Problem 1. (10pts) Consider the permutations

$$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix} \text{ and } \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 2 & 3 & 4 \end{pmatrix}.$$

Are they conjugates? If so, find a permutation τ such that $\tau\rho\tau^{-1} = \sigma$.

Problem 1: $\rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix}$, $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 2 & 3 & 4 \end{pmatrix}$

In cycle notation: $\rho = (132)(45)$ and $\sigma = (1)(2543)$

Since ρ and σ don't have the same number of cycles: ρ has one 2 long and one 3 long cycle, while σ has one 1 long and 4 long cycle

$\Rightarrow$ ρ and σ are not conjugates and thus there doesn't exist τ such that $\tau\rho\tau^{-1} = \sigma$

To teaching assistants: This is a note from StackOverflow that explains how the conjugates work. It is a note for myself, you can skip it.

Given two permutations , I'm asked to answer is they are conjugate permutations .

The two permutations are : $\alpha = (12)(345)(78)$, $\beta = (162)(35)(89)$.

Definition: Two permutations $\sigma, \sigma' \in S_n$ are conjugate if exists $\tau \in S_n$ such that:
$\sigma' = \tau\sigma\tau^{-1} = (\tau(a_0), \tau(a_1)\dots\tau(a_k))$, where $\alpha = (a_0a_1\dots a_k)$.

Write the two permutations in full cycle notation, writing cycles from longest to shortest (cycles of the same length can be ordered arbitrarily, the starting number of cycle can be chosen arbitrarily from within the cycle).

In your example:

$$\begin{matrix} \alpha = (3,4,5)(1,2)(7,8)(6)(9) \\ \beta = (1,6,2)(3,5)(8,9)(4)(7) \end{matrix} \text{ so } \tau = \begin{bmatrix} 3 & 4 & 5 & 1 & 2 & 7 & 8 & 6 & 9 \\ 1 & 6 & 2 & 3 & 5 & 8 & 9 & 4 & 7 \end{bmatrix}$$

In other words, $\tau(3) = 1, \tau(4) = 6$, etc. You can convert τ to cycle notation in the obvious way by "tracing", $\tau = (3,1)(4,6)(5,2)(7,8,9) = (1,3)(2,5)(4,6)(7,8,9)$.

Another τ is found by:

$$\begin{matrix} \alpha = (3,4,5)(7,8)(2,1)(9)(6) \\ \beta = (1,6,2)(5,3)(8,9)(4)(7) \end{matrix} \text{ so } \tau = \begin{bmatrix} 3 & 4 & 5 & 7 & 8 & 2 & 1 & 9 & 6 \\ 1 & 6 & 2 & 5 & 3 & 8 & 9 & 4 & 7 \end{bmatrix}$$

and $\tau = (3,1,9,4,6,7,5,2,8)$.

Since you can reorder the cycles of the same length, and since you can "cycle" a cycle as much as you want, you actually get many different τ, in fact an entire coset of a centralizer. One way to calculate the centralizer of α is to apply this procedure when $\alpha = \beta$.

Problem 2. (10pts) The **perfect riffle shuffle** (or "*Faro shuffle*") of a deck consisting of $2n$ cards (for a fixed $n \in \mathbb{N}$) is a permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & 2n-1 & 2n \\ 1 & n+1 & 2 & n+2 & \dots & n & 2n \end{pmatrix} \in S_{2n}$$

that splits a deck of $2n$ cards into two piles and interleaves them (i.e. card in position 1 goes to position 1, card in position 2 goes to position $n+1$, card in position 3 goes to position 3, card in position 4 goes to position $n+2$, etc.). This is also called an out-shuffle, because it leaves the top card at the top and bottom card at the bottom. Thus we can write a formula:

$$\sigma(k) = \begin{cases} \frac{k+1}{2}, & k \text{ is odd} \\ n + \frac{k}{2}, & k \text{ is even} \end{cases}$$

Let $n = 3$, that is, we have a deck of 6 cards. Find the number of perfect riffle shuffles needed to return the deck to its original state. In order words, find some $N \in \mathbb{N}$ such that

$$\sigma^N = \underbrace{\sigma\sigma\sigma\dots\sigma\sigma}_{N \text{ times}} = e,$$

where $e \in S_{2n}$ is the identity permutation $e(i) = i$ for all $i \in \{1, 2, \dots, 2n\}$.

Hint: for a deck of 52 cards, that is, when $n = 26$, *this can be done with* 8 *shuffles, that is,* $\sigma^8 = e$ *(proof:* `https://www.youtube.com/watch?v=7lNk7bfkFq8` *), so it probably is less than 8 here with just 6 cards.*

Problem 2

The Faro shuffle is done on n = 3 or 6 cards. The shuffle done continuously on the deck of card can be outlined as follows

First shuffle : $\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 2 & 5 & 3 & 6 \end{pmatrix}$

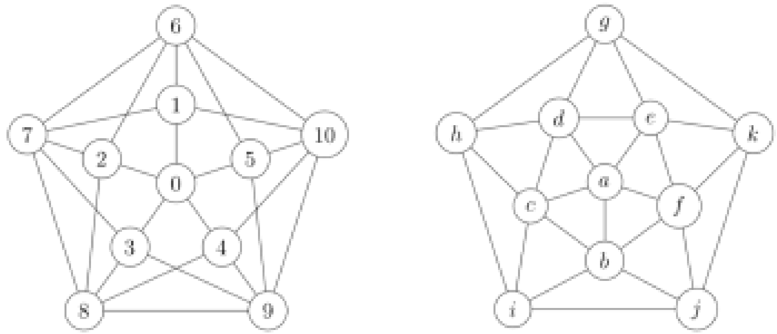
Second shuffle : $\sigma_2 = \begin{pmatrix} 1 & 4 & 2 & 5 & 3 & 6 \\ 1 & 5 & 4 & 3 & 2 & 6 \end{pmatrix}$

Third shuffle : $\sigma_3 = \begin{pmatrix} 1 & 5 & 4 & 3 & 2 & 6 \\ 1 & 3 & 5 & 2 & 4 & 6 \end{pmatrix}$

Fourth shuffle : $\sigma_4 = \begin{pmatrix} 1 & 3 & 5 & 2 & 4 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$ $\Rightarrow$ Return to the original order

$\Rightarrow \sigma^4 = e = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix} \Rightarrow N = 4$ (answer)

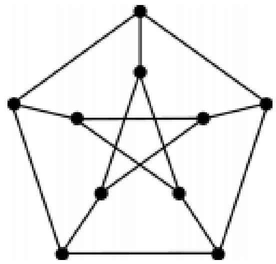
Problem 3. (10pts) The following figure shows two graphs with eleven vertices. The graph on the left has $V = \{0, 1, 2, \ldots, 10\}$, whereas the one on the right has nodes $V' = \{a, b, \ldots, k\}$. Are they isomorphic?



Two graphs G1 and G2 are isomorphic if there exists a matching between their vertices so that two vertices are connected by an edge in G1 if and only if corresponding vertices are connected by an edge in G2

Problem 3:
Let's call first graph as G and second graph as H
From the structure of G, there are ~~two~~ three classes of vertices
1st class: node 0, degree 5
2nd class: node 1, 2, 3, 4,5 degree ~~5~~4
3rd class: node 6, 7, 8, 9, 10, degree 5
Observation: node 0 is connected to all members of 2nd class
Each node in 2nd class is connected to node 0 and 3 members of 3rd class
Each node in 3rd class is connected to 3 members of 2nd class and 2 members of 3rd class
Now let's look at graph H
There is a group of nodes with degree 4 and has 5 members, which may only correspond to the 2nd class of graph G, which are g, h, i, j and k. From our analysis of graph G, node 0 is connected to all nodes in its 2nd class.
In graph H, there does not exist a single vertex of degree 5 that connects g, h, i, j and k together, hinting H doesn't have 1st class
=) G and H graphs have different structures =) They are not isomorphic (answer)

Problem 4. (10pts) Colour the following graph with the greedy algorithm.

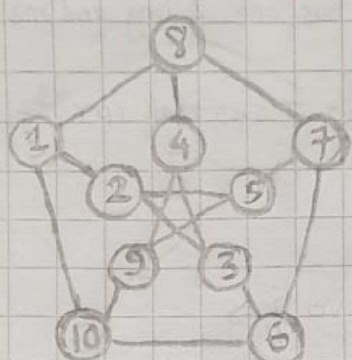


Can you find an ordering of the vertices such that the greedy algorithm colours the graph with 3 colors?

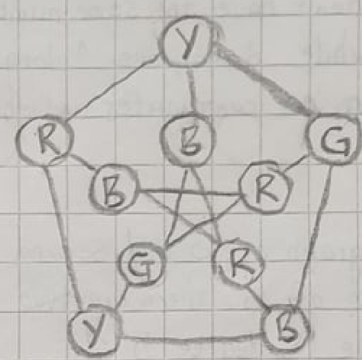
Problem 4: colour the graph with greedy algorithm

Let's define 4 colors : R (red), B (blue), G (green), Y (yellow)

- The order that the graph will need 4 colors are:

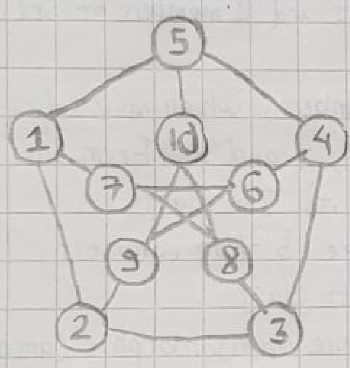


$\Rightarrow$

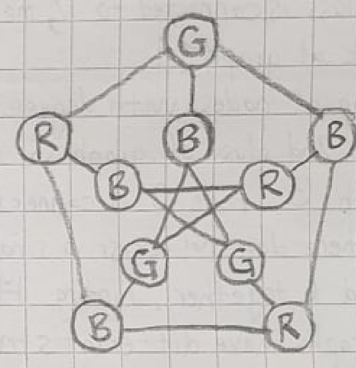


Ordering Coloring

- The ordering of vertices such that the greedy algorithm colours the graph with 3 colors is as follows



$\Rightarrow$



Ordering Coloring