Finite Element Method in Steady-State and Transient Heat Conduction

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Finite Element Method in Steady-State and Transient Heat Conduction

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Overview

Practical heat transfer problems are described by the partial differential equations with complex boundary conditions. Moreover, the irregular boundaries of the heat transfer region cause that it is difficult to find an analytical solution. With the advent of high performance computing, it is possible to solve even very complicated heat transfer problems in a numerical way. The numerical methods allow obtaining the approximate values of unknowns at discrete points (nodes). Various numerical methods have been developed and applied to solve numerous engineering problems – the finite difference method (FDM), the finite volume method (FVM), and the finite element method (FEM) are most frequently used is practice. The advantage of the FEM method is that the general-purpose computer program can be developed easily to analyze complicated heat transfer problems. Furthermore, the regions with irregular boundaries and complicated boundary conditions may be handled using this method.

This FEM method requires division of the problem domain into many subdomains for which the heat transfer problem is analyzed. Each subdomain is called the finite element; thus, the name of this method is the finite element method. The FEM method is widely described in the literature and used for solving, for example, the heat transfer problems [1–4], analyzing the behavior of structures [4–7], or even for predicting the fluid flow phenomena [3, 8].

Governing Equation

The application of the FEM method for solving heat conduction problems is presented for two-dimensional case. Thus, it is assumed that thickness measured in the plane perpendicular to the element face equals 1 m. The basic FEM equations for two-dimensional element are derived using Galerkin method. The governing equation for the heat conduction in solids is [1, 2]

$$c\rho\frac{\partial T}{\partial t}=\frac{\partial}{\partial x}\left(k_x\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y\frac{\partial T}{\partial y}\right)+q_v \quad (1)$$

where c denotes the specific heat capacity, ρ is the density, and anisotropic thermal conductivities in directions x and y are denoted as k_x and k_y. The initial conditions, necessary for solving (1), are

$$T(x, y, t)|_{t=0} = T_0(x, y) \tag{2}$$

Three different types of boundary conditions (3–5) are applied. The first type (Dirichlet) boundary condition assumes that the temperature value at the boundary region is known. For the second type (Neumann) boundary condition, the derivative of temperature at the boundary is known. The third type boundary condition (mixed boundary condition) combines the Dirichlet and the Neumann boundary condition.

For the heat transfer domain (see Fig. 1), the Dirichlet type boundary condition – the constant boundary temperature T_b known at the boundary region Γ_T – is given below

$$T|_{\Gamma_T} = T_b \tag{3}$$

Neumann type boundary condition – the constant heat flux q_b at the boundary Γ_q is given by

$$\left(k_x \frac{\partial T}{\partial x} n_x + k_y \frac{\partial T}{\partial y} n_y \right)\Bigg|_{\Gamma_q} = \mathbf{q}_b \cdot \mathbf{n} = q_{bn} \tag{4}$$

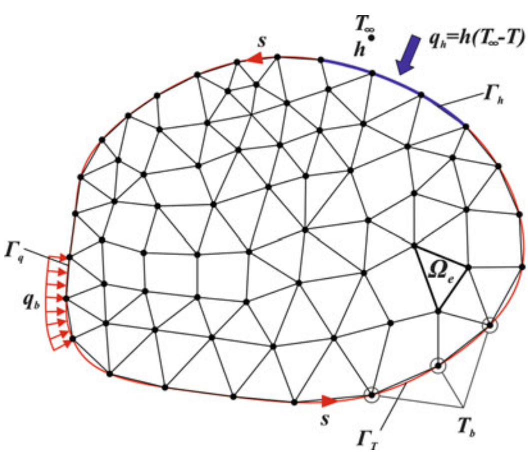
The mixed type boundary condition – the convective heat flux q_h with known heat transfer coefficient h and the bulk temperature T_∞ given at the boundary Γ_h is

$$\left(k_x \frac{\partial T}{\partial x} n_x + k_y \frac{\partial T}{\partial y} n_y \right)\Bigg|_{\Gamma_h} = h\left(T_\infty - T|_{\Gamma_h}\right) \tag{5}$$

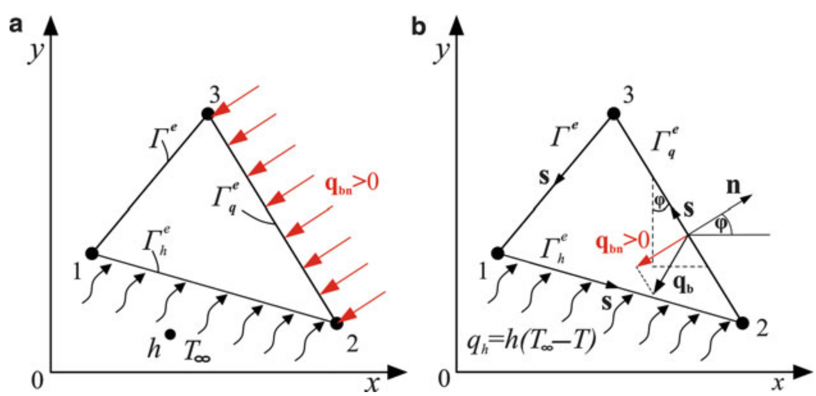
These three types of boundary conditions are illustrated in Fig. 1.

The heat flux q_b is in the inward normal direction, which is assumed to be positive. The normal vector $\mathbf{n}$ is a unit vector directed to the outside of the region (see Fig. 2), while its components are equal to directional cosines

$$n_x = \cos\varphi,\ n_y = \cos\left(\frac{\pi}{2} - \varphi\right) = \sin\varphi \tag{6}$$



Finite Element Method in Steady-State and Transient Heat Conduction, Fig. 1 Heat transfer domain with boundary conditions



Finite Element Method in Steady-State and Transient Heat Conduction, Fig. 2 The triangular finite element with node numbering in counterclockwise sequence (**a**) boundary conditions (**b**) sign convention of the heat flux vector **q**

where φ is the angle between the normal vector $\mathbf{n}$ and the horizontal direction x.

The thermal conductivity matrix is

$$[\mathbf{D}] = \begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix} \tag{7}$$

Column vector of the spatial temperature derivatives equals to

$$\{\mathbf{g}\} = \begin{Bmatrix} \dfrac{\partial T}{\partial x} \\ \dfrac{\partial T}{\partial y} \end{Bmatrix} \tag{8}$$

The components of the heat flux vector in x and y direction are

$$\begin{Bmatrix} q_x \\ q_y \end{Bmatrix} = -[\mathbf{D}]\{\mathbf{g}\} \tag{9}$$

The heat flux vector can be expressed as

$$\begin{aligned} \mathbf{q} &= -k_x \frac{\partial T}{\partial x}\mathbf{i} - k_y \frac{\partial T}{\partial y}\mathbf{j} = q_x\mathbf{i} + q_y\mathbf{j} \\ \mathbf{n} &= n_x\mathbf{i} + n_y\mathbf{j} \end{aligned} \tag{10}$$

The symbols $\mathbf{i}$ and $\mathbf{j}$ denote unit vectors in x and y direction. The normal component of the heat flux vector is

$$q_n = -\mathbf{q}\cdot\mathbf{n} = -\left(-k_x \frac{\partial T}{\partial x} n_x - k_y \frac{\partial T}{\partial y} n_y\right) \tag{11}$$

where q_n is the normal component of the vector $\mathbf{q}$ which is perpendicular to the isotherm and $\mathbf{n}$ is the outward unit normal vector.

Let us assume the heat flows into the element control domain with the heat flux vector directed inward (see Fig. 2b). Then the cosine of the angle between the vectors $\mathbf{q}$ and $\mathbf{n}$ is negative. Therefore, the dot product of these two vectors is negative. Thus, according to (11), the normal component of the boundary heat flux vector q_{bn} is positive. This means, that the heat is added into the element control domain. The temperature distribution inside the element Ω^e is approximated using the element shape functions $\phi_j^e(x,y)$. Each shape function corresponds to one node, and equals to 1 at the node location and 0 at the locations of other nodes. Inside the element domain, the shape function takes values between 0 and 1. The temperature in each element is expressed as

$$T^e(x,y) = \sum_{j=1}^{n} T_j^e \cdot \phi_j^e(x,y) = [\Phi^e]\{\mathbf{T}^e\} \tag{12}$$

where n relates to the number of nodes inside the element and the nodal temperatures are denoted as T_j^e $j = 1,\ldots,n$. For example, the symbol T_1^e relates to the temperature in the first node of the element.

The Galerkin method, which allows determining the nodal temperatures T_j^e $j = 1,\ldots,n$, for the element domain Ω^e, is written in the following form:

$$\begin{aligned} &\int_{\Omega^e} \left[c\rho \frac{\partial T^e}{\partial t} - \left[\frac{\partial}{\partial x}\left(k_x \frac{\partial T^e}{\partial x}\right) \right.\right. \\ &\left.\left. + \frac{\partial}{\partial y}\left(k_y \frac{\partial T^e}{\partial y}\right)\right] - q_v \right] \phi_i^e(x,y)\,dxdy = 0 \end{aligned} \tag{13}$$

Applying to the above integral, the Green's theorem in the form

$$\int_{\Omega^e} \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}\right) dxdy = \oint_{\Gamma^e} (Fdx + Gdx) \tag{14}$$

and assuming that the functions F and G equal to

$$F = -k_y \frac{\partial T^e}{\partial y}\phi_i^e \text{ and } G = k_x \frac{\partial T^e}{\partial x}\phi_i^e \tag{15}$$

Then, transforming second and third terms in (13) using (15) gives

$$\begin{aligned} &\int_{\Omega^e} \left[\frac{\partial}{\partial x}\left(k_x \frac{\partial T^e}{\partial x}\phi_i^e\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial T^e}{\partial y}\phi_i^e\right)\right] dxdy \\ &= \int_{\Gamma^e} \phi_i^e \left(-k_y \frac{\partial T^e}{\partial y} dx + k_x \frac{\partial T^e}{\partial x} dy\right) \end{aligned} \tag{16}$$

The left side of (16) is the integral over element domain Ω^e and the right-hand side denotes the integral over element boundary Γ^e. Integrating by parts the left side integrand of (16) yields

$$\int_{\Omega^e} \left[\frac{\partial}{\partial x}\left(k_x\frac{\partial T^e}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y\frac{\partial T^e}{\partial y}\right)\right]\phi_i^e dxdy$$
$$= -\int_{\Omega^e}\left(k_x\frac{\partial T^e}{\partial x}\frac{\partial \phi_i^e}{\partial x}+k_y\frac{\partial T^e}{\partial y}\frac{\partial \phi_i^e}{\partial y}\right)dxdy \quad (17)$$
$$+\int_{\Gamma^e}\phi_i^e\left(-k_y\frac{\partial T^e}{\partial y}dx+k_x\frac{\partial T^e}{\partial x}dy\right)$$

Taking into account (17) into (13), the following is obtained:

$$\int_{\Omega^e}\left(k_x\frac{\partial T^e}{\partial x}\frac{\partial \phi_i^e}{\partial x}+k_y\frac{\partial T^e}{\partial y}\frac{\partial \phi_i^e}{\partial y}\right)dxdy$$
$$= -\int_{\Omega^e} c\rho\phi_i^e\phi_j^e dxdy\frac{dT_j^e}{dt}+\int_{\Omega^e}\phi_i^e q_v dxdy$$
$$+\int_{\Gamma^e}\phi_i^e\left(-k_y\frac{\partial T^e}{\partial y}dx+k_x\frac{\partial T^e}{\partial x}dy\right) \quad (18)$$

According to Fig. 2, the increments of the components of boundary vector **s** are

$$-dx = ds\cdot\sin\varphi = n_y ds$$
$$dy = ds\cdot\cos\varphi = n_x ds \quad (19)$$

Including (11) and (19) into the boundary integral in (18), the following is obtained:

$$\int_{\Gamma^e}\phi_i^e\left(-k_y\frac{\partial T^e}{\partial y}dx+k_x\frac{\partial T^e}{\partial x}dy\right)$$
$$=\int_{\Gamma^e}\phi_i^e\left(k_y\frac{\partial T^e}{\partial y}n_y ds+k_x\frac{\partial T^e}{\partial x}n_x ds\right)=\int_{\Gamma^e}\phi_i^e q_n ds \quad (20)$$

Taking into account boundary conditions given by (4) and (5), (20) becomes

$$\int_{\Gamma^e}\phi_i^e\left(-k_y\frac{\partial T^e}{\partial y}dx+k_x\frac{\partial T^e}{\partial x}dy\right)$$
$$=\int_{\Gamma_q^e}\phi_i^e q_{bn} ds+\int_{\Gamma_h^e}\phi_i^e h(T_\infty - T^e)ds \quad (21)$$

Hence, (18) can be written as follows:

$$\int_{\Omega^e}\left(k_x\frac{\partial T^e}{\partial x}\frac{\partial \phi_i^e}{\partial x}+k_y\frac{\partial T^e}{\partial y}\frac{\partial \phi_i^e}{\partial y}\right)dxdy$$
$$= -\int_{\Omega^e} c\rho\phi_i^e\phi_j^e\frac{dT_j^e}{dt}dxdy+\int_{\Omega^e}\phi_i^e q_v dxdy$$
$$+\int_{\Gamma_q^e}\phi_i^e q_{bn} ds+\int_{\Gamma_h^\varepsilon}\phi_i^e h(T_\infty - T^e)ds \quad (22)$$

Equation (22) can be written in a more convenient form as

$$\sum_{j=1}^{n}\left(M_{ij}^e\frac{dT_j^e}{dt}+K_{ij}^e T_j^e\right)=f_{Q,i}^e+f_{q,i}^e+f_{\alpha,i},$$
$$i = 1,\ldots,n \quad (23)$$

The heat flux by conduction $K_{c,ij}^e$ and the convective flux $K_{h,ij}^e$ are incorporated in K_{ij}^e as follows:

$$K_{ij}^e = K_{c,ij}^e + K_{h,ij}^e \quad (24)$$

The other terms in (23) are

$$M_{ij}^e = \int_{\Omega^e} c\rho\ \phi_i\phi_j dxdy \quad (25)$$

$$K_{c,ij}^e = \int_{\Omega^e}\left(k_x\frac{\partial \phi_i^e}{\partial x}\frac{\partial \phi_j^e}{\partial x}+k_y\frac{\partial \phi_i^e}{\partial y}\frac{\partial \phi_j^e}{\partial y}\right)dxdy \quad (26)$$

$$K_{h,ij}^e = \int_{\Gamma_\alpha^e} h\phi_i^e\phi_j^e ds \quad (27)$$

$$f_{G,i}^e = \int_{\Omega^e} q_v \phi_i^e dxdy \tag{28}$$

$$f_{q,i}^e = \int_{\Gamma_q^e} q_{bn} \phi_i^e ds \tag{29}$$

and

$$f_{h,i}^e = \int_{\Gamma_h^e} hT_\infty \phi_i^e ds \tag{30}$$

Equation (23) can be expressed in a matrix form

$$[\mathbf{M}^e]\left\{\frac{d\ \mathbf{T}^e}{dt}\right\} + [\mathbf{K}^e]\{\mathbf{T}^e\} = \{\mathbf{f}^e\} \tag{31}$$

where the element capacitance matrix is denoted as $[\mathbf{M}^e]$ and term $[\mathbf{K}^e]$ relates to element stiffness matrix. The element load vector is expressed as $\{\mathbf{f}^e\}$. Defining the shape function matrix as

$$[\Phi^e] = \left[\phi_1^e,\ \phi_2^e, \ldots,\ \phi_n^e\right] \tag{32}$$

and the vector of nodal temperatures as

$$\{\mathbf{T}^e\} = \begin{Bmatrix} T_1^e \\ T_2^e \\ \cdots \\ T_n^e \end{Bmatrix} \tag{33}$$

Then the temperature inside the element is expressed as

$$T^e = [\Phi^e]\{\mathbf{T}^e\} \tag{34}$$

The matrix of temperature gradient is then defined as

$$\{\mathbf{g}^e\} = \begin{Bmatrix} \dfrac{\partial \mathbf{T}^e}{\partial x} \\ \dfrac{\partial \mathbf{T}^e}{\partial y} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial \phi_1^e}{\partial x} & \dfrac{\partial \phi_2^e}{\partial x} & \dfrac{\partial \phi_3^e}{\partial x} & \dfrac{\partial \phi_4^e}{\partial x} \\ \dfrac{\partial \phi_1^e}{\partial y} & \dfrac{\partial \phi_2^e}{\partial y} & \dfrac{\partial \phi_3^e}{\partial y} & \dfrac{\partial \phi_4^e}{\partial y} \end{bmatrix} \begin{Bmatrix} T_1^e \\ T_2^e \\ T_3^e \\ T_4^e \end{Bmatrix} = [\mathbf{B}^e]\{\mathbf{T}^e\} \tag{35}$$

where $[\mathbf{B}^e]$ is the spatial derivatives matrix.

The element stiffness matrix $[\mathbf{K}^e]$ can be written as a sum

$$[\mathbf{K}^e] = \left[\mathbf{K}_c^e\right] + \left[\mathbf{K}_h^e\right] \tag{36}$$

where element conductivity matrix is

$$\left[\mathbf{K}_c^e\right] = \int_{\Omega^e} [\mathbf{B}^e]^T [\mathbf{D}][\mathbf{B}^e]\ dxdy \tag{37}$$

and the convective flux matrix equals to

$$\left[\mathbf{K}_h^e\right] = \int_{\Gamma_h^e} h[\Phi^e]^T [\Phi^e] ds \tag{38}$$

The element capacitance matrix $[\mathbf{M}^e]$ is expressed as

$$[\mathbf{M}^e] = \int_{\Omega^e} c\rho[\Phi^e]^T [\Phi^e] dxdy \tag{39}$$

The vector of nodal loads $\{\mathbf{f}^e\}$ has the following components:

$$\{\mathbf{f}^e\} = \{\mathbf{f}_G^e\} + \left\{\mathbf{f}_q^e\right\} + \{\mathbf{f}_h^e\} \tag{40}$$

where the heat source vector is

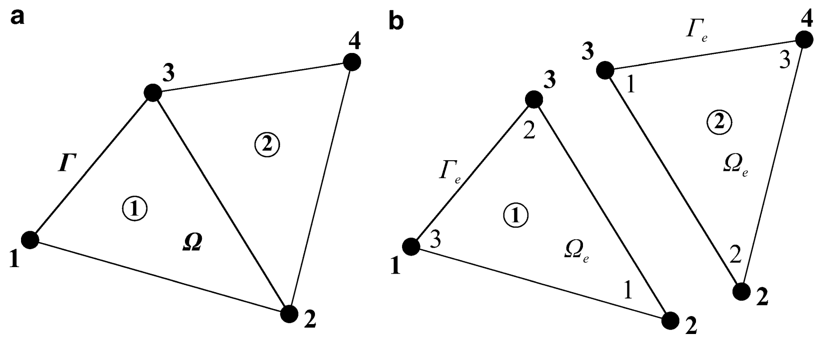
$$\{\mathbf{f}_G^e\} = \int_{\Omega^e} q_v [\Phi^e]^T dxdy \tag{41}$$

The boundary heat flux vector is expressed as

$$\left\{\mathbf{f}_q^e\right\} = \int_{\Gamma_q^e} q_{bn} [\Phi^e]^T ds, \tag{42}$$

and the convective flux vector is given by

$$\{\mathbf{f}_h^e\} = \int_{\Gamma_h^e} hT_\infty [\Phi^e]^T ds \tag{43}$$



Finite Element Method in Steady-State and Transient Heat Conduction, Fig. 3 Two triangular elements (**a**) global nodes (*bolded*) (**b**) local nodes (*normal font*)

Defining the vector of temperature derivatives with respect to time:

$$\{\dot{\mathbf{T}}^e] = \left\{\frac{d\mathbf{T}^e}{dt}\right\} = \begin{Bmatrix} \frac{dT_1^e}{dt} \\ \frac{dT_2^e}{dt} \\ \cdots \\ \frac{dT_n^e}{dt} \end{Bmatrix} \tag{44}$$

Equation (31) can be written in the following form:

$$[\mathbf{M}^e]\left\{\dot{\mathbf{T}}^e\right\} + [\mathbf{K}^e]\{\mathbf{T}^e\} = \{\mathbf{f}^e\} \tag{45}$$

Equation (45) is the Galerkin representation of the transient heat transfer equation. If the thermophysical properties like k_x, k_y, or c and ρ are temperature dependent, then (45) is nonlinear and requires an iterative procedure to find the solution. If the thermophysical properties inside the element domain are temperature independent, then (45) is linear and the solution is obtained using the direct or iterative methods for solving linear systems of equations.

The transient heat transfer equation (45) can be formulated for the all elements. For the common element nodes, the assembly procedure must be applied. The way in which the element matrices are assembled is presented in the next subsection.

Finite Element Assembly Procedure

The procedure is shown for two triangular elements presented in Fig. 3.

Finite Element Method in Steady-State and Transient Heat Conduction, Table 1 Nodal correspondence table

	Local node	Global node
Element 1	1	2
	2	3
	3	1
Element 2	1	3
	2	2
	3	4

The global node numbering is shown in Fig. 3a and the local node numbering is presented in Fig. 3b. Before the element assembly procedure, the nodal correspondence table (see Table 1) is created.

In this table, the correspondence of the local node with the global node for all elements is indicated. Next the stiffness (see (36)) and mass matrixes (see (39)) for both elements are created (see Fig. 4).

The node number in a circle indicates the global number of the node. The assembled global matrix is obtained by replacing the entries, indicated with local node number, with the corresponding global numbers. Then the entries from the element matrices which have the same global row and column numbers are added. For example, $K_{11}^{(1)}$ from the first element has the global (indicated by the number in circle) row index 2 and the global column index 2; therefore, it stands at the intersection of second row and second column of the global matrix [**K**]. Similarly, the global row number of the element $K_{22}^{(2)}$ is 2 as well as the global column number equals to 2. Hence, to assemble the global entry K_{22} the

Finite Element Method in Steady-State and Transient Heat Conduction, Fig. 4 The assembly procedure for element matrices [**K**] and [**M**]

$$
[\mathbf{K}^{e1}] = \begin{array}{cc} & \\ & \\ 2 & 1 \\ 3 & 2 \\ 1 & 3 \end{array}
\begin{array}{ccc} 2 & 3 & 1 \\ 1 & 2 & 3 \\ \multicolumn{3}{c}{\left[\begin{array}{ccc} K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} \\ K_{21}^{(1)} & K_{22}^{(1)} & K_{23}^{(1)} \\ K_{31}^{(1)} & K_{32}^{(1)} & K_{33}^{(1)} \end{array}\right]} \end{array}
\qquad
[\mathbf{K}^{e2}] = \begin{array}{cc} & \\ & \\ 3 & 1 \\ 2 & 2 \\ 4 & 3 \end{array}
\begin{array}{ccc} 3 & 2 & 4 \\ 1 & 2 & 3 \\ \multicolumn{3}{c}{\left[\begin{array}{ccc} K_{11}^{(2)} & K_{12}^{(2)} & K_{13}^{(2)} \\ K_{21}^{(2)} & K_{22}^{(2)} & K_{23}^{(2)} \\ K_{31}^{(2)} & K_{32}^{(2)} & K_{33}^{(2)} \end{array}\right]} \end{array}
$$

$$
[\mathbf{K}] = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \multicolumn{4}{c}{\left[\begin{array}{cccc} K_{33}^{(1)} & K_{31}^{(1)} & K_{32}^{(1)} & 0 \\ K_{13}^{(1)} & K_{11}^{(1)} + K_{22}^{(2)} & K_{12}^{(1)} + K_{21}^{(2)} & K_{23}^{(2)} \\ K_{23}^{(1)} & K_{21}^{(1)} + K_{12}^{(2)} & K_{22}^{(1)} + K_{11}^{(2)} & K_{13}^{(2)} \\ 0 & K_{32}^{(2)} & K_{31}^{(2)} & K_{33}^{(2)} \end{array}\right]} \end{array}
$$

$$
[\mathbf{M}^{e1}] = \begin{array}{cc} & \\ & \\ 2 & 1 \\ 3 & 2 \\ 1 & 3 \end{array}
\begin{array}{ccc} 2 & 3 & 1 \\ 1 & 2 & 3 \\ \multicolumn{3}{c}{\left[\begin{array}{ccc} M_{11}^{(1)} & M_{12}^{(1)} & M_{13}^{(1)} \\ M_{21}^{(1)} & M_{22}^{(1)} & M_{23}^{(1)} \\ M_{31}^{(1)} & M_{32}^{(1)} & M_{33}^{(1)} \end{array}\right]} \end{array}
\qquad
[\mathbf{M}^{e2}] = \begin{array}{cc} & \\ & \\ 3 & 1 \\ 2 & 2 \\ 4 & 3 \end{array}
\begin{array}{ccc} 3 & 2 & 4 \\ 1 & 2 & 3 \\ \multicolumn{3}{c}{\left[\begin{array}{ccc} M_{11}^{(2)} & M_{12}^{(2)} & M_{13}^{(2)} \\ M_{21}^{(2)} & M_{22}^{(2)} & M_{23}^{(2)} \\ M_{31}^{(2)} & M_{32}^{(2)} & M_{33}^{(2)} \end{array}\right]} \end{array}
$$

$$
[\mathbf{M}] = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \multicolumn{4}{c}{\left[\begin{array}{cccc} M_{33}^{(1)} & M_{31}^{(1)} & M_{32}^{(1)} & 0 \\ M_{13}^{(1)} & M_{11}^{(1)} + M_{22}^{(2)} & M_{12}^{(1)} + M_{21}^{(2)} & M_{23}^{(2)} \\ M_{23}^{(1)} & M_{21}^{(1)} + M_{12}^{(2)} & M_{22}^{(1)} + M_{11}^{(2)} & M_{13}^{(2)} \\ 0 & M_{32}^{(2)} & M_{31}^{(2)} & M_{33}^{(2)} \end{array}\right]} \end{array}
$$

entries of element matrices which have global row and column index equal to 2 must be algebraically added. Therefore, the element K_{22} is equal to $K_{22} = K_{11}^{(1)} + K_{22}^{(2)}$. If the entry located in the element matrix has the unique global row and column number, for example, $K_{32}^{(1)}$ which is assigned to the global row 1 and the global column 3, the global entry is replaced by this local entry. Thus, for example, the K_{13} is equal to $K_{32}^{(1)}$. The same procedure is applied in order to obtain the global capacitance matrix [**M**] and the global vector of nodal loads {**f**} (Figs. 4 and 5).

The procedure is intuitive and is equivalent to transforming the local element matrices into the global element matrices. Then these matrices are assembled. The assembly procedure avoids adding 0 terms that makes the algorithm effective for large and sparse matrices. Assembling the matrixes, given by (45), for all elements, the global system of ordinary differential equations (ODE) is obtained.

$$
\{\mathbf{f}^{e1}\} = \left\{\begin{array}{c} f_1^{(1)} \\ f_2^{(1)} \\ f_3^{(1)} \end{array}\right\} \begin{array}{c} 2 \\ 3 \\ 1 \end{array}
\qquad
\{\mathbf{f}^{e2}\} = \left\{\begin{array}{c} f_1^{(2)} \\ f_2^{(2)} \\ f_3^{(2)} \end{array}\right\} \begin{array}{c} 3 \\ 2 \\ 4 \end{array}
$$

$$
\{\mathbf{f}\} = \left\{\begin{array}{c} f_3^{(1)} \\ f_1^{(1)} + f_2^{(2)} \\ f_2^{(1)} + f_1^{(2)} \\ f_3^{(2)} \end{array}\right\} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}
$$

Finite Element Method in Steady-State and Transient Heat Conduction, Fig. 5 The assembly procedure for nodal load vector {**f**}

$$
[\mathbf{M}]\{\dot{\mathbf{T}}\} + [\mathbf{K}]\{\mathbf{T}\} = \{\mathbf{f}\} \tag{46}
$$

where [**M**] is the global capacitance matrix, [**K**] is the global stiffness matrix, vectors {**T**}, {**f**}, and $\{\dot{\mathbf{T}}\}$ denote the nodal temperatures, nodal loads, and the derivatives of the temperature with respect to time for all the nodes. For the discrete time points, the vector of nodal temperatures $\{\mathbf{T}\}$ is obtained by solving the system of ODE given by (46). In the next subsection, the solution procedure for (46) will be presented.

Transient Problem: Generalized Crank-Nicolson Method

The generalized Crank-Nicolson method, also known as θ *method*, is applied to numerically integrate of the system of ODE. According to the generalized trapezoidal approximation, the temperatures obtained at the time level $n + 1$ are

$$\{\mathbf{T}\}^{n+1} = \{\mathbf{T}\}^n + \left[(1-\theta)\{\dot{\mathbf{T}}\}^n + \theta\{\dot{\mathbf{T}}\}^{n+1}\right]\Delta t \tag{47}$$

where Δt is the time step between time levels n and $n + 1$ and the temporal parameter θ has the value between 0 and 1. For time levels denoted as n and $n + 1$, the global system of ODE – (46) – can be written as follows:

$$[\mathbf{M}]\{\dot{\mathbf{T}}\}^{n+1} + [\mathbf{K}]\{\mathbf{T}\}^{n+1} = \{\mathbf{f}\}^{n+1} \tag{48}$$

$$[\mathbf{M}]\{\dot{\mathbf{T}}\}^{n} + [\mathbf{K}]\{\mathbf{T}\}^{n} = \{\mathbf{f}\}^{n} \tag{49}$$

Multiplying both sides of (48) by θ and both sides of (49) by $(1 - \theta)$ yields

$$\theta\left([\mathbf{M}]\{\dot{\mathbf{T}}\}^{n+1} + [\mathbf{K}]\{\mathbf{T}\}^{n+1}\right) = \theta\{\mathbf{f}\}^{n+1} \tag{50}$$

and

$$(1-\theta)\left([\mathbf{M}]\{\dot{\mathbf{T}}\}^{n} + [\mathbf{K}]\{\mathbf{T}\}^{n}\right) = (1-\theta)\{\mathbf{f}\}^{n} \tag{51}$$

Equations (50) and (51) added algebraically give

$$[\mathbf{M}]\{(1-\theta)\dot{\mathbf{T}}^n + \theta\dot{\mathbf{T}}^{n+1}\} + [\mathbf{K}]\{(1-\theta)\mathbf{T}^n + \theta\mathbf{T}^{n+1}\} = (1-\theta)\{\mathbf{f}\}^n + \theta\{\mathbf{f}\}^{n+1} \tag{52}$$

Combining (47) and (52) leads to

$$[\mathbf{M}]\left\{\frac{\mathbf{T}^{n+1} - \mathbf{T}^n}{\Delta t}\right\} + [\mathbf{K}]\{(1-\theta)\mathbf{T}^n + \theta\mathbf{T}^{n+1}\} = (1-\theta)\{\mathbf{f}\}^n + \theta\{\mathbf{f}\}^{n+1} \tag{53}$$

Equation (53) can be rearranged as follows:

$$\left(\frac{1}{\Delta t}[\mathbf{M}] + \theta[\mathbf{K}]\right)\{\mathbf{T}\}^{n+1} = \left[\frac{1}{\Delta t}[\mathbf{M}] - (1-\theta)[\mathbf{K}]\right]\{\mathbf{T}\}^n + (1-\theta)\{\mathbf{f}\}^n + \theta\{\mathbf{f}\}^{n+1} \tag{54}$$

Equation (54) gives the nodal values of temperatures at the $n + 1$ time level. Note that the values of vector of nodal loads $\{\mathbf{f}\}$ for the time levels n and $n + 1$ must be known. The stability of solution for (54) depends on the value of parameter θ. If $\theta \geq ½$ then the solution is stable for the arbitrary chosen time step Δt. The large time step may cause the computational time low. Nevertheless, the solution can be inaccurate. Therefore, the size of the time step should be a compromise between the accuracy of temperature determination and the computational speed. By varying the parameter θ, different transient schemes are obtained.

$\theta = 0$ – Fully explicit scheme; conditionally stable – the time step Δt should be smaller than the reliable boundary value

$\theta = 1/2$ – Semi-implicit scheme – Crank-Nicolson method, unconditionally stable

$\theta = 2/3$ – Galerkin method, unconditionally stable

$\theta = 1$ – Fully implicit scheme, unconditionally stable

The explicit method ($\theta = 0$) is the first-order scheme. Hence, to ensure high accuracy of calculations, the time step size should be small. The allowable time step size decreases with the quotient A^e/a. Where A^e is the area of element and a is the length of element edge. Substituting $\theta = 0$ into (54), the formula from which nodal temperature vector $\{\mathbf{T}\}$ evaluated at time level $n + 1$ is obtained

$$\{\mathbf{T}\}^{n+1} = \{\mathbf{T}\}^n + \Delta t\left([\mathbf{M}]^{-1}\{\mathbf{f}\}^n - [\mathbf{M}]^{-1}[\mathbf{K}]\{\mathbf{T}\}^n\right) \tag{55}$$

Note that temperature vector is obtained directly by substituting the vectors of nodal temperature $\{\mathbf{T}\}^n$ and nodal loads $\{\mathbf{f}\}^n$ from previous step n. If the thermophysical properties like ρ, c, and k are constant, the global capacitance matrix $[\mathbf{M}]$ and global stiffness matrix $[\mathbf{K}]$ does not change for all the time levels. If the global capacitance matrix $[\mathbf{M}]$ is a diagonal matrix, it is unnecessary to solve the system of linear equations because the vector $\{\mathbf{T}\}^{n+1}$ is obtained in the time marching procedure given by (55). Therefore, the calculations are straightforward and easy to program. Despite the limitations of the time step Δt, the explicit method is frequently used, since it is very accurate for a small time steps, especially when the temperature change rate for a solid is large.

For the implicit method ($\theta = 1$), the system of linear algebraic equation must be solved at every time level using direct methods (Gaussian elimination method) or iterative methods (e.g., Gauss Seidel method, conjugate gradient method).

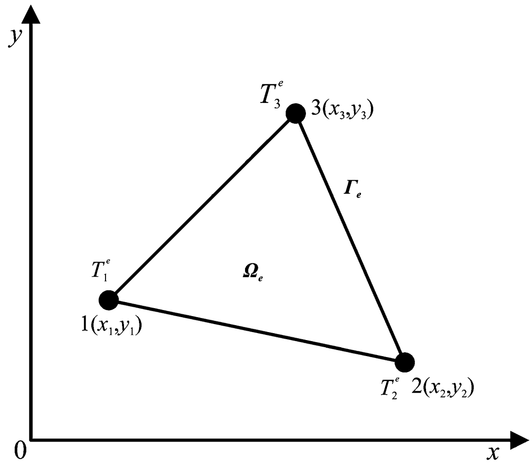
Crank-Nicolson method ($\theta = 1/2$) is the second-order scheme O[(Δt^2)]. This method is unconditionally stable. However, with large time steps, the solution exhibits oscillations (which do not occur in reality) and becomes less accurate. The Crank-Nicolson method is semi-implicit scheme; thus, the system of linear algebraic equations must be solved at every time level.

Temperature Approximation Inside Element

The temperature inside the element is approximated using the values of nodal temperatures and the shape functions. In this subsection, the temperature approximation inside three types of elements will be presented: linear triangular, bilinear rectangular, and axisymmetric triangular element. The element stiffness matrix $[\mathbf{K}^e]$, element mass matrix $[\mathbf{M}^e]$, and element load vector $\{\mathbf{f}^e\}$ will be calculated for all presented elements

Linear Triangular Element

The three node triangular element is presented in Fig. 6.



Finite Element Method in Steady-State and Transient Heat Conduction, Fig. 6 Linear triangular element

The temperature approximated inside the element domain Ω^e is given by

$$T^e(x,y) = \alpha_1^e + \alpha_2^e \cdot x + \alpha_3^e \cdot y \tag{56}$$

At the nodal locations, the temperature $T^e(x,y)$ is equal to

$$\begin{aligned} &T^e(x_1,y_1) = T_1^e, \quad T^e(x_2,y_2) = T_2^e, \\ &T^e(x_3,y_3) = T_3^e \end{aligned} \tag{57}$$

Substituting (57) to (56) gives

$$\begin{cases} \alpha_1^e + \alpha_2^e x_1 + \alpha_3^e y_1 = T_1^e \\ \alpha_1^e + \alpha_2^e x_2 + \alpha_3^e y_2 = T_2^e \\ \alpha_1^e + \alpha_2^e x_3 + \alpha_3^e y_3 = T_3^e \end{cases} \tag{58}$$

The coefficients $\alpha_1^e, \alpha_2^e, \alpha_3^e$ are determined by solving the system of algebraic equations given by (58)

$$\begin{aligned} \alpha_1^e &= \frac{1}{2A^e}\left[(x_2y_3 - x_3y_2)T_1^e + (x_3y_1 - x_1y_3)T_2^e \right. \\ &\quad \left. + (x_1y_2 - x_2y_1)T_3^e\right] \\ \alpha_2^e &= \frac{1}{2A^e}\left[(y_2 - y_3)T_1^e + (y_3 - y_1)T_2^e + (y_1 - y_2)T_3^e\right] \\ \alpha_3^e &= \frac{1}{2A^e}\left[(x_3 - x_2)T_1^e + (x_1 - x_3)T_2^e + (x_2 - x_1)T_3^e\right] \end{aligned} \tag{59}$$

where A^e is the area of element face

$$A^e = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} \qquad (60)$$

Substituting (59) into (56) and rearranging the terms because of nodal temperatures, one obtains

$$T^e(x,y) = \phi_1^e \cdot T_1^e + \phi_2^e \cdot T_2^e + \phi_3^e \cdot T_3^e \qquad (61)$$

where the shape functions $\phi_1^e,\ \phi_2^e,\ \phi_3^e$ are given by

$$\begin{aligned} \phi_1^e &= \frac{1}{2A^e}\left(a_1^e + b_1^e x + c_1^e y\right) \\ \phi_2^e &= \frac{1}{2A^e}\left(a_2^e + b_2^e x + c_2^e y\right) \\ \phi_3^e &= \frac{1}{2A^e}\left(a_3^e + b_3^e x + c_3^e y\right) \end{aligned} \qquad (62)$$

where

$$\begin{aligned} a_1^e &= x_2y_3 - x_3y_2, b_1^e = y_2 - y_3, c_1^e = x_3 - x_2 \\ a_2^e &= x_3y_1 - x_1y_3, b_2^e = y_3 - y_1, c_2^e = x_1 - x_3 \\ a_3^e &= x_1y_2 - x_2y_1, b_3^e = y_1 - y_2, c_3^e = x_2 - x_1 \end{aligned} \qquad (63)$$

The shape functions for all element types also satisfy condition

$$\phi_i^e(x_i, y_i) = \delta_{ij} \qquad (64)$$

and

$$\sum_{i=1}^{n} \phi_i^e = 1 \qquad (65)$$

Here the δ_{ij} is the Kronecker delta. That is,

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \qquad (66)$$

If the element shape functions are known, then the element conductivity matrix is

$$[\mathbf{K}_c^e] = \int_{\Omega^e} \left([\mathbf{B}]^T[\mathbf{D}][\mathbf{B}]\right) dxdy$$

$$[\mathbf{K}_c^e] = \int_{\Omega^e} \left(k_x \begin{Bmatrix} \frac{\partial \phi_1^e}{\partial x} \\ \frac{\partial \phi_2^e}{\partial x} \\ \frac{\partial \phi_3^e}{\partial x} \end{Bmatrix} \begin{Bmatrix} \frac{\partial \phi_1^e}{\partial x} & \frac{\partial \phi_2^e}{\partial x} & \frac{\partial \phi_3^e}{\partial x} \end{Bmatrix} + k_y \begin{Bmatrix} \frac{\partial \phi_1^e}{\partial y} \\ \frac{\partial \phi_2^e}{\partial y} \\ \frac{\partial \phi_3^e}{\partial y} \end{Bmatrix} \begin{Bmatrix} \frac{\partial \phi_1^e}{\partial y} & \frac{\partial \phi_2^e}{\partial y} & \frac{\partial \phi_3^e}{\partial y} \end{Bmatrix} \right) dxdy \qquad (67)$$

Substituting (62) into (67) and performing integration yields [1, 6]

$$[\mathbf{K}_c^e] = \begin{bmatrix} K_{11}^e & K_{12}^e & K_{13}^e \\ K_{21}^e & K_{22}^e & K_{23}^e \\ K_{31}^e & K_{32}^e & K_{33}^e \end{bmatrix} \qquad (68)$$

where

$$\begin{aligned} K_{11}^e &= \frac{k_x}{4A^e}\left(b_1^e\right)^2 + \frac{k_y}{4A^e}\left(c_1^e\right)^2 \\ K_{12}^e &= \frac{k_x}{4A^e}\left(b_1^e \cdot b_2^e\right) + \frac{k_y}{4A^e}\left(c_1^e \cdot c_2^e\right) \\ K_{13}^e &= \frac{k_x}{4A^e}\left(b_1^e \cdot b_3^e\right) + \frac{k_y}{4A^e}\left(c_1^e \cdot c_3^e\right) \\ K_{21}^e &= K_{12}^e \\ K_{22}^e &= \frac{k_x}{4A^e}\left(b_2^e\right)^2 + \frac{k_y}{4A^e}\left(c_2^e\right)^2 \\ K_{23}^e &= \frac{k_x}{4A^e}\left(b_2^e \cdot b_3^e\right) + \frac{k_y}{4A^e}\left(c_2^e \cdot c_3^e\right) \\ K_{31}^e &= K_{13}^e \\ K_{32}^e &= K_{23}^e \\ K_{33}^e &= \frac{k_x}{4A^e}\left(b_3^e\right)^2 + \frac{k_y}{4A^e}\left(c_3^e\right)^2 \end{aligned} \qquad (69)$$

The element capacitance matrix for linear triangle is given by

$$[\mathbf{M}^e] = \int_{\Omega^e} c\rho [\Phi^e]^T [\Phi^e]\, dxdy$$
$$= c\rho \int_{\Omega^e} \begin{Bmatrix} \phi_1^e \\ \phi_2^e \\ \phi_3^e \end{Bmatrix} \{\phi_1^e\ \phi_2^e\ \phi_3^e\} dxdy \tag{70}$$
$$[\mathbf{M}^e] = \frac{A^e}{12} c\rho \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Assuming uniform heat source q_v and constant heat flux q_{bn} at the boundary edge 1–2 (local element nodes), the components of element load vector are

$$\{\mathbf{f}_G^e\} = \int_{\Omega^e} q_v [\Phi^e]^T dxdy = \int_{\Omega^e} q_v \begin{Bmatrix} \phi_1^e \\ \phi_2^e \\ \phi_3^e \end{Bmatrix} dxdy$$
$$= q_v \cdot \frac{A^e}{3} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \tag{71}$$

and

$$\{\mathbf{f}_q^e\} = \int_{\Gamma_q^e} q_{bn} [\Phi^e]^T ds = \int_{\Gamma_q^e} q_{bn} \begin{Bmatrix} \phi_1^e \\ \phi_2^e \\ \phi_3^e \end{Bmatrix} ds$$
$$= q_{bn} \frac{l_{12}}{2} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \tag{72}$$

The boundary integrals in (72) are calculated as below

$$\int_0^l \phi_i^m \phi_j^n ds = l \frac{m!n!}{(m+n+1)!} \tag{73}$$

where l is the length of element edge at which boundary conditions are applied. As an example the boundary integral $\int_0^{l_{12}} q_{bn} \phi_1^e ds = \int_0^{l_{12}} q_{bn} \phi_1^1 \phi_1^0 ds$ $= q_{bn} \cdot l_{12} \frac{1!\cdot 0!}{(1+0+1)!} = q_{bn} \cdot \frac{l_{12}}{2}$

Note (see (72)) that when the heat flux boundary condition is applied on the element edge i-j, then the components of the element nodal loads vector $\{\mathbf{f}^e\}$, with indexes i and j, are equal to 1 and the other equal to 0.

If the convective boundary condition is applied along the edge 3-1, then the associated with convective flux element matrix $[\mathbf{K}_h^e]$ and element load vector $\{\mathbf{f}_h^e\}$ are calculated.

If the uniform heat transfer coefficient h and bulk temperature T_∞ are applied along the element edge 3–1, then

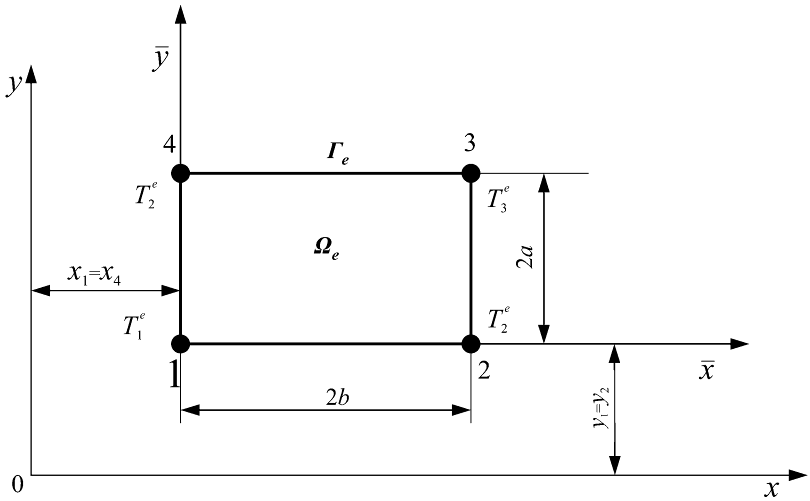
$$[\mathbf{K}_h^e] = \int_{\Gamma_h^e} h [\Phi^e]^T [\Phi^e] ds$$
$$= \int_0^{l_{31}} h \begin{Bmatrix} \phi_1^e \\ \phi_2^e \\ \phi_3^e \end{Bmatrix} \{\phi_1^e \quad \phi_2^e \quad \phi_3^e\} ds$$
$$= h \frac{l_{31}}{2} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix} \tag{74}$$

and

$$\{\mathbf{f}_h^e\} = \int_{\Gamma_h^e} h T_\infty [\Phi^e]^T ds = \int_0^{l_{13}} h T_\infty \begin{Bmatrix} \phi_1^e \\ \phi_2^e \\ \phi_3^e \end{Bmatrix} ds$$
$$= h \frac{T_\infty}{2} l_{13} \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix} \tag{75}$$

As explained previously, the heat flux (convective flux q_h) is interpolated into the nodes which form the boundary edge of element. The boundary integrals in (74) and (75) are calculated according to formula given in (73). Finally, according to (36), the element matrices $[\mathbf{K}_h^e]$ and $[\mathbf{K}^e]$ must be added to form the stiffness matrix of element $[\mathbf{K}^e]$

Finite Element Method in Steady-State and Transient Heat Conduction, Fig. 7 Bilinear rectangular element



$$[\mathbf{K}^e] = [\mathbf{K}_c^e] + [\mathbf{K}_h^e] = \begin{bmatrix} K_{11}^e & K_{12}^e & K_{13}^e \\ K_{21}^e & K_{22}^e & K_{23}^e \\ K_{31}^e & K_{32}^e & K_{33}^e \end{bmatrix} + h\frac{l_{31}}{2}\begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix} \tag{76}$$

Bilinear Rectangular Element

The bilinear rectangular element is shown in Fig. 7. The coordinate system with the origin at the location of first node is considered for the calculations of the temperature inside the element.

Temperature distribution inside the bilinear rectangular element may be approximated using the following function:

$$T^e(\bar{x}, \bar{y}) = \alpha_1^e + \alpha_2^e \cdot \bar{x} + \alpha_3^e \cdot \bar{y} + \alpha_4^e \cdot \bar{x} \cdot \bar{y} \tag{77}$$

The interpolation function $T^e(\bar{x}, \bar{y})$ should take into account the nodal variables at the four node locations. Therefore, substituting the values of $\bar{x}$ and $\bar{y}$ coordinates at each nodal point gives

$$\begin{aligned} T^e(0,0) &= T_1^e, \quad T^e(2b,0) = T_2^e \\ T^e(2b,2a) &= T_3^e, \quad T^e(0,2a) = T_4^e \end{aligned} \tag{78}$$

Substituting (78) into (77), the following system of equations is obtained:

$$\begin{cases} \alpha_1^e = T_1^e \\ \alpha_1^e + 2b \cdot \alpha_2^e = T_2^e \\ \alpha_1^e + 2b \cdot \alpha_2^e + 2a \cdot \alpha_3^e + 4ab \cdot \alpha_4^e = T_3^e \\ \alpha_1^e + 2a \cdot \alpha_3^e = T_4^e \end{cases} \tag{79}$$

The coefficients $\alpha_1^e, \ldots, \alpha_4^e$ are determined by solving the system of algebraic equations given by (79)

$$\begin{aligned} &\alpha_1^e = T_1^e, \qquad \alpha_2^e = \frac{1}{2b}(T_2^e - T_1^e) \\ &\alpha_3^e = \frac{1}{2a}(T_4^e - T_1^e), \qquad \alpha_4^e = \frac{1}{4ab}(T_1^e - T_2^e + T_3^e - T_4^e) \end{aligned} \tag{80}$$

Substituting (80) into (77) and rearranging the terms because of nodal temperatures, the function which interpolates the temperature inside the element is

$$T^e(\bar{x}, \bar{y}) = \phi_1^e \cdot T_1^e + \phi_2^e \cdot T_2^e + \phi_3^e \cdot T_3^e + \phi_4^e \cdot T_4^e \tag{81}$$

where ϕ_i^e is the shape function for bilinear rectangular element and it is given below

$$\begin{aligned} &\phi_1^e = \left(1 - \frac{\bar{x}}{2b}\right)\left(1 - \frac{\bar{y}}{2a}\right), \quad \phi_2^e = \frac{\bar{x}}{2b}\left(1 - \frac{\bar{y}}{2a}\right) \\ &\phi_3^e = \frac{\bar{x} \cdot \bar{y}}{4ab}, \qquad \phi_4^e = \frac{\bar{y}}{2a}\left(1 - \frac{\bar{x}}{2b}\right) \end{aligned} \tag{82}$$

The element conductivity matrix is obtained according to the following dependence:

$$[\mathbf{K}_c^e] = \int_{\Omega^e} \left([\mathbf{B}]^T[\mathbf{D}][\mathbf{B}]\right) d\bar{x}d\bar{y}$$

$$[\mathbf{K}_c^e] = \int_{\Omega^e} \left(k_x \begin{Bmatrix} \frac{\partial \phi_1^e}{\partial \bar{x}} \\ \frac{\partial \phi_2^e}{\partial \bar{x}} \\ \frac{\partial \phi_3^e}{\partial \bar{x}} \\ \frac{\partial \phi_4^e}{\partial \bar{x}} \end{Bmatrix} \left\{ \frac{\partial \phi_1^e}{\partial \bar{x}} \; \frac{\partial \phi_2^e}{\partial \bar{x}} \; \frac{\partial \phi_3^e}{\partial \bar{x}} \; \frac{\partial \phi_4^e}{\partial \bar{x}} \right\} + k_x \begin{Bmatrix} \frac{\partial \phi_1^e}{\partial \bar{y}} \\ \frac{\partial \phi_2^e}{\partial \bar{y}} \\ \frac{\partial \phi_3^e}{\partial \bar{y}} \\ \frac{\partial \phi_4^e}{\partial \bar{y}} \end{Bmatrix} \left\{ \frac{\partial \phi_1^e}{\partial \bar{y}} \; \frac{\partial \phi_2^e}{\partial \bar{y}} \; \frac{\partial \phi_3^e}{\partial \bar{y}} \; \frac{\partial \phi_4^e}{\partial \bar{y}} \right\} \right) d\bar{x}d\bar{y} \tag{83}$$

Substituting (82) into (83) and performing integration yields

$$[\mathbf{K}_c^e] = \begin{bmatrix} K_{11}^e & K_{12}^e & K_{13}^e & K_{14}^e \\ K_{21}^e & K_{22}^e & K_{23}^e & K_{24}^e \\ K_{31}^e & K_{32}^e & K_{33}^e & K_{34}^e \\ K_{41}^e & K_{42}^e & K_{43}^e & K_{44}^e \end{bmatrix} \tag{84}$$

The term K_{11}^e involves the calculation of spatial derivatives $\frac{\partial \phi_1^e}{\partial \bar{x}}$ and $\frac{\partial \phi_1^e}{\partial \bar{y}}$ of element shape function ϕ_1^e which are equal to

$$\frac{\partial \phi_1^e}{\partial \bar{x}} = -\frac{1}{2b}\left(1 - \frac{\bar{y}}{2a}\right), \quad \frac{\partial \phi_1^e}{\partial \bar{y}} = -\frac{1}{2a}\left(1 - \frac{\bar{x}}{2b}\right) \tag{85}$$

Thus, the first component of element stiffness matrix $[\mathbf{K}^e]$ is

$$\begin{aligned} K_{11}^e &= \int_{\Omega^e} \left(k_x \frac{\partial \phi_1^e}{\partial \bar{x}} \frac{\partial \phi_1^e}{\partial \bar{x}} + k_y \frac{\partial \phi_1^e}{\partial \bar{y}} \frac{\partial \phi_1^e}{\partial \bar{y}} \right) d\bar{x}d\bar{y} \\ &= k_x \int_0^{2a} \left[\int_0^{2b} \frac{1}{4b^2}\left(1 - \frac{\bar{y}}{2a}\right)^2 d\bar{x} \right] d\bar{y} \\ &\quad + k_y \int_0^{2a} \left[\int_0^{2b} \frac{1}{4a^2}\left(1 - \frac{\bar{x}}{2b}\right)^2 d\bar{x} \right] d\bar{y} \\ &= \frac{k_x}{3}\frac{a}{b} + \frac{k_y}{3}\frac{b}{a} \end{aligned} \tag{86}$$

The second component of element stiffness matrix $[\mathbf{K}^e]$ is calculated in the similar way. The spatial derivatives of the shape function ϕ_2^e are

$$\frac{\partial \phi_2^e}{\partial \bar{x}} = \frac{1}{2b}\left(1 - \frac{\bar{y}}{2a}\right), \quad \frac{\partial \phi_2^e}{\partial \bar{y}} = -\frac{\bar{x}}{4ab} \tag{87}$$

and the element K_{12}^e is obtained by following integration:

$$\begin{aligned} K_{12}^e &= \int_{\Omega^e} \left(k_x \frac{\partial \phi_1^e}{\partial \bar{x}} \frac{\partial \phi_2^e}{\partial \bar{x}} + k_y \frac{\partial \phi_1^e}{\partial \bar{y}} \frac{\partial \phi_2^e}{\partial \bar{y}} \right) d\bar{x}d\bar{y} \\ &= -k_x \int_0^{2a} \left[\int_0^{2b} \frac{1}{4b^2}\left(1 - \frac{\bar{y}}{2a}\right)^2 d\bar{x} \right] d\bar{y} \\ &\quad + k_y \int_0^{2a} \left[\int_0^{2b} \frac{1}{8a^2 b}\left(\bar{x} - \frac{\bar{x}^2}{2b}\right) d\bar{x} \right] d\bar{y} \\ &= -\frac{k_x}{3}\frac{a}{b} + \frac{k_y}{6}\frac{b}{a} \end{aligned} \tag{88}$$

Performing integration for all terms results in

$$[\mathbf{K}_c^e] = \frac{k_x}{6}\frac{a}{b} \begin{bmatrix} 2 & -2 & -1 & 1 \\ -2 & 2 & 1 & -1 \\ -1 & 1 & 2 & -2 \\ 1 & -1 & -2 & 2 \end{bmatrix} + \frac{k_y}{6}\frac{b}{a} \begin{bmatrix} 2 & 1 & -1 & -2 \\ 1 & 2 & -2 & -1 \\ -1 & -2 & 2 & 1 \\ -2 & -1 & 1 & 2 \end{bmatrix} \tag{89}$$

The element mass matrix for bilinear rectangular element is given by

$$[\mathbf{M}^e] = \int_{\Omega^e} c\rho[\Phi^e]^T [\Phi^e]\, dxdy = c\rho \int_{\Omega^e} \begin{Bmatrix} \phi_1^e \\ \phi_2^e \\ \phi_3^e \\ \phi_4^e \end{Bmatrix} \{\phi_1^e\ \phi_2^e\ \phi_3^e\ \phi_4^e\} dxdy$$

$$[\mathbf{M}^e] = \frac{A^e}{36} c\rho \begin{bmatrix} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{bmatrix} \tag{90}$$

Next the procedure of creating the vector of nodal loads for a bilinear rectangular element is presented. For the uniform heat source q_v and constant heat flux at the boundary edge 1–2 (local node numbering for element), the components of element load vector are

$$\{\mathbf{f}_G^e\} = \int_{\Omega^e} q_v [\Phi^e]^T dxdy = \int_{\Omega^e} q_v \begin{Bmatrix} \phi_1^e \\ \phi_2^e \\ \phi_3^e \\ \phi_4^e \end{Bmatrix} dxdy = q_v \cdot \frac{A^e}{4} \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix} = q_v \cdot ab \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix} \tag{91}$$

If the heat flux (second order) boundary conditions are applied on the element edge 1–2, then

$$\{\mathbf{f}_q^e\} = \int_{\Gamma_q^e} q_b [\Phi^e]^T ds = \int_{\Gamma_q^e} q_b \begin{Bmatrix} \phi_1^e \\ \phi_2^e \\ \phi_3^e \\ \phi_4^e \end{Bmatrix} ds = q_{bn} \frac{l_{12}}{2} \begin{Bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{Bmatrix} = q_{bn} \cdot b \begin{Bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{Bmatrix} \tag{92}$$

The boundary integrals in (94) are calculated as below

$$\int_0^{l_{12}} \{\Phi^T\} ds = \int_0^{l_{12}} \begin{Bmatrix} \phi_1^e(x,0) \\ \phi_2^e(x,0) \\ \phi_3^e(x,0) \\ \phi_4^e(x,0) \end{Bmatrix} ds = \int_0^{2b} \begin{Bmatrix} \phi_1^e(x,0) \\ \phi_2^e(x,0) \\ 0 \\ 0 \end{Bmatrix} d\bar{x}$$

$$= \int_0^{2b} \begin{Bmatrix} 1 - \frac{\bar{x}}{2b} \\ \frac{\bar{x}}{2b} \\ 0 \\ 0 \end{Bmatrix} d\bar{x} = \begin{Bmatrix} \bar{x} - \frac{\bar{x}^2}{4b}\Big|_0^{2b} \\ \frac{\bar{x}^2}{4b}\Big|_0^{2b} \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} b \\ b \\ 0 \\ 0 \end{Bmatrix} \tag{93}$$

Assuming that on the element edge 4–1 the convective boundary condition is applied, the associated element matrix $[\mathbf{K}_h^e]$ and element load vector $\{\mathbf{f}_h^e\}$ should be calculated.

If the uniform heat transfer coefficient h and bulk temperature T_∞ are applied along the element edge 4–1, then

$$[\mathbf{K}_h^e] = \int_{\Gamma_h^e} h[\Phi^e]^T [\Phi^e] ds = \int_0^{l_{41}} h \begin{Bmatrix} \phi_1^e \\ \phi_2^e \\ \phi_3^e \\ \phi_4^e \end{Bmatrix} \{\phi_1^e\quad \phi_2^e\quad \phi_3^e\quad \phi_4^e\} ds$$

$$[\mathbf{K}_h^e] = h \int_0^{2a} \frac{2a}{2} \begin{bmatrix} \phi_1^e\phi_1^e & 0 & 0 & \phi_1^e\phi_4^e \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \phi_4^e\phi_1^e & 0 & 0 & \phi_4^e\phi_4^e \end{bmatrix} dx = h\frac{l_{41}}{2} \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} = ha \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} \tag{94}$$

and

$$\{\mathbf{f}_h^e\} = \int_{\Gamma_h^e} hT_\infty [\Phi^e]^T ds = \int_0^{l_{41}} hT_\infty \begin{Bmatrix} \phi_1^e \\ \phi_2^e \\ \phi_3^e \\ \phi_4^e \end{Bmatrix} ds$$
$$= \int_0^a hT_\infty \begin{Bmatrix} \phi_1^e(0,y) \\ \phi_2^e(0,y) \\ \phi_3^e(0,y) \\ \phi_4^e(0,y) \end{Bmatrix} dy = hT_\infty a \begin{Bmatrix} \phi_1^e \\ 0 \\ 0 \\ \phi_4^e \end{Bmatrix} \quad (95)$$

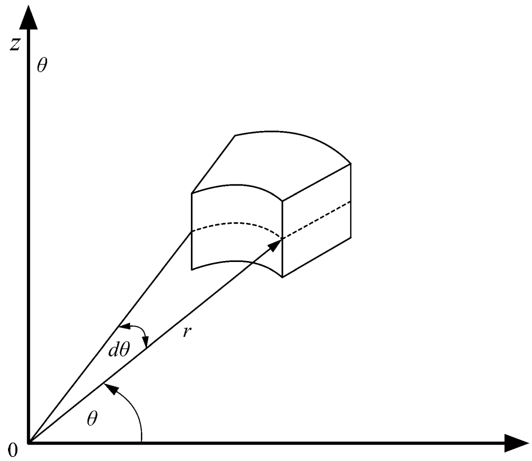
Because the shape functions are evaluated at the boundary, $x = 0$. Therefore, only the values of the shape functions corresponding to boundary nodes are different from 0 (see (94) and (95)). This property has all two-dimensional linear elements. The boundary integrals in (94) and (95) are calculated as in (93), only the length of element boundary is equal to a instead of b.

If the matrices $[\mathbf{K}_h^e]$ and $[\mathbf{K}_c^e]$ are known, it is necessary to combine them to form the stiffness matrix of element $[\mathbf{K}^e]$

$$[\mathbf{K}^e] = [\mathbf{K}_c^e] + [\mathbf{K}_h^e] = \begin{bmatrix} K_{11}^e & K_{12}^e & K_{13}^e & K_{14}^e \\ K_{21}^e & K_{22}^e & K_{23}^e & K_{24}^e \\ K_{31}^e & K_{32}^e & K_{33}^e & K_{34}^e \\ K_{41}^e & K_{42}^e & K_{43}^e & K_{44}^e \end{bmatrix} + ha \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} \quad (96)$$

Axisymmetric Linear Triangular Element

In the cylindrical coordinates, the transient heat transfer equation for solids can be written as follows [1]:



Finite Element Method in Steady-State and Transient Heat Conduction, Fig. 8 Cylindrical coordinate system

$$\rho c \frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(rk_r \frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta} \times \left(k_\theta \frac{\partial T}{\partial r}\right) + \frac{\partial}{\partial z}\left(k_z \frac{\partial T}{\partial z}\right) + q_v \quad (97)$$

where the variables r, ϕ, and z denote the radial, circumferential, and axial coordinates (see Fig. 8).

For the axisymmetric problem, the temperature is independent of circumferential coordinate ϕ. Also the loading and boundary conditions are axisymmetric. Then, after differentiating (97) is simplified to

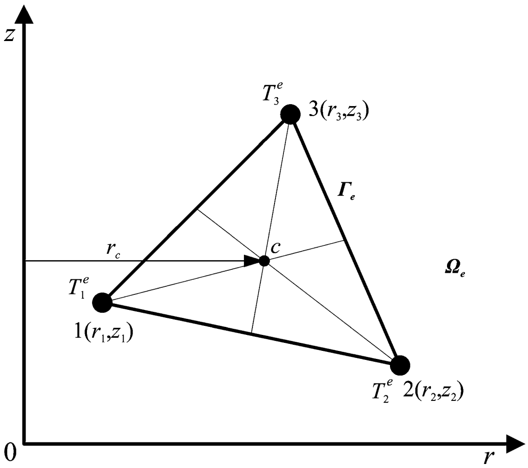
$$\rho c \frac{\partial T}{\partial t} = k_r \frac{\partial^2 T}{\partial r^2} + k_r \frac{1}{r}\frac{\partial T}{\partial r} + k_z \frac{\partial^2 T}{\partial z^2} + q_v \quad (98)$$

Approximating temperature according to (12) and applying Galerkin method to (98), one obtains

$$\int_{\Omega^e} \left[\rho c \frac{\partial T^e}{\partial t} - \left(k_r \frac{\partial^2 T^e}{\partial r^2} + k_r \frac{1}{r}\frac{\partial T^e}{\partial r} + k_z \frac{\partial^2 T^e}{\partial z^2}\right) - q_v\right] \times \phi_i^e(x,y) r dr dz d\theta = 0 \quad (99)$$

The integral of the term in parentheses can be expressed as

$$\int_{\Omega^e} \left(k_r \frac{\partial^2 T^e}{\partial r^2} + k_r \frac{1}{r}\frac{\partial T^e}{\partial r} + k_z \frac{\partial^2 T^e}{\partial z^2}\right) \phi_i^e(r,z) r dr dz d\theta = \int_{\Omega^e} \left(k_r \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T^e}{\partial r}\right) + k_z \frac{\partial^2 T^e}{\partial z^2}\right) \phi_i^e(r,z) r dr dz d\theta \quad (100)$$



Finite Element Method in Steady-State and Transient Heat Conduction, Fig. 9 Linear triangular axisymmetric element

Performing integration yields

$$\int_{\Omega^e}\left(k_r\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T^e}{\partial r}\right)+k_z\frac{\partial^2 T^e}{\partial z^2}\right)\phi_i^e(r,z)rdrdzd\theta$$
$$=2\pi\int_{\Omega^e}\left(k_r\frac{\partial}{\partial r}\left(r\frac{\partial T^e}{\partial r}\right)+k_z r\frac{\partial^2 T^e}{\partial z^2}\right)\phi_i^e(r,z)rdrdzd\theta \tag{101}$$

The integration by parts of left side of (101) yields

$$-2\pi\int_{\Omega^e} r\left(k_r\left(\frac{\partial\phi_i^e}{\partial r}\frac{\partial T^e}{\partial r}\right)+k_z\left(\frac{\partial\phi_i^e}{\partial z}\frac{\partial T^e}{\partial z}\right)\right)drdz$$
$$+2\pi\int_{\Gamma^e} r\phi_i^e\left(-k_r\frac{\partial T^e}{\partial r}dr+k_z\frac{\partial T^e}{\partial z}dz\right) \tag{102}$$

It is possible to observe that (102) is expressed only in r and z. The same shape functions can be used for both two-dimensional and axisymmetric analysis. However, in opposite to two-dimensional analysis, the axisymmetric formulation contains the r term within the integral. Taking into consideration the linear triangular axisymmetric element (see Fig. 9) and then interpolating the temperature inside the element according to (12), the element conductivity matrix is obtained

$$[\mathbf{K}_c^e]=2\pi\int_{\Omega^e} r\left(k_r\begin{Bmatrix}\frac{\partial\phi_1^e}{\partial r}\\ \frac{\partial\phi_2^e}{\partial r}\\ \frac{\partial\phi_3^e}{\partial r}\end{Bmatrix}\begin{Bmatrix}\frac{\partial\phi_1^e}{\partial r} & \frac{\partial\phi_2^e}{\partial r} & \frac{\partial\phi_3^e}{\partial r}\end{Bmatrix}+k_z\begin{Bmatrix}\frac{\partial\phi_1^e}{\partial z}\\ \frac{\partial\phi_2^e}{\partial z}\\ \frac{\partial\phi_3^e}{\partial z}\end{Bmatrix}\begin{Bmatrix}\frac{\partial\phi_1^e}{\partial z} & \frac{\partial\phi_2^e}{\partial z} & \frac{\partial\phi_3^e}{\partial z}\end{Bmatrix}\right)drdz \tag{103}$$

Here the shape functions ϕ_i^e are given by (62). However, the x and y coordinates must be replaced by r and z. The first moment of area of axis z is equal to

$$\int_{\Omega^e} rdrdz = A^e r_c = A^e\cdot\frac{1}{3}(r_1+r_2+r_3) \tag{104}$$

The symbol r_c relates to the gravity center of the triangular element. Because the derivatives of shape functions are independent of r and z after substituting (104) into (103) and performing integration over the element domain Ω, the conductivity matrix $[\mathbf{K}_c^e]$ is obtained [2, 6]

$$[\mathbf{K}^e]^A = 2\pi r_c\begin{bmatrix}K_{11}^e & K_{12}^e & K_{13}^e\\ K_{21}^e & K_{22}^e & K_{23}^e\\ K_{31}^e & K_{32}^e & K_{33}^e\end{bmatrix} \tag{105}$$

Here the components K_{ij}^e $i=1,..,3$ $j=1,..3$ are the same as for the stiffness matrix of linear triangular element given by (69). In a similar procedure, the element capacitance matrix for linear triangle is obtained

$$[\mathbf{M}^e]^A = \int_{\Omega^e} r\rho c[\Phi]^T[\Phi]drdz$$
$$= \int_{\Omega^e} r\rho c \begin{Bmatrix} \phi_1^e \\ \phi_2^e \\ \phi_3^e \end{Bmatrix} \{ \phi_1^e \quad \phi_2^e \quad \phi_3^e \} drdz$$
$$[\mathbf{M}^e]^A = \pi \cdot r_c \cdot \frac{A^e}{6} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \tag{106}$$

The boundary conditions at element boundary are applied in a similar way as for the two-dimensional triangular element. However, the boundary integrals also contain r (see (102)); therefore, it is necessary to interpolate the radial coordinate inside the element [2]

$$r^e(r,z) = \phi_1^e \cdot r_1^e + \phi_2^e \cdot r_2^e + \phi_3^e \cdot r_3^e \tag{107}$$

For the uniform heat source q_v and constant heat flux at the boundary edge 1–2 (local node numbering for element), the components of element load vector are

$$\{\mathbf{f}_G^e\}^A = \int_{\Omega^e} q_v[\Phi^e]^T[\Phi^e]\{\mathbf{r}\}drdz$$
$$= \frac{1}{6}\pi q_v A^e \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \tag{108}$$

If the heat flux (second order) boundary conditions applied on the element edge 1–2 are

$$\left\{\mathbf{f}_q^e\right\}^A = \int_{\Gamma_q^e} q_{bn}[\Phi^e]^T ds = \frac{\pi}{3} q_{bn} l_{12} \begin{Bmatrix} 2r_1^e + r_2^e \\ r_1^e + 2r_2^e \\ 0 \end{Bmatrix} \tag{109}$$

If the convection on the element edge 1–3 occurs, then the matrix associated with the convective type boundary condition $\left[\mathbf{K}_h^e\right]^A$ and the element load vector $\left\{\mathbf{f}_h^e\right\}^A$ must be calculated according to

$$\left[\mathbf{K}_h^e\right]^A = \int_{\Gamma_h^e} h[\Phi^e]^T[\Phi^e]\{\mathbf{r}\}ds$$
$$= \frac{\pi}{6} h l_{31} \begin{bmatrix} 3r_1^e + r_3^e & 0 & r_1^e + r_3^e \\ 0 & 0 & 0 \\ r_3^e + r_1^e & 0 & 3r_3^e + r_1^e \end{bmatrix} \tag{110}$$

and

$$\{\mathbf{f}_h^e\}^A = \int_{\Gamma_h^e} hT_\infty[\Phi^e]^T ds = \frac{\pi}{3} hT_\infty l_{31} \begin{Bmatrix} 2r_1^e + r_3^e \\ 0 \\ r_1^e + 2r_2^e \end{Bmatrix} \tag{111}$$

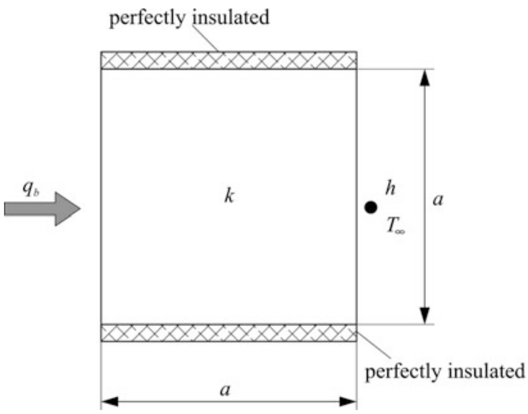
If the matrices $\left[\mathbf{K}_h^e\right]^A$ and $[\mathbf{K}_c^e]^A$ are known, it is necessary to combine them to form the stiffness matrix of element $[\mathbf{K}^e]^A$

$$[\mathbf{K}^e]^A = [\mathbf{K}_c^e]^A + [\mathbf{K}_h^e]^A = 2\pi r_c \begin{bmatrix} K_{11}^e & K_{12}^e & K_{13}^e \\ K_{21}^e & K_{22}^e & K_{23}^e \\ K_{31}^e & K_{32}^e & K_{33}^e \end{bmatrix} + \frac{\pi}{6} h l_{31} \begin{bmatrix} 3r_1^e + r_3^e & 0 & r_1^e + r_3^e \\ 0 & 0 & 0 \\ r_3^e + r_1^e & 0 & 3r_3^e + r_1^e \end{bmatrix} \tag{112}$$

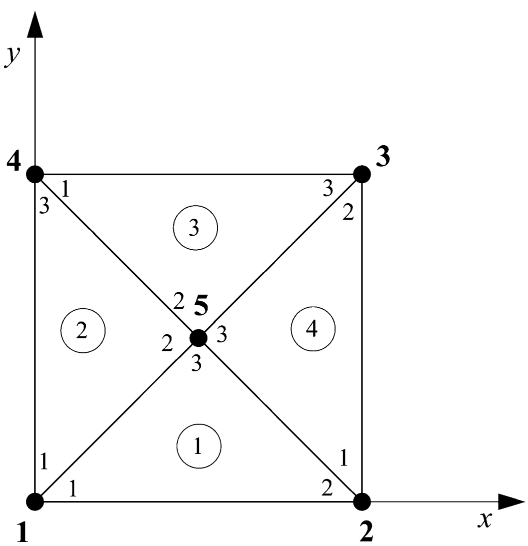
The application of finite element method for solving heat conduction problems will be illustrated in the examples.

Numerical Examples

Example 1. Determine the steady-state temperature distribution inside the infinite steel square rod (see Fig. 10), with the side length $a = 0.02$ m. Assume the thermal conductivity of $k_x = k_y = 50$ W/(m · K). The rod is insulated at the top and the bottom surface. On the left side surface of the rod, the heat flux is equal to $q_b = 200000 \, \frac{\mathrm{W}}{\mathrm{m}^2}$ On the right side of the domain, the convective boundary condition is applied with the heat transfer coefficient $h = 1{,}000$ W/ $(\mathrm{m}^2 \cdot \mathrm{K})$ and the bulk



Finite Element Method in Steady-State and Transient Heat Conduction, Fig. 10 The infinitely long rod with boundary conditions



Finite Element Method in Steady-State and Transient Heat Conduction, Fig. 11 Finite element discretization

temperature is equal to $T_\infty = 20$ °C. Use the linear triangular elements to solve this example. The temperature field is two dimensional; thus, the linear or rectangular elements can be applied to mode the heat transfer phenomenon. The square rod, divided into the finite elements, is shown in Fig. 11

The number in a circle indicates the element. To distinguish global and local node numbering, the global node numbers are marked in bold. The nodal coordinates are listed in Table 2

Finite Element Method in Steady-State and Transient Heat Conduction, Table 2 Nodal coordinates

Node number	x, m	y, m
1	0	0
2	0.02	0
3	0.02	0.02
4	0	0.02
5	0.01	0.01

Finite Element Method in Steady-State and Transient Heat Conduction, Table 3 Nodal connectivity table

	Local nodes			
Element number	N1	N2	N3	
1	1	2	5	Global nodes
2	1	5	4	
3	4	5	3	
4	2	3	5	

The node connectivity table (compare Fig. 11) is presented in Table 3.

First, the element matrices $[\mathbf{K}_c{}^e]$ and $[\mathbf{K}_h{}^e]$ and nodal load vector $\{\mathbf{f}\}$ for each element are determined. Assuming the isotropic thermal conductivity $k = k_x = k_y$, then the conductivity matrix of linear triangular element given by (69) has the form

$$[\mathbf{K}_c^e] = \frac{k}{4A^e}\begin{bmatrix} (b_1^e)^2 + (c_1^e)^2 & b_1^e b_2^e + c_1^e c_2^e & b_1^e b_3^e + c_1^e c_3^e \\ b_1^e b_2^e + c_1^e c_2^e & (b_2^e)^2 + (c_2^e)^2 & b_2^e b_3^e + c_2^e c_3^e \\ b_1^e b_3^e + c_1^e c_3^e & b_2^e b_3^e + c_2^e c_3^e & (b_3^e)^2 + (c_3^e)^2 \end{bmatrix} \tag{113}$$

The convective boundary condition is applied on the side formed from nodes 2 and 3, which corresponds to the edge 1–2 for element 4. Therefore, matrix $[K_h{}^e]$ is

$$[\mathbf{K}_h^{e4}] = \frac{h\, l_{12}}{6}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \frac{h\, a}{6}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{114}$$

The elements matrices $[K_c{}^e]$ and $[K_h{}^e]$ are

- Element 1

$$
\begin{aligned}
&x_1^{(1)} = 0\,\mathrm{m}, \quad y_1^{(1)} = 0\,\mathrm{m} \\
&x_2^{(1)} = 2\cdot 10^{-2}\,\mathrm{m}, \quad y_2^{(1)} = 0\,\mathrm{m} \\
&x_3^{(1)} = 1\cdot 10^{-2}\,\mathrm{m}, \quad y_3^{(1)} = 1\cdot 10^{-2}\mathrm{m} \\
&b_1^{(1)} = y_2^{(1)} - y_3^{(1)} = 0 - 1\cdot 10^{-2} = -1\cdot 10^{-2}\,\mathrm{m}, \\
&b_2^{(1)} = y_3^{(1)} - y_1^{(1)} = 1\cdot 10^{-2} - 0 = 1\cdot 10^{-2}\mathrm{m} \\
&b_3^{(1)} = y_1^{(1)} - y_2^{(1)} = 0 - 0 = 0\ \mathrm{m} \\
&c_1^{(1)} = x_3^{(1)} - x_2^{(1)} = 1\cdot 10^{-2} - 2\cdot 10^{-2} = -1\cdot 10^{-2}\,\mathrm{m}, \\
&c_2^{(1)} = x_1^{(1)} - x_3^{(1)} = 0 - 1\cdot 10^{-2} = -1\cdot 10^{-2}\mathrm{m} \\
&c_3^{(1)} = x_2^{(1)} - x_1^{(1)} = 2\cdot 10^{-2} - 0 = 2\cdot 10^{-2}\mathrm{m} \\
&A^{(1)} = \frac{1}{2}\cdot a\cdot\frac{a}{2} = \frac{1}{2}\cdot 0,02\cdot 0,01 = 1\cdot 10^{-4}\mathrm{m}^2
\end{aligned}
\tag{115}
$$

Hence, the conductivity matrix of element 1 calculated according to (113) is

$$
\left[\mathbf{K}_c^{e1}\right] = 50
\begin{array}{c}
\begin{array}{ccc} 1 & 2 & 5 \end{array} \\
\begin{bmatrix} 0.5 & 0 & -0.5 \\ 0 & 0.5 & -0.5 \\ -0.5 & -0.5 & 1 \end{bmatrix}
\end{array}
\begin{array}{c} \\ 1 \\ 2 \\ 5 \end{array}
\tag{116}
$$

The numbers above and next to the matrix denote the global node numbers.

- Element 2

$$
\begin{aligned}
&x_1^{(2)} = 0\ \mathrm{m}, \quad y_1^{(2)} = 0\ \mathrm{m}, \quad x_2^{(2)} = 1\cdot 10^{-2}\,\mathrm{m}, \\
&y_2^{(2)} = 1\cdot 10^{-2}\,\mathrm{m} \\
&x_3^{(2)} = 0\,\mathrm{m}, \quad y_3^{(2)} = 2\cdot 10^{-2}\,\mathrm{m} \\
&b_1^{(2)} = y_2^{(2)} - y_3^{(2)} = 1\cdot 10^{-2} - 2\cdot \\
&10^{-2} = -1\cdot 10^{-2}\ \mathrm{m}, \\
&b_2^{(2)} = y_3^{(2)} - y_1^{(2)} = 2\cdot 10^{-2} - 0 = 2\cdot 10^{-2}\ \mathrm{m} \\
&b_3^{(2)} = y_1^{(2)} - y_2^{(2)} = 0 - 1\cdot 10^{-2} = -1\cdot 10^{-2}\ \mathrm{m} \\
&c_1^{(2)} = x_3^{(2)} - x_2^{(2)} = 0 - 1\cdot 10^{-2} = -1\cdot 10^{-2}\ \mathrm{m}, \\
&c_2^{(2)} = x_1^{(2)} - x_3^{(2)} = 0 - 0 = 0\ \mathrm{m} \\
&c_3^{(2)} = x_2^{(2)} - x_1^{(2)} = 1\cdot 10^{-2} - 0 = 1\cdot 10^{-2}\ \mathrm{m} \\
&A^{(2)} = \frac{1}{2}\cdot a\cdot\frac{a}{2} = \frac{1}{2}\cdot 2\cdot 10^{-2}\cdot 1\cdot 10^{-2} = 1\cdot 10^{-4}\ \mathrm{m}^2
\end{aligned}
\tag{117}
$$

Thus, the conductivity matrix of element 2 is

$$
\left[\mathbf{K}_c^{e2}\right] = 50
\begin{array}{c}
\begin{array}{ccc} 1 & 5 & 4 \end{array} \\
\begin{bmatrix} 0.5 & -0.5 & 0 \\ -0.5 & 1 & -0.5 \\ 0 & -0.5 & 0.5 \end{bmatrix}
\end{array}
\begin{array}{c} \\ 1 \\ 5 \\ 4 \end{array}
\tag{118}
$$

- Element 3

$$
\begin{aligned}
&x_1^{(3)} = 0\,\mathrm{m}, \quad y_1^{(3)} = 2\cdot 10^{-2}\,\mathrm{m} \\
&x_2^{(3)} = 1\cdot 10^{-2}\,\mathrm{m}, \quad y_2^{(3)} = 1\cdot 10^{-2}\,\mathrm{m} \\
&x_3^{(3)} = 2\cdot 10^{-2}\,\mathrm{m}, \quad y_3^{(3)} = 2\cdot 10^{-2}\,\mathrm{m} \\
&b_1^{(3)} = y_2^{(3)} - y_3^{(3)} = 1\cdot 10^{-2} - 2\cdot 10^{-2} = -1\cdot 10^{-2}\,\mathrm{m}, \\
&b_2^{(3)} = y_3^{(3)} - y_1^{(3)} = 2\cdot 10^{-2} - 2\cdot 10^{-2} = 0\mathrm{m} \\
&b_3^{(3)} = y_1^{(3)} - y_2^{(3)} = 2\cdot 10^{-2} - 1\cdot 10^{-2} = 1\cdot 10^{-2}\,\mathrm{m} \\
&c_1^{(3)} = x_3^{(3)} - x_2^{(3)} = 2\cdot 10^{-2} - 1\cdot 10^{-2} = 1\cdot 10^{-2}\mathrm{m}, \\
&c_2^{(3)} = x_1^{(3)} - x_3^{(3)} = 0 - 2\cdot 10^{-2} = -2\cdot 10^{-2}\ \mathrm{m} \\
&c_3^{(3)} = x_2^{(3)} - x_1^{(3)} = 1\cdot 10^{-2} - 0 = 1\cdot 10^{-2}\mathrm{m} \\
&A^{(3)} = \frac{1}{2}\cdot a\cdot\frac{a}{2} = \frac{1}{2}\cdot 2\cdot 10^{-2}\cdot 1\cdot 10^{-2} = 1\cdot 10^{-4}\mathrm{m}^2
\end{aligned}
\tag{119}
$$

Hence, the conductivity matrix of element 3 is

$$
\left[\mathbf{K}_c^{e3}\right] = 50
\begin{array}{c}
\begin{array}{ccc} 4 & 5 & 3 \end{array} \\
\begin{bmatrix} 0.5 & -0.5 & 0 \\ -0.5 & 1 & -0.5 \\ 0 & -0.5 & 0.5 \end{bmatrix}
\end{array}
\begin{array}{c} \\ 4 \\ 5 \\ 3 \end{array}
\tag{120}
$$

- Element 4

$$
\begin{aligned}
&x_1^{(4)} = 2\cdot 10^{-2}\ \mathrm{m}, \quad y_1^{(4)} = 0\,\mathrm{m}, \\
&x_2^{(4)} = 2\cdot 10^{-2}\ \mathrm{m}, \\
&y_2^{(4)} = 2\cdot 10^{-2}\ \mathrm{m} \\
&x_3^{(4)} = 1\cdot 10^{-2}\ \mathrm{m}, \quad y_3^{(4)} = 1\cdot 10^{-2}\ \mathrm{m} \\
&b_1^{(4)} = y_2^{(4)} - y_3^{(4)} = 2\cdot 10^{-2} - 1\cdot 10^{-2} \\
&\quad = 1\cdot 10^{-2}\,\mathrm{m}, \quad b_2^{(4)} = y_3^{(4)} - y_1^{(4)} \\
&\quad = 1\cdot 10^{-2} - 0 = 1\cdot 10^{-2}\ \mathrm{m}
\end{aligned}
\tag{121}
$$

$$b_3^{(4)} = y_1^{(4)} - y_2^{(4)} = 0 - 2 \cdot 10^{-2} = -2 \cdot 10^{-2}\ \mathrm{m} c_1^{(4)} = x_3^{(4)} - x_2^{(4)} = 1 \cdot 10^{-2} - 2 \cdot 10^{-2} = -1 \cdot 10^{-2}\ \mathrm{m},$$

$$c_2^4 = x_1^4 - x_3^4 = 2 \cdot 10^{-2} - 1 \cdot 10^{-2} = 1 \cdot 10^{-2}\ \mathrm{m} c_3^{(4)} = x_2^{(4)} - x_1^{(4)} = 2 \cdot 10^{-2} - 2 \cdot 10^{-2} = 0\ \mathrm{m} A^{(4)} = \frac{1}{2} \cdot a \cdot \frac{a}{2} = \frac{1}{2} \cdot 2 \cdot 10^{-2} \cdot 1 \cdot 10^{-2} = 1 \cdot 10^{-4}\mathrm{m}^2$$

The conductivity matrix of element 4 is

$$\left[\mathbf{K}_c^{e4}\right] = 50 \begin{array}{c} \begin{array}{ccc} 2 & 3 & 5 \end{array} \\ \begin{bmatrix} 0.5 & 0 & -0.5 \\ 0 & 0.5 & -0.5 \\ -0.5 & -0.5 & 1 \end{bmatrix} \end{array} \begin{array}{c} 2 \\ 3 \\ 5 \end{array} \qquad (122)$$

The element matrix which arises from the convective heat transfer on the element edge 1–2 (or 2–3 for global node numbering) is

$$\left[\mathbf{K}_h^{e4}\right] = 3.33(3) \begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} 2 \\ 3 \\ \end{array} = \begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} 6.667 & 3.333 & 0 \\ 3.333 & 6.667 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} 2 \\ 3 \\ \end{array} \qquad (123)$$

Adding matrices $\left[\mathbf{K}_h^{e4}\right]$ and $\left[\mathbf{K}_c^{e4}\right]$ yields

$$\left[\mathbf{K}^{e4}\right] = \left[\mathbf{K}_c^{e4}\right] + \left[\mathbf{K}_h^{e4}\right] = \begin{array}{c} \begin{array}{ccc} 2 & 3 & 5 \end{array} \\ \begin{bmatrix} 31.667 & 3.333 & -25.0 \\ 3.333 & 31.667 & -25.0 \\ -25.0 & -25.0 & 50 \end{bmatrix} \end{array} \begin{array}{c} 2 \\ 3 \\ 5 \end{array} \qquad (124)$$

The assembly of element stiffness matrices $[\mathbf{K}^{\mathrm{e}}]$ into the global stiffness matrix $[\mathbf{K}]$ is performed below

$$K_{11} = K_{11}^{(1)} + K_{11}^{(2)} = 25 + 25 = 50\ \mathrm{W/(m \cdot K)}$$
$$K_{12} = K_{21}^{(1)} = 0,\quad K_{13} = 0,\quad K_{14} = K_{13}^{(2)} = 0$$
$$K_{15} = K_{31}^{(1)} + K_{21}^{(2)} = -25 + (-25) = -50\ \mathrm{W/(m \cdot K)}$$
$$K_{21} = K_{21}^{(1)} = 0,\ K_{22} = K_{22}^{(1)} + \left(K_{11}^{(4)} + K_{h,11}^{(4)}\right) = 25 + 31.6667 = 56.6667\ \mathrm{W/(m \cdot K)}$$
$$K_{23} = \left(\left[K_{21}^{(4)}\right] + \left[K_{h,21}^{(4)}\right]\right) = 3.333\ \mathrm{W/(m \cdot K)}$$
$$K_{24} = 0$$
$$K_{25} = K_{32}^{(1)} + K_{31}^{(4)} = -25 + (-25) = -50\ \mathrm{W/(m \cdot K)}$$
$$K_{31} = 0, K_{32} = \left(K_{12}^{(4)} + K_{h,12}^{(4)}\right) = 3.333\ \mathrm{W/(m \cdot K)}$$
$$K_{33} = K_{33}^{(3)} + \left(K_{22}^{(4)} + K_{h,22}^{(4)}\right) = 25 + 31{,}6667 = 56.6667\ \mathrm{W/(m \cdot K)}$$
$$K_{34} = K_{13}^{(3)} = 0\ \mathrm{W/(m \cdot K)}$$
$$K_{35} = K_{23}^{(3)} + K_{32}^{(4)} = -25 + (-25) = -50\ \mathrm{W/(m \cdot K)}$$
$$K_{41} = K_{13}^{(2)} = 0, K_{42} = 0,\quad K_{43} = K_{31}^{(3)} = 0,$$
$$K_{44} = K_{33}^{(2)} + K_{11}^{(3)} = 25 + 25 = 50\ \mathrm{W/(m \cdot K)}$$
$$K_{45} = K_{23}^{(2)} + K_{21}^{(3)} = -25 + (-25) = -50\ \mathrm{W/(m \cdot K)}$$
$$K_{51} = K_{13}^{(1)} + K_{12}^{(2)} = -25 + (-25) = -50\ \mathrm{W/(m \cdot K)}$$
$$K_{52} = K_{23}^{(1)} + K_{13}^{(4)} = -25 + (-25) = -50\ \mathrm{W/(m \cdot K)}$$
$$K_{53} = K_{32}^{(3)} + K_{23}^{(4)} = -25 + (-25) = -50\ \mathrm{W/(m \cdot K)}$$
$$K_{54} = K_{32}^{(2)} + K_{12}^{(3)} = -25 + (-25) = -50\ \mathrm{W/(m \cdot K)}$$
$$K_{55} = K_{33}^{(1)} + K_{22}^{(2)} + K_{22}^{(3)} + K_{33}^{(4)} = 50 + 50 + 50 + 50 = 200\ \mathrm{W/(m \cdot K)} \qquad (125)$$

If all the coefficients are known, then the global stiffness matrix is

$$[\mathbf{K}] = \begin{bmatrix} 50 & 0 & 0 & 0 & -50 \\ 0 & 56.667 & 3.333 & 0 & -50 \\ 0 & 3.333 & 56.667 & 0 & -50 \\ 0 & 0 & 0 & 50 & -50 \\ -50 & -50 & -50 & -50 & 200 \end{bmatrix} \mathrm{W/(m \cdot K)} \qquad (126)$$

The heat flux boundary condition is applied on the boundary edge 1–3 (1–4 if global node numbering) of element 2, thus

$$\left\{\mathbf{f}_q^{e2}\right\} = \begin{Bmatrix} f_1 \\ f_5 \\ f_4 \end{Bmatrix} = \frac{q_B l_{31}}{2} \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 2{,}000 \\ 0 \\ 2{,}000 \end{Bmatrix} \mathrm{W/m} \tag{127}$$

The convective boundary condition is applied on the boundary edge 1–2 (2–3 if global node numbering) of element 4, thus

$$\left\{\mathbf{f}_h^{e4}\right\} = \begin{Bmatrix} f_2 \\ f_3 \\ f_5 \end{Bmatrix} = \frac{hT_\infty l_{12}}{2} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 200 \\ 200 \\ 0 \end{Bmatrix} \mathrm{W/m} \tag{128}$$

Then the global vector of nodal loads is

$$\{\mathbf{f}\} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{Bmatrix} = \begin{Bmatrix} 2{,}000 \\ 200 \\ 200 \\ 2{,}000 \\ 0 \end{Bmatrix} \mathrm{W/m} \tag{129}$$

Assuming the steady-state heat conduction, if the global stiffness matrix $[\mathbf{K}]$ and global vector of nodal loads $\{\mathbf{f}\}$ are known, then the system of equations given by (46) simplifies to

$$[\mathbf{K}]\{\mathbf{T}\} = \{\mathbf{f}\} \tag{130}$$

Substituting (126) and (129) into (130) yields

$$\begin{bmatrix} 50 & 0 & 0 & 0 & -50 \\ 0 & 56.667 & 3.333 & 0 & -50 \\ 0 & 3.333 & 56.667 & 0 & -50 \\ 0 & 0 & 0 & 50 & -50 \\ -50 & -50 & -50 & -50 & 200 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} 2{,}000 \\ 200 \\ 200 \\ 2{,}000 \\ 0 \end{Bmatrix} \tag{131}$$

The nodal temperatures obtained by solving linear system given by (131) are

$$\{\mathbf{T}\} = \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} 300 \\ 220 \\ 220 \\ 300 \\ 260 \end{Bmatrix} {}^\circ\mathrm{C} \tag{132}$$

The rod is perfectly insulated at top and bottom surfaces; thus, the temperature field inside the rod is one dimensional. Thus, the nodal temperatures can be calculated using the analytical formulas [1]

$$\begin{aligned} T_1 = T_4 &= \frac{q_B \cdot a}{k} + \frac{q_B}{h} + T_\infty = \frac{200{,}000 \cdot 0.02}{50} \\ &\quad + \frac{200{,}000}{1{,}000} + 20 = 300\,{}^\circ\mathrm{C} \\ T_5 &= \frac{q_B \cdot a}{2k} + \frac{q_B}{h} + T_\infty = \frac{200{,}000 \cdot 0.02}{2 \cdot 50} \\ &\quad + \frac{200{,}000}{1{,}000} + 20 = 260\,{}^\circ\mathrm{C} \\ T_2 = T_3 &= \frac{q_B}{h} + T_\infty = \frac{200{,}000}{1{,}000} + 20 = 220\,{}^\circ\mathrm{C} \end{aligned} \tag{133}$$

The results obtained using the analytical and numerical methods (FEM) are the same.

Example 2. For the steel square rod shown in Example 1, determine the transient temperature distribution. Assume the step size Δt equal to 5 s, the initial nodal temperatures inside the steel rod $\{\mathbf{T}\}$ =20 °C and the end time $t_f = 1{,}000$ s.

Solve this problem using the implicit method. The specific heat capacity equals $c = 460$ J/(kg K) and density is $\rho = 7{,}860$ kg/m^3.

- The implicit (backward difference) method

For the backward difference method ($\theta = 1$ in (54)), the transient heat conduction equation can be written in the following form:

$$([\mathbf{M}] + \Delta t[\mathbf{K}])\{\mathbf{T}\}^{n+1} = \Delta t\{\mathbf{f}\}^{n+1} + [\mathbf{M}]\{\mathbf{T}\}^n \tag{134}$$

The global stiffness matrix [**K**] is given by (126) and the global nodal load vector {**f**} is constant in time, thus

$$[\mathbf{K}] = \begin{bmatrix} 50 & 0 & 0 & 0 & -50 \\ 0 & 56.667 & 3.333 & 0 & -50 \\ 0 & 3.333 & 56.667 & 0 & -50 \\ 0 & 0 & 0 & 50 & -50 \\ -50 & -50 & -50 & -50 & 200 \end{bmatrix} \mathrm{W/(m \cdot K)}$$

$$\{\mathbf{f}\}^n = \{\mathbf{f}\}^{n+1} = \{\mathbf{f}\}^{n+2} =, \ldots, = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{Bmatrix} = \begin{Bmatrix} 2{,}000 \\ 200 \\ 200 \\ 2{,}000 \\ 0 \end{Bmatrix} \mathrm{W/m} \tag{135}$$

The global capacitance matrix [**M**] is formed from the element capacitance matrices. Therefore, the element capacitance matrices must be found at first. The matrix [**M**^{e}] can be calculated from (70). Because all the elements have equal areas, then the matrix [**M**^{e}] is

$$[\mathbf{M}^e] = \frac{A^e}{12} c\rho \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \frac{1 \cdot 10^{-4}}{12} \cdot 460 \cdot 7{,}860 \cdot \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 60.26 & 30.13 & 30.13 \\ 30.13 & 60.26 & 30.13 \\ 30.13 & 30.13 & 60.26 \end{bmatrix} \tag{136}$$

Taking into account the global numbering of node, the element capacitance matrices are

- Elements 1 and 2

$$[\mathbf{M}^{e1}] = \begin{array}{c} \begin{matrix} 1 & \quad 2 & \quad 3 & 4 & \quad 5 \end{matrix} \\ \begin{bmatrix} 60.26 & 30.13 & 0 & 0 & 30.13 \\ 30.13 & 60.26 & 0 & 0 & 30.13 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 30.13 & 30.13 & 0 & 0 & 60.26 \end{bmatrix} \end{array} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix}$$

$$[\mathbf{M}^{e2}] = \begin{array}{c} \begin{matrix} 1 & \quad 2 & 3 & \quad 4 & \quad 5 \end{matrix} \\ \begin{bmatrix} 60.26 & 0 & 0 & 30.13 & 30.13 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 30.13 & 0 & 0 & 60.26 & 30.13 \\ 30.13 & 0 & 0 & 30.13 & 60.26 \end{bmatrix} \end{array} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \tag{137}$$

- Elements 3 and 4

$$[\mathbf{M}^{e3}] = \begin{array}{c} \begin{matrix} 1 & 2 & \quad 3 & \quad 4 & \quad 5 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 60.26 & 30.13 & 30.13 \\ 0 & 0 & 30.13 & 60.26 & 30.13 \\ 0 & 0 & 30.13 & 30.13 & 60.26 \end{bmatrix} \end{array} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix}$$

$$[\mathbf{M}^{e4}] = \begin{array}{c} \begin{matrix} 1 & \quad 2 & \quad 3 & 4 & \quad 5 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 60.26 & 30.13 & 0 & 30.13 \\ 0 & 30.13 & 60.26 & 0 & 30.13 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 30.13 & 30.13 & 0 & 60.26 \end{bmatrix} \end{array} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \tag{138}$$

Adding matrices all the element matrices given by (137) and (138) yields

$$[\mathbf{M}] = [\mathbf{M}^{e1}] + [\mathbf{M}^{e2}] + [\mathbf{M}^{e3}] + [\mathbf{M}^{e4}] = [\mathbf{M}^{e4}]$$

$$= \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 120.52 & 30.13 & 0 & 30.13 & 60.26 \\ 30.13 & 120.52 & 30.13 & 0 & 60.26 \\ 0 & 30.13 & 120.52 & 30.13 & 60.26 \\ 30.13 & 0 & 30.13 & 120.52 & 60.26 \\ 60.26 & 60.26 & 60.26 & 60.26 & 241.04 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \tag{139}$$

If the global stiffness matrix $[\mathbf{K}]$ and the global capacitance matrix $[\mathbf{M}]$ are known, then the initial conditions can be applied.

- Initial conditions, $t = 0$ s

The nodal temperatures are equal to the initial temperatures, thus

$$\{\mathbf{T}\}^{t=0s} = \begin{Bmatrix} 20 \\ 20 \\ 20 \\ 20 \\ 20 \end{Bmatrix} {}^\circ\mathrm{C} \tag{140}$$

Performing the first iteration, the temperature vector at time $t = 5$ s is obtained.

- Time step 1, $t = 5$ s

The vector of nodal temperatures from previous iteration equals to

$$\{\mathbf{T}\}^n = \{\mathbf{T}\}^{t=0s} = \begin{Bmatrix} 20 \\ 20 \\ 20 \\ 20 \\ 20 \end{Bmatrix} {}^\circ\mathrm{C} \tag{141}$$

Then substituting (139), (141), and (135) into (134), one obtains

$$([\mathbf{M}] + \Delta t[\mathbf{K}])\{\mathbf{T}\}^{n+1} = \Delta t\{\mathbf{f}\}^{n+1} + [\mathbf{M}]\{\mathbf{T}\}^n$$

$$\left(\begin{bmatrix} 120.52 & 30.13 & 0 & 30.13 & 60.26 \\ 30.13 & 120.52 & 30.13 & 0 & 60.26 \\ 0 & 30.13 & 120.52 & 30.13 & 60.26 \\ 30.13 & 0 & 30.13 & 120.52 & 60.26 \\ 60.26 & 60.26 & 60.26 & 60.26 & 241.04 \end{bmatrix} + 5.0 \cdot \begin{bmatrix} 50 & 0 & 0 & 0 & -50 \\ 0 & 56.667 & 3.333 & 0 & -50 \\ 0 & 3.333 & 56.667 & 0 & -50 \\ 0 & 0 & 0 & 50 & -50 \\ -50 & -50 & -50 & -50 & 200 \end{bmatrix} \right) \times \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = 5.0 \cdot \begin{Bmatrix} 2,000 \\ 200 \\ 200 \\ 2,000 \\ 0 \end{Bmatrix} + \begin{bmatrix} 120.52 & 30.13 & 0 & 30.13 & 60.26 \\ 30.13 & 120.52 & 30.13 & 0 & 60.26 \\ 0 & 30.13 & 120.52 & 30.13 & 60.26 \\ 30.13 & 0 & 30.13 & 120.52 & 60.26 \\ 60.26 & 60.26 & 60.26 & 60.26 & 241.04 \end{bmatrix} \times \begin{Bmatrix} 20 \\ 20 \\ 20 \\ 20 \\ 20 \end{Bmatrix} \tag{142}$$

Performing the matrix operation for the first time step, the following system of linear equations is obtained:

$$\begin{bmatrix} 370.52 & 30.13 & 0 & 30.13 & -189.74 \\ 30.13 & 403.85 & 46.80 & 0 & -189.74 \\ 0 & 46.80 & 403.85 & 30.13 & -189.74 \\ 30.13 & 0 & 30.13 & 370.52 & -189.74 \\ -189.74 & -189.74 & -189.74 & -189.74 & 1241.04 \end{bmatrix} \times \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} 14{,}820.8 \\ 5{,}820.8 \\ 5{,}820.8 \\ 14{,}820.8 \\ 9{,}641.6 \end{Bmatrix} \tag{143}$$

Solving the linear system given by (143) one obtains

$$\{\mathbf{T}\}^{n+1} = \{\mathbf{T}\}^{t=5\text{s}} = \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} 49.36 \\ 22.09 \\ 22.09 \\ 49.36 \\ 29.61 \end{Bmatrix} {}^\circ\text{C} \tag{144}$$

Next, the second step to obtain the temperature distribution at $t = 10$ s is performed.

- Time step 2, $t = 10$ s

The vector of nodal temperatures from previous time step equals

$$\{\mathbf{T}\}^{n} = \{\mathbf{T}\}^{t=5\text{s}} = \begin{Bmatrix} 49.36 \\ 22.09 \\ 22.09 \\ 49.36 \\ 29.61 \end{Bmatrix} {}^\circ\text{C} \tag{145}$$

Then substituting (139), (145), and (135) into (134), one obtains

$$([\mathbf{M}] + \Delta t[\mathbf{K}])\{\mathbf{T}\}^{n+1} = \Delta t\{\mathbf{f}\}^{n+1} + [\mathbf{M}]\{\mathbf{T}\}^{n}$$

$$\left(\begin{bmatrix} 120.52 & 30.13 & 0 & 30.13 & 60.26 \\ 30.13 & 120.52 & 30.13 & 0 & 60.26 \\ 0 & 30.13 & 120.52 & 30.13 & 60.26 \\ 30.13 & 0 & 30.13 & 120.52 & 60.26 \\ 60.26 & 60.26 & 60.26 & 60.26 & 241.04 \end{bmatrix} + 5.0 \cdot \begin{bmatrix} 50 & 0 & 0 & 0 & -50 \\ 0 & 56.667 & 3.333 & 0 & -50 \\ 0 & 3.333 & 56.667 & 0 & -50 \\ 0 & 0 & 0 & 50 & -50 \\ -50 & -50 & -50 & -50 & 200 \end{bmatrix} \right) \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix}$$
$$= 5.0 \cdot \begin{Bmatrix} 2{,}000 \\ 200 \\ 200 \\ 2{,}000 \\ 0 \end{Bmatrix} + \begin{bmatrix} 120.52 & 30.13 & 0 & 30.13 & 60.26 \\ 30.13 & 120.52 & 30.13 & 0 & 60.26 \\ 0 & 30.13 & 120.52 & 30.13 & 60.26 \\ 30.13 & 0 & 30.13 & 120.52 & 60.26 \\ 60.26 & 60.26 & 60.26 & 60.26 & 241.04 \end{bmatrix} \times \begin{Bmatrix} 49.36 \\ 22.09 \\ 22.09 \\ 49.36 \\ 29.61 \end{Bmatrix} \tag{146}$$

Performing the matrix operation for time step 2, one obtains

$$\begin{bmatrix} 370.52 & 30.13 & 0 & 30.13 & -189.74 \\ 30.13 & 403.85 & 46.80 & 0 & -189.74 \\ 0 & 46.80 & 403.85 & 30.13 & -189.74 \\ 30.13 & 0 & 30.13 & 370.52 & -189.74 \\ -189.74 & -189.74 & -189.74 & -189.74 & 1241.04 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} 19{,}885.36 \\ 7{,}598.72 \\ 7{,}598.72 \\ 19{,}885.36 \\ 15{,}748.13 \end{Bmatrix} \tag{147}$$

Solving the linear system given by (147), the nodal temperatures at time $t = 10$ s are obtained

$$\{\mathbf{T}\}^{n+1} = \{\mathbf{T}\}^{t=10\mathrm{s}} = \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} 67.53 \\ 30.28 \\ 30.28 \\ 67.53 \\ 42.60 \end{Bmatrix} {}^\circ\mathrm{C} \tag{148}$$

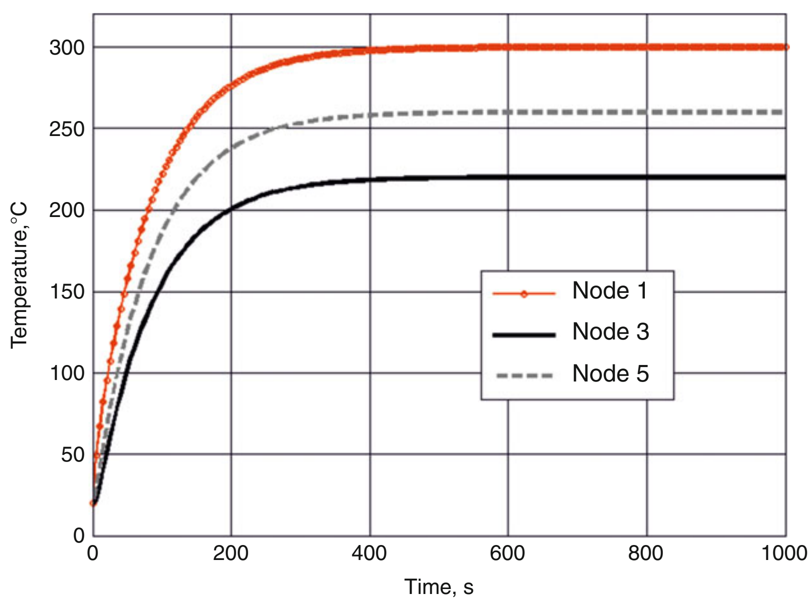
The nodal temperatures vector in next time step is obtained by increasing the value of t by Δt and taking into account that vector $\{\mathbf{T}\}^n$ is equal to the nodal temperatures vector obtained in previous time step. The transient temperature profiles for nodes 1, 3, 5 are presented in Fig. 12.

Example 3. Determine the steady-state temperature distribution inside the steel tube, with the thickness $a = 0.02$ m. The inner radius of the tube is $r_{in} = 0.1$ m. Assume the thermal conductivity of $k_x = k_y = 50$ W/(m · K). On the left side surface of the rod, the heat flux is equal to $q_b = 200,000 \frac{\mathrm{W}}{\mathrm{m}^2}$. On the right side of the domain, the convective boundary condition is applied with the heat transfer coefficient $h = 1000$ W/(m^2 · K) and the bulk temperature is equal to $T_\infty = 20$ °C. Use the axisymmetric linear triangular elements to solve this example.

Only the section of the tube is considered. The heat transfer domain and the boundary conditions are shown in Fig. 13.

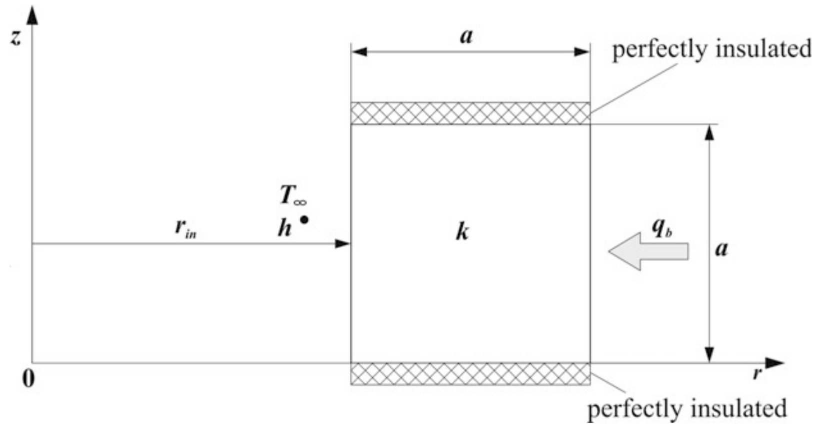
The discretization of the heat transfer domain is presented in Fig. 14.

First the mean radius r_c is calculated for all the elements

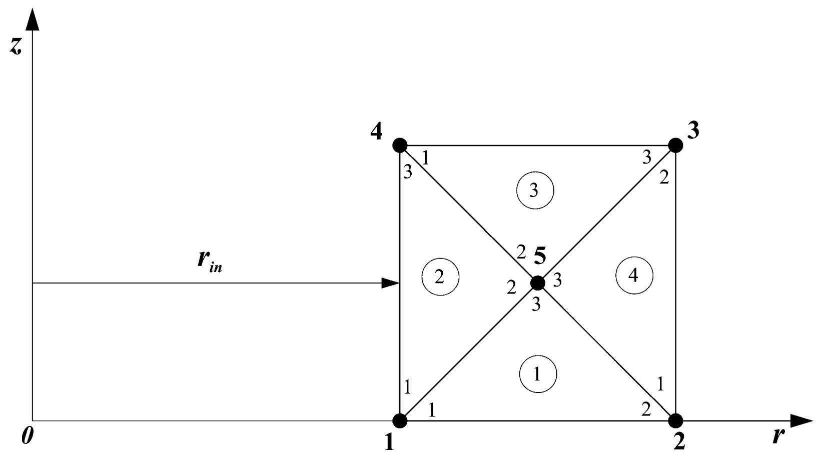


Finite Element Method in Steady-State and Transient Heat Conduction, Fig. 12 Transient temperature distribution for nodes 1, 3, 5 with $\Delta t = 5$ s

Finite Element Method in Steady-State and Transient Heat Conduction, Fig. 13 The section of tube



Finite Element Method in Steady-State and Transient Heat Conduction, Fig. 14 FEM discretization



$$
\begin{aligned}
r_c^{(1)} &= \frac{1}{3}\left(r_1^{(1)} + r_2^{(1)} + r_3^{(1)}\right) = \frac{1}{3}(r_1 + r_2 + r_5) \\
&= \frac{1}{3}(0.1 + 0.12 + 0.11) = 0.11 \text{ m} \\
r_c^{(2)} &= \frac{1}{3}\left(r_1^{(2)} + r_2^{(2)} + r_3^{(2)}\right) = \frac{1}{3}(r_1 + r_5 + r_4) \\
&= \frac{1}{3}(0.1 + 0.11 + 0.1) = 0.1033 \text{ m} \\
r_c^{(3)} &= \frac{1}{3}\left(r_1^{(3)} + r_2^{(3)} + r_3^{(3)}\right) = \frac{1}{3}(r_4 + r_5 + r_3) \\
&= \frac{1}{3}(0.1 + 0.11 + 0.12) = 0.11 \text{ m} \\
r_c^{(4)} &= \frac{1}{3}\left(r_1^{(4)} + r_2^{(4)} + r_3^{(4)}\right) = \frac{1}{3}(r_2 + r_3 + r_5) \\
&= \frac{1}{3}(0.12 + 0.12 + 0.11) = 0.11667 \text{ m}
\end{aligned} \tag{149}
$$

Note that the finite elements used in the discretization of the heat transfer domain have the same size as for the Example 1. Denoting the stiffness matrix of axisymmetric element as $\left[\mathbf{K}_c^e\right]^A$ and the conductivity matrix of the two-dimensional linear triangular element as $\left[\mathbf{K}_c^e\right]$ and using (105) and (113), one obtains

- Element 1

$$
\left[\mathbf{K}_c^{e1}\right]^A = 2\pi r_c^{(1)}\left[\mathbf{K}_c^{e1}\right]
$$

$$
= 2\pi \cdot 0.11 \cdot \begin{array}{c} \begin{array}{ccc} 1 & 2 & 5 \end{array} \\ \begin{bmatrix} 25 & 0 & -25 \\ 0 & 25 & -25 \\ -25 & -25 & 50 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 5 \end{array} \tag{150}
$$

$$
= \begin{array}{c} \begin{array}{ccc} 1 & 2 & 5 \end{array} \\ \begin{bmatrix} 17.28 & 0 & -17.28 \\ 0 & 17.28 & -17.28 \\ -17.28 & -17.28 & 34.56 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 5 \end{array}
$$

- Element 2

$$\left[\mathbf{K}_c^{e2}\right]^A = 2\pi r_c^{(2)}\left[\mathbf{K}_c^{e2}\right]$$

$$= 2\pi\cdot 0.1033\cdot \begin{array}{c} \begin{array}{ccc} 1 & 5 & 4 \end{array} \\ \begin{bmatrix} 25 & -25 & 0 \\ -25 & 50 & -25 \\ 0 & -25 & 25 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 5 \\ 4 \end{array} \qquad (151)$$

$$= \begin{array}{c} \begin{array}{ccc} 1 & 5 & 4 \end{array} \\ \begin{bmatrix} 16.23 & -16.23 & 0 \\ -16.23 & 32.46 & -16.23 \\ 0 & -16.23 & 16.23 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 5 \\ 4 \end{array}$$

- Element 3

$$\left[\mathbf{K}_c^{e3}\right]^A = 2\pi r_c^{(3)}\left[\mathbf{K}_c^{e3}\right]$$

$$= 2\pi\cdot 0.11\cdot \begin{array}{c} \begin{array}{ccc} 4 & 5 & 3 \end{array} \\ \begin{bmatrix} 25 & -25 & 0 \\ -25 & 50 & -25 \\ 0 & -25 & 25 \end{bmatrix} \end{array} \begin{array}{c} \\ 4 \\ 5 \\ 3 \end{array} \qquad (152)$$

$$= \begin{array}{c} \begin{array}{ccc} 4 & 5 & 3 \end{array} \\ \begin{bmatrix} 17.28 & -17.28 & 0 \\ -17.28 & 34.56 & -17.28 \\ 0 & -17.28 & 17.28 \end{bmatrix} \end{array} \begin{array}{c} \\ 4 \\ 5 \\ 3 \end{array}$$

- Element 4

$$\left[\mathbf{K}_c^{e4}\right]^A = 2\pi r_c^{(4)}\left[\mathbf{K}_c^{e4}\right]$$

$$= 2\pi\cdot 0.11667\cdot \begin{array}{c} \begin{array}{ccc} 2 & 3 & 5 \end{array} \\ \begin{bmatrix} 25 & 0 & -25 \\ 0 & 25 & -25 \\ -25 & -25 & 50 \end{bmatrix} \end{array} \begin{array}{c} \\ 2 \\ 3 \\ 5 \end{array} \qquad (153)$$

$$= \begin{array}{c} \begin{array}{ccc} 2 & 3 & 5 \end{array} \\ \begin{bmatrix} 18.33 & 0 & -18.33 \\ 0 & 18.33 & -18.33 \\ -18.33 & -18.33 & 36.66 \end{bmatrix} \end{array} \begin{array}{c} \\ 2 \\ 3 \\ 5 \end{array}$$

The convective boundary condition is applied on edge 1–3 (1–4 for global node numbering) of element 2. According to (110), the matrix $[\mathbf{K}_h^e]^A$ is

$$[\mathbf{K}_h^{e2}]^A = \frac{\pi}{6} h l_{31} \begin{bmatrix} 3r_1^e + r_3^e & 0 & r_1^e + r_3^e \\ 0 & 0 & 0 \\ r_3^e + r_1^e & 0 & 3r_3^e + r_1^e \end{bmatrix}$$

$$= \frac{\pi}{6} ha \begin{bmatrix} 3r_1 + r_4 & 0 & r_1 + r_4 \\ 0 & 0 & 0 \\ r_4 + r_1 & 0 & 3r_4 + r_1 \end{bmatrix}$$

$$= \frac{\pi}{6} ha \begin{bmatrix} 4r_{in} & 0 & 2r_{in} \\ 0 & 0 & 0 \\ 2r_{in} & 0 & 4r_{in} \end{bmatrix}$$

$$[\mathbf{K}_h^{e2}]^A = \frac{\pi}{3} ha \begin{bmatrix} 2r_{in} & 0 & r_{in} \\ 0 & 0 & 0 \\ r_{in} & 0 & 2r_{in} \end{bmatrix}$$

$$= \frac{\pi}{3}\cdot 1{,}000\cdot 0.02\cdot \begin{array}{c} \begin{array}{ccc} 1 & 5 & 4 \end{array} \\ \begin{bmatrix} 0.2 & 0 & 0.1 \\ 0 & 0 & 0 \\ 0.1 & 0 & 0.2 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 5 \\ 4 \end{array}$$

$$= \begin{array}{c} \begin{array}{ccc} 1 & 5 & 4 \end{array} \\ \begin{bmatrix} 4.19 & 0 & 2.095 \\ 0 & 0 & 0 \\ 2.095 & 0 & 4.19 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 5 \\ 4 \end{array} \qquad (154)$$

According to (111), the nodal load vector for element 2 is

$$\{\mathbf{f}_h^{e2}\}^A = \frac{\pi}{3} hT_\infty l_{31} \begin{Bmatrix} 2r_1^e + r_3^e \\ 0 \\ r_1^e + 2r_2^e \end{Bmatrix}$$

$$= \frac{\pi}{3} hT_\infty a \begin{Bmatrix} 2r_1 + r_4 \\ 0 \\ r_1 + 2r_4 \end{Bmatrix}$$

$$= \frac{\pi}{3} hT_\infty a \begin{Bmatrix} 3r_{in} \\ 0 \\ 3r_{in} \end{Bmatrix}$$

$$\{\mathbf{f}_h^{e2}\}^A = \begin{Bmatrix} f_1 \\ f_5 \\ f_4 \end{Bmatrix} = \frac{\pi}{3}\cdot 1,000\cdot 20\cdot 0.02\begin{Bmatrix} 3\cdot 0.1 \\ 0 \\ 3\cdot 0.1 \end{Bmatrix} = \begin{Bmatrix} 125.66 \\ 0 \\ 125.66 \end{Bmatrix}\mathrm{W/m} \tag{155}$$

The stiffness matrix for element 2 is obtained by adding matrices $[\mathbf{K}_c^{e2}]^A$ and $[\mathbf{K}_h^{e2}]^A$

$$[\mathbf{K}^{e2}]^A = [\mathbf{K}_c^{e2}]^A + [\mathbf{K}_h^{e2}]^A$$

$$= \begin{array}{c} \begin{matrix} 1 & \quad 5 & \quad 4 \end{matrix} \\ \begin{bmatrix} 16.23 & -16.23 & 0 \\ -16.23 & 32.46 & -16.23 \\ 0 & -16.23 & 16.23 \end{bmatrix} \end{array} \begin{matrix} 1 \\ 5 \\ 4 \end{matrix} + \begin{array}{c} \begin{matrix} 1 & 5 & 4 \end{matrix} \\ \begin{bmatrix} 4.19 & 0 & 2.095 \\ 0 & 0 & 0 \\ 2.095 & 0 & 4.19 \end{bmatrix} \end{array} \begin{matrix} 1 \\ 5 \\ 4 \end{matrix}$$

$$[\mathbf{K}^{e2}]^A = \begin{array}{c} \begin{matrix} 1 & \quad 5 & \quad 4 \end{matrix} \\ \begin{bmatrix} 20.42 & -16.23 & 2.095 \\ -16.23 & 32.46 & -16.23 \\ 2.095 & -16.23 & 20.42 \end{bmatrix} \end{array} \begin{matrix} 1 \\ 5 \\ 4 \end{matrix} \tag{156}$$

The heat flux boundary conditions are applied to the edge 1–2 (2–3 if global node numbering) of element 4; thus, according to (109), the nodal load vector for element 4 is

$$\left\{\mathbf{f}_q^{e4}\right\}^A = \frac{\pi}{3}q_b l_{12}\begin{Bmatrix} 2r_1^e + r_2^e \\ r_1^e + 2r_2^e \\ 0 \end{Bmatrix} = \frac{\pi}{3}q_b a\begin{Bmatrix} 2r_2 + r_3 \\ r_3 + 2r_2 \\ 0 \end{Bmatrix} = \frac{\pi}{3}q_b a\begin{Bmatrix} 3(r_{in} + a) \\ 3(r_{in} + a) \\ 0 \end{Bmatrix}$$

$$\left\{\mathbf{f}_q^{e4}\right\}^A = \begin{Bmatrix} f_2 \\ f_3 \\ f_5 \end{Bmatrix} = \frac{\pi}{3}\cdot 200,000\cdot 0.02\begin{Bmatrix} 3\cdot 0.12 \\ 3\cdot 0.12 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1,508 \\ 1,508 \\ 0 \end{Bmatrix}\mathrm{W/m} \tag{157}$$

Then the global vector of nodal loads is obtained by adding global forms of vectors $\{\mathbf{f}_h^{e2}\}^A$ and $\left\{\mathbf{f}_q^{e4}\right\}^A$.

$$\{\mathbf{f}\} = \{\mathbf{f}_h^{e2}\}^A + \left\{\mathbf{f}_q^{e4}\right\}^A = \begin{Bmatrix} 125.66 \\ 0 \\ 0 \\ 125.66 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 1,508 \\ 1,508 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 125.66 \\ 1,508 \\ 1,508 \\ 125.66 \\ 0 \end{Bmatrix}\mathrm{W/m} \tag{158}$$

Adding the global forms of element matrices, the global stiffness matrix is obtained

$$[\mathbf{K}^{e1}] = \begin{bmatrix} 17.28 & 0 & 0 & 0 & -17.28 \\ 0 & 17.28 & 0 & 0 & -17.28 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -17.28 & -17.28 & 0 & 0 & 34.56 \end{bmatrix}$$

$$[\mathbf{K}^{e2}] = \begin{bmatrix} 20.42 & 0 & 0 & 2.095 & -16.23 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2.095 & 0 & 0 & 20.42 & -16.23 \\ -16.23 & 0 & 0 & -16.23 & 32.46 \end{bmatrix}$$

$$[\mathbf{K}^{e3}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 17.28 & 0 & -17.28 \\ 0 & 0 & 0 & 17.28 & -17.28 \\ 0 & 0 & -17.28 & -17.28 & 34.56 \end{bmatrix}$$

$$[\mathbf{K}^{e4}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 18.33 & 0 & 0 & -18.33 \\ 0 & 0 & 18.33 & 0 & -18.33 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 18.33 & -18.33 & 0 & 36.66 \end{bmatrix}$$

$$[\mathbf{K}] = [\mathbf{K}^{e1}] + [\mathbf{K}^{e2}] + [\mathbf{K}^{e3}] + [\mathbf{K}^{e4}] = \begin{bmatrix} 37.7 & 0 & 0 & 2.1 & -33.51 \\ 0 & 35.61 & 0 & 0 & -35.61 \\ 0 & 0 & 35.61 & 0 & -35.61 \\ 2.1 & 0 & 0 & 37.7 & -33.51 \\ -33.51 & -35.61 & -35.61 & -33.51 & 138.24 \end{bmatrix} \mathrm{W/(m \cdot K)} \tag{159}$$

Then the nodal temperature vector is obtained by solving the linear system

$$[\mathbf{K}]\{\mathbf{T}\} = \{\mathbf{f}\}$$

$$\begin{bmatrix} 37.7 & 0 & 0 & 2.1 & -33.51 \\ 0 & 35.61 & 0 & 0 & -35.61 \\ 0 & 0 & 35.61 & 0 & -35.61 \\ 2.1 & 0 & 0 & 37.7 & -33.51 \\ -33.51 & -35.61 & -35.61 & -33.51 & 138.24 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} 125.66 \\ 1508 \\ 1508 \\ 125.66 \\ 0 \end{Bmatrix} \tag{160}$$

The nodal temperatures are

$$\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} 259.72 \\ 347.07 \\ 347.07 \\ 259.72 \\ 304.72 \end{Bmatrix} {}^\circ\mathrm{C} \tag{161}$$

Performing the calculations with eight digits precision, one obtains

$$\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} 260 \\ 347.35 \\ 347.35 \\ 260 \\ 305 \end{Bmatrix} {}^\circ\mathrm{C} \tag{162}$$

The temperatures obtained using analytical methods presented in [1] are

$$
\begin{aligned}
T &= T_\infty + \frac{q_b(r_{in}+a)}{k}\left(\ln\frac{r}{r_{in}} + \frac{k}{h\cdot r_{in}}\right), \text{ hence}\\
T_1 = T_4 &= T_\infty + \frac{q_b(r_{in}+a)}{k}\left(\ln\frac{r_{in}}{r_{in}} + \frac{k}{h\cdot r_{in}}\right)\\
&= 20 + \frac{2\cdot 10^5\cdot 0.12}{50}\left(\ln\frac{0.1}{0.1} + \frac{50}{1\cdot 10^3\cdot 0.1}\right) = 260\,^\circ\mathrm{C}\\
T_3 = T_2 &= T_\infty + \frac{q_b(r_{in}+a)}{k}\left(\ln\frac{(r_{in}+a)}{r_{in}} + \frac{k}{h\cdot r_{in}}\right)\\
&= 20 + \frac{2\cdot 10^5\cdot 0.12}{50}\left(\ln\frac{0.12}{0.1} + \frac{50}{1\cdot 10^3\cdot 0.1}\right) = 347.51\,^\circ\mathrm{C}\\
T_5 &= T_\infty + \frac{q_b(r_{in}+a)}{k}\left(\ln\frac{(r_{in}+\frac{a}{2})}{r_{in}} + \frac{k}{h\cdot r_{in}}\right)\\
&= 20 + \frac{2\cdot 10^5\cdot 0.12}{50}\left(\ln\frac{0.11}{0.1} + \frac{50}{1\cdot 10^3\cdot 0.1}\right) = 305.75\,^\circ\mathrm{C}
\end{aligned}
\tag{163}
$$

The numerical results differ slightly from the analytical results, because of coarse mesh. Therefore, the logarithmic temperature profile cannot be correctly approximated using linear shape functions. With the further refinement of the numerical grid, the solution approaches the analytical temperature profile.

Summary

In this entry, the Galerkin finite element method (FEM) for solving steady-state, transient, and axisymmetric heat conduction problems was presented. This numerical technique allows solving the heat transfer problems with irregular boundaries, which makes it widely used for solving practical engineering problems. The examples of steady-state and transient heat transfer problems, solved using FEM, were presented. Moreover, the axisymmetric analysis of the temperature distribution along the cross section of tube was performed. The numerical results, obtained using FEM, were well in accordance with analytical solutions. In addition to the numerical analysis shown in this entry, it is possible to apply FEM in solving nonlinear problems, with temperature-dependent material properties. Therefore, the finite element method is widely used for solving the practical engineering heat transfer problems, where analytical solution is unknown.

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Finite Element Simulation of the Fusion Welding of Metal Components Including Post-weld Heat Treatment

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Overview

Welding is a popular method of forming joints between metal components. The most common