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Linear algebra

Exercise sheet 9

Practice problems

The following problems are just for practice. Their solutions will be shown by the TAs during the exercise sessions. You are welcome to share your own solution during the sessions, if you want.

1. Let $\boldsymbol{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as

$$\boldsymbol{F}(\boldsymbol{x}) = \begin{bmatrix} 1 - x_1 - x_2 \\ x_1 x_2 \end{bmatrix}.$$

   Find the equilibrium points of the differential equation system

$$\boldsymbol{x}'(t) = \boldsymbol{F}(\boldsymbol{x}(t)). \tag{1}$$

   Linearise the system (1) close to the equilibrium points and compute the eigenvalues of the corresponding coefficient matrices. How do the solution curves starting close to the equilibrium points behave?

   Hint: Write the solution of the linearised system using the variable $\boldsymbol{y}(t) = \boldsymbol{x}(t) - \tilde{\boldsymbol{x}}$, where $\tilde{\boldsymbol{x}} \in \mathbb{R}^2$ is the equilibrium point, as $\boldsymbol{y}(t) = \mathrm{e}^{tA}\boldsymbol{y}_0 = V\mathrm{e}^{t\Lambda}V^{-1}\boldsymbol{y}_0$ for suitable $A = V\Lambda V^{-1}$ and deduce how the solution behaves when $t \to \infty$. Note that you do not have to explicitly find V or V^{-1}.

2. Let $A \in \mathbb{R}^{m\times n}$, $\boldsymbol{b} \in \mathbb{R}^m$ and define the function $f : \mathbb{R}^n \to \mathbb{R}$ as

$$f(\boldsymbol{x}) = \|A\boldsymbol{x} - \boldsymbol{b}\|_2^2.$$

   Prove by direct calculation that $\nabla f(\boldsymbol{x}) = \boldsymbol{0} \in \mathbb{R}^n$ if and only if $A^T A\boldsymbol{x} = A^T\boldsymbol{b}$.

   Hint: You may write $f(\boldsymbol{x}) = (\boldsymbol{x}^T A^T - \boldsymbol{b}^T)(A\boldsymbol{x} - \boldsymbol{b})$ in component form and evaluate the partial derivatives with respect to each coordinate x_ℓ.

Homework

Return the solutions to the following problems on MyCourses by Friday, May 28th, 18:00.

3. Let $\boldsymbol{y}, \boldsymbol{b} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$. Consider the minimisation problem

$$\min_{\alpha\in\mathbb{R}} \|\alpha\boldsymbol{y} - \boldsymbol{b}\|^2,$$

   where $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ is a norm induced by some inner product $\langle\cdot,\cdot\rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, that is, $\|\boldsymbol{z}\|^2 = \langle\boldsymbol{z},\boldsymbol{z}\rangle$ for any $\boldsymbol{z} \in \mathbb{R}^n$.

(a) Show that the minimiser is given by

$$\alpha = \frac{\langle \boldsymbol{y}, \boldsymbol{b} \rangle}{\|\boldsymbol{y}\|^2}.$$

(b) Let

$$\boldsymbol{y} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Determine α,

i. when $\|\cdot\|$ is the Euclidian norm,

ii. when $\|\cdot\|$ is induced by the inner product

$$\langle \boldsymbol{x}, \boldsymbol{z} \rangle = \boldsymbol{z}^T \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \boldsymbol{x}.$$

4. Let $A \in \mathbb{R}^{m \times n}$, $\boldsymbol{b} \in \mathbb{R}^m$, $\langle \cdot, \cdot \rangle : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ be some inner product in $\mathbb{R}^m$ and $\|\cdot\|^2 = \langle \cdot, \cdot \rangle$ the corresponding norm. Consider the least-squares problem: find $\boldsymbol{x} \in \mathbb{R}^n$ such that

$$\|A\boldsymbol{x} - \boldsymbol{b}\|^2 \tag{2}$$

is minimised.

(a) By Lemma 2.3 in Section 1, there exists a symmetric and positive definite matrix $M \in \mathbb{R}^{m \times m}$ such that $\langle \boldsymbol{y}, \boldsymbol{z} \rangle = \boldsymbol{z}^T M \boldsymbol{y}$ for all $\boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^m$. Decompose $M = Q \Lambda Q^T$. Show that each eigenvalue of M is positive and find a matrix $L \in \mathbb{R}^{m \times m}$ such that $M = LL^T$.

(b) Use (a) to reformulate the above problem as a least-squares problem posed in the Euclidian norm. That is, find $\tilde{A} \in \mathbb{R}^{m \times n}$ and $\tilde{\boldsymbol{b}} \in \mathbb{R}^m$ such that

$$\|A\boldsymbol{x} - \boldsymbol{b}\|^2 = \|\tilde{A}\boldsymbol{x} - \tilde{\boldsymbol{b}}\|_2^2$$

for all $\boldsymbol{x} \in \mathbb{R}^n$.

Hint: Give the matrix $\tilde{A}$ and vector $\tilde{\boldsymbol{b}}$ as a product of A, $\boldsymbol{b}$ and L. You do not have to define L explicitly, it may remain as a part of the solution.