Foundations on Continuum Mechanics - Week 3 - Kinetics

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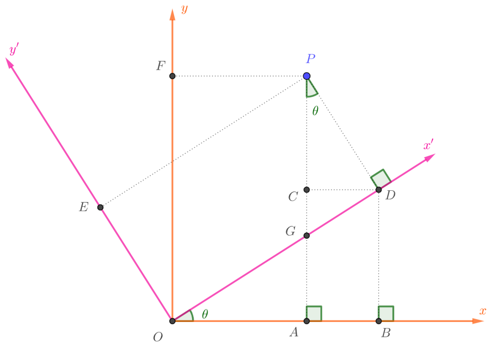
Transformation of coordinates

Transformation of coordinates - Rotation

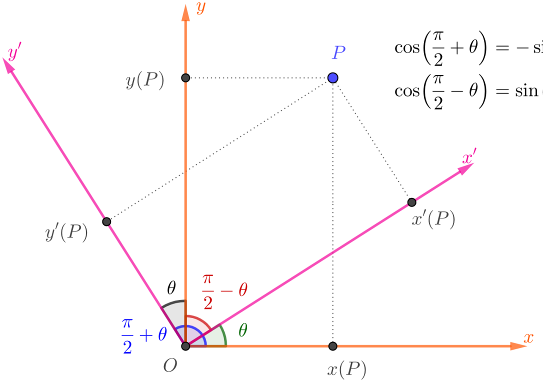


$$x(P) = \overline{OA} = \overline{FP}$$
$$y(P) = \overline{OF} = \overline{AP}$$
$$x(P) = \overline{OB} - \overline{AB}$$
$$\overset{\triangle}{OBD} \to \overline{OB} = x'(P)\cos\theta$$
$$\overset{\triangle}{PCD} \to \overline{AB} = \overline{CD} = y'(P)\sin\theta$$
$$x(P) = x'(P)\cos\theta - y'(P)\sin\theta$$
$$y(P) = \overline{AC} + \overline{CP}$$
$$\overset{\triangle}{OBD} \to \overline{AC} = \overline{DB} = x'(P)\sin\theta$$
$$\overset{\triangle}{PCD} \to \overline{PC} = y'(P)\cos\theta$$
$$y(P) = x'(P)\sin\theta + y'(P)\cos\theta$$

$$x'(P) = \overline{OD} = \overline{EP}$$
$$y'(P) = \overline{OE} = \overline{DP}$$
$$x(P) = \overline{OG} + \overline{GD}$$
$$\overset{\triangle}{OAG} \to \overline{OG} = \frac{x(P)}{\cos\theta}$$
$$\overset{\triangle}{PGD} \to \overline{GD} = \overline{GP}\sin\theta$$
$$\overset{\triangle}{OGD} \to \overline{GP} = y - \overline{OG}\sin\theta = y(P) - \frac{x(P)}{\cos\theta}\sin\theta$$
$$\overline{GD} = \left(y(P) - \frac{x(P)}{\cos\theta}\sin\theta\right)\sin\theta$$
$$x'(P) = \overline{OG} + \overline{GD} = \frac{x(P)}{\cos\theta} + \left(y(P) - \frac{x(P)}{\cos\theta}\sin\theta\right)\sin\theta$$
$$x'(P) = \frac{x(P)}{\cos\theta}\left(1 - \sin^2\theta\right) + y(P)\sin\theta$$
$$x'(P) = \frac{x(P)}{\cos\theta}\left(\cos^2\theta\right) + y(P)\sin\theta$$
$$x'(P) = x(P)\cos\theta + y(P)\sin\theta$$
$$y'(P) = \overline{PD}$$
$$\overset{\triangle}{PGD} \to \overline{PD} = \overline{GP}\cos\theta$$
$$y'(P) = \left(y(P) - \frac{x(P)}{\cos\theta}\sin\theta\right)\cos\theta$$
$$y'(P) = -x(P)\sin\theta + y(P)\cos\theta$$

$$x(P) = x'(P)\cos\theta - y'(P)\sin\theta \qquad x'(P) = x(P)\cos\theta + y(P)\sin\theta$$
$$y(P) = x'(P)\sin\theta + y'(P)\cos\theta \qquad y'(P) = -x(P)\sin\theta + y(P)\cos\theta$$

Transformation of coordinates - Rotation



$$\cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta$$
$$\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$$

$$x'(P) = x(P)\cos\theta + y(P)\sin\theta$$
$$y'(P) = -x(P)\sin\theta + y(P)\cos\theta$$
$$x'(P) = x(P)\cos(\widehat{x',x}) + y(P)\cos(\widehat{x',y})$$
$$y'(P) = x(P)\cos(\widehat{y',x}) + y(P)\cos(\widehat{y',y})$$
$$\cos(x',x) = \cos\theta$$
$$\cos(x',y) = \cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$$
$$\cos(y',x) = \cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta$$
$$\cos(y',y) = \cos\theta$$
$$x(P) = x'(P)\cos\theta - y'(P)\sin\theta$$
$$y(P) = x'(P)\sin\theta + y'(P)\cos\theta$$

$$x(P) = x'(P)\cos(\widehat{x,x'}) + y'(P)\cos(\widehat{x,y'})$$
$$y(P) = x'(P)\cos(\widehat{y,x'}) + y'(P)\cos(\widehat{y,y'})$$
$$\cos(\widehat{x,x'}) = \cos\theta$$
$$\cos(\widehat{x,y'}) = \cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta$$
$$\cos(\widehat{y,x'}) = \cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$$
$$\cos(\widehat{y,y'}) = \cos\theta$$

$x \to x_1$
$y \to x_2$
$x' \to x'_1$
$y' \to x'_2$

$x'_i = \beta_{ij}x_j$

$x_i = \beta_{ji}x'_j$

$$\beta_{ij} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\beta_{ji} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$(\beta_{ji}) = (\beta_{ij})^T$$
$$(\beta_{ji}) = (\beta_{ij})^{-1}$$
$$(\beta_{ij})^{-1} = (\beta_{ij})^T$$
$$(\beta_{ij})(\beta_{ij})^T = (\beta_{ij})(\beta_{ij})^{-1} = (\delta_{ij})$$
$$\beta_{ik}\beta_{jk} = \delta_{ij}$$

Transformation of coordinates - Rotation

The previous discussion can be extended to the three dimensions. Let $\vec{\boldsymbol{r}}$ be the position vector and the coordinate basis $\{\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3\}$ expressed as:

$$\vec{\boldsymbol{r}} = r_1\hat{\boldsymbol{e}}_1 + r_2\hat{\boldsymbol{e}}_2 + r_3\hat{\boldsymbol{e}}_3$$

We want to transform the position vector to another basis $\{\hat{\boldsymbol{e}}_1', \hat{\boldsymbol{e}}_2', \hat{\boldsymbol{e}}_3'\}$. Note that both coordinate basis are right-handed orthogonal, which means that:

$$\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j = \delta_{ij}; \quad \hat{\boldsymbol{e}}_i' \cdot \hat{\boldsymbol{e}}_j' = \delta_{ij}$$

In other words:

$$\hat{\boldsymbol{e}}_1 \cdot \hat{\boldsymbol{e}}_2 = \hat{\boldsymbol{e}}_2 \cdot \hat{\boldsymbol{e}}_3 = \hat{\boldsymbol{e}}_1 \cdot \hat{\boldsymbol{e}}_3 = 0; \quad \hat{\boldsymbol{e}}_1' \cdot \hat{\boldsymbol{e}}_2' = \hat{\boldsymbol{e}}_2' \cdot \hat{\boldsymbol{e}}_3' = \hat{\boldsymbol{e}}_1' \cdot \hat{\boldsymbol{e}}_3' = 0$$

The position vector $\vec{\boldsymbol{r}}$ in the coordinate basis can be written as

$$\vec{\boldsymbol{r}} = r_1'\hat{\boldsymbol{e}}_1' + r_2'\hat{\boldsymbol{e}}_2' + r_3'\hat{\boldsymbol{e}}_3'$$

Transformation of coordinates - Rotation

The components of the vector $\vec{\boldsymbol{r}}$ can be written:

$$r_i = \vec{\boldsymbol{r}} \cdot \hat{\boldsymbol{e}}_i; \quad r'_i = \vec{\boldsymbol{r}} \cdot \hat{\boldsymbol{e}}'_i$$

To this end,

$$\vec{\boldsymbol{r}} = r_i \hat{\boldsymbol{e}}_i = (\vec{\boldsymbol{r}} \cdot \hat{\boldsymbol{e}}_i)\hat{\boldsymbol{e}}_i; \quad \vec{\boldsymbol{r}} = r'_i \hat{\boldsymbol{e}}'_i = (\vec{\boldsymbol{r}} \cdot \hat{\boldsymbol{e}}'_i)\hat{\boldsymbol{e}}'_i$$

And therefore:

$$r'_i = \vec{\boldsymbol{r}} \cdot \hat{\boldsymbol{e}}'_i = r_j \hat{\boldsymbol{e}}_j \cdot \hat{\boldsymbol{e}}'_i \equiv \beta_{ij} r_j$$

where is the transformation matrix:

$$\beta_{ij} = \hat{\boldsymbol{e}}'_i \cdot \hat{\boldsymbol{e}}_j = \cos(\widehat{\hat{\boldsymbol{e}}'_i, \hat{\boldsymbol{e}}_j}), \text{ where } (\widehat{\hat{\boldsymbol{e}}'_i, \hat{\boldsymbol{e}}_j}) \text{ is the angle between } \hat{\boldsymbol{e}}'_i \text{ and } \hat{\boldsymbol{e}}_j$$

In matrix form:

$$\boldsymbol{B} = \begin{bmatrix} \hat{\boldsymbol{e}}'_1 \cdot \hat{\boldsymbol{e}}_1 & \hat{\boldsymbol{e}}'_1 \cdot \hat{\boldsymbol{e}}_2 & \hat{\boldsymbol{e}}'_1 \cdot \hat{\boldsymbol{e}}_3 \\ \hat{\boldsymbol{e}}'_2 \cdot \hat{\boldsymbol{e}}_1 & \hat{\boldsymbol{e}}'_2 \cdot \hat{\boldsymbol{e}}_2 & \hat{\boldsymbol{e}}'_2 \cdot \hat{\boldsymbol{e}}_3 \\ \hat{\boldsymbol{e}}'_3 \cdot \hat{\boldsymbol{e}}_1 & \hat{\boldsymbol{e}}'_3 \cdot \hat{\boldsymbol{e}}_2 & \hat{\boldsymbol{e}}'_3 \cdot \hat{\boldsymbol{e}}_3 \end{bmatrix} = \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{bmatrix}; \quad \det(\boldsymbol{B}) = 1$$

Transformation matrix - rotation

- The transformation matrix is NOT symmetric
- The transformation matrix is orthogonal $(\beta_{ij})(\beta_{ij})^T = (\beta_{ij})^T(\beta_{ij}) = (\delta_{ij})$
- The orthogonality leads to the conclusion: $(\beta_{ij})^T = (\beta_{ij})^{-1}$

Transformation matrix - second-order Tensor

A second-order tensor can be defined as:

$$\underline{\underline{\boldsymbol{A}}} = A_{ij}\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j = A'_{ij}\hat{\boldsymbol{e}}'_i\hat{\boldsymbol{e}}'_j$$

The unit base vectors are related:

$$\hat{\boldsymbol{e}}_i = \beta_{ji}\hat{\boldsymbol{e}}'_j; \quad \hat{\boldsymbol{e}}'_i = \beta_{ij}\hat{\boldsymbol{e}}_j; \quad \beta_{ij} = \hat{\boldsymbol{e}}'_i \cdot \hat{\boldsymbol{e}}_j$$

Therefore the components of a second-order tensor transform as follows:

$$A'_{kl} = \beta_{ki}\beta_{lj}A_{ij}; \quad \boldsymbol{A}' = \boldsymbol{B}\boldsymbol{A}\boldsymbol{B}^T$$

where:

$$\boldsymbol{B}^{-1} = \boldsymbol{B}^T; \quad \boldsymbol{B}\boldsymbol{B}^T = \boldsymbol{I}$$

Transformation rules

rank	from (x_1, x_2, x_3) to (x'_1, x'_2, x'_3)	from (x'_1, x'_2, x'_3) to (x_1, x_2, x_3)
0	$\alpha' = \alpha$	$\alpha = \alpha'$
1	$S'_i = \beta_{ij} S_j$	$S_i = \beta_{ji} S'_j$
2	$S'_{ij} = \beta_{ik}\beta_{jl} S_{kl}$	$S_{ij} = \beta_{ki}\beta_{lj} S'_{kl}$
3	$S'_{ijk} = \beta_{il}\beta_{jm}\beta_{kn} S_{lmn}$	$S_{ijk} = \beta_{li}\beta_{mj}\beta_{nk} S'_{lmn}$
4	$S'_{ijkl} = \beta_{im}\beta_{jn}\beta_{kp}\beta_{lq} S_{mnpq}$	$S_{ijkl} = \beta_{mi}\beta_{nj}\beta_{pk}\beta_{ql} S'_{mnpq}$

Example
MatLab

Kinetics

Stress

- The concept of stress is the cornerstone of Continuum Mechanics (CM)!
- It is the way CM determines the interaction between the one part of material with another one.
- We will show that in order to define a stress at a point we need 9 numbers, which can be arranged in a matrix.
- Assuming that the body-moment and the couple-stress does not exist, we conclude that the matrix is symmetric.
- Due to symmetry 6 independent components fully describe the state of stress at any point.
- A change of the frame of reference alters the stress components.
- The change of stress components under rotation of the frame of reference shows that obeys the tensor-transformation rule, [1].
- To this end, the stress is a tensor.
- When the stress tensor is known, by using the Cauchy's formula the stress vector acting on any surface can be computed!

Stress

- Stress, namely a measure of force per unit area, determines the capacity of a material to carry loads.
- In order to design a structure, the criterion that the structure can maintain a certain load (strength criterion) should be maintained.
- The other two criteria related to design are stiffness and stability (the 3-St altogether).
- The stress depends on the magnitude of the applied force, on the direction of the force and on the direction of the plane on which the stress is applied.

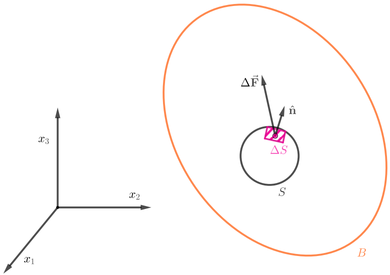
The idea of Stress

- In particle mechanics 2 types of interaction between particles are studied: (i) by collision and (ii) by action at a distance, [1].
- In CM we consider the interaction between different parts of the body.

The idea of Stress

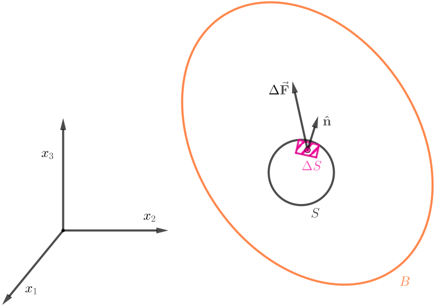
A material continuum B occupies a volume v. A surface S lies inside B. Our goal is to describe the interaction between the material outside S and the one inside S. There are two types of forces:

- Forces acting at a distance (e.g. gravitational, electromagnetic), which are called body forces
- Forces acting on the boundary S, which are called surface forces.



The idea of Stress

- A small area ΔS lies on the surface S.
- A unit vector $\hat{\boldsymbol{n}}$ normal to ΔS with direction outwards is drawn.
- The side to which the normal is pointing is positive and the other one is negative.
- The part on the positive side exerts a force $\Delta \vec{\boldsymbol{F}}$ to the material on the negative side.

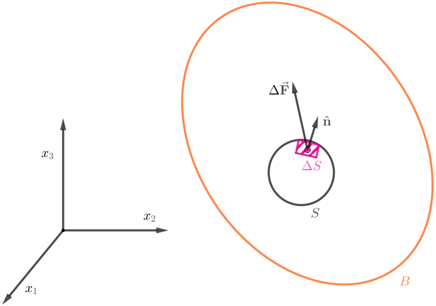


The idea of Stress

- We assume that as the area ΔS goes to zero, the ratio $\frac{\Delta \vec{\boldsymbol{F}}}{\Delta S}$ tends to the limit $\frac{d\vec{\boldsymbol{F}}}{dS}$ and the moment of the force on the surface ΔS about any point within the area vanishes in the limit. Then the limiting vector will be:

$$\vec{\boldsymbol{t}}(\hat{\boldsymbol{n}}) = \frac{d\vec{\boldsymbol{F}}}{dS}$$

and this vector $\vec{\boldsymbol{t}}(\hat{\boldsymbol{n}})$ is called traction or stress vector.



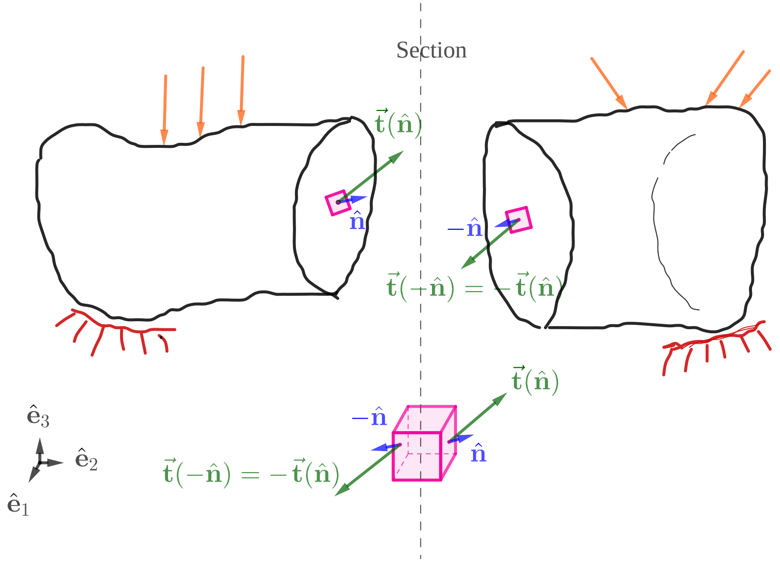
The idea of Stress

- The stress principle of Euler and Cauchy defines that upon a closed surface S inside a continuum, there exists a vector field, whose action on the interior material of surface S is equivalent to the action of the exterior material upon it, [1].
- The interaction between the two sides of the material of the surface ΔS is momentless.
- There are theories that assume that moments exist in the surface by introducing the concept of couple-stress.
- These theories will not be considered further due to the fact that the material that we are going to deal do not impose a couple on the surface.

Stress is a quantity that describes internal forces, namely **cohesion forces** between material points. **Augustin-Louis CAUCHY** showed that only a tensor, namely the **stress tensor**, can represent these internal forces.

Source:Wikipedia

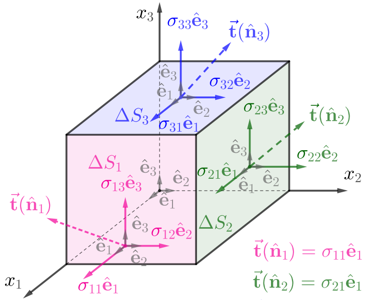
Action-Reaction



Stress Components - Cartesian system

The surface ΔS_i has an outer normal vector pointing in the positive direction of the x_i-axis, where $(i = 1, 2, 3)$. The normal of ΔS_i is along the positive direction x_i-axis. A stress vector $\vec{\boldsymbol{t}}(\hat{\boldsymbol{n}}_i)$ acts on ΔS_i and has three components:

$$\vec{\boldsymbol{t}}(\hat{\boldsymbol{n}}_i) = \Big((\vec{\boldsymbol{t}}(\hat{\boldsymbol{n}}_i) \cdot \hat{\boldsymbol{e}}_1)\hat{\boldsymbol{e}}_1, (\vec{\boldsymbol{t}}(\hat{\boldsymbol{n}}_i) \cdot \hat{\boldsymbol{e}}_2)\hat{\boldsymbol{e}}_2, (\vec{\boldsymbol{t}}(\hat{\boldsymbol{n}}_i) \cdot \hat{\boldsymbol{e}}_3)\hat{\boldsymbol{e}}_3 \Big) = (\sigma_{i1}\hat{\boldsymbol{e}}_1, \sigma_{i2}\hat{\boldsymbol{e}}_2, \sigma_{i3}\hat{\boldsymbol{e}}_3)$$



$\vec{\mathbf{t}}(\hat{\mathbf{n}}_1) = \sigma_{11}\hat{\mathbf{e}}_1 + \sigma_{12}\hat{\mathbf{e}}_2 + \sigma_{13}\hat{\mathbf{e}}_3$

$\vec{\mathbf{t}}(\hat{\mathbf{n}}_2) = \sigma_{21}\hat{\mathbf{e}}_1 + \sigma_{22}\hat{\mathbf{e}}_2 + \sigma_{23}\hat{\mathbf{e}}_3$

$\vec{\mathbf{t}}(\hat{\mathbf{n}}_3) = \sigma_{31}\hat{\mathbf{e}}_1 + \sigma_{32}\hat{\mathbf{e}}_2 + \sigma_{33}\hat{\mathbf{e}}_3$

Stress Components - Cartesian system

The stresses $\sigma_{11}, \sigma_{22}, \sigma_{33}$ are called normal stresses and the remaining ones are called shearing components.

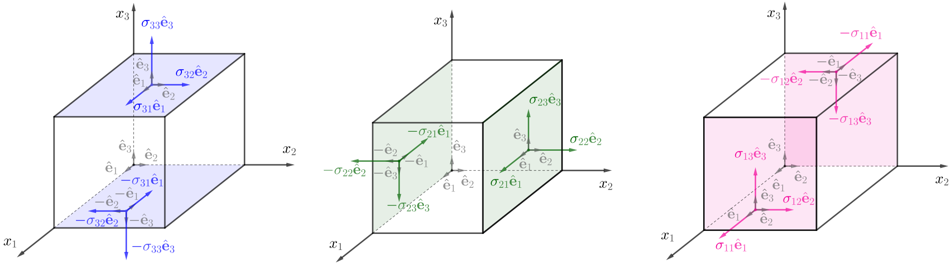
Components of stresses	1	2	3
Surface normal to x_1	σ_{11}	σ_{12}	σ_{13}
Surface normal to x_2	σ_{21}	σ_{22}	σ_{23}
Surface normal to x_3	σ_{31}	σ_{32}	σ_{33}

The stress tensor $\underline{\underline{\boldsymbol{\sigma}}}$ can be stored in a matrix in the Cartesian coordinate system as follows:

$$\underbrace{\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}}_{\text{scientific notation}} \equiv \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \equiv \underbrace{\begin{bmatrix} \sigma_{x} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{y} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{z} \end{bmatrix}}_{\text{engineering notation}}$$

Stress Components - Cartesian system

- The stress is **force per unit area** and the part of stress that lies on the positive side (is the side of the outer normal) of a surface element exerts the part of the stress on the negative side.
- If the outer normal of a surface points to the positive direction of x_i-axis then the component of stress vector is positive.
- If the positive outer normal points to the negative direction of x_i-axis then the stress vector will also point to the negative direction of x_i-axis.



What is a tensor?

- ‘We use the term tensor as synonym for the phrase "linear transformation from $\mathcal{V}$ into $\mathcal{V}$". A tensor $\underline{\underline{\boldsymbol{S}}}$ is a linear mapping of vectors to vectors’, [2]. Given a vector $\vec{\boldsymbol{u}}$ provides with

$$\vec{\boldsymbol{v}} = \underline{\underline{\boldsymbol{S}}} \cdot \vec{\boldsymbol{u}}$$

where $\vec{\boldsymbol{v}}$ is also a vector.
- A tensor can be thought as machine that is fed with with vectors as inputs and provides another vector as an output.
- The linearity of a tensor $\underline{\underline{\boldsymbol{S}}}$ is described by the requirements:

$$\underline{\underline{\boldsymbol{S}}} \cdot (\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}}) = \underline{\underline{\boldsymbol{S}}} \cdot \vec{\boldsymbol{u}} + \underline{\underline{\boldsymbol{S}}} \cdot \vec{\boldsymbol{v}}$$

$$\underline{\underline{\boldsymbol{S}}} \cdot (\alpha \vec{\boldsymbol{u}}) = \alpha \underline{\underline{\boldsymbol{S}}} \cdot \vec{\boldsymbol{u}}$$

What is a tensor?

- Two tensors $\underline{\underline{\boldsymbol{S}}}$ and $\underline{\underline{\boldsymbol{T}}}$ are equal if their outputs are the same whenever their inputs are equal, [2]:

$$\underline{\underline{\boldsymbol{S}}} = \underline{\underline{\boldsymbol{T}}} \text{ if and only if } \underline{\underline{\boldsymbol{S}}} \cdot \vec{\boldsymbol{v}} = \underline{\underline{\boldsymbol{T}}} \cdot \vec{\boldsymbol{v}} \text{ for all vectors } \vec{\boldsymbol{v}}$$

- A way to show that tensors $\underline{\underline{\boldsymbol{S}}}$ and $\underline{\underline{\boldsymbol{T}}}$ are equal is a consequence of:

$$\vec{\boldsymbol{a}} \cdot \underline{\underline{\boldsymbol{S}}} \cdot \vec{\boldsymbol{b}} = \vec{\boldsymbol{a}} \cdot \underline{\underline{\boldsymbol{T}}} \cdot \vec{\boldsymbol{b}} \text{ for all vectors } \vec{\boldsymbol{a}} \text{ and } \vec{\boldsymbol{b}} \text{ if and only if } \underline{\underline{\boldsymbol{S}}} = \underline{\underline{\boldsymbol{T}}}$$

- Tensors are generally defined by their actions on arbitrary vectors, [2]. For example:

$$(\underline{\underline{\boldsymbol{S}}} + \underline{\underline{\boldsymbol{T}}}) \cdot \vec{\boldsymbol{v}} = \underline{\underline{\boldsymbol{S}}} \cdot \vec{\boldsymbol{v}} + \underline{\underline{\boldsymbol{T}}} \cdot \vec{\boldsymbol{v}}$$

$$(\alpha \underline{\underline{\boldsymbol{S}}}) \cdot \vec{\boldsymbol{v}} = \alpha (\underline{\underline{\boldsymbol{S}}} \cdot \vec{\boldsymbol{v}})$$

Tensors - Dyads - Matrix form

- According to the definition:

$$\underline{\underline{\boldsymbol{A}}} = A_{ij}(\hat{\boldsymbol{e}}_i \otimes \hat{\boldsymbol{e}}_j) = A_{11}\hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_1 + A_{12}\hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_2 + A_{13}\hat{\boldsymbol{e}}_1\hat{\boldsymbol{e}}_3$$
$$+ A_{21}\hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_1 + A_{22}\hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_2 + A_{23}\hat{\boldsymbol{e}}_2\hat{\boldsymbol{e}}_3$$
$$+ A_{31}\hat{\boldsymbol{e}}_3\hat{\boldsymbol{e}}_1 + A_{32}\hat{\boldsymbol{e}}_3\hat{\boldsymbol{e}}_2 + A_{33}\hat{\boldsymbol{e}}_3\hat{\boldsymbol{e}}_3$$

- In matrix form the tensor $\underline{\underline{\boldsymbol{A}}}$ can be written as $\boldsymbol{A}$:

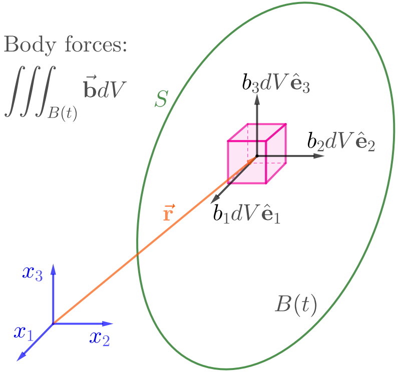
$$\boldsymbol{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} ; \quad \underline{\underline{\boldsymbol{A}}} \equiv \begin{bmatrix} \hat{\boldsymbol{e}}_1 \\ \hat{\boldsymbol{e}}_2 \\ \hat{\boldsymbol{e}}_3 \end{bmatrix}^T \boldsymbol{A} \begin{bmatrix} \hat{\boldsymbol{e}}_1 \\ \hat{\boldsymbol{e}}_2 \\ \hat{\boldsymbol{e}}_3 \end{bmatrix}$$

- The unit dyad can be defined as:

$$\underline{\underline{\boldsymbol{I}}} = \hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j = \hat{\boldsymbol{e}}_i \otimes \hat{\boldsymbol{e}}_j$$

Laws of motion

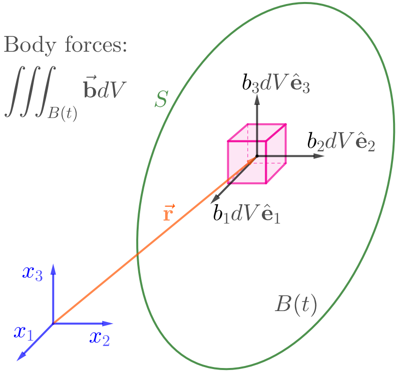
- CM is founded on Newton's laws of motion.
- Assume that we have a material body $B(t)$ at any time t.
- The position vector is $\vec{r}$ of a material point enclosed by an infinitesimal element of volume dV.



Laws of motion

- The density of the material is ρ, the velocity $\vec{\boldsymbol{v}}$.
- The mass of the infinitesimal element is ρdv and the linear momentum is $(\rho dV)\vec{\boldsymbol{v}}$
- The integration of the momentum over $B(t)$ gives the linear momentum at time t:

$$\mathcal{P} = \iiint_{B(t)} \vec{\boldsymbol{v}} \rho dV$$



Laws of motion

- The integration of the moment of the momentum over $B(t)$ at time t gives the angular momentum:
$$\mathcal{H} = \iiint_{B(t)} \vec{\boldsymbol{r}} \times \vec{\boldsymbol{v}} \rho dV$$
- According to Euler the Newton's laws for a continuum assert that the rate of change of the linear momentum is equal to the total force $\mathcal{F}$ acting on the body, [1]:
$$\dot{\mathcal{P}} = \mathcal{F}$$
- Also the rate of change of the moment of momentum is equal to the total applied torque $\mathcal{L}$ about the origin:
$$\dot{\mathcal{H}} = \mathcal{L}$$

Forces

- Two types of forces acting **externally** on bodies:
  1. Body forces, acting on volume elements (e.g. gravitational forces, electromagnetic forces).
  2. Surface forces, acting on surface elements.
- To derive body forces (e.g. a body $B(t)$ at time t bounded by surface S) we need to integrate over the volume (volume integral) of domain $B(t)$

$$\iiint_{B(t)} \vec{\boldsymbol{b}} dV$$

  where the vector $\vec{\boldsymbol{b}}$ has three components b_1, b_2, b_3 of the dimensions of forces.
- The surface forces can be denoted as:

$$\oiint_S \vec{\boldsymbol{t}}(\hat{\boldsymbol{n}})\, dS$$

Forces- Torques

- Finally, the total force acting upon a body $B(t)$ at time t closed by a surface S is:

$$\mathcal{F} = \oiint_S \vec{\boldsymbol{t}}(\hat{\boldsymbol{n}})\, dS + \iiint_{B(t)} \vec{\boldsymbol{b}}\, dV$$

  where $\vec{\boldsymbol{t}}(\hat{\boldsymbol{n}})$ is a stress vector acting on dS whose outer normal vector is $\hat{\boldsymbol{n}}$.
- The torque about the origin is given by:

$$\mathcal{L} = \oiint_S \vec{\boldsymbol{r}} \times \vec{\boldsymbol{t}}(\hat{\boldsymbol{n}})\, dS + \iiint_{B(t)} \vec{\boldsymbol{r}} \times \vec{\boldsymbol{b}}\, dV$$

Equations of motion

- The rate of change of the linear momentum $\dot{\mathcal{P}}$ is equal to the total force $\mathcal{F}$ acting on the body and the rate of change of the moment of momentum $\dot{\mathcal{H}}$ is equal to the total applied torque $\mathcal{L}$ about the origin, [1]:

$$\dot{\mathcal{P}} = \mathcal{F}; \quad \dot{\mathcal{H}} = \mathcal{L}$$

- The equations of motion can be written as:

$$\oiint_S \vec{\boldsymbol{t}}(\hat{\boldsymbol{n}})dS + \iiint_{B(t)} \vec{\boldsymbol{b}}dV = \frac{d}{dt}\iiint_{B(t)} \vec{\boldsymbol{v}}\rho dV$$

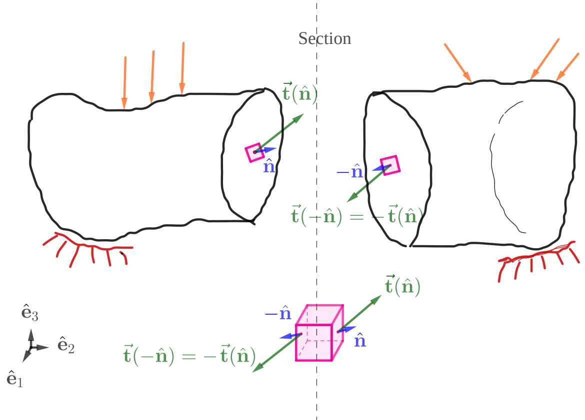
$$\oiint_S \vec{\boldsymbol{r}} \times \vec{\boldsymbol{t}}(\hat{\boldsymbol{n}})dS + \iiint_{B(t)} \vec{\boldsymbol{r}} \times \vec{\boldsymbol{b}}dV = \frac{d}{dt}\iiint_{B(t)} \vec{\boldsymbol{r}} \times \vec{\boldsymbol{v}}\rho dV$$

  The domain $B(t)$ consists of same material particles at all points and they form a continuum bounded by surface S

- The equations above apply to any material body.

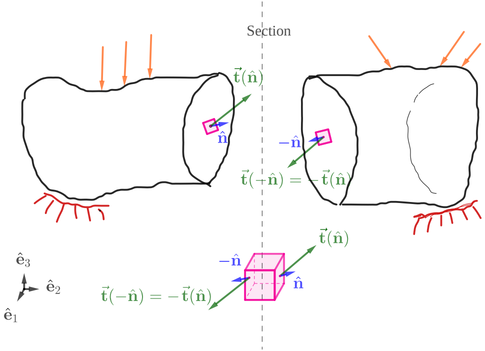
Internal forces

- Extending the concept of surface forces to the interior of the body: we split a body in two parts.
- We have two different bodies with different body forces.
- In addition, we have the INTERNAL FORCES that one body imposes to the other in order to act as a whole body.



Cauchy's postulates

1. Cauchy's postulate: The traction vector $\vec{\boldsymbol{t}}$ remains unchanged for any kind of surface passing through specific point P as far as the surfaces have the same normal $\hat{\boldsymbol{n}}$, [3]: $\vec{\boldsymbol{t}} = \vec{\boldsymbol{t}}(P, \hat{\boldsymbol{n}})$
2. Cauchy's lemma: The traction vectors acting at point P inside a body (internal forces!!!!!!) on the opposite sides of the split have equal magnitude and opposite directions, [3]: $\vec{\boldsymbol{t}}(P, \hat{\boldsymbol{n}}) = -\vec{\boldsymbol{t}}(P, -\hat{\boldsymbol{n}})$



Cauchy's formula

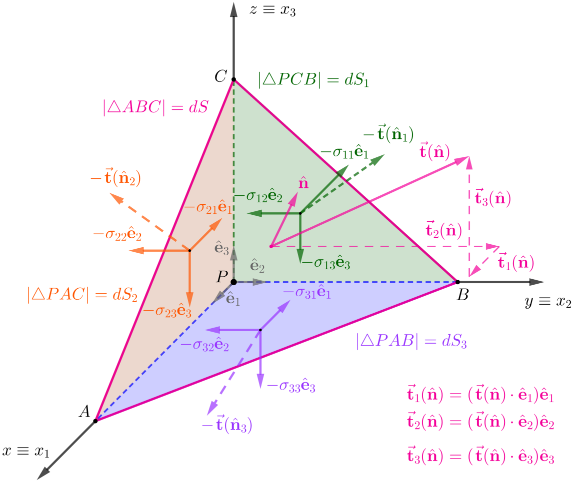
- The Cauchy's formula states that knowing the components of stress tensor σ_{ij} we can define the stress vector $\vec{\boldsymbol{t}}(\hat{\boldsymbol{n}})$ with components $t_i(\hat{\boldsymbol{n}})$ acting on any surface with unit outer normal $\hat{\boldsymbol{n}}$ with components n_i.
- There exists a spatial tensor field $\underline{\underline{\boldsymbol{\sigma}}}$, called the Cauchy stress, such that: $\vec{\boldsymbol{t}}(\hat{\boldsymbol{n}}) = \hat{\boldsymbol{n}} \cdot \underline{\underline{\boldsymbol{\sigma}}}$. From the equation can be observed that $\underline{\underline{\boldsymbol{\sigma}}}$ maps spatial vectors to spatial vectors.
- The equation in direct tension and index notation respectively can be written as follows:

$$\vec{\boldsymbol{t}}(\hat{\boldsymbol{n}}) = \hat{\boldsymbol{n}} \cdot \underline{\underline{\boldsymbol{\sigma}}}; \quad t_i(\hat{\boldsymbol{n}}) = \hat{n}_j \sigma_{ji}$$

- The stress vector $\vec{\boldsymbol{t}}(\hat{\boldsymbol{n}})$ denotes the surface force density of the resultant of the cohesion (internal microscopic) force keeping the two parts together
- The point of application of these forces together with the orientation of surface defines the sectioning surface

Cauchy Tetrahedron - Stress at a point

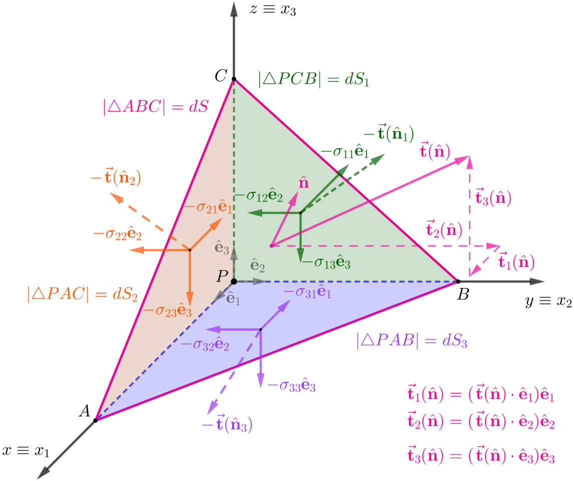
- Consider the infinitesimal tedrahedron which represents point P and, three orthogonal to each other planes (i.e. coordinate planes) and one plane with unit normal vector $\hat{\boldsymbol{n}}$.



$$\vec{\mathbf{t}}_1(\hat{\mathbf{n}}) = (\vec{\mathbf{t}}(\hat{\mathbf{n}}) \cdot \hat{\mathbf{e}}_1)\hat{\mathbf{e}}_1$$
$$\vec{\mathbf{t}}_2(\hat{\mathbf{n}}) = (\vec{\mathbf{t}}(\hat{\mathbf{n}}) \cdot \hat{\mathbf{e}}_2)\hat{\mathbf{e}}_2$$
$$\vec{\mathbf{t}}_3(\hat{\mathbf{n}}) = (\vec{\mathbf{t}}(\hat{\mathbf{n}}) \cdot \hat{\mathbf{e}}_3)\hat{\mathbf{e}}_3$$

Cauchy Tetrahedron - Stress at a point

- The area of the surface normal to $\hat{\boldsymbol{n}}$, namely $|\triangle ABC|$ is equal to dS, while the other areas, namely $dS_1 = |\triangle PCB|$, $dS_2 = |\triangle PAC|$, $dS_3 = |\triangle PAB|$, are the projections of dS on the different coordinate planes. To this end: $dS_i = dS(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{e}}_i) = dSn_i$



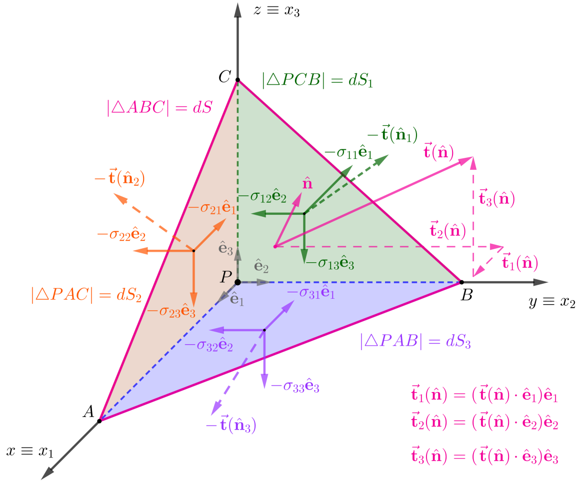
$\vec{\mathbf{t}}_1(\hat{\mathbf{n}}) = (\vec{\mathbf{t}}(\hat{\mathbf{n}}) \cdot \hat{\mathbf{e}}_1)\hat{\mathbf{e}}_1$

$\vec{\mathbf{t}}_2(\hat{\mathbf{n}}) = (\vec{\mathbf{t}}(\hat{\mathbf{n}}) \cdot \hat{\mathbf{e}}_2)\hat{\mathbf{e}}_2$

$\vec{\mathbf{t}}_3(\hat{\mathbf{n}}) = (\vec{\mathbf{t}}(\hat{\mathbf{n}}) \cdot \hat{\mathbf{e}}_3)\hat{\mathbf{e}}_3$

Cauchy Tetrahedron - Stress at a point

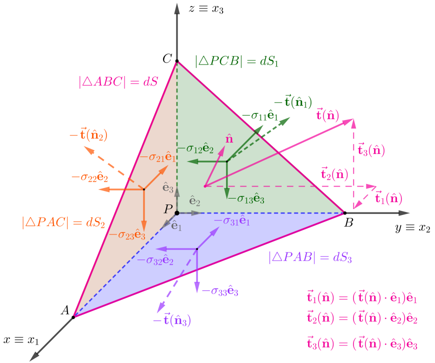
- The volume of the tetrahedron is equal to $dV = \frac{1}{3} h dS$, where h is the height of the vertex P to the base dS.



$$\vec{\mathbf{t}}_1(\hat{\mathbf{n}}) = (\vec{\mathbf{t}}(\hat{\mathbf{n}}) \cdot \hat{\mathbf{e}}_1)\hat{\mathbf{e}}_1$$
$$\vec{\mathbf{t}}_2(\hat{\mathbf{n}}) = (\vec{\mathbf{t}}(\hat{\mathbf{n}}) \cdot \hat{\mathbf{e}}_2)\hat{\mathbf{e}}_2$$
$$\vec{\mathbf{t}}_3(\hat{\mathbf{n}}) = (\vec{\mathbf{t}}(\hat{\mathbf{n}}) \cdot \hat{\mathbf{e}}_3)\hat{\mathbf{e}}_3$$

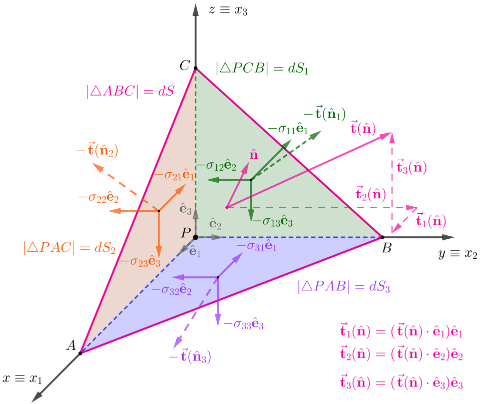
Cauchy Tetrahedron - Stress at a point

- The forces acting on the coordinate planes along the positive direction x_1 are: $(-\sigma_{11} + er_1)dS_1, (-\sigma_{21} + er_2)dS_2, (-\sigma_{31} + er_3)dS_3$.
- They are negative because the outer normal on the planes are opposing the positive directions of the coordinates axis.
- The er_i are inserted due to the fact that the traction vectors $\vec{\boldsymbol{t}}(\hat{\boldsymbol{n}}_i)$ act at a point slightly different than P.



Cauchy Tetrahedron - Stress at a point

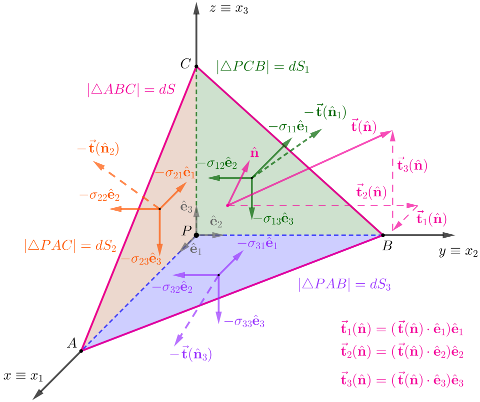
- The stress vector acting on dS will have a component $(t_1(\hat{\boldsymbol{n}}) + er)dS$ in x_1 direction
- The body force will be $(b_1 + er')dV$ and rate of change of the linear momentum is $\rho \dot{v}_1 dV$, where $\dot{v}_1$ is the component of acceleration along x_1 axis.



Cauchy Tetrahedron - Stress at a point

- The first equation of motion is:

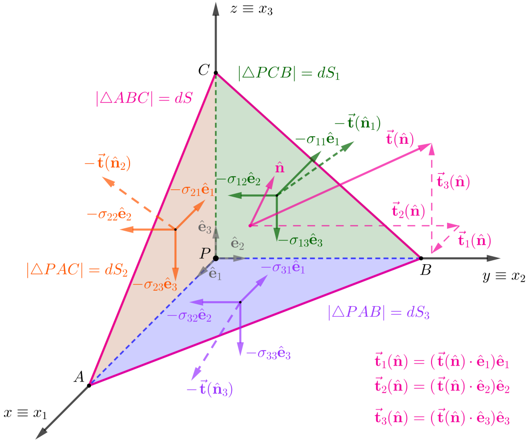
$$(-\sigma_{11} + er_1)n_1\,dS + (-\sigma_{21} + er_2)n_2\,dS + (-\sigma_{31} + er_3)n_3\,dS$$
$$+ (t_1(\hat{\boldsymbol{n}}) + er)dS + (b_1 + er')\frac{1}{3}hdS = \rho\dot{v}_1\frac{1}{3}hdS$$



Cauchy Tetrahedron - Stress at a point

- Dividing by dS and taking the limit $h \to 0$ and noting that $er_1, er_2, er_3, er, er'$ vanish with h we get:

$$t_1(\hat{\boldsymbol{n}}) = \sigma_{11}n_1 + \sigma_{21}n_2 + \sigma_{31}n_3$$

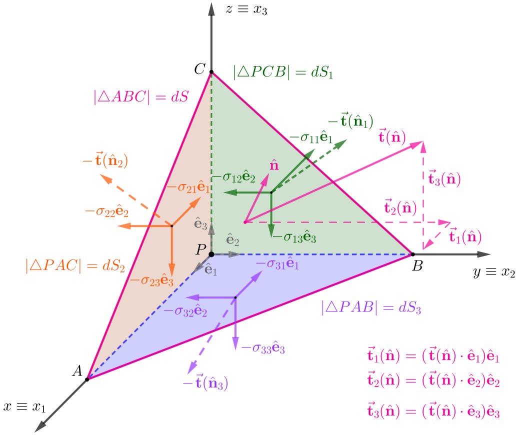


Cauchy Tetrahedron - Stress at a point

- Following the same process the other two components can be calculated:

$$t_2(\hat{\boldsymbol{n}}) = \sigma_{12}n_1 + \sigma_{22}n_2 + \sigma_{32}n_3$$

$$t_3(\hat{\boldsymbol{n}}) = \sigma_{13}n_1 + \sigma_{23}n_2 + \sigma_{33}n_3$$



$\vec{\mathbf{t}}_1(\hat{\mathbf{n}}) = (\vec{\mathbf{t}}(\hat{\mathbf{n}}) \cdot \hat{\mathrm{e}}_1)\hat{\mathrm{e}}_1$

$\vec{\mathbf{t}}_2(\hat{\mathbf{n}}) = (\vec{\mathbf{t}}(\hat{\mathbf{n}}) \cdot \hat{\mathrm{e}}_2)\hat{\mathrm{e}}_2$

$\vec{\mathbf{t}}_3(\hat{\mathbf{n}}) = (\vec{\mathbf{t}}(\hat{\mathbf{n}}) \cdot \hat{\mathrm{e}}_3)\hat{\mathrm{e}}_3$

Cauchy Tetrahedron - Stress at a point

- Cauchy's formula expressing relating the stress vector $\vec{\boldsymbol{t}}(\hat{\boldsymbol{n}}_i)$ at every coordinate plane i with the unit normal vector $\hat{\boldsymbol{n}}$ and the stress tensor $\underline{\underline{\boldsymbol{\sigma}}}$:

$$t_i(\hat{\boldsymbol{n}}) = n_j \sigma_{ji}$$

where σ_{ji} in Cartesian coordinate system can be stored in matrix form as follows

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}; \quad n_j = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

- The stress tensor can be written as in terms of the stress vectors on the coordinate planes $\vec{\boldsymbol{t}}(\hat{\boldsymbol{n}}_i)$:

$$\underline{\underline{\boldsymbol{\sigma}}} \equiv \hat{\boldsymbol{e}}_1 \vec{\boldsymbol{t}}(\hat{\boldsymbol{n}}_1) + \hat{\boldsymbol{e}}_2 \vec{\boldsymbol{t}}(\hat{\boldsymbol{n}}_2) + \hat{\boldsymbol{e}}_3 \vec{\boldsymbol{t}}(\hat{\boldsymbol{n}}_3)$$

- In Direct Tensor Notation we can write the Cauchy's formula as follows:

$$\vec{\boldsymbol{t}}(\hat{\boldsymbol{n}}) = \hat{\boldsymbol{n}} \cdot \underline{\underline{\boldsymbol{\sigma}}} = \underline{\underline{\boldsymbol{\sigma}}}^T \cdot \hat{\boldsymbol{n}}$$

$$\left\{ \begin{array}{c} t_1(\hat{\boldsymbol{n}}) \\ t_2(\hat{\boldsymbol{n}}) \\ t_3(\hat{\boldsymbol{n}}) \end{array} \right\} = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \left\{ \begin{array}{c} n_1 \\ n_2 \\ n_3 \end{array} \right\}$$

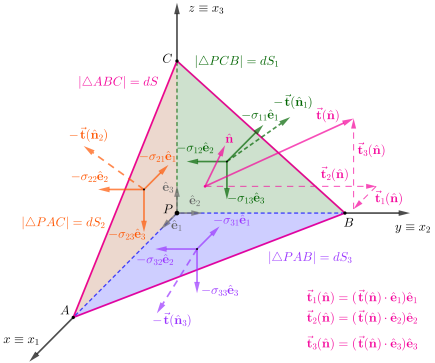
Cauchy Tetrahedron - Stress at a point

- The stress tensor can be written as in terms of the stress vectors on the coordinate planes $\vec{t}(\hat{n}_i)$:

$$\underline{\underline{\sigma}} \equiv \hat{e}_1 \vec{t}(\hat{n}_1) + \hat{e}_2 \vec{t}(\hat{n}_2) + \hat{e}_3 \vec{t}(\hat{n}_3)$$

- The stress vectors can be written as:

$$\vec{t}(\hat{n}_i) = \sigma_{i1}\hat{e}_1 + \sigma_{i2}\hat{e}_2 + \sigma_{i3}\hat{e}_3 = \sigma_{ij}\hat{e}_j$$



Cauchy Tetrahedron - Stress at a point

- The stress tensor can be written as in terms of the stress vectors on the coordinate planes $\vec{t}(\hat{\boldsymbol{n}}_i)$, in other words as a dyad:

$$\underline{\underline{\boldsymbol{\sigma}}} \equiv \hat{\boldsymbol{e}}_1 \vec{t}(\hat{\boldsymbol{n}}_1) + \hat{\boldsymbol{e}}_2 \vec{t}(\hat{\boldsymbol{n}}_2) + \hat{\boldsymbol{e}}_3 \vec{t}(\hat{\boldsymbol{n}}_3) = \hat{\boldsymbol{e}}_i \vec{t}(\hat{\boldsymbol{n}}_i) = \sigma_{ij} \hat{\boldsymbol{e}}_i \hat{\boldsymbol{e}}_j$$

- By expanding the last term:

$$\begin{aligned} \sigma_{ij} \hat{\boldsymbol{e}}_i \hat{\boldsymbol{e}}_j = \sigma_{11} \hat{\boldsymbol{e}}_1 \hat{\boldsymbol{e}}_1 + \sigma_{12} \hat{\boldsymbol{e}}_1 \hat{\boldsymbol{e}}_2 + \sigma_{13} \hat{\boldsymbol{e}}_1 \hat{\boldsymbol{e}}_3 + \\ \sigma_{21} \hat{\boldsymbol{e}}_2 \hat{\boldsymbol{e}}_1 + \sigma_{22} \hat{\boldsymbol{e}}_2 \hat{\boldsymbol{e}}_2 + \sigma_{23} \hat{\boldsymbol{e}}_2 \hat{\boldsymbol{e}}_3 + \\ \sigma_{31} \hat{\boldsymbol{e}}_3 \hat{\boldsymbol{e}}_1 + \sigma_{32} \hat{\boldsymbol{e}}_3 \hat{\boldsymbol{e}}_2 + \sigma_{33} \hat{\boldsymbol{e}}_3 \hat{\boldsymbol{e}}_3 \end{aligned}$$

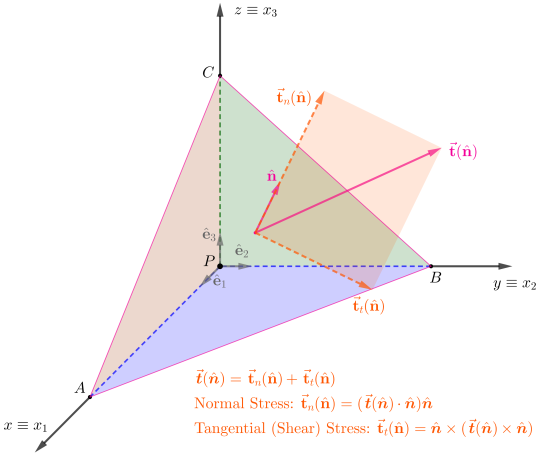
- In matrix form:

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

Cauchy Tetrahedron - Stress at a point

- The stress vector acting on surface with outer unit normal $\hat{\boldsymbol{n}}$ can be written as follows in terms of normal ($\vec{\mathbf{t}}_n(\hat{\mathbf{n}})$) and tangential stress components ($\vec{\mathbf{t}}_t(\hat{\mathbf{n}})$):

$$\vec{\boldsymbol{t}}(\hat{\boldsymbol{n}}) = \vec{\mathbf{t}}_n(\hat{\mathbf{n}}) + \vec{\mathbf{t}}_t(\hat{\mathbf{n}}) = (\vec{\boldsymbol{t}}(\hat{\boldsymbol{n}}) \cdot \hat{\boldsymbol{n}})\hat{\boldsymbol{n}} + \hat{\boldsymbol{n}} \times (\vec{\boldsymbol{t}}(\hat{\boldsymbol{n}}) \times \hat{\boldsymbol{n}})$$



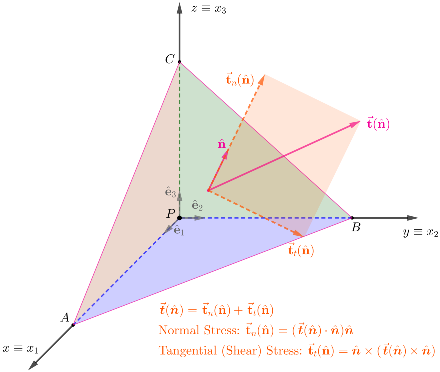
Cauchy Tetrahedron - Stress at a point

- The magnitude of the stress tensor normal to the plane (normal stress) can be calculated as:

$$|\vec{\mathbf{t}}_n(\hat{\mathbf{n}})| = \vec{\boldsymbol{t}}(\hat{\boldsymbol{n}}) \cdot \hat{\boldsymbol{n}} = t_i(\hat{\boldsymbol{n}})n_i = n_j\sigma_{ji}n_i$$

- The magnitude of the in plane stress, namely tangential stress, component (shear stress) can be found as:

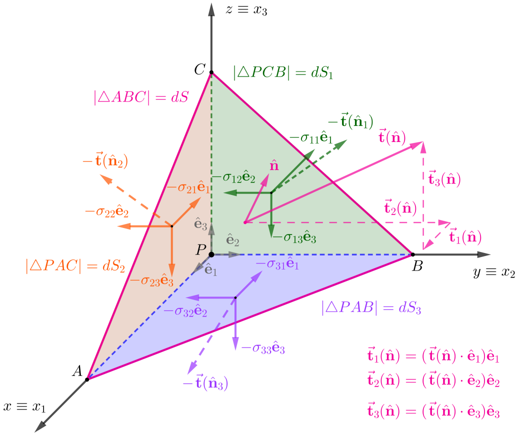
$$|\vec{\mathbf{t}}_t(\hat{\mathbf{n}})| = \sqrt{|\vec{\boldsymbol{t}}(\hat{\boldsymbol{n}})|^2 - |\vec{\mathbf{t}}_n(\hat{\mathbf{n}})|^2}$$



Cauchy Tetrahedron - Stress at a point

- In direct tensor notation the Cauchy's formula can be written:

$$\vec{\boldsymbol{t}}(\hat{\boldsymbol{n}}) = \hat{\boldsymbol{n}} \cdot \underline{\underline{\boldsymbol{\sigma}}} \equiv \underline{\underline{\boldsymbol{\sigma}}}^T \cdot \hat{\boldsymbol{n}}$$



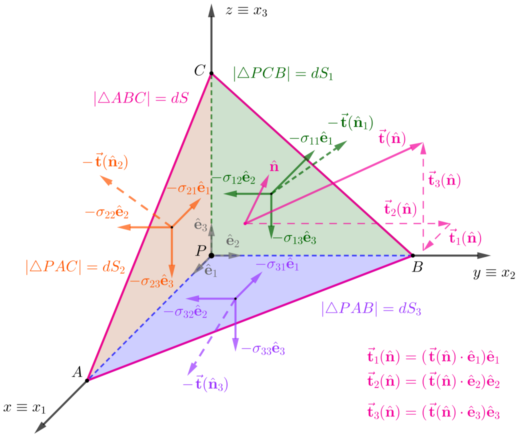
$\vec{\mathbf{t}}_1(\hat{\mathbf{n}}) = (\vec{\mathbf{t}}(\hat{\mathbf{n}}) \cdot \hat{\mathbf{e}}_1)\hat{\mathbf{e}}_1$

$\vec{\mathbf{t}}_2(\hat{\mathbf{n}}) = (\vec{\mathbf{t}}(\hat{\mathbf{n}}) \cdot \hat{\mathbf{e}}_2)\hat{\mathbf{e}}_2$

$\vec{\mathbf{t}}_3(\hat{\mathbf{n}}) = (\vec{\mathbf{t}}(\hat{\mathbf{n}}) \cdot \hat{\mathbf{e}}_3)\hat{\mathbf{e}}_3$

Cauchy Tetrahedron - Stress at a point

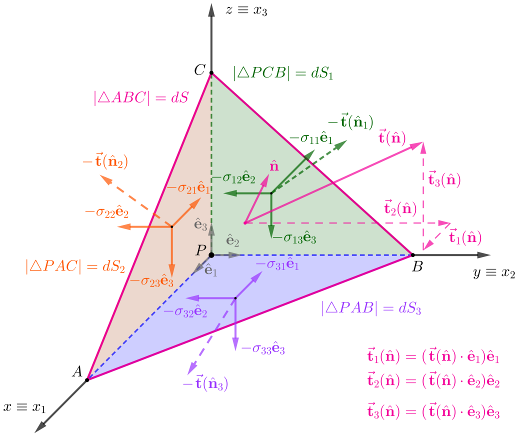
- Cauchy's formula states that nine components of stresses σ_{ij} are enough to define the traction across any surface element in a body, [1].
- The stress state in a body completely characterized by the quantities σ_{ij}, [1].



σ_{ij}

i - denotes the plane to which the component is acting

j - denotes the direction along which the stress component is acting



$$\vec{\mathbf{t}}_1(\hat{\mathbf{n}}) = (\vec{\mathbf{t}}(\hat{\mathbf{n}}) \cdot \hat{\mathbf{e}}_1)\hat{\mathbf{e}}_1$$

$$\vec{\mathbf{t}}_2(\hat{\mathbf{n}}) = (\vec{\mathbf{t}}(\hat{\mathbf{n}}) \cdot \hat{\mathbf{e}}_2)\hat{\mathbf{e}}_2$$

$$\vec{\mathbf{t}}_3(\hat{\mathbf{n}}) = (\vec{\mathbf{t}}(\hat{\mathbf{n}}) \cdot \hat{\mathbf{e}}_3)\hat{\mathbf{e}}_3$$

Conservation of Angular Momentum

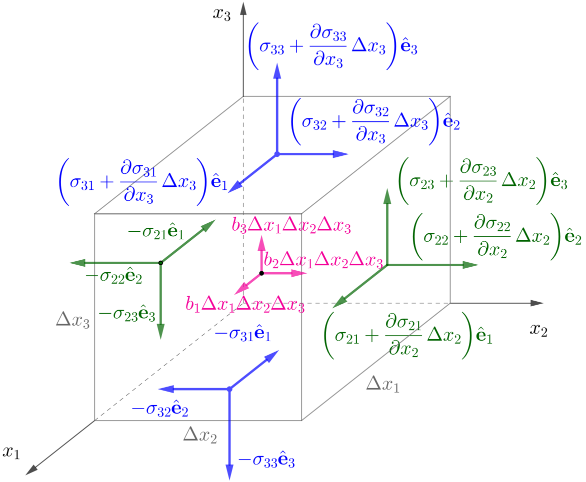
- The equations of motion can be written as:

$$\oiint_S \vec{\boldsymbol{t}}(\hat{\boldsymbol{n}})\,dS + \iiint_{B(t)} \vec{\boldsymbol{b}}\,dV = \frac{d}{dt}\iiint_{B(t)} \vec{\boldsymbol{v}}\rho\,dV$$
$$\oiint_S \vec{\boldsymbol{r}} \times \vec{\boldsymbol{t}}(\hat{\boldsymbol{n}})\,dS + \iiint_{B(t)} \vec{\boldsymbol{r}} \times \vec{\boldsymbol{b}}\,dV = \frac{d}{dt}\iiint_{B(t)} \vec{\boldsymbol{r}} \times \vec{\boldsymbol{v}}\rho\,dV$$

  The domain $B(t)$ consists of same material particles at all points and they form a continuum bounded by surface S
- We will transform the above equations to differential equations.

Conservation of Angular Momentum

To transform them to differential equations we use an infinitesimal parallelepiped:



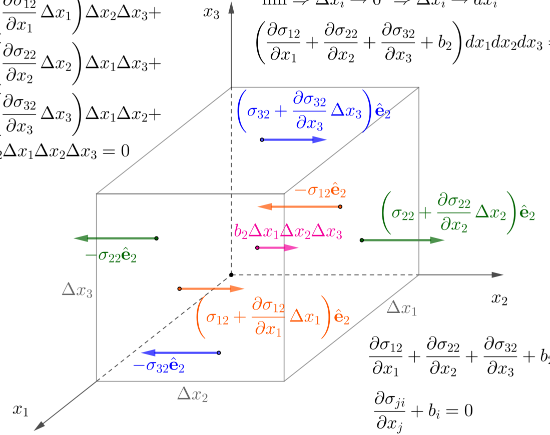
Conservation of Angular Momentum

The stresses acting along x_2-axis are the ones shown

$$\left(\frac{\partial \sigma_{12}}{\partial x_1}\Delta x_1\right)\Delta x_2 \Delta x_3 +$$
$$\left(\frac{\partial \sigma_{22}}{\partial x_2}\Delta x_2\right)\Delta x_1 \Delta x_3 +$$
$$\left(\frac{\partial \sigma_{32}}{\partial x_3}\Delta x_3\right)\Delta x_1 \Delta x_2 +$$
$$b_2 \Delta x_1 \Delta x_2 \Delta x_3 = 0$$

$$\lim \Rightarrow \Delta x_i \to 0^* \Rightarrow \Delta x_i \to dx_i$$

$$\left(\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3} + b_2\right) dx_1 dx_2 dx_3 = 0$$



$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3} + b_2 = 0$$

$$\frac{\partial \sigma_{ji}}{\partial x_j} + b_i = 0$$

Conservation of Angular Momentum

Moments around an axis parallel to x_2-axis that goes through the center of the cube.

$$\left(\sigma_{31} + \frac{\partial \sigma_{31}}{\partial x_3}\Delta x_3\right)\Delta x_1 \Delta x_2 \frac{\Delta x_3}{2} + \sigma_{31}\Delta x_1 \Delta x_2 \frac{\Delta x_3}{2} +$$

$$-\left(\sigma_{13} + \frac{\partial \sigma_{13}}{\partial x_1}\Delta x_1\right)\frac{\Delta x_1}{2}\Delta x_2 \Delta x_3 - \sigma_{13}\frac{\Delta x_1}{2}\Delta x_2 \Delta x_3 = 0$$

Ommiting fourth order Δx_i^4

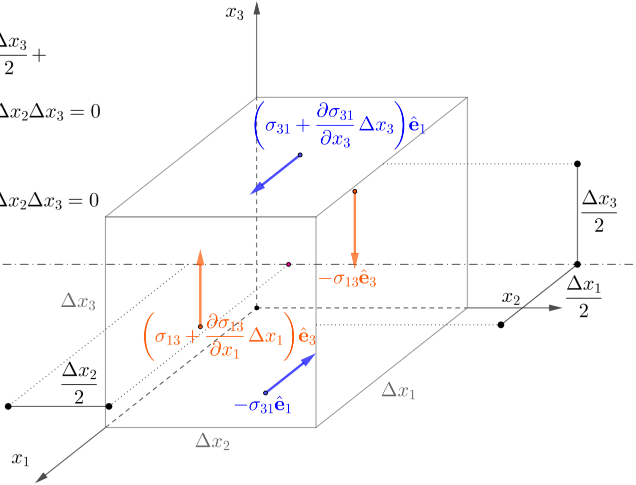
$$(\sigma_{31} - \sigma_{31})\frac{\Delta x_1 \Delta x_2 \Delta x_3}{2} = 0 \Leftrightarrow (\sigma_{31} - \sigma_{31})\Delta x_1 \Delta x_2 \Delta x_3 = 0$$

$\lim \Leftrightarrow \Delta x_i \to 0^* \Leftrightarrow \Delta x_i \to dx_i$

$$(\sigma_{31} - \sigma_{31})dx_1 dx_2 dx_3 = 0$$

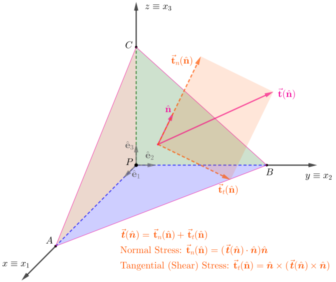
$$dx_1 dx_2 dx_3 \neq 0 \Leftrightarrow (\sigma_{31} - \sigma_{31}) = 0 \Leftrightarrow \sigma_{31} = \sigma_{13}$$

The stress tensor is symmetric
due the conservation of angular momentum



Compression Tension

- The traction (stress) vector can be analysed in two components, one normal to the plane $\vec{\boldsymbol{t}}_n(\hat{\boldsymbol{n}})$ and one tangential (shear) in the plane $\vec{\boldsymbol{t}}_t(\hat{\boldsymbol{n}})$
- The sense of $\vec{\boldsymbol{t}}_n(\hat{\boldsymbol{n}})$ with respect to the normal $\hat{\boldsymbol{n}}$ defines the character: positive means tension, negative compression.
- For the stress tensor $\underline{\underline{\boldsymbol{\sigma}}}$ the sign criterion:
  1. $\sigma_a > 0$ is tension, where $a \in \{x, y, z\}$
  2. $\sigma_a < 0$ is compression, where $a \in \{x, y, z\}$
  3. $\tau_{ab} > 0$ towards positive direction of b-axis
  4. $\tau_{ab} < 0$ towards negative direction of b-axis



Cauchy's equation of motion

- For any material volume we can define the **Cauchy equation of motion** (to be proven):

$$\vec{\boldsymbol{\nabla}} \cdot \underline{\underline{\boldsymbol{\sigma}}} + \rho \vec{\boldsymbol{b}} = \rho \vec{\boldsymbol{a}}; \quad \forall \vec{\boldsymbol{x}} \in V$$

$$\frac{\partial \sigma_{ij}}{\partial x_i} + \rho b_j = \rho a_j; \quad i, j \in \{1, 2, 3\}$$

  where $\vec{\boldsymbol{b}}(\vec{\boldsymbol{x}}, t)$ are the body forces, $\vec{\boldsymbol{a}}(\vec{\boldsymbol{x}}, t)$ is the acceleration vector, ρ is the mass density.
- The Cauchy equation of motion is derived from the **principle of linear momentum**.
- For a body in static equilibrium, we assume that $\vec{\boldsymbol{a}}(\vec{\boldsymbol{x}}, t) = 0$ (proven already).
- The equilibrium at the boundary can be written as:

$$\hat{\boldsymbol{n}}(\vec{\boldsymbol{x}}, t) \cdot \underline{\underline{\boldsymbol{\sigma}}}(\vec{\boldsymbol{x}}, t) = \vec{\boldsymbol{t}}(\vec{\boldsymbol{x}}, t); \quad \vec{\boldsymbol{x}} \forall \partial V$$

$$n_i \sigma_{ij} = t_j; \quad i, j \in \{1, 2, 3\}$$

Cauchy's equation of motion -Symmetry

- Due to symmetry of the stress tensor:
- The equation of motion can be written as:

$$\vec{\boldsymbol{\nabla}} \cdot \underline{\underline{\boldsymbol{\sigma}}} + \rho \vec{\boldsymbol{b}} = \underline{\underline{\boldsymbol{\sigma}}} \cdot \vec{\boldsymbol{\nabla}} + \rho \vec{\boldsymbol{b}} = \rho \vec{\boldsymbol{a}}; \quad \forall \vec{\boldsymbol{x}} \in V$$

$$\frac{\partial \sigma_{ij}}{\partial x_i} + \rho b_j = \frac{\partial \sigma_{ji}}{\partial x_i} + \rho b_j = \rho a_j; \quad i, j \in \{1, 2, 3\}$$

- The boundary conditions can be written as:

$$\hat{\boldsymbol{n}}(\vec{\boldsymbol{x}}, t) \cdot \underline{\underline{\boldsymbol{\sigma}}}(\vec{\boldsymbol{x}}, t) = \underline{\underline{\boldsymbol{\sigma}}}(\vec{\boldsymbol{x}}, t) \cdot \hat{\boldsymbol{n}}(\vec{\boldsymbol{x}}, t) = \vec{\boldsymbol{t}}(\vec{\boldsymbol{x}}, t); \quad \vec{\boldsymbol{x}} \forall \partial V$$

$$n_i \sigma_{ij} = \sigma_{ji} n_i = t_j; \quad i, j \in \{1, 2, 3\}$$

Principal stresses - Principal stress directions

- It is possible to choose a set of axes for which the shear stress components vanish (principal axes of stress)
- The three planes are called principal planes and they are perpendicular to each other
- The normal stress components are the principal stresses.

$$[\underline{\underline{\boldsymbol{\sigma}}}] = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} ; \quad \sigma_1 \geq \sigma_2 \geq \sigma_3$$

Principal stresses - Principal stress directions

- The Cauchy stress tensor is a symmetric 2^{nd} order tensor and therefore its eigenvalues are real numbers.
- For the eigenvalue λ and the eigenvector $\vec{\boldsymbol{v}}$:

$$\underline{\underline{\boldsymbol{\sigma}}} \cdot \vec{\boldsymbol{v}} = \lambda \vec{\boldsymbol{v}} \Leftrightarrow [\underline{\underline{\boldsymbol{\sigma}}} - \lambda \underline{\underline{\mathbf{1}}}] \cdot \vec{\boldsymbol{v}} = 0$$

Therefore:

$$\det[\underline{\underline{\boldsymbol{\sigma}}} - \lambda \underline{\underline{\mathbf{1}}}] = |\underline{\underline{\boldsymbol{\sigma}}} - \lambda \underline{\underline{\mathbf{1}}}| = 0$$

The characteristic equation can be written as follows:

$$\lambda^3 - I_1(\underline{\underline{\boldsymbol{\sigma}}})\lambda^2 + I_2(\underline{\underline{\boldsymbol{\sigma}}})\lambda - I_3(\underline{\underline{\boldsymbol{\sigma}}}) = 0$$

where $\lambda_1 \equiv \sigma_1, \lambda_2 \equiv \sigma_2, \lambda_3 \equiv \sigma_3$ and:

$$I_1 = tr\underline{\underline{\boldsymbol{\sigma}}} = \sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33};$$

$$I_2 = \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{vmatrix} + \begin{vmatrix} \sigma_{22} & \sigma_{23} \\ \sigma_{32} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{31} & \sigma_{33} \end{vmatrix} = \frac{1}{2}\left(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ji}\right) = \frac{1}{2}\left[(tr\underline{\underline{\boldsymbol{\sigma}}})^2 - tr(\underline{\underline{\boldsymbol{\sigma}}})^2\right]$$

$$I_3 = \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{vmatrix} = \det[\underline{\underline{\boldsymbol{\sigma}}}]$$

Invariants

The principal stresses are invariants with respect the coordinate system.
The principal stresses define the stress invariants I_1, I_2, I_3 as follows:

$$I_1 = tr(\underline{\underline{\boldsymbol{\sigma}}}) = \sigma_{ii} = \sigma_1 + \sigma_2 + \sigma_3$$

$$I_2 = \frac{1}{2}(\underline{\underline{\boldsymbol{\sigma}}} : \underline{\underline{\boldsymbol{\sigma}}} - I_1^2) = -(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3)$$

$$I_3 = \det(\underline{\underline{\boldsymbol{\sigma}}}) = \sigma_1\sigma_2\sigma_3$$

Mean stress - Mean pressure

- After defining the principal stresses $\sigma_1, \sigma_2, \sigma_3$
- The mean stress can be defined as:

$$\sigma_m = \frac{1}{3} tr(\underline{\underline{\boldsymbol{\sigma}}}) = \frac{1}{3}\sigma_{ii} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$$

- The mean pressure can be defined as:

$$\overline{p} = -\sigma_m = -\frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$$

- The hydrostatic state of stress is the following:

$$\sigma_1 = \sigma_2 = \sigma_3$$

In the hydrostatic state of stress the stress tensor is isotropic (its components are the same in any Cartesian coordinate system). Therefore any direction is the principal direction.

Hydrostatic and Deviatoric stress tensors

The Cauchy stress tensor can be split into two parts, the hydrostatic and the deviatoric part:

$$\underline{\underline{\boldsymbol{\sigma}}} = \underline{\underline{\boldsymbol{\sigma}}}_{hyd} + \underline{\underline{\boldsymbol{\sigma}}}_{dev}$$

The hydrostatic stress tensor can be defined as:

$$\underline{\underline{\boldsymbol{\sigma}}}_{hyd} = \sigma_m \underline{\underline{\mathbf{1}}}$$

The hydrostatic stress tensor changes the volume of the body.
The deviatoric stress tensor can be defined as follows:

$$\underline{\underline{\boldsymbol{\sigma}}}_{dev} = \underline{\underline{\boldsymbol{\sigma}}} - \underline{\underline{\boldsymbol{\sigma}}}_{hyd}$$

It tends to distort the volume of the body

Maximum shear stresses

The maximum shear stresses can be derived after the principal stresses have been derived:

$$\frac{|\sigma_1 - \sigma_2|}{2}; \quad \frac{|\sigma_1 - \sigma_3|}{2}; \quad \frac{|\sigma_2 - \sigma_3|}{2}$$

The direction of the shear stresses are 45^o between the principal directions

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