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Exercise 3: Consider the system of differential equations

$x'(t) = Ax(t)$ for $t > 0$, $x(0) = [1\ 0]^T$ where $A \in C^{2\times 2}$ and $x: R_+ \to C^2$. Solve by using matix exponential, when

a) $A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$

$\Rightarrow$ The solution is $x(t) = e^{At}x(0)$

$A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = XJX^{-1}$

$\Rightarrow x(t) = e^{At}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = Xe^{Jt}X^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\Rightarrow x(t) = \begin{bmatrix} e^{2t}t + e^{2t} \\ te^{2t} \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$

Find eigenvalues: $\det(A - \lambda I) = 0 \Rightarrow \det\begin{bmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{bmatrix} = 0$

$\Rightarrow (1-\lambda)^2 - 9 = 0 \Rightarrow (\lambda+2)(\lambda-4) = 0 \Rightarrow \lambda_1 = -2 \quad \lambda_2 = 4$

Find eigenvectors:

- $\lambda_1 = -2 \Rightarrow (A - \lambda_1 I)\cdot v = 0 \Rightarrow \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} v = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v = \text{span}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$

- $\lambda_2 = 4 \Rightarrow (A - \lambda_2 I)v = 0 \Rightarrow \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} v = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

Diagonalizing A: $A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}^{-1}$

$\Rightarrow x(t) = e^{At} x(0) = X e^{\Lambda t} X^{-1} x(0)$

$= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{4t} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}^{T} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\Rightarrow x(t) = \frac{1}{2} \begin{bmatrix} e^{-2t} + e^{4t} \\ -e^{-2t} + e^{4t} \end{bmatrix}$

Exercise 4: Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1+\epsilon \end{bmatrix}$

a) For which values of ϵ is the matrix diagonalizable?

We have: $\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 \\ 0 & 1+\epsilon-\lambda \end{bmatrix} = (1-\lambda)(1+\epsilon-\lambda) = 0$

We notice that when $\epsilon \neq 0$, there are two distinct eigenvalues that has algebraic multiplicity of 1 $\Rightarrow$ For all $\epsilon \neq 0$, the matrix is diagonalizable

Now let $\epsilon = 0$ to check the multipliaties: $(1-\lambda)(1-\lambda) = 0$

$\Rightarrow \lambda = 1$ has algebraic multiplicity of 2

The eigenspace: $(A - \lambda I)v = 0 \Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

$\Rightarrow \lambda = 1$ has geometric multiplicity of 1

Since $1 \neq 2 \Rightarrow$ at $\epsilon \neq 0$, the matrix is diagonalizable (answer)

b) Let ϵ be such that A is diagonalizable, find V and $\Lambda \in \mathbb{C}^{2\times 2}$ so that $A = V \Lambda V^{-1}$

From a), we have the eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 1 + \epsilon$

$\Rightarrow \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 1+\epsilon \end{bmatrix}$

At $\lambda_1 = 1 \Rightarrow (A - \lambda_1 I) v_1 = 0 \Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & \epsilon \end{bmatrix} v_1 = 0 \Rightarrow v_1 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

At $\lambda_2 = \epsilon + 1 \Rightarrow (A - \lambda_2 I) v_2 = 0 \Rightarrow \begin{bmatrix} -\epsilon & 1 \\ 0 & 0 \end{bmatrix} v_2 = 0 \Rightarrow v_2 = \text{span}\left(\begin{bmatrix} 1/\epsilon \\ 1 \end{bmatrix}\right)$

Scale columns of V so that first row of V is $[1 \ 1]$

$\Rightarrow V = \begin{bmatrix} 1 & 1 \\ 0 & \epsilon \end{bmatrix} \Rightarrow A = V \Lambda V^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & \epsilon \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \epsilon + 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & \epsilon \end{bmatrix}^{-1}$ (answer)