Nguyen Xuan Binh 887799 Exercise sheet 1

Exercise 3: Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & -2 & 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ -3 \\ 8 \end{bmatrix}$

Compute a basis for the null space $N(A)$ by hand and find all solutions to $Ax = b$

A in reduced echelon form:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

=) Null space of A

$$A.\vec{n} = 0 \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} n_1 - n_3 = 0 \\ n_2 - n_3 = 0 \end{cases} \Rightarrow N(A) = s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Solve $Ax = b$

Augmented

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 8 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & -3 \\ 0 & -2 & 2 & 8 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & -4 \\ 0 & -2 & 2 & 8 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & -1 & -3 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} x_1 - x_3 = -3 \\ x_2 - x_3 = -4 \end{cases} \Rightarrow \bar{x} = \begin{bmatrix} t-3 \\ t-4 \\ t \end{bmatrix}$$

$\Rightarrow$ All solutions to equation $Ax = b$ is

$$\bar{x} + N(A) = \begin{bmatrix} t-3 \\ t-4 \\ t \end{bmatrix} + s\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad t, s \in \mathbb{R}$$

Exercise 4: Let $0 \neq (a, b) \in \mathbb{R}^n$ and defind $ab^T \in \mathbb{R}^{n \times n}$. Show that

a) $R(A) = \text{span}\{a\}$ with $A = ab^T$

Let $a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ $\Rightarrow A = ab^T = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}\begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}$

$$\Rightarrow A = \xcancel{\begin{bmatrix} a_1b_1 & a_2b_1 & \cdots & a_{n-1}b_1 & a_nb_1 \\ a_1b_2 & a_2b_2 & \cdots & a_{n-1}b_2 & a_nb_2 \\ \vdots & & & & \\ a_1b_{n-1} & a_2b_{n-1} & \cdots & a_{n-1}b_{n-1} & a_nb_{n-1} \\ a_1b_n & a_2b_n & \cdots & a_{n-1}b_n & a_nb_n \end{bmatrix}} = \begin{bmatrix} a_1b_1 & a_1b_2 & \cdots & a_1b_{n-1} & a_1b_n \\ a_2b_1 & a_2b_2 & & a_2b_{n-1} & a_2b_n \\ \vdots & \vdots & & & \vdots \\ a_{n-1}b_1 & a_{n-1}b_2 & \cdots & a_{n-1}b_{n-1} & a_{n-1}b_n \\ a_nb_1 & a_nb_2 & \cdots & a_nb_{n-1} & a_nb_n \end{bmatrix}$$

We have: $\text{span}\{a\} = \alpha a$ with $\alpha \in \mathbb{R}^n$

We have $R(A) = \{y \in \mathbb{R}^n \mid y = Ax \text{ for } x \in \mathbb{R}^n\} = \text{span}(a_1, \ldots a_n)$

$= x_1(a_1b_1 \;\; a_2b_2 \;\; \cdots \;\; a_{n-1}b_1 \;\; a_nb_1)^T + \cdots$

$+ x_n(a_1b_n \;\; a_2b_n \;\; \cdots \;\; a_{n-1}b_n \;\; a_nb_n)^T$

$= x_1b_1a + x_2b_2a + \cdots + x_nb_na$

$= (x_1b_1 + x_2b_2 + \cdots + x_nb_n)a = \alpha a = \text{span}\{a\}$ (proven)

b) $N(A) = \{x \in \mathbb{R}^n \mid b^Tx = 0\}$

We have $N(A) = x$, where $Ax = 0$

$$Ax = 0 \Rightarrow \begin{bmatrix} a_1b_1 & a_1b_2 & \cdots & a_1b_{n-1} & a_1b_n \\ a_2b_1 & a_2b_2 & & a_2b_{n-1} & a_2b_n \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n-1}b_1 & a_{n-1}b_2 & \cdots & a_{n-1}b_{n-1} & a_{n-1}b_n \\ a_nb_1 & a_nb_2 & \cdots & a_nb_{n-1} & a_nb_n \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} a_1b_1x_1 + a_1b_2x_2 + \cdots + a_1b_{n-1}x_{n-1} + a_1b_nx_n = 0 \\ a_2b_1x_1 + a_2b_2x_2 + \cdots + a_2b_{n-1}x_{n-1} + a_2b_nx_n = 0 \\ \cdots \\ a_{n-1}b_1x_1 + a_{n-1}b_2x_2 + \cdots + a_{n-1}b_{n-1}x_{n-1} + a_{n-1}b_nx_n = 0 \\ a_nb_1x_1 + a_nb_2x_2 + \cdots + a_nb_{n-1}x_{n-1} + a_nb_nx_n = 0 \end{cases}$$

$$\Rightarrow \begin{cases} a_1(b_1x_1 + \cdots b_nx_n) = 0 \\ a_2(b_1x_1 + \cdots b_nx_n) = 0 \\ \vdots \\ a_n(b_1x_1 + \cdots b_nx_n) = 0 \end{cases} \text{Since } a \neq 0 \Rightarrow \begin{cases} b_1x_1 + \cdots + b_nx_n = 0 \\ b_1x_1 + \cdots + b_nx_n = 0 \\ \cdots \\ b_1x_1 + \cdots b_nx_n = 0 \end{cases} \Rightarrow b^Tx = 0$$

with $x \in \mathbb{R}^n$ (proven)