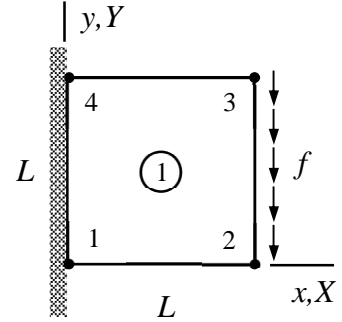


Name _____ Student number _____

Assignment 4

A thin slab is loaded by distributed force on its outer edge as shown in the figure. Determine the vertical displacement of the outer edge 2-3 by using a bi-linear interpolation to the nodal values. Edge 1-4 is welded to a rigid wall so that the displacements vanish. Thickness of the slab t , Young's modulus E , and Poisson's ratio ν are constants. Assume plane stress conditions. Simplify the setting with conditions $u_{Y3} = u_{Y2}$ and $u_{X3} = u_{X2} = 0$.



Solution template

Under the plane stress conditions, the virtual work densities (virtual works per unit area) of the thin slab model are given by

$$\delta w_{\Omega}^{\text{int}} = - \left\{ \begin{array}{c} \partial \delta u / \partial x \\ \partial \delta v / \partial y \\ \partial \delta u / \partial y + \partial \delta v / \partial x \end{array} \right\}^T t[E]_{\sigma} \left\{ \begin{array}{c} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{array} \right\} \text{ and } \delta w_{\Omega}^{\text{ext}} = \left\{ \begin{array}{c} \delta u \\ \delta v \end{array} \right\}^T \left\{ \begin{array}{c} f_x \\ f_y \end{array} \right\} \text{ where}$$

$$[E]_{\sigma} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}.$$

Expressions take into account the internal forces (stress) and the external area forces acting on the element domain. The external forces t_x and t_y (tractions per unit length in this case) acting on the element edges can be taken into account by a separate force element with the density expression (per unit length)

$$\delta w_{\partial\Omega}^{\text{ext}} = \left\{ \begin{array}{c} \delta u \\ \delta v \end{array} \right\}^T \left\{ \begin{array}{c} t_x \\ t_y \end{array} \right\}.$$

The approximation on the boundary is just the restriction of the element approximation to the boundary (corresponds to a linear two-node element).

Only the shape functions associated with nodes 2 and 3 are needed as the other nodes are fixed (displacement vanishes). By deducing the expression, i.e., combining the linear shape functions in the x -directions and y -directions

$$N_2 = \left(1 - \frac{y}{L}\right) \frac{x}{L} \quad \text{and} \quad N_3 = \frac{y}{L} \frac{x}{L}.$$

In terms of the vertical displacement component u_{Y2} of node 2, approximations to the displacement components and their derivatives are

$$u(x, y) = \mathbf{N}^T \mathbf{a} = 0 \Rightarrow \frac{\partial u}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} = 0,$$

$$v(x, y) = \mathbf{N}^T \mathbf{a} = u_{Y2} \frac{x}{L} \Rightarrow \frac{\partial v}{\partial x} = u_{Y2} \frac{1}{L} \quad \text{and} \quad \frac{\partial v}{\partial y} = 0.$$

Virtual work density of the internal forces simplifies to (when the approximations are substituted there)

$$\delta w_{\Omega}^{\text{int}} = - \left\{ \begin{array}{c} 0 \\ 0 \\ \delta u_{Y2} / L \end{array} \right\}^T \frac{Et}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \left\{ \begin{array}{c} 0 \\ 0 \\ u_{Y2} / L \end{array} \right\} = -\delta u_{Y2} \frac{1}{L^2} \frac{Et}{2(1+\nu)} u_{Y2}.$$

Virtual work density is constant in this case. Integration over the element gives the virtual work expression of internal forces

$$\delta W^{\text{int}} = \int_0^L \int_0^L \delta w_{\Omega}^{\text{int}} dx dy = -\delta u_{Y2} \frac{Et}{2(1+\nu)} u_{Y2}.$$

Virtual work expression of external distributed force components $t_x = 0$ and $t_y = -f$ is obtained as an integral over the edge defined by $x = L$. The restriction of approximation to $x = L$ is given by

$$u(L, y) = 0 \quad \text{and} \quad v(L, y) = u_{Y2}$$

so the virtual work density expression simplifies to

$$\delta w_{\partial\Omega}^{\text{ext}} = \left\{ \begin{array}{c} \delta u \\ \delta v \end{array} \right\}^T \left\{ \begin{array}{c} t_x \\ t_y \end{array} \right\} = -\delta u_{Y2} f$$

giving the virtual work expression

$$\delta W^{\text{ext}} = \int_0^L w_{\partial\Omega}^{\text{ext}} dy = -\delta u_{Y2} L f.$$

Virtual work expression is the sum of internal and external parts

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = -\delta u_{Y2} \left(\frac{Et}{2(1+\nu)} u_{Y2} + L f \right).$$

Principle of virtual work $\delta W = 0 \quad \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus in the form $\delta \mathbf{a}^T \mathbf{R} = 0 \quad \forall \delta \mathbf{a} \Leftrightarrow \mathbf{R} = 0$ give

$$\frac{Et}{2(1+\nu)}u_{Y2} + Lf = 0 \quad \Leftrightarrow \quad u_{Y2} = -\frac{Lf2(1+\nu)}{Et} = -\frac{Lf}{tG}.$$

