Aalto University

Linear algebra

Exercise sheet 5

Practice problems

The following problems are just for practice. Their solutions will be shown by the TAs during the exercise sessions. You are welcome to share your own solution during the sessions, if you want.

1. (a) Let $A \in \mathbb{R}^{n\times n}$ be an invertible matrix. Show that λ is an eigenvalue of A if and only if λ^{-1} is an eigenvalue of $A^{-1} \in \mathbb{R}^{n\times n}$. *Hint*: Show that each eigenvector of A is also an eigenvector of A^{-1} and vice versa.

   (b) Let $A \in \mathbb{R}^{n\times n}$ be an invertible matrix. Show that the matrices AA^T and A^TA have the same eigenvalues. *Hint*: Show that $\det(AA^T - \lambda I) = \det(A^TA - \lambda I)$.

   (c) Let $A \in \mathbb{R}^{n\times n}$ be an invertible matrix. Show that

$$\|A^{-1}\|_2^2 = \frac{1}{\lambda_{\min}(A^TA)},$$

   where $\lambda_{\min}(A^TA)$ is the smallest eigenvalue of the matrix A^TA. *Hint*: Use (a) and (b).

2. (a) Let λ be an eigenvalue of $A \in \mathbb{R}^{n\times n}$. Show that λ^2 is an eigenvalue of the matrix A^2.

   (b) Let λ be an eigenvalue of the matrix A^2, with $A \in \mathbb{R}^{n\times n}$. Show that $\sqrt{\lambda}$ or $-\sqrt{\lambda}$ is an eigenvalue of A. (For a complex number λ the notation $\sqrt{\lambda}$ stands for the main branch of the square root.) *Hint*: $\det(A^2 - \lambda I) = \det((A - \sqrt{\lambda}I)(A + \sqrt{\lambda}I)) = \det(A - \sqrt{\lambda}I)\det(A + \sqrt{\lambda}I)$.

   (c) Let $A \in \mathbb{R}^{n\times n}$ be symmetric. Show that

$$\|A\|_2 = |\lambda_{\text{amax}}(A)|,$$

   where $\lambda_{\text{amax}}(A) \in \mathbb{R}$ is an eigenvalue of A with largest absolute value. *Hint*: Use Lemma 2.5 from the lecture notes and parts (a), (b).

   (d) Let $A \in \mathbb{R}^{n\times n}$ be symmetric and invertible. Show that

$$\|A^{-1}\|_2 = \frac{1}{|\lambda_{\text{amin}}(A)|},$$

   where $\lambda_{\text{amin}}(A) \in \mathbb{R}$ is an eigenvalue of A with smallest absolute value. *Hint*: Use Problem 1(a) and (c).

Homework

Return the solutions to the following problems on MyCourses by Friday, May 14th, 18:00.

3. Let $A, B \in \mathbb{R}^{n\times n}$ be symmetric. In addition, assume that

$$|\lambda_{\text{amin}}(A)| \geq 2 \qquad \text{and} \qquad |\lambda_{\text{amax}}(B)| \leq 1,$$

where $\lambda_{\text{amin}}(A)$ is an eigenvalue of A with smallest absolute value and $\lambda_{\text{amax}}(B)$ is an eigenvalue of B with largest absolute value. Show that:

(a) $\|A^{-1}\|_2 \leq \frac{1}{2}$,

(b) the matrix $A + B$ is invertible,

(c) taking for granted the formula

$$\|(A+B)^{-1}\|_2 \leq \frac{\|A^{-1}\|_2}{1 - \|B\|_2\|A^{-1}\|_2},$$

a solution $\boldsymbol{x}$ to the equation $(A+B)\boldsymbol{x} = \boldsymbol{b}$ satisfies $\|\boldsymbol{x}\|_2 \leq \|\boldsymbol{b}\|_2$.

Hints: For (a), use Problem 2(d). For (b), use Theorem 3.1 and Problem 2. For (c), use Problem 2.

4. Let

$$A = \begin{bmatrix} 1+\epsilon^2 & 1 \\ 1 & 1+\epsilon^2 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{b} = \begin{bmatrix} 2 \\ 2-\delta \end{bmatrix},$$

where $\epsilon, \delta \in \mathbb{R}$ are free parameters.

(a) Compute $\kappa_2(A) = \|A\|_2\|A^{-1}\|_2$. What happens to the condition number $\kappa_2(A)$ when $\epsilon \to 0$?

(b) Compute $R(A)$ and $N(A)$, when $\epsilon \neq 0$. What about $\epsilon = 0$?

(c) For $\epsilon \neq 0$, solve $A\boldsymbol{x} = \boldsymbol{b}$ using the formula (Cramer's rule)

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

How does $\boldsymbol{x}$ behave, when $\delta = 0$ and $\epsilon \to 0$? What about $\delta \neq 0$ and $\epsilon \to 0$?

Hint: For (a), use Problem 2.