HOME WORK 1
SOLUTIONS

Problem 1. (10pts) Prove that if $n \in \mathbb{Z}$, then $n^2 + 3n + 4$ is even.

Solution: Proving by Direct proof we consider the two cases:

- Case 1: Assume n is even. This implies $n = 2k, k \in \mathbb{Z}$

$$\begin{aligned} \Longrightarrow n^2 + 3n + 4 &= 4k^2 + 6k + 4 \\ &= 2(2k^2 + 3k + 2) \\ &= 2m, m = 2k^2 + 3k + 2 \in \mathbb{Z} \end{aligned}$$

- Case 2: assume n is odd i.e $n = 2k + 1, k \in \mathbb{Z}$

$$\begin{aligned} \Longrightarrow n^2 + 3n + 4 &= 4k^2 + 10k + 8 \\ &= 2(2k^2 + 5k + 4) \\ &= 2m, m = 2k^2 + 5k + 4 \in \mathbb{Z} \end{aligned}$$

Problem 2. (10pts) Prove that there exists no integers a and b for which $24a+6b = 1$.

Solution: Proving by contradiction we suppose there exist integers a and b for which $24a + 6b = 1 \Longleftrightarrow 6(4a + 1) = 1 \Longrightarrow 4a + 1 = \frac{1}{6} \notin \mathbb{Z}$ which is a contradiction since $4a + 1 \in \mathbb{Z}$

Problem 3. (10pts) Prove by induction that for all $n \in \mathbb{Z}_+ = \{1, 2, \dots\}$ we have

$$\sum_{k=1}^{n} (-1)^k k^2 = \frac{(-1)^n (n+1)n}{2}.$$

Solution:

- Base case: Verify for $n = 1$

$$\text{LHS:} \sum_{k=1}^{n} (-1)^k k^2 = -1$$

$$\text{RHS:} \frac{(-1)^n (n+1)n}{2} = -1$$

Hence statement is true for the case $n = 1$.

- Assume the statement is true for n .i.e. $\sum_{k=1}^{n} (-1)^k k^2 = \frac{(-1)^n (n+1)n}{2}$.

- Verify if statement is true $n+1$.

$$\begin{aligned}\sum_{k=1}^{n+1}(-1)^k k^2 &= \sum_{k=1}^{n}(-1)^k k^2 + (-1)^{n+1}(n+1)^2 \\ &\overset{I.H}{=} \frac{(-1)^n(n+1)n}{2} + (-1)^{n+1}(n+1)^2 \\ &= \frac{(-1)^n(n+1)n + 2(-1)^{n+1}(n+1)^2}{2} \\ &= \frac{(-1)^{n+1}(n^2+3n+2)}{2} \\ &= \frac{(-1)^{n+1}(n+1)(n+2)}{2}\end{aligned}$$

Hence statement is true for $n+1$. Therefore statement is true $\forall n \in \mathbb{N}$.

Problem 4. (10pts) Define a relation $\sim$ on $\mathbb{R}$ by $a \sim b$ if and only if $a \leq b$. Check if $\sim$ is (i) reflexive, (ii) symmetric, and/or (iii) transitive, and prove it if it does. If it does not satisfy the property you are checking, give an example to show it.

Solution:

(i) Reflexive: $a \leq a \Longrightarrow a \sim a$ hence reflexive.
(ii) Symmetry: If $a \leq b$ it's not generally true that $b \leq a$ e.g $1 \leq 2$ but $2 \nleq 1$ hence the relation is not symmetric
(iii) Transitive: If $a \leq b$ and $b \leq c \Longrightarrow a \leq c$ hence transitive.