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Exercise 5B3:

- Leaf length has normal distribution with unknown mean & $\sigma = 2$
- Prior distribution for Θ is normal with $\mu_0 = 10$, $\sigma_0 = 1$
- Measure $\vec{x} = (9, 13, 14, 12, 17)$

a) Find posterior mean of Θ

We have : $m(\vec{x}) = \dfrac{9 + 13 + 14 + 12 + 17}{5} = 13$

$\Rightarrow$ The posterior mean of Θ is

$$\mu_1 = \frac{\frac{1}{\sigma_0^2}\mu_0 + \frac{n}{\sigma^2} m(\vec{x})}{\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}} = \frac{\frac{1}{1^2}.10 + \frac{5}{2^2}.13}{\frac{1}{1^2} + \frac{5}{2^2}} = \frac{\frac{105}{4}}{\frac{9}{4}} = \frac{35}{3} \approx 11.66$$

b) Find an interval that contains Θ with prob 90%

The posterior standard deviation is

$$\sigma_1 = \frac{1}{\sqrt{\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}}} = \frac{1}{\sqrt{\frac{1}{1^2} + \frac{5}{2^2}}} = \frac{2}{3}$$

The interval is Θ, at probability 90%

$$\Rightarrow P(\Theta = \mu_1 \pm c) = 0.9$$

$$\Rightarrow P\left(\frac{\Theta - \mu_1}{\sigma_1} = \pm \frac{c}{\sigma_1}\right) = 0.9$$

$$\Rightarrow P\left(|Z| \leq \frac{c}{\sigma_1}\right) = 0.9$$

$$\Rightarrow \frac{c}{\sigma_1} = 1.64 \Rightarrow c = 1.64\,\sigma_1 = 1.64 \times \frac{2}{3} \approx 1.0933$$

$$\Rightarrow \text{Interval } \Theta = \mu_1 \pm c = \frac{35}{3} \pm 1.0933 = [10.5733 , 12.76]$$

is the interval that includes leaf length at 90% probability

Exercise 5B4: (Dangerous Road)

- Number of accidents follow the Poisson distribution

$$f(x|\theta) = \frac{e^{-\theta}\theta^x}{x!}, \quad x = 0, 1, 2, \ldots \text{ (car accidents)}$$

- θ unknown, $\theta > 0$
- Prior distribution for unknown θ

θ	1	2	3
$f_\theta(\theta)$	0.25	0.5	0.25

a) There are 2 accidents in 1st month. Find posterior probabilities for $\theta = 1, 2, 3$

- Likelihood distribution

$$\theta = 1 \Rightarrow f_\theta(2|1) \hat{=} e^{-1}\frac{1^2}{2!} = e^{-1} = 0.184$$

$$\theta = 2 \Rightarrow f_\theta(2|2) \hat{=} e^{-2}\frac{2^2}{2!} = 2e^{-2} = 0.27$$

$$\theta = 3 \Rightarrow f_\theta(2|3) = e^{-3}\frac{3^2}{2!} = \frac{9}{2}e^{-3} = 0.224$$

θ	1	2	3
$f(2\|\theta)$	0.184	0.27	0.224

- Posterior distribution

$f_\theta(1) f(2|1) = 0.184 \times 0.25 = 0.046$

$f_\theta(2) f(2|2) = 0.27 \times 0.5 = 0.135$

$f_\theta(3) f(2|3) = 0.224 \times 0.25 = 0.056$

$c = 0.046 + 0.135 + 0.056 = 0.237$

Unnormalized

θ	1	2	3
$f(\theta)f(2\|\theta)$	0.046	0.135	0.056

Normalized

θ	1	2	3
$f(\theta)f(2\|\theta)$	0.194	0.57	0.236

b) After 2 accidents in 1st month, there are 0 in 2nd month. Find posterior distribution

- Likelihood distribution

$$\theta = 1 \Rightarrow f_\theta(2 \& 0 | 1) = \left(e^{-1}\frac{1^2}{2!}\right)\left(e^{-1}\frac{1^0}{0!}\right) = 0.0677$$

$$\theta = 2 \Rightarrow f_\theta(2 \& 0 | 2) = \left(e^{-2}\frac{2^2}{2!}\right)\left(e^{-2}\frac{2^0}{0!}\right) = 0.0366$$

$$\theta = 3 \Rightarrow f_\theta(2 \& 0 | 3) = \left(e^{-3}\frac{3^2}{2!}\right)\left(e^{-3}\frac{3^0}{0!}\right) = 0.0112$$

θ	1	2	3
$f(\theta)f(2\&0\|\theta)$	0.0677	0.0366	0.0112

- Posterior distribution

$f_\theta(1) f_\theta(2\&0|1) = 0.25 \times 0.0677 = 0.0169$

$f_\theta(2) f_\theta(2\&0|2) = 0.5 \times 0.0366 = 0.0183$

$f_\theta(3) f_\theta(2\&0|3) = 0.25 \times 0.0112 = 0.0028$

$c = 0.0169 + 0.0183 + 0.0028 = 0.038$

Unnormalized

θ	1	2	3
$f_\theta(\theta)f_\theta(2\&0\|\theta)$	0.0169	0.0183	0.0028

Normalized

θ	1	2	3
$f_\theta(\theta)f_\theta(2\&0\|\theta)$	0.4447	0.4816	0.0737