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Exercise 3: Let $\vec{q_1} = [0, 3, 2]^T$ $\vec{q_2} = [5, 1, -1]^T$ and $\vec{q_3} = [2, -2, 2]^T$ and define inner product as $(x, y) = x_1y_1 + 2x_2y_2 + 3x_3y_3$

a) Show that q_1, q_2, q_3 are orthogonal

We have $(\vec{q_1}\ \vec{q_2}) = 0.5 + 2(3.1) + 3(2.-1) = 0$

$(\vec{q_1}\ \vec{q_3}) = 0.2 + 2(3.-2) + 3(2.2) = 0$

$(\vec{q_2}\ \vec{q_3}) = 5.2 + 2(1\cdot(-2)) + 3(-1.2) = 0$

$\Rightarrow q_1, q_2, q_3$ are orthogonal with respect to inner product defined above

b) Represent the vectors $\vec{v} = [1, 2, 3]^T$ and $\vec{w} = [4, -2, 1]^T$ as a linear combination of the vectors q_1, q_2, q_3

- For $\vec{v}$:

$$\left[\begin{array}{ccc|c} 0 & 5 & 2 & 1 \\ 3 & 1 & -2 & 2 \\ 2 & -1 & 2 & 3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 3 & 1 & -2 & 2 \\ 0 & 5 & 2 & 1 \\ 2 & -1 & 2 & 3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ 0 & 5 & 2 & 1 \\ 2 & -1 & 2 & 3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ 0 & 1 & \frac{2}{5} & \frac{1}{5} \\ 0 & -\frac{2}{3} & \frac{10}{3} & \frac{5}{3} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ 0 & 1 & \frac{2}{5} & \frac{1}{5} \\ 0 & 0 & \frac{27}{5} & \frac{27}{10} \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1/3 & -2/3 & 2/3 \\ 0 & 1 & 2/5 & 1/5 \\ 0 & 0 & 1 & 1/2 \end{array}\right] \Rightarrow \bar{x} = \begin{bmatrix} 1 \\ 0 \\ 1/2 \end{bmatrix}$$

Null space:

$$\left[\begin{array}{ccc|c} 0 & 5 & 2 & 0 \\ 3 & 1 & -2 & 0 \\ 2 & -1 & 2 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow \text{trivial null-space}$$

$$\Rightarrow \vec{v} = \vec{q_1} + \frac{1}{2}\vec{q_3}$$

- For $\vec{w}$

$$\left[\begin{array}{ccc|c} 0 & 5 & 2 & 4 \\ 3 & 1 & -2 & -2 \\ 2 & -1 & 2 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{1}{5} \\ 0 & 1 & 0 & \frac{13}{50} \\ 0 & 0 & 1 & 11/12 \end{array}\right] \Rightarrow \bar{x} = \begin{bmatrix} -\frac{1}{5} \\ 13/50 \\ 11/12 \end{bmatrix}$$

Null space is trivial $\Rightarrow \vec{w} = -\frac{1}{5}\vec{q_1} + \frac{13}{50}\vec{q_2} + \frac{11}{12}\vec{q_3}$

Exercise 4: Let $x, y \in R^n$. Show that

a) $\|x\|_\infty \leqslant \|x\|_2 \leqslant \|x\|_1$

- We have: $\|x\|_\infty = \max\{|x_i|\} \Rightarrow \|x\|_\infty^2 = (\max\{|x_i|\})^2$

  $\Rightarrow \|x\|_\infty^2 \leqslant \sum_{i=1}^n |x_i|^2$ (1) (Range of x_i include the max and smaller components and they are always positive)

We have: $\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2} = \sqrt{\sum_{i=1}^n |x_i|^2}$

$\Rightarrow \|x\|_2^2 = \sum_{i=1}^n |x_i|^2$ (2)

From (1) and (2) $\Rightarrow \|x\|_\infty^2 \leqslant \|x\|_2^2 \Rightarrow \|x\|_\infty \leqslant \|x\|_2$ (proven)

- We have: $\|x\|_1 = \underline{\sum_{i=1}^n |x_i|} \Rightarrow \|x\|_1^2 = (|x_1| + |x_2| + \dots + |x_n|)^2$

  $\|x\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2} \Rightarrow \|x\|_2^2 = |x_1|^2 + |x_2|^2 + \dots + |x_n|^2$

We know that $(a+b)^2 = a^2 + b^2 + 2ab \geqslant a^2 + b^2$ $(a, b \geqslant 0)$ and it is true for further elements: $(a + b + \dots)^2 \geqslant a^2 + b^2 + \dots$

$\Rightarrow (|x_1| + |x_2| + \dots + |x_n|)^2 \geqslant |x_1|^2 + |x_2|^2 + \dots + |x_n|^2$

$\Rightarrow \|x\|_1 \geqslant \|x_2\|$ (proven) $\Rightarrow \|x\|_\infty \leqslant \|x\|_2 \leqslant \|x\|_1$

b) $\|x\|_1 \leqslant \sqrt{n}\,\|x\|_2 \leqslant n\,\|x\|_\infty$

- We have: $\|x\|_1 = \sum_{i=1}^n |x_1| = \sum_{i=1}^n |x_i| \cdot 1$. Applying Cauchy-Schwarz

$\|x\|_1 = \sum_{i=1}^n |x_i| \cdot 1 \leqslant \left(\sum_{i=i} |x_i|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^n 1^2\right)^{\frac{1}{2}} = \sqrt{n}\,\|x\|_2$ (1)

- We have $\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2} \leqslant \sqrt{\sum_{i=1}^n \|x\|_\infty^2}$ (Since $\|x\|_\infty$ is $\max(|x_i|)$)

  $= \sqrt{n\|x\|^2_\infty} = \sqrt{n}\,\|x\|_\infty$

  $\Rightarrow \sqrt{n}\,\|x\|_2 \leqslant n\|x\|_\infty$ (2)

From (1) (2) $\Rightarrow \|x\|_1 \leqslant \sqrt{n}\,\|x\|_2 \leqslant n\,\|x\|_\infty$ (proven)

c) $|x \cdot y| \leqslant \|x\|_\infty \|y\|_1$

We have: $\|x\|_\infty \|y\|_1 = \max\{|x_i|\} \sum_{i=1}^n |y_i|$

$= \max\{|x_i|\}|y_1| + \max\{|x_i|\}|y_2| + \dots + \max\{|x_i|\}|y_n|$

$|x \cdot y| = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$

Since $|x_i|$ and $|y_i|$ are always positive and $\max\{|x_i|\}$ is maximum positive value, it is guaranteed that $\max\{|x_i|\}|y_1| \geqslant x_1 y_1$, $\max\{|x_i|\}|y_2| \geqslant x_2 y_2$, ...

$\Rightarrow$ We can conclude that $|x \cdot y| \leqslant \|x\|_\infty \|y\|_1$