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Exercise 3: Let

$$\mathcal{V} = \text{span}\left\{\begin{bmatrix}1\\2\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\2\\1\end{bmatrix}\right\} \text{ and } \mathcal{W} = \text{span}\left\{\begin{bmatrix}2\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right\}, \text{ denote } V = \begin{bmatrix}1&1\\2&1\\1&2\\0&1\end{bmatrix} \text{ and } W = \begin{bmatrix}2&1\\1&1\\1&1\\0&1\end{bmatrix}$$

a) Find orthogonal projection matrix $Q \in R^{4\times4}$ such that $R(Q) = \mathcal{V}$

We have: $Px = Va \Rightarrow Px = V(V^TV)^{-1}V^Tx \Rightarrow P = V(V^TV)^{-1}V^T$

$\Rightarrow$ projection matrix: $Q = V(V^TV)^{-1}V^T$. Choose $V = \begin{bmatrix}1&2&1&0\\1&1&2&1\end{bmatrix}^T$

$$\Rightarrow Q = \begin{bmatrix}1&1\\2&1\\1&2\\0&1\end{bmatrix}\left(\begin{bmatrix}1&1\\2&1\\1&2\\0&1\end{bmatrix}^T\begin{bmatrix}1&1\\2&1\\1&2\\0&1\end{bmatrix}\right)^{-1}\begin{bmatrix}1&1\\2&1\\1&2\\0&1\end{bmatrix}^T = \begin{bmatrix}3/17&5/17&4/17&1/17\\5/17&14/17&1/17&-4/17\\4/17&1/17&11/17&7/17\\1/17&-4/17&7/17&6/17\end{bmatrix}$$

b) Using the matrixes V and W, give a formula for the projection matrix $P \in R^{4\times4}$ such that $R(P) = \mathcal{V}$ and $N(P) = \mathcal{W}$

If P is constructed so that it projects the direct sum $R^4 = \mathcal{V} \oplus \mathcal{W}$ to the first component, it means P makes the sum vanish the components from the second component

$\Rightarrow N(P)_{\text{onto } \mathcal{V}} = \mathcal{W}$

We have: $P_V x = Vz_V = [I, 0]\begin{bmatrix}z_V\\z_W\end{bmatrix} = [V, 0][V, W]^{-1}x$

$\Rightarrow P_V = [V, 0][V, W]^{-1}$ is the formula

$R(P_V) = R([V, 0][V, W]^{-1}) = \mathcal{V}$

$N(P_V) = N([V, 0][V, W]^{-1}) = R(I - [V, 0][V, W]^{-1}) = \mathcal{W}$

$$\Rightarrow P_V = [V, 0][V, W]^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -2/3 & 1/3 & 1 & -2/3 \\ -1 & 1 & 1 & -1 \\ -1 & 0 & 2 & -1 \\ -1/3 & -1/3 & 1 & -1/3 \end{bmatrix}$$

Exercise 4: Let $a, b \in R^3 \setminus \{0\}$ be such that $a^T b = 0$ and define the matrix $P = R^{3\times3}$ as

$$P = \frac{aa^T}{\|a\|_2^2} + \frac{bb^T}{\|b\|_2^2}$$

a) Is P a projection matrix ?

We have :
$$P^2 = \left(\frac{aa^T}{\|a\|_2^2} + \frac{bb^T}{\|b\|_2^2}\right)^2 = \frac{aa^Taa^T}{\|a\|_2^4} + \frac{bb^Tbb^T}{\|b\|_2^4} + 2\frac{aa^Tbb^T}{\|a\|_2^2\|b\|_2^2}$$

$$\Rightarrow P^2 = \frac{a(a^Ta)a^T}{\|a\|_2^4} + \frac{b(b^Tb)b^T}{\|b\|_2^4} + 2\frac{a(0)b^T}{\|a\|_2^2\|b\|_2^2}$$

We know that $\|x\|_2^2 = x^Tx$

$$\Rightarrow P^2 = \frac{a\|a\|_2^2a^T}{\|a\|_2^2\|a\|_2^2} + \frac{b\|b\|_2^2b^T}{\|b\|_2^2\|b\|_2^2} \Rightarrow P^2 = \frac{aa^T}{\|a\|_2^2} + \frac{bb^T}{\|b\|_2^2} = P$$

$\Rightarrow$ P is a projection matrix

* Is $I - P$ a projection matrix ?

We have :
$$(I-P)^2 = \left(I - \frac{aa^T}{\|a\|_2^2} - \frac{bb^T}{\|b\|_2^2}\right)^2 = I^2 + \underbrace{\left(\frac{aa^Taa^T}{\|a\|_2^2\|a\|_2^2} + \frac{bb^Tbb^T}{\|b\|_2^2\|b\|_2^2}\right)}_{\text{Proved above : } P^2 = P}$$
$$+ 2\underbrace{\frac{aa^Tbb^T}{\|a\|_2^2\|b\|_2^2}}_{=0} - 2I\underbrace{\left(\frac{aa^T}{\|a\|_2^2} + \frac{bb^T}{\|b\|_2^2}\right)}_{=P}$$

$$\Rightarrow (I-P)^2 = I + P + 0 - 2IP = I + P - 2P = I - P$$

$\Rightarrow$ $I - P$ is also a projection matrix

b) Is P an orthogonal projection ?

P is an orthogonal projection if $P = P^T$

We have :
$$(P)^T = \left(\frac{aa^T}{\|a\|_2^2} + \frac{bb^T}{\|b\|_2^2}\right)^T = \frac{(aa^T)^T}{\|a\|_2^2} + \frac{(bb^T)^T}{\|b\|_2^2}$$
$$= \frac{(a^T)^Ta^T}{\|a\|_2^2} + \frac{(b^T)^Tb^T}{\|b\|_2^2} = \frac{aa^T}{\|a\|_2^2} + \frac{bb^T}{\|b\|_2^2} = P$$

$\Rightarrow$ P is an orthogonal projection

c) Let $a = [1\ 1\ 1]^T$ and $b = [1, -2, 1]^T$. Find $c \in R^3$ such that

$$I - P = \frac{cc^T}{\|c\|_2^2}$$

We have: $I - P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{1^2+1^2+1^2}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}[1\ 1\ 1] - \frac{1}{1^2+(-2)^2+1^2}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}[1\ -2\ 1]$

$$\Rightarrow I - P = \begin{bmatrix} 1/2 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix}$$

Let $c = [x, y, z]^T \Rightarrow cc^T = \frac{1}{x^2+y^2+z^2}\begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix}$

$\Rightarrow y = 0$. Simplify the expression

$$cc^T = \frac{1}{x^2+z^2}\begin{bmatrix} x^2 & 0 & xz \\ 0 & 0 & 0 \\ xz & 0 & z^2 \end{bmatrix} \Rightarrow \begin{cases} x^2/x^2+z^2 = 1/2 \\ z^2/x^2+z^2 = 1/2 \\ xz/(x^2+z^2) = -1/2 \end{cases} \Rightarrow x = -z$$

$\Rightarrow c = \sigma[1\ 0\ -1]^T$ with $\sigma \in R$

* Other method: Since $R(I-P) = R(P)^\perp \Rightarrow c \perp a$ and $c \perp b$

Let $c = [x, y, z]^T$. $\left.\begin{aligned} c \perp a &\Rightarrow x + y + z = 0 \\ c \perp b &\Rightarrow x - 2y + z = 0 \end{aligned}\right\} \Rightarrow y = 0,\ x = -z$

$\Rightarrow c = \sigma[1\ 0\ -1]^T$ with $\sigma \in R$