Problem 1. (10pts) Prove that if $n \in \mathbb{Z}$, then $n^2 + 3n + 4$ is even.
Hint: Direct proof by cases Case 1: assume n is even and Case 2: assume n is odd.

Problem 1: Prove $n^2 + 3n + 4$ is even for $n \in \mathbb{Z}$ by direct proof
Since 4 is even $\Rightarrow$ We only need to prove $n^2 + 3n$ is even
- Case 1: assume n is even $\Rightarrow \exists k \in \mathbb{Z}$ so that $2k = n$
  $\Rightarrow n^2 + 3n = (2k)^2 + 3(2k) = 4k^2 + 6k = 2(2k^2 + 3k)$
  Because $2k^2 + 3k \in \mathbb{Z} \Rightarrow 2[(2k^2) + 3k]$ is also even
  $\Rightarrow n^2 + 3n$ is even (proven)
- Case 2: assume n is odd $\Rightarrow \exists k \in \mathbb{Z}$ so that $2k + 1 = n$
  $\Rightarrow n^2 + 3n = (2k+1)^2 + 3(k+1) = 4k^2 + 4k + 1 + 6k + 3$
  $= 4k^2 + 10k + 4 = 2(2k^2 + 5k + 2)$
  Because $2k^2 + 5k + 2 \in \mathbb{Z} \Rightarrow 2(2k^2 + 5k + 2)$ is also even
  $\Rightarrow n^2 + 3n$ is even (proven)

$\Rightarrow n^2 + 3n + 4$ is always even for $n \in \mathbb{Z}$

Problem 2. (10pts) Prove that there exists no integers a and b for which $24a+6b = 1$.
Hint: Use contradiction proof.

Problem 2: Prove that no integers a, b satisfies $24a + 6b = 1$ by contradiction
Suppose that there exists a, b such that $24a + 6b = 1$
We have: $6(4a + b) = 1$
$\Rightarrow \quad 4a + b = 1/6$
By the closure properties of integers, $4a + b$ is an integer as well because a and b are assumed to be integers. However, $1/6$ is not an integer
$\Rightarrow$ integer $=$ non-integer is a contradiction
$\Rightarrow$ No $a, b \in \mathbb{Z}$ satisfies $24a + 6b = 1$

Problem 3. (10pts) Prove by induction that for all $n \in \mathbb{Z}_+ = \{1, 2, \dots\}$ we have

$$\sum_{k=1}^{n} (-1)^k k^2 = \frac{(-1)^n (n+1) n}{2}.$$

Problem 3: Prove by induction for all $n \in \mathbb{Z}_+ = \{1, 2, \dots\}$ we have

$$\sum_{k=1}^{n} (-1)^k k^2 = \frac{(-1)^n (n+1) n}{2} \quad (*)$$

- Base case: $(n = 1)$

$$\sum_{k=1}^{1} (-1)^k k^2 = (-1)^1 1^2 = -1 = \frac{(-1)^1 (1+1) 1}{2} = -1$$

$\Rightarrow$ Base case is correct

- Induction step: Assume (IH) that $\sum_{k=1}^{m} (-1)^k k^2 = \frac{(-1)^m (m+1) m}{2}$

$$\sum_{k=1}^{m+1} (-1)^k k^2 \overset{\text{def}}{=} \sum_{k=1}^{m} (-1)^k k^2 + (-1)^{m+1} (m+1)^2$$

$$\overset{\text{IH}}{=} \frac{(-1)^m (m+1) m}{2} + (-1)^{m+1} (m+1)^2$$

$$= \frac{(m+1)\left[(-1)^m m + 2(-1)^{m+1}(m+1)\right]}{2}$$

$$= \frac{(m+1)(-1)^{m+1}\left[-m + 2(m+1)\right]}{2}$$

$$= \frac{(-1)^{m+1}(m+2)(m+1)}{2} = \frac{(-1)^{m+1}((m+1)+1)(m+1)}{2}$$

$\Rightarrow$ The statement is also true for $n = m+1$

$\Rightarrow$ Statement $(*)$ is true for all $n \in \mathbb{Z}_+$ (proven)

Problem 4. (10pts) Define a relation $\sim$ on $\mathbb{R}$ by $a \sim b$ if and only if $a \leq b$. Check if $\sim$ is (i) reflexive, (ii) symmetric, and/or (iii) transitive, and prove it if it does. If it does not satisfy the property you are checking, give an example to show it.

Problem 4: Define a relation $\sim$ on R by $a \sim b$ iif $a \leq b$

(i) is $\sim$ reflexive ?

Reflexivity: $\forall x \in A : x \sim x$

Or $\forall x \in R$, $x \leq x \Rightarrow$ Always correct $\Rightarrow$ $\sim$ is reflexive

(ii) is $\sim$ symmetric ?

Symmetry: $\forall x, y \in A : x \sim y \leftrightarrow y \sim x$

$\Rightarrow \forall a, b \in R : a \leq b \leftrightarrow b \leq a \Rightarrow$ Not correct

Let $a = 1$, $b = 2$, $a \leq b$ but not $b \leq a \Rightarrow$ $\sim$ is not symmetric

(iii) is $\sim$ transitive ?

Transitivity: $\forall x, y, z \in A : (x \sim y \land y \sim z) \to x \sim z$

$\Rightarrow \forall a, b, c \in R : a \leq b \land b \leq c \to a \leq c \Rightarrow$ Always correct

$\Rightarrow$ $\sim$ is transitive