Aalto-yliopisto

Linear algebra

Exercise sheet 2

Practice problems

The following problems are just for practice. Their solutions will be shown by the TAs during the exercise sessions. You are welcome to share your own solution during the sessions, if you want.

1. (a) Show that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ satisfy the three conditions in the definition of a norm.
   (b) Let $[1,2,3]^T \in \mathbb{R}^3$. Calculate $\|\boldsymbol{x}\|_1$, $\|\boldsymbol{x}\|_2$ and $\|\boldsymbol{x}\|_\infty$.

2. Let $W_1, W_2 \subseteq \mathbb{R}^n$ be subspaces with bases
$$\{\boldsymbol{q}_1,\ldots,\boldsymbol{q}_k\} \quad \text{and} \quad \{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_m\}, \quad \text{respectively.}$$
   Denote $Q = \begin{bmatrix}\boldsymbol{q}_1,\ldots,\boldsymbol{q}_k\end{bmatrix}$ and $V = \begin{bmatrix}\boldsymbol{v}_1,\ldots,\boldsymbol{v}_m\end{bmatrix}$.
   (a) Show that $W_1+W_2 = R(\begin{bmatrix}Q & V\end{bmatrix})$, where W_1+W_2 is by definition the vector subspace $\{\boldsymbol{x}+\boldsymbol{y} \mid \boldsymbol{x}\in W_1, \boldsymbol{y}\in W_2\}$ of $\mathbb{R}^n$.
   (b) Let $T:\mathbb{R}^{k+m}\to\mathbb{R}^k$ be such that
$$(T\boldsymbol{x})_i = x_i \quad \text{for } i=1,\ldots,k.$$
   Show that $W_1\cap W_2 = \{QT\boldsymbol{x} \mid \boldsymbol{x}\in N(\begin{bmatrix}Q & V\end{bmatrix})\}$.

Homework

Return the solutions to the following problems on MyCourses by Thursday, April 29th, 23:59.

3. Let $A\in\mathbb{R}^{m\times n}$.
   (a) Show that $\boldsymbol{y}^T\boldsymbol{x} = 0$ for any $\boldsymbol{x}\in N(A)$ and $\boldsymbol{y}\in R(A^T)$. (Hint: $\boldsymbol{y} = A^T\boldsymbol{z}$ for some $\boldsymbol{z}\in\mathbb{R}^m$.)
   (b) Let $\boldsymbol{x}\in\mathbb{R}^n$ be such that $\boldsymbol{y}^T\boldsymbol{x} = 0$ for any $\boldsymbol{y}\in R(A^T)$. Show that $\boldsymbol{x}\in N(A)$. (Hint: Choose $\boldsymbol{y} = A^TA\boldsymbol{x}$.)

4. Use Matlab to visualize the set
$$S := \left\{\boldsymbol{x}\in\mathbb{R}^2 \mid \|\boldsymbol{x}\|_* = 1\right\},$$

for $* = 1, 2$ or ∞. You may modify the function `plot_norm.m` found at the MyCourses page. Return both the script that you wrote and a printout of the resulting figure.

When $\|\boldsymbol{x}\|_2 = 1$, are $\|\boldsymbol{x}\|_1$ and $\|\boldsymbol{x}\|_\infty$ larger or smaller than 1? Which four vectors $\boldsymbol{x} \in \mathbb{R}^2$ satisfy

$$\|\boldsymbol{x}\|_1 = \|\boldsymbol{x}\|_2 = \|\boldsymbol{x}\|_\infty = 1?$$

Justify your answer based on the figure that you draw.