# Foundations on Continuum Mechanics - Week $2$ - Kinematics

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# Introduction

# Kinematics

# Kinematics

Kinematics:

1. Motion (equation of motion, rigid body motion)
2. Velocity
3. Acceleration
4. Deformation (deformation gradient tensor)
5. Displacement
6. Strain (strain tensor)
7. Stretch
8. Unit elongation
9. Area Variation
10. Volume variation
11. Volumetric strain
12. Infinitesimal Strain Theory

# Equation of Motion

# Material and Spatial points of a Configuration

- The continuous medium consists of particles.
- Material points are fixed points of the continuous medium (points of material).
- Spatial points are fixed points in space (points of space).
- Every material point can occupy only one point is space (classical mechanics, not quantum mechanics)
- Capital letters will be used to describe anything referred to continuum material.
- Lowercase letters will be used to describe anything related to space.
- Two special configurations are used:
  1. Reference or initial or undeformed
  2. Current or present or deformed
- The initial position of a given particle is defined by vector named $\vec{\mathbf{X}} = [X_1, X_2, X_3]^T$, which are the material coordinates of particle (label of particle).
- The current (present) position of the same particle is defined by the vector $\vec{\boldsymbol{x}} = [x_1, x_2, x_3]^T$, which are the spacial coordinates of the particle.

# Equation of motion

In order to relate initial configuration to the present configuration, the equation of motion is needed.

$$\vec{\boldsymbol{x}} = \phi(\text{Label of particle}, t) \equiv \vec{\boldsymbol{x}}(\text{Label of particle}, t)$$
$$x_i = \phi(\text{Label of particle}, t); \quad i \in \{1, 2, 3\}$$

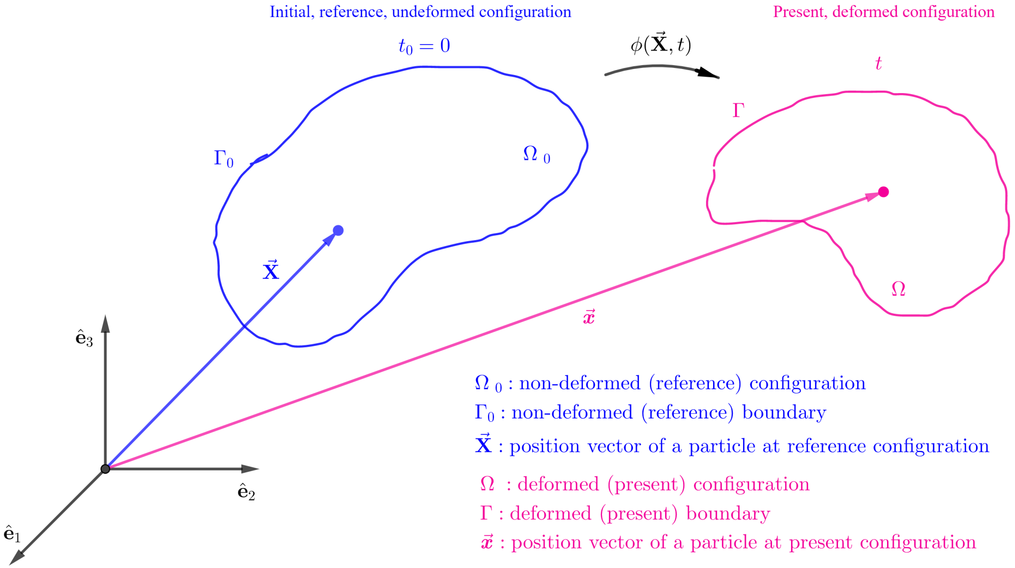
In order to label the particles the continuum medium

- we use the material coordinates $\vec{\mathbf{X}} = [X_1, X_2, X_3]^T$, to be able to refer to specific particle.
- Therefore:

$$\vec{\boldsymbol{x}} = \phi(\vec{\boldsymbol{X}}, t) \equiv \vec{\boldsymbol{x}}(\vec{\boldsymbol{X}}, t)$$
$$x_i = \phi_i(X_i, t); \quad i \in \{1, 2, 3\}$$

- The material coordinates identify the specific particle at the reference configuration. This is the canonical form of equation of motion, [2].

# Initial and Present Configuration




$\Omega_0$ : non-deformed (reference) configuration

$\Gamma_0$ : non-deformed (reference) boundary

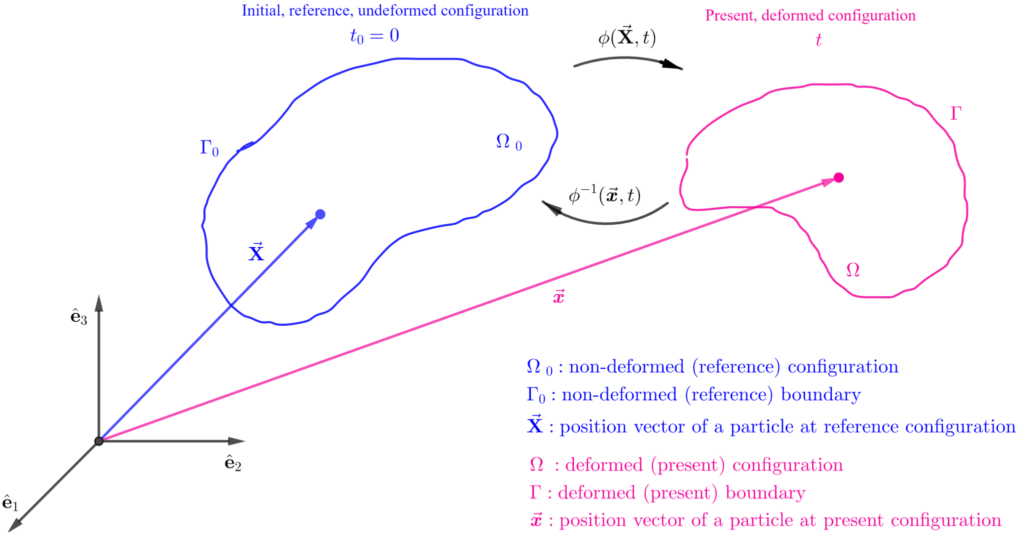
$\vec{\mathbf{X}}$ : position vector of a particle at reference configuration

$\Omega$ : deformed (present) configuration

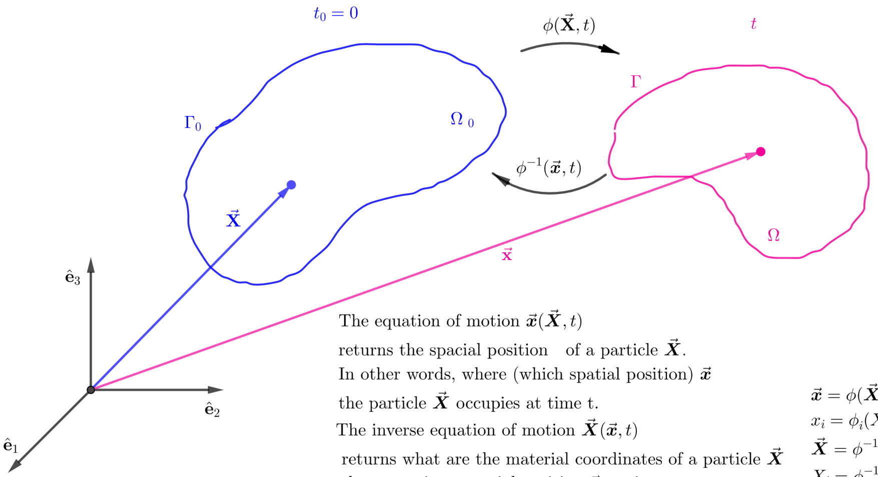
$\Gamma$ : deformed (present) boundary

$\vec{\boldsymbol{x}}$ : position vector of a particle at present configuration

# Initial and Present Configuration



# Equation of motion/Inverse Equation of Motion




The equation of motion $\vec{\boldsymbol{x}}(\vec{\boldsymbol{X}}, t)$
returns the spacial position of a particle $\vec{\boldsymbol{X}}$.
In other words, where (which spatial position) $\vec{\boldsymbol{x}}$
the particle $\vec{\boldsymbol{X}}$ occupies at time t.
The inverse equation of motion $\vec{\boldsymbol{X}}(\vec{\boldsymbol{x}}, t)$
returns what are the material coordinates of a particle $\vec{\boldsymbol{X}}$
that occupies a spacial position $\vec{\boldsymbol{x}}$ at time t.

$$\vec{\boldsymbol{x}} = \phi(\vec{\boldsymbol{X}}, t) \equiv \vec{\boldsymbol{x}}(\vec{\boldsymbol{X}}, t)$$
$$x_i = \phi_i(X_i, t); \quad i \in \{1, 2, 3\}$$
$$\vec{\boldsymbol{X}} = \phi^{-1}(\vec{\boldsymbol{x}}, t) \equiv \vec{\boldsymbol{X}}(\vec{\boldsymbol{x}}, t)$$
$$X_i = \phi_i^{-1}(x_i, t); \quad i \in \{1, 2, 3\}$$

# Restrictions for equations of motion

The equations $\phi()$ and $\phi^{-1}()$ to describe a motion, they need to fulfil the following conditions, [2]:

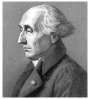
- Consistency: $\phi(\vec{\boldsymbol{X}}, 0) = \vec{\boldsymbol{X}}$.
- Continuity: $\phi(\vec{\boldsymbol{X}}, 0)$ is continuous with continuous derivatives.
- Biunovocity: to verify that two particles do NOT occupy at the same time the same spatial position, and that a particle does not occupy more than one spatial position simultaneously. Mathematically: the Jacobian should be different than zero, [2]: $J = \left| \frac{\partial \phi(\vec{\boldsymbol{X}}, t)}{\partial \vec{\boldsymbol{X}}} \right| = det \left[ \frac{\partial \phi_i}{\partial X_j} \right] \neq 0$
- Positive Jacobian (density is always positive, [2]): $J = \left| \frac{\partial \phi(\vec{\boldsymbol{X}}, t)}{\partial \vec{\boldsymbol{X}}} \right| = det \left[ \frac{\partial \phi_i}{\partial X_j} \right] > 0$

# Example 1

# Description of motion

The particles have properties and their mathematical description can be done in two different ways:

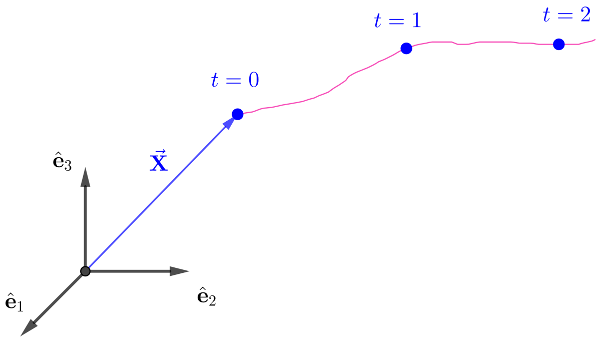
1. Material (Lagrangian) Description. We define the properties in terms of the particles (material point). We identify the particle and check its properties at every time. For example: what is the temperature of certain particle in time (we fix the material point). This description is used for solids.



2. Spatial (Eulerian) Description. We define the properties in terms of spatial point. We identify a specific spatial point and check the properties at this spatial point. For example: what is the temperature of the particles that pass through specific spatial point (we fix the spatial point). This description is used for fluids.

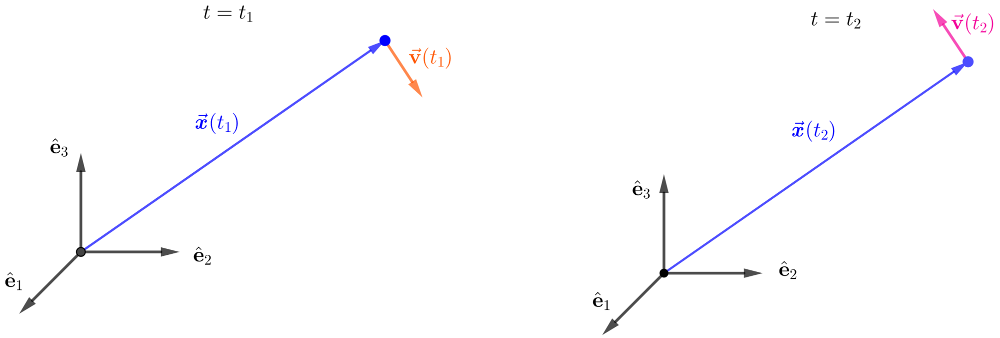
# Material (Lagrangian) Description

We use material coordinates and time to follow certain physical properties. We fix the material point (namely a particle) and we follow it as time evolves. In other words, what are the properties of the specific particle $\vec{X}$ in time. Typically used in Solid Mechanics.



# Spatial (Eulerian) Description

We use spatial coordinates and time to follow certain physical properties. We fix the spatial point and we follow how the properties change as time evolves. In other words, what are the properties at the specific $\vec{x}$ of the different particles that pass through this spatial point. Typically used in Fluid Mechanics.



# Example 2

# Time derivatives

The time derivatives of a property can be defined according to:

1. Material description $\Gamma(\vec{\boldsymbol{X}}, t)$, called material derivative or total. The property is changing with respect to time by following a specific particle in the continuum. Material derivative:
$$\frac{\partial \Gamma(\vec{\boldsymbol{X}}, t)}{\partial t} \tag{1}$$
denotes the partial time derivative of the material description of the property. It provides the change of the property of the specific particle.
2. Spatial description $\gamma(\vec{\boldsymbol{x}}, t)$, called spatial derivative or local. The property is changing with respect to time in a fixed position in space. The spatial derivative:
$$\frac{\partial \gamma(\vec{\boldsymbol{x}}, t)}{\partial t} \tag{2}$$
denotes the partial time derivative of the spatial description of the property. It provides the change of the property at the specific spatial point.

# Convective Derivative

- The equation of motion can be established as $\vec{\boldsymbol{x}}(\vec{\boldsymbol{X}}, t)$.
- Note that a property that is given in spatial description $\gamma(\vec{\boldsymbol{x}}, t)$, through the equation of motion can be written as $\gamma(\vec{\boldsymbol{x}}(\vec{\boldsymbol{X}}, t), t)$.
- In the case where the equation of motion is not available, the material derivative of spatial description can be calculated as (we $\frac{d}{dt}[\gamma(\vec{\boldsymbol{x}}, t)]$ to denote the material derivative, not $\partial$):

$$\frac{d}{dt}[\gamma(\vec{\boldsymbol{x}}, t)] = \frac{\partial\gamma(\vec{\boldsymbol{x}}, t)}{\partial t} + \frac{\partial\gamma}{\partial x_i}\frac{\partial x_i}{\partial t} = \frac{\partial\gamma(\vec{\boldsymbol{x}}, t)}{\partial t} + \frac{\partial\gamma}{\partial\vec{\boldsymbol{x}}}\cdot\frac{\partial\vec{\boldsymbol{x}}}{\partial t} =$$
$$= \frac{\partial\gamma(\vec{\boldsymbol{x}}, t)}{\partial t} + \vec{\boldsymbol{\nabla}}\gamma(\vec{\boldsymbol{x}}, t)\cdot\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}, t) = \frac{\partial\gamma(\vec{\boldsymbol{x}}, t)}{\partial t} + \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}, t)\cdot\vec{\boldsymbol{\nabla}}\gamma(\vec{\boldsymbol{x}}, t)$$

  where the term $\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}, t)\cdot\vec{\boldsymbol{\nabla}}\gamma(\vec{\boldsymbol{x}}, t)$ denotes convective range of change of the material property.
- Generally, for any material property $A$:

$$\underbrace{\frac{dA(\vec{\boldsymbol{x}}, t)}{dt}}_{\text{Material derivative}} = \underbrace{\frac{\partial A(\vec{\boldsymbol{x}}, t)}{\partial t}}_{\text{Local derivative}} + \underbrace{\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{\nabla}}A(\vec{\boldsymbol{x}}, t)}_{\text{Convective derivative}}$$

# Physical Meaning of Convective derivative

$$\underbrace{\frac{dA(\vec{\boldsymbol{x}}(\vec{\boldsymbol{X}},t),t)}{dt}}_{\text{Material derivative}} = \underbrace{\frac{\partial A(\vec{\boldsymbol{x}}(\vec{\boldsymbol{X}},t),t)}{\partial t}}_{\text{Local derivative}} + \underbrace{\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{\nabla}}A(\vec{\boldsymbol{x}}(\vec{\boldsymbol{X}},t),t)}_{\text{Convective derivative}}$$

If we take the material derivative of a spatial property $A(\vec{\boldsymbol{x}},t)$, but because time appears twice: one time in $A(\vec{\boldsymbol{x}},t)$ and one time in $\vec{\boldsymbol{x}}(\vec{\boldsymbol{X}},t)$, we need to derive the local and the convective term. The first term, the local, determines the change of the property in a given spatial point. The convective term is a consequence of the fact that the particles move. In general, the term convective is related with motion of the particles (when $\vec{\boldsymbol{v}}=0$ the convective part vanishes). The two terms together provide the material derivative of the property.

In other words, we can think that when a property is linked to a material point and can be transported with the motion with the material, this total derivative, is called material derivative.

In general, we could say that the total rage of change of a property at a material point is due to (i) its local change rate and (ii) rate of in-flow though the boundaries of the differential volume.

# Example 3

# Convective Derivative

- The convective derivative is defined as follows:

$$\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{\nabla}}(*)$$

  where $(*)$ is any tensorial quantity.
- Convection is related to motion.
- If there is no motion $\vec{\boldsymbol{v}}$ the material and local derivates coincide:

$$\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{\nabla}}(*) = 0 \Leftrightarrow \frac{d(*)}{dt} \equiv \frac{\partial(*)}{\partial t}$$

# Velocity

The material description of the velocity can be derived as follows:

$$\vec{\boldsymbol{V}}(\vec{\boldsymbol{X}}, t) = \frac{\partial \vec{\boldsymbol{x}}(\vec{\boldsymbol{X}}, t)}{\partial t}$$

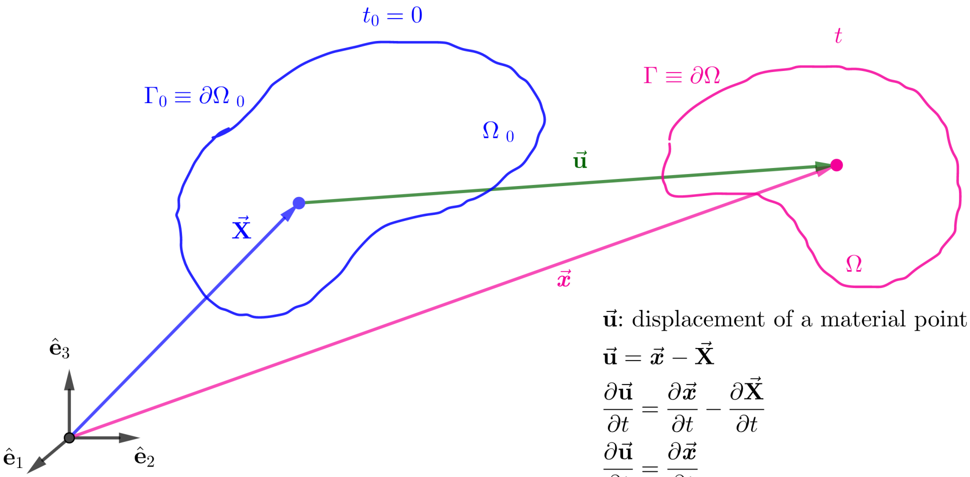
$$V_i(\vec{\boldsymbol{X}}, t) = \frac{\partial x_i(\vec{\boldsymbol{X}}, t)}{\partial t}; \quad i \in \{1, 2, 3\}$$

Note that the equation of motion in the form: $\vec{\boldsymbol{x}} = \phi(\vec{\boldsymbol{X}}, t) = \vec{\boldsymbol{x}}(\vec{\boldsymbol{X}}, t)$.
In spatial description the velocity can be given by using the inverse equation of motion:

$$\vec{\boldsymbol{V}}(\vec{\boldsymbol{X}}(\vec{\boldsymbol{x}}, t), t) \Leftrightarrow \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}, t)$$

Note that the concept of material derivative does not apply to the velocity because it is the time derivative of the equation of motion.

# Isn't the velocity the time derivative of the displacement?




$\vec{\mathbf{u}}$: displacement of a material point

$$\vec{\mathbf{u}} = \vec{\boldsymbol{x}} - \vec{\mathbf{X}}$$

$$\frac{\partial \vec{\mathbf{u}}}{\partial t} = \frac{\partial \vec{\boldsymbol{x}}}{\partial t} - \frac{\partial \vec{\mathbf{X}}}{\partial t}$$

$$\frac{\partial \vec{\mathbf{u}}}{\partial t} = \frac{\partial \vec{\boldsymbol{x}}}{\partial t}$$

$$\frac{\partial \vec{\mathbf{X}}}{\partial t} = 0$$

# Acceleration

The material description of the acceleration can be derived from the material time derivative of the velocity field:

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{X}}, t) = \frac{\partial \vec{\boldsymbol{V}}(\vec{\boldsymbol{X}}, t)}{\partial t}$$

$$A_i(\vec{\boldsymbol{X}}, t) = \frac{\partial V_i(\vec{\boldsymbol{X}}, t)}{\partial t}; \quad i \in \{1, 2, 3\}$$

Note that in the above equation for a specific material point $\vec{\boldsymbol{X}}$ is fixed.
In spatial description the acceleration can be given by using the inverse equation of motion:

$$\vec{\boldsymbol{A}}(\vec{\boldsymbol{X}}(\vec{\boldsymbol{x}}, t), t) \Leftrightarrow \vec{\boldsymbol{a}}(\vec{\boldsymbol{x}}, t)$$

Alternatively, the $\vec{\boldsymbol{a}}(\vec{\boldsymbol{x}}, t)$ can be derived through the material description of $\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}, t)$:

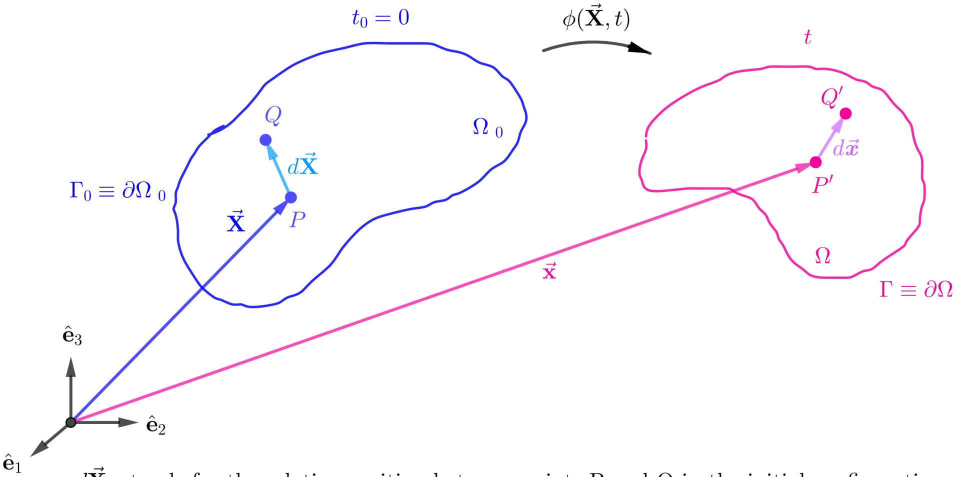
$$\vec{\boldsymbol{a}}(\vec{\boldsymbol{x}}, t) = \frac{d\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}, t)}{dt} = \frac{\partial \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}, t)}{\partial t} + \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}, t) \cdot \vec{\boldsymbol{\nabla}} \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}, t)$$

$$a_i(\vec{\boldsymbol{x}}, t) = \frac{dv_i(\vec{\boldsymbol{x}}, t)}{dt} = \frac{\partial v_i(\vec{\boldsymbol{x}}, t)}{\partial t} + v_j(\vec{\boldsymbol{x}}, t) \cdot \frac{\partial v_i}{\partial x_j}(\vec{\boldsymbol{x}}, t); \quad i \in \{1, 2, 3\}$$

# Example 4

# Deformation

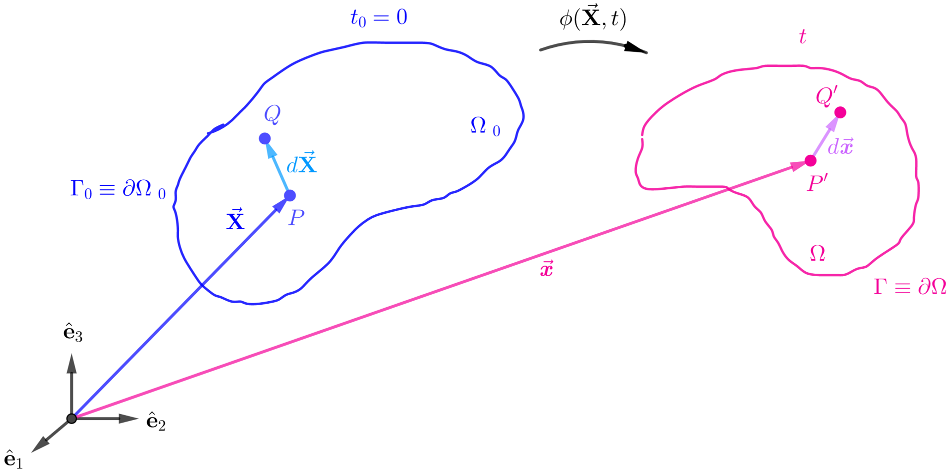
# Deformation




$d\vec{\mathbf{X}}$: stands for the relative position between points P and Q in the initial configuration at $t_0$

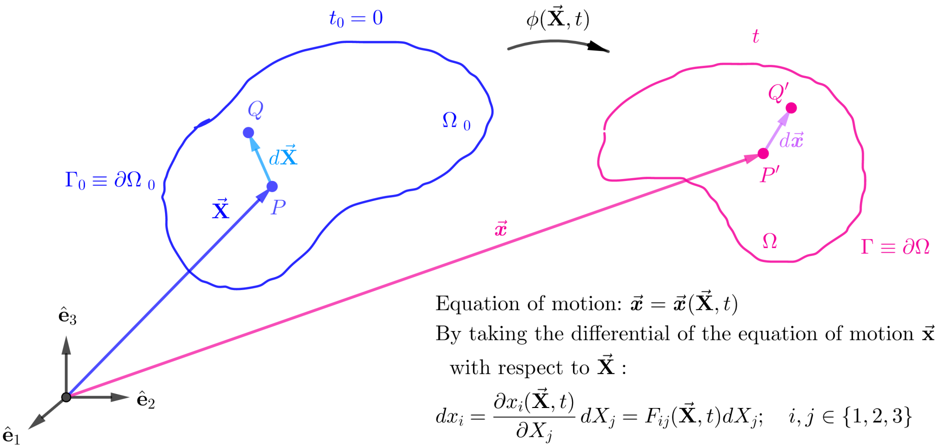
$d\vec{x}$: stands for the relative position between points P' and Q' in the present configuration at $t$

# Deformation




**Deformation** reflects how the transition of a body from the initial configuration to the present one, affects the **relative movement** between particles and the **change of size** and **shape** of the body itself.

# Deformation Gradient Tensor




Equation of motion: $\vec{x} = \vec{x}(\vec{\mathbf{X}}, t)$

By taking the differential of the equation of motion $\vec{\mathbf{x}}$ with respect to $\vec{\mathbf{X}}$ :

$$dx_i = \frac{\partial x_i(\vec{\mathbf{X}}, t)}{\partial X_j} dX_j = F_{ij}(\vec{\mathbf{X}}, t) dX_j; \quad i, j \in \{1, 2, 3\}$$

$$d\vec{x} = \frac{\partial \vec{x}(\vec{\mathbf{X}}, t)}{\partial \vec{\mathbf{X}}} \cdot d\vec{\mathbf{X}} = \underline{\underline{\mathbf{F}}} \cdot d\vec{\mathbf{X}}$$

$\underline{\underline{\mathbf{F}}}$: is the (material) **deformation gradient tensor**

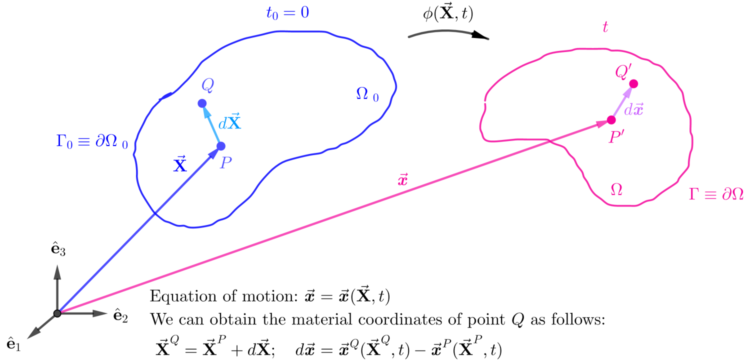
# Clarification

Note that the differential of equation of motion $\phi(\vec{\boldsymbol{X}}, t) \equiv \vec{\boldsymbol{x}}(\vec{\boldsymbol{X}}, t)$ would normally be:

$$dx_i(\vec{\boldsymbol{X}}, t) = \frac{\partial x_i}{\partial X_j} dX_j + \frac{\partial x_i}{\partial t} dt$$

The second term: $\frac{\partial x_i}{\partial t} dt$ vanishes due to the fact that we are investigating the motion at a specific time $t$ fixed.

# Deformation Gradient Tensor




Equation of motion: $\vec{\boldsymbol{x}} = \vec{\boldsymbol{x}}(\vec{\mathbf{X}}, t)$

We can obtain the material coordinates of point $Q$ as follows:

$$\vec{\mathbf{X}}^Q = \vec{\mathbf{X}}^P + d\vec{\mathbf{X}}; \quad d\vec{\boldsymbol{x}} = \vec{\boldsymbol{x}}^Q(\vec{\mathbf{X}}^Q, t) - \vec{\boldsymbol{x}}^P(\vec{\mathbf{X}}^P, t)$$

Note that $\vec{\boldsymbol{x}}^Q(\vec{\mathbf{X}}^Q, t) = \vec{\boldsymbol{x}}^P(\vec{\mathbf{X}}^P + d\vec{\mathbf{X}}, t) = \vec{\boldsymbol{x}}(\vec{\mathbf{X}} + d\vec{\mathbf{X}}, t)$

It follows that: $d\vec{\boldsymbol{x}} = \vec{\boldsymbol{x}}(\vec{\mathbf{X}} + d\vec{\mathbf{X}}, t) - \vec{\boldsymbol{x}}(\vec{\mathbf{X}}, t)$

By using Taylor series expansion

$$dx_i = \frac{\partial x_i(\vec{\mathbf{X}}, t)}{\partial X_j} dX_j + O(|dX_i|^2) \approx dx_i = \frac{\partial x_i(\vec{\mathbf{X}}, t)}{\partial X_j} dX_j = F_{ij}(\vec{\mathbf{X}}, t) dX_j; \quad i, j \in \{1, 2, 3\}$$

$$d\vec{\boldsymbol{x}} = \frac{\partial \vec{\boldsymbol{x}}(\vec{\mathbf{X}}, t)}{\partial \vec{\mathbf{X}}} \cdot d\vec{\mathbf{X}} = \underline{\underline{\mathbf{F}}} \cdot d\vec{\mathbf{X}}$$

# Deformation Gradient Tensor

The material deformation gradient tensor can be written as:

$$\underline{\underline{\boldsymbol{F}}}(\vec{\boldsymbol{X}}, t) = \vec{\boldsymbol{x}}(\vec{\boldsymbol{X}}, t) \otimes \vec{\boldsymbol{\nabla}}_X$$
$$F_{ij} = \frac{\partial x_i}{\partial X_j}; \quad i, j \in \{1, 2, 3\}$$

Note that while the symbol $\vec{\boldsymbol{\nabla}}_x$ operates on the spatial coordinates $\vec{\boldsymbol{x}}$, the symbol $\vec{\boldsymbol{\nabla}}_X$ operates on the material coordinates $\vec{\boldsymbol{X}}$ and therefore:

$$\vec{\boldsymbol{\nabla}}_X \equiv \frac{\partial}{\partial X_i} \hat{\boldsymbol{e}}_i$$

In matrix notation:

$$\nabla_X = \begin{bmatrix} \frac{\partial}{\partial X_1} \\ \frac{\partial}{\partial X_2} \\ \frac{\partial}{\partial X_3} \end{bmatrix}$$

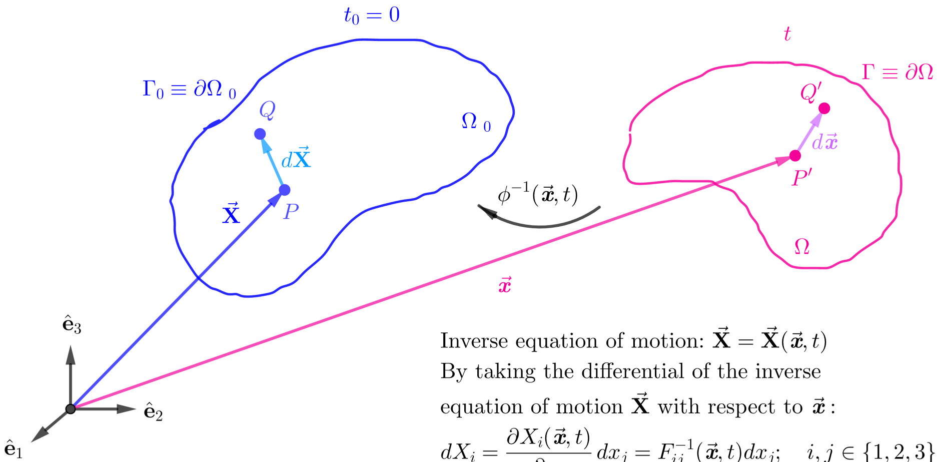
# Deformation Gradient Tensor

In matrix notation:

$$[F] = [x] \otimes [\nabla_X] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial X_1} & \frac{\partial}{\partial X_2} & \frac{\partial}{\partial X_3} \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix}$$

The deformation gradient tensor $\underline{\underline{\boldsymbol{F}}}$ is very important measure, due to the fact that defines the relative change in the infinitesimal region of the material point.

# Inverse Deformation Gradient Tensor




Inverse equation of motion: $\vec{\mathbf{X}} = \vec{\mathbf{X}}(\vec{\boldsymbol{x}}, t)$

By taking the differential of the inverse equation of motion $\vec{\mathbf{X}}$ with respect to $\vec{\boldsymbol{x}}$ :

$$dX_i = \frac{\partial X_i(\vec{\boldsymbol{x}}, t)}{\partial x_j} dx_j = F_{ij}^{-1}(\vec{\boldsymbol{x}}, t) dx_j; \quad i, j \in \{1, 2, 3\}$$

$$d\vec{\mathbf{X}} = \frac{\partial \vec{\mathbf{X}}(\vec{\boldsymbol{x}}, t)}{\partial \vec{\boldsymbol{x}}} \cdot d\vec{\boldsymbol{x}} = \underline{\underline{\mathbf{F}}}^{-1} \cdot d\vec{\boldsymbol{x}}$$

# Inverse Deformation Gradient Tensor

The spatial deformation gradient tensor can be written as:

$$\underline{\underline{\boldsymbol{F}}}^{-1}(\vec{\boldsymbol{x}}, t) = \vec{\boldsymbol{X}}(\vec{\boldsymbol{x}}, t) \otimes \vec{\boldsymbol{\nabla}}_x$$

$$F_{ij}^{-1} = \frac{dX_i}{dx_j}; \quad i, j \in \{1, 2, 3\}$$

Note that while the symbol $\vec{\boldsymbol{\nabla}}_x$ operates on the spatial coordinates $\vec{\boldsymbol{x}}$, the symbol $\vec{\boldsymbol{\nabla}}_X$ operates on the material coordinates $\vec{\boldsymbol{X}}$ and therefore:

$$\vec{\boldsymbol{\nabla}}_x \equiv \frac{\partial}{\partial x_i}\hat{\boldsymbol{e}}_i$$

In matrix notation:

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# Inverse Deformation Gradient Tensor

In matrix notation:

$$F^{-1} = X \otimes \nabla_x = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \end{bmatrix} = \begin{bmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_1}{\partial x_2} & \frac{\partial X_1}{\partial x_3} \\ \frac{\partial X_2}{\partial x_1} & \frac{\partial X_2}{\partial x_2} & \frac{\partial X_2}{\partial x_3} \\ \frac{\partial X_3}{\partial x_1} & \frac{\partial X_3}{\partial x_2} & \frac{\partial X_3}{\partial x_3} \end{bmatrix}$$

If we take the deformation gradient tensor $\underline{\underline{\boldsymbol{F}}}(\vec{\boldsymbol{X}}, t)$ inverse its matrix representation, and replace the material coordinates $\vec{\boldsymbol{X}}$ with the spatial coordinates $\vec{\boldsymbol{x}}$, we will obtain the matrix representation of the inverse deformation tensor $\underline{\underline{\boldsymbol{F}}}^{-1}(\vec{\boldsymbol{x}}, t)$. The proof is the following:

$$\underbrace{\frac{\partial x_i}{\partial X_k}}_{\underline{\underline{F}}} \underbrace{\frac{\partial X_k}{\partial x_j}}_{\underline{\underline{F}}^{-1}} = \frac{\partial x_i}{\partial x_j} = \delta_{ij} \Leftrightarrow \underline{\underline{\boldsymbol{F}}} \cdot \underline{\underline{\boldsymbol{F}}}^{-1} = \underline{\underline{\boldsymbol{F}}}^{-1} \cdot \underline{\underline{\boldsymbol{F}}} = \underline{\underline{\boldsymbol{1}}}$$

# Deformation Gradient Tensor - Properties

- If the deformation gradient tensor $\underline{\underline{\boldsymbol{F}}}$ does not depend on the space coordinates (in other words it has the same value for every material point), $\underline{\underline{\boldsymbol{F}}}(\vec{\boldsymbol{X}}, t) = \underline{\underline{\boldsymbol{F}}}(t)$ the deformation is homogenous (the deformation is constant, every part of the body deforms as a whole).

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- If there is no motion $\vec{\boldsymbol{X}} = \vec{\boldsymbol{x}}$ and $\underline{\underline{\boldsymbol{F}}}(\vec{\boldsymbol{X}}, t) = \underline{\underline{\boldsymbol{F}}}^{-1}(\vec{\boldsymbol{x}}, t) = \underline{\underline{\mathbf{1}}}$

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- The determinant of the deformation gradient tensor is the Jacobian

$$det([\underline{\underline{\boldsymbol{F}}}]) = \left| \left[ \frac{\partial \vec{\boldsymbol{x}}}{\partial \vec{\boldsymbol{X}}} \right] \right| = J$$

# Deformation Gradient Tensor - Properties

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- The determinant of the deformation gradient tensor is the Jacobian

$$det([\underline{\underline{\boldsymbol{F}}}]) = \left| \left[ \frac{\partial \vec{\boldsymbol{x}}}{\partial \vec{\boldsymbol{X}}} \right] \right| = J$$

- If the equation $\underline{\underline{\boldsymbol{F}}} \cdot d\vec{\boldsymbol{X}} = 0$ for $d\vec{\boldsymbol{X}} \neq 0$ means that in the deformed configuration the linear segment $d\vec{\boldsymbol{x}}$ from $d\vec{\boldsymbol{X}}$ reduces to zero. This is physically unrealistic which implies that $\underline{\underline{\boldsymbol{F}}}$ is a nonsingular tensor with its determinant different from zero: $J \neq 0$

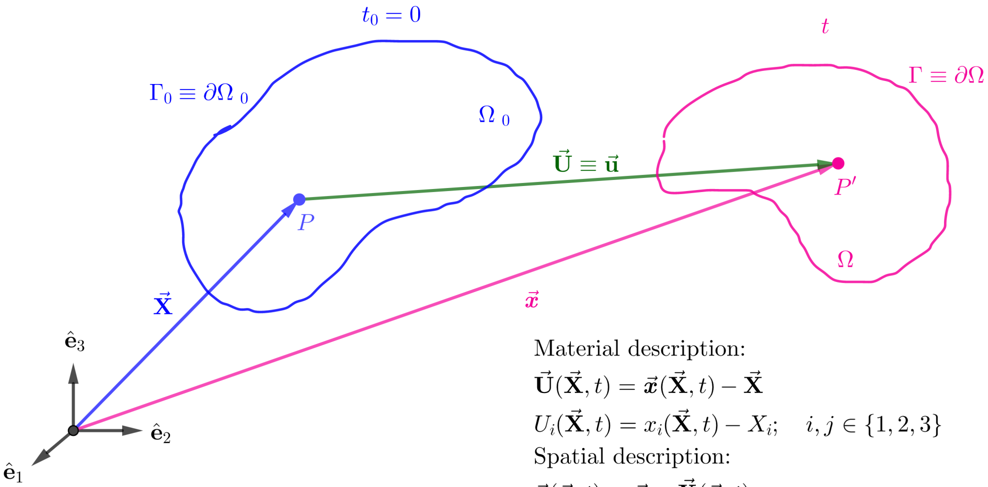
# Example 5

# Methods

- Numerical methods usually use either Lagrangian (material) or Eulerian (spatial) description.
- Examples of Lagrangian approach codes is the HEMP [3], which is based in the finite-difference method.
- Examples of Eulerian approach codes is the HELP [1], which is based in the finite-difference method.
- There are also hybrid methods that use both Lagrangian and Eulerian method.
- One of the most promising hybrid methods, is the Material Point Method (MPM), [4].
- Material Point Method - Application 1
- Material Point Method - Application 2

# Displacement

# Displacement




Material description:

$$\vec{\mathbf{U}}(\vec{\mathbf{X}}, t) = \vec{\boldsymbol{x}}(\vec{\mathbf{X}}, t) - \vec{\mathbf{X}}$$

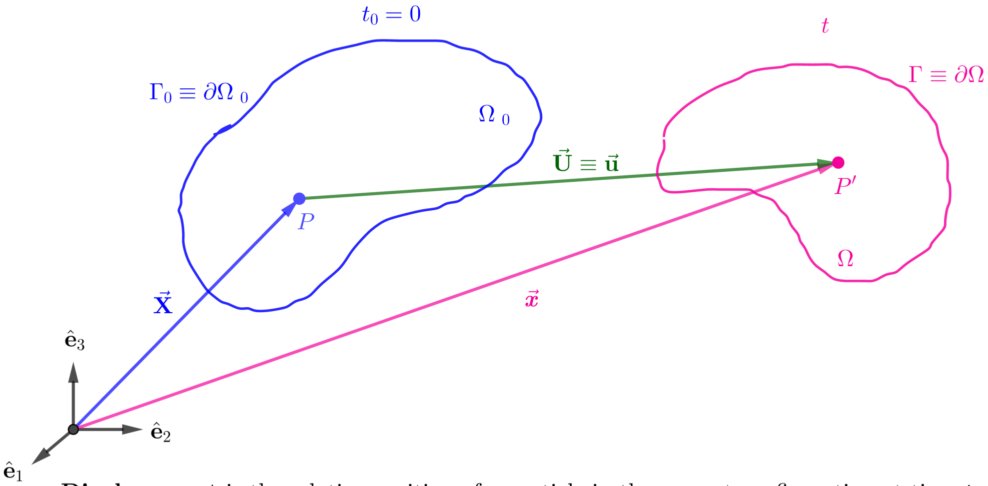
$$U_i(\vec{\mathbf{X}}, t) = x_i(\vec{\mathbf{X}}, t) - X_i; \quad i, j \in \{1, 2, 3\}$$

Spatial description:

$$\vec{\mathbf{u}}(\vec{\boldsymbol{x}}, t) = \vec{\boldsymbol{x}} - \vec{\mathbf{X}}(\vec{\boldsymbol{x}}, t)$$

$$u_i(\vec{\boldsymbol{x}}, t) = x_i - X_i(\vec{\boldsymbol{x}}, t); \quad i, j \in \{1, 2, 3\}$$

# Displacement




**Displacement** is the relative position of a particle in the current configuration at time $t$ with respect to the reference configuration at time $t_0$. The **displacement field** is the **vector field** of all displacements of the continuous medium.

# Displacement Gradient Tensor

The displacement vector is defined as follows:

$$\vec{\boldsymbol{U}}(\vec{\boldsymbol{X}}, t) = \vec{\boldsymbol{x}}(\vec{\boldsymbol{X}}, t) - \vec{\boldsymbol{X}}$$

$$U_i(\vec{\boldsymbol{X}}, t) = x_i(\vec{\boldsymbol{X}}, t) - X_i; \quad i, j \in \{1, 2, 3\}$$

Differentiating $\vec{\boldsymbol{U}}$ with respect to $\vec{\boldsymbol{X}}$:

$$\frac{\partial U_i(\vec{\boldsymbol{X}}, t)}{\partial X_j} = \underbrace{\frac{\partial x_i(\vec{\boldsymbol{X}}, t)}{\partial X_j}}_{F_{ij}} - \underbrace{\frac{\partial X_i}{\partial X_j}}_{\delta_{ij}} = F_{ij} - \delta_{ij} = D_{ij}$$

The material displacement gradient tensor can be defined:

$$D_{ij} = \frac{\partial U_i}{\partial X_j} = F_{ij} - \delta_{ij}; \quad i, j \in \{1, 2, 3\}$$

$$\underline{\underline{\boldsymbol{D}}}(\vec{\boldsymbol{X}}, t) = \vec{\boldsymbol{U}}(\vec{\boldsymbol{X}}, t) \otimes \vec{\boldsymbol{\nabla}}_X = \underline{\underline{\boldsymbol{F}}} - \underline{\underline{\boldsymbol{1}}}$$

## Displacement Gradient Tensor

The displacement vector is defined in terms of the spatial coordinates as follows:

$$\vec{\boldsymbol{u}}(\vec{\boldsymbol{x}}, t) = \vec{\boldsymbol{x}} - \vec{\boldsymbol{X}}(\vec{\boldsymbol{x}}, t)$$
$$u_i(\vec{\boldsymbol{x}}, t) = x_i - X_i(\vec{\boldsymbol{x}}, t); \quad i, j \in \{1, 2, 3\}$$

Differentiating $\vec{\boldsymbol{u}}$ with respect to $\vec{\boldsymbol{x}}$:

$$\frac{\partial u_i(\vec{\boldsymbol{x}}, t)}{\partial x_j} = \underbrace{\frac{\partial x_i}{\partial x_j}}_{\delta_{ij}} - \underbrace{\frac{\partial X_i(\vec{\boldsymbol{x}}, t)}{\partial x_j}}_{F_{ij}^{-1}} = \delta_{ij} - F_{ij}^{-1} = d_{ij}$$

The spatial displacement gradient tensor can be defined:

$$d_{ij} = \frac{\partial u_i}{\partial x_j} = \delta_{ij} - F_{ij}^{-1}; \quad i, j \in \{1, 2, 3\}$$
$$\underline{\underline{\boldsymbol{d}}}(\vec{\boldsymbol{x}}, t) = \vec{\boldsymbol{u}}(\vec{\boldsymbol{x}}, t) \otimes \vec{\boldsymbol{\nabla}}_x = \underline{\underline{\boldsymbol{1}}} - \underline{\underline{\boldsymbol{F}}}^{-1}$$

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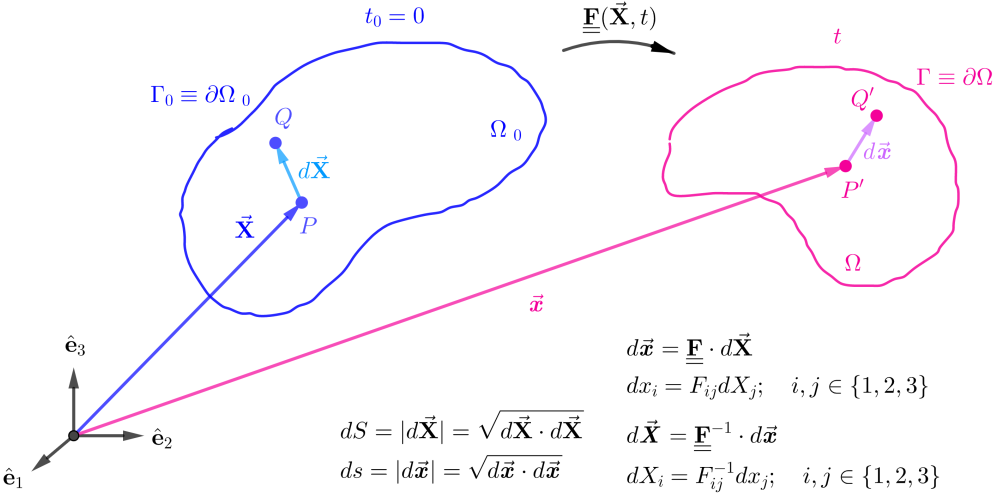
- Deformation is related to change of relative position, but strain is related to change of distances and angles.
- Strain is more appropriate engineering measure.
- Strains are responsible for generation of stresses.

# Strain




$$d\vec{\boldsymbol{x}} = \underline{\underline{\mathbf{F}}} \cdot d\vec{\mathbf{X}}$$

$$dx_i = F_{ij} dX_j; \quad i, j \in \{1, 2, 3\}$$

$$dS = |d\vec{\mathbf{X}}| = \sqrt{d\vec{\mathbf{X}} \cdot d\vec{\mathbf{X}}}$$

$$ds = |d\vec{\boldsymbol{x}}| = \sqrt{d\vec{\boldsymbol{x}} \cdot d\vec{\boldsymbol{x}}}$$

$$d\vec{\boldsymbol{X}} = \underline{\underline{\mathbf{F}}}^{-1} \cdot d\vec{\boldsymbol{x}}$$

$$dX_i = F_{ij}^{-1} dx_j; \quad i, j \in \{1, 2, 3\}$$

## Strain Tensor

Let's calculate the magnitudes of vectors $d\vec{\boldsymbol{x}}$ and $d\vec{\boldsymbol{X}}$, respectively, $ds$ and $dS$:

$$(ds)^2 = d\vec{\boldsymbol{x}} \cdot d\vec{\boldsymbol{x}} = \underline{\underline{\boldsymbol{F}}} \cdot d\vec{\boldsymbol{X}} \cdot \underline{\underline{\boldsymbol{F}}} \cdot d\vec{\boldsymbol{X}} = d\vec{\boldsymbol{X}} \cdot \underline{\underline{\boldsymbol{F}}}^T \cdot \underline{\underline{\boldsymbol{F}}} \cdot d\vec{\boldsymbol{X}} = d\vec{\boldsymbol{X}} \cdot \underline{\underline{\boldsymbol{F}}}^T \cdot \underline{\underline{\boldsymbol{F}}} \cdot d\vec{\boldsymbol{X}}$$
$$(ds)^2 = dx_i dx_i = F_{ij} dX_j F_{ik} dX_k = dX_j F_{ij} F_{ik} dX_k = dX_j F_{ji}^T F_{ik} dX_k$$

$$(dS)^2 = d\vec{\boldsymbol{X}} \cdot d\vec{\boldsymbol{X}} = \underline{\underline{\boldsymbol{F}}}^{-1} \cdot d\vec{\boldsymbol{x}} \cdot \underline{\underline{\boldsymbol{F}}}^{-1} \cdot d\vec{\boldsymbol{x}} = d\vec{\boldsymbol{x}} \cdot \underline{\underline{\boldsymbol{F}}}^{-T} \cdot \underline{\underline{\boldsymbol{F}}}^{-1} \cdot d\vec{\boldsymbol{x}} = d\vec{\boldsymbol{x}} \cdot \underline{\underline{\boldsymbol{F}}}^{-T} \cdot \underline{\underline{\boldsymbol{F}}}^{-1} \cdot d\vec{\boldsymbol{x}}$$
$$(dS)^2 = dX_i dX_i = F_{ij}^{-1} dx_j F_{ik}^{-1} dx_k = dx_j F_{ij}^{-1} F_{ik}^{-1} dx_k = dx_j F_{ji}^{-T} F_{ik}^{-1} dx_k$$

where:

$$[(*)^{-1}]^T = (*)^{-T}$$

Note the property: the transposed of a product is equal to the second term times the transposed of the second.

The information of distance between two points is contained in the deformation gradient tensor $\underline{\underline{\boldsymbol{F}}}$.

# Material Strain Tensor

We derived that:

$$(ds)^2 = d\vec{\boldsymbol{X}} \cdot \underline{\underline{\boldsymbol{F}}}^T \cdot \underline{\underline{\boldsymbol{F}}} \cdot d\vec{\boldsymbol{X}}; \quad (dS)^2 = d\vec{\boldsymbol{X}} \cdot d\vec{\boldsymbol{X}}$$

By taking the difference between the two squared distances:

$$\begin{aligned}(ds)^2-(dS)^2 = d\vec{\boldsymbol{X}}\cdot\underline{\underline{\boldsymbol{F}}}^T\cdot\underline{\underline{\boldsymbol{F}}}\cdot d\vec{\boldsymbol{X}}-d\vec{\boldsymbol{X}}\cdot d\vec{\boldsymbol{X}} = d\vec{\boldsymbol{X}}\cdot\underline{\underline{\boldsymbol{F}}}^T\cdot\underline{\underline{\boldsymbol{F}}}\cdot d\vec{\boldsymbol{X}}-d\vec{\boldsymbol{X}}\cdot\underline{\underline{\boldsymbol{1}}}\cdot d\vec{\boldsymbol{X}} &= d\vec{\boldsymbol{X}}\cdot(\underline{\underline{\boldsymbol{F}}}^T\cdot\underline{\underline{\boldsymbol{F}}}-\underline{\underline{\boldsymbol{1}}})\cdot d\vec{\boldsymbol{X}} \\ &= 2d\vec{\boldsymbol{X}} \cdot \underline{\underline{\boldsymbol{E}}} \cdot d\vec{\boldsymbol{X}}\end{aligned}$$

The Green-Lagrange or Material Strain Tensor can be defined as:

$$\underline{\underline{\boldsymbol{E}}}(\vec{\boldsymbol{X}}, t) = \frac{1}{2}(\underline{\underline{\boldsymbol{F}}}^T \cdot \underline{\underline{\boldsymbol{F}}} - \underline{\underline{\boldsymbol{1}}})$$

$$E_{ij}(\vec{\boldsymbol{X}}, t) = \frac{1}{2}(F_{ki}F_{kj} - \delta_{ij}) = \frac{1}{2}(F_{ik}^T F_{kj} - \delta_{ij}); \quad i, j \in \{1, 2, 3\}$$

The Material Strain Tensor is symmetric:

$$\underline{\underline{\boldsymbol{E}}}^T = \frac{1}{2}(\underline{\underline{\boldsymbol{F}}}^T \cdot \underline{\underline{\boldsymbol{F}}} - \underline{\underline{\boldsymbol{1}}})^T = \frac{1}{2}(\underline{\underline{\boldsymbol{F}}}^T \cdot (\underline{\underline{\boldsymbol{F}}}^T)^T - \underline{\underline{\boldsymbol{1}}}^T) = \frac{1}{2}(\underline{\underline{\boldsymbol{F}}}^T \cdot \underline{\underline{\boldsymbol{F}}} - \underline{\underline{\boldsymbol{1}}}) = \underline{\underline{\boldsymbol{E}}}$$

$$E_{ij} = E_{ji}; \quad i, j \in \{1, 2, 3\}$$

# Clarification

The dot product in index notation between a vector and a $2^{nd}$ order tensor can be written as:

$$(F_{ij}dX_j)^T = dX_jF_{ji}$$

where the term $F_{ji}$ is the transpose of $F_{ij}$. In direct tensor notation:

$$(\underline{\underline{\boldsymbol{F}}} \cdot d\vec{\boldsymbol{X}})^T = d\vec{\boldsymbol{X}} \cdot \underline{\underline{\boldsymbol{F}}}^T$$

In the case of dot product of two $2^{nd}$ order tensors:

$$(F_{ij}G_{jk})^T = G_{kj}F_{ji}$$

where the term $G_{kj}$ is the transpose of the term $G_{jk}$ and the term $F_{ji}$ is the transpose of the term $F_{ij}$.In direct tensor notation:

$$(\underline{\underline{\boldsymbol{F}}} \cdot \underline{\underline{\boldsymbol{G}}})^T = \underline{\underline{\boldsymbol{G}}}^T \cdot \underline{\underline{\boldsymbol{F}}}^T$$

# Spatial Strain Tensor

We derived that:

$$(ds)^2 = d\vec{\boldsymbol{x}} \cdot d\vec{\boldsymbol{x}}; \quad (dS)^2 = d\vec{\boldsymbol{x}} \cdot \underline{\underline{\boldsymbol{F}}}^{-T} \cdot \underline{\underline{\boldsymbol{F}}}^{-1} \cdot d\vec{\boldsymbol{x}}$$

By taking the difference between the two squared distances:

$$\begin{aligned}(ds)^2 - (dS)^2 &= d\vec{\boldsymbol{x}} \cdot d\vec{\boldsymbol{x}} - d\vec{\boldsymbol{x}} \cdot \underline{\underline{\boldsymbol{F}}}^{-T} \cdot \underline{\underline{\boldsymbol{F}}}^{-1} \cdot d\vec{\boldsymbol{x}} = d\vec{\boldsymbol{x}} \cdot \underline{\underline{\boldsymbol{1}}} \cdot d\vec{\boldsymbol{x}} - d\vec{\boldsymbol{x}} \cdot \underline{\underline{\boldsymbol{F}}}^{-T} \cdot \underline{\underline{\boldsymbol{F}}}^{-1} \cdot d\vec{\boldsymbol{x}} \\ &= d\vec{\boldsymbol{x}} \cdot (\underline{\underline{\boldsymbol{1}}} - \underline{\underline{\boldsymbol{F}}}^{-T} \cdot \underline{\underline{\boldsymbol{F}}}^{-1}) \cdot d\vec{\boldsymbol{x}} = 2 \cdot d\vec{\boldsymbol{x}} \cdot \underline{\underline{\boldsymbol{e}}} \cdot d\vec{\boldsymbol{x}}\end{aligned}$$

The Euler-Almansi or Spatial Strain Tensor can be defined as:

$$\underline{\underline{\boldsymbol{e}}}(\vec{\boldsymbol{x}}, t) = \frac{1}{2}(\underline{\underline{\boldsymbol{1}}} - \underline{\underline{\boldsymbol{F}}}^{-T} \cdot \underline{\underline{\boldsymbol{F}}}^{-1})$$

$$e_{ij}(\vec{\boldsymbol{x}}, t) = \frac{1}{2}(\delta_{ij} - F^{-1}_{ki} F^{-1}_{kj}) = \frac{1}{2}(\delta_{ij} - F^{-T}_{ik} F^{-1}_{kj}); \quad i, j \in \{1, 2, 3\}$$

The Spatial Strain Tensor is symmetric:

$$\underline{\underline{\boldsymbol{e}}}^T = \frac{1}{2}(\underline{\underline{\boldsymbol{1}}} - \underline{\underline{\boldsymbol{F}}}^{-T} \cdot \underline{\underline{\boldsymbol{F}}}^{-1})^T = \frac{1}{2}(\underline{\underline{\boldsymbol{1}}}^T - (\underline{\underline{\boldsymbol{F}}}^{-1})^T \cdot (\underline{\underline{\boldsymbol{F}}}^{-T})^T) = \frac{1}{2}(\underline{\underline{\boldsymbol{1}}} - \underline{\underline{\boldsymbol{F}}}^{-T} \cdot \underline{\underline{\boldsymbol{F}}}^{-1}) = \underline{\underline{\boldsymbol{e}}}$$

$$e_{ij} = e_{ji}; \quad i, j \in \{1, 2, 3\}$$

# Strain Tensors

The Material and the Spatial Tensors are not the same tensor:

$$(ds)^2 - (dS)^2 = 2d\vec{\boldsymbol{X}} \cdot \underline{\underline{\boldsymbol{E}}} \cdot d\vec{\boldsymbol{X}} = 2d\vec{\boldsymbol{x}} \cdot \underline{\underline{\boldsymbol{e}}} \cdot d\vec{\boldsymbol{x}}$$

They are not the material and spatial description of the same tensor, [2].
The material strain tensor is obtained in material coordinates. By replacing the inverse equation of motion in the material strain tensor:

$$\underline{\underline{\boldsymbol{E}}}(\vec{\boldsymbol{X}}(\vec{\boldsymbol{x}}, t), t) = \underline{\underline{\boldsymbol{E}}}(\vec{\boldsymbol{x}}, t)$$

which the material strain tensor in spatial coordinates.
The spatial strain tensor is obtained in spatial coordinates. By replacing the equation of motion the spatial strain tensor:

$$\underline{\underline{\boldsymbol{e}}}(\vec{\boldsymbol{x}}(\vec{\boldsymbol{X}}, t), t) = \underline{\underline{\boldsymbol{e}}}(\vec{\boldsymbol{X}}, t)$$

# Strain Tensors in Displacement terms

By replacing the following equation $\underline{\underline{\boldsymbol{F}}} = \underline{\underline{\boldsymbol{D}}} + \underline{\underline{\mathbf{1}}}$, where $\underline{\underline{\boldsymbol{D}}} = \vec{\boldsymbol{U}}(\vec{\boldsymbol{X}}, t) \otimes \vec{\boldsymbol{\nabla}}_X$, in terms of displacements in the material strain tensor $\underline{\underline{\boldsymbol{E}}} = \frac{1}{2}(\underline{\underline{\boldsymbol{F}}}^T \cdot \underline{\underline{\boldsymbol{F}}} - \underline{\underline{\mathbf{1}}})$:

$$\underline{\underline{\boldsymbol{E}}} = \frac{1}{2}[(\underline{\underline{\boldsymbol{D}}}^T + \underline{\underline{\mathbf{1}}}) \cdot (\underline{\underline{\boldsymbol{D}}} + \underline{\underline{\mathbf{1}}}) - \underline{\underline{\mathbf{1}}}] = \frac{1}{2}[\underline{\underline{\boldsymbol{D}}} + \underline{\underline{\boldsymbol{D}}}^T + \underline{\underline{\boldsymbol{D}}}^T \cdot \underline{\underline{\boldsymbol{D}}}]$$

$$E_{ij} = \frac{1}{2}\left[\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i} + \frac{\partial U_k}{\partial X_i}\frac{\partial U_k}{\partial X_j}\right]; \quad i, j, k \in \{1, 2, 3\}$$

Note that: $\underline{\underline{\boldsymbol{D}}}^T = \vec{\boldsymbol{\nabla}}_X \otimes \vec{\boldsymbol{U}}(\vec{\boldsymbol{X}}, t)$. By replacing the following equation $\underline{\underline{\boldsymbol{F}}}^{-1} = \underline{\underline{\mathbf{1}}} - \underline{\underline{\boldsymbol{d}}}$, where $\underline{\underline{\boldsymbol{d}}}(\vec{\boldsymbol{x}}, t) = \vec{\boldsymbol{u}}(\vec{\boldsymbol{x}}, t) \otimes \vec{\boldsymbol{\nabla}}_x$, in terms of displacements in the spatial strain tensor $\underline{\underline{\boldsymbol{e}}} = \frac{1}{2}(\underline{\underline{\mathbf{1}}} - \underline{\underline{\boldsymbol{F}}}^{-T} \cdot \underline{\underline{\boldsymbol{F}}}^{-1})$:

$$\underline{\underline{\boldsymbol{e}}} = \frac{1}{2}[\underline{\underline{\mathbf{1}}} - (\underline{\underline{\mathbf{1}}} - \underline{\underline{\boldsymbol{d}}}^T) \cdot (\underline{\underline{\mathbf{1}}} - \underline{\underline{\boldsymbol{d}}})] = \frac{1}{2}[\underline{\underline{\boldsymbol{d}}} + \underline{\underline{\boldsymbol{d}}}^T - \underline{\underline{\boldsymbol{d}}}^T \cdot \underline{\underline{\boldsymbol{d}}}]$$

$$e_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right]; \quad i, j, k \in \{1, 2, 3\}$$

## Strain Tensors - Normal Strains

Using the material strain tensor:

$$E_{ij} = \frac{1}{2}\left[\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i} + \frac{\partial U_k}{\partial X_i}\frac{\partial U_k}{\partial X_j}\right]; \qquad i, j, k \in \{1, 2, 3\}$$

the $E_{11}, E_{22}, E_{33}$, which are called **normal strains**, can be calculated in terms of displacements:

$$E_{11} = \frac{\partial U_1}{\partial X_1} + \frac{1}{2}\left[\left(\frac{\partial U_1}{\partial X_1}\right)^2 + \left(\frac{\partial U_2}{\partial X_1}\right)^2 + \left(\frac{\partial U_3}{\partial X_1}\right)^2\right]$$

$$E_{22} = \frac{\partial U_2}{\partial X_2} + \frac{1}{2}\left[\left(\frac{\partial U_1}{\partial X_2}\right)^2 + \left(\frac{\partial U_2}{\partial X_2}\right)^2 + \left(\frac{\partial U_3}{\partial X_2}\right)^2\right]$$

$$E_{33} = \frac{\partial U_3}{\partial X_3} + \frac{1}{2}\left[\left(\frac{\partial U_1}{\partial X_3}\right)^2 + \left(\frac{\partial U_2}{\partial X_3}\right)^2 + \left(\frac{\partial U_3}{\partial X_3}\right)^2\right]$$

# Strain Tensors - Shear Strains

Using the material strain tensor:

$$E_{ij} = \frac{1}{2}\left[\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i} + \frac{\partial U_k}{\partial X_i}\frac{\partial U_k}{\partial X_j}\right]; \quad i, j, k \in \{1, 2, 3\}$$

the $E_{12}, E_{13}, E_{23}$, which are called **shear strains**, can be calculated in terms of displacements:

$$E_{12} = \frac{1}{2}\left(\frac{\partial U_1}{\partial X_2} + \frac{\partial U_2}{\partial X_1} + \frac{\partial U_1}{\partial X_1}\frac{\partial U_1}{\partial X_2} + \frac{\partial U_2}{\partial X_1}\frac{\partial U_2}{\partial X_2} + \frac{\partial U_3}{\partial X_1}\frac{\partial U_3}{\partial X_2}\right)$$

$$E_{13} = \frac{1}{2}\left(\frac{\partial U_1}{\partial X_3} + \frac{\partial U_3}{\partial X_1} + \frac{\partial U_1}{\partial X_1}\frac{\partial U_1}{\partial X_3} + \frac{\partial U_2}{\partial X_1}\frac{\partial U_2}{\partial X_3} + \frac{\partial U_3}{\partial X_1}\frac{\partial U_3}{\partial X_3}\right)$$

$$E_{23} = \frac{1}{2}\left(\frac{\partial U_2}{\partial X_3} + \frac{\partial U_3}{\partial X_2} + \frac{\partial U_1}{\partial X_2}\frac{\partial U_1}{\partial X_3} + \frac{\partial U_2}{\partial X_2}\frac{\partial U_2}{\partial X_3} + \frac{\partial U_3}{\partial X_2}\frac{\partial U_3}{\partial X_3}\right)$$

# Stretch

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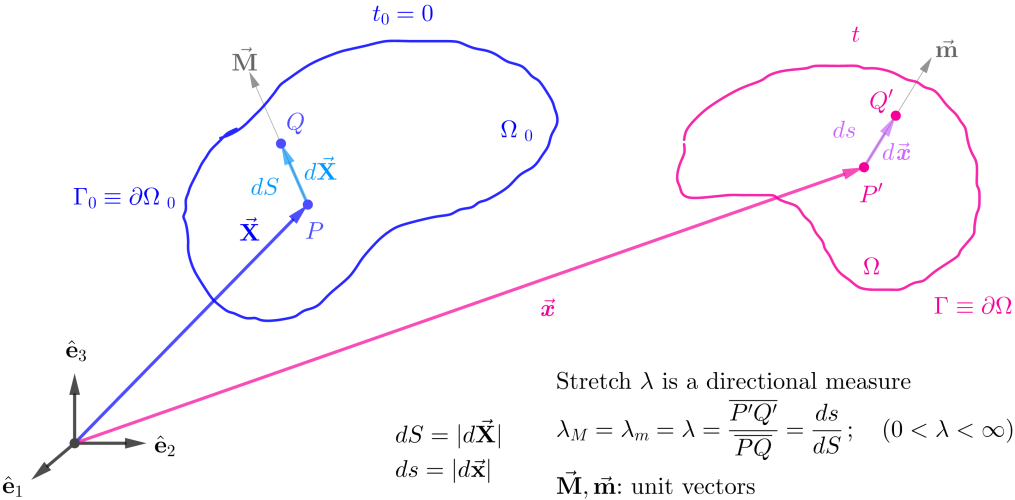


$$dS = |d\vec{\mathbf{X}}|$$
$$ds = |d\vec{\mathbf{x}}|$$

Stretch $\lambda$ is a directional measure

$$\lambda_M = \lambda_m = \lambda = \frac{\overline{P'Q'}}{\overline{PQ}} = \frac{ds}{dS}\,; \quad (0 < \lambda < \infty)$$

$\vec{\mathbf{M}}, \vec{\mathbf{m}}$: unit vectors

# Unit elongation

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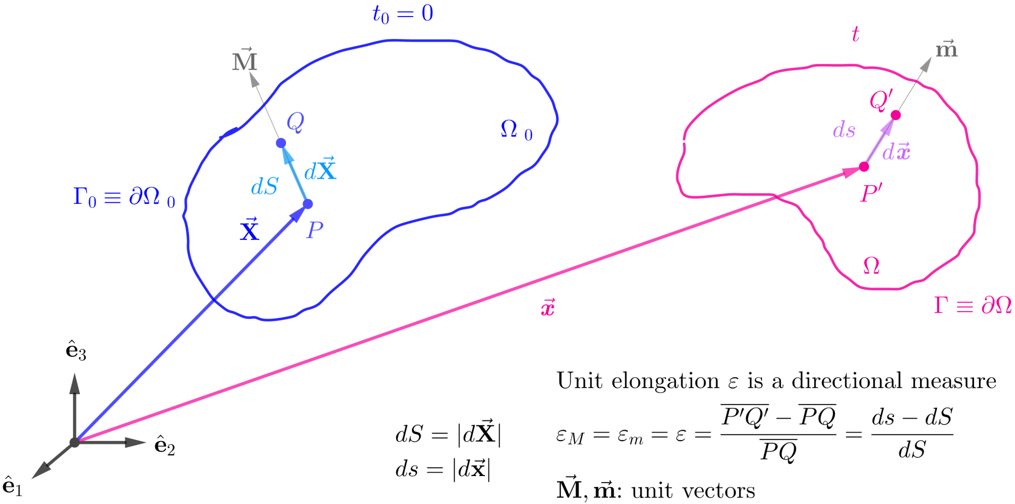


$$dS = |d\vec{\mathbf{X}}|$$
$$ds = |d\vec{\mathbf{x}}|$$

Unit elongation $\varepsilon$ is a directional measure

$$\varepsilon_M = \varepsilon_m = \varepsilon = \frac{\overline{P'Q'} - \overline{PQ}}{\overline{PQ}} = \frac{ds - dS}{dS}$$

$\vec{\mathbf{M}}, \vec{\mathbf{m}}$: unit vectors

## Stretch - Unit Elongation

- For the same direction the relationship between stretch $\lambda$ and unit elongation $\varepsilon$ is, [2]:

$$\varepsilon = \frac{ds - dS}{dS} = \underbrace{\frac{ds}{dS}}_{\lambda} - 1 = \lambda - 1;$$

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- If $\lambda > 1$ then $\varepsilon > 0$. This means that $ds > dS$ and that points $P$ and $Q$ have moved in time and their distance between them increased. $\lambda > 1$ means increased distance between the two points.

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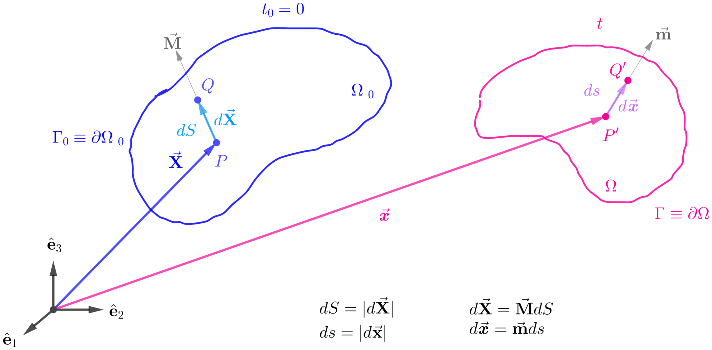
- If $\lambda > 1$ then $\varepsilon > 0$. This means that $ds > dS$ and that points $P$ and $Q$ have moved in time and their distance between them increased. $\lambda > 1$ means increased distance between the two points.
- If $\lambda < 1$ then $\varepsilon < 0$. This means that $ds < dS$ and that points $P$ and $Q$ have moved in time and their distance between them decreased. $\lambda < 1$ means decreased distance between the two points.

# Strain tensors - Stretch - Unit elongation

According to the definitions of strains:

$$(ds)^2 - (dS)^2 = 2d\vec{\boldsymbol{X}} \cdot \underline{\underline{\boldsymbol{E}}} \cdot d\vec{\boldsymbol{X}}; \quad d\vec{\boldsymbol{X}} = \vec{\boldsymbol{M}}dS$$

$$(ds)^2 - (dS)^2 = 2d\vec{\boldsymbol{x}} \cdot \underline{\underline{\boldsymbol{e}}} \cdot d\vec{\boldsymbol{x}}; \quad d\vec{\boldsymbol{x}} = \vec{\boldsymbol{m}}ds$$

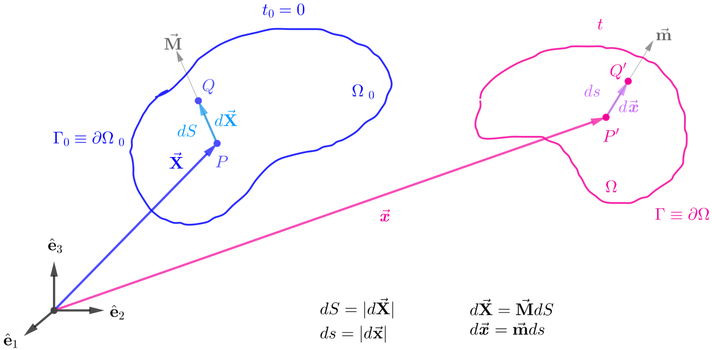


# Strain tensors - Stretch - Unit elongation

Using the definition of stretch $\lambda = \frac{ds}{dS}$:

$$(ds)^2 - (dS)^2 = 2(dS)^2 \vec{\boldsymbol{M}} \cdot \underline{\underline{\boldsymbol{E}}} \cdot \vec{\boldsymbol{M}} = \lambda^2 - 1 = 2\vec{\boldsymbol{M}} \cdot \underline{\underline{\boldsymbol{E}}} \cdot \vec{\boldsymbol{M}}$$

$$(ds)^2 - (dS)^2 = 2(ds)^2 \vec{\boldsymbol{m}} \cdot \underline{\underline{\boldsymbol{e}}} \cdot \vec{\boldsymbol{m}} = 1 - \frac{1}{\lambda^2} = 2\vec{\boldsymbol{m}} \cdot \underline{\underline{\boldsymbol{e}}} \cdot \vec{\boldsymbol{m}}$$

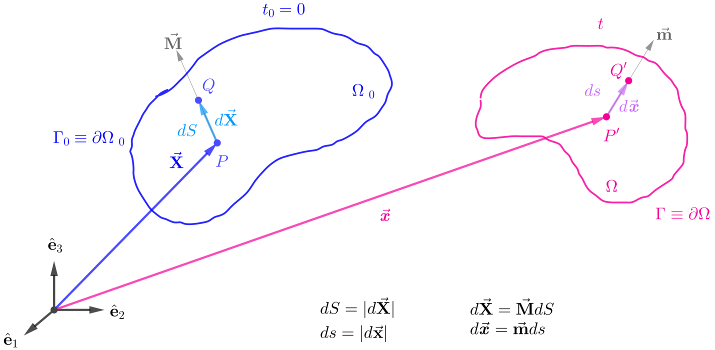


# Strain tensors - Stretch - Unit elongation

Using the relationship between stretch $\lambda$ and unit elongation $\varepsilon$, $\varepsilon = \lambda - 1$:

$$\lambda = \sqrt{1 + 2\vec{\boldsymbol{M}} \cdot \underline{\underline{\boldsymbol{E}}} \cdot \vec{\boldsymbol{M}}}; \quad \lambda = \frac{1}{\sqrt{1 - 2\vec{\boldsymbol{m}} \cdot \underline{\underline{\boldsymbol{e}}} \cdot \vec{\boldsymbol{m}}}}$$

$$\varepsilon = \sqrt{1 + 2\vec{\boldsymbol{M}} \cdot \underline{\underline{\boldsymbol{E}}} \cdot \vec{\boldsymbol{M}}} - 1; \quad \varepsilon = \frac{1}{\sqrt{1 - 2\vec{\boldsymbol{m}} \cdot \underline{\underline{\boldsymbol{e}}} \cdot \vec{\boldsymbol{m}}}} - 1$$




$dS = |d\vec{\mathbf{X}}|$ $\quad d\vec{\mathbf{X}} = \vec{\mathbf{M}}dS$

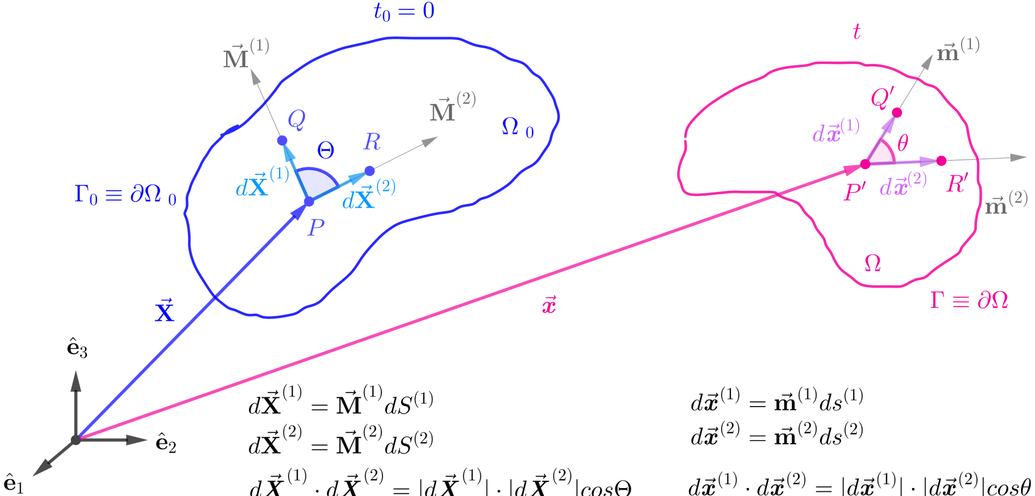
$ds = |d\vec{\mathbf{x}}|$ $\quad d\vec{x} = \vec{\mathbf{m}}ds$

Conclusion:

The strain tensors (material $\underline{\underline{\boldsymbol{E}}}$ and spatial $\underline{\underline{\boldsymbol{e}}}$) CONTAIN the information about **stretch** $\lambda$ and **unit elongation** $\varepsilon$.

# Angle Variation

# Angle Variation




$$d\vec{\mathbf{X}}^{(1)} = \vec{\mathbf{M}}^{(1)} dS^{(1)}$$
$$d\vec{\mathbf{X}}^{(2)} = \vec{\mathbf{M}}^{(2)} dS^{(2)}$$
$$d\vec{\boldsymbol{X}}^{(1)} \cdot d\vec{\boldsymbol{X}}^{(2)} = |d\vec{\boldsymbol{X}}^{(1)}| \cdot |d\vec{\boldsymbol{X}}^{(2)}| cos\Theta$$

$$d\vec{\boldsymbol{x}}^{(1)} = \vec{\mathbf{m}}^{(1)} ds^{(1)}$$
$$d\vec{\boldsymbol{x}}^{(2)} = \vec{\mathbf{m}}^{(2)} ds^{(2)}$$
$$d\vec{\boldsymbol{x}}^{(1)} \cdot d\vec{\boldsymbol{x}}^{(2)} = |d\vec{\boldsymbol{x}}^{(1)}| \cdot |d\vec{\boldsymbol{x}}^{(2)}| cos\theta$$

# Angle Variation

In material description:

$$d\vec{\boldsymbol{x}}^{(1)} \cdot d\vec{\boldsymbol{x}}^{(2)} = |d\vec{\boldsymbol{x}}^{(1)}| \cdot |d\vec{\boldsymbol{x}}^{(2)}| \cos\theta = ds^{(1)}\, ds^{(2)} \cos\theta$$

$$d\vec{\boldsymbol{x}}^{(1)} = \underline{\underline{\boldsymbol{F}}} \cdot d\vec{\boldsymbol{X}}^{(1)}; \quad d\vec{\boldsymbol{x}}^{(2)} = \underline{\underline{\boldsymbol{F}}} \cdot d\vec{\boldsymbol{X}}^{(2)}$$

$$d\vec{\boldsymbol{x}}^{(1)} \cdot d\vec{\boldsymbol{x}}^{(2)} = d\vec{\boldsymbol{X}}^{(1)} \cdot \underbrace{\underline{\underline{\boldsymbol{F}}}^T \cdot \underline{\underline{\boldsymbol{F}}}}_{2\underline{\underline{\boldsymbol{E}}}+\underline{\underline{\boldsymbol{1}}}} \cdot d\vec{\boldsymbol{X}}^{(2)}$$

$$d\vec{\boldsymbol{X}}^{(1)} = \vec{\boldsymbol{M}}^{(1)} \cdot dS^{(1)}; \quad d\vec{\boldsymbol{X}}^{(2)} = \vec{\boldsymbol{M}}^{(2)} \cdot dS^{(2)}$$

$$d\vec{\boldsymbol{x}}^{(1)} \cdot d\vec{\boldsymbol{x}}^{(2)} = \underbrace{dS^{(1)}}_{\frac{ds^{(1)}}{\lambda^{(1)}}} \vec{\boldsymbol{M}}^{(1)} \cdot (2\underline{\underline{\boldsymbol{E}}} + \underline{\underline{\boldsymbol{1}}}) \cdot \vec{\boldsymbol{M}}^{(2)} \underbrace{dS^{(2)}}_{\frac{ds^{(2)}}{\lambda^{(2)}}}$$

$$= ds^{(1)}\, ds^{(2)} \frac{1}{\lambda^{(1)}} \frac{1}{\lambda^{(2)}} \left( \vec{\boldsymbol{M}}^{(1)} \cdot (2\underline{\underline{\boldsymbol{E}}} + \underline{\underline{\boldsymbol{1}}}) \cdot \vec{\boldsymbol{M}}^{(2)} \right)$$

$$\lambda = \sqrt{1 + 2\vec{\boldsymbol{M}} \cdot \underline{\underline{\boldsymbol{E}}} \cdot \vec{\boldsymbol{M}}}$$

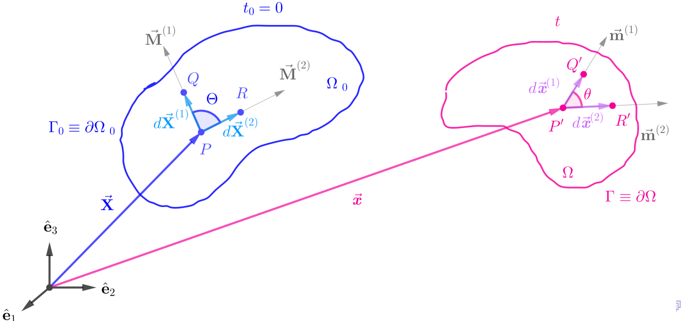
$$\frac{1}{\lambda^{(1)}} \frac{1}{\lambda^{(2)}} \left( \vec{\boldsymbol{M}}^{(1)} \cdot (2\underline{\underline{\boldsymbol{E}}} + \underline{\underline{\boldsymbol{1}}}) \cdot \vec{\boldsymbol{M}}^{(2)} \right) = \left( \frac{\vec{\boldsymbol{M}}^{(1)} \cdot (2\underline{\underline{\boldsymbol{E}}} + \underline{\underline{\boldsymbol{1}}}) \cdot \vec{\boldsymbol{M}}^{(2)}}{\sqrt{1 + 2\vec{\boldsymbol{M}}^{(1)} \cdot \underline{\underline{\boldsymbol{E}}} \cdot \vec{\boldsymbol{M}}^{(1)}}\sqrt{1 + 2\vec{\boldsymbol{M}}^{(2)} \cdot \underline{\underline{\boldsymbol{E}}} \cdot \vec{\boldsymbol{M}}^{(2)}}} \right)$$

# Angle Variation

$$d\vec{x}^{(1)} \cdot d\vec{x}^{(2)} = ds^{(1)} ds^{(2)} \left( \frac{\vec{M}^{(1)} \cdot (2\underline{\underline{E}} + \underline{\mathbf{1}}) \cdot \vec{M}^{(2)}}{\sqrt{1 + 2\vec{M}^{(1)} \cdot \underline{\underline{E}} \cdot \vec{M}^{(1)}} \sqrt{1 + 2\vec{M}^{(2)} \cdot \underline{\underline{E}} \cdot \vec{M}^{(2)}}} \right) = ds^{(1)} ds^{(2)} \cos\theta$$

$$\cos\theta = \left( \frac{\vec{M}^{(1)} \cdot (2\underline{\underline{E}} + \underline{\mathbf{1}}) \cdot \vec{M}^{(2)}}{\sqrt{1 + 2\vec{M}^{(1)} \cdot \underline{\underline{E}} \cdot \vec{M}^{(1)}} \sqrt{1 + 2\vec{M}^{(2)} \cdot \underline{\underline{E}} \cdot \vec{M}^{(2)}}} \right)$$

where $\vec{M}^{(1)}$, $\vec{M}^{(2)}$ are the unit vectors and $\underline{\underline{E}}$ is the material strain tensor.

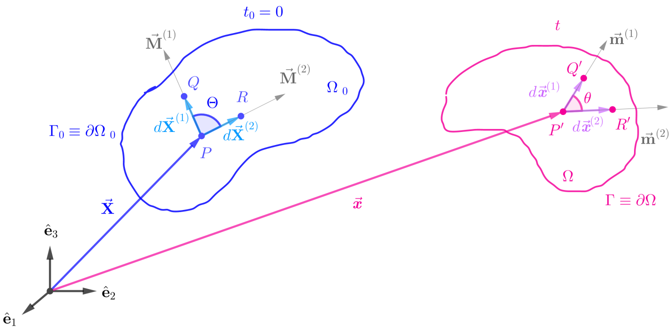


# Angle Variation

In spatial description:

$$\cos\Theta = \left( \frac{\vec{\boldsymbol{m}}^{(1)} \cdot (\underline{\underline{\mathbf{1}}} - 2\underline{\underline{\boldsymbol{e}}}) \cdot \vec{\boldsymbol{m}}^{(2)}}{\sqrt{1 - 2\vec{\boldsymbol{m}}^{(1)} \cdot \underline{\underline{\boldsymbol{e}}} \cdot \vec{\boldsymbol{m}}^{(1)}}\sqrt{1 - 2\vec{\boldsymbol{m}}^{(2)} \cdot \underline{\underline{\boldsymbol{e}}} \cdot \vec{\boldsymbol{m}}^{(2)}}} \right)$$

where $\vec{\boldsymbol{m}}^{(1)}$, $\vec{\boldsymbol{m}}^{(2)}$ are the unit vectors and $\underline{\underline{\boldsymbol{e}}}$ is the spatial strain tensor.



Conclusion:

The strain tensors (material $\underline{\underline{\boldsymbol{E}}}$ and spatial $\underline{\underline{\boldsymbol{e}}}$) CONTAIN the information about angle variation.

# Physical Interpretation of Strain Tensor

# Physical meaning of material strain tensor

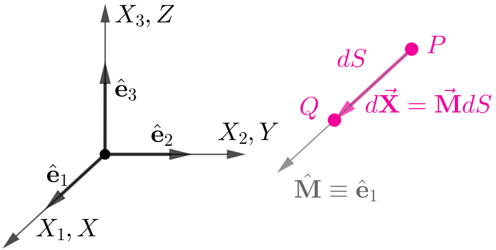
The material strain tensor can be given in matrix form:

$$\underline{\underline{\boldsymbol{E}}} \equiv \begin{bmatrix} E_{XX} & E_{XY} & E_{XZ} \\ E_{YX} & E_{YY} & E_{YZ} \\ E_{ZX} & E_{ZY} & E_{ZZ} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix}$$

Consider that we have a segment oriented along the X-axis (material coordinates). In this case the stretch will be:

$$\lambda = \sqrt{1 + 2\vec{\boldsymbol{M}} \cdot \underline{\underline{\boldsymbol{E}}} \cdot \vec{\boldsymbol{M}}} \equiv \sqrt{1 + 2 \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} E_{XX} & E_{XY} & E_{XZ} \\ E_{YX} & E_{YY} & E_{YZ} \\ E_{ZX} & E_{ZY} & E_{ZZ} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} = \sqrt{1 + 2E_{11}}$$

which is equivalent with stretching of the material in X-direction.



# Physical meaning of material strain tensor

The stretching of the material along the $X(X_1)$, $Y(X_2)$ and $Z(X_3)$ directions, [2]:

$$\lambda_1 = \sqrt{1+2E_{11}} \Leftrightarrow \varepsilon_1 = \lambda_1 - 1 = \sqrt{1+2E_{11}} - 1$$
$$\lambda_2 = \sqrt{1+2E_{22}} \Leftrightarrow \varepsilon_2 = \lambda_2 - 1 = \sqrt{1+2E_{22}} - 1$$
$$\lambda_3 = \sqrt{1+2E_{33}} \Leftrightarrow \varepsilon_3 = \lambda_3 - 1 = \sqrt{1+2E_{33}} - 1$$

The diagonal terms of the material strain tensor contain information about longitudinal strains

$$\underline{\underline{\boldsymbol{E}}} \equiv \begin{bmatrix} E_{XX} & E_{XY} & E_{XZ} \\ E_{YX} & E_{YY} & E_{YZ} \\ E_{ZX} & E_{ZY} & E_{ZZ} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix}$$

1. For $E_{XX} = 0$ no elongation along $X$ axis, $\varepsilon_X = \varepsilon_1 = 0$.

# Physical meaning of material strain tensor

The stretching of the material along the $X(X_1)$, $Y(X_2)$ and $Z(X_3)$ directions, [2]:

$$\lambda_1 = \sqrt{1+2E_{11}} \Leftrightarrow \varepsilon_1 = \lambda_1 - 1 = \sqrt{1+2E_{11}} - 1$$
$$\lambda_2 = \sqrt{1+2E_{22}} \Leftrightarrow \varepsilon_2 = \lambda_2 - 1 = \sqrt{1+2E_{22}} - 1$$
$$\lambda_3 = \sqrt{1+2E_{33}} \Leftrightarrow \varepsilon_3 = \lambda_3 - 1 = \sqrt{1+2E_{33}} - 1$$

The diagonal terms of the material strain tensor contain information about longitudinal strains

$$\underline{\underline{\boldsymbol{E}}} \equiv \begin{bmatrix} E_{XX} & E_{XY} & E_{XZ} \\ E_{YX} & E_{YY} & E_{YZ} \\ E_{ZX} & E_{ZY} & E_{ZZ} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix}$$

1. For $E_{XX} = 0$ no elongation along $X$ axis, $\varepsilon_X = \varepsilon_1 = 0$.
2. For $E_{YY} = 0$ no elongation along $Y$ axis, $\varepsilon_Y = \varepsilon_2 = 0$.

# Physical meaning of material strain tensor

The stretching of the material along the $X(X_1)$, $Y(X_2)$ and $Z(X_3)$ directions, [2]:

$$\lambda_1 = \sqrt{1+2E_{11}} \Leftrightarrow \varepsilon_1 = \lambda_1 - 1 = \sqrt{1+2E_{11}} - 1$$
$$\lambda_2 = \sqrt{1+2E_{22}} \Leftrightarrow \varepsilon_2 = \lambda_2 - 1 = \sqrt{1+2E_{22}} - 1$$
$$\lambda_3 = \sqrt{1+2E_{33}} \Leftrightarrow \varepsilon_3 = \lambda_3 - 1 = \sqrt{1+2E_{33}} - 1$$

The diagonal terms of the material strain tensor contain information about longitudinal strains

$$\underline{\underline{\boldsymbol{E}}} \equiv \begin{bmatrix} E_{XX} & E_{XY} & E_{XZ} \\ E_{YX} & E_{YY} & E_{YZ} \\ E_{ZX} & E_{ZY} & E_{ZZ} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix}$$

1. For $E_{XX} = 0$ no elongation along $X$ axis, $\varepsilon_X = \varepsilon_1 = 0$.
2. For $E_{YY} = 0$ no elongation along $Y$ axis, $\varepsilon_Y = \varepsilon_2 = 0$.
3. For $E_{ZZ} = 0$ no elongation along $Z$ axis, $\varepsilon_Z = \varepsilon_3 = 0$.

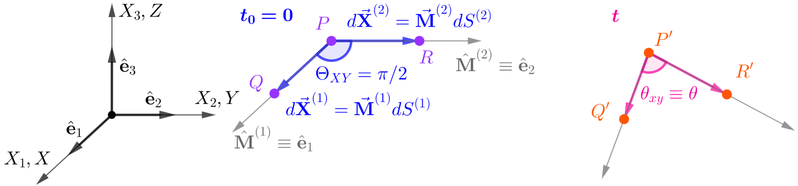
# Physical meaning of material strain tensor

Consider that we have two segments, one parallel to $X_1$-axis and one parallel to $X_2$-axis and the angle between the two segments is given by:

$$\cos\theta = \left( \frac{\vec{\boldsymbol{M}}^{(1)} \cdot (2\underline{\underline{\boldsymbol{E}}} + \underline{\underline{\mathbf{1}}}) \cdot \vec{\boldsymbol{M}}^{(2)}}{\sqrt{1 + 2\vec{\boldsymbol{M}}^{(1)} \cdot \underline{\underline{\boldsymbol{E}}} \cdot \vec{\boldsymbol{M}}^{(1)}}\sqrt{1 + 2\vec{\boldsymbol{M}}^{(2)} \cdot \underline{\underline{\boldsymbol{E}}} \cdot \vec{\boldsymbol{M}}^{(2)}}} \right)$$

Noting that: $\vec{\boldsymbol{M}}^{(1)} = \{1\ 0\ 0\}$, $\vec{\boldsymbol{M}}^{(2)} = \{0\ 1\ 0\}$, $\vec{\boldsymbol{M}}^{(1)} \cdot \vec{\boldsymbol{M}}^{(2)} = 0$, $\vec{\boldsymbol{M}}^{(1)} \cdot \underline{\underline{\boldsymbol{E}}} \cdot \vec{\boldsymbol{M}}^{(1)} = E_{11}$, $\vec{\boldsymbol{M}}^{(1)} \cdot \underline{\underline{\boldsymbol{E}}} \cdot \vec{\boldsymbol{M}}^{(2)} = E_{12}$, $\vec{\boldsymbol{M}}^{(2)} \cdot \underline{\underline{\boldsymbol{E}}} \cdot \vec{\boldsymbol{M}}^{(2)} = E_{22}$:

$$\cos\theta_{xy} = \frac{2E_{12}}{\sqrt{1 + 2E_{11}}\sqrt{1 + 2E_{22}}} \Leftrightarrow \theta_{xy} = \arccos\frac{2E_{XY}}{\sqrt{1 + 2E_{XX}}\sqrt{1 + 2E_{YY}}}$$



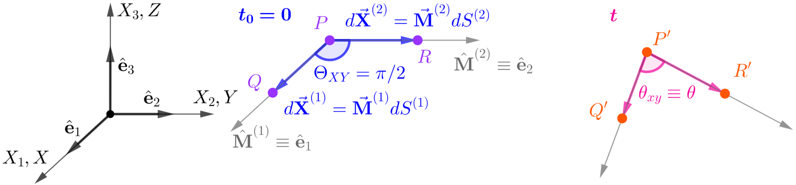
# Physical meaning of material strain tensor

$$\theta_{xy} = \frac{\pi}{2} - \arcsin \frac{2E_{XY}}{\sqrt{1+2E_{XX}}\sqrt{1+2E_{YY}}}$$

The angle increment can be derived as follows:

$$\Delta\Theta_{XY} = \theta_{xy} - \underbrace{\Theta_{XY}}_{\frac{\pi}{2}} = -\arcsin \frac{2E_{12}}{\sqrt{1+2E_{11}}\sqrt{1+2E_{22}}}$$

$$\Leftrightarrow \sin\Delta\Theta_{XY} = -\frac{2E_{XY}}{\sqrt{1+2E_{XX}}\sqrt{1+2E_{YY}}}$$



# Physical meaning of material strain tensor

Following the same process we can conclude:

$$\sin\Delta\Theta_{XY} = -\frac{2E_{XY}}{\sqrt{1+2E_{XX}}\sqrt{1+2E_{YY}}}$$

$$\sin\Delta\Theta_{XZ} = -\frac{2E_{XZ}}{\sqrt{1+2E_{XX}}\sqrt{1+2E_{ZZ}}}$$

$$\sin\Delta\Theta_{YZ} = -\frac{2E_{YZ}}{\sqrt{1+2E_{YY}}\sqrt{1+2E_{ZZ}}}$$

The angular strains $(E_{XY}, E_{XZ}, E_{YZ})$ of the material strain tensor contain information about the variation of angles that were oriented in $X, Y, Z$ directions:

$$\underline{\underline{\boldsymbol{E}}} \equiv \begin{bmatrix} E_{XX} & E_{XY} & E_{XZ} \\ E_{YX} & E_{YY} & E_{YZ} \\ E_{ZX} & E_{ZY} & E_{ZZ} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix}$$

1. For $E_{XY} = 0$ no angle variation between the $X$ and $Y$ axis.

# Physical meaning of material strain tensor

Following the same process we can conclude:

$$\sin \Delta\Theta_{XY} = -\frac{2E_{XY}}{\sqrt{1+2E_{XX}}\sqrt{1+2E_{YY}}}$$

$$\sin \Delta\Theta_{XZ} = -\frac{2E_{XZ}}{\sqrt{1+2E_{XX}}\sqrt{1+2E_{ZZ}}}$$

$$\sin \Delta\Theta_{YZ} = -\frac{2E_{YZ}}{\sqrt{1+2E_{YY}}\sqrt{1+2E_{ZZ}}}$$

The angular strains $(E_{XY}, E_{XZ}, E_{YZ})$ of the material strain tensor contain information about the variation of angles that were oriented in $X$, $Y$, $Z$ directions:

$$\underline{\underline{\boldsymbol{E}}} \equiv \begin{bmatrix} E_{XX} & E_{XY} & E_{XZ} \\ E_{YX} & E_{YY} & E_{YZ} \\ E_{ZX} & E_{ZY} & E_{ZZ} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix}$$

1. For $E_{XY} = 0$ no angle variation between the $X$ and $Y$ axis.
2. For $E_{XZ} = 0$ no angle variation between the $X$ and $Z$ axis.

# Physical meaning of material strain tensor

Following the same process we can conclude:

$$\sin \Delta\Theta_{XY} = -\frac{2E_{XY}}{\sqrt{1+2E_{XX}}\sqrt{1+2E_{YY}}}$$
$$\sin \Delta\Theta_{XZ} = -\frac{2E_{XZ}}{\sqrt{1+2E_{XX}}\sqrt{1+2E_{ZZ}}}$$
$$\sin \Delta\Theta_{YZ} = -\frac{2E_{YZ}}{\sqrt{1+2E_{YY}}\sqrt{1+2E_{ZZ}}}$$

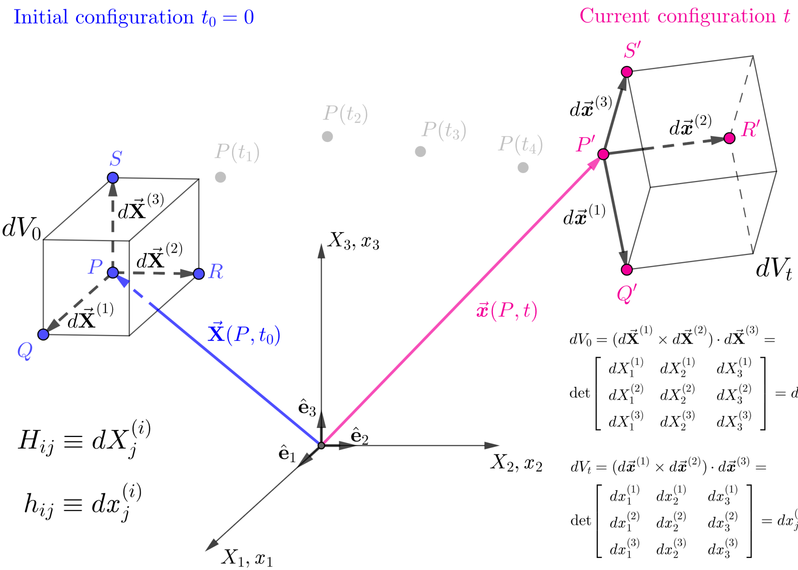
The angular strains $(E_{XY}, E_{XZ}, E_{YZ})$ of the material strain tensor contain information about the variation of angles that were oriented in $X$, $Y$, $Z$ directions:

$$\underline{\underline{\boldsymbol{E}}} \equiv \begin{bmatrix} E_{XX} & E_{XY} & E_{XZ} \\ E_{YX} & E_{YY} & E_{YZ} \\ E_{ZX} & E_{ZY} & E_{ZZ} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix}$$

1. For $E_{XY} = 0$ no angle variation between the $X$ and $Y$ axis.
2. For $E_{XZ} = 0$ no angle variation between the $X$ and $Z$ axis.
3. For $E_{YZ} = 0$ no angle variation between the $Y$ and $Z$ axis.

# Volume Variation

# Volume variation




$$dV_0 = (d\vec{\mathbf{X}}^{(1)} \times d\vec{\mathbf{X}}^{(2)}) \cdot d\vec{\mathbf{X}}^{(3)} = \det \begin{bmatrix} dX_1^{(1)} & dX_2^{(1)} & dX_3^{(1)} \\ dX_1^{(2)} & dX_2^{(2)} & dX_3^{(2)} \\ dX_1^{(3)} & dX_2^{(3)} & dX_3^{(3)} \end{bmatrix} = dX_j^{(i)}$$

$$dV_t = (d\vec{\boldsymbol{x}}^{(1)} \times d\vec{\boldsymbol{x}}^{(2)}) \cdot d\vec{\boldsymbol{x}}^{(3)} = \det \begin{bmatrix} dx_1^{(1)} & dx_2^{(1)} & dx_3^{(1)} \\ dx_1^{(2)} & dx_2^{(2)} & dx_3^{(2)} \\ dx_1^{(3)} & dx_2^{(3)} & dx_3^{(3)} \end{bmatrix} = dx_j^{(i)}$$

$$H_{ij} \equiv dX_j^{(i)}$$

$$h_{ij} \equiv dx_j^{(i)}$$

# Differential Volume

From the fundamental equation of deformation we have:

$$d\vec{\boldsymbol{x}}^{(i)} = \underline{\underline{\boldsymbol{F}}} \cdot d\vec{\boldsymbol{X}}^{(i)} \qquad i \in \{1,2,3\}$$

$$dx_j^{(i)} = F_{jk} dX_k^{(i)} \qquad i,j,k \in \{1,2,3\}$$

We also know that:

$$H_{ij} \equiv dX_j^{(i)}; \qquad h_{ij} \equiv dx_j^{(i)}$$

Combining the above equations:

$$h_{ij} = dx_j^{(i)} = F_{jk} dX_k^{(i)} = F_{jk} H_{ik} = H_{ik} F_{kj}^T \Leftrightarrow \boldsymbol{h} = \boldsymbol{H} \cdot \boldsymbol{F}^T$$

Therefore:

$$dV_t = \det[\boldsymbol{h}] = \det[\boldsymbol{H} \cdot \boldsymbol{F}^T] = \det[\boldsymbol{H}] \det[\boldsymbol{F}^T] = \det[\boldsymbol{F}] \underbrace{\det[\boldsymbol{H}]}_{dV_0} \Leftrightarrow dV_t = \det[\boldsymbol{F}] dV_0$$

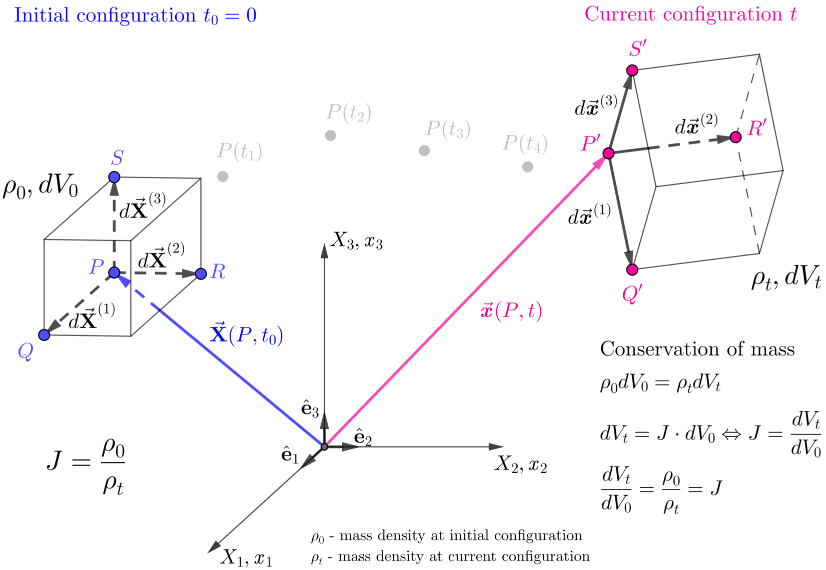
The determinant of the deformation gradient tensor is called the Jacobian $J(\vec{\boldsymbol{X}}, t)$:

$$J(\vec{\boldsymbol{X}}, t) = \det[\boldsymbol{F}(\vec{\boldsymbol{X}}, t)] > 0 \Leftrightarrow dV_t = J \cdot dV_0$$

## Volume variation - Physical meaning

**Since the material cannot have negative volume after deformation (if $J < 0$ then $dV_t < 0$ through $dV_t = J \cdot dV_0$)**
**No interpenetration of particles!!!**

# Jacobian - Physical meaning




Conservation of mass

$$\rho_0 dV_0 = \rho_t dV_t$$

$$dV_t = J \cdot dV_0 \Leftrightarrow J = \frac{dV_t}{dV_0}$$

$$\frac{dV_t}{dV_0} = \frac{\rho_0}{\rho_t} = J$$

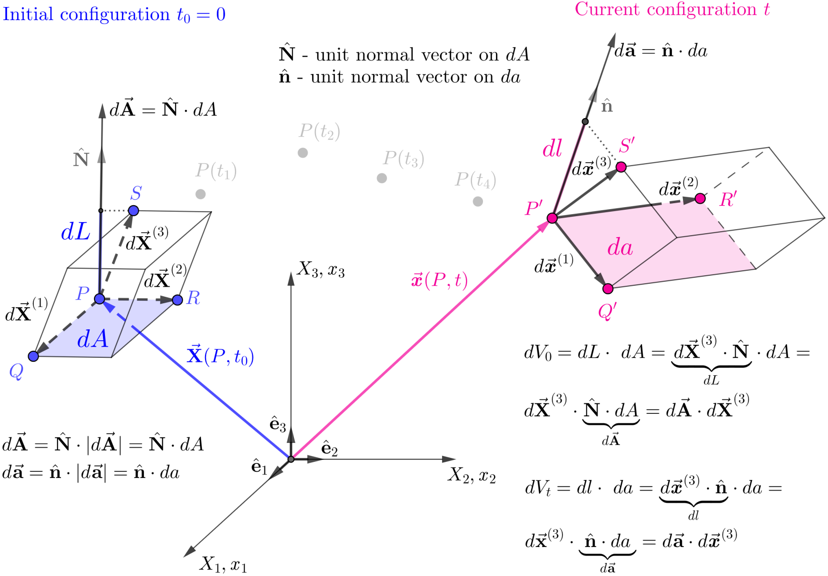
$$J = \frac{\rho_0}{\rho_t}$$

$\rho_0$ - mass density at initial configuration

$\rho_t$ - mass density at current configuration

# Area Variation

# Differential Area




$$d\vec{\mathbf{A}} = \hat{\mathbf{N}} \cdot |d\vec{\mathbf{A}}| = \hat{\mathbf{N}} \cdot dA$$

$$d\vec{\mathbf{a}} = \hat{\mathbf{n}} \cdot |d\vec{\mathbf{a}}| = \hat{\mathbf{n}} \cdot da$$

$$dV_0 = dL \cdot dA = \underbrace{d\vec{\mathbf{X}}^{(3)} \cdot \hat{\mathbf{N}}}_{dL} \cdot dA =$$

$$d\vec{\mathbf{X}}^{(3)} \cdot \underbrace{\hat{\mathbf{N}} \cdot dA}_{d\vec{\mathbf{A}}} = d\vec{\mathbf{A}} \cdot d\vec{\mathbf{X}}^{(3)}$$

$$dV_t = dl \cdot da = \underbrace{d\vec{\boldsymbol{x}}^{(3)} \cdot \hat{\mathbf{n}}}_{dl} \cdot da =$$

$$d\vec{\mathbf{x}}^{(3)} \cdot \underbrace{\hat{\mathbf{n}} \cdot da}_{d\vec{\mathbf{a}}} = d\vec{\mathbf{a}} \cdot d\vec{\boldsymbol{x}}^{(3)}$$

# Differential Area

We prove that:

$$dV_t = d\vec{\boldsymbol{a}} \cdot d\vec{\boldsymbol{x}}^{(3)}$$

From the fundamental equation of deformation:

$$d\vec{\boldsymbol{x}}^{(3)} = \underline{\underline{\boldsymbol{F}}} \cdot d\vec{\boldsymbol{X}}^{(3)}$$

From differential volume:

$$dV_t = \det[\mathbf{F}]dV_0$$

From differential area:

$$dV_0 = d\vec{\boldsymbol{A}} \cdot d\vec{\boldsymbol{X}}^{(3)}$$

By combining all equations:

$$\underbrace{dV_t}_{\det[\boldsymbol{F}]dV_0} = \det[\boldsymbol{F}]\underbrace{d\vec{\boldsymbol{A}} \cdot d\vec{\boldsymbol{X}}^{(3)}}_{dV_0} = \underbrace{d\vec{\boldsymbol{a}} \cdot \underline{\underline{\boldsymbol{F}}} \cdot d\vec{\boldsymbol{X}}^{(3)}}_{dV_t}; \quad \forall d\vec{\boldsymbol{X}}^{(3)} \Rightarrow \det[\boldsymbol{F}]d\vec{\boldsymbol{A}} = d\vec{\boldsymbol{a}} \cdot \underline{\underline{\boldsymbol{F}}}$$

We can conclude that (note that $d\vec{\boldsymbol{A}} = \hat{\boldsymbol{N}} \cdot dA$ and $d\vec{\boldsymbol{a}} = \hat{\boldsymbol{n}} \cdot da$):

$$d\vec{\boldsymbol{a}} = \det[\boldsymbol{F}] \cdot d\vec{\boldsymbol{A}} \cdot \underline{\underline{\boldsymbol{F}}}^{-1} \Rightarrow da \cdot \hat{\boldsymbol{n}} = \det[\boldsymbol{F}] \cdot \hat{\boldsymbol{N}} \cdot \underline{\underline{\boldsymbol{F}}}^{-1} \Rightarrow da = \det[\boldsymbol{F}] \cdot |\hat{\boldsymbol{N}} \cdot \underline{\underline{\boldsymbol{F}}}^{-1}|dA$$

# Volumetric Strain

# Volumetric Strain

The volumetric strain $e_v$ is the representative of unit elongation in terms of volume (not lengths as in unit elongation) and can be defined as follows:

$$e_v(\hat{\boldsymbol{X}}, t) \equiv \frac{dV(\hat{\boldsymbol{X}}, t) - dV(\hat{\boldsymbol{X}}, t_0)}{dV(\hat{\boldsymbol{X}}, t_0)} \equiv \frac{dV_t - dV_0}{dV_0}$$

Due to the fact that: $dV_t = \det[\boldsymbol{F}]dV_0$, we have:

$$e_v = \frac{\det[\boldsymbol{F}]dV_0 - dV_0}{dV_0} = \det[\boldsymbol{F}] - 1$$

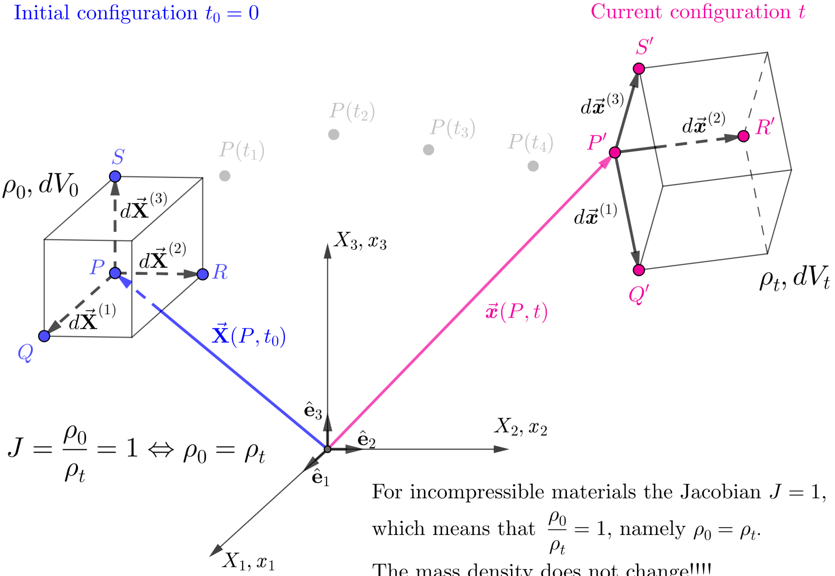
In other words:

$$e_v = J - 1$$

For incompressible materials (rubber (solid), water (liquid), all liquids can be considered incompressible to some extent), the property of incompressibility can be expressed as:

$$e_v = J - 1 = 0 \Rightarrow J = \det[\boldsymbol{F}] = 1$$

# Incompressible materials




$$J = \frac{\rho_0}{\rho_t} = 1 \Leftrightarrow \rho_0 = \rho_t$$

For incompressible materials the Jacobian $J = 1$, which means that $\frac{\rho_0}{\rho_t} = 1$, namely $\rho_0 = \rho_t$.

The mass density does not change!!!!

# Infinitesimal Strain Theory

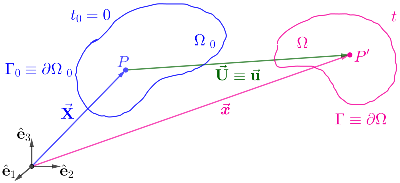
# Infinitesimal Strain Theory

The theory is based on the following assumptions:

1. Displacements are small with respect to the size of the object studied. The $|\vec{\boldsymbol{u}}| <<$(size of $\Omega_0$) and therefore the reference (initial) and current (deformed) configurations are considered to be the same $\Omega \approx \Omega_0$:

$$\vec{\boldsymbol{x}} = \vec{\boldsymbol{X}} + \vec{\boldsymbol{u}} \approx \vec{\boldsymbol{X}} \Leftrightarrow \vec{\boldsymbol{U}}(\vec{\boldsymbol{X}}, t) \equiv \vec{\boldsymbol{u}}(\vec{\boldsymbol{X}}, t) = \vec{\boldsymbol{u}}(\vec{\boldsymbol{x}}, t)$$

$$x_i = X_i + u_i \approx X_i \Leftrightarrow U_i(\vec{\boldsymbol{X}}, t) \equiv u_i(\vec{\boldsymbol{X}}, t) = u_i(\vec{\boldsymbol{x}}, t)$$



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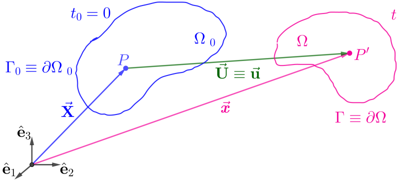
$$x_i = X_i + u_i \approx X_i \Leftrightarrow U_i(\vec{\boldsymbol{X}}, t) \equiv u_i(\vec{\boldsymbol{X}}, t) = u_i(\vec{\boldsymbol{x}}, t)$$

2. Displacements gradients (the change of displacements) are infinitesimal:

$$\left|\frac{\partial u_i}{\partial x_j}\right| << 1, \quad \forall i, j \in \{1, 2, 3\}$$



# Infinitesimal Strain Theory

- The material and spatial coordinates coincide, [2]: $\vec{\boldsymbol{x}} = \vec{\boldsymbol{X}} + \underbrace{\vec{\boldsymbol{u}}}_{\approx 0} \approx \vec{\boldsymbol{X}}$ But $\vec{\boldsymbol{u}}$ cannot be neglected when calculating the infinitesimal strain tensor $\varepsilon$.
- No difference between material and spatial operators.

$$\vec{\boldsymbol{\nabla}}_X = \frac{\partial}{\partial X_i} \otimes \hat{\boldsymbol{e}}_i = \frac{\partial}{\partial x_i} \otimes \hat{\boldsymbol{e}}_i = \vec{\boldsymbol{\nabla}}_x$$

$$\underline{\underline{\boldsymbol{D}}}(\vec{\boldsymbol{X}}, t) = \vec{\boldsymbol{U}}(\vec{\boldsymbol{X}}, t) \otimes \vec{\boldsymbol{\nabla}}_X = \vec{\boldsymbol{u}}(\vec{\boldsymbol{x}}, t) \otimes \vec{\boldsymbol{\nabla}}_x = \underline{\underline{\boldsymbol{d}}}(\vec{\boldsymbol{x}}, t)$$

- Local and material time derivatives coincide (no convective derivative):

$$\Gamma(\vec{\boldsymbol{X}}, t) \approxeq \Gamma(\vec{\boldsymbol{x}}, t) = \gamma(\vec{\boldsymbol{x}}, t) = \gamma(\vec{\boldsymbol{X}}, t)$$

$$\frac{d\gamma}{dt} = \frac{\partial \gamma(\vec{\boldsymbol{X}}, t)}{\partial t} = \frac{\partial \gamma(\vec{\boldsymbol{x}}, t)}{\partial t} = \dot{\gamma}$$

# Infinitesimal Strain Theory - Strain Tensors

The Green-Lagrange strain tensor defined as:

$$(ds)^2 - (dS)^2 = 2 \cdot d\vec{\boldsymbol{X}} \cdot \underline{\underline{\boldsymbol{E}}} \cdot d\vec{\boldsymbol{X}}$$

Can be simplified as follows:

$$\underline{\underline{\boldsymbol{E}}} = \frac{1}{2}(\underline{\underline{\boldsymbol{F}}}^T \cdot \underline{\underline{\boldsymbol{F}}} - \underline{\underline{\boldsymbol{1}}}) = \frac{1}{2}\left(\underline{\underline{\boldsymbol{D}}} + \underline{\underline{\boldsymbol{D}}}^T + \underbrace{\underline{\underline{\boldsymbol{D}}}^T \underline{\underline{\boldsymbol{D}}}}_{<<1}\right) \Leftrightarrow \underline{\underline{\boldsymbol{E}}} \approx \frac{1}{2}\left(\underline{\underline{\boldsymbol{D}}} + \underline{\underline{\boldsymbol{D}}}^T\right) = \frac{1}{2}\left(\underline{\underline{\boldsymbol{d}}} + \underline{\underline{\boldsymbol{d}}}^T\right) = \underline{\underline{\boldsymbol{\varepsilon}}}$$

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \underbrace{\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}}_{\left|\frac{\partial u_k}{\partial x_j}\right| << 1}\right) \Leftrightarrow E_{ij} \approx \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right); \quad i,j \in \{1,2,3\}$$

# Infinitesimal Strain Theory - Strain Tensors

The Euler-Almansi strain tensor defined as:

$$(ds)^2 - (dS)^2 = 2 \cdot d\vec{\boldsymbol{x}} \cdot \underline{\underline{\boldsymbol{e}}} \cdot d\vec{\boldsymbol{x}}$$

Can be simplified as follows:

$$\underline{\underline{\boldsymbol{e}}} = \frac{1}{2}(\underline{\underline{\boldsymbol{1}}} - \underline{\underline{\boldsymbol{F}}}^{-T} \cdot \underline{\underline{\boldsymbol{F}}}^{-1}) = \frac{1}{2}\left(\underline{\underline{\boldsymbol{d}}} + \underline{\underline{\boldsymbol{d}}}^T - \underbrace{\underline{\underline{\boldsymbol{d}}}^T \underline{\underline{\boldsymbol{d}}}}_{<<1}\right) \Leftrightarrow \underline{\underline{\boldsymbol{e}}} \approx \frac{1}{2}\left(\underline{\underline{\boldsymbol{d}}} + \underline{\underline{\boldsymbol{d}}}^T\right) = \frac{1}{2}\left(\underline{\underline{\boldsymbol{D}}} + \underline{\underline{\boldsymbol{D}}}^T\right) = \underline{\underline{\boldsymbol{\varepsilon}}}$$

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \underbrace{\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}}_{\left|\frac{\partial u_k}{\partial x_j}\right|<<1}\right) \Leftrightarrow e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \varepsilon_{ij}; \;\; i,j \in \{1,2,3\}$$

# Infinitesimal Strain Tensor

The infinitesimal strain tensor can be defined as follows:

$$\underline{\underline{\boldsymbol{\varepsilon}}} = \frac{1}{2}\left(\underline{\underline{\boldsymbol{d}}} + \underline{\underline{\boldsymbol{d}}}^T\right) = \frac{1}{2}\left(\underline{\underline{\boldsymbol{D}}} + \underline{\underline{\boldsymbol{D}}}^T\right) = \frac{1}{2}\left(\vec{\boldsymbol{u}} \otimes \vec{\boldsymbol{\nabla}} + \vec{\boldsymbol{\nabla}} \otimes \vec{\boldsymbol{u}}\right)$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right); \;\; i, j \in \{1, 2, 3\}$$

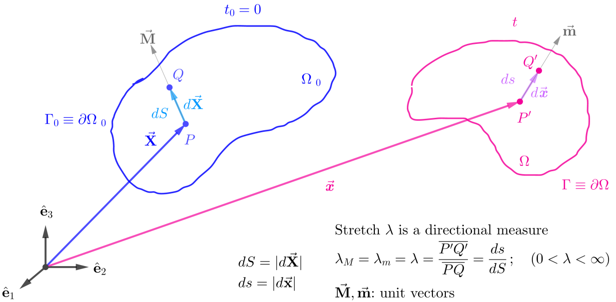
- The infinitesimal strain tensor is symmetric with infinitesimal components.
- It is not applicable to fluids!!!

# Infinitesimal Strain Theory - Stretch, Unit elongation

The stretch defined as ($\vec{\boldsymbol{M}}$ and $\vec{\boldsymbol{m}}$ are unit vectors in directions of $d\vec{\boldsymbol{X}}$ and $d\vec{\boldsymbol{x}}$ respectively):

$$\lambda_M(\vec{\boldsymbol{M}}) = \frac{ds}{dS} \Leftrightarrow \lambda_M(\vec{\boldsymbol{M}}) = \sqrt{1 + 2\vec{\boldsymbol{M}} \cdot \underline{\underline{\boldsymbol{E}}} \cdot \vec{\boldsymbol{M}}}$$

$$\lambda_m(\vec{\boldsymbol{m}}) = \frac{ds}{dS} \Leftrightarrow = \frac{1}{\sqrt{1 - 2\vec{\boldsymbol{m}} \cdot \underline{\underline{\boldsymbol{e}}} \cdot \vec{\boldsymbol{m}}}}$$



# Infinitesimal Strain Theory - Stretch, Unit elongation

Considering that $\underline{\underline{e}} \approx \underline{\underline{E}} \approx \underline{\underline{\varepsilon}}$ and by using the Taylor series expansion up to first order terms around $y = 0$, where $y = \vec{M} \cdot \underline{\underline{E}} \cdot \vec{M}$:

$$\lambda_M(y) = \sqrt{1 + 2y} \Leftrightarrow \lambda_M(y) = \lambda_M(0) + \frac{\frac{d\lambda_M}{dy}}{1!}(y - 0) = 1 + \frac{2}{2\sqrt{1 + 2 \cdot 0}} y = 1 + y$$

And therefore:

$$\lambda_M(\vec{M}) = 1 + \vec{M} \cdot \underline{\underline{E}} \cdot \vec{M}$$

Considering that $\underline{\underline{e}} \approx \underline{\underline{E}} \approx \underline{\underline{\varepsilon}}$ and by using the Taylor series expansion up to first order terms around $y = 0$, where $y = \vec{m} \cdot \underline{\underline{e}} \cdot \vec{m}$:

$$\lambda_m(y) = \frac{1}{\sqrt{1 - 2y}} \Leftrightarrow \lambda_m(y) = \lambda_m(0) + \frac{\frac{d\lambda_m}{dy}}{1!}(y - 0) = 1 + \left(-\frac{(-2)}{2}\right)(1 - 2 \cdot 0)^{-\frac{3}{2}} y = 1 + y$$

And therefore:

$$\lambda_m(\vec{m}) = 1 + \vec{m} \cdot \underline{\underline{e}} \cdot \vec{m}$$

# Infinitesimal Strain Theory - Stretch, Unit elongation

In the infinitesimal strain theory, $\vec{\boldsymbol{M}} = \vec{\boldsymbol{m}}$. Finally, the linearised stretch can be derived as follows:

$$\lambda = \frac{ds}{dS} \approx 1 + \vec{\boldsymbol{m}} \cdot \underline{\underline{\boldsymbol{\varepsilon}}} \cdot \vec{\boldsymbol{m}} = 1 + \vec{\boldsymbol{M}} \cdot \underline{\underline{\boldsymbol{\varepsilon}}} \cdot \vec{\boldsymbol{M}}$$

The linearised unit elongation can be derived as follows:

$$\varepsilon = \frac{ds - dS}{dS} = \lambda - 1 = \vec{\boldsymbol{m}} \cdot \underline{\underline{\boldsymbol{\varepsilon}}} \cdot \vec{\boldsymbol{m}}$$

# Physical Interpretation of Infinitesimal Strain Tensor

# Physical meaning of infinitesimal strain tensor

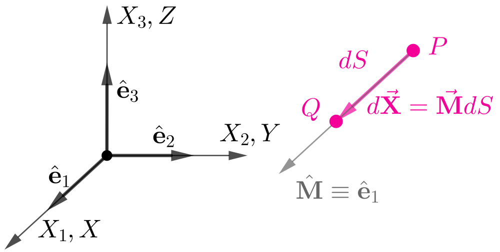
The infinitesimal strain tensor can be given in matrix form:

$$\underline{\underline{\varepsilon}} \equiv \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix}$$

Consider that we have a segment oriented along the X-axis (material coordinates). In this case the stretch will be:

$$\lambda = 1 + \vec{\boldsymbol{m}} \cdot \underline{\underline{\varepsilon}} \cdot \vec{\boldsymbol{m}} = 1 + \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1 + \varepsilon_{11}$$

which is equivalent with stretching of the material in X-direction.



# Physical meaning of infinitesimal strain tensor

The stretching and unit elongation of the material along the $x$, $y$ and $z$ directions, [2]:

$$\lambda_1 = 1 + \varepsilon_{11} \Leftrightarrow \varepsilon_1 = \lambda_1 - 1 = \varepsilon_{11}$$
$$\lambda_2 = 1 + \varepsilon_{22} \Leftrightarrow \varepsilon_2 = \lambda_2 - 1 = \varepsilon_{22}$$
$$\lambda_3 = 1 + \varepsilon_{33} \Leftrightarrow \varepsilon_3 = \lambda_3 - 1 = \varepsilon_{33}$$

The diagonal terms of the material strain tensor contain information about unit elongations in $x$, $y$ and $z$ directions:

$$\underline{\underline{\varepsilon}} \equiv \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix}$$

1. For $\varepsilon_{xx} = 0$ no elongation along $x$ axis, $\varepsilon_x = 0$.

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1. For $\varepsilon_{xx} = 0$ no elongation along $x$ axis, $\varepsilon_x = 0$.
2. For $\varepsilon_{yy} = 0$ no elongation along $y$ axis, $\varepsilon_y = 0$.

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1. For $\varepsilon_{xx} = 0$ no elongation along $x$ axis, $\varepsilon_x = 0$.
2. For $\varepsilon_{yy} = 0$ no elongation along $y$ axis, $\varepsilon_y = 0$.
3. For $\varepsilon_{zz} = 0$ no elongation along $z$ axis, $\varepsilon_z = 0$.

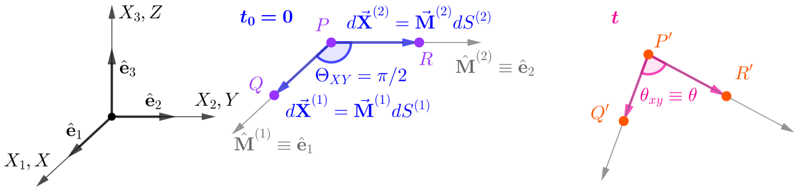
# Physical meaning of infinitesimal strain tensor

The angle between a segment parallel to $X_1$-axis ($X$) and a segment parallel to $X_2$-axis ($Y$) is given by angle $\Theta_{XY} = \frac{\pi}{2}$. Using:

$$\theta \equiv \theta_{xy} = \frac{\pi}{2} - \arcsin \frac{2E_{XY}}{\sqrt{1+2E_{XX}}\sqrt{1+2E_{YY}}}$$

Considering that $E_{XY} = \varepsilon_{xy}$, $E_{XX} = \varepsilon_{xx}$ and $E_{YY} = \varepsilon_{yy}$, then:

$$\theta \equiv \theta_{xy} = \frac{\pi}{2} - \arcsin \frac{2\varepsilon_{xy}}{\underbrace{\sqrt{1+2\varepsilon_{xx}}}_{\approx 1}\underbrace{\sqrt{1+2\varepsilon_{yy}}}_{\approx 1}} \approx \frac{\pi}{2} - \underbrace{\arcsin 2\varepsilon_{xy}}_{\approx 2\varepsilon_{xy}} = \frac{\pi}{2} - 2\varepsilon_{xy}$$



# Physical meaning of infinitesimal strain tensor

Knowing that:

$$\theta_{xy} = \frac{\pi}{2} - 2\varepsilon_{xy}$$

The increment of the final angle with respect to its orginal:

$$\Delta_{xy} = \theta_{xy} - \frac{\pi}{2} \approx \frac{\pi}{2} - 2\varepsilon_{xy} - \frac{\pi}{2} = -2\varepsilon_{xy}$$

Following the same process we can conclude that:

$$\varepsilon_{xy} = -\frac{1}{2}\Delta\theta_{xy}; \quad \varepsilon_{xz} = -\frac{1}{2}\Delta\theta_{xz}; \quad \varepsilon_{yz} = -\frac{1}{2}\Delta\theta_{yz}$$

We can conclude that the non-diagonal components of the infinitesimal strain tensor are equal to the semi-decrement of the deformation of the angles between segments initially oriented along x,y and z directions, [2].

# Infinitesimal Strain Tensor - Engineering Strains

$$\underline{\underline{\varepsilon}} \equiv \underbrace{\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix}}_{\text{Scientific Notation}} \equiv \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \equiv \underbrace{\begin{bmatrix} \varepsilon_{x} & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & \varepsilon_{y} & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \varepsilon_{z} \end{bmatrix}}_{\text{Engineering Notation}}$$

Note that:

- The infinitesimal strain tensor is symmetric.
- Positive longitudinal strains (diagonal terms in infinitesimal strain tensor) provides information about **increase** of segment length.
- Positive angular strains (non-diagonal terms in infinitesimal strain tensor) provides information about **decrease** of angles throughout the deformation process.
- Due to symmetry the infinitesimal strain tensor can be written in Voigt's notation as follows:

$$\underline{\underline{\varepsilon}} \equiv [\underbrace{\varepsilon_x, \varepsilon_y, \varepsilon_z,}_{\text{longitudinal}} \underbrace{\gamma_{xy}, \gamma_{xz}, \gamma_{yz}}_{\text{angular}}]^T$$

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