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Exercise 3: Let $y, b \in R^n \setminus \{0\}$. Consider the minimization problem $\min_{\alpha \in R} \|\alpha y - b\|^2$ where $\|\cdot\| : R^n \to R$ is a norm induced by some inner product $\langle \cdot , \cdot \rangle : R^n \times R^n \to R$, that is $\|z\|^2 = \langle z, z \rangle$ for any $z \in R^n$

a) Show that the minimizer is given by $\alpha = \frac{\langle y, b \rangle}{\|y\|^2}$

We have: $\|\alpha y - b\|^2 = \|\alpha y\|^2 + 2\langle \alpha y, -b \rangle + \|-b\|^2$

$$= \alpha^2 \|y\|^2 - 2\alpha \langle y, b \rangle + \|b\|^2$$

$$\Rightarrow \partial_\alpha \|\alpha y - b\|^2 = 2\alpha \|y\|^2 - 2\langle y, b \rangle = 0$$

$$\min_{\alpha \in R} \|\alpha y - b\|^2 \Rightarrow 2\alpha \|y\|^2 - 2\langle y, b \rangle = 0$$

$$\Rightarrow \alpha \|y\|^2 - \langle y, b \rangle = 0$$

$$\Rightarrow \alpha = \frac{\langle y, b \rangle}{\|y\|^2} \text{ (proven)}$$

b) Let $y = [1 \; 2]^T$ and $b = [1 \; 1]^T$. Determine α when

i. when $\|\cdot\|$ is the Euclidian norm

We have: $\langle y, b \rangle = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \times 1 + 2 \times 1 = 3$

$$\|y\|^2 = \left(\sqrt{1^2 + 2^2}\right)^2 = 5$$

$$\Rightarrow \alpha = \frac{\langle y, b \rangle}{\|y\|^2} = \frac{3}{5}$$

ii. when $\|\cdot\|$ is induced by the inner product

$$\langle x, z \rangle = z^T \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} x$$

We have: $\langle y, b\rangle = \begin{bmatrix}1\\1\end{bmatrix}^T \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = 8$

$$\|y\|^2 = \langle y, y\rangle = \begin{bmatrix}1\\2\end{bmatrix}^T \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = 13$$

$$\Rightarrow \alpha = \frac{\langle y, b\rangle}{\|y\|^2} = \frac{8}{13}$$

Exercise 4: Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $\langle\cdot,\cdot\rangle : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ be some inner product in $\mathbb{R}^m$ and $\|\cdot\|^2 = \langle\cdot,\cdot\rangle$ the corresponding norm. Consider the least-squares problem: find $x \in \mathbb{R}^n$ such that $\|Ax - b\|^2$ is minimized

a) Lemma: $\forall M \in \mathbb{R}^{m\times m}$ so that $\langle y, z\rangle = z^T M y$, where M is symmetric and positive definite. Show that eigenvalues of M are positive

Let λ_i be any eigenvalue of M and x are an arbitrary vector. Since M is positive definite $\Rightarrow$ $x^T M x > 0 \;\forall x \in \mathbb{R}^m$

λ_i is eigenvalue of $M \Rightarrow Mx = \lambda_i x$

$\Rightarrow x^T M x = x^T(\lambda_i x) = \lambda_i x^T x > 0$. Let x_i be element of x

$\Rightarrow \lambda_i x^T x = \lambda_i \sum_{i=1}^{n} x_i^2 > 0$. Since $\sum_{i=1}^{n} x_i^2 > 0 \Rightarrow \lambda_i > 0$

$\Rightarrow$ All eigenvalues of positive definite M are positive

- Find $L \in \mathbb{R}^{m\times m}$ such that $M = LL^T$

Since M is positive definite $\Rightarrow M = Q\Lambda Q^T$ where Λ is the diagonal matrix containing all the eigenvalues of M

$$\Lambda = \begin{bmatrix}\lambda_1 & & 0\\ & \lambda_2 & \\ 0 & & \lambda_n\end{bmatrix} \Rightarrow \Lambda^{\frac{1}{2}} = \begin{bmatrix}\sqrt{\lambda_1} & 0 & 0\\ 0 & \sqrt{\lambda_2} & 0\\ 0 & 0 & \sqrt{\lambda_n}\end{bmatrix}$$ $\Rightarrow \Lambda^{\frac{1}{2}}$ is also a diagonal matrix and is symmetric

$\Rightarrow M = Q\Lambda Q^T = Q(\Lambda^{\frac{1}{2}}\Lambda^{\frac{1}{2}})Q^T = (Q\Lambda^{\frac{1}{2}})(\Lambda^{\frac{1}{2}}Q^T)$

$\Rightarrow M = (Q\Lambda^{\frac{1}{2}})((\Lambda^{\frac{1}{2}})^T Q^T)$ (Since $\Lambda^{\frac{1}{2}}$ is symmetric)

$\Rightarrow M = (Q\Lambda^{\frac{1}{2}})(Q\Lambda^{\frac{1}{2}})^T$. Let $L = Q\Lambda^{\frac{1}{2}}$

$\Rightarrow M = LL^T$, where $L = Q\Lambda^{\frac{1}{2}}$ (Q is matrix of eigenvectors and Λ is matrix of eigenvalues of M)

b) Find $\tilde{A} \in \mathbb{R}^{m\times n}$ and $\tilde{b} \in \mathbb{R}^m$ such that

$$\|Ax - b\|^2 = \|\tilde{A}x - \tilde{b}\|_2^2$$

We have: $\|Ax-b\|^2 = \langle Ax-b, Ax-b\rangle = (Ax-b)^T M (Ax-b)$

$= (x^T A^T - b^T) L^T L (Ax-b)$ (L is symmetric)

$= (x^T A^T L^T L - b^T L^T L)(Ax-b)$

$= x^T A^T L^T L A x - x^T A^T L^T L b - b^T L^T L A x + b^T L^T L b$

In problem 2: $\|\tilde{A}x - \tilde{b}\|_2^2 = x^T \tilde{A}^T \tilde{A} x - x^T \tilde{A}^T \tilde{b} - \tilde{b}^T \tilde{A} x + \tilde{b}^T \tilde{b}$

We have: $\|Ax-b\|^2 = \|\tilde{A}x - \tilde{b}\|_2^2$

$$\Rightarrow \begin{cases} \tilde{A}^T\tilde{A} = A^T L^T L A \\ \tilde{A}^T \tilde{b} = A^T L^T L b \\ \tilde{b}^T \tilde{A} = b^T L^T L A \\ \tilde{b}^T \tilde{b} = b^T L^T L b \end{cases} \Rightarrow \begin{cases} \tilde{A} = LA \\ \tilde{A}^T = A^T L^T = (LA)^T \\ \tilde{b} = Lb \\ \tilde{b}^T = b^T L^T = (Lb)^T \end{cases}$$

$\Rightarrow \tilde{A} = LA$, $\tilde{b} = Lb$ (answer)