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Exercise 3: Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Compute by hand the QR decomposition of the matrix A and use it to solve the least-squares problem

$$\min_{x \in \mathbb{R}^2} \|Ax - b\|_2^2$$

- Gram-Schmidt orthogonalization process

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \vec{v}_1 = \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}; \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{u}_2 = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1 \times 1 + 1 \times 0 + 0 \times 1}{1 \times 1 + 1 \times 1 + 0 \times 0} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \vec{q}_1 = \frac{\vec{u}_1}{\|\vec{u}_1\|} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \quad \vec{q}_2 = \frac{\vec{u}_2}{\|\vec{u}_2\|} = \begin{bmatrix} \sqrt{6}/6 \\ -\sqrt{6}/6 \\ \sqrt{6}/3 \end{bmatrix} \Rightarrow Q = \begin{bmatrix} 1/\sqrt{2} & \sqrt{6}/6 \\ 1/\sqrt{2} & -\sqrt{6}/6 \\ 0 & \sqrt{6}/3 \end{bmatrix}$$

We have : $A = QR \Rightarrow Q^T A = Q^T Q R$

Since Q is matix of orthogonal basis $\Rightarrow Q^T Q = I$

$$\Rightarrow Q^T A = IR \Rightarrow R = Q^T A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ \sqrt{6}/6 & -\sqrt{6}/6 & \sqrt{6}/3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{6}/2 \end{bmatrix}$$

$\Rightarrow$ QR decomposition :

$$A = QR = \begin{bmatrix} 1/\sqrt{2} & \sqrt{6}/6 \\ 1/\sqrt{2} & -\sqrt{6}/6 \\ 0 & \sqrt{6}/3 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{6}/2 \end{bmatrix} \text{ (answer)}$$

- Least squares problem $\min_{x \in R^2} \|Ax - b\|_2^2$

We have : $Ax_{min} = b \Rightarrow A^T A x_{min} = A^T b$

$\Rightarrow (QR)^T QR\, x_{min} = (QR)^T b$

$\Rightarrow R^T Q^T Q R x_{min} = R^T Q^T b \Rightarrow R^T I R x_{min} = R^T Q^T b$. Since R is invertible

$\Rightarrow R x_{min} = Q^T b \Rightarrow x_{min} = R^{-1} Q^T b$

$$\Rightarrow x_{min} = \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{6}/2 \end{bmatrix}^{-1} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ \sqrt{6}/6 & -\sqrt{6}/6 & \sqrt{6}/3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix} \text{ (answer)}$$

Exercise 4: Let

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}$$

a) Find an orthonormal basis for $R(A)$ by using the modified Gram-Schmidt process

b) Find a QR decomposition of A by using the modified Gram-Schmidt process