1 Explorative Exercises

1.1

1.1.1

Example:
$f : \mathbb{N} \to B$, $B = \mathbb{N}_{\geq 1}$, $f(x) = x + 1$
Surjective function: $f^{-1} : B \to \mathbb{N}$, $f^{-1}(x) = x - 1$
If an injection exists, then $|\mathbb{N}| \leq |B|$. This condition is sufficient for there to exist a surjection from B to $\mathbb{N}$.

1.1.2

Example:
$f : \mathbb{N} \to B$, $B = \{1\}$, $f(x) = 1$
Surjective function: $g : B \to \mathbb{N}$, $g(1) = 1$
If a surjection exists, then $|\mathbb{N}| \geq |B|$. This condition is sufficient for there to exist an injection from B to $\mathbb{N}$.

1.1.3

This is true in general (Partition Principle)

1.2

1.2.1

For the first position we have n choices, $n - 1$ for the second one, $n - 2$ for the third one, and so on. The later number of choices are independent of the first choices, hence we get $n!$

1.2.2

We are given that $\binom{n}{k}$ is the number of ways to choose k elements out of a set of size n. Additionally, we know that $k!$ is the number of ways to order k people. Hence it easily follows that the number of ways to first choose k elements, then order then, and the order the remaining $n - k$ elements is $\binom{n}{k}k!(n-k)!$

1.2.3

Both the value $n!$ and $\binom{n}{k}k!(n-k)!$ count the number of ways to order n elements, although they have a different approach to it. Hence we have the equality

$$n! = \binom{n}{k}k!(n-k)!$$

Dividing this by $k!(n-k)!$, we get the final result.

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

1.2.4

$$\binom{n}{0} = 1 = \binom{n}{n}$$

These formulas do make sense. There is only one way to choose nothing out n, i.e. by choosing nothing. Similarly, there is only one way to choose everything, i.e. by choosing everything.

1.3

1.3.1

$\binom{n}{k}$ is the number of ways to choose k elements from a set of n. However, we can also count this in a different way. Let us fix a special element. Then there are two ways to choose k elements from n; either you choose the special element or you don't choose it.

If you do not choose the special element, then we have $\binom{n-1}{k}$ ways to choose k elements from the group without the special element. On the other hand, if we choose to have the special element, then we have to choose the remaining $k-1$ elements from a set of size $n-1$. This can be done in $\binom{n-1}{k-1}$ ways. So the number of ways to choose k elements from a set of size n is the sum of the previous two numbers.

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

1.3.2

$$\begin{aligned}\binom{n-1}{k} + \binom{n-1}{k-1} &= \frac{(n-1)!}{(n-k-1)!k!} + \frac{(n-1)!}{(n-k)!(k-1)!} \\ = \frac{(n-k)(n-1)!}{(n-k)!k!} + \frac{k(n-1)!}{(n-k)!k!} &= \frac{n(n-1)! - k(n-1)! + k(n-1)!}{(n-k)!k!} \\ &= \frac{n!}{k!(n-k)!} = \binom{n}{k}\end{aligned}$$

1.4

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

This sum computes the number of all possible ways to choose a team of size k, for all values of k between 0 and n, out of a group of n people. We get the same number by doing a binary choice on every person whether or not they are in the team. So we have 2 choices n times, i.e. 2^n.

2 Additional Exercises

2.1

For the number to be odd, its last digit must be odd. Hence we have five choices for the final digit. For the number to be consider a 5-digit number, the first digit has to be non-zero. So it has at most 9 choices. The other digits have no other restrictions aside from being distinct.

Let us choose these two digits first and then the middle digits in any order. There are two choices, either the first digit is odd and thus it limits the choices of the final digit, or it is even and it does not limit the last digit. These two cases contain $5*8*7*6*4$ and $4*8*7*6*5$ numbers, respectively (each multiplier corresponds to the number of choices for the digit in the corresponding position. The final answer is the sum of these two (equal) numbers, i.e. $2*4*8*7*6*5 = 13440$

2.2

Since every zero must be followed by one, the number will have eight "01" blocks. We are left with two ones that can be placed to the sequence with the "01" blocks freely. These positions can be picked in $\binom{10}{2}$ ways. Thus, there are $\binom{10}{2} = 45$ such numbers

2.3

We have to choose 3 men and 3 women to the committee. The choices of men are independent of the choices of women. Thus the number of solutions is

$$\binom{10}{3}\binom{15}{3} = 54600$$

2.4

Binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

From this we see that the coefficient of $x^i y^j$ in the expansion of $(x+y)^n$ is $\binom{n}{i}$

2.4.1

$\binom{5}{2} = 10$

2.4.2

$\binom{17}{8} = 24310$

2.4.3

Substituting $x' = 2x$ and $y' = 3y$, we get

$$\binom{17}{8}(x')^8(y')^9 = \binom{17}{8}2^8 3^9 x^8 y^9.$$

Thus the coefficient is $\binom{17}{8}2^8 3^9 = 122494394880$

2.5

2.5.1

For a relation on a set of size n, there are n^2 possible elements (a, b) in a relation. There are n elements of the form (a, a). For any element (a, b) we have in the relation, we also must have (a, a) and (b, b). So the number of possible additional elements, outside of the (a, a) elements, in the relation is $n^2 - n$. Thus we get

$$2^{n^2-n} = 2^{n(n-1)}$$

2.5.2

For symmetric relations we need to choose the pairs (a, b) and (b, a) to be in the relation. We have n choices for elements of the form (a, a). For the others we have for other possible elements we have $n^2 - n$ choices, but since we choose those elements in pairs, we have $(n^2 - n)/2$ choices for those. Thus the final answer is

$$2^n 2^{(n^2-n)/2} = 2^{n(n+1)/2}$$

2.5.3

Again, the number of elements of the form (a, a) is n and we have a binary choice for them. For all other pairs (a, b) and (b, a), we have three options either one of the is in the relation or neither of them is. The number of such pairs is $n(n-1)/2$ and thus our final answer is

$$2^n 3^{n(n-1)/2}$$

2.6

We want to show that among 101 integers there is a pair whose difference is divisible by 100. Let the pidgeonholes be the equivalence classes of integers modulo 100. If any two integers land in the same equivalence class, their difference must be divisible by 100. Since we have 101 integers, by the pidgeonhole principle at least two of them have to be in the same pidgeonhole, i.e. there are two integers who have the same remainder when divided by 100. The difference of these two integers is divisible by 100.

2.7

In an $n \times m$ chess-board there are $n+1$ horizontal lines and $m+1$ vertical lines. A rectangle on chess-board is nothing but a space enclosed by two vertical and two horizontal lines, that is to say that counting rectangles is equivalent to counting the ways to choose two horizontal and two vertical lines. The number of ways to do so is $\binom{n+1}{2}$ for horizontal and $\binom{m+1}{2}$ for vertical lines. Since these choices are independent, the final answer is

$$\binom{n+1}{2}\binom{m+1}{2}$$

2.8

$$\sum_{k=0}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$$

We will present a combinatorial argument for this equality. Note that

$$\sum_{k=0}^{n} k\binom{n}{k}^2 = \sum_{k=0}^{n} k\binom{n}{k}\binom{n}{n-k}.$$

Suppose we have two groups of size n, for example a group of teachers and a group students. Now suppose we have to form a team of teachers and students of size n with a team leader who is a teacher. Say we choose k teachers, then we must have $n-k$ students. We have $\binom{n}{k}$ choices for the teachers, $\binom{n}{n-k}$ choices for the students, and k choices for the team leader from the chosen teachers. So for a given k, there are $k\binom{n}{k}\binom{n}{n-k}$ way to form the team. To get the number of all possible teams, we sum over k.

On the other hand, the team formation can be done differently. If we pool the groups together, we have a total of $2n$ people. For choosing the team leader, who has to be a teacher, we have n choices. Now we have to choose $n-1$ members, who can be either teachers or students, from $2n-1$ people. Hence we get that the number of ways to do this is, $n\binom{2n-1}{n-1}$. Thus the claimed equality follows.

Problem 9. Proof that there is no bijection $\mathbb{N} \to P(\mathbb{N})$

Suppose we can list all the subsets of $\mathbb{N}$:
$\{1, 5, 13, 2, 7, ...\}$
$\{6, 5, 3, 9, 45, ...\}$
$\{45, 3, 6, 9, 5, ...\}$
etc.

Now change the i:th element in each set in any way. Now, the set consisting of all the changed elements cannot be in the original set $P(\mathbb{N})$, which creates a contradiction. Thus, $|\mathbb{N}| \neq |P(\mathbb{N})|$ and there cannot be a bijection.