CS-E4075 Special course on Gaussian processes: Session #1

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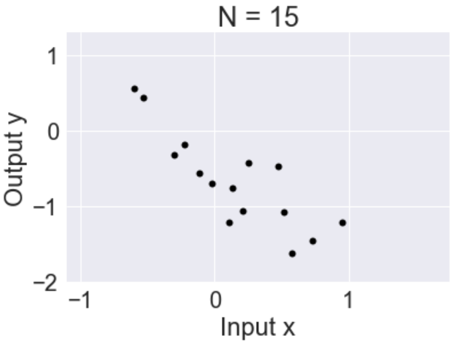
Agenda for today

1. Motivation for Gaussian processes
2. Course content, format, and evaluation
3. Warm up for Gaussian processes: Review of the multivariate Gaussian distribution
4. First assignment

Gaussian processes in a nutshell

- It's all about learning functions from data
- Suppose we are given a data set $\mathcal{D} = \{\boldsymbol{x}_n, y_n\}_{n=1}^{N}$



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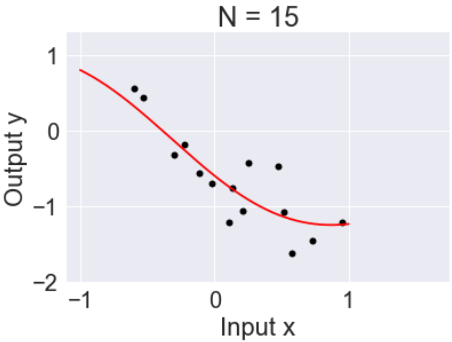

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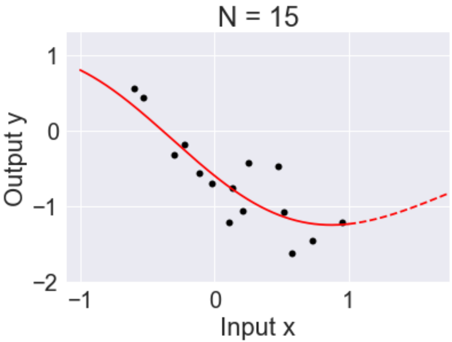

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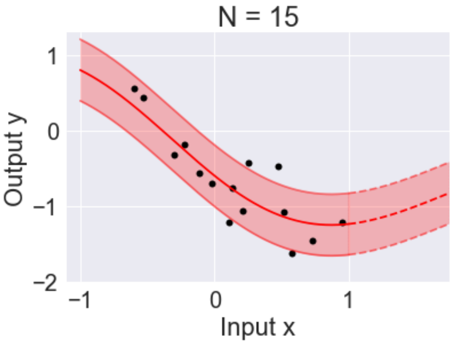

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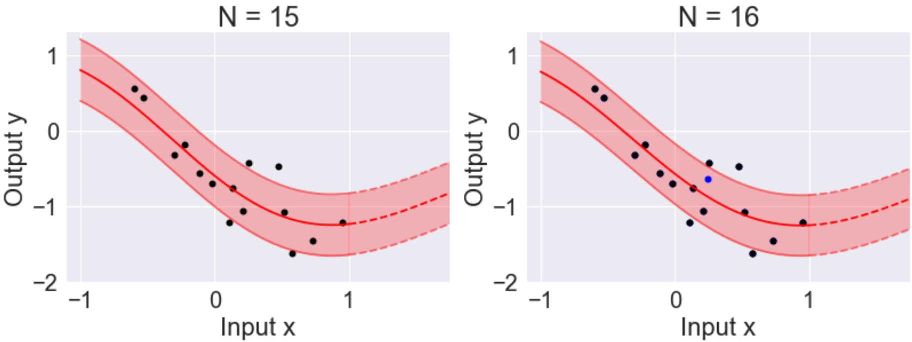

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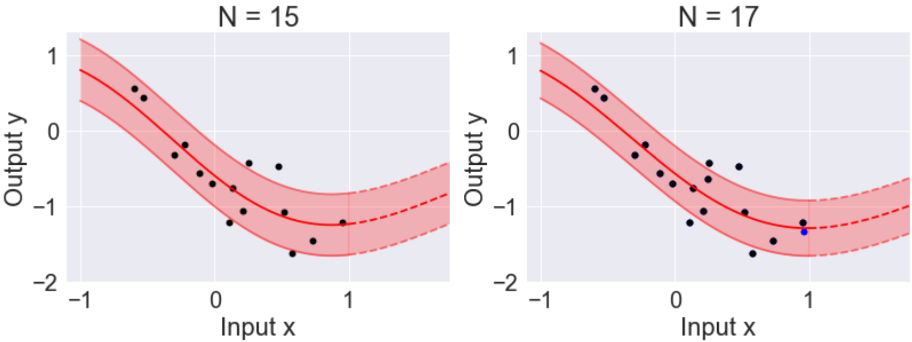

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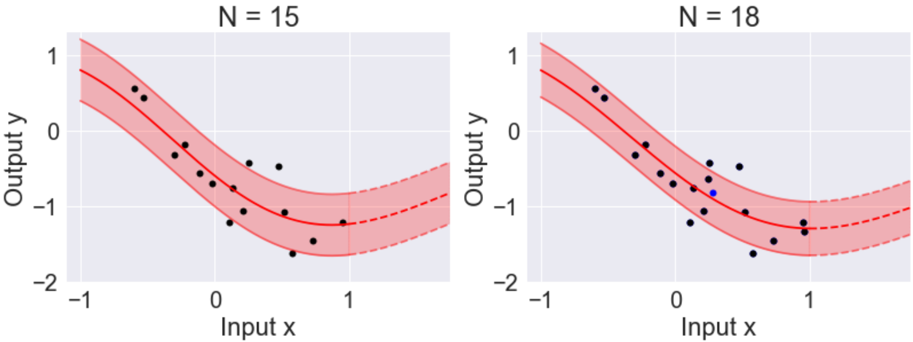

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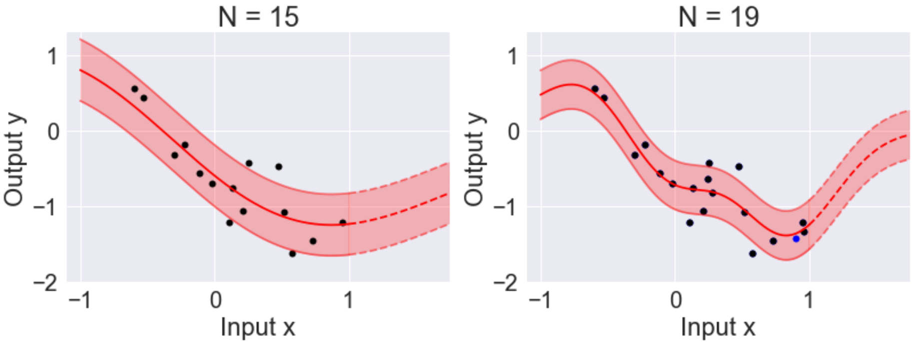

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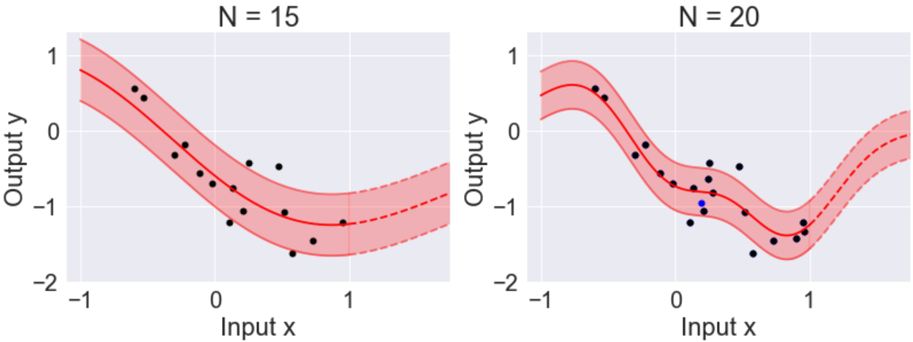

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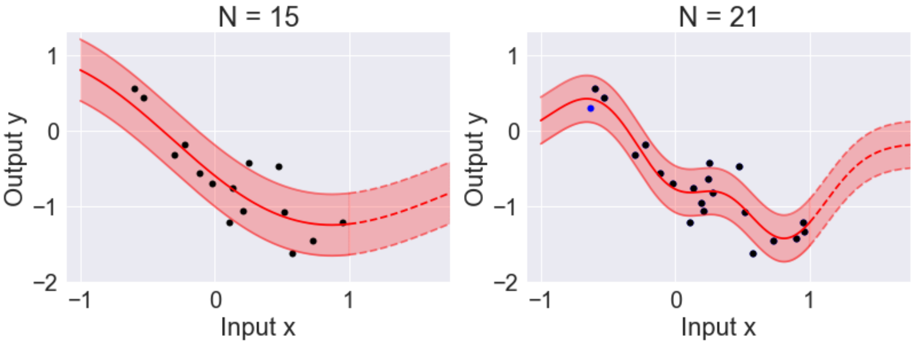

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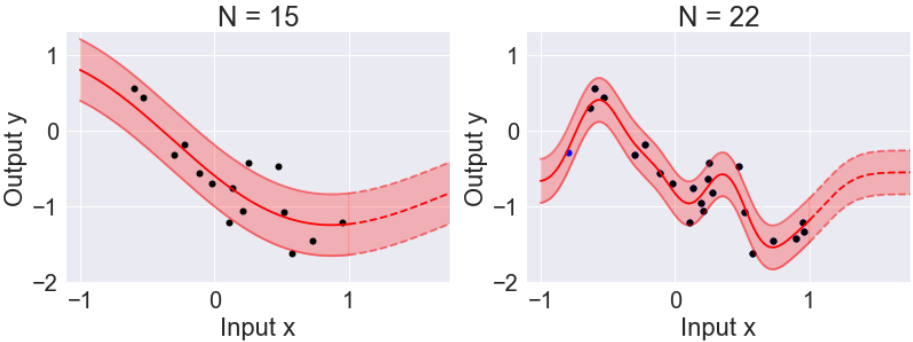

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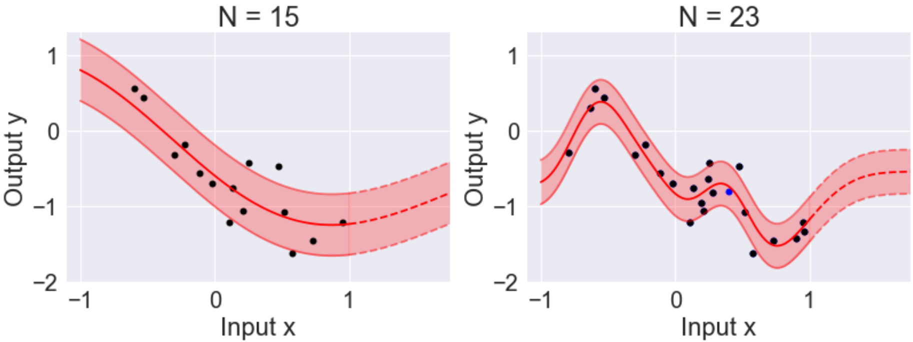

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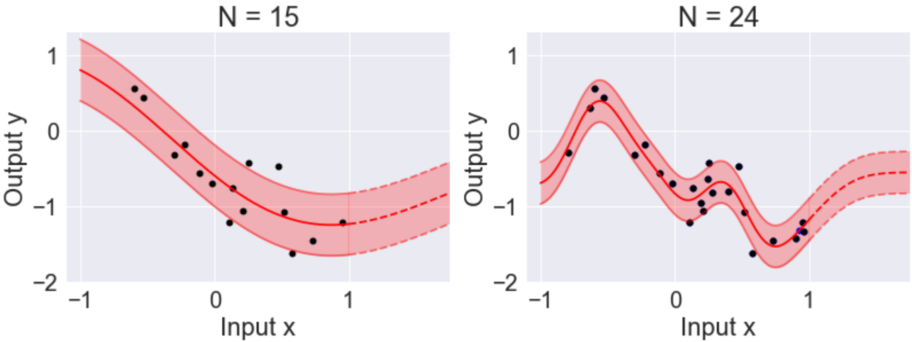

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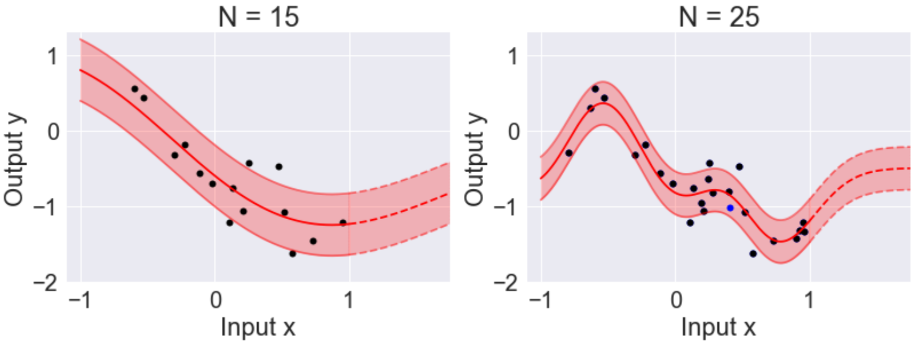

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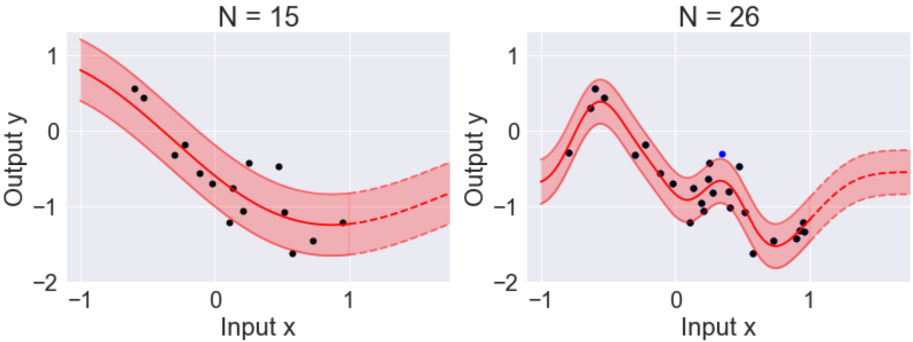

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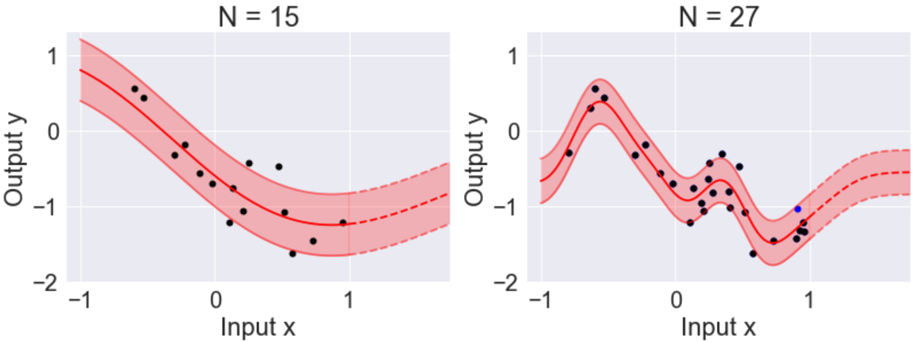

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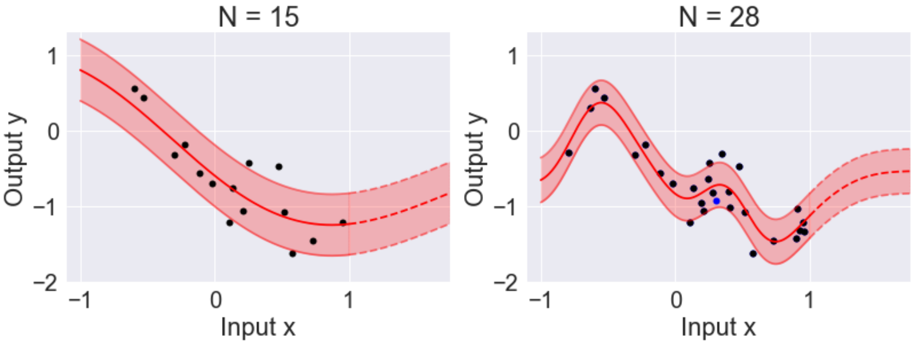

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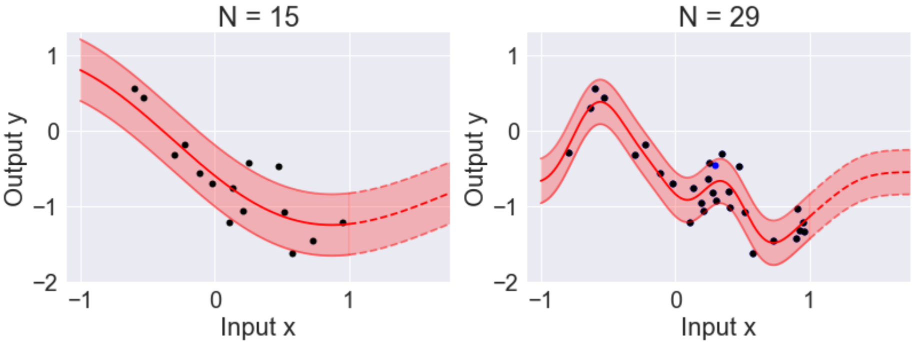

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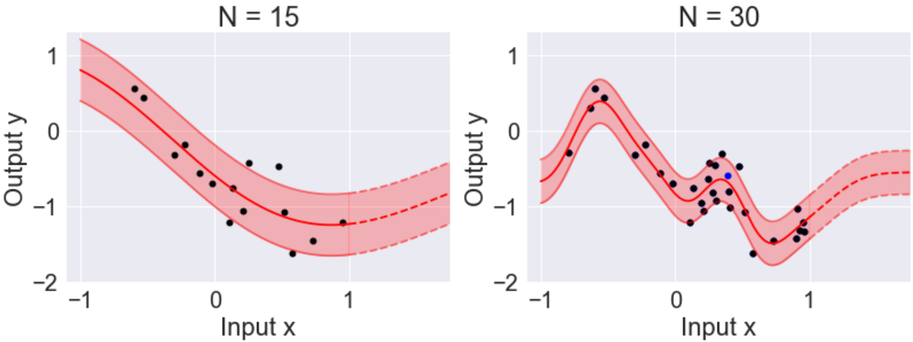

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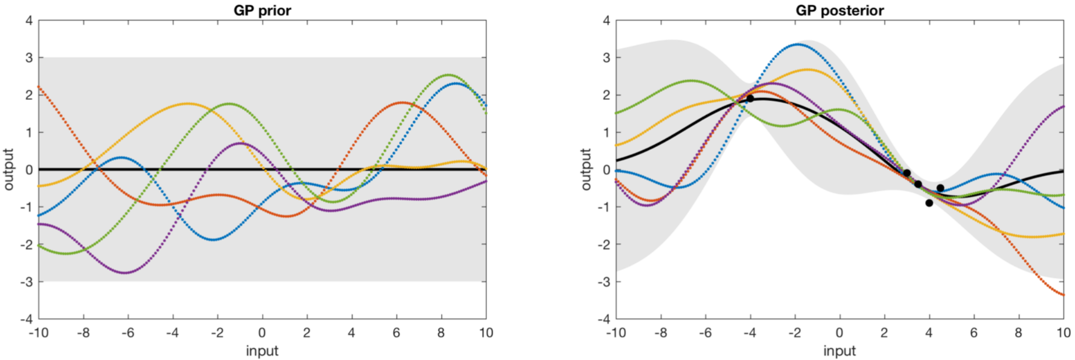
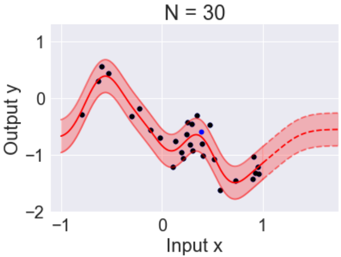
Gaussian process paradigm



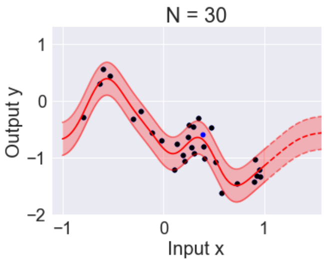
- What functions are probable before seeing the data?
  - How smooth function do we expect?
- What functions are probable after seeing the data?
- What is the probability of a single *function*, ie. $p(f(x))$
- How does the function *correlate*, ie. $cov[f(x), f(x')]$

Functions with different domains



The real line

Functions with different domains



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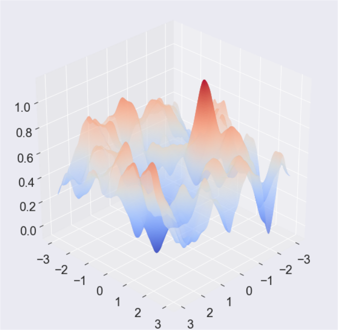

Higher dimensions

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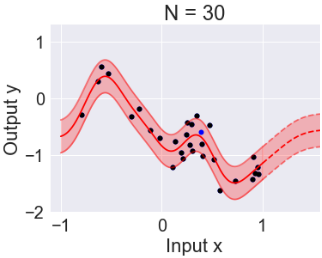

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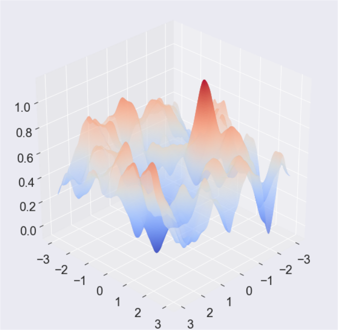

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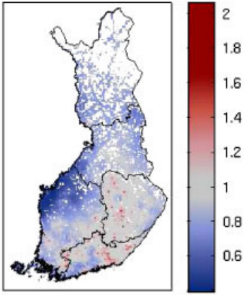
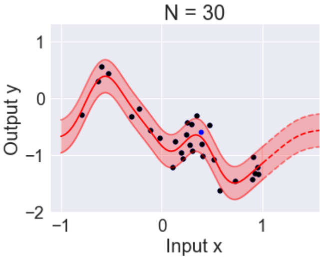


Finland

Functions with different domains



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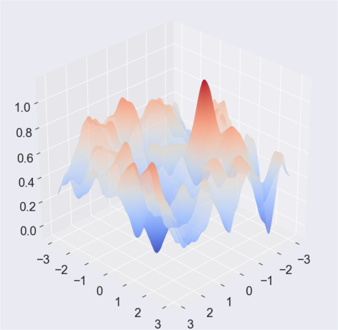

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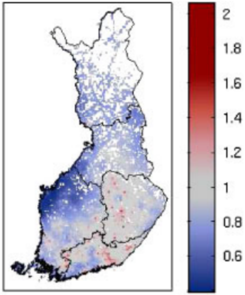

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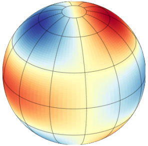
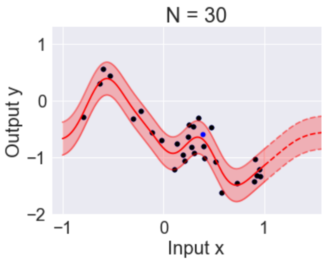


A sphere

Functions with different domains



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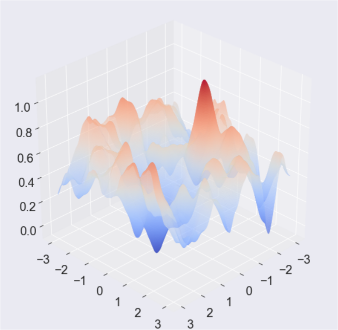

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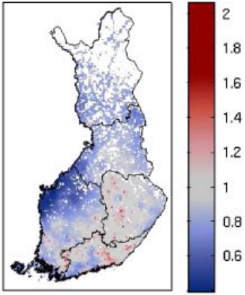

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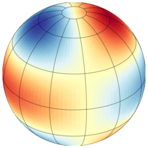

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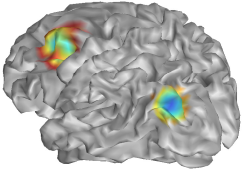
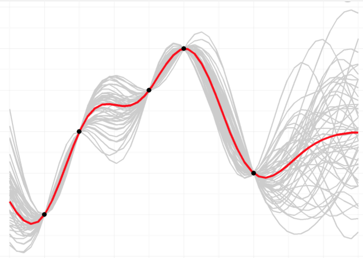


A human brain

Gaussian pro

Multitude of Gaussian processes applications



- Regression (supervised learning)
  - Time series analysis / dynamical models
  - EEG brain imaging
  - Survival analysis for cancer data
  - Predicting rainfall
  - Robot dynamics
  - Spatial modelling
- Classification (supervised learning)
  - Image recognition
  - Brain decoding
- Dimensionality reduction (unsupervised learning)
- Optimization of black box functions (Bayesian optimization)
- Numerical integration (Bayesian quadrature)
- Solving differential equations (probabilistic numerics)
- Experimental design / active learning
- Reinforcement learning

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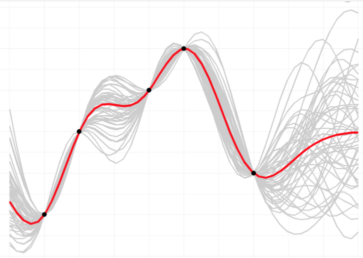
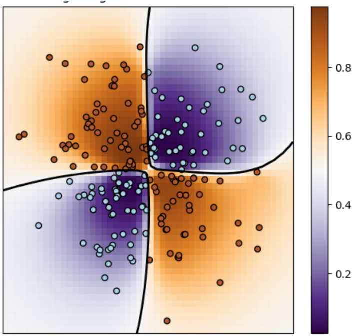




Course content

- The goal of the course is to introduce you to Gaussian processes, and to most important research advances
- We will cover
  1. ... Gaussian process regression & classification
  2. ... model selection
  3. ... approximate inference & how to speed up GPs
  4. ... spatio-temporal modelling
  5. ... latent modelling
  6. ... deep learning
  7. ... dynamical modelling

Format of the course

- The course will be based on
  - 12 lectures
  - 5 python notebook assignments
  - (optional) project work + presentation in groups of 1-4 persons

- To pass the course, you need to
  - complete and hand in exercises for 5 ECTS
  - attend exercise sessions
  - do project work for extra 2 ECTS

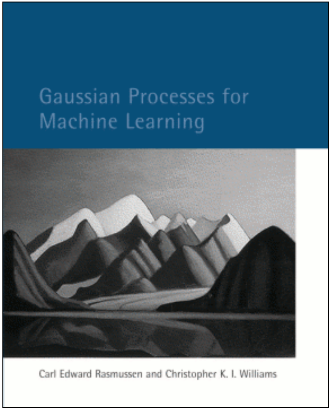
Course plan

Lectures

- Lecture 1: Warm up: Properties of the multivariate normal distribution
- Lecture 2: Linear Gaussian models and intro to Gaussian processes
- Lecture 3: Kernels and model selection
- Lecture 4: Inducing points method (.. or how to make GPs faster)
- Lecture 5: Latent modelling
- Lecture 6: Kernel learning (.. how to make GPs more flexible)
- Lecture 7: Convolution GPs (.. or how to handle images)
- Lecture 8: Deep GPs
- Lecture 9: Bayesian modelling
- Lecture 10: Spatio-temporal models
- Lecture 11: Dynamical modelling

Course material

- Lecture slides
- Exercises
- The book "Gaussian Processes for Machine Learning" by Rasmussen and Williams, MIT press, 2006, `gaussianprocess.org/gpml` (Free to download)



Assignments

Five assignments

- Released on mondays
- Deadline to complete and return following week wednesday (at mycourses)
- Present solutions at exercise session (following week) wednesday (12-14) and friday (12-14) [choose one]
- First sessions on jan 20th and 22th

Deadlines

- Assignment 1: due jan 20th (noon), sessions 20th/22th
- Assignment 2: due jan 27th (noon), sessions 27th/29th
- Assignment 3: due feb 3rd (noon), sessions 3rd/5th
- Assignment 4: due feb 10th (noon), sessions 10th/12th
- Assignment 5: due feb 17th (noon), sessions 17th/19th

Grading

- max 3 points per assignment, 1 extra point to attend either exercise session

No exam

Relation to other courses

- Designed as a 2nd / 1st year machine learning Msc course

Prerequisite assumed: basics of ML, eg.:

- CS-C3240 Machine Learning
- CS-E4710 Machine Learning: Supervised methods
- (CS-E3210 Machine learning: Basic principles)

Similar level courses

- CS-E5710 Bayesian Data Analysis (.. GPs are Bayesian)
- CS-E4820 Machine Learning: Advanced Probabilistic Methods (.. GPs are probabilistic) [Period III]
- CS-E4830 Kernel Methods in Machine Learning (.. GPs are probabilistic kernel methods) [Period IV]
- CS-E4890 Deep Learning (.. GPs can do probabilistic deep learning)
- CS-E4800 Artificial Intelligence (.. GPs are often practical for applied modelling)

The properties of the multivariate Gaussian distribution

The multivariate Gaussian distribution

- **Definition** A random vector $\boldsymbol{x} = [x_1, x_2, \cdots, x_D]^T$ is said to have the multivariate Gaussian distribution if all linear combinations of $\boldsymbol{x}$ are Gaussian distributed:

$$y = \boldsymbol{a}^{\boldsymbol{T}} \boldsymbol{x} = a_1 x_1 + a_2 x_2 + \cdots + a_D x_D \sim \mathcal{N}(m, v) \tag{1}$$

for all $\boldsymbol{a} \in \mathbb{R}^D$, where $\boldsymbol{a} \neq 0$

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- The multivariate Gaussian density for a variable $\boldsymbol{x} \in \mathbb{R}^D$:

$$\mathcal{N}(\boldsymbol{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma}) = (2\pi)^{-\frac{D}{2}} |\boldsymbol{\Sigma}|^{-\frac{1}{2}} \exp\left[-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right] \quad \in \mathbb{R}_{\geq 0} \tag{2}$$

$$\log \mathcal{N}(\boldsymbol{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma}) = -\frac{D}{2}\log 2\pi - \frac{1}{2}\log|\boldsymbol{\Sigma}| - \frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}) \quad \in \mathbb{R} \tag{3}$$

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$$\log \mathcal{N}(\boldsymbol{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma}) = -\frac{D}{2} \log 2\pi - \frac{1}{2} \log |\boldsymbol{\Sigma}| - \frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}) \quad \in \mathbb{R} \tag{3}$$

- Completely described by its parameters:
  - $\boldsymbol{\mu} \in \mathbb{R}^D$ is the mean vector
  - $\boldsymbol{\Sigma} \in \mathbb{R}^{D \times D}$ is the covariance matrix (positive definite)

The multivariate Gaussian distribution

- **Definition** A random vector $\boldsymbol{x} = [x_1, x_2, \cdots, x_D]^T$ is said to have the multivariate Gaussian distribution if all linear combinations of $\boldsymbol{x}$ are Gaussian distributed:

$$y = \boldsymbol{a}^T \boldsymbol{x} = a_1 x_1 + a_2 x_2 + \cdots + a_D x_D \sim \mathcal{N}(m, v) \tag{1}$$

  for all $\boldsymbol{a} \in \mathbb{R}^D$, where $\boldsymbol{a} \neq 0$

- The multivariate Gaussian density for a variable $\boldsymbol{x} \in \mathbb{R}^D$:

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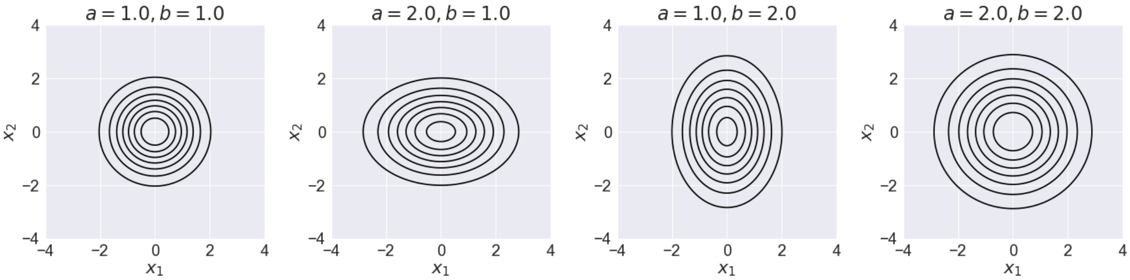
  - $\boldsymbol{\mu} \in \mathbb{R}^D$ is the mean vector
  - $\boldsymbol{\Sigma} \in \mathbb{R}^{D \times D}$ is the covariance matrix (positive definite)
- $(\boldsymbol{\Sigma})_{ij}$ is the covariance between the i'th and j'th elements x_i and x_j of $\boldsymbol{x}$

Interpretation of the covariance matrix - 2D examples

The diagonal of the covariance controls the scaling/marginal variances

$$\boldsymbol{\mu} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \boldsymbol{\Sigma} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \tag{4}$$



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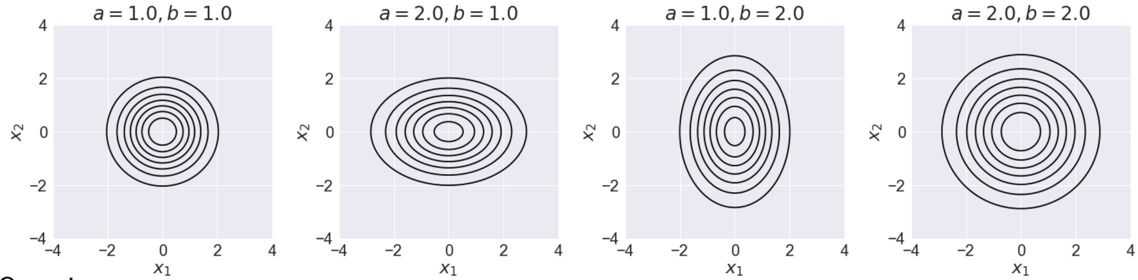


Questions:

1. If Σ is diagonal, then x_1 and x_2 are uncorrelated? True or false?
2. If Σ is diagonal, then x_1 and x_2 are independent? True or false?
3. What is the volume (integral) of density?
4. Which of the four densities has the highest peak and why?

The density at the mode

- The density is given by

$$\mathcal{N}\left(\boldsymbol{x} \middle| \boldsymbol{\mu}, \boldsymbol{\Sigma}\right) = (2\pi)^{-\frac{D}{2}} \left|\boldsymbol{\Sigma}\right|^{-\frac{1}{2}} \exp\left[-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{x}-\boldsymbol{\mu})\right] \tag{5}$$

- The mode (highest density value) is achieve at $\boldsymbol{x} = \boldsymbol{\mu}$

$$\mathcal{N}\left(\boldsymbol{\mu} \middle| \boldsymbol{\mu}, \boldsymbol{\Sigma}\right) = (2\pi)^{-\frac{D}{2}} \left|\boldsymbol{\Sigma}\right|^{-\frac{1}{2}} \tag{6}$$

- The determinant of the covariance is

$$\left|\boldsymbol{\Sigma}\right| = \det \begin{bmatrix} a & \rho \\ \rho & b \end{bmatrix} = ab - \rho^2 \tag{7}$$

- Therefore

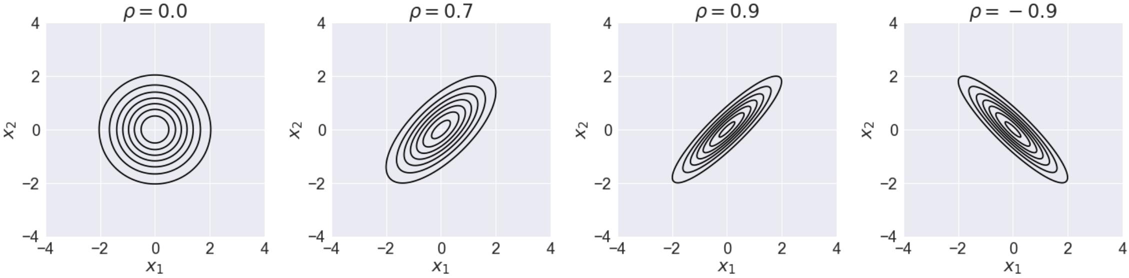
$$\mathcal{N}\left(\boldsymbol{\mu} \middle| \boldsymbol{\mu}, \boldsymbol{\Sigma}\right) = (2\pi)^{-\frac{D}{2}} \left|\boldsymbol{\Sigma}\right|^{-\frac{1}{2}} = \frac{(2\pi)^{-\frac{D}{2}}}{\sqrt{ab - \rho^2}} \tag{8}$$

Interpretation of the covariance matrix

The off-diagonals control the covariances:

$$(\boldsymbol{\Sigma})_{ij} = \text{cov}\left(x_i, x_j\right) = \mathbb{E}\left[x_i x_j\right] - \mu_i \mu_j \tag{9}$$

$$\boldsymbol{\mu} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \boldsymbol{\Sigma} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \tag{10}$$



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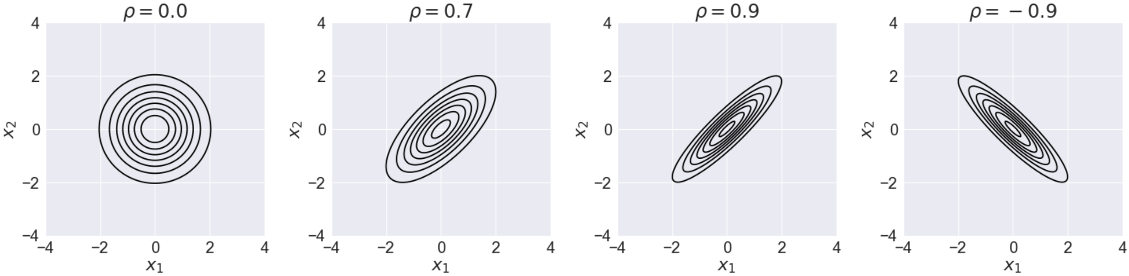

Question:

- Which of the four densities has the highest peak and why?

Interpretation of the covariance matrix

Covariance matrices must be symmetric:

$$(\boldsymbol{\Sigma})_{ij} = \text{cov}\left(x_i, x_j\right) = \text{cov}\left(x_j, x_i\right) = (\boldsymbol{\Sigma})_{ji} \tag{11}$$

Consider the following set of covariance matrices:

$$\boldsymbol{\Sigma} = \begin{bmatrix} a & c \\ c & b \end{bmatrix} \tag{12}$$

c is the covariance between x_1 and x_2. Can c take any values?

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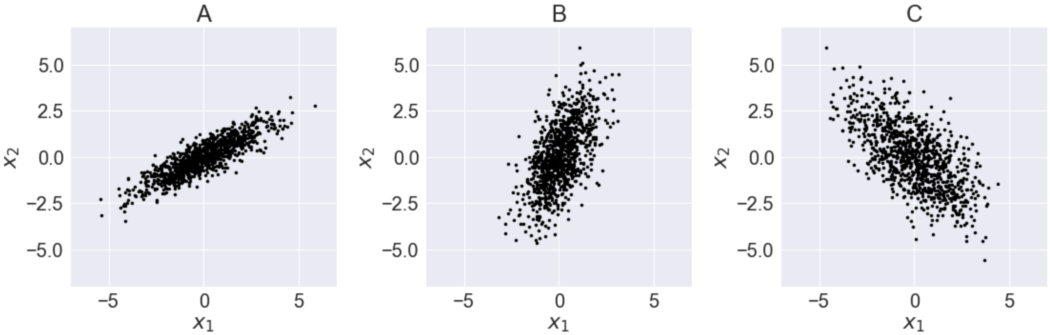
$\boldsymbol{\Sigma}$ must be positive definite

Interpretation of the covariance matrix

Determine which of the following 5 matrices are valid covariance matrices and match them to the set of samples below.

$$\mathbf{\Sigma}_1 = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} \qquad \mathbf{\Sigma}_2 = \begin{bmatrix} 3 & 2 \\ 1.5 & 3 \end{bmatrix} \qquad \mathbf{\Sigma}_3 = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$$

$$\mathbf{\Sigma}_4 = \begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix} \qquad \mathbf{\Sigma}_5 = \begin{bmatrix} 3 & 1.5 \\ 1.5 & 1 \end{bmatrix}$$



The multivariate Gaussian: Basic properties

- Gaussian distributions are closed under addition:

$$\boldsymbol{x}_1 \sim \mathcal{N}\left(\boldsymbol{m}_1, \boldsymbol{V}_1\right), \quad \boldsymbol{x}_2 \sim \mathcal{N}\left(\boldsymbol{m}_2, \boldsymbol{V}_2\right) \quad \Rightarrow \quad \boldsymbol{x}_1 + \boldsymbol{x}_2 \sim \mathcal{N}\left(\boldsymbol{m}_1 + \boldsymbol{m}_2, \boldsymbol{V}_1 + \boldsymbol{V}_2\right) \tag{14}$$

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- For any finite number of independent variables:

$$\boldsymbol{x}_i \sim \mathcal{N}(\boldsymbol{m}_i, \boldsymbol{V}_i) \quad \Rightarrow \quad \sum_i \boldsymbol{x}_i \sim \mathcal{N}\left(\sum_i \boldsymbol{m}_i, \sum_i \boldsymbol{V}_i\right) \tag{15}$$

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- Gaussian distributions are closed under affine transformations:

$$\boldsymbol{x} \sim \mathcal{N}(\boldsymbol{m}, \boldsymbol{V}), \quad \Rightarrow \quad \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b} \sim \mathcal{N}\left(\boldsymbol{A}\boldsymbol{m} + \boldsymbol{b}, \boldsymbol{A}\boldsymbol{V}\boldsymbol{A}^T\right) \tag{16}$$

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- Manipulating Gaussian distributions often boils down to linear algebra
- ‘Matrix cookbook’ (section 8) and Rasmussen book (Appendix A)

Question

… how to use the following two results

$$\boldsymbol{x}_i \sim \mathcal{N}(\boldsymbol{m}_i, \boldsymbol{V}_i) \quad \Rightarrow \qquad \sum_i \boldsymbol{x}_i \sim \mathcal{N}\left(\sum_i \boldsymbol{m}_i, \sum_i \boldsymbol{V}_i\right) \tag{17}$$

$$\boldsymbol{x} \sim \mathcal{N}(\boldsymbol{m}, \boldsymbol{V}) \quad \Rightarrow \qquad \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b} \sim \mathcal{N}\left(\boldsymbol{A}\boldsymbol{m} + \boldsymbol{b}, \boldsymbol{A}\boldsymbol{V}\boldsymbol{A}^T\right), \tag{18}$$

to calculate the distribution of $\boldsymbol{Y}$ in the following linear model?

$$\boldsymbol{Y} = \mu + \boldsymbol{X}\boldsymbol{w} + \epsilon, \tag{19}$$

where

$$\boldsymbol{w} \sim \mathcal{N}(\boldsymbol{m}, \boldsymbol{V}) \qquad \epsilon \sim \mathcal{N}(0, \sigma^2 \boldsymbol{I}) \tag{20}$$

Sampling from the multivariate Gaussian distribution

$$\boldsymbol{x} \sim \mathcal{N}(\boldsymbol{m}, \boldsymbol{V}) \quad \Rightarrow \quad \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b} \sim \mathcal{N}\left(\boldsymbol{A}\boldsymbol{m} + \boldsymbol{b}, \boldsymbol{A}\boldsymbol{V}\boldsymbol{A}^T\right) \tag{21}$$

- Suppose we know how to generate samples from a standardized univariate Gaussian distribution
- How can we use the above result to generate samples from an arbitrary multivariate Gaussian distribution $\boldsymbol{y} \sim \mathcal{N}(\boldsymbol{m}, \boldsymbol{V})$?

Sampling from the multivariate Gaussian distribution

$$\boldsymbol{x} \sim \mathcal{N}(\boldsymbol{m}, \boldsymbol{V}) \quad \Rightarrow \quad \boldsymbol{Ax} + \boldsymbol{b} \sim \mathcal{N}\left(\boldsymbol{Am} + \boldsymbol{b}, \boldsymbol{AVA}^T\right) \tag{21}$$

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  1. Compute the matrix square root of $\boldsymbol{V} = \boldsymbol{LL}^T$
  2. Generate a sample of $\boldsymbol{x}$ such that $x_i \sim \mathcal{N}(0, 1)$, i.e. $\boldsymbol{x} \sim \mathcal{N}(0, \boldsymbol{I})$
  3. Compute $\boldsymbol{y} = \boldsymbol{Lx} + \boldsymbol{m}$

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    3. Compute $\boldsymbol{y} = \boldsymbol{Lx} + \boldsymbol{m}$
- Why does it work?

$$\boldsymbol{y} = \boldsymbol{Lx} + \boldsymbol{m} \sim \mathcal{N}\left(\boldsymbol{L}0 + \boldsymbol{m}, \boldsymbol{LIL}^T\right) = \mathcal{N}(\boldsymbol{m}, \boldsymbol{V}) \tag{22}$$

The multivariate Gaussian: Marginalization

- Gaussian densities are closed on marginalization
- Let $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ be a partitioning of $\boldsymbol{x} = \boldsymbol{x}_1 \cup \boldsymbol{x}_2$, then

$$p(\boldsymbol{x}_1, \boldsymbol{x}_2) = \mathcal{N}\left(\begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} \middle| \begin{bmatrix} \boldsymbol{m}_1 \\ \boldsymbol{m}_2 \end{bmatrix}, \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix} \right) \tag{23}$$

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then

$$p(\boldsymbol{x}_1) = \int p(\boldsymbol{x}_1, \boldsymbol{x}_2) d\boldsymbol{x}_2 = \mathcal{N}\left(\boldsymbol{x}_1 | \boldsymbol{m}_1, \boldsymbol{\Sigma}_{11}\right) \tag{24}$$

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and

$$p(\boldsymbol{x}_2) = \int p(\boldsymbol{x}_1, \boldsymbol{x}_2) \mathrm{d}\boldsymbol{x}_1 = \mathcal{N}\left(\boldsymbol{x}_2 | \boldsymbol{m}_2, \Sigma_{22}\right) \tag{25}$$

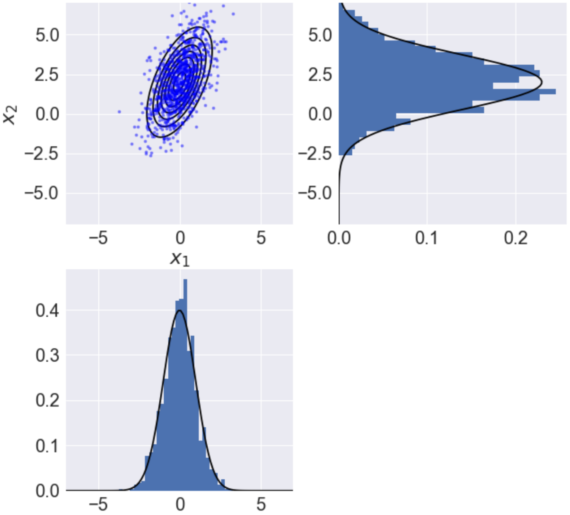
- The same is true for any partitioning

Marginalization example in 2D

$$\boldsymbol{x} \sim \mathcal{N}\left(\begin{bmatrix}0\\2\end{bmatrix}, \begin{bmatrix}1 & 1\\1 & 3\end{bmatrix}\right)$$

$$x_1 \sim \mathcal{N}(0, 1)$$

$$x_2 \sim \mathcal{N}(2, 3)$$



Conditioning

- Gaussian densities are closed under conditioning!
- Recall the definition of conditioning:

$$p(A|B) = \frac{p(A \cap B)}{p(B)} \tag{26}$$

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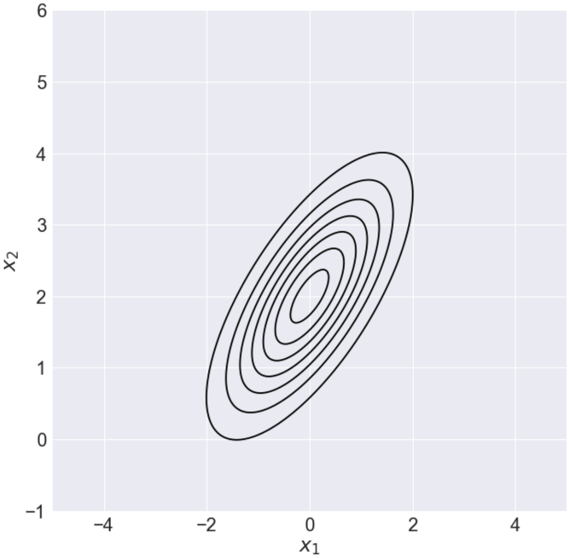
- The conditional of x_1 is given x_2 by:

$$p(\boldsymbol{x}_1|\boldsymbol{x}_2) = \mathcal{N}\left(\boldsymbol{x}_1 | \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}[\boldsymbol{x}_2 - \boldsymbol{m}_2] + \boldsymbol{m}_1, \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}\right) \tag{28}$$

  - $\boldsymbol{x}_1$ is a random variable, $\boldsymbol{x}_2$ is assigned a fixed value

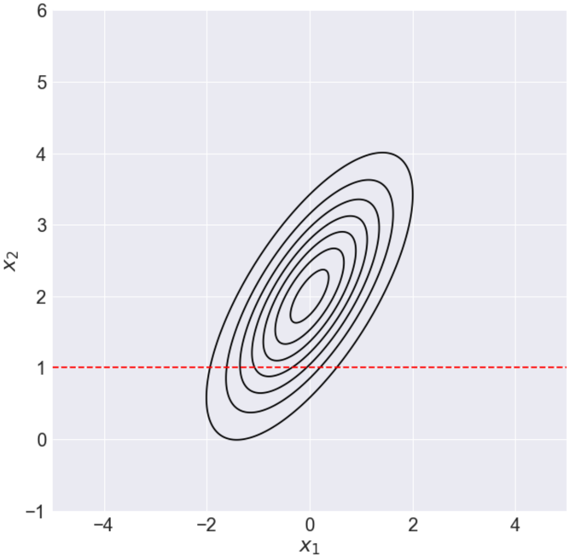
Conditioning example in 2D



- 2D example

$$\boldsymbol{\mu} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \qquad \boldsymbol{\Sigma} = \begin{bmatrix} 1 & 0.8 \\ 0.8 & 1 \end{bmatrix}$$

Conditioning example in 2D



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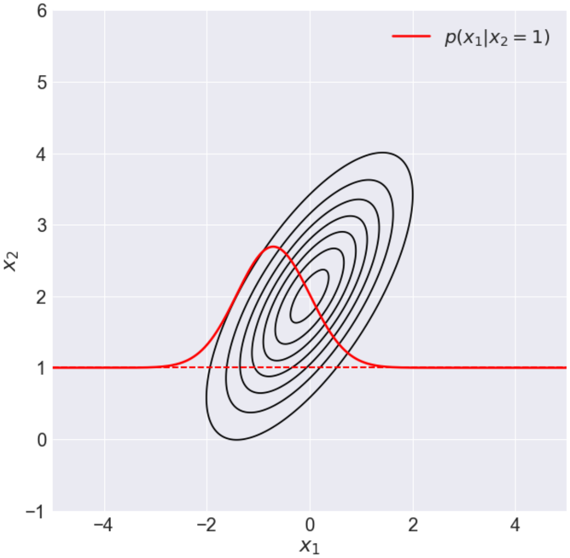
- Assume we observe $x_2 = 1$

Conditioning example in 2D



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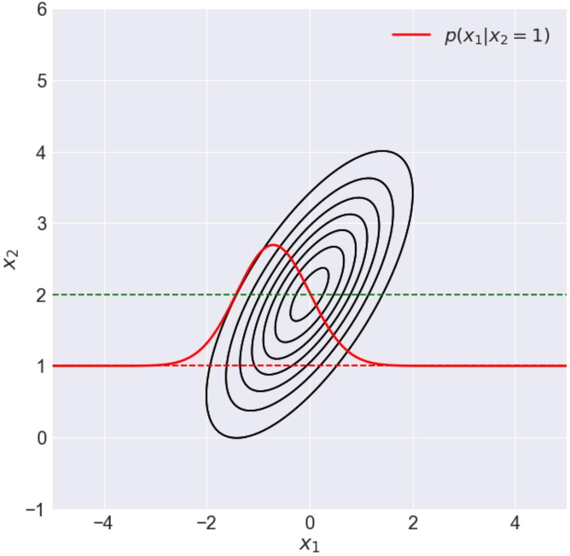
- Assume we observe $x_2 = 1$
- The conditional disitribution

$$p(x_1|x_2) = \mathcal{N}\left(x_1 \middle| -\frac{\sqrt{2}}{2}, \frac{1}{2}\right)$$

Conditioning example in 2D

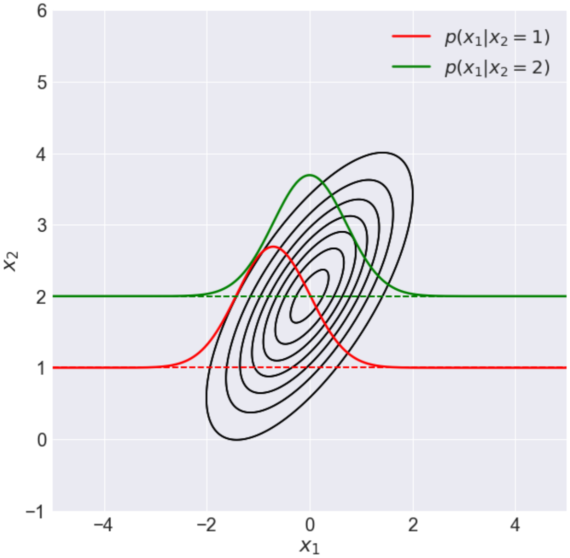


- 2D example

$$\boldsymbol{\mu} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \qquad \boldsymbol{\Sigma} = \begin{bmatrix} 1 & 0.8 \\ 0.8 & 1 \end{bmatrix}$$

- Assume we observe $x_2 = 2$

Conditioning example in 2D



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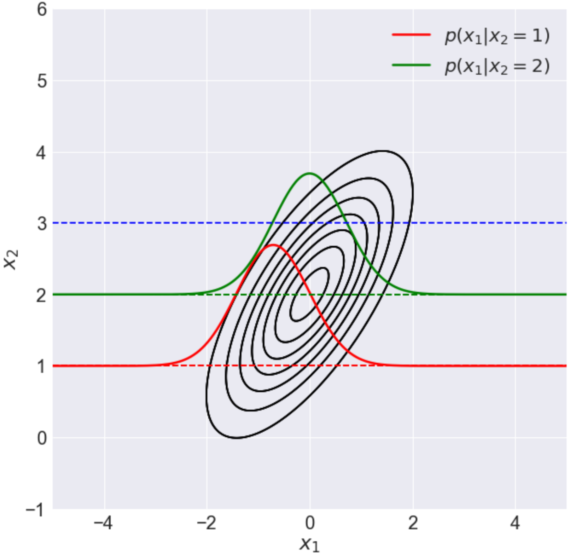
$$\boldsymbol{\mu} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \qquad \boldsymbol{\Sigma} = \begin{bmatrix} 1 & 0.8 \\ 0.8 & 1 \end{bmatrix}$$

- Assume we observe $x_2 = 2$
- The conditional disitribution

$$p(x_1|x_2) = \mathcal{N}\left(x_1|0, \frac{1}{2}\right)$$

Conditioning example in 2D

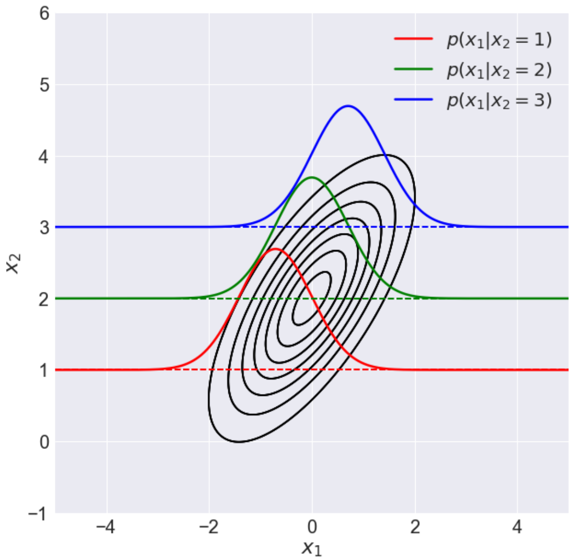


- 2D example

$$\boldsymbol{\mu} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \qquad \boldsymbol{\Sigma} = \begin{bmatrix} 1 & 0.8 \\ 0.8 & 1 \end{bmatrix}$$

- Assume we observe $x_2 = 3$

Conditioning example in 2D



- 2D example

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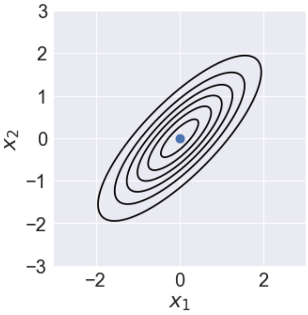
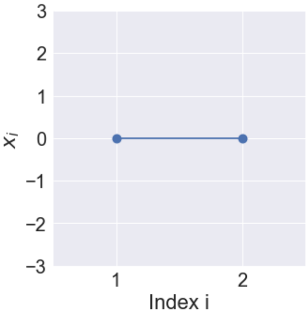
- Assume we observe $x_2 = 3$
- The conditional disitribution

$$p(x_1|x_2) = \mathcal{N}\left(x_1 \middle| \frac{\sqrt{2}}{2}, \frac{1}{2}\right)$$

- Visualizations in 2D

$$\Sigma = \begin{bmatrix} 1 & 0.8 \\ 0.8 & 1 \end{bmatrix}$$

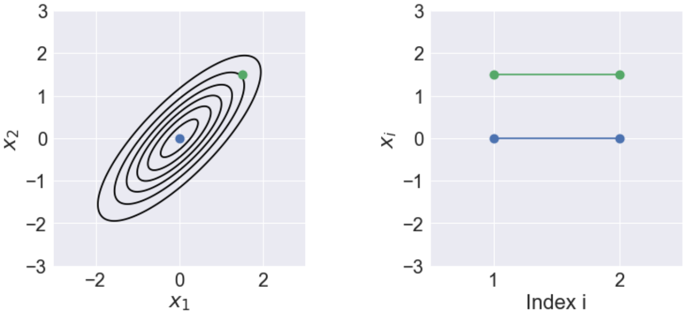




Visualizing samples in higher dimensions

- Visualizations in 2D

$$\boldsymbol{\Sigma} = \begin{bmatrix} 1 & 0.8 \\ 0.8 & 1 \end{bmatrix}$$



Visualizing samples in higher dimensions

- Visualizations in 2D

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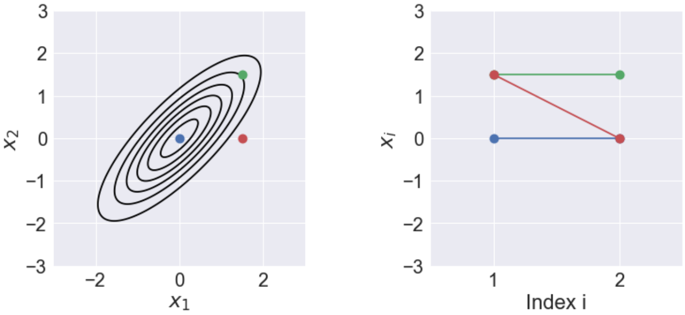

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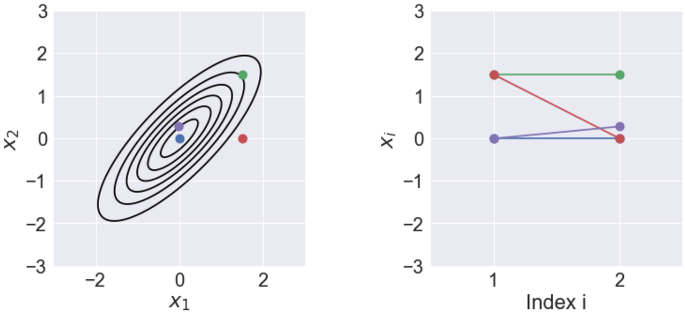

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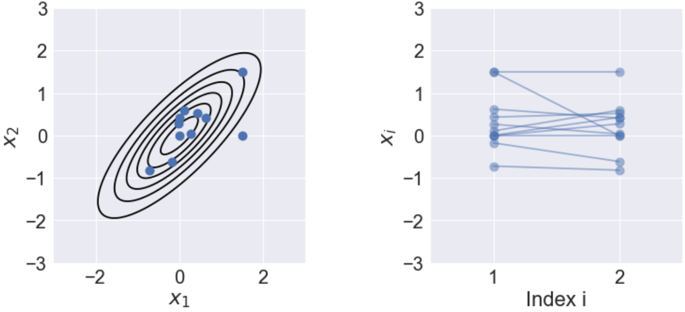
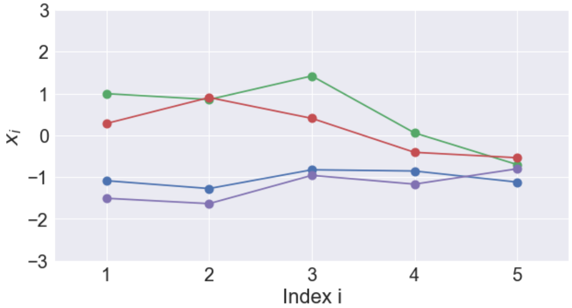


Visualizing samples in higher dimensions

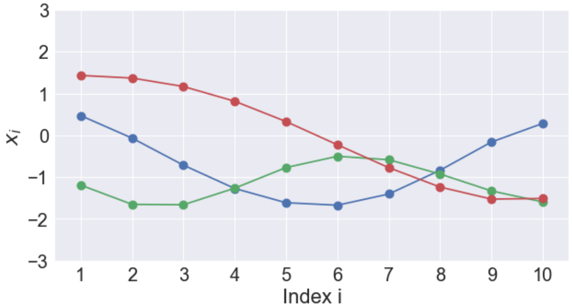
- Visualizations in 5D



$$\mathbf{\Sigma} = \begin{bmatrix} 1 & 0.8^1 & 0.8^2 & 0.8^3 & 0.8^4 \\ 0.8^1 & 1 & 0.8^1 & 0.8^2 & 0.8^3 \\ 0.8^2 & 0.8^1 & 1 & 0.8^1 & 0.8^2 \\ 0.8^3 & 0.8^2 & 0.8^1 & 1 & 0.8^1 \\ 0.8^4 & 0.8^3 & 0.8^2 & 0.8^1 & 1 \end{bmatrix}$$

Visualizing samples in higher dimensions

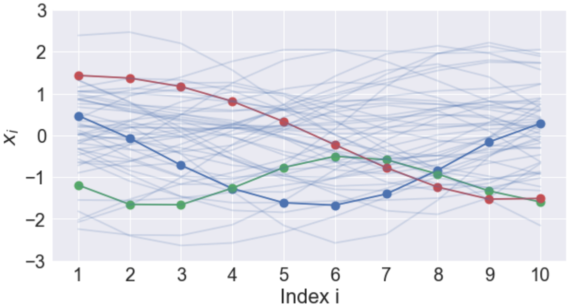
- Visualizations in 10D



$$\Sigma = \begin{bmatrix} 1 & 0.8^1 & 0.8^2 & \dots & 0.8^9 \\ 0.8^1 & 1 & 0.8^1 & & \vdots \\ 0.8^2 & 0.8^1 & 1 & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0.8^9 & \dots & \dots & \dots & 1 \end{bmatrix}$$

Visualizing samples in higher dimensions

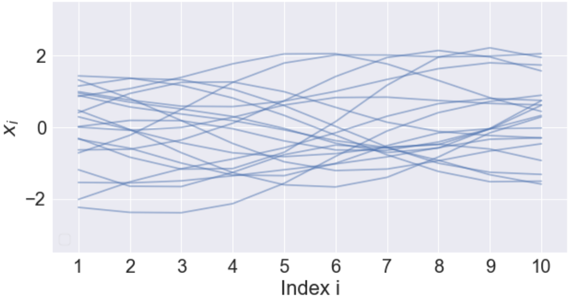
- Visualizations in 10D



$$\Sigma = \begin{bmatrix} 1 & 0.8^1 & 0.8^2 & \dots & 0.8^9 \\ 0.8^1 & 1 & 0.8^1 & & \vdots \\ 0.8^2 & 0.8^1 & 1 & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0.8^9 & \dots & \dots & \dots & 1 \end{bmatrix}$$

Back to conditioning

- So far, we have seen samples from the distribution $p(\boldsymbol{x}) = \mathcal{N}\left(\boldsymbol{x} \middle| 0, \boldsymbol{\Sigma}\right)$



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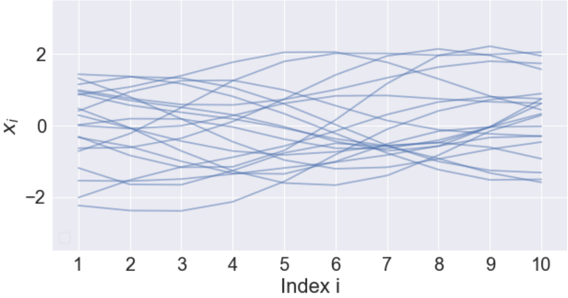

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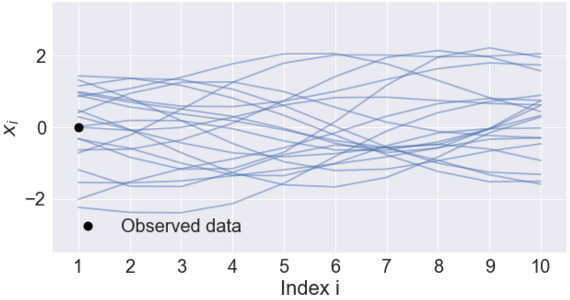

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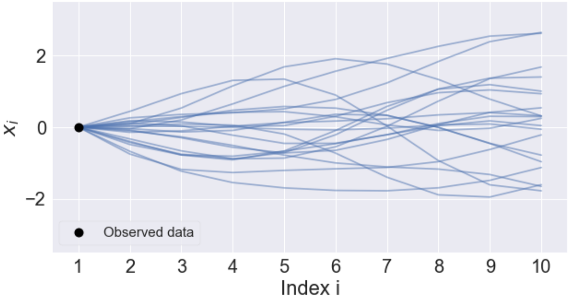

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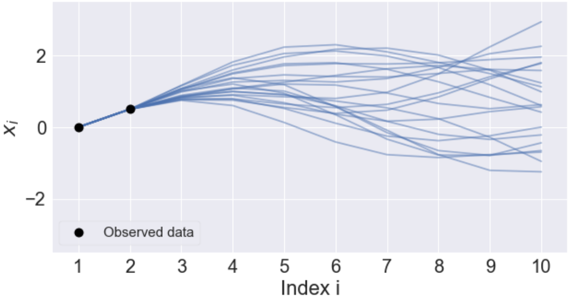

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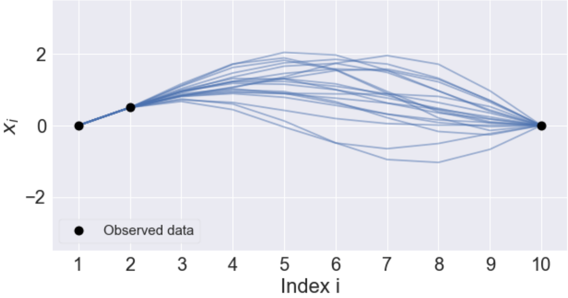

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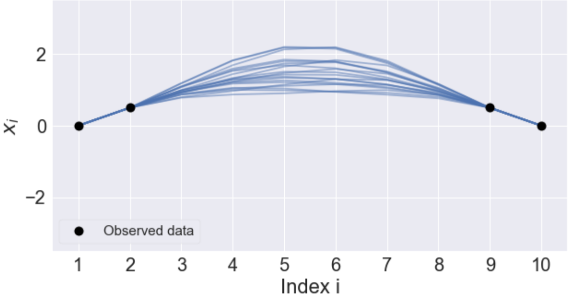

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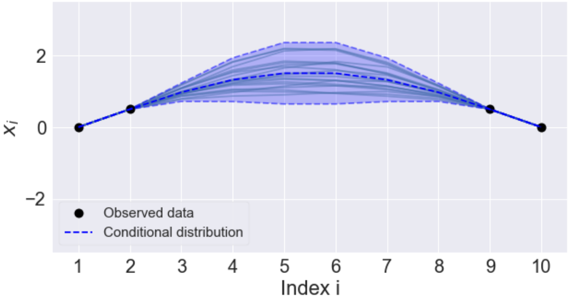
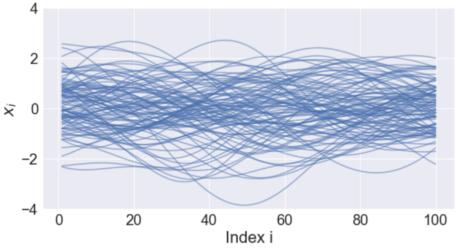


Back to conditioning II

- Let's now consider a case with $\boldsymbol{x} \in \mathbb{R}^{100}$ dimensions with 5 observations



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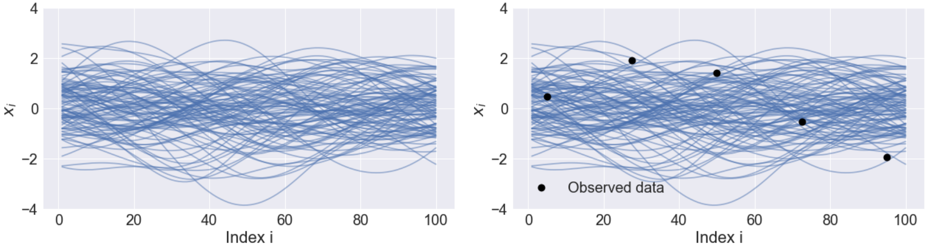

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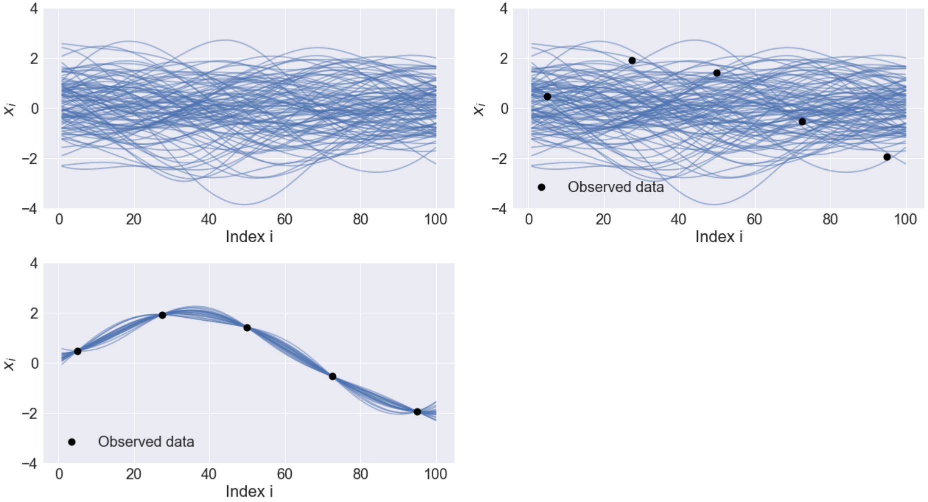

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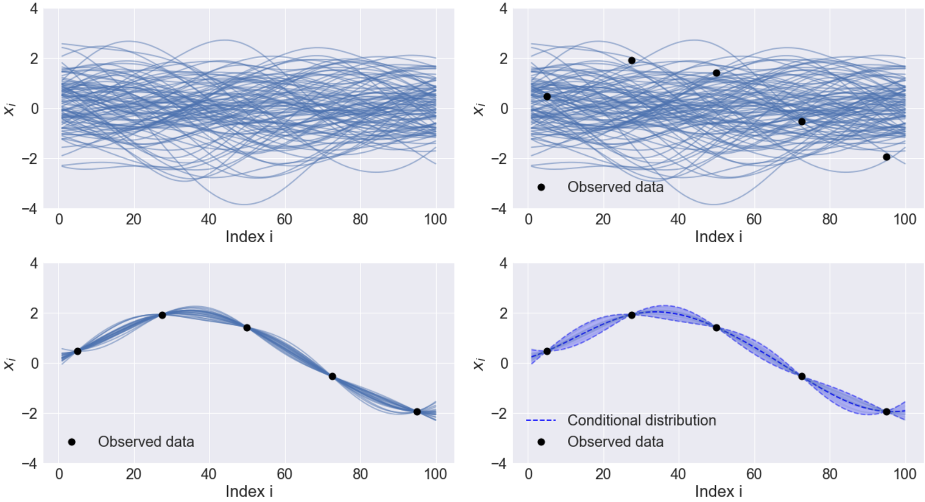

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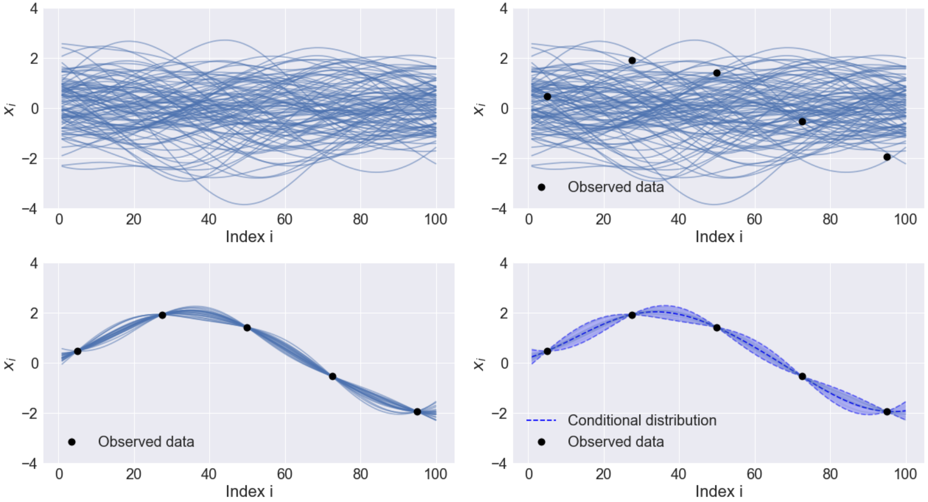

- Informally: We can think functions as vectors with infinite dimensions
- Using conditining in Gaussian distributions, we can do non-linear regression!

The end of todays lecture

- Next thursday 14th, 10pm
  - We will introduce Gaussian processes more formally
  - Read Chapter 1 & 2 of the Gaussian process book `gaussianprocess.org/gpml`
- Time to work: first assignment
  - Released today, deadline jan 20th, 12:00 (midday)
  - Reviews the basics of Bayesian inference and Gaussians
  - **Must be** handed in through MyCourses
  - Q&A sessions on 20th and 22th (grants extra point for being present!)