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Linear algebra
Exercise sheet 3

Practice problems

The following problems are just for practice. Their solutions will be shown by the TAs during the exercise sessions. You are welcome to share your own solution during the sessions, if you want.

1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that

$$f(\boldsymbol{x}) = \left(\boldsymbol{x}^T A \boldsymbol{x}\right)^{1/2} \quad \text{where} \quad A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

The matrix A can be decomposed as $A = U^T \Lambda U$, where

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

   (a) Show that $\boldsymbol{y}^T \Lambda \boldsymbol{y} \geq \boldsymbol{y}^T \boldsymbol{y}$ for all $\boldsymbol{y} \in \mathbb{R}^2$ and $\boldsymbol{x}^T A \boldsymbol{x} \geq \boldsymbol{x}^T \boldsymbol{x}$ for all $\boldsymbol{x} \in \mathbb{R}^2$. (*Hint:* Write $A = U^T \Lambda U$ and set $\boldsymbol{y} = U\boldsymbol{x}$.)

   (b) Show that $\boldsymbol{x}^T A \boldsymbol{y} \leq \left(\boldsymbol{x}^T A \boldsymbol{x}\right)^{1/2} \left(\boldsymbol{y}^T A \boldsymbol{y}\right)^{1/2}$. (*Hint:* Write $\boldsymbol{x}^T A \boldsymbol{y} = (\Lambda^{1/2} U \boldsymbol{x})^T (\Lambda^{1/2} U \boldsymbol{y})$ and use the Cauchy–Schwarz inequality: $|\boldsymbol{x}^T \boldsymbol{y}| \leq \|\boldsymbol{x}\|_2 \|\boldsymbol{y}\|_2$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$.)

   (c) Show that f satisfies the triangle inequality : $f(\boldsymbol{x} + \boldsymbol{y}) \leq f(\boldsymbol{x}) + f(\boldsymbol{y})$. Is f a norm?

2. Let $\boldsymbol{q}_1 = [0, 1, 1]^T$, $\boldsymbol{q}_2 = [1, 1, -1]^T$ and $\boldsymbol{q}_3 = [2, -1, 1]^T$.

   (a) Show that $\boldsymbol{q}_1$, $\boldsymbol{q}_2$ and $\boldsymbol{q}_3$ are mutually orthogonal with respect to the Euclidian inner product.

   (b) Represent the vectors $\boldsymbol{v} = [1, 2, 3]^T$ and $\boldsymbol{w} = [4, -2, 1]^T$ as a linear combination of the vectors $\boldsymbol{q}_1$, $\boldsymbol{q}_2$ and $\boldsymbol{q}_3$. (*Hint:* Write $\boldsymbol{v} = \alpha_1 \boldsymbol{q}_1 + \alpha_2 \boldsymbol{q}_2 + \alpha_3 \boldsymbol{q}_3$. Use the orthogonality to find the coefficients $\alpha_1, \alpha_2, \alpha_3$.)

Homework

Return the solutions to the following problems on MyCourses by Friday, May 7th, 18:00.

3. Let $\boldsymbol{q}_1 = [0, 3, 2]^T$, $\boldsymbol{q}_2 = [5, 1, -1]^T$ and $\boldsymbol{q}_3 = [2, -2, 2]^T$ and define the inner product

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = x_1 y_1 + 2 x_2 y_2 + 3 x_3 y_3. \tag{1}$$

(a) Show that $\boldsymbol{q}_1$, $\boldsymbol{q}_2$ and $\boldsymbol{q}_3$ are mutually orthogonal with respect to the inner product defined in (1).

(b) Represent the vectors $\boldsymbol{v} = [1, 2, 3]^T$ and $\boldsymbol{w} = [4, -2, 1]^T$ as a linear combination of the vectors $\boldsymbol{q}_1$, $\boldsymbol{q}_2$ and $\boldsymbol{q}_3$.

4. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. Show that:

   (a) $\|\boldsymbol{x}\|_\infty \leq \|\boldsymbol{x}\|_2 \leq \|\boldsymbol{x}\|_1$,

   (b) $\|\boldsymbol{x}\|_1 \leq \sqrt{n}\|\boldsymbol{x}\|_2 \leq n\|\boldsymbol{x}\|_\infty$,

   (c) $|\boldsymbol{x} \cdot \boldsymbol{y}| \leq \|\boldsymbol{x}\|_\infty \|\boldsymbol{y}\|_1$.

   (*Hint:* (b) Write $\|\boldsymbol{x}\|_1$ as an inner product between $\boldsymbol{x}$ and a vector containing elements -1 and 1. Apply the Cauchy–Schwarz inequality.)