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Thứ

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Final exam Matrix Algebra

Problem 1: Let

$$A = \begin{pmatrix} 6 & 16 & 21 \\ 10 & 0 & -15 \\ 4 & 4 & 14 \end{pmatrix}, \quad b = \begin{pmatrix} 10 \\ -50 \\ 40 \end{pmatrix}, \quad x = \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix}$$

a) We have

$$Ax = \begin{bmatrix} 6 & 16 & 21 \\ 10 & 0 & -15 \\ 4 & 4 & 14 \end{bmatrix} \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix} = \begin{bmatrix} 6.1 + 16(-5) + 21.4 \\ 10.1 + 0.(-5) + (-15).4 \\ 4.1 + 4.(-5) + 14.4 \end{bmatrix}$$

$$= \begin{bmatrix} 10 \\ -50 \\ 40 \end{bmatrix} = b \implies x \text{ is the solution of the equation}$$

$$Ax = b$$

b) We have: $10.x = b$

$\implies Ax = 10x = b \; (Ax = \lambda x)$

$\implies x$ is the eigenvector of matrix A and 10 is its eigenvalue

Problem 2: Solve the linear system $Ax = b$ for all admissible values of $\beta \in \mathbb{R}$, where

$$A = \begin{bmatrix} 2 & 3 & -1 & 4 \\ 3 & -1 & 0 & 1 \\ 1 & -4 & 1 & -2 \\ 2 & -2 & -2 & 5 \end{bmatrix} \qquad b = \begin{bmatrix} 16 \\ -5 \\ -22 \\ \beta \end{bmatrix}$$

▫ The reduced echelon form of A

Nguyen Xuan Binh 887799

$$\begin{bmatrix} 2 & 3 & -1 & 4 \\ 3 & -1 & 0 & 1 \\ 1 & -4 & 1 & -2 \\ 2 & -2 & -2 & 5 \end{bmatrix} \xrightarrow{R_4 = R_4 - R_1} \begin{bmatrix} 2 & 3 & -1 & 4 \\ 3 & -1 & 0 & 1 \\ 1 & -4 & 1 & -2 \\ 0 & -5 & -1 & 1 \end{bmatrix} \xrightarrow{R_2 = 3R_3 - R_2} \begin{bmatrix} 2 & 3 & -1 & 4 \\ 3 & -1 & 0 & 1 \\ 0 & -11 & 3 & -7 \\ 0 & -5 & -1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 = 2R_2 - 3R_1} \begin{bmatrix} 2 & 3 & -1 & 4 \\ 0 & -11 & 3 & -10 \\ 0 & -11 & 3 & -7 \\ 0 & -5 & -1 & 1 \end{bmatrix} \xrightarrow{R_3 = R_3 - R_2} \begin{bmatrix} 2 & 3 & -1 & 4 \\ 0 & -11 & 3 & -10 \\ 0 & 0 & 0 & 3 \\ 0 & -5 & -1 & 1 \end{bmatrix} \xrightarrow{R_1/2,\ R_2/-11,\ R_3/3} \begin{bmatrix} 1 & \frac{3}{2} & \frac{-1}{2} & 2 \\ 0 & 1 & -\frac{3}{11} & -\frac{10}{11} \\ 0 & 0 & 0 & 1 \\ 0 & -5 & -1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1 = R_1 - \frac{3}{2}R_2} \begin{bmatrix} 1 & 0 & \frac{-1}{11} & \frac{7}{11} \\ 0 & 1 & -\frac{3}{11} & \frac{10}{11} \\ 0 & 0 & 0 & 1 \\ 0 & -5 & -1 & 1 \end{bmatrix} \xrightarrow{R_4 = R_4 + 5R_2} \begin{bmatrix} 1 & 0 & \frac{-1}{11} & \frac{7}{11} \\ 0 & 1 & -\frac{3}{11} & \frac{10}{11} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{26}{11} & \frac{61}{11} \end{bmatrix} \xrightarrow{R_4 = R_4 \cdot \frac{-11}{26}} \begin{bmatrix} 1 & 0 & \frac{-1}{11} & \frac{7}{11} \\ 0 & 1 & -\frac{3}{11} & \frac{10}{11} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{-61}{26} \end{bmatrix}$$

$$\xrightarrow{R_1 = R_1 + \frac{1}{11}R_4} \begin{bmatrix} 1 & 0 & 0 & \frac{11}{26} \\ 0 & 1 & \frac{-3}{11} & \frac{10}{11} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{-61}{26} \end{bmatrix} \xrightarrow{R_2 = R_2 + \frac{3}{11}R_4} \begin{bmatrix} 1 & 0 & 0 & \frac{11}{26} \\ 0 & 1 & 0 & \frac{7}{26} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{-61}{26} \end{bmatrix} \xrightarrow{R_1 = R_1 - \frac{11}{26}R_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{7}{26} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{-61}{26} \end{bmatrix}$$

$$\xrightarrow{R_2 = R_2 - \frac{7}{26}R_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -\frac{61}{26} \end{bmatrix} \xrightarrow{R_4 = R_4 + \frac{61}{26}R_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Null space of A: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \ker(A) = [0\ 0\ 0\ 0]^T$ $\Rightarrow$ trivial solution

Nguyen Xuan Binh 887799

Thứ

Ngày

No.

Specific solution for A: $Ax = b$

$$\left[\begin{array}{cccc|c} 2 & 3 & -1 & 4 & 16 \\ 3 & -1 & 0 & 1 & -5 \\ 1 & -4 & 1 & -2 & -22 \\ 2 & -2 & -2 & 5 & \beta \end{array}\right]$$

Applying the rows combination like the rref above, we have

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{3-\beta}{26} \\ 0 & 1 & 0 & 0 & \frac{113-3\beta}{26} \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & -\frac{11\beta+175}{26} \end{array}\right] \Rightarrow \bar{x} = \left[\frac{3-\beta}{26}, \frac{113-3\beta}{26}, -\frac{11\beta+175}{26}, -1\right]^T$$

Since the null space is trivial, the solution to $Ax = \beta$ is

$$x = \left[\frac{3-\beta}{26}, \frac{113-3\beta}{26}, -\frac{11\beta+175}{26}, -1\right]^T \quad \forall \beta \in R$$

Problem 3: a) Find the LU - decomposition of

$$A = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow[\Rightarrow l_{21} = -\frac{1}{4}]{R_2 \to R_2 + \frac{1}{4}R_1} \begin{bmatrix} 4 & -1 & 0 \\ 0 & \frac{11}{4} & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\xrightarrow[l_{31} = 0]{R_3 \to R_3 + 0R_1} \begin{bmatrix} 4 & -1 & 0 \\ 0 & \frac{11}{4} & -1 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow[l_{32} = -\frac{4}{11}]{R_3 \to R_3 + \frac{4}{11}R_2} \begin{bmatrix} 4 & -1 & 0 \\ 0 & \frac{11}{4} & -1 \\ 0 & 0 & \frac{18}{11} \end{bmatrix} = U$$

Nguyen Xuan Binh 887799

Thứ
Ngày
No.

The lower matrix is $\begin{bmatrix} 1 & 0 & 0 \\ \frac{-1}{4} & 1 & 0 \\ 0 & -\frac{4}{11} & 1 \end{bmatrix}$

LU decomposition of A:

$$\underset{A}{\begin{bmatrix} 4 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ \frac{-1}{4} & 1 & 0 \\ 0 & -\frac{4}{11} & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 4 & -1 & 0 \\ 0 & \frac{11}{4} & -1 \\ 0 & 0 & \frac{18}{11} \end{bmatrix}}$$

b) Determinant of A

$$\det\left(\begin{bmatrix} 4 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}\right) = 4.[3.2 - (-1)(-1)] - (-1)[(-1).2 - (-1).0] + 0[(-1)(-1) - 3.0] = 18$$

So $\det(A) = 18$

Problem 4:

a) Show that the vectors $a_1 = (1\ 1\ 0)^T$, $a_2 = (0\ 1\ 1)^T$, $a_3 = (1\ 0\ 1)^T$ form a basis of R^3.

To prove that they form a basis of R^3, the vectors must be linearly independent

Let a, b, c be the scalar for a_1, a_2, a_3

$$a\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a + c \\ a + b \\ b + c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Nguyen Xuan Binh 887799

We have

$$\begin{cases} a + c = 0 \; (1) \\ a + b = 0 \; (2) \\ b + c = 0 \; (3) \end{cases} \quad \begin{matrix} (2) = \\ (2) - (1) \end{matrix} \Rightarrow \begin{cases} a + c = 0 \\ b - c = 0 \\ b + c = 0 \end{cases} \quad \begin{matrix} (3) = \\ (2) + (3) \end{matrix}$$

$$\Rightarrow \begin{cases} a + c = 0 \\ b - c = 0 \\ 2b = 0 \end{cases} \Rightarrow \begin{cases} a = 0 \\ c = 0 \\ b = 0 \end{cases}$$

Since $aa_1 + ba_2 + ca_2 = 0$ only satisfies with the only trivial solution $a = 0, b = 0, c = 0$

$\Rightarrow$ a_1, a_2, a_3 are linearly independent vectors

$\Rightarrow$ They form a basis of R^3

b) Find the coordinates of the vector $x = [1\ 2\ 3]^T$ in this basis

The coordinates of the vector x, t, is the solution to

$$\underset{\text{Basis}}{\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}} \underset{t}{\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}} = \underset{x}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}$$

$$\Rightarrow \begin{cases} t_1 + t_3 = 1 \\ t_1 + t_2 = 2 \\ t_2 + t_3 = 3 \end{cases} \Rightarrow \begin{cases} t_1 = 0 \\ t_2 = 2 \\ t_3 = 1 \end{cases} \Rightarrow t = [0\ 2\ 1]^T$$

$\Rightarrow$ The coordinates of the vector $x\,[1\ 2\ 3]^T$ in basis formed by a_1, a_2 and a_3 is $[0\ 2\ 1]^T$

Nguyen Xuan Binh 887799

Problem 5

a) Find the angle between the eigenvectors of the matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1+t \end{pmatrix}$ as a function of the parameter t

Let x be eigenvectors of A

We have : $Ax = \lambda x \Rightarrow \det(A - \lambda I) = 0$

$$\Rightarrow \det\begin{bmatrix} 1-\lambda & 1 \\ 0 & 1+t-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (1-\lambda)(1+t-\lambda) - 1.0 = 0$$

$$\Rightarrow 1 + t - \lambda - \lambda - \lambda t + \lambda^2 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda - \lambda t + t + 1 = 0$$

$$\Rightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 1 + t \end{cases}$$

- With $\lambda_1 = 1$, we have

$A - \lambda_1 I = \begin{bmatrix} 0 & 1 \\ 0 & t \end{bmatrix}$. Since $(A - \lambda_1 I)x_1 = 0$

$$\Rightarrow \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & t & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1 t} \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sigma \ (\sigma \in R)$$

- With $\lambda_2 = 1 + t$, we have

$A - \lambda_2 I = \begin{bmatrix} -t & 1 \\ 0 & 0 \end{bmatrix}$. Since $(A - \lambda_2 I)x_2 = 0$

$$\Rightarrow \left[\begin{array}{cc|c} -t & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow x_2 = \begin{bmatrix} 1 \\ t \end{bmatrix} \tau \ (\tau \in R)$$

Nguyen Xuan Binh 887799

Thứ

Ngày . . No.

▫ The angle between two eigenvectors :

$$\cos\theta = \frac{\bar{x}_1 . \bar{x}_2}{|\bar{x}_1||\bar{x}_2|} = \frac{1.1 + 0.t}{(\sqrt{1^2+0^2})(\sqrt{1^2+t^2})} = \frac{1}{\sqrt{t^2+1}}$$

$\Rightarrow \theta = \arccos\left(\frac{1}{\sqrt{t^2+1}}\right) \Rightarrow$ function of angle between the eigenvectors of matrix A

b) What is the relation between the angle and the linear independence of the eigenvectors ?

▫ If the eigenvectors are linearly dependent

$\Rightarrow x_2 = kx_1$

$$\Rightarrow \tau\begin{bmatrix}1\\t\end{bmatrix} = k\sigma\begin{bmatrix}1\\0\end{bmatrix} \Rightarrow \begin{cases}\tau = k\sigma\\ \tau t = 0\end{cases}$$

Excluding the trivial solution $\tau = \sigma = 0 \Rightarrow t = 0$

When $t = 0$, the angle between the eigenvectors is

$$\arccos\left(\frac{1}{\sqrt{0^2+1}}\right) = \arccos 1 = 0^\circ$$

$\Rightarrow$ When the eigenvectors are independent, the angle between them will be different from 0 degree, and when they are dependent, the angle between them will be 0 degree