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Exercise 3: Let $A, B \in \mathbb{R}^{n \times n}$ be symmetric. In addition, assume that

$$|\lambda_{amin}(A)| \geq 2 \text{ and } |\lambda_{amax}(B)| \leq 1$$

Show that

a) $\|A^{-1}\|_2 \leq 1/2$

We have : $\|A^{-1}\|_2 = \dfrac{1}{|\lambda_{amin}(A)|}$. Since $|\lambda_{amin}(A)| \geq 2$

$\Rightarrow \dfrac{1}{|\lambda_{amin}(A)|} \leq \dfrac{1}{2} \Rightarrow \|A^{-1}\|_2 \leq \dfrac{1}{2}$ (proven)

b) The matix $A + B$ is invertible

We have $\|B\|_2 = |\lambda_{amax}(B)| \leq 1$

$\|A^{-1}\|_2 \leq \dfrac{1}{2} \Rightarrow \dfrac{1}{\|A^{-1}\|_2} \geq 2$

$\Rightarrow \|B\|_2 \leq \dfrac{1}{\|A^{-1}\|_2}$. Since eigenvalues of A^{-1} and $B \neq 0 \Rightarrow A^{-1}$ and B are invertible

$\Rightarrow A + B$ is also invertible (proven)

c) Taking for granted the formula $\|(A+B)^{-1}\|_2 \leq \dfrac{\|A^{-1}\|_2}{1 - \|B\|_2 \|A^{-1}\|_2}$

Prove a solution x to the equation $(A+B)x = b$ satisfies $\|x\|_2 \leq \|b\|_2$

We have : $\dfrac{\|A^{-1}\|_2}{1 - \|B\|_2 \|A^{-1}\|_2} = \dfrac{1/|\lambda_{amin}(A)|}{1 - |\lambda_{amax}(B)| \dfrac{1}{|\lambda_{amin}(A)|}}$

$= \dfrac{1}{|\lambda_{amin}(A)| - |\lambda_{amax}(B)|}$. As $|\lambda_{amin}(A)| \geq 2$, $|\lambda_{amax}(B)| \leq 1$

$\Rightarrow |\lambda_{amin}(A)| - |\lambda_{amax}(B)| \geq 1 \Rightarrow \dfrac{1}{|\lambda_{amin}(A)| - |\lambda_{amax}(B)|} \leq 1$

$\Rightarrow \|(A+B)^{-1}\|_2 \leq 1$

We have: $(A+B)x = b \Rightarrow x = (A+B)^{-1} b$

$\Rightarrow \|x\|_2 = \|(A+B)^{-1} b\|_2 \leq \|(A+B)^{-1}\|_2 \|b\|_2$

As $\|(A+B)^{-1}\|_2 \leq 1 \Rightarrow \|(A+B)^{-1}\|_2 \|b\|_2 \leq \|b\|_2$ (Since 2 norm is strictly positive)

$\Rightarrow \|x\|_2 = \|(A+B)^{-1}\|_2 \|b\|_2 \leq \|b\|_2 \Rightarrow \|x\|_2 \leq \|b\|_2$ (proven)

Exercise 4:

Let $A = \begin{bmatrix} 1+\epsilon^2 & 1 \\ 1 & 1+\epsilon^2 \end{bmatrix}$ and $b = \begin{bmatrix} 2 \\ 2-\delta \end{bmatrix}$

a) Compute $k_2(A) = \|A\|_2 \|A^{-1}\|_2$. What happens to the condition number $k_2(A)$ when $\epsilon \to 0$?

We have: $\det(A - \lambda I) = \begin{bmatrix} \epsilon^2 + 1 - \lambda & 1 \\ 1 & \epsilon^2 + 1 - \lambda \end{bmatrix}$

$\Rightarrow \det(A - \lambda I) = (\lambda - \epsilon^2)(\lambda - (\epsilon^2 + 2)) = 0$

$\Rightarrow \begin{cases} \lambda_1 = \epsilon^2 \\ \lambda_2 = \epsilon^2 + 2 \end{cases}$

▫ $k_2(A) = \|A\|_2 \|A^{-1}\|_2 = \dfrac{|\lambda_{amax}(A)|}{|\lambda_{amin}(A)|} = \dfrac{\epsilon^2 + 2}{\epsilon^2} = 1 + \dfrac{2}{\epsilon^2}$

$\Rightarrow$ As $\epsilon \to 0$, $k_2(A) \to 1$

b) Compute $R(A)$ and $N(A)$ when $\epsilon \neq 0$. What about $\epsilon = 0$?

▫ For $\epsilon \neq 0 \Rightarrow \lambda_1$ and λ_2 are non zero

As A has non-zero eigenvalues $\Rightarrow A$ is an invertible matrix $\Rightarrow A^{-1}$ exist

We have: $R(A) = Ay$ for all $y \in R^n$. As A^{-1} exist, let x be a specific solution

$\Rightarrow x = Ay \Rightarrow A^{-1}x = A^{-1}Ay = A^{-1}x = y$

$\Rightarrow$ For all $x \in R^2$ there always exist a vector y so that $A^{-1}x = y$

$\Rightarrow R(A) = R^{2\times 2}$ for all possible matrices in dimension $R^{2\times 2}$

* Null space: $Ax = 0 \Rightarrow A^{-1}Ax = A^{-1}.0 \Rightarrow x = 0$

$\Rightarrow x = 0$ is the unique solution

$\Rightarrow N(A)$ has only trivial solution $x = 0$

▫ For $\epsilon = 0 \Rightarrow \lambda_1$ is a zero eigenvalue $\Rightarrow A$ is a non-invertible matrix

$\Rightarrow A^{-1}$ doesn't exist

$\epsilon = 0 \Rightarrow A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Let $y = [y_1\ y_2]^T$ be an arbitrary vector

$\Rightarrow Ay = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 + y_2 \\ y_1 + y_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}^T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = y^T \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\Rightarrow R(A) = \left\{ y^T \begin{bmatrix} 1 \\ 1 \end{bmatrix}, y \in R^{1\times 2} \right\}$

* Null space: $Ax = 0 \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow \begin{cases} x_1 + x_2 = 0 \\ x_1 + x_2 = 0 \end{cases} \Rightarrow x_1 = -x_2$

$\Rightarrow N(A) = \left\{ [1\ -1]s \cup [-1\ 1]t \mid s, t \in R \right\}$

c) For $\epsilon \neq 0$, solve $Ax = b$ using the Cramer's rule

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

We have: $Ax = b \Rightarrow \begin{bmatrix} 1+\epsilon^2 & 1 \\ 1 & 1+\epsilon^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2-\delta \end{bmatrix}$

(Since it is proved before that $\epsilon \neq 0$ means $N(A)$ is trivial, there's no need to find $N(A)$)

We have: $A^{-1} = \frac{1}{(\epsilon^2+1)^2 - 1}\begin{bmatrix} 1+\epsilon^2 & -1 \\ -1 & 1+\epsilon^2 \end{bmatrix}$

$\Rightarrow A^{-1}Ax = A^{-1}b \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{(\epsilon^2+1)^2-1}\begin{bmatrix} \epsilon^2+1 & -1 \\ -1 & \epsilon^2+1 \end{bmatrix}\begin{bmatrix} 2 \\ 2-\delta \end{bmatrix}$

$(x = A^{-1}b)$

$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{(\epsilon^2+1)^2-1}\begin{bmatrix} 2\epsilon^2+\delta \\ 2\epsilon^2 - \delta\epsilon^2 - \delta \end{bmatrix}$ (answer)

- When $\delta = 0 \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{(\epsilon^2+1)^2-1}\begin{bmatrix} 2\epsilon^2 \\ 2\epsilon^2 \end{bmatrix} = \left[\frac{2\epsilon^2}{\epsilon^4+2\epsilon^2}, \frac{2\epsilon^2}{\epsilon^4+2\epsilon^2}\right]^T$

$\Rightarrow \bar{x} = \left[\frac{2}{\epsilon^2+2}, \frac{2}{\epsilon^2+2}\right]^T \Rightarrow \lim_{\epsilon \to 0} \bar{x} = \left[\frac{2}{0^2+2}, \frac{2}{0^2+2}\right]^T = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

When $\delta = 0$, when $\epsilon \to 0 \Rightarrow \bar{x} \to [1 \ 1]^T$

- When $\delta \neq 0 \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \left[\frac{2\epsilon^2+\delta}{\epsilon^4+2\epsilon^2}, \frac{2\epsilon^2-\delta\epsilon^2-\delta}{\epsilon^4+2\epsilon^2}\right]^T$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \left[\frac{2}{\epsilon^2+2} + \frac{\delta}{\epsilon^4+2\epsilon^2}, \frac{2}{\epsilon^2+2} - \frac{\delta}{\epsilon^2+2} - \frac{\delta}{\epsilon^4+2\epsilon^2}\right]$$

	$\frac{2}{\epsilon^2+2}$	$\frac{\delta}{\epsilon^4+2\epsilon^2}$	$\frac{2}{\epsilon^2+2}$	$\frac{\delta}{\epsilon^2+2}$	$\frac{\delta}{\epsilon^4+2\epsilon^2}$
$\epsilon \to 0^+$	1	$+\infty$	1	$-\delta/2$	$-\infty$
$\epsilon \to 0^-$	1	$-\infty$	1	$-\delta/2$	$+\infty$

$\Rightarrow$ When $\delta \neq 0$, $\epsilon \to 0^+ \Rightarrow \bar{x} \to [+\infty \ -\infty]^T$

$\epsilon \to 0^- \Rightarrow \bar{x} \to [-\infty \ +\infty]^T$