MEC-E8007 Fracture Mechanics **Luc St-Pierre**

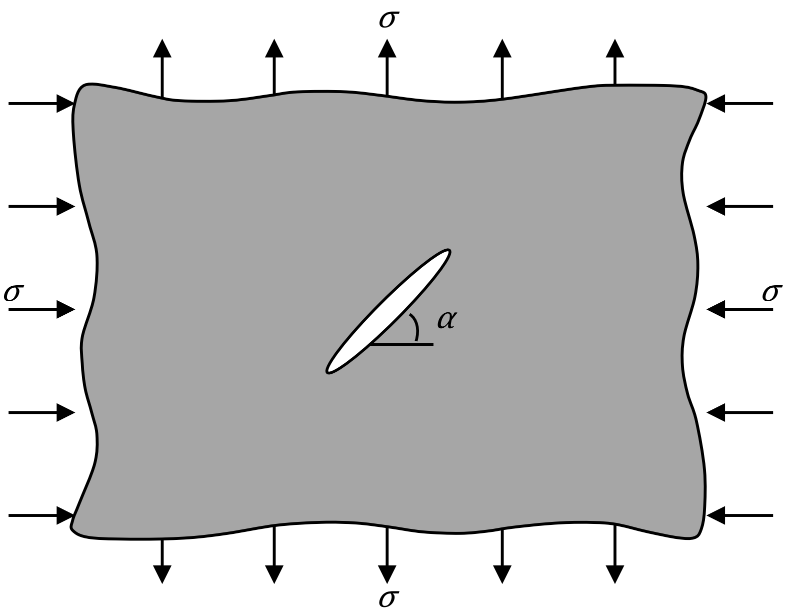
Solution 3

A? Problem 3.1

Consider the thin plate shown below with a crack of length $2a = 60\,\text{mm}$ at an angle $\alpha = 20°$.

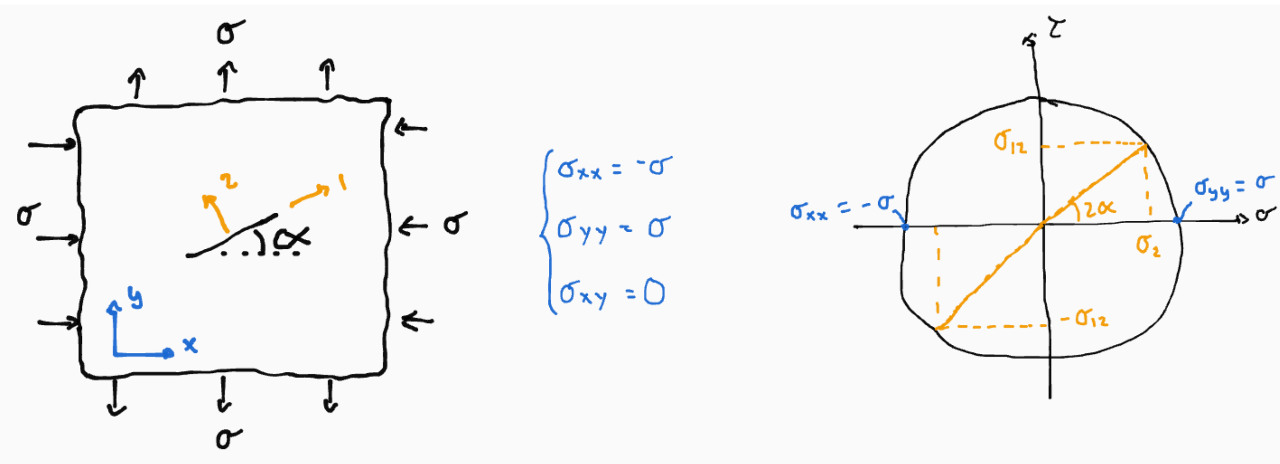
(a) Find expressions for the mode I and mode II stress intensity factors. Express your results as a function of the applied stress σ.

(b) Estimate the maximum stress σ that the plate can support provided that it is made from an aluminium alloy with a Young's modulus $E = 70\,\text{GPa}$ and a toughness $G_c = 12\,\text{kJ/m}^2$.



A! Solution

Part (a)



The Mohr circle for this configuration is shown above. The global stresses are:

$$\sigma_{xx} = -\sigma \qquad \sigma_{yy} = \sigma \qquad \sigma_{xy} = 0$$

Note that σ_{xx} is negative because it is a compressive stress. Using the Mohr circle, the stresses in the local reference frame $(1, 2)$ are:

$$\sigma_{11} = -\sigma\cos(2\alpha) = -\sigma\cos(2\cdot 20°) = -0.766\sigma$$
$$\sigma_{22} = \sigma\cos(2\alpha) = \sigma\cos(2\cdot 20°) = 0.766\sigma$$
$$\sigma_{12} = \sigma\sin(2\alpha) = \sigma\sin(2\cdot 20°) = 0.6428\sigma$$

The stress σ_{22} is opening the crack in mode I, whereas σ_{12} is loading the crack in mode II. Therefore, the stress intensity factors are:

$$K_I = \sigma_{22}\sqrt{\pi a} = 0.766\sigma\sqrt{\pi 0.03} = 0.2352\sigma$$
$$K_{II} = \sigma_{12}\sqrt{\pi a} = 0.6428\sigma\sqrt{\pi 0.03} = 0.1973\sigma$$

Part (b) The energy release rate G in plane stress (thin plate) for mixed-mode loading is given by:

$$\begin{aligned} G &= \frac{K_I^2}{E} + \frac{K_{II}^2}{E} \\ &= \frac{0.2352^2\sigma^2}{E} + \frac{0.1973^2\sigma^2}{E} \\ &= 0.0942\frac{\sigma^2}{E} \end{aligned}$$

The plate will fracture when $G = G_c$ and this gives:

$$G_c = G = 0.0942\frac{\sigma^2}{E} \quad\Longrightarrow\quad \sigma = \sqrt{\frac{EG_c}{0.0942}} = \sqrt{\frac{70e9\cdot 12e3}{0.0942}} = 94.4\,\text{MPa}$$

The maximum stress that the plate can support to avoid fracture is $\sigma = 94.4$ MPa.

A? Problem 3.2

A crack is loaded in a mixed-mode scenario where $K_I = K_{II}$. Find the direction θ, relative to the initial crack plane, in which the crack will propagate. Hint: don't hesitate to use a numerical approach to solve this equation.

A! Solution

The crack will propagate in the direction perpendicular to the maximum principal stress, and this corresponds to the orientation where $\sigma_{r\theta} = 0$. Here, the stress $\sigma_{r\theta}$ is obtained by adding the mode I and II contributions (taken from the datasheet), which gives:

$$\begin{aligned}
\sigma_{r\theta} &= \frac{K_I}{\sqrt{2\pi r}}\left[\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}\right] + \frac{K_{II}}{\sqrt{2\pi r}}\left[\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}\right] = 0 \\
\implies K_I &\left[\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}\right] + K_I\left[\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}\right] = 0 \qquad \text{since } K_I = K_{II} \\
\implies \sin&\frac{\theta}{2} + \sin\frac{3\theta}{2} + \cos\frac{\theta}{2} + 3\cos\frac{3\theta}{2} = 0
\end{aligned}$$

Solving this numerically with Matlab (for angles from $-\pi$ to π) returns three solutions:

$$\theta = \pi/2; \quad -0.9273; \quad \pi \qquad \text{corresponding to} \quad 90^\circ; \quad -53.1^\circ; \quad 180^\circ$$

The last solution is physically impossible, but to decide between the first two solutions, we need to look at which one gives a maximum (positive) value of $\sigma_{\theta\theta}$. The expression for $\sigma_{\theta\theta}$ is also taken from the datasheet and includes the mode I and II contributions:

$$\begin{aligned}
\sigma_{\theta\theta} &= \frac{K_I}{\sqrt{2\pi r}}\left[\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2}\right] - \frac{K_{II}}{\sqrt{2\pi r}}\left[\frac{3}{4}\sin\frac{\theta}{2} + \frac{3}{4}\sin\frac{3\theta}{2}\right] \qquad \text{where } K_I = K_{II} \\
&= \frac{K_I}{\sqrt{2\pi r}}\left(\left[\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{3\theta}{2}\right] - \left[\frac{3}{4}\sin\frac{\theta}{2} + \frac{3}{4}\sin\frac{3\theta}{2}\right]\right)
\end{aligned}$$

For $\theta = 90^\circ$, the term in parenthesis is equal to -0.7071, whereas for $\theta = -53.1^\circ$, it is 1.7889. Therefore, $\sigma_{\theta\theta}$ is maximum when $\theta = -53.1^\circ$ and the crack will grow in that direction.