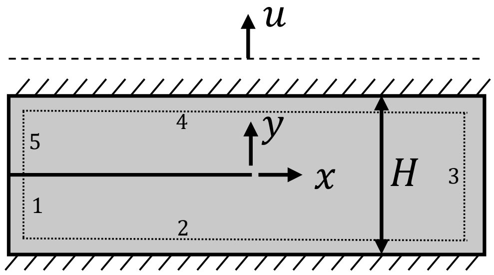
A? Question

An infinite strip of height H with a semi-infinite crack is rigidly clamped along its top and bottom faces, see below. Determine the J-integral for mode I loading, where top edge is moved upward by a prescribed displacement u. Assume that the material is linear elastic, isotropic, and under plane stress conditions.



A! Solution

In this case, the J-integral can be expressed as the sum of five segments labelled on the figure above:

$$J = J_1 + J_2 + J_3 + J_4 + J_5,$$

where each term is given by:

$$J = \int_\Gamma \left(w dy - t_i \frac{\partial u_i}{\partial x} ds \right).$$

Over segments 1 and 5, we have no deformations, $u_x = 0$ and u_y is a constant. This implies:

$$w = 0 \quad \text{and} \quad \frac{\partial u_i}{\partial x} = 0 \quad \Longrightarrow \quad J_1 = J_5 = 0.$$

Next, over segments 2 and 4 we have $dy = 0$, $u_x = 0$ (the top and bottom edges are clamped and cannot contract). Also, $u_y = 0$ for the bottom edge, whereas $u_y = u$ along the top edge. This implies:

$$dy = 0 \quad \text{and} \quad \frac{\partial u_i}{\partial x} = 0 \quad \Longrightarrow \quad J_2 = J_4 = 0$$

Finally, along segment 3, we have $dy \neq 0$ and $w \neq 0$, whereas the displacement $u_x = 0$ and $u_y = f(y)$, which implies $\frac{\partial u_i}{\partial x} = 0$. Therefore, we conclude that:

$$J = J_3 = \int_\Gamma \left(w dy - t_i \frac{\partial u_i}{\partial x} ds \right) = \int_{-H/2}^{H/2} w dy$$

For a linear elastic material the strain energy density $w = \frac{1}{2}\sigma_{ij}\epsilon_{ij}$. The stress and strain components are:

$$\sigma_{xx} \neq 0 \qquad \sigma_{yy} \neq 0 \qquad \sigma_{xy} = 0 \qquad \sigma_{zz} = 0$$

$$\epsilon_{xx} = 0 \qquad \epsilon_{yy} = \frac{u}{H} \qquad \epsilon_{xy} = 0 \qquad \epsilon_{zz} \neq 0$$

The stresses σ_{xx} and σ_{yy} can be obtained using Hooke's law. For a linear elastic isotropic material under plane stress, Hooke's law is:

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}\begin{bmatrix}\epsilon_{xx}\\ \epsilon_{yy}\\ \epsilon_{xy}\end{bmatrix}$$

Substituting in the values above, we find that:

$$\sigma_{yy} = \frac{E}{1-\nu^2}\epsilon_{yy} = \frac{Eu}{H(1-\nu^2)}$$

With this, we can compute the strain energy density w:

$$w = \frac{1}{2}\sigma_{ij}\epsilon_{ij} = \frac{1}{2}\sigma_{yy}\epsilon_{yy} = \frac{Eu^2}{2(1-\nu^2)H^2}$$

Finally, we can compute the J-integral using the above equation:

$$\begin{aligned} J &= \int_{-H/2}^{H/2} w dy \\ &= \int_{-H/2}^{H/2} \frac{Eu^2}{2(1-\nu^2)H^2} dy \\ &= \frac{Eu^2}{2(1-\nu^2)H^2}\left[y\right]_{-H/2}^{H/2} \\ &= \frac{Eu^2}{2(1-\nu^2)H} \end{aligned}$$

Since the material is linear elastic, $J = G$. From this, we can find the stress intensity factor K_I using:

$$K_I = \sqrt{EG} = \sqrt{EJ} = \sqrt{\frac{E^2u^2}{2(1-\nu^2)H}} = \sqrt{\frac{1}{2(1-\nu^2)H}}Eu$$

This is equal to the formula given in the datasheet.