Functions of form and points of integration finite elements

Summary:

One describes the geometry and the topology of the finite elements established in *Code_Aster* ; for each element of reference, the expression of the functions of form and the various families of points of integration as well as the associated weights are detailed.

Contents

1 Introduction........3
2 Linear elements: SE2, SE3 and SE4........4
3 Surface elements........5
3.1 Triangles: ELREFE TR3, TR6, TR7........5
3.2 Quadrangles: ELREFE QU4, QU8, QU9........9
4 Voluminal elements........12
4.1 Tetrahedrons: ELREFE TE4, T10........12
4.2 Pentahedrons: ELREFE PE6, P15, P18........14
4.3 Hexahedrons: ELREFE HE8 , H20 , H27........18
4.4 Pyramids: ELREFE PY5, P13........24
5 Bibliography........28
6 History of the versions of the document........28

1 Introduction

In *Code_Aster*, one calls "finite element", a triplet (phenomenon, modeling, type of mesh). There are three principal phenomena: MECHANICS, THERMICS and ACOUSTICS.
There exist many modelings; for example, for the phenomenon MECHANICS : 3D, C_PLAN, D_PLAN, AXIS, DKT, POU_D_E,...
For a given modeling (for example 3D) of a phenomenon (for example MECHANICS), there in general exist several finite elements: an element by type of mesh supported: HEXA8, HEXA20, PENTA6,...

With final, there thus exists of very many finite elements (more than 500 in July 2004).

On the other hand, the types of mesh are them of reduced number: POI1, SEG2, SEG3, SEG4, TRIA3, TRIA6, TRIA7, QUAD4, QUAD8, HEXA8, HEXA20, …, TETRA4, TETRA10.

In general, each finite element, to carry out its elementary calculations, uses the concepts of function of interpolation (or function of form) and of diagram of integration. In general also, these functions of form and these diagrams of integration are defined on an element known as "of reference" whose geometry is defined in an often called frame of reference: (ξ,η,ζ) . The passage of the element of reference to the real element is done thanks to a geometrical transformation which uses the same functions of interpolation. The element is then known as "isoparametric". These concepts are very well explained in [bib1].

The high number of finite elements of the code combined with the restricted number of the types of mesh, conduit to the fact that there exist several finite elements being based on the same type of mesh; for example the quadrilateral with 8 nodes called QUAD8 support more than 60 different finite elements.

In theory, each finite element can choose its functions of interpolation and its diagrams of integration as he hears it. But in practice, almost all the finite elements being based on the same type of mesh, use the same element of reference, the same functions of form and the same diagrams of integration. The goal of this document is to describe these various elements of reference.

For each element of reference (called in the continuation of the document ELREFE), one will indicate:

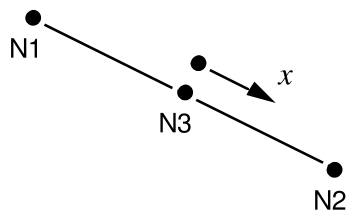
- the mesh support, the number of the nodes, their local classification and their coordinates,
- algebraical expressions of the functions of form and their derivative first (and sometimes seconds)
- families of points of integration which one will name. For each family, one will give the number of points, their coordinates and their "weights" of integration. The sum of these weights, must give the "volume" of the element of reference. For example, the "volume" of the quadrangle of reference ($-1\leq\xi\leq+1$, $1<\eta<+1$) is worth 4.

2 Linear elements: SE2, SE3 and SE4

SE2 : segment with 2 nodes
many nodes : 2
many nodes tops : 2

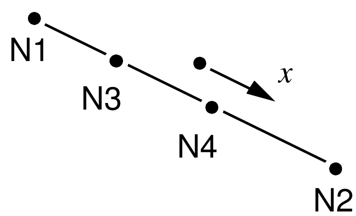
SE3 : segment with 3 nodes
many nodes : 3
many nodes tops : 2

	x
$N1$	-1.0
$N2$	1.0
$N3$	0.0



SE4 : segment with 4 nodes
many nodes : 4
many nodes tops : 2

	x
$N1$	-1.0
$N2$	1.0
$N3$	-1. /3.
$N4$	+1. /3.



functions of form of the segment with 2 nodes:

$$w_1(x)=0.5(1-x) \quad w_2(x)=0.5(1+x)$$

functions of form of the segment with 3 nodes:

$$w_1(x)=-0.5(1-x)x \quad w_2(x)=0.5(1+x)x \quad w_3(x)=(1+x)(1-x)$$

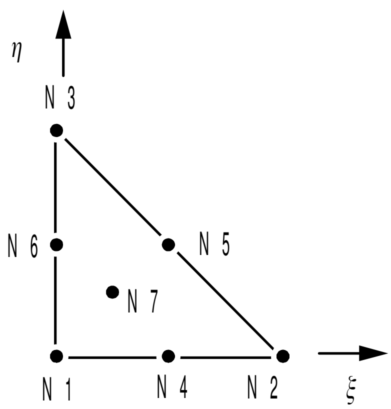
functions of form of the segment with 4 nodes:

$$w_1(x)=\frac{16}{9}(1-x)\left(x+\frac{1}{3}\right)\left(x-\frac{1}{3}\right)$$
$$w_2(x)=-\frac{16}{9}(1+x)\left(\frac{1}{3}-x\right)\left(x+\frac{1}{3}\right)$$
$$w_3(x)=\frac{16}{27}(x-1)(x+1)\left(x-\frac{1}{3}\right)$$
$$w_4(x)=-\frac{16}{27}(x-1)(x+1)\left(x+\frac{1}{3}\right)$$

Many points of integration	Not	x	Weight
1	1	0.0	2.0
2	1	0.577350269189626	1.0
	2	-0.577350269189626	1.0
3	1	-0.774596669241	0.55555…
	2	0.0	0.88888…
	3	0.774596669241	0.55555…
4	1	0.339981043584856	0.652145154862546
	2	-0.339981043584856	0.652145154862546
	3	0.861136311594053	0.347854845137454
	4	-0.861136311594053	0.347854845137454

3 Surface elements

3.1 Triangles: ELREFE TR3, TR6, TR7



Coordinates of the nodes:

	ξ	η
N1	0.0	0.0
N2	1.0	0.0
N3	0.0	1.0
N4	0.5	0.0
N5	0.5	0.5
N6	0.0	0.5
N7	1/3	1/3

Family	**Not**	ξ	η	**Weight**
FPG1	1	1/3	1/3	1/2
FPG3	1	1/6	1/6	1/6
	2	2/3	1/6	1/6
	3	1/6	2/3	1/6
FPG4	1	1/5	1/5	25/(24*4)
	2	3/5	1/5	25/(24*4)
	3	1/5	3/5	25/(24*4)
	4	1/3	1/3	-27/(24*4)
FPG6	1	B	B	P2
	2	1 – 2 B	B	P2
	3	B	1 – 2 B	P2
	4	has	1 – 2 has	P1
	5	has	has	P1
	6	1 – 2 has	has	P1
COT3	1	1/2	1/2	1/6
	2	0	1/2	1/6
	3	1/2	0	1/6

With `P1 = 0.11169079483905,` `P2 = 0.0549758718227661,`
`With = 0.445948490915965,` `B = 0.091576213509771`

Family	**Not**	ξ	η	**Weight**
FPG7	1	1/3	1/3	9/80
	2	With	With	P1
	3	1-2A	With	P1
	4	With	1-2A	P1
	5	B	B	P2
	6	1-2B	B	P2
	7	B	1-2B	P2

With
With = 0.470142064105115
B = 0.101286507323456
P1 = 0.066197076394253
P2 = 0.062969590272413

Family	**Not**	ξ	η	**Weight**
FPG12	1	With	With	P1
	2	1-2A	With	P1
	3	With	1-2A	P1
	4	B	B	P2
	5	1-2B	B	P2
	6	B	1-2B	P2
	7	C	D	P3
	8	D	C	P3
	9	1-C-D	C	P3
	10	1-C-D	D	P3
	11	C	1-C-D	P3
	12	D	1-C-D	P3

With
With = 0.063089014491502
B = 0.249286745170910
C = 0.310352451033785
D = 0.053145049844816
P1 = 0.025422453185103
P2 = 0.058393137863189
P3 = 0.041425537809187

<u>TR3</u> : triangle with 3 nodes
many nodes : 3
many nodes tops : 3

functions of form and derived first of the triangle with 3 nodes:

$\{N\}$	$\{\partial N/\partial \xi\}$	$\{\partial N/\partial \eta\}$
$1-\xi-\eta$	-1	-1
ξ	1	0
η	0	1

TR6 : triangle with 6 nodes
many nodes : 6
many nodes tops : 3

functions of form, derived first of the triangle with 6 nodes:

$\{N\}$	$\{\partial N/\partial\xi\}$	$\{\partial N/\partial\eta\}$
$-(1-\xi-\eta)(1-2(1-\xi-\eta))$	$1-4(1-\xi-\eta)$	$1-4(1-\xi-\eta)$
$-\xi(1-2\xi)$	$-1+4\xi$	0
$-\eta(1-2\eta)$	0	$-1+4\eta$
$4\xi(1-\xi-\eta)$	$4(1-2\xi-\eta)$	-4ξ
$4\xi\eta$	4η	4ξ
$4\eta(1-\xi-\eta)$	-4η	$4(1-\xi-2\eta)$

derived seconds from the triangle with 6 nodes:

$\{\partial^2 N/\partial\xi^2\}$	$\{\partial^2 N/\partial\xi\partial\eta\}$	$\{\partial^2 N/\partial\eta^2\}$
4	4	4
4	0	0
0	0	4
-8	-4	0
0	4	0
0	-4	-8

TR7 : triangle with 7 nodes
many nodes : 7
many nodes tops : 3

functions of form of the triangle with 7 nodes:

$\{N\}$
$1-3(\xi+\eta)+2(\xi^2+\eta^2)+7\xi\eta-3\xi\eta(\xi+\eta)$
$\xi(-1+2\xi+3\eta-3\eta(\xi+\eta))$
$\eta(-1+2\xi+3\eta-3\xi(\xi+\eta))$
$4\xi(1-\xi-4\eta+3\eta(\xi+\eta))$
$4\xi\eta(-2+3(\xi+\eta))$
$4\eta(1-4\xi-\eta+3\xi(\xi+\eta))$
$27\xi\eta(1-\xi-\eta)$

derived first from the triangle with 7 nodes:

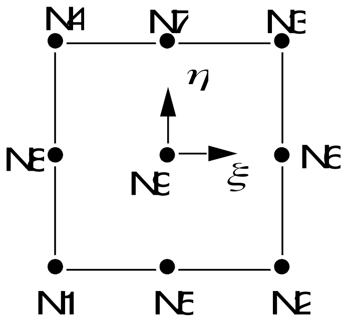
$\{\partial N/\partial\xi\}$	$\{\partial N/\partial\eta\}$
$-3+4\xi+7\eta-6\xi\eta-3\eta^2$	$-3+7\xi+4\eta-6\xi\eta-3\xi^2$
$-1+4\xi+3\eta-6\xi\eta-3\eta^2$	$3\xi(1-\xi-2\eta)$
$3\xi(1-2\eta-\xi)$	$-1+3\xi+4\eta-6\xi\eta-3\xi^2$
$4(1-2\xi-4\eta+6\xi\eta+3\eta^2)$	$4\xi(-4+3\xi+6\eta)$
$4\eta(-2+6\xi+3\eta)$	$4\xi(-2+3\xi+6\eta)$

$4\eta(-4+6\xi+3\eta)$	$4(-1-4\xi-2\eta+6\xi\eta+3\xi^2)$
$27\eta(1-2\xi-\eta)$	$27\xi(1-\xi-2\eta)$

derived seconds from the triangle with 7 nodes:

$\{\partial^2 N/\partial\xi^2\}$	$\{\partial^2 N/\partial\xi\partial\eta\}$	$\{\partial^2 N/\partial\eta^2\}$
$4-6\eta$	$7-6\xi-6\eta$	$4-6\xi$
$4-6\eta$	$3-6\xi-6\eta$	-6ξ
-6η	$3-6\xi-6\eta$	$4-6\xi$
$4(-2+6\eta)$	$4(-4+6\xi+6\eta)$	24ξ
24η	$4(-2+6\xi+6\eta)$	24ξ
24η	$4(-4+6\xi+6\eta)$	$4(-2+6\xi)$
-54η	$27(1-2\xi-2\eta)$	-54ξ

3.2 Quadrangles: `ELREFE QU4`, `QU8`, `QU9`



Coordinates of the nodes:

	ξ	η
N1	-1.0	-1.0
N2	1.0	-1.0
N3	1.0	1.0
N4	-1.0	1.0
N5	0.0	-1.0
N6	1.0	0.0
N7	0.0	1.0
N8	-1.0	0.0
N9	0.0	0.0

Family	**Not**	ξ	η	**Weight**
`FPG1`	1	0	0	4
`FPG4`	1	$-a$	$-a$	1.0
	2	a	$-a$	1.0
	3	a	a	1.0
	4	$-a$	a	1.0
		$a=1/\sqrt{3}$		

FPG9	1	$-a$	$-a$	25/81
	2	a	$-a$	25/81
	3	a	a	25/81
	4	$-a$	a	25/81
	5	0.0	$-a$	40/81
	6	a	0.0	40/81
	7	0.0	has	40/81
	8	$-a$	0.0	40/81
	9	0.0	0.0	64/81
		$a=0.774596669241483$		

QU4 : quadrangle with 4 nodes
many nodes : 4
many nodes tops : 4

functions of form, derived first and seconds of the quadrangle with 4 nodes:

$\{N\}$	$\{\partial N/\partial \xi\}$	$\{\partial N/\partial \eta\}$
$(1-\xi)(1-\eta)/4$	$-(1-\eta)/4$	$-(1-\xi)/4$
$(1+\xi)(1-\eta)/4$	$(1-\eta)/4$	$-(1+\xi)/4$
$(1+\xi)(1+\eta)/4$	$(1+\eta)/4$	$(1+\xi)/4$
$(1-\xi)(1+\eta)/4$	$-(1+\eta)/4$	$(1-\xi)/4$

$\{\partial^2 N/\partial \xi^2\}$	$\{\partial^2 N/\partial \xi \partial \eta\}$	$\{\partial^2 N/\partial \eta^2\}$
0	1/4	0
0	-1/4	0
0	1/4	0
0	-1/4	0

QU8 : quadrangle with 8 nodes
many nodes : 8
many nodes tops : 4

functions of form and derived first of the quadrangle with 8 nodes:

$\{N\}$	$\{\partial N/\partial \xi\}$	$\{\partial N/\partial \eta\}$
$(1-\xi)(1-\eta)(-1-\xi-\eta)/4$	$(1-\eta)(2\xi+\eta)/4$	$(1-\xi)(\xi+2\eta)/4$
$(1+\xi)(1-\eta)(-1+\xi-\eta)/4$	$(1-\eta)(2\xi-\eta)/4$	$-(1+\xi)(\xi-2\eta)/4$
$(1+\xi)(1+\eta)(-1+\xi+\eta)/4$	$(1+\eta)(2\xi+\eta)/4$	$(1+\xi)(\xi+2\eta)/4$
$(1-\xi)(1+\eta)(-1-\xi+\eta)/4$	$-(1+\eta)(-2\xi+\eta)/4$	$(1-\xi)(-\xi+2\eta)/4$
$(1-\xi^2)(1-\eta)/2$	$-\xi(1-\eta)$	$-(1-\xi^2)/2$
$(1+\xi)(1-\eta^2)/2$	$(1-\eta^2)/2$	$-\eta(1+\xi)$
$(1-\xi^2)(1+\eta)/2$	$-\xi(1+\eta)$	$(1-\xi^2)/2$
$(1-\xi)(1-\eta^2)/2$	$-(1-\eta^2)/2$	$-\eta(1-\xi)$

derived seconds from the quadrangle with 8 nodes:

$\{\partial^2 N/\partial \xi^2\}$	$\{\partial^2 N/\partial \xi \partial \eta\}$	$\{\partial^2 N/\partial \eta^2\}$
$(1-\eta)/2$	$(1-2\xi-2\eta)/4$	$(1-\xi)/2$
$(1-\eta)/2$	$-(1+2\xi-2\eta)/4$	$(1+\xi)/2$
$(1+\eta)/2$	$(1+2\xi+2\eta)/4$	$(1+\xi)/2$
$(1+\eta)/2$	$-(1-2\xi+2\eta)/4$	$(1-\xi)/2$
$-1+\eta$	ξ	0
0	$-\eta$	$-1-\xi$
$-1-\eta$	$-\xi$	0
0	η	$-1+\xi$

QU9 : quadrangle with 9 nodes
many nodes : 9
many nodes tops : 4

functions of form and derived first of the quadrangle with 9 nodes:

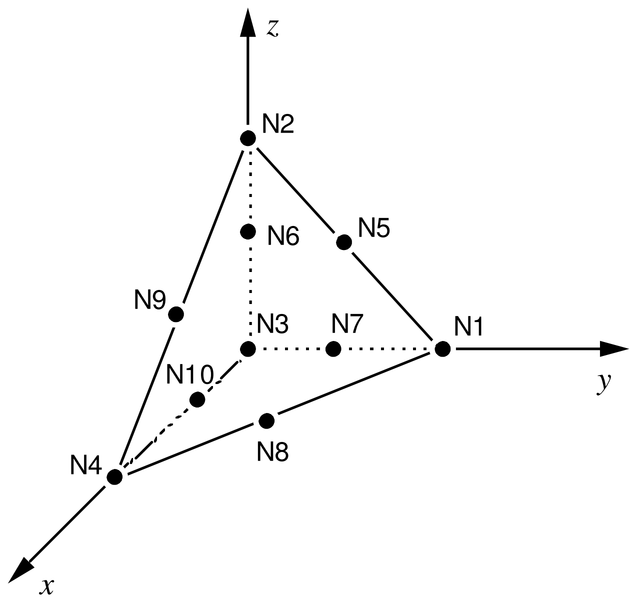
$\{N\}$	$\{\partial N/\partial \xi\}$	$\{\partial N/\partial \eta\}$
$\xi\eta(\xi-1)(\eta-1)/4$	$(2\xi-1)\eta(\eta-1)/4$	$\xi(\xi-1)(2\eta-1)/4$
$\xi\eta(\xi+1)(\eta-1)/4$	$(2\xi+1)\eta(\eta-1)/4$	$\xi(\xi+1)(2\eta-1)/4$
$\xi\eta(\xi+1)(\eta+1)/4$	$(2\xi+1)\eta(\eta+1)/4$	$\xi(\xi+1)(2\eta+1)/4$
$\xi\eta(\xi-1)(\eta+1)/4$	$(2\xi-1)\eta(\eta+1)/4$	$\xi(\xi-1)(2\eta+1)/4$
$(1-\xi^2)\eta(\eta-1)/2$	$-\xi\eta(\eta-1)$	$(1-\xi^2)(2\eta-1)/2$
$\xi(\xi+1)(1-\eta^2)/2$	$(2\xi+1)(1-\eta^2)/2$	$-\xi\eta(\xi+1)$
$(1-\xi^2)\eta(\eta+1)/2$	$-\xi\eta(\eta+1)$	$(1-\xi^2)(2\eta+1)/2$
$\xi(\xi-1)(1-\eta^2)/2$	$(2\xi-1)(1-\eta^2)/2$	$-\xi\eta(\xi-1)$
$(1-\xi^2)(1-\eta^2)$	$-2\xi(1-\eta^2)$	$-2\eta(1-\xi^2)$

derived seconds from the quadrangle with 9 nodes:

$\{\partial^2 N/\partial \xi^2\}$	$\{\partial^2 N/\partial \xi \partial \eta\}$	$\{\partial^2 N/\partial \eta^2\}$
$\eta(\eta-1)/2$	$(\xi-1/2)(\eta-1/2)$	$\xi(\xi-1)/2$
$\eta(\eta-1)/2$	$(\xi+1/2)(\eta-1/2)$	$\xi(\xi+1)/2$
$\eta(\eta+1)/2$	$(\xi+1/2)(\eta+1/2)$	$\xi(\xi+1)/2$
$\eta(\eta+1)/2$	$(\xi-1/2)(\eta+1/2)$	$\xi(\xi-1)/2$
$-\eta(\eta-1)$	$-\xi(2\eta-1)$	$1-\xi^2$
$1-\eta^2$	$-\eta(2\xi+1)$	$-\xi(\xi+1)$
$-\eta(\eta+1)$	$-\xi(2\eta+1)$	$1-\xi^2$
$1-\eta^2$	$-\eta(2\xi-1)$	$-\xi(\xi-1)$
$-2(1-\eta^2)$	$4\xi\eta$	$-2(1-\xi^2)$

4 Voluminal elements

4.1 Tetrahedrons: ELREFE TE4, T10



Coordinates of the nodes:

	x	y	z
$N1$	0.	1.	0.
$N2$	0.	0.	1.
$N3$	0.	0.	0.
$N4$	1.	0.	0.
$N5$	0.	0.5	0.5
$N6$	0.	0.	0.5
$N7$	0.	0.5	0.
$N8$	0.5	0.5	0.
$N9$	0.5	0.	0.5
$N10$	0.5	0.	0.

Functions of form:

Formula with 4 nodes

$$\begin{cases} w_1(x,y,z)=y \\ w_2(x,y,z)=z \\ w_3(x,y,z)=1-x-y-z \\ w_4(x,y,z)=x \end{cases}$$

Formula with 10 nodes

$$w_1=y(2y-1) \qquad w_6=4z(1-x-y-z)$$
$$w_2=z(2z-1) \qquad w_7=4y(1-x-y-z)$$
$$w_3=(1-x-y-z)(1-2x-2y-2z) \qquad w_8=4xy$$
$$w_4=x(2x-1) \qquad w_9=4xz$$
$$w_5=4yz \qquad w_{10}=4x(1-x-y-z)$$

Digital formula of integration:

Formula with 1 point, of order 1 in x, y, z : (FPG1)

Not	x	y	z	**Weight**
1	$1/4$	$1/4$	$1/4$	$1/6$

Formula at 4 points, of order 2 in x, y, z : (FPG4)

Not	x	y	z	**Weight**
1	a	a	a	$1/24$
2	a	a	b	$1/24$
3	a	b	a	$1/24$
4	b	a	a	$1/24$

with: $a=\dfrac{5-\sqrt{5}}{20}$, $b=\dfrac{5+3\sqrt{5}}{20}$

Formula at 5 points, of order 3 in x, y, z : (FPG5)

Not	x	y	z	**Weight**
1	a	a	a	$-2/15$
2	b	b	b	$3/40$
3	b	b	c	$3/40$
4	b	c	b	$3/40$
5	c	b	b	$3/40$

With: $a=0.25$, $b=\dfrac{1}{6}$, $c=0.5$

Formula at 15 points, of order 5 in x, y, z : (FPG15)

Not	x	y	z	**Weight**
1	a	a	a	$8/405$
2	b_1	b_1	b_1	$\dfrac{2665-14\sqrt{15}}{226800}$
3	b_1	b_1	c_1	
4	b_1	c_1	b_1	
5	c_1	b_1	b_1	

6	b_2	b_2	b_2	
7	b_2	b_2	c_2	$\frac{2665+14\sqrt{15}}{226800}$
8	b_2	c_2	b_2	
9	c_2	b_2	b_2	
10	d	d	e	
11	d	e	d	
12	e	d	d	$\frac{5}{567}$
13	d	e	e	
14	e	d	e	
15	e	e	d	

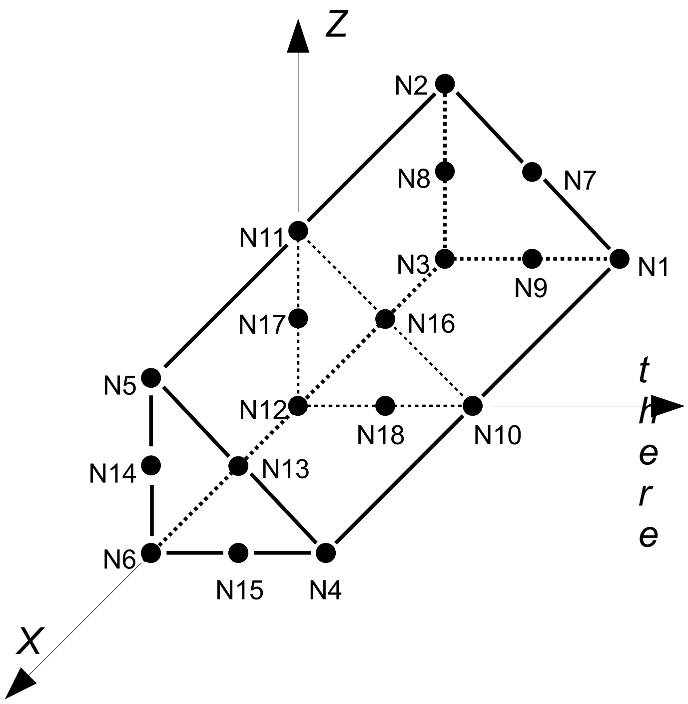
with:

$a=0.25$

$$b_1=\frac{7+\sqrt{15}}{34} \quad c_1=\frac{13-3\sqrt{15}}{34} \quad d=\frac{5-\sqrt{15}}{20}$$

$$b_2=\frac{7-\sqrt{15}}{34} \quad c_2=\frac{13+3\sqrt{15}}{34} \quad e=\frac{5+\sqrt{15}}{20}$$

4.2 Pentahedrons: `ELREFE PE6`, `P15`, `P18`



Coordinates of the nodes:

	x	y	z
$N1$	-1.	1.	0.
$N2$	-1.	0.	1.
$N3$	-1.	0.	0.
$N4$	1.	1.	0.
$N5$	1.	0.	1.
$N6$	1.	0.	0.
$N7$	-1.	0.5	0.5.
$N8$	-1.	0.	0.5.
$N9$	-1.	0.5	0.
$N10$	0.	1.	0.
$N11$	0.	0.	1.
$N12$	0.	0.	0.
$N13$	1.	0.5	0.5
$N14$	1.	0.	0.5
$N15$	1.	0.5	0.
$N16$	0.	0.5	0.5
$N17$	0.	0.	0.5
$N18$	0.	0.5	0.

Functions of form:

Formula with 6 nodes

$$w_1=\frac{1}{2}y(1-x) \qquad w_4=\frac{1}{2}y(x+1)$$
$$w_2=\frac{1}{2}z(1-x) \qquad w_5=\frac{1}{2}z(x+1)$$
$$w_3=\frac{1}{2}(1-y-z)(1-x) \qquad w_6=\frac{1}{2}(1-y-z)(x+1)$$

Formula with 15 nodes

$$w_1=y(1-x)(2y-2-x)/2$$
$$w_2=z(1-x)(2z-2-x)/2$$
$$w_3=(x-1)(1-y-z)(x+2y+2z)/2$$
$$w_4=y(1+x)(2y-2+x)/2$$
$$w_5=z(1+x)(2z-2+x)/2$$
$$w_6=(-x-1)(1-y-z)(-x+2y+2z)/2$$
$$w_7=2yz(1-x)$$
$$w_8=2z(1-y-z)(1-x)$$
$$w_9=2y(1-y-z)(1-x)$$
$$w_{10}=y(1-x^2)$$
$$w_{11}=z(1-x^2)$$
$$w_{12}=(1-y-z)(1-x^2)$$
$$w_{13}=2yz(1+x)$$
$$w_{14}=2z(1-y-z)(1+x)$$
$$w_{15}=2y(1-y-z)(1+x)$$

Formula with 18 nodes

$$w_1=xy(x-1)(2y-1)/2$$
$$w_2=xz(x-1)(2z-1)/2$$
$$w_3=x(x-1)(z+y-1)(2z+2y-1)/2$$
$$w_4=xy(x+1)(2y-1)/2$$
$$w_5=xz(x+1)(2z-1)/2$$
$$w_6=x(x+1)(z+y-1)(2z+2y-1)/2$$
$$w_7=2xyz(x-1)$$
$$w_8=-2xz(x-1)(z+y-1)$$
$$w_9=-2xy(x-1)(z+y-1)$$
$$w_{10}=y(1-x^2)(2y-1)$$
$$w_{11}=z(1-x^2)(2z-1)$$
$$w_{12}=(1-x^2)(z+y-1)(2z+2y-1)$$
$$w_{13}=2xyz(x+1)$$
$$w_{14}=-2xz(x+1)(z+y-1)$$
$$w_{15}=-2xy(x+1)(z+y-1)$$
$$w_{16}=4yz(1-x^2)$$
$$w_{17}=4z(x^2-1)(z+y-1)$$
$$w_{18}=4y(x^2-1)(z+y-1)$$

Formulas of digital integration at 6 points (order 3 in x , order 2 in y and z) (`FPG6`)

Not	x	y	z	Weight
1	$-1/\sqrt{3}$	0.5	0.5	$1/6$
2	$-1/\sqrt{3}$	0.	0.5	$1/6$
3	$-1/\sqrt{3}$	0.5	0.	$1/6$
4	$1/\sqrt{3}$	0.5	0.5	$1/6$
5	$1/\sqrt{3}$	0.	0.5	$1/6$
6	$1/\sqrt{3}$	0.5	0.	$1/6$

Digital formula of integration at 8 points: (`FPG8`)

2 points of Gauss in x (order 3).
4 points of Hammer in y and z (order 3).

Not	x	y	z	Weight
1	$-a$	$1/3$	$1/3$	$-27/96$
2	$-a$	0.6	0.2	$25/96$
3	$-a$	0.2	0.6	$25/96$
4	$-a$	0.2	0.2	$25/96$
5	$+a$	$1/3$	$1/3$	$-27/96$
6	$+a$	0.6	0.2	$25/96$
7	$+a$	0.2	0.6	$25/96$
8	$+a$	0.2	0.2	$25/96$

With $a=0.577350269189626$

Digital formula of integration at 21 points: (`FPG21`)

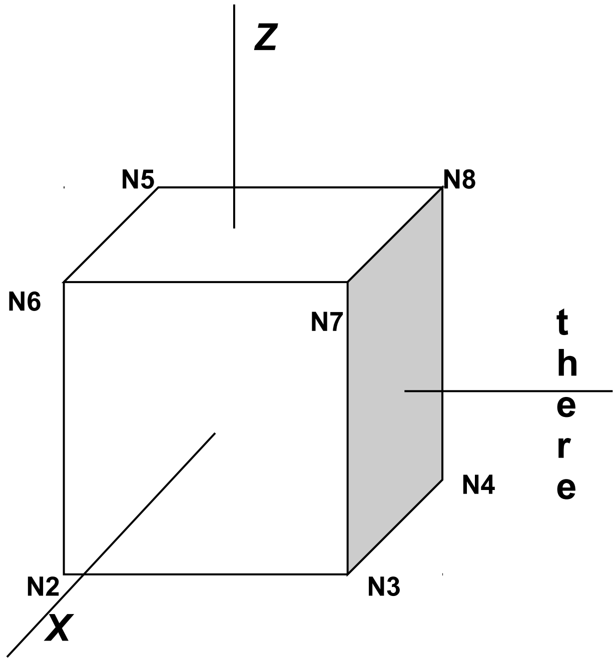
3 points of Gauss in x (order 5).
7 points of Hammer in y and z (order 5 in y and z).

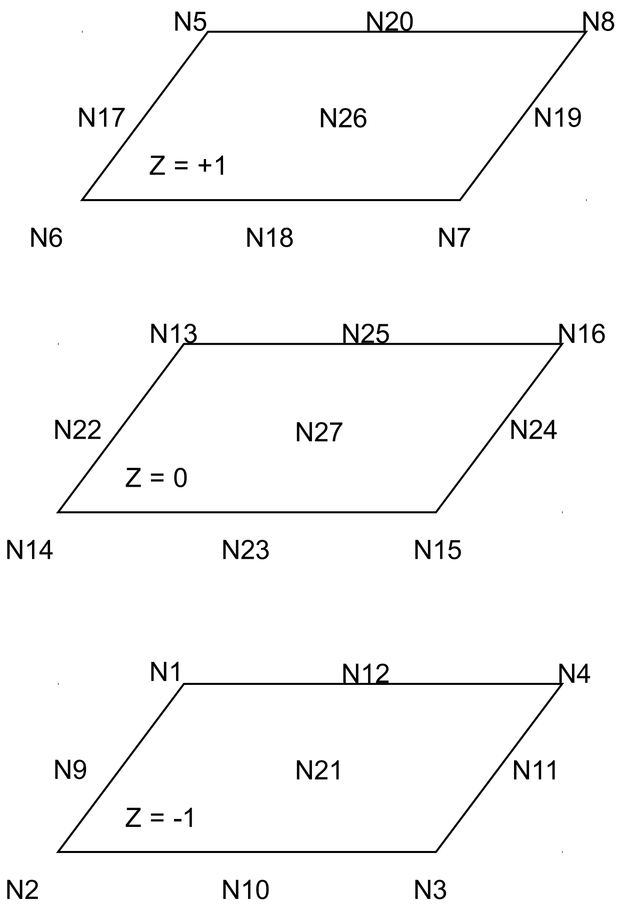
Not	x	y	z	Weight
1	$-\alpha$	$1/3$	$1/3$	$c_1\frac{9}{80}$
2 3 4	$-\alpha$ $-\alpha$ $-\alpha$	a $1-2a$ a	a a $1-2a$	$c_1\left(\frac{155+\sqrt{15}}{2400}\right)$
5 6 7	$-\alpha$ $-\alpha$ $-\alpha$	b $1-2b$ b	b b $1-2b$	$c_1\left(\frac{155-\sqrt{15}}{2400}\right)$
8	0.	$1/3$	$1/3$	$c_2\frac{9}{80}$
9 10 11	0. 0. 0.	a $1-2a$ a	a a $1-2a$	$c_2\left(\frac{155+\sqrt{15}}{2400}\right)$
12 13 14	0. 0. 0.	b $1-2b$ b	b b $1-2b$	$c_2\left(\frac{155-\sqrt{15}}{2400}\right)$
15	α	$1/3$	$1/3$	$c_1\frac{9}{80}$
16 17 18	α α α	b $1-2a$ a	a a $1-2a$	$c_1\left(\frac{155+\sqrt{15}}{2400}\right)$
19 20 21	α α α	b $1-2b$ b	b b $1-2b$	$c_1\left(\frac{155-\sqrt{15}}{2400}\right)$

with:

$\alpha=\sqrt{\frac{3}{5}}$ $\quad c_1=\frac{5}{9}$ $\quad c_2=\frac{8}{9}$ $\quad a=\frac{6+\sqrt{15}}{21}$ $\quad b=\frac{6-\sqrt{15}}{21}$

4.3 Hexahedrons: `ELREFE HE8 , H20 , H27`





Coordinates of the nodes:

	x	y	z
N1	-1.	-1.	-1.
N2	1.	-1.	-1.
N3	1.	1.	-1.
N4	-1.	1.	-1.
N5	-1.	-1.	1.
N6	1.	-1.	1.
N7	1.	1.	1.
N8	-1.	1.	1.
N9	0.	-1.	-1.
N10	1.	0.	-1.
N11	0.	1.	-1.
N12	-1.	0.	-1.
N13	-1.	-1.	0.
N14	1.	-1.	0.
N15	1.	1.	0.
N16	-1.	1.	0.
N17	0.	-1.	1.
N18	1.	0.	1.
N19	0.	1.	1.
N20	-1.	0.	1.
N21	0.	0.	-1.
N22	0.	-1.	0.
N23	1.	0.	0.
N24	0.	1.	0.
N25	-1.	0.	0.
N26	0.	0.	1.
N27	0.	0.	0.

Functions of form:

Formula with 8 nodes

$$w_1=\frac{1}{8}(1-x)(1-y)(1-z) \qquad w_5=\frac{1}{8}(1-x)(1-y)(1+z)$$
$$w_2=\frac{1}{8}(1+x)(1-y)(1-z) \qquad w_6=\frac{1}{8}(1+x)(1-y)(1+z)$$
$$w_3=\frac{1}{8}(1+x)(1+y)(1-z) \qquad w_7=\frac{1}{8}(1+x)(1+y)(1+z)$$
$$w_4=\frac{1}{8}(1-x)(1+y)(1-z) \qquad w_8=\frac{1}{8}(1-x)(1+y)(1+z)$$

Formula with 20 nodes

$$w_1=\frac{1}{8}(1-x)(1-y)(1-z)(-2-x-y-z) \qquad w_{11}=\frac{1}{4}(1-x^2)(1+y)(1-z)$$
$$w_2=\frac{1}{8}(1+x)(1-y)(1-z)(-2+x-y-z) \qquad w_{12}=\frac{1}{4}(1-y^2)(1-x)(1-z)$$
$$w_3=\frac{1}{8}(1+x)(1+y)(1-z)(-2+x+y-z) \qquad w_{13}=\frac{1}{4}(1-z^2)(1-x)(1-y)$$
$$w_4=\frac{1}{8}(1-x)(1+y)(1-z)(-2-x+y-z) \qquad w_{14}=\frac{1}{4}(1-z^2)(1+x)(1-y)$$
$$w_5=\frac{1}{8}(1-x)(1-y)(1+z)(-2-x-y+z) \qquad w_{15}=\frac{1}{4}(1-z^2)(1+x)(1+y)$$
$$w_6=\frac{1}{8}(1+x)(1-y)(1+z)(-2+x-y+z) \qquad w_{16}=\frac{1}{4}(1-z^2)(1-x)(1+y)$$
$$w_7=\frac{1}{8}(1+x)(1+y)(1+z)(-2+x+y+z) \qquad w_{17}=\frac{1}{4}(1-x^2)(1-y)(1+z)$$
$$w_8=\frac{1}{8}(1-x)(1+y)(1+z)(-2-x+y+z) \qquad w_{18}=\frac{1}{4}(1-y^2)(1+x)(1+z)$$
$$w_9=\frac{1}{4}(1-x^2)(1-y)(1-z) \qquad w_{19}=\frac{1}{4}(1-x^2)(1+y)(1+z)$$
$$w_{10}=\frac{1}{4}(1-y^2)(1+x)(1-z) \qquad w_{20}=\frac{1}{4}(1-y^2)(1-x)(1+z)$$

Formula with 27 nodes

$$w_1=\frac{1}{8}x(x-1)y(y-1)z(z-1)$$
$$w_2=\frac{1}{8}x(x+1)y(y-1)z(z-1)$$
$$w_3=\frac{1}{8}x(x+1)y(y+1)z(z-1)$$
$$w_4=\frac{1}{8}x(x-1)y(y+1)z(z-1)$$
$$w_5=\frac{1}{8}x(x-1)y(y-1)z(z+1)$$
$$w_6=\frac{1}{8}x(x+1)y(y-1)z(z+1)$$
$$w_7=\frac{1}{8}x(x+1)y(y+1)z(z+1)$$
$$w_8=\frac{1}{8}x(x-1)y(y+1)z(z+1)$$
$$w_9=\frac{1}{4}(1-x^2)y(y-1)z(z-1)$$
$$w_{10}=\frac{1}{4}x(x+1)(1-y^2)z(z-1)$$
$$w_{11}=\frac{1}{4}(1-x^2)y(y+1)z(z-1)$$
$$w_{12}=\frac{1}{4}x(x-1)(1-y^2)z(z-1)$$
$$w_{13}=\frac{1}{4}x(x-1)y(y-1)(1-z^2)$$
$$w_{14}=\frac{1}{4}x(x+1)y(y-1)(1-z^2)$$
$$w_{15}=\frac{1}{4}x(x+1)y(y+1)(1-z^2)$$
$$w_{16}=\frac{1}{4}x(x-1)y(y+1)(1-z^2)$$
$$w_{17}=\frac{1}{4}(1-x^2)y(y-1)z(z+1)$$
$$w_{18}=\frac{1}{4}x(x+1)(1-y^2)z(z+1)$$
$$w_{19}=\frac{1}{4}(1-x^2)y(y+1)z(z+1)$$
$$w_{20}=\frac{1}{4}x(x-1)(1-y^2)z(z+1)$$
$$w_{21}=\frac{1}{2}(1-x^2)(1-y^2)z(z-1)$$
$$w_{22}=\frac{1}{2}(1-x^2)y(y-1)(1-z^2)$$
$$w_{23}=\frac{1}{2}x(x+1)(1-y^2)(1-z^2)$$
$$w_{24}=\frac{1}{2}(1-x^2)y(y+1)(1-z^2)$$
$$w_{25}=\frac{1}{2}x(x-1)(1-y^2)(1-z^2)$$
$$w_{26}=\frac{1}{2}(1-x^2)(1-y^2)z(z+1)$$
$$w_{27}=(1-x^2)(1-y^2)(1-z^2)$$

Formula of squaring of Gauss at 2 points in each direction (order 3) (`FPG8`)

Not	x	y	z	**Weight**
1	$-1/\sqrt{3}$	$-1/\sqrt{3}$	$-1/\sqrt{3}$	1.
2	$-1/\sqrt{3}$	$-1/\sqrt{3}$	$1/\sqrt{3}$	1.
3	$-1/\sqrt{3}$	$1/\sqrt{3}$	$-1/\sqrt{3}$	1.
4	$-1/\sqrt{3}$	$1/\sqrt{3}$	$1/\sqrt{3}$	1.
5	$1/\sqrt{3}$	$-1/\sqrt{3}$	$-1/\sqrt{3}$	1.
6	$1/\sqrt{3}$	$-1/\sqrt{3}$	$1/\sqrt{3}$	1.
7	$1/\sqrt{3}$	$1/\sqrt{3}$	$-1/\sqrt{3}$	1.
8	$1/\sqrt{3}$	$1/\sqrt{3}$	$1/\sqrt{3}$	1.

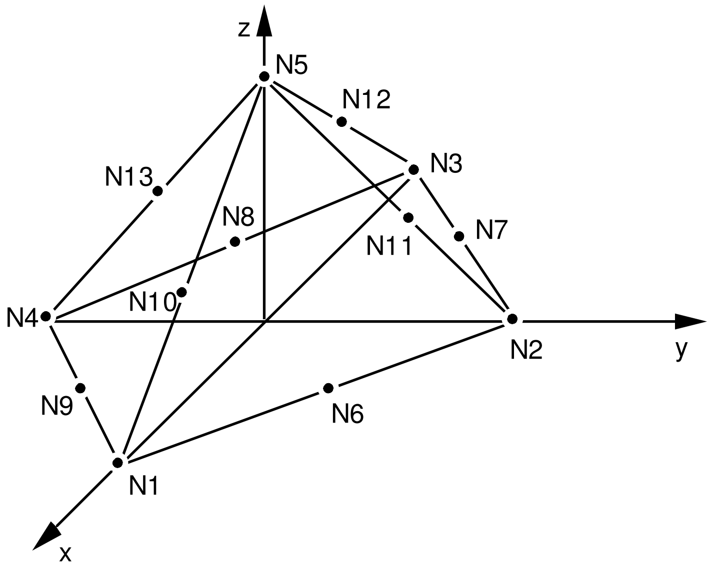
Formula of squaring of Gauss at 3 points in each direction (order 5): (FPG27)

Not	x	y	z	Weight
1	$-\alpha$	$-\alpha$	$-\alpha$	c_1^3
2	$-\alpha$	$-\alpha$	0.	$c_1^2 c_2$
3	$-\alpha$	$-\alpha$	α	c_1^3
4	$-\alpha$	0.	$-\alpha$	$c_1^2 c_2$
5	$-\alpha$	0.	0.	$c_1 c_2^2$
6	$-\alpha$	0.	α	$c_1^2 c_2$
7	$-\alpha$	α	$-\alpha$	c_1^3
8	$-\alpha$	α	0.	$c_1^2 c_2$
9	$-\alpha$	α	α	c_1^3
10	0.	$-\alpha$	$-\alpha$	$c_1^2 c_2$
11	0.	$-\alpha$	0.	$c_1 c_2^2$
12	0.	$-\alpha$	α	$c_1^2 c_2$
13	0.	0.	$-\alpha$	$c_1 c_2^2$
14	0.	0.	0.	c_2^3
15	0.	0.	α	$c_1 c_2^2$
16	0.	α	$-\alpha$	$c_1^2 c_2$
17	0.	α	0.	$c_1 c_2^2$
18	0.	α	α	$c_1^2 c_2$
19	α	$-\alpha$	$-\alpha$	c_1^3
20	α	$-\alpha$	0.	$c_1^2 c_2$
21	α	$-\alpha$	α	c_1^3
22	α	0.	$-\alpha$	$c_1^2 c_2$
23	α	0.	0.	$c_1 c_2^2$
24	α	0.	α	$c_1^2 c_2$
25	α	α	$-\alpha$	c_1^3
26	α	α	0.	$c_1^2 c_2$
27	α	α	α	c_1^3

with:

$$\alpha=\sqrt{\frac{3}{5}} \qquad c_1=\frac{5}{9} \qquad c_2=\frac{8}{9}$$

4.4 Pyramids: `ELREFE PY5`, `P13`



The square base is consisted the quadrangle $N_1 N_2 N_3 N_4$ and N_5 is the top of the pyramid.

	x	y	z
N_1	1.	0.	0.
N_2	0.	1.	0.
N_3	– 1.	0.	0.
N_4	0.	– 1.	0.
N_5	0.	0.	1.
N_6	0.5	0.5	0.
N_7	– 0.5	0.5	0.
N_8	– 0.5	– 0.5	0.
N_9	0.5	– 0.5	0.
N_{10}	0.5	0.	0.5
N_{11}	0.	0.5	0.5
N_{12}	– 0.5	0.	0.5
N_{13}	0.	– 0.5	0.5

Functions of form:

Formula with 5 nodes

$$w_1=\frac{(-x+y+z-1)(-x-y+z-1)}{4(1-z)}$$
$$w_2=\frac{(-x-y+z-1)(x-y+z-1)}{4(1-z)}$$
$$w_3=\frac{(x+y+z-1)(x-y+z-1)}{4(1-z)}$$
$$w_4=\frac{(x+y+z-1)(-x+y+z-1)}{4(1-z)}$$
$$w_5=z$$

Formula with 13 nodes

$$w_1=\frac{(-x+y+z-1)(-x-y+z-1)(x-0.5)}{2(1-z)}$$
$$w_2=\frac{(-x-y+z-1)(x-y+z-1)(y-0.5)}{2(1-z)}$$
$$w_3=\frac{(x-y+z-1)(x+y+z-1)(-x-0.5)}{2(1-z)}$$
$$w_4=\frac{(x+y+z-1)(-x+y+z-1)(-y-0.5)}{2(1-z)}$$
$$w_5=2z(z-0.5)$$
$$w_6=-\frac{(-x+y+z-1)(-x-y+z-1)(x-y+z-1)}{2(1-z)}$$
$$w_7=-\frac{(-x-y+z-1)(x-y+z-1)(x+y+z-1)}{2(1-z)}$$
$$w_8=-\frac{(x-y+z-1)(x+y+z-1)(-x+y+z-1)}{2(1-z)}$$
$$w_9=-\frac{(x+y+z-1)(-x+y+z-1)(-x-y+z-1)}{2(1-z)}$$
$$w_{10}=\frac{z(-x+y+z-1)(-x-y+z-1)}{1-z}$$
$$w_{11}=\frac{z(-x-y+z-1)(x-y+z-1)}{1-z}$$
$$w_{12}=\frac{z(x-y+z-1)(x+y+z-1)}{1-z}$$
$$w_{13}=\frac{z(x+y+z-1)(-x+y+z-1)}{1-z}$$

Digital formula of integration at 5 points (FPG5):

Not	x	y	z	Weight
1	0.5	0.	H 1	2/15
2	0.	0.5	H_1	2/15
3	– 0.5	0.	H_1	2/15
4	0.	– 0.5	H_1	2/15
5	0.	0.	H_2	2/15

with:

$$h_1 = 0.1531754163448146$$
$$h_2 = 0.6372983346207416$$

Digital formula of integration at 6 points (FPG6):

Not	x	y	z	Weight
1	has	0.	h_1	p_1
2	0.	a	h_1	p_1
3	$-a$	0.	h_1	p_1
4	0.	$-a$	h_1	p_1
5	0.	0.	h_2	p_2
6	0.	0.	h_3	p_3

with:

p_1 = 0.1024890634400000
p_2 = 0.1100000000000000
p_3 = 0.1467104129066667

a = 0.5702963741068025
h_1 = 0.1666666666666666
h_2 = 0.08063183038464675
h_3 = 0.6098484849057127

Digital formula of integration at 27 points (FPG27):

Not	x	y	z	Weight
1	0.	0.	$1/2$	a_1
2	$\frac{b_1}{2}(1-z)$	$\frac{b_1}{2}(1-z)$	$1/2$	b_6
3	$-\frac{b_1}{2}(1-z)$	$\frac{b_1}{2}(1-z)$	$1/2$	b_6
4	$-\frac{b_1}{2}(1-z)$	$-\frac{b_1}{2}(1-z)$	$1/2$	b_6
5	$\frac{b_1}{2}(1-z)$	$-\frac{b_1}{2}(1-z)$	$1/2$	b_6
6	0.	0.	$\frac{1-b_1}{2}$	b_6
7	0.	0.	$\frac{1+b_1}{2}$	b_6
8	$c_1(1-z)$	0.	$(1-c_1)/2$	c_8
9	0.	$c_1(1-z)$	$(1-c_1)/2$	c_8
10	$-c_1(1-z$	0.	$(1-c_1)/2$	c_8
11	0.	$-c_1(1-z)$	$(1-c_1)/2$	c_8
12	$c_1(1-z)$	0.	$(1+c_1)/2$	c_8
13	0.	$c_1(1-z)$	$(1+c_1)/2$	c_8
14	$-c_1(1-z)$	0.	$(1+c_1)/2$	c_8
15	0.	$-c_1(1-z)$	$(1+c_1)/2$	c_8
16	$\frac{d_1}{2}(1-z)$	$\frac{d_1}{2}(1-z)$	$(1-d_1)/2$	d_{12}
17	$-\frac{d_1}{2}(1-z)$	$\frac{d_1}{2}(1-z)$	$(1-d_1)/2$	d_{12}
18	$-\frac{d_1}{2}(1-z)$	$-\frac{d_1}{2}(1-z)$	$(1-d_1)/2$	d_{12}
19	$\frac{d_1}{2}(1-z)$	$-\frac{d_1}{2}(1-z)$	$(1-d_1)/2$	d_{12}
20	$d_1(1-z)$	0.	$1/2$	d_{12}
21	0.	$d_1(1-z)$	$1/2$	d_{12}
22	$-d_1(1-z)$	0.	$1/2$	d_{12}
23	0.	$-d_1(1-z)$	$1/2$	d_{12}
24	$\frac{d_1}{2}(1-z)$	$\frac{d_1}{2}(1-z)$	$(1+d_1)/2$	d_{12}
25	$-\frac{d_1}{2}(1-z)$	$\frac{d_1}{2}(1-z)$	$(1+d_1)/2$	d_{12}

26	$-\frac{d_1}{2}(1-z)$	$-\frac{d_1}{2}(1-z)$	$(1+d_1)/2$	d_{12}
27	$\frac{d_1}{2}(1-z)$	$-\frac{d_1}{2}(1-z)$	$(1+d_1)/2$	d_{12}

with:

a_1 = 0.788073483
b_6 = 0.499369002
b_1 = 0.848418011
c_8 = 0.478508449
c_1 = 0.652816472
d_{12} = 0.032303742
d_1 = 1.106412899

5 Bibliography

1 DHATT G., TOUZOT G.: A presentation of the finite element method 2ème edition. Editor: MALOINE S.A. Year 984

6 History of the versions of the document

Index Doc.	Version Aster	Author (S) or contributor (S), organization	Description of the modifications
E	8.4	J.Pellet, X.Desroches, EDF/R & D	Version 8 complete.
F	9.2	J.Pellet EDF/R & D /AMA	Correction concerning the HEXA27, cf drives REX 11036
F	9.4	J.Pellet EDF/R & D /AMA	Correction page 21 of the function of form w5 of the HEXA27 (card 12170)