Discrete Mathematics

Solutions For Exercise 2

March 5, 2022

Exploratory Exercises

Problem 1

Give an example of a function $f : \mathbf{Z} \to \mathbf{Z}$ that is

a) Injective but not surjective $f(x) = 2x$

$\star$ Injective: Suppose $\forall x, y \in \mathbb{Z} : f(x) = f(y) \implies 2x = 2y \implies x = y$ hence f is injective.

$\star$ Surjective: Let $y \in \mathbb{Z}$ and $f(x) = y \implies 2x = y \impliedby x = \frac{y}{2} \notin \mathbb{Z}$. Hence, f is not surjective.

b) Surjective but not injective. $f(x) = \lfloor \frac{x}{2} \rfloor$

$\star$ Injective: $f(0) = f(1) = 0$ but $0 \neq 1$ hence, f is not injective.

$\star$ Surjective: $f(2x) = x \forall x \in \mathbb{Z}$. Hence, f is surjective.

c) Both injective and surjective. $f(x) = x$. This is clear.

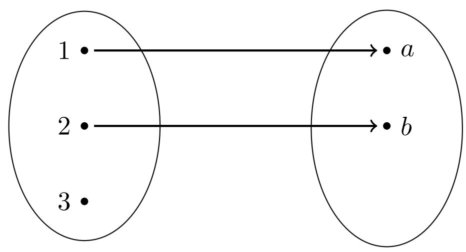
d) Neither injective nor surjective. $f(x) = x^2$

$\star$ Injective: $f(1) = f(-1)$ but $-1 \neq 1$ hence, f is not injective.

$\star$ Surjective: Let $y \in \mathbb{Z}$ and $f(x) = y \implies x = \sqrt{y} \notin \mathbb{Z}$ if y is not a square number. Hence, f is not surjective.

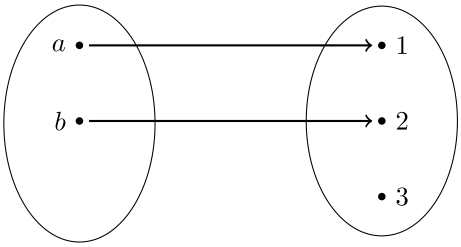
Problem 2

a) No there exist no injective map for the number of elements in the domain exceeds those of the codomain.



(b) No there exixs no surjective map.

(c) If there is an injection, $|A| \leq |B|$. And if the map is surjective, $|B| \leq |A|$



Problem 3

a) For the four examples above, determine if they are reflexive, transitive, and/or symmetric.

i) x and y are siblings: Obviously this is Symmetric, not Reflexive, it is transitive if we are only considering "full siblings." But if we consider half siblings, then it is not transitive(for example if A and B have the same mother but different fathers, and B and C have the same father but different mothers).

ii) x divides y.: $x = x.1 \rightarrow x|x$ so Reflexive. It is not symetric since $x|y \nRightarrow y|x$ e.g $1|2$ but $2 \nmid 1$. Lastly, it is Transitive since if $a|b \Rightarrow \exists k \in \mathbb{Z}$: $b = ak$ and $b|c \Rightarrow \exists l \in \mathbb{Z} : c = bl = akl = am$ where $m = kl \in \mathbb{Z}$.Therefore $a|c$

iii) $x < y$ in $\mathbb{R}$: Clearly Transitive. But not Reflexive since $1 \nless 1$ and not Symmetric since $0 < 1 \nRightarrow 1 < 0$.

iv) $x = y$ Cleary Relexive, Transitive and Symetric.

b) i) The relation $x = y$.

ii) The relation $R = \{(x, x), (y, y), (z, z), (x, y), (y, z)\}$

iii) $x \neq y$

iv) $x < y$

Additional Exercises

Problem 1

Prove that, if A, B and C are sets, then $(A \cup B) \times C = (A \times C) \cup (B \times C)$

- $(A \cup B) \times C \subseteq (A \times C) \cup (B \times C))$: Let $(x, y) \in (A \cup B) \times C \Rightarrow a \in A \cup B$ and $y \in C \Rightarrow x \in A$ and $y \in C$ or $x \in B$ and $y \in C$. Therefore $(x, y) \in A \times C$ or $(x, y) \in B \times C$. Hence $(x, y) \in (A \times C) \cup (B \times C))$.
- $(A \times C) \cup (B \times C)) \subseteq (A \cup B) \times C$: Let $(x, y) \in (A \times C) \cup (B \times C)) \Rightarrow (x, y) \in (A \times C)$ or $(x, y) \in (B \times C) \Rightarrow x \in A \cup B$ and $y \in C$.

Problem 2

Prove that $(\mathbb{Z} \times \mathbb{R}) \cap (\mathbb{R} \times \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}$

- $(\mathbb{Z}\times\mathbb{R})\cap(\mathbb{R}\times\mathbb{Z})\subseteq\mathbb{Z}\times\mathbb{Z}$: Let $(x,y)\in(\mathbb{Z}\times\mathbb{R})\cap(\mathbb{R}\times\mathbb{Z})\Rightarrow(x,y)\in(\mathbb{Z}\times\mathbb{R})$ and $(x,y)\in(\mathbb{R}\times\mathbb{Z})\Rightarrow(x,y)\in\mathbb{Z}\times\mathbb{Z}$ since x is an integer from the first product and y is an integer from the second.
- $\mathbb{Z}\times\mathbb{Z}\subseteq(\mathbb{Z}\times\mathbb{R})\cap(\mathbb{R}\times\mathbb{Z})$: Let $(x,y)\in\mathbb{Z}\times\mathbb{Z}\Rightarrow x\in\mathbb{Z}\subset\mathbb{R},\Rightarrow(x,y)\in\mathbb{R}\times\mathbb{Z}$ similarly, $y\in\mathbb{R}\Rightarrow(x,y)\in\mathbb{Z}\times\mathbb{R}$. Hence $(x,y)\in(\mathbb{Z}\times\mathbb{R})\cap(\mathbb{R}\times\mathbb{Z})$

Problem 3

$$f:\mathbb{R}\longrightarrow\mathbb{R} \qquad\qquad g:\mathbb{R}\longrightarrow\mathbb{R}$$
$$x\longmapsto 4x-3 \qquad\qquad y\longmapsto\frac{y+3}{4}$$

$$\begin{cases} f(g(y))=f(\frac{y+3}{4})=4(\frac{y+3}{4})-3=y \\ \\ g(f(x))=g(4x-3)=x \end{cases}$$

Problem 4

a) Clearly $\sim$ is Reflexive sinve $x^2-x^2=0\in\mathbb{Z}$. Also it is symmetric since the negative of an integer is also an integer. Lastly it is transitive for if $x^2-y^2\in\mathbb{Z}$ and $y^2-z^2\in\mathbb{Z}$, then $x^2-z^2=(x^2-y^2)+(y^2-z^2))\in\mathbb{Z}$. Hence $\sim$ is indeed an Equivalence relation.

b) $[0]=\{a|0Ra,\forall a\in\mathbb{R}\}=\{a|-a^2\in\mathbb{Z},\forall a\in\mathbb{R}\}=\{\dots,-\sqrt{3},-\sqrt{2},-1,0,1,\sqrt{2},\sqrt{3}\dots\}$

c) $[\frac{1}{3}]=\{a|\frac{1}{3}Ra,\forall a\in\mathbb{R}\}=\{a|\frac{1-(3a)^2}{9}\in\mathbb{Z},\forall a\in\mathbb{R}\}=\left\{\dots,-\frac{\sqrt{19)}}{3},-\frac{\sqrt{10}}{3},-\frac{1}{3},\frac{1}{3},\frac{\sqrt{10}}{3},\frac{\sqrt{19}}{3}\dots\right\}$

Problem 5

Prove by induction that for every $n\in\mathbb{N}$ holds $\displaystyle\sum_{i=1}^{n}\frac{i}{(i+1)!}=1-\frac{1}{(n+1)!}$

- Base Case Verify for $n=1$: $\displaystyle\sum_{i=1}^{n}\frac{i}{(i+1)!}=\frac{1}{2}=1-\frac{1}{2!}$ so True for $n=1$.
- I.H: Assume true for $n=k$. i.e $\displaystyle\sum_{i=1}^{k}\frac{i}{(i+1)!}=1-\frac{1}{(k+1)!}$
- Verify for $n=k+1$:

  $$\sum_{i=1}^{k+1}\frac{i}{(i+1)!}=\sum_{i=1}^{k}\frac{i}{(i+1)!}+\frac{k+1}{(k+2)!}=1-\frac{1}{(k+1)!}+\frac{k+1}{(k+2)!}=1-\frac{1}{(k+2)!}$$

  Hence statement is true for all $n\in\mathbb{N}$

Problem 6

The thrid point: In particular "Clearly, these colours must be the same, as the socks $s_2 \ldots s_n$ can only have one colour" is wrong when $n = 1$ (because then the set $s_2 \ldots s_n$ is empty). This highlights the fact that already if the induction step fails for one particular value of n, the induction proof collapses.

Problem 7

The statement is not true for the base case. i.e for $n = 1$, $1 \neq \frac{9}{8}$

Problem 8

By Double induction show that the Fibonacci numbers satisfy

$$f_{m+n+1} = f_m f_n + f_{m+1} f_{n+1} \ \forall m, n \geq 0.$$

Let the statement be denoted by $P(m, n)$

- Base Case $P(0, 0) : f_1 = 1 = f_1 f_1$ since $f_0 = 0, f_1 = 1$ So statement true.
- Assume $P(m, n)$ and $P(m, n-1)$ true $\forall m, n \geq 0$. We show that $P(m, n+1)$ is also true.

$$\begin{aligned} f_{m+n+2} = f_{m+n+1} + f_{m+n} &\overset{\text{I.H}}{=} f_m f_n + f_{m+1} f_{n+1} + f_{n-1} f_m + f_n f_{m+1} \\ &= f_m f_{n+2} + f_{m+1} f_{n+2} \end{aligned}$$

- Similarly, assume $P(m, n)$ and $P(m-1, n)$ true $\forall m, n \geq 0$ Show $P(m+1, n)$ is also true.

$$\begin{aligned} f_{m+n+2} = f_{m+n+1} + f_{m+n} &\overset{\text{I.H}}{=} f_m f_n + f_{m+1} f_{n+1} + f_n f_{m-1} + f_n f_{m+1} \\ &= f_n f_{m+2} + f_{n+1} f_{m+2} \end{aligned}$$

This implies $P(m, n)$ is true $\forall m, n \geq 0$