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Exercise 2A3

The density function : $f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$

a) Integrate the density function to find the CDF

▫ $x > 0$

$$F(x) = \int_0^x \lambda e^{-\lambda x} dx = \lambda \int_0^x e^{-\lambda x} dx$$

Let $t = -\lambda x \Rightarrow dt = -\lambda dx$

$$\Rightarrow \lambda \int_0^x e^{-\lambda x} dx = -\int_0^t e^t dt = -e^t \Big|_0^t = -e^t - (-e^0)$$
$$= 1 - e^t$$

Replace $t = -\lambda x \Rightarrow F(x) = 1 - e^{-\lambda x}$

▫ $x \leq 0$

$$F(x) = \int_x^0 0 \, dx = 0$$

$\Rightarrow$ CDF of density function : $F(x) = \begin{cases} 0, & x \leq 0 \\ 1 - e^{-\lambda x}, & x > 0 \end{cases}$

b) $A = \{X \leq 1\}$

We have: $P(X > t) = e^{-\lambda t}$

$\Rightarrow P(X \leq t) = 1 - e^{-\lambda t}$

$\Rightarrow P(A) = P(X \leq 1) = 1 - e^{-(0,5).1}$

$\approx 0,393469$: Probability of satellite's lifetime last equal or less than one year

▫ $B = \{X > 5\}$

$\Rightarrow P(B) = e^{-(0,5).5} \approx 0,082085$

$\Rightarrow$ Probability of satellite's lifetime lasting more than 5 years

▫ $C = \{5 < X \leq 6\}$

$\Rightarrow P(C) = P(X \leq 6) - P(X \leq 5)$

$= [1 - e^{-0.5.6}] - [1 - e^{-0,5.5}]$

$\approx 0,032298$

$\Rightarrow$ Probability of satellite's life time lasting more than 5 years but less than or equal 6 years

c) Conditional probability : $P(C|B) = \dfrac{P(C \cap B)}{P(B)}$

We have : $\left.\begin{array}{l} C = \{5 < X \leq 6\} \\ B = \{5 < X\} \end{array}\right\} \Rightarrow$ C is subset of B $\Rightarrow B \cap C = C$

$\Rightarrow P(C \cap B) = P(C)$

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$\Rightarrow P(C|B) = \dfrac{P(C)}{P(B)} = \dfrac{0,032298}{0,082085} \approx 0,393470$

$P(A) = 0,393469 \Rightarrow P(A) = P(C|B)$

This phenomenon is called memorylessness : regardless of how long the lifetime of the satellite lasts, the probability of the satellite ends its life time within the next t years is always the same

$P(X \leq 1) = P(5 < X \leq 6 | 5 < X)$

d) According to memorylessness of exponential distribution

$P(t+h \geq X > t | X > t) = P(X \leq h) = 1 - e^{-\lambda h}$

We know that at $h = 0$, the probability of satellite malfunctioning is 0

$\Rightarrow$ After $h = 0$, probability of satellite working is absolute certainty ($e^{-\lambda . 0} = 1$)

$P(t + 0,01 \geq X > t | X > t) = P(X \leq 0,01) = 1 - e^{-(0,5).(0,01)}$

$= 4,98752 . 10^{-3}$

$\lambda h = 0,5 . 0,01 = 5 . 10^{-3} \approx P(X \leq 0,01)$

$\Rightarrow \lambda$ is called the rate parameter because it denotes how the value of exponential distribution decreases in proportion with h exponentially. The smaller λ is, the more rapid the exponential distribution decreases

Exercise 2A4

$$f(x) = \begin{cases} cx(1-x), & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

a) Determine c. The density muss integrate to 1

$$\int_0^1 cx(1-x)\,dx = c\int_0^1 x(1-x)\,dx = c\int_0^1 (x - x^2)\,dx$$

$$= c \left| \frac{x^2}{2} - \frac{x^3}{3} \right|_0^1 = \frac{c}{6} = 1$$

$\Rightarrow c = 6$

b) Determine CDF or X and draw it

- For $0 \leq x \leq 1$

$$\int 6x(1-x)\,dx = 3x^2 - 2x^3$$

- For $x < 0$ and $x > 1$: $\int 0\,dx = 0$

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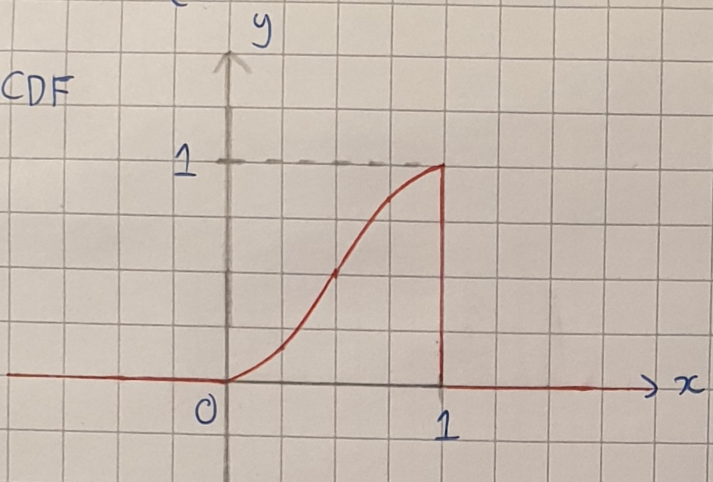
$$\Rightarrow \quad F(X) = \begin{cases} 3x^2 - 2x^3, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Graph of CDF of X



c) Calculate probability that the truth value is at least 0.75

We have: $P(0.75 \leq X \leq 1) = \int_{0.75}^{1} 6x(1-x)\,dx$

$$= 3x^2 - 2x^3 \Big|_{0.75}^{1} = 1 - \frac{27}{32}$$

$$= \frac{5}{32} = 0.15625$$

d) Find the mode of the distribution, the point where density $f(x)$ attains its maximum

$f(x) = 6x(1-x), \ 0 \leq x \leq 1$

$= 6x - 6x^2$

$\Rightarrow f'(x) = 6 - 12x, \quad f''(x) = -12$

$f'(x) = 0 \Rightarrow x = 0,5$

$\Rightarrow x = 0,5$ is the maximum

$\Rightarrow f(x) = 1,5$

$\Rightarrow$ Mode of the distribution is $x = 0,5$