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Foundations on Continuum Mechanics - Week 5 - Constitutive Equations - Elasticity

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Constitutive Equations

Constitutive Equations

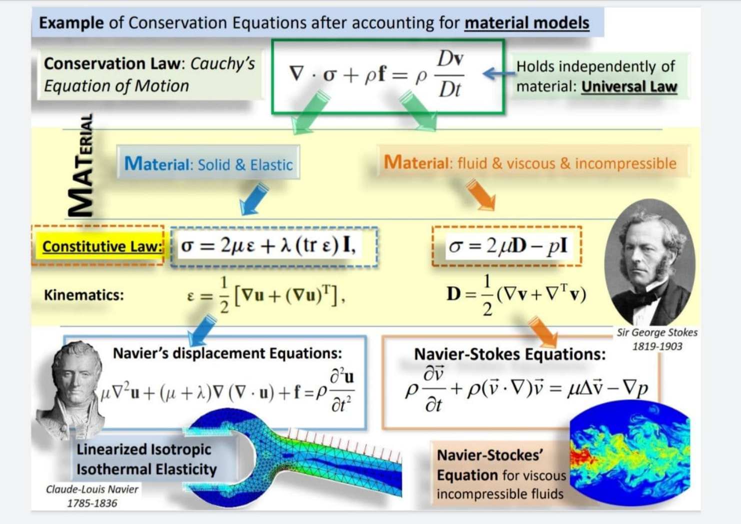
- So far, we have been dealing with principles that apply to all materials.
- To specify the behavior of certain type of material, we need to use the **constitutive equations**, [2].
- The constitutive equations defines the dependence of the stress tensor with the kinematic variables such as strain tensor or rate of deformation tensor, [2].
- Thermodynamic variables are also involved, but here will be mentioned briefly.
- The material behavior of real materials is very complex and diverse.
- The idea is to establish relationships that establish the most important features of the material.
- These equations are regarded as ideal materials.
- No real material will behave like ideal. It is just approximation.

Constitutive equations

- Some examples of the ideal materials are the following:
  - Linear Elastic Solids
  - Newtonian Viscous Fluids
- Constitutive equations should not depend on the coordinate system.
- They take the form of relationships between scalars, vectors and tensors.
- Constitutive equations should satisfy the dimensional homogeneity requirement: the dimensions of all terms
- Rigid-body motions do not affect the constitutive equations.
- The stress depends only on the change of shape and size.
- Materials are usually grouped as solids or fluids.
- The fluids are grouped as gases and liquids.
- The characteristic property of a fluid is that it cannot support shearing stresses.
- Solid can be in equilibrium under shear stress.
- Solids possess a natural configuration, while fluids don't.

Constitutive equations

Constitutive equations are derived from experimental observations.



Credit: Lecturer Dr. Djebar Baroudi

Linear Elasticity

Elasticity - Solids

- Linear elasticity hypothesis, [1]:
  - Infinitesimal strains and deformation framework
  - Existence of unstrained and unstressed reference frame
  - Isothermal, isentropic and adiabatic processes.

Infinitesimal strains

- The displacements are infinitesimal!!
- Material and spatial configurations are the same:

$$\vec{\boldsymbol{x}} = \vec{\boldsymbol{X}} + \vec{\boldsymbol{u}} \Rightarrow \vec{\boldsymbol{x}} \approx \vec{\boldsymbol{X}}$$

- Material and spatial operators and material and spatial properties COINCIDE:

$$\vec{\boldsymbol{x}} \approx \vec{\boldsymbol{X}} \Rightarrow \gamma(\vec{\boldsymbol{x}}, t) = \gamma(\vec{\boldsymbol{X}}, t) = \Gamma(\vec{\boldsymbol{x}}, t) = \Gamma(\vec{\boldsymbol{X}}, t)$$
$$\frac{\partial(*)}{\partial \vec{\boldsymbol{x}}} = \frac{\partial(*)}{\partial \vec{\boldsymbol{X}}} \Rightarrow \vec{\boldsymbol{\nabla}}_X = \vec{\boldsymbol{\nabla}}_x$$

- The deformation gradient tensor $\underline{\underline{\boldsymbol{F}}} \approx \frac{\partial \vec{\boldsymbol{x}}}{\partial \vec{\boldsymbol{X}}} \approx \underline{\underline{\mathbf{1}}} \Rightarrow |\underline{\underline{\boldsymbol{F}}}| \approx 1$ and therefore the current density is approximated by the density at the reference configuration: $\rho_0 = \rho_t |\underline{\underline{\boldsymbol{F}}}|$. In other words, the density is not unknown in linear elasticity.

Infinitesimal strains

- The displacement gradients are infinitesimal
- The strain tensors in material and spatial configurations provide the infinitesimal strain tensor:

$$\underline{\underline{\boldsymbol{E}}}(\vec{\boldsymbol{X}}, t) \approx \underline{\underline{\boldsymbol{e}}}(\vec{\boldsymbol{x}}, t) = \underline{\underline{\boldsymbol{\varepsilon}}}(\vec{\boldsymbol{x}}, t)$$

Unstrained-Unstressed reference state

- It is assumed that there exist a reference neutral (unstrained, unstressed) state, such that:

$$\underline{\underline{\boldsymbol{\varepsilon}}}_0(\vec{\boldsymbol{x}}) = \underline{\underline{\boldsymbol{\varepsilon}}}_0(\vec{\boldsymbol{x}}, t_0) = \underline{\underline{\mathbf{0}}}$$
$$\underline{\underline{\boldsymbol{\sigma}}}_0(\vec{\boldsymbol{x}}) = \underline{\underline{\boldsymbol{\sigma}}}_0(\vec{\boldsymbol{x}}, t_0) = \underline{\underline{\mathbf{0}}}$$

- Note that the neutral state is usually referred to the reference configuration.

Isothermal and adiabatic

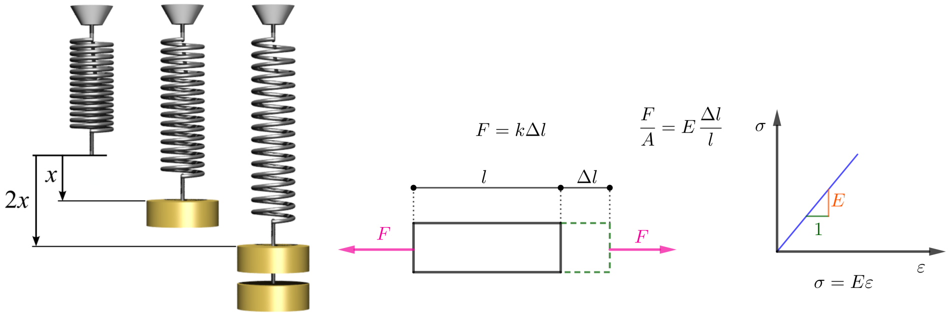
- Isothermal process implies that the temperature remains constant.
- Isentropic process implies that the entropy of the system remains constant.
- In an adiabatic process the net heat transfer entering the body is zero.

Linear Elasticity - Constitutive Equation

Hooke's Law

- Robert Hooke (1660): for small deformations the size of deformation is directly proportional to the deforming force:

$$F = k\Delta l$$



- For 1D elements, the Hooke's law states that strain is directly proportional to the stress.

Generalized Hooke's Law

- For the multidimensional case the proportionality can be generalized to the **generalized Hooke's Law**:

$$\underline{\underline{\boldsymbol{\sigma}}}(\vec{\boldsymbol{x}}, t) = \underline{\underline{\underline{\underline{\boldsymbol{C}}}}} : \underline{\underline{\boldsymbol{\varepsilon}}}$$

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}, \quad i, j, k, l \in \{1, 2, 3\}$$

- This is the constitutive equation of linear elastic material.
- The fourth order tensor $\underline{\underline{\underline{\underline{\boldsymbol{C}}}}}$ is the constitutive elastic constants tensor:
  - Has $3^4 = 81$ components
  - Has the following symmetries, that drop the number of components from 81 to 21:

$$C_{ijkl} = C_{jikl}$$
$$C_{ijkl} = C_{ijlk}$$
$$C_{ijkl} = C_{klij}$$

  - The components of the **constitutive elastic constants tensor** depend only on the material.

The current stress at a point depends on the current strain and not on the history of strain states at the point.

Elastic potential

Elastic Potential

The internal energy balance equation for ADIABATIC linear elastic solid is:

$$\underbrace{\frac{d}{dt}\int\int\int_V \rho_0 \; u \; dV}_{\text{internal energy}} = \int\int\int_V \frac{d(\rho_0 \;\; u)}{dt} \; dV$$

$$= \underbrace{\int\int\int_V \underline{\underline{\boldsymbol{\sigma}}} : \underbrace{\underline{\underline{\boldsymbol{d}}}}_{\underline{\underline{\dot{\boldsymbol{\varepsilon}}}}} \; dV}_{\text{stress power}} + \underbrace{\int\int\int_V \rho_0 \; r \; dV - \int\int_V \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{q}} \; dV}_{\text{heat variation}}$$

In local form:

$$\frac{d}{dt}(\rho_0 u) = \underline{\underline{\boldsymbol{\sigma}}} : \underline{\underline{\dot{\boldsymbol{\varepsilon}}}} + \underbrace{\left(\rho_0 \;\; r - \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{q}}\right)}_{=0}$$

where

- u is the specific internal energy (energy per unit of mass)
- r is the specific heat generated be internal sources
- $\vec{\boldsymbol{q}}$ is the heat conduction flux vector per unit surface

Elastic Potential

Note that the deformation rate tensor $\underline{\underline{\boldsymbol{d}}}$ is connected with the material strain tensor $\underline{\underline{\boldsymbol{E}}}$ through the material derivative as follows:

$$\dot{\underline{\underline{\boldsymbol{E}}}} = \underline{\underline{\boldsymbol{F}}}^T \cdot \underline{\underline{\boldsymbol{d}}} \cdot \underline{\underline{\boldsymbol{F}}}$$

In our case: $\dot{\underline{\underline{\boldsymbol{E}}}} = \dot{\underline{\underline{\boldsymbol{\varepsilon}}}}$ and $\underline{\underline{\boldsymbol{F}}} = \underline{\underline{\boldsymbol{1}}}$

Elastic Potential

The stress power per unit of volume is the differential of the internal energy density $\hat{u}$, or internal energy per unit of volume:

$$\frac{d}{dt}(\underbrace{\rho_0 u}_{\hat{u}}) = \frac{d\hat{u}(\vec{\boldsymbol{x}}, t)}{dt} = \dot{\hat{u}} = \underline{\underline{\boldsymbol{\sigma}}} : \underline{\underline{\dot{\boldsymbol{\varepsilon}}}}$$

By using the symmetry of $\underline{\underline{\underline{\underline{\boldsymbol{C}}}}}$ and $\underline{\underline{\boldsymbol{\sigma}}} = \underline{\underline{\underline{\underline{\boldsymbol{C}}}}} : \underline{\underline{\boldsymbol{\varepsilon}}}$ we can conclude that:

$$\frac{d\hat{u}}{dt} = \frac{d}{dt}\frac{1}{2}(\underline{\underline{\boldsymbol{\varepsilon}}} : \underline{\underline{\underline{\underline{\boldsymbol{C}}}}} : \underline{\underline{\boldsymbol{\varepsilon}}})$$

Elastic Potential

$$\frac{d\hat{u}}{dt} = \frac{d}{dt}\frac{1}{2}(\underline{\underline{\boldsymbol{\varepsilon}}} : \underline{\underline{\underline{\underline{\boldsymbol{C}}}}} : \underline{\underline{\boldsymbol{\varepsilon}}})$$

Consequence 1:

$$\int\int\int_V \frac{d}{dt}\hat{u}(\vec{\boldsymbol{x}}, t)\, dV = \frac{d}{dt}\int\int\int_V \hat{u}(\vec{\boldsymbol{x}}, t) = \frac{d}{dt}\hat{U}(\vec{\boldsymbol{x}}, t) = \underbrace{\int\int\int_V \underline{\underline{\boldsymbol{\sigma}}} : \underline{\underline{\dot{\boldsymbol{\varepsilon}}}}\ dV}_{\text{stress power}}$$

The stress power gives raise to the internal energy.
For elastic materials the deformation energy is the internal energy.
The internal energy in elastic material is the exact differential in elastic materials.

Elastic Potential

$$\frac{d\hat{u}}{dt} = \frac{d}{dt}\frac{1}{2}\left(\underline{\underline{\boldsymbol{\varepsilon}}} : \underline{\underline{\underline{\underline{\boldsymbol{C}}}}} : \underline{\underline{\boldsymbol{\varepsilon}}}\right)$$

Consequence 2:
By integrating the time derivative of the internal energy density

$$\hat{u}(\vec{\boldsymbol{x}}, t) = \frac{1}{2}\underline{\underline{\boldsymbol{\varepsilon}}}(\vec{\boldsymbol{x}}, t) : \underline{\underline{\underline{\underline{\boldsymbol{C}}}}}(\vec{\boldsymbol{x}}, t) : \underline{\underline{\boldsymbol{\varepsilon}}}(\vec{\boldsymbol{x}}, t) + \alpha(\vec{\boldsymbol{x}})$$

Assuming that the density of the internal energy vanishes at neutral reference state $\hat{u}(\vec{\boldsymbol{x}}, t_0) = 0, \quad \forall \vec{\boldsymbol{x}}$ we conclude that $\alpha(\vec{\boldsymbol{x}}) = 0$ and:

$$\hat{u}(\underline{\underline{\boldsymbol{\varepsilon}}}) = \frac{1}{2}\underbrace{\underline{\underline{\boldsymbol{\varepsilon}}} : \underline{\underline{\underline{\underline{\boldsymbol{C}}}}}}_{\underline{\underline{\boldsymbol{\sigma}}}} : \underline{\underline{\boldsymbol{\varepsilon}}} = \frac{1}{2}\underline{\underline{\boldsymbol{\sigma}}}(\underline{\underline{\boldsymbol{\varepsilon}}}) : \underline{\underline{\boldsymbol{\varepsilon}}}$$

Due to thermodynamic restrictions the elastic energy is always positive:

$$\hat{u}(\underline{\underline{\boldsymbol{\varepsilon}}}) = \frac{1}{2}\underline{\underline{\boldsymbol{\varepsilon}}} : \underline{\underline{\underline{\underline{\boldsymbol{C}}}}} : \underline{\underline{\boldsymbol{\varepsilon}}} > 0, \quad \forall \underline{\underline{\boldsymbol{\varepsilon}}} \neq \underline{\underline{\boldsymbol{0}}}$$

Elastic Potential

The internal energy density $\hat{u}$ defines a potential for the stress tensor and therefore is called elastic potential, [1]. The stress tensor can be derived as follows:

$$\frac{\partial \hat{u}(\underline{\underline{\boldsymbol{\varepsilon}}}(\vec{\boldsymbol{x}}, t))}{\partial \underline{\underline{\boldsymbol{\varepsilon}}}} = \frac{\partial}{\partial \underline{\underline{\boldsymbol{\varepsilon}}}} \frac{1}{2} (\underline{\underline{\boldsymbol{\varepsilon}}} : \underline{\underline{\underline{\underline{\boldsymbol{C}}}}} : \underline{\underline{\boldsymbol{\varepsilon}}}) = \frac{1}{2} \underline{\underline{\underline{\underline{\boldsymbol{C}}}}} : \underline{\underline{\boldsymbol{\varepsilon}}} + \frac{1}{2} \underline{\underline{\boldsymbol{\varepsilon}}} : \underline{\underline{\underline{\underline{\boldsymbol{C}}}}} = \frac{1}{2} (\underline{\underline{\boldsymbol{\sigma}}} + \underline{\underline{\boldsymbol{\sigma}}}^T) = \underline{\underline{\boldsymbol{\sigma}}} \Rightarrow \underline{\underline{\boldsymbol{\sigma}}} = \frac{\partial \hat{u}(\underline{\underline{\boldsymbol{\varepsilon}}}(\vec{\boldsymbol{x}}, t))}{\partial \underline{\underline{\boldsymbol{\varepsilon}}}}$$

The constitutive elastic constants tensor is the second derivative of the internal energy density with respect to the strain tensor field:

$$\frac{\partial \underline{\underline{\boldsymbol{\sigma}}}(\underline{\underline{\boldsymbol{\varepsilon}}})}{\partial \underline{\underline{\boldsymbol{\varepsilon}}}} = \frac{\partial^2 \hat{u}(\underline{\underline{\boldsymbol{\varepsilon}}})}{\partial \underline{\underline{\boldsymbol{\varepsilon}}} \otimes \partial \underline{\underline{\boldsymbol{\varepsilon}}}} = \frac{\partial (\underline{\underline{\underline{\underline{\boldsymbol{C}}}}} : \underline{\underline{\boldsymbol{\varepsilon}}})}{\partial \underline{\underline{\boldsymbol{\varepsilon}}}} = \underline{\underline{\underline{\underline{\boldsymbol{C}}}}}; \quad C_{ijkl} = \frac{\partial^2 \hat{u}(\underline{\underline{\boldsymbol{\varepsilon}}})}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}}$$

Isotropic Linear Elasticity

Isotropic Constitutive Elastic Constants Tensor

- Isotropic elastic materials have the same properties in all directions. Orthotropic material is the one that has different properties in orthogonal directions.
- All components of $\underline{\underline{\underline{\underline{\boldsymbol{C}}}}}$ are independent of the orientation of the Cartesian system.
- $\underline{\underline{\underline{\underline{\boldsymbol{C}}}}}$ is an isotropic tensor:

$$\underline{\underline{\underline{\underline{\boldsymbol{C}}}}} = \lambda \underline{\underline{\mathbf{1}}} \otimes \underline{\underline{\mathbf{1}}} + 2\mu \underline{\underline{\underline{\underline{\boldsymbol{I}}}}}$$

$$C_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \quad i, j, k, l \in \{1, 2, 3\}$$

- where $\underline{\underline{\underline{\underline{\boldsymbol{I}}}}}$ is the 4^{th} order unit tensor defined as $I_{ijkl} = \frac{1}{2}[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}]$
- λ and μ are scalar constants, namely Lamé parameters.
- Conclusion: The isotropy reduces the number of independent elastic constants from 21 to 2!!!!
- By changing the basis unit vectors the components of the tensor do not change!!!

Isotropic Constitutive Elastic Constants Tensor

- By introducing the isotropic constitutive elastic constants tensor $\underline{\underline{\underline{\underline{\boldsymbol{C}}}}} = \lambda \underline{\underline{\mathbf{1}}} \otimes \underline{\underline{\mathbf{1}}} + 2\mu \underline{\underline{\underline{\underline{\boldsymbol{I}}}}}$ into the Hooke's law $\underline{\underline{\boldsymbol{\sigma}}} = \underline{\underline{\underline{\underline{\boldsymbol{C}}}}} : \underline{\underline{\boldsymbol{\varepsilon}}}$

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} = (\lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}))\,\varepsilon_{kl} =$$

$$\lambda\delta_{ij} \underbrace{\delta_{kl}\varepsilon_{kl}}_{\varepsilon_{ll}=Tr(\underline{\underline{\boldsymbol{\varepsilon}}})} + 2\mu \underbrace{\left(\frac{1}{2}\underbrace{\delta_{ik}\delta_{jl}\varepsilon_{kl}}_{\varepsilon_{ij}} + \frac{1}{2}\underbrace{\delta_{il}\delta_{jk}\varepsilon_{kl}}_{\varepsilon_{ij}=\varepsilon_{ji}}\right)}_{=\frac{1}{2}\varepsilon_{ij}+\frac{1}{2}\varepsilon_{ij}=\varepsilon_{ij}} = \lambda Tr(\underline{\underline{\boldsymbol{\varepsilon}}})\delta_{ij} + 2\mu\varepsilon_{ij}$$

- Finally the isotropic linear constitutive equation (Hooke's law) will be:

$$\underline{\underline{\boldsymbol{\sigma}}} = \lambda Tr(\underline{\underline{\boldsymbol{\varepsilon}}})\underline{\underline{\mathbf{1}}} + 2\mu\underline{\underline{\boldsymbol{\varepsilon}}}$$

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{ll} + 2\mu\varepsilon_{ij}, \quad i,j \in \{1,2,3\}$$

Elastic potential

For the constitutive equation:

$$\underline{\underline{\boldsymbol{\sigma}}} = \lambda Tr(\underline{\underline{\boldsymbol{\varepsilon}}})\underline{\underline{\mathbf{1}}} + 2\mu\underline{\underline{\boldsymbol{\varepsilon}}}$$

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{ll} + 2\mu\varepsilon_{ij}, \quad i,j \in \{1,2,3\}$$

The internal energy density can be as follows:

$$\hat{u}(\underline{\underline{\boldsymbol{\varepsilon}}}) = \frac{1}{2}\underline{\underline{\boldsymbol{\sigma}}} : \underline{\underline{\boldsymbol{\varepsilon}}} = \frac{1}{2}\left(\lambda Tr(\underline{\underline{\boldsymbol{\varepsilon}}})\underline{\underline{\mathbf{1}}} + 2\mu\underline{\underline{\boldsymbol{\varepsilon}}}\right) : \underline{\underline{\boldsymbol{\varepsilon}}} =$$

$$= \frac{1}{2}\lambda Tr(\underline{\underline{\boldsymbol{\varepsilon}}})\underbrace{\underline{\underline{\mathbf{1}}} : \underline{\underline{\boldsymbol{\varepsilon}}}}_{Tr(\underline{\underline{\boldsymbol{\varepsilon}}})} + \frac{1}{2}2\mu\underline{\underline{\boldsymbol{\varepsilon}}} : \underline{\underline{\boldsymbol{\varepsilon}}} =$$

$$= \frac{1}{2}\lambda Tr^2(\underline{\underline{\boldsymbol{\varepsilon}}}) + \mu\underline{\underline{\boldsymbol{\varepsilon}}} : \underline{\underline{\boldsymbol{\varepsilon}}}$$

Note that the internal energy density is an elastic potential of the stress tensor, [1]:

$$\frac{\partial\hat{u}(\underline{\underline{\boldsymbol{\varepsilon}}})}{\partial\underline{\underline{\boldsymbol{\varepsilon}}}} = \underline{\underline{\boldsymbol{\sigma}}}(\underline{\underline{\boldsymbol{\varepsilon}}}) = \lambda Tr(\underline{\underline{\boldsymbol{\varepsilon}}})\underline{\underline{\mathbf{1}}} + 2\mu\underline{\underline{\boldsymbol{\varepsilon}}}$$

Inversion of the Constitutive Equation

1. $\underline{\underline{\varepsilon}}$ can be derived from Hooke's law:

$$\underline{\underline{\boldsymbol{\sigma}}} = \lambda Tr(\underline{\underline{\varepsilon}})\underline{\underline{\mathbf{1}}} + 2\mu\underline{\underline{\varepsilon}} \Rightarrow \underline{\underline{\varepsilon}} = \frac{1}{2\mu}\left(\underline{\underline{\boldsymbol{\sigma}}} - \lambda Tr(\underline{\underline{\varepsilon}})\underline{\underline{\mathbf{1}}}\right)$$

2. The trace of $\underline{\underline{\boldsymbol{\sigma}}}$ can be derived as:

$$Tr(\underline{\underline{\boldsymbol{\sigma}}}) = Tr(\lambda Tr(\underline{\underline{\varepsilon}})\underline{\underline{\mathbf{1}}} + 2\mu\underline{\underline{\varepsilon}}) = \lambda Tr(\underline{\underline{\varepsilon}}) \underbrace{Tr(\underline{\underline{\mathbf{1}}})}_{=3} + 2\mu Tr(\underline{\underline{\varepsilon}}) = (3\lambda + 2\mu) Tr(\underline{\underline{\varepsilon}})$$

3. The trace of $\underline{\underline{\varepsilon}}$ is obtained:

$$Tr(\underline{\underline{\varepsilon}}) = \frac{1}{(3\lambda + 2\mu)} Tr(\underline{\underline{\boldsymbol{\sigma}}})$$

4. The above expression is introduced in the first equation:

$$\underline{\underline{\varepsilon}} = \frac{1}{2\mu}\left(\underline{\underline{\boldsymbol{\sigma}}} - \lambda\frac{1}{(3\lambda + 2\mu)} Tr(\underline{\underline{\boldsymbol{\sigma}}})\underline{\underline{\mathbf{1}}}\right) \Rightarrow \underline{\underline{\varepsilon}} = -\frac{\lambda}{2\mu(3\lambda + 2\mu)} Tr(\underline{\underline{\boldsymbol{\sigma}}})\underline{\underline{\mathbf{1}}} + \frac{1}{2\mu}\underline{\underline{\boldsymbol{\sigma}}}$$

Inverse Isotropic Linear Elastic Constitutive Equation

- The Lamé parameters in terms of E and ν:

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \Rightarrow \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$
$$\nu = \frac{\lambda}{2(\lambda + \mu)} \Rightarrow \mu = G = \frac{E}{2(1+\nu)}$$

E is the modulus of elasticity (Young's modulus), ν is the Poisson's ratio and G is the shear modulus.

- The inverse constitutive equation (Inverse Hooke's Law) can be written as:

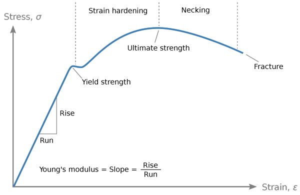
$$\underline{\underline{\boldsymbol{\varepsilon}}} = -\frac{\nu}{E} Tr(\underline{\underline{\boldsymbol{\sigma}}})\underline{\underline{\mathbf{1}}} + \frac{1+\nu}{E}\underline{\underline{\boldsymbol{\sigma}}}$$
$$\varepsilon_{ij} = -\frac{\nu}{E}\sigma_{ij}\delta_{ij} + \frac{1+\nu}{E}\sigma_{ij}, \quad i,j \in \{1,2,3\}$$

- In engineering notation:

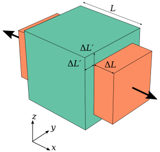
$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu(\sigma_y + \sigma_z)\right); \quad \gamma_{yz} = \frac{1}{G}\tau_{yz}$$
$$\varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu(\sigma_x + \sigma_z)\right); \quad \gamma_{xz} = \frac{1}{G}\tau_{xz}$$
$$\varepsilon_z = \frac{1}{E}\left(\sigma_z - \nu(\sigma_x + \sigma_y)\right); \quad \gamma_{xy} = \frac{1}{G}\tau_{xy}$$

Modulus of Elasticity E and Poisson's Ration ν

- Modulus of Elasticity $E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$ defines the stiffness of an elastic material body. More specifically, it is defined as the ration between the uniaxial stress over the uniaxial strain.



- Poisson's Ratio $\nu = \frac{\lambda}{2(\lambda+\mu)}$ defines the ration between transverse strain and the axial strain, when the solid is stretched uniaxially.



Example

Hydrostatic (Spherical) and Deviatoric parts of Hooke's Law

The stress tensor can be split into the spherical (hydrostatic or volumetric) part and the deviatoric:

$$\underline{\underline{\boldsymbol{\sigma}}}_{sph} = \underline{\underline{\boldsymbol{\sigma}}}_{hyd} = \sigma_m \underline{\underline{\mathbf{1}}} = \frac{1}{3} Tr(\underline{\underline{\boldsymbol{\sigma}}}) \underline{\underline{\mathbf{1}}}$$

$$\underline{\underline{\boldsymbol{\sigma}}}_{dev} = \underline{\underline{\boldsymbol{\sigma}}} - \sigma_m \underline{\underline{\mathbf{1}}}$$

$$\underline{\underline{\boldsymbol{\sigma}}} = \sigma_m \underline{\underline{\mathbf{1}}} + \underline{\underline{\boldsymbol{\sigma}}}_{dev}$$

On a similar way, the strain tensor can be written:

$$\underline{\underline{\boldsymbol{\varepsilon}}}_{sph} = \underline{\underline{\boldsymbol{\varepsilon}}}_{hyd} = \frac{1}{3} e \underline{\underline{\mathbf{1}}} = \frac{1}{3} Tr(\underline{\underline{\boldsymbol{\varepsilon}}}) \underline{\underline{\mathbf{1}}}$$

$$\underline{\underline{\boldsymbol{\varepsilon}}}_{dev} = \underline{\underline{\boldsymbol{\varepsilon}}} - \frac{1}{3} e \underline{\underline{\mathbf{1}}}$$

$$\underline{\underline{\boldsymbol{\varepsilon}}} = \frac{1}{3} e \underline{\underline{\mathbf{1}}} + \underline{\underline{\boldsymbol{\varepsilon}}}_{dev}$$

where $e = Tr(\underline{\underline{\boldsymbol{\varepsilon}}})$ is the volumetric strain

Hydrostatic (Spherical) and Deviatoric parts of Hooke's Law

Working on the volumetric strain $e = Tr(\underline{\underline{\boldsymbol{\varepsilon}}})$:

$$\underline{\underline{\boldsymbol{\varepsilon}}} = -\frac{\nu}{E}Tr(\underline{\underline{\boldsymbol{\sigma}}})\underline{\underline{\mathbf{1}}} + \frac{1+\nu}{E}\underline{\underline{\boldsymbol{\sigma}}} \Rightarrow Tr(\underline{\underline{\boldsymbol{\varepsilon}}}) = Tr\left(-\frac{\nu}{E}Tr(\underline{\underline{\boldsymbol{\sigma}}})\underline{\underline{\mathbf{1}}} + \frac{1+\nu}{E}\underline{\underline{\boldsymbol{\sigma}}}\right)$$

$$e = -\frac{\nu}{E}Tr(\underline{\underline{\boldsymbol{\sigma}}})\underbrace{Tr(\underline{\underline{\mathbf{1}}})}_{=3} + \frac{1+\nu}{E}\underbrace{Tr(\underline{\underline{\boldsymbol{\sigma}}})}_{3\sigma_m}$$

$$e = \frac{3(1-2\nu)}{E}\sigma_m \Rightarrow \sigma_m = \underbrace{\frac{E}{3(1-2\nu)}}_{K\text{:bulk modulus}} e$$

The bulk modulus K and the relationship between the spherical (volumetric) part of the stress tensor and the strain tensor, can be expressed as:

$$K = \lambda + \frac{2}{3}\mu = \frac{E}{3(1-2\nu)}; \quad \sigma_m = Ke$$

Hydrostatic (Spherical) and Deviatoric parts of Hooke's Law

Combining $\underline{\underline{\boldsymbol{\sigma}}} = \sigma_m \underline{\underline{\mathbf{1}}} + \underline{\underline{\boldsymbol{\sigma}}}_{dev}$ and $\underline{\underline{\boldsymbol{\varepsilon}}} = -\frac{\nu}{E} Tr(\underline{\underline{\boldsymbol{\sigma}}})\underline{\underline{\mathbf{1}}} + \frac{1+\nu}{E}\underline{\underline{\boldsymbol{\sigma}}}$:

$$\underline{\underline{\boldsymbol{\varepsilon}}} = -\frac{\nu}{E} Tr(\sigma_m \underline{\underline{\mathbf{1}}} + \underline{\underline{\boldsymbol{\sigma}}}_{dev})\underline{\underline{\mathbf{1}}} + \frac{1+\nu}{E}\left(\sigma_m \underline{\underline{\mathbf{1}}} + \underline{\underline{\boldsymbol{\sigma}}}_{dev}\right) =$$

$$= -\frac{\nu}{E}\sigma_m \underbrace{Tr(\underline{\underline{\mathbf{1}}})}_{=3}\underline{\underline{\mathbf{1}}} - \frac{\nu}{E}\underbrace{Tr(\underline{\underline{\boldsymbol{\sigma}}}_{dev})}_{=0}\underline{\underline{\mathbf{1}}} + \sigma_m \frac{1+\nu}{E}\underline{\underline{\mathbf{1}}} + \frac{1+\nu}{E}\underline{\underline{\boldsymbol{\sigma}}}_{dev} =$$

$$= \left(\frac{1+\nu}{E} - \frac{3\nu}{E}\right)\sigma_m \underline{\underline{\mathbf{1}}} + \frac{1+\nu}{E}\underline{\underline{\boldsymbol{\sigma}}}_{dev}$$

Considering that $\sigma_m = \frac{E}{3(1-2\nu)} e$:

$$\underline{\underline{\boldsymbol{\varepsilon}}} = \left(\frac{1-2\nu}{E}\right)\frac{1}{3}\frac{E}{(1-2\nu)} e\underline{\underline{\mathbf{1}}} + \frac{1+\nu}{E}\underline{\underline{\boldsymbol{\sigma}}}_{dev} = \frac{1}{3} e\underline{\underline{\mathbf{1}}} + \frac{1+\nu}{E}\underline{\underline{\boldsymbol{\sigma}}}_{dev} = \frac{1}{3} e\underline{\underline{\mathbf{1}}} + \underline{\underline{\boldsymbol{\varepsilon}}}_{dev}$$

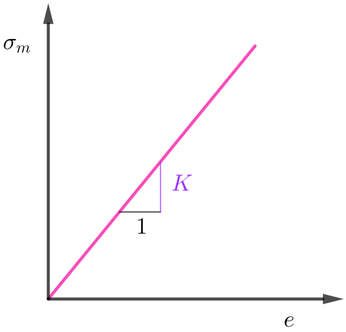
$$\underline{\underline{\boldsymbol{\varepsilon}}}_{dev} = \frac{1+\nu}{E}\underline{\underline{\boldsymbol{\sigma}}}_{dev}; \qquad \frac{1+\nu}{E} = \frac{1}{2\mu} = \frac{1}{2G}$$

The relationship between deviatoric stress and strain is: $\underline{\underline{\boldsymbol{\sigma}}}_{dev} = 2G\underline{\underline{\boldsymbol{\varepsilon}}}_{dev} \Rightarrow \sigma_{ij}^{dev} = 2G\varepsilon_{ij}^{dev},\ i,j \in \{1,2,3\}$

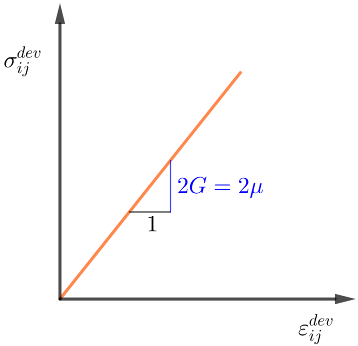
Hydrostatic (Spherical) and Deviatoric parts of Hooke's Law

There is a direct proportional relationship between deviatoric and spherical parts of strain tensor to the deviatoric and spherical parts of the stress tensor. The deviatoric components of the stress tensor are component by component proportional to the components of the deviatoric strain tensor:

$$\sigma_m = Ke$$



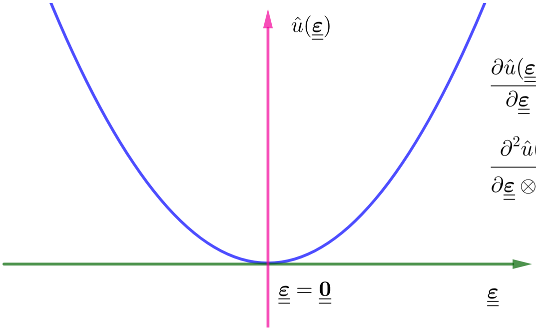
$$\sigma_{ij}^{dev} = 2G\varepsilon_{ij}^{dev}$$



Elastic Potential

Internal energy density $\hat{u}(\underline{\underline{\varepsilon}})$ defines a potential for the stress tensor:

$$\hat{u}(\underline{\underline{\boldsymbol{\varepsilon}}}) = \frac{1}{2}\underline{\underline{\boldsymbol{\varepsilon}}} : \underline{\underline{\underline{\underline{\boldsymbol{C}}}}} : \underline{\underline{\boldsymbol{\varepsilon}}} \Rightarrow \underline{\underline{\boldsymbol{\sigma}}} = \frac{\partial \hat{u}(\underline{\underline{\boldsymbol{\varepsilon}}})}{\partial \underline{\underline{\boldsymbol{\varepsilon}}}} = \underline{\underline{\underline{\underline{\boldsymbol{C}}}}} : \underline{\underline{\boldsymbol{\varepsilon}}}$$



$$\left.\frac{\partial \hat{u}(\underline{\underline{\boldsymbol{\varepsilon}}})}{\partial \underline{\underline{\boldsymbol{\varepsilon}}}}\right|_{\underline{\underline{\boldsymbol{\varepsilon}}}=\underline{\underline{\mathbf{0}}}} = \left.\left(\underline{\underline{\underline{\underline{\mathbf{C}}}}} : \underline{\underline{\boldsymbol{\varepsilon}}}\right)\right|_{\underline{\underline{\boldsymbol{\varepsilon}}}=\underline{\underline{\mathbf{0}}}} = \underline{\underline{\mathbf{0}}}$$

$$\left.\frac{\partial^2 \hat{u}(\underline{\underline{\boldsymbol{\varepsilon}}})}{\partial \underline{\underline{\boldsymbol{\varepsilon}}} \otimes \partial \underline{\underline{\boldsymbol{\varepsilon}}}}\right|_{\underline{\underline{\boldsymbol{\varepsilon}}}=\underline{\underline{\mathbf{0}}}} = \left.\underline{\underline{\underline{\underline{\mathbf{C}}}}}\right|_{\underline{\underline{\boldsymbol{\varepsilon}}}=\underline{\underline{\mathbf{0}}}} = \underline{\underline{\underline{\underline{\mathbf{C}}}}}$$

Note: the constitutive elastic constants tensor $\underline{\underline{\underline{\underline{\boldsymbol{C}}}}}$ is constant due to thermodynamic restrictions.

Elastic Potential

The elastic potential can be expressed in terms of spherical and deviatoric parts:

$$\hat{u}(\underline{\underline{\boldsymbol{\varepsilon}}}) = \frac{1}{2}\underline{\underline{\boldsymbol{\varepsilon}}} : \underline{\underline{\underline{\underline{\boldsymbol{C}}}}} : \underline{\underline{\boldsymbol{\varepsilon}}} = \frac{1}{2}\underline{\underline{\boldsymbol{\sigma}}} : \underline{\underline{\boldsymbol{\varepsilon}}} = \frac{1}{2}\underbrace{\left[\lambda Tr(\underline{\underline{\boldsymbol{\varepsilon}}})\underline{\underline{\mathbf{1}}} + 2\mu\underline{\underline{\boldsymbol{\varepsilon}}}\right]}_{=\underline{\underline{\boldsymbol{\sigma}}}} : \underline{\underline{\boldsymbol{\varepsilon}}} =$$

$$= \frac{1}{2}\lambda Tr(\underline{\underline{\boldsymbol{\varepsilon}}}) \underbrace{\underline{\underline{\mathbf{1}}} : \underline{\underline{\boldsymbol{\varepsilon}}}}_{=Tr(\underline{\underline{\boldsymbol{\varepsilon}}})=e} + \mu\underline{\underline{\boldsymbol{\varepsilon}}} : \underline{\underline{\boldsymbol{\varepsilon}}}$$

Note that:

$$\underline{\underline{\boldsymbol{\varepsilon}}} : \underline{\underline{\boldsymbol{\varepsilon}}} = \left(\frac{1}{3}e\underline{\underline{\mathbf{1}}} + \underline{\underline{\boldsymbol{\varepsilon}}}_{dev}\right) : \left(\frac{1}{3}e\underline{\underline{\mathbf{1}}} + \underline{\underline{\boldsymbol{\varepsilon}}}_{dev}\right) = \frac{1}{9}e^2\underbrace{\underline{\underline{\mathbf{1}}} : \underline{\underline{\mathbf{1}}}}_{=3} + \frac{2}{3}e\underbrace{\underline{\underline{\mathbf{1}}} : \underline{\underline{\boldsymbol{\varepsilon}}}_{dev}}_{=Tr(\underline{\underline{\boldsymbol{\varepsilon}}}_{dev})=0} + \underline{\underline{\boldsymbol{\varepsilon}}}_{dev} : \underline{\underline{\boldsymbol{\varepsilon}}}_{dev} = \frac{1}{3}e^2 + \underline{\underline{\boldsymbol{\varepsilon}}}_{dev} : \underline{\underline{\boldsymbol{\varepsilon}}}_{dev}$$

Finally:

$$\hat{u}(\underline{\underline{\boldsymbol{\varepsilon}}}) = \frac{1}{2}\lambda e^2 + \frac{1}{3}\mu e^2 + \mu\underline{\underline{\boldsymbol{\varepsilon}}}_{dev} : \underline{\underline{\boldsymbol{\varepsilon}}}_{dev} = \frac{1}{2}\underbrace{\left(\lambda + \frac{2}{3}\mu\right)}_{=K} + \mu\underline{\underline{\boldsymbol{\varepsilon}}}_{dev} : \underline{\underline{\boldsymbol{\varepsilon}}}_{dev}$$

$$\hat{u}(\underline{\underline{\boldsymbol{\varepsilon}}}) = \frac{1}{2}Ke^2 + \mu\underline{\underline{\boldsymbol{\varepsilon}}}_{dev} : \underline{\underline{\boldsymbol{\varepsilon}}}_{dev} = \frac{1}{2}Ke^2 + G\underline{\underline{\boldsymbol{\varepsilon}}}_{dev} : \underline{\underline{\boldsymbol{\varepsilon}}}_{dev} \geq 0$$

The last equation represents the elastic potential in terms of spherical and deviatoric terms.

Elastic Properties Limits

The expression holds true for any deformation process:

$$\hat{u}(\underline{\underline{\varepsilon}}) = \frac{1}{2}Ke^2 + G\underline{\underline{\varepsilon}}_{dev} : \underline{\underline{\varepsilon}}_{dev} \geq 0$$

For particular case of isotropic linear elastic material we have:

- Pure spherical (hydrostatic) deformation process:

$$\left.\begin{aligned} \underline{\underline{\varepsilon}}^{(1)} &= \tfrac{1}{3}e\underline{\underline{\mathbf{1}}} \\ \underline{\underline{\varepsilon}}^{dev(1)} &= \underline{\underline{\mathbf{0}}} \end{aligned}\right\} \Rightarrow \hat{u}^{(1)} = \frac{1}{2}Ke^2 \geq 0 \Rightarrow K > 0 \quad \text{bulk modulus}$$

- Pure deviatoric deformation process:

$$\left.\begin{aligned} \underline{\underline{\varepsilon}}^{(2)} &= \underline{\underline{\varepsilon}}^{(dev)} \\ e^{(2)} &= 0 \end{aligned}\right\} \Rightarrow \hat{u}^{(2)} = G\underline{\underline{\varepsilon}}_{dev} : \underline{\underline{\varepsilon}}_{dev} \geq 0 \Rightarrow G > 0 \quad \text{shear modulus}$$

Note that: $\underline{\underline{\varepsilon}}_{dev} : \underline{\underline{\varepsilon}}_{dev} = \varepsilon_{ij}^{(dev)}\varepsilon_{ij}^{(dev)} \geq 0$

Elastic Properties Limits

The bulk modulus K and shear modulus G are related with modulus of elasticity E and Poisson's ratio ν as follows:

$$K = \frac{E}{3(1-2\nu)} > 0; \quad G = \mu = \frac{E}{2(1+\nu)} > 0$$

- Poisson's ratio is non-negative

$$\left.\begin{array}{l} \frac{E}{2(1+\nu)} > 0 \\ \nu \geq 0 \end{array}\right\} \Rightarrow E \geq 0 \quad \text{modulus of elasticity}$$

- Therefore

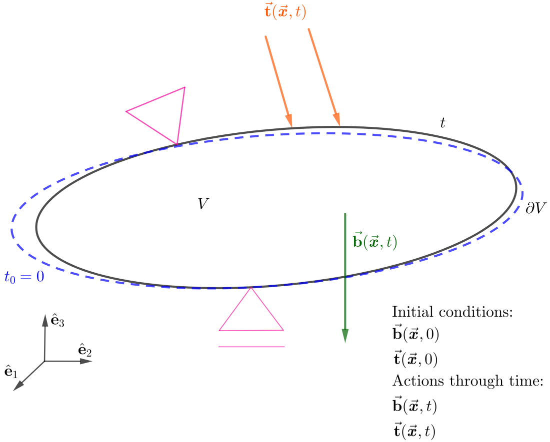
$$\left.\begin{array}{l} \frac{E}{3(1-2\nu)} > 0 \\ E \geq 0 \end{array}\right\} \Rightarrow 0 \leq \nu \leq \frac{1}{2} \quad \text{Poisson's ratio}$$

In some rare cases, materials can have negative Poisson's ratio. These materials are called auxetic materials.

The Linear Elastic Problem

Intro

The linear elastic solid is subjected to body forces and surface tractions:



The set of the elastic problem allows to obtain the displacements $\vec{u}(\vec{x}, t)$, strains $\underline{\underline{\varepsilon}}(\vec{x}, t)$ and stresses $\underline{\underline{\sigma}}(\vec{x}, t)$.

Governing Equations

The equations governing the problem:

1. Cauchy's Equation of Motion: Linear Momentum Balance

$$\vec{\boldsymbol{\nabla}} \cdot \underline{\underline{\boldsymbol{\sigma}}}(\vec{\boldsymbol{x}}, t) + \rho \vec{\boldsymbol{b}}(\vec{\boldsymbol{x}}, t) = \rho \frac{d\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}, t)}{dt}$$

2. Constitutive equation: Isotropic linear elastic

$$\underline{\underline{\boldsymbol{\sigma}}}(\vec{\boldsymbol{x}}, t) = \lambda Tr(\underline{\underline{\boldsymbol{\varepsilon}}})\underline{\underline{\mathbf{1}}} + 2\mu \underline{\underline{\boldsymbol{\varepsilon}}}$$

3. Geometric equation: Kinematic compatibility (relationship between displacement and strain)

$$\underline{\underline{\boldsymbol{\varepsilon}}}(\vec{\boldsymbol{x}}, t) = \frac{1}{2}\left(\vec{\boldsymbol{u}} \otimes \vec{\boldsymbol{\nabla}} + \vec{\boldsymbol{\nabla}} \otimes \vec{\boldsymbol{u}}\right)$$

A PDE system of 15 eqns with 15 unknowns: $\vec{\boldsymbol{u}}(\vec{\boldsymbol{x}}, t)$ 3 unknowns, $\underline{\underline{\boldsymbol{\varepsilon}}}(\vec{\boldsymbol{x}}, t)$ 6 unknowns, $\underline{\underline{\boldsymbol{\sigma}}}(\vec{\boldsymbol{x}}, t)$ 6 unknowns. The problem needs to be solved in the space $\mathbb{R}^3 \times \mathbb{R}$.

Boundary conditions

Boundary conditions in space affect the spatial arguments and are applied on the boundary Γ of the solid. They are divided in three parts:

- Prescribed displacements on Γ_u:

$$\vec{\boldsymbol{u}}(\vec{\boldsymbol{x}}, t) = \vec{\boldsymbol{u}}^*(\vec{\boldsymbol{x}}, t); \quad \forall \vec{\boldsymbol{x}} \in \Gamma_u \quad \forall t$$
$$u_i(\vec{\boldsymbol{x}}, t) = u_i^*(\vec{\boldsymbol{x}}, t); \quad i \in \{1, 2, 3\}$$

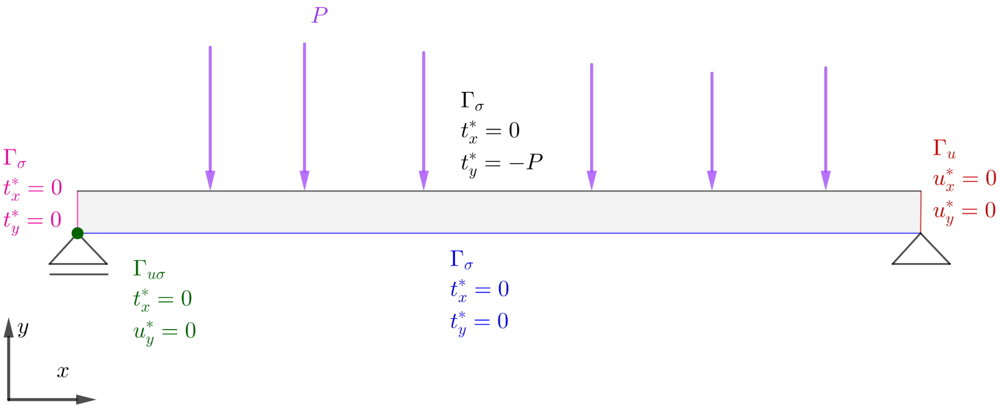
- Prescribed tractions on Γ_σ:

$$\underline{\underline{\boldsymbol{\sigma}}}(\vec{\boldsymbol{x}}, t) \cdot \hat{\boldsymbol{n}} = \vec{\boldsymbol{t}}^*(\vec{\boldsymbol{x}}, t); \quad \forall \vec{\boldsymbol{x}} \in \Gamma_\sigma \quad \forall t$$
$$\sigma_{ij}(\vec{\boldsymbol{x}}, t) n_j = t_i^*(\vec{\boldsymbol{x}}, t); \quad i, j \in \{1, 2, 3\}$$

- Prescribed displacements and stresses on $\Gamma_{u\sigma}$:

$$u_i(\vec{\boldsymbol{x}}, t) = u_i^*(\vec{\boldsymbol{x}}, t); \quad \forall \vec{\boldsymbol{x}} \in \Gamma_{u\sigma} \quad \forall t$$
$$\sigma_{jk}(\vec{\boldsymbol{x}}, t) n_k = t_j^*(\vec{\boldsymbol{x}}, t); \quad i, j, k \in \{1, 2, 3\}$$

Boundary conditions



Boundary conditions

- The boundary conditions with regard the time are called initial conditions.
- Affects the unknowns due to the fact that they are functions of $\vec{\boldsymbol{x}}$ and t.
- The unknown values are at $t = 0$:
  1. Initial displacements:
  $$\vec{\boldsymbol{u}}(\vec{\boldsymbol{x}}, 0) = \vec{\mathbf{0}}; \quad \forall \vec{\boldsymbol{x}} \in V$$
  2. Initial velocity:
  $$\frac{\partial \vec{\boldsymbol{u}}(\vec{\boldsymbol{x}}, 0)}{\partial t}|_{t=0} = \dot{\vec{\boldsymbol{u}}}(\vec{\boldsymbol{x}}, 0) = \vec{\boldsymbol{v}}_0(\vec{\boldsymbol{x}}); \quad \forall \vec{\boldsymbol{x}} \in V$$

Linear Elastic Problem

Knowing:

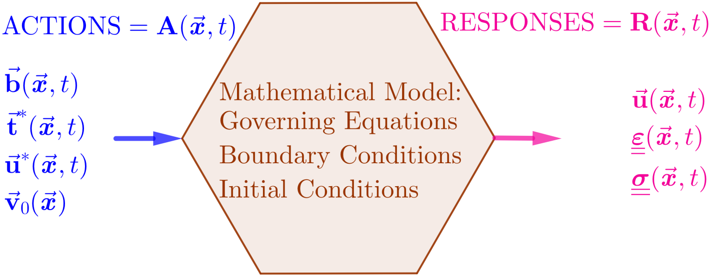
- $\vec{\boldsymbol{\nabla}} \cdot \underline{\underline{\boldsymbol{\sigma}}}(\vec{\boldsymbol{x}}, t) + \rho\vec{\boldsymbol{b}}(\vec{\boldsymbol{x}}, t) = \rho\frac{d\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}},t)}{dt}$ Cauchy's Equation of Motion
- $\underline{\underline{\boldsymbol{\sigma}}}(\vec{\boldsymbol{x}}, t) = \lambda Tr(\underline{\underline{\boldsymbol{\varepsilon}}})\underline{\underline{\mathbf{1}}} + 2\mu\underline{\underline{\boldsymbol{\varepsilon}}}$ Constitutive Equation
- $\underline{\underline{\boldsymbol{\varepsilon}}}(\vec{\boldsymbol{x}}, t) = \frac{1}{2}\left(\vec{\boldsymbol{u}} \otimes \vec{\boldsymbol{\nabla}} + \vec{\boldsymbol{\nabla}} \otimes \vec{\boldsymbol{u}}\right)$ Geometric Equation
- $\Gamma_u : \vec{\boldsymbol{u}} = \vec{\boldsymbol{u}}^*$, $\Gamma_\sigma : \vec{\boldsymbol{t}}^* = \underline{\underline{\boldsymbol{\sigma}}} \cdot \hat{\boldsymbol{n}}$ Boundary conditions
- $\vec{\boldsymbol{u}}(\vec{\boldsymbol{x}}, 0) = \vec{\mathbf{0}}$, $\dot{\vec{\boldsymbol{u}}}(\vec{\boldsymbol{x}}, 0) = \vec{\boldsymbol{v}}_0(\vec{\boldsymbol{x}})$ Initial conditions

Find:

the displacements $\vec{\boldsymbol{u}}(\vec{\boldsymbol{x}}, t)$, strains $\underline{\underline{\boldsymbol{\varepsilon}}}(\vec{\boldsymbol{x}}, t)$ and stresses $\underline{\underline{\boldsymbol{\sigma}}}(\vec{\boldsymbol{x}}, t)$.

Actions and Responses

- The linear elastic problem can be seen as a system (mathematical model built upon the governing equations, the boudnary and initial conditions) where the input are the actions and the output are the responses (displacement, strain, stresses)



- The dynamic problem can is integrated in the space $\mathbb{R}^3 \times \mathbb{R}$
- The quasi-static problem is integrated in the space $\mathbb{R}^3$

The Quasi-Static Problem

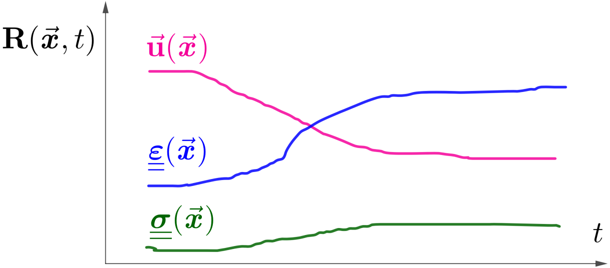
The Quasi-Static Problem

In the case where the acceleration term can be neglected, the problem can be reduced from dynamic to quasi-static:

$$\vec{\boldsymbol{a}} = \frac{\partial^2 \vec{\boldsymbol{u}}(\vec{\boldsymbol{x}}, t)}{\partial t^2} = \vec{\mathbf{0}}$$

More specifically, the assumption is valid in the cases where the application of the actions is introduced slowly and then

$$\frac{\partial^2 \boldsymbol{A}}{\partial t^2} \approx \mathbf{0} \Rightarrow \frac{\partial^2 \boldsymbol{R}}{\partial t^2} \approx \mathbf{0} \Rightarrow \frac{\partial^2 \vec{\boldsymbol{u}}(\vec{\boldsymbol{x}}, t)}{\partial t^2} = \vec{\mathbf{0}}$$



Quasi-Static Problem - Example

Zero-Gravity: Kinematic Pavilion

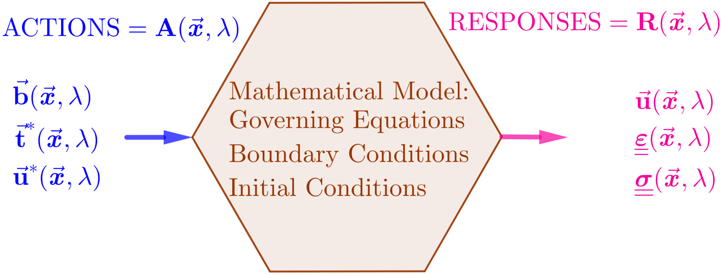
Quasi-Static Problem

Find the displacements $\vec{\boldsymbol{u}}(\vec{\boldsymbol{x}}, t)$, strains $\underline{\underline{\boldsymbol{\varepsilon}}}(\vec{\boldsymbol{x}}, t)$ and stresses $\underline{\underline{\boldsymbol{\sigma}}}(\vec{\boldsymbol{x}}, t)$ from:

- $\vec{\boldsymbol{\nabla}} \cdot \underline{\underline{\boldsymbol{\sigma}}}(\vec{\boldsymbol{x}}, t) + \rho \vec{\boldsymbol{b}}(\vec{\boldsymbol{x}}, t) = \vec{\boldsymbol{0}}$ Equilibrium Equation
- $\underline{\underline{\boldsymbol{\sigma}}}(\vec{\boldsymbol{x}}, t) = \lambda Tr(\underline{\underline{\boldsymbol{\varepsilon}}})\underline{\underline{\boldsymbol{1}}} + 2\mu \underline{\underline{\boldsymbol{\varepsilon}}}$ Constitutive Equation
- $\underline{\underline{\boldsymbol{\varepsilon}}}(\vec{\boldsymbol{x}}, t) = \frac{1}{2}\left(\vec{\boldsymbol{u}} \otimes \vec{\boldsymbol{\nabla}} + \vec{\boldsymbol{\nabla}} \otimes \vec{\boldsymbol{u}}\right)$ Geometric Equation
- $\Gamma_u : \vec{\boldsymbol{u}} = \vec{\boldsymbol{u}}^*$, $\Gamma_\sigma : \vec{\boldsymbol{t}}^* = \underline{\underline{\boldsymbol{\sigma}}} \cdot \hat{\boldsymbol{n}}$ Boundary conditions
- $\cancel{\vec{\boldsymbol{u}}(\vec{\boldsymbol{x}}, 0) = \vec{\boldsymbol{0}},\ \dot{\vec{\boldsymbol{u}}}(\vec{\boldsymbol{x}}, 0) = \vec{\boldsymbol{v}}_0(\vec{\boldsymbol{x}})}$ Initial conditions

The Quasi-Static Problem

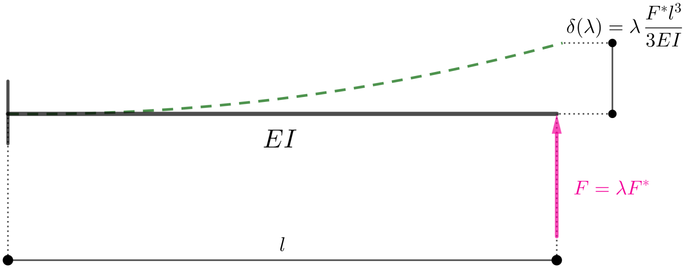
- The quasi-static linear elastic problem does not involve TIME DERIVATIVES. The time derivative describes the evolution of actions.



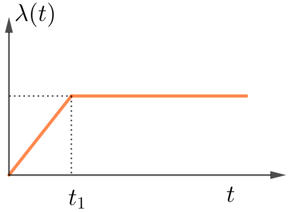
- For each value of actions $\boldsymbol{A}(\vec{x}, \lambda^*)$, with a characteristic value λ^* a response $\boldsymbol{R}(\vec{x}, \lambda^*)$ is derived
- By varying λ^*, a group of actions and the corresponding family of responses is obtained.

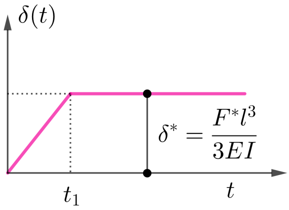
The Quasi-Static Problem - Example

- A cantilever is subjected to a force $F(t)$ at its tip. For the quasi-static problem, [1]:



- The response $\delta(t) = \delta\lambda(t)$ and at every time instant depends on the value of $\lambda(t)$





Solution of the Linear Elastic Problem

Solution of the Linear Elastic Problem

To solve the isotropic linear elastic problem posed, two approaches:

- Displacement formulation - Navier Equations
  Eliminate stress $\underline{\underline{\boldsymbol{\sigma}}}(\vec{\boldsymbol{x}}, t)$ and strain $\underline{\underline{\boldsymbol{\varepsilon}}}(\vec{\boldsymbol{x}}, t)$ from the general equations. Solve the system of 3 equations to find the 3 unknown components of $\vec{\boldsymbol{u}}(\vec{\boldsymbol{x}}, t)$:
  - Useful with displacement BCs
  - Avoids compatibility equations
  - Used in 3D problems
  - Basis of most numerical methods
- Stress formulation - Beltrami-Mitchell Equations
  Eliminate the displacement $\vec{\boldsymbol{u}}(\vec{\boldsymbol{x}}, t)$ and the strain $\underline{\underline{\boldsymbol{\varepsilon}}}(\vec{\boldsymbol{x}}, t)$ from the general equations. Solve the system of 6 equations to find the 6 unknown components of stress $\underline{\underline{\boldsymbol{\sigma}}}(\vec{\boldsymbol{x}}, t)$:
  - Useful when BCs are given in terms of stresses
  - Work with compatibility equations
  - Used in 2D problems
  - Can be used only in the quasi-static problem

Displacement Formulation

Displacement Formulation

The goal is to reduce the system to a system with only unknowns $\vec{\boldsymbol{u}}(\vec{\boldsymbol{x}}, t)$. Then the strain tensor $\underline{\underline{\boldsymbol{\varepsilon}}}(\vec{\boldsymbol{x}}, t)$ and the stress tensor $\underline{\underline{\boldsymbol{\sigma}}}(\vec{\boldsymbol{x}}, t)$ can be derived.

$\vec{\boldsymbol{\nabla}} \cdot \underline{\underline{\boldsymbol{\sigma}}}(\vec{\boldsymbol{x}}, t) + \rho_0 \vec{\boldsymbol{b}}(\vec{\boldsymbol{x}}, t) = \rho_0 \frac{d\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}, t)}{dt}$	Cauchy's Equation of motion
$\underline{\underline{\boldsymbol{\sigma}}}(\vec{\boldsymbol{x}}, t) = \lambda Tr(\underline{\underline{\boldsymbol{\varepsilon}}})\underline{\underline{\mathbf{1}}} + 2\mu\underline{\underline{\boldsymbol{\varepsilon}}}$	Constitutive Equation
$\underline{\underline{\boldsymbol{\varepsilon}}}(\vec{\boldsymbol{x}}, t) = \frac{1}{2}\left(\vec{\boldsymbol{u}} \otimes \vec{\boldsymbol{\nabla}} + \vec{\boldsymbol{\nabla}} \otimes \vec{\boldsymbol{u}}\right)$	Geometric Equation
$\Gamma_u : \vec{\boldsymbol{u}} = \vec{\boldsymbol{u}}^*$ $\Gamma_\sigma : \vec{\boldsymbol{t}}^* = \underline{\underline{\boldsymbol{\sigma}}} \cdot \hat{\boldsymbol{n}}$	Boundary conditions
$\vec{\boldsymbol{u}}(\vec{\boldsymbol{x}}, 0) = \vec{\mathbf{0}}$ $\dot{\vec{\boldsymbol{u}}}(\vec{\boldsymbol{x}}, 0) = \vec{\boldsymbol{v}}_0(\vec{\boldsymbol{x}})$	Initial conditions

Displacement Formulation

By introducing the constitutive equation into the Cauchy's eqaution of motion:

$$\left.\begin{array}{l} \underline{\underline{\boldsymbol{\sigma}}}(\vec{\boldsymbol{x}},t) = \lambda Tr(\underline{\underline{\boldsymbol{\varepsilon}}})\underline{\underline{\mathbf{1}}} + 2\mu\underline{\underline{\boldsymbol{\varepsilon}}} \\ \vec{\boldsymbol{\nabla}}\cdot\underline{\underline{\boldsymbol{\sigma}}}(\vec{\boldsymbol{x}},t) + \rho_0\vec{\boldsymbol{b}}(\vec{\boldsymbol{x}},t) = \rho_0\frac{\partial^2\vec{\boldsymbol{u}}(\vec{\boldsymbol{x}},t)}{\partial t^2} \end{array}\right\} \Rightarrow \lambda\vec{\boldsymbol{\nabla}}\cdot\left[Tr(\underline{\underline{\boldsymbol{\varepsilon}}})\underline{\underline{\mathbf{1}}}\right] + 2\mu\vec{\boldsymbol{\nabla}}\cdot\underline{\underline{\boldsymbol{\varepsilon}}} + \rho_0\vec{\boldsymbol{b}} = \rho_0\frac{\partial^2\vec{\boldsymbol{u}}(\vec{\boldsymbol{x}},t)}{\partial t^2}$$

The following identities hold:

$$\left[\vec{\boldsymbol{\nabla}}\cdot Tr(\underline{\underline{\boldsymbol{\varepsilon}}})\underline{\underline{\mathbf{1}}}\right]_i = \frac{\partial}{\partial x_j}\left(\varepsilon_{kk}\delta_{ij}\right) = \frac{\partial}{\partial x_j}\left[\frac{\partial u_k}{\partial x_k}\delta_{ij}\right] = \frac{\partial}{\partial x_i}\underbrace{\left[\frac{\partial u_k}{\partial x_k}\right]}_{=\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{u}}} = \underbrace{\frac{\partial}{\partial x_i}\left(\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{u}}\right)}_{=\vec{\boldsymbol{\nabla}}\left(\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{u}}\right)} = \left[\vec{\boldsymbol{\nabla}}\left(\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{u}}\right)\right]_i$$

$$i,j,k \in \{1,2,3\}$$

Finally, we conclude that:

$$\vec{\boldsymbol{\nabla}}\cdot\left(Tr(\underline{\underline{\boldsymbol{\varepsilon}}})\underline{\underline{\mathbf{1}}}\right) = \vec{\boldsymbol{\nabla}}\left(\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{u}}\right)$$

Displacement Formulation

By introducing the constitutive equation into the Cauchy's eqation of motion:

$$\left.\begin{array}{l} \underline{\underline{\boldsymbol{\sigma}}}(\vec{\boldsymbol{x}},t) = \lambda Tr(\underline{\underline{\boldsymbol{\varepsilon}}})\underline{\underline{\mathbf{1}}} + 2\mu\underline{\underline{\boldsymbol{\varepsilon}}} \\ \vec{\boldsymbol{\nabla}} \cdot \underline{\underline{\boldsymbol{\sigma}}}(\vec{\boldsymbol{x}},t) + \rho_0 \vec{\boldsymbol{b}}(\vec{\boldsymbol{x}},t) = \rho_0 \frac{\partial^2 \vec{\boldsymbol{u}}(\vec{\boldsymbol{x}},t)}{\partial t^2} \end{array}\right\} \Rightarrow \lambda \vec{\boldsymbol{\nabla}} \cdot \left[Tr(\underline{\underline{\boldsymbol{\varepsilon}}})\underline{\underline{\mathbf{1}}}\right] + 2\mu \vec{\boldsymbol{\nabla}} \cdot \underline{\underline{\boldsymbol{\varepsilon}}} + \rho_0 \vec{\boldsymbol{b}} = \rho_0 \frac{\partial^2 \vec{\boldsymbol{u}}(\vec{\boldsymbol{x}},t)}{\partial t^2}$$

The following identities hold:

$$\left[\vec{\boldsymbol{\nabla}} \cdot \underline{\underline{\boldsymbol{\varepsilon}}}\right]_i = \frac{\partial \varepsilon_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right] = \frac{1}{2}\underbrace{\frac{\partial^2 u_i}{\partial x_j \partial x_j}}_{=\left(\vec{\boldsymbol{\nabla}}^2 \vec{\boldsymbol{u}}\right)_i} + \frac{1}{2}\frac{\partial}{\partial x_i}\underbrace{\frac{\partial u_j}{\partial x_j}}_{=\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{u}}} =$$

$$= \frac{1}{2}\left(\vec{\boldsymbol{\nabla}}^2 \vec{\boldsymbol{u}}\right)_i + \frac{1}{2}\underbrace{\frac{\partial}{\partial x_i}(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{u}})}_{=\left(\vec{\boldsymbol{\nabla}}(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{u}})\right)_i} = \left[\frac{1}{2}\vec{\boldsymbol{\nabla}}^2 \vec{\boldsymbol{u}} + \frac{1}{2}\vec{\boldsymbol{\nabla}}\left(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{u}}\right)\right]_i \quad i,j \in \{1,2,3\}$$

Finally, we conclude that:

$$\vec{\boldsymbol{\nabla}} \cdot \underline{\underline{\boldsymbol{\varepsilon}}} = \frac{1}{2}\vec{\boldsymbol{\nabla}}^2 \vec{\boldsymbol{u}} + \frac{1}{2}\vec{\boldsymbol{\nabla}}\left(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{u}}\right)$$

Displacement Formulation - Navier Equations

By introducing the constitutive equation into the Cauchy's eqaution of motion:

$$\left.\begin{array}{l} \underline{\underline{\boldsymbol{\sigma}}}(\vec{\boldsymbol{x}},t) = \lambda Tr(\underline{\underline{\boldsymbol{\varepsilon}}})\underline{\underline{\mathbf{1}}} + 2\mu\underline{\underline{\boldsymbol{\varepsilon}}} \\ \vec{\boldsymbol{\nabla}}\cdot\underline{\underline{\boldsymbol{\sigma}}}(\vec{\boldsymbol{x}},t) + \rho_0\vec{\boldsymbol{b}}(\vec{\boldsymbol{x}},t) = \rho_0\frac{\partial^2\vec{\boldsymbol{u}}(\vec{\boldsymbol{x}},t)}{\partial t^2} \end{array}\right\} \Rightarrow \lambda\vec{\boldsymbol{\nabla}}\cdot\left[Tr(\underline{\underline{\boldsymbol{\varepsilon}}})\underline{\underline{\mathbf{1}}}\right] + 2\mu\vec{\boldsymbol{\nabla}}\cdot\underline{\underline{\boldsymbol{\varepsilon}}} + \rho_0\vec{\boldsymbol{b}} = \rho_0\frac{\partial^2\vec{\boldsymbol{u}}}{\partial t^2}$$

Replacing the identities:

$$\vec{\boldsymbol{\nabla}}\cdot\left(Tr(\underline{\underline{\boldsymbol{\varepsilon}}})\underline{\underline{\mathbf{1}}}\right) = \vec{\boldsymbol{\nabla}}\left(\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{u}}\right); \quad \vec{\boldsymbol{\nabla}}\cdot\underline{\underline{\boldsymbol{\varepsilon}}} = \frac{1}{2}\vec{\boldsymbol{\nabla}}^2\vec{\boldsymbol{u}} + \frac{1}{2}\vec{\boldsymbol{\nabla}}\left(\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{u}}\right)$$

Then:

$$\lambda\vec{\boldsymbol{\nabla}}\left(\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{u}}\right) + 2\mu\left(\frac{1}{2}\vec{\boldsymbol{\nabla}}^2\vec{\boldsymbol{u}} + \frac{1}{2}\vec{\boldsymbol{\nabla}}\left(\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{u}}\right)\right) + \rho_0\vec{\boldsymbol{b}} = \rho_0\frac{\partial^2\vec{\boldsymbol{u}}}{\partial t^2}$$

The **Navier Equations** are obtained as follows:

$$(\lambda+\mu)\vec{\boldsymbol{\nabla}}\left(\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{u}}\right) + \mu\vec{\boldsymbol{\nabla}}^2\vec{\boldsymbol{u}} + \rho_0\vec{\boldsymbol{b}} = \rho_0\frac{\partial^2\vec{\boldsymbol{u}}}{\partial t^2}$$

$$(\lambda+\mu)u_{j,ji} + \mu u_{i,jj} + \rho_0 b_i = \rho_0\ddot{u}_i \quad i,j\in\{1,2,3\}$$

$$(\lambda+\mu)u_{j,ji} + \mu u_{i,jj} + f_i = \rho_0\ddot{u}_i \quad i,j\in\{1,2,3\}$$

where $\rho_0 b_i = f_i$

Displacement Formulation - Boundary Conditions

By introducing the constitutive equation into the Cauchy's eqation of motion:

$$\left.\begin{array}{l} \underline{\underline{\boldsymbol{\sigma}}}(\vec{\boldsymbol{x}},t) = \lambda Tr(\underline{\underline{\boldsymbol{\varepsilon}}})\underline{\underline{\mathbf{1}}} + 2\mu\underline{\underline{\boldsymbol{\varepsilon}}} \\ \\ \vec{\boldsymbol{t}}^* = \underline{\underline{\boldsymbol{\sigma}}} \cdot \hat{\boldsymbol{n}} \end{array}\right\} \Rightarrow \vec{\boldsymbol{t}}^* = \lambda \underbrace{\left(Tr(\underline{\underline{\boldsymbol{\varepsilon}}})\right)}_{=\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{u}}} \hat{\boldsymbol{n}} + 2\mu \underbrace{\underline{\underline{\boldsymbol{\varepsilon}}}}_{=\frac{1}{2}\left(\vec{\boldsymbol{u}}\otimes\vec{\boldsymbol{\nabla}}+\vec{\boldsymbol{\nabla}}\otimes\vec{\boldsymbol{u}}\right)} \cdot \hat{\boldsymbol{n}}$$

$$\Rightarrow \vec{\boldsymbol{t}}^* = \lambda\left(\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{u}}\right)\hat{\boldsymbol{n}} + \mu\left(\vec{\boldsymbol{u}}\otimes\vec{\boldsymbol{\nabla}} + \vec{\boldsymbol{\nabla}}\otimes\vec{\boldsymbol{u}}\right)\cdot\hat{\boldsymbol{n}}$$

The boundary conditions are expressed as follows:

$$\left.\begin{array}{l} \vec{\boldsymbol{u}} = \vec{\boldsymbol{u}}^* \\ u_i = u_i^*; \quad i \in \{1,2,3\} \end{array}\right\} \Rightarrow \text{on } \Gamma_u$$

$$\left.\begin{array}{l} \lambda\left(\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{u}}\right)\hat{\boldsymbol{n}} + \mu\left(\vec{\boldsymbol{u}}\otimes\vec{\boldsymbol{\nabla}} + \vec{\boldsymbol{\nabla}}\otimes\vec{\boldsymbol{u}}\right)\cdot\hat{\boldsymbol{n}} = \vec{\boldsymbol{t}}^* \\ \lambda u_{k,k}n_i + \mu(u_{i,j}n_j + u_{j,i}n_j) = t_i^*; \quad i,j,k \in \{1,2,3\} \end{array}\right\} \Rightarrow \text{on } \Gamma_\sigma$$

Note that the initial conditions remain the same.

Stress Formulation

Stress Formulation

The goal is to reduce the system to a system with only unknowns $\underline{\underline{\boldsymbol{\sigma}}}(\vec{\boldsymbol{x}}, t)$. Then the strain tensor $\underline{\underline{\boldsymbol{\varepsilon}}}(\vec{\boldsymbol{x}}, t)$ and the displacement vector $\vec{\boldsymbol{u}}(\vec{\boldsymbol{x}}, t)$ can be derived.

$\vec{\boldsymbol{\nabla}} \cdot \underline{\underline{\boldsymbol{\sigma}}}(\vec{\boldsymbol{x}}, t) + \rho_0 \vec{\boldsymbol{b}}(\vec{\boldsymbol{x}}, t) = \vec{\boldsymbol{0}}$	Equilibrium Equation (Quasi-static problem)
$\underline{\underline{\boldsymbol{\varepsilon}}} = -\frac{\nu}{E} Tr(\underline{\underline{\boldsymbol{\sigma}}})\underline{\underline{\boldsymbol{1}}} + \frac{1+\nu}{E}\underline{\underline{\boldsymbol{\sigma}}}$	Inverse Constitutive Equation
$\underline{\underline{\boldsymbol{\varepsilon}}}(\vec{\boldsymbol{x}}, t) = \frac{1}{2}\left(\vec{\boldsymbol{u}} \otimes \vec{\boldsymbol{\nabla}} + \vec{\boldsymbol{\nabla}} \otimes \vec{\boldsymbol{u}}\right)$	Geometric Equation
$\Gamma_u : \vec{\boldsymbol{u}} = \vec{\boldsymbol{u}}^*$ $\Gamma_\sigma : \vec{\boldsymbol{t}}^* = \underline{\underline{\boldsymbol{\sigma}}} \cdot \hat{\boldsymbol{n}}$	Boundary conditions

Note that the time variable acts like a loading factor for the quasi-static problem.

Stress Formulation

Taking the successive derivatives of the geometric equation the displacement are eliminated:

$$\frac{\partial^2 \varepsilon_{ij}}{\partial x_k \partial x_l} + \frac{\partial^2 \varepsilon_{kl}}{\partial x_i \partial x_j} - \frac{\partial^2 \varepsilon_{ik}}{\partial x_j \partial x_l} - \frac{\partial^2 \varepsilon_{jl}}{\partial x_i \partial x_k} = 0 \quad i, j, k, l \in \{1, 2, 3\}$$

Including the inverse constitutive equation in the compatibility equations:

$$\varepsilon_{ij} = -\frac{\nu}{E}\sigma_{pp}\delta_{ij} + \frac{1+\nu}{E}\sigma_{ij}$$
$$\frac{\partial \sigma_{ij}}{\partial x_i} + \rho_0 b_j = 0$$

The **Beltrami-Michell Equations** are as follows:

$$\nabla^2 \sigma_{ij} + \frac{1}{1+\nu}\frac{\partial^2 \sigma_{kk}}{\partial x_i \partial x_j} = -\frac{\nu}{1-\nu}\delta_{ij}\frac{\partial(\rho_0 b_k)}{\partial x_k} - \frac{\partial(\rho_0 b_i)}{\partial x_j} - \frac{\partial(\rho_0 b_j)}{\partial x_i}; \quad i, j \in \{1, 2, 3\}$$
$$\sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{kk,ij} = -\frac{\nu}{1-\nu} f_{k,k}\delta_{ij} - f_{i,j} - f_{j,i}; \quad i, j \in \{1, 2, 3\}$$

where $\rho_0 b_i = f_i$. No longer used.

Stress Formulation

The boundary equations:

- Equilibrium equations: $\vec{\boldsymbol{\nabla}} \cdot \underline{\underline{\boldsymbol{\sigma}}} + \rho_0 \vec{\boldsymbol{b}} = \vec{\boldsymbol{0}}$
  A first order PDE system that can act as boundary conditions of the Beltrami-Michell Equations.
- Prescribed stresses on Γ_σ: $\underline{\underline{\boldsymbol{\sigma}}} \cdot \hat{\boldsymbol{n}} = \vec{\boldsymbol{t}}^*$

Stress Formulation

After the stress is derived, the strain field can be derived as follows:

$$\underline{\underline{\boldsymbol{\varepsilon}}}(\vec{\boldsymbol{x}}, t) = -\frac{\nu}{E} Tr(\underline{\underline{\boldsymbol{\sigma}}})\underline{\underline{\mathbf{1}}} + \frac{1+\nu}{E}\underline{\underline{\boldsymbol{\sigma}}}$$

The calculation of the displacement field demands that the geometric equations are aligned with the prescribed displacements on Γ_u:

$$\underline{\underline{\boldsymbol{\varepsilon}}} = \frac{1}{2}\left(\vec{\boldsymbol{u}}(\vec{\boldsymbol{x}}) \otimes \vec{\boldsymbol{\nabla}} + \vec{\boldsymbol{\nabla}} \otimes \vec{\boldsymbol{u}}(\vec{\boldsymbol{x}})\right) \quad \vec{\boldsymbol{x}} \in V$$
$$\vec{\boldsymbol{u}}(\vec{\boldsymbol{x}}) = \vec{\boldsymbol{u}}^*(\vec{\boldsymbol{x}}) \quad \forall \vec{\boldsymbol{x}} \in \Gamma_u$$

It is a disadvantage with respect to displacement formulation, the fact that integration is needed when using numerical methods for the linear elastic problem.

Uniqueness of the solution

The solution of the linear system is unique if:

- it is unique in strains and stresses.
- it is unique in displacements assuming that boundary conditions hold to avoid rigid body motions.

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