Nguyen Xuan Binh 887799 Linear Algebra final exam

Exercise 1: Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} \end{bmatrix}, \quad b = \begin{bmatrix} \pi \\ 1 \\ 1 \end{bmatrix}$$

a) QR decomposition: $u_1 = v_1 = [1\ 0\ 0]^T$

$$u_2 = v_2 - \frac{(v_2, u_1)}{\|u_1\|^2} u_1 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} - \frac{0+0+0}{1\times1+0+0} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow u_2 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$\Rightarrow Q = \left[\frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2}\right] = \begin{bmatrix} 1 & 0 \\ 0 & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} \end{bmatrix} = A$$

$A = QR \Rightarrow A = AR \Rightarrow R = A^TA$ (A is unitary matrix)

$$\Rightarrow R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \Rightarrow A = AI_2 \text{ (proven)}$$

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b) Set of solutions to the least squares problem $\min_{x \in \mathbb{R}^2} \|Ax - b\|_2$

We have: $Ax_{min} = b \Rightarrow A^T A x_{min} = A^T b \Rightarrow R x_{min} = Q^T b$

$\Rightarrow x_{min} = R^{-1} Q^T b = R^{-1} A^T b$

$\Rightarrow x_{min} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \pi \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$ (answer)

Exercise 3: Let $y \in \mathbb{R}$ be a real parameter : $B = \begin{bmatrix} 1 & y \\ y & 1 \end{bmatrix}$

a) We have: $Bv - \lambda v = 0 \Rightarrow \det \begin{bmatrix} 1-\lambda & y \\ y & 1-\lambda \end{bmatrix} = 0 \Rightarrow (1-\lambda)^2 - y^2 = 0$

$\Rightarrow \cancel{y^2} (1-\lambda-y)(1-\lambda+y) = 0 \Rightarrow \left[\begin{array}{l} \lambda_1 = 1+y \\ \lambda_2 = 1-y \end{array}\right.$ are the eigenvalues of B

For $\lambda_1 = 1+y \Rightarrow (B - \lambda_1 I) v_1 = 0 \Rightarrow \begin{bmatrix} -y & y \\ y & -y \end{bmatrix} v_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_1 = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

For $\lambda_2 = 1-y \Rightarrow (B - \lambda_2 I) v_2 = 0 \Rightarrow \begin{bmatrix} y & y \\ y & y \end{bmatrix} v_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_2 = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$

$\Rightarrow v_1$ and v_2 are the eigenvectors of B

- Matrix B is positive definite if for every nonzero column vector x, $x^T B x > 0$

Let $x = [x_1 \ x_2]^T$

$\Rightarrow x^T B x = [x_1 \ x_2] \begin{bmatrix} 1 & y \\ y & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + 2x_1 x_2 y + x_2^2$

We have: $x_1^2 > 0, x_2^2 > 0 \ \forall x_1, x_2 \neq 0$. The only condition left is $2x_1 x_2 y \geq 0$ $\forall y \in \mathbb{R}$. Because $x_1^2 + x_2^2$ guaranteed to be nonzero if $x_1 \neq 0, x_2 \neq 0$

$\Rightarrow 2x_1 x_2 y = 0$ can still make B positive definite $\Rightarrow y = 0$ to cancel out this term

$\Rightarrow$ For $y = 0$ the matrix B is symmetric positive definite : $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b) Let $x = [x_1 \ x_2]^T$ and $b = [\alpha \ \beta]^T$

$\Rightarrow Bx = b : \begin{bmatrix} 1 & y \\ y & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 + y x_2 \\ x_2 + y x_1 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

If $y = 0 \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \Rightarrow$ there's always unique solution $\Rightarrow y \neq 0$

If $y = 1 \Rightarrow \begin{bmatrix} x_1 + x_2 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \Rightarrow$ if $\alpha \neq \beta \Rightarrow$ no solution $\Rightarrow y = 1$

~~or $\alpha = \beta = 0 \Rightarrow$ There are two solutions~~

If $y = -1 \Rightarrow \begin{bmatrix} x_1 - x_2 \\ x_2 - x_1 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \Rightarrow$ if $\alpha \neq -\beta \Rightarrow$ no solution $\Rightarrow y = -1$

~~or $\alpha = -\beta = 0 \Rightarrow$ There are two solutions~~

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In fact, any $y \neq 0$ and ± 1, there will always be unique solution because the plane $z = x_1 + yx_2$ and $z = x_2 + yx_1$ intersect each other $\Rightarrow$ only $y = \pm 1$ satisfy

For $y = 1$, the null space of B is $\underset{\text{span}}{\{[-1 \;\; 1]^T\}}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

We have: $[-1 \;\; 1]^T \cdot [1 \;\; 1] = 0 \Rightarrow N(B)$ is orthogonal complement of $R(B)$ at $y = 1$

For $y = -1$, $N(B) = \text{span}\{[1;\ 1]^T\}$ and $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

We have: $[1 \;\; 1]^T \cdot [1 \;\; -1]^T = 0$ and $[1 \;\; 1]^T \cdot [-1 \;\; 1]^T = 0$

$\Rightarrow N(B)$ is also orthogonal complement of $R(B)$ at $y = -1$

c) Diagonalize B: We have already known the eigenvalues and eigenvectors from (a)

$$\Rightarrow B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1+y & 0 \\ 0 & 1-y \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1}$$

$$\Rightarrow e^B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{1+y} & 0 \\ 0 & e^{1-y} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} e^{1+y} & -e^{1-y} \\ e^{1+y} & e^{1-y} \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

$$\Rightarrow e^B = e \begin{bmatrix} (e^y + e^{-y})/2 & (e^y - e^{-y})/2 \\ (e^y - e^{-y})/2 & (e^y + e^{-y})/2 \end{bmatrix}$$

$$\Rightarrow e^B = e \begin{bmatrix} \cosh(y) & \sinh(y) \\ \sinh(y) & \cosh(y) \end{bmatrix} \text{ (proven)}$$

Exercise 4:

a) A matrix C is normal if $CC^* = C^*C$

Schur decomposition can help decompose a square matrix into another one that is unitarily equivalent and has an upper triangular form whose diagonal are the eigenvalues of the matrix

C can be decomposed as: $C = U\Lambda U^*$. For any $n \times n$ complex-valued matrix, there is an orthonormal basis that is in the form of upper triangular matrix

$\Rightarrow C^*C = (U\Lambda U^*)^* U\Lambda U^* = U\Lambda^* U^* U\Lambda U^* = U\Lambda^*\Lambda U^*$ (U is unitary matrix)

$CC^* = (U\Lambda U^*)(U\Lambda U^*)^* = U\Lambda U^* U\Lambda^* U^* = U\Lambda\Lambda^* U^*$

Λ is a diagonal matrix $\Rightarrow \Lambda^*\Lambda = \Lambda\Lambda^* \Rightarrow U\Lambda^*\Lambda U^* = U\Lambda\Lambda^* U^*$

$\Rightarrow C^*C = CC^* \Rightarrow C$ is normal matrix

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b) We know that $C = U \Lambda U^*$. Use properties of determinant, we have the characteristic polynomial: $p_C(\lambda) = \det(C - \lambda I) = \det(U\Lambda U^* - \lambda UU^*) = \det(U(\Lambda - \lambda I)U^*)$

$$= \det(U)\det(\Lambda - \lambda I)\det(U^*) = \det(U)\det(\Lambda - \lambda I)\det(U)^*$$

$$= \det(\Lambda - \lambda I) = p_\Lambda(\lambda)$$

$\Rightarrow$ C and Λ have the same eigenvalues

Since determinant of the upper triangular matrix $\Lambda - \lambda I$ is the product of its diagonal elements $\Rightarrow$ eigenvalues of A are on the diagonal of Λ

▫ We have: $\|C\|_2 = \max\limits_{x \neq 0} \dfrac{\|Cx\|_2}{\|x\|_2} = \sigma_{max}(C)$ (largest singular value of A)

Spectral theorem: when C is normal, $\sigma_{max}(C)$ equals to the largest modulus of C's eigenvalues $\Rightarrow \sigma_{max}(C) = |\lambda_{max}(C)|$ (proven in week 6: $\|A\|_2 = \sigma_1 / \sigma_{max}$)

$\Rightarrow \|C\|_2 = |\lambda_{max}(C)|$

c) Let $D = \begin{bmatrix} 3 & 9 \\ 7 & 6 \end{bmatrix}$. We have: $D^TD = \begin{bmatrix} 58 & 69 \\ 69 & 117 \end{bmatrix}$, $DD^T = \begin{bmatrix} 90 & 75 \\ 75 & 85 \end{bmatrix}$

$\Rightarrow$ A is not a normal matrix. Check whether $\|D\|_2 \neq |\lambda_{max}(D)|$

We have: $\|D\|_2 = \lambda_{max}(D^TD)^{1/2}$

$(D^TD - \lambda I)v = 0 \Rightarrow \det\begin{bmatrix} 58-\lambda & 69 \\ 69 & 117-\lambda \end{bmatrix} = 0 \Rightarrow \begin{cases} \lambda_1 = \dfrac{-5\sqrt{901}+175}{2} \\ \lambda_2 = \dfrac{5\sqrt{901}+175}{2} \end{cases}$

$\Rightarrow \|D\|_2 = \sqrt{\dfrac{5\sqrt{901}+175}{2}}$

We have: $(D - \lambda I)v = 0 \Rightarrow \det\begin{bmatrix} 58-\lambda & 69 \\ 69 & 117-\lambda \end{bmatrix} = 0 \Rightarrow \begin{cases} \lambda_1 = \dfrac{-3\sqrt{29}+9}{2} \\ \lambda_2 = \dfrac{3\sqrt{29}+9}{2} \end{cases}$

$\Rightarrow |\lambda_{max}(D)| = \dfrac{3\sqrt{29}+9}{2} \neq \|D\|_2$

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Exercise 2: $A \mapsto \|A\|$ a norm on $R^{n \times n}$ and $c_n > 0$

a) $c_n \|A\|$ is a matrix norm on $R^{n \times n}$ as it satisfies

- $c_n \|A\| \geq 0 \ \forall A \in R^{n \times n}$: Since $\|A\| \geq 0$ and $c_n > 0 \Rightarrow c_n \|A\| \geq 0$
- $\|A\|$ is norm $\Rightarrow \|\alpha A\| = |\alpha| \|A\| \Rightarrow c_n \|\alpha A\| = c_n |\alpha| \|A\|$
- $\|A\|$ is norm $\Rightarrow \|AA\| \leq \|A\| \|A\| \Rightarrow c_n \|AA\| \leq c_n \|A\| \|A\|$ since c_n is positive

  ↳ This argument also applies for $\|\text{matrix}\| \times \|\text{vector}\|$
- $\|A\| + \|A\| \geq \|A + A\| \Rightarrow c_n \|2A\| \leq c_n \|A\| + c_n \|A\|$

$\Rightarrow c_n \|A\|$ is a norm on $R^{n \times n}$

b) - Show that $\|A\|_{max} = \max_{i,j=1}^{n} |A_{ij}|$.

$\|A\|_{max}$ is a norm because it satisfies

- $\|\alpha A\|_{max} = \max_{i,j=1}^{n} |\alpha A_{ij}| = |\alpha| \max_{i,j=1}^{n} |\alpha A_{ij}| = |\alpha| \|A\|_{max}$
- $\|A\|_{max} = \max_{i,j=1}^{n} |A_{ij}| > 0$ ($\|A\|_{max} = 0$ when all elements are $0 \Rightarrow \|A\|_{max} = 0, A = 0$)
- $\|A\|_{max} + \|A\|_{max} = 2 \max_{i,j=1}^{n} |A_{ij}| = 2\|A\|_{max} = \|2A\|_{max}$

  $= \|A + A\|_{max}$

* Since $\|A\|_{max}$ is a norm $\Rightarrow \|A\|_{\mu} = n \|A\|_{max}$ is also a norm since n is positive (proof from (a))

c) We have: $|x \cdot y| = \left| \sum_{j=1}^{n} x_j y_j \right| \leq \sum_{j=1}^{n} |x_j y_j| = \sum_{j=1}^{n} |x_j| |y_j| \leq \sum_{j=1}^{n} (\max |x_i|) |y_i|$

$= \|x\|_\infty \sum_{j=1}^{n} |y_j| = \|x\|_\infty \|y\|_1$

We have: $\|x\|_1 \leq \sqrt{n} \|x\|_2 \leq n \|x\|_\infty \Rightarrow |x \cdot y| \leq n \|x\|_\infty \|y\|_\infty$

where $\|\cdot\|_\infty = \max_{i} \sum_{j=1}^{n} |a_{ij}|$

$\Rightarrow \|A\|_\mu \|B\|_\mu = n \|A\|_{max} \, n \|B\|_{max} = n^2 \|A\|_{max} \|B\|_{max}$

$\|AB\|_\mu = n \|AB_{max}\|$

For any vectors $\in A, B \Rightarrow |A_j \cdot B_j| \leq n \|A\|_\infty \|B\|_\infty \| \leq$

$\Rightarrow \|AB\|_\mu \leq \|A\|_\mu \|B\|_\mu$ (proven)

For $\|\cdot\|_{max}$, there are no n so $n^2 > 1 \Rightarrow \|\cdot\|_{max}$ is not multiplicative