MS-A0503 First course in probability and statistics

3B Statistical datasets

Jukka Kohonen

Deparment of mathematics and systems analysis
Aalto SCI

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Contents

What is statistics (as a science)?

- Applying and developing methods for studying random or uncertain phenomena in the *real world*.
  - The methods are based on the mathematical laws of probability.
  - Sources of uncertainty are many: random chance; unknown properties of the real world phenomenon; random sampling; measurement errors; missing data ...
  - ...generally, the same math applies.

- Roughly:
  - Probability theory tells: How a certain process produces data.
  - Statistics tells: What was the process that produced the data.

- Statistics is applicable whenever you have data; especially if there is any kind of uncertainty or randomness.
  - Most fields of engineering and business have data, so they can (and do) use statistics.

Two basic approaches of statistics

Descriptive statistics

Present and describe the data "as it is", either fully, or in a summary way.

- Tables ("the raw data")
- Graphs (visualization)
- Numerical summaries or "statistics" (e.g. average, minimum, maximum)

Statistical inference

Infer facts about the real phenomenon that lies "behind" the data.
Generalize, e.g. sample $\rightarrow$ population; or measurements $\rightarrow$ universal physical law.

- Stochastic models
- Parameter estimation
- Hypothesis testing

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Statistical data

Typically (not always), statistical data is in a table, the data frame, where

- rows correspond to units, e.g. people
- columns are the variables observed for each unit, e.g. height

Caveat: Different fields of science/engineering use different words. E.g. "units" may be "objects", "items", "data points" (geometrically thinking), "records" (in databases)

Depending on number of variables, we may call our data *univariate*, *bivariate*, *multivariate*.

Levels of measurement = What kind of values

- nominal scale: just distinct types or classes
  gender: {male, female}
  country of birth: {Finland, Sweden, Norway}
- ordinal scale: classes have meaningful order
  cloth size: { XS < S < M < L < XL }
  Likert scale: { str. disagree < disagree < neutral < agree < str. agree }
- numerical: values have arithmetic meaning
  - interval scale: differences $x - y$ are meaningful
    calendar dates, Celsius temperature
  - ratio scale: also quotients x/y are meaningful
    length, weight, distance, Kelvin temperature

Notes:

- all can be *represented* as numbers, e.g. Finland=1, Sweden=2, Norway=3, but arithmetic might not make sense.
- nominal sometimes called "qualitative", but other meanings
- this is *not* the discrete/continuous distinction. Numerical data can be well discrete; e.g. counts (frequencies)

Data set (Data frame)

- data set = sequence of elements (units) of the same type, e.g. numbers, identifiers, or lists of values (one for each variable)
- Often arranged in a table; (R terminology) "data frame"
- Order of units often not meaningful, so we could treat it as a set (or *multiset*, if many identical observations possible)

E.g. course feedback: ((12345A, 5, 1, 5), (98759K, 1, 5, 2), (33312K, 4, 4, 3), (23453B, 4, 4, 3), (21453U, 3, 3, 3))

One string variable (student id), three numerical variables (general satisfaction, workload, usefulness)

Student ID	General	Workload	Usefulness
12345A	5	1	5
98759K	1	5	2
33312K	4	4	3
23453B	4	4	3
21453U	3	3	5

5 units, 4 variables

Average and standard deviation

If we have univariate numerical data: $\vec{x} = (x_1, \ldots, x_n)$

Average (sample mean) $m(\vec{x}) = \frac{1}{n}\sum_{i=1}^{n} x_i$

Variance $\mathrm{var}(\vec{x}) = \frac{1}{n}\sum_{i=1}^{n}(x_i - m(\vec{x}))^2$

Standard deviation $\mathrm{sd}(\vec{x}) = \sqrt{\mathrm{var}(\vec{x})}$

Eg. $\vec{y} = (0, 0, 1, 1, 2, 2)$

$$m(\vec{y}) \ = \ \frac{1}{6}\Big(0 + 0 + 1 + 1 + 2 + 2\Big) \ = \ 1$$

$$\mathrm{var}(\vec{y}) \ = \ \frac{1}{6}\Big((0-1)^2+(0-1)^2+(1-1)^2+(1-1)^2+(2-1)^2+(2-1)^2\Big) \ = \ \frac{2}{3}$$

$$\mathrm{sd}(\vec{y}) \ = \ \sqrt{\frac{2}{3}} \ \approx \ 0.8165$$

Caveat: Sometimes $n-1$ used as divisor for variance and sd, for technical reasons; more about this later (in parameter estimation).

Example

Calculate sample mean and standard deviation for the following data sets

$$\begin{aligned}
\vec{x} &= (1, 1, 1, 1, 1), \\
\vec{y} &= (0, 0, 1, 1, 2, 2), \\
\vec{z} &= (0, 2, 0, 2, 0, 2, 0, 2, 0, 2), \\
\vec{w} &= (\underbrace{0, 0, 0, 0, \ldots, 0, 0, 0, 0}_{666666 \text{ times}}, 1000000, \underbrace{0, 0, \ldots, 0, 0}_{333333 \text{ times}}).
\end{aligned}$$

Dataset	Mean	SD
$\vec{x}$	1	0.0000
$\vec{y}$	1	0.8165
$\vec{z}$	1	1.0000
$\vec{w}$	1	999.9995

Average and standard deviation are *summaries*, they do not tell everything about the data. (Just like in probability distributions.)

Computing the summary statistics

Notation	Name	R	Python	Excel
$m(\vec{x})$	Average	`mean()`	`np.mean()`	`AVERAGE()`
$\mathrm{sd}(\vec{x})$	Standard deviation	`sqrt(1-1/n)*sd()`	`np.std()`	`STDEV.P()`
$\mathrm{sd}_s(\vec{x})$	Sample std.dev.	`sd()`	`np.std( ,ddof=1)`	`STDEV.S()`
$\mathrm{var}(\vec{x})$	Variance	`(1-1/n)*var()`	`np.var()`	`VAR.P()`
$\mathrm{var}_s(\vec{x})$	Sample variance	`var()`	`np.var( ,ddof=1)`	`VAR.S()`
$\mathrm{cov}(\vec{x}, \vec{y})$	Covariance	`(1-1/n)*cov()`	`np.cov( ,ddof=0)[0][1]`	`COVARIANCE.P()`
$\mathrm{cov}_s(\vec{x}, \vec{y})$	Sample covariance	`cov()`	`np.cov( ,ddof=1)[0][1]`	`COVARIANCE.S()`
$\mathrm{cor}(\vec{x}, \vec{y})$	Correlation	`cor()`	`np.corrcoef()[0][1]`	`CORREL()`
$q_{0.5}(\vec{x})$	Median	`median()`	`np.median()`	`MEDIAN()`
$q_{0.25}(\vec{x})$	Lower quartile	`quantile( ,.25)`	`np.quantile( ,.25)`	`PERCENTILE.INC( ,.25)`
$q_{0.75}(\vec{x})$	Upper quartile	`quantile( ,.75)`	`np.quantile( ,.75)`	`PERCENTILE.INC( ,.75)`

Caveat. Terminology and notation varies across sources. For technical reasons, many computer programs offer the so-called "unbiased" or "corrected" *sample variance* and *sample standard deviation*, where the divisor is $n-1$ instead of n. (Don't worry too much now – this will become clearer later.)

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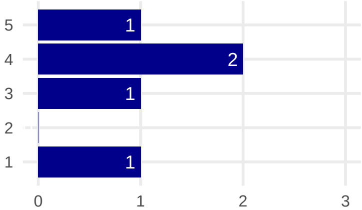
Counts = Frequencies of values

The count, or (absolute) frequency of a value x, in the univariate dataset $\vec{x}$, is

$$n_{\vec{x}}(x) \; = \; \#\{i : x_i = x\}$$

Course feedback, pick one variable "General" $\rightarrow$ univariate data $(5, 1, 4, 4, 3)$. Frequency as a table and a (horizontal) bar chart:

x	1	2	3	4	5
$n_{\vec{x}}(x)$	1	0	1	2	1



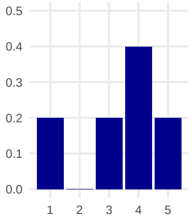
Relative frequencies

The proportion, or relative frequency of value x in dataset $\vec{x}$ is

$$f_{\vec{x}}(x) = \frac{n_{\vec{x}}(x)}{n} = \frac{\#\{i : x_i = x\}}{n}$$

Course feedback, pick "General", dataset $(5, 1, 4, 4, 3)$, relative frequencies as a table and (vertical) bar chart

x	1	2	3	4	5
$f_{\vec{x}}(x)$	$\frac{1}{5}$	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{5}$



Observation: $\sum_x f_{\vec{x}}(x) = 1$, thus $f_{\vec{x}}(x)$ is a probability distribution!
It is the empirical distribution of the dataset $\vec{x}$.

Empirical distribution

Proposition

If an element X is chosen uniformly at random, from the dataset $\vec{x} = (x_1, \ldots, x_n)$, then X is a discrete random variable, whose density corresponds to the empirical distribution: $f_X(x) = f_{\vec{x}}(x)$. Furthermore,

$$\mathbb{E}(X) = m(\vec{x}), \tag{1}$$

$$\mathrm{SD}(X) = \mathrm{sd}(\vec{x}), \tag{2}$$

$$\mathrm{Var}(X) = \mathrm{var}(\vec{x}). \tag{3}$$

Also, for any function g, we have

$$\mathbb{E}[g(X)] = \frac{1}{n}\sum_{i=1}^{n} g(x_i). \tag{4}$$

Example

For the dataset $\vec{y} = (0, 0, 1, 1, 2, 2)$, determine the empirical distribution, and its mean and standard deviation.

The relative frequencies are

y	0	1	2
$f_{\vec{y}}(y)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

If random variable Y has density $f_{\vec{y}}(y)$, then

$$\mathbb{E}(Y) \;=\; \sum_{y=0}^{2} y f_{\vec{y}}(y) \;=\; 0 \times \frac{1}{3} + 1 \times \frac{1}{3} + 2 \times \frac{1}{3} \;=\; 1,$$

$$\mathrm{Var}(Y) \;=\; \sum_{y=0}^{2} (y-1)^2 f_{\vec{y}}(y) \;=\; (0-1)^2 \times \frac{1}{3} + (1-1)^2 \times \frac{1}{3} + (2-1)^2 \times \frac{1}{3} \;=\; \frac{2}{3}$$

$$\implies m(\vec{y}) = \mathbb{E}(Y) = 1$$

$$\implies \mathrm{sd}(\vec{y}) = \sqrt{\mathrm{var}(\vec{y})} = \sqrt{\mathrm{Var}(Y)} = \sqrt{\tfrac{2}{3}} \approx 0.8165$$

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Binning (Grouping)

Eg. ages of all Finns 31.12.2015.
$n = 5\ 487\ 308$ units (data points)
Not probably good idea to draw as individual points (especially if ages are expressed in 1-day precision)
Let us group the data into bins.
(This might even be forced, if we only *have* counts and not individual data points.)

Age (yr)	Frequency
0–14	896 023
15–24	640 387
25–44	1 363 155
45–64	1 464 640
65–74	642 428
75–	480 675

How to draw a histogram (allowing unequal bin widths)

- One bar for each bin (interval of possible values)
- Bar width = width of the interval
- Bar height = relative frequency *divided* by width

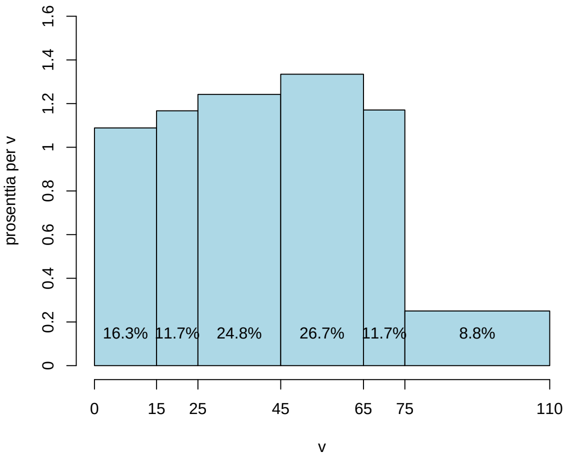
Example: Age distribution, first bar:

- Represents Finns with ages in interval $[0, 15)$
  Note: values *strictly* smaller than 15; age in whole years 0–14
- Bar width = 15 years
- Frequency = 896023, relative frequency $896023/5487308 \approx 16.3\%$
- Bar height = $16.3/15 \approx 1.09$ (unit: % per year).
- Then bar *area* is the relative frequency.

(Typically, we use equal-width intervals, but not always.)

Example: Histogram with unequal widths

Finnish age distribution 31.12.2015 [Source: Tilastokeskus]



Age (yr)	Frequency
0–14	896 023
15–24	640 387
25–44	1 363 155
45–64	1 464 640
65–74	642 428
75–	480 675

The bars are an *approximation* of the true density function.
Could you find the proportion of Finns in the 1-year interval $[13, 14)$?
What about the the interval $[109, 110)$ years? Would it be accurate?

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Bivariate data

Bivariate data = sequence (or multiset) of *pairs*

$$\vec{xy} \;=\; ((x_1, y_1), \ldots, (x_n, y_n)).$$

Alternatively, a pair $(\vec{x}, \vec{y})$, where $\vec{x} = (x_1, \ldots, x_n)$ and $\vec{y} = (y_1, \ldots, y_n)$ are univariate data (note: as ordered sequences, so we know which x_i and y_i belong together).

Course feedback: Two variables "General" and "Usefulness" composed as a bivariate dataset ((5,5), (1,2), (4,3), (4,3), (3,3))

Univariate statistics $m(\vec{x}), m(\vec{y}), \mathrm{sd}(\vec{x}), \mathrm{sd}(\vec{y})$ are surely useful, but they tell nothing about the dependence between variables. Covariance and correlation tell (some aspects of) dependence.

$$\mathrm{cov}(\vec{x}, \vec{y}) \;=\; \frac{1}{n} \sum_{i=1}^{n} (x_i - m(\vec{x}))(y_i - m(\vec{y}))$$

$$\mathrm{cor}(\vec{x}, \vec{y}) \;=\; \frac{\mathrm{cov}(\vec{x}, \vec{y})}{\mathrm{sd}(\vec{x})\,\mathrm{sd}(\vec{y})}$$

Example: Heights of father-son pairs

I	P	I	P	I	P	I	P	I	P
165	152	165	171	179	174	178	178	184	187
161	161	166	171	181	172	177	178	173	192
165	161	167	170	180	174	179	177	172	192
167	159	167	171	182	173	180	178	185	196
155	163	167	172	186	172	181	178	159	156
160	163	167	171	160	175	183	179	173	189
166	163	167	170	162	177	183	178	165	180
164	163	170	172	166	177	192	178	187	191
168	164	169	171	164	176	164	181	160	157
170	163	171	170	169	177	169	180	159	163
150	166	171	171	167	176	167	181	178	159
160	166	169	171	167	176	170	181	181	187
162	167	173	172	168	177	170	182	181	169
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I	P	I	P	I	P	I	P	I	P
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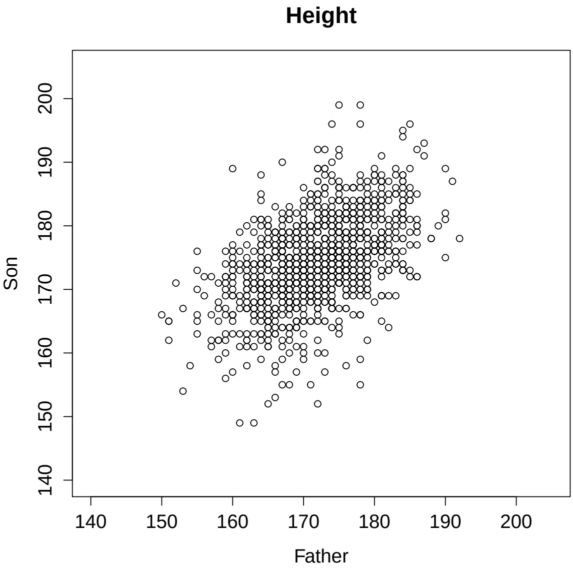
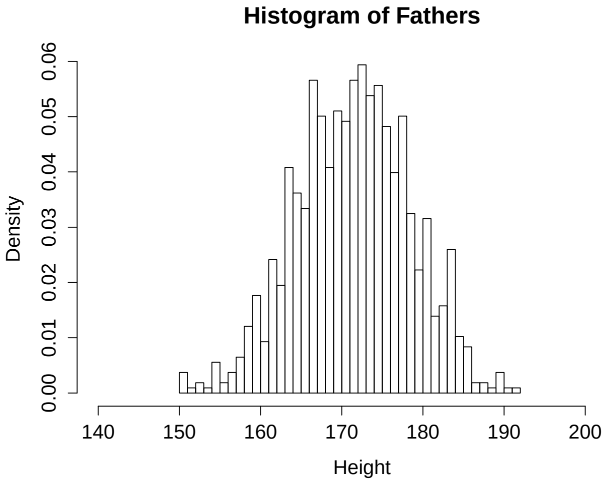
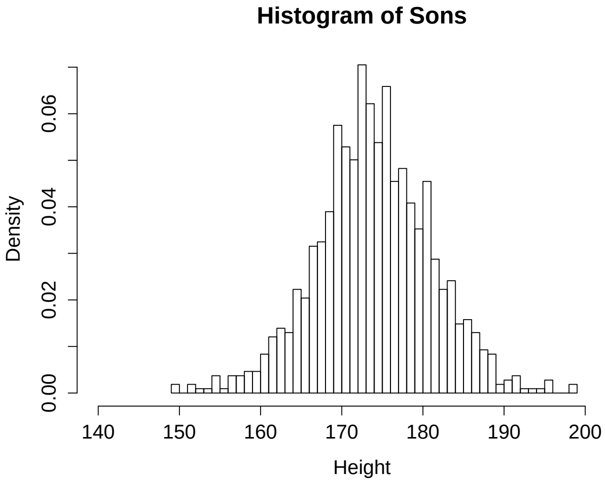
169	178	182	184	161	169	160	172	175	177
168	179	185	184	162	167	161	173	175	176
170	177	173	186	163	168	164	174	175	177
170	178	172	187	164	168	163	174	175	176
172	179	176	186	165	169	164	172	175	175
173	178	179	186	167	169	164	174	178	175
174	177	181	187	167	169	168	173	179	176
173	178	184	187	168	168	167	173	178	175
176	179	183	185	171	169	167	173	177	175
176	178	164	188	169	168	167	173	181	176
174	179	172	189	169	168	169	173	181	177
175	179	178	188	170	169	170	174	185	177
177	179	180	188	173	168	170	174	166	179
177	177	185	189	172	168	171	174	168	178
179	179	175	191	174	167	170	174	166	179
181	179	184	194	176	167	170	172	167	179

Table: 1000 pairs of heights (father, son) from Pearson's data. Note that this not just two data sets, but a set of *pairs* that belong together.

Scatterplot (scatter diagram)

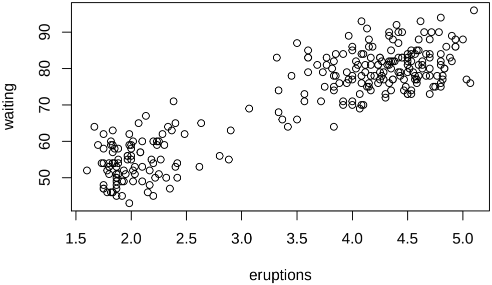



Histogram of Fathers
Density
0.00
0.01
0.02
0.03
0.04
0.05
0.06
140
150
160
170
180
190
200
Height


Histogram of Sons
Density
0.00
0.02
0.04
0.06
140
150
160
170
180
190
200
Height

Example: Eruptions of Old Faithful geysir

Scatterplot of 272 eruptions of *Old Faithful* (Yellowstone).
Two variables: eruption length and waiting time to next eruption.



Already a visual inspection reveals interesting patterns.

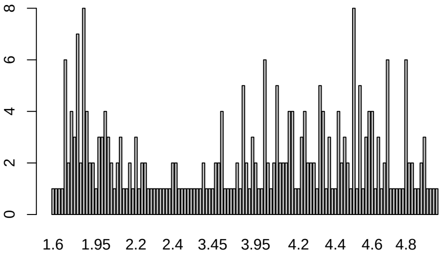
You can find the data in R, try `faithful` and `help("faithful")`

Old Faithful: bar chart of one variable...

We could try listing *all different values* of eruption length, and collect their frequencies (within $n = 272$)

x	1.6	1.667	1.7	1.733	1.75	...	5.1
$n_{\vec{x}}(x)$	1	1	1	1	6	...	1

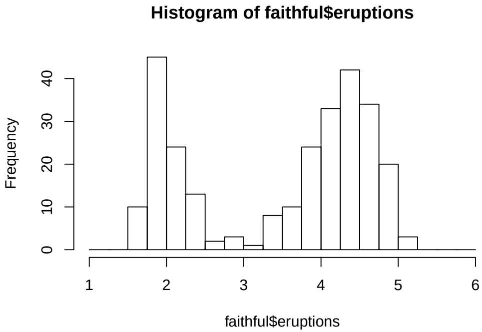
and draw a bar chart



Neither the table or the bar chart seems very informative.

Old Faithful: histogram

But if we group the data into 0.25-minute intervals such as $[2.00, 2.25)$, and plot the counts, we have a better picture of the distribution.



Try this on your own! What happens if you use more (finer) intervals? What if you use less (coarser) intervals?

Cross-tabulation (Contingency table)

The frequency of the pair (x, y), in the data set, is

$$n_{\vec{x}\vec{y}}(x, y) \;=\; \#\{i \;:\; x_i = x \;\text{ and }\; y_i = y\}$$

Course feedback: "General" and "Usefulness" as bivariate dataset ((5,5), (1,2), (4,3), (4,3), (3,3)) has this contingency table:

	y					
x	1	2	3	4	5	Sum
1	0	1	0	0	0	1
2	0	0	0	0	0	0
3	0	0	1	0	0	1
4	0	0	2	0	0	2
5	0	0	0	0	1	1
Sum	0	1	3	0	1	

Cross-tabulation of relative frequencies

The relative frequency of the pair (x, y) is

$$f_{\vec{x}\vec{y}}(x, y) \;=\; \frac{\#\{i \;:\; x_i = x \;\text{ and }\; y_i = y\}}{n}$$

	y					
x	1	2	3	4	5	Sum
1	0	$\frac{1}{5}$	0	0	0	$\frac{1}{5}$
2	0	0	0	0	0	0
3	0	0	$\frac{1}{5}$	0	0	$\frac{1}{5}$
4	0	0	$\frac{2}{5}$	0	0	$\frac{2}{5}$
5	0	0	0	0	$\frac{1}{5}$	$\frac{1}{5}$
Sum	0	$\frac{1}{5}$	$\frac{3}{5}$	0	$\frac{1}{5}$	

$\sum_{x,y} f_{\vec{x}\vec{y}}(x, y) = 1$, thus $f_{\vec{x}\vec{y}}(x, y)$ is a probability distribution.
$f_{\vec{x}\vec{y}}(x, y)$ is the empirical joint distribution of the dataset $\vec{x}\vec{y}$.

Empirical joint distribution

Proposition

If a pair (X, Y) is chosen uniformly at random, from the dataset $\vec{xy} = ((x_1, y_1), \ldots, (x_n, y_n))$, it is a discrete random variable whose (joint) density is the empirical distribution $f_{X,Y}(x, y) = f_{\vec{xy}}(x, y)$ and also

$$\begin{aligned} \mathbb{E}(X) &= m(\vec{x}), & \mathbb{E}(Y) &= m(\vec{y}), \\ \mathrm{SD}(X) &= \mathrm{sd}(\vec{x}), & \mathrm{SD}(Y) &= \mathrm{sd}(\vec{y}), \\ \mathrm{Var}(X) &= \mathrm{var}(\vec{x}), & \mathrm{Var}(Y) &= \mathrm{var}(\vec{y}), \end{aligned} \tag{5}$$

and

$$\mathrm{Cor}(X, Y) = \mathrm{cor}(\vec{x}, \vec{y}), \tag{6}$$

$$\mathrm{Cov}(X, Y) = \mathrm{cov}(\vec{x}, \vec{y}). \tag{7}$$

Also, for any two-argument function g, we have

$$\mathbb{E}[g(X, Y)] = \frac{1}{n} \sum_{i=1}^{n} g(x_i, y_i). \tag{8}$$

Contents

Quantiles (Percentiles)

Suppose data can be ordered from smallest to largest.
(OK for numerical or ordinal data; not for nominal data.)

If $0 < p < 1$, then the p-quantile (or $100p$-percentile) $Q(p)$ is roughly the point x such that *proportion* p of the data is smaller than x, and $1 - p$ is greater.

- $Q(0.25)$ is lower (first) quartile; 25% of data is below
- $Q(0.5)$ is median or second quartile
- $Q(0.75)$ is upper (third) quartile; 25% of data is above

R: `quantile(x,p)`, `summary(x)`, `median(x)`

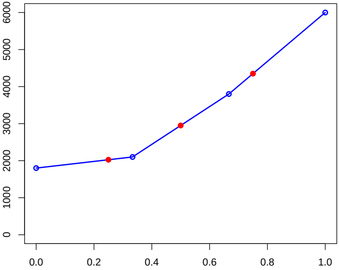
The "roughly" is because in finite data, you may not find exact quarters. There are some (varying) conventions for this.

Quantile function

One way to define the quantile function of dataset $(x_1, \ldots, x_n)$:

- Order the data as $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$
- Divide the horizontal unit interval $[0, 1]$ into equal parts, at points $p_k = (k-1)/(n-1)$, $k = 1, \ldots, n$
- Plot the points $(p_k, x_{(k)})$ and connect with lines

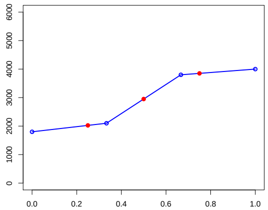
Example. Four salaries (eur/month): $3800, 1800, 2100, 6000$

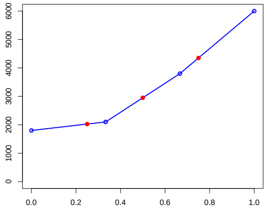


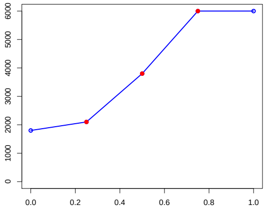
Quartiles = Evaluate the quantile function at 0.25, 0.50, 0.75

Example: Three small datasets

$$\vec{x} = (1800, 2100, 3800, 4000)$$
$$\vec{y} = (1800, 2100, 3800, 6000)$$
$$\vec{z} = (1800, 2100, 3800, 6000, 6000) \qquad (n \text{ not divisible by four})$$







$Q_x(0.50) = 2950$, $m(x) = 2925$

$Q_y(0.50) = 2950$, $m(x) = 3425$

$Q_z(0.50) = 3800$, $m(x) = 3940$

Sample, population and "population"

The finite dataset you have, the "sample", is often thought to "represent" qualities of a larger "population".

sample	population
Pearson's 1000 fathers and sons	All father-son pairs in . . . ?
1000 poll responses	Opinions of 5 million Finns now
272 eruptions of Old Faithful	All its eruptions (in future?)
Drug effect on 30 patients	Drug effect on future patients
100 rolls of a loaded die	Potential infinite sequence of rolls

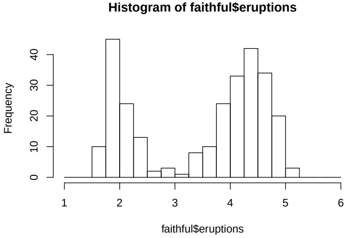
Population is statistical jargon for

- where your data came from
  (data-generating mechanism; data source)
- what you are trying to understand by looking at the data

Hence, terms such as *sample mean* and *population mean*.
The "population" may be quite concrete, or a figure of speech.

Old Faithful, once again

We have a *sample* of 272 eruption lengths. The real physical mechanism may be complicated, but perhaps we can think the lengths are as if they come from one particular distribution f. But what distribution?



One eruption length is a random variable X from this generating distribution or underlying distribution or true distribution. ("Population" if you want.)

Empirical distribution approximates the generating distribution. Why?
Answer: Think of the event $\{2.0 \leq X < 2.25\}$, its probability, and law of large numbers. We know *relative frequency* $\approx$ *probability of event*.

Next lecture is about inference.

From the data, we will estimate some parameters concerning the reality "behind" the data.