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Exercise 2: Let $A \in R^{n \times n}$ have singular values $\sigma_1, \ldots, \sigma_n$ such that $\sigma_1 \geqslant \sigma_2 \geqslant \ldots \geqslant \sigma_{n-1} \geqslant \sigma_n$. Show that

a) $\|A\|_2 = \sigma_1$

We have: $\|A\|_2^2 = \sup_x \frac{\|Ax\|_2^2}{\|x\|_2^2} = \sup_x \frac{(Ax)^T Ax}{x^T x} = \sup_x \frac{(USV^T x)^T USV^T x}{x^T x}$

$= \sup_x \frac{x^T V S^T U^T U S V^T x}{x^T x} = \sup_x \frac{x^T V S^T S V^T x}{x^T x}$ (U is unitary)

Let $y = Vx \Rightarrow y^T y = (Vx)^T Vx = x^T V^T V x = x^T x$ (V is unitary)

Let $D = S^T S$

$\Rightarrow$ D is a square matrix whose diagonal contain squares of singular values

$$D = \begin{bmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 \ddots & 0 \\ 0 & 0 & \sigma_n^2 \end{bmatrix}$$

$$\Rightarrow \|A\|_2^2 = \sup_y \frac{y^T D y}{y^T y} = \sup_y R(D, y) = \frac{\sum_{i=1}^n \sigma_i^2 |y_i|^2}{\sum_{i=1}^n |y_i|^2}$$

$$\Rightarrow \|A\|_2^2 \leq \frac{\sigma_1^2 \sum_{i=1}^n |y_i|^2}{\sum_{i=1}^n |y_i|^2} \text{ (Since } \sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \dots \geq \sigma_n)$$

$\Rightarrow \|A\|_2^2 \leq \sigma_1^2$, which ~~hap~~ equality happens at $y_1 = 1$, $y_2 = 0$, $y_3 = 0$, ... $y_n = 0$

Since the upper bound can be reached with vector y described above

$\Rightarrow \|A\|_2^2 = \sup_x \dfrac{\|Ax\|_2^2}{\|x\|_2^2} = \sigma_1^2$. Since 2-norm is positive and σ_i is positive

$\Rightarrow \|A\|_2 = \sigma_1$ (proven)

b) $\|A^{-1}\|_2 = \dfrac{1}{\sigma_n}$

We have: $$\|A^{-1}\|_2^2 = \frac{1}{\lambda_{min}(A^T A)} = \frac{1}{\inf_x \frac{\|Ax\|_2^2}{\|x\|_2^2}} = \frac{1}{\inf_y \frac{y^T D y}{y^T y}}$$

$$\inf_y \frac{y^T D y}{y^T y} = \inf_y R(D, y) = \frac{\sum_{i=1}^n \sigma_i^2 |y_i|^2}{\sum_{i=1}^n |y_i|^2} \geq \frac{\sigma_n^2 \sum_{i=1}^n |y_i|^2}{\sum_{i=1}^n |y_i|^2} = \sigma_n^2$$

(Since $\sigma_n \leq \sigma_{n-1} \leq \dots \leq \sigma_2 \leq \sigma_1$)

$\Rightarrow \|A^{-1}\|_2^2 = \dfrac{1}{\inf_y \frac{y^T D y}{y^T y}} = \dfrac{1}{\sigma_n^2}$. Since 2-norm and σ_i are positive

Equality happens when $y_1 = y_2 = \dots y_{n-1} = 0$, $y_n = 1$

$\Rightarrow \|A^{-1}\|_2 = \dfrac{1}{\sigma_n}$ (proven)

c) $k_2(A) = \dfrac{\sigma_1}{\sigma_n}$

We have: $k_2(A) = \|A\|_2 \|A^{-1}\|_2 = \sigma_1 \times \dfrac{1}{\sigma_n} = \dfrac{\sigma_1}{\sigma_n}$ (proven)