Problem 1. Let P be the set of all Finland's presidents, and let G be the set of all ordered pairs $(a, b) \in P \times P$ such that the president b succeeded president a in office. Is G the graph of a function? Explain your answer.

Let A,B,C,D be any random presidents of Finland.
Let G be the set of all ordered pairs (a, b) ∈ P × P such that the president B succeeded president A.
Hypothetically, let G = {(A, B), (B, A), (A,C), (C,D)}
=> G is not the graph of a function for two reasons
- First, the president can be in an office more than once. If he is not the latest incumbent president, it means he would have more than one successor. For example in G above, (A, B) and (A,C) means that both B and C are presidents after the term of A. Since a function cannot have more than one output for an input => G is not a graph of a function
- If a president has not been a president before and he is the latest incumbent president, then he would have no successor currently, which is D in the relation (C,D). A function strictly must have one output for any input in the domain, and since D does not have an output, G is not the graph of a function

Conclusion: G is not the graph of a function.

Problem 2. Find the domain and range of the function which assigns to each nonnegative integer its last digit.

This function maps a nonnegative integer to its last digit: f(x) = y
Domain is a non-negative number => **x ∈ N**, or the domain is the set of natural numbers
Range is only one digit. Since all natural numbers end in digits from 0 to 9, the range is simply
y ∈ {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}

Problem 3. Eight people are to be seated around a table; the chairs don't matter, only who is next to whom, but right and left are different. Two people, X and Y, cannot be seated next to each other. How many seating arrangements are possible?

There are 8 seats in total, X and Y cannot seat next to each other
=> Suppose that X sits in one chair. Of course Y cannot sit in the same chair with X. And also, Y cannot sit at the chairs left or right to X. So there are always 3 chairs that Y cannot sit wherever X sits => Y has only 5 seats available.
Let's say X can freely choose any seats he wants and the restriction depends on Y. The reverse is also equivalent to the above statement in terms of number of arrangements as well.
=> X has 8 different seats available to choose from. For each seat X chooses, Y has 5 options left. The total possible seating arrangements are therefore **8 x 5 = 40 arrangements**

Problem 4. Prove that for all $n \in \mathbb{N}$, $n \geq 9$, the following statement is true: for all $k \in \mathbb{N}$ with $0 \leq k \leq n$ we have

$$\binom{n}{k} < 2^{n-2}.$$

Hint: You can use induction in n with base case $n = 9$.

Problem 4: Prove that $n \in N, n \geqslant 9, \forall k \in N, 0 \leqslant k \leqslant n$, we have

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} < 2^{n-2}$$

Proof by induction:

- Base case : $n = 9$. From Pascal triangle, we know that $\binom{n}{k}$ is maximized when $k = n/2$ or $k = n/2 + 1$ when n is even, and $k = (n+1)/2$ when n is odd
  To prove for the base case, we need only to plug in $k = (9+1)/2 = 5$ as $\binom{9}{5}$ will be maximized

$$\binom{9}{5} = \frac{9!}{5!(9-5)!} = 126 < 2^{9-2} = 128 \Rightarrow \text{Base case } n = 9 \text{ is correct}$$

- Induction Step : assume $\binom{m}{k} < 2^{m-2}$, then

$$\binom{m+1}{k} = \frac{(m+1)!}{k!(m+1-k)!} = \frac{m!(m+1)}{k!(m-k)!(m+1-k)} \overset{IH}{<} 2^{m-2} \frac{m+1}{m+1-k}$$

We know before that $\binom{n}{k}$ is maximized when k is half way to n from 1
To maximize the left side, k will take the formula like above

- For even m, to maximize $(m+1)/(m+1-k)$, let $k = m/2$

$$\Rightarrow \frac{m+1}{m+1-m/2} = \frac{m+1}{\frac{m}{2}+1} = \frac{m+1}{\frac{1}{2}(m+1)+\frac{1}{2}} < 2$$

- For odd m, let $k = (m+1)/2$

$$\Rightarrow \frac{m+1}{m+1-\left(\frac{(m+1)}{2}\right)} = \frac{m+1}{\frac{1}{2}(m+1)} = 2$$

Therefore $\frac{m+1}{m+1-k} \leq 2$ when k is halfway of m

$$\Rightarrow \binom{m+1}{k} < 2^{m-2} \frac{m+1}{m+1-k} \leqslant 2^{m-2} \times 2 = 2^{m-1} = 2^{(m+1)-2}$$

$$\Rightarrow \binom{m+1}{k} < 2^{(m+1)-2}$$ (proven), the statement is true for $n = m+1$