Fracture mechanics

Seminar 3: Mixed-mode loading

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Learning outcomes

After this week, you will be able to:

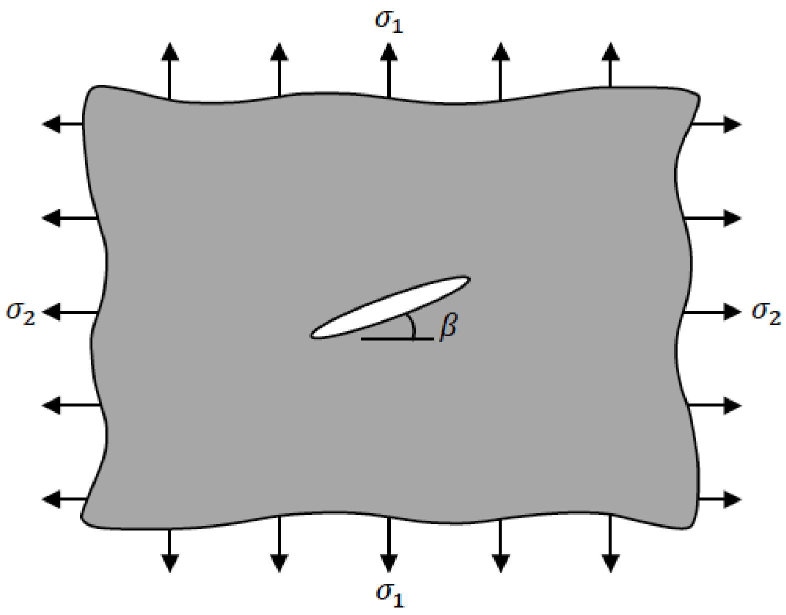
- Calculate the stress intensity factors K_I and K_{II} for mixed-mode loading.
- Predict when fracture will occur.
- Find the direction of crack propagation.

Mixed mode loading

Cracks are often loaded in a combination of mode I and II.

This raises three questions:

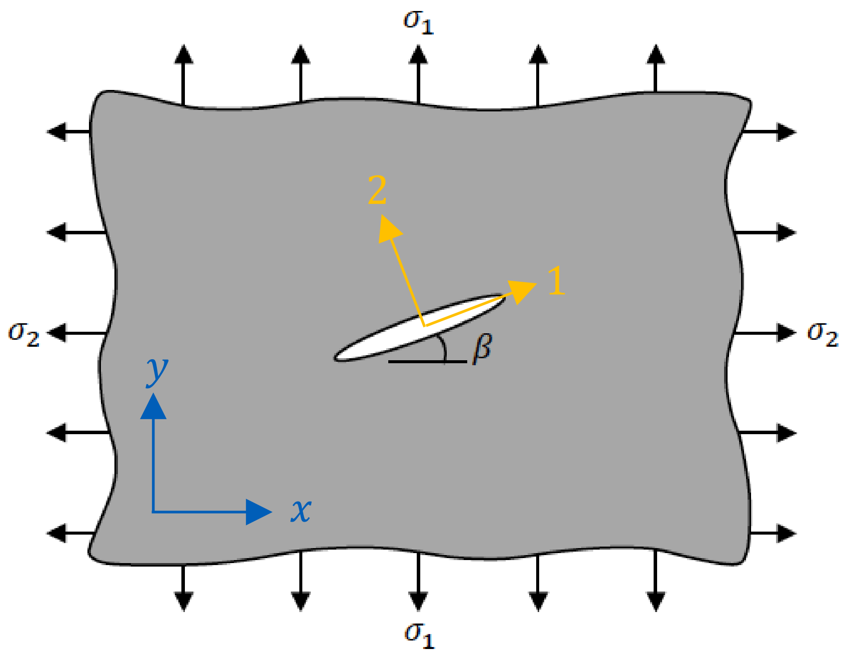
1. How can we compute K_I and K_{II}?
2. When will the crack propagate?
3. In which direction will the crack grow?



Computing K_I and K_{II}

Under mixed-mode loading, we can use Mohr’s circle to express the stress components in a reference frame aligned with the crack plane.

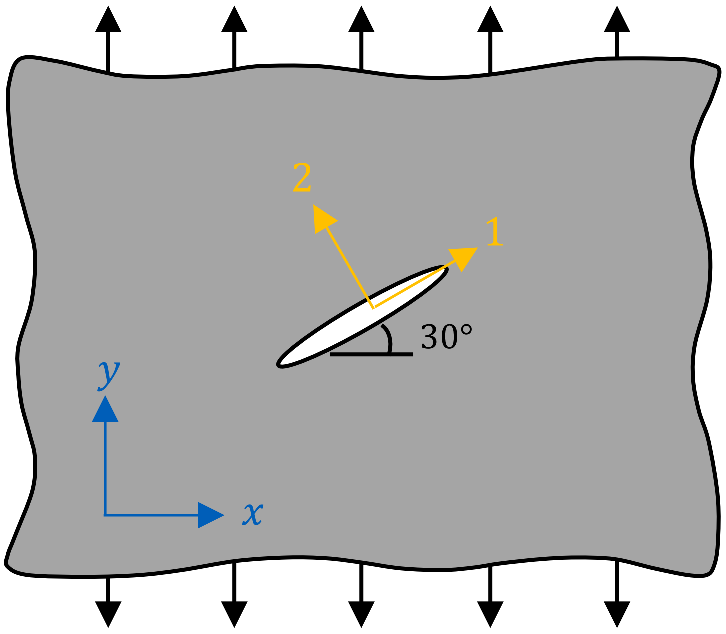
Let’s review the Mohr’s circle.



Mohr's circle in three steps

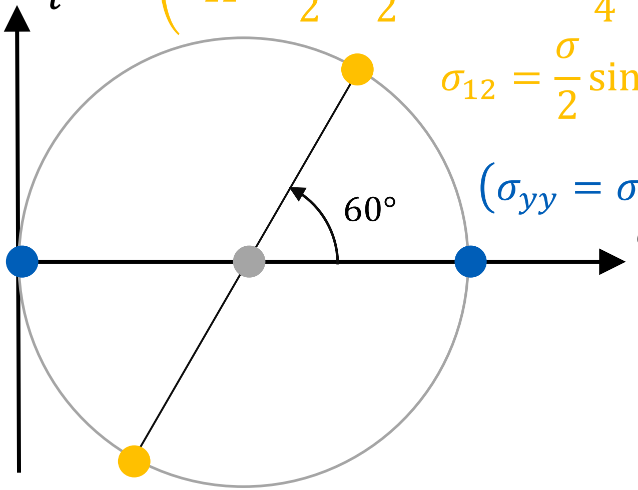
1. Write the stress components in the global reference frame.
    - Three components: $\sigma_{xx}; \sigma_{yy}; \sigma_{xy}$.
    - Be careful with signs: tension/compression.
2. Draw the circle using two points on σ vs τ axes:
    - First point: $\left(\sigma_{xx}; -\sigma_{xy}\right)$ and second point: $\left(\sigma_{yy}; \sigma_{xy}\right)$.
3. Rotate clockwise by 2θ to find $\sigma_{11}; \sigma_{22}; \sigma_{12}$.

Mohr's circle: example



$\sigma_{xx} = 0 \qquad \sigma_{yy} = \sigma \qquad \sigma_{xy} = 0$

$\left(\sigma_{22} = \frac{\sigma}{2} + \frac{\sigma}{2}\cos 60° = \frac{3}{4}\sigma;\ \sigma_{12} = \frac{\sigma}{2}\sin 60° = \frac{\sqrt{3}}{4}\sigma\right)$



$$K_I = \sigma_{22}\sqrt{\pi a} = \frac{3}{4}\sigma\sqrt{\pi a} \qquad K_{II} = \sigma_{12}\sqrt{\pi a} = \frac{\sqrt{3}}{4}\sigma\sqrt{\pi a}$$

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How to predict fracture

NO: you cannot add stress intensity factors for different modes:

$$K^{[\text{total}]} \neq K_I^{[\text{A}]} + K_{II}^{[\text{B}]} + K_{III}^{[\text{C}]} + \cdots$$

YES: you can add the energy release rate of different modes:

$$G = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'}$$

Where $E' = E$ in plane stress and $E' = E/(1-\nu^2)$. A simple fracture criterion is to set $G = G_c = K_{Ic}^2/E'$.

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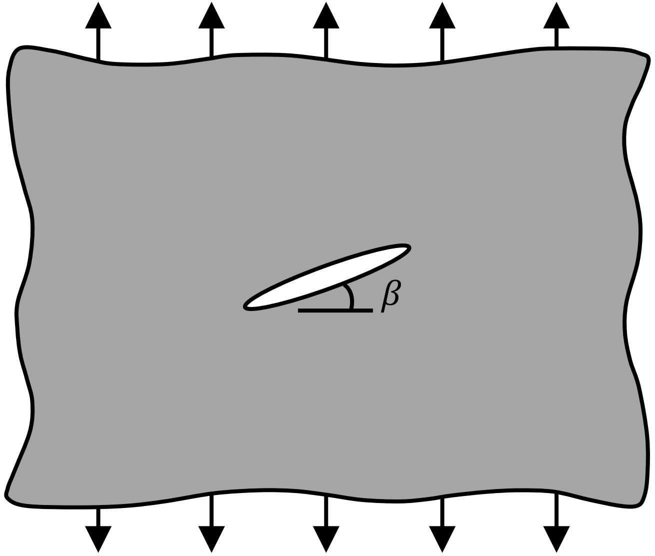
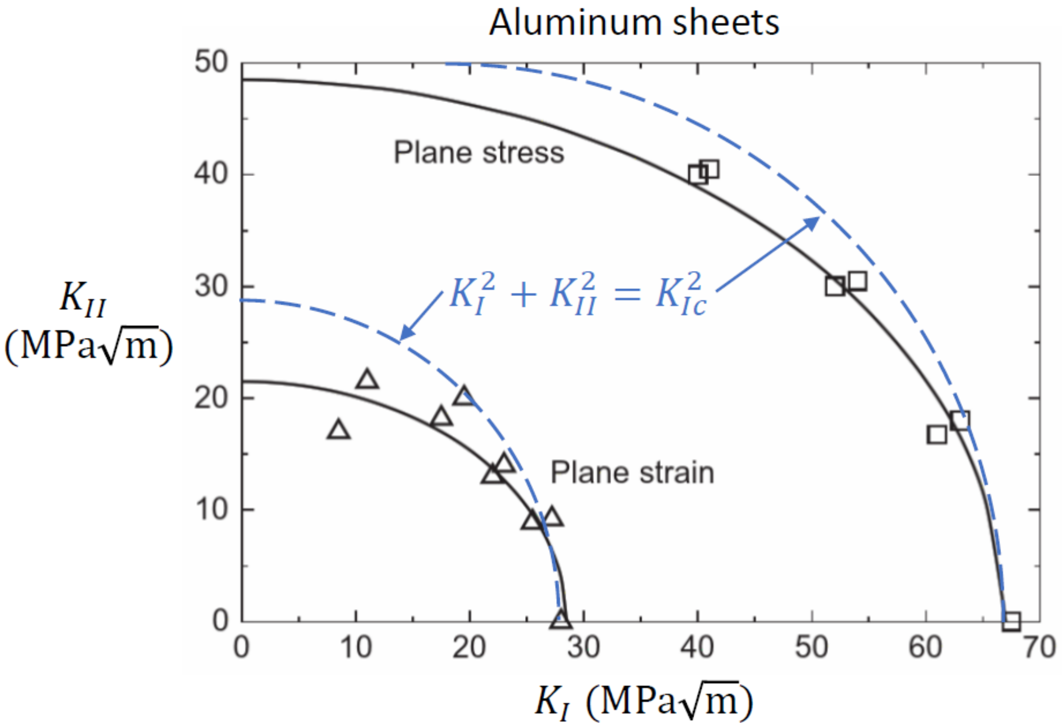
Fracture envelope

Multiple experiments are needed to form a fracture envelope.

Vary β to change K_I/K_{II}





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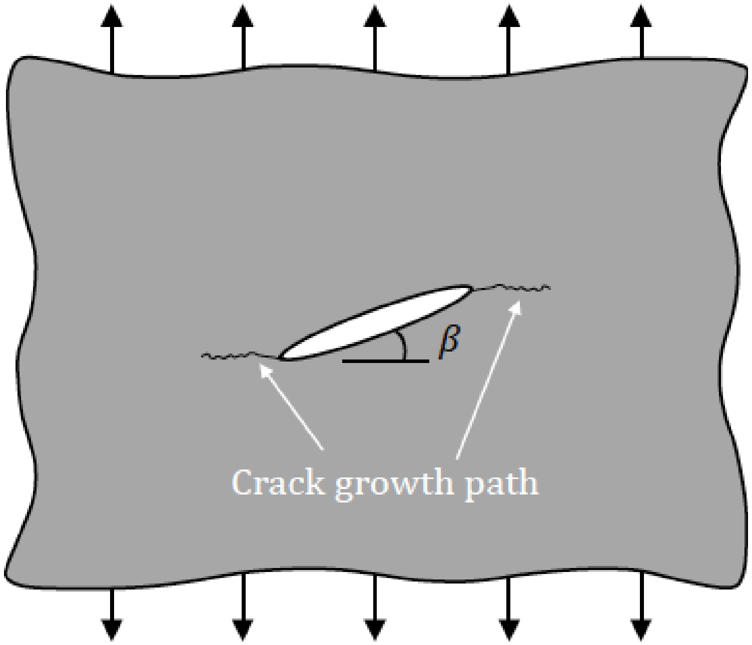
Sun et al. (2012) *Fracture mechanics.*

Angle of crack propagation

Cracks generally grow in the local mode I direction. This is where:

- $\sigma_{\theta\theta}$ is maximum,
- which is the same direction where $\sigma_{r\theta} = 0$.



Stress field under mixed-mode loading

Add stress components for modes I and II to get the total stress field close to the crack tip:

$$\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{5}{4}\cos\frac{\theta}{2} - \frac{1}{4}\cos\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\left(-\frac{5}{4}\sin\frac{\theta}{2} + \frac{3}{4}\sin\frac{3\theta}{2}\right)$$

$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2}\right) - \frac{K_{II}}{\sqrt{2\pi r}}\left(\frac{3}{4}\sin\frac{\theta}{2} + \frac{3}{4}\sin\frac{3\theta}{2}\right)$$

$$\sigma_{r\theta} = \underbrace{\frac{K_I}{\sqrt{2\pi r}}\left(\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}\right)}_{\sigma_{ij}\text{ for mode I (from datasheet)}} + \underbrace{\frac{K_{II}}{\sqrt{2\pi r}}\left(\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}\right)}_{\sigma_{ij}\text{ for mode II (from datasheet)}}$$

Angle of crack propagation

First, set $\sigma_{r\theta} = 0$, which gives:

$$0 = K_I\left(\sin\frac{\theta}{2} + \sin\frac{3\theta}{2}\right) + K_{II}\left(\cos\frac{\theta}{2} + 3\cos\frac{3\theta}{2}\right)$$

And solve for θ. You should find multiple values for θ.

Second, the crack will propagate in the direction θ corresponding to the maximum $\sigma_{\theta\theta}$.

$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2}\right) - \frac{K_{II}}{\sqrt{2\pi r}}\left(\frac{3}{4}\sin\frac{\theta}{2} + \frac{3}{4}\sin\frac{3\theta}{2}\right)$$

In summary

Under mixed-mode loading:

- find the stress intensity factors K_I and K_{II} using Mohr's circle,
- use the energy release rate G to predict when fracture will occur,
- the crack will propagate in the direction of $\max(\sigma_{\theta\theta})$, which corresponds to $\sigma_{r\theta} = 0$.

This concludes the part on Linear Elastic Fracture Mechanics (LEFM). Next week, we will introduce plasticity in our analysis of fracture.