Fracture Mechanics

Basic notions of solid mechanics

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April 19, 2022

Reviewing notions of solid mechanics

There are a few important notions of solid mechanics to review before we tackle fracture mechanics. These include:

- Plane stress/strain
- Hooke's law
- Equilibrium equations
- Compatibility equations
- Airy stress functions

If this brief review is insufficient, refer to the book by Timoshenko and Goodier, *Theory of Elasticity*.

Plane stress

In continuum mechanics, 2D problems are usually treated as plane stress or plane strain.

Plane stress, which is used for **thin** plates, assumes that $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$ so the stress tensor becomes:

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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while the strains are:

$$\epsilon_{ij} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_{33} \end{bmatrix} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & 0 \\ \epsilon_{12} & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_{33} \end{bmatrix}$$

note that $\epsilon_{33} \neq 0$!

Plane strain

Plane strain is used for **thick** structures and assumes that $\epsilon_{13} = \epsilon_{23} = \epsilon_{33} = 0$. Therefore, the strain tensor becomes:

$$\epsilon_{ij} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_{33} \end{bmatrix} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & 0 \\ \epsilon_{12} & \epsilon_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

while the stress tensor is:

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}$$

where for a linear isotropic material $\sigma_{33} = \nu(\sigma_{11} + \sigma_{22})$, where ν is the Poisson's ratio.

Hooke's law

For an isotropic linear elastic material, the strain and stress components are related as:

$$\begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \epsilon_{xy} \\ \epsilon_{xz} \\ \epsilon_{yz} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2+2\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2+2\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2+2\nu \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{bmatrix}$$

where E is the Young's modulus and ν is the Poisson's ratio.

Equilibrium equations

In 2D, the stress field should always respect equilibrium equations, and these are:

Cartesian coordinates:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0$$
$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0$$

Polar coordinates:

$$\frac{\partial}{\partial r}(r\sigma_{rr}) + \frac{\partial \sigma_{r\theta}}{\partial \theta} - \sigma_{\theta\theta} = 0$$
$$\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial}{\partial r}(r\sigma_{r\theta}) + \sigma_{r\theta} = 0$$

The above equations assume that body forces are negligible.

Strains and compatibility equation

In 2D, and using cartesian coordinates, the strains are defined as:

$$\epsilon_{xx} = \frac{\partial u}{\partial x} \qquad \epsilon_{yy} = \frac{\partial v}{\partial y} \qquad \epsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

where u and v are the displacements in the x and y directions, respectively.

Strains and compatibility equation

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where u and v are the displacements in the x and y directions, respectively. The three strain components are not independent since they are defined from only two values of displacement. They are related via the compatibility equation:

$$\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} = \frac{\partial^2 \epsilon_{xy}}{\partial x \partial y}$$

This equation can also be expressed in polar coordinates... but it is messy.

Compatibility equation in terms of stress

For an isotropic linear elastic material loaded in plane stress, the strain and stress components are related as:

$$\epsilon_{xx} = \frac{1}{E}(\sigma_{xx} - \nu\sigma_{yy}) \qquad \epsilon_{yy} = \frac{1}{E}(\sigma_{yy} - \nu\sigma_{xx})$$

$$\epsilon_{xy} = \frac{1}{G}\sigma_{xy} = \frac{2(1+\nu)}{E}\sigma_{xy}$$

Subtituting these in the compatibility equation (and doing some algebra) returns:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_{xx} + \sigma_{yy}) = 0$$

This compatibility equation in terms of stresses is also valid for plane strain.

Airy stress function

Therefore, in 2D, a valid stress field as to respect both equilibrium and compatibility equations:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0$$
$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0$$
$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_{xx} + \sigma_{yy}) = 0$$

Finding three unknown functions $(\sigma_{xx}, \sigma_{yy}, \sigma_{xy})$ that respect these three equations is difficult; however, this can be made easier by introducing the Airy stress function.

Airy stress function

The Airy stress function $\phi(x, y)$ has no physical meaning, it is simply a mathematical trick. The function is related to the stress components as:

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} \qquad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} \qquad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

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It can be shown that equilibirum and compatibility equations are both respected when:

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \quad \Rightarrow \quad \nabla^4 \phi = 0$$

We have simplified the problem: there is now one function ϕ and a single equation.

Airy stress function

In polar coordinates, the Airy stress function $\phi(r,\theta)$ is related to the stresses as:

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} \quad \sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2} \quad \sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)$$

and the compatibility equation $\nabla^4 \phi = 0$ becomes:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}\right) = 0$$

Fracture Mechanics

1. The stress intensity factor

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April 18, 2023

Learning outcomes for this week

After this week, you will be able to:

- Explain how the stress intensity factor is derived.
- Understand how the stress intensity factor can be used to predict fracture.
- Use the stress intensity factor to solve engineering problems.

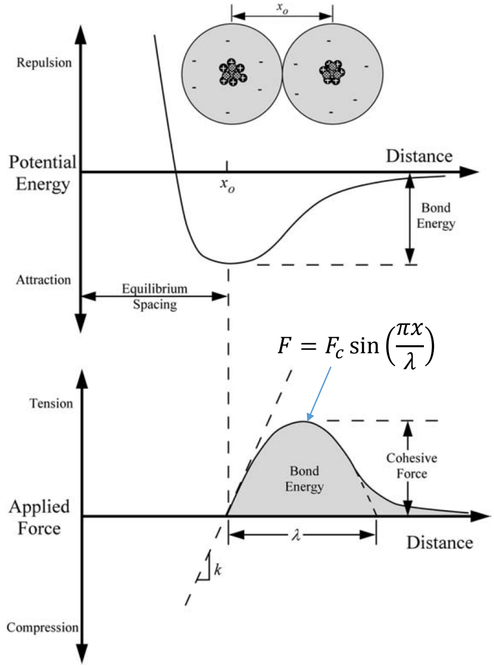
Outline

Outline

Let's begin with a question…

- How can we predict the strength of a material?
- Maybe a model based on interatomic forces could work.

Atomic perspective of fracture



The stiffness of the atomic bonds can be approximated as:

$$k = F_c \left(\frac{\pi}{\lambda}\right)$$

Dividing by the number of bonds per unit area gives:

$$\sigma_c = \frac{E\lambda}{\pi x_0} \approx \frac{E}{\pi}$$

Atomic perspective of fracture

This simple analysis predicts that the cohesive stress, *i.e.* the strength, of a material should be roughly:

$$\sigma_c \approx \frac{E}{\pi}$$

where E is the Young's modulus. Let's see if this is a good prediction. Glass has a Young's modulus $E = 68 - 74\,\mathrm{GPa}$, which gives us:

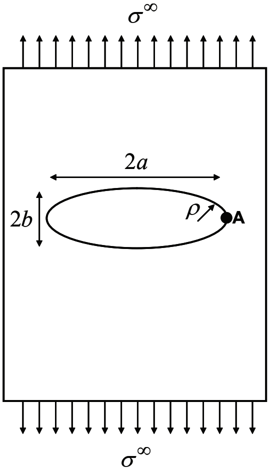
$$\sigma_c \approx \frac{E}{\pi} = 21 - 24\,\mathrm{GPa}$$

while the measured tensile strength is 0.045-0.155 GPa. Our prediction is too high by three orders of magnitudes!

Atomic perspective of fracture

- An atomic model significantly overpredicts the strength of materials.
- Conclusion: the strength of a material must be controlled by something else than atomic bonds.
- Hypothesis: materials might contain defects that lower their strength.
- Defects? these could be holes or cavities for example. Let's see what would be the effect of an elliptical hole.

Stress concentration around an elliptical hole



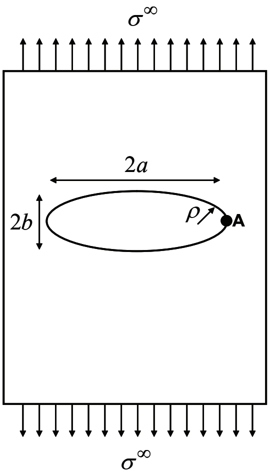
Inglis found that the stress concentration is:

$$\sigma_A = \sigma_{\max} = \sigma^\infty \left(1 + \frac{2a}{b}\right)$$

The hole resembles a sharp crack when $a \gg b$. It can be convenient to express this result as a function of the radius of curvature $\rho = b^2/a$, which gives:

$$\sigma_A = \sigma^\infty \left(1 + 2\sqrt{\frac{a}{\rho}}\right)$$

Stress concentration around an elliptical hole



With $\rho = b^2/a$, we found that:

$$\sigma_A = \sigma^\infty \left(1 + 2\sqrt{\frac{a}{\rho}}\right)$$
$$\Rightarrow \sigma_A \approx 2\sigma^\infty \sqrt{\frac{a}{\rho}}$$

This simple analysis shows that:

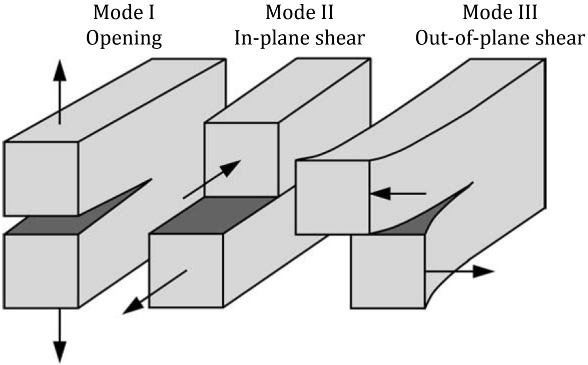
- σ_A scales as $\sigma^\infty \sqrt{a}$
- when $\rho \to 0$ we have $\sigma_A \to \infty$

Next, we will derive more precisely the stress field at the tip of a sharp crack.

Outline

Modes of crack loading

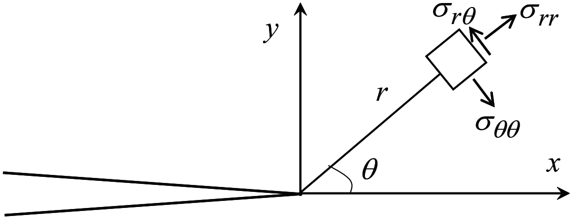
Three modes of loading can be applied to a crack:



A crack can be loaded by a single mode, or a combination of modes.

Stress field around a crack tip

Consider an infinitely large plate made from an isotropic linear elastic material. The plate contains a sharp crack which is loaded in mode I.



How does the stress field vary as $r \to 0$?

Williams (1957) found an analytical solution to this problem using Airy stress functions.

Stress field around a crack tip

Consider the following Airy stress function in polar coordinates:

$$\phi(r,\theta) = r^{(\lambda+1)} f(\theta)$$

To satisfy equilibrium and compatibility equations, the Airy stress function has to respect:

$$\nabla^4 \phi = 0 \quad \Rightarrow \quad \frac{d^4 f}{d\theta^4} + 2(\lambda^2+1)\frac{d^2 f}{d\theta^2} + (\lambda^2-1)^2 f = 0$$

This differential equation has the following solution:

$$f(\theta) = A\cos(\lambda-1)\theta + B\sin(\lambda-1)\theta + C\cos(\lambda+1)\theta + D\sin(\lambda+1)\theta$$

where A, B, C and D are unknown constants.

Stress field around a crack tip

Now that the Airy stress function $\phi = r^{(\lambda+1)} f(\theta)$ is known, we can write the stresses:

$$
\begin{aligned}
\sigma_{\theta\theta} &= \frac{\partial^2 \phi}{\partial r^2} \\
&= (\lambda+1)\lambda r^{\lambda-1}[A\cos(\lambda-1)\theta + B\sin(\lambda-1)\theta \ldots \\
&\qquad\qquad\qquad\qquad + C\cos(\lambda+1)\theta + D\sin(\lambda+1)\theta] \\
\sigma_{r\theta} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right) \\
&= \lambda r^{\lambda-1}[A(\lambda-1)\sin(\lambda-1)\theta - B(\lambda-1)\cos(\lambda-1)\theta \ldots \\
&\qquad\quad + C(\lambda+1)\sin(\lambda+1)\theta + D(\lambda+1)\cos(\lambda+1)\theta]
\end{aligned}
$$

We can simplify these expressions using the boundary conditions.

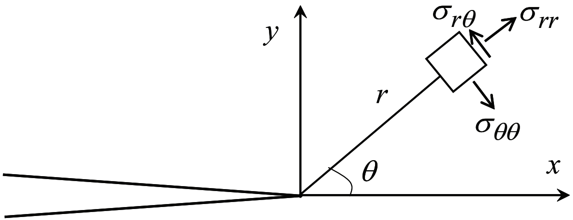
Stress field around a crack tip

The boundary conditions are such that the crack surfaces are stress-free:

$$\sigma_{\theta\theta}(\theta = \pi) = \sigma_{r\theta}(\theta = \pi) = 0$$

For mode I loading, the stresses should be symmetric in θ, meaning:

$$\sigma_{\theta\theta}(\theta) = \sigma_{\theta\theta}(-\theta)$$



Stress field around a crack tip

Previously, we found that:

$$\begin{aligned}\sigma_{\theta\theta} = (\lambda+1)\lambda r^{\lambda-1}[A\cos(\lambda-1)\theta + B\sin(\lambda-1)\theta \ldots \\ + C\cos(\lambda+1)\theta + D\sin(\lambda+1)\theta]\end{aligned}$$

To ensure that $\sigma_{\theta\theta}$ is an even function, the sin terms have to disappear, which implies that:

$$B = D = 0$$

With this result, the stresses now become:

$$\begin{aligned}\sigma_{\theta\theta} &= (\lambda+1)\lambda r^{\lambda-1}[A\cos(\lambda-1)\theta + C\cos(\lambda+1)\theta] \\ \sigma_{r\theta} &= \lambda r^{\lambda-1}[A(\lambda-1)\sin(\lambda-1)\theta + C(\lambda+1)\sin(\lambda+1)\theta]\end{aligned}$$

Stress field around a crack tip

Next, the boundary conditions $\sigma_{\theta\theta} = \sigma_{r\theta} = 0$ on $\theta = \pi$ imply:

$$\sigma_{\theta\theta}(\theta = \pi) = A\cos(\lambda - 1)\pi + C\cos(\lambda + 1)\pi = 0$$
$$\sigma_{r\theta}(\theta = \pi) = A(\lambda - 1)\sin(\lambda - 1)\pi + C(\lambda + 1)\sin(\lambda + 1)\pi = 0$$

which can be written in matrix form:

$$\begin{bmatrix} \cos(\lambda - 1)\pi & \cos(\lambda + 1)\pi \\ (\lambda - 1)\sin(\lambda - 1)\pi & (\lambda + 1)\sin(\lambda + 1)\pi \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Nontrivial solutions exist when the determinant of the matrix is zero.

$$(\lambda + 1)[\sin(\lambda + 1)\pi][\cos(\lambda - 1)\pi] - (\lambda - 1)[\sin(\lambda - 1)\pi][\cos(\lambda + 1)\pi] = 0$$

Stress field around a crack tip

We had:

$$(\lambda+1)[\sin(\lambda+1)\pi][\cos(\lambda-1)\pi]-(\lambda-1)[\sin(\lambda-1)\pi][\cos(\lambda+1)\pi]=0$$

which has the following solutions:

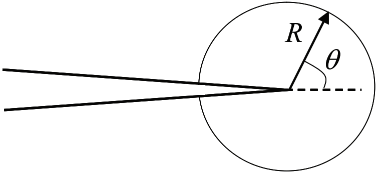
$$\sin(2\pi\lambda)=0 \quad \Rightarrow \quad \lambda=\ldots,-1,-\frac{1}{2},0,\frac{1}{2},1,\ldots$$

For each λ, a corresponding ratio A/C can be found, but not the absolute value of A and C (this is an eigenvalue/vector problem).

Note that there are physical restrictions on λ...

Stress field around a crack tip

One restriction is that the total energy W in a region near the crack tip should be finite:



$$W = \int_A \frac{1}{2}\frac{\sigma^2}{E} dA = \int_0^{2\pi}\int_0^R \frac{1}{2}\frac{\sigma^2}{E} r dr d\theta < \infty$$

where E is the Young's modulus. The stress scales as $\sigma \propto r^{\lambda-1}$, and therefore:

$$W \propto \int_0^R r^{2(\lambda-1)} r dr = \left| \frac{r^{2\lambda}}{2\lambda} \right|_{r=0}^{R} < \infty$$

And this implies that $\lambda > 0$.

Stress field around a crack tip

Therefore, the Airy stress function for mode I is:

$$\phi = r^{(\lambda+1)} f(\theta) = \sum r^{(\lambda_i+1)} \left[A_i \cos(\lambda_i - 1)\theta + C_i \cos(\lambda_i + 1)\theta\right]$$

where $\lambda_i = \frac{1}{2}, 1, \frac{3}{2}, \ldots$ and the ratio A_i/C_i is fixed for each value of λ_i. Finally, the stresses are obtained directly from:

$$\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2}$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)$$

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

Stress field around a crack tip

which gives us the following stresses for mode I:

$$\sigma_{\theta\theta} = \frac{A}{4} r^{-1/2} \left(3\cos\frac{\theta}{2} + \cos\frac{3\theta}{2} \right) + \mathcal{O}(r^0) + \mathcal{O}(r^{1/2}) + \dots$$
$$\sigma_{r\theta} = \frac{A}{4} r^{-1/2} \left(\sin\frac{\theta}{2} + \sin\frac{3\theta}{2} \right) + \dots$$
$$\sigma_{rr} = \frac{A}{4} r^{-1/2} \left(5\cos\frac{\theta}{2} - \cos\frac{3\theta}{2} \right) + \dots$$

- The constant C was removed since A/C is fixed for each value of λ.
- The higher order terms $\mathcal{O}(r^0), \mathcal{O}(r^{1/2}), \dots$ are needed to satisfy remote boundary conditions, but their contribution is negligible at the crack tip ($r \to 0$).

Stress field around a crack tip

Neglecting higher order terms and setting $A = \frac{K_I}{\sqrt{2\pi}}$, the equations become:

$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2}\right)$$

$$\sigma_{r\theta} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}\right)$$

$$\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{5}{4}\cos\frac{\theta}{2} - \frac{1}{4}\cos\frac{3\theta}{2}\right)$$

- K_I is called the stress intensity factor. The subscript I refers to mode I loading.
- This analysis does not give us the value of K_I. To find K_I, we need a specific boundary value problem.
- The magnitude of the stresses scales linearly with K_I.

Displacement field around a crack tip

Next, the strain components ϵ_{rr}, $\epsilon_{\theta\theta}$ and $\epsilon_{r\theta}$ can be obtained from Hooke's law.

Finally, we can integrate the strains to obtain the displacement field:

$$u_\theta = \frac{K_I}{4G}\sqrt{\frac{r}{2\pi}}\left[-(2\kappa+1)\sin\frac{\theta}{2} + \sin\frac{3\theta}{2}\right]$$
$$u_r = \frac{K_I}{4G}\sqrt{\frac{r}{2\pi}}\left[(2\kappa-1)\cos\frac{\theta}{2} - \cos\frac{3\theta}{2}\right]$$

$$\text{where} \quad \kappa = \begin{cases} \frac{3-\nu}{1+\nu} & \text{for plane stress} \\ 3-4\nu & \text{for plane strain} \end{cases}$$

and G is the shear modulus and ν is the Poisson's ratio.

Stress field around a crack tip

Let's look at $\sigma_{\theta\theta}$ along $\theta = 0$:

$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}} \left(\frac{3}{4} \cos \frac{\theta}{2} + \frac{1}{4} \cos \frac{3\theta}{2} \right) \quad \Rightarrow \quad \sigma_{\theta\theta}(\theta = 0) = \frac{K_I}{\sqrt{2\pi r}}$$

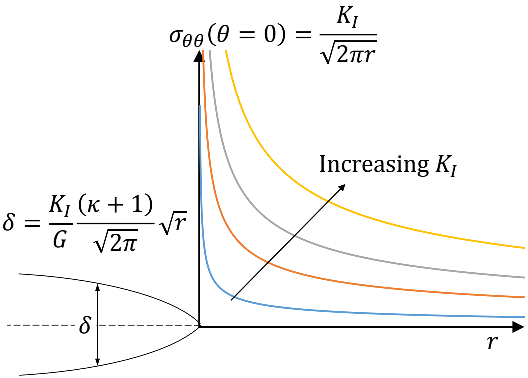
Otherwise, the crack opening profile δ can be obtained from the displacement field:

$$u_\theta = \frac{K_I}{4G} \sqrt{\frac{r}{2\pi}} \left[-(2\kappa + 1) \sin \frac{\theta}{2} + \sin \frac{3\theta}{2} \right]$$

$$\Rightarrow \quad \delta = -2u_\theta(\theta = \pi) = \frac{K_I}{G} \frac{(\kappa + 1)}{\sqrt{2\pi}} \sqrt{r}$$

Those quantities are illustrated next.

Stress field around a crack tip



- $\sigma_{\theta\theta}$ is linearly proportional to K_I.
- K_I needs to have units of Pa$\cdot\sqrt{\text{m}}$.
- Crack tip singularity: $\sigma_{\theta\theta} \to \infty$ when $r \to 0$.
- The opening profile δ is parabolic.

Stress field around a crack tip

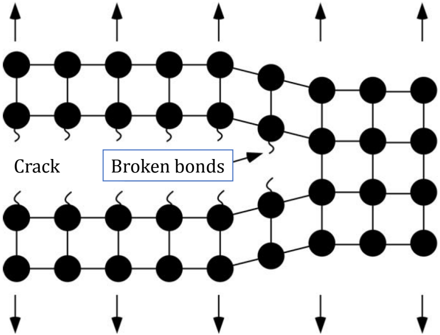
This analysis can be reapeated to find the stress and displacement fields for mode II and mode III loading. The results for all modes are summarised in the datasheet.

A few important questions remain unanswered:

- Stresses tend to infinity at $r = 0$, how is this possible?
- How can we compute the stress intensity factor K_I?
- When will the crack propagate?

Why stresses tend to infinity at $r = 0$?

In our modelling approach the stresses tend to infinity at $r = 0$ because we have considered the crack to be infinitely sharp, meaning that the radius of curvature $\rho \to 0$.



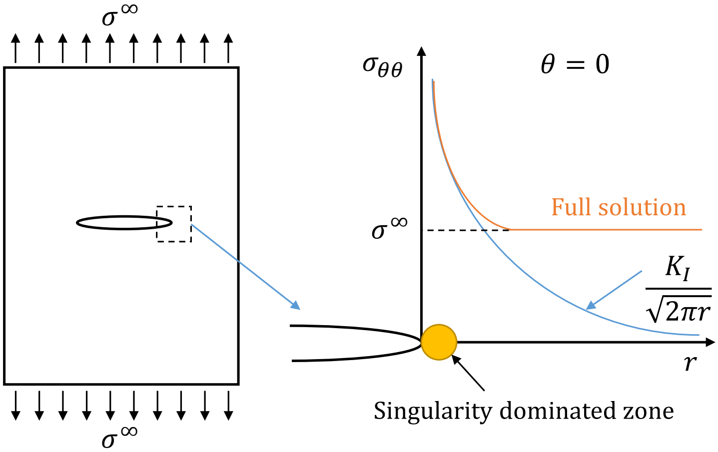
In reality, the crack tip has a non-zero curvature. For example, the radius of curvature has to be equal or greater than the spacing between atoms.

Computing the stress intensity factor

Finding the stress intensity factor is important because this single-parameter completely defines the stresses, strains and displacements close to the crack tip.

$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2}\right)$$
$$\sigma_{r\theta} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}\right)$$
$$\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{5}{4}\cos\frac{\theta}{2} - \frac{1}{4}\cos\frac{3\theta}{2}\right)$$
$$u_{\theta} = \frac{K_I}{4G}\sqrt{\frac{r}{2\pi}}\left[-(2\kappa+1)\sin\frac{\theta}{2} + \sin\frac{3\theta}{2}\right]$$
$$u_{r} = \frac{K_I}{4G}\sqrt{\frac{r}{2\pi}}\left[(2\kappa-1)\cos\frac{\theta}{2} - \cos\frac{3\theta}{2}\right]$$

Local and global behaviours



How does K_I vary with the remote applied stress σ^∞ and the geometry?

Examples of stress intensity factors

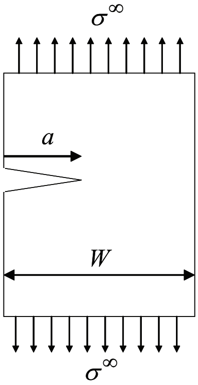
For a given geometry, the relation between the stess intensity factor and the applied load can be found:

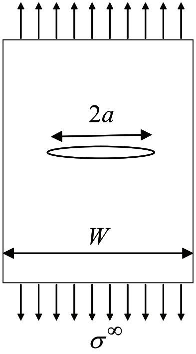
analytically for simple geometries. Results are summarised in many textbooks.

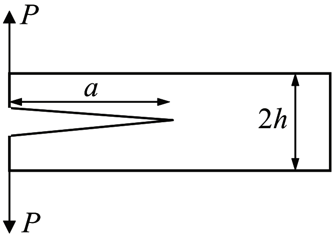
numerically for more complex geometries. This generally requires the use of the Finite Element Method.

We will derive analytical solutions next week, but for now, let's look at a few existing solutions.

Examples of stress intensity factors







$W >> a$

$K_I = 1.122\sigma^{\infty}\sqrt{\pi a}$

$K_I = Y\sigma^{\infty}\sqrt{\pi a}$

$Y \approx \left(\cos\frac{\pi a}{W}\right)^{-1/2}$

$K_I = 2\sqrt{3}Pah^{-3/2}$

More configurations are included in the datasheet.

When will a crack propagate?

Irwin (1957) postulated that fracture occurs when K_I reaches a critical value K_{Ic}, which is a material property called fracture toughness.

Material	K_{Ic} (MPa$\sqrt{\text{m}}$)
Low carbon steel alloys	40-80
Aluminum alloys	22-35
Titanium alloys	14-120
Wood (best orientation)	5-9
PMMA	0.7-1.6
Glass	0.6-0.8
Concrete	0.35-0.45

Testing methods to measure the fracture toughness will be covered later during the course.

Outline

Principle of superposition

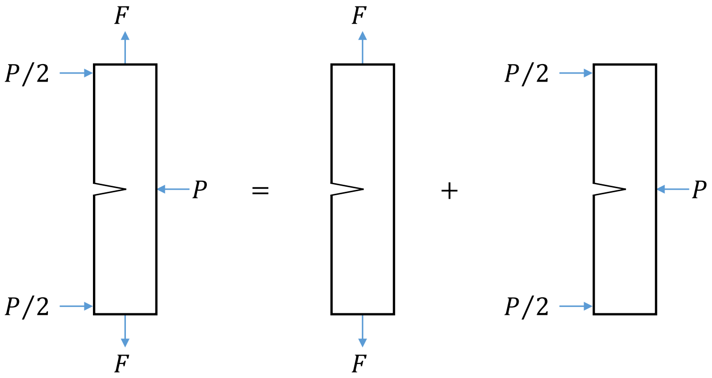
For linear elastic materials, individual components of stress, strain and displacement are additive. Likewise, the stress intensity factors are additive, as long as the mode of loading is the same:

$$K_I^{(\mathrm{total})} = K_I^{(\mathrm{A})} + K_I^{(\mathrm{B})} + K_I^{(\mathrm{C})} + \ldots$$

The stress intensity factors for different modes of loading should never be added together:

$$K_{(\mathrm{total})} \neq K_I + K_{II} + K_{III}$$

Principle of superposition: example



$$K_I^{(\text{total})} = K_I^{(\text{tension})} + K_I^{(\text{bending})}$$

Both $K_I^{(\text{tension})}$ and $K_I^{(\text{bending})}$ can be obtained from a handbook of stress intensity factors.

Summary

The stress intensity factor:

- quantifies the stress field at the crack tip,
- follows the principle of superposition for a given loading mode,
- can be used to predict fracture.

Fracture Mechanics

2. Energy release rate

Luc St-Pierre

April 18, 2023

Motivation

- Last week, we investigated fracture with an approach based on stress.
- We found that fracture occurs when the stress intensity factor K_I reaches a critical value, the fracture toughness K_{Ic}.
- This week, we will see how fracture can be analysed with an energy approach, and we will see how the energy and stress approaches are related.

Learning outcomes

After this week, you will be able to:

- Derive the energy release rate for simple geometries.
- Relate the energy release rate G to the stress intensity factor K_I.
- Predict when crack growth will be stable/unstable.

Outline

Energy release rate G

Relation between K and G

R-curves and stability

Outline

Energy release rate G

Relation between K and G

R-curves and stability

Energy balance

Griffith (1920) realised that there was a certain amount of energy consumed in extending a crack. The energy balance can be written in incremental form as:

$$\delta W = \delta U + G\delta A$$

where	W	is the work done by external forces;
	U	is the strain energy of the deformable body;
	G	is the energy release rate with units of $\mathrm{J/m^2}$;
	δA	is the crack extension in $\mathrm{m^2}$.

Therefore, when the crack advances by δA, the energy released is $G\delta A$.

Energy release rate

A definition for the energy release rate G can be obtained from the energy balance:

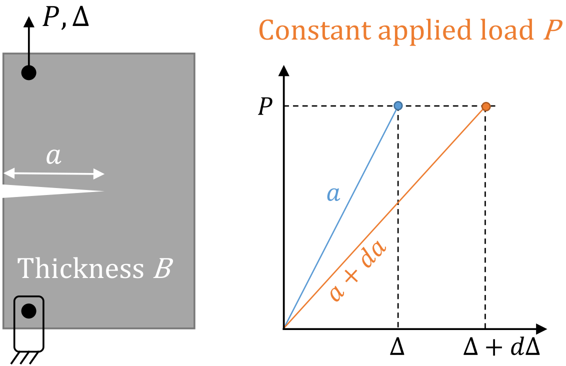
$$\delta W = \delta U + G\delta A \qquad \Rightarrow \qquad G = -\frac{\delta}{\delta A}(U - W)$$

Rewritting this in differential form gives:

$$G = -\frac{\partial \Pi}{\partial A} = -\frac{\partial}{\partial A}(U - W)$$

where $\Pi = U - W$ is called the potential energy of an elastic body. Let's see how we can use this formula.

Energy release rate - Load control

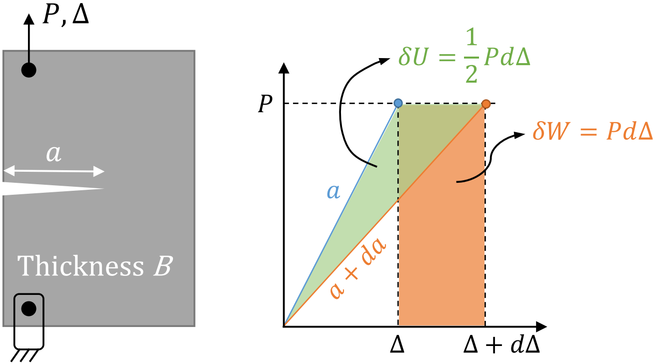


Let's introduce the compliance (inverse of stiffness):

$$C = \frac{\Delta}{P} \qquad \Rightarrow \qquad d\Delta = PdC$$

As the crack grows, the compliance C increases.

Energy release rate - Load control



From the energy balance $\delta W = \delta U + G\delta A$, we can write:

$$G = \frac{(\delta W - \delta U)}{Bda} = \frac{1}{Bda}\left(\frac{1}{2}Pd\Delta\right) = \frac{P^2}{2B}\frac{dC}{da}$$

Energy release rate - load control

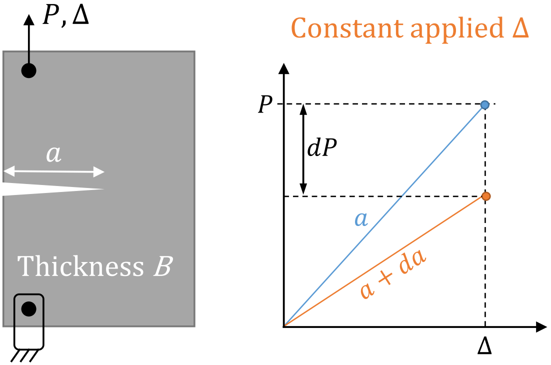
The same result can be obtained algebraically. The work done by external forces W, the strain energy U and the potential energy Π are given by:

$$W = P\Delta; \qquad U = \frac{1}{2}P\Delta; \qquad \Pi = U - W = -\frac{1}{2}P\Delta$$

respectively. The energy release rate is easily obtained by:

$$\begin{aligned}
G &= -\frac{\partial \Pi}{\partial A} && \text{where } \partial A = B\partial a \\
&= \frac{1}{B}\frac{\partial}{\partial a}\left(\frac{1}{2}P\Delta\right) && \text{where } \Delta = CP \\
&= \frac{1}{B}\frac{\partial}{\partial a}\left(\frac{1}{2}CP^2\right) \\
&= \frac{P^2}{2B}\frac{\partial C}{\partial a} && \text{since } P \text{ is constant}
\end{aligned}$$

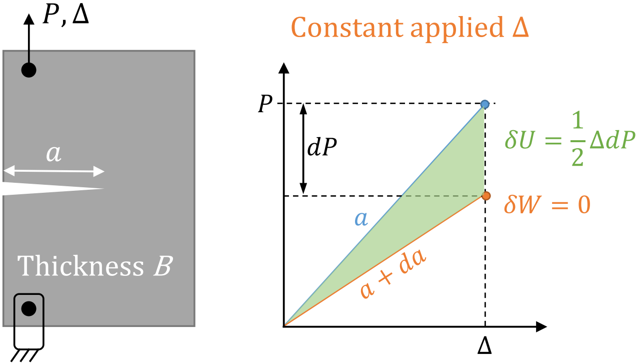
Energy release rate - displacement control



Since Δ is constant, we obtain:

$$C = \frac{\Delta}{P} \qquad \Rightarrow \qquad dC = \Delta \left(-\frac{1}{P^2} dP \right)$$

Energy release rate - displacement control



From the energy balance $\delta W = \delta U + G\delta A$, we can write:

$$G = \frac{(\delta W - \delta U)}{Bda} = -\frac{1}{Bda}\left(\frac{1}{2}\Delta dP\right) = \frac{P^2}{2B}\frac{dC}{da}$$

Energy release rate

In conclusion, we can use:

$$G = \frac{P^2}{2B}\frac{dC}{da}$$

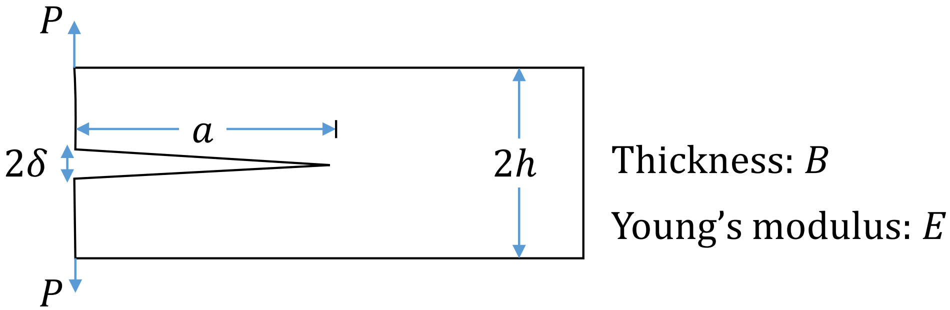
to compute G from experiments since the load P and the change in compliance $\frac{dC}{da}$ are both easy to measure.

Fracture will occur when:

$$G = G_c$$

where G_c is a material property called toughness. Both G and K are equivalent parameters: they quantify the loading intensity at the crack tip. We will see later how G and K are related.

Energy release rate: an example



Calculate the energy release rate G for the double cantilever beam shown here under an applied load P. Recall that from beam theory we have:

$$\delta = \frac{Pa^3}{3EI} \qquad \text{where} \qquad I = \frac{Bh^3}{12}$$

Energy release rate: an example

The work done by external forces W:

$$W = 2P\delta = \frac{8a^3P^2}{EBh^3} \qquad \text{where } \delta = \frac{Pa^3}{3EI} \text{ and } I = \frac{Bh^3}{12}$$

There is a factor of 2 because the total displacement is 2δ. The strain energy U is:

$$U = 2 \cdot \frac{1}{2} P\delta = \frac{4a^3P^2}{EBh^3}$$

The potential energy and the energy release rate are:

$$\Pi = U - W = -\frac{4a^3P^2}{EBh^3}$$

$$G = -\frac{\partial \Pi}{B \partial a} = \frac{1}{B}\frac{\partial}{\partial a}\left[\frac{4a^3P^2}{EBh^3}\right] = \frac{12P^2a^2}{EB^2h^3}$$

Energy release rate: an example

The same result can be obtained using the compliance method. Define:

$$C = \frac{2\delta}{P} = \frac{8a^3}{EBh^3} \qquad \text{where } \delta = \frac{Pa^3}{3EI} \text{ and } I = \frac{Bh^3}{12}$$

The energy release rate is:

$$G = \frac{P^2}{2B}\frac{\partial C}{\partial a} = \frac{P^2}{2B}\frac{\partial}{\partial a}\left[\frac{8a^3}{EBh^3}\right] = \frac{12P^2a^2}{EB^2h^3}$$

which is the same result that we obtained previously. Thus, fracture initiates when the applied load is:

$$P_c = \sqrt{\frac{EB^2h^3G_c}{12a^2}}$$

Energy release rate: an example

It is also possible to express G as a function of an applied displacement δ. We had:

$$G = \frac{12P^2a^2}{EB^2h^3} \qquad \text{where } \delta = \frac{Pa^3}{3EI} \text{ and } I = \frac{Bh^3}{12}$$

Substituting for the load P gives:

$$G = \frac{12a^2}{EB^2h^3}\left(\frac{EBh^3\delta}{4a^3}\right)^2 = \frac{3Eh^3\delta^2}{4a^4}$$

Therefore, the critical displacement causing fracture is:

$$\delta_c = a^2\sqrt{\frac{4G_c}{3Eh^3}}$$

Outline

Energy release rate G

Relation between K and G

R-curves and stability

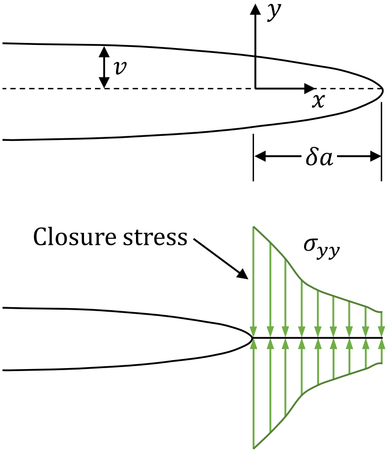
Relation between K and G

So far, we have introduced two parameters to predict fracture:

- the local stress intensity factor K and
- the global energy release rate G.

For linear elastic materials, these two parameters are uniquely related as we will demonstrate next.

Relation between K and G



Consider a crack of length $a + \delta a$ subjected to mode I loading.

Let's now close the crack, over a length δa, by applying a compressive stress σ_{yy}.

The work done to close the crack δU is related to G as:

$$G = \frac{\delta U}{B \delta a}$$

Relation between K and G

The work done to close the crack is given by:

$$\delta U = 2B \int_0^{\delta a} \frac{1}{2} \sigma_{yy}(x) v(x) dx$$

Both σ_{yy} and v can be obtained from the datasheet. For plane strain we have:

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi x}} \qquad \text{and} \qquad v = \frac{4(1-\nu^2)}{E} K_I \sqrt{\frac{\delta a - x}{2\pi}}$$

Substituting above gives:

$$\delta U = \frac{2B(1-\nu^2)}{\pi E} K_I^2 \int_0^{\delta a} \sqrt{\frac{\delta a - x}{x}} dx$$

which can be integrated with a change of variable: $x = \delta a \sin^2 u$.

Relation between K and G

After integration, we obtain:

$$\delta U = \frac{BK_I^2(1-\nu^2)}{E}\delta a$$

and the energy release rate becomes:

$$G = \frac{\delta U}{B\delta a} = \frac{K_I^2(1-\nu^2)}{E}$$

Therefore, for mode I under plane strain we have:

$$K_I^2 = \frac{EG_I}{1-\nu^2}$$

Relation between K and G

In summary, for mode I we have:

$$K_I^2 = \frac{E}{1-\nu^2} G_I \qquad \text{for plane strain}$$

$$K_I^2 = E G_I \qquad \text{for plane stress}$$

Repeating this procedure for mode II gives:

$$K_{II}^2 = \frac{E}{1-\nu^2} G_{II} \qquad \text{for plane strain}$$

$$K_{II}^2 = E G_{II} \qquad \text{for plane stress}$$

and for mode III we get:

$$K_{III}^2 = 2\mu G_{III}$$

where μ is the shear modulus.

A note on G under mixed mode loading

If the above anlysis is reapeated for mixed mode loading, we would find that for plane stress:

$$G = \frac{K_I^2}{E} + \frac{K_{II}^2}{E} + \frac{K_{III}^2}{2\mu} = G_I + G_{II} + G_{III}$$

However, we saw earlier that the stress intensity factor is not additive:

$$K_{\text{(total)}} \neq K_I + K_{II} + K_{III}$$

The energy release rate G, like energy, is a scalar and therefore the total G is the sum of all three modes. On the other hand, the stress intensity factor K is related to the stress tensor (not a scalar) where only individual components can be added together.

Outline

Energy release rate G

Relation between K and G

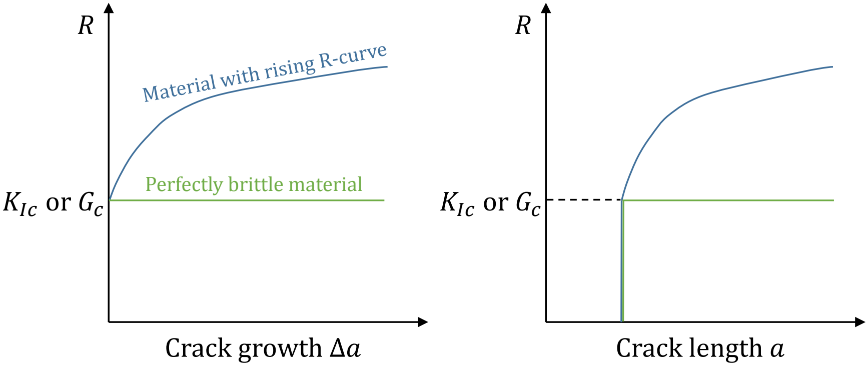
R-curves and stability

Resistance curve

- So far, we have considered the toughness G_c or fracture toughness K_{Ic} to be constant material properties.
- This is not always the case; the toughness can increase with crack propagation.
- A plot of toughness as a function of crack growth is called a Resistance curve (or R-curve).

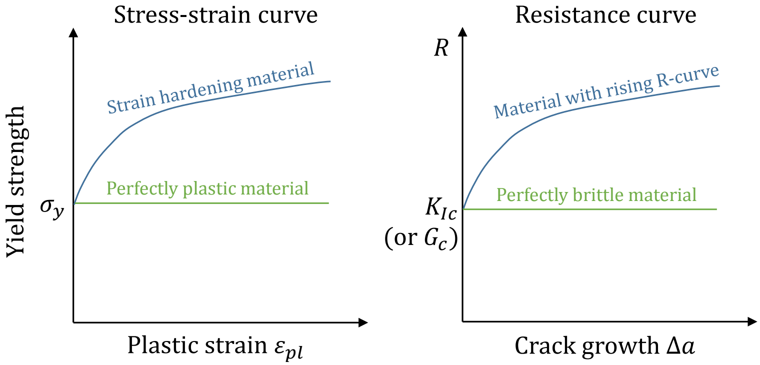
Resistance curve

Materials can have a rising R-curve because of plasticity or other phenomena that we will cover later.



Resistance curve

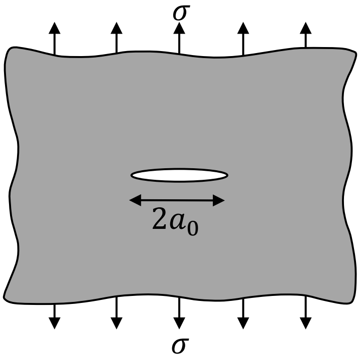
The concept of a resistance curve is similar to the stress-strain curve used to characterised plasticity.

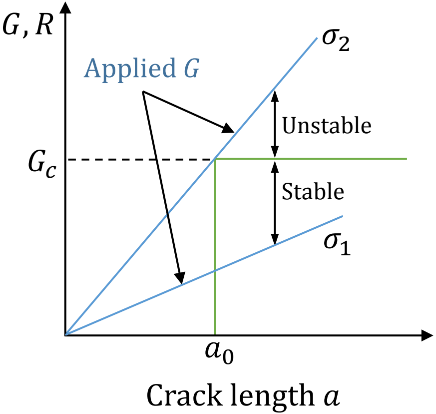


Instability: flat R-curve

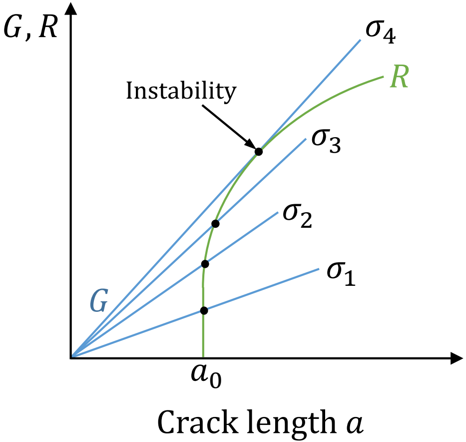
For a large plate with a through crack under plane stress, we have:

$$G = \frac{K_I^2}{E} = \frac{\pi\sigma^2 a}{E}$$





Instability: rising R-curve



- Stable crack growth for σ_2 and σ_3.
- Crack growth becomes unstable when $\sigma \geq \sigma_4$

For σ_2 and σ_3, the crack grows a small amount and stops since the resistance R increases at a faster rate than G.

Conditions for stability/instability

The conditions for stable crack growth can be expressed as:

$$G = R \qquad \text{and} \qquad \frac{dG}{da} \leq \frac{dR}{da}$$

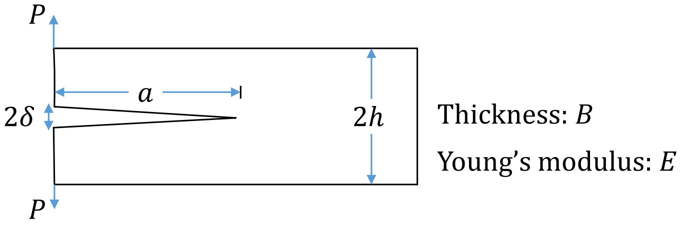
It follows that unstable crack growth occurs when:

$$G \geq R \qquad \text{and} \qquad \frac{dG}{da} > \frac{dR}{da}$$

These conditions hold true if the energy release rate G is replaced by the stress intensity factor K.

Load control vs displacement control

Let's investigate the crack growth stability for the double cantilever beam specimen under both load and displacement control. Assume that the material has a flat R-curve.



Load control vs displacement control

Under load control, the energy release rate is:

$$G = \frac{12P^2a^2}{EB^2h^3} \qquad \Rightarrow \qquad \frac{dG}{da} = \frac{24P^2a}{EB^2h^3} > 0$$

Therefore crack growth will be unstable if the material has a flat R-curve ($\frac{dR}{da} = 0$).

Otherwise, under displacement control, the energy release rate is:

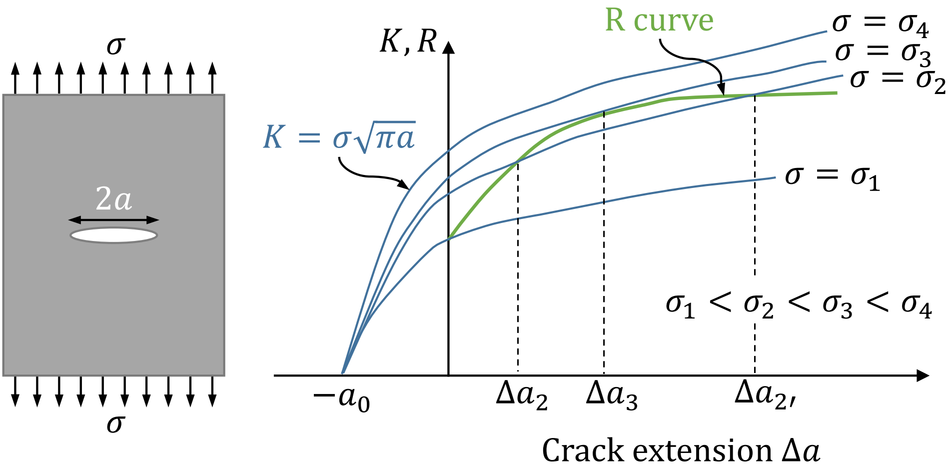
$$G = \frac{3Eh^3\delta^2}{4a^4} \qquad \Rightarrow \qquad \frac{dG}{da} = -\frac{3Eh^3\delta^2}{a^5} < 0$$

Even if the material has a flat R-curve, crack growth is stable under displacement control.

Load control vs displacement control

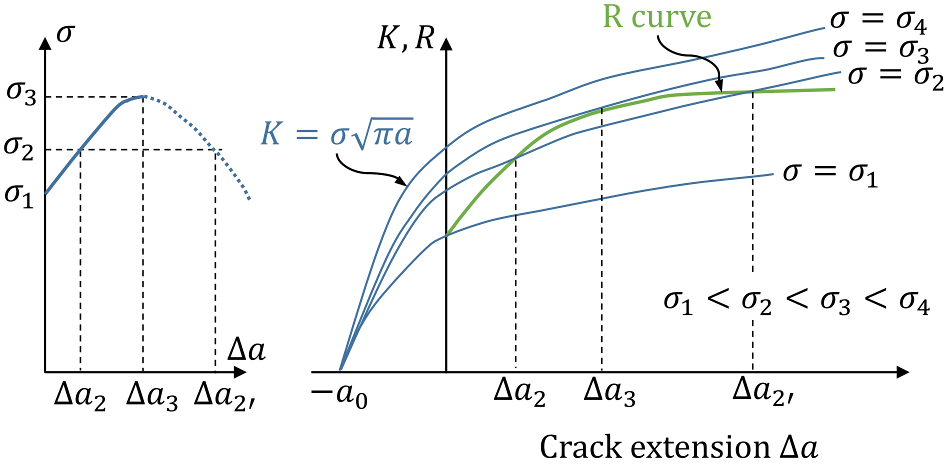
- In general, displacement control tends to be more stable then load control.
- This is convenient since most testing machines work in displacement control rather than load control.
- The geometry of most standardised test specimens is such that they exhibit stable crack growth under displacement control. This allows us to measure the R curve of the material.
- We will discuss testing methods later in the course.

Estimating the amount of stable crack extension



The crack will start to grow at σ_1, up to the maximum stress σ_3 resulting in a stable crack extension of Δa_3.

Estimating the amount of stable crack extension



From the R curve we can construct the σ vs Δa response. The first branch, up to the peak stress σ_3, will result in stable crack propagation. The decreasing branch (dashed line) is unstable under load control, and can only be obtained under displacement control.

Summary

The energy release rate G:

- can be used to predict fracture, and
- is related to the stress intensity factor.

We saw the conditions leading to stable/unstable crack growth.

The R-curve of is a material property and can be used to predict the amount of stable crack growth.

Fracture Mechanics

3. Mixed-mode loading

Luc St-Pierre

April 18, 2023

Learning outcomes for this week

After this week, you will be able to:

- Calculate the stress intensity factors K_I and K_{II} for mixed-mode loading.
- Predict when fracture will occur.
- Find the direction of crack propagation.

Outline

Evaluating K_I and K_{II}

When will fracture occur?

In which direction will the crack propagate?

Outline

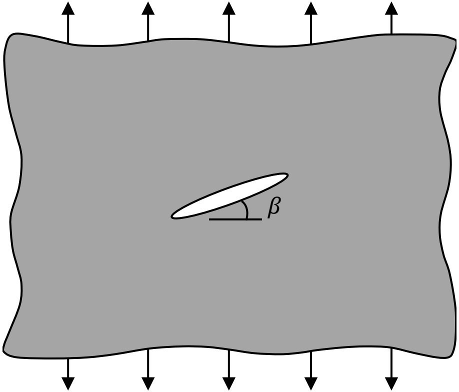
Evaluating K_I and K_{II}

When will fracture occur?

In which direction will the crack propagate?

Mixed-mode loading

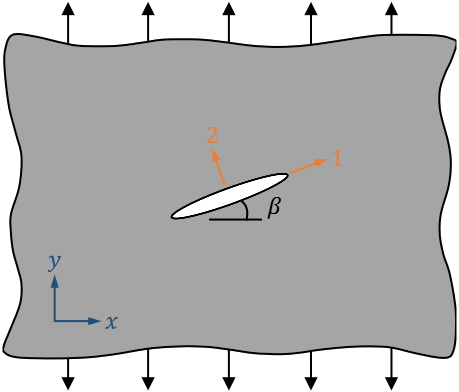
Often, cracks are at a certain angle from the loading direction. This means that the crack is loaded in modes I and II simultaneously.



To find K_I and K_{II}, we need to use Mohr's circle to express the stress components in a reference frame aligned with the crack plane.

Mohr's circle: procedure

Follow these three steps to create and use Mohr's circle:

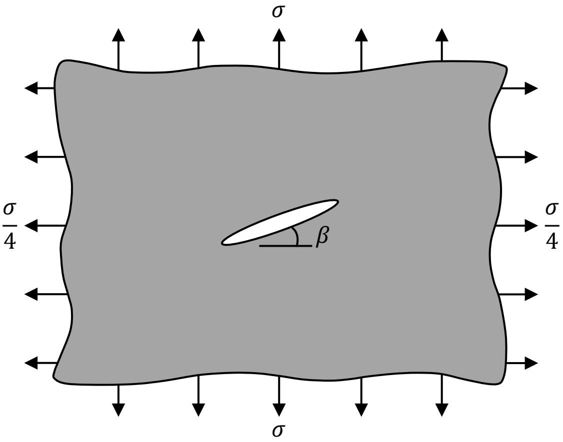


1. Write the stress components in the global reference frame (x, y).
   - Three components: σ_{xx}, σ_{yy}, and σ_{xy}.
   - Be careful with signs: tension/compression.
2. Draw the circle using two points on σ vs τ axes.
   - First point: $(\sigma_{xx}; -\sigma_{xy})$.
   - Second point: $(\sigma_{yy}; \sigma_{xy})$.
3. Rotate clockwise by 2β to find the local stress components: σ_{11}, σ_{22}, and σ_{12}.

Finally, we can calculate K_I with σ_{22}, and K_{II} using σ_{12}.

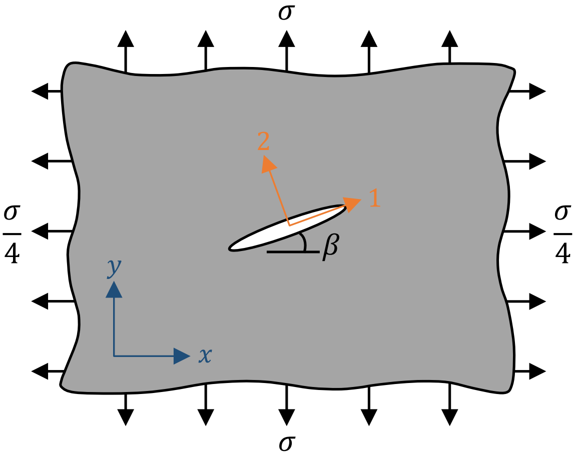
Example problem

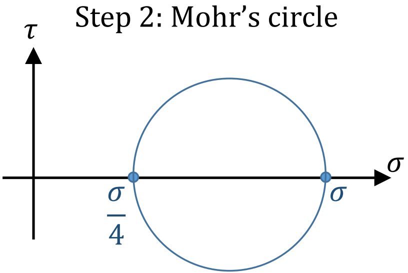
Find the stress intensity factors K_I and K_{II} for the biaxial loading case shown below, where the total crack length is $2a$. Express your results as functions of σ, β, and a.



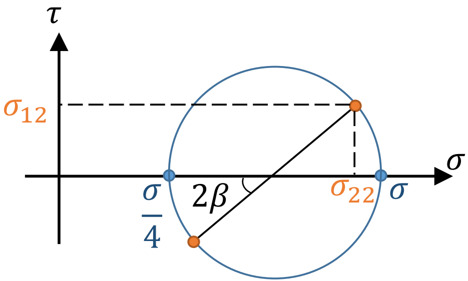
Example problem: solution

Step 1: the stress field in the (x, y) frame is $\sigma_{xx} = \sigma/4$, $\sigma_{yy} = \sigma$, and $\sigma_{xy} = 0$. Then, we can plot Mohr's circle with two points: $(\sigma_{xx}; -\sigma_{xy}) = (\sigma/4; 0)$ and $(\sigma_{yy}; \sigma_{xy}) = (\sigma, 0)$.





Example problem: solution



This circle has a radius:

$$r = \frac{\sigma - \sigma/4}{2} = \frac{3\sigma}{8}$$

and a centre:

$$c = \frac{\sigma}{4} + \frac{3\sigma}{8} = \frac{5\sigma}{8}$$

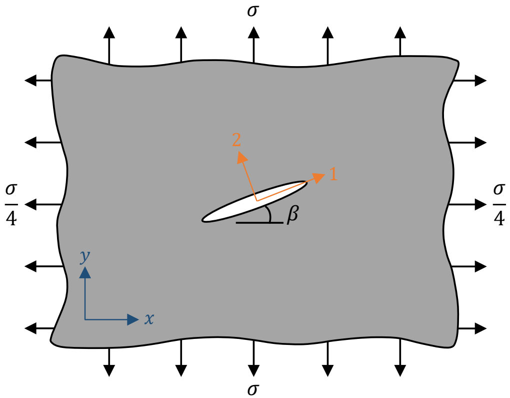
Then, the stress field in the (1,2) frame is:

$$\sigma_{11} = c - r\cos 2\beta = \frac{\sigma}{8}(5 - 3\cos 2\beta) \qquad \sigma_{22} = c + r\cos 2\beta = \frac{\sigma}{8}(5 + 3\cos 2\beta)$$

and

$$\sigma_{12} = r\sin 2\beta = \frac{3\sigma}{8}\sin 2\beta$$

Example problem: solution



We found:

$$\sigma_{11} = \frac{\sigma}{8}(5 - 3\cos 2\beta)$$
$$\sigma_{22} = \frac{\sigma}{8}(5 + 3\cos 2\beta)$$
$$\sigma_{12} = \frac{3\sigma}{8}\sin 2\beta$$

Then, the stress intensity factors K_I and K_{II} are given by:

$$K_I = \sigma_{22}\sqrt{\pi a} = \frac{\sigma}{8}(5 + 3\cos 2\beta)\sqrt{\pi a}$$
$$K_{II} = \sigma_{12}\sqrt{\pi a} = \frac{3\sigma}{8}\sin 2\beta\sqrt{\pi a}$$

Outline

Evaluating K_I and K_{II}

When will fracture occur?

In which direction will the crack propagate?

Mixed-mode fracture

As mentioned earlier, the stress intensity factor is **not** additive:

$$K_{\text{(total)}} \neq K_I + K_{II} + K_{III}$$

In contrast, the energy release rate G is the sum of each mode:

$$G = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{(1+\nu)K_{III}^2}{E}$$

where $E' = E$ for plane stress, and $E' = E/(1-\nu^2)$ for plane strain. A simple fracture criterion is obtained by setting G equal to the material's toughness:

$$G_{Ic} = \frac{K_{Ic}^2}{E'}$$

Mixed-mode fracture: a simple criterion

From previous slide, setting $G = G_{Ic}$ gives:

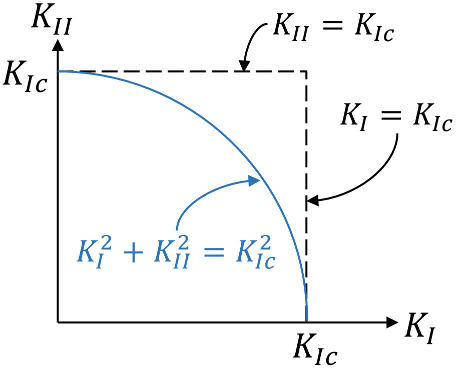
$$\frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{(1+\nu)K_{III}^2}{E} = \frac{K_{Ic}^2}{E'}$$

If $K_{III} = 0$, this simplifies to:

$$K_I^2 + K_{II}^2 = K_{Ic}^2$$

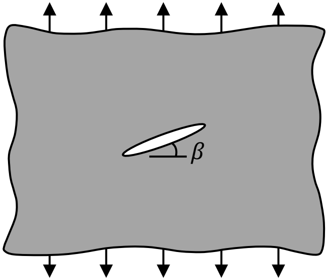
which gives a circular fracture envelope.

Is this equation validated experimentally?

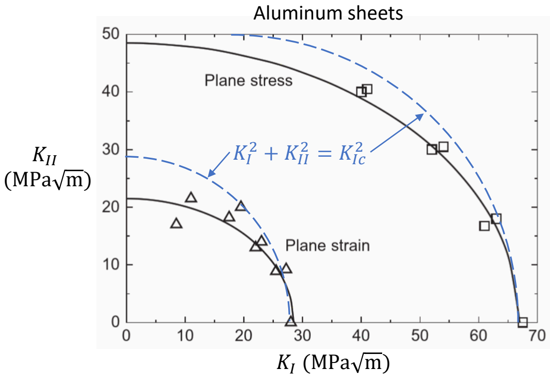


Mixed-mode fracture: an experimental approach

A fracture envelope can be obtained experimentally. Multiple tests are needed where β is varied to change K_I/K_{II}.



Sun et al. (2012) *Fracture mechanics.*



As shown here, the simple criterion $K_I^2 + K_{II}^2 = K_{Ic}^2$ is relatively close to experiments.

Outline

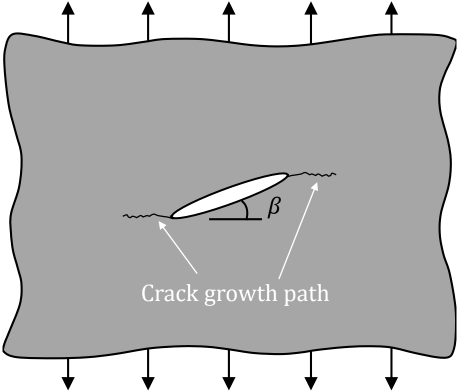
Evaluating K_I and K_{II}

When will fracture occur?

In which direction will the crack propagate?

Direction of crack propagation

For mixed-mode loading, experiments have shown that cracks grow in the direction of local mode I.



The local mode I direction is where $\sigma_{\theta\theta}$ is maximum, which is the same as the direction where $\sigma_{r\theta} = 0$.

Stress field for mixed-mode loading

The stress field close to the crack tip is obtained by adding the modes I and II contributions **for each stress component**, which gives:

$$\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{5}{4}\cos\frac{\theta}{2} - \frac{1}{4}\cos\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\left(-\frac{5}{4}\sin\frac{\theta}{2} + \frac{3}{4}\sin\frac{3\theta}{2}\right)$$
$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2}\right) - \frac{K_{II}}{\sqrt{2\pi r}}\left(\frac{3}{4}\sin\frac{\theta}{2} + \frac{3}{4}\sin\frac{3\theta}{2}\right)$$
$$\sigma_{r\theta} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\left(\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}\right)$$

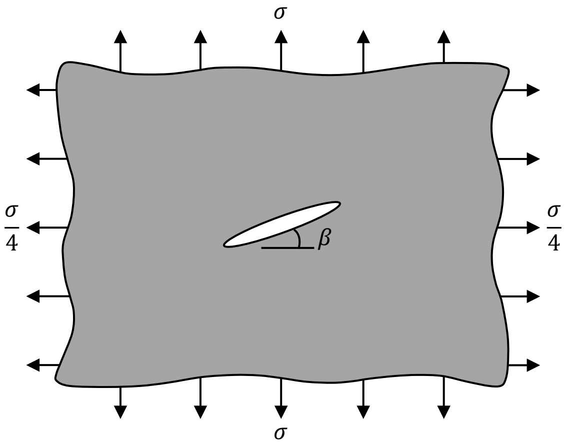
The direction of crack propagation is given by the angle θ for which $\sigma_{r\theta} = 0$. This gives the following equation:

$$0 = K_I\left(\sin\frac{\theta}{2} + \sin\frac{3\theta}{2}\right) + K_{II}\left(\cos\frac{\theta}{2} + 3\cos\frac{3\theta}{2}\right)$$

If you know K_I and K_{II}, you can solve for θ.

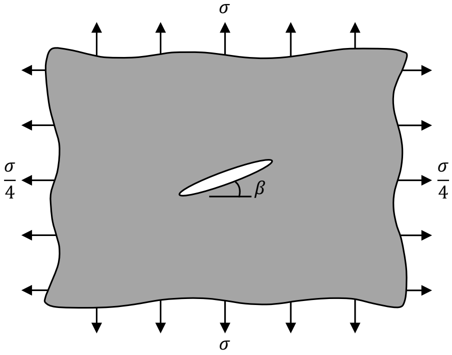
Example problem

Find the angle of crack propagation for the biaxial loading case shown below, provided that $\sigma = 10\,\mathrm{MPa}$ and $\beta = 20°$.



Example problem: solution

Find the angle of crack propagation for the biaxial loading case shown below, provided that $\sigma = 10\,\mathrm{MPa}$ and $\beta = 20°$.



On Slide 10, we found:

$$K_I = \frac{\sigma}{8}(5 + 3\cos 2\beta)\sqrt{\pi a} \qquad = 9.12\sqrt{\pi a}$$

$$K_{II} = \frac{3\sigma}{8}\sin 2\beta\sqrt{\pi a} \qquad = 2.41\sqrt{\pi a}$$

Example problem: solution

Substituting $K_I = 9.12\sqrt{\pi a}$ and $K_{II} = 2.41\sqrt{\pi a}$ in $\sigma_{r\theta} = 0$ returns:

$$
\begin{aligned}
\sigma_{r\theta} &= \frac{K_I}{\sqrt{2\pi r}}\left(\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}\right) + \frac{K_{II}}{\sqrt{2\pi r}}\left(\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}\right) \\
\implies 0 &= 9.12\left(\sin\frac{\theta}{2} + \sin\frac{3\theta}{2}\right) + 2.4\left(\cos\frac{\theta}{2} + 3\cos\frac{3\theta}{2}\right)
\end{aligned}
$$

And solving numerically, we find:

$$\theta = -26^\circ, \quad 130^\circ \quad \text{and} \quad 180^\circ$$

- The value of 180°is physically impossible.
- The other two values correspond to either a minimum or a maximum of $\sigma_{\theta\theta}$. The crack will propagate in the direction of maximum $\sigma_{\theta\theta}$.

Example problem: solution

We are left with $\theta = -26^\circ$ or 130°, and we are looking for the value corresponding to a maximum in $\sigma_{\theta\theta}$. This stress is given by (see Slide 17):

$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}}\left(\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2}\right) - \frac{K_{II}}{\sqrt{2\pi r}}\left(\frac{3}{4}\sin\frac{\theta}{2} + \frac{3}{4}\sin\frac{3\theta}{2}\right)$$

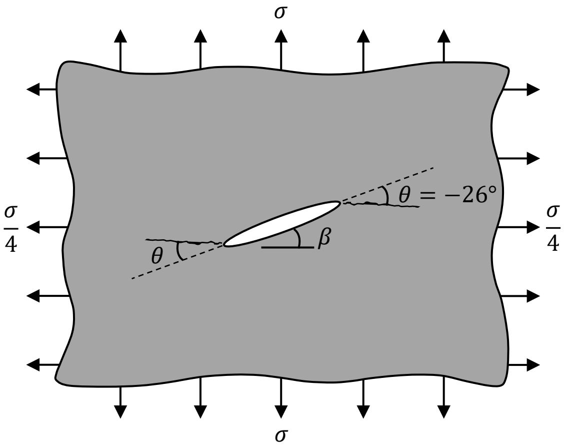
$$\text{for } \theta = -26^\circ \implies \sigma_{\theta\theta} = 9.98\sqrt{\frac{a}{2r}}$$

$$\text{for } \theta = 130^\circ \implies \sigma_{\theta\theta} = -0.48\sqrt{\frac{a}{2r}}$$

The maximum $\sigma_{\theta\theta}$ is obtained with $\theta = -26^\circ$; therefore, the crack will propagte in this direction.

Example problem: solution

Note that our solution, $\theta = -26°$, is in the local (crack) reference frame.



Summary

Under mixed-mode loading:

- find the stress intensity factors K_I and K_{II} using Mohr's circle,
- use the energy release rate G to predict when fracture will occur,
- the crack will propagate in the direction where $\sigma_{r\theta} = 0$.

Fracture Mechanics

4. Plasticity

Luc St-Pierre

April 20, 2023

Motivation

- So far, we investigated fracture using (i) stress and (ii) energy approaches, assuming that the material is linear elastic.
- This is refered to as Linear Elastic Fracture Mechanics (LEFM).
- Metals are not simply linear elastic, they also exhibit plasticity.
- If there is plasticity, can we still use LEFM?

Learning outcomes for this week

After this week, you will be able to:

- Estimate the size of the plastic zone.
- Evaluate when it is adequate to use Linear Elastic Fracture Mechanics (LEFM).
- Determine the transition flaw size.
- Describe the J-integral and understand when to use it.

Outline

- Plastic zone size
  - Irwin approach
  - Strip-yield model
- Plane stress vs plane strain: the effect of sheet thickness
- Transition flaw size
- Elastic-Plastic Fracture Mechanics
  - Crack tip opening displacement
  - J-integral
- Should you use K, G or J?

Outline

- Plastic zone size
  - Irwin approach
  - Strip-yield model
- Plane stress vs plane strain: the effect of sheet thickness
- Transition flaw size
- Elastic-Plastic Fracture Mechanics
  - Crack tip opening displacement
  - J-integral
- Should you use K, G or J?

Plastic zone size - First approximation

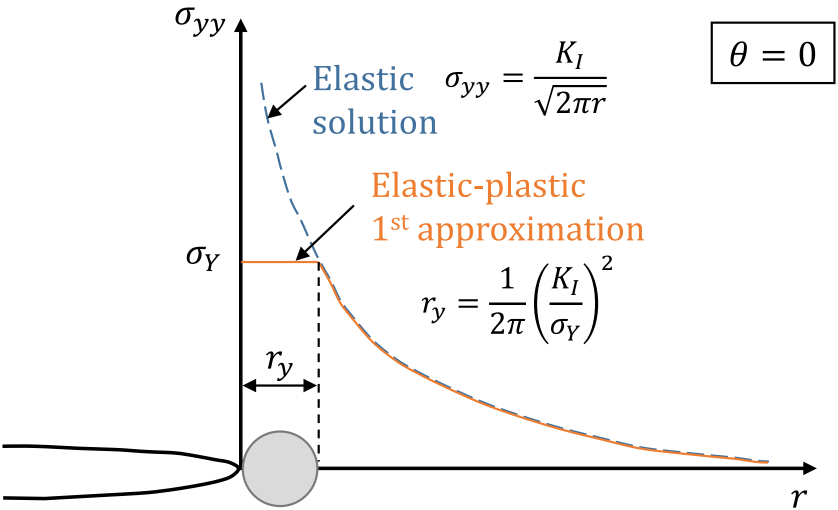
For a linear elastic material, the normal stress σ_{yy} on the crack plane ($\theta = 0$) is given by:

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}$$

which tends to infinity at $r = 0$. If the solid is **elastic perfectly plastic**, there will be a zone of plastic deformation close to the crack tip. For plane stress, the size of this plastic zone r_y can be estimated by setting $\sigma_{yy} = \sigma_Y$, which gives us:

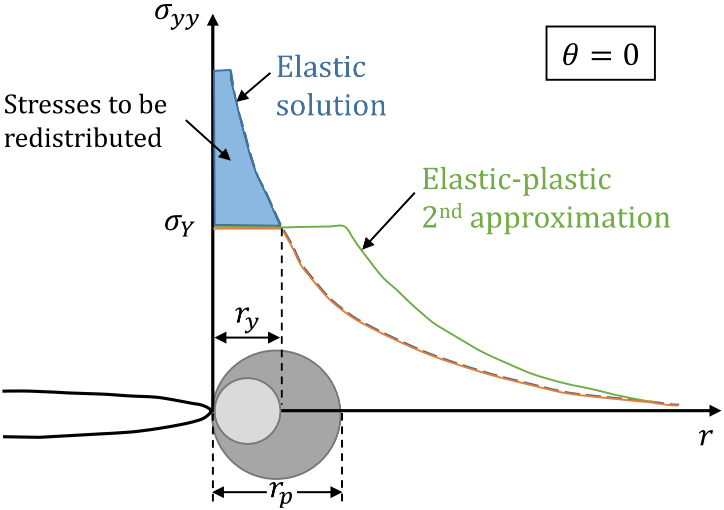
$$r_y = \frac{1}{2\pi} \left(\frac{K_I}{\sigma_Y} \right)^2$$

Plastic zone size - First approximation



Plastic zone size - Second approximation

There is a problem in the previous analysis; some stresses need to be redistributed to satisfy equilibrium.



Plastic zone size - Second approximation

Based on the previous figure, equilibrium is respected when:

$$\sigma_Y r_p = \int_0^{r_y} \sigma_{yy} dr = \int_0^{r_y} \frac{K_I}{\sqrt{2\pi r}} dr$$

after integrating, we get a better estimation of the plastic zone size r_p:

$$r_p = \frac{1}{\pi}\left(\frac{K_I}{\sigma_Y}\right)^2 = 2r_y$$

This result is two times larger than our first approximation.

This analysis was done for plane stress, does the results change for plane strain?

Yielding criterion

For a 3D stress state, we need a yielding criterion to predict the onset of plasticity. The von Mises criterion is:

$$\frac{1}{\sqrt{2}}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_1-\sigma_3)^2\right]^{1/2}=\sigma_Y$$

where $\sigma_1, \sigma_2, \sigma_3$ are the three principal stresses. The mode I stress field along the crack plane ($\theta = 0$) is:

$$\sigma_{yy}=\sigma_1=\frac{K_I}{\sqrt{2\pi r}} \qquad \sigma_{xx}=\sigma_2=\frac{K_I}{\sqrt{2\pi r}}=\sigma_{yy} \qquad \sigma_{xy}=0$$

$$\sigma_{zz}=\sigma_3=\begin{cases} 0 & \text{for plane stress} \\ \nu(\sigma_{xx}+\sigma_{yy})=2\nu\sigma_{yy} & \text{for plane strain} \end{cases}$$

Substituting in the von Mises criterion returns the yielding condition.

Yielding criterion - plastic zone size r_p

After substituting the stress field in the von Mises criterion, we obtain that yielding occurs when:

$$\sigma_{yy} = \sigma_Y \qquad \text{plane stress}$$

$$\sigma_{yy} = \frac{\sigma_Y}{1-2\nu} \approx 3\sigma_Y \qquad \text{plane strain}$$

where we assumed $\nu = 1/3$. This has an effect on the plastic zone size r_p:

$$r_p = \begin{cases} \frac{1}{\pi}\left(\frac{K_I}{\sigma_Y}\right)^2 & \text{plane stress} \\ \frac{1}{3\pi}\left(\frac{K_I}{\sigma_Y}\right)^2 & \text{plane strain} \end{cases}$$

Therefore, r_p is three times smaller in plane strain compared to plane stress.

Plastic zone shape

- Our previous estimate of the plastic zone size focused on the crack plane only, $\theta = 0$.
- This gives us a scalar r_y or r_p depending on the analysis.
- However, this does not give us the shape of the plastic zone: is it a cricle, an ellipse or something else?
- To find the plastic zone shape, we need to repeat the previous analysis for all values of θ to find the function $r_y(\theta)$.

Plastic zone shape

The first step is to compute the principal stresses using the Mohr's circle relationship:

$$\begin{aligned}
\sigma_1, \sigma_2 &= \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \left[\left(\frac{\sigma_{xx} - \sigma_{yy}}{2} \right)^2 + \sigma_{xy}^2 \right]^{1/2} \\
\sigma_3 &= 0 \qquad\qquad\qquad \text{for plane stress} \\
&= \nu(\sigma_1 + \sigma_2) \quad \text{for plane strain}
\end{aligned}$$

where $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$ are given in the datasheet.

Plastic zone shape

This gives us the following principal stresses for a mode I crack:

$$\begin{aligned}
\sigma_1 &= \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\right) \\
\sigma_2 &= \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\right) \\
\sigma_3 &= 0 && \text{for plane stress} \\
&= \frac{2\nu K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2} && \text{for plane strain}
\end{aligned}$$

The next step is to substitute them in the von Mises yielding criterion:

$$\frac{1}{\sqrt{2}}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2\right]^{1/2} = \sigma_Y$$

Plastic zone shape

Expressing r as a function of K_I, σ_Y and θ gives us the shape of the plastic zone size:

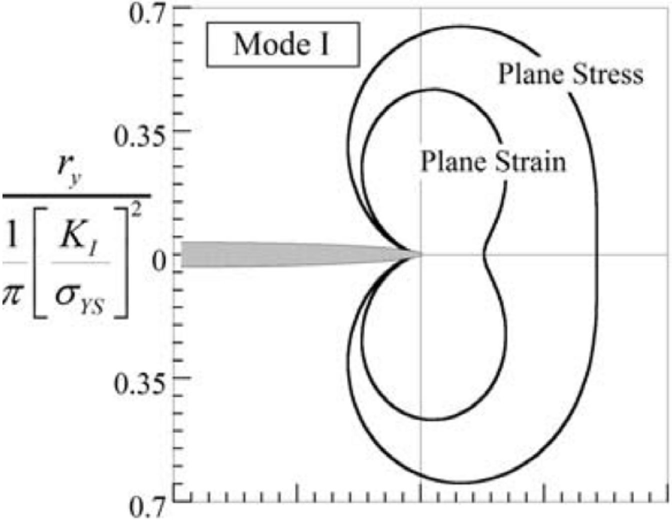
$$r_y(\theta) = \frac{1}{4\pi}\left(\frac{K_I}{\sigma_Y}\right)^2\left[1 + \cos\theta + \frac{3}{2}\sin^2\theta\right]$$

for plane stress, and:

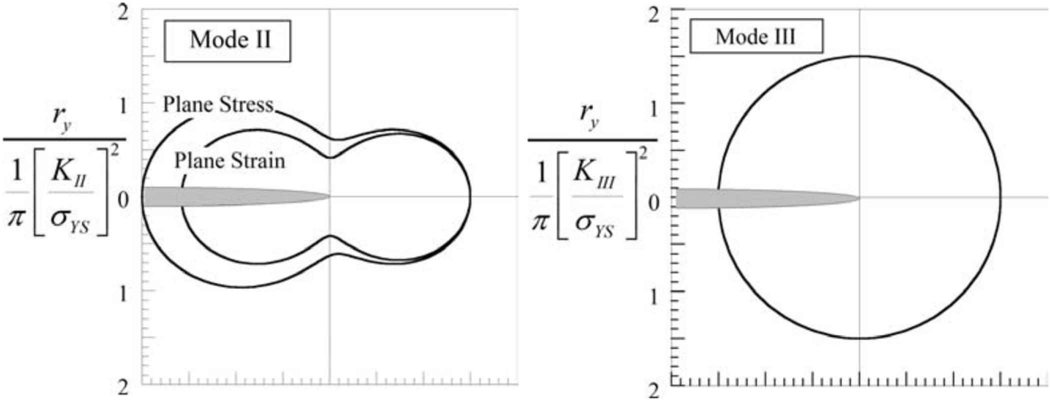
$$r_y(\theta) = \frac{1}{4\pi}\left(\frac{K_I}{\sigma_Y}\right)^2\left[(1-2\nu)^2(1+\cos\theta) + \frac{3}{2}\sin^2\theta\right]$$

for plane strain. These equations are plotted next.

Plastic zone shape: mode I



Plastic zone shape: modes II and III



Outline

Strip-yield model

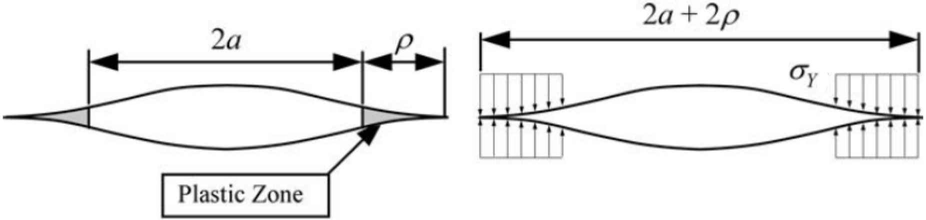
The stip-yield model, developed by Dugdale and Barenblatt independently, is an alternative way to estimate the plastic zone size.

Their model was developed for an infinitely large plate with a through crack under plane stress.

Their analysis is based on the principle of superposition: the model superimposes two elastic solutions to approximate an elastic-plastic behaviour.

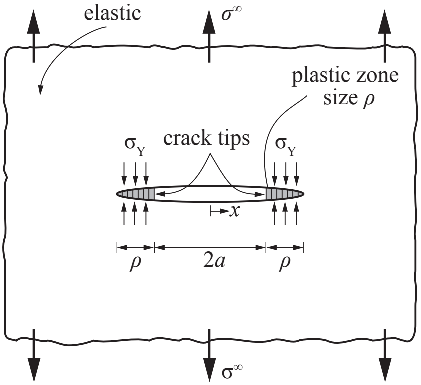
Strip-yield model

Consider a crack of length $2a$, where there is a plastic zone of size ρ at each crack tip.



The strip-yield model, replaces the physical crack of length $2a$ by a fictitious crack of length $2(a + \rho)$. At each crack tip, there is a closing stress σ_Y keeping a portion ρ of the crack closed.

Strip-yield model



Stresses are finite at the fictitious crack tip, which implies:

$$\text{at } x = a + \rho : \quad K_I^{(tot)} = K_I^{(\sigma^\infty)} + K_I^{(\sigma_Y)} = 0$$

Strip-yield model

For an infinitely large plate, $K_I^{(\sigma^\infty)}$ is given by:

$$K_I^{(\sigma^\infty)} = \sigma^\infty \sqrt{\pi(a+\rho)}$$

Otherwise, the contribution of the closure stress σ_Y is obtained by integrating the solution for a point force over a portion ρ. This gives:

$$K_I^{(\sigma_Y)} = -2\sigma_Y \sqrt{\frac{a+\rho}{\pi}} \arccos\left(\frac{a}{a+\rho}\right)$$

Next, we substituting these results in:

$$K_I^{(tot)} = K_I^{(\sigma^\infty)} + K_I^{(\sigma_Y)} = 0$$

Strip-yield model

to obtain:

$$\frac{a}{a+\rho} = \cos\left(\frac{\pi\sigma^\infty}{2\sigma_Y}\right)$$

The cosine function can be expressed as a Taylor series:

$$\frac{a}{a+\rho} = 1 - \frac{1}{2!}\left(\frac{\pi\sigma^\infty}{2\sigma_Y}\right)^2 + \frac{1}{4!}\left(\frac{\pi\sigma^\infty}{2\sigma_Y}\right)^4 - \frac{1}{6!}\left(\frac{\pi\sigma^\infty}{2\sigma_Y}\right)^6 + \dots$$

Keeping the first two terms only, and solving for the plastic zone size gives:

$$\rho = \frac{\pi}{8}\left(\frac{\sigma^\infty\sqrt{\pi a}}{\sigma_Y}\right)^2 = \frac{\pi}{8}\left(\frac{K_I}{\sigma_Y}\right)^2$$

Irwin approach vs strip-yield model

For plane stress, we now have two different estimates of the plastic zone size:

- Irwin approach:

$$r_p = \frac{1}{\pi}\left(\frac{K_I}{\sigma_Y}\right)^2 \approx 0.32\left(\frac{K_I}{\sigma_Y}\right)^2$$

- Strip-yield model:

$$\rho = \frac{\pi}{8}\left(\frac{K_I}{\sigma_Y}\right)^2 \approx 0.39\left(\frac{K_I}{\sigma_Y}\right)^2$$

Both approaches give similar results (20% difference.)

Validity of LEFM

So far, we have used LEFM to make different estimates of the plastic zone size, why?

If the plastic zone size is small then we conclude that LEFM applies and fracture will occur when $K_I = K_{Ic}$.

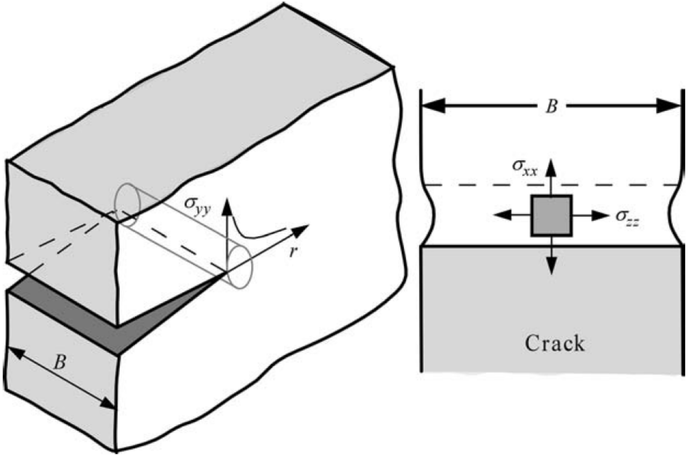
What is a small plastic zone size? If r_p is roughly an order of magnitude smaller than the crack length, that is small enough. We will define a more precise criterion when discussing testing methods.

Outline

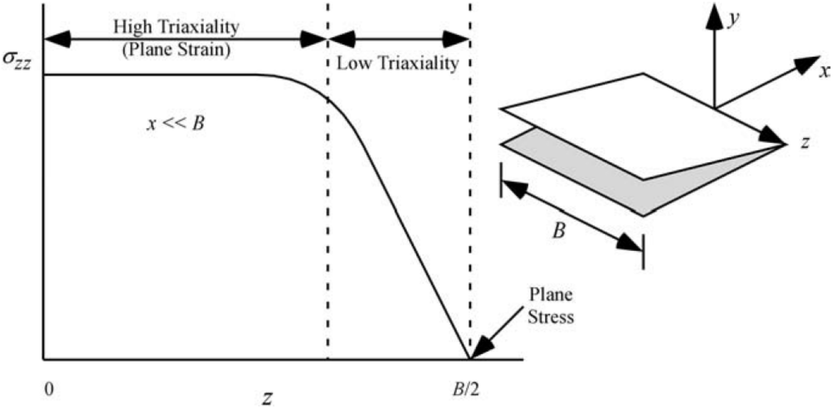
- Plastic zone size
  - Irwin approach
  - Strip-yield model
- Plane stress vs plane strain: the effect of sheet thickness
- Transition flaw size
- Elastic-Plastic Fracture Mechanics
  - Crack tip opening displacement
  - J-integral
- Should you use K, G or J?

Plane stress vs plane strain

The high normal stress σ_{yy} at the crack tip causes material near the surface to contract. However, material inside the specimen is constrained, resulting in a triaxial stress state.



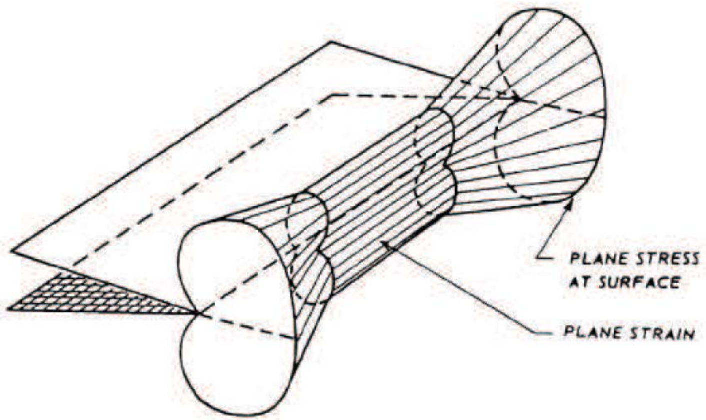
Triaxial stress state



The triaxial stress state leads to:

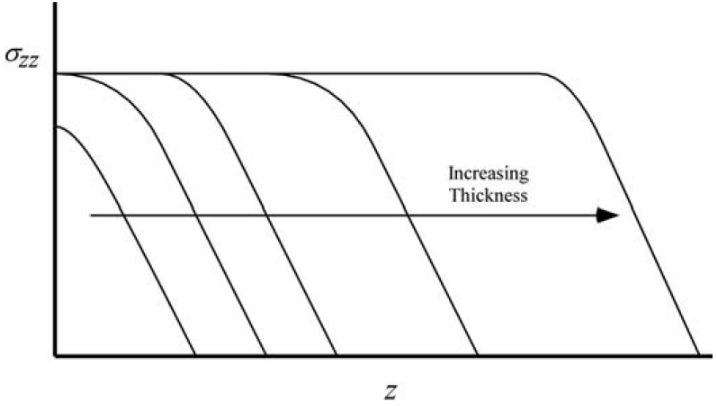
- plane strain conditions inside the specimen,
- plane stress on the surface.

Plastic zone size through the thickness



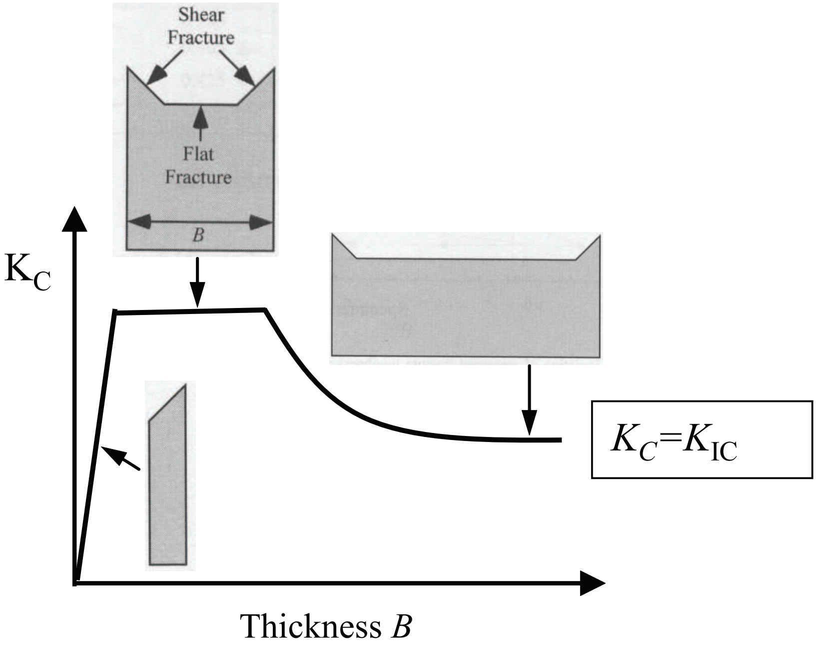
The change in triaxiality means that the plastic zone size varies through the thickness.

Triaxiality vs sheet thickness



- The sheet thickness has a strong effect on the state of triaxiality and the magnitude of σ_{zz}.
- Consequently, the sheet thickness has an effect on the measured fracture toughness.

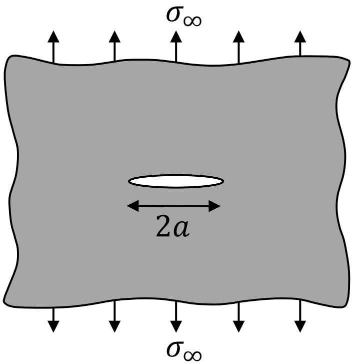
Fracture toughness vs sheet thickness



Outline

Plasticity vs Fracture

Consider that a plate made from an elastic-plastic material with a yield strength σ_Y and a fracture toughness K_{Ic}.



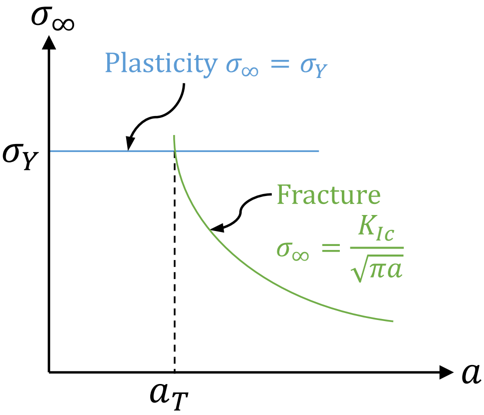
For a long crack, we would expect fracture to occur when:

$$\sigma_\infty = \frac{K_{Ic}}{\sqrt{\pi a}}$$

For short cracks, plasticity may occur before fracture:

$$\sigma_\infty = \sigma_Y$$

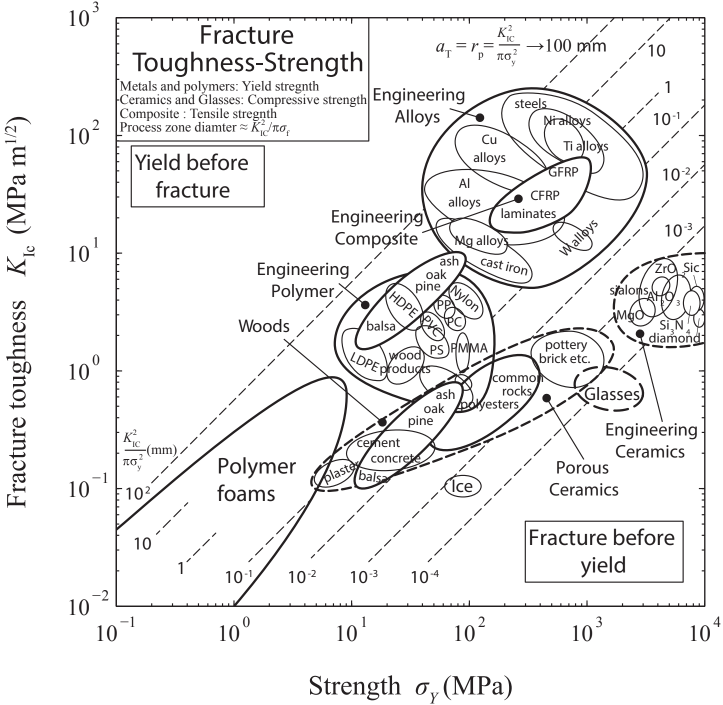
Plasticity vs Fracture



The transition flaw size a_T is given by:

$$a_T = \frac{1}{\pi}\left(\frac{K_{Ic}}{\sigma_Y}\right)^2$$

If this looks familiar it is because $a_T = r_p$, where r_p is the plastic zone size in plane stress, see slide 11.


Fracture Toughness-Strength
Metals and polymers: Yield stregnth
Ceramics and Glasses: Compressive strength
Composite : Tensile stregnth
Process zone diamter ≈ $K_{IC}^2/\pi\sigma_f$
Yield before fracture
Fracture before yield
$a_T = r_p = \frac{K_{IC}^2}{\pi\sigma_y^2} \rightarrow 100$ mm
$\frac{K_{IC}^2}{\pi\sigma_y^2}$(mm)
Engineering Alloys
steels
Ni alloys
Ti alloys
Cu alloys
Al alloys
GFRP
CFRP
laminates
Mg alloys
W alloys
cast iron
Engineering Composite
Engineering Polymer
ash oak pine
HDPE
Nylon
PP
PC
PVC
balsa
PS
PMMA
LDPE
wood products
Woods
polyesters
common rocks
pottery brick etc.
Glasses
cement concrete
plaster
balsa
Ice
Polymer foams
Porous Ceramics
Engineering Ceramics
ZrO2
SiC
sialons
Al2O3
MgO
Si3N4
diamond
Fracture toughness K_{Ic} (MPa m$^{1/2}$)
Strength σ_Y (MPa)

Transition flaw size

In conclusion:

- If a structure contains a crack of length $a \ll a_T$, it will fail by plastic collapse, $\sigma_\infty = \sigma_Y$.
- If a structure contains a crack of length $a \gg a_T$, it will fail by brittle fracture, $\sigma_\infty = K_{Ic}/\sqrt{\pi a}$.

Outline

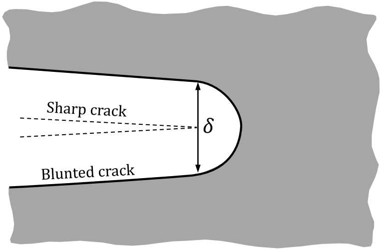
Elastic Plastic Fracture Mechanics

- When researchers tried to measure the fracture toughness of metals, they found that they were too tough to be characterised by LEFM.
- This lead to the development of Elastic Plastic Fracture Mechanics, where the material has a non-linear behaviour.
- We will introduce two new parameters: the crack tip opening displacement and the J-integral. Both parameters decribe the conditions at the crack tip of an elastic plastic material, and both can be used as a fracture criterion.

Outline

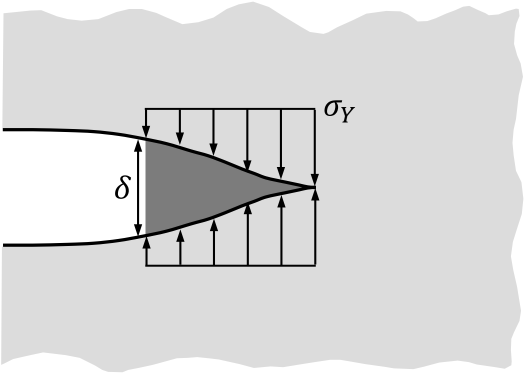
Crack tip opening displacement

Wells (1961) observed that in metals the crack faces moved apart significantly prior to fracture; plasticity was blunting an initially sharp crack.



The crack tip opening displacement (CTOD) is represented by δ on the figure. Tougher materials have a higher CTOD.

Estimating the CTOD using the strip-yield model



The CTOD can be estimated as the displacement at the end of the strip-yield zone. For a large plate with a through crack this definition gives us:

$$\delta = \frac{8\sigma_Y a}{\pi E} \ln \sec\left(\frac{\pi\sigma_\infty}{2\sigma_Y}\right)$$

Estimating the CTOD using the strip-yield model

In the previous equation, we can expand the $\ln \sec$ term as a Taylor series, which gives:

$$\begin{aligned}
\delta &= \frac{8\sigma_Y a}{\pi E}\left[\frac{1}{2}\left(\frac{\pi\sigma_\infty}{2\sigma_Y}\right)^2 + \frac{1}{12}\left(\frac{\pi\sigma_\infty}{2\sigma_Y}\right)^4 + \dots\right] \\
&= \frac{K_I^2}{\sigma_Y E}\left[1 + \frac{1}{6}\left(\frac{\pi\sigma_\infty}{2\sigma_Y}\right)^2 + \dots\right]
\end{aligned}$$

As $\sigma_\infty/\sigma_Y \to 0$, this becomes:

$$\delta = \frac{K_I^2}{\sigma_Y E} = \frac{G}{\sigma_Y}$$

Estimating the CTOD using the strip-yield model

The previous analysis assumed plane stress conditions and a perfectly-plastic material.

The relation can be expressed in a more general form as:

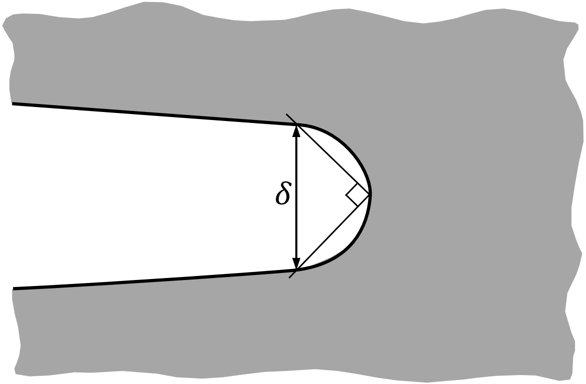
$$\delta = \frac{K_I^2}{m\sigma_Y E'} = \frac{G}{m\sigma_Y}$$

where,

$$\begin{array}{lll} \text{For plane stress:} & m = 1 \quad \text{and} & E' = E \\ \text{For plane strain:} & m = 2 \quad \text{and} & E' = E/(1-\nu^2) \end{array}$$

Alternative definitions of the CTOD

There are other definitions of the CTOD. One commonly used in Finite Element simulations is shown below.



The important thing to remember is that the CTOD is related to K_I and G; therefore, it can be used to predict fracture.

Outline

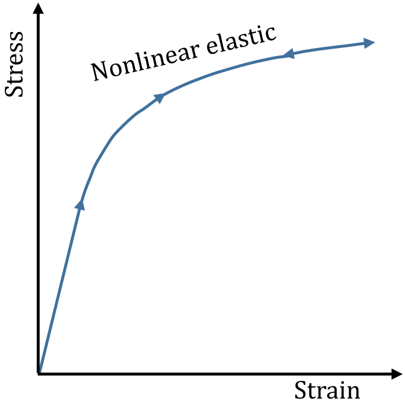
The advantages of the J-integral

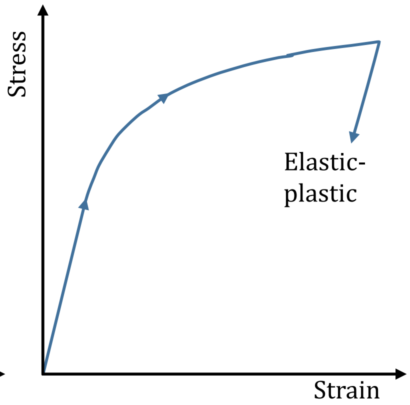
The J-integral is the most commonly used parameter to predict fracture in elastic-plastic materials. It is a versatile parameter since:

- its definition is similar to that of the energy release rate G; the J-integral also has units of J/m^2.
- the J-integral characterises the stress field at the crack tip of an elastic-plastic material. In this sense, it is similar to the stress intensity factor K for elastic solids.
- it can be expressed as a path-independent line integral, which makes it easy to implement in Finite Element codes.

An important note on the material behaviour

- In LEFM, the material is assumed to be linear elastic.
- In EPFM and all the work done on J-integral, the material is assumed to be nonlinear elastic. This is different from the elastic-plastic behaviour of metals.



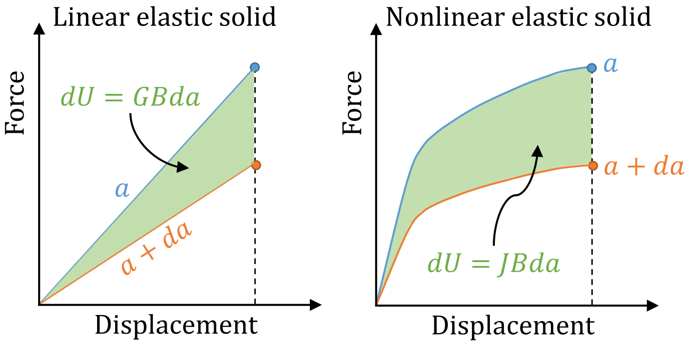


J as an energy release rate

The J-integral is defined in the same way as G:

$$J = -\frac{d\Pi}{dA} = -\frac{d\Pi}{Bda} \qquad \text{where} \qquad \Pi = U - W$$

This definition can be visualised as follows:



When the material is linear elastic, we find that $J = G$.

J as a stress intensity parameter

Hutchinson (1968) and Rice and Rosengren (1968) independently showed that J uniquely characterises the stress field at the crack tip.

They both assumed that the material follows a Ramberg-Osgood relationship:

$$\epsilon = \frac{\sigma}{E} + K\left(\frac{\sigma}{E}\right)^n$$

where

E	is the Young's modulus
K	is a dimensionless constant
n	is the strain hardening exponent

J as a stress intensity parameter

Assuming:

$$\epsilon = \frac{\sigma}{E} + K\left(\frac{\sigma}{E}\right)^n$$

Hutchinson, Rice and Rosengren showed that:

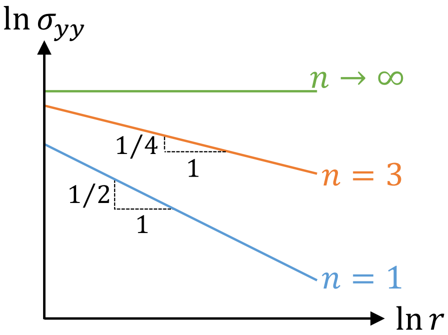
$$\sigma_{ij} \propto \left(\frac{J}{r}\right)^{\frac{1}{n+1}} \qquad \text{and} \qquad \epsilon_{ij} \propto \left(\frac{J}{r}\right)^{\frac{n}{n+1}}$$

- For a linear elastic material ($n = 1$) we recover the $1/\sqrt{r}$ relation.
- Otherwise, for a perfectly plastic solid ($n \to \infty$) we find that σ_{ij} is independent of r.

J as a stress intensity parameter

The stresses scale as:

$$\sigma_{ij} \propto \left(\frac{J}{r}\right)^{\frac{1}{n+1}}$$

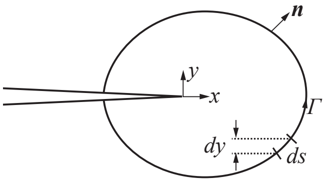


The zone where the stresses scale according to this relation is called the HRR field (for Hutchinson, Rice and Rosengren).

J as a contour integral

Rice (1968) showed that J can be computed as a contour integral:

$$J = \int_\Gamma \left(w dy - t_i \frac{\partial u_i}{\partial x} ds \right)$$

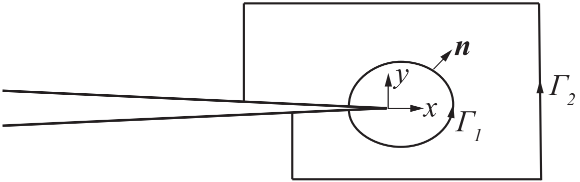


where

$w = \int_0^{\epsilon_{ij}} \sigma_{ij} d\epsilon_{ij}$	is the strain energy density
$t_i = \sigma_{ij} n_j$	is the traction vector
n_j	is the vector normal to the contour Γ
u_i	is the displacement vector
ds	is the length increment along the contour Γ

J as a contour integral

- The contour has to go from one crack surface to the other.
- The J-integral is path independent: any contour will give the same value of J

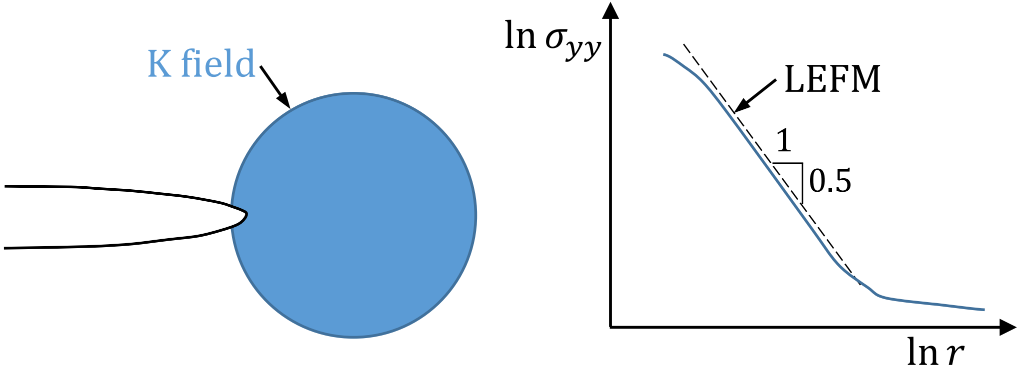


This means that: $J_{\Gamma_1} = J_{\Gamma_2}$.

Outline

Elastic regime

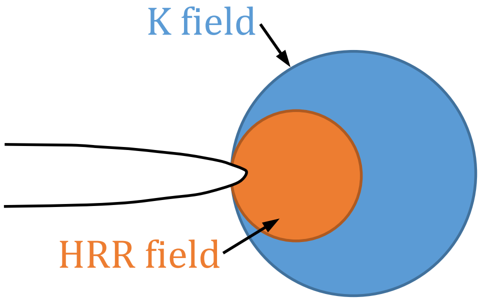
If the material is elastic, like glass, or if the stresses are within the elastic range, there is a K field surrounding the crack tip, where $\sigma_{ij} \propto 1/\sqrt{r}$.

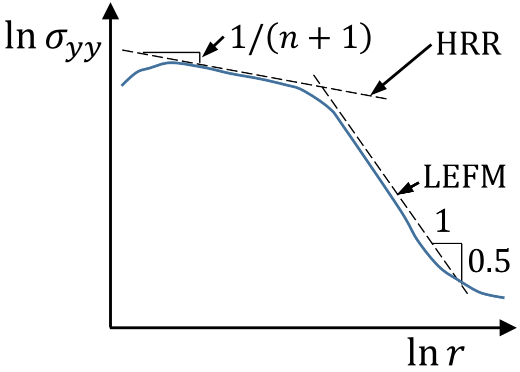


Fracture when $K_I = K_{Ic}$ and $J = G = G_{Ic}$.

Small scale yielding

As the loading is increased, there is a HRR field and a plastic zone surrounded by a larger K field.

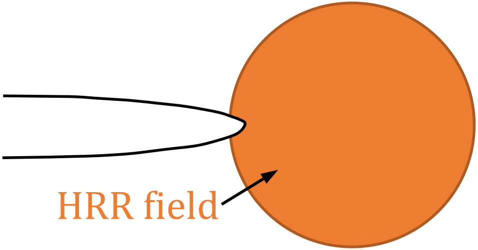


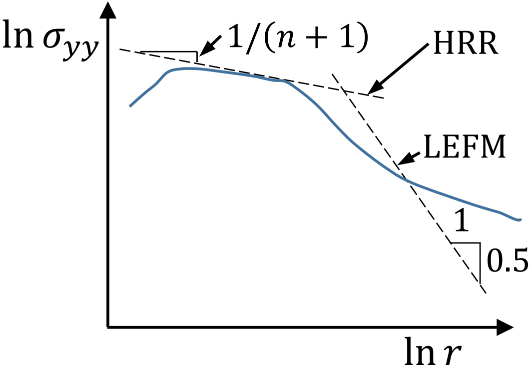


Here, LEFM still applies and fracture occurs when $K_I = K_{Ic}$ and $J = G = G_{Ic}$.

Large scale yielding

When the plastic zone becomes large compared to the dimensions of the structure, we have a HRR field but no more K field.





Here, EPFM is necessary and fracture occurs when $J_I = J_{Ic}$ (do not use K).

Summary

- We can estimate the size of the plastic zone r_p using Irwin's approach.
- Linear Elastic Fracture Mechanics (LEFM) is applicable if the plastic zone size is small ($r_p < a/10$).
- If the plastic zone size is large, it is necessary to use Elastic Plastic Fracture Mechanics (usually an approach based on the J-integral).

Fracture Mechanics

5. Testing and fracture mechanisms

Luc St-Pierre

April 19, 2023

Learning outcomes

After this section, you will be able to:

- Describe how fracture toughness tests are conducted.
- Explain fracture mechanisms in metals.
- Identify mechanisms leading to a rising R-curve.

Outline

Outline

Standard test methods

Fracture toughness tests are complex, consult the relevant standards:

ASTM E1820 Standard test method for measurement of fracture toughness.

ASTM E399 Standard test method for linear-elastic plane-strain fracture toughness K_{Ic} of metallic materials.

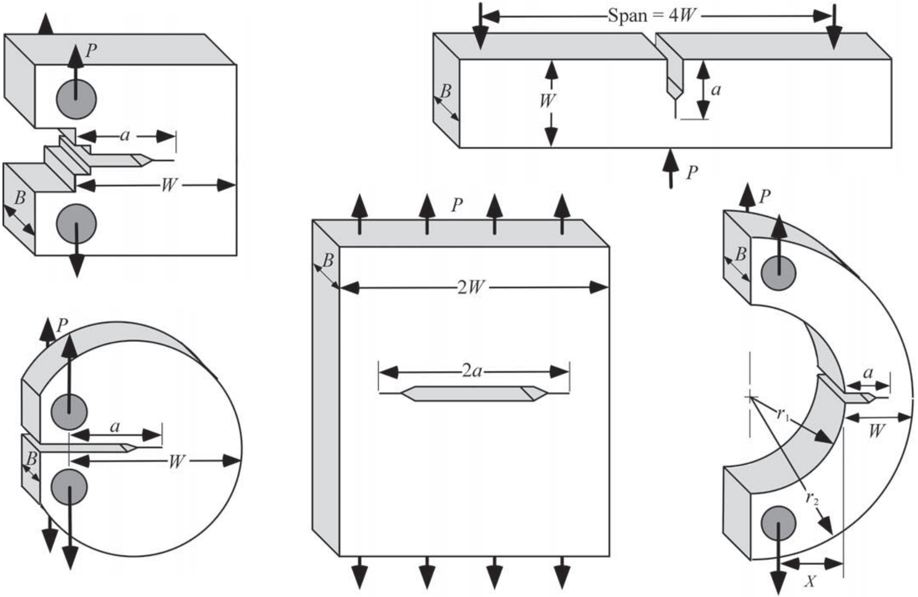
The first one, is the main reference and it covers K_{Ic}, J_{Ic} and R-curve measurements. The second one is an older version of the standard but you will often see it as a reference.

Outline

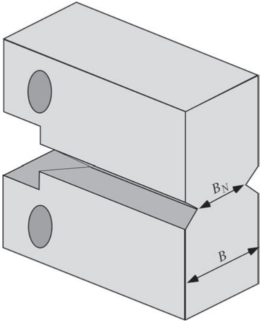
Specimen geometries

Compact Tension (CT)

Single-Edge-Notched Bend (SENB)



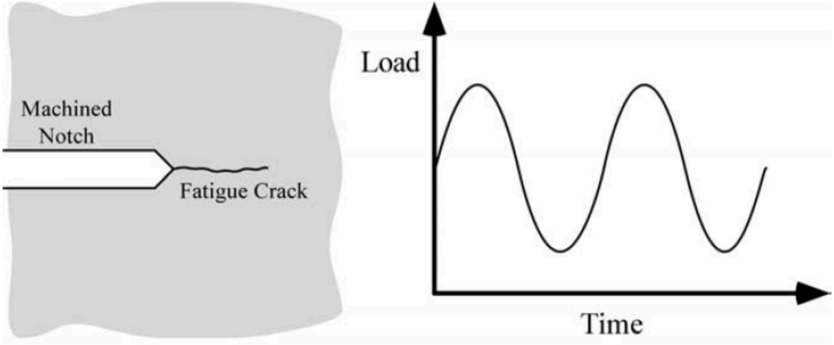
Side grooves



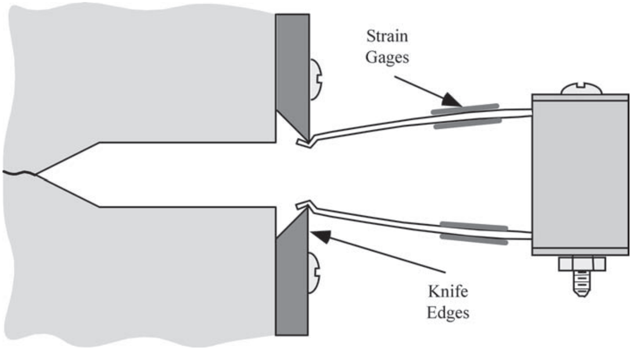
- Side grooves reduce the importance of the low triaxiality zones that exist close to the free surface.
- They help to produce a straighter crack front.
- They allow the crack to propagate in a straight line.

Fatigue precrack

From the initial notch, you need to grow a fatigue crack. For metals, there are no other way of producing a sharp crack.



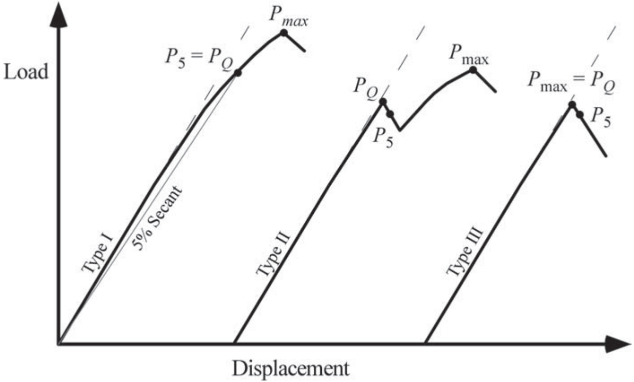
Instrumentation



- The applied force is measured by the testing machine.
- The most accurate way to measure displacement is at the crack mouth using a clip gauge.

Outline

Typical load-displacement curves



The stress intensity factor is given by:

$$K_Q = \frac{P_Q}{B\sqrt{W}} f\left(\frac{a}{W}\right)$$

Validity

For the test to be a valid fracture toughness measurement, $K_Q = K_{Ic}$, we need:

$$0.45 \leq \frac{a}{W} \leq 0.55$$
$$P_{max} \leq 1.10 P_Q$$
$$a, (W - a), B \geq 2.5 \left(\frac{K_{Ic}}{\sigma_Y} \right)^2$$

If these requirements are not satisfied, then the plastic zone is too large to use LEFM and you will need to perform a J_{Ic} test instead.

Size requirement for a K_{Ic} test

Consider a structural steel with $\sigma_Y = 240\,\text{MPa}$ and $K_{Ic} = 60\,\text{MPa}\sqrt{\text{m}}$, what would be the minimum thickness B to conduct a valid test:

$$B \geq 2.5\left(\frac{K_{Ic}}{\sigma_Y}\right)^2 = 2.5\left(\frac{60}{240}\right)^2 = 15.6\,\text{cm!}$$

This might not seem like a large number, but a SENB specimen with this thickness B would be about $600\,\text{kg}$!

For tough materials, it is almost impossible to do a K_{Ic} test, you need to go for a J_{Ic} test.

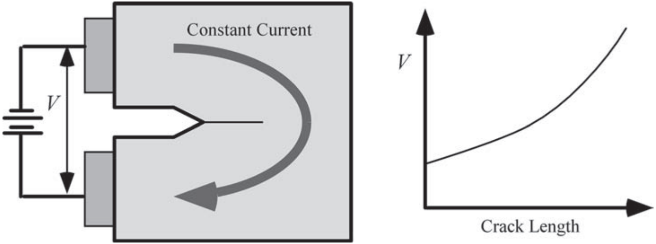
Outline

- Testing
  - Basic notions: geometry, precrack and instrumentation
  - Measuring K_{Ic}
  - Measuring J_{Ic} and the R-curve
- Fracture mechanisms in metals
  - Ductile fracture
  - Cleavage
  - Intergranular fracture
- Why metals have a rising R-curve?
- Toughening mechanisms in fibre-reinforced composites
- Increasing toughness: examples from research
  - Lattice materials
  - Carbon fibre composites

J-test

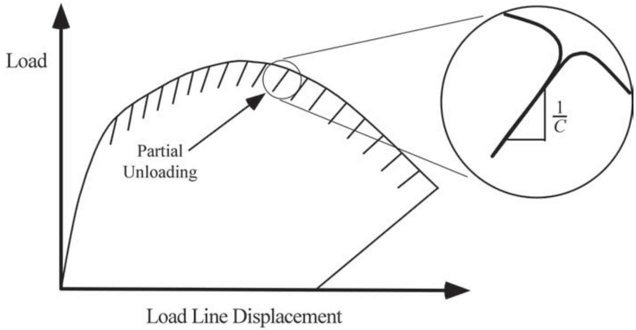
- The most commonly used geometry are the Compact-Tension (CT) and Single-Edge-Notch Bend (SENB) specimens, see slide 7.
- Again, fatigue precrack is necessary for metals.
- To plot the R-curve of the material, the crack extension needs to be measured during the test. This can be done using:
  - the potential drop technique or
  - the unloading compliance method
  - (visual observation might be possible but it is not recommended).

Measuring crack length: potential drop technique



- Advantages: accurate and offers continuous measurement during the test.
- Disadvantages: needs to be calibrated, requires extra equipment and works only with materials that conduct electricity.

Measuring crack length: unloading compliance



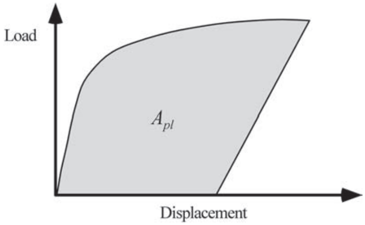
- **Advantages**: doesn't require additional equipment and the relationship between the crack length a and the compliance C is given for standard test geometries.
- **Disadvantages**: makes testing and data processing more tedious.

Computing J

The J-integral is given by:

$$J = J_{el} + J_{pl}$$

where the elastic part is:



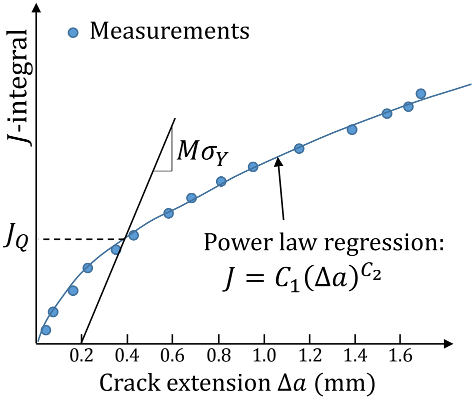
$$J_{el} = \frac{K^2(1-\nu^2)}{E} \quad \text{where} \quad K = \frac{P}{B\sqrt{W}} f\left(\frac{a}{W}\right)$$

and the plastic part is proportional to the load vs plastic displacement curve:

$$J_{pl} = \frac{\eta A_{pl}}{B b_0}$$

where the uncracked ligament $b_0 = W - a_0$ and η is a constant that depends on the specimen geometry.

Plotting the J vs Δa curve



1. Plot measurements on a J vs Δa curve.
2. Fit the data with a power law regression.
3. Find J_Q: the intersection between the power law regression and the 0.2 mm blunting line.

Test validity

The value of $J_Q = J_{Ic}$ if:

$$B, b_0 \geq \frac{25 J_Q}{\sigma_Y}$$

If this requirement is satisfied you can convert J_{Ic} to K_{Ic} with:

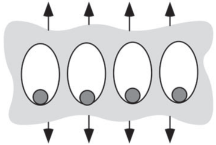
$$K_{Ic} = \sqrt{\frac{E J_{Ic}}{1 - \nu^2}}$$

With this procedure, you can have much smaller specimens and you get K_{Ic} and the R-curve with a single test.

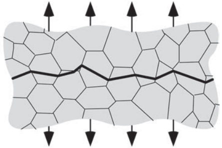
Outline

Fracture mechanisms in metals

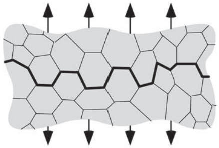
The most common fracture mechanisms in metals are:



Ductile fracture involves the nucleation, growth and coalescence of microvoids.



Cleavage is when the crack propagates along specific crystallographic planes.



Intergranular fracture is when the grain boundaries consitute the prefered crack path.

Outline

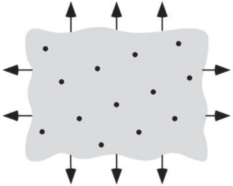
Ductile fracture

Ductile fracture is the most common fracture mechanisms in metals at room temperature. It involves three stages:

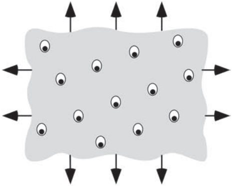
1. Inclusions break or debond from the surrounding matrix forming a void. This is called void nucleation.
2. These voids grow under plastic deformation.
3. Adjacent voids join together; this is known as void coalescence.

This process is illustrated next.

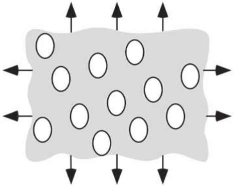
Ductile fracture process



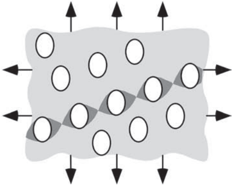
1. Inclusions in a ductile matrix



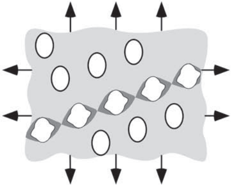
2. Void nucleation



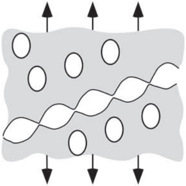
3. Void growth



4. Strain localisation between voids

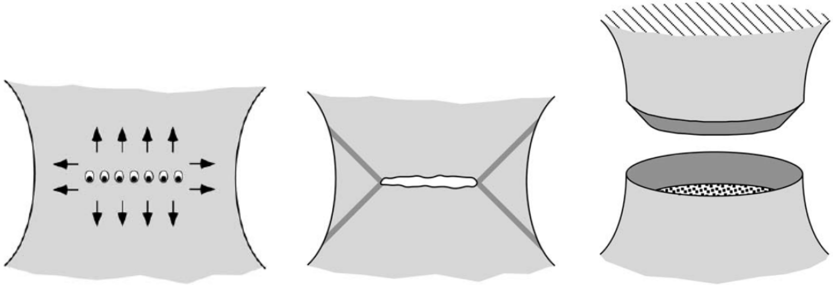


5. Necking between voids



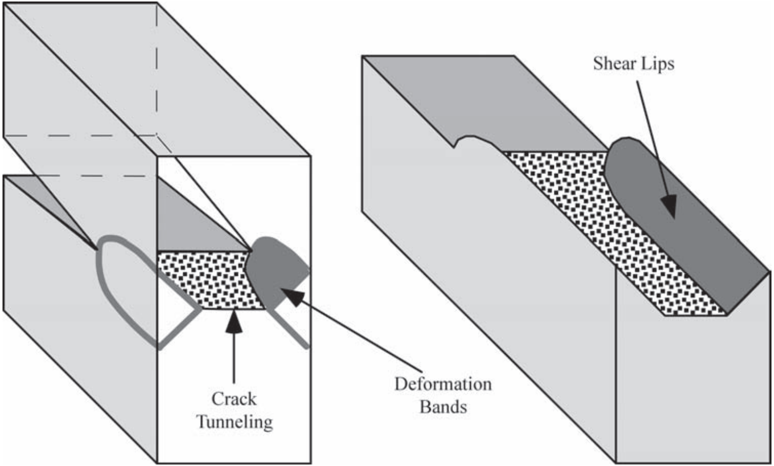
6. Void coalescence and fracture

Ductile fracture in tensile tests



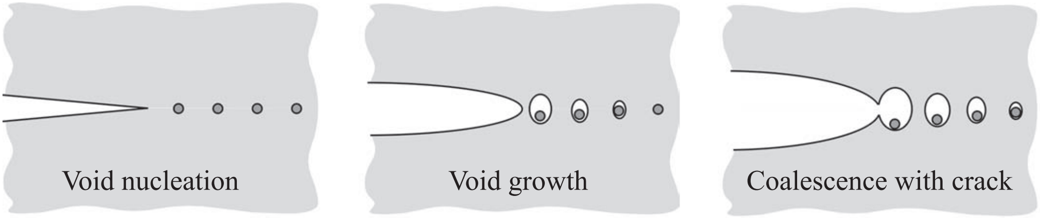
1. The triaxial stress state in the middle of the specimen promotes void nucleation and growth.
2. This leads to the formation of crack in the centre of the specimen. The crack leads to an accumulation of plastic strain along 45° bands.
3. This produces the cup and cone fracture surface observed in tensile tests.

Ductile crack propagation



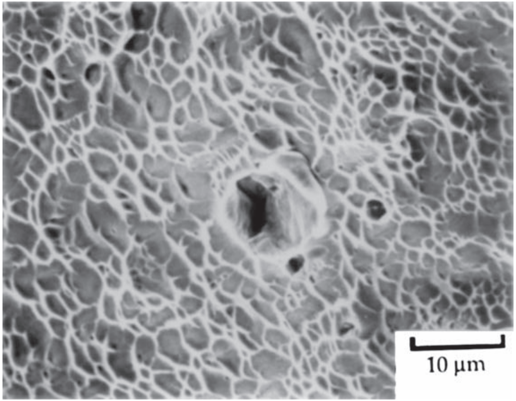
- Crack growth is flat and faster in the centre of the specimen. This is due to the higher stress triaxiality and known as crack tunneling.
- Closer to the free surfaces, the crack is often at 45°. These are called shear lips.

Ductile crack propagation



- Void nucleation and growth occurs at the crack tip because of the high stresses in this region.
- The crack grows when these voids link with the crack.
- Note how these voids also blunt the crack. This blunting effect is a cause of the rising R-curve of metals.

Fractography



Ductile fracture produces dimpled fracture surfaces. This is the fracture surface of a stainless steel.

Ductile fracture

There are multiple factors influencing the ductile fracture process:

- the volume fraction and dispersion of inclusions;
- the size and nature of inclusions;
- the stress state.

Should we remove all inclusions to increase toughness?

- Pure metal are very ductile, but have a very low yield strength so it is not a good idea to remove inclusions.
- Inclusions are also favorable for other properties such as manufacturability and resistance to corrosion.

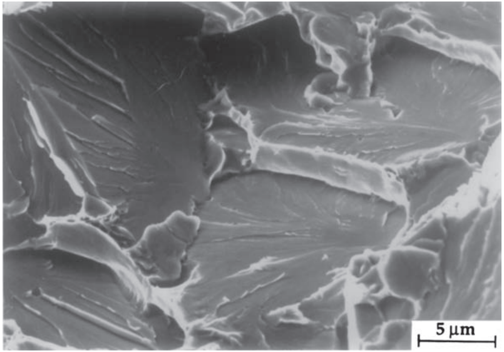
Outline

Cleavage

Cleavage is when a crack propagates rapidly along a particular crystallographic plane.

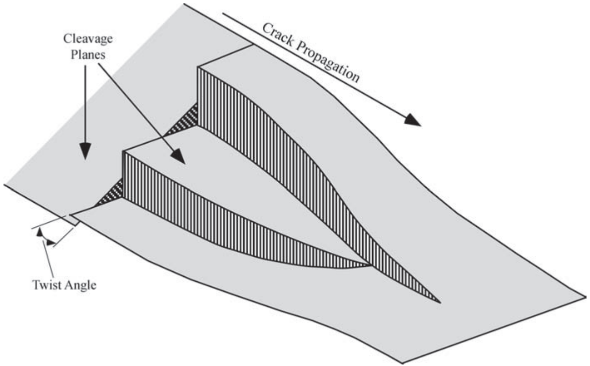
- Cleavage occurs along planes that have a low packing density (fewer bonds to break).
- Usually brittle, but may be preceeded by ductile crack growth.
- Cleavage typically happens when plasticity is restricted; for examples, at low temperatures.

Fractography



Cleavage produces multiple flat surfaces, where each facet corresponds to a grain. The thin lines on each facet are known as river patterns.

Formation of river patterns



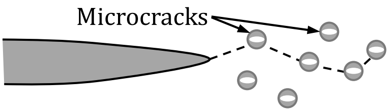
When a cleavage crack encounters a grain boundary, it has to change its orientation to follow the cleavage plane. This change in orientation produces the river patterns.

Cleavage

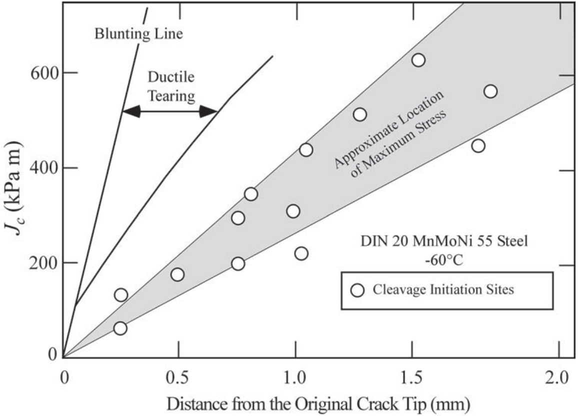
If cleavage is breaking atomic bonds along a specific crystallographic plane, can we capture this with an atomic model (see week 1)?

No, an atomic model would still be too strong.

For a cleavage crack to propagates, there needs to be additional microcracks providing a sufficient local stress concentration.



Cleavage fracture toughness



For cleavage, fracture toughness measurements are widely scattered. This is because the distance between the crack tip and the first large microcrack can vary significantly.

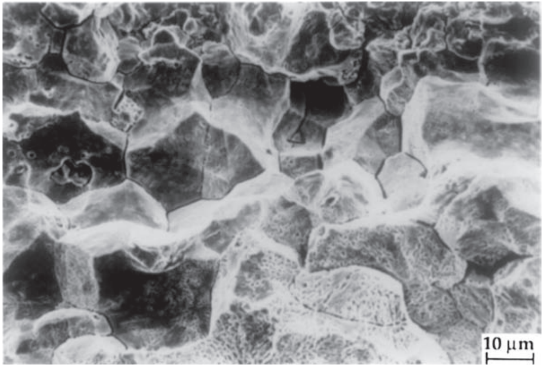
Outline

Intergranular fracture

Intergranular fracture is when the crack propagates along the grain boundaries. This is rare in metals, but it may occur in special circumstances such as when:

- the material is in a harsh environment, exposed to corrosion or radiation for example.
- a brittle phase accumulated at the grain boundaries.
- cavitation and cracking occured at the grain boundaries due to high temperatures.

Fractography

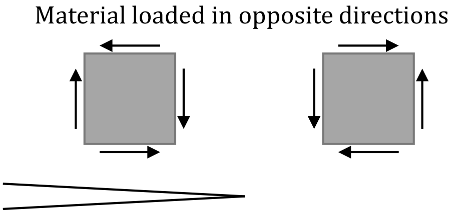


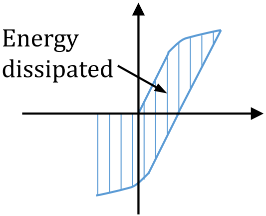
Intergranular fracture in a steel weld that was in contact with ammonia.

Outline

Why metals have a rising R-curve?

- Material ahead of the crack tip is loaded in one direction while material behind the crack is loaded in the other direction.
- When the crack advances, some material experiences a change in loading direction.
- This change dissipates energy by hysteresis.

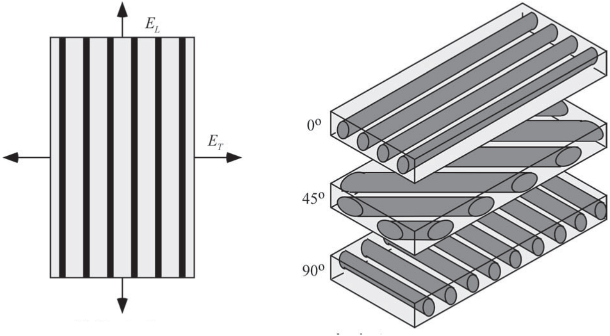




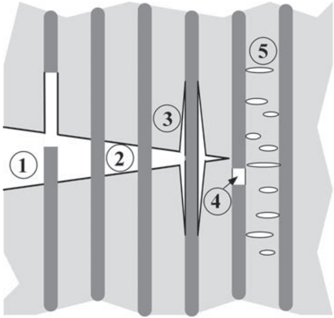
Outline

Fibre-reinforced composites

Carbon fibre reinforced composites are replacing metals in multiple lightweight applications. They exhibit very different fracture mechanisms than metals.



In-plane mechanisms



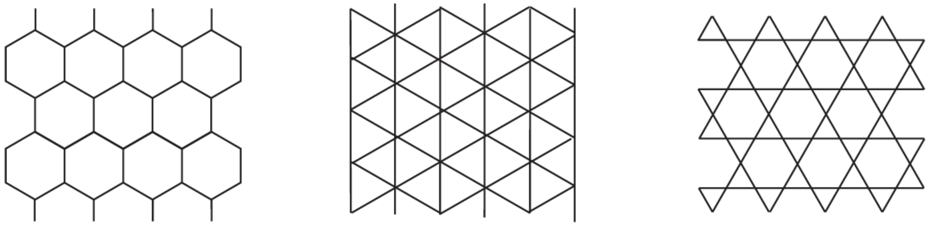
The most important mechanisms in plane are:

1. Fibre pull-out
2. Fibre bridging
3. Fibre/matrix debonding
4. Fibre failure
5. Matrix cracking

Outline

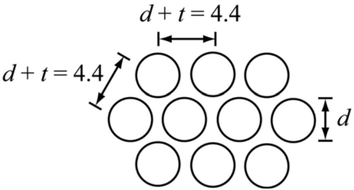
Outline

Lattice materials



- Lattices are interconnected arrays of beams.
- They can be made from any existing material.
- Combine high stiffness and high strength at low densities.

Geometry



Dimensions in mm

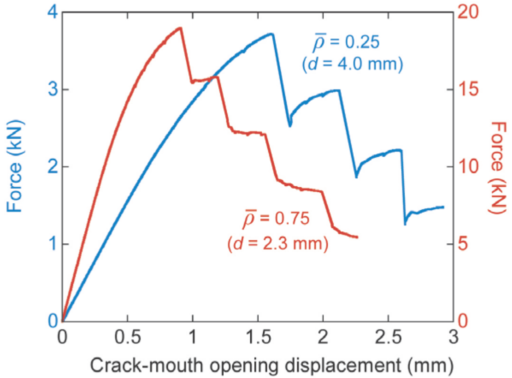
Relative density: $\bar{\rho} = 1 - \dfrac{\sqrt{3}\pi d^2}{6(d+t)^2}$

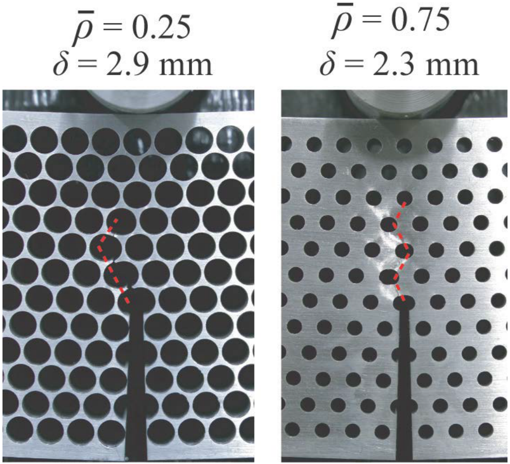
Dimensions of the specimens tested:

d (mm)	4.2	4.0	3.2	2.3	1.0
$\bar{\rho}$	0.17	0.25	0.52	0.75	0.95

Made by drilling holes in aluminum.

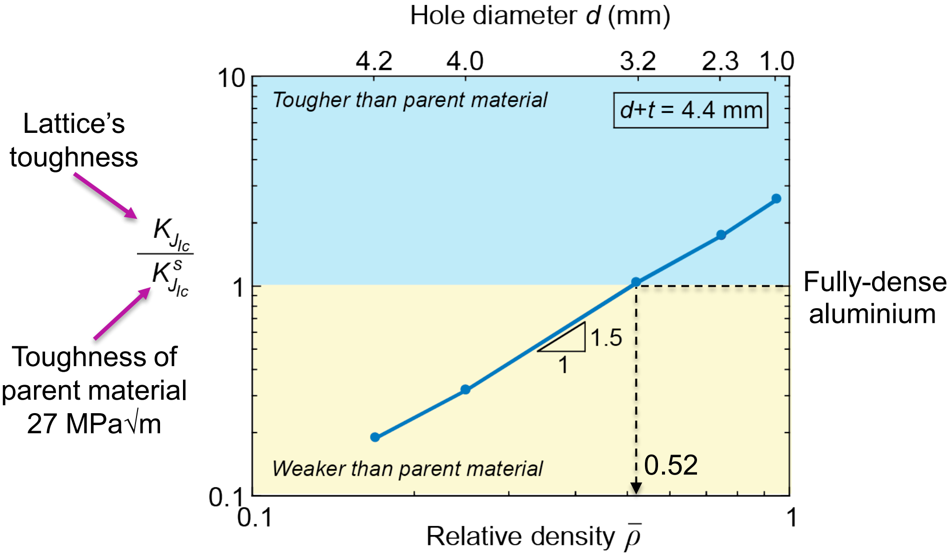
Toughness test





Each step in the load-displacement curve corresponds to a cell wall breaking.

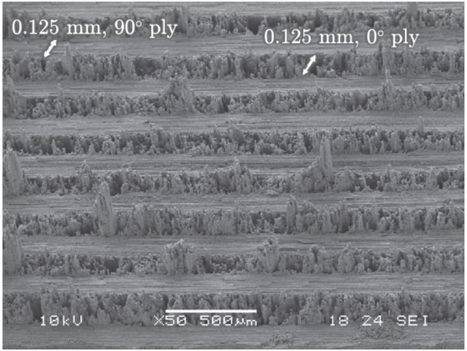
Results



- The lattice can be 50% lighter than aluminum and maintain the same toughness.

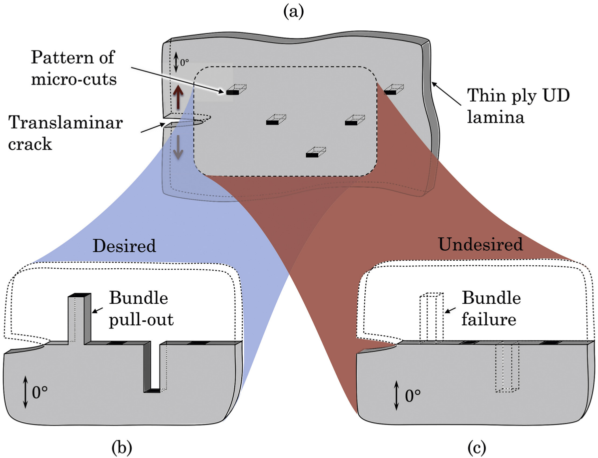
Outline

Motivation

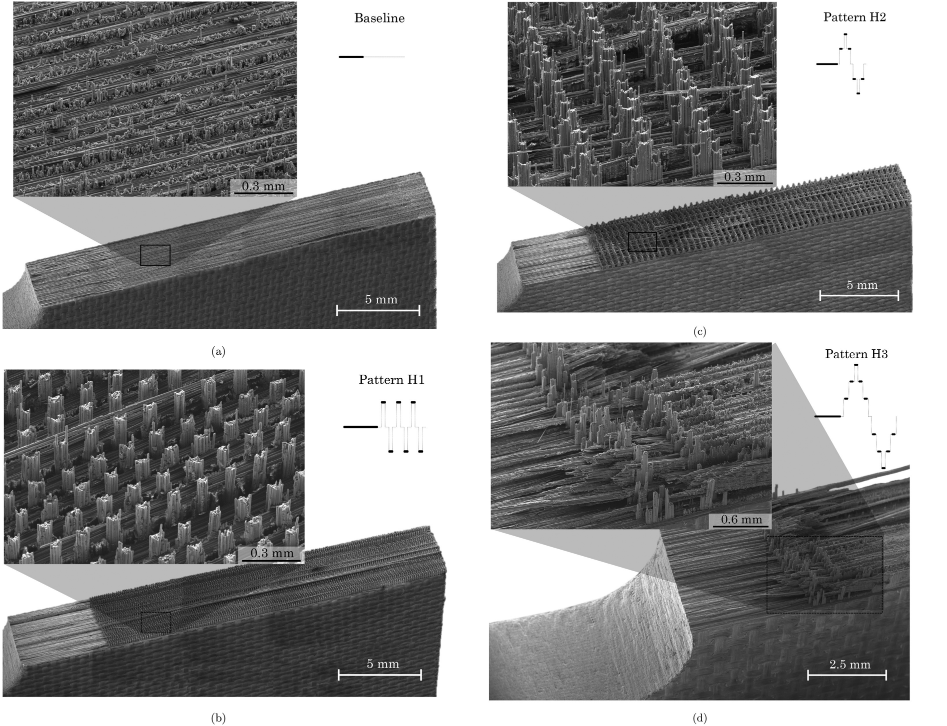


- A lot of energy is dissipated by fibre pull-out.
- Can we increase the pull-out length?

Method

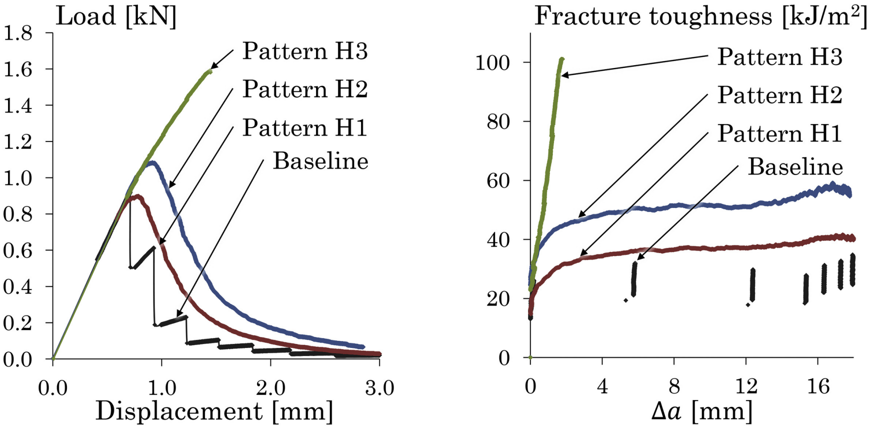


Provide a pattern of laser micro-cuts to guide the crack and increase the pull-out length.



Baseline
0.3 mm
5 mm
(a)
Pattern H1
0.3 mm
5 mm
(b)
Pattern H2
0.3 mm
5 mm
(c)
Pattern H3
0.6 mm
2.5 mm
(d)

Results



- Patterns H1 and H2 have stable crack growth while the baseline exhibit unstable crack growth.
- Increase in toughness of up to 200%.
- See Bullegas et al. (2016) Engineering the translaminar fracture behaviour of thin-ply composites.

Summary

- There is a strict protocol to measure the fracture toughness of metals: fatigue precrack, specimen geometry, and size requirements.
- Ductile fracture is the most common mechanism in metals. Cleavage and intergranular fracture can occur at low temperatures or in harsh environments.
- Several toughening mechanisms can lead to a rising R-curve: plasticity, crack tip blunting, fibre bridging, and/or fibre pull-out. These mechanisms can be exploited to design tougher materials.

Fracture Mechanics

6. Computational fracture mechanics

Luc St-Pierre

April 19, 2023

Learning outcomes

After this section, you will be able to:

- Describe the computational methods related to fracture mechanics.

Motivation

- The analytical methods covered so far allow us to tackle simple problems: 2D plane stress/strain with an isotropic linear elastic material.
- Numerical methods are needed to solve more complicated scenarios including:
  - 3D cracks inside complex geometries,
  - anisotropic materials (wood, composites),
  - cracks at the interface between two materials,
  - materials with plasticity or visco-elasticity.
- The numerical techniques presented below are all relying on the Finite Element Method (FEM).
- There are other ways to simulate fracture without FEM, but these will not be covered here.

Classification

Computational methods can be divided in two categories:

Stationary crack where you are trying to compute the stress intensity factors, the energy release rate or the J-integral.

Crack propagation where you are interested in predicting when and where the crack will propagate.

Of course, simulations for crack propagation require significantly more computational power than those for stationary cracks.

Outline

Stationary cracks

- Stress and displacement matching
- Energy release rate
- Contour integral

Crack propagation

- Cohesive surfaces and elements
- XFEM

Outline

Outline

Stress matching

Consider a crack loaded in mode I. Based on LEFM, the stress opening the crack σ_{yy} is given by (see datasheet):

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$

On the crack plane, $\theta = 0$, this reduces to:

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \quad \Longrightarrow \quad K_I = \sigma_{yy}\sqrt{2\pi r}$$

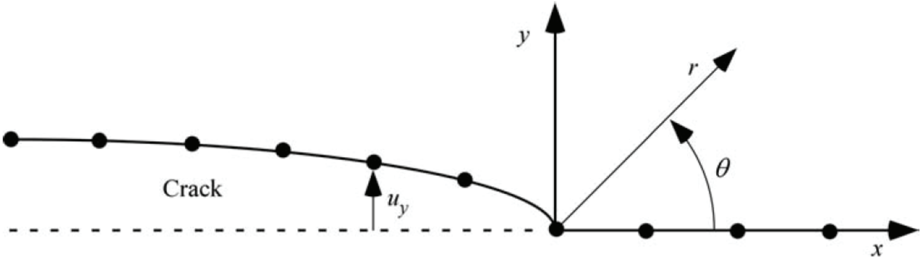
FE simulations will give you σ_{yy} for each element located a distance r from the crack tip. Therefore, you can use the above equation to compute K_I.

Displacement matching

Similarly, the displacement u_y of the crack surfaces at $\theta = \pi$ are given by:

$$u_y = \frac{4K_I}{E'}\sqrt{\frac{r}{2\pi}} \quad \Longrightarrow \quad K_I = \frac{E'u_y}{4}\sqrt{\frac{2\pi}{r}}$$

where $E' = E$ for plane stress and $E' = E/(1-\nu^2)$ for plane strain. Again, FE simulations will give you u_y and you can use this equation to get another estimate of K_I.



Stress and displacement matching

The results obtained above are valid very close to the crack tip. Therefore, to estimate K_I from FE simulations the expressions obtained earlier should be extrapolated to $r \to 0$. This gives us:

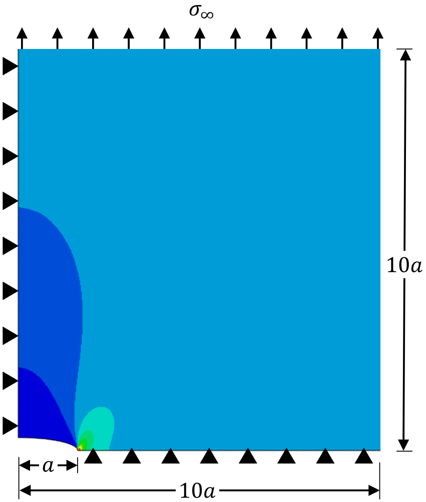
$$K_I = \lim_{r \to 0} \left[\sigma_{yy} \sqrt{2\pi r} \right]$$

and

$$K_I = \lim_{r \to 0} \left[\frac{E' u_y}{4} \sqrt{\frac{2\pi}{r}} \right]$$

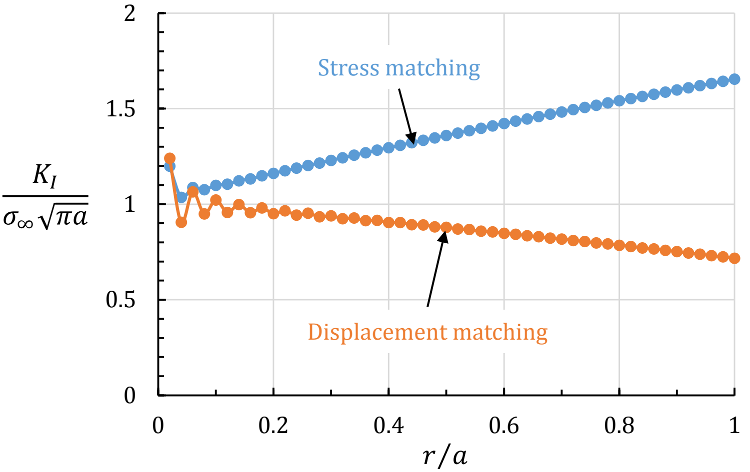
Example

To test this method, I have performed FE simulations on a large plate made from an elastic isotropic material.



Example

Both methods are accurate, but the first few elements show oscillations.



Stress and displacement matching

Advantages:

- simple to use.

Disadvantages:

- a fine mesh is required,
- not computed automatically by FE software,
- difficult to deal with mixed-mode scenarios.

Outline

Stationary cracks

Stress and displacement matching

Energy release rate

Contour integral

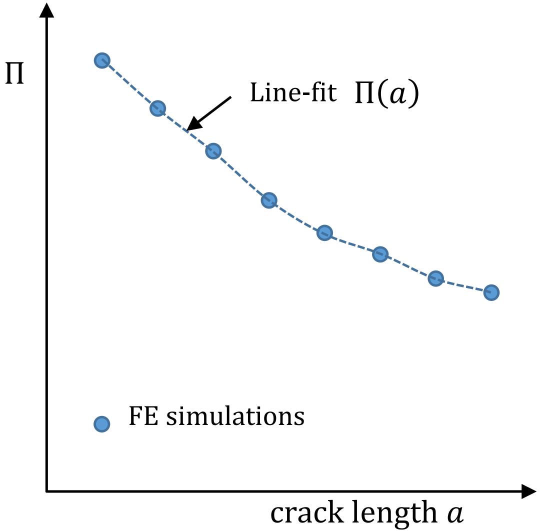
Crack propagation

Cohesive surfaces and elements

XFEM

Computing the energy release rate

Most FE codes can directly output the potential energy Π. If you run multiple simulations with different crack lengths a, you can plot the figure shown below.



The energy release rate is given by:

$$G = -\frac{\partial \Pi}{B \partial a}$$

Computing the energy release rate

Advantages:

- suitable for mixed-mode loading,
- energy is output directly from the FE software,

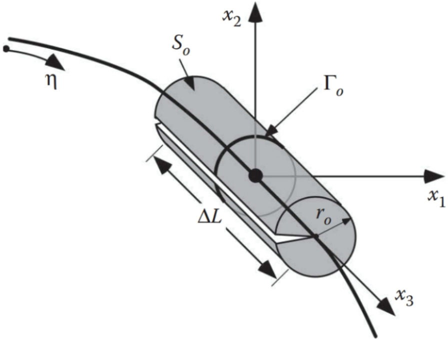
Disadvantages:

- doesn't separate the contribution of modes I, II or III,
- multiple simulations needed.
- may require a fine mesh to obtain accurate results.

Outline

Contour integral

The most advanced tool available in FE is the contour integral. It is based on the J-integral, and its definition has been extended to 3D cracks.



Contour integral

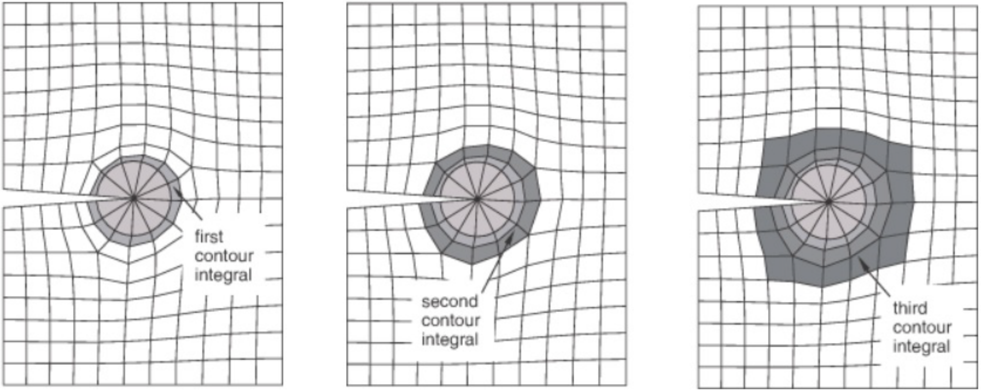
The theoretical formulation of the contour integral will not be covered here. If you want to compute this integral, the FE software will ask you to provide:

- the crack tip (2D) or the crack front (3D),
- the direction of crack propagation,
- the number of contours.

The first two are easy to understand, and the number of contours is shown next.

Number of contours

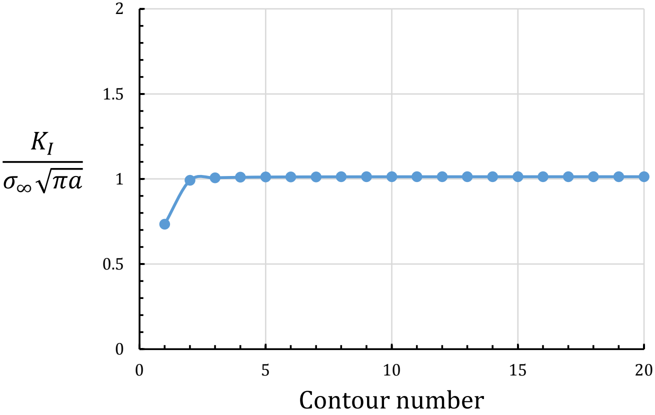
The number of contours is roughly the number of elements along the radius of the control domain.



As the number of contours is increased the value of the J-integral should converge.

Example

For the plate shown earlier, the contour integral gives accurate results with very few contours. The rate of convergence may depend on the mesh size.



Contour integral

Advantages:

- less mesh sensitive than other techniques,
- suitable for 2D and 3D cracks,
- the software may be able compute directly the stress intensity factors from the J-integral,

Disadvantages:

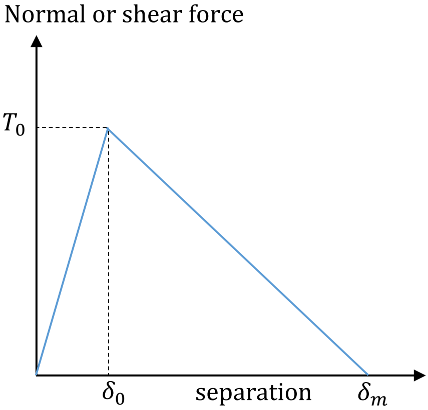
- not usually available with triangular and tetrahedral elements,
- may not be implemented in some FE codes.

Outline

Outline

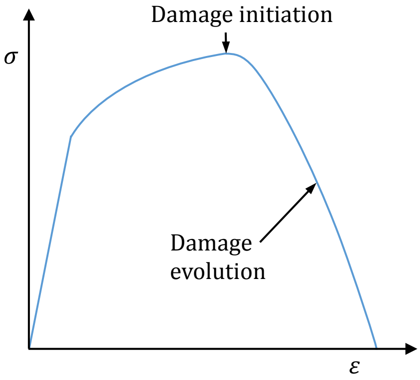
Cohesive surfaces

Cohesive surfaces are not elements, but they describe how two elements can be separated. Accordingly, they are defined by a traction-separation law, as shown below.



Cohesive elements - Progressive damage models

Cohesive elements (or elements with progressive damage) have a stress-strain curve that includes a softening law as shown below. The elements can be deleted when they reach the end of the damage evolution law.



Example

Have a look at the video *PlateHole.avi*. This is a simulation of a plate which includes a notch and a hole. Here are a few more details about the FE model:

- The material properties are those of aluminium. It includes plasticity and failure initiates at a plastic strain of 10%. Degradation is linear up to a strain of 15%.
- It is a 2D model in plane stress.
- In the movie, elements in red have failed completely but they are not deleted from the model. Elements in green have started to fail but they can still carry some load.

Cohesive surfaces and elements

Advantages:

- cohesive surfaces are excellent to model debonding between two different materials,
- the softening curve can be used to model fracture mechanisms such as crack bridging or microcracking,

Disadvantages:

- with cohesive surfaces, the crack can only propagate along these predefined surfaces,
- with cohesive elements, the crack tip has the shape of an element, which is not particularly sharp,
- in both cases, the results are often sensitive to the mesh size,
- convergence may be difficult.

Outline

XFEM

The Extended Finite Element Method is a numerical method based on FEM. This extension was developed to deal with discontinuities inside the elements, such as cracks, holes or changes in materials.

The theoretical framework of XFEM is beyond the scope of this course. Only a brief overview is given here.

XFEM

With the conventional FEM, the displacement vector $\mathbf{u}$ at any point x is given by:

$$\mathbf{u} = \sum_{I=1}^{N} N_I(x)\mathbf{u_I}$$

where N_I are the shape functions and $\mathbf{u_I}$ are the nodal displacements.

In XFEM, additional terms, called enrichment functions, are added to the above equation to model discontinuities inside the elements, such as cracks.

XFEM

Advantages:

- 2D and 3D cracks can be added without modifying the mesh,
- less mesh sensitive than other methods,
- can be used for stationary and propagating cracks,

Disadvantages:

- not implemented in all codes,
- a more complicated formulation that requires a more experienced user.

Summary

There is a range of computational methods related to fracture mechanics, and your selection should consider a number of factors:

- is it a stationary crack or a crack propagation problem?
- what are the materials involved? (some methods are limited to isotropic materials or structures made of a single material)
- which functionalities are available in your Finite Element software?