Foundations on Continuum Mechanics - Week 4 - Balance Principles

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Balance Principles

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Theories

- Every Theoretical Framework (**Theory**) is founded in hypotheses (**Principles**, Postulates).
- All theories contain **Principles** (that cannot be proven).
- The **Principles** (Postulates) are the building blocks of **Theorems**.
- The **Theorems** establish the Theory.

Balance Principles

The principles that relate the way concepts (like deformation and stress) vary are the following:

- The conservation/balance principles:
  1. Conservation of mass
  2. Linear momentum balance principle
  3. Angular momentum balance principle
  4. Energy balance principle
- The restriction principle: 2^{nd} thermodynamic law

The expression of the principles can be given:

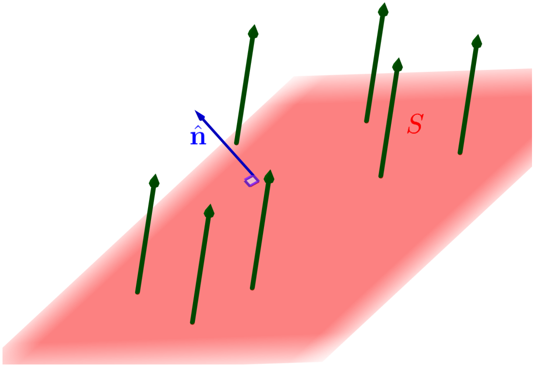
- Global (integral) form
- Local (strong) form (partial differential equations)

These principles are ALWAYS valid independent of material

Convective Flux

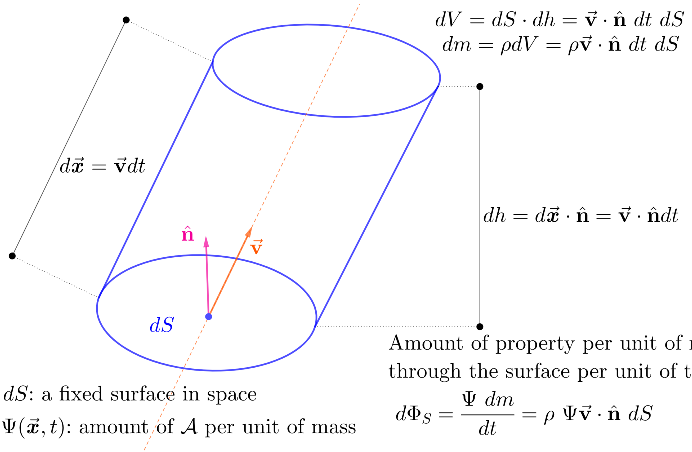
Convection

- Convection is associated with the change of a property in a spatial point due to the motion of mass particles (mass transport), [1].
- Properties are transported with the mass as it moves in space.
- The **convective flux** of a property $\mathcal{A}$ can be given as follows: $\Phi_S = \frac{\text{ammount of } \mathcal{A} \text{ through } S}{\text{unit of time}}$



Convective flux

- A property related to mass particles (sticks to it) $\mathcal{A}$ (any tensorial order)
- A volume of particles dV crosses the surface dS in the interval $[t, t+dt]$:



dS: a fixed surface in space

$\Psi(\vec{\boldsymbol{x}}, t)$: amount of $\mathcal{A}$ per unit of mass

Amount of property per unit of mass through the surface per unit of time:

$$d\Phi_S = \frac{\Psi\ dm}{dt} = \rho\ \Psi \vec{\mathbf{v}} \cdot \hat{\mathbf{n}}\ dS$$

Convective flux

- A property related to mass particles (sticks to it) $\mathcal{A}$ (any tensorial order)
- The content of $\mathcal{A}$ per unit mass $\Psi(\vec{\boldsymbol{x}}, t)$
- The convective flux of $\mathcal{A}$ through a SPATIAL surface S with unit normal $\hat{\boldsymbol{n}}$ is given by, [1]:

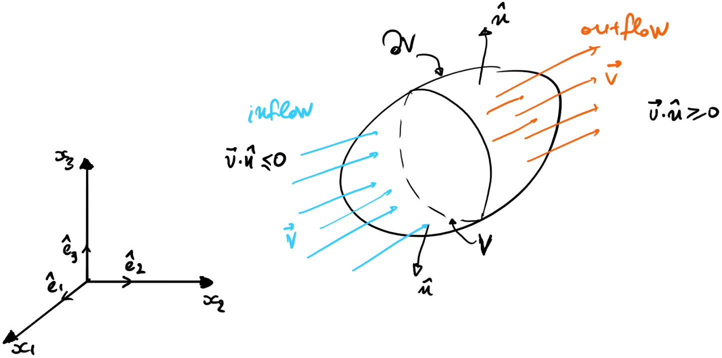
$$\Phi_S(t) = \int\int_S \rho \Psi \vec{\boldsymbol{v}} \cdot \hat{\boldsymbol{n}} dS$$

  where the $\vec{\boldsymbol{v}}$ is the velocity and ρ is the density.

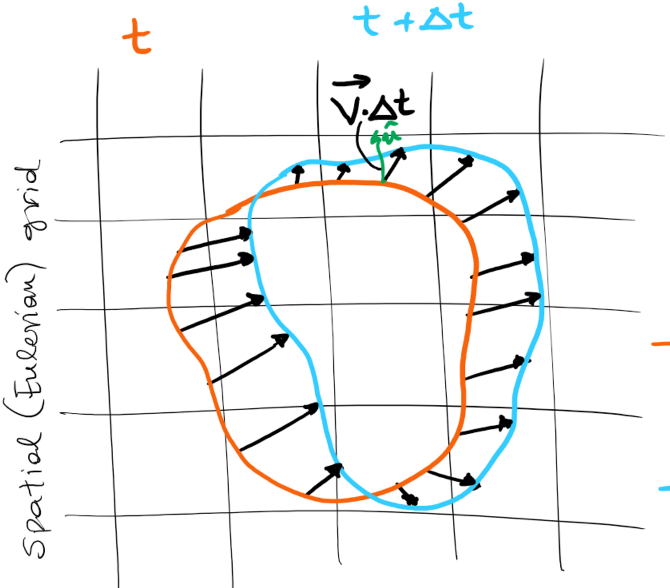
Convective flux - Closed surface

The net convective flux through a closed surface $S = \partial V$ of a volume V is:

$$\Phi_S(t) = \int\int_{\partial V} \rho \Psi \vec{\boldsymbol{v}} \cdot \hat{\boldsymbol{n}} dS = \text{outflow - inflow}$$



Convective flux



The convective flux of mass from surface

$\vec{v}\cdot\hat{n}\,\rho\, dS$

$\hat{n}$ - the unit normal of the surface

— surface that encloses a certain number of particles at t

— surface that encloses the same particles at t+Δt

Non-Convective flux

- The convective flux over material surface is ZERO.
- The non-convective flux: Properties are transported without transport of mass particles. Example: heat transfer by conduction, etc.
- The non-convective flux vector can be defined as:

$$\int\int_S \vec{\boldsymbol{q}} \cdot \vec{\boldsymbol{n}} dS$$

Local and Material Derivative of volume integral

Derivative of volume integral

- A property of a continuum can be named as $\mathcal{A}$
- The description of the property per unit volume, in other words its density, is characterized by $\mu(\vec{\boldsymbol{x}}, t)$
- The total amount of the property in a volume can be derived as follows, [1]:

$$Q(t) = \int\int\int_V \mu(\vec{\boldsymbol{x}}, t) dV$$

- The time derivative of the integral can be derived as:

$$\dot{Q}(t) = \lim_{\Delta t \to 0} \frac{Q(t + \Delta t) - Q(t)}{\Delta t}$$

- The relationship between μ and Ψ is as follows: $\mu = \rho\Psi$

Local derivative of volume integral

The volume integral is given by:

$$Q(t) = \int\int\int_V \mu(\vec{\boldsymbol{x}}, t) dV$$

The local derivative of the volume can be derived as:

$$\frac{\partial}{\partial t}\int\int\int_V \mu(\vec{\boldsymbol{x}}, t) dV = \lim_{\Delta t \to 0} \frac{\int\int\int_V \mu(\vec{\boldsymbol{x}}, \Delta t + t) dV - \int\int\int_V \mu(\vec{\boldsymbol{x}}, \Delta t) dV}{\Delta t}$$

We can compute as follows:

$$\frac{\partial}{\partial t}\int\int\int_V \mu(\vec{\boldsymbol{x}}, t) dV = \lim_{\Delta t \to 0} \frac{Q(t+\Delta t) - Q(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\int\int\int_V \mu(\vec{\boldsymbol{x}}, \Delta t + t) dV - \int\int\int_V \mu(\vec{\boldsymbol{x}}, \Delta t) dV}{\Delta t}$$

$$\lim_{\Delta t \to 0} \frac{\int\int\int_V \left(\mu(\vec{\boldsymbol{x}}, \Delta t + t) - \mu(\vec{\boldsymbol{x}}, \Delta t)\right) dV}{\Delta t} = \int\int\int_V \lim_{\Delta t \to 0} \frac{\mu(\vec{\boldsymbol{x}}, \Delta t + t) - \mu(\vec{\boldsymbol{x}}, \Delta t)}{\Delta t} dV =$$

$$= \int\int\int_V \frac{\partial \mu(\vec{\boldsymbol{x}}, \Delta t)}{\partial t} dV$$

Note that the volume is fixed in space!! $\vec{\boldsymbol{x}}$ is fixed!

Material derivative of volume integral

The volume integral is given by:

$$Q(t) = \int\int\int_V \mu(\vec{\boldsymbol{x}}, t)dV$$

The material derivative of $Q(t)$ will be:

$$\frac{d}{dt}\int\int\int_{V_t \equiv V} \mu(\vec{\boldsymbol{x}}, t)dV = \lim_{\Delta t \to 0} \frac{\int\int\int_{V(t+\Delta t)} \mu(\vec{\boldsymbol{x}}, t+\Delta t)dV - \int\int\int_{V(t)} \mu(\vec{\boldsymbol{x}}, t)dV}{\Delta t}$$

The material derivative will be:

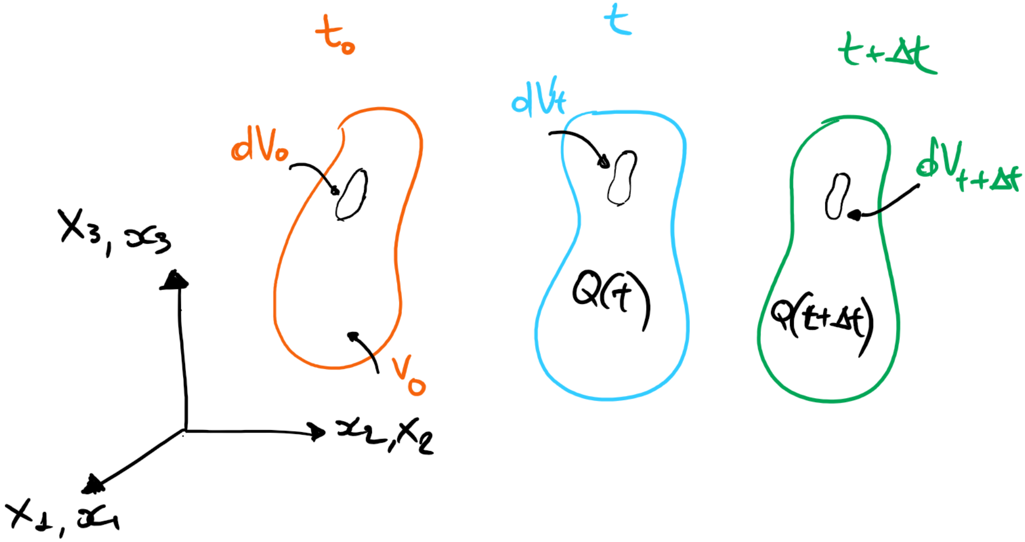
$$\frac{d}{dt}\int\int\int_{V_t \equiv V} \mu(\vec{\boldsymbol{x}}, t)dV = \frac{\partial}{\partial t}\int\int\int_V \mu dV + \int\int\int_V \vec{\boldsymbol{\nabla}} \cdot (\mu\vec{\boldsymbol{v}})dV = \int\int\int_V \left(\frac{\partial\mu}{\partial t} + \vec{\boldsymbol{\nabla}} \cdot (\mu\vec{\boldsymbol{v}})\right) dV =$$

$$\int\int\int_V \left(\underbrace{\frac{\partial\mu}{\partial t} + \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{\nabla}}\mu}_{\text{material derivative}} + \mu\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{v}}\right) dV = \int\int\int_V \left(\frac{d\mu}{dt} + \mu\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{v}}\right) dV$$

Note that the volume moves in space (material volume), it can deform and move. Two dependencies on time:

1. The integrand depends on time (property $\mu(\vec{\boldsymbol{x}}, t)$)
2. The domain of integration depends on time (V_t)

Material derivative of volume integral



Principle of Mass Conservation

Principle of Mass Conservation

- Postulate: the mass of a continuum body is CONSERVED
- The total mass $\mathcal{M}(t)$ of the system:

$$\mathcal{M}(t) = \mathcal{M}(t + \Delta t) > 0$$

- The mass can be defined as:

$$\mathcal{M}(t) = \int\int\int_{\Delta V_t} \rho(\vec{\boldsymbol{x}}, t) dV, \quad \forall \Delta V_t \in V_t$$

$$\mathcal{M}(t + \Delta t) = \int\int\int_{\Delta V_{t+\Delta t}} \rho(\vec{\boldsymbol{x}}, t + \Delta t) dV, \quad \forall \Delta V_{t+\Delta t} \in V_{t+\Delta t}$$

Conservation of Mass - Spatial form

- The material derivative of the mass $\mathcal{M}(t)$ for any region of the material volume can be written as:

$$\dot{\mathcal{M}}(t) = \lim_{\Delta t \to 0} \frac{\mathcal{M}(t+\Delta t) - \mathcal{M}(t)}{\Delta t} = \frac{d}{dt} \int\int\int_{\Delta V_t \in V_t \equiv V} \rho dV = 0, \quad \forall \Delta V \in V, \quad \forall t$$

- The global/integral spatial form of mass conservation can be written as, [1]:

$$\frac{d}{dt} \int\int\int_{V_t \equiv V} \mu(\vec{\boldsymbol{x}}, t) dV = \int\int\int_{V} \left(\frac{d\mu}{dt} + \mu \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{v}} \right) dV \Leftrightarrow$$

$$\frac{d}{dt} \int\int\int_{\Delta V_t \in V_t \equiv V} \rho(\vec{\boldsymbol{x}}, t) dV = \int\int\int_{\Delta V \in V} \left(\frac{d\rho}{dt} + \rho \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{v}} \right) dV = 0, \quad \forall \Delta V \in V, \forall t$$

- By localization process the local/differential spatial form can be obtained for $\Delta V \to dV(\vec{\boldsymbol{x}}, t)$:

$$\frac{d\rho(\vec{\boldsymbol{x}}, t)}{dt} + \left(\rho \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{v}} \right) (\vec{\boldsymbol{x}}, t) = \frac{\partial \rho(\vec{\boldsymbol{x}}, t)}{\partial t} + \vec{\boldsymbol{\nabla}} \cdot (\rho \vec{\boldsymbol{v}}) (\vec{\boldsymbol{x}}, t) = 0 \quad \forall \vec{\boldsymbol{x}} \in V, \forall t$$

- The equation is called the continuity equation.

Velocity gradient tensor

Two particles P, Q occupy the spatial points P', Q' at time t and their velocities are $\vec{\boldsymbol{v}}_P = \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}, t)$ and $\vec{\boldsymbol{v}}_Q = \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}} + d\vec{\boldsymbol{x}}, t)$:

$$d\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}, t) = \vec{\boldsymbol{v}}_Q - \vec{\boldsymbol{v}}_P = \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}} + d\vec{\boldsymbol{x}}, t) - \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}, t)$$

The differential is:

$$d\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}, t) = \frac{\partial \vec{\boldsymbol{v}}}{\partial \vec{\boldsymbol{x}}} \cdot d\vec{\boldsymbol{x}}$$

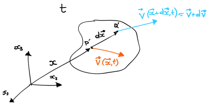
$$dv_i = \frac{\partial v_i}{\partial x_j} dx_j$$

The velocity gradient tensor can be defined as:

$$\underline{\boldsymbol{l}}(\vec{\boldsymbol{x}}, t) = \frac{\partial \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}, t)}{\partial \vec{\boldsymbol{x}}}$$

$$\underline{\boldsymbol{l}} = \vec{\boldsymbol{v}} \otimes \vec{\boldsymbol{\nabla}}$$

$$l_{ij} = \frac{\partial v_i}{\partial x_j}$$



Material time derivative of deformation gradient tensor

The differential of the gradient of deformation tensor $\underline{\underline{\boldsymbol{F}}}$ with respect to time:

$$F_{ij} = \frac{\partial x_i(\vec{\boldsymbol{X}}, t)}{\partial X_j} \Rightarrow \frac{dF_{ij}}{dt} = \frac{\partial}{\partial t}\frac{\partial x_i(\vec{\boldsymbol{X}}, t)}{\partial X_j} = \frac{\partial}{\partial X_j}\underbrace{\frac{\partial x_i(\vec{\boldsymbol{X}}, t)}{\partial t}}_{v_i}$$

Therefore:

$$\frac{dF_{ij}}{dt} = \frac{\partial v_i(\vec{\boldsymbol{X}}, t)}{\partial X_j} = \underbrace{\frac{\partial v_i(\vec{\boldsymbol{x}}(\vec{\boldsymbol{X}}, t))}{\partial X_k}}_{l_{ik}}\underbrace{\frac{x_k}{X_j}}_{F_{kj}} = l_{ik}F_{kj}$$

Finally:

$$\frac{d\underline{\underline{\boldsymbol{F}}}}{dt} \equiv \dot{\underline{\underline{\boldsymbol{F}}}} = \underline{\underline{\boldsymbol{l}}} \cdot \underline{\underline{\boldsymbol{F}}}$$

Volume differential

The volume differential $dV(\vec{\boldsymbol{X}}, t)$ related to a certain particle changes with time, [1]:

$$dV(\vec{\boldsymbol{X}}, t) = |\underline{\underline{\boldsymbol{F}}}(\vec{\boldsymbol{X}}, t)| dV_0(\vec{\boldsymbol{X}}) \Leftrightarrow \frac{d}{dt} dV(t) = \frac{|\underline{\underline{\boldsymbol{F}}}|}{dt} dV_0$$

The material derivative of the deformation gradient tensor can be derived as:

$$\frac{d|\underline{\underline{d\boldsymbol{F}}}|}{dt} = \frac{d|\underline{\underline{\boldsymbol{F}}}|}{dF_{ij}} \frac{dF_{ij}}{dt} = |\underline{\underline{\boldsymbol{F}}}| \cdot F_{ji}^{-1} \underbrace{\frac{dF_{ij}}{dt}}_{l_{ik} F_{kj}} = |\underline{\underline{\boldsymbol{F}}}| F_{ji}^{-1} l_{ik} F_{kj} = |\underline{\underline{\boldsymbol{F}}}| \underbrace{F_{kj} F_{ji}^{-1}}_{=\delta_{ki}} l_{ik}$$

$$= |\underline{\underline{\boldsymbol{F}}}| \delta_{ki} l_{ik} = |\underline{\underline{\boldsymbol{F}}}| l_{ii} = |\underline{\underline{\boldsymbol{F}}}| \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{v}} \Rightarrow \frac{|\underline{\underline{d\boldsymbol{F}}}|}{dt} = |\underline{\underline{\boldsymbol{F}}}| \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{v}}$$

Note by definition: $d|\underline{\underline{\boldsymbol{A}}}|/dA_{ij} = |\underline{\underline{\boldsymbol{A}}}| \cdot A_{ji}^{-1}$

Conservation of Mass - Material form

The global/integral material form of mass conservation principle can be written as:

$$\int\int\int_V \left(\frac{d\rho}{dt} + \rho\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{v}}\right) dV = \int\int\int_V \left(\frac{d\rho}{dt} + \rho\frac{1}{|\underline{\underline{\boldsymbol{F}}}|}\frac{d|\underline{\underline{\boldsymbol{F}}}|}{dt}\right) dV =$$

$$\int\int\int_V \frac{1}{|\underline{\underline{\boldsymbol{F}}}|}\underbrace{\left(|\underline{\underline{\boldsymbol{F}}}|\frac{d\rho}{dt} + \rho\frac{d|\underline{\underline{\boldsymbol{F}}}|}{dt}\right)}_{\frac{d}{dt}(\rho|\underline{\underline{\boldsymbol{F}}}|)} dV = \int\int\int_V \frac{1}{|\underline{\underline{\boldsymbol{F}}}|}\frac{d}{dt}\left(\rho|\underline{\underline{\boldsymbol{F}}}|\right)\underbrace{dV}_{|\underline{\underline{\boldsymbol{F}}}|dV_0} =$$

$$\int\int\int_{V_0} \frac{\partial}{\partial t}\left(\rho(\vec{\boldsymbol{X}},t)|\underline{\underline{\boldsymbol{F}}}(\vec{\boldsymbol{X}},t)|\right) dV_0 = 0, \quad \forall\Delta V_0 \in V_0, \forall t$$

The integration domain is the reference configuration volume V_0.
The local material form of mass conservation can be written as:

$$\frac{\partial}{\partial t}\left(\rho(\vec{\boldsymbol{X}},t)|\underline{\underline{\boldsymbol{F}}}(\vec{\boldsymbol{X}},t)|\right) = 0$$

Reynolds Transport Theorem

Reynolds Transport Theorem

Reynolds lemma states that for an arbitrary property $\mathcal{A}$ of a continuum with a spatial description per unit of mass $\psi(\vec{\boldsymbol{x}}, t)$ the material derivative of the amount of property $\mathcal{A}$ at time t for an arbitrary volume be expressed as:

$$\frac{d}{dt}\int\int\int_V \rho\psi dV = \int\int\int_V \rho\frac{d\psi}{dt}dV$$

The Reynolds transport theorem in integral form can be expressed as:

$$\frac{\partial}{\partial t}\int\int\int_V \rho\psi dV = \int\int\int_V \rho\frac{d\psi}{dt}dV - \int\int_{\partial V} \rho\psi\vec{\boldsymbol{v}}\cdot\hat{\boldsymbol{n}}dS$$

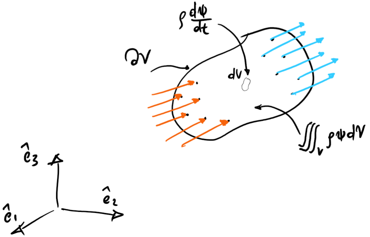
The local form of the Reynolds transport theorem can be expressed as:

$$\frac{\partial}{\partial t}(\rho\psi) = \rho\frac{d\psi}{dt} - \vec{\boldsymbol{\nabla}}\cdot(\rho\psi\vec{\boldsymbol{v}}), \quad \forall\vec{\boldsymbol{x}}\in V, \quad \forall t$$

Physical meaning of Reynolds Theorem

$$\frac{\partial}{\partial t}\int\int\int_V \rho\psi dV = \int\int\int_V \rho\frac{d\psi}{dt}dV - \int\int_{\partial V} \rho\psi\vec{\boldsymbol{v}}\cdot\hat{\boldsymbol{n}}dS$$

- Left-hand side denotes the rate of change of the total amount of $\mathcal{A}$ within the control volume V at time t.
- First term on right-hand side denotes the rate of change of $\mathcal{A}$ instantaneously possessed by the material in the control volume, [1].
- The second term on the right-hand side denotes the change due to convective flux of $\mathcal{A}$ through the boundary ∂V.



General Balance Equation

General Balance Equation

The general balance equation can be written as:

$$\frac{\partial}{\partial t}\int\int\int_V \rho\psi dV = \int\int\int \rho k_{\mathcal{A}} dV - \int\int_{\partial V} \rho\psi\vec{\boldsymbol{v}}\cdot\hat{\boldsymbol{n}}dS - \int\int_{\partial V} \vec{\boldsymbol{j}}_{\mathcal{A}}\cdot\hat{\boldsymbol{n}}dS$$

- The first term of the right-hand side denotes the generation of the property due to a source
- The second term of the right-hand side denotes the convective flux across the surface of the volume.
- The third term on the right-hand side is the non-convective flux across the surface.
- The local spatial form can be written as follows:

$$\rho\frac{d\psi}{dt} = \rho k_{\mathcal{A}} - \vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{j}}_{\mathcal{A}}$$

For only convective transport: $\vec{\boldsymbol{j}}_{\mathcal{A}} = 0$

Balance of Linear Momentum

Linear momentum in Classical Mechanics

- The 2^{nd} Newton's law for a discrete system of n particles states that the resulting force acting on a system of forces will be:

$$\vec{\boldsymbol{R}}(t) = \sum_{i=1}^{n} \vec{\boldsymbol{f}}_i = \sum_{i=1}^{n} m_i \vec{\boldsymbol{a}}_i = \sum_{i=1}^{n} m_i \frac{d\vec{\boldsymbol{v}}_i}{dt} = \frac{d}{dt} \sum_{i=1}^{n} m_i \vec{\boldsymbol{v}}_i = \frac{d\vec{\mathcal{P}}(t)}{dt}$$

  where $\vec{\mathcal{P}}(t)$ is the linear momentum.
- Special case: for a system in equilibrium $\vec{\boldsymbol{R}} = 0, \forall t$:

$$\frac{d\vec{\mathcal{P}}(t)}{dt} = 0 \Rightarrow \vec{\mathcal{P}}(t) = \text{constant}$$

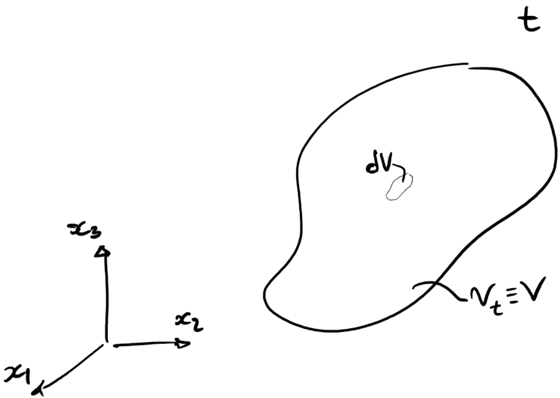
  Conservation of linear momentum.

Linear momentum in Continuum Mechanics

- The linear momentum of a material volume V of a continuum medium with mass $\mathcal{M}$ is :

$$\vec{\mathcal{P}}(t) = \int\int\int_{\mathcal{M}} \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}, t) d\mathcal{M} = \int\int\int_{V} \rho(\vec{\boldsymbol{x}}, t)\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}, t) dV$$

Note that $d\mathcal{M} = \rho dV$.



Balance of Linear Momentum

- The variation of the linear momentum over time is equal to the resultant force acting on the material volume:

$$\frac{d\vec{\mathcal{P}(t)}}{dt} = \frac{d}{dt}\int\int\int_V \rho\vec{\boldsymbol{v}}dV = \vec{\boldsymbol{R}}(t)$$

- The resultant forces consist of body $\vec{\boldsymbol{b}}$ and surface $\vec{\boldsymbol{t}}$ forces:

$$\vec{\boldsymbol{R}}(t) = \int\int\int_V \rho\vec{\boldsymbol{b}}dV + \int\int_{\partial V} \vec{\boldsymbol{t}}dS$$

- Special case: for a system in equilibrium $\vec{\boldsymbol{R}} = 0, \forall t$, linear momentum is conserved:

$$\frac{d\vec{\mathcal{P}}(t)}{dt} = 0 \Rightarrow \vec{\mathcal{P}}(t) = \text{constant}$$

Balance of Linear Momentum - Global form

- The global form of the linear momentum principle can be given as follows:

$$\vec{\boldsymbol{R}}(t) = \int\int\int_{\Delta V} \rho \vec{\boldsymbol{b}} dV + \int\int_{\partial \Delta V} \vec{\boldsymbol{t}} dS = \frac{d}{dt}\int\int\int_{\Delta V} \rho \vec{\boldsymbol{v}} dV = \frac{d\vec{\mathcal{P}}(t)}{dt}, \quad \forall \Delta V \in V, \quad \forall$$

- By using the Divergence Theorem to transform a surface integral to volume one, and using $\vec{\boldsymbol{t}} = \hat{\boldsymbol{n}} \cdot \underline{\underline{\boldsymbol{\sigma}}}$:

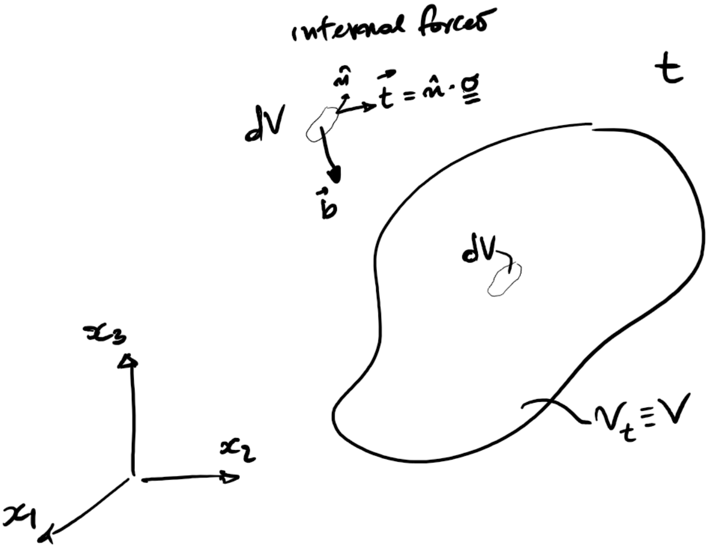
$$\int\int_{\partial \Delta V} \vec{\boldsymbol{t}} dS = \int\int_{\partial \Delta V} \hat{\boldsymbol{n}} \cdot \underline{\underline{\boldsymbol{\sigma}}} dS = \int\int\int_{V} \vec{\boldsymbol{\nabla}} \cdot \underline{\underline{\boldsymbol{\sigma}}} dV$$

- The global form can be written as:

$$\int\int\int_{\Delta V} \rho \vec{\boldsymbol{b}} dV + \int\int_{\partial \Delta V} \vec{\boldsymbol{t}} dS =$$

$$\int\int\int_{\Delta V} (\rho \vec{\boldsymbol{b}} + \vec{\boldsymbol{\nabla}} \cdot \underline{\underline{\boldsymbol{\sigma}}}) dV = \frac{d}{dt}\int\int\int_{\Delta V} \rho \vec{\boldsymbol{v}} dV \quad \forall \Delta V \in V, \quad \forall t$$

Clarification



Balance of Linear Momentum - Local form

- Using the Reynolds Lemma to the global form:

$$\int\int\int_{\Delta V} (\vec{\boldsymbol{\nabla}} \cdot \underline{\underline{\boldsymbol{\sigma}}} + \rho \vec{\boldsymbol{b}}) dV = \frac{d}{dt} \int\int\int_{\Delta V} \rho \vec{\boldsymbol{v}} dV = \int\int\int_{\Delta V} \rho \frac{d\vec{\boldsymbol{v}}}{dt} dV, \quad \forall \Delta V \in V, \quad \forall t$$

- Localizing the linear momentum we obtain the local spatial form:

$$\Delta V \to dV(\vec{\boldsymbol{x}}, t)$$

$$\vec{\boldsymbol{\nabla}} \cdot \underline{\underline{\boldsymbol{\sigma}}}(\vec{\boldsymbol{x}}, t) + \rho \vec{\boldsymbol{b}}(\vec{\boldsymbol{x}}, t) = \rho \frac{d\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}, t)}{dt}$$

Cauchy's equation of motion

$$\vec{\boldsymbol{\nabla}} \cdot \underline{\underline{\boldsymbol{\sigma}}}(\vec{\boldsymbol{x}}, t) + \rho \vec{\boldsymbol{b}}(\vec{\boldsymbol{x}}, t) = \rho \frac{d\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}, t)}{dt}$$

Cauchy's equation - Equilibrium

$$\vec{\boldsymbol{\nabla}} \cdot \underline{\underline{\boldsymbol{\sigma}}}(\vec{\boldsymbol{x}}, t) + \rho \vec{\boldsymbol{b}}(\vec{\boldsymbol{x}}, t) = 0$$

Balance of Angular Momentum

Angular Momentum in Classical Mechanics

- The 2^{nd} Newton's law for a discrete system of $n$particles states that the resulting force acting on a system of forces will be:

$$\vec{\boldsymbol{M}}_O(t) = \sum_{i=1}^{n} \vec{\boldsymbol{r}}_i \times \vec{\boldsymbol{f}}_i = \sum_{i=1}^{n} \vec{\boldsymbol{r}}_i \times m_i \frac{d\vec{\boldsymbol{v}}_i}{dt} = \frac{d}{dt} \sum_{i=1}^{n} \vec{\boldsymbol{r}}_i \times m_i \vec{\boldsymbol{v}}_i = \frac{d\vec{\mathcal{L}}(t)}{dt}$$

$$\vec{\boldsymbol{M}}_O(t) = \frac{d\vec{\mathcal{L}}(t)}{dt}$$

- Special case: for a system in equilibrium $\vec{\boldsymbol{M}}_0 = 0, \forall t$:

$$\frac{d\vec{\mathcal{L}}(t)}{dt} = 0 \Rightarrow \vec{\mathcal{L}}(t) = \text{constant}$$

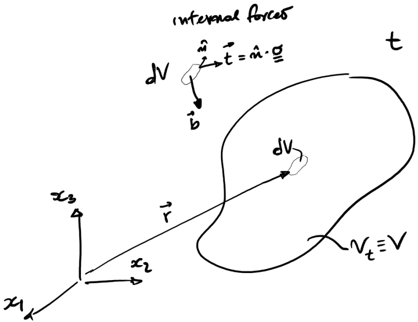
Conservation of angular momentum.

Angular momentum in Continuum Mechanics

- The angular momentum of a material volume V of a continuum medium with mass $\mathcal{M}$ is :

$$\vec{\mathcal{L}}(t) = \int\int\int_{\mathcal{M}} \vec{\boldsymbol{r}} \times \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}, t) d\mathcal{M} = \int\int\int_{V} \vec{\boldsymbol{x}} \times \rho(\vec{\boldsymbol{x}}, t)\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}, t) dV$$

Note that $d\mathcal{M} = \rho dV$.



Balance of Angular Momentum

- The variation of the linear momentum over time is equal to the resultant force acting on the material volume:

$$\frac{d\mathcal{L}(t)}{dt} = \frac{d}{dt}\int\int\int_V \vec{\boldsymbol{r}} \times \rho\vec{\boldsymbol{v}}dV = \vec{\boldsymbol{M}}_0(t)$$

- The torque consists of torque due to body $\vec{\boldsymbol{b}}$ and surface $\vec{\boldsymbol{t}}$ forces:

$$\vec{\boldsymbol{M}}_0(t) = \int\int\int_V \vec{\boldsymbol{r}} \times \rho\vec{\boldsymbol{b}}dV + \int\int_{\partial V} \vec{\boldsymbol{r}} \times \vec{\boldsymbol{t}}dS$$

Balance of Angular Momentum

- The global form of the angular momentum principle can be given as follows:

$$\int\int\int_V \vec{\boldsymbol{r}} \times \rho\vec{\boldsymbol{b}}dV + \int\int_{\partial V} \vec{\boldsymbol{r}} \times \vec{\boldsymbol{t}}dS = \frac{d}{dt}\int\int\int_V \vec{\boldsymbol{r}} \times \rho\vec{\boldsymbol{v}}dV$$

- By using the Divergence Theorem to transform a surface integral to volume one, and using $\vec{\boldsymbol{t}} = \hat{\boldsymbol{n}} \cdot \underline{\underline{\boldsymbol{\sigma}}}$:

$$\int\int_{\partial V} \vec{\boldsymbol{r}}\times\vec{\boldsymbol{t}}dS = \int\int_{\partial V} \vec{\boldsymbol{r}}\times\hat{\boldsymbol{n}}\cdot\underline{\underline{\boldsymbol{\sigma}}}dS = \int\int_{\partial V} (\vec{\boldsymbol{r}}\times\underline{\underline{\boldsymbol{\sigma}}}^T)\cdot\hat{\boldsymbol{n}}dS = \int\int\int_V (\vec{\boldsymbol{r}}\times\underline{\underline{\boldsymbol{\sigma}}}^T)\cdot\vec{\boldsymbol{\nabla}}dV$$

Balance of Angular Momentum - Proof

Note that $\vec{\boldsymbol{r}} \equiv \vec{\boldsymbol{x}}$:

$$\left[(\vec{\boldsymbol{r}} \times \underline{\underline{\boldsymbol{\sigma}}}^T) \cdot \vec{\boldsymbol{\nabla}}\right]_i \equiv (\epsilon_{ijk} x_j \underbrace{\sigma_{rk}}_{\sigma^T_{kr}}) \frac{\partial}{\partial x_r} = \frac{\partial}{\partial x_r}(\epsilon_{ijk} x_j \sigma_{rk}) = \epsilon_{ijk} \underbrace{\frac{\partial x_j}{\partial x_r}}_{\delta_{jr}} \sigma_{rk} + \underbrace{\epsilon_{ijk} x_j \frac{\partial \sigma_{rk}}{\partial x_r}}_{[\vec{\boldsymbol{r}} \times \vec{\boldsymbol{\nabla}} \cdot \underline{\underline{\boldsymbol{\sigma}}}]_i} \Leftrightarrow$$

$$\left[(\vec{\boldsymbol{r}} \times \underline{\underline{\boldsymbol{\sigma}}}^T) \cdot \vec{\boldsymbol{\nabla}}\right]_i = \underbrace{\epsilon_{ijk} \sigma_{jk}}_{m_i} + [\vec{\boldsymbol{r}} \times \vec{\boldsymbol{\nabla}} \cdot \underline{\underline{\boldsymbol{\sigma}}}]_i = [\vec{\boldsymbol{r}} \times \vec{\boldsymbol{\nabla}} \cdot \underline{\underline{\boldsymbol{\sigma}}}]_i + m_i$$

Balance of Angular Momentum - Global form

We use the Reynolds lemma of the right-hand side of global form equation:

$$\frac{d}{dt}\int\int\int_V \vec{\boldsymbol{r}} \times \rho \vec{\boldsymbol{v}} dV = \frac{d}{dt}\int\int\int_V \rho(\vec{\boldsymbol{r}} \times \vec{\boldsymbol{v}}) dV = \int\int\int_V \rho \frac{d}{dt}(\vec{\boldsymbol{r}} \times \vec{\boldsymbol{v}}) dV =$$

$$\int\int\int_V \rho \left(\underbrace{\frac{d\vec{\boldsymbol{r}}}{dt}}_{=\vec{\boldsymbol{v}}} \times \vec{\boldsymbol{v}} + \vec{\boldsymbol{r}} \times \frac{d\vec{\boldsymbol{v}}}{dt} \right) dV = \int\int\int_V \vec{\boldsymbol{r}} \times \rho \frac{d\vec{\boldsymbol{v}}}{dt} dV$$

The global form can finally be written as:

$$\int\int\int_V \left[\vec{\boldsymbol{r}} \times \left(\rho \vec{\boldsymbol{b}} + \vec{\boldsymbol{\nabla}} \cdot \underline{\underline{\boldsymbol{\sigma}}} \right) + \epsilon_{ijk}\sigma_{jk}\hat{\boldsymbol{e}}_i \right] dV = \int\int\int_V \vec{\boldsymbol{r}} \times \rho \frac{d\vec{\boldsymbol{v}}}{dt} dV$$

Balance of Angular Momentum - Local form

The local form can be written as:

$$\iiint_V \left[\vec{\boldsymbol{r}} \times \underbrace{\left(\rho \vec{\boldsymbol{b}} + \vec{\boldsymbol{\nabla}} \cdot \underline{\underline{\boldsymbol{\sigma}}} - \rho \frac{d\vec{\boldsymbol{v}}}{dt} \right)}_{=0 \text{ Cauchy's equation}} + \vec{\boldsymbol{m}} \right] dV = \iiint_V \vec{\boldsymbol{m}} dV = 0$$

Finally we have:

$$\vec{\boldsymbol{m}}(\vec{\boldsymbol{x}}, t) = \vec{\boldsymbol{0}} \Rightarrow m_i = \epsilon_{ijk}\sigma_{jk} = 0, \quad i, j, k \in \{1, 2, 3\}, \quad \forall \vec{\boldsymbol{x}} \in V, \quad \forall t$$

The proof of symmetry of stress tensor $\underline{\underline{\boldsymbol{\sigma}}}$:

$$i = 1 \rightarrow \epsilon_{123}\sigma_{23} + \epsilon_{132}\sigma_{32} = 0 \Rightarrow \sigma_{23} = \sigma_{32}$$
$$i = 2 \rightarrow \epsilon_{213}\sigma_{13} + \epsilon_{231}\sigma_{31} = 0 \Rightarrow \sigma_{13} = \sigma_{31}$$
$$i = 3 \rightarrow \epsilon_{321}\sigma_{21} + \epsilon_{312}\sigma_{12} = 0 \Rightarrow \sigma_{21} = \sigma_{12}$$
$$\underline{\underline{\boldsymbol{\sigma}}}(\vec{\boldsymbol{x}}, t) = \underline{\underline{\boldsymbol{\sigma}}}^T(\vec{\boldsymbol{x}}, t), \quad \forall \vec{\boldsymbol{x}} \in V, \quad \forall t$$

Power

Power

- Power $W(t)$ can be defined as the work (force$\times$distance) per unit of time!
- Power is scalar quantity.
- In some cases, the power can be expressed as the time derivative of the function energy $E(t)$:

$$W(t) = \frac{dE(t)}{dt}$$

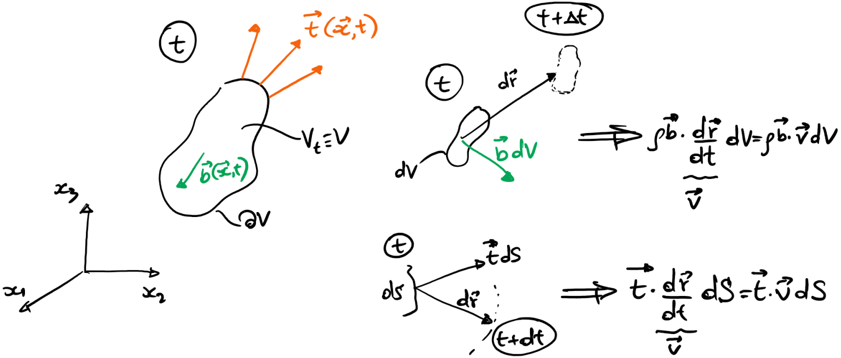
- The continuum medium absorbs the following types of power from the exterior:
  - Mechanical Power: implemented through mechanical actions, namely body and surface forces on the medium.
  - Thermal Power: heat entering the medium.

Mechanical Power

Mechanical Power

- The mechanical power is the word implemented due to body and surface forces per unit of time. In spatial form we can write:

$$P_e(t) = \int\int\int_V \rho \vec{b} \cdot \vec{v} dV + \int\int_{\partial V} \vec{t} \cdot \vec{v} dS$$



Mechanics Energy Balance

We know that $\vec{\boldsymbol{t}} = \hat{\boldsymbol{n}} \cdot \underline{\underline{\boldsymbol{\sigma}}}$ and through the divergence theorem we get:

$$\int\int_{\partial V} \vec{\boldsymbol{t}} \cdot \vec{\boldsymbol{v}} dS = \int\int_{\partial V} \hat{\boldsymbol{n}} \cdot (\underline{\underline{\boldsymbol{\sigma}}} \cdot \vec{\boldsymbol{v}}) dS =$$

$$= \int\int\int_{V} \vec{\boldsymbol{\nabla}} \cdot (\underline{\underline{\boldsymbol{\sigma}}} \cdot \vec{\boldsymbol{v}}) dV = \int\int\int_{V} \left[(\vec{\boldsymbol{\nabla}} \cdot \underline{\underline{\boldsymbol{\sigma}}}) \cdot \vec{\boldsymbol{v}} + \underline{\underline{\boldsymbol{\sigma}}} : \underbrace{(\vec{\boldsymbol{\nabla}} \vec{\boldsymbol{v}})}_{\underline{\underline{\boldsymbol{l}}}} \right] dV$$

where $\underline{\underline{\boldsymbol{l}}}$ is the spatial velocity gradient tensor.

- The identity holds: $\underline{\underline{\boldsymbol{l}}} = \underbrace{\frac{1}{2}\left(\underline{\underline{\boldsymbol{l}}} + \underline{\underline{\boldsymbol{l}}}^T\right)}_{\underline{\underline{\boldsymbol{d}}}} + \underbrace{\frac{1}{2}\left(\underline{\underline{\boldsymbol{l}}} - \underline{\underline{\boldsymbol{l}}}^T\right)}_{\underline{\underline{\boldsymbol{w}}}}$, where $\underline{\underline{\boldsymbol{d}}}$ is the symmetric part and $\underline{\underline{\boldsymbol{w}}}$ is the skew symmetric part:

$$\underline{\underline{\boldsymbol{\sigma}}} : \underline{\underline{\boldsymbol{l}}} = \underline{\underline{\boldsymbol{\sigma}}} : \underline{\underline{\boldsymbol{d}}} + \underbrace{\underline{\underline{\boldsymbol{\sigma}}} : \underline{\underline{\boldsymbol{w}}}}_{=0}$$

Mechanical Energy Balance

- Finally we conclude that:

$$\int\int_{\partial V} \vec{\boldsymbol{t}} \cdot \vec{\boldsymbol{v}} dS = \int\int\int_V (\vec{\boldsymbol{\nabla}} \cdot \underline{\underline{\boldsymbol{\sigma}}}) \cdot \vec{\boldsymbol{v}} dV + \int\int\int_V \underline{\underline{\boldsymbol{\sigma}}} : \underline{\underline{\boldsymbol{d}}} dV$$

Mechanical Energy Balance

- Collecting the terms the external mechanical power in spatial form can be:

$$
\begin{aligned}
P_e(t) &= \int\int\int_V \rho\vec{\boldsymbol{b}} \cdot \vec{\boldsymbol{v}} dV + \int\int\int_V (\vec{\boldsymbol{\nabla}} \cdot \underline{\underline{\boldsymbol{\sigma}}}) \cdot \vec{\boldsymbol{v}} dV + \int\int\int_V \underline{\underline{\boldsymbol{\sigma}}} : \underline{\underline{\boldsymbol{d}}} dV \\
&= \int\int\int_V \underbrace{\left(\rho\vec{\boldsymbol{b}} + \vec{\boldsymbol{\nabla}} \cdot \underline{\underline{\boldsymbol{\sigma}}}\right)}_{\rho\frac{d\vec{\boldsymbol{v}}}{dt}} \cdot \vec{\boldsymbol{v}} dV + \int\int\int_V \underline{\underline{\boldsymbol{\sigma}}} : \underline{\underline{\boldsymbol{d}}} dV \\
&= \frac{d}{dt} \int\int\int_V \rho \left(\frac{1}{2}\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}}\right) dV + \int\int\int_V \underline{\underline{\boldsymbol{\sigma}}} : \underline{\underline{\boldsymbol{d}}} dV
\end{aligned}
$$

Mechanical Energy Balance

$$P_e(t) = \frac{d}{dt}\int\int\int_V \rho\vec{\boldsymbol{b}}\cdot\vec{\boldsymbol{v}}dV + \int\int_{\partial V}\vec{\boldsymbol{t}}\cdot\vec{\boldsymbol{v}}dS = \underbrace{\int\int\int_V \rho\left(\frac{1}{2}\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{v}}\right)dV}_{K} + \underbrace{\int\int\int_V \underline{\underline{\boldsymbol{\sigma}}} : \underline{\underline{\boldsymbol{d}}}dV}_{P_\sigma}$$

where $P_e(t)$ is the external power entering the body, K is the kinetic energy and P_σ is the stress power.
Finally:

$$P_e(t) = \frac{d}{dt}K(t) + P_\sigma$$

The stress power P_σ is the part of mechanical power entering the system, [1], that does not change the kinetic energy. It is the work done by the stress per unit of time in the deformation process of the medium.

Thermal Power

External Thermal Power

Two ways to introduce heat in a continuum medium:

- Internal heat sources:

$$\int\int\int_V \rho \, r(\vec{\boldsymbol{x}}, t) \; dV = \frac{\text{heat generated by internal sources}}{\text{unit of time}}$$

- Non-convective heat transfer across the volume's surface:

$$-\int\int_{\partial V} \vec{\boldsymbol{q}}(\vec{\boldsymbol{x}}, t) \cdot \hat{\boldsymbol{n}} \; dS = \frac{\text{incoming heat}}{\text{unit of time}}$$

where $\vec{\boldsymbol{q}}$ is the heat conduction flux vector.

External Thermal Power

The external thermal power is heat coming in the medium per unit of time

- In spatial form:

$$Q_e(t) = \int\int\int_V \rho\ r\ dV - \int\int_{\partial V} \vec{\boldsymbol{q}} \cdot \hat{\boldsymbol{n}} dS = \int\int\int_V (\rho r - \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{q}}) dV$$

- $\vec{\boldsymbol{q}}(\vec{\boldsymbol{x}}, t)$ is the non-convective heat flux vector per unit of spatial surface
- $r(\vec{\boldsymbol{x}}, t)$ is the internal heat source rate per unit of mass

Total Power

The total power entering the continuum medium in spatial form:

$$P_e + Q_e = \int\int\int_V \rho \left(\frac{1}{2} \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}} \right) dV + \int\int\int_V \underline{\underline{\boldsymbol{\sigma}}} : \underline{\underline{\boldsymbol{d}}} \, dV + \int\int\int_V \rho \, r \, dV - \int\int_{\partial V} \vec{\boldsymbol{q}} \cdot \hat{\boldsymbol{n}} \, dS$$

First Law of Thermodynamics

First Law of Thermodynamics

Postulates:

1. There exist a function $E(t)$, the total energy of the system, with its material derivative equal to the total power entering teh system.

$$\frac{d}{dt}E(t) = P_e(t) + Q_e(t)$$

$$= \frac{d}{dt}\int\int\int_V \rho\left(\frac{1}{2}\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{v}}\right)dV + \int\int\int_V \underline{\underline{\boldsymbol{\sigma}}} : \underline{\underline{\boldsymbol{d}}}\; dV + \int\int\int_V \rho\; r\; dV - \int\int_{\partial V} \vec{\boldsymbol{q}}\cdot\hat{\boldsymbol{n}}\; dS$$

2. There is a function $U(t)$, the internal energy of the system, which which can be defined in terms of the specific internal energy $u(\vec{\boldsymbol{x}}, t)$:

$$U(t) = \int\int\int_V \rho\; u\; dV$$

The variation of the total energy of the system is:

$$\frac{d}{dt}E(t) = \frac{d}{dt}K(t) + \frac{d}{dt}U(t)$$

Global Form of the Internal Energy Balance

The expression for the total power into the first postulate gives:

$$\frac{d}{dt}E(t) = \underbrace{\frac{d}{dt}\int\int\int_V \rho\left(\frac{1}{2}\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{v}}\right)dV}_{K} + \int\int\int_V \underline{\underline{\boldsymbol{\sigma}}} : \underline{\underline{\boldsymbol{d}}}\, dV + \int\int\int_V \rho\, r\, dV - \int\int_{\partial V} \vec{\boldsymbol{q}}\cdot\hat{\boldsymbol{n}}\, dS$$

Therefore, the internal energy balance in global form can be formulated as follows:

$$\frac{d}{dt}U(t) = \frac{d}{dt}\int\int\int_V \rho\; u\; dV$$

$$= \underbrace{\int\int\int_V \underline{\underline{\boldsymbol{\sigma}}} : \underline{\underline{\boldsymbol{d}}}\; dV}_{\text{stress power } P_\sigma(t)} + \underbrace{\int\int\int_V \rho\; r\; dV - \int\int_{\partial V} \vec{\boldsymbol{q}}\cdot\hat{\boldsymbol{n}}\; dS}_{\text{external thermal power } Q_e(t)}$$

Local Form of the Internal Energy Balance

The internal energy of system will be formed as the internal energy balance in local form:

$$\rho \, \frac{du}{dt} = \underline{\underline{\boldsymbol{\sigma}}} : \underline{\underline{\boldsymbol{d}}} + \left(\rho \, r - \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{q}} \right), \quad \forall \vec{\boldsymbol{x}} \in V, \quad \forall t$$

The above equation is also know as the energy equation.
Note that u is the internal energy.

Second Law of Thermodynamics

Second Law of Thermodynamics

Postulates:

1. There exists a state function $\theta(\vec{\boldsymbol{x}}, t)$, absolute temperature, which is ALWAYS positive!!
2. A state function S exists that is called the entropy and has the following properties:
   - It can be defined in terms of a specific entropy (entropy per unit mass s):

$$S(t) = \int\int\int_V \rho \; s(\vec{\boldsymbol{x}}, t) \; dV$$

   - Fulfills the inequality:

$$\frac{d}{dt} S(t) = \frac{d}{dt} \int\int\int_V \rho \; s \; dV \geq \int\int\int_V \rho \frac{r}{\theta} \; dV - \int\int_{\partial V} \frac{\vec{\boldsymbol{q}}}{\theta} \cdot \hat{\boldsymbol{n}} \; dS$$

The last equation is the global form of the 2^{nd} Law of Thermodynamics.

Second Law of Thermodynamics

The rate of the total entropy of the system is equal or greater to the rate of heat per unit of temperature:

$$\frac{d}{dt}S(t) = \frac{d}{dt}\int\int\int_V \rho\ s\ dV \geq \int\int\int_V \rho\frac{r}{\theta}\ dV - \int\int_{\partial V}\frac{\vec{\boldsymbol{q}}}{\theta}\cdot\hat{\boldsymbol{n}}\ dS$$

$$Q_e(t) = \int\int\int_V \rho\ r\ dV - \int\int_{\partial V}\vec{\boldsymbol{q}}\cdot\hat{\boldsymbol{n}}\ dS$$

$$\Gamma_e(t) = \int\int\int_V \rho\frac{r}{\theta}\ dV - \int\int_{\partial V}\frac{\vec{\boldsymbol{q}}}{\theta}\cdot\hat{\boldsymbol{n}}\ dS$$

where $Q_e(t)$ is the rate of the total amount of the heat per unit time entering he system and $\Gamma_e(t)$ is the rate of the total heat per unit of absolute temperature per unit of time entering into the system.

Second Law of Thermodynamics

- The entropy S can be split into two parts:
  - $S^{(i)}$ is the part generated internally
  - $S^{(e)}$ is the part generated through the interaction with the exterior
- The second law of thermodynamics states that the $S^{(i)}$ never decreases with time.

Second Law of Thermodynamics

- The local spatial form of the second law of thermodynamics can be written as:

$$\rho \frac{ds^{(i)}}{dt} = \rho \frac{ds}{dt} - \left(\rho \frac{r}{\theta} - \vec{\boldsymbol{\nabla}} \cdot \left(\frac{\vec{\boldsymbol{q}}}{\theta} \right) \right) \geq 0, \quad \forall \vec{\boldsymbol{x}} \in V, \quad \forall t$$

- The equation above is also called the Cluasius-Duhem inequality.

Second Law of Thermodynamics

The Clausious-Plank inequality states that:

$$\left(\dot{s} - \frac{r}{\theta} + \frac{1}{\rho\theta}\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{q}}\right) \geq 0$$

The heat flow inequality states that:

$$-\frac{1}{\rho\theta^2}\vec{\boldsymbol{q}} \cdot \vec{\boldsymbol{\nabla}}\theta \geq 0$$

The internal generated entropy can be by local sources, first equation, of by thermal heat conduction, second equation.
An alternative form of the Clausius-Planck inequality in terms of the specific internal energy u can be written as:

$$-\rho(\dot{u} - \theta\dot{s}) + \underline{\underline{\boldsymbol{\sigma}}} : \underline{\underline{\boldsymbol{d}}} \geq 0$$

Governing Equations

Governing Equations in Spatial Form

$\dot{\rho} + \rho\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{v}} = 0$	Conservation of Mass Continuity Equation	1 eqn.	PDE
$\vec{\boldsymbol{\nabla}} \cdot \underline{\underline{\boldsymbol{\sigma}}} + \rho\vec{\boldsymbol{b}} = \rho\dot{\vec{\boldsymbol{v}}}$	Linear Momentum Balance Cauchy's Equation of motion	3 eqns.	PDE
$\underline{\underline{\boldsymbol{\sigma}}} = \underline{\underline{\boldsymbol{\sigma}}}^T$	Angular Momentum Balance Symmetry of Cauchy stress tensor	3 eqns.	ALG
$\rho\dot{u} = \underline{\underline{\boldsymbol{\sigma}}} : \underline{\underline{\boldsymbol{d}}} + \rho r - \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{q}}$	Energy Balance First Law of Thermodynamics	1 eqn.	PDE
$-\rho(\dot{u} - \theta\dot{s}) + \underline{\underline{\boldsymbol{\sigma}}} : \underline{\underline{\boldsymbol{d}}} \geq 0$ $-\frac{1}{\rho\theta^2}\vec{\boldsymbol{q}} \cdot \vec{\boldsymbol{\nabla}}\theta \geq 0$	Second Law of Thermodynamics Clausius-Plank Inequality Heat Flow Inequality	2 restrictions	PDE

PDE-Partial Differential Equation
ALG-ALGebraic Equation
In total 8 equations and 2 restrictions
General Equations for all materials

Governing Equations in Spatial Form

The unknown variables (scalars):

ρ	Density	1 var.
$\vec{\boldsymbol{v}}$	Velocity Vector Field	3 vars.
$\underline{\underline{\boldsymbol{\sigma}}}$	Cauchy's stress tensor field	9(6) vars.
u	Specific Internal Energy	1 var.
$\vec{\boldsymbol{q}}$	Heat flux per unit of surface vector field	3 vars.
θ	Absolute Temperature	1 var.
s	Specific Entropy	1 var.

In total 19 unknowns

11 equations are missing

Set of boundary conditions

Constitutive Equations in Spatial Form

The missing equations:

$\underline{\underline{\boldsymbol{\sigma}}} = \underline{\underline{\boldsymbol{\sigma}}}(\vec{\boldsymbol{v}}, \theta, \zeta)$	Thermo-Mechanical Constitutive Equations	6 eqns.
$s = s(\vec{\boldsymbol{v}}, \theta, \zeta)$	Entropy Constitutive Equation	1 eqn.
$\vec{\boldsymbol{q}} = \vec{\boldsymbol{q}}(\vec{\boldsymbol{v}}, \theta) = -k\vec{\boldsymbol{\nabla}}\theta$	Thermal Consitutive Equation Fourier's Law of Conduction	3 eqns.
$u = v(\rho, \vec{\boldsymbol{v}}, \theta, \zeta)$ $F_i(\rho, \theta, \zeta); \quad i \in \{1, 2, .., p\}$	Heat State Equation Kinetic State Equation	(1+p) eqns.

In total 19+p Equations

In total 19+p Unknowns

ζ denotes thermodynamic variables

Equations specific to each material

The strain tensor $\underline{\underline{\boldsymbol{\varepsilon}}}$ is not unknown, because it can be derived through equations of motion, namely $\underline{\underline{\boldsymbol{\varepsilon}}} = \underline{\underline{\boldsymbol{\varepsilon}}}(\vec{\boldsymbol{v}})$

Coupled Thermo-Mechanical Problem

$\dot{\rho} + \rho \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{v}} = 0$	Conservation of Mass Continuity Equation	1 eqn.
$\vec{\boldsymbol{\nabla}} \cdot \underline{\underline{\boldsymbol{\sigma}}} + \rho \vec{\boldsymbol{b}} = \rho \dot{\vec{\boldsymbol{v}}}$	Linear Momentum Balance Cauchy's Equation of motion	3 eqns.
$\underline{\underline{\boldsymbol{\sigma}}} = \underline{\underline{\boldsymbol{\sigma}}}(\underline{\underline{\boldsymbol{\varepsilon}}}\vec{\boldsymbol{v}}, \theta)$	Mechanical Consitituve Equations	6 eqns.
$\rho \dot{u} = \underline{\underline{\boldsymbol{\sigma}}} : \underline{\underline{\boldsymbol{d}}} + \rho r - \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{q}}$	Energy Balance First Law of Thermodynamics	1 eqn.
$-\rho(\dot{u} - \theta \dot{s}) + \underline{\underline{\boldsymbol{\sigma}}} : \underline{\underline{\boldsymbol{d}}} \geq 0$ $-\frac{1}{\rho \theta^2} \vec{\boldsymbol{q}} \cdot \vec{\boldsymbol{\nabla}} \theta \geq 0$	Second Law of Thermodynamics Clausius-Plank Inequality Heat Flow Inequality	2 restrictions

The Uncoupled Thermo-Mechanical Problem

The mechanical and thermal problem can be uncoupled if the temperature distribution $\theta(\vec{\boldsymbol{x}}, t)$ is known a priori or does not get involved in the constitutive equations
Then the mechanical problem can be solved independently.

Uncoupled Thermo-Mechanical Problem

Mechanical Problem:

$\dot{\rho} + \rho \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{v}} = 0$	Conservation of Mass Continuity Equation	1 eqn.
$\vec{\boldsymbol{\nabla}} \cdot \underline{\underline{\boldsymbol{\sigma}}} + \rho \vec{\boldsymbol{b}} = \rho \dot{\vec{\boldsymbol{v}}}$	Linear Momentum Balance Cauchy's Equation of motion	3 eqns.
$\underline{\underline{\boldsymbol{\sigma}}} = \underline{\underline{\boldsymbol{\sigma}}}(\underline{\underline{\boldsymbol{\varepsilon}}}\vec{\boldsymbol{v}}, \theta)$	Mechanical Consitituve Equations	6 eqns.

Thermal Problem:

$\rho \dot{u} = \underline{\underline{\boldsymbol{\sigma}}} : \underline{\underline{\boldsymbol{d}}} + \rho r - \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{q}}$	Energy Balance First Law of Thermodynamics	1 eqn.
$-\rho(\dot{u} - \theta \dot{s}) + \underline{\underline{\boldsymbol{\sigma}}} : \underline{\underline{\boldsymbol{d}}} \geq 0$ $-\frac{1}{\rho \theta^2} \vec{\boldsymbol{q}} \cdot \vec{\boldsymbol{\nabla}} \theta \geq 0$	Second Law of Thermodynamics Clausius-Plank Inequality Heat Flow Inequality	2 restrictions

Mechanical problem: 10 equations

Uncoupled Thermo-Mechanical Problem

Mechanical variables:

ρ	Density	1 var.
$\vec{\boldsymbol{v}}$	Velocity Vector Field	3 vars.
$\underline{\underline{\boldsymbol{\sigma}}}$	Cauchy's stress tensor field	9(6) vars.

Thermal variables:

u	Specific Internal Energy	1 var.
$\vec{\boldsymbol{q}}$	Heat flux per unit of surface vector field	3 vars.
θ	Absolute Temperature	1 var.
s	Specific Entropy	1 var.

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