$

or in a more concise form

$$\begin{Bmatrix} M_{x1} \\ M_{x2} \end{Bmatrix} = \frac{GJ}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix} - \frac{m_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

Consider the structure of the figure loaded by its own weight. Determine the displacement u_{X3} of the free end by using bar elements (1) and (2). The cross-sectional area of the bar (2) is twice that of bar (1). Acceleration by gravity g and material properties E and ρ are constants.



Solution

Let the material coordinate systems of both bars coincide with the structural system. The element and node tables are

	model	properties	geometry
1	BAR	{{E}, {A}, {-A g ρ, 0, 0}}	Line[{2, 3}]
2	BAR	{{E}, {2 A}, {-2 A g ρ, 0, 0}}	Line[{1, 2}]

	{X,Y,Z}	{u_X,u_Y,u_Z}	{θ_X,θ_Y,θ_Z}
1	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}
2	{L, 0, 0}	{uX[2], 0, 0}	{0, 0, 0}
3	{2 L, 0, 0}	{uX[3], 0, 0}	{0, 0, 0}

According to the recipe for assembly (build of the system equations), element contributions are first written in terms of the displacement and force components in the structural coordinate system (notice that gravity is acting in the direction of the negative x-axis):

$$\text{Bar 1}: \begin{Bmatrix} F_{X2}^1 \\ F_{X3}^1 \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} + \frac{gA\rho L}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix},$$

$$\text{Bar 2}: \begin{Bmatrix} F_{X1}^2 \\ F_{X2}^2 \end{Bmatrix} = \frac{E2A}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_{X2} \end{Bmatrix} + \frac{2gA\rho L}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

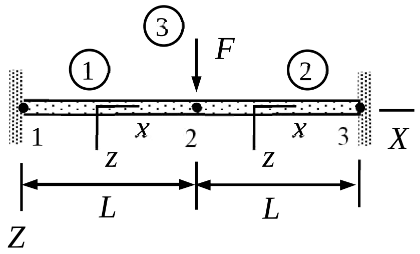
For equilibrium of nodes, sums of the internal forces and moments acting on the nodes need to vanish. In build of the system equations (minimal equation set) it is enough to consider the non-constrained directions of displacements at nodes 2 and 3:

$$\begin{Bmatrix} F_{X2}^1 + F_{X2}^2 \\ F_{X3}^1 \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} + \frac{gA\rho L}{2}\begin{Bmatrix} 3 \\ 1 \end{Bmatrix} = 0 \quad \Leftrightarrow$$

$$\begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} = -\frac{1}{2}\frac{gA\rho L^2}{EA}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}^{-1}\begin{Bmatrix} 3 \\ 1 \end{Bmatrix} = -\frac{1}{4}\frac{gA\rho L^2}{EA}\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}\begin{Bmatrix} 3 \\ 1 \end{Bmatrix} = -\frac{1}{2}\frac{gA\rho L^2}{EA}\begin{Bmatrix} 2 \\ 3 \end{Bmatrix} \quad \Leftrightarrow$$

$$u_{X2} = -\frac{g\rho L^2}{E} \text{ and } u_{X3} = -\frac{3}{2}\frac{g\rho L^2}{E}.$$

Determine the displacement u_{Z2} at node 2 of the beam structure shown. Use two beam elements of equal length. Assume that rotation $\theta_{Y2}=0$. Point force of magnitude F is acting on node 2. Young's modulus of the material E and the second moment of area I are constants.



Solution

In hand calculations, explicit forms of the node and element tables are not needed. In simple cases, the relationship between the displacement, rotation, force, and moment components of the material coordinate and structural coordinate systems can also be deduced from the figure. The beam and point force element contributions are

$$\begin{Bmatrix} F_{z1} \\ M_{y1} \\ F_{z2} \\ M_{y2} \end{Bmatrix} = \frac{EI_{yy}}{h^3}\begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix}\begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix} - \frac{f_z h}{12}\begin{Bmatrix} 6 \\ -h \\ 6 \\ h \end{Bmatrix} \text{ and } \begin{Bmatrix} F_{X1} \\ F_{Y1} \\ F_{Z1} \end{Bmatrix} = -\begin{Bmatrix} F_X \\ F_Y \\ F_Z \end{Bmatrix}.$$

Element contributions need to be written in terms of the displacement and rotation components of the structural coordinate system. The structure has just one-degree of freedom u_{Z2}.

Beam 1: $$\begin{Bmatrix} F_{Z1}^1 \\ M_{Y1}^1 \\ F_{Z2}^1 \\ M_{Y2}^1 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ u_{Z2} \\ 0 \end{Bmatrix},$$

Beam 2: $$\begin{Bmatrix} F_{Z2}^2 \\ M_{Y2}^2 \\ F_{Z3}^2 \\ M_{Y3}^2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ 0 \\ 0 \\ 0 \end{Bmatrix},$$ Force 3: $$\begin{Bmatrix} F_{X2}^3 \\ F_{Y2}^3 \\ F_{Z2}^3 \end{Bmatrix} = -\begin{Bmatrix} F \\ 0 \\ 0 \end{Bmatrix}.$$

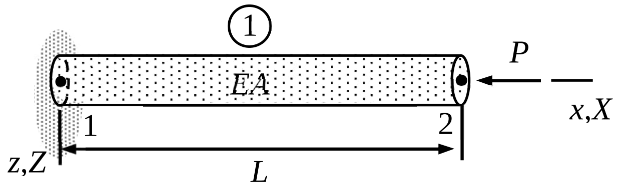
Assembly means writing the equilibrium equations of the nodes. In practice, the equations giving the displacements and rotations are obtained by summing the internal forces in directions where displacement and rotation components are not constrained. If point forces are considered as one node element, the sum is over the elements connected to a node.

$$\sum F_{Z2}^i = F_{Z2}^1 + F_{Z2}^2 + F_{Z2}^3 = 24\frac{EI}{L^3}u_{Z2} - F = 0 \quad \Leftrightarrow \quad u_{Z2} = \frac{1}{24}\frac{FL^3}{EI}.$$

Name________________________ Student number________________

Assignment 1

Consider the bar structure below and solve for the displacement u_{X2} at node 2. Left end of the bar (node 1) is fixed and the given external force P is acting on node 2. Young's modulus E and cross-sectional area A are constants.



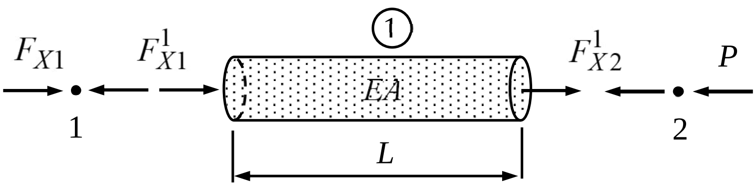
Solution template

The generic force-displacement relationship of a bar element

$$\begin{Bmatrix} F_{x1} \\ F_{x2} \end{Bmatrix} = \frac{EA}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

depends on the cross-sectional area A, Young's modulus E, bar length h, and force per unit length of the bar f_x in the direction of the $x-$axis.

Free body diagrams of the bar element and the two nodes. External given force P and the constraint force F_{X1} are acting on the nodes and the material and structural coordinate systems coincide:



Equilibrium equations of nodes 1 and 2, and the force-displacement relationship of element 1 (from the figure $u_{x1}^1 = u_{X1}$, $u_{x2}^1 = u_{X2}$, $F_{x1}^1 = F_{X1}^1$, and $F_{x2}^1 = F_{X2}^1$)

Node 1 : $\quad F_{X1} - F_{X1}^1 = 0$,

Node 2 : $\quad -F_{X2}^1 - P = 0$,

Bar 1 : $\quad \begin{Bmatrix} F_{X1}^1 \\ F_{X2}^1 \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_{X2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$

Equilibrium equations of node 2 can be solved for the displacement when the internal forces are first eliminated by using the bar element contribution. After elimination

Node 2 : $-\frac{EA}{L}u_{X2} - P = 0$.

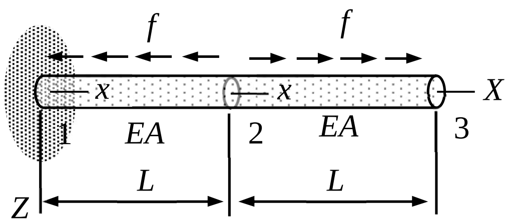
Therefore, solution to the unknown displacement

$$u_{X2} = -\frac{PL}{EA}.$$

Name________________________ Student number________________

Assignment 2

Consider the structure of the figure consisting of two bar elements loaded by piecewise constant distributed force changing its direction at the midpoint. First, write the element contributions. Second, assemble the system equations and solve for the unknown displacement components.



Solution template

The generic force-displacement relationship of a bar element (a template to be adopted to match the actual nodal numbers and material parameters of the bars in the structure)

$$\begin{Bmatrix} F_{x1} \\ F_{x2} \end{Bmatrix} = \frac{EA}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

depends on the cross-sectional area A, Young's modulus E, bar length h, and force per unit length of the bar f_x in the direction of the $x-$axis.

Bar element contributions need to be written in terms of the displacement and force components of the structural system (here the $x-$ and $X-$axes are aligned)

Bar 1: $$\begin{Bmatrix} F_{X1}^1 \\ F_{X2}^1 \end{Bmatrix} = \begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix}\begin{Bmatrix} 0 \\ u_{X2} \end{Bmatrix} - \begin{Bmatrix} -\frac{fL}{2} \\ -\frac{fL}{2} \end{Bmatrix}$$

Bar 2: $$\begin{Bmatrix} F_{X2}^2 \\ F_{X3}^2 \end{Bmatrix} = \begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} - \begin{Bmatrix} \frac{fL}{2} \\ \frac{fL}{2} \end{Bmatrix}$$

Sums of the forces acting on nodes 2 and 3 should vanish for the equilibrium so the equilibrium equations are given by

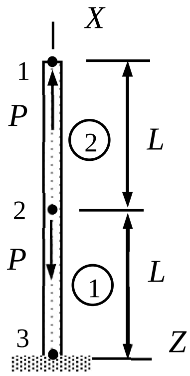
$$\begin{Bmatrix} F_{X2}^1 + F_{X2}^2 \\ F_{X3}^2 \end{Bmatrix} = \begin{bmatrix} 2\frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} - \begin{Bmatrix} 0 \\ \frac{fL}{2} \end{Bmatrix} = 0,$$

Solution to the two-equation system (by the matrix inversion method)

$$\begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} = \begin{bmatrix} \dfrac{L}{EA} & \dfrac{L}{EA} \\ \dfrac{L}{EA} & 2\dfrac{L}{EA} \end{bmatrix} \begin{Bmatrix} 0 \\ \dfrac{fL}{2} \end{Bmatrix} = \begin{Bmatrix} \dfrac{1}{2}\dfrac{fL^2}{EA} \\ \dfrac{fL^2}{EA} \end{Bmatrix}.$$

Name_________________________ Student number_________________

Assignment 3



The bar structure shown is loaded by two point forces of equal magnitude P but opposite directions. Determine the nodal displacements u_{X1} and u_{X2}. Cross-sectional area A and Young's modulus E are constants. Use two bar elements as indicated in the figure.

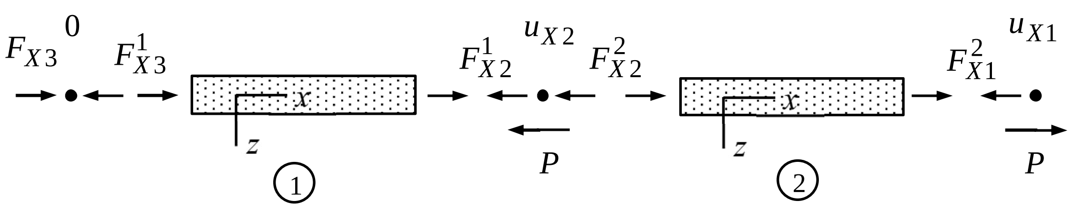
Solution template

The generic force-displacement relationship of a bar element

$$\begin{Bmatrix} F_{x1} \\ F_{x2} \end{Bmatrix} = \frac{EA}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

depends on the cross-sectional area A, Young's modulus E, bar length h, and force per unit length of the bar f_x in the direction of the $x-$axis.

Let us start with the free body diagram of the structure consisting of two bar elements (the structure is rotated clockwise just to save space).



Element contributions (notice that $f_x = 0$ and the force components of the material and structural systems coincide here) are:

bar 1 : $\begin{Bmatrix} F_{X3}^1 \\ F_{X2}^1 \end{Bmatrix} = \begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix}\begin{Bmatrix} 0 \\ u_{X2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ eq.1 eq.2

bar 2 : $\begin{Bmatrix} F_{X2}^2 \\ F_{X1}^2 \end{Bmatrix} = \begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{X1} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ eq.3 eq.4

Equilibrium equations of the nodes are:

node 1: $\sum F_X = P - F_{X1}^2 = 0$ eq.5

node 2: $\sum F_X = -F_{X2}^1 - F_{X2}^2 - P = 0$ eq.6

node 3: $\sum F_X = F_{X3} - F_{X3}^1 = 0$ eq.7

The outcome is 7 linear equations for the 2 displacements, 4 internal forces, and 1 constraint force. As the first step toward the solution (always), the internal forces are replaced in eq.5 and eq.6 (non-constrained nodes 1 and 2) by their expression given by eq.2, eq.3 and eq.4, to get the equilibrium equations of the nodes in terms of displacements:

node 1: $P - \frac{EA}{L}(u_{X1} - u_{X2}) = 0$

node 2: $-(\frac{EA}{L}u_{X2}) - \frac{EA}{L}(u_{X2} - u_{X1}) - P = 0$

After that, the unknown displacements follow from the system of linear equations for node 1 and 2. In matrix form (for example)

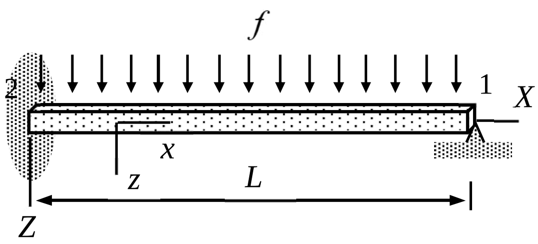
$$\begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & 2\frac{EA}{L} \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{X2} \end{Bmatrix} - \begin{Bmatrix} P \\ -P \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix} u_{X1} \\ u_{X2} \end{Bmatrix} = \begin{Bmatrix} PL/EA \\ 0 \end{Bmatrix}.$$

Use the code of MEC-E1050 to check your solution!

Name_________________________ Student number_________________

Assignment 4

Determine rotation θ_{Y1} of the bending beam shown at the support of the right end (use one element). The x-axis of the material coordinate system coincides with the neutral axis of the beam. Young's modulus E and the second moment of the cross-sectional area $I_{yy} = I$ are constants.

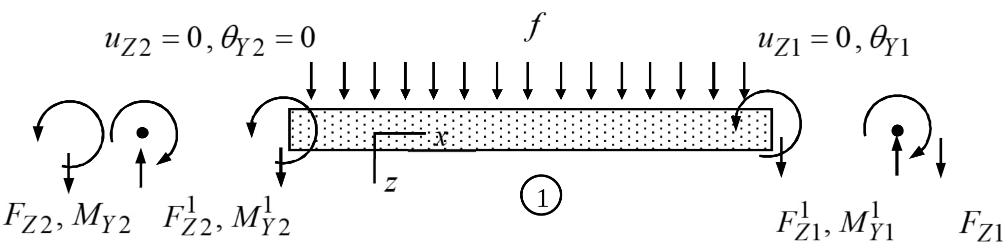


Solution template

The generic force-displacement relationship of a bending beam element

$$\begin{Bmatrix} F_{z1} \\ M_{y1} \\ F_{z2} \\ M_{y2} \end{Bmatrix} = \frac{EI_{yy}}{h^3} \begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix} - \frac{f_z h}{12} \begin{Bmatrix} 6 \\ -h \\ 6 \\ h \end{Bmatrix}$$

depends on the second moment of area I_{yy}, Young's modulus E, beam length h, and force per unit length f_z in the direction of the $z-$axis. Let us start with the free body diagram of the beam and the two nodes



When written in terms of displacement, rotation, force, and moment components in the structural system (the components of the material and structural systems coincide here), the beam element contribution becomes

Beam 1:
$$\begin{Bmatrix} F_{Z2}^1 \\ M_{Y2}^1 \\ F_{Z1}^1 \\ M_{Y1}^1 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \theta_{Y1} \end{Bmatrix} - \frac{fL}{12} \begin{Bmatrix} 6 \\ -L \\ 6 \\ L \end{Bmatrix} \quad \begin{matrix} \text{eq. 1} \\ \text{eq. 2} \\ \text{eq. 3} \\ \text{eq. 4} \end{matrix}$$

Equilibrium equations of the nodes are

Node 2 : $\sum F_Z = F_{Z2} - F_{Z2}^1 = 0$ eq. 5

$\sum M_Y = M_{Y2} - M_{Y2}^1 = 0$ eq. 6

Node 1 : $\sum F_Z = F_{Z1} - F_{Z1}^1 = 0$ eq. 7

$\sum M_Y = -M_{Y1}^1 = 0$ eq. 8

The outcome is a set of 8 linear equations for 1 rotation, 4 internal forces, and 3 constraint forces/moments. As the first step toward the solution, the internal forces in the node equilibrium equations (just those not depending on the constraint forces) are replaced by their expressions given by eq.1, eq.2 , eq.3 and eq.4. After that, the unknown displacements and rotations follow from the modified equilibrium equations. The moment equilibrium condition of node 1 (the only equation which does not depend on the constraint forces/moments) gives

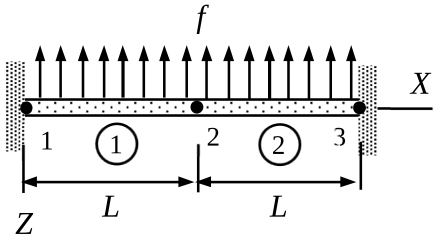
$$\frac{EI}{L^3} 4L^2 \theta_{Y1} - \frac{fL^2}{12} = 0 \quad \Leftrightarrow \quad \theta_{Y1} = \frac{1}{48} \frac{fL^3}{EI}.$$

Use the code of MEC-E1050 to check your answer!

Name________________________ Student number________________

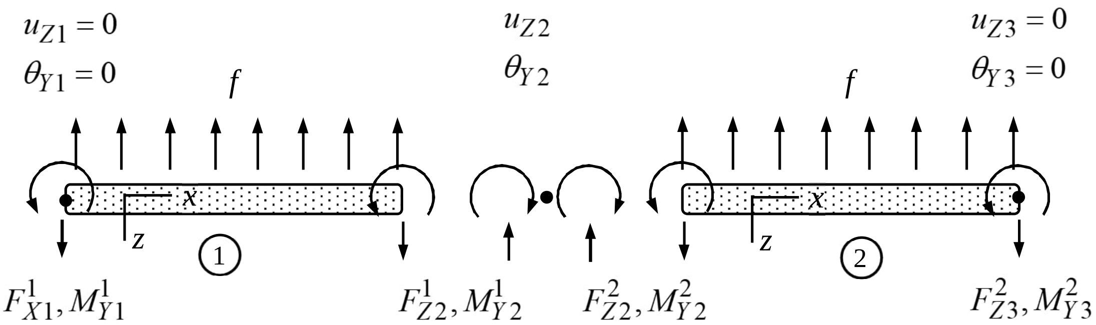
Assignment 5

External load f acting on the beam shown is constant. Determine displacement u_{Z2} and rotation θ_{Y2} of the mid-point (node 2). Young's modulus of the material and the second moments of area are E and I, respectively. Use two beam elements of equal length.



Solution

Only the displacement in the $Z-$direction and rotation in the $Y-$direction matter in the planar beam bending problem. From the figure, the non-zero displacement/rotation components are u_{Z2} and θ_{Y2}. Free body diagrams of the two bending beam elements and node 2 are (nodes 1 and 3 are constrained)



Element contributions of the two $xz-$plane bending beams (formulae collection) and the equilibrium equations of node 2 are (notice that the distributed force in the element contribution is the transverse component in the material system associated with beam)

$$\text{Beam 1:}\quad \begin{Bmatrix} F_{Z1}^1 \\ M_{Y1}^1 \\ F_{Z2}^1 \\ M_{Y2}^1 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ u_{Z2} \\ \theta_{Y2} \end{Bmatrix} + \frac{fL}{12}\begin{Bmatrix} 6 \\ -L \\ 6 \\ L \end{Bmatrix}, \quad (f_z = -f)$$

$$\text{Beam 2:}\quad \begin{Bmatrix} F_{Z2}^2 \\ M_{Y2}^2 \\ F_{Z3}^2 \\ M_{Y3}^2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \\ 0 \\ 0 \end{Bmatrix} + \frac{fL}{12}\begin{Bmatrix} 6 \\ -L \\ 6 \\ L \end{Bmatrix}, \quad (f_z = -f)$$

Node 2: $-F_{Z2}^1 - F_{Z2}^2 = 0$ and $-M_{Y2}^1 - M_{Y2}^2 = 0$.

Elimination of the internal forces from the two equilibrium equations of node 2 using the element contributions gives the forms

Node 2: $$-[\frac{EI}{L^3}(12u_{Z2}+6L\theta_{Y2})+6\frac{fL}{12}]-[\frac{EI}{L^3}(12u_{Z2}-6L\theta_{Y2})+6\frac{fL}{12}]=0 \quad \text{and}$$

$$-[\frac{EI}{L^3}(6Lu_{Z2}+4L^2\theta_{Y2})+L\frac{fL}{12}]-[\frac{EI}{L^3}(-6Lu_{Z2}+4L^2\theta_{Y2})-L\frac{fL}{12}]=0.$$

Matrix representation of the two equilibrium equations, containing u_{Z2} and θ_{Y2} as the unknowns, is

$$\frac{EI}{L^3}\begin{bmatrix}24 & 0\\ 0 & 8L^2\end{bmatrix}\begin{Bmatrix}u_{Z2}\\ \theta_{Y2}\end{Bmatrix}+fL\begin{Bmatrix}1\\ 0\end{Bmatrix}=0 \quad \Leftrightarrow$$

$$\begin{Bmatrix}u_{Z2}\\ \theta_{Y2}\end{Bmatrix}=-\frac{fL^4}{EI}\begin{bmatrix}24 & 0\\ 0 & 8L^2\end{bmatrix}^{-1}\begin{Bmatrix}1\\ 0\end{Bmatrix}=-\frac{fL^4}{EI}\begin{bmatrix}1/24 & 0\\ 0 & 1/(8L^2)\end{bmatrix}\begin{Bmatrix}1\\ 0\end{Bmatrix}=-\frac{1}{24}\frac{fL^4}{EI}\begin{Bmatrix}1\\ 0\end{Bmatrix} \quad \Leftrightarrow$$

$$u_{Z2}=-\frac{1}{24}\frac{fL^4}{EI} \quad \text{and} \quad \theta_{Y2}=0.$$

MEC-E1050

FINITE ELEMENT METHOD IN SOLIDS 2024

3 BAR AND BEAM STRUCTURES

LEARNING OUTCOMES

Students can solve the lecture problems, home problems, and exercise problems on the topics of the week:

- Bar truss displacements. Element contribution of a bar in the structural coordinate system.
- Principle of virtual work and the fundamental lemma of variation calculus. Virtual work expressions of elements and structures.
- Beam element loading modes and the element.
- Kinematic constraints, kinematic links, and boundary conditions.

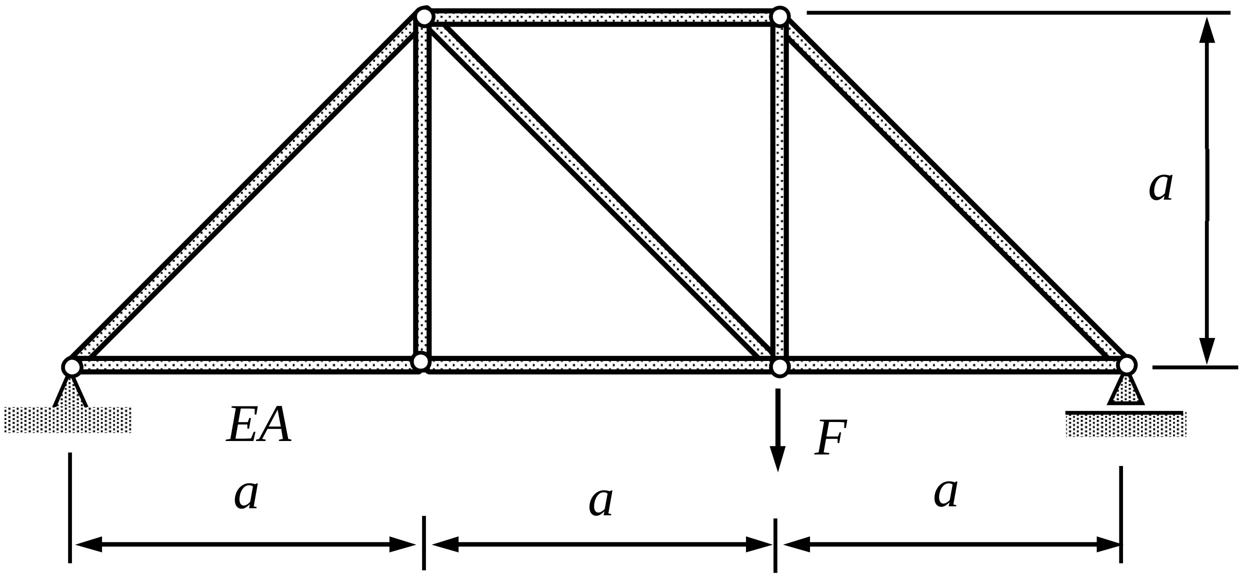
3.1 BAR TRUSS

Slender structural parts of a truss may act as bars or beams depending on the loading and the type of joints. If internal forces are aligned with the axes of the parts, a simple bar model may give satisfactory results!

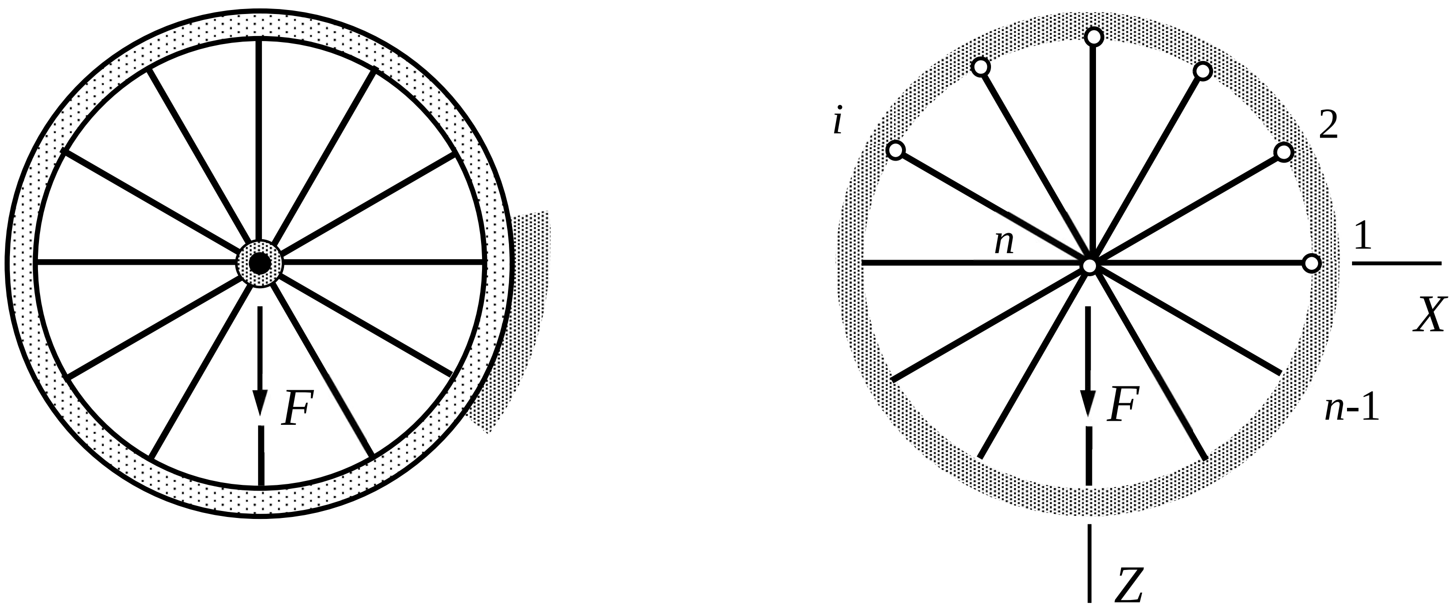
Typical applications are bridges, cranes, roof trusses etc.

BAR TRUSS

Bar truss consists of straight slender structural parts connected by cylindrical or spherical joints so that internal forces are aligned with the axes of the structural parts (a straight line between the joints). The unknowns are the nodal displacements. Rotations do not matter as they do not appear in the element contributions.

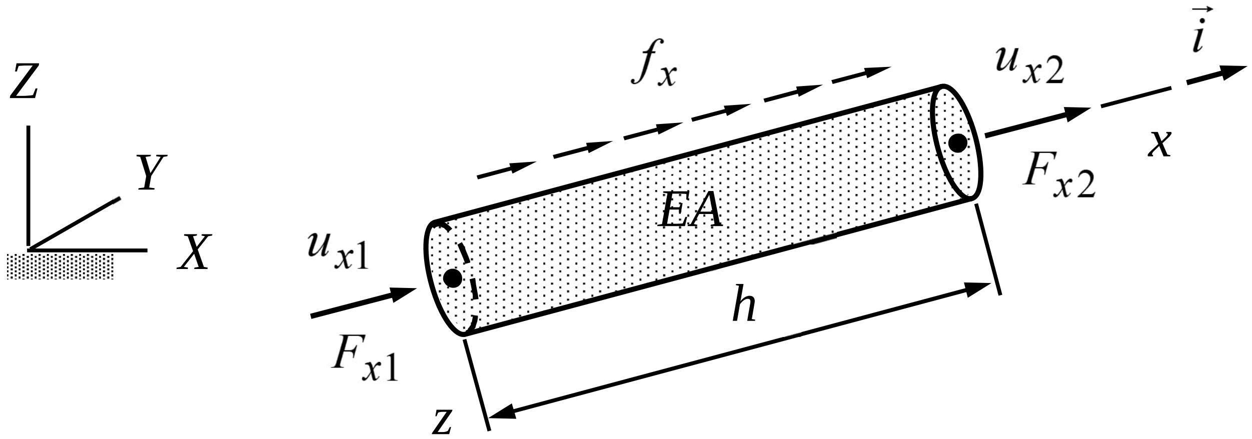


EXAMPLE 3.1. The outer rim and center of a wheel are assumed to be fully rigid. The center is fastened with 12 (diameter $d = 1\text{mm}$) steel ($E = 210\,\text{GN/m}^2$) rods (length $L = 300\,\text{mm}$). Using bar elements, calculate the displacement of the center when a load of $F = 1\text{kN}$ is placed (buckling does not occur)?



Answer $U_{Zn} = \dfrac{FL}{6EA} (= \dfrac{1}{525\pi} [\text{m}])$, when $n = 13$

BAR ELEMENT IN THE STRUCTURAL SYSTEM



$$\begin{Bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \end{Bmatrix} = \frac{EA}{h}\begin{bmatrix} \mathbf{i}\mathbf{i}^{\mathrm{T}} & -\mathbf{i}\mathbf{i}^{\mathrm{T}} \\ -\mathbf{i}\mathbf{i}^{\mathrm{T}} & \mathbf{i}\mathbf{i}^{\mathrm{T}} \end{bmatrix}\begin{Bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{Bmatrix} - \frac{f_x h}{2}\begin{Bmatrix} \mathbf{i} \\ \mathbf{i} \end{Bmatrix}, \text{ where } \mathbf{a} = \begin{Bmatrix} u_X \\ u_Y \\ u_Z \end{Bmatrix} \text{ and } \mathbf{R} = \begin{Bmatrix} F_X \\ F_Y \\ F_Z \end{Bmatrix}$$

The displacement components of the material system are expressed in terms of those in the structural system, which brings the orientation into the element contribution. Column matrix $\mathbf{i}$ contains the components of the unit vector $\vec{i}$ in the structural coordinate system!

- The starting point is the element contribution in terms of displacement and force components in the material system (the simplest representation)

$$\begin{Bmatrix} F_{x1} \\ F_{x2} \end{Bmatrix} = \frac{EA}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

- With notations $\mathbf{a} = \{u_X \quad u_Y \quad u_Z\}^{\mathrm{T}}$ and $\mathbf{R} = \{F_X \quad F_Y \quad F_Z\}^{\mathrm{T}}$ and taking into account that $F_y = F_z = 0$ for a bar

$$u_x = \mathbf{i}^{\mathrm{T}}\begin{Bmatrix} u_X \\ u_Y \\ u_Z \end{Bmatrix} = \mathbf{i}^{\mathrm{T}}\mathbf{a} \quad \text{and} \quad \mathbf{R} = \begin{Bmatrix} F_X \\ F_Y \\ F_Z \end{Bmatrix} = \mathbf{i}F_x \quad \Rightarrow$$

$$\begin{Bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \end{Bmatrix} = \begin{bmatrix} \mathbf{i} & \mathbf{0} \\ \mathbf{0} & \mathbf{i} \end{bmatrix}\begin{Bmatrix} F_{x1} \\ F_{x2} \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} = \begin{bmatrix} \mathbf{i}^{\mathrm{T}} & \mathbf{0} \\ \mathbf{0} & \mathbf{i}^{\mathrm{T}} \end{bmatrix}\begin{Bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{Bmatrix}.$$

- Therefore, element contribution in the structural system

$$\begin{bmatrix} \mathbf{i} & \mathbf{0} \\ \mathbf{0} & \mathbf{i} \end{bmatrix} \begin{Bmatrix} F_{x1} \\ F_{x2} \end{Bmatrix} = \frac{EA}{h} \begin{bmatrix} \mathbf{i} & \mathbf{0} \\ \mathbf{0} & \mathbf{i} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2} \begin{bmatrix} \mathbf{i} & \mathbf{0} \\ \mathbf{0} & \mathbf{i} \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \quad \Rightarrow$$

$$\begin{Bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \end{Bmatrix} = \frac{EA}{h} \begin{bmatrix} \mathbf{i} & \mathbf{0} \\ \mathbf{0} & \mathbf{i} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{i}^{\mathrm{T}} & \mathbf{0} \\ \mathbf{0} & \mathbf{i}^{\mathrm{T}} \end{bmatrix} \begin{Bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{Bmatrix} - \frac{f_x h}{2} \begin{bmatrix} \mathbf{i} & \mathbf{0} \\ \mathbf{0} & \mathbf{i} \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \quad \Rightarrow$$

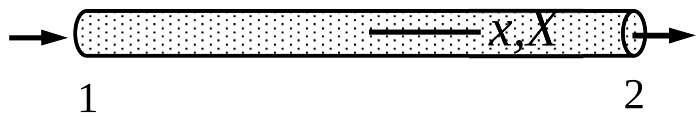
$$\begin{Bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \end{Bmatrix} = \frac{EA}{h} \begin{bmatrix} \mathbf{i}\mathbf{i}^{\mathrm{T}} & -\mathbf{i}\mathbf{i}^{\mathrm{T}} \\ -\mathbf{i}\mathbf{i}^{\mathrm{T}} & \mathbf{i}\mathbf{i}^{\mathrm{T}} \end{bmatrix} \begin{Bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{Bmatrix} - \frac{f_x h}{2} \begin{Bmatrix} \mathbf{i} \\ \mathbf{i} \end{Bmatrix} \quad \Leftarrow$$

contains also the orientation $\mathbf{i}$ of the bar.

- The actual size of the matrix etc. depends on the number of components in $\mathbf{i}$ (dimension of the problem). For example, assuming that the axes of the material and structural coordinate systems are aligned, the bar elements for the uni-axial (X-axis), and planar problems (XZ-plane) are

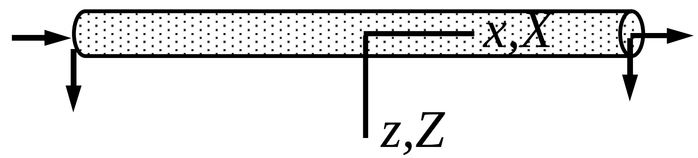
$$\begin{Bmatrix} F_{X1} \\ F_{X2} \end{Bmatrix} = \frac{EA}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{X2} \end{Bmatrix} - \frac{f_x h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix},$$

$$\mathbf{a} = \{u_X\}^{\mathrm{T}}, \mathbf{R} = \{F_X\}^{\mathrm{T}}$$



$$\begin{Bmatrix} F_{X1} \\ F_{Z1} \\ \hline F_{X2} \\ F_{Z2} \end{Bmatrix} = \frac{EA}{h}\left[\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]\begin{Bmatrix} u_{X1} \\ u_{Z1} \\ \hline u_{X2} \\ u_{Z2} \end{Bmatrix} - \frac{f_x h}{2}\begin{Bmatrix} 1 \\ 0 \\ \hline 1 \\ 0 \end{Bmatrix}.$$

$$\mathbf{a} = \{u_X \quad u_Z\}^{\mathrm{T}}, \mathbf{R} = \{F_X \quad F_Z\}^{\mathrm{T}}$$

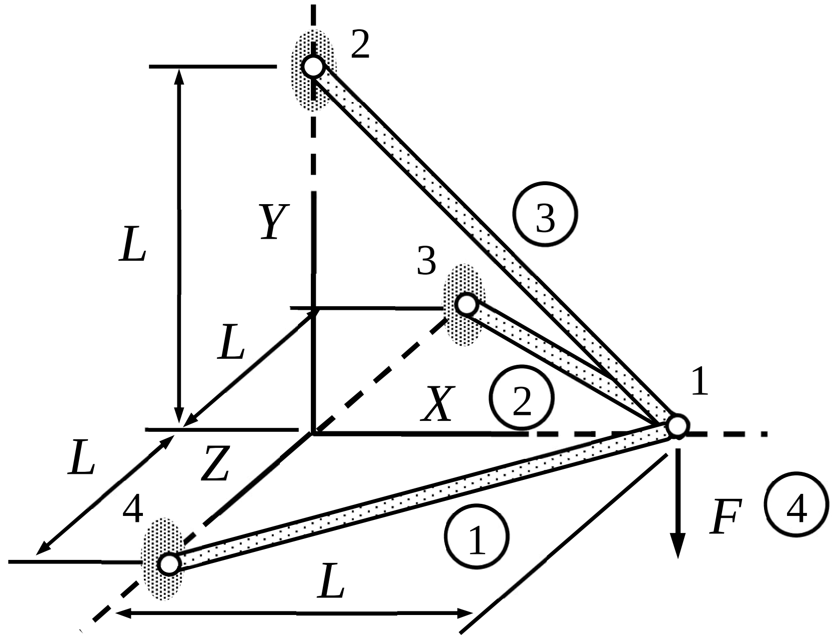


EXAMPLE 3.2. The nodes of a bar are at $(0,0,0)$ (node 1) and (L,L,L) (node 2) in the structural coordinate system and the positive x-axis is directed from node 1 to 2. Determine the element contribution $\mathbf{R}=\mathbf{Ka}-\mathbf{F}$ in the structural coordinate system if f_x and EA are constants.

Answer

$$\begin{Bmatrix} F_{X1} \\ F_{Y1} \\ F_{Z1} \\ \hline F_{X2} \\ F_{Y2} \\ F_{Z2} \end{Bmatrix} = \frac{EA}{3\sqrt{3}L} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ \hline -1 & -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 \end{array}\right] \begin{Bmatrix} u_{X1} \\ u_{Y1} \\ u_{Z1} \\ \hline u_{X2} \\ u_{Y2} \\ u_{Z2} \end{Bmatrix} - \frac{f_x L}{2} \begin{Bmatrix} 1 \\ 1 \\ 1 \\ \hline 1 \\ 1 \\ 1 \end{Bmatrix}$$

EXAMPLE 3.3. If the space truss of the figure is loaded by a vertical force F acting on node 1, determine the displacement of node 1. Assume that the displacement in Z-direction vanishes due to the symmetry i.e. $u_{Z1} = 0$. Young's modulus of the material E and the cross-sectional area A are constants. Gravity is negligible.



Answer $u_{X1} = -\sqrt{2}\dfrac{FL}{EA}$ and $u_{Y1} = -3\sqrt{2}\dfrac{FL}{EA}$

- The bar element contribution in the structural coordinate system

$$\begin{Bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \end{Bmatrix} = \frac{EA}{h}\begin{bmatrix} \mathbf{i}\mathbf{i}^{\mathrm{T}} & -\mathbf{i}\mathbf{i}^{\mathrm{T}} \\ -\mathbf{i}\mathbf{i}^{\mathrm{T}} & \mathbf{i}\mathbf{i}^{\mathrm{T}} \end{bmatrix}\begin{Bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{Bmatrix} - \frac{f_x h}{2}\begin{Bmatrix} \mathbf{i} \\ \mathbf{i} \end{Bmatrix}, \quad \mathbf{R} = \begin{Bmatrix} F_X \\ F_Y \\ F_Z \end{Bmatrix}, \text{ and } \mathbf{a} = \begin{Bmatrix} u_X \\ u_Y \\ u_Z \end{Bmatrix}$$

  is useful in hand calculations. The elements of $\mathbf{i}$ are the components of the unit vector $\vec{i}$ in the structural coordinate system which can be deduced from the figure:

$$\mathbf{i}^1 = \frac{1}{\sqrt{2}}\begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} \text{ and } h = \sqrt{2}L, \quad \mathbf{i}^2 = \frac{1}{\sqrt{2}}\begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix} \text{ and } h = \sqrt{2}L, \quad \mathbf{i}^3 = \frac{1}{\sqrt{2}}\begin{Bmatrix} 1 \\ -1 \\ 0 \end{Bmatrix} \text{ and } h = \sqrt{2}L.$$

- The element contributions of the three bars and one point force are

$$\text{Bar 1: } \left\{\begin{array}{c} F_{X4}^1 \\ F_{Y4}^1 \\ F_{Z4}^1 \\ \hdashline F_{X1}^1 \\ F_{Y1}^1 \\ F_{Z1}^1 \end{array}\right\} = \frac{EA}{\sqrt{8}L}\left[\begin{array}{ccc:ccc} 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & -1 \\ \hdashline -1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 1 \end{array}\right]\left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ \hdashline u_{X1} \\ u_{Y1} \\ 0 \end{array}\right\}, \left(\mathbf{i}^1 = \frac{1}{\sqrt{2}}\left\{\begin{array}{c} 1 \\ 0 \\ -1 \end{array}\right\}, \ h = \sqrt{2}L \right)$$

$$\text{Bar 2: } \left\{\begin{array}{c} F_{X3}^2 \\ F_{Y3}^2 \\ F_{Z3}^2 \\ \hdashline F_{X1}^2 \\ F_{Y1}^2 \\ F_{Z1}^2 \end{array}\right\} = \frac{EA}{\sqrt{8}L}\left[\begin{array}{ccc:ccc} 1 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 & -1 \\ \hdashline -1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 & 0 & 1 \end{array}\right]\left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ \hdashline u_{X1} \\ u_{Y1} \\ 0 \end{array}\right\}, \left(\mathbf{i}^2 = \frac{1}{\sqrt{2}}\left\{\begin{array}{c} 1 \\ 0 \\ 1 \end{array}\right\}, \ h = \sqrt{2}L \right)$$

Bar 3: $$\begin{Bmatrix} F_{X2}^3 \\ F_{Y2}^3 \\ F_{Z2}^3 \\ \hdashline F_{X1}^3 \\ F_{Y1}^3 \\ F_{Z1}^3 \end{Bmatrix} = \frac{EA}{\sqrt{8}L} \left[\begin{array}{ccc:ccc} 1 & -1 & 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline -1 & 1 & 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \hdashline u_{X1} \\ u_{Y1} \\ 0 \end{Bmatrix}, \quad (\mathbf{i}^3 = \frac{1}{\sqrt{2}} \begin{Bmatrix} 1 \\ -1 \\ 0 \end{Bmatrix}, \; h = \sqrt{2}L)$$

Force 4: $$\begin{Bmatrix} F_{X1}^4 \\ F_{Y1}^4 \\ F_{Z1}^4 \end{Bmatrix} = -\begin{Bmatrix} 0 \\ -F \\ 0 \end{Bmatrix}.$$

- In assembly, internal forces acting on the nodes are added to end up with the equilibrium equations for the nodes. To get the minimal system for the unknown displacement components, only the non-constrained directions are considered first (the remaining equilibrium equation can be used to get the solution to the constraint forces and element contributions to solution for the internal forces)

$$\begin{Bmatrix} F_{X1}^{1} \\ F_{Y1}^{1} \end{Bmatrix} + \begin{Bmatrix} F_{X1}^{2} \\ F_{Y1}^{2} \end{Bmatrix} + \begin{Bmatrix} F_{X1}^{3} \\ F_{Y1}^{3} \end{Bmatrix} + \begin{Bmatrix} F_{X1}^{4} \\ F_{Y1}^{4} \end{Bmatrix} = \frac{EA}{\sqrt{8}L} \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} + F \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = 0.$$

- The unknown displacement components follow as the solution of the system equations

$$\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} = -\sqrt{8}\frac{FL}{EA} \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = -\frac{\sqrt{8}}{2}\frac{FL}{EA} \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = -\sqrt{2}\frac{FL}{EA} \begin{Bmatrix} 1 \\ 3 \end{Bmatrix} \quad \Leftrightarrow$$

$$u_{X1} = -\sqrt{2}\frac{FL}{EA} \quad \text{and} \quad u_{Y1} = -3\sqrt{2}\frac{FL}{EA} .$$

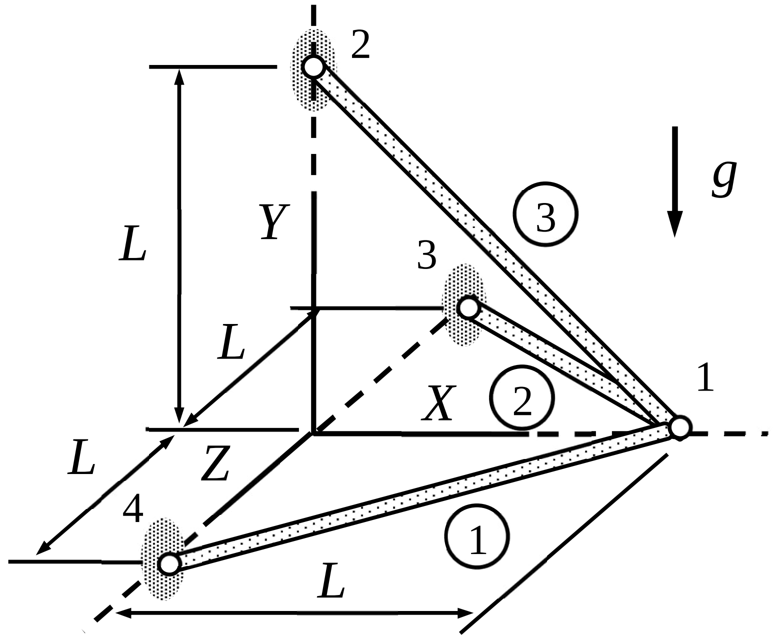
EFFECT OF WEIGHT

Weight $\vec{f} = \rho A\vec{g}$ acting on bars of a truss may not be aligned with the axes. Assuming that the joints are not capable for taking moments, the bar model may give a good picture about the internal forces if the weight is taken to act at the element nodes according to

$$\begin{Bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \end{Bmatrix} = \frac{EA}{h}\begin{bmatrix} \mathbf{i}\mathbf{i}^{\mathrm{T}} & -\mathbf{i}\mathbf{i}^{\mathrm{T}} \\ -\mathbf{i}\mathbf{i}^{\mathrm{T}} & \mathbf{i}\mathbf{i}^{\mathrm{T}} \end{bmatrix}\begin{Bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{Bmatrix} - \frac{h\rho A}{2}\begin{Bmatrix} \mathbf{g} \\ \mathbf{g} \end{Bmatrix}, \text{ where } \mathbf{a} = \begin{Bmatrix} u_X \\ u_Y \\ u_Z \end{Bmatrix}, \ \mathbf{R} = \begin{Bmatrix} F_X \\ F_Y \\ F_Z \end{Bmatrix} \text{ and } \ \mathbf{g} = \begin{Bmatrix} g_X \\ g_Y \\ g_Z \end{Bmatrix}.$$

Above, $\mathbf{g}$ contains the components of acceleration by gravity in the structural coordinate system. The elements of $\mathbf{i}$ are the components of the unit vector $\vec{i}$ in the structural coordinate system.

EXAMPLE 3.4. A space truss is loaded by its own weight. If the joints do not take any moments, determine the displacement of node 1. Young's modulus E, density ρ, and cross-sectional area A are constants. Use symmetry.



Answer $u_{X1} = -3\dfrac{\rho g L^2}{E}$ and $u_{Y1} = -9\dfrac{\rho g L^2}{E}$

- The Length of all the bars $h=\sqrt{2}L$. The unit vectors to the directions of the x-axes and the components of the acceleration by gravity are (figure)

$$\mathbf{i}^1=\frac{1}{\sqrt{2}}\begin{Bmatrix}1\\0\\-1\end{Bmatrix},\quad \mathbf{i}^2=\frac{1}{\sqrt{2}}\begin{Bmatrix}1\\0\\1\end{Bmatrix},\quad \mathbf{i}^3=\frac{1}{\sqrt{2}}\begin{Bmatrix}1\\-1\\0\end{Bmatrix},\text{ and }\mathbf{g}=\begin{Bmatrix}0\\-g\\0\end{Bmatrix}.$$

- The element contributions need to be expressed in the structural coordinate system:

$$\text{Bar 1:}\quad \begin{Bmatrix}F^1_{X4}\\F^1_{Y4}\\F^1_{Z4}\\\hdashline F^1_{X1}\\F^1_{Y1}\\F^1_{Z1}\end{Bmatrix}=\frac{EA}{\sqrt{8}L}\left[\begin{array}{ccc:ccc}1&0&-1&-1&0&1\\0&0&0&0&0&0\\-1&0&1&1&0&-1\\\hdashline -1&0&1&1&0&-1\\0&0&0&0&0&0\\1&0&-1&-1&0&1\end{array}\right]\begin{Bmatrix}0\\0\\0\\\hdashline u_{X1}\\u_{Y1}\\0\end{Bmatrix}-\frac{\rho ALg}{\sqrt{2}}\begin{Bmatrix}0\\-1\\0\\\hdashline 0\\-1\\0\end{Bmatrix},$$

$$
\text{Bar 2:}\quad \left\{\begin{array}{c} F_{X3}^{2} \\ F_{Y3}^{2} \\ F_{Z3}^{2} \\ \hline F_{X1}^{2} \\ F_{Y1}^{2} \\ F_{Z1}^{2} \end{array}\right\} = \frac{EA}{\sqrt{8}L}\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 & -1 \\ \hline -1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 & 0 & 1 \end{array}\right]\left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline u_{X1} \\ u_{Y1} \\ 0 \end{array}\right\} - \frac{\rho ALg}{\sqrt{2}}\left\{\begin{array}{c} 0 \\ -1 \\ 0 \\ \hline 0 \\ -1 \\ 0 \end{array}\right\},
$$

$$
\text{Bar 3:}\quad \left\{\begin{array}{c} F_{X2}^{3} \\ F_{Y2}^{3} \\ F_{Z2}^{3} \\ \hline F_{X1}^{3} \\ F_{Y1}^{3} \\ F_{Z1}^{3} \end{array}\right\} = \frac{EA}{\sqrt{8}L}\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -1 & 1 & 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]\left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline u_{X1} \\ u_{Y1} \\ 0 \end{array}\right\} - \frac{\rho ALg}{\sqrt{2}}\left\{\begin{array}{c} 0 \\ -1 \\ 0 \\ \hline 0 \\ -1 \\ 0 \end{array}\right\}.
$$

- Sum of the forces acting on node 1 need to vanish for the equilibrium

$$0=\begin{Bmatrix} F_{X1}^{1}+F_{X1}^{2}+F_{X1}^{3} \\ F_{Y1}^{1}+F_{Y1}^{2}+F_{Y1}^{3} \end{Bmatrix}=\frac{EA}{2\sqrt{2}L}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix}-\frac{\rho ALg}{\sqrt{2}}\begin{Bmatrix} 0 \\ -3 \end{Bmatrix}.$$

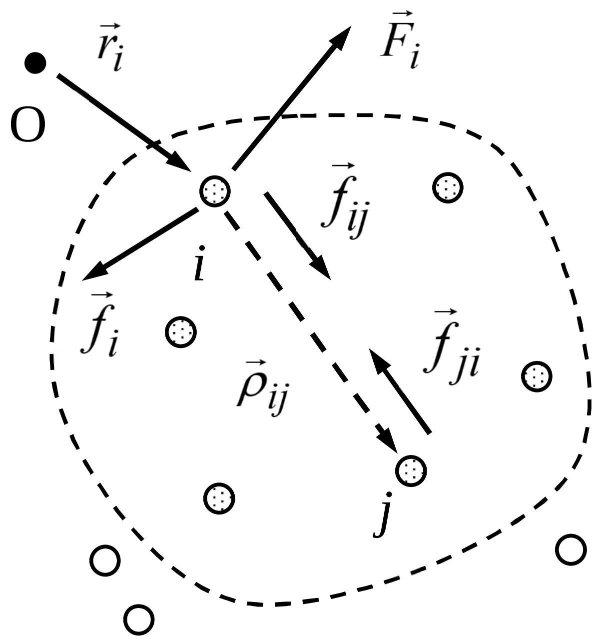
- The values of the unknown displacement components are obtained from the equation system

$$\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix}=2\frac{\rho L^2 g}{E}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}^{-1}\begin{Bmatrix} 0 \\ -3 \end{Bmatrix}=\frac{\rho g L^2}{E}\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}\begin{Bmatrix} 0 \\ -3 \end{Bmatrix}=\frac{\rho g L^2}{E}\begin{Bmatrix} -3 \\ -9 \end{Bmatrix} \quad \Leftrightarrow$$

$$u_{X1}=-\frac{3\rho g L^2}{E} \text{ and } u_{Y1}=-\frac{9\rho g L^2}{E}.$$

3.2 PRINCIPLE OF VIRTUAL WORK

Principle of virtual work is one of the equivalent forms of equilibrium equations (an important form).



Virtual work $\delta W = \delta W^{\text{ext}} + \delta W^{\text{int}} = 0 \quad \forall\, \delta \vec{r}_i$

External forces $\delta W^{\text{ext}} = \sum \delta \vec{r}_i \cdot \vec{F}_i$

Internal forces $\delta W^{\text{int}} = \sum \delta \vec{r}_i \cdot \vec{f}_i = -\sum \delta \rho_{ij} f_{ij}$

The principle is very useful, for example, in connection with kinematical constraints. As an example, $\delta W^{\text{int}} = 0$ for a rigid body, as the distances between particles are constants and therefore $\delta \rho_{ij} = 0$!

- The starting point is the equilibrium equations of a particle system

$$\sum_{i\in I} \delta\vec{r}_i \cdot (\vec{F}_i + \vec{f}_i) = \delta W^{\text{ext}} + \delta W^{\text{int}} = 0 \quad \forall\, \delta\vec{r}_i, \text{ where}$$

$$\delta W^{\text{ext}} = \sum_{i\in I} \delta\vec{r}_i \cdot \vec{F}_i \quad \text{and} \quad \delta W^{\text{int}} = \sum_{i\in I} \delta\vec{r}_i \cdot \vec{f}_i \ .$$

- The virtual work of internal forces can be written in a more concise form: Let us consider a typical pair (i, j) of particles:

$$W_{(i,j)}^{\text{int}} = \vec{f}_{ij} \cdot \delta\vec{r}_i + \vec{f}_{ji} \cdot \delta\vec{r}_j = \vec{f}_{ij} \cdot \delta(\vec{r}_i - \vec{r}_j) = -\vec{f}_{ij} \cdot \delta\vec{\rho}_{ij} = -f_{ij}\delta\rho_{ij},$$

where $\vec{\rho}_{ij} = \vec{r}_j - \vec{r}_i$ is the position of particle j relative to particle i. The expression for a body (a closed system of particles) is obtained as a sum over the particle pairs.

FUNDAMENTAL LEMMA OF VARIATION CALCULUS

The fundamental lemma of variation calculus in one form or another is an important tool in FEM. In MEC-E1050 the lemma tells how to deduce the equilibrium equations of a structure using a virtual work expression and the principle of virtual work:

- ☐ $u, v \in \mathbb{R} \quad : \quad vu = 0 \;\; \forall v \quad \Leftrightarrow \quad u = 0$
- ☐ $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n \quad : \quad \mathbf{v}^{\mathrm{T}}\mathbf{u} = 0 \;\; \forall \mathbf{v} \quad \Leftrightarrow \quad \mathbf{u} = 0$
- ☐ $u, v \in C^0(\Omega) \quad : \quad \int_\Omega uv d\Omega = 0 \;\; \forall v \quad \Leftrightarrow \quad u(x, y, \ldots) = 0 \;\; \text{in} \;\; \Omega$

In mechanics of the materials, variable or function v is often chosen as the kinematically admissible variation of the displacement field δu.

PRINCIPLE OF VIRTUAL WORK IN FEM

Principle of virtual work $\delta W = 0 \quad \forall \delta \mathbf{a}$ is just one form of equilibrium equations. In connection with MEC-E1050, the principle is a representation of the equilibrium equations of *nodes*.

$$\delta W = -\underbrace{\sum_{e \in E} \sum_i (\delta \vec{u}_i \cdot \vec{F}_i^e + \delta \vec{\theta}_i \cdot \vec{M}_i^e)}_{\text{forces from elements}} + \underbrace{\sum_{i \in I} (\delta \vec{u}_i \cdot \vec{F}_i + \delta \vec{\theta}_i \cdot \vec{M}_i)}_{\text{given forces acting on nodes}} \quad \Rightarrow$$

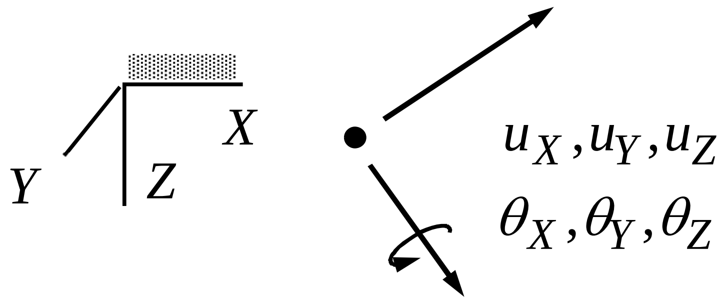
$$\delta W = \sum_{e \in E} \delta W^e = -\delta \mathbf{a}^{\mathrm{T}} (\mathbf{K}\mathbf{a} - \mathbf{F}).$$ (point forces treated as one-node elements)

The negative sign in the first term is due to the selection that the forces acting on the *elements* are positive in the direction of displacement. Therefore, according to Newton's 3rd law, forces acting on the nodes are negative.

FORCE ELEMENT CONTRIBUTION

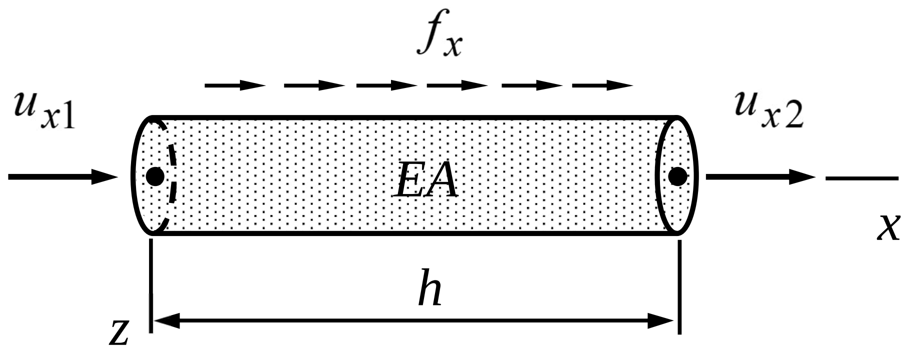
External point forces and moments are assumed to act on the joints. They are treated as elements associated with one node only. Virtual work expression is usually simplest in the structural coordinate system:

$$\delta W = \begin{Bmatrix} \delta u_X \\ \delta u_Y \\ \delta u_Z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} F_X \\ F_Y \\ F_Z \end{Bmatrix} + \begin{Bmatrix} \delta \theta_X \\ \delta \theta_Y \\ \delta \theta_Z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} M_X \\ M_Y \\ M_Z \end{Bmatrix}$$



Above, F_X, F_Y, F_Z and M_X, M_Y, M_Z are the given components. A rigid body can be modeled as a particle at the center of mass connected to the other joints of the body by rigid links!

BAR ELEMENT CONTRIBUTION



$$\delta W = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} (\frac{EA}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}), \text{ where } u_x = \mathbf{i}^{\mathrm{T}}\begin{Bmatrix} u_X \\ u_Y \\ u_Z \end{Bmatrix}.$$

Element contribution in its variational form is a scalar which simplifies assembly considerably. Mathematica code of MEC-E1050 uses the variational form of the element contribution!

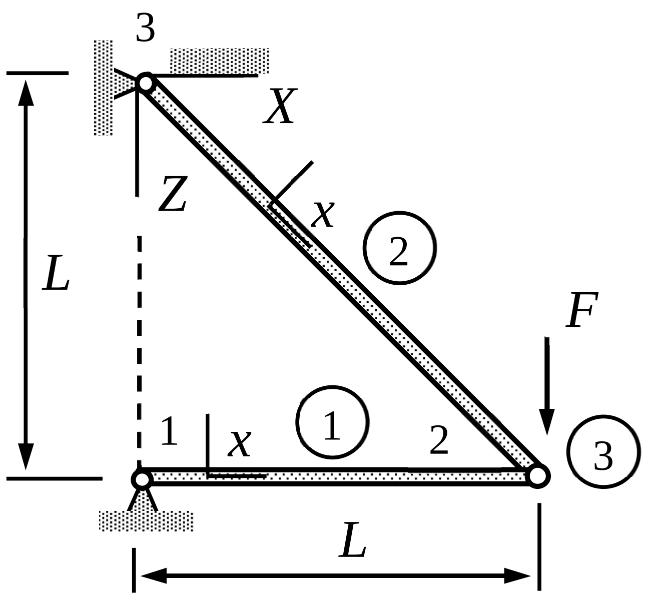
DISPLACEMENT ANALYSIS; THE IMPROVED RECIPE

- Express the nodal displacements and rotations u_x, u_y, u_z, $\theta_x, \theta_y, \theta_z$ of the material coordinate systems in terms of those in the structural coordinate system u_X, u_Y, u_Z, $\theta_X, \theta_Y, \theta_Z$ ($u_x = \{u_X \quad u_Y \quad u_Z\}\mathbf{i}$ etc.) and write down the element contributions $\delta W = -\delta\mathbf{a}^{\mathrm{T}}(\mathbf{Ka} - \mathbf{F})$.

- Sum the element contributions to end up with the virtual work expression of the structure $\delta W = \sum_{e \in E} \delta W^e$ (point forces can be considered as elements also). Re-structure to get the "standard" form $\delta W = -\delta\mathbf{a}^{\mathrm{T}}(\mathbf{Ka} - \mathbf{F})$.

- Use the principle of virtual work $\delta W = 0 \;\; \forall \delta\mathbf{a}$, the fundamental lemma of variation calculus for $\delta\mathbf{a} \in \mathbb{R}^n$, and solve for the dofs from the system equations $\mathbf{Ka} - \mathbf{F} = 0$.

EXAMPLE 3.5. A *bar truss* is loaded by a point force having magnitude F as shown in the figure. Derive the equilibrium equations and determine the nodal displacements. The cross-sectional area of bar 1 is A and that for bar 2 $\sqrt{8}A$. Young's modulus is E and weight is omitted. Use the principle of virtual work.



Answer $\dfrac{EA}{L}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{Z2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ F \end{Bmatrix} = 0 \quad \Leftrightarrow \quad u_{X2} = -\dfrac{FL}{EA}$ and $u_{Z2} = 2\dfrac{FL}{EA}$.

- Element contributions $\delta W = -\delta\mathbf{a}^{\mathrm{T}}(\mathbf{Ka}-\mathbf{F})$ to the virtual work expression of the structure are

Bar 1: $\delta W^{1} = -\begin{Bmatrix} 0 \\ \delta u_{X2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_{X2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}\right) = -\frac{EA}{L} u_{X2}\delta u_{X2}$,

Bar 2: $\delta W^{2} = -\frac{1}{\sqrt{2}}\begin{Bmatrix} 0 \\ \delta u_{X2}+\delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{E\sqrt{8}A}{\sqrt{2}L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{Bmatrix} 0 \\ u_{X2}+u_{Z2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}\right) \Leftrightarrow$

$$\delta W^{2} = -\frac{EA}{L}(\delta u_{X2}+\delta u_{Z2})(u_{X2}+u_{Z2}),$$

Force 3: $\delta W^{3} = \delta u_{Z2} F$.

- Virtual work expression is obtained as the sum of the element contributions

$$\delta W = -\frac{EA}{L}\delta u_{X2}u_{X2} - \frac{EA}{L}(\delta u_{X2}+\delta u_{Z2})(u_{X2}+u_{Z2}) + \delta u_{Z2}F \quad \Leftrightarrow$$

$$\delta W = -\delta u_{X2}(2\frac{EA}{L}u_{X2} + \frac{EA}{L}u_{Z2}) - \delta u_{Z2}(-F + \frac{EA}{L}u_{X2} + \frac{EA}{L}u_{Z2}) \quad \Leftrightarrow$$

$$\delta W = -\begin{Bmatrix}\delta u_{X2}\\ \delta u_{Z2}\end{Bmatrix}^{\mathrm{T}} (\frac{EA}{L}\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\begin{Bmatrix}u_{X2}\\ u_{Z2}\end{Bmatrix} - \begin{Bmatrix}0\\ F\end{Bmatrix}).$$

”standard” form

- Principle of virtual work $\delta W = 0 \ \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus imply

$$\frac{EA}{L}\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\begin{Bmatrix}u_{X2}\\ u_{Z2}\end{Bmatrix} - \begin{Bmatrix}0\\ F\end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix}u_{X2}\\ u_{Z2}\end{Bmatrix} = \frac{LF}{EA}\begin{Bmatrix}-1\\ 2\end{Bmatrix}.$$

3.3 BEAM ELEMENT CONTRIBUTION

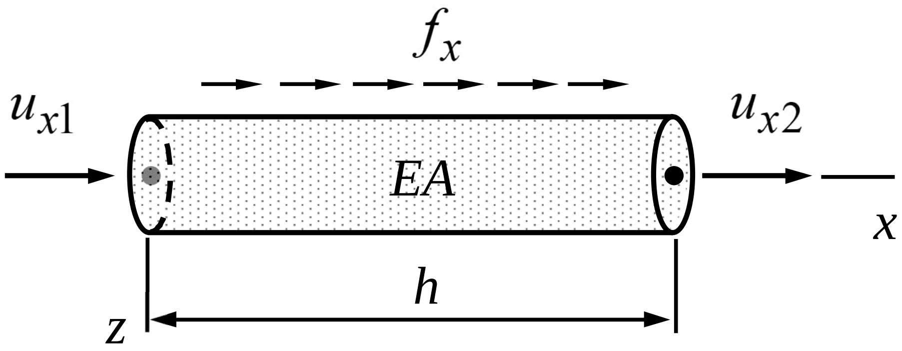
The beam element is obtained by combining the virtual work expressions of tension, torsion, and bending loading modes (b~bending, t~torsion, s~stretching)!

Virtual work of a structure: $\delta W = \sum_{e \in E} \delta W^e$

Virtual work of a beam: $\delta W^e = \sum_m \delta W_m^e = \delta W_{bxz}^e + \delta W_{bxy}^e + \delta W_{tx}^e + \delta W_{sx}^e$

In hand calculations, one starts with the expressions in the material coordinate system, expresses the nodal displacements and rotations in the structural coordinate system, and sums over the elements and loading modes. The remaining follows from the principle of virtual work $\delta W = 0 \;\; \forall \delta \mathbf{a}$.

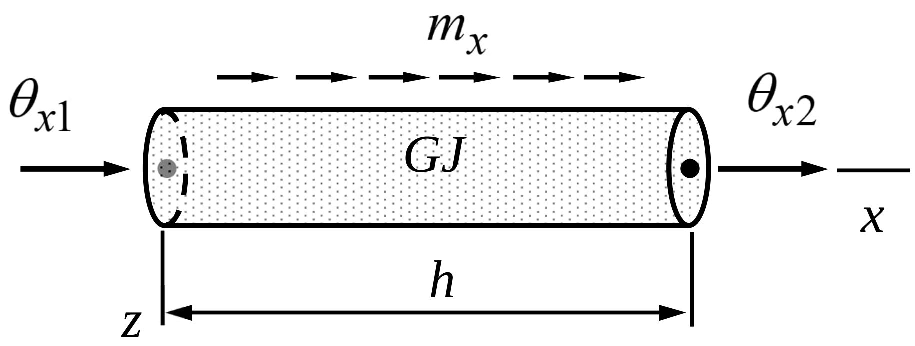
BAR MODE



$$\delta W = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}\right), \quad u_x = \mathbf{i}^{\mathrm{T}}\begin{Bmatrix} u_X \\ u_Y \\ u_Z \end{Bmatrix}, \text{ where } \mathbf{i} = \frac{1}{h}\begin{Bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{Bmatrix}.$$

Above, f_x and EA are assumed constants and the elements of the column matrix $\mathbf{i}$ are the components of the unit vector $\vec{i}$ in the structural coordinate system.

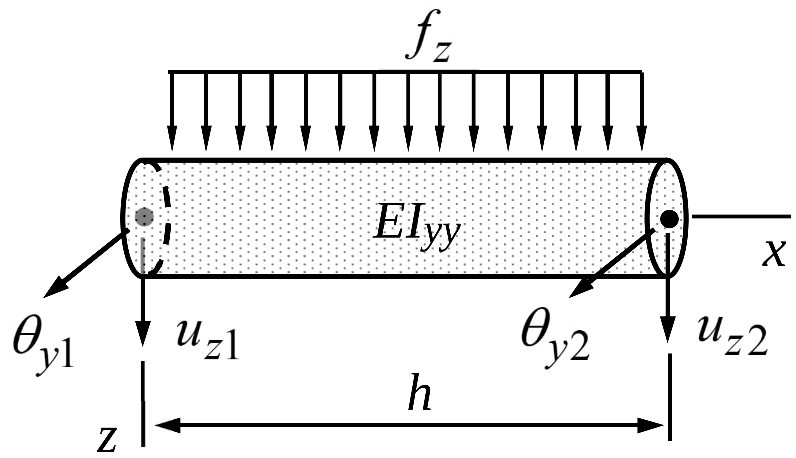
TORSION MODE



$$\delta W = -\begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{GJ}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix} - \frac{m_x h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}\right), \quad \theta_x = \mathbf{i}^{\mathrm{T}}\begin{Bmatrix} \theta_X \\ \theta_Y \\ \theta_Z \end{Bmatrix}, \quad \text{where } \mathbf{i} = \frac{1}{h}\begin{Bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{Bmatrix}.$$

Above, m_x and GJ are assumed constants and the elements of the column matrix **i** are the components of the unit vector $\vec{i}$ in the structural coordinate system.

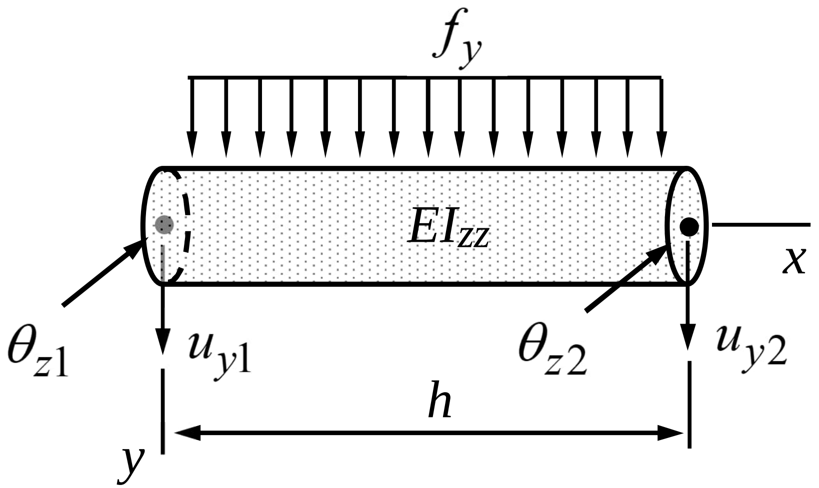
xz-PLANE BENDING MODE



$$\delta W=-\begin{Bmatrix}\delta u_{z1}\\ \delta\theta_{y1}\\ \hdashline \delta u_{z2}\\ \delta\theta_{y2}\end{Bmatrix}^{\mathrm{T}}\left(\frac{EI_{yy}}{h^3}\left[\begin{array}{cc:cc}12 & -6h & -12 & -6h\\ -6h & 4h^2 & 6h & 2h^2\\ \hdashline -12 & 6h & 12 & 6h\\ -6h & 2h^2 & 6h & 4h^2\end{array}\right]\begin{Bmatrix}u_{z1}\\ \theta_{y1}\\ \hdashline u_{z2}\\ \theta_{y2}\end{Bmatrix}-\frac{f_z h}{12}\begin{Bmatrix}6\\ -h\\ \hdashline 6\\ h\end{Bmatrix}\right),\ u_z=\mathbf{k}^{\mathrm{T}}\begin{Bmatrix}u_X\\ u_Y\\ u_Z\end{Bmatrix}\text{etc.}$$

Above, f_z and EI_{yy} are assumed to be constants and the elements of the column matrices $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the components of the unit vectors $\vec{i}$, $\vec{j}$ and $\vec{k}$ in the structural coordinate system.

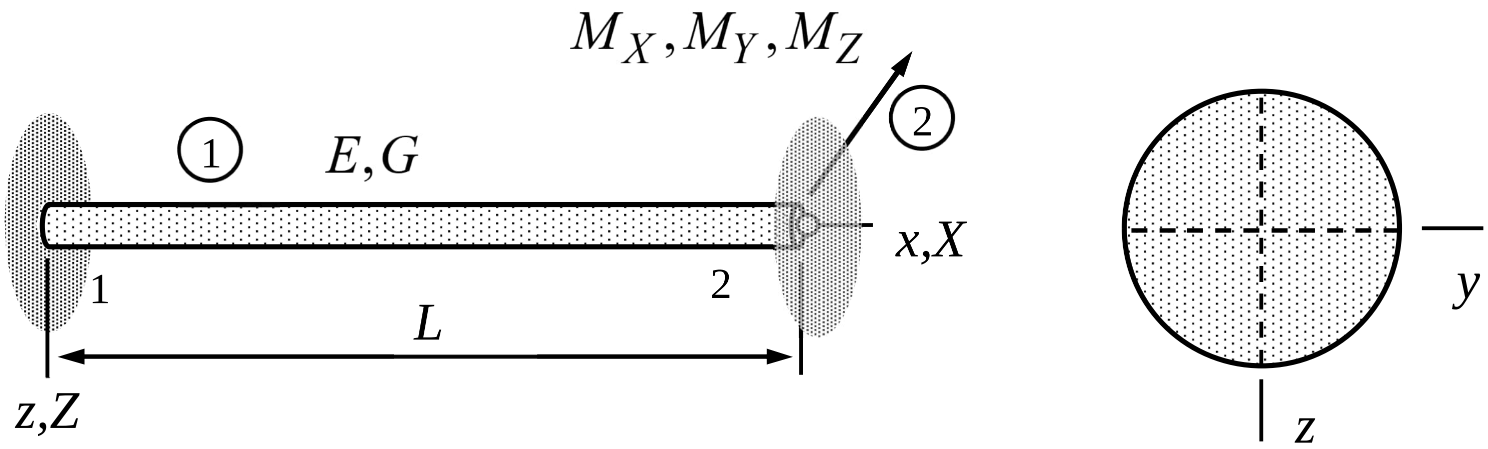
xy-PLANE BENDING MODE



$$\delta W = -\begin{Bmatrix} \delta u_{y1} \\ \delta\theta_{z1} \\ \delta u_{y2} \\ \delta\theta_{z2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EI_{zz}}{h^3} \left[\begin{array}{cc:cc} 12 & 6h & -12 & 6h \\ 6h & 4h^2 & -6h & 2h^2 \\ \hdashline -12 & -6h & 12 & -6h \\ 6h & 2h^2 & -6h & 4h^2 \end{array}\right] \begin{Bmatrix} u_{y1} \\ \theta_{z1} \\ u_{y2} \\ \theta_{z2} \end{Bmatrix} - \frac{f_y\, h}{12} \begin{Bmatrix} 6 \\ h \\ 6 \\ -h \end{Bmatrix} \right), \quad u_y = \mathbf{j}^{\mathrm{T}} \begin{Bmatrix} u_X \\ u_Y \\ u_Z \end{Bmatrix} \text{etc.}$$

Above, f_y and EI_{zz} are assumed to be constants and the column matrices **I**, **j** and **k** contain the components of the unit vectors $\vec{i}$, $\vec{j}$ and $\vec{k}$ in the structural coordinate system.

EXAMPLE 3.6. Consider the beam of the figure and determine the rotation of point 2 by using a generic beam element. The *x*-axis coincides with the geometrical axis, the spherical joint at point 2 is frictionless, and the components of the external moment acting on point 2 are M_X, M_Y, and M_Z. The second moments of area are $I_{yy} = I_{zz} = I$ and $J = 2I$.



Answer $\theta_{X2} = \dfrac{1}{2}\dfrac{M_X L}{GI}$, $\theta_{Y2} = \dfrac{1}{4}\dfrac{M_Y L}{EI}$, and $\theta_{Z2} = \dfrac{1}{4}\dfrac{M_Z L}{EI}$

- The element contribution consists of parts of the loading modes. The active degrees of freedom are rotations θ_{X2}, θ_{Y2} and θ_{Z2} (element and structural systems coincide here)

$$\delta W_{bxz}^{1} = -\begin{Bmatrix} 0 \\ 0 \\ \hline 0 \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EI}{L^3}\left[\begin{array}{cc|cc} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ \hline -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{array}\right]\begin{Bmatrix} 0 \\ 0 \\ \hline 0 \\ \theta_{Y2} \end{Bmatrix}\right) = -\delta\theta_{Y2}\frac{EI}{L^3}4L^2\theta_{Y2}$$

$$\delta W_{bxy}^{1} = -\begin{Bmatrix} 0 \\ 0 \\ \hline 0 \\ \delta\theta_{Z2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EI}{L^3}\left[\begin{array}{cc|cc} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ \hline -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{array}\right]\begin{Bmatrix} 0 \\ 0 \\ \hline 0 \\ \theta_{Z2} \end{Bmatrix}\right) = -\delta\theta_{Z2}4L^2\frac{EI}{L^3}\theta_{Z2}$$

$$\delta W_{tx}^{1} = -\begin{Bmatrix} 0 \\ \delta\theta_{X2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{2GI}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ \theta_{X2} \end{Bmatrix}\right) = -\delta\theta_{X2}\frac{G2I}{L}\theta_{X2} \quad (J = 2I)$$

$$\delta W^2 = \delta\theta_{X2} M_X + \delta\theta_{Y2} M_Y + \delta\theta_{Z2} M_Z .$$

- Virtual work expression of the structure is sum over the elements and their loading modes

$$\delta W = \sum_e \delta W^e = \delta W^1 + \delta W^2 = (\delta W^1_{bxz} + \delta W^1_{bxy} + \delta W^1_{tx}) + \delta W^2 \quad \Rightarrow$$

$$\delta W = -\delta\theta_{Y2} \frac{EI}{L^3} 4L^2 \theta_{Y2} - \delta\theta_{Z2} 4L^2 \frac{EI}{L^3} \theta_{Z2} - \delta\theta_{X2} \frac{2GI}{L} \theta_{X2} + \delta\theta_{X2} M_X +$$

$$\delta\theta_{Y2} M_Y + \delta\theta_{Z2} M_Z \qquad \Leftrightarrow$$

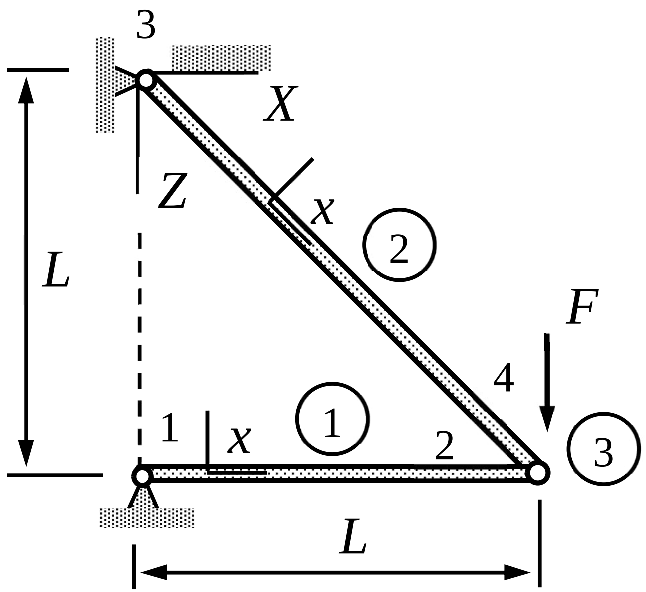
$$\delta W = -\delta\theta_{X2} \left(\frac{2GI}{L} \theta_{X2} - M_X\right) - \delta\theta_{Y2} \left(\frac{EI}{L^3} 4L^2 \theta_{Y2} - M_Y\right) - \delta\theta_{Z2} \left(4L^2 \frac{EI}{L^3} \theta_{Z2} - M_Z\right)$$

$$\delta W=-\begin{Bmatrix}\delta\theta_{X2}\\ \delta\theta_{Y2}\\ \delta\theta_{Z2}\end{Bmatrix}^{\mathrm{T}}\left(\frac{I}{L}\begin{bmatrix}2G & 0 & 0\\ 0 & 4E & 0\\ 0 & 0 & 4E\end{bmatrix}\begin{Bmatrix}\theta_{X2}\\ \theta_{Y2}\\ \theta_{Z2}\end{Bmatrix}-\begin{Bmatrix}M_X\\ M_Y\\ M_Z\end{Bmatrix}\right).$$

- Principle of virtual work $\delta W=0\;\;\forall\delta\mathbf{a}$ and the fundamental lemma of variation calculus in the form $\delta\mathbf{a}^{\mathrm{T}}\mathbf{R}=0\;\;\forall\delta\mathbf{a}\;\Leftrightarrow\;\mathbf{R}=0$ give

$$\frac{2I}{L}\begin{bmatrix}G & 0 & 0\\ 0 & 2E & 0\\ 0 & 0 & 2E\end{bmatrix}\begin{Bmatrix}\theta_{X2}\\ \theta_{Y2}\\ \theta_{Z2}\end{Bmatrix}-\begin{Bmatrix}M_X\\ M_Y\\ M_Z\end{Bmatrix}=0\quad\Leftrightarrow\quad\begin{Bmatrix}\theta_{X2}\\ \theta_{Y2}\\ \theta_{Z2}\end{Bmatrix}=\frac{L}{2I}\begin{Bmatrix}M_X G\\ M_Y/(2E)\\ M_Z/(2E)\end{Bmatrix}.$$ ←

EXAMPLE 3.7. A *beam truss* is loaded by a point force having magnitude F as shown in the figure. Determine the nodal displacements. The cross-sectional area of beam 1 is A and that for beam 2 $\sqrt{8}A$. Young's modulus is E and weight is omitted. Use the principle of virtual work.



Answer $u_{X2} = u_{X4} = -\dfrac{FL}{EA}$ and $u_{Z2} = u_{Z4} = 2\dfrac{FL}{EA}$.

- A joint is generated by using a duplicate node in the Mathematica code. The displacement components coincide at the nodes but the rotations may not:

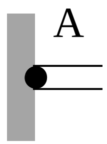
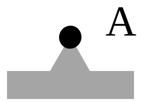
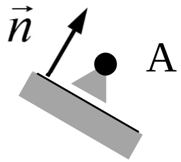
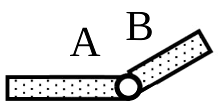
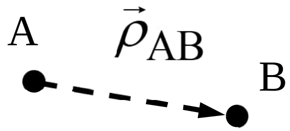
	model	properties	geometry
1	BEAM	$\{\{\mathbb{E}, G\}, \{A, \mathbb{I}, \mathbb{I}\}\}$	Line[{1, 2}]
2	BEAM	$\{\{\mathbb{E}, G\}, \{2\sqrt{2}\, A, \mathbb{I}, \mathbb{I}\}\}$	Line[{3, 4}]
3	FORCE	$\{0, 0, F\}$	Point[{4}]

	$\{X,Y,Z\}$	$\{u_X,u_Y,u_Z\}$	$\{\theta_X,\theta_Y,\theta_Z\}$
1	$\{0, 0, L\}$	$\{0, 0, 0\}$	$\{0, \theta Y[1], 0\}$
2	$\{L, 0, L\}$	$\{uX[2], 0, uZ[2]\}$	$\{0, \theta Y[2], 0\}$
3	$\{0, 0, 0\}$	$\{0, 0, 0\}$	$\{0, \theta Y[3], 0\}$
4	$\{L, 0, L\}$	$\{uX[2], 0, uZ[2]\}$	$\{0, \theta Y[4], 0\}$

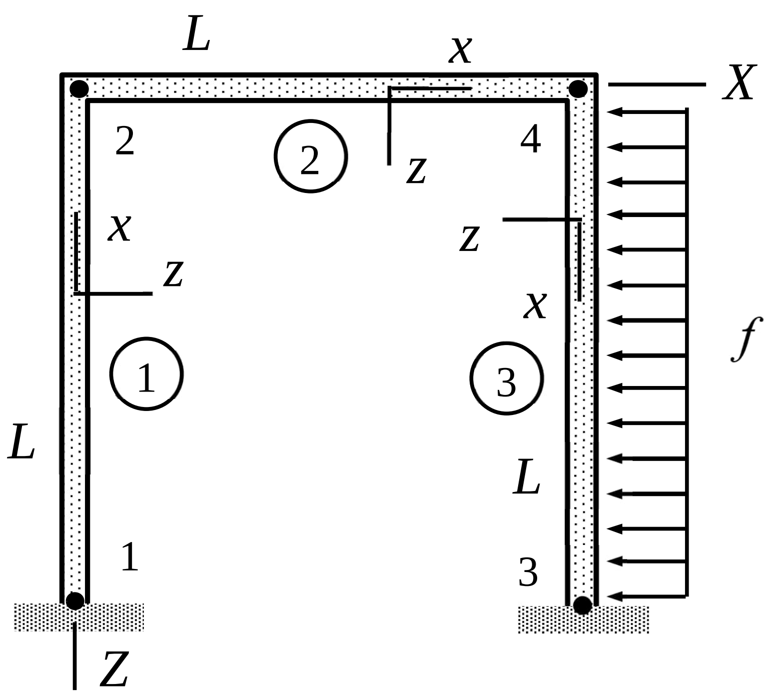
$$\left\{uX[2] \to -\frac{F\,L}{A\,\mathbb{E}},\ uZ[2] \to \frac{2\,F\,L}{A\,\mathbb{E}},\ \theta Y[1] \to -\frac{2\,F}{A\,\mathbb{E}},\right.$$
$$\left.\theta Y[2] \to -\frac{2\,F}{A\,\mathbb{E}},\ \theta Y[3] \to -\frac{3\,F}{2\,A\,\mathbb{E}},\ \theta Y[4] \to -\frac{3\,F}{2\,A\,\mathbb{E}}\right\}$$

Solution to the displacements is the same as with the bar model!

3.4 CONSTRAINTS AND LINKS

name	**symbol**	**equation**
clamped		$\vec{u}_{\mathrm{A}} = 0$ and $\vec{\theta}_{\mathrm{A}} = 0$
fixed		$\vec{u}_{\mathrm{A}} = 0$
slide		$\vec{n} \cdot \vec{u}_{\mathrm{A}} = 0$
joint		$\vec{u}_{\mathrm{B}} = \vec{u}_{\mathrm{A}}$
rigid		$\vec{u}_{\mathrm{B}} = \vec{u}_{\mathrm{A}} + \vec{\theta}_{\mathrm{A}} \times \vec{\rho}_{\mathrm{AB}}$ and $\vec{\theta}_{\mathrm{B}} = \vec{\theta}_{\mathrm{A}}$

EXAMPLE 3.8. Consider the beam truss of the figure. Determine the displacements and rotations of nodes 2 and 4. Assume that the beams are rigid in the axial directions so that the axial *strain* vanishes. Bending rigidity of the beams EI is constant.



Answer $u_{X2} = u_{X4} = -\dfrac{3}{112}\dfrac{fL^4}{EI}$, $\theta_{Y2} = \dfrac{19}{1008}\dfrac{fL^3}{EI}$, and $\theta_{Y4} = \dfrac{5}{1008}\dfrac{fL^3}{EI}$

- Only the bending in *XZ*-plane needs to be accounted for. The non-zero displacement and rotation components of the structure are u_{X2}, θ_{Y2}, and θ_{Y4}. As the axial strain of beam 2 vanishes, axial displacements satisfy $u_{X4} = u_{X2}$.

$$\delta W_{bxz}^{1} = -\left\{\begin{array}{c} 0 \\ 0 \\ \hdashline \delta u_{X2} \\ \delta\theta_{Y2} \end{array}\right\}^{\mathrm{T}} \left(\frac{EI}{L^3}\left[\begin{array}{cc:cc} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ \hdashline -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{array}\right]\left\{\begin{array}{c} 0 \\ 0 \\ \hdashline u_{X2} \\ \theta_{Y2} \end{array}\right\}\right) \quad (u_{z2} = u_{X2}, \theta_{y2} = \theta_{Y2})$$

$$\delta W_{bxz}^{2} = -\left\{\begin{array}{c} 0 \\ \delta\theta_{Y2} \\ \hdashline 0 \\ \delta\theta_{Y4} \end{array}\right\}^{\mathrm{T}} \left(\frac{EI}{L^3}\left[\begin{array}{cc:cc} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ \hdashline -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{array}\right]\left\{\begin{array}{c} 0 \\ \theta_{Y2} \\ \hdashline 0 \\ \theta_{Y4} \end{array}\right\}\right) \quad (\theta_{y1} = \theta_{Y2}, \theta_{y2} = \theta_{Y4})$$

$$\delta W_{bxz}^{2}=-\left\{\begin{array}{c}-\delta u_{X2}\\ \delta\theta_{Y4}\\ \hdashline 0\\ 0\end{array}\right\}^{\mathrm{T}}\left(\frac{EI}{L^{3}}\left[\begin{array}{cc:cc}12 & -6L & -12 & -6L\\ -6L & 4L^{2} & 6L & 2L^{2}\\ \hdashline -12 & 6L & 12 & 6L\\ -6L & 2L^{2} & 6L & 4L^{2}\end{array}\right]\left\{\begin{array}{c}-u_{X2}\\ \theta_{Y4}\\ \hdashline 0\\ 0\end{array}\right\}-\frac{fL}{12}\left\{\begin{array}{c}6\\ -L\\ \hdashline 6\\ L\end{array}\right\}\right)$$

$$(u_{z1}=-u_{X2})$$

- Virtual work expression of the structure is

$$\delta W=\delta W_{bxz}^{1}+\delta W_{bxz}^{2}+\delta W_{bxz}^{3}=-\left\{\begin{array}{c}\delta u_{X2}\\ \delta\theta_{Y2}\\ \delta\theta_{Y4}\end{array}\right\}^{\mathrm{T}}\left(\frac{EI}{L^{3}}\begin{bmatrix}24 & 6L & 6L\\ 6L & 8L^{2} & 2L^{2}\\ 6L & 2L^{2} & 8L^{2}\end{bmatrix}\left\{\begin{array}{c}u_{X2}\\ \theta_{Y2}\\ \theta_{Y4}\end{array}\right\}-\frac{fL}{12}\left\{\begin{array}{c}-6\\ 0\\ -L\end{array}\right\}\right).$$

- Principle of virtual work $\delta W=0\ \forall\delta\mathbf{a}$ and the fundamental lemma of variation calculus give

$$\frac{EI}{L^3}\begin{bmatrix} 24 & 6L & 6L \\ 6L & 8L^2 & 2L^2 \\ 6L & 2L^2 & 8L^2 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ \theta_{Y2} \\ \theta_{Y4} \end{Bmatrix} - \frac{fL}{12}\begin{Bmatrix} -6 \\ 0 \\ -L \end{Bmatrix} = 0 \Leftrightarrow \begin{Bmatrix} u_{X2} \\ \theta_{Y2} \\ \theta_{Y4} \end{Bmatrix} = \frac{fL^3}{1008EI}\begin{Bmatrix} -27L \\ 19 \\ 5 \end{Bmatrix}. \leftarrow$$

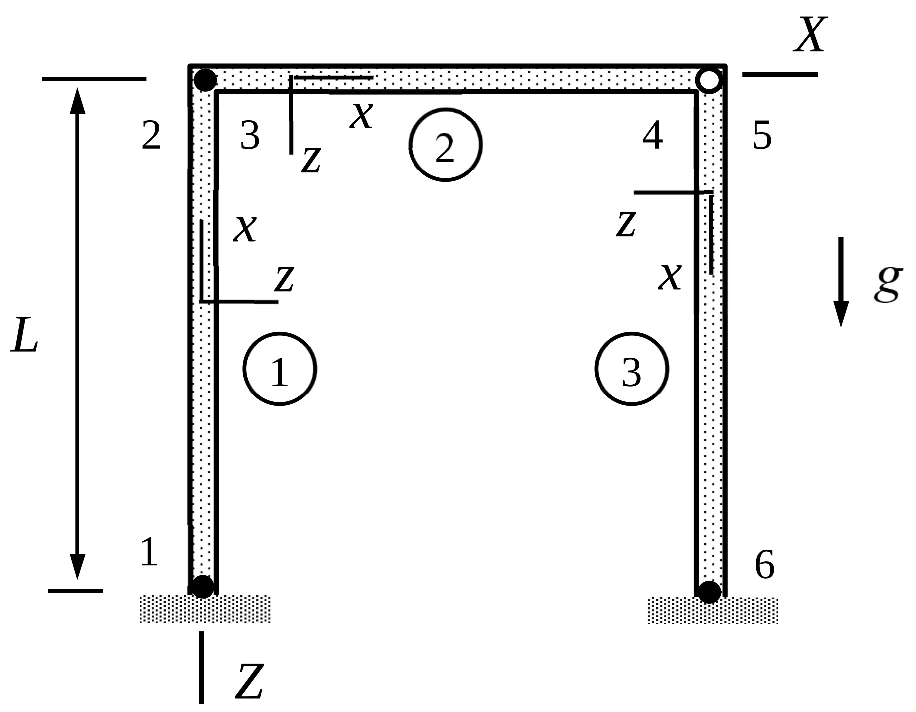
- In Mathematica code calculation, horizontal displacements of nodes 2 and 4 are forced to be same ($u_{X4} = u_{X2}$)

	model	properties	geometry
1	BEAM	{{E, G}, {A, I, I}}	Line[{1, 2}]
2	BEAM	{{E, G}, {A, I, I}}	Line[{2, 4}]
3	BEAM	{{E, G}, {A, I, I}, {-f, 0, 0}}	Line[{4, 3}]

	{X,Y,Z}	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	{0, 0, L}	{0, 0, 0}	{0, 0, 0}
2	{0, 0, 0}	{uX[2], 0, 0}	{0, θY[2], 0}
3	{L, 0, L}	{0, 0, 0}	{0, 0, 0}
4	{L, 0, 0}	{uX[2], 0, 0}	{0, θY[4], 0}

$$\left\{uX[2] \to -\frac{3\,f\,L^4}{112\,E\,I},\ \theta Y[2] \to \frac{19\,f\,L^3}{1008\,E\,I},\ \theta Y[4] \to \frac{5\,f\,L^3}{1008\,E\,I}\right\}$$

EXAMPLE 3.9. Consider the beam truss of the figure and displacements and rotations at nodes 2 (3) and 4 (5) modeled by using duplicate nodes. Write down the element tables by considering 4 (5) as a cylindrical frictionless joint.



- The structural parts can be joined by kinematical constraints. At nodes (black circle), displacement and rotation components coincide. At a joint (white circle), only displacement components need to coincide:

	model	properties	geometry
1	BEAM	{{E, G}, {A, I, I}, {0, 0, A g ρ}}	Line[{1, 2}]
2	BEAM	{{E, G}, {A, I, I}, {0, 0, A g ρ}}	Line[{3, 4}]
3	BEAM	{{E, G}, {A, I, I}, {0, 0, A g ρ}}	Line[{5, 6}]

	{X,Y,Z}	$\{u_X,u_Y,u_Z\}$	$\{\theta_X,\theta_Y,\theta_Z\}$
1	{0, 0, L}	{0, 0, 0}	{0, 0, 0}
2	{0, 0, 0}	{uX[2], 0, uZ[2]}	{0, θY[2], 0}
3	{0, 0, 0}	{uX[2], 0, uZ[2]}	{0, θY[2], 0}
4	{L, 0, 0}	{uX[4], 0, uZ[4]}	{0, θY[4], 0}
5	{L, 0, 0}	{uX[4], 0, uZ[4]}	{0, θY[5], 0}
6	{L, 0, L}	{0, 0, 0}	{0, 0, 0}

Solution to the problem is a bit lengthy so it is not given here (see the examples section of the Mathematica code).

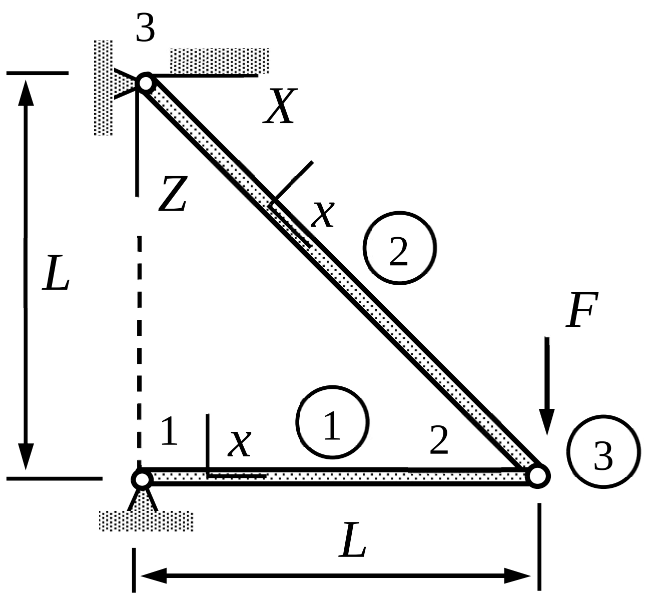
POINT CONSTRAINT CONTRIBUTION

Displacement and rotation constraints can be enforced by using a given value in calculations. Alternatively, one may use a one-node constraint element:

$$\delta W=\begin{Bmatrix}\delta u_X\\ \delta u_Y\\ \delta u_Z\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}F_X\\ F_Y\\ F_Z\end{Bmatrix}+\begin{Bmatrix}u_X-\underline{u}_X\\ u_Y-\underline{u}_Y\\ u_Z-\underline{u}_Z\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\delta F_X\\ \delta F_Y\\ \delta F_Z\end{Bmatrix}+\begin{Bmatrix}\delta\theta_X\\ \delta\theta_Y\\ \delta\theta_Z\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}M_X\\ M_Y\\ M_Z\end{Bmatrix}+\begin{Bmatrix}\theta_X-\underline{\theta}_X\\ \theta_Y-\underline{\theta}_Y\\ \theta_Z-\underline{\theta}_Z\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\delta M_X\\ \delta M_Y\\ \delta M_Z\end{Bmatrix}$$

Above, F_X,F_Y,F_Z and M_X,M_Y,M_Z are considered as unknown constraint forces/moments whenever the corresponding displacement/rotation should be constrained to the value indicated by an underline. Notice that the variation of a given quantity vanishes. Explicit constraint in this form can be used to find some of the internal forces in calculations based on the virtual work expressions.

EXAMPLE 3.10. A *bar truss* is loaded by a point force having magnitude F as shown in the figure. The cross-sectional area of bar 1 is A and that for bar 2 $\sqrt{8}A$. Young's modulus is E and weight is omitted. Determine the nodal displacements. Enforce the zero displacement conditions at nodes 1 and 3 by point constraints



Answer $u_{X2} = -\dfrac{FL}{EA}$ and $u_{Z2} = 2\dfrac{FL}{EA}$.

- An alternative way to enforce displacement/rotation constraints uses a one-node constraint element:

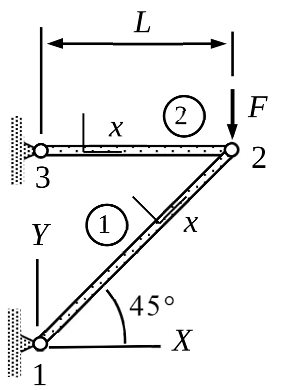
	model	properties	geometry
1	BAR	{{E}, {A}}	Line[{1, 2}]
2	BAR	$\{\{E\}, \{2\sqrt{2}\,A\}\}$	Line[{3, 2}]
3	FORCE	{0, 0, F}	Point[{2}]
4	RIGID	{{0, 0, 0}, {0, 0, 0}}	Point[{1}]
5	RIGID	{{0, 0, 0}, {0, 0, 0}}	Point[{3}]

	{X,Y,Z}	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	{0, 0, L}	{uX[1], 0, uZ[1]}	{0, 0, 0}
2	{L, 0, L}	{uX[2], 0, uZ[2]}	{0, 0, 0}
3	{0, 0, 0}	{uX[3], 0, uZ[3]}	{0, 0, 0}

$$\Big\{FX[1] \to F,\ FX[3] \to -F,\ FZ[1] \to 0,\ FZ[3] \to -F,\ uX[1] \to 0,\ uX[2] \to -\frac{F\,L}{A\,E},\ uX[3] \to 0,\ uZ[1] \to 0,\ uZ[2] \to \frac{2\,F\,L}{A\,E},\ uZ[3] \to 0\Big\}$$

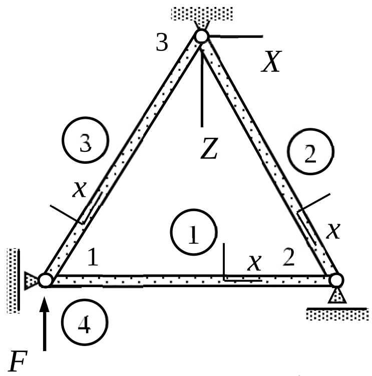
MEC-E1050 Finite Element Method in Solid, week 45/2024

1. Determine the horizontal and vertical displacements of node 2. Cross-sectional area of bar 1 is $\sqrt{2}A$ and that of bar 2 is A. Young's modulus of the material is E.



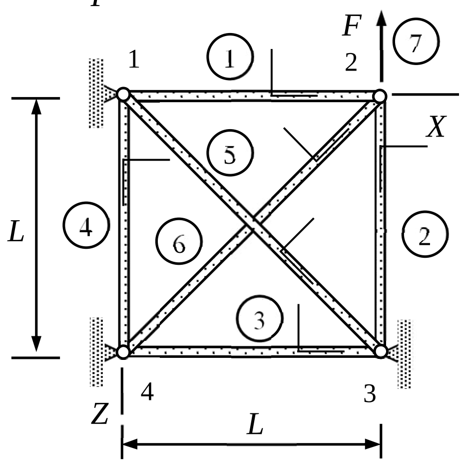
Answer $u_{X2} = \frac{FL}{EA}$, $u_{Y2} = -3\frac{FL}{EA}$

2. Determine the nodal displacements of the bar structure (3 bar elements and 1 force element) shown. The length of all the bars is L and the cross-sectional area A. Young's modulus of the material E is constant. First, write down the element table and node table. Second, assemble the system equations. Third, solve the equations for u_{Z1} and u_{X2}.



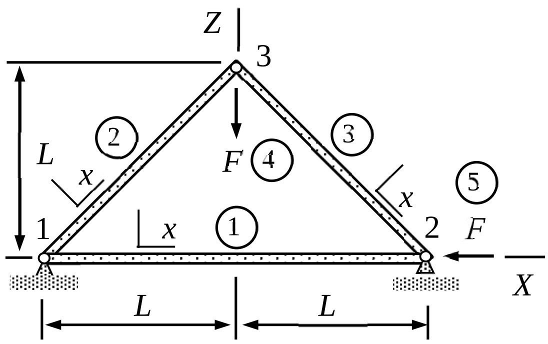
Answer $u_{Z1} = -\frac{4}{3}\frac{FL}{EA}$, $u_{X2} = 0$

3. Determine the nodal displacements when force F is acting on the structure as shown. The cross-sectional area of bars 1,2,3 and 4 is A and the cross-sectional area of bars 5 and 6 is $2\sqrt{2}A$. Young's modulus of the material is E. Use the principle of virtual work.



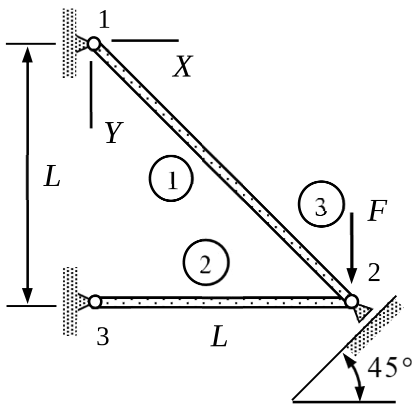
Answer $u_{X2} = -\frac{1}{3}\frac{FL}{EA}$, $u_{Z2} = -\frac{2}{3}\frac{FL}{EA}$

4. Consider the XZ-plane bar structure of the figure. Young´s modulus E is constant. Cross-sectional areas of bars 2 and 3 are $\sqrt{8}A$ and the cross-sectional area of bar 1 is $2A$. Determine the displacement components u_{X2}, u_{X3}, and u_{Z3}. Use the principle of virtual work.



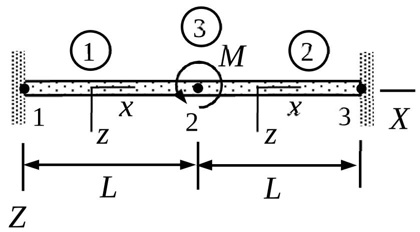
Answer $u_{X2} = -\frac{1}{2}\frac{FL}{EA}$, $u_{X3} = u_{Z3} = -\frac{1}{4}\frac{FL}{EA}$

5. Determine the nodal displacements, when force F is acting on the structure as shown in the figure. The cross-sectional area of all the bars is A and the Young's modulus of the material is E. Use the principle of virtual work.



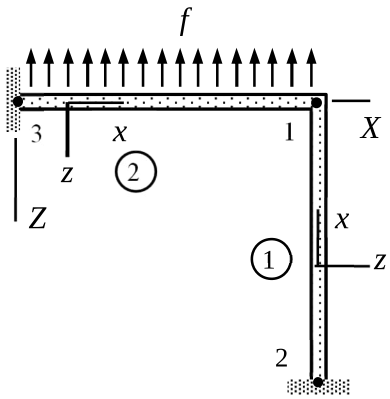
Answer $u_{X2} = -\dfrac{FL}{EA}$, $u_{Y2} = \dfrac{FL}{EA}$

6. Determine displacement u_{Z2} and rotation θ_{Y2} at point 2 of the structure shown. Use two beam elements of equal length. Point moment with magnitude M is acting on node 2. Young's modulus of the material E and the second moment of area I are constants. Use the principle of virtual work.



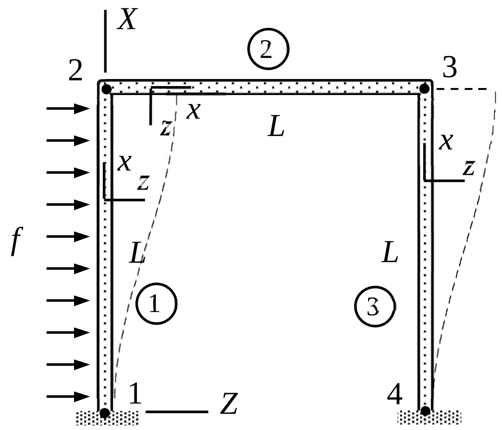
Answer $u_{Z2} = 0$, $\theta_{Y2} = \dfrac{1}{8}\dfrac{ML}{EI}$

7. Determine the rotation θ_{Y1} at node 1 of the structure shown. Use two beam elements of length L. Assume that the beams are inextensible in the axial directions. Young's modulus of the material E and the second moment of area I are constants. Use the principle of virtual work.



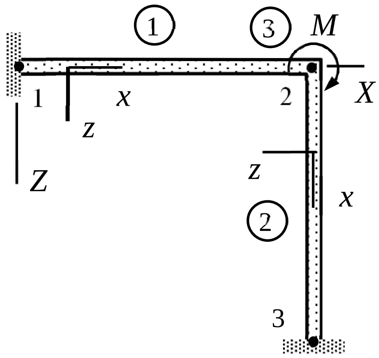
Answer $\theta_{Y1} = -\dfrac{1}{96}\dfrac{fL^3}{EI}$

8. The frame of the figure consists of a rigid body (2) and beams (1) and (3). Determine the non-zero displacements and rotations. The beams are identical and can be assumed as inextensible in the axial directions. Displacements are confined to the XZ – plane. Young's modulus E, second moment of area I, and distributed force f acting on element 1 are constants. Use the principle of virtual work.



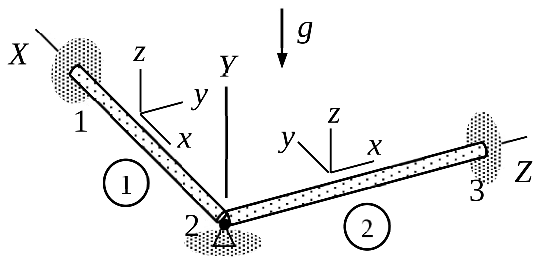
Answer $u_{Z2} = \dfrac{1}{48}\dfrac{fL^4}{EI}$

9. Determine the rotation θ_{Y2} at node 2 of the structure loaded by a point moment (magnitude M) acting on node 2. Use beam elements (1) and (2) of length L and a point moment element (3). Assume that the beams are inextensible in the axial directions. Young's modulus E and the second moment of area I are constants. Use the principle of virtual work.



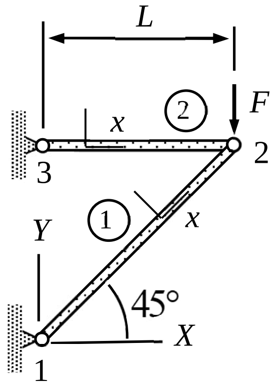
Answer $\theta_{Y2} = -\frac{1}{8}\frac{ML}{EI}$.

10. Beam 1 (length L) of the figure is loaded by its own weight and beam 2 (length L) is assumed weightless. Determine the rotation component θ_{Z2}. Moments of the cross section are A, $I_{yy} = I_{zz} = I$ and $J = 2I$. Young's modulus, shear modulus, and density of the material E, G and ρ are constants.



Answer $\theta_{Z2} = -\frac{AgL^3\rho}{24(G+2E)I}$

Determine the horizontal and vertical displacements of node 2. Cross-sectional area of bar 1 is $\sqrt{2}A$ and that of bar 2 is A. Young's modulus of the material is E.



Solution

Element contribution, written in terms of displacement components of the structural coordinate system,

$$\begin{Bmatrix}\mathbf{R}_1\\ \mathbf{R}_2\end{Bmatrix}=\frac{EA}{h}\begin{bmatrix}\mathbf{i}\mathbf{i}^{\mathrm{T}} & -\mathbf{i}\mathbf{i}^{\mathrm{T}}\\ -\mathbf{i}\mathbf{i}^{\mathrm{T}} & \mathbf{i}\mathbf{i}^{\mathrm{T}}\end{bmatrix}\begin{Bmatrix}\mathbf{a}_1\\ \mathbf{a}_2\end{Bmatrix}-\frac{f_x h}{2}\begin{Bmatrix}\mathbf{i}\\ \mathbf{i}\end{Bmatrix},\ \text{where } \mathbf{i}=\frac{1}{h}\begin{Bmatrix}\Delta X\\ \Delta Y\end{Bmatrix}$$

depends on the cross-sectional area A, Young's modulus E, bar length h, force per unit length f_x and the components of the basis vector $\vec{i}$ in the structural coordinate system (the components define the orientation).

Element contributions are first written in terms of the nodal displacements of the structural coordinate system (notice that the point force is treated as a one-node element)

Bar 1: $h=\sqrt{2}L$, $\mathbf{i}=\begin{Bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{Bmatrix}$, $\begin{Bmatrix}F_{X1}^1\\ F_{Y1}^1\\ F_{X2}^1\\ F_{Y2}^1\end{Bmatrix}=\frac{EA}{2L}\begin{bmatrix}1 & 1 & -1 & -1\\ 1 & 1 & -1 & -1\\ -1 & -1 & 1 & 1\\ -1 & -1 & 1 & 1\end{bmatrix}\begin{Bmatrix}0\\ 0\\ u_{X2}\\ u_{Y2}\end{Bmatrix}-\begin{Bmatrix}0\\ 0\\ 0\\ 0\end{Bmatrix}$,

Bar 2: $h=L$, $\mathbf{i}=\begin{Bmatrix}1\\ 0\end{Bmatrix}$, $\begin{Bmatrix}F_{X3}^2\\ F_{Y3}^2\\ F_{X2}^2\\ F_{Y2}^2\end{Bmatrix}=\frac{EA}{L}\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}0\\ 0\\ u_{X2}\\ u_{Y2}\end{Bmatrix}-\begin{Bmatrix}0\\ 0\\ 0\\ 0\end{Bmatrix}$,

Force 3: $\begin{Bmatrix}F_{X2}^3\\ F_{Y2}^3\end{Bmatrix}=-\begin{Bmatrix}0\\ -F\end{Bmatrix}$.

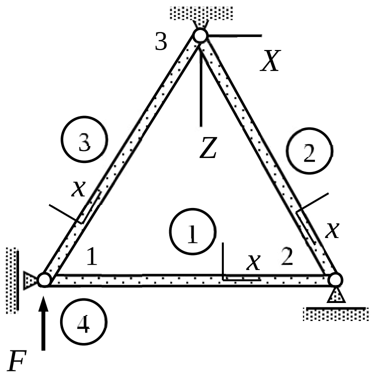
In assembly, the internal forces acting on the non-constrained directions are added to get the equilibrium equations of the structure. The unknown displacement components follow as the solution to the equilibrium equations:

$$\sum \begin{Bmatrix} F_{X2}^{e} \\ F_{Y2}^{e} \end{Bmatrix} = \begin{Bmatrix} F_{X2}^{1} \\ F_{Y2}^{1} \end{Bmatrix} + \begin{Bmatrix} F_{X2}^{2} \\ F_{Y2}^{2} \end{Bmatrix} + \begin{Bmatrix} F_{X2}^{3} \\ F_{Y2}^{3} \end{Bmatrix} = \frac{EA}{2L} \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} u_{X2} \\ u_{Y2} \end{Bmatrix} + \begin{Bmatrix} 0 \\ F \end{Bmatrix} = 0 \quad \Leftrightarrow$$

$u_{X2} = \dfrac{FL}{EA}$ and $u_{Y2} = -3\dfrac{FL}{EA}$. 🡸

Use the code of MEC-E1050 to check the solution!

Determine the nodal displacements of the bar structure (3 bar elements and 1 force element) shown. The length of all the bars is L and the cross-sectional area A. Young's modulus of the material E is constant. First, write down the element table and node table. Second, assemble the system equations. Third, solve the equations for u_{Z1} and u_{X2}.



Solution

Element and node tables contain the information needed in the displacement and stress analysis of the structures.

	model	properties	geometry
1	BAR	{{E}, {A}}	Line[{1, 2}]
2	BAR	{{E}, {A}}	Line[{2, 3}]
3	BAR	{{E}, {A}}	Line[{3, 1}]
4	FORCE	{0, 0, −F}	Point[{1}]

	{X,Y,Z}	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	$\{-\frac{L}{2}, 0, \frac{\sqrt{3}L}{2}\}$	{0, 0, uZ[1]}	{0, 0, 0}
2	$\{\frac{L}{2}, 0, \frac{\sqrt{3}L}{2}\}$	{uX[2], 0, 0}	{0, 0, 0}
3	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}

In hand calculations, with simple problems of a few elements, explicit forms of the tables are not necessary. Element contributions need to be written in terms of the displacement and force components of the structural system before assembly. Bar element contribution in terms of displacement and force components of the structural system

$$\begin{Bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \end{Bmatrix} = \frac{EA}{h}\begin{bmatrix} \mathbf{i}\mathbf{i}^{\mathrm{T}} & -\mathbf{i}\mathbf{i}^{\mathrm{T}} \\ -\mathbf{i}\mathbf{i}^{\mathrm{T}} & \mathbf{i}\mathbf{i}^{\mathrm{T}} \end{bmatrix}\begin{Bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{Bmatrix} - \frac{f_x h}{2}\begin{Bmatrix} \mathbf{i} \\ \mathbf{i} \end{Bmatrix}, \text{ in which } \mathbf{i} = \frac{1}{h}\begin{Bmatrix} \Delta X \\ \Delta Z \end{Bmatrix} \text{ (in this case)}$$

is convenient in truss calculations:

$$\text{Bar 1:} \quad \begin{Bmatrix} F_{X1}^1 \\ F_{Z1}^1 \\ F_{X2}^1 \\ F_{Z2}^1 \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ u_{Z1} \\ u_{X2} \\ 0 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (\mathbf{i} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix},\ h = L),$$

$$\text{Bar 2:} \quad \begin{Bmatrix} F_{X3}^2 \\ F_{Z3}^2 \\ F_{X2}^2 \\ F_{Z2}^2 \end{Bmatrix} = \frac{EA}{4L}\begin{bmatrix} 1 & \sqrt{3} & -1 & -\sqrt{3} \\ \sqrt{3} & 3 & -\sqrt{3} & -3 \\ -1 & -\sqrt{3} & 1 & \sqrt{3} \\ -\sqrt{3} & -3 & \sqrt{3} & 3 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ u_{X2} \\ 0 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (\mathbf{i} = \frac{1}{2}\begin{Bmatrix} 1 \\ \sqrt{3} \end{Bmatrix},\ h = L),$$

Bar 3: $$\begin{Bmatrix} F_{X1}^3 \\ F_{Z1}^3 \\ \hdashline F_{X3}^3 \\ F_{Z3}^3 \end{Bmatrix} = \frac{EA}{4L}\left[\begin{array}{cc:cc} 1 & -\sqrt{3} & -1 & \sqrt{3} \\ -\sqrt{3} & 3 & \sqrt{3} & -3 \\ \hdashline -1 & \sqrt{3} & 1 & -\sqrt{3} \\ \sqrt{3} & -3 & -\sqrt{3} & 3 \end{array}\right]\begin{Bmatrix} 0 \\ u_{Z1} \\ \hdashline 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ \hdashline 0 \\ 0 \end{Bmatrix} \quad (\mathbf{i} = \frac{1}{2}\begin{Bmatrix} 1 \\ -\sqrt{3} \end{Bmatrix},\ h=L),$$

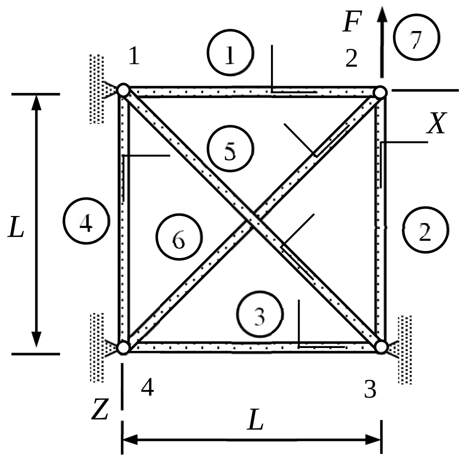
Force 4: $$\begin{Bmatrix} F_{X1}^4 \\ F_{Z1}^4 \end{Bmatrix} = -\begin{Bmatrix} 0 \\ -F \end{Bmatrix}.$$

Equilibrium equations of the nodes, giving displacements as their solution, are obtained by summing the internal forces acting on the nodes (non-constrained directions only):

$$\sum\begin{Bmatrix} F_{Z1}^e \\ F_{X2}^e \end{Bmatrix} = \begin{Bmatrix} F_{Z1}^1 + F_{Z1}^3 + F_{Z1}^4 \\ F_{X2}^1 + F_{X2}^2 \end{Bmatrix} = \begin{Bmatrix} 0 + \frac{3}{4}\frac{EA}{L}u_{Z1} + F \\ \frac{EA}{L}u_{X2} + \frac{1}{4}\frac{EA}{L}u_{X2} \end{Bmatrix} = \frac{1}{4}\frac{EA}{L}\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}\begin{Bmatrix} u_{Z1} \\ u_{X2} \end{Bmatrix} - \begin{Bmatrix} -F \\ 0 \end{Bmatrix} = 0 \quad \Rightarrow$$

$$\begin{Bmatrix} u_{Z1} \\ u_{X2} \end{Bmatrix} = 4\frac{L}{EA}\begin{bmatrix} 1/3 & 0 \\ 0 & 1/5 \end{bmatrix}\begin{Bmatrix} -F \\ 0 \end{Bmatrix} = -\frac{4}{3}\frac{FL}{EA}\begin{Bmatrix} 1 \\ 0 \end{Bmatrix}.$$

Determine the nodal displacements when force F is acting on the structure as shown. The cross-sectional area of bars 1,2,3 and 4 is A and the cross-sectional area of bars 5 and 6 is $2\sqrt{2}A$. Young's modulus of the material is E. Use the principle of virtual work.



Solution

Element and node tables contain the information needed in displacement and stress analysis of the structure. In hand calculations, it is often enough to complete the figure by the material coordinate systems and express the nodal displacements/rotations in terms symbols for the nodal displacements and rotations and/or values known a priori. The components in the material coordinate systems can also be deduced directly form the figure (in simple cases). Virtual work expression of the bar element is given by

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} (\frac{EA}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}) .$$

Nodal displacements/rotations of the structure are zeros except u_{X2} and u_{Z2}. Element contributions in their virtual work forms are (nodal displacements of the material coordinate system need to be expressed in terms of the structural system components)

Bar 1: $u_{x1} = 0$, $u_{x2} = u_{X2}$: $\quad \delta W^1 = -\delta u_{X2} \dfrac{EA}{L} u_{X2}$,

Bar 2: $u_{x2} = u_{Z2}$, $u_{x3} = 0$: $\quad \delta W^2 = -\delta u_{Z2} \dfrac{EA}{L} u_{Z2}$,

Bar 3: $u_{x4} = 0$ and $u_{x3} = 0$: $\quad \delta W^3 = 0$,

Bar 4: $u_{x1} = 0$ and $u_{x4} = 0$: $\quad \delta W^4 = 0$,

Bar 5: $u_{x1} = 0$ and $u_{x3} = 0$: $\quad \delta W^5 = 0$,

Bar 6: $u_{x4} = 0$, $u_{x2} = \dfrac{1}{\sqrt{2}}(u_{X2} - u_{Z2})$: $\quad \delta W^6 = -(\delta u_{X2} - \delta u_{Z2}) \dfrac{EA}{L}(u_{X2} - u_{Z2})$

Force 7: $\quad \delta W^7 = -\delta u_{Z2} F$.

Virtual work expression of the structure is sum of the element contributions

$$\delta W = \sum_e \delta W^e = \delta W^1 + \delta W^2 + \delta W^3 + \delta W^4 + \delta W^5 + \delta W^6 + \delta W^7 \;\Rightarrow$$

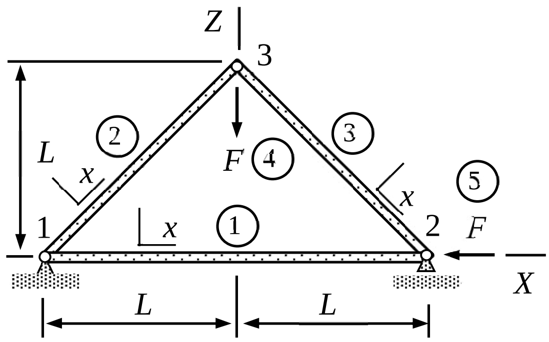
$$\delta W = -\delta u_{X2}\frac{EA}{L}u_{X2} - \delta u_{Z2}\frac{EA}{L}\mathrm{a}_2 + 0 + 0 + 0 - (\delta u_{X2} - \delta u_{Z2})\frac{EA}{L}(u_{X2} - u_{Z2}) - \delta u_{Z2}F \quad \Leftrightarrow$$

$$\delta W = -\begin{Bmatrix}\delta u_{X2}\\ \delta u_{Z2}\end{Bmatrix}^{\mathrm{T}}\left(\frac{EA}{L}\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}\begin{Bmatrix}u_{X2}\\ u_{Z2}\end{Bmatrix} - \begin{Bmatrix}0\\ -F\end{Bmatrix}\right).$$

Principle of virtual work $\delta W = 0 \; \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus in the form $\delta \mathbf{a}^{\mathrm{T}}\mathbf{R} = 0 \;\; \forall \delta \mathbf{a} \;\Leftrightarrow\; \mathbf{R} = 0$ imply

$$\frac{EA}{L}\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}\begin{Bmatrix}u_{X2}\\ u_{Z2}\end{Bmatrix} - \begin{Bmatrix}0\\ -F\end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix}u_{X2}\\ u_{Z2}\end{Bmatrix} = \frac{FL}{EA}\frac{1}{3}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{Bmatrix}0\\ -1\end{Bmatrix} = -\frac{FL}{EA}\begin{Bmatrix}1/3\\ 2/3\end{Bmatrix}. \;\leftarrow$$

Consider the XZ-plane bar structure of the figure. Young´s modulus E is constant. Cross-sectional areas of bars 2 and 3 are $\sqrt{8}A$ and the cross-sectional area of bar 1 is $2A$. Determine the displacement components u_{X2}, u_{X3}, and u_{Z3}. Use the principle of virtual work.



Solution

Virtual work expression of the bar element

$$\delta W=\delta W^{\mathrm{int}}+\delta W^{\mathrm{ext}}=-\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}(\frac{EA}{h}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix}-\frac{f_x h}{2}\begin{Bmatrix}1\\ 1\end{Bmatrix})$$

depends on the cross-sectional area A, Young's modulus E, bar length h, and force per unit length f_x.

External distributed force vanishes and the unknown displacement components of the structure are u_{X2}, u_{X3}, and u_{Z3}. Virtual work expression of the elements are

Bar 1: $u_{x1}=0$, $u_{x2}=u_{X2}$,

$$\delta W^1=-\begin{Bmatrix}0\\ \delta u_{X2}\end{Bmatrix}^{\mathrm{T}}\frac{E2A}{2L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}0\\ u_{X2}\end{Bmatrix}=-\begin{Bmatrix}\delta u_{X2}\\ \delta u_{X3}\\ \delta u_{Z3}\end{Bmatrix}^{\mathrm{T}}\frac{EA}{L}\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}u_{X2}\\ u_{X3}\\ u_{Z3}\end{Bmatrix},$$

Bar 2: $u_{x1}=0$, $u_{x3}=\frac{1}{\sqrt{2}}(u_{X3}+u_{Z3})$

$$\delta W^2=-\begin{Bmatrix}0\\ \delta u_{X3}+\delta u_{Z3}\end{Bmatrix}^{\mathrm{T}}\frac{E\sqrt{8}A}{\sqrt{8}L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}0\\ u_{X3}+u_{Z3}\end{Bmatrix}=-\begin{Bmatrix}\delta u_{X2}\\ \delta u_{X3}\\ \delta u_{Z3}\end{Bmatrix}^{\mathrm{T}}\frac{EA}{L}\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 1\\ 0 & 1 & 1\end{bmatrix}\begin{Bmatrix}u_{X2}\\ u_{X3}\\ u_{Z3}\end{Bmatrix}$$

Bar 3: $u_{x3}=\frac{1}{\sqrt{2}}(u_{X3}-u_{Z3})$, $u_{x2}=\frac{1}{\sqrt{2}}u_{X2}$

$$\delta W^3=-\begin{Bmatrix}\delta u_{X3}-\delta u_{Z3}\\ \delta u_{X2}\end{Bmatrix}^{\mathrm{T}}\frac{E\sqrt{8}A}{\sqrt{8}L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_{X3}-u_{Z3}\\ u_{X2}\end{Bmatrix}=-\begin{Bmatrix}\delta u_{X2}\\ \delta u_{X3}\\ \delta u_{Z3}\end{Bmatrix}^{\mathrm{T}}\frac{EA}{L}\begin{bmatrix}1 & -1 & 1\\ -1 & 1 & -1\\ 1 & -1 & 1\end{bmatrix}\begin{Bmatrix}u_{X2}\\ u_{X3}\\ u_{Z3}\end{Bmatrix}$$

Force 4: $\delta W^4=-\begin{Bmatrix}\delta u_{X2}\\ \delta u_{X3}\\ \delta u_{Z3}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}0\\ 0\\ F\end{Bmatrix}$

Force 5: $\delta W^5 = -\begin{Bmatrix} \delta u_{X2} \\ \delta u_{X3} \\ \delta u_{Z3} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} F \\ 0 \\ 0 \end{Bmatrix}$

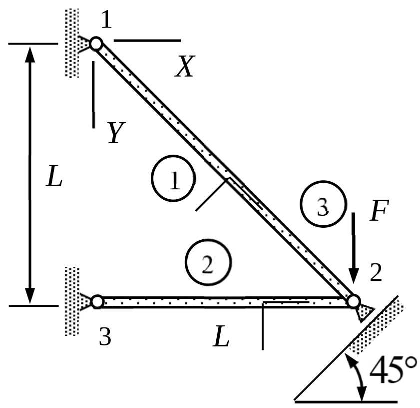
Virtual work expression of a structure is sum of the element contributions

$$\delta W = -\begin{Bmatrix} \delta u_{X2} \\ \delta u_{X3} \\ \delta u_{Z3} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{L}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{X3} \\ u_{Z3} \end{Bmatrix} + \begin{Bmatrix} F \\ 0 \\ F \end{Bmatrix}\right).$$

Principle of virtual work and the fundamental lemma of variation calculus imply the linear equation system and thereby the solution to the nodal displacements

$$\frac{EA}{L}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{X3} \\ u_{Z3} \end{Bmatrix} + F\begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix} u_{X2} \\ u_{X3} \\ u_{Z3} \end{Bmatrix} = \frac{FL}{EA}\begin{Bmatrix} -1/2 \\ -1/4 \\ -1/4 \end{Bmatrix}. \quad \Leftarrow$$

Determine the nodal displacements when force F is acting on the structure as shown in the figure. The cross-sectional area of all the bars is A and the Young's modulus of the material is E. Use the principle of virtual work.



Solution

Virtual work expression of the bar and force elements are given by

$$\delta W = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}\right) \text{ and } \delta W = \begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \\ \delta u_{Z1} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} F_X \\ F_Y \\ F_Z \end{Bmatrix}.$$

Nodal displacements of the structure are zeros except for node 2. Notice that the node is allowed to move only along the frictionless plane. The unit normal vector to the plane and displacement of node 2 (omitting the constraint first) are $\vec{n} = -(\vec{I} + \vec{J})/\sqrt{2}$, $\vec{u}_2 = u_{X2}\vec{I} + u_{Y2}\vec{J}$. Therefore (see the formulae collection "frictionless contact"):

$$\vec{n} \cdot \vec{u}_2 = -\frac{1}{\sqrt{2}}(\vec{I} + \vec{J}) \cdot (u_{X2}\vec{I} + u_{Y2}\vec{J}) = -\frac{1}{\sqrt{2}}(u_{X2} + u_{Y2}) = 0 \quad \Leftrightarrow \quad u_{Y2} = -u_{X2}.$$

Element contributions need to be written in terms of displacement components of the structural system. Due to the constraint at node 2, the unknown displacement components can be chosen to be u_{X2} (say).

Bar 1: $\quad u_{x1} = 0$ and $u_{x2} = 0 \quad \Rightarrow \quad \delta W^1 = 0$,

Bar 2: $\quad u_{x3} = 0$ and $u_{x2} = u_{X2} \quad \Rightarrow \quad \delta W^2 = -\delta u_{X2}\frac{EA}{L}u_{X2}$,

Force 3: $\quad \delta W^3 = \delta u_{Y2}F = -\delta u_{X2}F$.

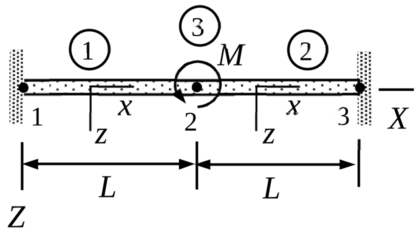
Virtual work expression of the structure is sum of the element contributions

$$\delta W = \sum_e \delta W^e = \delta W^1 + \delta W^2 + \delta W^3 = -\delta u_{X2}\frac{EA}{L}u_{X2} - \delta u_{X2}F = -\delta u_{X2}\left(\frac{EA}{L}u_{X2} + F\right)$$

Principle of virtual work $\delta W = 0 \; \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus in the form $\delta \mathbf{a}^{\mathrm{T}}\mathbf{R} = 0 \;\; \forall \delta \mathbf{a} \;\Leftrightarrow\; \mathbf{R} = 0$ imply

$$\frac{EA}{L}u_{X2} + F = 0 \quad \Leftrightarrow \quad u_{X2} = -\frac{FL}{EA} \quad \Rightarrow \quad u_{Y2} = -u_{X2} = \frac{FL}{EA}.$$

Determine displacement u_{Z2} and rotation θ_{Y2} at point 2 of the structure shown. Use two beam elements of equal length. Point moment with magnitude M is acting on node 2. Young's modulus of the material E and the second moment of area I are constants. Use the principle of virtual work.



Solution

Virtual work expression of the beam *xz*-plane bending and point moments elements are

$$\delta W = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} (\frac{EI_{yy}}{h^3}\begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix}\begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix} - \frac{f_z h}{12}\begin{Bmatrix} 6 \\ -h \\ 6 \\ h \end{Bmatrix}) \text{ and } \delta W = \begin{Bmatrix} \delta\theta_{X1} \\ \delta\theta_{Y1} \\ \delta\theta_{Z1} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} M_X \\ M_Y \\ M_Z \end{Bmatrix}.$$

Except u_{Z2} and θ_{Y2}, nodal displacements/rotations of the structure are zeros. The bar loading mode can be omitted as the axial loading and axial displacements vanish. Here, the axes of material and structural coordinate systems are aligned:

Beam 1: $u_{z1} = 0$, $\theta_{y1} = 0$, $u_{z2} = u_{Z2}$ and $\theta_{y2} = \theta_{Y2}$

$$\delta W^1 = -\begin{Bmatrix} 0 \\ 0 \\ \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ u_{Z2} \\ \theta_{Y2} \end{Bmatrix} = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3}\begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix}.$$

Beam 2: $u_{z2} = u_{Z2}$, $\theta_{y2} = \theta_{Y2}$, $u_{z3} = 0$ and $\theta_{y3} = 0$

$$\delta W^2 = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \\ 0 \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \\ 0 \\ 0 \end{Bmatrix} = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3}\begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix}.$$

Moment 3:

$$\delta W^3 = \begin{Bmatrix} \delta\theta_{X2} \\ \delta\theta_{Y2} \\ \delta\theta_{Z2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ M \\ 0 \end{Bmatrix} = \delta\theta_{Y2} M = \begin{Bmatrix} \delta\theta_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ M \end{Bmatrix}.$$

Virtual work expression of the structure is sum of the element contributions

$$\delta W = \sum_e \delta W^e = \delta W^1 + \delta W^2 + \delta W^3 \Rightarrow$$

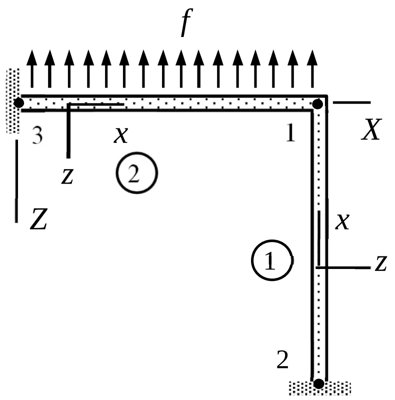
$$\delta W = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EI}{L^3}\begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix} + \frac{EI}{L^3}\begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ M \end{Bmatrix}\right) \quad \Leftrightarrow$$

$$\delta W = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EI}{L^3}\begin{bmatrix} 24 & 0 \\ 0 & 8L^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ M \end{Bmatrix}\right).$$

Principle of virtual work $\delta W = 0 \; \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus in the form $\delta \mathbf{a}^{\mathrm{T}} \mathbf{R} = 0 \;\; \forall \delta \mathbf{a} \;\Leftrightarrow\; \mathbf{R} = 0$ imply

$$\frac{EI}{L^3}\begin{bmatrix} 24 & 0 \\ 0 & 8L^2 \end{bmatrix}\begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ M \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix} = \frac{1}{8}\frac{ML}{EI}\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}.$$

Determine the rotation θ_{Y1} at node 1 of the structure shown. Use two beam elements of length L. Assume that the beams are inextensible in the axial directions. Young's modulus of the material E and the second moment of area I are constants. Use the principle of virtual work.



Solution

Virtual work expression of the xz-plane bending beam element

$$\delta W=-\begin{Bmatrix}\delta u_{z1}\\ \delta\theta_{y1}\\ \delta u_{z2}\\ \delta\theta_{y2}\end{Bmatrix}^{\mathrm{T}}\left(\frac{EI_{yy}}{h^3}\begin{bmatrix}12 & -6h & -12 & -6h\\ -6h & 4h^2 & 6h & 2h^2\\ -12 & 6h & 12 & 6h\\ -6h & 2h^2 & 6h & 4h^2\end{bmatrix}\begin{Bmatrix}u_{z1}\\ \theta_{y1}\\ u_{z2}\\ \theta_{y2}\end{Bmatrix}-\frac{f_z h}{12}\begin{Bmatrix}6\\ -h\\ 6\\ h\end{Bmatrix}\right)$$

depends on the second moment of cross-section area I_{yy}, Young's modulus E, beam length h, and force per unit length f_z.

The displacement and rotation components of the material coordinate system need to be expressed in terms of the components of the structural system. As beams are inextensible in the axial directions, the structure has only the rotation degree of freedom θ_{Y1} and it is enough to consider bending in the $xz-$ plane only.

Beam 1: $u_{z2}=0$, $\theta_{y2}=0$, $u_{z1}=0$, $\theta_{y1}=\theta_{Y1}$, and $f_z=0$

$$\delta W^1=-\begin{Bmatrix}0\\ 0\\ 0\\ \delta\theta_{Y1}\end{Bmatrix}^{\mathrm{T}}\left(\frac{EI}{L^3}\begin{bmatrix}12 & -6L & -12 & -6L\\ -6L & 4L^2 & 6L & 2L^2\\ -12 & 6L & 12 & 6L\\ -6L & 2L^2 & 6L & 4L^2\end{bmatrix}\begin{Bmatrix}0\\ 0\\ 0\\ \theta_{Y1}\end{Bmatrix}-\frac{0L}{12}\begin{Bmatrix}6\\ -L\\ 6\\ L\end{Bmatrix}\right)=-\delta\theta_{Y1}\frac{EI}{L^3}4L^2\theta_{Y1}$$

Beam 2: $u_{z3}=0$, $\theta_{y3}=0$, $u_{z1}=0$, $\theta_{y1}=\theta_{Y1}$, and $f_z=-f$

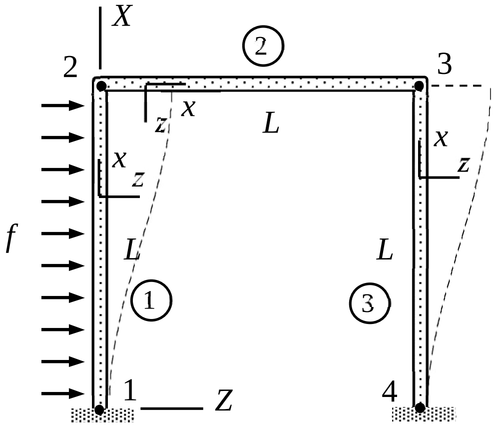
$$\delta W^2=-\begin{Bmatrix}0\\ 0\\ 0\\ \delta\theta_{Y1}\end{Bmatrix}^{\mathrm{T}}\left(\frac{EI}{L^3}\begin{bmatrix}12 & -6L & -12 & -6L\\ -6L & 4L^2 & 6L & 2L^2\\ -12 & 6L & 12 & 6L\\ -6L & 2L^2 & 6L & 4L^2\end{bmatrix}\begin{Bmatrix}0\\ 0\\ 0\\ \theta_{Y1}\end{Bmatrix}-\frac{-fL}{12}\begin{Bmatrix}6\\ -L\\ 6\\ L\end{Bmatrix}\right)=-\delta\theta_{Y1}\left(\frac{EI}{L^3}4L^2\theta_{Y1}+\frac{fL^2}{12}\right)$$

Virtual work expression of the structure is obtained by summing the element contributions. After that, the virtual work expression is rearranged into the standard form (similar to the virtual work expression of an element):

$$\delta W = \delta W^1 + \delta W^2 = -\delta\theta_{Y1} \frac{EI}{L^3} 4L^2 \theta_{Y1} - \delta\theta_{Y1} \left(\frac{EI}{L^3} 4L^2 \theta_{Y1} + \frac{fL^2}{12}\right) = -\delta\theta_{Y1} \left(8\frac{EI}{L} \theta_{Y1} + \frac{fL^2}{12}\right).$$

Principle of virtual work $\delta W = 0 \quad \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus imply the equilibrium equation and thereby solution to θ_{Y1}

$$8\frac{EI}{L}\theta_{Y1} + \frac{f\,L^2}{12} = 0 \quad \Leftrightarrow \quad \theta_{Y1} = -\frac{1}{96}\frac{f\,L^3}{EI}. \quad \leftarrow$$



The frame of the figure consists of a rigid body (2) and beams (1) and (3). Determine the non-zero displacements and rotations. The beams are identical and can be assumed as inextensible in the axial directions. Displacements are confined to the $XZ-$plane. Young's modulus E, second moment of area I, and the distributed force f acting on element 1 are constants. Use the principle of virtual work.

Solution

As element 2 is a rigid body and the other beams are inextensible in the axial directions, only the horizontal displacement components $u_{Z3} = u_{Z2}$ are non-zeros. Element contributions to the virtual work expression are

$$\delta W^1 = -\begin{Bmatrix} 0 \\ 0 \\ \delta u_{Z2} \\ 0 \end{Bmatrix}^{\mathrm{T}} \left(\frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ u_{Z2} \\ 0 \end{Bmatrix} - \frac{fL}{12}\begin{Bmatrix} 6 \\ -L \\ 6 \\ L \end{Bmatrix}\right) = -\delta u_{Z2}\left(12\frac{EI}{L^3}u_{Z2} - \frac{fL}{2}\right),$$

$$\delta W^2 = 0$$

$$\delta W^3 = -\begin{Bmatrix} 0 \\ 0 \\ \delta u_{Z2} \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ u_{Z2} \\ 0 \end{Bmatrix} = -\delta u_{Z2} 12\frac{EI}{L^3}u_{Z2}$$

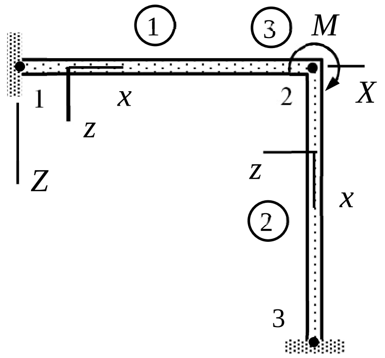
Virtual work expression of structure is the sum of element contributions

$$\delta W = \delta W^1 + \delta W^2 + \delta W^3 = -\delta u_{Z2}\left(24\frac{EI}{L^3}u_{Z2} - \frac{fL}{2}\right)$$

Principle of virtual work $\delta W = 0 \ \forall \delta \mathrm{a}$ and the fundamental lemma of variation calculus imply

$$u_{Z2} = \frac{1}{48}\frac{fL^4}{EI} \ .$$

Determine the rotation θ_{Y2} at node 2 of the structure loaded by a point moment (magnitude M) acting on node 2. Use beam elements (1) and (2) of length L and a point moment element (3). Assume that the beams are inextensible in the axial directions. Young's modulus E and the second moment of area I are constants. Use the principle of virtual work.



Solution

In a planar problem, torsion and out-plane bending deformation modes can be omitted. As beams are assumed to be inextensible in the axial direction and there are no axial distributed forces, the bar mode virtual work expression vanishes. Virtual work expressions of the beam xz-plane bending element and point force/moment elements are given by

$$\delta W=-\begin{Bmatrix}\delta u_{z1}\\ \delta\theta_{y1}\\ \delta u_{z2}\\ \delta\theta_{y2}\end{Bmatrix}^{\mathrm{T}}(\frac{EI_{yy}}{h^3}\begin{bmatrix}12 & -6h & -12 & -6h\\ -6h & 4h^2 & 6h & 2h^2\\ -12 & 6h & 12 & 6h\\ -6h & 2h^2 & 6h & 4h^2\end{bmatrix}\begin{Bmatrix}u_{z1}\\ \theta_{y1}\\ u_{z2}\\ \theta_{y2}\end{Bmatrix}-\frac{f_z h}{12}\begin{Bmatrix}6\\ -h\\ 6\\ h\end{Bmatrix}),$$

$$\delta W=\begin{Bmatrix}\delta u_{X1}\\ \delta u_{Y1}\\ \delta u_{Z1}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}F_{X1}\\ F_{Y1}\\ F_{Z1}\end{Bmatrix}+\begin{Bmatrix}\delta\theta_{X1}\\ \delta\theta_{Y1}\\ \delta\theta_{Z1}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}M_{X1}\\ M_{Y1}\\ M_{Z1}\end{Bmatrix}.$$

Nodal displacements/rotations of the structure are clearly zeros except those of node 2. Displacement of node 2 vanishes as both beams are inextensible in the axial directions. Therefore, the only non-zero displacement/rotation component of the structure is θ_{Y2}.

Beam 1: $u_{z1}=0$, $\theta_{y1}=0$, $u_{z2}=0$, and $\theta_{y2}=\theta_{Y2}$

$$\delta W^1=-\begin{Bmatrix}0\\ 0\\ 0\\ \delta\theta_{Y2}\end{Bmatrix}^{\mathrm{T}}\frac{EI}{L^3}\begin{bmatrix}12 & -6L & -12 & -6L\\ -6L & 4L^2 & 6L & 2L^2\\ -12 & 6L & 12 & 6L\\ -6L & 2L^2 & 6L & 4L^2\end{bmatrix}\begin{Bmatrix}0\\ 0\\ 0\\ \theta_{Y2}\end{Bmatrix}=-\delta\theta_{Y2}4\frac{EI}{L}\theta_{Y2}.$$

Beam 2: $u_{z2}=0$, $\theta_{y2}=\theta_{Y2}$, $u_{z3}=0$, and $\theta_{y3}=0$

$$\delta W^2=-\begin{Bmatrix}0\\ \delta\theta_{Y2}\\ 0\\ 0\end{Bmatrix}^{\mathrm{T}}\frac{EI}{L^3}\begin{bmatrix}12 & -6L & -12 & -6L\\ -6L & 4L^2 & 6L & 2L^2\\ -12 & 6L & 12 & 6L\\ -6L & 2L^2 & 6L & 4L^2\end{bmatrix}\begin{Bmatrix}0\\ \theta_{Y2}\\ 0\\ 0\end{Bmatrix}=-\delta\theta_{Y2}4\frac{EI}{L}\theta_{Y2}.$$

Point moment 3:

$$\delta W^3 = \begin{Bmatrix} \delta u_{X2} \\ \delta u_{Y2} \\ \delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} F_{X2} \\ F_{Y2} \\ F_{Z2} \end{Bmatrix} + \begin{Bmatrix} \delta\theta_{X2} \\ \delta\theta_{Y2} \\ \delta\theta_{Z2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} M_{X2} \\ M_{Y2} \\ M_{Z2} \end{Bmatrix} = -\delta\theta_{Y2} M \, .$$

Virtual work expression of the structure is sum of the element contributions

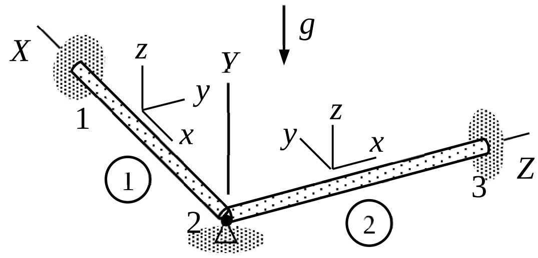
$$\delta W = \sum\nolimits_e \delta W^e = \delta W^1 + \delta W^2 + \delta W^3 \ \Rightarrow$$

$$\delta W = -\delta\theta_{Y2} 4 \frac{EI}{L} \theta_{Y2} - \delta\theta_{Y2} 4 \frac{EI}{L} \theta_{Y2} + 0 - \delta\theta_{Y2} M = -\delta\theta_{Y2} (8 \frac{EI}{L} \theta_{Y2} + M) \, .$$

Principle of virtual work $\delta W = 0 \; \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus in the form $\delta \mathbf{a}^{\mathrm{T}} \mathbf{R} = 0 \;\; \forall \delta \mathbf{a} \;\Leftrightarrow\; \mathbf{R} = 0$ imply

$$8 \frac{EI}{L} \theta_{Y2} + M = 0 \quad \Leftrightarrow \quad \theta_{Y2} = -\frac{1}{8} \frac{ML}{EI} \, .$$

Beam 1 (length L) of the figure is loaded by its own weight and beam 2 (length L) is assumed weightless. Determine the rotation component θ_{Z2}. Moments of the cross section are A, $I_{yy} = I_{zz} = I$ and $J = 2I$. Young's modulus, shear modulus, and density of the material E, G and ρ are constants.



Solution

The only non-zero nodal displacement/rotation of the structure is θ_{Z2}. The virtual work expression of the structure is the sum of the virtual work expressions of the elements which consist of parts coming from bending, torsion etc. For beam 1 $\theta_{y2} = \theta_{Z2}$ and $f_z = -A\rho g$. The non-zero contribution to the virtual work expression comes from bending in $xz-$ plane:

$$\delta W^{\text{int}} = -\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta\theta_{Z2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \theta_{Z2} \end{Bmatrix} = -\delta\theta_{Z2} \frac{4EI}{L} \theta_{Z2},$$

$$\delta W^{\text{ext}} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta\theta_{Z2} \end{Bmatrix}^{\mathrm{T}} \frac{(-A\rho g) L}{12} \begin{Bmatrix} 6 \\ -L \\ 6 \\ L \end{Bmatrix} = -\delta\theta_{Z2} \frac{A\rho g\, L^2}{12}.$$

For beam 2 $\theta_{x2} = \theta_{Z2}$ and the non-zero contribution to the virtual work expression comes from the torsion mode

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta\theta_{Z2} \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{G2I}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \theta_{Z2} \\ 0 \end{Bmatrix} = -\delta\theta_{Z2} \frac{2GI}{L} \theta_{Z2}.$$

Virtual work expression of the structure is the sum of the element contributions

$$\delta W = -\delta\theta_{Z2} \frac{4EI}{L} \theta_{Z2} - \delta\theta_{Z2} \frac{A\rho g\, L^2}{12} - \delta\theta_{Z2} \frac{2GI}{L} \theta_{Z2} = -\delta\theta_{Z2} (2I \frac{2E+G}{L} \theta_{Z2} + \frac{A\rho g\, L^2}{12})$$

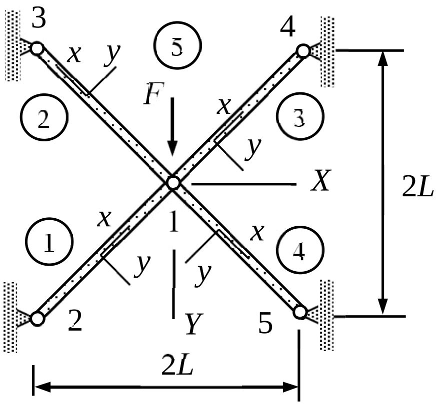
Principle of virtual work $\delta W = 0 \quad \forall \delta \mathrm{a}$ and the fundamental lemma of variation calculus give

$$\delta W = -\delta\theta_{Z2} (2I \frac{2E+G}{L} \theta_{Z2} + \frac{A\rho g\, L^2}{12}) = 0 \qquad \forall\, \delta\theta_{Z2} \quad \Leftrightarrow$$

$$2I \frac{2E+G}{L} \theta_{Z2} + \frac{A\rho g\, L^2}{12} = 0 \quad \Leftrightarrow \quad \theta_{Z2} = -\frac{\rho g A L^3}{24I(2E+G)}.$$

Name__________________________ Student number__________________

Assignment 1



Determine the element contributions of bars 2 and 3 of the structure shown using the bar element contribution for the structural coordinate system. Cross-sectional area of all the bars is $\sqrt{8}A$ and Young's modulus E.

Solution template

In the structural coordinate system, the element contribution of a bar is given by

$$\begin{Bmatrix}\mathbf{R}_1\\ \mathbf{R}_2\end{Bmatrix}=\frac{EA}{h}\begin{bmatrix}\mathbf{i}\mathbf{i}^{\mathrm{T}} & -\mathbf{i}\mathbf{i}^{\mathrm{T}}\\ -\mathbf{i}\mathbf{i}^{\mathrm{T}} & \mathbf{i}\mathbf{i}^{\mathrm{T}}\end{bmatrix}\begin{Bmatrix}\mathbf{a}_1\\ \mathbf{a}_2\end{Bmatrix}-\frac{f_x h}{2}\begin{Bmatrix}\mathbf{i}\\ \mathbf{i}\end{Bmatrix}, \text{ in which } \mathbf{i}=\frac{1}{h}\begin{Bmatrix}\Delta X\\ \Delta Y\end{Bmatrix}.$$

Above, $\mathbf{i}$ consists of components of the unit vector $\vec{i}$ of the material coordinate system expressed in the structural coordinate system, h is the length of the bar element, and components ΔX, ΔY are the differences of the structural coordinates of the element end points.

The quantities in the element contribution of bar 2 are given by

$$h=\sqrt{2}L, \quad \mathbf{i}=\begin{Bmatrix}-1/\sqrt{2}\\ -1/\sqrt{2}\end{Bmatrix}, \text{ and } \mathbf{i}\mathbf{i}^{\mathrm{T}}=\begin{bmatrix}1/2 & 1/2\\ 1/2 & 1/2\end{bmatrix}, \text{ therefore}$$

$$\begin{Bmatrix}F_{X1}\\ F_{Y1}\\ F_{X3}\\ F_{Y3}\end{Bmatrix}=\frac{EA}{L}\begin{bmatrix}1 & 1 & -1 & -1\\ 1 & 1 & -1 & -1\\ -1 & -1 & 1 & 1\\ -1 & -1 & 1 & 1\end{bmatrix}\begin{Bmatrix}u_{X1}\\ u_{Y1}\\ 0\\ 0\end{Bmatrix}-\begin{Bmatrix}0\\ 0\\ 0\\ 0\end{Bmatrix}.$$

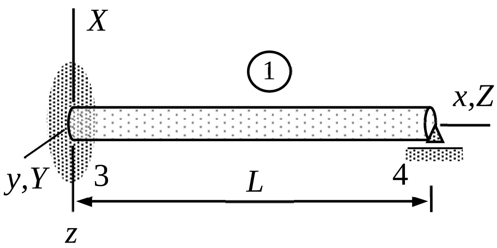
The quantities in the element contribution of bar 3 are given by

$$h=\sqrt{2}L, \quad \mathbf{i}=\begin{Bmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\end{Bmatrix}, \text{ and } \mathbf{i}\mathbf{i}^{\mathrm{T}}=\begin{bmatrix}1/2 & -1/2\\ -1/2 & 1/2\end{bmatrix}, \text{ therefore}$$

$$\begin{Bmatrix} F_{X1} \\ F_{Y1} \\ \hdashline F_{X4} \\ F_{Y4} \end{Bmatrix} = \frac{EA}{L} \left[\begin{array}{cc:cc} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ \hdashline -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{array}\right] \begin{Bmatrix} u_{X1} \\ u_{Y1} \\ \hdashline 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ \hdashline 0 \\ 0 \end{Bmatrix}.$$

Name________________________ Student number________________

Assignment 2



Determine the non-zero virtual work expressions for the deformation modes of the beam shown. The non-zero nodal displacements/rotations are θ_{Y4} and u_{Z4}.

Solution template

Virtual work expressions of the bending and bar deformation modes of the beam element in *xz*-plane are

$$\delta W=-\begin{Bmatrix}\delta u_{z1}\\ \delta\theta_{y1}\\ \delta u_{z2}\\ \delta\theta_{y2}\end{Bmatrix}^{\mathrm{T}}\left(\frac{EI_{yy}}{h^3}\begin{bmatrix}12 & -6h & -12 & -6h\\ -6h & 4h^2 & 6h & 2h^2\\ -12 & 6h & 12 & 6h\\ -6h & 2h^2 & 6h & 4h^2\end{bmatrix}\begin{Bmatrix}u_{z1}\\ \theta_{y1}\\ u_{z2}\\ \theta_{y2}\end{Bmatrix}-\frac{f_z h}{12}\begin{Bmatrix}6\\ -h\\ 6\\ h\end{Bmatrix}\right),$$

$$\delta W=-\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\left(\frac{EA}{h}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix}-\frac{f_x h}{2}\begin{Bmatrix}1\\ 1\end{Bmatrix}\right),$$

in which $I_{yy}=I$ is the second moment of area, E is the Young's modulus, A is the cross-sectional area, and h is the length of the beam. Distributed (force per unit length) external force components f_x and f_z are assumed to be constants.

The displacement and rotation components of the material coordinate system are first expressed in terms of those in the structural coordinate system. Notice that the node numbers 1,2 of the *element template* are replaced by node numbers 3,4 of the *actual element*. Hence, for the element (superscripts in u_{x3}^1 etc. are omitted for simplicity)

$u_{x3}=0$, $\quad u_{z3}=0$, $\quad \theta_{y3}=0$,

$u_{x4}=u_{Z4}$, $\quad u_{z4}=0$, $\quad \theta_{y4}=\theta_{Y4}$.

Virtual work expression for the bar mode:

$$\delta W=-\begin{Bmatrix}0\\ \delta u_{Z4}\end{Bmatrix}^{\mathrm{T}}\left(\frac{EA}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}0\\ u_{Z4}\end{Bmatrix}-\begin{Bmatrix}0\\ 0\end{Bmatrix}\right) \quad \Rightarrow \quad \text{(simplify)}$$

$$\delta W=-\delta u_{Z4}\frac{EA}{L}u_{Z4}.$$

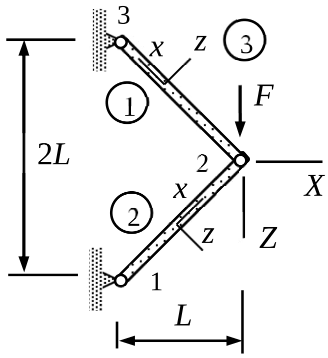
Virtual work expression for the bending mode in *xz*-plane:

$$\delta W = -\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta\theta_{Y4} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \theta_{Y4} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \right) \quad \Rightarrow$$

$$\delta W = -\delta\theta_{Y4} 4 \frac{EI}{L} \theta_{Y4} .$$

Name________________________ Student number_________________

Assignment 3



Determine horizontal and vertical displacements of node 2 of the bar structure shown. The cross-sectional area of the bars and Young's modulus of the material are $\sqrt{2}A$ and E.

Solution template

Element contribution written in terms of displacement components of the structural coordinate system

$$\begin{Bmatrix}\mathbf{R}_1\\ \mathbf{R}_2\end{Bmatrix}=\frac{EA}{h}\begin{bmatrix}\mathbf{i}\mathbf{i}^{\mathrm{T}} & -\mathbf{i}\mathbf{i}^{\mathrm{T}}\\ -\mathbf{i}\mathbf{i}^{\mathrm{T}} & \mathbf{i}\mathbf{i}^{\mathrm{T}}\end{bmatrix}\begin{Bmatrix}\mathbf{a}_1\\ \mathbf{a}_2\end{Bmatrix}-\frac{f_x h}{2}\begin{Bmatrix}\mathbf{i}\\ \mathbf{i}\end{Bmatrix},\ \text{where } \mathbf{i}=\frac{1}{h}\begin{Bmatrix}\Delta X\\ \Delta Z\end{Bmatrix}=\frac{1}{h}\begin{Bmatrix}X_2-X_1\\ Z_2-Z_1\end{Bmatrix}$$

depends on the cross-sectional area A, Young's modulus E, bar length h, force per unit length of the bar f_x in the direction of the $x-$axis, and the components of the basis vector $\vec{i}$ in the structural coordinate system.

Element contributions are first written in terms of the nodal displacements of the structural coordinate system (notice that the point force is treated as a one-node element)

Bar 1: $h=\sqrt{2}L$, $\mathbf{i}=\begin{Bmatrix}-\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{Bmatrix}$, $\begin{Bmatrix}F_{X2}^1\\ F_{Z2}^1\\ F_{X3}^1\\ F_{Z3}^1\end{Bmatrix}=\frac{EA}{2L}\begin{bmatrix}1 & 1 & -1 & -1\\ 1 & 1 & -1 & -1\\ -1 & -1 & 1 & 1\\ -1 & -1 & 1 & 1\end{bmatrix}\begin{Bmatrix}u_{X2}\\ u_{Z2}\\ 0\\ 0\end{Bmatrix}-\begin{Bmatrix}0\\ 0\\ 0\\ 0\end{Bmatrix}$,

Bar 2: $h=\sqrt{2}L$, $\mathbf{i}=\begin{Bmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{Bmatrix}$, $\begin{Bmatrix}F_{X1}^2\\ F_{Z1}^2\\ F_{X2}^2\\ F_{Z2}^2\end{Bmatrix}=\frac{EA}{2L}\begin{bmatrix}1 & -1 & -1 & 1\\ -1 & 1 & 1 & -1\\ -1 & 1 & 1 & -1\\ 1 & -1 & -1 & 1\end{bmatrix}\begin{Bmatrix}0\\ 0\\ u_{X2}\\ u_{Z2}\end{Bmatrix}-\begin{Bmatrix}0\\ 0\\ 0\\ 0\end{Bmatrix}$,

Force 3: $\begin{Bmatrix}F_{X2}^3\\ F_{Z2}^3\end{Bmatrix}=-\begin{Bmatrix}0\\ F\end{Bmatrix}$.

In assembly of the system equations, the forces acting on the non-constrained node 2 are added to get the equilibrium equations in terms of displacement components

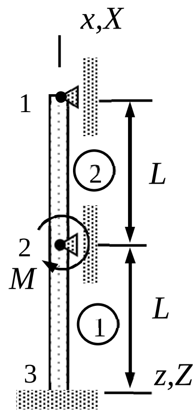
$$\sum \begin{Bmatrix} F_{X2}^{e} \\ F_{Z2}^{e} \end{Bmatrix} = \begin{Bmatrix} F_{X2}^{1} \\ F_{Z2}^{1} \end{Bmatrix} + \begin{Bmatrix} F_{X2}^{2} \\ F_{Z2}^{2} \end{Bmatrix} + \begin{Bmatrix} F_{X2}^{3} \\ F_{Z2}^{3} \end{Bmatrix} = \frac{EA}{2L} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} u_{X2} \\ u_{Z2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ F \end{Bmatrix} = 0 .$$

The unknown displacement components are obtained as the solution to the equilibrium equations

$u_{X2} = 0$ and $u_{Z2} = \dfrac{FL}{EA}$.

Use the code of MEC-E1050 to check your answer!

Name________________________ Student number________________



Assignment 4

Beam structure of the figure is loaded by a point moment acting on node 2. Determine the rotations θ_{Y1} and θ_{Y2} by using two beam bending elements. Displacements are confined to the XZ-plane. The cross-section properties of the beam A, I and Young's modulus of the material E are constants.

Solution template

Virtual work expression for the displacement analysis consists of parts coming from internal and external forces. For the beam bending mode in xz-plane, the element contribution is

$$\delta W=-\begin{Bmatrix}\delta u_{z1}\\ \delta\theta_{y1}\\ \delta u_{z2}\\ \delta\theta_{y2}\end{Bmatrix}^{\mathrm{T}}\left(\frac{EI_{yy}}{h^3}\begin{bmatrix}12 & -6h & -12 & -6h\\ -6h & 4h^2 & 6h & 2h^2\\ -12 & 6h & 12 & 6h\\ -6h & 2h^2 & 6h & 4h^2\end{bmatrix}\begin{Bmatrix}u_{z1}\\ \theta_{y1}\\ u_{z2}\\ \theta_{y2}\end{Bmatrix}-\frac{f_z h}{12}\begin{Bmatrix}6\\ -h\\ 6\\ h\end{Bmatrix}\right).$$

The element contribution of the point force/moment follows from the definition or work and is given by

$$\delta W=\begin{Bmatrix}\delta u_{X1}\\ \delta u_{Y1}\\ \delta u_{Z1}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}F_X\\ F_Y\\ F_Z\end{Bmatrix}+\begin{Bmatrix}\delta\theta_{X1}\\ \delta\theta_{Y1}\\ \delta\theta_{Z1}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}M_X\\ M_Y\\ M_Z\end{Bmatrix}.$$

For beam 1, the element contribution simplifies to

$$\delta W^1=-\begin{Bmatrix}0\\ 0\\ 0\\ \delta\theta_{Y2}\end{Bmatrix}^{\mathrm{T}}\frac{EI}{L^3}\begin{bmatrix}12 & -6L & -12 & -6L\\ -6L & 4L^2 & 6L & 2L^2\\ -12 & 6L & 12 & 6L\\ -6L & 2L^2 & 6L & 4L^2\end{bmatrix}\begin{Bmatrix}0\\ 0\\ 0\\ \theta_{Y2}\end{Bmatrix}=-\delta\theta_{Y2}4\frac{EI}{L}\theta_{Y2}.$$

For beam 2, the element contribution is given by

$$\delta W^2=-\begin{Bmatrix}0\\ \delta\theta_{Y2}\\ 0\\ \delta\theta_{Y1}\end{Bmatrix}^{\mathrm{T}}\frac{EI}{L^3}\begin{bmatrix}12 & -6L & -12 & -6L\\ -6L & 4L^2 & 6L & 2L^2\\ -12 & 6L & 12 & 6L\\ -6L & 2L^2 & 6L & 4L^2\end{bmatrix}\begin{Bmatrix}0\\ \theta_{Y2}\\ 0\\ \theta_{Y1}\end{Bmatrix}=-\begin{Bmatrix}\delta\theta_{Y2}\\ \delta\theta_{Y1}\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}4\frac{EI}{L} & 2\frac{EI}{L}\\ 2\frac{EI}{L} & 4\frac{EI}{L}\end{bmatrix}\begin{Bmatrix}\theta_{Y2}\\ \theta_{Y1}\end{Bmatrix}.$$

Virtual work expression of the point moment (considered as element 3) takes the form

$$\delta W^3 = -M\delta\theta_{Y2}.$$

Virtual work expression of structure is sum of the element contributions. In the standard form

$$\delta W = \delta W^1 + \delta W^2 + \delta W^3 = -\begin{Bmatrix} \delta\theta_{Y2} \\ \delta\theta_{Y1} \end{Bmatrix}^{\mathrm{T}} (\begin{bmatrix} 8\dfrac{EI}{L} & 2\dfrac{EI}{L} \\ 2\dfrac{EI}{L} & 4\dfrac{EI}{L} \end{bmatrix} \begin{Bmatrix} \theta_{Y2} \\ \theta_{Y1} \end{Bmatrix} - \begin{Bmatrix} -M \\ 0 \end{Bmatrix}).$$

Principle of virtual work and the fundamental lemma of variation calculus imply that

$$\begin{bmatrix} 8\dfrac{EI}{L} & 2\dfrac{EI}{L} \\ 2\dfrac{EI}{L} & 4\dfrac{EI}{L} \end{bmatrix} \begin{Bmatrix} \theta_{Y2} \\ \theta_{Y1} \end{Bmatrix} - \begin{Bmatrix} -M \\ 0 \end{Bmatrix} = 0.$$

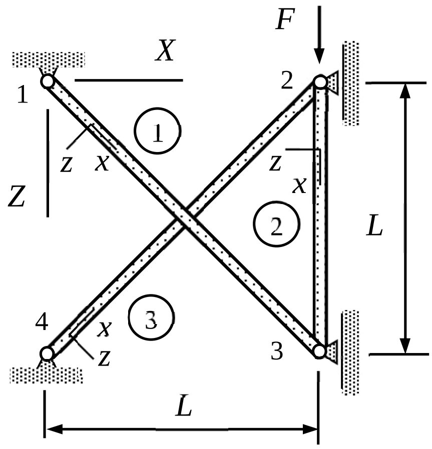
Solution to the linear equation system is given by

$$\theta_{Y2} = -\frac{1}{7}\frac{ML}{EI} \quad \text{and} \quad \theta_{Y1} = \frac{1}{14}\frac{ML}{EI}.$$

Use the code of MEC-E1050 to check your solution!

Name________________________ Student number________________

Assignment 5



The bar truss shown in loaded by vertical force F at node 2. If bar 2 is inextensible, determine the non-zero displacements of the nodes. The cross-sectional area of bar 1 and 3 is A and that of bar 2 is $\sqrt{2}A$. Young's modulus E of the material is constant. Use the principle of virtual work.

Solution

In the material coordinate system, virtual work expression of the bar model is given by

$$\delta W=-\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}(\frac{EA}{h}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix}-\frac{f_x h}{2}\begin{Bmatrix}1\\ 1\end{Bmatrix}).$$

As bar 2 is inextensible, nodal displacements in the axial direction coincide so $u_{Z2}=u_{Z3}$. In terms of the nodal displacement components of the structural system, axial displacements of bar 1 are $u_{x1}=0$ and $u_{x3}=u_{Z3}/\sqrt{2}$. Length of the bar $h=\sqrt{2}L$, cross-sectional area is A, and the external distributed force $f_x=0$. Therefore

$$\delta W^1=-\begin{Bmatrix}0\\ \delta u_{Z3}/\sqrt{2}\end{Bmatrix}^{\mathrm{T}}(\frac{EA}{\sqrt{2}L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}0\\ u_{Z3}/\sqrt{2}\end{Bmatrix})=-\delta u_{Z3}\frac{EA}{2\sqrt{2}L}u_{Z3}.$$

Axial displacements of bar 2 are $u_{x2}=u_{Z2}=u_{Z3}$ and $u_{x3}=u_{Z3}$. Length of the bar $h=L$, cross-sectional area is $\sqrt{2}A$, and the external distributed force $f_x=0$. Therefore

$$\delta W^2=-\begin{Bmatrix}\delta u_{Z3}\\ \delta u_{Z3}\end{Bmatrix}^{\mathrm{T}}(\frac{E\sqrt{2}A}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_{Z3}\\ u_{Z3}\end{Bmatrix})=0.$$

In terms of the nodal displacement components of the structural system, axial displacements of bar 3 $u_{x4}=0$ and $u_{x2}=-u_{Z2}/\sqrt{2}=-u_{Z3}/\sqrt{2}$, Length of the bar $h=\sqrt{2}L$, cross-sectional area is A, and the external distributed force $f_x=0$. Therefore

$$\delta W^3=-\begin{Bmatrix}0\\ -\delta u_{Z3}/\sqrt{2}\end{Bmatrix}^{\mathrm{T}}(\frac{EA}{\sqrt{2}L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}0\\ -u_{Z3}/\sqrt{2}\end{Bmatrix})=-\delta u_{Z3}\frac{EA}{2\sqrt{2}L}u_{Z3}.$$

Virtual work expression of the external force follows, for example, from the definition of work

$$\delta W^4=\delta u_{Z2}F=\delta u_{Z3}F\ .$$

Virtual work expression of the structure is the sum of element contributions

$$\delta W = -\delta u_{Z3} \frac{EA}{2\sqrt{2}L} u_{Z3} + 0 - \delta u_{Z3} \frac{EA}{2\sqrt{2}L} u_{Z3} + \delta u_{Z3} F = -\delta u_{Z3} (\frac{EA}{\sqrt{2}L} u_{Z3} - F) \; .$$

Principle of virtual work and the fundamental lemma of variation calculus imply

$$\frac{EA}{\sqrt{2}L} u_{Z3} - F = 0 \quad \Leftrightarrow \quad u_{Z3} = \sqrt{2} \frac{LF}{EA} \; .$$

MEC-E1050

FINITE ELEMENT METHOD IN SOLIDS 2024

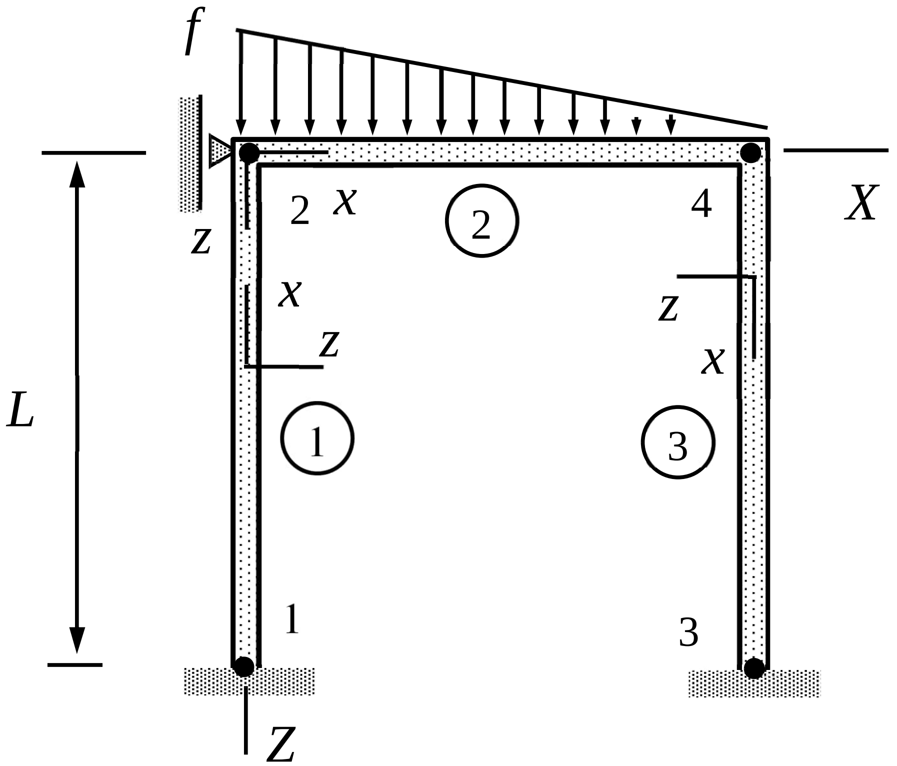
4 ELEMENT CONTRIBUTIONS

LEARNING OUTCOMES

Students are able to solve the lecture problems, home problems, and exercise problems on the topics of the week:

- ☐ The basic building blocks of element contributions: virtual work density and element interpolant (to the nodal values).
- ☐ Derivation of the beam element contribution starting with the basic building blocks
- ☐ Element interpolant and shape functions

EXAMPLE 4.1. Consider the beam truss of the figure. Determine the displacements and rotations of nodes 2 and 4. Assume that the beams are rigid in the axial directions. Cross-sections and lengths are the same and Young's modulus E is constant.



Answer $\theta_{Y2} = -\dfrac{7}{900}\dfrac{fL^3}{EI}$ and $\theta_{Y4} = \dfrac{11}{1800}\dfrac{fL^3}{EI}$

- The Mathematica code solution is given by (f_z is specified by its nodal values)

	model	properties	geometry
1	BEAM	{{E, G}, {A, I, I}}	Line[{1, 2}]
2	BEAM	{{E, G}, {A, I, I}, {0, 0, {f, 0}}}	Line[{2, 4}]
3	BEAM	{{E, G}, {A, I, I}}	Line[{4, 3}]

	$\{X,Y,Z\}$	$\{u_X,u_Y,u_Z\}$	$\{\theta_X,\theta_Y,\theta_Z\}$
1	{0, 0, L}	{0, 0, 0}	{0, 0, 0}
2	{0, 0, 0}	{0, 0, 0}	{0, θY[2], 0}
3	{L, 0, L}	{0, 0, 0}	{0, 0, 0}
4	{L, 0, 0}	{0, 0, 0}	{0, θY[4], 0}

$$\left\{\theta Y[2] \to -\frac{7 f L^3}{900 E I},\ \theta Y[4] \to \frac{11 f L^3}{1800 E I}\right\}$$

Parameters of the problem can be functions of x. Then, derivation of the element contribution by using the exact solution may not be practical and, with 2D/3D elements for plates etc., impossible.

4.1 VIRTUAL WORK EXPRESSION

To find the virtual work expression of an element without recourse to the exact solution of a boundary value problem (which may not be available due to non-constant material properties, distributed forces etc.)

- Start with the basic building blocks: virtual work density for the model and a polynomial interpolant to nodal displacements and rotations.

- Substitute the interpolant to the virtual work density expression and integrate the density over the mathematical domain occupied by the element (the density represents virtual work per unit length, area etc.).

- Rearrange to get the standard form $\delta W = -\delta \mathbf{a}^{\mathrm{T}}(\mathbf{K}\mathbf{a} - \mathbf{F})$.

VIRTUAL WORK DENSITY

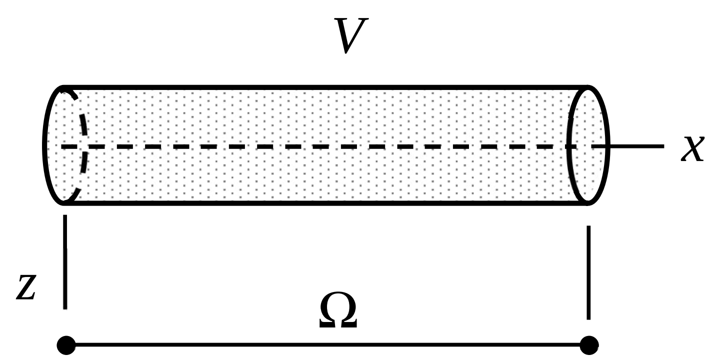
Virtual work densities are concise representations of engineering models (bar, beam, plate, shell, etc.). For the four loading modes of the beam model virtual (density = virtual work per unit length)

Bar: $\delta w_\Omega = -\frac{d\delta u}{dx} EA \frac{du}{dx} + \delta u f_x$

Torsion: $\delta w_\Omega = -\frac{d\delta\phi}{dx} GJ \frac{d\phi}{dx} + \delta\phi m_x$

Bending (*xz*): $\delta w_\Omega = -\frac{d^2\delta w}{dx^2} EI_{yy} \frac{d^2\delta w}{dx^2} + \delta w f_z$

Bending (*xy*): $\delta w_\Omega = -\frac{d^2\delta v}{dx^2} EI_{zz} \frac{d^2 v}{dx^2} + \delta v f_y$



STRUCTURE ANALYSIS; IMPROVED RECIPE

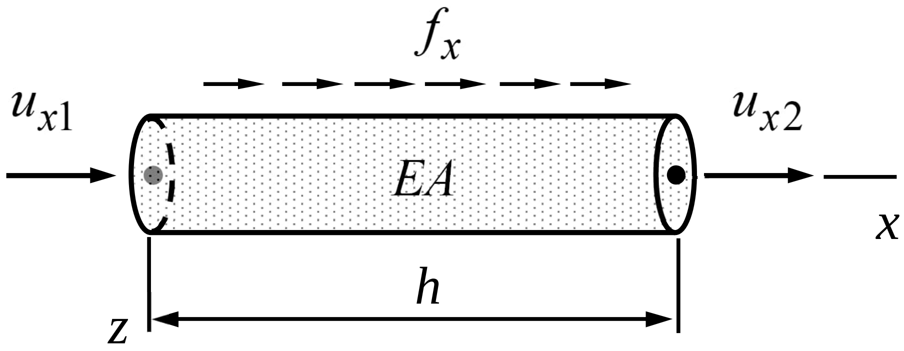
- Derive the element contributions δW^e from *virtual work density of the model* and *polynomial interpolation* of the nodal displacements and rotations in the material coordinate system. a new step

- Express the nodal displacements and rotations of the material coordinate system in terms of those in the structural coordinate system.

- Sum the element contributions over the elements and their loading modes to end up with the virtual work expression $\delta W = \sum_{e \in E} \delta W^e = \sum_{e \in E} (\sum_m \delta W_m^e)$ of structure. Restructure to get the form $\delta W = -\delta \mathbf{a}^{\mathrm{T}} (\mathbf{K}\mathbf{a} - \mathbf{F})$

- Use the principle of virtual work $\delta W = 0 \ \forall \delta \mathbf{a}$, fundamental lemma of variation calculus for $\delta \mathbf{a} \in \mathbb{R}^n$, and solve the dofs from $\mathbf{K}\mathbf{a} - \mathbf{F} = 0$.

4.2 BAR MODE

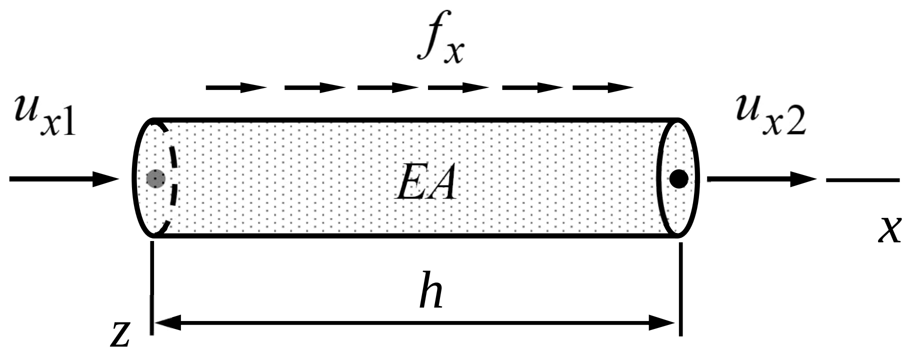


Virtual work density: $\delta w_\Omega = \delta w_\Omega^{\text{int}} + \delta w_\Omega^{\text{ext}} = -\dfrac{d\delta u}{dx} EA \dfrac{du}{dx} + \delta u f_x$

Linear interpolant: $u(x) = \mathbf{N}^{\mathrm{T}}\mathbf{a} \equiv \dfrac{1}{h}\begin{Bmatrix} h-x \\ x \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix}$

Cross-sectional area A, Young's modulus E, and force per unit length f_x (acting on the x-axis) may depend on position. Virtual work density depends only on the model but the interpolant (or approximation) can be chosen in various ways!

BAR ELEMENT CONTRIBUTION



$$\delta W = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}\right),\; u_x = \mathbf{i}^{\mathrm{T}}\begin{Bmatrix} u_X \\ u_Y \\ u_Z \end{Bmatrix},\text{ where } \mathbf{i} = \frac{1}{h}\begin{Bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{Bmatrix}$$

Above, f_x and EA are assumed constants and the elements of matrix **i** **(**$1\times1,\ 2\times1,\ 3\times1$**)** are the components of the unit vector $\vec{i}$ in the structural coordinate system. The algorithm of Mathematica code is based on element contributions in its variational form!

- First, element interpolant $u=\mathbf{N}^{\mathrm{T}}\mathbf{a}$ and its variation $\delta u=\mathbf{N}^{\mathrm{T}}\delta\mathbf{a}=\delta\mathbf{a}^{\mathrm{T}}\mathbf{N}$ are substituted into the virtual work expression to get (here $\Omega=]0,h[$ and $d\Omega=dx$)

$$\delta W=\int_0^h(-\frac{d\delta u}{dx}EA\frac{du}{dx}+\delta u f_x)dx \qquad \Rightarrow$$

$$\delta W=-\int_0^h\delta\mathbf{a}^{\mathrm{T}}\frac{d\mathbf{N}}{dx}EA\frac{d\mathbf{N}^{\mathrm{T}}}{dx}\mathbf{a}dx+\int_0^h\delta\mathbf{a}^{\mathrm{T}}\mathbf{N}f_x dx \qquad \Leftrightarrow$$

$$\delta W=-\delta\mathbf{a}^{\mathrm{T}}(\int_0^h\frac{d\mathbf{N}}{dx}EA\frac{d\mathbf{N}^{\mathrm{T}}}{dx}dx\mathbf{a}-\int_0^h\mathbf{N}f_x dx). \qquad \Leftarrow$$

- If the interpolant is taken to be linear, shape functions and the nodal values are given by

$$\mathbf{N}=\frac{1}{h}\begin{Bmatrix}h-x\\x\end{Bmatrix},\quad \frac{d}{dx}\mathbf{N}=\frac{1}{h}\begin{Bmatrix}-1\\1\end{Bmatrix},\quad \mathbf{a}=\begin{Bmatrix}u_{x1}\\u_{x2}\end{Bmatrix},\text{ and }\delta\mathbf{a}=\begin{Bmatrix}\delta u_{x1}\\\delta u_{x2}\end{Bmatrix}$$

- Assuming that Young's modulus E, cross-sectional area A, and the distributed force f_x are constants, integration over the element domain gives (the expressions of the shape functions need to be substituted now)

$$\delta W = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \left(\int_0^h \frac{1}{h} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} EA \frac{1}{h} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}^{\mathrm{T}} dx \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \int_0^h \frac{1}{h} \begin{Bmatrix} h-x \\ x \end{Bmatrix} f_x dx \right) \quad \Leftrightarrow$$

$$\delta W = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \right). \quad \Leftarrow$$

Derivation out of virtual work densities works also when Young's modulus E, cross-sectional area A, and the distributed force f_x are not constants!

EXAMPLE 4.2 Consider the bar model and a piecewise linear interpolant of the nodal values. Determine the *equivalent nodal forces* $\mathbf{F}$ of the element contribution $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}}$, in which $\delta W^{\text{int}} = -\delta \mathbf{a}^{\mathrm{T}} \mathbf{K} \mathbf{a}$ and $\delta W^{\text{ext}} = \delta \mathbf{a}^{\mathrm{T}} \mathbf{F}$, when length of the element is h and

(a) f_x is constant,

(b) f_x is piecewise linear $f_x = \mathbf{N}^{\mathrm{T}} \mathbf{f}$, where the nodal values are $\mathbf{f}^{\mathrm{T}} = \{ f_{x1} \quad f_{x2} \}$,

(c) $f_x = F_x \delta(\frac{x}{h} - \frac{1}{2})$, where δ is the Dirac-delta and F_x is a point force.

Answer (a) $\mathbf{F} = \frac{f_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$ (b) $\mathbf{F} = \frac{h}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} f_{x1} \\ f_{x2} \end{Bmatrix}$ (c) $\mathbf{F} = \frac{F_x}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

- The equivalent nodal forces are obtained by using $u = \mathbf{N}^{\mathrm{T}}\mathbf{a}$ in the virtual work expression of the external forces

$$\delta W^{\text{ext}} = \int_0^h \delta u f_x dx, \;\; \delta u = \delta \mathbf{a}^{\mathrm{T}}\mathbf{N} \text{ and } \mathbf{N} = \frac{1}{h}\begin{Bmatrix} h-x \\ x \end{Bmatrix} \Rightarrow \;\; \mathbf{F} = \int_0^h \frac{1}{h}\begin{Bmatrix} h-x \\ x \end{Bmatrix} f_x dx \;\; \Leftarrow$$

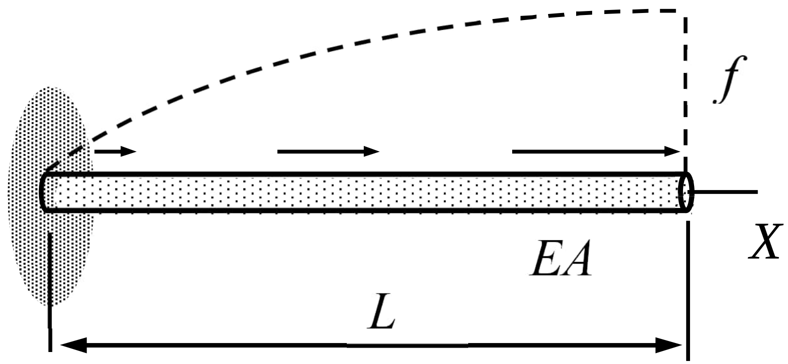
- With the constant, linear and Dirac delta distributions

$$\mathbf{F} = \int_0^h \frac{1}{h}\begin{Bmatrix} h-x \\ x \end{Bmatrix} f_x dx = f_x \int_0^h \frac{1}{h}\begin{Bmatrix} h-x \\ x \end{Bmatrix} dx = \frac{f_x h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \;\; \Leftarrow$$

$$\mathbf{F} = \int_0^h \frac{1}{h}\begin{Bmatrix} h-x \\ x \end{Bmatrix} f_x dx = \int_0^h \frac{1}{h}\begin{Bmatrix} h-x \\ x \end{Bmatrix}\begin{Bmatrix} h-x \\ x \end{Bmatrix}^{\mathrm{T}} dx \begin{Bmatrix} f_{x1} \\ f_{x2} \end{Bmatrix} = \frac{h}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} f_{x1} \\ f_{x2} \end{Bmatrix} \;\; \Leftarrow$$

$$\mathbf{F} = \int_0^h \frac{1}{h}\begin{Bmatrix} h-x \\ x \end{Bmatrix} F_x \delta(x - \frac{h}{2}) dx = \frac{F_x}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \;\; \Leftarrow$$

EXAMPLE 4.3 The bar of the figure (*EA* is constant) is loaded by a quadratic distributed force $f_X = f\xi(2-\xi)$ where $\xi = x/L$. Determine the displacement at the free end by the finite element method. Use one, two, and four elements of equal lengths.



Answer $u_{X2} = \dfrac{5}{12}\dfrac{fL^2}{EA}$ no matter the number of elements (exact $u(L) = \dfrac{5}{12}\dfrac{fL^2}{EA}$)

- Distributed force f_x, Young's modulus E, and the cross-sectional area A may depend on x. In Mathematica code a quadratic distributed force f_x is defined by its values on the nodes and at the midpoint.

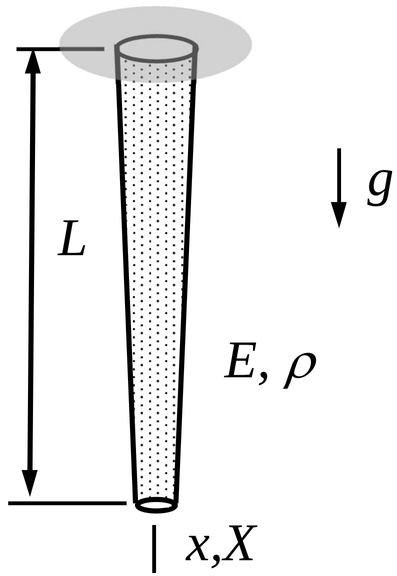
	model	properties	geometry
1	BAR	$\{\{E\},\{A\},\{\{0,\frac{3f}{4},f\},0,0\}\}$	Line[{1, 2}]

	$\{X,Y,Z\}$	$\{u_X,u_Y,u_Z\}$	$\{\theta_X,\theta_Y,\theta_Z\}$
1	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}
2	{L, 0, 0}	{uX[2], 0, 0}	{0, 0, 0}

$$\left\{\frac{5fL^2}{12AE},\frac{5fL^2}{12AE},\frac{5fL^2}{12AE}\right\}$$

Above, the problem has been solved three times with 1, 2, and 4 elements and displacements at the free end are given as a list (see the Mathematica notebook for the details).

EXAMPLE 4.4 The cross-sectional area of a bar is given by $A/A_0 = 1 - x/(2L)$. Assuming that the approximation of displacement u is (piecewise) linear, Young's modulus E and density ρ of the material are constants and distributed loading f_x is due to the gravity, determine the displacement at the free end of the bar. Use two elements of equal length.



Answer $u(L) = \dfrac{29}{70}\dfrac{g\rho L^2}{E}$ ... (exact $u(L) = \dfrac{3 - \log 4}{4}\dfrac{g\rho L^2}{E}$, error 2.7%)

- Element interpolants of displacement, cross-sectional area (in terms of its nodal values), and weight per unit length are here

$$u(x)=\frac{1}{h}\{h-x \quad x\}\begin{Bmatrix}u_{x1}\\u_{x2}\end{Bmatrix} \;\Rightarrow\; \frac{du}{dx}=\frac{1}{h}\{-1 \quad 1\}\begin{Bmatrix}u_{x1}\\u_{x2}\end{Bmatrix} \text{ and } \frac{d\delta u}{dx}=\frac{1}{h}\{-1 \quad 1\}\begin{Bmatrix}\delta u_{x1}\\\delta u_{x2}\end{Bmatrix},$$

$$A(x)=\frac{1}{h}\{h-x \quad x\}\begin{Bmatrix}A_1\\A_2\end{Bmatrix} \;\Rightarrow\; f_x=\rho g A=\frac{\rho g}{h}\{h-x \quad x\}\begin{Bmatrix}A_1\\A_2\end{Bmatrix}.$$

- Virtual works of internal and external forces per unit length of a bar are given by

$$\delta w_\Omega^{\text{int}}=-\begin{Bmatrix}\delta u_{x1}\\\delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\frac{EA}{h^2}\begin{Bmatrix}-1\\1\end{Bmatrix}\{-1 \quad 1\}\begin{Bmatrix}u_{x1}\\u_{x2}\end{Bmatrix}=-\begin{Bmatrix}\delta u_{x1}\\\delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\frac{EA}{h^2}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{Bmatrix}u_{x1}\\u_{x2}\end{Bmatrix} \;\Rightarrow$$

$$\delta w_\Omega^{\text{int}}=-\begin{Bmatrix}\delta u_{x1}\\\delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\frac{E}{h^3}\{h-x \quad x\}\begin{Bmatrix}A_1\\A_2\end{Bmatrix}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{Bmatrix}u_{x1}\\u_{x2}\end{Bmatrix},$$

$$\delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{1}{h^2} \begin{Bmatrix} h-x \\ x \end{Bmatrix} \begin{Bmatrix} h-x & x \end{Bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} \rho g \,.$$

- Element contribution of a typical element is obtained as integral over the domain occupied by the element

$$\delta W^{\text{int}} = \int_0^h \delta w_{\Omega}^{\text{int}} dx = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{E(A_1 + A_2)}{2h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix},$$

$$\delta W^{\text{ext}} = \int_0^h \delta w_{\Omega}^{\text{ext}} dx = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{\rho g h}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} \quad \Rightarrow$$

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{E}{h} \frac{A_1 + A_2}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{\rho g h}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} \right)$$

- Element contributions of the two elements

$$\delta W^1 = -\begin{Bmatrix} 0 \\ \delta u_{X2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{7EA_0}{4L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_{X2} \end{Bmatrix} - \frac{\rho g L A_0}{48}\begin{Bmatrix} 11 \\ 10 \end{Bmatrix}\right) \quad (A_1 = A_0,\ A_2 = \frac{3A_0}{4}),$$

$$\delta W^2 = -\begin{Bmatrix} \delta u_{X2} \\ \delta u_{X3} \end{Bmatrix}^{\mathrm{T}} \left(\frac{5EA_0}{4L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} - \frac{\rho g L A_0}{48}\begin{Bmatrix} 8 \\ 7 \end{Bmatrix}\right) \quad (A_1 = \frac{3A_0}{4},\ A_2 = \frac{A_0}{2}).$$

- Virtual work of the structure is the sum over elements $\delta W = \sum \delta W^e = \delta W^1 + \delta W^2$

$$\delta W = -\delta u_{X2}\left(\frac{7EA_0}{4L}u_{X2} - \frac{A_0 \rho g L}{48}10\right) - \begin{Bmatrix} \delta u_{X2} \\ \delta u_{X3} \end{Bmatrix}^{\mathrm{T}} \left(\frac{A_0}{4L}\begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} - \frac{\rho g L A_0}{48}\begin{Bmatrix} 8 \\ 7 \end{Bmatrix}\right)$$

$$\delta W = -\begin{Bmatrix} \delta u_{X2} \\ \delta u_{X3} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA_0}{4L}\begin{bmatrix} 12 & -5 \\ -5 & 5 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} - \frac{\rho g L A_0}{48}\begin{Bmatrix} 18 \\ 7 \end{Bmatrix}\right)$$

- Principle of virtual work and the fundamental lemma of variation calculus give

$$\delta W = -\begin{Bmatrix} \delta u_{X2} \\ \delta u_{X3} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA_0}{4L} \begin{bmatrix} 12 & -5 \\ -5 & 5 \end{bmatrix} \begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} - \frac{\rho g L A_0}{48} \begin{Bmatrix} 18 \\ 7 \end{Bmatrix} \right) = 0 \quad \forall \begin{Bmatrix} \delta u_{X2} \\ \delta u_{X3} \end{Bmatrix} \quad \Leftrightarrow$$

$$\frac{EA_0}{4L} \begin{bmatrix} 12 & -5 \\ -5 & 5 \end{bmatrix} \begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} - \frac{\rho g L A_0}{48} \begin{Bmatrix} 18 \\ 7 \end{Bmatrix} = 0 \quad \Leftrightarrow$$

$$\begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} = \frac{\rho g L^2}{12E} \begin{bmatrix} 12 & -5 \\ -5 & 5 \end{bmatrix}^{-1} \begin{Bmatrix} 18 \\ 7 \end{Bmatrix} = \frac{\rho g L^2}{12 \cdot 35E} \begin{bmatrix} 5 & 5 \\ 5 & 12 \end{bmatrix} \begin{Bmatrix} 18 \\ 7 \end{Bmatrix} = \frac{\rho g L^2}{E} \begin{Bmatrix} 25/84 \\ 29/70 \end{Bmatrix}. \leftarrow$$

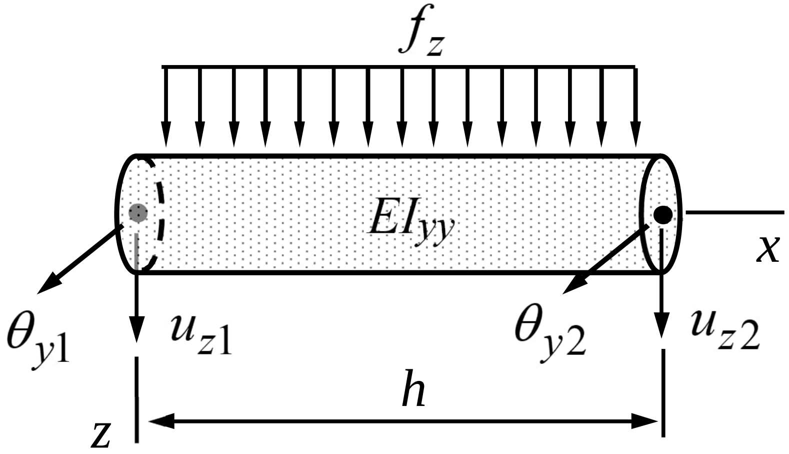
- In the Mathematica code of the course, the given quantities may vary linearly

	model	properties	geometry
1	BAR	$\{\{\mathbb{E}\},\{\{A0,\frac{3\,A0}{4}\}\},\{\{A0\,g\,\rho,\frac{3\,A0\,g\,\rho}{4}\},0,0\}\}$	Line[{1, 2}]
2	BAR	$\{\{\mathbb{E}\},\{\{\frac{3\,A0}{4},\frac{A0}{2}\}\},\{\{\frac{3\,A0\,g\,\rho}{4},\frac{A0\,g\,\rho}{2}\},0,0\}\}$	Line[{2, 3}]

	$\{X,Y,Z\}$	$\{u_X,u_Y,u_Z\}$	$\{\theta_X,\theta_Y,\theta_Z\}$
1	$\{0,0,0\}$	$\{0,0,0\}$	$\{0,0,0\}$
2	$\{\frac{L}{2},0,0\}$	$\{uX[2],0,0\}$	$\{0,0,0\}$
3	$\{L,0,0\}$	$\{uX[3],0,0\}$	$\{0,0,0\}$

$$\left\{uX[2]\to\frac{25\,g\,L^2\,\rho}{84\,\mathbb{E}},\ uX[3]\to\frac{29\,g\,L^2\,\rho}{70\,\mathbb{E}}\right\}$$

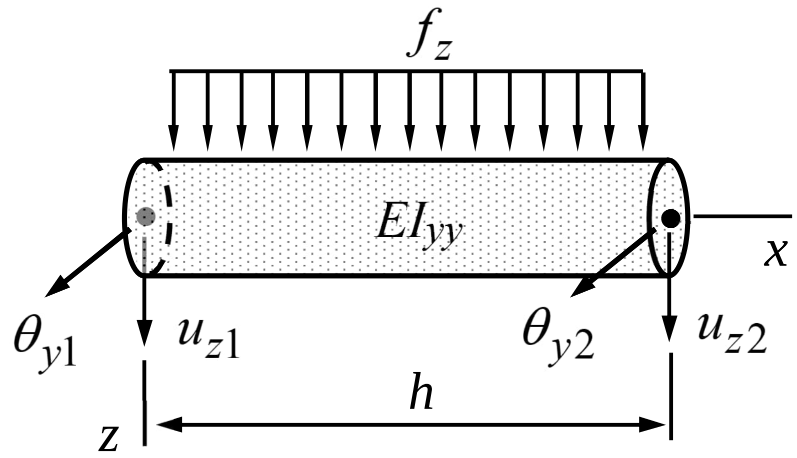
4.3 BENDING MODE



Virtual work density: $\delta w_\Omega = \delta w_\Omega^{\text{int}} + \delta w_\Omega^{\text{ext}} = -\dfrac{d^2 \delta w}{dx^2} EI_{yy} \dfrac{d^2 w}{dx^2} + \delta w f_z$

Cubic interpolant: $w(x) = \begin{Bmatrix} (1-\xi)^2(1+2\xi) \\ -h(1-\xi)^2\xi \\ \hdashline (3-2\xi)\xi^2 \\ -h\xi^2(\xi-1) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ \hdashline u_{z2} \\ \theta_{y2} \end{Bmatrix}$ where $\xi = \dfrac{x}{h}$.

BEAM BENDING ELEMENT



$$\delta W = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \hdashline \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EI_{yy}}{h^3} \left[\begin{array}{cc:cc} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ \hdashline -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{array} \right] \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ \hdashline u_{z2} \\ \theta_{y2} \end{Bmatrix} - \frac{f_z h}{12} \begin{Bmatrix} 6 \\ -h \\ \hdashline 6 \\ h \end{Bmatrix} \right), \quad u_z = \mathbf{k}^{\mathrm{T}} \begin{Bmatrix} u_X \\ u_Y \\ u_Z \end{Bmatrix} \text{ etc.}$$

Above, f_z and EI_{yy} are assumed to be constants and the elements of matrices **i**, **j** and **k** **(** 1×1, 2×1, 3×1**)** are the components of the unit vectors $\vec{i}, \vec{j}$ and $\vec{k}$ in the structural coordinate system.

- First, element interpolant (approximation) $w = \mathbf{N}^{\mathrm{T}}\mathbf{a}$ and its variation $\delta w = \delta\mathbf{a}^{\mathrm{T}}\mathbf{N}$ are substituted into the virtual work expression to get

$$\delta W = \int_0^h \left(-\frac{d^2\delta w}{dx^2} EI_{yy} \frac{d^2 w}{dx^2} + \delta w f_z\right) dx \qquad \Rightarrow$$

$$\delta W = -\int_0^h \delta\mathbf{a}^{\mathrm{T}} \frac{d^2\mathbf{N}}{dx^2} EI_{yy} \frac{d^2\mathbf{N}^{\mathrm{T}}}{dx^2} \mathbf{a} dx + \int_0^h \delta\mathbf{a}^{\mathrm{T}} \mathbf{N} f_z dx \qquad \Leftrightarrow$$

$$\delta W = -\delta\mathbf{a}^{\mathrm{T}} \left(\int_0^h \frac{d^2\mathbf{N}}{dx^2} EI_{yy} \frac{d^2\mathbf{N}^{\mathrm{T}}}{dx^2} dx\, \mathbf{a} - \int_0^h \mathbf{N} f_z dx\right). \qquad \Leftarrow$$

- The shape function expressions and their second derivatives are (Mathematica is useful in the calculations)

$$\mathbf{N}=\begin{Bmatrix}(1-x/h)^2(1+2x/h)\\ -h(1-x/h)^2x/h\\ \hdashline (3-2x/h)(x/h)^2\\ -h(x/h)^2(x/h-1)\end{Bmatrix} \text{ and } \frac{d^2\mathbf{N}}{dx^2}=\frac{1}{h^2}\begin{Bmatrix}6(2x/h-1)\\ -2(3x/h-2)h\\ \hdashline 6(1-2x/h)\\ -(3x/h-1)h\end{Bmatrix}.$$

- In the next step, the shape function expressions are substituted into the virtual work expression. Integration over the domain occupied by the element gives the element contribution. A derivation along these lines is valid also when the given functions are not constants!

EXAMPLE 4.5 The integral representation of *equivalent nodal forces* is $\mathbf{F}=\int_{\Omega^e}\mathbf{N}f_z d\Omega$. Determine the equivalent nodal forces of a beam element for (a) $f_z=\text{const.}$, (b) $f_z = f_{z1}(1-\xi)+\xi f_{z2}$, and (c) $f_z=F_z\delta(x-h/2)$ (Dirac delta at the midpoint), when

$$\mathbf{N}=\left\{(1-\xi)^2(1+2\xi) \;\; -h(1-\xi)^2\xi \;\; (3-2\xi)\xi^2 \;\; -h\xi^2(\xi-1)\right\}^{\mathrm{T}}$$

Answer $\mathbf{F}=\dfrac{hf_z}{12}\begin{Bmatrix} 6 \\ -h \\ 6 \\ h \end{Bmatrix}$, $\mathbf{F}=\dfrac{hf_{z1}}{60}\begin{Bmatrix} 21 \\ -3h \\ 9 \\ 2h \end{Bmatrix}+\dfrac{hf_{z2}}{60}\begin{Bmatrix} 9 \\ -2h \\ 21 \\ 3h \end{Bmatrix}$, and $\mathbf{F}=\dfrac{F_z}{8}\begin{Bmatrix} 4 \\ -h \\ 4 \\ h \end{Bmatrix}$

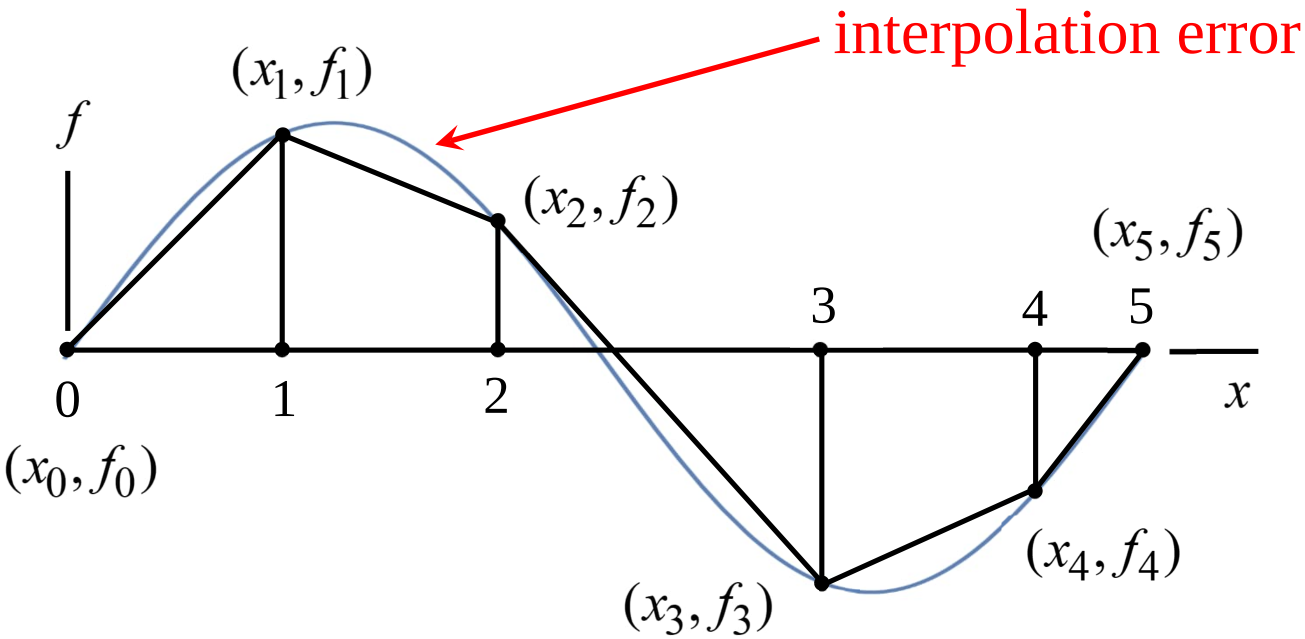
- $$\mathbf{F} = \int_0^h \begin{Bmatrix} (1-x/h)^2(1+2x/h) \\ -h(1-x/h)^2 x/h \\ \hdashline (3-2x/h)(x/h)^2 \\ -h(x/h)^2(x/h-1) \end{Bmatrix} f_z\,dx = \frac{hf_z}{12}\begin{Bmatrix} 6 \\ -h \\ \hdashline 6 \\ h \end{Bmatrix} \Leftarrow$$

- $$\mathbf{F} = \int_0^h \begin{Bmatrix} (1-x/h)^2(1+2x/h) \\ -h(1-x/h)^2 x/h \\ \hdashline (3-2x/h)(x/h)^2 \\ -h(x/h)^2(x/h-1) \end{Bmatrix} [(1-\frac{x}{h}) f_{z1} + \frac{x}{h} f_{z2}]dx = \frac{hf_{z1}}{60}\begin{Bmatrix} 21 \\ -3h \\ 9 \\ 2h \end{Bmatrix} + \frac{hf_{z2}}{60}\begin{Bmatrix} 9 \\ -2h \\ 21 \\ 3h \end{Bmatrix} \Leftarrow$$

- $$\mathbf{F} = \int_0^h \begin{Bmatrix} (1-x/h)^2(1+2x/h) \\ -h(1-x/h)^2 x/h \\ \hdashline (3-2x/h)(x/h)^2 \\ -h(x/h)^2(x/h-1) \end{Bmatrix} F_z \delta(x-\frac{h}{2})dx = \frac{F_z}{8}\begin{Bmatrix} 4 \\ -h \\ 4 \\ h \end{Bmatrix} \Leftarrow$$

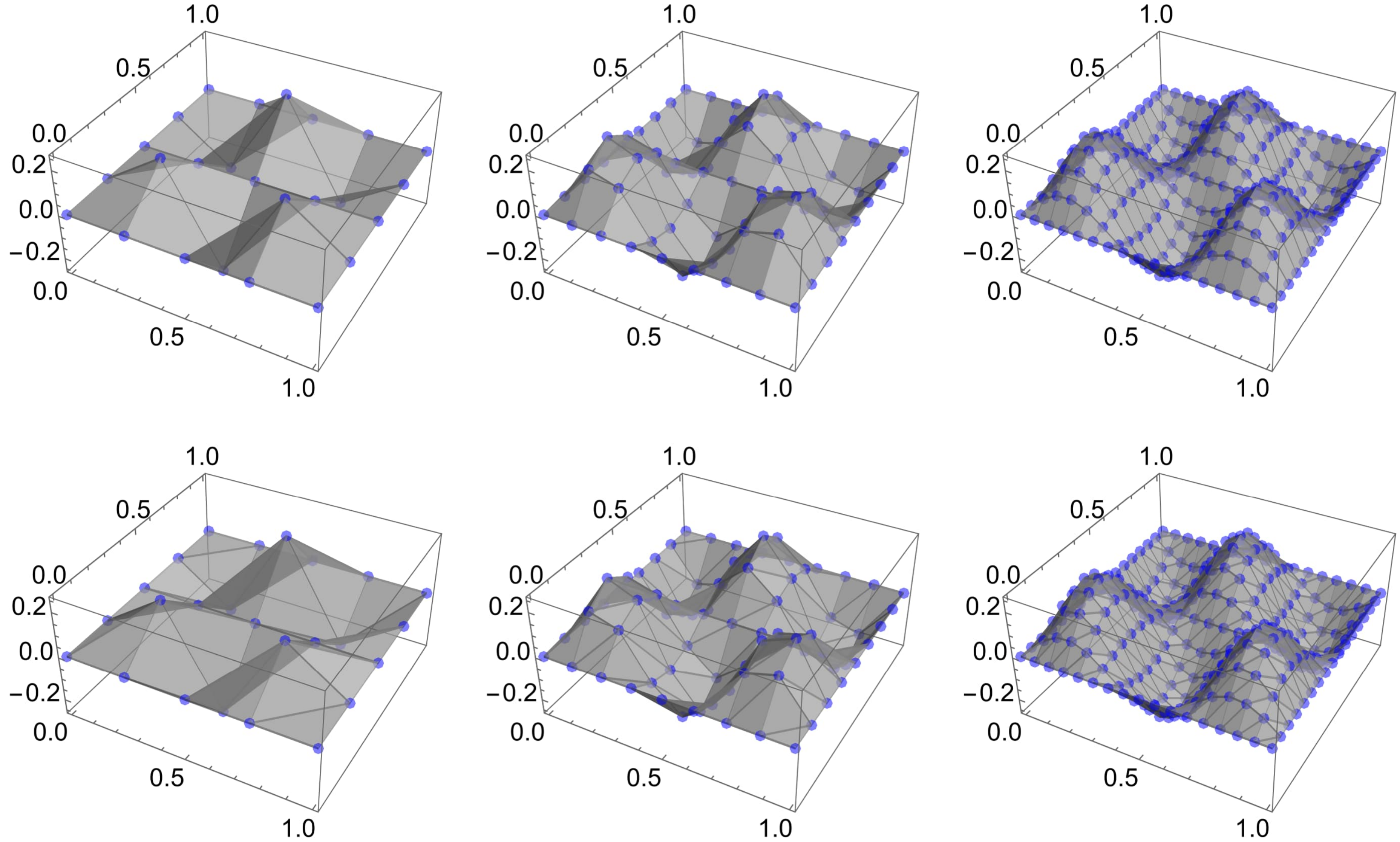
4.4 INTERPOLATION

Piecewise linear interpolant to nodal values $\{(x_0, f_0),(x_1, f_1),\ldots,(x_n, f_n)\}$ gives the simplest continuous polynomial approximation to $f(x)$.

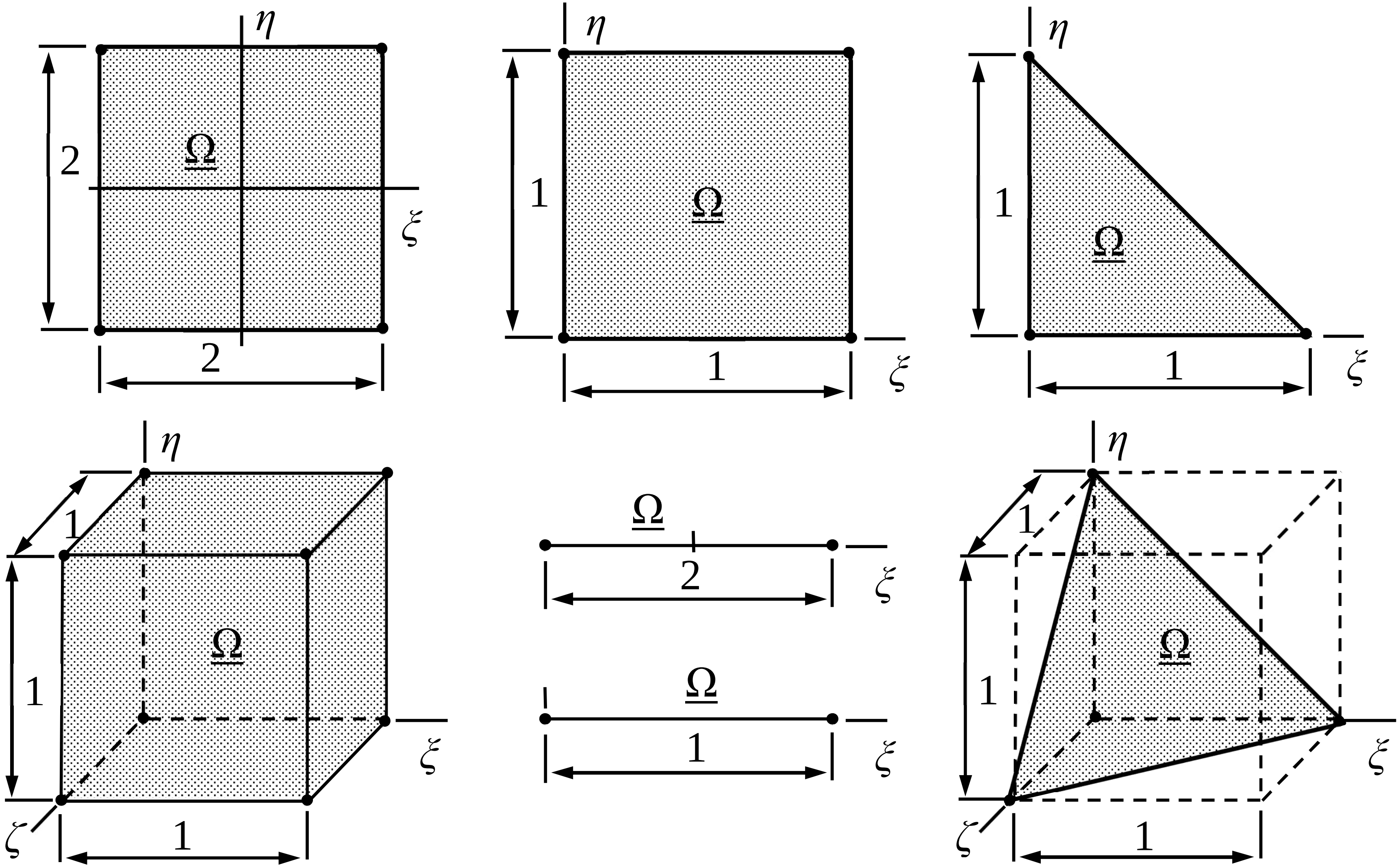


Interpolation with piecewise linear polynomials extends straightforwardly to more dimensions, higher order polynomials, and divisions of the domain into elements.

EXAMPLE 4.6 Interpolants of $f(x,y)/F = \sin(2\pi x/L)\sin(\pi y/L)/4$ on square domain $(x,y)/L \in [0,1]\times[0,1]$ with triangle and rectangle elements of increasing number.



ELEMENTS



SHAPE FUNCTIONS

Shape functions are used to interpolate the nodal values inside the elements. The shape function N_i of node i in element Ω^e

- ☐ is the lowest order polynomial taking the value 1 at node i and the value 0 at all the other nodes of the element.
- ☐ shape functions should satisfy the previous condition on each edge (as an example, shape function should be linear on an edge of two nodes)
- ☐ Sum of the shape functions of an element should be 1.

The shape functions can often be deduced directly by using the conditions above or/and by using the Lagrange interpolation polynomials.

Lagrange interpolation polynomial $p_n(x)$ of degree n and its error formula are for dataset $\{(x_0, f_0),(x_1, f_1),\ldots(x_n, f_n)\}$

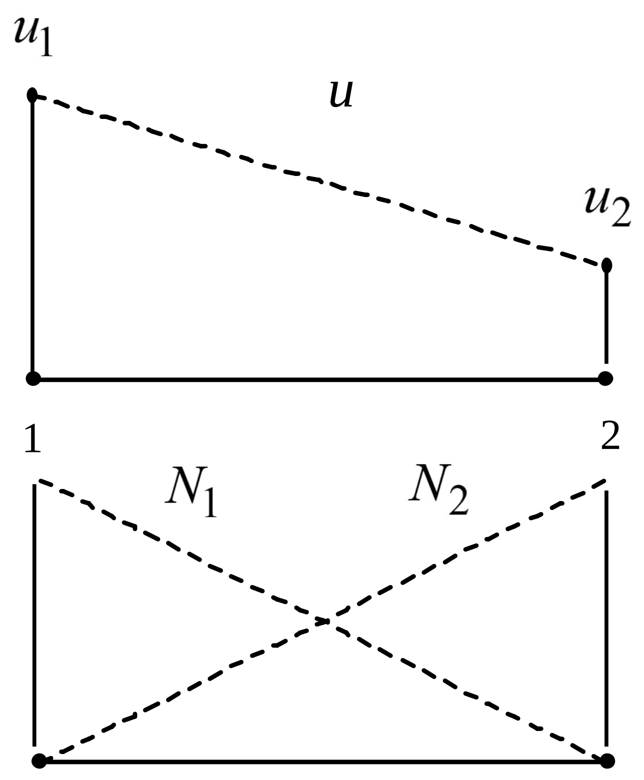
$$p_n(x) = \sum_{i\in\{0,1,\ldots,n\}} f_i \Pi_{j\in\{0,1,\ldots i-1,i+1,\ldots,n\}} \frac{x - x_j}{x_i - x_j},$$

$$f(x) - p_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi)\Pi_{i\in\{0,1,\ldots,n\}}(x - x_i) \ .$$

Notice the removal of index i in the product term inside the sum of the interpolation formula.

LINEAR SHAPE FUNCTIONS

Piecewise linear approximation in one dimension is continuous in Ω and a first order polynomial inside the elements. In element Ω^e



Approximation: $u = \mathbf{N}^{\mathrm{T}}\mathbf{a}$

Nodal values: $\mathbf{a} = \{u_1 \;\; u_2\}^{\mathrm{T}}$

Shape functions: $\mathbf{N} = \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} = \begin{Bmatrix} 1-\xi \\ \xi \end{Bmatrix}$ where $\xi = \dfrac{x}{h}$

Piecewise linear approximation is the simplest choice e.g. for the bar model.

- The method based on combining given polynomials gives (use of the scaled coordinate ξ simplifies the expressions)

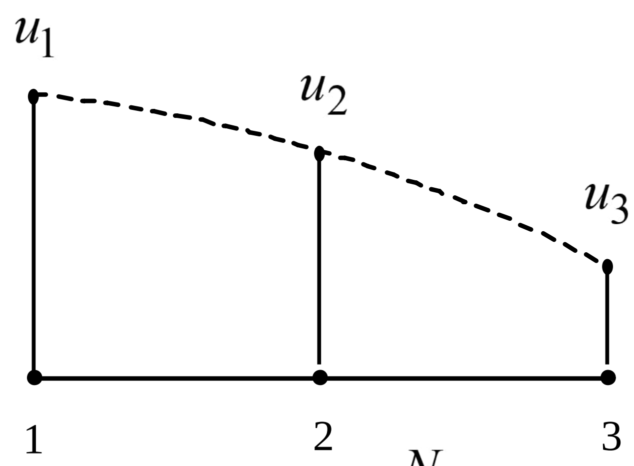
$$\mathbf{N}=\begin{bmatrix}1 & 1\\ x_1 & x_2\end{bmatrix}^{-1}\begin{Bmatrix}1\\ x\end{Bmatrix}=\frac{1}{h}\begin{bmatrix}x_2 & -1\\ -x_1 & 1\end{bmatrix}\begin{Bmatrix}1\\ x\end{Bmatrix}=\frac{1}{h}\begin{Bmatrix}x_2-x\\ x-x_1\end{Bmatrix}=\begin{Bmatrix}1-\xi\\ \xi\end{Bmatrix},\quad \text{where}\quad \xi=\frac{x}{h}$$

- The virtual work expression (e.g. of the bar model) contains integrals of the shape functions in certain combinations. The most common are (here $\Omega^e=]x_1,x_2[$, $d\Omega=dx$, and $h=|x_2-x_1|$)

$$\int_{\Omega^e}\mathbf{N}d\Omega=\frac{h}{2}\begin{Bmatrix}1\\ 1\end{Bmatrix},\quad \int_{\Omega^e}\mathbf{N}\mathbf{N}^{\mathrm{T}}d\Omega=\frac{h}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix},\quad \text{and}\quad \int_{\Omega^e}\frac{d\mathbf{N}}{dx}\frac{d\mathbf{N}}{dx}^{\mathrm{T}}d\Omega=\frac{1}{h}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}.$$

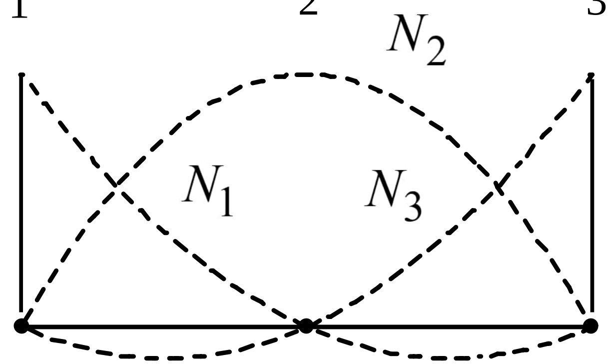
QUADRATIC SHAPE FUNCTIONS

Piecewise quadratic approximation in one dimension is continuous in Ω and a second order polynomial inside the elements. In element Ω^e



Approximation: $u = \mathbf{N}^{\mathrm{T}}\mathbf{a}$

Nodal values: $\mathbf{a} = \{u_1 \;\; u_2 \;\; u_3\}^{\mathrm{T}}$



Shape functions: $\mathbf{N} = \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} = \begin{Bmatrix} 1-3\xi+2\xi^2 \\ 4\xi(1-\xi) \\ \xi(2\xi-1) \end{Bmatrix}, \quad \xi = \dfrac{x}{h}$

More nodes can be used to generate higher order approximations!

- Derivation based on the Lagrange interpolation polynomials is convenient in the one-dimensional case. The idea is to write a polynomial vanishing on some set of points and scale the expression to take the value one at a certain point. In terms of $\xi = x/h$

$$N_1 = \frac{(\xi - 1/2)(\xi - 1)}{(0-1/2)(0-1)} = (2\xi - 1)(\xi - 1) \quad \text{and} \quad N_2 = \frac{(\xi - 0)(\xi - 1)}{(1/2-0)(1/2-1)} = 4\xi(1-\xi) \quad \text{etc.}$$

- Some integrals of the virtual work expression are given by

$$\int_0^h \mathbf{N}dx = \frac{h}{6}\begin{Bmatrix}1\\4\\1\end{Bmatrix}, \quad \int_0^h \mathbf{N}\mathbf{N}^{\mathrm{T}}dx = \frac{h}{30}\begin{bmatrix}4 & 2 & -1\\2 & 16 & 2\\-1 & 2 & 4\end{bmatrix}, \quad \int_0^h \frac{d\mathbf{N}}{dx}\frac{d\mathbf{N}}{dx}^{\mathrm{T}}dx = \frac{1}{3h}\begin{bmatrix}7 & -8 & 1\\-8 & 16 & -8\\1 & -8 & 7\end{bmatrix}.$$

EXAMPLE 4.7 Find the virtual work expression $\delta W = \int_{\Omega} \delta w d\Omega$ of a bar element, when $\delta w_{\Omega} = -(d\delta u / dx)EA(du / dx) + \delta u f_x$, the shape functions are quadratic (a three-node element) and the force per unit length is (a) $f_x = \text{constant}$ (b) $f_x = F_x \delta(\xi - 1/2)$. The length of the element is h.

Answer:

(a) $$\delta W = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \\ \delta u_{x3} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{3h}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \\ u_{x3} \end{Bmatrix} - \frac{f_x h}{6}\begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix}\right)$$

(b) $$\delta W = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \\ \delta u_{x3} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{3h}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \\ u_{x3} \end{Bmatrix} - F_x \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}\right)$$

- The quadratic shape functions of a three-node element can be obtained e.g. by using the Lagrange interpolation polynomials ($\xi = x/h$)

$$\mathbf{N} = \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} = \begin{Bmatrix} 1-3x/h+2(x/h)^2 \\ 4x/h-4(x/h)^2 \\ 2(x/h)^2 - x/h \end{Bmatrix} \quad \Rightarrow \quad \frac{d\mathbf{N}}{dx} = \begin{Bmatrix} dN_1/dx \\ dN_2/dx \\ dN_3/dx \end{Bmatrix} = \frac{1}{h}\begin{Bmatrix} -3+4(x/h) \\ 4-8(x/h) \\ 4(x/h)-1 \end{Bmatrix}.$$

- Approximation, its derivative and variations needed in the virtual work density are

$$\frac{du}{dx} = \begin{Bmatrix} dN_1/dx \\ dN_2/dx \\ dN_3/dx \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{x1} \\ u_{x2} \\ u_{x3} \end{Bmatrix}, \quad \frac{d\delta u}{dx} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \\ \delta u_{x3} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} dN_1/dx \\ dN_2/dx \\ dN_3/dx \end{Bmatrix}, \text{ and } \delta u = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \\ \delta u_{x3} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix}.$$

- When the approximation is substituted there, virtual work density takes the form

$$\delta w = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \\ \delta u_{x3} \end{Bmatrix}^{\mathrm{T}} (EA \begin{bmatrix} \frac{dN_1}{dx}\frac{dN_1}{dx} & \frac{dN_1}{dx}\frac{dN_2}{dx} & \frac{dN_1}{dx}\frac{dN_3}{dx} \\ \frac{dN_2}{dx}\frac{dN_1}{dx} & \frac{dN_2}{dx}\frac{dN_2}{dx} & \frac{dN_2}{dx}\frac{dN_3}{dx} \\ \frac{dN_3}{dx}\frac{dN_1}{dx} & \frac{dN_3}{dx}\frac{dN_2}{dx} & \frac{dN_3}{dx}\frac{dN_3}{dx} \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \\ u_{x3} \end{Bmatrix} - \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} f_x)$$

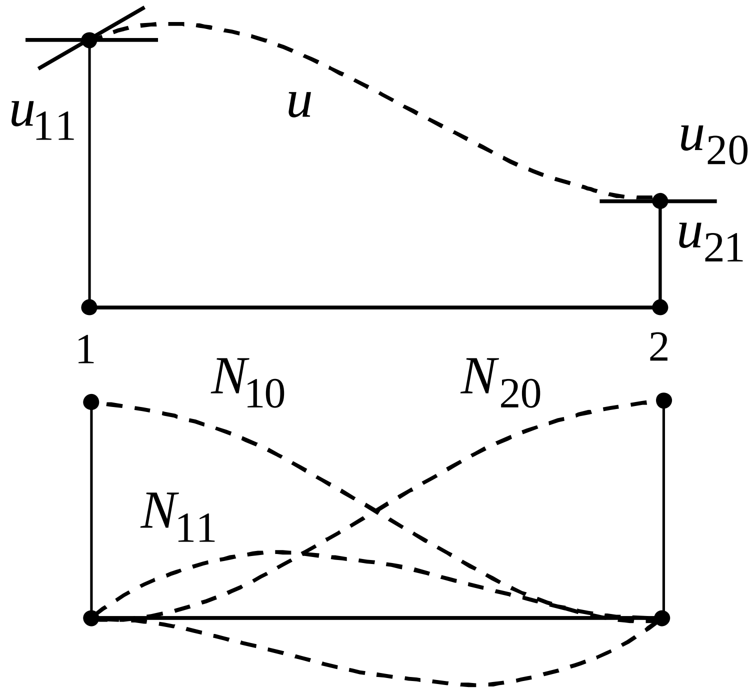
- Virtual work of the external volume force is given by integral $\delta W = \int_0^h \delta w dx$. If $f_x = \text{constant}$ or $f_x = F_x \delta(\xi - 1/2)$, the outcome is

$$\delta W = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \\ \delta u_{x3} \end{Bmatrix}^{\mathrm{T}} (\frac{EA}{3h} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \\ u_{x3} \end{Bmatrix} - \frac{f_x h}{6} \begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix}) \leftarrow$$

$$\delta W = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \\ \delta u_{x3} \end{Bmatrix}^{\mathrm{T}} (\frac{EA}{3h} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \\ u_{x3} \end{Bmatrix} - F_x \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}) \leftarrow$$

CUBIC SHAPE FUNCTIONS

Piecewise cubic approximation has continuous derivatives up to the first order in Ω and is a third order polynomial inside the elements.



Approximation: $u = \mathbf{N}^{\mathrm{T}}\mathbf{a}$

Nodal values: $\mathbf{a} = \{u_1 \;\; du_1/dx \mid u_2 \;\; du_2/dx\}$

Shape functions: $\mathbf{N} = \begin{Bmatrix} N_{10} \\ N_{11} \\ \hdashline N_{20} \\ N_{21} \end{Bmatrix} = \begin{Bmatrix} (1-\xi)^2(1+2\xi) \\ h(1-\xi)^2\xi \\ \hdashline (3-2\xi)\xi^2 \\ h\xi^2(\xi-1) \end{Bmatrix}$

In xz −plane bending $u = u_z$ and $du/dx = -\theta_y$, in xy −plane $u = u_y$ and $du/dx = \theta_z$.

- In one-dimensional case, the brute force approach works. Let us collect the coefficients of the monomials of the shape functions into a matrix, and use the definition of the shape functions

$$\begin{Bmatrix} N_{10} \\ N_{11} \\ N_{20} \\ N_{21} \end{Bmatrix} = [A] \begin{Bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{Bmatrix} \quad \Rightarrow \quad \left[\begin{array}{c:c:c:c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] = [A] \left[\begin{array}{c:c:c:c} 1 & 0 & 1 & 0 \\ 0 & 1 & h & 1 \\ 0 & 0 & h^2 & 2h \\ 0 & 0 & h^3 & 3h^2 \end{array}\right] \Rightarrow$$

$$\begin{Bmatrix} N_{10} \\ N_{11} \\ N_{20} \\ N_{21} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & h & 1 \\ 0 & 0 & h^2 & 2h \\ 0 & 0 & h^3 & 3h^2 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{Bmatrix} = \begin{Bmatrix} (-1+\xi)^2(1+2\xi) \\ h(-1+\xi)^2\xi \\ (3-2\xi)\xi^2 \\ h(-1+\xi)\xi^2 \end{Bmatrix}, \text{ where } \quad \xi = \frac{x}{h}.$$

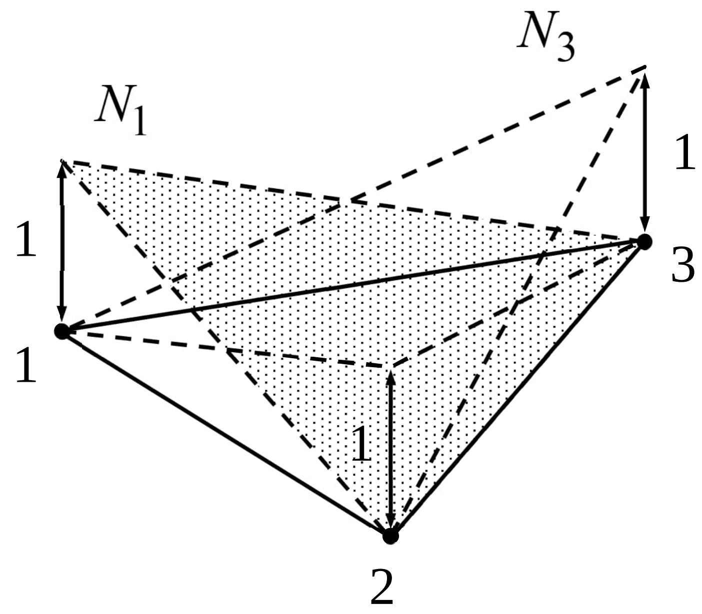
LINEAR SHAPE FUNCTIONS

Piecewise linear approximation in two-dimension is continuous in Ω and linear inside the elements of triangle shape. In element Ω^e

Approximation: $u = \mathbf{N}^{\mathrm{T}}\mathbf{a}$

Nodal values: $\mathbf{a} = \{u_1 \;\; u_2 \;\; u_3\}^{\mathrm{T}}$

Shape functions: $\mathbf{N} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}$



Triangle element is the simplest element in two dimensions. Division of any 2D domain into triangles is always possible, which makes the element quite useful.

- Let $\mathbf{N}=\{N_1 \;\; N_2 \;\; N_3\}^{\mathrm{T}}$ be the shape functions taking the value one at the vertices $\mathbf{x}_1=(x_1,y_1)$, $\mathbf{x}_2=(x_2,y_2)$ and $\mathbf{x}_3=(x_3,y_3)$, respectively. Then

$$\mathbf{N}=\begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}^{-1}\begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}=\begin{Bmatrix} \dfrac{x_2 y - x_3 y - x y_2 + x_3 y_2 + x y_3 - x_2 y_3}{x_2 y_1 - x_3 y_1 - x_1 y_2 + x_3 y_2 + x_1 y_3 - x_2 y_3} \\ \dfrac{-x_1 y + x_3 y + x y_1 - x_3 y_1 - x y_3 + x_1 y_3}{x_2 y_1 - x_3 y_1 - x_1 y_2 + x_3 y_2 + x_1 y_3 - x_2 y_3} \\ \dfrac{x_1 y - x_2 y - x y_1 + x_2 y_1 + x y_2 - x_1 y_2}{x_2 y_1 - x_3 y_1 - x_1 y_2 + x_3 y_2 + x_1 y_3 - x_2 y_3} \end{Bmatrix}. \;\; \Leftarrow$$

- Some integrals needed, e.g., in the virtual work expression of the thin slab model, are

$$\int_{\Omega^e} \mathbf{N}d\Omega=\frac{A}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}, \;\; \int_{\Omega^e} \mathbf{N}\mathbf{N}^{\mathrm{T}}d\Omega=\frac{A}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

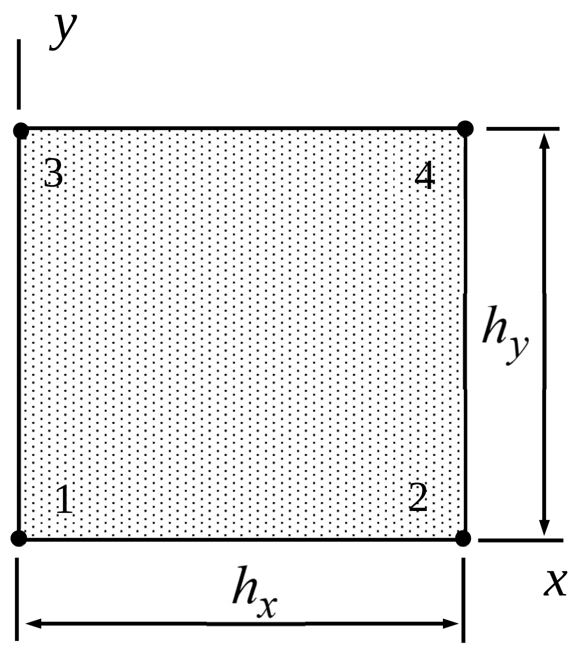
BI-LINEAR SHAPE FUNCTIONS

Bilinear approximation in two dimensions is continuous on Ω and linear with respect to both coordinates inside the elements of rectangular shape. In element Ω^e and notation $\xi = x / h_x$, $\eta = y / h_y$

Approximation: $u = \mathbf{N}^{\mathrm{T}}\mathbf{a}$

Nodal values: $\mathbf{a} = \{u_1 \ \ u_2 \ \ u_3 \ \ u_4\}^{\mathrm{T}}$

Shape functions: $\mathbf{N} = \begin{Bmatrix} (1-\xi)(1-\eta) \\ \xi(1-\eta) \\ (1-\xi)\eta \\ \xi\eta \end{Bmatrix}$

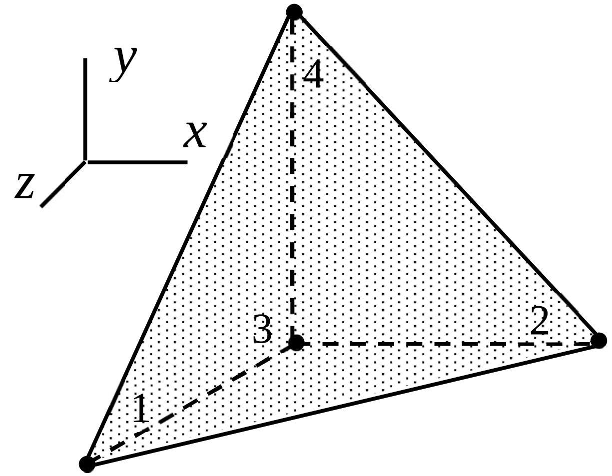


The ordering of the node numbers varies in literature.

LINEAR SHAPE FUNCTIONS

Piecewise linear approximation in three dimensions is continuous in Ω and a linear polynomial inside the tetrahedron elements. In a typical element Ω^e

Shape functions: $$\mathbf{N} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \\ y \\ z \end{Bmatrix}$$

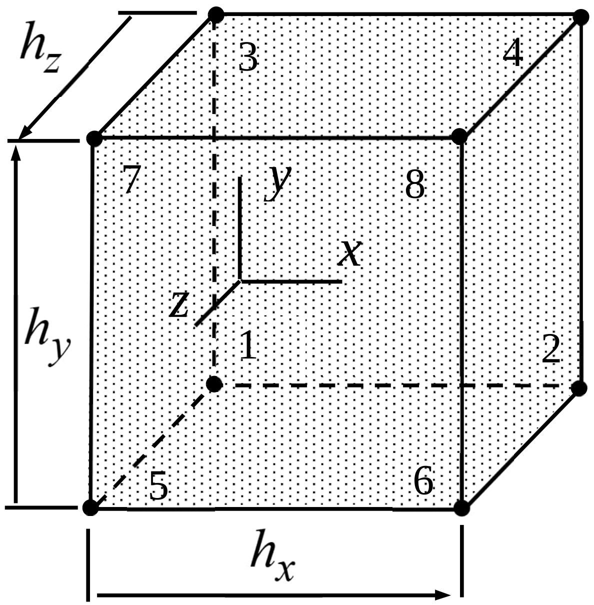


Tetrahedron is the simplest element in three dimensions. Division of any 3D domain into tetrahedrons is always possible, which makes also this element quite useful in practice.

TRI-LINEAR SHAPE FUNCTIONS FOR A BODY

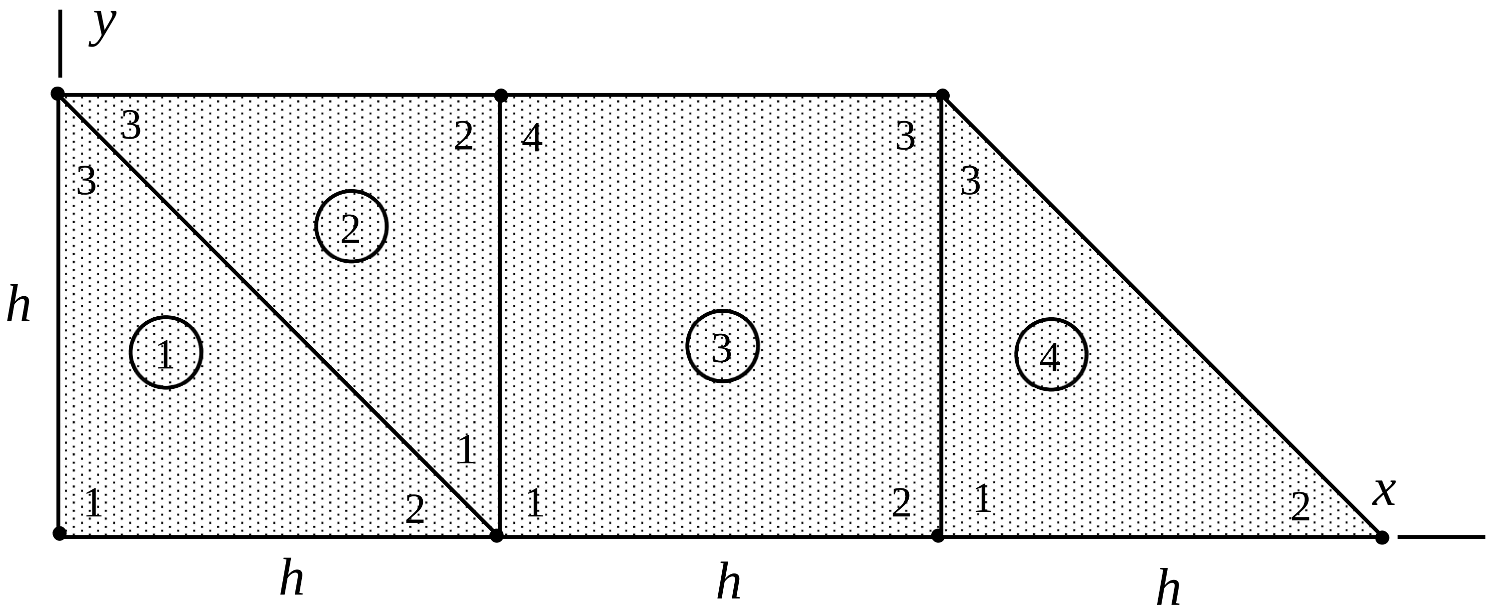
Approximation is continuous on Ω and tri-linear inside an element. In a typical element and with notations $\xi = (x - x_1)/h_x$, $\eta = (y - y_1)/h_y$, and $\zeta = (z - z_1)/h_z$,

$$\mathbf{N} = \begin{Bmatrix} (1-\xi)(1-\eta)(1-\zeta) \\ \xi(1-\eta)(1-\zeta) \\ (1-\xi)\eta(1-\zeta) \\ \xi\eta(1-\zeta) \\ (1-\xi)(1-\eta)\zeta \\ \xi(1-\eta)\zeta \\ (1-\xi)\eta\zeta \\ \xi\eta\zeta \end{Bmatrix}$$



Bi-linear and tri-linear shape functions of 2D and 3D cases are products of the linear shape functions of the 1D case.

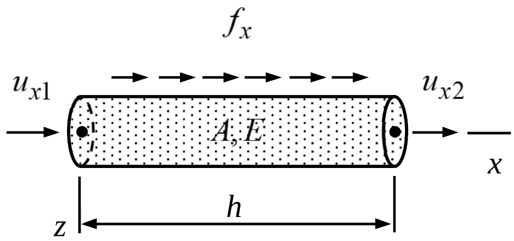
EXAMPLE Consider the structure of the figure consisting of triangle and quadrilateral elements. Write down the shape functions of the elements in the $xy-$coordinates (the sum of the shape functions of an element is always 1).



Answer $\mathbf{N}^1 = \frac{1}{h}\begin{Bmatrix} h-x-y \\ x \\ y \end{Bmatrix}$, $\mathbf{N}^2 = \frac{1}{h}\begin{Bmatrix} h-y \\ x+y-h \\ h-x \end{Bmatrix}$, $\mathbf{N}^3 = \frac{1}{h^2}\begin{Bmatrix} (2h-x)(h-y) \\ (x-h)(h-y) \\ (x-h)y \\ (2h-x)y \end{Bmatrix}$, $\mathbf{N}^4 = ?$

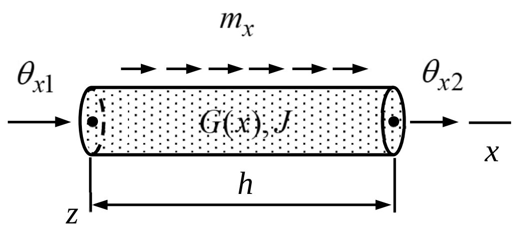
MEC-E1050 Finite Element Method in Solids, week 46/2024

1. Consider a bar element when A and E are constants and $f_x = (1 - x/h) f_{x1} + (x/h) f_{x2}$ is the linear distributed force. Derive the virtual work expression of linear bar element. Use the virtual work density expression $\delta w_\Omega = -(d\delta u/dx)EA(du/dx) + \delta u f_x$ and approximation $u = (1 - x/h)u_{x1} + (x/h)u_{x2}$.



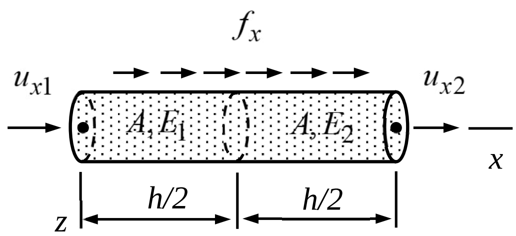
Answer $\delta W = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{h}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} f_{x1} \\ f_{x2} \end{Bmatrix} \right)$

2. Derive the virtual work expression of a torsion bar, when J and m_x are constants and shear modulus G is linear and defined by the nodal values G_1 and G_2. Use approximation $\phi = (1 - x/h)\theta_{x1} + (x/h)\theta_{x2}$. Virtual work density of the torsion bar model is $\delta w_\Omega = -(d\delta\phi/dx)GJ(d\phi/dx) + \delta\phi m_x$.



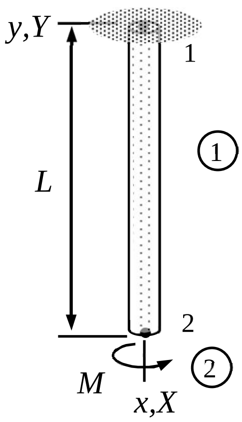
Answer $\delta W = -\begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{G_1 + G_2}{2} \frac{J}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix} - \frac{m_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \right)$

3. Consider a bar element having constant A and f_x and a piecewise constant E as shown in the figure. Derive the virtual work expression of the element by using the virtual work density expression $\delta w_\Omega = -(d\delta u/dx)EA(du/dx) + \delta u f_x$ of the bar model and interpolant $u = (1 - x/h)u_{x1} + (x/h)u_{x2}$ to the nodal displacements.



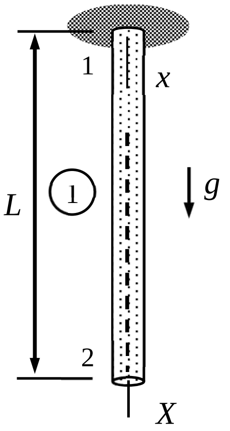
Answer $\delta W = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{E_1 + E_2}{2} \frac{A}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \right)$

4. Consider the torsion bar (1) of the figure loaded by torque M (2) acting on the free end. Determine the rotation θ_{X2} at the free end, if the polar moment J is constant and shear modulus G varies linearly so that the values at the nodes are G_1 and G_2. Start with the virtual work density $\delta w_\Omega = -(d\delta\phi/dx)GJ(d\phi/dx) + \delta\phi m_x$ and use linear approximation to rotation (a linear two-node element).



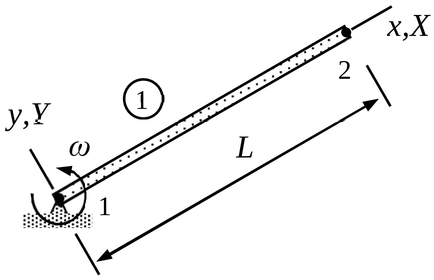
Answer $\theta_{X2} = -2\frac{ML}{(G_1 + G_2)J}$

5. Consider a bar of length L loaded by its own weight (figure). Determine the displacement u_{X2} at the free end. Start with the virtual work density expression $\delta w_{\Omega} = -(d\delta u/dx)EA(du/dx) + \delta u f_x$ and approximation $u = (1 - x/L)u_{x1} + (x/L)u_{x2}$. Cross-sectional area A, acceleration by gravity g, and material properties E and ρ are constants.



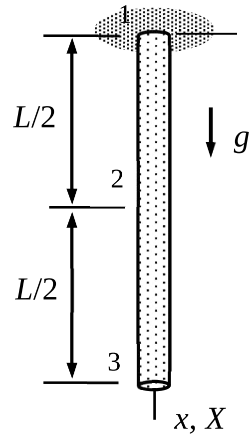
Answer $u_{X2} = \dfrac{\rho g L^2}{2E}$

6. Structural coordinate system and the bar shown are rotating in a plane with a constant angular speed $\omega_Z = \omega$. Material properties E, ρ and the cross-sectional area A are constants. Determine the nodal displacement u_{X2} at the free end using just one linear element. The *volume force* due to the rotation is given by $\vec{f} = -\rho\vec{a} = -\rho\vec{\omega}\times(\vec{\omega}\times\vec{r})$ in which $\vec{\omega} = \omega\vec{k}$ and $\vec{r} = x\vec{i}$.



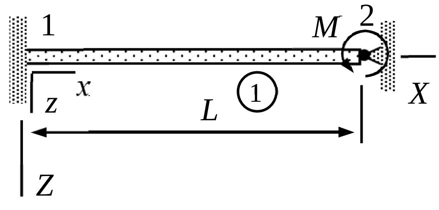
Answer $u_{X2} = \dfrac{1}{3}\dfrac{\rho\omega^2 L^3}{E}$

7. Consider the bar of the figure loaded by its own weight. Determine the displacement of the free end with one element. Use virtual work density expression $\delta w_{\Omega} = -(d\delta u/dx)EA(du/dx) + \delta u f_x$ and quadratic approximation $u = (1 - 3\xi + 2\xi^2)u_{x1} + 4\xi(1-\xi)u_{x2} + \xi(2\xi - 1)u_{x3}$ in which $\xi = x/L$. Cross-sectional area of the bar A, acceleration by gravity g, and material properties E and ρ are constants.



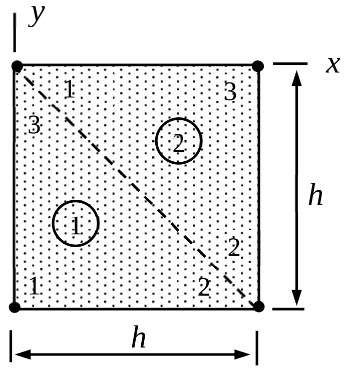
Answer $u_{X3} = \dfrac{1}{2}\dfrac{\rho g L^2}{E}$

8. Determine the rotation θ_{Y2} of the beam shown at the support of the right end which allows rotation but not transverse displacement. Young's modulus E of the material and second moment of cross-section $I_{yy} = I$ are constants. Use the virtual work density of beam bending mode $\delta w_{\Omega} = -(d^2\delta w/dx^2)EI_{yy}(d^2w/dx^2) + \delta w f_z$ and cubic approximation to the transverse displacement.



Answer $\theta_{Y2} = \dfrac{1}{4}\dfrac{ML}{EI}$

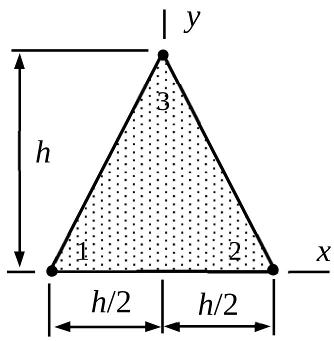
9. Deduce the shape functions of the triangle elements 1 and 2 shown in the figure in terms of the (material) xy – coordinates.



Answer $\mathbf{N}^1 = \frac{1}{h}\begin{Bmatrix} -y-x \\ x \\ h+y \end{Bmatrix}$ and $\mathbf{N}^2 = \frac{1}{h}\begin{Bmatrix} h-x \\ -y \\ x+y \end{Bmatrix}$

10. Using a linear interpolant to the nodal values, determine

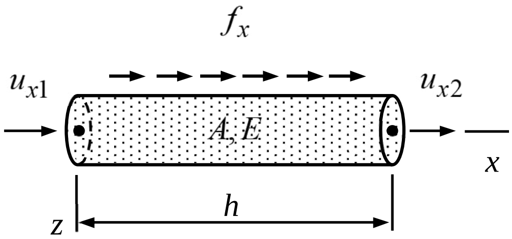
$$u_x, \ \frac{\partial u_x}{\partial x}, \ \frac{\partial u_x}{\partial y}, \ \text{and} \ I = \int_{\Omega^e} u_x d\Omega,$$



for the element shown. The nodal values of the displacement component $u_x(x, y)$ are $u_{x1} = \mathrm{a}$, $u_{x2} = -\mathrm{a}$, and $u_{x3} = 2\mathrm{a}$.

Answer $u_x = 2\frac{\mathrm{a}}{h}(y-x)$, $\frac{\partial u_x}{\partial x} = -2\frac{\mathrm{a}}{h}$, $\frac{\partial u_x}{\partial y} = 2\frac{\mathrm{a}}{h}$, $I = \frac{1}{3}\mathrm{a}h^2$

Consider a bar element when A and E are constants and $f_x = (1-x/h) f_{x1} + (x/h) f_{x2}$ is the linear distributed force. Derive the virtual work expression of linear bar element. Use the virtual work density expression $\delta w_\Omega = -(d\delta u/dx)EA(du/dx) + \delta u f_x$ and approximation $u = (1-x/h)u_{x1} + (x/h)u_{x2}$.



Solution

The concise representation of the element contribution consists of a virtual work density expression and approximations to the displacement and rotation components. Approximations are first substituted into the density expression which is followed by integration over the domain occupied by the element (line segment, triangle etc.). Here the two building blocks are

$$\delta w_\Omega = -\frac{d\delta u}{dx} EA \frac{du}{dx} + \delta u f_x \quad \text{and} \quad u = (1-\frac{x}{h})u_{x1} + \frac{x}{h} u_{x2}.$$

The quantities needed in the virtual work density are the axial displacement, variation of the axial displacement, and variation of the derivative of the axial displacement

$$u = \begin{Bmatrix} 1-x/h \\ x/h \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} \quad \Rightarrow \quad \delta u = \begin{Bmatrix} 1-x/h \\ x/L \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 1-x/h \\ x/h \end{Bmatrix},$$

$$\frac{du}{dx} = \begin{Bmatrix} -1/h \\ 1/h \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} \quad \Rightarrow \quad \frac{d\delta u}{dx} = \begin{Bmatrix} -1/h \\ 1/h \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} -1/h \\ 1/h \end{Bmatrix}.$$

When the approximation is substituted there, virtual work densities of the internal and external forces take the forms

$$\delta w_\Omega^{\mathrm{int}} = -\frac{d\delta u}{dx} EA \frac{du}{dx} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} EA/h^2 & -EA/h^2 \\ -EA/h^2 & EA/h^2 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix},$$

$$\delta w_\Omega^{\mathrm{ext}} = \delta u f_x = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} (1-x/h)^2 & (1-x/h)(x/h) \\ (1-x/h)(x/h) & (x/h)^2 \end{bmatrix} \begin{Bmatrix} f_{x1} \\ f_{x2} \end{Bmatrix}.$$

Integration over the element gives the virtual work expressions of the internal and external forces

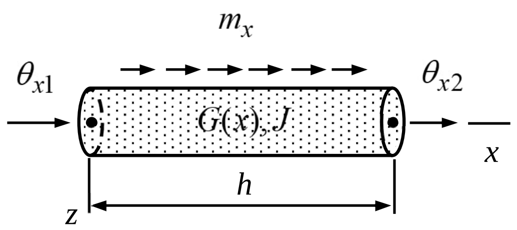
$$\delta W^{\mathrm{int}} = \int_0^h \delta w_\Omega^{\mathrm{int}} dx = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} EA/h & -EA/h \\ -EA/h & EA/h \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix},$$

$$\delta W^{\mathrm{ext}} = \int_0^h \delta w_\Omega^{\mathrm{ext}} dx = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 2h/6 & h/6 \\ h/6 & 2h/6 \end{bmatrix} \begin{Bmatrix} f_{x1} \\ f_{x2} \end{Bmatrix}.$$

Virtual work expression of bar element is the sum of internal and external parts

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \left(\begin{bmatrix} EA/h & -EA/h \\ -EA/h & EA/h \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \begin{bmatrix} 2h/6 & h/6 \\ h/6 & 2h/6 \end{bmatrix} \begin{Bmatrix} f_{x1} \\ f_{x2} \end{Bmatrix} \right).$$

Derive the virtual work expression of a torsion bar, when J and m_x are constants and shear modulus G is linear and defined by the nodal values G_1 and G_2. Use approximation $\phi=(1-x/h)\theta_{x1}+(x/h)\theta_{x2}$. Virtual work density of the torsion bar model is $\delta w_\Omega=-\delta(d\phi/dx)GJ(d\phi/dx)+\delta\phi m_x$.



Solution

Virtual work density of the torsion bar model

$$\delta w_\Omega=\delta w_\Omega^{\text{int}}+\delta w_\Omega^{\text{ext}}=-\frac{d\delta\phi}{dx}GJ\frac{d\phi}{dx}+\delta\phi m_x$$

depends on the polar moment of area J, shear modulus G, and moment per unit length m_x. Expression is valid also when data are not constant. Virtual work expression (element contribution)

$$\delta W=\int_\Omega \delta w_\Omega d\Omega$$

is integral of the density over the domain $\Omega=]0,h[$ occupied by the element.

Assuming that the origin of the material coordinate system is placed at node 1, linear approximation (and its derivatives as well as their variations) to the axial rotation ϕ take the forms

$$\phi=\begin{Bmatrix}1-x/h\\ x/h\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\theta_{x1}\\ \theta_{x2}\end{Bmatrix},\ \frac{d\phi}{dx}=\frac{1}{h}\begin{Bmatrix}-1\\ 1\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\theta_{x1}\\ \theta_{x2}\end{Bmatrix},\ \delta\phi=\begin{Bmatrix}\delta\theta_{x1}\\ \delta\theta_{x2}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}1-x/h\\ x/h\end{Bmatrix},\ \frac{d\delta\phi}{dx}=\begin{Bmatrix}\delta\theta_{x1}\\ \delta\theta_{x2}\end{Bmatrix}^{\mathrm{T}}\frac{1}{h}\begin{Bmatrix}-1\\ 1\end{Bmatrix}.$$

As the shear modulus of the material is known to be linear and defined by its nodal values

$$G=\begin{Bmatrix}1-x/h\\ x/h\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}G_1\\ G_2\end{Bmatrix}=(1-\frac{x}{h})G_1+\frac{x}{h}G_2.$$

When the approximation to ϕ and expression for G are substituted there, virtual work densities of the internal and external forces take the forms

$$\delta w_\Omega^{\text{int}}=-\frac{d\delta\phi}{dx}GJ\frac{d\phi}{dx}\quad\Rightarrow$$

$$\delta w_\Omega^{\text{int}}=-\begin{Bmatrix}\delta\theta_{x1}\\ \delta\theta_{x2}\end{Bmatrix}^{\mathrm{T}}\frac{1}{h}\begin{Bmatrix}-1\\ 1\end{Bmatrix}GJ\frac{1}{h}\begin{Bmatrix}-1\\ 1\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\theta_{x1}\\ \theta_{x2}\end{Bmatrix}=-\begin{Bmatrix}\delta\theta_{x1}\\ \delta\theta_{x2}\end{Bmatrix}^{\mathrm{T}}\frac{GJ}{h^2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}\theta_{x1}\\ \theta_{x2}\end{Bmatrix}\quad\Leftrightarrow$$

$$\delta w_\Omega^{\text{int}}=-\begin{Bmatrix}\delta\theta_{x1}\\ \delta\theta_{x2}\end{Bmatrix}^{\mathrm{T}}[(1-\frac{x}{h})G_1+\frac{x}{h}G_2]\frac{J}{h^2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}\theta_{x1}\\ \theta_{x2}\end{Bmatrix},$$

and

$$\delta w_{\Omega}^{\text{ext}} = \delta\phi m_x \quad \Rightarrow \quad \delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 1-x/h \\ x/h \end{Bmatrix} m_x \,.$$

Virtual work expressions are integrals of the virtual work densities over the domain Ω occupied by the element (here $\Omega =]0,h[$)

$$\delta W^{\text{int}} = \int_0^h \delta w_{\Omega}^{\text{int}} dx \quad \Rightarrow$$

$$\delta W^{\text{int}} = -\int_0^h \begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}^{\mathrm{T}} [(1-\frac{x}{h})G_1 + \frac{x}{h}G_2]\frac{J}{h^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix} dx \quad \Leftrightarrow$$

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{G_1 + G_2}{2}\frac{J}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix}.$$

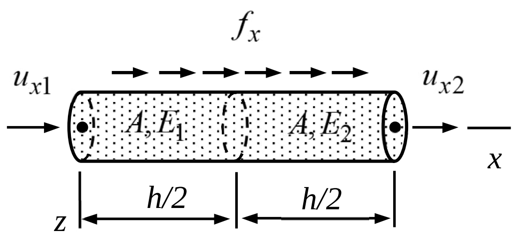
$$\delta W^{\text{ext}} = \int_{\Omega} \delta w_{\Omega}^{\text{ext}} d\Omega \quad \Rightarrow$$

$$\delta W^{\text{ext}} = \int_0^h \begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 1-x/h \\ x/h \end{Bmatrix} m_x dx = \begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{m_x h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

Virtual work expression of the element is the sum of internal and external parts

$$\delta W = -\begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}^{\mathrm{T}} (\frac{G_1 + G_2}{2}\frac{J}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix} - \frac{m_x h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}) \,. \quad \leftarrow$$

Consider a bar element having constant A and f_x and a piecewise constant E as shown in the figure. Derive the virtual work expression of the element by using the virtual work density expression $\delta w_\Omega = -(d\delta u/dx)EA(du/dx)+\delta u f_x$ of the bar model and interpolant $u=(1-x/h)u_{x1}+(x/h)u_{x2}$ to the nodal displacements.



Solution

Virtual work densities of the internal and external distributed forces of the bar model

$$\delta w_\Omega^{\text{int}} = -\frac{d\delta u}{dx} EA \frac{du}{dx} \quad \text{and} \quad \delta w_\Omega^{\text{ext}} = \delta u f_x$$

depend on the cross-sectional area A, Young's modulus E, and force per unit length f_x. Expressions are valid also when the data (A, E, f_x) are not constants. Virtual work expressions

$$\delta W^{\text{int}} = \int_\Omega \delta w_\Omega^{\text{int}} d\Omega \quad \text{and} \quad \delta W^{\text{ext}} = \int_\Omega \delta w_\Omega^{\text{ext}} d\Omega$$

are integrals of the densities over the domain $\Omega =]0,h[$. As a virtual work expression of the bar element with a varying Young's modulus is not available in formulae collection, it needs to be calculated from scratch. Here

$$E = \begin{cases} E_1 & 0 \le x \le h/2 \\ E_2 & h/2 < x \le h \end{cases}.$$

Assuming that the origin of the material coordinate system is placed at node 1, linear approximation (and its derivatives as well as their variations) to the axial displacement ϕ and Young's modulus E are given by

$$u = \begin{Bmatrix} 1-x/h \\ x/h \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} \quad \Rightarrow \quad \frac{du}{dx} = \frac{1}{h}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix}, \quad \text{so}$$

$$\delta u = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 1-x/h \\ x/h \end{Bmatrix} \quad \text{and} \quad \frac{d\delta u}{dx} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{1}{h}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}.$$

When the approximation to u and the expression of E are substituted there, virtual work densities of the internal and external forces take the forms

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{1}{h}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix} E_1 A \frac{1}{h}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{E_1 A}{h^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} \quad 0 \le x \le \frac{h}{2},$$

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{1}{h}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix} E_2 A \frac{1}{h}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{E_2 A}{h^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} \quad \frac{h}{2} < x \le h \,.$$

$$\delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} f_x \begin{Bmatrix} 1 - x/h \\ x/h \end{Bmatrix}.$$

Virtual work expressions are integrals of the virtual work densities over the domain Ω occupied by the element. Here $\Omega =]0, h[$ needs to be divided into two parts since the integrand is piecewise constant

$$\delta W^{\text{int}} = \int_0^h \delta w_{\Omega}^{\text{int}} dx = \int_0^{h/2} \delta w_{\Omega}^{\text{int}} dx + \int_{h/2}^{h} \delta w_{\Omega}^{\text{int}} dx \quad \text{in which}$$

$$\int_0^{h/2} \delta w_{\Omega}^{\text{int}} dx = -\int_0^{h/2} \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{E_1 A}{h^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} dx = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{E_1 A}{2h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix},$$

$$\int_{h/2}^{h} \delta w_{\Omega}^{\text{int}} dx = -\int_{h/2}^{h} \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{E_2 A}{h^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} dx = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{E_2 A}{2h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix}.$$

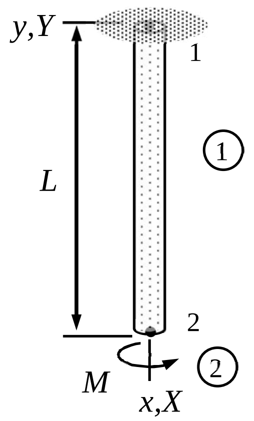
and

$$\delta W^{\text{ext}} = \int_{\Omega} \delta w_{\Omega}^{\text{ext}} d\Omega \quad \Rightarrow \quad \delta W^{\text{ext}} = \int_0^h \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} f_x \begin{Bmatrix} 1 - x/h \\ x/h \end{Bmatrix} dx = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{f_x h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

Virtual work expression of the element is the sum of internal and external parts

$$\delta W = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{E_1 A}{2h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{E_2 A}{2h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} + \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{f_x h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \quad \Rightarrow$$

$$\delta W = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{E_1 + E_2}{2}\frac{A}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \frac{f_x h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}\right).$$



Consider the torsion bar (1) of the figure loaded by torque M (2) acting on the free end. Determine the rotation θ_{X2} at the free end, if the polar moment J is constant and shear modulus G varies linearly so that the values at the nodes are G_1 and G_2. Start with the virtual work density $\delta w_\Omega = -\delta(d\phi / dx)GJ(d\phi / dx) + \delta\phi m_x$ and use linear approximation to rotation (a linear two-node element).

Solution

Virtual work densities of the internal and external distributed forces of the torsion bar model

$$\delta w_\Omega^{\text{int}} = -\frac{d\delta\phi}{dx} GJ \frac{d\phi}{dx} \quad \text{and} \quad \delta w_\Omega^{\text{ext}} = \delta\phi m_x$$

depend on the polar moment J, shear modulus G, and moment per unit length m_x. Virtual work expression is obtained as integrals

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} \quad \text{where} \quad \delta W^{\text{int}} = \int_\Omega \delta w_\Omega^{\text{int}} d\Omega \quad \text{and} \quad \delta W^{\text{ext}} = \int_\Omega \delta w_\Omega^{\text{ext}} d\Omega$$

over the element. In this case $\Omega =]0, L[$ (just one element of length L) and $d\Omega = dx$. As “ready-to-use” virtual work expression of a torsion bar with varying shear modulus is not available in the formulae collection of the course, it needs to be calculated by using the virtual work density and approximation/interpolant to rotation.

Assuming that the origin of the material coordinate system is placed at node 1 and coincides with the structural system (for simplicity), linear approximation to the axial rotation ϕ and the expression of the shear modulus G are

$$\phi = \begin{Bmatrix} 1 - x/L \\ x/L \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ \theta_{X2} \end{Bmatrix} = \frac{x}{L}\theta_{X2} \text{ and } G = \begin{Bmatrix} 1 - x/L \\ x/L \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} G_1 \\ G_2 \end{Bmatrix} = (1 - \frac{x}{L})G_1 + \frac{x}{L}G_2 .$$

When the approximation to ϕ and the expression of the shear modulus G are substituted there, virtual work density of internal forces becomes (external part coming from the distributed moment vanishes here)

$$\delta w_\Omega^{\text{int}} = -\frac{d\delta\phi}{dx} GJ \frac{d\phi}{dx} = -\frac{1}{L}\delta\theta_{X2}[(1 - \frac{x}{L})G_1 + \frac{x}{L}G_2]J\frac{1}{L}\theta_{X2} .$$

Virtual work expression is the integral of virtual work density over domain Ω occupied by element (here $\Omega =]0, L[$)

$$\delta W^{\text{int}} = \int_0^L \delta w_\Omega^{\text{int}} dx = -\frac{1}{L}\delta\theta_{X2}(\frac{L}{2}G_1 + \frac{L}{2}G_2)J\frac{1}{L}\theta_{X2} = -\delta\theta_{X2}\frac{G_1 + G_2}{2}\frac{J}{L}\theta_{X2} \quad \Rightarrow$$

$$\delta W^1 = -\delta\theta_{X2}\frac{G_1+G_2}{2}\frac{J}{L}\theta_{X2}.$$

Virtual work expression of the point force/moment is available in the formulae collection (the definition of work can also be used)

$$\delta W^{\text{ext}} = \begin{Bmatrix}\delta u_{Xi}\\ \delta u_{Yi}\\ \delta u_{Zi}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}F_{Xi}\\ F_{Yi}\\ F_{Zi}\end{Bmatrix}+\begin{Bmatrix}\delta\theta_{Xi}\\ \delta\theta_{Yi}\\ \delta\theta_{Zi}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}M_{Xi}\\ M_{Yi}\\ M_{Zi}\end{Bmatrix} = -\delta\theta_{X2}M \quad\Rightarrow\quad \delta W^2 = -\delta\theta_{X2}M\,.$$

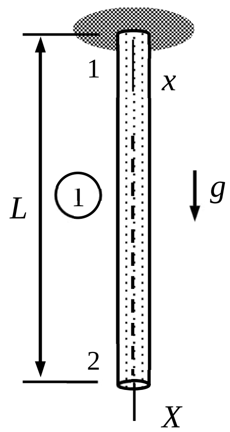
Virtual work expression of the structure is sum of the element contributions

$$\delta W = \delta W^1 + \delta W^2 = -\delta\theta_{X2}\left(\frac{G_1+G_2}{2}\frac{J}{L}\theta_{X2}+M\right).$$

Principle of virtual work $\delta W = 0 \;\forall\delta\mathbf{a}$ and the fundamental lemma of variation calculus $\delta\mathbf{a}^{\mathrm{T}}\mathbf{F}=0 \;\forall\delta\mathbf{a} \;\Leftrightarrow\; \mathbf{F}=0$ imply

$$\frac{G_1+G_2}{2}\frac{J}{L}\theta_{X2}-(-M)=0 \quad\Leftrightarrow\quad \theta_{X2} = -2\frac{ML}{(G_1+G_2)J}\,.$$

Consider a bar of length L loaded by its own weight (figure). Determine the displacement u_{X2} at the free end. Start with the virtual work density expression $\delta w_\Omega = -(d\delta u/dx)EA(du/dx)+\delta u f_x$ and approximation $u=(1-x/L)u_{x1}+(x/L)u_{x2}$. Cross-sectional area A, acceleration by gravity g and material properties E and ρ are constants.



Solution

The concise representation of the element contribution consists of a virtual work density expression and approximations to the displacement and rotation components. Approximations are just substituted into the density expression followed by integration over the domain occupied by the element (line segment, triangle etc.). Here the two building blocks are

$$\delta w_\Omega = -\frac{d\delta u}{dx}EA\frac{du}{dx}+\delta u f_x \quad \text{and} \quad u=(1-\frac{x}{L})u_{x1}+\frac{x}{L}u_{x2}.$$

The quantities needed in the virtual work density are the axial displacement, variation of the axial displacement, and variation of the derivative of the axial displacement

$$u=\begin{Bmatrix}1-x/L\\ x/L\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix} \quad\Rightarrow\quad \delta u=\begin{Bmatrix}1-x/L\\ x/L\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}=\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}1-x/L\\ x/L\end{Bmatrix},$$

$$\frac{du}{dx}=\frac{1}{L}\begin{Bmatrix}-1\\ 1\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix} \quad\Rightarrow\quad \frac{d\delta u}{dx}=\frac{1}{L}\begin{Bmatrix}-1\\ 1\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}=\frac{1}{L}\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}-1\\ 1\end{Bmatrix}.$$

When the approximation is substituted there, virtual work density expression of the bar model takes the form

$$\delta w_\Omega = -\frac{d\delta u}{dx}EA\frac{du}{dx}+\delta u f_x = -\frac{1}{L}\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}-1\\ 1\end{Bmatrix}EA\frac{1}{L}\begin{Bmatrix}-1\\ 1\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix}+\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}1-x/L\\ x/L\end{Bmatrix}f \quad\Leftrightarrow$$

$$\delta w_\Omega = -\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}(\begin{Bmatrix}-1\\ 1\end{Bmatrix}\frac{1}{L}EA\frac{1}{L}\begin{Bmatrix}-1\\ 1\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix}-\begin{Bmatrix}1-x/L\\ x/L\end{Bmatrix}f) \quad\Leftrightarrow$$

$$\delta w_\Omega = -\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}(\frac{EA}{L^2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix}-\begin{Bmatrix}1-x/L\\ x/L\end{Bmatrix}f)\ .$$

Finally, integration over the element gives the virtual work expression of the bar element

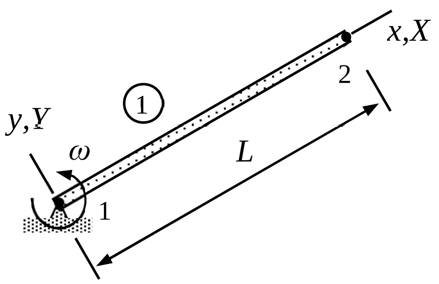
$$\delta W=\int_0^L \delta w_\Omega dx = -\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}(\frac{EA}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix}-\frac{fL}{2}\begin{Bmatrix}1\\ 1\end{Bmatrix}).$$

Finding the displacement of the free end follows the usual lines. Here, $f_x = \rho g A$, $u_{x1} = u_{X1} = 0$, and $u_{x2} = u_{X2}$

$$\delta W = -\begin{Bmatrix} 0 \\ \delta u_{X2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ u_{X2} \end{Bmatrix} - \frac{\rho g A L}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \right) = -\delta u_{X2} \left(\frac{EA}{L} u_{X2} - \frac{\rho g A L}{2} \right) = 0 \quad \forall \delta u_{X2} \quad \Leftrightarrow$$

$$\frac{EA}{L} u_{X2} - \frac{\rho g A L}{2} = 0 \quad \Leftrightarrow \quad u_{X2} = \frac{\rho g L^2}{2E}. \quad \leftarrow$$

Structural coordinate system and the bar shown are rotating in a plane with a constant angular speed $\omega_Z = \omega$. Material properties E, ρ and the cross-sectional area A are constants. Determine the nodal displacement u_{X2} at the free end using just one linear element. The *volume force* due to the rotation is given by $\vec{f} = -\rho\vec{a} = -\rho\vec{\omega}\times(\vec{\omega}\times\vec{r})$ in which $\vec{\omega} = \omega\vec{k}$ and $\vec{r} = x\vec{i}$.



Solution

Virtual work densities of the internal and external distributed forces of the bar model

$$\delta w_\Omega = -\frac{d\delta u}{dx}EA\frac{du}{dx} + \delta u f_x \quad \text{and} \quad \delta w_\Omega^{\text{ext}} = \delta u f_x$$

depend on the cross-sectional area A, Young's modulus E, and external force per unit length f_x. Virtual work expressions

$$\delta W^{\text{int}} = \int_\Omega \delta w_\Omega^{\text{int}} d\Omega \quad \text{and} \quad \delta W^{\text{ext}} = \int_\Omega \delta w_\Omega^{\text{ext}} d\Omega$$

are integrals of the densities over the domain $\Omega =]0, h[$ occupied by the element. Principle of virtual work corresponds to equilibrium equations that are valid in their simple form ($\vec{F} = m\vec{a}$ with $\vec{a} = 0$ etc.) in an inertial frame of reference. The correct form for a coordinate system which rotates with a constant angular speed $\vec{\omega}$ in the manner shown in the figure ($\vec{F} = m[\vec{a}_r + \vec{\omega}\times(\vec{\omega}\times\vec{r})]$ with $\vec{a}_r = 0$) gives rise to a force per unit volume in the same manner as gravity is acting on the body. In the present case, (material system and structural system coincide) force per unit length in the direction of the $x-$ axis takes the form

$$f_x = \vec{f}\cdot\vec{i} = -\rho A[\omega\vec{k}\times(\omega\vec{k}\times x\vec{i})]\cdot\vec{i} = \rho A\omega^2 x .$$

Virtual work expression of bar element with a varying distributed external force is not available in formulae collection and it needs to be calculated from scratch. According to the formulae collection

$$u = \begin{Bmatrix} 1-\xi \\ \xi \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} \quad \text{in which} \quad \xi = \frac{x}{h} .$$

Assuming that the origin of the material coordinate system is placed at node 1 and coincides with the structural system, approximation to the axial displacement and its derivatives and variations are (In hand calculations, it is advantageous to use information about boundary conditions etc. as soon as possible.)

$$u = \frac{1}{L}\begin{Bmatrix} L-x \\ x \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ u_{X2} \end{Bmatrix} = \frac{x}{L}u_{X2} \quad \text{and} \quad \delta u = \frac{x}{L}\delta u_{X2} ,$$

$$\frac{du}{dx}=\frac{1}{L}\begin{Bmatrix}-1\\1\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}0\\u_{X2}\end{Bmatrix}=\frac{1}{L}u_{X2} \quad \text{and} \quad \delta\frac{du}{dx}=\frac{1}{L}\delta u_{X2}\,.$$

When the approximation to u is substituted there, virtual work densities of the internal and external forces take the forms

$$\delta w_{\Omega}^{\text{int}}=-\frac{1}{L}\delta u_{X2}EA\frac{1}{L}u_{X2} \quad \text{and} \quad \delta w_{\Omega}^{\text{ext}}=\frac{x}{L}\delta u_{X2}\rho A\omega^{2}x\,.$$

Virtual work expressions are integrals of the densities over the domain Ω occupied by the element

$$\delta W^{\text{int}}=\int_{0}^{L}\delta w_{\Omega}^{\text{int}}dx=-\delta u_{X2}\frac{EA}{L}u_{X2}\,,$$

$$\delta W^{\text{ext}}=\int_{0}^{L}\delta W^{\text{int}}dx=\delta u_{X2}\frac{1}{3}L^{2}\rho A\omega^{2}\,.$$

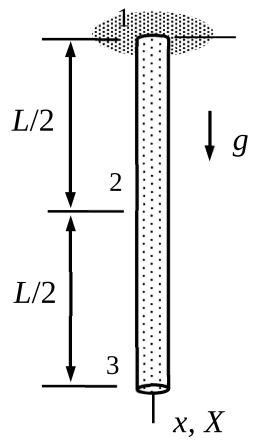
Virtual work expression of the element is sum of the internal and external parts

$$\delta W=\delta W^{\text{int}}+\delta W^{\text{ext}}=-\delta u_{X2}\left(\frac{EA}{L}u_{X2}-\frac{1}{3}L^{2}\rho A\omega^{2}\right).$$

Principle of virtual work $\delta W=0\ \forall\delta\mathbf{a}$ and the fundamental lemma of variation calculus in the form $\delta\mathbf{a}^{\mathrm{T}}\mathbf{R}=0\ \ \forall\delta\mathbf{a}\ \Leftrightarrow\ \mathbf{R}=0$ imply

$$\frac{EA}{L}u_{X2}-\frac{1}{3}L^{2}\rho A\omega^{2}=0 \quad \Leftrightarrow \quad u_{X2}=\frac{1}{3}\frac{L^{3}\rho\omega^{2}}{E}\,.$$

Consider the bar of the figure loaded by its own weight. Determine the displacement of the free end with one element. Use virtual work density expression $\delta w_\Omega = -\delta(du/dx)EA(du/dx) + \delta u f_x$ and quadratic approximation $u = (1-3\xi+2\xi^2)u_{x1} + 4\xi(1-\xi)u_{x2} + \xi(2\xi-1)u_{x3}$ in which $\xi = x/L$. Cross-sectional area of the bar A, acceleration by gravity g, and material properties E and ρ are constants.



Solution

Virtual work densities of the internal and external distributed forces of the bar model

$$\delta w_\Omega^{\text{int}} = -\frac{d\delta u}{dx} EA \frac{du}{dx} \quad \text{and} \quad \delta w_\Omega^{\text{ext}} = \delta u f_x$$

depend on the cross-sectional area A, Young's modulus E, and force per unit length f_x. Virtual work expressions

$$\delta W^{\text{int}} = \int_\Omega \delta w_\Omega^{\text{int}} d\Omega \quad \text{and} \quad \delta W^{\text{ext}} = \int_\Omega \delta w_\Omega^{\text{ext}} d\Omega$$

are obtained as integrals over the domain $\Omega =]0, h[$ occupied by the element (here $h = L$).

Virtual work expression of the bar element, when approximation is quadratic (a three-node element), is not available in formulae collection and it needs to be calculated from scratch. In hand calculations, it is advantageous to use information about boundary conditions etc. as soon as possible. In the present case, force per unit length is due to the weight

$$f_x = \rho g A .$$

Assuming that the origin of the material coordinate system is placed at node 1 and coincides with the structural system, quadratic approximation to the axial displacement u (see the formula collection) and its derivatives and variations are given by

$$u = \begin{Bmatrix} 1-3\xi+2\xi^2 \\ 4\xi(1-\xi) \\ \xi(2\xi-1) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{x1} \\ u_{x2} \\ u_{x3} \end{Bmatrix} = \begin{Bmatrix} 1-3\xi+2\xi^2 \\ 4\xi(1-\xi) \\ \xi(2\xi-1) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ u_{X2} \\ u_{X3} \end{Bmatrix} = \begin{Bmatrix} 4\frac{x}{L} - 4(\frac{x}{L})^2 \\ 2(\frac{x}{L})^2 - \frac{x}{L} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix} \quad \Rightarrow$$

$$\delta u = \begin{Bmatrix} 4\frac{x}{L} - 4(\frac{x}{L})^2 \\ 2(\frac{x}{L})^2 - \frac{x}{L} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \delta u_{X2} \\ \delta u_{X3} \end{Bmatrix}, \quad \frac{du}{dx} = \frac{1}{L} \begin{Bmatrix} 4-8\frac{x}{L} \\ 4\frac{x}{L}-1 \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{X2} \\ u_{X3} \end{Bmatrix}, \text{ and } \frac{d\delta u}{dx} = \begin{Bmatrix} \delta u_{X2} \\ \delta u_{X3} \end{Bmatrix}^{\mathrm{T}} \frac{1}{L} \begin{Bmatrix} 4-8\frac{x}{L} \\ 4\frac{x}{L}-1 \end{Bmatrix}.$$

When the approximation to u is substituted there, virtual work densities of the internal and external forces become

$$\delta w_{\Omega}^{\text{int}}=-\frac{d\delta u}{dx}EA\frac{du}{dx}=-\begin{Bmatrix}\delta u_{X2}\\ \delta u_{X3}\end{Bmatrix}^{\mathrm{T}}\frac{1}{L^2}\begin{Bmatrix}4L-8x\\ 4x-L\end{Bmatrix}EA\frac{1}{L^2}\begin{Bmatrix}4L-8x\\ 4x-L\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}u_{X2}\\ u_{X3}\end{Bmatrix}\quad\Leftrightarrow$$

$$\delta w_{\Omega}^{\text{int}}=-\begin{Bmatrix}\delta u_{X2}\\ \delta u_{X3}\end{Bmatrix}^{\mathrm{T}}\frac{EA}{L^4}\begin{bmatrix}(4L-8x)^2 & (4L-8x)(4x-L)\\ (4x-L)(4L-8x) & (4x-L)^2\end{bmatrix}\begin{Bmatrix}u_{X2}\\ u_{X3}\end{Bmatrix},$$

$$\delta w_{\Omega}^{\text{ext}}=\delta u f_x=\begin{Bmatrix}\delta u_{X2}\\ \delta u_{X3}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}4x/L-4(x/L)^2\\ 2(x/L)^2-x/L\end{Bmatrix}\rho gA.$$

Virtual work expressions are integrals of the virtual work densities over the domain Ω occupied by the element

$$\delta W^{\text{int}}=\int_0^L \delta w_{\Omega}^{\text{int}}dx\ \Rightarrow$$

$$\delta W^{\text{int}}=-\begin{Bmatrix}\delta u_{X2}\\ \delta u_{X3}\end{Bmatrix}^{\mathrm{T}}\frac{EA}{L^4}\int_0^L\begin{bmatrix}(4L-8x)^2 & (4L-8x)(4x-L)\\ (4x-L)(4L-8x) & (4x-L)^2\end{bmatrix}dx\begin{Bmatrix}u_{X2}\\ u_{X3}\end{Bmatrix}\quad\Leftrightarrow$$

$$\delta W^{\text{int}}=-\begin{Bmatrix}\delta u_{X2}\\ \delta u_{X3}\end{Bmatrix}^{\mathrm{T}}\frac{EA}{3L}\begin{bmatrix}16 & -8\\ -8 & 7\end{bmatrix}\begin{Bmatrix}u_{X2}\\ u_{X3}\end{Bmatrix},$$

$$\delta W^{\text{ext}}=\int_{\Omega}\delta w_{\Omega}^{\text{ext}}d\Omega\ \Rightarrow$$

$$\delta W^{\text{ext}}=\int_{\Omega}\delta w_{\Omega}^{\text{ext}}d\Omega=\begin{Bmatrix}\delta u_{X2}\\ \delta u_{X3}\end{Bmatrix}^{\mathrm{T}}\int_0^L\frac{1}{L^2}\begin{Bmatrix}4xL-4x^2\\ 2x^2-xL\end{Bmatrix}dx\rho gA=\begin{Bmatrix}\delta u_{X2}\\ \delta u_{X3}\end{Bmatrix}^{\mathrm{T}}\frac{\rho gAL}{6}\begin{Bmatrix}4\\ 1\end{Bmatrix}.$$

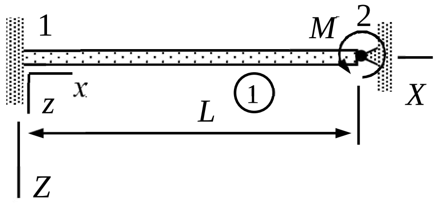
Virtual work expression of the element is the sum of internal and external parts

$$\delta W=\delta W^{\text{int}}+\delta W^{\text{ext}}=-\begin{Bmatrix}\delta u_{X2}\\ \delta u_{X3}\end{Bmatrix}^{\mathrm{T}}(\frac{EA}{3L}\begin{bmatrix}16 & -8\\ -8 & 7\end{bmatrix}\begin{Bmatrix}u_{X2}\\ u_{X3}\end{Bmatrix}-\frac{\rho gAL}{6}\begin{Bmatrix}4\\ 1\end{Bmatrix}).$$

Principle of virtual work $\delta W=0\ \forall\delta\mathbf{a}$ and the fundamental lemma of variation calculus in the form $\delta\mathbf{a}^{\mathrm{T}}\mathbf{R}=0\ \ \forall\delta\mathbf{a}\ \Leftrightarrow\ \mathbf{R}=0$ imply

$$\frac{EA}{3L}\begin{bmatrix}16 & -8\\ -8 & 7\end{bmatrix}\begin{Bmatrix}u_{X2}\\ u_{X3}\end{Bmatrix}-\frac{\rho gAL}{6}\begin{Bmatrix}4\\ 1\end{Bmatrix}=0\quad\Leftrightarrow\quad\begin{Bmatrix}u_{X2}\\ u_{X3}\end{Bmatrix}=\frac{\rho gL^2}{6E}\frac{1}{48}\begin{bmatrix}7 & 8\\ 8 & 16\end{bmatrix}\begin{Bmatrix}4\\ 1\end{Bmatrix}=\frac{1}{2}\frac{\rho gL^2}{E}\begin{Bmatrix}3/4\\ 1\end{Bmatrix}.$$

Determine the rotation θ_{Y2} of the beam shown at the support of the right end which allows rotation but not transverse displacement. Young's modulus E of the material and second moment of cross-section $I_{yy} = I$ are constants. Use the virtual work density of beam bending mode $\delta w_\Omega = -\delta(d^2 w/dx^2) EI_{yy}(d^2 w/dx^2) + \delta w f_z$ and cubic approximation to the transverse displacement.



Solution

In the $xz-$plane problem bending problem, when x-axis is chosen to coincide with the neutral axis, virtual work densities of the beam bending mode are

$$\delta w_\Omega^{\text{int}} = -\frac{d^2\delta w}{dx^2} EI_{yy} \frac{d^2 w}{dx^2} \quad \text{and} \quad \delta w_\Omega^{\text{ext}} = \delta w f_z .$$

Approximation is the first thing to be considered. The left end of the beam is clamped and the right end support does not allow transverse displacement. As only $\theta_{y2} = \theta_{Y2}$ is non-zero, approximation to w simplifies into the form (see the formulae collection for the cubic beam bending approximation)

$$w = \begin{Bmatrix} (1-\xi)^2(1+2\xi) \\ L(1-\xi)^2\xi \\ (3-2\xi)\xi^2 \\ L\xi^2(\xi-1) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -\theta_{Y2} \end{Bmatrix} = L(\frac{x}{L})^2(1-\frac{x}{L})\theta_{Y2} \quad \Rightarrow \quad \frac{d^2 w}{dx^2} = \frac{1}{L}(2-6\frac{x}{L})\theta_{Y2} \text{ and}$$

$$\delta w = L(\frac{x}{L})^2(1-\frac{x}{L})\delta\theta_{Y2} \quad \Rightarrow \quad \frac{d^2\delta w}{dx^2} = \frac{1}{L}(2-6\frac{x}{L})\delta\theta_{Y2} .$$

When the approximation is substituted there, virtual work density takes the form (external distributed force vanishes)

$$\delta w_\Omega = -\frac{d^2\delta w}{dx^2} EI_{yy} \frac{d^2 w}{dx^2} = -\delta\theta_{Y2} \frac{EI}{L^4} \theta_{Y2} (2L-6x)^2 .$$

Integration over the domain $\Omega =]0, L[$ gives the virtual work expressions (beam is considered as element 1)

$$\delta W^1 = \int_0^L \delta w_\Omega dx = -\delta\theta_{Y2} 4\frac{EI}{L}\theta_{Y2} .$$

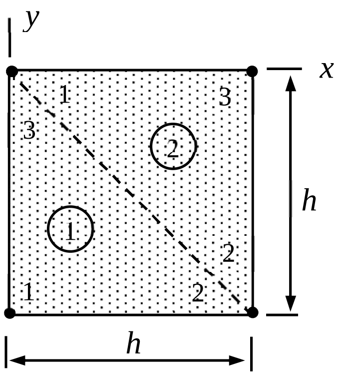
The external moment gives the contribution (element 2)

$$\delta W^2 = \delta\theta_{Y2} M .$$

Principle of virtual work $\delta W = \delta W^1 + \delta W^2 = 0 \quad \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus imply the solution

$$\delta W = -\delta\theta_{Y2}(4\frac{EI}{L}\theta_{Y2} - M) = 0 \quad \forall \delta u_{Z2} \quad \Leftrightarrow \quad 4\frac{EI}{L}\theta_{Y2} - M = 0 \quad \Leftrightarrow \quad \theta_{Y2} = \frac{1}{4}\frac{ML}{EI} \ .$$

Derive the shape functions of the triangle elements 1 and 2 shown in the figure in terms of the (material) xy – coordinates.



Solution

Shape functions of the linear triangle element are given by the simple formula

$$\mathbf{N} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}$$

in which the subscripts refer to coordinates of the three nodes. Order of the node numbers does not matter:

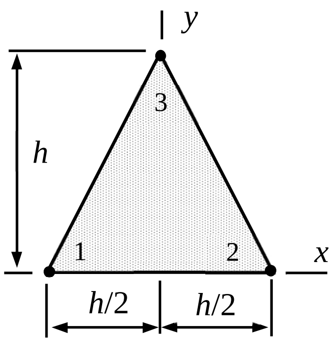
$$\mathbf{N}^1 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & h & 0 \\ -h & -h & 0 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} = \frac{1}{h^2} \begin{bmatrix} 0 & 0 & h^2 \\ -h & h & 0 \\ -h & 0 & h \end{bmatrix}^{\mathrm{T}} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} = \frac{1}{h} \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ h & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} = \frac{1}{h} \begin{Bmatrix} -y-x \\ x \\ h+y \end{Bmatrix}, \quad \leftarrow$$

$$\mathbf{N}^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & h & h \\ 0 & -h & 0 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} = \frac{1}{h^2} \begin{bmatrix} h^2 & 0 & 0 \\ -h & 0 & h \\ 0 & -h & h \end{bmatrix}^{\mathrm{T}} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} = \frac{1}{h} \begin{bmatrix} h & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} = \frac{1}{h} \begin{Bmatrix} h-x \\ -y \\ x+y \end{Bmatrix}. \quad \leftarrow$$

Using a linear interpolant to the nodal values, determine

$$u_x, \ \frac{\partial u_x}{\partial x}, \ \frac{\partial u_x}{\partial y}, \ \text{and } I = \int_{\Omega^e} u_x d\Omega,$$

for the element shown. The nodal values of the displacement component $u_x(x, y)$ are $u_{x1} = \mathrm{a}$, $u_{x2} = -\mathrm{a}$, and $u_{x3} = 2\mathrm{a}$.



Solution

The shape functions of a three-node triangular element in xy – coordinates are linear and therefore of the form

$$N_i = a_i + b_i x + c_i y = \{1 \quad x \quad y\}\begin{Bmatrix} a_i \\ b_i \\ c_i \end{Bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

as the basis functions should take the value 1 at their own nodes and vanish at all the other nodes. The closed form solution to the shape functions is given by

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}^{\mathrm{T}} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} = \frac{1}{\det}\begin{Bmatrix} x_3(y - y_2) + x(y_2 - y_3) + x_2(-y + y_3) \\ x_3(-y + y_1) + x_1(y - y_3) + x(-y_1 + y_3) \\ x_2(y - y_1) + x(y_1 - y_2) + x_1(-y + y_2) \end{Bmatrix}$$

where $\det = -x_2 y_1 + x_3 y_1 + x_1 y_2 - x_3 y_2 - x_1 y_3 + x_2 y_3$. Often, the linear shape functions can be deduced directly from a figure. However, the generic expressions work also when intuition does not:

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} -h/2 \\ h/2 \\ 0 \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ h \end{Bmatrix} \quad \Leftrightarrow \quad \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} = \frac{1}{2h}\begin{Bmatrix} h - 2x - y \\ h + 2x - y \\ 2y \end{Bmatrix}.$$

With the given nodal values, element interpolant (approximation) becomes

$$u_x = \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{x1} \\ u_{x2} \\ u_{x3} \end{Bmatrix} = \left(\frac{1}{2} - \frac{x}{h} - \frac{y}{2h}\right)\mathrm{a} + \left(\frac{1}{2} + \frac{x}{h} - \frac{y}{2h}\right)(-\mathrm{a}) + \frac{y}{h}2\mathrm{a} = 2\frac{\mathrm{a}}{h}(y - x).$$

Thus

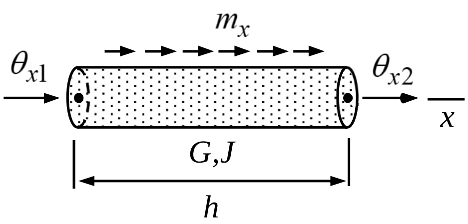
$$\frac{\partial u_x}{\partial x} = -2\frac{\mathrm{a}}{h},$$

$$\frac{\partial u_x}{\partial y} = 2\frac{\mathrm{a}}{h},$$

$$\int_{\Omega^e} u_x d\Omega = \int_0^h \left[\int_{(y-h)/2}^{(h-y)/2} 2\frac{\mathrm{a}}{h}(y-x)dx\right]dy = \int_0^h 2\frac{\mathrm{a}}{h} y(h-y)dy = \frac{1}{3}\mathrm{a}h^2 .$$

Name______________________ Student number________________

Assignment 1



Derive the virtual work expression $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}}$ of the bar element (length h) if the approximation is linear (a two-node element) and J , G and m_x are constants.

Solution template

Virtual work densities of the internal and external forces of the torsion bar model are

$$\delta w_{\Omega}^{\text{int}} = -\frac{d\delta\phi}{dx} GJ \frac{d\phi}{dx} \quad \text{and} \quad \delta w_{\Omega}^{\text{ext}} = \delta\phi m_x ,$$

in which J is the second moment of area with respect to x-axis, G is the shear modulus, and m_x is the external moment per unit length.

Let us start with the linear approximation to the rotation angle. The origin of the material coordinate system is at node 1 and the length of the bar is h.

$$\phi = \frac{1}{h}\begin{Bmatrix} h-x \\ x \end{Bmatrix}^{\text{T}} \begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix}, \quad \frac{d\phi}{dx} = \frac{1}{h}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}^{\text{T}} \begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix},$$

$$\delta\phi = \frac{1}{h}\begin{Bmatrix} h-x \\ x \end{Bmatrix}^{\text{T}} \begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}, \quad \frac{d\delta\phi}{dx} = \frac{1}{h}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}^{\text{T}} \begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}.$$

When the approximation is substituted there, virtual works of internal and external forces per unit take the forms

$$\delta w_{\Omega}^{\text{int}} = -\frac{d\delta\phi}{dx} GJ \frac{d\phi}{dx} = -\begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}^{\text{T}} \frac{GJ}{h^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix},$$

$$\delta w_{\Omega}^{\text{ext}} = \delta\phi m_x = \begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}^{\text{T}} \frac{m_x}{h} \begin{Bmatrix} h-x \\ x \end{Bmatrix}.$$

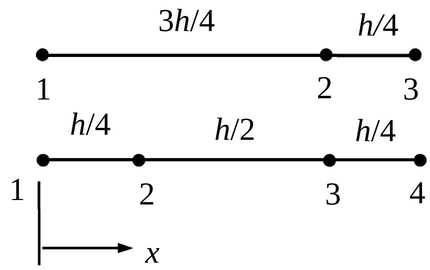
Virtual work expressions are integrals of the densities over the length. Then, virtual work expressions of the bar element are given by

$$\delta W^{\text{int}} = \int_0^h \delta w_{\Omega}^{\text{int}} dx = -\begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}^{\text{T}} \frac{GJ}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix}, \quad \leftarrow$$

$$\delta W^{\text{ext}} = \int_0^h \delta w_\Omega^{\text{ext}} dx = \begin{Bmatrix} \delta\theta_{x1} \\ \delta\theta_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{m_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

Name______________________ Student number________________

Assignment 2



Write down the shape function expressions of the elements shown in the figure. Deduce the expressions using the Lagrange interpolation polynomial.

Solution template

Derivation with the Lagrange interpolation polynomials is convenient in the one-dimensional case. The idea is to write a polynomial vanishing on some set of points followed by scaling for the value one at a certain point. In the first case of three-node element

$$N_1(x)=\frac{(x-\frac{3}{4}h)(x-h)}{(0-\frac{3}{4}h)(0-h)}=\frac{4}{3h^2}(x-\frac{3}{4}h)(x-h) \ ,$$

$$N_2(x)=\frac{(x)(x-h)}{(\frac{3}{4}h)(\frac{3}{4}h-h)}=-\frac{16}{3h^2}(x)(x-h) \ ,$$

$$N_3(x)=\frac{(x-0)(x-\frac{3}{4}h)}{(h-0)(h-\frac{3}{4}h)}=\frac{4}{h^2}(x)(x-\frac{3}{4}h) \, .$$

Considering the four-node element

$$N_1(x)=\frac{(x-\frac{1}{4}h)(x-\frac{3}{4}h)(x-h)}{(0-\frac{1}{4}h)(0-\frac{3}{4}h)(0-h)}=-\frac{16}{3h^3}(x-\frac{1}{4}h)(x-\frac{3}{4}h)(x-h) \, ,$$

$$N_2(x)=\frac{(x-0)(x-\frac{3}{4}h)(x-h)}{(\frac{1}{4}h-0)(\frac{1}{4}h-\frac{3}{4}h)(\frac{1}{4}h-h)}=\frac{32}{3h^3}(x-0)(x-\frac{3}{4}h)(x-h) \, ,$$

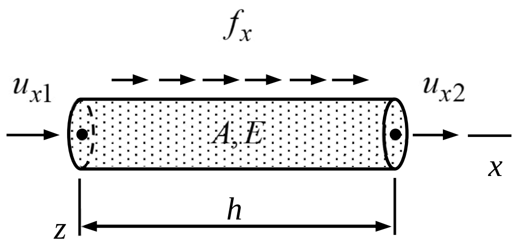
$$N_3(x)=\frac{(x-0)(x-\frac{1}{4}h)(x-h)}{(\frac{3}{4}h-0)(\frac{3}{4}h-\frac{1}{4}h)(\frac{3}{4}h-h)}=-\frac{32}{3h^3}(x-0)(x-\frac{1}{4}h)(x-h) \, ,$$

$$N_4(x) = \frac{(x-0)(x-\frac{1}{4}h)(x-\frac{3}{4}h)}{(h-0)(h-\frac{1}{4}h)(h-\frac{3}{4}h)} = -\frac{16}{3h^3}(x-0)(x-\frac{1}{4}h)(x-\frac{3}{4}h)\,.$$

Name_________________________ Student number_________________

Assignment 3

Consider a bar element when A and E are constants. Distributed external force f_x is piecewise constant with values $f_x = f_{x1}$ when $x < h/2$ and $f_x = f_{x2}$ when $x > h/2$. Derive the virtual work expression of a linear bar element. Use the virtual work density expression $\delta w_\Omega = -(d\delta u/dx)EA(du/dx) + \delta u f_x$ and approximation $u = (1 - x/h)u_{x1} + (x/h)u_{x2}$.



Solution template

The concise representation of the element contribution consists of a virtual work density expression and approximations to the displacement and rotation components. Approximations are first substituted into the density expression which is followed by integration over the domain occupied by the element (line segment, triangle etc.). For the two-node bar element the two building blocks are

$$\delta w_\Omega = -\frac{d\delta u}{dx} EA \frac{du}{dx} + \delta u f_x \quad \text{and} \quad u = (1 - \frac{x}{h})u_{x1} + \frac{x}{h}u_{x2}.$$

The quantities needed in the virtual work density are the axial displacement, variation of the axial displacement, and variation of the derivative of the axial displacement

$$u = \begin{Bmatrix} 1 - x/h \\ x/h \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} \Rightarrow \quad \delta u = \begin{Bmatrix} 1 - x/h \\ x/L \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 1 - x/h \\ x/h \end{Bmatrix},$$

$$\frac{du}{dx} = \begin{Bmatrix} -1/h \\ 1/h \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} \Rightarrow \quad \frac{d\delta u}{dx} = \begin{Bmatrix} -1/h \\ 1/h \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} -1/h \\ 1/h \end{Bmatrix}.$$

When the approximation is substituted there, virtual work densities of the internal and external forces take the forms

$$\delta w_\Omega^{\text{int}} = -\frac{d\delta u}{dx} EA \frac{du}{dx} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} EA/h^2 & -EA/h^2 \\ -EA/h^2 & EA/h^2 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix},$$

$$\delta w_\Omega^{\text{ext}} = \delta u f_x = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 1 - x/h \\ x/h \end{Bmatrix} f_x \quad \text{where} \quad f_x = \begin{cases} f_{x1} & x < h/2 \\ f_{x2} & x > h/2 \end{cases}$$

Integration over the element gives the virtual work expressions of the internal and external forces

$$\delta W^{\text{int}} = \int_0^h \delta w_\Omega^{\text{int}} dx = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} EA/h & -EA/h \\ -EA/h & EA/h \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix},$$

$$\delta W^{\text{ext}} = \int_0^{h/2} \delta w_\Omega^{\text{ext}} dx + \int_{h/2}^{h} \delta w_\Omega^{\text{ext}} dx = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 3h/8 & h/8 \\ h/8 & 3h/8 \end{bmatrix} \begin{Bmatrix} f_{x1} \\ f_{x2} \end{Bmatrix}.$$

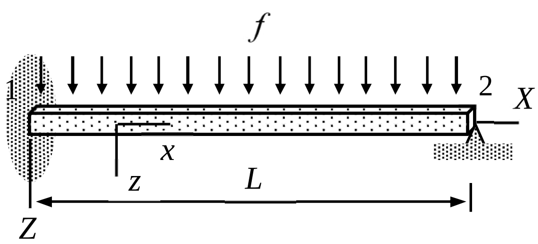
Virtual work expression of bar element is the sum of internal and external parts

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \left(\begin{bmatrix} EA/h & -EA/h \\ -EA/h & EA/h \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} - \begin{bmatrix} 3h/8 & h/8 \\ h/8 & 3h/8 \end{bmatrix} \begin{Bmatrix} f_{x1} \\ f_{x2} \end{Bmatrix} \right).$$

Name_________________________ Student number_________________

Assignment 4

Determine rotation θ_{Y2} of the bending beam shown at the support of the right end (use one element). The x-axis of the material coordinate system coincides with the neutral axis of the beam. Young's modulus E of the material and the second moment of cross-section $I_{yy} = I$ are constants. Use the virtual work density of the beam $xz-$plane bending mode and a cubic approximation to the transverse displacement.



Solution template

In the $xz-$plane problem bending problem, when x-axis is chosen to coincide with the neutral axis, virtual work densities of the Bernoulli beam model are

$$\delta w_{\Omega}^{\text{int}} = -\frac{d^2\delta w}{dx^2} EI_{yy} \frac{d^2 w}{dx^2} \quad \text{and} \quad \delta w_{\Omega}^{\text{ext}} = \delta w f_z \, .$$

Approximation is the first thing to be considered. The left end of the beam is clamped and the right end support allows only rotation. As only $\theta_{y2} = \theta_{Y2}$ is non-zero, approximation to w in terms of $\xi = x/L$ simplifies into the form (see the formulae collection for the cubic beam bending approximation)

$$w(\xi) = \begin{Bmatrix} (1-\xi)^2(1+2\xi) \\ L(1-\xi)^2\xi \\ \hdashline (3-2\xi)\xi^2 \\ L\xi^2(\xi-1) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{z1} \\ -\theta_{y1} \\ \hdashline u_{z2} \\ -\theta_{y2} \end{Bmatrix} = \begin{Bmatrix} (1-\xi)^2(1+2\xi) \\ L(1-\xi)^2\xi \\ \hdashline (3-2\xi)\xi^2 \\ L\xi^2(\xi-1) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ 0 \\ \hdashline 0 \\ -\theta_{Y2} \end{Bmatrix} = L\xi^2(1-\xi)\theta_{Y2} \qquad \Rightarrow$$

$$w(x) = \frac{1}{L^2} x^2 (L-x)\theta_{Y2} \qquad \Rightarrow \qquad \frac{d^2 w}{dx^2} = \frac{1}{L^2}(2L-6x)\theta_{Y2} \quad \text{so}$$

$$\delta w(x) = \frac{1}{L^2} x^2 (L-x)\delta\theta_{Y2} \quad \text{and} \quad \frac{d^2\delta w}{dx^2} = \frac{1}{L^2}(2L-6x)\delta\theta_{Y2} \, .$$

When the approximation is substituted there, virtual work densities of the internal and external forces (external distributed force is constant) simplify to

$$\delta w_{\Omega}^{\text{int}} = -\frac{d^2\delta w}{dx^2} EI_{yy} \frac{d^2 w}{dx^2} = -\delta\theta_{Y2} \frac{EI}{L^4} (2L - 6x)^2 \theta_{Y2} ,$$

$$\delta w_{\Omega}^{\text{ext}} = \delta w f_z = \delta\theta_{Y2} \frac{1}{L^2} x^2 (L - x) f \ .$$

Integration over the domain $\Omega =]0, L[$ gives the virtual work expressions of the internal and external forces

$$\delta W^{\text{int}} = \int_0^L \delta w_{\Omega}^{\text{int}} dx = -\delta\theta_{Y2} 4 \frac{EI}{L} \theta_{Y2} ,$$

$$\delta W^{\text{ext}} = \int_0^L \delta w_{\Omega}^{\text{ext}} dx = \delta\theta_{Y2} \frac{1}{12} L^2 f \ .$$

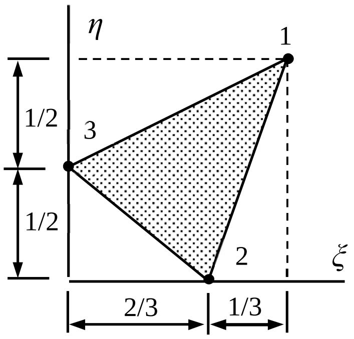
Principle of virtual work $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = 0 \quad \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus imply the solution

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = -\delta\theta_{Y2} (4 \frac{EI}{L} \theta_{Y2} - \frac{1}{12} L^2 f) = 0 \quad \forall \delta\theta_{Y2} \quad \Leftrightarrow$$

$$4 \frac{EI}{L} \theta_{Y2} - \frac{1}{12} L^2 f = 0 \quad \Leftrightarrow \quad \theta_{Y2} = \frac{1}{48} \frac{L^3 f}{EI} .$$

Name________________________ Student number________________

Assignment 5



Derive the shape function expressions for the three-node element shown in terms of the scaled coordinates $\xi = x/h$ and $\eta = y/h$. Also, write down the shape functions in terms of the material coordinates x and y.

Solution

Shape functions of the linear triangle element are given by the simple formula

$$\mathbf{N} = \begin{bmatrix} 1 & 1 & 1 \\ \xi_1 & \xi_2 & \xi_3 \\ \eta_1 & \eta_2 & \eta_3 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ \xi \\ \eta \end{Bmatrix}$$

in which the subscripts refer to coordinates of the three nodes. Columns of the matrix contain the coordinates of nodes and indexing does not matter. With the present triangle (using the determinant rule for the inverse)

$$\mathbf{N} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2/3 & 0 \\ 1 & 0 & 1/2 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ \xi \\ \eta \end{Bmatrix} = \frac{1}{1/3-1/2-2/3} \begin{bmatrix} 1/3 & -1/2 & -2/3 \\ -1/2 & -1/2 & 1 \\ -2/3 & 1 & -1/3 \end{bmatrix}^{\mathrm{T}} \begin{Bmatrix} 1 \\ \xi \\ \eta \end{Bmatrix} \Leftrightarrow$$

$$\mathbf{N} = -\frac{6}{5} \begin{bmatrix} 1/3 & -1/2 & -2/3 \\ -1/2 & -1/2 & 1 \\ -2/3 & 1 & -1/3 \end{bmatrix} \begin{Bmatrix} 1 \\ \xi \\ \eta \end{Bmatrix} = \frac{1}{5} \begin{bmatrix} -2 & 3 & 4 \\ 3 & 3 & -6 \\ 4 & -6 & 2 \end{bmatrix} \begin{Bmatrix} 1 \\ \xi \\ \eta \end{Bmatrix} = \frac{1}{5} \begin{Bmatrix} -2+3\xi+4\eta \\ 3+3\xi-6\eta \\ 4-6\xi+2\eta \end{Bmatrix}. \quad \leftarrow$$

Representations in terms of the material coordinates follow by substituting the relationships $\xi = x/h$ and $\eta = y/h$

$$\mathbf{N} = \frac{1}{5} \begin{Bmatrix} -2+3x/h+4y/h \\ 3+3x/h-6y/h \\ 4-6x/h+2y/h \end{Bmatrix} = \frac{1}{5h} \begin{Bmatrix} -2h+3x+4y \\ 3h+3x-6y \\ 4h-6x+2y \end{Bmatrix}. \quad \leftarrow$$

MEC-E1050

FINITE ELEMENT METHOD IN SOLIDS 2024

Week 47-0

5 VIRTUAL WORK DENSITY

LEARNING OUTCOMES

Students are able to solve the lecture problems, home problems, and exercise problems on the topics of the week:

- ☐ The concepts, quantities, and equations of linear elasticity theory.
- ☐ Principle of virtual work for linear elasticity and virtual work densities.
- ☐ Virtual work densities of the solid, thin slab, bar, and torsion models.

BALANCE LAWS OF MECHANICS

Balance of mass (def. of a body or a material volume) Mass of a body is constant

Balance of linear momentum (Newton 2) The rate of change of linear momentum within a material volume equals the external force resultant acting on the material volume.

Balance of angular momentum (Cor. of Newton 2) The rate of change of angular momentum within a material volume equals the external moment resultant acting on the material volume.

Balance of energy (Thermodynamics 1)

Entropy growth (Thermodynamics 2)

5.1 LINEAR ELASTICITY

In the usual setting, a reference solution $(\vec{\vec{\sigma}}^\circ, \vec{u}^\circ, V^\circ)$ with $\vec{u}^\circ = 0$ is assumed known. The goal is to find a new solution $(\vec{\vec{\sigma}}, \vec{u}, V)$ corresponding to a slightly changed setting.

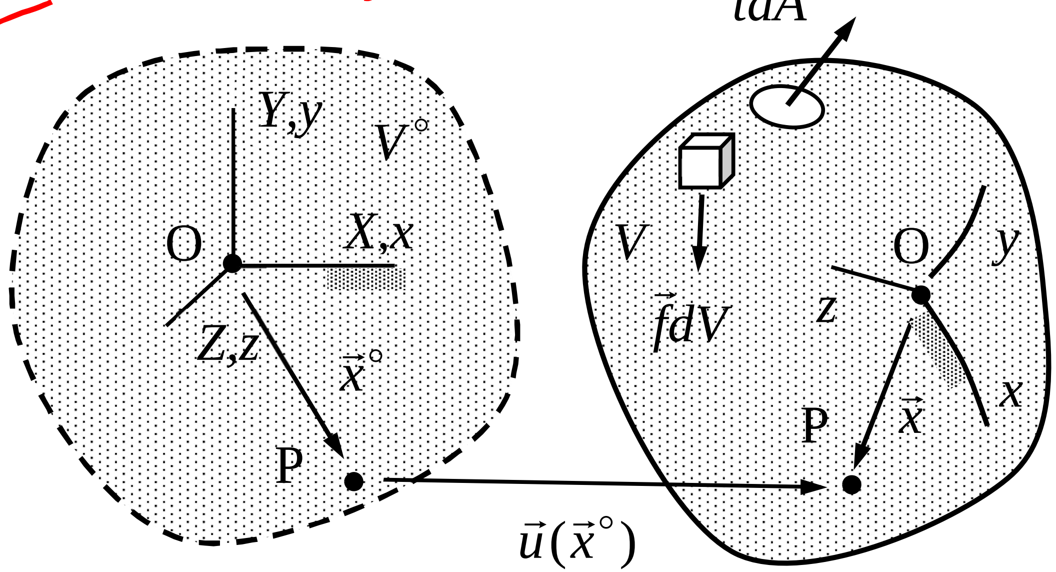
deformed body

$$\dot{m} = 0 \quad : \quad \rho^\circ = J\rho$$

$$\dot{\vec{p}} = \vec{F} \quad : \quad \nabla \cdot \vec{\vec{\sigma}} + \vec{f} = 0 \quad \vec{r} \in V$$

$$\dot{\vec{p}} = \vec{F} \quad : \quad \vec{n} \cdot \vec{\vec{\sigma}} = \vec{t} \quad \vec{r} \in \partial V_t$$

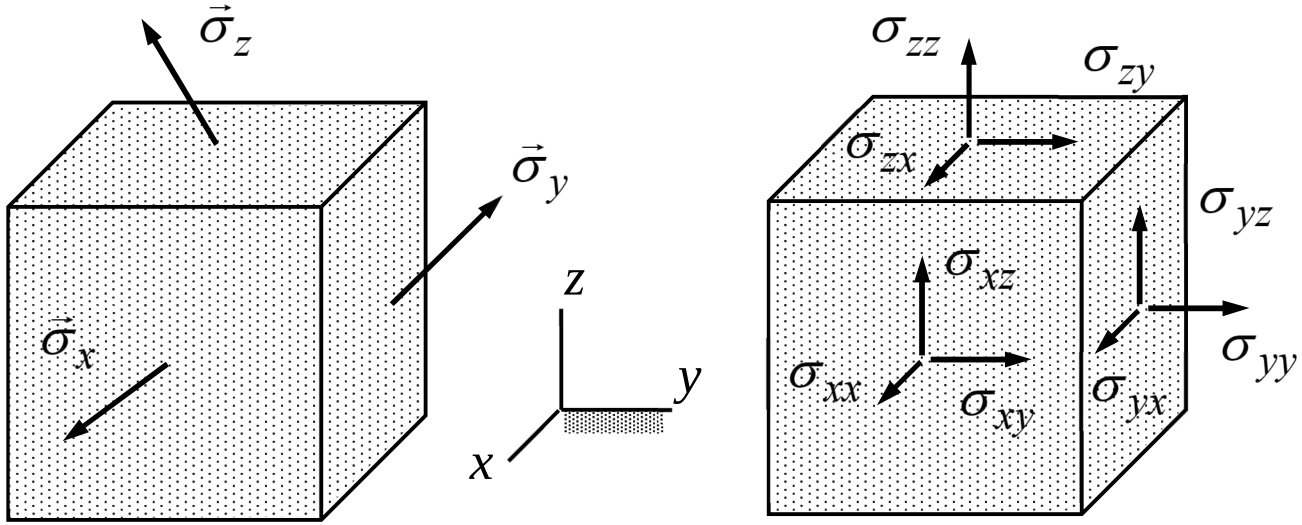
$$\dot{\vec{L}} = \vec{M} \quad : \quad \vec{\vec{\sigma}} = \vec{\vec{\sigma}}_{\mathrm{c}} \quad \vec{r} \in V$$



A constitutive equation of type $g(\vec{\vec{\sigma}}, \vec{u}) = 0$, bringing the material details into the model, and displacement boundary conditions are also needed.

TRACTION AND STRESS

Traction vector $\vec{\sigma} = \lim \Delta\vec{F} / \Delta A$ describes the internal force acting on a surface element in classical continuum mechanics. Stress tensor $\vec{\vec{\sigma}}$ describes all internal forces acting on a material element. The quantities are related by $\vec{\sigma} = \vec{n} \cdot \vec{\vec{\sigma}}$ where $\vec{n}$ is the unit normal to a material surface.



The first index of a stress component refers to the direction of the surface normal and the second that of the force component (on opposite sides directions are the opposite).

GENERALIZED HOOKE'S LAW

The isotropic homogeneous material material model $g(\vec{\sigma},\vec{u})=0$ of the present course can be expressed, e.g., in its compliance form as

Strain-stress: $\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} = \dfrac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} = [E]^{-1}\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix}$ and $\begin{Bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \dfrac{1}{G}\begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix}$

Strain-displacement: $\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} = \begin{Bmatrix} \partial u_x / \partial x \\ \partial u_y / \partial y \\ \partial u_z / \partial z \end{Bmatrix}$ and $\begin{Bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{Bmatrix} \partial u_x / \partial y + \partial u_y / \partial x \\ \partial u_y / \partial z + \partial u_z / \partial y \\ \partial u_z / \partial x + \partial u_x / \partial z \end{Bmatrix}$

Above, E is the Young's modulus, ν the Poisson's ratio, and $G = E/(2+2\nu)$ the shear modulus. Strain and stress are assumed to be symmetric.

MATERIAL PARAMETERS

Material	ρ [kg / m^3]	E [GN / m^2]	ν [1]
Steel	7800	210	0.3
Aluminum	2700	70	0.33
Copper	8900	120	0.34
Glass	2500	60	0.23
Granite	2700	65	0.23
Birch	600	16	-
Rubber	900	10^{-2}	0.5
Concrete	2300	25	0.1

EXAMPLE. Determine the *stress-strain* relationship of linear isotropic material subjected to (a) $xy-$plane stress and (b) $x-$axis stress (uni-axial) conditions. Start with the generic *strain-stress* relationships

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \frac{1}{G} \begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix}.$$

Answer: (a) $\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \end{Bmatrix} = \dfrac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \end{Bmatrix}$ and $\sigma_{xy} = G\gamma_{xy}$ **(**b) $\sigma_{xx} = E\varepsilon_{xx}$

- In $xy-$plane stress, the stress components having at least one z as an index vanish (notice that the corresponding strain components need not to vanish)

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ 0 \end{Bmatrix} \text{ and } \begin{Bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \frac{1}{G} \begin{Bmatrix} \sigma_{xy} \\ 0 \\ 0 \end{Bmatrix} \Rightarrow$$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \end{Bmatrix} \text{ and } \sigma_{xy} = G\gamma_{xy} \Leftarrow$$

- In the $x-$axis stress, components having y or z as an index vanish:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ 0 \\ 0 \end{Bmatrix} \text{ and } \begin{Bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \frac{1}{G} \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \Rightarrow \sigma_{xx} = E\varepsilon_{xx} \Leftarrow$$

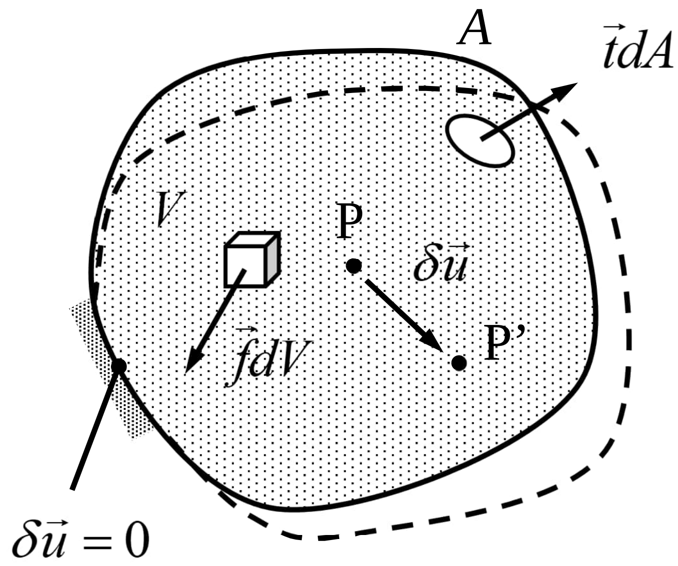
EXAMPLE Determine the *stress-strain* relationship of linearly elastic isotropic material subjected to (a) $xy-$plane stress and (b) $xy-$plane strain conditions. Start with the generic *strain-stress* relationship.

Answer: (a) $$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{Bmatrix}=[E]_\sigma\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{Bmatrix} \text{ where } [E]_\sigma=\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & (1-\nu)/2\end{bmatrix}$$

(b) $$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{Bmatrix}=[E]_\varepsilon\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{Bmatrix} \text{ with } [E]_\varepsilon=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & (1-2\nu)/2\end{bmatrix}$$

5.2 PRINCIPLE OF VIRTUAL WORK

Principle of virtual work $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = 0 \;\; \forall \delta\vec{u}$ is just one form of the balance laws of continuum mechanics. It is important due to its wide applicability and physical meanings of the terms.



$$\delta W^{\text{int}} = \int_V \delta w_V^{\text{int}} dV = -\int_V (\vec{\vec{\sigma}} : \delta\vec{\vec{\varepsilon}}_{\text{c}}) dV$$

$$\delta W_V^{\text{ext}} = \int_V \delta w_V^{\text{ext}} dV = \int_V (\vec{f} \cdot \delta\vec{u}) dV$$

$$\delta W_A^{\text{ext}} = \int_A \delta w_A^{\text{ext}} dA = \int_A (\vec{t} \cdot \delta\vec{u}) dA$$

The details of the final expressions may vary case by case, but the but the starting point is always the generic expressions above!

- Let us consider the balance law of momentum in its local form. Multiplication by the variation of the displacement, integration over the domain, and integration by parts give

$$\nabla\cdot\vec{\vec{\sigma}}+\vec{f}=0 \quad \vec{r}\in V \quad \Leftrightarrow$$

$$\int_V (\nabla\cdot\vec{\vec{\sigma}}+\vec{f})\cdot\delta\vec{u}dV=\int_V (-\vec{\vec{\sigma}}:(\nabla\delta\vec{u})_{\mathrm{c}}+\vec{f}\cdot\delta\vec{u})dV+\int_A (\vec{n}\cdot\vec{\vec{\sigma}}\cdot\delta\vec{u})dA=0 \quad \forall\delta\vec{u}$$

- Balance of momentum $\vec{t}=\vec{n}\cdot\vec{\vec{\sigma}}$ written for the boundary, local form of moment of momentum $\vec{\vec{\sigma}}=\vec{\vec{\sigma}}_{\mathrm{c}}$, and the definition of linear strain $2\vec{\vec{\varepsilon}}=\nabla\vec{u}+(\nabla\vec{u})_{\mathrm{c}}$ give the final form

$$\delta W=-\int_V (\vec{\vec{\sigma}}:\delta\vec{\vec{\varepsilon}}_{\mathrm{c}})dV+\int_V (\vec{f}\cdot\delta\vec{u})dV+\int_A (\vec{t}\cdot\delta\vec{u})dA=0 \quad \forall\delta\vec{u}\in U\,.$$

DEFINITIONS AND NOTATIONS

- **Domain** $\Omega \subset \mathbb{R}^n$**, boundary** $\partial\Omega$, and subset of the boundary $\partial\Omega_t$, $\partial\Omega_u$, etc.

- **Function sets:** $C^0(\Omega)$ (continuous functions on Ω), $C^1(\Omega)$, $L_2(\Omega)$ etc.

- **Notations** $\exists$ ~ "exists" & $\forall$ ~ "for all" & $\vee$ ~ "or" & $\wedge$ ~ "and"

- **Fundamental theorem of calculus** (integration by parts) $u, v \in C^0(\Omega)$

$$\int_\Omega u \frac{\partial v}{\partial \alpha} d\Omega = \int_{\partial\Omega} (n_\alpha uv) d\Gamma - \int_\Omega v \frac{\partial u}{\partial \alpha} d\Omega \quad \alpha \in \{x, y, z, \ldots\}$$

- **Fundamental lemma of variation calculus** $u, v \in C^0(\Omega)$

$$\int_\Omega uv d\Omega = 0 \ \forall v \Leftrightarrow u = 0 \text{ in } \Omega$$

VIRTUAL WORK DENSITIES

Virtual work densities of the internal forces, external volume forces, and external surface forces are (subscripts V and A denote virtual work per unit volume and area, respectively)

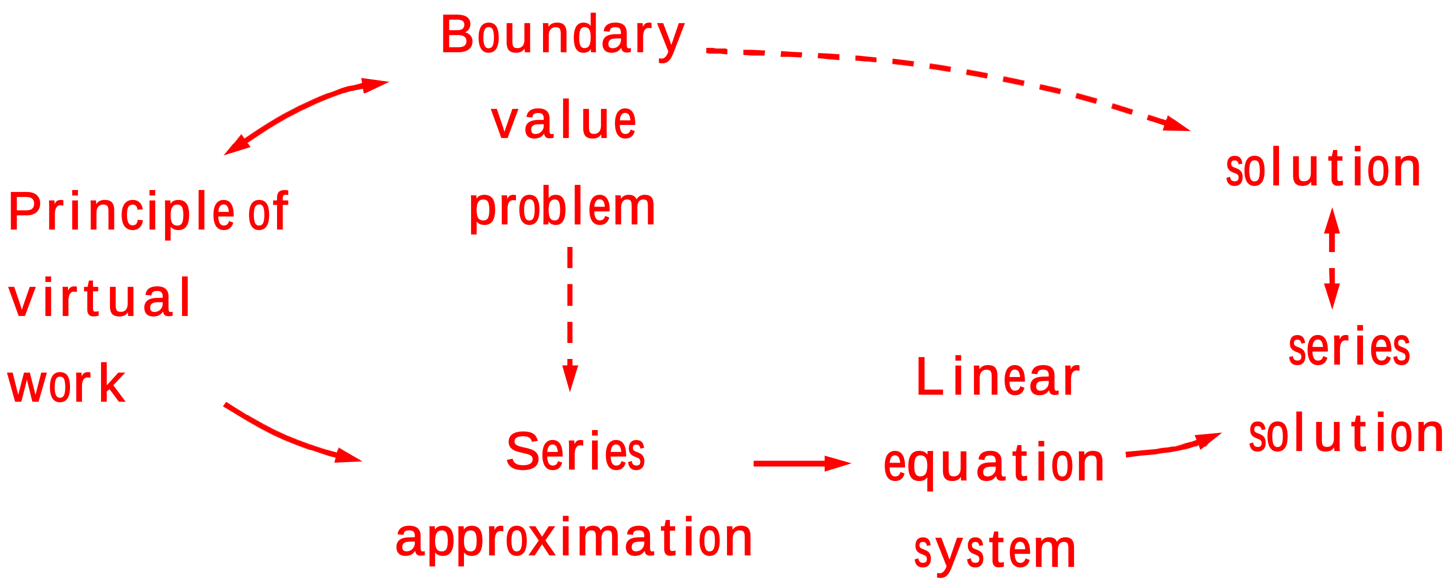
Internal forces: $$\delta w_V^{\text{int}} = -\begin{Bmatrix} \delta\varepsilon_{xx} \\ \delta\varepsilon_{yy} \\ \delta\varepsilon_{zz} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} - \begin{Bmatrix} \delta\gamma_{xy} \\ \delta\gamma_{yz} \\ \delta\gamma_{zx} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix}$$

External forces: $$\delta w_V^{\text{ext}} = \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix}, \quad \delta w_A^{\text{ext}} = \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix}.$$

Virtual work densities consist of terms containing kinematic quantities and their “work conjugates”!

PRINCIPLE OF VIRTUAL WORK IN MECHANICS

The common dimension reduction (engineering) models like beam, plate, shell etc. models have their origin in the principle of virtual work priciple. The principle is also the starting point for numerical solution methods of various types: A series approximation is substituted there to end up with an algebraic equations system for the unknown parameters.



5.3 ENGINEERING MODELS

Engineering (dimension reduction) models are defined concisely by their *kinematic* and *kinetic* assumptions. The rest is pure mathematics based on the principle of virtual work.

Bar: $\vec{u}(x,y,z)=\vec{u}_0(x)$ $\sigma_{xx}\neq 0$ only

Beam: $\vec{u}(x,y,z)=\vec{u}_0(x)+\vec{\theta}_0(x)\times\vec{\rho}(y,z)$ $\sigma_{yy}=\sigma_{zz}=0$

Curved beam: $\vec{u}(s,n,b)=\vec{u}_0(s)+\vec{\theta}_0(s)\times\vec{\rho}(n,b)$ $\sigma_{nn}=\sigma_{bb}=0$

Thin slab: $\vec{u}(x,y,z)=\vec{u}_0(x,y)$ $\sigma_{zz}=\sigma_{yz}=\sigma_{zx}=0$

Membrane: $\vec{u}(s,n,b)=\vec{u}(s,n)$, $\sigma_{ss}\neq 0$, $\sigma_{nn}\neq 0$, $\sigma_{sn}\neq 0$ only

Plate: $\vec{u}(x,y,z)=\vec{u}(x,y)+\vec{\theta}(x,y)\times\vec{\rho}(z)$ $\sigma_{zz}=0$

Shell: $\vec{u}(s,n,b)=\vec{u}(s,n)+\vec{\theta}(s,n)\times\vec{\rho}(b)$ $\sigma_{bb}=0$

VIRTUAL WORK DENSITY OF A MODEL

Virtual work density δw serves as a concise representation of the model in the recipe for the element contribution. To derive the virtual work density (bar, beam plate, shell etc.)

- ☐ Start with the virtual work expression $\delta W = \int_V \delta w_V dV$ of an elastic body. Use the kinematical and kinetic assumptions of the model to simplify δw_V. After that,

- ☐ integrate over the small dimension(s) to end up with expression $\delta W = \int_\Omega \delta w_\Omega d\Omega$, where the remaining integral is over the mathematical solution domain Ω. Then, virtual work density of the model is δw_Ω.

The dimension of domain Ω is smaller than that of the physical domain due to the integration over the small dimensions.

5.4 SOLID MODEL

Solid model does not contain any assumptions in addition to those of the generic linear elasticity theory. Therefore

$$\delta w_V^{\text{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta w/\partial z \end{Bmatrix}^{\mathrm{T}} [E] \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial w/\partial z \end{Bmatrix} - \begin{Bmatrix} \partial\delta u/\partial y + \partial\delta v/\partial x \\ \partial\delta v/\partial z + \partial\delta w/\partial y \\ \partial\delta w/\partial x + \partial\delta u/\partial z \end{Bmatrix}^{\mathrm{T}} G \begin{Bmatrix} \partial u/\partial y + \partial v/\partial x \\ \partial v/\partial z + \partial w/\partial y \\ \partial w/\partial x + \partial u/\partial z \end{Bmatrix},$$

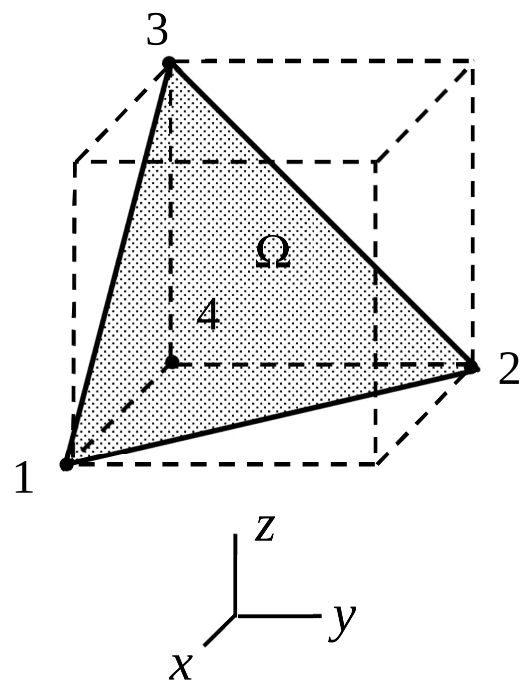
$$\delta w_V^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix}, \quad \delta w_A^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix}.$$

The simplest element is a four-node tetrahedron with linear interpolations to the three displacement components $u(x,y,z)$, $v(x,y,z)$, and $w(x,y,z)$.

- Usually, the approximations to the components are of the same type. For example, the linear element interpolant for a tetrahedron element shape is given by

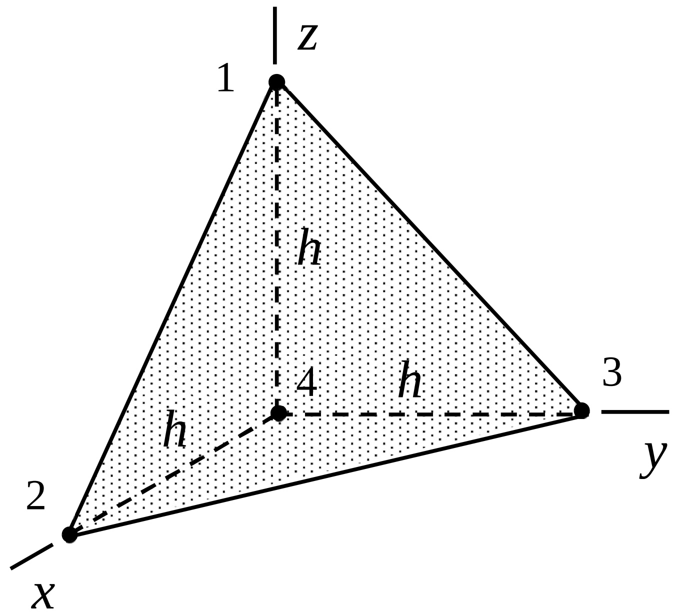
$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} u_{x1} & u_{x2} & u_{x3} & u_{x4} \\ u_{y1} & u_{y2} & u_{y3} & u_{y4} \\ u_{z1} & u_{z2} & u_{z3} & u_{z4} \end{bmatrix} \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{Bmatrix} \text{ where}$$

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \\ y \\ z \end{Bmatrix}.$$



The solid model works with any geometry but use of plate, shell, beam, bar etc. models may mean huge savings in computational complexity as dimension of the mathematical solution domain is smaller than 3 (physical dimensions)!

EXAMPLE 5.1 Compute the virtual work expression of external volume force with the components $f_x = \text{constant}$ and $f_y = f_z = 0$. Consider the tetrahedron element shown and assume that the shape functions are linear (a four-node tetrahedron element).



Answer: $$\delta W_V^{\text{ext}} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \\ \delta u_{x3} \\ \delta u_{x4} \end{Bmatrix}^{\mathrm{T}} \frac{f_x h^3}{24} \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}$$

- The linear shape functions can be deduced directly from the figure $N_1 = z/h$, $N_2 = x/h$, $N_3 = y/h$ and $N_4 = 1 - x/h - y/h - z/h$ (sum of the shape functions is 1). Therefore, the approximation is given by

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{Bmatrix} N_1 u_{x1} + N_2 u_{x2} + N_3 u_{x3} + N_4 u_{x4} \\ N_1 u_{y1} + N_2 u_{y2} + N_3 u_{y3} + N_4 u_{y4} \\ N_1 u_{z1} + N_2 u_{z2} + N_3 u_{z3} + N_4 u_{z4} \end{Bmatrix} = \begin{bmatrix} u_{x1} & u_{x2} & u_{x3} & u_{x4} \\ u_{y1} & u_{y2} & u_{y3} & u_{y4} \\ u_{z1} & u_{z2} & u_{z3} & u_{z4} \end{bmatrix} \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{Bmatrix}.$$

- Virtual work density of the external volume forces is

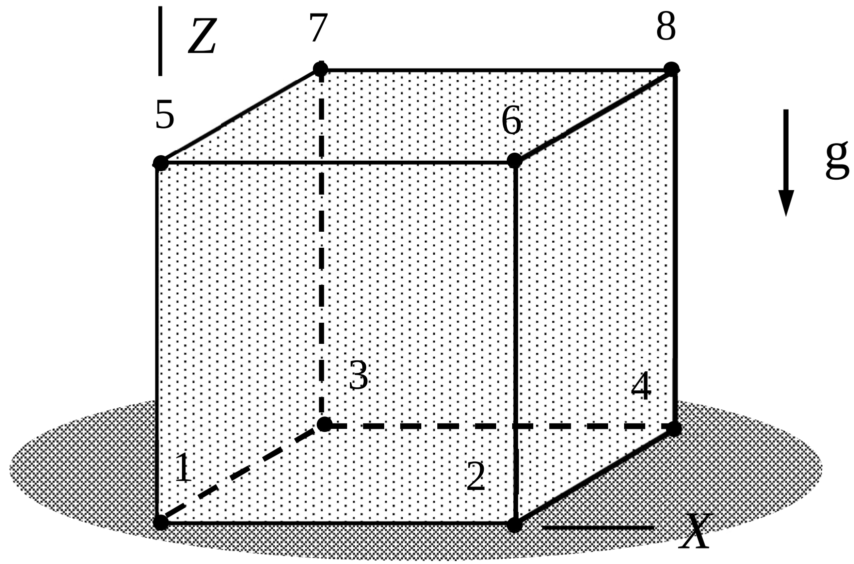
$$\delta w_V^{\mathrm{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \delta u_{x1} & \delta u_{y1} & \delta u_{z1} \\ \delta u_{x2} & \delta u_{y2} & \delta u_{z2} \\ \delta u_{x3} & \delta u_{y3} & \delta u_{z3} \\ \delta u_{x4} & \delta u_{y4} & \delta u_{z4} \end{bmatrix} \begin{Bmatrix} f_x \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \\ \delta u_{x3} \\ \delta u_{x4} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{Bmatrix} f_x$$

- Virtual work expression of the external volume force is obtained as an integral over the volume:

$$\delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \\ \delta u_{x3} \\ \delta u_{x4} \end{Bmatrix}^{\mathrm{T}} \int_V \begin{Bmatrix} z/h \\ x/h \\ y/h \\ 1-x/h-y/h-z/h \end{Bmatrix} f_x dV = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \\ \delta u_{x3} \\ \delta u_{x4} \end{Bmatrix}^{\mathrm{T}} \frac{f_x h^3}{24} \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}. \quad \Leftarrow$$

The explicit form of the virtual work expression for a generic shape is too complicated to be practical (due to the large number of the geometric parameters involved).

EXAMPLE 5.2 A concrete cube of edge length L, density ρ, and elastic properties E, ν is subjected to its own weight on a horizontal floor. Calculate the displacement of the top surface with one hexahedron element and tri-linear approximation. Assume that displacement components in $X-$ and $Y-$directions vanish, $u_{Z5} = u_{Z6} = u_{Z7} = u_{Z8}$, and that the bottom surface is fixed.



Answer: $u_{Z5} = -\frac{1}{2}\frac{\rho g L^2}{E}\frac{1-\nu-2\nu^2}{1-\nu}$

- Let the material coordinate system coincide with the structural system. The shape functions for the upper surface nodes can be deduced directly from the figure. Approximations to the displacement components are $(\xi = x/L,\ \eta = y/L,\ \zeta = z/L)$

$$u = 0,\ v = 0,\ \text{and}\ \ w = \begin{Bmatrix} (1-\xi)(1-\eta)\zeta \\ \xi(1-\eta)\zeta \\ (1-\xi)\eta\zeta \\ \xi\eta\zeta \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{Z5} \\ u_{Z5} \\ u_{Z5} \\ u_{Z5} \end{Bmatrix} = \frac{z}{L} u_{Z5},\ \text{giving}\ \frac{\partial w}{\partial z} = \frac{1}{L} u_{Z5}\ .$$

- When the approximation is substituted there, the virtual work densities of the internal forces, external forces, and their sum simplify to

$$\delta w_V^{\mathrm{int}} = -\begin{Bmatrix} 0 \\ 0 \\ \partial\delta w/\partial z \end{Bmatrix}^{\mathrm{T}} \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \partial w/\partial z \end{Bmatrix},$$

$$\delta w_V^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ z/L\delta u_{Z5} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ 0 \\ -\rho g \end{Bmatrix},$$

$$\delta w_V = \delta w_V^{\text{int}} + \delta w_V^{\text{ext}} = -\delta u_{Z5}[\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\frac{u_{Z5}}{L^2} + \frac{z}{L}\rho g].$$

- Virtual work expression is obtained as integral of the density over the volume:

$$\delta W = \int_V \delta w_V dV = -\delta u_{Z5}[\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}Lu_{Z5} + \frac{L^3}{2}\rho g].$$

- Finally, principle of virtual work $\delta W = 0 \ \forall \delta u_{Z5}$ implies that

$$u_{Z5} = -\frac{1}{2}\frac{\rho g L^2}{E}\frac{1-\nu-2\nu^2}{1-\nu}.$$

5.5 THIN SLAB MODEL

Thin slab model is the in-plane mode of the plate model and also the solid model in two dimensions. The elasticity matrices $[E]_\sigma$ and $[E]_\varepsilon$ for the plane stress and plane strain versions differ.

Internal forces: $$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial\delta u / \partial x \\ \partial\delta v / \partial y \\ \partial\delta u / \partial y + \partial\delta v / \partial x \end{Bmatrix}^{\mathrm{T}} t[E]_\sigma \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix},$$

External forces: $$\delta w_\Omega^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix}, \quad \delta w_{\partial\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} t_x \\ t_y \end{Bmatrix}.$$

Although the virtual work density $\delta w_{\partial\Omega}^{\text{ext}}$ for the external line force t_x, t_y acting on the edges belongs to the thin slab, it will be treated separately (like point forces/moments)!

- Thin slab is a body which is thin in one dimension. The kinematic assumptions of the thin slab model $u_x = u(x,y)$ and $u_y = v(x,y)$ give the (non-zero) strain components

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix}.$$

- The kinetic assumptions of the plane stress version $\sigma_{zz} = \sigma_{yz} = \sigma_{zx} = 0$ (these are replaced by kinematic assumptions $\varepsilon_{zz} = \gamma_{yz} = \gamma_{zx} = 0$ in the plane strain version) and the generalized Hooke's law imply the stress-strain relationship

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \equiv [E]_\sigma \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}.$$

- Therefore, the generic virtual work densities (per unit volume) simplify first to

$$\delta w_V^{\text{int}} = -\begin{Bmatrix} \delta\varepsilon_{xx} \\ \delta\varepsilon_{yy} \\ \delta\varepsilon_{zz} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} - \begin{Bmatrix} \delta\gamma_{xy} \\ \delta\gamma_{yz} \\ \delta\gamma_{zx} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = -\begin{Bmatrix} \delta\varepsilon_{xx} \\ \delta\varepsilon_{yy} \\ \delta\gamma_{xy} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} \quad \Rightarrow$$

$$\delta w_V^{\text{int}} = -\begin{Bmatrix} \delta\varepsilon_{xx} \\ \delta\varepsilon_{yy} \\ \delta\gamma_{xy} \end{Bmatrix}^{\mathrm{T}} [E]_\sigma \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = -\begin{Bmatrix} \partial\delta u / \partial x \\ \partial\delta v / \partial y \\ \partial\delta u / \partial y + \partial\delta v / \partial x \end{Bmatrix}^{\mathrm{T}} [E]_\sigma \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix},$$

$$\delta w_V^{\text{ext}} = \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix} \text{ and } \delta w_A^{\text{ext}} = \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} t_x \\ t_y \end{Bmatrix}.$$

- Integration over the volume is performed in two steps: first over the thickness with $z \in [z_-, z_+]$ $(t = |z_+ - z_-|)$ and after that over the mid-plane $(x, y) \in \Omega$. As the virtual work density of internal forces does not depend on z and $dV = dzd\Omega$

$$\delta W^{\text{int}} = \int_\Omega (\int_{z_-}^{z_+} \delta w_V^{\text{int}} dz) d\Omega = \int_\Omega \delta w_\Omega^{\text{int}} d\Omega \quad \text{in which}$$

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial\delta u / \partial x \\ \partial\delta v / \partial y \\ \partial\delta u / \partial y + \partial\delta v / \partial x \end{Bmatrix}^{\mathrm{T}} t[E]_\sigma \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix}. \quad \leftarrow$$

- The contributions for the external forces follow in the same manner. Boundary of the body is divided into the lower, upper and edge parts A_-, A_+, $S = \partial\Omega \times [z_-, z_+]$, Surface area elements are $dA = d\Omega$ (upper and lower surfaces) and $dA = dzds$ (edge). The volume force acting on V and the surface forces on A_- and A_+ give

$$\delta W^{\text{ext}} = \int_{\Omega} \left(\int_{z_-}^{z_+} \delta w_V^{\text{ext}} dz + \sum_{z \in \{z_-, z_+\}} \delta w_A^{\text{ext}}\right) d\Omega = \int_{\Omega} \delta w_{\Omega}^{\text{ext}} d\Omega \quad \text{in which}$$

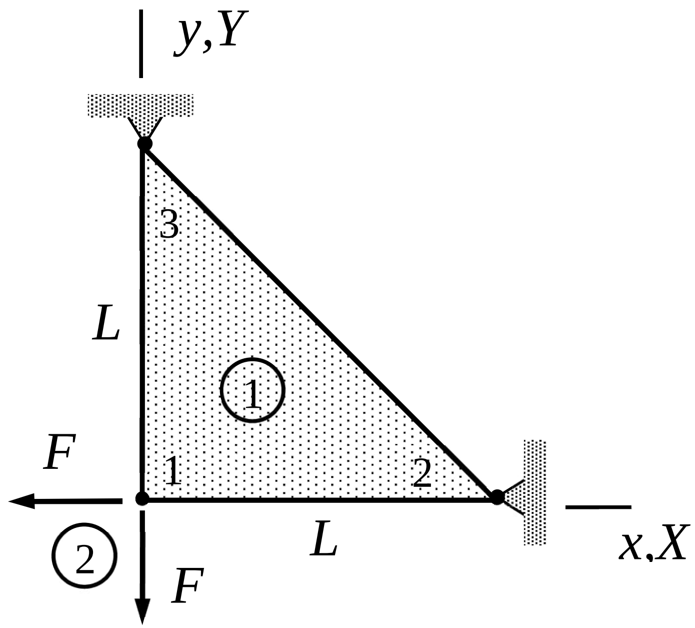
$$\delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \left(\int_{z_-}^{z_+} \delta w_V^{\text{ext}} dz \begin{Bmatrix} f_x \\ f_y \end{Bmatrix} + \sum_{z \in \{z_-, z_+\}} \begin{Bmatrix} t_x \\ t_y \end{Bmatrix}\right) = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix}. \quad \Leftarrow$$

- The contribution from the remaining boundary part $S = \partial\Omega \times [z_-, z_+]$

$$\delta W^{\text{ext}} = \int_{\partial\Omega} \int_{z_-}^{z_+} \delta w_A^{\text{ext}} dz ds = \int_{\partial\Omega} \delta w_{\partial\Omega}^{\text{ext}} ds \quad \text{where} \quad \delta w_{\partial\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} t_x \\ t_y \end{Bmatrix} \quad \Leftarrow$$

is part of the virtual work expression for the thin slab model. In practice, the distributed boundary force is taken into account by force elements by using the restriction of the element approximation to the boundary. With a linear or bilinear element, distributed force an element edge gives rise to a two-node force element.

EXAMPLE 5.3 Derive the virtual work expression for the linear triangle element shown. Young's modulus E, Poisson's ratio ν, and thickness t are constants. Distributed external force vanishes. Assume plane strain conditions. Also determine the displacement of node 1 when the force components acting on the node are as shown in the figure.



Answer: $\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} = -\dfrac{F}{Et}\dfrac{(1+\nu)(1-2\nu)}{1-\nu}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

- Nodes 2 are 3 are fixed so the non-zero displacement components are u_{X1} and u_{Y1}. Linear shape functions $N_1 = (L - x - y)/L$, $N_2 = x/L$ and $N_3 = y/L$ are easy to deduce from the figure. Therefore

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \frac{L-x-y}{L} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} \quad \text{so} \quad \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial x \end{Bmatrix} = -\frac{1}{L} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} \text{ and } \begin{Bmatrix} \partial u/\partial y \\ \partial v/\partial y \end{Bmatrix} = -\frac{1}{L} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix}.$$

- Virtual work density of internal forces is given by

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \\ \delta u_{X1} + \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \frac{1}{L^2} \frac{tE}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{Y1} \\ u_{X1} + u_{Y1} \end{Bmatrix}.$$

- Integration over the triangular domain gives (integrand is constant)

$$\delta W^{1}=-\begin{Bmatrix}\delta u_{X1}\\ \delta u_{Y1}\\ \delta u_{X1}+\delta u_{Y1}\end{Bmatrix}^{\mathrm{T}}\frac{1}{2}\frac{tE}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & (1-2\nu)/2\end{bmatrix}\begin{Bmatrix}u_{X1}\\ u_{Y1}\\ u_{X1}+u_{Y1}\end{Bmatrix}\Leftrightarrow$$

$$\delta W^{1}=-\frac{Et}{4(1+\nu)(1-2\nu)}\begin{Bmatrix}\delta u_{X1}\\ \delta u_{Y1}\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}3-4\nu & 1\\ 1 & 3-4\nu\end{bmatrix}\begin{Bmatrix}u_{X1}\\ u_{Y1}\end{Bmatrix}, \quad \leftarrow$$

$$\delta W^{2}=\begin{Bmatrix}\delta u_{X1}\\ \delta u_{Y1}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}-F\\ -F\end{Bmatrix}. \quad \leftarrow$$

- Principle of virtual work in the form $\delta W=\delta W^{1}+\delta W^{2}=0 \ \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus give

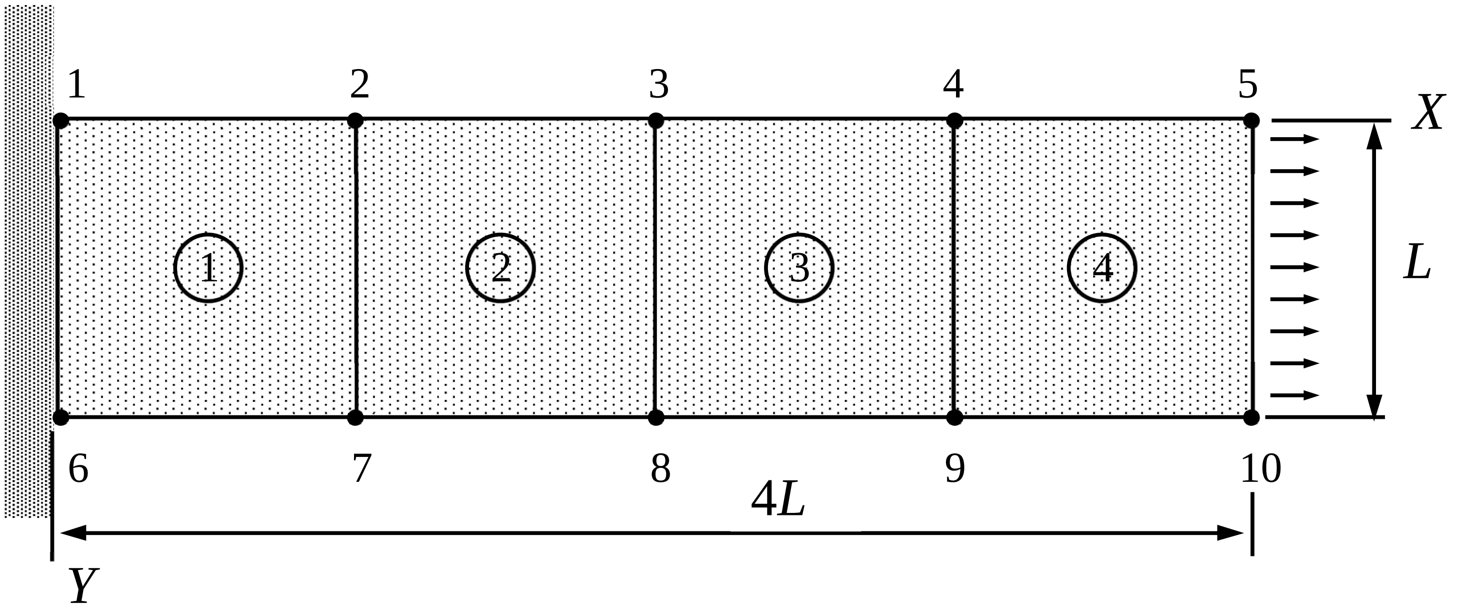
$$\delta W=-\begin{Bmatrix}\delta u_{X1}\\ \delta u_{Y1}\end{Bmatrix}^{\mathrm{T}}\left(\frac{Et}{4(1+\nu)(1-2\nu)}\begin{bmatrix}3-4\nu & 1\\ 1 & 3-4\nu\end{bmatrix}\begin{Bmatrix}u_{X1}\\ u_{Y1}\end{Bmatrix}+F\begin{Bmatrix}1\\ 1\end{Bmatrix}\right)=0 \quad \forall\begin{Bmatrix}\delta u_{X1}\\ \delta u_{Y1}\end{Bmatrix} \Rightarrow$$

$$\frac{Et}{4(1+\nu)(1-2\nu)}\begin{bmatrix} 3-4\nu & 1 \\ 1 & 3-4\nu \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} + F\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = 0 \quad \Leftrightarrow$$

$$\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} = -\frac{F}{Et}\frac{(1+\nu)(1-2\nu)}{1-\nu}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}. \qquad \Leftarrow$$

NOTICE: The point forces acting on a thin slab should be considered as “equivalent nodal forces” i.e. just representations of distributed forces acting on the edges with some selection of the element division. Therefore, refinement of the mesh requires a new set of equivalent nodal forces. Under the action of an actual point force, exact solution to the displacement becomes non-bounded so also the numerical solution to the displacement at the point of action increases without a bound, when the mesh is refined.

EXAMPLE 5.4 A thin slab is loaded by an evenly distributed traction having the resultant F as shown. Calculate the displacement at the midpoint of edge 5-10 by using bi-linear approximation in each element and the plane-stress assumption. Thickness of the slab is t. Material parameters E and $\nu = 1/3$ are constants.



Answer: $\dfrac{u_{X5} + u_{X10}}{2} = \dfrac{6856}{1731}\dfrac{F}{Et} \approx 3.96072\dfrac{F}{Et}$ (bar model or $\nu = 0$ gives $4\dfrac{F}{Et}$)

- Mathematica solution can be obtained with the problem description tables

	model	properties	geometry
1	PLANE	{{E, ν}, {t}}	Polygon[{1, 2, 7, 6}]
2	PLANE	{{E, ν}, {t}}	Polygon[{2, 3, 8, 7}]
3	PLANE	{{E, ν}, {t}}	Polygon[{3, 4, 9, 8}]
4	PLANE	{{E, ν}, {t}}	Polygon[{4, 5, 10, 9}]
5	FORCE	$\{\{\frac{F}{L}, \frac{F}{L}\}, \{0, 0\}, \{0, 0\}\}$	Line[{5, 10}]

	{X,Y,Z}	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}
2	{L, 0, 0}	{uX[2], uY[2], 0}	{0, 0, 0}
3	{2 L, 0, 0}	{uX[3], uY[3], 0}	{0, 0, 0}
4	{3 L, 0, 0}	{uX[4], uY[4], 0}	{0, 0, 0}
5	{4 L, 0, 0}	{uX[5], uY[5], 0}	{0, 0, 0}
6	{0, L, 0}	{0, 0, 0}	{0, 0, 0}
7	{L, L, 0}	{uX[7], uY[7], 0}	{0, 0, 0}
8	{2 L, L, 0}	{uX[8], uY[8], 0}	{0, 0, 0}
9	{3 L, L, 0}	{uX[9], uY[9], 0}	{0, 0, 0}
10	{4 L, L, 0}	{uX[10], uY[10], 0}	{0, 0, 0}

5.6 BAR MODEL

Bar model is one of the loading modes of the beam model and the solid model in one dimension. Virtual work densities of the model are given by

Internal forces: $\delta w_{\Omega}^{\text{int}} = -\frac{d\delta u}{dx} EA \frac{du}{dx}$

External forces: $\delta w_{\Omega}^{\text{ext}} = \delta u f_x, \quad \delta w_{\partial\Omega}^{\text{ext}} = \delta u F_x.$

Although the virtual work density $\delta w_{\partial\Omega}^{\text{ext}}$ for the external point force F_x acting on the edges (here on the nodes) belongs to the bar model, it will be treated separately by forces/moments elements.

- A bar is a thin body in two dimensions. The kinematic and kinetic assumptions of the bar model $\vec{u}(x,y,z)=\vec{u}(x)$ and only $\sigma_{xx} \neq 0$ imply the non-zero strain and stress components

$$\varepsilon_{xx} = \frac{du}{dx} \quad \text{and} \quad \sigma_{xx} = E\varepsilon_{xx} = E\frac{du}{dx}.$$

- Therefore, virtual work densities of the solid model simplify to

$$\delta w_V^{\text{int}} = -\begin{Bmatrix} \delta\varepsilon_{xx} \\ \delta\varepsilon_{yy} \\ \delta\varepsilon_{zz} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} - \begin{Bmatrix} \delta\gamma_{xy} \\ \delta\gamma_{yz} \\ \delta\gamma_{zx} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = -\frac{d\delta u}{dx} E \frac{du}{dx},$$

$$\delta w_V^{\text{ext}} = \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix} = f_x \delta u \quad \text{and} \quad \delta w_A^{\text{ext}} = \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix} = t_x \delta u .$$

- Integration over the body consists of integration over the cross-section (small dimensions of the bar and beam models) and integration over the length

$$\delta W^{\text{int}} = \int_V \delta w_V^{\text{int}} dV = \int_\Omega (\int_A \delta w_V^{\text{int}} dA) d\Omega = \int_\Omega \delta w_\Omega^{\text{int}} d\Omega \quad \text{in which}$$

$$\delta w_\Omega^{\text{int}} = -\frac{d\delta u}{dx} EA \frac{du}{dx} . \quad \Leftarrow$$

- The contributions of the external forces are obtained in the same manner. Considering first the volume force and the surface force acting on the circumferential part (in the final form, f_x denotes force per unit length although the symbol is the same as for the volume force)

$$\delta W^{\text{ext}} = \int_V \delta w_V^{\text{ext}} dV + \int_A \delta w_A^{\text{ext}} dA = \int_\Omega \delta w_\Omega^{\text{ext}} d\Omega \text{ in which}$$

$$\delta w_\Omega^{\text{ext}} = \delta u (\int_A f_x dA + \int_S t_x ds) = \delta u f_x . \;\leftarrow$$

- Surface forces on the remaining area (end surfaces) give (in the final form, F_x is force acting at an end point in the direction of the axis)

$$\delta W^{\text{ext}} = \sum_{\partial\Omega} \int_A \delta w_A^{\text{ext}} dA = \sum_{\partial\Omega} \delta u [\int_A t_x dA] = \sum_{\partial\Omega} \delta u F_x .$$

Virtual work of traction at the end surfaces belongs to the bar model but the contribution is taken into account by a force element of one node.

5.7 TORSION MODEL

Torsion model is one of the loading modes of beam. Virtual work densities of the model are given by

Internal forces: $\delta w_{\Omega}^{\text{int}} = -\dfrac{d\delta\phi}{dx} GJ \dfrac{d\phi}{dx}$

External forces: $\delta w_{\Omega}^{\text{ext}} = \delta\phi m_x, \;\; \delta w_{\partial\Omega}^{\text{ext}} = \delta\phi M_x$

Although the virtual work density $\delta w_{\partial\Omega}^{\text{ext}}$ for the external point moment M_x acting on the edges (here on the nodes) belongs to the torsion model, it will be treated separately by moment elements.

- The kinematic assumptions of the torsion model $u_x = 0$, $u_y = -z\phi(x)$ and $u_z = y\phi(x)$ follow from the kinematic assumption of the beam model when only $\phi(x) \neq 0$. The strain-displacement relationships and the generalized Hooke's law give

$$\gamma_{xy} = -z\frac{d\phi}{dx},\ \ \gamma_{zx} = y\frac{d\phi}{dx},\ \ \sigma_{xy} = G\gamma_{xy} = -Gz\frac{d\phi}{dx},\ \text{and}\ \sigma_{zx} = G\gamma_{zx} = Gy\frac{d\phi}{dx}.$$

- Virtual work densities of internal and external forces follow from the generic expressions

$$\delta w_V^{\text{int}} = -\begin{Bmatrix} \delta\varepsilon_{xx} \\ \delta\varepsilon_{yy} \\ \delta\varepsilon_{zz} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} - \begin{Bmatrix} \delta\gamma_{xy} \\ \delta\gamma_{yz} \\ \delta\gamma_{zx} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = -\frac{d\delta\phi}{dx} G(z^2 + y^2)\frac{d\phi}{dx}$$

$$\delta w_V^{\text{ext}} = \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix} = \delta\phi(yf_z - zf_y) \quad \text{and} \quad \delta w_A^{\text{ext}} = \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix} = \delta\phi(yt_z - zt_y).$$

- Virtual work expressions are integrals over the volume divided here as integrals over the cross-section and length. Assuming that the shear modulus is constant

$$\delta W^{\text{int}} = \int_V \delta w_V^{\text{int}} dV = \int_\Omega (\int_A \delta w_V^{\text{int}} dA) d\Omega = \int_\Omega \delta w_\Omega^{\text{int}} d\Omega \quad \text{in which}$$

$$\delta w_\Omega^{\text{int}} = -\frac{d\delta\phi}{dx} \int_A (z^2 + y^2) G dA \frac{d\phi}{dx} = -\frac{d\delta\phi}{dx} GJ \frac{d\phi}{dx} . \quad \Leftarrow$$

The geometrical quantity $J = \int_A (z^2 + y^2) dA$ is called as the polar moment of the cross-section.

- The contribution from the external forces are obtained in the same manner (volume forces in V and surface forces on the entire A have to be accounted for). The volume and the circumferential area give

$$\delta W^{\text{ext}} = \int_V \delta w_V^{\text{ext}} dV + \int_A \delta w_A^{\text{ext}} dA = \int_\Omega \delta w_\Omega^{\text{ext}} d\Omega \quad \text{in which}$$

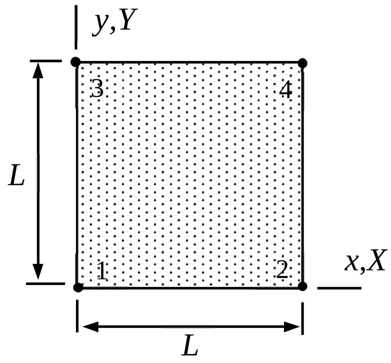
$$\delta w_\Omega^{\text{ext}} = \delta\phi[\int_A (yf_z - zf_y)dA + \int_S (yt_z - zt_y)ds] = \delta\phi m_x \ . \ \leftarrow$$

In the final form above, m_x is the moment per unit length.

- Surface forces on the remaining area (end surfaces) give rise to external moments M_x acting at the ends. These point moments are treated by using one-node force-moment elements.

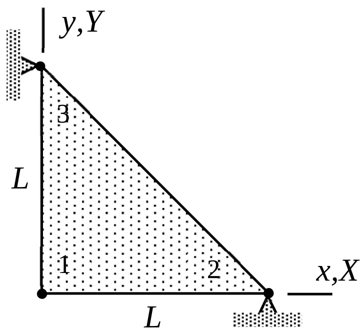
MEC-E1050 Finite Element Method in Solid, week 47/2024

1. Determine stress components at the midpoint of element shown if u_{Y2} is non-zero and the other nodal displacements are zeros. The approximations to the displacement components u, v are bi-linear. The material parameters E, ν and thickness t are constants. Use the strain-displacement and stress-strain relationship of linearly elastic isotropic material and assume plane stress conditions.



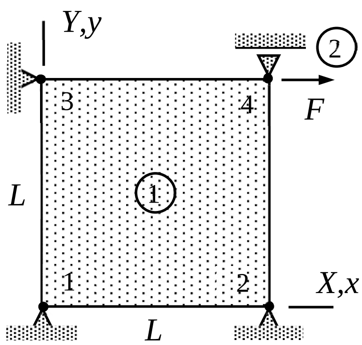
Answer $\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{u_{Y2}}{2L} \frac{E}{1-\nu^2} \begin{Bmatrix} -\nu \\ -1 \\ (1-\nu)/2 \end{Bmatrix}$

2. Determine the stress components σ_{xx}, σ_{yy}, and σ_{xy} of the triangle element shown in terms of the displacement components u_{X1}, u_{Y1} of node 1. Assume plane-strain conditions and use linear approximation to displacement components. The material parameters E, ν and thickness t are constants.



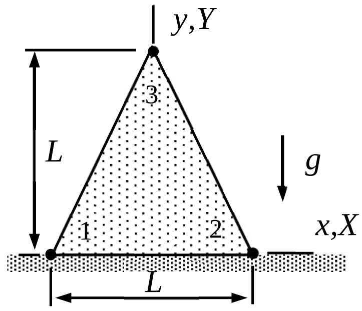
Answer $\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = -\frac{E}{L(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu \\ \nu & 1-\nu \\ (1-2\nu)/2 & (1-2\nu)/2 \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix}$

3. A thin slab (1) of square shape is loaded by a point force (2) as shown in the figure. Derive the relationship between the force F and the displacement u_{X4} of its point of action. Young's modulus E, Poisson's ratio ν, and thickness of the slab t are constants. External distributed forces vanish. Assume plane stress conditions and use a bilinear approximation.



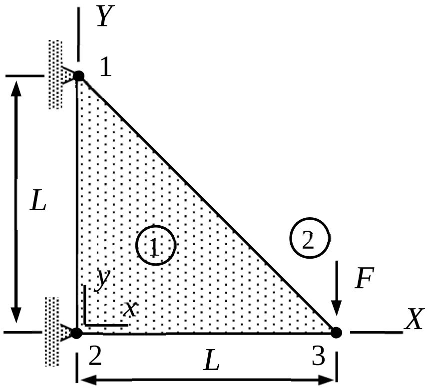
Answer $u_{X4} = \frac{6F}{Et} \frac{1-\nu^2}{3-\nu}$

4. A long wall having triangular cross-section, and made of homogeneous, isotropic, linearly elastic material, is subjected to its own weight. Material properties E,ν,ρ are constants. Determine the displacement components u_{X3} and u_{Y3} of node 3. Nodes 1 and 2 are fixed. Use just one three-node element and assume plane strain conditions.



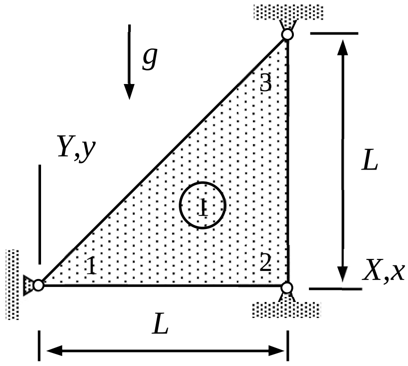
Answer $u_{X3}=0$, $u_{Y3}=-\frac{1}{3}\frac{(1+\nu)(1-2\nu)}{1-\nu}\frac{\rho g L^2}{E}$

5. A thin triangular slab of thickness t is loaded by a point force at node 3. Nodes 1 and 2 are fixed. Derive the virtual work expression δW of the structure in terms of u_{X3} and u_{Y3}, and solve for the nodal displacements. Approximation is linear and material parameters E and ν are constants. Assume plane-stress conditions.



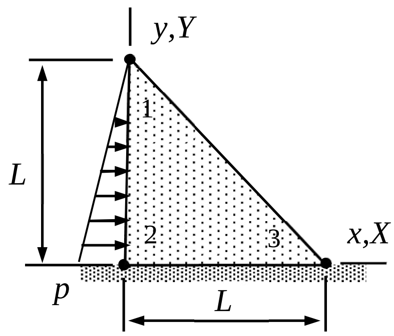
Answer $u_{X3}=0$, $u_{Y3}=-4\frac{F}{tE}(1+\nu)$

6. A thin triangular slab (assume plane stress conditions) loaded by its own weight is allowed to move vertically at node 1 and nodes 2 and 3 are fixed. Find the displacement u_{Y1}. Material parameters E, ν, ρ and thickness t of the slab are constants.



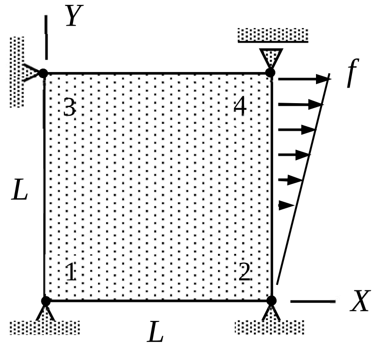
Answer $u_{Y1}=-\frac{2}{3}(1+\nu)\frac{\rho g L^2}{E}$

7. A long dam of homogeneous, isotropic, linearly elastic material, is subjected to water pressure on one side. Material properties E and ν are constants. Determine the displacement components u_{X1} and u_{Y1} of node 1. Nodes 2 and 3 are fixed. Use a three-node element and assume plane strain conditions. Consider a slab of thickness t in calculations. The peak value of the linearly varying pressure is p.



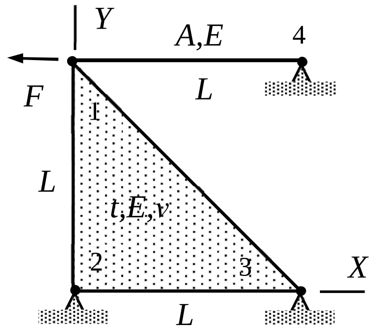
Answer $u_{X1}=\frac{2}{3}\frac{pL}{E}(1+\nu)$, $u_{Y1}=0$

8. A thin slab is loaded by a distributed force as shown. Derive the relationship between the force peak value f and displacement u_{X4}. Young's modulus E, Poisson's ratio ν, and thickness of the slab t are constants. Assume plane stress conditions and use the virtual work density of the thin slab and a bilinear approximation.



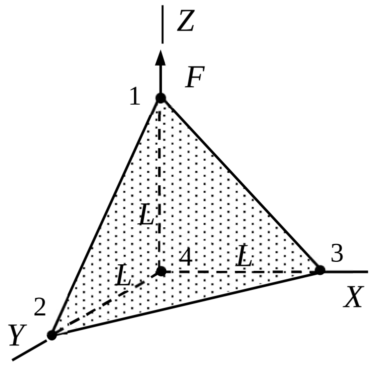
Answer $u_{X4} = 2\dfrac{fL}{Et}\dfrac{1-\nu^2}{3-\nu}$

9. A structure, consisting of a thin slab and a bar, is loaded by a horizontal force F acting on node 1. Material properties are E and ν, thickness of the slab is t and the cross-sectional area of the bar is A. Determine displacement of node 1 u_{X1} and u_{Y1} by using a linear bar element and a linear plane-stress element.



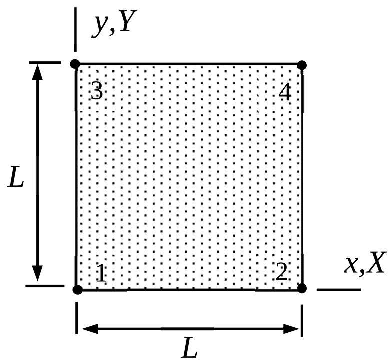
Answer $u_{X1} = -4\dfrac{L(1+\nu)}{Lt+4A(1+\nu)}\dfrac{F}{E}$ and $u_{Y1} = 0$

10. Point force F is acting on node 1 of the tetrahedron element of the figure. Nodes 2, 3 and 4 are fixed so that the displacement components are zeros. Determine displacement u_{Z1} of node 1 if $u_{X1} = u_{Y1} = 0$. Material properties E and ν are constants. Use linear approximation.



Answer $u_{Z1} = 6\dfrac{(1+\nu)(1-2\nu)}{1-\nu}\dfrac{F}{EL}$

Determine stress components at the midpoint of element shown if u_{Y2} is non-zero and the other nodal displacements are zeros. The approximations to the displacement components u,v are bi-linear. The material parameters E, ν and thickness t are constants. Use the strain-displacement and stress-strain relationship of linearly elastic isotropic material and assume plane stress conditions.



Solution

Under the plane-stress condition, the stress-strain and strain-displacement relationships of isotropic linearly elastic material are

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = [E]_\sigma \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \text{ with } \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix} \text{ and } [E]_\sigma = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}.$$

The material parameters are Young's modulus E and Poisson's ratio ν. The relationships can be used to calculate stress out of the given displacement components.

Element approximation of the present case simplifies to (shape functions can be deduced from the figure with $\xi = x/L$ and $\eta = y/L$)

$$u = \begin{Bmatrix} (1-\xi)(1-\eta) \\ \xi(1-\eta) \\ (1-\xi)\eta \\ \xi\eta \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = 0 \quad \text{and} \quad v = \begin{Bmatrix} (1-\xi)(1-\eta) \\ \xi(1-\eta) \\ (1-\xi)\eta \\ \xi\eta \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ u_{Y2} \\ 0 \\ 0 \end{Bmatrix} = \frac{x}{L}(1-\frac{y}{L})u_{Y2} \quad \Rightarrow$$

$$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial x} = \frac{1}{L}(1-\frac{y}{L})u_{Y2}, \text{ and } \frac{\partial v}{\partial y} = -\frac{x}{L}\frac{1}{L}u_{Y2}.$$

Strain components follow from the strain-displacement relationship

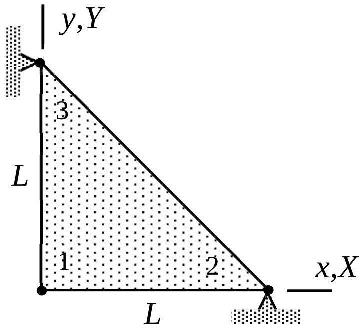
$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix} = \frac{u_{Y2}}{L^2} \begin{Bmatrix} 0 \\ -x \\ L-y \end{Bmatrix}.$$

After that, stress components follow from the stress-strain relationship

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \frac{u_{Y2}}{L^2} \begin{Bmatrix} 0 \\ -x \\ L-y \end{Bmatrix} = \frac{u_{Y2}}{L^2} \frac{E}{1-\nu^2} \begin{Bmatrix} -\nu x \\ -x \\ \frac{1-\nu}{2}(L-y) \end{Bmatrix}.$$

Evaluation at the midpoint $x = L/2$ and $y = L/2$ gives

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{u_{Y2}}{2L} \frac{E}{1-\nu^2} \begin{Bmatrix} -\nu \\ -1 \\ (1-\nu)/2 \end{Bmatrix}.$$



Determine the stress components σ_{xx}, σ_{yy}, and σ_{xy} of the triangle element shown in terms of the displacement components u_{X1}, u_{Y1} of node 1. Assume plane-strain conditions and use linear approximation to displacement components. The material parameters E, ν and thickness t are constants.

Solution

Under the plane-stress condition, the stress-strain and strain-displacement relationships of isotropic linearly elastic material and the matrix of elastic properties are

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{Bmatrix}=[E]_{\varepsilon}\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{Bmatrix},\quad \begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{Bmatrix}=\begin{Bmatrix}\partial u/\partial x\\ \partial v/\partial y\\ \partial u/\partial y+\partial v/\partial x\end{Bmatrix} \text{ and}$$

$$[E]_{\varepsilon}=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & (1-2\nu)/2\end{bmatrix} \text{ (from the formulae collection).}$$

The material parameters are Young's modulus E and Poisson's ratio ν. The relationships can be used to calculate stress out of the given displacement components.

Let us start with the approximation. Nodes 2 and 3 are fixed, u_{X1} and u_{Y1}. The shape function expressions can be deduced from the figure:

$$N_2 = x/L,\ \ N_3 = y/L \quad\Rightarrow\quad N_1 = 1-N_2-N_3 = 1-x/L-y/L \quad\Rightarrow$$

$$u=\frac{1}{L}\begin{Bmatrix}L-x-y\\ x\\ y\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}u_{X1}\\ 0\\ 0\end{Bmatrix}=(1-\frac{x}{L}-\frac{y}{L})u_{X1} \text{ and } v=\frac{1}{L}\begin{Bmatrix}L-x-y\\ x\\ y\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}u_{Y1}\\ 0\\ 0\end{Bmatrix}=(1-\frac{x}{L}-\frac{y}{L})u_{Y1} \quad\Rightarrow$$

$$\frac{\partial u}{\partial x}=-\frac{1}{L}u_{X1},\ \frac{\partial u}{\partial y}=-\frac{1}{L}u_{X1},\ \frac{\partial v}{\partial x}=-\frac{1}{L}u_{Y1},\ \text{and } \frac{\partial v}{\partial y}=-\frac{1}{L}u_{Y1}.$$

Strain components follow from the strain-displacement relationship

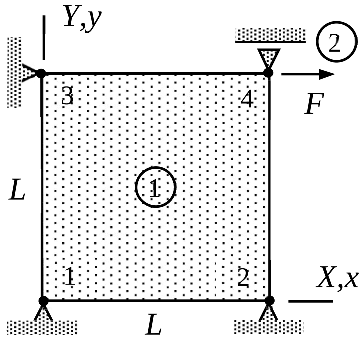
$$\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{Bmatrix}=\begin{Bmatrix}\partial u/\partial x\\ \partial v/\partial y\\ \partial u/\partial y+\partial v/\partial x\end{Bmatrix}=-\frac{1}{L}\begin{Bmatrix}u_{X1}\\ u_{Y1}\\ u_{X1}+u_{Y1}\end{Bmatrix}=-\frac{1}{L}\begin{bmatrix}1 & 0\\ 0 & 1\\ 1 & 1\end{bmatrix}\begin{Bmatrix}u_{X1}\\ u_{Y1}\end{Bmatrix}.$$

After that, stress components follow from the stress-strain relationship

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = [E]_{\varepsilon} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix} \left(-\frac{1}{L} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix}\right) \quad \Leftrightarrow$$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = -\frac{E}{L(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu \\ \nu & 1-\nu \\ (1-2\nu)/2 & (1-2\nu)/2 \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix}.$$

A thin slab (1) of square shape is loaded by a point force (2) as shown in the figure. Derive the relationship between the force F and the displacement u_{X4} of its point of action. Young's modulus E, Poisson's ratio ν, and thickness of the slab t are constants. External distributed forces vanish. Assume plane stress conditions and use a bilinear approximation.



Solution

Let us start with the shape functions of element 1 and approximations. As nodes 1, 2, and 3 are fixed, it is enough to deduce the shape function of node 4

$$N_4 = \frac{xy}{L^2} \text{ .}$$

Approximations to the displacement components and their derivatives with respect to x and y are

$$u = \frac{xy}{L^2} u_{X4}, \quad \frac{\partial u}{\partial x} = \frac{y}{L^2} u_{X4}, \text{ and } \frac{\partial u}{\partial y} = \frac{x}{L^2} u_{X4}$$

$$v = 0, \quad \frac{\partial v}{\partial x} = 0, \text{ and } \frac{\partial v}{\partial y} = 0 .$$

When the approximations are substituted there, the virtual work density of thin slab model simplifies to (plane stress conditions, only the internal part is needed)

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta u / \partial x \\ \partial\delta v / \partial y \\ \partial\delta u / \partial y + \partial\delta v / \partial x \end{Bmatrix}^{\mathrm{T}} \frac{tE}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix} \quad \Rightarrow$$

$$\delta w_{\Omega}^{\text{int}} = -\delta u_{X4} \frac{tE}{1-\nu^2} \frac{1}{L^4} (y^2 + \frac{1-\nu}{2} x^2) u_{X4} .$$

Integration over the domain occupied by the element gives the element contribution

$$\delta W^1 = \int_0^L \int_0^L \delta w_{\Omega}^{\text{int}} dxdy = -\delta u_{X4} \frac{Et}{6} \frac{3-\nu}{1-\nu^2} u_{X4} .$$

Virtual work expression of the point force (element 2) follows from the definition of work

$$\delta W^2 = \delta u_{X4} F \text{ .}$$

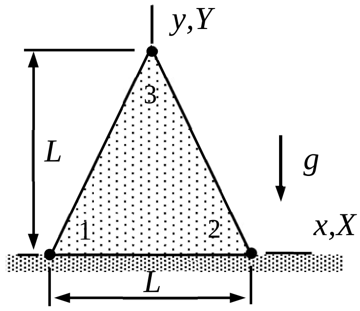
Virtual work expression of a structure is the sum of element contributions

$$\delta W = \delta W^1 + \delta W^2 = \delta u_{X4} (-\frac{Et}{6} \frac{3-\nu}{1-\nu^2} u_{X4} + F) .$$

Finally, principle of virtual work in the form $\delta W = 0 \quad \forall \delta \mathrm{a}$ and the fundamental lemma of variation calculus imply that

$$u_{X4} = \frac{6F}{Et}\frac{1-\nu^2}{3-\nu}.$$

A long wall having triangular cross-section, and made of homogeneous, isotropic, linearly elastic material, is subjected to its own weight. Material properties E, ν, ρ are constants. Determine the displacement components u_{X3} and u_{Y3} of node 3. Nodes 1 and 2 are fixed. Use a three-node element and assume plane strain conditions.



Solution

Under the plane strain conditions, the virtual work densities (virtual works per unit area) of the thin slab model

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta u/\partial y + \partial\delta v/\partial x \end{Bmatrix}^{\mathrm{T}} t[E]_\varepsilon \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix} \text{ and } \delta w_\Omega^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix} \text{ where}$$

$$[E]_\varepsilon = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix}$$

take into account the internal forces (stress) and external forces acting on the element domain. Notice that the components f_x and f_y are external forces per unit area. Distributed forces on the boundaries and point forces are taken into account by separate force elements.

Shape function $N_3 = y/L$ of node 3 can be deduced from the figure. Linear approximations to the displacement components are

$$u = N_3 u_{X3} = \frac{y}{L} u_{X3} \quad \Rightarrow \quad \frac{\partial u}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{1}{L} u_{X3},$$

$$v = N_3 u_{Y3} = \frac{y}{L} u_{Y3} \quad \Rightarrow \quad \frac{\partial v}{\partial x} = 0 \quad \text{and} \quad \frac{\partial v}{\partial y} = \frac{1}{L} u_{Y3}\ .$$

Virtual work of internal forces under the plane strain conditions with $G = E/(2+2\nu)$

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta u/\partial y + \partial\delta v/\partial x \end{Bmatrix}^{\mathrm{T}} \frac{tE}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix} \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix} \Rightarrow$$

$$\delta w_\Omega^{\text{int}} = -\frac{1}{L^2}\frac{Et}{1+\nu}\left(\frac{1-\nu}{1-2\nu} u_{Y3}\delta u_{Y3} + \frac{1}{2} u_{X3}\delta u_{X3}\right) \quad \Rightarrow$$

$$\delta W^{\text{int}} = \int_\Omega \delta w_\Omega^{\text{int}} d\Omega = \delta w_\Omega^{\text{int}} \frac{L^2}{2} = -\frac{1}{2}\frac{Et}{1+\nu}\left(\frac{1-\nu}{1-2\nu} u_{Y3}\delta u_{Y3} + \frac{1}{2} u_{X3}\delta u_{X3}\right).$$

Force density due to gravity is given by $f_x = 0$ and $f_y = -\rho g t$. Virtual work of external forces

$$\delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\text{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix} = f_y \delta v = -\rho g t \frac{y}{L} \delta u_{Y3} \quad \Rightarrow$$

$$\delta W^{\text{ext}} = \int_{\Omega} \delta w_{\Omega}^{\text{ext}} d\Omega = -\int_0^L \int_{(y-L)/2}^{(L-y)/2} \rho g t \frac{y}{L} \delta u_{Y3}) dx dy = -\rho g t \delta u_{Y3} \int_0^L (L-y) \frac{y}{L} dy \quad \Rightarrow$$

$$\delta W^{\text{ext}} = -\rho g t \delta u_{Y3} \int_0^L (L-y) \frac{y}{L} dy = -\frac{1}{6} L^2 \rho g t \delta u_{Y3} .$$

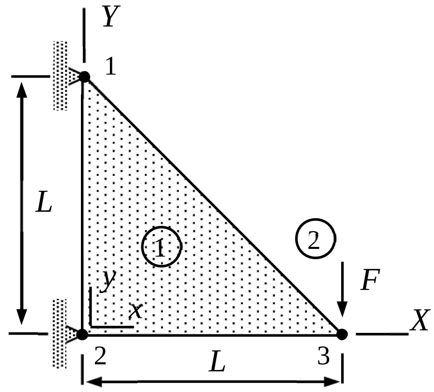
Virtual work expression in the sum of the internal and external parts

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = -\frac{1}{2} \frac{Et}{1+\nu} (\frac{1-\nu}{1-2\nu} u_{Y3} \delta u_{Y3} + \frac{1}{2} u_{X3} \delta u_{X3}) - \frac{1}{6} L^2 \rho g t \delta u_{Y3} .$$

Principle of virtual work $\delta W = 0$ and the basic lemma of variational calculus imply

$$u_{X3} = 0 \quad \text{and} \quad u_{Y3} = -\frac{1}{3}(1+\nu) \frac{1-2\nu}{1-\nu} L^2 \frac{\rho g}{E} .$$

A thin triangular slab of thickness t is loaded by a point force at node 3. Nodes 1 and 2 are fixed. Derive the virtual work expression δW of the structure in terms of u_{X3} and u_{Y3}, and solve for the nodal displacements. Approximation is linear and material parameters E and ν are constants. Assume plane stress conditions.



Solution

The virtual work densities (virtual works per unit area) of the thin slab model under the plane stress conditions

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta u / \partial x \\ \partial\delta v / \partial y \\ \partial\delta u / \partial y + \partial\delta v / \partial x \end{Bmatrix}^{\mathrm{T}} t[E]_{\sigma} \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix} \text{ and } \delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix} \text{where}$$

$$[E]_{\sigma} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}$$

take into account the internal forces (stress), external forces acting on the element domain, and external forces acting on the edges. Notice that the components f_x and f_y are external forces per unit area. The forces acting on the element edges are taken into account by separate force elements.

Expressions of linear shape functions in material $xy-$coordinates can be deduced from the figure. Only the shape function of node 3 is actually needed:

$$N_3 = \frac{x}{L}, \quad N_1 = \frac{y}{L} \text{ , and } N_2 = 1 - N_1 - N_3 = 1 - \frac{x}{L} - \frac{y}{L} \quad \Rightarrow$$

$$u = N_1 0 + N_2 0 + N_3 u_{X3} = \frac{x}{L} u_{X3} \quad \Rightarrow \quad \frac{\partial u}{\partial x} = \frac{1}{L} u_{X3} \quad \text{and} \quad \frac{\partial u}{\partial y} = 0,$$

$$v = N_1 0 + N_2 0 + N_3 u_{Y3} = \frac{x}{L} u_{Y3} \quad \Rightarrow \quad \frac{\partial v}{\partial x} = \frac{1}{L} u_{Y3} \quad \text{and} \quad \frac{\partial v}{\partial y} = 0.$$

When the approximation is substituted there, virtual work expression of internal forces per unit area simplifies to

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \delta u_{X3} \\ 0 \\ \delta u_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{1}{L} \frac{tE}{2(1-\nu^2)} \begin{bmatrix} 2 & 2\nu & 0 \\ 2\nu & 2 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix} \frac{1}{L} \begin{Bmatrix} u_{X3} \\ 0 \\ u_{Y3} \end{Bmatrix} = -\begin{Bmatrix} \delta u_{X3} \\ \delta u_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{tE}{2L^2(1-\nu^2)} \begin{bmatrix} 2 & 0 \\ 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} u_{X3} \\ u_{Y3} \end{Bmatrix}$$

As the integrand is constant, integration over the triangular domain gives

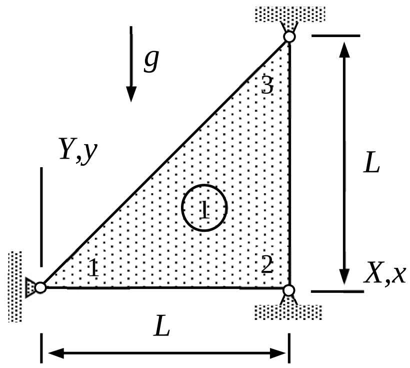
$$\delta W^{\text{int}} = \int_A \delta w_\Omega^{\text{int}} \mathrm{d}A = \delta w_\Omega^{\text{int}} \frac{L^2}{2} = -\begin{Bmatrix} \delta u_{X3} \\ \delta u_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{tE}{4(1-\nu^2)} \begin{bmatrix} 2 & 0 \\ 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} u_{X3} \\ u_{Y3} \end{Bmatrix}.$$

If also the point force is accounted for, the virtual work expression of the structure takes the form

$$\delta W = -\begin{Bmatrix} \delta u_{X3} \\ \delta u_{Y3} \end{Bmatrix}^{\mathrm{T}} \left(\frac{tE}{4(1-\nu^2)} \begin{bmatrix} 2 & 0 \\ 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} u_{X3} \\ u_{Y3} \end{Bmatrix} + \begin{Bmatrix} 0 \\ F \end{Bmatrix} \right).$$

Principle of virtual work $\delta W = 0 \; \forall \delta \mathrm{a}$ and the fundamental lemma of variation calculus give

$$\frac{tE}{4(1-\nu^2)} \begin{bmatrix} 2 & 0 \\ 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} u_{X3} \\ u_{Y3} \end{Bmatrix} + \begin{Bmatrix} 0 \\ F \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix} u_{X3} \\ u_{Y3} \end{Bmatrix} = -4(1+\nu) \frac{F}{tE} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}.$$



A thin triangular slab (assume plane stress conditions) loaded by its own weight is allowed to move vertically at node 1 and nodes 2 and 3 are fixed. Find the displacement u_{Y1}. Material parameters E, ν, ρ and thickness t of the slab are constants.

Solution

For the plane stress conditions and the thin-slab model, virtual work density of internal and external volume forces are given by

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta u/\partial y + \partial\delta v/\partial x \end{Bmatrix}^{\mathrm{T}} t[E]_{\sigma} \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix}, \text{ where } [E]_{\sigma} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix},$$

$$\delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix}.$$

Let us start with the approximations. Only the shape function of node 1 is needed as the other nodes are fixed. By using linearity and conditions $N_1(0,0) = 1$, $N_1(L,0) = N_1(0,L) = 0$

$$N_1(x,y) = 1 - \frac{x}{L}.$$

Displacement components simplify to

$$u = 0 \quad \Rightarrow \quad \frac{\partial u}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} = 0,$$

$$v = (1 - \frac{x}{L})u_{Y1} \quad \Rightarrow \quad \frac{\partial v}{\partial x} = -\frac{u_{Y1}}{L} \quad \text{and} \quad \frac{\partial v}{\partial y} = 0.$$

When approximations are substituted there, virtual work density simplifies to

$$\delta w_{\Omega}^{\text{int}} = -\delta u_{Y1}\frac{1}{L^2}\frac{tE}{2+2\nu}u_{Y1}.$$

Integration over the domain gives the virtual work expression. As the integrand is constant

$$\delta W^{\text{int}} = \int_{\Omega} \delta w_{\Omega}^{\text{int}} dA = \frac{L^2}{2}\delta w_{\Omega}^{\text{int}} = -\delta u_{Y1}\frac{1}{4}\frac{tE}{1+\nu}u_{Y1}.$$

Virtual work expression of the external volume force due to gravity takes the form

$$\delta W^{\text{ext}} = \int_{\Omega} \delta w_{\Omega}^{\text{ext}} dA = -\int_0^L \int_0^x (1 - \frac{x}{L})\delta u_{Y1} t\rho g dy dx = -\delta u_{Y1}\frac{1}{6}t\rho g L^2.$$

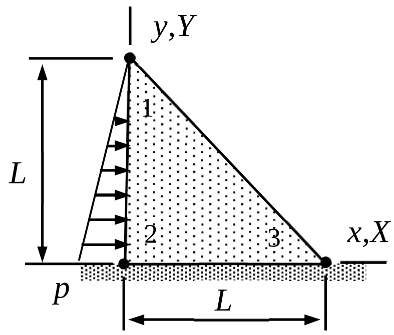
Virtual work expression of the thin slab is sum of the internal and external parts

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = -\delta u_{Y1}(\frac{1}{4}\frac{tE}{1+\nu}u_{Y1} + \frac{1}{6}t\rho g L^2) \,.$$

Principle of virtual work $\delta W = 0 \; \forall \delta \mathrm{a}$ and the fundamental lemma of variation calculus give

$$\frac{1}{4}\frac{tE}{1+\nu}u_{Y1} + \frac{1}{6}t\rho g L^2 = 0 \quad \Leftrightarrow \quad u_{Y1} = -\frac{2}{3}(1+\nu)\frac{\rho g L^2}{E} \,.$$

A long dam of homogeneous, isotropic, linearly elastic material, is subjected to water pressure on one side. Material properties E and ν are constants. Determine the displacement components u_{X1} and u_{Y1} of node 1. Nodes 2 and 3 are fixed. Use a three-node element and assume plane strain conditions. Consider a slab of thickness t in calculations. The peak value of the linearly varying pressure is p .



Solution

Under the plane strain conditions, the virtual work densities of thin slab are

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta u / \partial x \\ \partial\delta v / \partial y \\ \partial\delta u / \partial y + \partial\delta v / \partial x \end{Bmatrix}^{\mathrm{T}} t[E]_{\varepsilon} \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix} \text{ and } \delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix} \text{ where}$$

$$[E]_{\varepsilon} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix}.$$

The external forces t_x and t_y (force per unit length in this case) acting on the element edges can be taken into account by a separate force element with the density expression (per unit length)

$$\delta w_{\partial\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} t_x \\ t_y \end{Bmatrix}$$

although the expression is actually part of the thin slab model. The approximation on the boundary is just the restriction of the element approximation to the boundary.

Only the shape function for node 1 is needed as the other nodes are fixed (displacement vanishes). In terms of the displacement components u_{X1} and u_{Y1} of node 1, element approximations of the displacement components and their derivatives are

$$u = \frac{y}{L} u_{X1} \quad \Rightarrow \quad \frac{\partial u}{\partial x} = 0 \text{ and } \frac{\partial u}{\partial y} = \frac{1}{L} u_{X1},$$

$$v = \frac{y}{L} u_{Y1} \quad \Rightarrow \quad \frac{\partial v}{\partial x} = 0 \text{ and } \frac{\partial v}{\partial y} = \frac{1}{L} u_{Y1}.$$

When the approximation is substituted there, the virtual work densities simplify to

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} 0 \\ \delta u_{Y1} / L \\ \delta u_{X1} / L \end{Bmatrix}^{\mathrm{T}} \frac{Et}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix} \begin{Bmatrix} 0 \\ u_{Y1} / L \\ u_{X1} / L \end{Bmatrix} \quad \Rightarrow$$

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \dfrac{Et}{2(1+\nu)L^2} & 0 \\ 0 & \dfrac{Et(1-\nu)}{(1+\nu)(1-2\nu)L^2} \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix},$$

$$\delta w_{\partial\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} t_x \\ t_y \end{Bmatrix} = \begin{Bmatrix} \delta u_{X1} y / L \\ \delta u_{Y1} y / L \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} pt(1-y/L) \\ 0 \end{Bmatrix} = \begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} pt(1-y/L)y/L \\ 0 \end{Bmatrix}.$$

Integrations over the element and edge 2-1 give the virtual work expressions (notice that the virtual work density of internal forces is constant)

$$\delta W^{\text{int}} = \int_{\Omega} \delta w_{\Omega}^{\text{int}} d\Omega = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \dfrac{Et}{4(1+\nu)} & 0 \\ 0 & \dfrac{Et(1-\nu)}{2(1+\nu)(1-2\nu)} \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix},$$

$$\delta W^{\text{ext}} = \int_0^L \delta w_{\partial\Omega}^{\text{ext}} dy = \begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} ptL/6 \\ 0 \end{Bmatrix}.$$

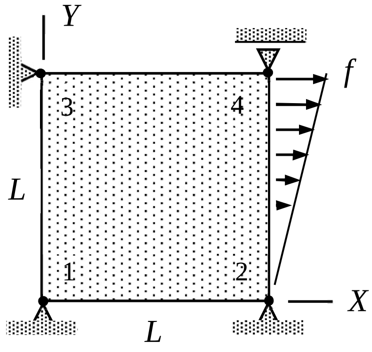
Principle of virtual work $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = 0 \quad \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus give

$$\delta W = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \left(\begin{bmatrix} \dfrac{Et}{4(1+\nu)} & 0 \\ 0 & \dfrac{Et(1-\nu)}{2(1+\nu)(1-2\nu)} \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} - \begin{Bmatrix} ptL/6 \\ 0 \end{Bmatrix} \right) = 0 \qquad \Rightarrow$$

$$\begin{bmatrix} \dfrac{Et}{4(1+\nu)} & 0 \\ 0 & \dfrac{Et(1-\nu)}{2(1+\nu)(1-2\nu)} \end{bmatrix} \begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} - \begin{Bmatrix} ptL/6 \\ 0 \end{Bmatrix} = 0 \qquad \Leftrightarrow$$

$u_{X1} = \dfrac{2}{3}\dfrac{pL}{E}(1+\nu)$ and $u_{Y1} = 0$.

A thin slab is loaded by a distributed force as shown. Derive the relationship between the force peak value f and displacement u_{X4}. Young's modulus E, Poisson's ratio ν, and thickness of the slab t are constants. Assume plane-stress conditions and use the virtual work density of the thin slab and a bilinear approximation.



Solution

Under the plane stress conditions, the virtual work densities (virtual works per unit area) of the thin slab model

$$\delta w_{\Omega}^{\mathrm{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta u/\partial y + \partial\delta v/\partial x \end{Bmatrix}^{\mathrm{T}} t[E]_{\sigma} \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix} \text{ and } \delta w_{\Omega}^{\mathrm{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix} \text{where}$$

$$[E]_{\sigma} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}$$

take into account the internal forces (stress) and the external area forces acting on the element domain. The external forces t_x and t_y (tractions per unit length in this case) acting on the element edges can be taken into account by a separate force element with the density expression (per unit length)

$$\delta w_{\partial\Omega}^{\mathrm{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} t_x \\ t_y \end{Bmatrix}$$

although the expression is actually part of the thin slab model. The approximation on the boundary is just the restriction of the element approximation to the boundary.

Only the shape function associated with node 4 is needed as the other nodes are fixed (displacement vanishes). In terms of the displacement component u_{X4} of node 4, approximations to the displacement components and their derivatives are

$$u = \mathbf{N}^{\mathrm{T}}\mathbf{a} = u_{X4}\frac{x}{L}\frac{y}{L} \quad \Rightarrow \quad \frac{\partial u}{\partial x} = u_{X4}\frac{1}{L}\frac{y}{L} \quad \text{and} \quad \frac{\partial u}{\partial y} = u_{X4}\frac{x}{L}\frac{1}{L},$$

$$v = 0 \quad \Rightarrow \quad \frac{\partial v}{\partial x} = 0 \quad \text{and} \quad \frac{\partial v}{\partial y} = 0.$$

Virtual work density of the internal forces simplifies to (when the approximations are substituted there)

$$\delta w_{\Omega}^{\text{int}} = -\delta u_{X4} \begin{Bmatrix} y \\ 0 \\ x \end{Bmatrix}^{\mathrm{T}} \frac{1}{L^4} \frac{Et}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} y \\ 0 \\ x \end{Bmatrix} u_{X4} = -\delta u_{X4} \frac{1}{L^4} \frac{Et}{1-\nu^2} (y^2 + x^2 \frac{1-\nu}{2}) u_{X4}.$$

Integration over the element gives the virtual work expression of internal forces

$$\delta W^{\text{int}} = \int_0^L \int_0^L \delta w_{\Omega}^{\text{int}} dxdy = -\delta u_{X4} \frac{1}{L^4} \frac{Et}{1-\nu^2} (L^4 \frac{1}{3} + L^4 \frac{1}{3} \frac{1-\nu}{2}) u_{X4} \quad \Rightarrow$$

$$\delta W^{\text{int}} = -\delta u_{X4} \frac{Et}{1-\nu^2} \frac{3-\nu}{6} u_{X4}.$$

Virtual work expression of external forces t_x and t_y is obtained as an integral over the edge defined by $x = L$. The restriction of approximation to $x = L$ and the linear distribution $t_x = fy/L$ give

$$\delta w_{\partial\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} t_x \\ t_y \end{Bmatrix} = \delta u_{X4} \frac{y}{L} (f \frac{y}{L}) \quad \Rightarrow \quad \delta W^{\text{ext}} = \int_0^L w_{\partial\Omega}^{\text{ext}} dy = \frac{fL}{3} \delta u_{X4}.$$

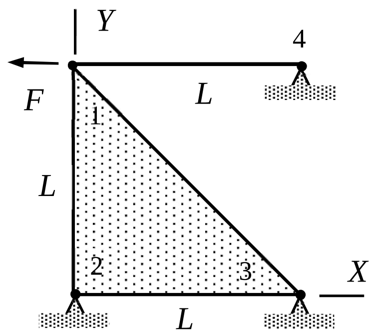
Virtual work expression is the sum of internal and external parts

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = -\delta u_{X4} (\frac{Et}{1-\nu^2} \frac{3-\nu}{6} u_{X4} - \frac{fL}{3}).$$

Principle of virtual work $\delta W = 0 \ \ \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus in the form $\delta \mathrm{a} R = 0 \ \ \forall \delta \mathrm{a} \ \Leftrightarrow \ R = 0$ give

$$\frac{Et}{1-\nu^2} \frac{3-\nu}{6} u_{X4} - \frac{fL}{3} = 0 \quad \Leftrightarrow \quad u_{X4} = 2 \frac{fL}{Et} \frac{1-\nu^2}{3-\nu}.$$

A structure, consisting of a thin slab and a bar, is loaded by a horizontal force F acting on node 1. Material properties are E and ν, thickness of the slab is t, and the cross-sectional area of the bar A are constants. Determine displacement components u_{X1} and u_{Y1} of node 1 by using a linear bar element and a linear plane-stress element.



Solution

Under the plane stress conditions, the virtual work densities (virtual works per unit area) of the thin slab model

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta u/\partial y + \partial\delta v/\partial x \end{Bmatrix}^{\mathrm{T}} t[E]_{\sigma} \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix} \text{ and } \delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix} \text{where}$$

$$[E]_{\sigma} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}$$

take into account the internal forces (stress) and external forces acting on the element domain. Notice that the components f_x and f_y are external forces per unit area. Forces acting on the element edges can be taken into account by separate force elements.

Element contribution for the thin slab needs to be derived from approximation and virtual work densities. Approximations to the displacement components depend only on the shape function associated with node 1 as the other nodes are fixed (displacement vanishes). In terms of the displacement components u_{X1} and u_{Y1}

$$u = u_{X1}\frac{y}{L} \quad \Rightarrow \quad \frac{\partial u}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} = u_{X1}\frac{1}{L},$$

$$v = u_{Y1}\frac{y}{L} \quad \Rightarrow \quad \frac{\partial v}{\partial x} = 0 \quad \text{and} \quad \frac{\partial v}{\partial y} = u_{Y1}\frac{1}{L}.$$

Virtual work density of the internal forces simplifies to (when the approximations are substituted there)

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} 0 \\ \delta u_{Y1} \\ \delta u_{X1} \end{Bmatrix}^{\mathrm{T}} \frac{1}{L^2}\frac{Et}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\begin{Bmatrix} 0 \\ u_{Y1} \\ u_{X1} \end{Bmatrix} = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \frac{1}{L^2}\frac{Et}{1-\nu^2}\begin{bmatrix} (1-\nu)/2 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix}$$

Virtual work expression is the integral of density over the domain occupied by the element (note that the virtual work density is constant in this case). Therefore

$$\delta W^1 = \int_\Omega \delta w_\Omega^{\text{int}} d\Omega = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \frac{1}{2}\frac{Et}{1-\nu^2}\begin{bmatrix} (1-\nu)/2 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix}.$$

Virtual work expression of the bar element is given in the formula collection with $u_{x1} = u_{X1}$ and $u_{x2} = 0$

$$\delta W^2 = -\begin{Bmatrix} \delta u_{X1} \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X1} \\ 0 \end{Bmatrix} = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \frac{EA}{L}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix}.$$

Virtual work expression of the point force follows e.g. directly from the definition (force multiplied by the virtual displacement in its direction)

$$\delta W^3 = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} F \\ 0 \end{Bmatrix}.$$

Virtual work expression of the structure is the sum of element contributions $\delta W = \delta W^1 + \delta W^2 + \delta W^3$

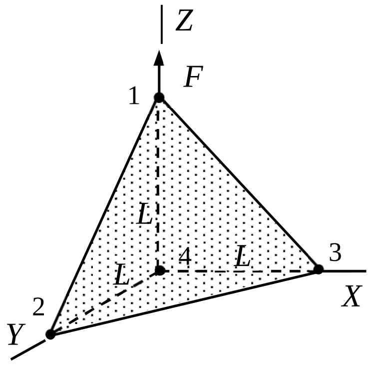
$$\delta W = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \left(\frac{1}{2}\frac{Et}{1-\nu^2}\begin{bmatrix} (1-\nu)/2 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} + \frac{EA}{L}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} + \begin{Bmatrix} F \\ 0 \end{Bmatrix}\right) \quad \Leftrightarrow$$

$$\delta W = -\begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \end{Bmatrix}^{\mathrm{T}} \left(\begin{bmatrix} \frac{1}{4}\frac{Et}{1+\nu} + \frac{EA}{L} & 0 \\ 0 & \frac{1}{2}\frac{Et}{1-\nu^2} \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} + \begin{Bmatrix} F \\ 0 \end{Bmatrix}\right).$$

Principle of virtual work $\delta W = 0 \quad \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus give

$$\begin{bmatrix} \frac{1}{4}\frac{Et}{1+\nu} + \frac{EA}{L} & 0 \\ 0 & \frac{1}{2}\frac{Et}{1-\nu^2} \end{bmatrix}\begin{Bmatrix} u_{X1} \\ u_{Y1} \end{Bmatrix} + \begin{Bmatrix} F \\ 0 \end{Bmatrix} = 0 \quad \Leftrightarrow \quad u_{X1} = -\frac{4(1+\nu)L}{tL + 4(1+\nu)A}\frac{F}{E} \quad \text{and} \quad u_{Y1} = 0.$$

Point force F is acting on node 1 of the tetrahedron element of the figure. Nodes 2, 3 and 4 are fixed so that the displacement components are zeros. Determine displacement u_{Z1} of node 1 if $u_{X1} = u_{Y1} = 0$. Material properties E and ν are constants. Use linear approximation.



Solution

Virtual work density of the solid model is (only internal forces in this problem)

$$\delta w_V^{\text{int}} = -\begin{Bmatrix} \partial\delta u / \partial x \\ \partial\delta v / \partial y \\ \partial\delta w / \partial z \end{Bmatrix}^{\mathrm{T}} [E] \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial w / \partial z \end{Bmatrix} - \begin{Bmatrix} \partial\delta u / \partial y + \partial\delta v / \partial x \\ \partial\delta v / \partial z + \partial\delta w / \partial y \\ \partial\delta w / \partial x + \partial\delta u / \partial z \end{Bmatrix}^{\mathrm{T}} G \begin{Bmatrix} \partial u / \partial y + \partial v / \partial x \\ \partial v / \partial z + \partial w / \partial y \\ \partial w / \partial x + \partial u / \partial z \end{Bmatrix} \text{ where}$$

$$[E] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix} \text{ and } G = \frac{E}{2+2\nu} .$$

Approximations to the displacement components depend only on the shape function associated with node 1 as the other nodes are fixed (displacement vanishes). In addition, the only non-zero displacement component is u_{Z1}. Here, $u = v = 0$ and

$$w = \mathbf{N}^{\mathrm{T}} \mathbf{a} = u_{Z1} \frac{z}{L} \quad \Rightarrow \quad \frac{\partial w}{\partial x} = 0 , \ \frac{\partial w}{\partial y} = 0 , \text{ and } \frac{\partial w}{\partial z} = u_{Z1} \frac{1}{L} .$$

Virtual work density of the internal forces simplifies to (when the approximations are substituted there)

$$\delta w_V^{\text{int}} = -\begin{Bmatrix} 0 \\ 0 \\ \delta u_{Z1} / L \end{Bmatrix}^{\mathrm{T}} \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_{Z1} / L \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{E}{2+2\nu} \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \Leftrightarrow$$

$$\delta w_V^{\text{int}} = -\delta u_{Z1} \frac{1}{L^2} \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} u_{Z1} .$$

Virtual work expression of the body element is integral of the density over the domain occupied by the element (note that the virtual work density is constant and volume $V = L^3 / 6$)

$$\delta W^1 = -\delta u_{Z1} \frac{L}{6} \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} u_{Z1} .$$

Virtual work expression of the given force follows, e.g., directly from the definition: force multiplied by the virtual displacement in its direction.

$$\delta W^2 = \delta u_{Z1} F \ .$$

Virtual work expression of the structure is the sum of element contributions

$$\delta W = \delta W^1 + \delta W^2 = -\delta u_{Z1} \left(\frac{L}{6} \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} u_{Z1} - F \right).$$

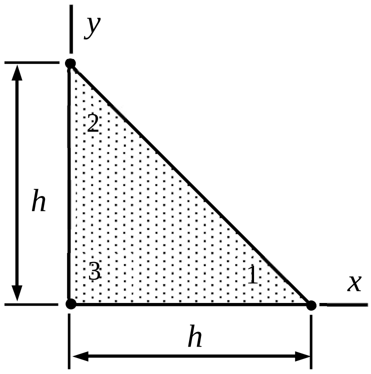
Principle of virtual work $\delta W = 0 \quad \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus give

$$\delta W = -\delta u_{Z1} \left(\frac{L}{6} \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} u_{Z1} - F \right) = 0 \quad \forall \delta u_{Z1} \quad \Leftrightarrow \quad \frac{L}{6} \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} u_{Z1} - F = 0 \quad \Leftrightarrow$$

$$u_{Z1} = 6 \frac{(1+\nu)(1-2\nu)}{1-\nu} \frac{F}{EL} .$$

Name________________________ Student number________________

Assignment 1



Calculate the strain and stress components of the element shown with the thin-slab model in the $xy-$plane, if the use of FEM gives the displacement components $u_{x1} = k_1 h$, $u_{x2} = -k_3 h$, $u_{x3} = 0$ and $u_{y1} = k_3 h$, $u_{y2} = k_2 h$, $u_{y3} = 0$ in which k_1, k_2, and k_3 are constants.

Solution template

The strain-displacement and strain-stress relationships of the thin-slab model are given by

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}.$$

Let us start with interpolation of the nodal values inside the element. Shape functions in terms of x, y and element size h (may be deduced using the definition: simplest possible polynomials taking the value one in one node and vanishing at all the other nodes)

$$N_1 = \frac{x}{h}, \qquad N_2 = \frac{y}{h}, \qquad N_3 = 1 - \frac{x}{h} - \frac{y}{h}.$$

Displacement components u and v in the x and y directions in terms of the shape functions and the nodal values

$$u = \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} k_1 h \\ -k_3 h \\ 0 \end{Bmatrix} = k_1 x - k_3 y, \quad v = \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} k_3 h \\ k_2 h \\ 0 \end{Bmatrix} = k_3 x + k_2 y.$$

Derivatives of u and v with respect to x and y

$$\frac{\partial u}{\partial x} = k_1, \quad \frac{\partial u}{\partial y} = -k_3, \quad \frac{\partial v}{\partial x} = k_3, \quad \frac{\partial v}{\partial y} = k_2.$$

Strain components follow from the strain definition

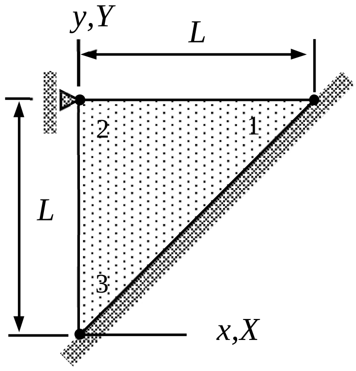
$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix} = \begin{Bmatrix} k_1 \\ k_2 \\ 0 \end{Bmatrix}.$$

Stress components follow from the stress-strain relationship

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{Bmatrix} k_1 + \nu k_2 \\ k_2 + \nu k_1 \\ 0 \end{Bmatrix}.$$

Name________________________ Student number_________________

Assignment 2



Consider the linear triangle element shown. Nodes 1 and 3 are fixed and the non-zero vertical displacement of node 2 is denoted by u_{Y2}. Determine the virtual work expression of internal forces using the virtual work density of the thin-slab model.

Solution template

Virtual work density of internal forces of the thin-slab model is given by

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta u/\partial y + \partial\delta v/\partial x \end{Bmatrix}^{\mathrm{T}} t[E]_\sigma \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix} \text{ where } [E]_\sigma = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}.$$

Shape functions in terms of x, y and element size L

$$N_1 = \frac{x}{L}, \qquad N_3 = 1 - \frac{y}{L}, \qquad N_2 = 1 - N_1 - N_3 = \frac{y}{L} - \frac{x}{L}.$$

Displacement components

$$u = \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = 0, \quad v = \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ u_{Y2} \\ 0 \end{Bmatrix} = u_{Y2}\frac{y-x}{L}.$$

Derivatives of u and v with respect to x and y

$$\frac{\partial u}{\partial x} = 0, \qquad \frac{\partial u}{\partial y} = 0, \qquad \frac{\partial v}{\partial x} = -\frac{u_{Y2}}{L}, \qquad \frac{\partial v}{\partial y} = \frac{u_{Y2}}{L}.$$

Virtual work density simplifies to

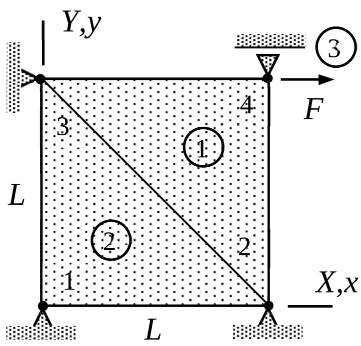
$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} 0 \\ \delta u_{Y2}/L \\ -\delta u_{Y2}/L \end{Bmatrix}^{\mathrm{T}} \frac{tE}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\begin{Bmatrix} 0 \\ u_{Y2}/L \\ -u_{Y2}/L \end{Bmatrix} = -\delta u_{Y2}\frac{tE}{2L^2}u_{Y2}\frac{3-\nu}{1-\nu^2}.$$

Virtual work expression is obtained as integral over the element (notice that integrand is constant)

$$\delta W = \int_\Omega \delta w_\Omega^{\text{int}} d\Omega = -\delta u_{Y2}\frac{tE}{4}\frac{3-\nu}{1-\nu^2}u_{Y2}. \quad \leftarrow$$

Name________________________ Student number________________

Assignment 3



A thin slab of square shape is loaded by a point force as shown. Derive the relationship between the force F and the displacement u_{X4} of its point of action. Young's modulus E, Poisson's ratio ν, and thickness of the slab t are constants. External distributed forces vanish. Assume plane stress conditions and use two linear triangle elements.

Solution template

Under the plane stress conditions, the virtual work densities (virtual works per unit area) of the thin slab model are given by

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta u/\partial y + \partial\delta v/\partial x \end{Bmatrix}^{\text{T}} t[E]_{\sigma} \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix} \text{ and } \delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\text{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix} \text{ where}$$

$$[E]_{\sigma} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}.$$

All nodes of element 2 are fixed so its element contribution vanishes and it is enough to consider only element 1 and 3. Let us start with the shape functions of element 1. As two of the edges are aligned with the coordinate axes, deducing the shape function expressions is not too difficult

$$N_3 = 1 - \frac{x}{L}, \quad N_2 = 1 - \frac{y}{L}, \text{ and } \quad N_4 = 1 - N_2 - N_3 = \frac{x}{L} + \frac{y}{L} - 1.$$

Approximations to the displacement components and their derivatives with respect to x and y are

$$u(x,y) = \mathbf{N}^{\text{T}}\mathbf{a} = (\frac{x}{L} + \frac{y}{L} - 1)u_{X4}, \quad \frac{\partial u}{\partial x} = \frac{1}{L}u_{X4}, \text{ and } \frac{\partial u}{\partial y} = \frac{1}{L}u_{X4},$$

$$v(x,y) = \mathbf{N}^{\text{T}}\mathbf{a} = 0, \quad \frac{\partial v}{\partial x} = 0, \text{ and } \frac{\partial v}{\partial y} = 0.$$

When the approximations are substituted there, the virtual work density of thin slab model simplifies to (plane stress conditions, only the internal part is needed)

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \delta u_{X4}/L \\ 0 \\ \delta u_{X4}/L \end{Bmatrix}^{\mathrm{T}} \frac{tE}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} u_{X4}/L \\ 0 \\ u_{X4}/L \end{Bmatrix} = -\delta u_{X4} \frac{tE}{1-\nu^2} \frac{1}{L^2} \frac{3-\nu}{2} u_{X4} \ .$$

Integration over the domain occupied by the element gives the element contribution (notice that the integrand is constant so it is enough to multiply by the area of the triangle)

$$\delta W^1 = \int_{\Omega} \delta w_{\Omega}^{\text{int}} dA = -\delta u_{X4} \frac{1}{4} tE \frac{3-\nu}{1-\nu^2} u_{X4} .$$

Virtual work expression of the point force (element 3) follows from the definition of work

$$\delta W^3 = \delta u_{X4} F \ .$$

Virtual work expression of a structure is the sum of element contributions

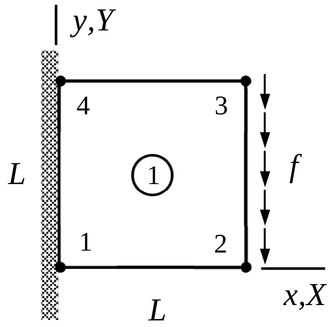
$$\delta W = \delta W^1 + \delta W^3 = -\delta u_{X4} \left(\frac{1}{4} tE \frac{3-\nu}{1-\nu^2} u_{X4} - F \right).$$

Principle of virtual work $\delta W = 0 \ \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus in the form $\delta \mathbf{a}^{\mathrm{T}} \mathbf{R} = 0 \ \forall \delta \mathbf{a} \ \Leftrightarrow \ \mathbf{R} = 0$ give

$$\frac{1}{4} tE \frac{3-\nu}{1-\nu^2} u_{X4} - F = 0 \qquad \Leftrightarrow \qquad u_{X4} = \frac{4F}{Et} \frac{1-\nu^2}{3-\nu} .$$

Name________________________ Student number________________

Assignment 4



A thin slab is loaded by distributed force on its outer edge as shown in the figure. Determine the vertical displacement of the outer edge 2-3 by using a bi-linear interpolation to the nodal values. Edge 1-4 is welded to a rigid wall so that the displacements vanish. Thickness of the slab *t*, Young's modulus *E*, and Poisson's ratio ν are constants. Assume plane stress conditions. Simplify the setting with conditions $u_{Y3} = u_{Y2}$ and $u_{X3} = u_{X2} = 0$.

Solution template

Under the plane stress conditions, the virtual work densities (virtual works per unit area) of the thin slab model are given by

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta u/\partial y + \partial\delta v/\partial x \end{Bmatrix}^{\mathrm{T}} t[E]_{\sigma} \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix} \text{ and } \delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix} \text{ where}$$

$$[E]_{\sigma} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}.$$

Expressions take into account the internal forces (stress) and the external area forces acting on the element domain. The external forces t_x and t_y (tractions per unit length in this case) acting on the element edges can be taken into account by a separate force element with the density expression (per unit length)

$$\delta w_{\partial\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} t_x \\ t_y \end{Bmatrix}.$$

The approximation on the boundary is just the restriction of the element approximation to the boundary (corresponds to a linear two-node element).

Only the shape functions associated with nodes 2 and 3 are needed as the other nodes are fixed (displacement vanishes). By deducing the expression, i.e., combining the linear shape functions in the $x-$directions and $y-$directions

$$N_2 = (1-\frac{y}{L})\frac{x}{L} \quad \text{and} \quad N_3 = \frac{y}{L}\frac{x}{L}.$$

In terms of the vertical displacement component u_{Y2} of node 2, approximations to the displacement components and their derivatives are

$$u(x,y)=\mathbf{N}^{\mathrm{T}}\mathbf{a}=0 \quad\Rightarrow\quad \frac{\partial u}{\partial x}=0 \quad\text{and}\quad \frac{\partial u}{\partial y}=0\,,$$

$$v(x,y)=\mathbf{N}^{\mathrm{T}}\mathbf{a}=u_{Y2}\frac{x}{L} \quad\Rightarrow\quad \frac{\partial v}{\partial x}=u_{Y2}\frac{1}{L} \quad\text{and}\quad \frac{\partial v}{\partial y}=0\,.$$

Virtual work density of the internal forces simplifies to (when the approximations are substituted there)

$$\delta w_{\Omega}^{\text{int}}=-\begin{Bmatrix} 0 \\ 0 \\ \delta u_{Y2}/L \end{Bmatrix}^{\mathrm{T}} \frac{Et}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ u_{Y2}/L \end{Bmatrix}=-\delta u_{Y2}\frac{1}{L^2}\frac{Et}{2(1+\nu)}u_{Y2}\,.$$

Virtual work density is constant in this case. Integration over the element gives the virtual work expression of internal forces

$$\delta W^{\text{int}}=\int_0^L\int_0^L \delta w_{\Omega}^{\text{int}}dxdy=-\delta u_{Y2}\frac{Et}{2(1+\nu)}u_{Y2}\,.$$

Virtual work expression of external distributed force components $t_x=0$ and $t_y=-f$ is obtained as an integral over the edge defined by $x=L$. The restriction of approximation to $x=L$ is given by

$$u(L,y)=0 \quad\text{and}\quad v(L,y)=u_{Y2}$$

so the virtual work density expression simplifies to

$$\delta w_{\partial\Omega}^{\text{ext}}=\begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix} t_x \\ t_y \end{Bmatrix}=-\delta u_{Y2}f$$

giving the virtual work expression

$$\delta W^{\text{ext}}=\int_0^L w_{\partial\Omega}^{\text{ext}}dy=-\,\delta u_{Y2}Lf\,.$$

Virtual work expression is the sum of internal and external parts

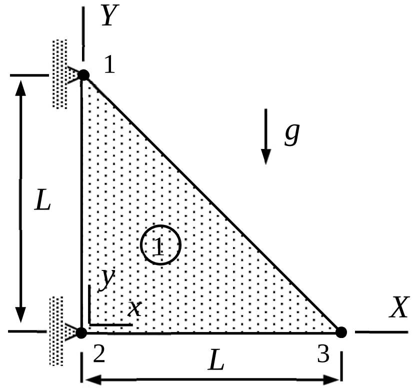
$$\delta W=\delta W^{\text{int}}+\delta W^{\text{ext}}=-\delta u_{Y2}\Big(\frac{Et}{2(1+\nu)}u_{Y2}+Lf\Big)\,.$$

Principle of virtual work $\delta W=0\;\;\forall\delta\mathbf{a}$ and the fundamental lemma of variation calculus in the form $\delta\mathbf{a}^{\mathrm{T}}\mathbf{R}=0\;\;\forall\delta\mathbf{a}\;\;\Leftrightarrow\;\;\mathbf{R}=0$ give

$$\frac{Et}{2(1+\nu)}u_{Y2}+Lf=0 \quad \Leftrightarrow \quad u_{Y2}=-\frac{Lf\,2(1+\nu)}{Et}=-\frac{Lf}{tG}.$$

Name_________________________ Student number_________________

Assignment 5



A thin triangular slab of thickness t is loaded by its own weight. Derive the virtual work expression δW of the structure and solve for the nodal displacements u_{X3} and u_{Y3}. Approximation is linear and elasticity parameters E, ν and density ρ are constants. Assume plane stress conditions.

Solution

The virtual work densities (virtual works per unit area) of the thin slab model under the plane stress conditions

$$\delta w_{\Omega}^{\mathrm{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta u/\partial y + \partial\delta v/\partial x \end{Bmatrix}^{\mathrm{T}} t[E]_{\sigma} \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix} \text{ and } \delta w_{\Omega}^{\mathrm{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix} \text{ where}$$

$$[E]_{\sigma} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}$$

take into account the internal forces (stress), external forces acting on the element domain, and external forces acting on the edges. Notice that the components f_x and f_y are external forces per unit area.

Expressions of linear shape functions in material xy – coordinates can be deduced from the figure. Only the shape function $N_3 = x/L$ of node 3 is actually needed. Hence

$$u = \frac{x}{L}u_{X3} \quad \Rightarrow \quad \frac{\partial u}{\partial x} = \frac{1}{L}u_{X3} \quad \text{and} \quad \frac{\partial u}{\partial y} = 0\,,$$

$$v = \frac{x}{L}u_{Y3} \quad \Rightarrow \quad \frac{\partial v}{\partial x} = \frac{1}{L}u_{Y3} \quad \text{and} \quad \frac{\partial v}{\partial y} = 0\,.$$

When the approximation is substituted there, virtual work expression of internal forces per unit area simplifies to

$$\delta w_{\Omega}^{\mathrm{int}} = -\begin{Bmatrix} \delta u_{X3} \\ 0 \\ \delta u_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{1}{L}\frac{tE}{2(1-\nu^2)}\begin{bmatrix} 2 & 2\nu & 0 \\ 2\nu & 2 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix}\frac{1}{L}\begin{Bmatrix} u_{X3} \\ 0 \\ u_{Y3} \end{Bmatrix} \quad \Leftrightarrow$$

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \delta u_{X3} \\ \delta u_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{tE}{2L^2(1-\nu^2)} \begin{bmatrix} 2 & 0 \\ 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} u_{X3} \\ u_{Y3} \end{Bmatrix}.$$

As the integrand is constant, integration over the triangular domain gives

$$\delta W^{\text{int}} = \int_A \delta w_{\Omega}^{\text{int}} \mathrm{d}A = \delta w_{\Omega}^{\text{int}} \frac{L^2}{2} = -\begin{Bmatrix} \delta u_{X3} \\ \delta u_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{tE}{4(1-\nu^2)} \begin{bmatrix} 2 & 0 \\ 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} u_{X3} \\ u_{Y3} \end{Bmatrix}.$$

In the virtual work density of the external forces $f_x = 0$ and $f_y = -\rho g t$ so

$$\delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix} = -\rho g t \frac{x}{L} \delta u_{Y3}.$$

Integration over the domain occupied by the element gives

$$\delta W^{\text{ext}} = \int_A \delta w_{\Omega}^{\text{ext}} \mathrm{d}A = \int_0^L \left(\int_0^{L-x} -\rho g t \frac{x}{L} \delta u_{Y3} dy\right) dx = -\frac{\rho g t L^2}{6} \delta u_{Y3} = -\begin{Bmatrix} \delta u_{X3} \\ \delta u_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{\rho g t L^2}{6} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}.$$

Virtual work expression of the structure takes the form

$$\delta W = -\begin{Bmatrix} \delta u_{X3} \\ \delta u_{Y3} \end{Bmatrix}^{\mathrm{T}} \left(\frac{tE}{4(1-\nu^2)} \begin{bmatrix} 2 & 0 \\ 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} u_{X3} \\ u_{Y3} \end{Bmatrix} + \frac{\rho g t L^2}{6} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}\right).$$

Principle of virtual work $\delta W = 0 \ \forall \delta \mathrm{a}$ and the fundamental lemma of variation calculus give

$$\frac{tE}{4(1-\nu^2)} \begin{bmatrix} 2 & 0 \\ 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} u_{X3} \\ u_{Y3} \end{Bmatrix} + \frac{\rho g t L^2}{6} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix} u_{X3} \\ u_{Y3} \end{Bmatrix} = -\frac{4}{6} \frac{\rho g L^2}{E} (1+\nu) \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}. \quad \leftarrow$$

MEC-E1050

FINITE ELEMENT METHOD IN SOLIDS 2024

Week 48-0

6 BEAM AND PLATE MODELS

LEARNING OUTCOMES

Students are able to solve the lecture problems, home problems, and exercise problems on the topics of the week:

- ◻ Virtual work densities of the Bernoulli and Timoshenko beam models
- ◻ Displacement analysis by beam elements
- ◻ Virtual work densities of the Reissner-Mindlin and Kirchhoff plate models
- ◻ Displacement analysis by plate models

6.1 CONTINUOUS APPROXIMATIONS

Virtual work density expressions can be used with various approximation types in line, rectangle, circular, etc. domains. Valid selections for a simply supported Kirchhoff plate in bending on a rectangle domain $\Omega = [0, L] \times [0, H]$ are, e.g.,

- □ Polynomial basis approximation $w(x, y) = a_0 xy(x - L)(y - H)$

- □ Double sine series approximation $w(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} \sin(i\pi \frac{x}{L}) \sin(j\pi \frac{y}{H})$

Although parameters a_0, a_{ij} etc. of the continuous approximations on Ω may not be displacements of certain points, the recipe for finding their values is the same as for the nodal values and an approximation based on element interpolants on Ω^e ($\Omega = \cup \Omega^e$ and $\cap \Omega^e = \varnothing$).

CONTINUOUS SERIES SOLUTION

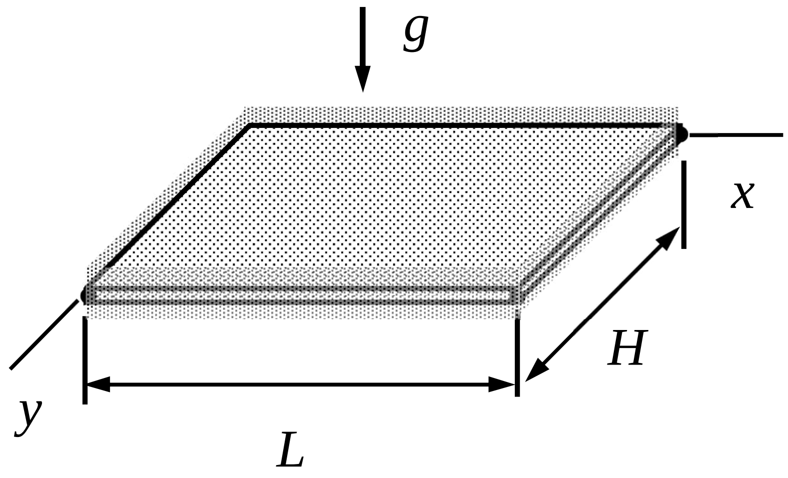
To find an approximate solution with a continuous series approximation for displacements/rotations and the virtual work density of a model

- Start with a linear combination of given functions with unknown coefficients (weights) $a_0, a_1, \ldots, a_n$. The series should satisfy the displacement/rotation boundary conditions no matter the coefficients.

- Substitute the series into the virtual work density expressions and continue with the recipe of the course to find the values of the coefficients.

Examples of useful function sets are polynomials of increasing order, harmonic functions of decreasing wavelength, etc. Mathematically, the function set should be complete so that the interpolation error reduces in the number of terms.

EXAMPLE 6.1 Consider pure bending of a rectangle Kirchhoff plate $\Omega=(0,L)\times(0,H)$. Derive the series solution $w(x,y)=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij}\sin(i\pi\frac{x}{L})\sin(j\pi\frac{y}{H})$ by considering the coefficients a_{ij} as the unknowns of the virtual work expression. Thickness t, Young's modulus E, and Poisson's ratio ν, and distributed load $f_z=\rho tg$ in direction of $z-$axis are constants.



Answer $a_{ij}=16\frac{f}{D}\frac{1}{ij\pi^2}/[(\frac{i\pi}{L})^2+(\frac{j\pi}{H})^2]^2 \quad i,j\in\{1,3,5,\ldots\},\ a_{ij}=0$ otherwise

- Shape functions need not to be polynomials. The well-known double sine-series solution to plate bending problem on a rectangle is an example of this theme. The solution uses the orthogonality properties of the sine and cosine functions (like)

$$\int_0^L \sin(i\pi\frac{x}{L})\sin(j\pi\frac{x}{L})dx = \delta_{ij}\frac{L}{2} \quad \text{and} \quad \int_0^L \sin(i\pi\frac{x}{L})dx = \frac{L}{i\pi}[1-(-1)^i]$$

$$\int_0^H \sin(i\pi\frac{y}{H})\sin(j\pi\frac{y}{H})dy = \delta_{ij}\frac{H}{2} \quad \text{and} \quad \int_0^H \sin(i\pi\frac{y}{H})dy = \frac{H}{i\pi}[1-(-1)^i]$$

- When the series approximation is substituted there, virtual work expression becomes a variational expression for the unknown coefficients. Using then orthogonality of the sines and cosines on $\Omega = (0,L)\times(0,H)$, virtual work expressions of the internal and external forces boil down to

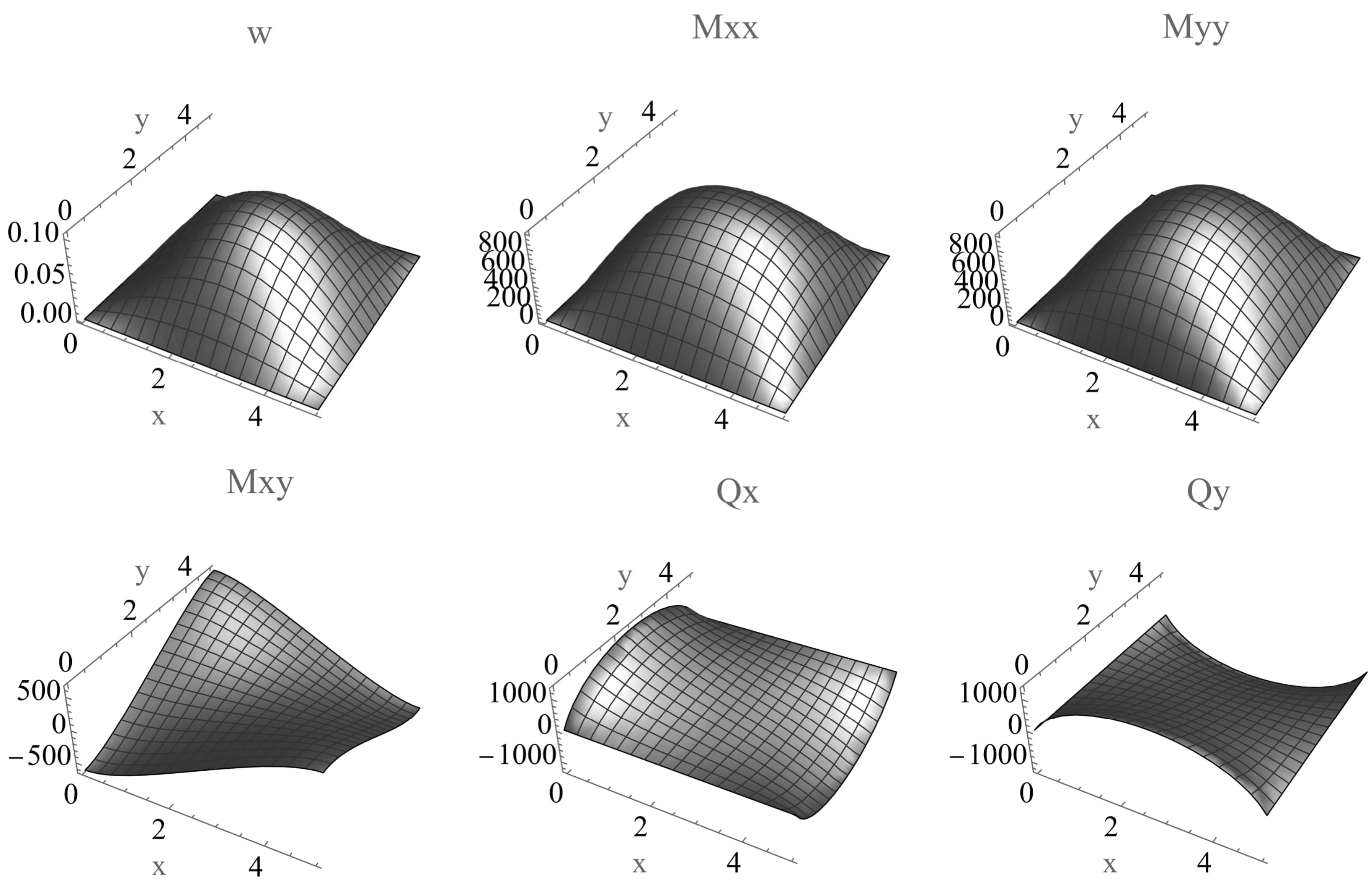
$$\delta W^{\text{int}} = -\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \delta a_{ij} \frac{Et^3}{12(1-\nu^2)} \frac{LH}{4} [(\frac{i\pi}{L})^2 + (\frac{j\pi}{H})^2]^2 a_{ij},$$

$$\delta W^{\text{ext}} = -\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \delta a_{ij} f_{ij}, \text{ where } \; f_{ij} = \int_0^L \int_0^H f(x,y) \sin(i\pi \frac{x}{L}) \sin(j\pi \frac{y}{H}) dxdy.$$

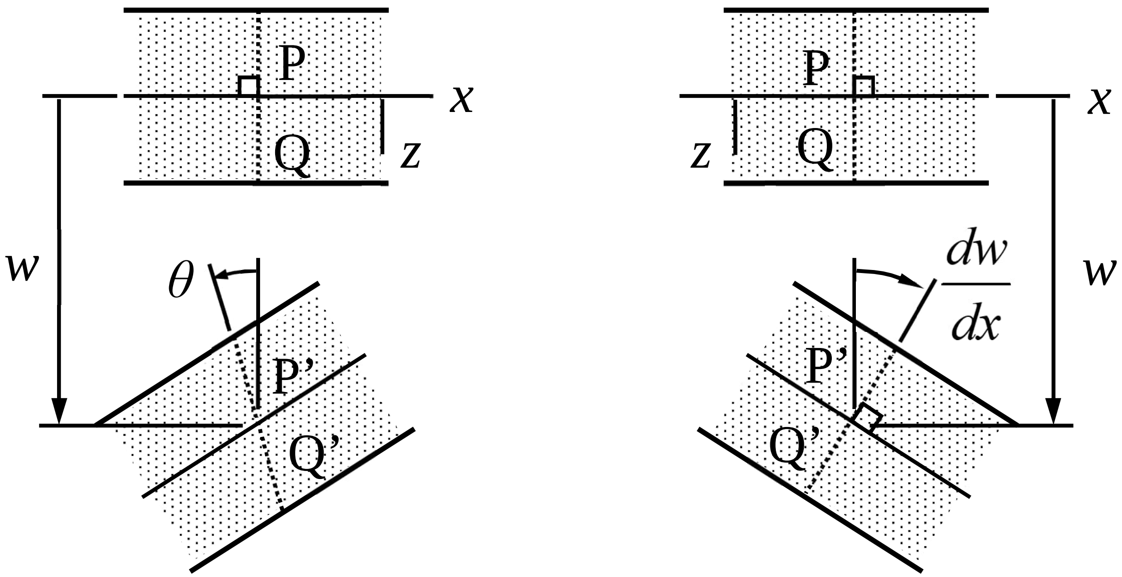
- As the terms are not connected in the virtual work expression (the matrix of the equation system implied by the principle of virtual work is diagonal), the fundamental lemma of variation calculus implies that

$$a_{ij} = 16 \frac{f}{D} \frac{1}{ij\pi^2} / [(\frac{i\pi}{L})^2 + (\frac{j\pi}{H})^2]^2 \quad i, j \in \{1,3,5,\ldots\}. \leftarrow$$

- With $L = H = 5\,\text{m}$, $t = 1\,\text{cm}$, $E = 210\,\text{GPa}$, $\nu = 0.3$, $\rho = 8000 \frac{\text{kg}}{\text{m}^3}$, $g = 9.81 \frac{\text{m}}{\text{s}^2}$, and 100 terms.


w
Mxx
Myy
Mxy
Qx
Qy
x
y
0
2
4
0.10
0.05
0.00
800
600
400
200
500
−500
1000
−1000

6.2 BEAM MODEL



Timoshenko **Bernoulli**

Normal planes to the (material) axis of beam remain planes (Timoshenko) and normal to the axis (Bernoulli) in deformation. Mathematically $\vec{u}_{\mathrm{Q}} = \vec{u}_{\mathrm{P}} + \vec{\theta} \times \vec{\rho}_{\mathrm{PQ}}$ (rigid body motion with translation point P). In addition, normal stress in small dimensions vanishes.

- In terms of the displacement components $u(x)$, $v(x)$, $w(x)$ and rotation components $\phi(x)$, $\theta(x)$, $\psi(x)$ of the translation point, the Timoshenko model displacement components are ($\vec{u} = (u\vec{i} + v\vec{j} + w\vec{k}) + (\phi\vec{i} + \theta\vec{j} + \psi\vec{k}) \times (y\vec{j} + z\vec{k})$)

$$u_x(x,y,z) = u(x) + \theta(x)z - \psi(x)y,$$

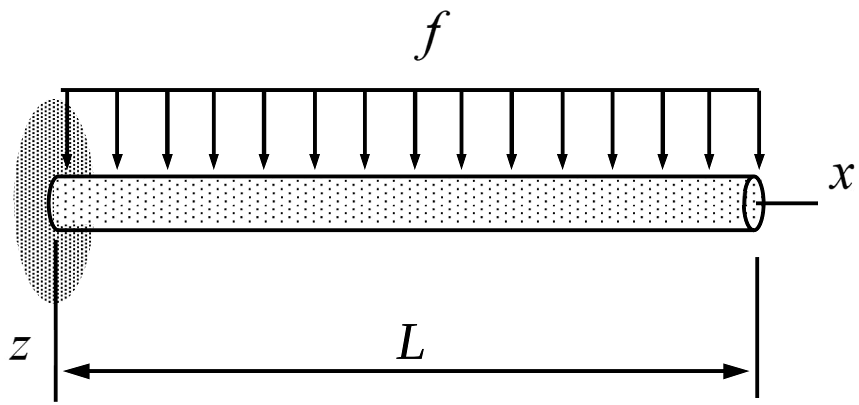
$$u_y(x,y,z) = v(x) - \phi(x)z,$$

$$u_z(x,y,z) = w(x) + \phi(x)y$$

In Bernoulli model, additionally $\theta = -dw/dx$ and $\psi = dv/dx$ so that normal planes remain normal to the axis.

- The kinetic assumption of the beam model means that $\sigma_{zz} = \sigma_{yy} = 0$.

EXAMPLE 6.2 Consider the beam of length L shown. Material properties E and G, cross-section properties A and I, and loading f are constants. Determine the deflection and rotation ($\theta = -dw/dx$) at the free end according to the Bernoulli beam model.



Answer $w(L) = \dfrac{fL^4}{8EI}$ and $\theta(L) = -\dfrac{dw}{dx}(L) = -\dfrac{fL^3}{6EI}$

- Mathematica solution according to the Bernoulli model is obtained with the problem description:

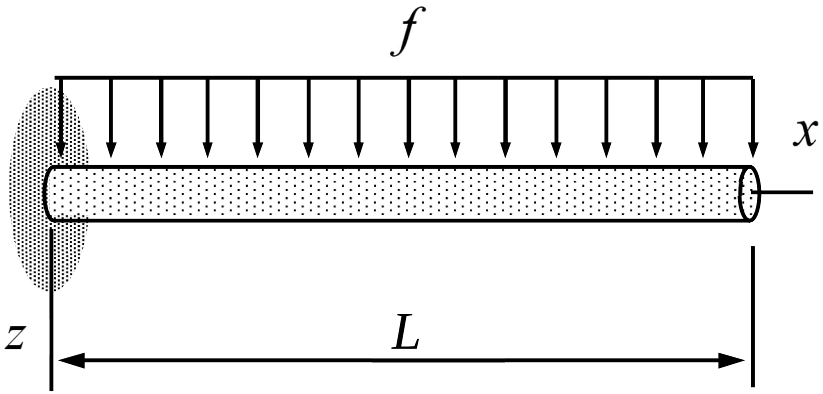
	model	properties	geometry
1	BEAM	{{E, G}, {A, Iy, Iz}, {0, 0, f}}	Line[{1, 2}]

	$\{X,Y,Z\}$	$\{u_X,u_Y,u_Z\}$	$\{\theta_X,\theta_Y,\theta_Z\}$
1	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}
2	{L, 0, 0}	{0, 0, uZ[2]}	{0, θY[2], 0}

$$\left\{uZ[2] \to \frac{f L^4}{8 E Iy}, \theta Y[2] \to -\frac{f L^3}{6 E Iy}\right\}$$

EXAMPLE 6.3 Consider the beam of length L shown. Material properties E and G, cross-section properties A and I, and loading f are constants. Determine the deflection and rotation at the free end according to the Timoshenko beam model.

"Timoshenko effect"
$\sim 1+(t/L)^2$



Answer $w(L)=\frac{1}{8}\frac{fL^4}{EI}(1+\frac{4EI}{GAL^2})$ and $\theta(L)=-\frac{1}{6}\frac{fL^3}{EI}$

- Mathematica solution according to the Timoshenko model is obtained with the problem description:

	model	properties	geometry
1	BEAMT	{{E, G}, {A, I, I}, {0, 0, f}}	Line[{1, 2}]

	$\{X, Y, Z\}$	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}
2	{L, 0, 0}	{0, 0, uZ[2]}	{0, θY[2], 0}

$$\left\{ uZ[2] \to \frac{1}{8} f L^2 \left(\frac{4}{A G} + \frac{L^2}{E I} \right), \ \theta Y[2] \to -\frac{f L^3}{6 E I} \right\}$$

BERNOULLI BEAM VIRTUAL WORK DENSITY

Bernoulli beam element combines the bar, torsion, and the *xz*-plane and *xy*-plane bending modes

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} d\delta u / dx \\ d^2\delta v / dx^2 \\ d^2\delta w / dx^2 \end{Bmatrix}^{\mathrm{T}} E \begin{bmatrix} A & -S_z & -S_y \\ -S_z & I_{zz} & I_{zy} \\ -S_y & I_{yz} & I_{yy} \end{bmatrix} \begin{Bmatrix} du / dx \\ d^2 v / dx^2 \\ d^2 w / dx^2 \end{Bmatrix} - \frac{d\delta\phi}{dx} GI_{rr} \frac{d\phi}{dx},$$

$$\delta w_\Omega^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} + \begin{Bmatrix} \delta\phi \\ -d\delta w / dx \\ d\delta v / dx \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} m_x \\ m_y \\ m_z \end{Bmatrix}.$$

Bar and bending modes are connected unless the first moments S_z, S_y and the cross moment I_{zy} (off-diagonal terms of the matrix) of the cross-section vanish.

- If the loading modes are not connected, the simplest element interpolants (approximations) to u and ϕ are linear and those for v and w cubic ($\xi = x/h$):

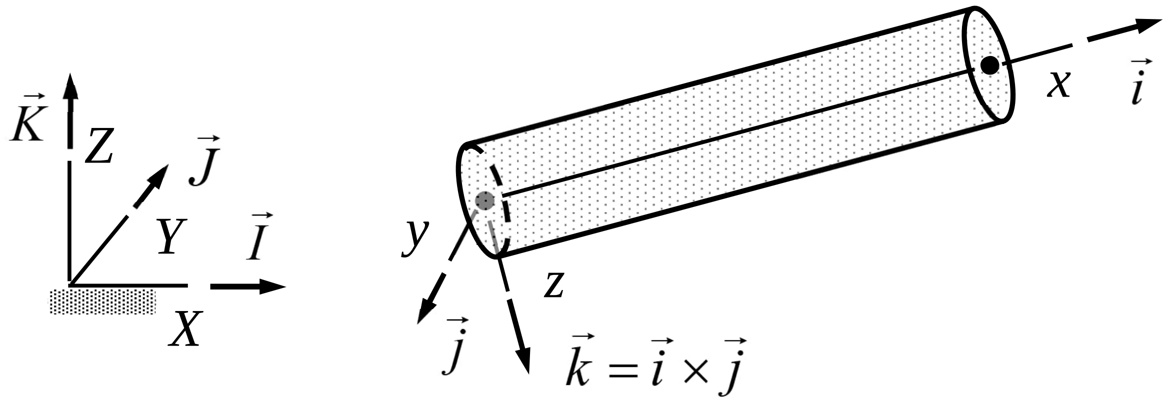
$$u(x) = \begin{Bmatrix} 1-\xi \\ \xi \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} \quad \text{and} \quad \phi(x) = \begin{Bmatrix} 1-\xi \\ \xi \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} \theta_{x1} \\ \theta_{x2} \end{Bmatrix},$$

$$v(x) = \begin{Bmatrix} (1-\xi)^2(1+2\xi) \\ h(1-\xi)^2\xi \\ \hdashline (3-2\xi)\xi^2 \\ h\xi^2(\xi-1) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \\ \hdashline u_{y2} \\ \theta_{z2} \end{Bmatrix} \quad \text{and} \quad w(x) = \begin{Bmatrix} (1-\xi)^2(1+2\xi) \\ h(1-\xi)^2\xi \\ \hdashline (3-2\xi)\xi^2 \\ h\xi^2(\xi-1) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{z1} \\ -\theta_{y1} \\ \hdashline u_{z2} \\ -\theta_{y2} \end{Bmatrix}.$$

If the loading modes are connected, a quadratic three-node interpolant (approximation) to u is needed. Therefore, the clever selection $S_y = S_z = 0$ and $I_{yz} = 0$ of the material coordinate system simplifies calculations a lot.

BEAM COORDINATE SYSTEM

The *x*-axis of the material system is aligned with the axis of the body. The coordinates of end nodes define the components of $\vec{i}$. The orientation of $\vec{j}$ is one of the geometrical parameters of the element contribution and it has to be given in the same manner as the moments of area.



NOTICE: Mathematica code assumes that the *y* and *Y* axes are aligned, i.e., $\vec{j} = \vec{J}$ unless the direction of *y*-axis is specified explicitly in the beam element description.

MOMENTS OF CROSS-SECTION

Cross-section geometry of a beam have effect on the constitutive equation through moments of area (material is assumed to be homogeneous):

Zero moment: $A = \int dA$

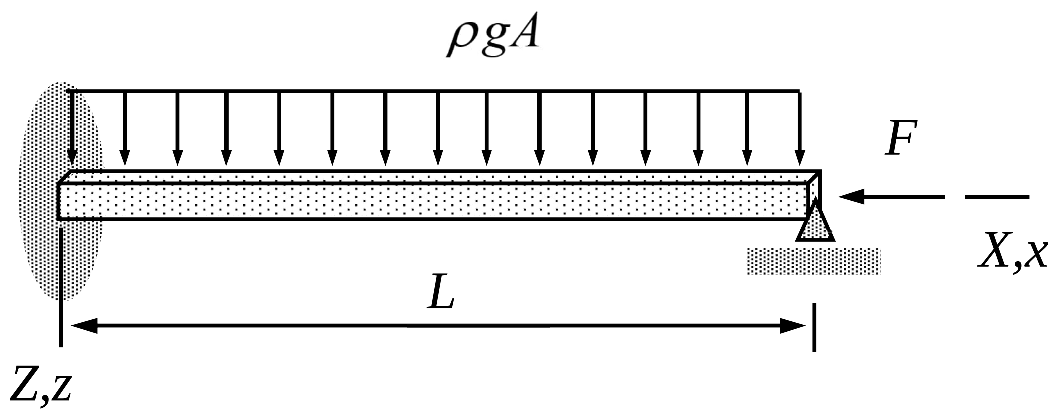
First moments: $S_z = \int ydA$ and $S_y = \int zdA$

Second moments: $I_{zz} = \int y^2 dA$, $I_{yy} = \int z^2 dA$, and $I_{yz} = \int yzdA$

Polar moment: $I_{rr} = \int y^2 + z^2 dA = I_{zz} + I_{yy}$

The moments depend on the selections of the material coordinate system. The origin and orientation can always be chosen so that $S_z = S_y = I_{yz} = 0$.

EXAMPLE 6.4 The beam of the figure is loaded by its own weight and a point force acting on the right end. Determine the displacement and rotation of the right end starting with the virtual density of the Bernoulli beam model. The x-axis of the material coordinate system is placed at the geometric centroid of the rectangle cross-section. Beam properties A, I_{yy}, I_{zz} and E of the planar problem are constants.



Answer: $u_{X2} = -\dfrac{FL}{EA}$ and $\theta_{Y2} = \dfrac{1}{48}\dfrac{\rho g A L^3}{EI_{yy}}$

- The left end of the beam is clamped and the right end simply supported. As the material and structural coordinate systems coincide $u_{x2}=u_{X2}$ and $\theta_{y2}=\theta_{Y2}$, the approximations of u and w simplify to

$$\begin{Bmatrix} u \\ w \end{Bmatrix} = \begin{Bmatrix} x/Lu_{X2} \\ L(x/L)^2(1-x/L)\theta_{Y2} \end{Bmatrix} \quad \Rightarrow \quad \begin{Bmatrix} du/dx \\ d^2w/dx^2 \end{Bmatrix} = \frac{1}{L}\begin{Bmatrix} u_{X2} \\ (2-6x/L)\theta_{Y2} \end{Bmatrix}.$$

- The moments of cross-section $S_y = S_z = 0$, I_{yy}, I_{zz} and $I_{yz} = 0$. As here $v = \phi = 0$, $f_x = f_y = 0$ and $m_x = m_y = m_z = 0$, virtual work densities take the forms

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} d\delta u/dx \\ d^2\delta w/dx^2 \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} EA & 0 \\ 0 & EI_{yy} \end{bmatrix} \begin{Bmatrix} du/dx \\ d^2w/dx^2 \end{Bmatrix} \text{ and } \delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ \rho g A \end{Bmatrix} \quad \Rightarrow$$

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \delta u_{X2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{E}{L^2} \begin{bmatrix} A & 0 \\ 0 & I_{yy}(2-6x/L)^2 \end{bmatrix} \begin{Bmatrix} u_{X2} \\ \theta_{Y2} \end{Bmatrix},$$

$$\delta w_{\Omega}^{\mathrm{ext}} = \begin{Bmatrix} \delta u_{X2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ L(x/L)^2(1-x/L)\rho g A \end{Bmatrix}.$$

- Integrations over the domain $\Omega =]0, L[$ give the virtual work expressions

$$\delta W^{\mathrm{int}} = \int_{\Omega} \delta w_{\Omega}^{\mathrm{int}} d\Omega = -\begin{Bmatrix} \delta u_{X2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{E}{L} \begin{bmatrix} A & 0 \\ 0 & 4I_{yy} \end{bmatrix} \begin{Bmatrix} u_{X2} \\ \theta_{Y2} \end{Bmatrix},$$

$$\delta W^{\mathrm{ext}} = \int_{\Omega} \delta w_{\Omega}^{\mathrm{ext}} d\Omega = \begin{Bmatrix} \delta u_{X2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{\rho g A L^2}{12} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \quad \Rightarrow$$

$$\delta W^{1} = \delta W^{\mathrm{int}} + \delta W^{\mathrm{ext}} = -\begin{Bmatrix} \delta u_{X2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{E}{L} \begin{bmatrix} A & 0 \\ 0 & 4I_{yy} \end{bmatrix} \begin{Bmatrix} u_{X2} \\ \theta_{Y2} \end{Bmatrix} - \frac{\rho g A L^2}{12} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}\right).$$

- Virtual work expression of the point force follows from definition of work (or from the expression of formulae collection)

$$\delta W^2 = \begin{Bmatrix} \delta u_{X2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} -F \\ 0 \end{Bmatrix}.$$

- Principle of virtual work $\delta W = \delta W^1 + \delta W^2 = 0 \quad \forall \delta\mathbf{a}$ and the fundamental lemma of variation calculus give

$$\delta W = -\begin{Bmatrix} \delta u_{X2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{E}{L}\begin{bmatrix} A & 0 \\ 0 & 4I_{yy} \end{bmatrix}\begin{Bmatrix} u_{X2} \\ \theta_{Y2} \end{Bmatrix} - \begin{Bmatrix} -F \\ \rho g A L^2/12 \end{Bmatrix}\right) \quad \forall \begin{Bmatrix} \delta u_{X2} \\ \delta\theta_{Y2} \end{Bmatrix} \quad \Leftrightarrow$$

$$\frac{E}{L}\begin{bmatrix} A & 0 \\ 0 & 4I \end{bmatrix}\begin{Bmatrix} u_{X2} \\ \theta_{Y2} \end{Bmatrix} - \begin{Bmatrix} -F \\ \rho g A L^2/12 \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix} u_{X2} \\ \theta_{Y2} \end{Bmatrix} = \begin{Bmatrix} -LF/EA \\ \rho g A L^3/(48 E I_{yy}) \end{Bmatrix}. \leftarrow$$

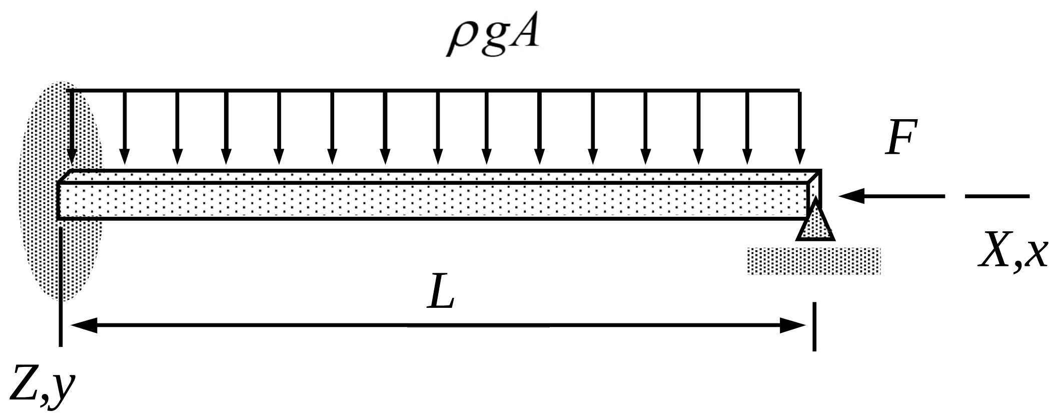
- Solution by the Mathematica code is obtained with the following problem description tables

	model	properties	geometry
1	BEAM	$\{\{\mathbb{E}, G\}, \{A, \mathbb{I}yy, \mathbb{I}zz\}, \{0, 0, A\,g\,\rho\}\}$	Line[{1, 2}]
2	FORCE	$\{-F, 0, 0\}$	Point[{2}]

	$\{X,Y,Z\}$	$\{u_X,u_Y,u_Z\}$	$\{\theta_X,\theta_Y,\theta_Z\}$
1	$\{0, 0, 0\}$	$\{0, 0, 0\}$	$\{0, 0, 0\}$
2	$\{L, 0, 0\}$	$\{uX[2], 0, 0\}$	$\{0, \theta Y[2], 0\}$

$$\left\{uX[2] \to -\frac{F\,L}{A\,\mathbb{E}},\ \theta Y[2] \to \frac{A\,g\,L^3\,\rho}{48\,\mathbb{E}\,\mathbb{I}yy}\right\}$$

EXAMPLE 6.5 The Bernoulli beam of the figure is loaded by its own weight and a point force acting on the right end. Determine the displacement and rotation of the right end starting with the virtual density of the Bernoulli beam model. The x-axis of the material coordinate system is placed at the geometric centroid of the rectangle cross-section. Beam properties A, I_{yy}, I_{zz} and E are constants.



Answer: $u_{X2} = -\dfrac{FL}{EA}$ and $\theta_{Y2} = \dfrac{1}{48}\dfrac{\rho g A L^3}{EI_{zz}}$

- Beam element definition of the Mathematica code requires the orientation of the $y-$axis unless $y-$ and $Y-$axes are aligned. Orientation is given by additional parameter defining the components of $\vec{j}$ in the structural coordinate system:

	model	properties	geometry
1	BEAM	{{E, G}, {A, Iyy, Izz, {0, 0, 1}}, {0, 0, A g ρ}}	Line[{1, 2}]
2	FORCE	{-F, 0, 0}	Point[{2}]

	$\{X,Y,Z\}$	$\{u_X,u_Y,u_Z\}$	$\{\theta_X,\theta_Y,\theta_Z\}$
1	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}
2	{L, 0, 0}	{uX[2], 0, 0}	{0, θY[2], 0}

$$\left\{uX[2] \to -\frac{F\,L}{A\,E},\ \theta Y[2] \to \frac{A\,g\,L^3\,\rho}{48\,Izz\,E}\right\}$$

TIMOSHENKO BEAM VIRTUAL WORK DENSITY

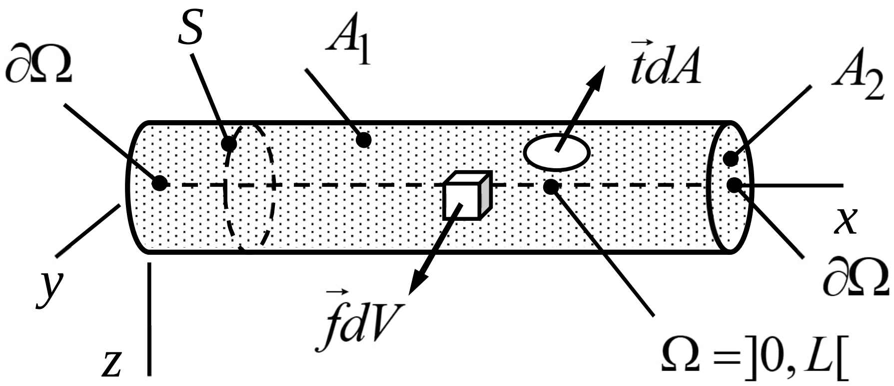
Timoshenko beam model takes into account transverse displacements due to shear. As the assumptions are less severe than those of the Bernoulli beam model, modelling error is smaller.

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} d\delta u/dx \\ d\delta\psi/dx \\ -d\delta\theta/dx \end{Bmatrix}^{\mathrm{T}} E \begin{bmatrix} A & -S_z & -S_y \\ -S_z & I_{zz} & I_{zy} \\ -S_y & I_{yz} & I_{yy} \end{bmatrix} \begin{Bmatrix} du/dx \\ d\psi/dx \\ -d\theta/dx \end{Bmatrix} - \begin{Bmatrix} -\delta\psi + d\delta v/dx \\ \delta\theta + d\delta w/dx \\ \delta d\phi/dx \end{Bmatrix}^{\mathrm{T}} \times$$

$$\times G \begin{bmatrix} A & 0 & -S_y \\ 0 & A & S_z \\ -S_y & S_z & I_{rr} \end{bmatrix} \begin{Bmatrix} -\psi + dv/dx \\ \theta + dw/dx \\ d\phi/dx \end{Bmatrix}, \quad \delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} + \begin{Bmatrix} \delta\phi \\ \delta\theta \\ \delta\psi \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} m_x \\ m_y \\ m_z \end{Bmatrix}.$$

If $S_z = S_y = 0$ and $I_{yz} = 0$, bar, torsion, and bending modes contribute to the virtual work expression as if they were separate bar, torsion and bending elements.

- Beam is a thin body in two dimensions



- The kinematic assumption of the Timoshenko beam model and definition of strain give the displacement and the non-zero strain components ($R_\psi = -\psi + dv/dx$, $R_\theta = \theta + dw/dx$)

$$\begin{Bmatrix} u_x(x,y,z) \\ u_y(x,y,z) \\ u_z(x,y,z) \end{Bmatrix} = \begin{Bmatrix} u(x) + z\theta(x) - y\psi(x) \\ v(x) - z\phi(x) \\ w(x) + y\phi(x) \end{Bmatrix} \Rightarrow$$

$$\varepsilon_{xx} = \frac{du}{dx} + z\frac{d\theta}{dx} - y\frac{d\psi}{dx} \quad \text{and} \quad \begin{Bmatrix} \gamma_{xy} \\ \gamma_{zx} \end{Bmatrix} = \begin{Bmatrix} -\psi + dv/dx - zd\phi/dx \\ \theta + dw/dx + yd\phi/dx \end{Bmatrix}.$$

- The kinetic assumptions $\sigma_{zz} = \sigma_{yy} = 0$ and the generalized Hooke's law give the non-zero stress components

$$\sigma_{xx} = E\varepsilon_{xx} = E(\frac{du}{dx} + z\frac{d\theta}{dx} - y\frac{d\psi}{dx}) \quad \text{and} \quad \begin{Bmatrix} \sigma_{xy} \\ \sigma_{zx} \end{Bmatrix} = G\begin{Bmatrix} -\psi + dv/dx - zd\phi/dx \\ \theta + dw/dx + yd\phi/dx \end{Bmatrix}.$$

- With notation $r^2 = y^2 + z^2$ the generic expressions for the virtual work densities per unit volume simplify to (some manipulations are needed here)

$$\delta w_V^{\text{int}} = -\begin{Bmatrix} d\delta u/dx \\ d\delta\psi/dx \\ -d\delta\theta/dx \end{Bmatrix}^{\mathrm{T}} E\begin{bmatrix} 1 & -y & -z \\ -y & yy & yz \\ -z & zy & zz \end{bmatrix}\begin{Bmatrix} du/dx \\ d\psi/dx \\ -d\theta/dx \end{Bmatrix} - \begin{Bmatrix} -\delta\psi + d\delta v/dx \\ \delta\theta + d\delta w/dx \\ \delta d\phi/dx \end{Bmatrix}^{\mathrm{T}} \times$$

$$\times G\begin{bmatrix} 1 & 0 & -z \\ 0 & 1 & y \\ -z & y & r^2 \end{bmatrix}\begin{Bmatrix} -\psi + dv/dx \\ \theta + dw/dx \\ d\phi/dx \end{Bmatrix},$$

$$\delta w_V^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\text{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} + \begin{Bmatrix} \delta\phi \\ \delta\theta \\ \delta\psi \end{Bmatrix}^{\text{T}} \begin{Bmatrix} -zf_y + yf_z \\ zf_x \\ -yf_x \end{Bmatrix},$$

$$\delta w_A^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\text{T}} \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix} + \begin{Bmatrix} \delta\phi \\ \delta\theta \\ \delta\psi \end{Bmatrix}^{\text{T}} \begin{Bmatrix} -zt_y + yt_z \\ zt_x \\ -yt_x \end{Bmatrix}.$$

- Virtual work density of the internal forces is obtained as an integral over the small dimensions which is the cross-section (the volume element $dV = dAd\Omega$).

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} d\delta u/dx \\ d\delta\psi/dx \\ -d\delta\theta/dx \end{Bmatrix}^{\mathrm{T}} E \begin{bmatrix} A & -S_z & -S_y \\ -S_z & I_{zz} & I_{zy} \\ -S_y & I_{yz} & I_{yy} \end{bmatrix} \begin{Bmatrix} du/dx \\ d\psi/dx \\ -d\theta/dx \end{Bmatrix} - \begin{Bmatrix} -\delta\psi + d\delta v/dx \\ \delta\theta + d\delta w/dx \\ d\delta\phi/dx \end{Bmatrix}^{\mathrm{T}} \times$$

$$\times G \begin{bmatrix} A & 0 & -S_y \\ 0 & A & S_z \\ -S_y & S_z & I_{rr} \end{bmatrix} \begin{Bmatrix} -\psi + dv/dx \\ \theta + dw/dx \\ d\phi/dx \end{Bmatrix}.$$

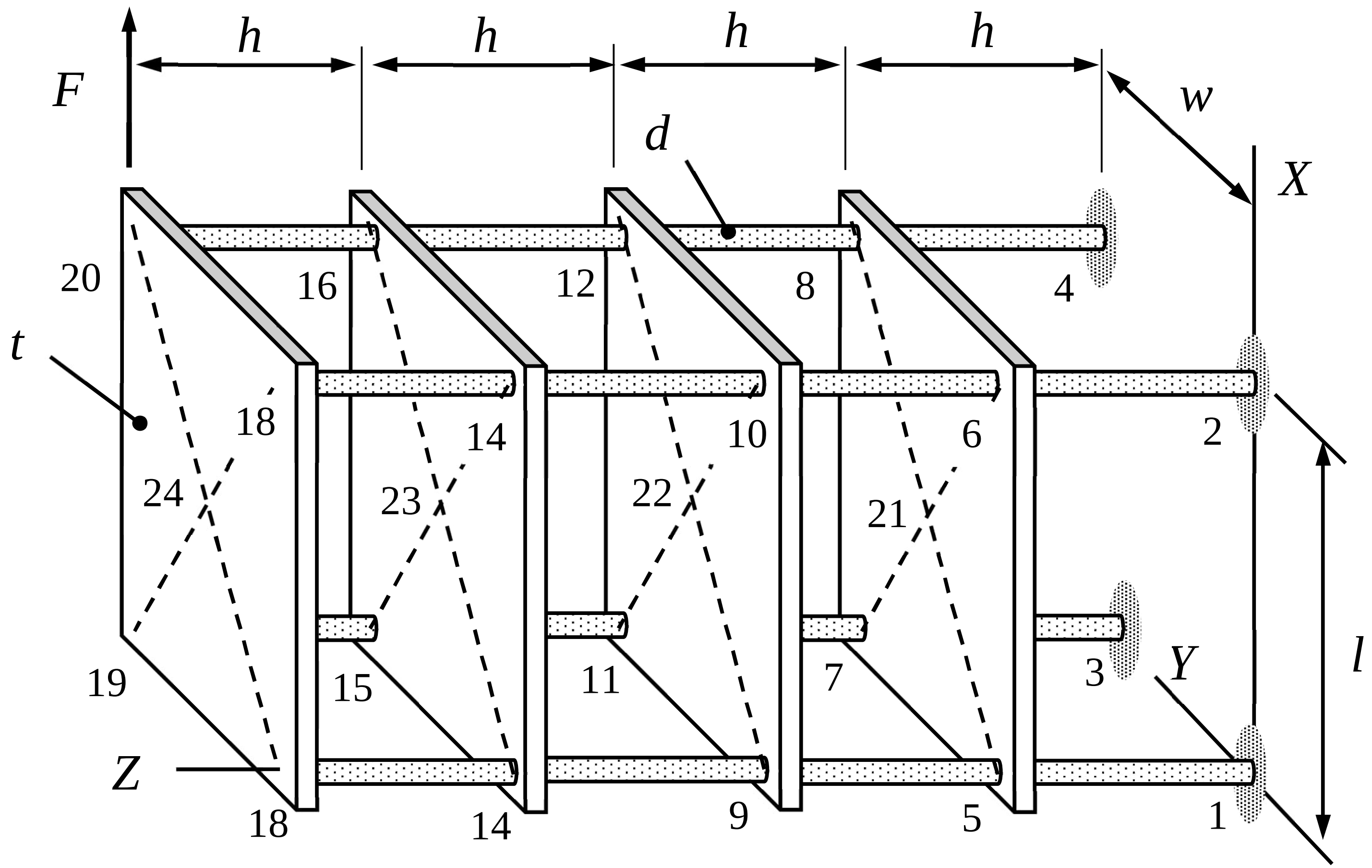
- The contributions coming from the external forces follow in the same manner. Assuming that the volume force is constant (in an element) and that the surface forces are acting on the end surfaces only, the expressions become

$$\delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} + \begin{Bmatrix} \delta\phi \\ \delta\theta \\ \delta\psi \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} m_x \\ m_y \\ m_z \end{Bmatrix} \text{ and } \delta W^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix} + \begin{Bmatrix} \delta\phi \\ \delta\theta \\ \delta\psi \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} M_x \\ M_y \\ M_z \end{Bmatrix}.$$

The last contribution is taken care of by a point force element in the Mathematica code.

IMPORTANT. The simplest possible linear approximation to the displacement and rotation components does not give a good numerical method unless numerical tricks like under-integration etc. are applied. To avoid numerical problems, approximations should be chosen cubic even if the exact solution is a simple polynomial! The Mathematica code uses a cubic approximation to all the unknowns and static condensation to end up with a two-node element.

EXAMPLE 6.6 A structure is modeled by using 16 beams and 4 rigid bodies. Assuming that a point force with the magnitude F is acting as shown in the figure, determine the displacement of the point of action in the direction of the force.



Week 48-32

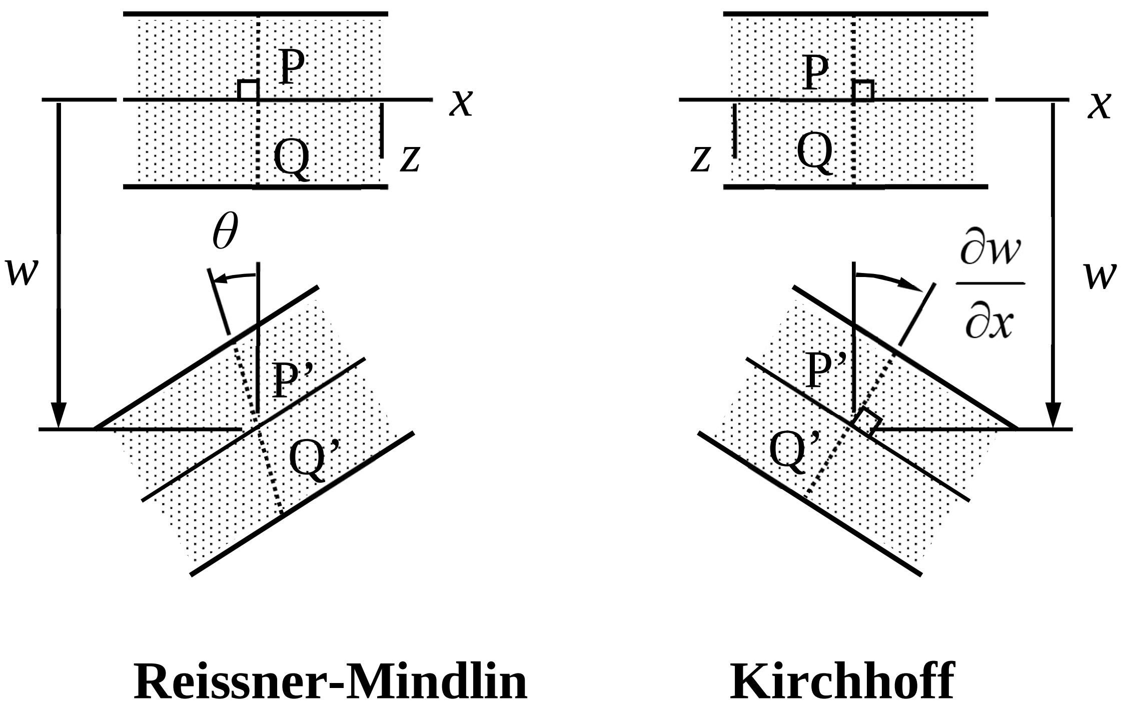
- Rigid bodies can be modelled by using a one node force element and rigid links with the other nodes. The problem description tables are given in the examples section of the Mathematica solver. The displacement of node 20 in the direction of X −axis as given by the solver is

$$u_{X20} = \frac{16Fh^3}{3\pi d^4 E}\left(\frac{64d^2 + 4l^2}{d^2 + 4l^2} + \frac{3w^2}{2h^2 G/E + 3l^2 + 3w^2}\right)$$

- If $E = 210 \cdot 10^3 \,\text{N/mm}^2$, $G = 80 \cdot 10^3 \,\text{N/mm}^2$, $d = 6.9\,\text{mm}$, $l = 408\,\text{mm}$, $w = 263\,\text{mm}$, $h = 170\,\text{mm}$, $F = 69\,\text{N}$, the displacement

$$u_{X20} = 1.56\,\text{mm}.$$

6.3 PLATE MODEL



Straight line segments perpendicular to the reference-plane remain straight in deformation (Reissner-Mindlin) and perpendicular to the reference-plane (Kirchhoff). In addition, transverse normal stress component is negligible.

- Normal line segments to the reference-plane move as rigid bodies. In terms of the displacement components $u(x,y)$, $v(x,y)$, $w(x,y)$ and rotation components $\phi(x,y)$, $\theta(x,y)$ of the translation point at the reference-plane, the displacement components are given by $\vec{u} = (u\vec{i} + v\vec{j} + w\vec{k}) + (\phi\vec{i} + \theta\vec{j}) \times z\vec{k}$). In component form

$$u_x(x,y,z) = u(x,y) + \theta(x,y)z,$$

$$u_y(x,y,z) = v(x,y) - \phi(x,y)z,$$

$$u_z(x,y,z) = w(x,y).$$

Rotation component in the z-direction is missing!

In the Kirchhoff model $u(x,y)$, $v(x,y)$, $w(x,y)$ and $\phi = \partial w / \partial y$ and $\theta = -\partial w / \partial x$ define the displacement field.

KIRCHHOFF PLATE VIRTUAL WORK DENSITY

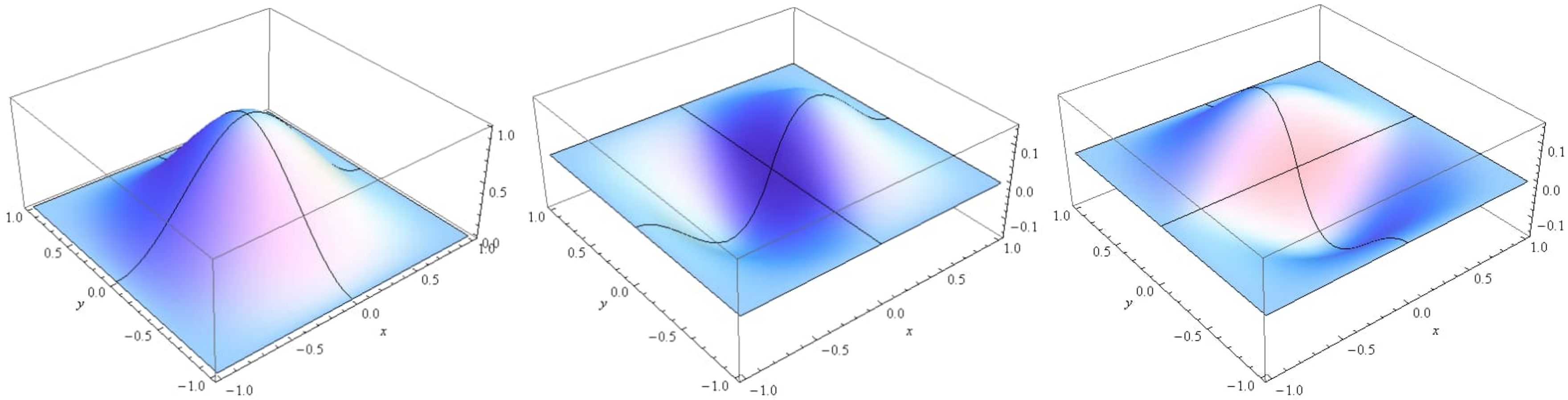
Virtual work densities combine the plane-stress thin slab and the plate bending modes which disconnect if the first moment of thickness vanishes. Virtual work densities of the bending mode of the Kirchhoff plate are

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial^2 \delta w / \partial x^2 \\ \partial^2 \delta w / \partial y^2 \\ 2\partial^2 \delta w / \partial x \partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12} [E]_{\sigma} \begin{Bmatrix} \partial^2 w / \partial x^2 \\ \partial^2 w / \partial y^2 \\ 2\partial^2 w / \partial x \partial y \end{Bmatrix}, \quad \left(\begin{Bmatrix} \phi \\ \theta \end{Bmatrix} = \begin{Bmatrix} \partial w / \partial y \\ -\partial w / \partial x \end{Bmatrix} \right)$$

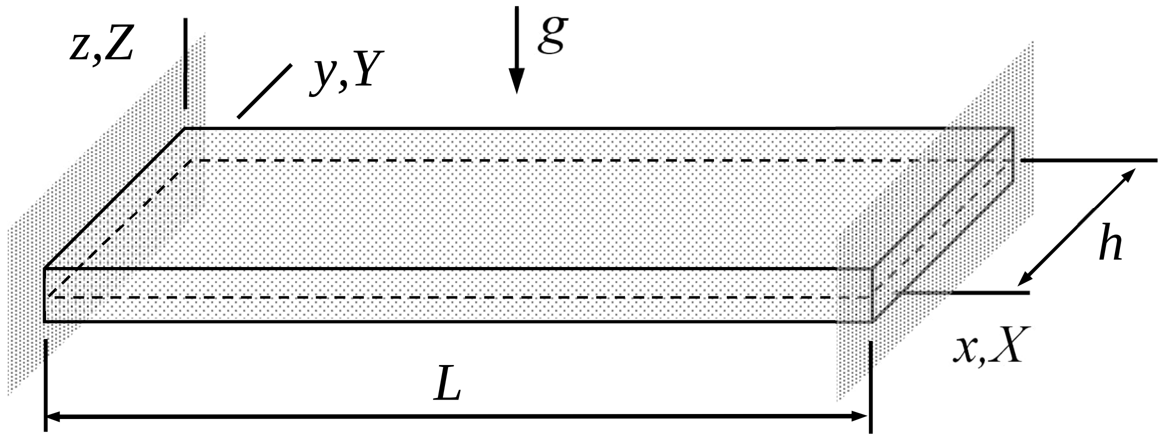
$$\delta w_{\Omega}^{\text{ext}} = \delta w f_z .$$

The planar solution domain (reference-plane) can be represented by triangular or rectangular elements. Interpolation of displacement components $w(x, y)$ should be continuous and have also continuous derivatives at the element interfaces.

- The severe continuity requirement of the approximation at the element interfaces is problematic in practice and cannot be satisfied with a simple interpolation of the nodal values. The figure illustrates the shape functions corresponding to displacement and rotation at a typical node in a patch of 4 square elements. The shape functions vanish outside the patch. In the course, Kirchhoff model is used only in calculations with domains of one element (no interfaces - no problems).



EXAMPLE 6.7 Consider a plate strip loaded by its own weight. Determine the deflection w according to the Kirchhoff model. Thickness, length of the plate are t, L, and h, respectively. Density ρ, Young's modulus E, and Poisson's ratio ν are constants. Use the one parameter approximation $w(x) = a_0(1 - x/L)^2(x/L)^2$.



Answer: $$w = -\frac{\rho g L^4}{2Et^2}(1-\nu^2)(1-\frac{x}{L})^2(\frac{x}{L})^2$$

- Approximation satisfies the boundary conditions ‘a priori’ and contains a free parameter a_0 (not associated with a node) to be solved by using the principle of virtual work:

$$w = a_0 (1-\frac{x}{L})^2 (\frac{x}{L})^2 \quad \Rightarrow \quad \frac{\partial^2 w}{\partial x^2} = a_0 \frac{2}{L^2}[1-6\frac{x}{L}+6(\frac{x}{L})^2] \text{ and } \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial x \partial y} = 0.$$

- When the approximation is substituted there, virtual work densities simplify to

$$\delta w_{\Omega}^{\text{int}} = -a_0 \delta a_0 \frac{Et^3}{3(1-\nu^2)} \frac{1}{L^4}[1-6\frac{x}{L}+6(\frac{x}{L})^2]^2,$$

$$\delta w_{\Omega}^{\text{ext}} = -\delta a_0 (1-\frac{x}{L})^2 (\frac{x}{L})^2 \rho g t.$$

- Integrations over the domain $\Omega =]0,L[\times]0,h[$ give the virtual works of internal and external forces

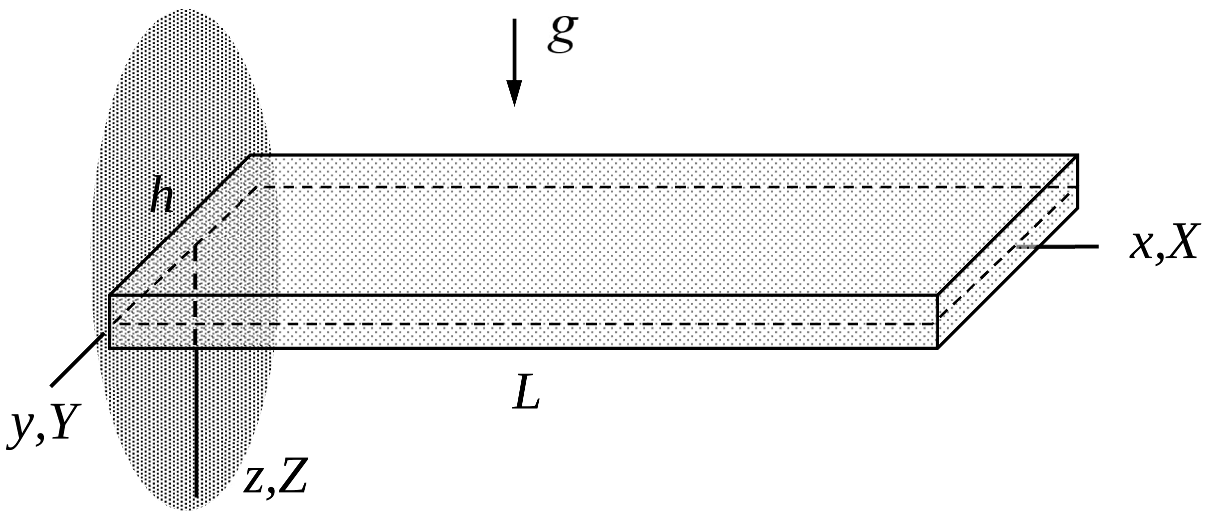
$$\delta W^{\text{int}} = \int_{\Omega} \delta w_{\Omega}^{\text{int}} d\Omega = -a_0 \delta a_0 \frac{1}{15} \frac{hEt^3}{L^3(1-\nu^2)},$$

$$\delta W^{\text{ext}} = \int_{\Omega} \delta w_{\Omega}^{\text{ext}} d\Omega = -\delta a_0 \frac{1}{30} \rho g t L h.$$

- Principle of virtual work $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = 0 \;\; \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus give finally $\forall \delta a_0$

$$\delta W = -\delta a_0 \left(\frac{1}{15} \frac{hEt^3}{L^3(1-\nu^2)} a_0 + \frac{1}{30} \rho g t L h\right) = 0 \quad \Leftrightarrow \quad a_0 = -\frac{1}{2} \frac{\rho g t L^4}{E t^2}(1-\nu^2).$$

EXAMPLE 6.8. A rectangular plate is loaded by its own weight. Determine the deflection of the plate at the free end by using the Kirchhoff plate model and one element. Thickness, width, and length of the plate are t, h and L, respectively. Density ρ, Young's modulus E, and Poisson's ratio ν of the material are constants. Assume that deflection w depends only on x.



Answer: $w(L) = u_{Z2} = \dfrac{3}{2}\dfrac{g\rho L^4}{Et^2}(1-\nu^2)$ (Bernoulli beam $w(L) = \dfrac{3}{2}\dfrac{g\rho L^4}{Et^2}$)

- As the solution is assumed to depend on x only and the material and structural coordinate systems coincide, one may use the cubic approximation of the Bernoulli beam model (bending in $xz-$plane and $\xi = x/L$). Let us denote the displacement and rotation at the free end by $u_{z2} = u_{Z2}$ and $\theta_{y2} = \theta_{Y2}$ to get

$$w = \begin{Bmatrix} (1-\xi)^2(1+2\xi) \\ L(1-\xi)^2\xi \\ \hdashline (3-2\xi)\xi^2 \\ L\xi^2(\xi-1) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ 0 \\ \hdashline u_{Z2} \\ -\theta_{Y2} \end{Bmatrix} = \begin{Bmatrix} (3-2x/L)(x/L)^2 \\ -L(x/L)^2(x/L-1) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix} \Rightarrow$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{L^2}\begin{Bmatrix} 6(L-2x)/L \\ -2(L-3x) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix} \text{ and } \frac{\partial^2 \delta w}{\partial x^2} = \begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{1}{L^2}\begin{Bmatrix} 6(L-2x)/L \\ -2(L-3x) \end{Bmatrix}.$$

- When the approximation is substituted there, virtual work densities simplify to

$$\delta w_\Omega^{\text{int}} = -\frac{Et^3}{12(1-\nu^2)L^4}\begin{Bmatrix}\delta u_{Z2}\\ \delta\theta_{Y2}\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}\frac{1}{L^2}(6L-12x)^2 & \frac{1}{L}(2L-6x)(6L-12x)\\ \frac{1}{L}(2L-6x)(6L-12x) & (2L-6x)^2\end{bmatrix}\begin{Bmatrix}u_{Z2}\\ \theta_{Y2}\end{Bmatrix}$$

$$\delta w_\Omega^{\text{ext}} = \begin{Bmatrix}\delta u_{Z2}\\ \delta\theta_{Y2}\end{Bmatrix}^{\mathrm{T}}\rho g t\begin{Bmatrix}(3-2x/L)(x/L)^2\\ -L(x/L)^2(x/L-1)\end{Bmatrix}.$$

- Integrations over the domain $\Omega =]0, L[\times]0, h[$ give the virtual works of internal and external forces

$$\delta W^{\text{int}} = \int_\Omega \delta w_\Omega^{\text{int}} d\Omega = -\frac{Et^3 h}{12(1-\nu^2)L^3}\begin{Bmatrix}\delta u_{Z2}\\ \delta\theta_{Y2}\end{Bmatrix}^{\mathrm{T}}\begin{bmatrix}12 & 6L\\ 6L & 4L^2\end{bmatrix}\begin{Bmatrix}u_{Z2}\\ \theta_{Y2}\end{Bmatrix},$$

$$\delta W^{\text{ext}} = \int_\Omega \delta w_\Omega^{\text{ext}} d\Omega = \begin{Bmatrix}\delta u_{Z2}\\ \delta\theta_{Y2}\end{Bmatrix}^{\mathrm{T}}\frac{\rho g t h L}{12}\begin{Bmatrix}6\\ L\end{Bmatrix}.$$

- Principle of virtual work $\delta W=\delta W^{\text{int}}+\delta W^{\text{ext}}=0\ \ \forall\delta\mathbf{a}$ and the fundamental lemma of variation calculus give finally

$$\delta W=-\begin{Bmatrix}\delta u_{Z2}\\ \delta\theta_{Y2}\end{Bmatrix}^{\mathrm{T}}\left(\frac{Et^3h}{12(1-\nu^2)L^3}\begin{bmatrix}12 & 6L\\ 6L & 4L^2\end{bmatrix}\begin{Bmatrix}u_{Z2}\\ \theta_{Y2}\end{Bmatrix}-\frac{\rho gthL}{12}\begin{Bmatrix}6\\ L\end{Bmatrix}\right)=0\quad\forall\begin{Bmatrix}\delta u_{Z2}\\ \delta\theta_{Y2}\end{Bmatrix}\quad\Leftrightarrow$$

$$\frac{Et^3h}{12(1-\nu^2)L^3}\begin{bmatrix}12 & 6L\\ 6L & 4L^2\end{bmatrix}\begin{Bmatrix}u_{Z2}\\ \theta_{Y2}\end{Bmatrix}-\frac{\rho gthL}{12}\begin{Bmatrix}6\\ L\end{Bmatrix}=0\quad\Leftrightarrow$$

$$\begin{Bmatrix}u_{Z2}\\ \theta_{Y2}\end{Bmatrix}=\frac{\rho gt(1-\nu^2)L^4}{Et^3}\begin{Bmatrix}3/2\\ -2/L\end{Bmatrix}.$$

A more detailed analysis may give dependence on $y-$coordinate which was excluded by the displacement assumption of the simplified analysis.

REISSNER-MINDLIN PLATE VIRTUAL WORK DENSITY

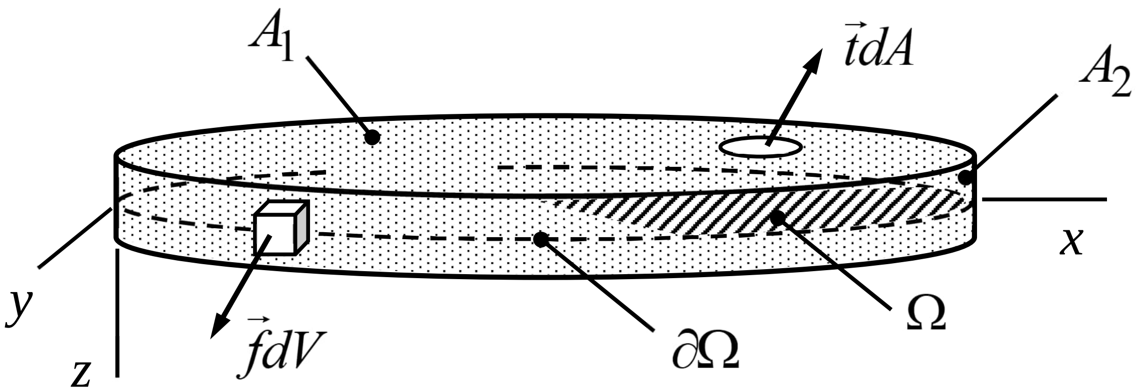
Virtual work densities combine the thin slab and plate bending modes which disconnect if the first moment of thickness vanishes. Virtual work densities of the Reissner-Mindlin plate bending mode are

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} -\partial\delta\theta/\partial x \\ \partial\delta\phi/\partial y \\ \partial\delta\phi/\partial x - \partial\delta\theta/\partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12}[E]_{\sigma} \begin{Bmatrix} -\partial\theta/\partial x \\ \partial\phi/\partial y \\ \partial\phi/\partial x - \partial\theta/\partial y \end{Bmatrix} - \begin{Bmatrix} \delta\gamma_x \\ \delta\gamma_y \end{Bmatrix}^{\mathrm{T}} tG \begin{Bmatrix} \gamma_x \\ \gamma_y \end{Bmatrix}$$

$$\delta w_{\Omega}^{\text{ext}} = \delta w f_z. \qquad \text{(shear strain expressions } \begin{Bmatrix} \gamma_x \\ \gamma_y \end{Bmatrix} = \begin{Bmatrix} \partial w/\partial x + \theta \\ \partial w/\partial y - \phi \end{Bmatrix}\text{)}$$

The planar solution domain can be represented by triangular or rectangular elements. Interpolation of displacement and rotation components $w(x,y)$, $\phi(x,y)$, and $\theta(x,y)$ should be continuous at the element interfaces.

- Plate is a thin body in one-dimension



- Normals to the reference plane (not necessarily the symmetry or mid-plane) remain straight in deformation. Kinematic assumption $\vec{u} = (u\vec{i} + v\vec{j} + w\vec{k}) + (\phi\vec{i} + \theta\vec{j}) \times z\vec{k}$ gives the displacement components and strains

$$\begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} = \begin{Bmatrix} u(x,y) \\ v(x,y) \\ w(x,y) \end{Bmatrix} + z \begin{Bmatrix} \theta(x,y) \\ -\phi(x,y) \\ 0 \end{Bmatrix} \quad \Rightarrow$$

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix} + z \begin{Bmatrix} \partial \theta / \partial x \\ -\partial \phi / \partial y \\ \partial \theta / \partial y - \partial \phi / \partial x \end{Bmatrix} \text{ and } \begin{Bmatrix} \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{Bmatrix} \partial w / \partial y - \phi \\ \partial w / \partial x + \theta \end{Bmatrix}.$$

- The constitutive equations of a linearly elastic isotropic material and kinetic assumption $\sigma_{zz} = 0$ give the non-zero stress components

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \text{ and } \begin{Bmatrix} \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = G \begin{Bmatrix} \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}.$$

- The generic expression of δw_V^{int} simplifies to a sum of thin slab, bending, transverse shear and interaction parts of which the last vanishes if the material is homogeneous and the reference plane coincides with the symmetry plane. With that assumption

$$\delta w_V^{\text{int}} = -\begin{Bmatrix} \partial\delta u / \partial x \\ \partial\delta v / \partial y \\ \partial\delta u / \partial y + \partial\delta v / \partial x \end{Bmatrix}^{\mathrm{T}} [E]_\sigma \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix}, \quad \text{thin slab part}$$

$$\delta w_V^{\text{int}} = -\begin{Bmatrix} \partial\delta\theta / \partial x \\ -\partial\delta\phi / \partial y \\ \partial\delta\theta / \partial y - \partial\delta\phi / \partial x \end{Bmatrix}^{\mathrm{T}} z^2 [E]_\sigma \begin{Bmatrix} \partial\theta / \partial x \\ -\partial\phi / \partial y \\ \partial\theta / \partial y - \partial\phi / \partial x \end{Bmatrix}, \quad \text{bending part}$$

$$\delta w_V^{\text{int}} = -\begin{Bmatrix} \partial\delta w / \partial y - \delta\phi \\ \partial\delta w / \partial x + \delta\theta \end{Bmatrix}^{\mathrm{T}} G \begin{Bmatrix} \partial w / \partial y - \phi \\ \partial w / \partial x + \theta \end{Bmatrix}. \quad \text{shear part}$$

- The generic expressions of δw_V^{ext} and δw_A^{ext} simplify to

$$\delta w_V^{\text{ext}} = \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} = \left(\begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} + z \begin{Bmatrix} \delta\theta \\ -\delta\phi \\ 0 \end{Bmatrix}^{\mathrm{T}} \right) \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix},$$

$$\delta w_A^{\text{ext}} = \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix} = \left(\begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} + z \begin{Bmatrix} \delta\theta \\ -\delta\phi \\ 0 \end{Bmatrix}^{\mathrm{T}} \right) \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix}.$$

- The virtual work of internal forces is obtained as integral over the domain occupied by the body (here the volume element $dV = dzd\Omega$). If $z \in [-t/2, t/2]$

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial\delta u / \partial x \\ \partial\delta v / \partial y \\ \partial\delta u / \partial y + \partial\delta v / \partial x \end{Bmatrix}^{\mathrm{T}} t[E]_\sigma \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix},$$

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta\theta / \partial x \\ -\partial\delta\phi / \partial y \\ \partial\delta\theta / \partial y - \partial\delta\phi / \partial x \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12} [E]_{\sigma} \begin{Bmatrix} \partial\theta / \partial x \\ -\partial\phi / \partial y \\ \partial\theta / \partial y - \partial\phi / \partial x \end{Bmatrix},$$

$$\delta w_{V}^{\text{int}} = -\begin{Bmatrix} \partial\delta w / \partial y - \delta\phi \\ \partial\delta w / \partial x + \delta\theta \end{Bmatrix}^{\mathrm{T}} tG \begin{Bmatrix} \partial w / \partial y - \phi \\ \partial w / \partial x + \theta \end{Bmatrix}.$$

- The contributions coming from the external forces follow in the same manner. Assuming that the volume force is constant (in an element), the surface forces do not act on the top and bottom surfaces, the expression simplifies to

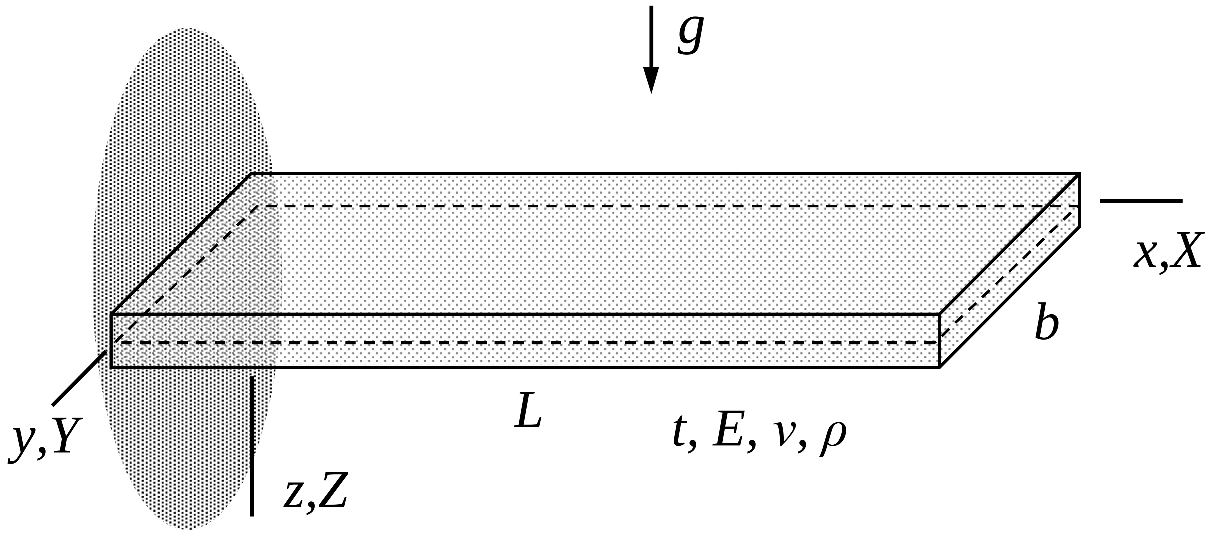
$$\delta W^{\text{ext}} = \int_{\Omega} \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} d\Omega + \int_{\partial\Omega} \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix} d\Gamma. \quad \leftarrow$$

- As the virtual work expression contains only first derivatives, the approximation should be continuous. Continuity requirement does not introduce any problems here and one may choose e.g.

$$w=\begin{Bmatrix}(1-\xi)(1-\eta)\\ \xi(1-\eta)\\ (1-\xi)\eta\\ \xi\eta\end{Bmatrix}\begin{Bmatrix}w_1\\ w_2\\ w_3\\ w_4\end{Bmatrix},\ \phi=\begin{Bmatrix}(1-\xi)(1-\eta)\\ \xi(1-\eta)\\ (1-\xi)\eta\\ \xi\eta\end{Bmatrix}\begin{Bmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{Bmatrix},\text{ and }\ \theta=\begin{Bmatrix}(1-\xi)(1-\eta)\\ \xi(1-\eta)\\ (1-\xi)\eta\\ \xi\eta\end{Bmatrix}\begin{Bmatrix}\theta_1\\ \theta_2\\ \theta_3\\ \theta_4\end{Bmatrix}.$$

IMPORTANT. Reissner-Mindlin plate model shares the numerical difficulties of the Timoshenko beam model and, in practice, finite element methods using low order approximations, e.g. a linear approximation on a triangle, suffer from severe shear locking that can be avoided only with carefully designed tricks.

EXAMPLE 6.9 A rectangular plate is loaded by its own weight. Determine the deflection of the plate at the free end by using the Reissner-Mindlin plate model with bi-linear, bi-quadratic and bi-cubic approximations. Thickness, width, and length of the plate are t, b and L, respectively. Density ρ, Young's modulus E, and Poisson's ratio ν of the material are constants. Consider finally the limit $G \to \infty$.



Answer: $w(L) = u_{Z2} = u_{Z4} = \dfrac{3}{2}\dfrac{g\rho L^4}{Et^2}(1-\nu^2)$ (Bernoulli beam $w(L) = \dfrac{3}{2}\dfrac{g\rho L^4}{Et^2}$)

- The solutions given by the Mathematica code of the course are

Bi-linear: $w(L) = 0$

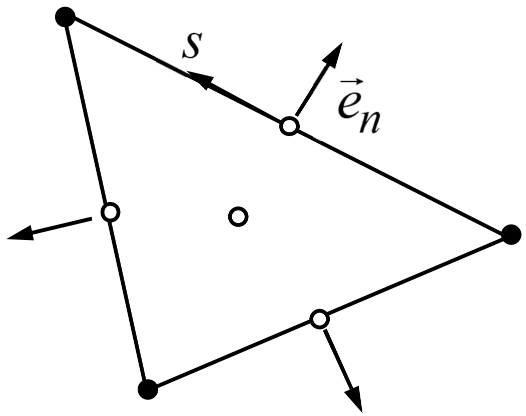
Bi-quadratic: $w(L) = \dfrac{g\rho L^4}{Et^2}(1-\nu^2)$

Bi-cubic: $w(L) = \dfrac{3}{2}\dfrac{g\rho L^4}{Et^2}(1-\nu^2)$

Therefore, approximations to the unknown functions should be cubic for a precise prediction, in this particular case. For example, a linear non-locking triangle element needs reconsideration of the shear terms.

MITC3 TRIANGLE

MITC3 (Mixed Interpolation of Tensorial Components) method modifies the shear part of virtual work expression by introducing additional rotation components (underbar) related with the original ones by conditions ($\vec{\gamma} = \nabla w - \vec{e} \times \vec{\theta}$ and $\underline{\vec{\gamma}} = \nabla w - \vec{e} \times \underline{\vec{\theta}}$)

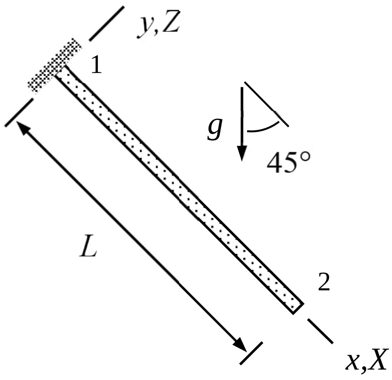


On the edges: $\dfrac{d}{ds}\underline{\theta}_n = 0$,

At the edge midpoints: $\theta_n - \underline{\theta}_n = 0$,

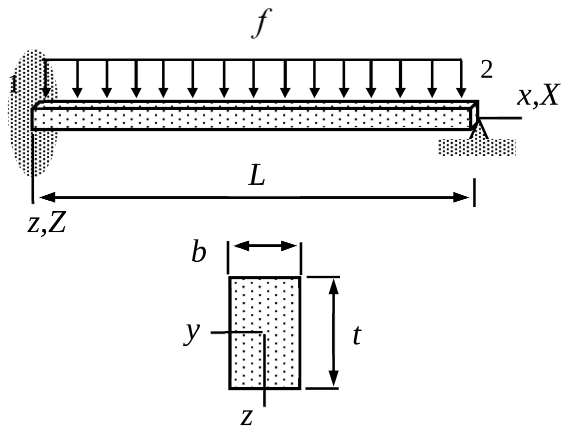
where $\vec{e} \times \vec{e}_n = \vec{e}_s$, $\vec{e}_n = \vec{e}_s \times \vec{e}$ etc. The outcome is a 6 linear equation system which can be solved for the coefficient of $\underline{\vec{\theta}}$ in terms of those of $\vec{\theta}$ (shear correction factor $\kappa = t^2 / (t^2 + \alpha h^2)$ is also needed with an appropriate value of the tuning parameter α).

MEC-E1050 Finite Element Method in Solid, week 48/2024



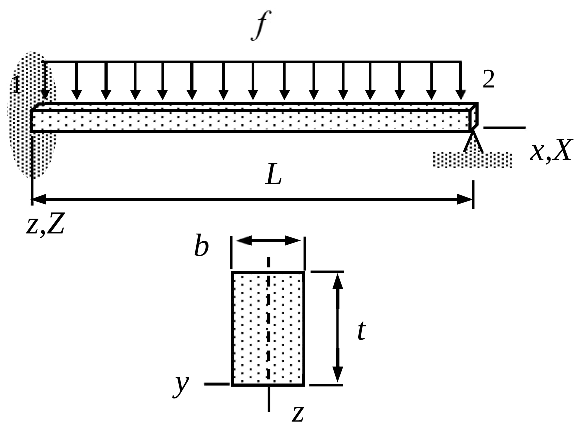
1. Consider a cantilever in *xy*-plane loaded by its own weight. Determine the displacement and rotation of the free end. Density ρ, Young's modulus E, Poisson's ratio ν are constants, and cross-section is rectangle of side length *t*. Use one element and Bernoulli beam model with the bar and bending modes.

   Answer $u_{X2} = \frac{1}{2\sqrt{2}}\frac{\rho g L^2}{E}$, $u_{Z2} = -\frac{3}{2\sqrt{2}}\frac{\rho g L^4}{Et^2}$, $\theta_{Y2} = \sqrt{2}\frac{\rho g L^3}{Et^2}$.



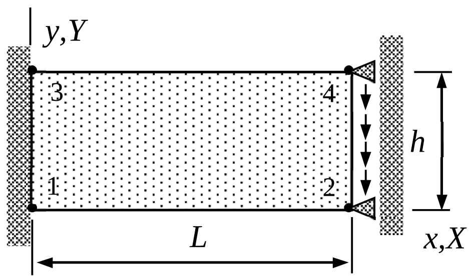
2. Determine rotation of the Bernoulli beam of the figure at the support of the right end (use one element). The neutral axis coincides with the *x*-axis of the material coordinate system and the support does not allow displacement *at the x-axis*. Material property *E* is constant.

   Answer $\theta_{Y2} = \frac{1}{4}\frac{fL^3}{Ebt^3}$



3. Determine the rotation of the Bernoulli beam in the figure at node 2. The *x*-axis of the material coordinate system is placed as shown and the support at node 2 does not allow displacement *at the x-axis*. Young's modulus of the material *E* is constant. Use quadratic approximation (three nodes) to the axial displacement u.

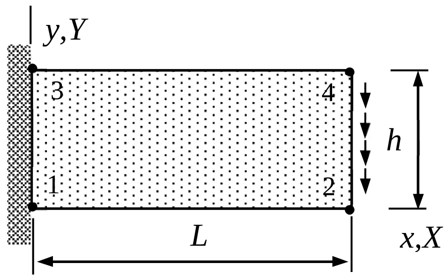
   Answer $\theta_{Y2} = \frac{1}{7}\frac{fL^3}{Ebt^3}$



4. A plate is loaded in its plane by shear force *F* distributed evenly as shown in the figure. Determine the displacement at the free end. Use thin-slab mode virtual work density of the plate model and a four-node element. Material properties *E*, ν, ρ and thickness *t* are constants. Assume that $u_{Y4} = u_{Y2}$.

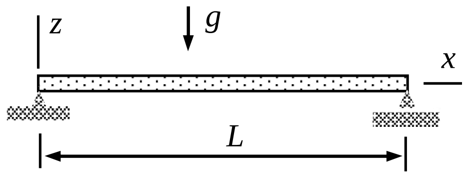
   Answer $u_{Y2} = -2\frac{LF}{htE}(1+\nu) = -\frac{LF}{htG}$

5. A plate is loaded in its plane by shear force F distributed evenly as shown. Determine the displacement of the free end. Use the virtual work density expressions of the thin-slab mode of the plate model and a four-node element. Material properties E, $\nu = 0$, ρ and thickness t are constants. Assume that $u_{Y4} = u_{Y2}$ and $u_{X4} = -u_{X2}$ and consider the slender plate limit $h / L \ll 1$.



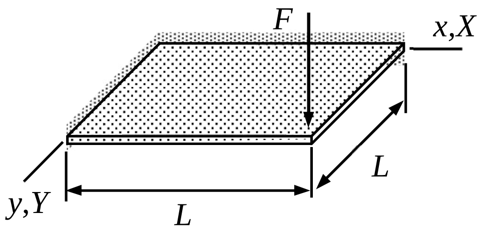
Answer $u_{X2} = -6\dfrac{F}{tE}$, $u_{Y2} = -8\dfrac{FL}{tEh}$

6. Consider the plate strip shown loaded by its own weight. Thickness, length and width of the plate are t, L, and H, respectively. Density ρ, Young's modulus E, and Poisson's ratio ν are constants. Find an approximation to the transverse displacement w of the plate using series $w = \mathrm{a}_0(1 - x/L)(x/L)$ (just one term of a series) in which a_0 is an unknown parameter.



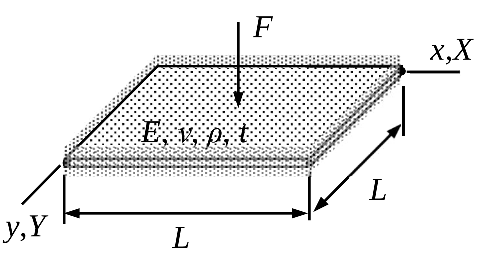
Answer $w(x,y) = -\dfrac{1}{2}\dfrac{L^4\rho g}{t^2 E}(1-\nu^2)(1-\dfrac{x}{L})\dfrac{x}{L}$

7. A plate, loaded by point force F acting at the free corner, is simply supported on two edges and free on the other two edges as shown in the figure. Determine the parameter a_0 of approximation $w(x,y) = a_0(x/L)(y/L)$ and displacement at the center point. Use the virtual work density of the plate bending mode with constant E, ν, ρ and t.



Answer $a_0 = 6\dfrac{FL^2}{Et^3}(1+\nu)$, $w(\dfrac{L}{2},\dfrac{L}{2}) = \dfrac{3}{2}\dfrac{FL^2}{Et^3}(1+\nu)$

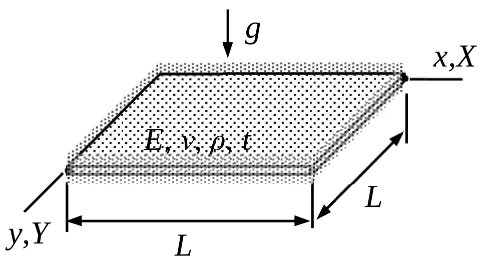
8. A simply supported plate is loaded by force F acting at the center as shown in the figure. Determine the displacement $w(x,y)$ by using the principle of virtual work. Consider the plate bending mode only and use approximation $w = a_0 \sin(\pi x/L)\sin(\pi y/L)$ in which a_0 is a parameter. Material properties E, ν, ρ and thickness t are constants. The shape functions of the approximation satisfy, e.g.,



$$\int_0^L \sin(i\pi\frac{x}{L})\sin(j\pi\frac{x}{L})dx = \frac{L}{2}\delta_{ij}\,.$$

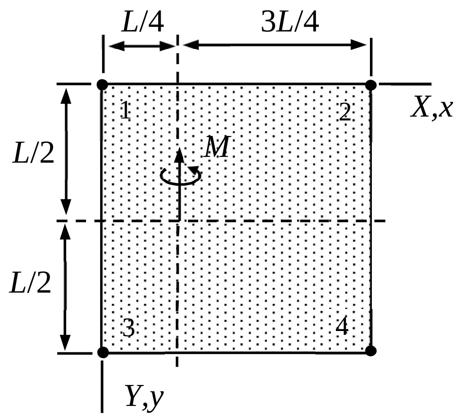
Answer $w(x,y) = \frac{12}{\pi^4}\frac{FL^2}{Et^3}(1-\nu^2)\sin(\pi\frac{x}{L})\sin(\pi\frac{y}{L})$

9. A simply supported plate is loaded by its own weight as shown. Use the bending mode virtual work density of the plate model to find the displacement. Use approximation $w = a_0(1-\xi)\xi(1-\eta)\eta$ in which a_0 is the parameter to be determined and the scaled coordinates $\xi = x/L$ and $\eta = y/L$. Material properties E, ν, ρ and thickness t are constants.



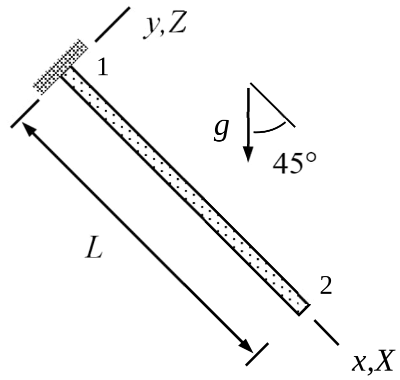
Answer $w = \frac{15}{22}\frac{g\rho L^4(1-\nu^2)}{Et^2}(1-\frac{x}{L})\frac{x}{L}(1-\frac{y}{L})\frac{y}{L}$

10. At point $x = L/4$ and $y = L/2$ of a 4-noded plate element there is a point moment with magnitude M. Determine the virtual work expression δW^{ext} of the moment for a Reissner-Mindlin plate element. Assume that nodes 1,2,4 are fixed and that the approximations to all unknown functions are bi-linear.



Answer $\delta W^{\mathrm{ext}} = -\frac{3}{8}M\delta\theta_{Y3}$.

Consider a cantilever in *xy*-plane loaded by its own weight. Determine the displacement and rotation of the free end. Density ρ, Young's modulus E, Poisson's ratio ν are constants, and cross-section is rectangle of side length *t*. Use one element and Bernoulli beam model with the bar and bending modes.



Solution

Assuming that the material coordinate system is chosen so that the bending and stretching modes decouple, the two modes can be taken into account as if they were separate elements. Therefore, one may use the virtual work expressions for the beam *xy*-plane bending and bar modes of the formulae collection

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{y1} \\ \delta\theta_{z1} \\ \delta u_{y2} \\ \delta\theta_{z2} \end{Bmatrix}^{\mathrm{T}} \frac{EI_{zz}}{h^3} \begin{bmatrix} 12 & 6h & -12 & 6h \\ 6h & 4h^2 & -6h & 2h^2 \\ -12 & -6h & 12 & -6h \\ 6h & 2h^2 & -6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \\ u_{y2} \\ \theta_{z2} \end{Bmatrix}, \quad \delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{y1} \\ \delta\theta_{z1} \\ \delta u_{y2} \\ \delta\theta_{z2} \end{Bmatrix}^{\mathrm{T}} \frac{f_y h}{12} \begin{Bmatrix} 6 \\ h \\ 6 \\ -h \end{Bmatrix},$$

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix}, \quad \delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\mathrm{T}} \frac{f_x h}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

The nodal displacements and rotations of the material coordinate systems need to be expressed in terms of those of the structural coordinate system. By using the figure

$$u_{x1} = 0, \quad u_{y1} = 0, \quad \theta_{z1} = 0,$$

$$u_{x2} = u_{X2}, \quad u_{y2} = u_{Z2}, \quad \theta_{z2} = -\theta_{Y2}.$$

The cross-section properties and the distributed force (per unit length) components in the material coordinate system are

$$A = t^2, \quad I_{zz} = \frac{1}{12}t^4, \quad f_x = \frac{1}{\sqrt{2}}t^2\rho g, \quad f_y = -\frac{1}{\sqrt{2}}t^2\rho g.$$

When these relationships are used in the element contribution of the beam bending mode, the generic expressions simplify to

$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \dfrac{Et^4}{L^3} & \dfrac{Et^4}{2L^2} \\ \dfrac{Et^4}{2L^2} & \dfrac{Et^4}{3L} \end{bmatrix} \begin{Bmatrix} u_{Z2} \\ \theta_{Y2} \end{Bmatrix} \quad \text{and} \quad \delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} -\dfrac{1}{2\sqrt{2}}t^2\rho g L \\ -\dfrac{1}{12\sqrt{2}}t^2\rho g L^2 \end{Bmatrix}.$$

The bar mode expressions take the forms

$$\delta W^{\text{int}} = -\delta u_{X2} \frac{Et^2}{L} u_{X2} \quad \text{and} \quad \delta W^{\text{ext}} = \delta u_{X2} \frac{1}{2\sqrt{2}} t^2 \rho g L \,.$$

Virtual work expression is the sum of the mode expressions

$$\delta W = -\begin{Bmatrix} \delta u_{X2} \\ \delta u_{Z2} \\ \delta \theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \left(\begin{bmatrix} \dfrac{Et^2}{L} & 0 & 0 \\ 0 & \dfrac{Et^4}{L^3} & \dfrac{Et^4}{2L^2} \\ 0 & \dfrac{Et^4}{2L^2} & \dfrac{Et^4}{3L} \end{bmatrix} \begin{Bmatrix} u_{X2} \\ u_{Z2} \\ \theta_{Y2} \end{Bmatrix} - \begin{Bmatrix} \dfrac{1}{2\sqrt{2}} t^2 \rho g L \\ -\dfrac{1}{2\sqrt{2}} t^2 \rho g L \\ -\dfrac{1}{12\sqrt{2}} t^2 \rho g L^2 \end{Bmatrix} \right).$$

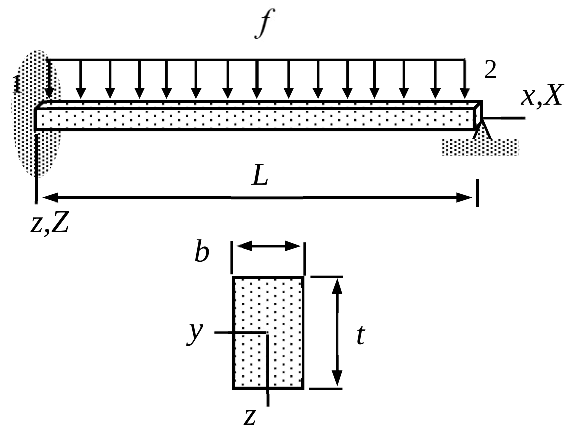
Principle of virtual work and the fundamental lemma of variational calculus imply the linear equation system

$$\begin{bmatrix} \dfrac{Et^2}{L} & 0 & 0 \\ 0 & \dfrac{Et^4}{L^3} & \dfrac{Et^4}{2L^2} \\ 0 & \dfrac{Et^4}{2L^2} & \dfrac{Et^4}{3L} \end{bmatrix} \begin{Bmatrix} u_{X2} \\ u_{Z2} \\ \theta_{Y2} \end{Bmatrix} - \begin{Bmatrix} \dfrac{1}{2\sqrt{2}} t^2 \rho g L \\ -\dfrac{1}{2\sqrt{2}} t^2 \rho g L \\ -\dfrac{1}{12\sqrt{2}} t^2 \rho g L^2 \end{Bmatrix} = 0 \,.$$

The first equation is not connected to the second and third. Therefore, the solution can be found without inverting the 3-by-3 matrix (the bending mode equations are connected so a 2-by-2 matrix needs to be inverted)

$$u_{X2} = \frac{1}{2\sqrt{2}} \frac{\rho g L^2}{E}, \quad u_{Z2} = -\frac{3}{2\sqrt{2}} \frac{\rho g L^4}{Et^2} \quad \text{and} \quad \theta_{Y2} = \sqrt{2} \frac{\rho g L^3}{Et^2} \,.$$

Determine rotation of the Bernoulli beam of the figure at the support of the right end (use one element). The neutral axis coincides with the x-axis of the material coordinate system and the support does not allow displacement *at the x-axis*. Material property E is constant.



Solution

Virtual work densities of the Bernoulli beam model taking into account all the modes

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} d\delta u/dx \\ d^2\delta v/dx^2 \\ d^2\delta w/dx^2 \end{Bmatrix}^{\text{T}} E \begin{bmatrix} A & -S_z & -S_y \\ -S_z & I_{zz} & I_{zy} \\ -S_y & I_{yz} & I_{yy} \end{bmatrix} \begin{Bmatrix} du/dx \\ d^2v/dx^2 \\ d^2w/dx^2 \end{Bmatrix} - \frac{d\delta\phi}{dx} GI_{rr} \frac{d\phi}{dx},$$

$$\delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\text{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} + \begin{Bmatrix} \delta\phi \\ -d\delta w/dx \\ d\delta v/dx \end{Bmatrix}^{\text{T}} \begin{Bmatrix} m_x \\ m_y \\ m_z \end{Bmatrix}$$

depend on the material properties E, G and on the moments of area A, S_y, S_z, I_{yy}, I_{zz}, I_{yz}, and $I_{rr} = I_{yy} + I_{zz}$. Expressions take the simplest form when x-axis is chosen to coincide with the neutral axis and y and z are symmetry axes of the cross-section.

Approximations to the unknown functions is the first thing to be considered. The left end of the beam is clamped and the right end simply supported. As the x-axis coincides with the neutral axis and the beam is not loaded in the direction of its axis, only the transverse displacement needs to be considered. Approximation to the transverse displacement w simplifies to

$$w = (\frac{x}{L})^2(1-\frac{x}{L})\theta_{Y2} \quad \Rightarrow \quad \frac{d^2w}{dx^2} = \frac{1}{L}(2-6\frac{x}{L})\theta_{Y2}.$$

Virtual work density depends on the moments of cross-section $A = bt$, $S_y = 0$, $S_z = 0$ and $I_{yy} = I = bt^3/12$. When the approximation is substituted there, virtual work densities simplify to

$$\delta w_{\Omega}^{\text{int}} = -\frac{d^2\delta w}{dx^2} EI \frac{d^2w}{dx^2} = -\delta\theta_{Y2} \frac{EI}{L^2}(2-6\frac{x}{L})^2\theta_{Y2},$$

$$\delta w_{\Omega}^{\text{ext}} = \delta w f_z = \delta\theta_{Y2} L(\frac{x}{L})^2(1-\frac{x}{L}) f.$$

Virtual work expressions are integrals of the densities over the element domain $\Omega =]0, L[$

$$\delta W^{\text{int}} = \int_{\Omega} \delta w_{\Omega}^{\text{int}} d\Omega = -\delta\theta_{Y2} 4\frac{EI}{L}\theta_{Y2},$$

$$\delta W^{\text{ext}} = \int_{\Omega} \delta w_{\Omega}^{\text{ext}} d\Omega = \delta\theta_{Y2} \frac{fL^2}{12} \;.$$

Principle of virtual work $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = 0 \quad \forall \delta\mathbf{a}$ and the fundamental lemma of variation calculus give

$$\delta W = -\delta\theta_{Y2}(4\frac{EI}{L}\theta_{Y2} - \frac{fL^2}{12}) = 0 \quad \forall \delta\theta_{Y2} \qquad \Leftrightarrow \qquad 4\frac{EI}{L}\theta_{Y2} - \frac{fL^2}{12} = 0 \quad \Leftrightarrow$$

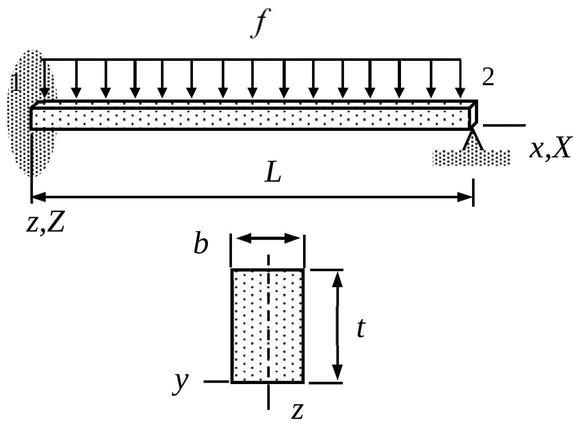
$$\theta_{Y2} = \frac{1}{48}\frac{fL^3}{EI} = \frac{1}{4}\frac{fL^3}{Ebt^3} \;. \qquad \leftarrow$$

Solution follows also from the virtual work expression of the formulae collection

$$\delta W = -\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} (\frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \theta_{Y2} \end{Bmatrix} - \frac{fL}{12}\begin{Bmatrix} 6 \\ -L \\ 6 \\ L \end{Bmatrix}) = -\delta\theta_{Y2}(\frac{EI}{L^3}4L^2\theta_{Y2} - \frac{fL^2}{12})$$

which is valid when the *x*-axis of the material coordinate system coincides with the neutral axis of the beam (and *y*- and *z*- axes are symmetry axes of the cross-section):

Determine the rotation of the Bernoulli beam in the figure at node 2. The *x*-axis of the material coordinate system is placed as shown and the support at node 2 does not allow displacement *at the x-axis*. Young's modulus of the material E is constant. Use quadratic approximation (three nodes) to the axial displacement u.



Solution

Virtual work densities of the Bernoulli beam model

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} d\delta u/dx \\ d^2\delta v/dx^2 \\ d^2\delta w/dx^2 \end{Bmatrix}^{\mathrm{T}} E \begin{bmatrix} A & -S_z & -S_y \\ -S_z & I_{zz} & I_{zy} \\ -S_y & I_{yz} & I_{yy} \end{bmatrix} \begin{Bmatrix} du/dx \\ d^2v/dx^2 \\ d^2w/dx^2 \end{Bmatrix} - \frac{d\delta\phi}{dx} GI_{rr} \frac{d\phi}{dx},$$

$$\delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} + \begin{Bmatrix} \delta\phi \\ -d\delta w/dx \\ d\delta v/dx \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} m_x \\ m_y \\ m_z \end{Bmatrix}$$

depend on the material properties E, G and on the moments of the area A, S_y, S_z, I_{yy}, I_{zz}, I_{yz}, and $I_{rr} = I_{yy} + I_{zz}$.

The left end of the beam is clamped and the right end simply supported. The additional node 3 for the quadratic approximation is places at the mid-point of the beam. Approximations to u and w become

$$u = \begin{Bmatrix} 1-3\xi+2\xi^2 \\ 4\xi(1-\xi) \\ \xi(2\xi-1) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ u_{X3} \\ 0 \end{Bmatrix} = 4\frac{x}{L}(1-\frac{x}{L})u_{X3} \quad \Rightarrow \quad \frac{du}{dx} = 4\frac{1}{L}(1-2\frac{x}{L})u_{X3}.$$

$$w = \begin{Bmatrix} (1-\xi)^2(1+2\xi) \\ L(1-\xi)^2\xi \\ \hdashline (3-2\xi)\xi^2 \\ L\xi^2(\xi-1) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ 0 \\ \hdashline 0 \\ -\theta_{Y2} \end{Bmatrix} = L(\frac{x}{L})^2(1-\frac{x}{L})\theta_{Y2} \quad \Rightarrow \quad \frac{d^2w}{dx^2} = \frac{1}{L}2(1-3\frac{x}{L})\theta_{Y2}.$$

Virtual work densities depend on the moments of cross-section

$$A = bt, \; S_z = 0, \; S = S_y = \int_{-t}^{0} zbdz = -bt^2/2 \text{ and } I = I_{yy} = \int_{-t}^{0} z^2bdz = bt^3/3.$$

When the approximations are substituted there, virtual work densities simplify to

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} d\delta u / dx \\ d^2\delta w / dx^2 \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} EA & -ES \\ -ES & EI \end{bmatrix} \begin{Bmatrix} du / dx \\ d^2 w / dx^2 \end{Bmatrix} \quad \text{and} \quad \delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ \rho g A \end{Bmatrix} \quad \Rightarrow$$

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \delta u_{X3} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{4E}{L^2} \begin{bmatrix} A4(1-2x/L)^2 & -S2(1-3x/L)(1-2x/L) \\ -S2(1-3x/L)(1-2x/L) & I(1-3x/L)^2 \end{bmatrix} \begin{Bmatrix} u_{X3} \\ \theta_{Y2} \end{Bmatrix},$$

$$\delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u_{X3} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ L(x/L)^2(1-x/L)f \end{Bmatrix}.$$

Virtual work expressions are integrals of the densities over the mathematical solution domain. Integrations over $\Omega =]0, L[$ give the virtual work expressions

$$\delta W^{\text{int}} = \int_{\Omega} \delta w_{\Omega}^{\text{int}} d\Omega = -\begin{Bmatrix} \delta u_{X3} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} 4\frac{E}{L} \begin{bmatrix} 4A/3 & -S \\ -S & I \end{bmatrix} \begin{Bmatrix} u_{X3} \\ \theta_{Y2} \end{Bmatrix},$$

$$\delta W^{\text{ext}} = \int_{\Omega} \delta w_{\Omega}^{\text{ext}} d\Omega = \begin{Bmatrix} \delta u_{X3} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{fL^2}{12} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}.$$

Principle of virtual work $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = 0 \quad \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus give

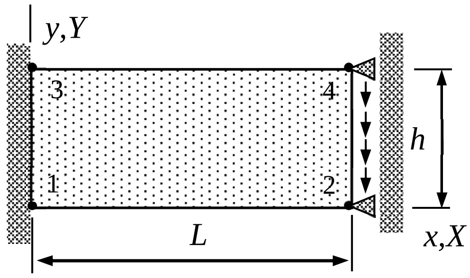
$$\delta W = -\begin{Bmatrix} \delta u_{X3} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} (4\frac{Ebt}{L} \begin{bmatrix} 4/3 & t/2 \\ t/2 & t^2/3 \end{bmatrix} \begin{Bmatrix} u_{X3} \\ \theta_{Y2} \end{Bmatrix} - \frac{fL^2}{12} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}) = 0 \quad \forall \begin{Bmatrix} \delta u_{X3} \\ \delta\theta_{Y2} \end{Bmatrix} \quad \Leftrightarrow$$

$$4\frac{Ebt}{L} \begin{bmatrix} 4/3 & t/2 \\ t/2 & t^2/3 \end{bmatrix} \begin{Bmatrix} u_{X3} \\ \theta_{Y2} \end{Bmatrix} - \frac{fL^2}{12} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = 0 \quad \Leftrightarrow$$

$$\begin{Bmatrix} u_{X3} \\ \theta_{Y2} \end{Bmatrix} = \frac{fL^3}{48Ebt} \begin{bmatrix} 4/3 & t/2 \\ t/2 & t^2/3 \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \frac{fL^3}{48Ebt} \frac{1}{7t^2} \begin{bmatrix} 12t^2 & -18t \\ -18t & 48 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \frac{f}{48Eb} (\frac{L}{t})^3 \frac{1}{7} \begin{Bmatrix} -18t \\ 48 \end{Bmatrix} \quad \Rightarrow$$

$$\theta_{Y2} = \frac{1}{7} \frac{fL^3}{Ebt^3}. \quad \leftarrow$$

A plate is loaded in its plane by shear force F distributed evenly as shown in the figure. Determine the displacement at the free end. Use thin-slab mode virtual work density of the plate model and a four-node element. Material properties E, ν, ρ and thickness t are constants. Assume that $u_{Y4} = u_{Y2}$.



Solution

Assuming that the material coordinate system is chosen so that the plate bending and thin slab modes decouple, virtual work density of plate is the sum of the thin-slab and plate bending mode virtual work densities. Here the bending part vanishes. The thin-slab expressions are

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta u/\partial y + \partial\delta v/\partial x \end{Bmatrix}^{\mathrm{T}} t[E]_{\sigma} \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix}, \quad \delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix}$$

and $\delta w_{\partial\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} t_x \\ t_y \end{Bmatrix}$.

Elasticity matrix of the plane stress case is given in the formulae collection

$$[E]_{\sigma} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}.$$

Let us choose the material and structural coordinate systems to coincide. Approximations to the in-plane displacements are (the shape functions can be deduced from the figure)

$$u = \begin{Bmatrix} (1-x/L)(1-y/h) \\ (x/L)(1-y/h) \\ (1-x/L)(y/h) \\ (x/L)(y/h) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = 0 \quad \text{and} \quad v = \begin{Bmatrix} (1-x/L)(1-y/h) \\ (x/L)(1-y/h) \\ (1-x/L)(y/h) \\ (x/L)(y/h) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ u_{Y2} \\ 0 \\ u_{Y2} \end{Bmatrix} = \frac{x}{L} u_{Y2}.$$

When the approximations are substituted there, virtual work densities of internal forces and external surface forces simplify to

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} 0 \\ 0 \\ \delta u_{Y2}/L \end{Bmatrix}^{\mathrm{T}} \frac{tE}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_{Y2}/L \end{Bmatrix} = -\delta u_{Y2} \frac{1}{L^2} \frac{tE}{2(1+\nu)} u_{Y2},$$

$$\delta w_{\partial\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ t_y \end{Bmatrix} = \begin{Bmatrix} 0 \\ \delta u_{Y2} x/L \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ -F/h \end{Bmatrix} = -\delta u_{Y2} \frac{F}{h} \frac{x}{L}.$$

Virtual work expressions are integrals of the densities over the corresponding domains

$$\delta W^{\text{int}} = \int_0^h \int_0^L \delta w_\Omega^{\text{int}} dxdy = -\delta u_{Y2} \frac{h}{L} \frac{tE}{2(1+\nu)} u_{Y2} \quad \Rightarrow$$

$$\delta W^{\text{ext}} = \int_0^h \delta w_{\partial\Omega}^{\text{ext}} dy = -\delta u_{Y2} F \, .$$ (notice that $x = L$ on edge 2-4)

Therefore

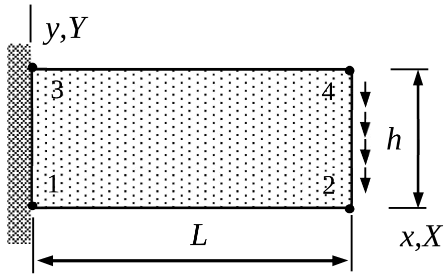
$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = -\delta u_{Y2} \left(\frac{h}{L} \frac{tE}{2(1+\nu)} u_{Y2} + F \right).$$

Principle of virtual work $\delta W = 0 \; \forall \delta \mathrm{a}$ and the fundamental lemma of variation calculus give

$$\delta W = -\delta u_{Y2} \left(\frac{h}{L} \frac{tE}{2(1+\nu)} u_{Y2} + F \right) = 0 \quad \forall \delta u_{Y2} \quad \Rightarrow \quad \frac{h}{L} \frac{tE}{2(1+\nu)} u_{Y2} + F = 0 \quad \Leftrightarrow$$

$$u_{Y2} = -\frac{LF}{ht} \frac{2(1+\nu)}{E} \, .$$

A plate is loaded in its plane by shear force F distributed evenly as shown. Determine the displacement of the free end. Use the virtual work density expressions of the thin-slab mode of the plate model and a four-node element. Material properties E, $\nu = 0$, ρ and thickness t are constants. Assume that $u_{Y4} = u_{Y2}$ and $u_{X4} = -u_{X2}$. Consider the slender plate limit $h/L \ll 1$.



Solution

Assuming that the material coordinate system is chosen so that the plate bending and thin slab modes decouple, virtual work density of plate is the sum of the thin-slab and plate bending modes. Here the bending part vanishes and only the thin slab part

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta u/\partial x \\ \partial\delta v/\partial y \\ \partial\delta u/\partial y + \partial\delta v/\partial x \end{Bmatrix}^{\mathrm{T}} t[E]_{\sigma} \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix}, \quad \delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix}$$

and $\delta w_{\partial\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} t_x \\ t_y \end{Bmatrix}$.

is needed. Elasticity matrix of the plane stress case is given in the formulae collection

$$[E]_{\sigma} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}.$$

Let us choose the material and structural coordinate systems to coincide. Approximation to the in-plane displacement was chosen to be bilinear so that the displacement components are (the shape functions can be deduced from the figure)

$$u = \begin{Bmatrix} (1-x/L)(1-y/h) \\ (x/L)(1-y/h) \\ (1-x/L)(y/h) \\ (x/L)(y/h) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ u_{X2} \\ 0 \\ -u_{X2} \end{Bmatrix} = \frac{x}{hL}(h-2y)u_{X2}, \quad \frac{\partial u}{\partial x} = \frac{h-2y}{hL}u_{X2}, \quad \frac{\partial u}{\partial y} = -2\frac{x}{hL}u_{X2}$$

$$v = \begin{Bmatrix} (1-x/L)(1-y/h) \\ (x/L)(1-y/h) \\ (1-x/L)(y/h) \\ (x/L)(y/h) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ u_{Y2} \\ 0 \\ u_{Y2} \end{Bmatrix} = \frac{x}{L}u_{Y2}, \quad \frac{\partial v}{\partial x} = \frac{1}{L}u_{Y2}, \text{ and } \frac{\partial v}{\partial y} = 0.$$

When the approximation is substituted there, virtual work densities of internal forces and external surface forces simplify to ($\nu = 0$)

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \dfrac{h-2y}{hL}\delta u_{X2} \\ 0 \\ -2\dfrac{x}{hL}\delta u_{X2} + \dfrac{1}{L}\delta u_{Y2} \end{Bmatrix}^{\mathrm{T}} tE \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{Bmatrix} \dfrac{h-2y}{hL}u_{X2} \\ 0 \\ -2\dfrac{x}{hL}u_{X2} + \dfrac{1}{L}u_{Y2} \end{Bmatrix} \quad \Leftrightarrow$$

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \delta u_{X2} \\ \delta u_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{tE}{L^2} \begin{bmatrix} (\dfrac{h-2y}{h})^2 + 2(\dfrac{x}{h})^2 & -\dfrac{x}{h} \\ -\dfrac{x}{h} & \dfrac{1}{2} \end{bmatrix} \begin{Bmatrix} u_{X2} \\ u_{Y2} \end{Bmatrix}.$$

$$\delta w_{\partial\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ t_y \end{Bmatrix} = \begin{Bmatrix} 0 \\ \delta u_{Y2} x / L \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ -F/h \end{Bmatrix} = -\delta u_{Y2} \frac{F}{h}\frac{x}{L}.$$

Virtual work expressions are integrals of the density over the corresponding domains

$$\delta W^{\text{int}} = \int_0^h \int_0^L \delta w_{\Omega}^{\text{int}} dxdy = -\begin{Bmatrix} \delta u_{X2} \\ \delta u_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{tEL}{h} \begin{bmatrix} \dfrac{7}{3}\dfrac{h^2}{L^2} + \dfrac{2}{3} & -\dfrac{1}{2}\dfrac{h}{L} \\ -\dfrac{1}{2}\dfrac{h}{L} & \dfrac{1}{2}\dfrac{h^2}{L^2} \end{bmatrix} \begin{Bmatrix} u_{X2} \\ u_{Y2} \end{Bmatrix},$$

$$\delta W^{\text{ext}} = \int_0^h \delta w_{\partial\Omega}^{\text{ext}} dy = -\delta u_{Y2} F = -\begin{Bmatrix} \delta\theta_{Z2} \\ \delta u_{Y2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ F \end{Bmatrix}.$$

Therefore with $h \ll L$ (so that $h^2 \ll L^2$ in the 1-1 term of the matrix)

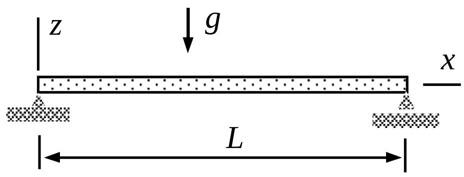
$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = -\begin{Bmatrix} \delta u_{X2} \\ \delta u_{Y2} \end{Bmatrix}^{\mathrm{T}} (tE \begin{bmatrix} \dfrac{2}{3}\dfrac{L}{h} & -\dfrac{1}{2} \\ -\dfrac{1}{2} & \dfrac{1}{2}\dfrac{h}{L} \end{bmatrix} \begin{Bmatrix} u_{X2} \\ u_{Y2} \end{Bmatrix} + \begin{Bmatrix} 0 \\ F \end{Bmatrix}).$$

Principle of virtual work $\delta W = 0 \; \forall \delta \mathrm{a}$ and the fundamental lemma of variation calculus give

$$\begin{Bmatrix} u_{X2} \\ u_{Y2} \end{Bmatrix} = -\frac{F}{tE} \begin{bmatrix} \dfrac{2}{3}\dfrac{L}{h} & -\dfrac{1}{2} \\ -\dfrac{1}{2} & \dfrac{1}{2}\dfrac{h}{L} \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = -12\frac{F}{tE} \begin{bmatrix} \dfrac{1}{2}\dfrac{h}{L} & \dfrac{1}{2} \\ \dfrac{1}{2} & \dfrac{2}{3}\dfrac{L}{h} \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = -12\frac{F}{tE} \begin{Bmatrix} \dfrac{1}{2} \\ \dfrac{2}{3}\dfrac{L}{h} \end{Bmatrix} \quad \Leftrightarrow$$

$$u_{X2} = -6\frac{F}{tE} \quad \text{and} \quad u_{Y2} = -8\frac{FL}{tEh}.$$

Consider the plate strip shown loaded by its own weight. Thickness, length and width of the plate are t, L, and H, respectively. Density ρ, Young's modulus E, and Poisson's ratio ν are constants. Find an approximation to the transverse displacement w of the plate using series $w = a_0(1 - x/L)(x/L)$ (just one term of a series) in which a_0 is an unknown parameter.



Solution

Assuming that the material coordinate system is chosen so that the plate bending and thin slab modes decouple, virtual work densities of the Kirchhoff plate model are given by

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial^2 \delta w / \partial x^2 \\ \partial^2 \delta w / \partial y^2 \\ 2\partial^2 \delta w / \partial x \partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12} [E]_\sigma \begin{Bmatrix} \partial^2 w / \partial x^2 \\ \partial^2 w / \partial y^2 \\ 2\partial^2 w / \partial x \partial y \end{Bmatrix} \quad \text{and} \quad \delta w_\Omega^{\text{ext}} = \delta w f_z \, .$$

in which the elasticity matrix of plane stress

$$[E]_\sigma = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}.$$

Approximation to the transverse displacement (notice that the polynomial shape is known and variation of displacement is through the multiplier)

$$w = a_0(1 - \frac{x}{L})(\frac{x}{L}) \quad \Rightarrow \quad \frac{\partial^2 w}{\partial x^2} = -a_0 \frac{2}{L^2}, \quad \frac{\partial^2 w}{\partial x \partial y} = 0, \quad \frac{\partial^2 w}{\partial y^2} = 0$$

$$\delta w = \delta a_0 (1 - \frac{x}{L})(\frac{x}{L}) \quad \Rightarrow \quad \frac{\partial^2 \delta w}{\partial x^2} = -\delta a_0 \frac{2}{L^2}, \quad \frac{\partial^2 \delta w}{\partial x \partial y} = 0, \quad \frac{\partial^2 \delta w}{\partial y^2} = 0.$$

When the approximation is substituted there, virtual work density simplifies to

$$\delta w_\Omega = \delta w_\Omega^{\text{int}} + \delta w_\Omega^{\text{ext}} = -\delta a_0 \frac{2}{L^2} \frac{t^3}{12} \frac{E}{1-\nu^2} a_0 \frac{2}{L^2} - \delta a_0 (1 - \frac{x}{L})(\frac{x}{L}) g \rho t \, .$$

Integration over the element gives

$$\delta W = \int_0^H \int_0^L \delta w_\Omega dx dy = -\delta a_0 H (\frac{4}{L^3} \frac{t^3 E}{12(1-\nu^2)} a_0 + \rho g t \frac{1}{6}) \, .$$

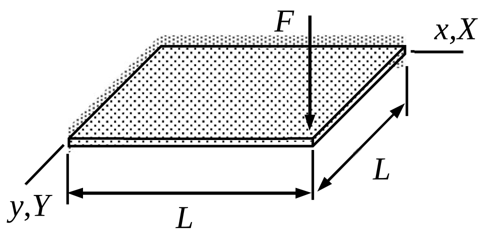
Principle of virtual work $\delta W = 0 \ \forall \delta a_0$ and the fundamental lemma of variation calculus give solution to the parameter

$$a_0 = -\frac{1}{2}\frac{L^4 \rho g}{t^2 E}(1-\nu^2) .$$

Therefore, approximation to the transverse displacement is given by

$$w(x,y) = -\frac{1}{2}\frac{L^4 \rho g}{t^2 E}(1-\nu^2)(1-\frac{x}{L})\frac{x}{L} .$$

A Kirchhoff plate, loaded by point force F acting at the free corner, is simply supported on two edges and free on the other two edges as shown in the figure. Determine the parameter a_0 of approximation $w(x,y)=a_0(x/L)(y/L)$ and displacement at the center point. Use the virtual work density of the Kirchhoff plate model with constant E, ν, ρ and t.



Solution

Assuming that the material coordinate system is chosen so that the plate bending and thin slab modes decouple, it is enough to consider the virtual work densities of the bending mode only

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial^2\delta w/\partial x^2 \\ \partial^2\delta w/\partial y^2 \\ 2\partial^2\delta w/\partial x\partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12}[E]_\sigma \begin{Bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ 2\partial^2 w/\partial x\partial y \end{Bmatrix}, \quad \delta w_\Omega^{\text{ext}} = \delta w f_z \,.$$

in which the elasticity matrix of plane stress

$$[E]_\sigma = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}.$$

In the present case, distributed force vanishes i.e. $f_z = 0$ and the point force is taken into account by a point force element.

Approximation to the transverse displacement is chosen to be (a_0 is not associated with any point but it just a parameter of the approximation)

$$w(x,y) = a_0 \frac{x}{L}\frac{y}{L} \quad \Rightarrow \quad \frac{\partial^2 w}{\partial x^2} = 0, \; \frac{\partial^2 w}{\partial y^2} = 0, \text{ and } \frac{\partial^2 w}{\partial x \partial y} = \frac{1}{L^2} a_0 \,.$$

When the approximation is substituted there, virtual work density of internal forces simplifies to

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial^2\delta w/\partial x^2 \\ \partial^2\delta w/\partial y^2 \\ 2\partial^2\delta w/\partial x\partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3 E}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ 2\partial^2 w/\partial x\partial y \end{Bmatrix} = -\delta a_0 \frac{Et^3}{6(1+\nu)}\frac{1}{L^4} a_0 \,,$$

Virtual work expression of the plate bending element (element 1 here) is integral of the virtual work density over the domain occupied by the element

$$\delta W^1 = \int_0^L \int_0^L \delta w_\Omega^{\text{int}} dxdy = -\delta a_0 \frac{Et^3}{6(1+\nu)} \frac{1}{L^2} a_0 \,.$$

Virtual work expression of the point force (element 2 here) follows from the definition of work (notice the use of virtual displacement of the point of action $x = y = L$)

$$\delta W^2 = \delta w(L,L)F = \delta a_0 F \; .$$

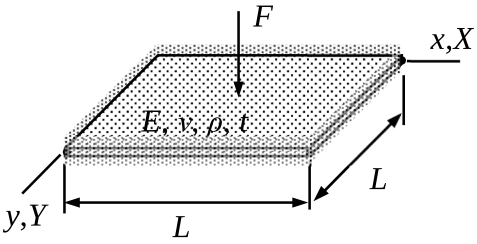
Principle of virtual work and the fundamental lemma of variation calculus give

$$\delta W = \delta W^1 + \delta W^2 = -\delta a_0 \left(\frac{Et^3}{6(1+\nu)}\frac{1}{L^2}a_0 - F\right) = 0 \quad \Rightarrow \quad a_0 = 6(1+\nu)\frac{FL^2}{Et^3} \; .$$

Displacement at the center point

$$w\left(\frac{L}{2},\frac{L}{2}\right) = a_0 \frac{1}{4} = \frac{3}{2}(1+\nu)\frac{FL^2}{Et^3} \, .$$

A simply supported plate is loaded by force F acting at the center as shown in the figure. Determine the displacement $w(x,y)$ by using the principle of virtual work. Consider the plate bending mode only and use approximation $w = a_0 \sin(\pi x/L)\sin(\pi y/L)$ in which a_0 is a parameter. Material properties E, ν, ρ and thickness t are constants. The shape functions of the approximation satisfy, e.g.,



$$\int_0^L \sin(i\pi\frac{x}{L})\sin(j\pi\frac{x}{L})dx = \frac{L}{2}\delta_{ij}\,.$$

Solution

Virtual work density of the internal forces is given by

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial^2\delta w/\partial x^2 \\ \partial^2\delta w/\partial y^2 \\ 2\partial^2\delta w/\partial x\partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12}[E]_\sigma \begin{Bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ 2\partial^2 w/\partial x\partial y \end{Bmatrix} \text{ where } [E]_\sigma = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}.$$

Approximation to the transverse displacement and its derivatives

$$w = a_0\sin(\pi\frac{x}{L})\sin(\pi\frac{y}{L}) \quad \Rightarrow$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial y^2} = -a_0(\frac{\pi}{L})^2\sin(\pi\frac{x}{L})\sin(\pi\frac{y}{L}) \text{ and } \frac{\partial^2 w}{\partial x\partial y} = a_0(\frac{\pi}{L})^2\cos(\pi\frac{x}{L})\cos(\pi\frac{y}{L})\,.$$

When the approximation is substituted there, virtual work density of internal forces simplifies to

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial^2\delta w/\partial x^2 \\ \partial^2\delta w/\partial y^2 \\ 2\partial^2\delta w/\partial x\partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3 E}{12(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\begin{Bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ 2\partial^2 w/\partial x\partial y \end{Bmatrix} \quad \Rightarrow$$

$$\delta w_\Omega^{\text{int}} = -\delta a_0\frac{t^3 E}{12(1-\nu^2)}(\frac{\pi}{L})^4 2[\sin^2(\frac{\pi x}{L})\sin^2(\frac{\pi y}{L})(1+\nu) + (1-\nu)\cos^2(\frac{\pi x}{L})\cos^2(\frac{\pi y}{L})]a_0\,.$$

Virtual work expression of the plate bending element (element 1 here) is integral of the virtual work density over the domain occupied by the element

$$\delta W^1 = \int_0^L\int_0^L \delta w_\Omega^{\text{int}} dxdy = -\delta a_0 4\frac{t^3 E}{12(1-\nu^2)}(\frac{\pi}{L})^4(\frac{L}{2})^2 a_0\,.$$

Virtual work expression of the point force (element 2 here) is given by the definition of work

$$\delta W^2 = \delta w(\frac{L}{2},\frac{L}{2})F = \delta a_0 F \ .$$

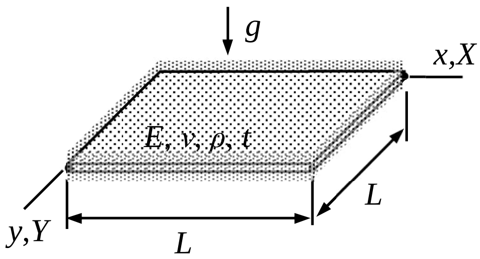
Principle of virtual work and the fundamental lemma of variation calculus give

$$\delta W = \delta W^1 + \delta W^2 = -\delta a_0 (\frac{1}{3}\frac{t^3 E}{1-\nu^2}(\frac{\pi}{L})^4 (\frac{L}{2})^2 a_0 - F) \quad \forall \delta a_0 \quad \Leftrightarrow \quad a_0 = \frac{12}{\pi^4}\frac{FL^2}{Et^3}(1-\nu^2) \,.$$

Displacement

$$w(x,y) = \frac{12}{\pi^4}\frac{FL^2}{Et^3}(1-\nu^2)\sin(\pi\frac{x}{L})\sin(\pi\frac{y}{L}) \,.$$

A simply supported plate is loaded by its own weight as shown. Use the bending mode virtual work density of the plate model to find the displacement. Use approximation $w = a_0(1-\xi)\xi(1-\eta)\eta$ in which a_0 is the parameter to be determined and the scaled coordinates $\xi = x/L$ and $\eta = y/L$. Material properties E, ν, ρ and thickness t are constants.



Solution

Assuming that the material coordinate system is chosen so that the plate bending and thin slab modes decouple, it is enough to consider the virtual work densities of the bending mode only

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial^2\delta w/\partial x^2 \\ \partial^2\delta w/\partial y^2 \\ 2\partial^2\delta w/\partial x\partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12}[E]_{\sigma} \begin{Bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ 2\partial^2 w/\partial x\partial y \end{Bmatrix}, \ \delta w_{\Omega}^{\text{ext}} = \delta w f_z \, .$$

in which the elasticity matrix of plane stress

$$[E]_{\sigma} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}.$$

The one parameter approximation to the transverse displacement gives

$$w(x,y) = a_0(1-\frac{x}{L})\frac{x}{L}(1-\frac{y}{L})\frac{y}{L} \quad \Rightarrow$$

$$\frac{\partial^2 w}{\partial x^2} = -a_0\frac{2}{L^2}(1-\frac{y}{L})\frac{y}{L}, \ \frac{\partial^2 w}{\partial y^2} = -a_0\frac{2}{L^2}(1-\frac{x}{L})\frac{x}{L}, \text{ and } \frac{\partial^2 w}{\partial x\partial y} = a_0\frac{1}{L^2}(1-2\frac{x}{L})(1-2\frac{y}{L}) \, .$$

When the approximation is substituted there, virtual work densities of internal and external forces take the forms

$$\delta w_{\Omega}^{\text{int}} = -\delta a_0 \begin{Bmatrix} -(1-\frac{y}{L})\frac{y}{L} \\ -(1-\frac{x}{L})\frac{x}{L} \\ (1-2\frac{x}{L})(1-2\frac{y}{L}) \end{Bmatrix}^{\mathrm{T}} \frac{4}{L^4}\frac{t^3}{12}\frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} -(1-\frac{y}{L})\frac{y}{L} \\ -(1-\frac{x}{L})\frac{x}{L} \\ (1-2\frac{x}{L})(1-2\frac{y}{L}) \end{Bmatrix} a_0 \quad \Leftrightarrow$$

$$\delta w_{\Omega}^{\text{int}} = -\delta a_0 \frac{4}{L^4}\frac{t^3}{12}\frac{E}{1-\nu^2}[(1-\frac{y}{L})^2(\frac{y}{L})^2 + (1-\frac{x}{L})^2(\frac{x}{L})^2 + 2\nu(1-\frac{y}{L})(\frac{y}{L})(1-\frac{x}{L})(\frac{x}{L}) +$$

$$\frac{1-\nu}{2}(1-2\frac{x}{L})^2(1-2\frac{y}{L})^2]\mathrm{a}_0,$$

$$\delta w_{\Omega}^{\text{ext}} = \delta w f_z = \delta a_0 (1-\frac{x}{L})\frac{x}{L}(1-\frac{y}{L})\frac{y}{L}\rho g t \ .$$

Virtual work expression of the plate bending element (element 1 here) is integral of the virtual work density over the domain occupied by the element

$$\delta W^{\text{int}} = \int_0^L \int_0^L \delta w_{\Omega}^{\text{int}} dxdy = -\delta a_0 \frac{1}{L^4}\frac{t^3}{3}\frac{E}{1-\nu^2}(\frac{L^2}{30} + \frac{L^2}{30} + 2\nu\frac{L^2}{36} + \frac{1-\nu}{2}\frac{L^2}{9})a_0 \quad \Leftrightarrow$$

$$\delta W^{\text{int}} = -\delta a_0 \frac{1}{L^2}\frac{Et^3}{1-\nu^2}\frac{11}{270}a_0 \, ,$$

$$\delta W^{\text{ext}} = \int_0^L \int_0^L \delta w_{\Omega}^{\text{ext}} dxdy = \delta a_0 \frac{L^2}{36}\rho g t \ .$$

Therefore

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = -\delta a_0 \frac{1}{L^2}\frac{Et^3}{1-\nu^2}\frac{11}{270}a_0 + \delta a_0 \frac{L^2}{36}\rho g t \quad \Leftrightarrow$$

$$\delta W = -\delta a_0 (\frac{1}{L^2}\frac{Et^3}{1-\nu^2}\frac{11}{270}a_0 - \frac{L^2}{36}\rho g t) .$$

Principle of virtual work and the fundamental lemma of variation calculus give

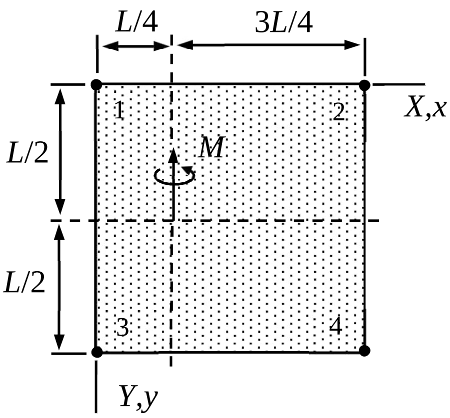
$$\delta W = -\delta a_0 (\frac{1}{L^2}\frac{Et^3}{1-\nu^2}\frac{11}{270}a_0 - \frac{L^2}{36}\rho g t) = 0 \quad \forall \delta a_0 \quad \Leftrightarrow$$

$$\frac{1}{L^2}\frac{Et^3}{1-\nu^2}\frac{11}{270}a_0 - \frac{L^2}{36}\rho g t = 0 \quad \Leftrightarrow \quad a_0 = \frac{15}{22}\frac{g\rho L^4}{Et^2}(1-\nu^2) \ .$$

Displacement

$$w(x,y) = \frac{15}{22}\frac{g\rho L^4}{Et^2}(1-\nu^2)(1-\frac{x}{L})\frac{x}{L}(1-\frac{y}{L})\frac{y}{L} .$$

At point $x = L/4$ and $y = L/2$ of a 4-noded plate element there is a point moment with magnitude M . Determine the virtual work expression δW^{ext} of the moment for a Reissner-Mindlin plate element. Assume that nodes 1,2,4 are fixed and that the approximations to all unknown functions are bi-linear.



Solution

In the present course, point forces and moment are taken into account by one node element. Virtual work expression follows from the definition "force multiplied by virtual displacement in its direction" and "point moment multiplied by the virtual rotation in its direction".

Virtual rotation at the point of action depends on the bilinear rotation component approximation in the *y*-direction for

$$\delta\theta(x,y) = \begin{Bmatrix} (1-x/L)(1-y/L) \\ (x/L)(1-y/L) \\ (1-x/L)(y/L) \\ (x/L)(y/L) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ 0 \\ \delta\theta_{Y3} \\ 0 \end{Bmatrix} = (1-\frac{x}{L})\frac{y}{L}\delta\theta_{Y3} \quad \Rightarrow$$

$$\delta\theta(\frac{L}{4},\frac{L}{2}) = \frac{3}{4}\frac{1}{2}\delta\theta_{Y3} = \frac{3}{8}\frac{1}{2}\delta\theta_{Y3}.$$

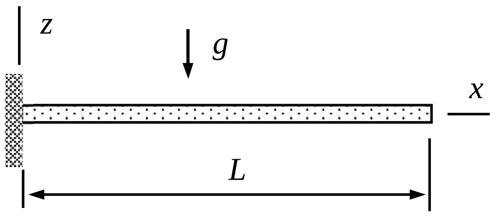
Virtual work expression from the definition "point moment multiplied by the virtual rotation in its direction"

$$\delta W^{\text{ext}} = \delta\theta(\frac{L}{4},\frac{L}{2})(-M) = -\frac{3}{8}M\delta\theta_{Y3}. \leftarrow$$

Name________________________ Student number________________

Assignment 1

Consider the beam loaded by its own weight as shown in the figure. Thickness, width, and length of the beam are t, b, and L, respectively. Density ρ, Young's modulus E, and Poisson's ratio ν are constants. Find the unknown parameter a_0 of the assumed transverse displacement $w = a_0 x^2$. The origin of the material coordinate system is placed at the symmetry axes of the rectangular cross section.



Solution template

Virtual work density expressions of the beam bending mode are

$$\delta w_\Omega^{\text{int}} = -\frac{d^2\delta w}{dx^2} EI_{yy} \frac{d^2 w}{dx^2} \quad \text{and} \quad \delta w_\Omega^{\text{ext}} = \delta w f_z ,$$

in which f_z is the $z-$component of the external force per unit length, E is the Young's modulus of the material, and I_{yy} the second moment of area with respect to the area centroid.

With the displacement assumption $w = a_0 x^2$ and the expression of I_{yy}, virtual work densities simplify to

$$\delta w_\Omega^{\text{int}} = -\frac{d^2\delta w}{dx^2} EI_{yy} \frac{d^2 w}{dx^2} = -\delta a_0 \frac{Ebt^3}{3} a_0 ,$$

$$\delta w_\Omega^{\text{ext}} = \delta w f_z = -\delta a_0 x^2 g \rho t b .$$

Integration over the length of the beam gives the virtual work expressions

$$\delta W^{\text{int}} = \int_0^L \delta w_\Omega^{\text{int}} dx = -\delta a_0 \frac{Ebt^3 L}{3} a_0 ,$$

$$\delta W^{\text{ext}} = \int_0^L \delta w_\Omega^{\text{ext}} dx = -\delta a_0 \frac{1}{3} L^3 g \rho t b .$$

Finally, principle of virtual work with $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}}$ and the fundamental lemma of variation calculus imply the solution

$$\delta W = -\delta a_0 \left(\frac{Ebt^3 L}{3} a_0 + \frac{1}{3} L^3 g \rho t b\right) = 0 \quad \forall \delta a_0 \quad \Leftrightarrow$$

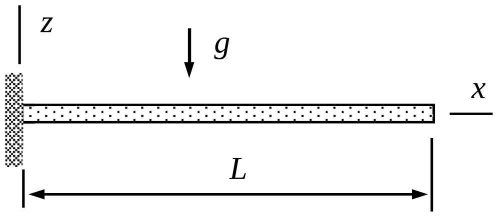
$$\frac{Ebt^3L}{3}a_0 + \frac{1}{3}L^3 g\rho tb = 0 \quad \Leftrightarrow$$

$$a_0 = -(\frac{L}{t})^2 \frac{g\rho}{E}.$$

Name______________________ Student number________________

Assignment 2

Consider the plate strip loaded by its own weight as shown in the figure. Thickness, width, and length of the plate are t, b, and L, respectively. Density ρ, Young's modulus E, and Poisson's ratio ν are constants. Find the unknown parameter a_0 of the assumed transverse displacement $w = a_0 x^2$. The origin of the material coordinate system is placed at the symmetry plane of the plate.



Solution template

Virtual work density expressions of the plate bending mode are

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial^2 \delta w / \partial x^2 \\ \partial^2 \delta w / \partial y^2 \\ 2\partial^2 \delta w / \partial x \partial y \end{Bmatrix}^{\text{T}} \frac{t^3}{12}[E]_\sigma \begin{Bmatrix} \partial^2 w / \partial x^2 \\ \partial^2 w / \partial y^2 \\ 2\partial^2 w / \partial x \partial y \end{Bmatrix} \quad \text{and} \quad \delta w_\Omega^{\text{ext}} = \delta w f_z .$$

in which f_z is the $z-$component of the distributed force per unit area, t is the thickness of the plate, and the elasticity matrix of plane stress is given by

$$[E]_\sigma = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}.$$

Under the displacement assumption $w = a_0 x^2$, virtual work densities of the plate model simplify to

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial^2 \delta w / \partial x^2 \\ \partial^2 \delta w / \partial y^2 \\ 2\partial^2 \delta w / \partial x \partial y \end{Bmatrix}^{\text{T}} \frac{t^3}{12}[E]_\sigma \begin{Bmatrix} \partial^2 w / \partial x^2 \\ \partial^2 w / \partial y^2 \\ 2\partial^2 w / \partial x \partial y \end{Bmatrix} = -\delta a_0 \frac{Et^3}{3(1-\nu^2)} a_0 ,$$

$$\delta w_\Omega^{\text{ext}} = \delta w f_z = -\delta a_0 x^2 g \rho t .$$

Integration over the area of the plate symmetry plane gives the virtual work expressions

$$\delta W^{\text{int}} = \int_0^L \int_0^b \delta w_\Omega^{\text{int}} dy dx = -\delta a_0 \frac{LEbt^3}{3(1-\nu^2)} a_0 ,$$

$$\delta W^{\text{ext}} = \int_0^L \int_0^b \delta w_\Omega^{\text{ext}} dydx = -\delta a_0 \frac{bL^3}{3} g\rho t \,.$$

Principle of virtual work with $\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}}$ and the fundamental lemma of variation calculus imply the solution

$$\delta W = -\delta a_0 (\frac{LbEt^3}{3(1-\nu^2)} a_0 + \frac{bL^3}{3} g\rho t) = 0 \quad \forall \delta a_0 \quad \Leftrightarrow$$

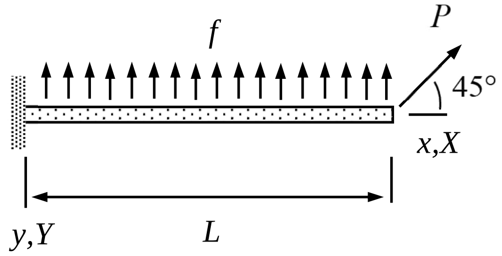
$$\frac{LbEt^3}{3(1-\nu^2)} a_0 + \frac{bL^3}{3} g\rho t = 0 \quad \Leftrightarrow$$

$$a_0 = -(1-\nu^2)(\frac{L}{t})^2 \frac{g\rho}{E} \,.$$

Name________________________ Student number________________

Assignment 3

Consider a cantilever in xy-plane loaded by distributed force f and point force $P=\sqrt{2}fL$. Determine the displacement and rotation of the free end. Young's modulus E and Poisson's ratio ν are constants. Cross-section is a rectangle of side length t. Assume that the neutral axis coincides with the x-axis of the material coordinate system. Use the bar and bending modes of the beam model.



Solution template

Assuming that the material coordinate system is chosen so that the bending and stretching modes decouple, the two modes can be taken into account as if they were separate elements. Therefore, one may use the virtual work expressions for the beam xy-plane bending and bar modes of the formulae collection

$$\delta W^{\text{int}}=-\begin{Bmatrix}\delta u_{y1}\\ \delta\theta_{z1}\\ \delta u_{y2}\\ \delta\theta_{z2}\end{Bmatrix}^{\mathrm{T}}\frac{EI_{zz}}{h^3}\begin{bmatrix}12 & 6h & -12 & 6h\\ 6h & 4h^2 & -6h & 2h^2\\ -12 & -6h & 12 & -6h\\ 6h & 2h^2 & -6h & 4h^2\end{bmatrix}\begin{Bmatrix}u_{y1}\\ \theta_{z1}\\ u_{y2}\\ \theta_{z2}\end{Bmatrix},\quad \delta W^{\text{ext}}=\begin{Bmatrix}\delta u_{y1}\\ \delta\theta_{z1}\\ \delta u_{y2}\\ \delta\theta_{z2}\end{Bmatrix}^{\mathrm{T}}\frac{f_y h}{12}\begin{Bmatrix}6\\ h\\ 6\\ -h\end{Bmatrix},$$

$$\delta W^{\text{int}}=-\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\frac{EA}{h}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix},\quad \delta W^{\text{ext}}=\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\frac{f_x h}{2}\begin{Bmatrix}1\\ 1\end{Bmatrix}.$$

The nodal displacements and rotations of the material coordinate systems need to be expressed in terms of those of the structural coordinate system. By using the figure

$u_{x1}=0$, $u_{y1}=0$, $\theta_{z1}=0$,

$u_{x2}=u_{X2}$, $u_{y2}=u_{Y2}$, $\theta_{z2}=\theta_{Z2}$.

The cross-section properties and the distributed force (per unit length) components in the material coordinate system are

$$A=t^2,\quad I_{zz}=\frac{1}{12}t^4,\quad f_x=0,\quad f_y=-f\,.$$

When the relationships are used in the element contribution of the beam bending mode the generic expressions simplify to

$$\delta W^1 = -\begin{Bmatrix} \delta u_{Y2} \\ \delta\theta_{Z2} \end{Bmatrix}^{\mathrm{T}} \left(\begin{bmatrix} \frac{Et^4}{L^3} & -\frac{Et^4}{2L^2} \\ -\frac{Et^4}{2L^2} & \frac{Et^4}{3L} \end{bmatrix} \begin{Bmatrix} u_{Y2} \\ \theta_{Z2} \end{Bmatrix} - \begin{Bmatrix} -\frac{fL}{2} \\ \frac{fL^2}{12} \end{Bmatrix} \right).$$

The bar mode expression takes the form

$$\delta W^2 = -\delta u_{X2} \frac{Et^2}{L} u_{X2}.$$

Virtual work expression of the point force

$$\delta W^3 = \begin{Bmatrix} \delta u_{X2} \\ \delta u_{Y2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} fL \\ -fL \end{Bmatrix}.$$

Virtual work expression is the sum of the element/mode contributions

$$\delta W = -\begin{Bmatrix} \delta u_{X2} \\ \delta u_{Y2} \\ \delta\theta_{Z2} \end{Bmatrix}^{\mathrm{T}} \left(\begin{bmatrix} \frac{Et^2}{L} & 0 & 0 \\ 0 & \frac{Et^4}{L^3} & -\frac{Et^4}{2L^2} \\ 0 & -\frac{Et^4}{2L^2} & \frac{Et^4}{3L} \end{bmatrix} \begin{Bmatrix} u_{X2} \\ u_{Y2} \\ \theta_{Z2} \end{Bmatrix} - \begin{Bmatrix} fL \\ -\frac{3}{2}fL \\ \frac{1}{12}fL^2 \end{Bmatrix} \right).$$

Principle of virtual work and the fundamental lemma of variational calculus imply the linear equations system

$$\begin{bmatrix} \frac{Et^2}{L} & 0 & 0 \\ 0 & \frac{Et^4}{L^3} & -\frac{Et^4}{2L^2} \\ 0 & -\frac{Et^4}{2L^2} & \frac{Et^4}{3L} \end{bmatrix} \begin{Bmatrix} u_{X2} \\ u_{Y2} \\ \theta_{Z2} \end{Bmatrix} - \begin{Bmatrix} fL \\ -\frac{3}{2}fL \\ \frac{1}{12}fL^2 \end{Bmatrix} = 0.$$

The first equation is not connected to the second and third. Therefore, the solution can be found without inverting the 3-by-3 matrix (the bending mode equations are connected so a 2-by-2 matrix needs to be inverted)

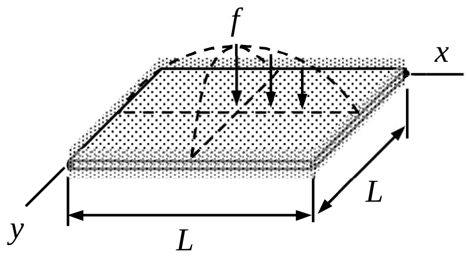
$$u_{X2} = \frac{f}{E}\left(\frac{L}{t}\right)^2,$$

$$\begin{Bmatrix} u_{Y2} \\ \theta_{Z2} \end{Bmatrix} = \begin{bmatrix} \dfrac{Et^4}{L^3} & -\dfrac{Et^4}{2L^2} \\ -\dfrac{Et^4}{2L^2} & \dfrac{Et^4}{3L} \end{bmatrix}^{-1} \begin{Bmatrix} -\dfrac{3}{2} f L \\ \dfrac{1}{12} f L^2 \end{Bmatrix} = \begin{bmatrix} 4\dfrac{L^3}{t^4 E} & 6\dfrac{L^2}{t^4 E} \\ 6\dfrac{L^2}{t^4 E} & 12\dfrac{L}{t^4 E} \end{bmatrix} \begin{Bmatrix} -\dfrac{3}{2} f L \\ \dfrac{1}{12} f L^2 \end{Bmatrix} = \begin{Bmatrix} -\dfrac{11}{2}\dfrac{f}{E}(\dfrac{L}{t})^4 \\ -8\dfrac{f}{Et}(\dfrac{L}{t})^3 \end{Bmatrix}.$$

Name__________________________ Student number__________________

Assignment 4

A simply supported plate is loaded by distributed force $f_z = f\sin(\pi x/L)\sin(\pi y/L)$ as shown in the figure. Determine the displacement $w(x,y)$ by using the principle of virtual work. Consider the plate bending mode only and use approximation $w = a_0\sin(\pi x/L)\sin(\pi y/L)$ in which a_0 is a parameter. Material properties E, ν, ρ and thickness t are constants. The shape functions of the approximation satisfy, e.g.,



$$\int_0^L \sin(i\pi\frac{x}{L})\sin(j\pi\frac{x}{L})dx = \frac{L}{2}\delta_{ij}\,.$$

Solution template

Assuming that the material coordinate system is chosen so that the plate bending and thin slab modes decouple, virtual work densities of the Kirchhoff plate model are given by

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial^2\delta w/\partial x^2 \\ \partial^2\delta w/\partial y^2 \\ 2\partial^2\delta w/\partial x\partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12}[E]_\sigma \begin{Bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ 2\partial^2 w/\partial x\partial y \end{Bmatrix} \quad \text{and} \quad \delta w_\Omega^{\text{ext}} = \delta w f_z\,.$$

in which the elasticity matrix of plane stress

$$[E]_\sigma = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}.$$

Approximation to the transverse displacement and its derivatives

$$w = a_0\sin(\pi\frac{x}{L})\sin(\pi\frac{y}{L}) \quad \Rightarrow$$

$$\frac{\partial^2 w}{\partial x^2} = -a_0(\frac{\pi}{L})^2\sin(\pi\frac{x}{L})\sin(\pi\frac{y}{L})\,,$$

$$\frac{\partial^2 w}{\partial y^2} = -a_0(\frac{\pi}{L})^2\sin(\pi\frac{x}{L})\sin(\pi\frac{y}{L})\,,$$

$$\frac{\partial^2 w}{\partial x \partial y} = a_0 (\frac{\pi}{L})^2 \cos(\pi \frac{x}{L}) \cos(\pi \frac{y}{L}) .$$

When the approximation and the expression for the distributed force are substituted there, virtual work densities simplify to

$$\delta w_{\Omega}^{\text{int}} = -\delta a_0 \frac{t^3 E}{12(1-\nu^2)} (\frac{\pi}{L})^4 2[\sin^2(\frac{\pi x}{L}) \sin^2(\frac{\pi y}{L})(1+\nu) + (1-\nu)\cos^2(\frac{\pi x}{L})\cos^2(\frac{\pi y}{L})] a_0 ,$$

$$\delta w_{\Omega}^{\text{ext}} = \delta a_0 f \sin^2(\pi \frac{x}{L}) \sin^2(\pi \frac{y}{L}) .$$

Virtual work expressions are integrals of the virtual work densities over the domain occupied by the element

$$\delta W^{\text{int}} = \int_0^L \int_0^L \delta w_{\Omega}^{\text{int}} dx dy = -\delta a_0 4 \frac{t^3 E}{12(1-\nu^2)} (\frac{\pi}{L})^4 (\frac{L}{2})^2 a_0 ,$$

$$\delta W^{\text{ext}} = \int_0^L \int_0^L \delta w_{\Omega}^{\text{ext}} dx dy = \delta a_0 f (\frac{L}{2})^2 .$$

Principle of virtual work and the fundamental lemma of variation calculus give

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = -\delta a_0 [4 \frac{t^3 E}{12(1-\nu^2)} (\frac{\pi}{L})^4 (\frac{L}{2})^2 a_0 - f (\frac{L}{2})^2] = 0 \quad \forall \delta a_0 \quad \Leftrightarrow$$

$$a_0 = 3\pi^4 \frac{f L^4}{t^3 E} (1-\nu^2) .$$

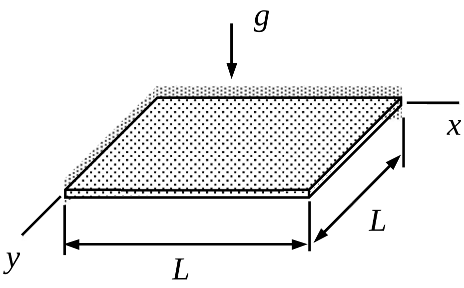
Displacement

$$w(x,y) = \frac{3}{\pi^4} \frac{f L^4}{t^3 E} (1-\nu^2) \sin(\pi \frac{x}{L}) \sin(\pi \frac{y}{L}) .$$

Name_________________________ Student number__________________

Assignment 5

A Kirchhoff plate of thickness t is loaded by its own weight. The plate is clamped on the edges where $x=0$ or $y=0$ and free on the other two edges. Material parameters E, $\nu=0$, and ρ are constants. Determine the parameter a_0 of the approximation $w(x,y)=a_0x^2y^2$. Use the principle of virtual work in form $\delta W=0 \quad \forall \delta a_0 \in \mathbb{R}$.



Solution

Virtual work densities of the plate bending mode are given by

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial^2\delta w/\partial x^2 \\ \partial^2\delta w/\partial y^2 \\ 2\partial^2\delta w/(\partial x\partial y) \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12}[E]_\sigma \begin{Bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ 2\partial^2 w/(\partial x\partial y) \end{Bmatrix} \text{ and } \delta w^{\text{ext}} = \delta w f_z ,$$

in which f_z is the $z-$component of the distributed force per unit area and the elasticity matrix of plane stress

$$[E]_\sigma = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}.$$

Approximation to the transverse displacement and its derivatives are

$$w(x,y)=a_0x^2y^2 \quad \Rightarrow \quad \frac{\partial^2 w}{\partial x^2}=2a_0y^2, \quad \frac{\partial^2 w}{\partial y^2}=2a_0x^2, \quad \text{and} \quad \frac{\partial^2 w}{\partial x\partial y}=4a_0xy .$$

When the approximation is substituted there, virtual work densities simplify to

$$\delta w_\Omega^{\text{int}} = -2\delta a_0 \begin{Bmatrix} y^2 \\ x^2 \\ 4xy \end{Bmatrix}^{\mathrm{T}} \frac{t^3E}{12}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}\begin{Bmatrix} y^2 \\ x^2 \\ 4xy \end{Bmatrix} 2a_0 = -\delta a_0 \frac{t^3E}{3}(y^4+x^4+8x^2y^2)a_0 ,$$

$$\delta w_\Omega^{\text{ext}} = \delta w f_z = \delta a_0 x^2y^2\rho g t .$$

Virtual work expressions are integrals of the virtual work densities over the mathematical solution domain (here mid-plane)

$$\delta W^{\text{int}} = \int_0^L \int_0^L \delta w_\Omega^{\text{int}} dxdy = -\delta a_0 \frac{t^3 E}{3} L^6 \frac{58}{45} a_0 \,,$$

$$\delta W^{\text{ext}} = \int_0^L \int_0^L \delta w_\Omega^{\text{ext}} dxdy = \delta a_0 \frac{1}{9} L^6 \rho g t \,.$$

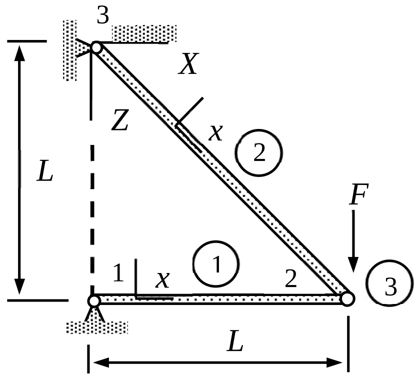
Principle of virtual work $\delta W = 0 \ \forall \delta \text{a}$ and the fundamental lemma of variation calculus give

$$\delta W = -\delta a_0 (\frac{t^3 E}{3} L^6 \frac{58}{45} a_0 - \frac{1}{9} L^6 \rho g t) = 0 \quad \forall \delta a_0 \quad \Leftrightarrow \quad \frac{t^3 E}{3} L^6 \frac{58}{45} a_0 - \frac{1}{9} L^6 \rho g t = 0 \quad \Leftrightarrow$$

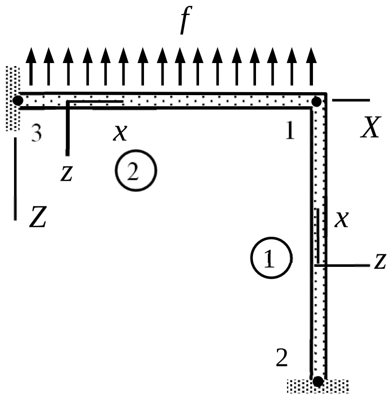
$$a_0 = \frac{15}{58} \frac{\rho g}{t^2 E} \,.$$

MEC-E1050 Finite Element Method in Solids, onsite exam 02.12.2024

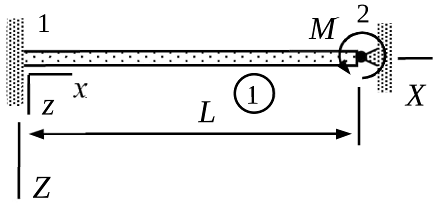
1. A *bar truss* is loaded by a point force having magnitude F as shown in the figure. Derive the equilibrium equations and determine the nodal displacements. The cross-sectional area of bar 1 is A and that for bar 2 $\sqrt{8}A$. Young's modulus is E and weight is omitted. Use the principle of virtual work.



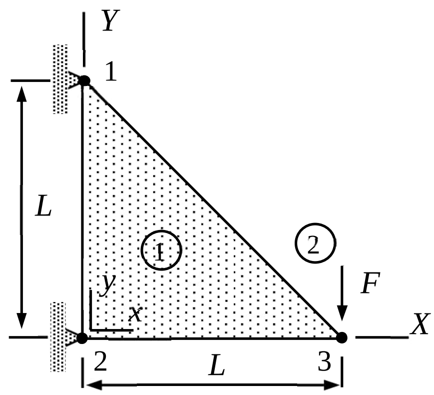
2. Determine the rotation θ_{Y1} at node 1 of the structure shown. Use two beam elements of length L. Assume that the beams are inextensible in the axial directions. Young's modulus of the material E and the second moment of area I are constants. Use the principle of virtual work.



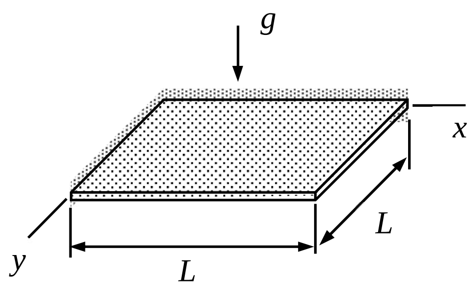
3. Determine the rotation θ_{Y2} of the beam shown at the support of the right end which allows rotation but not transverse displacement. Young's modulus E of the material and second moment of cross-section $I_{yy} = I$ are constants. Use the virtual work density of beam bending mode $\delta w_{\Omega} = -(d^2\delta w / dx^2)EI_{yy}(d^2 w / dx^2) + \delta w f_z$ and cubic approximation to the transverse displacement.



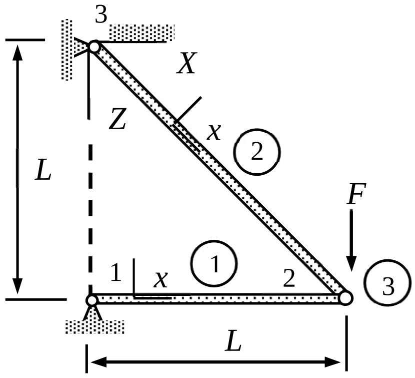
4. A thin triangular slab of thickness t is loaded by a point force at node 3. Nodes 1 and 2 are fixed. Derive the virtual work expression δW of the structure in terms of u_{X3} and u_{Y3}, and solve for the nodal displacements. Approximation is linear and material parameters E and ν are constants. Assume plane-stress conditions.



5. A Kirchhoff plate of thickness t is loaded by its own weight. The plate is clamped on the edges where $x = 0$ or $y = 0$ and free on the other two edges. Material parameters E, $\nu = 0$, and ρ are constants. Determine the parameter a_0 of the approximation $w(x, y) = a_0 x^2 y^2$. Use the principle of virtual work in form $\delta W = 0 \ \forall \delta a_0 \in \mathbb{R}$.



A *bar truss* is loaded by a point force having magnitude F as shown in the figure. Derive the equilibrium equations and determine the nodal displacements. The cross-sectional area of bar 1 is A and that for bar 2 $\sqrt{8}A$. Young's modulus is E and weight is omitted. Use the principle of virtual work.



Solution

4p Element contributions $\delta W = -\delta \mathbf{a}^{\mathrm{T}}(\mathbf{Ka}-\mathbf{F})$ to the virtual work expression of the structure are

Bar 1: $\delta W^1 = -\begin{Bmatrix} 0 \\ \delta u_{X2} \end{Bmatrix}^{\mathrm{T}} (\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_{X2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}) = -\frac{EA}{L}u_{X2}\delta u_{X2}$,

Bar 2: $\delta W^2 = -\frac{1}{\sqrt{2}}\begin{Bmatrix} 0 \\ \delta u_{X2}+\delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} (\frac{E\sqrt{8}A}{\sqrt{2}L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{Bmatrix} 0 \\ u_{X2}+u_{Z2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}) \quad \Leftrightarrow$

$$\delta W^2 = -\frac{EA}{L}(\delta u_{X2}+\delta u_{Z2})(u_{X2}+u_{Z2}),$$

Force 3: $\delta W^3 = \delta u_{Z2}F$.

2p Virtual work expression is the sum of the element contributions

$$\delta W = -\frac{EA}{L}\delta u_{X2}u_{X2} - \frac{EA}{L}(\delta u_{X2}+\delta u_{Z2})(u_{X2}+u_{Z2}) + \delta u_{Z2}F \quad \Leftrightarrow$$

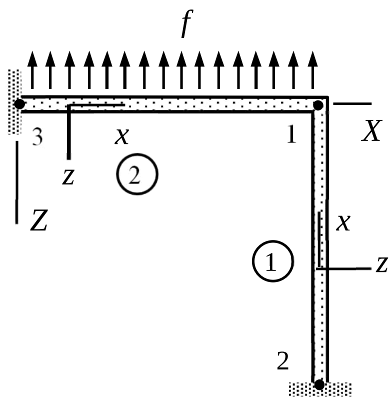
$$\delta W = -\delta u_{X2}(2\frac{EA}{L}u_{X2}+\frac{EA}{L}u_{Z2}) - \delta u_{Z2}(-F+\frac{EA}{L}u_{X2}+\frac{EA}{L}u_{Z2}) \quad \Leftrightarrow$$

$$\delta W = -\begin{Bmatrix} \delta u_{X2} \\ \delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} (\frac{EA}{L}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{Z2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ F \end{Bmatrix}).$$

Principle of virtual work $\delta W = 0 \quad \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus imply

$$\frac{EA}{L}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{Z2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ F \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix} u_{X2} \\ u_{Z2} \end{Bmatrix} = \frac{LF}{EA}\begin{Bmatrix} -1 \\ 2 \end{Bmatrix}.$$

Determine the rotation θ_{Y1} at node 1 of the structure shown. Use two beam elements of length L. Assume that the beams are inextensible in the axial directions. Young's modulus of the material E and the second moment of area I are constants. Use the principle of virtual work.



Solution

Virtual work expression of the xz-plane bending beam element

$$\delta W=-\begin{Bmatrix}\delta u_{z1}\\ \delta\theta_{y1}\\ \delta u_{z2}\\ \delta\theta_{y2}\end{Bmatrix}^{\mathrm{T}}\left(\frac{EI_{yy}}{h^3}\begin{bmatrix}12 & -6h & -12 & -6h\\ -6h & 4h^2 & 6h & 2h^2\\ -12 & 6h & 12 & 6h\\ -6h & 2h^2 & 6h & 4h^2\end{bmatrix}\begin{Bmatrix}u_{z1}\\ \theta_{y1}\\ u_{z2}\\ \theta_{y2}\end{Bmatrix}-\frac{f_z h}{12}\begin{Bmatrix}6\\ -h\\ 6\\ h\end{Bmatrix}\right)$$

depends on the second moment of cross-section area I_{yy}, Young's modulus E, beam length h, and force per unit length f_z.

The displacement and rotation components of the material coordinate system need to be expressed in terms of the components of the structural system. As beams are inextensible in the axial directions, the structure has only the rotation degree of freedom θ_{Y1} and it is enough to consider bending in the $xz-$ plane only.

2p Beam 1: $u_{z2}=0$, $\theta_{y2}=0$, $u_{z1}=0$, $\theta_{y1}=\theta_{Y1}$, and $f_z=0$

$$\delta W^1=-\begin{Bmatrix}0\\ 0\\ 0\\ \delta\theta_{Y1}\end{Bmatrix}^{\mathrm{T}}\left(\frac{EI}{L^3}\begin{bmatrix}12 & -6L & -12 & -6L\\ -6L & 4L^2 & 6L & 2L^2\\ -12 & 6L & 12 & 6L\\ -6L & 2L^2 & 6L & 4L^2\end{bmatrix}\begin{Bmatrix}0\\ 0\\ 0\\ \theta_{Y1}\end{Bmatrix}-\frac{0\,L}{12}\begin{Bmatrix}6\\ -L\\ 6\\ L\end{Bmatrix}\right)=-\delta\theta_{Y1}\frac{EI}{L^3}4L^2\theta_{Y1}$$

2p Beam 2: $u_{z3}=0$, $\theta_{y3}=0$, $u_{z1}=0$, $\theta_{y1}=\theta_{Y1}$, and $f_z=-f$

$$\delta W^2=-\begin{Bmatrix}0\\ 0\\ 0\\ \delta\theta_{Y1}\end{Bmatrix}^{\mathrm{T}}\left(\frac{EI}{L^3}\begin{bmatrix}12 & -6L & -12 & -6L\\ -6L & 4L^2 & 6L & 2L^2\\ -12 & 6L & 12 & 6L\\ -6L & 2L^2 & 6L & 4L^2\end{bmatrix}\begin{Bmatrix}0\\ 0\\ 0\\ \theta_{Y1}\end{Bmatrix}-\frac{-fL}{12}\begin{Bmatrix}6\\ -L\\ 6\\ L\end{Bmatrix}\right)=-\delta\theta_{Y1}\left(\frac{EI}{L^3}4L^2\theta_{Y1}+\frac{fL^2}{12}\right)$$

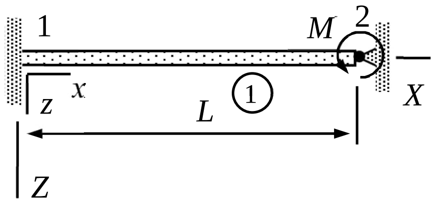
2p Virtual work expression of the structure is obtained by summing the element contributions. After that, the virtual work expression is rearranged into the standard form (similar to the virtual work expression of an element):

$$\delta W=\delta W^1+\delta W^2=-\delta\theta_{Y1}\frac{EI}{L^3}4L^2\theta_{Y1}-\delta\theta_{Y1}\left(\frac{EI}{L^3}4L^2\theta_{Y1}+\frac{fL^2}{12}\right)=-\delta\theta_{Y1}\left(8\frac{EI}{L}\theta_{Y1}+\frac{fL^2}{12}\right).$$

Principle of virtual work $\delta W = 0 \quad \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus imply the equilibrium equation and thereby solution to θ_{Y1}

$$8\frac{EI}{L}\theta_{Y1} + \frac{f\,L^2}{12} = 0 \quad \Leftrightarrow \quad \theta_{Y1} = -\frac{1}{96}\frac{f\,L^3}{EI}.$$

Determine the rotation θ_{Y2} of the beam shown at the support of the right end which allows rotation but not transverse displacement. Young's modulus E of the material and second moment of cross-section $I_{yy} = I$ are constants. Use the virtual work density of beam bending mode $\delta w_\Omega = -\delta(d^2w/dx^2)EI_{yy}(d^2w/dx^2) + \delta w f_z$ and cubic approximation to the transverse displacement.



Solution

In the $xz-$plane problem bending problem, when x-axis is chosen to coincide with the neutral axis, virtual work densities of the beam bending mode are

$$\delta w_\Omega^{\text{int}} = -\frac{d^2\delta w}{dx^2} EI_{yy} \frac{d^2 w}{dx^2} \quad \text{and} \quad \delta w_\Omega^{\text{ext}} = \delta w f_z .$$

2p Approximation is the first thing to be considered. The left end of the beam is clamped and the right end support does not allow transverse displacement. As only $\theta_{y2} = \theta_{Y2}$ is non-zero, approximation to w simplifies into the form (see the formulae collection for the cubic beam bending approximation)

$$w = \begin{Bmatrix} (1-\xi)^2(1+2\xi) \\ L(1-\xi)^2\xi \\ (3-2\xi)\xi^2 \\ L\xi^2(\xi-1) \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -\theta_{Y2} \end{Bmatrix} = L(\frac{x}{L})^2(1-\frac{x}{L})\theta_{Y2} \quad \Rightarrow \quad \frac{d^2w}{dx^2} = \frac{1}{L}(2-6\frac{x}{L})\theta_{Y2} \text{ and}$$

$$\delta w = L(\frac{x}{L})^2(1-\frac{x}{L})\delta\theta_{Y2} \quad \Rightarrow \quad \frac{d^2\delta w}{dx^2} = \frac{1}{L}(2-6\frac{x}{L})\delta\theta_{Y2} .$$

3p When the approximation is substituted there, virtual work density takes the form (external distributed force vanishes)

$$\delta w_\Omega = -\frac{d^2\delta w}{dx^2} EI_{yy} \frac{d^2w}{dx^2} = -\delta\theta_{Y2}\frac{EI}{L^4}\theta_{Y2}(2L-6x)^2 .$$

Integration over the domain $\Omega =]0, L[$ gives the virtual work expressions (beam is considered as element 1)

$$\delta W^1 = \int_0^L \delta w_\Omega dx = -\delta\theta_{Y2} 4\frac{EI}{L}\theta_{Y2} .$$

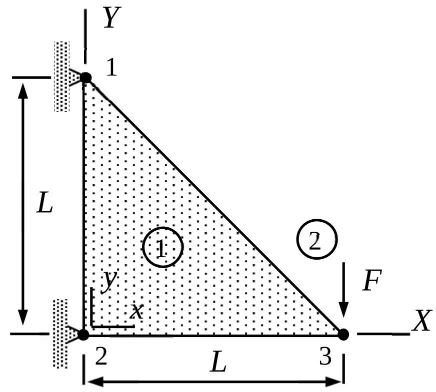
The external moment gives the contribution (element 2)

$$\delta W^2 = \delta\theta_{Y2} M .$$

1p Principle of virtual work $\delta W = \delta W^1 + \delta W^2 = 0 \quad \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus imply the solution

$$\delta W = -\delta\theta_{Y2}(4\frac{EI}{L}\theta_{Y2} - M) = 0 \quad \forall\,\delta u_{Z2} \quad \Leftrightarrow \quad 4\frac{EI}{L}\theta_{Y2} - M = 0 \quad \Leftrightarrow \quad \theta_{Y2} = \frac{1}{4}\frac{ML}{EI} \ .$$

A thin triangular slab of thickness t is loaded by a point force at node 3. Nodes 1 and 2 are fixed. Derive the virtual work expression δW of the structure in terms of u_{X3} and u_{Y3}, and solve for the nodal displacements. Approximation is linear and material parameters E and ν are constants. Assume plane stress conditions.



Solution

The virtual work densities (virtual works per unit area) of the thin slab model under the plane stress conditions

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial \delta u / \partial x \\ \partial \delta v / \partial y \\ \partial \delta u / \partial y + \partial \delta v / \partial x \end{Bmatrix}^{\mathrm{T}} t[E]_{\sigma} \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix} \text{ and } \delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix} \text{where}$$

$$[E]_{\sigma} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}$$

take into account the internal forces (stress), external forces acting on the element domain, and external forces acting on the edges. Notice that the components f_x and f_y are external forces per unit area. The forces acting on the element edges are taken into account by separate force elements.

2p Expressions of linear shape functions in material $xy-$coordinates can be deduced from the figure. Only the shape function of node 3 is actually needed:

$$N_3 = \frac{x}{L}, \quad N_1 = \frac{y}{L}, \text{ and } N_2 = 1 - N_1 - N_3 = 1 - \frac{x}{L} - \frac{y}{L} \quad \Rightarrow$$

$$u = N_1 0 + N_2 0 + N_3 u_{X3} = \frac{x}{L} u_{X3} \quad \Rightarrow \quad \frac{\partial u}{\partial x} = \frac{1}{L} u_{X3} \quad \text{and} \quad \frac{\partial u}{\partial y} = 0,$$

$$v = N_1 0 + N_2 0 + N_3 u_{Y3} = \frac{x}{L} u_{Y3} \quad \Rightarrow \quad \frac{\partial v}{\partial x} = \frac{1}{L} u_{Y3} \quad \text{and} \quad \frac{\partial v}{\partial y} = 0.$$

2p When the approximation is substituted there, virtual work expression of internal forces per unit area simplifies to

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \delta u_{X3} \\ 0 \\ \delta u_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{1}{L} \frac{tE}{2(1-\nu^2)} \begin{bmatrix} 2 & 2\nu & 0 \\ 2\nu & 2 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix} \frac{1}{L} \begin{Bmatrix} u_{X3} \\ 0 \\ u_{Y3} \end{Bmatrix} = -\begin{Bmatrix} \delta u_{X3} \\ \delta u_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{tE}{2L^2(1-\nu^2)} \begin{bmatrix} 2 & 0 \\ 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} u_{X3} \\ u_{Y3} \end{Bmatrix}$$

As the integrand is constant, integration over the triangular domain gives

$$\delta W^{\text{int}} = \int_A \delta w_\Omega^{\text{int}} dA = \delta w_\Omega^{\text{int}} \frac{L^2}{2} = -\begin{Bmatrix} \delta u_{X3} \\ \delta u_{Y3} \end{Bmatrix}^{\mathrm{T}} \frac{tE}{4(1-\nu^2)} \begin{bmatrix} 2 & 0 \\ 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} u_{X3} \\ u_{Y3} \end{Bmatrix}.$$

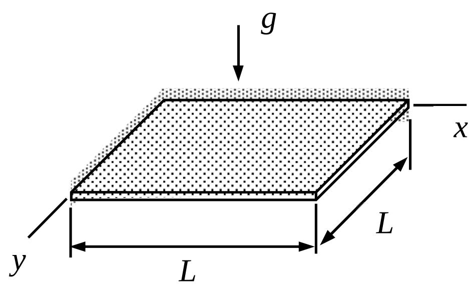
2p If also the point force is accounted for, the virtual work expression of the structure takes the form

$$\delta W = -\begin{Bmatrix} \delta u_{X3} \\ \delta u_{Y3} \end{Bmatrix}^{\mathrm{T}} \left(\frac{tE}{4(1-\nu^2)} \begin{bmatrix} 2 & 0 \\ 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} u_{X3} \\ u_{Y3} \end{Bmatrix} + \begin{Bmatrix} 0 \\ F \end{Bmatrix} \right). \leftarrow$$

Principle of virtual work $\delta W = 0 \; \forall \delta \mathrm{a}$ and the fundamental lemma of variation calculus give

$$\frac{tE}{4(1-\nu^2)} \begin{bmatrix} 2 & 0 \\ 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} u_{X3} \\ u_{Y3} \end{Bmatrix} + \begin{Bmatrix} 0 \\ F \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix} u_{X3} \\ u_{Y3} \end{Bmatrix} = -4(1+\nu)\frac{F}{tE} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}. \qquad \leftarrow$$

A Kirchhoff plate of thickness t is loaded by its own weight. The plate is clamped on the edges where $x=0$ or $y=0$ and free on the other two edges. Material parameters E, $\nu=0$, and ρ are constants. Determine the parameter a_0 of the approximation $w(x,y)=a_0x^2y^2$. Use the principle of virtual work in form $\delta W=0\quad\forall\delta a_0\in\mathbb{R}$.



Solution

Virtual work densities of the plate bending mode are given by

$$\delta w_\Omega^{\text{int}}=-\begin{Bmatrix}\partial^2\delta w/\partial x^2\\ \partial^2\delta w/\partial y^2\\ 2\partial^2\delta w/(\partial x\partial y)\end{Bmatrix}^{\mathrm{T}}\frac{t^3}{12}[E]_\sigma\begin{Bmatrix}\partial^2 w/\partial x^2\\ \partial^2 w/\partial y^2\\ 2\partial^2 w/(\partial x\partial y)\end{Bmatrix}\text{ and }\delta w^{\text{ext}}=\delta w f_z,$$

in which f_z is the $z-$component of the distributed force per unit area and the elasticity matrix of plane stress

$$[E]_\sigma=\frac{E}{1-\nu^2}\begin{bmatrix}1&\nu&0\\ \nu&1&0\\ 0&0&(1-\nu)/2\end{bmatrix}.$$

2p Approximation to the transverse displacement and its derivatives are

$$w(x,y)=a_0x^2y^2\quad\Rightarrow\quad\frac{\partial^2 w}{\partial x^2}=2a_0y^2,\quad\frac{\partial^2 w}{\partial y^2}=2a_0x^2,\quad\text{and}\quad\frac{\partial^2 w}{\partial x\partial y}=4a_0xy.$$

When the approximation is substituted there, virtual work densities simplify to

$$\delta w_\Omega^{\text{int}}=-2\delta a_0\begin{Bmatrix}y^2\\ x^2\\ 4xy\end{Bmatrix}^{\mathrm{T}}\frac{t^3E}{12}\begin{bmatrix}1&0&0\\ 0&1&0\\ 0&0&1/2\end{bmatrix}\begin{Bmatrix}y^2\\ x^2\\ 4xy\end{Bmatrix}2a_0=-\delta a_0\frac{t^3E}{3}(y^4+x^4+8x^2y^2)a_0,$$

$$\delta w_\Omega^{\text{ext}}=\delta w f_z=\delta a_0x^2y^2\rho g t.$$

2p Virtual work expressions are integrals of the virtual work densities over the mathematical solution domain (here mid-plane)

$$\delta W^{\text{int}}=\int_0^L\int_0^L\delta w_\Omega^{\text{int}}dxdy=-\delta a_0\frac{t^3E}{3}L^6\frac{58}{45}a_0,$$

$$\delta W^{\text{ext}}=\int_0^L\int_0^L\delta w_\Omega^{\text{ext}}dxdy=\delta a_0\frac{1}{9}L^6\rho g t.$$

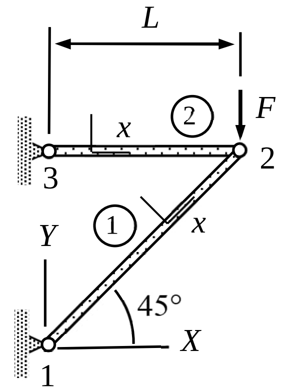
2p Principle of virtual work $\delta W=0\ \forall\delta a$ and the fundamental lemma of variation calculus give

$$\delta W = -\delta a_0 \left(\frac{t^3 E}{3} L^6 \frac{58}{45} a_0 - \frac{1}{9} L^6 \rho g t\right) = 0 \quad \forall \delta a_0 \quad \Leftrightarrow \quad \frac{t^3 E}{3} L^6 \frac{58}{45} a_0 - \frac{1}{9} L^6 \rho g t = 0 \quad \Leftrightarrow$$

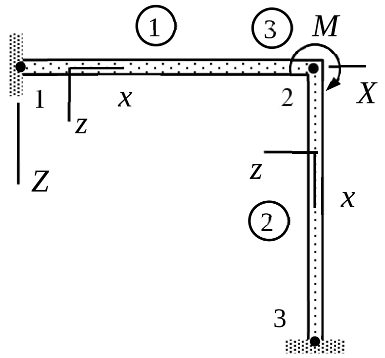
$$a_0 = \frac{15}{58} \frac{\rho g}{t^2 E} \ .$$

MEC-E1050 Finite Element Method in Solids, onsite exam 04.12.2023

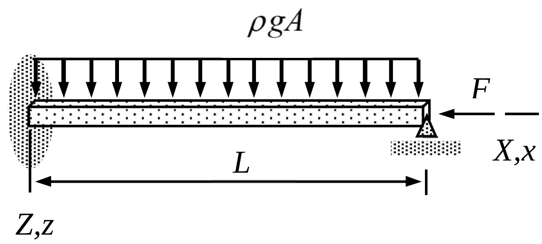
1. Determine the horizontal and vertical displacements of node 2. Cross-sectional area of bar 1 is $\sqrt{2}A$ and that of bar 2 is A. Young's modulus of the material is E.



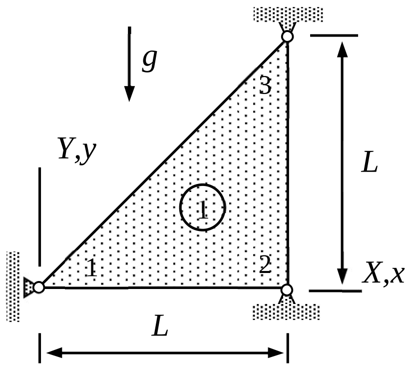
2. Determine the rotation θ_{Y2} at node 2 of the structure loaded by a point moment (magnitude M) acting on node 2. Use beam elements (1) and (2) of length L and a point moment element (3). Assume that the beams are inextensible in the axial directions. Young's modulus E and the second moment of area I are constants. Use the principle of virtual work.



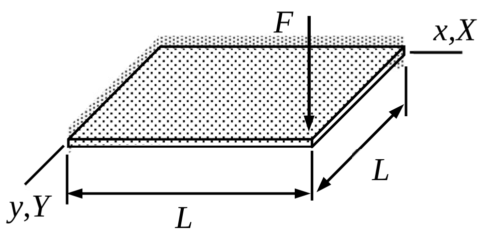
3. The beam of the figure is loaded by its own weight and a point force acting on the right end. Determine the displacement and rotation of the right end starting with the virtual density of the Bernoulli beam model. The x-axis of the material coordinate system is placed at the geometric centroid of the rectangle cross-section. Beam properties A, I_{yy}, I_{zz} and E of the planar problem are constants.



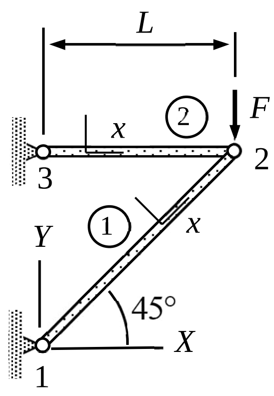
4. A thin triangular slab (assume plane stress conditions) loaded by its own weight is allowed to move vertically at node 1 and nodes 2 and 3 are fixed. Find the displacement u_{Y1}. Material parameters E, ν, ρ and thickness t of the slab are constants.



5. A plate, loaded by point force F acting at the free corner, is simply supported on two edges and free on the other two edges as shown in the figure. Determine the parameter a_0 of approximation $w(x,y) = a_0(x/L)(y/L)$ and displacement at the center point. Use the virtual work density of the plate bending mode with constant E, ν, ρ and t.



Determine the horizontal and vertical displacements of node 2. Cross-sectional area of bar 1 is $\sqrt{2}A$ and that of bar 2 is A. Young's modulus of the material is E.



Solution (option 1)

Element contribution, written in terms of displacement components of the structural coordinate system,

$$\begin{Bmatrix}\mathbf{R}_1\\ \mathbf{R}_2\end{Bmatrix}=\frac{EA}{h}\begin{bmatrix}\mathbf{i}\mathbf{i}^{\mathrm{T}} & -\mathbf{i}\mathbf{i}^{\mathrm{T}}\\ -\mathbf{i}\mathbf{i}^{\mathrm{T}} & \mathbf{i}\mathbf{i}^{\mathrm{T}}\end{bmatrix}\begin{Bmatrix}\mathbf{a}_1\\ \mathbf{a}_2\end{Bmatrix}-\frac{f_x h}{2}\begin{Bmatrix}\mathbf{i}\\ \mathbf{i}\end{Bmatrix}, \text{ where } \mathbf{i}=\frac{1}{h}\begin{Bmatrix}\Delta X\\ \Delta Y\end{Bmatrix}$$

depends on the cross-sectional area A, Young's modulus E, bar length h, force per unit length f_x and the components of the basis vector $\vec{i}$ in the structural coordinate system (the components define the orientation).

4p Element contributions are first written in terms of the nodal displacements of the structural coordinate system (notice that the point force is treated as a one-node element)

Bar 1: $h=\sqrt{2}L$, $\mathbf{i}=\begin{Bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{Bmatrix}$, $\begin{Bmatrix}F_{X1}^1\\ F_{Y1}^1\\ F_{X2}^1\\ F_{Y2}^1\end{Bmatrix}=\frac{EA}{2L}\begin{bmatrix}1 & 1 & -1 & -1\\ 1 & 1 & -1 & -1\\ -1 & -1 & 1 & 1\\ -1 & -1 & 1 & 1\end{bmatrix}\begin{Bmatrix}0\\ 0\\ u_{X2}\\ u_{Y2}\end{Bmatrix}-\begin{Bmatrix}0\\ 0\\ 0\\ 0\end{Bmatrix}$,

Bar 2: $h=L$, $\mathbf{i}=\begin{Bmatrix}1\\ 0\end{Bmatrix}$, $\begin{Bmatrix}F_{X3}^2\\ F_{Y3}^2\\ F_{X2}^2\\ F_{Y2}^2\end{Bmatrix}=\frac{EA}{L}\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}0\\ 0\\ u_{X2}\\ u_{Y2}\end{Bmatrix}-\begin{Bmatrix}0\\ 0\\ 0\\ 0\end{Bmatrix}$,

Force 3: $\begin{Bmatrix}F_{X2}^3\\ F_{Y2}^3\end{Bmatrix}=-\begin{Bmatrix}0\\ -F\end{Bmatrix}$.

2p In assembly, the internal forces acting on the non-constrained directions are added to get the equilibrium equations of the structure. The unknown displacement components follow as the solution to the equilibrium equations:

$$\sum\begin{Bmatrix}F_{X2}^e\\ F_{Y2}^e\end{Bmatrix}=\begin{Bmatrix}F_{X2}^1\\ F_{Y2}^1\end{Bmatrix}+\begin{Bmatrix}F_{X2}^2\\ F_{Y2}^2\end{Bmatrix}+\begin{Bmatrix}F_{X2}^3\\ F_{Y2}^3\end{Bmatrix}=\frac{EA}{2L}\begin{bmatrix}3 & 1\\ 1 & 1\end{bmatrix}\begin{Bmatrix}u_{X2}\\ u_{Y2}\end{Bmatrix}+\begin{Bmatrix}0\\ F\end{Bmatrix}=0 \quad \Leftrightarrow$$

$$u_{X2}=\frac{FL}{EA} \quad \text{and} \quad u_{Y2}=-3\frac{FL}{EA}.$$

Solution (option 2)

In the material coordinate system, virtual work expression of the bar model is given by

$$\delta W=-\begin{Bmatrix}\delta u_{x1}\\ \delta u_{x2}\end{Bmatrix}^{\mathrm{T}}\left(\frac{EA}{h}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_{x1}\\ u_{x2}\end{Bmatrix}-\frac{f_x h}{2}\begin{Bmatrix}1\\ 1\end{Bmatrix}\right).$$

4p In terms of the nodal displacement components of the structural system, axial displacements of bar 1 are $u_{x1}=0$ and $u_{x2}=(u_{X2}+u_{Y2})/\sqrt{2}$. Length of the bar $h=\sqrt{2}L$, cross-sectional area is $\sqrt{2}A$, and the external distributed force $f_x=0$. Therefore

$$\delta W^1=-\begin{Bmatrix}0\\ (\delta u_{X2}+\delta u_{Y2})/\sqrt{2}\end{Bmatrix}^{\mathrm{T}}\frac{E\sqrt{2}A}{\sqrt{2}L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}0\\ (u_{X2}+u_{Y2})/\sqrt{2}\end{Bmatrix}=-\begin{Bmatrix}\delta u_{X2}\\ \delta u_{Y2}\end{Bmatrix}^{\mathrm{T}}\frac{EA}{2L}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{Bmatrix}u_{X2}\\ u_{Y2}\end{Bmatrix}$$

Axial displacements of bar 2 are $u_{x3}=0$ and $u_{x2}=u_{X2}$. Length of the bar $h=L$, cross-sectional area is A, and the external distributed force $f_x=0$. Hence

$$\delta W^2=-\begin{Bmatrix}0\\ \delta u_{X2}\end{Bmatrix}^{\mathrm{T}}\frac{EA}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}0\\ u_{X2}\end{Bmatrix}=-\begin{Bmatrix}\delta u_{X2}\\ \delta u_{Y2}\end{Bmatrix}^{\mathrm{T}}\frac{EA}{L}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{Bmatrix}u_{X2}\\ u_{Y2}\end{Bmatrix}.$$

Virtual work expression of the external force follows, for example, from the definition of work

$$\delta W^3=-\delta u_{Y2}F=-\begin{Bmatrix}\delta u_{X2}\\ \delta u_{Y2}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}0\\ F\end{Bmatrix}.$$

2p Virtual work expression of the structure is the sum of element contributions

$$\delta W=-\begin{Bmatrix}\delta u_{X2}\\ \delta u_{Y2}\end{Bmatrix}^{\mathrm{T}}\frac{EA}{2L}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{Bmatrix}u_{X2}\\ u_{Y2}\end{Bmatrix}-\begin{Bmatrix}\delta u_{X2}\\ \delta u_{Y2}\end{Bmatrix}^{\mathrm{T}}\frac{EA}{2L}\begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix}\begin{Bmatrix}u_{X2}\\ u_{Y2}\end{Bmatrix}-\begin{Bmatrix}\delta u_{X2}\\ \delta u_{Y2}\end{Bmatrix}^{\mathrm{T}}\begin{Bmatrix}0\\ F\end{Bmatrix}$$

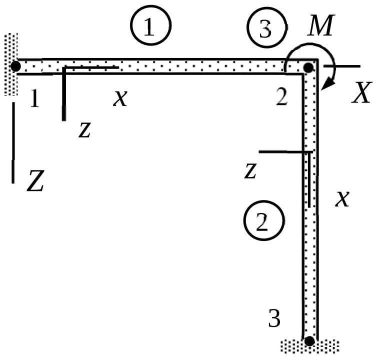
or when written in the standard form

$$\delta W=-\begin{Bmatrix}\delta u_{X2}\\ \delta u_{Y2}\end{Bmatrix}^{\mathrm{T}}\left(\frac{EA}{2L}\begin{bmatrix}3 & 1\\ 1 & 1\end{bmatrix}\begin{Bmatrix}u_{X2}\\ u_{Y2}\end{Bmatrix}+\begin{Bmatrix}0\\ F\end{Bmatrix}\right).$$

Principle of virtual work and the fundamental lemma of variation calculus imply

$$\frac{EA}{2L}\begin{bmatrix}3 & 1\\ 1 & 1\end{bmatrix}\begin{Bmatrix}u_{X2}\\ u_{Y2}\end{Bmatrix}+\begin{Bmatrix}0\\ F\end{Bmatrix}=0 \quad \Leftrightarrow \quad \begin{Bmatrix}u_{X2}\\ u_{Y2}\end{Bmatrix}=-2\frac{FL}{EA}\frac{1}{2}\begin{bmatrix}1 & -1\\ -1 & 3\end{bmatrix}\begin{Bmatrix}0\\ 1\end{Bmatrix}=-\frac{FL}{EA}\begin{Bmatrix}-1\\ 3\end{Bmatrix}.$$

Determine the rotation θ_{Y2} at node 2 of the structure loaded by a point moment (magnitude M) acting on node 2. Use beam elements (1) and (2) of length L and a point moment element (3). Assume that the beams are inextensible in the axial directions. Young's modulus E and the second moment of area I are constants. Use the principle of virtual work.



Solution

In a planar problem, torsion and out-plane bending deformation modes can be omitted. As beams are assumed to be inextensible in the axial direction and there are no axial distributed forces, the bar mode virtual work expression vanishes. Virtual work expressions of the beam xz-plane bending element and point force/moment elements are given by

$$\delta W = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EI_{yy}}{h^3} \begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix} - \frac{f_z h}{12} \begin{Bmatrix} 6 \\ -h \\ 6 \\ h \end{Bmatrix} \right),$$

$$\delta W = \begin{Bmatrix} \delta u_{X1} \\ \delta u_{Y1} \\ \delta u_{Z1} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} F_{X1} \\ F_{Y1} \\ F_{Z1} \end{Bmatrix} + \begin{Bmatrix} \delta\theta_{X1} \\ \delta\theta_{Y1} \\ \delta\theta_{Z1} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} M_{X1} \\ M_{Y1} \\ M_{Z1} \end{Bmatrix}.$$

4p Nodal displacements/rotations of the structure are clearly zeros except those of node 2. Displacement of node 2 vanishes as both beams are inextensible in the axial directions. Therefore, the only non-zero displacement/rotation component of the structure is θ_{Y2}.

Beam 1: $u_{z1} = 0$, $\theta_{y1} = 0$, $u_{z2} = 0$, and $\theta_{y2} = \theta_{Y2}$

$$\delta W^1 = -\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \theta_{Y2} \end{Bmatrix} = -\delta\theta_{Y2} 4\frac{EI}{L}\theta_{Y2}.$$

Beam 2: $u_{z2} = 0$, $\theta_{y2} = \theta_{Y2}$, $u_{z3} = 0$, and $\theta_{y3} = 0$

$$\delta W^2 = -\begin{Bmatrix} 0 \\ \delta\theta_{Y2} \\ 0 \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L^3} \begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ \theta_{Y2} \\ 0 \\ 0 \end{Bmatrix} = -\delta\theta_{Y2} 4\frac{EI}{L}\theta_{Y2}.$$

Point moment 3:

$$\delta W^3 = \begin{Bmatrix} \delta u_{X2} \\ \delta u_{Y2} \\ \delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} F_{X2} \\ F_{Y2} \\ F_{Z2} \end{Bmatrix} + \begin{Bmatrix} \delta\theta_{X2} \\ \delta\theta_{Y2} \\ \delta\theta_{Z2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} M_{X2} \\ M_{Y2} \\ M_{Z2} \end{Bmatrix} = -\delta\theta_{Y2} M \,.$$

2p Virtual work expression of the structure is sum of the element contributions

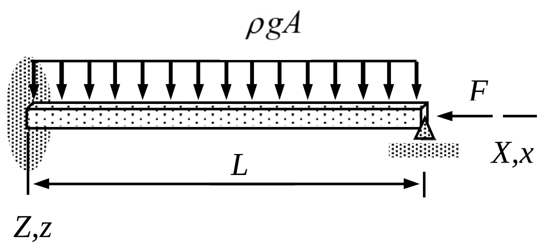
$$\delta W = \sum\nolimits_e \delta W^e = \delta W^1 + \delta W^2 + \delta W^3 \Rightarrow$$

$$\delta W = -\delta\theta_{Y2} 4\frac{EI}{L}\theta_{Y2} - \delta\theta_{Y2} 4\frac{EI}{L}\theta_{Y2} + 0 - \delta\theta_{Y2} M = -\delta\theta_{Y2}(8\frac{EI}{L}\theta_{Y2} + M)\,.$$

Principle of virtual work $\delta W = 0 \; \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus in the form $\delta \mathbf{a}^{\mathrm{T}} \mathbf{R} = 0 \;\; \forall \delta \mathbf{a} \;\Leftrightarrow\; \mathbf{R} = 0$ imply

$$8\frac{EI}{L}\theta_{Y2} + M = 0 \quad \Leftrightarrow \quad \theta_{Y2} = -\frac{1}{8}\frac{ML}{EI}\,. \;\leftarrow$$

The beam of the figure is loaded by its own weight and a point force acting on the right end. Determine the displacement and rotation of the right end starting with the virtual density of the Bernoulli beam model. The *x*-axis of the material coordinate system is placed at the geometric centroid of the rectangle cross-section. Beam properties A, I_{yy}, I_{zz} and E of the planar problem are constants.



Solution

2p The left end of the beam is clamped and the right end simply supported. As the material and structural coordinate systems coincide $u_{x2} = u_{X2}$ and $\theta_{y2} = \theta_{Y2}$, the approximations of u and w simplify to

$$\begin{Bmatrix} u \\ w \end{Bmatrix} = \begin{Bmatrix} x/L u_{X2} \\ L(x/L)^2(1-x/L)\theta_{Y2} \end{Bmatrix} \quad \Rightarrow \quad \begin{Bmatrix} du/dx \\ d^2w/dx^2 \end{Bmatrix} = \frac{1}{L}\begin{Bmatrix} u_{X2} \\ (2-6x/L)\theta_{Y2} \end{Bmatrix}.$$

3p The moments of cross-section $S_y = S_z = 0$, I_{yy}, I_{zz} and $I_{yz} = 0$. As here $v = \phi = 0$, $f_x = f_y = 0$ and $m_x = m_y = m_z = 0$, virtual work densities take the forms

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} d\delta u/dx \\ d^2\delta w/dx^2 \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} EA & 0 \\ 0 & EI_{yy} \end{bmatrix} \begin{Bmatrix} du/dx \\ d^2w/dx^2 \end{Bmatrix} \quad \text{and} \quad \delta w_\Omega^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta w \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ \rho g A \end{Bmatrix} \quad \Rightarrow$$

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \delta u_{X2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{E}{L^2} \begin{bmatrix} A & 0 \\ 0 & I_{yy}(2-6x/L)^2 \end{bmatrix} \begin{Bmatrix} u_{X2} \\ \theta_{Y2} \end{Bmatrix},$$

$$\delta w_\Omega^{\text{ext}} = \begin{Bmatrix} \delta u_{X2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ L(x/L)^2(1-x/L)\rho g A \end{Bmatrix}.$$

Integrations over the domain $\Omega =]0, L[$ give the virtual work expressions

$$\delta W^{\text{int}} = \int_\Omega \delta w_\Omega^{\text{int}} d\Omega = -\begin{Bmatrix} \delta u_{X2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{E}{L} \begin{bmatrix} A & 0 \\ 0 & 4I_{yy} \end{bmatrix} \begin{Bmatrix} u_{X2} \\ \theta_{Y2} \end{Bmatrix},$$

$$\delta W^{\text{ext}} = \int_\Omega \delta w_\Omega^{\text{ext}} d\Omega = \begin{Bmatrix} \delta u_{X2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{\rho g A L^2}{12} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \quad \Rightarrow$$

$$\delta W^1 = \delta W^{\text{int}} + \delta W^{\text{ext}} = -\begin{Bmatrix} \delta u_{X2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{E}{L} \begin{bmatrix} A & 0 \\ 0 & 4I_{yy} \end{bmatrix} \begin{Bmatrix} u_{X2} \\ \theta_{Y2} \end{Bmatrix} - \frac{\rho g A L^2}{12} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}\right).$$

Virtual work expression of the point force follows from definition of work (or from the expression of formulae collection)

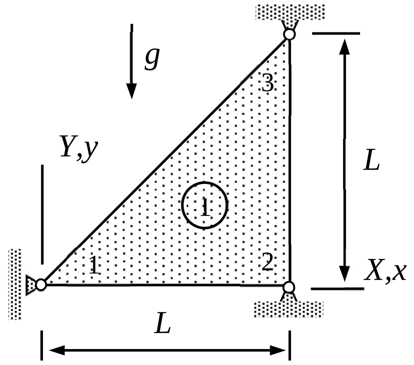
$$\delta W^2 = \begin{Bmatrix} \delta u_{X2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} -F \\ 0 \end{Bmatrix}.$$

1p Principle of virtual work $\delta W = \delta W^1 + \delta W^2 = 0 \quad \forall \delta\mathbf{a}$ and the fundamental lemma of variation calculus give

$$\delta W = -\begin{Bmatrix} \delta u_{X2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} (\frac{E}{L}\begin{bmatrix} A & 0 \\ 0 & 4I_{yy} \end{bmatrix}\begin{Bmatrix} u_{X2} \\ \theta_{Y2} \end{Bmatrix} - \begin{Bmatrix} -F \\ \rho g A L^2/12 \end{Bmatrix}) \quad \forall \begin{Bmatrix} \delta u_{X2} \\ \delta\theta_{Y2} \end{Bmatrix} \quad \Leftrightarrow$$

$$\frac{E}{L}\begin{bmatrix} A & 0 \\ 0 & 4I \end{bmatrix}\begin{Bmatrix} u_{X2} \\ \theta_{Y2} \end{Bmatrix} - \begin{Bmatrix} -F \\ \rho g A L^2/12 \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix} u_{X2} \\ \theta_{Y2} \end{Bmatrix} = \begin{Bmatrix} -LF/EA \\ \rho g A L^3/(48EI_{yy}) \end{Bmatrix}. \leftarrow$$

A thin triangular slab (assume plane stress conditions) loaded by its own weight is allowed to move vertically at node 1 and nodes 2 and 3 are fixed. Find the displacement u_{Y1}. Material parameters E, ν, ρ and thickness t of the slab are constants.



Solution

For the plane stress conditions and the thin-slab model, virtual work density of internal and external volume forces are given by

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial\delta u / \partial x \\ \partial\delta v / \partial y \\ \partial\delta u / \partial y + \partial\delta v / \partial x \end{Bmatrix}^{\mathrm{T}} t[E]_{\sigma} \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix}, \text{ where } [E]_{\sigma} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix},$$

$$\delta w_{\Omega}^{\text{ext}} = \begin{Bmatrix} \delta u \\ \delta v \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix}.$$

2p Let us start with the approximations. Only the shape function of node 1 is needed as the other nodes are fixed. By using linearity and conditions $N_1(0,0) = 1$, $N_1(L,0) = N_1(0,L) = 0$

$$N_1(x,y) = 1 - \frac{x}{L}.$$

Displacement components simplify to

$$u = 0 \quad \Rightarrow \quad \frac{\partial u}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} = 0,$$

$$v = (1-\frac{x}{L})u_{Y1} \quad \Rightarrow \quad \frac{\partial v}{\partial x} = -\frac{u_{Y1}}{L} \quad \text{and} \quad \frac{\partial v}{\partial y} = 0.$$

4p When approximations are substituted there, virtual work density simplifies to

$$\delta w_{\Omega}^{\text{int}} = -\delta u_{Y1} \frac{1}{L^2} \frac{tE}{2+2\nu} u_{Y1}.$$

Integration over the domain gives the virtual work expression. As the integrand is constant

$$\delta W^{\text{int}} = \int_{\Omega} \delta w_{\Omega}^{\text{int}} dA = \frac{L^2}{2} \delta w_{\Omega}^{\text{int}} = -\delta u_{Y1} \frac{1}{4} \frac{tE}{1+\nu} u_{Y1}.$$

Virtual work expression of the external volume force due to gravity takes the form

$$\delta W^{\text{ext}} = \int_{\Omega} \delta w_{\Omega}^{\text{ext}} dA = -\int_0^L \int_0^x (1-\frac{x}{L}) \delta u_{Y1} t\rho g dy dx = -\delta u_{Y1} \frac{1}{6} t\rho g L^2.$$

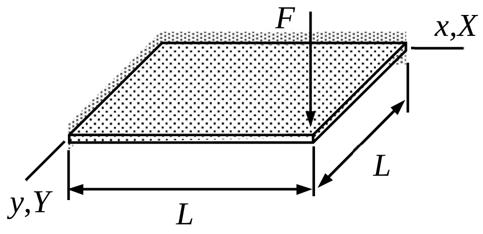
Virtual work expression of the thin slab is sum of the internal and external parts

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = -\delta u_{Y1}\left(\frac{1}{4}\frac{tE}{1+\nu}u_{Y1} + \frac{1}{6}t\rho g L^2\right).$$

Principle of virtual work $\delta W = 0 \;\forall \delta \mathrm{a}$ and the fundamental lemma of variation calculus give

$$\frac{1}{4}\frac{tE}{1+\nu}u_{Y1} + \frac{1}{6}t\rho g L^2 = 0 \quad \Leftrightarrow \quad u_{Y1} = -\frac{2}{3}(1+\nu)\frac{\rho g L^2}{E}.$$

A Kirchhoff plate, loaded by point force F acting at the free corner, is simply supported on two edges and free on the other two edges as shown in the figure. Determine the parameter a_0 of approximation $w(x,y)=a_0(x/L)(y/L)$ and displacement at the center point. Use the virtual work density of the Kirchhoff plate model with constant E, ν, ρ and t.



Solution

Assuming that the material coordinate system is chosen so that the plate bending and thin slab modes decouple, it is enough to consider the virtual work densities of the bending mode only

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial^2\delta w/\partial x^2 \\ \partial^2\delta w/\partial y^2 \\ 2\partial^2\delta w/\partial x\partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3}{12}[E]_\sigma \begin{Bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ 2\partial^2 w/\partial x\partial y \end{Bmatrix}, \ \delta w_\Omega^{\text{ext}} = \delta w f_z .$$

in which the elasticity matrix of plane stress

$$[E]_\sigma = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}.$$

In the present case, distributed force vanishes i.e. $f_z = 0$ and the point force is taken into account by a point force element.

4p Approximation to the transverse displacement is chosen to be (a_0 is not associated with any point but it just a parameter of the approximation)

$$w(x,y) = a_0 \frac{x}{L}\frac{y}{L} \quad \Rightarrow \quad \frac{\partial^2 w}{\partial x^2} = 0, \ \frac{\partial^2 w}{\partial y^2} = 0, \text{ and } \frac{\partial^2 w}{\partial x\partial y} = \frac{1}{L^2}a_0 .$$

When the approximation is substituted there, virtual work density of internal forces simplifies to

$$\delta w_\Omega^{\text{int}} = -\begin{Bmatrix} \partial^2\delta w/\partial x^2 \\ \partial^2\delta w/\partial y^2 \\ 2\partial^2\delta w/\partial x\partial y \end{Bmatrix}^{\mathrm{T}} \frac{t^3 E}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \partial^2 w/\partial x^2 \\ \partial^2 w/\partial y^2 \\ 2\partial^2 w/\partial x\partial y \end{Bmatrix} = -\delta a_0 \frac{Et^3}{6(1+\nu)}\frac{1}{L^4}a_0 ,$$

Virtual work expression of the plate bending element (element 1 here) is integral of the virtual work density over the domain occupied by the element

$$\delta W^1 = \int_0^L \int_0^L \delta w_\Omega^{\text{int}} dxdy = -\delta a_0 \frac{Et^3}{6(1+\nu)}\frac{1}{L^2}a_0 .$$

Virtual work expression of the point force (element 2 here) follows from the definition of work (notice the use of virtual displacement of the point of action $x = y = L$)

$$\delta W^2 = \delta w(L,L)F = \delta a_0 F \, .$$

2p Principle of virtual work and the fundamental lemma of variation calculus give

$$\delta W = \delta W^1 + \delta W^2 = -\delta a_0 \left(\frac{Et^3}{6(1+\nu)} \frac{1}{L^2} a_0 - F \right) = 0 \quad \Rightarrow \quad a_0 = 6(1+\nu) \frac{FL^2}{Et^3} \; .$$

Displacement at the center point

$$w\left(\frac{L}{2}, \frac{L}{2}\right) = a_0 \frac{1}{4} = \frac{3}{2}(1+\nu) \frac{FL^2}{Et^3} \, . \quad \leftarrow$$