Foundations on Continuum Mechanics - Week 1

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Introduction

Athanasios A. Markou

- Bachelor, Master in Aristotle University of Thessaloniki, Greece
- **PhD** in Aristotle University, Greece and University of Catania, Italy
- Postdoc in Norwegian Geotechnical Institute (NGI), Norway
- Joined Aalto May-2018
- Background in Earthquake Engineering, Structural Engineering
- Responsible teacher:
  - Fundamentals of Structural Design (M) Building Technology
  - Continuum Mechanics (B) Computational Engineering
- Co-teacher:
  - Numerical Methods in Engineering (B) Computational Engineering,
  - Computer-aided tools in engineering (B)
  - Informed Structures (M)
  - ARTS-ENG (B)
- **My office**: Room 227 | Rakentajanaukio 4A

Course

Course Organization

Organization of the **course**:

1. **Mathematical** preliminaries (Tensor Algebra and Analysis)
2. **Kinematics**
3. **Kinetics**
4. **Balance principles**
5. Constitutive equations, **Solids**
6. Constitutive equations, **Fluids**

Passing the course

Submit ALL weekly assignments (6):
Grade more than 50%
Assignments will be announced Wednesday evening and you will need to submit them until next week's Wednesday midnight (deadline)
Cut-off day will be Thursday midnight, after which you can no longer submit the assignment
Submitting the assignment after Wednesday midnight will result in downgrading of 20%
The grading is as follows:
$50\% - 59.99\%$ - grade 1
$60\% - 69.99\%$ - grade 2
$70\% - 79.99\%$ - grade 3
$80\% - 89.99\%$ - grade 4
$90\% - 100\%$ - grade 5
No Exam

Introduce yourselves

**Name,
Bachelor Program,
How is the online teaching experience so far?,
What do you expect from the course?**

Mechanics

Mechanics

- **Mechanics** is a branch of physics **CONCERNED** with the **motion** of **matter** under **forces** that cause such motion and the relationship between the three of them, namely **motion**, **matter** and **force**, [3].
- More specifically, **mechanics** deals with the concepts of:
  1. matter
  2. **space**
  3. time
  4. force
  5. **energy**
- **Mechanics** can be split into:
  1. **Theoretical**
  2. **Applied**
  3. Computational

Mechanics

'The classical nature of mechanics reflects its `greatness`*: Ever old and ever new, it continues to pour out for us understanding and application, linking a changing world to unchanged law.'*, C. Truesdell, [14].

'Mechanics does not study natural things directly. Instead, it considers bodies, which are ***mathematical concepts*** *designed to abstract some common features of many natural things. One such feature is the mass assigned to each body. Always, a natural body is at any one instant found to occupy a set of places; that set is the shape of that body at that instant. The theory of places, which is called* **`geometry`**, *was created long ago and thus lies ready to hand for application in mechanics. The* ***change of shape*** *undergone by a body* ***from one instant to another*** *is called the* ***motion*** *of that body, and description of motion, or* **`kinematics`**, *is the* ***second part*** *of the foundation of mechanics. Third, motions of bodies are conceived as resulting from or at least being invariably accompanied by the action of forces. Thus,* **`mechanics`** *provides a mathematical model, or, better, an infinite class of models, for certain aspects of nature.'*, C. Truesdell, [14].

Matter

Matter

- **Matter** consists of **atoms**, which consist of subatomic **particles**. Therefore **matter** is **discontinuous**.
- The **scales** of MATERIAL SCIENCE can be classified as follows, [15]:
  1. Meter level $(1 \times 10^{0}\mathrm{m})$
  2. Millimeter level $(1 \times 10^{-3}\mathrm{m})$
  3. Micrometer level $(1 \times 10^{-6}\mathrm{m})$
  4. Nanometer level $(1 \times 10^{-9}\mathrm{m})$
- Multi-scale material **mechanics** can be **grouped** as, [15]:
  1. Structural mechanics
  2. **Macro Mechanics**
  3. **Meso** Mechanics
  4. `Micro` Mechanics
  5. `Nano` Mechanics

Continuum Medium

Continuum medium

- **Continuum medium** can be defined as a **matter** that its physical properties are **independent** of its **`actual size`**, [10].
- As we split the MEDIUM to smaller and smaller pieces, its properties will ***not change***, [6].
- In other words, the theory of **continuum medium** **neglects** the `atomic` structure of matter.
- **Continuum medium** consists of **material points** that are tiny for the '*human eye*', [5].

Material points

- **Fundamental** for the theory of continuum mechanics is the concept of **material point**, [5].
- **Material point** is a region in space that is, [5]:
  1. small from `macroscopic` point of view
  2. large from `microscopic` point of view
- **Material point** appears as point to an **engineer**, [5].
- ***Material point*** contains large number of molecules, so that we can speak of large `thermodynamic` system, [5].
- In **solids**, material points remain in the same neighborhood throughout their entire life, **apart** from the case of **cracking**, [1]. In **fluids** the previous statement does NOT hold true.

Continuum Mechanics

Continuum Mechanics

'*The* ***response of a material*** *ignoring relativistic phenomena is dictated by its atoms, the behavior of which is governed by* QUANTUM MECHANICS. ***Quantum mechanics*** *involves hundreds of atoms over a time of* NANOSECONDS. *The* ***opposite extreme*** *of quantum mechanics is* ***continuum mechanics***, *which ignores completely the discontinuous nature of matter, which results in a theory with continuously varying fields. With the exception of electromagnetic phenomena, all courses in Engineering curricula are applications of simplified versions of* ***Continuum Mechanics*** *and* ***Thermodynamics***.', Tadmor, E.B., Miller, R.E. and Elliott, R.S., [13].

Continuum Mechanics

'*The analysis of the* ***kinematic and mechanical behavior*** *of materials modeled on the* **continuum assumption** *is what we know as* ***continuum mechanics****. There are* ***two main themes*** *into which the* TOPICS *of continuum mechanics are divided. In the first, emphasis is on the derivation of fundamental equations which are* `valid for all continuous media`*. These equations are based upon* ***universal laws of physics*** *such as the* ***conservation of mass, the principles of energy and momentum****, etc. In the second, the* ***focus*** *of attention is on the development of so-called* `constitutive equations` *characterizing the behavior of* SPECIFIC IDEALIZED MATERIALS*, the perfectly* ***elastic solid and the viscous fluid*** *being the best known examples.*', Mase, G.T., Smelser, R.E. and Mase, G.E., [9].

Why learn Continuum Mechanics?

'*`Continuum Mechanics`* *and* ***Thermodynamics*** *form the fundamental theory lying at the heart of* ***many disciplines in science and engineering***', Tadmor, E.B., Miller, R.E. and Elliott, R.S., [13].

'*First, it may not be all that obvious to the students that there is a common foundation for the Continuum Mechanics is* ***a foundation for all our theories****, theories for bars, beams, etc. Second, there are a lot of* ***other important cases*** ***that have not been covered by the theories already learned****, and if the student encounters such theories later it is imperative that he or she knows the* **foundation for any sound continuum mechanical theory***, including the* ***specialized*** *ones. Otherwise, the consequences may be* ***dire***.', Byskov, E. [1]

Continuum Mechanics

'*Through **several centuries** there has been a lively interaction between **mathematics** and **mechanics**. On the one side, **mechanics has used mathematics** to formulate the **basic laws** and to apply them to a host of problems that call for the quantitative prediction of the **consequences of some action**. On the other side, the `needs of mechanics` have stimulated the development of `mathematical concepts`. DIFFERENTIAL CALCULUS grew out of the needs of NEWTONIAN DYNAMICS; **vector algebra** was developed as a means to describe **force systems**; **vector analysis**, to study **velocity fields** and **force fields**; and the **calculus of variations** has evolved from the **energy principles of mechanics**.*', Flügge, W., [2].

Continuum Mechanics

'*The* ***mechanics of continuum medium*** *is a that branch of mechanics concerned with the* ***stresses in solids, liquids and gases*** *and the* ***deformation or flow*** *of these materials. The adjective* `continuous` *refers to the* **simplifying concept** *underlying the analysis: we* ***disregard the molecular structure*** *of matter and picture it as being without gaps or empty spaces. We further suppose that all the* ***mathematical functions*** *entering the theory are* *continuous functions*, *except possibly at a finite number of interior surfaces separating regions of continuity. This statement implies that the* ***derivatives of the functions are continuous too***, *if they enter the theory, since* *all functions* *entering the theory are assumed* ***continuous***. *This* HYPOTHETICAL ***continuous material*** *we call a* ***continuous medium or continuum***', Malvern, L.E. [8].

Continuum Mechanics

- Continuum Mechanics deals with **mechanical behavior** of 2 kinds of **matter**:
  1. **Solids**
  2. **Fluids**
     - 2.1 **Liquids**
     - 2.2 **Gases**
- Continuum Mechanics deals with 2 kinds of **equations**:
  1. GENERAL (BALANCE) PRINCIPLES
  2. **Constitutive equations**
- Continuum Mechanics deals with 2 kinds of **problems**:
  1. **Formulation of constitutive equations**
  2. Solve **constitutive equations along with general principles** subjected to **initial and boundary conditions**

Continuum Mechanics

'*Material behavior* *is dictated by many different processes, occurring on vastly different length and time scales, that interact in* ***complex ways*** *to give the overall material response. Further, these processes have traditionally been studied by different researchers, from different fields, using different theories and tools. For example, the* **bonding between individual atoms** *making up a material is studied by physicists using* ***quantum mechanics****, while the* ***macroscopic deformation of materials*** *falls within the domain of engineers who use* ***continuum mechanics****. In the end, a* ***multiscale modeling approach*** *capable of predicting the behavior of materials at the macroscopic scale but built on the quantum foundations of atomic bonding requires a deep understanding of topics from a broad range of disciplines and the connections between them.*
These include ***quantum mechanics****,* ***statistical mechanics*** *and* ***materials science****, as well as* ***continuum mechanics*** *and* ***thermodynamics***.', Tadmor, E.B., Miller, R.E. and Elliott, R.S., [13].

Continuum Mechanics - General Principles

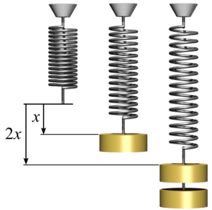
The **general principles** hold true for any kind of material.

In other words, they are **material independent**.

- The **balance (general) principles** include:
  1. **Conservation of mass**
  2. **Balance of linear momentum**
  3. **Balance of angular momentum**
  4. **First law of thermodynamics (conservation of energy)**
- The **general (balance) principles** appear in **two forms**:
  1. **Integral form** for finite volume of material
  2. **Differential form** for differential volume of material

Continuum Mechanics - Constitutive Equations

- **Constitutive equations** describe behavior of **idealized materials**.
- Two examples of idealized materials:
  - **Linear Elasticity** (Solids): a mechanical load is applied to a solid body and causes deformation, which **disappears after removal of the load** and therefore there is **no residual/permanent deformation**. In other words, the stress is linearly proportional to strain.



Source:Wikipedia:Hooke's Law

  - **Linear Viscosity** (Fluids): the stress is a **linear expression** of the **rates of change of length and angle**.

Continuum Mechanics - Solution

- Due to the **continuum assumption** we can use
- **Calculus** (mathematical study of continuous change)
- and therefore we end up with `physical problems` governed by **partial differential equations**
- and in order to have **uniqueness of solution** we need:
  1. **boundary conditions**
  2. **initial conditions (dynamics)**
- There are 2 kinds of **solutions** of the differential equations:
  1. `analytical`
  2. **numerical**
- The **numerical methods** try to transform the **continuous problem to discrete**.

  Some **numerical methods** used to solve the partial differential equations are:
  1. Finite Element Method (**FEM**)
  2. Finite Differences Method (**FDM**)
  3. Finite Volume Method (**FVM**)
  4. Boundary Element Method (**BEM**)

Continuum Mechanics - Solution

*'It is sometimes said that **solutions to engineering problems** obtained through the* SIMPLIFYING ASSUMPTIONS *of the engineering **theory of beams** are only **approximations** to the **'exact' solutions** of the* `theory of elasticity`*. There may be **exact mathematical solutions** to the equations formulated in **elasticity** or in other branches of **continuum mechanics**, but **the equations themselves** are **not exact descriptions of nature**. In this respect the difference between the **elementary theory** and the **advanced theory** is one of **degree** rather than of **kind**. When the **elementary** theory is formulated consistently and logically, it is just **as respectable as** the **advanced** theory from a mathematical or logical point of view. And from a practical point of view it is just as good in those areas where its **predictions** agree closely enough with **experience**. The **bounds** of applicability of these elementary theories are determined **by experience**, either from **experimental verification**, or from comparisons with **predictions** of the more **advanced continuum theories**.'*, Malvern, L.E. [8].

Mathematical Preliminaries

Mathematical preliminaries - Tensors

- In order to maintain their universality physical laws, should be independent of the observer, [8].
- In other words, the physical law should be independent of the coordinate system.
- To this end, the physical laws should be expressed by tensor equations, [8].
- Tensors are mathematical entities represented to any coordinate system by a set of quantities, namely their components.
- The law of transformation is used to define a tensor.
- The tensor transformation is linear and homogeneous, [9].
- In other words, tensor transformations are invariant under coordinate transformation, [9].
- In 3-dimensional Euclidean space the number of components of a tensor are 3^n, where n is the **order of tensor**, [9].
- We define the four-dimensional continuum by using Euclidean space (x,y,z) and time (t).

Tensors - order

- Mathematical entities with only magnitude are called scalars, namely **tensors of order 0** (zeroth-order), and they have $3^0 = 1$ components.
- Scalars have magnitude, but no direction.
- Some examples of scalars are the mass and the energy of a system.
- **Tensors of order 1 ($n = 1$)** (first-order) have 3 components ($3^1 = 3$) and they are called **vectors** and have both magnitude and direction.
- Some examples of **vectors** are the **forces**, the **velocity**, the **area**.
- **Tensors of order 2 ($n = 2$) (dyadic)** have 9 components ($3^2 = 9$) and they have magnitude and two directions.
- Some examples of **second-order tensors** are the **stress** and the **strain**.
- Stress for example has two directions: one for force, one for area and a magnitude: the magnitude of the stress.

Tensor Algebra

Tensor Algebra

- Lightface Latin and Greek letters denote **scalars**.
- Boldface lowercase Latin and Greek letters denote **vectors**, except from the letters $\boldsymbol{o}$, $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, which are used for **points**.
- Boldface uppercase Latin and Greek letters denote **tensors**, but letters $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{Z}$ are used for **points**.
-

Entity	Direct Tensor Notation	Matrix Notation
scalar	u	
vector	$\vec{\boldsymbol{u}}$	$\boldsymbol{u} = [u]$
tensor[1]	$\underline{\underline{\boldsymbol{U}}}$	$\boldsymbol{U} = [U]$

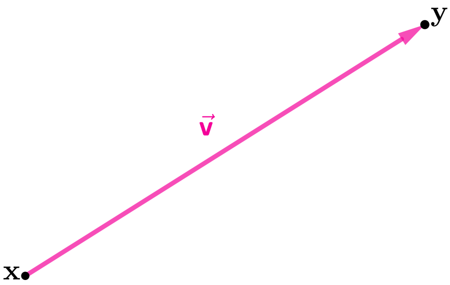
- Note that also **tensor notation** is independent of the **coordinate system** (we will use Cartesian) and there is one to one relationship between tensor and a matrix ONLY after the **coordinate system has been determined**.

[1] of rank/order 2

Point, Vector

- We use three-dimensional Euclidean space $\mathcal{E}$
- The term **point** is used for elements in $\mathcal{E}$
- The term **vector** is used for elements in vector space $\mathcal{V}$
- The difference between two points $\boldsymbol{x}$ and $\boldsymbol{y}$ defines the vector:

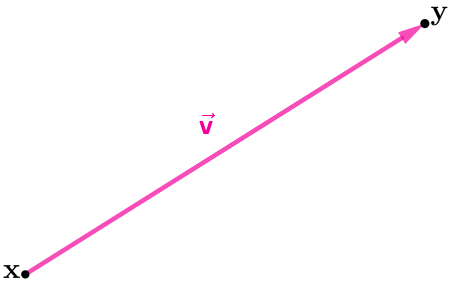
$$\vec{\boldsymbol{v}} = \boldsymbol{y} - \boldsymbol{x}$$



Vector - matrix form

- **Vectors** are stored in column **matrices** (arrays) and each line represents its **components**. The vector $\vec{v}$ is stored in the column matrix $\boldsymbol{v}$:

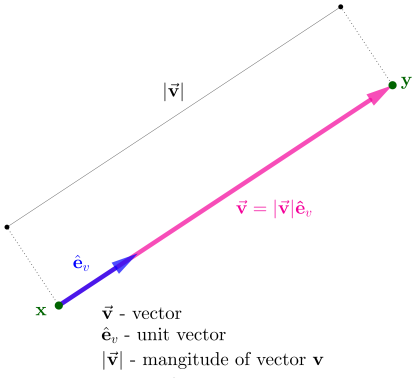
$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} y_1 - x_1 \\ y_2 - x_2 \\ y_3 - x_3 \end{bmatrix}$$

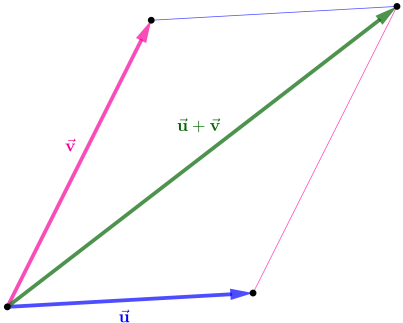


Vector

The characteristics of **vectors** are the following:

1. **vectors** have magnitude and direction. The magnitude can be defined by the Euclidean norm:
$$|\vec{\boldsymbol{v}}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$
2. two **vectors** are compounded in accordance to the parallelogram law, [12]

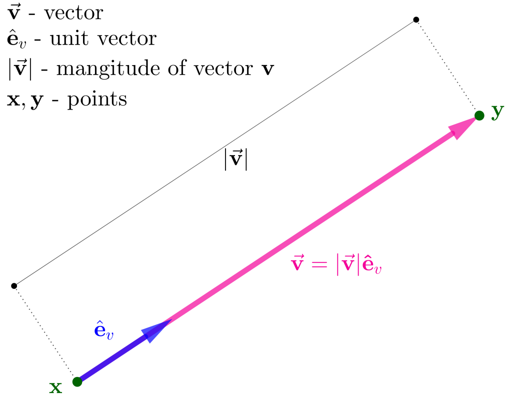




Unit Vector

The unit vector is denoted as $\hat{e}$ and has a magnitude equal to 1. The unit vector of a vector $\vec{v}$ is defined as:

$$\hat{\boldsymbol{e}}_v = \frac{\vec{\boldsymbol{v}}}{|\vec{\boldsymbol{v}}|}; \quad |\hat{\boldsymbol{e}}_v| = 1$$



Vector addition

Let $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}$ be any vector, then the following properties hold, [11]:

- Commutative property:

$$\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}} = \vec{\boldsymbol{v}} + \vec{\boldsymbol{u}}$$

- Associative property:

$$(\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}}) + \vec{\boldsymbol{w}} = \vec{\boldsymbol{u}} + (\vec{\boldsymbol{v}} + \vec{\boldsymbol{w}})$$

- Existence of zero vector $\vec{\mathbf{0}}$:

$$\vec{\boldsymbol{u}} + \vec{\mathbf{0}} = \vec{\boldsymbol{u}}$$

- For every vector $\vec{\boldsymbol{u}}$, there exists a vector such that:

$$\vec{\boldsymbol{u}} + (-\vec{\boldsymbol{u}}) = \vec{\mathbf{0}}$$

- The $\vec{\mathbf{0}}$ has zero magnitude ($|\vec{\mathbf{0}}| = 0$) and not direction.

Vector multiplied by a scalar

Let $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}$ be any vector and α, β be any real scalars, [11]:

- Associative property:

$$\alpha\left(\beta \vec{\boldsymbol{u}}\right) = \left(\alpha\beta\right) \vec{\boldsymbol{u}}$$

- Distributive scalar addition:

$$\left(\alpha + \beta\right) \vec{\boldsymbol{u}} = \alpha \vec{\boldsymbol{u}} + \beta \vec{\boldsymbol{u}}$$

- Distributive vector addition:

$$\alpha\left(\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}}\right) = \alpha \vec{\boldsymbol{u}} + \alpha \vec{\boldsymbol{v}}$$

- For every vector $\vec{\boldsymbol{u}}$, there exists vectors such that:

$$1 \vec{\boldsymbol{u}} = \vec{\boldsymbol{u}} 1 = \vec{\boldsymbol{u}}; \quad 0 \vec{\boldsymbol{u}} = \vec{\boldsymbol{0}}$$

Vector multiplied by a scalar

- Two vectors are equal when they have the same magnitude and same direction.
- The position of a vector in space can be arbitrary. There are applications that the point of location of a vector is important (moment of a force). In this case, the vector is called localized or bound vector, [11].
- Two vectors $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ are linear dependent if their summation after their multiplication with non-zero scalars (α, β) provides the zero vector $\vec{\mathbf{0}}$:

$$\alpha\vec{\boldsymbol{u}} + \beta\vec{\boldsymbol{v}} = \vec{\mathbf{0}}$$

- If two vectors are linearly dependent are called **collinear**. If three vectors are linearly dependent, they are called **coplanar**.
- Generally, the vectors $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}$ are **linear dependent** if the following expression is satisfied, when not all the coefficients (α, β, γ) are zero:

$$\alpha\vec{\boldsymbol{u}} + \beta\vec{\boldsymbol{v}} + \gamma\vec{\boldsymbol{v}} = \vec{\mathbf{0}}$$

Cartesian frame of reference

We will use the Cartesian coordinate system. The unit vectors in each of the three direction are:

$$(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3)$$

with magnitude equal to:

$$|\hat{\boldsymbol{e}}_1| = |\hat{\boldsymbol{e}}_2| = |\hat{\boldsymbol{e}}_3| = 1$$

The unit vectors are orthogonal:

$$\hat{\boldsymbol{e}}_1 \cdot \hat{\boldsymbol{e}}_2 = \hat{\boldsymbol{e}}_2 \cdot \hat{\boldsymbol{e}}_3 = \hat{\boldsymbol{e}}_2 \cdot \hat{\boldsymbol{e}}_1 = 0$$

Also for the unit vectors the following relationships hold:

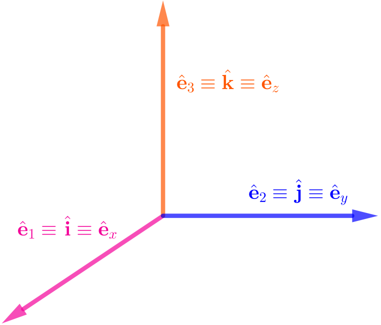
$$\hat{\boldsymbol{e}}_1 \times \hat{\boldsymbol{e}}_2 = \hat{\boldsymbol{e}}_3; \quad \hat{\boldsymbol{e}}_2 \times \hat{\boldsymbol{e}}_3 = \hat{\boldsymbol{e}}_1; \quad \hat{\boldsymbol{e}}_3 \times \hat{\boldsymbol{e}}_1 = \hat{\boldsymbol{e}}_2$$

Cartesian frame of reference

The set $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3)$ is called basis or orthonormal. A basis system can be also non orthonormal, but in that case it can be denoted without hats: $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$.
Any vector in 3-dimensional Euclidean space can be written as a linear combination of the orthonormal basis:

$$\vec{\boldsymbol{u}} = u_1 \hat{\boldsymbol{e}}_1 + u_2 \hat{\boldsymbol{e}}_2 + u_3 \hat{\boldsymbol{e}}_3$$

The components of the vector $\vec{\boldsymbol{u}}$ are $u_1 \hat{\boldsymbol{e}}_1$, $u_2 \hat{\boldsymbol{e}}_2$, $u_3 \hat{\boldsymbol{e}}_3$.



Index notation (Tensor Component notation)

The component expression of a vector $\vec{\boldsymbol{u}} = u_1\hat{\boldsymbol{e}}_1 + u_2\hat{\boldsymbol{e}}_2 + u_3\hat{\boldsymbol{e}}_3$ can be written by using the following sum:

$$\vec{\boldsymbol{u}} = \sum_{i=1}^{3} u_i\hat{\boldsymbol{e}}_i$$

By omitting the summation symbol and using the summation convention:

$$\vec{\boldsymbol{u}} = u_i\hat{\boldsymbol{e}}_i$$

The convention was suggested by Einstein and it allows to write expressions to compact form. The repeated index is called dummy and can be replaced by another symbol:

$$\vec{\boldsymbol{u}} = u_i\hat{\boldsymbol{e}}_i = u_j\hat{\boldsymbol{e}}_j = u_k\hat{\boldsymbol{e}}_k$$

No index can appear more than two times. Free index appears in every expression apart from the case of scalars and it runs from 1 to 3.

Kronecker delta and permutation symbol

The dot product between two orthonormal vectors can be defined:

$$\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j = \delta_{ij}$$

where

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

The term δ_{ij} is called Kronecker delta (unit tensor). The permutation symbol ϵ_{ijk} (permutation 3^{rd} order tensor) can be defined as:

$$\epsilon_{ijk} = \begin{cases} 1, & \text{if they are in cyclic order: } ijk,\ kij,\ jki \\ -1, & \text{if they are not in cyclic order: } ikj,\ kji,\ jik \\ 0, & \text{if any of } i, j, k \text{ are repeated} \end{cases}$$

The identity between the Kronecker delta and the permutation symbol:

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

The indices i, j, k can take values from 1 to 3 for 3D Cartesian coordinate system.

Index notation (Tensor Component notation) summary

Einstein Summation convention (repeated index):

$$u_i v_i = \sum_{i=1}^{3} u_i v_i = u_1 v_1 + u_2 v_2 + u_3 v_3; \quad i \text{ is a 'dummy' index}$$

$$A_{ij} v_j = \sum_{j=1}^{3} A_{ij} v_j = A_{i1} v_1 + A_{i2} v_2 + A_{i3} v_3; \quad j \text{ is a 'dummy' index, } i \text{ is a 'free' index}$$

- An index can only appear up to two times in a monomial
- A 'mute' ('dummy') index is an index that disappears after the summation takes place (its name can be changed arbitrarily)
- A 'talking' (free) index is not repeated in the same monomial (its name cannot be changed arbitrarily)

Kronecker delta - Permutation symbol

The Kronecker delta is called also substitution operator due to the following property:

$$\delta_{ij} v_j = v_i$$

Try to prove it! For the permutation symbol the following hold true:

$$\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}$$
$$\epsilon_{ijk} = -\epsilon_{ikj} = -\epsilon_{jik} = -\epsilon_{kji}$$

The permutation can also be written in terms of its indices:

$$\epsilon_{ijk} = \frac{1}{2}(i-j)(j-k)(k-i)$$

Index notation (Tensor Component notation) summary

Index notation (Tensor Component notation) guidelines:

- Sum over all repeated indices
- Expand free indices, cover all combinations
- A free index is not repeated in the same monomial (its name cannot be changed arbitrarily)
- The number of free indices indicate the order of the tensor
- A comma means differentiation:

$$u_{i,i} = \frac{\partial u_i}{\partial x_i} = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i}; \quad u_{i,jj} = \frac{\partial^2 u_i}{\partial x_j \partial x_j} = \sum_{j=1}^{3} \frac{\partial^2 u_i}{\partial x_j^2}; \quad A_{ij,j} = \frac{\partial A_{ij}}{\partial x_j} = \sum_{j=1}^{3} \frac{\partial A_{ij}}{\partial x_j}$$

Dot Product

- The **dot product**, also called **inner product** or **scalar product**, of two vectors $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ is denoted as in **direct tensor notation**:

$$\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}}$$

- In index notation, we can write:

$$\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} = u_i \hat{\boldsymbol{e}}_i \cdot v_j \hat{\boldsymbol{e}}_j = u_i v_j \hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j = u_i v_j \delta_{ij} = u_i v_i$$

- We can define the **magnitude** of a vector as:

$$|\vec{\boldsymbol{u}}| = \sqrt{\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{u}}}$$

  The can magnitude can also be denoted as $\|\vec{\boldsymbol{u}}\|$, or u. The square of vector $\vec{\boldsymbol{u}}$:

$$|\vec{\boldsymbol{u}}|^2 = \vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{u}}$$

- The **angle** $\theta = \angle(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}})$ between nonzero vectors $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ can be defined as:

$$cos(\theta) = \frac{\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}}}{|\vec{\boldsymbol{u}}||\vec{\boldsymbol{v}}|}; \quad (0 \leq \theta \leq \pi)$$

  and:

$$\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} = |\vec{\boldsymbol{u}}||\vec{\boldsymbol{v}}| cos(\theta)$$

Dot Product - Matrix Notation

- The **dot product**, of two vectors $\vec{u}$ and $\vec{v}$, namely $\vec{u} \cdot \vec{v}$, is denoted in **matrix notation** as

$$[u]^T[v] = \boldsymbol{u}^T\boldsymbol{v} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = u_1v_1 + u_2v_2 + u_3v_3$$

where $\boldsymbol{u}^T$ denotes the transpose of column array $\boldsymbol{u}$.

- The **magnitude** of a vector can be defined in **matrix notation** by the Euclidean norm:

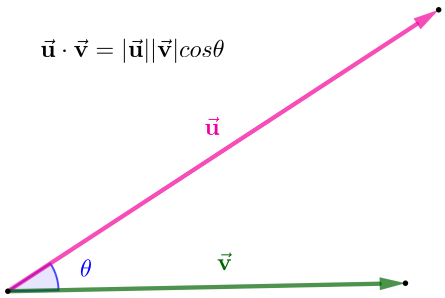
$$\|\boldsymbol{u}\| = \sqrt{\boldsymbol{u}^T\boldsymbol{u}} = \sqrt{\begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}} = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

Dot Product: Direct Tensor Notation - Matrix Notation

The **dot product** of the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in two different notations:

Algebraic Operation	Direct Tensor Notation	Matrix Notation
Dot product	$\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}}$	$\boldsymbol{u}^T \boldsymbol{v}$

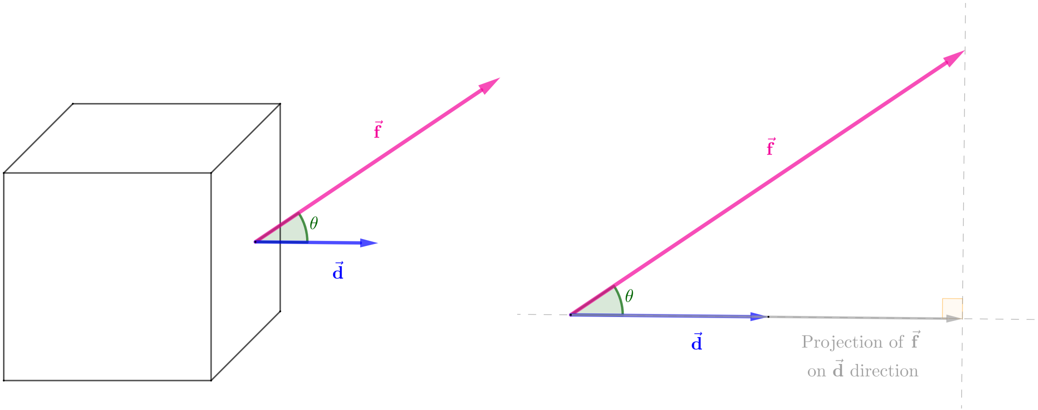
Note that **vectors** are stored in **column matrices**.



Dot Product - Application

The work done by a force vector $\vec{f}$ of a mass moved with displacement vector $\vec{d}$ can be calculated by the dot product, [11]:

$$\vec{f} \cdot \vec{d} = |\vec{f}||\vec{d}|cos\theta$$

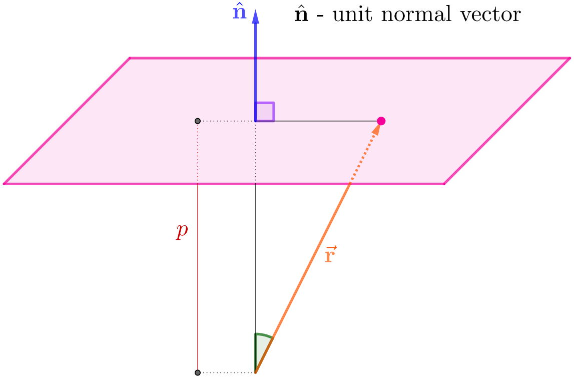


Dot Product - Application

Equation of a plane: $\vec{\mathbf{r}} \cdot \hat{\mathbf{n}} = p$

$\vec{\mathbf{r}}$ - position vector

$\hat{\mathbf{n}}$ - unit normal vector



Dot Product - Properties

The following properties hold for the dot product, [11]:

- Commutative property:

$$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$$

- For two vectors that are perpendicular to each other:

$$\vec{u} \cdot \vec{v} = 0$$

- For two vectors parallel to each other:

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|$$

- The projection of a vector $\vec{u}$ in any direction $\hat{e}$ is given:

$$(\vec{u} \cdot \hat{e})\, \hat{e}$$

  For a Cartesian coordinate system the components along the three basis axes $\hat{e}_1 = \hat{e}_x$, $\hat{e}_2 = \hat{e}_y$ and $\hat{e}_3 = \hat{e}_z$ are:

$$\vec{u_1} = (\vec{u} \cdot \hat{e}_1)\, \hat{e}_1; \quad \vec{u_2} = (\vec{u} \cdot \hat{e}_2)\, \hat{e}_2; \quad \vec{u_3} = (\vec{u} \cdot \hat{e}_3)\, \hat{e}_3;$$

- The distributive property:

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$$

Direction cosines

For a vector $\vec{\boldsymbol{v}}$:

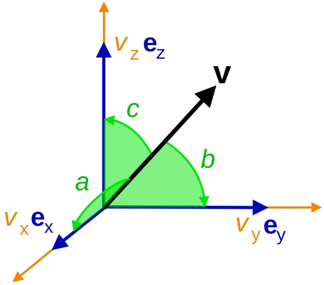
$$\vec{\boldsymbol{v}} = v_x \hat{\boldsymbol{e}}_x + v_y \hat{\boldsymbol{e}}_y + v_z \hat{\boldsymbol{e}}_z$$

the direction cosines between the Cartesian basis and the vector can be defined through the dot product as following:

$$cosa = \frac{\vec{\boldsymbol{v}} \cdot \hat{\boldsymbol{e}}_x}{|\vec{\boldsymbol{v}}|} = \frac{v_x}{\sqrt{v_x^2 + v_y^2 + v_z^2}}; \quad cosb = \frac{\vec{\boldsymbol{v}} \cdot \hat{\boldsymbol{e}}_y}{|\vec{\boldsymbol{v}}|} = \frac{v_y}{\sqrt{v_x^2 + v_y^2 + v_z^2}}; \quad cosc = \frac{\vec{\boldsymbol{v}} \cdot \hat{\boldsymbol{e}}_z}{|\vec{\boldsymbol{v}}|} = \frac{v_z}{\sqrt{v_x^2 + v_y^2 + v_z^2}}$$

And follows that:

$$cos^2 a + cos^2 b + cos^2 c = 1$$



Source: Wikipedia

Projection Vector

The projection vector of $\vec{\boldsymbol{u}}$ to vector $\vec{\boldsymbol{v}} = |\vec{\boldsymbol{v}}|\hat{e}_v$ denoted as $\vec{\boldsymbol{u}}_v$:

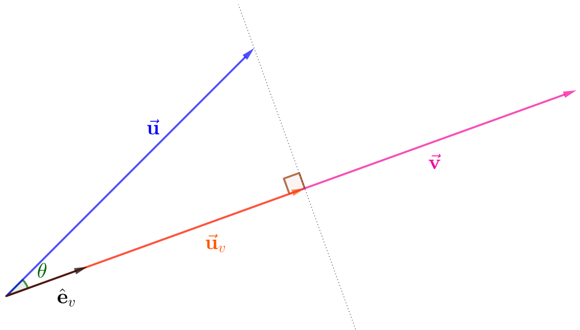
$$\vec{\boldsymbol{u}}_v = |\vec{\boldsymbol{u}}_v|\hat{\boldsymbol{e}}_v$$

The magnitude:

$$|\vec{\boldsymbol{u}}_v| = \vec{\boldsymbol{u}} \cdot \hat{\boldsymbol{e}}_v = \frac{\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}}}{|\vec{\boldsymbol{v}}|}$$

Finally, the projection vector can be defined as:

$$\vec{\boldsymbol{u}}_v = \frac{\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}}}{|\vec{\boldsymbol{v}}|}\hat{\boldsymbol{e}}_v = \frac{\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}}}{|\vec{\boldsymbol{v}}|}\frac{\vec{\boldsymbol{v}}}{|\vec{\boldsymbol{v}}|} = \frac{\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}}}{|\vec{\boldsymbol{v}}|^2}\vec{\boldsymbol{v}} = (\vec{\boldsymbol{u}} \cdot \hat{\boldsymbol{e}}_v)\hat{\boldsymbol{e}}_v$$

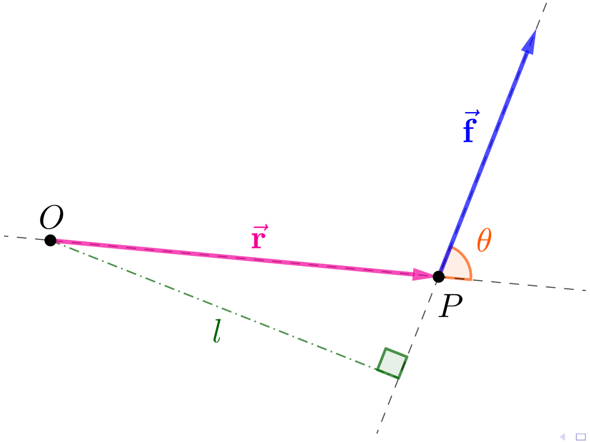


Cross (vector) Product

The moment $\vec{\boldsymbol{m}}$ of force $\vec{\boldsymbol{f}}$ around point O acting at point P can be calculated as:

$$|\vec{\boldsymbol{m}}| = |\vec{\boldsymbol{f}}|l; \quad l = |\vec{\boldsymbol{r}}|sin\theta$$

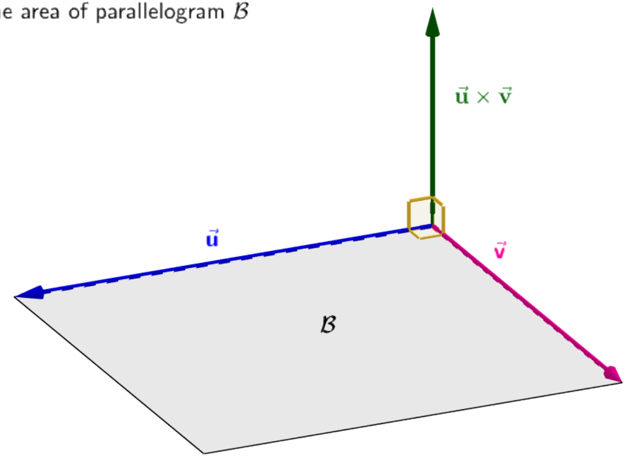
$$\vec{\boldsymbol{m}} = |\vec{\boldsymbol{m}}|\hat{\boldsymbol{e}}_m = |\vec{\boldsymbol{f}}||\vec{\boldsymbol{r}}|sin\theta = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{f}}$$



Cross (vector) Product

$\dfrac{\vec{\mathbf{u}} \times \vec{\mathbf{v}}}{|\vec{\mathbf{u}} \times \vec{\mathbf{v}}|}$ shows the direction (unit vector) of vector $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$

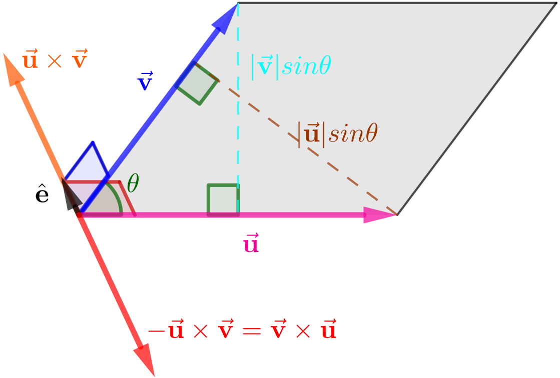
The magnitude of vector $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ represents the area of parallelogram $\mathcal{B}$



Cross (vector) Product

$$\hat{\boldsymbol{e}} = \frac{\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}}{|\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}|}; \quad |\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}| = |\vec{\boldsymbol{u}}||\vec{\boldsymbol{v}}| sin\theta; \quad \vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}} = |\vec{\boldsymbol{u}}||\vec{\boldsymbol{v}}| sin\theta\, \hat{\boldsymbol{e}}$$

$$\theta = \angle(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}); \quad -\theta = \angle(\vec{\boldsymbol{v}}, \vec{\boldsymbol{u}}); \quad 0 \leq \theta \leq \pi$$



Cross (vector) Product

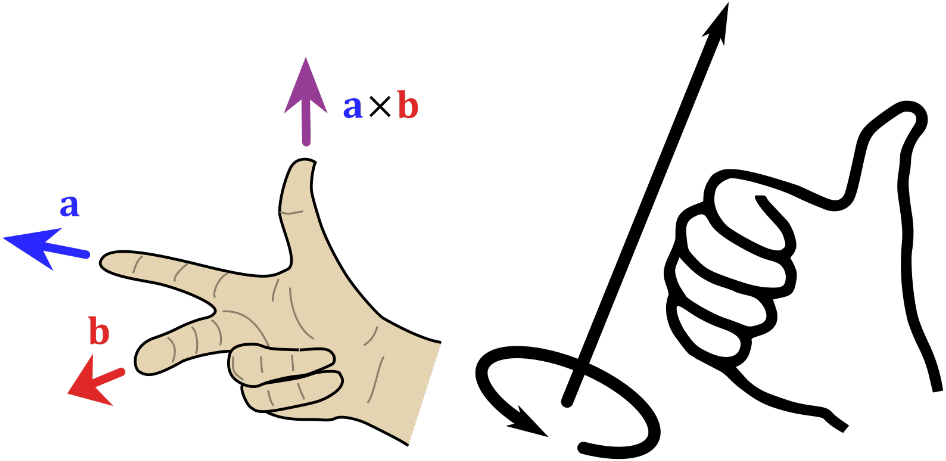
The cross product of two vectors $\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}$ in Cartesian coordinate system can be defined as the determinant

$$\vec{\boldsymbol{w}} = \begin{vmatrix} \vec{\boldsymbol{e}}_1 & \vec{\boldsymbol{e}}_2 & \vec{\boldsymbol{e}}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \vec{\boldsymbol{e}}_1 \left(u_2 v_3 - u_3 v_2\right) + \vec{\boldsymbol{e}}_2 \left(u_3 v_1 - u_1 v_3\right) + \vec{\boldsymbol{e}}_3 \left(u_1 v_2 - u_2 v_1\right)$$

In index notation:

$$\vec{\boldsymbol{w}} = \vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}} = w_i \hat{\boldsymbol{e}}_i = \epsilon_{ijk} u_j v_k \hat{\boldsymbol{e}}_i \Leftrightarrow w_i = \epsilon_{ijk} u_j v_k$$

Vector - Right-hand rule



Source: WikiPedia

Cross (vector) Product - Properties

- The product $\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}$ is not equal to $\vec{\boldsymbol{v}} \times \vec{\boldsymbol{u}}$. It is actually:

$$\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}} \equiv -\vec{\boldsymbol{v}} \times \vec{\boldsymbol{u}}$$

  In other words, cross product does not commute, [11].

- If vectors $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ are parallel ($\theta = 0$ or $\theta = \pi$) and $sin\theta = 0$:

$$\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}} = \vec{\boldsymbol{0}}$$

- The distributive property holds:

$$(\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}}) \times \vec{\boldsymbol{w}} = (\vec{\boldsymbol{u}} \times \vec{\boldsymbol{w}}) + (\vec{\boldsymbol{v}} \times \vec{\boldsymbol{w}})$$

Triple Products

- Multiplication of vector $\vec{\boldsymbol{u}}$ by a scalar $\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}}$:

$$\vec{\boldsymbol{u}}\,(\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}})$$

- The scalar triple product:

$$\vec{\boldsymbol{u}} \cdot (\vec{\boldsymbol{v}} \times \vec{\boldsymbol{w}})$$

- The vector triple product:

$$\vec{\boldsymbol{u}} \times (\vec{\boldsymbol{v}} \times \vec{\boldsymbol{w}})$$

Triple Products

- The dot and cross product can be interchanged:

$$\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} \times \vec{\boldsymbol{w}} = \vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}}$$

- Cyclical permutation of the order of vectors does not change the result:

$$\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} \times \vec{\boldsymbol{w}} = \vec{\boldsymbol{w}} \cdot \vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}} = \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}} \times \vec{\boldsymbol{u}}$$

- If the cyclic order changes, the sign changes as well:

$$\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} \times \vec{\boldsymbol{w}} = -\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{w}} \times \vec{\boldsymbol{v}} = -\vec{\boldsymbol{w}} \cdot \vec{\boldsymbol{v}} \times \vec{\boldsymbol{u}} = -\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{u}} \times \vec{\boldsymbol{w}}$$

- For three vectors to be coplanar the following relationship should hold:

$$\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} \times \vec{\boldsymbol{w}} = 0$$

Triple Products

- The vector triple product $\vec{u} \times (\vec{v} \times \vec{w})$ is a vector that is normal to vector $\vec{u}$ and vector $\vec{v} \times \vec{w}$. The vector $\vec{v} \times \vec{w}$ is perpendicular to the plane formed by $\vec{v}$ and $\vec{w}$, [11]. Therefore the vector $\vec{u} \times (\vec{v} \times \vec{w})$ lies in the plane formed by $\vec{v}$ and $\vec{w}$ and it can be expressed as a linear combination of those two vectors:

$$\vec{u} \times (\vec{v} \times \vec{w}) = \alpha \vec{v} + \beta \vec{w}$$

$$\alpha = \vec{u} \cdot \vec{w}; \quad \beta = -\vec{u} \cdot \vec{v}$$

$$\vec{u} \times (\vec{v} \times \vec{w}) = \vec{u} \cdot \vec{w} \vec{v} - \vec{u} \cdot \vec{v} \vec{w}$$

- The same holds true :

$$(\vec{u} \times \vec{v}) \times \vec{w} = \gamma \vec{u} + \zeta \vec{v}$$

Triple product - Matrix notation

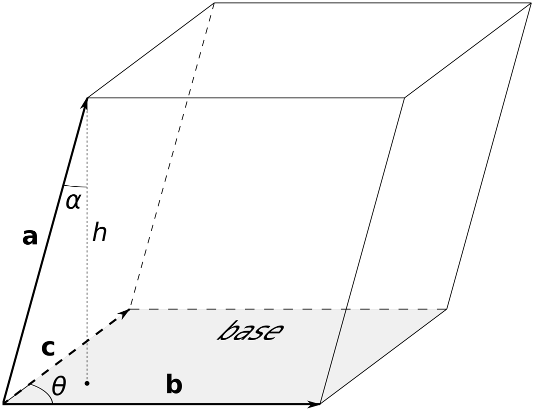
The triple product of two vectors $\vec{\boldsymbol{w}} \cdot (\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}})$ in Cartesian coordinate system can be defined as the determinant

$$\begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = w_1 \left(u_2 v_3 - u_3 v_2\right) + w_2 \left(u_3 v_1 - u_1 v_3\right) + w_3 \left(u_1 v_2 - u_2 v_1\right)$$

Physical meaning - scalar triple product

The scalar triple product provides with the volume of the parallelepiped:

$$\vec{\boldsymbol{a}} \cdot \left(\vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}}\right)$$



What is a tensor?

- ‘We use the term tensor as synonym for the phrase ”linear transformation from $\mathcal{V}$ into $\mathcal{V}$”. A tensor $\underline{\underline{\boldsymbol{S}}}$ is a linear mapping of vectors to vectors’, [4]. Given a vector $\vec{\boldsymbol{u}}$ provides with

$$\vec{\boldsymbol{v}} = \underline{\underline{\boldsymbol{S}}} \cdot \vec{\boldsymbol{u}}$$

  where $\vec{\boldsymbol{v}}$ is also a vector.
- A tensor can be thought as machine that is fed with with vectors as inputs and provides another vector as an output.
- The linearity of a tensor $\underline{\underline{\boldsymbol{S}}}$ is described by the requirements:

$$\underline{\underline{\boldsymbol{S}}} \cdot (\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}}) = \underline{\underline{\boldsymbol{S}}} \cdot \vec{\boldsymbol{u}} + \underline{\underline{\boldsymbol{S}}} \cdot \vec{\boldsymbol{v}}$$

$$\underline{\underline{\boldsymbol{S}}} \cdot (\alpha \vec{\boldsymbol{u}}) = \alpha \underline{\underline{\boldsymbol{S}}} \cdot \vec{\boldsymbol{u}}$$

What is a tensor?

- Two tensors $\underline{\underline{\boldsymbol{S}}}$ and $\underline{\underline{\boldsymbol{T}}}$ are equal if their outputs are the same whenever their inputs are equal, [4]:

$$\underline{\underline{\boldsymbol{S}}} = \underline{\underline{\boldsymbol{T}}} \text{ if and only if } \underline{\underline{\boldsymbol{S}}} \cdot \vec{\boldsymbol{v}} = \underline{\underline{\boldsymbol{T}}} \cdot \vec{\boldsymbol{v}} \text{ for all vectors } \vec{\boldsymbol{v}}$$

- A way to show that tensors $\underline{\underline{\boldsymbol{S}}}$ and $\underline{\underline{\boldsymbol{T}}}$ are equal is a consequence of:

$$\vec{\boldsymbol{a}} \cdot \underline{\underline{\boldsymbol{S}}} \cdot \vec{\boldsymbol{b}} = \vec{\boldsymbol{a}} \cdot \underline{\underline{\boldsymbol{T}}} \cdot \vec{\boldsymbol{b}} \text{ for all vectors } \vec{\boldsymbol{a}} \text{ and } \vec{\boldsymbol{b}} \text{ if and only if } \underline{\underline{\boldsymbol{S}}} = \underline{\underline{\boldsymbol{T}}}$$

- Tensors are generally defined by their actions on arbitrary vectors, [4]. For example:

$$(\underline{\underline{\boldsymbol{S}}} + \underline{\underline{\boldsymbol{T}}}) \cdot \vec{\boldsymbol{v}} = \underline{\underline{\boldsymbol{S}}} \cdot \vec{\boldsymbol{v}} + \underline{\underline{\boldsymbol{T}}} \cdot \vec{\boldsymbol{v}}$$

$$(\alpha \underline{\underline{\boldsymbol{S}}}) \cdot \vec{\boldsymbol{v}} = \alpha (\underline{\underline{\boldsymbol{S}}} \cdot \vec{\boldsymbol{v}})$$

Tensors - Dyads - Outer/Tensor product

- Tensors of order 2, namely dyads, are mathematical objects that endow with a magnitude and two directions. They fulfill the vector addition and scalar multiplication properties.
- An example of a dyad is a stress vector $\vec{t}$ (which is a force at unit area) acting on a surface apart from the magnitude and the orientation of the stress vector, depends as well on the orientation of the plane $\hat{n}$ that acts upon.
- Due to the fact that these quantities require two directions are called dyads, or second-order tensor. To this end, dyad is defined as two vectors standing side by side acting as a unit, [11]. A linear combination of dyads is called dyadic. The tensor product (outer product) of two vectors is denoted in direct tensor notation as:

$$\vec{\boldsymbol{u}}\vec{\boldsymbol{v}} = \vec{\boldsymbol{u}} \otimes \vec{\boldsymbol{v}} = \underline{\underline{\boldsymbol{A}}}$$

- The dot product of dyad with a vector gives another vector:

$$\underline{\underline{\boldsymbol{A}}} \cdot \vec{\boldsymbol{u}} = \vec{\boldsymbol{w}}$$

- The dot product of dyad with itself is a dyad denoted as:

$$\underline{\underline{\boldsymbol{A}}} \cdot \underline{\underline{\boldsymbol{A}}} = \underline{\underline{\boldsymbol{A}}}^2$$

- The outer/tensor product can be defined in index notation:

$$\underline{\underline{\boldsymbol{A}}} = \vec{\boldsymbol{u}} \otimes \vec{\boldsymbol{v}} \Leftrightarrow A_{ij} = u_i v_j$$

Tensors - Dyads

- The properties of dyads are as follows (α, β are scalars):

$$(\vec{\boldsymbol{u}} \otimes \vec{\boldsymbol{v}}) \cdot \vec{\boldsymbol{w}} = \vec{\boldsymbol{u}}(\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}}) \equiv \vec{\boldsymbol{u}} \otimes (\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{w}})$$

$$\vec{\boldsymbol{u}} \otimes (\alpha \vec{\boldsymbol{v}} + \beta \vec{\boldsymbol{w}}) = \alpha \vec{\boldsymbol{u}} \otimes \vec{\boldsymbol{v}} + \beta \vec{\boldsymbol{u}} \otimes \vec{\boldsymbol{w}}$$

$$(\alpha \vec{\boldsymbol{v}} \otimes \vec{\boldsymbol{u}} + \beta \vec{\boldsymbol{w}} \otimes \vec{\boldsymbol{c}}) \cdot \vec{\boldsymbol{d}} = \alpha(\vec{\boldsymbol{v}} \otimes \vec{\boldsymbol{u}}) \cdot \vec{\boldsymbol{d}} + \beta(\vec{\boldsymbol{w}} \otimes \vec{\boldsymbol{c}}) \cdot \vec{\boldsymbol{d}} = \alpha[\vec{\boldsymbol{v}} \otimes (\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{d}})] + \beta[\vec{\boldsymbol{w}} \otimes (\vec{\boldsymbol{c}} \cdot \vec{\boldsymbol{d}})]$$

  Dyad does not contain the commutative property: $\vec{\boldsymbol{u}} \otimes \vec{\boldsymbol{v}} \neq \vec{\boldsymbol{v}} \otimes \vec{\boldsymbol{u}}$

- In Cartesian system the dyad can be defined as:

$$\underline{\underline{\boldsymbol{A}}} = \vec{\boldsymbol{u}} \otimes \vec{\boldsymbol{v}} = (u_i \hat{\boldsymbol{e}}_i) \otimes (v_j \hat{\boldsymbol{e}}_j) = u_i v_j (\hat{\boldsymbol{e}}_i \otimes \hat{\boldsymbol{e}}_j) = A_{ij}(\hat{\boldsymbol{e}}_i \otimes \hat{\boldsymbol{e}}_j)$$

- Higher order tensors can be written as follows in Cartesian coordinates:

$$\underline{\underline{\underline{\boldsymbol{B}}}} = B_{ijk}(\hat{\boldsymbol{e}}_i \otimes \hat{\boldsymbol{e}}_j \otimes \hat{\boldsymbol{e}}_k); \quad \text{third-order tensor}$$

$$\underline{\underline{\underline{\underline{\boldsymbol{C}}}}} = C_{ijkl}(\hat{\boldsymbol{e}}_i \otimes \hat{\boldsymbol{e}}_j \otimes \hat{\boldsymbol{e}}_k \otimes \hat{\boldsymbol{e}}_l); \quad \text{fourth-order tensor}$$

- The double dot (scalar) product between two dyads $\underline{\underline{\boldsymbol{A}}} = \vec{\boldsymbol{u}} \otimes \vec{\boldsymbol{v}}$, $\underline{\underline{\boldsymbol{B}}} = \vec{\boldsymbol{c}} \otimes \vec{\boldsymbol{d}}$ is defined as, [11]:

$$\underline{\underline{\boldsymbol{A}}} : \underline{\underline{\boldsymbol{B}}} = (\vec{\boldsymbol{u}} \otimes \vec{\boldsymbol{v}}) : (\vec{\boldsymbol{c}} \otimes \vec{\boldsymbol{d}}) = (\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{d}})(\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{c}})$$

Tensors - Dyads - Matrix form

- According to the definition:

$$\underline{\underline{\boldsymbol{A}}} = A_{ij}(\hat{\boldsymbol{e}}_i \otimes \hat{\boldsymbol{e}}_j) = A_{11}\hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_1 + A_{12}\hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_2 + A_{13}\hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_3$$
$$+ A_{21}\hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_1 + A_{22}\hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_2 + A_{23}\hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_3$$
$$+ A_{31}\hat{\boldsymbol{e}}_3\hat{\boldsymbol{e}}_1 + A_{32}\hat{\boldsymbol{e}}_3\hat{\boldsymbol{e}}_2 + A_{33}\hat{\boldsymbol{e}}_3\hat{\boldsymbol{e}}_3$$

- In matrix form the tensor $\underline{\underline{\boldsymbol{A}}}$ can be written as $\boldsymbol{A}$:

$$\boldsymbol{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}; \quad \underline{\underline{\boldsymbol{A}}} \equiv \begin{bmatrix} \hat{\boldsymbol{e}}_1 \\ \hat{\boldsymbol{e}}_2 \\ \hat{\boldsymbol{e}}_3 \end{bmatrix}^T \boldsymbol{A} \begin{bmatrix} \hat{\boldsymbol{e}}_1 \\ \hat{\boldsymbol{e}}_2 \\ \hat{\boldsymbol{e}}_3 \end{bmatrix}$$

- The unit dyad can be defined as:

$$\underline{\underline{\boldsymbol{I}}} = \hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j = \hat{\boldsymbol{e}}_i \otimes \hat{\boldsymbol{e}}_j$$

Matrices - Properties

Matrices have the following properties:

1. Addition is commutative:
$$\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$$
2. Addition is associative:
$$\boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C}) = (\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C}$$
3. There exist a matrix $\mathbf{0}$, called zero matrix, such that:
$$\boldsymbol{A} + \mathbf{0} = \mathbf{0} + \boldsymbol{A} = \boldsymbol{A}$$
4. For each matrix $\boldsymbol{A}$ there exists a matrix $-\boldsymbol{A}$ such that:
$$\boldsymbol{A} - \boldsymbol{A} = \mathbf{0}$$
5. Addition is distributive with respect to scalar multiplication:
$$\alpha\,(\boldsymbol{A} + \boldsymbol{B}) = \alpha\boldsymbol{A} + \alpha\boldsymbol{B}$$
6. Addition is distributive with respect to matrix multiplication:
$$\alpha\,(\boldsymbol{A} + \boldsymbol{B})\,\boldsymbol{C} = \boldsymbol{A}\boldsymbol{C} + \boldsymbol{B}\boldsymbol{C}$$

Matrices - Symmetric and Skew Symmetric

The transpose of a matrix $\boldsymbol{A}$ is obtained by interchanging rows with lines and is denoted as $\boldsymbol{A}^T$. The following properties hold for transpose matrices:

1. $$\left(\boldsymbol{A}^T\right)^T = \boldsymbol{A}$$
2. $$(\boldsymbol{A} + \boldsymbol{B})^T = \boldsymbol{A}^T + \boldsymbol{B}^T$$
3. A square matrix is symmetric if $\boldsymbol{A}^T = \boldsymbol{A}$, or in index notation: $a_{ij} = a_{ji}$
4. A square matrix is skew symmetric if $\boldsymbol{A}^T = -\boldsymbol{A}$, or in index notation: $a_{ij} = -a_{ji}$. The diagonal elements of a skew symmetric matrix are zero.

Matrices - Multiplication

The matrix $\boldsymbol{A}$ denotes a $m * n$ matrix, while matrix $\boldsymbol{B}$ denotes a $p \times q$ matrix

- The product $\boldsymbol{AB}$ can be only determined if $n = p$. In other words, if number of the columns of matrix $\boldsymbol{A}$ is equal to the number of rows of matrix $\boldsymbol{B}$. The product will be a $m * q$ matrix. Also the product $\boldsymbol{BA}$ is defined if $q = m$.
- If the product $\boldsymbol{AB}$ is defined the product he product $\boldsymbol{BA}$, may or may not be defined. If both are defined, then they are not necessarily of equal size. The sizes will be equal only if the matrices are square.
- Matrix multiplication is not commutative. In other words, $\boldsymbol{AB} \neq \boldsymbol{BA}$
- The square matrix $\boldsymbol{A}$ is defined as normal if $\boldsymbol{A}\boldsymbol{A}^T = \boldsymbol{A}^T\boldsymbol{A}$
- If $\boldsymbol{A}$ is a square matrix, the powers of $\boldsymbol{A}$ can be defined as: $\boldsymbol{A}^2 = \boldsymbol{AA}$, $\boldsymbol{A}^3 = \boldsymbol{A}^2\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^2$
- Matrix multiplication is associative: $(\boldsymbol{AB})\,\boldsymbol{C} = \boldsymbol{A}\,(\boldsymbol{BC})$
- The product of any matrix with the identity results in the matrix itself.
- The transpose of matrix $(\boldsymbol{AB})^T = \boldsymbol{B}^T\boldsymbol{A}^T$

Matrices - Inverse

- If $\boldsymbol{A}$ and $\boldsymbol{B}$ are square matrices and $\boldsymbol{AB} = \boldsymbol{BA} = \boldsymbol{I}$, then $\boldsymbol{B}$ is the inverse of matrix $\boldsymbol{A}$.
- If both matrices $\boldsymbol{B}$ and $\boldsymbol{C}$ are inverse of matrix $\boldsymbol{A}$ then:

$$\boldsymbol{AB} = \boldsymbol{BA} = \boldsymbol{AC} = \boldsymbol{CA} = \boldsymbol{I}$$

- Matrix multiplication is associative:

$$\boldsymbol{BAC} = (\boldsymbol{BA})\boldsymbol{C} = \boldsymbol{IC} = \boldsymbol{C} = \boldsymbol{B}(\boldsymbol{AC}) = \boldsymbol{BI} = \boldsymbol{B}$$

  which shows that $\boldsymbol{B} = \boldsymbol{C}$
- The inverse of matrix $\boldsymbol{A}$ is denoted as $\boldsymbol{A}^{-1}$. If matrix $\boldsymbol{A}$ does not have an inverse is called singular. If $\boldsymbol{A}$ is nonsingular, then $(\boldsymbol{A}^{-1})^T = (\boldsymbol{A}^T)^{-1}$.

Matrices - Determinant

- In order to associate a scalar with matrix $\boldsymbol{A}$ and indicates if a matrix singular or not, we use the determinant:

$$det\boldsymbol{A} = |\boldsymbol{A}| = |a_{ij}|$$

- The determinant of a 2 b 2 matrix can be defined as:

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}; \quad |\boldsymbol{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

- For a 3 by 3 matrix:

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

the determinant can be defined as:

$$|\boldsymbol{A}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{23}a_{31} - a_{21}a_{33}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Determinant - Properties

- The equations hold true: $det(\boldsymbol{AB}) = det\boldsymbol{A}\,det\boldsymbol{B}$ and $det\boldsymbol{A}^T = det\boldsymbol{A}$
- For α a scalar and n the order of matrix:

$$det(\alpha\boldsymbol{A}) = \alpha^n det\boldsymbol{A}$$

- If matrix $\boldsymbol{A}'$ is obtained from matrix $\boldsymbol{A}$ by multiplied a row or a column by a scalar α then:

$$det\boldsymbol{A}' = \alpha\, det\boldsymbol{A}$$

- If matrix $\boldsymbol{A}'$ is obtained from matrix $\boldsymbol{A}$ by exchanging two rows or columns then:

$$det\boldsymbol{A}' = -det\boldsymbol{A}$$

- If matrix $\boldsymbol{A}$ has two rows or columns linear dependent (one is a scalar multiple of the other one) then:

$$det\boldsymbol{A} = 0$$

- If matrix $\boldsymbol{A}'$ is obtained from matrix $\boldsymbol{A}$ by adding a multiple of one row or column to another then:

$$det\boldsymbol{A}' = det\boldsymbol{A}$$

Determinant

- The determinant can be written in index notation as follows:

$$|\boldsymbol{A}| = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} |A_{ij}|$$

where $|A_{ij}|$ is the determinant of a matrix obtained by deleting the ith row and the jth column.

- The determinant can also be written as:

$$|\boldsymbol{A}| = \sum_{i=1}^{n} a_{ij} cof_{ij}(\boldsymbol{A})$$

where cof_{ij} is called the cofactor:

$$cof_{ij}(\boldsymbol{A}) \equiv (-1)^{i+j} A_{ij}(\boldsymbol{A})$$

- The inverse of a matrix $\boldsymbol{A}$ can be defined as:

$$\boldsymbol{A} = \frac{1}{det\boldsymbol{A}} Adj\boldsymbol{A}$$

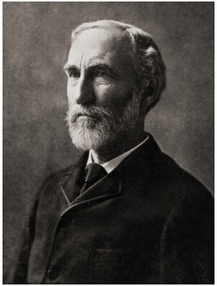
where $Adj\boldsymbol{A}$ is the matrix that each element is replaced by the cofactor cof_{ij}.

Examples of tensor and matrix notation

Direct tensor notation	Tensor Component notation	Matrix notation
$\alpha = \vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}}$	$\alpha = u_i v_i$	$\alpha = \boldsymbol{u}^T \boldsymbol{v}$
$\underline{\underline{\boldsymbol{A}}} = \vec{\boldsymbol{u}} \otimes \vec{\boldsymbol{v}} \equiv \vec{\boldsymbol{u}}\vec{\boldsymbol{v}}$	$A_{ij} = u_i v_j$	$\boldsymbol{A} = \boldsymbol{u}\boldsymbol{v}^T$
$\vec{\boldsymbol{u}} = \underline{\underline{\boldsymbol{A}}} \cdot \vec{\boldsymbol{v}}$	$u_i = A_{ij} v_j$	$\boldsymbol{u} = \boldsymbol{A}\boldsymbol{v}$
$\vec{\boldsymbol{u}} = \vec{\boldsymbol{v}} \cdot \underline{\underline{\boldsymbol{A}}}$	$u_i = v_i A_{ij}$	$\boldsymbol{u} = \boldsymbol{v}^T \boldsymbol{A}$

Tensor Analysis

Tensor Fields

- In continuum mechanics we have mathematical objects called fields, [7].
- A field is a function of three spatial coordinates and time, [7].
- Fields are classified as scalar, vector and tensor with different number of components each.
- Scalar field has one component, vector field has three and a tensor field of rank r has 3^r components.
- 

Josiah Willard Gibbs

Tensor Fields

Field	Direct Tensor Notation	Tensor Component Notation
scalar	$\phi = \phi(\vec{\boldsymbol{x}}, t)$	
vector	$\vec{\boldsymbol{u}} = \vec{\boldsymbol{u}}(\vec{\boldsymbol{x}}, t)$	$u_i = u_i(\vec{\boldsymbol{x}}, t)$
tensor[2]	$\underline{\underline{\boldsymbol{U}}} = \underline{\underline{\boldsymbol{U}}}(\vec{\boldsymbol{x}}, t)$	$T_{ij} = T_{ij}(\vec{\boldsymbol{x}}, t)$

[2] of rank/order 2

Spatial derivatives - Nabla (del) operator - Gradient

The nabla (del) operator $\vec{\boldsymbol{\nabla}}$ is used to to calculate spatial derivatives:

$$\vec{\boldsymbol{\nabla}} \equiv \frac{\partial}{\partial \vec{\boldsymbol{x}}} = \left(\frac{\partial}{\partial x} \hat{\boldsymbol{e}}_x, \frac{\partial}{\partial y} \hat{\boldsymbol{e}}_y, \frac{\partial}{\partial z} \hat{\boldsymbol{e}}_z \right) = \left(\frac{\partial}{\partial x_1} \hat{\boldsymbol{e}}_1, \frac{\partial}{\partial x_2} \hat{\boldsymbol{e}}_2, \frac{\partial}{\partial x_3} \hat{\boldsymbol{e}}_3 \right) = \frac{\partial}{\partial x_i} \hat{\boldsymbol{e}}_i$$

On a scalar field $\phi(\vec{\boldsymbol{x}})$ as a function of position vector $\vec{\boldsymbol{x}}$, the nabla operator creates a vector field called the gradient:

$$\vec{\boldsymbol{\nabla}} \otimes \phi \equiv \vec{\boldsymbol{\nabla}} \phi = \frac{\partial \phi}{\partial \vec{\boldsymbol{x}}} = \left(\frac{\partial \phi}{\partial x} \hat{\boldsymbol{e}}_x, \frac{\partial \phi}{\partial y} \hat{\boldsymbol{e}}_y, \frac{\partial \phi}{\partial z} \hat{\boldsymbol{e}}_z \right) = \left(\frac{\partial \phi}{\partial x_1} \hat{\boldsymbol{e}}_1, \frac{\partial \phi}{\partial x_2} \hat{\boldsymbol{e}}_2, \frac{\partial \phi}{\partial x_3} \hat{\boldsymbol{e}}_3 \right) = \frac{\partial \phi}{\partial x_i} \hat{\boldsymbol{e}}_i$$

Gradient can be used to calculate the difference in field values in nearby points. The first order of the coordinate differentials $d\vec{\boldsymbol{x}} = (dx, dy, dz)$ gives:

$$\phi(\vec{\boldsymbol{x}} + d\vec{\boldsymbol{x}}) - \phi(\vec{\boldsymbol{x}}) = dx \frac{\partial \phi}{\partial x} + dy \frac{\partial \phi}{\partial y} + dz \frac{\partial \phi}{\partial z} \Leftrightarrow d\phi(\vec{\boldsymbol{x}}) = d\vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{\nabla}} \phi(\vec{\boldsymbol{x}})$$

Gradient

Usually, scalar field is pictured by means of surfaces with constant value, also known as contour surfaces. The gradient at any point is always orthogonal to the contour surface containing this point, [7]. Both $\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{x}} + d\vec{\boldsymbol{x}}$ lie in the contour surface, the differential must vanish:

$$d\vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{\nabla}} \phi(\vec{\boldsymbol{x}}) = 0$$

All differentials $d\vec{\boldsymbol{x}}$ are tangential to the contour surface. From vanishing the dot product follows that the gradient $\vec{\boldsymbol{\nabla}} \phi$ should be orthogonal to the surface.
The gradient of a vector field can be defined as follows:

$$\vec{\boldsymbol{\nabla}} \otimes \vec{\boldsymbol{u}} \equiv \vec{\boldsymbol{\nabla}} \vec{\boldsymbol{u}} = \frac{\partial u_j}{\partial x_i} \hat{\boldsymbol{e}}_i \otimes \hat{\boldsymbol{e}}_j$$

In index notation:

$$\vec{\boldsymbol{\nabla}} \otimes \vec{\boldsymbol{u}} \equiv \vec{\boldsymbol{\nabla}} \vec{\boldsymbol{u}} = \frac{\partial}{\partial x_i} u_j = \frac{\partial u_j}{\partial x_i}$$

Gradient - Application

In order to find vector normal to a plane, the gradient of the equation of the plane will provide the vector normal to it. The equation of a plane is given as follows:

$$\alpha x + \beta y + \gamma z = \phi$$

where $\alpha, \beta, \gamma, \phi$ are constants. The gradient of ϕ will provide the vector normal (perpendicular) to the plane:

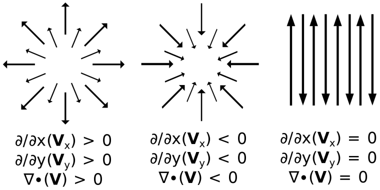
$$\vec{\boldsymbol{\nabla}} \phi = \alpha \hat{\boldsymbol{e}}_1 + \beta \hat{\boldsymbol{e}}_2 + \gamma \hat{\boldsymbol{e}}_3$$

Divergence

The dot product between the vector field $\vec{\boldsymbol{u}}(\vec{\boldsymbol{x}})$ and $\vec{\boldsymbol{\nabla}}$ is called divergence of the vector field:

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{u}}(\vec{\boldsymbol{x}}) = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = \frac{\partial u_{x_1}}{\partial x_1} + \frac{\partial u_{x_2}}{\partial x_2} + \frac{\partial u_{x_3}}{\partial x_3}$$

By plotting a vector field with positive divergence by means of arrows the arrows have tendency to diverge from each other, while for negative they converge.



Divergence

Divergence of gradient vector - Laplacian

By taking the divergence of a gradient vector

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{\nabla}} \phi = \nabla^2 \phi$$

The notation ∇^2 is called Laplacian operator. In Cartesian systems the Laplacian takes the following form:

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2} = \frac{\partial^2 \phi}{\partial x_i \partial x_i}$$

The Laplacian is a scalar that appears often in physical phenomena.

Curl (Rotation)

Using nabla $\vec{\boldsymbol{\nabla}}$ in cross product with the vector field $\vec{\boldsymbol{u}}(\vec{\boldsymbol{x}})$ the vector field called the curl of $\vec{\boldsymbol{u}}(\vec{\boldsymbol{x}})$:

$$\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{u}}(\vec{\boldsymbol{x}})$$

In matrix notation the curl of vector $\vec{\boldsymbol{u}}$ can be written as:

$$\begin{vmatrix} \vec{\boldsymbol{e}}_1 & \vec{\boldsymbol{e}}_2 & \vec{\boldsymbol{e}}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ u_1 & u_2 & u_3 \end{vmatrix} = \vec{\boldsymbol{e}}_1 \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \right) + \vec{\boldsymbol{e}}_2 \left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \right) + \vec{\boldsymbol{e}}_3 \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right)$$

In index notation it will be as follows:

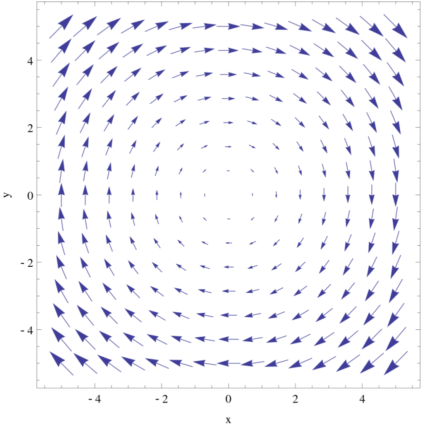
$$\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{u}}(\vec{\boldsymbol{x}}) = \epsilon_{ijk} \frac{\partial u_k}{\partial x_j} \hat{\boldsymbol{e}}_i = \epsilon_{ijk} u_{k,j} \hat{\boldsymbol{e}}_i$$

The curl of a 2^{nd} order tensor $\underline{\underline{\boldsymbol{A}}}$ is:

$$\vec{\boldsymbol{\nabla}} \times \underline{\underline{\boldsymbol{A}}} = \epsilon_{ijk} \frac{\partial A_{kl}}{\partial x_j} \hat{\boldsymbol{e}}_i \otimes \hat{\boldsymbol{e}}_l$$

Curl

By plotting a vector field with non-vanishing curl in terms of small arrows, the arrows will have the tendency to circulate.



Curl

Divergence-Gradient-Curl

Entity $\star$	Divergence $\vec{\boldsymbol{\nabla}} \cdot \star$	Gradient $\vec{\boldsymbol{\nabla}} \star$	Curl $\vec{\boldsymbol{\nabla}} \times \star$
Scalar		Vector	
Vector	Scalar	Second-order tensor	Vector
Second-order tensor	Vector	Third-order tensor	Second-order tensor

Gradient-Divergence-Curl theorems

Ω is a region bounded by closed surface Γ, ds is a differential surface element and $\hat{\boldsymbol{n}}$ is unit normal vector and dv is differential volume, then:

$$\int_\Omega \nabla\phi dv = \oint_\Gamma \hat{\boldsymbol{n}}\phi ds \quad \text{Gradient Theorem}$$

$$\int_\Omega \nabla\cdot\vec{\boldsymbol{u}} dv = \oint_\Gamma \hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{u}} ds \quad \text{Divergence Theorem}$$

$$\int_\Omega \nabla\times\vec{\boldsymbol{u}} dv = \oint_\Gamma \hat{\boldsymbol{n}}\times\vec{\boldsymbol{u}} ds \quad \text{Curl Theorem}$$

Vector operations in tensor and index notation

$\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}, \vec{\boldsymbol{w}}$ are vectors, ϕ is a scalar and $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ are Cartesian unit vectors

Direct Tensor Notation	Tensor Component (Index) Notation
$\vec{\boldsymbol{u}}$	$u_i\hat{\boldsymbol{e}}_i$
$\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}}$	$u_i v_i$
$\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}$	$\epsilon_{ijk} u_i v_j \hat{\boldsymbol{e}}_k$
$\vec{\boldsymbol{u}} \cdot (\vec{\boldsymbol{v}} \times \vec{\boldsymbol{w}})$	$\epsilon_{ijk} u_i v_j w_k$
$\vec{\boldsymbol{u}} \times (\vec{\boldsymbol{v}} \times \vec{\boldsymbol{w}}) = \vec{\boldsymbol{v}}(\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{w}}) - \vec{\boldsymbol{w}}(\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}})$	$\epsilon_{ijk}\epsilon_{klm} u_j v_l w_m \hat{\boldsymbol{e}}_i$
$\vec{\boldsymbol{\nabla}}\phi$	$\frac{\partial \phi}{\partial x_i}\hat{\boldsymbol{e}}_i$
$\vec{\boldsymbol{\nabla}}\vec{\boldsymbol{u}}$	$\frac{\partial u_j}{\partial x_i}\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j$
$\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{u}}$	$\frac{\partial u_i}{\partial x_i}$
$\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{u}}$	$\epsilon_{ijk}\frac{\partial u_j}{\partial x_i}\hat{\boldsymbol{e}}_k$
$\vec{\boldsymbol{\nabla}} \cdot (\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}) = \vec{\boldsymbol{v}} \cdot (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{u}}) - \vec{\boldsymbol{u}} \cdot (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{v}})$	$\epsilon_{ijk}\frac{\partial}{\partial x_i}(u_j v_k)$
$\vec{\boldsymbol{\nabla}} \cdot (\phi\vec{\boldsymbol{u}}) = \phi\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{u}} + \vec{\boldsymbol{\nabla}}\phi \cdot \vec{\boldsymbol{u}}$	$\frac{\partial}{\partial x_i}(\phi u_i)$
$\vec{\boldsymbol{\nabla}} \times (\phi\vec{\boldsymbol{u}}) = \vec{\boldsymbol{\nabla}}\phi \times \vec{\boldsymbol{u}} + \phi\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{u}}$	$\epsilon_{ijk}\frac{\partial}{\partial x_j}(\phi u_k)\hat{\boldsymbol{e}}_i$

Operators

Products	Operators
Dot/Scalar	$\cdot$
Cross/Vector	$\times$
Outer/Tensor	$\otimes$[3]
Double Dot Vertical	$:$
Double Dot Horizontal	$\cdot\cdot$

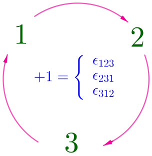
[3] the outer product can also be omitted, e.g. $\vec{u} \otimes \vec{v} \equiv \vec{u}\vec{v}$

Dot/Scalar Product

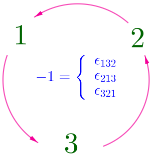
$$
\begin{array}{llll}
\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} & = u_i v_i & = \alpha & \text{scalar} \\
\underline{\underline{\boldsymbol{A}}} \cdot \vec{\boldsymbol{u}} & = A_{ij} u_j & = v_i & \text{vector} \\
\vec{\boldsymbol{u}} \cdot \underline{\underline{\boldsymbol{A}}} & = u_i A_{ij} & = v_j & \text{vector} \\
\underline{\underline{\boldsymbol{A}}} \cdot \underline{\underline{\boldsymbol{B}}} & = A_{ij} B_{jk} & = C_{ik} & \text{tensor } (2^{nd}) \\
\underline{\underline{\boldsymbol{A}}} \cdot \underline{\underline{\underline{\boldsymbol{D}}}} & = A_{ij} D_{jkl} & = E_{ikl} & \text{tensor } (3^{rd}) \\
\underline{\underline{\underline{\boldsymbol{D}}}} \cdot \underline{\underline{\boldsymbol{A}}} & = D_{ijk} A_{kl} & = E_{ijl} & \text{tensor } (3^{rd}) \\
\underline{\underline{\underline{\boldsymbol{D}}}} \cdot \underline{\underline{\underline{\boldsymbol{E}}}} & = D_{ijk} E_{klm} & = F_{ijlm} & \text{tensor } (4^{th})
\end{array}
$$

Cross/Vector Product

$$
\begin{array}{llll}
\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}} & = \epsilon_{ijk} u_j v_k & = w_i & \text{vector} \\
\underline{\underline{\boldsymbol{A}}} \times \vec{\boldsymbol{u}} & = \epsilon_{ijk} A_{jl} u_k & = B_{il} & \text{tensor } (2^{nd}) \\
\vec{\boldsymbol{u}} \times \underline{\underline{\boldsymbol{A}}} & = \epsilon_{ijk} u_j A_{kl} & = B_{il} & \text{tensor } (2^{nd}) \\
\underline{\underline{\boldsymbol{A}}} \times \underline{\underline{\boldsymbol{B}}} & = \epsilon_{ijk} A_{jl} B_{km} & = C_{ilm} & \text{tensor } (3^{rd}) \\
\underline{\underline{\boldsymbol{A}}} \times \underline{\underline{\underline{\boldsymbol{D}}}} & = \epsilon_{ijk} A_{jl} D_{kmn} & = E_{ilmn} & \text{tensor } (4^{th}) \\
\underline{\underline{\underline{\boldsymbol{D}}}} \times \underline{\underline{\boldsymbol{A}}} & = \epsilon_{ijk} D_{jlm} A_{kn} & = E_{ilmn} & \text{tensor } (4^{th}) \\
\underline{\underline{\underline{\boldsymbol{D}}}} \times \underline{\underline{\underline{\boldsymbol{F}}}} & = \epsilon_{ijk} D_{jlm} F_{kno} & = G_{ilmno} & \text{tensor } (5^{th})
\end{array}
$$



$$+1 = \begin{cases} \epsilon_{123} \\ \epsilon_{231} \\ \epsilon_{312} \end{cases}$$



$$-1 = \begin{cases} \epsilon_{132} \\ \epsilon_{213} \\ \epsilon_{321} \end{cases}$$

For all other combinations of i, j, k, the permutation tensor ϵ_{ijk} is equal to zero (i.e. if an index is repeated at least two times).

Outer/Tensor Product

$$
\begin{aligned}
\vec{\boldsymbol{u}} \otimes \vec{\boldsymbol{v}} &\equiv \vec{\boldsymbol{u}}\ \vec{\boldsymbol{v}} && = u_i v_j && = A_{ij} && \text{tensor } (2^{nd}) \\
\underline{\underline{\boldsymbol{A}}} \otimes \vec{\boldsymbol{u}} &\equiv \underline{\underline{\boldsymbol{A}}}\ \vec{\boldsymbol{u}} && = A_{ij} u_k && = B_{ijk} && \text{tensor } (3^{rd}) \\
\vec{\boldsymbol{u}} \otimes \underline{\underline{\boldsymbol{A}}} &\equiv \vec{\boldsymbol{u}}\ \underline{\underline{\boldsymbol{A}}} && = u_i A_{jk} && = B_{ijk} && \text{tensor } (3^{rd}) \\
\underline{\underline{\boldsymbol{A}}} \otimes \underline{\underline{\boldsymbol{B}}} &\equiv \underline{\underline{\boldsymbol{A}}}\ \underline{\underline{\boldsymbol{B}}} && = A_{ij} B_{kl} && = C_{ijkl} && \text{tensor } (4^{th}) \\
\underline{\underline{\boldsymbol{A}}} \otimes \underline{\underline{\underline{\boldsymbol{D}}}} &\equiv \underline{\underline{\boldsymbol{A}}}\ \underline{\underline{\underline{\boldsymbol{D}}}} && = A_{ij} D_{klm} && = E_{ijklm} && \text{tensor } (5^{th}) \\
\underline{\underline{\underline{\boldsymbol{D}}}} \otimes \underline{\underline{\boldsymbol{A}}} &\equiv \underline{\underline{\underline{\boldsymbol{D}}}}\ \underline{\underline{\boldsymbol{A}}} && = D_{ijk} A_{lm} && = E_{ijklm} && \text{tensor } (5^{th}) \\
\underline{\underline{\underline{\boldsymbol{D}}}} \otimes \underline{\underline{\underline{\boldsymbol{F}}}} &\equiv \underline{\underline{\underline{\boldsymbol{D}}}}\ \underline{\underline{\underline{\boldsymbol{F}}}} && = D_{ijk} F_{lmn} && = G_{ijklmn} && \text{tensor } (6^{th})
\end{aligned}
$$

Double Dot Vertical Product

$$
\begin{array}{llll}
\underline{\underline{\boldsymbol{A}}} : \underline{\underline{\boldsymbol{B}}} = A_{ij}B_{ij} & = \alpha & \text{scalar} \\
\underline{\underline{\boldsymbol{A}}} : \underline{\underline{\underline{\boldsymbol{C}}}} = A_{ij}C_{ijk} & = v_k & \text{vector} \\
\underline{\underline{\underline{\boldsymbol{D}}}} : \underline{\underline{\boldsymbol{A}}} = C_{ijk}A_{jk} & = v_i & \text{vector} \\
\underline{\underline{\underline{\boldsymbol{C}}}} : \underline{\underline{\underline{\boldsymbol{D}}}} = C_{ijk}D_{jkl} & = E_{il} & \text{tensor } (2^{nd}) \\
\underline{\underline{\underline{\underline{\boldsymbol{F}}}}} : \underline{\underline{\underline{\boldsymbol{G}}}} = F_{ijkl}G_{klm} & = G_{ijm} & \text{tensor } (3^{rd}) \\
\underline{\underline{\underline{\underline{\boldsymbol{F}}}}} : \underline{\underline{\underline{\underline{\boldsymbol{H}}}}} = F_{ijkl}H_{klmn} & = G_{ijmn} & \text{tensor } (4^{th})
\end{array}
$$

Double Dot Horizontal Product

$$
\begin{array}{llll}
\underline{\underline{\boldsymbol{A}}} \cdot\cdot \underline{\underline{\boldsymbol{B}}} = A_{ij} B_{ji} & = \alpha & \text{scalar} \\
\underline{\underline{\boldsymbol{A}}} \cdot\cdot \underline{\underline{\underline{\boldsymbol{C}}}} = A_{ij} C_{jik} & = v_k & \text{vector} \\
\underline{\underline{\underline{\boldsymbol{D}}}} \cdot\cdot \underline{\underline{\boldsymbol{A}}} = C_{ijk} A_{kj} & = v_i & \text{vector} \\
\underline{\underline{\underline{\boldsymbol{C}}}} \cdot\cdot \underline{\underline{\underline{\boldsymbol{D}}}} = C_{ijk} D_{kjl} & = E_{il} & \text{tensor } (2^{nd}) \\
\underline{\underline{\underline{\underline{\boldsymbol{F}}}}} \cdot\cdot \underline{\underline{\underline{\boldsymbol{G}}}} = F_{ijkl} G_{lkm} & = G_{ijm} & \text{tensor } (3^{rd}) \\
\underline{\underline{\underline{\underline{\boldsymbol{F}}}}} \cdot\cdot \underline{\underline{\underline{\underline{\boldsymbol{H}}}}} = F_{ijkl} H_{lkmn} & = G_{ijmn} & \text{tensor } (4^{th})
\end{array}
$$

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