Nguyen Xuan Binh 887799 Assignment Week 6

Exercise 1: In order for the displacement field $\vec{u}$ to be consistent with the Navier equations, define the constants C_1, C_2, C_3. The displacements $\vec{u}$ and the body forces $\vec{f}$ are given as follows

$$\vec{f} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} -6Gx_2x_3 \\ 2Gx_1x_3 \\ 10Gx_1x_2 \end{bmatrix} ; \quad \vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} C_1 x_1^2 x_2 x_3 \\ C_2 x_1 x_2^2 x_3 \\ C_3 x_1 x_2 x_3^2 \end{bmatrix}, \quad v = 1/4$$

The Navier equation in equilibrium: $(\lambda+\mu)\vec{\nabla}(\vec{\nabla}\cdot\vec{u}) + \mu\vec{\nabla}^2\vec{u} + f = \rho\dfrac{d\vec{u}}{dt}$

We have: $\dfrac{d\vec{u}}{dt} = 0$

$$\vec{\nabla}\vec{u} = \begin{bmatrix} 2C_1x_1x_2x_3 \\ 2C_2x_1x_2x_3 \\ 2C_3x_1x_2x_3 \end{bmatrix} \Rightarrow \vec{\nabla}^2\vec{u} = \begin{bmatrix} 2C_1x_2x_3 \\ 2C_2x_1x_3 \\ 2C_3x_1x_2 \end{bmatrix}$$

$$\vec{\nabla}\cdot\vec{u} = 2(C_1+C_2+C_3)x_1x_2x_3 \Rightarrow \vec{\nabla}(\vec{\nabla}\cdot\vec{u}) = \begin{bmatrix} 2(C_1+C_2+C_3)x_2x_3 \\ 2(C_1+C_2+C_3)x_1x_3 \\ 2(C_1+C_2+C_3)x_1x_2 \end{bmatrix}$$

$$\lambda + \mu = \frac{vE}{(1+v)(1-2v)} + \frac{E}{2(1+v)} = \frac{2vE}{2(1+v)(1-2v)} + \frac{E(1-2v)}{2(1+v)(1-2v)}$$

$$\Rightarrow \lambda+\mu = \frac{E}{2(1+v)(1-2v)} = \frac{G}{1-2v} \quad \text{and } \mu = G$$

$\Rightarrow$ Navier equation:
$$\frac{G}{1-2v}\begin{bmatrix} 2(C_1+C_2+C_3)x_2x_3 \\ 2(C_1+C_2+C_3)x_1x_3 \\ 2(C_1+C_2+C_3)x_1x_2 \end{bmatrix} + G\begin{bmatrix} 2C_1x_2x_3 \\ 2C_2x_1x_3 \\ 2C_3x_1x_2 \end{bmatrix} + G\begin{bmatrix} -6x_2x_3 \\ 2x_1x_3 \\ 10x_1x_2 \end{bmatrix} = 0$$

The equation has G in similar terms $\Rightarrow$ G can be omitted

$$\Rightarrow \frac{1}{1-2v} = \frac{1}{1-2\cdot(1/4)} = 2$$

$$\Rightarrow \begin{bmatrix} 4(C_1+C_2+C_3)x_2x_3 + 2C_1x_2x_3 - 6x_2x_3 \\ 4(C_1+C_2+C_3)x_1x_3 + 2C_2x_1x_3 + 2x_1x_3 \\ 4(C_1+C_2+C_3)x_1x_2 + 2C_3x_1x_2 + 10x_1x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} (6C_1+4C_2+4C_3-6)x_2x_3 \\ (4C_1+6C_2+4C_3+2)x_1x_3 \\ (4C_1+4C_2+6C_3+10)x_1x_2 \end{bmatrix} = 0 \Rightarrow \begin{cases} 6C_1+4C_2+4C_3-6=0 \\ 4C_1+6C_2+4C_3+2=0 \\ 4C_1+4C_2+6C_3+10=0 \end{cases}$$

In matrix notation:
$$\begin{bmatrix} 6 & 4 & 4 \\ 4 & 6 & 4 \\ 4 & 4 & 6 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} - \begin{bmatrix} 6 \\ -2 \\ -10 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 6 & 4 & 4 \\ 4 & 6 & 4 \\ 4 & 4 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ -2 \\ -10 \end{bmatrix} = \begin{bmatrix} 27/7 \\ -1/7 \\ -29/7 \end{bmatrix}$$

$\Rightarrow$ The displacement field is $\vec{u} = \begin{bmatrix} 27/7\, x_1^2 x_2 x_3 \\ -1/7\, x_1 x_2^2 x_3 \\ -29/7\, x_1 x_2 x_3^2 \end{bmatrix}$

Exercise 2: Use the Beltrami - Michell equations in index form, to write six independent equations for the case of constant body force.

The Beltrami - Michell equations in index form of constant body force is

$$\sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{kk,ij} = 0$$

Let $\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$ whose elements depend on displacement field : x_1, x_2, x_3

Since $\underline{\underline{\sigma}}$ is symmetric $\Rightarrow \sigma_{12} = \sigma_{21}$, $\sigma_{13} = \sigma_{31}$, $\sigma_{23} = \sigma_{32}$

$\Rightarrow ij \in \{11; 12; 23; 13; 22; 33\}$ and $kk \in \{11; 22; 33\}$. We notice that for each ij component, kk is repeated for 3 different equations of the same terms ij

$$\sigma_{11,11} + \frac{1}{1+\nu}\sigma_{11,11} = \frac{\partial^2}{\partial x_1^2}\sigma_{\underline{11}} + \frac{1}{1+\nu}\frac{1}{\partial x_{\underline{1}}^2}\sigma_{11} = 0$$

$$\sigma_{11,22} + \frac{1}{1+\nu}\sigma_{22,11} = \frac{\partial^2}{\partial x_2^2}\sigma_{\underline{11}} + \frac{1}{1+\nu}\frac{1}{\partial x_{\underline{1}}^2}\sigma_{22} = 0$$

$$\sigma_{11,33} + \frac{1}{1+\nu}\sigma_{33,11} = \frac{\partial^2}{\partial x_3^2}\sigma_{\underline{11}} + \frac{1}{1+\nu}\frac{1}{\partial x_{\underline{1}}^2}\sigma_{33} = 0$$

$$\Rightarrow \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)\sigma_{\underline{11}} + \frac{1}{1+\nu}\frac{1}{\partial x_{\underline{1}}^2}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = 0$$

$\sigma_{kk,**} = \sigma_{11} + \sigma_{22} + \sigma_{33}$ and $\Rightarrow \sigma_{**,kk} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$

The underlined index is ij-component

$\Rightarrow$ Repeat this process 6 times for each ij, we derive six independent equations as follows:

Eq 1: $$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)\sigma_{11} + \frac{1}{1+\nu}\frac{\partial^2}{\partial x_1^2}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = 0 \quad (ij = 11)$$

Eq 2: $$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)\sigma_{22} + \frac{1}{1+\nu}\frac{\partial^2}{\partial x_2^2}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = 0 \quad (ij = 22)$$

Eq 3: $$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)\sigma_{33} + \frac{1}{1+\nu}\frac{\partial^2}{\partial x_3^2}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = 0 \quad (ij = 33)$$

Eq 4: $$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)\sigma_{12} + \frac{1}{1+\nu}\frac{\partial^2}{\partial x_1 \partial x_2}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = 0 \quad (ij = 12)$$

Eq 5: $$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)\sigma_{23} + \frac{1}{1+\nu}\frac{\partial^2}{\partial x_2 \partial x_3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = 0 \quad (ij = 23)$$

Eq 6: $$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)\sigma_{13} + \frac{1}{1+\nu}\frac{\partial^2}{\partial x_1 x_3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = 0 \quad (ij = 13)$$

Exercise 3: Consider the case of a steady flow (at every fixed location nothing changes with time) of an incompressible viscous fluid in the direction of $\hat{e}_3$

a) Demonstrate that the velocity field is: $v_1 = 0$, $v_2 = 0$, $v_3 = v(x_1, x_2)$

The flow is steady, so nothing changes with time

$\Rightarrow \frac{d\rho}{dt} = 0$ and $\frac{dv}{dt} = 0 \Rightarrow v_i = v_i(x_1, x_2, x_3)$ (v is not a function of t)

The continuity equation: $\frac{d\rho}{dt} + \rho(\vec{\nabla} \cdot \vec{v}) = 0$. Since $\frac{d\rho}{dt} = 0$

$\Rightarrow \rho(\vec{\nabla} \cdot \vec{v}) = 0$. Density is non-zero $\Rightarrow \vec{\nabla} \cdot \vec{v} = 0$

$\Rightarrow \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = 0 \Rightarrow$ For $\frac{\partial v_i}{\partial x_i}$ to vanish, v_i must not contain x_i

$$\Rightarrow \frac{\partial v_1(x_2, x_3)}{\partial x_1} + \frac{\partial v_2(x_1, x_3)}{\partial x_2} + \frac{\partial v_3(x_1, x_2)}{\partial x_3} = 0$$

However, we know that the fluid flows in direction of $\hat{e}_3$, so there is no change of velocity in $\hat{e}_1$ and $\hat{e}_2$ direction $\Rightarrow v_1 = v_2 = 0$

Leaving the only component v_3 left $\Rightarrow v_3 = v(x_1, x_2)$

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b) For $v(x_1, x_2) = kx_2$, define the normal and shear stresses on the plane with normal in the direction of $\hat{e}_2 + \hat{e}_3$ as a function of the pressure p and the viscosity μ

Because the fluid is incompressible, its stress tensor can be given as

$\underline{\underline{\sigma}} = -p\underline{\underline{1}} + 2\mu\underline{\underline{d}}$ where $\underline{\underline{d}}$ is deformation rate tensor

We have: $\underline{\underline{d}} = \frac{1}{2}(\vec{\nabla}\vec{v} + (\vec{\nabla}\vec{v})^T)$

$$\vec{\nabla}\vec{v} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & k & 0 \end{bmatrix} \Rightarrow \underline{\underline{d}} = \frac{1}{2}(\vec{\nabla}\vec{v} + (\vec{\nabla}\vec{v})^T) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & k/2 \\ 0 & k/2 & 0 \end{bmatrix}$$

$$\Rightarrow \underline{\underline{\sigma}} = -p\underline{\underline{1}} + 2\mu\underline{\underline{d}} = -p\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2\mu\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & k/2 \\ 0 & k/2 & 0 \end{bmatrix} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & \mu \\ 0 & \mu & -p \end{bmatrix}$$

(k unit is $m^{-1}s^{-1}$ and μ unit is $m \cdot Pa \cdot s$)

Normal is in direction of $\hat{e}_2 + \hat{e}_3 \Rightarrow \hat{n} = \frac{1}{\sqrt{2}}(0, 1, 1)$

The stress vector along this direction

$$\vec{t} = \underline{\underline{\sigma}} \times \hat{n} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & \mu \\ 0 & \mu & -p \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{\mu - p}{\sqrt{2}} \\ \frac{\mu - p}{\sqrt{2}} \end{bmatrix} \Rightarrow |\vec{t}|^2 = (\mu - p)^2$$

Magnitude of normal stress

$$|\vec{t}_n| = \vec{n} \cdot \vec{t} = 0 \times 0 + \frac{1}{\sqrt{2}} \times \frac{\mu - p}{\sqrt{2}} + \frac{1}{\sqrt{2}} \times \frac{\mu - p}{\sqrt{2}} = \mu - p \text{ (answer)}$$

Its direction: $\vec{t}_n = (\mu - p) \times \frac{1}{\sqrt{2}}(\hat{e}_2 + \hat{e}_3) = \frac{\mu - p}{\sqrt{2}}\hat{e}_2 + \frac{\mu - p}{\sqrt{2}}\hat{e}_3$

Magnitude of the shear stress

$$|\vec{t}_t| = \sqrt{|\vec{t}|^2 - |\vec{t}_n|^2} = \sqrt{(\mu-p)^2 - (\mu-p)^2} = 0 \text{ (answer)}$$

$\Rightarrow$ The shear stress vanishes

c) Determine the plane on which the total normal stress are given by p

The normal stress is given as ($\hat{n} = [x_1\ x_2\ x_3]^T$ is normal vector of the plane)

$$\begin{bmatrix} -p & 0 & 0 \\ 0 & -p & \mu \\ 0 & \mu & -p \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -px_1 \\ -px_2 + \mu x_3 \\ \mu x_2 - p x_3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow |\vec{t}_n| = -p(x_1^2 + x_2^2 + x_3^2) + 2\mu x_2 x_3$$

The total normal stresses are only given by p $\Rightarrow$ μ vanishes $\Rightarrow$ $x_2 x_3 = 0$

$$\Rightarrow \begin{cases} x_2 = 0 \\ x_3 = 0 \\ x_2 = x_3 = 0 \end{cases}$$

$\Rightarrow$ The planes satisfy this condition has normal in direction of $\hat{e}_1$, $a\hat{e}_1 + b\hat{e}_2$, $a\hat{e}_1 + b\hat{e}_3$ ($a, b \neq 0$, $a, b \in R$)

Exercise 4: The viscosity of incompressible viscous fluid is $\mu = 0.96\ Pa.s$. The velocity field is given by:

$$v_1 = k(x_1^2 - x_2^2)\ ;\ v_2 = -2kx_1x_2\ ;\ v_3 = 0$$

where $k = \frac{1}{m.s}$. At the point with coordinates $(1, 2, 1)\,m$, for the plane with normal in the direction of $\hat{e}_1$:

a) Define the excess of the normal compressive force on top of the pressure p.

Velocity gradient tensor: $\underline{\underline{L}} = \begin{bmatrix} 2kx_1 & -2kx_2 & 0 \\ -2kx_2 & -2kx_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \underline{\underline{L}}$ is symmetric

The stress tensor: $\underline{\underline{\sigma}} = -p\underline{\underline{1}} + 2\mu\underline{\underline{d}} \Rightarrow \underline{\underline{\sigma}} = -p1 + 2\mu\left(\frac{1}{2}(L^T + L)\right)$

$\Rightarrow \underline{\underline{\sigma}} = -p\underline{\underline{1}} + 2\mu L$ (Since L is symmetric)

$$\Rightarrow \underline{\underline{\sigma}} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix} + 2 \times 0.96 \begin{bmatrix} 2x_1 & -2x_2 & 0 \\ -2x_2 & -2x_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -p + 3.84x_1 & -3.84x_2 & 0 \\ -3.84x_2 & -p - 3.84x_1 & 0 \\ 0 & 0 & -p \end{bmatrix} \text{Pa}$$

- The stress vector in direction of $\hat{e}_1$

$$\vec{t} = \underline{\underline{\sigma}} \cdot \hat{n} = \begin{bmatrix} -p + 3.84x_1 & -3.84x_2 & 0 \\ -3.84x_2 & -p - 3.84x_1 & 0 \\ 0 & 0 & -p \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -p + 3.84x_1 \\ -3.84x_2 \\ 0 \end{bmatrix}$$

- The normal stress's magnitude

$|\vec{t}_n| = \vec{t} \cdot \hat{n} = -p + 3.84x_1$

At the point $(1, 2, 1)$ m, the excess of normal compressive force on top of the pressure p is $3.84 \times 1 = 3.84$ Pa

b) Determine the magnitude of the shear stress

We have: $|\vec{t}|^2 = (-p + 3.84x_1)^2 + (-3.84x_2)^2$

$|\vec{t}_n|^2 = (-p + 3.84x_1)^2$

$\Rightarrow |\vec{t}_t| = \sqrt{|\vec{t}|^2 - |\vec{t}_n|^2} = \sqrt{(-3.84x_2)^2} = 3.84x_2$

At point $(1, 2, 1)$m, the magnitude of shearing stress is $3.84 \times 2 = 7.68$ Pa (answer)