Aalto University

Linear algebra

Exercise sheet 5 / Model solutions

1. (a) Let $A \in \mathbb{R}^{n\times n}$ be an invertible matrix. Show that λ is an eigenvalue of A if and only if λ^{-1} is an eigenvalue of $A^{-1} \in \mathbb{R}^{n\times n}$. *Hint*: Show that each eigenvector of A is also an eigenvector of A^{-1} and vice versa.

   (b) Let $A \in \mathbb{R}^{n\times n}$ be an invertible matrix. Show that the matrices AA^T and A^TA have the same eigenvalues. *Hint*: Show that $\det(AA^T - \lambda I) = \det(A^TA - \lambda I)$.

   (c) Let $A \in \mathbb{R}^{n\times n}$ be an invertible matrix. Show that

$$\|A^{-1}\|_2^2 = \frac{1}{\lambda_{\min}(A^TA)},$$

   where $\lambda_{\min}(A^TA)$ is the smallest eigenvalue of matrix A^TA. *Hint*: Use (a) and (b).

Solution.

(a) Note first that $\lambda \neq 0$ as otherwise A would not be invertible. Suppose $A\boldsymbol{v} = \lambda\boldsymbol{v}$ for a nonzero $\boldsymbol{v}$. Multiplying this equation on the left by $\lambda^{-1}A^{-1}$ we get

$$\lambda^{-1}\boldsymbol{v} = A^{-1}\boldsymbol{v}.$$

Conversely, suppose $A^{-1}\boldsymbol{w} = \lambda^{-1}\boldsymbol{w}$, and multiply on the left by λA to conclude that $\lambda\boldsymbol{w} = A\boldsymbol{w}$.

(b) *Method 1:* Follow the hint which was provided. By direct calculation, we have

$$\begin{aligned}\det(AA^T - \lambda I) &= \det(AA^T - \lambda I)\det(A^{-1}A)\\ &= \det(A^{-1})\det(AA^T - \lambda I)\det(A)\\ &= \det\left(A^{-1}(AA^T - \lambda I)A\right)\\ &= \det(A^TA - \lambda I).\end{aligned}$$

The eigenvalues are the roots of the characteristic polynomial, and this completes the proof. Notice that this proof can be generalized: we can consider any other square matrix B in place of A^T. We only needed A to be invertible for this argument to work.

Method 2: Suppose that for a nonzero $\boldsymbol{v}$ we have $A^TA\boldsymbol{v} = \lambda\boldsymbol{v}$; let also $\boldsymbol{w} = A\boldsymbol{v}$ and note that $\boldsymbol{w}$ is nonzero (else, $\boldsymbol{v} \in N(A)$, which contradicts the assumption that A is invertible). Then, multiplying by A on the left, we have that $\lambda\boldsymbol{w} = AA^T\boldsymbol{w}$. Therefore, if λ is an eigenvalue of A^TA with eigenvector $\boldsymbol{v}$, then it is an eigenvalue of AA^T with eigenvector $A\boldsymbol{v}$. Similarly, one can show that if λ is an eigenvalue of AA^T with eigenvector $\boldsymbol{w}$ then it is eigenvalue of A^TA with eigenvector $A^T\boldsymbol{w}$.

IMPORTANT: This proof is not valid when either A or A^T has a nontrivial null space (for example if A is square but not invertible, or rectangular), because a crucial step fails!

(c) From results in the lecture notes,

$$\|A^{-1}\|_2^2 = \lambda_{\max}((A^{-1})^T A^{-1}).$$

Using item (a) the right hand side is equal to $1/\lambda_{\min}(AA^T)$, which using item (b) is in turn equal to $1/\lambda_{\min}(A^T A)$.

2. (a) Let λ be an eigenvalue of $A \in \mathbb{R}^{n\times n}$. Show that λ^2 is an eigenvalue of the matrix A^2.

(b) Let λ be an eigenvalue of the matrix A^2, with $A \in \mathbb{R}^{n\times n}$. Show that $\sqrt{\lambda}$ or $-\sqrt{\lambda}$ is an eigenvalue of A. (For a complex number λ the notation $\sqrt{\lambda}$ stands for the main branch of the square root.) *Hint*: $\det(A^2 - \lambda I) = \det((A - \sqrt{\lambda}I)(A + \sqrt{\lambda}I)) = \det(A - \sqrt{\lambda}I)\det(A + \sqrt{\lambda}I)$.

(c) Let $A \in \mathbb{R}^{n\times n}$ be symmetric. Show that

$$\|A\|_2 = |\lambda_{\mathrm{amax}}(A)|,$$

where $\lambda_{\mathrm{amax}}(A) \in \mathbb{R}$ is an eigenvalue of A with largest absolute value. *Hint*: Use Lemma 2.5 from the lecture notes and parts (a), (b).

(d) Let $A \in \mathbb{R}^{n\times n}$ be symmetric and invertible. Show that

$$\|A^{-1}\|_2 = \frac{1}{|\lambda_{\mathrm{amin}}(A)|},$$

where $\lambda_{\mathrm{amin}}(A) \in \mathbb{R}$ is an eigenvalue of A with smallest absolute value. *Hint*: Use Problem 1(a) and (c).

Solution.

(a) Assume that $A\boldsymbol{v} = \lambda\boldsymbol{v}$ for some nonzero $\boldsymbol{v}$. Then, $A^2\boldsymbol{v} = A(A\boldsymbol{v}) = \lambda A\boldsymbol{v} = \lambda^2\boldsymbol{v}$.

(b) By the hint, if λ is an eigenvalue of A^2 then we have

$$\det(A - \sqrt{\lambda}I) = 0 \quad \text{or} \quad \det(A + \sqrt{\lambda}I) = 0$$

(or both).

(c) If A is symmetric then $A^T A = A^2$. Hence, by item (a),

$$\|A\|_2^2 = \lambda_{\max}(A^T A) = \lambda_{\mathrm{amax}}(A)^2.$$

Taking square roots the statement follows.

(d) This follows immediately by the hint.

3. Let $A, B \in \mathbb{R}^{n\times n}$ be symmetric. In addition, assume that

$$|\lambda_{\mathrm{amin}}(A)| \geq 2 \qquad \text{and} \qquad |\lambda_{\mathrm{amax}}(B)| \leq 1,$$

where $\lambda_{\mathrm{amin}}(A)$ is an eigenvalue of A with smallest absolute value and $\lambda_{\mathrm{amax}}(B)$ is an eigenvalue of B with largest absolute value. Show that:

(a) $\|A^{-1}\|_2 \le \frac{1}{2}$,

(b) The matrix $A + B$ is invertible.

(c) Taking for granted the formula

$$\|(A+B)^{-1}\|_2 \le \frac{\|A^{-1}\|_2}{1 - \|B\|_2\|A^{-1}\|_2},$$

a solution $\boldsymbol{x}$ to the equation $(A+B)\boldsymbol{x} = \boldsymbol{b}$ satisfies $\|\boldsymbol{x}\|_2 \le \|\boldsymbol{b}\|_2$.

Hints: For (a), use Problem 2(d). For (b), use Theorem 3.1 and Problem 2. For (c), use Problem 2.

Solution.

(a) By Problem 2(b) and by assumption,

$$\|A^{-1}\|_2 = \frac{1}{|\lambda_{\mathrm{amin}}(A)|} \le \frac{1}{2}.$$

(b) Since A is invertible, $A + B$ is also invertible if

$$\|B\|_2 < \underbrace{1/\|A^{-1}\|_2}_{\ge 2}$$

(see the proof of Theorem 3.1). On the other hand, by Problem 2(a), $\|B\|_2 \le |\lambda_{\mathrm{amax}}(B)| \le 1$, so that $A + B$ is invertible.

(c) By the given formula, by Problem 2 and by the assumptions, we get

$$\begin{aligned}\|(A+B)^{-1}\|_2 &\le \frac{\|A^{-1}\|_2}{1 - \|B\|_2\|A^{-1}\|_2} \\ &\le \frac{1/2}{1 - \|B\|_2\|A^{-1}\|_2} \le \frac{1/2}{1/2} = 1,\end{aligned}$$

since $\|B\|_2\|A^{-1}\|_2 \le 1/2$. If we have $(A+B)\boldsymbol{x} = \boldsymbol{b}$, which is equivalent to $\boldsymbol{x} = (A+B)^{-1}\boldsymbol{b}$, then we get

$$\|\boldsymbol{x}\|_2 = \|(A+B)^{-1}\boldsymbol{b}\|_2 \le \underbrace{\|(A+B)^{-1}\|_2}_{\le 1}\|\boldsymbol{b}\|_2 \le \|\boldsymbol{b}\|_2.$$

4. Let

$$A = \begin{bmatrix} 1+\epsilon^2 & 1 \\ 1 & 1+\epsilon^2 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{b} = \begin{bmatrix} 2 \\ 2-\delta \end{bmatrix},$$

where $\epsilon, \delta \in \mathbb{R}$ are free parameters.

(a) Compute $\kappa_2(A) = \|A\|_2\|A^{-1}\|_2$. What happens to the condition number $\kappa_2(A)$ when $\epsilon \to 0$?

(b) Compute $R(A)$ and $N(A)$, when $\epsilon \neq 0$. What about $\epsilon = 0$?

(c) For $\epsilon \neq 0$, solve $A\boldsymbol{x} = \boldsymbol{b}$ using the formula (Cramer's rule)

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

How does $\boldsymbol{x}$ behave, when $\delta = 0$ and $\epsilon \to 0$? What about $\delta \neq 0$ and $\epsilon \to 0$?

Hint: For (a), use Problem 2.

Solution.

(a) By definition, $\kappa_2(A) = \|A\|_2 \|A^{-1}\|_2$. Problem 2 gives easy expressions for $\|A\|_2$ and $\|A^{-1}\|_2$, so our goal is to use those instead of the definition. The eigenvalues of A are quick to compute: we set equal to zero the determinant

$$\begin{vmatrix} 1+\epsilon^2-\lambda & 1 \\ 1 & 1+\epsilon^2-\lambda \end{vmatrix} = (1+\epsilon^2-\lambda)^2 - 1,$$

so that we get $(1+\epsilon^2-\lambda)^2 = 1$, meaning that $1+\epsilon^2-\lambda = \pm 1$. So the two eigenvalues are $\lambda = \epsilon^2$ and $\lambda = \epsilon^2 + 2$. Using the same notation as Problem 2, we then get

$$\kappa_2(A) = \frac{|\lambda_{\text{amax}}(A)|}{|\lambda_{\text{amin}}(A)|} = \frac{\epsilon^2+2}{\epsilon^2}.$$

This goes to $+\infty$ for $\epsilon \to 0$.

(b) When $\epsilon \neq 0$, we have $\det(A) = 2\epsilon^2 + \epsilon^4 > 0$, hence A is invertible (it follows also from part (a)). So then $R(A) = \mathbb{R}^2$ and $N(A) = \{\mathbf{0}\}$. When $\epsilon = 0$, then we easily see that $R(A) = \operatorname{span}\{[1 \;\; 1]^T\}$ and also $N(A) = \operatorname{span}\{[1 \;\; -1]^T\}$.

(c) By Cramer's rule, we get

$$A^{-1} = \frac{1}{2\epsilon^2+\epsilon^4} \begin{bmatrix} 1+\epsilon^2 & -1 \\ -1 & 1+\epsilon^2 \end{bmatrix}.$$

Mutiplying $A\boldsymbol{x} = \boldsymbol{b}$ by A^{-1} on the left, we get

$$\boldsymbol{x} = \frac{1}{2\epsilon^2+\epsilon^4} \begin{bmatrix} 2\epsilon^2+\delta \\ \epsilon^2(2-\delta)-\delta \end{bmatrix}.$$

For $\delta = 0$ and $\epsilon \to 0$, this is

$$\boldsymbol{x} = \begin{bmatrix} \frac{2}{2+\epsilon^2} \\ \frac{2}{2+\epsilon^2} \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

whereas for $\delta \neq 0$ we get

$$\boldsymbol{x} = \begin{bmatrix} \frac{2\epsilon^2+\delta}{2\epsilon^2+\epsilon^4} \\ \frac{\epsilon^2(2-\delta)-\delta}{2\epsilon^2+\epsilon^4} \end{bmatrix} \to \begin{bmatrix} \pm\infty \\ \mp\infty \end{bmatrix},$$

where the sign depends on the sign of δ.