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Exercise 1:

a)

1 2	3 4 5 6		
1 2	3 4 5 6		
1 2	3 4 5 6		
Lotta	Peter		

Suppose that the game lasts 3 matches, even if we know that a player has already won in the first 2 rounds. For Lotta to win, the dice must have $X_1 = \{1; 2\}$ and $X_2 = \{1 .. 2\}$ and Peter is $X_1 = X_2 = \{3 ... 6\}$ in the first two rounds. Since the winner ~~player~~ is decided, the dice in third round doesn't matter

=) If the game lasts 2 rounds, the total number of 2-round matches is

$2 \times 2 \times 6 + 4 \times 4 \times 6 = 120$ matches

For the winner to be decided in 3 rounds, there are 4 main sequences

L: Lotta P: Peter

$L \to P \to L$ $P \to L \to P$ $L \to P \to P$ $P \to L \to L$

=) Total number of 3-round matches is

$2 \times 4 \times 2 + 4 \times 2 \times 4 + 2 \times 4 \times 4 + 4 \times 2 \times 2 = 96$ matches

Total number of all 3-round matches including 2-rounds is $6^3 = 216$

=)

Round (r)	2	3
$P(X = r)$	$\frac{120}{216}$	$\frac{96}{216}$

=) Expected value of number of rounds: $E(r) = \frac{120}{216} \times 2 + \frac{96}{216} \times 3 = \frac{22}{9}$

≈ 2.44 rounds (Answer)

b) Number of matches that Lotta wins is

$2 \times 2 \times 6 + 2 \times 4 \times 2 + 4 \times 2 \times 2 = 56$

Number of matches that Peter wins is

$4 \times 4 \times 6 + 4 \times 2 \times 4 + 2 \times 4 \times 4 = 160$

=) Probability that Lotta wins is $\frac{56}{216} = \frac{7}{27}$

Probability that Peter wins is $\frac{160}{216} = \frac{20}{27}$

Since Peter contributes 20 euros to the pot, the amount Lotta should contribute in proportion to her chance of winning is $20 : \frac{20}{27} \times \frac{7}{27} = 7$ euros (Answer)

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c) Since Lotta wins the first match, there are 3 main sequences

$L \to L$ (L wins) $\quad L \to P \to L$ (L wins) $\quad L \to P \to P$ (P wins)

There are 2 rounds left so $6^2 = 36$ matches possible

L wins: $2 \times 6 + 4 \times 2 = 20$ matches

P wins: $4 \times 4 = 16$ matches

$\Rightarrow L$ receives $\frac{20}{36} = \frac{5}{9}$ the prize of the pot

$\Rightarrow P$ receives $\frac{16}{36} = \frac{4}{9}$ the prize of the pot

Exercise 2:

The first digit of my student number is 8

$\Rightarrow T \in [100, 158]$

a) $X = \frac{1}{2} a T^2$. Since $a = 2m/s^2 \Rightarrow X = T^2$

Since T is uniformly distributed in $I = [100, 158]$

$$\Rightarrow f_T(t) = \begin{cases} \frac{1}{58}, & \text{for } t \in [100; 158] \\ 0, & \text{otherwise} \end{cases}$$

$\Rightarrow$ The expected value of distance travelled is

$$E(X) = E(T^2) = \int_{100}^{158} t^2 f_T(t)\,dt = \int_{100}^{158} t^2 \frac{1}{58}\,dt$$

$$= \frac{1}{58}\left(t^3/3 \Big|_{100}^{158} \right) = \frac{50764}{3} \approx 16921.33$$

b) $X = T^2$

$$\Rightarrow F_X(t) = P(T^2 \leq t) = P(T \leq t^{1/2}) = \int_{100}^{t^{\frac{1}{2}}} \frac{1}{58}\,dt$$

$$\Rightarrow \text{CDF}: f_X(x) = \begin{cases} 0, & \cancel{t}\, x < 100 \\ \frac{1}{58} x^{\frac{1}{2}} - \frac{100}{58}, & x \in [100, 158] \\ 1, & x > 158 \end{cases}$$

$\Rightarrow$ Probability of $X \geq 20000$ is

$$1 - \left(\frac{1}{58} \cdot 20000^{\frac{1}{2}} - \frac{100}{58} \right) = 0.2858$$

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c) We have : $E(X^2) = E(T^4) = \int_{100}^{158} t^4 f_T(t)\,dt = \int_{100}^{158} t^4 \cdot \frac{1}{58} f_T(t)\,dt$

$= \frac{158^5 - 10^{10}}{290}$

$\Rightarrow SD(X) = \sqrt{E[X^2] - (E(X))^2} = \sqrt{\left(\frac{158^5 - 10^{10}}{290}\right) - \left(\frac{50764}{3}\right)^2}$

≈ 4327 (answer)

d) Probability that exactly 3 of the ships travel more than 20000 m is
$0.2858^3 \simeq 0.0233$ (Since these ships are independent)

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Exercise 3: Prior density table

$f_\Theta(\theta)$	0.5	0.6	0.4
$P(\Theta=\theta)$	$\frac{98}{100}$	$\frac{1}{100}$	$\frac{1}{100}$

a) What is the probability of obtaining only heads when the coin is tossed 30 + s times

My first student digit is 8 $\Rightarrow$ 38 times

Probability of obtaining a head in one toss is

Law of total probability: $0.5 \times \frac{98}{100} + 0.6 \cdot \frac{1}{100} + 0.4 \cdot \frac{1}{100} = 0.5$

Since the coin is not put back $\Rightarrow$ θ never changes

$\Rightarrow$ Probability of 38 heads in a row is : $0.5^{38} \times \frac{98}{100} + 0.6^{38} \cdot \frac{1}{100} + 0.4^{38} \cdot \frac{1}{100}$

b) Posterior distribution table after 38 tosses

Prior θ	Likelihood	Product	Posterior (normalized)
$\theta_1 = 0.98$	0.5^{38}	$0,98 . 0,5^{38}$	0.0876
$\theta_2 = 0.01$	0.6^{38}	$0,01 . 0,6^{38}$	0.9124
$\theta_3 = 0.01$	0.4^{38}	$0,01 . 0,4^{38}$	$1.8566 . 10^{-7}$

$c = \sum \text{product} = 4.0697 . 10^{-11}$

c) We have $\theta = 0.6$ (The coin with 0.6 probability of heads) is the mode/MAP estimate of the posterior distribution

d) The mean of ~~of~~ the posterior distribution is

$0.0876 \times 0.5 + 0.9124 \times 0.6 + 1.8566 . 10^{-7} \times 0.4$

≈ 0.5912

e) As we know that a coin has had 30 heads, it will have the prior density as the posterior distribution in (b) for the next tosses

Prior θ	Likelihood	Product
$\theta_1 = 0.0876$	0.5^3	0.0109
$\theta_2 = 0.9124$	0.6^3	0.1990
$\theta_3 = 1.8566 . 10^{-7}$	0.4^3	$1.1882 . 10^{-8}$

$\Rightarrow P(33 \text{ heads}) = \sum \text{product} = 0.0109 + 0.1990 + 1.1882 . 10^{-8}$

$= 0.2099$

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Exercise 4:

a) X is the result from rolling an ordinary 6-sided die

$\Rightarrow \mu_X = \frac{1}{6}(1+2+3+4+5+6) = 3.5$

$Var(X) = \frac{1}{6}\left(\sum_{i=1}^{n=6}(|i - 3.5|)^2\right) = \frac{35}{12}$

$\Rightarrow \sigma_X = \sqrt{\frac{35}{12}} = \frac{\sqrt{105}}{6}$

$\Rightarrow P(|X - 3.5| \leq \sqrt{2}\,\sigma_X) = P\left(|X - 3.5| \leq \sqrt{2}.\frac{\sqrt{105}}{6}\right)$

$= P\left(|X - 3.5| \leq \frac{\sqrt{210}}{6}\right) = P(|X - 3.5| \leq 2.4152)$

Only $X = \{2; 3; 4; 5\}$ satisfies the equation

$\Rightarrow P(|X - 3.5| \leq 2.4152) = \frac{2}{3} > \frac{1}{2}$ (proved)

b) $X \sim Exp(2) \Rightarrow \begin{cases} E(X) = \frac{1}{2} \\ SD(X) = \frac{1}{2} \end{cases}$

$\Rightarrow P\left(\left|X - \frac{1}{2}\right| \leq \sqrt{2}.\frac{1}{2}\right) = P\left(\left|X - \frac{1}{2}\right| \leq 0.7071\right)$

$= P\left(X - \frac{1}{2} \leq 0.7071 \,\middle|\, X \geq \frac{1}{2}\right) + P\left(\frac{1}{2} - X \leq 0.7071 \,\middle|\, 0 \leq x < \frac{1}{2}\right)$

$= P\left(X \leq 1.207 \,\middle|\, X \geq \frac{1}{2}\right) + P\left(X \geq 0.2071 \,\middle|\, 0 \leq x < \frac{1}{2}\right)$

$= P\left(\frac{1}{2} \leq X \leq 1.207\right) + P\left(0.2071 \leq x < \frac{1}{2}\right)$

Omit

$\cancel{= 1 - e^{-2\times 1.207} - [1-(1-e^{-2\times 0.5})] + 1 - e^{-2.0.5} - [1-(1-e^{-2\times 0.2071})]}$

$\cancel{= 0.91054 - [1 - 0.\quad] + 0.63212 -}$

$\cancel{= 0.27842}$

$= P(0.2071 \leq X \leq 1.207)$

$= 1 - e^{-2.1.207} - (1 - e^{-2.0.207})$

$= 0.5715 > \frac{1}{2}$ (proven)

c) $X \sim N(\mu, \sigma^2) = N(3, 5^2)$, $\mu = 3$ and $\sigma = 5$

$\Rightarrow P(|X - 3| \leq \sqrt{2} . 5) = P(|X-3| \leq 7.071)$

$\Rightarrow P(-4.071 \leq X \leq 10.071)$

$\Rightarrow P(-4.071 \leq \mu + \sigma Z \leq 10.071)$

$\Rightarrow P(-4.071 \leq 3 + 5Z \leq 10.071)$

$\Rightarrow P(-1.414 \leq Z \leq 1.414) = \phi(1.414) - \phi(-1.414)$

$= 0.921 - 0.07$

$= 0.851 > \frac{1}{2}$ (proven)