Aalto-yliopisto

Linear algebra

Exercise sheet 8

Practice problems

The following problems are just for practice. Their solutions will be shown by the TAs during the exercise sessions. You are welcome to share your own solution during the sessions, if you want.

1. For any $A \in \mathbb{C}^{n\times n}$ and $\lambda \in \mathbb{C}$ it holds that

$$(\lambda I + A)^k = \sum_{i=0}^{k} \binom{k}{i} \lambda^{k-i} A^i, \qquad k \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}.$$

   (a) Let $T \in \mathbb{C}^{n\times n}$ be a *nilpotent matrix*. That is, there exists some $p \in \mathbb{N}$ such that $T^p = \mathbf{0} \in \mathbb{C}^{n\times n}$. Show that

$$(\lambda I + T)^k = \sum_{i=0}^{\min\{k,p-1\}} \binom{k}{i} \lambda^{k-i} T^i.$$

   (b) Let

$$B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

   Write an explicit formula for B^k, for general $k \in \mathbb{N}$.

2. Let

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}.$$

   (a) Is the matrix A diagonalizable ?

   (b) Compute e^{tA} using the Jordan decomposition

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{-1}.$$

Hints: (a) Use Example 1.1 in section 2. (b) Use Problem 1(b). Note that $\sum_{k=1}^{\infty} \frac{1}{k!} k t^k 2^{k-1} = te^{2t}$.

Homework

Return the solutions to the following problems on MyCourses by Friday, May 21st, 18:00.

3. Consider the system of differential equations

$$\boldsymbol{x}'(t) = A\boldsymbol{x}(t) \quad \text{for } t > 0, \qquad \boldsymbol{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{1}$$

where $A \in \mathbb{C}^{2\times 2}$ and $\boldsymbol{x} : \mathbb{R}_+ \to \mathbb{C}^2$. Solve (1) by using the matrix exponential, when

(a)

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix},$$

(b)

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}.$$

Hints: (a) Use Problem 2(b). (b) Observe that A is symmetric.

4. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1+\epsilon \end{bmatrix},$$

where $\epsilon \in \mathbb{R}$.

(a) For which values of ϵ is the matrix A diagonalizable?

(b) Let ϵ be such that A is diagonalizable. Find an invertible $V \in \mathbb{C}^{2\times 2}$ and a diagonal matrix $\Lambda \in \mathbb{C}^{2\times 2}$ so that $A = V\Lambda V^{-1}$. Scale the columns of V so that the first row of V is $\begin{bmatrix} 1 & 1 \end{bmatrix}$.

(c) Compute the condition number $\kappa_2(V)$ using the Matlab function `cond`. Plot the condition number as a function of ϵ on the intervall $\epsilon \in [10^{-4}, 1]$. Use semilogarithmic scale, see `help semilogy`. What happens when A is very close to a non-diagonalizable matrix?

(d) Set $\epsilon = 0$ and try to compute V and Λ using the Matlab function `eig`. What is the condition number $\kappa_2(V)$? Is the diagonalization given by Matlab plausible? (Compare the result to (a).)

Hints: (a) If a (2×2)-matrix has two distinct eigenvalues, it is diagonalizable (see Section 2, Theorem 1.1 of the lecture notes); if this is not the case, one has to check that the geometric and algebraic multiplicities of each eigenvalue meet. (b) Note that Λ and V depend on the parameter ϵ.