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Exercise 3: Consider the function $\|\cdot\|_{P_1} : R^2 \to R$ defined as follows: for any $\vec{x} = (x_1, x_2) \in R^2$, set $\|x\|^2_{P_1} = \frac{1}{3}x_1^2 + x_1x_2 + x_2^2$

a) Find symmetric matrix $A \in R^{2\times 2}$ such that: $\|x\|^2_{P_1} = x^TAx$ for all $x \in R^2$

We have: $\|x\|^2_{P_1} = x^TAx$

$$\Rightarrow \|x\|^2_{P_1} = [x_1 \; x_2]\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [x_1 \; x_2]\begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix}$$

$$\Rightarrow \|x\|^2_{P_1} = \frac{1}{3}x_1^2 + x_1x_2 + x_2^2 = a_{11}x_1^2 + a_{12}x_1x_2 + a_{21}x_1x_2 + a_{22}x_2^2$$
$$= a_{11}x_1^2 + (a_{12} + a_{21})x_1x_2 + a_{22}x_2^2$$

$$\Rightarrow a_{11} = \frac{1}{3}, \; a_{22} = 1, \; a_{12} + a_{21} = 1$$

$\Rightarrow A = \begin{bmatrix} \frac{1}{3} & a_1 \\ a_2 & 1 \end{bmatrix}$ so that $a_1 + a_2 = 1$ (For ex $A = \begin{bmatrix} 1/3 & 2/5 \\ 3/5 & 1 \end{bmatrix}$)

Since A is symmetric $\Rightarrow a_1 = a_2$. Since $a_1 + a_2 = 1 \Rightarrow a_1 = a_2 = 1/2$

$\Rightarrow A = \begin{bmatrix} 1/3 & 1/2 \\ 1/2 & 1 \end{bmatrix}$ (answer)

b) Show that A above is positive definite

We have: $x^TAx = \frac{1}{3}x_1^2 + x_1x_2 + x_2^2 = \left(\frac{\sqrt{3}}{3}x_1\right)^2 + 2\cdot\frac{\sqrt{3}}{3}\frac{\sqrt{3}}{2}x_1x_2 + \left(\frac{\sqrt{3}}{2}x_2\right)^2 + \frac{1}{4}x_2^2$

$$= \left(\frac{\sqrt{3}}{3}x_1 + \frac{\sqrt{3}}{2}x_2\right)^2 + \frac{1}{4}x_2^2 > 0, \forall x_1, x_2 \neq 0$$

$\Rightarrow$ Matrix A is positive definite. Another proof: since $\lambda_{min}(A) = \frac{-\sqrt{13}+4}{6} > 0$

$\Rightarrow x^TAx \geq \lambda_{min}(A)\|x\|_2^2 > 0 \; \forall x \neq 0$

c) Show that $\|\cdot\|_{P_1} : R^2 \to R$ is a norm

For any inner product, there exists a symmetric positive definite matrix $A \in R^{n\times n}$ such that $(x, y) = y^TAx$

We have: $\|x\|^2_{P_1} = x^TAx = (x, x)$, implying that $\|\cdot\|_{P_1}$ is a norm.

The inner product (x, x) always exist since A is a positive symmetric definite proven in (b)

$\Rightarrow \|\cdot\|_{P_1}$ is a norm

Exercise 4: Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

Eigenvalues of A: $\det(A - \lambda I) = \det\begin{pmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{pmatrix}$

$\Rightarrow \det(A - \lambda I) = -\lambda^3 + 6\lambda^2 - 10\lambda + 4 = (-\lambda - 2)(\lambda + \sqrt{2} - 2)(\lambda - \sqrt{2} - 2) = 0$

$\Rightarrow \lambda_1 = 2, \ \lambda_2 = -\sqrt{2} + 2, \ \lambda_3 = \sqrt{2} + 2$

$\Rightarrow \lambda_{max} = \lambda_3 = \sqrt{2} + 2 \approx 3.414213562$

Corresponding eigenvector of λ_{max} is

$v = \text{null space}(A - \lambda_{max} I) = N\begin{pmatrix} -\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{pmatrix} \Rightarrow v = \text{span}\left(\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}\right)$