Nguyen Xuan Binh 887799 - Assignment Week 3 - Kinetics

Exercise 1: The stress state at a body at a point is given in MPa

$$[\sigma] = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 4 & 0 \\ 3 & 0 & -1 \end{bmatrix}$$

a) Find the traction (stress) vector at a point in the plane whose normal is in the direction of $2\hat{e}_1 + 2\hat{e}_2 + \hat{e}_3$

The stress vector is: $\vec{t}(n) = \sigma(x)\,\vec{n}(x) = 1/3 \begin{bmatrix} 2 & -1 & 3 \\ -1 & 4 & 0 \\ 3 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 2 \\ 5/3 \end{bmatrix}$ (normal $\vec{n} = \frac{1}{3}(2,2,1)$)

$\Rightarrow \vec{t}(n) = 5/3\,\hat{e}_1 + 2\hat{e}_2 + 5/3\,\hat{e}_3$ (MPa)

b) Determine the magnitude of the normal and the shearing stresses on this plane

Magnitude of the normal stress

$|\vec{t}_n(\hat{n})| = \vec{t}(n) \cdot \vec{n}(x) = 5/3 \cdot \frac{2}{3} + 2 \times \frac{2}{3} + 5/3 \times 1/3 = 3$ (answer)

Magnitude of the stress vector: $||T|| = \sqrt{\left(\frac{5}{3}\right)^2 + 2^2 + \left(\frac{5}{3}\right)^2} = \sqrt{\frac{86}{9}}$

$\Rightarrow$ Magnitude of shearing stress : $|\vec{t}_t(\hat{n})| = \sqrt{|\vec{t}(\hat{n})|^2 - |t_n(\hat{n})|^2}$
$= \sqrt{\frac{86}{9} - 3^2} = \frac{\sqrt{5}}{3}$ (answer)

$\Rightarrow |\vec{t}_n(\hat{n})| = 3 \quad , \quad |\vec{t}_t(\hat{n})| = \frac{\sqrt{5}}{3}$ (MPa)

c) Find the traction vector of a point parallel to the plane $x_1 - 2x_2 + 3x_3 = 4$
$\Rightarrow$ The traction vector has normal in direction of $\hat{e}_1 - 2\hat{e}_2 + 3\hat{e}_3$
Unit normal $\sqrt{1^2 + (-2)^2 + 3^2}^{-1}$ $\vec{n} = \frac{1}{\sqrt{14}}(1, -2, 3)$

The traction vector is $\vec{t}(\hat{n}) = \sigma(x)\,\vec{n}(x) = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 4 & 0 \\ 3 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{14} \\ -2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix} = \begin{bmatrix} 13/\sqrt{14} \\ -9/\sqrt{14} \\ 0 \end{bmatrix}$ (answer)
$\Rightarrow \vec{t}(n) = 13/\sqrt{14}\,\hat{e}_1 - 9/\sqrt{14}\,\hat{e}_2$ (MPa)

d) Find normal & shearing stresses on this plane
Magnitude of the normal stress
$|\vec{t}_n(\vec{n})| = \vec{t}(\vec{n}) \cdot \vec{n}(x) = \frac{13}{\sqrt{14}} \cdot \frac{1}{\sqrt{14}} + \frac{-9}{\sqrt{14}} \cdot \frac{-2}{\sqrt{14}} + 0 \cdot \frac{3}{\sqrt{14}} = \frac{31}{14}$ (answer)

Magnitude of stress vector squared : $|\vec{t}(\hat{n})|^2 = \left(\frac{13}{\sqrt{14}}\right)^2 + \left(\frac{-9}{\sqrt{14}}\right)^2 = \frac{125}{7}$

Magnitude of the shearing stress
$|\vec{t}_t(\vec{n})| = \sqrt{|\vec{t}(\hat{n})|^2 - |\vec{t}_n(\hat{n})|^2} = \sqrt{\frac{125}{7} - \left(\frac{31}{14}\right)^2} = \sqrt{\frac{2539}{196}}$ (answer) ≈ 3.599 MPa

Exercise 2: Justify if each statement are T or F

a) Symmetry of stress tensor is not valid if the body has an angular acceleration
This statement is true. The stress tensor is symmetric due to conservation of angular momentum. If the stress tensor is asymmetric, there would be a net rotation and the body starts to gain angular acceleration

b) On the plane of maximum normal stress, the shearing stress is always zero
This statement is true. Plane of maximum normal stress is the principle plane that only consists of normal stresses caused along the principle plane and any component of shear stress is 0

c) On the plane of maximum shearing stress, the normal stress is always zero
This statement is false. The reason is that on the plane of maximum shearing stress, the normal stress is half the difference between maximum and normal stress, which is not necessarily zero

d) A plane ~~of maximum~~ with its normal in the direction of $\hat{e}_1 + 2\hat{e}_2 - 2\hat{e}_3$ has a traction vector $\vec{t} = 50\hat{e}_1 + 100\hat{e}_2 - 100\hat{e}_3$ MPa. This plane is a principal plane
Unit normal : $\sqrt{1^2 + 2^2 + (-2)^2} = 3 \Rightarrow \vec{n} = \frac{1}{3}(1, 2, -2)$

$\Rightarrow$ Magnitude of normal stress: $|\vec{t}_n(\vec{n})| = \vec{t}(\vec{n}) \cdot \vec{n}(x) = \frac{1}{3} \cdot 50 + \frac{2}{3} \cdot 100 - \frac{2}{3} \cdot (-100)$

$\Rightarrow |\vec{t}_n(\vec{n})| = 150$

Magnitude of shearing stress: $|\vec{t}_t(\vec{n})| = \sqrt{22500 - (150)^2} = 0$

$\Rightarrow$ This statement is true since principal plane has 0 shear stress

e) The following matrix represent a Cauchy stress tensor $[\sigma] = \begin{bmatrix} 100 & 100 & 0 \\ 200 & 10 & 0 \\ 0 & 0 & 20 \end{bmatrix}$

This statement is false because the tensor is not symmetric

Exercise 3: The stress tensor is given at a point in MPa: $[\sigma] = \begin{bmatrix} 0 & 100 & 0 \\ 100 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

a) What is the magnitude of the shearing stress of the plane whose normal is in direction of $\hat{e}_1 + \hat{e}_2$

The normal unit is $1/\sqrt{2}(\hat{e}_1 + \hat{e}_2)$

The traction vector: $\vec{t}_n(\vec{n}) = \begin{bmatrix} 0 & 100 & 0 \\ 100 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 50/\sqrt{2} \\ 50/\sqrt{2} \\ 0 \end{bmatrix}$

$\Rightarrow$ Magnitude of normal stress: $\vec{t}_n(\vec{n}) \cdot \vec{n}(x) = \frac{50}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \frac{50}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = 50$

$\Rightarrow$ Magnitude of shearing stress: $\vec{t}_t(\vec{n}) = \sqrt{50^2 - 50^2} = 0$

b) Find the maximum and minimum normal stresses and the planes on which they act

To attain maximum & minimum normal stress, the plane should be in principal state, which means the plane should be orthogonal to normal stress.

We have: $\det[\sigma - \lambda \underline{\underline{1}}] = \det \begin{bmatrix} -\lambda & 100 & 0 \\ 100 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix} = -\lambda^3 - 100(-100\lambda)$

Characteristic polynomial: $-\lambda^3 + 10000\lambda \Rightarrow \lambda_1 = 0, \lambda_2 = 100, \lambda_3 = -100$

We have: $[\sigma]\hat{n} = \lambda\hat{n} \rightarrow$ value of normal stress is λ since $\hat{n}$ is unit vector

$\Rightarrow$ Maximum normal stress is 100 and minimum normal stress is -100 MPa

▫ Direction of maximum and minimum normal stress

* $\vec{t}_n(\vec{n})_{max} = 100 \Rightarrow \begin{bmatrix} -100 & 100 & 0 \\ 100 & -100 & 0 \\ 0 & 0 & -100 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 100n_1 \\ 100n_2 \\ 100n_3 \end{bmatrix} \Rightarrow \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right)$

$\Rightarrow$ The plane $\vec{t}_n(\vec{n})_{max}$ act on is $ax + ay = C$ $(C \in \mathbb{R})(a \in \mathbb{R}, a \neq 0)$

* For $\vec{t}_n(\vec{n})_{min} = -100 \Rightarrow \begin{bmatrix} 100 & 100 & 0 \\ 100 & 100 & 0 \\ 0 & 0 & 100 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} -100n_1 \\ -100n_2 \\ -100n_3 \end{bmatrix} \Rightarrow \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \text{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right)$

$\Rightarrow$ The plane $\vec{t}_n(\vec{n})_{min}$ act on is $-ax + ay = C$ $(C \in \mathbb{R})(a \in \mathbb{R}, a \neq 0)$

c) Find the maximum shear stress and the plane on which it acts

We have: $\vec{t}_{shear}(\vec{n})_{max} = \dfrac{\vec{t}_n(\vec{n})_{max} - \vec{t}_n(\vec{n})_{min}}{2} = \dfrac{100 - (-100)}{2} = 100$ MPa

The plane it acts on is $45°$ angle on the planes of maximum and minimum normal stress

$\Rightarrow$ The planes are $x = 0$ and $y = \dfrac{C}{a}$ $(C \in \mathbb{R}, a \in \mathbb{R})$ and planes parallel to those planes

Exercise 4: With a reference to a Cartesian system (x_1, x_2, x_3), the components of a stress tensor are defined in MPa:

$$[\sigma] = \begin{bmatrix} 200 & 400 & 300 \\ 400 & 0 & 0 \\ 300 & 0 & -100 \end{bmatrix}$$

$\phi = x_1 + 2x_2 + 2x_3 \Rightarrow$ unit normal vector: $1/3\,(1, 2, 2)$

$\Rightarrow$ Traction vector: $\vec{t} = \begin{bmatrix} 200 & 400 & 300 \\ 400 & 0 & 0 \\ 300 & 0 & -100 \end{bmatrix} \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 1600/3 \\ 400/3 \\ 100/3 \end{bmatrix} = \dfrac{1}{3} \begin{bmatrix} 1600 \\ 400 \\ 100 \end{bmatrix}$

$\Rightarrow \vec{t} = \dfrac{1600}{3} x_1 + \dfrac{400}{3} x_2 + \dfrac{100}{3} x_3$ MPa $\Rightarrow \|\vec{t}\| = \sqrt{910000/3} \approx 550.75$ MPa

- The normal component: $\vec{t} \cdot \vec{n} = \dfrac{1600}{3} \cdot \dfrac{1}{3} + \dfrac{400}{3} \cdot \dfrac{2}{3} + \dfrac{100}{3} \cdot \dfrac{2}{3} = \dfrac{2600}{9}$ MPa
- The tangential component: $\vec{t}_t = \sqrt{\vec{t}^2 - \vec{t}_n^2} = \sqrt{\dfrac{910000}{3} - \left(\dfrac{2600}{9}\right)^2}$

$\Rightarrow \vec{t}_t = \sqrt{\dfrac{17810000}{81}} \approx 468.9$ MPa

Exercise 5: The distribution of stress inside a body is

$$[\sigma] = \begin{bmatrix} -P + \rho g y & 0 & 0 \\ 0 & -P + \rho g y & 0 \\ 0 & 0 & -P + \rho g y \end{bmatrix}$$

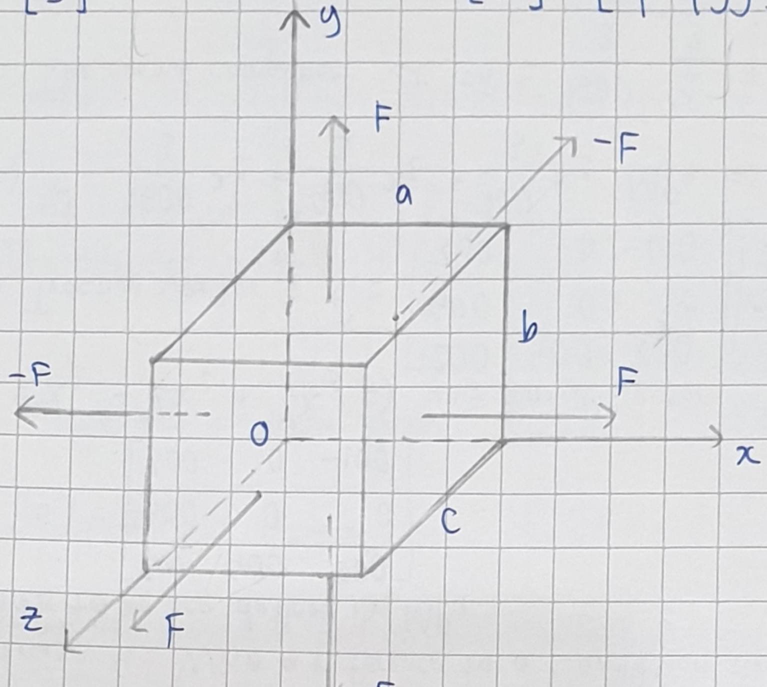
where P, ρ, g are constants and y is the coordinate of x, y, z Cartesian

a) The normal vector and their corresponding stress vector of each face is

+) $\begin{bmatrix}1\\0\\0\end{bmatrix} \Rightarrow$ face $x = a \Rightarrow \vec{t} = [\sigma]\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}-p+\rho g y\\0\\0\end{bmatrix}$; $\begin{bmatrix}-1\\0\\0\end{bmatrix}$ face $x=0 \Rightarrow \vec{t} = [\sigma]\begin{bmatrix}-1\\0\\0\end{bmatrix} = \begin{bmatrix}p-\rho g y\\0\\0\end{bmatrix}$

+) $\begin{bmatrix}0\\1\\0\end{bmatrix} \Rightarrow$ face $y = b \Rightarrow \vec{t} = [\sigma]\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\-p+\rho g y\\0\end{bmatrix}$; $\begin{bmatrix}0\\-1\\0\end{bmatrix}$ face $y=0 \Rightarrow \vec{t} = [\sigma]\begin{bmatrix}0\\-1\\0\end{bmatrix} = \begin{bmatrix}0\\p-\rho g y\\0\end{bmatrix}$

+) $\begin{bmatrix}0\\0\\1\end{bmatrix} \Rightarrow$ face $z = c \Rightarrow \vec{t} = [\sigma]\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\-p+\rho g y\end{bmatrix}$; $\begin{bmatrix}0\\0\\-1\end{bmatrix} \Rightarrow$ face $z=0 \Rightarrow \vec{t} = [\sigma]\begin{bmatrix}0\\0\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\p-\rho g y\end{bmatrix}$



$(F = -p + \rho g y$
$-F = p - \rho g y)$

b) Resultant force acting on faces $y = 0$ and $x = 0$

Magnitude : $F_R = \sqrt{(-F)^2 + (-F)^2} = F\sqrt{2} = (-p + \rho g y)\sqrt{2}$

Direction: $\cos^{-1}\left(\frac{F_R}{F}\right) = 45^0 \Rightarrow$ in direction of $-\hat{x} - \hat{y}$ from origin