Aalto University

Linear algebra

Exercise sheet 4

Practice problems

The following problems are just for practice. Their solutions will be shown by the TAs during the exercise sessions. You are welcome to share your own solution during the sessions, if you want.

1. Let $\boldsymbol{a} \in \mathbb{R}^n$ and $A = \boldsymbol{a}\boldsymbol{a}^T$.

   (a) Give a geometrical interpretation to the mapping $\boldsymbol{x} \mapsto A\boldsymbol{x}$. Fix $n = 2$, $\boldsymbol{a} = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ and draw the set
   $$\{A\boldsymbol{x} \mid \boldsymbol{x} \in \mathbb{R}^2, \|\boldsymbol{x}\|_2 = 1\}.$$

   (b) Show that $\|A\|_2 = \|\boldsymbol{a}\|_2^2$.

   (c) Fix $n = 2$, $\boldsymbol{a} = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$. Calculate $\|A\|_2$, $\|A\|_F$ and $\|A\|_{max}$.

2. Let $A, B, X \in \mathbb{R}^{n\times n}$, $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$ and $\|\cdot\|$ some vector norm (and the same notation is used for the corresponding operator norm). Show that:

   (a) $\|\boldsymbol{a} + \boldsymbol{b}\| \geq \big|\|\boldsymbol{a}\| - \|\boldsymbol{b}\|\big|$, (so-called "reverse triangle inequality")

   (b) $\|A + B\| \geq \big|\|A\| - \|B\|\big|$,

   (c) The matrix $I - X$ is invertible, if $\|X\| < 1$.

   Hint: In (c) assume that there exists $\mathbf{0} \neq \boldsymbol{x} \in N(I - X)$ and argue by contradiction.

Homework

Return the solutions to the following problems on MyCourses by Friday, May 7th, 18:00.

3. Let $\boldsymbol{x} \in \mathbb{R}^n$, $A, B \in \mathbb{R}^{n\times n}$ and $\|\cdot\|$ be some vector norm (same notation is used for the corresponding operator norm). Show that
   $$\|A\boldsymbol{x}\| \leq \|A\|\|\boldsymbol{x}\|, \qquad \|AB\| \leq \|A\|\|B\| \qquad \text{and} \qquad \|I\| = 1. \tag{1}$$

   Show by counterexample that $\|\cdot\|_{\max}$ and $\|\cdot\|_F$ are not operator norms.

Hint: Find matrices $A, B \in \mathbb{R}^{2\times 2}$ such that some of the properties in (1) do not hold. Note that $\|I\|_{\max} = 1$. In addition, the Frobenius norm satisfies the second inequality in (1): let $A^T = [\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n]$ and $B = [\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n]$, so that

$$\begin{aligned}\|AB\|_F^2 &= \sum_{i,j=1}^{n} \left(\boldsymbol{a}_i^T \boldsymbol{b}_j\right)^2 \leq \sum_{i,j=1}^{n} \|\boldsymbol{a}_i\|_2^2 \|\boldsymbol{b}_j\|_2^2 = \sum_{i=1}^{n} \|\boldsymbol{a}_i\|_2^2 \sum_{j=1}^{n} \|\boldsymbol{b}_j\|_2^2 \\ &= \sum_{i,k=1}^{n} |a_{ki}|^2 \sum_{j,l=1}^{n} |b_{lj}|^2 = \|A\|_F^2 \|B\|_F^2\end{aligned}$$

where the Cauchy–Schwarz inequality was used.

4. Let $A \in \mathbb{R}^{m\times n}$.

   (a) Show that
   $$N(A^T A) = N(A).$$
   That is, the null space of A can be determined by computing eigenvectors corresponding to the zero eigenvalue for $A^T A$.

   (b) Let
   $$A = \begin{bmatrix} 1 & 1 \\ \epsilon & 0 \\ 0 & \epsilon \end{bmatrix},$$
   where $\epsilon \in \mathbb{R}$. Compute the eigenvalues for the matrix $A^T A$ by hand. For which values of ϵ is the null-space of A trivial?

   (c) Determine $N(A)$ using Matlab, when $\epsilon = 1, 10^{-6}$ and 10^{-9}. Compute eigenvalues and corresponding eigenvectors for $A^T A$ numerically. Does Matlab agree with you on the dimension of $N(A)$ for each value of ϵ? Include the script that you used and the computed eigenvalues and vectors to your solution.

Hints: (a) The inclusion $N(A) \subset N(A^T A)$ can be easily proven. To prove that $N(A^T A) \subset N(A)$, multiply $A^T A$ from both sides with $\boldsymbol{x}$ and interpret the result as $\|A\boldsymbol{x}\|_2^2$. (c) Try out the commands `help eig` and `format long` in Matlab.