Problem 1. (10pts) How many integers from 1 to 60 are multiples of 2 or 3 but not both?

Problem ~~3~~ 1: We can use integer division for this problem

- First: $60 \div 2 = 30$ multiples of 2 in range $[1, 60] = m_2$
- Secondly: $60 \div 3 = 20$ multiples of 3 in range $[1, 60] = m_3$

Multiples of 2 and 3 means multiples of 6

$\Rightarrow 60 \div 6 = 10$ multiples of 6 in range $[1, 60] = m_6$

Since multiples of 6 appear both in multiples of 2 and multiples of 3

$\Rightarrow$ Num of integers in range $[1, 60]$ that are multiples of 2 and 3 but not both are given by:

$n = m_2 + m_3 - 2 * m_6 = 30 + 20 - 2 \times 10 = 30$ numbers (answer)

Problem 2. (10pts) Consider the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 7 & 3 & 6 & 4 & 2 & 1 & 5 & 8 \end{pmatrix}$$

a) Write it as a product of disjoint cycles.

b) Write it as a product of transpositions.

Problem 2: Consider permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 7 & 3 & 6 & 4 & 2 & 1 & 5 & 8 \end{pmatrix}$

a) Write it as product of disjoint cycles

1^{st} cycle: $1 \to 9 \to 8 \to 5 \to 4 \to 6 \to 2 \to 7 \to 1$

2^{nd} cycle: $3 \to 3$

$\Rightarrow$ Permutation as product of cycles: $(1985 4627)(3)$ or just (19854627)

b) Write it as product of transpositions

Every permutation $\pi \in S_n$ as a product of transpositions are written as $(1, 2), (1, 3), \dots (1, n)$ or $(1, 2), (2, 3), \dots (n-1, n)$

$\Rightarrow$ Permutation as product of transpositions

$\pi = (1, 9)(9, 8)(8, 5)(5, 4)(4, 6)(6, 2)(2, 7)$

Problem 3. (10pts) In how many ways can we rearrange the letters in the word

"knackered"

(a) with no restrictions?
(b) if the first and last letter must be vowels?

Problem 3: The word "knackered" has 2 sets of repeated elements which are 2 k and 2 e. The word has 9 letters

a) How many arrangements without restriction?

$\Rightarrow$ num of arrangements $= \dfrac{9!}{2!\,2!} = 90720$ different arrangments

b) If the first and last letter must be vowels?

There are 3 different possibilities of first and last vowels: a - e, e - a, e - e

The middle part of the word thus have 7 letters left. However it has 2k as well

$\Rightarrow$ num of arrangements: $3 \times \dfrac{7!}{2!} = 7560$ different arrangements

Problem 4. (10pts) How many ways are there to tile dominos (with size 2×1) on a grid of size 2×20?

(This is a modified job interview question for a Quantitative Researcher position in a London based research firm, © G-Research

Hint *(just a suggestion): Experiment with first* 2×1, 2×2, 2×3, 2×4, *etc. sized grids and try to come up with a way to relate*

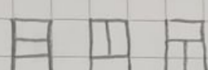
$$a_n = \text{ number of ways to tile a grid of size } 2 \times n$$

to the previous terms a_{n-1} *and* a_{n-2}. *Apply this recursive relation then until you reach* a_{20}.*)*

Problem 4:

- For grid 2×1:  $\Rightarrow f(1) = 1$
- For grid 2×2: $\Rightarrow f(2) = 2$

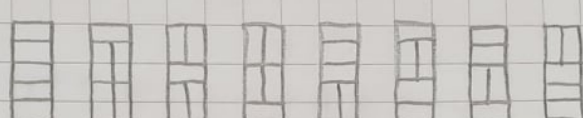
- For grid 2×3

 $\Rightarrow f(3) = 3$

- For grid 2×4

 $\Rightarrow f(4) = 5$

- For grid 2×5

 $\Rightarrow f(5) = 8$

The sequence starts to resemble the Fibonacci sequence, where $f(n) = f(n-1) + f(n-2)$ and base case $f(1) = 1$ and $f(2) = 2$

$\Rightarrow f(20) = a_n = 10946$ number of ways to tile in grid size 2×20

(answer)