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Linear algebra

Exercise sheet 7

Practice problems

The following problems are just for practice. Their solutions will be shown by the TAs during the exercise sessions. You are welcome to share your own solution during the sessions, if you want.

1. Let $A \in \mathbb{C}^{n\times n}$ be a Hermitian matrix. Show that

   (a) $\|\boldsymbol{x}\|_2^2 = \boldsymbol{x}^*\boldsymbol{x} = \sum_{i=1}^n |\alpha_i|^2$,

   (b) $\boldsymbol{x}^*A\boldsymbol{x} = \sum_{i=1}^n \lambda_i|\alpha_i|^2$,

   (c) $\|A\boldsymbol{x}\|_2^2 = \sum_{i=1}^n \lambda_i^2|\alpha_i|^2$,

   where $\alpha_i \in \mathbb{C}$, for $i = 1, \dots, n$, are the coordinates of the vector $\boldsymbol{x} \in \mathbb{C}^n$ in the orthonormal eigenbasis of the matrix A and $\lambda_i \in \mathbb{R}$, for $i = 1, \dots, n$ are the corresponding eigenvalues (repeated by their algebraic multiplicity). That is, one has $\boldsymbol{x} = \sum_{i=1}^n \alpha_i\boldsymbol{v}_i$, where the vectors $\boldsymbol{v}_i$, for $i = 1, \dots, n$, satisfy $A\boldsymbol{v}_i = \lambda_i\boldsymbol{v}_i$, $\|\boldsymbol{v}_i\|_2 = 1$ and $\boldsymbol{v}_i^*\boldsymbol{v}_j = 0$ if $i \neq j$.

2. Let $A \in \mathbb{C}^{n\times n}$ be a Hermitian matrix.

   (a) Prove the second half of Theorem 2.2 from the lecture notes:

   $$\lambda_{\max}(A) = \max_{0\neq\boldsymbol{x}\in\mathbb{C}^n} R(A,\boldsymbol{x}),$$

   where $\lambda_{\max}(A) \in \mathbb{R}$ is the largest eigenvalue of the matrix A.

   (b) Show that for each $\boldsymbol{x} \in \mathbb{C}^n$ it holds that

   $$\lambda_{\min}(A)\|\boldsymbol{x}\|_2^2 \leq \boldsymbol{x}^*A\boldsymbol{x} \leq \lambda_{\max}(A)\|\boldsymbol{x}\|_2^2,$$

   where $\lambda_{\min}(A), \lambda_{\max}(A) \in \mathbb{R}$ are the smallest and the largest eigenvalue of A.

Homework

Return the solutions to the following problems on MyCourses by Friday, May 21st, 18:00.

3. Consider the function $\|\cdot\|_{P^1} : \mathbb{R}^2 \to \mathbb{R}$ defined as follows: for any $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$, set

   $$\|\boldsymbol{x}\|_{P^1}^2 = \frac{1}{3}x_1^2 + x_1x_2 + x_2^2.$$

(a) Find a symmetric matrix $A \in \mathbb{R}^{2\times 2}$ such that

$$\|\boldsymbol{x}\|_{P^1}^2 = \boldsymbol{x}^T A \boldsymbol{x}, \qquad \text{for all } \boldsymbol{x} \in \mathbb{R}^2.$$

(b) Show that A above is positive definite (see Definition 2.4 from the lecture notes).

(c) Show that $\|\cdot\|_{P^1} : \mathbb{R}^2 \to \mathbb{R}$ is a norm.

Hints: (b) Use Problem 2(b). (c) Use Lemma 2.3. from the lecture notes.

4. The eigenvalues of a matrix $A \in \mathbb{C}^{n\times n}$ are usually not computed in practice by finding the zeros of the characteristic polynomial. One way to approximate the eigenvector corresponding to the eigenvalue of a diagonalizable matrix A with largest absolute value is power iteration,

$$\boldsymbol{x}_{i+1} = \frac{A\boldsymbol{x}_i}{\|A\boldsymbol{x}_i\|_2}, \qquad i = 0, 1, 2, \ldots,$$

where $\boldsymbol{x}_0 \in \mathbb{C}^n$ is some initial guess. Approximation for the eigenvalue of A with largest absolute value is computed as

$$\mu_i = R(A, \boldsymbol{x}_i), \qquad i = 0, 1, 2, \ldots \tag{1}$$

The intuition behind power iteration is that repeated multiplication by A turns the vector slowly towards the eigenvector corresponding to the eigenvalue of A with largest absolute value (unless $\boldsymbol{x}_0$ is orthogonal to it).

Let now

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

Choose $\boldsymbol{x}_0 = [1, 0, 0]^T$. Compute an approximation for the eigenvalue of A with largest absolute value $\lambda_{\text{amax}}(A)$ and the corresponding eigenvector by using power iteration. Plot the error $|\lambda_{\text{amax}}(A) - \mu_i|$ as a function of the index $i = 0, 1, \ldots, 10$.