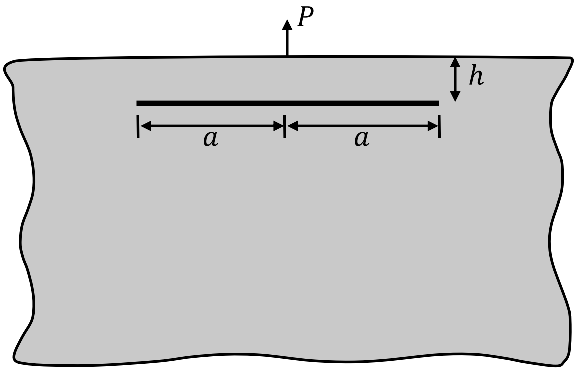
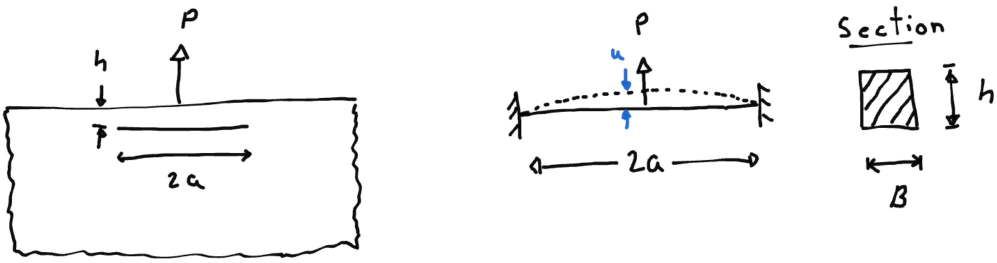
### A? Question

A crack of length $2a$ is parallel to the edge of a thin semi-infinite plate subjected to a point force $P$. Find the energy release rate $G$ and the stress intensity factor $K_I$, provided that the out-of-plane thickness is $B$ and the material has a Young's modulus $E$. Hint: assume that the material above the crack deforms like a beam clamped at both ends.




### A! Solution




The deflection of a beam clamped at both ends is given by:

$$u = \frac{P(2a)^3}{192EI} = \frac{8Pa^3}{192E} \cdot \frac{12}{Bh^3} = \frac{Pa^3}{2EBh^3}$$

Therefore, the compliance is:

$$C = \frac{u}{P} = \frac{a^3}{2EBh^3}$$

Using the compliance, the energy release rate becomes:

$$G = \frac{P^2}{2B}\frac{dC}{da} = \frac{P^2}{2B} \cdot \frac{3a^2}{2EBh^3} = \frac{3P^2a^2}{4EB^2h^3}$$

The plate is thin so we can assume plane stress, and the stress intensity factor is:

$$K_I = \sqrt{EG} = \sqrt{\frac{3P^2a^2}{4B^2h^3}} = \frac{\sqrt{3}Pa}{2Bh^{3/2}}$$

### A? Question

The R-curve of a high strength steel ($E = 210\,\text{GPa}$) can be expressed as:

$$R = 0.1 + 3(a - a_0)^{0.1},$$

where $R$ is given in MJ/m$^2$, whereas $a$ and $a_0$ are in m. A large thin plate, made from this material, has a centre crack of length $2a_0 = 60\,\text{mm}$. Find the amount of stable crack growth $\Delta a$ and the stress $\sigma_\infty$ at which fracture will become unstable.

### A! Solution

The energy release rate for this geometry is:

$$G = \frac{K_I^2}{E} = \frac{(\sigma_\infty \sqrt{\pi a})^2}{E} = \frac{\sigma_\infty^2 \pi a}{E},$$

where we have assumed that we are under plane stress since the plate is thin. The moment at which fracture will become unstable is when:

$$G = R \qquad \text{and} \qquad \frac{dG}{da} = \frac{dR}{da}$$

The first condition gives us:

$$G = R \quad \Rightarrow \quad \frac{\sigma_\infty^2 \pi a}{E} = 0.1 + 3(a - a_0)^{0.1} \tag{1}$$

whereas the second condition returns:

$$\frac{dG}{da} = \frac{dR}{da} \quad \Rightarrow \quad \frac{\sigma_\infty^2 \pi}{E} = 3 \cdot 0.1(a - a_0)^{-0.9} \tag{2}$$

Note that we now have a system of two equations and two unknowns: $\sigma_\infty$ and $a$. To solve the system, we can substitute Eq. (2) into (1) to obtain:

$$3 \cdot 0.1(a - a_0)^{-0.9} \cdot a = 0.1 + 3(a - a_0)^{0.1} \quad \text{where} \quad a_0 = 0.030 \text{ m}$$

This is difficult to solve analytically, but using Matlab we find:

$$a = 0.0331 \text{ m} \quad \Rightarrow \quad \Delta a = a - a_0 = 3.1 \text{ mm}$$

Next, we can substitute this result in Eq. (1) or (2) to solve for $\sigma_\infty$. Using Eq. (2), we find:

$$\sigma_\infty = \sqrt{\frac{E}{\pi} 3 \cdot 0.1(a - a_0)^{-0.9}} = \sqrt{\frac{210000}{\pi} \cdot 3 \cdot 0.1(0.0031)^{-0.9}} = 1905 \text{ MPa}$$

This value of $\sigma_\infty$ is rather high, even for a high strength steel, but that is sufficient to demonstrate how to calculate the amount of stable crack growth and the stress at which fracture will become unstable. The methodology is simple, but you have to be extremely careful with units to make sure the value of $G$ is in the same units as the value of $R$.

### A? Question

The energy release rate $G$ for a double cantilever beam (slide 14) with an applied force $P$ is given by:

$$G = \frac{12P^2a^2}{EB^2h^3},$$

which shows that $G$ increases with increasing crack length $a$. Explain how you could redesign the geometry to ensure that $G$ remains constant with increasing crack length $a$.

### A! Solution

This could be done by varying the thickness $h$ of each beam along their length, such that the compliance $C$ becomes a linear function of the crack length $a$. If $C \propto a$ then, $dC/da$ will be a constant and the energy release rate $G$ will be independent of the crack length $a$.