

# optimization

January 18, 2023

## 1 Importing the libraries

```
[35]: # Importing libraries
import import_ipynb
import numpy as np
import time
import math
import copy
import matplotlib.pyplot as plt
from matplotlib.pyplot import figure
from matplotlib import ticker
```

## 2 Generate the positive definite matrix A, the vector b in the range of A and the optimal solution x\*

```
[36]: # Method of generating a positive semidefinite matrix
# 1. Generate a random square matrix
# 2. multiply it by its own transposition
# 3. we have obtained a positive semi-definite matrix.
def generateRandomPositiveSemidefiniteMatrix(size, scaleDown):
    randomMatrix = np.random.rand(size, size)
    positive_semidefinite_matrix = np.matmul(randomMatrix, randomMatrix.
    ↪transpose()) / scaleDown
    return positive_semidefinite_matrix

# Generate linearly spacing vector from 1 to size for x*
def generateLinearOptimalX(size):
    return np.arange(1, size + 1, 1).astype(int)

# A matrix is positive semidefinite if all of its eigenvalues are nonnegative
# Note: this matrix should be symmetric
def isPositiveSemidefinite(A):
    return np.all(np.linalg.eigvals(A) >= 0)

# A matrix does not have an inverse if its determinant is equal to 0
def inverse(A):
```

```
return np.linalg.inv(A)
```

- 3 There are three different test cases: the small scale with dimension of 10, the large scale with dimension of 100 and the huge scale with the dimension of 1000

```
[50]: scales = ["small", "large", "huge"]

for scale in scales:

    if scale == "small":
        size = 10
        scaleDown = 1
    elif scale == "large":
        size = 100
        scaleDown = 10
    elif scale == "huge":
        size = 1000
        scaleDown = 100
    print(f"Data generation for the {scale} scale test. Dimension: {size}")

    start = time.time()
    A = generateRandomPositiveSemidefiniteMatrix(size, scaleDown)
    end = time.time()

    print("The matrix A is")
    print(A)

    print("\nTime required to generate the matrix A is")
    print(f"{end - start} seconds")

    # print(f"Is the matrix A positive semidefinite?:  

    ↪{isPositiveSemidefinite(A)}")

    start = time.time()
    invA = inverse(A)
    end = time.time()

    print("\nThe inverse of matrix A is")
    print(invA)

    print("\nTime required to invert the matrix A is")
    print(f"{end - start} seconds")
```

```

# The optimal solution is designed as the natural numbers: 1,2,3,4,5 and so
on
x_opt = generateLinearOptimalX(size)
print("\nThe optimal solution x* is")
print(x_opt)

# b is in the range of A
# In other words, b is a linear combination of the columns of matrix A
b = A.dot(x_opt)
print("\nThe vector b is")
print(b)
print("\n\n")

# So we have the identities:
#  $Ax^* = b$  or  $Ax - b = 0$ 
#  $A^{-1}b = x^*$ 

np.save(f"data/{scale}Matrix.npy", A)
np.save(f"data/{scale}Vector.npy", b)
np.save(f"data/{scale}Solution.npy", x_opt)

```

Data generation for the small scale test. Dimension: 10

The matrix A is

```

[[5.18739286 3.82006574 3.02467891 4.67019152 4.4263327 3.85433794
 3.29062272 3.21637545 3.68674808 2.44198635]
 [3.82006574 3.95598735 2.96634602 3.74129605 3.34534356 2.84943095
 2.56262218 2.73734196 3.29195817 2.18187658]
 [3.02467891 2.96634602 2.83309499 2.97713416 2.88903791 2.4030659
 2.41896517 2.51807188 2.29166691 1.45231162]
 [4.67019152 3.74129605 2.97713416 4.79083468 4.09810121 3.76307602
 3.28039034 3.1383191 3.33488093 2.4079852 ]
 [4.4263327 3.34534356 2.88903791 4.09810121 4.58298378 3.45154963
 2.71213933 3.02208139 3.06182015 1.88265053]
 [3.85433794 2.84943095 2.4030659 3.76307602 3.45154963 3.25389305
 2.68453657 2.45483449 2.70832725 1.83173519]
 [3.29062272 2.56262218 2.41896517 3.28039034 2.71213933 2.68453657
 2.93555239 2.44009243 2.2225287 1.65173892]
 [3.21637545 2.73734196 2.51807188 3.1383191 3.02208139 2.45483449
 2.44009243 2.54214198 2.1846371 1.47177919]
 [3.68674808 3.29195817 2.29166691 3.33488093 3.06182015 2.70832725
 2.2225287 2.1846371 3.49206426 1.76574634]
 [2.44198635 2.18187658 1.45231162 2.4079852 1.88265053 1.83173519
 1.65173892 1.47177919 1.76574634 1.97690738]]

```

Time required to generate the matrix A is

0.0010020732879638672 seconds

The inverse of matrix A is

```
[[ 6.31369782  2.03077804  3.30784445  2.05512423  0.36795623
 -4.62446685  1.55283315 -8.91786292 -3.60674967 -2.47592768]
 [ 2.03077804  9.03651895 -3.97172084 -1.46664953  1.98667402
 -0.30673841  4.40954976 -6.20830477 -5.1248167 -3.87031225]
 [ 3.30784445 -3.97172084  9.37011006  5.16454873  0.70198202
 -5.92456378 -0.3777639 -10.04340755 -0.35261707  0.05186895]
 [ 2.05512423 -1.46664953  5.16454873  7.43147417  1.52950198
 -7.38450401  1.12373609 -9.37834034 -1.00926741 -1.43566414]
 [ 0.36795623  1.98667402  0.70198202  1.52950198  4.06457578
 -4.36667293  3.85996682 -7.56401338 -1.54658383 -1.0630487 ]
 [-4.62446685 -0.30673841 -5.92456378 -7.38450401 -4.36667293
 12.91607466 -5.0174079 15.37804544  2.37932681  2.20718637]
 [ 1.55283315  4.40954976 -0.3777639  1.12373609  3.85996682
 -5.0174079  6.74968797 -10.03170498 -2.86005041 -2.51956657]
 [-8.91786292 -6.20830477 -10.04340755 -9.37834034 -7.56401338
 15.37804544 -10.03170498 33.22340482  8.04006146  6.09008591]
 [-3.60674967 -5.1248167 -0.35261707 -1.00926741 -1.54658383
 2.37932681 -2.86005041  8.04006146  5.00478556  2.80176085]
 [-2.47592768 -3.87031225  0.05186895 -1.43566414 -1.0630487
 2.20718637 -2.51956657  6.09008591  2.80176085  3.58237677]]
```

Time required to invert the matrix A is

0.0009872913360595703 seconds

The optimal solution x\* is

```
[ 1  2  3  4  5  6  7  8  9 10]
```

The vector b is

```
[192.20597715 160.7030465 130.45422714 185.47955251 169.34526813
149.71853723 135.04951123 130.43527391 144.16316175 100.19547579]
```

Data generation for the large scale test. Dimension: 100

The matrix A is

```
[[3.13994183 2.488769 2.61732092 ... 2.32355668 2.47360974 2.40710816]
 [2.488769 3.86331209 2.75016916 ... 2.51931198 2.72744451 2.82935441]
 [2.61732092 2.75016916 3.45267036 ... 2.5659738 2.60370428 2.51052948]
 ...
 [2.32355668 2.51931198 2.5659738 ... 3.14748759 2.35775558 2.41655926]
 [2.47360974 2.72744451 2.60370428 ... 2.35775558 3.35645681 2.5015893 ]
 [2.40710816 2.82935441 2.51052948 ... 2.41655926 2.5015893 3.44387547]]
```

Time required to generate the matrix A is

0.0 seconds

The inverse of matrix A is

```
[[ 18442.25324954 -13676.98079602 -24967.96961572 ... 9012.18119031
  -10628.70876097 -5349.44837691]
 [-13676.98079597 10330.50762925 18635.67831259 ... -6682.64986148
  8059.02048141 3930.35998728]
 [-24967.96961574 18635.67831267 33951.48959648 ... -12237.03885599
  14465.66155191 7213.92584786]
 ...
 [ 9012.18119033 -6682.64986152 -12237.03885602 ... 4434.71391424
  -5162.65309222 -2605.73539014]
 [-10628.70876088 8059.02048137 14465.66155178 ... -5162.65309216
  6369.35043999 3059.75612469]
 [ -5349.44837692 3930.3599873 7213.92584787 ... -2605.73539014
  3059.75612472 1577.84389255]]
```

Time required to invert the matrix A is

0.0039980411529541016 seconds

The optimal solution x\* is

```
[ 1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18
 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36
 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54
 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72
 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90
 91 92 93 94 95 96 97 98 99 100]
```

The vector b is

```
[11951.4115316 13668.11060914 13128.3439263 13446.41130468
 11396.74358665 11536.42040716 11988.35414693 12879.58711637
 11583.57842884 12845.35112076 11614.20913591 13216.49209787
 13229.22664941 13513.64521322 11989.24061182 12044.72498256
 12472.99432521 12577.17238309 11386.44514139 13589.6475785
 13342.42551611 12432.27946437 13539.67697297 14194.83510356
 13067.33397387 12472.70724337 13109.32227699 10339.41769487
 12179.19629227 14106.66874556 12536.63839371 12443.16129939
 12666.12477877 11941.17426665 13158.52635285 13153.74186299
 12730.74100369 12515.77518577 12487.5578692 12614.73106332
 11893.75445949 13837.71474911 12020.31168545 11480.49611585
 11769.8904268 12605.2935294 12347.12761509 13052.13205585
 13309.06504383 12644.22308362 12631.98078663 12961.90355795
 13034.46437815 12244.63325669 13129.16772386 11740.37817754
 12774.00697391 13193.42876032 11782.56774397 12176.45877321
 12688.60833572 12503.98770786 13188.4320737 12397.85396559
 13301.72422593 12280.79083091 11602.52164541 12624.5807516
 13842.09258105 12080.52747419 13046.84671288 14717.04320886
 12694.69158636 12399.5084073 13335.03107025 11627.10817166
 12600.18097037 11275.94272285 13181.0396423 13639.14539868
 12757.78370794 13104.98871563 12445.35888241 13974.87166027]
```

```

11060.44962511 12371.85740991 11157.31066671 11766.00586107
12787.45699238 13889.46329881 13396.38475157 11996.09469735
13505.67298862 11887.65152997 11545.07427836 11605.48391919
13260.68733659 12215.80212647 12881.82173727 12650.63877369]

```

Data generation for the huge scale test. Dimension: 1000

The matrix A is

```

[[3.46670326 2.4664962 2.57697622 ... 2.60897989 2.48262417 2.52327387]
 [2.4664962 3.26569944 2.42777349 ... 2.49552908 2.39568779 2.40236126]
 [2.57697622 2.42777349 3.38609734 ... 2.5507125 2.48939967 2.43790291]
 ...
 [2.60897989 2.49552908 2.5507125 ... 3.42260011 2.48801867 2.45307578]
 [2.48262417 2.39568779 2.48939967 ... 2.48801867 3.16346634 2.38766003]
 [2.52327387 2.40236126 2.43790291 ... 2.45307578 2.38766003 3.1789813 ]]

```

Time required to generate the matrix A is

0.02899932861328125 seconds

The inverse of matrix A is

```

[[ 1255.45872873 -718.41413243 -359.88817163 ... 1025.38260481
   -513.6637602 373.04889329]
 [ -718.41413204 2158.58014172 1720.56453956 ... -1194.13527399
   401.98400636 -897.54651358]
 [ -359.88817123 1720.56453939 1605.42339669 ... -798.26995276
   144.57037194 -783.63724997]
 ...
 [ 1025.3826046 -1194.13527413 -798.269953 ... 1208.3547055
   -440.81361876 610.37214533]
 [ -513.6637602 401.98400657 144.57037215 ... -440.81361887
   510.17392994 -151.45545363]
 [ 373.04889304 -897.54651343 -783.63724993 ... 610.37214515
   -151.4554535 585.27254052]]

```

Time required to invert the matrix A is

0.10899591445922852 seconds

The optimal solution x\* is

```

[ 1  2  3  4  5  6  7  8  9 10 11 12 13 14
 15 16 17 18 19 20 21 22 23 24 25 26 27 28
 29 30 31 32 33 34 35 36 37 38 39 40 41 42
 43 44 45 46 47 48 49 50 51 52 53 54 55 56
 57 58 59 60 61 62 63 64 65 66 67 68 69 70
 71 72 73 74 75 76 77 78 79 80 81 82 83 84
 85 86 87 88 89 90 91 92 93 94 95 96 97 98
 99 100 101 102 103 104 105 106 107 108 109 110 111 112]

```

113	114	115	116	117	118	119	120	121	122	123	124	125	126
127	128	129	130	131	132	133	134	135	136	137	138	139	140
141	142	143	144	145	146	147	148	149	150	151	152	153	154
155	156	157	158	159	160	161	162	163	164	165	166	167	168
169	170	171	172	173	174	175	176	177	178	179	180	181	182
183	184	185	186	187	188	189	190	191	192	193	194	195	196
197	198	199	200	201	202	203	204	205	206	207	208	209	210
211	212	213	214	215	216	217	218	219	220	221	222	223	224
225	226	227	228	229	230	231	232	233	234	235	236	237	238
239	240	241	242	243	244	245	246	247	248	249	250	251	252
253	254	255	256	257	258	259	260	261	262	263	264	265	266
267	268	269	270	271	272	273	274	275	276	277	278	279	280
281	282	283	284	285	286	287	288	289	290	291	292	293	294
295	296	297	298	299	300	301	302	303	304	305	306	307	308
309	310	311	312	313	314	315	316	317	318	319	320	321	322
323	324	325	326	327	328	329	330	331	332	333	334	335	336
337	338	339	340	341	342	343	344	345	346	347	348	349	350
351	352	353	354	355	356	357	358	359	360	361	362	363	364
365	366	367	368	369	370	371	372	373	374	375	376	377	378
379	380	381	382	383	384	385	386	387	388	389	390	391	392
393	394	395	396	397	398	399	400	401	402	403	404	405	406
407	408	409	410	411	412	413	414	415	416	417	418	419	420
421	422	423	424	425	426	427	428	429	430	431	432	433	434
435	436	437	438	439	440	441	442	443	444	445	446	447	448
449	450	451	452	453	454	455	456	457	458	459	460	461	462
463	464	465	466	467	468	469	470	471	472	473	474	475	476
477	478	479	480	481	482	483	484	485	486	487	488	489	490
491	492	493	494	495	496	497	498	499	500	501	502	503	504
505	506	507	508	509	510	511	512	513	514	515	516	517	518
519	520	521	522	523	524	525	526	527	528	529	530	531	532
533	534	535	536	537	538	539	540	541	542	543	544	545	546
547	548	549	550	551	552	553	554	555	556	557	558	559	560
561	562	563	564	565	566	567	568	569	570	571	572	573	574
575	576	577	578	579	580	581	582	583	584	585	586	587	588
589	590	591	592	593	594	595	596	597	598	599	600	601	602
603	604	605	606	607	608	609	610	611	612	613	614	615	616
617	618	619	620	621	622	623	624	625	626	627	628	629	630
631	632	633	634	635	636	637	638	639	640	641	642	643	644
645	646	647	648	649	650	651	652	653	654	655	656	657	658
659	660	661	662	663	664	665	666	667	668	669	670	671	672
673	674	675	676	677	678	679	680	681	682	683	684	685	686
687	688	689	690	691	692	693	694	695	696	697	698	699	700
701	702	703	704	705	706	707	708	709	710	711	712	713	714
715	716	717	718	719	720	721	722	723	724	725	726	727	728
729	730	731	732	733	734	735	736	737	738	739	740	741	742
743	744	745	746	747	748	749	750	751	752	753	754	755	756
757	758	759	760	761	762	763	764	765	766	767	768	769	770
771	772	773	774	775	776	777	778	779	780	781	782	783	784

785	786	787	788	789	790	791	792	793	794	795	796	797	798
799	800	801	802	803	804	805	806	807	808	809	810	811	812
813	814	815	816	817	818	819	820	821	822	823	824	825	826
827	828	829	830	831	832	833	834	835	836	837	838	839	840
841	842	843	844	845	846	847	848	849	850	851	852	853	854
855	856	857	858	859	860	861	862	863	864	865	866	867	868
869	870	871	872	873	874	875	876	877	878	879	880	881	882
883	884	885	886	887	888	889	890	891	892	893	894	895	896
897	898	899	900	901	902	903	904	905	906	907	908	909	910
911	912	913	914	915	916	917	918	919	920	921	922	923	924
925	926	927	928	929	930	931	932	933	934	935	936	937	938
939	940	941	942	943	944	945	946	947	948	949	950	951	952
953	954	955	956	957	958	959	960	961	962	963	964	965	966
967	968	969	970	971	972	973	974	975	976	977	978	979	980
981	982	983	984	985	986	987	988	989	990	991	992	993	994
995	996	997	998	999	1000								

The vector  $b$  is

[1288148.38605397	1225895.69176978	1257473.64841123	1228835.82255788
1224077.01871562	1212517.82389932	1240859.74005241	1253152.65647997
1240228.57636776	1270103.38490047	1264142.82338179	1251688.46063073
1254482.1363195	1225112.43408274	1263940.76502782	1255410.04538622
1239677.20774742	1235349.11386752	1231038.64020359	1242211.03679888
1231434.4916032	1274326.9892582	1255441.89928393	1197434.74024783
1274248.85359364	1240688.57269685	1215064.08463919	1240491.20378502
1268013.52287767	1222841.25313544	1231853.95927474	1272321.56748619
1231225.74554436	1248468.52281984	1232173.80067627	1262733.3429281
1242232.63935319	1238660.79503354	1218555.36147668	1265515.51709145
1262823.83753068	1251383.99191741	1245206.85638745	1244614.05808569
1257808.12575237	1276288.56409817	1250555.11610416	1291540.50838127
1267898.33174838	1221586.73837982	1261762.54012527	1219170.18772827
1247321.56926833	1248703.99586329	1249072.05168852	1215396.66210849
1246632.51585232	1274542.09354422	1262926.11340021	1251961.20389832
1251023.67083806	1242339.57914703	1255491.32801905	1218716.64152746
1248261.53484672	1250685.32733516	1226141.03617422	1232635.91286844
1272435.87572042	1250252.59078934	1242675.28187014	1236688.21629416
1243897.67962365	1245703.51495165	1233543.19954982	1288941.57185538
1198509.41844045	1260015.08577748	1254602.76998612	1229639.00630556
1255115.99097223	1253405.58882617	1197644.35695534	1266849.14852753
1254736.53656355	1229982.91288287	1248720.51392522	1229144.46429352
1288155.68776751	1248370.74605938	1263127.8497193	1258886.07518881
1281759.45916786	1199323.11761293	1234355.40813537	1273824.50105719
1233894.33898138	1214474.94791131	1244721.46139526	1244794.07319837
1245518.74462949	1225879.98228785	1274658.40637994	1241826.17857897
1282226.21055016	1251817.69963363	1238609.12132459	1254127.06105312
1240814.5583992	1251757.55090083	1262885.0717024	1249320.99152669
1246609.59108422	1216496.92673508	1257602.62152679	1293186.23603885
1294909.63188488	1252813.35505656	1272304.8774921	1251163.37527779



1216502.1299694	1266594.11851374	1284842.36537209	1281542.01729402
1299318.24333636	1240977.16137476	1271431.76179625	1266595.13012004
1248368.59211165	1255175.30590977	1246129.51102496	1278275.05902438
1242771.35691271	1249081.12162635	1235199.20504633	1259758.82793643
1224921.03358947	1252958.92046671	1240983.89255875	1216864.81553027
1250556.82683249	1281189.66226206	1184858.67823268	1244733.86092426
1240304.03386651	1227183.69727283	1268157.53192575	1251992.55711503
1221678.88880842	1221405.35449494	1229307.99907886	1290766.17083046
1279512.71764187	1218397.9016137	1254437.0508617	1246750.86567303
1232762.78964127	1264343.55922994	1215153.31614295	1266446.41736942
1228031.43709566	1233881.55471041	1272917.14523503	1286065.26981243
1289965.17220237	1266871.57380324	1269237.70700397	1221383.35448964
1260886.75002073	1216883.7180023	1227148.79668091	1214490.31047607
1261017.20865857	1274011.54610594	1272537.12011991	1266013.70244475
1255996.57321239	1220372.30196754	1268530.37470999	1242853.69042061
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1262283.20536914 1267764.8649733 1220009.4662269 1216843.34192144]

```

## 4 Helper functions

```

[54]: # Returns a vector, which is the result of the gradient
def gradient(A, b, x):
    return A.dot(x) - b

# Returns a scalar, which is the norm of the result above
def gradientNorm(A, b, x):
    return np.linalg.norm(A.dot(x) - b)

# Returns a scalar, which is the norm of the difference between x and x*
def differenceNorm(x, x_opt):
    return np.linalg.norm(x - x_opt)

```

```

# Returns a scalar, which is the norm of x
def norm(x):
    return np.linalg.norm(x)

```

```

[55]: # Plotting the difference norms  $\log \|x - x^*\|/2$ 
def plotDifferenceNorms(scale, maxIter, tolerance, algorithmName, algorithm,
    logBase):
    A = np.load(f"data/{scale}Matrix.npy", allow_pickle=True)
    b = np.load(f"data/{scale}Vector.npy", allow_pickle=True)
    x_opt = np.load(f"data/{scale}Solution.npy", allow_pickle=True)

    print(f"\nThe {scale} scale problem is chosen. The matrix A and vector b
    dimension is {b.size}")
    print(f"The number of maximum iterations is {maxIter}. The allowed
    tolerance for gradient norm is {tolerance}")

    start = time.time()
    x_opt_algo, x_iterations_algo, stoppingReason = algorithm(A, b, maxIter,
    tolerance)
    end = time.time()

    print(f"\nThe {algorithmName} algorithm runs in {end - start} seconds")
    print("Reason of stopping")
    print(stoppingReason)

    if scale == "huge":
        print(f"\nFirst 100 values in the optimal solution x found by
        {algorithmName} algorithm")
        print(x_opt_algo[0:100])
        print("\nFirst 100 values in the theoretical optimal solution x*")
        print(x_opt[0:100])
    else:
        print(f"\nThe optimal solution x found by {algorithmName} algorithm")
        print(x_opt_algo)
        print("\nThe theoretical optimal solution x*")
        print(x_opt)

    differenceNorms = []
    for x_sol in x_iterations_algo:
        differenceNorms.append(differenceNorm(x_sol, x_opt))
    differenceNorms = np.array(differenceNorms)

    figure(figsize=(8, 6), dpi=80)

    size = 16
    iterations = np.arange(0, differenceNorms.size, 1)

```

```

plt.plot(iterations, differenceNorms, label = f"Experimental convergence_
↪rate")
plt.title(f"Convergence rate of\n{algorithmName} algorithm\n{scale} scale_
↪problem - dimension: {b.size}", size=size + 4)
plt.xticks(fontsize=size)
plt.yticks(fontsize=size)
# Plotting the log graph in base 2
plt.yscale('log', base=2)
plt.xlabel("Iterations", size=size)
plt.ylabel(r'$\log||x-x*||_2$', size=size)
plt.legend(loc=1, frameon=False, fontsize=size + 2)
plt.show()

```

## 5 Task 1: Gradient Descent Algorithm

### 5.1 Gradient descent algorithm implementation

```

[56]: def gradientDescent(A, b, maxIters = 5000, epsilon = 10e-5):
    # Dimension of A and b
    dim = b.size
    # initial random vector x filled with the mean of matrix A, with length_
    ↪equal to the dimension
    x = np.repeat(np.mean(A), dim)
    # The Lipschitz constant, which is the maximum eigenvalue of A
    L = np.max(np.linalg.eigvals(A))
    # The step size is alpha = 1/L
    alpha = 1/L
    # currentIteration
    iter = 1
    # Saving the results
    x_iterations = [x]
    # The main iteration loop
    while (gradientNorm(A, b, x) > epsilon and iter <= maxIters):
        x = x - alpha * gradient(A, b, x)
        x_iterations.append(x)
        iter += 1
    # Stopping reason (max iteration exceeded or gradient norm smaller than the_
    ↪tolerance epsilon)
    if iter > maxIters:
        stoppingReason = f"Max iterations ({maxIters}) exceeded"
    else:
        stoppingReason = f"Gradient norm smaller than {epsilon}\nCompleted_
        ↪iteration: {iter}"
    return (x, x_iterations, stoppingReason)

```

## 5.2 Gradient descent algorithm convergence rate analysis

```
[57]: # There are three different scales: small, large and huge
scales = ["small", "large", "huge"]
# Number of maximum iterations of the three scales small, large and huge
maxIters = [20000, 40000, 60000]
# Tolerance of the gradient norm of the three scales small, large and huge
tolerances = [10e-5, 10e-3, 10e-1]
# The algorithm
algorithmName = "gradient descent"
algorithm = gradientDescent
# The logarithm base
logBase = 2
# Running optimization for the three scales small, large and huge
for i in range(0,3):
    plotDifferenceNorms(scales[i], maxIters[i], tolerances[i], algorithmName,
    ↪algorithm, logBase)
```

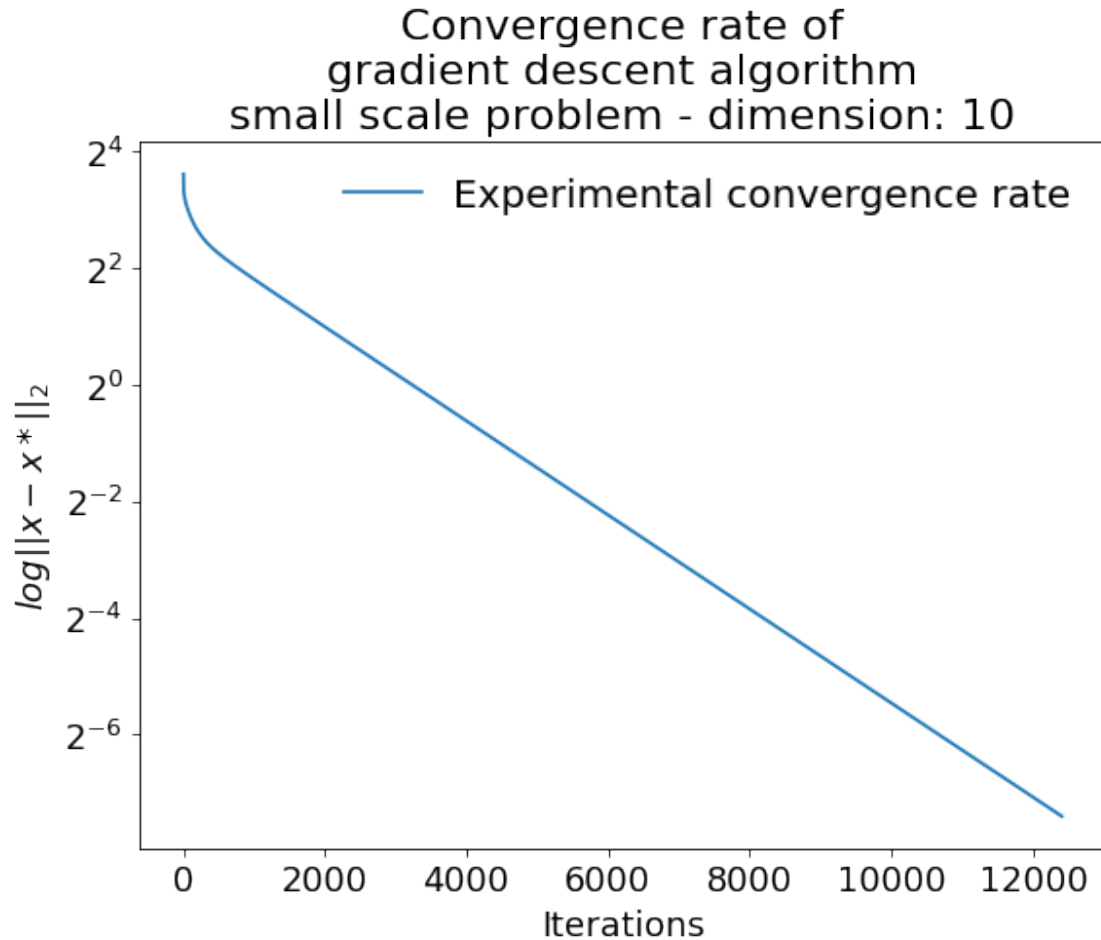
The small scale problem is chosen. The matrix A and vector b dimension is 10  
The number of maximum iterations is 20000. The allowed tolerance for gradient  
norm is 0.0001

The gradient descent algorithm runs in 0.11102747917175293 seconds  
Reason of stopping  
Gradient norm smaller than 0.0001  
Completed iteration: 12410

The optimal solution x found by gradient descent algorithm  
[1.00126508 2.00079503 3.00132805 4.00138349 5.00101498 5.99769032  
7.00135383 7.99564111 8.99891374 9.99914945]

The theoretical optimal solution x\*  
[ 1 2 3 4 5 6 7 8 9 10]





The large scale problem is chosen. The matrix A and vector b dimension is 100  
The number of maximum iterations is 40000. The allowed tolerance for gradient norm is 0.01

The gradient descent algorithm runs in 9.819963932037354 seconds

Reason of stopping

Max iterations (40000) exceeded

The optimal solution x found by gradient descent algorithm

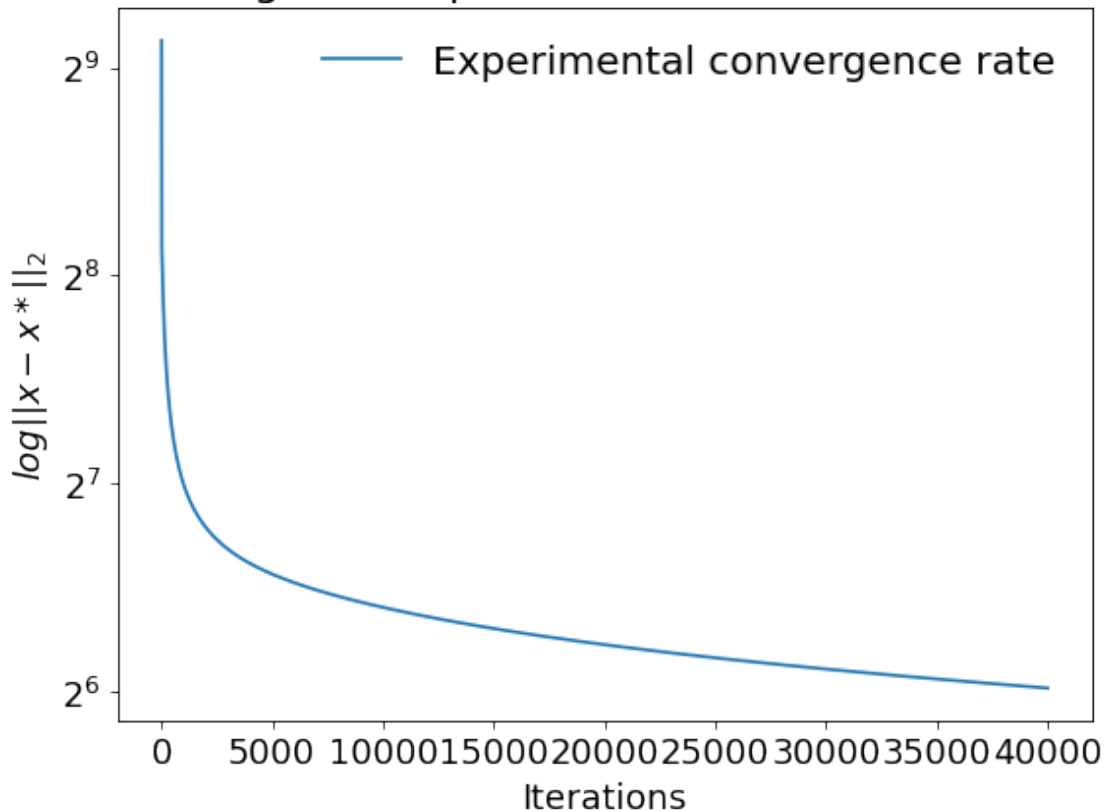
```
[ 10.80954396  13.53080539  12.34048827  13.56010399  13.21795864
   3.44248581   2.58263856   9.56551187   2.56131046  10.9542098
   2.02486301  17.76882364  17.15608743  14.20430406  16.40691544
   8.04793459  22.89717879  33.23654019  17.13214426  23.06360859
   8.69197351  21.62729296  28.15997518  25.24696302  19.44788055
  28.4998845   31.10315707  32.34107394  23.97768023  28.70397213
  33.47529818  31.31788127  30.57097049  42.9701813   25.36785319
  44.08458021  42.57848789  49.70319644  40.79161149  43.17582171]
```

39.66958338	29.48079611	53.8984904	43.90625763	46.05410526
45.78675598	37.05648298	58.2827156	54.04590147	44.93695454
48.81566026	44.28675882	49.047388	60.45390524	61.11938949
40.53789776	60.62875042	56.28938643	62.1533439	61.76770693
51.48532466	67.58310946	61.32562353	62.64173655	66.11893653
62.32141118	63.33040059	67.5100942	76.88171982	70.84572134
68.48750716	79.83034661	67.32404466	68.93852863	76.16173337
76.48988536	82.8672848	77.91749532	80.45729634	71.02831119
82.99051374	85.51164549	89.89223927	83.19374159	73.46159411
89.37990293	76.8168497	91.13527585	91.98504286	94.51285716
76.8489968	81.06639423	85.75055827	82.75699504	92.51526601
95.51839141	89.29033727	95.65619384	102.39389886	89.53180617]

The theoretical optimal solution  $x^*$

```
[ 1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18
 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36
 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54
 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72
 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90
 91 92 93 94 95 96 97 98 99 100]
```

Convergence rate of  
gradient descent algorithm  
large scale problem - dimension: 100



The huge scale problem is chosen. The matrix A and vector b dimension is 1000  
The number of maximum iterations is 60000. The allowed tolerance for gradient norm is 1.0

The gradient descent algorithm runs in 34.8260064125061 seconds

Reason of stopping

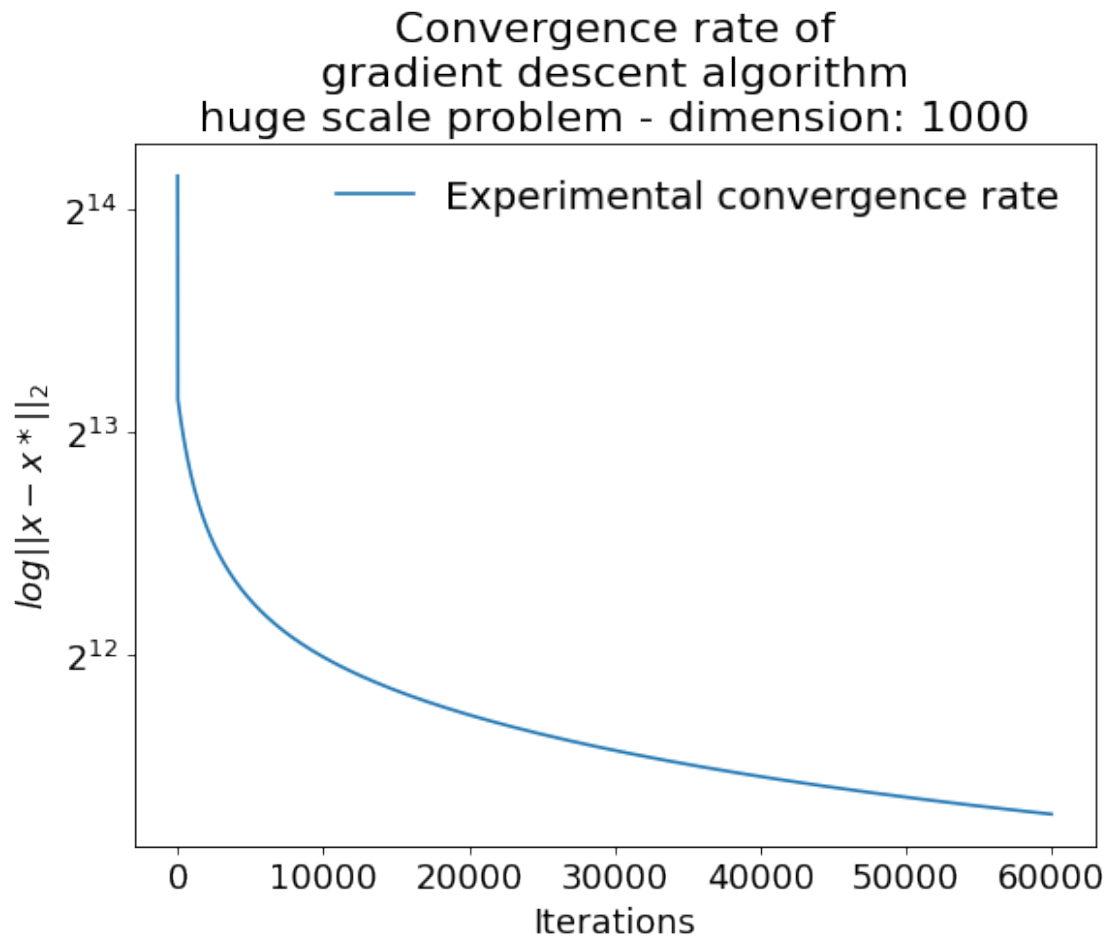
Max iterations (60000) exceeded

First 100 values in the optimal solution x found by gradient descent algorithm

```
[ -10.63174488  63.37143999 157.67587662 -28.98040714  18.44819265
   8.89392715  91.04566023  42.1484773  106.46289367  56.4020265
  70.9809846  67.83963515  82.22141957  70.79467462 163.73643644
  17.01598209 -52.40830661  49.13319062  45.59715111 149.41525211
  43.00752814  44.7106021  -21.74792399 127.86263676  74.14244684
  79.51530711  22.52373491 -24.35930537 -18.73706392 -23.63218186
  -3.24740986 115.55521407 139.73619666 133.63803682  46.38450882
 -60.23245944 192.40687214  57.71963766  62.39052301 -38.6454732
  41.17525845  31.5090981  71.09865821  45.91672493 103.90332947
 153.50563552 -13.61070701 114.34910011 240.60198667 165.59520295
  97.3492586  147.7839864  108.61395491  83.05104859 142.45615019
  92.53214458  68.65573783  80.94962884 154.63271263 121.71339939
 167.58529837 104.6968628  11.07101534  97.61481586  89.13633389
 128.60756142 188.79503521  98.14144784  87.77590542 101.83752068
  -1.78828288 108.89490175 160.52246442 191.82613482 141.4979839
 181.23537129 190.12705397  10.9273383  224.73054108  55.03595881
  88.36713869  79.19878383 175.2532021  108.02181236 205.83864386
 159.01782838  46.4736954  134.76585137  15.2735539  149.76101792
 127.14084231  50.26364769 208.6255157  220.10366406  59.03353063
 126.83309631  40.4653204  152.97359151 266.45642179 136.7430323 ]
```

First 100 values in the theoretical optimal solution x\*

```
[ 1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18
 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36
 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54
 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72
 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90
 91 92 93 94 95 96 97 98 99 100]
```



From the three graphs, it appears that the implemented gradient descent has a sublinear to nearly linear convergence rate.

I use base 2 logarithm because the convergence rate of gradient descent only decreases linearly, as base 10 is big and the graph will not show any information on the y-axis

For the small scale problem, gradient descent returns optimal solution close to  $x^*$

For the large scale problem, gradient descent returns suboptimal solution that has the same pattern as  $x^*$

For the huge scale problem, gradient descent returns suboptimal solution that is still far from  $x^*$  because the dimension is too large (1000)

## 6 Task 2: Conjugate Gradient Algorithm

### 6.1 Conjugate gradient algorithm implementation

```
[58]: def conjugateGradient(A, b, maxIters = 5000, epsilon = 10e-5, period = 100):
    # Dimension of A and b
    dim = b.size
    # initial random vector x filled with the mean of matrix A, with length
    ↪ equal to the dimension
    x = np.repeat(np.mean(A), dim)
    # The Lipschitz constant
    L = np.max(np.linalg.eigvals(A))
    # The step size is alpha = 1/L
    alpha = 1/L
    # currentIteration
    iter = 1
    # Saving the results
    x_iterations = [x]

    gradients = []
    gradientNorms = []

    rhos = []

    while (gradientNorm(A, b, x) > epsilon and iter <= maxIters):
        if (iter % period == 1):
            gradientNorms.append(gradientNorm(A,b,x))
            gradientVector = gradient(A, b, x)
            gradients.append(gradientVector)
            rho = - gradientVector
            rhos.append(rho)
            x = x + alpha * rho
            x_iterations.append(x)
        else:
            gradientNorms.append(gradientNorm(A,b,x))
            gradients.append(gradient(A, b, x))
            delta = (gradientNorms[iter - 1] ** 2)/(gradientNorms[iter - 2] **
    ↪ 2)

            rho = - gradients[iter - 1] + delta * rhos[iter - 2]
            rhos.append(rho)
            x = x + alpha * rho
            x_iterations.append(x)
            iter += 1

    if iter > maxIters:
        stoppingReason = f"Max iterations ({maxIters}) exceeded"
    else:
```

```

        stoppingReason = f"Gradient norm smaller than {epsilon}\nCompleted_
↪iteration: {iter}"
    return (x, x_iterations, stoppingReason)

```

## 6.2 Conjugate gradient algorithm convergence rate analysis

```

[59]: # There are three different scales: small, large and huge
scales = ["small", "large", "huge"]
# Number of maximum iterations of the three scales small, large and huge
maxIters = [20000, 40000, 60000]
# Tolerance of the gradient norm of the three scales small, large and huge
tolerances = [10e-5, 10e-3, 10e-1]
# The algorithm
algorithmName = "conjugate gradient"
algorithm = conjugateGradient
# The logarithm base
logBase = 2
# Running optimization for the three scales small, large and huge
for i in range(0,3):
    plotDifferenceNorms(scales[i], maxIters[i], tolerances[i], algorithmName,
↪algorithm, logBase)

```

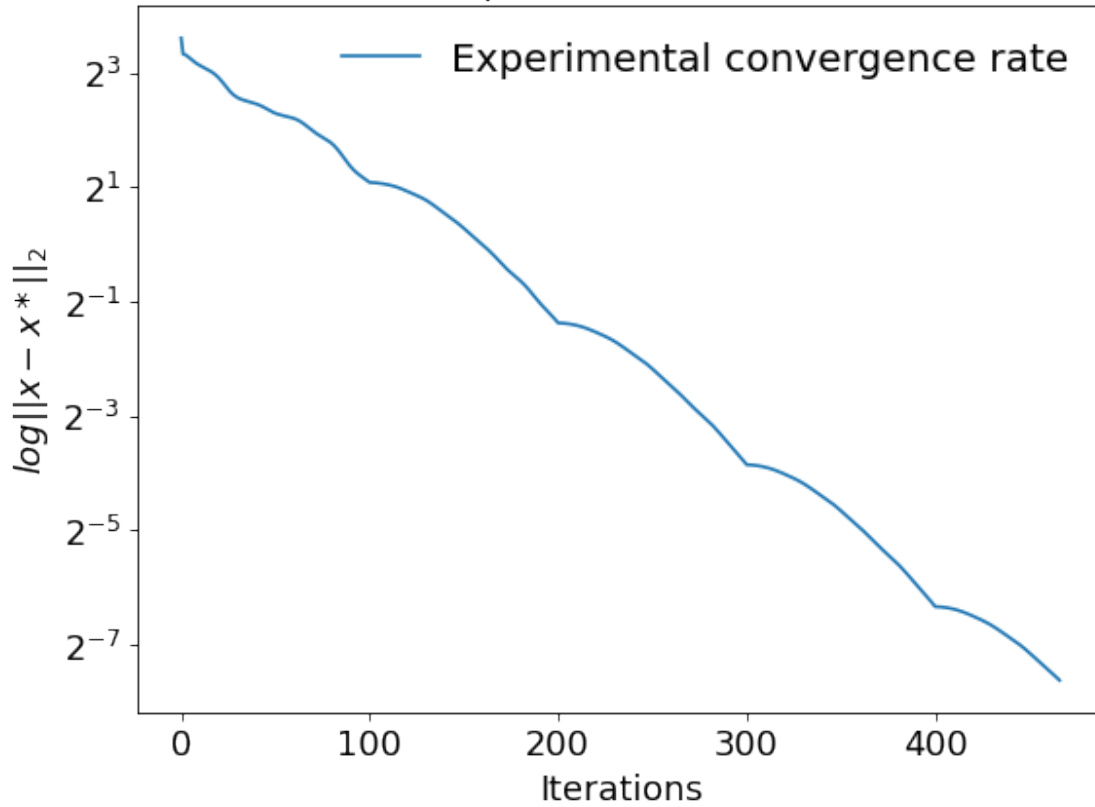
The small scale problem is chosen. The matrix A and vector b dimension is 10  
The number of maximum iterations is 20000. The allowed tolerance for gradient norm is 0.0001

The conjugate gradient algorithm runs in 0.009025812149047852 seconds  
Reason of stopping  
Gradient norm smaller than 0.0001  
Completed iteration: 467

The optimal solution x found by conjugate gradient algorithm  
[1.00113074 2.00040352 3.0014233 4.00135845 5.00073744 5.99799636  
7.00095232 7.99627049 8.99914574 9.99935844]

The theoretical optimal solution x\*  
[ 1 2 3 4 5 6 7 8 9 10]

### Convergence rate of conjugate gradient algorithm small scale problem - dimension: 10



The large scale problem is chosen. The matrix A and vector b dimension is 100  
The number of maximum iterations is 40000. The allowed tolerance for gradient norm is 0.01

The conjugate gradient algorithm runs in 3.840991973876953 seconds  
Reason of stopping  
Gradient norm smaller than 0.01  
Completed iteration: 9823

The optimal solution x found by conjugate gradient algorithm

```
[ 1.11845736  3.82449596  5.59523166  5.98512795  8.37384602  5.92841262
  8.13534334  8.18346144  9.21724186 11.52938575  7.16282571 12.55292507
  9.99412557 13.55998946 15.8113756  14.63876449 18.55548775 21.29674155
 18.01564307 20.73430025 19.27528084 20.65206849 21.40484399 23.25767325
 24.43084158 25.58133179 28.70759418 29.75083553 27.78756451 31.44894691
 32.68742246 31.56959598 34.3672904  34.67444323 32.67273952 35.97099353
 37.24588129 39.96257566 38.08324313 40.15028579 43.89269533 39.40561199]
```

```

45.94381356 43.37593535 44.08142957 46.09508346 45.84955739 49.46344864
51.58919538 48.07725847 50.62617696 49.38565347 52.32973617 55.90379266
55.66569541 53.74651079 55.55973899 57.54897615 60.39739274 61.23529851
57.30760211 60.38931943 62.88273663 64.75815727 67.12739098 65.71134963
67.20393953 68.95358952 69.47680588 72.13896062 71.84075926 72.55925089
72.22615688 72.27349425 76.78555679 75.72601205 78.06025144 79.28783052
79.23312947 78.84645431 79.93175      83.09755769 83.87617387 81.99399407
82.79711543 85.13748553 85.7513178  88.2150248  91.28535614 92.50966152
87.40856966 89.53616214 89.92482951 92.74008419 94.46197493 96.9215144
95.15131279 97.24234619 99.95992433 99.46343368]

```

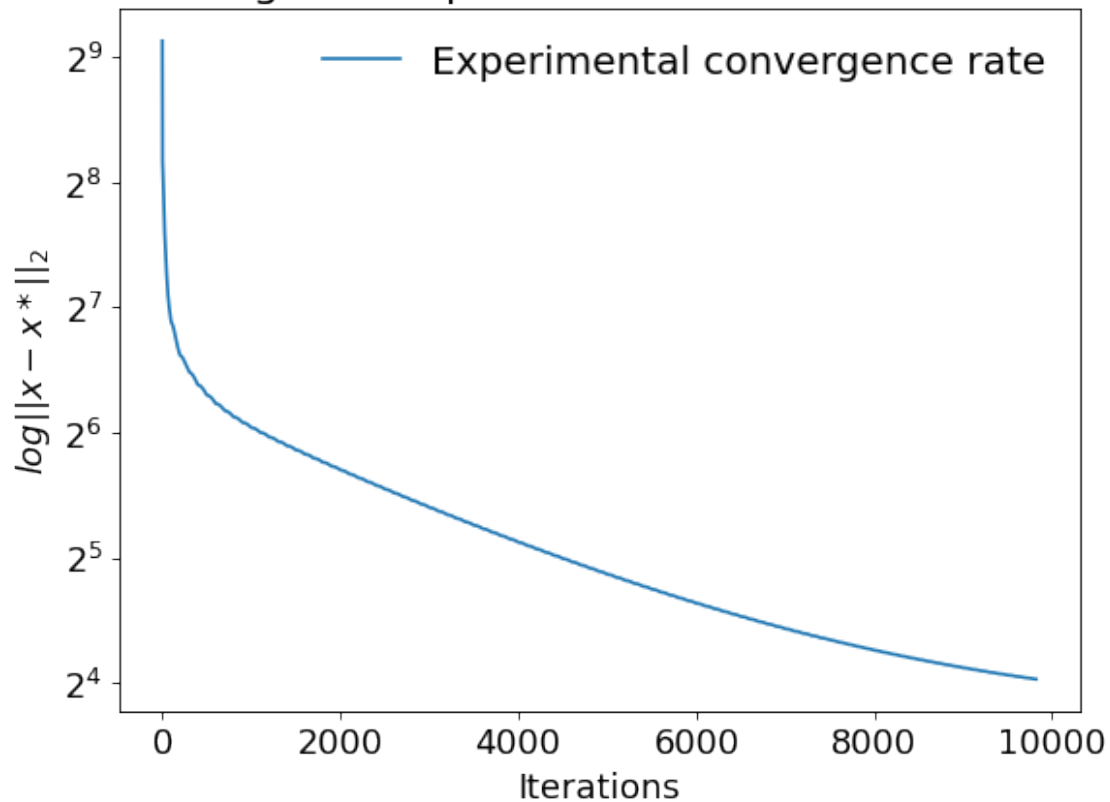
The theoretical optimal solution  $x^*$

```

[ 1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18
 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36
 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54
 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72
 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90
 91 92 93 94 95 96 97 98 99 100]

```

Convergence rate of  
conjugate gradient algorithm  
large scale problem - dimension: 100





The huge scale problem is chosen. The matrix A and vector b dimension is 1000  
The number of maximum iterations is 60000. The allowed tolerance for gradient norm is 1.0

The conjugate gradient algorithm runs in 22.06999945640564 seconds

Reason of stopping

Gradient norm smaller than 1.0

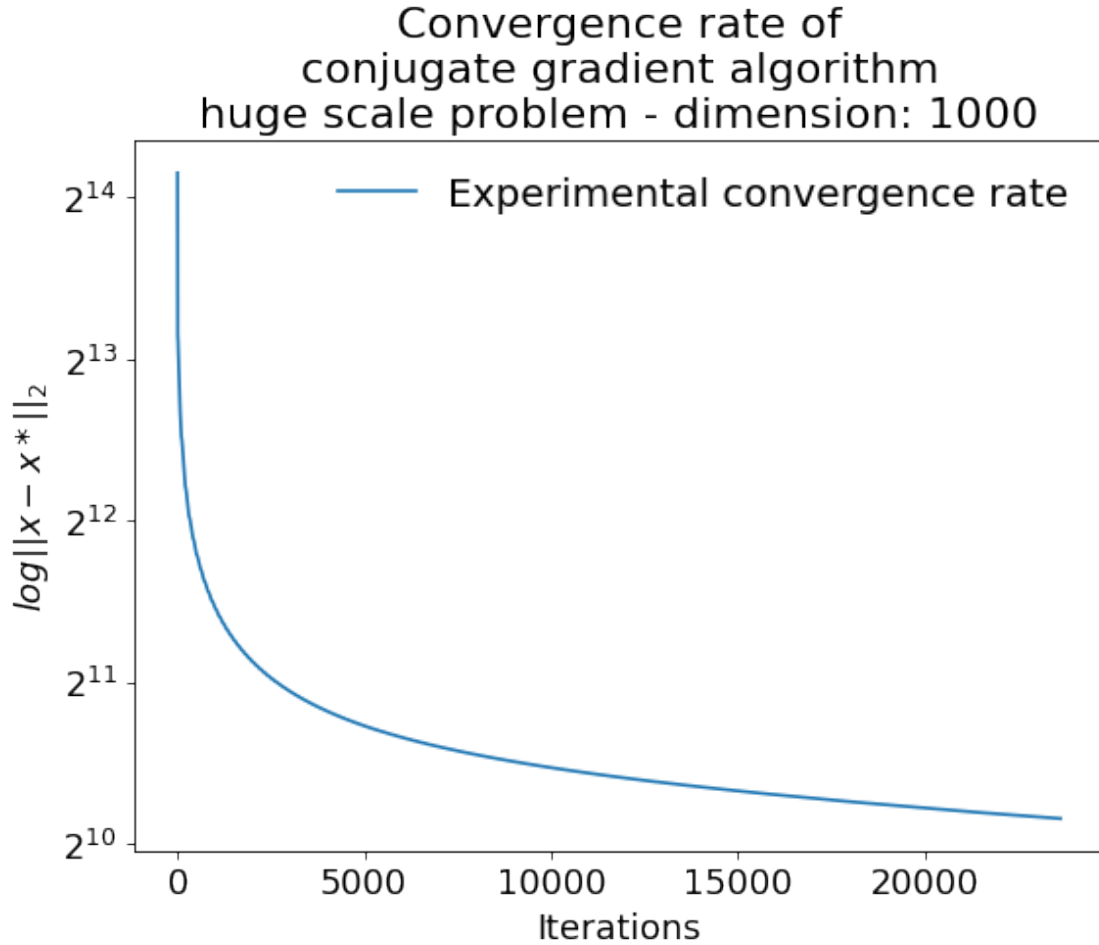
Completed iteration: 23630

First 100 values in the optimal solution x found by conjugate gradient algorithm

```
[-3.32597261e+01  2.67394699e+01  3.65614952e+01 -1.66945474e+01
 5.76708378e+00  3.58056494e+01 -1.59294870e+01  2.36735775e+01
 1.97478852e+01  5.82344457e+01  2.99316448e+01 -1.91287839e+01
 2.75501216e+01  1.61975741e+01  5.30509243e+01  7.93120223e-02
 1.73846033e+01  5.95173660e+00  5.26046964e+01  8.68809282e+01
 2.31772120e+01 -1.91380419e+01  1.73136072e+01  3.51953234e+01
 1.25673999e+02  3.15630144e+01 -7.46888261e+00  1.66473683e+01
 4.20094226e+01  3.72441865e+00 -1.91348219e+01  5.74741411e+01
 9.67722766e+01  6.69688270e+01  6.33407970e+01  2.21848958e+01
 9.39510118e+01  1.86679066e+01  5.59229062e+01  4.88220892e+01
 7.29302020e+01  2.64264203e+01  1.39717552e+01  4.15625854e+01
-2.12819961e+00  6.33192822e+01 -3.43799578e+01  6.68174526e+01
 9.18657530e+01  8.82029264e+01  5.46540702e+01  9.95163880e+01
 4.46939205e+01  2.14902166e+01  7.38928585e+01  2.17458532e+01
 7.12783135e+01  5.27827875e+01  3.99669150e+01  6.01247285e+01
 9.34323829e+01  1.04346879e+02  2.34800243e+01  2.48187161e+01
 4.22327986e+01  1.04362547e+02  7.90315636e+01  4.89505753e+01
 6.47104836e+01  9.93002052e+01  2.72200086e+01  1.95567312e+01
 7.93460548e+01  8.49954654e+01  6.51297542e+01  1.19231358e+02
 4.49252558e+01  9.74449310e+01  1.26617145e+02  8.64696118e+01
 7.20525822e+01  7.69059731e+01  1.12038938e+02  8.23099986e+01
 1.47370210e+02  1.33486088e+02  8.88785454e+01  1.19244692e+02
 5.78003204e+01  1.41824208e+02  9.58324929e+01  5.55414266e+01
 9.22953647e+01  1.12927704e+02  9.83759588e+01  7.84487926e+01
 7.10830502e+01  1.74305455e+02  1.16923181e+02  9.17655116e+01]
```

First 100 values in the theoretical optimal solution x\*

```
[ 1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18
 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36
 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54
 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72
 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90
 91 92 93 94 95 96 97 98 99 100]
```



From the three graphs, it appears that the implemented conjugate gradient has a superlinear convergence rate.

For the small scale problem, conjugate gradient returns optimal solution close to  $x^*$

For the large scale problem, conjugate gradient returns a solution whose value has an increasing trend like  $x^*$

For the huge scale problem, conjugate gradient returns suboptimal solution, although it still has an increasing trend like  $x^*$

By increasing trend, I mean I have deliberately chosen  $x^*$  to be increasing natural numbers to easily test the result returned by the algorithms

The conjugate gradient also shows noticeable turbulence during the early iterations where the errors start to reduce significantly

## 7 Task 3: FISTA algorithm

FISTA is acronym of “Fast Iterative Shrinkage-Thresholding Algorithm”

## 7.1 FISTA algorithm implementation

```
[44]: def FISTA(A, b, maxIters = 5000, epsilon = 10e-5):
    # Dimension of A and b
    dim = b.size
    # initial random vector x0 filled with the mean of matrix A, with length
    ↪ equal to the dimension
    x = np.repeat(np.mean(A), dim)
    # assign y1 equals to x0
    y = x
    # assign t equals to 1
    t = 1
    # currentIteration
    iter = 1
    # Saving the results
    x_iterations = [x]
    y_iterations = []
    t_iterations = []

    gradients = []

    # The Lipschitz constant
    L = np.max(np.linalg.eigvals(A))
    # The step size is alpha = 1/L
    alpha = 1/L

    while (gradientNorm(A, b, x) > epsilon and iter <= maxIters):
        # For example, this is the first iteration, where k = 1 (iter = 1)
        # Saving the previous x, which is x0
        previous_x = x
        # x is now x1, y is now y1
        x = y - alpha * gradient(A, b, y)
        x_iterations.append(x)
        # Saving the previous t, which is t1
        previous_t = t
        # t is now t2 and the latter t is still t1
        t = 1/2 * (1 + math.sqrt(1 + 4 * (previous_t ** 2)))
        # y is now y2, x is x1 and x_previous is x0
        y = x + (previous_t - 1)/t * (x - previous_x)
        iter += 1

    if iter > maxIters:
        stoppingReason = f"Max iterations ({maxIters}) exceeded"
    else:
        stoppingReason = f"Gradient norm smaller than {epsilon}\nCompleted,
    ↪ iteration: {iter - 1}"
```

```
return (x, x_iterations, stoppingReason)
```

## 7.2 FISTA algorithm convergence rate analysis

```
[60]: # There are three different scales: small, large and huge
scales = ["small", "large", "huge"]
# Number of maximum iterations of the three scales small, large and huge
maxIters = [20000, 40000, 60000]
# Tolerance of the gradient norm of the three scales small, large and huge
tolerances = [10e-5, 10e-3, 10e-1]
# The algorithm
algorithmName = "FISTA"
algorithm = FISTA
# The logarithm base
logBase = 2
# Running optimization for the three scales small, large and huge
for i in range(0,3):
    plotDifferenceNorms(scales[i], maxIters[i], tolerances[i], algorithmName,
    ↪algorithm, logBase)
```

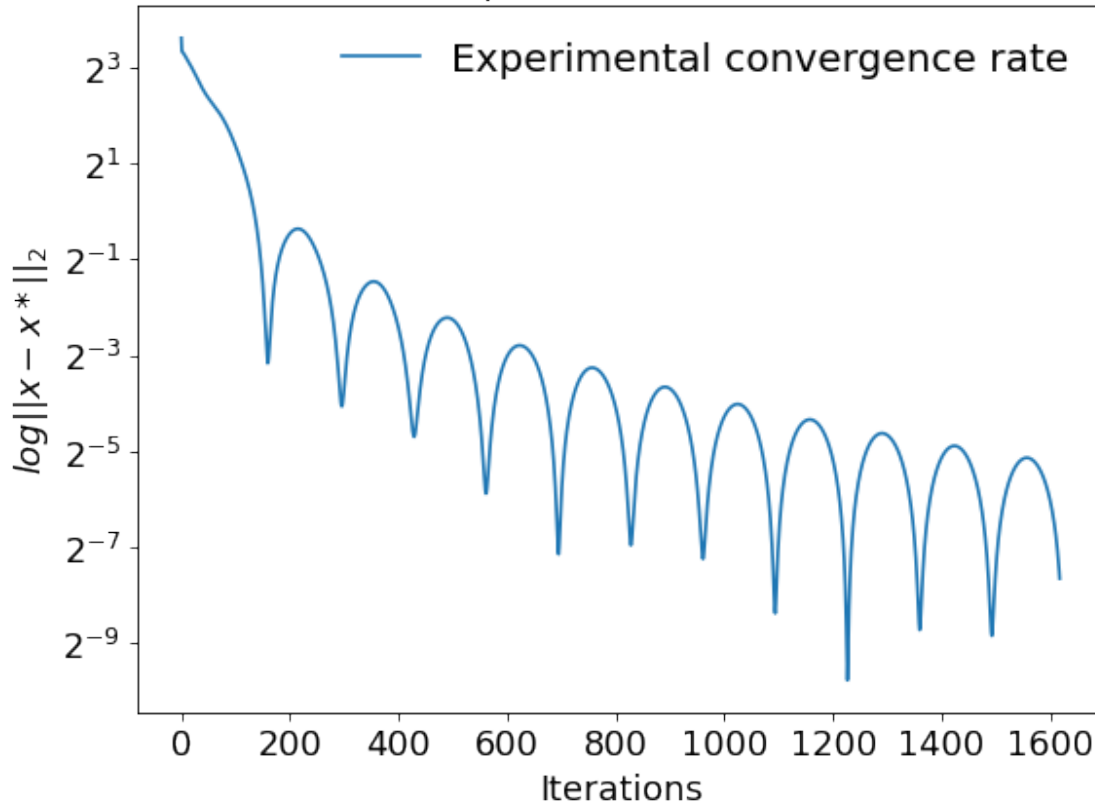
The small scale problem is chosen. The matrix A and vector b dimension is 10  
The number of maximum iterations is 20000. The allowed tolerance for gradient norm is 0.0001

The FISTA algorithm runs in 0.021035194396972656 seconds  
Reason of stopping  
Gradient norm smaller than 0.0001  
Completed iteration: 1618

The optimal solution x found by FISTA algorithm  
[ 0.99882646 1.99899609 2.99909342 3.99902631 4.99919153 6.00173722  
 6.99881059 8.00366691 9.00111356 10.00085021]

The theoretical optimal solution x\*  
[ 1 2 3 4 5 6 7 8 9 10]

# Convergence rate of FISTA algorithm small scale problem - dimension: 10



The large scale problem is chosen. The matrix A and vector b dimension is 100  
The number of maximum iterations is 40000. The allowed tolerance for gradient norm is 0.01

The FISTA algorithm runs in 1.0489630699157715 seconds

Reason of stopping

Gradient norm smaller than 0.01

Completed iteration: 4038

The optimal solution x found by FISTA algorithm

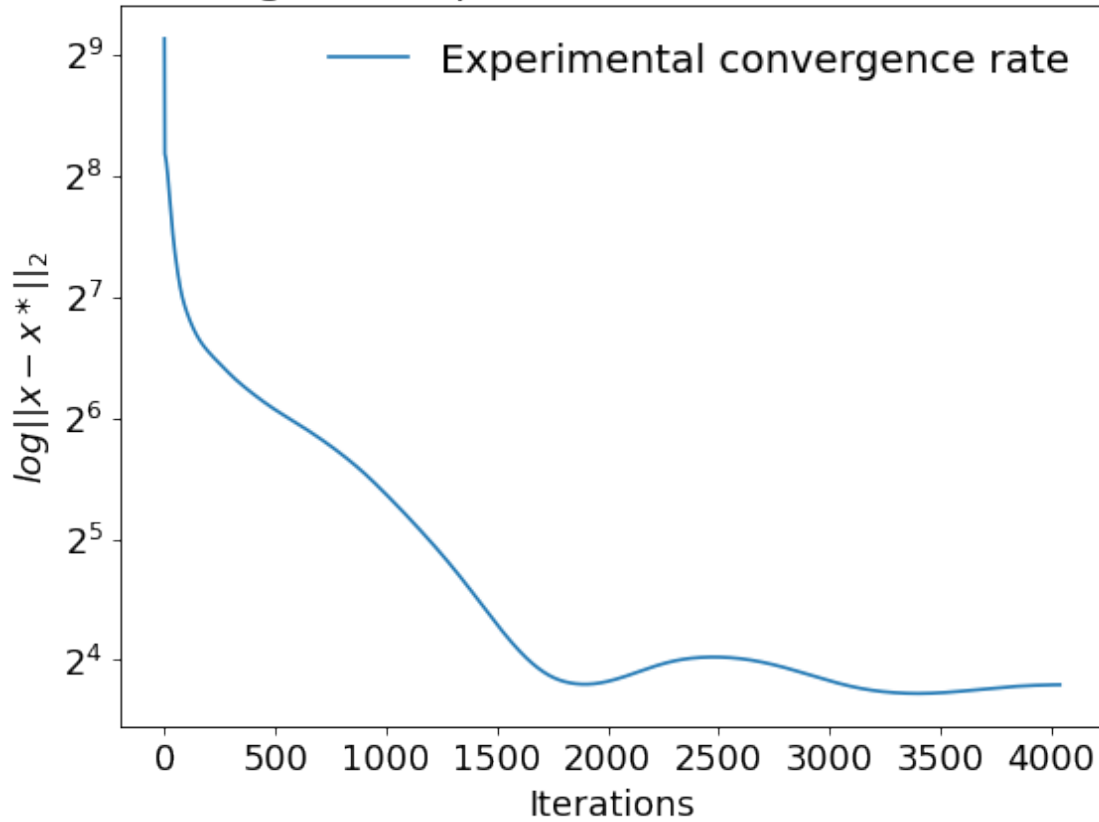
0.5804197	3.49937851	4.89739875	5.43814026	8.2733384
6.06932763	8.32561155	8.58557714	9.73348927	10.82446087
7.71904481	11.20116919	9.67236023	13.36241332	15.35118423
14.87964507	18.04439859	20.17234125	18.63427131	21.34265901
20.38616586	21.05935953	21.35136413	22.7698276	24.30630488
25.21044793	29.10143877	29.66149567	27.76431124	31.91013713
32.62781383	31.5158694	34.72057296	33.51675469	33.61008684

35.43708197	36.33179422	39.16351096	37.97249797	40.45407114
43.7780257	40.50797794	44.62097971	43.4433036	43.96178159
46.02935254	46.97377021	48.10345962	50.94346539	48.11515001
50.84876113	49.6817762	53.09498024	55.95326384	54.91944472
54.82495121	55.21809438	58.09844605	60.46227026	60.56368964
58.39584443	59.86170489	63.25100034	64.80830732	66.97287797
65.72819593	67.79311612	68.50607027	69.32864347	72.1111604
72.09514606	71.7748174	72.73075049	72.75332264	76.26977995
75.97086927	77.44400962	79.29045818	79.3294741	79.52988738
79.95864334	82.40012622	83.03507198	81.83079594	82.85343074
84.56631005	85.96829517	88.76676183	91.23441379	91.96725468
88.30332248	90.51121035	90.4400783	93.8698661	94.96714562
96.84842306	95.92304513	97.5761641	100.18765082	99.86925561]

The theoretical optimal solution x\*

[	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	
37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	
55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	
73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	
91	92	93	94	95	96	97	98	99	100]									

# Convergence rate of FISTA algorithm large scale problem - dimension: 100



The huge scale problem is chosen. The matrix A and vector b dimension is 1000  
The number of maximum iterations is 60000. The allowed tolerance for gradient norm is 1.0

The FISTA algorithm runs in 8.035976886749268 seconds

Reason of stopping

Gradient norm smaller than 1.0

Completed iteration: 10184

First 100 values in the optimal solution x found by FISTA algorithm

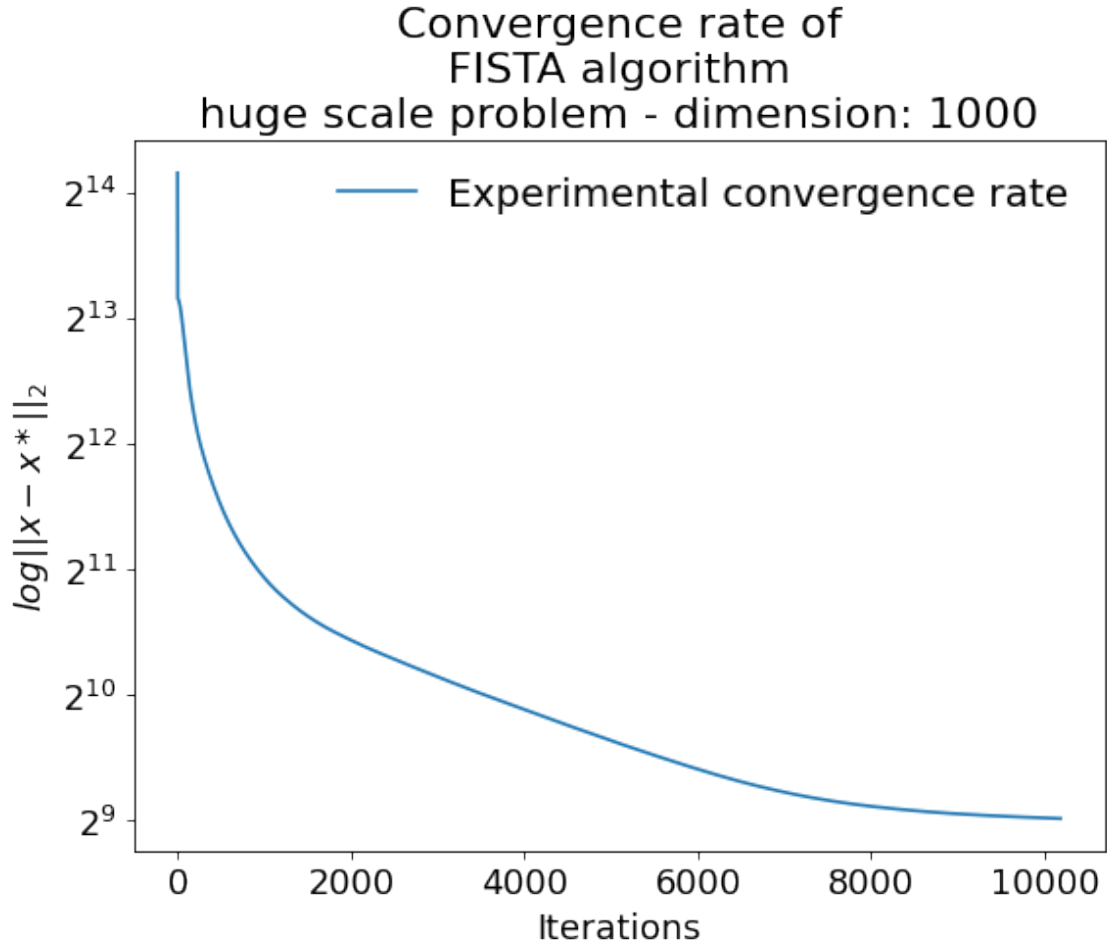
```
[ 3.17758067 13.90709143  6.2257751  12.74674775 -9.2641401
 25.04514651  1.18279701 11.11102729 -1.00895173 50.32358794
 32.52995236  7.83888725 15.39044038  9.99965363 15.17581813
 -6.08194271 39.45570317 22.68464105 27.96035342 33.06690366
 20.47455608  9.90012816 18.38430427 39.78134304 28.19009002
 50.51040477 26.38980085 54.63117852 45.14140792 31.76175477
```

20.50274681	23.66084756	67.93184858	43.61313347	37.79765888
-4.35895841	51.81728362	41.89768425	35.45183519	46.97417003
29.65361469	36.69807236	68.0127728	36.24465202	23.11418608
33.22279616	40.23792797	43.87409123	58.45726856	68.22955575
58.75432045	87.97692044	42.83803203	75.64180086	61.89088511
69.27824164	36.82995876	45.11893975	38.04609984	39.35089089
58.3395861	57.85263588	56.70727358	37.46462927	75.12688082
70.72628701	63.43135767	78.83189416	69.61501011	88.20094158
44.76383585	65.59348403	41.42195482	67.78668395	54.35791682
82.6495478	58.77283997	95.346273	100.99343729	77.5348936
80.272615	114.00730891	70.52956246	87.26150746	92.56950051
104.87676885	99.67191676	77.24307112	82.37433418	99.40422848
85.03055606	87.16685688	114.83056539	81.10002986	105.36691563
103.84455875	94.18861197	88.77848881	107.25604763	97.46812361]

First 100 values in the theoretical optimal solution x\*

[	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	
37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	
55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	
73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	
91	92	93	94	95	96	97	98	99	100]									





From the three graphs, it appears that the implemented FISTA has a superlinear convergence rate.

For the small scale problem, FISTA returns optimal solution close to  $x^*$

For the large scale problem, FISTA returns optimal solution that is also close to  $x^*$

For the huge scale problem, FISTA returns suboptimal solution again, because the dimension is too large (1000)

FISTA has unique pattern, whose errors rise and fall in a periodic manner. However, this behavior is only observed in small dimension problems (10). For large dimensions like 100 and 1000, FISTA doesn't seem to show this periodic error behavior

## 8 Task 4: Coordinate Descent Algorithm

### 8.1 Deterministic (cyclic) coordinate descent algorithm implementation

```
[46]: def coordinateDescent(A, b, maxIters = 5000, epsilon = 10e-5, period = 100):  
    # Dimension of A and b  
    dim = b.size  
    # initial random vector x filled with the mean of matrix A, with length  
    # equal to the dimension  
    x = np.repeat(np.mean(A), dim)  
    # The Lipschitz constant  
    L = np.max(np.linalg.eigvals(A))  
    # The step size is alpha = 1/L  
    alpha = 1/L  
    # currentIteration  
    iter = 1  
    # Saving the results  
    x_iterations = []  
  
    while (gradientNorm(A, b, x) > epsilon and iter <= maxIters):  
        for index in range(0, dim):  
            x_totalGradient = x - alpha * gradient(A, b, x)  
            x_partialGradient = copy.deepcopy(x)  
            x_partialGradient[index] = x_totalGradient[index]  
            x = copy.deepcopy(x_partialGradient)  
            x_iterations.append(x)  
            iter += 1  
  
    if iter > maxIters:  
        stoppingReason = f"Max iterations ({maxIters}) exceeded"  
    else:  
        stoppingReason = f"Gradient norm smaller than {epsilon}"  
    return (x, x_iterations, stoppingReason)
```

### 8.2 Coordinate descent algorithm convergence rate analysis

```
[61]: # There are three different scales: small, large and huge  
scales = ["small", "large", "huge"]  
# Number of maximum iterations of the three scales small, large and huge  
maxIters = [20000, 40000, 60000]  
# Tolerance of the gradient norm of the three scales small, large and huge  
tolerances = [10e-5, 10e-3, 10e-1]  
# The algorithm  
algorithmName = "coordinate descent"  
algorithm = coordinateDescent  
# The logarithm base  
logBase = 2
```

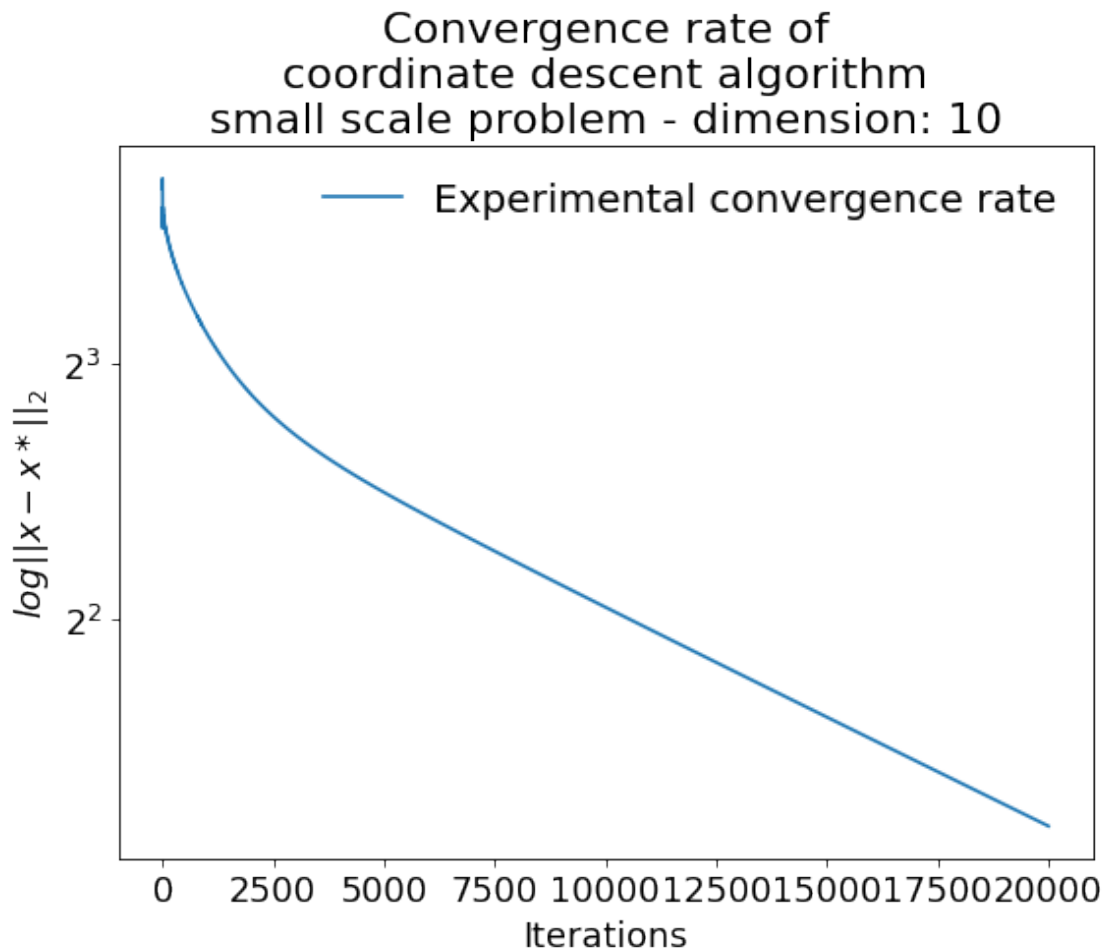
```
# Running optimization for the three scales small, large and huge
for i in range(0,3):
    plotDifferenceNorms(scales[i], maxIters[i], tolerances[i], algorithmName,
    ↪algorithm, logBase)
```

The small scale problem is chosen. The matrix A and vector b dimension is 10  
 The number of maximum iterations is 20000. The allowed tolerance for gradient norm is 0.0001

The coordinate descent algorithm runs in 0.1549978256225586 seconds  
 Reason of stopping  
 Max iterations (20000) exceeded

The optimal solution x found by coordinate descent algorithm  
 [1.47699082 2.35781817 3.46787914 4.50929597 5.40624933 5.13076524  
 7.5507039 6.31390472 8.56043183 9.65492076]

The theoretical optimal solution x\*  
 [ 1 2 3 4 5 6 7 8 9 10]



The large scale problem is chosen. The matrix A and vector b dimension is 100  
The number of maximum iterations is 40000. The allowed tolerance for gradient norm is 0.01

The coordinate descent algorithm runs in 5.4509971141815186 seconds  
Reason of stopping  
Max iterations (40000) exceeded

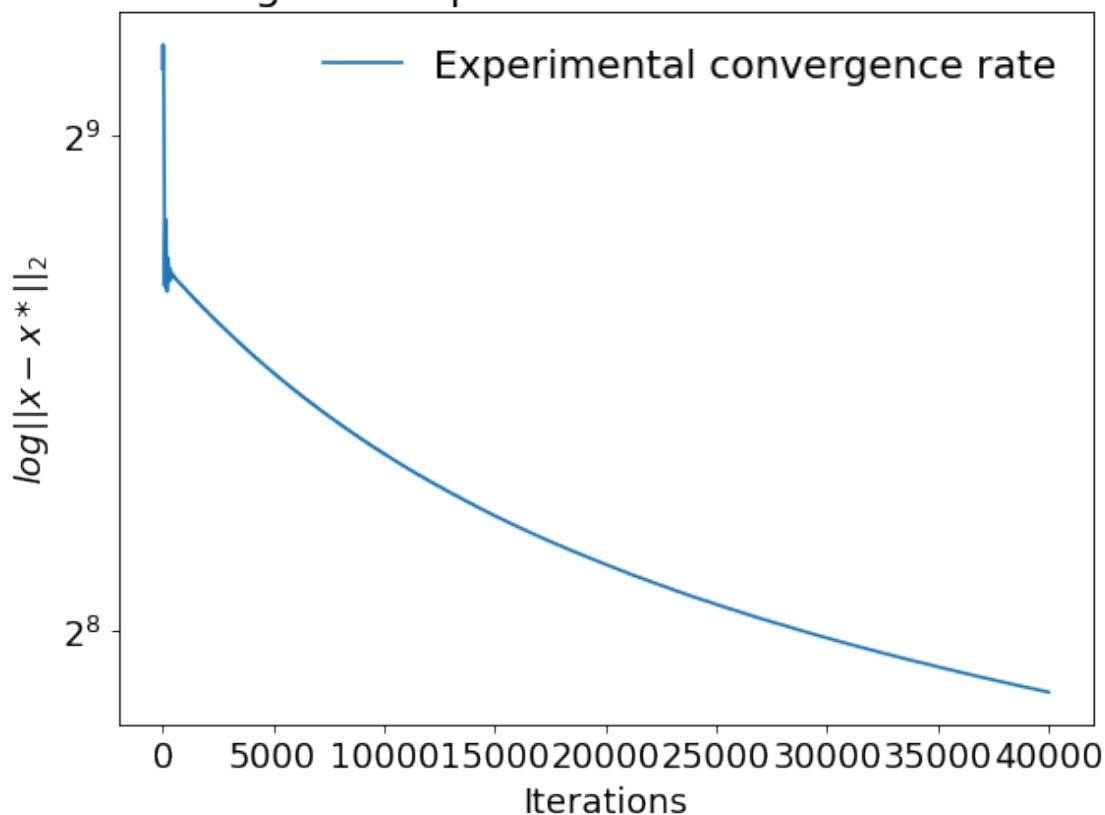
The optimal solution x found by coordinate descent algorithm

```
[18.4854829 44.25797993 33.84664296 39.61234692 43.26652519 -0.82146242
15.00451965 52.11998119 10.76369715 25.1453549 28.87342704 37.68531833
44.19373803 45.315784 24.7130248 39.4887291 39.58141494 57.33326627
26.06927215 39.95258182 15.22478807 4.79245542 66.18112869 48.69361618
18.27821879 50.73646993 47.50715397 32.42348842 34.52312645 50.71650713
60.63219618 42.21262062 41.34583668 45.13040413 25.36149809 65.30948681
75.59908436 82.54055454 85.61344422 67.17576163 39.51059327 36.51296661
45.63694754 41.39579049 54.17594465 39.74886786 47.9720558 71.45948733
62.09519968 44.51460762 39.8757539 51.34473849 57.75974209 53.2257247
49.44691577 20.17204556 44.4482648 50.61497979 68.89810189 45.26713231
39.99975888 28.42594745 60.15426635 45.38487869 41.32665787 55.27151341
62.98501958 53.96928178 75.53946547 65.40910437 73.01644848 68.73756688
45.00888811 61.54728233 95.44033156 57.37198932 57.90282469 44.17543557
52.0856743 58.60408173 66.043381 69.46903856 81.10761356 82.18584216
41.08230713 65.37505301 53.73202672 66.5458011 41.94942278 77.84410287
66.05262032 41.59800608 54.61866748 58.0782905 76.20893878 48.2623992
75.94760826 65.24842958 85.82354167 67.84013826]
```

The theoretical optimal solution x\*

```
[ 1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18
19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36
37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54
55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72
73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90
91 92 93 94 95 96 97 98 99 100]
```

# Convergence rate of coordinate descent algorithm large scale problem - dimension: 100



The huge scale problem is chosen. The matrix A and vector b dimension is 1000  
The number of maximum iterations is 60000. The allowed tolerance for gradient norm is 1.0

The coordinate descent algorithm runs in 20.09000062942505 seconds

Reason of stopping

Max iterations (60000) exceeded

First 100 values in the optimal solution x found by coordinate descent algorithm

```
[786.36500442 759.42701392 783.74774341 744.80662387 753.08825931
742.78808719 760.18211188 777.27424516 757.91373402 758.12745557
766.10996105 750.37865797 766.10471747 764.91040497 760.78579612
758.11719986 747.67318075 749.62924301 746.1362748 750.95497023
752.17386221 778.96218249 747.4290864 714.79019894 758.97535057
747.10343406 727.02081922 750.73859498 750.97628318 738.76970634
736.07352565 764.37012303 739.62066681 739.31938756 739.60957661
```

```

749.5229503 739.1625242 737.6346086 715.51621968 753.15318817
747.93541246 744.19403141 727.84812933 723.57128145 747.46835544
765.6173821 741.40360752 762.00495038 756.29983561 719.92925462
755.82107351 713.52233949 734.80493195 724.90254274 739.17386901
716.23402562 728.28426787 744.69790816 741.56694161 730.99598664
730.83657932 718.91793017 718.70380229 708.53268377 723.53976133
723.48207872 707.61470858 717.52670474 721.07172183 722.39170492
711.9685472 723.31235392 727.61018016 726.06507322 708.27536654
734.64569696 699.04454468 708.1519943 720.78399487 691.34651768
720.93266727 709.69029447 687.15949397 738.30665392 712.75056441
710.18703696 705.38177504 705.07495572 723.14854648 714.25198636
727.63531076 702.062696 730.23588956 674.97107686 679.81614971
707.92066626 703.61102605 687.32613191 710.72097096 705.03773584]

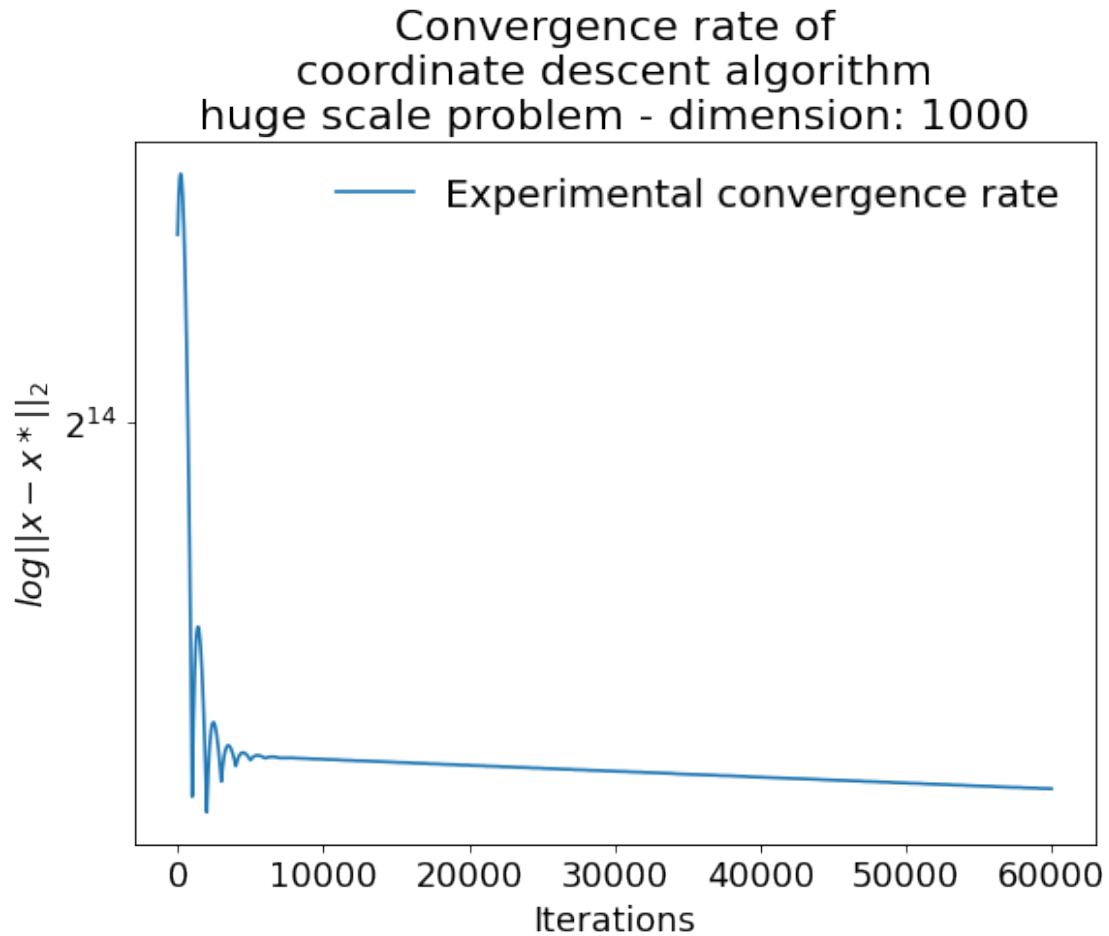
```

First 100 values in the theoretical optimal solution  $x^*$

```

[ 1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18
 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36
 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54
 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72
 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90
 91 92 93 94 95 96 97 98 99 100]

```



From the three graphs, it appears that the implemented coordinate descent has a sublinear convergence rate.

For the small scale problem, coordinate descent returns optimal solution close to  $x^*$

For the large scale problem, coordinate descent returns suboptimal solution that has the same pattern as  $x^*$

For the huge scale problem, coordinate descent returns suboptimal solution that is still far from  $x^*$ , again because the dimension is too large (1000)

Coordinate descent error graphs seem to resemble a wavelength gradually flattening out from the earliest iterations until later iterations

## 9 Task 5: Comparison between the algorithms

```
[64]: # Plotting the difference norms  $\log \|x - x^*\|_2$ 
def plotDifferenceNormsMultipleAlgorithms(scale, maxIter, tolerance,
    algorithmNames, algorithms, logBase):

    A = np.load(f"data/{scale}Matrix.npy", allow_pickle=True)
    # print("The matrix A")
    # print(A)

    b = np.load(f"data/{scale}Vector.npy", allow_pickle=True)
    # print("\nThe vector b")
    # print(b)
    x_opt = np.load(f"data/{scale}Solution.npy", allow_pickle=True)

    print(f"\nThe {scale} scale problem is chosen. The matrix A and vector b
    dimension is {b.size}")
    print(f"The number of maximum iterations is {maxIter}. The allowed
    tolerance for gradient norm is {tolerance}")

    if scale == "huge":
        print("\nFirst 100 values in the theoretical optimal solution x*")
        print(x_opt[0:100])
    else:
        print("\nThe theoretical optimal solution x*")
        print(x_opt)

    figure(figsize=(8, 6), dpi=80)

    for i in range(0, len(algorithms)):
        start = time.time()
        x_opt_algo, x_iterations_algo, stoppingReason = algorithms[i](A, b,
        maxIter, tolerance)
        end = time.time()

        differenceNorms = []
        for x_sol in x_iterations_algo:
            differenceNorms.append(differenceNorm(x_sol, x_opt))
        differenceNorms = np.array(differenceNorms)

        iterations = np.arange(0, differenceNorms.size, 1)
        plt.plot(iterations, differenceNorms, label = algorithmNames[i])#,
        marker='.', markersize=5)

    size = 16
```



```

plt.title(f"Convergence rate comparison\nbetween optimization_
↳algorithms\n{scale} scale problem - dimension: {b.size}", size=size + 4)
plt.xticks(fontsize=size)
plt.yticks(fontsize=size)
plt.yscale('log',base=logBase)
plt.xlabel("Iterations", size=size)
plt.ylabel(r'$\log||x-x*||_2$', size=size)

plt.legend(loc=4, frameon=False, fontsize=size, ncol=1)
plt.show()

```

```

[65]: # There are three different scales: small, large and huge
scales = ["small", "large", "huge"]
# Number of maximum iterations of the three scales small, large and huge
maxIters = [5000, 10000, 20000]
# Tolerance of the gradient norm of the three scales small, large and huge
tolerances = [10e-5, 10e-3, 10e-1]
# The algorithm
algorithmNames = ["gradient descent", "conjugate gradient", "FISTA",
↳"coordinate descent"]
algorithms = [gradientDescent, conjugateGradient, FISTA, coordinateDescent]
# The logarithm base
logBase = 2
for i in range(0,3):
    plotDifferenceNormsMutipleAlgorithms(scales[i], maxIters[i], tolerances[i],
↳algorithmNames, algorithms, logBase)

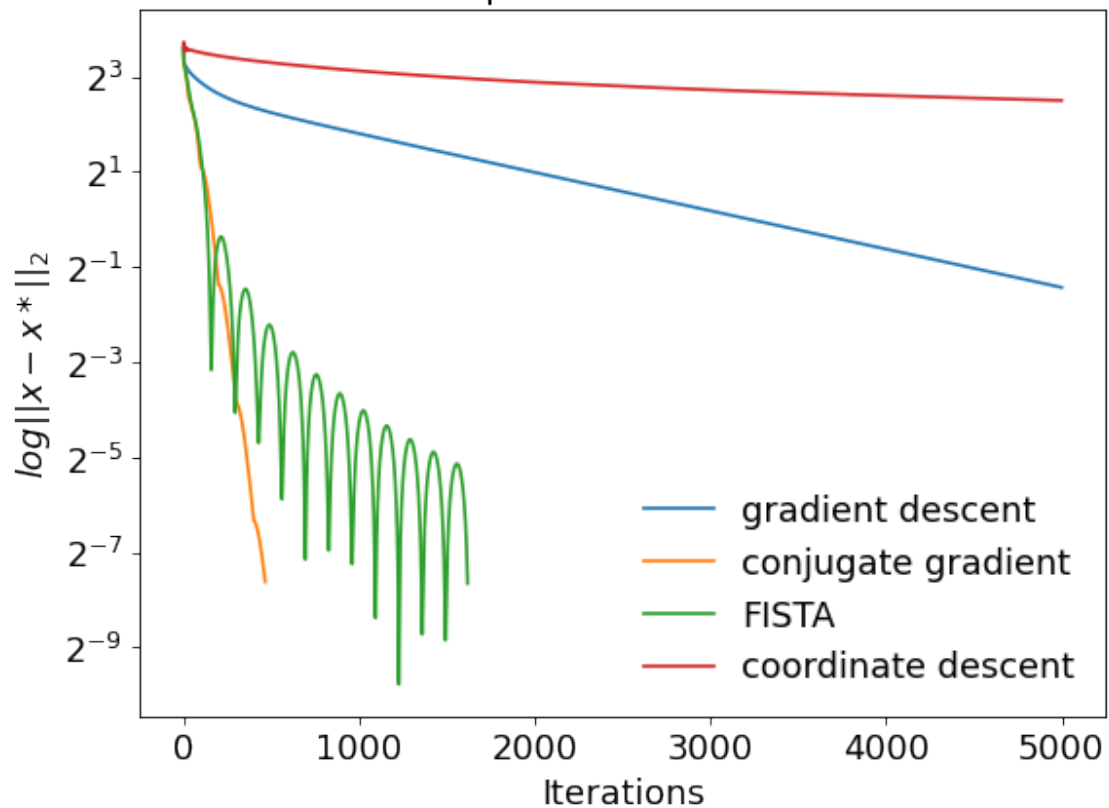
```

The small scale problem is chosen. The matrix A and vector b dimension is 10  
The number of maximum iterations is 5000. The allowed tolerance for gradient norm is 0.0001

The theoretical optimal solution x\*

1	2	3	4	5	6	7	8	9	10
---	---	---	---	---	---	---	---	---	----

# Convergence rate comparison between optimization algorithms small scale problem - dimension: 10

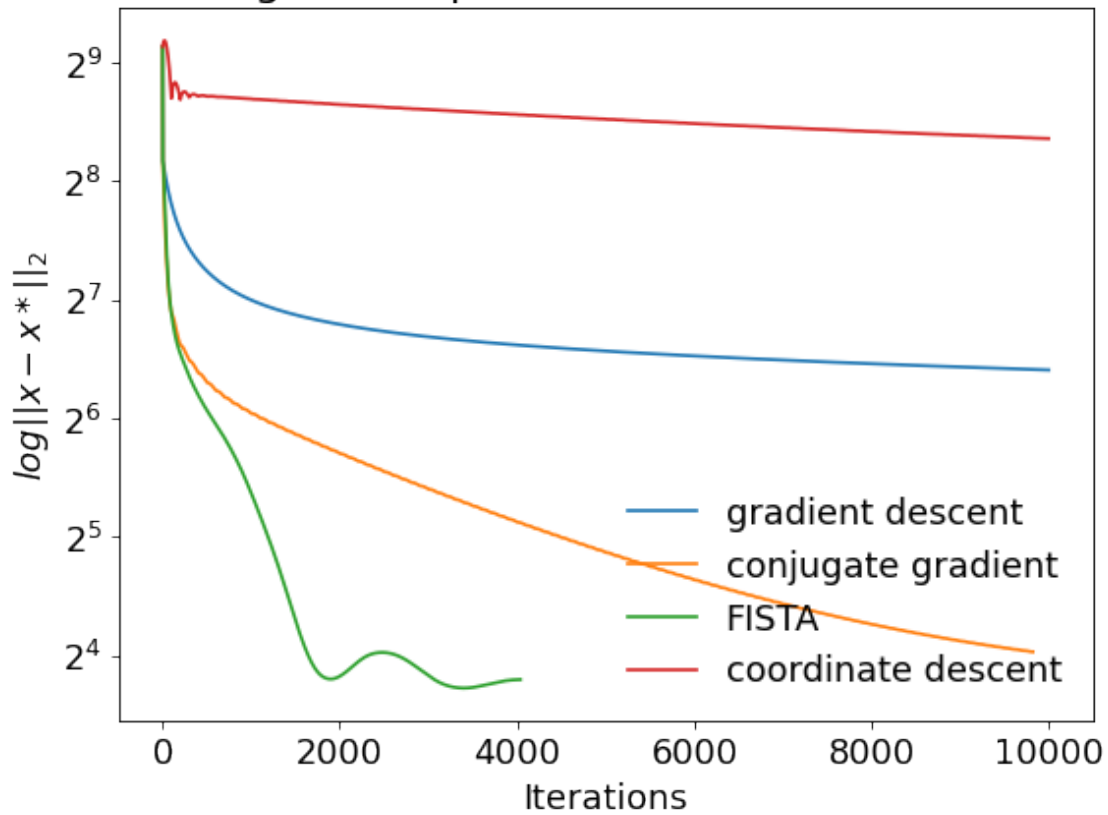


The large scale problem is chosen. The matrix A and vector b dimension is 100  
The number of maximum iterations is 10000. The allowed tolerance for gradient norm is 0.01

The theoretical optimal solution  $x^*$

```
[ 1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18
 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36
 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54
 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72
 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90
 91 92 93 94 95 96 97 98 99 100]
```

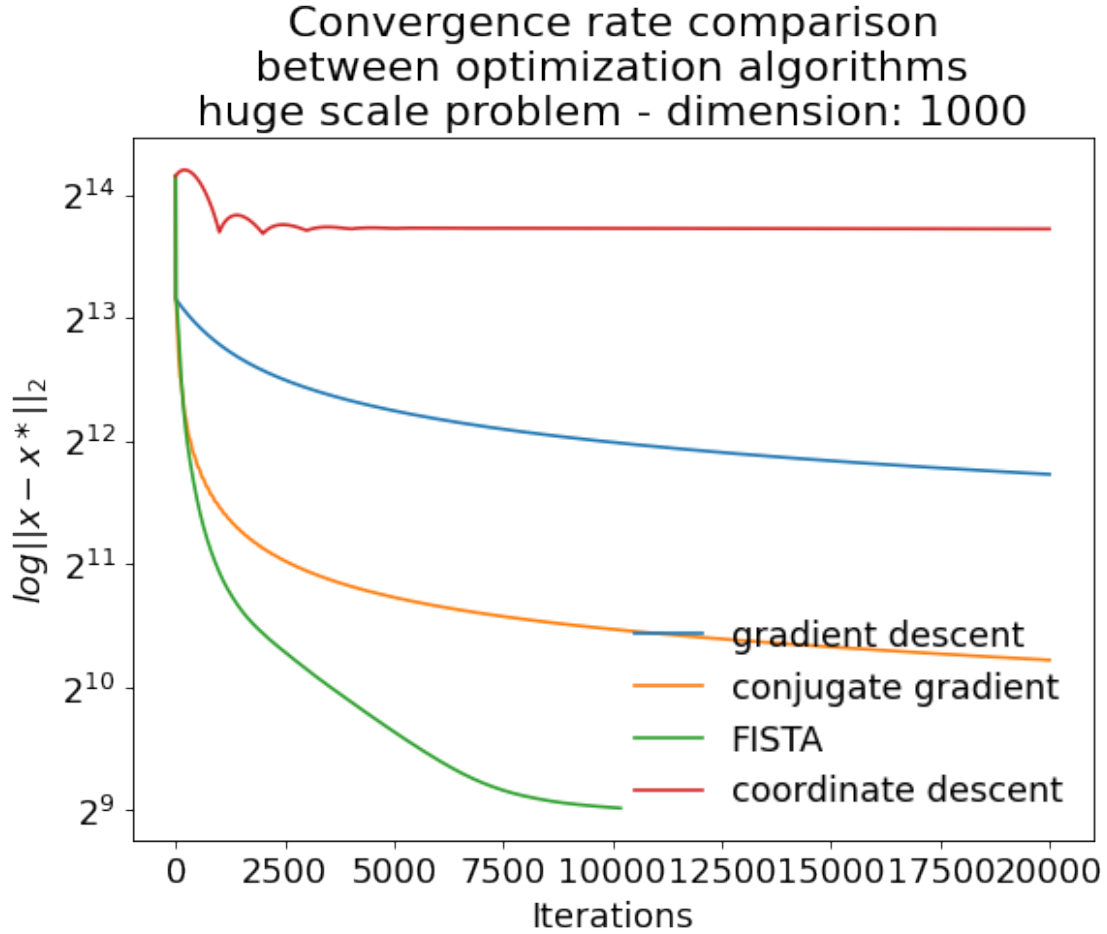
# Convergence rate comparison between optimization algorithms large scale problem - dimension: 100



The huge scale problem is chosen. The matrix A and vector b dimension is 1000  
The number of maximum iterations is 20000. The allowed tolerance for gradient norm is 1.0

First 100 values in the theoretical optimal solution  $x^*$

```
[ 1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18
 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36
 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54
 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72
 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90
 91 92 93 94 95 96 97 98 99 100]
```



From the three comparison graphs, we can finally conclude the performance of each algorithms

1. Gradient descent: Normal linear convergence rate in small dimension and slightly sublinear convergence rate in higher dimensions
2. Conjugate gradient: Fast superlinear convergence rate in all dimensions
3. FISTA: extremely fast superlinear convergence rate in all dimensions
4. Coordinate descent: Slow sublinear convergence rate in all dimensions

The speed of convergence rankings are therefore:

- Small dimension (10): conjugate gradient > FISTA > gradient descent > coordinate descent
- Large dimension (100): FISTA > conjugate gradient > gradient descent > coordinate descent
- Huge dimension (1000): FISTA > conjugate gradient > gradient descent > coordinate descent

Gradient descent is popular in many ML algorithms and solvers in deep learning. Particularly, stochastic gradient descent is much more useful in batches training, where updating the training performance with the whole data is expensive or impossible.

Conjugate gradient is applicable to sparse systems that are too large to be handled by a direct

implementation or other direct methods such as the Cholesky decomposition

FISTA is the fastest algorithm and is robust against large dimensions, making it highly suitable for solving many optimization problems involving a large number of parameters.

Coordinate descent should be used for problems where individual updates are much easier than the whole updates of all components, such as LASSO method in ML. Therefore, coordinate descent is useful in distributed optimization problem.

Visually, gradient descent is going in straight line in a Euclidean map towards the optimum, while coordinate descent follows along only one variable at a time like a stair case, which means it traverse the Manhattan distance towards the optimum. As a result, the gradient descent strictly converges faster than coordinate descent because straight line distance is always larger than the Manhattan distance.

Conclusion: We should use FISTA for large scale problems and conjugate gradient for small scale problems, if applicable, thanks to their fast convergence speed. If not, (stochastic) gradient descent is highly recommended, as it has been implemented for many existing problems. If individual updating is much easier than total updates, coordinate descent is the most suitable algorithm