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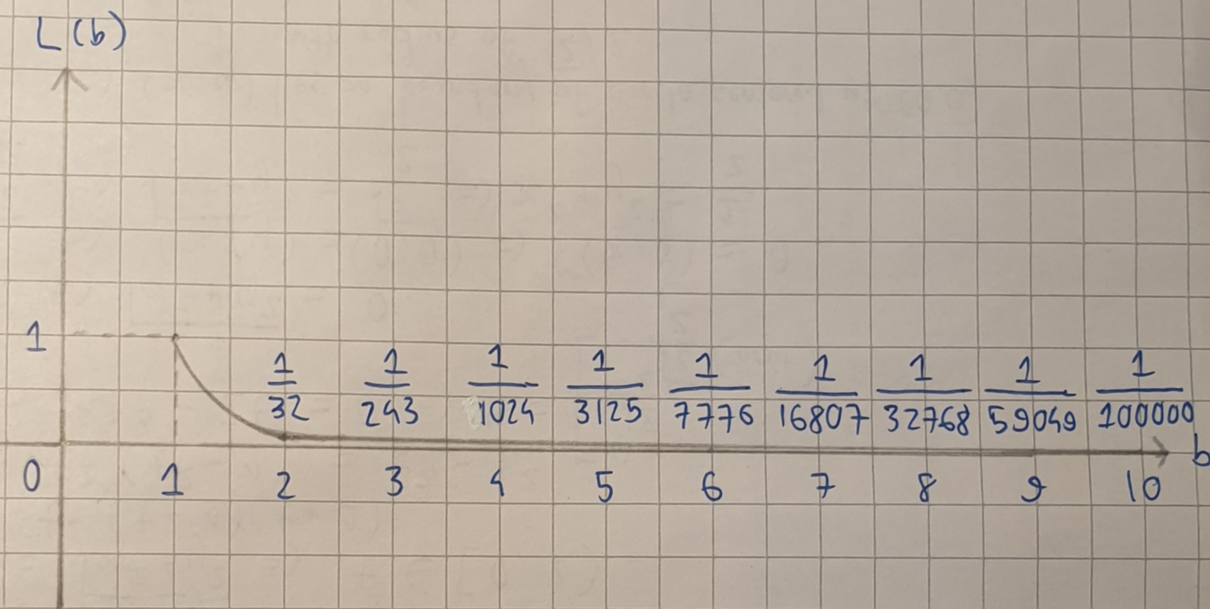
Exercise 4B3: Continuous uniform distribution

a) The likelihood function is

$$L(b) = \prod_{i=1}^{5} f_b(x_i) = \begin{cases} \left(\frac{1}{b}\right)^5, & 0 \leq x_i \leq b \\ 0, & \text{otherwise} \end{cases}$$

Plotting $L(b)$ from 1 to 10



The function is a logarithmic function because the bigger b is, the chance of getting a result is decreased logarithmically

b) We know that if x_i lies in $[0, b]$, the probability totally depends on b. If exists one x_i lies outside the range, the probability is 0

$$L(b) = \begin{cases} \left(\frac{1}{b}\right)^5, & x_i \in [0, b] \\ 0, & x_i \notin [0, b] \end{cases}$$

$\Rightarrow$ The smaller b is, the bigger $L(b)$ is. However, b must be bigger than any x_i

$$\Rightarrow L(b)_{max} \Rightarrow b = \max(x_1, \ldots x_5)$$
$$= \max(1.3, 1.9, 3.6, 1.1, 5.1) = 5.1$$

$\Rightarrow$ Maximum likelihood estimate of the data is $\hat{b} = 5.1 \Rightarrow L(b) = \frac{1}{(5.1)^5}$

c) Generalize to any data: $\hat{b} = \max(\vec{x}) = \max(x_1, x_2, \ldots, x_n)$ is the maximum likelihood estimate of b

d) Since $f_b(x)$ is uniformly distributed over $[0, b]$, the expected value is

$$E(X_1) = \frac{b}{2}$$

Since the set has only 1 data X_1, $\hat{b} = \max\{X_1\} = X_1$

$$\Rightarrow E(\hat{b}) = E(X_1) = \frac{b}{2}$$

Since $E(\hat{b}) = \frac{b}{2} \neq b \Rightarrow$ Estimator $\hat{b}$ is biased

e) The expected value of the defined estimator is

$$E(\tilde{b}(\vec{X})) = E\left(\frac{2}{n}\sum_{i=1}^{n}\vec{X}\right) = E\left(\frac{2}{n}(X_1 + X_2 + \dots + X_n)\right)$$

$$= \frac{2}{n} E(X_1 + X_2 + \dots + X_n)$$

$$= \frac{2}{n}\left[E(X_1) + E(X_2) + \dots E(X_n)\right]$$

$$= \frac{2}{n}\left[\frac{b}{2} + \frac{b}{2} + \dots \frac{b}{2}\right] \left(\frac{b}{2} \text{ n times}\right)$$

$$= \frac{2}{n} \cdot \frac{nb}{2} = b$$

Since $E(\tilde{b}(\vec{X})) = b \Rightarrow$ Estimator $\tilde{b}(\vec{x})$ is unbiased

This estimator is not reasonable since it equates to original b which means it can still contain some X_i whose value is bigger than b

f) $E(\tilde{b}(\vec{x})) = E\left(\frac{2}{3}(2 + 3 + 16)\right)$

$$= E(14) = 14 = b$$

The estimate is not reasonable since the data set has 16 which is bigger than $b = 14$

$\Rightarrow$ according to $L(b)$, this data set's probability is 0 ($16 \notin [0, 14]$

Exercise 4B4:

Let the observations $x_1 = 5$, $x_2 = 3$, $x_3 = 10$. The likelihood function is

$$L(p) = \prod_{i=1}^{3} p(1-p)^{x_i} = p^3 (1-p)^{x_1+x_2+x_3}$$

$$\Rightarrow \ell(p) = \log\left(p^3 (1-p)^{x_1+x_2+x_3}\right)$$
$$= \log(p^3) + \log\left[(1-p)^{x_1+x_2+x_3}\right]$$
$$= 3\log(p) + (x_1+x_2+x_3)\log(1-p)$$

$$\Rightarrow \ell'(p) = \frac{3}{p} - \frac{x_1+x_2+x_3}{1-p}$$

$$\Rightarrow \ell''(p) = -\frac{3}{p^2} - \frac{x_1+x_2+x_3}{(1-p)^2}$$

We have: $\ell'(p) = 0 \Rightarrow \frac{3}{p} - \frac{5+3+10}{1-p} = 0$

$$\Rightarrow \frac{3}{p} - \frac{18}{1-p} = 0$$

$$\Rightarrow 3(1-p) - 18p = 0 \Rightarrow 3 - 3p - 18p = 0 \Rightarrow p = \frac{1}{7}$$

We have: $\ell''\left(\frac{1}{7}\right) = -\frac{3}{\left(\frac{1}{7}\right)^2} - \frac{18}{\left(\frac{6}{7}\right)^2} = -\frac{342}{2} < 0$

$\Rightarrow$ maximum likelihood estimate for parameter p is $\hat{p} = \frac{1}{7}$

Since $p = \frac{1}{7}$ $\Rightarrow$ There is a probability of $\frac{1}{7}$ in succeeding in an experiment

Example: Helsinki has 631694 people, 1/7 of which (90242 people) supports the KOK party. We go to the street and ask random people.

First trial we ask 5 people and they are not supporters, until the 6^{th} person is

Second trial we ask 3 people and they are not supporters, until the 4^{th} person is

Third trial we ask 10 people and they are not supporters, until the 11^{th} person is

$\Rightarrow$ In order for this scenario to most likely occur, $\hat{p} = \frac{1}{7}$