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Linear algebra
Exercise sheet 12

Practice problem

The following problem is just for practice. Its solution will be shown by the TAs during the exercise sessions. You are welcome to share your own solution during the sessions, if you want.

1. Let $A \in \mathbb{R}^{m\times n}$ have singular value decomposition $A = U\Sigma V^T$, where $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ are orthogonal (i.e., $U^T U = I \in \mathbb{R}^{m\times m}$ and $V^T V = I \in \mathbb{R}^{n\times n}$) and $\Sigma \in \mathbb{R}^{m\times n}$ is the diagonal matrix such that $\Sigma_{ii} = \sigma_i$, where σ_i are the singular values of A. The Moore–Penrose pseudoinverse of A is defined as

$$A^\dagger = V\Sigma^\dagger U^T \in \mathbb{R}^{n\times m},$$

where $\Sigma^\dagger \in \mathbb{R}^{n\times m}$ is the diagonal matrix with entries

$$\Sigma^\dagger_{ii} = \begin{cases} \sigma_i^{-1} & \text{when } \sigma_i > 0, \\ 0 & \text{when } \sigma_i = 0. \end{cases}$$

   (a) Show that $P = AA^\dagger \in \mathbb{R}^{m\times m}$ is an orthogonal projection to the subspace $R(A)$.

   (b) Show that $\boldsymbol{x}^\dagger = A^\dagger \boldsymbol{b} \in \mathbb{R}^n$ is a solution to the equation

$$A\boldsymbol{x} = P\boldsymbol{b}.$$

   (c) Show that $\boldsymbol{x}^\dagger$ is a solution to the least-squares problem

$$\min_{\boldsymbol{x}\in\mathbb{R}^n} \|A\boldsymbol{x} - \boldsymbol{b}\|_2^2.$$

Homework

Return the solution to the following problem on MyCourses by Wednesday, the 2nd of June, 18:00.

2. Let $A \in \mathbb{R}^{n\times n}$ have singular values $\sigma_1, \dots, \sigma_n$ such that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$. Show that:

   (a) $\|A\|_2 = \sigma_1$,

   (b) $\|A^{-1}\|_2 = \frac{1}{\sigma_n}$ (when A is invertible),

   (c) $\kappa_2(A) = \frac{\sigma_1}{\sigma_n}$ (when A is invertible).

Good luck for the exam and enjoy your summer!