MS-A0503 First course in probability and statistics

2B Standard deviation and correlation

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Contents

Expectation tells only about location of distribution

For a random number X, the expected value (mean) $\mu = \mathbb{E}(X)$:

- is the probability-weighted average of X's possible values, $\sum_x x\,f(x)$ or $\int x\,f(x)dx$
- is roughly a central location of the distribution
- is the approximate average of many independent random numbers that are distributed like X
- tells nothing about the width of the distribution

Example

Some discrete distributions with the same expectation 1:

k	1
$\mathbb{P}(X = k)$	1

k	0	1	2
$\mathbb{P}(Z = k)$	$\frac{1}{2}$	0	$\frac{1}{2}$

k	0	1	2
$\mathbb{P}(Y = k)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

k	0	1000000
$\mathbb{P}(W = k)$	0.999999	0.000001

How to measure the difference of X from its expectation?

The absolute difference of X from its mean $\mu = \mathbb{E}(X)$ is a random variable $|X - \mu|$.

If in dice-rolling ($\mu = 3.5$) we obtain $X = 2$, then $X - \mu = -1.5$.

The *mean absolute difference* $\mathbb{E}|X - \mu|$:

- is an approximation to the average $\frac{1}{n}\sum_{i=1}^{n}|X_i - \mu|$, from a large number of independent random numbers distributed like X
- is mathematically slightly inconvenient, because (among other things) the function $x \mapsto |x|$ is not differentiable at zero.

What if we instead use the squared difference $(X - \mu)^2$

Variance

If X has mean $\mu = \mathbb{E}(X)$, then the *squared difference* of X from the mean is a random number $(X - \mu)^2$.

If in dice rolling ($\mu = 3.5$) we obtain $X = 2$, then squared difference is $(2 - 3.5)^2 = (-1.5)^2 = 2.25$.

The expectation *of the squared difference* is called the variance of the random number X: $\mathrm{Var}(X) = \mathbb{E}[(X - \mu)^2]$:

- approximates average $\frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2$ in many repetitions
- is mathematically convenient, (among other things) because the squaring function $x \mapsto x^2$ has derivatives of all orders
- has the units of *squared* something

	X	$\mathrm{Var}(X)$
Height	m	m^2
Time	s	s^2
Sales	EUR	EUR^2

We go back to the original units by taking the square root.

Standard deviation

Standard deviation, $\mathrm{SD}(X) = \sqrt{\mathbb{E}[(X-\mu)^2]}$ is the *expectation* of the square-difference, returned back to original scale by square root.

Other notations also exist, like $\mathbb{D}(X)$.

It measures:

- (roughly, in cumbersome square-squareroot-way) how much realizations of X are expected to differ from their mean
- width of the distribution of X

For discrete distributions:

$$\mu = \sum_x x\, f(x)$$

$$\mathrm{SD}(X) = \sqrt{\sum_x (x-\mu)^2 f(x)}$$

For continuous distributions:

$$\mu = \int x\, f(x) dx$$

$$\mathrm{SD}(X) = \sqrt{\int (x-\mu)^2\, f(x) dx}$$

Example. Some distributions with mean 1

What are the standard deviations of X, Y, Z?

k	1
$\mathbb{P}(X = k)$	1

k	0	1	2
$\mathbb{P}(Y = k)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

k	0	2
$\mathbb{P}(Z = k)$	$\frac{1}{2}$	$\frac{1}{2}$

$$\mathrm{SD}(X) = \sqrt{\sum_k (k-\mu)^2 f_X(k)} = \sqrt{(1-1)^2 \times 1} = 0.$$

$$\mathrm{SD}(Y) = \sqrt{(0-1)^2 \times \frac{1}{3} + (1-1)^2 \times \frac{1}{3} + (2-1)^2 \times \frac{1}{3}} = \sqrt{\frac{2}{3}} \approx 0.82.$$

$$\mathrm{SD}(Z) = \sqrt{(0-1)^2 \times \frac{1}{2} + (1-1)^2 \times 0 + (2-1)^2 \times \frac{1}{2}} = 1.$$

Standard deviation: Alternative (equivalent) formula

Fact

If X has mean $\mu = \mathbb{E}(X)$, then it is also true that

$$\mathsf{SD}(X) \;=\; \sqrt{\mathsf{Var}(X)} \;=\; \sqrt{\mathbb{E}(X^2) - \mu^2}.$$

(This is convenient for calculation, if $\mathbb{E}(X^2)$ is easy to calculate.)

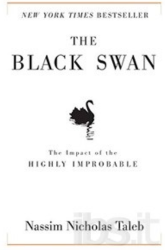
Proof.

$$\begin{aligned} \mathsf{Var}(X) \;=\; \mathbb{E}[(X-\mu)^2] \;&=\; \mathbb{E}[X^2 - 2\mu X + \mu^2] \\ &=\; \mathbb{E}[X^2] - \mathbb{E}[2\mu X] + \mathbb{E}[\mu^2] \\ &=\; \mathbb{E}[X^2] - 2\mu\mathbb{E}[X] + \mu^2 \\ &=\; \mathbb{E}[X^2] - \mu^2 \end{aligned}$$

$$\Longrightarrow \qquad \mathsf{SD}(X) \;=\; \sqrt{\mathsf{Var}(X)} \;=\; \sqrt{\mathbb{E}[X^2] - \mu^2}$$

$\square$

Example: Black swan — Two-valued distribution



k	0	10^6
$\mathbb{P}(X = k)$	$1 - 10^{-6}$	10^{-6}

$$\mu = \mathbb{E}(X) = 1$$

Calculate the standard deviation.

Method 1 (straight from the definition):

$$\begin{aligned} \mathrm{SD}(X) &= \sqrt{\sum_x (x-\mu)^2 f(x)} \\ &= \sqrt{(0-1)^2 \times (1-10^{-6}) + (10^6-1)^2 \times 10^{-6}} \approx 1000. \end{aligned}$$

Method 2 (alternative formula):

$$\mathbb{E}(X^2) = \sum_x x^2 f(x) = 0^2 \times (1-10^{-6}) + (10^6)^2 \times 10^{-6} = 10^6.$$

$$\implies \quad \mathrm{SD}(X) = \sqrt{\mathbb{E}(X^2) - \mu^2} = \sqrt{10^6 - 1^2} \approx 1000.$$

Example: Metro — Continuous uniform distribution

Waiting time X is uniformly distributed in interval $[0, 10]$. Then it has mean $\mu = 5$ (minutes). What is the standard distribution?

Method 1 (from definition):

$$\mathrm{SD}(X) = \sqrt{\int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx} = \sqrt{\int_{0}^{10} (x-5)^2 \frac{1}{10} dx} = \cdots$$

Method 2 (by alternative formula):

$$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx = \int_{0}^{10} x^2 \frac{1}{10} dx = \frac{1}{10}\left[\frac{1}{3}x^3\right]_0^{10} \approx 33.33.$$

$$\Longrightarrow \quad \mathrm{SD}(X) = \sqrt{\mathbb{E}(X^2) - \mu^2} = \sqrt{33.33 - 5^2} \approx 2.89 \text{ minutes.}$$

Std. dev. of shifted and scaled random numbers

Fact (Previous lecture)

(i) $\mathbb{E}(a) = a$.

(ii) $\mathbb{E}(bX) = b\mathbb{E}(X)$.

(iii) $\mathbb{E}(a + bX) = a + b\mathbb{E}(X)$.

Fact

(i) $\mathrm{SD}(a) = 0$.

(ii) $\mathrm{SD}(bX) = |b|\,\mathrm{SD}(X)$.

(iii) $\mathrm{SD}(a + bX) = |b|\,\mathrm{SD}(X)$.

Proof.

$$\begin{aligned}
\mathrm{Var}(a + bX) &= \mathbb{E}[\big(a + bX - \mathbb{E}[a + bX]\big)^2] \\
&= \mathbb{E}[\big(a + bX - a - b\mu\big)^2] \\
&= \mathbb{E}[\big(bX - b\mu\big)^2] \\
&= \mathbb{E}[b^2(X - \mu)^2] \;=\; b^2\mathbb{E}[(X - \mu)^2] \;=\; b^2\,\mathrm{Var}(X),
\end{aligned}$$

$$\mathrm{SD}(a + bX) \;=\; \sqrt{\mathrm{Var}(a + bX)} \;=\; \sqrt{b^2\,\mathrm{Var}(X)} \;=\; |b|\,\mathrm{SD}(X).$$

This proves (iii). Items (i) and (ii) follow as special cases. $\square$

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Chebyshev's inequality: probability of large differences

Fact (Chebyshev's inequality)

For any random variable that has mean μ *and standard deviation* σ*, it is true that the event* $\{X = \mu \pm 2\sigma\} = \{X \in [\mu - 2\sigma,\ \mu + 2\sigma]\}$ *has probability at least*

$$\mathbb{P}(X = \mu \pm 2\sigma) \geq \frac{3}{4}.$$

Pafnuty Chebyshev
1821–1894

More generally $\mathbb{P}(X = \mu \pm r\sigma) \geq 1 - \frac{1}{r^2}$ for any $r \geq 1$.

- X is rather probably ($\geq 75\%$) *within* two std. deviations from its mean
- X is very probably ($\geq 99\%$) within ten std. deviations from its mean

Chebyshev's inequality gives a *lower bound* for the "near mean" probability. For particular distributions, the real probability may be larger. (For "tail probability" we have an *upper bound*.)

Example: Document lengths

In a certain journal, word counts of articles have mean 1000 and standard deviation 200. We don't know the exact distribution. Is it probable that a randomly chosen article's word count is

(a) within $[600, 1400]$? (two std.dev. from mean)

(b) within $[800, 1200]$? (one std.dev. from mean)

Solution

(a) From Chebyshev's inequality

$$\mathbb{P}(X \in [600, 1400]) \;=\; \mathbb{P}(X = \mu \pm 2\sigma) \;\geq\; 1 - \frac{1}{2^2} \;=\; 75\%,$$

so at least 75% of articles are like this.

(b) Here Chebyshev says nothing very useful. All it says is

$$\mathbb{P}(X \in [800, 1200]) \;=\; \mathbb{P}(X = \mu \pm \sigma) \;\geq\; 1 - \frac{1}{1^2} \;=\; 0.$$

We would need better information about the actual distribution.

Example: Document lengths (take two)

In a certain journal, word counts of articles have mean 1000 and standard deviation 200. We also happen to know they are normally distributed. Is it probable that a randomly chosen article's word count is

(a) within $[600, 1400]$ (two std.dev. from mean)

(b) within $[800, 1200]$ (one std.dev. from mean)

Solution

(a) From the CDF of normal distribution (e.g. in R: `1-2*pnorm(-2)`)

$$\mathbb{P}(X \in [600, 1400]) = \mathbb{P}(X = \mu \pm 2\sigma) = \mathbb{P}(\frac{X - \mu}{\sigma} = 0 \pm 2) \approx 95\%.$$

(b) From the CDF of normal distribution (e.g. in R: `1-2*pnorm(-1)`)

$$\mathbb{P}(X \in [800, 1200]) = \mathbb{P}(X = \mu \pm \sigma) = \mathbb{P}\left(\frac{X - \mu}{\sigma} = 0 \pm 1\right) \approx 68\%.$$

We got much higher probabilities because we knew the distribution.

Example: Document lengths (take three)

In a certain journal, word counts of articles have mean 1000 and standard deviation 200; in fact, they have distribution

k	750	1000	1250
$\mathbb{P}(X = k)$	32%	36%	32%

Is it probable that a randomly chosen article's word count is

(a) within $[600, 1400]$ (two std.dev. from mean)

(b) within $[800, 1200]$ (one std.dev. from mean)

Solution

Directly from the distribution table, we see that the word count is

(a) *certainly* (100%) within $[600, 1400]$

(b) but not very probably (only 36%) within $[800, 1200]$

Food for thought: How was this example generated? We wanted a distribution that has SD=200, and two possible values symmetric around the mean. But how to choose their probabilities so that we get the SD we wanted?

Proving Chebyshev (continuous; dicrete similar)

Let $r > 0$. Suppose X has density $f(x)$, mean μ and standard deviation σ. Let MID be the interval $[\mu - r\sigma, \mu + r\sigma]$ and TAIL its complement. Now

$$\begin{aligned}
\mathrm{Var}(X) = \sigma^2 &= \int_{\mathbb{R}} (x-\mu)^2 f(x) dx = \int_{\mathrm{MID}} (\dots) + \int_{\mathrm{TAIL}} (\dots) \\
&\geq \int_{\mathrm{TAIL}} (x-\mu)^2 f(x) dx \quad \geq \quad \int_{\mathrm{TAIL}} (r\sigma)^2 f(x) dx \\
&= r^2\sigma^2 \int_{\mathrm{TAIL}} f(x) dx = r^2\sigma^2\, \mathbb{P}(x \in \mathrm{TAIL}).
\end{aligned}$$

Cancel σ^2 and move r^2 to other side:

$$\mathbb{P}(X \in \mathrm{TAIL}) \leq \frac{1}{r^2}.$$

Note: From Chebyshev, one can actually prove the (Weak) Law of Large Numbers. One extra ingredient is needed, namely the variance of a sum; see next lecture and https://en.wikipedia.org/wiki/Law_of_large_numbers

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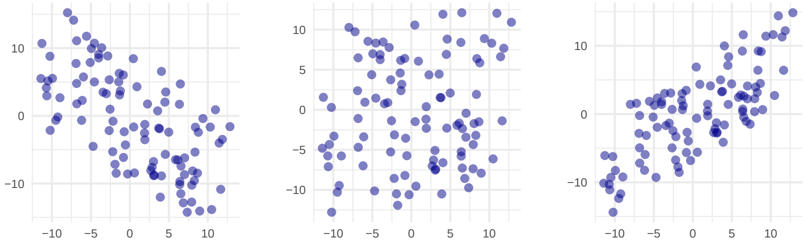
Shape of the joint distribution

Standard deviation measures the dispersion of *one* r.v. around its mean.

For two random variables, we would like to know X and Y typically differ (from their means) *to the same direction* and how strong this effect is.



Covariance

$\text{Cov}(X, Y) = \mathbb{E}[(X - \mu_X)(Y - \mu_Y)]$, measures how strongly X and Y vary in the same direction.

Discrete

$$\sum_x \sum_y (x - \mu_X)(y - \mu_Y)\, f(x, y)$$

Continuous

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mu_X)(y - \mu_Y)\, f(x, y) dxdy.$$

The covariance

- is > 0, if $X - \mu_X$ and $Y - \mu_Y$ have often the same sign
- is < 0, if $X - \mu_X$ and $Y - \mu_Y$ have often opposite signs
- like variance, is in square units (m^2, s^2, EUR^2, ...)

But now we do not want to take the square root (why)? (Can be neg.)

Covariance: Alternative formula

Often more convenient in calculations than the definition.

Fact

$\mathrm{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$.

Proof.

$$
\begin{aligned}
\mathrm{Cov}(X, Y) &= \mathbb{E}[(X - \mu_X)(Y - \mu_Y)] \\
&= \mathbb{E}[XY - \mu_X Y - \mu_Y X + \mu_X \mu_Y] \\
&= \mathbb{E}[XY] - \mu_X \mathbb{E}[Y] - \mu_Y \mathbb{E}[X] + \mathbb{E}[\mu_X \mu_Y] \\
&= \mathbb{E}[XY] - \mu_X \mu_Y - \mu_Y \mu_X + \mu_X \mu_Y \\
&= \mathbb{E}[XY] - \mu_X \mu_Y .
\end{aligned}
$$

$\square$

Symmetry and linearity of covariance

Fact

The covariance $\mathrm{Cov}(X, Y)$ *is symmetric and linear in each of its arguments:*

$\mathrm{Cov}(Y, X) = \mathrm{Cov}(X, Y)$
$\mathrm{Cov}(X_1 + X_2, Y) = \mathrm{Cov}(X_1, Y) + \mathrm{Cov}(X_2, Y).$
$\mathrm{Cov}(X, Y_1 + Y_2) = \mathrm{Cov}(X, Y_1) + \mathrm{Cov}(X, Y_2).$
$\mathrm{Cov}(aX, Y) = a\,\mathrm{Cov}(X, Y)$

More generally:

$$\mathrm{Cov}\left(\sum_{i=1}^{m} a_i X_i, \sum_{j=1}^{n} b_j Y_j\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_i b_j\,\mathrm{Cov}(X_i, Y_j)$$

Proving linearity of covariance

Let's denote $Y = \sum_{j=1}^{n} b_j Y_j$. Using the "alternative formula" of covariance, and linearity of expectation,

$$\begin{aligned}
\mathrm{Cov}\left(\sum_i a_i X_i, Y\right) &= \mathbb{E}[(\sum_i a_i X_i) Y] - \mathbb{E}[(\sum_i a_i X_i)]\mathbb{E}[Y] \\
&= \sum_i a_i \mathbb{E}[X_i Y] - \left(\sum_i a_i \mathbb{E}[X_i]\right) \mathbb{E}[Y] \\
&= \sum_i a_i \mathbb{E}[X_i Y] - \sum_i a_i \mathbb{E}[X_i]\mathbb{E}[Y] \\
&= \sum_i a_i \left(\mathbb{E}[X_i Y] - \mathbb{E}[X_i]\mathbb{E}[Y]\right) = \sum_i a_i \,\mathrm{Cov}(X_i, Y).
\end{aligned}$$

By symmetry and the above, we obtain

$$\begin{aligned}
\sum_i a_i \,\mathrm{Cov}(X_i, Y) &= \sum_i a_i \,\mathrm{Cov}(Y, X_i) \\
&= \sum_i a_i \,\mathrm{Cov}(\sum_j b_j Y_j, X_i) \\
&= \sum_i a_i \sum_j b_j \,\mathrm{Cov}(Y_j, X_i) \\
&= \sum_i \sum_j a_i b_j \,\mathrm{Cov}(X_i, Y_j).
\end{aligned}$$

Covariance: Summary

The covariance of random variables X and Y is

$$\text{Cov}(X, Y) = \mathbb{E}[(X - \mu_X)(Y - \mu_Y)] = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

where $\mu_X = \mathbb{E}(X)$ ja $\mu_Y = \mathbb{E}(Y)$.

Discrete

$$\sum_x \sum_y (x - \mu_X)(y - \mu_Y) f(x, y)$$

Continuous

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mu_X)(y - \mu_Y) f(x, y) dx dy.$$

Covariance is symmetric and linear:

$$\text{Cov}(Y, X) = \text{Cov}(X, Y)$$

$$\text{Cov}\left(\sum_{i=1}^{m} a_i X_i, \sum_{j=1}^{n} b_j Y_j\right) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_i b_j \, \text{Cov}(X_i, Y_j)$$

Correlation (coefficient)

It would be awkward to "normalize" covariance by square root (because covariance can be negative).

Also, we would like to know the covariance *relative* to the scaling of the two variables. (Think what happens to covariance if both variables multiplied by 1000.)

Here we apply a different kind of normalization ...

Correlation (coefficient)

$$\mathrm{Cor}(X, Y) \;=\; \frac{\mathrm{Cov}(X, Y)}{\mathrm{SD}(X)\,\mathrm{SD}(Y)}$$

measures how X and Y vary jointly, in *normalized* units.

Independent random numbers are uncorrelated

Fact

If X and Y are (stochastically) independent, then $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$ *and* $\mathrm{Cor}(X, Y) = 0$.

Proof.

In the discrete case:

$$\begin{aligned}
\mathbb{E}(XY) &= \sum_x \sum_y xy \; f_{X,Y}(x, y) \\
&= \sum_x \sum_y xy \; f_X(x) f_Y(y) \\
&= \left(\sum_x x \; f_X(x) \right) \left(\sum_y y \; f_Y(y) \right) \; = \; \mathbb{E}(X)\mathbb{E}(Y).
\end{aligned}$$

Applying the covariance formula
$\mathrm{Cov}(X, Y) \; = \; \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) \; = \; \mathbb{E}(X)\mathbb{E}(Y) - \mathbb{E}(X)\mathbb{E}(Y) \; = \; 0.$
Thus also $\mathrm{Cor}(X, Y) = 0$. $\square$

Example. Two binary random variables

X and Y are both uniformly distributed among two values $\{-1, +1\}$.

More over $\mathbb{P}(X = +1, Y = +1) \;=\; c$.

Find joint distribution and correlation.

		Y: -1	Y: $+1$	Sum
X	-1	c	$\frac{1}{2} - c$	$\frac{1}{2}$
	$+1$	$\frac{1}{2} - c$	c	$\frac{1}{2}$
Sum		$\frac{1}{2}$	$\frac{1}{2}$	

$\mathbb{E}(X) = 0$
$\mathbb{E}(X^2) = (-1)^2 \times \frac{1}{2} + (+1)^2 \times \frac{1}{2} = 1$
$\mathrm{SD}(X) = \sqrt{\mathbb{E}(X^2) - (\mathbb{E}(X))^2} = \sqrt{1 - 0^2} = 1$

$\mathbb{E}(Y) = \mathbb{E}(X) = 0$, $\mathrm{SD}(Y) = \mathrm{SD}(X) = 1$.

$\mathbb{E}(XY) \;=\; (-1)^2 \times c \;+\; 2 \times (-1)(+1) \times (\frac{1}{2} - c) \;+\; (+1)^2 c \;=\; 4c - 1$
$\mathrm{Cov}(X, Y) \;=\; \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) \;=\; 4c - 1$

$$\mathrm{Cor}(X, Y) \;=\; \frac{\mathrm{Cov}(X, Y)}{\mathrm{SD}(X)\,\mathrm{SD}(Y)} \;=\; 4c - 1$$

Example. Linear *deterministic* dependence

Suppose we have two random variables X, Y such that always $Y = a + bX$ (exactly!), and X has some distribution with mean $\mathbb{E}(X) = \mu$ and standard deviation $\mathrm{SD}(X) = \sigma$.

Calculate the correlation of X and Y.

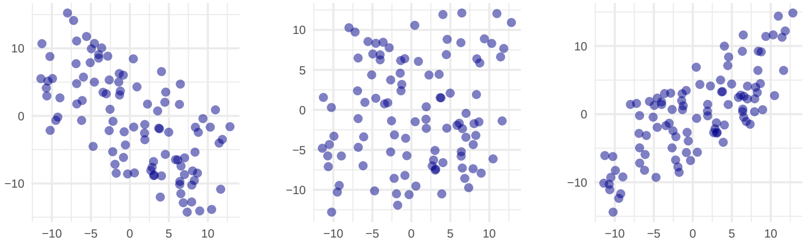
$$\mathrm{Cov}(X, Y) \;=\; \mathrm{Cov}(X, a + bX) \;=\; \mathrm{Cov}(X, a) + \mathrm{Cov}(X, bX) \;=\; b\,\mathrm{Var}(X).$$

$$\mathrm{SD}(Y) \;=\; \mathrm{SD}(a + bX) \;=\; |b|\,\mathrm{SD}(X)$$

$$\mathrm{Cor}(X, Y) \;=\; \frac{\mathrm{Cov}(X, Y)}{\mathrm{SD}(X)\,\mathrm{SD}(Y)} \;=\; \frac{b\,\mathrm{Var}(X)}{|b|\,\mathrm{SD}(X)^2} \;=\; \frac{b}{|b|}.$$

$$\mathrm{Cor}(X, Y) \;=\; \begin{cases} +1, & \text{if } b > 0, \\ 0, & \text{if } b = 0, \\ -1, & \text{if } b < 0. \end{cases}$$

(x, y) pairs drawn from some correlated distributions



$\rho = -0.60$ $\rho = 0.28$ $\rho = 0.80$

Next lecture is about sums of (many) random variables, and normal approximation...