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Problem Sheet 1

Exercise 1: Three vectors are given $(0\ 2\ {-4}\ 8)^T$, $(6\ 12\ 3\ 3)^T$, $(2\ 5\ {-1}\ 5)^T$. Is the vector $(-2\ 0\ {-9}\ 15)^T$ a linear combination of previous three? If so, is the combination unique?

Linear combination of the three vectors

$$a\begin{pmatrix}0\\2\\-4\\8\end{pmatrix} + b\begin{pmatrix}6\\12\\3\\3\end{pmatrix} + c\begin{pmatrix}2\\5\\-1\\5\end{pmatrix} = \begin{pmatrix}6b+2c\\2a+12b+5c\\-4a+3b-c\\8a+3b+5c\end{pmatrix}$$

With a, b, c scalar multiples. If there exists a, b, c so that

$$\begin{cases} 6b + 2c = -2 & R_1 \\ 2a + 12b + 5c = 0 & R_2 \\ -4a + 3b - c = -9 & R_3 \\ 8a + 3b + 5c = 15 & R_4 \end{cases}$$

then $(-2\ 0\ {-9}\ 15)^T$ is their linear combination

$$\begin{bmatrix}0&6&2&-2\\2&12&5&0\\-4&3&-1&-9\\8&3&5&15\end{bmatrix} \quad R_2 \to 2R_2 + R_3 \Rightarrow \begin{bmatrix}0&6&2&-2\\0&27&9&-9\\-4&3&-1&-9\\8&3&5&15\end{bmatrix}$$

$\Rightarrow R_2 \to 4{,}5R_1 + R_2 \Rightarrow$

$R_3 \to R_4 + 2R_3 - 1{,}5R_2$

$R_4 \to R_4 + 2R_3 - 1{,}5R_2$

$$\begin{bmatrix}0&6&2&-2\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

Since $0a + 0b + 0c = 0 \Rightarrow$ There are infinite solutions

$\Rightarrow (-2\ 0\ {-9}\ 15)^T$ is the linear combination, but is not unique

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Exercise 2: Three vectors are given: $a_1 = (1\ 1\ 1)^T$ $a_2 = (1\ 1\ 0)^T$, $a_3 = (1\ 0\ 0)^T$. Show that the vector $b = (-2\ 0\ -9)^T$ a linear combination of the previous three. If $a_2 = (\alpha\ \beta\ 0)^T$, is it possible to find α, β so that b can't be linear combination of a_1, a_2, a_3?

▫ Linear combination of the three vectors

$$i\begin{pmatrix}1\\1\\1\end{pmatrix} + j\begin{pmatrix}1\\1\\0\end{pmatrix} + k\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}i+j+k\\i+j\\i\end{pmatrix}$$

We have

$$\begin{cases} i+j+k = -2 \\ i+j = 0 \\ i = -9 \end{cases} \Rightarrow \begin{cases} k = -2 \\ j = 9 \\ i = -9 \end{cases}$$

$\Rightarrow b = (-2\ 0\ -9)^T$ is a unique linear combination of the three vectors

▫ If $a_2 = (\alpha\ \beta\ 0)^T$, the linear combination is

$$i\begin{pmatrix}1\\1\\1\end{pmatrix} + j\begin{pmatrix}\alpha\\\beta\\0\end{pmatrix} + k\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}i+\alpha j+k\\i+\beta j\\i\end{pmatrix}$$

We have

$$\begin{cases} i+\alpha j+k = -2 \\ i+\beta j = 0 \\ i = -9 \end{cases} \Rightarrow \begin{cases} \alpha j + k = 7 \\ \beta j = 9 \\ i = -9 \end{cases}$$

For $\beta j \neq 9\ \forall j \in R \Rightarrow \beta = 0$

$\Rightarrow$ For b to not be linear combination of a_1, a_2, a_3, $\beta = 0$ and α can be any real number

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Exercise 3: Using Gaussian elimination, solve the linear systems of equation, or show that the solutions do not exist

a) $\begin{cases} x + y - z = 9 \\ 8y + 6z = -6 \\ -2x + 4y - 6z = 40 \end{cases}$

The augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 9 \\ 0 & 8 & 6 & -6 \\ -2 & 4 & -6 & 40 \end{array}\right] : R_2 \to R_2/8 \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 9 \\ 0 & 1 & 3/4 & -3/4 \\ -2 & 4 & -6 & 40 \end{array}\right] R_3 \to R_3 + 2R_1$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 9 \\ 0 & 1 & \frac{3}{4} & -\frac{3}{4} \\ 0 & 6 & -8 & 58 \end{array}\right] R_3 \to 6R_2 - R_3 \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 9 \\ 0 & 1 & \frac{3}{4} & -\frac{3}{4} \\ 0 & 0 & \frac{25}{2} & -\frac{125}{2} \end{array}\right] \Rightarrow R_3 \to R_3 \cdot \frac{2}{25}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 9 \\ 0 & 1 & \frac{3}{4} & -\frac{3}{4} \\ 0 & 0 & 1 & -5 \end{array}\right] \Rightarrow \begin{cases} x + y - z = 9 \\ y + \frac{3}{4}z = -\frac{3}{4} \\ z = -5 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 3 \\ z = -5 \end{cases}$$

b) $\begin{cases} 4y + 3z = 8 \\ 2x - z = 2 \\ 3x + 2y = 5 \end{cases}$

The augmented matrix

$$\left[\begin{array}{ccc|c} 0 & 4 & 3 & 8 \\ 2 & 0 & -1 & 2 \\ 3 & 2 & 0 & 5 \end{array}\right] R_2 \to 2R_3 - 3R_2 \Rightarrow \left[\begin{array}{ccc|c} 0 & 4 & 3 & 8 \\ 0 & 4 & 3 & 4 \\ 3 & 2 & 0 & 5 \end{array}\right]$$

Since $4y + 3z = 8$ & $4y + 3z = 4 \Rightarrow$ There're no solutions

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Exercise 4

a) $A = \begin{pmatrix} 1 & 2 \\ -2 & 3 \\ 1 & -1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & -2 & -4 \\ -1 & 4 & 9 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$

Product of CA

$$CA = \begin{bmatrix} 1.1 + (-2).(-2) + (-4).1 & 1.2 + (-2).3 + (-4)(-1) \\ -1.1 + 4.(-2) + 9.1 & -1.2 + 4.3 + 9.(-1) \end{bmatrix}$$

$$\Rightarrow CA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Multiplying C from the left for the equations $Ax = b$

$$CAx = CB$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x = \begin{bmatrix} 1 & -2 & -4 \\ -1 & 4 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x = \begin{bmatrix} -25 \\ 56 \end{bmatrix}$$ Suppose $x = \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} -25 \\ 56 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1.x_{11} + 0x_{21} \\ 0x_{21} + 1x_{21} \end{bmatrix} = \begin{bmatrix} -25 \\ 56 \end{bmatrix} \Rightarrow \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} -25 \\ 56 \end{bmatrix}$$

There fore, ~~solution of~~ vector x is $x = \begin{bmatrix} -25 \\ 56 \end{bmatrix}$

Verify : $Ax = B$

$$\Rightarrow \begin{pmatrix} 1 & 2 \\ -2 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -25 \\ 56 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} \Rightarrow \begin{pmatrix} 87 \\ 218 \\ -81 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$$ (Incorrect)

$\Rightarrow$ Vector $x = \begin{bmatrix} -25 \\ 56 \end{bmatrix}$ is not the solution of $Ax = B$

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b) $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}$, $C = \frac{1}{10}\begin{pmatrix} 10 & 5 & 0 & -5 \\ -3 & -1 & 1 & 3 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$

Product of CA

$$CA = \begin{pmatrix} 1 + \frac{1}{2} + 0 - \frac{1}{2} & 1 + 1 + 0 - 2 \\ -\frac{3}{10} - \frac{1}{10} + \frac{1}{10} + \frac{3}{10} & -\frac{3}{10} - \frac{2}{10} + \frac{3}{10} + \frac{12}{10} \end{pmatrix}$$

$$\Rightarrow CA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Multiplying C from the left for the equation $Ax = b$

$$CAx = CB$$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x = \begin{bmatrix} 1 & \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{3}{10} & -\frac{1}{10} & \frac{1}{10} & \frac{3}{10} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow x = \begin{pmatrix} 1 \\ -\frac{2}{5} \end{pmatrix}$$

Verify : $Ax = B$

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ -\frac{2}{5} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{3}{5} \\ \frac{1}{5} \\ -\frac{1}{5} \\ -\frac{3}{5} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \text{(Not Correct)}$$

$\Rightarrow$ Vector $x = \begin{pmatrix} 1 \\ -\frac{2}{5} \end{pmatrix}$ is not the solution of $Ax = B$

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Exercise 5 : Find $(AB)^k$, $k \in \mathbb{N}$ when

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \\ 2 & 3 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & -2 & -2 \\ 0 & 8 & 4 \\ 0 & -5 & -2 \end{pmatrix}$$

▫ We have :

$$AB = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \\ 2 & 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 0 & -2 & -2 \\ 0 & 8 & 4 \\ 0 & -5 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$AB^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$AB^3 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$AB^4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$AB^5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow (AB)^k$, $k \to \infty$, $k \in \mathbb{N}$ will be

$$(AB)^k = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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Exercise 6: Let A and B be matrices of size 10×10 with elements $\alpha_{ij} = i + j$, $\beta_{ij} = i - j$. Express the elements γ_{ij} of the product matrix $C = AB$ as functions of the indeces i & j

We have $\underset{m \times n}{A} . \underset{n \times p}{B} = (Ab_1 \; Ab_2 \cdots Ab_p) = \underset{m \times p}{C}$

With inner products: $\underset{m \times p}{C} = (\gamma_{ij})$, $\gamma_{ij} = \sum_{k=1}^{n} \alpha_{ik} \cdot \beta_{kj}$

Since the matrix has size $10 \times 10 \Rightarrow n = 10$

We have: $\alpha_{ij} = i + j \Rightarrow \alpha_{ik} = i + k$

$\beta_{ij} = i - j \Rightarrow \beta_{kj} = k - j$

Therefore, the function of γ_{ij} of indices i & j is

$$\underset{10 \times 10}{C} = (\gamma_{ij}), \quad \gamma_{ij} = \sum_{k=1}^{10} (i + k)(k - j)$$