Nguyen Xuan Binh 887799 Assignment Week 1

Exercise 1: Prove that for any vectors $\vec{u}$, $\vec{v}$, the following holds true

$$|\vec{u}-\vec{v}|^2 + |\vec{u}+\vec{v}|^2 = 2(|\vec{u}|^2 + |\vec{v}|^2)$$

We have $\vec{u}^2 = |\vec{u}||\vec{u}|\cos 0° = |\vec{u}|^2 \Leftrightarrow |\vec{u}|^2 = \vec{u}^2$

$$\Rightarrow |\vec{u}-\vec{v}|^2 + |\vec{u}+\vec{v}|^2 = (\vec{u}-\vec{v})(\vec{u}-\vec{v}) + (\vec{u}+\vec{v})(\vec{u}+\vec{v})$$

$$= \vec{u}^2 - 2\vec{u}\vec{v} + \vec{v}^2 + \vec{u}^2 + 2\vec{u}\vec{v} + \vec{v}^2$$

$$= 2(\vec{u}^2 + \vec{v}^2)$$

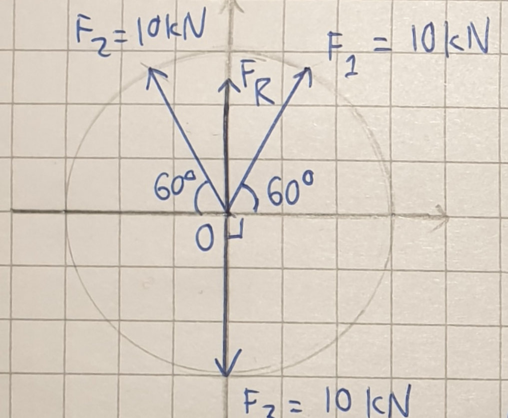
$$= 2(|\vec{u}|^2 + |\vec{v}|^2) \text{ (proven)}$$

Exercise 2: Find a unit vector in the direction of $\vec{u} = -3\hat{e}_1 + 4\hat{e}_2 + 5\hat{e}_3$

Magnitude of $\vec{u}$: $|\vec{u}| = \sqrt{(-3)^2 + 4^2 + 5^2} = 5\sqrt{2}$

$\Rightarrow$ unit vector: $\vec{v} = \dfrac{\vec{u}}{|\vec{u}|} = -\dfrac{3}{5\sqrt{2}}\hat{e}_1 + \dfrac{2\sqrt{2}}{5}\hat{e}_2 + \dfrac{\sqrt{2}}{2}\hat{e}_3$

Exercise 3: Find the magnitude and direction of resultant force of 3 coplanar forces in the xy plane with unit vectors $\hat{e}_1, \hat{e}_2$ of 10 kN each acting outward on a body at the origin and making angles of $60°, 120°, 270°$ with x-axis with unit vector equal to $\hat{e}_1$



The components of the resultant force are

▫ $(F_R)_x = F_1 \cos 60° - F_2 \cos 60°$
$= 10 \cos 60° - 10 \cos 60° = 0$

▫ $(F_R)_y = F_1 \sin 60° + F_2 \sin 60° - F_3$
$= 10 \sin 60° + 10 \sin 60° - 10$
$= 10\sqrt{3} - 10$

$\Rightarrow$ Magnitude of $\vec{F_R} = 10\sqrt{3} - 10$

Orientation of F_R respecting to y-axis

$\theta = \tan^{-1}\left(\dfrac{(F_R)_x}{(F_R)_y}\right) = \tan^{-1}\left(\dfrac{0}{10\sqrt{3}-10}\right) = 0°$

$\Rightarrow$ F_R is pointing upwards along y-axis and making angle of $90°$ with x-axis

Exercise 4: Find equation of the plane going through $A(1,0,2)$, $B(0,1,-1)$, $C(2,2,3)$

We have: $\vec{AB} = (-1, 1, -3)$ $\vec{AC} = (1, 2, 1)$

$\Rightarrow$ normal vector to plane $= \vec{AB} \times \vec{AC} = (7, -2, -3)$

$\Rightarrow$ plane: $7x - 2y - 3z + d = 0$ (*)

Replace $A(1,0,2)$ into (*) $\Rightarrow d = -1$

$\Rightarrow$ The plane passing through 3 points A, B, C is $7x - 2y - 3z - 1 = 0$

Exercise 5: Prove $\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$

We have: $\varepsilon_{ijk} = \hat{e}_1 \cdot (\hat{e}_j \times \hat{e}_k) = \det(\hat{e}_i, \hat{e}_j, \hat{e}_k)$

$\Rightarrow \varepsilon_{ijk}\varepsilon_{imn} = \det(\hat{e}_i, \hat{e}_j, \hat{e}_k)\det\begin{pmatrix} e_i^T \\ e_m^T \\ e_n^T \end{pmatrix} = \det\begin{pmatrix} e_i^T e_i & e_i^T e_j & e_i^T e_k \\ e_m^T e_i & e_m^T e_j & e_m^T e_k \\ e_n^T e_i & e_n^T e_j & e_n^T e_k \end{pmatrix}$

$= \det\begin{pmatrix} \delta_{ii} & \delta_{ij} & \delta_{ik} \\ \delta_{mi} & \delta_{mj} & \delta_{mk} \\ \delta_{ni} & \delta_{nj} & \delta_{nk} \end{pmatrix} = \det\begin{pmatrix} 1 & 0 & 0 \\ \delta_{mi} & \delta_{mj} & \delta_{mk} \\ \delta_{ni} & \delta_{nj} & \delta_{nk} \end{pmatrix}$

$= 1(\delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj}) - 0 + 0 = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$ (proven)

Exercise 6: Prove $\vec{\nabla} \times (\vec{\nabla} \times \vec{u}) = \vec{\nabla} \otimes (\vec{\nabla} \cdot \vec{u}) - (\vec{\nabla} \cdot \vec{\nabla})\vec{u}$

Using direct notation: According to triple products

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{u}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{u}) - (\vec{\nabla} \cdot \vec{\nabla})\vec{u}$$

According to properties of dyads: $\vec{u} \otimes (\vec{v} \cdot \vec{w}) = \vec{u}(\vec{v} \cdot \vec{w})$

$$\Rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{u}) = \vec{\nabla} \otimes (\vec{\nabla} \cdot \vec{u}) - (\vec{\nabla} \cdot \vec{\nabla})\vec{u} \text{ (proven)}$$

Using index notation: $\vec{\nabla} \times (\vec{\nabla} \times \vec{u}) = (\vec{\nabla}_i \hat{e}_i) \times (\epsilon_{jkl} \vec{\nabla}_j \vec{u}_k \hat{e}_l)$

$$= \epsilon_{jkl} \vec{\nabla}_i \vec{\nabla}_j \vec{u}_k (\hat{e}_i \times \hat{e}_l) = \epsilon_{jkl} \vec{\nabla}_i \vec{\nabla}_j \vec{u}_k \epsilon_{ilm} \hat{e}_m$$

$$= \epsilon_{jkl} \epsilon_{mil} \vec{\nabla}_i \vec{\nabla}_j \vec{u}_k \hat{e}_m = (\delta_{jm}\delta_{ki} - \delta_{ji}\delta_{km}) \vec{\nabla}_i \vec{\nabla}_j \vec{u}_k \hat{e}_m$$

$$= \delta_{jm}\delta_{ki} \vec{\nabla}_i \vec{\nabla}_j \vec{u}_k \hat{e}_m - \delta_{ji}\delta_{km} \vec{\nabla}_i \vec{\nabla}_j \vec{u}_k \hat{e}_m$$

$$= \vec{\nabla}_i \vec{\nabla}_j \vec{u}_i \hat{e}_j - \vec{\nabla}_i \vec{\nabla}_i \vec{u}_k \hat{e}_k$$

$$= (\vec{\nabla}_i \vec{u}_i)(\vec{\nabla}_j \hat{e}_j) - (\vec{\nabla}_i \vec{\nabla}_i)(\vec{u}_k \hat{e}_k)$$

According to outer / tensor product: $A = \vec{u} \otimes \vec{v} \Leftrightarrow A_{ij} = u_i v_j$

$$\Rightarrow \vec{\nabla} \otimes (\vec{\nabla} \cdot \vec{u}) = (\vec{\nabla}_j \hat{e}_j)(\vec{\nabla}_i \vec{u}_i)$$

$$\Rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{u}) = (\vec{\nabla}_i \vec{u}_i)(\vec{\nabla}_j \hat{e}_j) - (\vec{\nabla}_i \vec{\nabla}_i)(\vec{u}_k \hat{e}_k)$$

$$= \vec{\nabla} \otimes (\vec{\nabla} \cdot \vec{u}) - (\vec{\nabla} \cdot \vec{\nabla})\vec{u} \text{ (proven)}$$

The term $(\vec{\nabla} \cdot \vec{\nabla})$ is the Laplace operator

The term $\vec{\nabla} \times (\vec{\nabla} \times \vec{u})$ is called Curl(Curl $\vec{u}$))

The term $\vec{\nabla} \otimes (\vec{\nabla} \cdot \vec{u})$ is equivalent with $\vec{\nabla}(\vec{\nabla} \cdot \vec{u})$