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Linear algebra

Exercise sheet 11

Practice problems

The following problems are just for practice. Their solutions will be shown by the TAs during the exercise sessions. You are welcome to share your own solution during the sessions, if you want.

1. Let $\boldsymbol{a}_1 = [1, 1, 0]^T$ and $\boldsymbol{a}_2 = [1, 0, 1]^T$.

   (a) Find two vectors $\boldsymbol{q}_1, \boldsymbol{q}_2 \in \mathbb{R}^3$ such that $\operatorname{span}(\boldsymbol{q}_1, \boldsymbol{q}_2) = \operatorname{span}(\boldsymbol{a}_1, \boldsymbol{a}_2)$ and such that $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ are orthogonal with respect to the Euclidian inner product.

   (b) Find two vectors $\boldsymbol{q}_1, \boldsymbol{q}_2 \in \mathbb{R}^3$ such that $\operatorname{span}(\boldsymbol{q}_1, \boldsymbol{q}_2) = \operatorname{span}(\boldsymbol{a}_1, \boldsymbol{a}_2)$ and such that $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ are orthogonal with respect to the inner product defined as

$$\left\langle \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \right\rangle = x_1y_1 + 2x_2y_2 + 3x_3y_3.$$

2. Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ -1 & 1 & 0 \end{bmatrix}$$

   and use the Euclidian inner product to measure orthogonality in this poblem.

   (a) Modify the Gram–Schmidt orthogonalisation process so that you can use it to find an orthonormal basis for a possibly linearly dependent set of vectors.

   (b) Find an orthonormal basis for $R(A)$.

Homework

Return the solutions to the following problems on MyCourses by Wednesday, June 2nd, 18:00.

3. Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Compute by hand a QR decomposition of the matrix A and use it to solve the least-squares problem

$$\min_{\boldsymbol{x}\in\mathbb{R}^2} \|A\boldsymbol{x} - \boldsymbol{b}\|_2^2.$$

That is, solve the equation $R\boldsymbol{x} = Q^T\boldsymbol{b}$.

Hint: Problem 1(a).

4. Let

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}.$$

   (a) Find an orthonormal basis for $R(A)$ by using the modified Gram–Schmidt process.

   (b) Find a QR decomposition of A by using the modified Gram–Schmidt process.

Use Matlab and the function `my_gsmith.m` given on page 51 of the lecture notes.