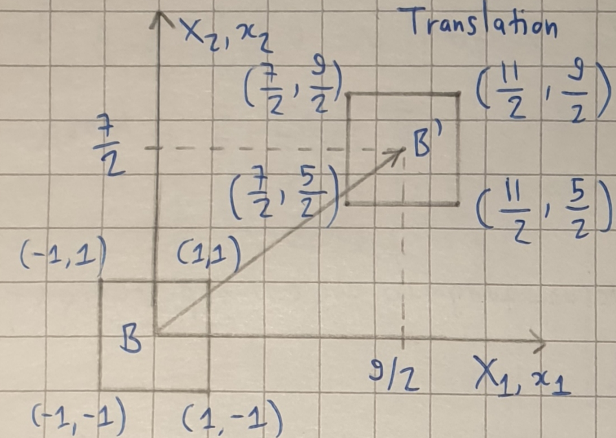
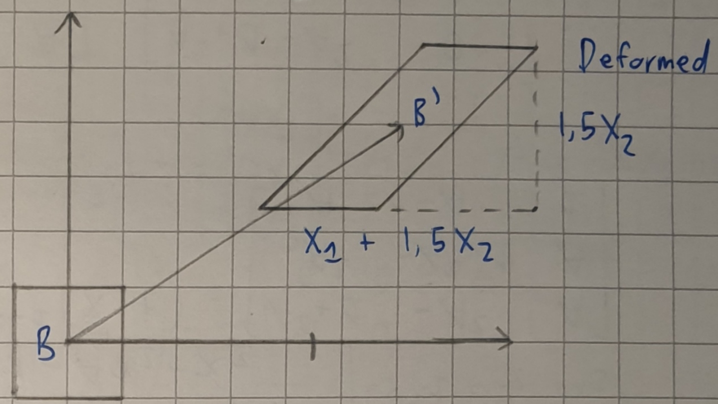
Nguyen Xuan Binh 887799 Assignment Week 2

Exercise 1

a) Sketch the deformed configuration of B

$$\vec{x} = \begin{cases} x_1 = \frac{1}{4}(18 + 4X_1 + 6X_2) \Rightarrow x_1 = X_1 + 1{,}5X_2 + 4{,}5 \\ x_2 = \frac{1}{4}(14 + 6X_2) \Rightarrow x_2 = 1{,}5X_2 + 3{,}5 \end{cases}$$





b) Compute the components of the deformation gradient tensor (material) $\underline{\underline{F}}$

Deformation gradient tensor : $[\underline{\underline{F}}] = \begin{bmatrix} \frac{\partial x_1}{X_1} & \frac{\partial x_1}{X_2} \\ \frac{\partial x_2}{X_1} & \frac{\partial x_2}{X_2} \end{bmatrix} = \begin{bmatrix} 1 & 1{,}5 \\ 0 & 1{,}5 \end{bmatrix} = \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & \frac{3}{2} \end{bmatrix}$

$\Rightarrow$ Components : $F_1 = \begin{bmatrix} 1 \\ 3/2 \end{bmatrix}$, $F_2 = \begin{bmatrix} 0 \\ 3/2 \end{bmatrix}$

Inverse deformation gradient tensor $[\underline{\underline{F}}]^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & \frac{2}{3} \end{bmatrix}$

$\Rightarrow$ Components : $F_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $F_2 = \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix}$

c) The material strain tensor

$$\underline{\underline{E}} = \frac{1}{2}(\underline{\underline{F}}^T \cdot \underline{\underline{F}} - \underline{\underline{1}}) = \frac{1}{2}\left(\begin{bmatrix} 1 & 0 \\ 3/2 & 3/2 \end{bmatrix}\begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 3/2 \end{bmatrix} - \underline{\underline{1}}\right) = \frac{1}{2}\left(\begin{bmatrix} 1 & 3/2 \\ 3/2 & 9/2 \end{bmatrix} - \underline{\underline{1}}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 3/2 \\ \frac{3}{2} & 7/2 \end{bmatrix} = \begin{bmatrix} 0 & 3/4 \\ 3/4 & 7/4 \end{bmatrix} \text{ (Answer)}$$

The spatial strain tensor

$$\underline{\underline{e}} = \frac{1}{2}(\underline{\underline{1}} - \underline{\underline{F}}^{-T} \cdot \underline{\underline{F}}^{-1}) = \frac{1}{2}\left(\underline{\underline{1}} - \begin{bmatrix} 1 & 0 \\ -1 & 2/3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 2/3 \end{bmatrix}\right) = \frac{1}{2}\left(\underline{\underline{1}} - \begin{bmatrix} 1 & -1 \\ -1 & 13/9 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 0 & -1 \\ -1 & 4/9 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{2}{9} \end{bmatrix}$$

d) Compute the stretch and unit elongation in material descriptions of the sides of square block

(i) $(-1, 1)$ and $(-1, -1) \Rightarrow \vec{v} = [0, -2]^T \Rightarrow dS = |\vec{v}| = 2$

We have $\begin{bmatrix} 1 & 3/2 \\ 0 & 3/2 \end{bmatrix}\begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \end{bmatrix} \Rightarrow ds = \sqrt{(-3)^2 + (-3)^2} = 3\sqrt{2}$

$$\Rightarrow \begin{cases} \text{Stretch: } \lambda = \dfrac{ds}{dS} = \dfrac{3\sqrt{2}}{2} \approx 2.121 \\ \text{Unit elongation: } \varepsilon = \dfrac{ds - dS}{dS} = \dfrac{3\sqrt{2} - 2}{2} \approx 1.121 \end{cases}$$

(ii) $(-1, 1)$ and $(1, -1) \Rightarrow \vec{v} = [2, -2] \Rightarrow |\vec{v}| = dS = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$

We have: $\begin{bmatrix} 1 & 3/2 \\ 0 & 3/2 \end{bmatrix} \begin{bmatrix} -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \Rightarrow ds = \sqrt{1^2 + 3^2} = \sqrt{10}$

$$\Rightarrow \begin{cases} \text{Stretch: } \lambda = \dfrac{ds}{dS} = \dfrac{\sqrt{10}}{2\sqrt{2}} = \dfrac{\sqrt{5}}{2} \approx 1.11 \\ \text{Unit elongation } \varepsilon = \dfrac{ds - dS}{dS} = \dfrac{\sqrt{10} - 2\sqrt{2}}{2\sqrt{2}} = \dfrac{\sqrt{5}}{2} - 1 \approx 0.11 \end{cases}$$

Exercise 2: Let $\begin{cases} x_1 = X_1 - AX_3 \\ x_2 = X_2 - AX_3 \\ x_3 = -AX_1 + AX_2 + X_3 \end{cases}$

a) Obtain deformation gradient tensor (material)

$$\underline{\underline{F}}(\vec{X}, t) = [x_i][\vec{\nabla}_i]^T = \begin{bmatrix} X_1 - AX_3 \\ X_2 - AX_3 \\ -AX_1 + AX_2 + X_3 \end{bmatrix} \left[\frac{\partial}{\partial X_1}, \frac{\partial}{\partial X_2}, \frac{\partial}{\partial X_3}\right] = \begin{bmatrix} 1 & 0 & -A \\ 0 & 1 & -A \\ -A & A & 1 \end{bmatrix}$$

b) We have $\begin{cases} x_1 = X_1 - AX_3 \\ x_2 = X_2 - AX_3 \\ x_3 = -AX_1 + AX_2 + X_3 \end{cases} \Rightarrow \begin{cases} x_1 - x_2 = X_1 - X_2 \\ x_3 = -A(X_1 - X_2) + X_3 \\ x_2 = X_2 - AX_3 \end{cases}$

$$\Rightarrow \begin{cases} X_1 = x_1 + A(x_3 + A(x_1 - x_2)) \\ X_2 = x_2 + A(x_3 + A(x_1 - x_2)) \\ X_3 = A(x_1 - x_2) + x_3 \end{cases} \Rightarrow \begin{cases} X_1 = (A^2 + 1)x_1 - A^2 x_2 + Ax_3 \\ X_2 = A^2 x_1 + (1 - A^2)x_2 + Ax_3 \\ X_3 = Ax_1 - Ax_2 + x_3 \end{cases}$$

$$\Rightarrow \underline{\underline{F}}^{-1}(\vec{X}, t) = [X_i][\vec{\nabla}_{x_i}]^T = \begin{bmatrix} (A^2+1)x_1 - A^2 x_2 + Ax_3 \\ A^2 x_1 + (1 - A^2)x_2 + Ax_3 \\ Ax_1 - Ax_2 + x_3 \end{bmatrix} \left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right]$$

$\Rightarrow$ Spatial deformation gradient tensor: $\underline{\underline{F}}^{-1}(\vec{X}, t) = \begin{bmatrix} A^2+1 & -A^2 & A \\ A^2 & -A^2+1 & A \\ A & -A & 1 \end{bmatrix}$

Verify if $\underline{\underline{F}} \cdot \underline{\underline{F}}^{-1} = \underline{\underline{1}}$

$$\underline{\underline{F}} \cdot \underline{\underline{F}}^{-1} = \begin{bmatrix} 1 & 0 & -A \\ 0 & 1 & -A \\ -A & A & 1 \end{bmatrix} \begin{bmatrix} A^2+1 & -A^2 & A \\ A^2 & -A^2+1 & A \\ A & -A & 1 \end{bmatrix} = \begin{bmatrix} A^2 + 1 - A^2 & 0 & 0 \\ 0 & -A^2 \cdot 0 + 1 - A^2 - A^2 & 0 \\ 0 & 0 & -A^2 + A^2 + 1 \end{bmatrix}$$

$$\Rightarrow \underline{\underline{F}} \cdot \underline{\underline{F}}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underline{\underline{1}} \text{ (verified)}$$

c) Displacement

$$\vec{u}(\vec{x},t) = \vec{x} - \vec{X}(\vec{x},t) = \begin{bmatrix} X_1 - AX_3 \\ X_2 - AX_3 \\ -AX_1 + AX_2 + X_3 \end{bmatrix} - \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} -AX_3 \\ -AX_3 \\ -AX_1 + AX_2 \end{bmatrix}$$

$\Rightarrow$ When $A = 0$, there is no motion at all, and the closer $A \to 0$, the motion constitutes an infinitesimal strain tensor

The infinitesimal strain tensor

$$\underline{\underline{\varepsilon}} = \frac{1}{2}(\underline{\underline{d}} + \underline{\underline{d}}^T) = \frac{1}{2}(\vec{u} \otimes \overleftarrow{\nabla} + \vec{\nabla} \otimes \vec{u})$$

$$= \frac{1}{2}\left(\begin{bmatrix} -AX_3 \\ -AX_3 \\ -AX_1 + AX_2 \end{bmatrix}\begin{bmatrix} \frac{\partial}{\partial X_1}, \frac{\partial}{\partial X_2}, \frac{\partial}{\partial X_3} \end{bmatrix} + \begin{bmatrix} \partial/\partial X_1 \\ \partial/\partial X_2 \\ \partial/\partial X_3 \end{bmatrix}\begin{bmatrix} -AX_3 & -AX_3 & -AX_1 + AX_2 \end{bmatrix}\right)$$

$$= \frac{1}{2}\left(\begin{bmatrix} 0 & 0 & -A \\ 0 & 0 & -A \\ -A & A & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -A \\ 0 & 0 & A \\ -A & -A & 0 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 0 & 0 & -2A \\ 0 & 0 & 0 \\ -2A & 0 & 0 \end{bmatrix}$$

d) Obtain the material and spatial strain tensors

Material strain tensors

$$\underline{\underline{E}}(\vec{X},t) = \frac{1}{2}[\underline{\underline{F}}^T \cdot \underline{\underline{F}} - \underline{\underline{1}}] = \frac{1}{2}\left(\begin{bmatrix} 1 & 0 & -A \\ 0 & 1 & A \\ -A & -A & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -A \\ 0 & 1 & -A \\ -A & A & 1 \end{bmatrix} - \underline{\underline{1}}\right)$$

$$= \frac{1}{2}\left(\begin{bmatrix} A^2+1 & -A^2 & -2A \\ -A^2 & A^2+1 & 0 \\ -2A & 0 & 2A^2+1 \end{bmatrix} - \underline{\underline{1}}\right) = \frac{1}{2}\begin{bmatrix} A^2 & -A^2 & -2A \\ -A^2 & A^2 & 0 \\ -2A & 0 & 2A^2 \end{bmatrix}$$

Spatial strain tensors

$$\underline{\underline{e}}(\vec{X},t) = \frac{1}{2}[\underline{\underline{1}} - \underline{\underline{F}}^{-T} \cdot \underline{\underline{F}}^{-1}] = \frac{1}{2}\left(\underline{\underline{1}} - \begin{bmatrix} A^2+1 & A^2 & A \\ -A^2 & -A^2+1 & -A \\ A & A & 1 \end{bmatrix}\begin{bmatrix} A^2+1 & -A^2 & A \\ A^2 & -A^2+1 & A \\ A & -A & 1 \end{bmatrix}\right)$$

$$= \frac{1}{2}\left(\underline{\underline{1}} - \begin{bmatrix} 2A^4+3A^2+1 & -2A^4-A^2 & 2A^3+2A \\ -2A^4-A^2 & 2A^4-A^2+1 & -2A^3 \\ 2A^3+2A & -2A^3 & 2A^2+1 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 2A^4+3A^2 & -2A^4-A^2 & 2A^3+2A \\ -2A^4-A^2 & 2A^4-2A^2 & -2A^3 \\ 2A^3+2A & -2A^3 & 2A^2 \end{bmatrix}$$

e) The order of material strain tensors is 2 while spatial strain tensor is 4, meanwhile the order of infinitesimal strain tensor is 1

$\Rightarrow$ According to infinitesimal hypothesis, if the displacement is infinitesimally small, then the displacement can be regarded as not occuring. This is why the order of infinitesimal strain theory is lower than material and spatial strain tensors

Exercise 3: Let $\vec{x}(\vec{X},t) = \begin{cases} X + Yt^2 \\ Y(1+t) \\ Ze^t \end{cases}$

- The material deformation gradient tensor

$$\underline{\underline{F}}(\vec{X},t) = [x_i][\nabla_{X_i}]^T = \begin{bmatrix} X + Yt^2 \\ Y(1+t) \\ Ze^t \end{bmatrix}\left[\frac{\partial}{\partial X}, \frac{\partial}{\partial Y}, \frac{\partial}{\partial Z}\right] = \begin{bmatrix} 1 & t^2 & 0 \\ 0 & 1+t & 0 \\ 0 & 0 & e^t \end{bmatrix}$$

- The spatial deformation gradient tensor

$$\begin{cases} x_1 = X + Yt^2 \\ x_2 = Y(1+t) \\ x_3 = Ze^t \end{cases} \Rightarrow \begin{cases} X = x_1 - t^2 (x_2/(1+t)) \\ Y = \dfrac{x_2}{1+t} \\ Z = \dfrac{1}{e^t} x_3 \end{cases}$$

$$\Rightarrow \underline{\underline{F}}^{-1}(X,t) = [X_i][\nabla x_i]^T = \begin{bmatrix} x_1 - \dfrac{x_2 t^2}{1+t} \\ \dfrac{x_2}{1+t} \\ \dfrac{1}{e^t} x_3 \end{bmatrix}\left[\frac{\partial}{\partial x_1} \quad \frac{\partial}{\partial x_2} \quad \frac{\partial}{\partial x_3}\right] = \begin{bmatrix} 1 & -\dfrac{t^2}{1+t} & 0 \\ 0 & \dfrac{1}{1+t} & 0 \\ 0 & 0 & \dfrac{1}{et} \end{bmatrix}$$

Exercise 4: The spatial velocity field components are given as follows

$$\begin{cases} v_1 = x_1 \\ v_2 = \dfrac{x_3}{2t+3} \\ v_3 = 0 \end{cases}$$

Find equation of motion of the particle, which was at the reference configuration at (X_1, X_2, X_3)

$$\begin{cases} v_1 = x_1 \\ v_2 = x_3/(2t+3) \\ v_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = v_1 \\ x_3 = v_2(2t+3) \\ x_2 = 0 \end{cases} \Rightarrow \text{Equation of motion} \begin{cases} x_1 = X_1 \\ x_2 = 0 \\ x_3 = X\ (2t+3) \end{cases}$$

Exercise 5: Consider $\vec{x} = \begin{cases} x_1 = X_1 \\ x_2 = X_2 + \frac{t}{2} X_3 \\ x_3 = X_3 + \frac{t}{2} X_2 \end{cases}$

a) Is this motion possible ?

- Consistency: $\phi(\vec{X},0) = \begin{bmatrix} X_1 \\ X_2 + 0X_3 \\ X_3 + 0X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \vec{X}$ (✓)
- Continuity: linear function of t is always continuous with continuous derivatives (✓)
- Biunivocity: $J = \det\left[\dfrac{\partial \phi_i}{\partial X_j}\right] = \det\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & t/2 \\ 0 & t/2 & 1 \end{bmatrix} = 1\left(1^2 - \dfrac{t^2}{4}\right) = 1 - \dfrac{t^2}{4}$

At $t = 0 \Rightarrow \det\left[\dfrac{\partial \phi_i}{\partial X_i}\right] = 0 \Rightarrow$ motion is not biunivocity

- Positive Jacobian

With $t > 2 \Rightarrow$ Jacobian determinant $< 0 \Rightarrow$ negative density

$\Rightarrow$ This motion is not possible

b) Obtain velocity components

- material descriptions:

$$\vec{V}(\vec{x}, t) = \frac{\partial \vec{x}(\vec{X}, t)}{\partial t} = \begin{bmatrix} \frac{\partial}{\partial t}(X_1) \\ \frac{\partial}{\partial t}(X_2 + \frac{t}{2} X_3) \\ \frac{\partial}{\partial t}(X_3 + \frac{t}{2} X_2) \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{X_3}{2} \\ \frac{X_2}{2} \end{bmatrix}$$

- spatial descriptions

$$\begin{cases} x_1 = X_1 \\ x_2 = X_2 + t/2\, X_3 \\ x_3 = X_3 + t/2\, X_2 \end{cases} \Rightarrow \begin{cases} X_1 = x_1 \\ X_2 = \frac{-2t}{-t^2+4} x_3 + \frac{4}{4-t^2} x_2 \\ X_3 = -\frac{2t}{4-t^2} x_2 + \frac{4}{4-t^2} x_3 \end{cases}$$

$$\Rightarrow \vec{v}(\vec{X}, t) = \frac{\partial \vec{X}(\vec{x}, t)}{\partial t} = \begin{bmatrix} 0 \\ -2x_3 \frac{(4+t^2)}{(4-t^2)^2} + 8x_2 \frac{t}{(4-t^2)^2} \\ -2x_2 \frac{(4+t^2)}{(4-t^2)^2} + 8x_3 \frac{t}{(4-t^2)^2} \end{bmatrix}$$

c) Define conditions when material is incompressible

The incompressibility condition: $e = J - 1 = 0 \Rightarrow J = |\underline{\underline{F}}| = 1$

$$\Rightarrow \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & t/2 \\ 0 & t/2 & 1 \end{bmatrix} = 1 - \frac{t^2}{4} = 0 \Rightarrow t = 2 \text{ is the condition}$$