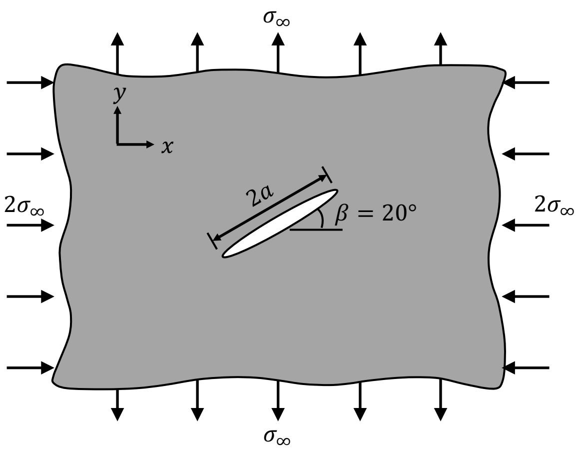
## A? Question

A large plate contains a central crack of length $2a$ at an angle $\beta = 20°$ from the horizontal. The plate is loaded in tension by a stress $\sigma_\infty$ in the vertical direction, and in compression in the horizontal direction by a stress $2\sigma_\infty$, see below.

(a) Find the stress intensity factors $K_I$ and $K_{II}$. Express your results as a function of $\sigma_\infty$ and $a$.

(b) Find the angle of crack propagation.

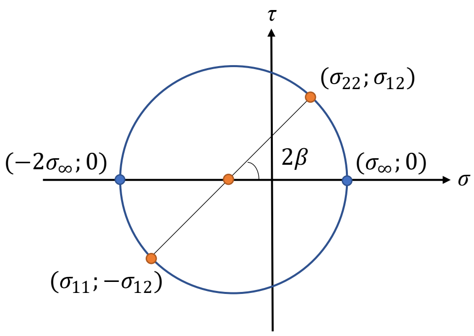



## A! Solution

The stress field in the global reference frame is:

$$\sigma_{xx} = -2\sigma_\infty \qquad \sigma_{yy} = \sigma_\infty \qquad \sigma_{xy} = 0.$$

Mohr's circle for this stress field is:




where the centre $c$ and radius $r$ of the circle are:

$$c = \frac{\sigma_\infty - 2\sigma_\infty}{2} = -\frac{\sigma_\infty}{2} \qquad \text{and} \qquad r = \frac{\sigma_\infty + 2\sigma_\infty}{2} = \frac{3\sigma_\infty}{2}.$$

The stresses $\sigma_{22}$ and $\sigma_{12}$ in the local reference frame are given by:

$$\sigma_{22} = c + r\cos(2\beta) = -\frac{\sigma_\infty}{2} + \frac{3\sigma_\infty}{2}\cos(2\beta) = 0.6491\sigma_\infty,$$
$$\sigma_{12} = r\sin(2\beta) = \frac{3\sigma_\infty}{2}\sin(2\beta) = 0.9642\sigma_\infty.$$

Finally, the stress intensity factors are given by:

$$K_I = \sigma_{22}\sqrt{\pi a} = 0.6491\sigma_\infty\sqrt{\pi a},$$
$$K_{II} = \sigma_{12}\sqrt{\pi a} = 0.9642\sigma_\infty\sqrt{\pi a}.$$

To find the angle of crack propagation, we set $\sigma_{r\theta} = 0$, and this gives:

$$\sigma_{r\theta} = \frac{K_I}{\sqrt{2\pi r}}\left[\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}\right] + \frac{K_{II}}{\sqrt{2\pi r}}\left[\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}\right] = 0$$
$$\implies K_I\left[\sin\frac{\theta}{2} + \sin\frac{3\theta}{2}\right] + K_{II}\left[\cos\frac{\theta}{2} + 3\cos\frac{3\theta}{2}\right] = 0$$
$$\implies 0.6491\left[\sin\frac{\theta}{2} + \sin\frac{3\theta}{2}\right] + 0.9642\left[\cos\frac{\theta}{2} + 3\cos\frac{3\theta}{2}\right] = 0$$
$$\implies \theta = -1.0188 \quad \text{or} \quad 1.4603 \quad \text{or} \quad \pi$$

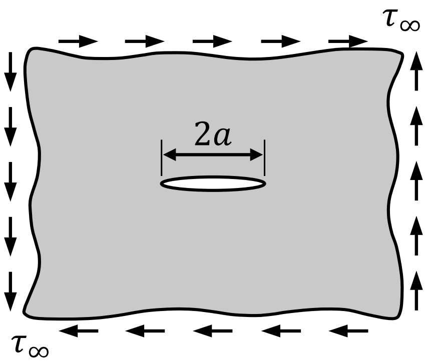
The correct angle $\theta$ is the one corresponding to the maximum $\sigma_{\theta\theta}$. Plotting

$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}}\left[\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2}\right] - \frac{K_{II}}{\sqrt{2\pi r}}\left[\frac{3}{4}\sin\frac{\theta}{2} + \frac{3}{4}\sin\frac{3\theta}{2}\right]$$

shows that $\theta = -1.0188$ corresponds to a maximum in $\sigma_{\theta\theta}$, whereas $\theta = 1.4603$ corresponds to a minimum in $\sigma_{\theta\theta}$. Therefore, the crack will propagate along $\theta = -1.0188 = -58.4°$.

**A? Question**

Find the direction of crack propagation for pure mode II loading. You might find this trigonometric identity useful: $\cos(3\alpha) = 4\cos^3\alpha - 3\cos\alpha$.




**A! Solution**

When the plate is loaded in mode II the shear stress $\sigma_{r\theta}$ close to the crack tip is obtained from the datasheet and is equal to:

$$\sigma_{r\theta} = \frac{K_{II}}{\sqrt{2\pi r}}\left(\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}\right)$$

The direction of crack propagation is the direction where $\sigma_{r\theta} = 0$, which gives:

$$\sigma_{r\theta} = \frac{K_{II}}{\sqrt{2\pi r}}\left(\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}\right) = 0$$
$$\Rightarrow \cos\frac{\theta}{2} + 3\cos\frac{3\theta}{2} = 0$$

Using the trigonometric identity: $\cos(3\alpha) = 4\cos^3\alpha - 3\cos\alpha$, the above expression becomes:

$$\cos\frac{\theta}{2} + 12\cos^3\frac{\theta}{2} - 9\cos\frac{\theta}{2} = 0$$
$$\Rightarrow 12\cos^3\frac{\theta}{2} = 8\cos\frac{\theta}{2}$$
$$\Rightarrow \cos^2\frac{\theta}{2} = \frac{2}{3}$$
$$\Rightarrow \cos\frac{\theta}{2} = \pm\sqrt{\frac{2}{3}}$$
$$\Rightarrow \theta = \pm 70.5^\circ$$

To determine which angle is the correct solution, we need to check which one gives the maximum value of $\sigma_{\theta\theta}$. The expression for $\sigma_{\theta\theta}$ is also obtained from the datasheet and equal to:

$$\sigma_{\theta\theta} = -\frac{K_{II}}{\sqrt{2\pi r}}\left(\frac{3}{4}\sin\frac{\theta}{2} + \frac{3}{4}\sin\frac{3\theta}{2}\right)$$

Substituting our two possible solutions $\theta = \pm 70.5^\circ$ returns:

$$\sigma_{\theta\theta} = -\frac{1.15 K_{II}}{\sqrt{2\pi r}} < 0 \quad \text{for } \theta = 70.5^\circ$$
$$\sigma_{\theta\theta} = \frac{1.15 K_{II}}{\sqrt{2\pi r}} > 0 \quad \text{for } \theta = -70.5^\circ$$

The maximum value of $\sigma_{\theta\theta}$ is obtained when $\theta = -70.5^\circ$ and therefore this will be the direction of crack propagation.