Aalto-yliopisto

Linear algebra

Exercise sheet 10

Practice problems

The following problems are just for practice. Their solutions will be shown by the TAs during the exercise sessions. You are welcome to share your own solution during the sessions, if you want.

1. (a) Let $\mathcal{V} \subset \mathbb{R}^n$ be a subspace and $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ an *orthonormal basis* for $\mathcal{V}$ with respect to the inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. Show that for any $\boldsymbol{x} \in \mathbb{R}^n$ it holds that

$$\boldsymbol{x}^{\perp} := \boldsymbol{x} - \sum_{i=1}^{k} \langle \boldsymbol{x}, \boldsymbol{v}_i \rangle \boldsymbol{v}_i \in \mathcal{V}^{\perp},$$

where $\mathcal{V}^{\perp} \subset \mathbb{R}^n$ is the orthogonal complement of $\mathcal{V}$ with respect to the inner product $\langle \cdot, \cdot \rangle$, see Definition 2.3 on page 45 of the lecture notes. (This proves that equation (75) there does define a direct sum $\mathbb{R}^n = \mathcal{V} \oplus \mathcal{V}^{\perp}$.)

(b) Prove that any projection matrix $P \in \mathbb{R}^{n \times n}$ is an orthogonal projection in the Euclidian inner product, if $P^T = P$.

Hints: (a) Write an arbitrary vector of the subspace $\mathcal{V}$ in the form $\boldsymbol{v} = \sum_{i=1}^{k} \alpha_i \boldsymbol{v}_i$ and show by direct calculation that $\langle \boldsymbol{x}^{\perp}, \boldsymbol{v} \rangle = 0$. (b) With a projection P we may write the direct sum $\mathbb{R}^n = R(P) \oplus R(I - P)$; check equation (72) from the lecture notes. You have to show that $R(I - P) = R(P)^{\perp}$. Since moreover $\mathbb{R}^n = R(P) \oplus R(P)^{\perp}$, it's actually enough to prove that $R(I - P) \subseteq R(P)^{\perp}$.

2. Let

$$\mathcal{V} = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

(a) Find an orthogonal projection P such that $R(P) = \mathcal{V}$.

(b) Choose some other basis for the subspace $\mathcal{V}$. Compute P again in this new basis.

Homework

Return the solutions to the following problems on MyCourses by Friday, May 28th, 18:00.

3. Let

$$\mathcal{V} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad \mathcal{W} = \operatorname{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Denote

$$V = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

You may assume known that $\mathbb{R}^4 = \mathcal{V} \oplus \mathcal{W}$.

(a) Using the matrix V give a formula for the orthogonal projection matrix $Q \in \mathbb{R}^{4\times4}$ such that $R(Q) = \mathcal{V}$.

(b) Using the matrices V and W give a formula for the projection matrix $P \in \mathbb{R}^{4\times4}$ such that $R(P) = \mathcal{V}$ and $N(P) = \mathcal{W}$.

(c) Compute the projection matrices in (a) and (b) using Matlab. Check that $Q^2 = Q$, $P^2 = P$, $Q^T(I - Q) = 0$, $PV = V$ and $PW = 0$ (within floating point accuracy).

Hints: (b) If P is constructed so that it projects the direct sum $\mathbb{R}^4 = \mathcal{V} \oplus \mathcal{W}$ to the first component, what is the null space of P?

4. Let $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3 \setminus \{\boldsymbol{0}\}$ be such that $\boldsymbol{a}^T\boldsymbol{b} = 0$ and define the matrix $P \in \mathbb{R}^{3\times3}$ as

$$P = \frac{\boldsymbol{a}\boldsymbol{a}^T}{\|\boldsymbol{a}\|_2^2} + \frac{\boldsymbol{b}\boldsymbol{b}^T}{\|\boldsymbol{b}\|_2^2}.$$

(a) Is P a projection matrix? How about $I - P$, where I is the identity matrix?

(b) Is P an orthogonal projection?

(c) Let $\boldsymbol{a} = [1, 1, 1]^T$ and $\boldsymbol{b} = [1, -2, 1]^T$. Find $\boldsymbol{c} \in \mathbb{R}^3$ such that

$$I - P = \frac{\boldsymbol{c}\boldsymbol{c}^T}{\|\boldsymbol{c}\|_2^2}.$$

Hints: (a) See Definition 2.1 on page 45 of the lecture notes. (b) Use problem 1(b). (c) The vector $\boldsymbol{c}$ has to be orthogonal to $\boldsymbol{a}$ and $\boldsymbol{b}$, since $R(I - P) = R(P)^\perp$.