EXERCISE SET 5 SOLUTIONS,
MS-A0402, FOUNDATIONS OF DISCRETE MATHEMATICS

Homework

The written solutions to the homework problems should be handed in on MyCourses by Monday 4.4., 12:00. You are allowed and encouraged to discuss the exercises with your fellow students, but everyone should write down their own solutions.

Problem 1. (10pts) Consider the permutations

$$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix} \text{ and } \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 2 & 3 & 4 \end{pmatrix}.$$

Are they conjugates? If so, find a permutation τ such that $\tau\rho\tau^{-1} = \sigma$.

Solution 1. We note that $\rho = (132)(45)$ and $\sigma = (1)(2543)$. The permutation ρ is composed of a 3-cycle and a 2-cycle and the permutation σ is composed of a 1-cycle and a 4-cycle. For two conjugate permutations, the permutations must have the same amount of cycles and for each cycle of a certain length in one of the permutations there should exist a cycle of an equal length in the other permutation. This is clearly not the case for the two given permutations and hence they are not conjugates.

Problem 2. (10pts) The **perfect riffle shuffle** (or "*Faro shuffle*") of a deck consisting of $2n$ cards (for a fixed $n \in \mathbb{N}$) is a permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & 2n-1 & 2n \\ 1 & n+1 & 2 & n+2 & \dots & n & 2n \end{pmatrix} \in S_{2n}$$

that splits a deck of $2n$ cards into two piles and interleaves them (i.e. card in position 1 goes to position 1, card in position 2 goes to position $n+1$, card in position 3 goes to position 3, card in position 4 goes to position $n+2$, etc.). This is also called an out-shuffle, because it leaves the top card at the top and bottom card at the bottom. Thus we can write a formula:

$$\sigma(k) = \begin{cases} \frac{k+1}{2}, & k \text{ is odd} \\ n + \frac{k}{2}, & k \text{ is even} \end{cases}$$

Let $n = 3$, that is, we have a deck of 6 cards. Find the number of perfect riffle shuffles needed to return the deck to its original state. In order words, find some $N \in \mathbb{N}$ such that

$$\sigma^N = \underbrace{\sigma\sigma\sigma \dots \sigma\sigma}_{N \text{ times}} = e,$$

where $e \in S_{2n}$ is the identity permutation $e(i) = i$ for all $i \in \{1, 2, \dots, 2n\}$.

Hint: for a deck of 52 cards, that is, when $n = 26$, *this can be done with* 8 *shuffles, that is,* $\sigma^8 = e$ *(proof:* `https://www.youtube.com/watch?v=7lNk7bfkFq8` *), so it probably is less than* 8 *here with just 6 cards.*

Solution 2. When $n = 3$, $\sigma \in S_6$. Using the given formula we get $\sigma(1) = 1$, $\sigma(2) = 4$, $\sigma(3) = 2$, $\sigma(4) = 5$, $\sigma(5) = 3$, $\sigma(6) = 6$. Hence in cycle notation we have $\sigma = (2453)$. We then look for the order (N such that $\sigma^N = e$) of this σ. For any cycle, the order corresponds to the length of the cycle. Hence $N = 4$.

Problem 3. (10pts) The following figure shows two graphs with eleven vertices. The graph on the left has $V = \{0, 1, 2, \dots, 10\}$, whereas the one on the right has nodes $V' = \{a, b, \dots, k\}$. Are they isomorphic?

Solution 3. Let us consider the degrees of the vertices in each graph. The (vertex, degree) pairs of V are:

$$V \times D_V = \{(0, 5), (1, 4), (2, 4), (3, 4), (4, 4), (5, 4), (6, 5), (7, 5), (8, 5), (9, 5), (10, 5)\}$$

And for V' these pairs are:

$$V' \times D_{V'} = \{(a, 5), (b, 5), (c, 5), (d, 5), (e, 5), (f, 5), (g, 4), (h, 4), (i, 4), (j, 4), (k, 4)\}$$

Assume then there exists a bijection $\phi : \{0, ..., 10\} \to \{a, ..., k\}$. By noting the listed degrees of each vertex we have the following:

- Elements from $\{1, 2, 3, 4, 5\}$ map to $\{g, h, i, j, k\}$.
- Elements from $\{0, 6, 7, 8, 9, 10\}$ map to $\{a, b, c, d, e, f\}$.

We then note that in the first graph the vertex 0 is connected to 5 edges each with degree of 4. Hence in the other graph each neighbour of $\phi(0)$, that is any of $\phi(1), \phi(2), \phi(3), \phi(4), \phi(5)$, must have a degree of 4 as well for bijectivity to hold. However in the set $\{a, b, c, d, e, f\}$ (to which $\phi(0)$ must map to) there is:

- One element such that all the neighbours have a degree of 5.
- Five elements such that two of the neighbours have degree of 4 and three of the neighbours have a degree of 5.

Hence there is no $\phi(0)$ (no matter how this element is chosen) such that all the neighbours of $\phi(0)$ have degree of 4. This is a contradiction, giving that the assumption of bijective ϕ is false. Hence the two graphs are not isomorphic.

Problem 4. (10pts) Colour the following graph with the greedy algorithm.

Can you find an ordering of the vertices such that the greedy algorithm colours the graph with 3 colors?

Solution 4. Label the colors 1,2,3 by Red, Green, Blue. Then by the greedy algorithm we get (for example) the following coloring.