Nguyen Xuan Binh 887799 Assignment Week 4

Exercise 1: The equation of motion is given as follows $\begin{cases} x_1 = X_1 + ctX_3 \\ x_2 = X_2 + ctX_3 \\ x_3 = X_3 - ct(X_1 + X_2) \end{cases}$

Calculate the mass density at the current configuration ($\rho$) as a function of the mass density at the reference configuration ($\rho_0$)

We have the change of Volume: $\frac{V}{V_0} = J = \left| \frac{\partial \phi(\vec{X}, t)}{\partial \vec{X}} \right| = \begin{bmatrix} 1 & 0 & ct \\ 0 & 1 & ct \\ -ct & -ct & 1 \end{bmatrix}$

$\Rightarrow \frac{V}{V_0} = \det(J) = 2c^2t^2 + 1$

Since mass of the object is conserved $\Rightarrow m_0 = m \Rightarrow \rho_0 V_0 = \rho V$

$\Rightarrow \frac{\rho_0}{\rho} = \frac{V}{V_0} = 2c^2t^2 + 1 \Rightarrow \rho = \frac{\rho_0}{2c^2t^2 + 1}$ (answer)

Exercise 2: The velocity field is given $\begin{cases} v_1 = ax_1 - bx_2 \\ v_2 = bx_1 - ax_2 \\ v_3 = c\sqrt{x_1^2 + x_2^2} \end{cases}$ where a, b, c are constants.

- Under which conditions the continuity equation is fulfilled

The continuity equation: $\frac{\partial \rho(\vec{x}, t)}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}(\vec{x}, t)) = 0$

$\Rightarrow \frac{\partial \rho(\vec{x}, t)}{\partial t} + \frac{\partial(\rho v_1)}{\partial x_1} + \frac{\partial(\rho v_2)}{\partial x_2} + \frac{\partial(\rho v_3)}{\partial x_3} = 0$

$\Rightarrow \frac{\partial \rho(\vec{x}, t)}{\partial t} + a + (-a) + 0 = 0 \Rightarrow \frac{\partial \rho(\vec{x}, t)}{\partial t} = 0$

$\Rightarrow$ When density is constant value, the continuity equation is fulfilled

- Conclusion: the density of the material is constant, meaning it is incompressible. ($V = V_0$) $\Rightarrow J = 1$. The material is likely to be fluids or rubber.

Exercise 3: The stress tensor field is given in component form:

$$\sigma_{ij} = b \begin{bmatrix} x_1^2 x_2 & (a^2 - x_2^2)x_1 & 0 \\ (a^2 - x_2^2)x_1 & \frac{1}{3}(x_2^3 - 3a^2x_2) & 0 \\ 0 & 0 & 2ax_3^2 \end{bmatrix}$$

In order to achieve equilibrium, find the specific body force $\vec{b}$ (per unit mass)

Cauchy's equation - Equilibrium: $\vec{\nabla} \cdot \underline{\underline{\sigma}}(\vec{x}, t) + \rho\vec{b}(\vec{x}, t) = 0$

$$\Rightarrow \vec{\nabla} \cdot \begin{bmatrix} bx_1^2 x_2 & bx_1(a^2 - x_2^2) & 0 \\ bx_1(a^2 - x_2^2) & 1/3\, b(x_2^3 - 3a^2 x_2) & 0 \\ 0 & 0 & 2abx_3^2 \end{bmatrix} + \rho\vec{b}(\vec{x}, t) = 0$$

$$\Rightarrow \begin{bmatrix} 2bx_1x_2 - 2bx_1x_2 + 0 \\ b(a^2 - x_2^2) + b(x_2^2 - a^2) + 0 \\ 0 + 0 + 4abx_3 \end{bmatrix} + \rho\vec{b}(\vec{x}, t) = 0$$

$$\Rightarrow \begin{bmatrix} 0 \\ 0 \\ 4abx_3 \end{bmatrix} + \rho\vec{b}(\vec{x}, t) = 0 \Rightarrow \vec{b}(\vec{x}, t) = \begin{bmatrix} 0 \\ 0 \\ -\frac{4abx_3}{\rho} \end{bmatrix} = -\frac{4abx_3}{\rho}\hat{e}_3 \text{ (answer)}$$

Exercise 4: The velocity field components are given: $\begin{cases} v_1 = x_1 x_3 \\ v_2 = x_2^2 t \\ v_3 = x_2 x_3 t \end{cases}$

and $\sigma_{ij} = a \begin{bmatrix} x_2 x_1 & -x_2 x_3 & 0 \\ -x_2 x_3 & x_2^2 & -x_2 \\ 0 & -x_2 & x_3^2 \end{bmatrix}$

In order for the principle of conservation of linear momentum holds

Cauchy's equation of motion: $\vec{\nabla} \cdot \underline{\underline{\sigma}}(\vec{x}, t) + \rho\vec{b}(\vec{x}, t) = \rho \frac{d\vec{v}(\vec{x}, t)}{dt}$

$$\Rightarrow \vec{\nabla} \cdot a \begin{bmatrix} x_1 x_2 & -x_2 x_3 & 0 \\ -x_2 x_3 & x_2^2 & -x_2 \\ 0 & -x_2 & x_3^2 \end{bmatrix} + \rho\vec{b}(\vec{x}, t) = \rho \frac{d}{dt} \begin{bmatrix} x_1 x_3 \\ x_2^2 t \\ x_2 x_3 t \end{bmatrix}$$

$$\Rightarrow a \begin{bmatrix} x_2 - x_3 + 0 \\ 0 + 2x_2 + 0 \\ 0 - 1 + 2x_3 \end{bmatrix} + \rho\vec{b}(\vec{x}, t) = \rho \begin{bmatrix} 0 \\ x_2^2 \\ x_2 x_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a(x_2 - x_3) \\ 2ax_2 \\ a(2x_3 - 1) \end{bmatrix} + \rho\vec{b}(\vec{x}, t) = \rho \begin{bmatrix} 0 \\ x_2^2 \\ x_2 x_3 \end{bmatrix} \Rightarrow \frac{1}{\rho} \begin{bmatrix} a(x_2 - x_3) \\ 2ax_2 \\ a(2x_3 - 1) \end{bmatrix} + \vec{b}(\vec{x}, t) = \begin{bmatrix} 0 \\ x_2^2 \\ x_2 x_3 \end{bmatrix}$$

$$\Rightarrow \vec{b}(\vec{x}, t) = \begin{bmatrix} 0 \\ x_2^2 \\ x_2 x_3 \end{bmatrix} - \frac{1}{\rho} \begin{bmatrix} a(x_2 - x_3) \\ 2ax_2 \\ a(2x_3 - 1) \end{bmatrix} = \begin{bmatrix} -a/\rho\,(x_2 - x_3) \\ x_2^2 - \frac{2a}{\rho} x_2 \\ x_2 x_3 - \frac{a}{\rho}(2x_3 - 1) \end{bmatrix}$$

$$= -\frac{a}{\rho}(x_2 - x_3)\hat{e}_1 + \left(x_2^2 - \frac{2a}{\rho} x_2\right)\hat{e}_2 + \left(x_2 x_3 - \frac{a}{\rho}(2x_3 - 1)\right)\hat{e}_3$$

(answer)