A? Problem 4.1

A test was done to measure the fracture toughness of a thin polymer plate. The geometry had a central crack of length $2a = 50\,\text{mm}$, and the plate was tested by applying a tensile stress σ_∞ in the direction normal to the crack.

(a) If the plate failed at a stress $\sigma_\infty = 3\,\text{MPa}$, evaluate the fracture toughness K_{Ic} of the material.

(b) Provided that the polymer has a yield strength $\sigma_Y = 30\,\text{MPa}$, estimate the size of the plastic zone at the crack tip. Is it adequate to use Linear Elastic Fracture Mechanics to compute K_{Ic} in this case?

A! Solution

Part (a). For a large plate with a central crack, setting the stress intensity factor K_I equal to the fracture toughness K_{Ic} gives:

$$K_I = \sigma_\infty \sqrt{\pi a} = K_{Ic}$$
$$\implies K_{Ic} = \sigma_\infty \sqrt{\pi a} = 3 \cdot \sqrt{\pi \cdot 0.025} = 0.84\,\text{MPa}\sqrt{\text{m}}$$

Part (b). The plate is thin so we assume plane stress conditions when evaluating the size of the plastic zone. This gives:

$$d_p = \frac{1}{\pi}\left(\frac{K_{Ic}}{\sigma_Y}\right)^2 = \frac{1}{\pi}\left(\frac{0.84}{30}\right)^2 = 0.25\,\text{mm}$$

The size of the plastic zone is very small: two orders of magnitude smaller than the crack length. Therefore, it is adequate to use Linear Elastic Fracture Mechanics (LEFM in this situation. In general, the plastic zone can be considered sufficiently small to use LEFM if $d_p < a/10$.

A? Problem 4.2

A cylindrical pressure vessel with closed ends has a radius $R = 1\,\text{m}$ and wall thickness $t = 40\,\text{mm}$, and is subjected to an internal pressure p. The vessel must be designed safely against failure by yielding (according to the von Mises yield criterion) and fracture. Three steels with the following values of yield stress σ_Y and fracture toughness K_{Ic} are being considered for constructing the vessel.

Steel	σ_Y (MPa)	K_{Ic} (MPa$\sqrt{\text{m}}$)
A: 4340	860	100
B: 4335	1300	70
C: 350 Maraging	1550	55

Fracture of the vessel is caused by a long axial surface crack of depth a (you can assume an infinite plate with an edge crack). The vessel should be designed with a safety factor $S = 2$ against yielding and fracture. For each steel:

(a) Plot the maximum permissible pressure p as a function of the crack depth a.

(b) Calculate the maximum permissible crack depth a for an operating pressure $p = 12\,\text{MPa}$.

(c) Calculate the failure pressure p for a minimum detectable crack depth $a = 1\,\text{mm}$.

A! Solution

Part (a). For a thin-walled cylindrical pressure vessel, the hoop stress $\sigma_{\theta\theta}$ and the longitudinal stress σ_{zz} are:

$$\sigma_{\theta\theta} = \frac{pR}{t} \qquad \text{and} \qquad \sigma_{zz} = \frac{pR}{2t}$$

The von Mises yielding criterion, taking into account the safety factor S, can be expressed as:

$$\frac{1}{\sqrt{2}}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2\right]^{1/2} = \frac{\sigma_Y}{S}$$

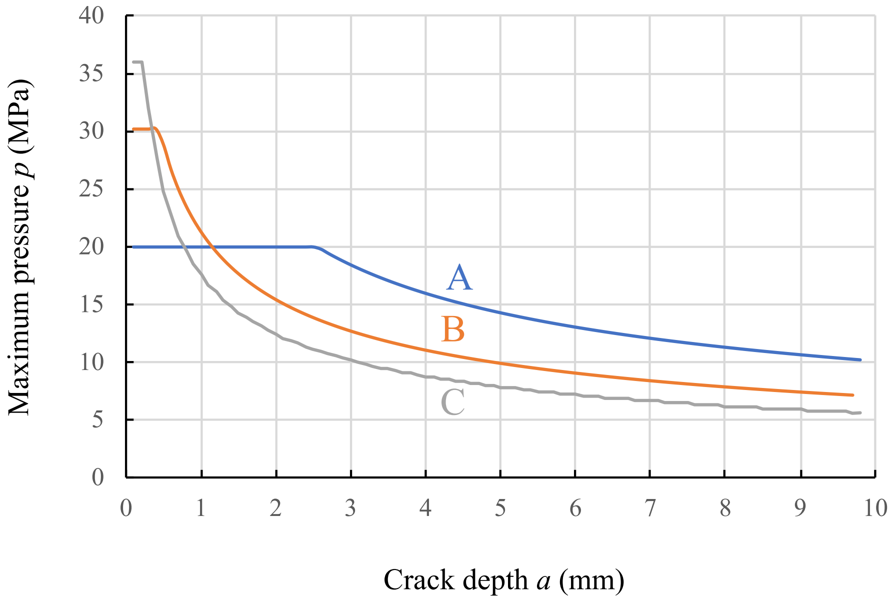
where the principal stresses are $\sigma_1 = \sigma_{\theta\theta}$; $\sigma_2 = \sigma_{zz}$ and $\sigma_3 = 0$. Substituting in the equation above gives:

$$\begin{aligned}
&\frac{1}{\sqrt{2}}\left[\left(\frac{pR}{t} - \frac{pR}{2t}\right)^2 + \left(\frac{pR}{2t}\right)^2 + \left(\frac{pR}{t}\right)^2\right]^{1/2} = \frac{\sigma_Y}{S} \\
&\implies \frac{1}{\sqrt{2}}\left[\frac{3p^2R^2}{2t^2}\right]^{1/2} = \frac{\sigma_Y}{S} \\
&\implies \frac{\sqrt{3}pR}{2t} = \frac{\sigma_Y}{S} \\
&\implies p = \frac{2\sigma_Y t}{\sqrt{3}SR} = \frac{2 \cdot 0.04}{\sqrt{3} \cdot 2 \cdot 1}\sigma_Y = 0.0231\sigma_Y \qquad (1)
\end{aligned}$$

This gives the maximum allowable pressure to avoid yielding. Next, we need to ensure that we also avoid fracture. Assuming an infinite plate with an edge crack loaded by the hoop stress and including the safety factor in our analysis, the criterion for fracture is:

$$\begin{aligned} & K_I = \frac{K_{Ic}}{S} \\ \implies & 1.12\sigma_{\theta\theta}\sqrt{\pi a} = \frac{K_{Ic}}{S} \\ \implies & 1.12\frac{pR}{t}\sqrt{\pi a} = \frac{K_{Ic}}{S} \\ \implies & p = \frac{K_{Ic}t}{1.12SR\sqrt{\pi a}} = \frac{0.04}{1.12 \cdot 2 \cdot 1 \cdot \sqrt{\pi}}\frac{K_{Ic}}{\sqrt{a}} = 0.0101\frac{K_{Ic}}{\sqrt{a}} \end{aligned} \tag{2}$$

This is the maximum pressure to avoid fracture. The maximum allowable pressure to avoid both yielding and fracture is the minimum value of Eq. (1) and (2), and this is plotted below.



Part (b). For $p = 12$ MPa, all steels fail by fracture and the maximum crack length can be obtained from Eq. (2). This gives:

A: $a = 7.05$ mm B: $a = 3.45$ mm C: $a = 2.13$ mm

Part (b). For a fixed crack length $a = 1$ mm, the maximum allowable pressure for each steel is:

A: $p = 19.9$ MPa B: $p = 22.3$ MPa C: $p = 17.5$ MPa

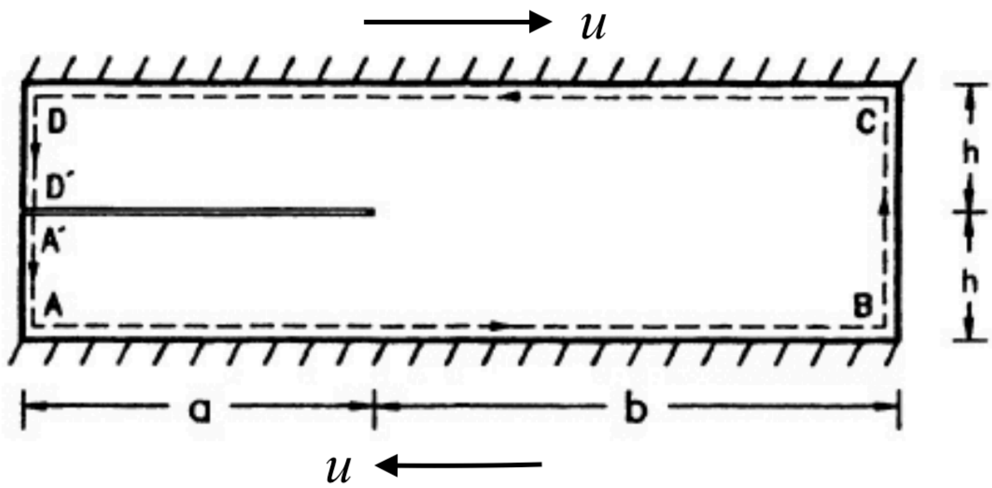
In conclusion, using a material with a high fracture toughness is beneficial to resist long cracks. However, materials with a high yield strength (and a low fracture toughness) can tolerate much higher stresses if cracks are short.

A? Problem 4.3

An infinitely wide strip, of height $2h$ and with a semi-infinite crack, is rigidly clamped along its top and bottom faces, see below. The strip is loaded in mode II with a prescribed displacement u as shown below. Assuming that the material is linear elastic and isotropic, show that the value of the J-integral for this scenario is given by:

$$J = \frac{Gu^2}{h} = \frac{Eu^2}{2(1+\nu)h}$$

where $G = \frac{E}{2(1+\nu)}$ is the shear modulus.



A! Solution

The definition of the J integral is:

$$J = \int_{\Gamma} \left(w dy - t_i \frac{\partial u_i}{\partial x} ds \right)$$

and the contour Γ can be divided in five segments as:

$$J = J_{AA'} + J_{AB} + J_{BC} + J_{CD} + J_{DD'}$$

Let's define x and y in the horizontal and vertical directions, respectively. Along segments AB and CD, we have no variations in y and the displacement field will be constant with x so:

$$dy = 0 \qquad \text{and} \qquad \frac{\partial u_1}{\partial x} = \frac{\partial u_2}{\partial x} = 0 \qquad \Longrightarrow J_{AB} = J_{BC} = 0$$

Along segments AA' and DD', we have no stress and the contour is vertical therefore:

$$w = 0 \qquad \text{and} \qquad \frac{\partial u_1}{\partial x} = \frac{\partial u_2}{\partial x} = 0 \qquad \Longrightarrow J_{AA'} = J_{DD'} = 0$$

Consequently, $J = J_{BC}$. Segment BC is vertical therefore:

$$\frac{\partial u_1}{\partial x} = \frac{\partial u_2}{\partial x} = 0$$

and then the J integral is:

$$\begin{aligned}
J &= J_{BC} \\
&= \int_{-h}^{h} w dy \\
&= \int_{-h}^{h} \frac{1}{2}\sigma_{xy}\epsilon_{xy} dy \qquad \text{since there is only shear} \\
&= \int_{-h}^{h} \frac{1}{2} G\epsilon_{xy}^2 dy \qquad \text{where } G \text{ is the shear modulus} \\
&= \frac{G}{2}\int_{-h}^{h} \left(\frac{u}{h}\right)^2 dy \qquad \text{the shear strain is } \epsilon_{xy} = u/h \\
&= \frac{Gu^2}{2h^2}\left[y\right]_{-h}^{h} \\
&= \frac{Gu^2}{h} \\
&= \frac{Eu^2}{2(1+\nu)h} \qquad \text{since } E = 2G(1+\nu) \text{ for isotropic materials}
\end{aligned}$$