EXERCISE SET 3,
MS-A0402, FOUNDATIONS OF DISCRETE MATHEMATICS

Explorative exercises

I recommend to study the explorative problems before the first lecture of the week.

Problem 1. Let B be an arbitrary set.

a) Let $f : \mathbb{N} \to B$ be an injective function. (Think of a few concrete examples.) Can you use f to construct a surjective function $B \to \mathbb{N}$?
b) Let $g : \mathbb{N} \to B$ be a surjective function. (Think of a few concrete examples.) Can you use g to construct an injective function $B \to \mathbb{N}$?
c) Can you do the same if $\mathbb{N}$ is replaced by an arbitrary set A?

Problem 2. The number of ways to select k elements out of a set of size n is denoted $\binom{n}{k}$

a) Argue that the number of ways to order a set of size n is

$$n! = n(n-1)(n-2)\cdots 2 \cdot 1.$$

b) Argue that the number of ways to first select k elements out of a set of size n, then order these elements, and then order the remaining $n-k$ elements, is

$$\binom{n}{k} k!(n-k)!$$

c) Conclude that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

d) What does this formula say when $k = 0$, or when $k = n$? Do these formulas make sense?

Problem 3. Prove that the *binomial coefficients* $\binom{n}{k}$ satisfy the identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

a) Using the interpretation of $\binom{n}{k}$ as the number of combinations of k out of n elements.
b) Using the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Problem 4. What is the sum $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n}$? If you do not see the answer immediately, first compute the sum for $n = 3$, $n = 4$, $n = 5$, etc.. Can you explain this phenomenon combinatorially?

Homework

The written solutions to the homework problems should be handed in on MyCourses by Monday 21.3 at 12:00. You are allowed and encouraged to discuss the exercises with your fellow students, but everyone should write down their own solutions.

Problem 1. Let P be the set of all Finland's presidents, and let G be the set of all ordered pairs $(a, b) \in P \times P$ such that the president b succeeded president a in office. Is G the graph of a function? Explain your answer.

Problem 2. Find the domain and range of the function which assigns to each nonnegative integer its last digit.

Problem 3. Eight people are to be seated around a table; the chairs don't matter, only who is next to whom, but right and left are different. Two people, X and Y, cannot be seated next to each other. How many seating arrangements are possible?

Problem 4. Prove that for all $n \in \mathbb{N}$, $n \geq 9$, the following statement is true: for all $k \in \mathbb{N}$ with $0 \leq k \leq n$ we have

$$\binom{n}{k} < 2^{n-2}.$$

Hint: You can use induction in n with base case $n = 9$.

Additional problems

These do not need to be returned for marking.

Problem 1. How many odd 5-digit numbers (in the decimal system) have all their digits different?

Problem 2. How many bit strings contain exactly eight 0s and ten 1s if every 0 must be immediately followed by a 1?

Problem 3. Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have the same number of men and women?

Problem 4.

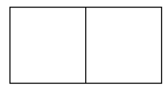
a) What is the coefficient of x^2y^3 in the expansion of $(x + y)^5$?
b) What is the coefficient of x^8y^9 in the expansion of $(x + y)^{17}$?
c) What is the coefficient of x^8y^9 in the expansion of $(2x + 3y)^{17}$?

Problem 5.

a) How many relations on $\{1, 2, \ldots, n\}$ are reflexive?
b) How many relations on $\{1, 2, \ldots, n\}$ are symmetric?
c) How many relations on $\{1, 2, \ldots, n\}$ are antisymmetric?

Problem 6. The *pigeonhole principle* is the following very simple but surprisingly useful observation: If a set with n elements ("pigeons") is partitioned into m parts ("pigeonholes"), where $m < n$, then at least one of the pigeonholes contains at least two pigeons. Use this to show that, among 101 integers, there is a pair whose difference is divisible by 100.

Problem 7. How many rectangles are bounded by the straight lines on a "chess-board" of size $n \times m$? (Example: On a board of size 1×2, like below, there are 3 rectangles: the left one, the right one, and the entire board.)



Problem 8. Prove the identity

$$\sum_{k=0}^{n} k \binom{n}{k}^2 = n \binom{2n-1}{n-1}.$$

Problem 9. Prove that there is no bijection $\mathbb{N} \to P(\mathbb{N})$. Hint: Imitate the proof that there is no bijection $\mathbb{N} \to \mathbb{R}$.