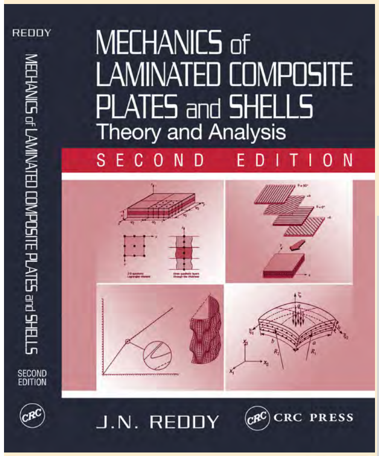
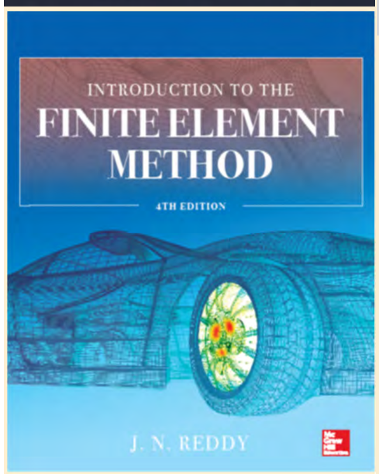
THEORY AND ANALYSIS OF LAMINATED COMPOSITE AND FUNCTIONALLY GRADED STRUCTURES





5-8 September 2023

Department of Mechanical Engineering

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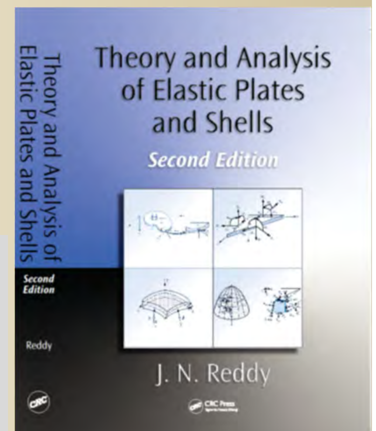
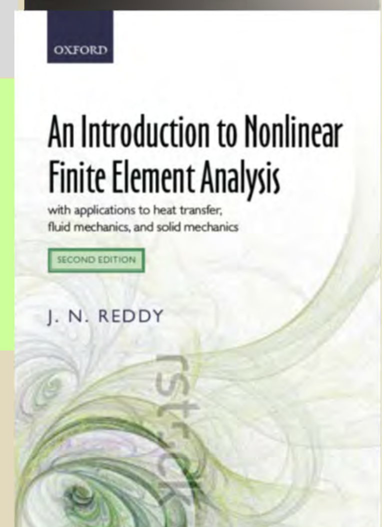




Course Material*
(Copy of the Overheads)

* This document contains a copy of the overheads used in the course. Much of the material used in the course comes from the instructor's books, *Mechanics of Laminated Composite Plates and Shells* (2nd ed., CRC Press, 2004) and *Theories and Analyses of Beams and Axisymmetric Circular Plates* (CRC Press, 2022); other material comes from the research publications of the instructor.

Some Simple Rules to Follow during the Course

- Turn off your audio unless you are asking a question

- Ask questions when you have a doubt or not understand something I explained.

GENERAL REMARKS

Engineering is a problem-solving discipline, and it require an understanding of complex systems and phenomena that occurs in the systems.

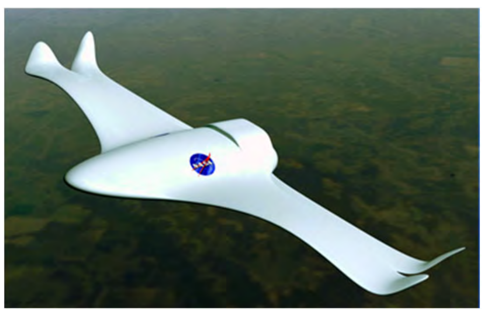
The study of natural phenomena involves:

(1) developing mathematical models,
(2) conducting physical experiments,
(3) carrying out numerical simulations, and
(4) designing and building systems for human needs.

REAL-WORLD PROBLEMS ARE COMPLEX

Nanocomposite based car

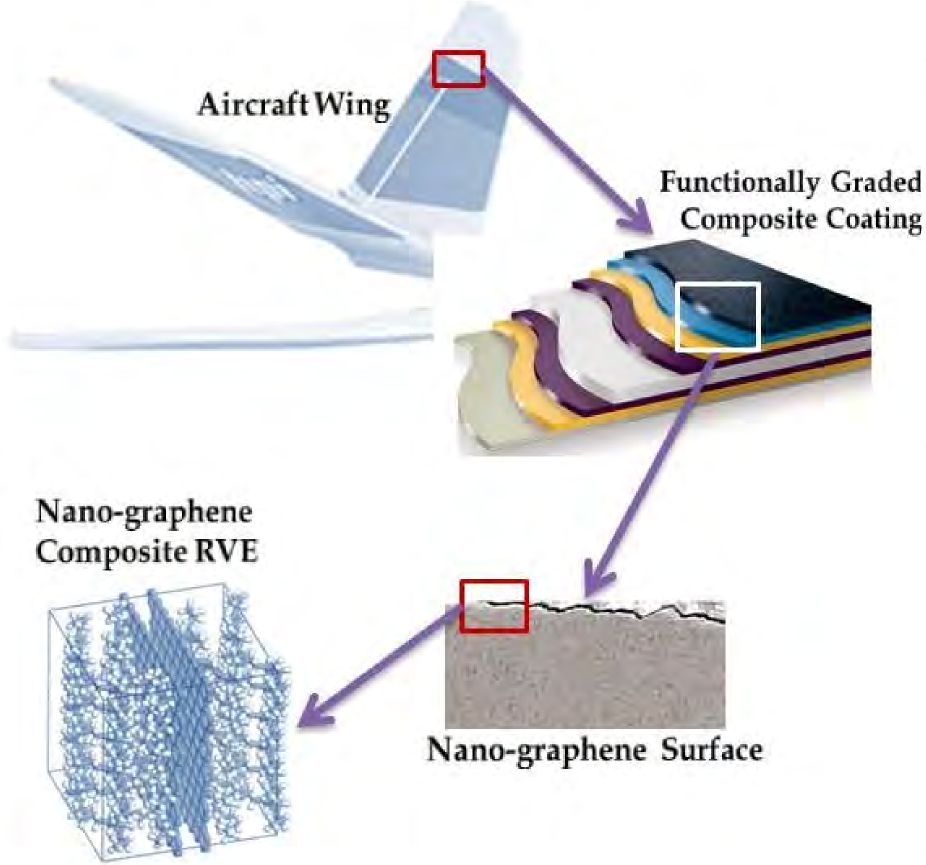
Nanotube-enhanced amphibious aircraft designed by Harbor Composites



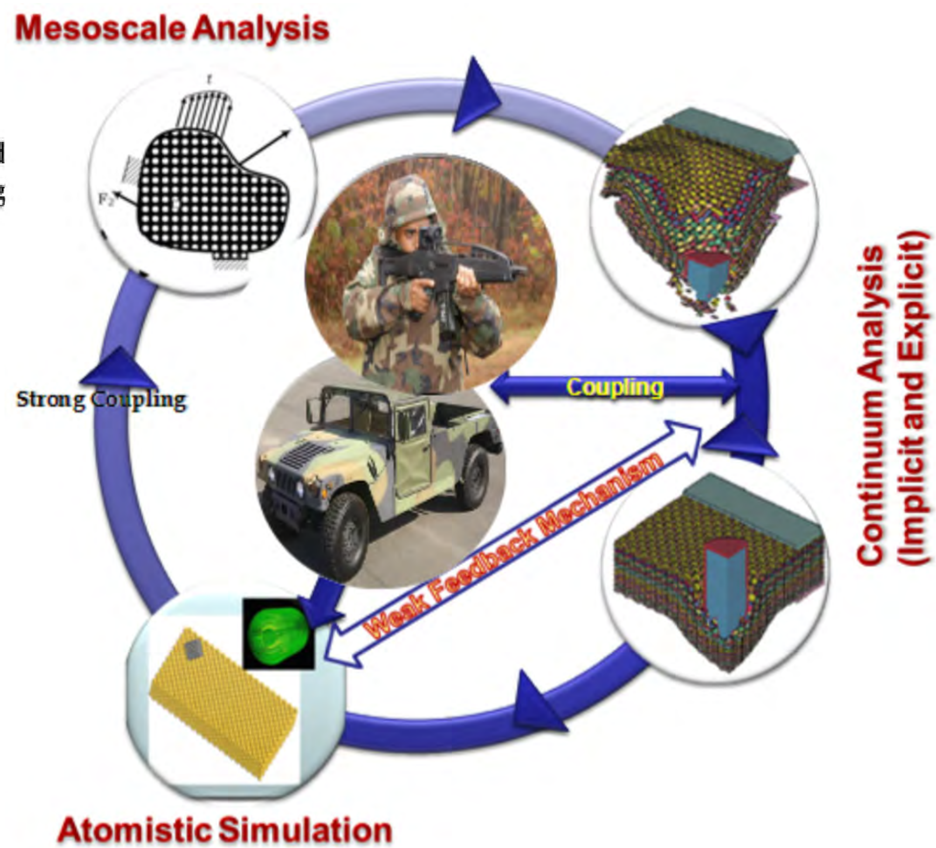
Nanocomposites in morphing aircraft - NASA

F-35 Lightning II aircraft to integrate structural nanocomposites in non-load bearing airframe components.

Complicated Geometries, Coupled Phenomena, and Multifunctional Material Challenges



Computational Analysis - Multiscale Framework



Impact mitigation

GENERAL REMARKS (continued)

- **Everything engineers do is model physical phenomena that occurs in nature. Numerical modeling requires mathematical and computational models of the physics studied.**
- **Engineering subjects provide concepts to formulate, analyze, and make a decision (towards designing and building). They are not an exact science. In fact, there is no "complete" mathematical model of anything we study.**
- **We can only try to "improve" on what we already know (often, based on the objective of the study).**
- **Only two things that matter in engineering: (1) Reliable functionality (or know the probability of failure) and (2) cost of the product.**

GENERAL REMARKS (continued)

Mathematical Model Development and Computer Simulations **continues to be a major component of engineering analysis, design and manufacturing (CAE and CAM).**

Numerical simulations facilitate investigations into the use of alternative materials and configurations. They reduce and/or replace prototype testing and associated costs.

A good understanding of the phenomena modeled and the computer method used to simulate the process is essential for the design and manufacturing of complex systems.

Therefore, learn the subjects as well as possible – not to just to carry some task with the help of a computer program and without understanding – but gain understanding and knowledge to be creative and investigative in your works.

GENERAL REMARKS (summary statements)

- Our works must be built on sound mechanics foundation (wisdom to see details).
- We must develop robust computational tools that make use of advances made in theoretical developments and numerical methods.
- We must seek physically meaningful experimental validations to understand and predict the risks of failure (i.e., understand what is happening and use it to assess risk of failure).

Everything we do as engineers is: model physical phenomenon.

COURSE CONTENTS

First Day (Sept 5)

- Composite Materials: An Introduction
- Anisotropic Elasticity and FGM materials
- Laminate Structural Theories (CLPT and FSDT)
- Interaction Session

Second Day (Sept 6)

- Analytical Solution Methods – Navier Solution
- Numerical examples of bending, vibration, and buckling
- Finite Element Models
- Numerical Results and Discussion
- Interaction Session

COURSE CONTENTS

Third Day (Sept 7)

- FGM Structures (Beams)
- FGM Structures (Plates)
- Finite Element Models
- Numerical Results and Discussion
- Third-order and layerwise theories
- Interaction Session

Fourth Day (Sept 8)

- Continuum Shell Elements
- A Robust Shell Finite Element
- Failures in Composites and Design Considerations
- Overview of the Course
- Interaction Session

ACKNOWLEDGEMENTS

Much of the material presented herein is taken from the instructor's books and research papers available in the open literature.

The presentation of this course is made possible by the support and encouragement of

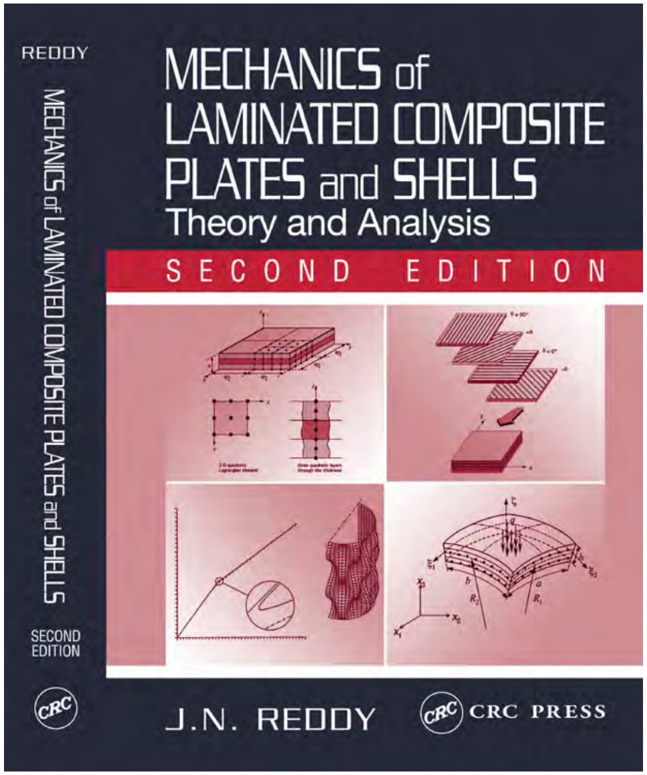
Professor Jani Romanoff

The instructor (JN Reddy) is very grateful to him for the support, coordination, and friendship.

Thanks to you for attending the course.

COMPOSITE MATERIALS: An Introduction

- Composite Materials - Definition
- Classification of Composites
- Advantages and Disadvantages of Composites
- The Role of Stress Analysis
- Analysis of Composite Structures
- Mechanical Characterization



COMPOSITE MATERIALS-Definition

Definition: **Two or more materials combined on a *macroscopic* scale to form a useful third material**

Properties to be Improved: **Strength, stiffness, weight, fatigue life, wear resistance, thermal insulation, thermal conductivity, corrosion resistance, acoustical insulation, etc.**

CLASSIFICATION OF COMPOSITE MATERIALS

Fibrous composites: Fibers in a matrix

Particulate composites: Particles in a matrix

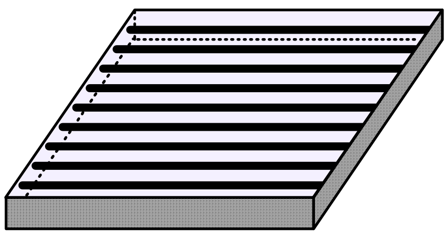
Combinations of above: Reinforced fiber-reinforced composites

Textile Composites: (a) Woven composites
(b) Braided composites

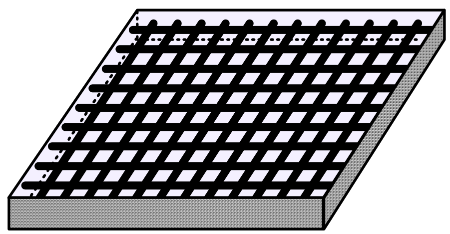
Laminated composites: Layers of various materials (nano-composites)

The principal difference between the braided and other fabric-processing methods is that woven fabrics are formed by orthogonal interlacing of yarns and knitted fabrics are formed by inter-looping yarns, whereas conventional braiding forms nonorthogonal, multidirectional (typically, two- or three-directional) fabrics without any loops.

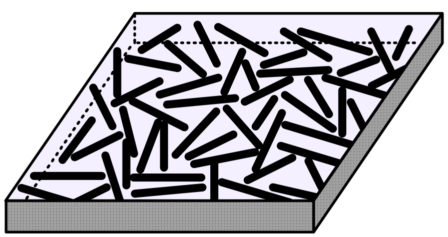
Composite Laminates



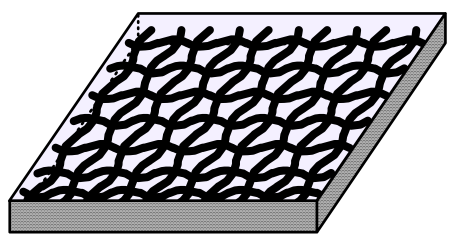
(a) Unidirectional



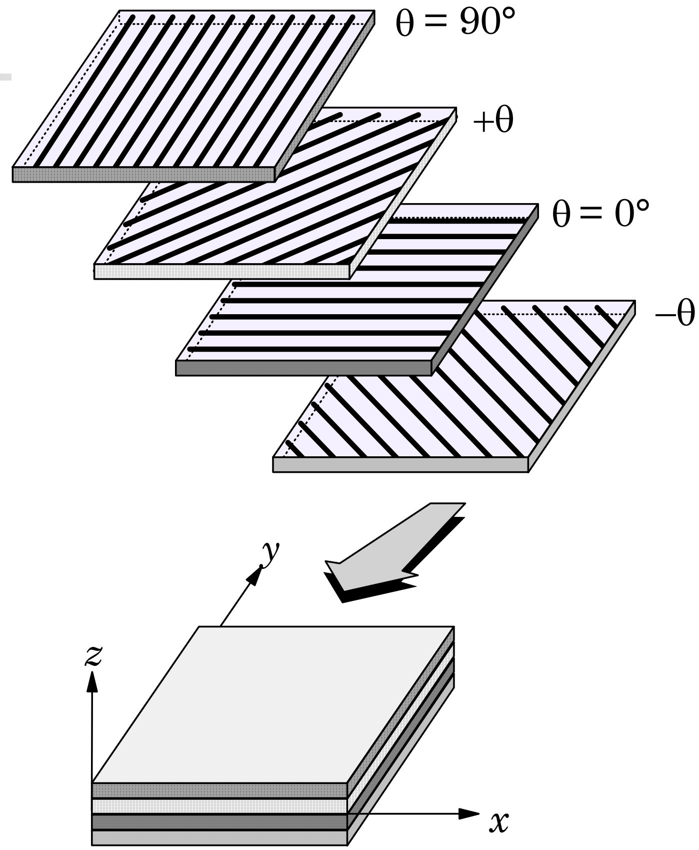
(b) Bi–directional

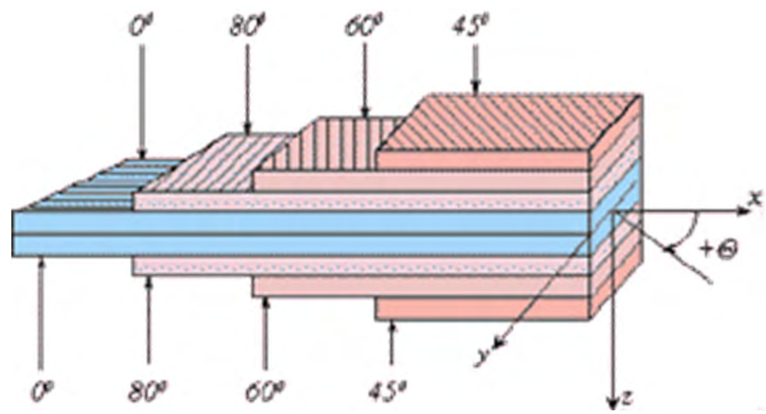


(c) Discontinuous fiber



(d) Woven





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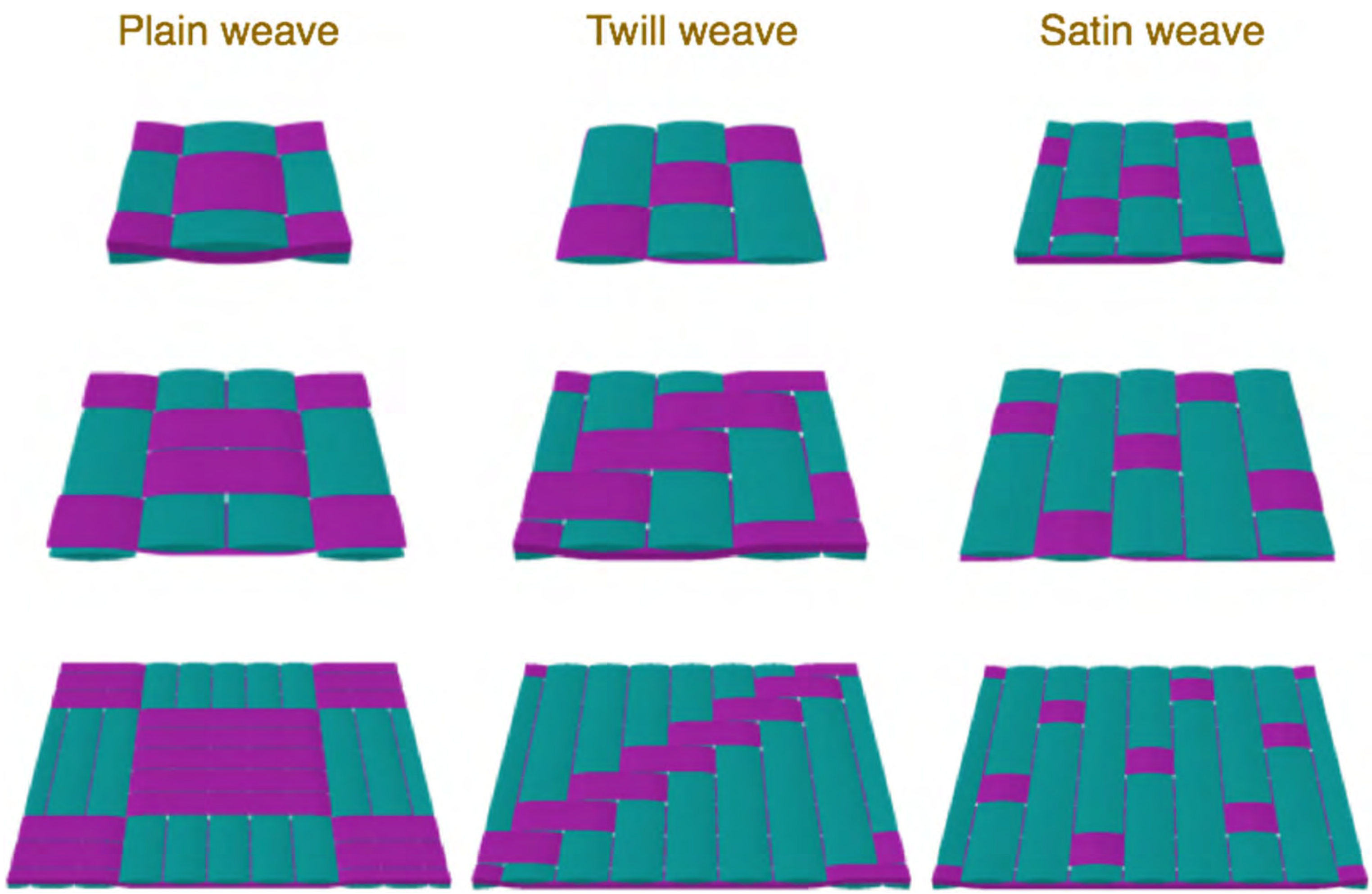
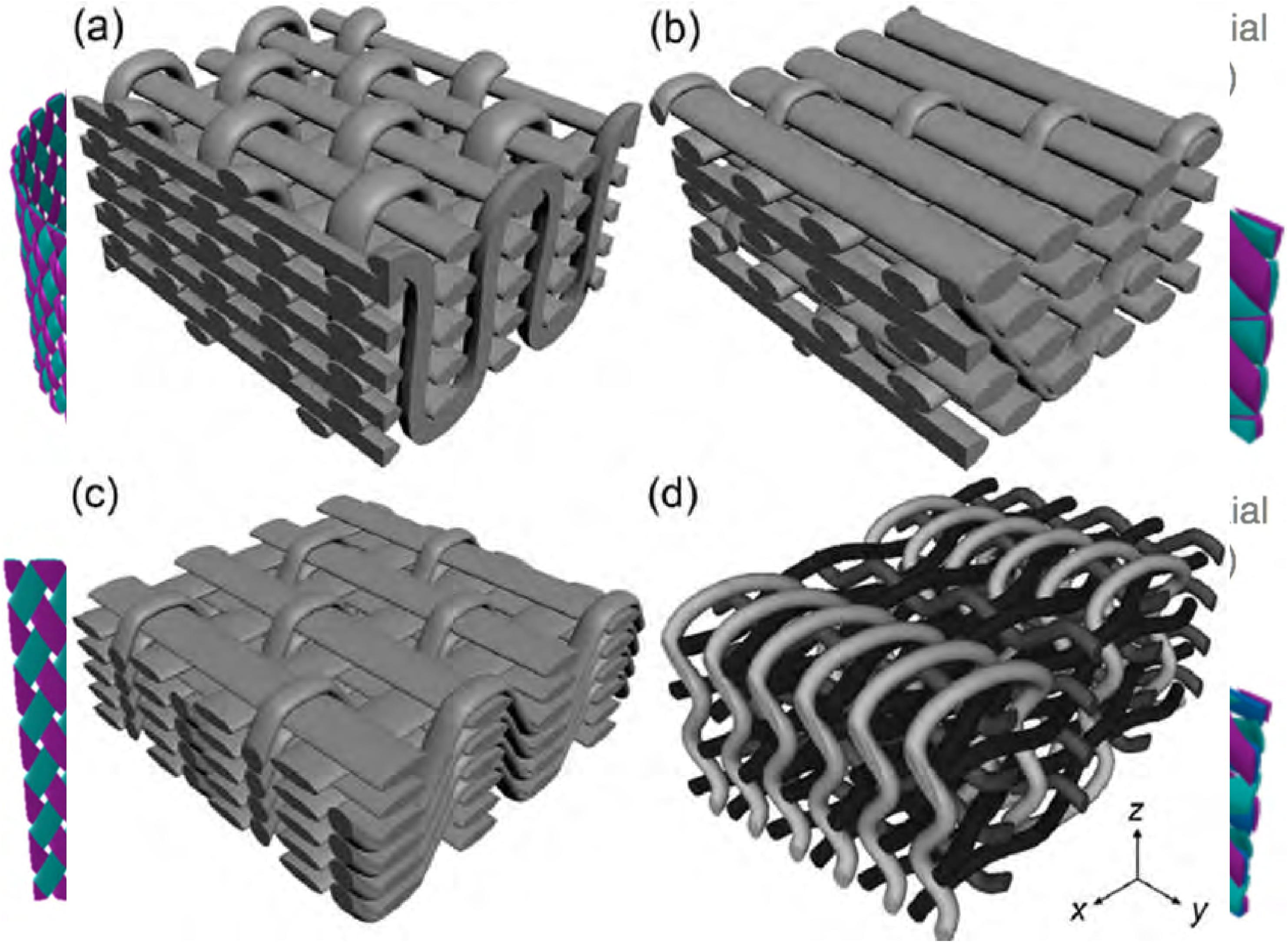
Woven Composites



 https://www.xcomposites.com/index.html

Braided Composites



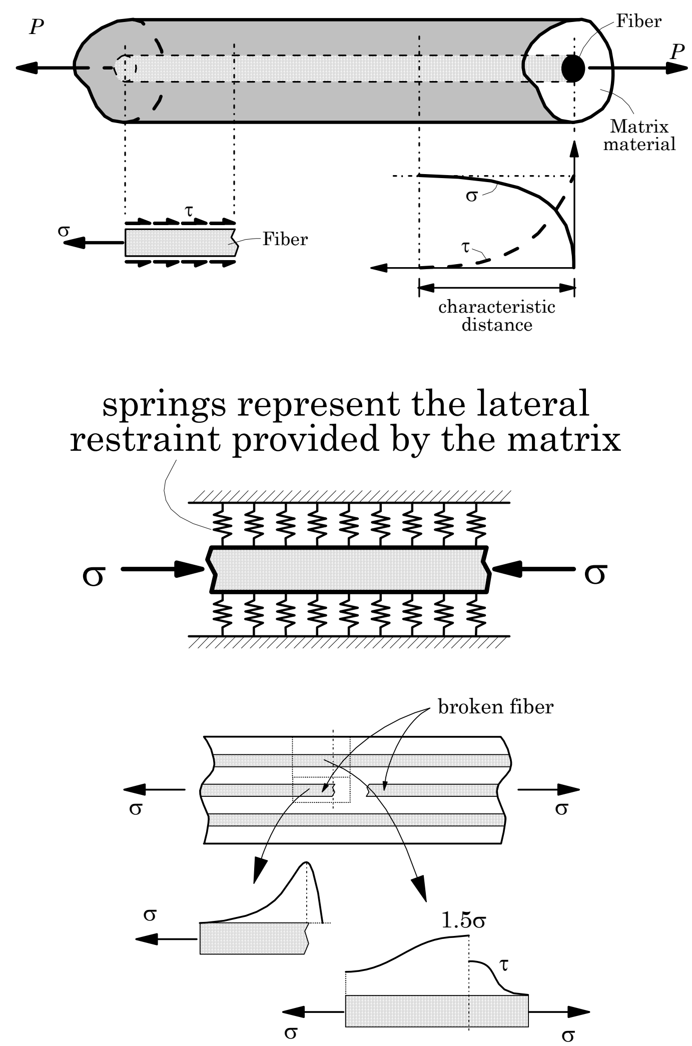
https://www.xcomposites.com/index.html

Fiber-reinforced Composite Materials: Constituents

Fiber: Load-carrying agent

Matrix: Supports and protects fibers, and transfers load between broken fibers

Lamina: Basic building block; flat or curved arrangement of unidirectional or woven fibers in a matrix



(c)

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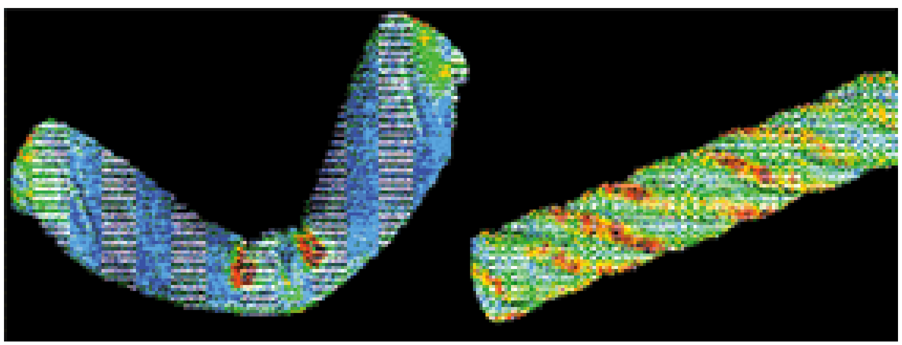
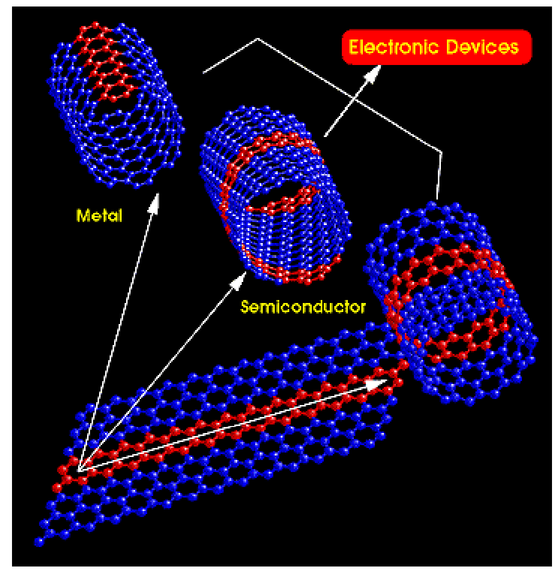
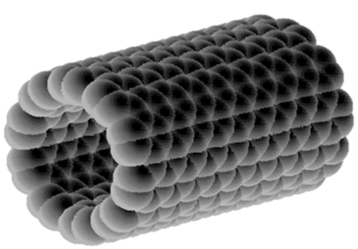
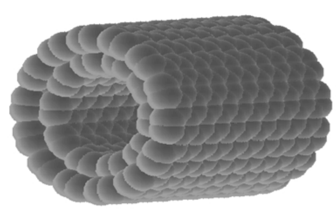
Nanocomposites: Carbon Nanotubes

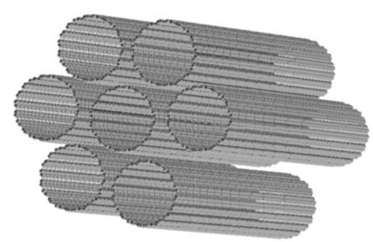
Carbon nanotubes can be viewed as a sheet of graphite that has been rolled in to a single tube. Carbon nanotubes can be single- or multi-walled.







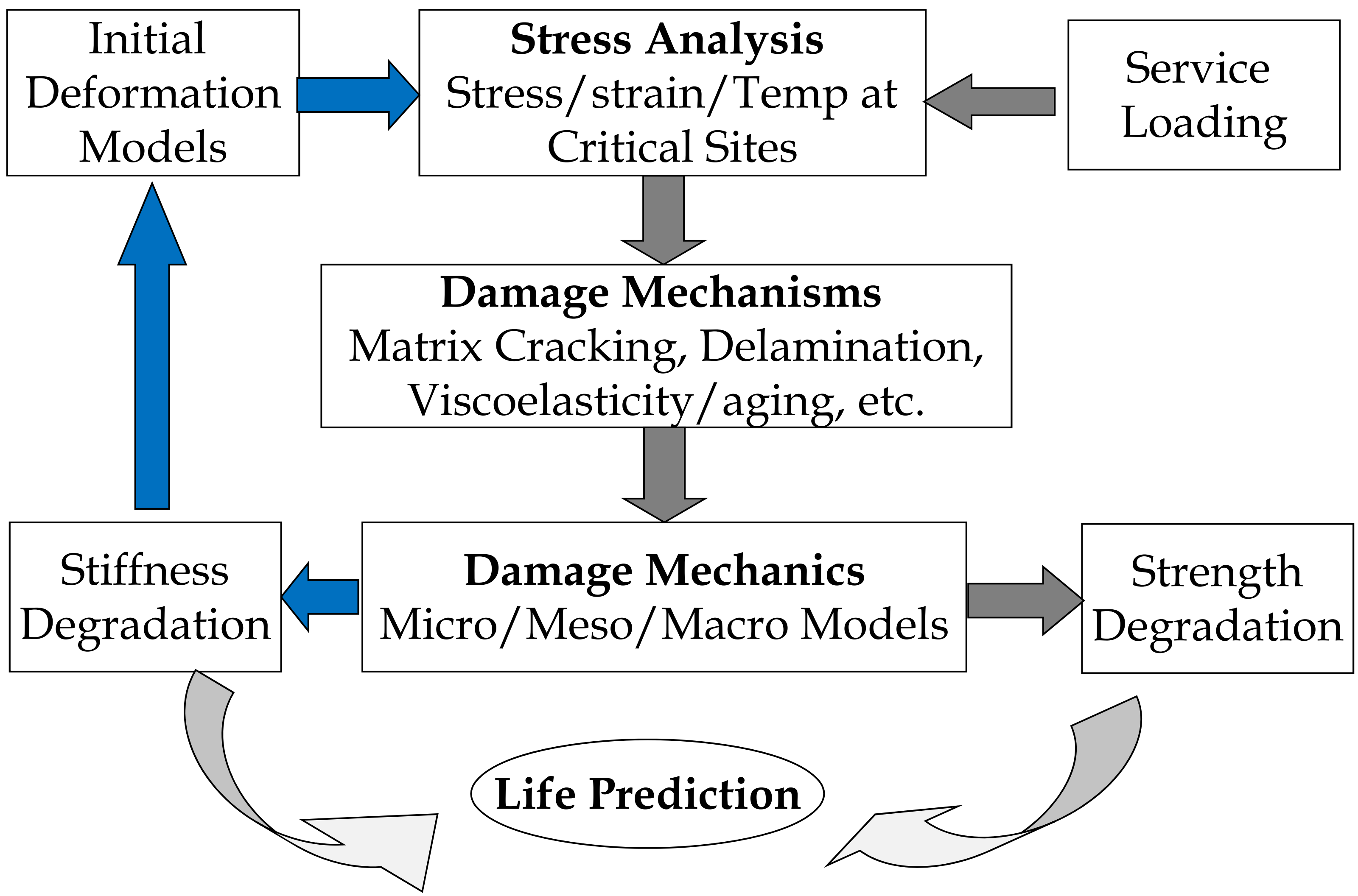




ADVANTAGES/DISADVANTAGES OF COMPOSITES

Advantages	*Disadvantages*
• Weight reduction • High strength or stiffness to weight ratio • Tailorable properties • Can tailor strength or stiffness in the load direction • Longer life (no corrosion) • Lower manufacturing costs because of less part count. • Inherent damping. • Increased (or decreased) thermal or electrical conductivity	• Cost of raw material and fabrication • Transverse properties may be weak. • Matrix is weak, low toughness • Reuse and disposal may be difficult • Difficult to attach. • Analysis is difficult. • Matrix subjected to environment degradation.

THE ROLE OF STRESS ANALYSIS



STUDY AREAS IN COMPOSITES

Characterization of material properties:

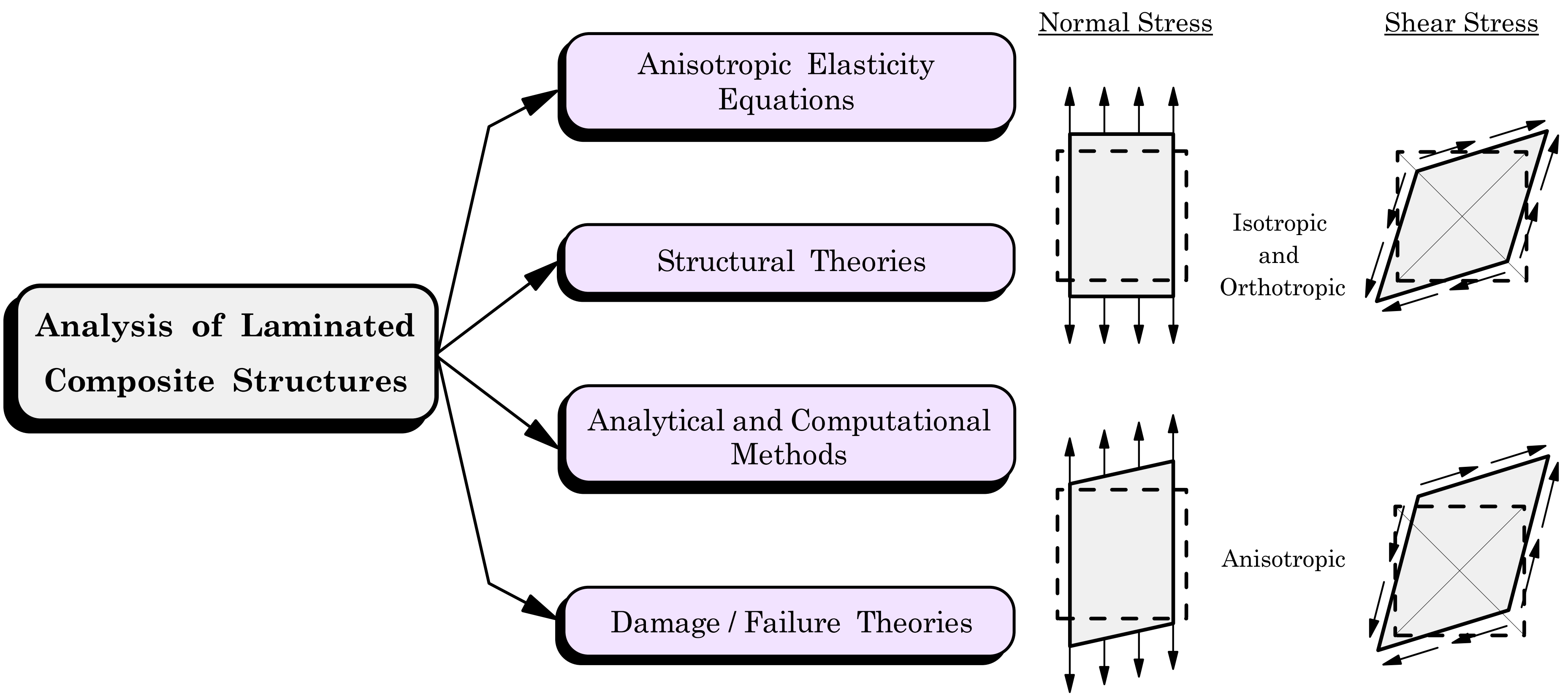
Static, dynamic, fatigue, fracture, temperature, moisture, electrical, etc.

Analyses:

Directional nature of response; different load-response; coupling (shear-extension, bend-twist, bending-extension); many layers versus one.

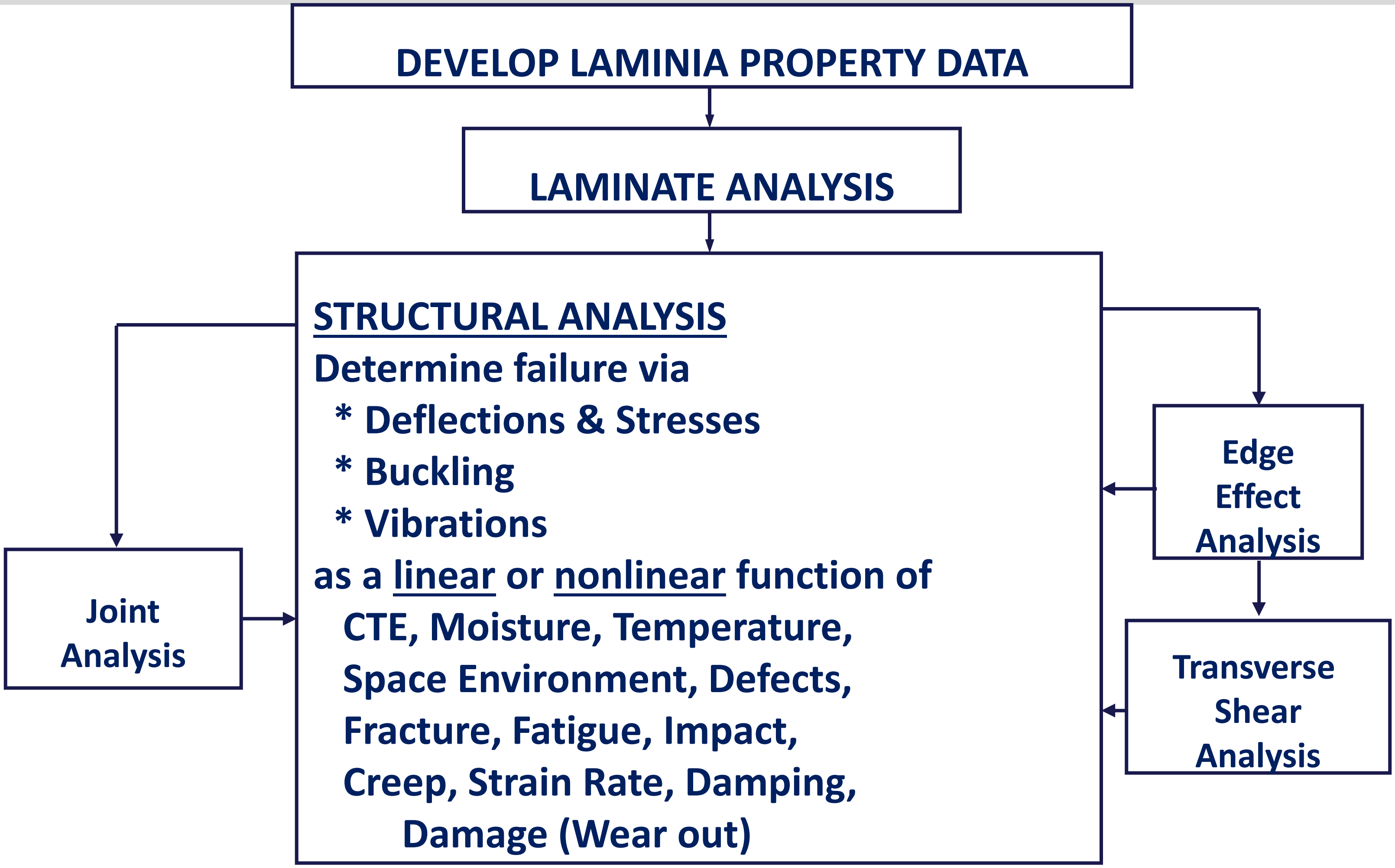
- Bonded and bolted joints
- Holes in laminates
- Fracture mechanics
- Optimization
- Interlaminar stresses
- Nonlinear material behavior

ANALYSIS OF COMPOSITE STRUCTURES: THE BIG PICTURE



Mechanics of Composites 12

ANALYSIS OF COMPOSITE STRUCTURES: Sequence of Steps Involved



ELEMENTS OF ANALYSIS

Determine pertinent structural response parameters:

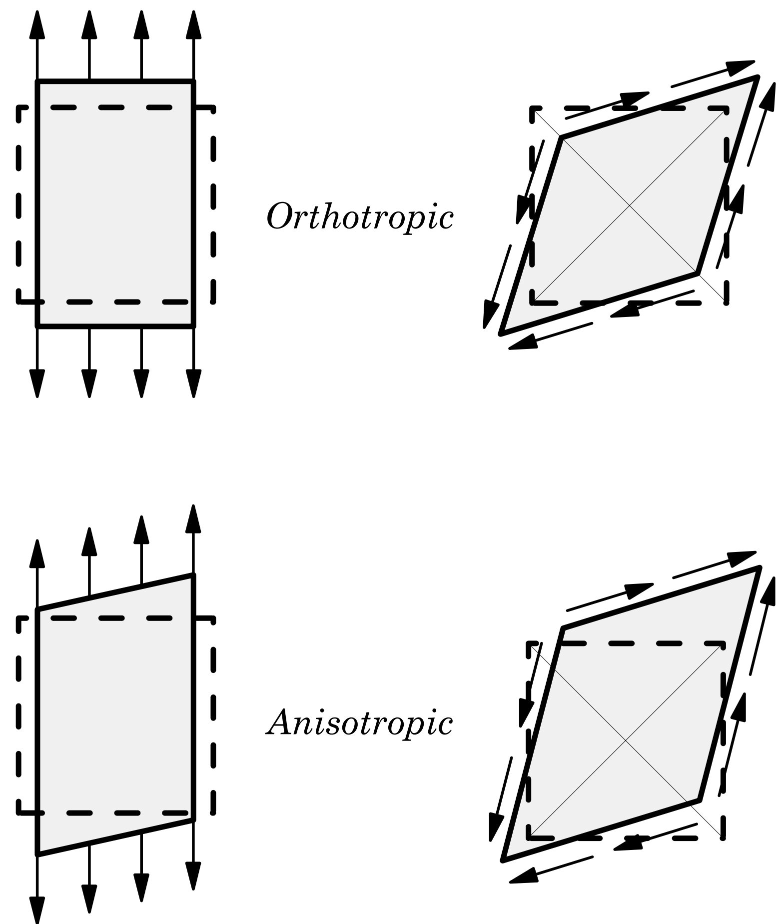
- Deflections
- Stresses
  - Full-field stresses
  - Stresses around cutouts
  - Stresses around defects
- Buckling loads
- Vibration frequencies

ANISOTROPIC ELASTICITY

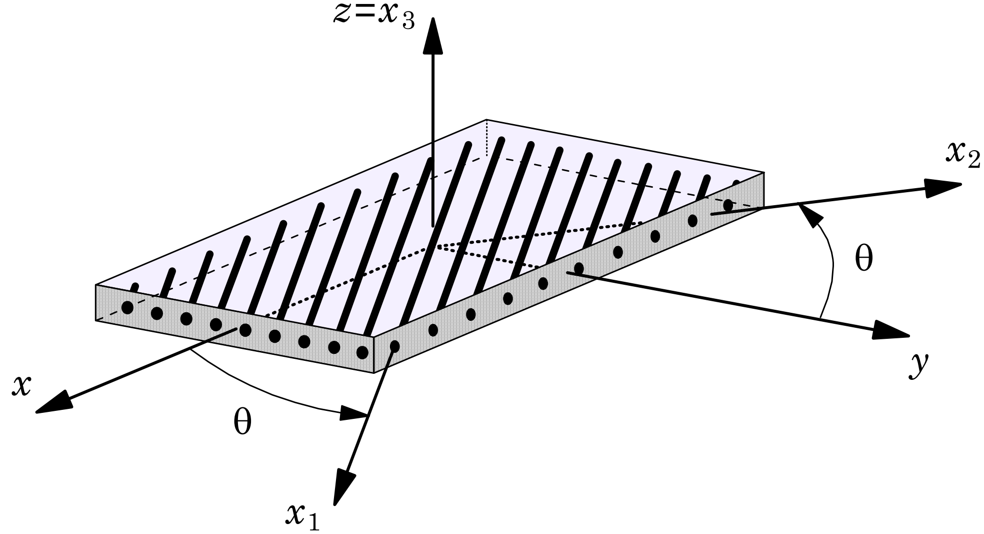
Mechanical Behavior of Isotropic, Orthotropic, and Anisotropic Materials

Normal Stress

Shear Stress



Transformation of stresses, strains, and material stiffnesses from lamina coordinates (x_1, x_2, x_3) to structural coordinates (x,y,z) is required.



Strain-Stress Relations:
Generalized Hooke's Law

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix}$$

$$\frac{\nu_{21}}{E_2} = \frac{\nu_{12}}{E_1}; \quad \frac{\nu_{31}}{E_3} = \frac{\nu_{13}}{E_1}; \quad \frac{\nu_{32}}{E_3} = \frac{\nu_{23}}{E_2}$$

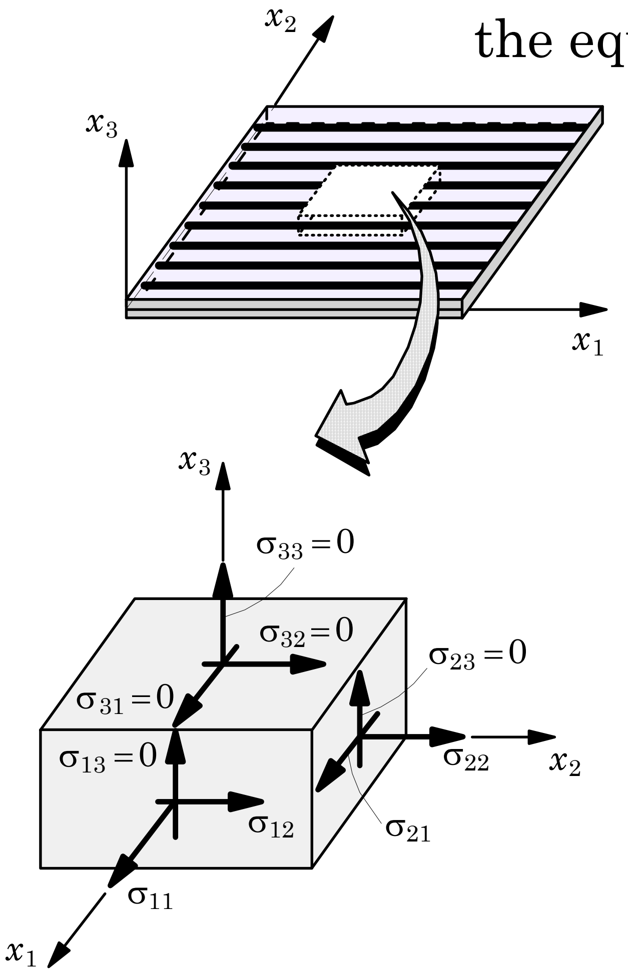
$$\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j} \quad (\text{no sum on } \ i, j)$$

There are nine independent material constants:

$$E_1, \ E_2, \ E_3, \ G_{23}, \ G_{13}, \ G_{12}, \ \nu_{12}, \ \nu_{13}, \ \nu_{23}$$

Plane-Stress Reduced State

Begin with the **strain-stress** relations and set to the transverse stress components to zero. Then invert the equations to obtain the stress-strain relations.



$$\sigma_3 = 0,\ \sigma_4 = 0,\ \sigma_5 = 0 \quad (\sigma_{33} = 0,\ \sigma_{32} = 0,\ \sigma_{31} = 0)$$

$$\varepsilon_3 = S_{13}\sigma_1 + S_{23}\sigma_2, \quad \varepsilon_4 = 0,\ \varepsilon_5 = 0$$

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix}$$

$$= \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix}$$

Constitutive Equations of the Plane-Stress State

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix}$$

$$Q_{11} = \frac{S_{22}}{S_{11}S_{22} - S_{12}^2} = \frac{E_1}{1 - \nu_{12}\nu_{21}}$$

$$Q_{12} = \frac{S_{12}}{S_{11}S_{22} - S_{12}^2} = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}}$$

$$Q_{22} = \frac{S_{11}}{S_{11}S_{22} - S_{12}^2} = \frac{E_2}{1 - \nu_{12}\nu_{21}}$$

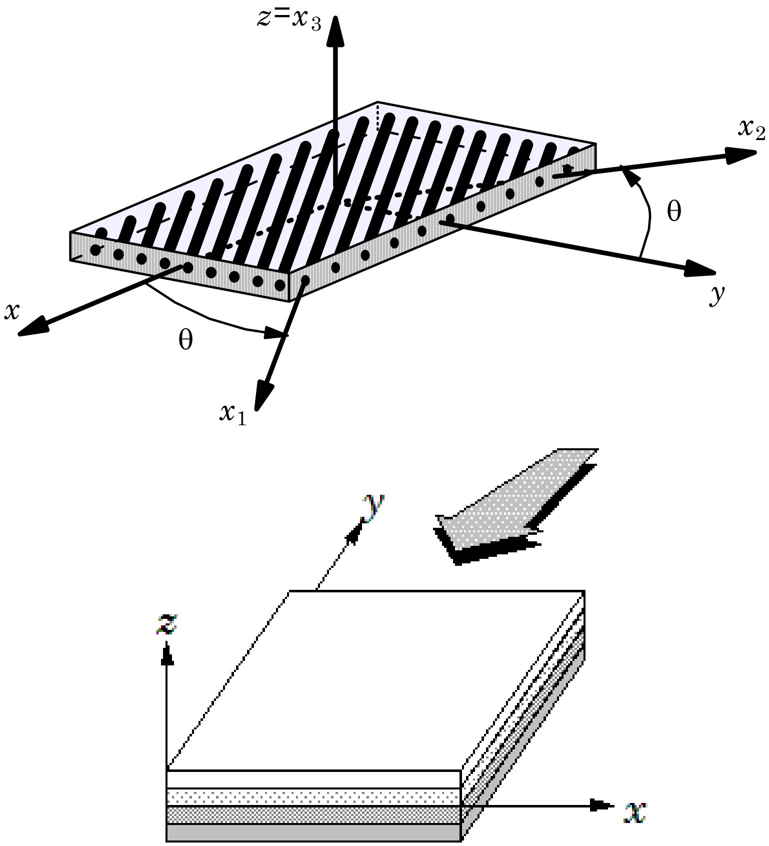
$$Q_{66} = \frac{1}{S_{66}} = G_{12}$$

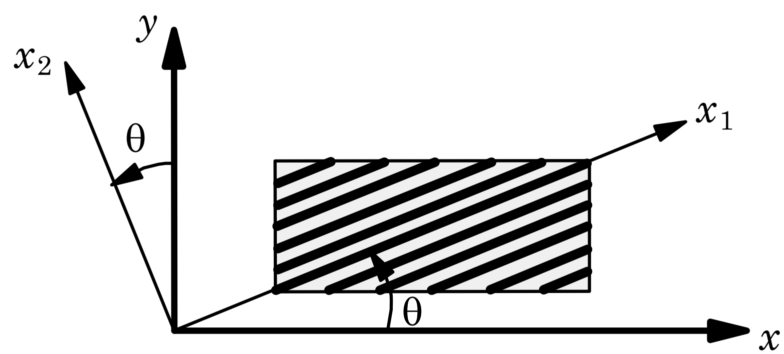
$$E_1, \quad E_2, \quad \nu_{12}, \quad G_{12}$$

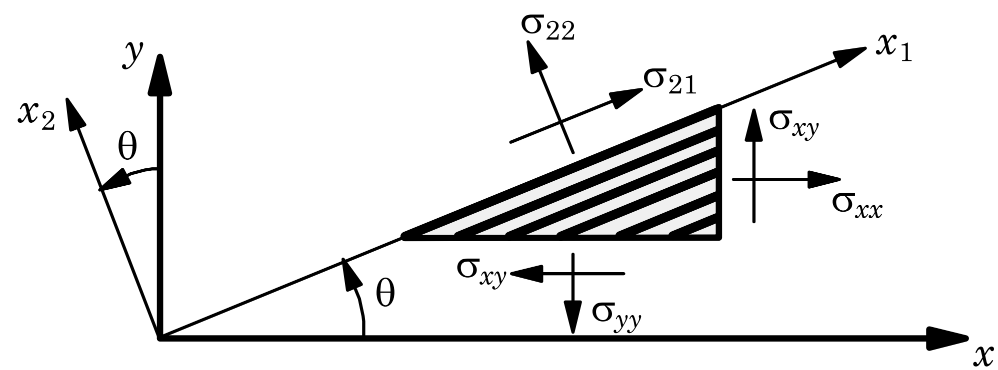
Strain-Stress Relations in Structural Coordinates

(x_1, x_2, x_3) = Material coordinates

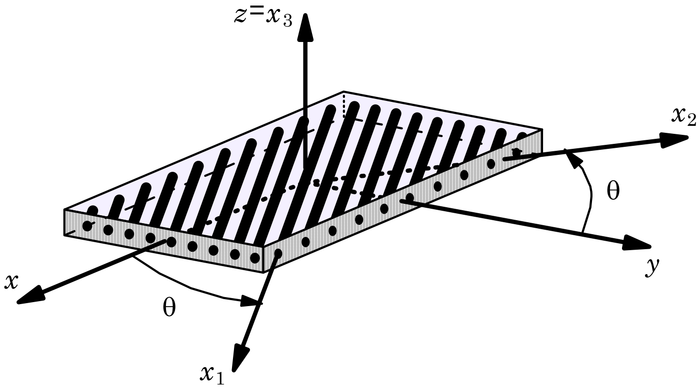
(x, y, z) = Structural coordinates







Strain-Stress Relations in Structural Coordinates

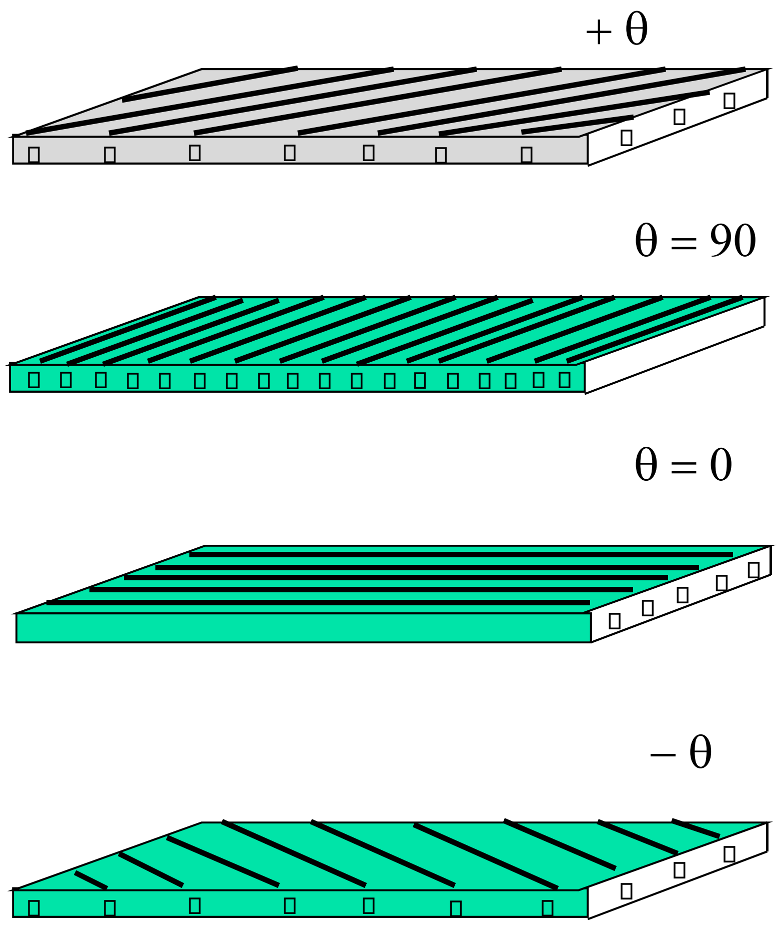


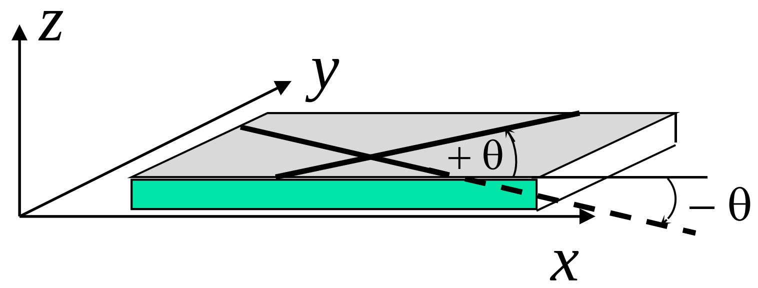
$$\left\{\begin{matrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{matrix}\right\} = \begin{bmatrix}\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16}\\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26}\\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}\end{bmatrix} \left\{\begin{matrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{matrix}\right\}$$

$$\left\{\begin{matrix}\sigma_{yz}\\ \sigma_{xz}\end{matrix}\right\} = \begin{bmatrix}\bar{Q}_{44} & \bar{Q}_{45}\\ \bar{Q}_{45} & \bar{Q}_{55}\end{bmatrix} \left\{\begin{matrix}\gamma_{yz}\\ \gamma_{xz}\end{matrix}\right\}$$

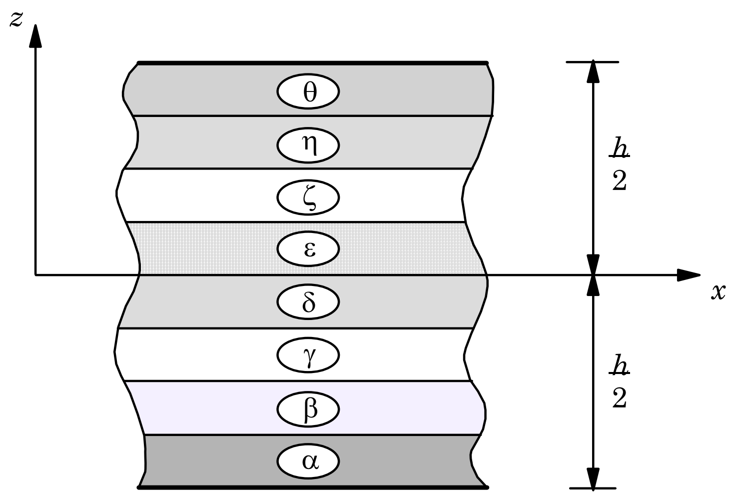
$$\bar{Q}_{11} = Q_{11}\cos^4\theta + 2(Q_{12} + 2Q_{66})\sin^2\theta\ \cos^2\theta + Q_{22}\sin^4\theta$$
$$\bar{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66})\sin^2\theta\ \cos^2\theta + Q_{12}(\sin^4\theta + \cos^4\theta)$$
$$\bar{Q}_{22} = Q_{11}\sin^4\theta + 2(Q_{12} + 2Q_{66})\sin^2\theta\ \cos^2\theta + Q_{22}\cos^4\theta$$
$$\bar{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66})\sin\theta\ \cos^3\theta + (Q_{12} - Q_{22} + 2Q_{66})\sin^3\theta\ \cos\theta$$
$$\bar{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66})\sin^3\theta\cos\theta + (Q_{12} - Q_{22} + 2Q_{66})\sin\theta\ \cos^3\theta$$
$$\bar{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})\sin^2\theta\ \cos^2\theta + Q_{66}(\sin^4\theta + \cos^4\theta)$$
$$\bar{Q}_{44} = Q_{44}\cos^2\theta + Q_{55}\sin^2\theta$$
$$\bar{Q}_{45} = (Q_{55} - Q_{44})\cos\theta\sin\theta$$
$$\bar{Q}_{55} = Q_{55}\cos^2\theta + Q_{44}\sin^2\theta$$

Lamination Scheme -Notation

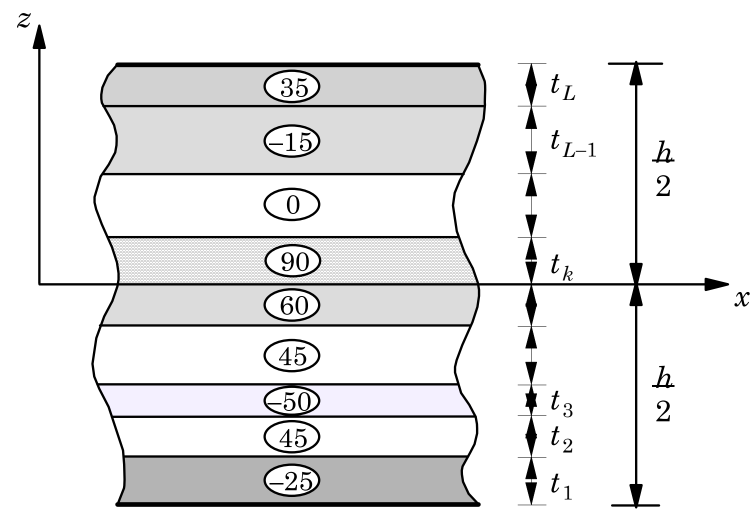




Laminate with *general* stacking sequence

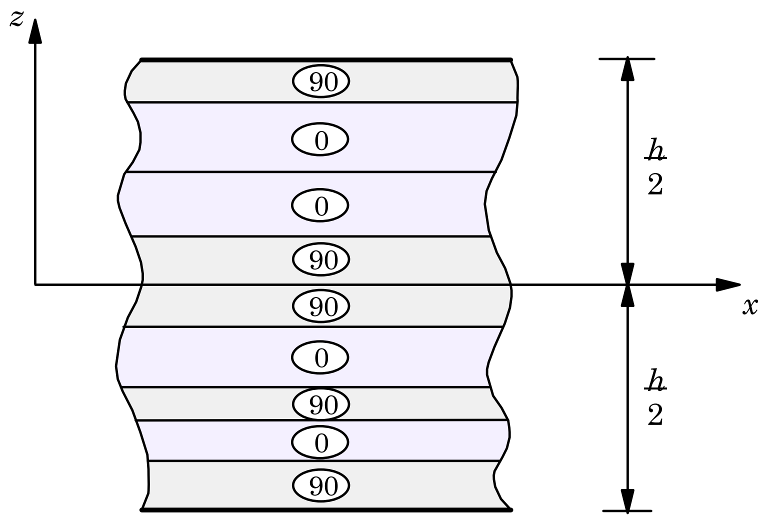


A general angle-ply laminate

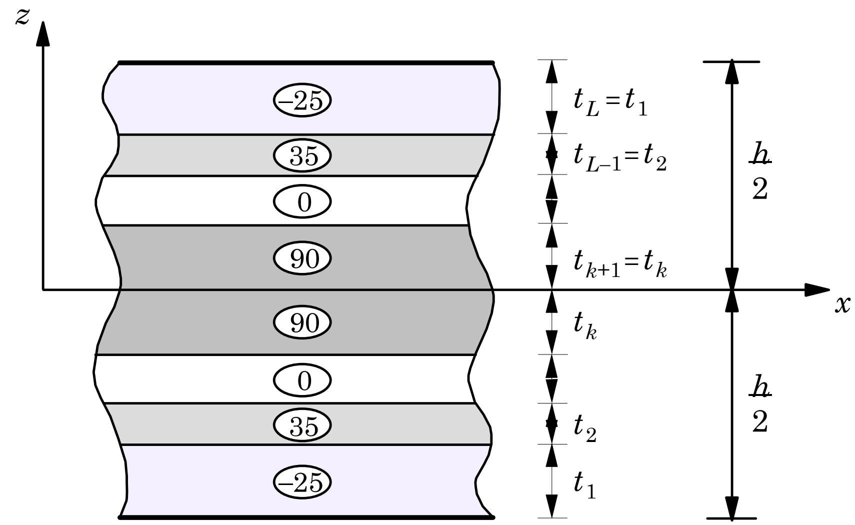


Different types of Laminates

A *cross-ply* laminate



A *symmetric* laminate



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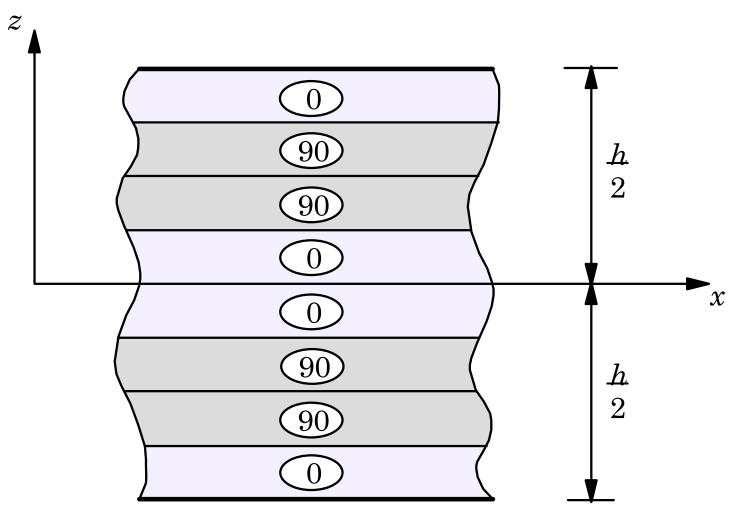
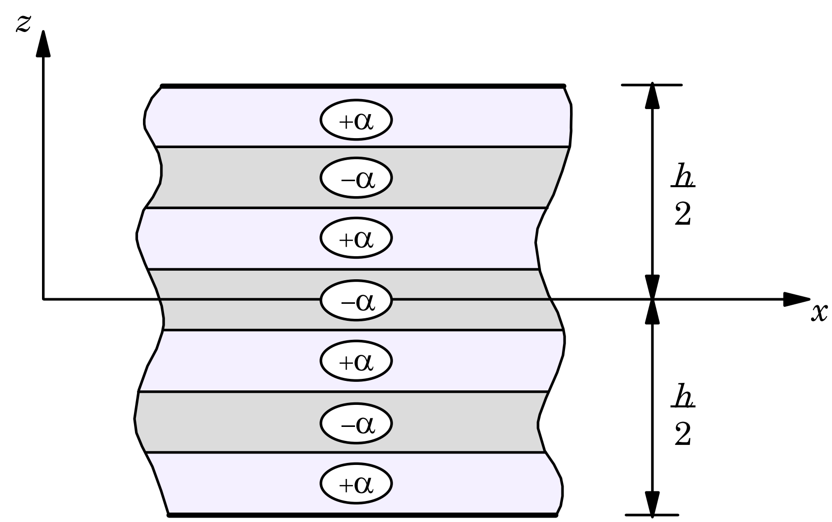
Different types of Laminates

A *symmetric cross-ply* laminate

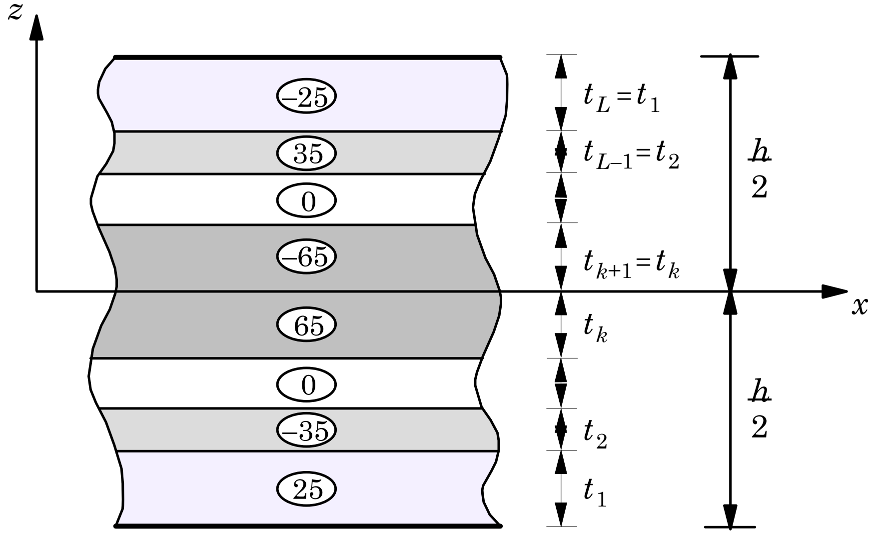


A *symmetric angle-ply* laminate

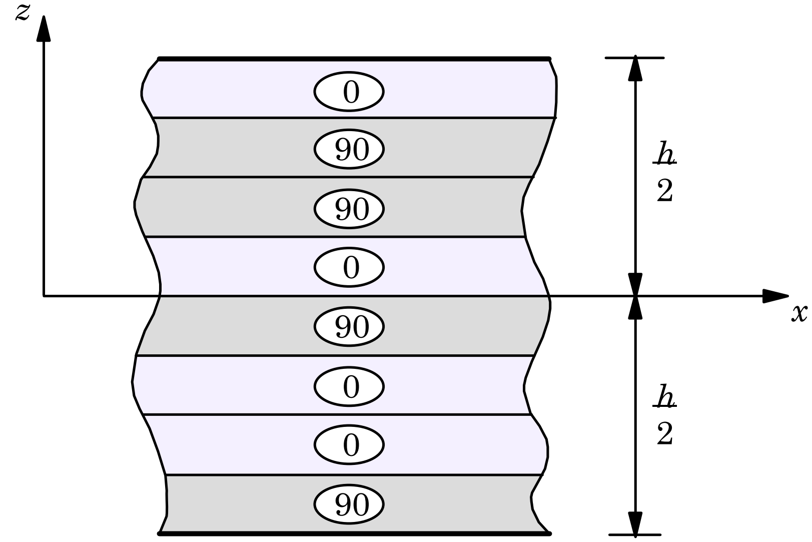


Different types of Laminates

An *antisymmetric* laminate

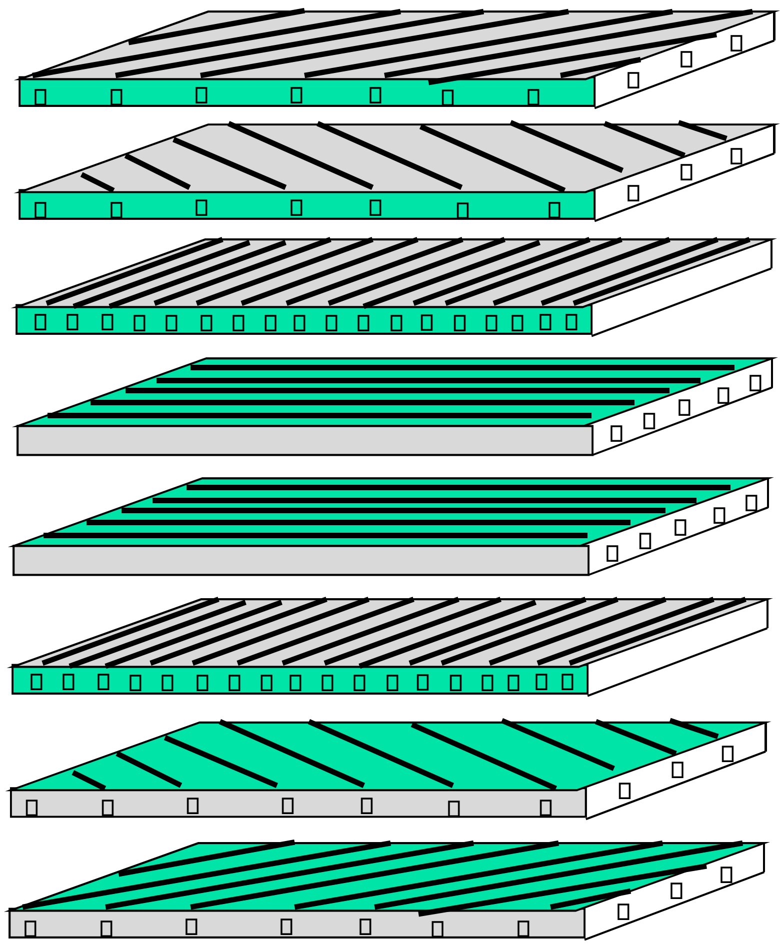


An *antisymmetric cross-ply* laminate



JN Reddy

Lamination Scheme - Notation



$\theta = +45$

$\theta = -45$

$\theta = 90$

$\theta = 0$

$\theta = 0$

$\theta = 90$

$\theta = -45$

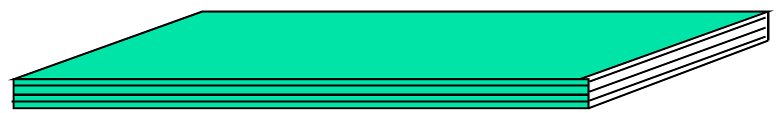
$\theta = +45$

$[\pm 45/90/0]_s$

Effect of the Lamination Scheme on Laminate Stiffness

Layer No.	Angle	Thickness	Mat. Type
1	0.	.5000E-02	1
2	90.	.5000E-02	1
3	90.	.5000E-02	1
4	0.	.5000E-02	1

$[0/90]_s$



Youngs modulus, E1 = 0.30000E+08
Youngs modulus, E2 = 0.30000E+07
Shear modulus, G12 = 0.15000E+07
Shear modulus, G13 = 0.15000E+07
Shear modulus, G23 = 0.15000E+07
Poissons ratio, NU12 = 0.25000E+00

Laminate stiffnesses, [A]:

0.33208E+06	0.15094E+05	0.45056E-12
0.15094E+05	0.33208E+06	0.16186E-10
0.45056E-12	0.16186E-10	0.30000E+05

Laminate stiffnesses, [B]:

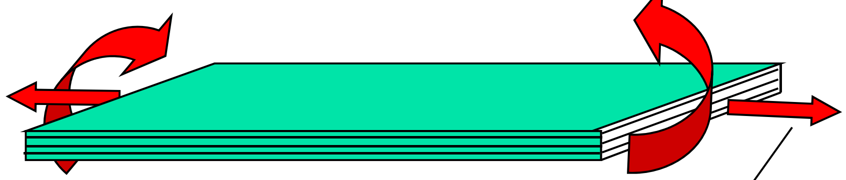
0.00000E+00	-0.35527E-14	0.00000E+00
-0.35527E-14	0.00000E+00	0.00000E+00
0.00000E+00	0.00000E+00	0.71054E-14

Laminate stiffnesses, [D]:

0.17862E+02	0.50314E+00	0.37547E-17
0.50314E+00	0.42767E+01	0.13488E-15
0.37547E-17	0.13488E-15	0.10000E+01

Effect of the Lamination Scheme on Laminate Stiffness

Effect of the Lamination Scheme



Specified inplane strain, EXX0 .... = 0.10000E-02
Specified inplane strain, EYY0 ....= 0.00000E+00
Specified inplane strain, EXY0 ....= 0.00000E+00
Specified bending strain, EXX1 .... = 0.10000E+00
Specified bending strain, EYY1 .... = 0.00000E+00
Specified bending strain, EXY1 .... = 0.00000E+00

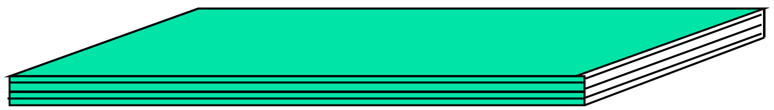
Inplane force, Nxx .... = 0.33208E+03
Inplane force, Nyy .... = 0.15094E+02
Inplane force, Nxy .... = 0.45056E-15
Bending moment, Mxx .... = 0.17862E+01
Bending moment, Myy .... = 0.50314E-01
Bending moment, Mxy ....= 0.37547E-18

JN Reddy

Effect of the Lamination Scheme on Laminate Stiffness

Layer No.	Angle	Thickness	Mat. Type
1	90.	.5000E-02	1
2	0.	.5000E-02	1
3	0.	.5000E-02	1
4	90.	.5000E-02	1

$[90/0]_s$



Youngs modulus, E1 = 0.30000E+08
Youngs modulus, E2 = 0.30000E+07
Shear modulus, G12 = 0.15000E+07
Shear modulus, G13 = 0.15000E+07
Shear modulus, G23 = 0.15000E+07
Poissons ratio, NU12 = 0.25000E+00

Laminate stiffnesses, [A]:

0.33208E+06	0.15094E+05	0.45056E-12
0.15094E+05	0.33208E+06	0.16186E-10
0.45056E-12	0.16186E-10	0.30000E+05

Laminate stiffnesses, [B]:

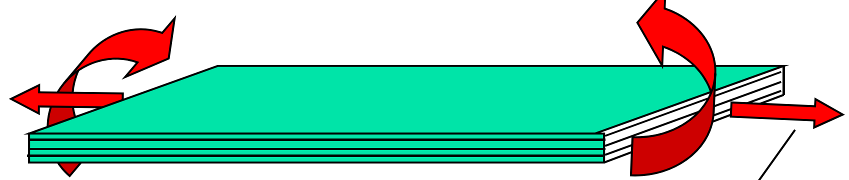
0.00000E+00	-0.35527E-14	0.00000E+00
-0.35527E-14	0.00000E+00	0.00000E+00
0.00000E+00	0.00000E+00	0.71054E-14

Laminate stiffnesses, [D]:

0.42767E+01	0.50314E+00	0.26283E-16
0.50314E+00	0.17862E+02	0.94416E-15
0.26283E-16	0.94416E-15	0.10000E+01

Effect of the Lamination Scheme on Laminate Stiffness

$[90/0]_s$



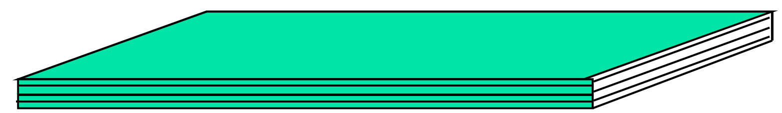
Specified inplane strain, EXX0 .... = 0.10000E-02
Specified inplane strain, EYY0 ....= 0.00000E+00
Specified inplane strain, EXY0 ....= 0.00000E+00
Specified bending strain, EXX1 .... = 0.10000E+00
Specified bending strain, EYY1 .... = 0.00000E+00
Specified bending strain, EXY1 .... = 0.00000E+00

Inplane force, Nxx .... = 0.33208E+03
Inplane force, Nyy .... = 0.15094E+02
Inplane force, Nxy .... = 0.45056E-15
Bending moment, Mxx .... = 0.42767E+00
Bending moment, Myy ....= 0.50314E-01
Bending moment, Mxy ....= 0.26283E-17

Effect of the Lamination Scheme on Laminate Stiffness

Layer No.	Angle	Thickness	Mat. Type
1	0.	.5000E-02	1
2	90.	.5000E-02	1
3	0.	.5000E-02	1
4	90.	.5000E-02	1

$[0/90]_{as} = [0/90/0/90]$



Youngs modulus, E1 = 0.30000E+08
Youngs modulus, E2 = 0.30000E+07
Shear modulus, G12 = 0.15000E+07
Shear modulus, G13 = 0.15000E+07
Shear modulus, G23 = 0.15000E+07
Poissons ratio, NU12 = 0.25000E+00

Laminate stiffnesses, [A]:

0.33208E+06	0.15094E+05	0.45056E-12
0.15094E+05	0.33208E+06	0.16186E-10
0.45056E-12	0.16186E-10	0.30000E+05

Laminate stiffnesses, [B]:

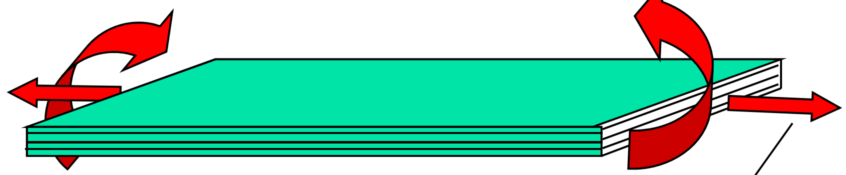
-0.67925E+03	-0.35527E-14	0.11264E-14
-0.35527E-14	0.67925E+03	0.40464E-13
0.11264E-14	0.40464E-13	0.71054E-14

Laminate stiffnesses, [D]:

0.11069E+02	0.50314E+00	0.15019E-16
0.50314E+00	0.11069E+02	0.53952E-15
0.15019E-16	0.53952E-15	0.10000E+01

Effect of the Lamination Scheme on Laminate Stiffness

(0/90/0/90)



Specified inplane strain, EXX0 .... = 0.10000E-02
Specified inplane strain, EYY0 ....= 0.00000E+00
Specified inplane strain, EXY0 ....= 0.00000E+00
Specified bending strain, EXX1 .... = 0.10000E+00
Specified bending strain, EYY1 .... = 0.00000E+00
Specified bending strain, EXY1 .... = 0.00000E+00

<u>EXX1 = 0.10000E+00</u>

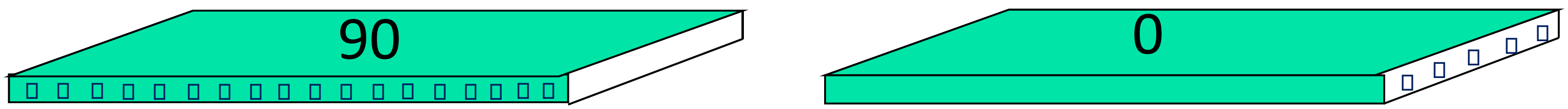
Inplane force, Nxx ...... = 0.26415E+03
Inplane force, Nyy ...... = 0.15094E+02
Inplane force, Nxy ...... = 0.56320E-15
Bending moment, Mxx = 0.42767E+00
Bending moment, Myy = 0.50314E-01
Bending moment, Mxy = 0.26283E-17

<u>EXX1 = 0.00000E+00</u>

Inplane force, Nxx ..... = 0.33208E+03
Inplane force, Nyy ...... = 0.15094E+02
Inplane force, Nxy ...... = 0.45056E-15
Bending moment, Mxx = -0.67925E+00
Bending moment, Myy = -0.35527E-17
Bending moment, Mxy = 0.11264E-17

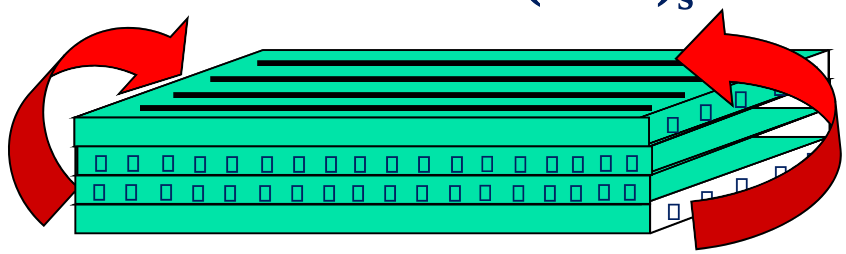
EFFECT OF THE LAMINATION SCHEME

on Extensional and Bending Deformations



No change in extensional response

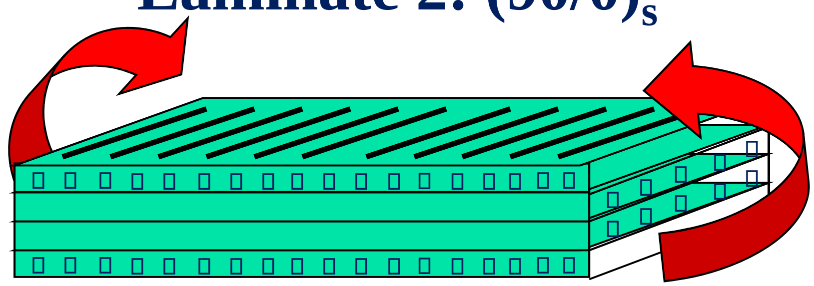
Laminate 1: $(0/90)_s$



$(A_{ij})_1 = (A_{ij})_2$

Stiffer in bending

Laminate 2: $(90/0)_s$



$(D_{ij})_1 > (D_{ij})_2$

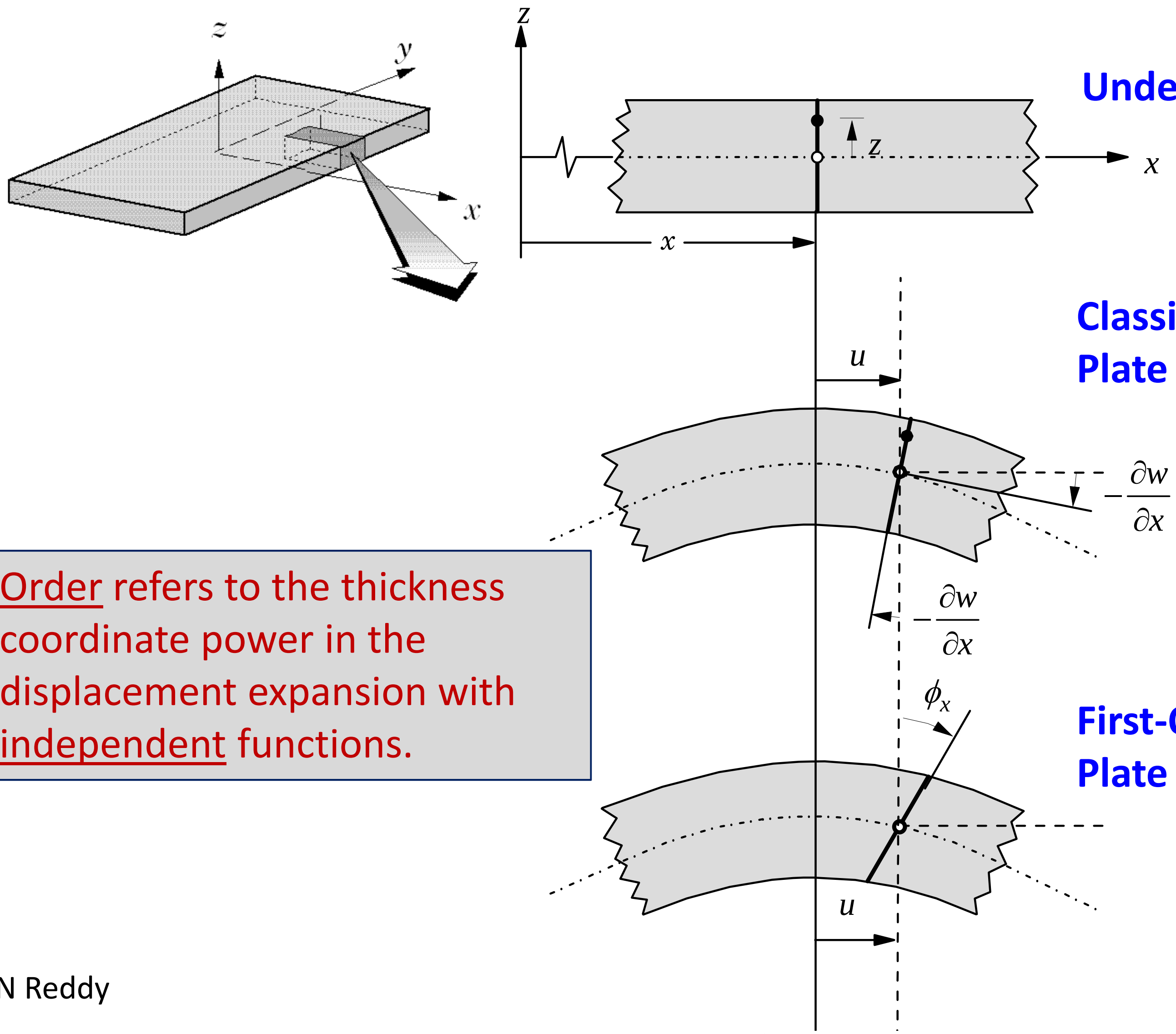
Softer in bending

STRUCTURAL THEORIES OF PLATES (Classical and First-Order Plate Theories)

CONTENTS OF THE LECTURE

- Kinematics of deformation of the CPT and FSDT
- Classical Plate Theory
  - Displacement field
  - Strains
  - Equations of motion
- First-Order Shear Deformation Theory
  - Displacement field
  - Strains
  - Equations of motion

Kinematics of Deformation



Undeformed Edge

Classical (Kirchhoff) Plate Theory (CPT)

First-Order (Mindlin) Plate Theory (FST)

Order refers to the thickness coordinate power in the displacement expansion with independent functions.

DISPLACEMENT FIELDS OF VARIOUS THEORIES

Classical Laminate Plate Theory (CLPT)

$$u_1(x,y,z,t) = u(x,y,t) + z\theta_x(x,y,t) \quad , \quad \theta_x = -(\partial w / \partial x)$$
$$u_2(x,y,z,t) = v(x,y,t) + z\theta_y(x,y,t) \quad , \quad \theta_y = -(\partial w / \partial y)$$
$$u_3(x,y,z,t) = w(x,y,t)$$

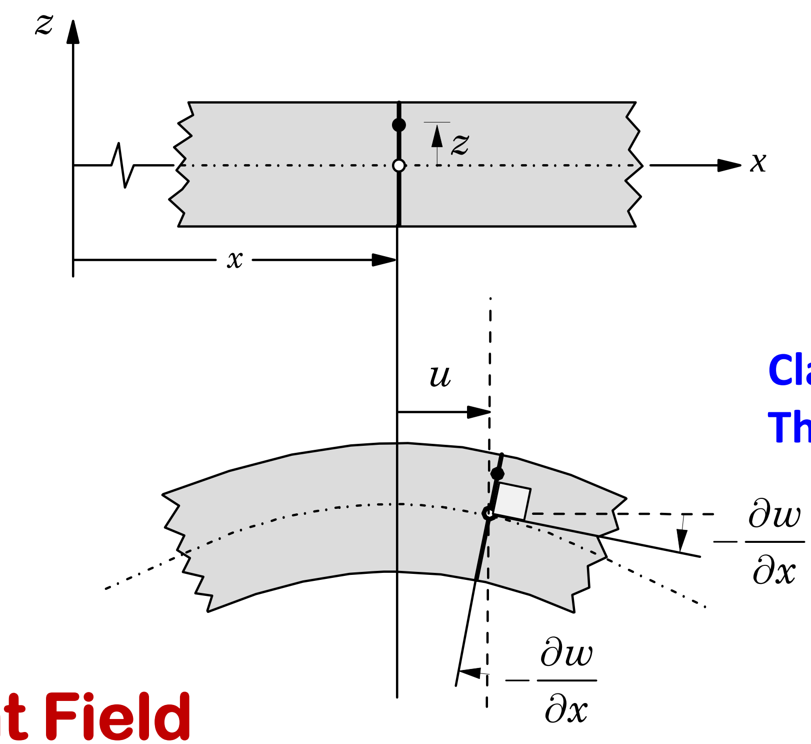
First-order Shear Deformation Theory (FSDT)

$$u_1(x,y,z,t) = u(x,y,t) + z\phi_x(x,y,t)$$
$$u_2(x,y,z,t) = v(x,y,t) + z\phi_y(x,y,t)$$
$$u_3(x,y,z,t) = w(x,y,t)$$

Higher-order Shear Deformation Theories (HSDT)

$$u_1(x,y,z,t) = \sum_{i=0}^{M} z^i\, \phi_1^{(i)}(x,y,t)$$
$$u_2(x,y,z,t) = \sum_{i=0}^{M} z^i\, \phi_2^{(i)}(x,y,t)$$
$$u_3(x,y,z,t) = \sum_{i=0}^{N} z^i\, \phi_3^{(i)}(x,y,t)$$

CLASSICAL LAMINATE PLATE THEORY



Displacement Field

$$u_1(x,y,z,t) = u(x,y,t) + z\theta_x(x,y,t) \quad , \quad \theta_x = -\frac{\partial w}{\partial x}$$

$$u_2(x,y,z,t) = v(x,y,t) + z\theta_y(x,y,t) \quad , \quad \theta_y = -\frac{\partial w}{\partial y}$$

$$u_3(x,y,z,t) = w(x,y,t)$$

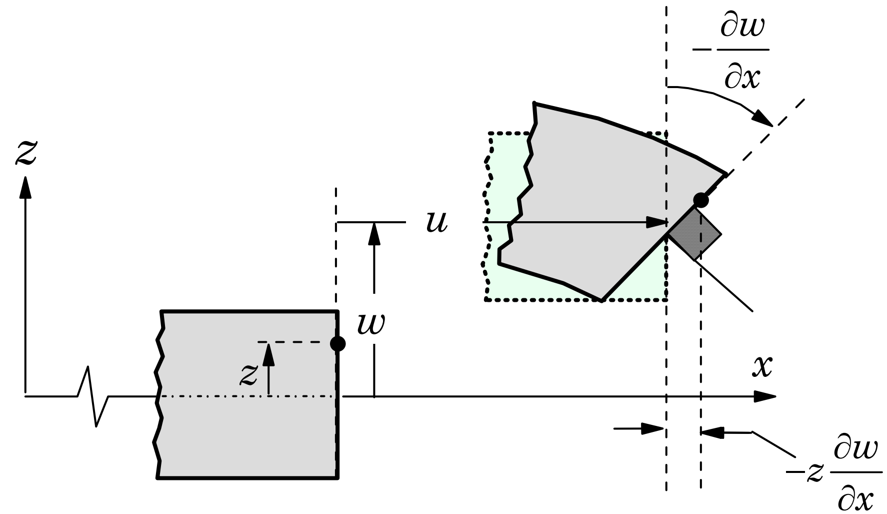
Kinematics of Deformation
in the Classical Plate Theory (CPT)

Displacement field

$$u_1(x,y,z,t) = u(x,y,t) - z\frac{\partial w}{\partial x}$$

$$u_2(x,y,z,t) = v(x,y,t) - z\frac{\partial w}{\partial y}$$

$$u_3(x,y,z,t) = w(x,y,t)$$

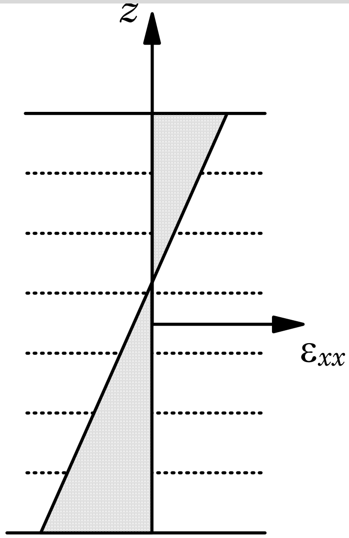


Nonzero linear strains

$$\varepsilon_{xx} = \frac{\partial u_1}{\partial x} = \frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2}, \quad \varepsilon_{yy} = \frac{\partial u_2}{\partial y} = \frac{\partial v}{\partial y} - z\frac{\partial^2 w}{\partial y^2}$$

$$\gamma_{xy} = \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2z\frac{\partial^2 w}{\partial x \partial y}$$

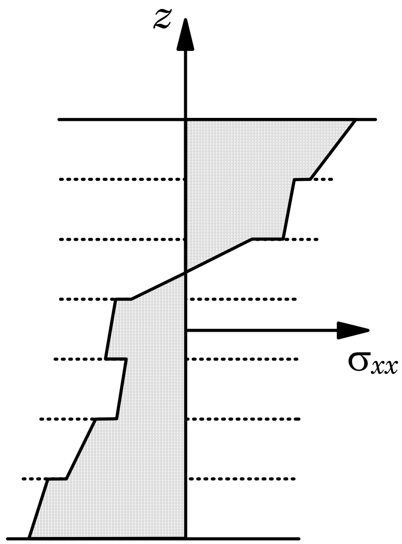
STRAIN AND STRESS DISTRIBUTIONS through the Thickness of the Laminate



Strain distribution is *continuous* through laminate thickness

$$\varepsilon_{xx} = \frac{\partial u_1}{\partial x} = \frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2}, \quad \varepsilon_{yy} = \frac{\partial u_2}{\partial y} = \frac{\partial v}{\partial y} - z\frac{\partial^2 w}{\partial y^2}$$

$$\gamma_{xy} = \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2z\frac{\partial^2 w}{\partial x \partial y}$$



Stress distribution is *discontinuous* through laminate thickness

$$\sigma_{xx} = Q_{11}\,\varepsilon_{xx} + Q_{12}\,\varepsilon_{yy}$$

CLASSICAL LAMINATE PLATE THEORY

Nonlinear Strains accounting for moderate rotations

Nonlinear strains

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j}\right)$$

$$\varepsilon_{xx} = \frac{\partial u_1}{\partial x_1} + \frac{1}{2}\frac{\partial u_m}{\partial x_1}\frac{\partial u_m}{\partial x_1} = \frac{\partial u_1}{\partial x_1} + \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1}\right)^2 + \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1}\right)^2$$

Strain Field

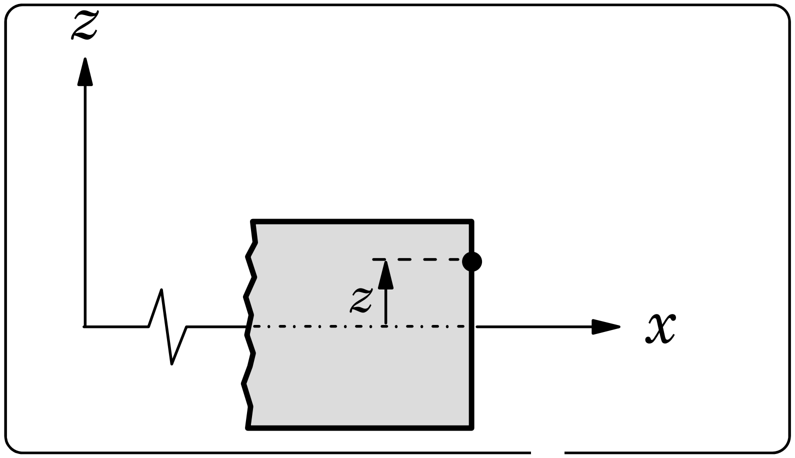
$$\varepsilon_{xx} = \frac{\partial u_1}{\partial x} + \tfrac{1}{2}\left(\frac{\partial u_3}{\partial x}\right)^2 = \frac{\partial u}{\partial x} + \tfrac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + z\frac{\partial \theta_x}{\partial x}$$

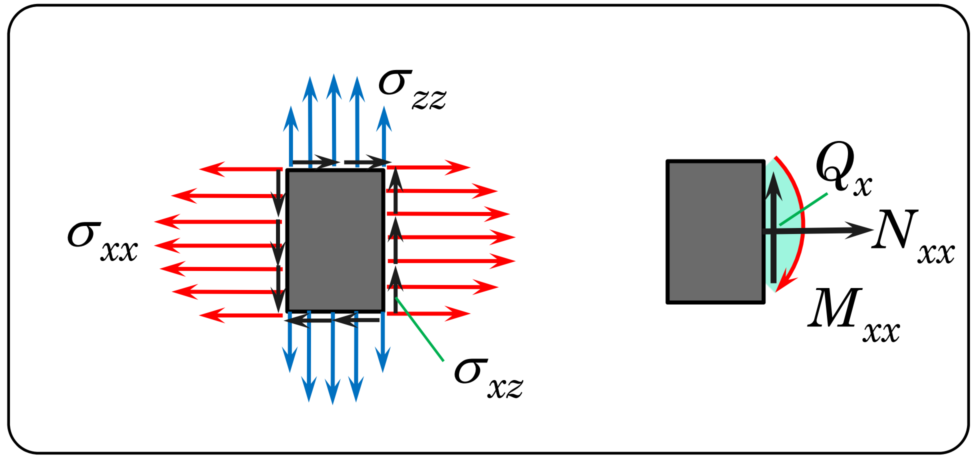
$$\varepsilon_{yy} = \frac{\partial u_2}{\partial y} + \tfrac{1}{2}\left(\frac{\partial u_3}{\partial y}\right)^2 = \frac{\partial v}{\partial y} + \tfrac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + z\frac{\partial \theta_y}{\partial y}$$

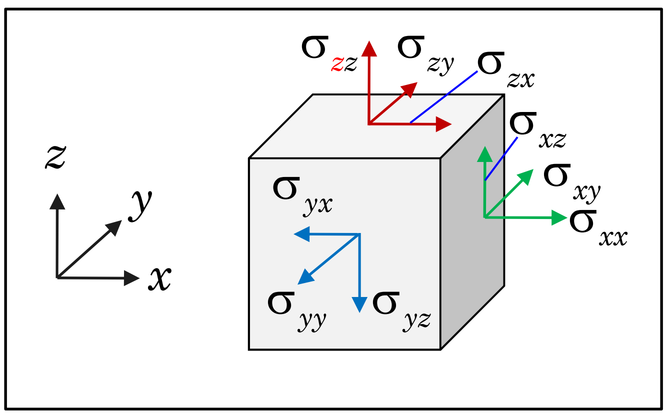
$$\gamma_{xy} = \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} + \frac{\partial u_3}{\partial x}\frac{\partial u_3}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} + z\left(\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x}\right)$$

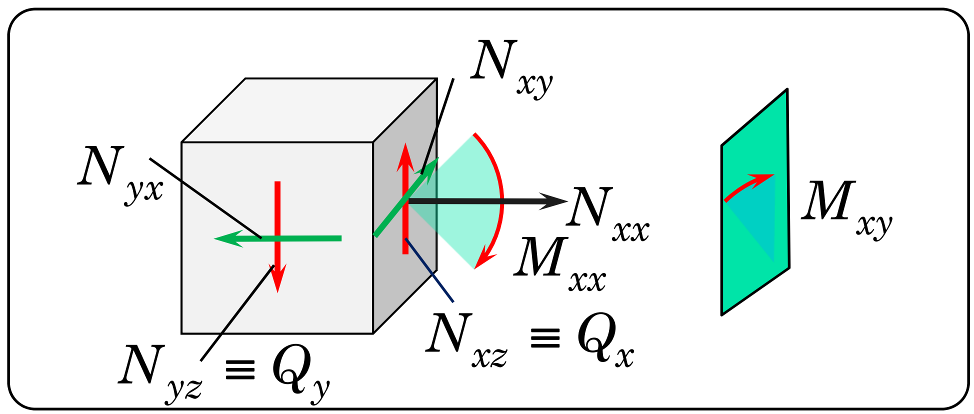
$$\gamma_{xz} = 0\ , \quad \gamma_{yz} = 0, \quad \theta_x = -\frac{\partial w}{\partial x}\ , \quad \theta_y = -\frac{\partial w}{\partial y}$$

SIGN CONVENTION FOR STRESS RESULTANTS







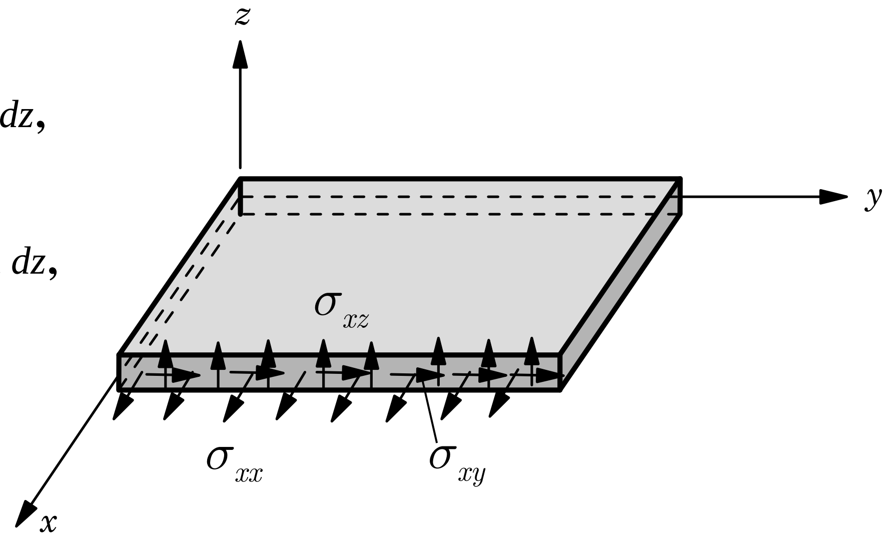


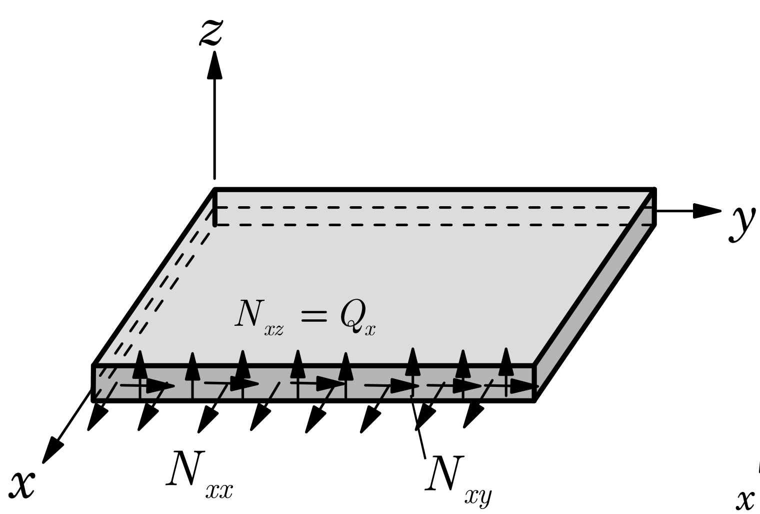
Stresses and Stress Resultants on an edge of a Plate

$$N_{xx} = \int_{-h/2}^{h/2} \sigma_{xx}\, dz, \quad N_{xy} = \int_{-h/2}^{h/2} \sigma_{xy}\, dz, \quad N_{yy} = \int_{-h/2}^{h/2} \sigma_{yy}\, dz,$$

$$M_{xx} = \int_{-h/2}^{h/2} z\sigma_{xx}\, dz, \; M_{xy} = \int_{-h/2}^{h/2} z\sigma_{xy}\, dz, \; M_{yy} = \int_{-h/2}^{h/2} z\sigma_{yy}\, dz,$$

$$N_{xz} = \int_{-h/2}^{h/2} \sigma_{xz}\, dz \equiv Q_x, \qquad N_{yz} = \int_{-h/2}^{h/2} \sigma_{yz}\, dz \equiv Q_y$$





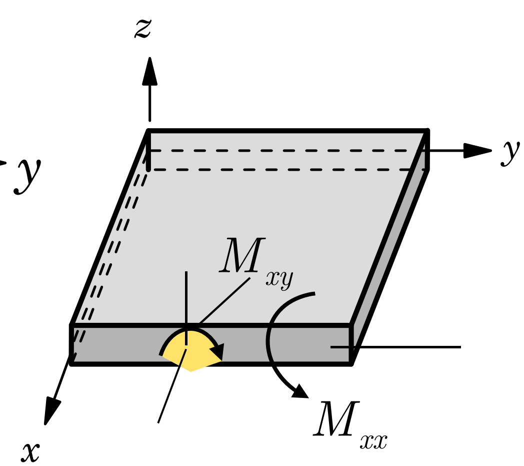
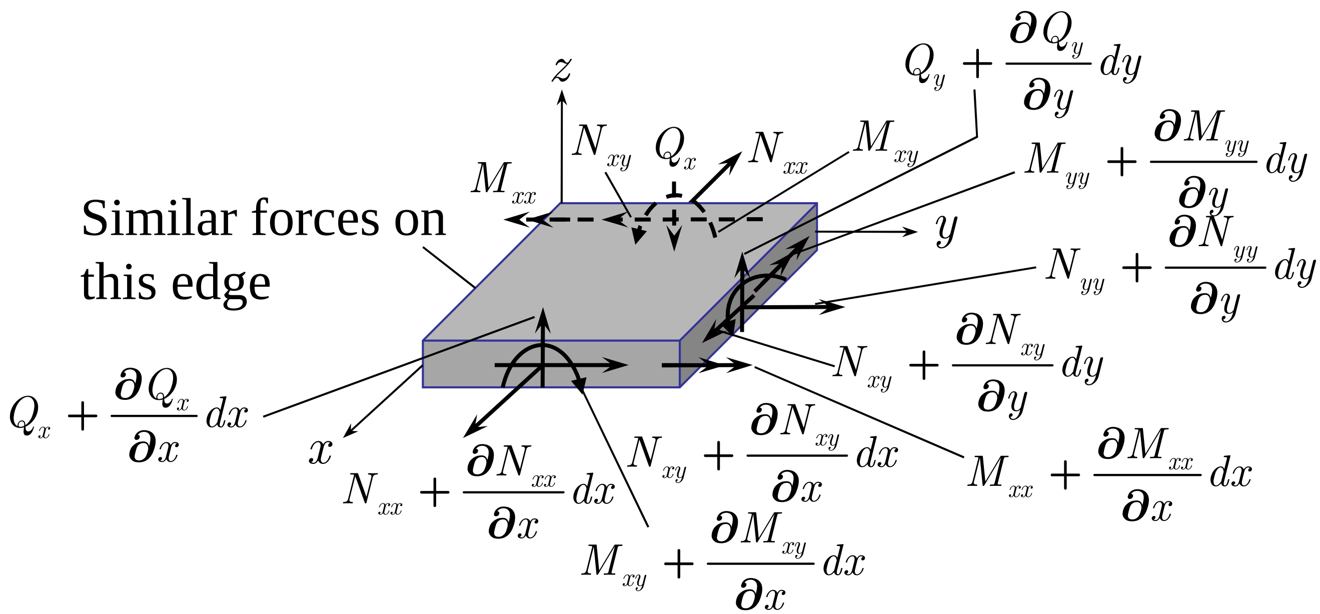


Plate bending: 9

EQUATIONS OF MOTION

Element of dimensions dx, dy, and h



Equations of motion (CPT)

$$\sum F_x = 0: \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} + f_x = I_0 \frac{\partial^2 u}{\partial t^2}$$

$$\sum F_y = 0: \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} + f_y = I_0 \frac{\partial^2 v}{\partial t^2}$$

$$\sum F_z = 0: \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = I_0 \frac{\partial^2 w}{\partial t^2}$$

$$\sum M_y = 0: \frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = -I_2 \frac{\partial^3 w}{\partial t^2 \partial x}$$

$$\sum M_x = 0: \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - Q_y = -I_2 \frac{\partial^3 w}{\partial t^2 \partial y}$$

Plate bending: 10

Equations of Motion of the CLPT

$$\frac{\partial N_{xx}}{\partial x}+\frac{\partial N_{xy}}{\partial y}+f_x=I_0\frac{\partial^2 u}{\partial t^2}+I_1\frac{\partial^2 \theta_x}{\partial t^2} \qquad (1)$$

$$\frac{\partial N_{xy}}{\partial x}+\frac{\partial N_{yy}}{\partial y}+f_y=I_0\frac{\partial^2 v}{\partial t^2}+I_1\frac{\partial^2 \theta_y}{\partial t^2} \qquad (2)$$

$$\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+\frac{\partial}{\partial x}\left(N_{xx}\frac{\partial w}{\partial x}+N_{xy}\frac{\partial w}{\partial y}\right)$$

$$+\frac{\partial}{\partial y}\left(N_{xy}\frac{\partial w}{\partial x}+N_{yy}\frac{\partial w}{\partial y}\right)+q=I_0\frac{\partial^2 w}{\partial t^2} \qquad (3)$$

$$\frac{\partial M_{xx}}{\partial x}+\frac{\partial M_{xy}}{\partial y}-Q_x=I_2\frac{\partial^2 \theta_x}{\partial t^2}+I_1\frac{\partial^2 u}{\partial t^2} \qquad (4)$$

$$\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_{yy}}{\partial y}-Q_y=I_2\frac{\partial^2 \theta_y}{\partial t^2}+I_1\frac{\partial^2 v}{\partial t^2} \qquad (5)$$

GOVERNING EQUATIONS
OF THE CLASSICAL PLATE THEORY (CPT)

Equations of Motion

$$\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} + f_x = I_0 \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} + f_y = I_0 \frac{\partial^2 v}{\partial t^2}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y}\right) + q = -I_2 \frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + I_0 \frac{\partial^2 w}{\partial t^2}$$

Plate Constitutive Equations

$$N_{xx} = A_{11}\frac{\partial u}{\partial x} + A_{12}\frac{\partial v}{\partial y} - N_{xx}^T, \qquad N_{yy} = A_{12}\frac{\partial u}{\partial x} + A_{22}\frac{\partial v}{\partial y} - N_{yy}^T$$

$$N_{xy} = A_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \qquad M_{xx} = -D_{11}\frac{\partial^2 w}{\partial x^2} - D_{12}\frac{\partial^2 w}{\partial y^2} - M_{xx}^T$$

$$M_{yy} = -D_{12}\frac{\partial^2 w}{\partial x^2} - D_{22}\frac{\partial^2 w}{\partial y^2} - M_{yy}^T, \qquad M_{xy} = -2D_{66}\frac{\partial^2 w}{\partial x \partial y}$$

Plate bending: 12

Stress Resultant-Displacement Relations

$$N_{xx} = \int_{-h/2}^{h/2} \sigma_{xx}\, dz = \int_{-h/2}^{h/2} \left\{ \bar{Q}_{11}\left(\frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2}\right) + \bar{Q}_{12}\left(\frac{\partial v}{\partial y} - z\frac{\partial^2 w}{\partial y^2}\right)\right\} dz$$

$$+ \int_{-h/2}^{h/2} \bar{Q}_{16}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2z\frac{\partial^2 w}{\partial x \partial y}\right) dz$$

$$= A_{11}\left[\frac{\partial u}{\partial x}\right] + A_{12}\left[\frac{\partial v}{\partial y}\right] + A_{16}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right]$$

$$- B_{11}\frac{\partial^2 w}{\partial x^2} - B_{12}\frac{\partial^2 w}{\partial y^2} - 2B_{16}\frac{\partial^2 w}{\partial x \partial y}$$

$$M_{xx} = \int_{-h/2}^{h/2} \sigma_{xx}\, z\, dz = \int_{-h/2}^{h/2} z\left[\bar{Q}_{11}\left(\frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2}\right) + \bar{Q}_{12}\left(\frac{\partial v}{\partial y} - z\frac{\partial^2 w}{\partial y^2}\right)\right] dz$$

$$+ \int_{-h/2}^{h/2} z\,\bar{Q}_{16}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2z\frac{\partial^2 w}{\partial x \partial y}\right) dz$$

$$= B_{11}\left[\frac{\partial u}{\partial x}\right] + B_{12}\left[\frac{\partial v}{\partial y}\right] + B_{16}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$- D_{11}\frac{\partial^2 w}{\partial x^2} - D_{12}\frac{\partial^2 w}{\partial y^2} - 2D_{16}\frac{\partial^2 w}{\partial x \partial y}$$

- **For composite laminates, the integration through the plate thickness is replaced with integration through each layer of the plate (i.e., will have A, B, and D matrices that represent the lamination scheme)**

Stress Resultant-Displacement Relations (nonlinear)

$$N_{xx} = \int_{-h/2}^{h/2} \sigma_{xx}\, dz = \int_{-h/2}^{h/2} \left\{ \bar{Q}_{11} \left(\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - z\frac{\partial^2 w}{\partial x^2} \right) + \bar{Q}_{12} \left(\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - z\frac{\partial^2 w}{\partial y^2} \right) \right\} dz$$

$$+ \int_{-h/2}^{h/2} \bar{Q}_{16} \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} - 2z\frac{\partial^2 w}{\partial x \partial y} \right) dz$$

$$= A_{11}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \right] + A_{12}\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \right] + A_{16}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \right]$$

$$- B_{11}\frac{\partial^2 w}{\partial x^2} - B_{12}\frac{\partial^2 w}{\partial y^2} - 2B_{16}\frac{\partial^2 w}{\partial x \partial y}$$

$$M_{xx} = \int_{-h/2}^{h/2} \sigma_{xx}\, z\, dz = \int_{-h/2}^{h/2} z \left[\bar{Q}_{11} \left(\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - z\frac{\partial^2 w}{\partial x^2} \right) + \bar{Q}_{12} \left(\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - z\frac{\partial^2 w}{\partial y^2} \right) \right] dz$$

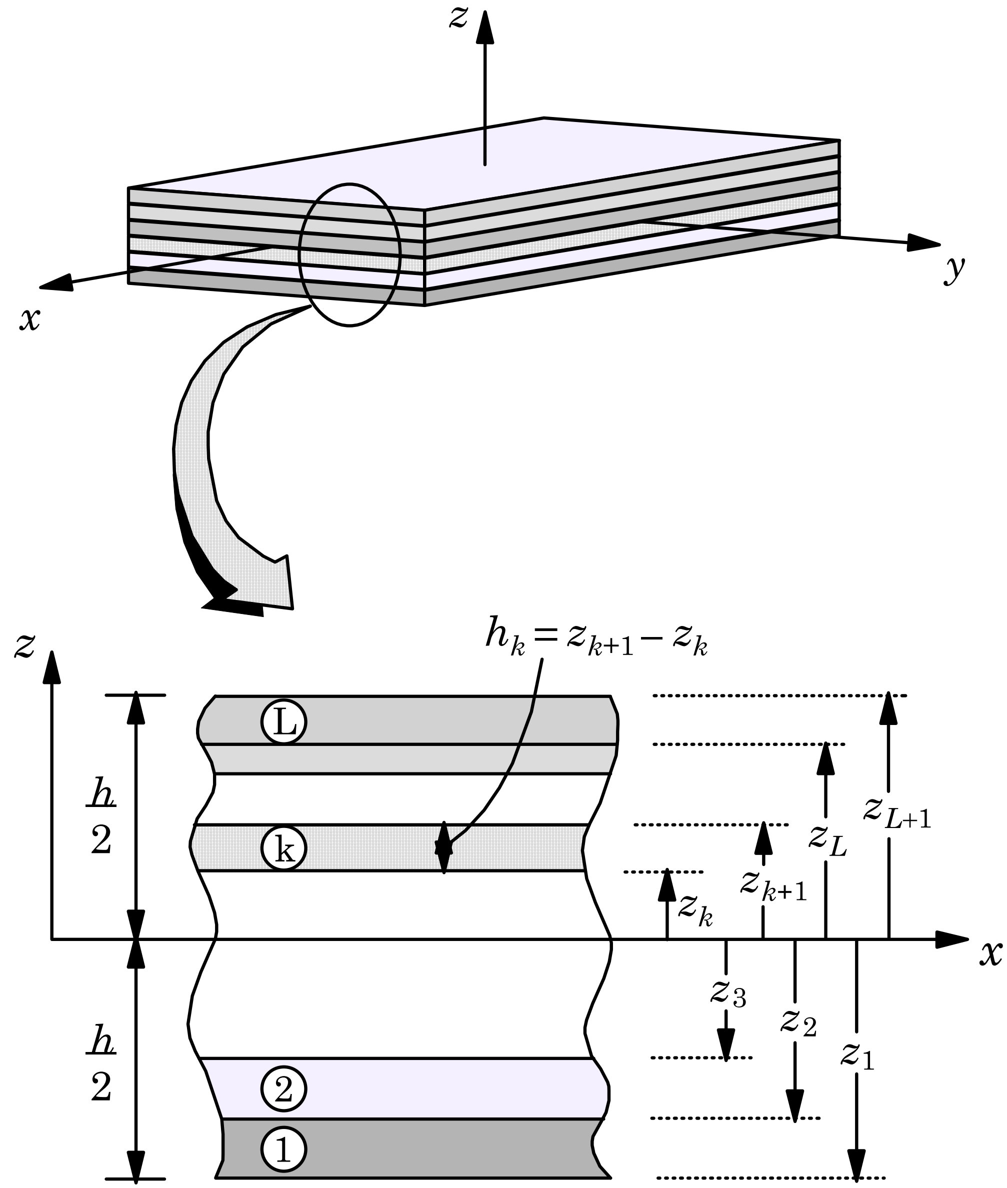
$$+ \int_{-h/2}^{h/2} z\bar{Q}_{16} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} - 2z\frac{\partial^2 w}{\partial x \partial y} \right) dz$$

$$= B_{11}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \right] + B_{12}\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \right] + B_{16}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \right)$$

$$- D_{11}\frac{\partial^2 w}{\partial x^2} - D_{12}\frac{\partial^2 w}{\partial y^2} - 2D_{16}\frac{\partial^2 w}{\partial x \partial y}$$

LAMINATION SCHEME AND NOTATION (REVISITED)

Notation and Coordinate System Used in a Laminate Analysis

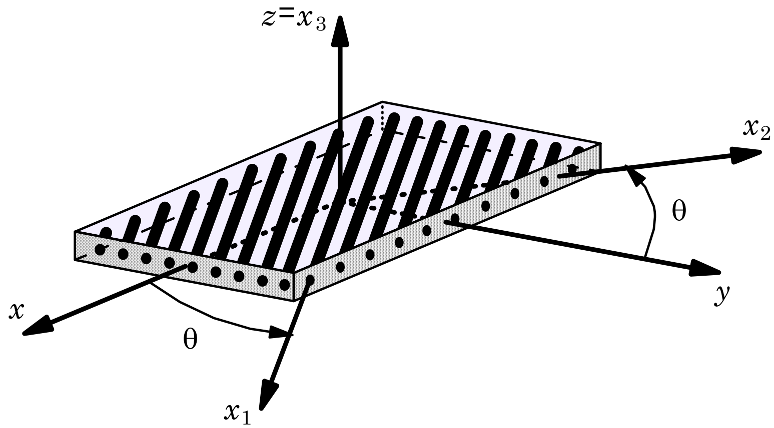


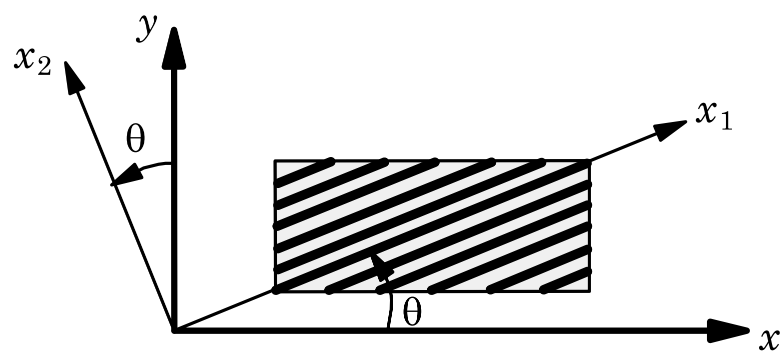
Layers are numbered in the +ve z direction

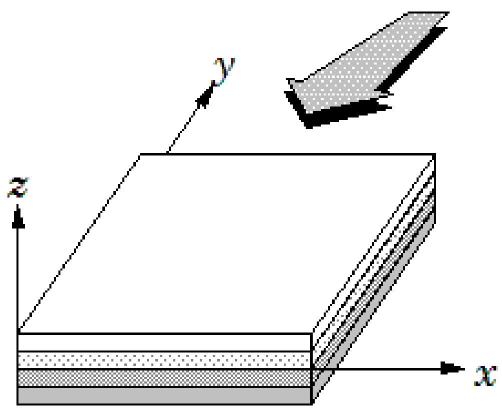
Strain-Stress Relations
in Structural Coordinates

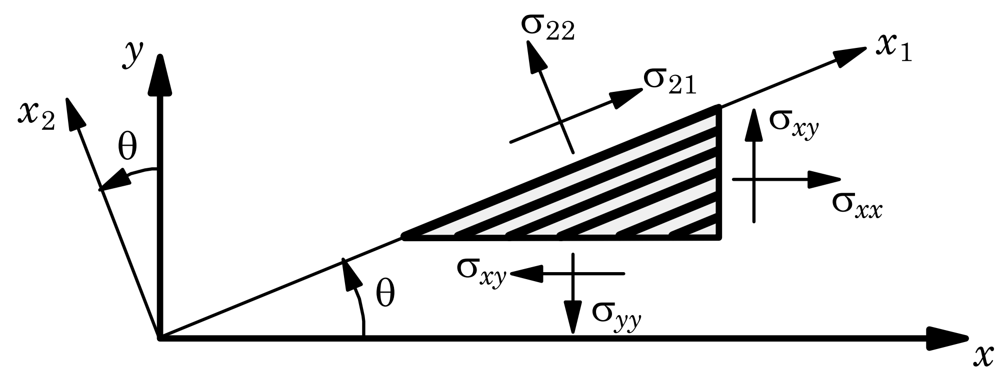
(x_1, x_2, x_3) = Material coordinates

(x, y, z) = Structural coordinates

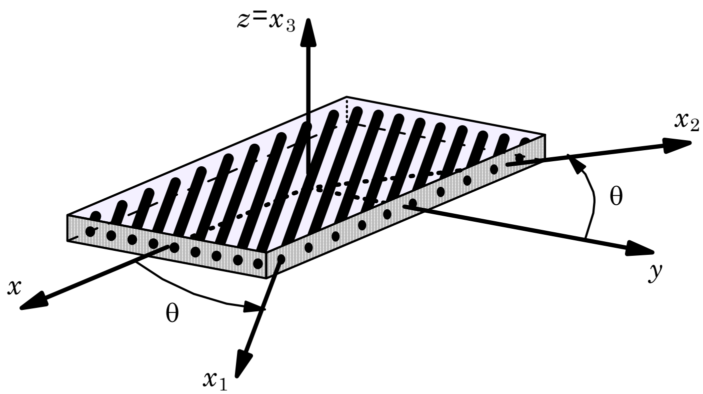








Strain-Stress Relations in Structural Coordinates



$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}$$

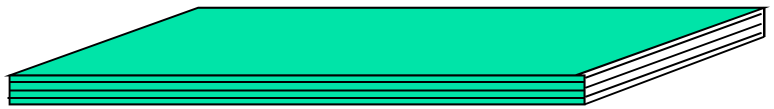
$$\begin{Bmatrix} \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{44} & \bar{Q}_{45} \\ \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}$$

$$\bar{Q}_{11} = Q_{11}\cos^4\theta + 2(Q_{12} + 2Q_{66})\sin^2\theta\ \cos^2\theta + Q_{22}\sin^4\theta$$
$$\bar{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66})\sin^2\theta\ \cos^2\theta + Q_{12}(\sin^4\theta + \cos^4\theta)$$
$$\bar{Q}_{22} = Q_{11}\sin^4\theta + 2(Q_{12} + 2Q_{66})\sin^2\theta\ \cos^2\theta + Q_{22}\cos^4\theta$$
$$\bar{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66})\sin\theta\ \cos^3\theta + (Q_{12} - Q_{22} + 2Q_{66})\sin^3\theta\ \cos\theta$$
$$\bar{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66})\sin^3\theta\cos\theta + (Q_{12} - Q_{22} + 2Q_{66})\sin\theta\ \cos^3\theta$$
$$\bar{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})\sin^2\theta\ \cos^2\theta + Q_{66}(\sin^4\theta + \cos^4\theta)$$
$$\bar{Q}_{44} = Q_{44}\cos^2\theta + Q_{55}\sin^2\theta$$
$$\bar{Q}_{45} = (Q_{55} - Q_{44})\cos\theta\sin\theta$$
$$\bar{Q}_{55} = Q_{55}\cos^2\theta + Q_{44}\sin^2\theta$$

Effect of the Lamination Scheme on Laminate Stiffness

Layer No.	Angle	Thickness	Mat. Type
1	0.	.5000E-02	1
2	90.	.5000E-02	1
3	90.	.5000E-02	1
4	0.	.5000E-02	1

$[0/90]_s$



Youngs modulus, E1 = 0.30000E+08
Youngs modulus, E2 = 0.30000E+07
Shear modulus, G12 = 0.15000E+07
Shear modulus, G13 = 0.15000E+07
Shear modulus, G23 = 0.15000E+07
Poissons ratio, NU12 = 0.25000E+00

Laminate stiffnesses, [B]:

0.00000E+00	-0.35527E-14	0.00000E+00
-0.35527E-14	0.00000E+00	0.00000E+00
0.00000E+00	0.00000E+00	0.71054E-14

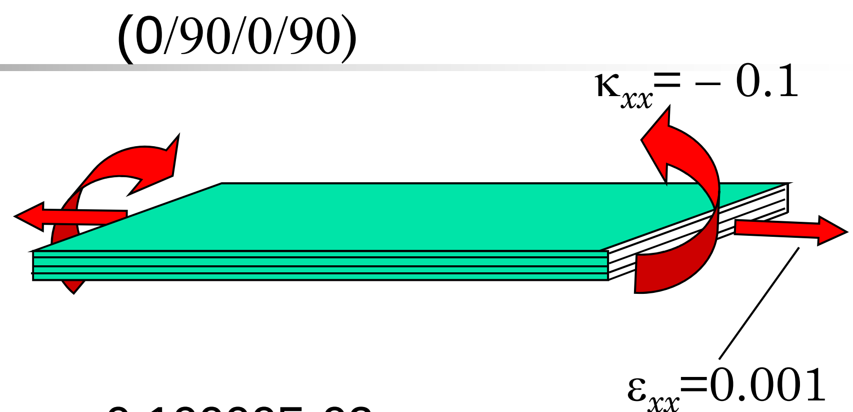
Laminate stiffnesses, [A]:

0.33208E+06	0.15094E+05	0.45056E-12
0.15094E+05	0.33208E+06	0.16186E-10
0.45056E-12	0.16186E-10	0.30000E+05

Laminate stiffnesses, [D]:

0.17862E+02	0.50314E+00	0.37547E-17
0.50314E+00	0.42767E+01	0.13488E-15
0.37547E-17	0.13488E-15	0.10000E+01

Effect of the Lamination Scheme



Specified inplane strain, EXX0 .... = 0.10000E-02
Specified inplane strain, EYY0 ....= 0.00000E+00
Specified inplane strain, EXY0 ....= 0.00000E+00
Specified bending strain, EXX1 .... = 0.10000E+00
Specified bending strain, EYY1 .... = 0.00000E+00
Specified bending strain, EXY1 .... = 0.00000E+00

<u>EXX1 = 0.10000E+00</u>

Inplane force, Nxx ...... = 0.26415E+03
Inplane force, Nyy ...... = 0.15094E+02
Inplane force, Nxy ...... = 0.56320E-15
Bending moment, Mxx = 0.42767E+00
Bending moment, Myy = 0.50314E-01
Bending moment, Mxy = 0.26283E-17

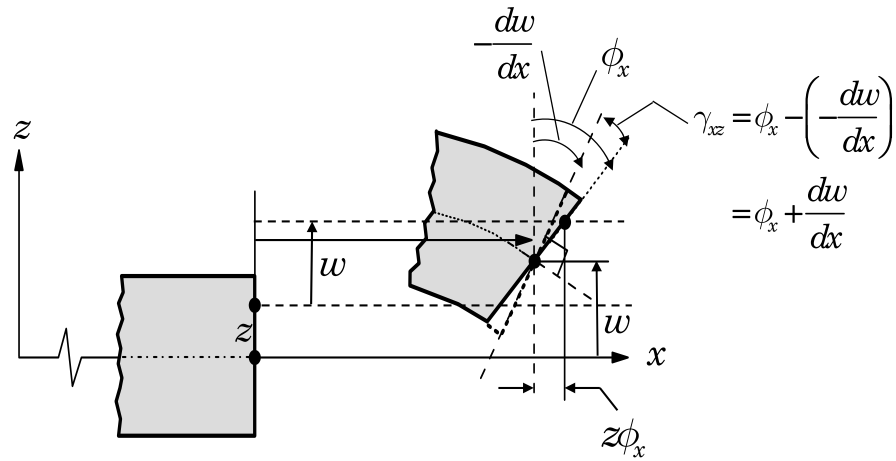
<u>EXX1 = 0.00000E+00</u>

Inplane force, Nxx ..... = 0.33208E+03
Inplane force, Nyy ...... = 0.15094E+02
Inplane force, Nxy ...... = 0.45056E-15
Bending moment, Mxx = -0.67925E+00
Bending moment, Myy = -0.35527E-17
Bending moment, Mxy = 0.11264E-17

FIRST-ORDER SHEAR DEFORMATION THEORY (FSDT)

Displacement Field

$$u_1(x,y,z,t) = u(x,y,t) + z\phi_x(x,y,t)$$
$$u_2(x,y,z,t) = v(x,y,t) + z\phi_y(x,y,t)$$
$$u_3(x,y,z,t) = w(x,y,t)$$



Linearized strains

$$\varepsilon_{xx} = \frac{\partial u_1}{\partial x} = \frac{\partial u}{\partial x} + z\frac{\partial \phi_x}{\partial x},$$

$$\varepsilon_{yy} = \frac{\partial u_2}{\partial y} = \frac{\partial v}{\partial y} + z\frac{\partial \phi_y}{\partial y}$$

$$\gamma_{xy} = \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} + \frac{\partial u_3}{\partial x}\frac{\partial u_3}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + z\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right)$$

$$\gamma_{xz} = \frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x} = \phi_x + \frac{\partial w}{\partial x}, \quad \gamma_{yz} = \frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y} = \phi_y + \frac{\partial w}{\partial y}$$

EQUATIONS OF MOTION OF FSDT

$$\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} + f_x = I_0 \frac{\partial^2 u}{\partial t^2} + I_1 \frac{\partial^2 \phi_x}{\partial t^2} \qquad (u,\ N_n)$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} + f_y = I_0 \frac{\partial^2 v}{\partial t^2} + I_1 \frac{\partial^2 \phi_y}{\partial t^2} \qquad (v,\ N_{ns})$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial}{\partial x}\left(N_{xx}\frac{\partial w}{\partial x} + N_{xy}\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial y}\left(N_{xy}\frac{\partial w}{\partial x} + N_{yy}\frac{\partial w}{\partial y}\right) + q = I_0 \frac{\partial^2 w}{\partial t^2} \qquad (w,\ Q_n)$$

$$\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = I_2 \frac{\partial^2 \phi_x}{\partial t^2} + I_1 \frac{\partial^2 u}{\partial t^2} \qquad (\phi_n,\ M_n)$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - Q_y = I_2 \frac{\partial^2 \phi_y}{\partial t^2} + I_1 \frac{\partial^2 v}{\partial t^2} \qquad (\phi_{ns},\ M_{ns})$$

$$N_n = N_{xx} n_x + N_{xy} n_y; \qquad N_{ns} = N_{xy} n_x + N_{yy} n_y$$

$$M_n = M_{xx} n_x + M_{xy} n_y; \qquad M_{ns} = M_{xy} n_x + M_{yy} n_y; \qquad Q_n = Q_x n_x + Q_y n_y$$

THE FIRST-ORDER SHEAR DEFORMATION THEORY
Stress Resultants (linear)

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix}$$

$$\sigma_{xz} = 2G_{13}\, \varepsilon_{xz}, \quad \sigma_{yz} = 2G_{23}\, \varepsilon_{xz}$$

$$N_{xx} = A_{11}\frac{\partial u}{\partial x} + A_{12}\frac{\partial v}{\partial y} + A_{16}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + B_{11}\frac{\partial \phi_x}{\partial x} + B_{12}\frac{\partial \phi_y}{\partial y} + B_{16}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right)$$

$$N_{yy} = A_{12}\frac{\partial u}{\partial x} + A_{22}\frac{\partial v}{\partial y} + A_{26}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + B_{12}\frac{\partial \phi_x}{\partial x} + B_{22}\frac{\partial \phi_y}{\partial y} + B_{26}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right)$$

$$M_{xx} = B_{11}\frac{\partial u}{\partial x} + B_{12}\frac{\partial v}{\partial y} + B_{16}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + D_{11}\frac{\partial \phi_x}{\partial x} + D_{12}\frac{\partial \phi_y}{\partial y} + D_{16}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right)$$

$$M_{yy} = B_{12}\frac{\partial u}{\partial x} + B_{22}\frac{\partial v}{\partial y} + B_{26}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + D_{12}\frac{\partial \phi_x}{\partial x} + D_{22}\frac{\partial \phi_y}{\partial y} + D_{26}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right)$$

$$M_{xy} = B_{16}\frac{\partial u}{\partial x} + B_{26}\frac{\partial v}{\partial y} + B_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + D_{16}\frac{\partial \phi_x}{\partial x} + D_{26}\frac{\partial \phi_y}{\partial y} + D_{66}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right)$$

$$Q_x = K_s A_{55}\left(\phi_x + \frac{\partial w}{\partial x}\right) + K_s A_{45}\left(\phi_y + \frac{\partial w}{\partial y}\right); \quad Q_y = K_s A_{45}\left(\phi_x + \frac{\partial w}{\partial x}\right) + K_s A_{44}\left(\phi_y + \frac{\partial w}{\partial y}\right)$$

THE FIRST-ORDER SHEAR DEFORMATION THEORY
Stress Resultants (Nonlinear)

$$N_{xx} = A_{11}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + A_{12}\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] + A_{16}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)$$
$$+ B_{11}\frac{\partial \phi_x}{\partial x} + B_{12}\frac{\partial \phi_y}{\partial y} + B_{16}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) - N_{xx}^T$$

$$N_{yy} = A_{12}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + A_{22}\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] + A_{26}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)$$
$$+ B_{12}\frac{\partial \phi_x}{\partial x} + B_{22}\frac{\partial \phi_y}{\partial y} + B_{26}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) - N_{yy}^T$$

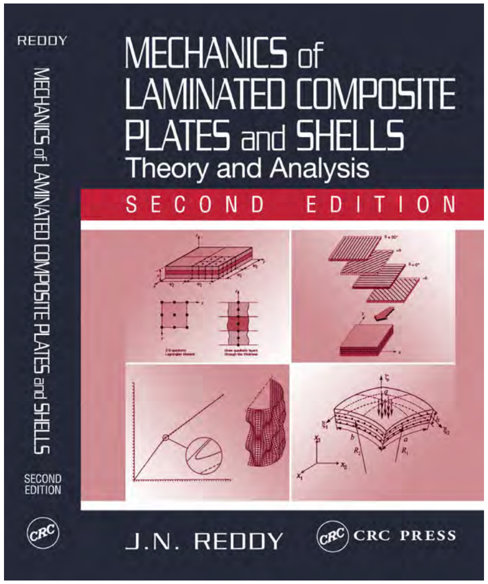
$$M_{xx} = B_{11}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + B_{12}\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] + B_{16}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - D_{11}\frac{\partial^2 w}{\partial x^2} - D_{12}\frac{\partial^2 w}{\partial y^2} - 2D_{16}\frac{\partial^2 w}{\partial x \partial y} - M_{xx}^T$$

$$M_{yy} = B_{12}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + B_{22}\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] + B_{26}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - D_{12}\frac{\partial^2 w}{\partial x^2} - D_{22}\frac{\partial^2 w}{\partial y^2} - 2D_{26}\frac{\partial^2 w}{\partial x \partial y} - M_{yy}^T$$

$$M_{xy} = B_{16}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + B_{26}\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] + B_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - D_{16}\frac{\partial^2 w}{\partial x^2} - D_{26}\frac{\partial^2 w}{\partial y^2} - 2D_{66}\frac{\partial^2 w}{\partial x \partial y} - M_{xy}^T$$

$$Q_x = K_s A_{55}\left(\phi_x + \frac{\partial w}{\partial x}\right) + K_s A_{45}\left(\phi_y + \frac{\partial w}{\partial y}\right); \; Q_y = K_s A_{45}\left(\phi_x + \frac{\partial w}{\partial x}\right) + K_s A_{44}\left(\phi_y + \frac{\partial w}{\partial y}\right)$$

ANALYSIS OF LAMINATED PLATES (Classical and First-Order Plate Theories)



CONTENTS OF THE LECTURE

- Solution procedures
- The Novier Solution Procedure
- Numerical results based on the Navier solution procedure
- Numerical Results

Equations of Equilibrium for bending in terms of Displacements (FSDT)

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0$$

$$Q_x = \int_{-h/2}^{h/2} Q_{55}\left(\phi_x + \frac{\partial w}{\partial x}\right)dz = A_{55}\left(\phi_x + \frac{\partial w}{\partial x}\right), \quad Q_y = \int_{-h/2}^{h/2} Q_{44}\left(\phi_y + \frac{\partial w}{\partial y}\right)dz = A_{44}\left(\phi_y + \frac{\partial w}{\partial y}\right)$$

$$A_{55}\frac{\partial}{\partial x}\left(\phi_x + \frac{\partial w}{\partial x}\right) + A_{44}\frac{\partial}{\partial y}\left(\phi_y + \frac{\partial w}{\partial y}\right) + q = 0 \Rightarrow A_{55}\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) + A_{44}\left(\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2}\right) + q = 0$$

$$\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0 \Rightarrow D_{11}\frac{\partial^2 \phi_x}{\partial x^2} + D_{12}\frac{\partial^2 \phi_y}{\partial x \partial y} + D_{66}\left(\frac{\partial^2 \phi_x}{\partial y^2} + \frac{\partial^2 \phi_y}{\partial x \partial y}\right) - A_{55}\left(\phi_x + \frac{\partial w}{\partial x}\right) = 0$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - Q_y = 0 \Rightarrow D_{12}\frac{\partial^2 \phi_x}{\partial x \partial y} + D_{22}\frac{\partial^2 \phi_y}{\partial y^2} + D_{66}\left(\frac{\partial^2 \phi_x}{\partial x \partial y} + \frac{\partial^2 \phi_y}{\partial x^2}\right) - A_{44}\left(\phi_y + \frac{\partial w}{\partial y}\right) = 0$$

Navier Solution of Simply Supported ORTHOTROPIC PLATES (bending only)

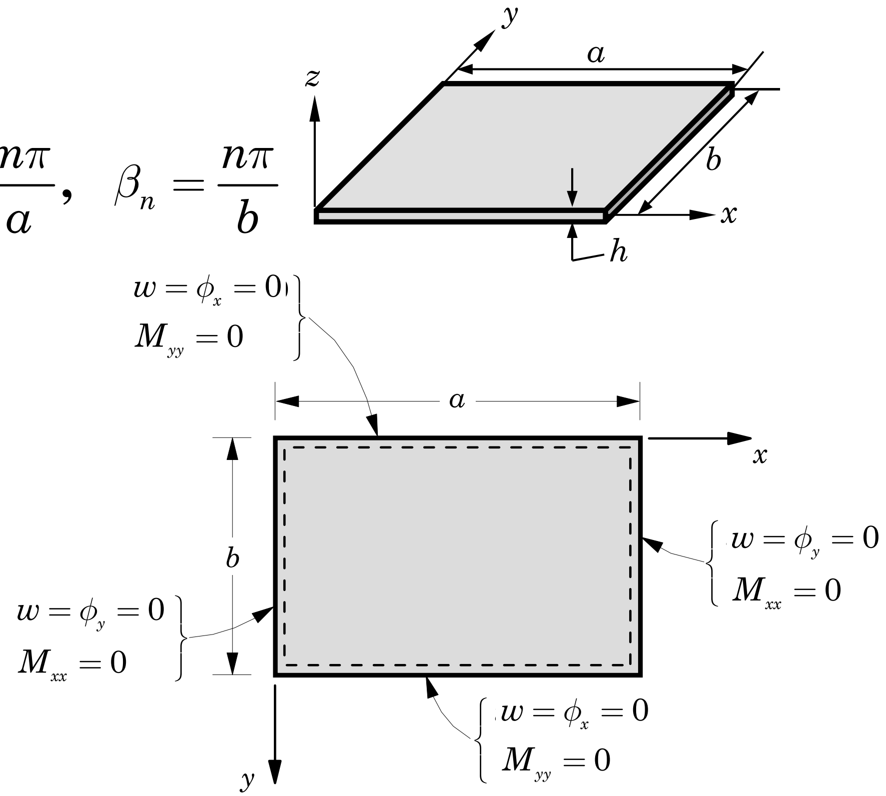
$$w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn} \sin\alpha_m x \sin\beta_n y$$

$$\phi_x(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} X_{mn} \cos\alpha_m x \sin\beta_n y \qquad \alpha_m = \frac{m\pi}{a}, \quad \beta_n = \frac{n\pi}{b}$$

$$\phi_y(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} Y_{mn} \sin\alpha_m x \cos\beta_n y$$

$$q(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} Q_{mn} \sin\alpha_m x \sin\beta_n y$$

$$Q_{mn} = \frac{4}{ab}\int_0^a \int_0^b q(x,y) \sin\alpha_m x \sin\beta_n y \, dxdy$$



Substitution of the expansions into the equations of equilibrium give the following algebraic equations for the coefficients of the expansion (as explained in more detail on the next slide):

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{bmatrix} \begin{Bmatrix} W_{mn} \\ X_{mn} \\ Y_{mn} \end{Bmatrix} = \begin{Bmatrix} Q_{mn} \\ 0 \\ 0 \end{Bmatrix}$$

Equations of Equilibrium for bending in terms of the parameters W_{mn}, Y_{mn}, W_{mn}, and Q_{mn}

$$A_{55}\left(\frac{\partial \phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+A_{44}\left(\frac{\partial \phi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)+q=0$$

$$\sum\sum \sin\alpha_m x\,\sin\beta_n y\left[A_{55}\left(-\alpha_m X_{mn}-\alpha_m^2 W_{mn}\right)+A_{44}\left(-\beta_n Y_{mn}-\beta_m^2 W_{mn}\right)+Q_{mn}\right]=0$$

$$S_{11}W_{mn}+S_{12}X_{mn}+S_{13}Y_{mn}=Q_{mn}$$

$$D_{11}\frac{\partial^2 \phi_x}{\partial x^2}+D_{12}\frac{\partial^2 \phi_y}{\partial x\partial y}+D_{66}\left(\frac{\partial^2 \phi_x}{\partial y^2}+\frac{\partial^2 \phi_y}{\partial y\partial x}\right)-A_{55}\left(\phi_x+\frac{\partial w}{\partial x}\right)=0$$

$$\sum\sum \cos\alpha_m x\,\sin\beta_n y\left[-D_{11}\alpha_m^2 X_{mn}-D_{12}\alpha_m\beta_n Y_{mn}-D_{66}\left(\beta_n^2 X_{mn}+\alpha_m\beta_n Y_{mn}\right)-A_{55}\left(X_{mn}+\alpha_m W_{mn}\right)\right]=0$$

$$S_{21}W_{mn}+S_{22}X_{mn}+S_{23}Y_{mn}=0$$

$$D_{12}\frac{\partial^2 \phi_x}{\partial x\partial y}+D_{22}\frac{\partial^2 \phi_y}{\partial y^2}+D_{66}\left(\frac{\partial^2 \phi_x}{\partial x\partial y}+\frac{\partial^2 \phi_y}{\partial x^2}\right)-A_{44}\left(\phi_y+\frac{\partial w}{\partial y}\right)=0$$

$$\sum\sum \sin\alpha_m x\,\cos\beta_n y\left[-D_{12}\alpha_m\beta_n X_{mn}-D_{22}\beta_n^2 Y_{mn}-D_{66}\left(\alpha_m\beta_n X_{mn}+\alpha_n^2 Y_{mn}\right)-A_{44}\left(Y_{mn}+\beta_n W_{mn}\right)\right]=0$$

$$S_{31}W_{mn}+S_{32}X_{mn}+S_{33}Y_{mn}=0$$

Navier Solution of Simply Supported Orthotropic Plates (continued)

where

$$s_{11} = (A_{55}\alpha_m^2 + A_{44}\beta_n^2),\ s_{12} = A_{55}\alpha_m,\ \ s_{13} = A_{44}\beta_n,$$

$$s_{22} = (D_{11}\alpha_m^2 + D_{66}\beta_n^2 + A_{55}),\ s_{23} = (D_{12} + D_{66})\alpha_m\beta_n,$$

$$s_{33} = (D_{66}\alpha_m^2 + D_{22}\beta_n^2 + A_{44})$$

The solution becomes

$$W_{mn} = \frac{b_0 Q_{mn}}{b_{mn}},\ \ X_{mn} = \frac{b_1 W_{mn}}{b_0},\ Y_{mn} = \frac{b_2 W_{mn}}{b_0}$$

$$b_{mn} = s_{11}b_0 + s_{12}b_1 + s_{13}b_2\ ,\quad b_0 = s_{22}s_{33} - s_{23}s_{23},$$

$$b_1 = s_{23}s_{13} - s_{12}s_{33},\qquad b_2 = s_{12}s_{23} - s_{22}s_{13},$$

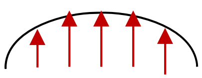
$$M_{xx} = -\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(D_{11}\alpha_m X_{mn} + D_{12}\beta_n Y_{mn}\right)\sin\alpha_m x\ \sin\beta_n y,$$

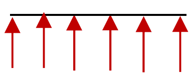
$$M_{yy} = -\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(D_{12}\alpha_m X_{mn} + D_{22}\beta Y_{mn}\right)\sin\alpha_m x\sin\beta_n y,$$

$$M_{xy} = D_{66}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\beta_n X_{mn} + \alpha_m Y_{mn}\right)\cos\alpha_m x\ \cos\beta_n y$$

Navier Solution of Simply Supported Orthotropic Plates

$$E_1 = 25E_2,\ G_{12} = G_{13} = 0.5E_2, G_{23} = 0.2E_2, \nu_{12} = 0.25, K_s = 5/6$$





Load	$\frac{b}{h}$	$\hat{w}$	$\bar{\sigma}_{xx}$	$\bar{\sigma}_{yy}$	$\bar{\sigma}_{xy}$	$\bar{\sigma}_{xz}$	$\bar{\sigma}_{yz}$
Isotropic plates ($\nu = 0.25$)							
SL	10	0.2702	0.1900	0.1900	0.1140	0.1910	0.1910
						0.2387	0.2387†
	20	0.2600	0.1900	0.1900	0.1140	0.1910	0.1910
	50	0.2572	0.1900	0.1900	0.1140	0.1910	0.1910
	100	0.2568	0.1900	0.1900	0.1140	0.1910	0.1910
	CPT	0.2566	0.1900	0.1900	0.1140	—	—
UL (19)	10	0.4259	0.2762	0.2762	0.2085	0.3927	0.3927
						0.4909	0.4909†
	20	0.4111	0.2762	0.2762	0.2085	0.3927	0.3927
	50	0.4070	0.2762	0.2762	0.2085	0.3927	0.3927
	100	0.4060	0.2762	0.2762	0.2085	0.3927	0.3927
	CPT	0.4062	0.2762	0.2762	0.2085	—	—

$$\bar{w} = w_0\left(E_2h^3/b^4q_0\right), \hat{w} = w_0\left(D_{22}/b^4q_0\right)\times 10^2,\ \bar{\sigma}_{xx} = \sigma_{xx}\left(h^2/b^2q_0\right),\ \ \bar{\sigma}_{yy} = \sigma_{yy}\left(h^2/b^2q_0\right),$$

$$\bar{\sigma}_{xy} = \sigma_{xy}\left(h^2/b^2q_0\right),\ \ \bar{\sigma}_{xz} = \sigma_{xz}\left(h/bq_0\right),\ \bar{\sigma}_{yz} = \sigma_{yz}\left(h/bq_0\right)$$

$$\bar{\sigma}_{xx}(a/2, b/2, \frac{h}{2}), \bar{\sigma}_{yy}(a/2, b/2, \frac{h}{2}), \bar{\sigma}_{xy}(a, b, -\frac{h}{2}), \bar{\sigma}_{xz}(0, b/2, \frac{h}{2}), \bar{\sigma}_{yz}(a/2, 0, \frac{h}{2})$$

Navier Solution of Simply Supported Orthotropic Plates

Load	$\frac{a}{h}$	$\bar{w}$	$\bar{\sigma}_{xx}$	$\bar{\sigma}_{yy}$	$\bar{\sigma}_{xy}$	$\bar{\sigma}_{xz}$	$\bar{\sigma}_{yz}$
Orthotropic plates							
SL	10	0.0533	0.5248	0.0338	0.0246	0.3452	0.0367
						0.4315	0.0459
	20	0.0404	0.5350	0.0286	0.0222	0.3501	0.0319
						0.4376	0.0399
	50	0.0367	0.5380	0.0270	0.0214	0.3515	0.0304
						0.4394	0.0380
	100	0.0362	0.5385	0.0267	0.0213	0.3517	0.0302
						0.4397	0.0377
	CPT	0.0360	0.5387	0.0267	0.0213	—	—
						0.4398	0.0376
UL (19)	10	0.0795	0.7706	0.0352	0.0539	0.6147	0.1529
						0.7684	0.1911
	20	0.0607	0.7828	0.0272	0.0487	0.6194	0.1466
						0.7742	0.1833
	50	0.0553	0.7860	0.0249	0.0468	0.6207	0.1452
						0.7756	0.1814
	100	0.0545	0.7865	0.0245	0.0464	0.6206	0.1449
						0.7757	0.1812
	CPT	0.0543	0.7866	0.0244	0.0463	—	—
						0.7758	0.1811

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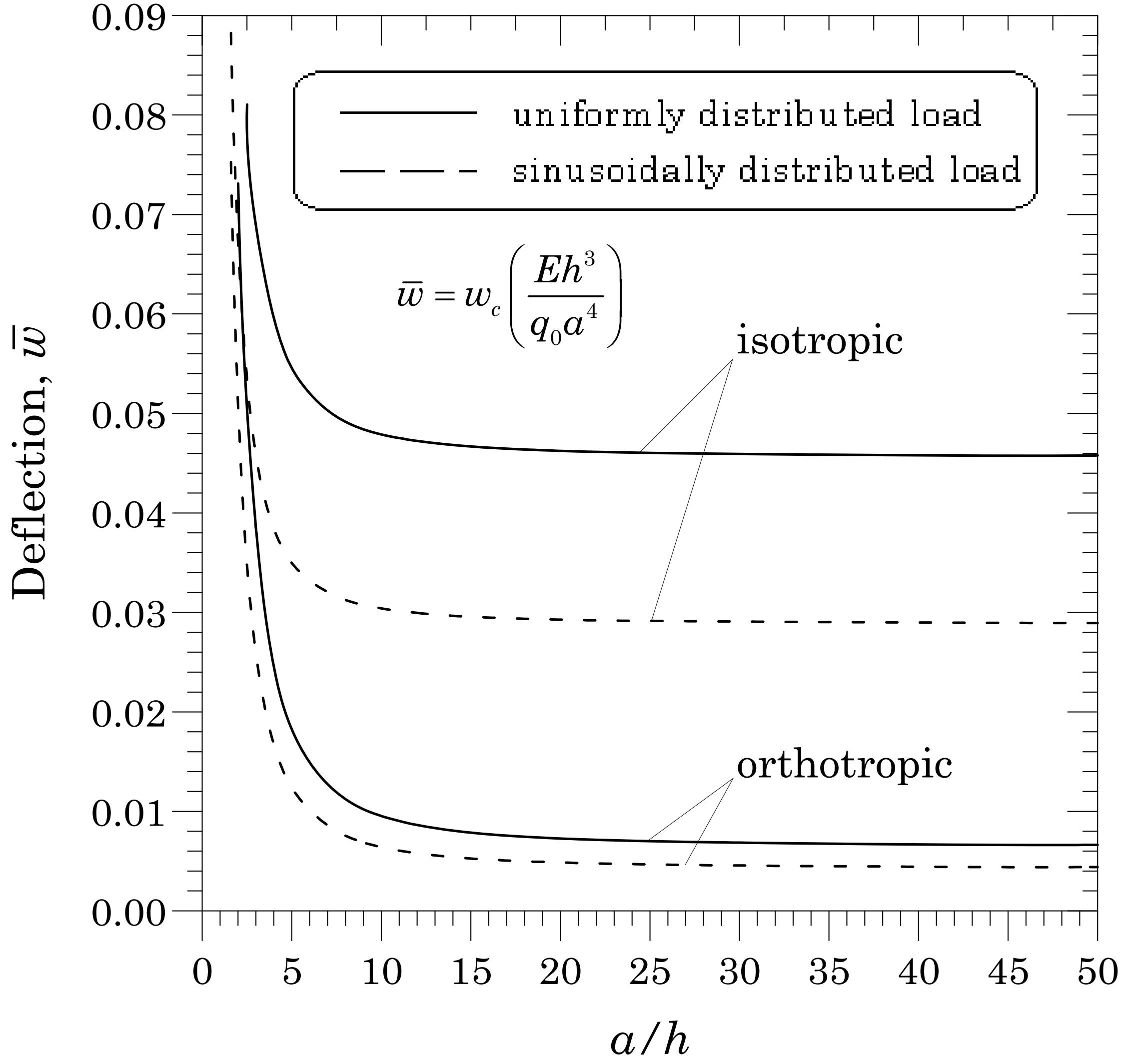
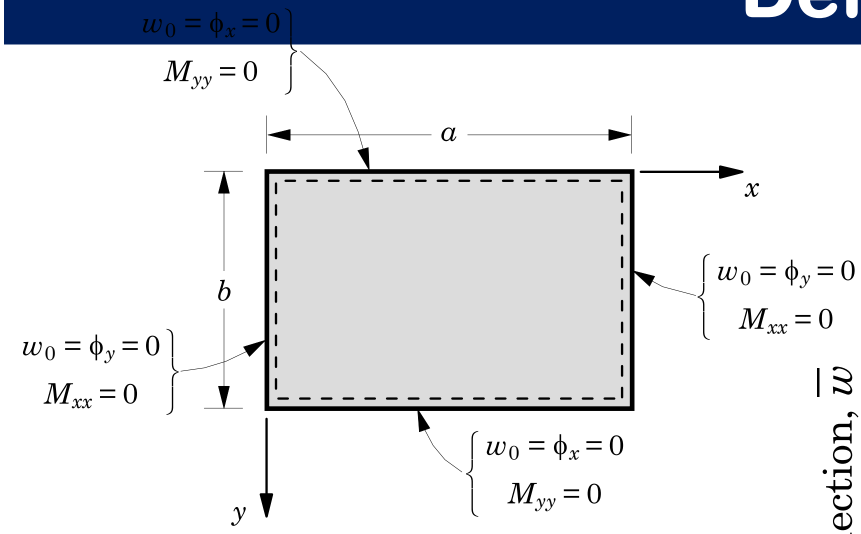
Navier Solution of Simply Supported Plates



Nondimensional center transverse deflection (w) versus side-to-thickness ratio (a/h) for simply supported square plates.

Effect of Shear Deformation on Bending Deformation



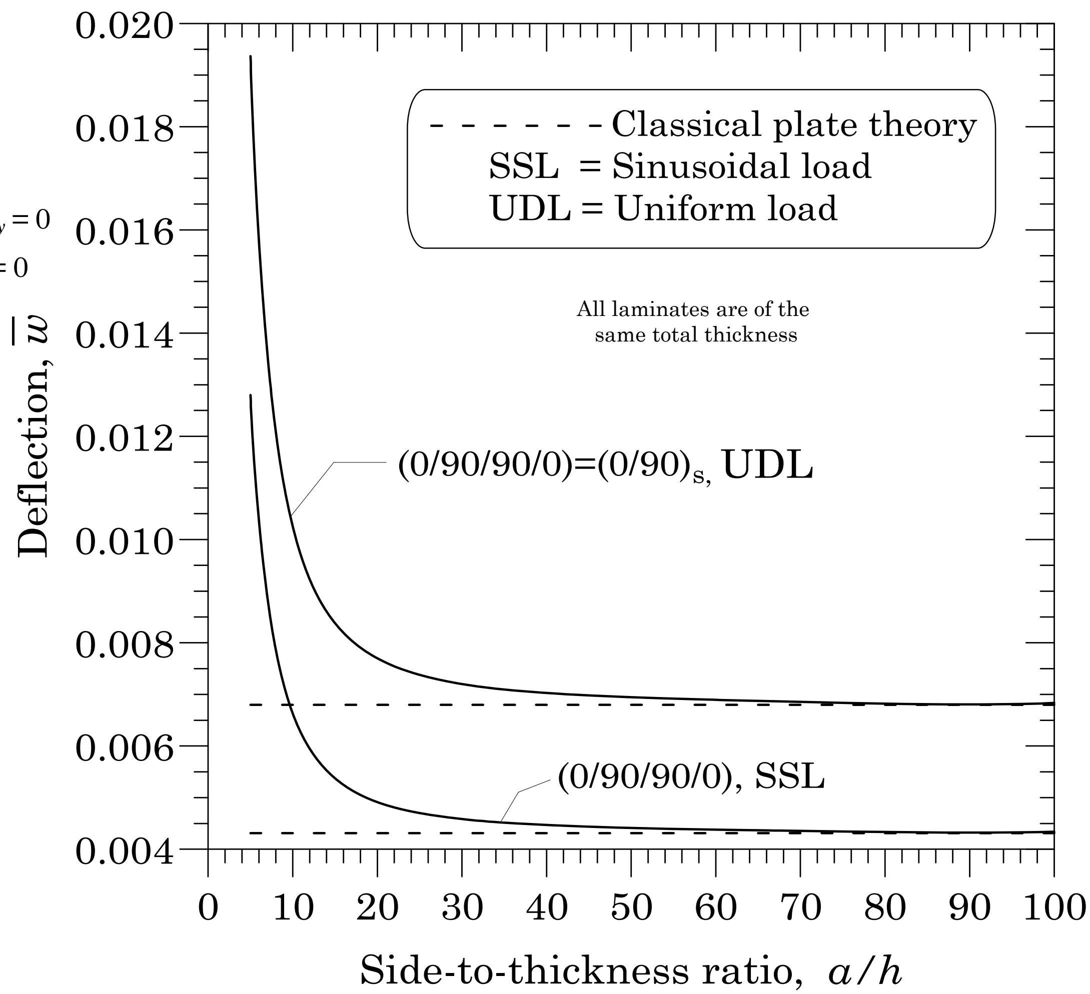
$$E_1 = 25E_2,$$

$$G_{12} = G_{13} = 0.5E_2,$$

$$G_{23} = 0.2E_2,$$

$$\nu_{12} = 0.25$$

$$\bar{w} = w\left(\frac{E_2 h^3}{q_0 a^4}\right)$$

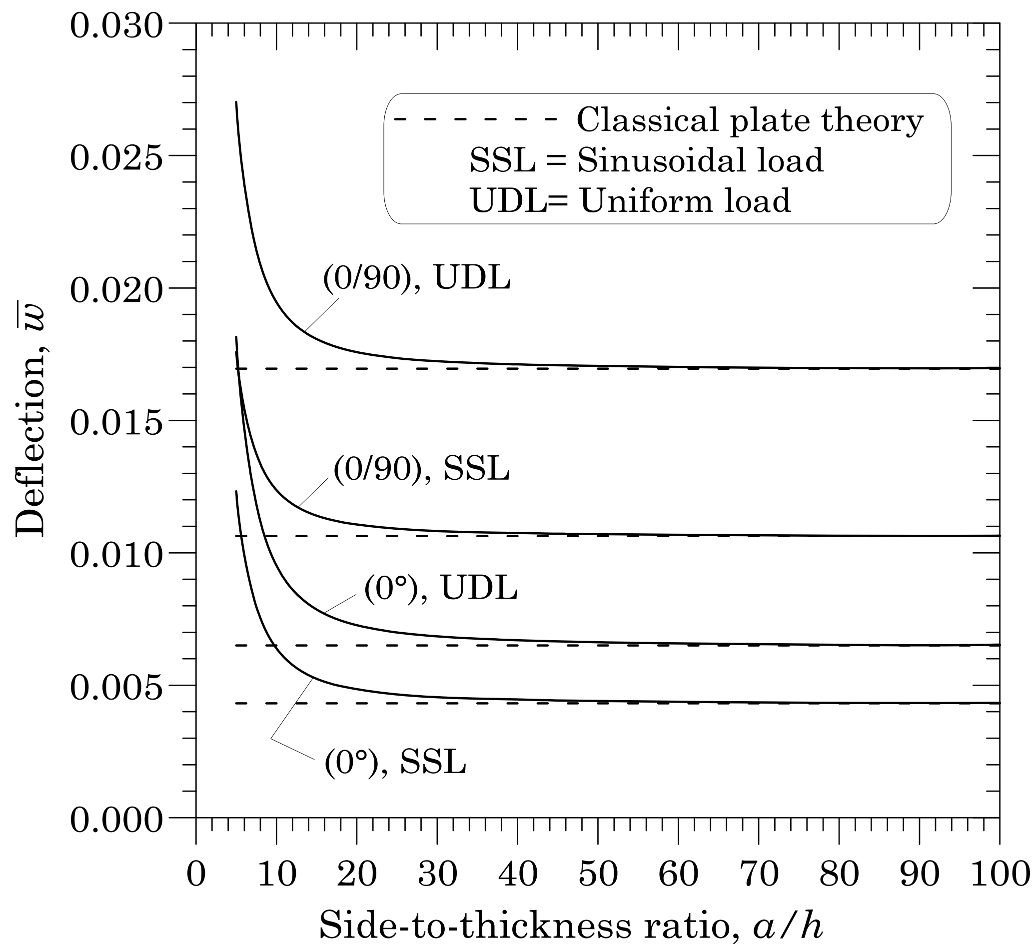


Effect of Shear Deformation on Bending Deformation

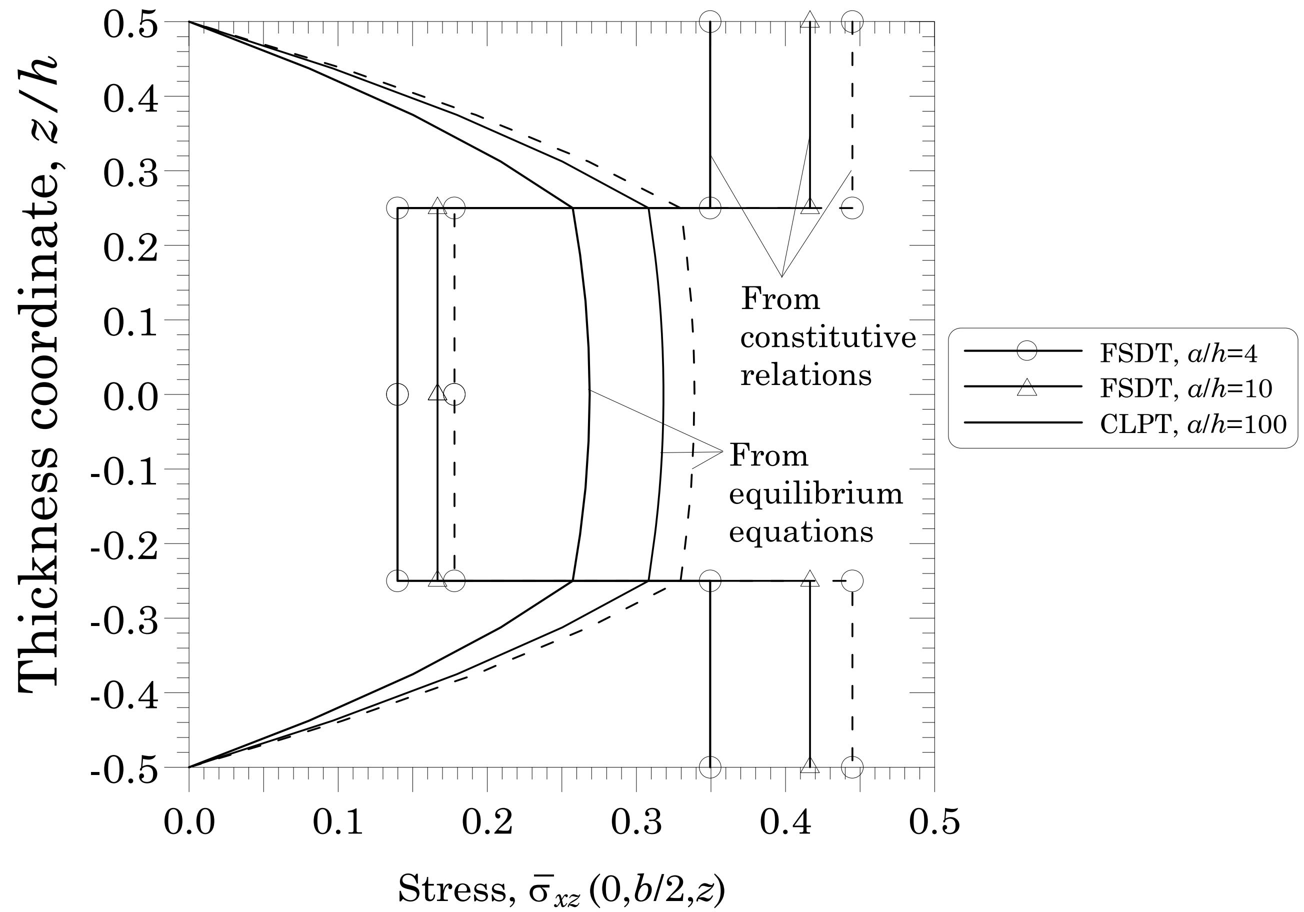
$$E_1 = 25E_2,$$
$$G_{12} = G_{13} = 0.5E_2,$$
$$G_{23} = 0.2E_2,$$
$$\nu_{12} = 0.25$$

$$\bar{w} = w\left(\frac{E_2 h^3}{q_0 a^4}\right)$$

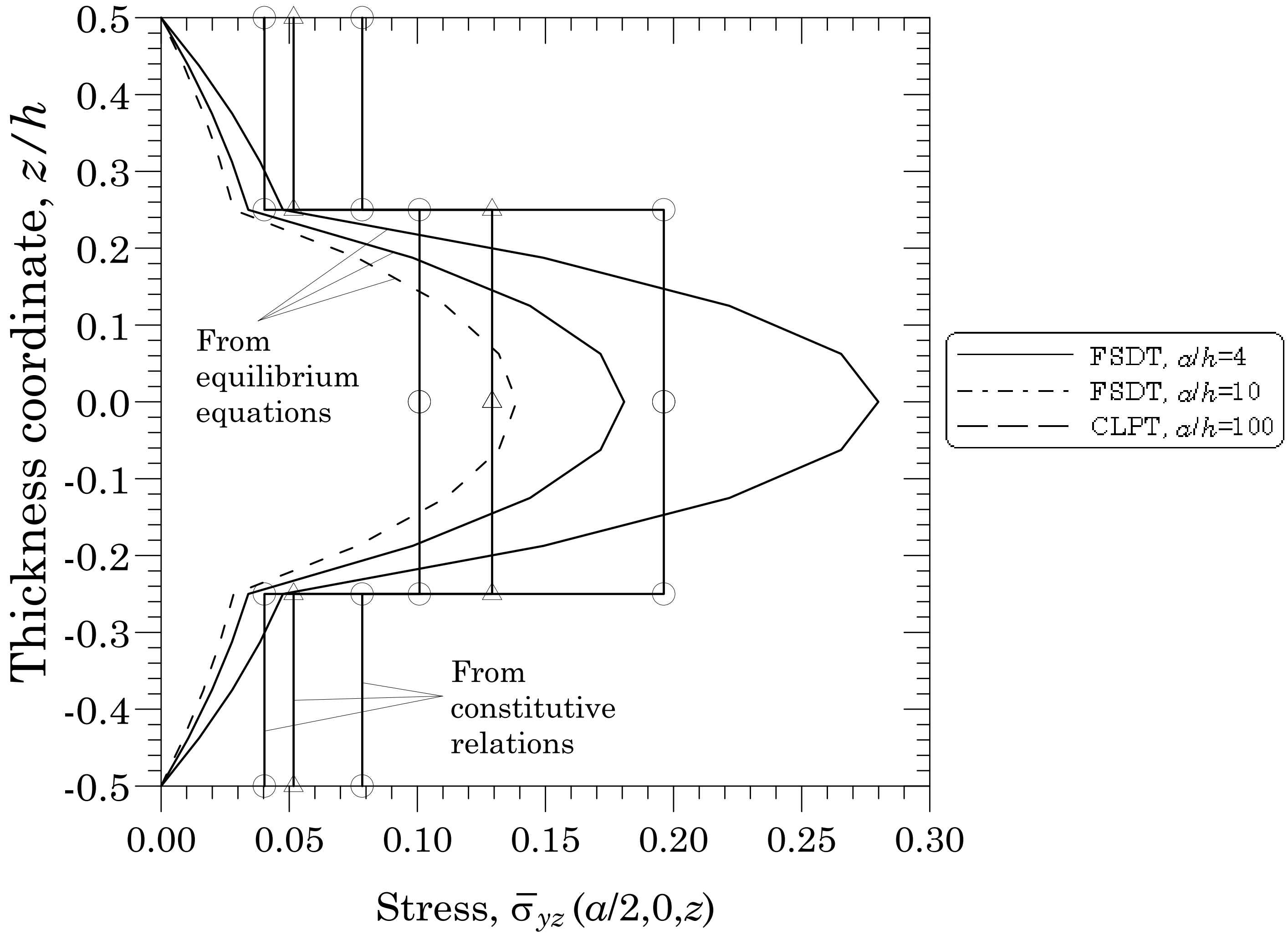
All laminates are of the same total thickness



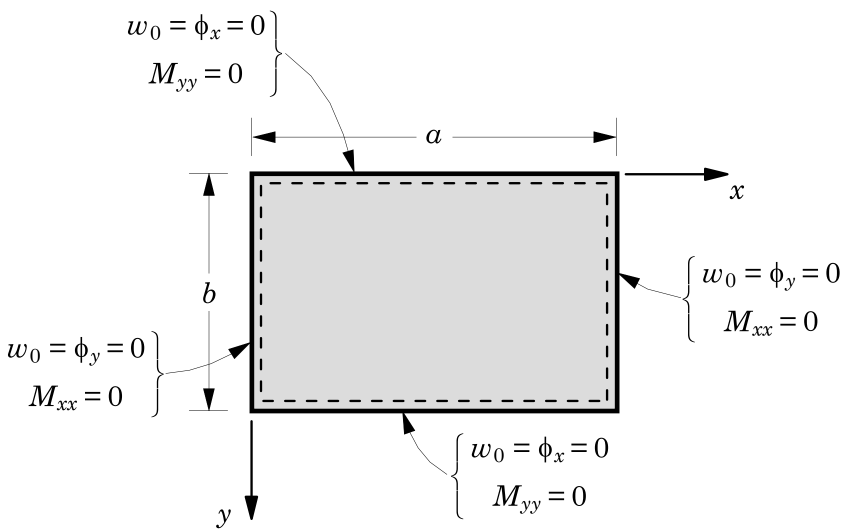
Transverse Shear Stresses (0/90/90/0)



Transverse Shear Stresses (0/90/90/0)



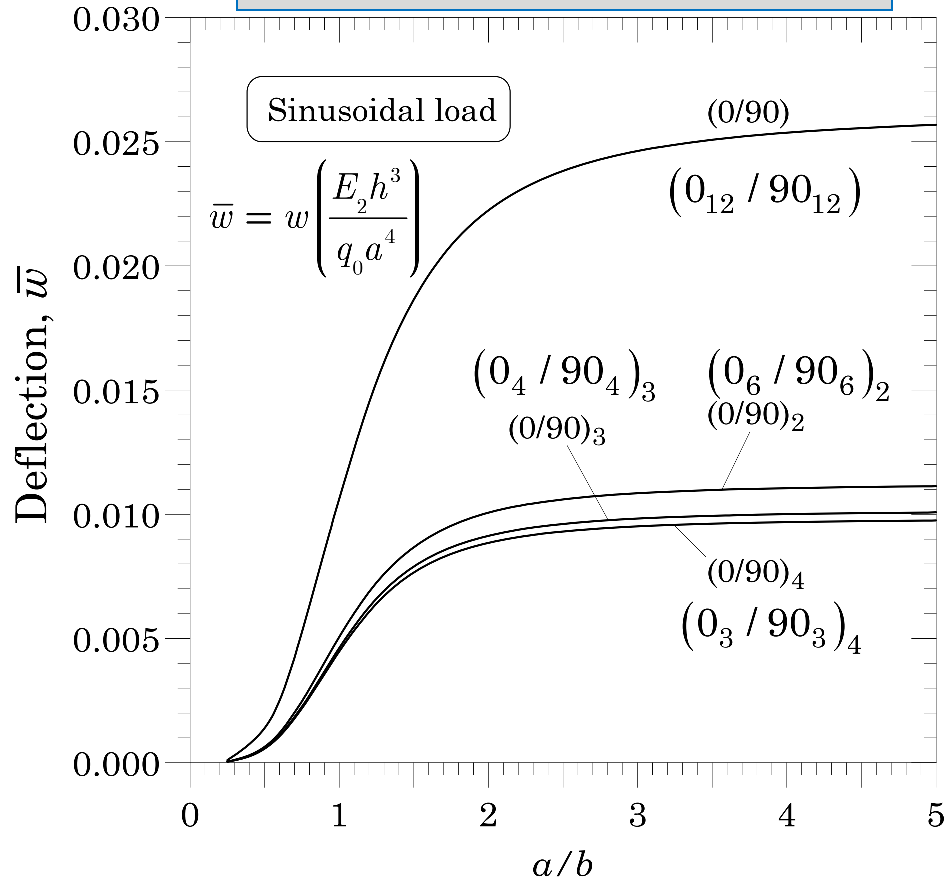
ANALYTICAL SOLUTIONS (continued)
BENDING OF CROSS-PLY PLATES -CLPT



SS-1 Boundary Conditions

$$E_1 = 25E_2,\ G_{12} = G_{13} = 0.5E_2,$$
$$G_{23} = 0.2E_2,\ \nu_{12} = 0.25$$

The *total thickness* of all laminates is the same



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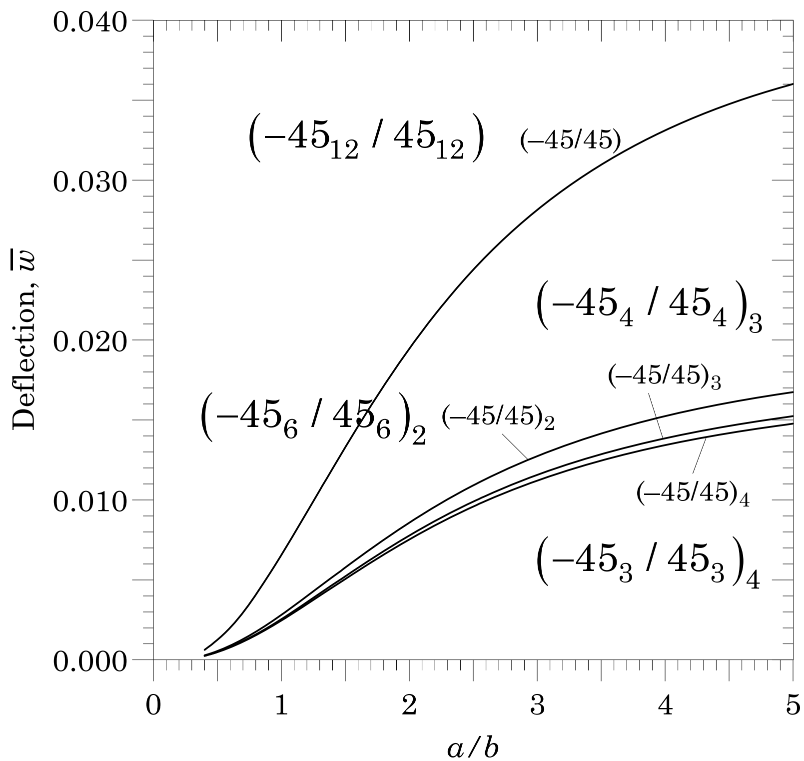
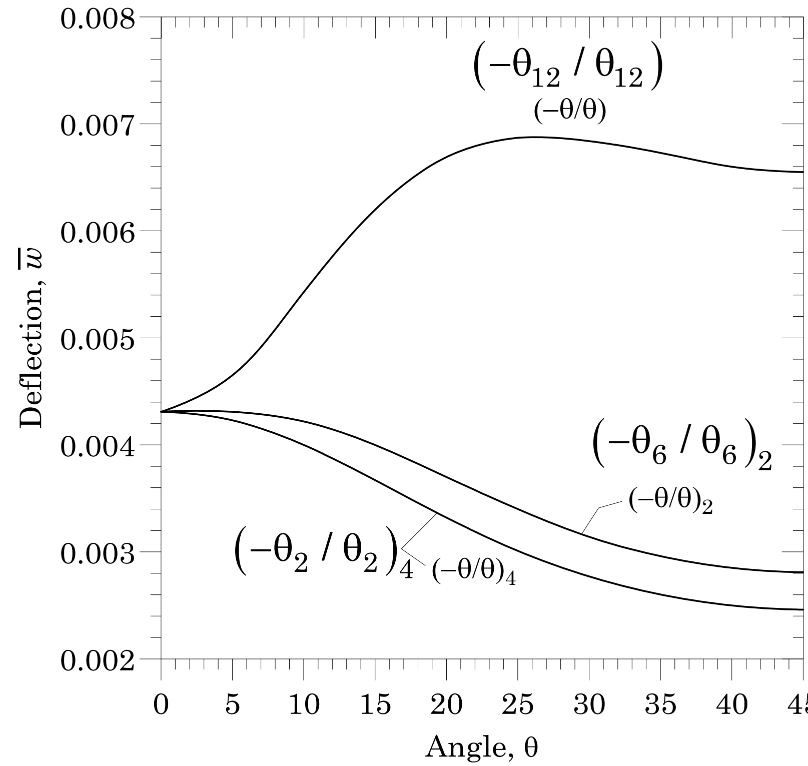
ANALYTICAL SOLUTIONS (continued)
BENDING OF ANGLE-PLY PLATES -CLPT

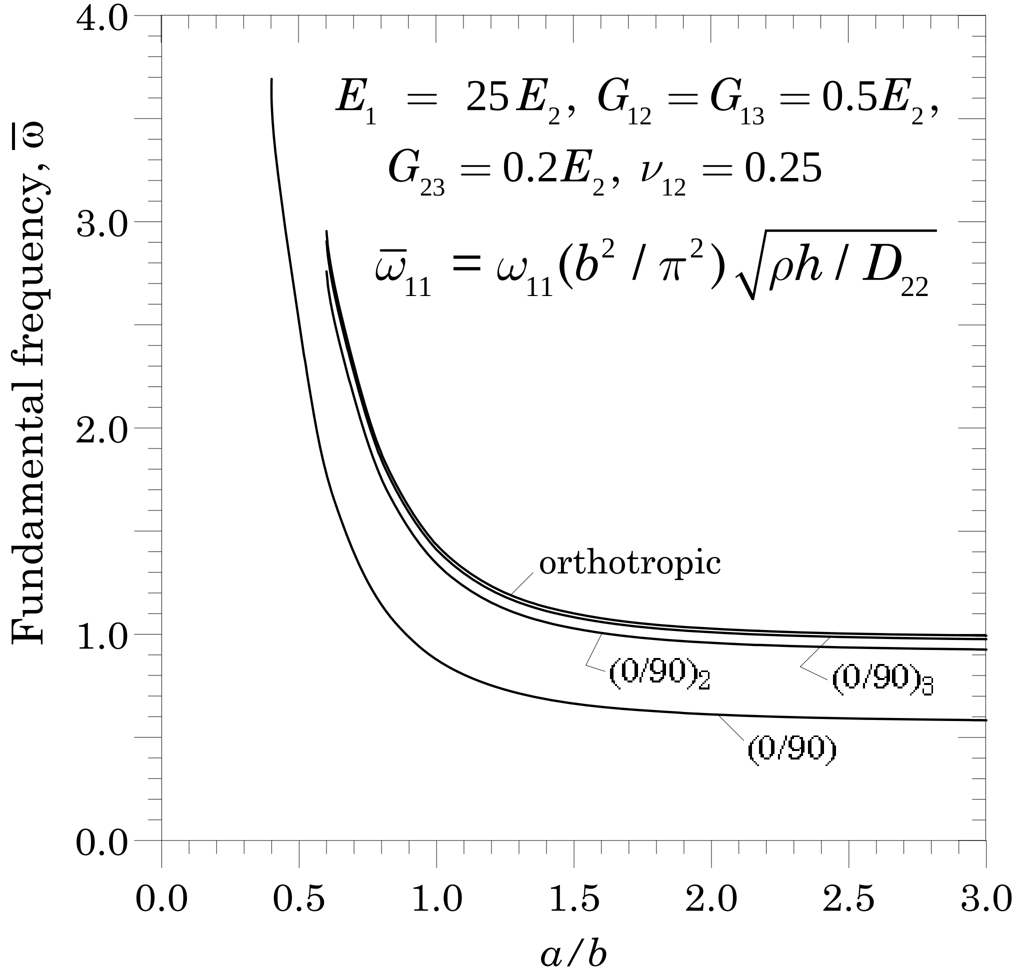




$$E_1 = 25E_2,\ G_{12} = G_{13} = 0.5E_2,$$
$$G_{23} = 0.2E_2,\ \nu_{12} = 0.25$$

$$\bar{w} = w\left(\frac{E_2 h^3}{q_0 a^4}\right)$$

ANALYTICAL SOLUTIONS (continued)
VIBRATION OF CROSS-PLY PLATES -CLPT



$$(0/90) = (0_{12}/90_{12})$$

$$(0/90)_2 = (0_6/90_6)_2$$

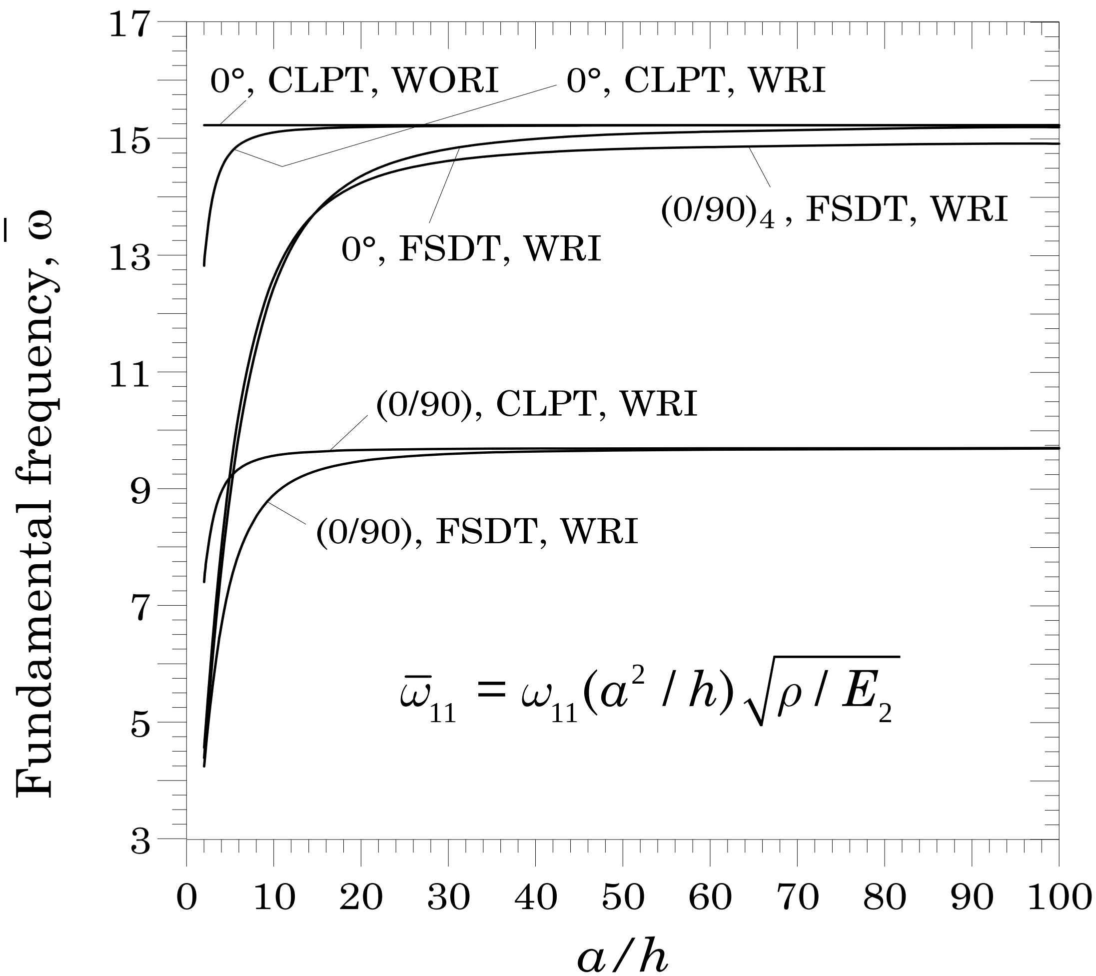
$$(0/90)_4 = (0_3/90_3)_4$$

$$(0/90)_3 = (0_4/90_4)_3$$

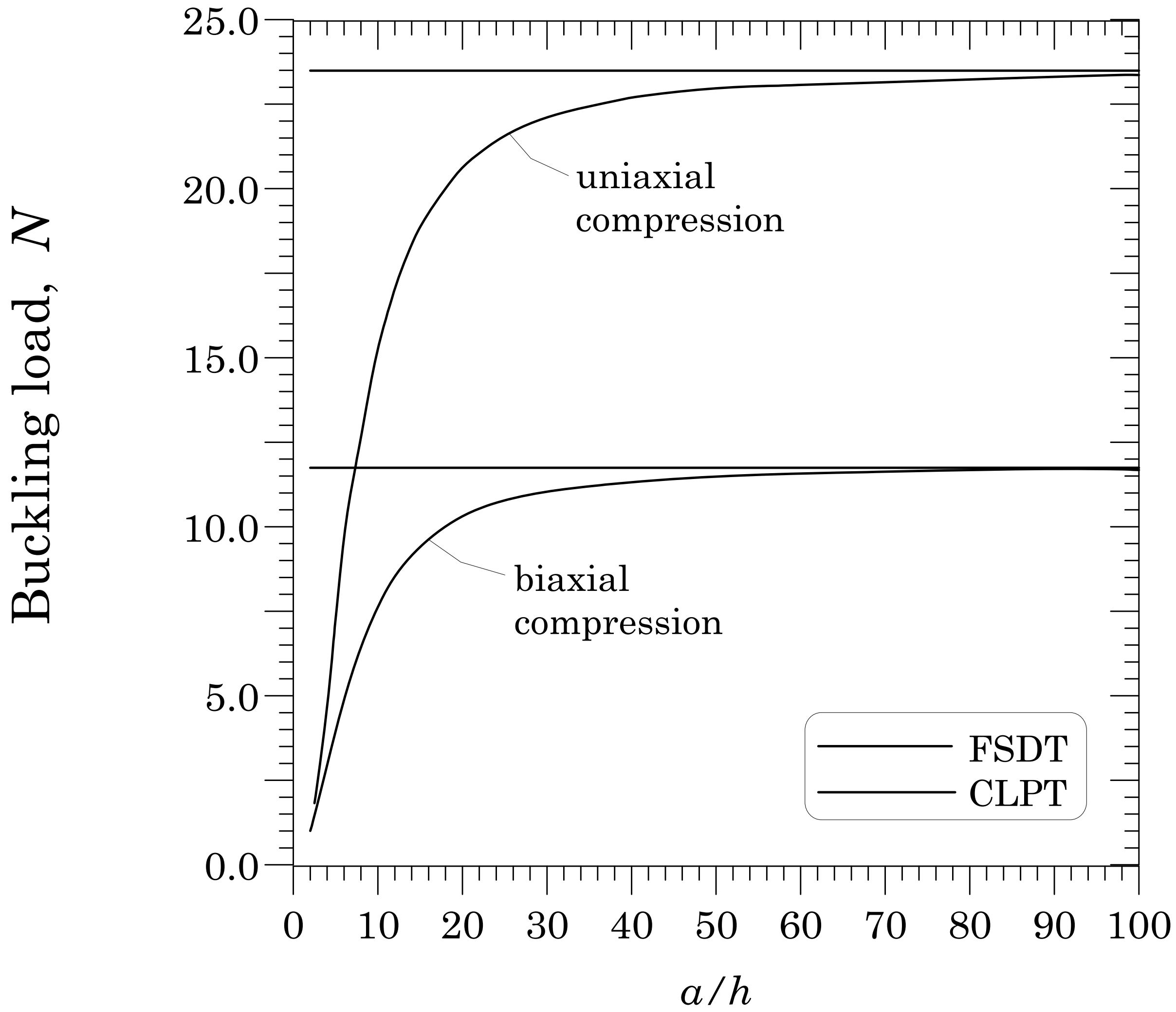
Effect of Shear Deformation on Vibration Frequencies

$$E_1 = 25E_2,$$
$$G_{12} = G_{13} = 0.5E_2,$$
$$G_{23} = 0.2E_2,$$
$$\nu_{12} = 0.25$$

All laminates are of the same total thickness



Effect of Shear Deformation on Buckling Loads

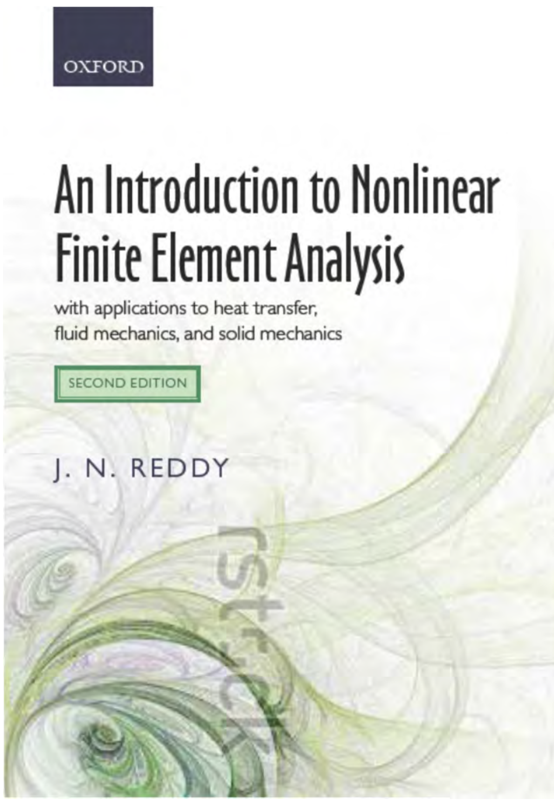


SOME REMARKS

- The effect of shear deformation is to: (a) increase deflections, (b) reduce buckling loads and natural frequencies. This is due to the fact that the classical plate theory, due to the restrictive kinematic assumptions made, over represents the stiffness of the structure.
- In practice, analytical or variational methods can be applied only to simple geometries circular and rectangular plates.
- The finite element method is the most commonly used method for the solution of plate and shell problems.

NONLINEAR ANALYSIS OF THE EULER-BERNOULLI BEAMS

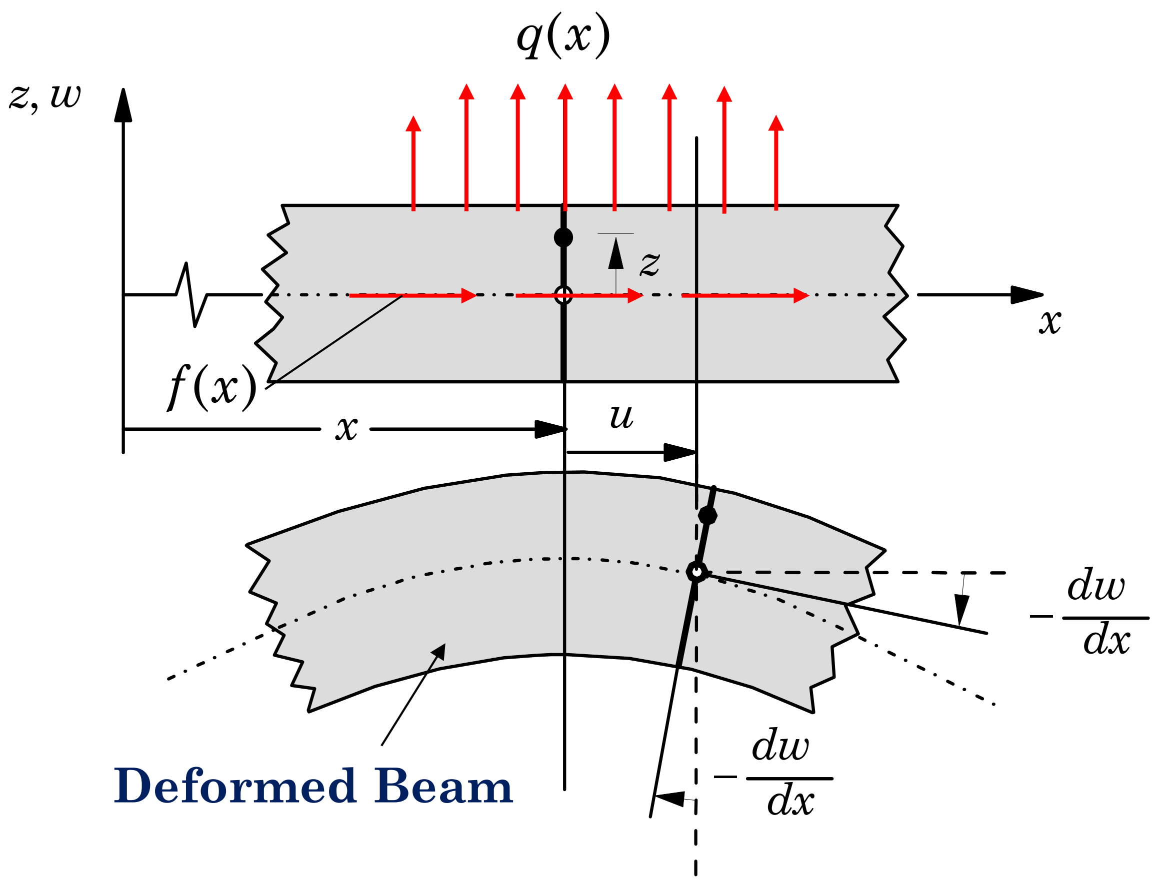
Chapter 5



CONTENTS

- Governing Equations
- Weak Forms
- Finite element models
- Numerical examples

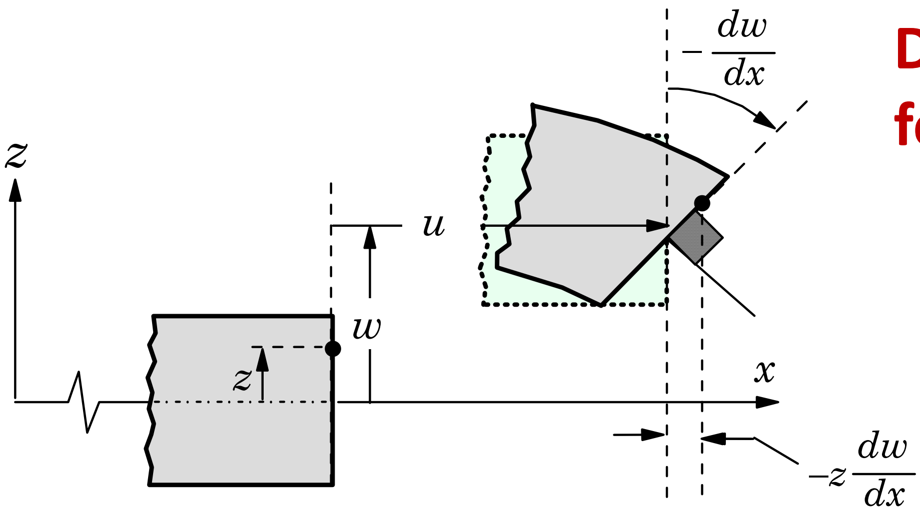
THE EULER-BERNOULLI BEAM THEORY (development of governing equations)



Undeformed Beam

Euler-Bernoulli Beam Theory (EBT)
Straightness, inextensibility, and normality

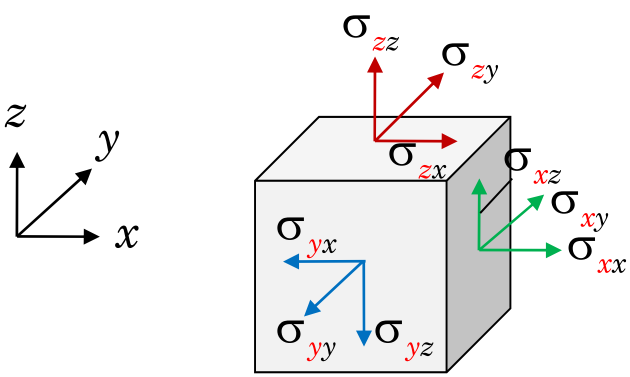
Kinematics of Deformation in the Euler-Bernoulli Beam Theory (EBT)



Displacement field (vector form)

$$\mathbf{u} = (u + z\theta_x)\hat{\mathbf{e}}_1 + w\hat{\mathbf{e}}_3,$$

$$\theta_x = -\frac{dw}{dx}$$



Notation for stress components

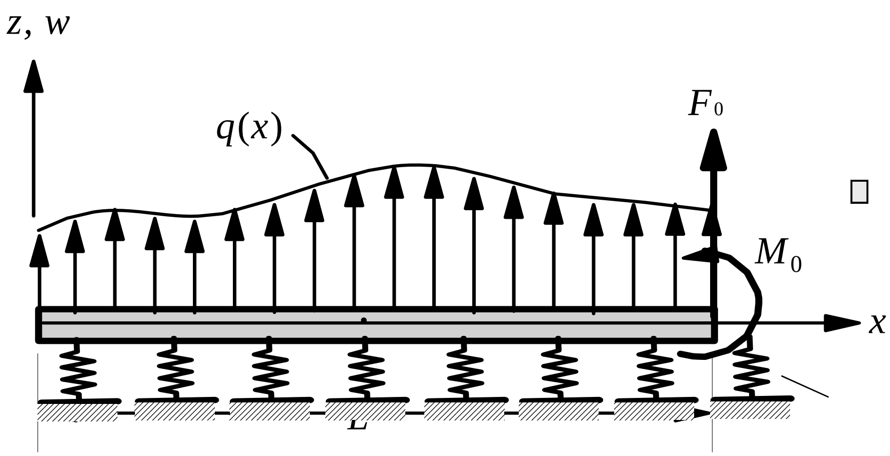
Displacement components

$$u_1(x,z) = u - z\frac{dw}{dx}$$

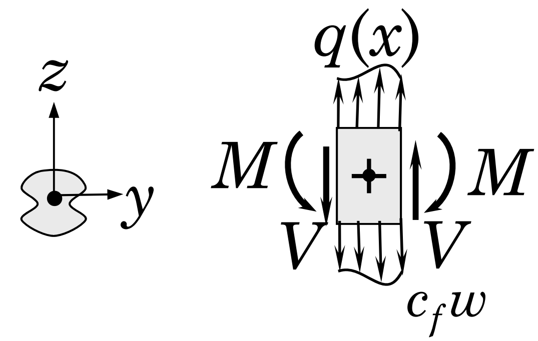
$$u_2 = 0,$$

$$u_3(x,z) = w(x)$$

NONLINEAR ANALYSIS OF EULER-BERNOULLI BEAMS



Beam cross section



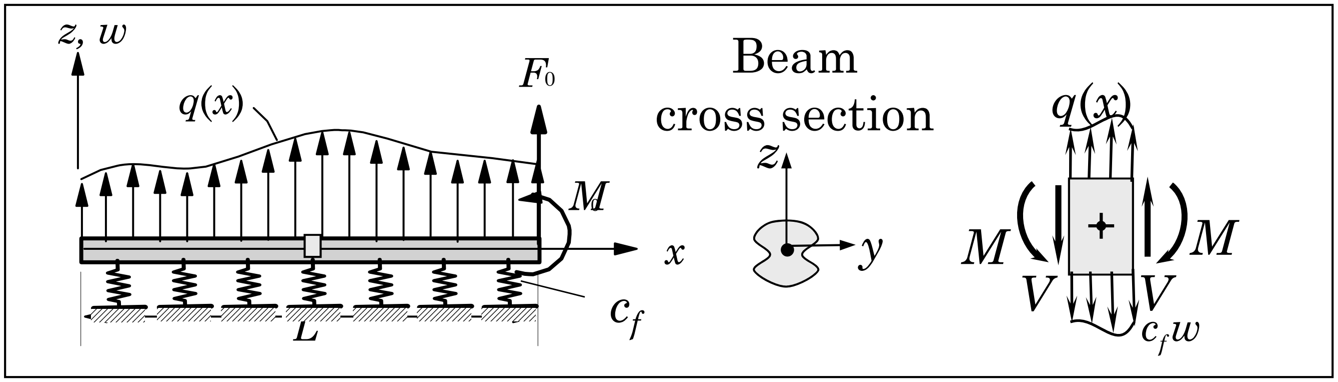
➢ **Displacements and strain-displacement relations**

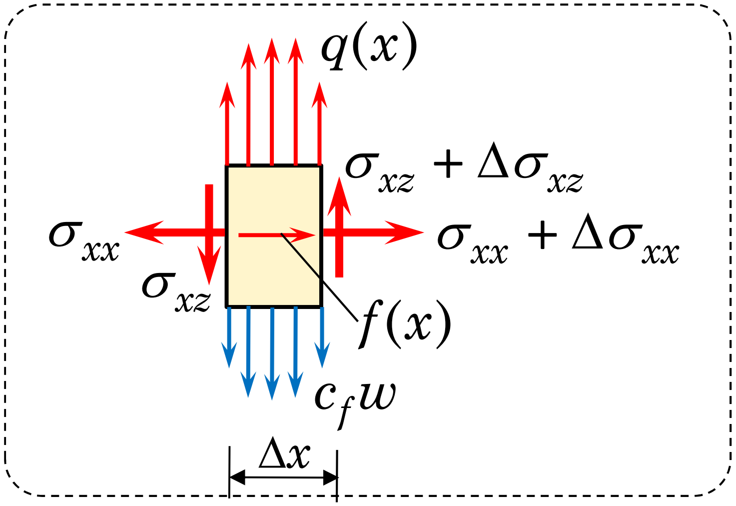
$$u_1(x,z) = u - z\frac{dw}{dx}, \quad u_2 = 0, \quad u_3 = w(x)$$

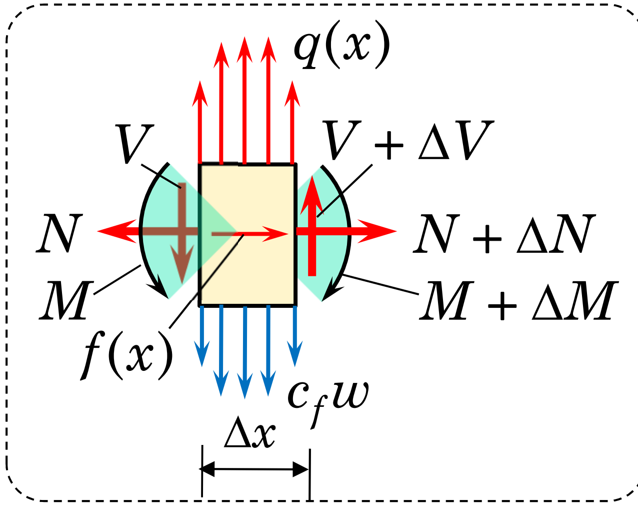
$$\varepsilon_{xx} = \frac{\partial u_1}{\partial x} + \frac{1}{2}\left(\frac{\partial u_3}{\partial x}\right)^2 = \frac{du}{dx} \boxed{+ \frac{1}{2}\left(\frac{dw}{dx}\right)^2} - z\frac{d^2w}{dx^2},$$

Nonlinear Problems (1-D) : 4

EQUILIBRIUM EQUATIONS







Definition of stress resultants

$$N = \int_A \sigma_{xx}\ dA,\ \ M = \int_A \sigma_{xx} \cdot z\ dA,\ \ V = \int_A \sigma_{xz}\ dA.$$

NONLINEAR ANALYSIS OF EULER-BERNOULLI BEAMS

Equilibrium equations

$$\frac{dN}{dx}+f=0,\quad \frac{dM}{dx}-V=0,\quad \frac{dV}{dx}+\frac{d}{dx}\left(N\frac{dw}{dx}\right)+q=0$$

$$\boxed{\frac{dN}{dx}+f=0,\qquad \frac{d^2M}{dx^2}+\frac{d}{dx}\left(N\frac{dw}{dx}\right)+q=0}$$

Stress resultants in terms of deflection

$$N=\int_A \sigma_{xx}\ dA=\int_A\left[E\left\{\frac{du}{dx}+\tfrac{1}{2}\left(\frac{dw}{dx}\right)^2\right\}-Ez\frac{d^2w}{dx^2}\right]dA\ =EA\left[\frac{du}{dx}+\tfrac{1}{2}\left(\frac{dw}{dx}\right)^2\right]$$

$$M=\int_A \sigma_{xx}\times z\ dA=\int_A\left[E\left\{\frac{du}{dx}+\tfrac{1}{2}\left(\frac{dw}{dx}\right)^2\right\}-Ez\frac{d^2w}{dx^2}\right]z\ dA=-EI\frac{d^2w}{dx^2}$$

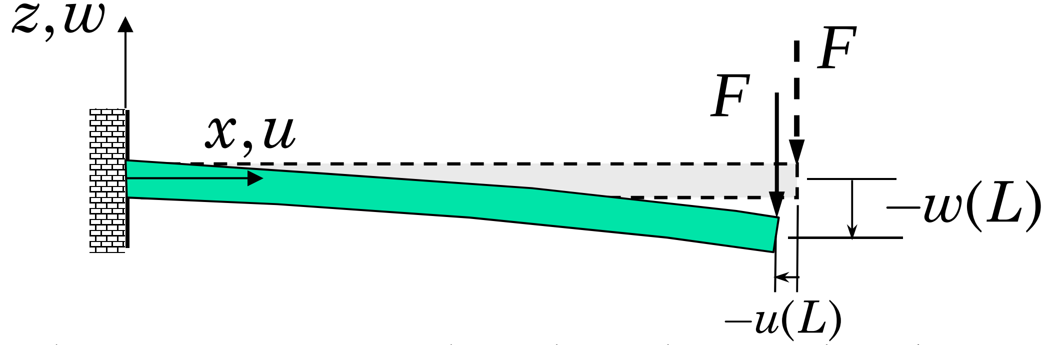
$$V=\frac{dM}{dx}=\frac{d}{dx}\left(-EI\frac{d^2w}{dx^2}\right)$$

NONLINEAR ANALYSIS OF EULER-BERNOULLI BEAMS

➢ **Equilibrium equations in terms of displacements (*u*,*w*)**

$$\frac{d}{dx}\left\{EA\left[\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]\right\}-f=0$$

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)-\frac{d}{dx}\left(\frac{dw}{dx}EA\left[\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]\right)-q=0$$



➢ Clearly, transverse load induces both axial displacement u and transverse displacement w.

EULER-BERNOULLI BEAM THEORY (continued)

➢ Weak forms

$$0=\int_{x_a}^{x_b} v_1\left(-\frac{dN}{dx}-f\right)dx=\int_{x_a}^{x_b}\left(\frac{dv_1}{dx}N-v_1 f\right)dx-v_1(x_a)[-N(x_a)]-v_1(x_b)N(x_b)$$

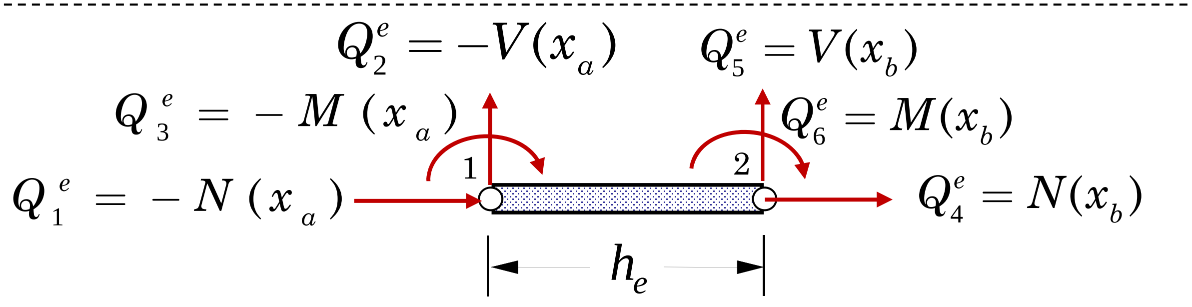
$$=\int_{x_a}^{x_b}\left(\frac{dv_1}{dx}N-v_1 f\right)dx-v_1(x_a)Q_1-v_1(x_b)Q_4$$

$$N=EA\left[\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]$$

$$M=-EI\frac{d^2w}{dx^2}$$

$$0=\int_{x_a}^{x_b} v_2\left[-\frac{d^2M}{dx^2}-\frac{d}{dx}\left(N\frac{dw}{dx}\right)-q\right]dx$$

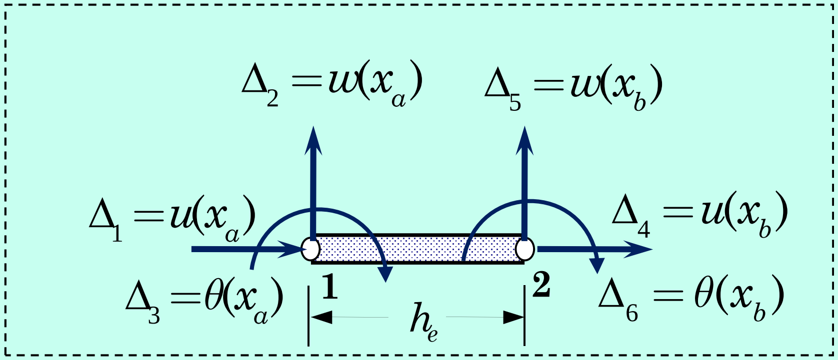
$$=\int_{x_a}^{x_b}\left[EI\frac{d^2v_2}{dx^2}\frac{d^2w}{dx^2}+\frac{dv_2}{dx}\left(N\frac{dw}{dx}\right)-v_2 q\right]dx-v_2(x_a)Q_2-\left(-\frac{dv_2}{dx}\right)_{x_a}Q_3-v_2(x_b)Q_5-\left(-\frac{dv_2}{dx}\right)_{x_b}Q_6$$



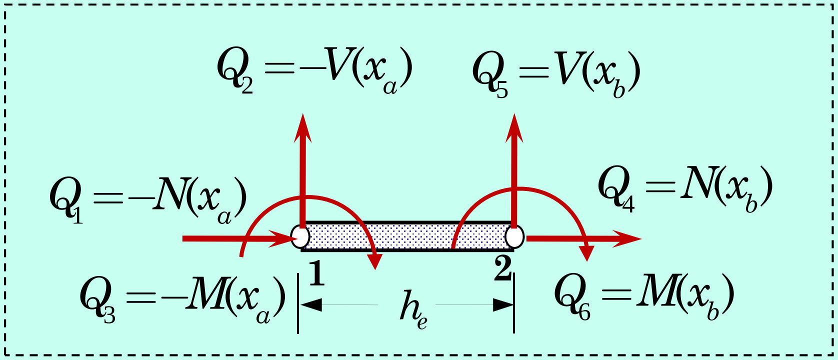
BEAM ELEMENT DEGREES OF FREEDOM

(notation for the displacement and force degrees of freedom)

Generalized displacements



Generalized forces



FINITE ELEMENT APPROXIMATION

Primary variables (serve as the nodal variables that must be continuous across elements) $u,\, w,\; \theta = -\dfrac{dw}{dx}$

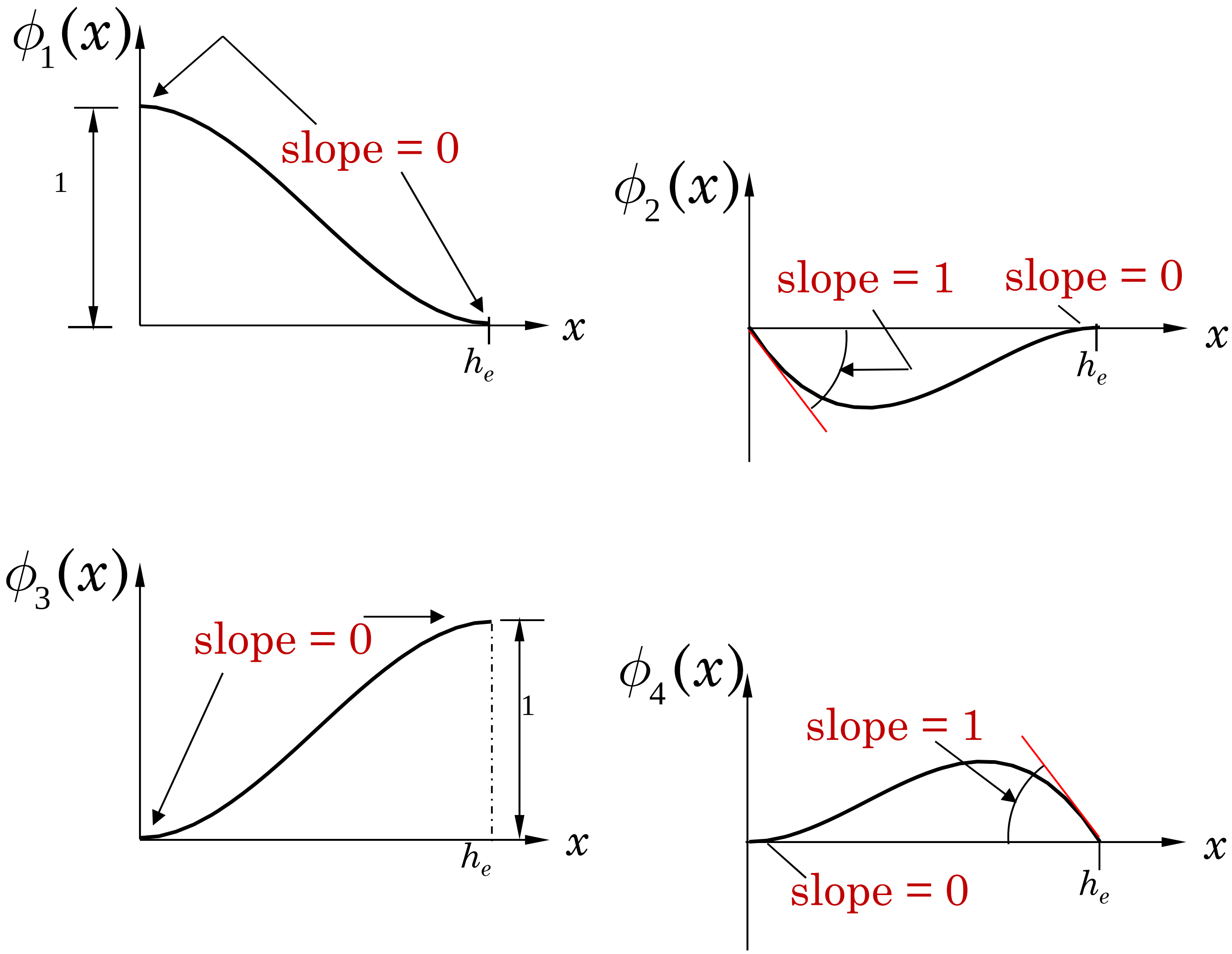
$$w(x) \approx \sum_{j=1}^{4} \bar{\Delta}_j \phi_j(x), \quad u(x) \approx \sum_{j=1}^{n} u_j \psi_j(x),$$

Hermite cubic interpolation functions

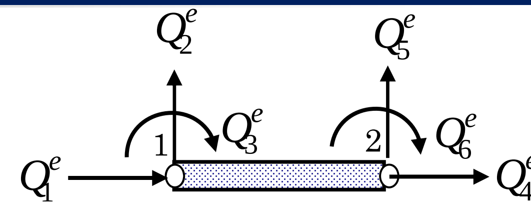
$$\phi_1(\bar{x}) = 1 - 3\left(\frac{\bar{x}}{h}\right)^2 + 2\left(\frac{\bar{x}}{h}\right)^3, \quad \phi_2(\bar{x}) = -\bar{x}\left(1 - \frac{\bar{x}}{h}\right)^2$$

$$\phi_3(\bar{x}) = 3\left(\frac{\bar{x}}{h}\right)^2 - 2\left(\frac{\bar{x}}{h}\right)^3, \qquad \phi_4(\bar{x}) = -\bar{x}\left[\left(\frac{\bar{x}}{h}\right)^2 - \frac{\bar{x}}{h}\right]$$

HERMITE CUBIC INTERPOLATION FUNCTIONS $\phi_i(x)$

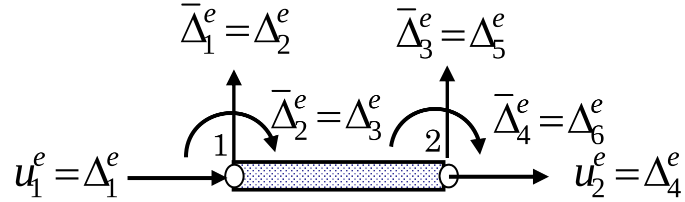


FINITE ELEMENT MODEL



➢ Finite Element Equations

$$u(x) \approx \sum_{j=1}^{2} u_j \psi_j(x), \quad w(x) \approx \sum_{j=1}^{4} \bar{\Delta}_j \phi_j(x),$$



$$\begin{bmatrix} [K^{11}] & [K^{12}] \\ [K^{21}] & [K^{22}] \end{bmatrix} \begin{Bmatrix} \{u\} \\ \{\bar{\Delta}\} \end{Bmatrix} = \begin{Bmatrix} \{F^1\} \\ \{F^2\} \end{Bmatrix}$$

$$K_{ij}^{11} = \int_{x_a}^{x_b} EA \frac{d\psi_i}{dx} \frac{d\psi_j}{dx} dx, \quad K_{ij}^{12} = \boxed{\frac{1}{2} \int_{x_a}^{x_b} EA \frac{dw}{dx} \frac{d\psi_i}{dx} \frac{d\phi_j}{dx} dx,}$$

$$K_{ij}^{21} \boxed{= \int_{x_a}^{x_b} EA \frac{dw}{dx} \frac{d\phi_i}{dx} \frac{d\psi_j}{dx} dx,} \quad F_i^1 = \int_{x_a}^{x_b} f \psi_i \, dx + \psi_i(x_a) Q_1 + \psi_i(x_b) Q_4$$

$$K_{ij}^{22} = \int_{x_a}^{x_b} EI \frac{d^2\phi_i}{dx^2} \frac{d^2\phi_j}{dx^2} dx + \boxed{\int_{x_a}^{x_b} EA \left(\frac{dw}{dx}\right)^2 \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} dx,}$$

$$F_i^2 = \int_{x_a}^{x_b} q \psi_i \, dx + \phi_i(x_a) Q_2 + \phi_i(x_b) Q_5 + \left(-\frac{d\phi_i}{dx}\right)_{x_a} Q_3 + \left(-\frac{d\phi_i}{dx}\right)_{x_b} Q_6$$

MEMBRANE LOCKING

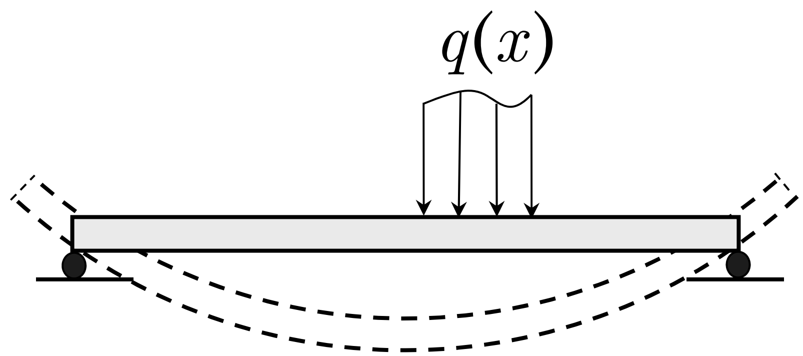
Membrane strain

$$\varepsilon_{xx}^{0} = \frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2$$

$$\varepsilon_{xx}^{0} = \frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 = 0$$

$$\Rightarrow \quad \frac{du}{dx} = -\frac{1}{2}\left(\frac{dw}{dx}\right)^2$$

Beam on roller supports



Remedy

$$\Rightarrow \quad \text{make } \left(\frac{dw}{dx}\right)^2 \text{ to behave like a constant}$$

SOLUTION OF EQUATIONS – A REVIEW
Direct Iteration

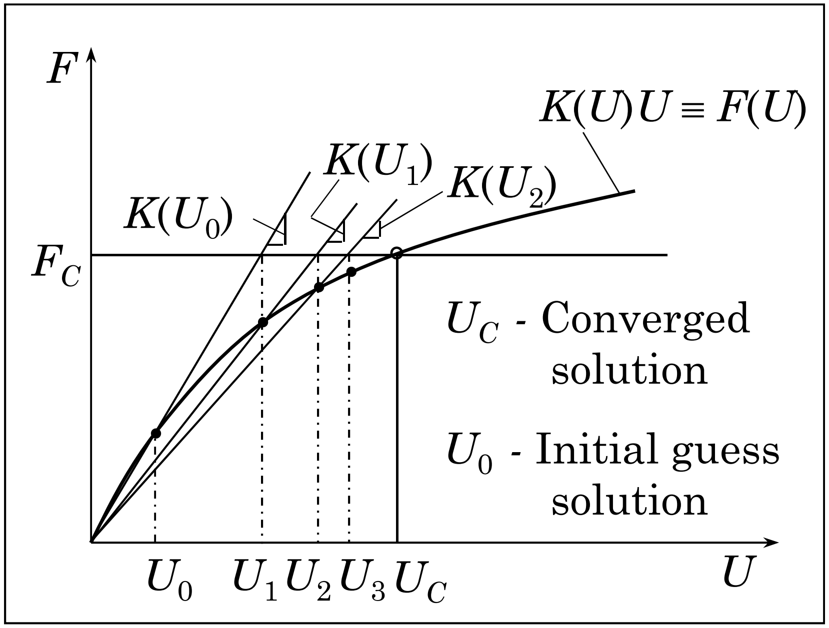
Non-Linear Finite Element Model

$$[K^e(\Delta^e)]\{\Delta^e\} = \{F^e\} \Rightarrow \text{assembled } \; [K(U)]\{U\} = \{F\}$$

Direct Iteration Method

Solution $\{U\}^r$ at r^{th} iteration is known and solve for $\{U\}^{r+1}$

$$\boxed{[K(\{U\}^r)]\{U\}^{r+1} = \{F\}}$$



ERROR CHECK – A REVIEW
Direct Iteration

Direct Iteration Method

Solution $\{U\}^r$ at r^{th} iteration is known and solve for $\{U\}^{r+1}$

$$[K(\{U\}^r)]\{U\}^{r+1} = \{F\}$$

Convergence Criterion

$$\varepsilon = \sqrt{\frac{\sum_{I=1}^{NEQ} \left(U_I^r - U_I^{r+1}\right)^2}{\sum_{I=1}^{NEQ} \left(U_I^{r+1}\right)^2}} \leq \text{specified tolerance}$$

SOLUTION OF EQUATIONS – A REVIEW
Newton Iteration

Taylor's series

$$\text{Residual,}\quad \{R\} \equiv [K(\{U\}^r)]\{U\}^{r+1} - \{F\}^r$$

$$\{R(U^{r+1})\} = \{R(U^r)\} + (U^{r+1} - U^r)\left[\frac{\partial R}{\partial U}\right]^r + \frac{1}{2!}(U^{r+1} - U^r)^2\left[\frac{\partial^2 R}{\partial U^2}\right]^r + \cdots$$

$$\approx \{R(U^r)\} + (U^{r+1} - U^r)\left[\frac{\partial R}{\partial U}\right]^r + O(\delta U)^2, \quad \boxed{\delta U = U^{r+1} - U^r}$$

Requiring the residual $\{R\}^{r+1}$ to be zero at the $r+1^{\text{st}}$ iteration, we have

$$\boxed{[K^{\tan}(\{U\}^r)]\{\delta U\} = -\{R\}^r = \{F\}^r - [K(U^r)]^r\{U\}^r}$$

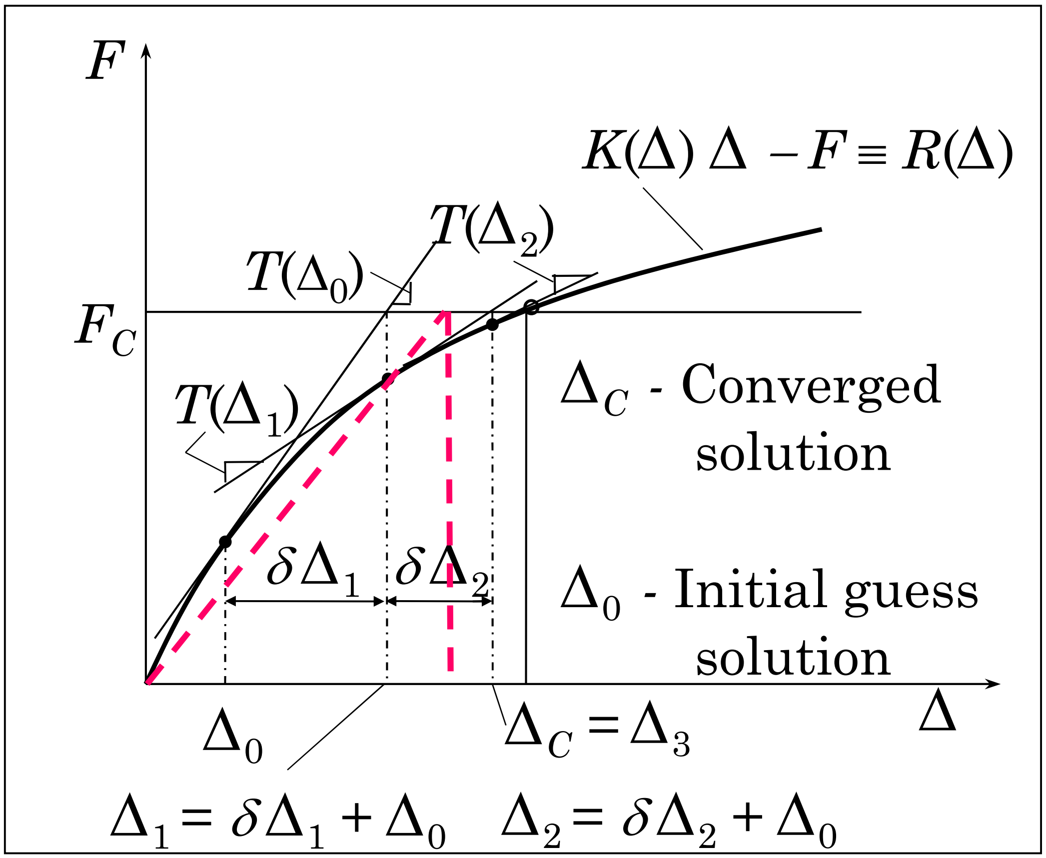
The tangent matrix at the element level is

$$\left(K_{ij}^{\alpha\beta}\right)^{\tan} = \frac{\partial R_i^\alpha}{\partial \Delta_j^\beta} = \frac{\partial}{\partial \Delta_j^\beta}\left(\sum_{\gamma=1}^{2}\sum_{p=1}^{n} K_{ip}^{\alpha\lambda}\Delta_p^\gamma - F_i^\alpha\right)$$

SOLUTION OF EQUATIONS – A REVIEW
Newton Iteration

$$T_{ij}^{\alpha\beta} = \frac{\partial R_i^{\alpha}}{\partial \Delta_j^{\beta}} = \frac{\partial}{\partial \Delta_j^{\beta}}\left(\sum_{\gamma=1}^{2}\sum_{p=1}^{n_\beta} K_{ip}^{\alpha\gamma}\Delta_p^{\gamma} - F_i^{\alpha}\right) = K_{ij}^{\alpha\beta} + \sum_{\gamma=1}^{2}\sum_{p=1}^{n}\frac{\partial K_{ip}^{\alpha\gamma}}{\partial \Delta_j^{\beta}}\Delta_p^{\gamma} \equiv T_{ij}^{\alpha\beta}$$

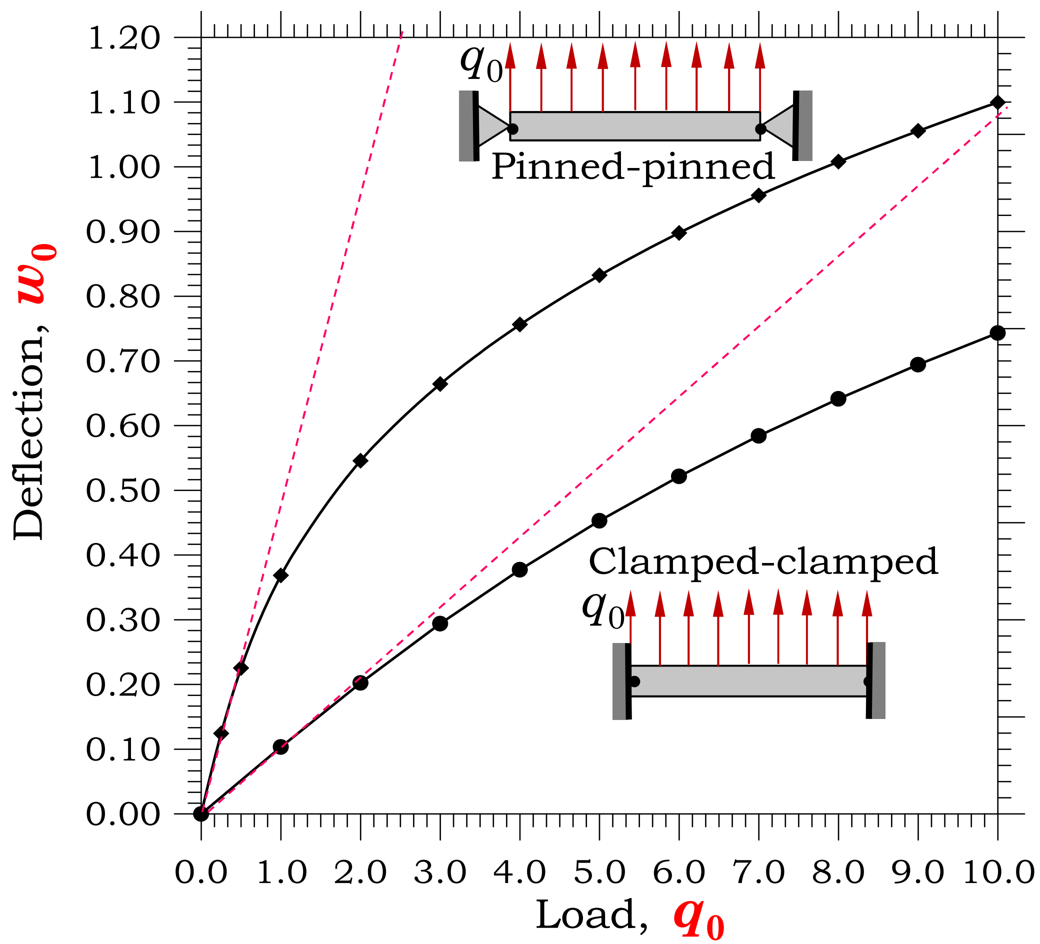
$$[T(\{\Delta\}^r)]\{\delta\Delta\} = \{F\}^r - [K(\Delta^r)]^r\{\Delta\}^r, \quad \{\Delta\}^{r+1} = \{\Delta\}^r + \{\delta\Delta\}$$



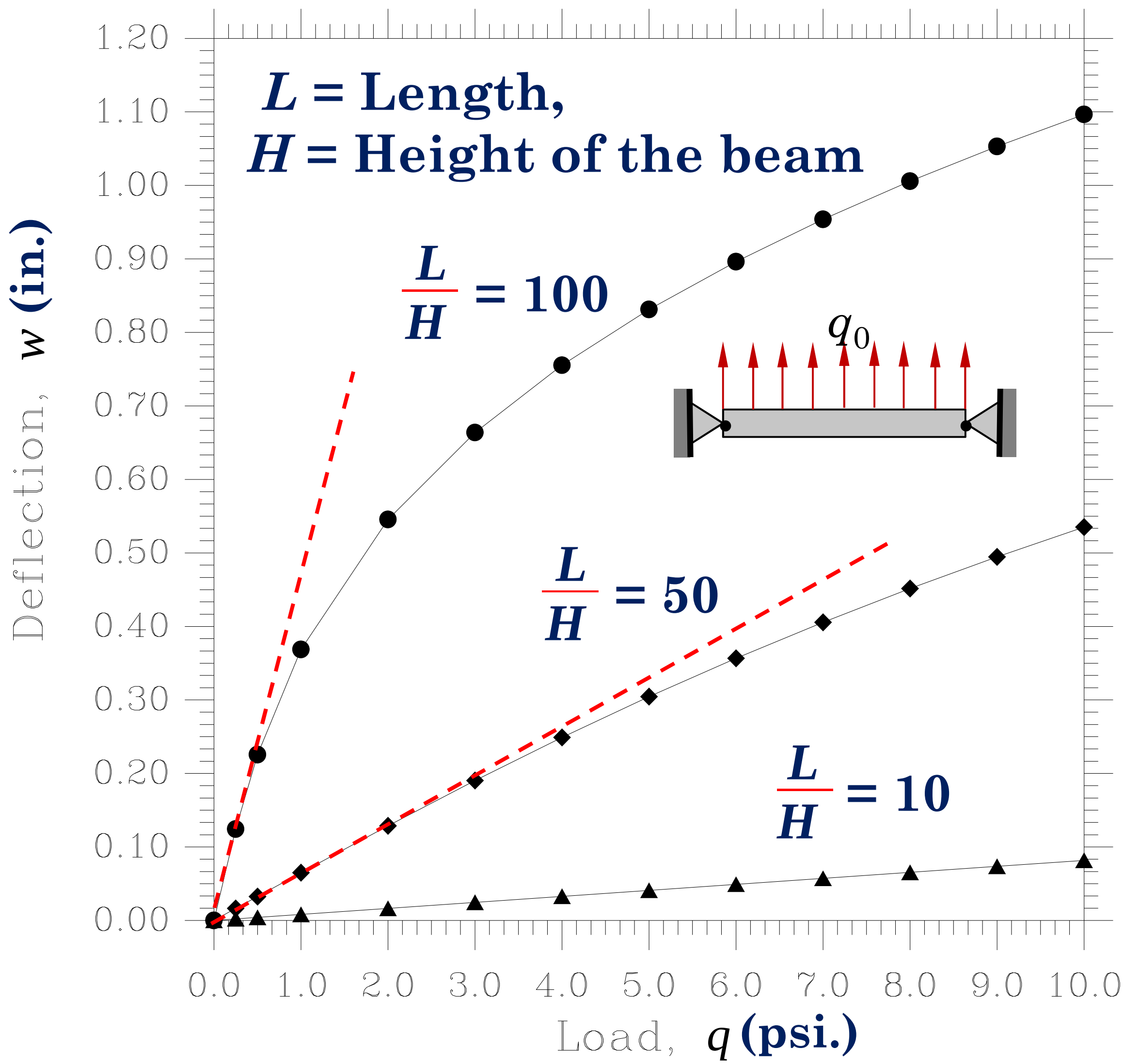
Nonlinear Problems: (1-D) - 17

NUMERICAL EXAMPLES

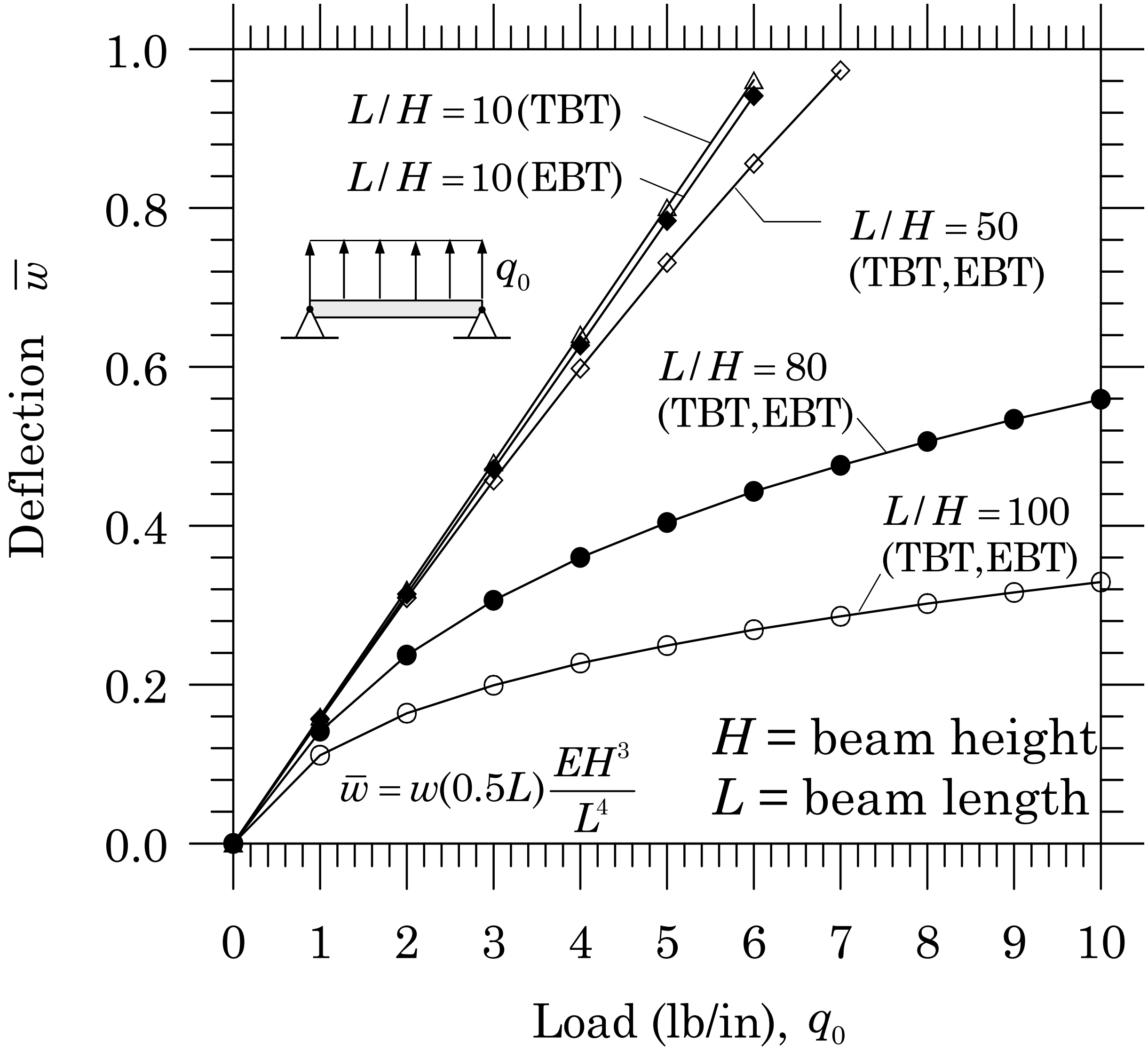
Pinned-pinned beam (EBT)



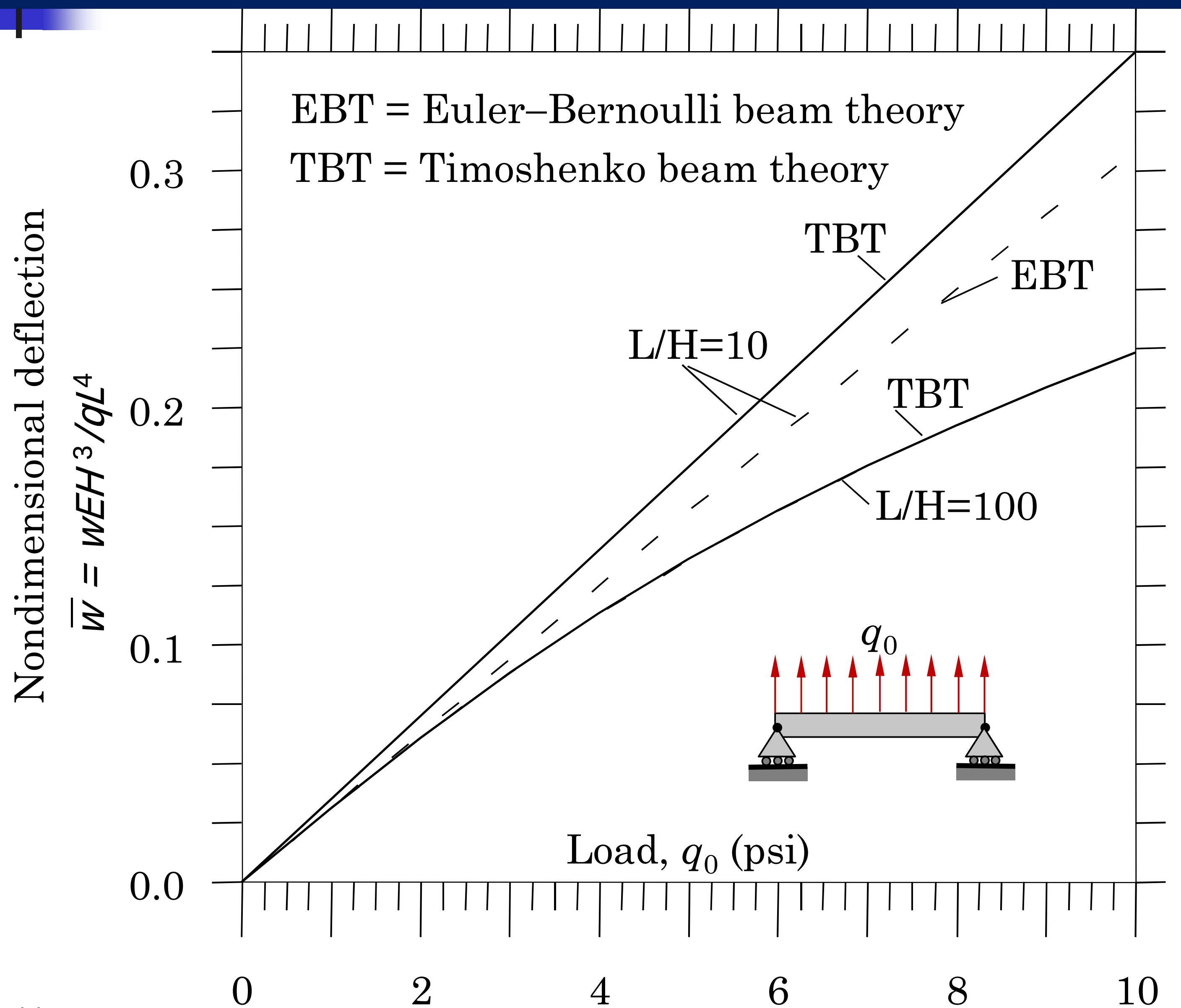
Pinned-pinned beam (TBT)



Pinned-pinned beam (EBT, TBT)



Hinged-Hinged beam (EBT and TBT)

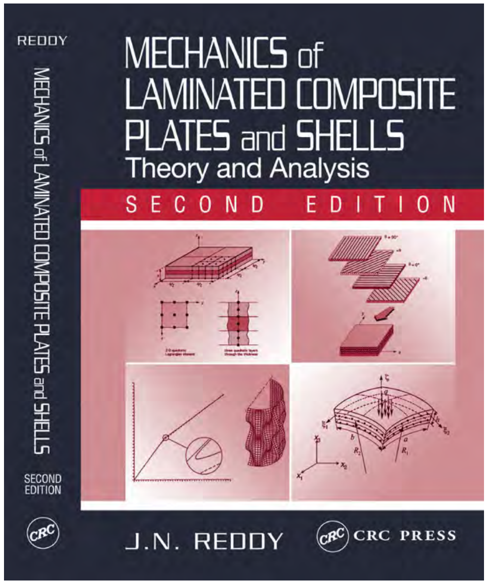


SUMMARY OF THE LECTURE

In this lecture we have covered the following topics:

- **Derived the governing equations of the** Euler-Bernoulli beam theory
- **Developed Weak forms of the EBT**
- **Developed Finite element models of EBT and TBT**
- **Discussed** membrane locking **(due to the geometric nonlinearity)**
- **Discussed examples**

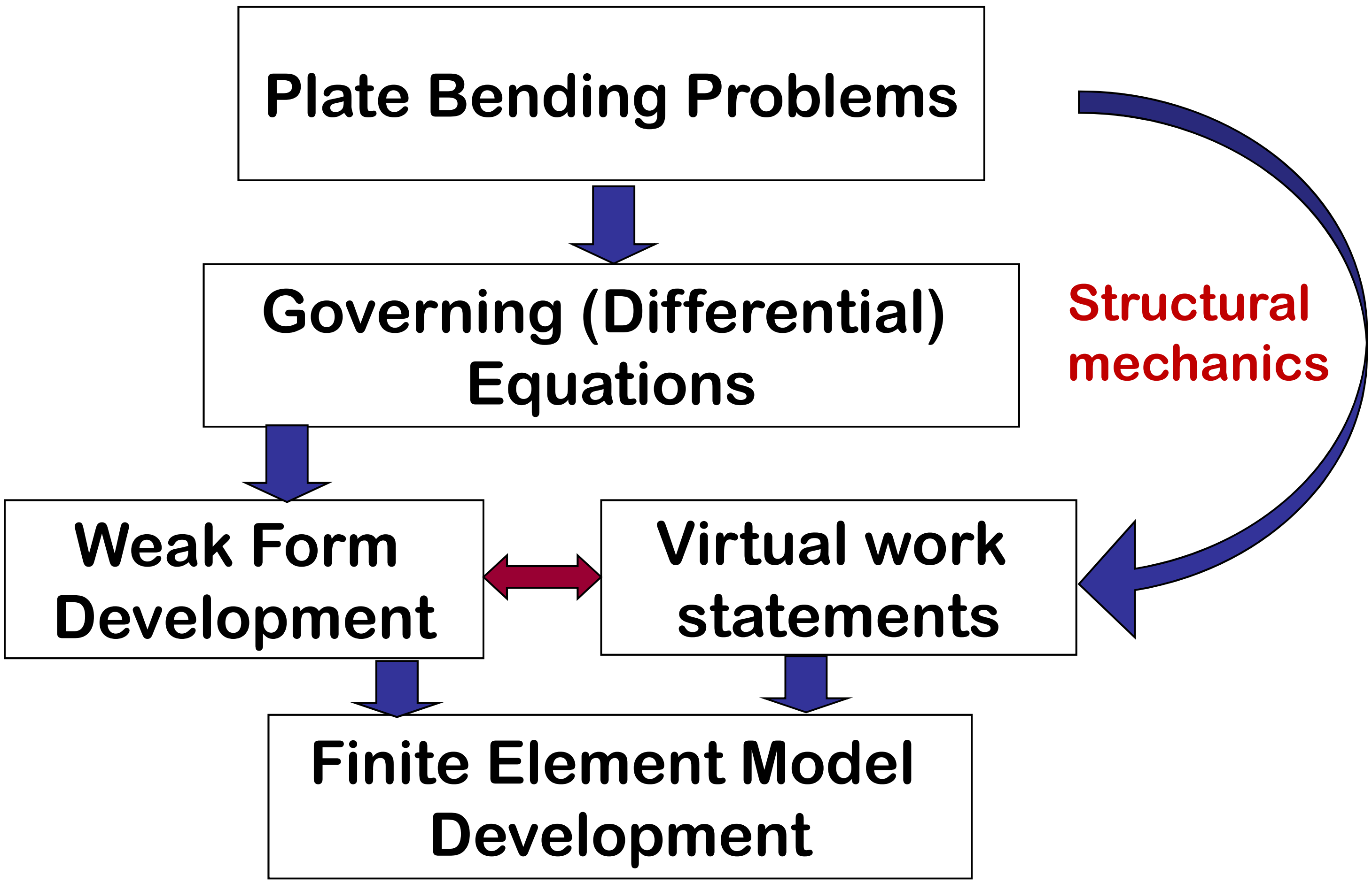
ANALYSIS OF LAMINATED PLATES (Classical and First-Order Plate Theories)



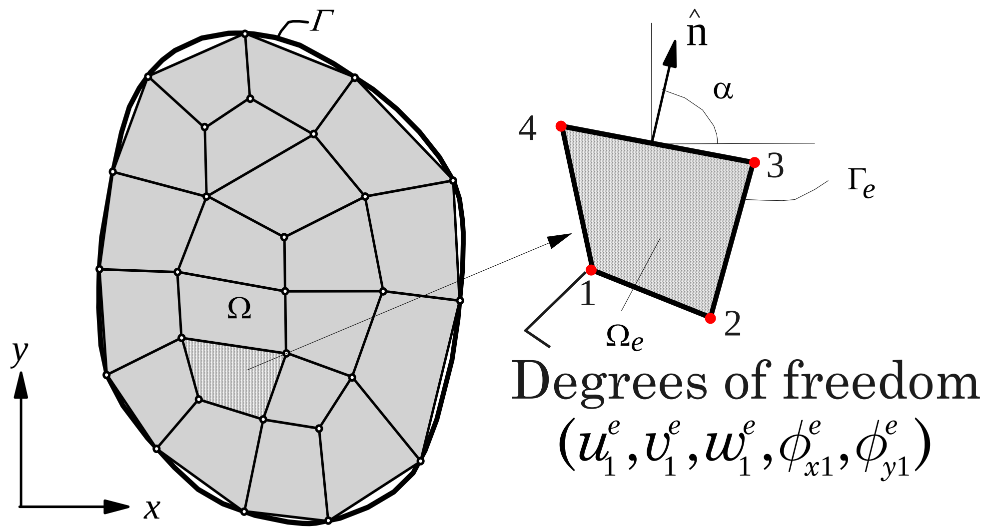
CONTENTS OF THE LECTURE

- Major steps of the finite element analysis
- Weak forms from the principle of virtual displacements
- Weak forms from governing equations
- Finite element model development
- Solution of nonlinear equations
- Numerical results

Major Steps of Finite Element Model Development



Finite Element Discretization



$$(u_1^e, v_1^e, w_1^e, \phi_{x1}^e, \phi_{y1}^e)$$

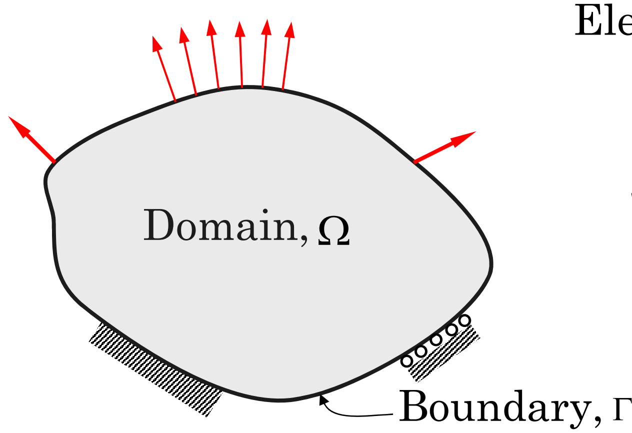
1. Spatial approximation (semidiscretization)

$$\mathbf{M}\ddot{\Delta} + \mathbf{K}(\mathbf{w})\Delta = \mathbf{F}$$

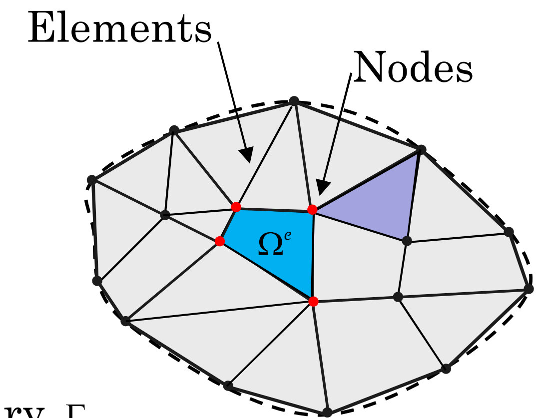
2. Time approximation (full discretization)

$$\hat{\mathbf{K}}(\mathbf{w}_{s+1})\,\Delta_{s+1} = \hat{\mathbf{F}}_{s,s+1}$$

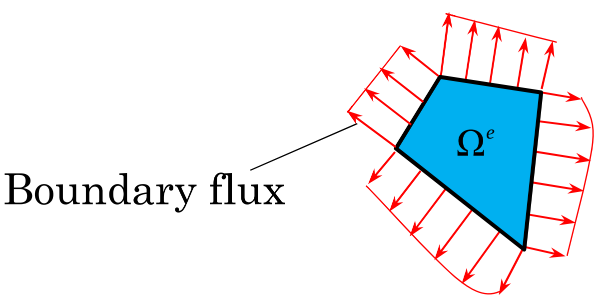
2D Finite Element Discretization



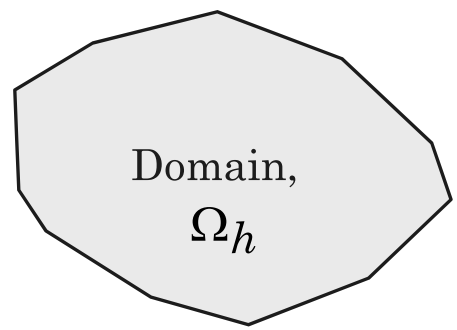
(a) Given domain



(b) Finite element mesh

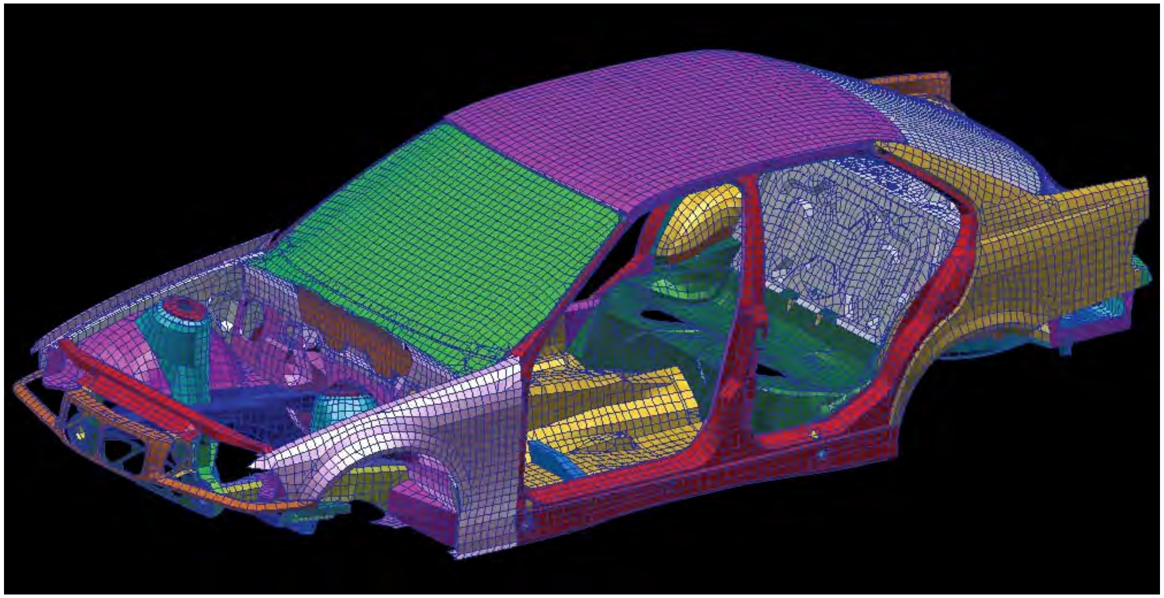


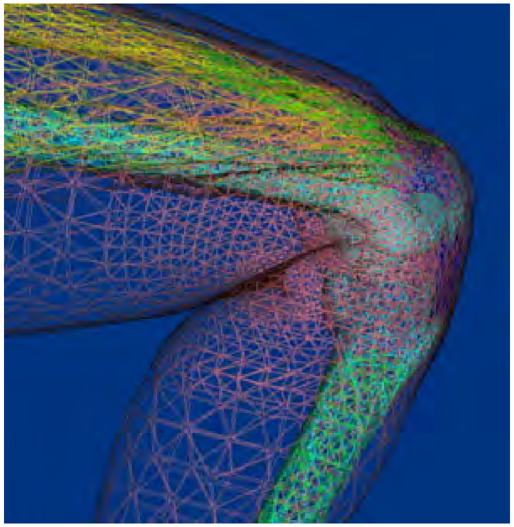
(c) Typical element with boundary fluxes

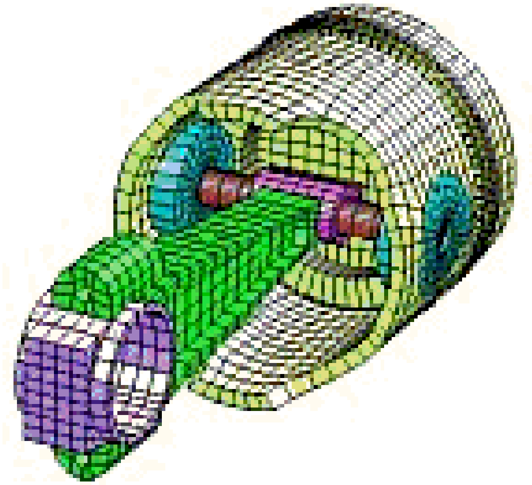


(d) Discretized domain

Some Examples of Real-World Finite Element Discretizations







WEAK FORMS OVER A TYPICAL ELEMENT

Principle of Virtual Displacements (PVD) over an element (accounting for the geometric nonlinearity in the von Karman sense):

$$
\begin{aligned}
0 = \int_{\Omega^e} \Bigg\{ & N_{xx}\left(\frac{\partial \delta u}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x}\right) + N_{yy}\left(\frac{\partial \delta v}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial y}\right) \\
& + N_{xy}\left(\frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial x}\right) \\
& + M_{xx}\left(\frac{\partial \delta \phi_x}{\partial x}\right) + M_{yy}\left(\frac{\partial \delta \phi_y}{\partial y}\right) + M_{xy}\left(\frac{\partial \delta \phi_y}{\partial x} + \frac{\partial \delta \phi_x}{\partial y}\right) \\
& + Q_x\left(\delta\phi_x + \frac{\partial \delta w}{\partial x}\right) + Q_y\left(\delta\phi_y + \frac{\partial \delta w}{\partial y}\right) - f_x\delta u - f_y\delta v - q\,\delta w \Bigg\} dxdy \\
& - \int_{\Gamma^e} \left(N_n\delta u_n + N_{ns}\delta u_{ns} + Q_n\delta w + M_n\,\delta\phi_n + M_{ns}\,\delta\phi_{ns}\right) ds
\end{aligned}
$$

WEAK FORMS USING THE PVD

$$0=\int_{\Omega^e}\left[N_{xx}\left(\frac{\partial \delta u}{\partial x}\right)+N_{xy}\left(\frac{\partial \delta u}{\partial y}\right)-f_x\delta u\right]dxdy-\int_{\Gamma^e}N_n\delta u\,ds$$

$$0=\int_{\Omega^e}\left[N_{xy}\left(\frac{\partial \delta v}{\partial x}\right)+N_{yy}\left(\frac{\partial \delta v}{\partial y}\right)-f_y\delta v\right]dxdy-\int_{\Gamma^e}N_{ns}\delta v\,ds$$

$$0=\int_{\Omega^e}\left\{N_{xx}\left(\frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x}\right)+N_{yy}\left(\frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial y}\right)+N_{xy}\left(\frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial y}+\frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial x}\right)\right.$$

$$\left.+Q_x\left(\frac{\partial \delta w}{\partial x}\right)+Q_y\left(\frac{\partial \delta w}{\partial y}\right)-q\,\delta w\right\}dxdy-\int_{\Gamma^e}Q_n\delta w\,ds$$

$$0=\int_{\Omega^e}\left[M_{xx}\left(\frac{\partial \delta\phi_x}{\partial x}\right)+M_{xy}\left(\frac{\partial \delta\phi_x}{\partial y}\right)+Q_x\,\delta\phi_x\right]dxdy-\int_{\Gamma^e}M_n\,\delta\phi_x\,ds$$

$$0=\int_{\Omega^e}\left[M_{xy}\left(\frac{\partial \delta\phi_y}{\partial x}\right)+M_{yy}\left(\frac{\partial \delta\phi_y}{\partial y}\right)+Q_y\,\delta\phi_y\right]dxdy-\int_{\Gamma^e}M_{ns}\,\delta\phi_y\,ds$$

FINITE ELEMENT MODELS OF (FSDT)

Weak Forms from the Governing Equations

$$0 = -\int_{\Omega^e} v_1 \left(\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} + f_x - I_0 \frac{\partial^2 u}{\partial t^2} \right) dxdy$$

$$= \int_{\Omega^e} \left(\frac{\partial v_1}{\partial x} N_{xx} + \frac{\partial v_1}{\partial y} N_{xy} - v_1 f_x + I_0 v_1 \frac{\partial^2 u}{\partial t^2} \right) dxdy - \oint_{\Gamma^e} v_1 (N_{xx} n_x + N_{xy} n_y) ds$$

$$\boxed{= \int_{\Omega^e} \left(\frac{\partial v_1}{\partial x} N_{xx} + \frac{\partial v_1}{\partial y} N_{xy} - v_1 f_x + I_0 v_1 \frac{\partial^2 u}{\partial t^2} \right) dxdy - \oint_{\Gamma^e} v_1 N_{nn} ds}$$

$$0 = -\int_{\Omega^e} v_2 \left(\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} + f_y - I_0 \frac{\partial^2 v}{\partial t^2} \right) dxdy$$

$$= \int_{\Omega^e} \left(\frac{\partial v_2}{\partial x} N_{xy} + \frac{\partial v_2}{\partial y} N_{yy} - v_2 f_y + I_0 v_2 \frac{\partial^2 v}{\partial t^2} \right) dxdy - \oint_{\Gamma^e} v_2 (N_{xy} n_x + N_{yy} n_y) ds$$

$$\boxed{= \int_{\Omega^e} \left(\frac{\partial v_2}{\partial x} N_{xy} + \frac{\partial v_2}{\partial y} N_{yy} - v_2 f_y + I_0 v_2 \frac{\partial^2 v}{\partial t^2} \right) dxdy - \oint_{\Gamma^e} v_2 N_{ns} ds}$$

Plate bending: 8

FINITE ELEMENT MODELS OF (FSDT) (continued)

Weak Forms (continued)

$$0 = \int_{\Omega^e} \left(\frac{\partial v_3}{\partial x} Q_x + \frac{\partial v_3}{\partial y} Q_y + v_3 N + I_0 v_3 \frac{\partial^2 w}{\partial t^2} \right) dxdy - \oint_{\Gamma^e} v_3 V ds$$

$$0 = \int_{\Omega^e} \left(\frac{\partial v_4}{\partial x} M_{xx} + \frac{\partial v_4}{\partial y} M_{xy} + v_4 Q_x + I_2 v_4 \frac{\partial^2 \phi_x}{\partial t^2} \right) dxdy - \oint_{\Gamma^e} v_4 M_{nn} ds$$

$$0 = \int_{\Omega^e} \left(\frac{\partial v_5}{\partial x} M_{xy} + \frac{\partial v_5}{\partial y} M_{yy} + v_4 Q_y + I_2 v_5 \frac{\partial^2 \phi_y}{\partial t^2} \right) dxdy - \oint_{\Gamma^e} v_4 M_{ns} ds$$

$$Q_n = Q_x n_x + Q_y n_y; \quad M_{nn} = M_{xx} n_x + M_{xy} n_y; \quad M_{ns} = M_{xy} n_x + M_{yy} n_y$$

FINITE ELEMENT MODELS OF FSDT

Finite element approximations

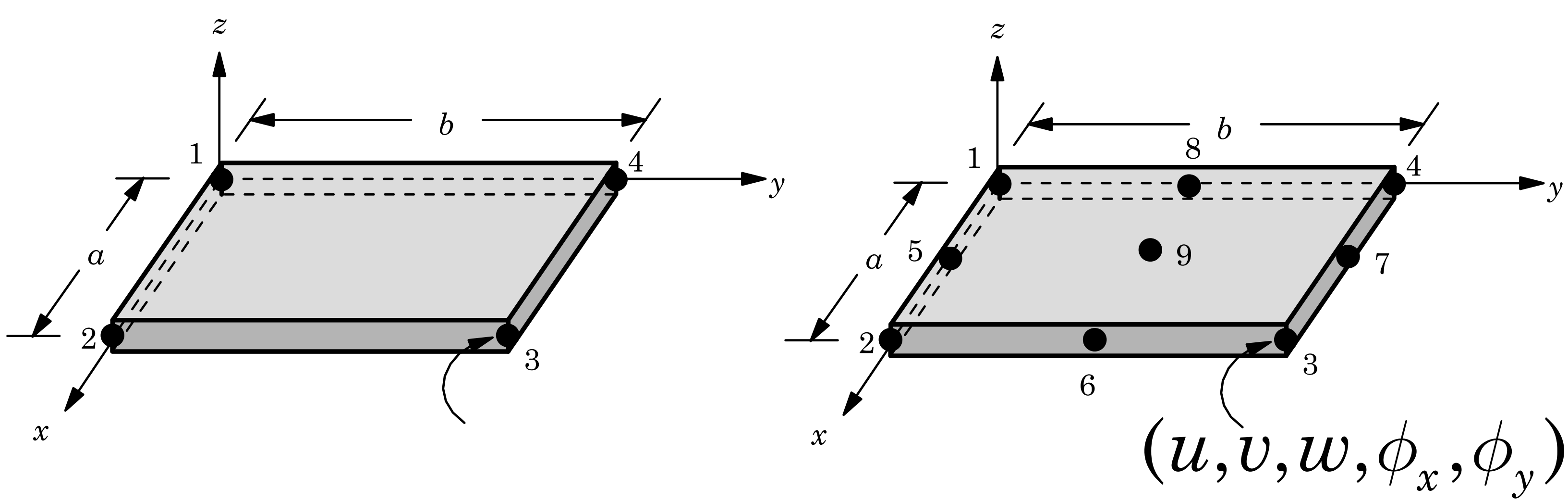
$$u(x,y,t) \approx \sum_{j=1}^{n} u_j^e(t)\,\psi_j^e(x,y)$$

$$v(x,y,t) \approx \sum_{j=1}^{n} v_j^e(t)\,\psi_j^e(x,y)$$

$$w(x,y,t) \approx \sum_{j=1}^{n} w_j^e(t)\,\psi_j^e(x,y)$$

$$\phi_x(x,y,t) \approx \sum_{j=1}^{n} \phi_{xj}^e(t)\,\psi_j^e(x,y)$$

$$\phi_y(x,y,t) \approx \sum_{j=1}^{n} \phi_{yj}^e(t)\,\psi_j^e(x,y)$$



SEMIDISCRETE FINITE ELEMENT MODEL

$$\begin{bmatrix} \mathbf{M}^{11} & \mathbf{0} & \mathbf{0} & \mathbf{M}^{14} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{22} & \mathbf{0} & \mathbf{0} & \mathbf{M}^{25} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}^{33} & \mathbf{0} & \mathbf{0} \\ \mathbf{M}^{41} & \mathbf{0} & \mathbf{0} & \mathbf{M}^{44} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{52} & \mathbf{0} & \mathbf{0} & \mathbf{M}^{55} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{u}} \\ \ddot{\mathbf{v}} \\ \ddot{\mathbf{w}} \\ \ddot{\Phi}_x \\ \ddot{\Phi}_y \end{Bmatrix} + \begin{bmatrix} \mathbf{K}^{11} & \mathbf{K}^{12} & \mathbf{K}^{13} & \mathbf{K}^{14} & \mathbf{K}^{15} \\ \mathbf{K}^{21} & \mathbf{K}^{22} & \mathbf{K}^{23} & \mathbf{K}^{24} & \mathbf{K}^{25} \\ \mathbf{K}^{31} & \mathbf{K}^{32} & \mathbf{K}^{33} & \mathbf{K}^{34} & \mathbf{K}^{35} \\ \mathbf{K}^{41} & \mathbf{K}^{42} & \mathbf{K}^{43} & \mathbf{K}^{44} & \mathbf{K}^{45} \\ \mathbf{K}^{51} & \mathbf{K}^{52} & \mathbf{K}^{53} & \mathbf{K}^{54} & \mathbf{K}^{55} \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \\ \Phi_x \\ \Phi_y \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}^1 \\ \mathbf{F}^2 \\ \mathbf{F}^3 \\ \mathbf{F}^4 \\ \mathbf{F}^5 \end{Bmatrix}$$

$$M_{ij}^{11} = \int_{\Omega^e} I_0 \psi_i^e \psi_j^e \; dxdy, \;\; M_{ij}^{14} = M_{ij}^{41} = \int_{\Omega^e} I_1 \psi_i^e \psi_j^e \; dxdy$$

$$M_{ij}^{22} = \int_{\Omega^e} I_0 \psi_i^e \psi_j^e \; dxdy, \;\; M_{ij}^{25} = M_{ij}^{25} = \int_{\Omega^e} I_1 \psi_i^e \psi_j^e \; dxdy$$

$$M_{ij}^{33} = \int_{\Omega^e} I_0 \psi_i^e \psi_j^e \; dxdy, \;\; M_{ij}^{44} = M_{ij}^{55} = \int_{\Omega^e} I_2 \psi_i^e \psi_j^e \; dxdy$$

TYPICAL FINITE ELEMENT COEFFICIENTS

$$K_{ij}^{11} = \int_{\Omega^e} \left[A_{11} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + A_{66} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} + A_{16} \left(\frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial y} + \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial x} \right) \right] dxdy$$

$$K_{ij}^{12} = \int_{\Omega^e} \left[A_{12} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial y} + A_{66} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial x} + A_{26} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} + A_{16} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} \right] dxdy = K_{ji}^{21}$$

$$K_{ij}^{13} = \frac{1}{2} \int_{\Omega^e} \left\{ \frac{\partial \psi_i}{\partial x} \left[A_{11} \frac{\partial w_0}{\partial x} \frac{\partial \psi_j}{\partial x} + A_{12} \frac{\partial w_0}{\partial y} \frac{\partial \psi_j}{\partial y} + A_{16} \left(\frac{\partial w_0}{\partial x} \frac{\partial \psi_j}{\partial y} + \frac{\partial w_0}{\partial y} \frac{\partial \psi_j}{\partial x} \right) \right] \right.$$

$$\left. + \frac{\partial \psi_i}{\partial y} \left[A_{16} \frac{\partial w_0}{\partial x} \frac{\partial \psi_j}{\partial x} + A_{26} \frac{\partial w_0}{\partial y} \frac{\partial \psi_j}{\partial y} + A_{66} \left(\frac{\partial w_0}{\partial x} \frac{\partial \psi_j}{\partial y} + \frac{\partial w_0}{\partial y} \frac{\partial \psi_j}{\partial x} \right) \right] \right\} dxdy$$

$$K_{ij}^{14} = \int_{\Omega^e} \left[\frac{\partial \psi_i}{\partial x} \left(B_{11} \frac{\partial \psi_j}{\partial x} + B_{16} \frac{\partial \psi_j}{\partial y} \right) + \frac{\partial \psi_i}{\partial y} \left(B_{16} \frac{\partial \psi_j}{\partial x} + B_{66} \frac{\partial \psi_j}{\partial y} \right) \right] dxdy = K_{ji}^{41}$$

$$K_{ij}^{15} = \int_{\Omega^e} \left[\frac{\partial \psi_i}{\partial x} \left(B_{16} \frac{\partial \psi_j}{\partial x} + B_{12} \frac{\partial \psi_j}{\partial y} \right) + \frac{\partial \psi_i}{\partial y} \left(B_{66} \frac{\partial \psi_j}{\partial x} + B_{26} \frac{\partial \psi_j}{\partial y} \right) \right] dxdy = K_{ji}^{51}$$

TYPICAL FINITE ELEMENT COEFFICIENTS (continued)

$$K_{ij}^{22} = \int_{\Omega^e} \left[A_{22} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} + A_{66} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + A_{26} \left(\frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial y} + \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial x} \right) \right] dxdy$$

$$\begin{aligned} K_{ij}^{35} = K_s \int_{\Omega^e} \underline{\left(A_{45} \frac{\partial \psi_i}{\partial x} \psi_j + A_{44} \frac{\partial \psi_i}{\partial y} \psi_j \right)} dxdy + \int_{\Omega^e} \Bigg\{ \frac{\partial \psi_i}{\partial x} \Bigg[\frac{\partial w_0}{\partial x} \left(B_{12} \frac{\partial \psi_j^{(3)}}{\partial y} + B_{16} \frac{\partial \psi_j^{(3)}}{\partial x} \right) \\ + \frac{\partial w_0}{\partial y} \left(B_{26} \frac{\partial \psi_j}{\partial y} + B_{66} \frac{\partial \psi_j}{\partial x} \right) \Bigg] + \frac{\partial \psi_i}{\partial y} \Bigg[\frac{\partial w_0}{\partial y} \left(B_{22} \frac{\partial \psi_j}{\partial y} + B_{26} \frac{\partial \psi_j}{\partial x} \right) \\ + \frac{\partial w_0}{\partial x} \left(B_{26} \frac{\partial \psi_j}{\partial y} + B_{66} \frac{\partial \psi_j}{\partial x} \right) \Bigg] \Bigg\} dxdy \end{aligned}$$

$$K_{ij}^{44} = \int_{\Omega^e} \left[\frac{\partial \psi_i}{\partial x} \left(D_{11} \frac{\partial \psi_j}{\partial x} + D_{16} \frac{\partial \psi_j}{\partial y} \right) + \frac{\partial \psi_i}{\partial y} \left(D_{16} \frac{\partial \psi_j}{\partial x} + D_{66} \frac{\partial \psi_j}{\partial y} \right) + \underline{K_s A_{55} \psi_i \psi_j} \right] dxdy$$

$$K_{ij}^{45} = \int_{\Omega^e} \left[\frac{\partial \psi_i}{\partial x} \left(D_{16} \frac{\partial \psi_j}{\partial x} + D_{12} \frac{\partial \psi_j}{\partial y} \right) + \frac{\partial \psi_i}{\partial y} \left(D_{66} \frac{\partial \psi_j}{\partial x} + D_{26} \frac{\partial \psi_j}{\partial y} \right) + \underline{K_s A_{45} \psi_i \psi_j} \right] dxdy = K_{ji}^{54}$$

Plate bending: 13

Stiffness Coefficients (typical)

$$K_{ij}^{22} = \int_{\Omega^e} \left[\frac{\partial \psi_i}{\partial x} \left(A_{66} \frac{\partial \psi_j}{\partial x} \right) + \frac{\partial \psi_i}{\partial y} \left(A_{22} \frac{\partial \psi_j}{\partial y} \right) \right] dxdy$$

$$K_{ij}^{31} = \int_{\Omega^e} \left[\frac{\partial \psi_i}{\partial x} \left(A_{11} \frac{\partial w}{\partial x} \frac{\partial \psi_j}{\partial x} + A_{66} \frac{\partial w}{\partial y} \frac{\partial \psi_j}{\partial y} \right) + \frac{\partial \psi_i}{\partial y} \left(A_{66} \frac{\partial w}{\partial x} \frac{\partial \psi_j}{\partial y} + A_{12} \frac{\partial w}{\partial y} \frac{\partial \psi_j}{\partial x} \right) \right] dxdy;$$

$$K_{ij}^{34} = \int_{\Omega^e} \frac{\partial \psi_i}{\partial x} \left(A_{55} \psi_j \right) dxdy; \quad K_{ij}^{35} = \int_{\Omega^e} \frac{\partial \psi_i}{\partial y} \left(A_{44} \psi_j \right) dxdy$$

The finite element model is of the form

$$\mathbf{M}\ddot{\Delta} + \mathbf{K}\Delta = \mathbf{F}$$

The fully discretized form of the is given by

$$\hat{\mathbf{K}}(\Delta^{s+1})\Delta^{s+1} = \hat{\mathbf{F}}^{s,s+1}$$

FINITE ELEMENT MODELS OF FSDT (continued)

Fully Discretized Finite Element Model

$$\hat{\mathbf{K}}^{s+1}(\Delta^{s+1})\Delta^{s+1} = \hat{\mathbf{F}}^{s+1}, \quad \text{where } \hat{\mathbf{K}}^{s+1} \equiv \mathbf{K}^{s+1} + a_3\mathbf{M}^{s+1}$$

$$\hat{\mathbf{F}}^{s+1} \equiv \mathbf{F}^{s+1} + \mathbf{M}^{s+1}\left(a_3\Delta^s + a_4\dot{\Delta}^s + a_5\ddot{\Delta}^s\right)$$

$$a_1 = \alpha\Delta t,\ a_2 = (1-\alpha)\Delta t,\ a_3 = \frac{2}{\gamma(\Delta t)^2},\ a_4 = \Delta t\, a_3,\ a_5 = \frac{1}{\gamma} - 1$$

Computation of Velocities and Accelerations

At the end of each time step, we update the velocity and acceleration as follows:

$$\ddot{\Delta}^{s+1} = a_3(\Delta^{s+1} - \Delta^s) - a_4\dot{\Delta}^s - a_5\ddot{\Delta}^s$$

$$\dot{\Delta}^{s+1} = \dot{\Delta}^s + a_2\ddot{\Delta}^s + a_1\ddot{\Delta}^{s+1}$$

Plate bending: 15

Shear and Membrane Locking

Shear Locking

Use reduced integration to evaluate all *shear* stiffnesses (i.e., all K_{ij} that contain transverse shear terms)

Membrane Locking

Use reduced integration to evaluate all *membrane* stiffnesses (i.e., all K_{ij} that contain von Kármán nonlinear terms)

SOLUTION OF NONLINEAR EQUATIONS
Direct Iteration

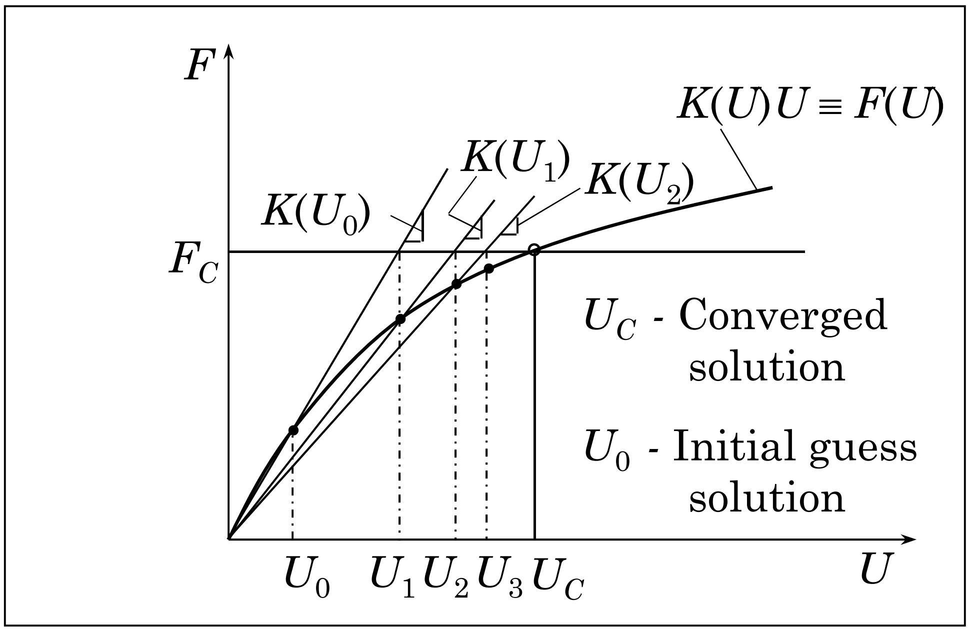
Non-Linear Finite Element Model

$$\hat{\mathbf{K}}^e(\Delta^e_{s+1})\Delta^e_{s+1} = \hat{\mathbf{F}}^e \Rightarrow \text{assembled} \ \ \hat{\mathbf{K}}(\mathbf{U}_{s+1})\mathbf{U}_{s+1} = \hat{\mathbf{F}}$$

Direct Iteration Method

When the solution $\mathbf{U}^r_{s+1}$ at $r^{\mathbf{th}}$ iteration is known, solve for $\mathbf{U}^{r+1}_{s+1}$

$$\boxed{\hat{\mathbf{K}}(\mathbf{U}^r_{s+1})\mathbf{U}^{r+1}_{s+1} = \hat{\mathbf{F}}}$$



Hat on K and subscript s+1 are omitted for brevity

Nonlinear plate bending : 17

Convergence Criterion

$$\varepsilon = \sqrt{\frac{\sum_{I=1}^{NEQ} \left(U_I^r - U_I^{r+1}\right)^2}{\sum_{I=1}^{NEQ} \left(U_I^{r+1}\right)^2}} \leq \text{specified tolerance}$$

SOLUTION OF NONLINEAR EQUATIONS
Newton-Raphson Iteration

Taylor's series

$$\text{Residual,}\quad \mathbf{R}(\mathbf{U}_{s+1}) \equiv \hat{\mathbf{K}}(\mathbf{U}_{s+1})\mathbf{U}_{s+1} - \hat{\mathbf{F}}$$

$$\mathbf{R}(\mathbf{U}^{r+1}) = \mathbf{R}(\mathbf{U}^{r}) + (\mathbf{U}^{r+1} - \mathbf{U}^{r})\left[\frac{\partial \mathbf{R}}{\partial \mathbf{U}}\right]^{r} + \frac{1}{2!}(\mathbf{U}^{r+1} - \mathbf{U}^{r})^{2}\left[\frac{\partial^{2}\,\mathrm{R}}{\partial \mathbf{U}^{2}}\right]^{r} + \cdots$$

$$\approx \mathbf{R}(\mathbf{U}^{r}) + (\mathbf{U}^{r+1} - \mathbf{U}^{r})\left[\frac{\partial \mathbf{R}}{\partial \mathbf{U}}\right]^{r} + O(\delta\mathbf{U})^{2},\quad \delta\mathbf{U} = \mathbf{U}^{r+1} - \mathbf{U}^{r}$$

Requiring the residual $\mathbf{R}^{r+1}$ to be zero at the $r+1^{\mathbf{st}}$ iteration, we have

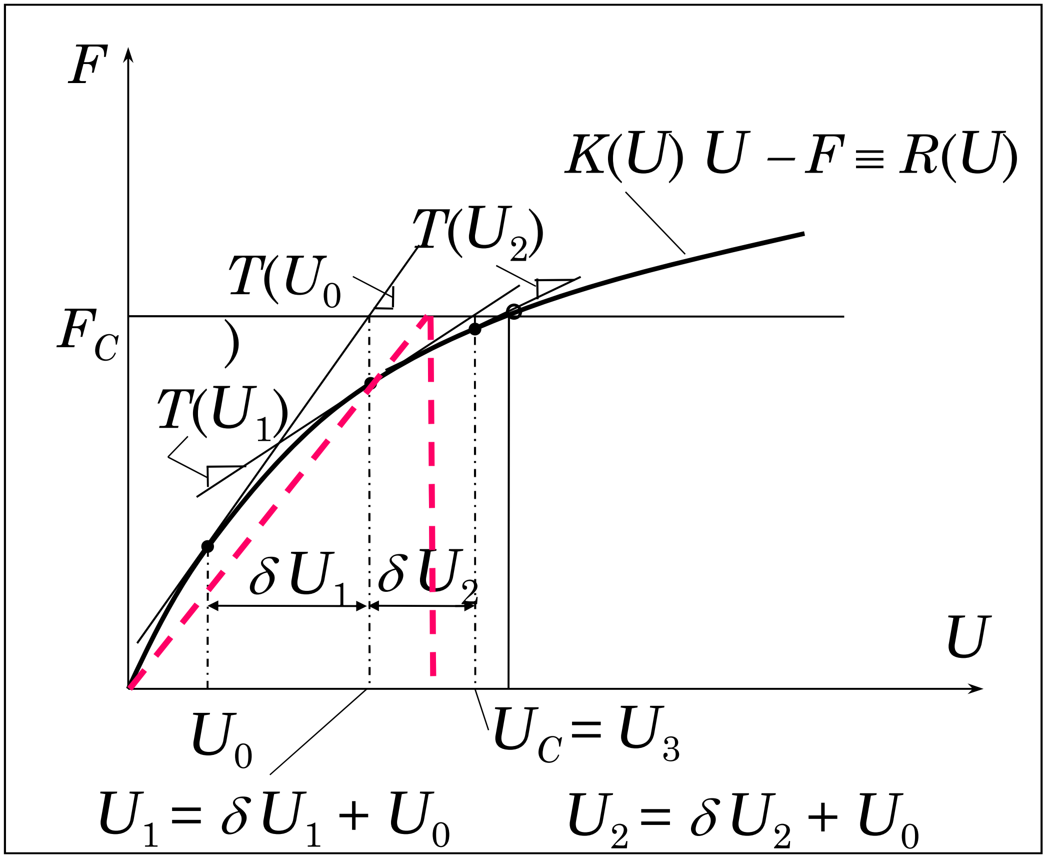
$$\boxed{\hat{\mathbf{K}}^{\mathbf{tan}}(\mathbf{U}^{r}_{s+1})\delta\mathbf{U}_{s+1} = -\mathbf{R}^{r}(\mathbf{U}^{r}_{s+1}) = \hat{\mathbf{F}}^{r} - \hat{\mathbf{K}}(\mathbf{U}^{r}_{s+1})\mathbf{U}^{r}_{s+1}}$$

The tangent matrix at the element level is

$$\left(\hat{K}^{\alpha\beta}_{ij}\right)^{\tan} = \frac{\partial R^{\alpha}_{i}}{\partial \Delta^{\beta}_{j}} = \frac{\partial}{\partial \Delta^{\beta}_{j}}\left(\sum_{\beta=1}^{5}\sum_{p=1}^{n}\hat{K}^{\alpha\beta}_{ip}\Delta^{\beta}_{p} - \hat{F}^{\alpha}_{i}\right)$$

SOLUTION OF NONLINEAR EQUATIONS
Newton-Raphson Iteration (continued)

$$\mathbf{T}(\mathbf{U}^r_{s+1})\delta\mathbf{U}_{s+1} = \mathbf{F}^r_{s+1} - \mathbf{K}(\mathbf{U}^r_{s+1})\mathbf{U}^r_{s+1}, \quad \mathbf{U}^{r+1}_{s+1} = \mathbf{U}^r_{s+1} + \delta\mathbf{U}_{s+1}$$



U_C - Converged solution

U_0 - Initial guess solution

Post-Computation of Strains

We compute the required derivatives at the **reduced integration points:**

Derivatives of the generalized displacements

$$\frac{\partial u}{\partial x}=\sum_{j=1}^{m}u_j\frac{\partial\psi_j}{\partial x},\quad \frac{\partial v}{\partial y}=\sum_{j=1}^{m}v_j\frac{\partial\psi_j}{\partial y}$$

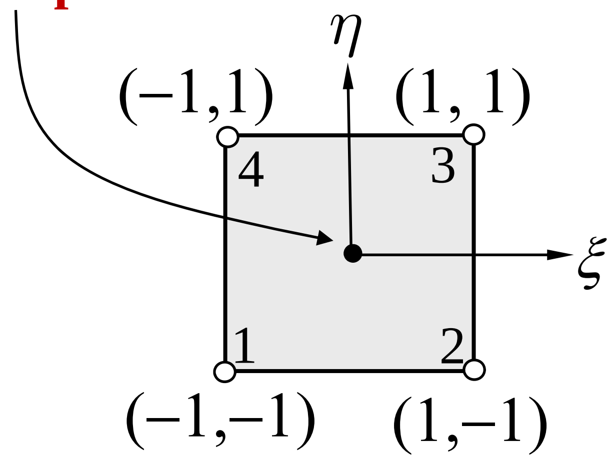
$$\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=\sum_{j=1}^{m}\left(u_j\frac{\partial\psi_j}{\partial y}+v_j\frac{\partial\psi_j}{\partial x}\right)$$

$$\frac{\partial w}{\partial x}=\sum_{j=1}^{m}w_j\frac{\partial\psi_j}{\partial x},\quad \frac{\partial w}{\partial y}=\sum_{j=1}^{m}w_j\frac{\partial\psi_j}{\partial y}$$

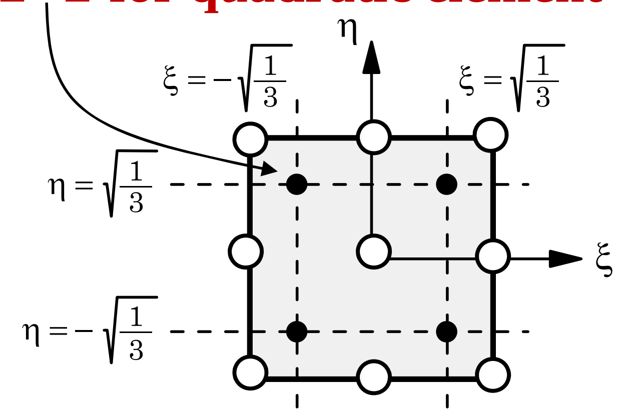
$$\frac{\partial \phi_x}{\partial x}=\sum_{j=1}^{m}S_j^{(1)}\frac{\partial\psi_j}{\partial x},\quad \frac{\partial \phi_y}{\partial y}=\sum_{j=1}^{m}S_j^{(2)}\frac{\partial\psi_j}{\partial y}$$

$$\frac{\partial \phi_x}{\partial y}+\frac{\partial \phi_y}{\partial x}=\sum_{j=1}^{m}\left(S_j^1\frac{\partial\psi_j}{\partial y}+S_j^2\frac{\partial\psi_j}{\partial x}\right)$$

One-point for linear element



2×2 for quadratic element



Post-Computation of Stresses

Stresses

$$\sigma_{xx} = Q_{11}\varepsilon_{xx} + Q_{12}\varepsilon_{yy}, \;\; \sigma_{xy} = Q_{12}\varepsilon_{xx} + Q_{22}\varepsilon_{yy},$$

$$\sigma_{xy} = 2Q_{66}\varepsilon_{xy}, \;\; \sigma_{xz} = 2Q_{55}\varepsilon_{xz}, \; \sigma_{yz} = 2Q_{44}\varepsilon_{yz}$$

$$Q_{11} = \frac{E_1}{1-\upsilon_{12}\upsilon_{21}}, \;\; Q_{22} = \frac{E_2}{1-\upsilon_{12}\upsilon_{21}},$$

$$Q_{12} = \frac{\upsilon_{12}E_2}{1-\upsilon_{12}\upsilon_{21}}, \;\; Q_{44} = G_{23}, \;\; Q_{55} = G_{13}, \;\; Q_{66} = G_{12}$$

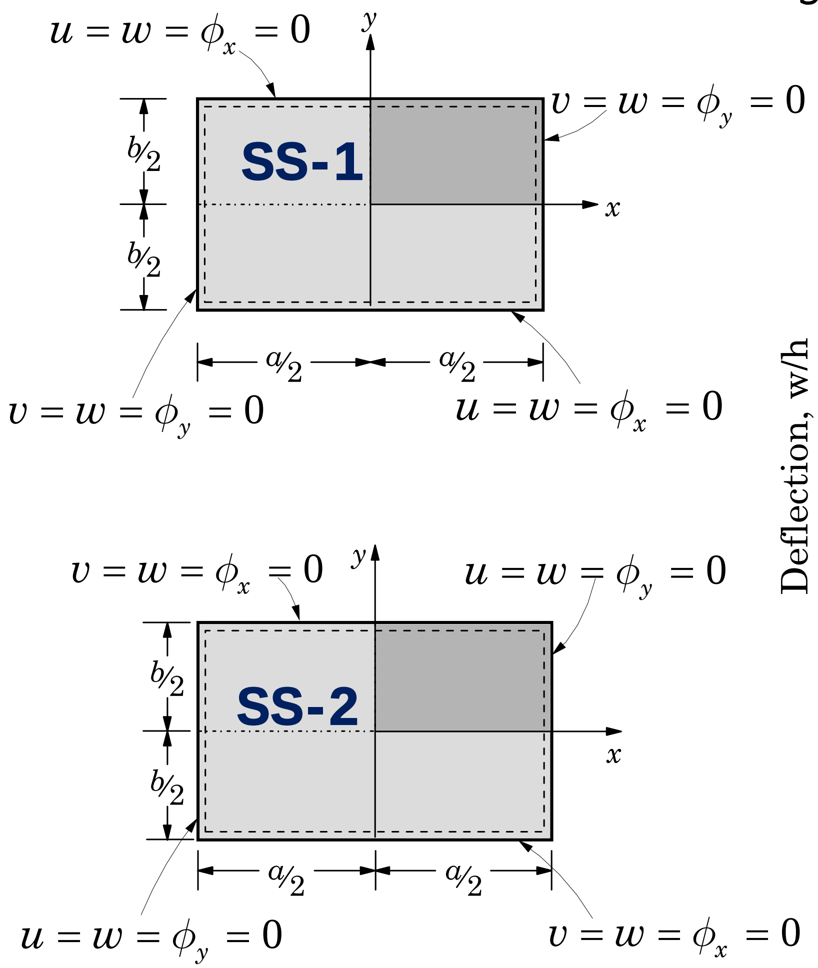
Strains

$$\varepsilon_{xx} = \frac{\partial u_1}{\partial x} + \tfrac{1}{2}\left(\frac{\partial u_3}{\partial x}\right)^2 = \frac{\partial u}{\partial x} + \tfrac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + z\frac{\partial \phi_x}{\partial x}$$

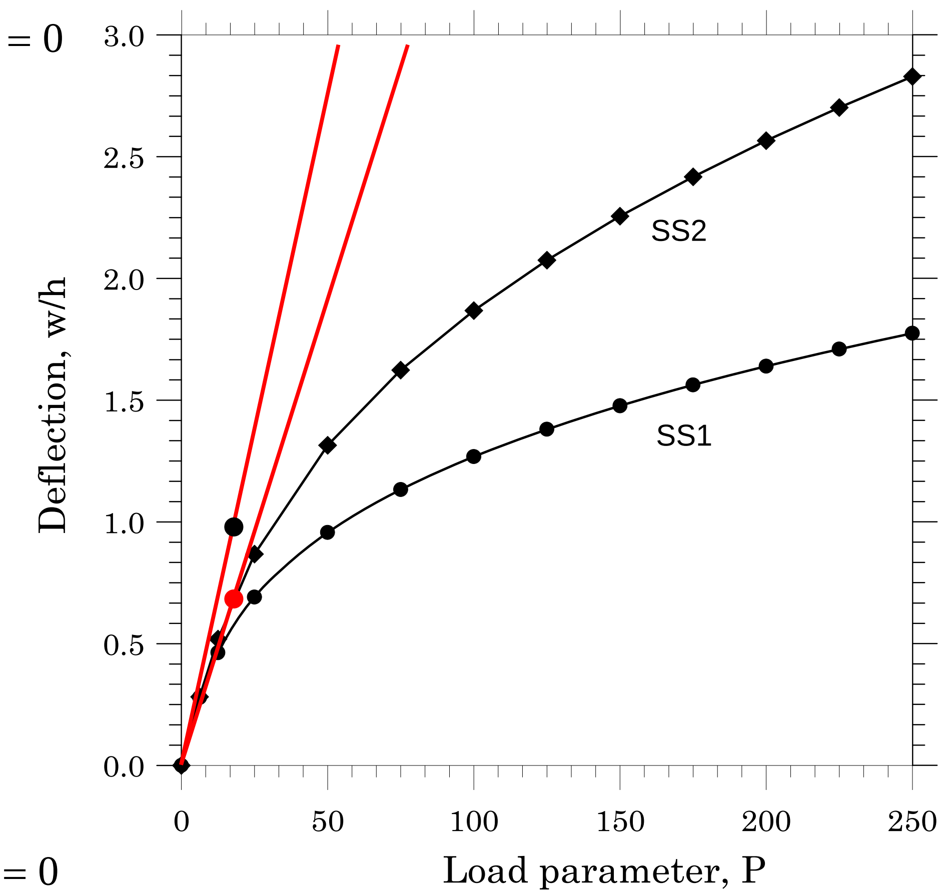
$$\varepsilon_{yy} = \frac{\partial u_2}{\partial y} + \tfrac{1}{2}\left(\frac{\partial u_3}{\partial y}\right)^2 = \frac{\partial v}{\partial y} + \tfrac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + z\frac{\partial \phi_y}{\partial y}$$

$$\gamma_{xy} = \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} + \frac{\partial u_3}{\partial x}\frac{\partial u_3}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} + z\left(\frac{\partial \phi_y}{\partial x} + \frac{\partial \phi_x}{\partial y}\right)$$

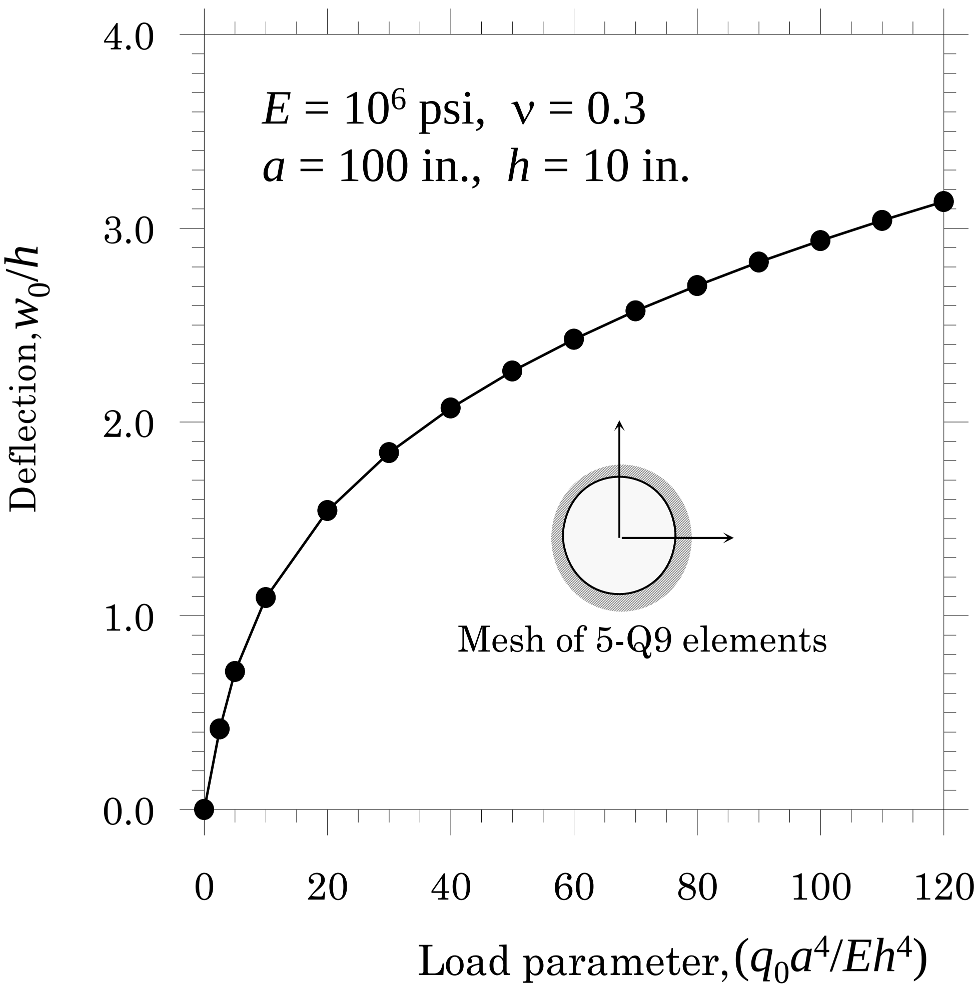
Simply Supported Plate (SS-1)

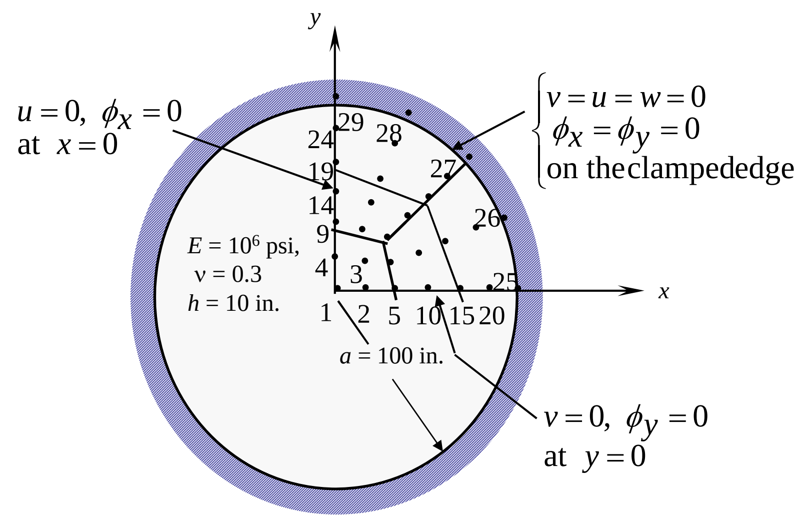


Deflection versus load parameter for simply supported (SS1) plate under uniformly distributed load.

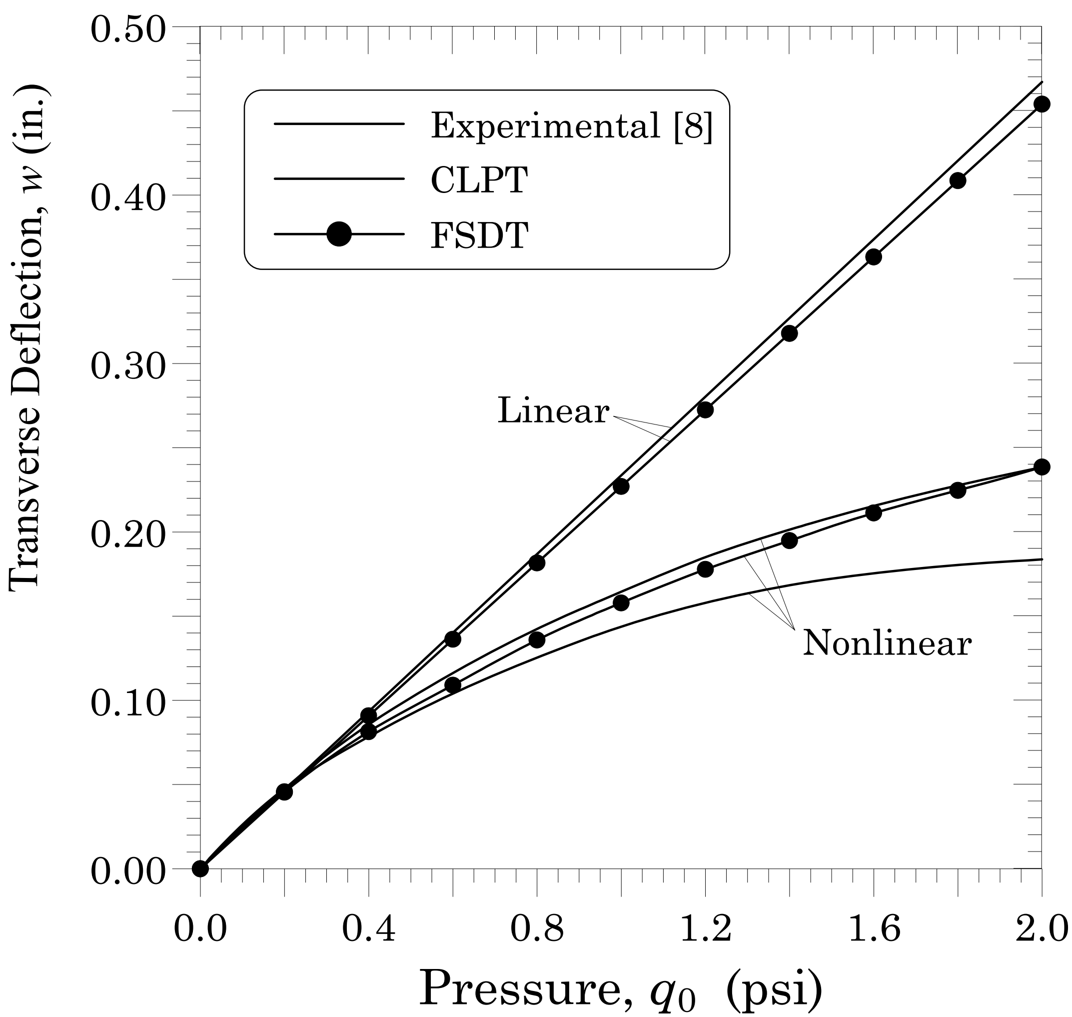


Clamped Circular Plate under UDL





Simply Supported (SS2) Orthotropic* Plate



Geometry and Material Properties

$a = b = 12$ in, $h = 0.138$ in

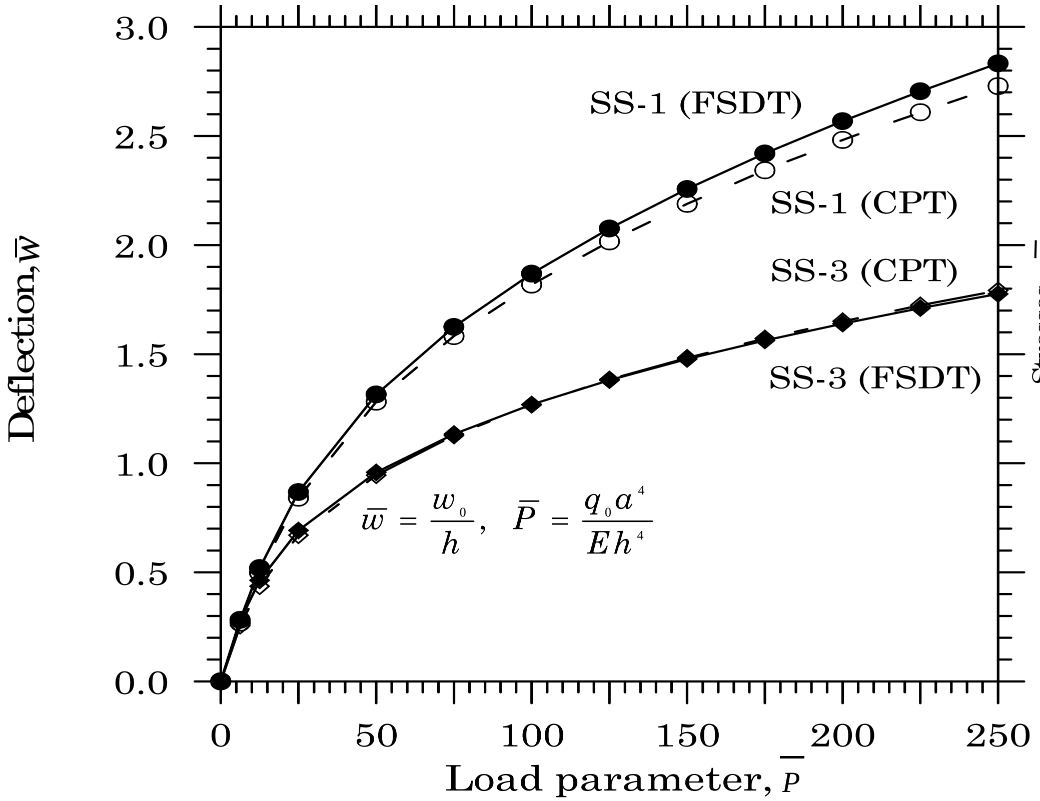
$E_1 = 3\times10^6$ psi, $E_2 = 1.28\times10^6$ psi

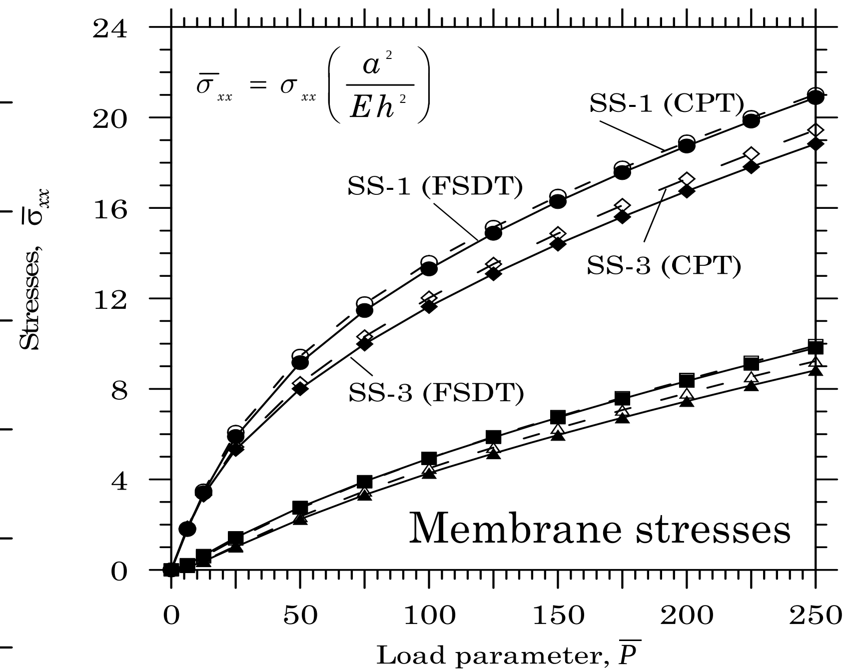
$G_{12} = G_{23} = G_{13} = 0.37\times10^6$ psi

$\nu_{12} = \nu_{23} = \nu_{13} = 0.32$

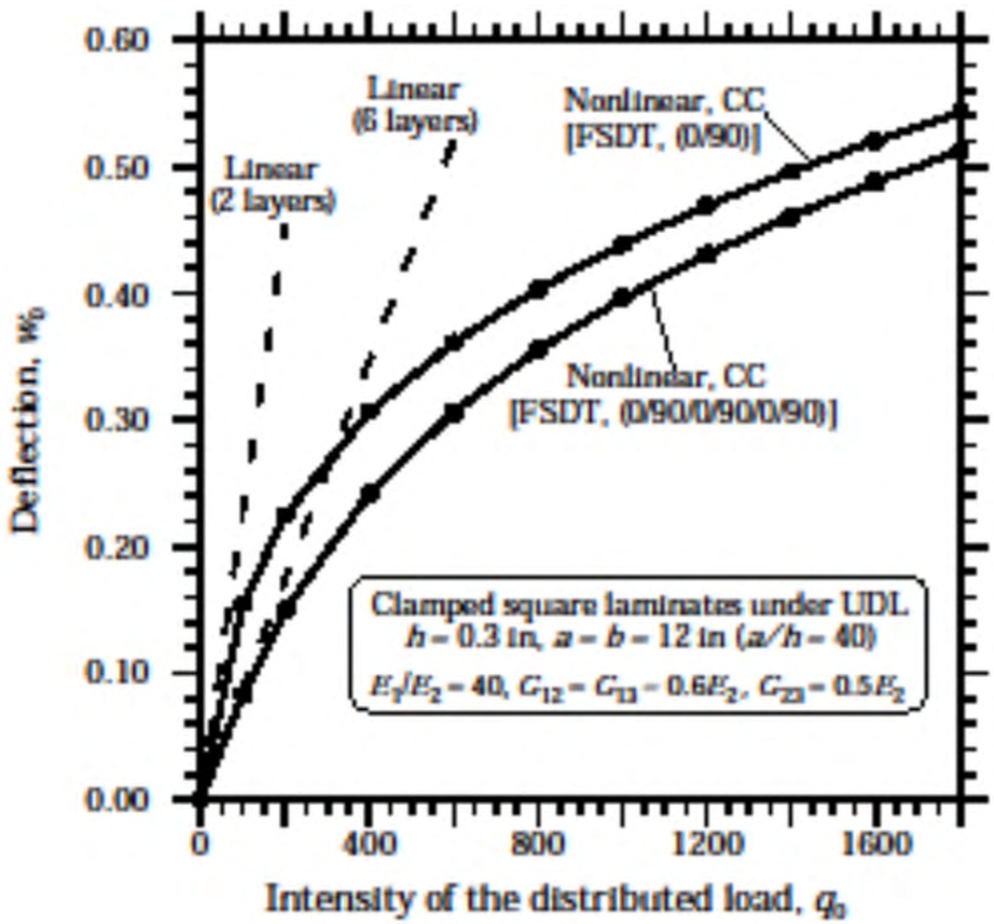
[8] Zaghloul, S. A. and Kennedy, J. B., ``Nonlinear Behavior of Symmetrically Laminated Plates,” *Journal of Applied Mechanics*, 42, 234-236, 1975.

Deflection vs. load parameter for plates under uniformly distributed load





Deflection vs. load parameter for composite plates under uniformly distributed load



JN Reddy

SOME REMARKS

- The effect of shear deformation is to: (a) increase deflections, (b) reduce buckling loads and natural frequencies. This is due to the fact that the classical plate theory, due to the restrictive kinematic assumptions made, over represents the stiffness of the structure.
- In practice, analytical or variational methods can be applied only to simple geometries circular and rectangular plates.
- The finite element method is the most commonly used method for the solution of plate and shell problems.

FUNCTIONALLY GRADED STRUCTURES: BACKGROUND

- Fiber-reinforced composites have a mismatch of mechanical properties across an interface due to two discrete materials bonded together. As a result, the constituents of fiber-matrix composites are prone to debonding at extremely high thermo-mechanical loading.
- Further, cracks are likely to initiate at the interfaces and grow into weaker material sections.
- Additional problems include the presence of residual stresses due to the difference in coefficients of thermal expansion of the fiber and matrix in the composite materials.

BACKGROUND (continued)

- Functionally graded materials are inhomogeneous materials in which the material properties are varied continuously from point to point.
- The problems of composite laminates can be avoided or reduced by the use of FGMs. This gradation in properties of the material reduces thermal stresses, residual stresses, and stress concentration factors.
- Furthermore, the gradual change of mechanical properties can be tailored to different applications and working environments.

BACKGROUND (continued)

- For example, a plate structure used as a thermal barrier may be graded through the plate thickness from ceramic on the face of the plate that is exposed to high temperature to metal on the other face.
- A mixture of the ceramic and a metal with a continuously varying volume fraction can be manufactured, which eliminates interface problems.
- The ceramic constituent of the material provides the high temperature resistance due to its low thermal conductivity. The ductile metal constituent, on the other hand, prevents fracture caused by stresses due to high temperature gradient in a very short period of time.

A TYPICAL METAL-CERAMIC FGM for thermal barrier applications

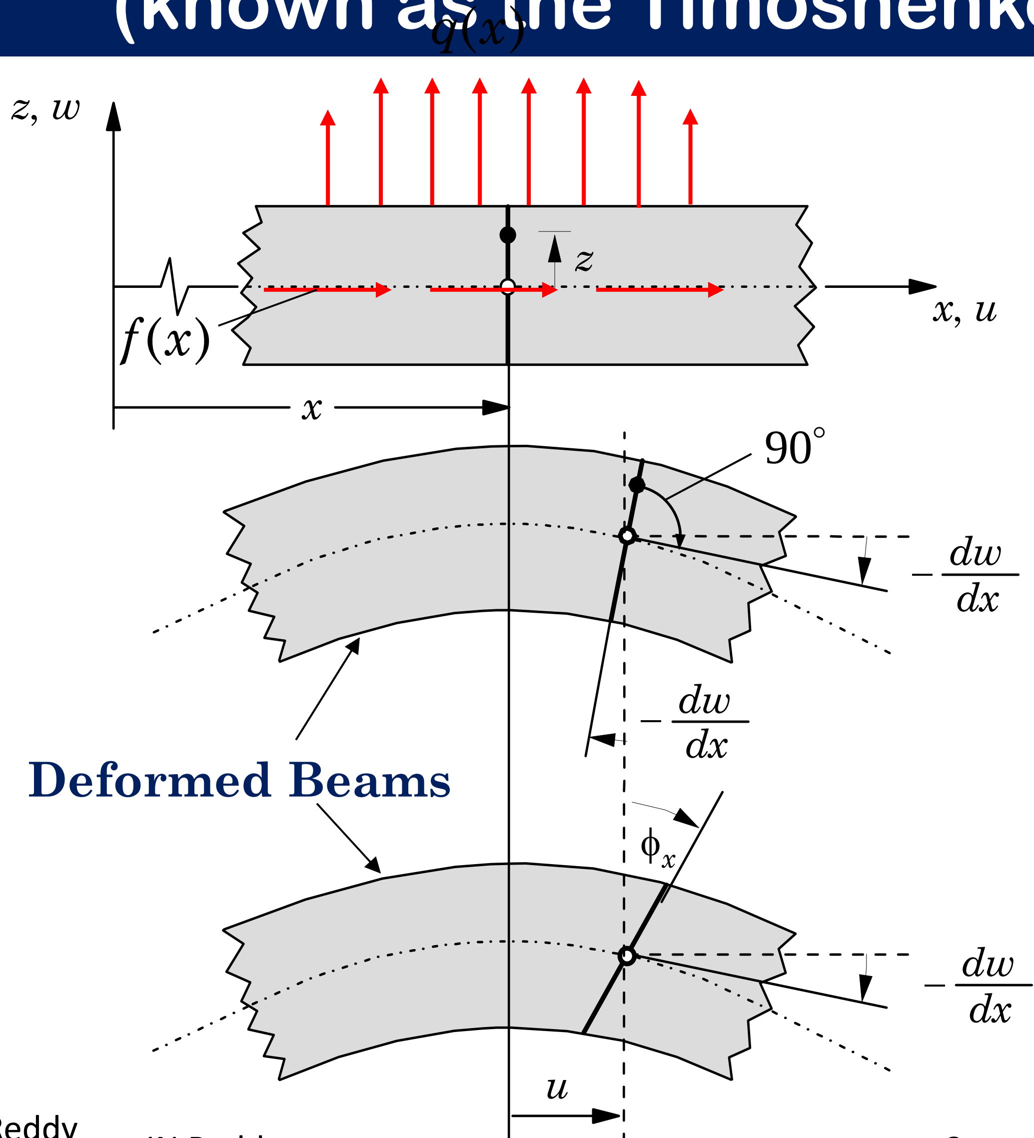
High temperature side	Ceramic	Heat resistant; good anti-oxidant property; low thermal conductivity
Low temperature side	Metal	Mechanical strength; high thermal conductivity; high fracture toughness
In between	Ceramic & metal	Effective thermal stress relaxation throughout

FUNCTIONALLY GRADED BEAMS

Conventional theories of functionally graded straight beams

- **Euler-Bernoulli beam theory**
- **Timoshenko beam theory**

Kinematics of a Shear Deformation Beam Theory (known as the Timoshenko beam theory)

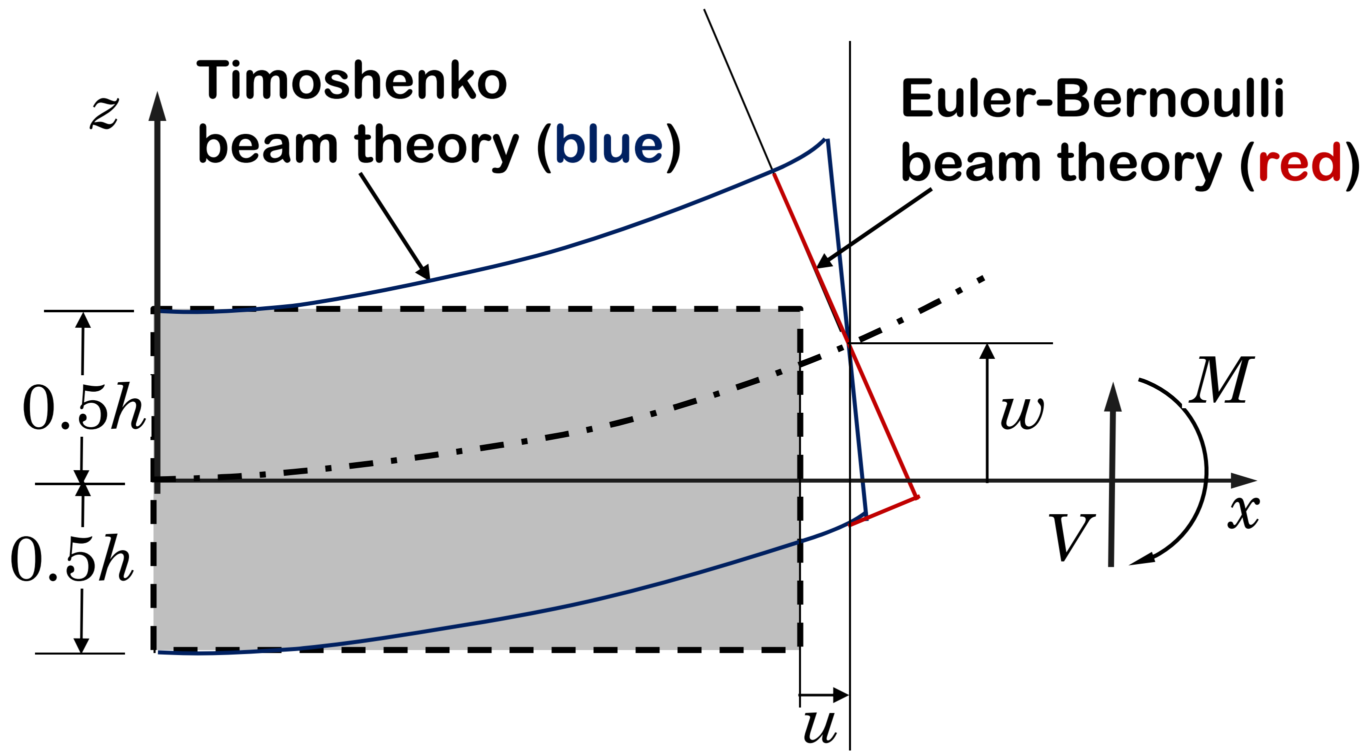


Undeformed Beam

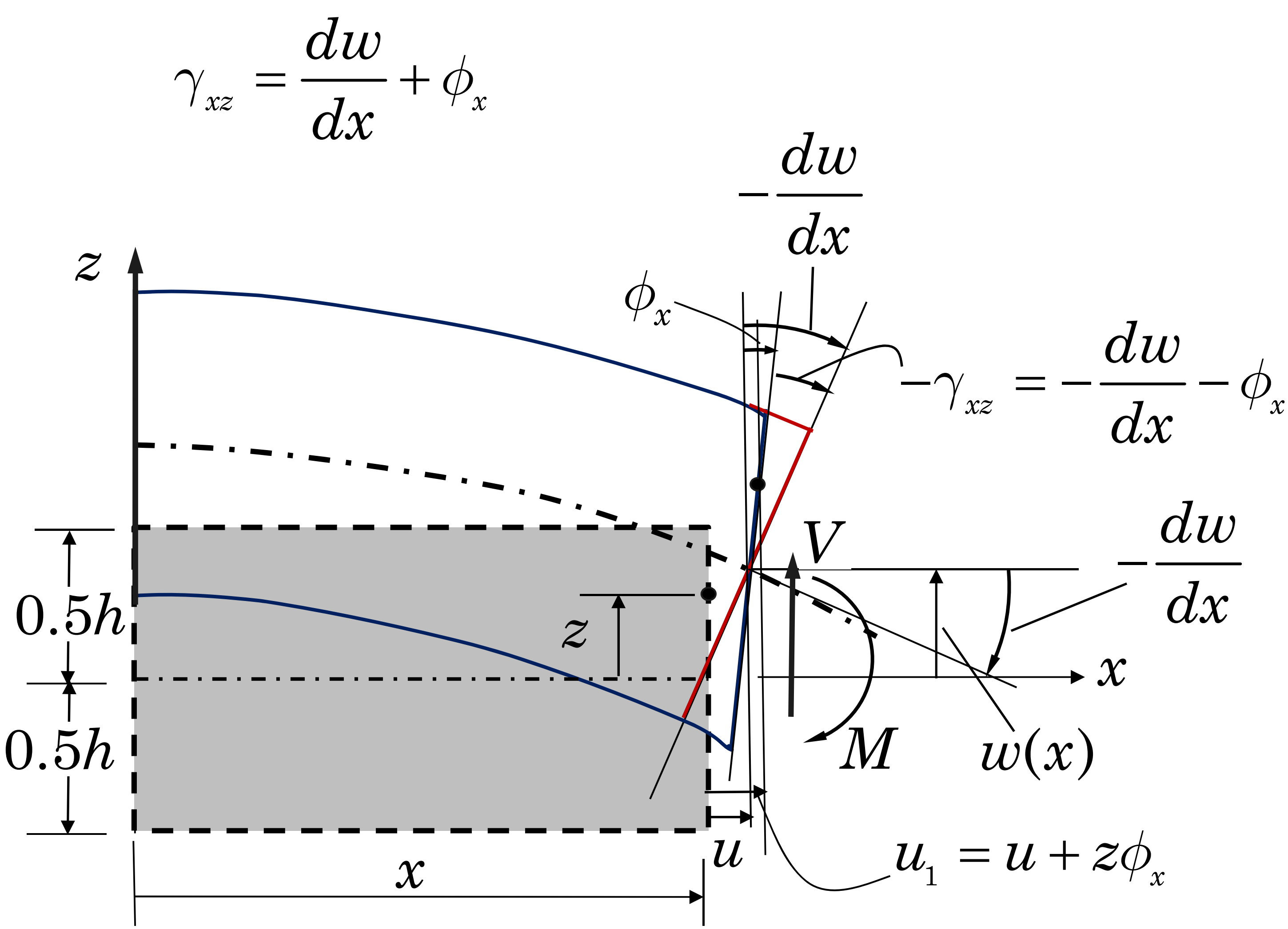
Euler-Bernoulli Beam Theory (EBT)
Straightness, inextensibility, and normality

Timoshenko Beam Theory (TBT)
Straightness and inextensibility

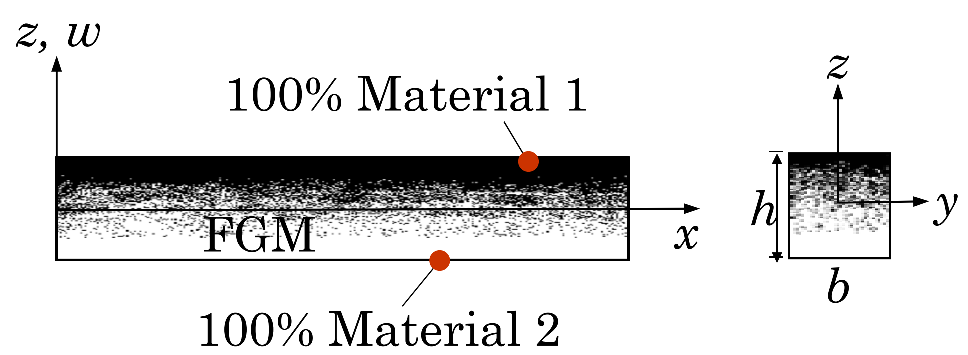
Kinematics of the Timoshenko Beam Theory



Kinematics of the Timoshenko Beam Theory



MATERIAL VARIATION THROUGH BEAM HEIGHT



$$P(z,T) = \left[P_c(T) - P_m(T)\right] f(z) + P_m(T), \quad f(z) = \left(\frac{1}{2} + \frac{z}{h}\right)^n$$

$$P_\alpha(T) = c_0\left(c_{-1}T^{-1} + 1 + c_1 T + c_2 T^2 + c_3 T^3\right), \ \alpha = c \text{ or } m$$

NONLINEAR ANALYSIS OF EULER-BERNOULLI BEAMS

Equilibrium equations

$$\frac{dN}{dx} + f = 0, \quad \frac{dM}{dx} - V = 0, \quad \frac{dV}{dx} + \frac{d}{dx}\left(N\frac{dw}{dx}\right) + q = 0$$

$$\boxed{\frac{dN}{dx} + f = 0, \qquad \frac{d^2M}{dx^2} + \frac{d}{dx}\left(N\frac{dw}{dx}\right) + q = 0}$$

Stress resultants in terms of deflection

$$N = \int_A \sigma_{xx}\ dA = \int_A \left[E(z)\left\{\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right\} - E(z)z\frac{d^2w}{dx^2}\right] dA$$

$$M = \int_A \sigma_{xx} \times z\ dA = \int_A \left[E(z)\left\{\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right\} - E(z)z\frac{d^2w}{dx^2}\right] z\ dA$$

$$V = \frac{dM}{dx}$$

Constitutive Relations for FGM Beams

Euler-Bernoulli Beam Theory

$$N_{xx} = \int_A \sigma_{xx}\, dA = \int_A E(z)\left(\frac{du}{dx} - z\frac{d^2w}{dx^2}\right) dA$$

$$= A_{xx}\frac{du}{dx} - B_{xx}\frac{d^2w}{dx^2}$$

$$M_{xx} = \int_A \sigma_{xx}\cdot z\, dA = \int_A E(z)\left(\frac{du}{dx} - z\frac{d^2w}{dx^2}\right) z\, dA$$

$$= B_{xx}\frac{du}{dx} - D_{xx}\frac{d^2w}{dx^2}$$

$$A_{xx} = \int_A E(z)\, dA,\;\; B_{xx} = \int_A zE(z)\, dA,\; D_{xx} = \int_A z^2E(z)\, dA$$

We now see that the axial displacements are coupled to the bending displacements.

NONLINEAR ANALYSIS OF Timoshenko Beams

Equilibrium equations

$$\frac{dN}{dx}+f=0,\quad \frac{dM}{dx}-V=0,\quad \frac{dV}{dx}+\frac{d}{dx}\left(N\frac{dw}{dx}\right)+q=0$$

Stress resultants in terms of deflection

$$N=\int_A \sigma_{xx}\ dA=\int_A\left[E(z)\left\{\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right\}+E(z)z\frac{d\phi_x}{dx}\right]dA$$

$$M=\int_A \sigma_{xx}\times z\ dA=\int_A\left[E(z)\left\{\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right\}+E(z)z\frac{d\phi_x}{dx}\right]z\ dA$$

$$V=K_s\int_A \sigma_{xz}\ dA=K_s\int_A G(z)\left(\phi_x+\frac{dw}{dx}\right)dA$$

Constitutive Relations for FGM Beams

Timoshenko Beam Theory

$$N_{xx} = \int_A \sigma_{xx}\ dA = \int_A E(z)\left(\frac{du}{dx} + z\frac{d\phi_x}{dx}\right) dA$$

$$= A_{xx}\frac{du}{dx} + B_{xx}\frac{d\phi_x}{dx}$$

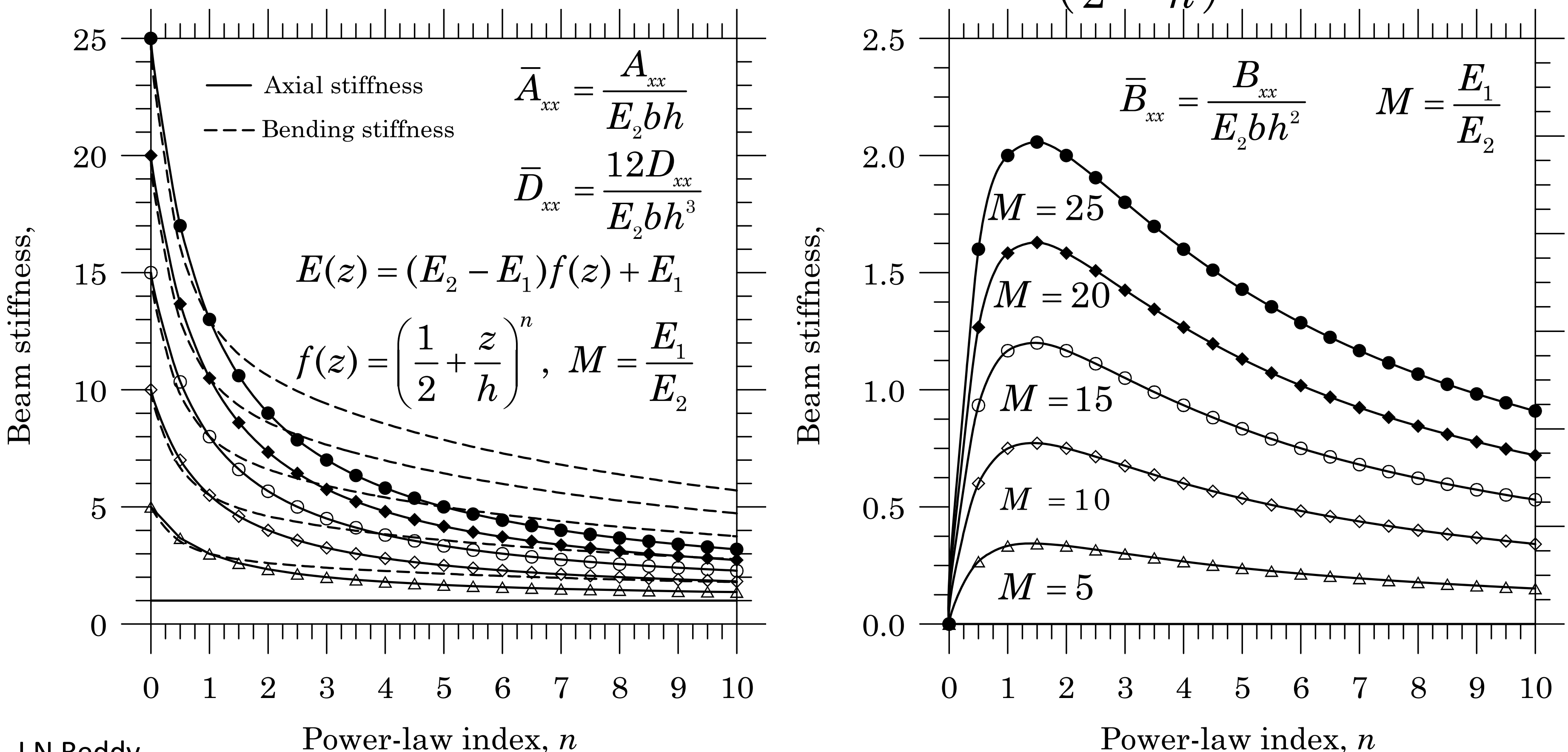
$$M_{xx} = \int_A \sigma_{xx} \cdot z\ dA = \int_A E(z)\left(\frac{du}{dx} + z\frac{d\phi_x}{dx}\right) z\ dA$$

$$= B_{xx}\frac{du}{dx} + D_{xx}\frac{d\phi_x}{dx}$$

$$A_{xx} = \int_A E(z)\ dA,\ \ B_{xx} = \int_A zE(z)\ dA,\ \ D_{xx} = \int_A z^2E(z)\ dA$$

VARIATIONS OF STRUCTURAL STIFFNESSES WITH POWER-LAW INDEX *n*

$$E(z) = \left[E_1 - E_2\right] f(z) + E_2, \quad f(z) = \left(\frac{1}{2} + \frac{z}{h}\right)^n$$



J N Reddy
JN Reddy
Beams 14
Beams:

TIMOSHENKO BEAM THEORY

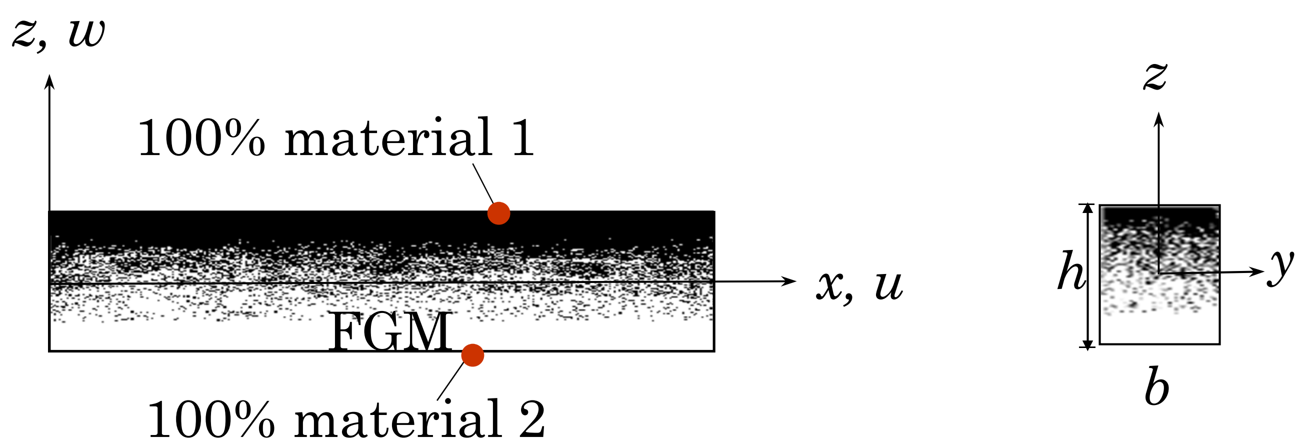
Governing equations (nonlinear) in terms of the generalized displacements

$$-\frac{d}{dx}\left\{A_{xx}\left[\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]+B_{xx}\frac{d\phi_x}{dx}\right\}-f=0$$

$$-\frac{d}{dx}\left[S_{xz}\left(\phi_x+\frac{dw}{dx}\right)\right]-\frac{d}{dx}\left[\frac{dw}{dx}\left\{A_{xx}\left[\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]+B_{xx}\frac{d\phi_x}{dx}\right\}\right]-q=0$$

$$-\frac{d}{dx}\left\{B_{xx}\left[\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]+D_{xx}\frac{d\phi_x}{dx}\right\}+S_{xz}\left(\phi_x+\frac{dw}{dx}\right)=0$$

FUNCTIONALLY GRADED BEAMS



Property variation (FGM)

$$E(z) = [E_1 - E_2] f(z) + E_2, \;\; f(z) = \left(\frac{1}{2} + \frac{z}{h}\right)^n$$

$$E_1 = 14.4 \text{ GPa}$$
$$E_2 = 1.44 \text{ GPa}$$
$$\nu = 0.38$$

$$h = 17.6 \times 10^{-6} \text{ m}$$
$$b = 2h, L = 20h,$$

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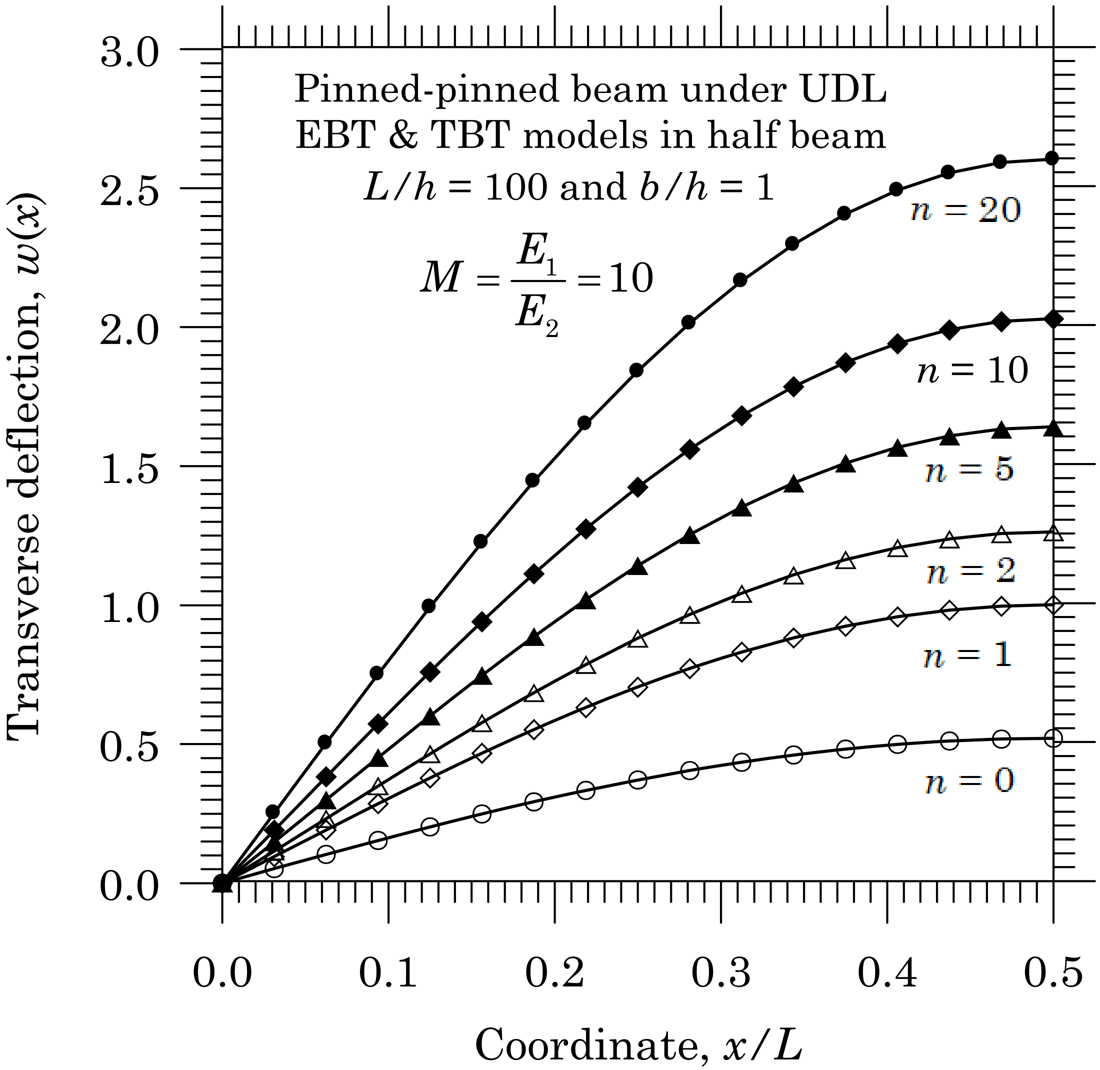
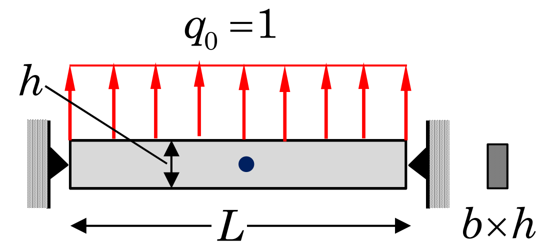
NUMERICAL EXAMPLES

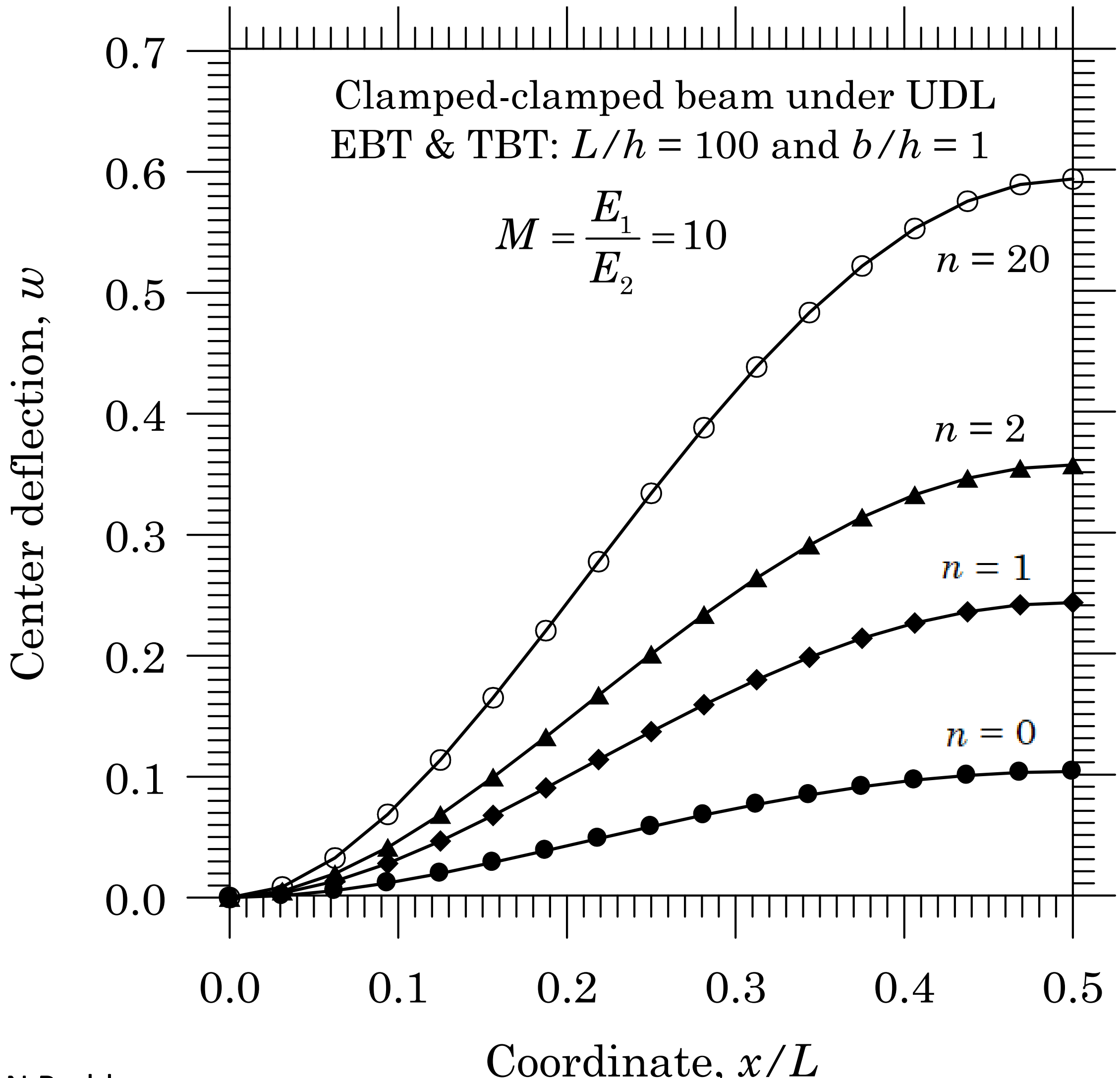


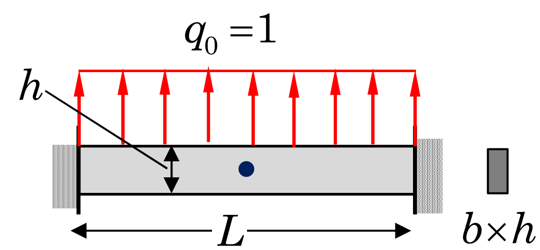


$$E(z) = \left[E_1 - E_2\right] f(z) + E_2$$

$$f(z) = \left(\frac{1}{2} + \frac{z}{h}\right)^n$$

NUMERICAL EXAMPLES

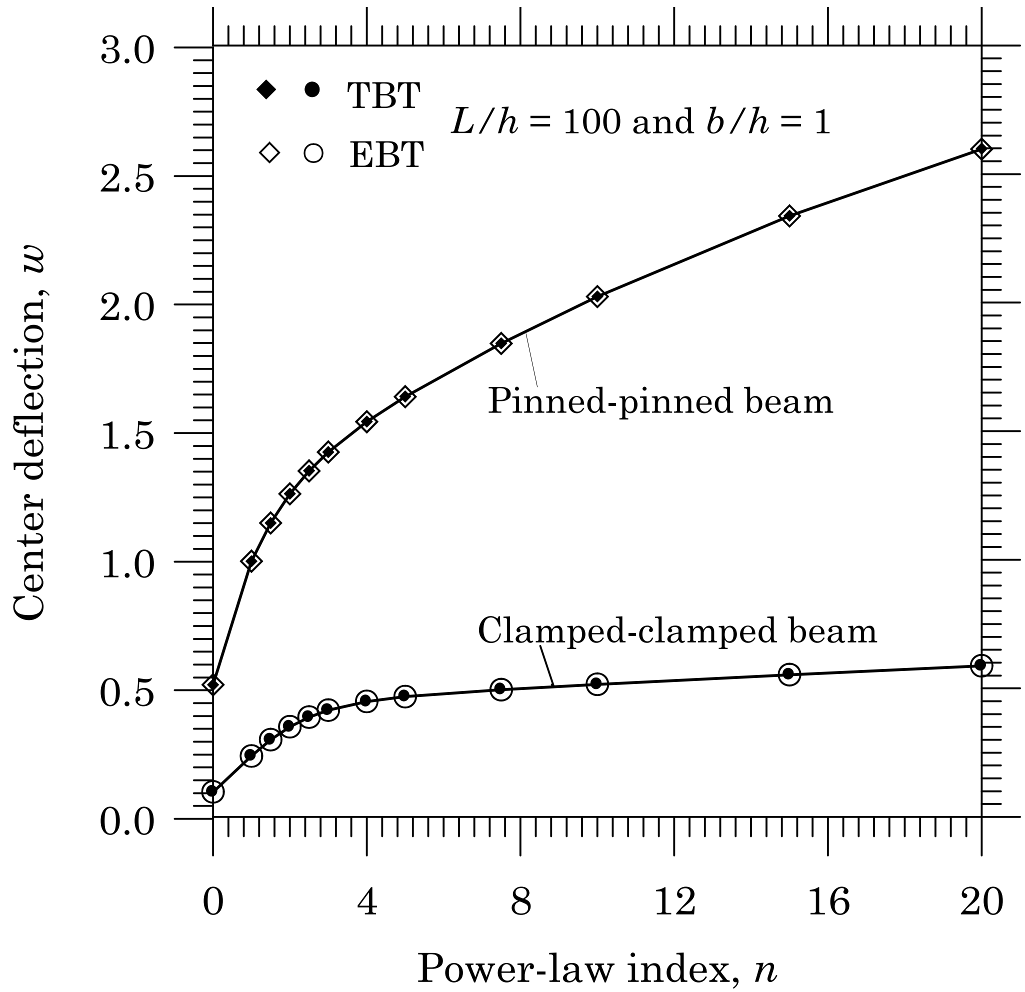




$$E(z) = \left[E_1 - E_2\right] f(z) + E_2$$

$$f(z) = \left(\frac{1}{2} + \frac{z}{h}\right)^n$$

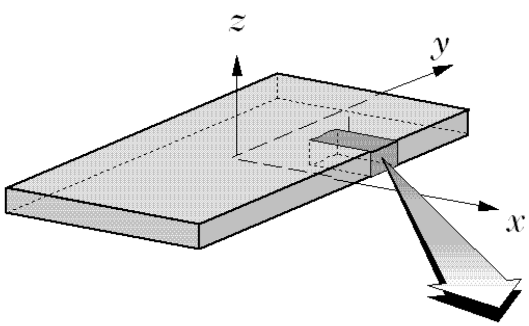
NUMERICAL EXAMPLES

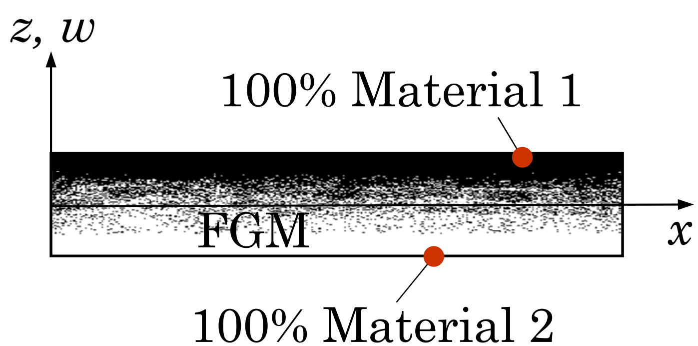


FUNCTIONALLY GRADED PLATES (First-Order Plate Theory)

CONTENTS OF THE LECTURE

- Governing equations
- FGM plate constitutive realtions
- Numerical Results





$$E(z) = \left[E_1 - E_2\right] f(z) + E_2, \quad f(z) = \left(\frac{1}{2} + \frac{z}{h}\right)^n$$

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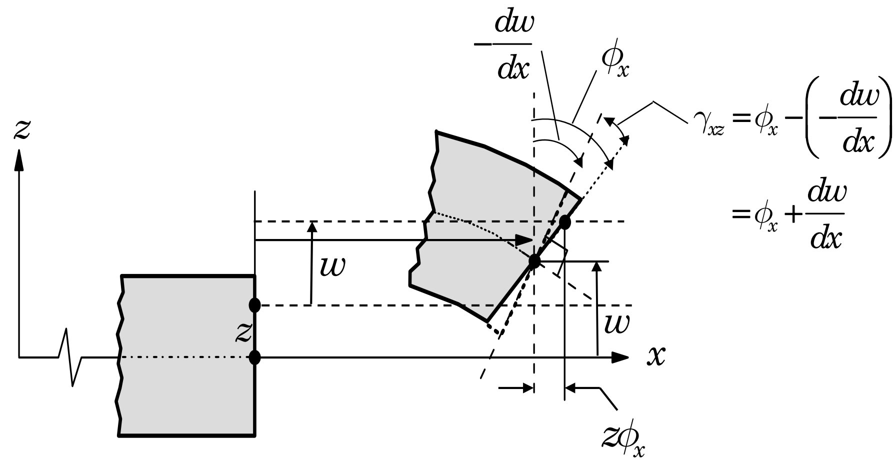
Beams:

FIRST-ORDER SHEAR DEFORMATION THEORY (FSDT)

Displacement Field

$$u_1(x,y,z,t) = u(x,y,t) + z\phi_x(x,y,t)$$
$$u_2(x,y,z,t) = v(x,y,t) + z\phi_y(x,y,t)$$
$$u_3(x,y,z,t) = w(x,y,t)$$



Linearized strains

$$\varepsilon_{xx} = \frac{\partial u_1}{\partial x} = \frac{\partial u}{\partial x} + z\frac{\partial \phi_x}{\partial x},$$

$$\varepsilon_{yy} = \frac{\partial u_2}{\partial y} = \frac{\partial v}{\partial y} + z\frac{\partial \phi_y}{\partial y}$$

$$\gamma_{xy} = \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} + \frac{\partial u_3}{\partial x}\frac{\partial u_3}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + z\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right)$$

$$\gamma_{xz} = \frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x} = \phi_x + \frac{\partial w}{\partial x}, \quad \gamma_{yz} = \frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y} = \phi_y + \frac{\partial w}{\partial y}$$

EQUATIONS OF MOTION OF FSDT

$$\frac{\partial N_{xx}}{\partial x}+\frac{\partial N_{xy}}{\partial y}+f_x=I_0\frac{\partial^2 u}{\partial t^2}+I_1\frac{\partial^2 \phi_x}{\partial t^2} \qquad (u,\ N_n)$$

$$\frac{\partial N_{xy}}{\partial x}+\frac{\partial N_{yy}}{\partial y}+f_y=I_0\frac{\partial^2 v}{\partial t^2}+I_1\frac{\partial^2 \phi_y}{\partial t^2} \qquad (v,\ N_{ns})$$

$$\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+\frac{\partial}{\partial x}\left(N_{xx}\frac{\partial w}{\partial x}+N_{xy}\frac{\partial w}{\partial y}\right)$$

$$+\frac{\partial}{\partial y}\left(N_{xy}\frac{\partial w}{\partial x}+N_{yy}\frac{\partial w}{\partial y}\right)+q=I_0\frac{\partial^2 w}{\partial t^2} \qquad (w,\ Q_n)$$

$$\frac{\partial M_{xx}}{\partial x}+\frac{\partial M_{xy}}{\partial y}-Q_x=I_2\frac{\partial^2 \phi_x}{\partial t^2}+I_1\frac{\partial^2 u}{\partial t^2} \qquad (\phi_n,\ M_n)$$

$$\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_{yy}}{\partial y}-Q_y=I_2\frac{\partial^2 \phi_y}{\partial t^2}+I_1\frac{\partial^2 v}{\partial t^2} \qquad (\phi_{ns},\ M_{ns})$$

$$N_n=N_{xx}n_x+N_{xy}n_y;\qquad N_{ns}=N_{xy}n_x+N_{yy}n_y$$

$$M_n=M_{xx}n_x+M_{xy}n_y;\qquad M_{ns}=M_{xy}n_x+M_{yy}n_y;\quad Q_n=Q_xn_x+Q_yn_y$$

J N Reddy

Beams:

Stress Resultant-Displacement Relations (nonlinear)

$$
N_{xx} = \int_{-h/2}^{h/2} \sigma_{xx}\, dz = \int_{-h/2}^{h/2} \left\{ Q_{11}\left(\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - z\frac{\partial^2 w}{\partial x^2} \right) + Q_{12}\left(\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - z\frac{\partial^2 w}{\partial y^2} \right) \right\} dz
$$

$$
+ \int_{-h/2}^{h/2} Q_{16}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} - 2z\frac{\partial^2 w}{\partial x \partial y} \right) dz
$$

$$
= A_{11}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \right] + A_{12}\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \right] + A_{16}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \right]
$$

$$
- B_{11}\frac{\partial^2 w}{\partial x^2} - B_{12}\frac{\partial^2 w}{\partial y^2} - 2B_{16}\frac{\partial^2 w}{\partial x \partial y}
$$

$$
M_{xx} = \int_{-h/2}^{h/2} \sigma_{xx}\, z\, dz = \int_{-h/2}^{h/2} z\left[Q_{11}\left(\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - z\frac{\partial^2 w}{\partial x^2} \right) + Q_{12}\left(\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - z\frac{\partial^2 w}{\partial y^2} \right) \right] dz
$$

$$
+ \int_{-h/2}^{h/2} z\, Q_{16}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} - 2z\frac{\partial^2 w}{\partial x \partial y} \right) dz
$$

$$
= B_{11}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \right] + B_{12}\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \right] + B_{16}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \right)
$$

$$
- D_{11}\frac{\partial^2 w}{\partial x^2} - D_{12}\frac{\partial^2 w}{\partial y^2} - 2D_{16}\frac{\partial^2 w}{\partial x \partial y}
$$

THE FIRST-ORDER SHEAR DEFORMATION THEORY
Stress Resultants (linear)

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix}$$

$$Q_{11} = \frac{E(z)}{1-\nu_2} \quad Q_{22} = \frac{E(z)}{1-\nu^2}$$

$$Q_{12} = \frac{\nu E(z)}{1-\nu_2} \quad Q_{66} = G(z)$$

$$N_{xx} = A_{11}\frac{\partial u}{\partial x} + A_{12}\frac{\partial v}{\partial y} + A_{16}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + B_{11}\frac{\partial \phi_x}{\partial x} + B_{12}\frac{\partial \phi_y}{\partial y} + B_{16}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right)$$

$$N_{yy} = A_{12}\frac{\partial u}{\partial x} + A_{22}\frac{\partial v}{\partial y} + A_{26}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + B_{12}\frac{\partial \phi_x}{\partial x} + B_{22}\frac{\partial \phi_y}{\partial y} + B_{26}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right)$$

$$M_{xx} = B_{11}\frac{\partial u}{\partial x} + B_{12}\frac{\partial v}{\partial y} + B_{16}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + D_{11}\frac{\partial \phi_x}{\partial x} + D_{12}\frac{\partial \phi_y}{\partial y} + D_{16}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right)$$

$$M_{yy} = B_{12}\frac{\partial u}{\partial x} + B_{22}\frac{\partial v}{\partial y} + B_{26}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + D_{12}\frac{\partial \phi_x}{\partial x} + D_{22}\frac{\partial \phi_y}{\partial y} + D_{26}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right)$$

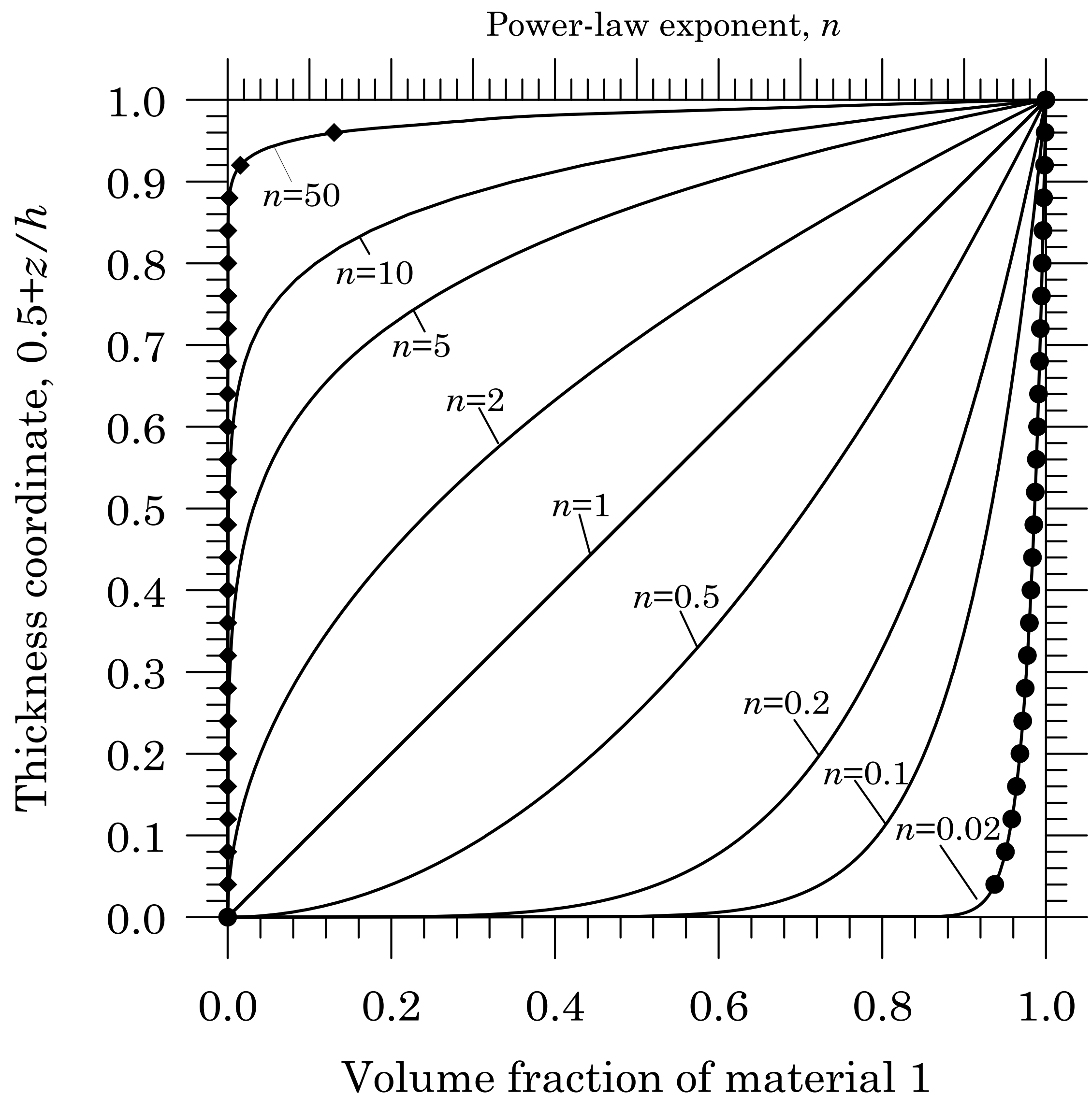
$$M_{xy} = B_{16}\frac{\partial u}{\partial x} + B_{26}\frac{\partial v}{\partial y} + B_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + D_{16}\frac{\partial \phi_x}{\partial x} + D_{26}\frac{\partial \phi_y}{\partial y} + D_{66}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right)$$

$$Q_x = K_s A_{55}\left(\phi_x + \frac{\partial w}{\partial x}\right) + K_s A_{45}\left(\phi_y + \frac{\partial w}{\partial y}\right); \; Q_y = K_s A_{45}\left(\phi_x + \frac{\partial w}{\partial x}\right) + K_s A_{44}\left(\phi_y + \frac{\partial w}{\partial y}\right)$$

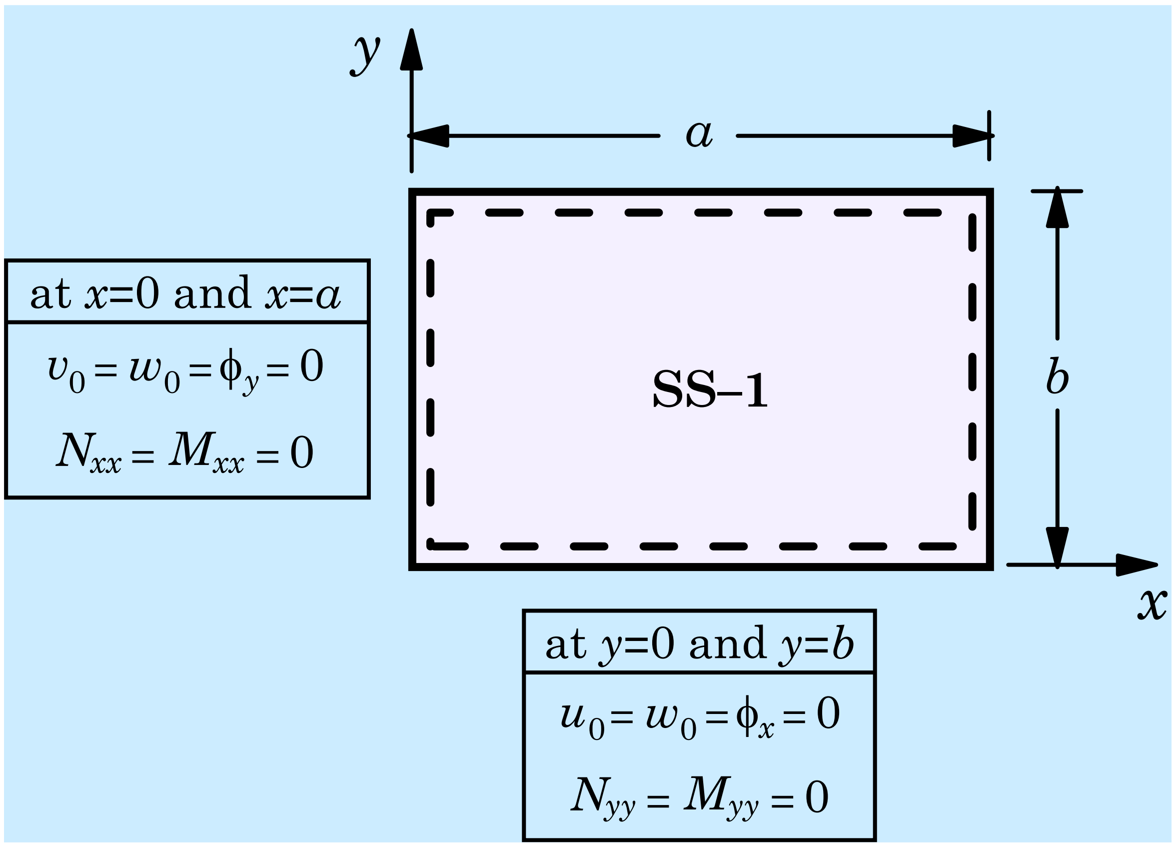
J N Reddy

Beams:

Variation of metal volume fraction through the plate thickness



Boundary Conditions of a Simply Supported Plate



Aluminum and Zirconia Material Properties

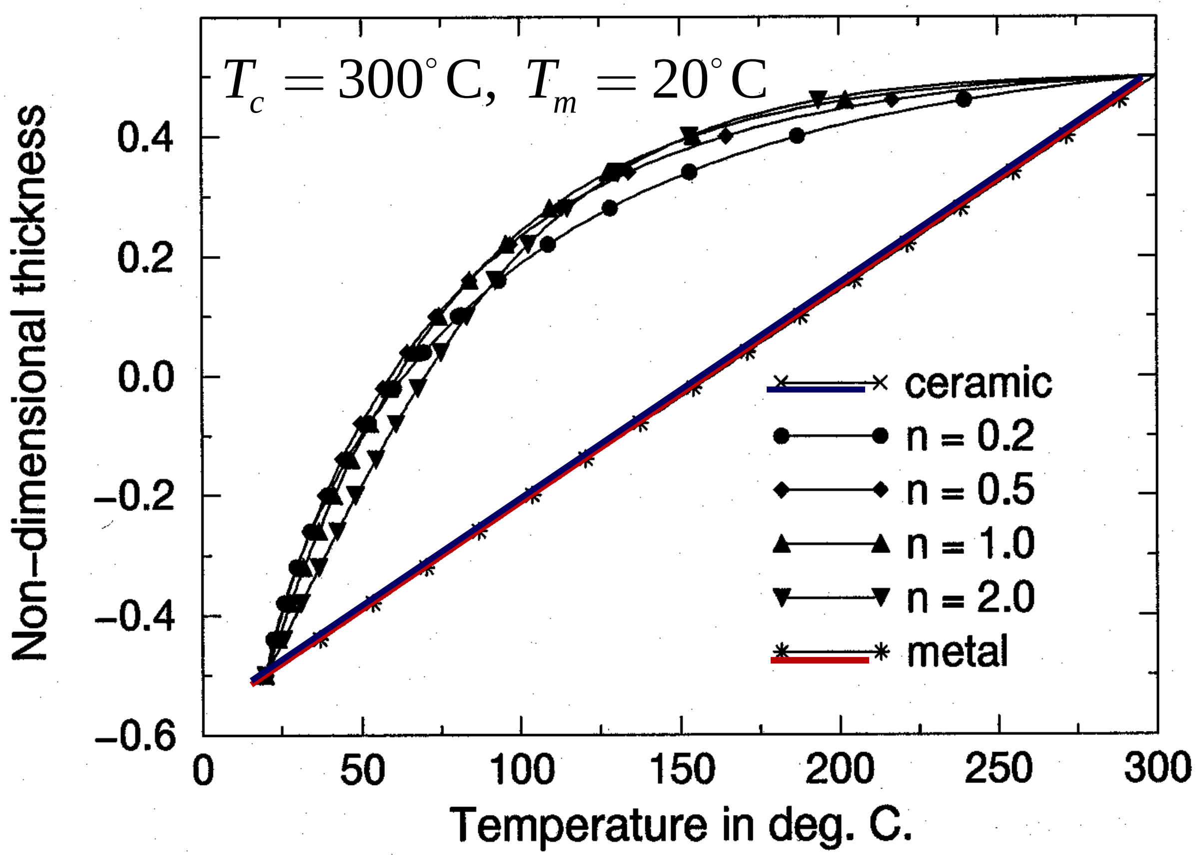
Aluminum (bottom surface)

$$E_1 = 70 \text{ Gpa}, \ \nu = 0.3, \ \ \rho = 2707 \text{ kg/m}^3,$$
$$k = 204 \text{ W/(m.K)}, \ \alpha = 23 \ \times 10^{-6} \ / \ ^{\text{o}}\text{C}$$

Zirconia (top surface)

$$E_1 = 151 \text{ Gpa}, \ \nu = 0.3, \ \ \rho = 3000 \text{ kg/m}^3,$$
$$k = 2.09 \text{ W/(m.K)}, \ \alpha = 23 \ \times 10^{-5} \ / \ ^{\text{o}}\text{C}$$

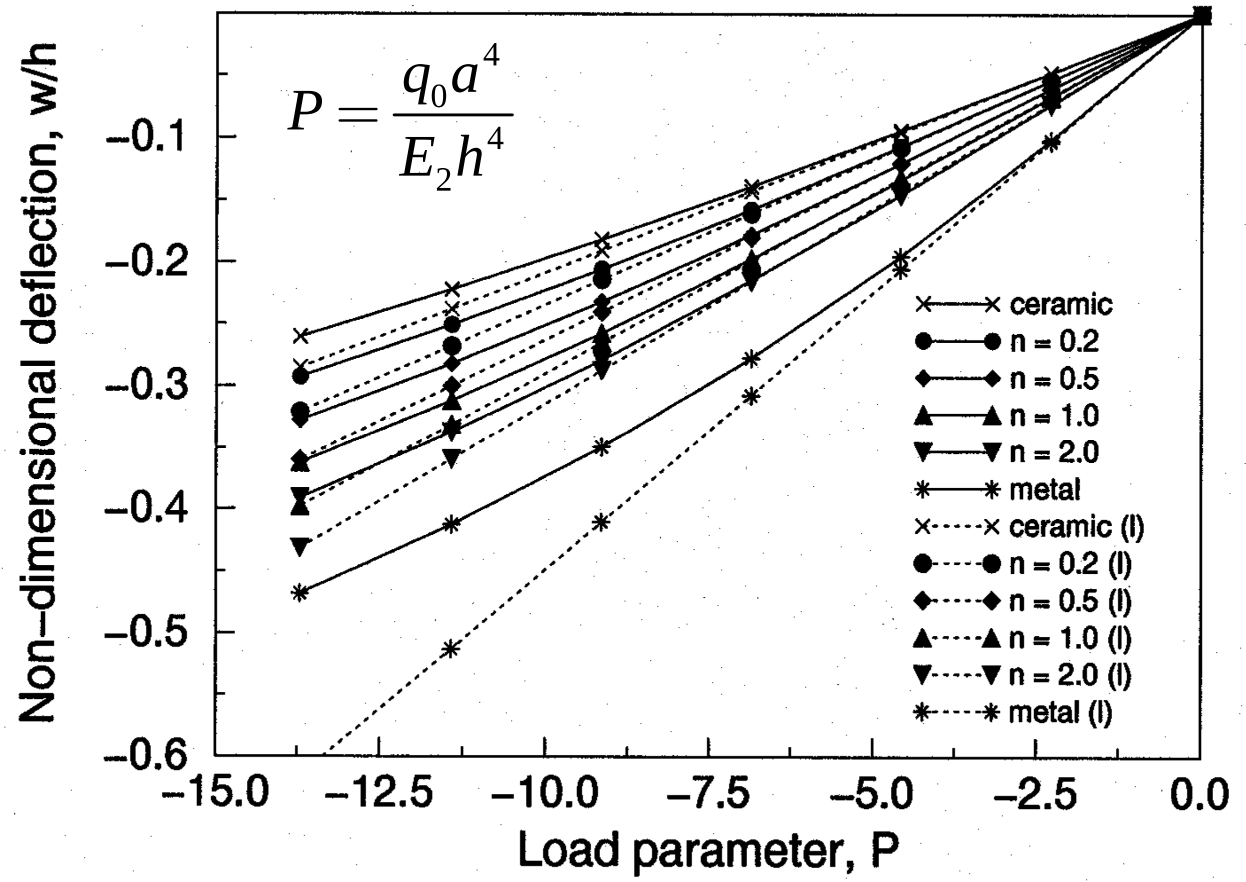
Temperature variation through the thickness of the plate



aluminum-zirconia FGM plate

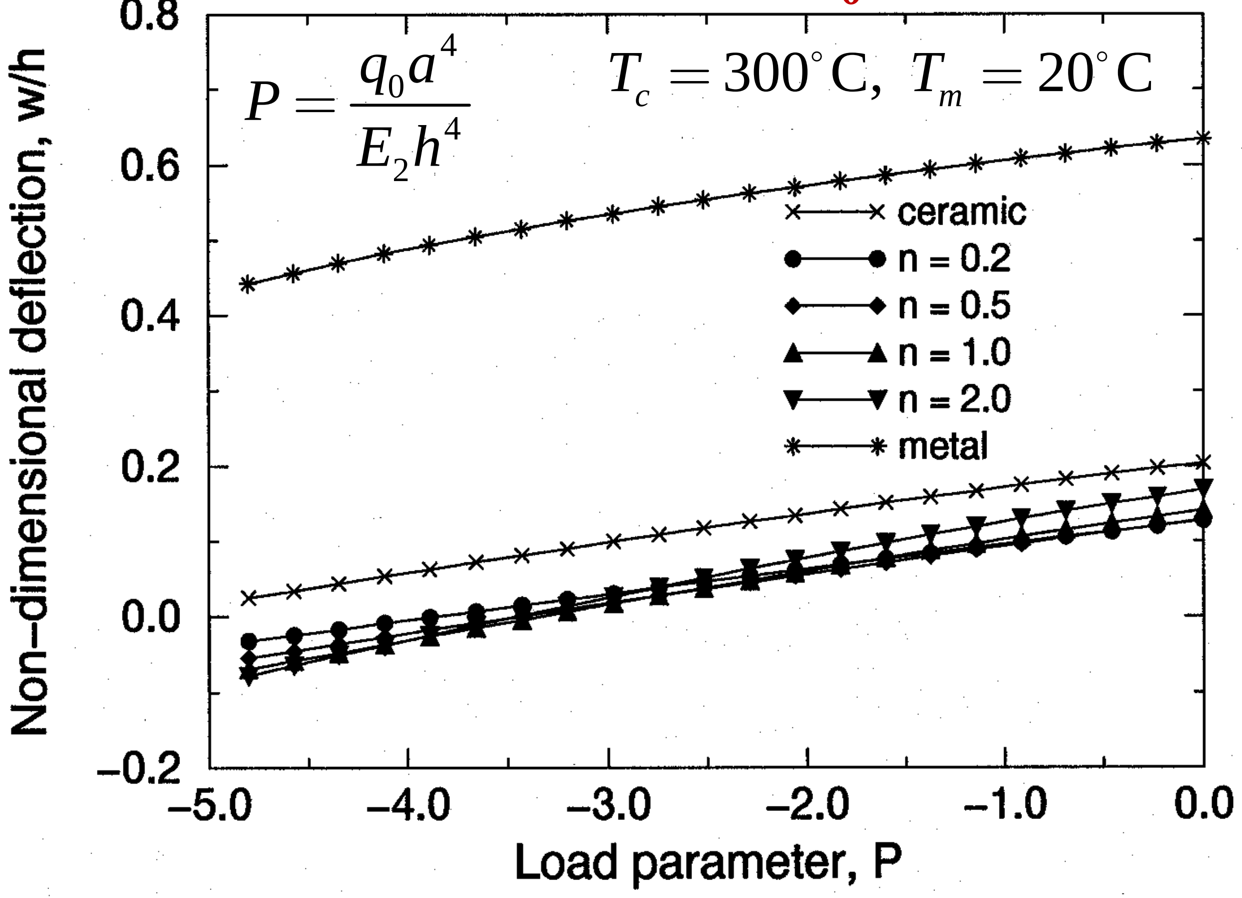
Center deflection vs load parameter for a simply supported FGM plate under uniform pressure

$a = 0.2\text{m},\ h = 0.01\text{m},\ q_0 = -10^4\ \text{N/m}^2$

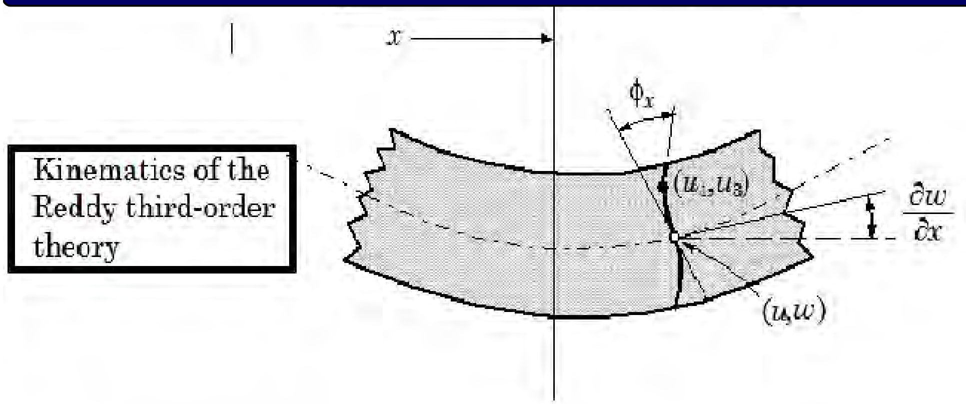


Center deflection vs load parameter for a simply supported FGM plate under uniform pressure and temperature variation

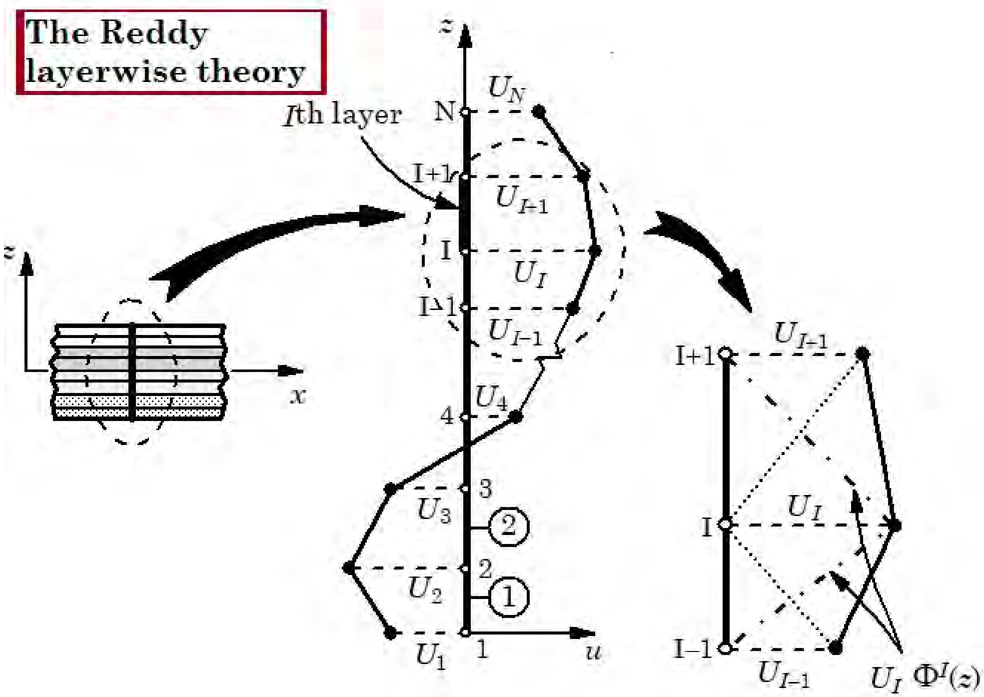
$a = 0.2\text{m},\ h = 0.01\text{m},\ q_0 = -10^4\ \text{N/m}^2$



THIRD-ORDER SHEAR DEFORMATION AND LAYERWISE THEORIES OF PLATES



The third-order shear deformation theory

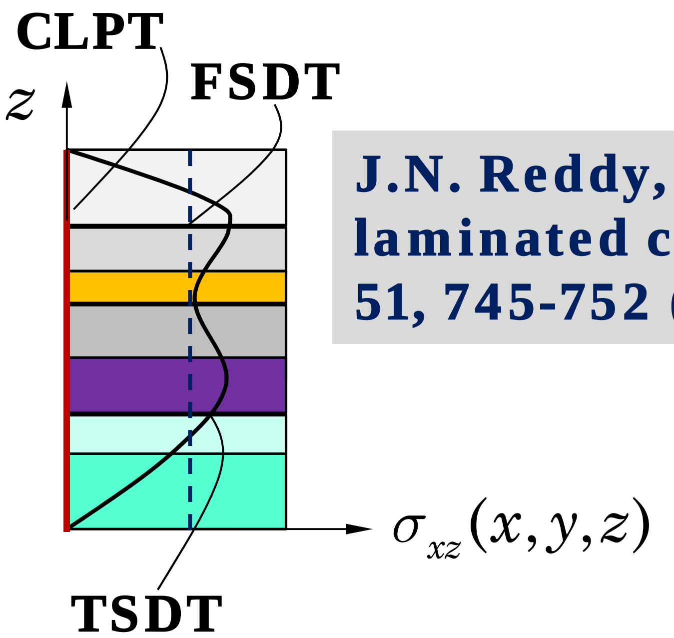


The layerwise laminate theory

CONTENTS OF THE LECTURE

- Kinematics of deformation of the TSDT
  - Displacement field
  - Strains
  - Equations of motion
  - Numerical results
- Layerwise Theory
  - Displacement field
  - Strains
  - Equations of motion
  - Numerical results
- Summary of the lecture

THIRD-ORDER SHEAR DEFORMATION THEORY



J.N. Reddy, "A simple higher-order theory for laminated composite plates," *J. of Applied Mechanics*, 51, 745-752 (1984). (3560 citations)

Transverse shear stress

$$\sigma_{xz}(x,y,z) = Q_{55}\gamma_{xz} + Q_{45}\gamma_{yz}$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}; \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

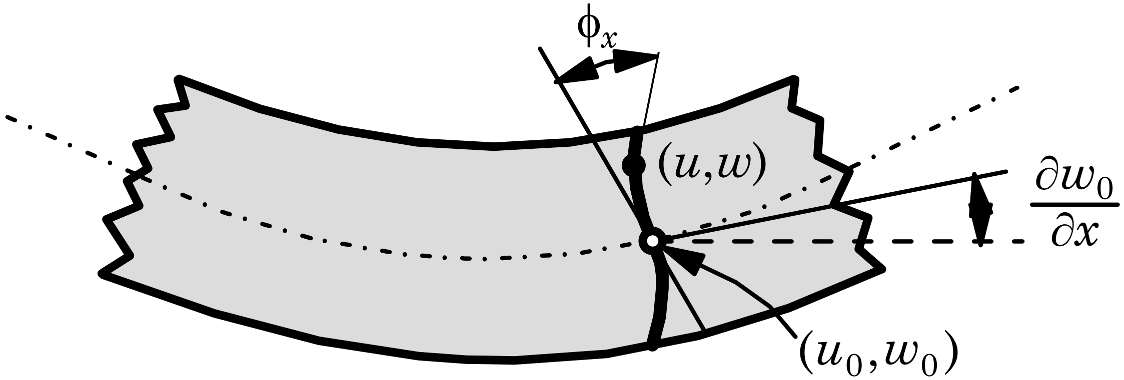
Displacement field

$$u(x,y,z) = u_0(x,y) + z\phi_x(x,y) + z^2\theta_x(x,y) + z^3\lambda_x(x,y)$$

$$v(x,y,z) = v_0(x,y) + z\phi_y(x,y) + z^2\theta_y(x,y) + z^3\lambda_y(x,y)$$

$$w(x,y,z) = w_0(x,y)$$

Third-Order Shear Deformation Plate Theory (TSDT) –Displacement Field



Assumed Displacement Field

$$u_1(x,y,z,t) = u(x,y,t) + z\phi_x(x,y,t) + z^2\psi_x(x,y,t) + z^3\theta_x(x,y,t)$$

$$u_2(x,y,z,t) = v(x,y,t) + z\phi_y(x,y,t) + z^2\psi_y(x,y,t) + z^3\theta_y(x,y,t)$$

$$u_3(x,y,z,t) = w(x,y,t)$$

Third-Order Shear Deformation Plate Theory (TSDT) –Displacement Field

Reduction of the Displacement Field : Require that the top and bottom faces of the plate are free of shear stress

$$\sigma_{xz}(x,y,\pm\frac{h}{2},t)=\sigma_{yz}(x,y,\pm\frac{h}{2},t)=0$$

$$\Rightarrow \gamma_{xz}(x,y,\pm\frac{h}{2},t)=\gamma_{yz}(x,y,\pm\frac{h}{2},t)=0$$

$$\gamma_{xz}=\phi_x+\frac{\partial w}{\partial x}+2z\psi_x+3z^2\theta_x$$

$$\phi_x+\frac{\partial w}{\partial x}-h\psi_x+\frac{3h^2}{4}\theta_x=0,\ \ \phi_x+\frac{\partial w}{\partial x}+h\psi_x+\frac{3h^2}{4}\theta_x=0$$

$$\Rightarrow \boxed{\theta_x=-\frac{4}{3h^2}\left(\phi_x+\frac{\partial w}{\partial x}\right),\qquad \psi_x=0}$$

Displacement Field of the *Reddy Third-Order Laminate Plate Theory*

Displacement Field

$$u_1(x,y,z,t) = u(x,y,t) + z\phi_x(x,y,t) + z^3\left(-\frac{4}{3h^2}\right)\left(\phi_x + \frac{\partial w}{\partial x}\right)$$

$$u_2(x,y,z,t) = v(x,y,t) + z\phi_y(x,y,t) + z^3\left(-\frac{4}{3h^2}\right)\left(\phi_y + \frac{\partial w}{\partial y}\right)$$

$$u_3(x,y,z,t) = w(x,y,t)$$

Strain Field

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix}^{(0)} + z\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix}^{(1)} + z^3\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix}^{(3)}$$

$$\begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}^{(0)} + z^2\begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}^{(2)}$$

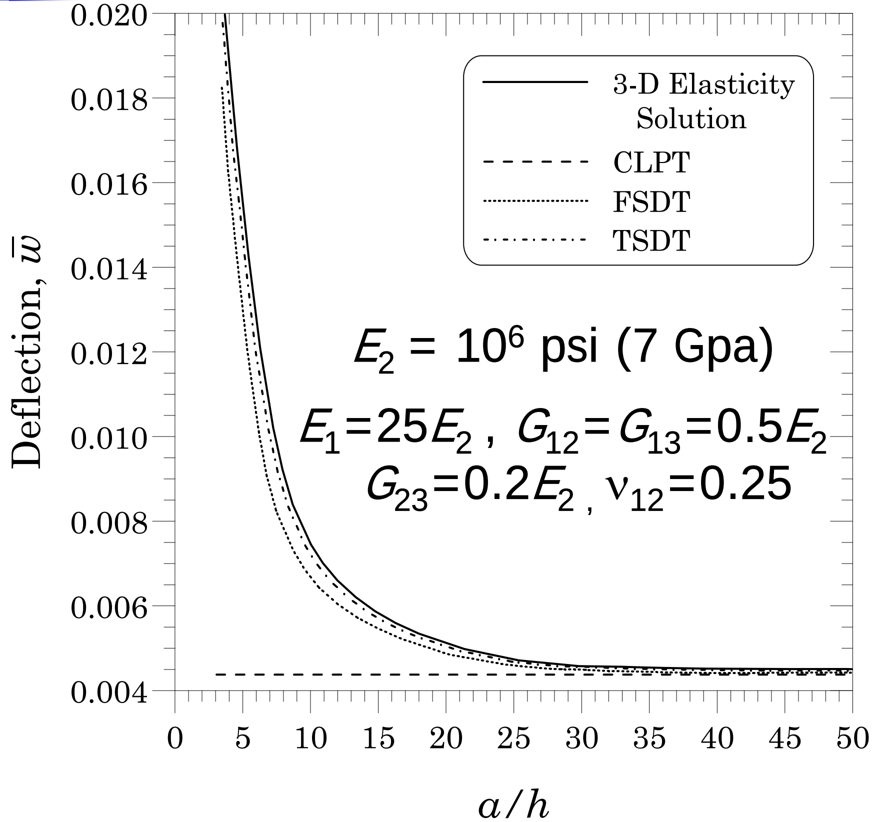
Quadratic variation

Strain Field of the *Reddy Third-Order Laminate Plate Theory-cont.*

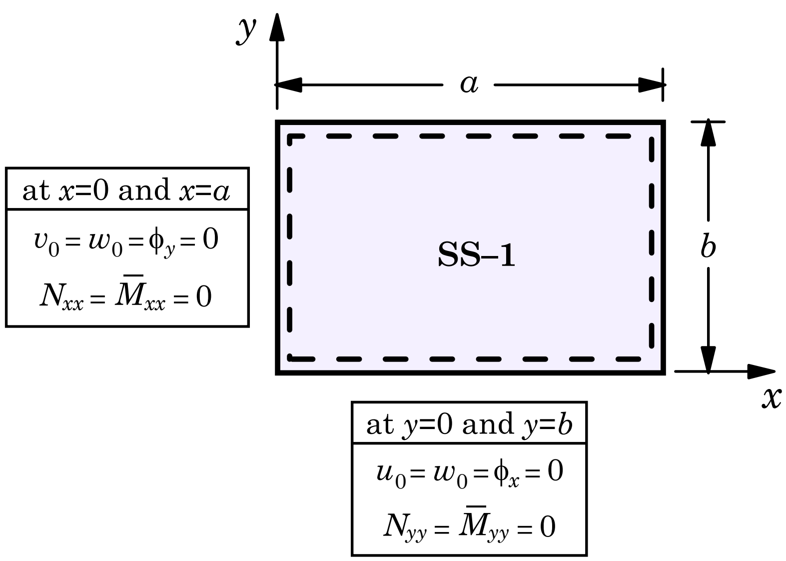
$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix}^{(0)} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} + \frac{1}{2}\left(\dfrac{\partial w}{\partial x}\right)^2 \\ \dfrac{\partial v}{\partial y} + \frac{1}{2}\left(\dfrac{\partial w}{\partial y}\right)^2 \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} + \dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} \end{Bmatrix}, \quad \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix}^{(1)} = \begin{Bmatrix} \dfrac{\partial \phi_x}{\partial x} \\ \dfrac{\partial \phi_y}{\partial y} \\ \dfrac{\partial \phi_x}{\partial y} + \dfrac{\partial \phi_y}{\partial x} \end{Bmatrix}, \quad c_1 = \frac{4}{3h^2}$$

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix}^{(3)} = -c_1 \begin{Bmatrix} \dfrac{\partial \phi_x}{\partial x} + \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial \phi_y}{\partial y} + \dfrac{\partial^2 w}{\partial y^2} \\ \dfrac{\partial \phi_x}{\partial y} + \dfrac{\partial \phi_y}{\partial x} + 2\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix}, \quad \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}^{(0)} = -\frac{1}{3c_1}\begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}^{(2)} = \begin{Bmatrix} \phi_x + \dfrac{\partial w}{\partial x} \\ \phi_y + \dfrac{\partial w}{\partial y} \end{Bmatrix}$$

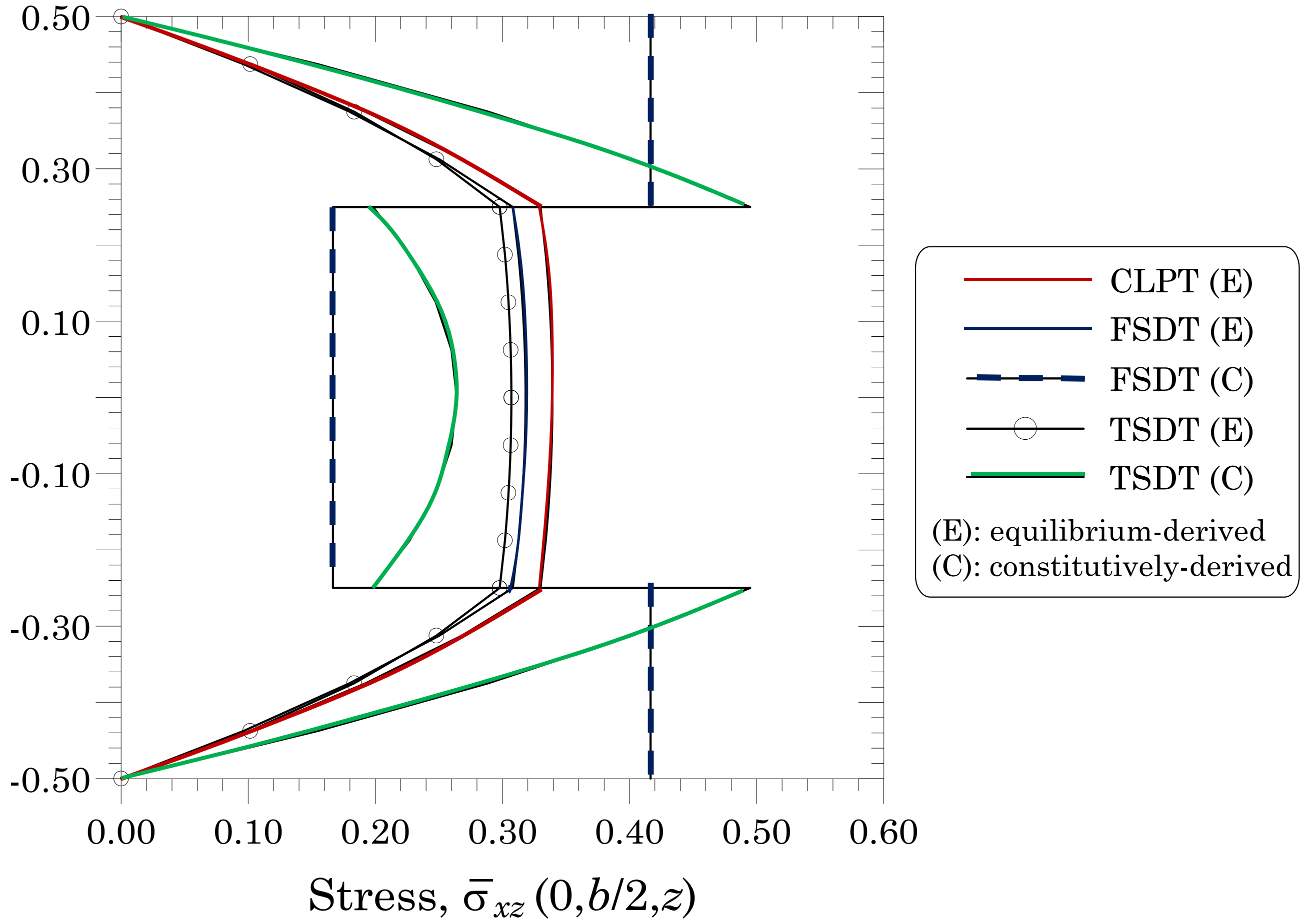
Bending of a symmetric cross-ply (0/90/90/0) laminate under uniformly distributed load



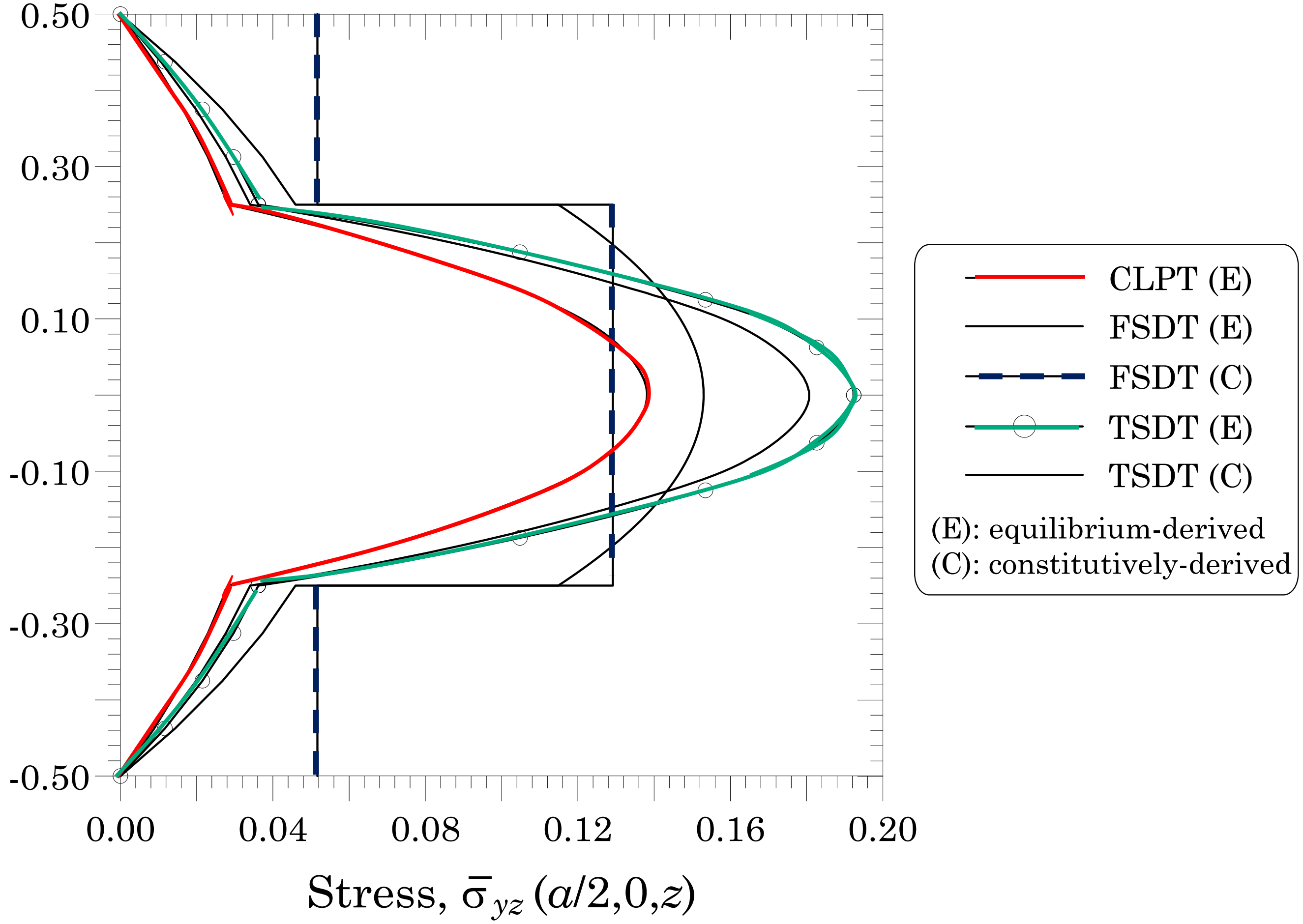
SS-1 Boundary Conditions



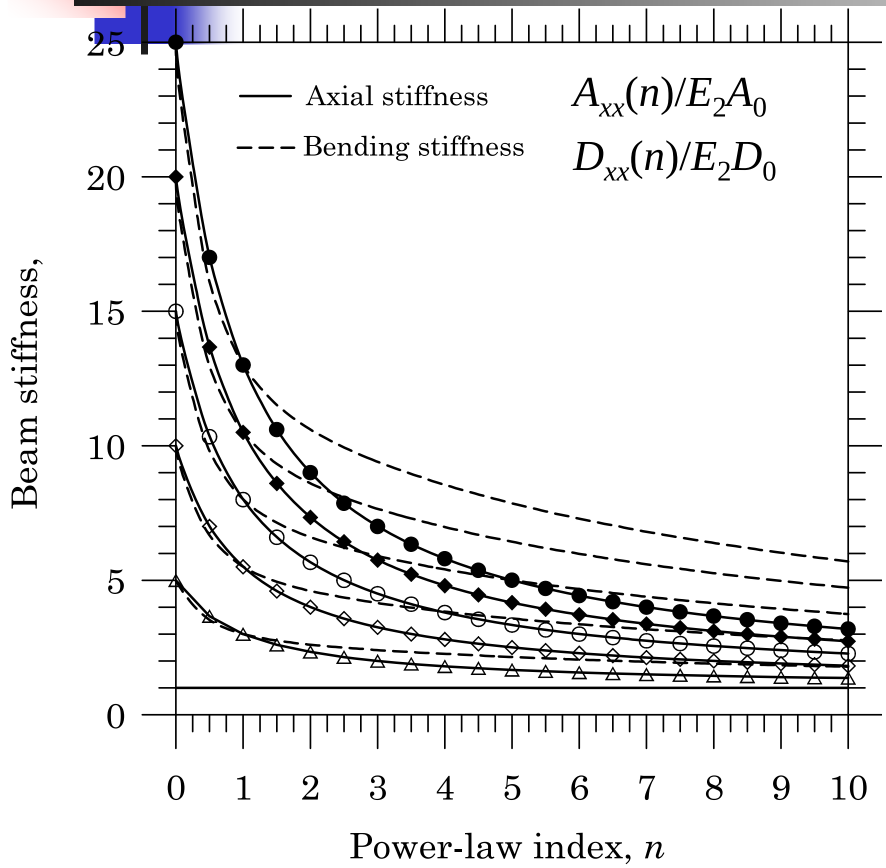
Bending of a symmetric cross-ply (0/90/90/0) laminate under uniformly distributed load

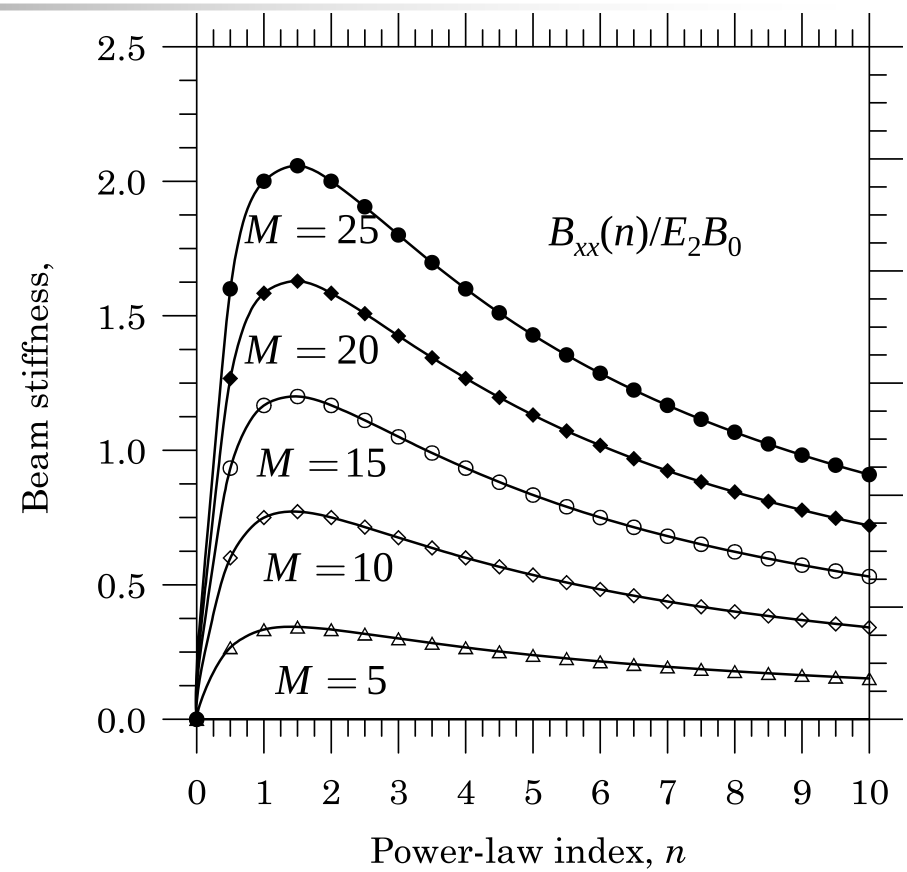


Bending of a symmetric cross-ply (0/90/90/0) laminate under uniformly distributed load



VARIATION OF THE PLATE STIFFNESSES of the FGM Beams





Numerical Results for the FGM Plates

$a = 20h,\ b = 20h,$

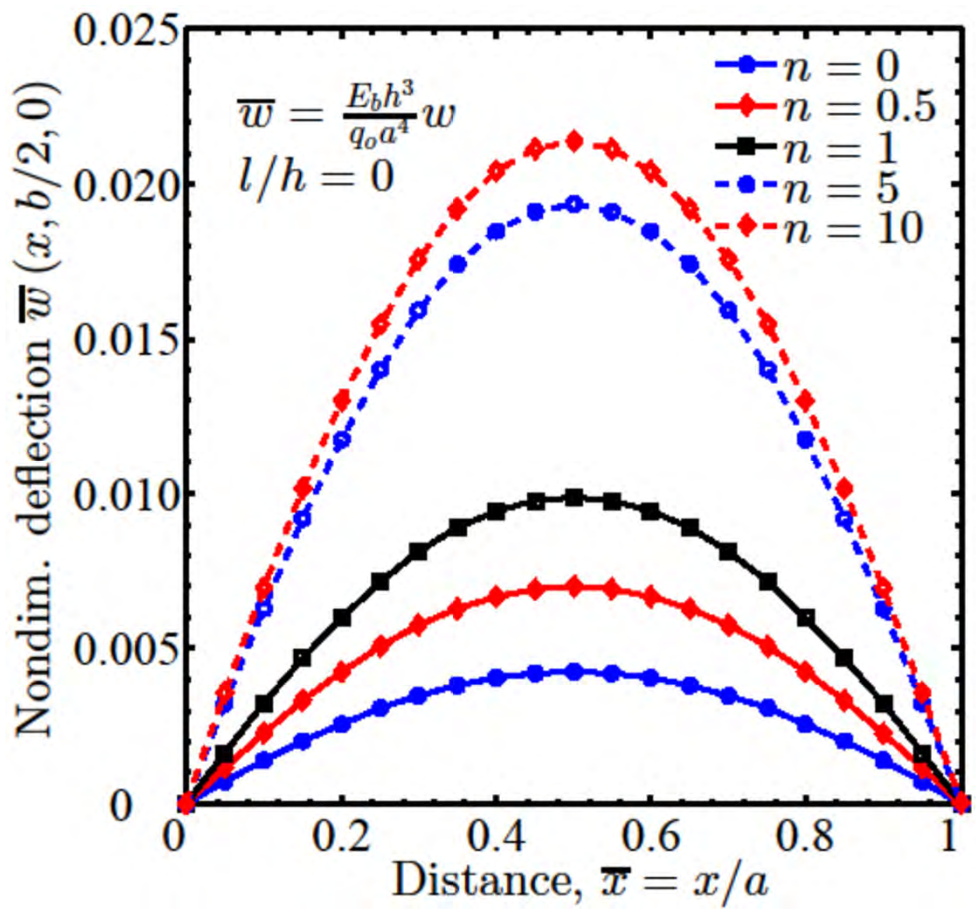
$h = 17.6\times10^{-6}\,\text{m}$

$E_t = 14.4\,\text{GPa},$

$E_b = 1.44\,\text{GPa}$

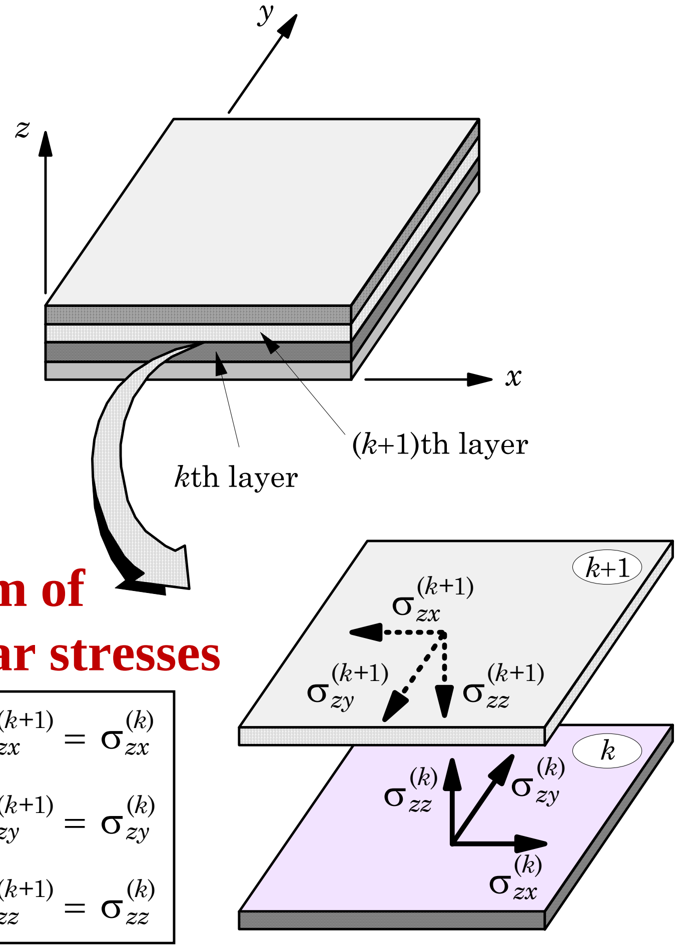
$\rho_t = 12.2\times10^{3}\,kg/m,$

$\rho_b = 1.22\times10^{3}\,kg/m$



Non-dimensional deflection $\bar{w}(x, b/2, 0)$ versus *x/a* distance along a FGM simply supported plate with various values of the power-law index, *n*

LAYERWISE THEORY: Background

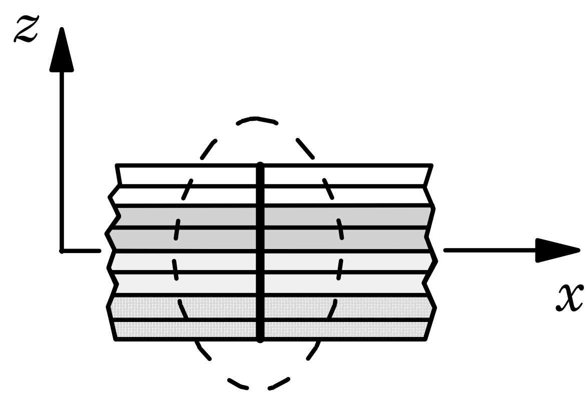


Equilibrium of interlaminar stresses

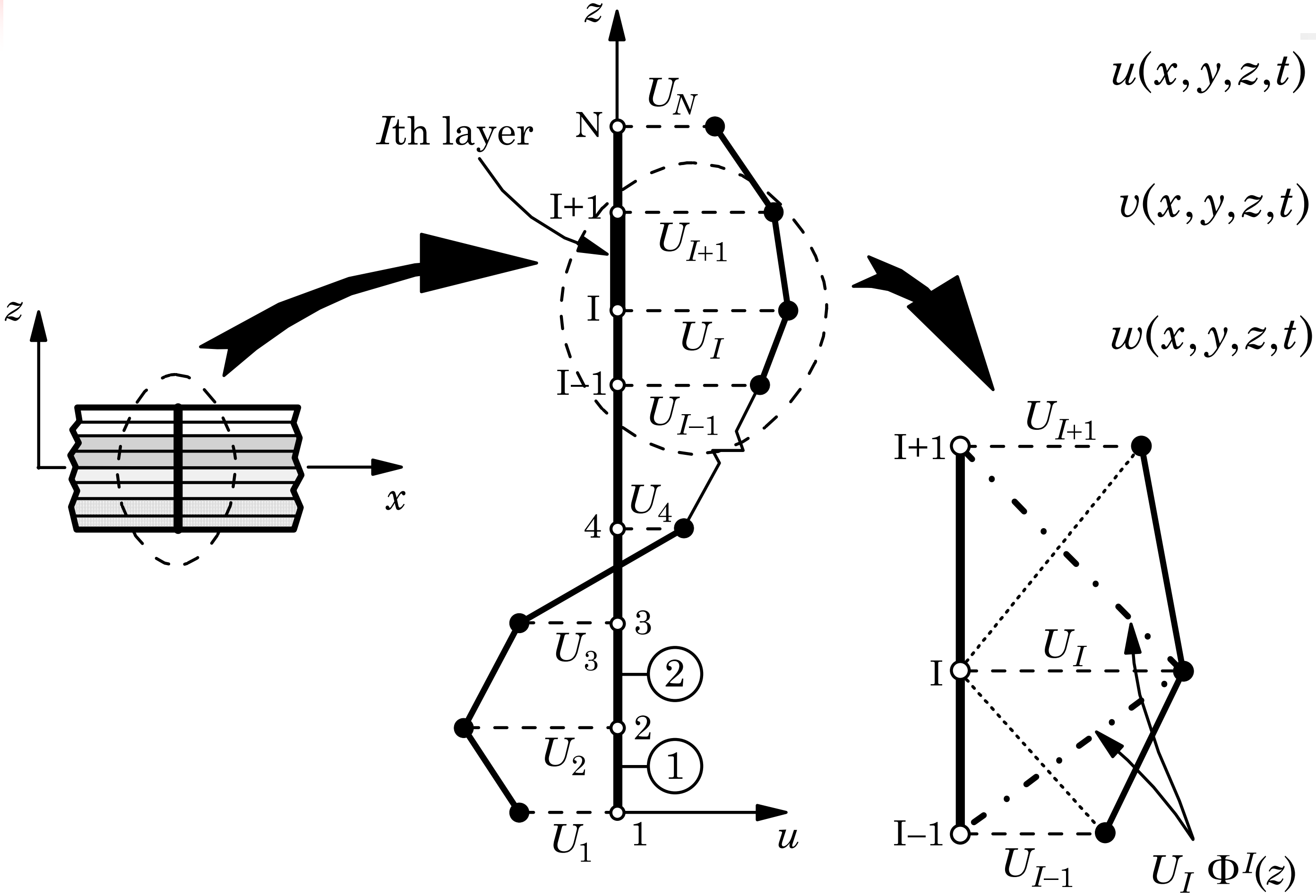
$$\sigma_{zx}^{(k+1)} = \sigma_{zx}^{(k)}$$

$$\sigma_{zy}^{(k+1)} = \sigma_{zy}^{(k)}$$

$$\sigma_{zz}^{(k+1)} = \sigma_{zz}^{(k)}$$



Layerwise Kinematic Model:
Basic Idea



$$u(x,y,z,t) = \sum_{I=1}^{N} U_I(x,y,t)\Phi^I(z)$$

$$v(x,y,z,t) = \sum_{I=1}^{N} V_I(x,y,t)\Phi^I(z)$$

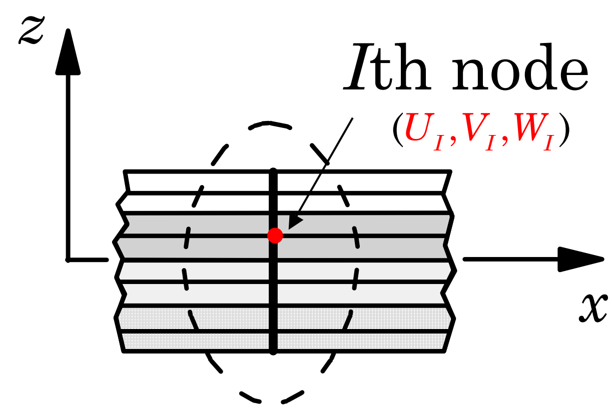
$$w(x,y,z,t) = \sum_{I=1}^{M} W_I(x,y,t)\Psi^I(z)$$

Layerwise Kinematic Model: Governing Equations

$$\frac{\partial N_{xx}^I}{\partial x}+\frac{\partial N_{xy}^I}{\partial y}-Q_x^I=\sum_{J=1}^{N} m^{IJ}\frac{\partial^2 U_J}{\partial t^2}$$

$$\frac{\partial N_{xy}^I}{\partial x}+\frac{\partial N_{yy}^I}{\partial y}-Q_y^I=\sum_{J=1}^{N} m^{IJ}\frac{\partial^2 V_J}{\partial t^2}$$

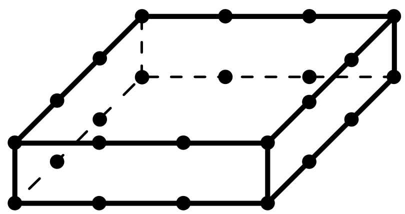
$$\frac{\partial \tilde{Q}_x^I}{\partial x}+\frac{\partial \tilde{Q}_y^I}{\partial y}-\tilde{Q}_z^I+q_b\delta_{I1}+q_t\delta_{IM}=\sum_{J=1}^{M} \tilde{I}^{IJ}\frac{\partial^2 W_J}{\partial t^2}$$



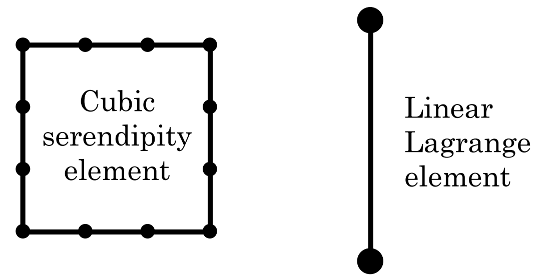
$$\begin{Bmatrix} N_{xx}^I \\ N_{yy}^I \\ N_{xy}^I \end{Bmatrix}=\sum_{k=1}^{N_e}\int_{z_b^k}^{z_t^k}\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}^{(k)}\Phi^I\,dz=\sum_{k=1}^{N_e}\int_{z_b^k}^{z_t^k}\begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{16} \\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{26} \\ \bar{C}_{16} & \bar{C}_{26} & \bar{C}_{66} \end{bmatrix}^{(k)}\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}^{(k)}\Phi^I\,dz$$

Layerwise Kinematic Model

Conventional 3D | Layerwise 2D + 1D

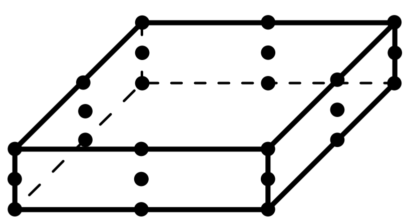


(1a)

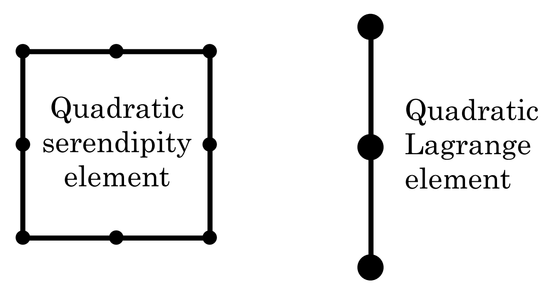


(in-plane) (through thickness)

(1b)



(2a)

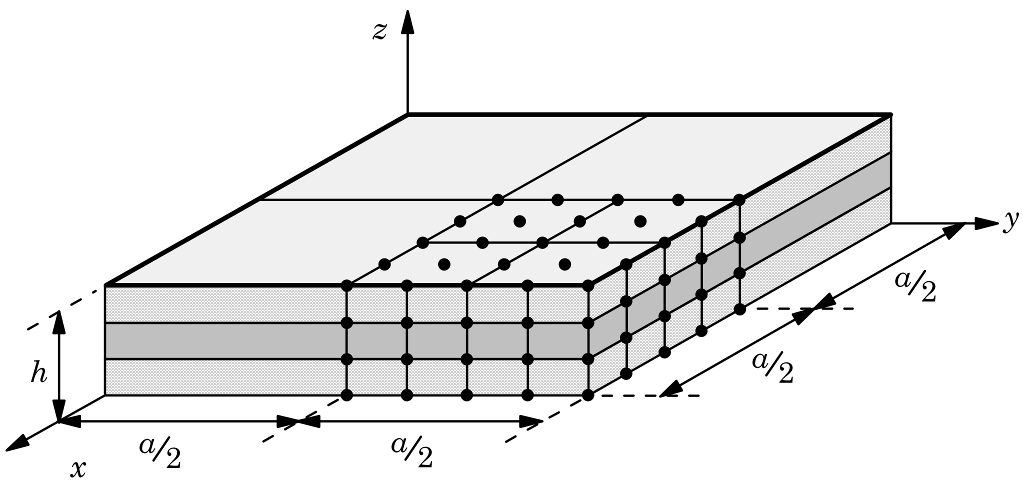


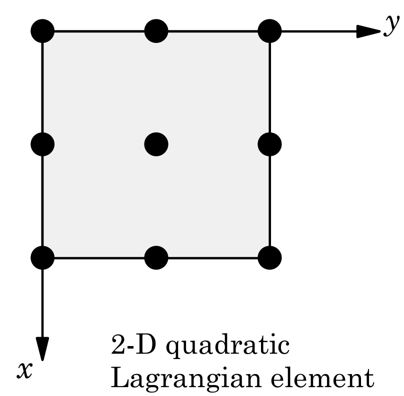
(in-plane) (through thickness)

(2b)

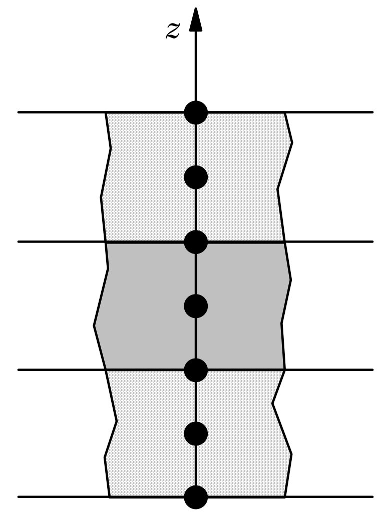
Layerwise Kinematic Model

3D modeling with 2D & 1D elements





2-D quadratic
Lagrangian element



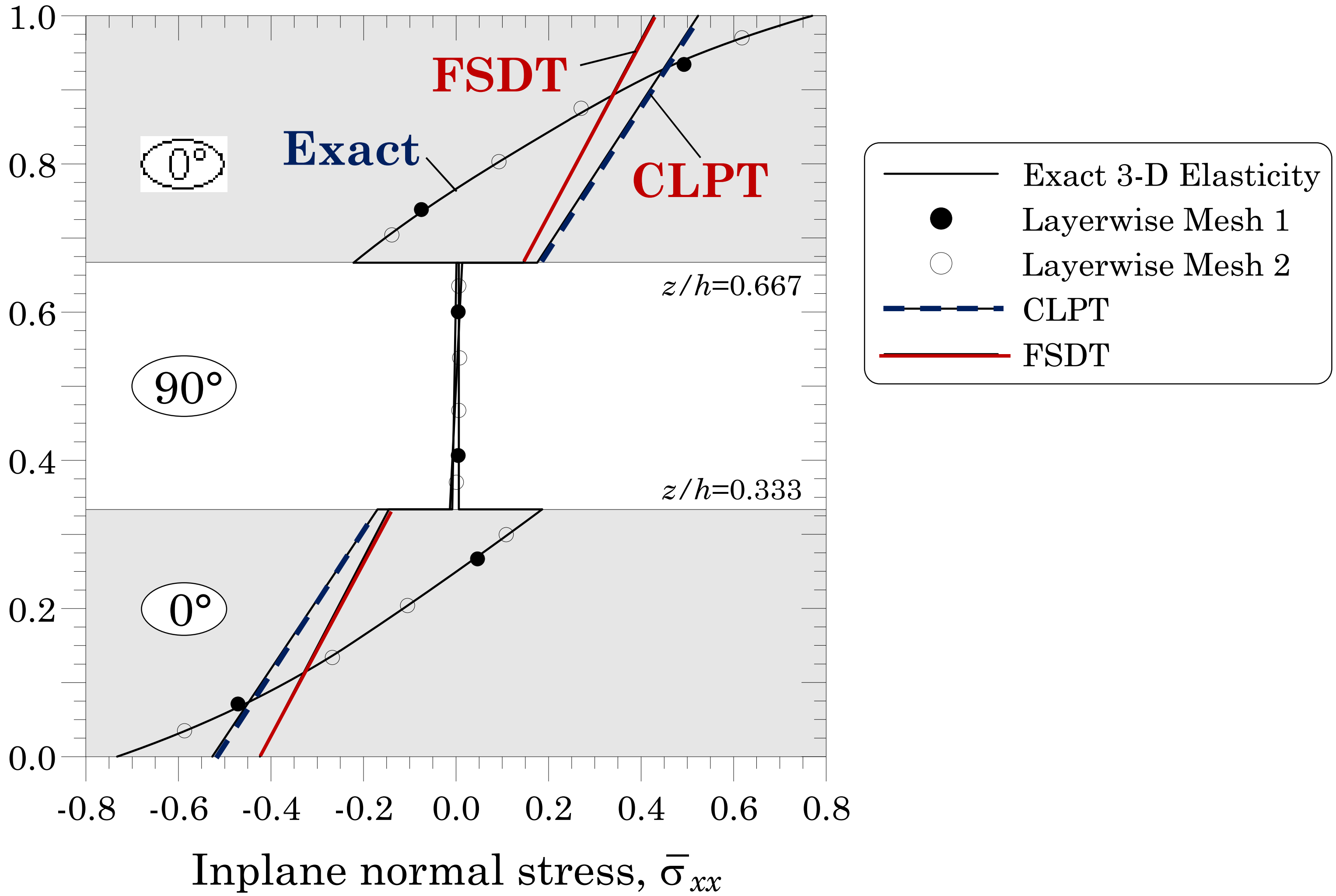
three quadratic layers
through the thickness

Table: Comparison of the number of operations needed to form the element sti®ness matrices for equivalent elements in the conventional 3-D format and the layerwise 2-D format. Full quadrature is used in all.

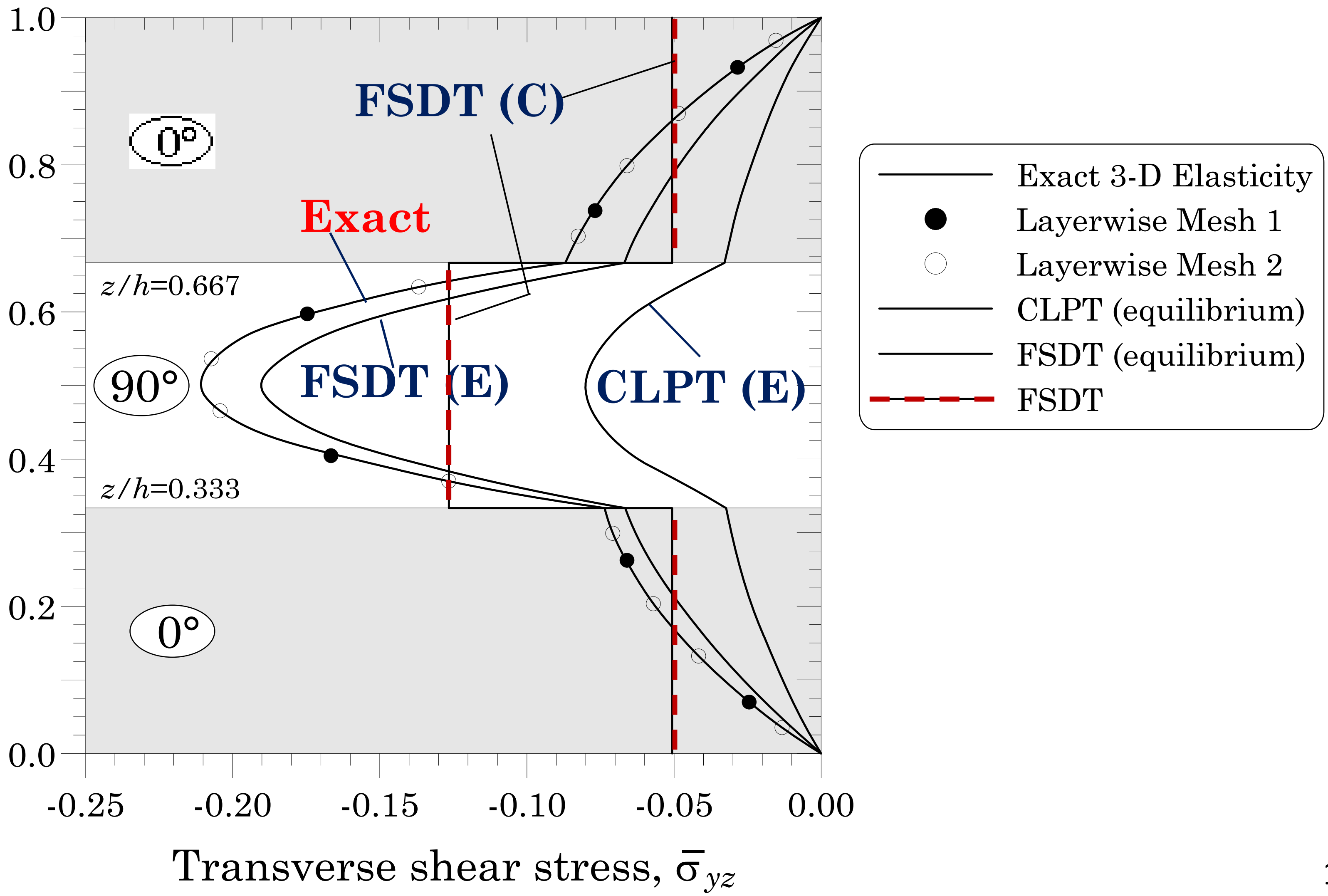
Element Type[y]	Multipli.	Addition	Assignments
1a (3-D)	1,116,000	677,000	511,000
1b (LWPT)	423,000	370,000	106,000
2a (3-D)	1,182,000	819,000	374,000
2b (LWPT)	284,000	270,000	69,000

[y] Element 1a: 72 degrees of freedom, 24-node 3-D isoparametric hexahedron with cubic in-plane interpolation and linear transverse interpolation.
Element 1b: 72 degrees of freedom, E12{L1 layerwise element.
Element 2a: 81 degrees of freedom, 27-node 3-D isoparametric hexahedron with quadratic interpolation in all three directions.
Element 2b: 81 degrees of freedom, E9{Q1 layerwise element.

Inplane stresses predicted by the Layerwise Theory



Transverse shear stresses predicted by the Layerwise Theory



CLOSING REMARKS

- The third-order shear deformation theory (TSDT) removes the normality and straightness assumptions and represents the deformed transverse lines to be cubic curves through the plate thickness.

The TSDT represents the transverse shear stresses to vary quadratically through the thickness and thus does not require shear correction coefficients.

The layerwise theory is a 3-D elasticity theory but exploits the layered nature of the laminate to simplify the governing equations, and it has no thickness aspect ratio restriction.

SUMMARY OF THE LECTURE

In this lecture, we have discussed the following topics:

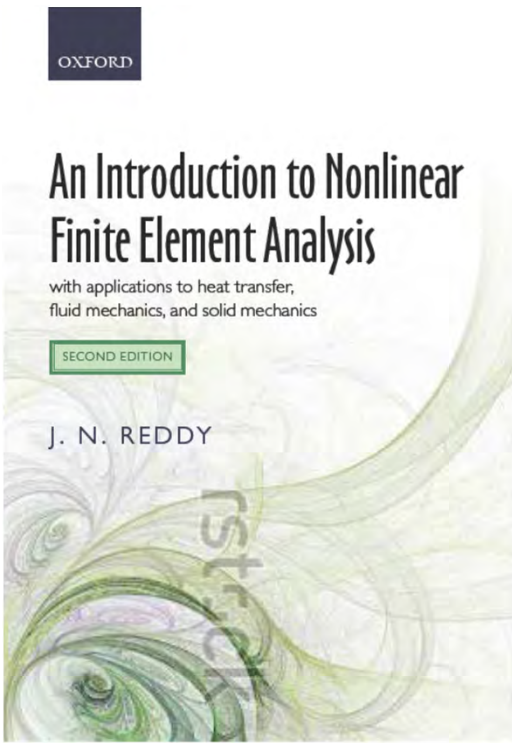
Third-order Shear Deformation Plate Theory

- Kinematics of the theory
- Development of governing equations
- Numerical results

Layerwise Laminate Theory

- The basic idea
- Equations of motion
- Numerical results

FINITE ELEMENTS FOR the analysis of laminated composite shells



CONTENTS

- General Introduction
- Continuum shell element
- Post-buckling of composite panels

SHELL FINITE ELEMENTS

Degenerate Continuum Shell Elements

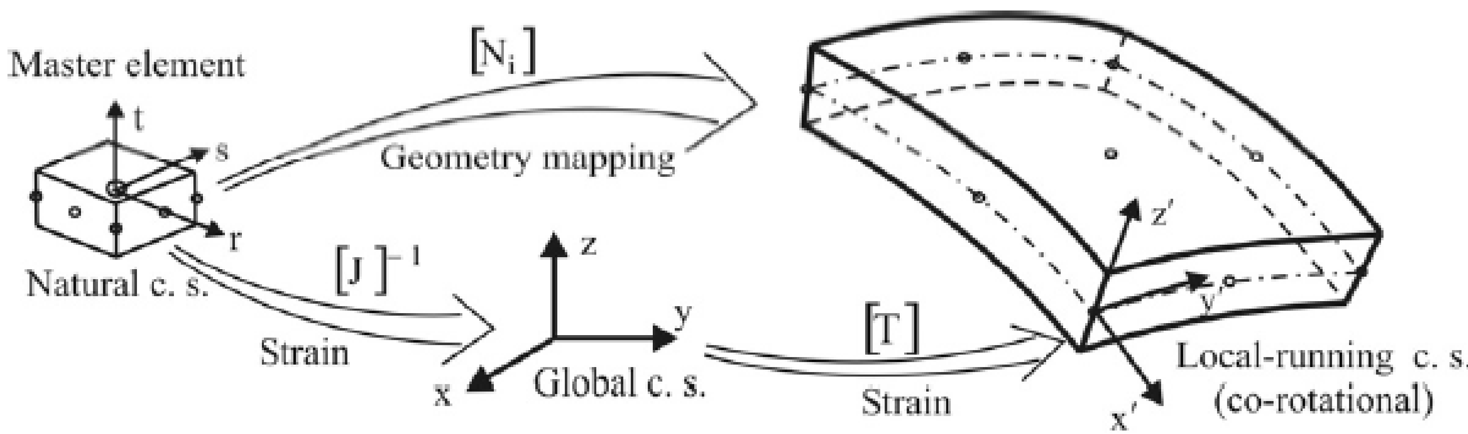
It is obtained by a degeneration process from the 3D continuum equations. Less expensive than the solid element but more expensive than the shell-theory element. Also exhibit locking problems.

Shell Theory Elements

Based on a curvilinear description of the continuum using an specific shell theory. Analytical integration of energy terms over the thickness. Also exhibit locking problems.

CONTINUUM SHELL FINITE ELEMENTS

The degenerate continuum shell element approach was first developed by Ahmad et al. from a three-dimensional solid element by a process which the 3D elasticity equations are expressed in terms of mid-surface nodal variables. It uses displacements and rotational DOFs.



CONTINUUM SHELL FINITE ELEMENTS

Assumptions are:

1. Fibers remain straight
2. The stress normal to the midsurface vanishes (the plane stress condition)

Discretization of the reference surface by using Cartesian coordinates

$$X_i = \sum_{k=1}^{n} \psi_k(\xi,\eta)\left[\frac{1+\zeta}{2}(X_i^k)_{\text{TOP}} + \frac{1-\zeta}{2}(X_i^k)_{\text{BOTTOM}}\right]$$

where

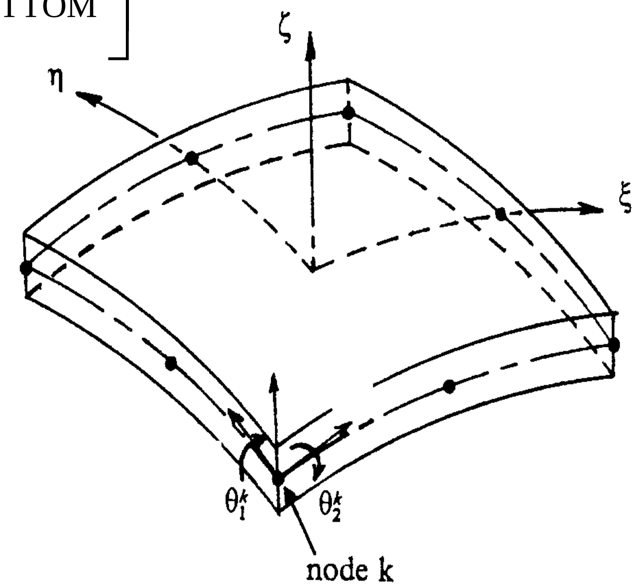
$\psi_{\mathrm{k}}(\xi,\eta)$: Lagrangian interpolation functions

(ξ,η,ζ): Natural coordinates

CONTINUUM SHELL FINITE ELEMENTS

$$x_i = \sum_{k=1}^{n} \psi_k(\xi,\eta)\left[\frac{1+\zeta}{2}(x_i^k)_{\text{TOP}} + \frac{1-\zeta}{2}(x_i^k)_{\text{BOTTOM}}\right]$$

Discretization of the final configuration of the reference surface

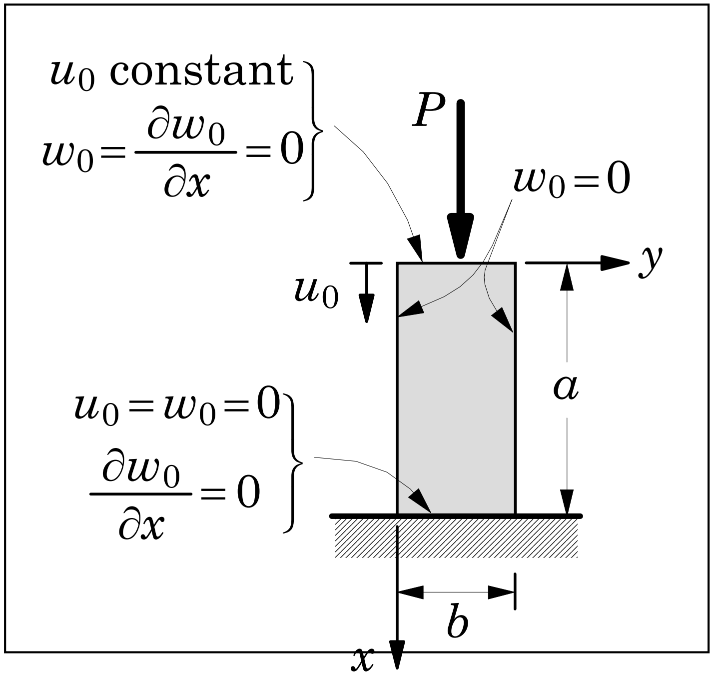


Equivalent to the discretization of the displacement of the reference surface as

$$u_i = \sum_{k=1}^{n} \psi_{\text{k}}(\xi,\eta)\left[u_i^k + \frac{\zeta}{2} h_k \left(\mathrm{e}_{3i}^k\right)\right]$$

POSTBUCKLING AND FAILURE ANALYSIS

- Nonlinear FE analysis
- Comparison with experimental results of Starnes and Rouse
- Progressive failure analysis



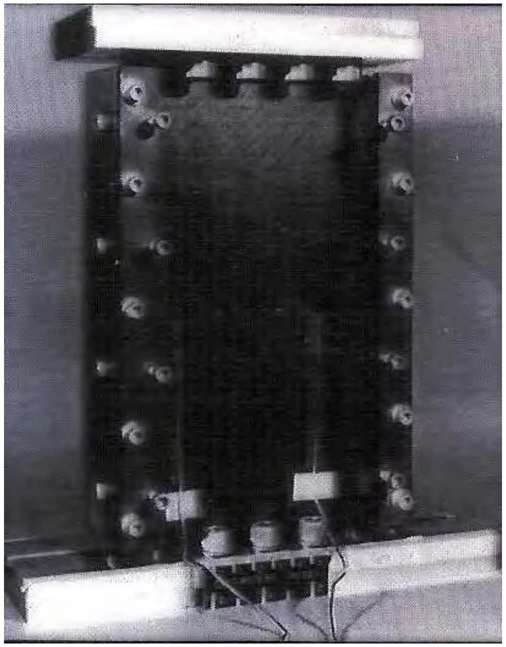
a = 50.8 cm (20 in.), b = 17.8 cm (7 in.), h_k = 0.14 mm (0.0055 in.)

24-ply laminate: $[45/-45/0_2/45/-45/0_2/45/-45/0/90]_s$

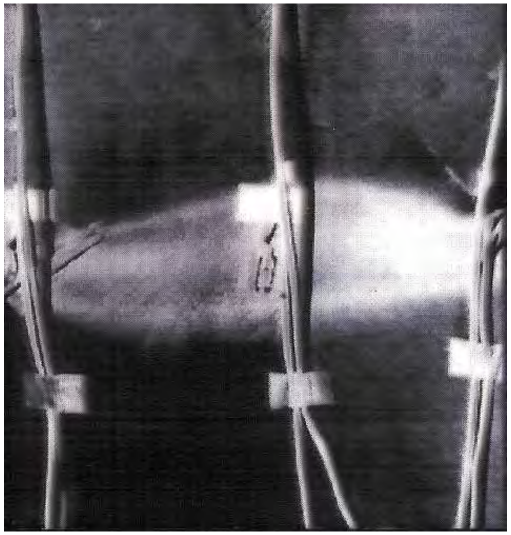
E_1 = 131 Gpa (19,000 ksi), E_2 = 13 Gpa (1,890 ksi),

G_{12} = 6.4 Gpa (930 ksi), ν_{12} = 0.38 (graphite-epoxy)

Experimental Setup and Failure Region (Starnes & Rouse, NASA Langley)



(a) Typical panel with test fixture



(b) A transverse shear failure mode

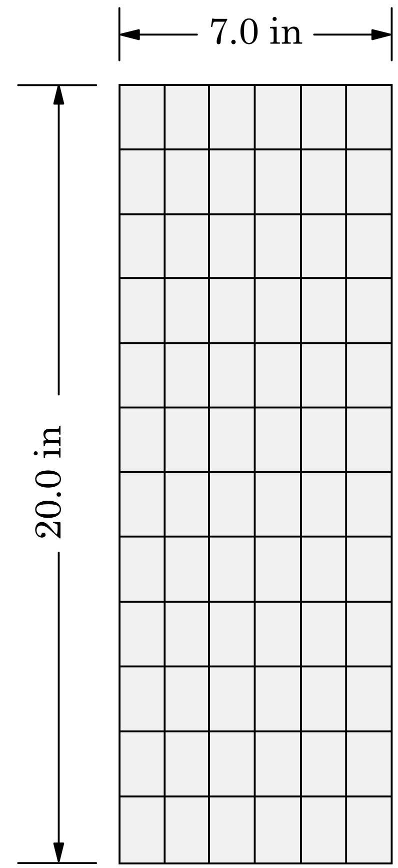
POST-BUCKLING OF A COMPOSITE PANEL UNDER IN-PLANE LOADING

Finite Element Models

- Mesh has six elements per buckling mode half wave in each direction

- Elements used:
  (1) Four-node C^1 -based flat shell element (**STAGS**)
  (2) Nine-node continuum shell element (**Chao & Reddy**)
  (3) For-node and nine-node ANS shell elements (**Park & Stanley**)

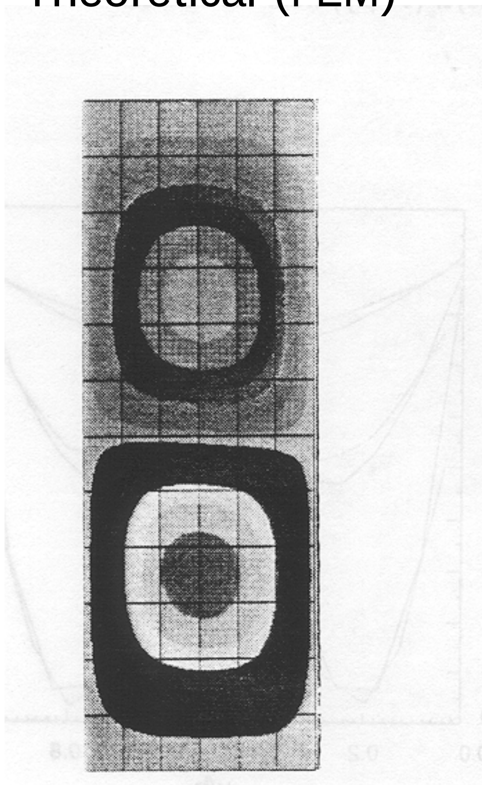
- All meshes have 72 elements (91 nodes for the meshes with four-node elements and 325 nodes for meshes with nine-node elements).



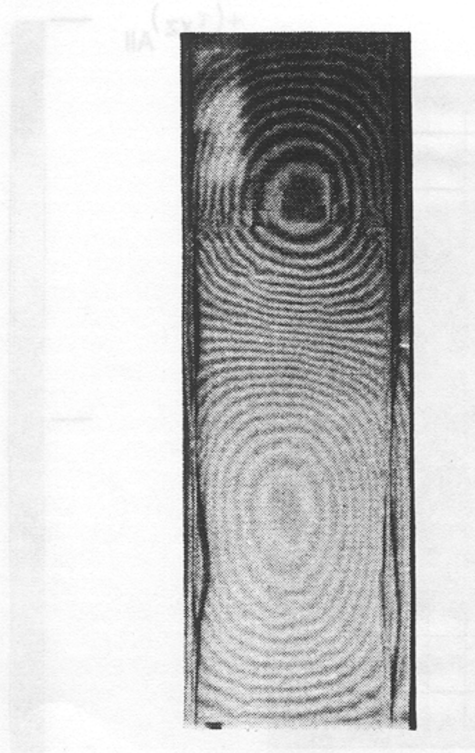
Continuum Shell Element

Comparison of the Experimental (Moire) and Analytical Out-of-Plane Deflection Patterns

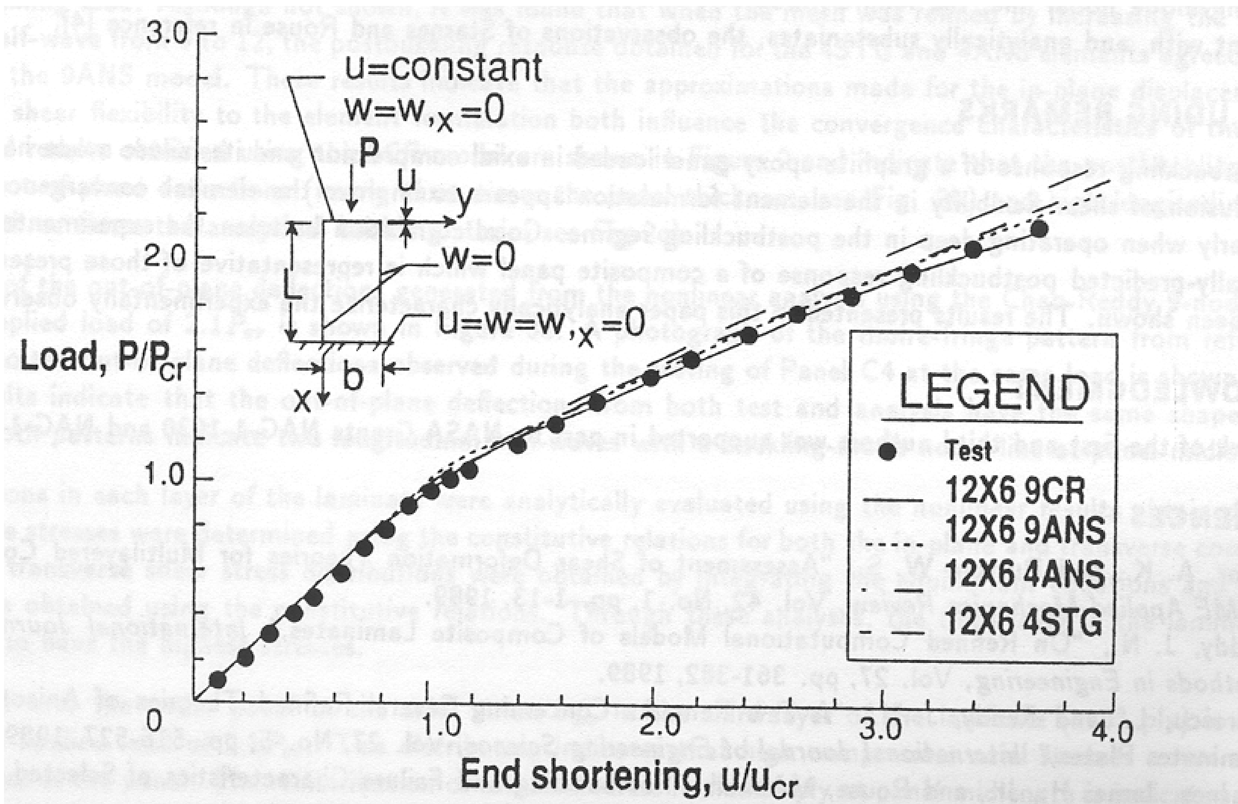
Theoretical (FEM)



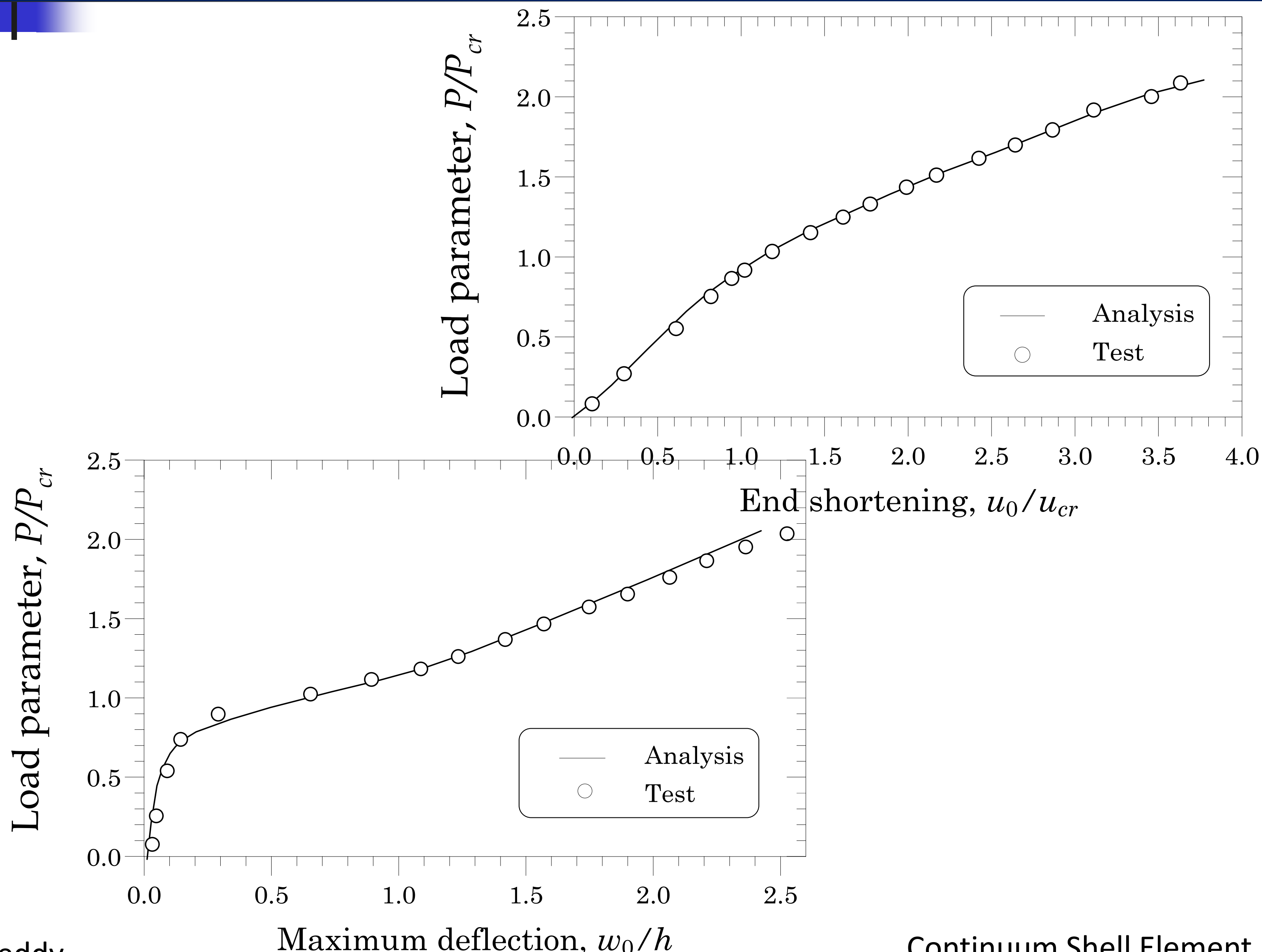
Experimental



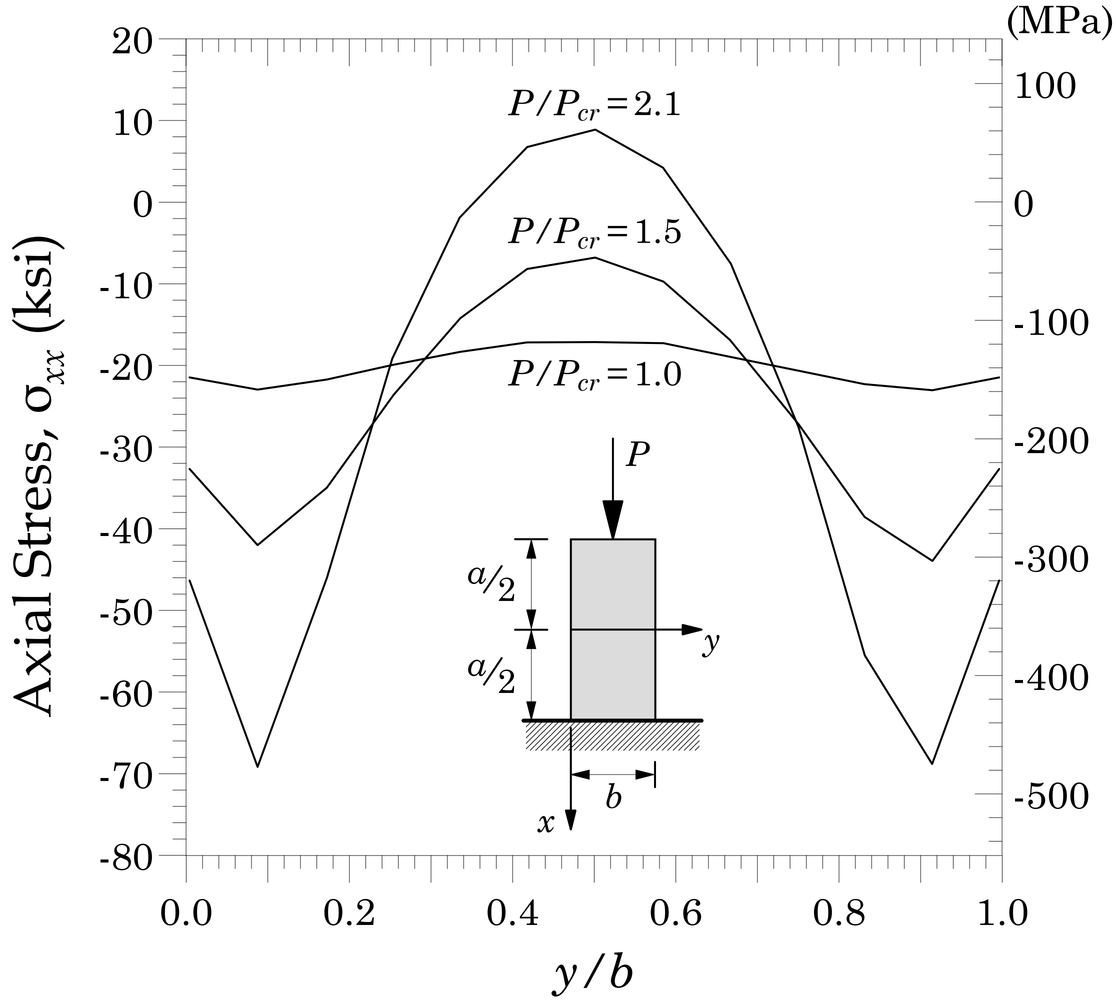
Comparison of the Experimental and Analytical Solutions for End Shortening



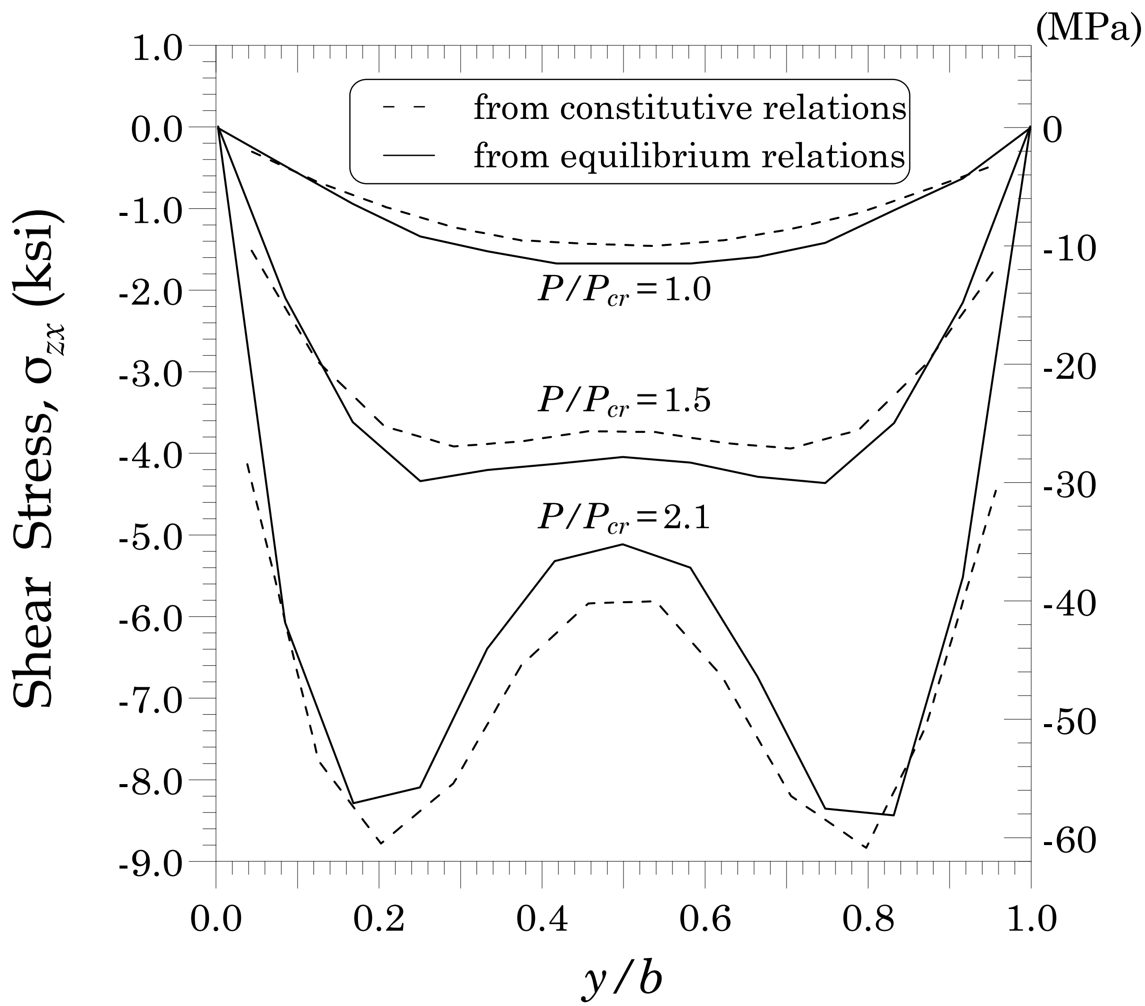
Postbuckling of a Composite Panel



Postbuckling of a Composite Panel (Stress Distributions)

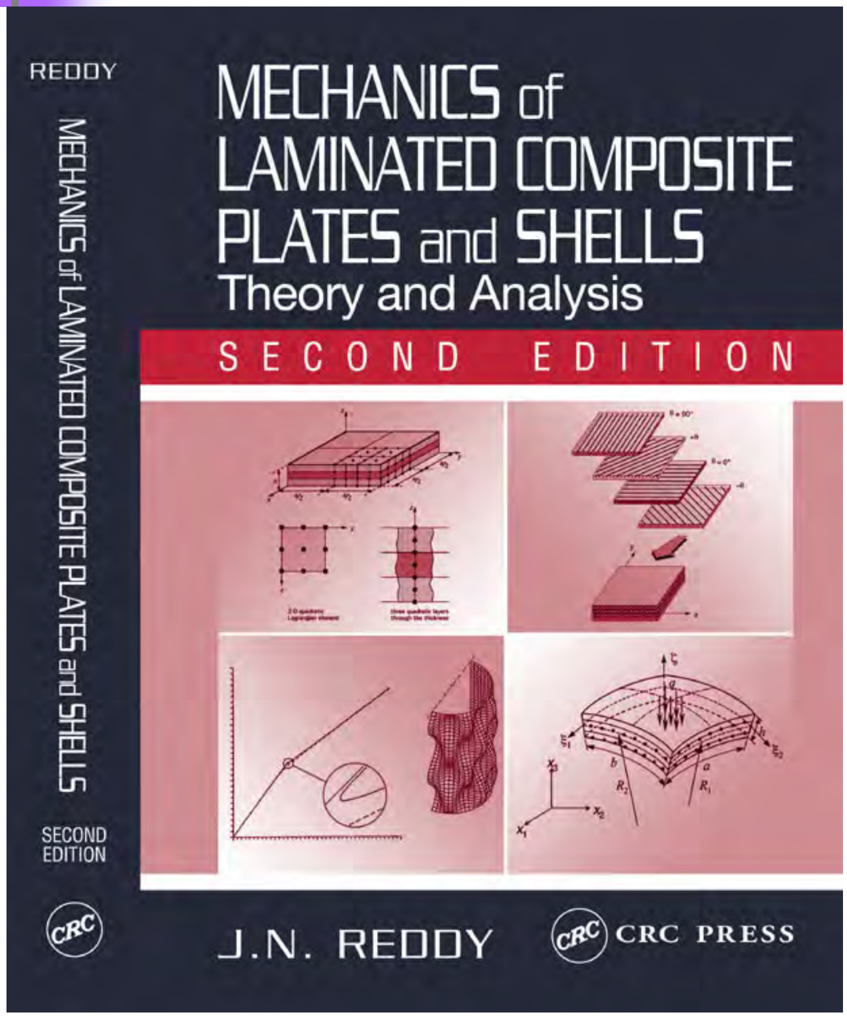


Postbuckling of a Composite Panel (Stress Distributions)



A ROBUST SHELL FINITE ELEMENT FOR THE ANALYSIS OF LAMINATED COMPOSITE AND FUNCTIONALLY GRADED STRUCTURES

- General introduction
- A 7-parameter shell finite element
  - Isotropic and functionally graded elastic shells
  - Multi-layered elastic composite laminates
- Numerical examples: benchmark problems
- Concluding remarks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Theory and Analysis
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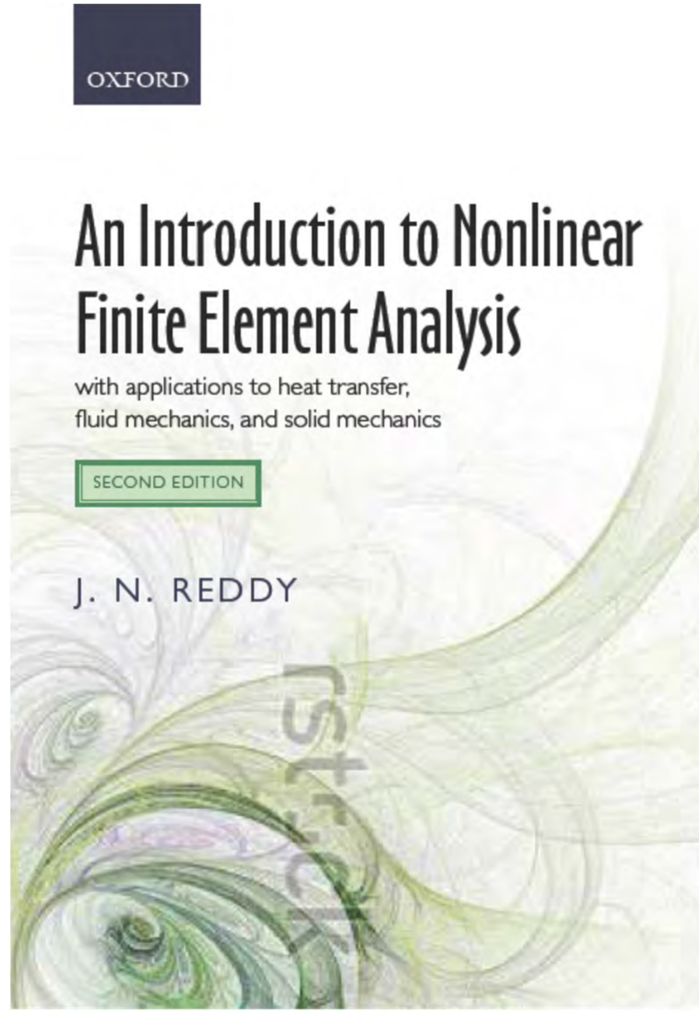
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with applications to heat transfer, fluid mechanics, and solid mechanics
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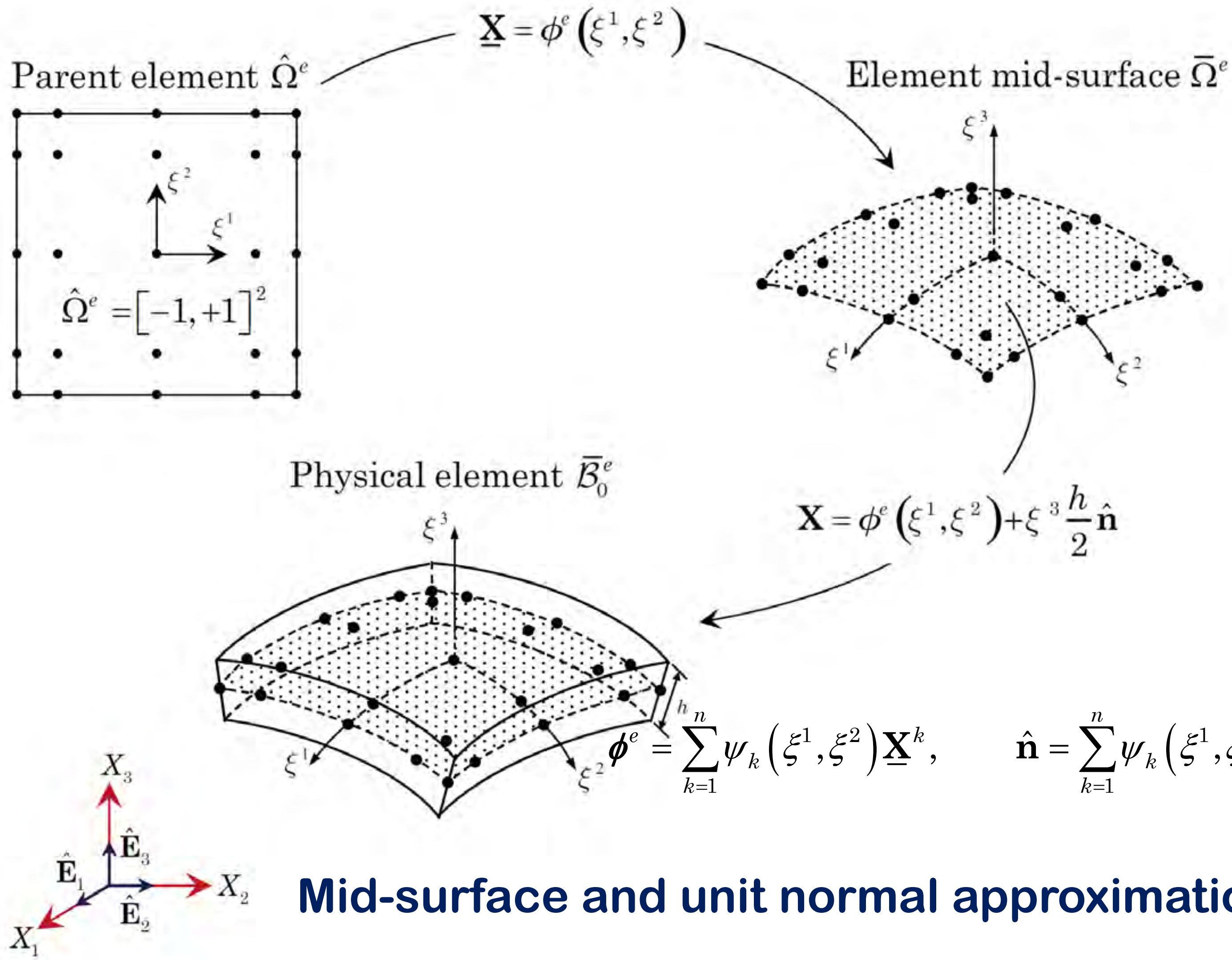
A 7-Parameter Shell Formulation

- **Present shell model is based on an improved shear deformation shell theory with 7 independent parameters**
- **Notable features of the 7-parameter formulation**
  - **Avoids the use of a rotation tensor**
  - **Thickness stretching is considered**
  - **Three-dimensional constitutive equations are used**
- **Notable features of present implementation**
  - **Utilization of spectral/*hp* finite element technology to represent the differential geometry and avoid locking**
  - **Applicability to FGM and laminated composite shells**
  - **Applicability to geometrically linear and nonlinear analysis**

A 7-PARAMETER SHELL FORMULATION

Parameterization of the Approximate 3-D Geometry



$$\boldsymbol{\phi}^e = \sum_{k=1}^{n} \psi_k(\xi^1, \xi^2)\underline{\mathbf{X}}^k, \qquad \hat{\mathbf{n}} = \sum_{k=1}^{n} \psi_k(\xi^1, \xi^2)\hat{\mathbf{n}}^k$$

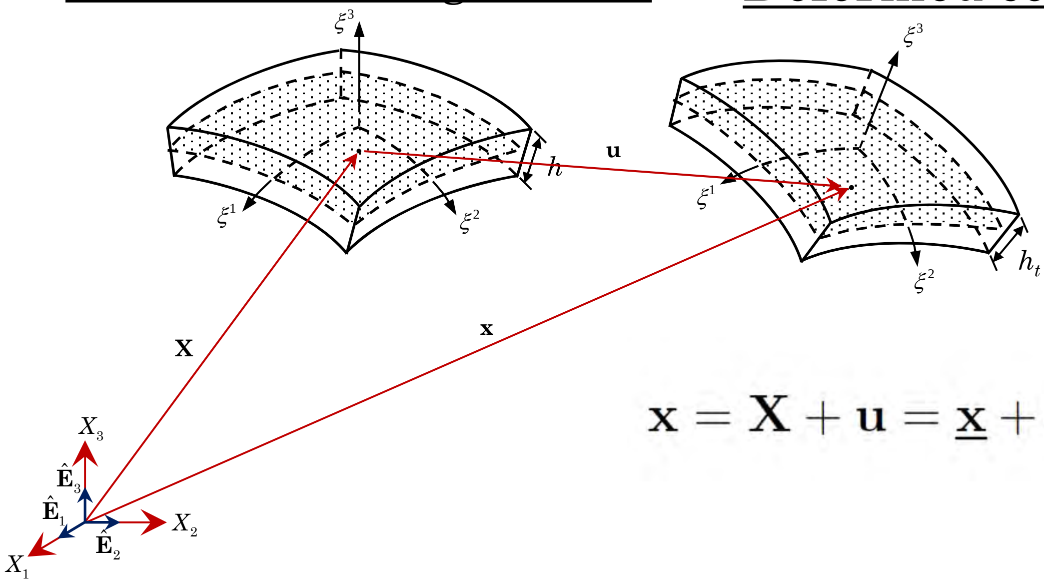
Mid-surface and unit normal approximations

A 7-PARAMETER SHELL FORMULATION:
Assumed Displacement Field

$$\boxed{\mathbf{u}(\xi^i) = \underline{\mathbf{u}}(\xi^\alpha) + \xi^3 \frac{h}{2}\boldsymbol{\varphi}(\xi^\alpha) + (\xi^3)^2 \frac{h}{2}\boldsymbol{\psi}(\xi^\alpha)}$$

$$\underline{\mathbf{u}}(\xi^\alpha) = \underline{u}_i(\xi^\alpha)\hat{\mathbf{E}}_i, \qquad \boldsymbol{\varphi}(\xi^\alpha) = \varphi_i(\xi^\alpha)\hat{\mathbf{E}}_i, \qquad \boldsymbol{\psi}(\xi^\alpha) = \Psi(\xi^\alpha)\hat{\mathbf{n}}(\xi^\alpha)$$

<u>Reference configuration</u> <u>Deformed configuration</u>



$$\mathbf{x} = \mathbf{X} + \mathbf{u} = \underline{\mathbf{x}} + \xi^3 \frac{h}{2}\hat{\mathbf{n}} + (\xi^3)^2 \frac{h}{2}\Psi\hat{\mathbf{n}}$$

JN Reddy

A 7-Parameter Shell Formulation:

Green-Lagrange Strain Tensor

- **Contravariant basis representation of Green-Lagrange**

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^{\mathrm{T}} \cdot \mathbf{F} - \mathbf{I}\right) = \frac{1}{2}\left(\mathbf{u}_{,i} \cdot \mathbf{g}_j + \mathbf{g}_i \cdot \mathbf{u}_{,j} + \mathbf{u}_{,i} \cdot \mathbf{u}_{,j}\right)\mathbf{g}^i\mathbf{g}^j$$

$$E_{ij}(\xi^m) = \varepsilon_{ij}^{(0)} + \xi^3\varepsilon_{ij}^{(1)} + \underbrace{\underline{(\xi^3)^2\varepsilon_{ij}^{(2)} + (\xi^3)^3\varepsilon_{ij}^{(3)} + (\xi^3)^4\varepsilon_{ij}^{(4)}}}_{\textit{neglected}}$$

- **Covariant components of the Green-Lagrange strain tensor**

$$\varepsilon_{\alpha\beta}^{(0)} = \frac{1}{2}\left(\underline{\mathbf{u}}_{,\alpha} \cdot \mathbf{a}_\beta + \mathbf{a}_\alpha \cdot \underline{\mathbf{u}}_{,\beta} + \underline{\mathbf{u}}_{,\alpha} \cdot \underline{\mathbf{u}}_{,\beta}\right)$$

$$\varepsilon_{\alpha\beta}^{(1)} = \frac{h}{4}\left[\underline{\mathbf{u}}_{,\alpha} \cdot (\hat{\mathbf{n}}_{,\beta} + \boldsymbol{\varphi}_{,\beta}) + (\hat{\mathbf{n}}_{,\alpha} + \boldsymbol{\varphi}_{,\alpha}) \cdot \underline{\mathbf{u}}_{,\beta} + \boldsymbol{\varphi}_{,\alpha} \cdot \mathbf{a}_\beta + \mathbf{a}_\alpha \cdot \boldsymbol{\varphi}_{,\beta}\right]$$

$$\varepsilon_{\alpha 3}^{(0)} = \frac{h}{4}\left[\underline{\mathbf{u}}_{,\alpha} \cdot (\hat{\mathbf{n}} + \boldsymbol{\varphi}) + \mathbf{a}_\alpha \cdot \boldsymbol{\varphi}\right]$$

$$\varepsilon_{\alpha 3}^{(1)} = \frac{h}{2}\left\{\frac{h}{4}\left[\boldsymbol{\varphi}_{,\alpha} \cdot \hat{\mathbf{n}} + (\hat{\mathbf{n}}_{,\alpha} + \boldsymbol{\varphi}_{,\alpha}) \cdot \boldsymbol{\varphi}\right] + (\mathbf{a}_\alpha + \underline{\mathbf{u}}_{,\alpha}) \cdot \boldsymbol{\psi}\right\}$$

$$\varepsilon_{33}^{(0)} = \frac{h^2}{8}\left(2\hat{\mathbf{n}} + \boldsymbol{\varphi}\right) \cdot \boldsymbol{\varphi}$$

$$\varepsilon_{33}^{(1)} = \frac{h^2}{2}\left(\hat{\mathbf{n}} + \boldsymbol{\varphi}\right) \cdot \boldsymbol{\psi}$$

<u>Covariant basis vectors</u>

$$\mathbf{g}_\alpha = \mathbf{a}_\alpha + \xi^3\frac{h}{2}\hat{\mathbf{n}}_{,\alpha}, \qquad \mathbf{g}_3 = \frac{h}{2}\hat{\mathbf{n}}$$

$$\mathbf{a}_\alpha = \frac{\partial \underline{\mathbf{X}}}{\partial \xi^\alpha} \equiv \underline{\mathbf{X}}_{,\alpha}$$

A 7-Parameter Shell Formulation

Constitutive Equations: General Discussion

- **We assume a linear relation between S and E**

$$\mathbf{S} = \mathbb{C} : \mathbf{E} \quad \text{where } \mathbb{C} = \mathbb{C}^{ijkl}\mathbf{g}_i\mathbf{g}_j\mathbf{g}_k\mathbf{g}_l \text{ is the fourth-order elasticity tensor}$$

$$S^{ij} = \mathbb{C}^{ijkl}E_{kl}$$

Second Piola-Kirchhoff stress: $\mathbf{S} = J_{\mathbf{F}}\mathbf{F}^{-1}\cdot\sigma\cdot\mathbf{F}^{-\mathrm{T}}$

- **The constitutive equation may be expressed in matrix form as**

$$\begin{Bmatrix} S^{11} \\ S^{22} \\ S^{33} \\ S^{23} \\ S^{13} \\ S^{12} \end{Bmatrix} = \begin{bmatrix} \mathbb{C}^{1111} & \mathbb{C}^{1122} & \mathbb{C}^{1133} & \mathbb{C}^{1123} & \mathbb{C}^{1113} & \mathbb{C}^{1112} \\ \mathbb{C}^{1122} & \mathbb{C}^{2222} & \mathbb{C}^{2233} & \mathbb{C}^{2223} & \mathbb{C}^{2213} & \mathbb{C}^{2212} \\ \mathbb{C}^{1133} & \mathbb{C}^{2233} & \mathbb{C}^{3333} & \mathbb{C}^{3323} & \mathbb{C}^{3313} & \mathbb{C}^{3312} \\ \mathbb{C}^{1123} & \mathbb{C}^{2223} & \mathbb{C}^{3323} & \mathbb{C}^{2323} & \mathbb{C}^{2313} & \mathbb{C}^{2312} \\ \mathbb{C}^{1113} & \mathbb{C}^{2213} & \mathbb{C}^{3313} & \mathbb{C}^{2313} & \mathbb{C}^{1313} & \mathbb{C}^{1312} \\ \mathbb{C}^{1112} & \mathbb{C}^{2212} & \mathbb{C}^{3312} & \mathbb{C}^{2312} & \mathbb{C}^{1312} & \mathbb{C}^{1212} \end{bmatrix} \begin{Bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{23} \\ 2E_{13} \\ 2E_{12} \end{Bmatrix}$$

- **The elasticity tensor C depends on the material symmetry and/or homogeneity of the reference configuration**

A 7-Parameter Shell Formulation

Constitutive Equations: Isotropic and FGM

- For an isotropic material the elasticity tensor is of the form

$$\mathbb{C} = \lambda \mathbf{I}\mathbf{I} + 2\mu \mathbb{I} \qquad \mathbb{C}^{ijkl} = \lambda g^{ij} g^{kl} + \mu (g^{ik} g^{jl} + g^{il} g^{jk})$$

$$g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j \qquad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \qquad \mu = \frac{E}{2(1+\nu)}$$

- For functionally graded materials (FGM) we allow the Young's modulus to vary in accordance with a power-law expression

$$E(\xi^3) = (E^+ - E^-) f^+(\xi^3) + E^-$$

$$f^+(\xi^3) = \left(\frac{\xi^3 + 1}{2}\right)^n$$

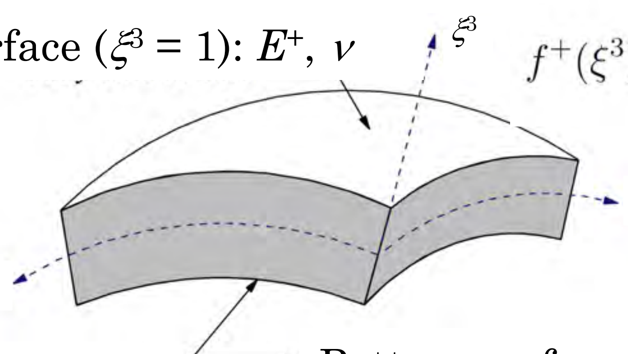


Fig. A functionally graded isotropic shell.

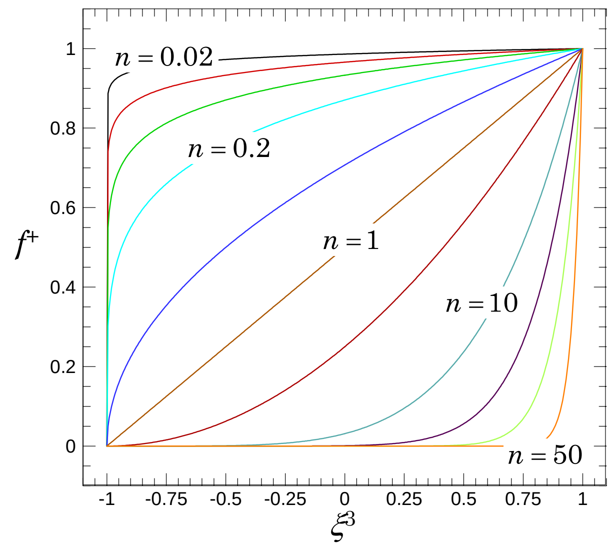
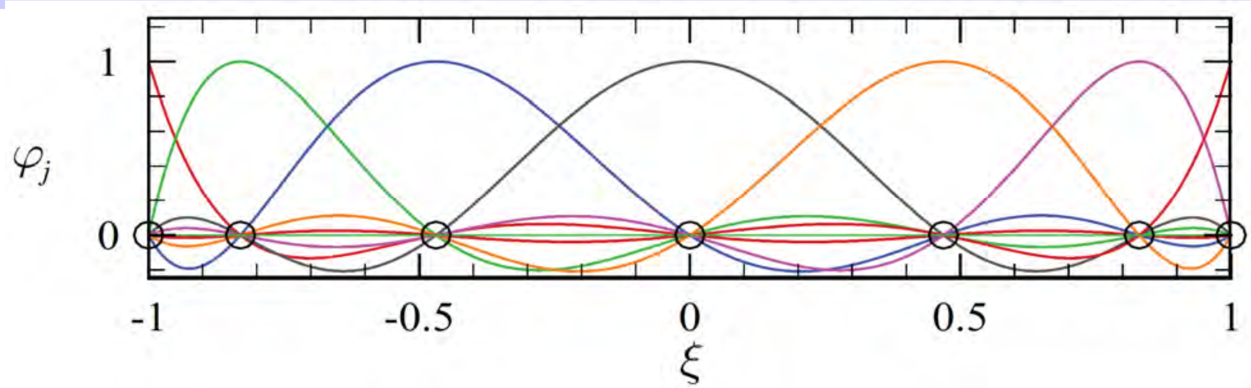


Fig. Volume fraction f^+ vs. ξ^3.

Spectral/*hp* Finite Element Technology

Basis Functions (Finite Element Interpolants)



High-order C^0 basis functions (case shown is for a p-level of 6).

- **High-order C^0 spectral interpolation functions in 1-D**

$$\varphi_j(\xi) = \frac{(\xi-1)(\xi+1)L_p'(\xi)}{p(p+1)L_p(\xi)(\xi-\xi_j)} = \prod_{i=1,i\neq j}^{p+1} \frac{\xi-\xi_i}{\xi_j-\xi_i},$$

$L_p(\xi)$: p^{th} order Legendre polynomial

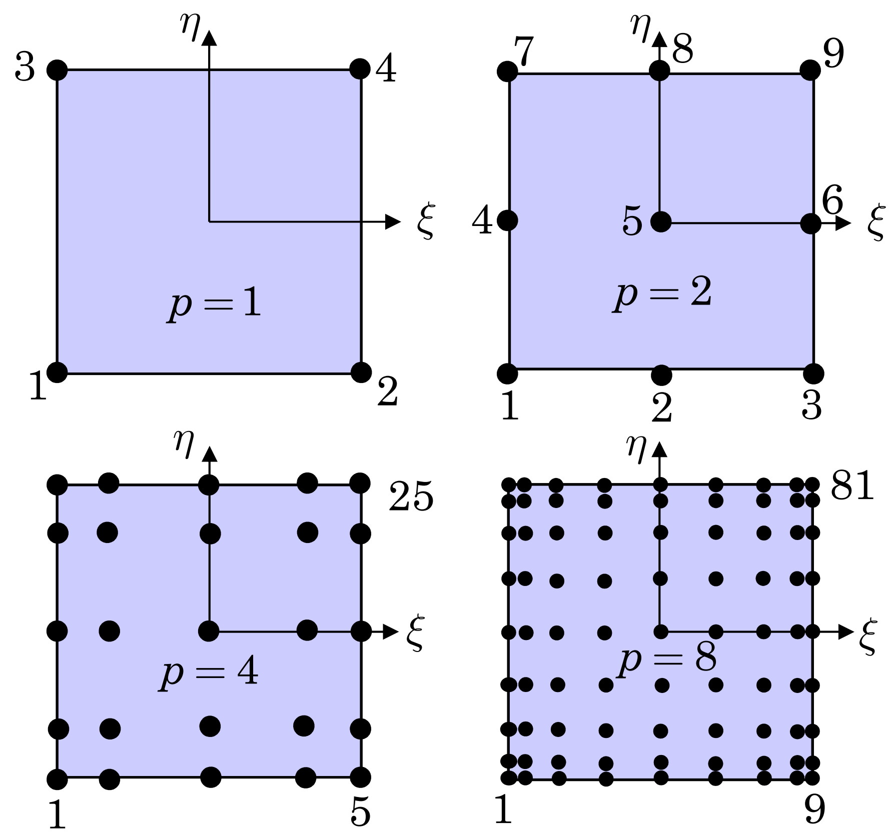
$\{\xi_j\}_{j=1}^{p+1}$: GLL points

- **Multi-dimensional interpolants: constructed from tensor products**

$$\left.\begin{aligned} \psi_i(\xi,\eta) &= \varphi_j(\xi)\varphi_k(\eta) \\ i &= j+(k-1)(p+1) \end{aligned}\right\} \Rightarrow \quad u(x,y) \cong u_{hp}(x,y) = \sum_{i=1}^{(p+1)^2} \Delta_i^e \psi_i(\xi,\eta)$$

Spectral/*hp* Finite Element Technology

Representative Two-Dimensional Master Elements



Examples of various high-order two-dimensional quadrilateral elements used in the present formulation (a) a 4-node element, *p* = 1; (b) a 9-node element, *p* = 2; (c) a 25-nod element, *p* = 4; and (d) an 81-node element, *p* = 8.

Spectral/*hp* Finite Element Technology

Improving Numerical Efficiency: Static Condensation

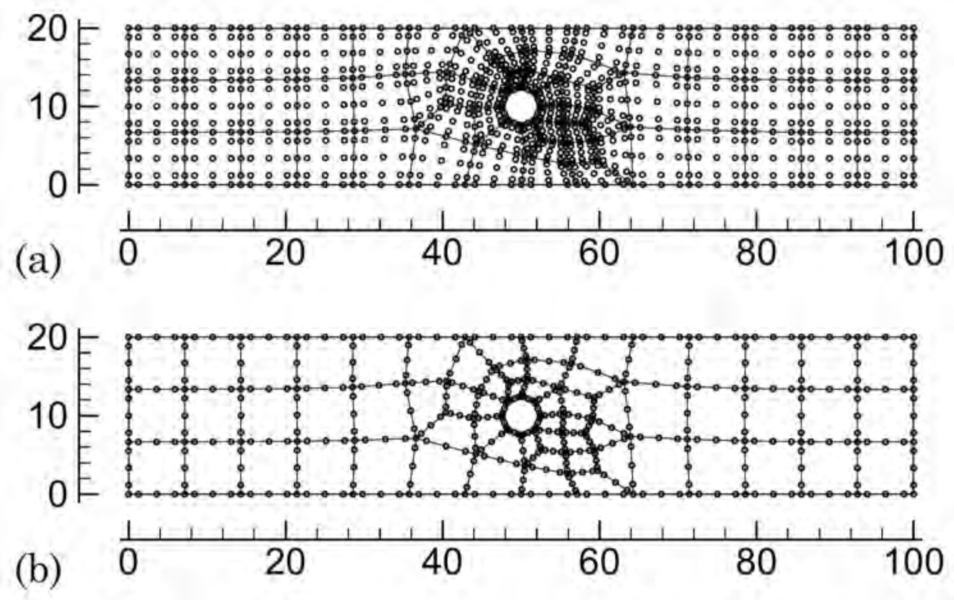


Fig 1. A high-order spectral/*hp* finite element discretization (*p*-level of 4) of a 2-D region: (a) finite element mesh showing elements and nodes and (b) a statically condensed version of the same mesh showing the elements and nodes.

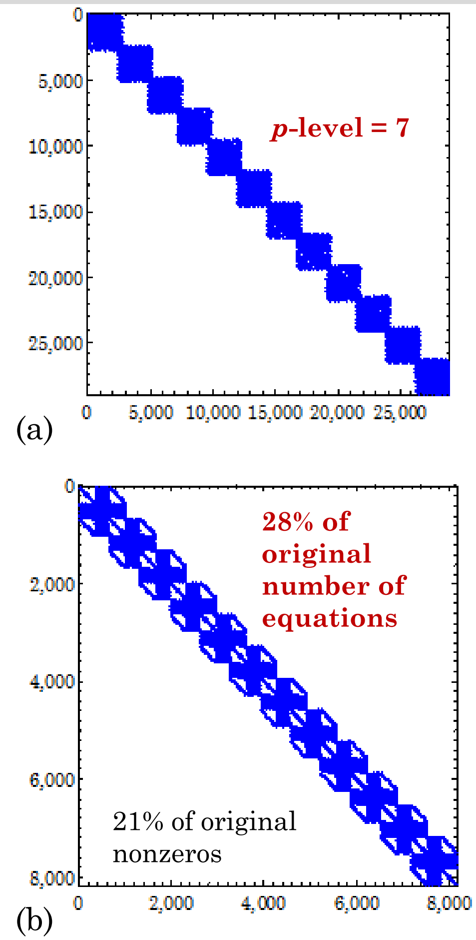


Fig 2. Sparsity patterns for: (a) a high-order finite element mesh and (b) the same high-order mesh using static condensation.

System memory requirements for low-order and high-order finite element problems are similar for two-dimensional problems.

Example 1: Post-buckling of a plate strip

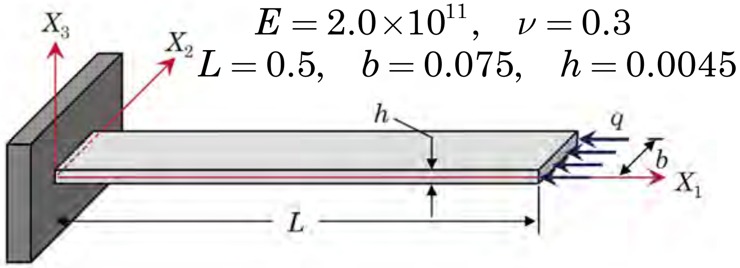


Fig . A axially loaded cantilevered plate strip.

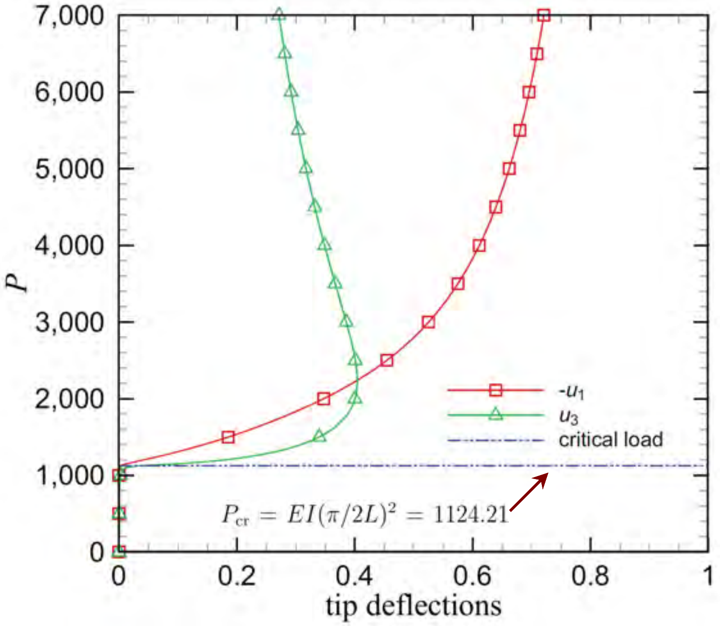


Fig. Tip deflections vs. applied load $P = q\times b$.

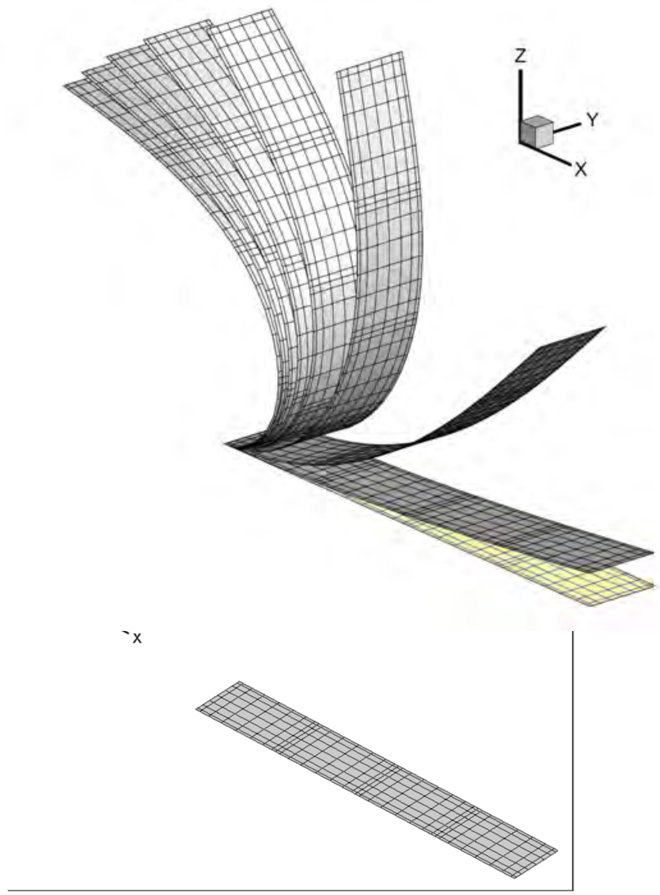


Fig. Deformed configurations of post-buckled plate strip.

Example 2: An annular plate with a slit under end transverse load

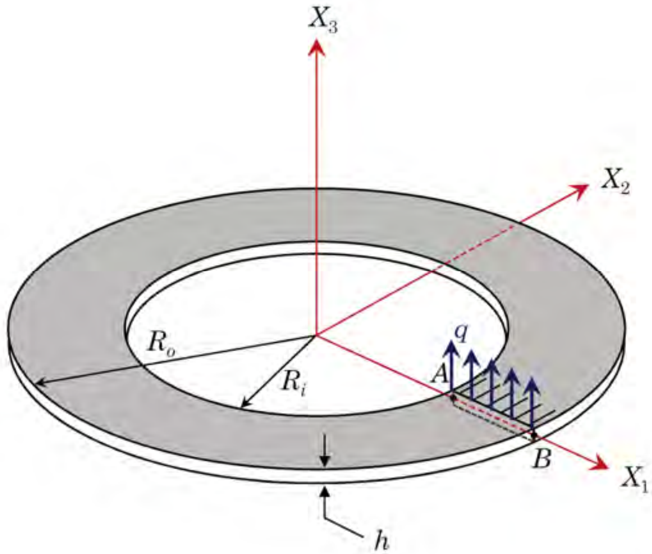
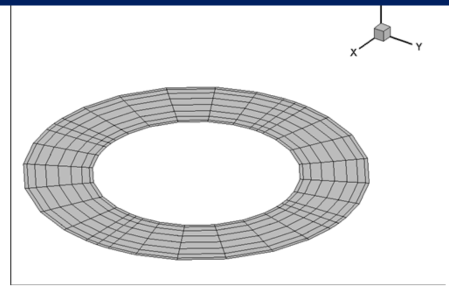


Fig. Geometry and loading for annular plate problem.



$R_i = 6,\ R_o = 10$ and $h = 0.03$

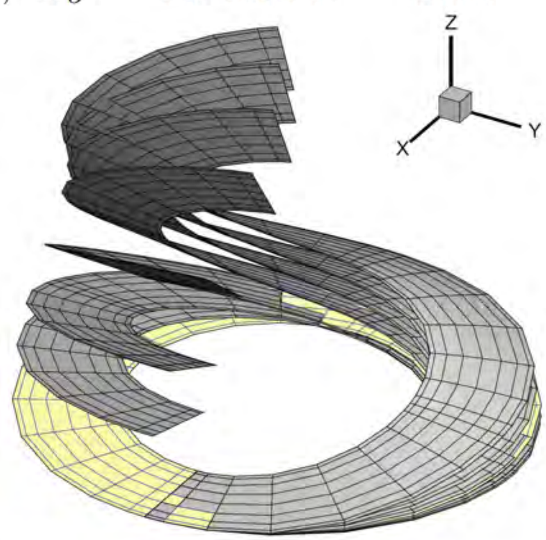


Fig. Deformed configurations of the isotropic slit annular plate.

Isotropic: $E = 21.0 \times 10^6, \quad \nu = 0.0$

$$\text{Orthotropic:} \begin{cases} E_1 = 20.0 \times 10^6, & E_2 = E_3 = 6.0 \times 10^6 \\ G_{23} = 2.4 \times 10^6, & G_{13} = G_{12} = 3.0 \times 10^6 \\ \nu_{23} = 0.25, & \nu_{13} = \nu_{12} = 0.3 \end{cases}$$

Example 2: Slit annular plate under end loading
(continued)

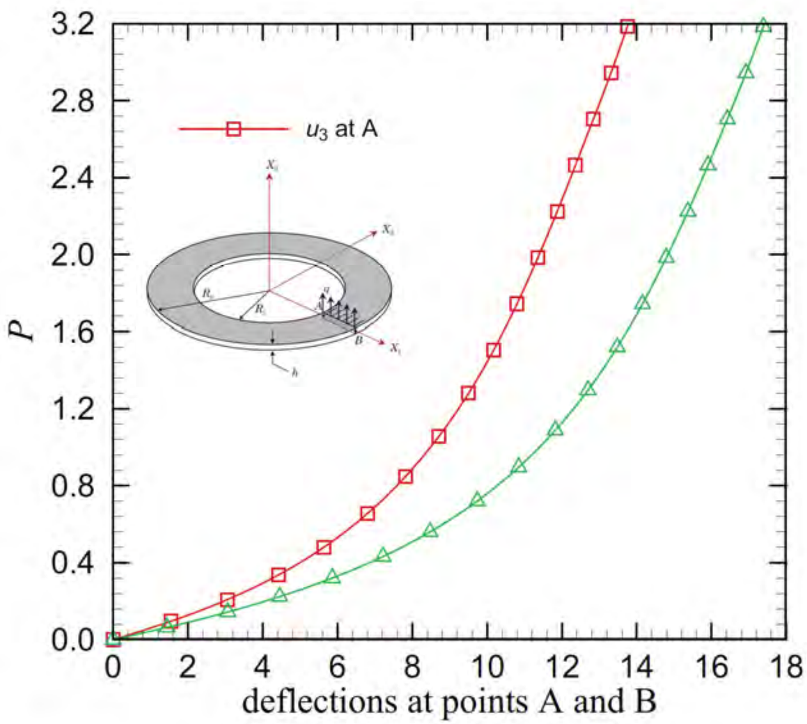


Fig. Vertical tip deflections at points A and B vs. shear force P for the isotropic slit annular plate.

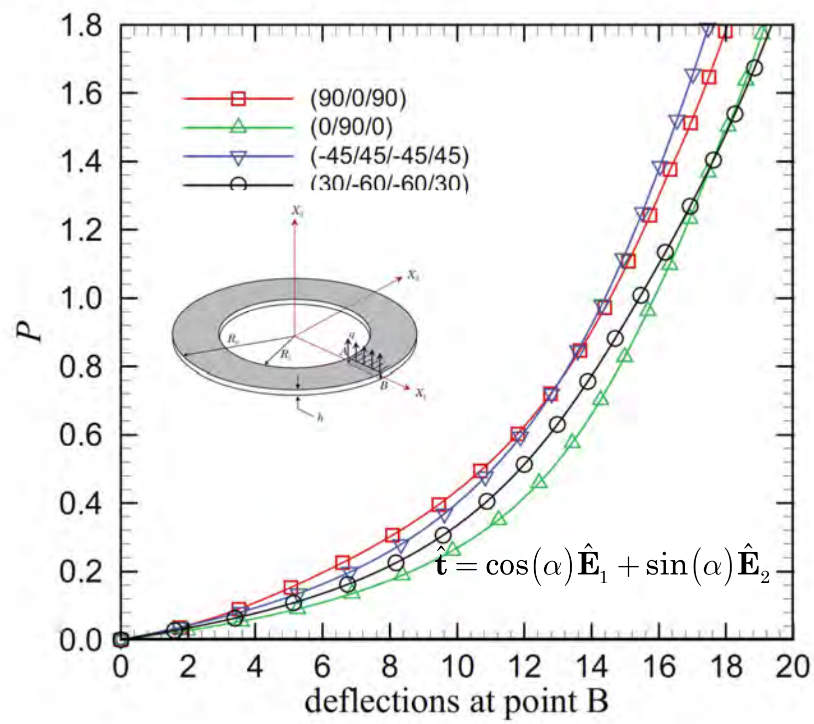


Fig. Vertical tip deflections at point B vs. shear force P for the various composite slit annular plates.

The computed deflections are in excellent agreement with the tabulated results of Sze et al. for the isotropic case and Arciniega and Reddy for each lamination scheme.

Example 3: Cylindrical panel under a point load

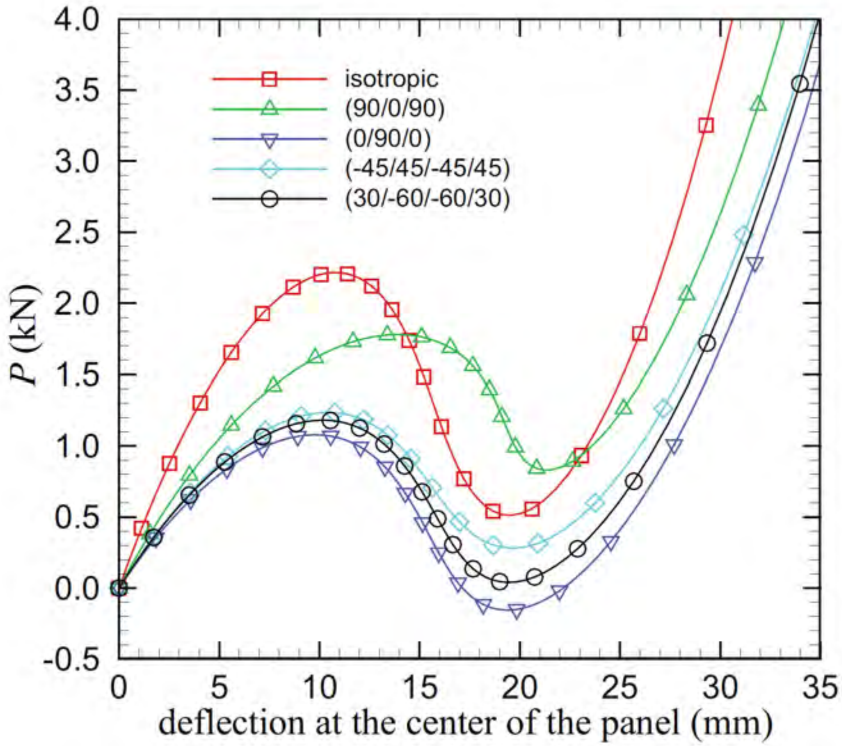


Fig. Vertical deflections of isotropic and laminated composite panels (cases shown are for h = 12.7 mm).

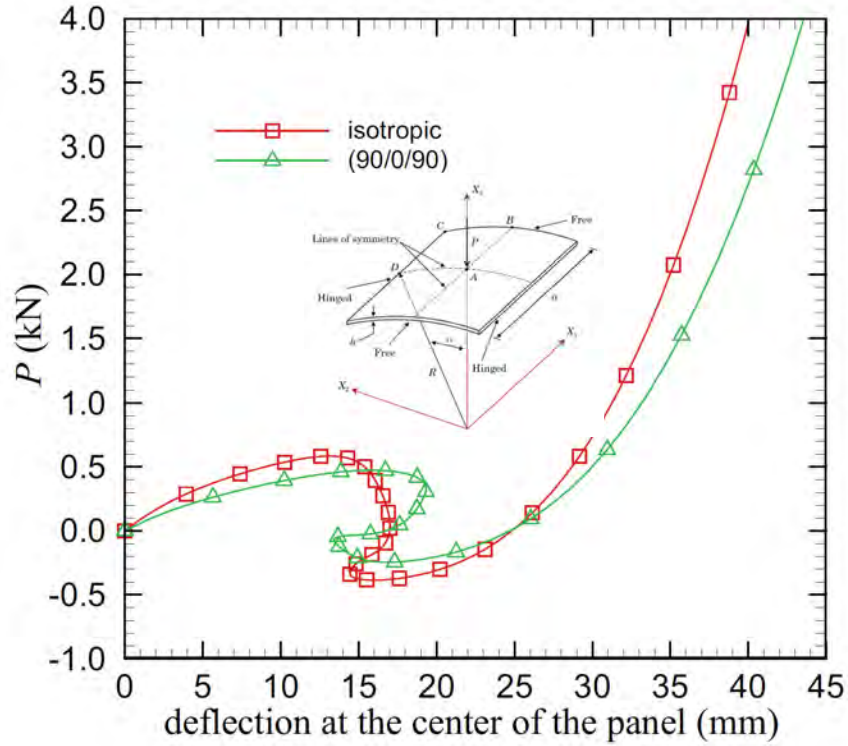
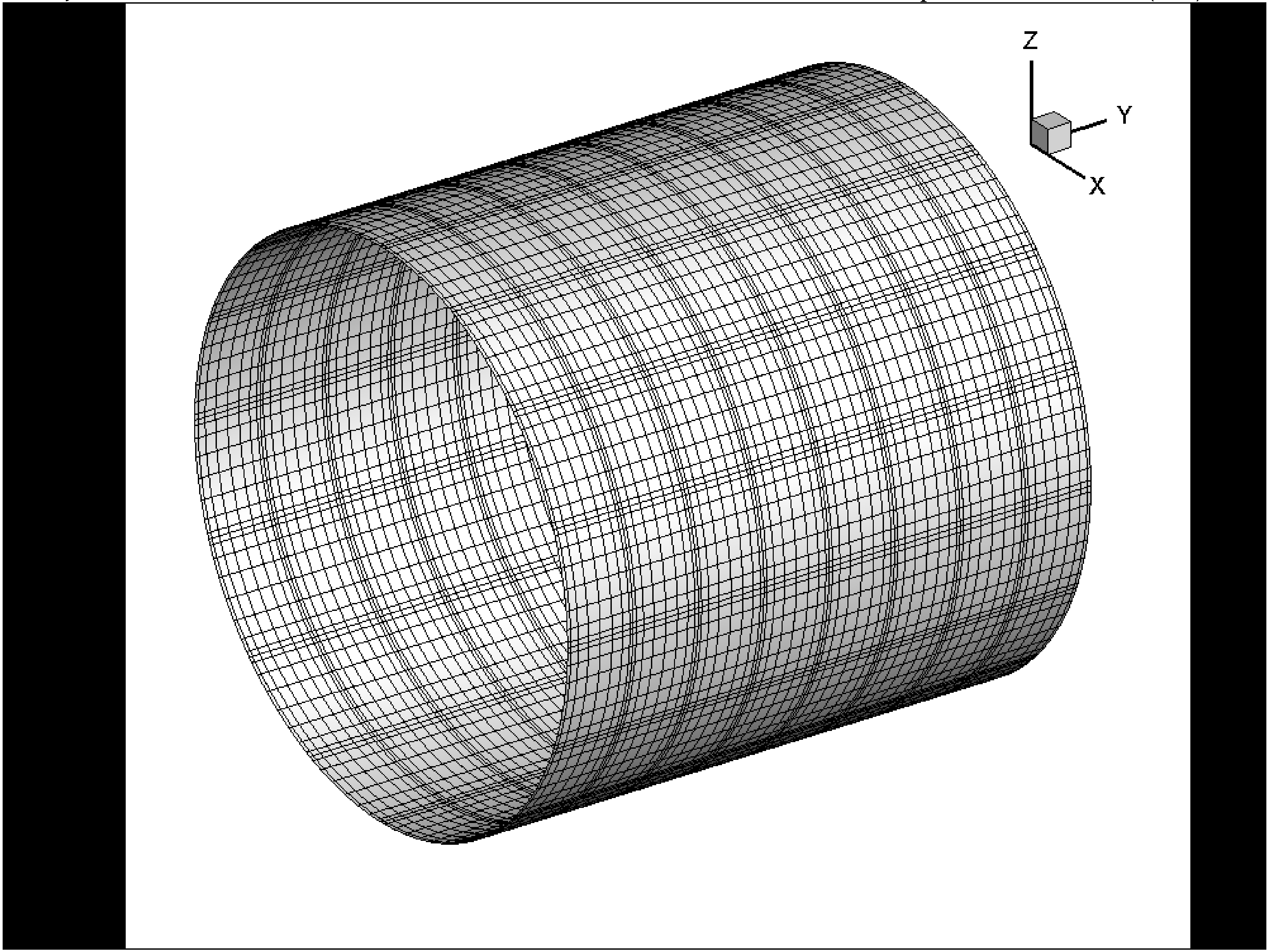


Fig. Vertical deflections of isotropic and laminated composite panels (cases shown are for h = 6.35 mm).

The computed deflections are in excellent agreement with the tabulated results of Sze et al. [121] and Arciniega and Reddy [4].


Z
Y
X

Example 4: A Pinched Composite Hyperboloid

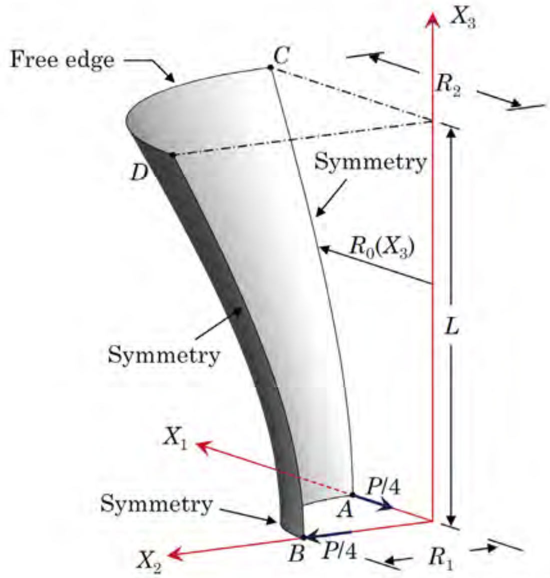


Fig. Computational domain.

Characterization mid-surface nodal coordinates

$$\underline{\mathbf{X}} = R_0(\omega^2)\left[\cos(\pi\omega^1/2)\hat{\mathbf{E}}_1 + \sin(\pi\omega^1/2)\hat{\mathbf{E}}_2\right] + L\omega^2\hat{\mathbf{E}}_3$$

$$R_0(\omega^2) = R_1\sqrt{1 + (L\omega^2/C)^2}$$

$$(\omega^1, \omega^2) \in [0,1]^2$$

Characterization of nodal normal and tangents

$$\hat{\mathbf{n}} = \frac{\partial\underline{\mathbf{X}}/\partial\omega^1 \times \partial\underline{\mathbf{X}}/\partial\omega^2}{||\partial\underline{\mathbf{X}}/\partial\omega^1 \times \partial\underline{\mathbf{X}}/\partial\omega^2||}$$

$$\hat{\mathbf{t}} = -\sin(\pi\omega^1/2)\hat{\mathbf{E}}_1 + \cos(\pi\omega^1/2)\hat{\mathbf{E}}_2$$

Geometric parameters

$$R_1 = 7.5,\ C = 20/\sqrt{3},\ L = 20.0 \text{ and } h = 0.04$$

Orthotropic material properties

$$E_1 = 40.0\times 10^6, \qquad E_2 = E_3 = 1.0\times 10^6$$

$$G_{23} = 0.6\times 10^6, \qquad G_{13} = G_{12} = 0.6\times 10^6$$

$$\nu_{23} = 0.25, \qquad \nu_{13} = \nu_{12} = 0.25$$

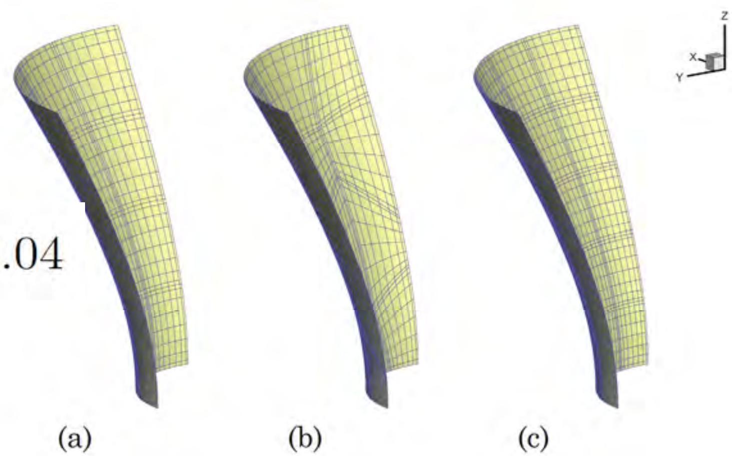
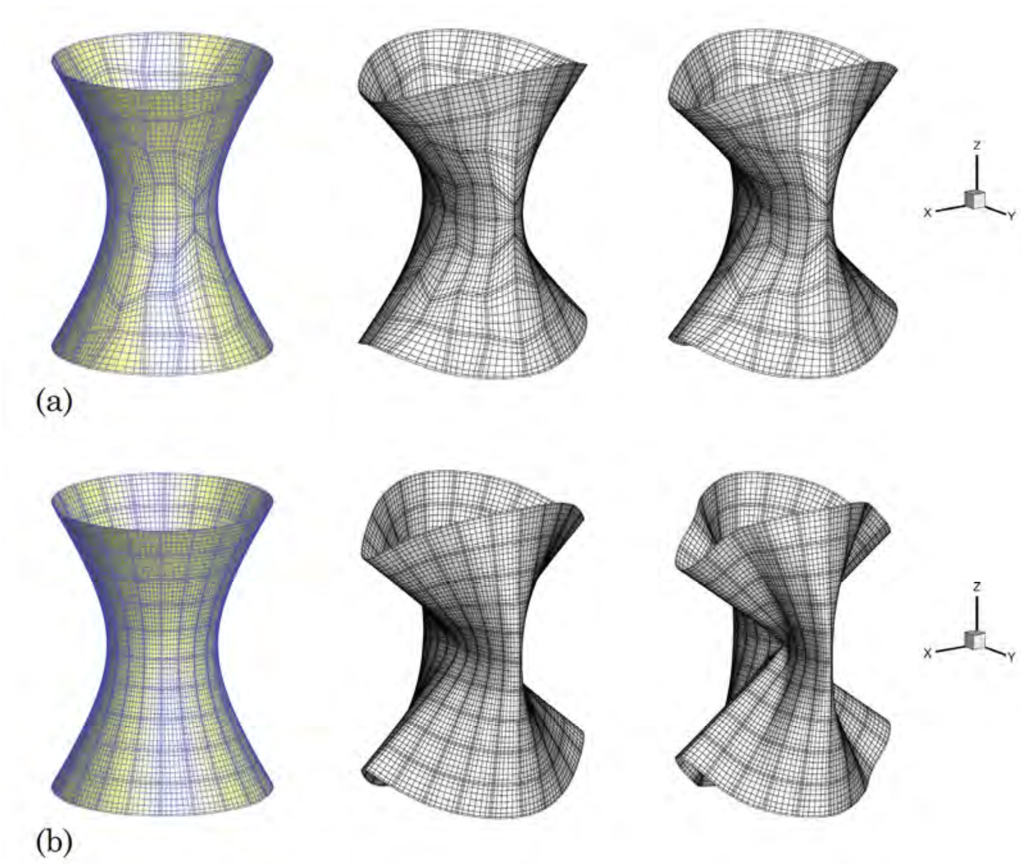
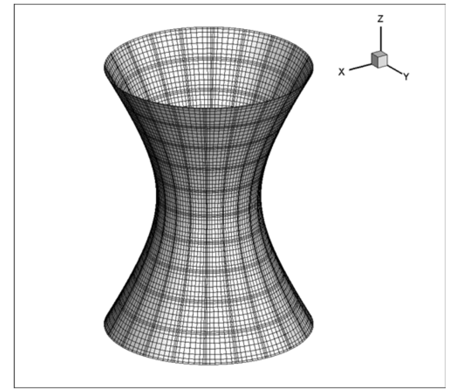


Fig. Finite element meshes: (a) 4×4 structured, (b) 4×4 un-structured and (c) 5×5 structured.

A Pinched Composite Hyperboloid (cont.)





Undeformed and various deformed configurations of the hyperboloid: (a) stacking sequence (0°/90°/0°): P = 0, 250 and 500 (from left to right) and (b) stacking sequence (90°/0°/ 90°): P = 0, 250 and 500 (from left to right).

SUMMARY AND CONCLUSIONS

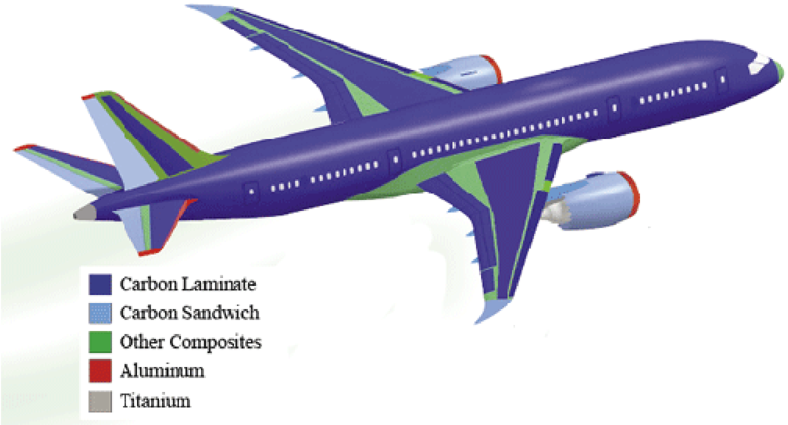
A 7-parameter shell element for finite deformation analysis is presented with the following features:

- **Numerical integration through the shell thickness allowed us to analyze isotropic, functionally graded and laminated composite shells without introducing any thin-shell type approximations.**
- **The displacement-based finite element approximation with isoparametric formulation of the differential geometry of the shell did not negatively impact the numerical results.**
- **The element was shown to be completely locking free (when a sufficiently high *p*-level was adopted) and insensitive to severe geometric distortions**
- **The introduction of a discrete tangent vector allowed for a simple and effective means of numerically simulating laminated composite shells**

FAILURES AND DESIGN OF COMPOSITE MATERIALS: A BRIEF DISCUSSION

Composites in Aerospace Structures

- Boeing 787 – more than 50% structure is composites



- Composite components are approximately 15% of structural weight for civil aircraft.
- For military aircrafts and helicopters, it is 40% of structural weight.
- Earlier use of fibrous composites in aerospace are because of the potential for lighter structures as it affects fuel consumption, performance, and payload

DESIGN REQUIREMENTS AND OBJECTIVE FOR AEROSPACE VEHICLES

Product	Structural item	Primary structural requirements	Primary design objectives
Aircraft	Airframe	Compressive strength Damage tolerance Joint strength Durability	Minimum weight Maximum service
Helicopter	Rotor blades	Tensile strength Stiffness Fatigue life	Minimum weight Maximum service life
	Understructure	Stiffness Energy absorption	Minimum weight Crashworthiness

DESIGN REQUIREMENTS AND OBJECTIVE FOR AEROSPACE AND UNDERWATER VEHICLES

Product	Structural item	Primary structural requirements	Primary design objectives
Rocket motor	Motor cases nozzles	Tensile strength Resistance to elevated temperature	Minimum weight Survivability at 2000 C
Satellite	Rotor blades	Stiffness Low thermal expansion	Minimum weight Dimensional stability
Marine (sub-mercibles)	Understructure	Compression strength and stability Joint integrity	Minimum weight Maximum depth

CHOICE OF COMPOSITE MATERIALS

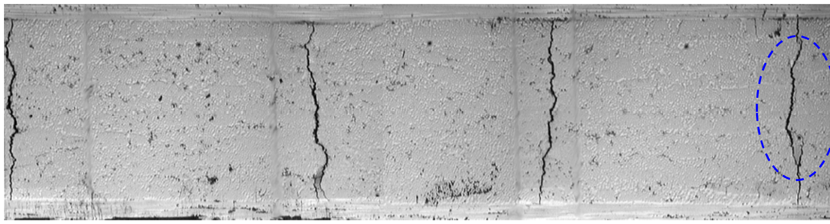
Reason for use	Material selected	Application/driver
Lower inertia, less deflection	High strength carbon/graphite-epoxy	Industrial rolls.
Light weight, damage tolerance	High strength carbon/graphite, hybrids, epoxy	Trucks and buses to reduce environment pollutions.
More reproducible complex surface	High strength or high modulus carbon, graphite epoxy	High special aircraft.

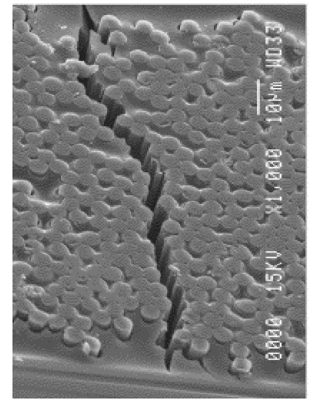
DAMAGE IN COMPOSITES

- Multiple matrix cracks, interfacial disbonds, delaminations, fiber breaks, microbuckled fibers, and more
- Multiple orientations
- Multiple scales of damage entities
- Multiple rates of evolution
- Multiple effects on material response+

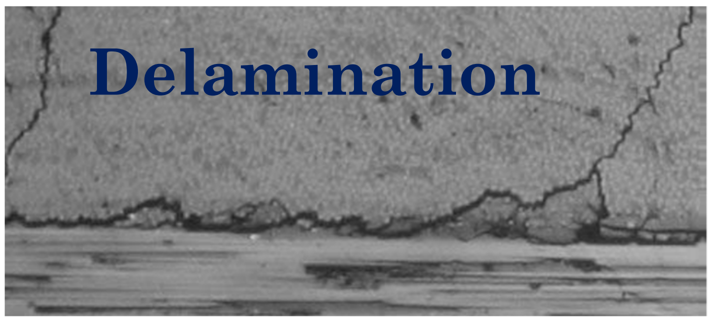
Damage Mechanisms in Composites

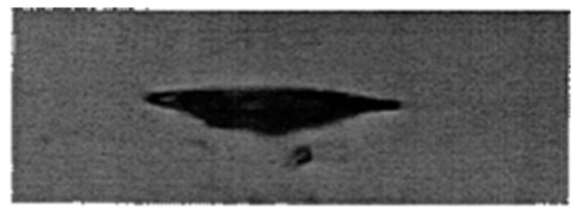
Multiple Matrix Cracking



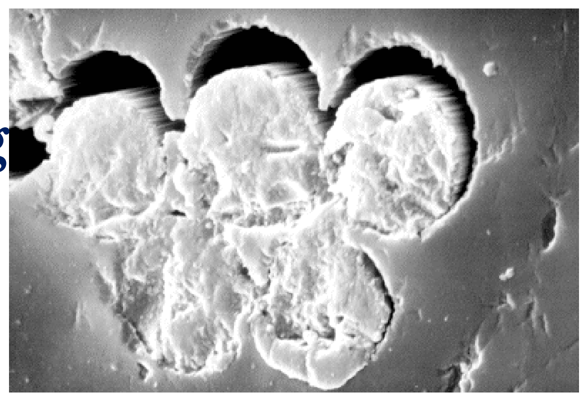


Delamination

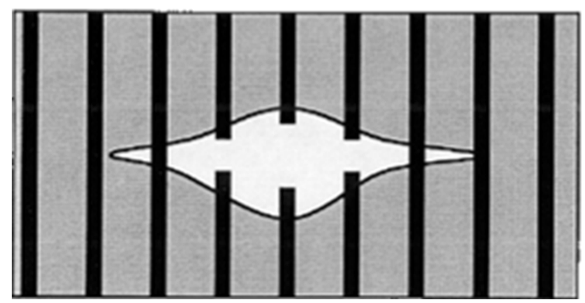




Debonding



Fiber breakage

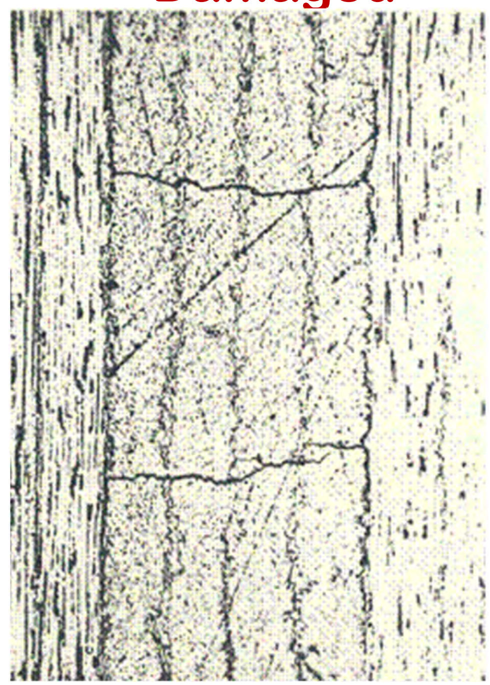


Cross-Ply Laminate in Monotonic Tension

Undamaged

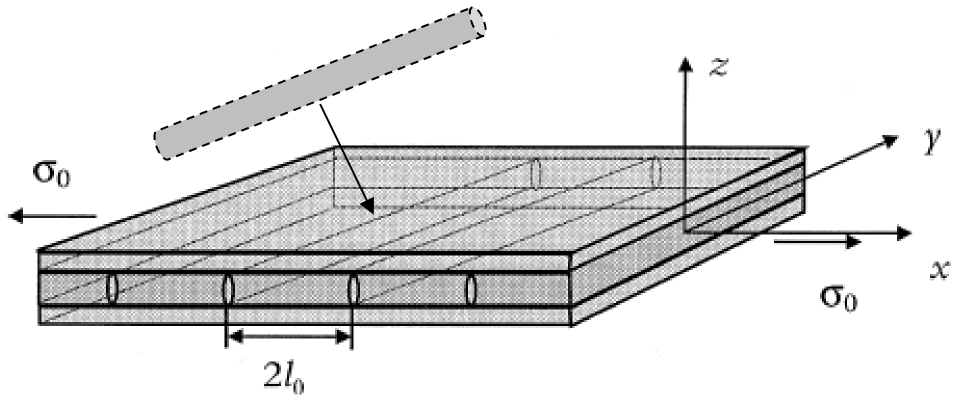
0^0 90^0

Damaged



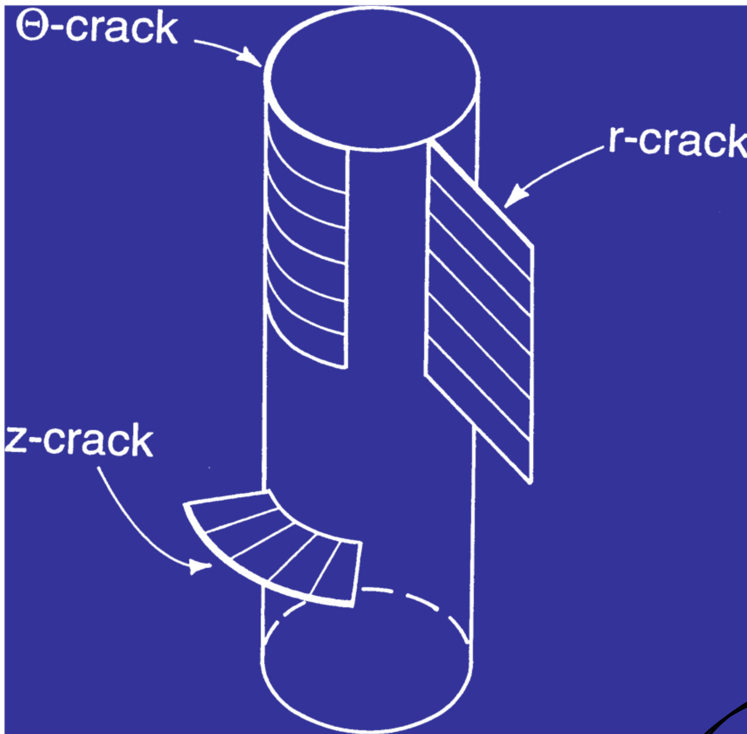
0^0 90^0

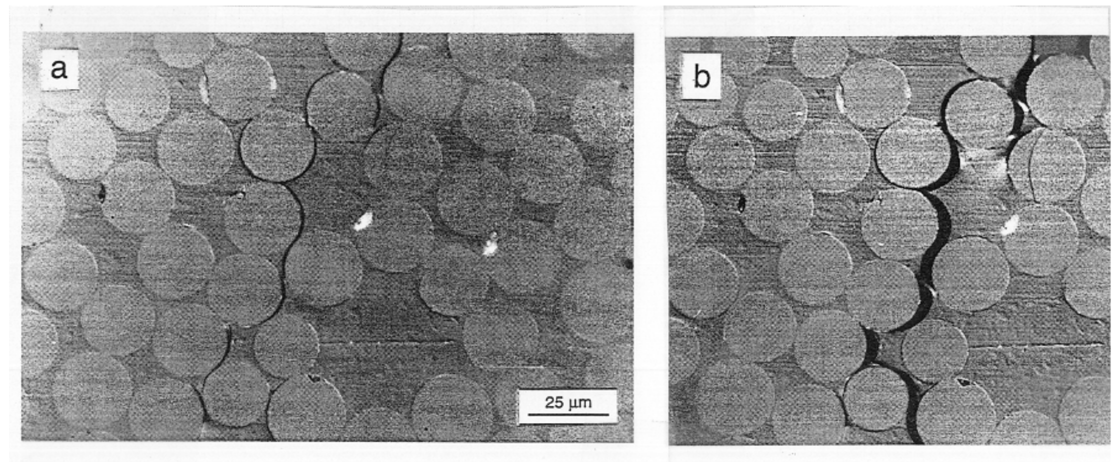
Multiple Matrix Cracking



- Usually first mode of damage in composites
- An array of cracks appear spanning whole laminate width and thickness of the cracked layer
- May not cause final failure
- However, it can cause significant degradation in stiffness properties
- May induce other severe damage modes, such as delamination, etc.

Crack Formation under Loading Transverse to fibers





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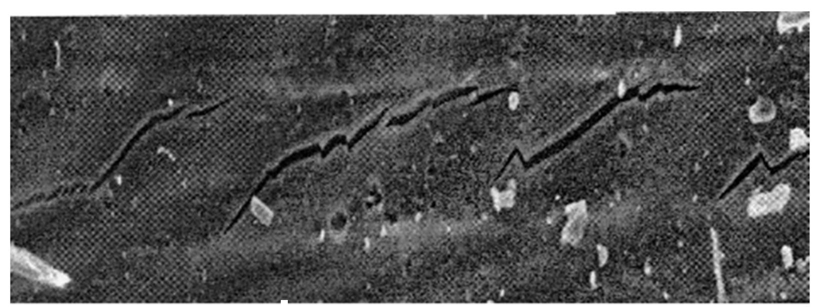
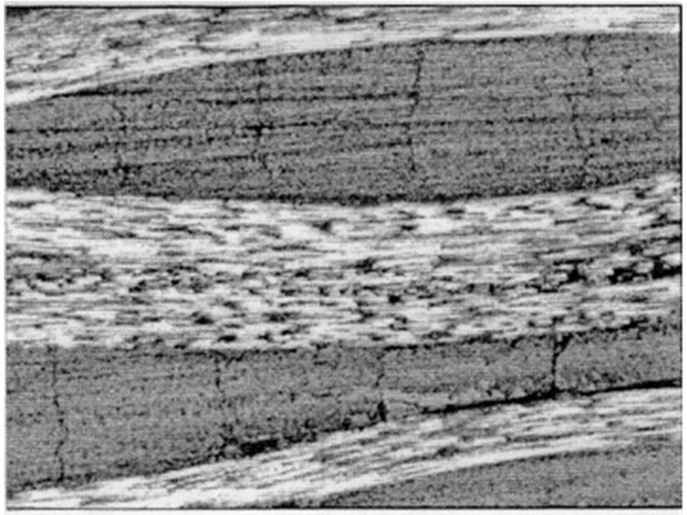
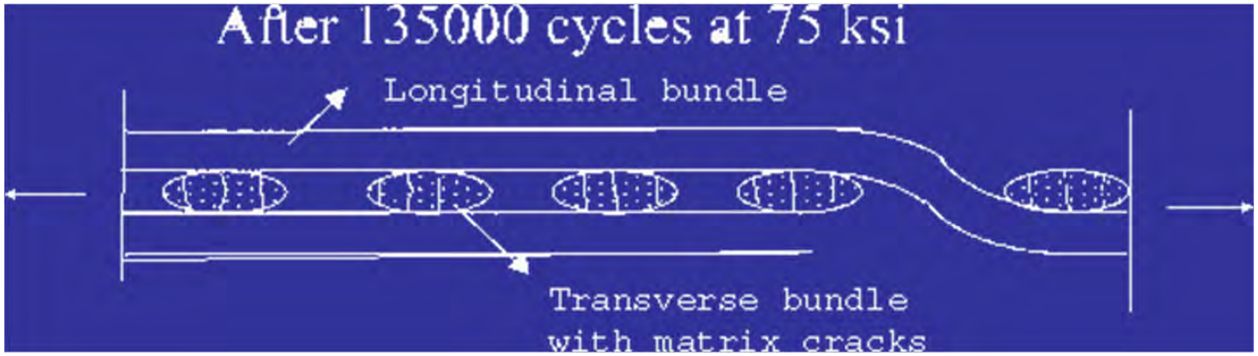


Shear induced cracking
At fiber/matrix interface

0.1mm

Damage Mechanisms in Woven Laminates





DESIGN WITH COMPOSITE MATERIALS

- **Basic Objective**

  To use the compellingly attractive properties of composites to get structures of lower weight, higher strength, & stiffness, etc.

- **Problem Areas**
  - **Behavioral characteristics intimidate**
  - **Analysts usually overemphasize importance**
  - **Designers must overcome or avoid**
  - **Not all are important simultaneously**
  - **If not avoided or controlled, structure will not perform correctly**

ELEMENTS OF DESIGN

- Analysis: If I can’t analyze it, I can’t design it.
- Manufacturing: If I can’t make it, the design is wasted.
- Materials: Did I choose the ‘best’ material ?
- Configuration: What alternative configuration would be better ?

ANALYSIS VERSUS DESIGN

- **Analysis: Determination of the *behavioral response* exhibited by a particular structural configuration under specific loads (*what loads does the structure take ?)***

- **Design: The process of altering dimensions, shapes, and materials to find the best (optimum) structural configuration to perform a specific function (what is the *best* structure to take the load ?)**

COMPOSITE STRUCTURAL DESIGN

- *Given*: loads (transverse, in-plane, excitation frequencies)
- *Required*: Find the laminate structural *configuration* necessary to carry those loads
- Design is *not* a deterministic process; it is an iterative procedure of selecting a *configuration*.

Anisotropic Analysis

- Shear Coupling
  - $A_{16} = A_{26} = 0$ for $[0°/\pm\theta/90°]_s$
  - Easy to include A_{16} & A_{26}
- Twist Coupling
  - D_{16} & D_{26} NOT zero for any general laminate more complicated than a cross-ply
  - Easy to include D_{16} & D_{26} in analysis
  - D_{16} & D_{26} go to zero as the number of alternations of $\pm\theta$ layers increases
- Both Effects are Negligibly Small
  - If the number of alternations of $\pm\theta$ is high enough

Nonlinear Material Behavior

- Matrix controlled properties (E_2 & G_{12}) nonlinear.
- Laminate behavior often and usually linear even with $\pm 45^{\circ}$ layers.
- Behavior of large structures appears linear.
- Behavior around cutouts seems linear.
- Operating in a nonlinear behavior range might be undesirable. Therefore, it is possible to "design out" nonlinear behavior.

Failure Prediction

Are the structural response parameters within design bounds ?

- Deflections too high?
- Buckling loads too close?
- Vibration frequencies near resonance?
- Stresses too high? Less than strength?
- Joints (can loads be transmitted?)

FAILURE THEORIES

- Maximum Stress Criterion
- Maximum Strain Criterion
- Tsai-Hill Criterion
- Hoffman Criterion
- Tsai-Wu Criterion
- Hashin Criterion

FAILURE THEORIES (continued)

- The failure criteria consist of parameters that must be experimentally determined. Often, these parameters are difficult to determine with certainty.

- When an allowable is exceeded in a given layer, the engineering constants responsible for the particular mode of failure are degraded (i.e., reduced) by a predetermined magnitude.

FAILURE THEORIES (continued)

- The damage modes are dependent on loading, stacking sequence, and specimen geometry.
- There are many proposed theories to predict the on-set of failures and their progression.
- Most of the failure criteria are based on the stress state in a lamina.
- An accurate kinematic model of the laminate is necessary to determine three-dimensional stress and strain fields.

FAILURE THEORIES (continued)

- The reduction of stiffnesses is another area where care and compromise should support the reality of the situation.
- The stiffness reduction is carried out within a layer and within a single finite element at selected integration points. Thus, stiffness reduction is made point wise, and it is not as drastic as discarding an entire layer.
- In the case of delaminations, the failure criterion can only determine the on-set of delaminations. However, the growth of delaminations cannot be modeled unless the kinematic model is three-dimensional.

FAILURE THEORIES (continued)

- The strain criterion is a more accurate measure since it is directly obtained from the strain gages.
- The stress based data is difficult to generate, due to the statistical variations in obtaining stresses from strains. The major problem is that the constitutive matrix is not known precisely at the strain level. Since damage accumulation is cumulative, and failure is usually progressive rather than catastrophic and instantaneous, error introduced through assumed material stiffnesses can grow with local steps.

SUMMARY OF THE COURSE

First Day (Sept 5)

- Composite Materials: An Introduction
- Anisotropic Elasticity and FGM materials
- Laminate Structural Theories (CLPT and FSDT)
- Interaction Session

Second Day (Sept 6)

- Analytical Solution Methods – Navier Solution
- Numerical examples of bending, vibration, and buckling
- Finite Element Models
- Numerical Results and Discussion
- Interaction Session

COURSE CONTENTS

Third Day (Sept 7)

- FGM Structures (Beams)
- FGM Structures (Plates)
- Finite Element Models
- Numerical Results and Discussion
- Third-order and layerwise theories
- Interaction Session

Fourth Day (Sept 8)

- Continuum Shell Elements
- A Robust Shell Finite Element
- Failures in Composites and Design Considerations
- Overview of the Course
- Interaction Session

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About the Instructor: Professor J. N. Reddy

Dr. Reddy is a Distinguished Professor, Regents' Professor, and inaugural holder of the O'Donnell Foundation Chair IV in Mechanical Engineering at Texas A&M University, College Station, Texas. Professor Reddy is one of the most well-known mechanics researchers and educators of our times. His pioneering works on the development of shear deformation theories (that bear his name in the literature as the Reddy third-order plate theory and the Reddy layerwise theory) have had a major impact and have led to new research developments and applications. Some of the ideas on shear deformation theories and penalty finite element models of fluid flows have been implemented into commercial finite element computer programs like ABAQUS, NISA, and HyperXtrude. In recent years, Reddy's research has focused on the development of locking-free shell finite elements and nonlocal and non-classical continuum mechanics problems involving couple stresses, damage, and fracture in solids. He is one of the original top 100 ISI Highly Cited Researchers with 95,000 citations and h-index of 115 as per Google Scholar.

Professor Reddy has authored and coauthored 24 books (several with second, third, and fourth editions and two in print) on a wide variety of mechanics topics, beginning with variational principles and methods, mathematical theory of finite elements, engineering analysis, linear and nonlinear finite elements, finite elements in heat transfer and fluid dynamics, mechanics of composite materials and structures, plates and shells, continuum mechanics, mechanics of materials, and finite element and finite volume methods. An especially strong point of Professor Reddy's books is the clarity and physical insight of explanations of even the most difficult topics through relevant engineering examples but without compromising on the mathematical rigor. His classic authoritative text book, *An Introduction to the Finite Element Method*, has appeared as the 4th edition in 2019 and translated in to Persian and Chinese. It is not an exaggeration to say that a majority of engineers and researchers for the last 4 decades have learned the finite element method from his book. His books are often self-contained and they provide strong foundation for mechanics concepts and numerical simulations. Another strong point of his books is the introduction of current developments into the conventional contents.

Professor Reddy earned a number of significant (and some are the highest) honors from various professional societies. The most significant honors include: the Worcester Reed Warner Medal, the Charles Russ Richards Memorial Award, ASME Honorary Member, the ASME Medal, and the **S. P. Timoshenko Medal** from the American Society of Mechanical Engineers; the Raymond D. Mindlin, Nathan M. Newmark, and Theodore von Karman Medals from the American Society of Civil Engineers; Award for Excellence in the Field of Composites and Distinguished Research Award from the American Society for Composites; the Computational Solid Mechanics award (now known as the Ted Belytschko award), the John von Neumann Medal from US Association for Computational Mechanics; the O. C. Zienkiewicz Award from the International Association of Computational Mechanics; Prager Medal from the Society of Engineering Science, the **IACM (Gauss-Newton) Medal** from the International Association of Computational Mechanics, the Michael Païdoussis Medal from Royal Society of Canada, and the Leonardo da Vinci Award from the European Academy of Sciences. He received Honoris Causa (honorary degrees) from Technical University of Lisbon, Portugal and Odlar Yurdu University, Azerbaijan. Professor Reddy is a member of the US National Academy of Engineering and a foreign fellow of the Indian National Academy of Engineering, the Canadian Academy of Engineering, the Chinese Academy of Engineering, the Brazilian National Academy of Engineering, the Royal Academy of Engineering of Spain, the European Academy of Sciences, and the European Academy of Sciences and Arts.