Fracture Mechanics

Basic notions of solid mechanics

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Reviewing notions of solid mechanics

There are a few important notions of solid mechanics to review before we tackle fracture mechanics. These include:

- Plane stress/strain
- Hooke's law
- Equilibrium equations
- Compatibility equations
- Airy stress functions

If this brief review is insufficient, refer to the book by Timoshenko and Goodier, *Theory of Elasticity*.

Plane stress

In continuum mechanics, 2D problems are usually treated as plane stress or plane strain.

Plane stress, which is used for **thin** plates, assumes that $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$ so the stress tensor becomes:

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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while the strains are:

$$\epsilon_{ij} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_{33} \end{bmatrix} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & 0 \\ \epsilon_{12} & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_{33} \end{bmatrix}$$

note that $\epsilon_{33} \neq 0$!

Plane strain

Plane strain is used for **thick** structures and assumes that $\epsilon_{13} = \epsilon_{23} = \epsilon_{33} = 0$. Therefore, the strain tensor becomes:

$$\epsilon_{ij} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_{33} \end{bmatrix} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & 0 \\ \epsilon_{12} & \epsilon_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

while the stress tensor is:

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where for a linear isotropic material $\sigma_{33} = \nu(\sigma_{11} + \sigma_{22})$, where ν is the Poisson's ratio.

Hooke's law

For an isotropic linear elastic material, the strain and stress components are related as:

$$\begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \epsilon_{xy} \\ \epsilon_{xz} \\ \epsilon_{yz} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2+2\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2+2\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2+2\nu \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{bmatrix}$$

where E is the Young's modulus and ν is the Poisson's ratio.

Equilibrium equations

In 2D, the stress field should always respect equilibrium equations, and these are:

Cartesian coordinates:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0$$
$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0$$

Polar coordinates:

$$\frac{\partial}{\partial r}(r\sigma_{rr}) + \frac{\partial \sigma_{r\theta}}{\partial \theta} - \sigma_{\theta\theta} = 0$$
$$\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial}{\partial r}(r\sigma_{r\theta}) + \sigma_{r\theta} = 0$$

The above equations assume that body forces are negligible.

Strains and compatibility equation

In 2D, and using cartesian coordinates, the strains are defined as:

$$\epsilon_{xx} = \frac{\partial u}{\partial x} \qquad \epsilon_{yy} = \frac{\partial v}{\partial y} \qquad \epsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

where u and v are the displacements in the x and y directions, respectively.

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where u and v are the displacements in the x and y directions, respectively. The three strain components are not independent since they are defined from only two values of displacement. They are related via the compatibility equation:

$$\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} = \frac{\partial^2 \epsilon_{xy}}{\partial x \partial y}$$

This equation can also be expressed in polar coordinates... but it is messy.

Compatibility equation in terms of stress

For an isotropic linear elastic material loaded in plane stress, the strain and stress components are related as:

$$\epsilon_{xx} = \frac{1}{E}(\sigma_{xx} - \nu\sigma_{yy}) \qquad \epsilon_{yy} = \frac{1}{E}(\sigma_{yy} - \nu\sigma_{xx})$$

$$\epsilon_{xy} = \frac{1}{G}\sigma_{xy} = \frac{2(1+\nu)}{E}\sigma_{xy}$$

Subtituting these in the compatibility equation (and doing some algebra) returns:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_{xx} + \sigma_{yy}) = 0$$

This compatibility equation in terms of stresses is also valid for plane strain.

Airy stress function

Therefore, in 2D, a valid stress field as to respect both equilibrium and compatibility equations:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0$$
$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0$$
$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_{xx} + \sigma_{yy}) = 0$$

Finding three unknown functions $(\sigma_{xx}, \sigma_{yy}, \sigma_{xy})$ that respect these three equations is difficult; however, this can be made easier by introducing the Airy stress function.

Airy stress function

The Airy stress function $\phi(x, y)$ has no physical meaning, it is simply a mathematical trick. The function is related to the stress components as:

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} \qquad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} \qquad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

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It can be shown that equilibirum and compatibility equations are both respected when:

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \quad \Rightarrow \quad \nabla^4 \phi = 0$$

We have simplified the problem: there is now one function ϕ and a single equation.

Airy stress function

In polar coordinates, the Airy stress function $\phi(r,\theta)$ is related to the stresses as:

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} \quad \sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} \quad \sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)$$

and the compatibility equation $\nabla^4\phi = 0$ becomes:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right) = 0$$