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Exercise 3:

a) $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \Rightarrow \det(A - \lambda I) = \begin{bmatrix} 1-\lambda & 0 & 3 \\ 0 & 1-\lambda & 0 \\ 3 & 0 & 1-\lambda \end{bmatrix}$

$\Rightarrow$ Characteristic polynomial: $(1-\lambda)^3 + 3(-3(1-\lambda)) = -\lambda^3 + 3\lambda^2 + 6\lambda - 8$

$\Rightarrow p_A(\lambda) = -(\lambda - 1)(\lambda + 2)(\lambda - 4) = 0$

$\Rightarrow$ The eigen values are: $\lambda_1 = 1$; $\lambda_2 = -2$; $\lambda_3 = 4$

▫ At $\lambda_1 = 1$

$\Rightarrow (A - I)v = 0 \Rightarrow \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0 \Rightarrow v = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)$ (eigenvector of λ_1)

Rank of null space is 1 and $\lambda_1 = 1$ appear only once

$\Rightarrow$ Geometric and algebraic multiplicity of $\lambda_1 = 1$ is both 1

▫ At $\lambda_2 = -2$

$\Rightarrow (A + 2I)v = 0 \Rightarrow \begin{bmatrix} 3 & 0 & 3 \\ 0 & 3 & 0 \\ 3 & 0 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0 \Rightarrow v = \text{span}\left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right)$ (eigenvector of λ_2)

Rank of null space is 1 and $\lambda_2 = -2$ appear only once

$\Rightarrow$ Geometric and algebraic multiplicity of $\lambda_2 = -2$ are both 1

▫ At $\lambda_3 = 4$

$\Rightarrow (A - 4I)v = 0 \Rightarrow \begin{bmatrix} -3 & 0 & 3 \\ 0 & -3 & 0 \\ 3 & 0 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0 \Rightarrow v = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$ (eigenvector of λ_3)

Rank of null space is 1 and $\lambda_3 = 4$ appear only once

$\Rightarrow$ Geometric & algebraic multiplicity of $\lambda_3 = 4$ are both 1

b) $B = \begin{bmatrix} 1 & -\sqrt{2} & 3 \\ 0 & -2 & 0 \\ 3 & \sqrt{2} & 1 \end{bmatrix} \Rightarrow \det(B - \lambda I) = \det \begin{bmatrix} 1-\lambda & -\sqrt{2} & 3 \\ 0 & -2-\lambda & 0 \\ 3 & \sqrt{2} & 1-\lambda \end{bmatrix}$

$\Rightarrow$ Characteristic polynomial : $-(1-\lambda)^2(2+\lambda) + 3(0 - 3(-2-\lambda))$

$\Rightarrow p_B(\lambda) = -(\lambda+2)^2(\lambda - 4) = 0$

$\Rightarrow$ The eigenvalues are $\lambda_1 = -2$, $\lambda_2 = 4$

▫ At $\lambda_1 = -2$

$\Rightarrow (B + 2I)v = 0 \Rightarrow \begin{bmatrix} 3 & -\sqrt{2} & 3 \\ 0 & 0 & 0 \\ 3 & \sqrt{2} & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0 \Rightarrow v = \text{span}\left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right)$ (eigenvector of λ_1)

Rank of null space is 1 and $\lambda_1 = -2$ appear twice

$\Rightarrow$ Geometric multiplicity of λ_1 is 1 and algebraic multiplicity of λ_1 is 2

▫ At $\lambda_2 = 4$

$\Rightarrow (B - 4I)v = 0 \Rightarrow \begin{bmatrix} -3 & -\sqrt{2} & 3 \\ 0 & -6 & 0 \\ 3 & \sqrt{2} & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0 \Rightarrow v = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$ (eigenvector of λ_2)

Rank of null space is 1 and $\lambda_2 = 4$ appear once

$\Rightarrow$ Geometric and algebraic multiplicity of $\lambda_2 = 4$ are both 1

Exercise 4: $(F_n)_{n=1,2,\ldots}$ is a Fibonacci sequence

Let $x_n = [F_n, F_{n-1}]^T$ for $n = 2, 3, \ldots$

a) Find matrix $A \in R^{2\times 2}$ such that $x_n = Ax_{n-1}$, $n > 2$

We have: $x_n = A x_{n-1}$

$$\Rightarrow \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix}.$$ We know that $F_n = F_{n-1} + F_{n-2}$

$$\Rightarrow \begin{bmatrix} F_{n-1} + F_{n-2} \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} a_{11} F_{n-1} + a_{12} F_{n-2} \\ a_{21} F_{n-1} + a_{22} F_{n-2} \end{bmatrix} \Rightarrow A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

b) Let $X \in R^{2\times 2}$ and diagonal matrix $\Lambda \in R^{2\times 2}$ be such that $A = X\Lambda X^{-1}$. Show that

$$x_n = X\Lambda^{n-2} X^{-1} x_2, \quad n \geq 2$$

We have: $x_n = Ax_{n-1} = X\Lambda X^{-1} x_{n-1}$

$$= X\Lambda X^{-1}(A x_{n-2}) = X\Lambda X^{-1}(X\Lambda X^{-1} x_{n-2})$$

$$= X\Lambda^2 X^{-1} x_{n-2} = X\Lambda^2 X^{-1}(X\Lambda X^{-1} x_{n-3})$$

$$= X\Lambda^3 X^{-1} x_{n-3}$$

$\Rightarrow$ We notice that $x_n = X\Lambda^m X^{-1} x_{n-m}$. We can find that in the original RHS

$X\Lambda^{n-2} X^{-1} x_2 \Rightarrow n - m = 2 \Rightarrow m = n-2$

$\Rightarrow x^n = X\Lambda^m X^{-1} x_{n-m} = X\Lambda^{(n-2)} X^{-1} x_{n-(n-2)}$

$\Rightarrow x^n = X\Lambda^{n-2} X^{-1} x_2$ (proven)