A? Question

A large but thin steel plate has developed a central through crack of 70 mm in length after being subjected to cyclic loading. If the steel has a fracture toughness $K_{Ic} = 50\,\text{MPa}\sqrt{\text{m}}$, calculate the maximum stress that the plate can support when loaded in tension perpendicularly to the crack plane.

A! Solution

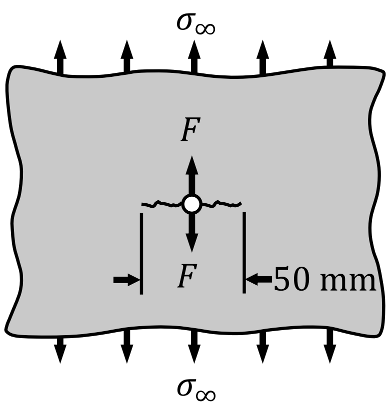
The plate will fracture when the stress intensity factor K_I is equal to the fracture toughness K_{Ic}. For a large with a central through crack, the stress intensity factor is $K_I = \sigma_\infty\sqrt{\pi a}$ (see datasheet). Therefore, we have:

$$K_{Ic} = K_I = \sigma_\infty\sqrt{\pi a} \quad \Rightarrow \quad \sigma_\infty = \frac{K_{Ic}}{\sqrt{\pi a}} = \frac{50}{\sqrt{\pi 0.070/2}} = 150.8\,\text{MPa}$$

This problem is not particularly complicated, but there are two elements to be careful about. First, be careful with units: crack lengths are often given in mm, but fracture toughness has units of $\text{MPa}\sqrt{\text{m}}$, where the length scale is in m. Second, the notation used in the datasheet (and in most textbooks) is that central cracks have a length $2a$, whereas edge cracks have a length a. Therefore, in this problem the crack length is $2a = 70\,\text{mm}$.

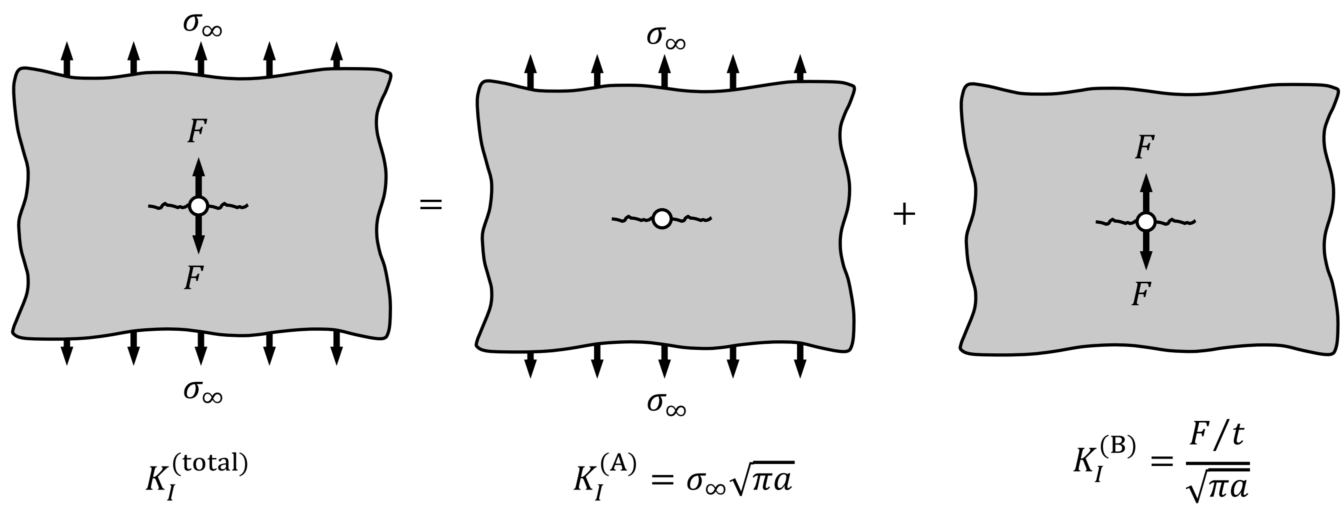
A? Question

A large plate, containing a small hole, has developed cracks on both sides as shown below. In addition, a nail was inserted in the hole and this is generating a force F because the nail was larger than the opening. The plate (with the nail) was then tested in tension and fractured at an applied stress $\sigma_\infty = 10\,\text{MPa}$. Calculate the force F (in N) provided that the plate had a thickness $t = 3\,\text{mm}$ and was made from a polymer with a fracture toughness $K_{Ic} = 3\,\text{MPa}\sqrt{\text{m}}$.



A! Solution

The stress intensity factor for the plate with the nail has to be obtained with the principle of superposition.



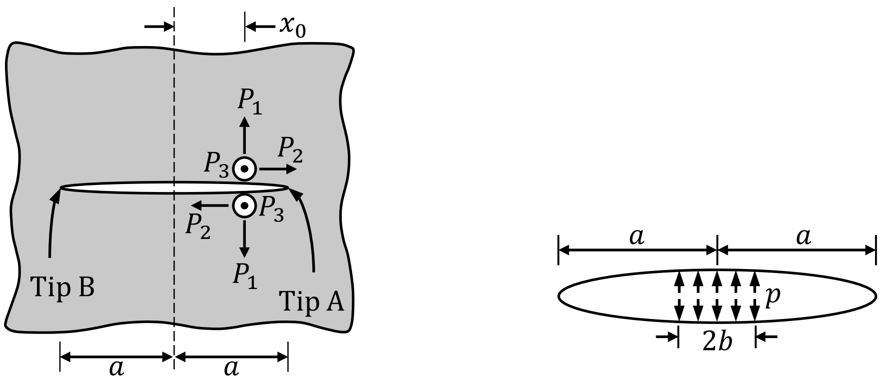
Where K_I^{A} and K_I^{B} were obtained from the datasheet. Note that K_I^{B} requires a force per unit depth and this is why we have F/t at the numerator. At the moment of fracture we have:

$$K_I^{\text{total}} = K_I^{\text{A}} + K_I^{\text{B}} = K_{Ic}$$
$$\Rightarrow \sigma_\infty\sqrt{\pi a} + \frac{F/t}{\sqrt{\pi a}} = K_{Ic}$$
$$\Rightarrow F = t(K_{Ic} - \sigma_\infty\sqrt{\pi a})\sqrt{\pi a} = 0.003(3\text{e}6 - 10\text{e}6\sqrt{\pi \cdot 0.025})\sqrt{\pi \cdot 0.025} = 166\,\text{N}$$

A? Question

Use the stress intensity factor for a point force (geometry shown on the left) and the principle of superposition to prove that the stress intensity factor for a localised pressure (image on the right) is:

$$K_I = \frac{2}{\sqrt{\pi}} p\sqrt{a}\arcsin\frac{b}{a}$$



A! Solution

The stress intensity factor for a point force can be used to derive K_I for a localised pressure by recognising that $P_1 = pdx$. Look carefully at the solution for a point force in the datasheet; there are two formulas depending if the force is on the same side as the crack tip or on opposite sides. Based on the stress intensity factor for a point force, K_I for the localised pressure can be obtained by integrating:

$$
\begin{aligned}
K_I &= \int_0^b \frac{pdx}{\sqrt{\pi a}}\sqrt{\frac{a+x}{a-x}} + \int_0^b \frac{pdx}{\sqrt{\pi a}}\sqrt{\frac{a-x}{a+x}} \\
&= \frac{p}{\sqrt{\pi a}} \int_0^b \left(\sqrt{\frac{a+x}{a-x}} + \sqrt{\frac{a-x}{a+x}}\right) dx \\
&= \frac{p}{\sqrt{\pi a}} \int_0^b \frac{(a+x)+(a-x)}{\sqrt{(a-x)(a+x)}} dx \\
&= \frac{2pa}{\sqrt{\pi a}} \int_0^b \frac{dx}{\sqrt{a^2-x^2}} \\
&= \frac{2}{\sqrt{\pi}} p\sqrt{a} \left[\arcsin\frac{x}{a}\right]_0^b \\
&= \frac{2}{\sqrt{\pi}} p\sqrt{a} \arcsin\frac{b}{a}
\end{aligned}
$$

This is the same answer given in the datasheet for a localised pressure. This is not a particularly easy integral to solve but the most important element in this course is that are able to come up with the correct integral to solve i.e. write the first line. You can always use Matlab or online integral tables to help you solve the integral.

A? Question

In the lecture notes, the stress field around a crack tip (slides 13-23) was derived for a crack loaded in mode I. What would you have to do differently if the crack was loaded in mode II (instead of mode I)?

A! Solution

In mode II, the shear stress should be symmetric: $\sigma_{r\theta}(\theta) = \sigma_{r\theta}(-\theta)$. This implies that the constants $A = C = 0$ whereas $B \neq 0$ and $D \neq 0$, see slide 15.

A? Question

In the lecture notes, all stresses tend to infinity at the crack tip when $r \to 0$ (see slide 23). If stresses are infinitely high, explain why a component does not necessarily fracture when there is a crack.

A! Solution

The theoretical analysis considers an infinitely sharp crack, whereas in reality, the crack tip has a finite radius (it is incredibly small but not zero). When the crack tip has a finite radius, the stresses also become finite (not going to infinity anymore).

A? Question

In the analysis of the stress field, we found multiple values of $\lambda = 1/2; 1; 3/2; \ldots$ (see slide 21) Why did we keep only $\lambda = 1/2$? Could the other terms be useful?

A! Solution

We kept $\lambda = 1/2$ because this term gives the highest stresses close to the crack tip $r \to 0$. Other terms are useful to respect boundary conditions when the problem includes dimensions and external loads.

A? Problem 1.1

A thick plate of aluminium alloy, 200 mm wide, contains an edge crack of 60 mm in length. The plate is loaded by a tensile stress perpendicular to the crack plane. The plate fractures in a brittle way at an applied stress of 40 MPa.

(a) Determine the fracture toughness K_{Ic} of the material.

(b) What would be the fracture stress if the plate was wide enough to assume an infinite width?

A! Solution

Part (a). The plate fractured at $\sigma_\infty = 40\,\text{MPa}$; therefore at that moment, we have $K_I = K_{Ic}$. For this configuration, we can find in the datasheet that the stress intensity factor K_I is given by:

$$\begin{aligned}
K_{Ic} &= K_I \\
&= \sigma_\infty \sqrt{\pi a} \left[1.12 - 0.23 \frac{a}{W} + 10.6 \left(\frac{a}{W}\right)^2 - 21.7 \left(\frac{a}{W}\right)^3 + 30.4 \left(\frac{a}{W}\right)^4 \right] \\
&= 40\,\text{MPa} \cdot \sqrt{\pi 0.060\,\text{m}} \cdot [1.6653] \\
&= 28.9\,\text{MPa}\sqrt{\text{m}}
\end{aligned}$$

The fracture toughness of this aluminium alloy is $K_{Ic} = 28.9\,\text{MPa}\sqrt{\text{m}}$.

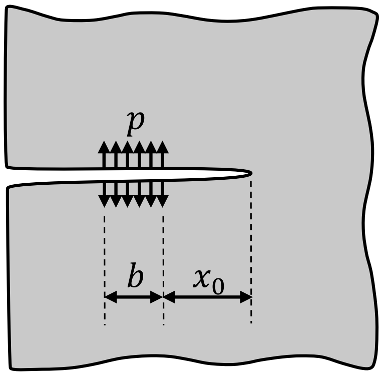
Part (b). If the plate was infinitely large then the stress intensity factor would be given by $K_I = 1.12\sigma_\infty\sqrt{\pi a}$. Fracture would occur when:

$$\begin{aligned}
& K_{Ic} = K_I = 1.12\sigma_\infty\sqrt{\pi a} \\
\implies \; & \sigma_\infty = \frac{K_{Ic}}{1.12\sqrt{\pi a}} = \frac{28.9}{1.12\sqrt{\pi 0.060}} = 59.5\,\text{MPa}
\end{aligned}$$

If the plate was infinitely large it would fracture at $\sigma_\infty = 59.5\,\text{MPa}$.

A? Problem 1.2

Find the stress intensity factor K_I for an edge crack loaded by a pressure p over a portion b as shown below. Hint: you will have to integrate the solution for a point force.



A! Solution

The stress intensity factor for a point force on an edge crack is (see datasheet):

$$K_I = \frac{2P}{\sqrt{2\pi x_0}}$$

where P is a force per unit depth and x_0 is the distance between the force and the crack tip. Therefore, the solution for a pressure p, over a portion b, can be obtained by integrating:

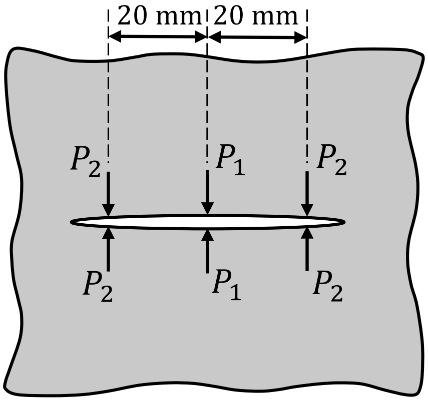
$$\begin{aligned} K_I &= \int_{x_0}^{x_0+b} \frac{2p dx}{\sqrt{2\pi x}} \\ &= \frac{2p}{\sqrt{2\pi}} \int_{x_0}^{x_0+b} \frac{dx}{\sqrt{x}} \\ &= \frac{2p}{\sqrt{2\pi}} \left[2\sqrt{x}\right]_{x_0}^{x_0+b} \\ &= \frac{4p}{\sqrt{2\pi}} \left(\sqrt{x_0+b} - \sqrt{x_0}\right) \end{aligned}$$

A? Problem 1.3

A thin polymer plate is fabricated by casting. The process creates a central crack of length $2a = 50\,\text{mm}$. The plate is then tested by applying a tensile stress σ_∞ in the direction normal to the crack plane.

(a) If the plate failed at a stress $\sigma_\infty = 5\,\text{MPa}$, evaluate the fracture toughness K_{Ic} of the material.

(b) Another plate is produced from the same material, but this time copper wires are introduced to act as reinforcements. These wires have a 20 mm spacing, and one of them crosses the central crack exactly through the middle. These wires can be assumed to create local forces closing the crack as shown in the figure below (where $P_1 = 50\,\text{kN/m}$ and $P_2 = 30\,\text{kN/m}$). Determine the value of σ_∞ that will trigger fracture.

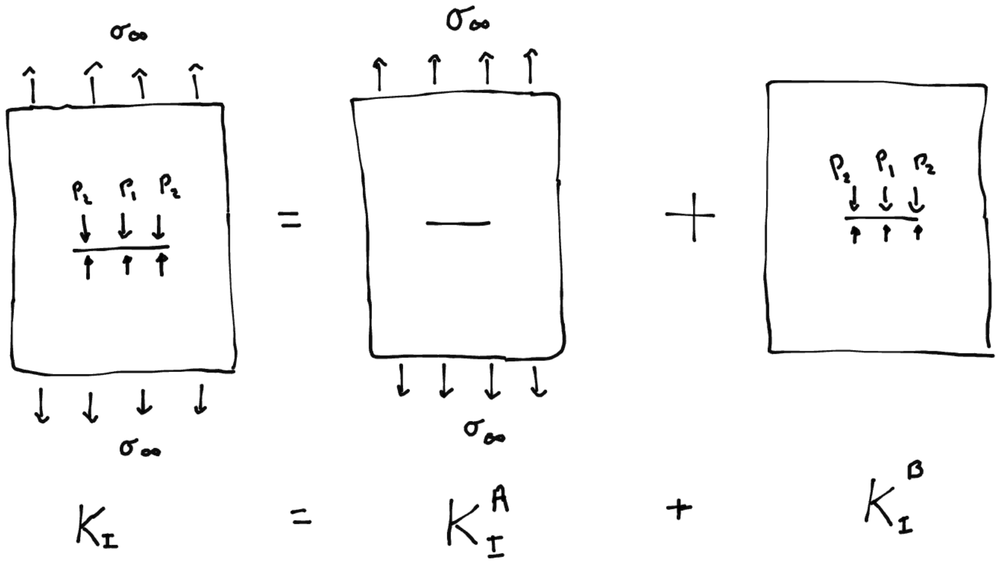


A! Solution

Part (a). The plate failed when the stress intensity factor K_I reached the fracture toughness K_{Ic}. Therefore, we have:

$$K_{Ic} = K_I = \sigma_\infty\sqrt{\pi a} = 5\,\text{MPa} \cdot \sqrt{\pi 0.025\,\text{m}} = 1.4012\,\text{MPa}\sqrt{\text{m}}$$

Part (b). The stress intensity factor K_I can be obtained using the principle of superposition:



where K_I^B will be negative because the forces are closing the crack. Using the datasheet, K_I^B is given by:

$$
\begin{aligned}
K_I^B &= \frac{-P_1}{\sqrt{\pi a}} + \frac{-P_2}{\sqrt{\pi a}} \cdot \sqrt{\frac{a+x_0}{a-x_0}} + \frac{-P_2}{\sqrt{\pi a}} \cdot \sqrt{\frac{a-x_0}{a+x_0}} \\
&= \frac{-50000}{\sqrt{\pi 0.025}} + \frac{-30000}{\sqrt{\pi 0.025}} \cdot \sqrt{\frac{25+20}{25-20}} + \frac{-30000}{\sqrt{\pi 0.025}} \cdot \sqrt{\frac{25-20}{25+20}} \\
&= -0.1784 - 0.3211 - 0.0357 \\
&= -0.5352 \,\text{MPa}\sqrt{\text{m}}
\end{aligned}
$$

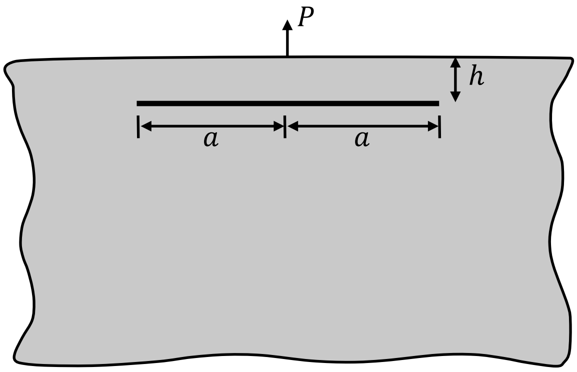
Note that the second and third terms in K_I^B are not equal: the second represents the contribution of P_2 close to the crack tip, whereas the third is the contribution of the furthest P_2. The plate will fracture when $K_I = K_{Ic}$ which gives:

$$
\begin{aligned}
& K_I = K_I^A + K_I^B = K_{Ic} \\
\implies \; & \sigma_\infty \sqrt{\pi a} + K_I^B = K_{Ic} \\
\implies \; & \sigma_\infty = \frac{K_{Ic} - K_I^B}{\sqrt{\pi a}} = \frac{1.4012 + 0.5352}{\sqrt{\pi \cdot 0.025}} = 6.9 \,\text{MPa}
\end{aligned}
$$

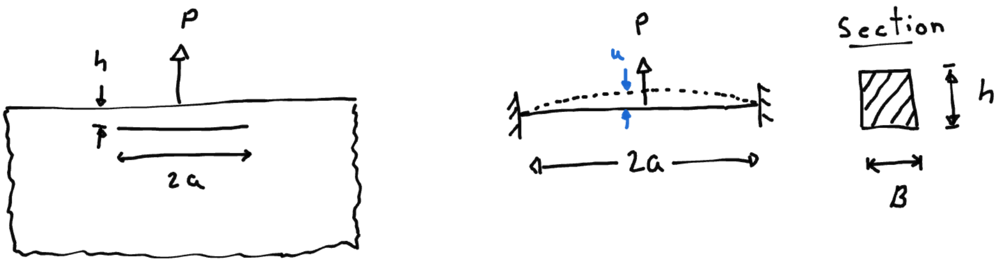
Adding the wires will increase the fracture stress to 6.9 MPa.

A? Question

A crack of length $2a$ is parallel to the edge of a thin semi-infinite plate subjected to a point force P. Find the energy release rate G and the stress intensity factor K_I, provided that the out-of-plane thickness is B and the material has a Young's modulus E. Hint: assume that the material above the crack deforms like a beam clamped at both ends.



A! Solution



The deflection of a beam clamped at both ends is given by:

$$u = \frac{P(2a)^3}{192EI} = \frac{8Pa^3}{192E} \cdot \frac{12}{Bh^3} = \frac{Pa^3}{2EBh^3}$$

Therefore, the compliance is:

$$C = \frac{u}{P} = \frac{a^3}{2EBh^3}$$

Using the compliance, the energy release rate becomes:

$$G = \frac{P^2}{2B}\frac{dC}{da} = \frac{P^2}{2B} \cdot \frac{3a^2}{2EBh^3} = \frac{3P^2a^2}{4EB^2h^3}$$

The plate is thin so we can assume plane stress, and the stress intensity factor is:

$$K_I = \sqrt{EG} = \sqrt{\frac{3P^2a^2}{4B^2h^3}} = \frac{\sqrt{3}Pa}{2Bh^{3/2}}$$

A? Question

The R-curve of a high strength steel ($E = 210\,\text{GPa}$) can be expressed as:

$$R = 0.1 + 3(a - a_0)^{0.1},$$

where R is given in MJ/m^2, whereas a and a_0 are in m. A large thin plate, made from this material, has a centre crack of length $2a_0 = 60\,\text{mm}$. Find the amount of stable crack growth Δa and the stress σ_∞ at which fracture will become unstable.

A! Solution

The energy release rate for this geometry is:

$$G = \frac{K_I^2}{E} = \frac{(\sigma_\infty\sqrt{\pi a})^2}{E} = \frac{\sigma_\infty^2 \pi a}{E},$$

where we have assumed that we are under plane stress since the plate is thin. The moment at which fracture will become unstable is when:

$$G = R \qquad \text{and} \qquad \frac{dG}{da} = \frac{dR}{da}$$

The first condition gives us:

$$G = R \quad \Rightarrow \quad \frac{\sigma_\infty^2 \pi a}{E} = 0.1 + 3(a - a_0)^{0.1} \tag{1}$$

whereas the second condition returns:

$$\frac{dG}{da} = \frac{dR}{da} \quad \Rightarrow \quad \frac{\sigma_\infty^2 \pi}{E} = 3 \cdot 0.1(a - a_0)^{-0.9} \tag{2}$$

Note that we now have a system of two equations and two unknowns: σ_∞ and a. To solve the system, we can substitute Eq. (2) into (1) to obtain:

$$3 \cdot 0.1(a - a_0)^{-0.9} \cdot a = 0.1 + 3(a - a_0)^{0.1} \quad \text{where} \quad a_0 = 0.030\ \text{m}$$

This is difficult to solve analytically, but using Matlab we find:

$$a = 0.0331\ \text{m} \quad \Rightarrow \quad \Delta a = a - a_0 = 3.1\ \text{mm}$$

Next, we can substitute this result in Eq. (1) or (2) to solve for σ_∞. Using Eq. (2), we find:

$$\sigma_\infty = \sqrt{\frac{E}{\pi} 3 \cdot 0.1(a - a_0)^{-0.9}} = \sqrt{\frac{210000}{\pi} \cdot 3 \cdot 0.1(0.0031)^{-0.9}} = 1905\ \text{MPa}$$

This value of σ_∞ is rather high, even for a high strength steel, but that is sufficient to demonstrate how to calculate the amount of stable crack growth and the stress at which fracture will become unstable. The methodology is simple, but you have to be extremely careful with units to make sure the value of G is in the same units as the value of R.

A? Question

The energy release rate G for a double cantilever beam (slide 14) with an applied force P is given by:

$$G = \frac{12P^2a^2}{EB^2h^3},$$

which shows that G increases with increasing crack length a. Explain how you could redesign the geometry to ensure that G remains constant with increasing crack length a.

A! Solution

This could be done by varying the thickness h of each beam along their length, such that the compliance C becomes a linear function of the crack length a. If $C \propto a$ then, dC/da will be a constant and the energy release rate G will be independent of the crack length a.

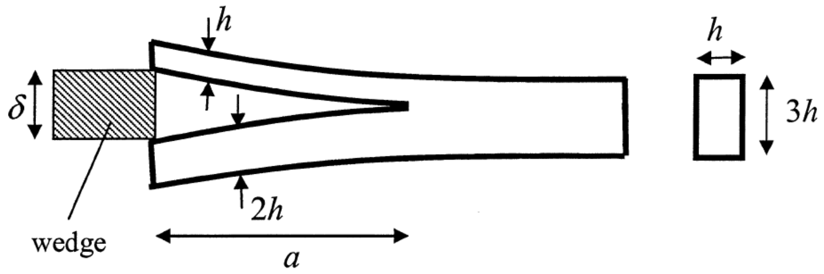
A? Problem 2.1

Wooden chop-sticks have the geometry shown below, where the arms are opened by a wedge of height δ. The wood has a linear elastic behaviour with a Young's modulus E.

(a) Determine the compliance of each arm of the chop-sticks.

(b) Calculate the energy release rate G.

(c) Will crack growth be stable or unstable? Assume that the material has a flat R-curve.



A! Solution

Part (a). According to beam theory, the deflection u of a cantilever beam, subjected to a force P at its extremity, is:

$$u = \frac{PL^3}{3EI}$$

where L is the beam's length, E is the Young's modulus and $I = bt^3/12$ for a rectangular cross-section of thickess t and width b. Here, the top and bottom arms have a different I, and these are:

$$I_{top} = \frac{h \cdot h^3}{12} = \frac{h^4}{12} \quad \text{and} \quad I_{bot} = \frac{h(2h)^3}{12} = \frac{2h^4}{3}$$

The wedge will create a force P on each arm, but their deflections (u_{top}; u_{bot}) will be different. Their compliance C are given by:

$$C_{top} = \frac{u_{top}}{P} = \frac{a^3}{3EI_{top}} = \frac{a^3}{3E} \cdot \frac{12}{h^4} = \frac{4a^3}{Eh^4}$$

$$C_{bot} = \frac{u_{bot}}{P} = \frac{a^3}{3EI_{bot}} = \frac{a^3}{3E} \cdot \frac{3}{2h^4} = \frac{a^3}{2Eh^4}$$

Part (b). The compliance C of system is:

$$C = \frac{\delta}{P} = \frac{u_{top} + u_{bot}}{P} = C_{top} + C_{bot} = \frac{4a^3}{Eh^4} + \frac{a^3}{2Eh^4} = \frac{9a^3}{2Eh^4}$$

From this, we can get an expression for the force P as a function of δ:

$$C = \frac{\delta}{P} = \frac{9a^3}{2Eh^4} \implies P = \frac{2Eh^4\delta}{9a^3}$$

Finally, we can get the energy release rate G using the compliance formula:

$$\begin{aligned} G &= \frac{P^2}{2h} \cdot \frac{dC}{da} \\ &= \frac{1}{2h} \cdot \left[\frac{2Eh^4\delta}{9a^3}\right]^2 \cdot \left[\frac{9}{2Eh^4} \cdot 3a^2\right] \\ &= \frac{1}{2h} \cdot \frac{4E^2h^8\delta^2}{81a^6} \cdot \frac{27a^2}{2Eh^4} \\ &= \frac{Eh^3\delta^2}{3a^4} \end{aligned}$$

Note that in this question, we have an applied displacement δ, and not an applied force P. Therefore, the final answer for the energy release rate G should be a function of the applied displacement δ only (the force P should not appear in G). Of course, this would be the opposite if the problem had an applied force P instead of a displacement.

Part (c). Crack growth will be stable if $dG/da < 0$, and otherwise, unstable. Computing dG/da gives:

$$\frac{dG}{da} = \frac{d}{da}\left[\frac{Eh^3\delta^2}{a^4}\right] = \frac{Eh^3\delta^2}{3} \cdot (-4a^{-5}) = -\frac{4Eh^3\delta^2}{3a^5} < 0$$

Therefore, crack growth will be stable.

A? Problem 2.2

The R-curve for a steel alloy is given by:

$$R = \frac{K_{Ic}^2}{E} + \frac{1}{2}\sqrt{\Delta a}$$

where R is in MJ/m^2, the crack extension Δa is in meters, $K_{Ic} = 95\,\text{MPa}\sqrt{\text{m}}$ and $E = 210000\,\text{MPa}$. A large but thin plate is made from this material and contains a centre crack of length $2a_0 = 40\,\text{mm}$.

(a) Show that this plate allows a maximum stable crack growth of 6.3 mm at both tips.

(b) Calculate the critical stress σ_c at which unstable fracture will occur.

A! Solution

Part (a). The energy release rate G for a thin (plane stress) plate with a centre crack is:

$$G = \frac{K_I^2}{E} = \frac{\pi\sigma^2 a}{E} = \frac{\pi\sigma^2(a_0 + \Delta a)}{E}$$

Units are important in this problem. With dimensional analysis we can show that if a and Δa are in m; and σ and E are in MPa; then, G is in MJ/m^2 (same units as the R curve given in the question). Unstable fracture will occur when two conditions are met:

$$G = R \qquad \text{and} \qquad \frac{dG}{d(\Delta a)} = \frac{dR}{d(\Delta a)}$$

The first condition gives:

$$G = R \quad \Longrightarrow \quad \frac{\pi\sigma^2(a_0 + \Delta a)}{E} = \frac{K_{Ic}^2}{E} + \frac{1}{2}\sqrt{\Delta a} \tag{1}$$

And the second condition gives:

$$\frac{dG}{d(\Delta a)} = \frac{dR}{d(\Delta a)} \quad \Longrightarrow \quad \frac{\pi\sigma^2}{E} = \frac{1}{4\sqrt{\Delta a}} \tag{2}$$

Substituting (2) in (1) gives:

$$\begin{aligned}
&\frac{1}{4\sqrt{\Delta a}} \cdot (a_0 + \Delta a) = \frac{K_{Ic}^2}{E} + \frac{1}{2}\sqrt{\Delta a} \\
&\Longrightarrow\; a_0 + \Delta a = \frac{4K_{Ic}^2}{E}\sqrt{\Delta a} + 2\Delta a \\
&\Longrightarrow\; 0 = -a_0 + \frac{4K_{Ic}^2}{E}\sqrt{\Delta a} + \Delta a \\
&\Longrightarrow\; 0 = -0.020 + 0.1719\sqrt{\Delta a} + \Delta a
\end{aligned}$$

This is a quadratic equation of the form $ax^2 + bx + c = 0$, where $x = \sqrt{\Delta a}$. Solving this returns:

$$\sqrt{\Delta a} = -0.2514 \quad \text{and} \quad 0.0795$$

The negative answer is impossible; therefore, the stable crack growth up to failure is:

$$\Delta a = 0.0795^2 = 6.3\,\text{mm}$$

Part (b). We can substitute Δa in (2) to find the critical stress σ_c at failure:

$$\frac{\pi\sigma_c^2}{E} = \frac{1}{4\sqrt{\Delta a}}$$

$$\implies \sigma_c = \frac{1}{2}\sqrt{\frac{E}{\pi\sqrt{\Delta a}}} = \frac{1}{2}\sqrt{\frac{210000}{\pi\sqrt{0.0063}}} = 458\,\text{MPa}$$

A? Problem 2.3

The following data were obtained from a series of tests conducted on pre-cracked specimens with a thickness $B = 10\,\text{mm}$.

Crack length a (mm)	Compliance C (mm/kN)	Critical load P (kN)
50.0	0.100	10.00
66.7	0.143	8.75
84.2	0.202	7.80
102.7	0.279	7.00
119.5	0.359	6.55

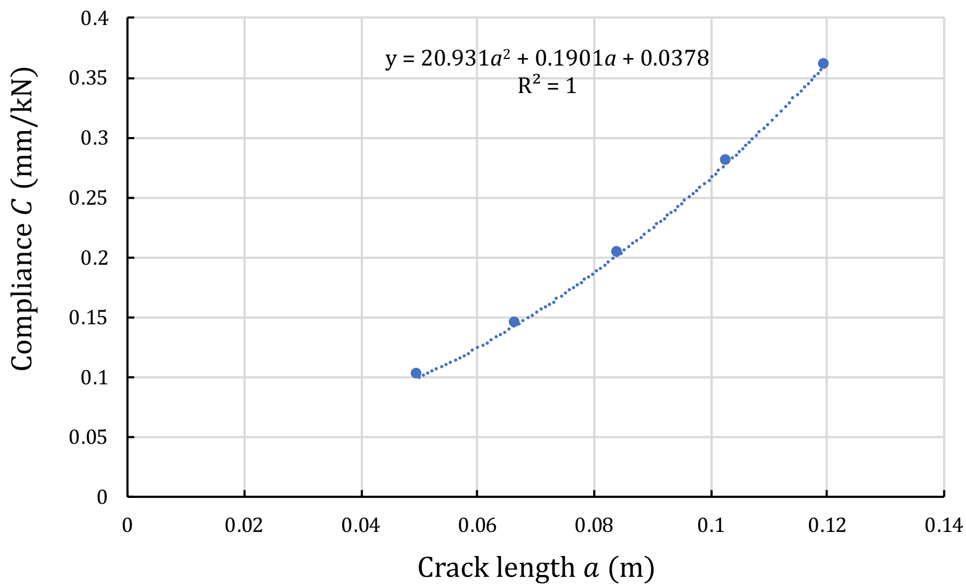
Where P is the critical load at fracture. All load-displacement records were linearly elastic up to fracture. Determine the critical energy release rate G_c for this material.

A! Solution

The energy release rate G for each test can be calculated using the compliance formula:

$$G = \frac{P^2}{2B}\frac{dC}{da}.$$

This requires to evaluate the derivative of the compliance with respect to a. To do so, we can plot the compliance C as a function of the crack length a, and fit the data with a trendline as shown below. Here, I used a second order polynomial (because it is accurate and easy to differentiate) but other equations could be used too.



For each data point, we can compute the energy release rate using:

$$G = \frac{P^2}{2B}\frac{dC}{da} = \frac{P^2}{2 \cdot 0.010}\left[\frac{20.931 \cdot 2 \cdot a + 0.1901}{10^6}\right]$$

where the division by 10^6 is simply to ensure that the units of G are in J/m^2. The values for each data point are given in the table below:

Crack length a (mm)	Compliance C (mm/kN)	Critical load P (kN)	G_c (kJ/m^2)
50.0	0.100	10.00	11.4
66.7	0.143	8.75	11.4
84.2	0.202	7.80	11.3
102.7	0.279	7.00	11.0
119.5	0.359	6.55	11.1

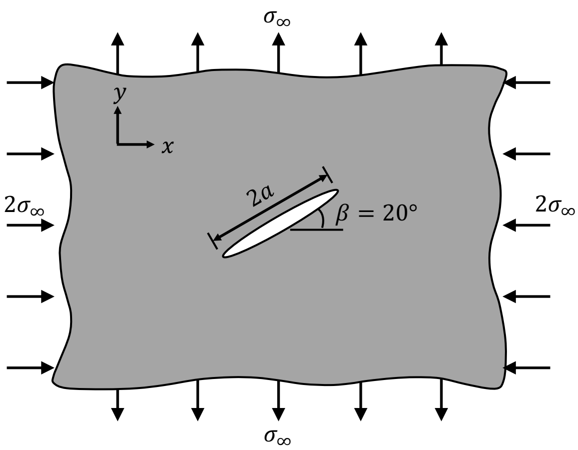
From these five tests, we can conclude that the toughness, or critical energy release rate, is $G_c \approx 11$ kJ/m^2.

A? Question

A large plate contains a central crack of length $2a$ at an angle $\beta = 20°$ from the horizontal. The plate is loaded in tension by a stress σ_∞ in the vertical direction, and in compression in the horizontal direction by a stress $2\sigma_\infty$, see below.

(a) Find the stress intensity factors K_I and K_{II}. Express your results as a function of σ_∞ and a.

(b) Find the angle of crack propagation.

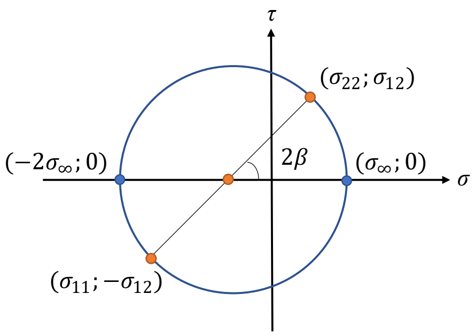


A! Solution

The stress field in the global reference frame is:

$$\sigma_{xx} = -2\sigma_\infty \qquad \sigma_{yy} = \sigma_\infty \qquad \sigma_{xy} = 0.$$

Mohr's circle for this stress field is:



where the centre c and radius r of the circle are:

$$c = \frac{\sigma_\infty - 2\sigma_\infty}{2} = -\frac{\sigma_\infty}{2} \qquad \text{and} \qquad r = \frac{\sigma_\infty + 2\sigma_\infty}{2} = \frac{3\sigma_\infty}{2}.$$

The stresses σ_{22} and σ_{12} in the local reference frame are given by:

$$\sigma_{22} = c + r\cos(2\beta) = -\frac{\sigma_\infty}{2} + \frac{3\sigma_\infty}{2}\cos(2\beta) = 0.6491\sigma_\infty,$$

$$\sigma_{12} = r\sin(2\beta) = \frac{3\sigma_\infty}{2}\sin(2\beta) = 0.9642\sigma_\infty.$$

Finally, the stress intensity factors are given by:

$$K_I = \sigma_{22}\sqrt{\pi a} = 0.6491\sigma_\infty\sqrt{\pi a},$$
$$K_{II} = \sigma_{12}\sqrt{\pi a} = 0.9642\sigma_\infty\sqrt{\pi a}.$$

To find the angle of crack propagation, we set $\sigma_{r\theta} = 0$, and this gives:

$$\begin{aligned}
&\sigma_{r\theta} = \frac{K_I}{\sqrt{2\pi r}}\left[\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}\right] + \frac{K_{II}}{\sqrt{2\pi r}}\left[\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}\right] = 0\\
&\implies K_I\left[\sin\frac{\theta}{2} + \sin\frac{3\theta}{2}\right] + K_{II}\left[\cos\frac{\theta}{2} + 3\cos\frac{3\theta}{2}\right] = 0\\
&\implies 0.6491\left[\sin\frac{\theta}{2} + \sin\frac{3\theta}{2}\right] + 0.9642\left[\cos\frac{\theta}{2} + 3\cos\frac{3\theta}{2}\right] = 0\\
&\implies \theta = -1.0188 \quad \text{or} \quad 1.4603 \quad \text{or} \quad \pi
\end{aligned}$$

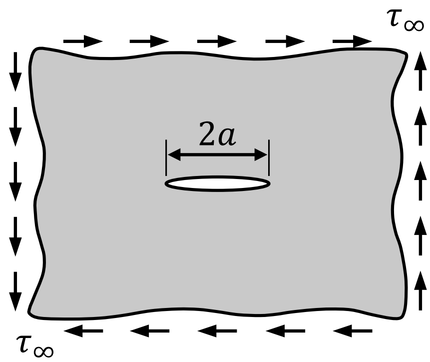
The correct angle θ is the one corresponding to the maximum $\sigma_{\theta\theta}$. Plotting

$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}}\left[\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2}\right] - \frac{K_{II}}{\sqrt{2\pi r}}\left[\frac{3}{4}\sin\frac{\theta}{2} + \frac{3}{4}\sin\frac{3\theta}{2}\right]$$

shows that $\theta = -1.0188$ corresponds to a maximum in $\sigma_{\theta\theta}$, whereas $\theta = 1.4603$ corresponds to a minimum in $\sigma_{\theta\theta}$. Therefore, the crack will propagate along $\theta = -1.0188 = -58.4°$.

A? Question

Find the direction of crack propagation for pure mode II loading. You might find this trigonometric identity useful: $\cos(3\alpha) = 4\cos^3\alpha - 3\cos\alpha$.



A! Solution

When the plate is loaded in mode II the shear stress $\sigma_{r\theta}$ close to the crack tip is obtained from the datasheet and is equal to:

$$\sigma_{r\theta} = \frac{K_{II}}{\sqrt{2\pi r}}\left(\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}\right)$$

The direction of crack propagation is the direction where $\sigma_{r\theta} = 0$, which gives:

$$\sigma_{r\theta} = \frac{K_{II}}{\sqrt{2\pi r}}\left(\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}\right) = 0$$
$$\Rightarrow \cos\frac{\theta}{2} + 3\cos\frac{3\theta}{2} = 0$$

Using the trigonometric identity: $\cos(3\alpha) = 4\cos^3\alpha - 3\cos\alpha$, the above expression becomes:

$$\cos\frac{\theta}{2} + 12\cos^3\frac{\theta}{2} - 9\cos\frac{\theta}{2} = 0$$
$$\Rightarrow 12\cos^3\frac{\theta}{2} = 8\cos\frac{\theta}{2}$$
$$\Rightarrow \cos^2\frac{\theta}{2} = \frac{2}{3}$$
$$\Rightarrow \cos\frac{\theta}{2} = \pm\sqrt{\frac{2}{3}}$$
$$\Rightarrow \theta = \pm 70.5^\circ$$

To determine which angle is the correct solution, we need to check which one gives the maximum value of $\sigma_{\theta\theta}$. The expression for $\sigma_{\theta\theta}$ is also obtained from the datasheet and equal to:

$$\sigma_{\theta\theta} = -\frac{K_{II}}{\sqrt{2\pi r}}\left(\frac{3}{4}\sin\frac{\theta}{2} + \frac{3}{4}\sin\frac{3\theta}{2}\right)$$

Substituting our two possible solutions $\theta = \pm 70.5^\circ$ returns:

$$\sigma_{\theta\theta} = -\frac{1.15 K_{II}}{\sqrt{2\pi r}} < 0 \quad \text{for } \theta = 70.5^\circ$$
$$\sigma_{\theta\theta} = \frac{1.15 K_{II}}{\sqrt{2\pi r}} > 0 \quad \text{for } \theta = -70.5^\circ$$

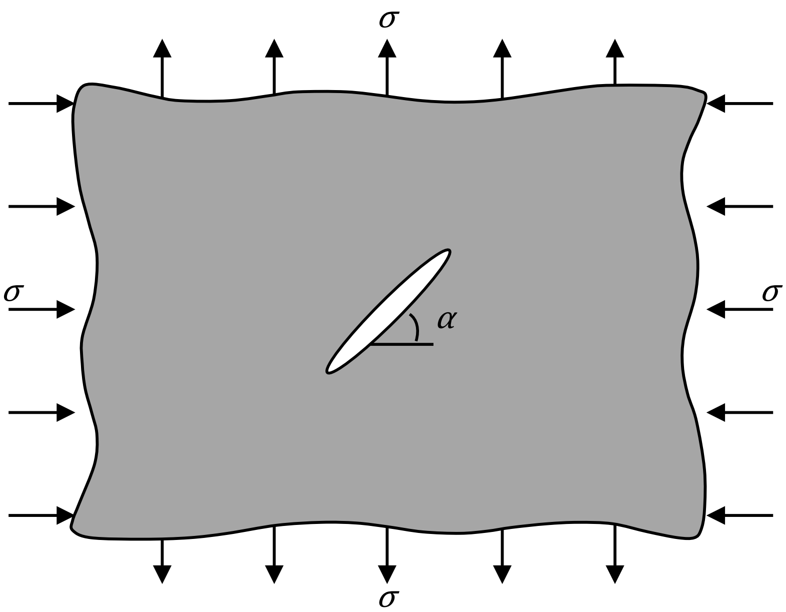
The maximum value of $\sigma_{\theta\theta}$ is obtained when $\theta = -70.5^\circ$ and therefore this will be the direction of crack propagation.

A? Problem 3.1

Consider the thin plate shown below with a crack of length $2a = 60$ mm at an angle $\alpha = 20°$.

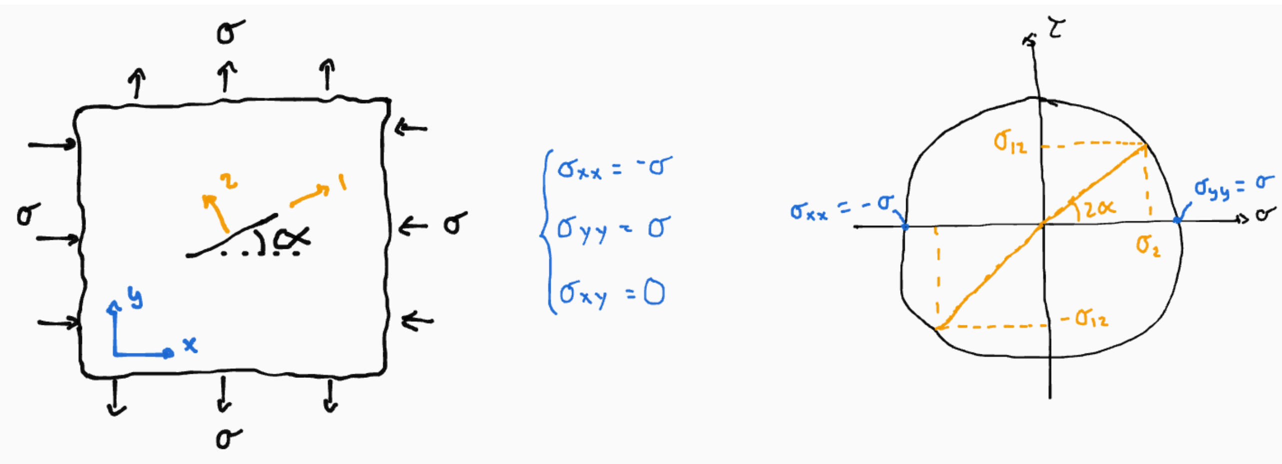
(a) Find expressions for the mode I and mode II stress intensity factors. Express your results as a function of the applied stress σ.

(b) Estimate the maximum stress σ that the plate can support provided that it is made from an aluminium alloy with a Young's modulus $E = 70$ GPa and a toughness $G_c = 12$ kJ/m^2.



A! Solution

Part (a)



The Mohr circle for this configuration is shown above. The global stresses are:

$$\sigma_{xx} = -\sigma \qquad \sigma_{yy} = \sigma \qquad \sigma_{xy} = 0$$

Note that σ_{xx} is negative because it is a compressive stress. Using the Mohr circle, the stresses in the local reference frame $(1, 2)$ are:

$$\sigma_{11} = -\sigma\cos(2\alpha) = -\sigma\cos(2\cdot 20°) = -0.766\sigma$$
$$\sigma_{22} = \sigma\cos(2\alpha) = \sigma\cos(2\cdot 20°) = 0.766\sigma$$
$$\sigma_{12} = \sigma\sin(2\alpha) = \sigma\sin(2\cdot 20°) = 0.6428\sigma$$

The stress σ_{22} is opening the crack in mode I, whereas σ_{12} is loading the crack in mode II. Therefore, the stress intensity factors are:

$$K_I = \sigma_{22}\sqrt{\pi a} = 0.766\sigma\sqrt{\pi 0.03} = 0.2352\sigma$$
$$K_{II} = \sigma_{12}\sqrt{\pi a} = 0.6428\sigma\sqrt{\pi 0.03} = 0.1973\sigma$$

Part (b) The energy release rate G in plane stress (thin plate) for mixed-mode loading is given by:

$$\begin{aligned} G &= \frac{K_I^2}{E} + \frac{K_{II}^2}{E} \\ &= \frac{0.2352^2\sigma^2}{E} + \frac{0.1973^2\sigma^2}{E} \\ &= 0.0942\frac{\sigma^2}{E} \end{aligned}$$

The plate will fracture when $G = G_c$ and this gives:

$$G_c = G = 0.0942\frac{\sigma^2}{E} \quad\implies\quad \sigma = \sqrt{\frac{EG_c}{0.0942}} = \sqrt{\frac{70e9\cdot 12e3}{0.0942}} = 94.4\,\text{MPa}$$

The maximum stress that the plate can support to avoid fracture is $\sigma = 94.4$ MPa.

A? Problem 3.2

A crack is loaded in a mixed-mode scenario where $K_I = K_{II}$. Find the direction θ, relative to the initial crack plane, in which the crack will propagate. Hint: don't hesitate to use a numerical approach to solve this equation.

A! Solution

The crack will propagate in the direction perpendicular to the maximum principal stress, and this corresponds to the orientation where $\sigma_{r\theta} = 0$. Here, the stress $\sigma_{r\theta}$ is obtained by adding the mode I and II contributions (taken from the datasheet), which gives:

$$\sigma_{r\theta} = \frac{K_I}{\sqrt{2\pi r}} \left[\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}\right] + \frac{K_{II}}{\sqrt{2\pi r}} \left[\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}\right] = 0$$

$$\implies K_I \left[\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}\right] + K_I \left[\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}\right] = 0 \qquad \text{since } K_I = K_{II}$$

$$\implies \sin\frac{\theta}{2} + \sin\frac{3\theta}{2} + \cos\frac{\theta}{2} + 3\cos\frac{3\theta}{2} = 0$$

Solving this numerically with Matlab (for angles from $-\pi$ to π) returns three solutions:

$$\theta = \pi/2; \quad -0.9273; \quad \pi \qquad \text{corresponding to} \quad 90^\circ; \quad -53.1^\circ; \quad 180^\circ$$

The last solution is physically impossible, but to decide between the first two solutions, we need to look at which one gives a maximum (positive) value of $\sigma_{\theta\theta}$. The expression for $\sigma_{\theta\theta}$ is also taken from the datasheet and includes the mode I and II contributions:

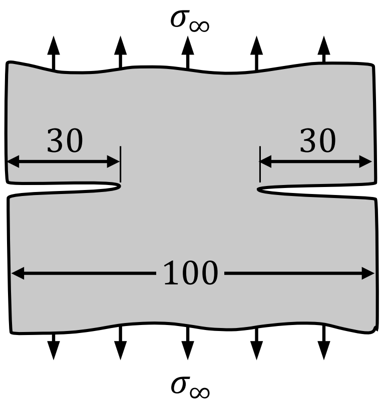
$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}} \left[\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2}\right] - \frac{K_{II}}{\sqrt{2\pi r}} \left[\frac{3}{4}\sin\frac{\theta}{2} + \frac{3}{4}\sin\frac{3\theta}{2}\right] \qquad \text{where } K_I = K_{II}$$

$$= \frac{K_I}{\sqrt{2\pi r}} \left(\left[\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2}\right] - \left[\frac{3}{4}\sin\frac{\theta}{2} + \frac{3}{4}\sin\frac{3\theta}{2}\right]\right)$$

For $\theta = 90^\circ$, the term in parenthesis is equal to -0.7071, whereas for $\theta = -53.1^\circ$, it is 1.7889. Therefore, $\sigma_{\theta\theta}$ is maximum when $\theta = -53.1^\circ$ and the crack will grow in that direction.

A? Question

You are responsible of designing a new product made from a ductile polymer. The material testing division of your company provided you with a tensile test and a fracture toughness test, conducted on a thin plate with the geometry below. They found that $E = 3\,\text{GPa}$, $\sigma_Y = 40\,\text{MPa}$, and $K_{Ic} = 15\,\text{MPa}\sqrt{\text{m}}$. Can you trust this value of K_{Ic}?



All dimensions in mm

A! Solution

The plastic zone size under plane stress is given by:

$$d_p = \frac{1}{\pi}\left(\frac{K_{Ic}}{\sigma_Y}\right)^2 = \frac{1}{\pi}\left(\frac{15}{40}\right)^2 = 44.7\,\text{mm}$$

This is larger than the crack length $a = 30\,\text{mm}$ and the unbroken ligament (40 mm). Therefore, the plastic zone size is too large to use LEFM. The consequence is that you shouldn't truss this value of K_{Ic}, the J-integral should be used instead. (It would be possible to use LEFM if $d_p \leq a/10$).

A? Question

Demonstrate that the shape of the plastic zone in mode III is a circle of radius:

$$r_y = \frac{3}{2\pi}\left(\frac{K_{III}}{\sigma_Y}\right)^2,$$

centered at the crack tip. (Hint: search for an expression of the von Mises yielding criterion that includes all 6 components of the stress tensor, instead of only the 3 principal stresses).

A! Solution

The stress field at the crack tip in mode III is (taken from the datasheet):

$$\sigma_{xz} = -\frac{K_{III}}{\sqrt{2\pi r}}\sin\frac{\theta}{2} \qquad \sigma_{yz} = \frac{K_{III}}{\sqrt{2\pi r}}\cos\frac{\theta}{2} \qquad \sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = 0$$

Substituting this stress field in the von Mises criterion gives:

$$\frac{1}{\sqrt{2}}\left[(\sigma_{xx}-\sigma_{yy})^2 + (\sigma_{yy}-\sigma_{zz})^2 + (\sigma_{zz}-\sigma_{xx})^2 + 6(\sigma_{xy}^2 + \sigma_{xz}^2 + \sigma_{yz}^2)\right]^{1/2} = \sigma_Y$$

$$\implies \frac{1}{\sqrt{2}}\left[\frac{6K_{III}^2}{2\pi r}\left(\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}\right)\right]^{1/2} = \sigma_Y$$

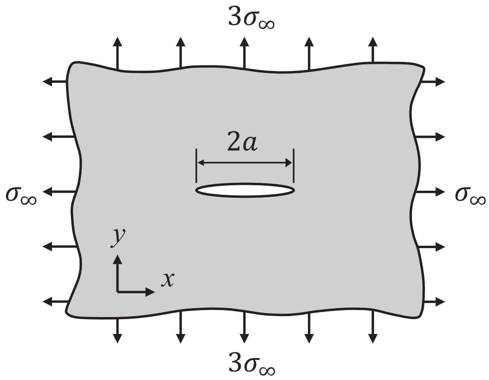
$$\implies \frac{1}{\sqrt{2}}\left[\frac{6K_{III}^2}{2\pi r}\right]^{1/2} = \sigma_Y$$

$$\implies \frac{6K_{III}^2}{2\pi r} = 2\sigma_Y^2$$

$$\implies r = \frac{3}{2\pi}\left(\frac{K_{III}}{\sigma_Y}\right)^2$$

A? Question

A thin aluminium plate, with $\sigma_{ys} = 320\,\text{MPa}$ and $K_{Ic} = 30\,\text{MPa}\sqrt{\text{m}}$, is subjected to bi-axial loading as shown below. Plot the maximum allowable stress σ_∞, as a function of a, to avoid both yielding and fracture.



A! Solution

The global stress field for this plate is:

$$\sigma_{xx} = \sigma_\infty, \qquad \sigma_{yy} = 3\sigma_\infty, \qquad \sigma_{zz} = \sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0.$$

Substituting this stress field in the von Mises criterion gives us the load that will cause yielding:

$$\frac{1}{\sqrt{2}}\left[(\sigma_{xx}-\sigma_{yy})^2 + (\sigma_{yy}-\sigma_{zz})^2 + (\sigma_{zz}-\sigma_{xx})^2 + 6(\sigma_{xy}^2 + \sigma_{xz}^2 + \sigma_{yz}^2)\right]^{1/2} = \sigma_{ys}$$

$$\implies \frac{1}{\sqrt{2}}\left[4\sigma_\infty^2 + 9\sigma_\infty^2 + \sigma_\infty^2 +\right]^{1/2} = \sigma_{ys}$$

$$\implies \sigma_\infty = \frac{\sigma_{ys}}{\sqrt{7}} \qquad \text{yielding}$$

Otherwise, fracture will occur when $K_I = K_{Ic}$, which gives:

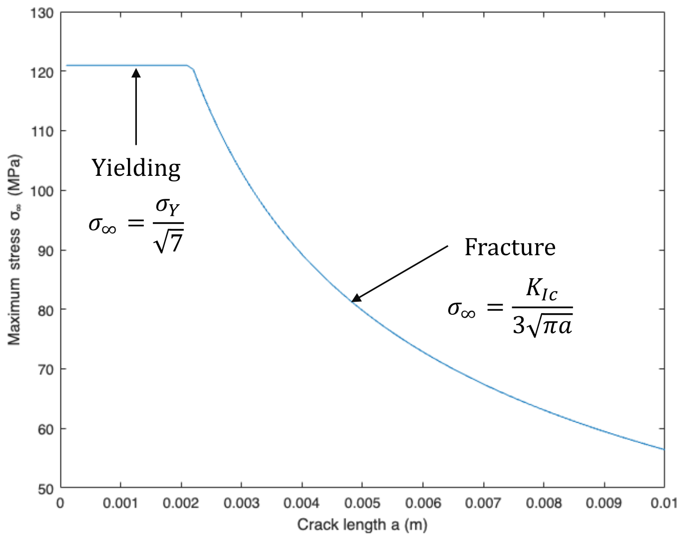
$$3\sigma_\infty\sqrt{\pi a} = K_{Ic}$$

$$\implies \sigma_\infty = \frac{K_{Ic}}{3\sqrt{\pi a}} \qquad \text{fracture}$$

The maximum allowable stress σ_∞ to avoid both yielding and fracture is therefore:

$$\sigma_\infty = \min\left(\frac{\sigma_{ys}}{\sqrt{7}}; \frac{K_{Ic}}{3\sqrt{\pi a}}\right)$$

This is plotted below as a function of the crack length a.



A? Question

Is the transition flaw size a material property?

A! Solution

The transition flaw size is not a material property. The example above shows that the transition flaw size depends on the geometry of the structure. A material property, such as the yield strength or fracture toughness, is independent of geometry.

A? Problem 4.1

A test was done to measure the fracture toughness of a thin polymer plate. The geometry had a central crack of length $2a = 50\,\text{mm}$, and the plate was tested by applying a tensile stress σ_∞ in the direction normal to the crack.

(a) If the plate failed at a stress $\sigma_\infty = 3\,\text{MPa}$, evaluate the fracture toughness K_{Ic} of the material.

(b) Provided that the polymer has a yield strength $\sigma_Y = 30\,\text{MPa}$, estimate the size of the plastic zone at the crack tip. Is it adequate to use Linear Elastic Fracture Mechanics to compute K_{Ic} in this case?

A! Solution

Part (a). For a large plate with a central crack, setting the stress intensity factor K_I equal to the fracture toughness K_{Ic} gives:

$$\begin{aligned} &K_I = \sigma_\infty \sqrt{\pi a} = K_{Ic} \\ &\implies K_{Ic} = \sigma_\infty \sqrt{\pi a} = 3 \cdot \sqrt{\pi \cdot 0.025} = 0.84\,\text{MPa}\sqrt{\text{m}} \end{aligned}$$

Part (b). The plate is thin so we assume plane stress conditions when evaluating the size of the plastic zone. This gives:

$$d_p = \frac{1}{\pi}\left(\frac{K_{Ic}}{\sigma_Y}\right)^2 = \frac{1}{\pi}\left(\frac{0.84}{30}\right)^2 = 0.25\,\text{mm}$$

The size of the plastic zone is very small: two orders of magnitude smaller than the crack length. Therefore, it is adequate to use Linear Elastic Fracture Mechanics (LEFM in this situation. In general, the plastic zone can be considered sufficiently small to use LEFM if $d_p < a/10$.

A? Problem 4.2

A cylindrical pressure vessel with closed ends has a radius $R = 1\,\text{m}$ and wall thickness $t = 40\,\text{mm}$, and is subjected to an internal pressure p. The vessel must be designed safely against failure by yielding (according to the von Mises yield criterion) and fracture. Three steels with the following values of yield stress σ_Y and fracture toughness K_{Ic} are being considered for constructing the vessel.

Steel	σ_Y (MPa)	K_{Ic} (MPa$\sqrt{\text{m}}$)
A: 4340	860	100
B: 4335	1300	70
C: 350 Maraging	1550	55

Fracture of the vessel is caused by a long axial surface crack of depth a (you can assume an infinite plate with an edge crack). The vessel should be designed with a safety factor $S = 2$ against yielding and fracture. For each steel:

(a) Plot the maximum permissible pressure p as a function of the crack depth a.

(b) Calculate the maximum permissible crack depth a for an operating pressure $p = 12\,\text{MPa}$.

(c) Calculate the failure pressure p for a minimum detectable crack depth $a = 1\,\text{mm}$.

A! Solution

Part (a). For a thin-walled cylindrical pressure vessel, the hoop stress $\sigma_{\theta\theta}$ and the longitudinal stress σ_{zz} are:

$$\sigma_{\theta\theta} = \frac{pR}{t} \qquad \text{and} \qquad \sigma_{zz} = \frac{pR}{2t}$$

The von Mises yielding criterion, taking into account the safety factor S, can be expressed as:

$$\frac{1}{\sqrt{2}}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_1-\sigma_3)^2\right]^{1/2} = \frac{\sigma_Y}{S}$$

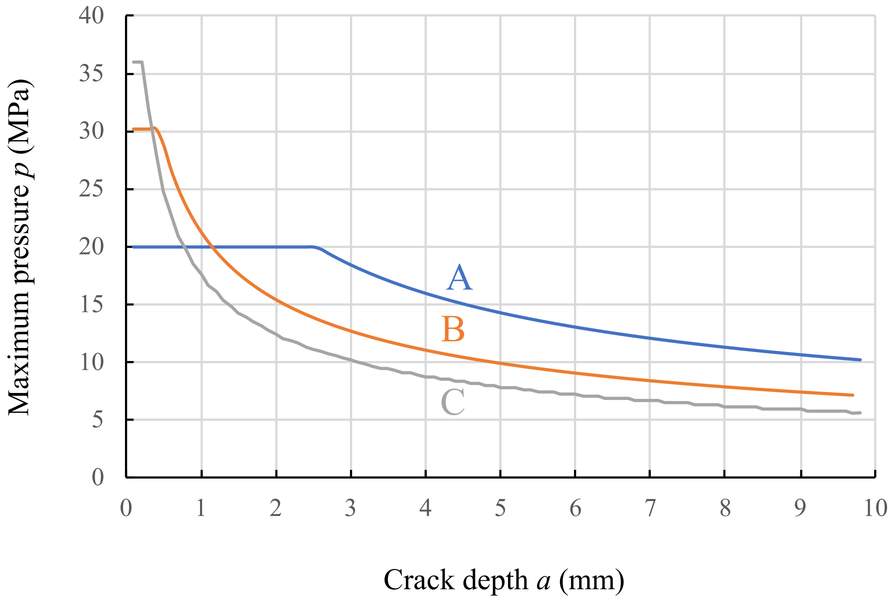
where the principal stresses are $\sigma_1 = \sigma_{\theta\theta}$; $\sigma_2 = \sigma_{zz}$ and $\sigma_3 = 0$. Substituting in the equation above gives:

$$\begin{aligned}
&\frac{1}{\sqrt{2}}\left[\left(\frac{pR}{t}-\frac{pR}{2t}\right)^2+\left(\frac{pR}{2t}\right)^2+\left(\frac{pR}{t}\right)^2\right]^{1/2} = \frac{\sigma_Y}{S}\\
\implies &\frac{1}{\sqrt{2}}\left[\frac{3p^2R^2}{2t^2}\right]^{1/2} = \frac{\sigma_Y}{S}\\
\implies &\frac{\sqrt{3}pR}{2t} = \frac{\sigma_Y}{S}\\
\implies &p = \frac{2\sigma_Y t}{\sqrt{3}SR} = \frac{2\cdot 0.04}{\sqrt{3}\cdot 2\cdot 1}\sigma_Y = 0.0231\sigma_Y \qquad (1)
\end{aligned}$$

This gives the maximum allowable pressure to avoid yielding. Next, we need to ensure that we also avoid fracture. Assuming an infinite plate with an edge crack loaded by the hoop stress and including the safety factor in our analysis, the criterion for fracture is:

$$\begin{aligned} & K_I = \frac{K_{Ic}}{S} \\ \implies & 1.12\sigma_{\theta\theta}\sqrt{\pi a} = \frac{K_{Ic}}{S} \\ \implies & 1.12\frac{pR}{t}\sqrt{\pi a} = \frac{K_{Ic}}{S} \\ \implies & p = \frac{K_{Ic}t}{1.12SR\sqrt{\pi a}} = \frac{0.04}{1.12 \cdot 2 \cdot 1 \cdot \sqrt{\pi}}\frac{K_{Ic}}{\sqrt{a}} = 0.0101\frac{K_{Ic}}{\sqrt{a}} \end{aligned} \tag{2}$$

This is the maximum pressure to avoid fracture. The maximum allowable pressure to avoid both yielding and fracture is the minimum value of Eq. (1) and (2), and this is plotted below.



Part (b). For $p = 12\,\text{MPa}$, all steels fail by fracture and the maximum crack length can be obtained from Eq. (2). This gives:

A: $a = 7.05\,\text{mm}$ B: $a = 3.45\,\text{mm}$ C: $a = 2.13\,\text{mm}$

Part (b). For a fixed crack length $a = 1\,\text{mm}$, the maximum allowable pressure for each steel is:

A: $p = 19.9\,\text{MPa}$ B: $p = 22.3\,\text{MPa}$ C: $p = 17.5\,\text{MPa}$

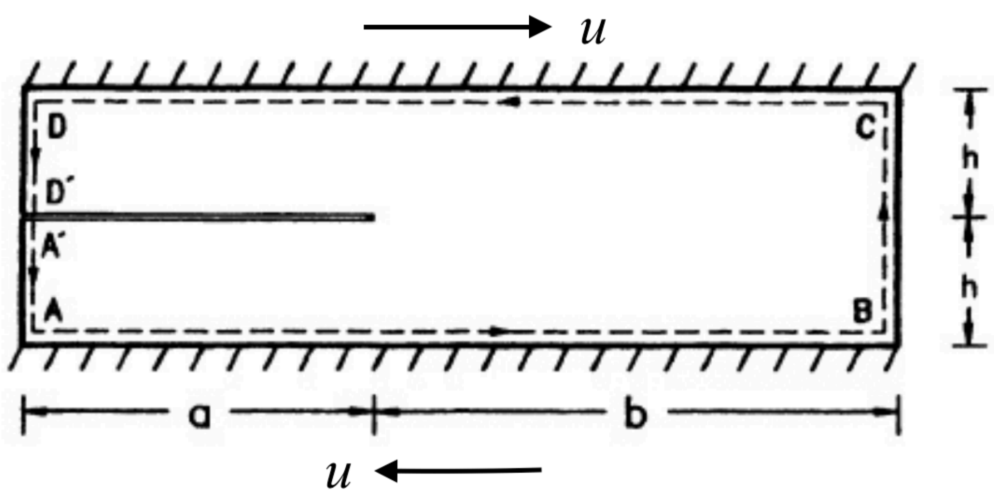
In conclusion, using a material with a high fracture toughness is beneficial to resist long cracks. However, materials with a high yield strength (and a low fracture toughness) can tolerate much higher stresses if cracks are short.

A? Problem 4.3

An infinitely wide strip, of height $2h$ and with a semi-infinite crack, is rigidly clamped along its top and bottom faces, see below. The strip is loaded in mode II with a prescribed displacement u as shown below. Assuming that the material is linear elastic and isotropic, show that the value of the J-integral for this scenario is given by:

$$J = \frac{Gu^2}{h} = \frac{Eu^2}{2(1+\nu)h}$$

where $G = \frac{E}{2(1+\nu)}$ is the shear modulus.



A! Solution

The definition of the J integral is:

$$J = \int_{\Gamma} \left(w dy - t_i \frac{\partial u_i}{\partial x} ds \right)$$

and the contour Γ can be divided in five segments as:

$$J = J_{AA'} + J_{AB} + J_{BC} + J_{CD} + J_{DD'}$$

Let's define x and y in the horizontal and vertical directions, respectively. Along segments AB and CD, we have no variations in y and the displacement field will be constant with x so:

$$dy = 0 \qquad \text{and} \qquad \frac{\partial u_1}{\partial x} = \frac{\partial u_2}{\partial x} = 0 \qquad \Longrightarrow \; J_{AB} = J_{BC} = 0$$

Along segments AA' and DD', we have no stress and the contour is vertical therefore:

$$w = 0 \qquad \text{and} \qquad \frac{\partial u_1}{\partial x} = \frac{\partial u_2}{\partial x} = 0 \qquad \Longrightarrow \; J_{AA'} = J_{DD'} = 0$$

Consequently, $J = J_{BC}$. Segment BC is vertical therefore:

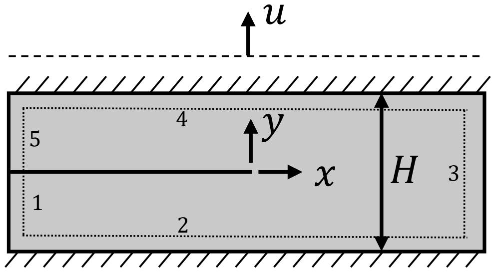
$$\frac{\partial u_1}{\partial x} = \frac{\partial u_2}{\partial x} = 0$$

and then the J integral is:

$$\begin{aligned}
J &= J_{BC} \\
&= \int_{-h}^{h} w dy \\
&= \int_{-h}^{h} \frac{1}{2}\sigma_{xy}\epsilon_{xy} dy && \text{since there is only shear} \\
&= \int_{-h}^{h} \frac{1}{2} G\epsilon_{xy}^2 dy && \text{where } G \text{ is the shear modulus} \\
&= \frac{G}{2}\int_{-h}^{h} \left(\frac{u}{h}\right)^2 dy && \text{the shear strain is } \epsilon_{xy} = u/h \\
&= \frac{Gu^2}{2h^2}\left[y\right]_{-h}^{h} \\
&= \frac{Gu^2}{h} \\
&= \frac{Eu^2}{2(1+\nu)h} && \text{since } E = 2G(1+\nu) \text{ for isotropic materials}
\end{aligned}$$

A? Question

An infinite strip of height H with a semi-infinite crack is rigidly clamped along its top and bottom faces, see below. Determine the J-integral for mode I loading, where top edge is moved upward by a prescribed displacement u. Assume that the material is linear elastic, isotropic, and under plane stress conditions.



A! Solution

In this case, the J-integral can be expressed as the sum of five segments labelled on the figure above:

$$J = J_1 + J_2 + J_3 + J_4 + J_5,$$

where each term is given by:

$$J = \int_\Gamma \left(w dy - t_i \frac{\partial u_i}{\partial x} ds \right).$$

Over segments 1 and 5, we have no deformations, $u_x = 0$ and u_y is a constant. This implies:

$$w = 0 \quad \text{and} \quad \frac{\partial u_i}{\partial x} = 0 \quad \implies \quad J_1 = J_5 = 0.$$

Next, over segments 2 and 4 we have $dy = 0$, $u_x = 0$ (the top and bottom edges are clamped and cannot contract). Also, $u_y = 0$ for the bottom edge, whereas $u_y = u$ along the top edge. This implies:

$$dy = 0 \quad \text{and} \quad \frac{\partial u_i}{\partial x} = 0 \quad \implies \quad J_2 = J_4 = 0$$

Finally, along segment 3, we have $dy \neq 0$ and $w \neq 0$, whereas the displacement $u_x = 0$ and $u_y = f(y)$, which implies $\frac{\partial u_i}{\partial x} = 0$. Therefore, we conclude that:

$$J = J_3 = \int_\Gamma \left(w dy - t_i \frac{\partial u_i}{\partial x} ds \right) = \int_{-H/2}^{H/2} w dy$$

For a linear elastic material the strain energy density $w = \frac{1}{2}\sigma_{ij}\epsilon_{ij}$. The stress and strain components are:

$$\sigma_{xx} \neq 0 \qquad \sigma_{yy} \neq 0 \qquad \sigma_{xy} = 0 \qquad \sigma_{zz} = 0$$

$$\epsilon_{xx} = 0 \qquad \epsilon_{yy} = \frac{u}{H} \qquad \epsilon_{xy} = 0 \qquad \epsilon_{zz} \neq 0$$

The stresses σ_{xx} and σ_{yy} can be obtained using Hooke's law. For a linear elastic isotropic material under plane stress, Hooke's law is:

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}\begin{bmatrix}\epsilon_{xx}\\ \epsilon_{yy}\\ \epsilon_{xy}\end{bmatrix}$$

Substituting in the values above, we find that:

$$\sigma_{yy} = \frac{E}{1-\nu^2}\epsilon_{yy} = \frac{Eu}{H(1-\nu^2)}$$

With this, we can compute the strain energy density w:

$$w = \frac{1}{2}\sigma_{ij}\epsilon_{ij} = \frac{1}{2}\sigma_{yy}\epsilon_{yy} = \frac{Eu^2}{2(1-\nu^2)H^2}$$

Finally, we can compute the J-integral using the above equation:

$$\begin{aligned} J &= \int_{-H/2}^{H/2} w dy \\ &= \int_{-H/2}^{H/2} \frac{Eu^2}{2(1-\nu^2)H^2} dy \\ &= \frac{Eu^2}{2(1-\nu^2)H^2}\left[y\right]_{-H/2}^{H/2} \\ &= \frac{Eu^2}{2(1-\nu^2)H} \end{aligned}$$

Since the material is linear elastic, $J = G$. From this, we can find the stress intensity factor K_I using:

$$K_I = \sqrt{EG} = \sqrt{EJ} = \sqrt{\frac{E^2u^2}{2(1-\nu^2)H}} = \sqrt{\frac{1}{2(1-\nu^2)H}}Eu$$

This is equal to the formula given in the datasheet.