MS-A0001 - Matrix Algebra, 26.10.2020-08.12.2020

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Time taken	4 hours 44 mins
Grade	**2.00** out of 2.00 (**100**%)

Quiz navigation

1 2

Finish review

Question **1**

Flag question Mark 1.00 out of 1.00 Correct

The eigenpairs = (eigenvalue, eigenvector) of matrix A are $(6, [2,1]^T)$ and $(9, [3,2]^T)$.

What is matrix A?

-3	18
-6	18

Your last answer was interpreted as follows:

$$\begin{bmatrix} -3 & 18 \\ -6 & 18 \end{bmatrix}$$

Your answer is correct!

Marks for this submission: 1.00/1.00.

Worked solution:

The eigenvalues and vectors give rise to equations

$$\mathrm{D} = \begin{bmatrix} 6 & 0 \\ 0 & 9 \end{bmatrix} \text{ ja } \mathrm{P} = [\mathbf{v}_1|\mathbf{v}_2] = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}.$$

Inverting the matrix P we have

$$\mathrm{P}^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}.$$

Matrix A is now the product of these matrices:

$$\mathrm{A} = \mathrm{PDP}^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 6 & 0 \\ 0 & 9 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 18 \\ -6 & 18 \end{bmatrix}.$$

A correct answer is $\begin{bmatrix} -3 & 18 \\ -6 & 18 \end{bmatrix}$.

Question **2**

Flag question Mark 1.00 out of 1.00 Correct

Calculate A^5 using the eigenvalue decomposition (diagonalization) $\mathrm{A} = \mathrm{PDP}^{-1}$ when

$$\mathrm{A} = \begin{bmatrix} 1 & 8 \\ 18 & -6 \end{bmatrix}.$$

Also give the eigenvalues λ_1 and λ_2.

Eigenvalues:

$\lambda_1 =$ -15

Your last answer was interpreted as follows:

$$-15$$

$\lambda_2 =$ 10

Your last answer was interpreted as follows:

$$10$$

The matrix:

$\mathrm{A}^5 =$

-209375	275000
618750	-450000

Your last answer was interpreted as follows:

$$\begin{bmatrix} -209375 & 275000 \\ 618750 & -450000 \end{bmatrix}$$

Your answer is correct!

Your answer is correct!

Eigenvalue λ_2 is correct!

Marks for this submission: 0.50/0.50.

Your answer is correct!

Matrix A^5 is correct!

Marks for this submission: 0.50/0.50.

Worked solution:

Letäs begin by calculating the eigenvalues of A. The eigenvalue equation is

$$\mathrm{A}\mathbf{x} = \lambda\mathbf{x},$$

which can be simplified to

$$(\lambda\mathrm{I} - \mathrm{A})\,\mathbf{x} = \mathbf{0}.$$

The characteristic polynomial of A is the determinant of matrix $\lambda\mathrm{I} - \mathrm{A}$. This matrix is

$$\lambda\mathrm{I} - \mathrm{A} = \begin{bmatrix} \lambda - 1 & -8 \\ -18 & \lambda + 6 \end{bmatrix}$$

And its determinant is

$$\begin{aligned} & \begin{vmatrix} \lambda - 1 & -8 \\ -18 & \lambda + 6 \end{vmatrix} \\ = \; & (\lambda - 1) \cdot (\lambda + 6) - (-8) \cdot (-18) \\ = \; & \lambda^2 + 5 \cdot \lambda - 150 \end{aligned}$$

The characteristic equation is $\lambda^2 + 5 \cdot \lambda - 150 = 0$ and its roots are $\lambda_1 = -15$ and $\lambda_2 = 10$. These are the eigenvalues of the matrix. The diagonal matrix D for the eigenvalue decomposition is then

$$\mathrm{D} = \begin{bmatrix} -15 & 0 \\ 0 & 10 \end{bmatrix}.$$

Next we find out what the eigenvectors are by solving equations $(\lambda\mathrm{I} - \mathrm{A})\,\mathbf{x} = \mathbf{0}$ for both values of λ. First we do $\lambda = \lambda_1 = -15$. The matrix is

$$\mathrm{B} = (\lambda\mathrm{I} - \mathrm{A}) = \begin{bmatrix} -16 & -8 \\ -18 & -9 \end{bmatrix}.$$

And now we have the equation

$$\underbrace{\begin{bmatrix} -16 & -8 \\ -18 & -9 \end{bmatrix}}_{=\mathrm{B}}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{=\mathbf{x}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We form the augmented matrix and solve it using row operations

$$\begin{aligned} & \begin{bmatrix} -16 & -8 & 0 \\ -18 & -9 & 0 \end{bmatrix} \\ \xrightarrow{-\frac{1}{16}R_1} & \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ -18 & -9 & 0 \end{bmatrix} \\ \xrightarrow{R_2 + 18R_1} & \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

By interpreting the augmented matrix we have

$$x_1 + \frac{1}{2}x_2 = 0$$

and

$$\begin{cases} x_1 = -\frac{1}{2}t \\ x_2 = t. \end{cases}$$

One eigenvector corresponding to $\lambda_1 = -15$ is then

$$\mathbf{x}_1 = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}.$$

Similarly we discover one eigenvector corresponding to $\lambda_2 = 10$:

$$\mathbf{x}_2 = \begin{bmatrix} \frac{8}{9} \\ 1 \end{bmatrix}.$$

Now we are able to form matrix P for the diagonalization.

$$\mathrm{P} = [\mathbf{x}_1\ \mathbf{x}_2] = \begin{bmatrix} -\frac{1}{2} & \frac{8}{9} \\ 1 & 1 \end{bmatrix}$$

The only thing left is to compute the inverse of P. The inverse can be computed for example using Gauss-Jordan elimination.

$$\mathrm{P}^{-1} = \begin{bmatrix} -\frac{18}{25} & \frac{16}{25} \\ \frac{18}{25} & \frac{9}{25} \end{bmatrix}$$

Let's check that we have the correct matrices for the diagonalization:

$$\begin{aligned} \mathrm{PDP}^{-1} &= \begin{bmatrix} -\frac{1}{2} & \frac{8}{9} \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -15 & 0 \\ 0 & 10 \end{bmatrix}\begin{bmatrix} -\frac{18}{25} & \frac{16}{25} \\ \frac{18}{25} & \frac{9}{25} \end{bmatrix} \\ &= \begin{bmatrix} \frac{15}{2} & \frac{80}{9} \\ -15 & 10 \end{bmatrix}\begin{bmatrix} -\frac{18}{25} & \frac{16}{25} \\ \frac{18}{25} & \frac{9}{25} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 8 \\ 18 & -6 \end{bmatrix} \\ &= \mathrm{A} \end{aligned}$$

Lastly we compute A^5.

$$\begin{aligned} \mathrm{A}^5 &= \left(\mathrm{PDP}^{-1}\right)^5 \\ &= \mathrm{PD}^5\mathrm{P}^{-1} \\ &= \begin{bmatrix} -\frac{1}{2} & \frac{8}{9} \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -15 & 0 \\ 0 & 10 \end{bmatrix}^5\begin{bmatrix} -\frac{18}{25} & \frac{16}{25} \\ \frac{18}{25} & \frac{9}{25} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{2} & \frac{8}{9} \\ 1 & 1 \end{bmatrix}\begin{bmatrix} (-15)^5 & 0 \\ 0 & 10^5 \end{bmatrix}\begin{bmatrix} -\frac{18}{25} & \frac{16}{25} \\ \frac{18}{25} & \frac{9}{25} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{2} & \frac{8}{9} \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -759375 & 0 \\ 0 & 100000 \end{bmatrix}\begin{bmatrix} -\frac{18}{25} & \frac{16}{25} \\ \frac{18}{25} & \frac{9}{25} \end{bmatrix} \\ &= \begin{bmatrix} \frac{759375}{2} & \frac{800000}{9} \\ -759375 & 100000 \end{bmatrix}\begin{bmatrix} -\frac{18}{25} & \frac{16}{25} \\ \frac{18}{25} & \frac{9}{25} \end{bmatrix} \\ &= \begin{bmatrix} -209375 & 275000 \\ 618750 & -450000 \end{bmatrix} \end{aligned}$$

A correct answer is -15, which can be typed in as follows: `-15`

A correct answer is 10, which can be typed in as follows: `10`

A correct answer is $\begin{bmatrix} -209375 & 275000 \\ 618750 & -450000 \end{bmatrix}$.