Fracture Mechanics

3. Mixed-mode loading

Luc St-Pierre

April 18, 2023

Learning outcomes for this week

After this week, you will be able to:

- Calculate the stress intensity factors K_I and K_{II} for mixed-mode loading.
- Predict when fracture will occur.
- Find the direction of crack propagation.

Outline

Evaluating K_I and K_{II}

When will fracture occur?

In which direction will the crack propagate?

Outline

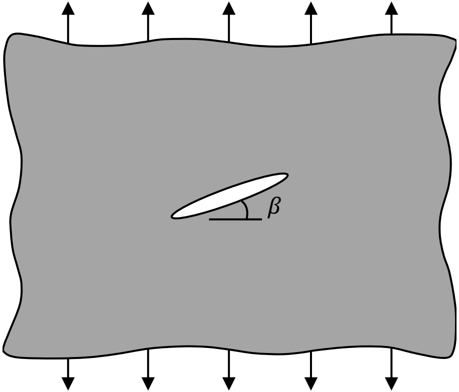
Evaluating K_I and K_{II}

When will fracture occur?

In which direction will the crack propagate?

Mixed-mode loading

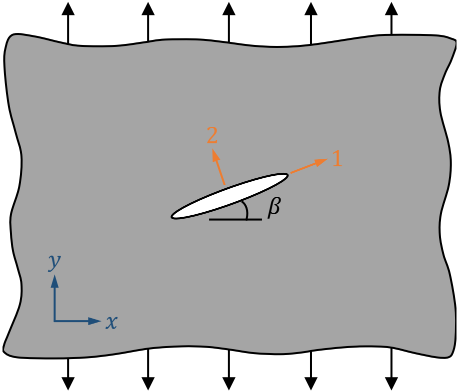
Often, cracks are at a certain angle from the loading direction. This means that the crack is loaded in modes I and II simultaneously.



To find K_I and K_{II}, we need to use Mohr's circle to express the stress components in a reference frame aligned with the crack plane.

Mohr's circle: procedure

Follow these three steps to create and use Mohr's circle:

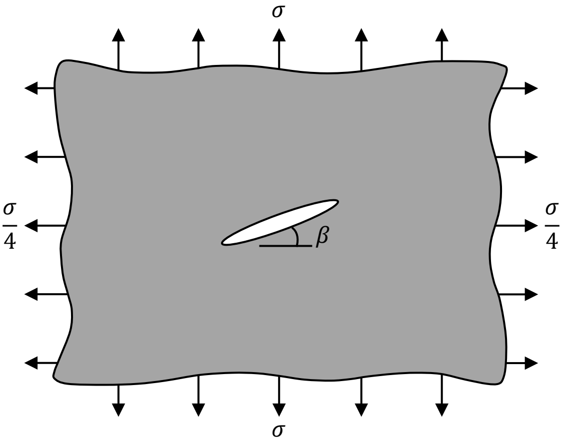


1. Write the stress components in the global reference frame (x, y).
   - Three components: σ_{xx}, σ_{yy}, and σ_{xy}.
   - Be careful with signs: tension/compression.
2. Draw the circle using two points on σ vs τ axes.
   - First point: $(\sigma_{xx}; -\sigma_{xy})$.
   - Second point: $(\sigma_{yy}; \sigma_{xy})$.
3. Rotate clockwise by 2β to find the local stress components: σ_{11}, σ_{22}, and σ_{12}.

Finally, we can calculate K_I with σ_{22}, and K_{II} using σ_{12}.

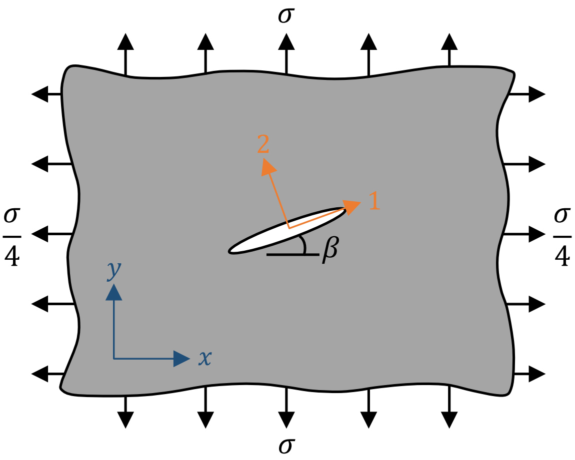
Example problem

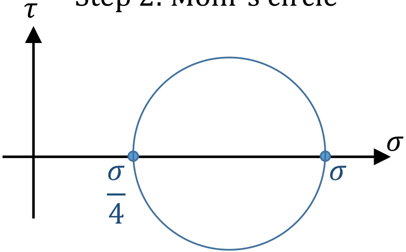
Find the stress intensity factors K_I and K_{II} for the biaxial loading case shown below, where the total crack length is $2a$. Express your results as functions of σ, β, and a.



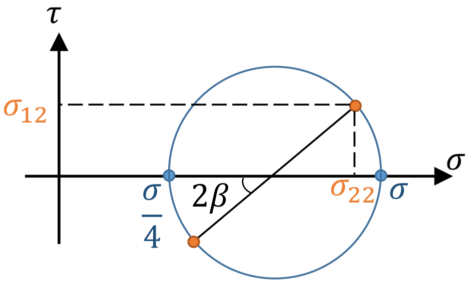
Example problem: solution

Step 1: the stress field in the (x, y) frame is $\sigma_{xx} = \sigma/4$, $\sigma_{yy} = \sigma$, and $\sigma_{xy} = 0$. Then, we can plot Mohr's circle with two points: $(\sigma_{xx}; -\sigma_{xy}) = (\sigma/4; 0)$ and $(\sigma_{yy}; \sigma_{xy}) = (\sigma, 0)$.





Example problem: solution



This circle has a radius:

$$r = \frac{\sigma - \sigma/4}{2} = \frac{3\sigma}{8}$$

and a centre:

$$c = \frac{\sigma}{4} + \frac{3\sigma}{8} = \frac{5\sigma}{8}$$

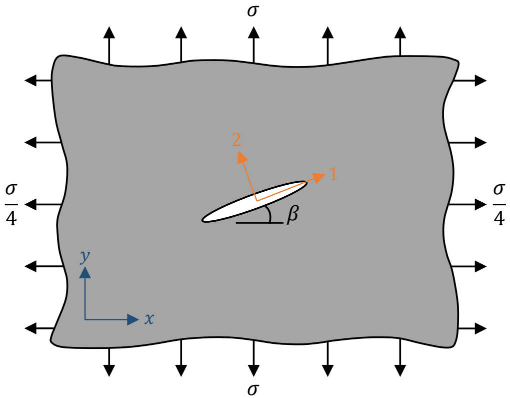
Then, the stress field in the (1,2) frame is:

$$\sigma_{11} = c - r\cos 2\beta = \frac{\sigma}{8}(5 - 3\cos 2\beta) \qquad \sigma_{22} = c + r\cos 2\beta = \frac{\sigma}{8}(5 + 3\cos 2\beta)$$

and

$$\sigma_{12} = r\sin 2\beta = \frac{3\sigma}{8}\sin 2\beta$$

Example problem: solution



We found:

$$\sigma_{11} = \frac{\sigma}{8}(5 - 3\cos 2\beta)$$
$$\sigma_{22} = \frac{\sigma}{8}(5 + 3\cos 2\beta)$$
$$\sigma_{12} = \frac{3\sigma}{8}\sin 2\beta$$

Then, the stress intensity factors K_I and K_{II} are given by:

$$K_I = \sigma_{22}\sqrt{\pi a} = \frac{\sigma}{8}(5 + 3\cos 2\beta)\sqrt{\pi a}$$
$$K_{II} = \sigma_{12}\sqrt{\pi a} = \frac{3\sigma}{8}\sin 2\beta\sqrt{\pi a}$$

Outline

Evaluating K_I and K_{II}

When will fracture occur?

In which direction will the crack propagate?

Mixed-mode fracture

As mentioned earlier, the stress intensity factor is **not** additive:

$$K_{(\text{total})} \neq K_I + K_{II} + K_{III}$$

In contrast, the energy release rate G is the sum of each mode:

$$G = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{(1+\nu)K_{III}^2}{E}$$

where $E' = E$ for plane stress, and $E' = E/(1-\nu^2)$ for plane strain. A simple fracture criterion is obtained by setting G equal to the material's toughness:

$$G_{Ic} = \frac{K_{Ic}^2}{E'}$$

Mixed-mode fracture: a simple criterion

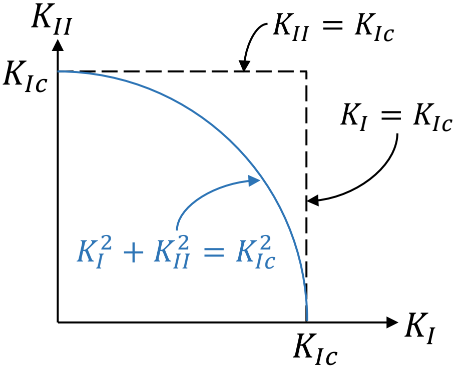
From previous slide, setting $G = G_{Ic}$ gives:

$$\frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{(1+\nu)K_{III}^2}{E} = \frac{K_{Ic}^2}{E'}$$

If $K_{III} = 0$, this simplifies to:

$$K_I^2 + K_{II}^2 = K_{Ic}^2$$

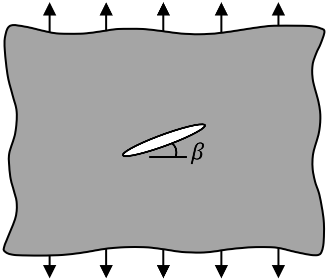
which gives a circular fracture envelope.



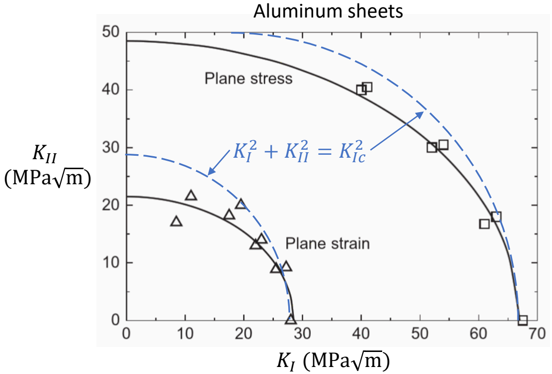
Is this equation validated experimentally?

Mixed-mode fracture: an experimental approach

A fracture envelope can be obtained experimentally. Multiple tests are needed where β is varied to change K_I/K_{II}.



Sun et al. (2012) *Fracture mechanics.*



As shown here, the simple criterion $K_I^2 + K_{II}^2 = K_{Ic}^2$ is relatively close to experiments.

Outline

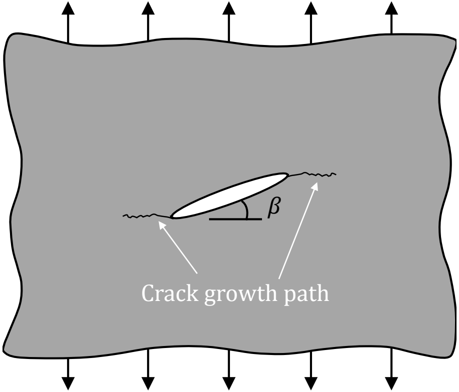
Evaluating K_I and K_{II}

When will fracture occur?

In which direction will the crack propagate?

Direction of crack propagation

For mixed-mode loading, experiments have shown that cracks grow in the direction of local mode I.



The local mode I direction is where $\sigma_{\theta\theta}$ is maximum, which is the same as the direction where $\sigma_{r\theta} = 0$.

Stress field for mixed-mode loading

The stress field close to the crack tip is obtained by adding the modes I and II contributions **for each stress component**, which gives:

$$\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{5}{4}\cos\frac{\theta}{2} - \frac{1}{4}\cos\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\left(-\frac{5}{4}\sin\frac{\theta}{2} + \frac{3}{4}\sin\frac{3\theta}{2}\right)$$

$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2}\right) - \frac{K_{II}}{\sqrt{2\pi r}}\left(\frac{3}{4}\sin\frac{\theta}{2} + \frac{3}{4}\sin\frac{3\theta}{2}\right)$$

$$\sigma_{r\theta} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\left(\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}\right)$$

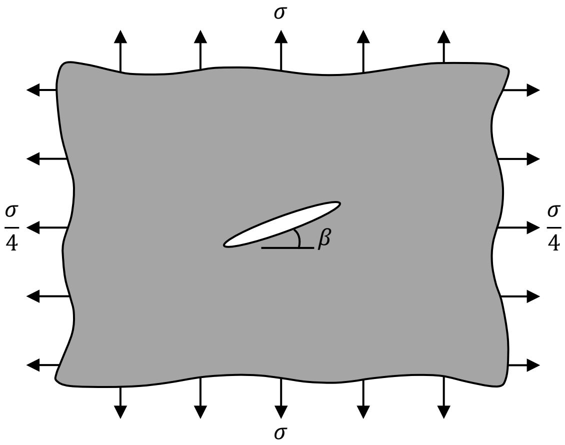
The direction of crack propagation is given by the angle θ for which $\sigma_{r\theta} = 0$. This gives the following equation:

$$0 = K_I\left(\sin\frac{\theta}{2} + \sin\frac{3\theta}{2}\right) + K_{II}\left(\cos\frac{\theta}{2} + 3\cos\frac{3\theta}{2}\right)$$

If you know K_I and K_{II}, you can solve for θ.

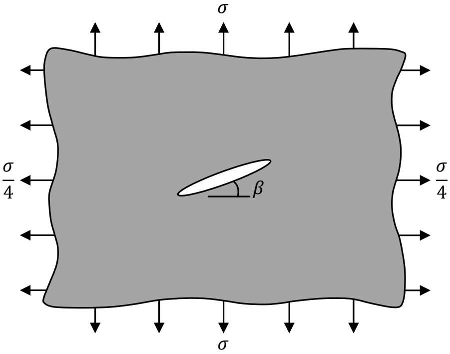
Example problem

Find the angle of crack propagation for the biaxial loading case shown below, provided that $\sigma = 10\,\mathrm{MPa}$ and $\beta = 20°$.



Example problem: solution

Find the angle of crack propagation for the biaxial loading case shown below, provided that $\sigma = 10\,\mathrm{MPa}$ and $\beta = 20°$.



On Slide 10, we found:

$$K_I = \frac{\sigma}{8}(5 + 3\cos 2\beta)\sqrt{\pi a} \qquad = 9.12\sqrt{\pi a}$$

$$K_{II} = \frac{3\sigma}{8}\sin 2\beta\sqrt{\pi a} \qquad = 2.41\sqrt{\pi a}$$

Example problem: solution

Substituting $K_I = 9.12\sqrt{\pi a}$ and $K_{II} = 2.41\sqrt{\pi a}$ in $\sigma_{r\theta} = 0$ returns:

$$\sigma_{r\theta} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\left(\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}\right)$$

$$\implies 0 = 9.12\left(\sin\frac{\theta}{2} + \sin\frac{3\theta}{2}\right) + 2.4\left(\cos\frac{\theta}{2} + 3\cos\frac{3\theta}{2}\right)$$

And solving numerically, we find:

$$\theta = -26^\circ, \quad 130^\circ \quad \text{and} \quad 180^\circ$$

- The value of 180°is physically impossible.
- The other two values correspond to either a minimum or a maximum of $\sigma_{\theta\theta}$. The crack will propagate in the direction of maximum $\sigma_{\theta\theta}$.

Example problem: solution

We are left with $\theta = -26^\circ$ or 130°, and we are looking for the value corresponding to a maximum in $\sigma_{\theta\theta}$. This stress is given by (see Slide 17):

$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2}\right) - \frac{K_{II}}{\sqrt{2\pi r}}\left(\frac{3}{4}\sin\frac{\theta}{2} + \frac{3}{4}\sin\frac{3\theta}{2}\right)$$

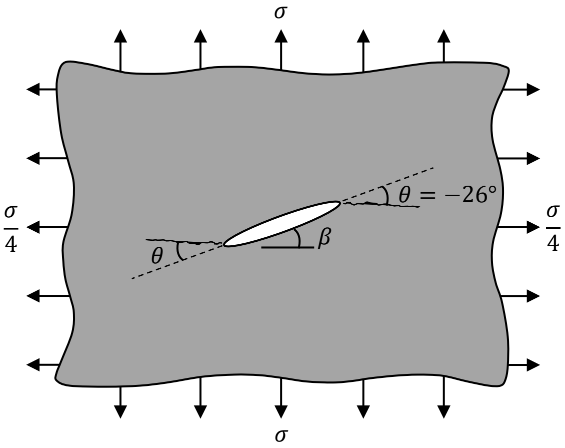
$$\text{for } \theta = -26^\circ \implies \sigma_{\theta\theta} = 9.98\sqrt{\frac{a}{2r}}$$

$$\text{for } \theta = 130^\circ \implies \sigma_{\theta\theta} = -0.48\sqrt{\frac{a}{2r}}$$

The maximum $\sigma_{\theta\theta}$ is obtained with $\theta = -26^\circ$; therefore, the crack will propagte in this direction.

Example problem: solution

Note that our solution, $\theta = -26°$, is in the local (crack) reference frame.



Summary

Under mixed-mode loading:

- find the stress intensity factors K_I and K_{II} using Mohr's circle,
- use the energy release rate G to predict when fracture will occur,
- the crack will propagate in the direction where $\sigma_{r\theta} = 0$.