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Matrix Algebra Problem Sheet 5

Exercise 1: Find the eigen values and eigenvectors of

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

Characteristic polynomial of A: $Ax = \lambda x$

$\Leftrightarrow \det(A - \lambda I) = 0$

$$\Rightarrow \det \begin{bmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 1 & 3-\lambda \end{bmatrix} = 0$$

$\Rightarrow (3-\lambda)[(3-\lambda)^2 - 1] - 1.[(3-\lambda) - 1]$

$+ 1[1 - (3-\lambda)] = 0$

$\Rightarrow (3-\lambda)(\lambda^2 - 6\lambda + 8) + \lambda - 2 + 1 - 3 + \lambda = 0$

$\Rightarrow 3\lambda^2 - 18\lambda + 24 - \lambda^3 + 6\lambda^2 - 8\lambda + 2\lambda - 4 = 0$

$\Rightarrow -\lambda^3 + 9\lambda^2 - 24\lambda + 20 = 0$

$\Rightarrow \left[\begin{array}{l} \lambda_1 = 2 \\ \lambda_2 = 5 \end{array}\right.$

▫ For $\lambda_1 = 2$

$\Rightarrow A - \lambda_1 I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ We have $(A - \lambda_1 I)x_1 = 0$

Gaussian elimination

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow t_1 + t_2 + t_3 = 0$$

$\Rightarrow x_1 = (-t_2 - t_3 \quad t_2 \quad t_3)^T, \ t_2, t_3 \in R$

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One specific solution of x_1: $(3 \quad 1 \quad -4)^T$

- For $\lambda_2 = 5$

$$\Rightarrow A - \lambda_2 I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$. We have $(A - \lambda_2 I)x_2 = 0$

Solve system of linear equations

$$\begin{cases} -2s_1 + s_2 + s_3 = 0 \\ s_1 - 2s_2 + s_3 = 0 \\ s_1 + s_2 - 2s_3 = 0 \end{cases} \Rightarrow \begin{cases} s_1 - s_3 = 0 \\ s_2 - s_3 = 0 \end{cases}$$

$\Rightarrow x_2 = (s_3 \quad s_3 \quad s_3)^T, s_3 \in R$

One specific solution x_2: $(2 \quad 2 \quad 2)^T$

Answer:

Eigenpair 1: $(2, [-t_2 - t_3, t_2, t_3]^T (t_2, t_3 \in R))$

Eigenpair 2: $(5, [s_3, s_3, s_3]^T (s_3 \in R))$

Exercise 2: The matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ -2 & 5 & -1 \end{bmatrix}$$

has eigenvectors $u_1 = [1 \ 1 \ 1]^T$, $u_2 = [1 \ 0 \ -1]^T$, $u_3 = [0 \ 0 \ 1]^T$.

Let $x = [2 \ 1 \ 1]^T$. Find $A^{11}x$

We have: $Au_1 = \lambda_1 u_1$ since λ and u are eigenpair

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ -2 & 5 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\Rightarrow \lambda_1 = 2$

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We have: $Au_2 = \lambda_2 u_2$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ -2 & 5 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \lambda_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} . \lambda_2$$

$\Rightarrow \lambda_2 = 1$

We have: $Au_3 = \lambda_3 u_3$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ -2 & 5 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \lambda_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \lambda_3$$

$\Rightarrow \lambda_3 = -1$

Diagonalizing matrix A : $A = SDS^{-1}$

$$S = (u_1 \; u_2 \; u_3) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix} \Rightarrow S^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\Rightarrow A = \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & -2 & 1 \end{bmatrix}}_{S^{-1}}$$

□ We have: $Au = \lambda u$

$\Rightarrow A^2 u = A . Au = A . \lambda u = \lambda^2 u$

$\Rightarrow A^{11} u = \lambda^{11} u$

$\Rightarrow$ The eigen value of A^{11} is λ^{11}. The eigen vectors of A^{11} and A are the same.

We have: $A^{11} = \underbrace{(SDS^{-1})(SDS^{-1})(SDS^{-1}) \ldots (SDS^{-1})}_{11 \text{ times}}$

$\Rightarrow A^{11} = SD^{11}S^{-1}$

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We have : $A^{11} = SD^{11}S^{-1}$

Since $D = \begin{bmatrix} x_1 & & 0 \\ & x_2 \ddots & \\ 0 & & x_n \end{bmatrix} \Rightarrow D^n = \begin{bmatrix} (x_1)^n & & 0 \\ & (x_2)^n \ddots & \\ 0 & & (x_n)^n \end{bmatrix}$

$$\Rightarrow D^{11} = \begin{bmatrix} 2^{11} & 0 & 0 \\ 0 & 1^{11} & 0 \\ 0 & 0 & (-1)^{11} \end{bmatrix} \Rightarrow D^{11} = \begin{bmatrix} 2048 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\Rightarrow A^{11} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2048 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\Rightarrow A^{11} = \begin{bmatrix} 1 & 2047 & 0 \\ 0 & 2048 & 0 \\ -2 & 2051 & -1 \end{bmatrix}$$

Then $A^{11}x = \begin{bmatrix} 1 & 2047 & 0 \\ 0 & 2048 & 0 \\ -2 & 2051 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2049 \\ 2048 \\ 2046 \end{bmatrix}$

Exercise 3 : Is the matrix

$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & -2 & 0 & 3 \end{bmatrix}$ diagonalizable ? If so, find the similarity transform

We have : $\det(A - \lambda I) = 0$

$$\Rightarrow \det \begin{bmatrix} 2-\lambda & 0 & 0 & 0 \\ 0 & -1-\lambda & 0 & 2 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & -2 & 0 & 3-\lambda \end{bmatrix} = 0$$

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Since the top row has all 0 except 2, the determinant is

$$(2-\lambda).\det\begin{bmatrix} -1-\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ -2 & 0 & 3-\lambda \end{bmatrix} = 0$$

$\Rightarrow (2-\lambda)[(-1-\lambda)(2-\lambda)(3-\lambda) + 2(0+2(2-\lambda))] = 0$

$\Rightarrow (2-\lambda)[(-1-\lambda)(2-\lambda)(3-\lambda)] + 8 - 4\lambda = 0$

$\Rightarrow \lambda^4 - 6\lambda^3 + 13\lambda^2 - 12\lambda + 4 = 0$

$\Rightarrow (\lambda-1)^2(\lambda-2)^2 = 0$

$\Rightarrow \left[\begin{array}{l} \lambda_1 = 1 \text{ (multiplicity : 2)} \\ \lambda_2 = 2 \text{ (multiplicity : 2)} \end{array}\right.$

Since λ_1 and λ_2 has multiplicities greater than one, some of eigenvectors of A are linearly dependent and therefore this matrix is not diagonalizable. To allow diagonalization, the number of eigenvectors must equal the matrix dimensions.

Since A is non-diagonalizable, there doesn't exist a matrix P such that $P^{-1}AP = D \Rightarrow$ no similarity transform for A to become a digagonal matrix. A can still transform with generic P

Exercise 4: Find the eigenvalues and eigenvectors of

a) $A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$ b) $B = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

and form an orthonormal basis using the eigenvectors

a) $A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$

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$\det(A - \lambda I) = 0 \Rightarrow \det \begin{bmatrix} 4-\lambda & 2 & 2 \\ 2 & 1-\lambda & 1 \\ 2 & 1 & 1-\lambda \end{bmatrix} = 0$

$\Rightarrow (4-\lambda)[(1-\lambda)^2 - 1] - 2[2(1-\lambda) - 2]$
$\quad + 2[2 - 2(1-\lambda)] = 0$

$\Rightarrow (4-\lambda)[\lambda^2 - 2\lambda] + 4\lambda + 4\lambda = 0$

$\Rightarrow 4\lambda^2 - 8\lambda - \lambda^3 + 2\lambda^2 + 8\lambda = 0$

$\Rightarrow -\lambda^3 + 6\lambda^2 = 0$

$\Rightarrow \left[\begin{array}{l} \lambda_1 = 0 \\ \lambda_2 = 6 \end{array}\right.$

□ For $\lambda_1 = 0$

$\Rightarrow A - \lambda_1 I = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$. We have $(A - \lambda_1 I) u_1 = 0$

$\Rightarrow \left[\begin{array}{ccc|c} 4 & 2 & 2 & 0 \\ 2 & 1 & 1 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 4 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\Rightarrow 2t_1 + t_2 + t_3 = 0 \Rightarrow u_1 = \left(\frac{t_2 + t_3}{-2}, t_2, t_3\right)$,
$t_2, t_3 \in R$

One specific solution for u_1 : $(-3, 2, 4)^T$

□ For $\lambda_2 = 6$

$\Rightarrow A - \lambda_2 I = \begin{bmatrix} -2 & 2 & 2 \\ 2 & -5 & 1 \\ 2 & 1 & -5 \end{bmatrix}$. We have $(A - \lambda_2 I) u_2 = 0$

$\Rightarrow \left[\begin{array}{ccc|c} -2 & 2 & 2 & 0 \\ 2 & -5 & 1 & 0 \\ 2 & 1 & -5 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

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$$\Rightarrow \begin{cases} s_1 - 2s_3 = 0 \\ s_2 - s_3 = 0 \end{cases} \Rightarrow u_2 = (2s_3, s_3, s_3)^T, s_3 \in R$$

One specific solution for u_2 : $(4, 2, 2)^T$

▫ The eigenvectors are

$$u_1 = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} t_2 + \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix} t_3, \quad u_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} s_3$$

We have $(-1/2\ 1\ 0)^T \perp (2\ 1\ 1)^T$

$(-1/2\ 0\ 1)^T \perp (2\ 1\ 1)^T$

But $(-1/2\ 1\ 0)^T$ is not orthogonal to $(-1/2\ 1\ 0)^T$

We have:

$$\begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} \cdot \left[\begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} t_2 + \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix} t_3 \right] = 0$$

$$\Rightarrow \frac{5}{4} t_2 + \frac{1}{4} t_3 = 0. \text{ Let } t_2 = 1 \Rightarrow t_3 = -5$$

$$\Rightarrow \vec{v} = (-1/2\ 1\ 0)^T (1) + (-1/2\ 0\ 1)^T . (-5) = 0$$

$$\Rightarrow \vec{v} = (2\ 1\ -5)^T$$

$\Rightarrow$ Orthonormal basis using eigen vectors $\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix} \right\}$

Normalization

$$\hat{v}_1 = \frac{|\ 2\ 1\ 1\ |^T}{\sqrt{2^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$\hat{v}_2 = \frac{|-\frac{1}{2}\ 1\ 0\ |^T}{\sqrt{(\frac{1}{2})^2 + 1^2}} = \frac{\sqrt{5}}{2} \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}$$

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$$\hat{v}_3 = \frac{|1 \; 0 \; -5|^T}{\sqrt{1^2 + (-5)^2}} = \frac{1}{\sqrt{26}} \begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix}$$

$\Rightarrow$ Orthonormal basis $= [\hat{v}_1 \; \hat{v}_2 \; \hat{v}_3]$

b)
$$B = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\det(B - \lambda I) = 0 \Rightarrow \det \begin{bmatrix} 1-\lambda & 2 & 0 \\ 2 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (1-\lambda)[(1-\lambda)^2 - 1] - 2[2(1-\lambda)] = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2 - 2\lambda) - 4 + 4\lambda = 0$$

$$\Rightarrow \lambda^2 - 2\lambda - \lambda^3 + 2\lambda^2 - 4 + 4\lambda = 0$$

$$\Rightarrow -\lambda^3 + 3\lambda^2 + 2\lambda - 4 = 0$$

$$\Rightarrow \left[\begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = \sqrt{5} + 1 \\ \lambda_3 = -\sqrt{5} + 1 \end{array}\right.$$

- For $\lambda_1 = 1$

$$\Rightarrow A - \lambda_1 I = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow (A - \lambda_1 I) u_1 = 0$$

$$\Rightarrow \left[\begin{array}{ccc|c} 0 & 2 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{cases} 2t_1 + t_3 = 0 \\ t_2 = 0 \end{cases} \Rightarrow u_1 = (t, 0, -2t)^T, \; t \neq 0$$

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- For $\lambda_2 = \sqrt{5} + 1$

$$\Rightarrow A - \lambda_2 I = \begin{bmatrix} -\sqrt{5} & 2 & 0 \\ 2 & -\sqrt{5} & 1 \\ 0 & 1 & -\sqrt{5} \end{bmatrix} \Rightarrow (A - \lambda_2 I) u_2 = 0$$

$$\Rightarrow \left[\begin{array}{ccc|c} -\sqrt{5} & 2 & 0 & 0 \\ 2 & -\sqrt{5} & 1 & 0 \\ 0 & 1 & -\sqrt{5} & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -\sqrt{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{cases} r_1 - 2r_3 = 0 \\ r_2 - \sqrt{5} r_3 = 0 \end{cases} \Rightarrow u_2 = [2r, \sqrt{5}r, r]^T, r \neq 0$$

- For $\lambda_3 = -\sqrt{5} + 1$

$$\Rightarrow A - \lambda_3 I = \begin{bmatrix} \sqrt{5} & 2 & 0 \\ 2 & \sqrt{5} & 1 \\ 0 & 1 & \sqrt{5} \end{bmatrix} \Rightarrow (A - \lambda_3) u_3 = 0$$

$$\Rightarrow \left[\begin{array}{ccc|c} \sqrt{5} & 2 & 0 & 0 \\ 2 & \sqrt{5} & 1 & 0 \\ 0 & 1 & \sqrt{5} & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & \sqrt{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{cases} s_1 - 2s_3 = 0 \\ s_2 + \sqrt{5} s_3 = 0 \end{cases} \Rightarrow u_3 = [2s, -\sqrt{5}s, s]^T, s \neq 0$$

- The eigen vectors are

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} t \quad u_2 = \begin{bmatrix} 2 \\ \sqrt{5} \\ 1 \end{bmatrix} r \quad u_3 = \begin{bmatrix} 2 \\ -\sqrt{5} \\ 1 \end{bmatrix} s$$

Since $u_1 \perp u_2$, $u_2 \perp u_3$, $u_1 \perp u_3$

$\Rightarrow$ Orthonormal basis using eigen vectors $\Rightarrow \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \begin{bmatrix} 2 \\ \sqrt{5} \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ -\sqrt{5} \\ 1 \end{bmatrix} \right\}$

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Normalization

$$\hat{v}_1 = \frac{|1\ 0\ -2|^T}{\sqrt{1^2 + (-2)^2}} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

$$\hat{v}_2 = \frac{|2\ \sqrt{5}\ 1|^T}{\sqrt{2^2 + 5 + 1^2}} = \frac{1}{\sqrt{10}}\begin{bmatrix} 2 \\ \sqrt{5} \\ 1 \end{bmatrix}$$

$$\hat{v}_3 = \frac{|2\ -\sqrt{5}\ 1|^T}{\sqrt{2^2 + 5 + 1}} = \frac{1}{\sqrt{10}}\begin{vmatrix} 2 \\ -\sqrt{5} \\ 1 \end{vmatrix}$$

$\Rightarrow$ Orthonormal basis $= [\hat{v}_1\ \hat{v}_2\ \hat{v}_3]$