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Matrix Algebra Problem Sheet 3

Exercise 1: Solve $Ax = b$, when

$$A = \begin{pmatrix} 1 & 3 & 5 & -2 \\ 3 & -2 & -7 & 5 \\ 2 & 1 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 11 \\ 0 \\ 3 \end{pmatrix}$$

The augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 3 & 5 & -2 & 11 \\ 3 & -2 & -7 & 5 & 0 \\ 2 & 1 & 0 & 1 & 3 \end{array}\right] \quad \text{and} \quad \bar{x} = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}$$

Use system of linear equations to solve

$$\begin{cases} x + 3y + 5z - 2t = 11 & R_1 \\ 3x - 2y - 7z + 5t = 0 & R_2 \\ 2x + y + t = 3 & R_3 \end{cases}$$

We have: $3R_1 - R_2 \Rightarrow 11y + 22z - 11t = 33$

$\Rightarrow y + 2z - t = 3 \quad R_4$

$R_4 + R_3 \Rightarrow 2x + 2y + 2z = 6$

$\Rightarrow x + y + z = 3 \quad R_5$

$R_2 + 2R_3 \Rightarrow 7x - 7z + 7t = 6$

$\Rightarrow x - z + t = 6/7 \quad R_6$

$R_5 + R_6 \Rightarrow 2x + y + t = 27/7$

Since R_3: $2x + y + t = 3$

$\Rightarrow$ This system of linear equations has no solution

$\Rightarrow$ There's no $\bar{x}$ such that $Ax = b$

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Exercise 2: Let

$$A = \begin{pmatrix} 1 & -1 & 3 \\ 2 & 3 & 1 \\ 5 & 4 & \alpha \\ 3 & 2 & 4 \end{pmatrix}, \quad b = \begin{pmatrix} 7 \\ -1 \\ 8 \\ \beta \end{pmatrix}$$

Find the solutions of $Ax = b$ for all $\alpha, \beta \in R$

The augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 7 \\ 2 & 3 & 1 & -1 \\ 5 & 4 & \alpha & 8 \\ 3 & 2 & 4 & \beta \end{array}\right] \text{ and } \bar{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

The system of linear equations

$$\begin{cases} x - y + 3z = 7 \\ 2x + 3y + z = -1 \\ 5x + 4y + \alpha z = 8 \\ 3x + 2y + 4z = \beta \end{cases}$$

Since there are 4 vectors in this R^3 matrix $\Rightarrow$ the system is linearly dependent

Reduced row echolon form

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 7 \\ 2 & 3 & 1 & -1 \\ 5 & 4 & \alpha & 8 \\ 3 & 2 & 4 & \beta \end{array}\right] \xrightarrow{R_2 \to 2R_1 - R_2} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 7 \\ 0 & -5 & 5 & 15 \\ 5 & 4 & \alpha & 8 \\ 3 & 2 & 4 & \beta \end{array}\right] \xrightarrow{R_3 \to R_3 - 5R_1}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 7 \\ 0 & -5 & 5 & 15 \\ 0 & 9 & (\alpha - 15) & -27 \\ 3 & 2 & 4 & \beta \end{array}\right] \xrightarrow[R_3 \to 5R_3 + 9R_2]{R_2 \to -R_2/5} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 7 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 5(\alpha - 15) & 0 \\ 3 & 2 & 4 & \beta \end{array}\right]$$

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$$\xrightarrow[\substack{R_4 \to R_4 - 5R_2}]{\substack{R_3 \to R_3/5 \\ R_4 \to R_4 - 3R_1}} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 7 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & (\alpha-15) & 0 \\ 0 & 0 & 0 & \beta - 6 \end{array}\right]$$

From the matrix, we can see that $0x + 0y + 0z = \beta - 6$

$\Rightarrow$ If $\beta = 6$, there will be infinite solutions, but if $\beta \neq 6 \Rightarrow$ there is no solution. From the first three rows, we have

$$\begin{cases} x - y + 3z = 7 \\ y - z = -3 \\ (\alpha - 15)z = 0 \end{cases}$$

If $\alpha \neq 15 \Rightarrow \bar{x} = [4 \ -3 \ 0]^T$

If $\alpha = 15$. Let z be any real number k

$$\Rightarrow \begin{cases} x - y + 3k = 7 \\ y + k = -3 \end{cases} \Rightarrow \begin{cases} x + 3 + k + 3k = 7 \\ y = -3 - k \end{cases}$$

$$\Rightarrow \begin{cases} x = -4k + 4 \\ y = -3 - k \\ z = k \end{cases} \Rightarrow \bar{x} = \begin{bmatrix} -4k + 4 \\ -3 - k \\ k \end{bmatrix}$$

Answer: If $\alpha \in R$, $\beta \neq 6 \Rightarrow$ no solution

$\alpha \neq 15$, $\beta = 6 \Rightarrow \bar{x} = [4 \ -3 \ 0]^T$

$\alpha = 15$, $\beta = 6$

$$\Rightarrow \bar{x} = \begin{bmatrix} -4k + 4 \\ -3 - k \\ k \end{bmatrix} \text{ with } k \in R$$

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Exercise 3: Under which conditions on $\alpha, \beta, \gamma, \delta$, two vecto(rs) $\alpha x + \beta y$ and $\gamma x + \delta y$ are linearly independent, if the vector x and y are

a) Linearly independent ?

b) Not linearly independent ?

a) Let vector $u = \alpha x + \beta y$ and $v = \gamma x + \delta y$

Since x and y are linearly independent, u and v are vectors lying in the span of x and y => u and v lies on the same plane

Two vectors are independent if they are not parallel.
If u and v are parallel => u or v is just scalar multiple of the other. Consider if u and v are parallel, there will exists k so that $u = kv$

$$\Rightarrow \alpha x + \beta y = k(\gamma x + \delta y)$$

$$\Rightarrow \alpha x + \beta y = k\gamma x + k\delta y$$

$$\Rightarrow \begin{cases} \alpha = k\gamma \\ \beta = k\delta \end{cases} \text{ (Since } x \text{ and } y \text{ are independent)}$$

$$\Rightarrow \frac{\alpha}{\gamma} = \frac{\beta}{\delta} = k$$

Therefore, when x and y are independent, for u and v to be linearly independent, $\frac{\alpha}{\gamma} \neq \frac{\beta}{\delta}$. If $\frac{\alpha}{\gamma} = \frac{\beta}{\delta}$, there will exists "$k$" and thus u and v are parallel so they are not linearly independent anymore.

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b) Let $u = \alpha x + \beta y$ and $v = \gamma x + \delta y$

If x and y are linearly dependent, they are parallel vectors.

$\Rightarrow$ There exists k so that $x = ky$

Replace $x = ky$ into u and v, we have

$$u = \alpha k y + \beta y = (\alpha k + \beta) y$$

$$v = \gamma k y + \delta y = (\gamma k + \delta) y$$

Since u and v are just scalar multiples of y $\Rightarrow$ when x and y are not linearly independent, u and v are always dependent without exception. With $\alpha, \beta, \gamma, \delta \in R$, u and v will never be linearly independent

Exercise 4: Let $a_1, \dots a_p$ linearly independent. Determine if the set is linearly independent or dependent

a) $a_1, a_2 + a_1, a_3 + a_2, \dots, a_p + a_{p-1}$

Let $u_1 = a_1$, $u_2 = a_2 + a_1 \dots u_p = a_p + a_{p+1}$

We have: $r_1 u_1 + r_2 u_2 + \dots r_p u_p$

$$= r_1 a_1 + r_2 (a_2 + a_1) + \dots + r_p (a_p + a_{p+1})$$

$$= (r_1 + r_2) a_1 + (r_2 + r_3) a_2 + \dots + r_p a_p$$

Since $a_1, \dots a_p$ are independent $\Rightarrow$ a_i does not contain vector 0 so $u_1 + u_2 + \dots + u_p$ will be a non zero vector. For this sum to equal 0

$$\Rightarrow \begin{cases} r_1 + r_2 = 0 \\ r_2 + r_3 = 0 \\ \dots \\ r_{p-1} + r_p = 0 \\ r_p = 0 \end{cases} \Rightarrow \begin{cases} r_1 = 0 \\ r_2 = 0 \\ \dots \\ r_{p-1} = 0 \\ r_p = 0 \end{cases}$$

$\Rightarrow$ The trivial solution is the only solution to $u_1 + u_2 + \dots + u_p = 0$

$\Rightarrow$ This set of vectors is linearly independent

b) $a_1 - a_p, a_2 - a_1, a_3 - a_2, \dots, a_p - a_{p-1}$

Let $u_1 = a_1 - a_p$, $u_2 = a_2 - a_1 \dots u_p = a_p - a_{p-1}$

If all the vectors are summed together, each vector is multiplied by the same scalar r, we have

$ru_1 + ru_2 + ru_3 + \dots + ru_p = 0$ (The vectors are cancelled out)

Since there exist $r \neq 0$ satisfying this equation

$\Rightarrow$ This set is a linearly dependent set of vectors

c) $a_1, \dots, a_{j-1}, a_j + \lambda a_k, a_{j+1}, \dots, a_p$, where $k \neq j$

This vector set starts with $u_1 = a_1$, $u_2 = a_2 + \lambda a_k$, $u_3 = a_3$, $u_4 = a_4 + \lambda a_k, \dots, a_p$

$\Rightarrow$ This set has odd number of vectors

For this set, we need to prove that

- a_i is independent from a_{i+2m}, that is, odd vectors are independent. Since odd vectors are just simple vectors $a_1, a_3, a_5, a_7, \dots a_p$, according to the precondition, they are already ~~de~~ independent.
- a_i is independent from $a_{i+1} + \lambda a_k$, that is, all odd vectors are independent from even numbered vectors

We have : $a_i \rightarrow a_{i+1} + \lambda a_k \rightarrow a_{i+1}$

odd $\underline{u_i}$ even $\underline{u_{i+1}}$ odd $\underline{u_{i+2}}$

If $a_k = a_{i+1}$, then the even numbered vector will be scalar multiple of the next vector: $(1+k)\alpha_{i+1}$ and a_{i+1} are dependent. However, since $k \neq j$, or $k \neq i+1$, the even numbered vector will never be scalar multiple of any

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dd numbered vectors $\Rightarrow$ add and even numbered vectors are independent

□ $a_{i+1} + \lambda a_k$ is independent of $a_{i+2m+1} + \lambda a_k$, that is, all even vectors are independent

Since a_{i+1} and a_{i+2m+1} are single vectors, they are ~~de~~ independent. Even if a_k of $a_{i+1} + \lambda a_k$ and $a_{i+2m+1} + \lambda a_k$ are the same ~~x~~ a, they would still be two independent vectors

For example; let $a_k = a_5$. For two random even numbered vectors, we have: $a_2 + \lambda a_5$ and $a_6 + \lambda a_5$

Solve $r_2(a_2 + \lambda a_5) + r_6(a_6 + \lambda a_5) = 0$ (1)

$\Rightarrow r_2 a_2 \cancel{r} + r_6 a_6 + \lambda(r_2 + r_6) a_5 = 0$

Since a_2, a_6 and a_5 are independent $\Rightarrow r_2 = 0, r_6 = 0$ is the only solution $\Rightarrow$ (1) only has the trivial solution

$\Rightarrow a_2 + \lambda a_5$ and $a_6 + \lambda a_5$ are independent

Conclusion: since in this set, all odd numbered vectors u_1, u_3, u_5, ... u_p are independent, all even numbered vectors u_2, u_4, u_6, ... u_{p-1} are independent and odd and even numbered vectors are independent of each other: a_i, a_{i+1}

$\Rightarrow$ This vector set is linearly independent