EXERCISE SET 4,
MS-A0402, FOUNDATIONS OF DISCRETE MATHEMATICS

HOMEWORK SOLUTIONS

Problem 1. How many integers from 1 to 60 are multiples of 2 or 3 but not both?

There are $\lfloor \frac{60}{2} \rfloor = 30$ numbers that are divisible by two.
There are $\lfloor \frac{60}{3} \rfloor = 20$ numbers that are divisible by three.
There are $\lfloor \frac{60}{2 \cdot 3} \rfloor = \lfloor \frac{60}{6} \rfloor = 10$ numbers that are divisible by both two and three.
As the numbers divisible by six are also divisible by both two and three, the total number of integers between 1 to 60 that are multiples of 2 or 3 but not both is $30 - 10 + 20 - 10 = 30$

Problem 2. Consider the permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 7 & 3 & 6 & 4 & 2 & 1 & 5 & 8 \end{pmatrix}$$

a) Write it as a product of disjoint cycles.

The first cycle can be obtained from: $\pi(1) = 9$, $\pi(9) = 8$, $\pi(8) = 5$, $\pi(5) = 4$, $\pi(4) = 6$, $\pi(6) = 2$, $\pi(2) = 7$, $\pi(7) = 1$. It is (1985462). The only other cycle is (3) because $\pi(3) = 3$. Thus, $\pi = (1985462)(3)$.

b) Write it as a product of transpositions.

Transpositions are cycles of length 2 (e.g. (ab)). Transpositions can be obtained in many ways, for example using $(a_1a_2...a_k) = (a_1a_k)(a_1a_{k-1})...(a_1a_2)$. Thus, one example is $\pi = (12)(16)(14)(15)(18)(19)$

Problem 3. In how many ways can we rearrange the letters in the word "knackered"

a) with no restrictions?

There are nine letters that can be arranged in 9! ways. However, as there are two k:s and two e:s, we need to eliminate the duplicates. As there are 2! ways to order the k:s and 2! ways to order the e:s, we get $\frac{9!}{2!2!} = 90720$ unique rearrangements.

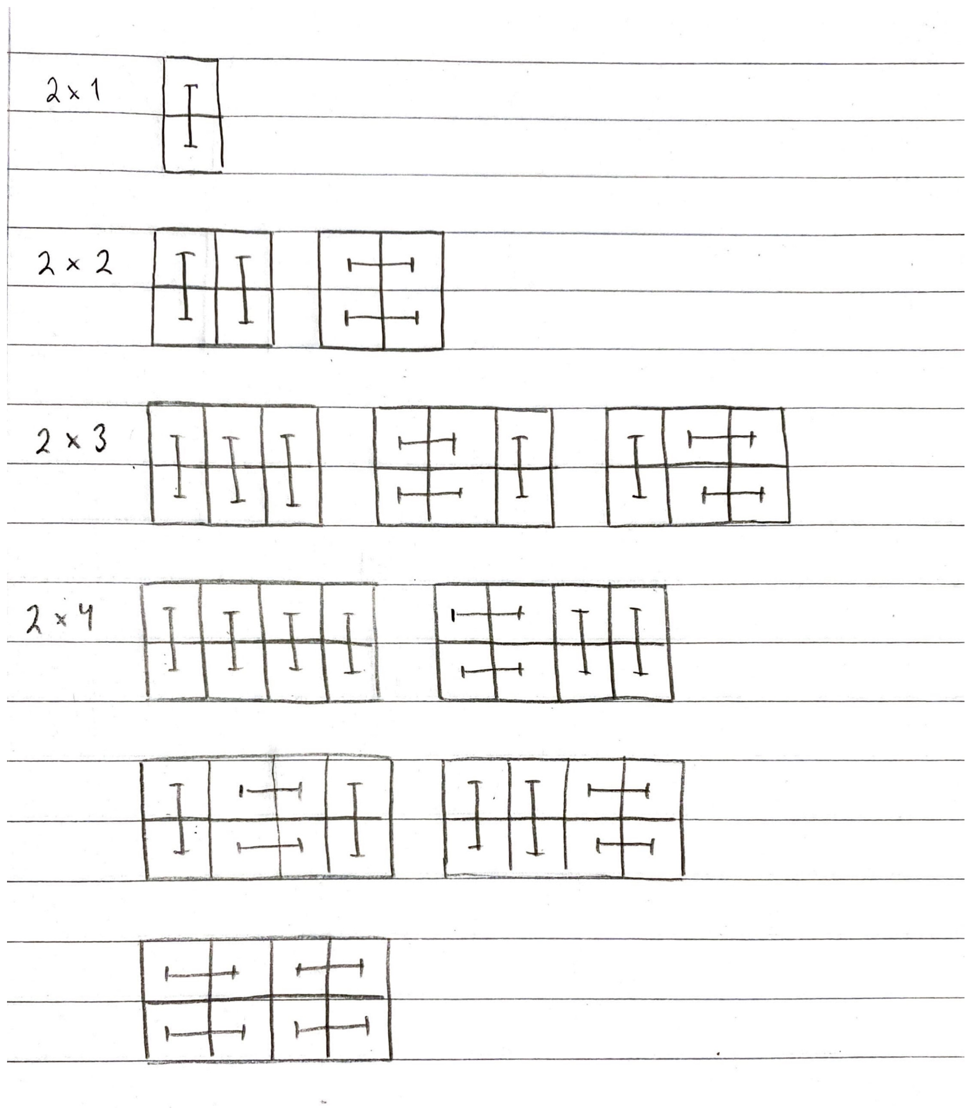
Another way to approach this problem is to first choose two positions for k:s, which can be done in $\binom{9}{2}$ ways. Then choosing two positions for e:s, which can be done in $\binom{7}{2}$ ways. Last, the rest of the digits can be rearranged in 5! ways. This gives the same result as above $\binom{9}{2}\binom{7}{2}5! = 90720$

b) if the first and last letter must be vowels?

First, there are three possibilities for the first letter. Then there are two possibilities for the last letter. The rest of the letters can be arranged freely in 7! ways. To eliminate the duplicates, we must again divide the possible number of arrangements by 2!2!. This gives a total of $\frac{3 \cdot 2 \cdot 7!}{2!2!} = 7560$ rearrangements.

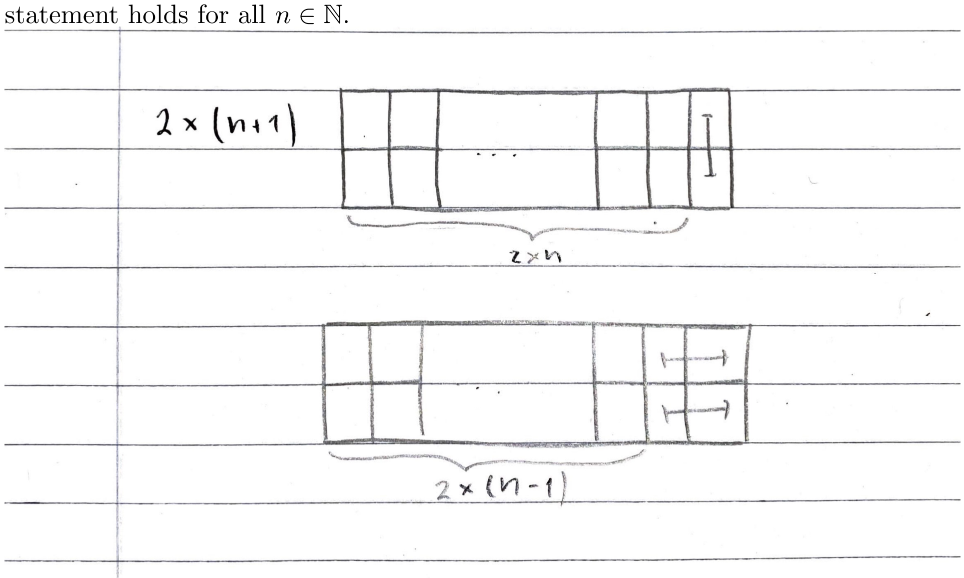
Problem 4. How many ways are there to tile dominos (with size 2×1) on a grid of size $2 \times n$?

Experimenting with small values of n:



Denoting the number of tilings on a $2 \times n$ grid by $T(n)$. We notice that it looks like $T(n) = T(n-1) + T(n-2)$ (also known as Fibonacci sequence). This relationship can be proven by induction.

(1) Base case shown above
(2) Assume that the statement holds for arbitrary $n = k$
(3) We see that tiling of $2 \times (n+1)$ grid can be obtained in two ways; either by adding a vertical domino to any tiling of size $2 \times n$ grid or by adding two horizontal dominos to any tiling of size $2 \times (n-1)$ grid. It follows that the statement holds for all $n \in \mathbb{N}$.



Using this formula we get 10946 tilings on a 2×20 grid.