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Exercise 3: Let $x \in R^n$, $A, B \in R^{n\times n}$ and $\|\cdot\|$ be some vector norm. Show that

▫ $\|Ax\| \leq \|A\|\cdot\|x\|$

We have: $\|A\| = \max_{y\neq 0} \frac{\|Ay\|}{\|y\|}$ With all $y \in R^n$ satisfies such that $\frac{\|Ay\|}{\|y\|}$ reaches its maximum $\Rightarrow$ Other vectors will be x

$\Rightarrow \max_{y\neq 0} \frac{\|Ay\|}{\|y\|} \geq \frac{\|Ax\|}{\|x\|}_{x\neq 0} \Rightarrow \|A\| \geq \frac{\|Ax\|}{\|x\|} \Rightarrow \|Ax\| \leq \|A\|\|x\|$ (proven)

▫ $\|AB\| \leq \|A\|\cdot\|B\|$

We have: $\|AB\| = \max_{\|y\|=1} \|ABy\|$ At this moment By becomes a vector. Utilize the inequality above

$\Rightarrow \|AB\| \leq \max_{\|y\|=1} \|A\|\cdot\|By\| = \max_{\|y\|=1} \|By\|\,\|A\| = \|B\|\,\|A\|$

$\Rightarrow \|AB\| \leq \|A\|\cdot\|B\|$

▫ $\|I\| = 1$

We have: $\|I\| = \max_{y\neq 0} \frac{\|Iy\|}{\|y\|} = \max_{y\neq 0} \frac{\|y\|}{\|y\|}$ (vector multiplied by I will not change)

$= \max_{y\neq 0} 1 = 1 \Rightarrow \|I\| = 1$ (proven)

▫ Show by counterexample that $\|\cdot\|_{max}$ and $\|\cdot\|_F$ are not operator norms

* We have: $\|I\|_F = \left(\sum_{i=1}^{n}\sum_{j=1}^{n} |I_{ij}|^2\right)^{\frac{1}{2}}$, with $I_{ij} = 0$ when $i \neq j$
$= 1$ when $i = j$

$= \left[n\,(1^2)\right]^{\frac{1}{2}} = \sqrt{n}$

$\Rightarrow \|I\|_F = 1$ occurs only when I's dimension is just 1×1

$\Rightarrow \|\cdot\|_F$ is not an operator norm

* Let $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 5 \\ 7 & 6 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 11 & 0 \\ 29 & 28 \end{bmatrix}$

$\Rightarrow \|A\|_{max} = 3 \quad \|B\|_{max} = 7 \quad \|AB\|_{max} = 29$

We have $\|AB\|_{max} = 29 > \|A\|_{max}\|B\|_{max} = 3.7 = 21$ when for operator norms it should be $\|AB\|_{max} \leq \|A\|_{max}\|B\|_{max}$

$\Rightarrow \|\cdot\|_{max}$ is not an operator norm

Exercise 4: Let $A \in \mathbb{R}^{m \times n}$

a) Show that $N(A^TA) = N(A)$

Let x be null space of $A \Rightarrow Ax = 0$. Anything multiplied with Ax will be 0

$\Rightarrow A^TAx = 0 \Rightarrow (A^TA)x = 0$

$\Rightarrow x \in N(A^TA) \Rightarrow N(A) \subset N(A^TA)$

~~$\Rightarrow N(A^TA) = N(A)$ (proven)~~ (More proof after (b)!)

b) Let $A = \begin{bmatrix} 1 & 1 \\ \epsilon & 0 \\ 0 & \epsilon \end{bmatrix}$ where $\epsilon \in \mathbb{R} \Rightarrow A^T = \begin{bmatrix} 1 & \epsilon & 0 \\ 1 & 0 & \epsilon \end{bmatrix}$

$\Rightarrow A^TA = \begin{bmatrix} 1+\epsilon^2 & 1 \\ 1 & 1+\epsilon^2 \end{bmatrix} \Rightarrow \det(A^TA - \lambda I) = \det\begin{bmatrix} \epsilon^2 - \lambda + 1 & 1 \\ 1 & \epsilon^2 - \lambda + 1 \end{bmatrix}$

$\Rightarrow$ Characteristic polynomial: $p_A = (\epsilon^2 - \lambda + 1)^2 - 1 = 0$

$\Rightarrow (\epsilon^2 - \lambda)^2 + 2(\epsilon^2 - \lambda) + 1 - 1 = 0$

$\Rightarrow (\epsilon^2 - \lambda + 2)(\epsilon^2 - \lambda) = 0 \Rightarrow$ eigenvalues of A^TA: $\begin{cases} \lambda = \epsilon^2 \\ \lambda = \epsilon^2 + 2 \end{cases}$

For $N(A)$ to be trivial $\Rightarrow N(A^TA)$ must be trivial $\Rightarrow N(A^TA)$ corresponds to non-zero eigenvalues

$\Rightarrow \begin{cases} \lambda = \epsilon^2 \neq 0 \Rightarrow \epsilon \neq 0 \\ \lambda = \epsilon^2 + 2 \neq 0 \Rightarrow \text{always true since } \epsilon^2 + 2 \geq 2 \end{cases}$

For all $\epsilon \neq 0 \in \mathbb{R}$, the null space of A is trivial

* a) Provided proof: Let $x \in N(A^TA) \Rightarrow A^TAx = 0$

$\Rightarrow x^TA^TAx = 0$

$\Rightarrow (Ax)^T(Ax) = 0$

$\Rightarrow Ax = 0$

$\Rightarrow x \in N(A) \Rightarrow N(A^TA) \subset N(A)$

Since $N(A) \subset N(A^TA)$ and $N(A^TA) \subset N(A) \Rightarrow N(A) = N(A^TA)$ (proven)

c) Results reported below. Matlab agrees with me on dimension of $N(A)$ for all ϵ