MS-A0001 - Matrix Algebra, 26.10.2020-08.12.2020

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Question **1**

Flag question   Mark 1.00 out of 1.00   Correct

Investigate the linear dependence of vectors

$$\begin{aligned}\mathbf{a} &= \begin{bmatrix}2 & -6 & -6 & 6\end{bmatrix}\\ \mathbf{b} &= \begin{bmatrix}-2 & 5 & 4 & 5\end{bmatrix}\\ \mathbf{c} &= \begin{bmatrix}-8 & 22 & 20 & -2\end{bmatrix}\end{aligned}$$

by choosing the correct alternative. Also compute the determinant of the matrix constructed from the row vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d} = \begin{bmatrix}-7 & -9 & 7 & -1\end{bmatrix}$.

Hint: Can the result from the first part be used in the second part?

(1) I'm in no mood for another Gauss-Jordan exercise.
(2) In this case it cannot be determined.
(3) The vectors are linearly independent.
(4) The vectors are linearly dependent.

Answer: 4

Determinant: 0

Your answer is correct!

Marks for this submission: 1.00/1.00.

Worked solution:

Let's first investigate whether the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly independent. Vectors $\mathbf{v}_1, \ \dots \ , \mathbf{v}_n$ are linearly independent if equation

$$\sum_{i=1}^{n} \alpha_i \mathbf{v}_i = \alpha_1 \mathbf{v}_1 + \ \dots \ \alpha_n \mathbf{v}_n = \mathbf{0} \qquad \alpha_i \in \mathbb{R}$$

has no other solution except the trivial solution $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. Let's write up the equation for the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$:

$$\begin{aligned} \alpha_1\mathbf{a} + \alpha_2\mathbf{b} + \alpha_3\mathbf{c} &= \mathbf{0}\\ \iff \alpha_1\begin{bmatrix}2\\-6\\-6\\6\end{bmatrix}^{\mathsf{T}} + \alpha_2\begin{bmatrix}-2\\5\\4\\5\end{bmatrix}^{\mathsf{T}} + \alpha_3\begin{bmatrix}-8\\22\\20\\-2\end{bmatrix}^{\mathsf{T}} &= \begin{bmatrix}0\\0\\0\\0\end{bmatrix}^{\mathsf{T}}\\ \iff \begin{bmatrix}2\cdot\alpha_1\\-6\cdot\alpha_1\\-6\cdot\alpha_1\\6\cdot\alpha_1\end{bmatrix}^{\mathsf{T}} + \begin{bmatrix}-2\cdot\alpha_2\\5\cdot\alpha_2\\4\cdot\alpha_2\\5\cdot\alpha_2\end{bmatrix}^{\mathsf{T}} + \begin{bmatrix}-8\cdot\alpha_3\\22\cdot\alpha_3\\20\cdot\alpha_3\\-2\cdot\alpha_3\end{bmatrix}^{\mathsf{T}} &= \begin{bmatrix}0\\0\\0\\0\end{bmatrix}^{\mathsf{T}}\\ \iff \begin{bmatrix}-2\cdot\alpha_2 - 8\cdot\alpha_3 + 2\cdot\alpha_1\\5\cdot\alpha_2 + 22\cdot\alpha_3 - 6\cdot\alpha_1\\4\cdot\alpha_2 + 20\cdot\alpha_3 - 6\cdot\alpha_1\\5\cdot\alpha_2 - 2\cdot\alpha_3 + 6\cdot\alpha_1\end{bmatrix}^{\mathsf{T}} &= \begin{bmatrix}0\\0\\0\\0\end{bmatrix}^{\mathsf{T}} \end{aligned}$$

So now we have the system

$$\begin{cases} -2\cdot\alpha_2 - 8\cdot\alpha_3 + 2\cdot\alpha_1 = 0\\ 5\cdot\alpha_2 + 22\cdot\alpha_3 - 6\cdot\alpha_1 = 0\\ 4\cdot\alpha_2 + 20\cdot\alpha_3 - 6\cdot\alpha_1 = 0\\ 5\cdot\alpha_2 - 2\cdot\alpha_3 + 6\cdot\alpha_1 = 0 \end{cases}.$$

The coefficient matrix, variable vector and parameter vector corresponding to this system are

$$\mathrm{A} = \begin{bmatrix}2 & -2 & -8\\-6 & 5 & 22\\-6 & 4 & 20\\6 & 5 & -2\end{bmatrix} \quad \mathbf{x} = \begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix} \quad \mathbf{b} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

and in matrix form the system is

$$\underbrace{\begin{bmatrix}2 & -2 & -8\\-6 & 5 & 22\\-6 & 4 & 20\\6 & 5 & -2\end{bmatrix}}_{=\mathrm{A}} \underbrace{\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix}}_{=\mathbf{x}} = \underbrace{\begin{bmatrix}0\\0\\0\\0\end{bmatrix}}_{=\mathbf{b}}.$$

Now let's form the augmented matrix $\mathrm{B} = [\mathrm{A}|\mathbf{b}]$ and calculate its rref.

$$\begin{aligned} \mathrm{B} &= \begin{bmatrix}2 & -2 & -8 & 0\\-6 & 5 & 22 & 0\\-6 & 4 & 20 & 0\\6 & 5 & -2 & 0\end{bmatrix}\\ \mathrm{rref(B)} &= \begin{bmatrix}1 & 0 & -2 & 0\\0 & 1 & 2 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix} \end{aligned}$$

Interpreting the rref back to a system of equations we have

$$\begin{aligned} &\begin{cases}\alpha_1 - 2\cdot\alpha_3 = 0\\ \alpha_2 + 2\cdot\alpha_3 = 0\end{cases}\\ \iff &\begin{cases}\alpha_1 = 2\cdot\alpha_3\\ \alpha_2 = -2\cdot\alpha_3\end{cases}\\ \iff &\begin{cases}\alpha_1 = 2t\\ \alpha_2 = -2t\ ,\ t\in\mathbb{R}\\ \alpha_3 = t\end{cases} \end{aligned}$$

From this we see that there are infinitely many solutions which means the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly dependent. Now we also know that the set $\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$ is also linearly dependent because a set of linearly dependent vectors will stay linearly dependent when any vector is added.

Let's investigate the determinent next. The matrix in question is

$$\begin{bmatrix}\mathbf{a}\\\hline\mathbf{b}\\\hline\mathbf{c}\\\hline\mathbf{d}\end{bmatrix} = \begin{bmatrix}2 & -6 & -6 & 6\\-2 & 5 & 4 & 5\\-8 & 22 & 20 & -2\\-7 & -9 & 7 & -1\end{bmatrix}.$$

We also know that the rows of the matrix are linearly dependent. We could compute the determinant mechanically but there is also an easier way in this case. If either the rows or columns of a matrix form a linearly dependent set the determinant of the matrix must be zero. Therefore the determinant of the matrix in question is zero.

A correct answer is 4.

A correct answer is 0, which can be typed in as follows: `0`

Question **2**

Flag question   Mark 1.00 out of 1.00   Correct

Let $\mathbf{a} = (4, \alpha, -2)$ and $\mathbf{b} = (\beta, 7, 7)$ where α and β are unknown real constants. We know that the vectors $\mathbf{a}$ and $\mathbf{b}$ are linearly independent.

What can we say about the solutions to equation

$$x_1\mathbf{a} + x_2\mathbf{b} = \mathbf{0}$$

based on the given information?

1 The equation has no solution.

2 There are no real solutions but at least one complex solution can be found.

3 The only solution to the equation is $(x_1, x_2) = (0, 0)$.

4 The equation has infinitely many solutions.

5 One solution to the equation is $x_1 = \mathbf{a}^{-1}$, $x_2 = -\mathbf{b}^{-1}$.

Give the number corresponding to the correct statement.

Answer: 3

Also give values to the constants α and β such that the vectors $\mathbf{a}$ and $\mathbf{b}$ truly are linearly independent.

$\alpha =$ -1

$\beta =$ -15

Your answer is correct!

Your answer is correct!
You have chosen the correct statement.
Marks for this submission: 0.50/0.50.

Your answer is correct!
You have chosen appropriate values for α and β.

Marks for this submission: 0.50/0.50.

Worked solution:

According to the definition vectors $\mathbf{v}_1, \ \dots \ , \mathbf{v}_n$ are linearly independent if and only if equation

$$x_1\mathbf{v}_1 + \ldots + x_n\mathbf{v}_n = \sum_{i=1}^{n} x_i\mathbf{v}_i = \mathbf{o}$$

has no other solution except for the trivial solution $x_1 = x_2 = \ldots = x_n = 0$.

Let's write the definition for the vectors in question. There are only two vectors so the equation is

$$x_1\mathbf{a} + x_2\mathbf{b} = \mathbf{0}.$$

It was mentioned in the assignment that vectors $\mathbf{a}$ and $\mathbf{b}$ are known to be linearly independent. Now according to the definition the equation has only the trivial solution $x_1 = x_2 = 0$.

In the case of two vectors linear independence means that the vectors are not parallel ergo they are not scalar multiples of each other. So now we should pick values for α and β such that the vectors are not parallel. Let's try to pick such values that the vectors are perpendicular to each other because in that case we know that they are not parallel. Two vectors are perpendicular if their dot product is zero so let's investigate the dot product of the given vectors.

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= 0\\ \Leftrightarrow 4\cdot\beta + 7\cdot\alpha - 14 &= 0\end{aligned}$$

Solving this equation for α we get

$$\alpha = \frac{-4\cdot\beta + 14}{7}.$$

Now let's pick $\beta = 0$. Now we see that $\alpha = 2$. Now we may also check that the dot product is indeed zero:

$$\mathbf{a}\cdot\mathbf{b} = 0 + 14 - 14 = 0.$$

A correct answer is 3.

A correct answer is 2, which can be typed in as follows: `2`

A correct answer is 0, which can be typed in as follows: `0`

Finish review

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