Fracture Mechanics

2. Energy release rate

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Motivation

- Last week, we investigated fracture with an approach based on stress.
- We found that fracture occurs when the stress intensity factor K_I reaches a critical value, the fracture toughness K_{Ic}.
- This week, we will see how fracture can be analysed with an energy approach, and we will see how the energy and stress approaches are related.

Learning outcomes

After this week, you will be able to:

- Derive the energy release rate for simple geometries.
- Relate the energy release rate G to the stress intensity factor K_I.
- Predict when crack growth will be stable/unstable.

Outline

Energy release rate G

Relation between K and G

R-curves and stability

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Energy balance

Griffith (1920) realised that there was a certain amount of energy consumed in extending a crack. The energy balance can be written in incremental form as:

$$\delta W = \delta U + G\delta A$$

where	W	is the work done by external forces;
	U	is the strain energy of the deformable body;
	G	is the energy release rate with units of $\mathrm{J/m^2}$;
	δA	is the crack extension in m^2.

Therefore, when the crack advances by δA, the energy released is $G\delta A$.

Energy release rate

A definition for the energy release rate G can be obtained from the energy balance:

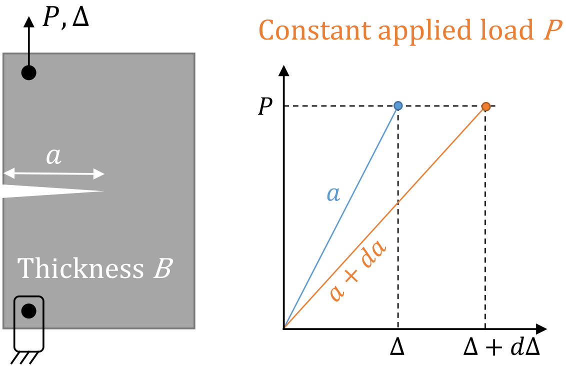
$$\delta W = \delta U + G\delta A \qquad \Rightarrow \qquad G = -\frac{\delta}{\delta A}(U - W)$$

Rewritting this in differential form gives:

$$G = -\frac{\partial \Pi}{\partial A} = -\frac{\partial}{\partial A}(U - W)$$

where $\Pi = U - W$ is called the potential energy of an elastic body. Let's see how we can use this formula.

Energy release rate - Load control

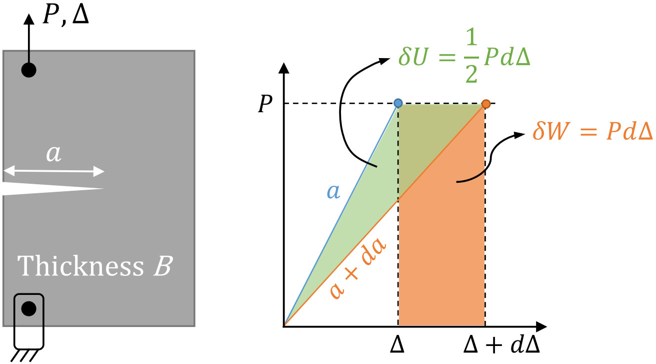


Let's introduce the compliance (inverse of stiffness):

$$C = \frac{\Delta}{P} \qquad \Rightarrow \qquad d\Delta = P dC$$

As the crack grows, the compliance C increases.

Energy release rate - Load control



From the energy balance $\delta W = \delta U + G\delta A$, we can write:

$$G = \frac{(\delta W - \delta U)}{Bda} = \frac{1}{Bda}\left(\frac{1}{2}Pd\Delta\right) = \frac{P^2}{2B}\frac{dC}{da}$$

Energy release rate - load control

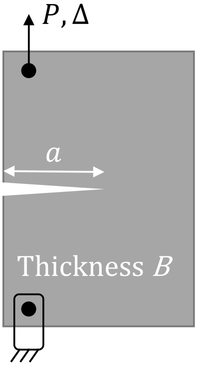
The same result can be obtained algebraically. The work done by external forces W, the strain energy U and the potential energy Π are given by:

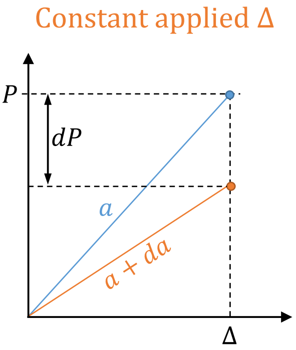
$$W = P\Delta; \qquad U = \frac{1}{2}P\Delta; \qquad \Pi = U - W = -\frac{1}{2}P\Delta$$

respectively. The energy release rate is easily obtained by:

$$\begin{aligned}
G &= -\frac{\partial \Pi}{\partial A} && \text{where } \partial A = B\partial a \\
&= \frac{1}{B}\frac{\partial}{\partial a}\left(\frac{1}{2}P\Delta\right) && \text{where } \Delta = CP \\
&= \frac{1}{B}\frac{\partial}{\partial a}\left(\frac{1}{2}CP^2\right) \\
&= \frac{P^2}{2B}\frac{\partial C}{\partial a} && \text{since } P \text{ is constant}
\end{aligned}$$

Energy release rate - displacement control

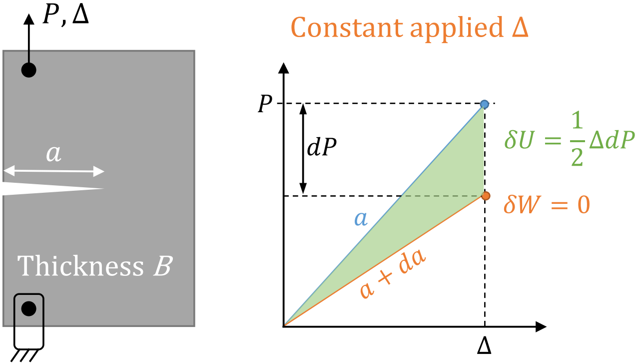




Since Δ is constant, we obtain:

$$C = \frac{\Delta}{P} \qquad \Rightarrow \qquad dC = \Delta\left(-\frac{1}{P^2}dP\right)$$

Energy release rate - displacement control



From the energy balance $\delta W = \delta U + G\delta A$, we can write:

$$G = \frac{(\delta W - \delta U)}{Bda} = -\frac{1}{Bda}\left(\frac{1}{2}\Delta dP\right) = \frac{P^2}{2B}\frac{dC}{da}$$

Energy release rate

In conclusion, we can use:

$$G = \frac{P^2}{2B}\frac{dC}{da}$$

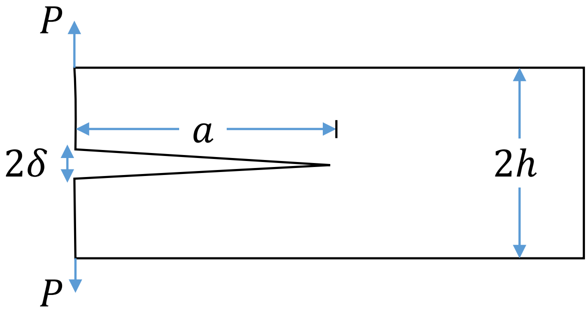
to compute G from experiments since the load P and the change in compliance $\frac{dC}{da}$ are both easy to measure.

Fracture will occur when:

$$G = G_c$$

where G_c is a material property called toughness. Both G and K are equivalent parameters: they quantify the loading intensity at the crack tip. We will see later how G and K are related.

Energy release rate: an example



Thickness: B

Young's modulus: E

Calculate the energy release rate G for the double cantilever beam shown here under an applied load P. Recall that from beam theory we have:

$$\delta = \frac{Pa^3}{3EI} \qquad \text{where} \qquad I = \frac{Bh^3}{12}$$

Energy release rate: an example

The work done by external forces W:

$$W = 2P\delta = \frac{8a^3P^2}{EBh^3} \qquad \text{where } \delta = \frac{Pa^3}{3EI} \text{ and } I = \frac{Bh^3}{12}$$

There is a factor of 2 because the total displacement is 2δ. The strain energy U is:

$$U = 2 \cdot \frac{1}{2}P\delta = \frac{4a^3P^2}{EBh^3}$$

The potential energy and the energy release rate are:

$$\Pi = U - W = -\frac{4a^3P^2}{EBh^3}$$

$$G = -\frac{\partial \Pi}{B\partial a} = \frac{1}{B}\frac{\partial}{\partial a}\left[\frac{4a^3P^2}{EBh^3}\right] = \frac{12P^2a^2}{EB^2h^3}$$

Energy release rate: an example

The same result can be obtained using the compliance method. Define:

$$C = \frac{2\delta}{P} = \frac{8a^3}{EBh^3} \qquad \text{where } \delta = \frac{Pa^3}{3EI} \text{ and } I = \frac{Bh^3}{12}$$

The energy release rate is:

$$G = \frac{P^2}{2B}\frac{\partial C}{\partial a} = \frac{P^2}{2B}\frac{\partial}{\partial a}\left[\frac{8a^3}{EBh^3}\right] = \frac{12P^2a^2}{EB^2h^3}$$

which is the same result that we obtained previously. Thus, fracture initiates when the applied load is:

$$P_c = \sqrt{\frac{EB^2h^3G_c}{12a^2}}$$

Energy release rate: an example

It is also possible to express G as a function of an applied displacement δ. We had:

$$G = \frac{12P^2a^2}{EB^2h^3} \qquad \text{where } \delta = \frac{Pa^3}{3EI} \text{ and } I = \frac{Bh^3}{12}$$

Substituting for the load P gives:

$$G = \frac{12a^2}{EB^2h^3}\left(\frac{EBh^3\delta}{4a^3}\right)^2 = \frac{3Eh^3\delta^2}{4a^4}$$

Therefore, the critical displacement causing fracture is:

$$\delta_c = a^2\sqrt{\frac{4G_c}{3Eh^3}}$$

Outline

Energy release rate G

Relation between K and G

R-curves and stability

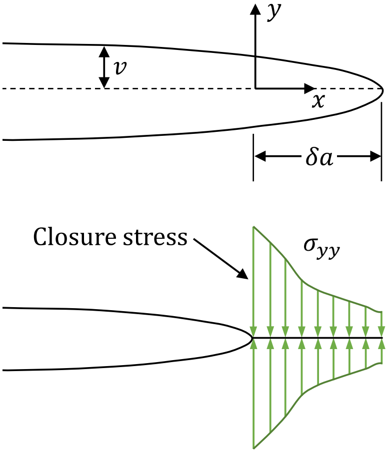
Relation between K and G

So far, we have introduced two parameters to predict fracture:

- the local stress intensity factor K and
- the global energy release rate G.

For linear elastic materials, these two parameters are uniquely related as we will demonstrate next.

Relation between K and G



Consider a crack of length $a + \delta a$ subjected to mode I loading.

Let's now close the crack, over a length δa, by applying a compressive stress σ_{yy}.

The work done to close the crack δU is related to G as:

$$G = \frac{\delta U}{B \delta a}$$

Relation between K and G

The work done to close the crack is given by:

$$\delta U = 2B \int_0^{\delta a} \frac{1}{2} \sigma_{yy}(x) v(x) dx$$

Both σ_{yy} and v can be obtained from the datasheet. For plane strain we have:

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi x}} \qquad \text{and} \qquad v = \frac{4(1-\nu^2)}{E} K_I \sqrt{\frac{\delta a - x}{2\pi}}$$

Substituting above gives:

$$\delta U = \frac{2B(1-\nu^2)}{\pi E} K_I^2 \int_0^{\delta a} \sqrt{\frac{\delta a - x}{x}} dx$$

which can be integrated with a change of variable: $x = \delta a \sin^2 u$.

Relation between K and G

After integration, we obtain:

$$\delta U = \frac{BK_I^2(1-\nu^2)}{E}\delta a$$

and the energy release rate becomes:

$$G = \frac{\delta U}{B\delta a} = \frac{K_I^2(1-\nu^2)}{E}$$

Therefore, for mode I under plane strain we have:

$$K_I^2 = \frac{EG_I}{1-\nu^2}$$

Relation between K and G

In summary, for mode I we have:

$$K_I^2 = \frac{E}{1-\nu^2} G_I \qquad \text{for plane strain}$$
$$K_I^2 = E G_I \qquad \text{for plane stress}$$

Repeating this procedure for mode II gives:

$$K_{II}^2 = \frac{E}{1-\nu^2} G_{II} \qquad \text{for plane strain}$$
$$K_{II}^2 = E G_{II} \qquad \text{for plane stress}$$

and for mode III we get:

$$K_{III}^2 = 2\mu G_{III}$$

where μ is the shear modulus.

A note on G under mixed mode loading

If the above anlysis is reapeated for mixed mode loading, we would find that for plane stress:

$$G = \frac{K_I^2}{E} + \frac{K_{II}^2}{E} + \frac{K_{III}^2}{2\mu} = G_I + G_{II} + G_{III}$$

However, we saw earlier that the stress intensity factor is not additive:

$$K_{(\text{total})} \neq K_I + K_{II} + K_{III}$$

The energy release rate G, like energy, is a scalar and therefore the total G is the sum of all three modes. On the other hand, the stress intensity factor K is related to the stress tensor (not a scalar) where only individual components can be added together.

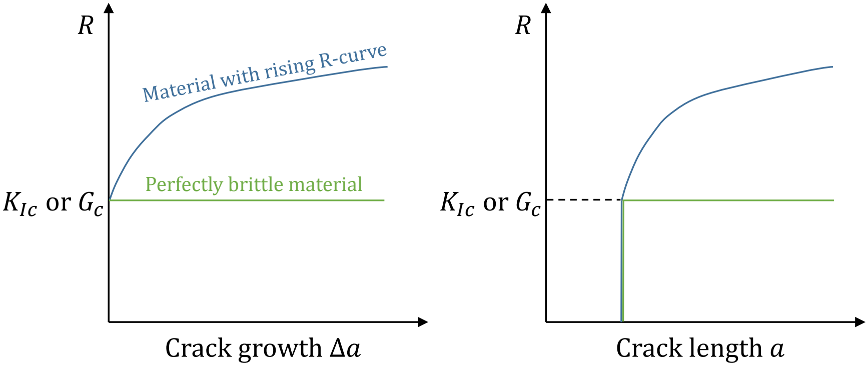
Outline

Resistance curve

- So far, we have considered the toughness G_c or fracture toughness K_{Ic} to be constant material properties.
- This is not always the case; the toughness can increase with crack propagation.
- A plot of toughness as a function of crack growth is called a Resistance curve (or R-curve).

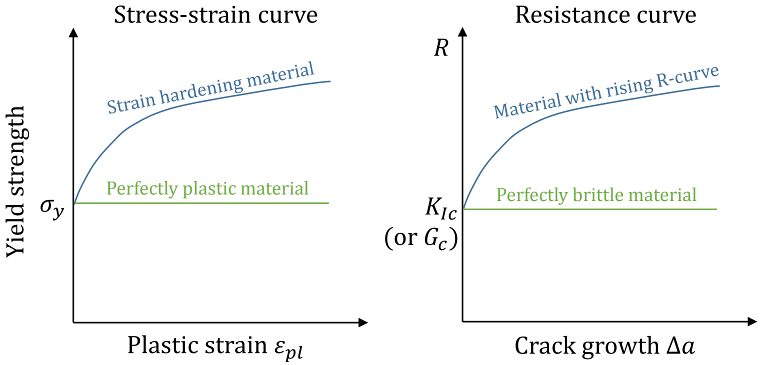
Resistance curve

Materials can have a rising R-curve because of plasticity or other phenomena that we will cover later.



Resistance curve

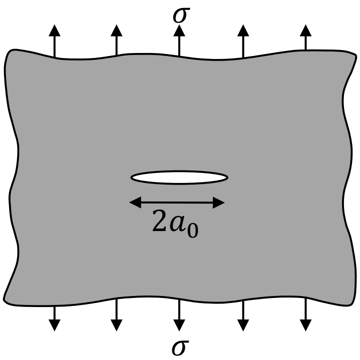
The concept of a resistance curve is similar to the stress-strain curve used to characterised plasticity.

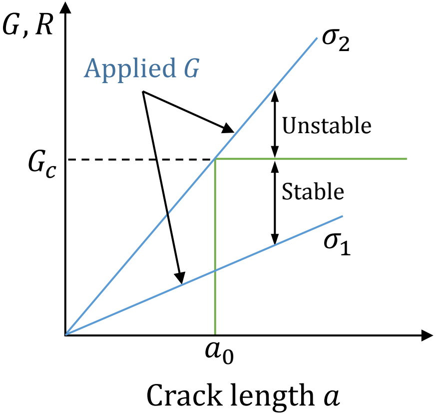


Instability: flat R-curve

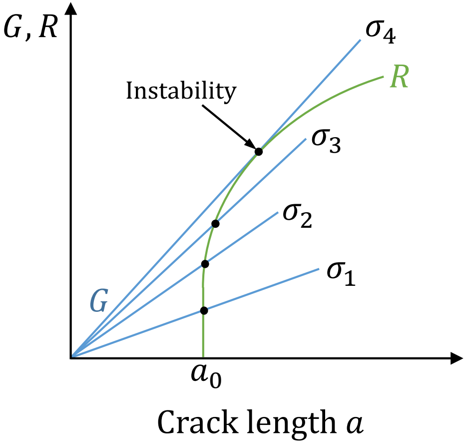
For a large plate with a through crack under plane stress, we have:

$$G = \frac{K_I^2}{E} = \frac{\pi \sigma^2 a}{E}$$





Instability: rising R-curve



- Stable crack growth for σ_2 and σ_3.
- Crack growth becomes unstable when $\sigma \geq \sigma_4$

For σ_2 and σ_3, the crack grows a small amount and stops since the resistance R increases at a faster rate than G.

Conditions for stability/instability

The conditions for stable crack growth can be expressed as:

$$G = R \qquad \text{and} \qquad \frac{dG}{da} \leq \frac{dR}{da}$$

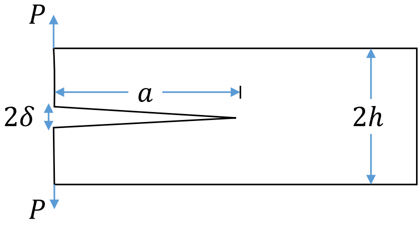
It follows that unstable crack growth occurs when:

$$G \geq R \qquad \text{and} \qquad \frac{dG}{da} > \frac{dR}{da}$$

These conditions hold true if the energy release rate G is replaced by the stress intensity factor K.

Load control vs displacement control

Let's investigate the crack growth stability for the double cantilever beam specimen under both load and displacement control. Assume that the material has a flat R-curve.



Thickness: B

Young's modulus: E

Load control vs displacement control

Under load control, the energy release rate is:

$$G = \frac{12P^2a^2}{EB^2h^3} \qquad \Rightarrow \qquad \frac{dG}{da} = \frac{24P^2a}{EB^2h^3} > 0$$

Therefore crack growth will be unstable if the material has a flat R-curve ($\frac{dR}{da} = 0$).

Otherwise, under displacement control, the energy release rate is:

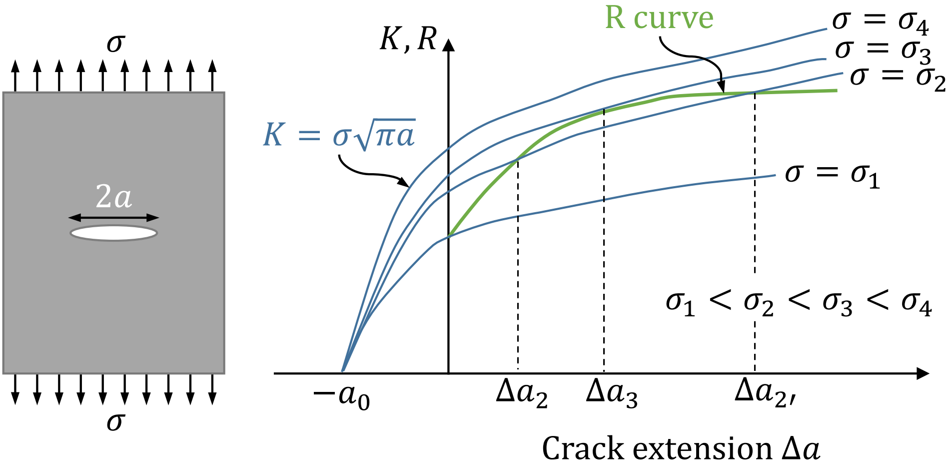
$$G = \frac{3Eh^3\delta^2}{4a^4} \qquad \Rightarrow \qquad \frac{dG}{da} = -\frac{3Eh^3\delta^2}{a^5} < 0$$

Even if the material has a flat R-curve, crack growth is stable under displacement control.

Load control vs displacement control

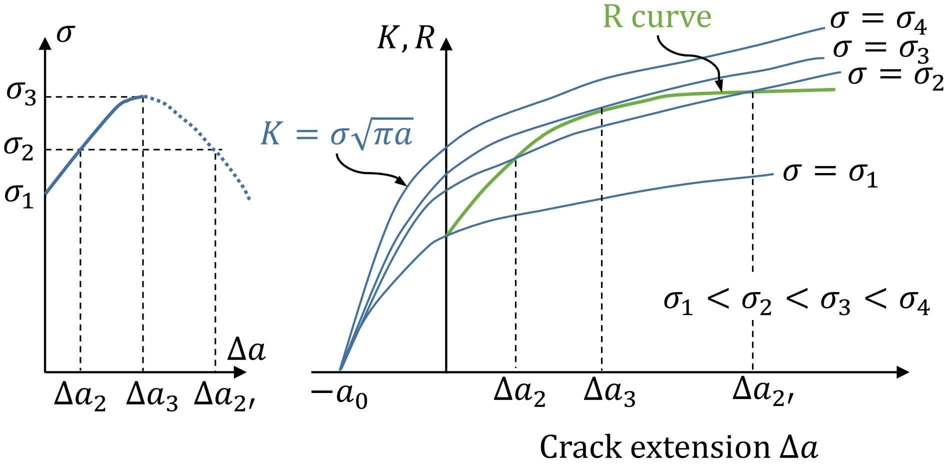
- In general, displacement control tends to be more stable then load control.
- This is convenient since most testing machines work in displacement control rather than load control.
- The geometry of most standardised test specimens is such that they exhibit stable crack growth under displacement control. This allows us to measure the R curve of the material.
- We will discuss testing methods later in the course.

Estimating the amount of stable crack extension



The crack will start to grow at σ_1, up to the maximum stress σ_3 resulting in a stable crack extension of Δa_3.

Estimating the amount of stable crack extension



From the R curve we can construct the σ vs Δa response. The first branch, up to the peak stress σ_3, will result in stable crack propagation. The decreasing branch (dashed line) is unstable under load control, and can only be obtained under displacement control.

Summary

The energy release rate G:

- can be used to predict fracture, and
- is related to the stress intensity factor.

We saw the conditions leading to stable/unstable crack growth.

The R-curve of is a material property and can be used to predict the amount of stable crack growth.