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Linear algebra

Exercise sheet 1 / Model solutions

1. Consider the vectors $\boldsymbol{x} = [2, 3, 4]^T, \boldsymbol{y} = [1, 0, 2]^T$ and $\boldsymbol{z} = [0, 1, 0]^T$ in $\mathbb{R}^3$.

   (a) Are the vectors $\boldsymbol{x}, \boldsymbol{y}$ and $\boldsymbol{z}$ linearly dependent?

   (b) Find a vector $\boldsymbol{w} \in \mathbb{R}^3$ such that $\boldsymbol{x}, \boldsymbol{y}$ and $\boldsymbol{w}$ are linearly independent.

   (c) Find a vector $\boldsymbol{v} \in \mathbb{R}^3$ such that $\boldsymbol{x}, \boldsymbol{z}$ and $\boldsymbol{v}$ are linearly independent.

   Solution.

   (a) Yes, the three vectors are linearly dependent. To see this, let $A = [\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{x}]$ and let us find a nonzero coefficient vector $\boldsymbol{\alpha} \in \mathbb{R}^3$ such that $A\boldsymbol{\alpha} = \boldsymbol{0}$. Gaussian elimination yields that this linear system is equivalent to

   $$\begin{cases} \alpha_1 + 2\alpha_3 = 0 \\ \alpha_2 + 3\alpha_3 = 0 \end{cases}$$

   which has solution $\alpha = [-2t, -3t, t]^T$ where $t \in \mathbb{R}$. So we can find a nonzero $\boldsymbol{\alpha}$ and the three vectors are linearly dependent.

   (b) One way to achieve this is to impose that the determinant of $\begin{bmatrix} 2 & 1 & w_1 \\ 3 & 0 & w_2 \\ 4 & 2 & w_3 \end{bmatrix}$ is nonzero. This yields $2w_1 \neq w_3$, so for example we can pick $\boldsymbol{w} = [1, 0, 1]^T$.

   (c) This part can be solved with the same method as part (b). We obtain the condition $v_3 - 2v_1 \neq 0$; one possible solution is $\boldsymbol{v} = [1, 0, 1]^T$.

2. Let $\boldsymbol{x} = [1, 2, 3]^T$. Represent $\boldsymbol{x}$ as a linear combination of the basis vectors

   (a) $\boldsymbol{e}_1 = [1, 0, 0]^T$, $\boldsymbol{e}_2 = [0, 1, 0]^T$ and $\boldsymbol{e}_3 = [0, 0, 1]^T$ (so-called *cartesian* basis vectors),

   (b) $\boldsymbol{q}_1 = [1, 1, 0]^T$, $\boldsymbol{q}_2 = [1, 0, 1]^T$ and $\boldsymbol{q}_3 = [1, 1, 1]^T$,

   (c) $\boldsymbol{v}_1 = [-1, 1, -1]^T$, $\boldsymbol{v}_2 = [1, 2, 2]^T$ and $\boldsymbol{v}_3 = [1, -2, 1]^T$.

   Solution.

   (a) Clearly $\boldsymbol{x} = \boldsymbol{e}_1 + 2\boldsymbol{e}_2 + 3\boldsymbol{e}_3$.

   (b) If $Q = [\boldsymbol{q}_1 \ \boldsymbol{q}_2 \ \boldsymbol{q}_3]$, we must solve the linear system $Q\boldsymbol{\alpha} = \boldsymbol{x}$. Any method (for instance Gaussian Elimination) yields as solution $\boldsymbol{\alpha} = [-2, -1, 4]^T$, so $\boldsymbol{x} = -2\boldsymbol{q}_1 - \boldsymbol{q}_2 + 4\boldsymbol{q}_3$.

   (c) This question can be solved similarly to part (b). The solution is $\boldsymbol{x} = 4\boldsymbol{v}_1 + 2\boldsymbol{v}_2 + 3\boldsymbol{v}_3$.

3. Let

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & -2 & 2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ -3 \\ 8 \end{bmatrix}.$$

Compute a basis for the nullspace $N(A)$ by hand and find all solutions to the equation

$$A\boldsymbol{x} = \boldsymbol{b}.$$

Solution. The set $N(A)$ is defined as

$$N(A) := \{\boldsymbol{x} \in \mathbb{R}^3 \mid A\boldsymbol{x} = \boldsymbol{0}\}.$$

Hence, a basis for $N(A)$ can be found by solving the linear system $A\boldsymbol{x} = \boldsymbol{0}$ by Gaussian elimination:

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & -2 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -2 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Back substitution gives

$$\boldsymbol{x} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad N(A) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right).$$

Gaussian elimination is used to find the solutions to $A\boldsymbol{x} = \boldsymbol{b}$:

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & -3 \\ 0 & -2 & 2 & 8 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & -4 \\ 0 & -2 & 2 & 8 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Back substitution gives

$$\boldsymbol{x} = \begin{bmatrix} -3 \\ -4 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

4. Let $\boldsymbol{0} \neq \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$ and define $A = \boldsymbol{a}\boldsymbol{b}^T \in \mathbb{R}^{n\times n}$. Show that

   (a) $R(A) = \operatorname{span}\{\boldsymbol{a}\}$,

   (b) $N(A) = \big\{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{b}^T\boldsymbol{x} = 0\big\}$.

Solution. To prove that two sets V and W are equal, first pick $v \in V$ and show that $v \in W$. Then pick $w \in W$ and show that $w \in V$.

   (a) Let $\boldsymbol{x} \in R(A)$. Then there exists $\boldsymbol{z} \in \mathbb{R}^n$ such that $\boldsymbol{x} = \boldsymbol{a}\boldsymbol{b}^T\boldsymbol{z}$. Denote $\alpha = \boldsymbol{b}^T\boldsymbol{z}$ so that $\boldsymbol{x} = \alpha\boldsymbol{a} \in \operatorname{span}\{\boldsymbol{a}\}$.
   Let $\boldsymbol{x} \in \operatorname{span}\{\boldsymbol{a}\}$. Choose $\boldsymbol{z} = \frac{\boldsymbol{b}}{\|\boldsymbol{b}\|_2^2}$, so that $\boldsymbol{a}\boldsymbol{b}^T\boldsymbol{z} = \boldsymbol{a}$. Hence $\boldsymbol{x} \in R(A)$.

   (b) Let $\boldsymbol{v} \in N(A)$. This means that $\boldsymbol{a}\boldsymbol{b}^T\boldsymbol{v} = \boldsymbol{0}$. Because $\boldsymbol{a} \neq \boldsymbol{0}$, we have that $\boldsymbol{b}^T\boldsymbol{v} = 0$ and $\boldsymbol{v} \in \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{b}^T\boldsymbol{x} = 0\}$.
   Let $\boldsymbol{v} \in \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{b}^T\boldsymbol{x} = 0\}$. Then $\boldsymbol{a}\boldsymbol{b}^T\boldsymbol{v} = \boldsymbol{0}$. Hence, $\boldsymbol{v} \in N(A)$.