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Foundations on Continuum Mechanics - Week 6 - Constitutive Equations - Fluids

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Fluids

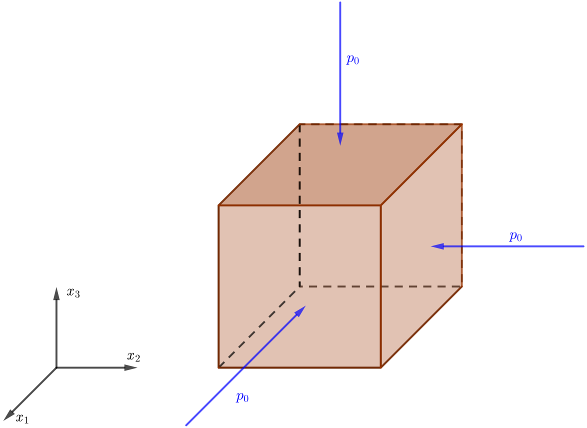
What is fluid?

Fluids can be classified in:

- Ideal (inviscid) fluids:
  - Called also perfect fluid
  - Water is an example of inviscid fluid
  - Resists only compressive stresses (pressure)
  - No resistance in fluid moves
- Real (viscous) fluids:
  - Has viscous (friction) behavior
  - Honey is an example of viscous fluid
  - Resistance in fluid moves

Pascal's Law

For a fluid at rest, the pressure acts on all directions.



Consequences of Pascal's Law

Consequences:

- No shear stresses at rest
- Only normal stresses due to pressure

The stress is isotropic at rest and of the form:

$$\underline{\underline{\boldsymbol{\sigma}}} = -p_0 \underline{\underline{\mathbf{1}}}$$

$$\sigma_{ij} = -p_0 \delta_{ij} \quad i, j \in \{1, 2, 3\}$$

p_0 denotes the hydrostatic pressure.

Pressure Concepts

Consequences:

- Hydrostatic pressure p_0: normal compressive stress on a fluid.
- Mean pressure $\overline{p}$:

$$\overline{p} = -\sigma_m = -\frac{1}{3}Tr(\underline{\underline{\boldsymbol{\sigma}}})$$

  $Tr(\underline{\underline{\boldsymbol{\sigma}}})$ is an invariant and therefore σ_m and $\overline{p}$ are also.

- Thermodynamic pressure p: used in constitutive equations. It is related to density and temperature through the kinetic equation of state:

$$F(\rho, p, \theta) = 0$$

  For a fluid at rest: $p_0 = \overline{p} = p$

Pressure Concepts

Consequences:

- Barotropic fluid is the fluid in which the pressure depends only on the density, [1]:

$$F(\rho, p) = 0 \Rightarrow p = f(\rho)$$

- Incompressible fluid, when the density is constant (special case of barometric):

$$F(\rho, p, \theta) \equiv F(\rho) = \rho - k = 0 \Rightarrow p = f(\rho) \Rightarrow \rho = k = const.$$

Constitutive Equations

Governing Equations: Thermo-mechanical problem

$\dot{\rho} + \rho \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{v}} = 0$	Conservation of Mass Continuity Equation	1 eqn.	PDE
$\vec{\boldsymbol{\nabla}} \cdot \underline{\underline{\boldsymbol{\sigma}}} + \rho \vec{\boldsymbol{b}} = \rho \dot{\vec{\boldsymbol{v}}}$	Linear Momentum Balance Cauchy's Equation of motion	3 eqns.	PDE
$\underline{\underline{\boldsymbol{\sigma}}} = \underline{\underline{\boldsymbol{\sigma}}}^T$	Angular Momentum Balance Symmetry of Cauchy stress tensor	3 eqns.	ALG
$\rho \dot{u} = \underline{\underline{\boldsymbol{\sigma}}} : \underline{\underline{\boldsymbol{d}}} + \rho\ r - \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{q}}$	Energy Balance First Law of Thermodynamics	1 eqn.	PDE
$-\rho(\dot{u} - \theta \dot{s}) + \underline{\underline{\boldsymbol{\sigma}}} : \underline{\underline{\boldsymbol{d}}} \geq 0$ $-\frac{1}{\rho \theta^2} \vec{\boldsymbol{q}} \cdot \vec{\boldsymbol{\nabla}} \theta \geq 0$	Second Law of Thermodynamics Clausius-Plank Inequality Heat Flow Inequality	2 restrictions	PDE

8 PDE and 2 restrictions

19 unknown scalars: $\rho, \vec{\boldsymbol{v}}, u, \underline{\underline{\boldsymbol{\sigma}}}, \vec{\boldsymbol{q}}, \theta, s$

Constitutive Equations: Thermo-mechanical problem

The Constitutive Equations:

$\underline{\underline{\boldsymbol{\sigma}}} = \underline{\underline{\boldsymbol{\sigma}}}(\vec{\boldsymbol{v}}, \theta, \zeta)$	Thermo-Mechanical Constitutive Equations	6 eqns.
$s = s(\vec{\boldsymbol{v}}, \theta, \zeta)$	Entropy Constitutive Equation	1 eqn.
$\vec{\boldsymbol{q}} = \vec{\boldsymbol{q}}(\vec{\boldsymbol{v}}, \theta) = -k\vec{\boldsymbol{\nabla}}\theta$	Thermal Constitutive Equation Fourier's Law of Conduction	3 eqns.
$u = v(\rho, \vec{\boldsymbol{v}}, \theta, \zeta)$ $F_i(\rho, \theta, \zeta); \quad i \in \{1, 2, .., p\}$	Heat State Equation Kinetic State Equation	(1+p) eqns.

Constitutive Equations

The constitutive equations along with governing equations can used to solve the problem
In fluid mechanics can be grouped as:

- Thermo-mechanical constitutive equations:

$$\underline{\underline{\boldsymbol{\sigma}}} = -p\underline{\underline{\mathbf{1}}} + \underline{\underline{\boldsymbol{f}}}(\underline{\underline{\boldsymbol{d}}}, \rho, \theta)$$

$$\sigma_{ij} = -p\delta_{ij} + f_{ij}(\underline{\underline{\boldsymbol{d}}}, \rho, \theta); \quad i, j \in \{1, 2, 3\}$$

- Caloric equation of state

$$u = g(\rho, \theta)$$

- Entropy constitutive equation

$$s = s(\underline{\underline{\boldsymbol{d}}}, \rho, \theta)$$

- Fourier's Law:

$$\vec{\boldsymbol{q}} = -k\vec{\boldsymbol{\nabla}}\theta$$

$$q_i = -k\frac{\partial \theta}{\partial x_i} \quad i, j \in \{1, 2, 3\}$$

- Kinetic equation of state:

$$F(\rho, p, \theta) = 0$$

Viscous Fluid Models

General form of the thermo-mechanical constitutive equations:

$$\underline{\underline{\boldsymbol{\sigma}}} = -p\underline{\underline{\mathbf{1}}} + \underline{\underline{\boldsymbol{f}}}(\underline{\underline{\boldsymbol{d}}}, \rho, \theta)$$

$$\sigma_{ij} = -p\delta_{ij} + f_{ij}(\underline{\underline{\boldsymbol{d}}}, \rho, \theta); \quad i, j \in \{1, 2, 3\}$$

Fluids are classified as:

- Perfect fluid (no viscosity), $\underline{\underline{\boldsymbol{f}}}(\underline{\underline{\boldsymbol{d}}}, \rho, \theta) = 0 \Rightarrow \underline{\underline{\boldsymbol{\sigma}}} = -p\underline{\underline{\mathbf{1}}}$.
- Newtonian fluid (viscous), $\underline{\underline{\boldsymbol{f}}}(\underline{\underline{\boldsymbol{d}}}, \rho, \theta)$ is a linear function of strain rate.
- Stokesian fluid (viscous), $\underline{\underline{\boldsymbol{f}}}(\underline{\underline{\boldsymbol{d}}}, \rho, \theta)$ is a non-linear function of its arguments, [1].

Note that $\underline{\underline{\boldsymbol{d}}} = \frac{1}{2}\left(\vec{\boldsymbol{v}} \otimes \vec{\boldsymbol{\nabla}} + \vec{\boldsymbol{\nabla}} \otimes \vec{\boldsymbol{v}}\right)$ is the symmetric part of the velocity gradient tensor $\underline{\underline{\boldsymbol{d}}}$.

Constitutive Equations: Newtonian Fluids

Constitutive equations:

$$\underline{\underline{\boldsymbol{\sigma}}} = -p\underline{\underline{\mathbf{1}}} + \underline{\underline{\underline{\underline{\boldsymbol{C}}}}} : \underline{\underline{\boldsymbol{d}}}$$

$$\sigma_{ij} = -p\delta_{ij} + C_{ijkl}d_{kl}; \quad i, j \in \{1, 2, 3\}$$

where $\underline{\underline{\underline{\underline{\boldsymbol{D}}}}}$ is a 4^{th} order constant VISCOUS constitutive tensor.

For an isotropic material the viscous constitutive tensor becomes:

$$\underline{\underline{\underline{\underline{\boldsymbol{C}}}}} = \lambda\underline{\underline{\mathbf{1}}} \otimes \underline{\underline{\mathbf{1}}} + 2\mu\underline{\underline{\underline{\underline{\boldsymbol{I}}}}}$$

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

Finally in the constitutive equation:

$$\underline{\underline{\boldsymbol{\sigma}}} = -p\underline{\underline{\mathbf{1}}} + \lambda Tr(\underline{\underline{\boldsymbol{d}}})\underline{\underline{\mathbf{1}}} + 2\mu\underline{\underline{\boldsymbol{d}}}$$

$$\sigma_{ij} = -p\delta_{ij} + \lambda d_{ll}\delta_{ij} + \mu d_{ij}; \quad i, j \in \{1, 2, 3\}$$

Note that λ and μ can vary with ρ and θ.

Dissipative and Recoverable Parts of Stress Tensor

The mechanical energy balance:

$$\underbrace{P_e(t)}_{\text{mechanical power}} = \int\int\int_V \rho\vec{\boldsymbol{b}}\cdot\vec{\boldsymbol{v}}dV + \int\int_{\partial V} \vec{\boldsymbol{t}}\cdot\vec{\boldsymbol{v}}dS = \underbrace{\frac{d}{dt}\int\int\int_V \frac{1}{2}\rho v^2 dV}_{\text{kinetic energy}} + \underbrace{\int\int\int_V \underline{\underline{\boldsymbol{\sigma}}} : \underline{\underline{\boldsymbol{d}}} dV}_{\text{stress power}}$$

Then:

$$P_e(t) = \frac{d}{dt}K(t) + P_\sigma$$

Note that a rigid body will have zero stress power.
The stress power is mechanical in the system, which is not spent in changing the kinetic energy, [1].

Dissipative and Recoverable Stress

Knowing that:

$$\int\int\int_V \underline{\underline{\boldsymbol{\sigma}}} : \underline{\underline{\boldsymbol{d}}}\, dV; \quad \underline{\underline{\boldsymbol{d}}} = \frac{1}{3} Tr(\underline{\underline{\boldsymbol{d}}})\underline{\underline{\mathbf{1}}} + \underline{\underline{\boldsymbol{d}}}_{dev}; \quad \underline{\underline{\boldsymbol{\sigma}}} = -\overline{p}\underline{\underline{\mathbf{1}}} + \underline{\underline{\boldsymbol{\sigma}}}_{dev}; \quad \overline{p} = -\frac{1}{3} Tr(\underline{\underline{\boldsymbol{\sigma}}})$$

Then:

$$\underline{\underline{\boldsymbol{\sigma}}} : \underline{\underline{\boldsymbol{d}}} = \left(-\overline{p}\underline{\underline{\mathbf{1}}} + \underline{\underline{\boldsymbol{\sigma}}}_{dev}\right) : \left(\frac{1}{3} Tr(\underline{\underline{\boldsymbol{d}}})\underline{\underline{\mathbf{1}}} + \underline{\underline{\boldsymbol{d}}}_{dev}\right) =$$

$$= -\frac{1}{3}\overline{p}\,Tr(\underline{\underline{\boldsymbol{d}}})\underbrace{\underline{\underline{\mathbf{1}}} : \underline{\underline{\mathbf{1}}}}_{=3} + \underline{\underline{\boldsymbol{\sigma}}}_{dev} : \underline{\underline{\boldsymbol{d}}}_{dev} - \overline{p}\underbrace{\underline{\underline{\mathbf{1}}} : \underline{\underline{\boldsymbol{d}}}_{dev}}_{=Tr(\underline{\underline{\boldsymbol{d}}}_{dev})=0} + \frac{1}{3} Tr(\underline{\underline{\boldsymbol{d}}})\underbrace{\underline{\underline{\boldsymbol{\sigma}}}_{dev} : \underline{\underline{\mathbf{1}}}}_{Tr(\underline{\underline{\boldsymbol{\sigma}}}_{dev})=0} = -\overline{p}\,Tr(\underline{\underline{\boldsymbol{d}}}) + \underline{\underline{\boldsymbol{\sigma}}}_{dev} : \underline{\underline{\boldsymbol{d}}}_{dev}$$

Noting that $\underline{\underline{\boldsymbol{\sigma}}}_{dev} = 2\mu\underline{\underline{\boldsymbol{d}}}_{dev}$ and $\overline{p} = p - \kappa\, Tr(\underline{\underline{\boldsymbol{d}}})$, we have:

$$\underline{\underline{\boldsymbol{\sigma}}} : \underline{\underline{\boldsymbol{d}}} = \underbrace{-p\ Tr(\underline{\underline{\boldsymbol{d}}})}_{\text{Recoverable Power } W_R} \underbrace{+\kappa\, Tr^2(\underline{\underline{\boldsymbol{d}}}) + 2\mu\underline{\underline{\boldsymbol{d}}}_{dev} : \underline{\underline{\boldsymbol{d}}}_{dev}}_{\text{Dissipative Power } 2\,W_D}$$

where $\kappa = \lambda + \frac{2}{3}\mu$ is the bulk viscosity.

Dissipative and Recoverable Parts of Stress Tensor

In the Cauchy stress tensor the dissipative and recoverable parts can be split as follows:

$$\underline{\underline{\boldsymbol{\sigma}}} = \underbrace{-p\underline{\underline{\mathbf{1}}}}_{\text{Recoverable Part } \underline{\underline{\boldsymbol{\sigma}}}_R} \underbrace{+\lambda Tr(\underline{\underline{\boldsymbol{d}}})\underline{\underline{\mathbf{1}}} + 2\mu\underline{\underline{\boldsymbol{d}}}}_{\text{Dissipative Part } \underline{\underline{\boldsymbol{\sigma}}}_D}$$

More specifically, the recoverable and the dissipative part can be written:

$$W_R = -p\,Tr(\underline{\underline{\boldsymbol{d}}}) = -p\underline{\underline{\mathbf{1}}} : \underline{\underline{\boldsymbol{d}}} = \underline{\underline{\boldsymbol{\sigma}}}_R : \underline{\underline{\boldsymbol{d}}}$$

$$2\,W_D = \kappa\,Tr^2(\underline{\underline{\boldsymbol{d}}}) + 2\mu\underline{\underline{\boldsymbol{d}}}_{dev} : \underline{\underline{\boldsymbol{d}}}_{dev} = \underline{\underline{\boldsymbol{\sigma}}}_D : \underline{\underline{\boldsymbol{d}}}$$

Note that for incompressible fluid: $W_R = -p\,Tr(\underline{\underline{\boldsymbol{d}}}) = 0$.
Due to second principle of thermodynamics:

- the dissipative part $2\,W_D \geq 0$,
- the bulk viscosity $\kappa = \lambda + \frac{2}{3}\mu \geq 0$
- the shear viscosity $\mu \geq 0$.

Fluid Mechanics

Governing Equations

$\dot{\rho} + \rho \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{v}} = 0$	Conservation of Mass Continuity Equation	1 eqn.	PDE
$\vec{\boldsymbol{\nabla}} \cdot \underline{\underline{\boldsymbol{\sigma}}} + \rho \vec{\boldsymbol{b}} = \rho \dot{\vec{\boldsymbol{v}}}$	Linear Momentum Balance Cauchy's Equation of motion	3 eqns.	PDE
$\underline{\underline{\boldsymbol{\sigma}}} = \underline{\underline{\boldsymbol{\sigma}}}^T$	Angular Momentum Balance Symmetry of Cauchy stress tensor	3 eqns.	ALG
$\rho \dot{u} = \underline{\underline{\boldsymbol{\sigma}}} : \underline{\underline{\boldsymbol{d}}} + \rho\ r - \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{q}}$	Energy Balance First Law of Thermodynamics	1 eqn.	PDE
$-\rho(\dot{u} - \theta \dot{s}) + \underline{\underline{\boldsymbol{\sigma}}} : \underline{\underline{\boldsymbol{d}}} \geq 0$ $-\frac{1}{\rho \theta^2} \vec{\boldsymbol{q}} \cdot \vec{\boldsymbol{\nabla}} \theta \geq 0$	Second Law of Thermodynamics Clausius-Plank Inequality Heat Flow Inequality	2 restrictions	PDE

8 Eqns and 2 restrictions

Constitutive Equations: Newtonian Fluids

The 12 missing equations:

$\underline{\underline{\boldsymbol{\sigma}}} = -p\underline{\underline{\mathbf{1}}} + \lambda Tr(\underline{\underline{\boldsymbol{d}}})\underline{\underline{\mathbf{1}}} + 2\mu\underline{\underline{\boldsymbol{d}}}$	Thermo-Mechanical Constitutive Equations	6 eqns.
$s = s(\vec{\boldsymbol{v}}, \theta, \zeta)$	Entropy Constitutive Equation	1 eqn.
$\vec{\boldsymbol{q}} = \vec{\boldsymbol{q}}(\vec{\boldsymbol{v}}, \theta) = -k\vec{\boldsymbol{\nabla}}\theta$	Thermal Consitutive Equation Fourier's Law of Conduction	3 eqns.
$u = v(\rho, \theta, \zeta)$ $F(\rho, p, \theta) = 0$	Caloric State Equation Kinetic State Equation	2 eqns.

In total: 20 Eqns. with 20 unknowns:

$\rho \to 1,\ \vec{\boldsymbol{v}} \to 3,\ \underline{\underline{\boldsymbol{\sigma}}} \to 9,\ u \to 1,\ \vec{\boldsymbol{q}} \to 3,\ \theta \to 1,\ s \to 1,\ p \to 1$

Constitutive Equations: Barotropic Fluids

A barotropic fluid is defined by the kinetic state equation, not depend on temperature θ:

$$F(\rho, p) = 0 \Rightarrow \rho = \rho(p)$$

The uncoupled problem becomes:

$\dot{\rho} + \rho \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{v}} = 0$	Conservation of Mass Continuity Equation	1 eqn.
$\vec{\boldsymbol{\nabla}} \cdot \underline{\underline{\boldsymbol{\sigma}}} + \rho \vec{\boldsymbol{b}} = \rho \dot{\vec{\boldsymbol{v}}}$	Linear Momentum Balance Cauchy's Equation of motion	3 eqns.
$\underline{\underline{\boldsymbol{\sigma}}} = -p\underline{\underline{\mathbf{1}}} + \lambda Tr(\underline{\underline{\boldsymbol{d}}})\underline{\underline{\mathbf{1}}} + 2\mu \underline{\underline{\boldsymbol{d}}}$	Thermo-Mechanical Constitutive Equations	6 eqns.
$\rho = \rho(p)$	Kinetic State Equation	1 eqn.

11 scalar unknowns: $\rho(1), \vec{\boldsymbol{v}}(3), \underline{\underline{\boldsymbol{\sigma}}}(6), p(1)$

Hydrostatics

Hydrostatic stress state

- Uniform velocity:

$$\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}, t) \equiv \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}) \Rightarrow \vec{\boldsymbol{\nabla}}\vec{\boldsymbol{v}} = \vec{\boldsymbol{\nabla}} \otimes \vec{\boldsymbol{v}} = \vec{\boldsymbol{v}} \otimes \vec{\boldsymbol{\nabla}} = \underline{\underline{\mathbf{0}}}$$

$$\underline{\underline{\boldsymbol{d}}} = \frac{1}{2}\left(\vec{\boldsymbol{v}} \otimes \vec{\boldsymbol{\nabla}} + \vec{\boldsymbol{\nabla}} \otimes \vec{\boldsymbol{v}}\right) = \underline{\underline{\mathbf{0}}}$$

Therefore, the hydrostatic stress state can be defines as:

$$\underline{\underline{\boldsymbol{\sigma}}} = -p\underline{\underline{\mathbf{1}}} + \lambda \underbrace{Tr(\underline{\underline{\boldsymbol{d}}})}_{=0}\underline{\underline{\mathbf{1}}} + 2\mu \underbrace{\underline{\underline{\boldsymbol{d}}}}_{=\underline{\underline{\mathbf{0}}}} \Rightarrow \underline{\underline{\boldsymbol{\sigma}}} = -p\underline{\underline{\mathbf{1}}} \Rightarrow \overline{p} = p$$

- Uniform and stationary velocity $\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}, t) =$ constant:

$$\vec{\boldsymbol{a}} = \frac{d\vec{\boldsymbol{v}}}{dt} = \frac{\partial \vec{\boldsymbol{v}}}{\partial t} + \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{\nabla}}\vec{\boldsymbol{v}} = \vec{\mathbf{0}}$$

There the hydrostatic case can be derived as:

$$\underline{\underline{\boldsymbol{\sigma}}} = -p_0\underline{\underline{\mathbf{1}}} \Rightarrow Tr(\underline{\underline{\boldsymbol{\sigma}}}) = -3p_0 \Rightarrow \overline{p} = p = p_0$$

- Fluid at rest $\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}, t) =$ constant $= \vec{\mathbf{0}}$. Hydrostatic (from Greek and means water at rest).

Hydrostatic Problem

Hydrostatic problem $\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}, t) =$ constant is defined as:

$\dot{\rho} + \rho \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{v}} = 0 \Rightarrow \rho(\vec{\boldsymbol{X}}, t) = \rho_0(\vec{\boldsymbol{X}})$	Conservation of Mass Continuity Equation	1 eqn.
$\vec{\boldsymbol{\nabla}} \cdot \underline{\underline{\boldsymbol{\sigma}}} + \rho \vec{\boldsymbol{b}} = \rho \dot{\vec{\boldsymbol{v}}} \Rightarrow \vec{\boldsymbol{\nabla}} \cdot \underline{\underline{\boldsymbol{\sigma}}} + \rho \vec{\boldsymbol{b}} = \vec{\boldsymbol{0}}$	Linear Momentum Balance Cauchy's Equation of motion	3 eqns.
$\underline{\underline{\boldsymbol{\sigma}}} = -p\underline{\underline{\mathbf{1}}} + \lambda Tr(\underline{\underline{\boldsymbol{d}}})\underline{\underline{\mathbf{1}}} + 2\mu \underline{\underline{\boldsymbol{d}}} \Rightarrow \underline{\underline{\boldsymbol{\sigma}}} = -p\underline{\underline{\mathbf{1}}}$	Thermo-Mechanical Constitutive Equations	6 eqns.

Introducing the constitutive equation in the Cauchy equation, we get the fundamental equation of hydrostatics:

$$\underline{\underline{\boldsymbol{\sigma}}} = -p\underline{\underline{\mathbf{1}}} \Rightarrow \vec{\boldsymbol{\nabla}} \cdot (-p\underline{\underline{\mathbf{1}}}) = -\vec{\boldsymbol{\nabla}} p_0 \to \begin{cases} -\vec{\boldsymbol{\nabla}} p_0 + \rho_0 \vec{\boldsymbol{b}} = \vec{\boldsymbol{0}} \\ \frac{\partial p_0}{\partial x_i} + \rho_0 b_i = 0; \quad i. \in \{1, 2, 3\} \end{cases}$$

We know the ρ_0 we find the pressure p from above equation, then we define the stress $\underline{\underline{\boldsymbol{\sigma}}}$

Barotropic Perfect Fluids

Barotropic Perfect Fluids

A perfect fluid is a Newtonian fluid without viscosity ($\mu = \lambda = 0$):

$$\underline{\underline{\boldsymbol{\sigma}}} = -p\underline{\underline{\mathbf{1}}} + \lambda Tr(\underline{\underline{\boldsymbol{d}}})\underline{\underline{\mathbf{1}}} + 2\mu\underline{\underline{\boldsymbol{d}}} \Rightarrow \underline{\underline{\boldsymbol{\sigma}}} = -p\underline{\underline{\mathbf{1}}} \Rightarrow \underline{\underline{\boldsymbol{\sigma}}} = -p\underline{\underline{\mathbf{1}}}$$

at hydrostatic stress state. Therefore:

$$\vec{\boldsymbol{\nabla}} \cdot \underline{\underline{\boldsymbol{\sigma}}} = -\vec{\boldsymbol{\nabla}} p$$

$$\underline{\underline{\boldsymbol{\sigma}}} : \underline{\underline{\boldsymbol{d}}} = -p\underline{\underline{\mathbf{1}}} : \underline{\underline{\boldsymbol{d}}} = -p\,Tr(\underline{\underline{\boldsymbol{d}}})$$

In a barotropic fluid temperature does not affect the kinetic state equation:

$$F(\rho, p) = 0 \Rightarrow \rho = \rho(p)$$

Note that most liquids can be assumed as barotropic (but not perfect). Some gases under certain circumstances.

Barotropic Perfect Fluids: Field Equations

Hydrostatic problem $\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}, t) =$ constant is defined as:

$\dot{\rho} + \rho \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{v}} = 0$	Conservation of Mass Continuity Equation	1 eqn.
$\vec{\boldsymbol{\nabla}} \cdot \underline{\underline{\boldsymbol{\sigma}}} + \rho \vec{\boldsymbol{b}} = \rho \dot{\vec{\boldsymbol{v}}} \Rightarrow -\vec{\boldsymbol{\nabla}} p + \rho \vec{\boldsymbol{b}} = \rho \dot{\vec{\boldsymbol{v}}}$	Linear Momentum Balance Euler's Equation	3 eqns.
$F(\rho, p) = 0 \Rightarrow \rho = \rho(p)$	Kinetic State Equation	1 eqn.

There are 5 scalar quantities unknowns: $\rho, \vec{\boldsymbol{v}}, p$

Bernoilli's Trinomial

Consider a barotropic fluid under potential body forces:

$$\phi(\vec{\boldsymbol{x}}, t) = gz \Rightarrow -\vec{\boldsymbol{\nabla}}\phi(\vec{\boldsymbol{x}}, t) = -\left[\frac{\partial \phi}{\partial x}\ \frac{\partial \phi}{\partial y}\ \frac{\partial \phi}{\partial z}\right]^T = \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix}$$

The following lemmas hold:

- Lemma 1. For barotropic fluids there exist a function $P(\vec{\boldsymbol{x}}, t) = \hat{P}(p(\vec{\boldsymbol{x}}, t))$ such that.
$$\vec{\boldsymbol{\nabla}} p = \rho \vec{\boldsymbol{\nabla}} P$$
- Lemma 2. The convective term of the acceleration can be expressed as:
$$\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{\nabla}} \vec{\boldsymbol{v}} = 2\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{v}} + \vec{\boldsymbol{\nabla}} \left(\frac{1}{2} v^2\right)$$
where $2\vec{\boldsymbol{\omega}} = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{v}}$ is the vorticity vector.

Bernoilli's Trinomial

Using the Euler's equation:

$$-\vec{\boldsymbol{\nabla}} p + \rho \vec{\boldsymbol{b}} = \rho \vec{\boldsymbol{v}} \rightarrow -\frac{1}{\rho} \vec{\boldsymbol{\nabla}} p + \vec{\boldsymbol{b}} = \frac{d\vec{\boldsymbol{v}}}{dt}$$

$$\frac{1}{\rho} \vec{\boldsymbol{\nabla}} p = \vec{\boldsymbol{\nabla}} P; \vec{\boldsymbol{b}} = \vec{\boldsymbol{\nabla}} \phi$$

$$\vec{\boldsymbol{v}} \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{v}}$$

Combining the equations:

$$-\vec{\boldsymbol{\nabla}} P - \vec{\boldsymbol{\nabla}} \phi = \frac{\partial \vec{\boldsymbol{v}}}{\partial t} + 2\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{v}} + \vec{\boldsymbol{\nabla}} \left(\frac{1}{2} v^2 \right)$$

Finally we can derive the equation of motion for a barotropic perfect fluid:

$$-\vec{\boldsymbol{\nabla}} \underbrace{\left[P + \phi + \frac{1}{2} v^2 \right]}_{\text{Bernoulli's Trinomial}} = \frac{\partial \vec{\boldsymbol{v}}}{\partial t} + 2\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{v}}$$

Newtonian Viscous Fluids

Governing Equations

The general fluid mechanics problem:

$\dot{\rho} + \rho \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{v}} = 0$	Conservation of Mass Continuity Equation	1 eqn.
$\vec{\boldsymbol{\nabla}} \cdot \underline{\underline{\boldsymbol{\sigma}}} + \rho \vec{\boldsymbol{b}} = \rho \dot{\vec{\boldsymbol{v}}}$	Linear Momentum Balance Cauchy's Equation of motion	3 eqns.
$\rho \dot{u} = \underline{\underline{\boldsymbol{\sigma}}} : \underline{\underline{\boldsymbol{d}}} + \rho r - \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{q}}$	Energy Balance First law of Thermodynamics	1 eqns.
$\underline{\underline{\boldsymbol{\sigma}}} = -p\underline{\underline{\mathbf{1}}} + \lambda Tr(\underline{\underline{\boldsymbol{d}}})\underline{\underline{\mathbf{1}}} + 2\mu \underline{\underline{\boldsymbol{d}}}$	Mechanical Constitutive Equations	6 eqns.
$s = s(\underline{\underline{\boldsymbol{d}}}, \theta, \rho)$	Entropy Constitutive equation	1 eqn.
$\vec{\boldsymbol{q}} = -K \vec{\boldsymbol{\nabla}} \theta$	Thermal constitutive equation. Fourier Law of conduction	3 eqns.
$u = u(\rho, \theta) \quad F(\rho, \theta, p) = 0$	Caloric and Kinetic State	2 eqns.

17 scalar unknowns: $\rho(1), \vec{\boldsymbol{v}}(3), \underline{\underline{\boldsymbol{\sigma}}}(6), u(1), \vec{\boldsymbol{q}}(3), \theta(1), s(1), p(1)$

Navier-Stokes equations

Consider the lemmas:

- Lemma 1:

$$\vec{\boldsymbol{\nabla}} \cdot \underline{\underline{\boldsymbol{d}}} = \frac{1}{2}\Delta\vec{\boldsymbol{v}} + \frac{1}{2}\vec{\boldsymbol{\nabla}}(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{v}})$$

  where $\underline{\underline{\boldsymbol{d}}}$ is the deformation rate tensor.

- Lemma 2:

$$\vec{\boldsymbol{\nabla}} \cdot (a\underline{\underline{\mathbf{1}}}) = \vec{\boldsymbol{\nabla}} a$$

  where $a(\vec{\boldsymbol{x}}, t)$ is a scalar fuction

Introducing the constitutive equation of stress in the term of Cauchy's equation ($Tr(\underline{\underline{\boldsymbol{d}}}) = \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{v}}$):

$$\vec{\boldsymbol{\nabla}} \cdot \underline{\underline{\boldsymbol{\sigma}}} = \vec{\boldsymbol{\nabla}} \cdot (-p\underline{\underline{\mathbf{1}}} + \lambda Tr(\underline{\underline{\boldsymbol{d}}})\underline{\underline{\mathbf{1}}} + 2\mu\underline{\underline{\boldsymbol{d}}}) = -\vec{\boldsymbol{\nabla}} p + \lambda \underbrace{\vec{\boldsymbol{\nabla}}(Tr(\underline{\underline{\boldsymbol{d}}}))}_{\vec{\boldsymbol{\nabla}}(\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{v}})} + \mu\Delta\vec{\boldsymbol{v}} + \mu\vec{\boldsymbol{\nabla}}(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{v}})$$

Navier-Stokes equations

The linear momentum balance equation can be arranged as follows:

$$\vec{\boldsymbol{\nabla}} \cdot \underline{\underline{\boldsymbol{\sigma}}} + \rho \vec{\boldsymbol{b}} = \rho \dot{\vec{\boldsymbol{v}}} \Rightarrow -\vec{\boldsymbol{\nabla}} p + (\lambda + \mu)\vec{\boldsymbol{\nabla}}(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{v}}) + \mu \Delta \vec{\boldsymbol{v}} + \rho \vec{\boldsymbol{b}} = \rho \frac{d\vec{\boldsymbol{v}}}{dt}$$

Finally the NAVIER-STOKES equations can be written as:

$$-\vec{\boldsymbol{\nabla}} p + (\lambda + \mu)\vec{\boldsymbol{\nabla}}(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{v}}) + \mu \Delta \vec{\boldsymbol{v}} + \rho \vec{\boldsymbol{b}} = \rho \frac{d\vec{\boldsymbol{v}}}{dt}$$

$$-\frac{\partial p}{\partial x_i} + (\lambda + \mu)\frac{\partial^2 v_j}{\partial x_i \partial x_j} + \mu \frac{\partial^2 v_i}{\partial x_i \partial x_j} + \rho b_i = \rho \frac{d v_i}{dt}, \quad 1, j \in \{1, 2, 3\}$$

The Navier-Stokes equations are the equation of motion (balance of linear momentum) written in terms of velocities.

There are 4 unknowns: $\vec{\boldsymbol{v}}(3), p(1)$ and 3 equations.

For incompressible fluids the $(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{v}}) = 0$ and $\underline{\underline{\boldsymbol{\sigma}}} = -p\underline{\underline{\mathbf{1}}} + 2\mu \underline{\underline{\boldsymbol{d}}}$

Energy Equations

The energy balance equation can be given as, [1]:

$$\rho \frac{du}{dt} = -p\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{v}} + \rho\ r + \vec{\boldsymbol{\nabla}} \cdot (K\vec{\boldsymbol{\nabla}}\theta) + \underbrace{\kappa\, Tr^2(\underline{\underline{\boldsymbol{d}}}) + 2\mu \underline{\underline{\boldsymbol{d}}}_{dev} : \underline{\underline{\boldsymbol{d}}}_{dev}}_{\text{Dissipative Power } 2\,W_D}$$

$$\rho \frac{du}{dt} = -p\frac{\partial v_i}{\partial x_i} + \rho\ r + \frac{\partial}{\partial x_i}(K\frac{\partial \theta}{\partial x_i}) + \kappa \left(\frac{\partial v_i}{\partial x_i}\right)^2 + 2\mu\, d_{ij}^{dev} d_{ij}^{dev}; \quad i,j \in \{1,2,3\}$$

The energy equation is just the energy balance in terms of velocity and pressure.

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