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Problem Sheet 2

Exercise 1: Find the decomposition $A = LU$, given

$$A = \begin{pmatrix} -1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix}$$

We have:

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}}_{L} \times \overbrace{\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}}^{U}$$

$$A = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{matrix} R_2 \to R_2 + R_1 \\ \Rightarrow l_{21} = -1 \end{matrix} = \begin{bmatrix} -1 & 1 & -1 \\ 0 & 2 & 0 \\ -1 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & -1 \\ 0 & 2 & 0 \\ -1 & 1 & 2 \end{bmatrix} \begin{matrix} R_3 \to R_3 - R_1 \\ \Rightarrow l_{31} = 1 \end{matrix} = \begin{bmatrix} -1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{matrix} R_3 \to R_3 + 0R_2 \\ \Rightarrow l_{32} = 0 \end{matrix} = \begin{bmatrix} -1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = U$$

We have: $l_{21} = -1$, $l_{31} = 1$, $l_{32} = 0$

$$\Rightarrow L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

The decomposition $A = LU$ is

$$\overset{A}{\begin{bmatrix} -1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix}} = \overset{L}{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}} \overset{U}{\begin{bmatrix} -1 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}}$$

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Exercise 2: Find all decompositions $PA = LU$ of

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & c & 1 \\ 0 & 1 & 1 \end{bmatrix} \text{ for all } c \in R$$

For the decompositions $PA = LU$ to be defined, the pivots of U must be defined and not equal to 0

- Apply Gaussian elimination to A to find U

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & c & 1 \\ 0 & 1 & 1 \end{bmatrix} R_2 - 3R_1 \rightarrow R_2 \Rightarrow E_{21} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & (c-6) & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow l_{21} = 3$$

$$E_{21} \begin{bmatrix} 1 & 2 & 0 \\ 0 & (c-6) & 1 \\ 0 & 1 & 1 \end{bmatrix} R_3 \rightarrow R_3 - \frac{1}{c-6} R_2 \Rightarrow E_{32} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & (c-6) & 1 \\ 0 & 0 & \frac{c-7}{c-6} \end{bmatrix}$$

$$\Rightarrow l_{32} = \frac{1}{c-6}$$

$$\Rightarrow U = \begin{bmatrix} 1 & 2 & 0 \\ 0 & (c-6) & 1 \\ 0 & 0 & \frac{c-7}{c-6} \end{bmatrix} \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & \frac{1}{c-6} & 1 \end{bmatrix}$$

The pivot of U have $c - 6$ & $\frac{c-7}{c-6}$

$$\Rightarrow \begin{cases} c \neq 6 \\ c \neq 7 \end{cases} \quad (\text{For this case, } P = I)$$

To overcome this, we swap row 2 and row 3

$$P_{23} A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 3 & c & 1 \end{bmatrix} \quad R_3 \rightarrow R_3 - 3R_1 \Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & (c-6) & 1 \end{bmatrix}$$

$$l_{31} = 3$$

$$\Rightarrow R_3 \rightarrow R_3 - (c-6) R_2 \Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & (7-c) \end{bmatrix} = U$$

$$l_{32} = c - 6$$

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$$u = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & (7-c) \end{bmatrix} \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & (6-c) & 1 \end{bmatrix}$$

We have : $7 - c \neq 0 \Rightarrow c \neq 7$

At $c = 6 \Rightarrow$

$$u = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

Since the rows can't be swapped anymore, we come to conclusion : $IA = LU$ $(c \neq 6,\ c \neq 7)$

$P_{23}A = LU$ (with $c = 6$)

Verify : if $c = 7$

$$\Rightarrow \det \begin{bmatrix} 1 & 2 & 0 \\ 3 & 7 & 1 \\ 0 & 1 & 1 \end{bmatrix} = 0$$

$\Rightarrow$ with $c = 7$, A can't be decomposed into $PA = LU$

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Exercise 3: Choose any vector x and set $r = x^T x$ and $u = \frac{x}{\sqrt{r}}$. Form $H = I - 2uu^T$ and its inverse H^{-1}. Is H either symmetric or orthogonal or both? Furthermore, show that $HH = I$ independent of x & give a geometric interpretation of the related mapping.

- We have: $H^T = (I - 2uu^T)^T$

$$= I^T - 2(uu^T)^T$$
$$= I - 2(u^T)^T(u^T)$$
$$= I - 2uu^T = H$$

Since $H^T = H \Rightarrow H$ is symmetric independent of x

- As $H = H^T \Rightarrow H^T H = HH$

$$= (I - 2uu^T)(I - 2uu^T)$$

$$\Rightarrow H^T H = I.I - I.2uu^T - 2uu^T.I + 4(uu^T)(uu^T)$$
$$H^T H = I - 2uu^T - 2uu^T + 4(uu^T)(uu^T) \quad (1)$$

- We have: $r^T = (x^T x)^T$

$$= x^T (x^T)^T = x^T x = r$$

Since $r^T = r \Rightarrow r$ is symmetric independent of x

If r is symmetric $\Rightarrow \sqrt{r}$ is also symmetric as well

We have: $u^T u = \left(\frac{x}{\sqrt{r}}\right)^T \left(\frac{x}{\sqrt{r}}\right)$

$$= (x(\sqrt{r})^{-1})^T (x(\sqrt{r})^{-1})$$
$$= ((\sqrt{r})^{-1})^T x^T x (\sqrt{r})^{-1}$$
$$= ((\sqrt{r})^T)^{-1} . r . (\sqrt{r})^{-1}$$
$$= (\sqrt{r})^{-1} . r (\sqrt{r})^{-1}$$
$$= (\sqrt{r})^{-1} \sqrt{r} . \sqrt{r} . (\sqrt{r})^{-1}$$
$$= I . I = I$$

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- From (1): $H^T H = HH = I - 4uu^T + 4u(u^T u)u^T$
$$= I - 4uu^T + 4u.\,I.\,u^T$$
$$= I - 4uu^T + 4uu^T = I$$

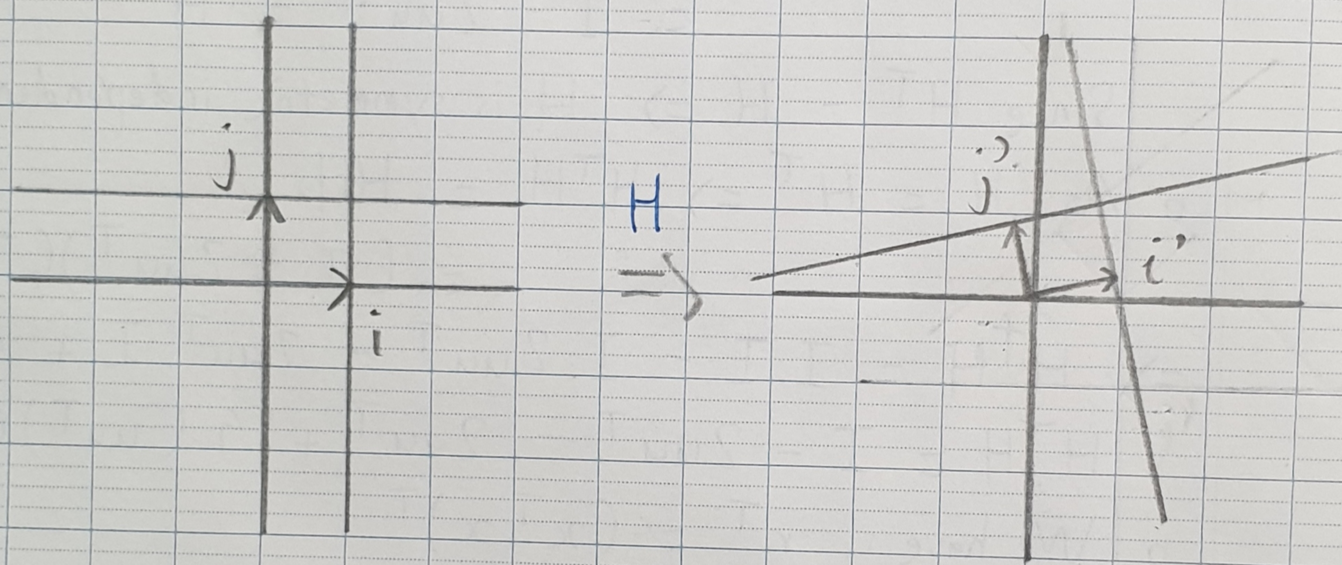
Since $H^T H = I \Rightarrow$ H is orthogonal ($H^T = H^{-1}$)

With all vectors x, HH is always equal to I

Conclusion: H is both symmetric and orthogonal

$HH = I$ is independent of x

- Geometric interpretation of orthogonal matrix H



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Exercise 4: Find $M.M^{-1}$

4) $M = A - UW^{-1}V$

$M^{-1} = A^{-1} + A^{-1}U(W - VA^{-1}U)^{-1}VA^{-1}$

$MM^{-1} = (A - UW^{-1}V)[A^{-1} + A^{-1}U(W - VA^{-1}U)^{-1}VA^{-1}]$

$= A.A^{-1} + AA^{-1}U(W - VA^{-1}U)^{-1}VA^{-1}$
$- UW^{-1}VA^{-1} - UW^{-1}VA^{-1}U(W - VA^{-1}U)^{-1}VA^{-1}$

$= I + U(W - VA^{-1}U)^{-1}VA^{-1} - UW^{-1}VA^{-1}$
$- UW^{-1}VA^{-1}U(W - VA^{-1}U)^{-1}VA^{-1}$

We have: $VA^{-1}U(W - VA^{-1}U)^{-1}$

$= (W + VA^{-1}U - W)(W - VA^{-1}U)^{-1}$

$= W(W - VA^{-1}U)^{-1} + (VA^{-1}U - W)(W - VA^{-1}U)^{-1}$

$= W(W - VA^{-1}U)^{-1} - (W - VA^{-1}U)(W - VA^{-1}U)^{-1}$

$= W(W - VA^{-1}U)^{-1} - I$

$\Rightarrow MM^{-1} = I + U(W - VA^{-1}U)^{-1}VA^{-1} - UW^{-1}VA^{-1}$
$- UW^{-1}[W(W - VA^{-1}U)^{-1} - I]VA^{-1}$

$= I + U(W - VA^{-1}U)^{-1}VA^{-1} - UW^{-1}VA^{-1}$
$- UW^{-1}[W(W - VA^{-1}U)^{-1}VA^{-1} - VA^{-1}]$

$= I + U(W - VA^{-1}U)^{-1}VA^{-1} - UW^{-1}VA^{-1}$
$- U(W - VA^{-1}U)^{-1}VA^{-1} + UW^{-1}VA^{-1}$

$= I$

$\Rightarrow$ This will be the form to solve / prove (1)(2)(3)

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3) $M = I - UV \quad M^{-1} = I_n + U(I_m - VU)^{-1}V$

Since (4) is already proven true, we can substitute identity matrix into (4)

$M = A - UW^{-1}V = I - UV$

$\Rightarrow A = I_n,\ W = I_m$

$\Rightarrow M^{-1} = (I_n)^{-1} + (I_n)^{-1}U(I - V(I_m)^{-1}U)^{-1}V(I_n)^{-1}$

$= I_n + U(I_m - VU^{-1})V$ (Correct)

$\Rightarrow M.M^{-1} = I$

2) $M = A - uv^T \quad M^{-1} = A^{-1} + A^{-1}uv^TA^{-1}/(1 - v^TA^{-1}u)$

We have $M = A - uv^T = A - UW^{-1}V$

$\Rightarrow U = u,\ V = v^T,\ W = 1$

$\Rightarrow M^{-1} = A^{-1} + A^{-1}u(1 - v^TA^{-1}u)^{-1}v^TA^{-1}$

$= A^{-1} + \dfrac{A^{-1}u}{(1 - v^TA^{-1}u)} \cdot v^TA^{-1}$

$= A^{-1} + \dfrac{A^{-1}uv^TA^{-1}}{(1 - v^TA^{-1}u)}$ (Correct)

$\Rightarrow M.M^{-1} = I$

1) $M = I - uv^T$ and $M^{-1} = I + uv^T/(1 - v^Tu)$

We have $M = I - uv^T = A - UW^{-1}V$

$\Rightarrow A = I,\ U = u,\ V = v^T,\ W = 1$

$\Rightarrow M^{-1} = (I)^{-1} + (I)^{-1}u(1 - v^T(I)^{-1}u)^{-1}v^T(I)^{-1}$

$= I + u(1 - v^Tu)^{-1}v^T$

$= I + \dfrac{u}{(1 - v^Tu)}v^T = I + \dfrac{uv^T}{(1 - v^Tu)}$ (Correct)

$\Rightarrow M.M^{-1} = I$

The notation M^{-1} is justified in all cases

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