HOMEWORK 6 SOLUTIONS, MS-A0402, FOUNDATIONS OF DISCRETE MATHEMATICS

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Homework

The written solutions to the homework problems should be handed in on MyCourses by Monday 11.4., 12:00. You are allowed and encouraged to discuss the exercises with your fellow students, but everyone should write down their own solutions.

Problem 1. (10pts) Does the following Diophantine equation

$$20x + 10y = 65.$$

have solutions $x, y \in \mathbb{N}$? If yes, find all the solutions. If not, justify your answer.

Solution 1. We compute

$$\gcd(20, 10) = \gcd(20 - 10, 10) = \gcd(10, 10) = \gcd(10 - 10, 10) = \gcd(0, 10) = 10$$

by the Euclidean algorithm. As 10 does not divide 65, the equation has no integer solutions.

Problem 2. (10pts) Does the following Diophantine equation

$$20x + 16y = 500.$$

have solutions $x, y \in \mathbb{N}$? If yes, find all the solutions. If not, justify your answer.

Solution 2. We compute $\gcd(20, 16)$ using the Euclidean algorithm.

$$\gcd(20, 16) = \gcd(20 - 1 \cdot 16, 16) = \gcd(4, 16) = \gcd(4, 16 - 4 \cdot 4) = \gcd(4, 0) = 4$$

Clearly 4 divides 500 so we expect the equation to have integer solutions. We can use the steps of the algorithm to write $500 = 125 \cdot 4$ as a linear combination of 20 and 16.

$$500 = 125 \cdot 4 = 125 \cdot (20 - 1 \cdot 16) = 20 \cdot 125 + 16 \cdot (-125)$$

So a particular solution is $x_0 = 125$ and $y_0 = -125$. Let $u = x - x_0$ and $v = y - y_0$ (x and y are some solution of the original equation). Then we must have the following.

$$20u + 16v = 0$$

This is a homogeneous Diophantine equation with solutions of the form $u = \frac{16}{4}k = 4k$ and $v = -\frac{20}{4}k = -5k$ with $k \in \mathbf{Z}$. We hence get that $x = u + x_0 = 4k + 125$ and $y = v + y_0 = -5k - 125$. Finally, we want only solutions such that $x, y \in \mathbb{N}$ so we get restrictions for k from $4k + 125 \geq 0$ and $-5k - 125 \geq 0$. Getting bounds for k from each inequality and combining these gives $-31.25 \leq k - 25$, meaning $-31 \leq k \leq -25$ as $k \in \mathbb{Z}$. The set of possible solution pairs (x, y) is hence

$$\{(4k + 125, -5k - 125) : k \in \mathbb{Z},\ -31 \leq k \leq -25\}.$$

Problem 3. (10pts) How many integers less than 22220 are relatively prime to 22220?

Solution 3. We note that $22220 = 10 \cdot 2222 = 2 \cdot 5 \cdot 2 \cdot 1111 = 2^2 \cdot 5 \cdot 11 \cdot 101$. As these are prime numbers and the prime factorization is always unique we can use Euler's totient function, which gives as the amount of relative primes

$$\phi(22220) = 22220 \cdot \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{5}\right)\left(1 - \frac{1}{11}\right)\left(1 - \frac{1}{101}\right) = 8000.$$

Problem 4. (10pts) Compute the last two digits of 2022^{2022}.

Solution 4. Evaluating the last two digits corresponds to calculating $2022^{2022} \mod 100$. We first note that $2022 \equiv 22 \mod 100$ and hence $2022^{2022} \equiv 22^{2022} \mod 100$. Our goal is to apply Euler's theorem to simplify the large power. We have $\phi(100) = 40$ and hence write

$$\begin{aligned} 2022^{2022} &\equiv 22^{2022} = 2^{2022}11^{2022} = 2^{101\cdot 20+2} \cdot (11^{40})^{50} \cdot 11^{22} \mod 100 \\ &= (2^{101\cdot 20+2} \mod 100) \cdot ((11^{40})^{50} \mod 100) \cdot (11^{22} \mod 100) \end{aligned}$$

The problem hence reduces to evaluating the three factors of the above product in mod 100.

(1) The leftmost part reduces to $2^2 = 4$ as powers of two repeat in a cycle in mod 100. The pattern can be found from a table and shown by induction, which is omitted here.
(2) The middle part is equal to 1, which follows from Euler's theorem by remembering that $\phi(100) = 40$ and noting that 11 and 100 are coprime i.e. $\gcd(11, 100) = 1$.
(3) The right part $11^{22} = (11^2)^{11} \equiv 21^{11} \mod 100$ is something we can calculate by writing the exponent as a sum of powers of 2 (in binary) i.e. $11 = 2^0 + 2^1 + 2^3 = 1 + 2 + 8$. We then have the following.

$$\begin{aligned} 21^2 &\equiv 441 \equiv 41 \mod 100 \\ 21^4 &\equiv 41 \cdot 41 = 1681 \equiv 81 \mod 100 \\ 21^8 &\equiv 81 \cdot 81 = 6561 \equiv 61 \mod 100 \end{aligned}$$

This gives $11^{22} \equiv 21^{11} = 21^1 \cdot 21^2 \cdot 21^8 = 21 \cdot 41 \cdot 61 = 52521 \equiv 21 \mod 100$.

Combining these results finally gives $2022^{2022} \equiv= 4 \cdot 1 \cdot 21 = 84 \mod 100$. The last two digits are hence 8 and 4.