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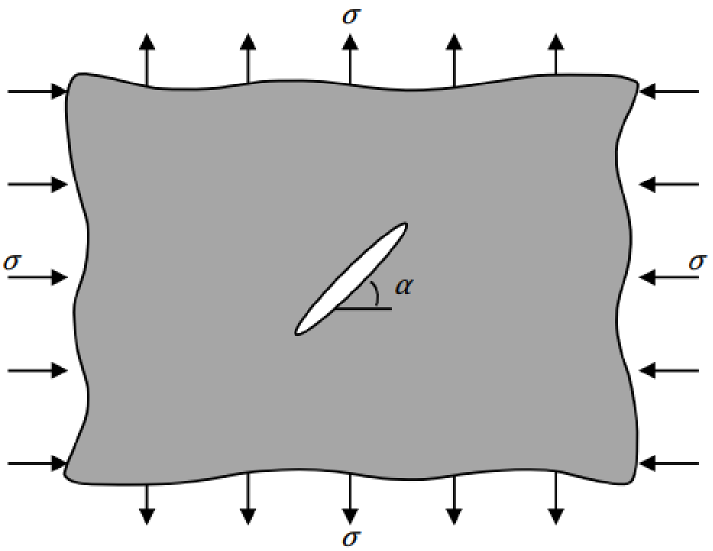
Fracture Mechanics Assignment 3

A? Problem 3.1 (5 pts)

Consider the thin plate shown below with a crack of length $2a = 60$ mm at an angle $\alpha = 20°$.



(a) Find expressions for the mode I and mode II stress intensity factors. Express your results as a function of the applied stress σ.

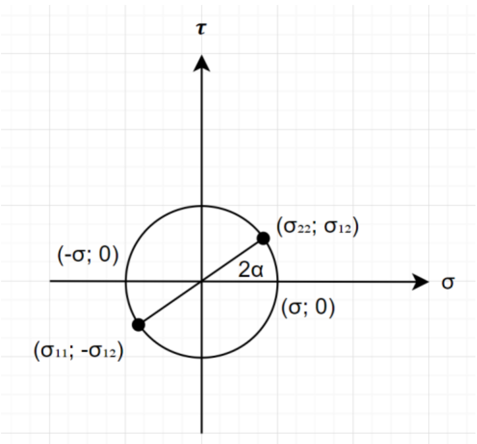
The stress field in the global reference frame is:

$$\sigma_{xx} = -\sigma,\ \sigma_{yy} = \sigma,\ \sigma_{xy} = 0$$

The first and second points on the Mohr circle are: $\left(\sigma_{xx}, -\sigma_{xy}\right)$ and $\left(\sigma_{yy}, -\sigma_{xy}\right)$

⇨ The two points are $\left(-\sigma, 0\right)$ and $\left(\sigma, 0\right)$

The Mohr's circle for this stress field is:



where the centre c and radius r of the circle are: $c = 0$ and $r = \sigma$

The stresses σ_{22} and σ_{11} in the local reference frame are given by:

$\sigma_{22} = c + r\cos(2\alpha) = 0 + \sigma\cos(2\times 20°) = 0.766\sigma$ (answer)

$\sigma_{11} = r\sin(2\alpha) = \sigma\sin(2\times 20°) = 0.643\sigma$ (answer)

Finally, the stress intensity factors are given by:

$$K_I = \sigma_{22}\sqrt{\pi a} = 0.766\sigma\sqrt{\pi a} = 0.2352\sigma$$
$$K_{II} = \sigma_{12}\sqrt{\pi a} = 0.643\sigma\sqrt{\pi a} = 0.1973\sigma$$

(b) Estimate the maximum stress σ that the plate can support provided that it is made from an aluminium alloy with a Young's modulus $E = 70\,\text{GPa}$ and a toughness $G_c = 12\,\text{kJ/m}^2$.

The energy release rate G is the sum of each mode:

$$G = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{(1+\nu)K_{III}^2}{E}$$

A fracture criterion is obtained by setting *G* equal to the material's toughness:

$G_C = \dfrac{K_{Ic}^2}{E'}$ (1). The plate is also conditioned as thin

⇨ plane stress condition is assumed so $E' = E = 70GPa = 70e9Pa$

Additionally, we have: $\dfrac{K_I^2}{E'} + \dfrac{K_{II}^2}{E'} + \dfrac{(1+\nu)K_{III}^2}{E} = \dfrac{K_{Ic}^2}{E'}$. In this exercise, $K_{III} = 0$

⇨ $\dfrac{K_{Ic}^2}{E'} = \dfrac{K_I^2}{E'} + \dfrac{K_{II}^2}{E'} \Rightarrow K_{Ic}^2 = K_I^2 + K_{II}^2 = \left(0.766\sigma_c\sqrt{\pi a}\right)^2 + \left(0.643\sigma_c\sqrt{\pi a}\right)^2 = \sigma_c^2\pi a$

Plug in equation (1), we have:

$$G_C = \frac{K_{Ic}^2}{E} \Rightarrow G_C = \frac{\sigma_c^2\pi a}{E} \Rightarrow \sigma_c = \sqrt{\frac{EG_C}{\pi a}} = \sqrt{\frac{70GPa\cdot 12kJ/m^2}{\pi(30mm)}}$$

⇨ $\sigma_c = \sqrt{\dfrac{70e9Pa\times 12e3Pa\cdot m}{\pi(0.03m)}} = 94406974Pa = 94.4MPa$ (answer)

A? Problem 3.2 (5 pts)

A crack is loaded in a mixed-mode scenario where $K_I = K_{II}$. Find the direction θ, relative to the initial crack plane, in which the crack will propagate. Hint: don't hesitate to use a numerical approach to solve this equation.

To find the angle of crack propagation, we set $\sigma_{r\theta} = 0$, and this gives:

$$\sigma_{r\theta} = \frac{K_I}{\sqrt{2\pi r}}\left[\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}\right] + \frac{K_{II}}{\sqrt{2\pi r}}\left[\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}\right] = 0$$

$$\Rightarrow \quad K_I\left[\sin\frac{\theta}{2} + \sin\frac{3\theta}{2}\right] + K_{II}\left[\cos\frac{\theta}{2} + 3\cos\frac{3\theta}{2}\right] = 0$$

We are also given the information that $K_I = K_{II}$

$$\Rightarrow \quad \sin\frac{\theta}{2} + \sin\frac{3\theta}{2} + \cos\frac{\theta}{2} + 3\cos\frac{3\theta}{2} = 0$$

We can apply these two identities to simplify the equation

sin 3x = 3 sin x – 4 sin³ x

cos 3x = 4cos³ x – 3 cos x

$$\Rightarrow \quad \sin\frac{\theta}{2} - \sin^3\frac{\theta}{2} + 3\cos^3\frac{\theta}{2} - 2\cos\frac{\theta}{2} = 0$$

Let $x = \frac{\theta}{2}$. Replace this into the equation: $\sin x - \sin^3 x + 3\cos^3 x - 2\cos x = 0$

We can further factorize the equation as:

$\cos x(\sin x\cos x + 3\cos^2 x - 2) = 0$. # Numerical solution applies here

$$\Rightarrow \quad \cos x(\sin x\cos x + 3\cos^2 x - 2\cos^2 x - 2\sin^2 x) = 0$$

$$\Rightarrow \quad \cos x(\sin x\cos x + \cos^2 x - 2\sin^2 x) = 0$$

$$\Rightarrow \quad \cos x(\sin x\cos x + 2\cos^2 x - \cos^2 x - 2\sin^2 x) = 0$$

$$\Rightarrow \quad \cos x(\cos x(\sin x - \cos x) + 2(\cos^2 x - \sin^2 x)) = 0$$

$$\Rightarrow \quad \cos x(\cos x(\sin x - \cos x) + 2(\cos x - \sin x)(\cos x + \sin x)) = 0$$

$$\Rightarrow \quad \cos x(\cos x(\sin x - \cos x) - 2(\sin x - \cos x)(\cos x + \sin x)) = 0$$

$$\Rightarrow \quad \cos x(\sin x - \cos x)(\cos x - 2(\cos x + \sin x)) = 0$$

$$\Rightarrow \quad \cos x(\sin x - \cos x)^2 = 0$$

We can either have $\sin x - \cos x = 0$ => $x = \pi/4 \Rightarrow \theta = \pi/2$

or $\cos x = 0 \Rightarrow x = \pi/2 \Rightarrow \theta = \pi$

Numerical solution in MATLAB

```
% Define the function to be solved
f = @(x) sin(x)*cos(x) + 3*cos(x)^2 - 2;

% Use the fzero function to find the first root of the function
x0 = 0; % initial guess
x1 = fzero(f, x0);

% Display the first solution
fprintf('The first solution is x1 = %f radians\n', x1);
```

```
The first solution is x1 = -0.463648 radians
```

```
% Use the fzero function to find the second root of the function
x0 = pi/2; % initial guess
x2 = fzero(f, x0);

% Display the second solution
fprintf('The second solution is x2 = %f radians\n', x2);
```

```
The second solution is x2 = 0.785398 radians
```

From the numerical solutions

$x = -0.4636 \Rightarrow \theta = -0.9272$ radian

$x = 0.7853 \Rightarrow \theta = 1.5706 = \pi / 2$ radian

The correct angle θ is the one corresponding to the maximum $\sigma_{\theta\theta}$. Plotting

$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}}\left[\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2}\right] - \frac{K_{II}}{\sqrt{2\pi r}}\left[\frac{3}{4}\sin\frac{\theta}{2} + \frac{3}{4}\sin\frac{3\theta}{2}\right]$$

We are also given the information that $K_I = K_{II}$, so we need to maximize this quantity

$$\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2} - \frac{3}{4}\sin\frac{\theta}{2} - \frac{3}{4}\sin\frac{3\theta}{2}$$

Plugging the solutions above:

$$\theta = \pi \Rightarrow \frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2} - \frac{3}{4}\sin\frac{\theta}{2} - \frac{3}{4}\sin\frac{3\theta}{2} = 0$$

$$\theta = -0.9272 \Rightarrow \frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2} - \frac{3}{4}\sin\frac{\theta}{2} - \frac{3}{4}\sin\frac{3\theta}{2} = 1.788$$

$$\theta = \pi / 2 \Rightarrow \frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2} - \frac{3}{4}\sin\frac{\theta}{2} - \frac{3}{4}\sin\frac{3\theta}{2} = -0.707$$

Therefore, it shows that $\theta = -0.9272$ corresponds to a maximum in $\sigma_{\theta\theta}$, whereas $\theta = 1.5706$ corresponds to a minimum in $\sigma_{\theta\theta}$. Therefore, the crack will propagate along $\theta = -0.9272 = -53.12°$ (answer)