Aalto University

Linear algebra

Exercise sheet 6 / Model solutions

1. Consider the matrices $A, M \in \mathbb{C}^{n\times n}$ and the *generalized eigenvalue problem*: find a pair $(\lambda, \boldsymbol{x}) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{\mathbf{0}\})$ such that

$$A\boldsymbol{x} = \lambda M\boldsymbol{x}. \tag{1}$$

(a) Derive a polynomial equation that defines the *generalized eigenvalues* λ. How can the *generalized eigenvector* $\boldsymbol{x}$ corresponding to λ be defined?

(b) Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \text{and} \qquad M = I.$$

Find all solutions to (1) in this case.

(c) Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$

Find all solutions to (1) in this case.

Solution.

(a) The definition is equivalent to $(\lambda M - A)\boldsymbol{x} = \mathbf{0}$, and hence we can define eigenvalues as the solutions to $\det(\lambda M - A) = 0$. (Note that this makes sense as the left hand side cannot be the zero polynomial, as when $\lambda \to \infty$ it behaves like $\lambda^n \det M$ and $\det M \neq 0$. And of course $\det(\lambda M - A)$ is a polynomial in λ.)

Eigenvectors can be defined as nonzero vectors of $N(\lambda M - A)$.

(b) Using item (a), we seek for the roots of $\det(\lambda M - A) = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3)$. This gives that the generalized eigenvalues are 1 and 3. Corresponding eigenvectors are $\begin{bmatrix} 1 & -1 \end{bmatrix}^T$ and $\begin{bmatrix} 1 & 1 \end{bmatrix}^T$ respectively.

(c) Using item (a), we seek for the roots of $\det(\lambda M - A) = 8\lambda^2 - 10\lambda + 3$. This gives that the generalized eigenvalues are $1/2$ and $3/4$. Corresponding eigenvectors are $\begin{bmatrix} 1 & -1 \end{bmatrix}^T$ and $\begin{bmatrix} 1 & 1 \end{bmatrix}^T$ respectively.

It's not a random conincidence that the eigenvectors in parts (b) and (c) are the same. This is due to the fact that the matrix M in part (c) is actually equal to $A + I$. So, assuming that λ and $\boldsymbol{x}$ satisfy $A\boldsymbol{x} = \lambda M\boldsymbol{x}$ one may write this as $A\boldsymbol{x} = \lambda(A + I)\boldsymbol{x}$, which gives $(1 - \lambda)A\boldsymbol{x} = \lambda\boldsymbol{x}$, and for $\lambda \neq 0$ (which indeed we have, in part (c)), this is equivalent to having $A\boldsymbol{x} = \frac{\lambda}{1-\lambda}\boldsymbol{x}$, which gives a corresponding eigenvalue (with the same eigenvector $\boldsymbol{x}$) for part (b).

2. Let
$$C = \begin{bmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{bmatrix}$$

   (a) Compute the characteristic polynomial of C.

   (b) Consider the polynomial
$$p(x) = 2x^3 + 4x^2 + 6x + 8.$$
   Using (a) find an eigenvalue problem that can be used to compute the roots of p.

Solution.

   (a) Expanding by the first column iteratively, one can find that the characteristic polynomial is $p_C(\lambda) = -\lambda^3 - a_2\lambda^2 - a_1\lambda - a_0$.

   (b) Note that the roots of p are the same as those of $\frac{p}{-2}$. By item (a), we can compute the latter by finding the eigenvalues of
$$\begin{bmatrix} 0 & 0 & -4 \\ 1 & 0 & -3 \\ 0 & 1 & -2 \end{bmatrix}.$$

3. Compute the eigenvalues and eigenvectors for the matrices
$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & -\sqrt{2} & 3 \\ 0 & -2 & 0 \\ 3 & \sqrt{2} & 1 \end{bmatrix}$$

In both cases, find the characteristic polynomial and determine the geometric and algebraic multiplicity for each eigenvalue.

Hint: Use the sub-determinat rule with respect to the middle row or column.

Solution. The algebraic and geometric multiplicity of an eigenvalue are defined as follows. Let λ be an eigenvalue of a matrix M.

- If λ is a root with multiplicity k of the characteristic polynomial fo M, then $m_a(\lambda) = k$ is the *algebraic multiplicity* of the eigenvalue λ.
- The *geometric multiplicity* $m_g(\lambda)$ of λ is the number of corresponding linearly independent eigenvectors, that is, the dimension of the eigenspace corresponding to λ.

   (a) Consider the matrix
$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

The characteristic polynomial of A is

$$\begin{aligned}\det(A-\lambda I) &= (1-\lambda)\det\begin{bmatrix}1-\lambda & 3\\ 3 & 1-\lambda\end{bmatrix}\\ &= (1-\lambda)(\lambda^2-2\lambda-8)\\ &= (1-\lambda)(\lambda-4)(\lambda+2),\end{aligned}$$

meaning that the eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = 4$ and $\lambda_3 = -2$. Let us then compute corresponding eigenvectors for them.

$\lambda_1 = 1$: Assuming $(A - 1I)\boldsymbol{x} = 0$ we get

$$\begin{bmatrix}0 & 0 & 3\\ 0 & 0 & 0\\ 3 & 0 & 0\end{bmatrix}\boldsymbol{x} = 0,$$

which means that $x_3 = 0$ and $x_1 = 0$, so that

$$\boldsymbol{x} = \begin{bmatrix}0\\ t\\ 0\end{bmatrix}, \quad t \in \mathbb{R}.$$

$\lambda_2 = 4$: Assuming $(A - 4I)\boldsymbol{x} = 0$ we get

$$\begin{bmatrix}-3 & 0 & 3\\ 0 & -3 & 0\\ 3 & 0 & -3\end{bmatrix}\boldsymbol{x} = 0,$$

which means that $x_2 = 0$ and $x_1 = x_3$, so that

$$\boldsymbol{x} = \begin{bmatrix}t\\ 0\\ t\end{bmatrix}, \quad t \in \mathbb{R}.$$

$\lambda_3 = -2$: Assuming $(A + 2I)\boldsymbol{x} = 0$ we get

$$\begin{bmatrix}3 & 0 & 3\\ 0 & 3 & 0\\ 3 & 0 & 3\end{bmatrix}\boldsymbol{x} = 0,$$

which means that $x_2 = 0$ and $x_1 = -x_3$, so that

$$\boldsymbol{x} = \begin{bmatrix}t\\ 0\\ -t\end{bmatrix}, \quad t \in \mathbb{R}.$$

Each of the eigenvalues of A is a simple root of the characteristic polynomial, that is, the algebraic multiplicity is 1 for all eigenvalues of A. For each eigenvalue only one linearly independent eigenvector was found, hence the geometric multiplicity is 1 for all eigenvalues of A.

Note. In general it holds $1 \leq m_g(\lambda) \leq m_a(\lambda)$, which implies that if the algebraic multiplicity of some eigenvalue is 1, then also its geometric multiplicity is necessarily 1.

(b) Let us now consider

$$B = \begin{bmatrix} 1 & -\sqrt{2} & 3 \\ 0 & -2 & 0 \\ 3 & \sqrt{2} & 1 \end{bmatrix}.$$

The characteristic polynomial of B is

$$\begin{aligned} \det(B - \lambda I) &= (-2-\lambda)\det\begin{bmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{bmatrix} \\ &= -(2+\lambda)((1-\lambda)^2 - 9) \\ &= -(\lambda+2)^2(\lambda-4), \end{aligned}$$

so that the eigenvalues of A are $\lambda_1 = 4$ and $\lambda_2 = -2$. Since $\lambda_1 = 4$ is a simple root, its algebraic multiplicity is 1. On the other hand $\lambda_2 = -2$ is a double root, and its algebraic multiplicity is hence 2. Let's compute the eigenvectors corresponding to the eigenvalues.

$\lambda_1 = 4$: Assuming $(B - 4I)\boldsymbol{x} = 0$ we get

$$\begin{bmatrix} -3 & -\sqrt{2} & 3 \\ 0 & -6 & 0 \\ 3 & \sqrt{2} & -3 \end{bmatrix} \boldsymbol{x} = 0,$$

which means that $-6x_2 = 0$ and $-3x_1 - \sqrt{2}x_2 + 3x_3 = 0$. Equivalently $x_2 = 0$ and $x_1 = x_3$, so that

$$\boldsymbol{x} = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix}, \quad t \in \mathbb{R}.$$

The eigenvalue λ_1 has one linearly independent eigenvector, that is, the geometric multiplicity is 1, in formulas $m_g(\lambda_1) = m_a(\lambda_1) = 1$.

$\lambda_2 = -2$: Assuming $(B + 2I)\boldsymbol{x} = 0$ we get

$$\begin{bmatrix} 3 & -\sqrt{2} & 3 \\ 0 & 0 & 0 \\ 3 & \sqrt{2} & 3 \end{bmatrix} \boldsymbol{x} = 0,$$

or equivalently

$$\begin{bmatrix} 3 & -\sqrt{2} & 3 \\ 0 & 0 & 0 \\ 0 & 2\sqrt{2} & 0 \end{bmatrix} \boldsymbol{x} = 0,$$

so that $2\sqrt{2}x_2 = 0$ and $3x_1 - \sqrt{2}x_2 + 3x_3 = 0$. This means that $x_2 = 0$ and $x_1 = -x_3$, so that we get

$$\boldsymbol{x} = \begin{bmatrix} t \\ 0 \\ -t \end{bmatrix}, \quad t \in \mathbb{R}.$$

Also the eigenvalue λ_2 has only one linearly independent eigenvector, so that the geometric multiplicity of λ_2 is 1. So then we have $1 = m_g(\lambda_2) < m_a(\lambda_2) = 2$.

4. The Fibonacci sequence, $(F_n)_{n=1,2,\dots}$, is defined using the recursion

$$F_n = \begin{cases} 1, & n = 1, \\ 1, & n = 2, \\ F_{n-1} + F_{n-2}, & n > 2. \end{cases}$$

Let $\boldsymbol{x}_n = [F_n, F_{n-1}]^T$, $n = 2, 3, \dots$

(a) Find a matrix $A \in \mathbb{R}^{2\times 2}$ such that

$$\boldsymbol{x}_n = A\boldsymbol{x}_{n-1}, \qquad n > 2.$$

(b) Let $X \in \mathbb{R}^{2\times 2}$ and the diagonal matrix $\Lambda \in \mathbb{R}^{2\times 2}$ be such that $A = X\Lambda X^{-1}$. Show that

$$\boldsymbol{x}_n = X\Lambda^{n-2}X^{-1}\boldsymbol{x}_2, \qquad n \geq 2.$$

Compute X and Λ in Matlab using `[X,L] = eig(A)`. Compute also F_{10}, F_{20} and F_{30}.

Solution. When $k > 2$, for some matrix A we have $\boldsymbol{x}_n = A\boldsymbol{x}_{n-1}$ if and only if

$$\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix} = \begin{bmatrix} a_{11}F_{n-1} + a_{12}F_{n-2} \\ a_{21}F_{n-1} + a_{22}F_{n-2} \end{bmatrix},$$

if and only if

$$\begin{cases} F_n &= a_{11}F_{n-1} + a_{12}F_{n-2} \\ F_{n-1} &= a_{21}F_{n-1} + a_{22}F_{n-2}. \end{cases}$$

By definition of F_n, the first equation in the system holds when $a_{11} = 1$ and $a_{12} = 1$, and the second equation holds when $a_{21} = 1$ and $a_{22} = 0$. So the matrix we are looking for is

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

The matrix A has factorization $A = X\Lambda X^{-1}$, where X contains the eigenvectors of A as columns, and Λ contains the eigenvalues of A on the diagonal. By using the equality

$\boldsymbol{x}_n = A\boldsymbol{x}_{n-1}$ recursively and the factorization of A, we get

$$\begin{aligned}\boldsymbol{x}_n &= A\boldsymbol{x}_{n-1} = AA\boldsymbol{x}_{n-2} = AAA\boldsymbol{x}_{n-3} = \ldots = A^{n-2}\boldsymbol{x}_2 \\ &= (X\Lambda X^{-1})^{n-2}\boldsymbol{x}_2 \\ &= \underbrace{(X\Lambda X^{-1})(X\Lambda X^{-1})\ldots(X\Lambda X^{-1})}_{n-2 \text{ copies}}\boldsymbol{x}_2 \\ &= X\Lambda^{n-2}X^{-1}\boldsymbol{x}_2.\end{aligned}$$

The eigenvalues of A and their corresponding eigenvectors are

$$\lambda_1 = \frac{1}{2}(1+\sqrt{5}), \qquad \boldsymbol{x}_1 = \begin{bmatrix}\frac{1}{2}(1+\sqrt{5}) \\ 1\end{bmatrix},$$

$$\lambda_2 = \frac{1}{2}(1-\sqrt{5}), \qquad \boldsymbol{x}_2 = \begin{bmatrix}\frac{1}{2}(1-\sqrt{5}) \\ 1\end{bmatrix}.$$

That is,

$$X = \begin{bmatrix}\frac{1}{2}(1+\sqrt{5}) & \frac{1}{2}(1-\sqrt{5}) \\ 1 & 1\end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix}\frac{1}{2}(1+\sqrt{5}) & 0 \\ 0 & \frac{1}{2}(1-\sqrt{5})\end{bmatrix}$$

Let us then compute the asked values of F_n using the above matrices and equalities (and the computer), remembering that $\boldsymbol{x}_n = \begin{bmatrix}F_n \\ F_{n-1}\end{bmatrix}$.

$$\begin{aligned}\begin{bmatrix}F_{10} \\ F_9\end{bmatrix} &= \boldsymbol{x}_{10} = X\Lambda^8X^{-1}\boldsymbol{x}_2 \\ &= \begin{bmatrix}\frac{1}{2}(1+\sqrt{5}) & \frac{1}{2}(1-\sqrt{5}) \\ 1 & 1\end{bmatrix}\begin{bmatrix}(\frac{1}{2}(1+\sqrt{5}))^8 & 0 \\ 0 & (\frac{1}{2}(1-\sqrt{5}))^8\end{bmatrix}\begin{bmatrix}\frac{1}{2}(1+\sqrt{5}) & \frac{1}{2}(1-\sqrt{5}) \\ 1 & 1\end{bmatrix}^{-1}\begin{bmatrix}1 \\ 1\end{bmatrix} \\ &= \begin{bmatrix}55 \\ 34\end{bmatrix}\end{aligned}$$

$$\begin{aligned}\begin{bmatrix}F_{20} \\ F_{19}\end{bmatrix} &= \boldsymbol{x}_{20} = X\Lambda^{18}X^{-1}\boldsymbol{x}_2 \\ &= \begin{bmatrix}\frac{1}{2}(1+\sqrt{5}) & \frac{1}{2}(1-\sqrt{5}) \\ 1 & 1\end{bmatrix}\begin{bmatrix}(\frac{1}{2}(1+\sqrt{5}))^{18} & 0 \\ 0 & (\frac{1}{2}(1-\sqrt{5}))^{18}\end{bmatrix}\begin{bmatrix}\frac{1}{2}(1+\sqrt{5}) & \frac{1}{2}(1-\sqrt{5}) \\ 1 & 1\end{bmatrix}^{-1}\begin{bmatrix}1 \\ 1\end{bmatrix} \\ &= \begin{bmatrix}6765 \\ 4181\end{bmatrix}\end{aligned}$$

$$
\begin{aligned}
\begin{bmatrix} F_{30} \\ F_{29} \end{bmatrix} &= \boldsymbol{x}_{30} = X\Lambda^{28}X^{-1}\boldsymbol{x}_2 \\
&= \begin{bmatrix} \frac{1}{2}(1+\sqrt{5}) & \frac{1}{2}(1-\sqrt{5}) \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (\frac{1}{2}(1+\sqrt{5}))^{28} & 0 \\ 0 & (\frac{1}{2}(1-\sqrt{5}))^{28} \end{bmatrix} \begin{bmatrix} \frac{1}{2}(1+\sqrt{5}) & \frac{1}{2}(1-\sqrt{5}) \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 832040 \\ 514229 \end{bmatrix}
\end{aligned}
$$

We find then $F_{10} = 55$, $F_{20} = 6765$ and $F_{30} = 832040$.