Discrete Mathematics

Exercise sheet 5

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Explorative exercises

Problem 1.

A graph G = (V, E) is a set of nodes V together with a set of edges E. Therefore we can represent the given graphs G and H as the following.

$G =(V_G, E_G) = (\{u1, u2, u3, u4), \{(u1, u2), (u1, u3), (u3, u4), (u2, u4)\})$

$H = (V_H, E_H) = (\{v1, v2, v3, v4),\{(v1, v4), (v1, v3), (v3, v2), (v4, v2)\})$

These two graphs are isomorphic given there exists a bijection $\phi : V_G \to V_H$ such that there is an edge between u_i and u_j if and only if there is an edge between $\phi(u_i)$ and $\phi(u_j)$.

Now it can be intuitively seen that the given graphs are the same, with v_4 corresponding to u_2 and v_2 corresponding to u_4, hence let us construct a the bijection: $\phi = \{(u_1, v_1), (u_2, v_4), (u_3, v_3), (u_4, v_2)\}$.

Now lets write H in terms of this bijection.

$H = (V_H, E_H) = (\{\phi (u1), \phi(u4), \phi (u3), \phi(u2)\},\{(\phi (u1), \phi (u2)), (\phi(u1), \phi(u3)), (\phi(u3), \phi(u4)), (\phi(u2), \phi(u4))\})$

Now it can clearly be seen that there is an edge between u_i and u_j if and only if there is an edge between $\phi(u_i)$ and $\phi(u_j)$, and hence the graphs are isomorphic.

Problem 2.

All the graphs have 6 vertices, and 9 edges, with each vertice being associated with 3 edges, and hence we can not use those details to dismiss the possibility of isomorphism, but let us consider the shape of the graphs, or rather the cycles in the graphs.

The graph a) has cycles of length 3, for example (1, 2, 3) and (4, 5, 6). So too does graph c), for example (1, 2, 3) and (6, 4, 5). However no cycle of size 3 can be found in b), hence it is clear that b) is not isomorphic to a) and c).

A = ({1, 2, 3, 4, 5, 6}, {(1, 2), (1, 4), (1, 3), (2, 3), (2, 5), (3, 6), (4, 5), (4, 6), (5, 6)}

C = ({C1, C2, C3, C4, C5, C6}, {(C1, C2), (C1, C3), (C1, C6), (C2, C3), (C2, C5), (C3, C4), (C4, C5), (C4, C6), (C5, C6)}

Now let us have a bijection $\phi = \{(1, C1), (2, C2), (3, C3), (4, C6), (5, C5), (6, C4)\}$

Then C can be written as

$C = (\{\phi(1), \phi(2), \phi(3), \phi(6), \phi(5), \phi(4)\}, \{(\phi(1), \phi(2)), (\phi(1), \phi(3)), (\phi(1), \phi(4)), (\phi(2), \phi(3)), (\phi(2), \phi(5)), (\phi(3), \phi(6)), (\phi(6), \phi(5)), (\phi(6), \phi(4)), (\phi(5), \phi(4))\}$

From which it can be seen that **a) and c) are isomorphic.**

Problem 3

a)

if n = 1 clearly there are 0 edges.

If n = 2 clearly there is 1 edge.

Since it is true for some n lets assume it true for n, that is lets assume a tree with n vertices has n-1 edges.

Now lets prove it for n + 1:

Let there be a tree A of n vertices (and hence n – 1 edges). Let there also be another tree B with 1 vertices (and hence 0 edges). Now let there be drawn an edge between any one vertice in A and the sole vertice in B. The graph created is now connected and acyclic, since A and B are acyclic, and there is only one path from A to B, and hence no cycle is possible. Hence the new graph is a tree, and the number of vertices is $|A| + |B| = n + 1$, and the number of edges is the sum of the edges in A and B and the single edge we created. Hence it is

$n-1 + 0 + 1 = n = (n + 1) - 1$

One can not add another edge from a vertice of A to the sole vertice of B without creating a

cycle, since if another one were to be added then there would be 2 distinct paths to the vertice of B and since A is a tree there must be a path from the vertices which connect to B to each other, and hence a cycle would exist.

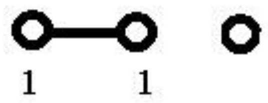
And hence by induction a tree of n vertices has exactly n – 1 edges.

b)

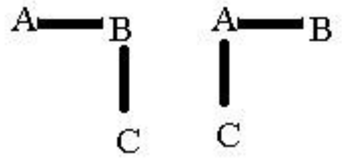
Let the degree of a vertice be the number of edges associated with the vertice. Now any configuration of a tree is isomorphic to another, if:

For each vertice of degree n in A with the neighbors of degrees $\{m1, ..., m_n\}$ there is a vertice of degree n in B with neighbors of degrees $\{m1, ..., m_n\}$

since by mapping the vertices of the same degree with the same neighbors to each other we can create a bijection such that there is an edge between u_i and u_j if and only if there is an edge between $\phi(u_i)$ and $\phi(u_j)$. Now let there be a tree of 2 vertices. Clearly any such tree only has vertices of degree 1



Now we have 2 choices, and regardless of which one of the 2 vertices we connect the third vertice to, we get a tree with 1 vertice of degree 2, and 2 of degree 1.



The first graph can be represented

A = ({A, B, C}, {(A, B), (C, B)} And the second

B = ({A, B, C}, {(C, A), (B, A)}

Now let ϕ = {(B, A), (A, C), (C, B)}

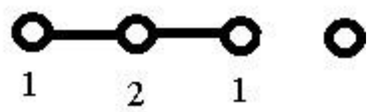
Now the edges of B can be represented as

{(ϕ(A), ϕ(B)), (ϕ(C), ϕ(B))}

And hence it is clear that the trees are isomorphic, and that all trees of 3 vertices are isomorphic.

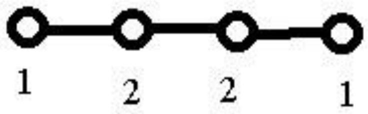
c)

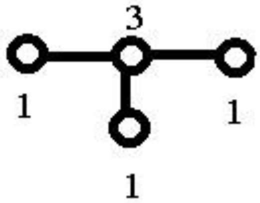
Let there be a tree of 3 vertices. There is only 1 such tree as all trees of 3 vertices are isomorphic as shown in part b)



Now we have 3 choices of where to connect the fourth vertice.

If we were to connect the fourth vertice to either of the vertices of degree 1, we would get a tree with 2 vertices of degree one, and 2 of degree 2, with each vertice of degree 1 having a neighbor of degree 2, and each vertice of degree 2 having a neighbor of degree 1 and 2, and hence choosing either of them would result in a tree that is isomorphic to the tree produced by the other choice. If however we connect it to the vertice of degree 2 we get a tree that has 3 vertices of degree 1, and 1 vertice of degree 3. Hence there are only 2 non-isomorphic trees of 4 vertices.

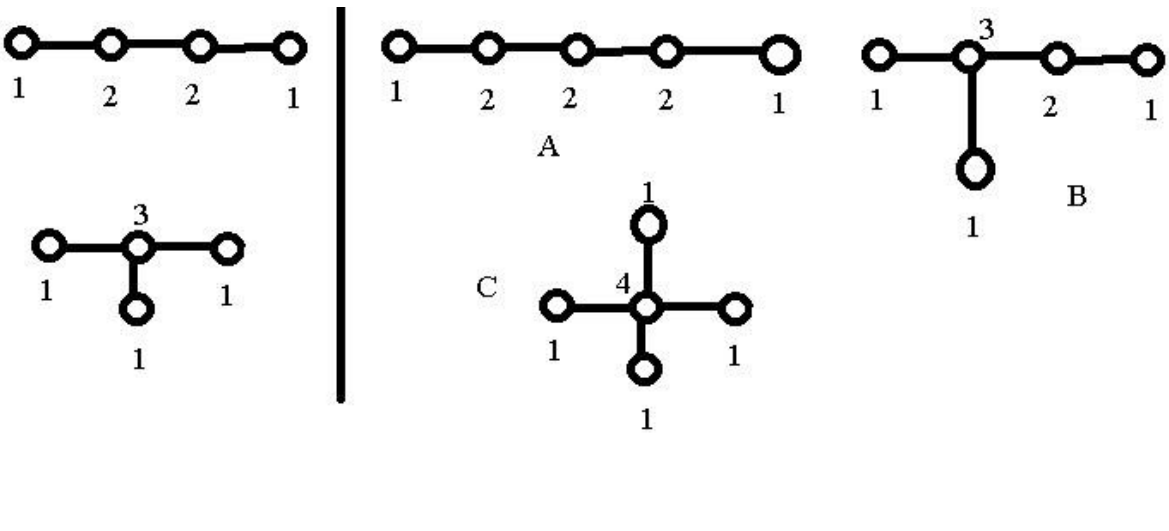




d)

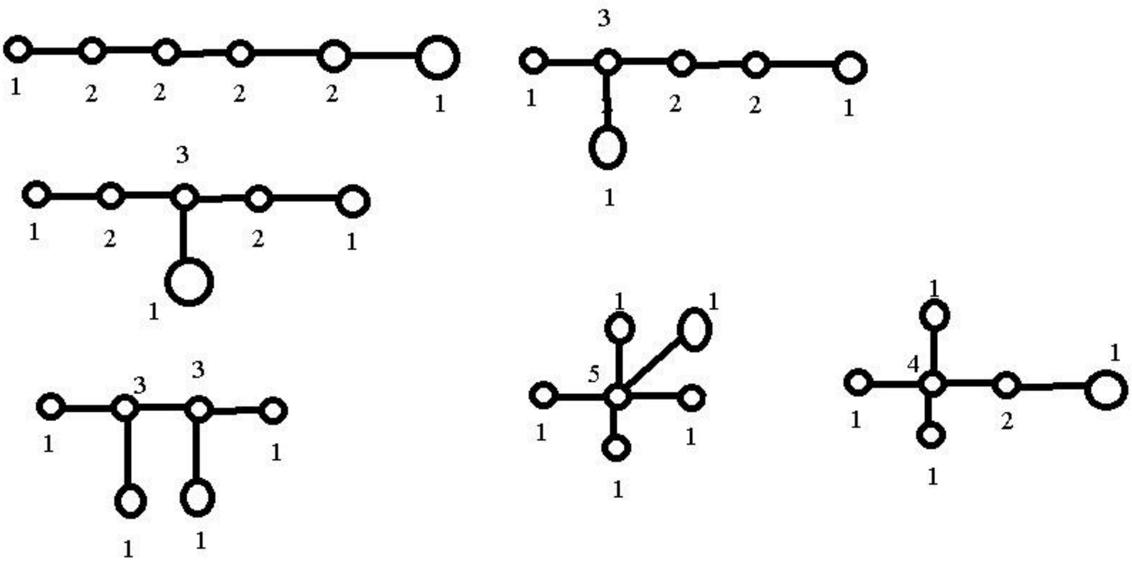
Given the 2 possible non-isomorphic trees of 4 vertices we get the following choices of where to attach the 5th vertice: Either we pick the first non-isomorphic tree of 4 vertices in which case we can either attach it to a vertice of degree 1 resulting in A below, or of degree 2, resulting in B. The choice of which vertice of degree 2 or 1 we attach it to does not matter due to symmetry.

Or we pick the second 4 vertice non-isomorphic tree, and get 2 choices, either attach to the vertice of degree 3, giving us C, or attach to one of the vertices of degree 1, giving us a tree with 1 vertice of degree 3 with neighbors of degrees {1, 1, 2}, 1 of degree 2 with neighbors of degrees {3, 1}, and 3 of degree 1, 2 of which have a neighbor of degree 3, and 1 which has a neighbor of degree 2, which would be isomorphic to B. Hence A, B and C are the only nonisomorphic trees of degree 5.



e)

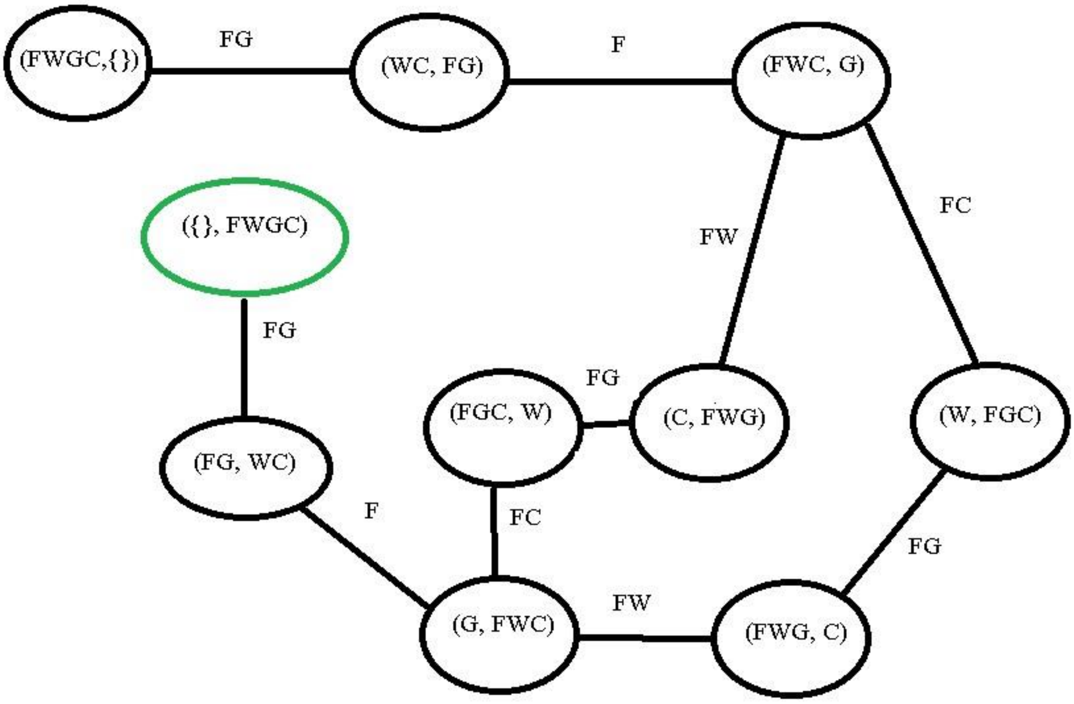
Following the same principles as in the previous parts the only possible non-isomorphic trees of degree 6 are the following:



Additional exercises

Problem 1

a) and b)



c)

The edges of the graph correspond to possible moves, and hence if we can find a path from the start to the goal we can extract from the edges on the path the steps to solve the puzzle.

d)

FG -> F -> FW -> FG -> FC -> F -> FG

And

FG -> F -> FC -> FG -> FW -> F -> FG

Problem 2

We can construct all such permutations by first choosing which corner 1 map to which gives us 4 choices, and then choosing which corner 2 will map to, which gives us 2 choices since 2 must be a neighbor of 1, after which 4 and 3 are locked in place, as 4 must be the remaining neighbor of 1, and 3 must then take the last available corner.

Hence there are 4 * 2 * 1 * 1 = 8 possible permutations in

D_4. **b)**

Let us prove it by writing out all the possible permutations as products of ρ and π.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = p^4 = p^0 = I = \begin{matrix} 4 & 3 \\ 1 & 2 \end{matrix} = \pi^2$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = p = \begin{matrix} 3 & 2 \\ 4 & 1 \end{matrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = p^2 = \begin{matrix} 2 & 1 \\ 3 & 4 \end{matrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} = p^3 = \begin{matrix} 1 & 4 \\ 2 & 3 \end{matrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} = \pi = \begin{matrix} 4 & 1 \\ 3 & 2 \end{matrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = p\pi = \begin{matrix} 1 & 2 \\ 4 & 3 \end{matrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} = p^2\pi = \begin{matrix} 2 & 3 \\ 1 & 4 \end{matrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = p^3\pi = \begin{matrix} 3 & 4 \\ 2 & 1 \end{matrix}$$

Problem 3

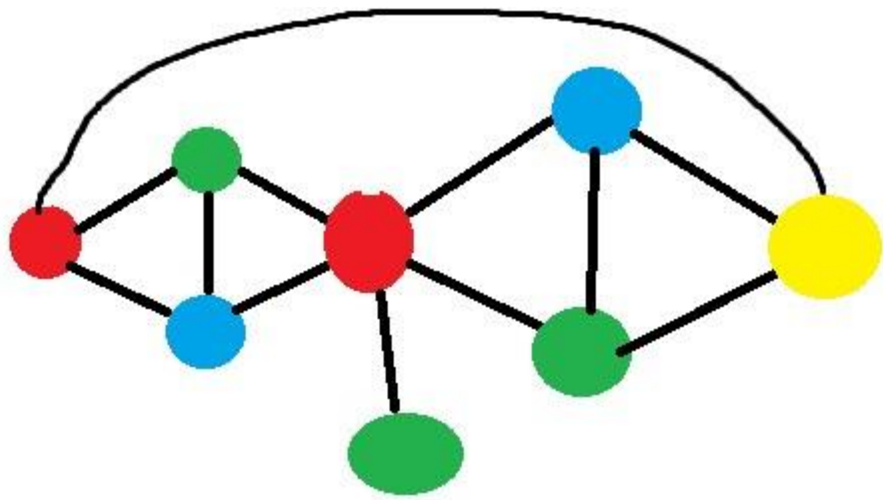
The shortest path from a to z is the following sequence:

a -> c -> d -> e -> g -> z

Problem 4

Chromatic number is the least amount of distinct colours required to color a graph so that no adjacent vertices have the same color.

A clique is a complete subgraph of a graph, that is a subgraph where each vertice has an edge to each other vertice.



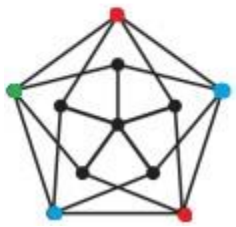
Here the chromatic number is 4, the largest clique is 3 and the largest degree is 5. 3 < 4 < 5.

Problem 5

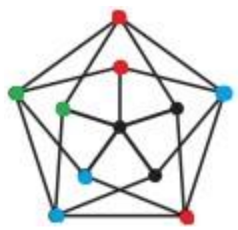
Let χ denote the chromatic number of the graph.

Clearly there is an odd length cycle in the graph, and hence χ ≥ 3 since colors in a cycle must alternate, but alternating colors in an odd length cycle will result in the first and last vertice to be colored to have the same color despite being adjacent, and hence coloring any odd length cycle requires atleast 3 colors.

Let us color the outlined cycle with the minimum 3 colors.



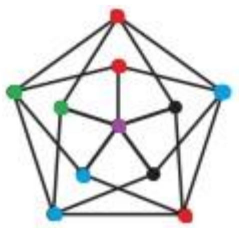
Now observer that using 3 colors this fixes the colors of the following vertices.



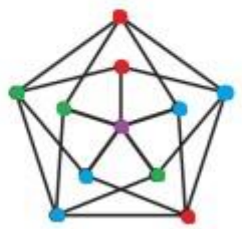
The red one must be red since it is adjacent to a blue and a green vertice, and similarly the green one must be green and the blue one must be blue.

Due to symmetry regardless of how we color the outside cycle 3 of the vertices in the middle will be fixed to different colors.

Now observe that the purple vertice can not be one of the three colors chosen, as it will always be adjacent to each of those colors. Hence $\chi \geq 4$.



Here is a coloring of the graph with 4 colors.



Since such a coloring is possible $4 \geq \chi$ and hence $4 \geq \chi \geq 4$

Hence the chromatic number is 4.

Problem 6

Since there are 5 total couples there are 10 total people. Since each person does not shake his own hand, nor their spouses, each person shakes from 0 to 8 hands. Since all 9 answers were different the answers must be 0, 1, ..., 8. It must be that the person who shook 0 hands is married to the person who shook 8 hands, since the person who shook 8 hands shook hands with everyone but themselves and their spouse, since there were 10 people total, and hence none of the other people could be the one who shook no hands, since everyone else has at least shook the hand of the person who shook 8 hands. Hence he shook hands with the people who answered 1, 2, 3, 4, 5, 6 and 7 and Mr X.

If we then consider the people who shook 1 to 7 hands we observe the person who shook 1 hand must have shook the hand of the person who shook 8 hands, and hence did not shake the hand of the person who shook 7 hands. Now since the person who shook 7 hands shook the hand of the person who shook 8 hands there are 6 handshakes unaccounted for. He cant have shook the people who shook 0 or 1 hands, and hence he must have shook the hands of the people who shook 2, 3, 4, 5, and 6 hands. That leaves 1 unaccounted handshake, which means he must have shook hands with Mr X. It also follows 1 must be 7s spouse, since otherwise 7 would not be able to shake all the other hands, as the alternative is one of those is 7s spouse.

Similarly it must be that 2 is 6's spouse and that 6 also shook Mr. X's hand. And that 3 is 5's spouse and 5 too shook Mr. X's hand. The only person left without a spouse is 4, which means 4 must be professor X, since only professor X's spouse is not in the group of people who answered, since Mr X was the one asking. Since nobody shakes their spouses hand it follows that Mr X shook 4 hands in total, those being the hands of 8, 7, 6, and 5.

Therefore the answer is Mr X shook 4 hands.

Problem 7

a)

Lets assume the converse is true for some party, that no 2 people know the same number of guests.

Then each of the **n** guests must know a distinct number of guests between 0 and **n - 1**. (or 1 to n if we assume each person to know themselves)

Since there are n guests and n distinct numbers to choose from, it must be that for each x $\in[0, \text{n-1}]$ there must be a guest that knows exactly x guests. However the guest that knows n-1 guests must know every single other guest at the party, and hence they must also know the person who knows 0 guests, and since we are assuming the knowing relation to be symmetric it must be then that the person who knows 0 must also know the person who knows n-1, but this leads to a contradiction since the person who knows 0 people cant know anyone.

Hence it must be that at least 2 people know the same number of guests, and the statement is true.

b)

Then each of the **n** guests must know a distinct number of guests between 0 and **n - 1**. (or 1 to n if we assume each person to know themselves)

Now it is possible to ennumerate each guest with a distinct number of known guests, since there are n guests and n distinct numbers of guests that each guest can know.

For example let there be a party with the guests {a, b, c}. Since the knowing relation is not symmetric we can then define it as the following:

R = {(a, b), (a, c), (b, c)}

Now 2 knows 2 people: b and c, b knows 1 person: c and c knows 0 people. Hence there exists a party where no 2 people know the same number of guests, and hence the statement is false.