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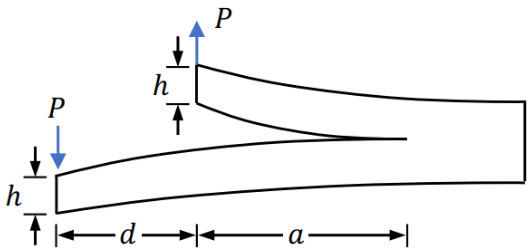
## Fracture Mechanics Final Exam

**A? Problem 1 (12 pts)**

A component, with two arms of different lengths, is loaded by a constant force $P$ as shown below. The component has an out-of-plane thickness $B$ and is made from a solid with a Young's modulus $E$.




(a) Determine the compliance of the system.

From the beam theory of cantilever maximum displacement, we have:

$\delta = \dfrac{PL^3}{3EI} \Rightarrow P = \dfrac{3EI}{\delta L^3}$, where L is the length of the beam and $\delta$ is maximum displacement

For the upper arm of the component, we have:

- Inertia: $I_{top} = \dfrac{Bh^3}{12}$
- Compliance: $C_{top} = \dfrac{\delta_{top}}{P} = \dfrac{Pa^3}{3EI_{top}P} = \dfrac{a^3}{3EI_{top}} = \dfrac{a^3}{3E}\dfrac{12}{Bh^3} = \dfrac{4a^3}{EBh^3}$ (answer)

For the lower arm of the component, we have:

- Inertia: $I_{bot} = \dfrac{Bh^3}{12}$
- Compliance: $C_{bot} = \dfrac{\delta_{bot}}{P} = \dfrac{P(a+d)^3}{3EI_{bot}P} = \dfrac{(a+d)^3}{3EI_{bot}} = \dfrac{(a+d)^3}{3E}\dfrac{12}{Bh^3} = \dfrac{4(a+d)^3}{EBh^3}$ (answer)

The compliance of the whole system is

$$C = \frac{\delta}{P} = \frac{\delta_{top} + \delta_{bot}}{P} = C_{top} + C_{bot} = \frac{4a^3}{EBh^3} + \frac{4(a+d)^3}{EBh^3} = \frac{4\left(a^3 + (a+d)^3\right)}{EBh^3} \quad \text{(answer)}$$

(b) Calculate the energy release rate $G$.

The energy release rate of the whole component system is

$$G = \frac{P^2}{2B}\frac{dC}{da} = \frac{P^2}{2B}\frac{d}{da}\left(\frac{4\left(a^3+(a+d)^3\right)}{EBh^3}\right) = \frac{P^2}{2B}\frac{d}{da}\left(\frac{4\left(2a^3+3a^2d+3ad^2+d^3\right)}{EBh^3}\right)$$

$$\Rightarrow \quad G = \frac{P^2}{2B}\left(\frac{4\left(6a^2+6ad+3d^2\right)}{EBh^3}\right) = \frac{P^2\left(24a^2+24ad+12d^2\right)}{2EB^2h^3} \quad \text{(answer)}$$

(c) Will crack growth be stable or unstable? Assume that the material has a flat R-curve.

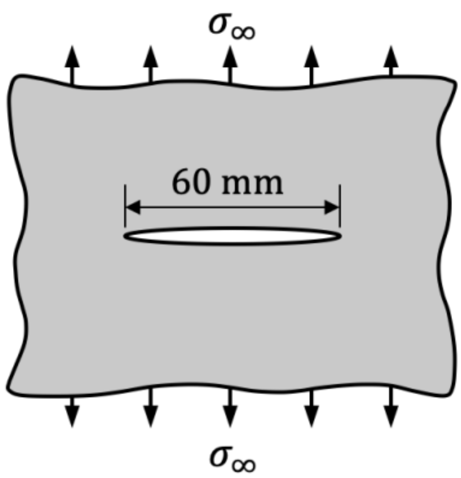
Under displacement control, the energy release rate is:

$$G = \frac{P^2\left(24a^2+24ad+12d^2\right)}{2EB^2h^3} \Rightarrow \frac{dG}{da} = \frac{d}{da}\left[\frac{P^2\left(24a^2+24ad+12d^2\right)}{2EB^2h^3}\right] = \frac{P^2\left(48a+24d\right)}{2EB^2h^3} > 0$$

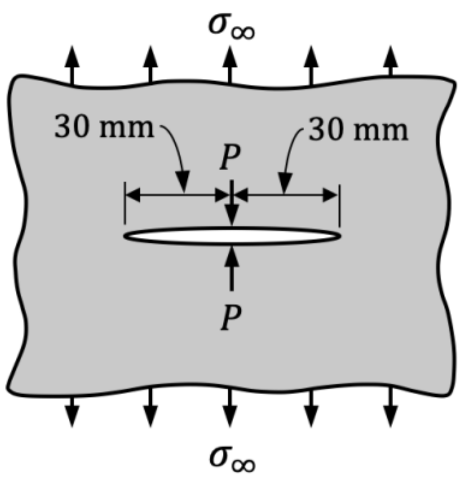
Assuming that the material has a flat R-curve, crack growth is unstable under displacement control because of $\frac{dG}{da} > 0$ (answer)

**A? Problem 2 (12 pts)**

A thin aluminium plate of thickness $t = 3$ mm has a central crack of length $2a = 60$ mm as a consequence of the manufacturing process. The plate is then tested by applying a tensile stress $\sigma_\infty$ in the direction normal to the crack.




Part (a,b)




Part (c)

(a) If the plate failed at a stress $\sigma_\infty = 90$ MPa, evaluate the fracture toughness $K_{Ic}$ of the material.

According to the formula in of fracture toughness:

$$K_{Ic} = \sigma_\infty\sqrt{\pi a} = 90MPa\cdot\sqrt{\pi(0.03m)} = 27.63MPa\sqrt{m} \quad \text{(answer)}$$

(b) Provided that this aluminium alloy has a yield strength $\sigma_Y = 350$ MPa, is it adequate to use Linear Elastic Fracture Mechanics?

The aluminum plate is thin => The plane stress condition is assumed

Therefore, the plastic zone size under plane stress is:

$$d_p = \frac{1}{\pi}\left(\frac{K_{Ic}}{\sigma_Y}\right)^2 = \frac{1}{\pi}\left(\frac{27.63MPa\sqrt{m}}{350MPa}\right)^2 = 0.001983m = 1.983mm$$

If the plastic zone size $d_p$ is roughly an order of magnitude smaller than the crack length ($10d_p < a$), we conclude that Linear Elastic Fracture Mechanics (LEFM) applies, and fracture will occur when $K_I = K_{Ic}$. Since $10d_p = 19.83mm < a = 30mm$, LEFM can be applied here.

(c) Another plate is produced from the same material, but this time it is reinforced by a wire creating a force $P$ closing the crack (see figure below). Calculate the force $P$, in N, needed to increase the fracture stress to $\sigma_\infty = 100$ MPa.

The stress intensity factor when the fracture stress is increased to $\sigma_\infty = 100MPa$ becomes:

$$K_I^{[\sigma_\infty]} = \sigma_\infty\sqrt{\pi a} = 100MPa\cdot\sqrt{\pi(0.03m)} = 30.7MPa\sqrt{m}$$

By principal of superposition, the fracture toughness with wire reinforcement becomes

$K_{Ic} = K_I^{[\sigma_\infty]} - K_I^{[wire]}$, where negative sign of $K_I^{[wire]}$ indicates compression on the aluminum plate

$$\Rightarrow \quad K_I^{[wire]} = K_I^{[\sigma_\infty]} - K_{Ic} = 30.7MPa\sqrt{m} - 27.63MPa\sqrt{m} = 3.07MPa\sqrt{m}$$

The stress intensity induced by the wire is:

$K_I^{[wire]} = \frac{P/t}{\sqrt{\pi a}}$, which is a reduced formula from $K_I^{[wire]} = \frac{P/t}{\sqrt{\pi a}}\sqrt{\frac{a+x_0}{a-x_0}}$ where $x_0 = 0$

$$\Rightarrow \quad P = K_I^{[wire]} t\sqrt{\pi a} = 3.07\cdot10^6 Pa\sqrt{m}(0.003m)\sqrt{\pi(0.03m)} = 2827.45N$$

The force P needed to increase fracture stress to $\sigma_\infty = 100MPa$ is $2827.45N$ (answer)

**A? Problem 3 (12 pts)**

A steel grade has an elastic modulus $E = 207$ GPa and the R-curve:

$$R = C\sqrt{a - a_0},$$

where $a_0$ is the initial crack size and $C = 2.2\cdot10^5$ J/m$^{5/2}$. Note that $R$ has units of J/m$^2$ and crack length is in m. Consider a thin and wide plate with a through central crack ($a \ll W$) that is made from this material. If this plate has an initial crack length $2a_0 = 50.8$ mm and is loaded by a tensile stress $\sigma_\infty$ perpendicular to the crack plane, compute the amount of stable crack growth and the stress $\sigma_\infty$ at which unstable fracture occurs.

The plate is thin and large => The plane stress condition is assumed. The energy release rate is

- $$K_I^2 = EG \Rightarrow G = \frac{K_I^2}{E} = \frac{\left(\sigma_\infty \sqrt{\pi a}\right)^2}{E} = \frac{\sigma_\infty^2 \pi a}{E}$$

The moment at which fracture will become unstable is when:

(I) $G = R$ and (II) $\frac{dG}{da} = \frac{dR}{da}$

The first condition gives us:

$$G = R \Rightarrow \frac{\sigma_\infty^2 \pi a}{E} = C\left(a - a_0\right)^{1/2} = C\left(\Delta a\right)^{1/2} \text{ (I)}$$

Whereas the second condition returns:

$$\frac{dG}{da} = \frac{dR}{da} \Rightarrow \frac{\sigma_\infty^2 \pi}{E} = \frac{d}{da}\left(C\left(a - a_0\right)^{1/2}\right) = \frac{1}{2}C\left(a - a_0\right)^{-1/2} = \frac{1}{2}C\left(\Delta a\right)^{-1/2} \text{ (II)}$$

Combining two equations, we have the following equality:

$$\frac{1}{a}\left(C\left(a - a_0\right)^{1/2}\right) = \frac{1}{2}C\left(a - a_0\right)^{-1/2} \Rightarrow \frac{1}{a}\left(a - a_0\right)^{1/2} = \frac{1}{2}\left(a - a_0\right)^{-1/2}$$

- $$\frac{2}{a}\left(a - a_0\right)^{1/2} = \left(a - a_0\right)^{-1/2} \Rightarrow \left(a - a_0\right)^{-1} = \frac{2}{a}$$
- $2\left(a - a_0\right) = a$. Replace $a_0 = 0.0254m$ into the equation
- $2\left(a - 0.0254\right) = a \Rightarrow a = 0.0508m$

Therefore, the amount of stable crack growth is $\Delta a = a - a_0 = 0.0254m$ (answer)

We can substitute the stable crack growth into the second equation to obtain the critical stress

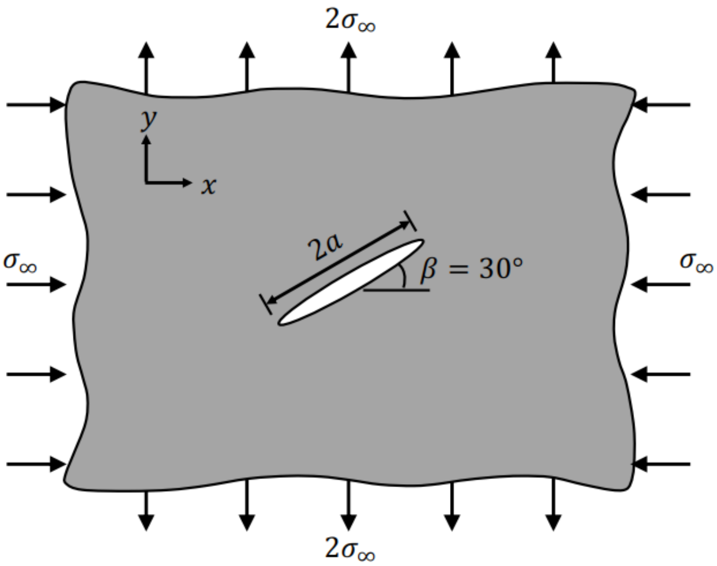
$$\frac{\sigma_c^2 \pi}{E} = \frac{1}{2}C\left(\Delta a\right)^{-1/2} \Rightarrow \sigma_c = \sqrt{\frac{1}{2}\frac{E}{\pi}C\left(\Delta a\right)^{-1/2}}$$

- $$\sigma_c = \sqrt{\frac{1}{2}\frac{207 \cdot 10^9 Pa}{\pi}(220000J / m^{5/2})\left(0.0254m\right)^{-1/2}} = 213254473Pa$$

Therefore, the stress at which unstable fracture occurs is $\sigma_c = 213.254MPa$ (answer)

**A? Problem 4 (12 pts)**

A large plate contains a central crack of length $2a$ at an angle $\beta = 30°$ from the horizontal. The plate is loaded in tension by a stress $2\sigma_\infty$ in the vertical direction, and in compression in the horizontal direction by a stress $\sigma_\infty$, see below. Find the stress intensity factors $K_I$ and $K_{II}$. Express your results as a function of $\sigma_\infty$ and $a$.



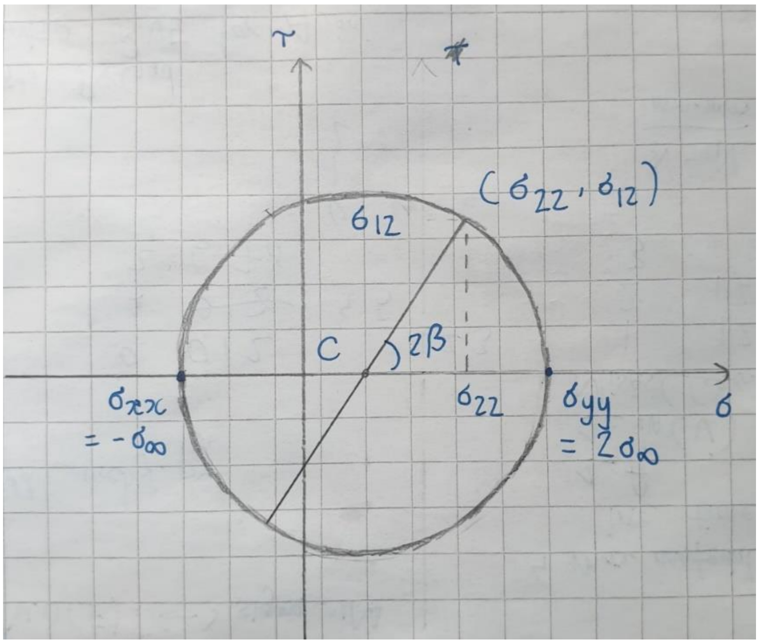

The stress field in the global reference frame is:

$$\sigma_{xx} = -\sigma_\infty,\ \sigma_{yy} = 2\sigma_\infty,\ \sigma_{xy} = 0$$

The first and second points on the Mohr circle are: $(\sigma_{xx}, -\sigma_{xy})$ and $(\sigma_{yy}, -\sigma_{xy})$

⇨ The two points are $(-\sigma_\infty, 0)$ and $(2\sigma_\infty, 0)$

The Mohr's circle for this stress field is:



where the center C and radius R of the circle are: $c = 0.5\sigma_\infty$ and $r = 1.5\sigma_\infty$

The stresses $\sigma_{22}$ and $\sigma_{12}$ in the local reference frame are given by:

$$\sigma_{22} = c + r\cos(2\beta) = 0.5\sigma_\infty + 1.5\sigma_\infty \cos(2\times 30°) = 1.25\sigma_\infty$$
$$\sigma_{12} = r\sin(2\beta) = 1.5\sigma_\infty \sin(2\times 30°) = 1.299\sigma_\infty$$

Finally, the stress intensity factors as a function of $\sigma_\infty$ and $a$ are given by:

$$K_I = \sigma_{22}\sqrt{\pi a} = 1.25\sigma_\infty\sqrt{\pi a} = 2.2155\sigma_\infty\sqrt{a}$$
$$K_{II} = \sigma_{12}\sqrt{\pi a} = 1.299\sigma_\infty\sqrt{\pi a} = 2.3024\sigma_\infty\sqrt{a}$$
(answer)

**A? Problem 5 (12 pts)**

A crack is loaded in a mixed-mode scenario where $K_I = 2K_{II}$. Find the direction $\theta$, relative to the existing crack plane, along which the crack will propagate.

To find the angle of crack propagation, we set $\sigma_{r\theta} = 0$, and this gives:

$$\sigma_{r\theta} = \frac{K_I}{\sqrt{2\pi r}}\left[\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}\right] + \frac{K_{II}}{\sqrt{2\pi r}}\left[\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}\right] = 0$$

⇨ $K_I\left[\sin\frac{\theta}{2} + \sin\frac{3\theta}{2}\right] + K_{II}\left[\cos\frac{\theta}{2} + 3\cos\frac{3\theta}{2}\right] = 0$

We are also given the information that $K_I = 2K_{II}$

⇨ $2\sin\frac{\theta}{2} + 2\sin\frac{3\theta}{2} + \cos\frac{\theta}{2} + 3\cos\frac{3\theta}{2} = 0$. Let $x = \frac{\theta}{2}$. Replace this into the equation

⇨ $2\sin x + 2\sin(3x) + \cos x + 3\cos(3x) = 0$

We can apply these two identities to simplify the equation

sin 3x = 3 sin x – 4 sin³ x

cos 3x = 4cos³ x – 3 cos x

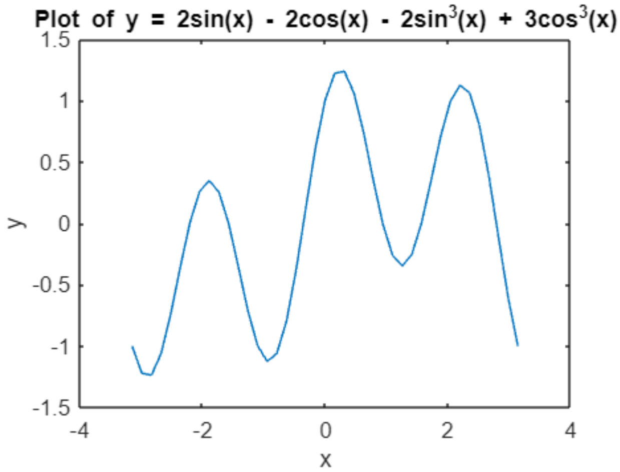
⇨ $2\sin x + 2\left[3\sin x - 4\sin^3 x\right] + \cos x + 3\left[4\cos^3 x - 3\cos x\right] = 0$

⇨ $2\sin x + 6\sin x - 8\sin^3 x + \cos x + 12\cos^3 x - 9\cos x = 0$

⇨ $8\sin x - 8\cos x - 8\sin^3 x + 12\cos^3 x = 0$

⇨ $2\sin x - 2\cos x - 2\sin^3 x + 3\cos^3 x = 0$

Plot the trigonometric function in MATLAB and we can observe that it has six roots in $[-\pi, \pi]$. Now we can use numerical solutions to find the six roots of the function

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The first solution is x = -2.202711 radians
    -2.2027
The second solution is x = -1.570796 radians
    -1.5708
The third solution is x = -0.350879 radians
    -0.3509

The fourth solution is x = 0.938882 radians
    0.9389

The fifth solution is x = 1.570796 radians
    1.5708

The sixth solution is x = 2.790713 radians
    2.7907
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The correct angle $\theta$ is the one corresponding to the maximum $\sigma_{\theta\theta}$

$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}}\left[\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2}\right] - \frac{K_{II}}{\sqrt{2\pi r}}\left[\frac{3}{4}\sin\frac{\theta}{2} + \frac{3}{4}\sin\frac{3\theta}{2}\right]$$

We are also given the information that $K_I = 2K_{II}$, so we need to maximize this quantity

$$\frac{3}{2}\cos\frac{\theta}{2} + \frac{1}{2}\cos\frac{3\theta}{2} - \frac{3}{4}\sin\frac{\theta}{2} - \frac{3}{4}\sin\frac{3\theta}{2}$$

Plugging the solutions above:

$\theta/2 = -2.202$ => $\sigma_{\theta\theta} = 0.432$ $\quad$ $\theta/2 = -1.57$ => $\sigma_{\theta\theta} = 0$

$\theta/2 = -0.35$ => $\sigma_{\theta\theta} = 2.565$ $\quad$ $\theta/2 = 0.938$ => $\sigma_{\theta\theta} = -0.432$

$\theta/2 = 1.57$ => $\sigma_{\theta\theta} = 0$ $\quad$ $\theta/2 = 2.79$ => $\sigma_{\theta\theta} = -2.565$

Therefore, it shows that $\theta/2 = -0.35$ corresponds to a maximum $\sigma_{\theta\theta}$. Therefore, the crack will propagate along $\theta = -0.7$ radian $= -40.1°$ (answer)