Fracture Mechanics

1. The stress intensity factor

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April 18, 2023

Learning outcomes for this week

After this week, you will be able to:

- Explain how the stress intensity factor is derived.
- Understand how the stress intensity factor can be used to predict fracture.
- Use the stress intensity factor to solve engineering problems.

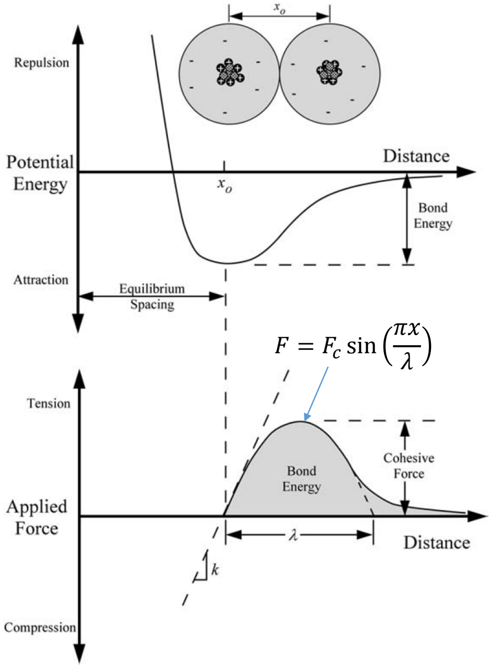
Outline

Outline

Let's begin with a question...

- How can we predict the strength of a material?
- Maybe a model based on interatomic forces could work.

Atomic perspective of fracture



The stiffness of the atomic bonds can be approximated as:

$$k = F_c \left(\frac{\pi}{\lambda}\right)$$

Dividing by the number of bonds per unit area gives:

$$\sigma_c = \frac{E\lambda}{\pi x_0} \approx \frac{E}{\pi}$$

Atomic perspective of fracture

This simple analysis predicts that the cohesive stress, *i.e.* the strength, of a material should be roughly:

$$\sigma_c \approx \frac{E}{\pi}$$

where E is the Young's modulus. Let's see if this is a good prediction. Glass has a Young's modulus $E = 68 - 74\,\mathrm{GPa}$, which gives us:

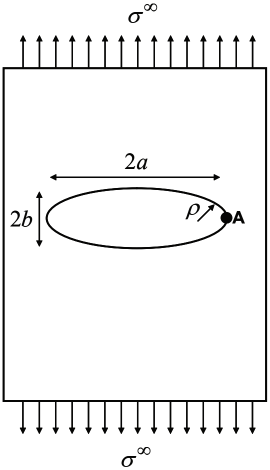
$$\sigma_c \approx \frac{E}{\pi} = 21 - 24\,\mathrm{GPa}$$

while the measured tensile strength is 0.045-0.155 GPa. Our prediction is too high by three orders of magnitudes!

Atomic perspective of fracture

- An atomic model significantly overpredicts the strength of materials.
- Conclusion: the strength of a material must be controlled by something else than atomic bonds.
- Hypothesis: materials might contain defects that lower their strength.
- Defects? these could be holes or cavities for example. Let's see what would be the effect of an elliptical hole.

Stress concentration around an elliptical hole



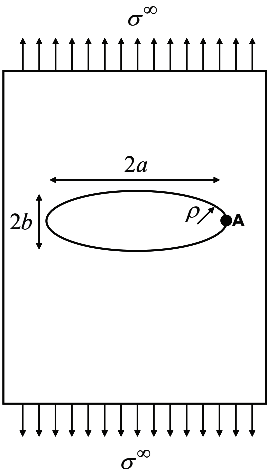
Inglis found that the stress concentration is:

$$\sigma_A = \sigma_{\max} = \sigma^\infty \left(1 + \frac{2a}{b}\right)$$

The hole resembles a sharp crack when $a \gg b$. It can be convenient to express this result as a function of the radius of curvature $\rho = b^2/a$, which gives:

$$\sigma_A = \sigma^\infty \left(1 + 2\sqrt{\frac{a}{\rho}}\right)$$

Stress concentration around an elliptical hole



With $\rho = b^2/a$, we found that:

$$\sigma_A = \sigma^\infty \left(1 + 2\sqrt{\frac{a}{\rho}}\right)$$
$$\Rightarrow \sigma_A \approx 2\sigma^\infty \sqrt{\frac{a}{\rho}}$$

This simple analysis shows that:

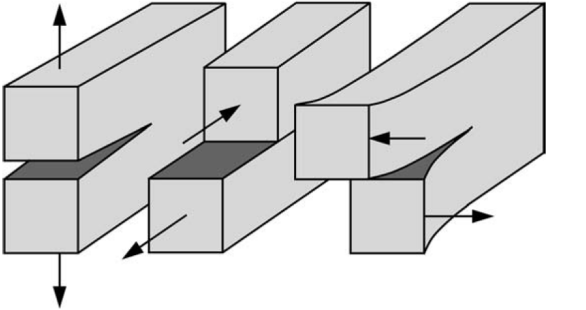
- σ_A scales as $\sigma^\infty \sqrt{a}$
- when $\rho \to 0$ we have $\sigma_A \to \infty$

Next, we will derive more precisely the stress field at the tip of a sharp crack.

Outline

Modes of crack loading

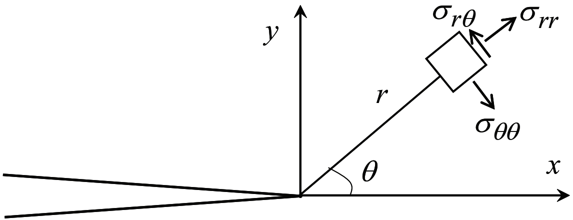
Three modes of loading can be applied to a crack:



A crack can be loaded by a single mode, or a combination of modes.

Stress field around a crack tip

Consider an infinitely large plate made from an isotropic linear elastic material. The plate contains a sharp crack which is loaded in mode I.



How does the stress field vary as $r \to 0$?

Williams (1957) found an analytical solution to this problem using Airy stress functions.

Stress field around a crack tip

Consider the following Airy stress function in polar coordinates:

$$\phi(r,\theta) = r^{(\lambda+1)} f(\theta)$$

To satisfy equilibrium and compatibility equations, the Airy stress function has to respect:

$$\nabla^4\phi = 0 \quad \Rightarrow \quad \frac{d^4 f}{d\theta^4} + 2(\lambda^2+1)\frac{d^2 f}{d\theta^2} + (\lambda^2-1)^2 f = 0$$

This differential equation has the following solution:

$$f(\theta) = A\cos(\lambda-1)\theta + B\sin(\lambda-1)\theta + C\cos(\lambda+1)\theta + D\sin(\lambda+1)\theta$$

where A, B, C and D are unknown constants.

Stress field around a crack tip

Now that the Airy stress function $\phi = r^{(\lambda+1)} f(\theta)$ is known, we can write the stresses:

$$
\begin{aligned}
\sigma_{\theta\theta} &= \frac{\partial^2 \phi}{\partial r^2} \\
&= (\lambda+1)\lambda r^{\lambda-1}[A\cos(\lambda-1)\theta + B\sin(\lambda-1)\theta \ldots \\
&\qquad\qquad\qquad\qquad + C\cos(\lambda+1)\theta + D\sin(\lambda+1)\theta] \\
\sigma_{r\theta} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right) \\
&= \lambda r^{\lambda-1}[A(\lambda-1)\sin(\lambda-1)\theta - B(\lambda-1)\cos(\lambda-1)\theta \ldots \\
&\qquad + C(\lambda+1)\sin(\lambda+1)\theta + D(\lambda+1)\cos(\lambda+1)\theta]
\end{aligned}
$$

We can simplify these expressions using the boundary conditions.

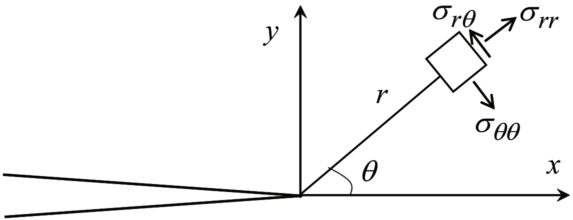
Stress field around a crack tip

The boundary conditions are such that the crack surfaces are stress-free:

$$\sigma_{\theta\theta}(\theta = \pi) = \sigma_{r\theta}(\theta = \pi) = 0$$

For mode I loading, the stresses should be symmetric in θ, meaning:

$$\sigma_{\theta\theta}(\theta) = \sigma_{\theta\theta}(-\theta)$$



Stress field around a crack tip

Previously, we found that:

$$\sigma_{\theta\theta} = (\lambda + 1)\lambda r^{\lambda-1}[A\cos(\lambda - 1)\theta + B\sin(\lambda - 1)\theta \dots \\ + C\cos(\lambda + 1)\theta + D\sin(\lambda + 1)\theta]$$

To ensure that $\sigma_{\theta\theta}$ is an even function, the sin terms have to disappear, which implies that:

$$B = D = 0$$

With this result, the stresses now become:

$$\sigma_{\theta\theta} = (\lambda + 1)\lambda r^{\lambda-1}[A\cos(\lambda - 1)\theta + C\cos(\lambda + 1)\theta]$$
$$\sigma_{r\theta} = \lambda r^{\lambda-1}[A(\lambda - 1)\sin(\lambda - 1)\theta + C(\lambda + 1)\sin(\lambda + 1)\theta]$$

Stress field around a crack tip

Next, the boundary conditions $\sigma_{\theta\theta} = \sigma_{r\theta} = 0$ on $\theta = \pi$ imply:

$$\begin{aligned}
\sigma_{\theta\theta}(\theta = \pi) &= A\cos(\lambda - 1)\pi + C\cos(\lambda + 1)\pi = 0 \\
\sigma_{r\theta}(\theta = \pi) &= A(\lambda - 1)\sin(\lambda - 1)\pi + C(\lambda + 1)\sin(\lambda + 1)\pi = 0
\end{aligned}$$

which can be written in matrix form:

$$\begin{bmatrix} \cos(\lambda - 1)\pi & \cos(\lambda + 1)\pi \\ (\lambda - 1)\sin(\lambda - 1)\pi & (\lambda + 1)\sin(\lambda + 1)\pi \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Nontrivial solutions exist when the determinant of the matrix is zero.

$$(\lambda + 1)[\sin(\lambda + 1)\pi][\cos(\lambda - 1)\pi] - (\lambda - 1)[\sin(\lambda - 1)\pi][\cos(\lambda + 1)\pi] = 0$$

Stress field around a crack tip

We had:

$$(\lambda+1)[\sin(\lambda+1)\pi][\cos(\lambda-1)\pi]-(\lambda-1)[\sin(\lambda-1)\pi][\cos(\lambda+1)\pi]=0$$

which has the following solutions:

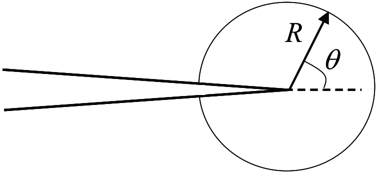
$$\sin(2\pi\lambda)=0 \quad \Rightarrow \quad \lambda=\ldots,-1,-\frac{1}{2},0,\frac{1}{2},1,\ldots$$

For each λ, a corresponding ratio A/C can be found, but not the absolute value of A and C (this is an eigenvalue/vector problem).

Note that there are physical restrictions on λ...

Stress field around a crack tip

One restriction is that the total energy W in a region near the crack tip should be finite:



$$W = \int_A \frac{1}{2}\frac{\sigma^2}{E} dA = \int_0^{2\pi}\int_0^R \frac{1}{2}\frac{\sigma^2}{E} r dr d\theta < \infty$$

where E is the Young's modulus. The stress scales as $\sigma \propto r^{\lambda-1}$, and therefore:

$$W \propto \int_0^R r^{2(\lambda-1)} r dr = \left| \frac{r^{2\lambda}}{2\lambda} \right|_{r=0}^{R} < \infty$$

And this implies that $\lambda > 0$.

Stress field around a crack tip

Therefore, the Airy stress function for mode I is:

$$\phi = r^{(\lambda+1)} f(\theta) = \sum r^{(\lambda_i+1)} \left[A_i \cos(\lambda_i - 1)\theta + C_i \cos(\lambda_i + 1)\theta\right]$$

where $\lambda_i = \frac{1}{2}, 1, \frac{3}{2}, \ldots$ and the ratio A_i/C_i is fixed for each value of λ_i. Finally, the stresses are obtained directly from:

$$\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2}$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)$$

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

Stress field around a crack tip

which gives us the following stresses for mode I:

$$\begin{aligned}
\sigma_{\theta\theta} &= \frac{A}{4} r^{-1/2} \left(3\cos\frac{\theta}{2} + \cos\frac{3\theta}{2} \right) + \mathcal{O}(r^0) + \mathcal{O}(r^{1/2}) + \dots \\
\sigma_{r\theta} &= \frac{A}{4} r^{-1/2} \left(\sin\frac{\theta}{2} + \sin\frac{3\theta}{2} \right) + \dots \\
\sigma_{rr} &= \frac{A}{4} r^{-1/2} \left(5\cos\frac{\theta}{2} - \cos\frac{3\theta}{2} \right) + \dots
\end{aligned}$$

- The constant C was removed since A/C is fixed for each value of λ.
- The higher order terms $\mathcal{O}(r^0), \mathcal{O}(r^{1/2}), \dots$ are needed to satisfy remote boundary conditions, but their contribution is negligible at the crack tip ($r \to 0$).

Stress field around a crack tip

Neglecting higher order terms and setting $A = \frac{K_I}{\sqrt{2\pi}}$, the equations become:

$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}} \left(\frac{3}{4} \cos\frac{\theta}{2} + \frac{1}{4} \cos\frac{3\theta}{2} \right)$$

$$\sigma_{r\theta} = \frac{K_I}{\sqrt{2\pi r}} \left(\frac{1}{4} \sin\frac{\theta}{2} + \frac{1}{4} \sin\frac{3\theta}{2} \right)$$

$$\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}} \left(\frac{5}{4} \cos\frac{\theta}{2} - \frac{1}{4} \cos\frac{3\theta}{2} \right)$$

- K_I is called the stress intensity factor. The subscript I refers to mode I loading.
- This analysis does not give us the value of K_I. To find K_I, we need a specific boundary value problem.
- The magnitude of the stresses scales linearly with K_I.

Displacement field around a crack tip

Next, the strain components ϵ_{rr}, $\epsilon_{\theta\theta}$ and $\epsilon_{r\theta}$ can be obtained from Hooke's law.

Finally, we can integrate the strains to obtain the displacement field:

$$u_\theta = \frac{K_I}{4G}\sqrt{\frac{r}{2\pi}}\left[-(2\kappa+1)\sin\frac{\theta}{2} + \sin\frac{3\theta}{2}\right]$$
$$u_r = \frac{K_I}{4G}\sqrt{\frac{r}{2\pi}}\left[(2\kappa-1)\cos\frac{\theta}{2} - \cos\frac{3\theta}{2}\right]$$

$$\text{where} \quad \kappa = \begin{cases} \frac{3-\nu}{1+\nu} & \text{for plane stress} \\ 3-4\nu & \text{for plane strain} \end{cases}$$

and G is the shear modulus and ν is the Poisson's ratio.

Stress field around a crack tip

Let's look at $\sigma_{\theta\theta}$ along $\theta = 0$:

$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}} \left(\frac{3}{4} \cos \frac{\theta}{2} + \frac{1}{4} \cos \frac{3\theta}{2} \right) \quad \Rightarrow \quad \sigma_{\theta\theta}(\theta = 0) = \frac{K_I}{\sqrt{2\pi r}}$$

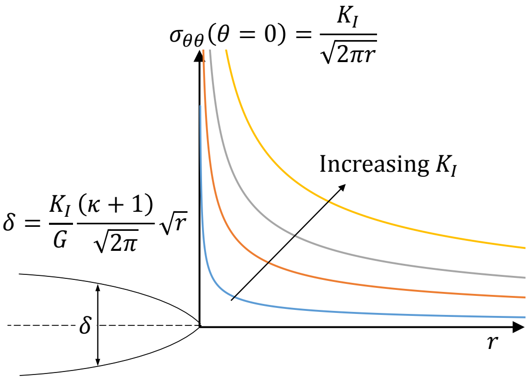
Otherwise, the crack opening profile δ can be obtained from the displacement field:

$$u_\theta = \frac{K_I}{4G} \sqrt{\frac{r}{2\pi}} \left[-(2\kappa + 1) \sin \frac{\theta}{2} + \sin \frac{3\theta}{2} \right]$$

$$\Rightarrow \quad \delta = -2u_\theta(\theta = \pi) = \frac{K_I}{G} \frac{(\kappa + 1)}{\sqrt{2\pi}} \sqrt{r}$$

Those quantities are illustrated next.

Stress field around a crack tip



- $\sigma_{\theta\theta}$ is linearly proportional to K_I.
- K_I needs to have units of Pa$\cdot\sqrt{\text{m}}$.
- Crack tip singularity: $\sigma_{\theta\theta} \to \infty$ when $r \to 0$.
- The opening profile δ is parabolic.

Stress field around a crack tip

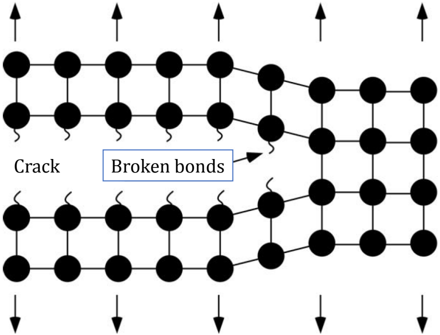
This analysis can be reapeated to find the stress and displacement fields for mode II and mode III loading. The results for all modes are summarised in the datasheet.

A few important questions remain unanswered:

- Stresses tend to infinity at $r = 0$, how is this possible?
- How can we compute the stress intensity factor K_I?
- When will the crack propagate?

Why stresses tend to infinity at $r = 0$?

In our modelling approach the stresses tend to infinity at $r = 0$ because we have considered the crack to be infinitely sharp, meaning that the radius of curvature $\rho \to 0$.



In reality, the crack tip has a non-zero curvature. For example, the radius of curvature has to be equal or greater than the spacing between atoms.

Computing the stress intensity factor

Finding the stress intensity factor is important because this single-parameter completely defines the stresses, strains and displacements close to the crack tip.

$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}} \left(\frac{3}{4} \cos \frac{\theta}{2} + \frac{1}{4} \cos \frac{3\theta}{2} \right)$$

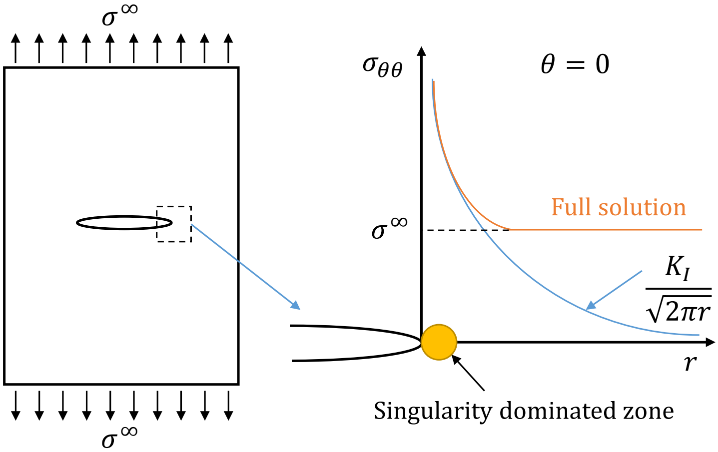
$$\sigma_{r\theta} = \frac{K_I}{\sqrt{2\pi r}} \left(\frac{1}{4} \sin \frac{\theta}{2} + \frac{1}{4} \sin \frac{3\theta}{2} \right)$$

$$\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}} \left(\frac{5}{4} \cos \frac{\theta}{2} - \frac{1}{4} \cos \frac{3\theta}{2} \right)$$

$$u_\theta = \frac{K_I}{4G} \sqrt{\frac{r}{2\pi}} \left[-(2\kappa + 1) \sin \frac{\theta}{2} + \sin \frac{3\theta}{2} \right]$$

$$u_r = \frac{K_I}{4G} \sqrt{\frac{r}{2\pi}} \left[(2\kappa - 1) \cos \frac{\theta}{2} - \cos \frac{3\theta}{2} \right]$$

Local and global behaviours



How does K_I vary with the remote applied stress σ^∞ and the geometry?

Examples of stress intensity factors

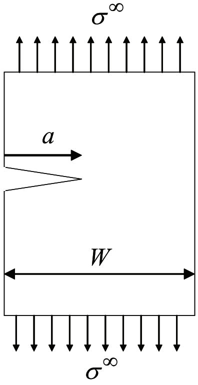
For a given geometry, the relation between the stess intensity factor and the applied load can be found:

analytically for simple geometries. Results are summarised in many textbooks.

numerically for more complex geometries. This generally requires the use of the Finite Element Method.

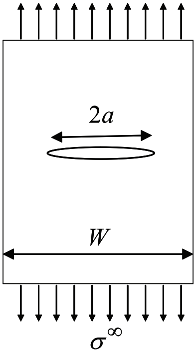
We will derive analytical solutions next week, but for now, let's look at a few existing solutions.

Examples of stress intensity factors



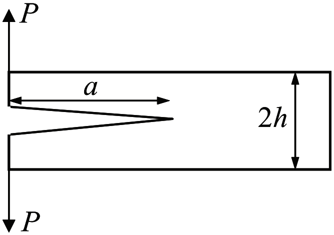
$W >> a$

$K_I = 1.122\sigma^\infty\sqrt{\pi a}$



$K_I = Y\sigma^\infty\sqrt{\pi a}$

$$Y \approx \left(\cos\frac{\pi a}{W}\right)^{-1/2}$$



$K_I = 2\sqrt{3}Pah^{-3/2}$

More configurations are included in the datasheet.

When will a crack propagate?

Irwin (1957) postulated that fracture occurs when K_I reaches a critical value K_{Ic}, which is a material property called fracture toughness.

Material	K_{Ic} (MPa$\sqrt{\text{m}}$)
Low carbon steel alloys	40-80
Aluminum alloys	22-35
Titanium alloys	14-120
Wood (best orientation)	5-9
PMMA	0.7-1.6
Glass	0.6-0.8
Concrete	0.35-0.45

Testing methods to measure the fracture toughness will be covered later during the course.

Outline

Principle of superposition

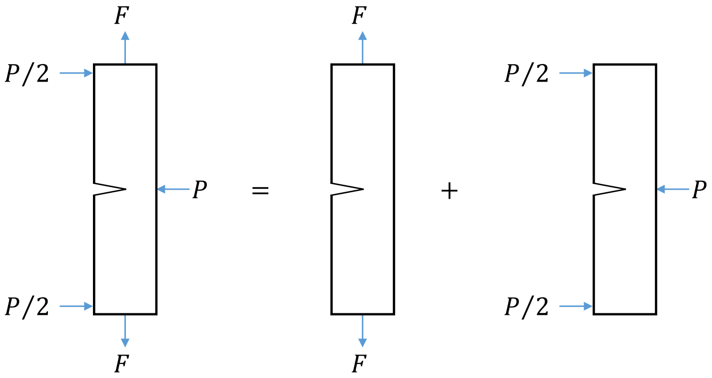
For linear elastic materials, individual components of stress, strain and displacement are additive. Likewise, the stress intensity factors are additive, as long as the mode of loading is the same:

$$K_I^{(\text{total})} = K_I^{(\text{A})} + K_I^{(\text{B})} + K_I^{(\text{C})} + \ldots$$

The stress intensity factors for different modes of loading should never be added together:

$$K_{(\text{total})} \neq K_I + K_{II} + K_{III}$$

Principle of superposition: example



$$K_I^{(\mathrm{total})} = K_I^{(\mathrm{tension})} + K_I^{(\mathrm{bending})}$$

Both $K_I^{(\mathrm{tension})}$ and $K_I^{(\mathrm{bending})}$ can be obtained from a handbook of stress intensity factors.

Summary

The stress intensity factor:

- quantifies the stress field at the crack tip,
- follows the principle of superposition for a given loading mode,
- can be used to predict fracture.