Aalto University

Linear algebra

Exercise sheet 8 / Model solutions

1. For any $A \in \mathbb{C}^{n\times n}$ and $\lambda \in \mathbb{C}$ it holds that

$$(\lambda I + A)^k = \sum_{i=0}^{k} \binom{k}{i} \lambda^{k-i} A^i, \qquad k \in \mathbb{N}_0 = \{0, 1, 2, 3, \dots\}.$$

   (a) Let $T \in \mathbb{C}^{n\times n}$ be a *nilpotent matrix*. That is, there exists some $p \in \mathbb{N}$ such that $T^p = \mathbf{0} \in \mathbb{C}^{n\times n}$. Show that

$$(\lambda I + T)^k = \sum_{i=0}^{\min\{k,p-1\}} \binom{k}{i} \lambda^{k-i} T^i.$$

   (b) Let

$$B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

   Write an explicit formula for B^k, for general $k \in \mathbb{N}$.

Solution.

   (a) This is immediate by the given formula and the assumption on p.

   (b) Observe that

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

   Hence, using item (a) with $\lambda = 2$, for any $k \geq 1$ it holds

$$B^k = \sum_{i=0}^{\min\{k,1\}} \binom{k}{i} 2^{k-i} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^i = 2^k I + \binom{k}{1} 2^{k-1} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2^k & k2^{k-1} \\ 0 & 2^k \end{bmatrix}.$$

2. Let

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}.$$

   (a) Is the matrix A diagonalizable ?

   (b) Compute e^{tA} using the Jordan decomposition

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{-1}.$$

Hints: (a) Use Example 1.1 in section 2. (b) Use Problem 1(b). Note that $\sum_{k=1}^{\infty} \frac{1}{k!} k t^k 2^{k-1} = te^{2t}$.

Solution.

(a) No. Indeed, $\det(A - \lambda I) = (\lambda - 2)^2$ but $N(A - 2I) = \operatorname{span}([1\ 1]^T)$. This means that 2 is an eigenvalue of A with algebraic multiplicity 2 but geometric multiplicty 1.

(b) Let

$$X = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

Then, by item (b) of Problem 1,

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k = X \sum_{k=0}^{\infty} \frac{t^k}{k!} B^k X^{-1} = X \left(I + \sum_{k=1}^{\infty} \begin{bmatrix} \frac{(2t)^k}{k!} & t\frac{(2t)^{k-1}}{(k-1)!} \\ 0 & \frac{(2t)^k}{k!} \end{bmatrix} \right) X^{-1}.$$

Summing entry by entry, this is in turn equal to

$$X \begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix} X^{-1} = \begin{bmatrix} (t+1)e^{2t} & -te^{2t} \\ te^{2t} & (1-t)e^{2t} \end{bmatrix}.$$

3. Consider the system of differential equations

$$\boldsymbol{x}'(t) = A\boldsymbol{x}(t) \quad \text{for } t > 0, \qquad \boldsymbol{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{1}$$

where $A \in \mathbb{C}^{2\times 2}$ and $\boldsymbol{x} : \mathbb{R}_+ \to \mathbb{C}^2$. Solve (1) by using the matrix exponential, when

(a)

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix},$$

(b)

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}.$$

Hints: (a) Use Problem 2(b). (b) Observe that A is symmetric.

Solution. From the lectures we know that the solution of (1) is

$$\boldsymbol{x}(t) = e^{tA}\boldsymbol{x}(0) = e^{tA} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

(a) By Problem 2(b) we know that

$$e^{tA} = \begin{bmatrix} (1+t)e^{2t} & -te^{2t} \\ te^{2t} & (1-t)e^{2t} \end{bmatrix},$$

hence

$$\boldsymbol{x}(t) = e^{tA} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} (1+t)e^{2t} \\ te^{2t} \end{bmatrix} = \begin{bmatrix} 1+t \\ t \end{bmatrix} e^{2t}.$$

(b) Let us diagonalize A. We know that this happens unitarily and with real eigenvalues, because $A^* = A$. We have the characteristc polynomial

$$\begin{aligned} p_A(A) = \det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} \\ &= \lambda^2 - 2\lambda - 8. \end{aligned}$$

Therefore the eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = 4$. Let us now compute the eigenvectors. We have $\boldsymbol{v} \in E_{-2} = N(A + 2I)$ if and only if

$$\begin{bmatrix} 1+2 & 3 \\ 3 & 1+2 \end{bmatrix} \boldsymbol{v} = \mathbf{0},$$

which means that $\boldsymbol{v} = \alpha \begin{bmatrix} 1 & -1 \end{bmatrix}^T$, for any $\alpha \in \mathbb{R}$. Similarly, we have $\boldsymbol{v} \in E_4 = N(A - 4I)$ if and only if $\boldsymbol{v} = \alpha \begin{bmatrix} 1 & 1 \end{bmatrix}^T$, for any $\alpha \in \mathbb{R}$. Let us pick unitary eigenvectors:

$$\boldsymbol{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \in E_{-2}, \qquad \boldsymbol{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in E_4.$$

Since A is symmetric, $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are orthogonal with respect to the euclidean inner product (eays). So, if we set

$$V = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix},$$

then

$$V^{-1} = V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

We can diagonalize the matrix A in the form

$$A = V \begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix} V^T = V \Lambda V^T.$$

And now we can easily compute

$$\begin{aligned} e^{tA} = V e^{t\Lambda} V^T &= V \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{4t} \end{bmatrix} V^T \\ &= \left(\frac{1}{\sqrt{2}} \right)^2 \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{4t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} & -e^{-2t} \\ e^{4t} & e^{4t} \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} e^{-2t} + e^{4t} & -e^{-2t} + e^{4t} \\ -e^{-2t} + e^{4t} & e^{-2t} + e^{4t} \end{bmatrix}. \end{aligned}$$

And then the solution to (1) in this case is

$$\boldsymbol{x}(t) = e^{tA}\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}e^{-2t}+e^{4t}\\-e^{-2t}+e^{4t}\end{bmatrix}.$$

4. Consider the matrix

$$A = \begin{bmatrix}1 & 1\\0 & 1+\epsilon\end{bmatrix},$$

where $\epsilon \in \mathbb{R}$.

(a) For which values of ϵ is the matrix A diagonalizable?

(b) Let ϵ be such that A is diagonalizable. Find an invertible $V \in \mathbb{C}^{2\times 2}$ and a diagonal matrix $\Lambda \in \mathbb{C}^{2\times 2}$ so that $A = V\Lambda V^{-1}$. Scale the columns of V so that the first row of V is $\begin{bmatrix}1 & 1\end{bmatrix}$.

(c) Compute the condition number $\kappa_2(V)$ using the Matlab function `cond`. Plot the condition number as a function of ϵ on the intervall $\epsilon \in [10^{-4}, 1]$. Use semilogarithmic scale, see `help semilogy`. What happens when A is very close to a non-diagonalizable matrix?

(d) Set $\epsilon = 0$ and try to compute V and Λ using the Matlab function `eig`. What is the condition number $\kappa_2(V)$? Is the diagonalization given by Matlab plausible? (Compare the result to (a).)

Hints: (a) If a (2×2)-matrix has two distinct eigenvalues, it is diagonalizable (see Section 2, Theorem 1.1 of the lecture notes); if this is not the case, one has to check that the geometric and algebraic multiplicities of each eigenvalue meet. (b) Note that Λ and V depend on the parameter ϵ.

Solution.

(a,b) The eigenvalues of an upper triangular matrix can be seen directly on the diagonal: they are $\lambda_1 = 1$ and $\lambda_2 = 1+\epsilon$. The matrix is surely diagonalizable when the two eigenvalues are distinct, that is, when $\epsilon \neq 0$ (in which case the algebraic multiplicity of both eigenvalues is 1). Let's assume now that $\epsilon \neq 0$. Let's find the eigenvectors for $\lambda_1 = 1$:

$$(A - 1I)\boldsymbol{x} = \begin{bmatrix}0 & 1\\0 & \epsilon\end{bmatrix}\boldsymbol{x} = 0 \qquad \Rightarrow \qquad \boldsymbol{x} = t\begin{bmatrix}1\\0\end{bmatrix}, \quad t \in \mathbb{R}.$$

And eigenvectors for $\lambda_2 = 1+\epsilon$:

$$(A - (1+\epsilon)I)\boldsymbol{x} = \begin{bmatrix}-\epsilon & 1\\0 & 0\end{bmatrix}\boldsymbol{x} = 0 \qquad \Rightarrow \qquad \boldsymbol{x} = t\begin{bmatrix}1\\\epsilon\end{bmatrix}, \quad t \in \mathbb{R}.$$

So a diagonalization is

$$A = X\Lambda X^{-1} = \begin{bmatrix}1 & 1\\0 & \epsilon\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1+\epsilon\end{bmatrix}\begin{bmatrix}1 & -\frac{1}{\epsilon}\\0 & \frac{1}{\epsilon}\end{bmatrix}.$$

In case $\epsilon = 0$,we have $\lambda_1 = \lambda_2 = 1$, and we get:

$$(A - 1I)\boldsymbol{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \boldsymbol{x} = 0 \qquad \Rightarrow \qquad \boldsymbol{x} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad t \in \mathbb{R}.$$

The geometric multiplicity is then 1, hence the matrix is not diagonalizable.

(c) The condition number of the matrix X is

$$\kappa_2(X) = ||X||_2||X^{-1}||_2 = \sigma_{\max}\sigma_{\min}^{-1}.$$

Let us compute the squares of the singular values: since we have

$$X^T X = \begin{bmatrix} 1 & 0 \\ 1 & \epsilon \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & \epsilon \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1+\epsilon^2 \end{bmatrix},$$

we get $\det(X^T X - \lambda I) = (1 - \lambda)(1 + \epsilon^2 - \lambda) - 1$, and by setting this qual to zero we get $\lambda^2 - (2 + \epsilon^2)\lambda + \epsilon^2 = 0$, whose solutions are

$$\lambda = \frac{2 + \epsilon^2 \pm \sqrt{\epsilon^4 + 4}}{2}.$$

And hence we get

$$\kappa_2(X) = \sigma_{\max}\sigma_{\min}^{-1} = \sqrt{\frac{2 + \epsilon^2 + \sqrt{\epsilon^4 + 4}}{2 + \epsilon^2 - \sqrt{\epsilon^4 + 4}}} = \sqrt{\frac{8 + 4\epsilon^2 + 2\epsilon^4 + (4 + 2\epsilon^2)\sqrt{\epsilon^4 + 4}}{4\epsilon^2}}.$$

And then we get $\kappa_2(X) \approx \sqrt{\frac{4}{\epsilon^2}} = \frac{2}{|\epsilon|}$ for small values of ϵ. Hence the condition number grows to $+\infty$ when the matrix gets close to being non-diagonalizable.

(d) The following is Matlab code which includes the output.

```
>> A=[1 1 ; 0 1];
>> [V,D]=eig(A)

V =

    1.0000   -1.0000
         0    0.0000


D =

     1     0
     0     1

>> cond(V)
```

```
ans =

   9.0072e+15

>> A=[1 1 ; 0 1+0.01];
>> [V,D]=eig(A)

V =

    1.0000    1.0000
         0    0.0100


D =

    1.0000         0
         0    1.0100

>> cond(V)

ans =

200.0050
```

The results look as expected.