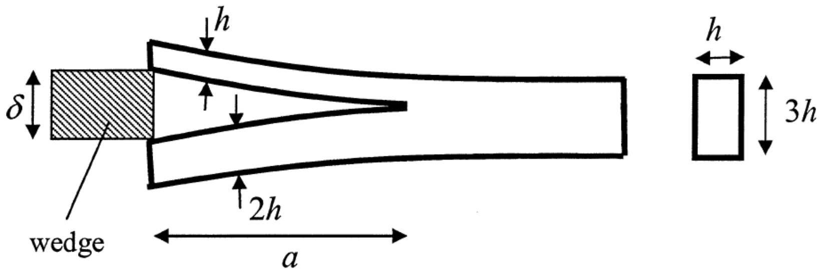
A? Problem 2.1

Wooden chop-sticks have the geometry shown below, where the arms are opened by a wedge of height δ. The wood has a linear elastic behaviour with a Young's modulus E.

(a) Determine the compliance of each arm of the chop-sticks.

(b) Calculate the energy release rate G.

(c) Will crack growth be stable or unstable? Assume that the material has a flat R-curve.



A! Solution

Part (a). According to beam theory, the deflection u of a cantilever beam, subjected to a force P at its extremity, is:

$$u = \frac{PL^3}{3EI}$$

where L is the beam's length, E is the Young's modulus and $I = bt^3/12$ for a rectangular cross-section of thickess t and width b. Here, the top and bottom arms have a different I, and these are:

$$I_{top} = \frac{h \cdot h^3}{12} = \frac{h^4}{12} \quad \text{and} \quad I_{bot} = \frac{h(2h)^3}{12} = \frac{2h^4}{3}$$

The wedge will create a force P on each arm, but their deflections ($u_{top}; u_{bot}$) will be different. Their compliance C are given by:

$$C_{top} = \frac{u_{top}}{P} = \frac{a^3}{3EI_{top}} = \frac{a^3}{3E} \cdot \frac{12}{h^4} = \frac{4a^3}{Eh^4}$$
$$C_{bot} = \frac{u_{bot}}{P} = \frac{a^3}{3EI_{bot}} = \frac{a^3}{3E} \cdot \frac{3}{2h^4} = \frac{a^3}{2Eh^4}$$

Part (b). The compliance C of system is:

$$C = \frac{\delta}{P} = \frac{u_{top} + u_{bot}}{P} = C_{top} + C_{bot} = \frac{4a^3}{Eh^4} + \frac{a^3}{2Eh^4} = \frac{9a^3}{2Eh^4}$$

From this, we can get an expression for the force P as a function of δ:

$$C = \frac{\delta}{P} = \frac{9a^3}{2Eh^4} \implies P = \frac{2Eh^4\delta}{9a^3}$$

Finally, we can get the energy release rate G using the compliance formula:

$$\begin{aligned} G &= \frac{P^2}{2h} \cdot \frac{dC}{da} \\ &= \frac{1}{2h} \cdot \left[\frac{2Eh^4\delta}{9a^3} \right]^2 \cdot \left[\frac{9}{2Eh^4} \cdot 3a^2 \right] \\ &= \frac{1}{2h} \cdot \frac{4E^2h^8\delta^2}{81a^6} \cdot \frac{27a^2}{2Eh^4} \\ &= \frac{Eh^3\delta^2}{3a^4} \end{aligned}$$

Note that in this question, we have an applied displacement δ, and not an applied force P. Therefore, the final answer for the energy release rate G should be a function of the applied displacement δ only (the force P should not appear in G). Of course, this would be the opposite if the problem had an applied force P instead of a displacement.

Part (c). Crack growth will be stable if $dG/da < 0$, and otherwise, unstable. Computing dG/da gives:

$$\frac{dG}{da} = \frac{d}{da} \left[\frac{Eh^3\delta^2}{a^4} \right] = \frac{Eh^3\delta^2}{3} \cdot (-4a^{-5}) = -\frac{4Eh^3\delta^2}{3a^5} < 0$$

Therefore, crack growth will be stable.

A? Problem 2.2

The R-curve for a steel alloy is given by:

$$R = \frac{K_{Ic}^2}{E} + \frac{1}{2}\sqrt{\Delta a}$$

where R is in MJ/m^2, the crack extension Δa is in meters, $K_{Ic} = 95\,\text{MPa}\sqrt{\text{m}}$ and $E = 210000\,\text{MPa}$. A large but thin plate is made from this material and contains a centre crack of length $2a_0 = 40\,\text{mm}$.

(a) Show that this plate allows a maximum stable crack growth of 6.3 mm at both tips.

(b) Calculate the critical stress σ_c at which unstable fracture will occur.

A! Solution

Part (a). The energy release rate G for a thin (plane stress) plate with a centre crack is:

$$G = \frac{K_I^2}{E} = \frac{\pi\sigma^2 a}{E} = \frac{\pi\sigma^2(a_0 + \Delta a)}{E}$$

Units are important in this problem. With dimensional analysis we can show that if a and Δa are in m; and σ and E are in MPa; then, G is in MJ/m^2 (same units as the R curve given in the question). Unstable fracture will occur when two conditions are met:

$$G = R \qquad \text{and} \qquad \frac{dG}{d(\Delta a)} = \frac{dR}{d(\Delta a)}$$

The first condition gives:

$$G = R \quad \Longrightarrow \quad \frac{\pi\sigma^2(a_0 + \Delta a)}{E} = \frac{K_{Ic}^2}{E} + \frac{1}{2}\sqrt{\Delta a} \tag{1}$$

And the second condition gives:

$$\frac{dG}{d(\Delta a)} = \frac{dR}{d(\Delta a)} \quad \Longrightarrow \quad \frac{\pi\sigma^2}{E} = \frac{1}{4\sqrt{\Delta a}} \tag{2}$$

Substituting (2) in (1) gives:

$$\begin{aligned}
&\frac{1}{4\sqrt{\Delta a}} \cdot (a_0 + \Delta a) = \frac{K_{Ic}^2}{E} + \frac{1}{2}\sqrt{\Delta a} \\
&\Longrightarrow\ a_0 + \Delta a = \frac{4K_{Ic}^2}{E}\sqrt{\Delta a} + 2\Delta a \\
&\Longrightarrow\ 0 = -a_0 + \frac{4K_{Ic}^2}{E}\sqrt{\Delta a} + \Delta a \\
&\Longrightarrow\ 0 = -0.020 + 0.1719\sqrt{\Delta a} + \Delta a
\end{aligned}$$

This is a quadratic equation of the form $ax^2 + bx + c = 0$, where $x = \sqrt{\Delta a}$. Solving this returns:

$$\sqrt{\Delta a} = -0.2514 \quad \text{and} \quad 0.0795$$

The negative answer is impossible; therefore, the stable crack growth up to failure is:

$$\Delta a = 0.0795^2 = 6.3\,\text{mm}$$

Part (b). We can substitute Δa in (2) to find the critical stress σ_c at failure:

$$\frac{\pi \sigma_c^2}{E} = \frac{1}{4\sqrt{\Delta a}}$$
$$\implies \sigma_c = \frac{1}{2}\sqrt{\frac{E}{\pi\sqrt{\Delta a}}} = \frac{1}{2}\sqrt{\frac{210000}{\pi\sqrt{0.0063}}} = 458\,\text{MPa}$$

A? Problem 2.3

The following data were obtained from a series of tests conducted on pre-cracked specimens with a thickness $B = 10\,\text{mm}$.

Crack length a (mm)	Compliance C (mm/kN)	Critical load P (kN)
50.0	0.100	10.00
66.7	0.143	8.75
84.2	0.202	7.80
102.7	0.279	7.00
119.5	0.359	6.55

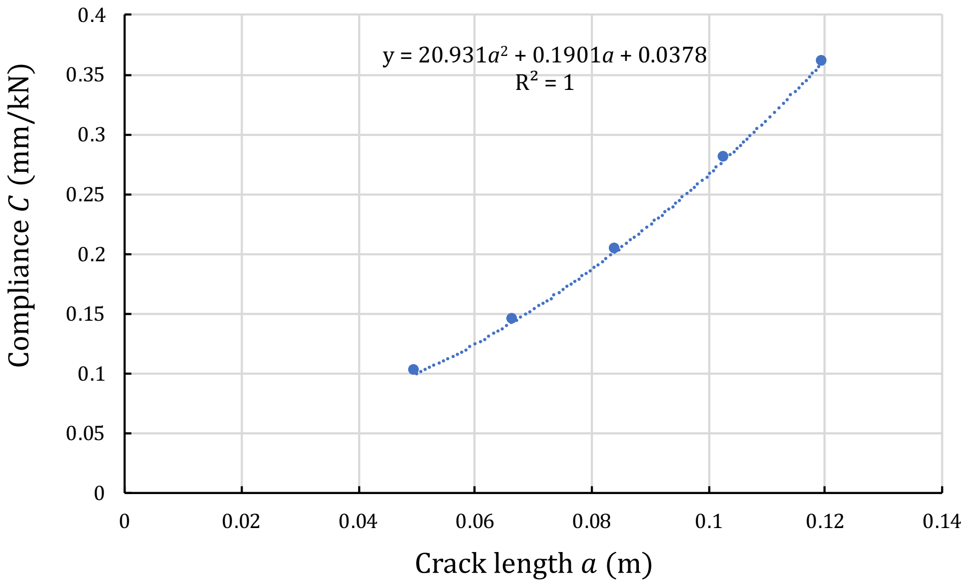
Where P is the critical load at fracture. All load-displacement records were linearly elastic up to fracture. Determine the critical energy release rate G_c for this material.

A! Solution

The energy release rate G for each test can be calculated using the compliance formula:

$$G = \frac{P^2}{2B}\frac{dC}{da}.$$

This requires to evaluate the derivative of the compliance with respect to a. To do so, we can plot the compliance C as a function of the crack length a, and fit the data with a trendline as shown below. Here, I used a second order polynomial (because it is accurate and easy to differentiate) but other equations could be used too.



For each data point, we can compute the energy release rate using:

$$G = \frac{P^2}{2B}\frac{dC}{da} = \frac{P^2}{2 \cdot 0.010}\left[\frac{20.931 \cdot 2 \cdot a + 0.1901}{10^6}\right]$$

where the division by 10^6 is simply to ensure that the units of G are in J/m^2. The values for each data point are given in the table below:

Crack length a (mm)	Compliance C (mm/kN)	Critical load P (kN)	G_c (kJ/m^2)
50.0	0.100	10.00	11.4
66.7	0.143	8.75	11.4
84.2	0.202	7.80	11.3
102.7	0.279	7.00	11.0
119.5	0.359	6.55	11.1

From these five tests, we can conclude that the toughness, or critical energy release rate, is $G_c \approx 11\,\text{kJ/m}^2$.