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Linear algebra
Exercise sheet 6

Practice problems

The following problems are just for practice. Their solutions will be shown by the TAs during the exercise sessions. You are welcome to share your own solution during the sessions, if you want.

1. Consider the matrices $A, M \in \mathbb{C}^{n\times n}$ and the *generalized eigenvalue problem*: find a pair $(\lambda, \boldsymbol{x}) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{\mathbf{0}\})$ such that

$$A\boldsymbol{x} = \lambda M\boldsymbol{x}. \tag{1}$$

   (a) Derive a polynomial equation that defines the *generalized eigenvalues* λ. How can the *generalized eigenvector* $\boldsymbol{x}$ corresponding to λ be defined?

   (b) Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \text{and} \qquad M = I.$$

   Find all solutions to (1) in this case.

   (c) Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$

   Find all solutions to (1) in this case.

2. Let

$$C = \begin{bmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{bmatrix}.$$

   (a) Compute the characteristic polynomial of C.

   (b) Consider the polynomial

$$p(x) = 2x^3 + 4x^2 + 6x + 8.$$

   Using (a) find an eigenvalue problem that can be used to compute the roots of p.

Homework

Return the solutions to the following problems on MyCourses by Friday, May 14th, 18:00.

3. Compute the eigenvalues and eigenvectors for the matrices

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 1 & -\sqrt{2} & 3 \\ 0 & -2 & 0 \\ 3 & \sqrt{2} & 1 \end{bmatrix}.$$

In both cases, find the characteristic polynomial and determine the geometric and algebraic multiplicity for each eigenvalue.

Hint: Use the sub-determinat rule with respect to the middle row or column.

4. The Fibonacci sequence $(F_n)_{n=1,2,\ldots}$ is defined recursively as

$$F_n = \begin{cases} 1, & n = 1, \\ 1, & n = 2, \\ F_{n-1} + F_{n-2}, & n > 2. \end{cases}$$

Let $\boldsymbol{x}_n = [F_n, F_{n-1}]^T$, for $n = 2, 3, \ldots$

(a) Find a matrix $A \in \mathbb{R}^{2\times 2}$ such that

$$\boldsymbol{x}_n = A\boldsymbol{x}_{n-1}, \qquad n > 2.$$

(b) Let $X \in \mathbb{R}^{2\times 2}$ and the diagonal matrix $\Lambda \in \mathbb{R}^{2\times 2}$ be such that $A = X\Lambda X^{-1}$. Show that

$$\boldsymbol{x}_n = X\Lambda^{n-2}X^{-1}\boldsymbol{x}_2, \qquad n \geq 2.$$

Compute X and Λ in Matlab using `[X,L] = eig(A)`. Compute also F_{10}, F_{20} and F_{30}.