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Linear algebra

Exercise sheet 2 / Model solutions

1. (a) Show that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ satisfy the three conditions in the definition of a norm. (See Definition 2.1 in the Lecture notes.)

   (b) Let $\boldsymbol{x} = [1, 2, 3]^T \in \mathbb{R}^3$. Calculate $\|\boldsymbol{x}\|_1$, $\|\boldsymbol{x}\|_2$ and $\|\boldsymbol{x}\|_\infty$.

   Solution.

   (a) For the 1-norm, $\|\boldsymbol{x}\|_1$, being the sum of the absolute values of the components of $\boldsymbol{x}$, is clearly never negative and zero if and only if $\boldsymbol{x} = \boldsymbol{0}$. Moreover,

   $$\|\alpha\boldsymbol{x}\|_1 = \sum_{i=1}^{n} |\alpha x_i| = |\alpha| \sum_{i=1}^{n} |x_i| = |\alpha|\|\boldsymbol{x}\|_1.$$

   Lastly,

   $$\|\boldsymbol{x}+\boldsymbol{y}\|_1 = \sum_{i=1}^{n} |x_i + y_i| \leq \sum_{i=1}^{n} |x_i| + |y_i| = \|\boldsymbol{x}\|_1\|\boldsymbol{y}\|_1.$$

   For the ∞-norm, $\|\boldsymbol{x}\|_\infty$, being the maximum of the absolute values of the components of $\boldsymbol{x}$, is clearly never negative and zero if and only if $\boldsymbol{x} = \boldsymbol{0}$. Moreover,

   $$\|\alpha\boldsymbol{x}\|_\infty = \max_{i=1,\dots,n} |\alpha x_i| = |\alpha| \max_{i=1,\dots,n} |x_i| = |\alpha|\|\mathbf{x}\|_\infty.$$

   Lastly,

   $$\|\boldsymbol{x}+\boldsymbol{y}\|_\infty = \max_{i=1,\dots,n} |x_i + y_i| \leq \max_{i=1,\dots,n} |x_i| + |y_i| = \|\boldsymbol{x}\|_1\|\boldsymbol{y}\|_\infty.$$

   (b) We have

   $$\begin{aligned} \|\boldsymbol{x}\|_1 &= |1| + |2| + |3| = 6, \\ \|\boldsymbol{x}\|_2 &= \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}, \\ \|\boldsymbol{x}\|_\infty &= \max\{|1|, |2|, |3|\} = 3. \end{aligned}$$

2. Let $W_1, W_2 \subset \mathbb{R}^n$ be subspaces with bases

   $$\{\boldsymbol{q}_1, \dots, \boldsymbol{q}_k\} \quad \text{and} \quad \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_m\}, \quad \text{respectively.}$$

   Denote $Q = \begin{bmatrix}\boldsymbol{q}_1, \dots, \boldsymbol{q}_k\end{bmatrix}$ and $V = \begin{bmatrix}\boldsymbol{v}_1, \dots, \boldsymbol{v}_m\end{bmatrix}$.

   (a) Show that $W_1 + W_2 = R(\begin{bmatrix}Q & V\end{bmatrix})$, where we define $W_1 + W_2$ as the vector subspace $\{\boldsymbol{x} + \boldsymbol{y} \mid \boldsymbol{x} \in W_1, \boldsymbol{y} \in W_2\}$ of $\mathbb{R}^n$.

(b) Let $T : \mathbb{R}^{k+m} \to \mathbb{R}^k$ be such that

$$(T\boldsymbol{x})_i = x_i \quad \text{for } i = 1, \ldots, k.$$

Show that $W_1 \cap W_2 = \{QT\boldsymbol{x} \mid \boldsymbol{x} \in N(\begin{bmatrix} Q & V \end{bmatrix})\}$.

Solution.

(a) By definition, the range of a matrix is the linear span of its columns. The sum $W_1 + W_2$ is equal to the span of $W_1 \cup W_2$. This proves the equality.

(b) If $\boldsymbol{y} \in W_1 \cap W_2$ then $\boldsymbol{y} = Q\boldsymbol{a} = V\boldsymbol{b}$ for some vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. If we set

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{a} \\ -\boldsymbol{b} \end{bmatrix},$$

then we get $[Q\ V]\boldsymbol{x} = \boldsymbol{0}$, if and only if $\boldsymbol{x} \in N([Q\ V])$. Clearly, $T\boldsymbol{x} = \boldsymbol{a}$ and so $QT\boldsymbol{x} = \boldsymbol{y}$. This shows that $W_1 \cap W_2 \subseteq N([Q\ V])$.
Conversely, let $\boldsymbol{x} \in N([Q\ V])$ and write

$$\boldsymbol{x} = \begin{bmatrix} T\boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix},$$

for some $\boldsymbol{y} \in \mathbb{R}^m$. Then we have that $QT\boldsymbol{x} = -V\boldsymbol{y}$, and hence $QT\boldsymbol{x} \in W_1 \cap W_2$. This proves $N([Q\ V]) \subseteq W_1 \cap W_2$ and concludes the proof.

3. Let $A \in \mathbb{R}^{m \times n}$.

(a) Show that $\boldsymbol{y}^T\boldsymbol{x} = 0$ for any $\boldsymbol{x} \in N(A)$ and $\boldsymbol{y} \in R(A^T)$. (Hint: $\boldsymbol{y} = A^T\boldsymbol{z}$ for some $\boldsymbol{z} \in \mathbb{R}^m$.)

(b) Let $\boldsymbol{x} \in \mathbb{R}^n$ be such that $\boldsymbol{y}^T\boldsymbol{x} = 0$ for any $\boldsymbol{y} \in R(A^T)$. Show that $\boldsymbol{x} \in N(A)$. (Hint: Choose $\boldsymbol{y} = A^TA\boldsymbol{x}$.)

Solution.

(a) By definition of $R(A^T)$ there exists $\boldsymbol{z} \in \mathbb{R}^m$ such that $\boldsymbol{y} = A^T\boldsymbol{z}$. Hence,

$$\boldsymbol{y}^T\boldsymbol{x} = \left(A^T\boldsymbol{z}\right)^T\boldsymbol{x}.$$

Using the calculation rules for the transpose $(AB)^T = (B^TA^T)$ and $(A^T)^T = A$, one gets

$$\left(A^T\boldsymbol{z}\right)^T\boldsymbol{x} = \boldsymbol{z}^T\left(A^T\right)^T\boldsymbol{x} = \boldsymbol{z}^TA\boldsymbol{x}.$$

By noticing that $\boldsymbol{x} \in N(A)$, so that $A\boldsymbol{x} = \boldsymbol{0}$, the proof is complete.

(b) Choosing $\boldsymbol{y} = A^T A\boldsymbol{x}$ gives

$$\boldsymbol{y}^T\boldsymbol{x} = \left(A^T A\boldsymbol{x}\right)^T \boldsymbol{x}.$$

Using calculation rules of the transpose,

$$\left(A^T A\boldsymbol{x}\right)^T \boldsymbol{x} = \boldsymbol{x}^T A^T A\boldsymbol{x} = \|A\boldsymbol{x}\|_2^2.$$

Hence, by assumption, $\|A\boldsymbol{x}\|_2^2 = 0$. By the properties of a norm, this implies that $A\boldsymbol{x} = \mathbf{0}$, thus $\boldsymbol{x} \in N(A)$.

4. Use Matlab to visualize the set

$$S := \{\boldsymbol{x} \in \mathbb{R}^2 \mid \|\boldsymbol{x}\|_* = 1\},$$

for $* = 1, 2$ or ∞. You may modify the function `plot_norm.m` found at the MyCourses page. Return both the script that you wrote and a printout of the resulting figure.

When $\|\boldsymbol{x}\|_2 = 1$, are $\|\boldsymbol{x}\|_1$ and $\|\boldsymbol{x}\|_\infty$ larger or smaller that one? Which four vectors $\boldsymbol{x} \in \mathbb{R}^2$ satisfy

$$\|\boldsymbol{x}\|_1 = \|\boldsymbol{x}\|_2 = \|\boldsymbol{x}\|_\infty = 1?$$

Justify your answer based on the figure that you draw.

Solution: The challenge in the visualization of the set S is to find a parametric presentation for it. Any $\boldsymbol{x} \in \mathbb{R}^2$ can be written as

$$\boldsymbol{x} = r\boldsymbol{v}_\theta, \quad \text{where} \quad \boldsymbol{v}_\theta = \begin{bmatrix}\sin\theta \\ \cos\theta\end{bmatrix}, \quad r \in \mathbb{R}, r \geq 0 \text{ and } \theta \in [0, 2\pi].$$

Now, one has

$$S = \{r\boldsymbol{v}_\theta \mid r \in \mathbb{R},\ r \geq 0,\ \theta \in [0, 2\pi] \text{ and } \|r\boldsymbol{v}_\theta\|_* = 1\}.$$

By the properties of a norm, $r\|\boldsymbol{v}_\theta\|_* = 1$. This implies that $r = 1/\|\boldsymbol{v}_\theta\|_*$ and thus

$$S = \left\{\frac{\boldsymbol{v}_\theta}{\|\boldsymbol{v}_\theta\|} \mid \theta \in [0, 2\pi]\right\}.$$

The set S is now easy to draw. See the file `norms.eps`. Here follows the code.

```
N = 1000;
t = linspace(0,2*pi,N);

x = cos(t);
y = sin(t);

figure;

for p=[1,2,Inf]
```

```
for i=1:N

    v = [x(i) ; y(i)];

    rho = 1./norm(v,p);

    xplot(i) = rho*x(i);
    yplot(i) = rho*y(i);

end



hold on;
plot(xplot,yplot);

end
```

By the figure one gets $\|\boldsymbol{x}\|_\infty \leq \|\boldsymbol{x}\|_2 \leq \|\boldsymbol{x}\|_1$, where the equality holds for $\boldsymbol{x} = \pm\boldsymbol{e}_1$ and $\boldsymbol{x} = \pm\boldsymbol{e}_2$.