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Matrix Algebra Problem Sheet 4

Exercise 1: Consider an affine mapping $F: x \mapsto \hat{A}_1 x + b_1$ where $\hat{A}_1$ as above and $b_1 = \left(\sqrt{3} - \frac{1}{2},\ 4 - \frac{3\sqrt{3}}{2}\right)^T$.
Classify the transform / mapping precisely, i.e., determine at least one fixed point x_1. Draw in the same picture both G and its image $H_1 = \hat{A}_1 G + B_1$, where $B_1 = (b_1\ b_1\ b_1 \dots b_1)$

□ We have $\hat{A}_1 = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$

Easily noticed that $\sin^{-1}\left(\frac{1}{2}\right) = 30^\circ$ and $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 30^\circ$

$\Rightarrow \hat{A}_1 = \begin{pmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{pmatrix}$

$\Rightarrow \hat{A}_1$ is the rotation matrix that will make object rotate 30° counterclockwise around the origin

□ Let $x_1 = (1\ 1)^T \Rightarrow x_1' = \hat{A}_1 x + b_1$

$$\Rightarrow x_1' = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \sqrt{3} - 1/2 \\ 4 - \frac{3\sqrt{3}}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-1+\sqrt{3}}{2} \\ \frac{1+\sqrt{3}}{2} \end{pmatrix} + \begin{pmatrix} \sqrt{3} - 1/2 \\ 4 - 3\sqrt{3}/2 \end{pmatrix} = \begin{pmatrix} \frac{-2+3\sqrt{3}}{2} \\ \frac{9-2\sqrt{3}}{2} \end{pmatrix} \approx \begin{pmatrix} 1{,}598 \\ 2{,}768 \end{pmatrix}$$

Mapping of $G \to H_1$

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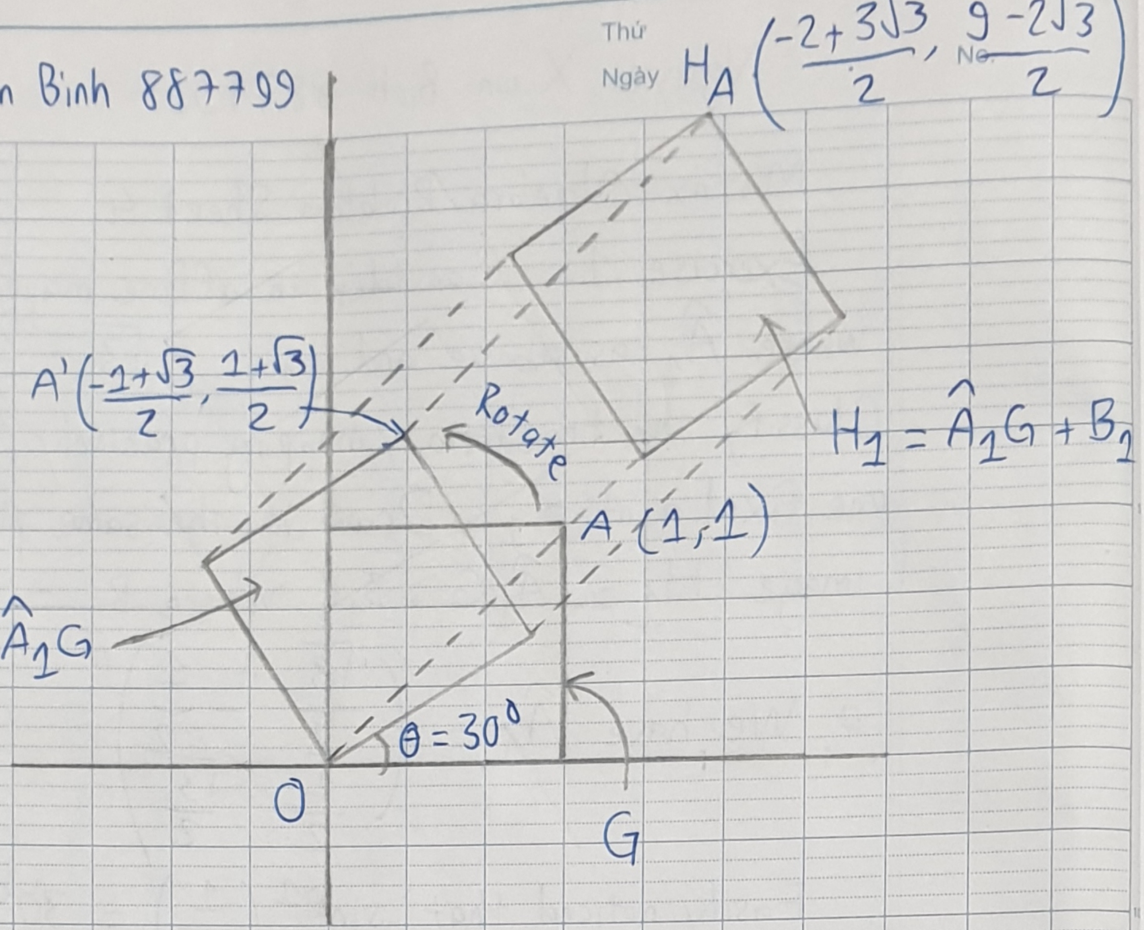

Exercise 2: Consider an affine mapping $F: x \mapsto \hat{A}_2 x + b_2$ where $\hat{A}_2$ as above and $b_2 = \left(-\frac{4}{5}, -\frac{8}{5}\right)^T$. Classify the transform (or mapping) precisely, i.e, determine at least one fixed point x_2. Draw in the same picture both G and its image $H_2 = \hat{A}_2 G + B_2$, where $B_2 = (b_2\ b_2\ b_2 \dots b_2)$

- We have $\hat{A}_2 = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ -4 & -3 \end{pmatrix} = \begin{pmatrix} 0,6 & -0,8 \\ -0,8 & -0,6 \end{pmatrix}$

It can be seen that $\hat{A}_2 = \begin{pmatrix} 0,6 & 0,8 \\ -0,8 & 0,6 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\Rightarrow \hat{A}_2 = \begin{pmatrix} \cos 53,13^0 & \sin 53,13^0 \\ -\sin 53,13^0 & \cos 53,13^0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\Rightarrow \hat{A}_2$ is both reflection and rotation matrix. First it will flip the object around x axis and rotates the object $53,13^0$ clockwise around the origin

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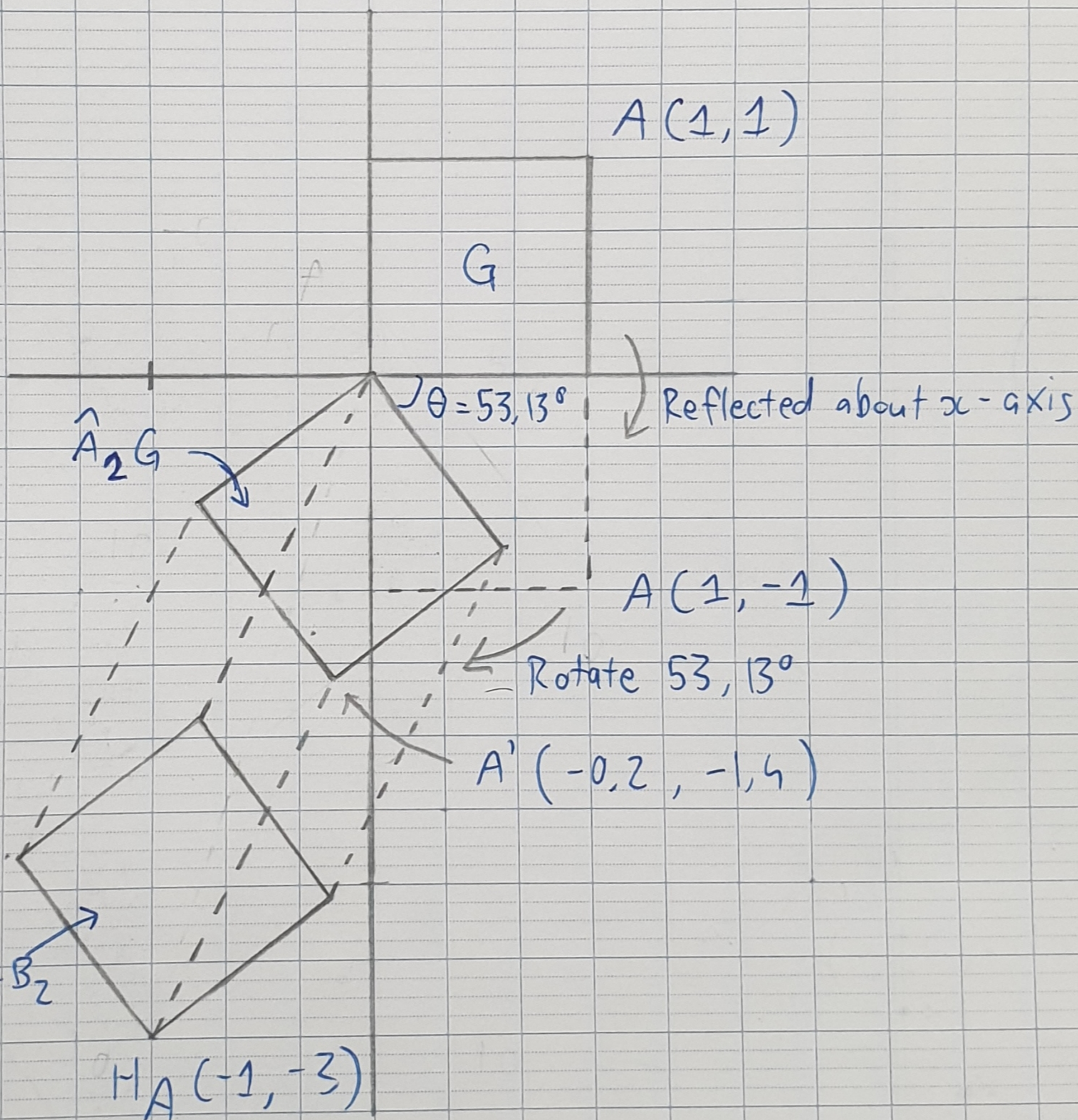
Let $x_2 = (1 \quad 1)^T \Rightarrow x_2' = \hat{A}_2 x_2 + b_2$

$$\Rightarrow x_2' = \begin{pmatrix} 0,6 & -0,8 \\ -0,8 & -0,6 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -4/5 \\ -8/5 \end{pmatrix}$$

$$= \begin{pmatrix} -0,2 \\ -1,4 \end{pmatrix} + \begin{pmatrix} -4/5 \\ -8/5 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$$

Mapping of $G \to H_1$



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Exercise 3: Under which conditions on parameters α and β the following determinants are equal to zero?

a) $A = \begin{vmatrix} \alpha+\beta & \alpha+2\beta & \alpha+3\beta \\ \alpha+3\beta & \alpha+\beta & \alpha+2\beta \\ \alpha+2\beta & \alpha+3\beta & \alpha+\beta \end{vmatrix}$

The determinant of this matrix is:

Let $\alpha+\beta = a$, $\alpha+2\beta = b$, $\alpha+3\beta = c$

$\Rightarrow \det(A) = a(a^2 - bc) - b(ac - b^2) + c(c^2 - ab)$

$= a^3 - abc + b^3 - abc + c^3 - abc$

$= a^3 + b^3 + c^3 - 3abc$

We have: $\det(A) = 0 \Rightarrow a^3 + b^3 + c^3 - 3abc = 0$

$\Leftrightarrow a^3 + b^3 + c^3 - 3abc = 0$

$\Rightarrow (a+b)^3 - 3ab(a+b) + c^3 - 3abc = 0$

$\Rightarrow (a+b)^3 + c^3 - 3ab(a+b+c) = 0$

$\Rightarrow (a+b+c)[(a+b)^2 - (a+b).c + c^2] - 3ab(a+b+c) = 0$

$\Rightarrow (a+b+c)(a^2+b^2+c^2+2ab-ac-bc) - 3ab(a+b+c) = 0$

$\Rightarrow (a+b+c)(a^2+b^2+c^2+2ab-ac-bc-3ab) = 0$

$\Rightarrow (a+b+c)(a^2+b^2+c^2-ab-ac-bc) = 0$

$\Rightarrow (a+b+c)(2a^2+2b^2+2c^2-2ab-2ac-2bc) = 0$

$\Rightarrow (a+b+c)[(a-b)^2 + (b-c)^2 + (c-d)^2] = 0$

$\Rightarrow \left[\begin{array}{l} a+b+c = 0 \ (1) \\ a = b = c \ (2) \end{array}\right.$

From (2): $\alpha+\beta = \alpha+2\beta = \alpha+3\beta$

$\Rightarrow$ No solution

From (1) $\Rightarrow (\alpha+\beta) + (\alpha+2\beta) + (\alpha+3\beta) = 0$

$\Rightarrow 3\alpha + 6\beta = 0 \Rightarrow \alpha + 2\beta = 0$

$\det(A) = 0$ if & only if $\alpha = -2\beta$

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b) $B = \begin{bmatrix} 2 & \alpha & 2 \\ 1 & \alpha^2 & \alpha \\ \alpha & \alpha^3 & 1 \end{bmatrix}$

The determinant of this matrix is

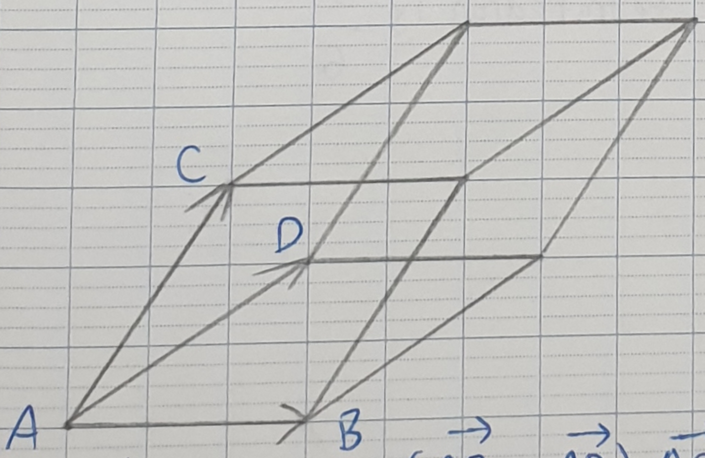
- $\det(B) = 2(\alpha^2 - \alpha^4) - \alpha(1 - \alpha^2) + 2(\alpha^3 - \alpha^3)$

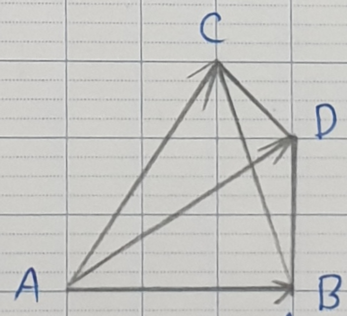
$$= -2\alpha^4 + 2\alpha^2 + \alpha^3 - \alpha$$
$$= -\alpha(2\alpha^3 - \alpha^2 - 2\alpha + 1)$$
$$= -\alpha[\alpha^2(2\alpha - 1) - (2\alpha - 1)]$$
$$= -\alpha(\alpha^2 - 1)(2\alpha - 1)$$
$$= -\alpha(\alpha + 1)(\alpha - 1)(2\alpha - 1)$$

- We have : $\det(B) = 0$

$$\Rightarrow \left[\begin{array}{l} \alpha = 0 \\ \alpha = 1 \\ \alpha = -1 \\ \alpha = \frac{1}{2} \end{array}\right. \Rightarrow \det(B) = 0 \text{ if } \alpha = 0, 1, -1 \text{ and } \frac{1}{2}$$

Exercise 4: A tetrahedron T is defined by its corners points $A(-1, -2, 4)$, $B(5, -1, 0)$, $C(2, -3, 6)$, $D(1, -1, 1)$. Find the volume of T





$V_{parallelpipep} = (\vec{AB} \times \vec{AD}) . \vec{AC}$

$V_{tetrahedron} = \frac{1}{6} V_{parallelpipep}$

$\Rightarrow V_{tetrahedron} = \frac{1}{6} (\vec{AB} \times \vec{AD}) \vec{AC}$

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We have : $A(-1,-2,4)$, $B(5,-1,0)$

$\Rightarrow \vec{AB} = (6, 1, -4)$

$A(-1,-2,4)$, $C(2,-3,6)$

$\Rightarrow \vec{AC} = (3,-1,2)$

$A(-1,-2,4)$, $D(1,-1,1)$

$\Rightarrow \vec{AD} = (2,1,-3)$

$\Rightarrow Volume_T = \frac{1}{6}(\vec{AB} \times \vec{AC}).\vec{AD}$

$$\vec{AB} \times \vec{AC} = det\left(\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 6 & 1 & -4 \\ 3 & -1 & 2 \end{vmatrix}\right)$$

$$= \hat{i}(2-4) - \hat{j}(12+12) + \hat{k}(-6-3)$$

$$= -2\hat{i} - 24\hat{j} - 9\hat{k}$$

$$(\vec{AB} \times \vec{AC}).\vec{AD} = (-2).2 - 24.1 - 9.(-3)$$

$$= -1$$

$\Rightarrow \vec{AD}$ is opposite direction from $(\vec{AB} \times \vec{AC})$ and volume of the parallelpipep is $|-1| = 1$

$\Rightarrow Volume_T = \frac{1}{6}.1 = \frac{1}{6}$

The volume of the tetrahedron is $\frac{1}{6}$