Fracture Mechanics

4. Plasticity

Luc St-Pierre

April 20, 2023

Motivation

- So far, we investigated fracture using (i) stress and (ii) energy approaches, assuming that the material is linear elastic.
- This is refered to as Linear Elastic Fracture Mechanics (LEFM).
- Metals are not simply linear elastic, they also exhibit plasticity.
- If there is plasticity, can we still use LEFM?

Learning outcomes for this week

After this week, you will be able to:

- Estimate the size of the plastic zone.
- Evaluate when it is adequate to use Linear Elastic Fracture Mechanics (LEFM).
- Determine the transition flaw size.
- Describe the J-integral and understand when to use it.

Outline

Plastic zone size
- Irwin approach
- Strip-yield model

Plane stress vs plane strain: the effect of sheet thickness

Transition flaw size

Elastic-Plastic Fracture Mechanics
- Crack tip opening displacement
- J-integral

Should you use K, G or J?

Outline

- Plastic zone size
  - Irwin approach
  - Strip-yield model
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- Should you use K, G or J?

Plastic zone size - First approximation

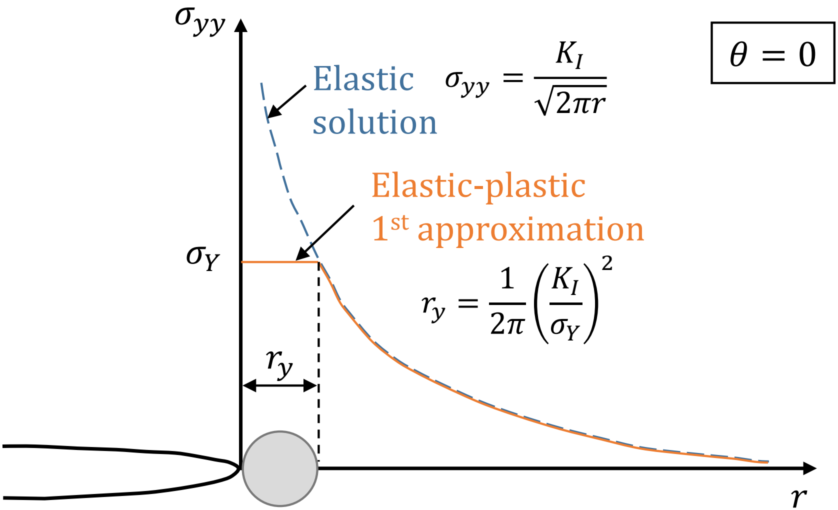
For a linear elastic material, the normal stress σ_{yy} on the crack plane ($\theta = 0$) is given by:

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}$$

which tends to infinity at $r = 0$. If the solid is **elastic perfectly plastic**, there will be a zone of plastic deformation close to the crack tip. For plane stress, the size of this plastic zone r_y can be estimated by setting $\sigma_{yy} = \sigma_Y$, which gives us:

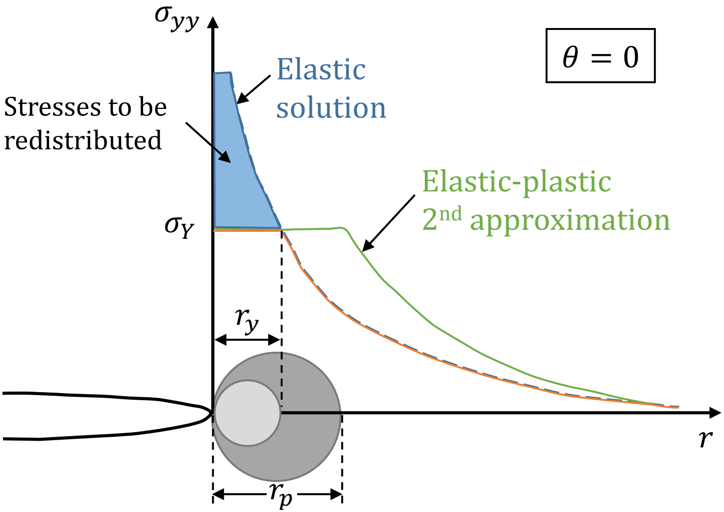
$$r_y = \frac{1}{2\pi}\left(\frac{K_I}{\sigma_Y}\right)^2$$

Plastic zone size - First approximation



Plastic zone size - Second approximation

There is a problem in the previous analysis; some stresses need to be redistributed to satisfy equilibrium.



Plastic zone size - Second approximation

Based on the previous figure, equilibrium is respected when:

$$\sigma_Y r_p = \int_0^{r_y} \sigma_{yy} dr = \int_0^{r_y} \frac{K_I}{\sqrt{2\pi r}} dr$$

after integrating, we get a better estimation of the plastic zone size r_p:

$$r_p = \frac{1}{\pi} \left(\frac{K_I}{\sigma_Y} \right)^2 = 2r_y$$

This result is two times larger than our first approximation.

This analysis was done for plane stress, does the results change for plane strain?

Yielding criterion

For a 3D stress state, we need a yielding criterion to predict the onset of plasticity. The von Mises criterion is:

$$\frac{1}{\sqrt{2}}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_1-\sigma_3)^2\right]^{1/2}=\sigma_Y$$

where $\sigma_1, \sigma_2, \sigma_3$ are the three principal stresses. The mode I stress field along the crack plane ($\theta = 0$) is:

$$\sigma_{yy}=\sigma_1=\frac{K_I}{\sqrt{2\pi r}} \qquad \sigma_{xx}=\sigma_2=\frac{K_I}{\sqrt{2\pi r}}=\sigma_{yy} \qquad \sigma_{xy}=0$$

$$\sigma_{zz}=\sigma_3=\begin{cases} 0 & \text{for plane stress} \\ \nu(\sigma_{xx}+\sigma_{yy})=2\nu\sigma_{yy} & \text{for plane strain} \end{cases}$$

Substituting in the von Mises criterion returns the yielding condition.

Yielding criterion - plastic zone size r_p

After substituting the stress field in the von Mises criterion, we obtain that yielding occurs when:

$$\sigma_{yy} = \sigma_Y \qquad \text{plane stress}$$

$$\sigma_{yy} = \frac{\sigma_Y}{1-2\nu} \approx 3\sigma_Y \qquad \text{plane strain}$$

where we assumed $\nu = 1/3$. This has an effect on the plastic zone size r_p:

$$r_p = \begin{cases} \frac{1}{\pi}\left(\frac{K_I}{\sigma_Y}\right)^2 & \text{plane stress} \\ \frac{1}{3\pi}\left(\frac{K_I}{\sigma_Y}\right)^2 & \text{plane strain} \end{cases}$$

Therefore, r_p is three times smaller in plane strain compared to plane stress.

Plastic zone shape

- Our previous estimate of the plastic zone size focused on the crack plane only, $\theta = 0$.
- This gives us a scalar r_y or r_p depending on the analysis.
- However, this does not give us the shape of the plastic zone: is it a cricle, an ellipse or something else?
- To find the plastic zone shape, we need to repeat the previous analysis for all values of θ to find the function $r_y(\theta)$.

Plastic zone shape

The first step is to compute the principal stresses using the Mohr's circle relationship:

$$\begin{aligned}
\sigma_1, \sigma_2 &= \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \left[\left(\frac{\sigma_{xx} - \sigma_{yy}}{2} \right)^2 + \sigma_{xy}^2 \right]^{1/2} \\
\sigma_3 &= 0 \qquad\qquad\qquad \text{for plane stress} \\
&= \nu(\sigma_1 + \sigma_2) \quad \text{for plane strain}
\end{aligned}$$

where $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$ are given in the datasheet.

Plastic zone shape

This gives us the following principal stresses for a mode I crack:

$$\begin{aligned}
\sigma_1 &= \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\right) \\
\sigma_2 &= \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\right) \\
\sigma_3 &= 0 && \text{for plane stress} \\
&= \frac{2\nu K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2} && \text{for plane strain}
\end{aligned}$$

The next step is to substitute them in the von Mises yielding criterion:

$$\frac{1}{\sqrt{2}}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_1-\sigma_3)^2\right]^{1/2}=\sigma_Y$$

Plastic zone shape

Expressing r as a function of K_I, σ_Y and θ gives us the shape of the plastic zone size:

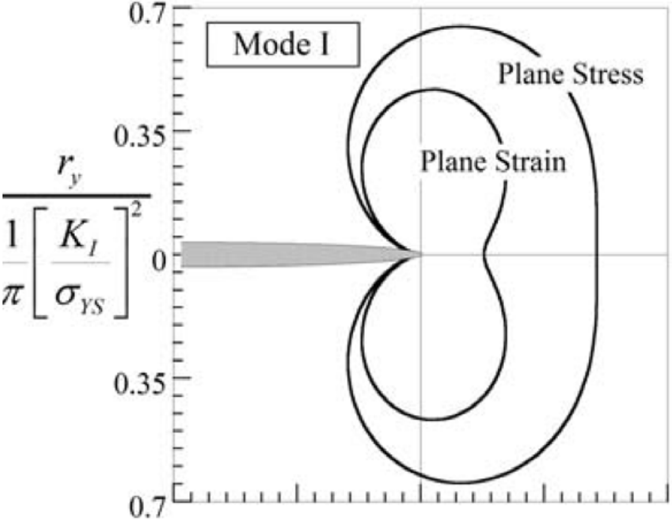
$$r_y(\theta) = \frac{1}{4\pi}\left(\frac{K_I}{\sigma_Y}\right)^2 \left[1 + \cos\theta + \frac{3}{2}\sin^2\theta\right]$$

for plane stress, and:

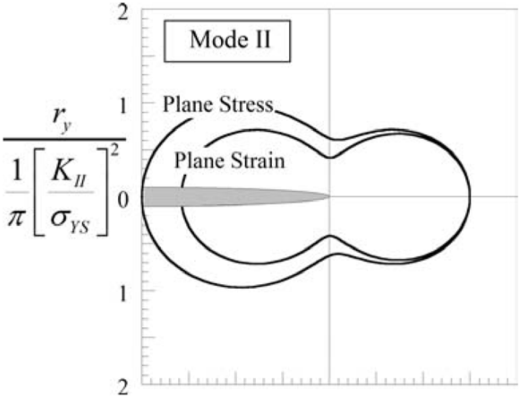
$$r_y(\theta) = \frac{1}{4\pi}\left(\frac{K_I}{\sigma_Y}\right)^2 \left[(1-2\nu)^2(1 + \cos\theta) + \frac{3}{2}\sin^2\theta\right]$$

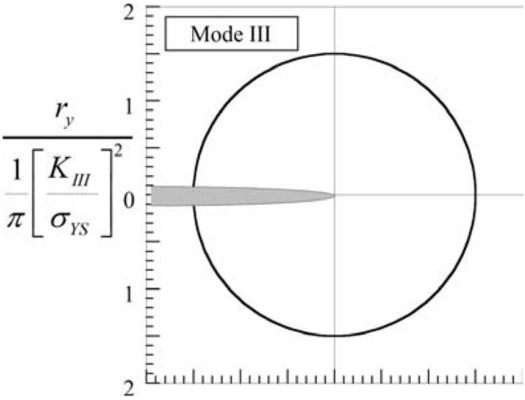
for plane strain. These equations are plotted next.

Plastic zone shape: mode I



Plastic zone shape: modes II and III





Outline

- Plastic zone size
  - Irwin approach
  - Strip-yield model
- Plane stress vs plane strain: the effect of sheet thickness
- Transition flaw size
- Elastic-Plastic Fracture Mechanics
  - Crack tip opening displacement
  - J-integral
- Should you use K, G or J?

Strip-yield model

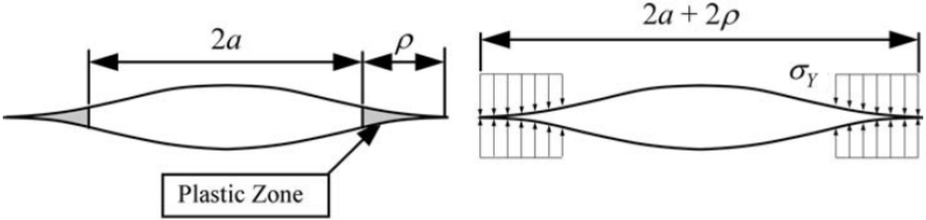
The stip-yield model, developed by Dugdale and Barenblatt independently, is an alternative way to estimate the plastic zone size.

Their model was developed for an infinitely large plate with a through crack under plane stress.

Their analysis is based on the principle of superposition: the model superimposes two elastic solutions to approximate an elastic-plastic behaviour.

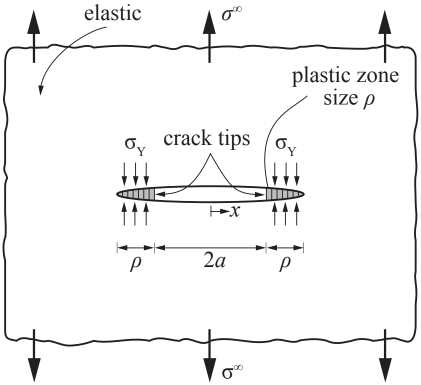
Strip-yield model

Consider a crack of length $2a$, where there is a plastic zone of size ρ at each crack tip.



The strip-yield model, replaces the physical crack of length $2a$ by a fictitious crack of length $2(a+\rho)$. At each crack tip, there is a closing stress σ_Y keeping a portion ρ of the crack closed.

Strip-yield model



Stresses are finite at the fictitious crack tip, which implies:

$$\text{at } x = a + \rho : \quad K_I^{(tot)} = K_I^{(\sigma^\infty)} + K_I^{(\sigma_Y)} = 0$$

Strip-yield model

For an infinitely large plate, $K_I^{(\sigma^\infty)}$ is given by:

$$K_I^{(\sigma^\infty)} = \sigma^\infty \sqrt{\pi(a+\rho)}$$

Otherwise, the contribution of the closure stress σ_Y is obtained by integrating the solution for a point force over a portion ρ. This gives:

$$K_I^{(\sigma_Y)} = -2\sigma_Y \sqrt{\frac{a+\rho}{\pi}} \arccos\left(\frac{a}{a+\rho}\right)$$

Next, we substituting these results in:

$$K_I^{(tot)} = K_I^{(\sigma^\infty)} + K_I^{(\sigma_Y)} = 0$$

Strip-yield model

to obtain:

$$\frac{a}{a+\rho} = \cos\left(\frac{\pi\sigma^{\infty}}{2\sigma_Y}\right)$$

The cosine function can be expressed as a Taylor series:

$$\frac{a}{a+\rho} = 1 - \frac{1}{2!}\left(\frac{\pi\sigma^{\infty}}{2\sigma_Y}\right)^2 + \frac{1}{4!}\left(\frac{\pi\sigma^{\infty}}{2\sigma_Y}\right)^4 - \frac{1}{6!}\left(\frac{\pi\sigma^{\infty}}{2\sigma_Y}\right)^6 + \dots$$

Keeping the first two terms only, and solving for the plastic zone size gives:

$$\rho = \frac{\pi}{8}\left(\frac{\sigma^{\infty}\sqrt{\pi a}}{\sigma_Y}\right)^2 = \frac{\pi}{8}\left(\frac{K_I}{\sigma_Y}\right)^2$$

Irwin approach vs strip-yield model

For plane stress, we now have two different estimates of the plastic zone size:

- Irwin approach:

$$r_p = \frac{1}{\pi}\left(\frac{K_I}{\sigma_Y}\right)^2 \approx 0.32\left(\frac{K_I}{\sigma_Y}\right)^2$$

- Strip-yield model:

$$\rho = \frac{\pi}{8}\left(\frac{K_I}{\sigma_Y}\right)^2 \approx 0.39\left(\frac{K_I}{\sigma_Y}\right)^2$$

Both approaches give similar results (20% difference.)

Validity of LEFM

So far, we have used LEFM to make different estimates of the plastic zone size, why?

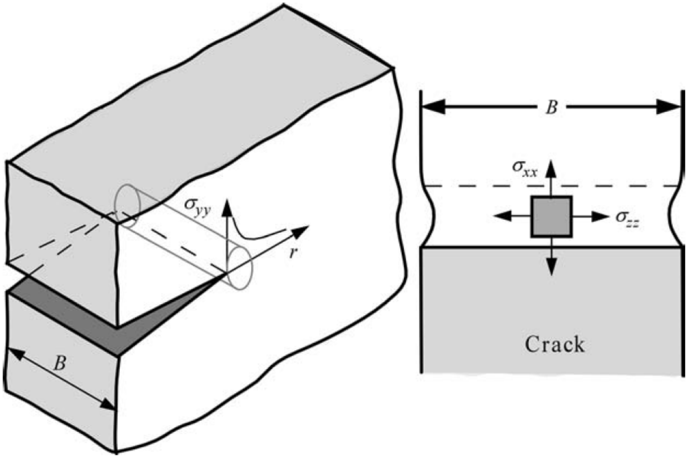
If the plastic zone size is small then we conclude that LEFM applies and fracture will occur when $K_I = K_{Ic}$.

What is a small plastic zone size? If r_p is roughly an order of magnitude smaller than the crack length, that is small enough. We will define a more precise criterion when discussing testing methods.

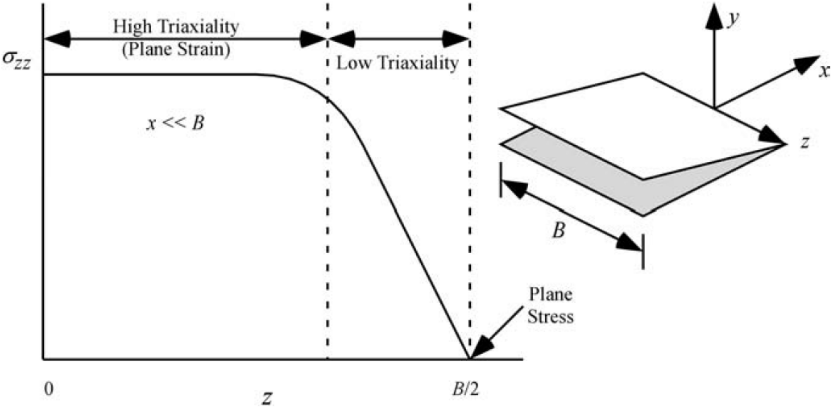
Outline

Plane stress vs plane strain

The high normal stress σ_{yy} at the crack tip causes material near the surface to contract. However, material inside the specimen is constrained, resulting in a triaxial stress state.



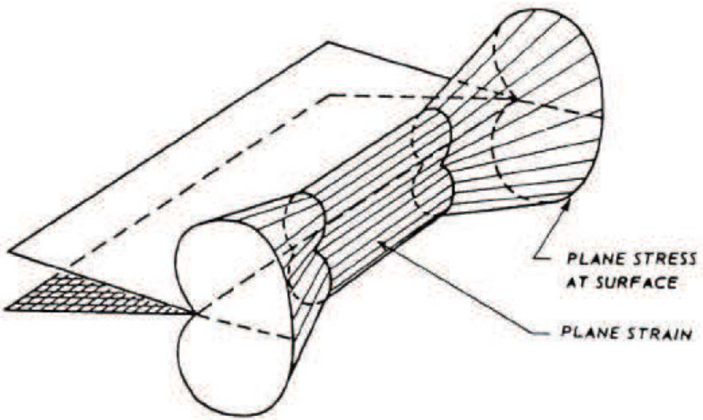
Triaxial stress state



The triaxial stress state leads to:

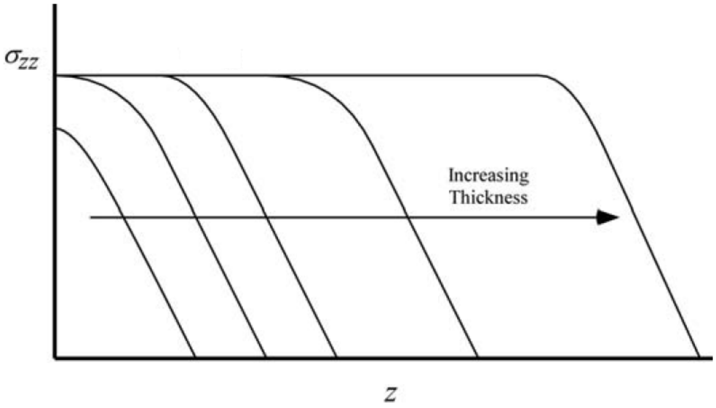
- plane strain conditions inside the specimen,
- plane stress on the surface.

Plastic zone size through the thickness



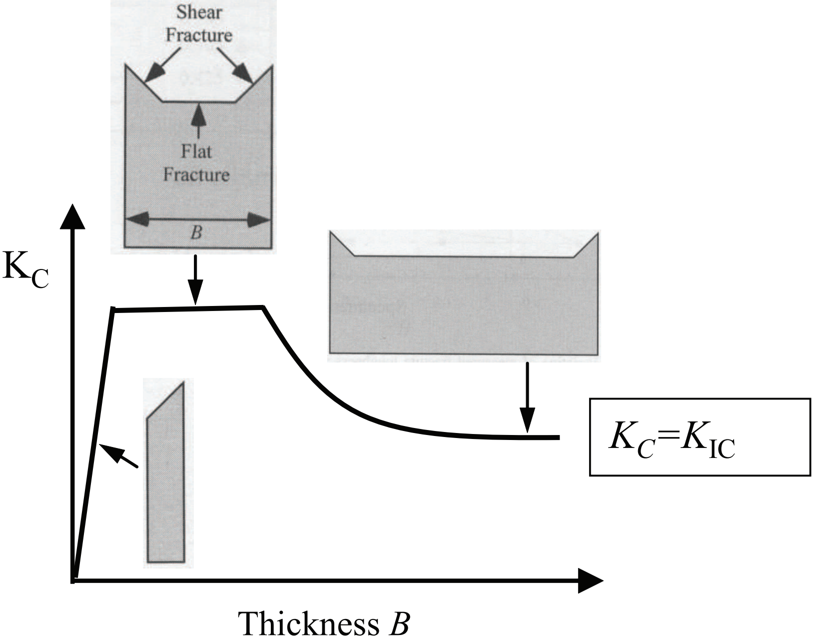
The change in triaxiality means that the plastic zone size varies through the thickness.

Triaxiality vs sheet thickness



- The sheet thickness has a strong effect on the state of triaxiality and the magnitude of σ_{zz}.
- Consequently, the sheet thickness has an effect on the measured fracture toughness.

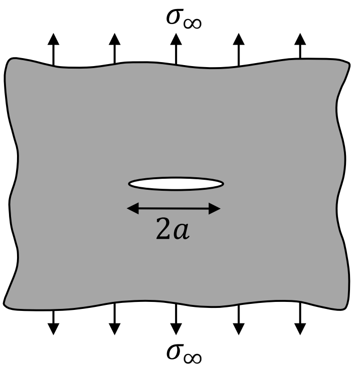
Fracture toughness vs sheet thickness



Outline

Plasticity vs Fracture

Consider that a plate made from an elastic-plastic material with a yield strength σ_Y and a fracture toughness K_{Ic}.



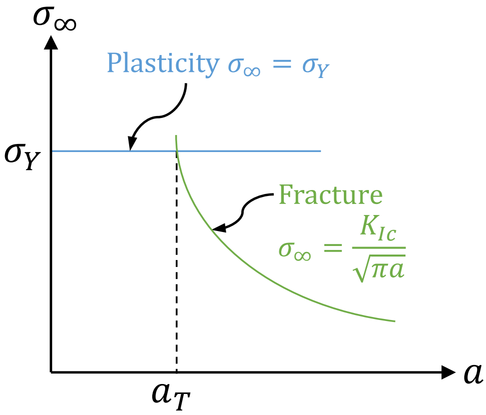
For a long crack, we would expect fracture to occur when:

$$\sigma_\infty = \frac{K_{Ic}}{\sqrt{\pi a}}$$

For short cracks, plasticity may occur before fracture:

$$\sigma_\infty = \sigma_Y$$

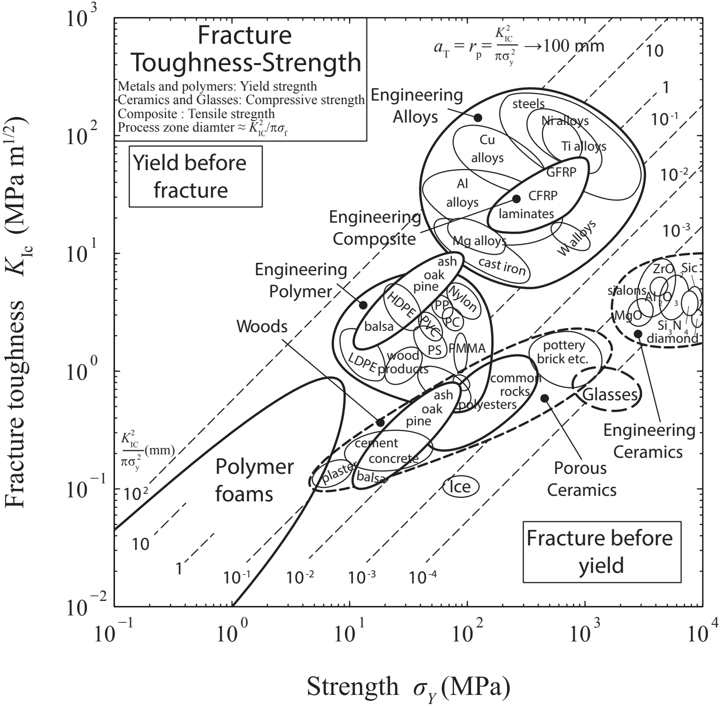
Plasticity vs Fracture



The transition flaw size a_T is given by:

$$a_T = \frac{1}{\pi}\left(\frac{K_{Ic}}{\sigma_Y}\right)^2$$

If this looks familiar it is because $a_T = r_p$, where r_p is the plastic zone size in plane stress, see slide 11.


Fracture Toughness-Strength
Metals and polymers: Yield stregnth
Ceramics and Glasses: Compressive strength
Composite : Tensile stregnth
Process zone diamter ≈ $K_{IC}^2/\pi\sigma_f$
$a_T = r_p = \frac{K_{IC}^2}{\pi\sigma_y^2} \rightarrow 100$ mm
Yield before fracture
Fracture before yield
Engineering Alloys
steels
Ni alloys
Ti alloys
Cu alloys
GFRP
Al alloys
CFRP
laminates
Engineering Composite
Mg alloys
W alloys
cast iron
Engineering Polymer
ash
oak
pine
HDPE
PP
Nylon
PVC
PC
balsa
PS
PMMA
LDPE
wood products
Woods
polyesters
common rocks
pottery brick etc.
Glasses
cement concrete
plaster
Ice
Polymer foams
Porous Ceramics
Engineering Ceramics
ZrO2
Sic
sialons
Al2O3
MgO
Si3N4
diamond
$\frac{K_{IC}^2}{\pi\sigma_y^2}$(mm)
Fracture toughness K_{Ic} (MPa m$^{1/2}$)
Strength σ_Y (MPa)

Transition flaw size

In conclusion:

- If a structure contains a crack of length $a \ll a_T$, it will fail by plastic collapse, $\sigma_\infty = \sigma_Y$.
- If a structure contains a crack of length $a \gg a_T$, it will fail by brittle fracture, $\sigma_\infty = K_{Ic}/\sqrt{\pi a}$.

Outline

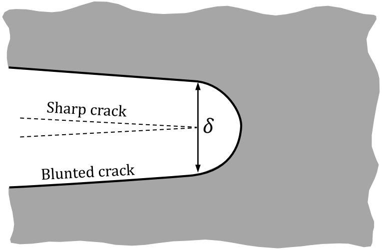
Elastic Plastic Fracture Mechanics

- When researchers tried to measure the fracture toughness of metals, they found that they were too tough to be characterised by LEFM.
- This lead to the development of Elastic Plastic Fracture Mechanics, where the material has a non-linear behaviour.
- We will introduce two new parameters: the crack tip opening displacement and the J-integral. Both parameters decribe the conditions at the crack tip of an elastic plastic material, and both can be used as a fracture criterion.

Outline

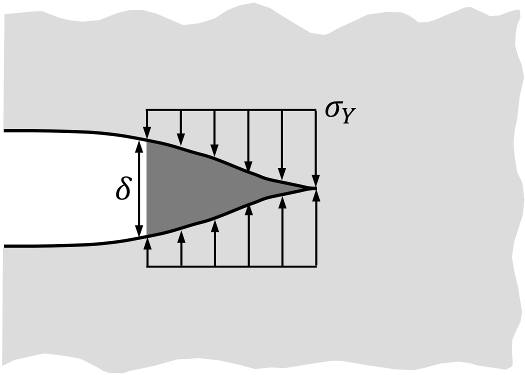
Crack tip opening displacement

Wells (1961) observed that in metals the crack faces moved apart significantly prior to fracture; plasticity was blunting an initially sharp crack.



The crack tip opening displacement (CTOD) is represented by δ on the figure. Tougher materials have a higher CTOD.

Estimating the CTOD using the strip-yield model



The CTOD can be estimated as the displacement at the end of the strip-yield zone. For a large plate with a through crack this definition gives us:

$$\delta = \frac{8\sigma_Y a}{\pi E} \ln \sec\left(\frac{\pi\sigma_\infty}{2\sigma_Y}\right)$$

Estimating the CTOD using the strip-yield model

In the previous equation, we can expand the $\ln\sec$ term as a Taylor series, which gives:

$$\begin{aligned}\delta &= \frac{8\sigma_Y a}{\pi E}\left[\frac{1}{2}\left(\frac{\pi\sigma_\infty}{2\sigma_Y}\right)^2 + \frac{1}{12}\left(\frac{\pi\sigma_\infty}{2\sigma_Y}\right)^4 + \dots\right] \\ &= \frac{K_I^2}{\sigma_Y E}\left[1 + \frac{1}{6}\left(\frac{\pi\sigma_\infty}{2\sigma_Y}\right)^2 + \dots\right]\end{aligned}$$

As $\sigma_\infty/\sigma_Y \to 0$, this becomes:

$$\delta = \frac{K_I^2}{\sigma_Y E} = \frac{G}{\sigma_Y}$$

Estimating the CTOD using the strip-yield model

The previous analysis assumed plane stress conditions and a perfectly-plastic material.

The relation can be expressed in a more general form as:

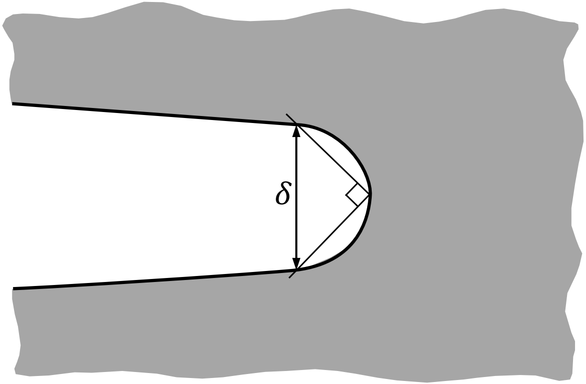
$$\delta = \frac{K_I^2}{m\sigma_Y E'} = \frac{G}{m\sigma_Y}$$

where,

$$\begin{aligned} &\text{For plane stress:} \quad m = 1 \quad \text{and} \quad E' = E \\ &\text{For plane strain:} \quad m = 2 \quad \text{and} \quad E' = E/(1-\nu^2) \end{aligned}$$

Alternative definitions of the CTOD

There are other definitions of the CTOD. One commonly used in Finite Element simulations is shown below.



The important thing to remember is that the CTOD is related to K_I and G; therefore, it can be used to predict fracture.

Outline

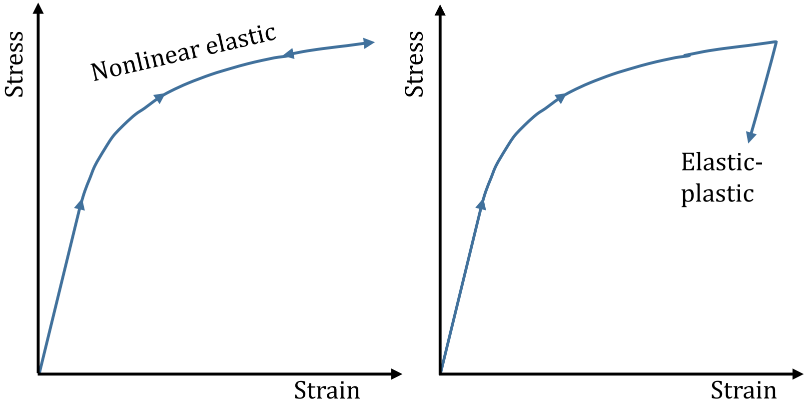
The advantages of the J-integral

The J-integral is the most commonly used parameter to predict fracture in elastic-plastic materials. It is a versatile parameter since:

- its definition is similar to that of the energy release rate G; the J-integral also has units of $\mathrm{J/m^2}$.
- the J-integral characterises the stress field at the crack tip of an elastic-plastic material. In this sense, it is similar to the stress intensity factor K for elastic solids.
- it can be expressed as a path-independent line integral, which makes it easy to implement in Finite Element codes.

An important note on the material behaviour

- In LEFM, the material is assumed to be linear elastic.
- In EPFM and all the work done on J-integral, the material is assumed to be nonlinear elastic. This is different from the elastic-plastic behaviour of metals.

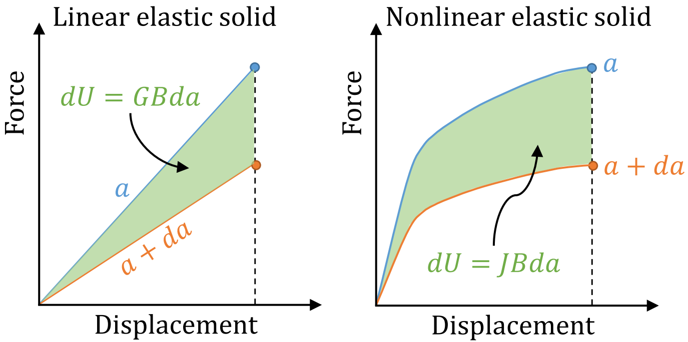


J as an energy release rate

The J-integral is defined in the same way as G:

$$J = -\frac{d\Pi}{dA} = -\frac{d\Pi}{Bda} \qquad \text{where} \qquad \Pi = U - W$$

This definition can be visualised as follows:



When the material is linear elastic, we find that $J = G$.

J as a stress intensity parameter

Hutchinson (1968) and Rice and Rosengren (1968) independently showed that J uniquely characterises the stress field at the crack tip.

They both assumed that the material follows a Ramberg-Osgood relationship:

$$\epsilon = \frac{\sigma}{E} + K\left(\frac{\sigma}{E}\right)^n$$

where

E	is the Young's modulus
K	is a dimensionless constant
n	is the strain hardening exponent

J as a stress intensity parameter

Assuming:

$$\epsilon = \frac{\sigma}{E} + K\left(\frac{\sigma}{E}\right)^n$$

Hutchinson, Rice and Rosengren showed that:

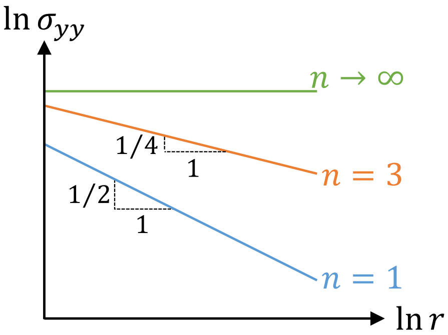
$$\sigma_{ij} \propto \left(\frac{J}{r}\right)^{\frac{1}{n+1}} \qquad \text{and} \qquad \epsilon_{ij} \propto \left(\frac{J}{r}\right)^{\frac{n}{n+1}}$$

- For a linear elastic material ($n = 1$) we recover the $1/\sqrt{r}$ relation.
- Otherwise, for a perfectly plastic solid ($n \to \infty$) we find that σ_{ij} is independent of r.

J as a stress intensity parameter

The stresses scale as:

$$\sigma_{ij} \propto \left(\frac{J}{r}\right)^{\frac{1}{n+1}}$$

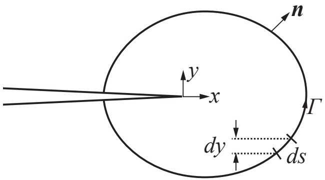


The zone where the stresses scale according to this relation is called the HRR field (for Hutchinson, Rice and Rosengren).

J as a contour integral

Rice (1968) showed that J can be computed as a contour integral:

$$J = \int_{\Gamma} \left(w dy - t_i \frac{\partial u_i}{\partial x} ds \right)$$

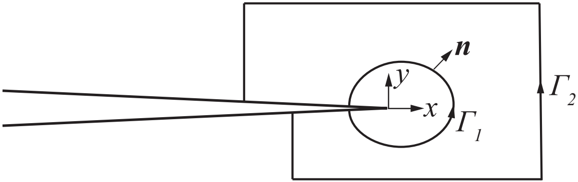


where

$w = \int_0^{\epsilon_{ij}} \sigma_{ij} d\epsilon_{ij}$	is the strain energy density
$t_i = \sigma_{ij} n_j$	is the traction vector
n_j	is the vector normal to the contour Γ
u_i	is the displacement vector
ds	is the length increment along the contour Γ

J as a contour integral

- The contour has to go from one crack surface to the other.
- The J-integral is path independent: any contour will give the same value of J

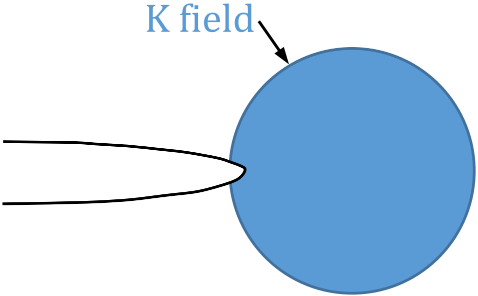


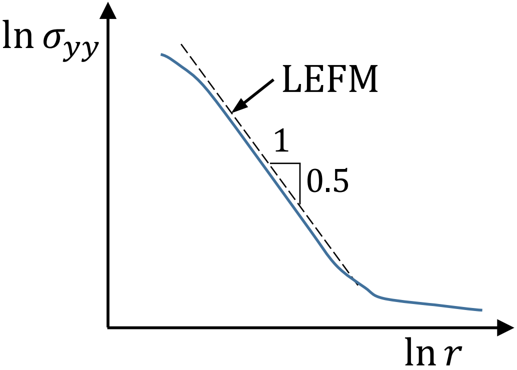
This means that: $J_{\Gamma_1} = J_{\Gamma_2}$.

Outline

Elastic regime

If the material is elastic, like glass, or if the stresses are within the elastic range, there is a K field surrounding the crack tip, where $\sigma_{ij} \propto 1/\sqrt{r}$.

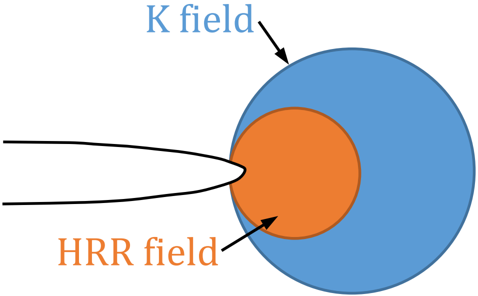


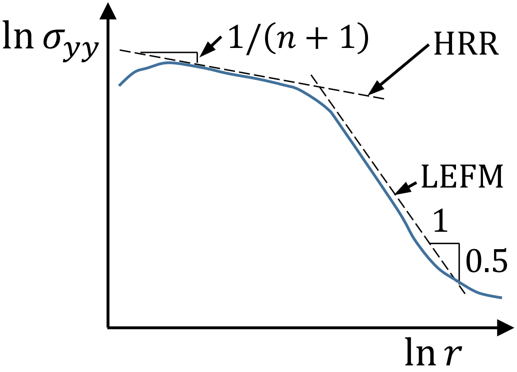


Fracture when $K_I = K_{Ic}$ and $J = G = G_{Ic}$.

Small scale yielding

As the loading is increased, there is a HRR field and a plastic zone surrounded by a larger K field.

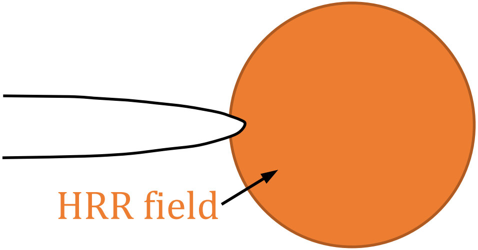


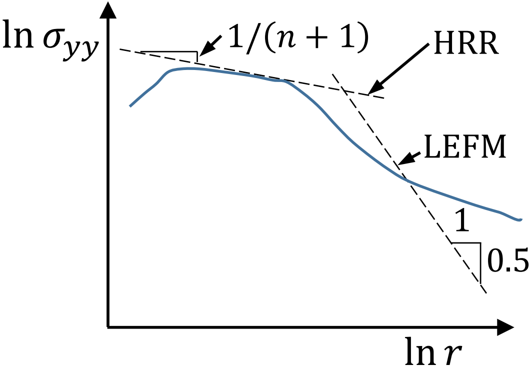


Here, LEFM still applies and fracture occurs when $K_I = K_{Ic}$ and $J = G = G_{Ic}$.

Large scale yielding

When the plastic zone becomes large compared to the dimensions of the structure, we have a HRR field but no more K field.





Here, EPFM is necessary and fracture occurs when $J_I = J_{Ic}$ (do not use K).

Summary

- We can estimate the size of the plastic zone r_p using Irwin's approach.
- Linear Elastic Fracture Mechanics (LEFM) is applicable if the plastic zone size is small ($r_p < a/10$).
- If the plastic zone size is large, it is necessary to use Elastic Plastic Fracture Mechanics (usually an approach based on the J-integral).