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Exercise 3A3 : Predicting temperatures

Equation: $\underset{\text{(tomorrow temp)}}{T_1} = \underset{\text{(today temp)}}{T_0} + \Delta T$ ← change in temp $\left.\begin{matrix} T_0 \\ \Delta T \end{matrix}\right\}$ independent

$E(T_0) = \mu$, $Var(T_0) = \sigma^2$, $E(\Delta T) = 0$ $Var(\Delta T) = \theta^2$

μ, σ, θ are known, $\sigma > 0$, $\theta \geq 0$

a) Find $E(T_1)$

$$E(T_1) = E(T_0 + \Delta T) = E(T_0) + E(\Delta T) = \mu + 0 = \mu$$

b) Find $SD(T_1)$

$SD(T_1) = SD(T_0 + \Delta T) = \sqrt{Var(T_0 + \Delta T)}$

Since T_0 and ΔT are independent

$\Rightarrow \sqrt{Var(T_0 + \Delta T)} = \sqrt{Var(T_0) + Var(\Delta T)}$

$= \sqrt{\sigma^2 + \theta^2}$

$\Rightarrow SD(T_1) = \sqrt{\sigma^2 + \theta^2}$

c) Find $Cov(T_1, T_0)$

$Cov(T_1, T_0) = Cov(T_0 + \Delta T, T_0)$

$= Cov(T_0, T_0) + Cov(T_0, \Delta T)$

$= Var(T_0) + 0$ (Since T_0 and ΔT are independent)

$= \sigma^2$

d) Find $Cor(T_1, T_0)$

$$Cor(T_1, T_0) = \frac{Cov(T_1, T_0)}{SD(T_1)\,SD(T_0)} = \frac{\sigma^2}{\sqrt{\sigma^2+\theta^2}\cdot\sigma} = \frac{\sigma}{\sqrt{\sigma^2+\theta^2}}$$

$\Rightarrow$ If θ becomes very small or $\theta = 0 \Rightarrow Cor(T_1, T_0) \to \sigma$

If θ is much larger than $\sigma \Rightarrow Cor(T_1, T_0) \to 0$

Exercise 3A4: Minimizing last function

a) Find the mean $\mu = E(X)$

$$E(X) = \int_0^2 x f(x)dx = \int_0^2 x \cdot \frac{x}{2} dx = \frac{1}{2}\int_0^2 x^2 dx$$

$$= \frac{1}{2} \cdot \frac{x^3}{3}\Big|_0^2 = \frac{4}{3}$$

Find the median m, such that $P(X \leq m) = \frac{1}{2}$

$$P(X \leq m) = \int_0^m \frac{x}{2} dx = \frac{1}{2} \Rightarrow \frac{x^2}{4}\Big|_0^m = \frac{1}{2} \Rightarrow m = \sqrt{2}$$

b) Find $E(X^2)$ and $SD(X)$

$$E(X^2) = \int_0^2 x^2 \frac{x}{2} dx = \int_0^2 \frac{1}{2} x^3 dx = \frac{1}{2} \cdot \frac{x^4}{4}\Big|_0^2 = 2$$

$$\Rightarrow Var(X) = E(X^2) - (E(X))^2 = 2 - \left(\frac{4}{3}\right)^2 = \frac{2}{9}$$

$$\Rightarrow SD(X) = \sqrt{Var(X)} = \frac{\sqrt{2}}{3}$$

c)
$$q(c) = E((X-c)^2) = E((X^2 - 2Xc + c^2))$$
$$= E(X^2) - E(2Xc) + E(c^2)$$
$$= E(X^2) - 2c\,E(X) + c^2 \text{ (since } c \text{ is a constant)}$$
$$= 2 - 2c \cdot \frac{4}{3} + c^2 = c^2 - \frac{8}{3}c + 2$$

$\Rightarrow q(c) = c^2 - \frac{8}{3}c + 2 \Rightarrow$ Shape of $q(c)$ is a parabola

$$q(c) = c^2 - 2\cdot\frac{4}{3}\cdot c + \left(\frac{4}{3}\right)^2 + \frac{2}{9} = \left(c - \frac{4}{3}\right)^2 + \frac{2}{9} \geq \frac{2}{9}$$

$\Rightarrow c = \frac{4}{3}$ so that $q(c)$ is minimized

We discover that $c = \frac{4}{3} = \mu$

d) $\ell(c) = E(|X-c|) = \int_0^2 |x-c| \frac{x}{2} dx$

$= \int_0^c (c-x)\frac{x}{2}dx + \int_c^2 (x-c)\frac{x}{2}dx$

$= \frac{c^3}{12} + \frac{1}{12}(c-2)^2(c+4)$

$= \frac{c^3}{6} - c + \frac{4}{3} \Rightarrow \ell(c)$ doesn't have minimum, but does have extrema

$\ell'(c) = \frac{c^2}{2} = 0 \Rightarrow c = \pm\sqrt{2}$

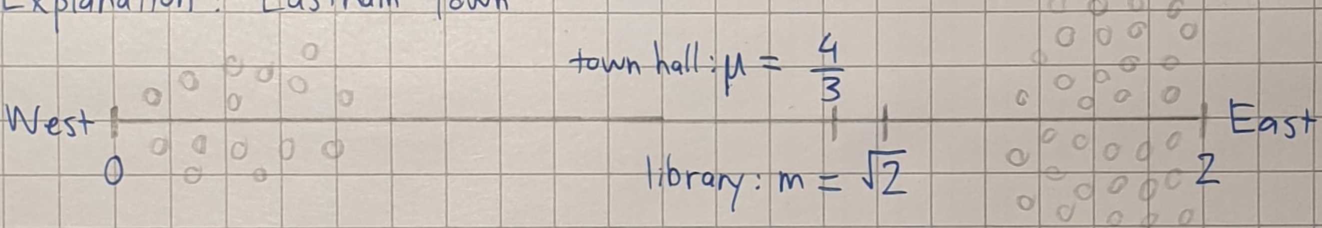
$\ell''(c) = 2c \Rightarrow \ell(c)$ reach minima at $c = \sqrt{2}$ and maxima at $c = -\sqrt{2}$

Since $c = \sqrt{2}$, it satisfies that $0 \leq x \leq 2$

$\Rightarrow$ at $c = \sqrt{2}$, $\ell(c)$ is minimized on the interval $[0; 2]$

$\Rightarrow$ We discover that $c = \sqrt{2} = m$

Explanation: Eastham Town



The location Abel chose is smallest squared distances to all households. Since the distance is squared, houses far from the town hall in the West side makes the average larger although its numbers are less than the East side, thus pulling the town hall closer to the west than the library

The location Bertha choose is smallest linear distances to all households. Since the distance is the same, the houses in the east outnumber those in the west, thus making the library closer to the east side than the town hall