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# A Continuous Damage Mechanics Model for Ductile Fracture

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*A model of isotropic ductile plastic damage based on a continuum damage variable, on the effective stress concept and on thermodynamics is derived. The damage is linear with equivalent strain and shows a large influence of triaxiality by means of a damage equivalent stress. Identification for several metals is made by means of elasticity modulus change induced by damage. A comparison with the McClintock and Rice-Tracey models and with some experiments is presented for the influence of triaxiality on the strain to rupture.*

## 1 Introduction

The phenomenon of initiation and growth of cavities and microcracks induced by large deformations in metals and called "ductile plastic damage," has been extensively studied by means of micro-mechanics analysis. In the pioneer works of McClintock [1], Rice and Tracey [2] and subsequent studies, defects are taken into account by analyzing their geometry in a continuous matrix using the procedure of the mechanics of continuous media.

At that microscale, a good representation of physical mechanisms can be introduced, but difficulties arise when these analyses have to be included in large scale structures to predict ductile failures. The main reason is the lack of accuracy of local stress calculations for the microscale level.

Between that microscale level, say  $10^{-3}$ - $10^{-2}$  mm, and the structure scale level, say  $10^2$ - $10^3$  mm, there exists a macroscale level of constitutive equations for strain behavior. The continuous damage mechanics approach deals within that macroscale defining a damage variable as an effective surface density of cracks or cavity intersections with a plane. Of course, at that macroscale it is difficult to introduce much physics but on the other hand this damage variable is easy to introduce in structural calculation, especially with the concept of effective stress. This stress, written as the mean density of forces acting on the elementary surface that effectively resists, has been introduced by Kachanov in 1958 to model creep rupture [3]. This has been the starting point of continuous damage mechanics developed further for dissipation and low cycle fatigue in metals (Lemaitre 1971 [4]), for coupling between damage and creep (Leckie 1974 [5]), between damage and cyclic creep (Hult [6]), for high cycle fatigue (Chaboche 1974 [7]), for creep fatigue interaction (Lemaitre-Chaboche 1974 [8]). . . Later, the thermodynamics of irreversible processes provided the necessary scientific basis to justify continuous damage mechanics as a theory (Chaboche 1978 [9], Lemaitre-Chaboche 1978 [10], Murakami [11], Cordebois-Sidoroff [12] Krajcinovic [13]) with many developments (see for examples papers presented at the Euromech

Colloquium on "Damage Mechanics" held in Cachan (France) in 1981). Models presented here are in the framework of that thermodynamics which gives the possibility of identifying the damage by means of its coupling with elasticity.

## 2 Elements of Continuous Damage Mechanics

As many developments of the theory of continuous damage mechanics have already been published [9, 10, 12], only the principal features used to build a new model of ductile plastic damage are summarized here.

**2.1 Damage Variable.** Consider a damaged body in which a volume element at macroscale level has been isolated (Fig. 1).

Let  $S$  be the overall section area of that element defined by its normal  $\mathbf{n}$ . In that section the microcracks and cavities have intersections of different shapes, let  $S_D$  be their total area.

Let  $\tilde{S}$  be the effective resisting area taking into account this area  $S_D$ , the microstress concentrations in the vicinity of discontinuities and the interactions between closed defects.

$$\tilde{S} < S - S_D$$

The concept of effective stress associated to the hypothesis of strain equivalence described below avoids the calculation of  $\tilde{S}$  and, by definition, the damage variable  $D$  associated with the normal  $\mathbf{n}$  is:

$$D_n = \frac{S - \tilde{S}}{S}$$

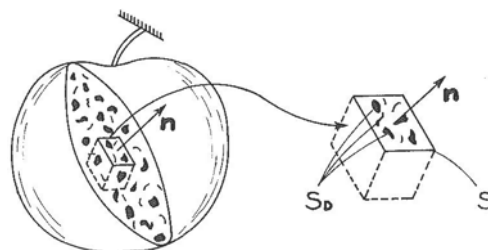


Fig. 1 Damaged element

From a physical point of view the variable  $D_n$  is the corrected area of cracks and cavities per unit surface cut by a plane perpendicular to  $\mathbf{n}$ .

From a mathematical point of view, as  $S$  approaches zero, then  $D_n$  is the corrected surface density of discontinuities in the body relative to the normal  $\mathbf{n}$ .

$D_n = 0$  corresponds to the undamaged state;

$D_n = 1$  corresponds to rupture of the element into two parts;

$0 < D_n < 1$  characterizes the damaged state.

**Hypothesis of Isotropy.** In the general case, cracks and voids are oriented and  $D_n$  is a function of  $\mathbf{n}$ . This leads to an intrinsic variable of damage which can be a second order tensor [12] or a fourth order tensor [9] depending upon the hypothesis made. In this paper we restrict ourselves to isotropic damage, the cracks and voids being equally distributed in all directions.  $D_n$  does not depend upon  $\mathbf{n}$  and the intrinsic damage variable is the scalar  $D$ .

**Concept of Effective Stress.** If  $\mathbf{F}$  is the load acting on the section  $S$  of the element considered in Fig. 1,  $\mathbf{T} = \mathbf{F}/S$  is the usual stress vector which leads to the Cauchy stress tensor  $\boldsymbol{\sigma}$  ( $\boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{T}$ ). The quantity  $\bar{S} = S(1-D)$  is the effective area which effectively carries the load  $\mathbf{F}$ . By definition:

$$\bar{\mathbf{T}} = \frac{\mathbf{F}}{\bar{S}} = \frac{\mathbf{T}}{1-D}$$

is the effective stress vector which, since  $D$  is a scalar, leads to the effective stress tensor ( $\bar{\boldsymbol{\sigma}} \cdot \mathbf{n} = \bar{\mathbf{T}}$ )

$$\left[ \bar{\boldsymbol{\sigma}} = \frac{\boldsymbol{\sigma}}{1-D} \right]$$

Beside the hypothesis of isotropy, we assume here that the mechanical effects of cavities and microcracks are the same both in tension and compression. As it is generally not the case in practice, this limits the applicability of the theory to those cases where "compressions" are small.

**Hypothesis of Strain Equivalence [4].** It is assumed that the strain behavior is modified by damage only through the effective stress:

The strain behavior of a damaged material is represented by constitutive equations of the virgin material (without any damage) in the potential of which the stress is simply replaced by the effective stress.

Examples:

-one dimensional linear elasticity of a damage material:

$$\epsilon_e = \frac{\bar{\sigma}}{E} = \frac{\sigma}{(1-D)E}$$

$\epsilon_e$  being the elastic strain and  $E$  the Young's modulus.

-Ramberg-Osgood equation for plastic strain-hardening evolution:

$$\epsilon_p = \left( \frac{\bar{\sigma}}{K} \right)^M = \left[ \frac{\sigma}{(1-D)K} \right]^M$$

where  $\epsilon_p$  is the plastic strain and  $K$  and  $M$  material coefficients.

**2.2 Thermodynamics.** In order to model elasticity, thermal effects, plasticity and damage within the hypothesis of isotropy for the three phenomena, the following variables have to be introduced [10]:

| Observable variables                  | Internal variables                | Associated variables                      |
|---------------------------------------|-----------------------------------|---|
| Elastic strain tensor<br>$\epsilon^e$ |                                   | stress tensor<br>$\sigma$                 |
| Temperature<br>$T$                    |                                   | entropy<br>$s$                            |
|                                       | Accumulated plastic strain<br>$p$ | Radius of yield surface: $R_y + R$<br>$R$ |
|                                       | Damage<br>$D$                     | Damage strain energy release rate<br>$y$  |

$p$  is defined by:

$$\dot{p} = \frac{dp}{dt} = \left( \frac{2}{3} \dot{\epsilon}^p : \dot{\epsilon}^p \right)^{1/2}$$

$\epsilon^p$  is the Euler-Almansi plastic strain tensor in large deformation theory, its rate being defined from the total strain rate  $\dot{\epsilon}$  by:

$$\dot{\epsilon}^p = \dot{\epsilon} - \dot{\epsilon}^e$$

**Thermodynamic Potential.** Taking the free-energy  $\psi$  as thermodynamic potential, it is assumed that it is a convex function of all observable and internal variables. Using the hypothesis that the elasticity and plasticity behaviors are uncoupled gives:

$$\psi = \psi_e(\epsilon^e, T, D) + \psi_p(T, p)$$

In order to obtain linear elasticity coupled with damage by means of the effective stress,  $\psi_e$  must be quadratic in  $\epsilon^e$  and linear in  $(1-D)$ . If  $\mathbf{a}$  is the fourth order tensor of elasticity and  $\rho$  the density

$$\psi_e = \frac{1}{2\rho} \mathbf{a} : \epsilon^e : \epsilon^e (1-D)$$

The damaged elasticity law is:

$$\boldsymbol{\sigma} = \rho \frac{\partial \psi_e}{\partial \epsilon^e} = \mathbf{a} : \epsilon^e (1-D)$$

and the variable  $y$  associated with  $D$ , by the power dissipated ( $-y \dot{D}$ ) in the phenomenon of damage, is defined by:

$$y = \rho \frac{\partial \psi_e}{\partial D} = - \frac{1}{2} \mathbf{a} : \epsilon^e : \epsilon^e$$

**Damage Criterion.** The density of elastic strain energy  $W_e$  being defined as:

$$dW_e = \boldsymbol{\sigma} : d\epsilon^e$$

if we replace  $d\epsilon^e$  by its value taken from the damaged elasticity law written for  $d\boldsymbol{\sigma} = 0$  at constant temperature one

\*The symbol  $( : )$  means the tensorial product contracted on two indices.

can see that  $-y$  is one half of the variation of  $W_e$  due to an infinitesimal increase of damage at constant stress and temperature. This gives for  $-y$  the name of “damage strain energy release rate” (as  $G$  in fracture mechanics!) [9].

$$-y = \frac{1}{2} \frac{dW_e}{dD} \Big|_{\sigma, T}$$

The expressions of  $y$  and  $W_e$  show that:

$$-y = \frac{W_e}{1-D}$$

$W_e$  is calculated as the sum of shear strain energy and volume dilatation energy with the tensor of elasticity written in terms of Young's modulus  $E$  and Poisson's ratio  $\nu$ , that is the following relations between elastic strain deviator  $\mathbf{e}^e$  and stress deviator  $\mathbf{s}$ , the hydrostatic strain  $\epsilon_H^e = 1/3 \text{tr}(\mathbf{e}^e)$  and the hydrostatic stress  $\sigma_H = 1/3 \text{tr}(\boldsymbol{\sigma})$ :

$$\mathbf{e}^e = \frac{1+\nu}{E} \frac{\mathbf{s}}{1-D}, \quad \epsilon_H^e = \frac{1-2\nu}{E} \frac{\sigma_H}{1-D}$$

We obtain

$$-y = \frac{1}{2} \left[ \frac{1+\nu}{E} \frac{\mathbf{s} : \mathbf{s}}{(1-D)^2} + 3 \frac{1-2\nu}{E} \frac{\sigma_H^2}{(1-D)^2} \right]$$

or with the Von Mises equivalent stress for plasticity

$$\sigma_{eq} = \left( \frac{3}{2} \mathbf{s} : \mathbf{s} \right)^{1/2}$$

$$-y = \frac{\sigma_{eq}^2}{2E(1-D)^2} \left[ \frac{2}{3} (1+\nu) + 3(1-2\nu) \left( \frac{\sigma_H}{\sigma_{eq}} \right)^2 \right]$$

This quantity can be calculated for an “equivalent” one-dimensional case defined by its stress  $\sigma^*$ , giving the same value of  $y$ .

$$\sigma_{eq} = \sigma^*, \quad \sigma_H = \frac{1}{3} \sigma^*$$

$$-y = \frac{\sigma^{*2}}{2E(1-D)^2}$$

As  $y$  is the variable associated with  $D$ , it means that evolution of  $D$  is governed by values of  $y$ ; by analogy with the Von Mises equivalent stress for plasticity, the quantity:

$$\left| \sigma^* = \sigma_{eq} \left[ \frac{2}{3} (1+\nu) + 3(1-2\nu) \left( \frac{\sigma_H}{\sigma_{eq}} \right)^2 \right]^{1/2} \right|$$

is called damage equivalent stress and can act as a criterion for damage just as  $\sigma_{eq}$  acts as a criterion for plasticity [14].  $\bar{\sigma}^* = \sigma^* / (1-D)$  is the damage equivalent effective stress and:

$$\left| -y = \frac{\bar{\sigma}^{*2}}{2E} \right|$$

It is interesting to note that  $\sigma^*$  is equal to the Von Mises equivalent stress multiplied by a factor function of the triaxiality ratio  $\sigma_H / \sigma_{eq}$  which is very important for damage evolution as shown by many experimental or theoretical studies [15–17].

**Dissipation, Rupture Criterion.** Within the hypothesis of uncoupling between intrinsic mechanical and thermal dissipations, the second law of thermodynamics imposes the condition of mechanical dissipation being positive,

$$\sigma : \dot{\epsilon}^p - R\dot{p} - y\dot{D} \geq 0$$

The processes of plasticity and damage may be independent, then we must have separately:

$$\sigma : \dot{\epsilon}^p - R\dot{p} \geq 0 \quad \text{and} \quad -y\dot{D} \geq 0$$

( $-y$ ) being positive  $\dot{D}$  must be positive which is a trivial

result! ( $-y\dot{D}$ ) is the energy dissipated within the damage process for decohesion of the material. By analogy with the toughness criterion in fracture mechanics one may postulate the following rupture criterion:

The damage process gives rise to initiation of a macrocrack for a critical value of  $-y$ , that value  $y_c$  being a characteristic for each material.

$$-y = y_c \rightarrow \text{crack initiation}$$

This corresponds to a critical value of the damage variable  $D$  which can be calculated from the uniaxial case for the rupture conditions:

$$\left. \begin{array}{l} -y = y_c \\ \sigma = \sigma_R \\ D = D_c \end{array} \right\} y_c = \frac{\sigma_R^2}{2E(1-D_c)^2}$$

$$\bar{\sigma}_R = \frac{\sigma_R}{1-D_c} = (2Ey_c)^{1/2} \quad \text{is the condition of brittle fracture of Orowan.}$$

$$D_c = 1 - \frac{\sigma_R}{(2Ey_c)^{1/2}} \quad \text{is the critical value of damage at macrocrack initiation.}$$

Many experiments have shown that:

$$.2 \leq D_c \leq .8$$

depending upon the materials.

**Potential of Dissipation.** In order to derive constitutive equations for evaluation of dissipative variables, the existence of a potential of dissipation is assumed: a scalar convex function of flux variables ( $\dot{\epsilon}^p$ ,  $\dot{p}$ ,  $\dot{D}$ , and the heat flux  $\mathbf{q}$ ) the state variables acting as parameters [18].

$$\varphi(\dot{\epsilon}^p, \dot{p}, \dot{D}, \mathbf{q}; \epsilon^e, T, p, D)$$

Other equivalent potentials can be obtained by means of the Legendre-Fenchel transform, in particular the partial transform changing  $\dot{D}$  to its dual variable  $y$ ;

$$\varphi^*(\dot{\epsilon}^p, \dot{p}, y, \mathbf{q}; \epsilon^e, T, p, D)$$

The constitutive equation for damage evolution  $D$  is given by the normality property of that potential:

$$\left| \dot{D} = - \frac{\partial \varphi^*}{\partial y} \right|$$

### 3 Models for Ductile Plastic Evolution

Restricting ourselves to isotropic plasticity and isotropic damage, mathematical models are of a scalar nature. Ductile plastic damage, as plasticity, is a phenomenon which does not depend explicitly upon time.

Within these hypothesis the main features of ductile plastic damage can be described by a potential of dissipation restricted to three variables

$$\varphi^*(y, \dot{p}, T)$$

written as a power function of  $y$  for convenience and linear in  $\dot{p}$  to ensure the non explicit dependency of  $D$  with time:

$$\varphi^* = \frac{S_0}{(s_0 + 1)} \left( \frac{-y}{S_0} \right)^{s_0 + 1} \dot{p}$$

where  $s_0$  and  $S_0$  are material and temperature dependent.

The complementary law of evolution of damage derives from  $\varphi^*$  by

$$\dot{D} = - \frac{\partial \varphi^*}{\partial y} = \left( \frac{-y}{S_0} \right)^{s_0 + 1} \dot{p}$$

### 3.1 One Dimensional Models Written in Terms of Stress.

It can be shown that several models proposed in the past from phenomenological considerations may be derived from the above general constitutive equation:

In the one-dimensional case of monotonic loading defined by the stress  $\sigma$  and the plastic strain  $\epsilon_p$ :

$$\dot{p} = \dot{\epsilon}_p, \quad -\dot{\gamma} = \frac{\dot{\sigma}^2}{2E}.$$

With

$$\bar{\sigma} = \frac{\sigma}{(1-D)}$$

then follows

$$\dot{D} = \left( \frac{\bar{\sigma}^2}{2ES_0} \right)^{s_0} \dot{\epsilon}_p$$

Replacing  $\dot{\epsilon}_p$  by its value taken from the Ramberg-Osgood law of hardening coupled with damage by means of the effective stress:

$$\epsilon^p = \left( \frac{\bar{\sigma}}{K} \right)^M \quad \text{or} \quad \dot{\epsilon}_p = \frac{M}{K} \left( \frac{\bar{\sigma}}{K} \right)^{M-1} \dot{\bar{\sigma}}$$

gives:

$$\dot{D} = \frac{\bar{\sigma}^{(2s_0 + M - 1)}}{\frac{K^M}{M} (2ES_0)^{s_0}} \dot{\bar{\sigma}}$$

or, with change of notation:

$$dD = \left( \frac{\bar{\sigma}}{S_1} \right)^{s_1} d\bar{\sigma}$$

This is the model proposed by Broberg [19]. Usually a damage threshold does exist such that:

$$\sigma < \sigma_D \rightarrow \dot{D} \equiv 0$$

Then a more realistic model has been proposed by Dufailly and Lemaitre [20]:

$$dD = \left\langle \frac{\sigma - \sigma_D}{S_2(1-D)} \right\rangle^{s_2} \frac{d\sigma}{S_2}$$

with:  $\langle x \rangle = x$  if  $x > 0$ ,  $\langle x \rangle = 0$  if  $x \leq 0$

Here  $\sigma_D$ ,  $S_2$ , and  $s_2$  are material constants identified from tension tests in which damage is derived from elasticity modulus change as explained in section 3.2. (Identification).

These models written in term of stress are difficult to apply in the range of large strains, where usually ductile plastic damage occurs, because the stress does not vary very much as the material becomes almost perfectly plastic close to rupture. Then a model written in terms of strains is more suitable especially for metal forming calculations.

**3.2 A Three Dimensional Model Written in Terms of Strains.** Ductile plastic damage generally occurs with large deformations and in metal forming calculations. Large strain theory must then be used.

Now  $\epsilon$  is the Green Lagrange strain tensor,  
 $\sigma$  is the Cauchy stress tensor,  
 $\epsilon^e$  is the elastic strain tensor defined with respect to the unstressed state.

Due to the large strain hypothesis the damage is written as function of total strain instead of plastic strain. Then:

$$\dot{p} = \left( \frac{2}{3} \dot{\epsilon} : \dot{\epsilon} \right)^{1/2}$$

**Formulation.** From the potential of dissipation already chosen in section 3:

$$\dot{D} = \left( \frac{-\dot{\gamma}}{S_0} \right)^{s_0} \dot{p}$$

In the expression for  $\dot{\gamma}$ :

$$-\dot{\gamma} = \frac{\sigma_{eq}^2}{2E(1-D)^2} \left[ \frac{2}{3} (1+\nu) + 3(1-2\nu) \left( \frac{\sigma_{II}}{\sigma_{eq}} \right)^2 \right]$$

replace  $\sigma_{eq}$  by its value, taken from the Ramberg-Osgood hardening law coupled with damage and written for the three-dimensional case:

$$p = \left[ \frac{\sigma_{eq}}{(1-D)K} \right]^M \quad \text{or} \quad \frac{\sigma_{eq}}{1-D} = Kp^{\frac{1}{M}}$$

Then

$$\dot{D} = \left( \frac{K^2}{2ES_0} \left[ \frac{2}{3} (1+\nu) + 3(1-2\nu) \left( \frac{\sigma_{II}}{\sigma_{eq}} \right)^2 \right] p^{\frac{2}{M}} \right)^{s_0} \dot{p}$$

This is the general constitutive equation for ductile plastic damage.

**Integration in the Particular Case of Radial Loading.** In most engineering applications the loading is such that the directions of principal stresses (different in each point  $M$ ) are constant with time through the process and may be described by:

$$\sigma_{(M,i)} = \alpha_{(i)} \sigma_{r(M)} \quad (\alpha \text{ scalar})$$

Within this hypothesis, the triaxiality ratio  $\sigma_{II}/\sigma_{eq}$  is constant with time and it is possible, by integration, to obtain a simple relation between the actual value of damage  $D$  and the accumulated plastic strain  $p$ . If  $p_D$  is the damage strain threshold:

$$p < p_D \rightarrow \dot{D} = 0$$

Integration yields:

$$D = \left( \frac{K^2}{2ES_0} \left[ \frac{2}{3} (1+\nu) + 3(1-2\nu) \left( \frac{\sigma_{II}}{\sigma_{eq}} \right)^2 \right] \right)^{s_0} \frac{M}{2s_0 + M} \left\langle p^{\frac{2s_0 + M}{M}} - p_D^{\frac{2s_0 + M}{M}} \right\rangle$$

This expression can be written in a simpler fashion by introducing the rupture strain  $p_R$  as a function of the triaxiality ratio  $\sigma_{II}/\sigma_{eq}$  corresponding to the intrinsic value of damage at failure  $D_c$  which we assume to be a material property:

$$p = p_R \rightarrow D = D_c$$

$$D_c = \left( \frac{K^2}{2ES_0} \left[ \frac{2}{3} (1+\nu) + 3(1-2\nu) \left( \frac{\sigma_{II}}{\sigma_{eq}} \right)^2 \right] \right)^{s_0} \frac{M}{2s_0 + M} \left( p_R^{\frac{2s_0 + M}{M}} - p_D^{\frac{2s_0 + M}{M}} \right)$$

Dividing  $D$  by  $D_c$  yields:

$$D = D_c \left\langle \frac{p^{\frac{2s_0 + M}{M}} - p_D^{\frac{2s_0 + M}{M}}}{p_R^{\frac{2s_0 + M}{M}} - p_D^{\frac{2s_0 + M}{M}}} \right\rangle$$

In the range of large deformations in metals, the hardening exponent  $M$  is usually very high (a perfectly plastic material corresponds to  $M = \infty$ ). Otherwise, identifications of one-dimensional models described in section 3.1. show that the  $s_0$  coefficient is of order of magnitude of unity then  $(2s_0 + M)/M$  is of order unity.

$p_D$  and  $p_R$  depend upon the triaxiality ratio but it is physically admissible to assume that this dependence is the

same for both quantities and, that the ratio  $p_D/p_R$  does not depend upon triaxiality, and is equal to its value in the one-dimensional case:

$$\frac{p_D}{p_R} = \frac{\epsilon_D}{\epsilon_R}$$

where  $\epsilon_D$  and  $\epsilon_R$  are the one-dimensional strain at damage threshold and at failure.

Then 
$$D = D_c \left\langle \frac{p \frac{\epsilon_R}{p_R} - \epsilon_D}{\epsilon_R - \epsilon_D} \right\rangle$$

The accumulated strain at rupture  $p_R$  can also be expressed as a function of  $\epsilon_R$ ,  $\sigma_H/\sigma_{eq}$  and of its value in the one-dimensional case:

$$\frac{\sigma_H}{\sigma_{eq}} = \frac{1}{3}$$

From the expression of  $D_c$  with  $(2s_0 + M)/M = 1$  follows:

$$\begin{aligned} p_R \left( \frac{\sigma_H}{\sigma_{eq}} = \frac{1}{3} \right) \\ = \epsilon_R = \frac{2ES_0 D_c}{K^2 \left( 1 - \frac{\epsilon_D}{\epsilon_R} \right)} \left[ \frac{2}{3} (1 + \nu) + 3(1 - 2\nu) \left( \frac{1}{9} \right) \right]^{-s_0} \end{aligned}$$

and

$$p_R \left( \frac{\sigma_H}{\sigma_{eq}} \right) = \epsilon_R \left[ \frac{2}{3} (1 + \nu) + 3(1 - 2\nu) \left( \frac{\sigma_H}{\sigma_{eq}} \right)^2 \right]^{-s_0}$$

$s_0$  being of the order of 1 as will be checked in section 4.1, the final equation is:

$$D = D_c \left\langle \frac{p \left[ \frac{2}{3} (1 + \nu) + 3(1 - 2\nu) \left( \frac{\sigma_H}{\sigma_{eq}} \right)^2 \right] - \epsilon_D}{\epsilon_R - \epsilon_D} \right\rangle$$

In the particular case of one dimension

$$D = D_c \left\langle \frac{\epsilon - \epsilon_D}{\epsilon_R - \epsilon_D} \right\rangle$$

which is very simple!

With the above expression for  $p_R$  the equation for  $D_c$  can be written in a very simple way using  $s_0 = 1$  and  $(2s_0 + M)/M = 1$ :

$$D_c = \frac{K^2}{2ES_0} \frac{\epsilon_R}{p_R} (p_R - p_D) = \frac{K^2}{2ES_0} (\epsilon_R - \epsilon_D)$$

This allows to replace  $K^2/2ES_0$  by  $D_c/(\epsilon_R - \epsilon_D)$  in the differential constitutive equation for  $\dot{D}$  to obtain the final results.

|  |  |
|--|--|
| Differential model<br>valid for any loading path<br>$\dot{p} = \left( \frac{2}{3} \dot{\epsilon} : \dot{\epsilon} \right)^{1/2}$ | $\dot{D} = \frac{D_c}{\epsilon_R - \epsilon_D} \left[ \frac{2}{3} (1 + \nu) + 3(1 - 2\nu) \left( \frac{\sigma_H}{\sigma_{eq}} \right)^2 \right] p^{2/M} \dot{p}$                     |
| Model valid for radial<br>loading only<br>$p = \left( \frac{2}{3} \epsilon : \epsilon \right)^{1/2}$                             | $D = \frac{D_c}{\epsilon_R - \epsilon_D} \left\langle p \left[ \frac{2}{3} (1 + \nu) + 3(1 - 2\nu) \left( \frac{\sigma_H}{\sigma_{eq}} \right)^2 \right] - \epsilon_D \right\rangle$ |

These models depend upon material constants  $\epsilon_D$ ,  $\epsilon_R$ ,  $D_c$  for damage properties and Poisson's ratio  $\nu$ . The first one which also depends upon hardening exponent  $M$  has to be integrated in each particular case of history of loading represented by  $\sigma_H/\sigma_{eq}(t)$  and  $p(t)$  to obtain the damage evolution  $D(t)$ . The second, only valid for radial or proportional loading gives directly the damage  $D$  as a function of  $t$  when  $\sigma_H/\sigma_{eq}$  is known.

**Identification of Parameters.** Identification of such models consists in the quantitative evaluation of the three coefficients  $\epsilon_D$ ,  $\epsilon_R$  and  $D_c$  characteristic of each material at each temperature considered. ( $M$  is known from the uniaxial hardening curve.)

$\epsilon_D$  and  $D_c$  need a measurement of damage which is somewhat difficult due to the fact that damage does not affect very much any measurable quantity far from the rupture condition.

Let us return to the definition of  $D$  and the effective stress concept applied to elasticity: it is possible to measure  $D$  through the variations of the elasticity modulus [21].

Writing again the damage elasticity law:

$$\bar{\sigma} = E \epsilon_e \quad \text{or} \quad \sigma = E(1 - D) \epsilon_e$$

$E$  being the Young's modulus of undamaged material, the quantity

$$E(1 - D) = \bar{E}$$

can be considered as the elasticity modulus of the damaged material.  $E$  being known and  $\bar{E}$  measured by a special technique described below, then the damage  $D$  is evaluated as:

$$D = 1 - \frac{\bar{E}}{E}$$

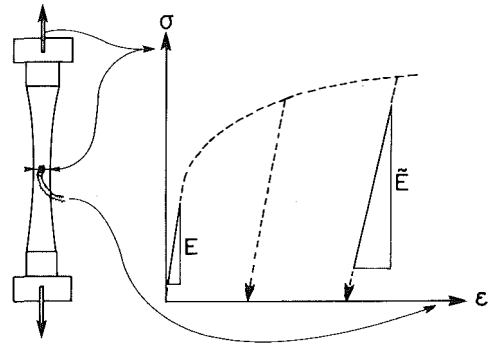
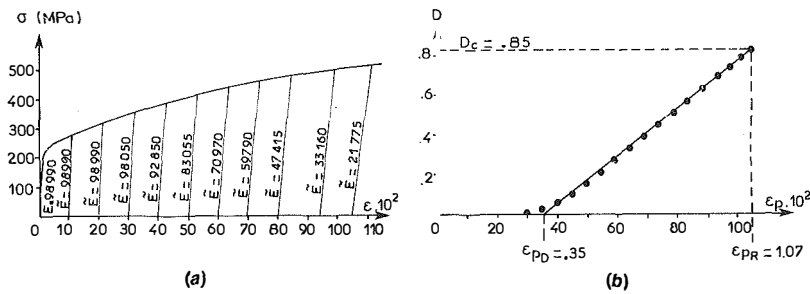


Fig. 2 Measure of damaged elasticity modulus

The damage elasticity modulus  $\bar{E}$  can be measured through tension tests but as damage is always localized in a very small region of the specimen some special precautions are needed. They are described in reference [22]. For the static method, a specimen of the shape given in Fig. 2 is needed. Roughly, ductile plastic damage begins when necking starts. As a



consequence, rapid change of geometry occurs and the local strain in the most damaged region must be measured through very small strain gages, say  $.5 \times .5$  mm. As these gages have a maximum strain amplitude of 10 to 15 percent then, if the damage has to be measured up to strain of 50 or 100 percent or more, the gages have to be changed, which interrupts the tests. Last point, a better accuracy is obtained if  $E$  is measured during unloading as shown in Fig. 2. With these precautions taken, a relative accuracy of 5 percent may be expected for  $D$ . When texture induces a variation of the elasticity modulus this is generally for small values of strain far below the damage threshold.  $\epsilon_D$  is then defined by the value of  $\epsilon$  for which the derivative  $dD/d\epsilon$  begins to be positive. The damage is of course supposed to be zero for  $\epsilon < \epsilon_D$ .

An example is shown in Fig. 3 for copper (99.9 percent) at room temperature, the variation of damage elasticity modulus is shown in Fig. 3(a) and the evolution of damage, deduced from  $D = 1 - \bar{E}/E$ , is plotted against the true strain  $\epsilon_v = \log(1 + \epsilon)$  in Fig. 3(b).

In the range of normal scatter  $D$  is linear in  $\epsilon$  as predicated by the model.

$\epsilon_D$  is the value of  $\epsilon$  where the best fit straight line of experimental data cuts the  $\epsilon$  axis.

$D_c$  is the value of  $D$  for the strain at rupture  $\epsilon_R$ .

## 4 Applications

The two main properties of the model is the linearity of  $D$  with strain and the influence of triaxiality given by the factor

$$\left[ \frac{2}{3} (1 + \nu) + 3(1 - 2\nu) \left( \frac{\sigma_H}{\sigma_{pg}} \right)^2 \right]$$

The first set of applications will check the first property and the second application will give some elements of the second.

**4.1 Ductile Plastic Damage Characteristics of Several Metals.** The method of identification described above has been applied to several materials. The results are given below in Fig. 4 with the characteristic values of  $\epsilon_D$ ,  $\epsilon_R$ , and  $D_c$ . The linearity of  $D$  with  $\epsilon$  is again well verified in these six examples.

**4.2 Influence of Triaxiality on Strain to Failure.** A way to check the model with regard to the triaxiality effect is to compare the strain to failure predicted by  $p = p_R$  when  $D$  reaches its critical value  $D_c$  with the one predicted by the McClintock or Rice and Tracey models for growth of voids and with experiments.

The strain to rupture  $p_R$  for any value of the triaxiality ratio has already been calculated to derive the model. Dividing  $p_R$  by  $\epsilon_R$  (the strain to rupture in the one-dimensional case) yields (with  $s_0 = 1$ ).

$$\frac{p_R}{\epsilon_R} = \left[ \frac{2}{3} (1 + \nu) + 3(1 - 2\nu) \left( \frac{\sigma_H}{\sigma_{eq}} \right)^2 \right]^{-1}$$

Let us now calculate the same ratio with the McClintock or Rice and Tracey models [1, 2]. These models predict the growth of a void of radius  $R$  in a plastic matrix by

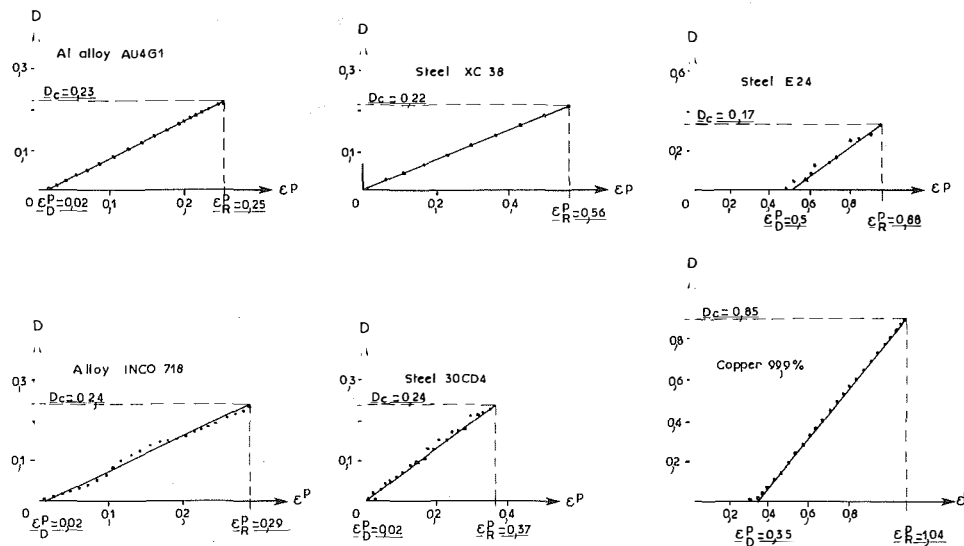
$$\frac{dR}{R} = B \exp\left(C \frac{\sigma_H}{\sigma_{eq}}\right) dp$$

where  $B$  and  $C = 1.5$  are coefficients determined by the theory.

Assume an initial value of the radius  $R_0$  which grows after a threshold value of the equivalent strain  $p_D$  is reached.

$$p < p_D \rightarrow R = R_0$$

Assume that failure occurs when a critical value of the radius  $R_c$  is reached [23]



$$R = R_c \rightarrow p = p_R$$

and integrate the differential equation for constant ratio  $\sigma_H/\sigma_{eq}$  between these two limits.

$$\text{Log} \frac{R_c}{R_0} = B(p_R - p_D) \exp\left(C \frac{\sigma_H}{\sigma_{eq}}\right)$$

or

$$R_c = R_0 \exp\left(B[p_R - p_D] \exp\left(C \frac{\sigma_H}{\sigma_{eq}}\right)\right)$$

Dividing by the same expression written for the one-dimensional case:

$$R_c = R_0 \exp\left(B[\epsilon_R - \epsilon_D] \exp \frac{C}{3}\right)$$

after some mathematical manipulations, we obtain:

$$\frac{p_R}{\epsilon_R} = \frac{p_D}{\epsilon_R} + \frac{1 - \frac{\epsilon_D}{\epsilon_R}}{\exp C \left( \frac{\sigma_H}{\sigma_{eq}} - \frac{1}{3} \right)}$$

The corresponding values of  $p_R/\epsilon_R$  for  $C = 1.5$  are plotted against the triaxiality ratio  $\sigma_H/\sigma_{eq}$  in Fig. 5. The two dashed lines correspond to extreme values of the ratios  $p_D/\epsilon_R$  and  $\epsilon_D/\epsilon_R$  that experiments on metal may show. That is:

$$\frac{p_D}{\epsilon_R} = \frac{\epsilon_D}{\epsilon_R} = 0 \quad (\text{no threshold})$$

$$\frac{p_D}{\epsilon_R} \approx \frac{\epsilon_D}{\epsilon_R} = .2$$

Values of  $p_R/\epsilon_R$  given by the model described in this paper are also plotted on the figure for the two values of  $\nu$ ,  $\nu = .25$  and  $\nu = .33$ .

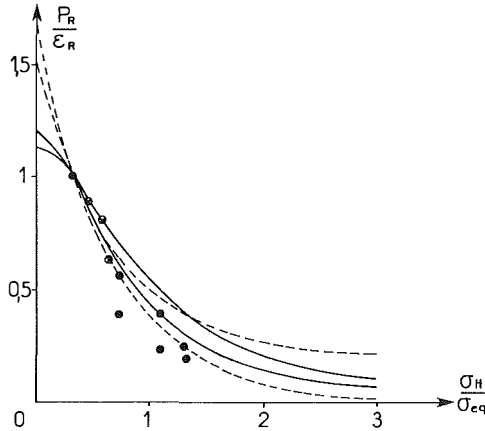


Fig. 5 Influence of triaxiality on strain to rupture •• A508 steel  
⊗ ⊗ H Steel

////// Domaine cover by McClintock-R.T. model  
————— Domaine cover by present model.

Except for very small values of the ratio  $\sigma_H/\sigma_{eq}$ , the domain of this model (between the two solid lines) covers the domain of the McClintock or Rice and Tracey models (between the two dashed lines). This strong influence of triaxiality ratio is also in accordance with results of reference [16] and [17]. Also plotted in Fig. 5 are some points representing experimental results from reference [23] obtained with notched specimens of A508 steel at room temperature and from reference [24] obtained with deep-drawing of sheets of "H" steel.

Considering the simplicity of the model and the scatter of tests, correlation may be considered to be good, and provides a proof of the validity of the triaxiality factor:

$$\left[ \frac{3}{2} (1 + \nu) + 3(1 - 2\nu) \left( \frac{\sigma_H}{\sigma_{eq}} \right)^2 \right]$$

## 5 Conclusion

The integrated model of ductile plastic damage developed on a thermodynamic and effective stress concept basis is linear in strain and shows a very strong effect of triaxiality as do the McClintock and Rice and Tracey models. Its range of validity is limited by the hypothesis of isotropy of damage and isotropy of plasticity, and also by the hypothesis of constant triaxiality ratio during loading, that is radial loading in the sense of plasticity (approximately constant principal directions of stresses), which is a very common case in metal forming. In more general cases of loading, the differential model written in terms of a continuum mechanics variable of damage is easy to apply together with plasticity equations coupled with damage in any type of structural step by step calculations such as the finite element method to predict the state of damage and ductile fracture.

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