EXERCISE SET 3,
MS-A0402, FOUNDATIONS OF DISCRETE MATHEMATICS

Homework solutions

Problem 1. Let P be the set of all Finland's presidents, and let G be the set of all ordered pairs $(a, b) \in P \times P$ such that the president b succeeded president a in office. Is G the graph of a function? Explain your answer.

The graph of a function $f : X \to Y$ is defined as

$$G(f) = \{(x, f(x)) : x \in X\}$$

Now let $p_{Niinistö} \in P$ be the current president of Finland. As there is no pair $(p_{Niinistö}, p) \in P \times P$, we see that G cannot be a graph of a function from P to $P \times P$. However, it is a graph of a function from $(P \backslash \{p_{Niinistö}\})$ to $P \times P$ because then all required ordered pairs exist in G. Also notice that the function only maps each president to one pair $(a, b) \in P \times P$, which is a requirement of a function.

Problem 2. Find the domain and range of the function which assigns to each non-negative integer its last digit.

Domain (where the function maps from): Non-negative integers ($\mathbb{Z}_{\geq 0}$)
Range (where the function maps to): All possible last digits of an integer i.e. $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Problem 3. Eight people are to be seated around a table; the chairs don't matter, only who is next to whom, but right and left are different. Two people, X and Y, cannot be seated next to each other. How many seating arrangements are possible?

First, we can place X to any position in the table. As the chairs do not matter, placing X to any chair is considered equivalent. Thus, there is 1 way to do so. Then, we can place Y to any of the remaining seats that is not next to X. We have 8-3=5 possibilities to do so. After that, the rest of the people can be positioned in the remaining 6 seats, giving 6! possible arrangements. Now the total number of arrangements is $5 \cdot 6! = 3600$.

Problem 4. Prove that for all $n \in \mathbb{N}$, $n \geq 9$, the following statement is true: for all $k \in \mathbb{N}$ with $0 \leq k \leq n$ we have

$$\binom{n}{k} < 2^{n-2}.$$

Proof by induction:

1. Base case ($n = 9$)

$$\binom{9}{k} \leq \max \binom{9}{k} = 126 < 128 = 2^{9-2}$$

2. Induction assumption:

Assume that the statement holds for all $k \in \mathbb{N}$ with $0 \leq k \leq n$ when $n = m - 1$ i.e.

$$\binom{m-1}{k} < 2^{m-1-2}$$

3. Proof that the statement also holds for for all $k \in \mathbb{N}$ with $0 \leq k \leq n$ when $n = m$

First, notice that

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Then, by the induction assumption

$$\binom{n-1}{k} < 2^{m-1-2} \text{ and } \binom{n-1}{k-1} < 2^{m-1-2}$$

Thus,

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} < 2 \cdot 2^{m-1-2} = 2^{m-2}$$