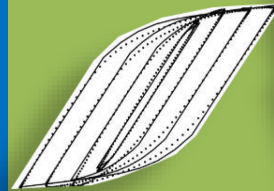
Material modelling in civil engineering

Department of Civil and Structural Engineering

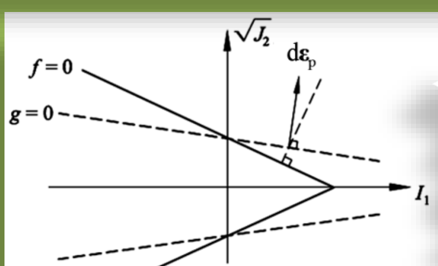
Djebar BAROUDI, PhD.
Lecturer @ Aalto-university
Civil Engineering Department

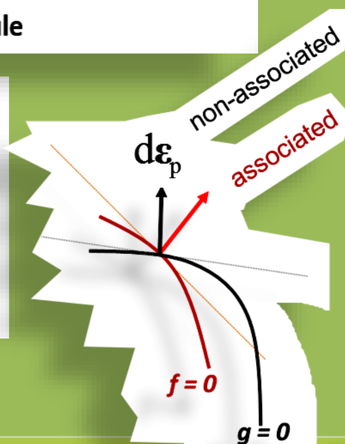
Version 27.5.2021



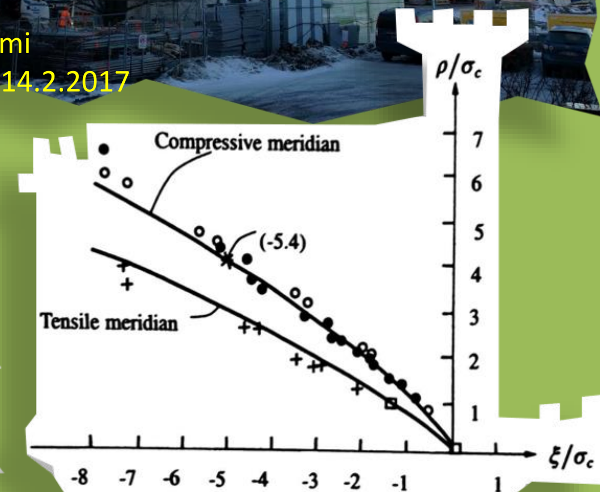
- **Plasticity – plastisuus**
  Engineering
- **Failure hypotheses - lujuushypoteesit**

$f(\boldsymbol{\sigma}) = \sqrt{J_2} + \alpha I_1 - k$	**Drucker-Prager yield criterion**
$g(\boldsymbol{\sigma}) = \sqrt{J_2} + \beta I_1 - k$	**Flow rule**





$$A\frac{J_2}{\sigma_c} + \Lambda\sqrt{J_2} + BI_1 - \sigma_c = 0,$$

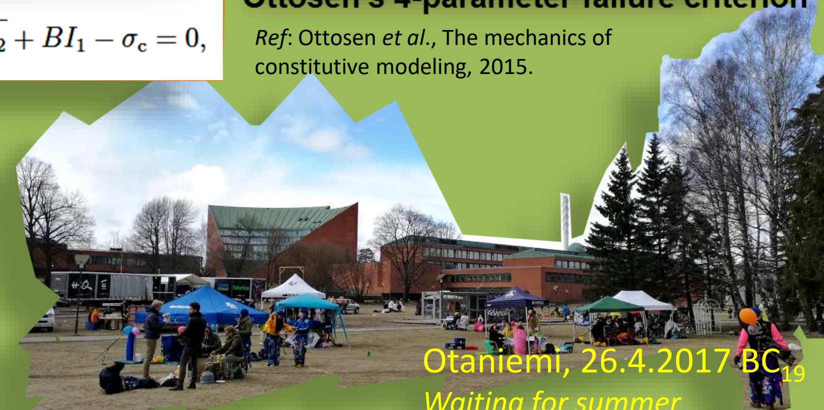


Ottosen's 4-parameter failure criterion

Ref: Ottosen *et al.*, The mechanics of constitutive modeling, 2015.

<u>**This material is not a collection of lecture-slides.**</u> **NB!**

It represents an illustrated reading supporting material for me and for you. I hope it will activate student's interest for the subject. The topics treated in this intensive course cannot be avoided by structural engineers; they will wait for you hidden inside the FE-software you will use to perform advanced structural analysis. This course may be the right place for first contact them in a friendly learning environment.



Djebar BAROUDI, PhD.
University Lecturer @ Aalto-university
Civil Engineering Department

Supporting Material in MyCourses

0. **INTRODUTION**
1. **ELASTICITY**
2. **VISCOELASTICITY** (+ basics of creep)
3. **PLASTICITY**
4. DAMAGE

Reading – **Textbooks:**

- Lemaitre and Chaboche – *Mechanics of Solid Materials*
- Ottosen & Ristinmaa – *The Mechanics of Constitutive Modeling*
- W.F. Chen, D.J. Han – *Plasticity for Structural Engineers*

(only *chapters* 1-5)

Classical Textbooks: <u>(not obligatory)</u>
L. M. Kachanov
Fundamentals of the theory of Plasticity

Lemaitre & Chaboche textbook as an **e-book**:
http://proquestcombo.safaribooksonline.com.libproxy.aalto.fi/book/physics/9781107384712

Otaniemi, 30.4.2017

Literature & textbooks

Learning Supporting Material in MyCourses

0. **INTRODUTION**
1. **ELASTICITY**
2. **VISCOELASTICITY** (& some basics of creep)
3. **PLASTICITY** (& Failure Hypotheses)
4. **DAMAGE** → Guest lecture

Reading:

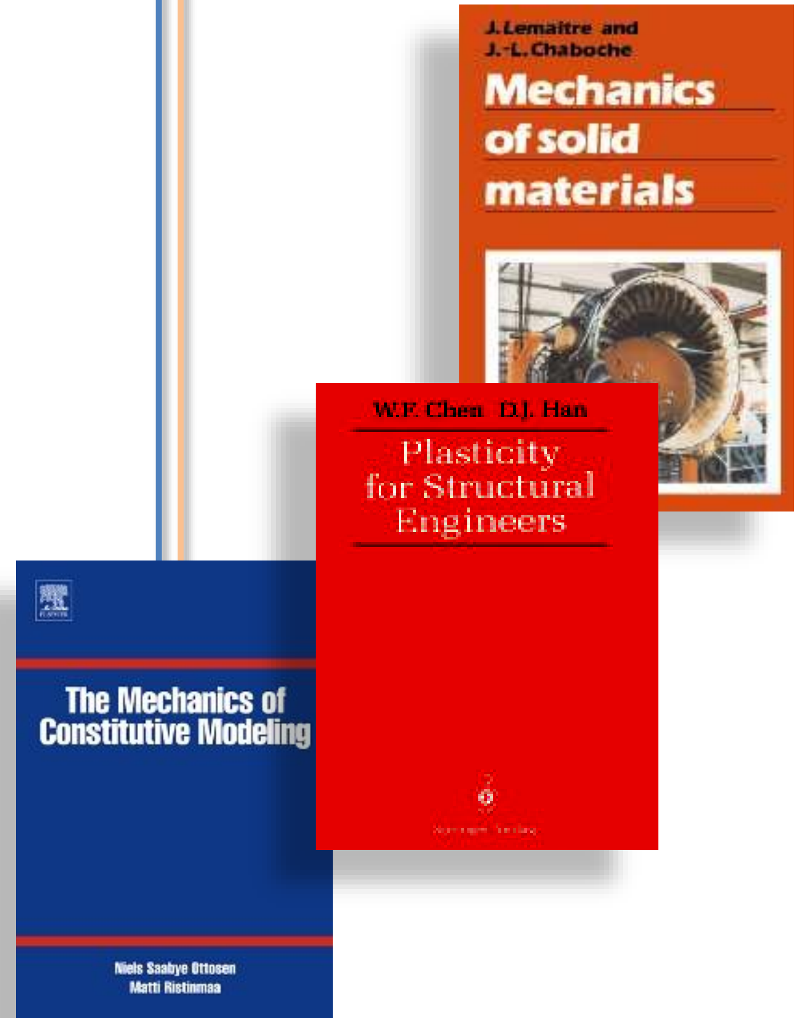
- Lemaitre, chapter 5. – **Plasticity**

  or

- W.F. Chen, D.J. Han, chapters 1-5 **Plasticity for Structural Engineers**

  or

- Ottosen & Ristinmaa, chapters 8…12 – **Plasticity**



Not directly related to this course, however, worth to visit:
Structures & Structural Analysis & Structural Mechanics:
https://www.colorado.edu/engineering/CAS/courses.d/Structures.d/

Lemaitre & Chaboche textbook as an **e-book**:
http://proquestcombo.safaribooksonline.com.libproxy.aalto.fi/book/physics/9781107384712

A must reading:

Erittäin ytimekkäitä - rutikuiva

Very concise and covers entirely our course

Prof. **Reijo Kouhia's lecture notes:**

Introduction to materials modelling , 103 pp.

Maximal Minimalism: Less than 20 pp.

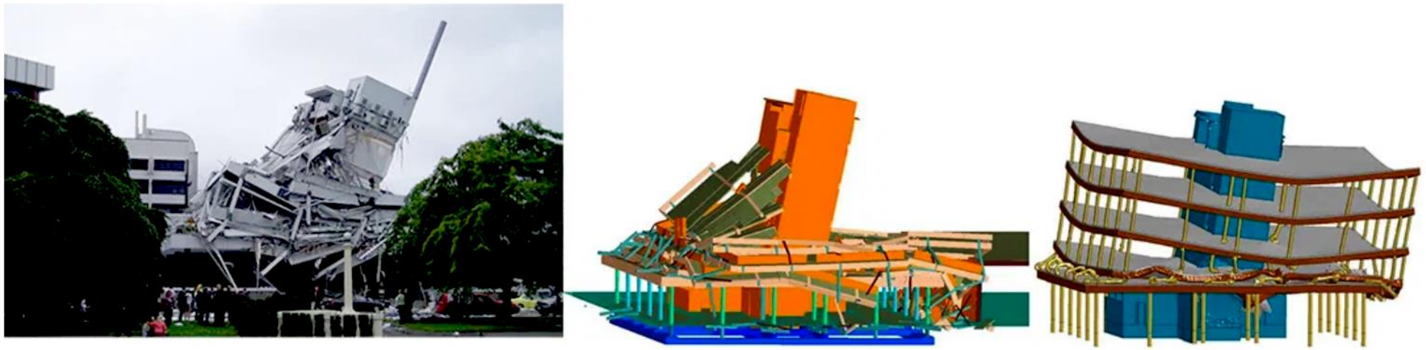
These pdfs are in MyCourses

CONTENTS

Must visit sites:

https://www.appliedelementmethod.org/ (AEM)

Numerical prediction of progressive collapse of buildings due to extreme loading
Applied finite element AEM is a ingenious combination (hybrid) of continuous finite element technology FEM and discrete elements DEM technology. FEM performs for regions with no discontinuities and DEM performs better when discontinuities and separation starts.



Pyne Gould Collapse AEM FEM

Reliability of collapse simulation – Comparing finite and applied element method at different levels

Must visit sites:

Example of content:

http://solidmechanics.org/index.html

Summarizes physical laws, mathematical models, and algorithms that are used to predict the response of materials and structures to mechanical or thermal loading

3. Constitutive Equations: Relations between Stress and Strain

3.1 General Requirements for Constitutive Equations

3.2 Linear Elastic Material Behavior

3.2.1 Isotropic, linear elastic material behavior
3.2.2 Stress—strain relations for isotropic, linear elastic materials
3.2.3 Reduced stress-strain equations for plane deformation of isotropic solids
3.2.4 Representative values for density, and elastic constants of isotropic solids
3.2.5 Other elastic constants – bulk, shear and Lame modulus
3.2.6 Physical Interpretation of the elastic constants for isotropic materials
3.2.7 Strain energy density for isotropic solids
3.2.8 Stress-strain relation for a general anisotropic linear elastic material
3.2.9 Physical Interpretation of the anisotropic elastic constants.
3.2.10 Strain energy density for anisotropic, linear elastic solids
3.2.11 Basis change formulas for the anisotropic elastic constants
3.2.12 The effect of material symmetry on anisotropic stress-strain relations
3.2.13 Stress-strain relations for linear elastic orthotropic materials
3.2.14 Stress-strain relations for linear elastic transversely isotropic material
3.2.15 Values for elastic constants of transversely isotropic crystals
3.2.16 Linear elastic stress-strain relations for cubic materials
3.2.17 Representative values for elastic properties of cubic crystals

3.3 Hypoelasticity - elasticity with nonlinear stress-strain behavior

3.4 Generalized Hooke's law – elastic materials subjected to small stretches but large rotations

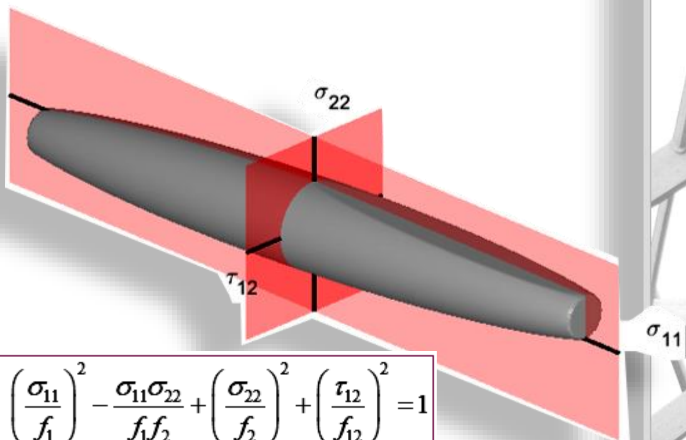
3.5 Hyperelasticity - time independent behavior of rubbers and foams subjected to large strains

3.5.1 Deformation measures used in finite elasticity
3.5.2 Stress measures used in finite elasticity
3.5.3 Calculating stress-strain relations from the strain energy density
3.5.4 A note on perfectly incompressible materials
3.5.5 Specific forms of the strain energy density
3.5.6 Calibrating nonlinear elasticity models
3.5.7 Representative values of material properties for rubbers and foams

Guest lecture year 2020:
18 MAI 2018 klo 14-16, zoom

Norris criterion (Norris 1962)

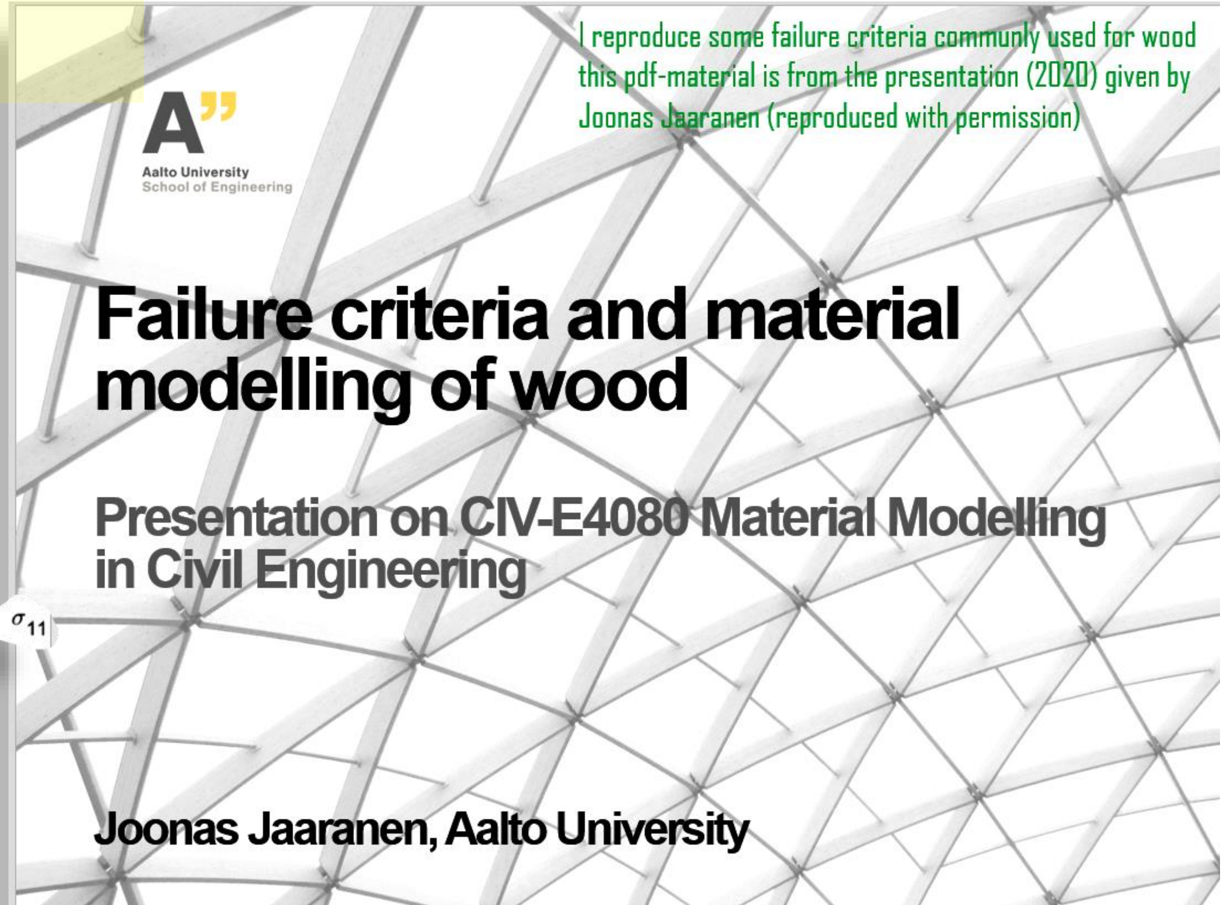
- **Multi-surface approach**
- **No distinction tension/compression**

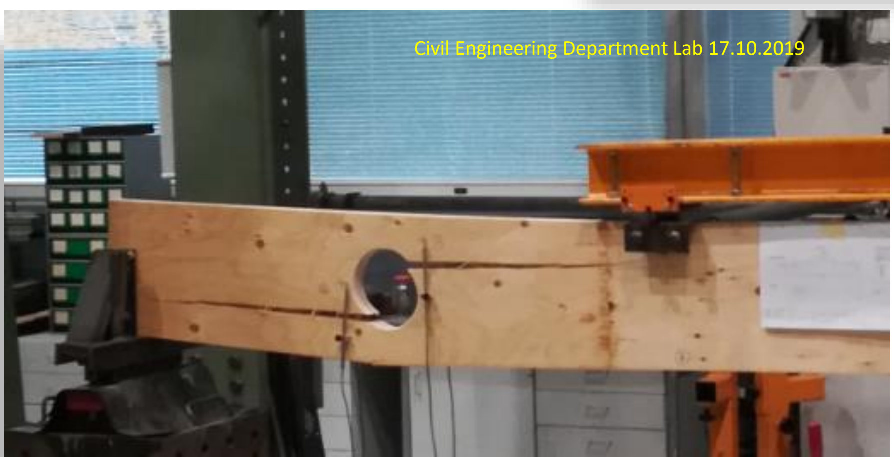


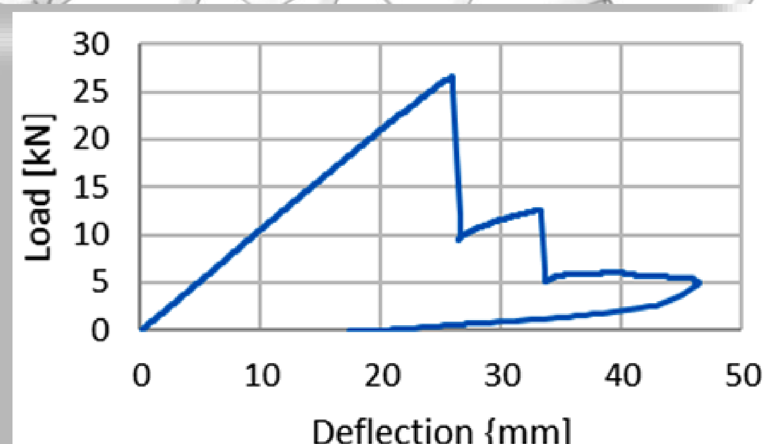
$$\left(\frac{\sigma_{11}}{f_1}\right)^2 - \frac{\sigma_{11}\sigma_{22}}{f_1 f_2} + \left(\frac{\sigma_{22}}{f_2}\right)^2 + \left(\frac{\tau_{12}}{f_{12}}\right)^2 = 1$$

$$\left(\frac{\sigma_{11}}{f_1}\right)^2 = 1$$

$$\left(\frac{\sigma_{22}}{f_2}\right)^2 = 1$$







Guest lecture from 2021:

17 MAI 2021 klo 14-16 in the air

Introduction to creep modelling

Reijo Kouhia

Tampere University, Structural Mechanics

May 18, 2021

- Phenomenology of creep
- Application areas
- Macroscopical modelling of creep
  - Viscoelasticity
  - Viscoplasticity
  - Creep
  - Continuum damage mechanics
- Some rules of thumb
- Example case
- Selected literature

Final constitutive equations

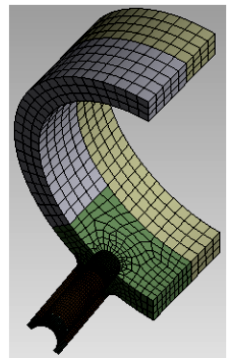
$$\boldsymbol{\sigma} = \omega \boldsymbol{C}_{\rm e} : \boldsymbol{\varepsilon}_{\rm e},$$

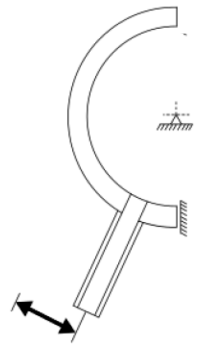
$$\dot{\boldsymbol{\varepsilon}}_{\rm c} = \frac{h_{\rm c}}{t_{\rm c}} \left(\frac{\bar{\sigma}}{\omega \sigma_{\rm rc}} \right)^p \frac{\partial \bar{\sigma}}{\partial \boldsymbol{\sigma}},$$

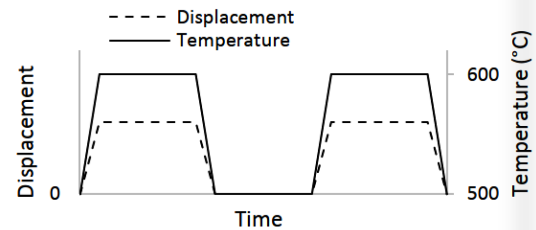
$$\boldsymbol{q} = -\boldsymbol{\lambda}\, \mathrm{grad} T.$$

The integrity rates

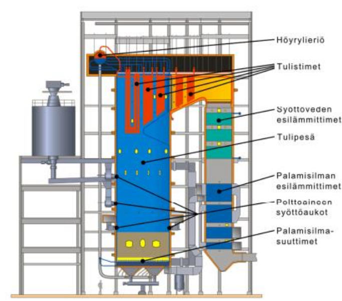
$$\dot{\omega} = -\frac{h_{\rm c}}{(1+k+p)\tau_{\rm d}\, \omega^k} \left(\frac{Y}{Y_{\rm r}} \right)^{\frac{1}{2}p}$$

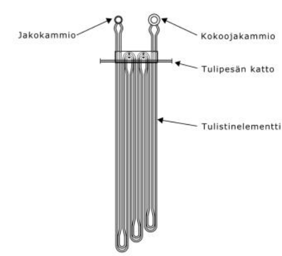


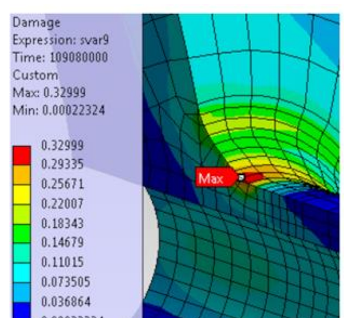




Important in the analysis of engines, power plant boilers & superheaters etc.







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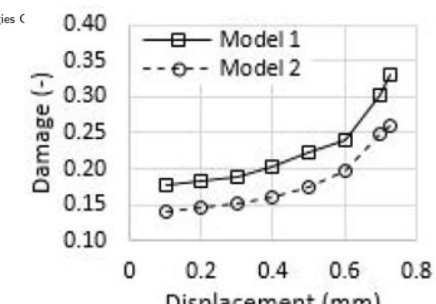


Guest lecture from year 2018:
2 MAI 2018 klo 14-16, R2

Tampereen yliopisto
Tampere University

Numerical modelling of quasi-brittle materials with polygonal finite elements

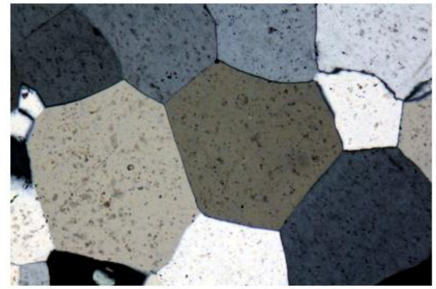
Timo Saksala, Docent صكسلا تمو
Academy Research Fellow
Civil Engineering, Tampere University

Visiting lecture at Aalto University, May 2, 20
At 14:15—16, Lecture hall R2
Rakentajanaukio 4 A, Otaniemi, Espoo

We wish you welcome – *coffee at 4 pm*

Jarkko **Niiranen**, Assistant Professor
Djebar **Baroudi**, Lecturer
Department of **Civil Engineering**,
School of Engineering,
Aalto University

Polygonal texture in quartz*



- **Physics of brittle behavior**
- **Constitutive modeling:** combined viscoplasticity damage model with contact
- **Numerical modeling:** Finite Element Method with special polygonal elements, Cutting-Plane method, impact and contact
- **Application:** progressive damage in rock, fracture analysis, rock dynamics, …

Real experiment

Mathematical Model

Viscoplastic model Hoek-Brown criterion:

$$f_{\mathrm{HB}}(\boldsymbol{\sigma}, \lambda, \dot{\lambda}) = (\sigma_1 - \sigma_3)^2 + \sigma_c(\lambda, \dot{\lambda})^2 \left(\frac{\sigma_1}{\sigma_t(\lambda, \dot{\lambda})} - 1 \right)$$

$$\sigma_t = \sigma_{t0} + h_t \lambda + s_t \dot{\lambda},$$
$$\sigma_c = \sigma_{c0} + h_c \lambda + s_c \dot{\lambda}.$$

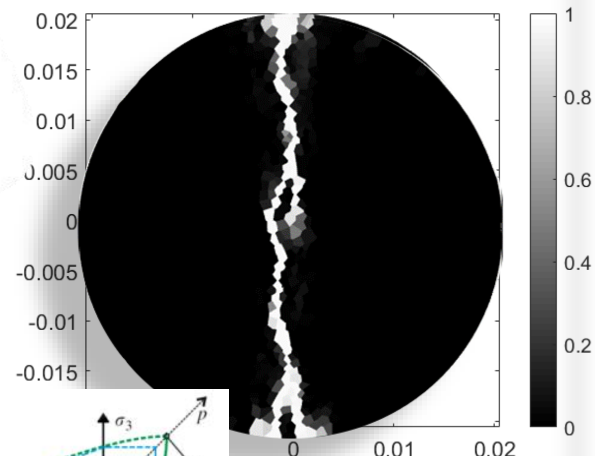
Damage model

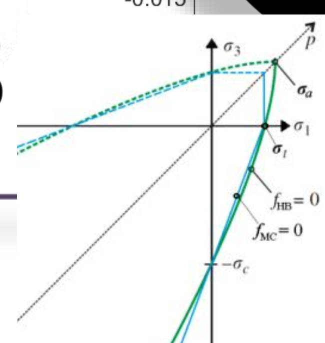
$$\omega_t = A_t (1 - \exp(\beta_t \epsilon^{\mathrm{vp}}_{\mathrm{eqvt}})),$$
$$\omega_c = A_c(\sigma_{\mathrm{conf}})(1 - \exp(\beta_c(\sigma_{\mathrm{conf}}) \epsilon^{\mathrm{vp}}_{\mathrm{eqvc}})),$$

$$\dot{\epsilon}^{\mathrm{vp}}_{\mathrm{eqvt}} = \sqrt{\sum_{i=1}^{3} \langle \dot{\epsilon}^{\mathrm{vp}}_i \rangle_+^2},$$

$$\dot{\epsilon}^{\mathrm{vp}}_{\mathrm{eqvc}} = \sqrt{\tfrac{2}{3} \dot{\boldsymbol{\epsilon}}^{\mathrm{vp}}_{\mathrm{dev}} : \dot{\boldsymbol{\epsilon}}^{\mathrm{vp}}_{\mathrm{dev}}}.$$

$$\boldsymbol{\sigma} = (1 - \omega_t)\bar{\boldsymbol{\sigma}}_+ + (1 - \omega_c)\bar{\boldsymbol{\sigma}}_-,$$

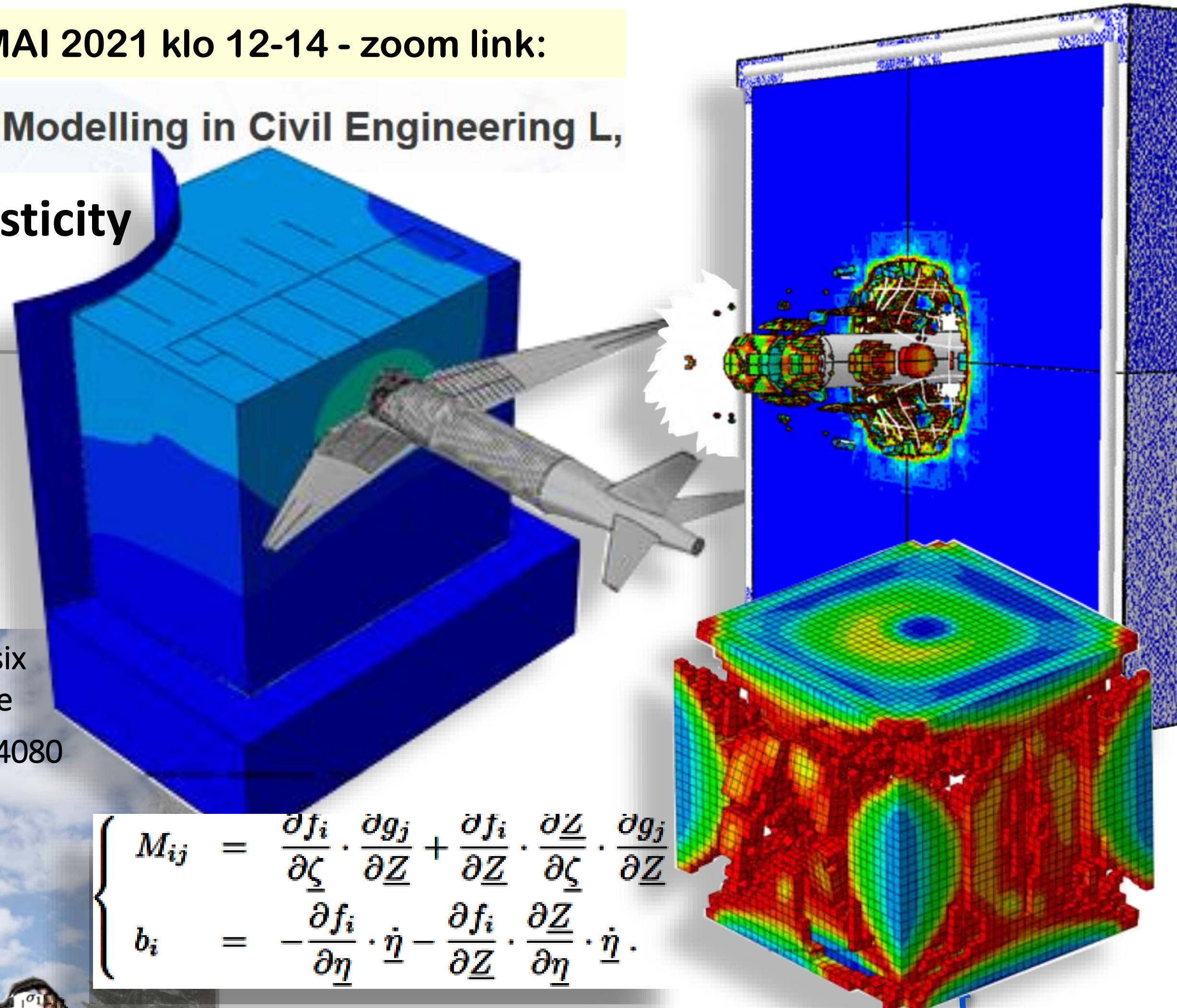




Guest X-pert e-lecture: **25 MAI 2021 klo 12-14 - zoom link:**

Course: **CIV-E4080 - Material Modelling in Civil Engineering L,**

Continuum Damage Plasticity for Concrete Modeling



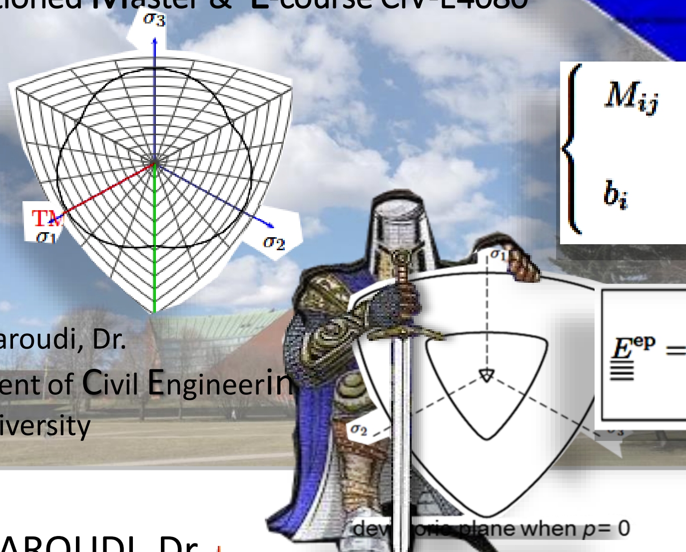
Guest-lecture kindly offered by

Dr. **Alexis Φedoroff**

Scientist @ VTT

25 MAI 2021 klo 12-14

- The topic of this lecture is one of the six topics which are adressed in the above mentioned Master & L-course CIV-E4080



$$\begin{cases} M_{ij} = \dfrac{\partial f_i}{\partial \underline{\zeta}} \cdot \dfrac{\partial g_j}{\partial \underline{Z}} + \dfrac{\partial f_i}{\partial \underline{Z}} \cdot \dfrac{\partial \underline{Z}}{\partial \underline{\zeta}} \cdot \dfrac{\partial g_j}{\partial \underline{Z}} \\ b_i = -\dfrac{\partial f_i}{\partial \underline{\eta}} \cdot \dot{\underline{\eta}} - \dfrac{\partial f_i}{\partial \underline{Z}} \cdot \dfrac{\partial \underline{Z}}{\partial \underline{\eta}} \cdot \dot{\underline{\eta}}\,. \end{cases}$$

$$\underline{\underline{E}}^{\mathrm{ep}} = \underline{\underline{E}}(d) + \frac{\left(\underline{\underline{E}}(0) : (\underline{\epsilon} - \underline{\epsilon}^{\mathrm{P}})\right) \otimes \left((\underline{\epsilon} - \underline{\epsilon}^{\mathrm{P}}) : \underline{\underline{E}}(0)\right)}{\frac{\partial f_2}{\partial d} / \frac{\partial f_2}{\partial Y}} - \frac{\left(\underline{\underline{E}}(0) : \underline{n}\right) \otimes \left(\underline{n} : \underline{\underline{E}}(0)\right)}{\frac{1}{1-d}\,\underline{n} : \underline{\underline{E}}(0) : \underline{n} + \underline{n} : \frac{\partial H}{\partial \underline{\chi}} : \underline{n} + \frac{\partial R}{\partial \pi}}$$

Djebar Baroudi, Dr.

Department of Civil Engineerin

Aalto-University

Teacher:

Djebar BAROUDI, Dr.

$$A\frac{J_2}{\sigma_c} + \Lambda\sqrt{J_2} + BI_1 - \sigma_c = 0,$$

Tangent vector to the yield surface:

$$\underline{\underline{M}}\,\dot{\underline{\lambda}} = \underline{b}$$

$$\dot{\underline{\gamma}} = \begin{pmatrix} \dfrac{\partial \underline{Z}}{\partial \underline{\zeta}} \cdot \dot{\underline{\zeta}} + \dfrac{\partial \underline{Z}}{\partial \underline{\eta}} \cdot \dot{\underline{\eta}} \\ \dot{\underline{\zeta}} \\ \dot{\underline{\eta}} \end{pmatrix}$$

May be: Guest lecture year 2021: **18? MAI 2018 klo 14-16, R2   (not yet confirmed)**

Continuum Damage Plasticity for Concrete Modeling

Guest-lecture offered by

CONCRETE²

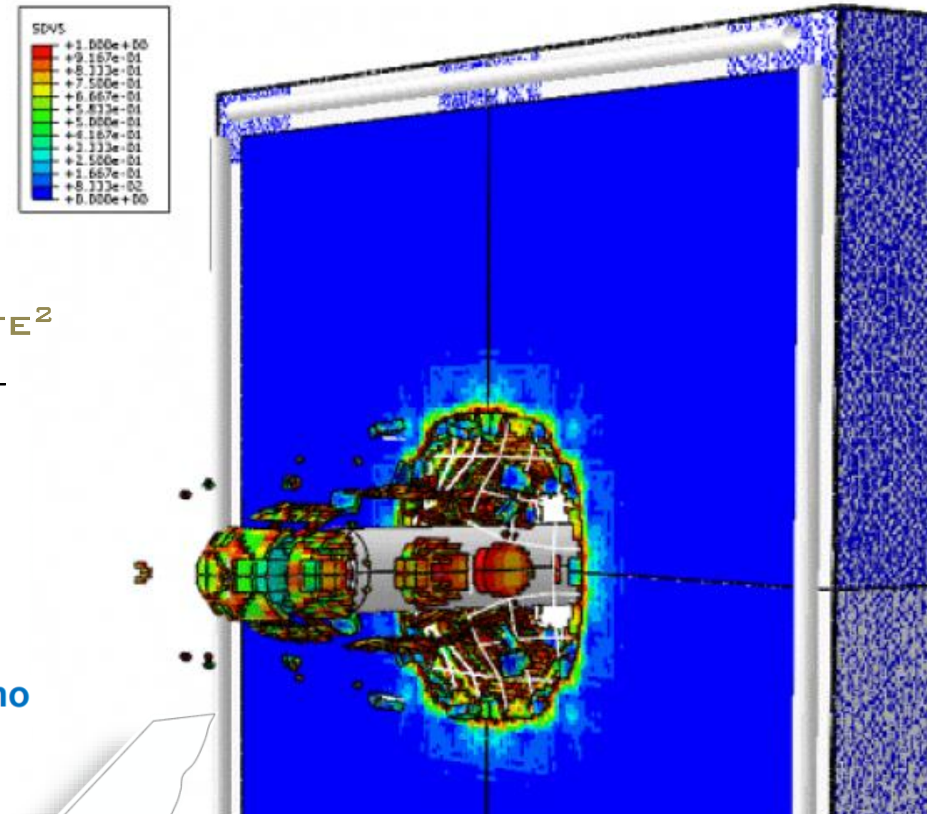
Dr. **Alexis Φedoroff** – Civil Engineer & Scientist @ VTT

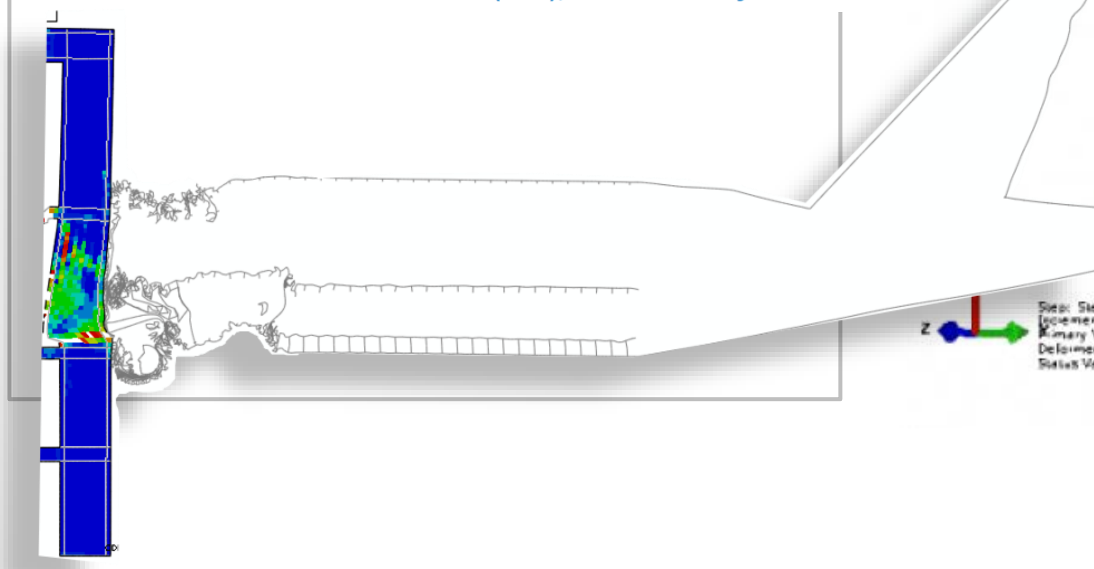
A physically motivated element deletion criterion

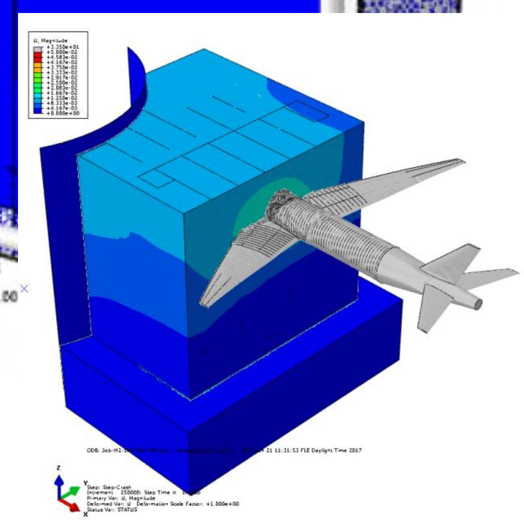
for the concrete damage plasticity model

Alexis Fedoroff, Juha Kuutti, and Arja Saarenheimo

Technical reseach centre of Finland (VTT), Nuclear safety.







Guest-lecture offered by

Dr. Alexis Φedoroff –

Civil Engineer & Scientist @ VTT

Concrete Damage Plasticity with applications

An introductory course to Abaqus CDP material modelling

- Pre-requisites: Continuum thermodynamics
  - ✓ Conservation laws,
  - ✓ I & II principles of thermodynamics, Clausius-Duhem inequality
  - ✓ Maximum dissipation postulate
  - ✓ Consistency condition
  - ✓ Solving the Lagrange multipliers
- Introduction to the Abaqus CDP material model
  - ✓ Thermodynamic potential, yield potential, flow potential
  - ✓ Concrete behaviour in uniaxial tests, hardening laws
  - ✓ Evolution of the yield surface, expressions of the normal vectors
  - ✓ Evolution of stiffness degradation
  - ✓ Incremental stress-strain relationship
- Examples of application of the Abaqus CDP model

Constitutive equations

- The identity $\gamma = \gamma_{\text{def}}$ must hold for all states $\left(\underline{\underline{\sigma}}, \underline{q}, \underline{Z}, \underline{\underline{\epsilon}}^e, \dot{T}\right)$:
  - ➢ Elasticity rule: $\underline{\underline{\sigma}} = \rho \frac{\partial \psi}{\partial \underline{\underline{\epsilon}}^e} = -\rho \frac{\partial \psi}{\partial \underline{\underline{\epsilon}}^p}$
  - ➢ Entropy relation: $s_* = \rho \frac{\partial \psi}{\partial T}$
  - ➢ Plastic flow rule: $\dot{\underline{\underline{\epsilon}}}^p = \frac{\partial \varphi}{\partial \underline{\underline{\sigma}}}$
  - ➢ Internal variable flow rule: $\dot{\underline{\zeta}} = -\frac{\partial \varphi}{\partial \underline{Z}}$
  - ➢ Temperature flow rule: $\underline{\nabla} \ln T = -\frac{\partial \varphi}{\partial \underline{q}}$
- The dissipation rate simplifies to:

$$\boxed{\gamma = \underline{\underline{\sigma}} : \dot{\underline{\underline{\epsilon}}}^p - \underline{Z} \cdot \dot{\underline{\zeta}} - \underline{q} \cdot \underline{\nabla} \ln T = \gamma^M + \gamma^T}$$

Guest-lecture kindly offered by

Dr. **Alexis Φedoroff**
Scientist @ VTT

Example: Mises plasticity with isotropic hardening and scalar damage

- Flux variables: $\boldsymbol{\eta} = \left\{\underline{\underline{\varepsilon}}\right\}$ and $\boldsymbol{\xi} = \left\{\underline{\underline{\varepsilon}}^p, -\varepsilon_{eq}^p, -d\right\}$. Force variable $\boldsymbol{X} = -\rho\frac{\partial\psi}{\partial\xi} = \left\{\underline{\underline{\sigma}}, \sigma_{\mathrm{y}}, Y\right\}$.
- Helmholz potential $\rho\psi\left(\underline{\underline{\varepsilon}}^e, \varepsilon_{eq}^p, d\right) = \frac{1}{2}\underline{\underline{\varepsilon}}^e : (1-d)\mathbb{E} : \underline{\underline{\varepsilon}}^e + \frac{1}{2}H{\varepsilon_{eq}^p}^2$, whence:

$$\boldsymbol{X} = \begin{pmatrix} (1-d)\mathbb{E} : \underline{\underline{\varepsilon}}^e \\ H\varepsilon_{eq}^p \\ -\frac{1}{2}\underline{\underline{\varepsilon}}^e : \mathbb{E} : \underline{\underline{\varepsilon}}^e \end{pmatrix}, \qquad \frac{\partial \boldsymbol{X}}{\partial \boldsymbol{\xi}} = \begin{pmatrix} -(1-d)\mathbb{E} & 0 & \mathbb{E} : \underline{\underline{\varepsilon}}^e \\ 0 & -H & 0 \\ \mathbb{E} : \underline{\underline{\varepsilon}}^e & 0 & 0 \end{pmatrix}$$

- Yield functions $f_1\left(\underline{\underline{\sigma}}, \sigma_{\mathrm{y}}; d\right) = \frac{1}{1-d}\left(\sqrt{3J_2\left(\underline{\underline{\sigma}}\right)} - \sigma_{\mathrm{y}}\right)$, $f_2(Y; d) = \frac{1}{2}\frac{Y^2}{k(1-d)}$

$$\frac{\partial f_1}{\partial \boldsymbol{X}} = \frac{1}{1-d}\begin{pmatrix} \underline{\underline{n}} \\ -1 \\ f_1 \end{pmatrix}, \quad \frac{\partial f_1}{\partial \boldsymbol{\xi}} = \frac{1}{1-d}\begin{pmatrix} 0 \\ 0 \\ -f_1 \end{pmatrix}, \quad \frac{\partial f_2}{\partial \boldsymbol{X}} = \frac{1}{1-d}\begin{pmatrix} \frac{-1}{2(1-d)}\underline{\underline{\varepsilon}}^e \\ 0 \\ Y/k \end{pmatrix}, \quad \frac{\partial f_2}{\partial \boldsymbol{\xi}} = \frac{1}{1-d}\begin{pmatrix} 0 \\ 0 \\ -f_2 \end{pmatrix}$$

Guest lecture from year 2018: **17 MAI 2018**

Guest-lecture offered by

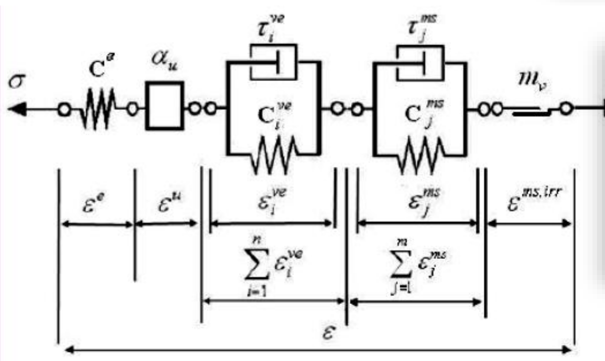
Dr **Stefania FORTINO** – Senior Scientist @ VTT

Hygro-mechanical behaviour of wooden structural elements

Stefania Fortino

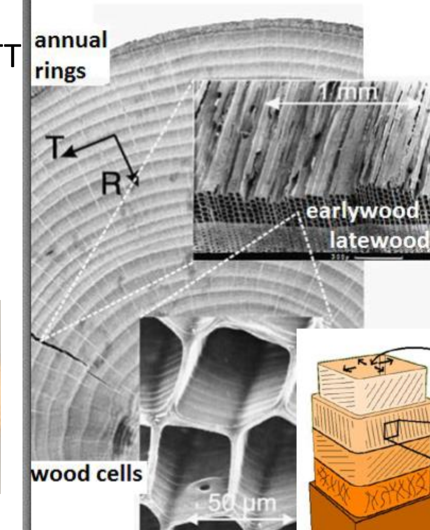
Senior Scientist

VTT Technical Research Centre of Finland

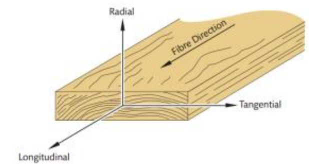


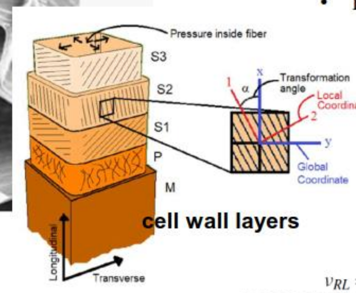
CIV-E4080 - Material Modelling in Civil Engineering L,

Anisotropy and multiscale material structure



- The mechanical properties of wood depend on its material structure from annual rings to wood cells and cell wall layers

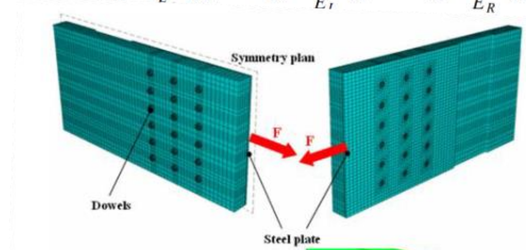


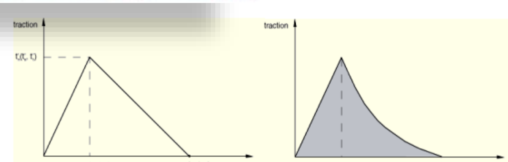


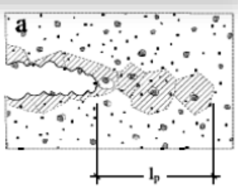
- Orthotropic material
- Elastic compliance matrix:

$$\mathbf{C} = \begin{bmatrix} \frac{1}{E_R} & \frac{-\nu_{TR}}{E_T} & \frac{-\nu_{LR}}{E_L} & 0 & 0 & 0 \\ \frac{-\nu_{RT}}{E_R} & \frac{1}{E_T} & \frac{-\nu_{LT}}{E_L} & 0 & 0 & 0 \\ \frac{-\nu_{RL}}{E_R} & \frac{-\nu_{TL}}{E_T} & \frac{1}{E_L} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{RT}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{RL}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{TL}} \end{bmatrix}$$

$$\nu_{RL} = \frac{E_R}{E_L}\nu_{LR}, \qquad \nu_{TL} = \frac{E_T}{E_L}\nu_{LT}, \qquad \nu_{TR} = \frac{E_T}{E_R}\nu_{RT}$$



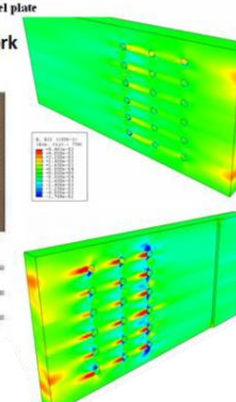




xponential softening (suitable for parametric study and for wood)

$$D = 1 - \left\{\frac{\delta_m^o}{\delta_m^{\max}}\right\}\left\{1 - \frac{1 - \exp\left(-\alpha\left(\frac{\delta_m^{\max} - \delta_m^o}{\delta_m^f - \delta_m^o}\right)\right)}{1 - \exp(-\alpha)}\right\}. \qquad \delta_m = \sqrt{\langle\delta_n\rangle^2 + \delta_s^2 + \delta_t^2}.$$

Example of computational work for MSc theses



$$\psi(T, u, \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_i^{ve}, \boldsymbol{\varepsilon}_j^{ms}) = \phi(T, u) + \frac{1}{2}\boldsymbol{\varepsilon}^e : \mathbf{C}^e : \boldsymbol{\varepsilon}^e + \frac{1}{2}\sum_{i=1}^{n} \boldsymbol{\varepsilon}_i^{ve} : \mathbf{C}_i^{ve} : \boldsymbol{\varepsilon}_i^{ve} + \frac{1}{2}\sum_{j=1}^{m} \boldsymbol{\varepsilon}_j^{ms} : \mathbf{C}_j^{ms} : \boldsymbol{\varepsilon}_j^{ms}$$

Content

Motivation

- Course of materials modelling in other universities
- Literature & textbooks
- Some historical notes on engineering plasticity
- Motivation: How engineering Plasticity is seen in Abaqus, Ansys, Lusas?
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- Examples of Failure of Structures
- What is failure? Types of failures, failure envelopes and failure criteria

Plasticity

Failure hypothesis or Yield criteria

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- Some examples of solved problems in Plasticity
  - ✓ **Plastic limit load and displacement-force relation in bending**

Continued…

Engineering Plasticity

Classical theory – fundamentals

The three ingredient of the classical plasticity theory

- **Yield criteria**
- **Flow rule**
- **Hardening rule**

Yield Criteria

Pressure independent Yield criteria

- Tresca yield Criterion
- Von Mises yield Criterion

Pressure dependent Yield criteria

- Mohr-Coulomb Criterion
- Drucker-Prager Criterion
- Ottosen (1977) developed a 4-parameters failure criterion for concrete
- Hoek–Brown failure criterion
- The Cam-Clay model

(good to know) Example of material Behavior of Clay and Silt in Otaniemi

Other types of failure criteria

- Maximum Principle Stress Criteria (Rankine)
- Maximum Principal strain (St. Venant)

Anisotropic yield criteria

…

Appendices

- **Appendix 1**: Stress invariants
- **Appendix 2**: Recommended compulsory reading

- **Plasticity – plastisuus**
- **Failure hypotheses - lujuushypoteesit**

Continued

Hardening – notions

- Hardening Rules
- Examples of simple rheological models for Rate-independent plasticity
- Examples of hardening rules in Abaqus – how they looks like?

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Flow rule & Consistency condition

Plastic strain increment

Principle of maximum plastic work

Normality rule

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Associative and Non-associate Plasticity

Convexity of the criterion

Normality of the plastic flow

Some application examples of associated and non-associated plasticity

Incremental Stress-Strain Relationships

Example of a flow rule for isotropic hardening

Examples of hardening rules in Abaqus – how it looks like?

EOL

Literature & textbooks

Learning Supporting Material in MyCourses

0. INTRODUTION
1. ELASTICITY
2. VISCOELASTICITY (& some basics of creep)
3. PLASTICITY (& Failure Hypotheses)

Reading:

- Lemaitre, **chapter 5.** – **Plasticity**

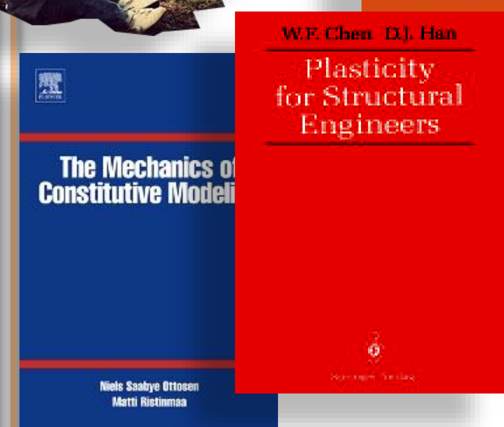
or

- W.F. Chen, D.J. Han, **chapters 1-5** **Plasticity for Structural Engineers**

or

- Ottosen & Ristinmaa, **chapters 8…12** – **Plasticity**

Detailed content on next slide



Content

Plasticity
Failure hypothesis or Yield criteria
Plasticity Isotropic & Isothermal Rate-Independent
- Examples
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- Plastic basic behavior in simple tension & compression

Modelling of uniaxial behavior in plasticity – simplified models
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continued…

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Example of a flow rule for isotropic hardening
Examples of hardening rules in Abaqus – how it looks like?
EOL

In one word: it is question of modeling the material behavior beyond its domain of elasticity
(Plasticity is one kind of material non-linearity)

Lemaitre & Chaboche textbook as an **e-book**:
http://proquestcombo.safaribooksonline.com.libproxy.aalto.fi/book/physics/9781107384712

Course of **materials modelling** in other universities

TECHNISCHE UNIVERSITÄT DARMSTADT

Technische Universität Darmstadt

9.10.2016

13-02-0003-vl Werkstoffmechanik

Veranstaltungsdetails

Lehrende: Prof. Dr.-Ing. Michael Vormwald; Dipl.-Ing. Melanie Fiedler

Veranstaltungsart: Vorlesung

Orga-Einheit: FB13 Bau- und Umweltingenieurwissenschaften

Anzeige im Stundenplan: Werkstm. (V)

Fach:

Anrechenbar für:

Semesterwochenstunden: 3

Unterrichtssprache: Deutsch

Min. | Max. Teilnehmerzahl: - | -

Lehrinhalte:

- Klassifizierung der Phänomene des Deformations- und Festigk
- Lineare Elastizität
- Isotropie, Anisotropie (Orthotropie, transversale Isotropie)
- Elastoplastizität
- Idealplastizität, Isotrope und kinematische Verfestigung
- Viskoelastizität, Viskoplastizität
- Werkstoffgesetze für Stahl, Beton, Glas, Holz, Kunststoffe und Geomaterialien
- Numerische Umsetzung

Today subject

... for comparison of the courses contents at **two universities to show the relevance of such course content for CIV-engineers**

Aalto-university,
summer 2016

Content:

Djebar BAROUDI, PhD,
Aalto-University

- **Elasticity – kimmoisuus tai elastisuus**
  (linear, hyper-elasticity, isotropy, anisotropy, orthotropy)
- **Viscoelasticity - viskoelastisuus**
- **Viscoplasticity** or creep – **viskoplastisuus** ... viruminen
- **Failure hypotheses - lujuushypoteesit**
- **Plasticity - plastisuus**
  associative, non-associative
- **Damage - vauriotuminen**
  damage-plasticity *ex.* Concrete Damage Plasticity, Model in Abaqus

if time...

Ref: thanks go to an exchange student for providing the course content-list above

GOTO slide 24 if you are in a hurry

Reminding the full content of this course

1.5.2017 Helsinki

Hyvää Wappua 2017

We lose two lectures! Will be replaced today (3.5.2018)

→ skip

CONTENT

1. INTRODUTION
2. **ELASTICITY**
3. VISCOELASTICITY
4. PLASTICITY

Elasticity in Solids

Definitions

Thermodynamical framework

Elastic Solids

Isothermal Cauchy-elastic material

Green-Elastic or Hyper-elastic Materials

Examples of Non-Linear Elasticity

Hysteresis during loading and unloading

Equations of Elasticity

Minimum Master level

Material Symmetries

Degree of symmetry

Linear Elasticity – Matrix Formulation

Anisotropy

Isotropy

Limits on Elastic Parameters Values

Orthotropy

Transversal isotropy

Limits on Elastic Parameters Values

Nonlinear isotropic Hooke formulation

Generalized Hooke's Law – Examples of problems

Orthotropic case – A worked example

Good to know: layered composite (transverse orthotropy)

Transformation of Stress and Strain Components

Example exercises for training

…

Nonlinear isotropic Hooke formulation

Some general aspects

Why splitting volumetric and deviatoric (shearing)?

Thermo-elasticity

Minimum L-level

Hyperelasticity

Rubber or rubber-like Elasticity

Terminology and some definitions

Thermodynamics of rubber – enthalpic and entropic forces

Some classical models

Neo-Hookean model

Mooney-Rivlin model

Yeoh model

Ogden model

Example of Rubber Elasticity In Abaqus

W. Gilbert's experiment

On thermodynamics of elastomers

Homework

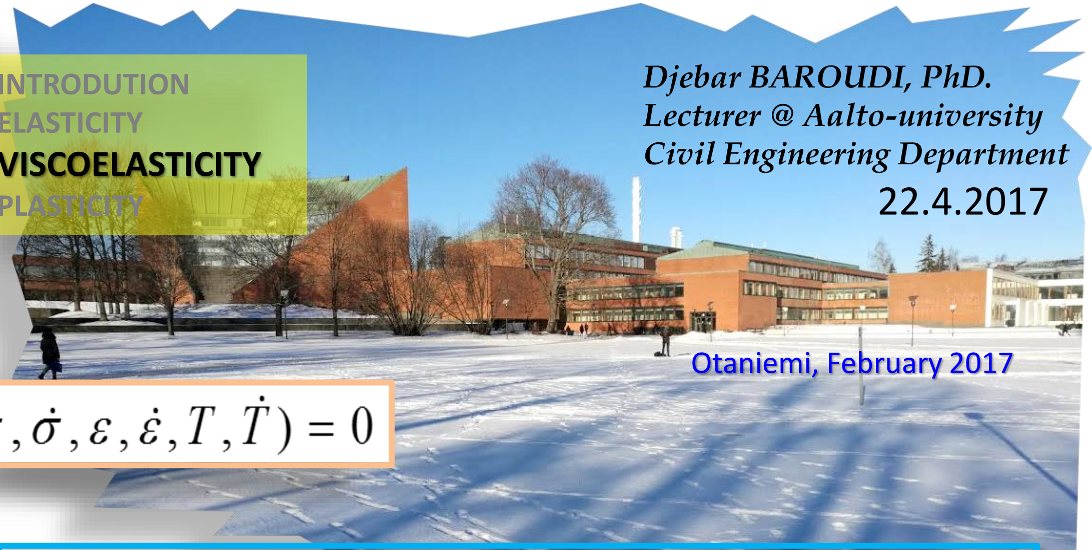
Appendix 1
Stress invariants (Recall)

Appendix 2
On Thermodynamics of Rubber
Enthalpic and Entropic forces

"The mountains **flowed** before the Lord"
(The Song of **Deborah**, Bible)
דְּבוֹרָה

1. INTRODUCTION
2. ELASTICITY
3. **VISCOELASTICITY**
4. PLASTICITY

Djebar BAROUDI, PhD.
Lecturer @ Aalto-university
Civil Engineering Department
22.4.2017



Viscoelasticity

$$\mathcal{F}(\sigma, \dot{\sigma}, \varepsilon, \dot{\varepsilon}, T, \dot{T}) = 0$$

Content

- **Experimental observations: evidence of viscoelastic behavior**
- **Stress relaxation at constant strain**
- **Creep at constant stress**
- **Strain-rate dependence**
- **Constitutive models in the rate form**:
  - Maxwell model
  - Kelvin-Voight model
  - Standard linear solid model
  - Burgers model
  - Generalized Maxwell model
  - Kelvin chain model

Reading: Textbooks

- Lemaitre and Chaboche – *Mechanics of Solid Materials*. **Chapter 4.3**
- Ottosen & Ristinmaa – *Introduction to time-dependent material behaviour.* **Chapter 14**

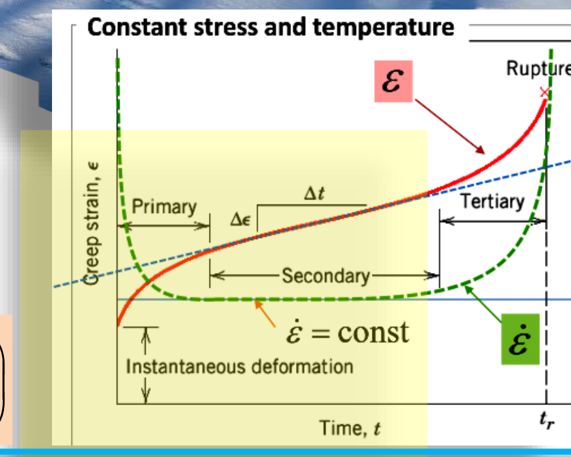
Lemaitre & Chaboche textbook as an **e-book**:
http://proquestcombo.safaribooksonline.com.libproxy.aalto.fi/book/physics/9781107384712

Primary creep:

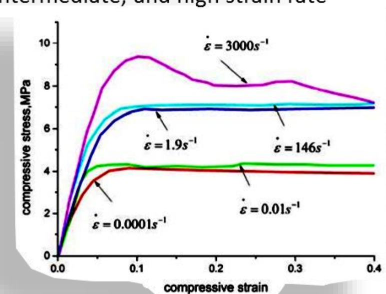
$$\varepsilon = A(\sigma) \cdot t^{1/\beta},$$
$$2 < \beta < 4$$
$$\varepsilon = A(\sigma) \cdot [t / t_{\mathrm{REF}}]^{1/\beta},$$

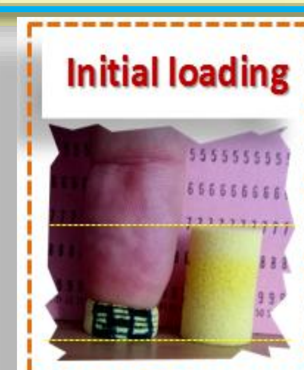
Secondary creep:

$$\dot{\varepsilon} = K_2 \left[\frac{\sigma}{\sigma_{\mathrm{Ref}}} \right]^n \exp\left(-\frac{Q_c}{RT} \right)$$



Experimental: Compressive responses of balsa wood at static, intermediate, and high strain rate





Partial recovery

Engineering **Plasticity**

Content

Motivation

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Engineering Plasticity

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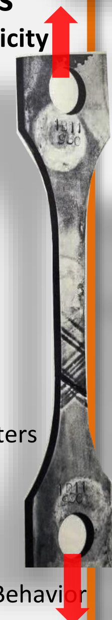
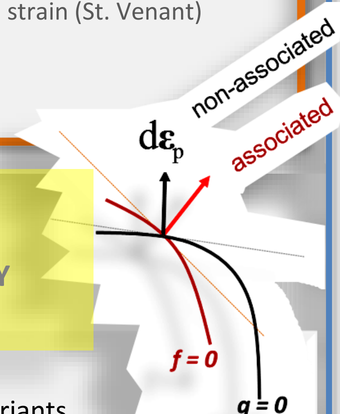
Anisotropic yield criteria

…



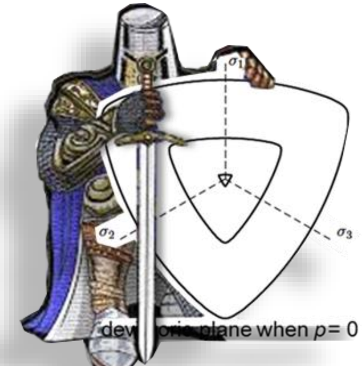


1. INTRODUCTION
2. ELASTICITY
3. VISCOELASTICITY
4. **PLASTICITY**

Appendices

Appendix 1: Stress invariants

Appendix 2: Recommended compulsory reading



$$A\frac{J_2}{\sigma_c} + \Lambda\sqrt{J_2} + BI_1 - \sigma_c = 0,$$

Continued

Hardening – notions

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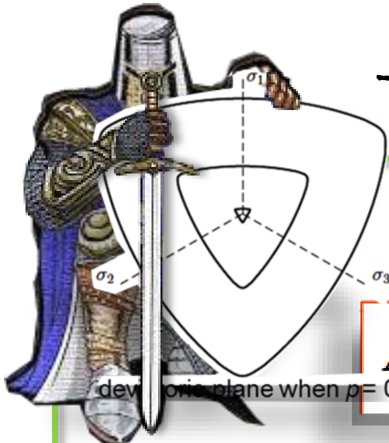
EOL

The story about engineering plasticity



$$A\frac{J_2}{\sigma_c} + \Lambda\sqrt{J_2} + BI_1 - \sigma_c = 0,$$

BEGIN

The story

{

START,

GO on,

keep going …

… till the end

THEN

STOP

}

END

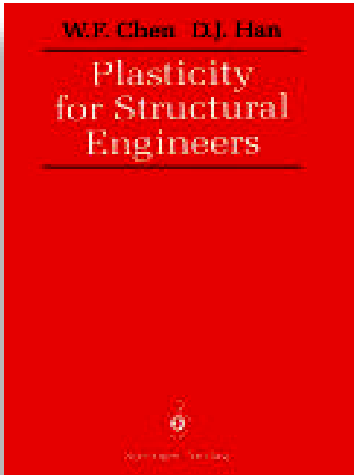
SOUAD MASSI

Raoui (Storyteller)

ya rawi hki hkaya
madabik tkoun rwaya,
hkili 3ala nass zman
hkili 3ala alf lila w lila
w3ala loundja bin lghoula
w3la wlid soultan
hadjitek, majitek,

The Storyteller

Djebar BAROUDI



Wai Fah Chen, D.J. Han
Springer New York, 2012 - Technology & Engineering - 606 pages

Textbook – Engineering Plasticity

(**very good and complete** for Structural Eng. Students and others….)

our course

Table of Contents

Wai Fah Chen, D.J. Han. **Plasticity** ***for Structural Engineers.*** Springer New York, 2012

Plasticity for Structural Engineers is a practical work that provides engineers and students in structural engineering or structural mechanics with the background needed to make the transition from fundamental theory to computer implementation and engineering practice

Key Features

- Outlines the finite element implementation of the generalized stress-strain relations for the solution of practical steel and concrete structural problems
- Provides fully-worked examples, end-of-chapter problems, answers to selected problems, and clear illustrations and tables
- Features important constitutive equations for structural materials and applications to steel and concrete structures
- Includes examples of the most useful constitutive models and analytical methods

Good to know if one wants to study 'difficult' subjects:

Ref: https://www.sciencedaily.com/releases/2017/04/170425092429.htm (27.4.2017)

SD Health Tech Enviro Society Quirky

Science News *from research organizations*

Movie research results: Multitasking overloads the brain

The brain works most efficiently when it can focus on a single task for a longer period of time

Date: April 25, 2017

Source: Aalto University

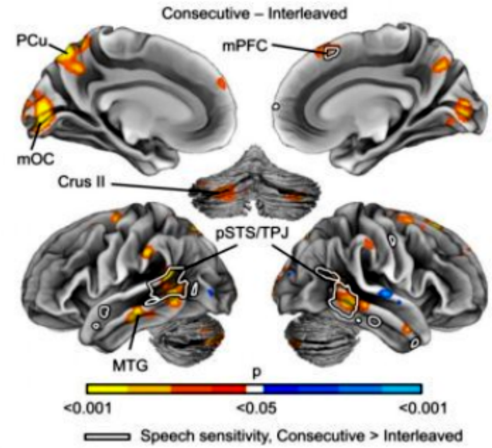
Summary: Previous research shows that multitasking, which means performing several tasks at the same time, reduces productivity by as much as 40%. Now a group of researchers specializing in brain imaging has found that changing tasks too frequently interferes with brain activity. This may explain why the end result is worse than when a person focuses on one task at a time.

The brain works most efficiently when it can focus on a single task for a longer period of time.

Neural Mechanisms for Integrating Consecutive and Interleaved Natural Events

Juha M. Lahnakoski,[1,2,3]* **Iiro P. Jääskeläinen**,[1] **Mikko Sams**,[1] and **Lauri Nummenmaa**[4,5]

[1]*Department of Neuroscience and Biomedical Engineering (NBE), School of Science, Aalto University, Espoo, FI-00076, Finland*
[2]*Advanced Magnetic Imaging (AMI) Centre, Aalto NeuroImaging, School of Science, Aalto University, Espoo, FI-00076, Finland*
[3]*Independent Max Planck Research Group for Social Neuroscience, Max Planck Institute of Psychiatry, Munich, DE-80804, Germany*
[4]*Turku PET Centre, University of Turku, Turku, FI-20521, Finland*
[5]*Department of Psychology, University of Turku, Turku, FI-20521, Finland*



The subjects brain areas functioned more smoothly when they watched the films in longer segments.

Credit: Juha Lahnakoski

The brain works most efficiently when it can focus on a single task for a longer period of time.

Previous research shows that multitasking, which means performing several tasks at the same time, reduces productivity by as much as 40%. Now a group of researchers specialising in brain imaging has found that changing tasks too frequently interferes with brain activity. This may explain why the end result is worse than when a person focuses on one task at a time.

'We used functional magnetic resonance imaging to measure different brain areas of our research subjects while they watched short segments of the Star Wars, Indiana Jones and James Bond movies,' explains Aalto University Associate Professor Iiro Jääskeläinen.

Journal Reference:

1. Juha M. Lahnakoski, Iiro P. Jääskeläinen, Mikko Sams, Lauri Nummenmaa. **Neural mechanisms for integrating consecutive and interleaved natural events**. *Human Brain Mapping*, 2017; DOI: 10.1002/hbm.23591

CONST nweeks = 6

iw= 0;

Aalto University

BEGIN

iw = 3

END

Content:

Djebar BAROUDI, PhD.
Aalto-University

- **Elasticity** – **kimmoisuus tai elastisuus**
  (linear, hyper-elasticity, isotropy, anisotropy, orthotropy)
- **Viscoelasticity** - **viskoelastisuus**
- **Viscoplasticity** or creep – **viskoplastisuus** … **viruminen**
- **Failure hypotheses** - **lujuushypoteesit**
- **Plasticity** - **plastisuus**
  associative, non-associative
- **Damage** - **vauriotuminen**
  damage-plasticity *ex.* Concrete Damage Plasticity, Model in Abaqus

← If time…

← this year
Guest Lectures

About material modelling

In general, modelling or choosing an adequate material model for a real material is challenging and is definitely not a simple task. The behaviour of real materials is complex. Even for such usual material like steel, many aspects of its mechanical behaviour remains not well-known.

It is quite impossible, for a specific material, to develop a universal model giving the response under all possible conditions of use (or loading). That is why, in every problem, one has to choose a simplest model reproducing with a relevant accuracy a specific aspect of the behaviour one wants to capture. Therefore, we account in the most simplest way, for the key physics needed to capture, with satisfactory precision, the response spectra we are interested in, in a specific problem. In oe word, the question itself, to which we are trying to find an answer, defines the needed or necessary complexity level of the model.

Imagine, the earth in its trajectory around the sun. You need only to determine this trajectory. Thus, the simplest mechanics is to reduce the earth with all the diversity that it holds , to a simple mathematical point having the total mass of the Earth. Then applying Newton's equations of motion, you will deduce the Kepler's laws about the trajectory. A more simpler working model cannot exist. Now, if you are interested, to make weather casting (to determine the weather of tomorrow based on measurements of today), then you need to introduce more complexity to the previous model. A material point is not enough for that. You should account for the motion of the atmosphere (air), temperature, pressure, velocity, input sun-energy, etc... the simplest model in this case is represented by the Navier-Stockes equations. If now, you need to determine the weather, for longer time, then you must account for the effect of oceans, seas and so on. This example, is just to show that the question defines the level necessary level of complexity of the model to develop or to use.

Equivalently, if I am interested in the elastic behaviour of steel for small strains, then Hooke's law is sufficient. If you, need to determine the plastic limit load of a member made of steel, then ideal-plastic model is sufficient. Now, if you have loading-unloading and reverse loading cycles, may be you need to integrate hardening to your elasto-plastic model. Now, if you need to account for effects of temperature, then thermally activated creep, or in other words, thermo-visco-plasticity should be added to the model. Then, if you are interested in fatigue failure, related to life-time of the steel member under low-cycles of high-cycles loading, then you should construct accounting for damage and damage accumulation. So, we see that for the same material, there is a multitude of models.

6 Engineering plasticity

6.1 Preliminary words

In structural mechanics or, equivalently, in strength of materials, traditionally, a macroscopic view of the deformation phenomena of a body is adopted. Consequently, the resulting stresses are also a macroscopic resultant of internal cohesion forces resisting such macroscopic deformation. This macroscopic framework has been proven through its wide spectrum of successful applications to be very fruitful and efficient in traditional engineering sciences and practices. Such framework is also known as *phenomenological*.

Engineering plasticity is one such phenomenological model which is of great importance, for instance, in civil and mechanical engineering, cold forming of metal and so on. In such theories, we consider macroscopic *plastic deformations* without having to know the subsequent microscopic processes from which they result, and develop internally operational and consistent theories of deformations and the resulting stresses. Successful such theories are the *classical engineering plasticity* that you are now studying in this notes. In theses models, you will read about *yield criteria*, *flow rule* and *hardening rule* in *mathematical terms* in forms of equations. Such set of equations enhance the self-confidence of our engineer that everything is under control since he controls the equations themselves. No doubt about that!

However, it is an intellectual curiosity and a professional duty, even for a 100-% pure structural engineer, to understand what are the microscopic processes or mechanisms from which these macroscopic plastic deformations result. In the following, I will make a short summary of that as regard to *metal plasticity*. For curious students, please refer to good courses or books on *material sciences* and specially for solids[8].

[8] Richard J. D. Tilley, *Understanding solids - The science of materials*. 2^{nd} Ed., Wiley, 2013.

6.2 What is plasticity at microscopical level?

As already pointed previously, I mean *metal plasticity* as regard to our engineering application. We try to understand, by opening the *macroscopic black box* of the plastic (macroscopic) strain $\epsilon^{(p)}$ and understand the origin of the microscopic motion from which it is the result.

It is an experimental fact that pure metals deform more than the metallic bond permits. This mystery is explained by the presence of *defects* in the crystals. (going without saying that metals are polycrystals (Fig. 5)). These defects allow the deformation to occur without breaking the *metallic bonds*. These defects are known as *dislocations*.

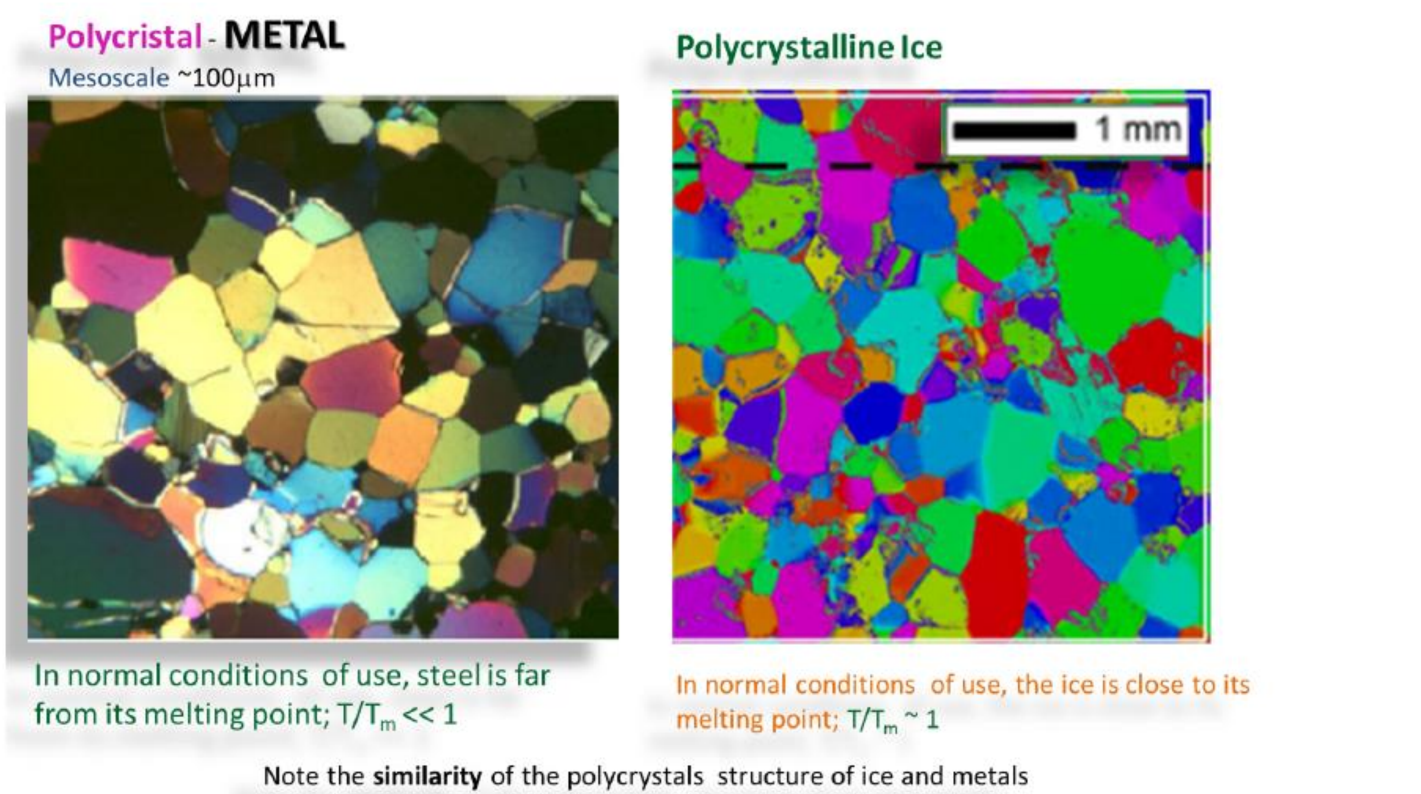


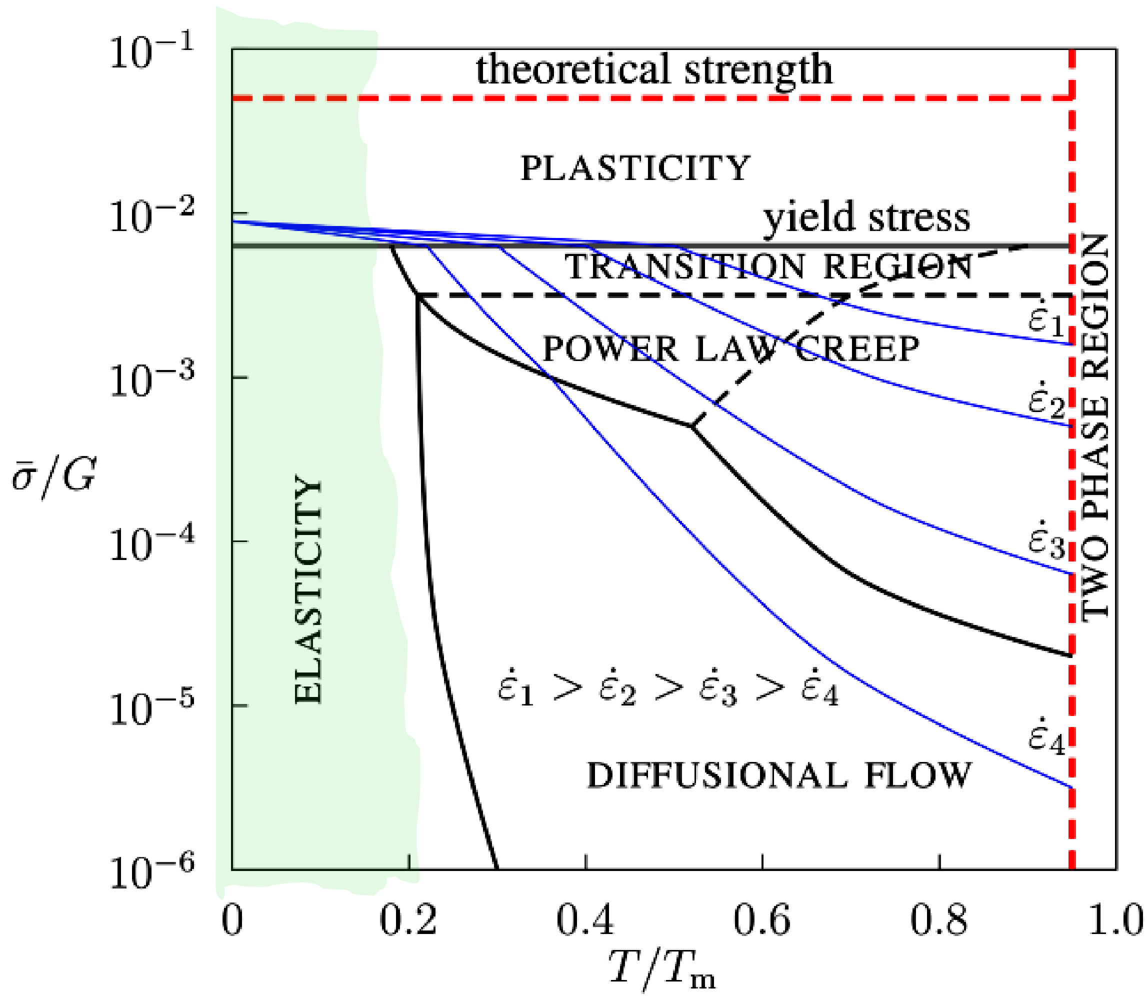
Figure 5: Metal and ice micro-structure under polarized light. The different colors correspond to different orientations in the mono-crystals (grains).

Ductility of metals: Metals can be deformed easily without breaking. They retain the deformed shape permanently when stressed beyond some threshold level. This property, results from the crystal structure of the metals which are roughly speaking, formed by packing less or more spherical atoms. Under stress these planes can roll each over the other easily resulting in deformation without braking the strong metallic bonds. Such permanent deformations are what is macroscopically called *plastic deformations*. On the other hand, for example, ceramics under the same stress conditions that the above metal, will fracture in a brittle manner with much less deformations.

This plasticity property of metals results from the presence of linear defects named *edge dislocations* which move freely in the solid under application of stress.

Dislocations: Edge dislocations consist of extra half-planes of inserted atoms into the crystal. It is these dislocations that allow metals to deform plastically, like in low carbon steels, for example. When a dislocation movement is prevented or made harder to occur, the material becomes *hard* and can even become brittle like cast iron, for instance.

Schematic deformation mechanism map for a metal alloy.

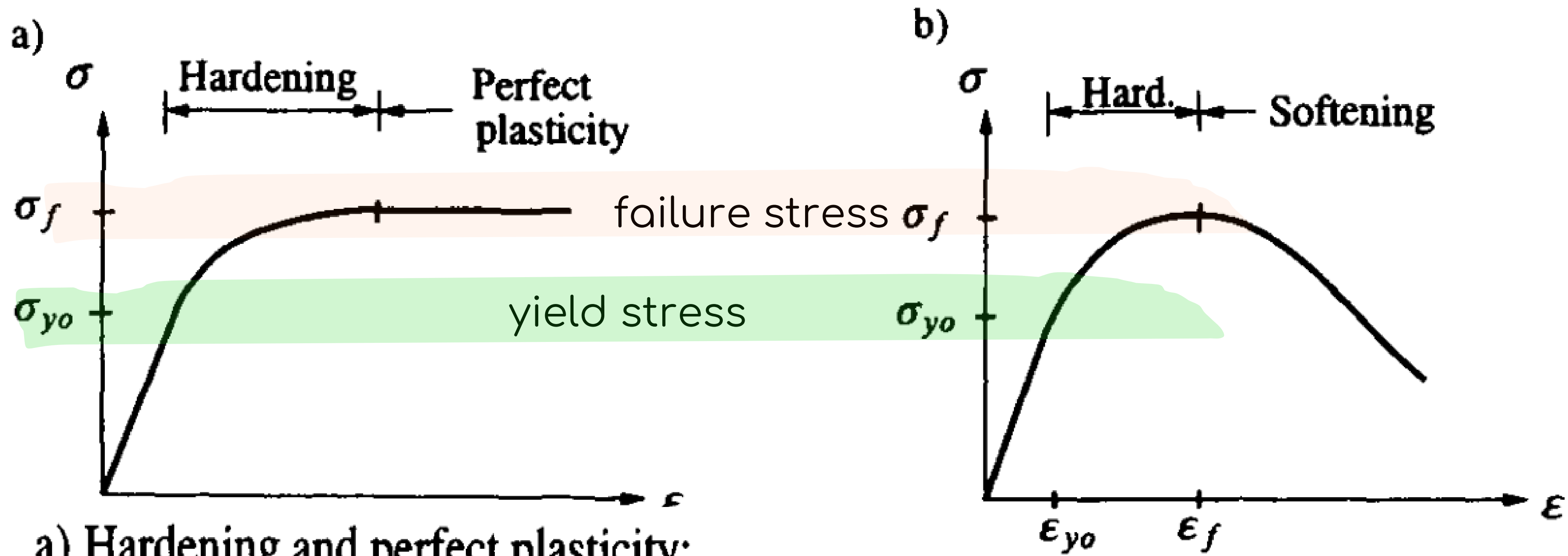


Temparature (K) / melting temperature (K)

H.J. Frost, M.F. Ashby. *Deformation-Mechanism Maps: The Plasticity and Creep of metals and Ceramics*, Pergamon-Pressm 1982. http://engineering.dartmouth.edu/defmech/

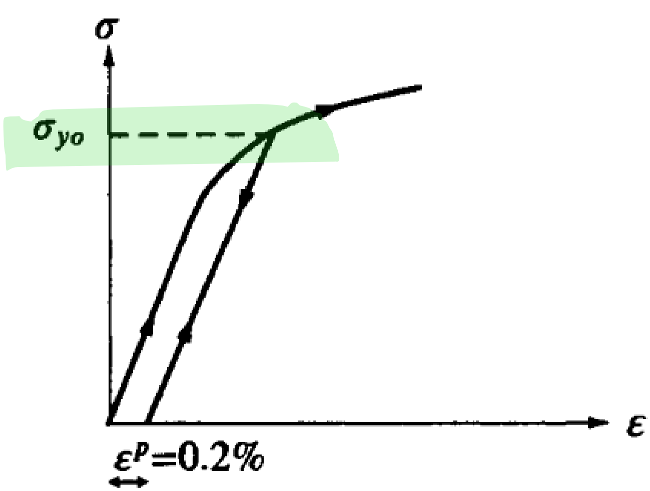
Ductile steel and ductile metals

Concrete, rocks, soils, ...



a) Hardening and perfect plasticity;

b) hardening and softening plasticity.

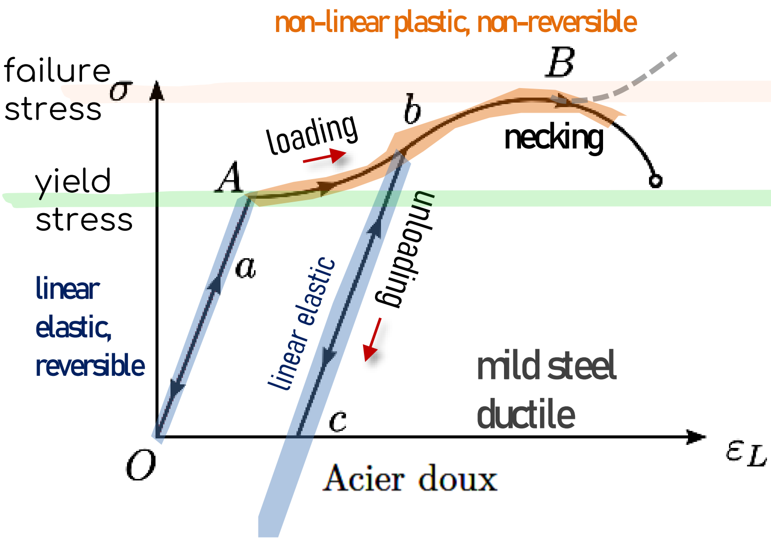


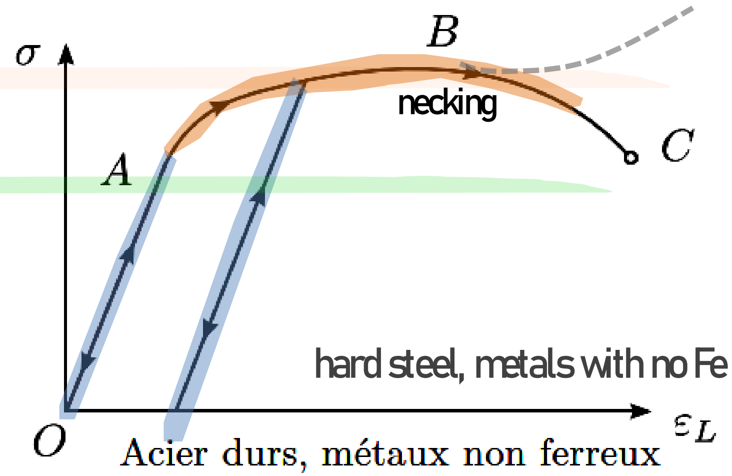
For general stress states, the conditions for failure or initial yielding are called *failure* or *initial yield criteria* respectively. Since they can be treated in a unified manner, we will often simply use the word criterion. We will see that stress invariants play an extremely important role in failure and yield criteria

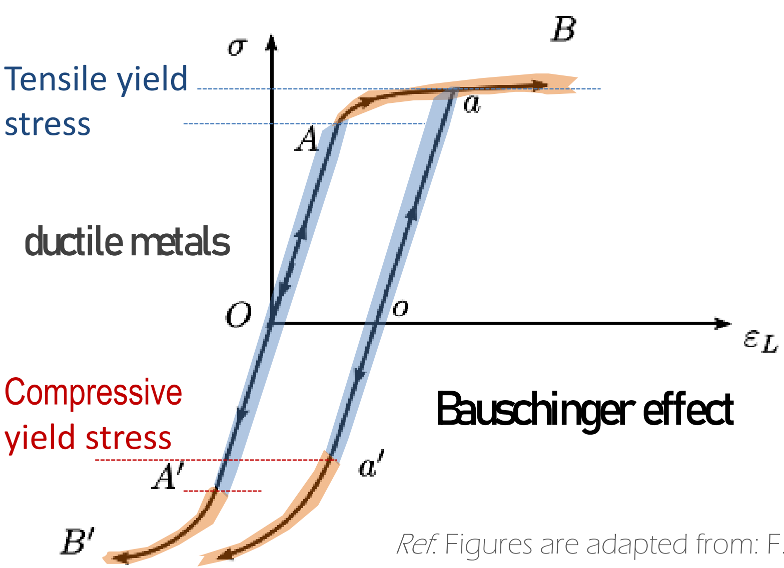
Ref. Ottosen *et al.*

Engineering Plasticity in metals

Metals at ambient temperature & quasi-satic loading

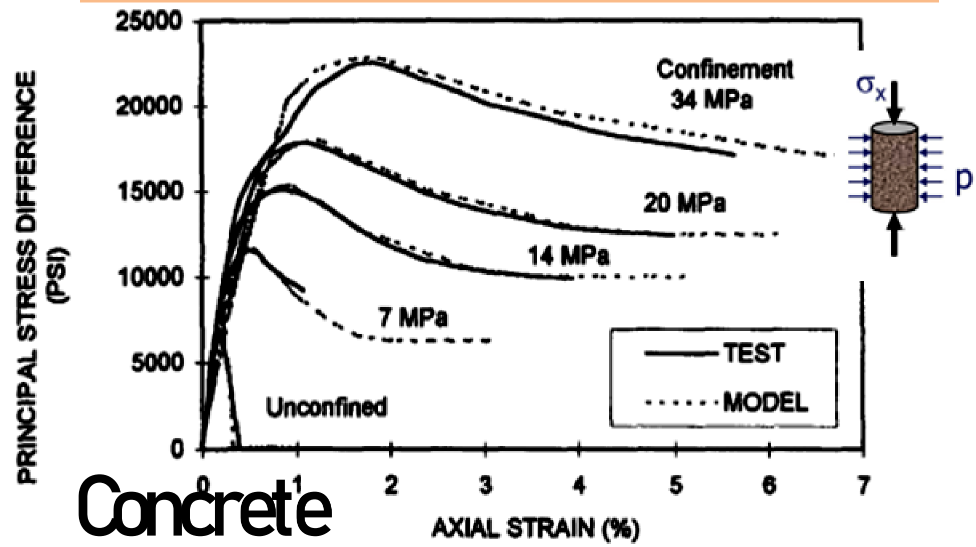






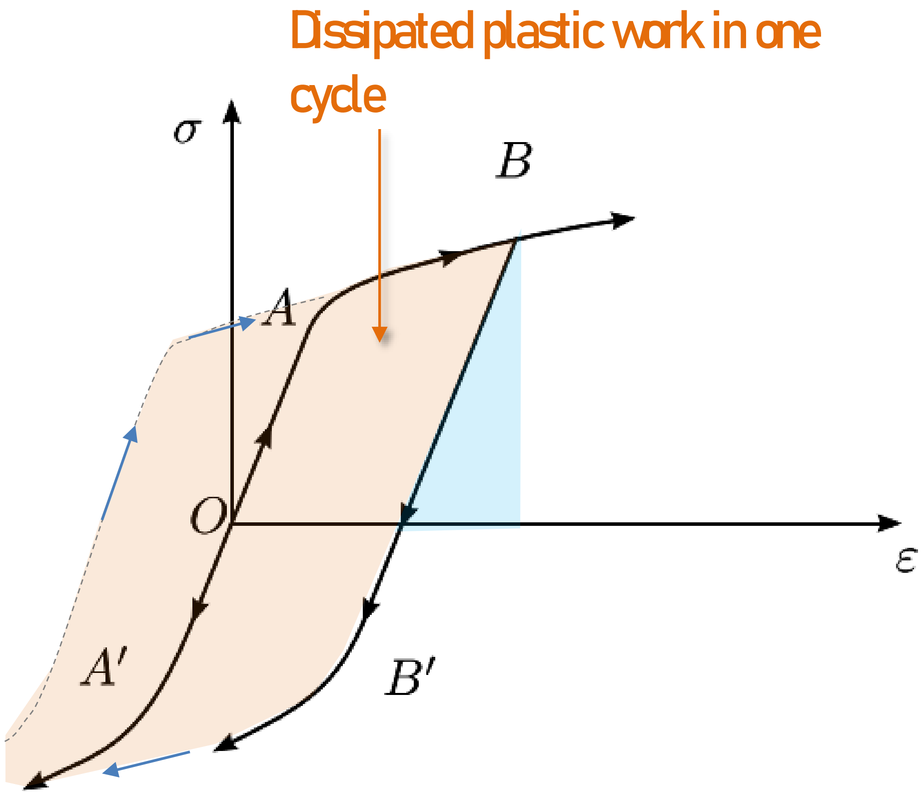
Ref. Figures are adapted from: F. Sidoroff *Mécanique des milieux continus* École central de Lyon.

Concrete – tri-axial test (monotonic loading)



Plasticitic behaviour in metals

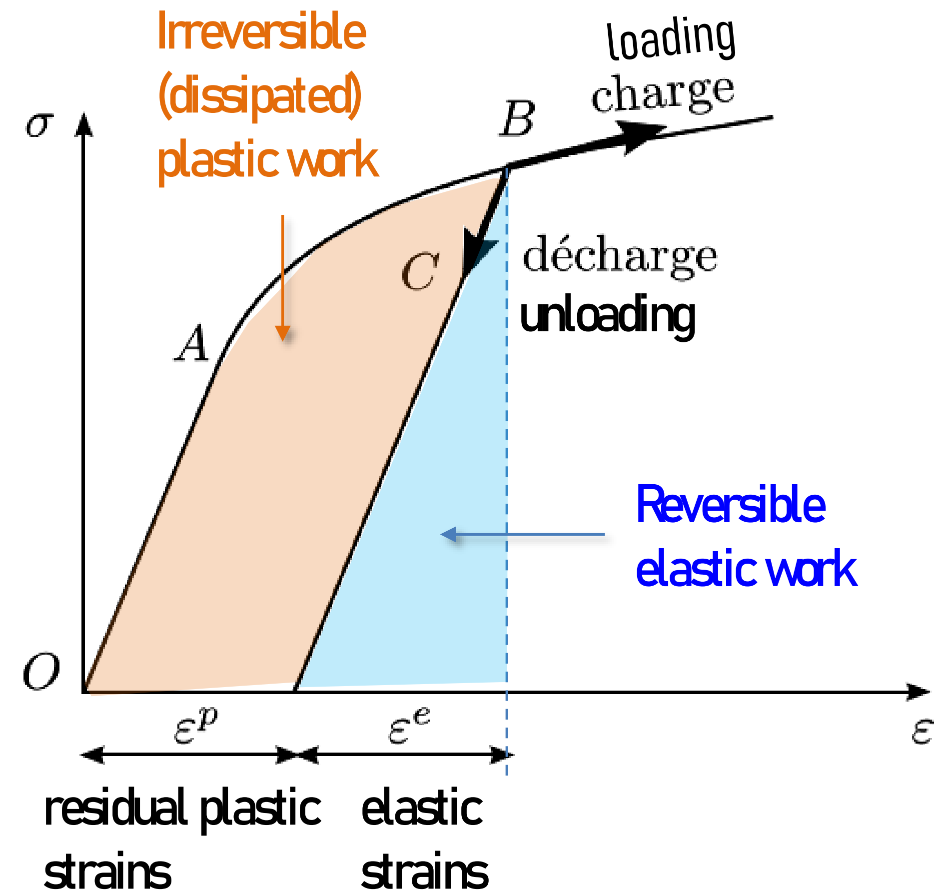
a) Cyclic loading.



$$\sigma_{A'} < \sigma < \sigma_A \qquad \sigma = E\varepsilon$$

$$\sigma_{B'} < \sigma < \sigma_B \qquad d\varepsilon^p = 0$$

b) Loading-unloading



$$\varepsilon = \varepsilon^e + \varepsilon^p$$

Internal friction coefficient

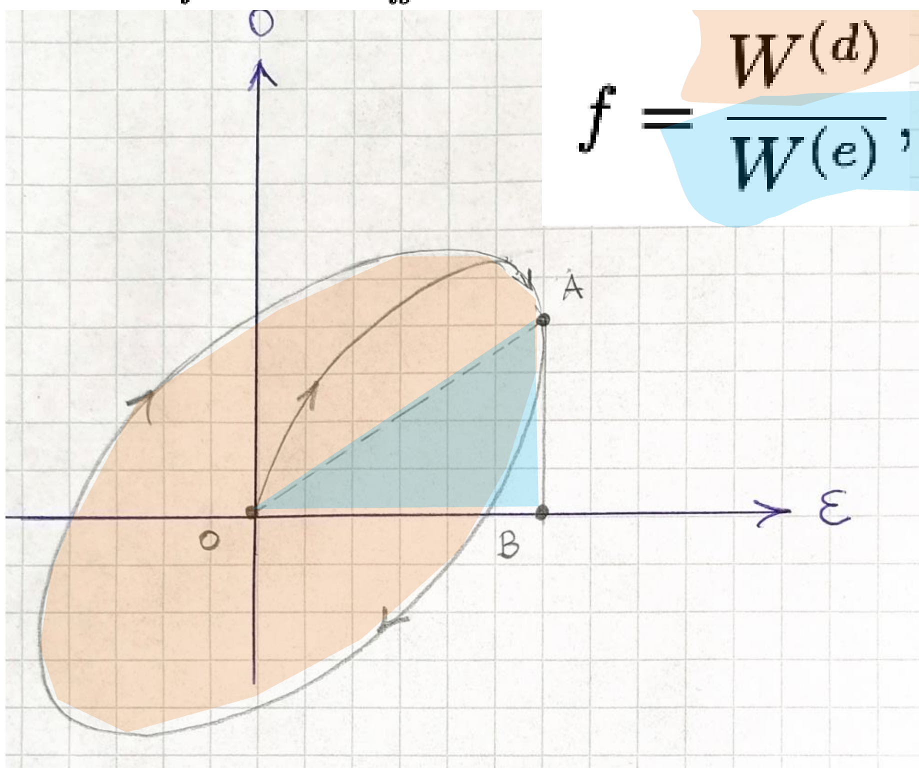
When a structure is dynamically loaded, it starts to response according to the laws of dynamics of deforming bodies. Assume the structure remains undamaged, and that at some time $t > t_0$ the load is removed completely. Despite the external forces removal, the structure continues to vibrate with decreasing amplitude until it reaches again a static equilibrium where all vibrations stop. Such vibrations are called *free vibrations*.

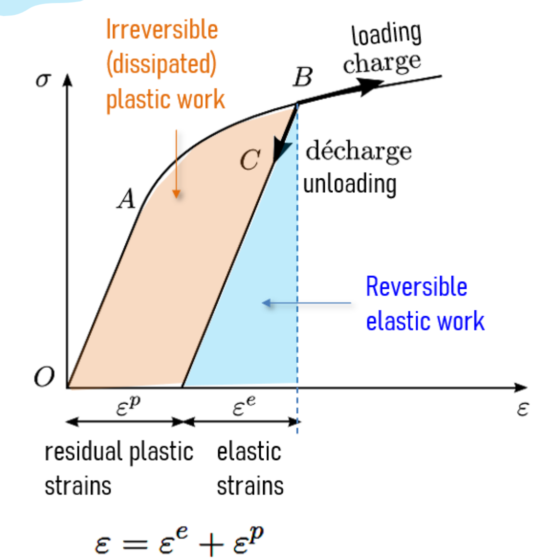
The processes *damping* these vibrations are the *dissipative* forces in the system: *internal* and *external*. External are those resulting from friction between parts of the structure and the surrounding, at supports, joints, aero-elastic forces, interface-soil structure, etc. The *internal friction* processes are related to the dissipative processes occurring with the deforming material itself; like viscous-type dissipation, plasticity. The resulting internal friction behaviour can be characterised by defining *internal friction coefficient*

$$f = \frac{W^{(d)}}{W^{(e)}},$$

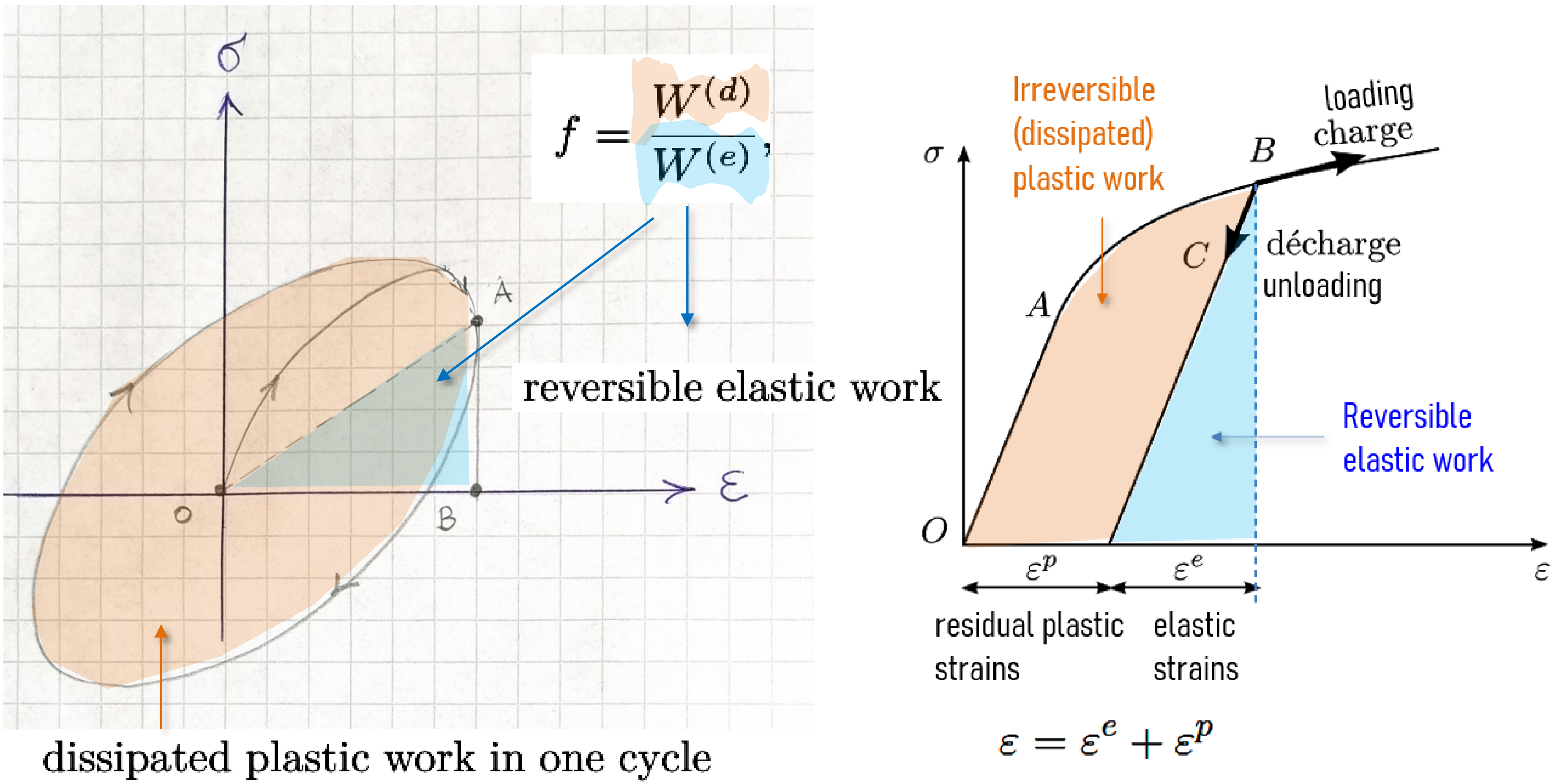
$W^{(d)}$ dissipated plastic work in one cycle

$W^{(e)}$ reversible elastic work





Internal friction coefficient



The internal friction coefficient f depends on the excitation frequency and on temperature. It remains practically independent of amplitudes for sufficiently small amplitudes, naturlich.

For metals, at normal temperatures, f is very small $(10^{-5} \ldots 10^{-4})$. For polymers, it becomes significant. For instance, for rubbers (vulcanised) , it can be as large as 0.6 at 16 Hz and low temperature (-20 deg. Celsius).

Internal friction and logarithm decrement in damping

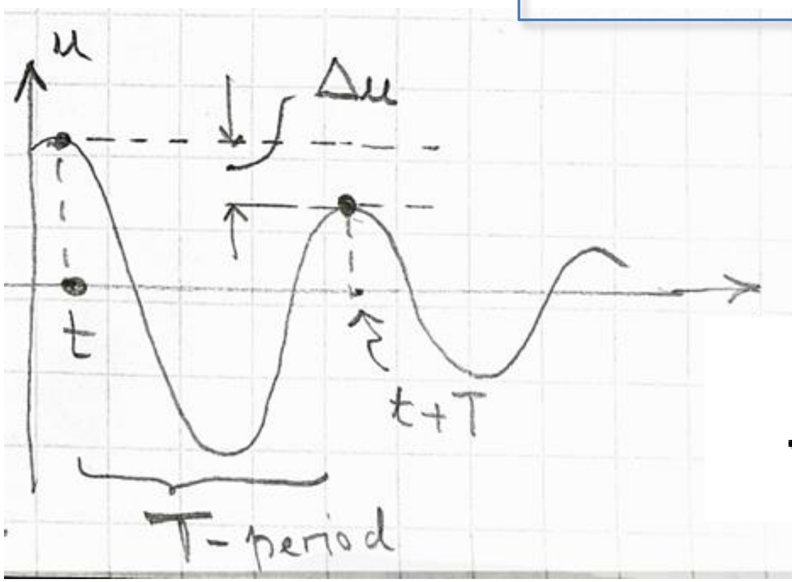
There is a close relation between the internal friction coefficient f and the e logarithmic decrement coefficient δ of damped free vibrations (lower part of Fig. 6).

During damped free vibrations, the ratio of two successive maximum amplitudes of vibrations (or strains) defines

$$\delta = \Delta u(t)/u(t), \quad \Delta u = u(t+T) - u(t)$$

period

Let's goo back to energy view points: the work dissipated within such period T is equal to the elastic work which is not restored between these two maximum amplitudes, namely $W^{(d)} = 1/2k \cdot [\Delta u]^2$. The stored elastic work is simple $W^{(e)} = 1/2ku^2$, where k being the relevant elasticity coefficient of the system



$$f = \frac{W^{(d)}}{W^{(e)}} = \frac{1/2k \cdot [\Delta(u)]^2}{1/2k \cdot u^2} = \frac{[\Delta u]^2}{u^2} \equiv \delta^2.$$

Classical theories of plasticity

There is two classes of such theories for modelling *engineering plasticity*. The first one is known as *total strain* or *deformation* theory of Hencky[8], Ilyushin[9] and also Nadai[10]. The second one is called *incremental* or *flow* theory of Prandtl[11]-Reuss[12].

Usually, incremental plasticity theory is used in structural and mechanical engineering where plastic strains are designed to remain small (may be of order of elastic strains for safety).

On the contrary, total strain plasticity theory is suitable for large deformations like those of encountered in cold forming of metals or in performing classical limit-plastic analysis where ideal plasticity (no hardening) plasticity is used.

[8] Henky H. *Zeit Angew Math Mech, 1924*, **4**, *323*.
[9] Ilyushin A. A. *Prik Matem Mekh, AN SSSR, 1946*, **10**, *347*
[10] Nadai A. *Plasticity: A Mechanics of the Plastic State of Matter*, 1931, McGraw-Hill, New York.
[11] Prandtl L. *Proc. 1st Int Congr Appl Mech. Delft*, 1924, p. 43.
[12] Reuss A. *Zeit Agrew Math Mech*, 1930, **10**, 266.

Classical theories of plasticity

1. **Hencky-Illyushin** - the deformation theory with no-hardening plasticity uses *total strains*. The total strain, under yielding, is decomposed into a deviatoric and volumetric part. The volumetric part results from mean normal stress and the deviatoric (shear deformation) part results from shear stress. The total strain tensor ϵ'_{ij} being defined as

$$\epsilon'_{ij} = \phi\sigma'_{ij} + \frac{\sigma_{kk}}{9K}\delta_{ij}$$

where $\phi > 0$ for loading and $\phi = 0$, for unloading and $E = 3K(1 - 2\nu)$. Assuming incompressibility ($\nu = 1/2$) in the above equation, simplifies enormously the total strain theory of plasticity and results in the good approximation

$$\epsilon'_{ij} = \phi\sigma'_{ij} - \phi\frac{\sigma_{kk}}{3}\delta_{ij}$$

Classical theories of plasticity

2. **Prandtl-Reuss** - In the incremental theory of plasticity, the total strain increment is decomposed into *elastic* (reversible) and *plastic* (irreversible) components

$$d\epsilon_{ij} = d\epsilon_{ij}^{(e)} + d\epsilon_{ij}^{(p)}$$

where, essentially speaking, the plastic component $d\epsilon_{ij}^{(p)}$ being determined from a *flow rule* as

$$d\epsilon_{ij}^{(p)} = d\lambda \cdot \partial f / \partial \sigma_{ij}, \text{ associative}$$

and the plastic constraint parameter $d\lambda$ being determined from the condition that during additional plastic flow, the actual stress point σ remains on the yield surface f. For the non-associative flow, the flow rule becomes

$$d\epsilon_{ij}^{(p)} = d\lambda \cdot \partial g / \partial \sigma_{ij}, \text{ non-associative}$$

where the plastic potential function g being close to the yield function f. Both potential functions are scalar valued.

Both potential functions f and g are scalar valued. The deviation from normality of the plastic increment for granular-type materials, results from 'plastic' volume increase at failure and is encoded into a *dilation* or more properly called *dilatation* angle into g related to the internal friction coefficient of he material. For metals, it's known that the plastic flow occurs with no appreciable volume change in normal pressure conditions (roughly, less than 10). It is this incompressibility of the plastic flow that results in the normality flow rule for the plastic deformation increment.

Failure criteria

In this course, we assume the plastic deformations to remain small --> *incremental plasticity*

What one means by *failure* is application and material dependent.

Generally speaking, here *yielding*, for metals, is associated with failure. So, the material fails when it yields. For other materials, engineering materials, failure should be clearly defined.

Classical plasticity

Elastic and plastic deformation

When loaded, a body deforms elastically. Upon full unloading, the body recovers its initial shape and volume. The elastic deformation is said to be *reversible*. Elasticity is then synonym for *reversibility* of deformations.

On the contrary, *plastic deformations*, are *irreversible*. This is the most visible feature of plasticity. In this case, in some parts of the body, being deforming by the loading, a combination of the resulting stresses reaches a material dependent threshold value leading to local yielding. Now, even when the load is completely removed, *permanent strains will remain*. These residual strains are called *plastic permanent strains*.

To model the time-independent *elasto-plastic* response of a material, one need to account, as sub-models, a set of three equations which *closes* the constitutive model:

1. **Yield criterion** - (myötö-ehto) delimits elastic and plastic regions
2. **Flow rule** - (myötö-sääntö) defines how plastic strains evolve
3. **Hardening rule** - (myötö-lujittumis-sääntö) defines how the yield criterion, itself, evolves with evolving of plastic strains.

Time-dependent *plasticity* can be simply modelled using various visco-plasticity formulations.

Failure criteria

Suppose we have a know stress state around a material point P expressed by the stress tensor $\boldsymbol{\sigma}$, or equivalently, its components $\sigma_{i,j}$. The question: *under what stress conditions, the material around the point P fails?* Here we assume that yielding leads to failure.

In general, the *failure criterion*, which should be written in terms of *stress invariants*, is usually expressed by oe of the equivalent conditions

$$
\begin{aligned}
f(\sigma_1, \sigma_2, \sigma_3) &= 0, \\
f(I_1, I_2, I_3) &= 0, \\
f(I_1, J_2, J_3) &= 0, \\
f(I_1, J_2, \cos(3\theta)) &= 0,
\end{aligned}
$$

<-- the failure surface has a clear geometrical representation

$$f(\sigma_1, \sigma_2, \sigma_2, k; \kappa) = 0$$

or also expressed in terms of stress invariants I_1, J_2 and J_3 instead of principle stresses. The threshold stress $k = k(\kappa)$ can depend on some internal parameter κ, like cumulated effective plastic strain of plastic work, is yield stress σ_y in uni-axial tension or τ_y yield shear stress in pure shear test.

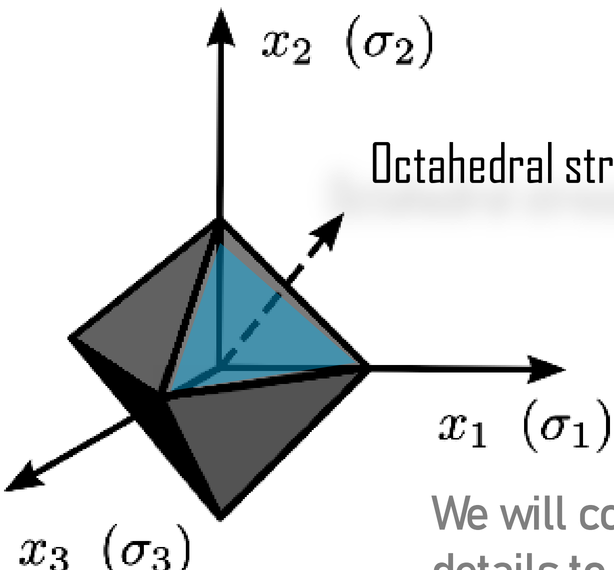
In the above, the two last set of invariants, are the most usually used in practice.

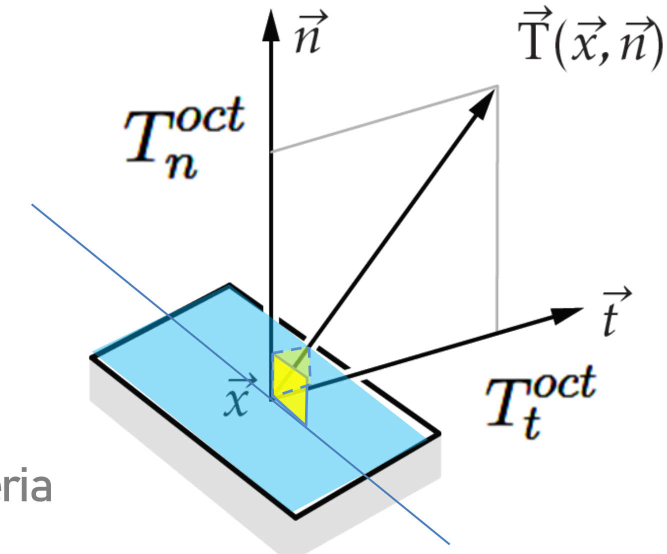
$$J_2 = \frac{1}{2} s_{ij} s_{ji} \; ; \quad \cos 3\theta = \frac{3\sqrt{3}}{2} \frac{J_3}{J_2^{3/2}} \; ; \quad J_3 = \frac{1}{3} s_{ij} s_{jk} s_{ki}$$

Failure criteria **example**: yielding of metals, for instance

$$T_n^{oct} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} = \frac{I_1}{3}$$

$$T_t^{oct} = \frac{1}{9}\left\{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right\} = -\frac{2}{3}J_2$$





We will come later back and in details to these two failure criteria

von Mises $\quad T_t^{oct} < \sigma_y$

Tresca $\quad \sup T_t = \dfrac{\sigma_1 - \sigma_3}{2} < \tau_y$

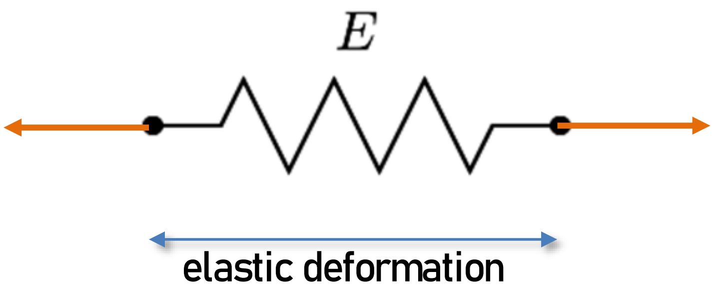
$\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$

threshold values for yielding in various situation (to be determined from tension and pure shear tests)

maximum shear stress

Basic rheological models for plasticity

Hookean element for elasticity

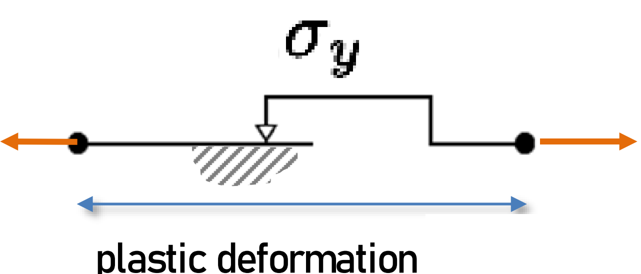


$$\sigma = E\varepsilon$$

Various plasticity and visco-plasticity material behaviours can be captured combining these basic rheological models

Friction element for plasticity

In the basic rheological friction model, the Coulomb's sliding friction coefficient μ corresponds to the threshold stress value at yield; the yield stress σ_y



$$\begin{cases} \dot\varepsilon = 0 & \text{if } |\sigma| < \sigma_y \\ \dot\varepsilon > 0 & \text{if } |\sigma| = \sigma_y \\ \dot\varepsilon < 0 & \text{if } |\sigma| = -\sigma_y \end{cases}$$

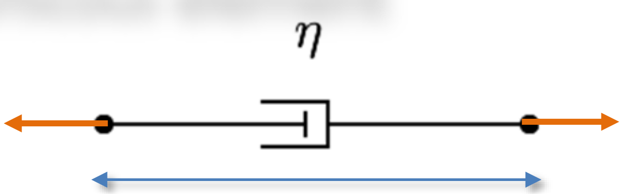
does not slide, the element is blocked

slides, fully developped friction

slides, fully developped friction

$$\sigma = \sigma_y \cdot \frac{\dot\epsilon}{|\dot\epsilon|}$$

Viscous element



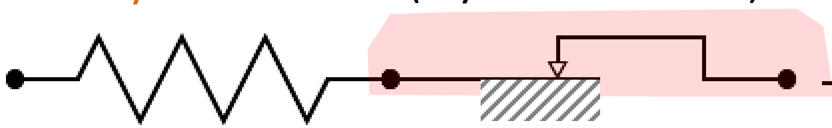
$$\sigma = \eta\dot\varepsilon$$

Rheological models for plasticity & visco-plasticity

Simple rheological models

Rate independent Plasticity

friction element
or kidding threshold block
(= yield threshold)

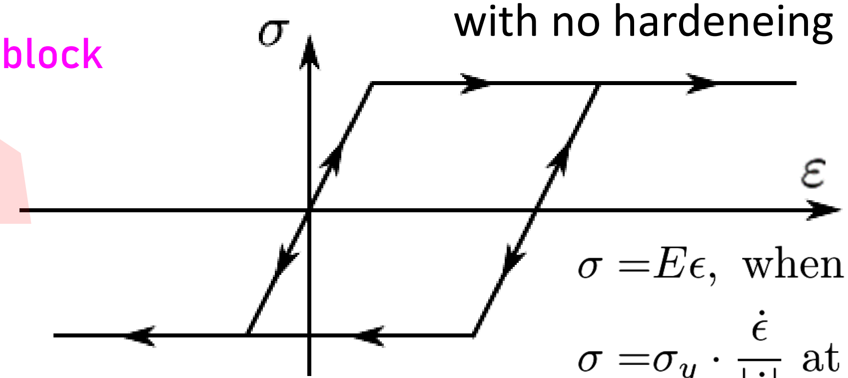


elastic deformation

plastic deformation

total deformation

with no hardeneing

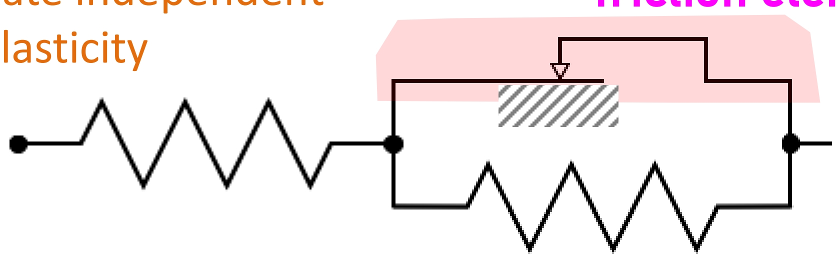


$$\sigma = E\epsilon, \text{ when } |\sigma| < \sigma_y$$

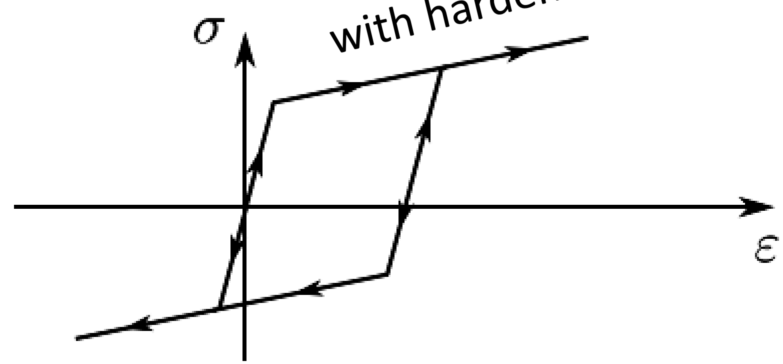
$$\sigma = \sigma_y \cdot \frac{\dot{\epsilon}}{|\dot{\epsilon}|} \text{ at yielding}$$

Rate independent Plasticity

friction element



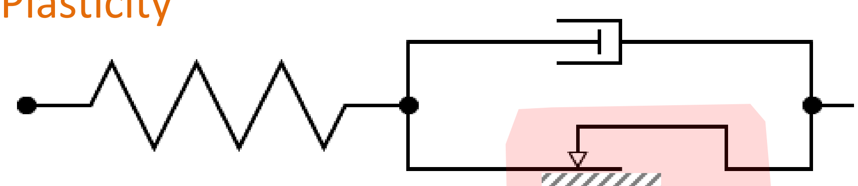
with hardeneing



ductile metals...

Rate dependent Plasticity

bingham model



visco-plasticity

friction element

visco-plastic deformation

asphalts, some pastes, 'pâtes', teeth paste, ...

$$\sigma = \begin{cases} E\varepsilon, & |\sigma| < \mu \equiv \sigma_Y \\ \mu + \eta\dot{\varepsilon}, & |\sigma| \geq \mu > 0 \end{cases} \qquad \mu \equiv \sigma_Y$$

Example from research of use of rheological models for visco-plasticity 1(2)

An elastic, plastic, viscous model for slow shear of a liquid foam

Philippe Marmottant and François Graner
Laboratoire de Spectrométrie Physique, CNRS-Université Grenoble I,

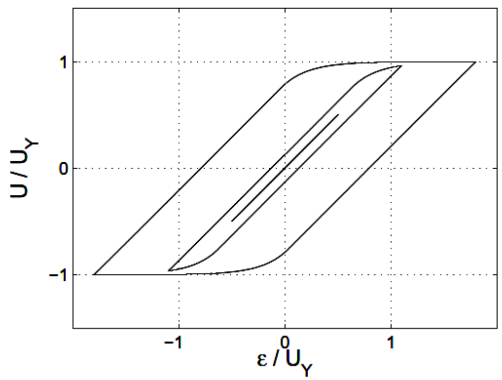
Article *in* The European Physical Journal E · September 2007
DOI: 10.1140/epje/i2006-10193-x · Source: PubMed

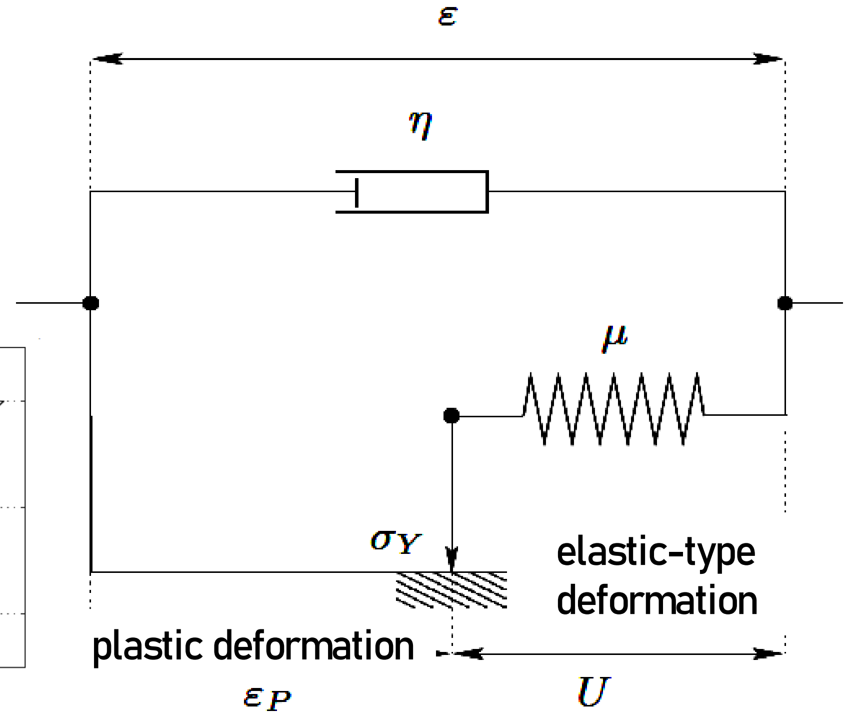
Various plasticity and visco-plasticity material behaviours can be captured combining these basic rheological models

a polymeric model

$$\sigma = \sigma^{\mathrm{el}} + \sigma^{\mathrm{vis}}$$
$$= \mu U + \eta \dot{\varepsilon}.$$

$$\dot{\varepsilon} = \frac{\mathrm{d}U}{\mathrm{d}t} + \dot{\varepsilon}_P.$$





We suggest a scalar model for deformation and flow of an amorphous material such as a foam or an emulsion. To describe elastic, plastic and viscous behaviours, we use three scalar variables: elastic deformation, plastic deformation rate and total deformation rate; and three material specific parameters: shear modulus, yield deformation and viscosity. We obtain equations valid for different types of deformations and flows slower than the relaxation rate towards mechanical equilibrium. In particular, they are valid both in transient or steady flow regimes, even at large elastic deformation. We discuss why viscosity can be relevant even in this slow shear (often called "quasi-static") limit. Predictions of the storage and loss moduli agree with the experimental literature, and explain with simple arguments the non-linear large amplitude trends.

Example from research of use of rheological models for visco-plasticity 2(2)

An elastic, plastic, viscous model for slow shear of a liquid foam

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a polymeric model

$$\sigma = \sigma^{el} + \sigma^{vis} = \mu U + \eta \dot{\varepsilon}.$$

Oscillating shear

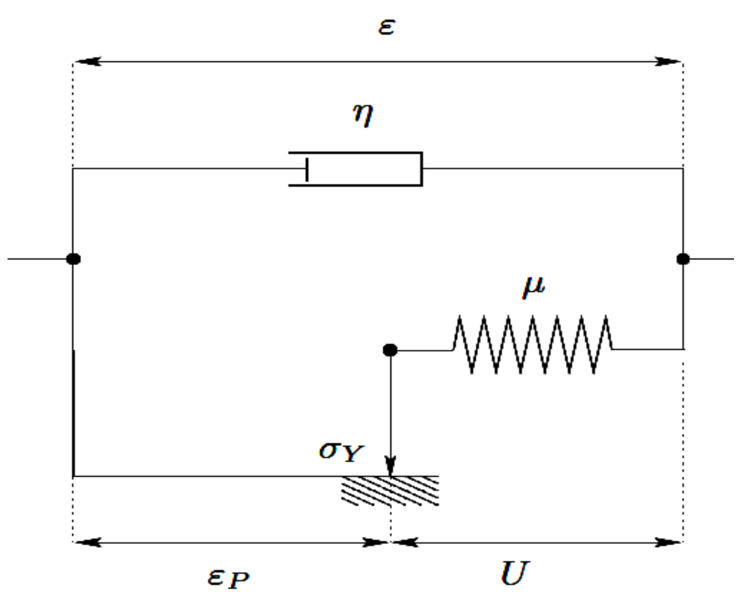
$$\varepsilon = \gamma \sin(\omega t)$$

Storage and loss moduli: predictions

$$\sigma^* = (G' + iG'')\varepsilon^*$$

storage

loss modulus



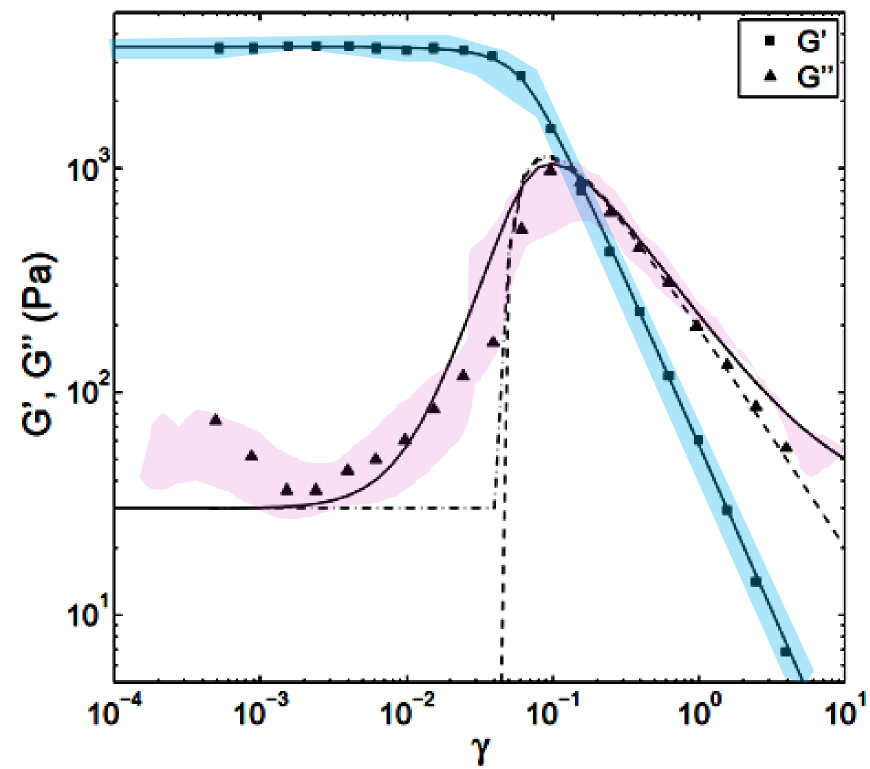


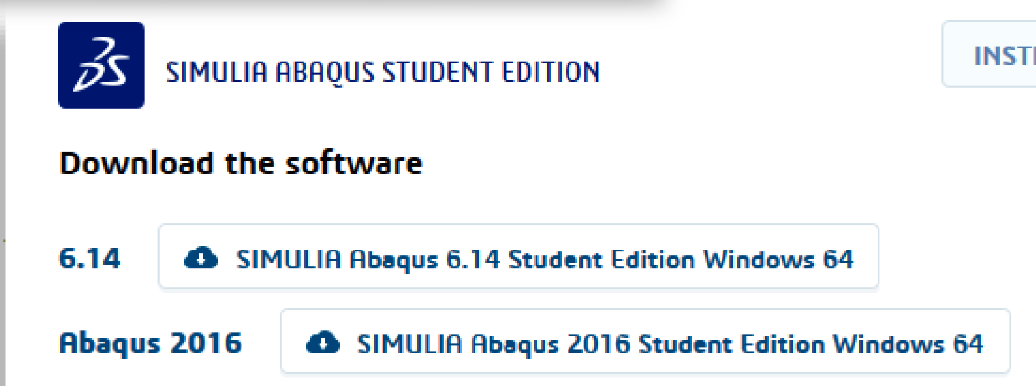
FIG. 9: Storage and loss moduli *versus* strain amplitude for a monodisperse emulsion. Symbols: experimental G' (circles) and G'' (triangles) in a close-packed emulsion (Fig. 1 of ref. [43]), fraction of the continuous phase 20%, droplet size 0.53 μm, oscillation pulsation $\omega = 1$ rad s^{-1}). Lines: models for G' (solid line), and for G'' with an abrupt transition (dashed line), with viscosity (dash-dotted line), with viscosity and a smooth yield function $h = (U/U_Y)^2$, $U_y = 0$ (solid line). Model parameters: shear modulus $\mu = 1.7\ 10^3$ Pa, yield deformation $U_Y = 0.045$, viscosity $\eta = 30$ Pa.s.

Motivation

How engineering Plasticity is 'written' in Abaqus, Ansys, Lusas, ...?

CIV-students should be able to read correctly the theory manuals and to exchange using exact terminology with experts!

Get Abaqus Free Student Version



Abaqus – theory Manuel … to be able to read it … and to understand the physics behind the material models you are using

https://www.3ds.com/products-services/simulia/products/abaqus/

ISOTROPIC ELASTO-PLASTICITY

The elasticity can be written in volumetric and deviatoric components as follows.
Volumetric:

$$p = -K\varepsilon_{vol}, \quad (4.3.\ldots$$

where

$$p = -\frac{1}{3}\,\mathrm{trace}(\boldsymbol{\sigma})$$

is the equivalent pressure stress.
Deviatoric:

$$\mathbf{S} = 2G\,\mathbf{e}^{el},$$

where $\mathbf{S}$ is the deviatoric stress,

$$\mathbf{S} = \boldsymbol{\sigma} + p\mathbf{I}.$$

The flow rule is

$$d\mathbf{e}^{pl} = d\bar{e}^{pl}\,\mathbf{n},$$

$$\mathbf{n} = \frac{3}{2}\frac{\mathbf{S}}{q},$$

$$q = \sqrt{\frac{3}{2}\mathbf{S}:\mathbf{S}},$$

…) equivalent plastic strain rate.
…uires that the material satisfy a uniaxial-stress plastic-…
…dependent, this is the yield condition:

$$q = \sigma^0,$$

…d is defined by the …

ABAQUS 6.14
THEORY GUIDE

SIMULIA ABAQUS STUDENT EDITION

INSTI…

Download the software

6.14 — SIMULIA Abaqus 6.14 Student Edition Windows 64

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GET Abaqus Free Student Version

$$f(\boldsymbol{\sigma}) = q,$$

$$q = \sqrt{\frac{3}{2}\mathbf{S}:\mathbf{S}},$$

$$\mathbf{S} = \boldsymbol{\sigma} - \frac{1}{3}\mathrm{trace}(\boldsymbol{\sigma})\mathbf{I} = \boldsymbol{\sigma} - \frac{1}{3}\mathbf{II}:(\boldsymbol{\sigma}).$$

…circle in the plane normal to the hydrostatic axis in…

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} = \frac{1}{q}\frac{3}{2}\mathbf{S},$$

$$\frac{\partial^2 f}{\partial\boldsymbol{\sigma}\partial\boldsymbol{\sigma}} = \frac{1}{q}\left(\frac{3}{2}\mathfrak{I} - \frac{1}{2}\mathbf{II} - \frac{\partial f}{\partial\boldsymbol{\sigma}}\frac{\partial f}{\partial\boldsymbol{\sigma}}\right)$$

…urth-order …or.
…unction … of the Mises function to allow…

$$\sigma_e^2 = \frac{3}{2}s_{ij}:s_{ij} = \frac{3}{2}\mathbf{s}:\mathbf{s} \equiv 3J_2$$

Lusas

http://www.lusas.com/

LUSAS

Engineering analysis and design software

- Model
  - Elasto-Plastic Models
    - General Considerations
    - Integration of Plasticity Models
    - Temperature Dependent Properties
    - Plasticity Models
  - Plasticity Models For Beams And Shells
  - Interface Models
  - Damage
  - Viscoelastic Models
  - Multi-Crack Concrete Model
  - Resin Cure Model
  - Shrinkage
  - Concrete Creep and Shrinkage Models
  - Generic Polymer Material
  - Joint Models
  - Field Models
  - Composite Models
  - Rubber Models

Engineering analysis and design software

Any structure. One solu

- Resin Cure Model
- Shrinkage
- Concrete Creep and Shrinkage Models
- Generic Polymer Material
- Joint Models
- Field Models
- Composite Models
- Rubber Models
- Volumetric Crushing Model
- Delamination Interface Model
- Load And Boundary Conditions
- Post-Processing Facilities
  - Nodal Extrapolation
  - Wood-Armer Reinforcement
  - Strain Energy And Plastic Work

Material models – an example

Theory Manual Volume 1

the plastic strains are integrated according to a strict interpretation of the flow rules governing their evolution. Recent developments in numerical analysis have re-interpreted the classic laws in a search for greater numerical efficiency and have led to the concept of "consistency" of formulation. These methods have the advantage of improved stability for large load steps and quadratic convergence in the global Newton-Raphson iterations. Within the framework of the consistent formulation, time dependent inelastic straining has also been modelled.

The models available within LUSAS are as follows:

Plasticity in Lusas

The models available within LUSAS are as follows:

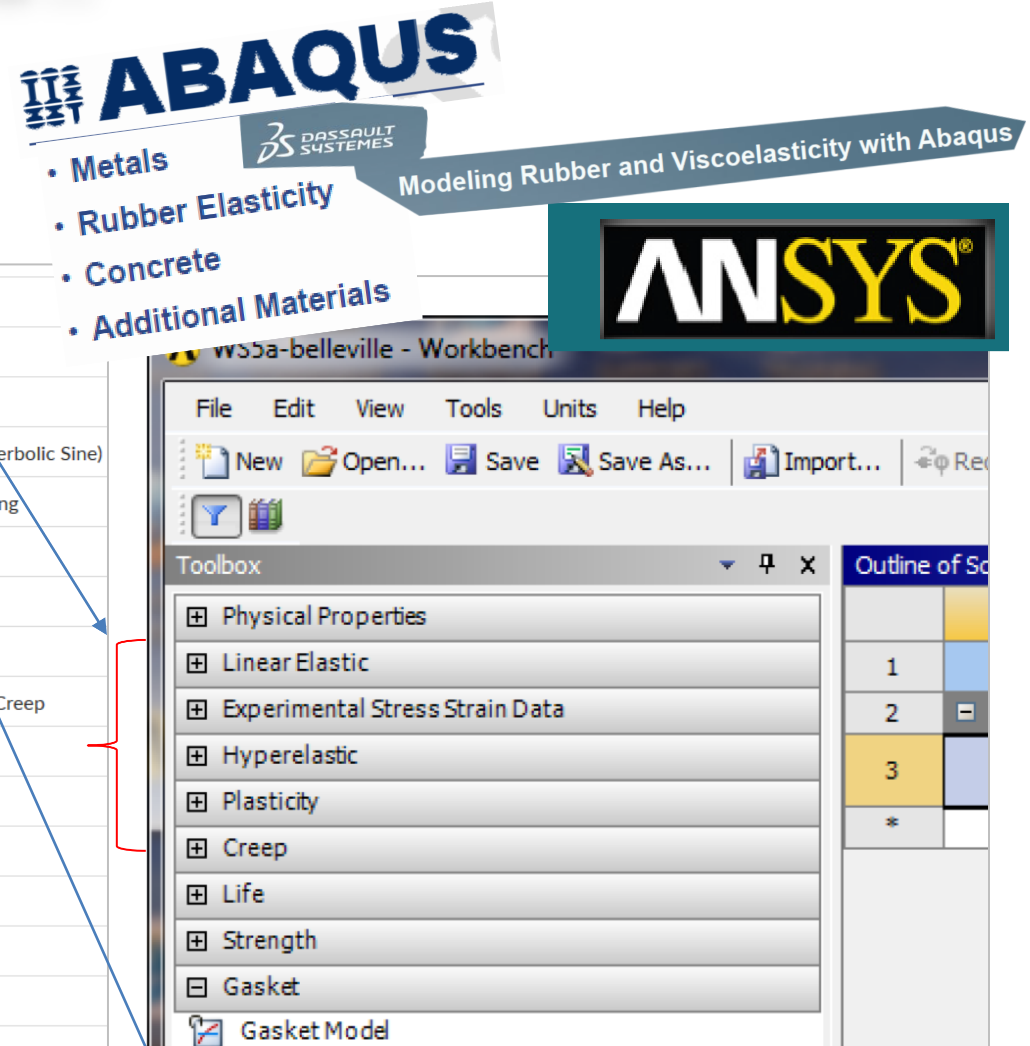
- Continuum formulation
  - von Mises, Tresca, Mohr-Coulomb and Drucker-Prager
  - stress resultant models for beams and shells
  - models for sliding interfaces
- Consistent formulation
  - Rate independent formulations, e.g. Hoffman, modified von Mises, Hill, von Mises
  - Rate dependent formulations, e.g. Uniaxial creep laws generalised to multiaxial states

- the constitutive response is independent of the rate of deformation or straining,
- the elastic response is not affected by the plastic deformation,

Material models: the user – engineers - **should know the *basics principles and basic physics* behind them …**

COMSOL

Hyperelastic	Plasticity and Viscoplasticity	Creep
Arruda-Boyce	Anand	Coble
Blatz-Ko	Kinematic Plastic Hardening	Deviatoric
Damping	Isotropic Plastic Hardening	Garofalo (Hyperbolic Sine)
Gao	Large Strain Plasticity	Nabarro-Herring
Gent	Orthotropic Hill Plasticity	Norton
Large Strain Plasticity	Perfectly Plastic Hardening	Norton-Bailey
Mooney-Rivlin (two, five and nine parameters)	Thermal Expansion	Potential
Murnaghan	Tresca Yield Criterion	User-defined Creep
Neo-Hookean	User-defined Plasticity	Volumetric
Ogden	von Mises Yield Criterion	Weertman
St Venant-Kirchhoff		
Storakers		
Thermal Expansion		
User-defined Hyperelastic Material		
Varga		
Yeoh		



… in order to design safe structures

Materials means here → Engineering Materials

A!jatuksia

L'Intelligence, ce n'est pas seulement la résolution de problème, c'est aussi la capacité à bien gérer les exceptions*.

Intelligence is not only problem solving, it is also the ability to cope with unexpected circumstances*, too.

Koehalli – rakennusosasto, February 2017
CIV's main experimental hall

DI, Antti Valkonen, end of 2016 or 2017! He wrote:

…

"Minulla oli helpottunut olo tutkinnon suorittamisen jälkeen . . . Tunne, että olin saanut lukutaitoon verrattavan kyvyn, jonka jälkeen oppiminen voi alkaa...."

....

*Find good solutions for ***unexpected*** problems in real time.

Cf. **Appendix 2**

for

Recommended additional literature

Historical notes on Plasticity**

Classical work and applications of *plasticity concepts* as *failure criteria* and *flow rule*, as all civil engineering students should remember, goes back to soil mechanics – **Coulomb** (1773) and **Rankine** (1853), in the problem of evaluating *earth pressure on retaining structures* giving emergence to concepts as limit equilibrium and limit analysis.

Experimental observations of plastic deformation and metal forming by **Coulomb** and **Tresca** in France. **Tresca** (1864-1872) introduced the first yield condition. Some fundamental investigations into plasticity in Germany under **Prandtl**.

The foundation of Classical Theory of Plasticity goes back to the fundamental work of **Hill** (1950) and **Koiter** (1960).

P. W. **Bridgman** was an American physicist who won the 1946 Nobel Prize in Physics for his work on *the physics of high pressures*. Some of his experimental results (as a side product) showed that plastic flow of metals under hydrostatic pressures occurs without plastic volume change. This result influenced the work of **Hill** (1950) in developing the fundaments of classical plasticity starting from this incompressibility of plastic flow observation.

Got interested, read more from:

Kozo Osakada
History of plasticity and metal forming analysis

Journal of Materials Processing Technology 210 (2010) 1436–1454

**This historical note is not from the cited reference.

This topic we had during the 1st lecture → skip

Cf. **Appendix 1**

Stress Invariants (Recall)

Recall

Elements of solid mechanics

Limited to the necessary minimum to follow this course

Stress Invariants

It is Natural to express yield conditions in a **coordinate free form** because they are independent of tthe observer … for this reason stress and strain Invariants are used.

Principal Stresses and Principal Planes

Why?

- Determination of **maximum normal stresses and shear stresses** at a point is of considerable interest

Why?

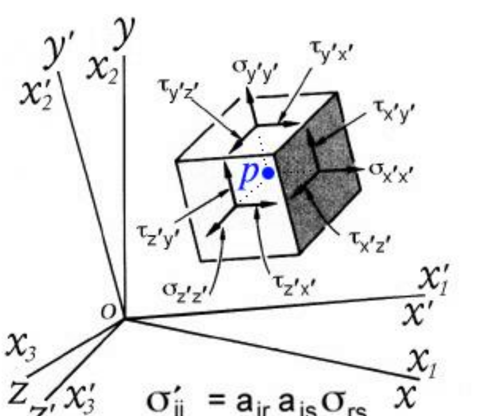
- Many *failure criteria* are expressed using **stress**/strain **invariants**

Stress tensor (or matrix)

$$(\boldsymbol{\sigma} - \lambda \mathbf{I}) \cdot \hat{\mathbf{n}} = 0$$

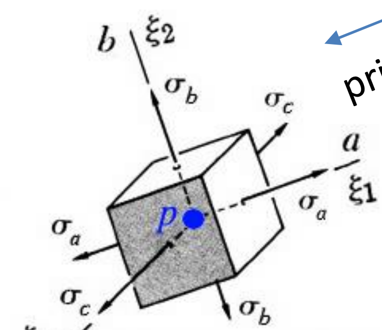
principal stresses **principal directions**

The principal axes: solely **normal stresses** and **no shear stresses** appear in sections perpendicular to these axes



$\sigma'_{ij} = a_{ir}\, a_{js}\, \sigma_{rs}$

General rotational stress transformation using tensor notation.

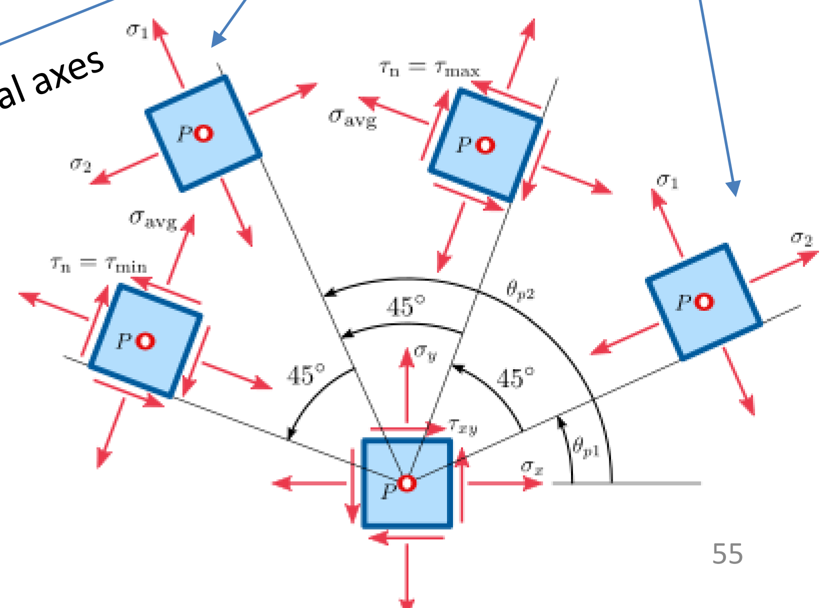


Principal 3D stress state: Special rotational orientation

$\xi_i = x'_i = a_{ir}\, x_r$

where all shear stresses go to zero

principal axes



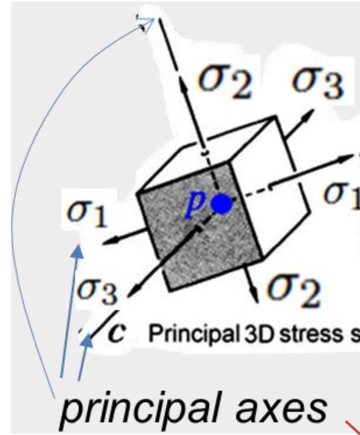
Principal Stresses and Principal Planes

The same holds for strains

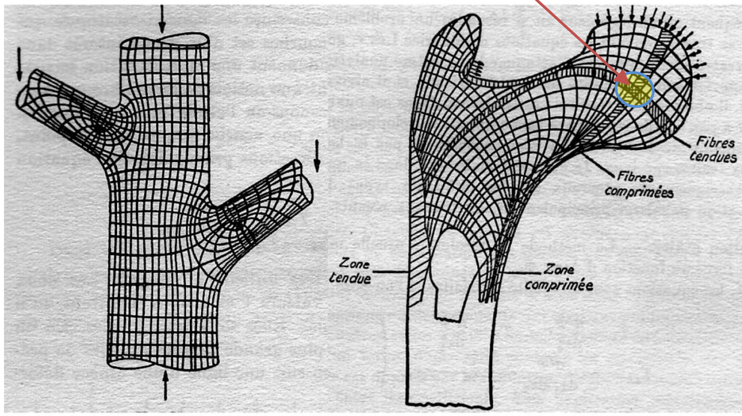
For **any arbitrary state of stress**, we can find a **set of orthogonal planes** on **which only normal stresses act** and **the shearing stresses are zero**

- Called Principal Planes and the normal stresses acting on these planes are Principal Stresses denoted as
- They are ordered such that : $\boxed{\sigma_1 > \sigma_2 > \sigma_3}$
- **Maximum shear stresses** are in planes forming **an angle of 45 deg. with Principal Planes**

$$\boxed{\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2}}$$



$$\sigma = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$



· Isostatiques des neuds du bois et d'une tête de fémur

Stress Invariants 1(2)

Most tensors used in engineering are symmetric 3×3. For this case the invariants can be calculated as:

$$\mathrm{I}_A = \mathrm{tr}(\mathbf{A})$$
$$\mathrm{II}_A = \frac{1}{2}\left((\mathrm{tr}\mathbf{A})^2 - \mathrm{tr}(\mathbf{AA})\right)$$
$$\mathrm{III}_A = \det(\mathbf{A})$$

Invariants

Stress tensor (or matrix)

$$(\boldsymbol{\sigma} - \lambda \mathbf{I}) \cdot \hat{\mathbf{n}} = \mathbf{0} \quad \Rightarrow \quad \sigma^3 - I_\sigma \sigma^2 - II_\sigma \sigma - III_\sigma = 0$$

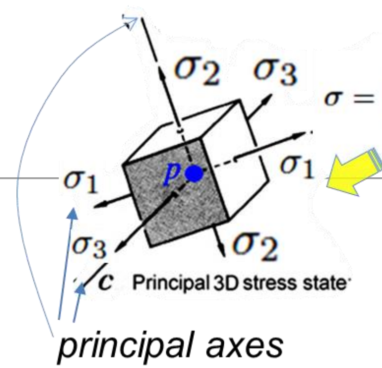
principal directions

principal stresses

Solutions give the principal stresses $\sigma_1, \sigma_2, \sigma_3$

Each principal stress corresponds to a principal direction

$$\sigma = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$



Also denoted as: I_1, I_2, I_3

I_1: $$I_\sigma = \sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33} \ ,$$

I_2: $$\begin{aligned} II_\sigma &= (\sigma_{ij}\sigma_{ij} - \sigma_{ii}\sigma_{jj})/2 \\ &= -(\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11}) + \sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2 \ , \end{aligned}$$

I_3: $$III_\sigma = \det(\sigma_{ij}) = \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{vmatrix} \ .$$

The same formula are valid for strain invariants too

Stress Invariants in terms of principle stresses

$$\boxed{(\boldsymbol{\sigma} - \lambda \mathbf{I}) \cdot \hat{\mathbf{n}} = \mathbf{0}} \Rightarrow \boxed{\sigma^3 - I_\sigma \sigma^2 - II_\sigma \sigma - III_\sigma = 0}$$

principal directions

principal stresses

Solutions give the principal stresses $\sigma_1, \sigma_2, \sigma_3$
Each principal stress corresponds to a principal direction

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$

Also denoted as: I_1, I_2, I_3

$$I_1 \equiv \mathrm{tr}(\boldsymbol{\sigma}) = \sigma_1 + \sigma_2 + \sigma_3$$

$$I_2 \equiv II_\sigma = \frac{1}{2}\left[\mathrm{tr}(\boldsymbol{\sigma})^2 - \mathrm{tr}(\boldsymbol{\sigma}^2)\right] \Rightarrow$$

$$\boxed{\begin{aligned} I_1 &= \sigma_1 + \sigma_2 + \sigma_3 \\ I_2 &= \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \\ I_3 &= \sigma_1\sigma_2\sigma_3 \end{aligned}}$$

$$I_3 \equiv III_\sigma = \det(\boldsymbol{\sigma})$$

Most tensors used in engineering are symmetric 3×3.
For this case the invariants can be calculated as:

$$\mathrm{I}_A = \mathrm{tr}(\mathbf{A})$$

$$\mathrm{II}_A = \frac{1}{2}\left((\mathrm{tr}\mathbf{A})^2 - \mathrm{tr}(\mathbf{A}\mathbf{A})\right)$$

$$\mathrm{III}_A = \det(\mathbf{A})$$

Invariant - Invariantti

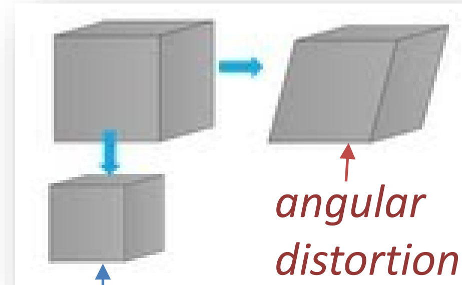
Deviatoric stress

Jännitysdeviaattori

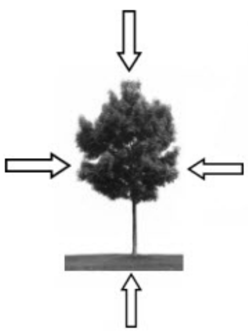
$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix} + \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix}$$

- additive decomposition of stress tensor – very useful

$$\sigma_{ij} = \frac{\sigma_{kk}}{3}\delta_{ij} + s_{ij} \qquad \text{or} \qquad \boldsymbol{\sigma} = \sigma_m \boldsymbol{I} + \boldsymbol{s} \;.$$

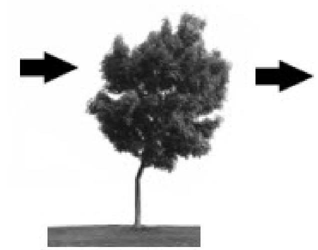


volume change



Mean normal stress $-p \equiv \sigma_m$ or also called ***hydrostatic stress state***

Deviatoric stress $\mathbf{s}$ characterizes deviation of the stress state from a hydrostatic state

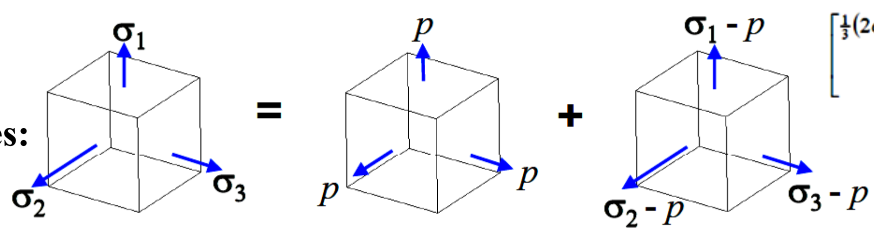


$$\boldsymbol{\sigma} \quad = \quad -p\mathbf{I} \quad + \quad \mathbf{s}$$

$$\sigma_m = \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3}$$

$$\begin{bmatrix} \frac{1}{3}(2\sigma_{11} - \sigma_{22} - \sigma_{33}) & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \frac{1}{3}(2\sigma_{22} - \sigma_{11} - \sigma_{33}) & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \frac{1}{3}(2\sigma_{33} - \sigma_{11} - \sigma_{22}) \end{bmatrix}$$

Expressed in principle stresses:



: $\sigma_1 \neq \sigma_2 \neq \sigma_3$)

hydrostatic stress
volume change only

deviatoric stress *- jännitysdeviaattori*
angular distortion only

Stress Invariants 2(2)

Also denoted as: J_1, J_2, J_3

$$J_1 = 0,$$

$$J_2 = \frac{1}{3}(I_1^2 - 3I_2),$$

$$J_3 = \frac{1}{27}(2I_1^3 - 9I_1 I_2 + 27 I_3).$$

Invariants of the **deviatoric** stress:

J_1: $$I_s = 0\ ,$$

$\frac{1}{2}\mathrm{tr}(\boldsymbol{s}^2)$

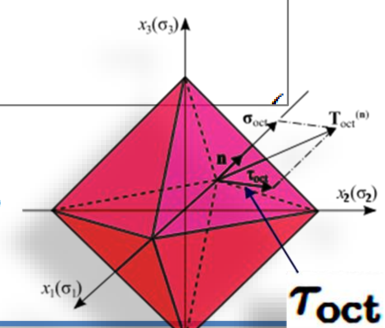
J_2: $$II_s = \frac{1}{2} s_{ij} s_{ij} = \frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$$

$$= \frac{1}{6}\left[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2\right] + \sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2$$

J_3: $$III_s = \frac{1}{3} s_{ij} s_{jk} s_{ki}$$

$\frac{1}{3}\mathrm{tr}(\boldsymbol{s}^3)$

$$\tau_{oct} = \sqrt{\frac{2}{3} J_2} \longleftrightarrow II_s = \frac{3}{2}\tau_{\mathrm{oct}}^2.$$

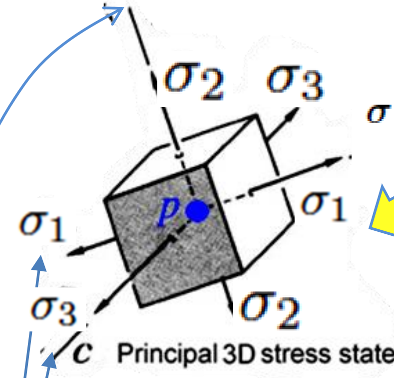


- For instance, the *equivalent stress* (von Mises stress) – *vertailujännitys* - $\tau_{oct} = \sqrt{\frac{2}{3} J_2} \leftrightarrow$ commonly used in solid mechanics is .... = $\sigma_e = \sqrt{3J_2}$ (which is up to a constant coefficient a weighted maximum shear stress on the octahedral planes)

$$\sigma_e \propto \tau_{oct}$$

Maximum shear stress

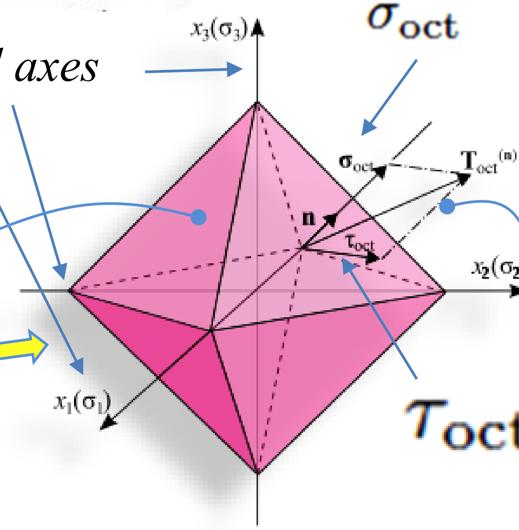
- ***Extreme shear*** stresses appear in sections with normal is perpendicular to one *principal axis* and forms ***angles of 45 deg.*** with the remaining two axes



$$\sigma = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$

Octopod

principal axes



σ_{oct}

τ_{oct}

direction cosines = $|1/\sqrt{3}|$

Octahedral stresses

Octa = eight = 8; we have 8 such planes

- Normal and shear stress in cross sections whose normal forms an equal angle with all three (3) principal axes:

$$\begin{cases} \sigma_{\text{oct}} = \dfrac{\sigma_1+\sigma_2+\sigma_3}{3} = \dfrac{\sigma_{ii}}{3} = \dfrac{I_\sigma}{3}\,, \\ \tau_{\text{oct}} = \dfrac{1}{3}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}\,. \end{cases}$$

Mean normal stress $-p \equiv \sigma_m$ or also called ***hydrostatic stress state***

$$\sigma_n = t \cdot n = t_i n_i = \sigma_{ij} n_i n_j .$$

$$n \equiv n_{\text{okt}} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\sigma_e = \sqrt{\frac{3}{2}} \cdot \tau_{\text{oct}} \approx 1.2\tau_{\text{oct}}$$

Octahedral shear stress component

$$\tau_{oct} = \sqrt{\frac{2}{3} J_2} \leftrightarrow \sigma_e = \sqrt{3J_2}$$

Equivalent stress or von Mises stress

Strain invariants

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \,. \qquad \boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix}$$

$$(\partial u_i / \partial X_j \ll 1)$$

Engineering strains

Same formula as for the stress tensor:

$$I_\varepsilon,\ II_\varepsilon,\ III_\varepsilon$$

$$I_\varepsilon = \varepsilon_V = \varepsilon_{kk} = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \,.$$

volumetric strain (relative volume change)

decomposition: $$\varepsilon_{ij} = \frac{\varepsilon_{kk}}{3}\delta_{ij} + e_{ij} \qquad \text{or} \qquad \boldsymbol{\varepsilon} = \frac{\varepsilon_V}{3}\boldsymbol{I} + \boldsymbol{e}$$

distortion = deviator

second invariant of the deviator: $$II_e = \frac{1}{2} e_{ij} e_{ij} = \frac{1}{6}\left[(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2\right] .$$

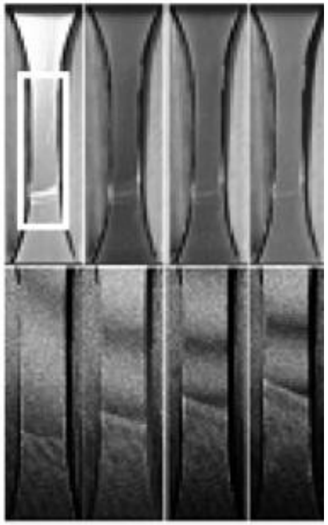
Classical Failure Hypotheses

Lujuushypoteesit

Here we mean: Failure hypothesis of MATERIALS
– materiaalien murtohypoteesit

Reference:
TECHNICAL BRIEFS
An Optical Interferometric Band as an Indicator of Plastic Deformation Front
Sanichiro Yoshida et al. *J. Appl. Mech* 72(5), 792-794 (Feb 07, 2005) (3 pages)

Lüders bands



Failure of Structures

Lämmittelyksi…

Mostly structural failure even though there is also material failure too. Often we cannot dissociate the two

This is not the scope of our course!

We are concerned in

Failure of Engineering Materials










Failure of Structures

Mostly structural failure even though there is also material failure too. Often we cannot dissociate the two

The Best, 10 points goe to ...

Slope Stability Failure

(got interrested? Take corses from soil mechanics, geo-engineering, ...)

Note that, the building is over dimensioned (ylimoitettu) since it survived such serious ...

What is Failure?

Types of failures

Failure Envelope
&
Failure Criteria

The three ingredients of engineering plasticity theory

- Yielding criteria myötöehto
- Flow rule myötösääntö
- Hardening rule lujittumisehto

$$\sigma_e = \sqrt{3J_2}$$

$$J_2 = \frac{1}{2} s_{ij} s_{ij} = \frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$$

What is Failure? Types of failure

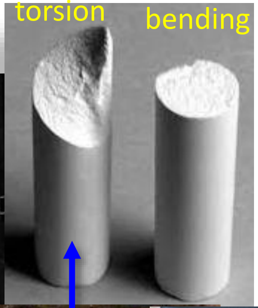
- approach : Macroscopic & Phenomenological
- only **Material failure** is considered

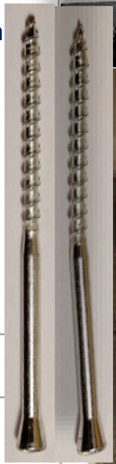
Materiaalin vaurioehdot tai -hypoteesit

Failure hypotheses

Lujuushypoteesit

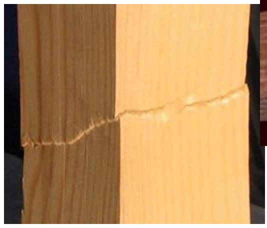
- ***Strength or failure hypotheses*** *are quantified criteria about the conditions* under which a *material fails* consequently to action of external loads
- ***Lujuushypoteesi*** vastaa kvantitatiivisesti (siis kaavalla) kysymykseen: ”Missä (jännitys-)olosuhteissa kappaleen lujuus ylitetään?”
- Their basis are experiments conducted under specific and often simple loading conditions





Failure criteria

are functions - in stress or strain space - **which separate "failed" states from "unfailed" states**

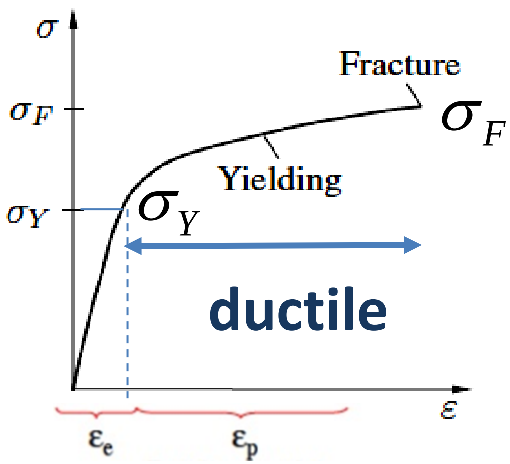


Soil Failure (slope stability)

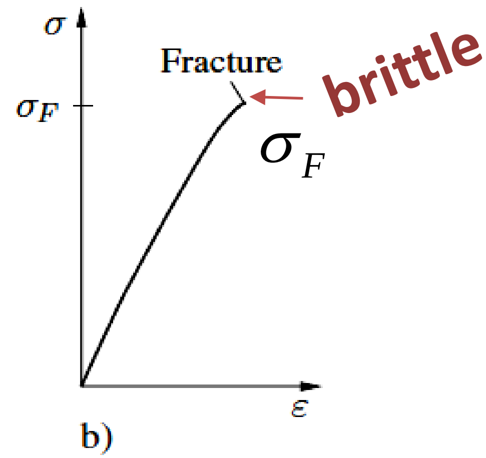
Ductile *versus* Brittle material 1(2)

The failure of a material is classified into:

ductile failure - *yield*



brittle failure - *fracture*



Ductile material: has a well defined **yield point** – ***ductile failure*** on yielding (**plasticity**)

Brittle material: has a no yield point & *sudden failure – brittle failure*

The ultimate stress at fracture will be reached only after sufficiently large inelastic deformations – *thus people have time to escape safely*

No significant inelastic deformations occur prior to Fracture – thus people doe not gat prior signs before failure

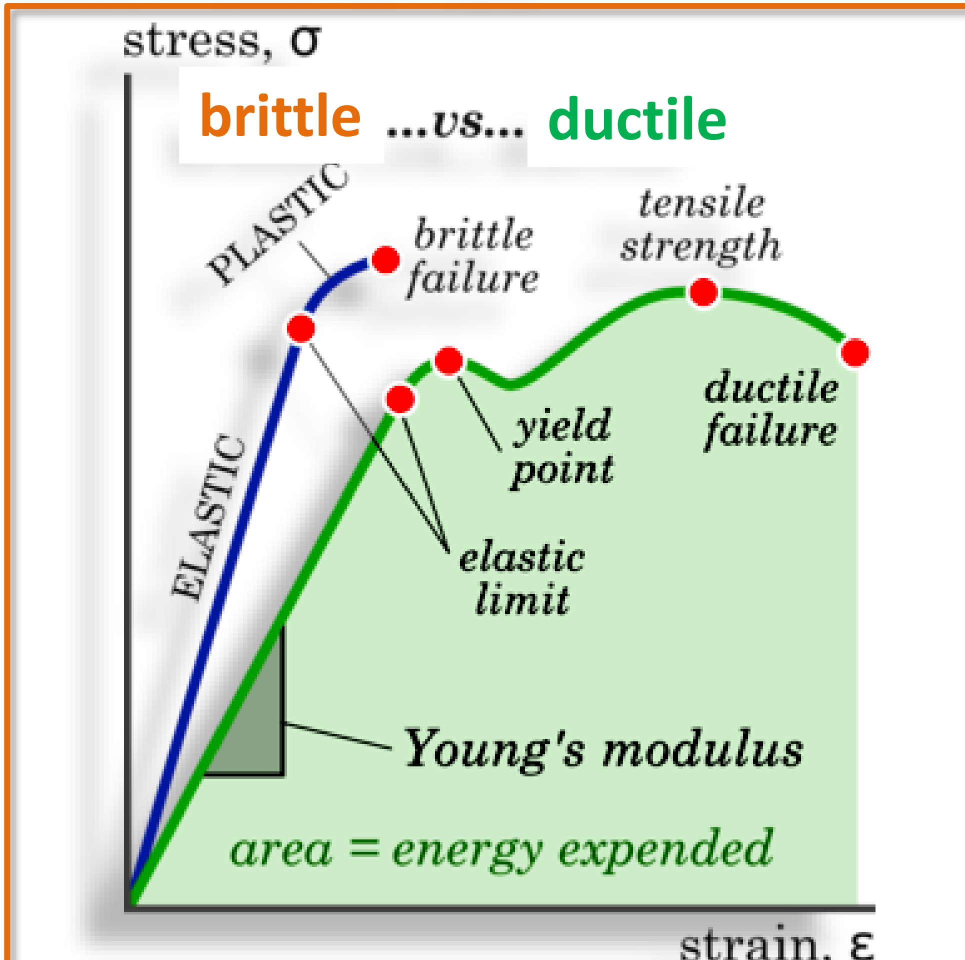
Ductile *versus* Brittle material 2(2)

Ductile failure

yield

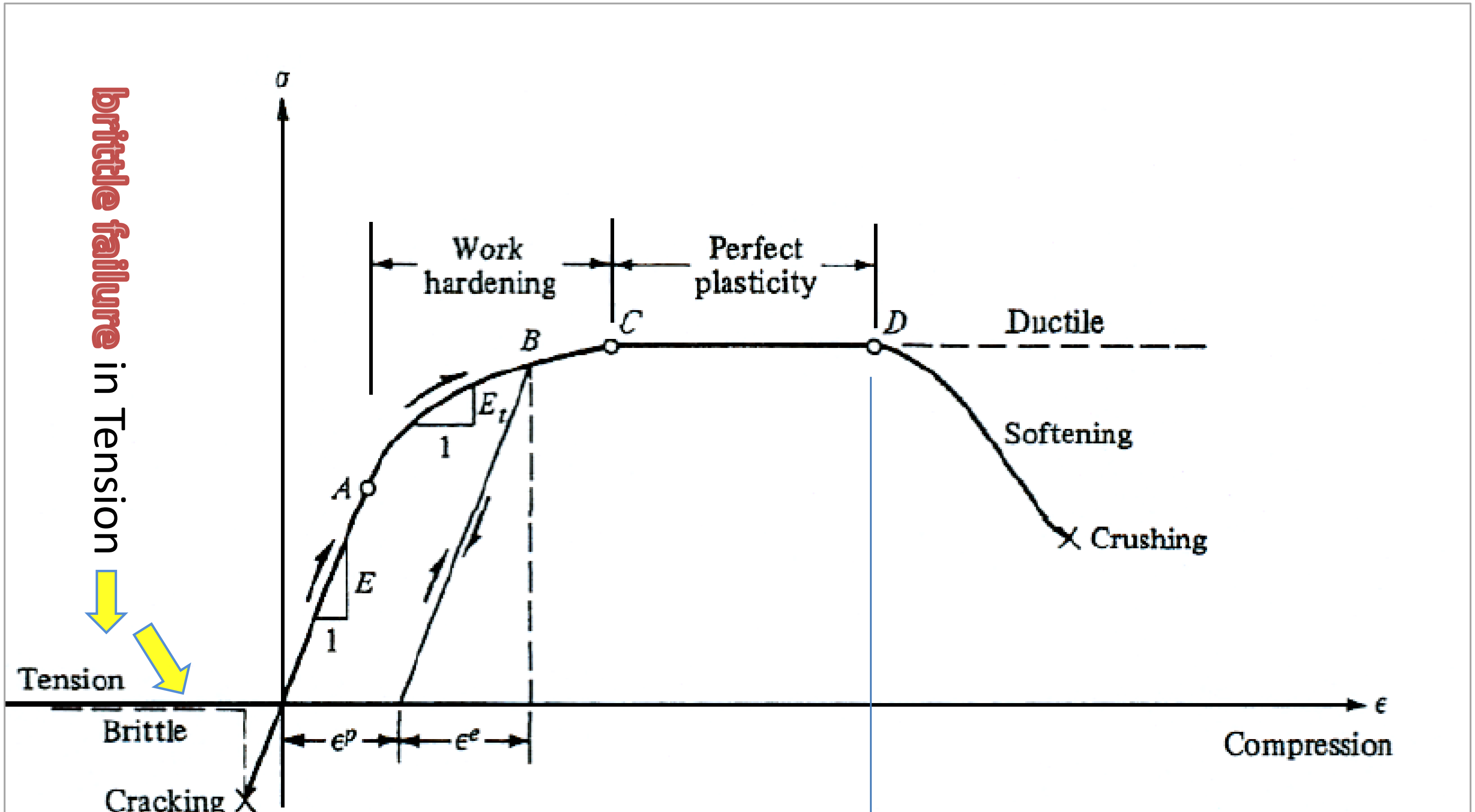
Brittle failure

fracture



Ductile material: has a well defined **yield point** – ***ductile failure*** on yielding (**plasticity**)	Brittle material: has a no yield point & *sudden failure – brittle failure*
The ultimate stress at fracture will be reached only after sufficiently large inelastic deformations – *thus people have time to escape safely*	*No significant inelastic deformations occur prior to Fracture – thus people doe not gat prior signs before failure*

Ductile *versus* Brittle material 2(2)



Uniaxial stress-strain curve, pre- and post failure regimes, (Chen, 1982)

Tension

Fragility

Compression

Ductility

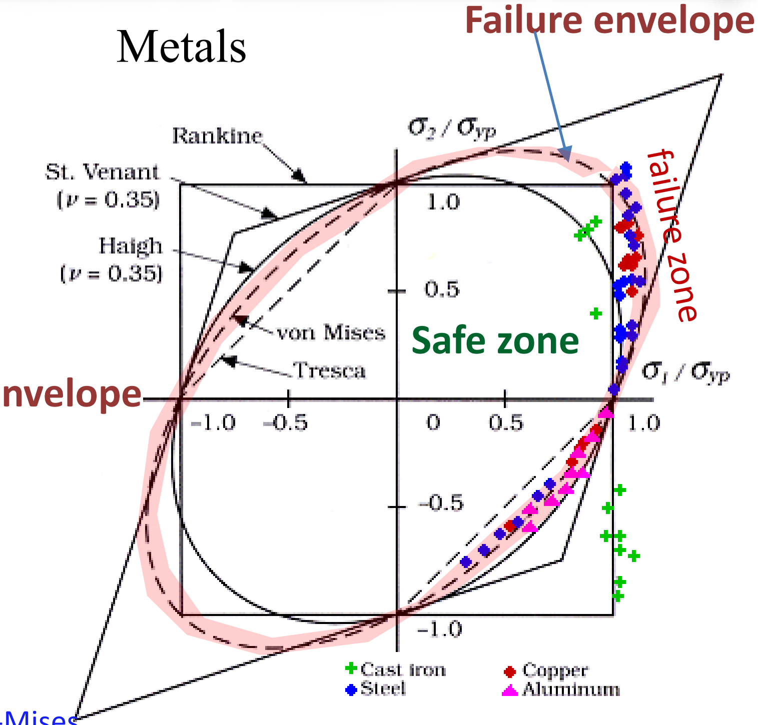
Strength Hypotheses – experimental evidence of existance of limiting failure surface

Failure criteria are functions in stress or strain space which separate "failed" states from "unfailed" states

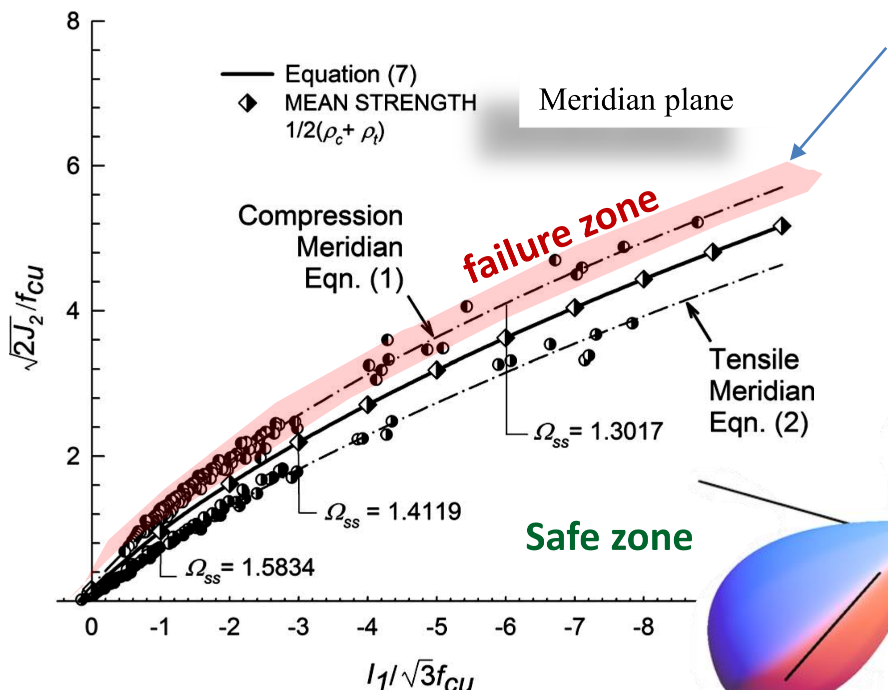
$$F(\sigma_{ij}) = 0$$

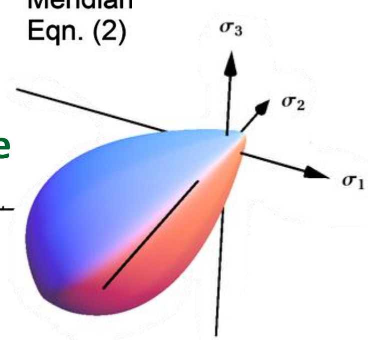
Failed – on the boundaries

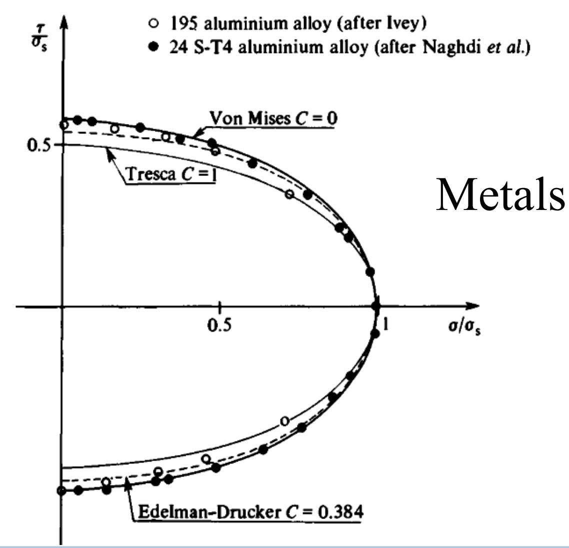
Unfailed – inside the envelope



Experimental: Mean concrete strength from CONCRETE compression and tension data



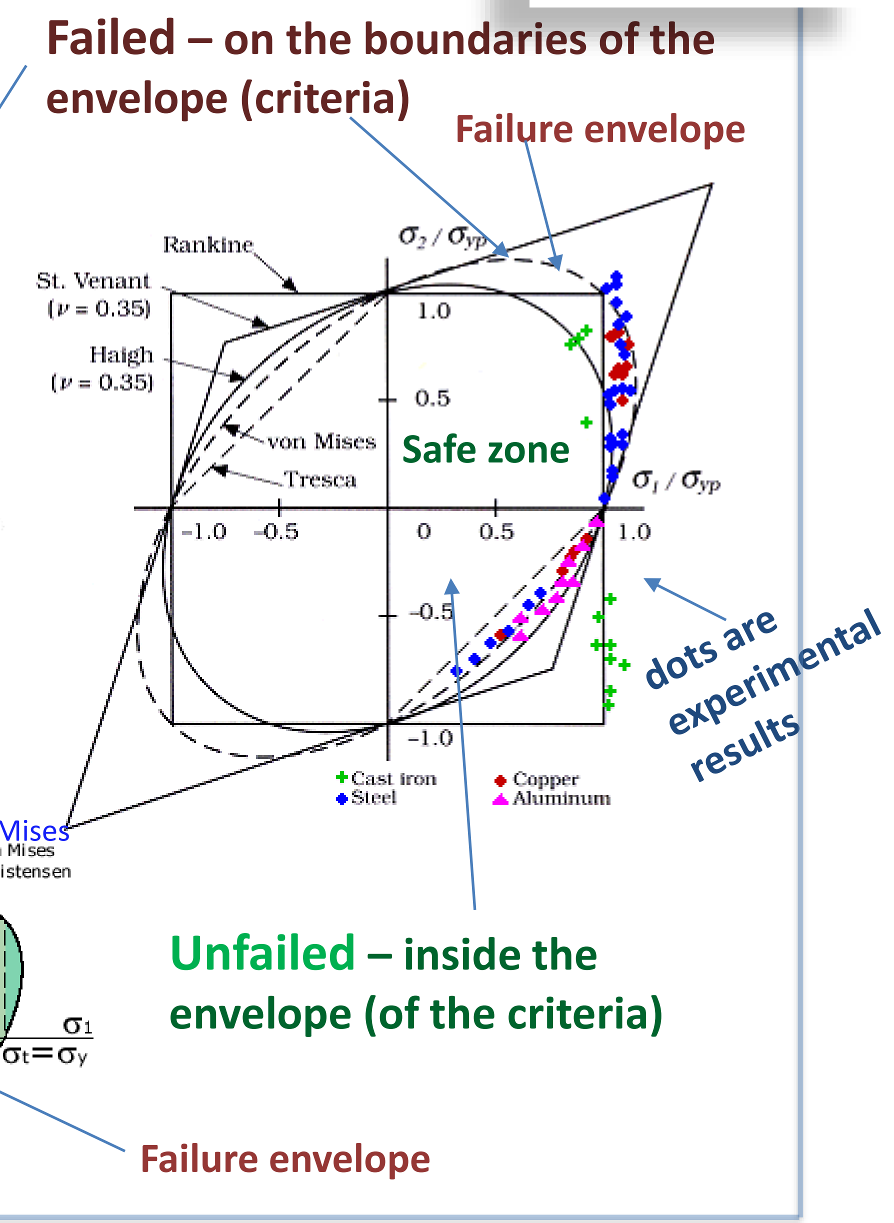


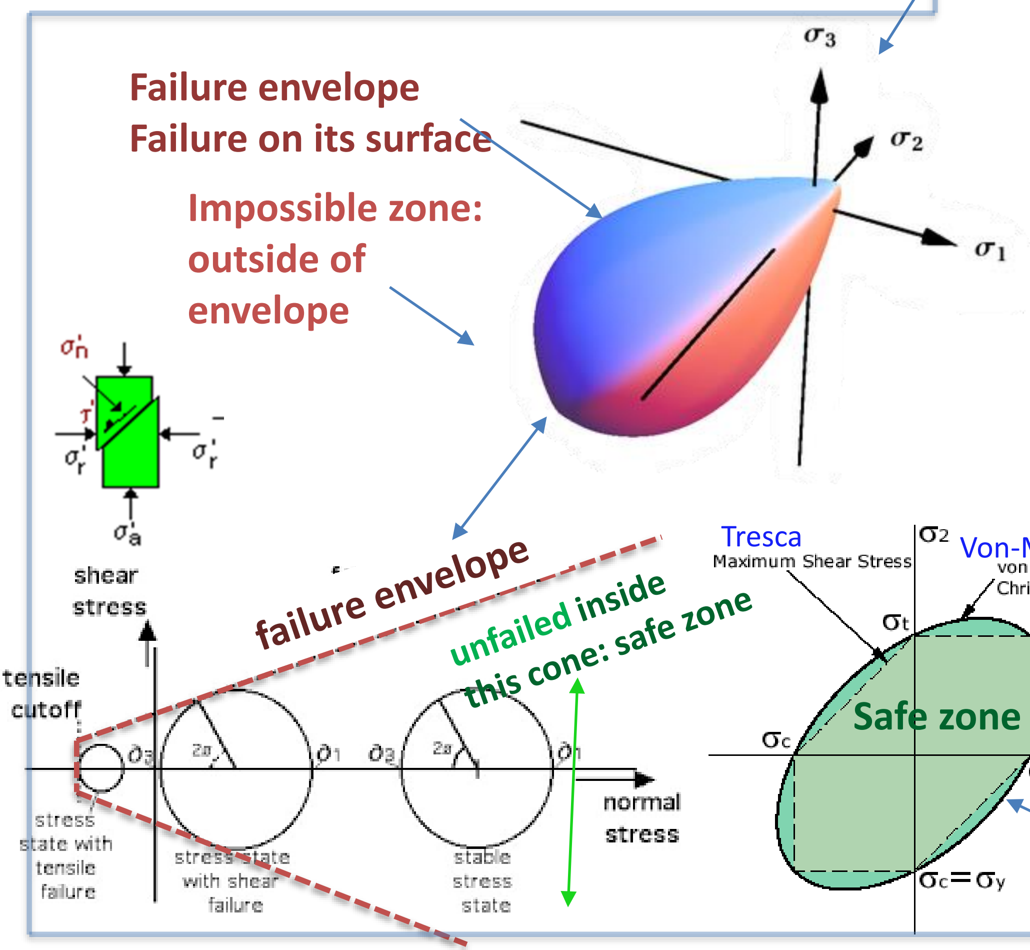


Strength Hypotheses

$$F(\sigma_{ij}) = 0$$

Failure criteria **are functions in stress or strain space which separate "failed" states from "unfailed" states**





Strength Hypotheses

Failure criteria **are functions in stress or strain space which separate "failed" states from "unfailed" states**

Failure criteria:

$$F(\sigma_{ij}) = 0$$

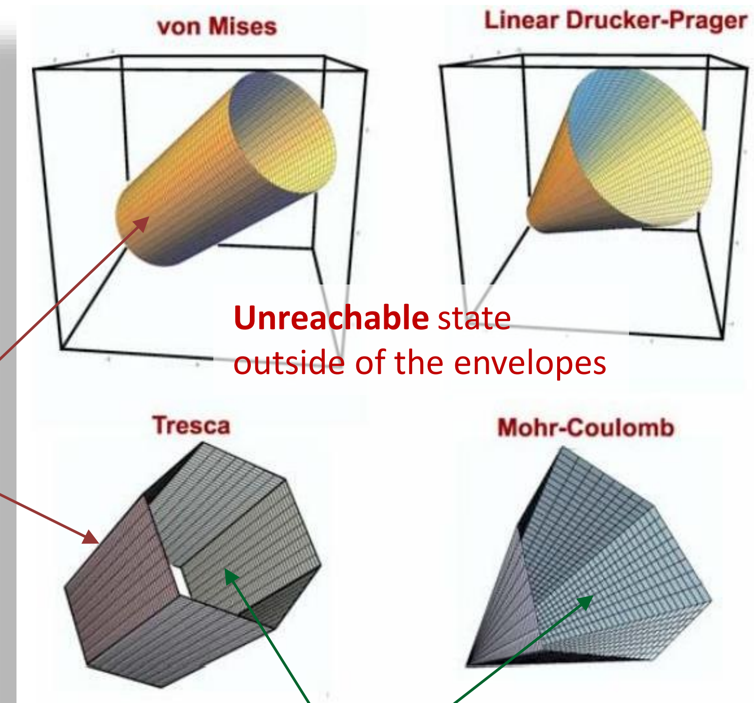
or

$$G(\varepsilon_{ij}) = 0$$

Failure envelope

= failure criteria

= failure function



Unfailed – inside the envelope

safe zone

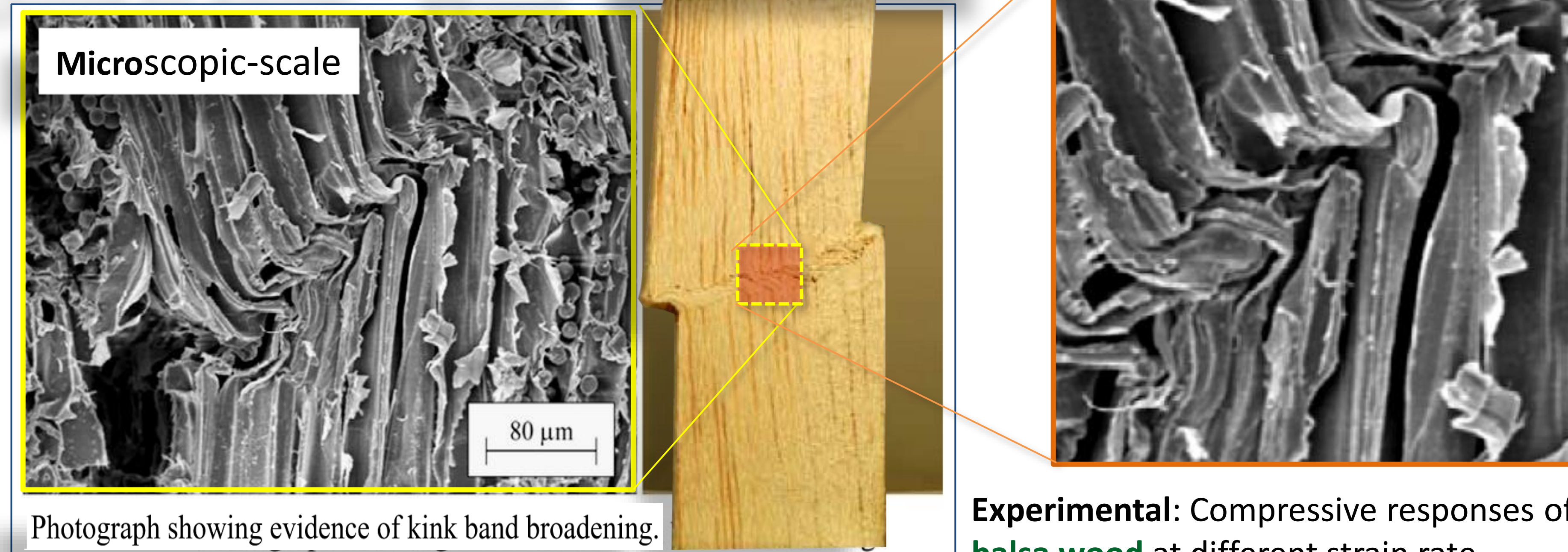
The **failure condition**, as for *instance the* ***yield*** *or fracture condition*, **corresponds** to a ***failure surface*** in the 6-dimensional **stress space** or alternatively, **in the 3-dimensional space of principal stresses** or expressed as functions of stress or strain Invariants

$I_1,\ J_2\ \&\ \cos(3\theta)$

$$f(\sigma_1, \sigma_2, \sigma_3; \kappa_1, \kappa_2, \ldots, \kappa_n) = 0$$

The basis of strength or failure hypothesis (for example *yield criteria*) are experiments conducted under specific and often more or less simple loading conditions

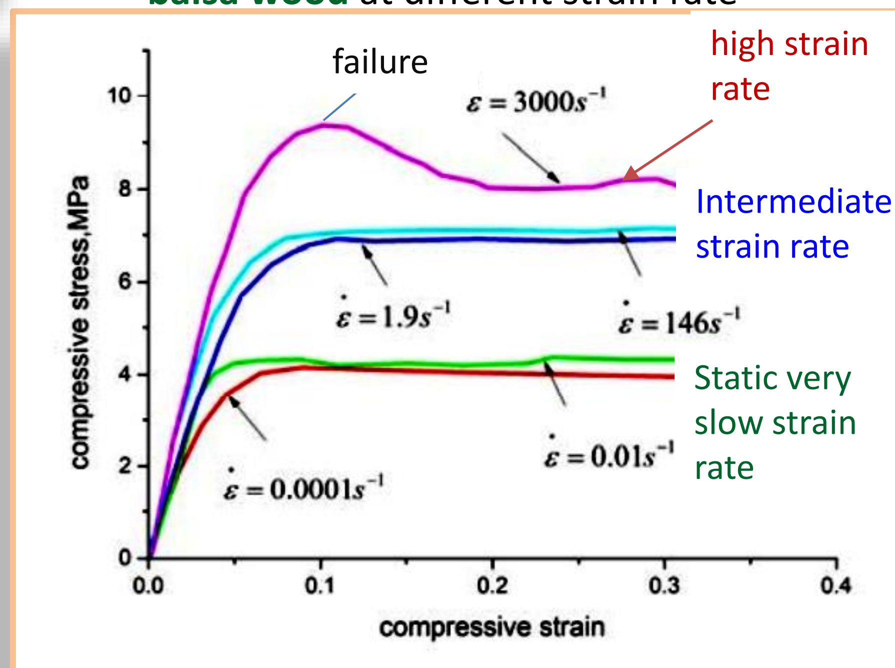
Good to know… Wood



Photograph showing evidence of kink band broadening.

Ref: *Advances in the study of mechanical properties and constitutive law in the field of wood research* S Zhao, , J X Zhao and G Z Han, 2016 Global Conference on Polymer and Composite Materials (PCM 2016) *Materials Science and Engineering* **137 (2016)** 012036 doi:10.1088/1757-899X/137/1/012036

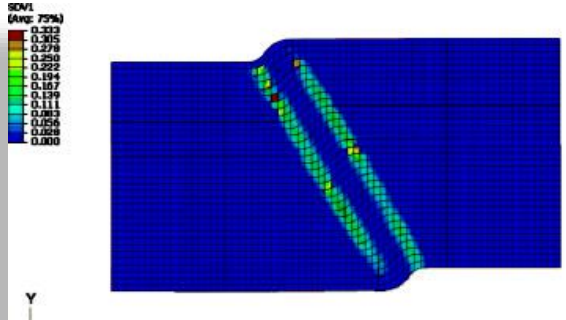
Experimental: Compressive responses of **balsa wood** at different strain rate



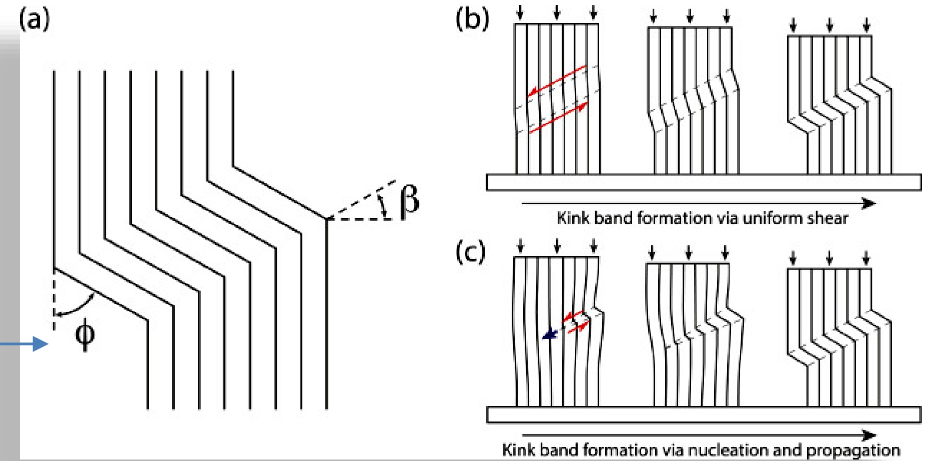
The complex mechanical behaviour of wood cannot be treated in this short course.

Wood deserves a complete course and an expertise in the field that I have not.

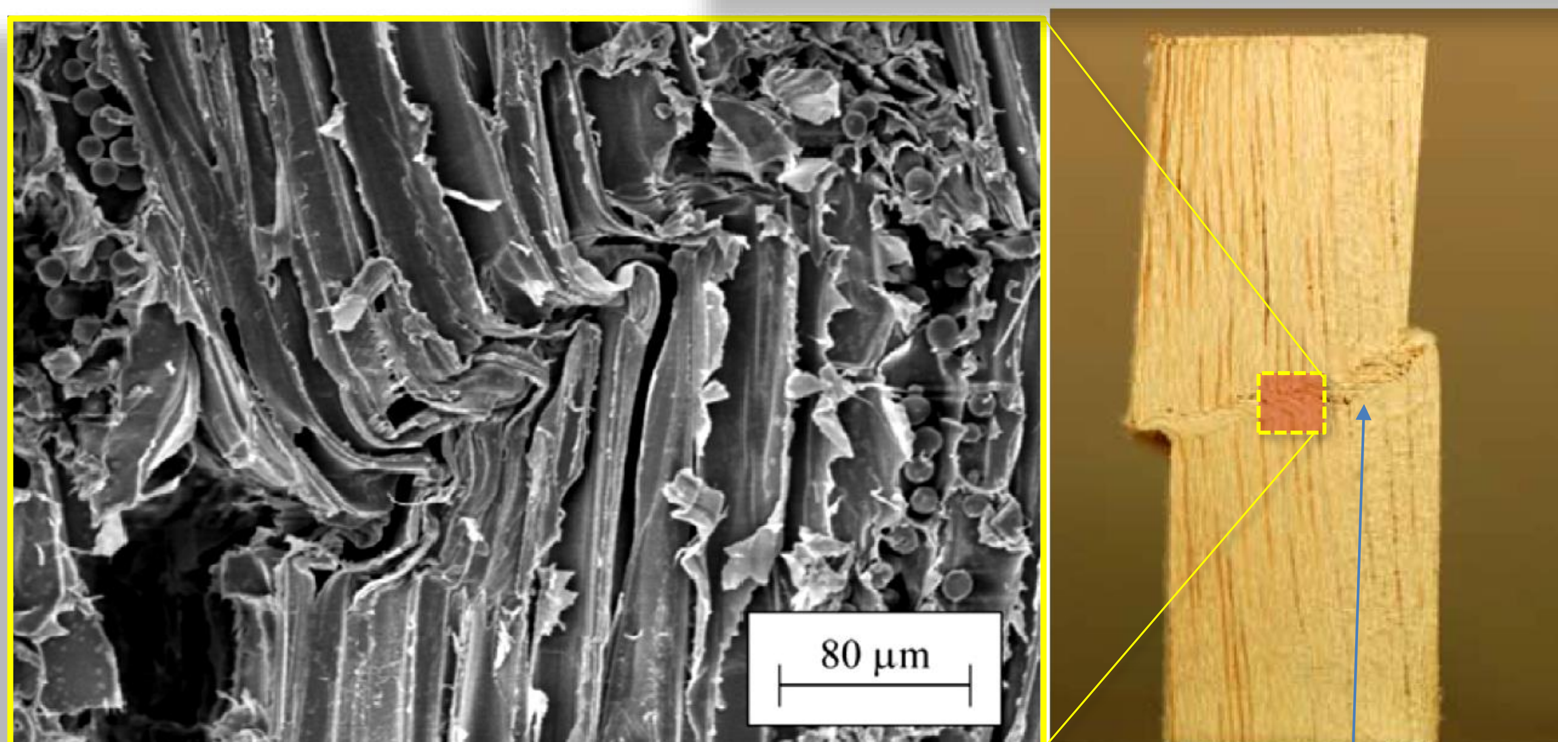
Wood



Ref: https://www.google.fi/search?q=Rwood+band+kink+broadening&client=firefox-b-ab&dcr=0&source=lnms&tbm=isch&sa=X&ved=0ahUKEwjfkrKy1qLaAhUFlCwKHXwbCVAQ_AUICigB&biw=1391&bih=1003#imgrc=5Ifu1GlYIgpAAM:



Good to know…



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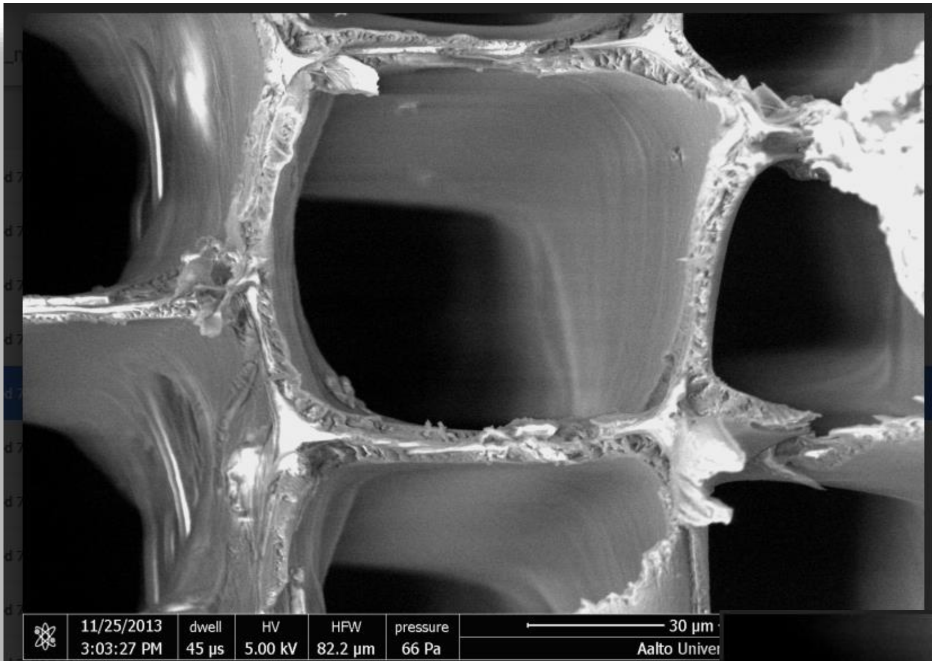
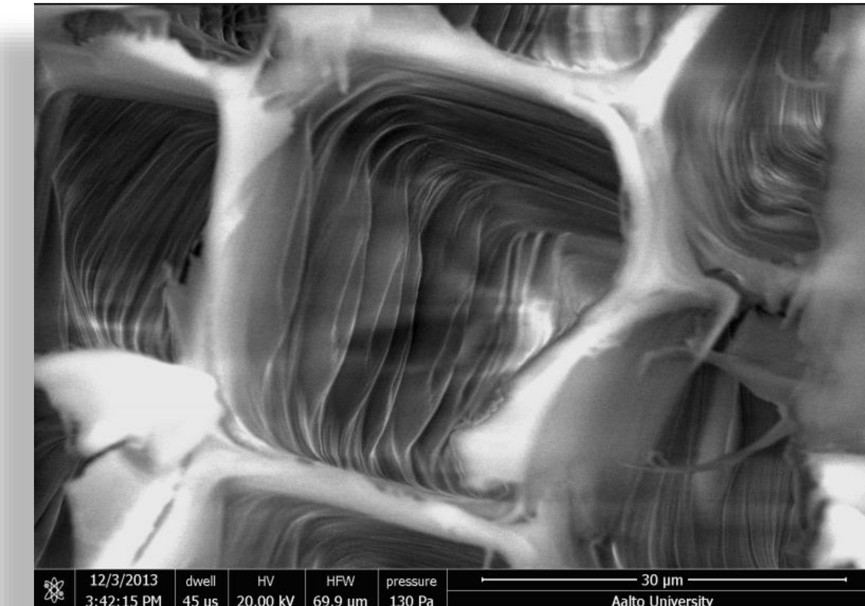
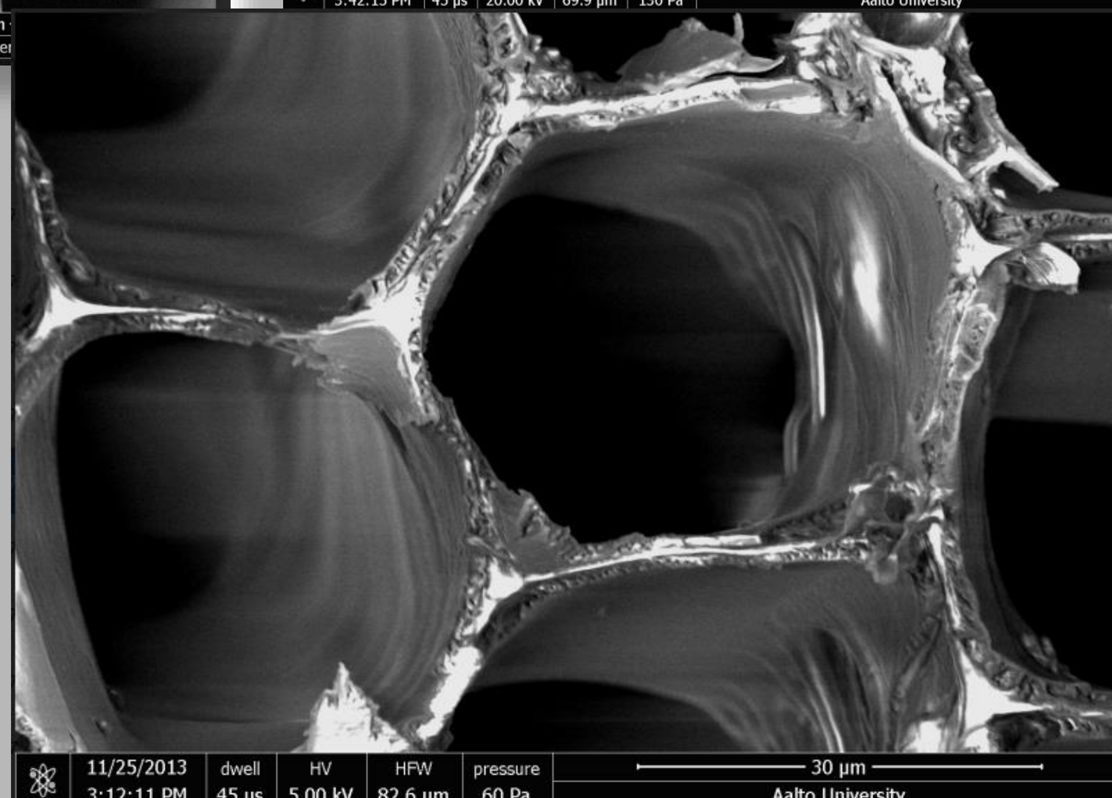




Wood - microstructure

Some SEM-micrographs
by Dr. Djebar Baroudi & Dr. Andrzej Cwirzen

Good to know...



Synthesis of the **experimental** observations related to plastic behaviour

What general can be conclude from all these experimental observations on plastic behaviour at constant normal room temperatures and with no damaging loading ?

At leas for metals, one can say:

1. **Thermodynamic *irreversibility* phenomenon related to plasticity is seen *via* the existence of plastic strains**
2. These plastic strains are practically strain-rate independent, and …
3. **The resulting plastic volume change is practically zero (incompressible plastic flow** in metals; experimental evidence will be shown in further slides – Bridgman experiments)
4. Existence of a threshold or plastic-yield surface (such surface is represented by a *convex* hypersurface in the stress space. To conclude this much more intellectual work is needed. Does not follow from a simple observation)
5. This yield-surface possibly depends on plastic-strain history (hardening); experimental evidence will be shown in further slides

Plasticity

Yield criteria

Isotropic & Isothermal Rate-Independent*

Plasticity deals with yielding of materials under complex stress states

Yield, residual deformations

Readings

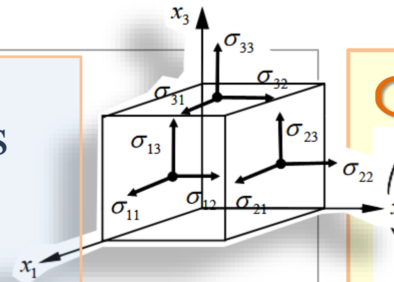
- Lemaitre, chapter 5. – **Plasticity** or
- Chen & Han, chapters 1 – 5 (complete)
- Ottosen & Ristinmaa, chapter 8…12 – *Plasticity*

- **Can have a look**
- Dietmar Gross, chapter 1.3.3
- Primer, Chapter 4
- *Elective*: Lubliner, Section 3.3 *Yield Criteria, Flow Rules and Hardening*

* 1. Isotropic & Isothermal Rate-Independent plasticity is the subject of this course. A quantitative review of commonly used *isotropic yield criteria* will be done.

2. For anisotropic cases (*i.e.*, plastic behavior having a directional dependency), there exist several anisotropic yield criteria – these are not the subject of this course.

Plasticity & Yield criteria



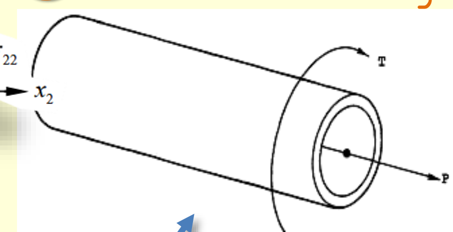
Plasticity deals with Yielding of materials under Complex Stress States

Consider a body under loading of gradually increasing magnitude resulting in a given combined stress state.

The initial small deformation of the body is entirely *elastic* and the body recovers its shape entirely under complete unloading.

If one continues loading then *for a certain combination of the applied stresses*, plastic deformation appears and thus the material ***yields***.

Combined loading



Thin metallic tube

Internal pressure, torque and axial loading result in a 3D-stress state

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

$$\sigma_{ij} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

The Question: for **which stress combination** the metal of the tube will yield?

A Law defining the Limit of the Elastic Behavior under any combination of stresses is called Yield Criterion. ?

Since the **physical yield is independent** on which coordinate system the stress tensor is written (*i.e.*, independent on who is observing the yield), the **yield criterion** should be **a function of the Stress Invariants** in the isotropic case. However, for anisotropic cases, principal directions of anisotropy will be involved.

Failure Hypotheses or more precisely Yield Criteria

myötöehto

Terminologia: Lujuushypoteesit
Strength hypotheses

Also known as Strength Hypotheses

- **Distortional energy** – Von Mises **yield** condition
  (as metals – plastic yield insensitive to pressure)
- **Maximum shear stress** – TRESCA **yield** condition [1864]

pressure insensitive yield (metals)

- Maximum Principal stress – **RANKINE** (1820–1872), [1850]
  LAMÉ (1795–1870) & **NAVIER** (1785–1836)

brittle materials

- Maximum Principal strain – **SAINT-VENANT** (1797–1886) & **BACH** (1889)
- Maximum strain energy – **BELTRAMI** (1835-1900)
- **Mohr-Coulomb hypothesis** … for granular and geological materials
  materials having pressure dependent yield
- Octahedral shear Stress theory or
  Drucker-Prager hypothesis - **DRUCKER** (1918-2001) & **PRAGER** (1903-1980)

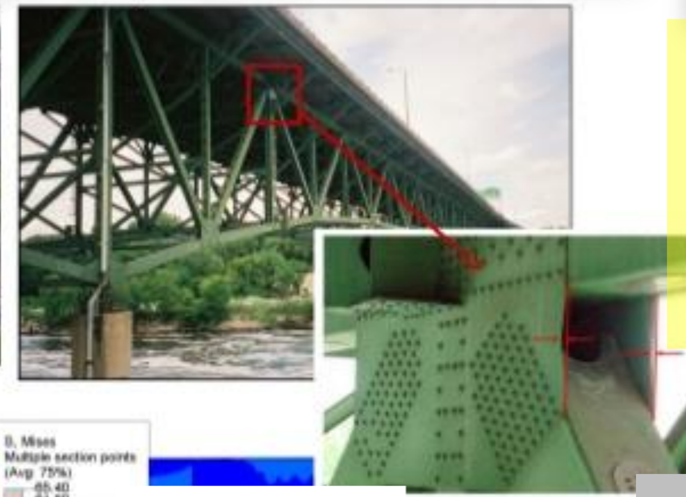
materials with pressure-sensitive yield

… initially developed for granular and geological materials

These criteria above are some of *commonly used* (engineering) isotropic **yield criteria**. This course will provide a quantitative review

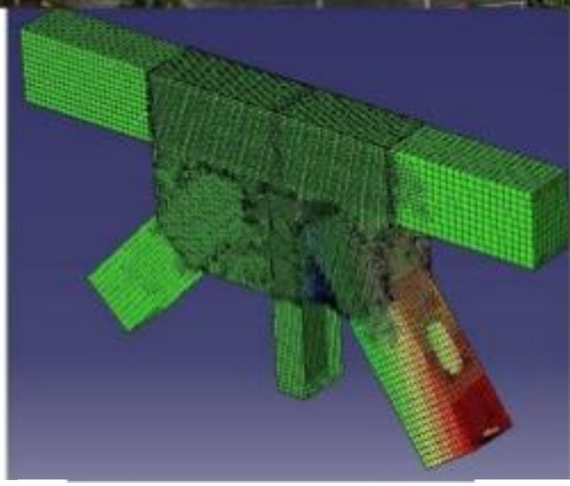
Collapse of the I-35W Bridge, Minneapolis, Minnesota, 2007

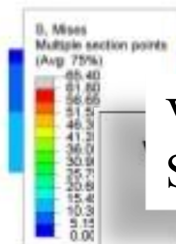
Plasticity



Accident cause: ! steel gusset plates that were undersized and inadequate to support the intended load of the bridge as the loading had increased over time!

$$F = J_2 - k^2 = 0$$





Von MISES Stress

plastic yielding: orange and red shading exceeds ***yield stress***

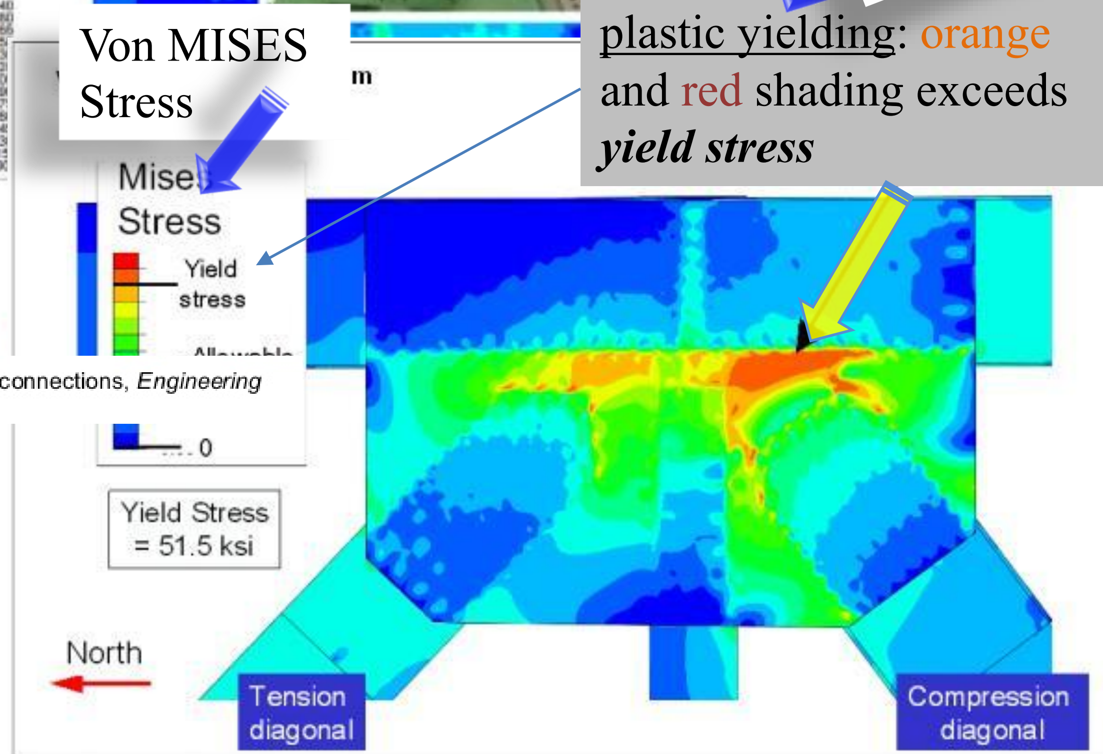


Figure 11. Mises stress contours under the dead load of the original 1967 bridge.

Crosti, C. & Duthinh, D. (2014) "A nonlinear model for gusset plate connections, *Engineering Structures*, V. 62-63, pp.135-147.

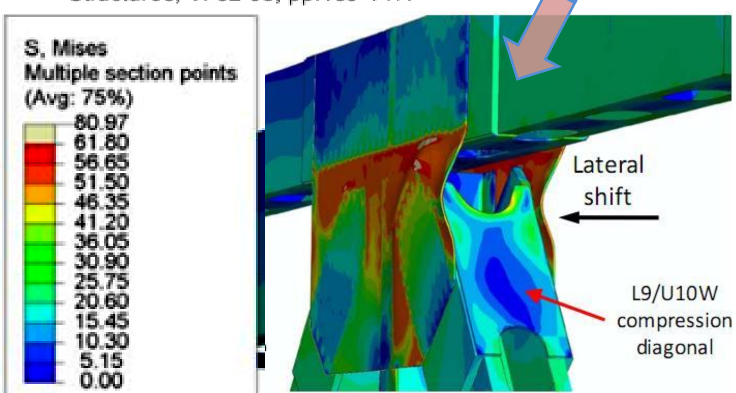
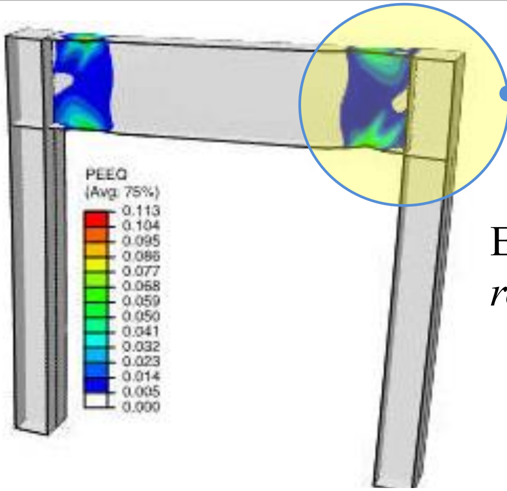


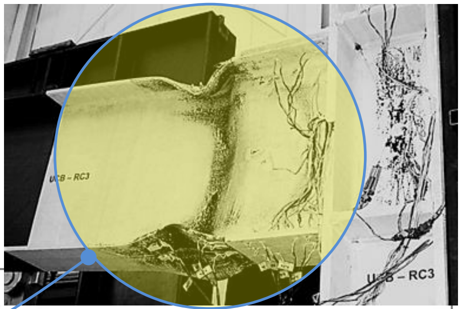
Figure 14. Indicating the lateral shift of the L9/U10W diagonal, with Mises stress contours.

Engineering Plasticity

One effect of plastic yielding are **remaining plastic deformations** after unloading

Plasticity: theory of time-independent irreversible deformations

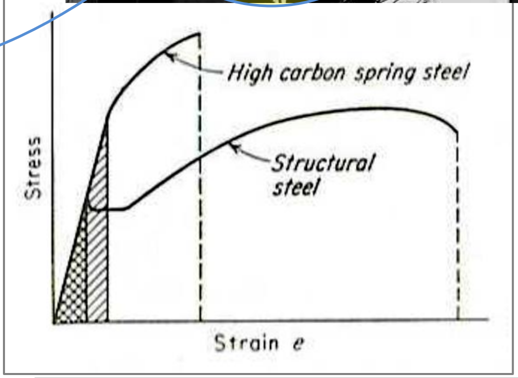




$$\Delta\varepsilon = \Delta\varepsilon^{(e)} + \Delta\varepsilon^{(p)}$$

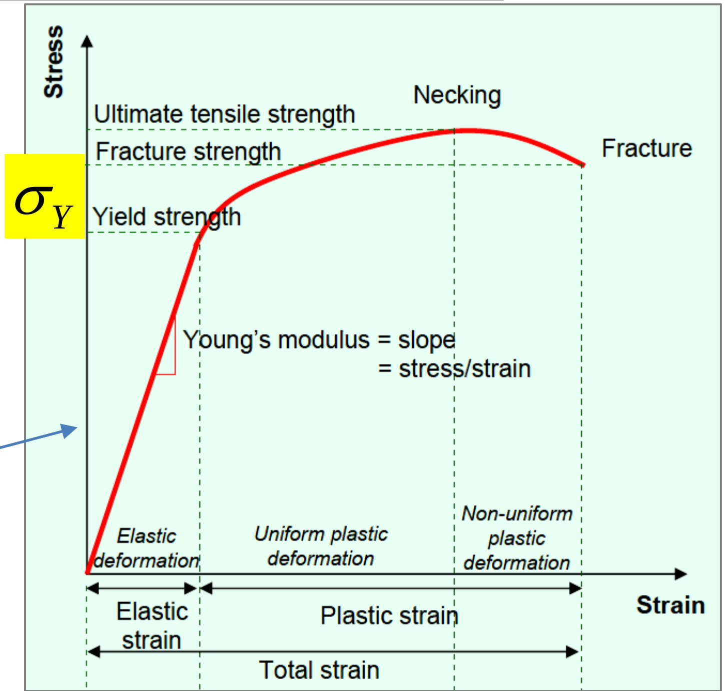
Elastic increment
reversible

Plastic increment
irreversible



The three ingredients of engineering plasticity theory

- Yielding criteria — myötöehto
- Flow rule — myötösääntö
- Hardening rule — lujittumisehto



- If stresses or strains **exceed a certain limit**, *plastic yielding* may be observed, especially in metallic materials (here one have a uniaxial tensile test)
- under arbitrary combination of stresses a more general yield criteria is needed

Plasticity

Plasticity: theory of isothermal time-independent *irreversible deformations*

Domain of validity and use:

❖ **Metal and alloys**:

- mainly movement of dislocations:

The total strain (increment) can be partitioned into reversible (elastic) and irreversible (plastic)

- Restrictions: - **Low temperature** $T < T_{melting} / 4$
  - **Non-damaging loads**

…otherwise → viscoplasticity, thermally activated creep

… further recrystallization…too complex

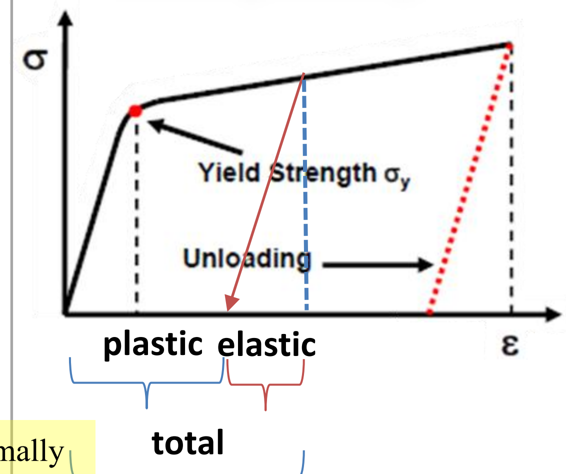
❖ **Soils:** Restrictions: occurrence of slip surfaces caused by instability

❖ Polymers and Wood: irreversible deformations are better accounted by **visco-elasticity** (however, for temperatures close to melting temperatures→ thermoplastics)

❖ **Concrete**: irreversible deformations are due mainly to micro-cracks therefore a *model coupling elasticity or plasticity* and *damage* may be preferable
(for longtime responce, creepresults in irrevesible deformations)

in soils plasticity or permanent shearing deformation occurs when constituting particles slide (& relatively rotate) over one another (*cohesive* and *frictional*)

Elasto-plasticity:



$$\Delta\varepsilon = \Delta\varepsilon^{(e)} + \Delta\varepsilon^{(p)}$$

Elastic increment
reversible

Plastic increment
irreversible

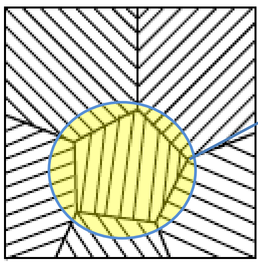
Plasticity -- some basic physics

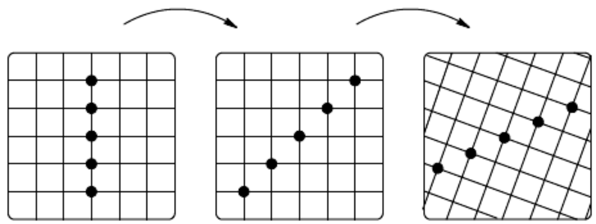
(*cf.* to some basics textbooks)

plastic yielding → remaining plastic deformations after unloading

- **Metal and alloys**: mainly movement of dislocations
  (crystal plasticity)
- The total strain can be partitioned into reversible (elastic) and irreversible (plastic) $\varepsilon^{(p)} + \varepsilon^{(e)}$

σ_Y ***Yield stress***





Microstructure of a single crystal showing plastic deformation followed by elastic deformation

Elasto-plasticity:

total strain: $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p$,

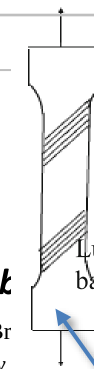
Variations in the inter-atomic distances without changes of place

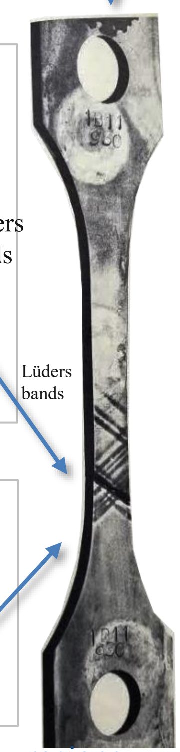
'slip' movements along crystal planes with modifications of inter-atomic bonds direction of the largest resolved shear stress (*distortion=shear*)

Metal and alloys

In Metal & metal alloys: ***volume change*** *during* ***plastic deformation*** is normally **negligible** in isotropic case (~***incompressib***

[*ref. Bridgman* experiments: Br W., 1952, *Studies in Large plastic Flow Fracture With Special Emphasis on the Effects of Hydrostatic Pressure*, McGraw-Hill, New York]





The **total strain** is **partitioned** into reversible (**elastic**) and irreversible (**plastic**) :

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^e + \dot{\boldsymbol{\varepsilon}}^p$$

$$\mathrm{d}\boldsymbol{\varepsilon} = \mathrm{d}\boldsymbol{\varepsilon}^e + \mathrm{d}\boldsymbol{\varepsilon}^p$$

Plastic deformation depends on loading history → use incremental or rate form

Need for a *flow rule* to compute this increment

$$\boldsymbol{\sigma} = \boldsymbol{C} : \boldsymbol{\varepsilon}^e = \boldsymbol{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p)$$

Elasticity

Dislocation := motion of atoms in the crystal structure rearranging themselves to have new neighbors

Clay or fine-grained soil: For these highly porous materials, the *inelastic* deformation has both ***distortional*** and ***volumetric*** components

Stainless-steel at 315 °C. Diagonal lines are slip bands, i.e., regions with high degree of plastic slip (*ref.* Shanley, Strength of materials, 1957)

Should use true strain when ~few percent larger … (large deformation)

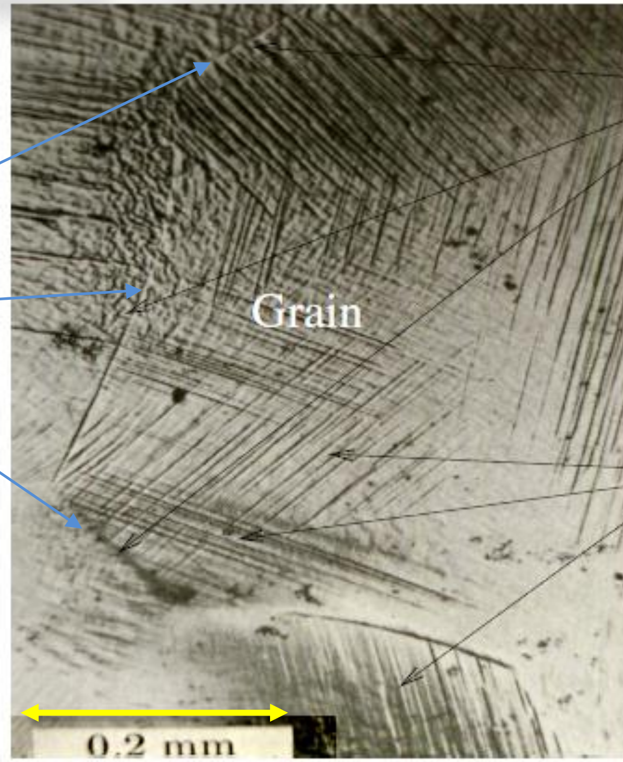
Illustration of slip planes in a grain of a metal (a monocristal)

Plastic Deformation of Polycrystalline Materials

G ~48 GPa

Metal: Copper

Grain boundaries



Joints de grain

Grain joints

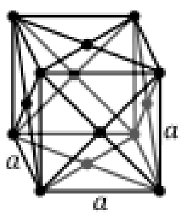
Note that we see <u>three</u> slip directions per grain

Trace de plans de glissement

Trace of slip movements along crystal planes

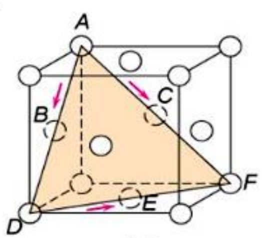
Note that the slip orientations changes with the grain orientations

Copper: Crystal structure

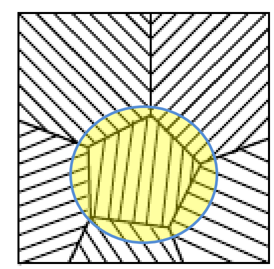


face-centered cubic (fcc)

Cu: <u>**three**</u> slip-planes FCC {111} = Direction of easiest slippage



□ **Slip plane - plane allowing easiest slippage**

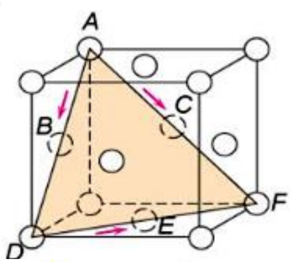


Dislocation := motion of atoms in the crystal structure rearranging themselves to have new neighbors

Ref: *Rupture et Plasticité, lecture notes by* Pierre Suquet

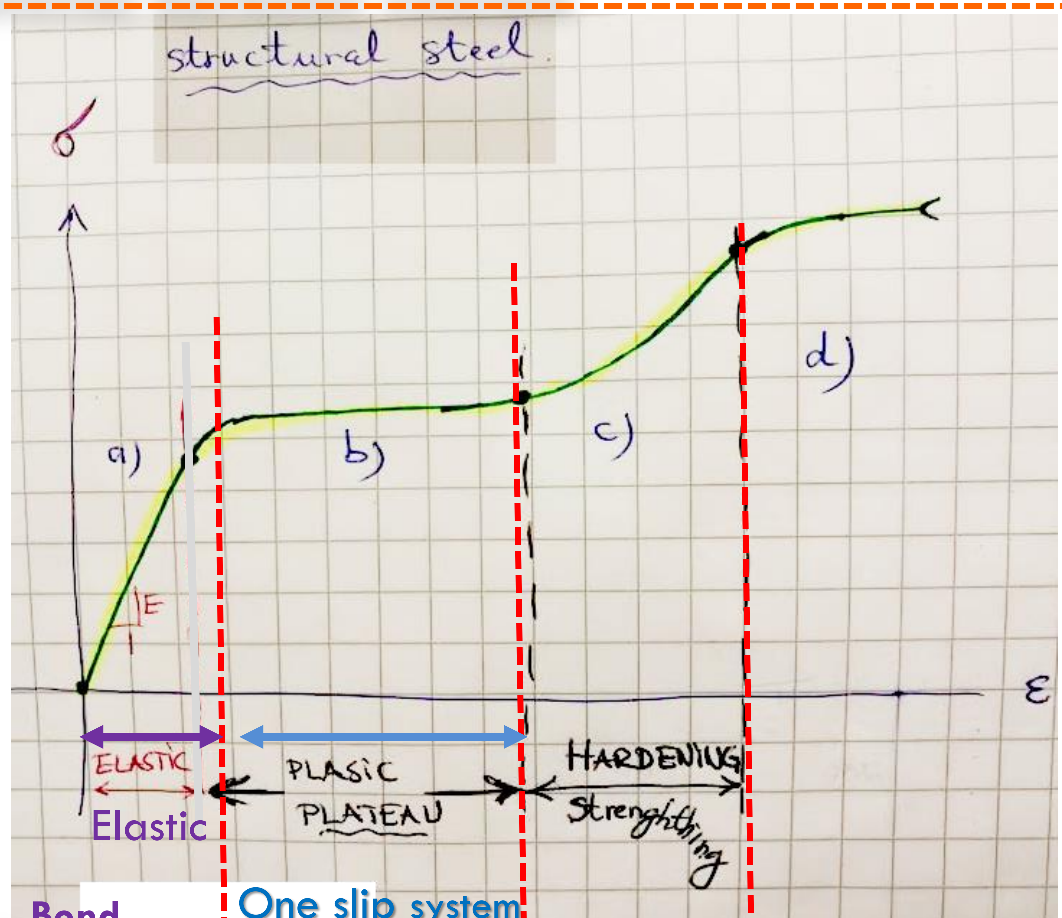
Plastic Deformation of Polycrystalline Materials

- Random crystallographic orientations of the numerous grains, the direction of slip varies from one grain to another ➔ **deformation and slip is complex**
- Photomicrograph of a polycrystalline copper specimen
  - ☐ Before deformation, the surface was **polished**
  - ☐ Slip lines visible
  - ☐ Two sets of parallel yet intersecting **sets of lines** ➔ It appears that two slip systems operated
  - ☐ The difference in alignment of the slip lines for the several grains ➔ variation in grain orientation



☐ **Slip plane - plane allowing easiest slippage**

Should be known to you from earlier material bachelor courses



Bond stretching → Elastic

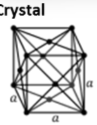
One slip system get activated

Slip on **one** slip system → '**easy** slip' or '**glide**' → **plasticity plateau**

Multiple slip systems get activated → hardening

Material fractures in some extreme orientations of grains (crystals)

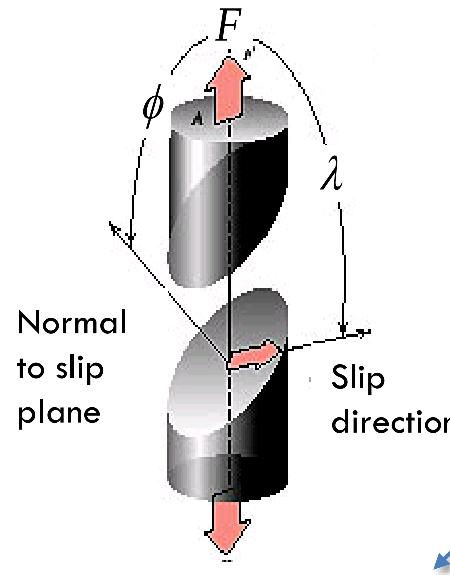
Copper: Crystal structure



face-centered cubic (fcc)

Cu: three slip-planes FCC {111}

Def: slip system = slip plane & slip direction



Stainless-steel at 315 °C. Diagonal lines are slip bands, i.e., regions with high degree of plastic slip (ref. Shanley, Strength of materials, 1957)

Largest resolved shear stress

(in a single crystal):

$$\tau_{\max} = \frac{F_s}{A_s} = \frac{F\cos\lambda}{A/\cos\phi} = \sigma[\cos\lambda\cos\phi]_{\max}$$

Schmidt factor

Therefore, the minimum stress to begin yielding (= slip) occurs when slip direction $\lambda = \phi = 45\,\text{deg}$.

Good to know

polycrystals

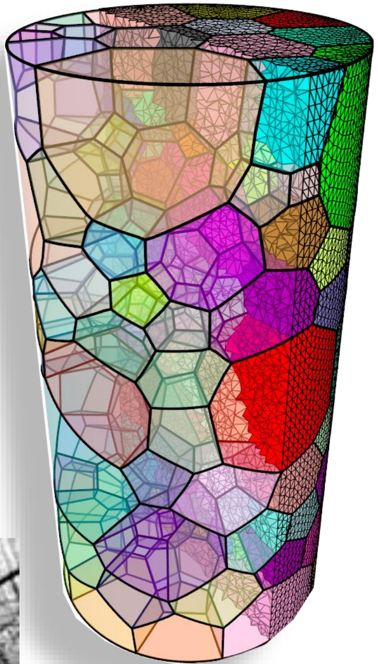
- Hall-Petch Equation: For many materials, Yield strength varies with grain size as

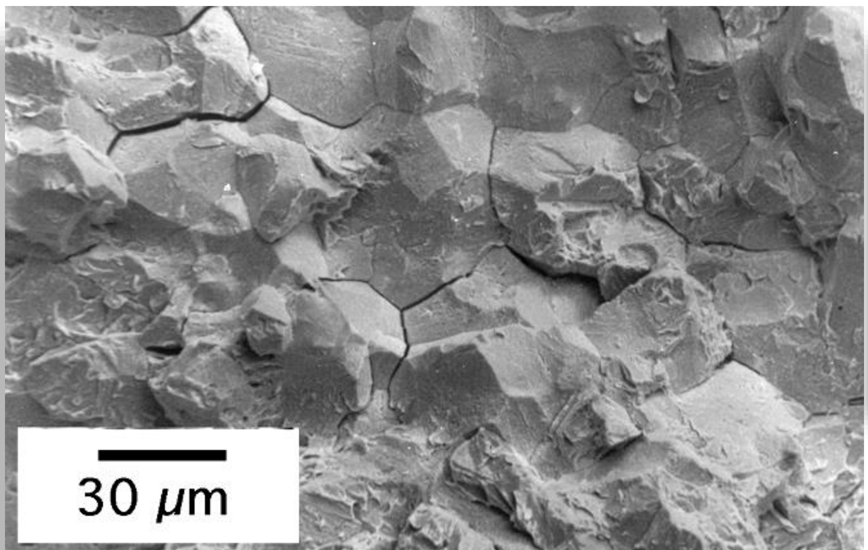
$$\sigma_y = \sigma_0 + k_y d^{-1/2}$$

d: average grain diameter

σ_0 and k_y are material constants

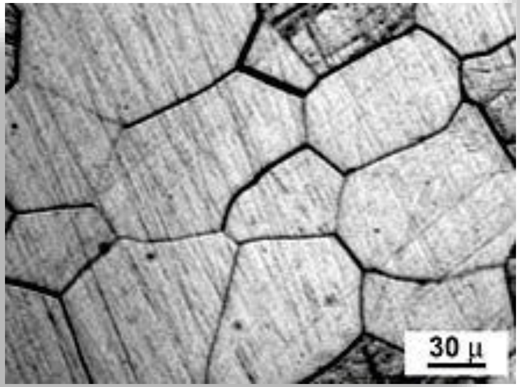
This a good approximation for moderate grain size.





Micrograph – **Steel** grain boundary failure

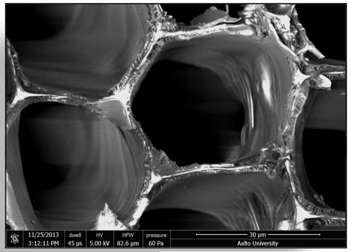
https://www.phase-trans.msm.cam.ac.uk/2008/Steel_Microstructure/SM.html



Micrograph of a polycrystalline **Metal**; grain boundaries evidenced by acid etching.

https://en.wikipedia.org/wiki/Grain_boundary

Wood – for comparison



30 μm

SEM-micrographs by Dr. D. Baroudi & Dr. A. Cwirzen

Got interested! Go to some basic textbook on material science of engineering materials to learn more…

To have an idea of the order of magnitude …

Typical tensile strengths

Tensile test

Material	Yield strength (MPa)	Ultimate tensile strength (MPa)
Steel, structural ASTM A36 steel	250	400–550
Steel, 1090 mild	247	841
Chromium-vanadium steel AISI 6150	620	940
Human skin	15	20

https://www.google.com/search?q=yield+stress+steel&ie=utf-8&oe=utf-8&client=firefox-b-ab

Effect of Hydrostatic Pressure

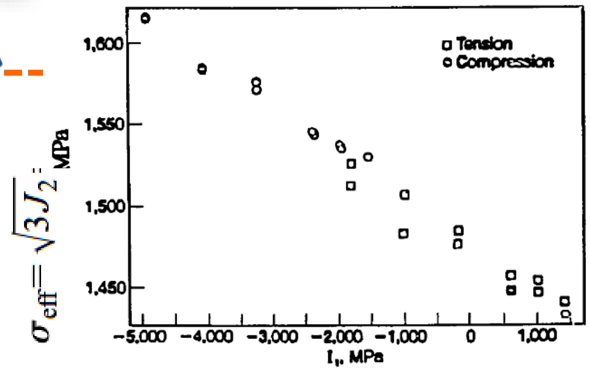
Good to know 1(3)

Effects of hydrostatic pressure up to ~1 GPa on the yield strength of four steels [3].

About “In Metals *volume change during plastic deformation* is normally negligible (~*incompre*

Reading pause ...

Experiments by Bridgman *et al*., (see [1] and [2]), have shown that under hydrostatic pressures, materials as metals, are practically incompressible (for not very high pressures) when undergoing yield even for very large plastic strain changes. The only volumetric change bein elastic. /Bridgman won the **Nobel Prize in Physics** (1946) for his work on the physics of high pressure



Effective Stress σ_{eff} versus I_1 for 4330 steel

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3$$

Therefore, **a metal** is assumed to have incompressible plastic strains.

- *no influence of hydrostatic pressure on yielding*
- and *incompressibility for plastic strain*
- The only volumetric change being elastic

These two observations are the basis of the *classical theory of plasticity*

In addition, Bridgman found increase of hydrostatic pressure increased the ductility (much larger strains before fracture were obtained with increased hydrostatic pressure).

However, later work by Bridgman showed that There is a definite effect of hydrostatic pressure on the strain hardening curves of mild steel.

Influence of pressure on shear modulus: Between 0 and 10 GPA, the shear modulus changed by about 2% for coiled steel and nickel (Bridgeman’s experiments) / shear modulus *G* encodes the *influence of hydrostatic pressure on yielding /*

[1] Bridgman, P. W., 1952, *Studies in Large Plastic Flow and Fracture With Special Emphasis on the Effects of Hydrostatic Pressure*, McGraw-Hill, New York

[3] Spitzig, W. A., Sober, R. J., and Richmond, O., 1976, ‘‘The Effect of Hydrostatic Pressure on the Deformation Behavior of Maraging and HY-80 Steels and Its Implications for Plasticity Theory,’’ *Metall. Trans. A*, **7A,** Nov., pp. 1703–1710.

[2] Bridgman, P. W., 1947, “The Effect of Hydrostatic Pressure on the Fracture of Brittle Substances,” *J. Appl. Phys*., **18**, p. 246.

Effect of Hydrostatic Pressure

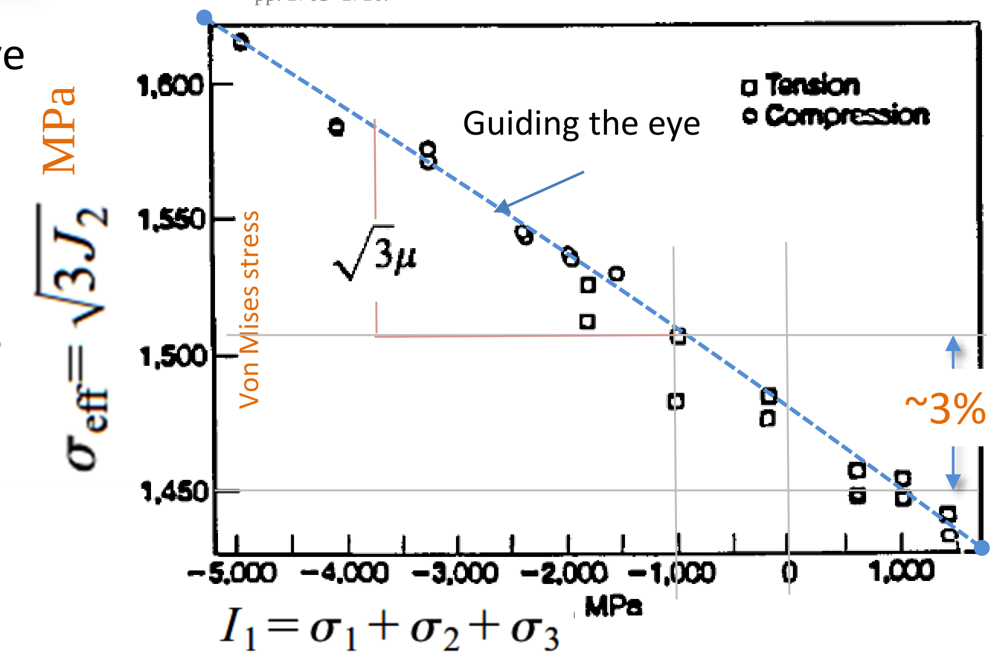
Good to know 2(3)

About "In Metals *volume change during plastic deformation* is normally negligible (~*incompressible*)"

Effects of hydrostatic pressure up to 1,100 MPa on the yield strength of four steels [3].

[3] Spitzig, W. A., Sober, R. J., and Richmond, O., 1976, "The Effect of Hydrostatic Pressure on the Deformation Behavior of Maraging and HY-80 Steels and Its Implications for Plasticity Theory," *Metall. Trans. A*, **7A,** Nov., pp. 1703–1710.

- Experiments by Bridgman *et al.* have shown that under hydrostatic pressures, materials as metals, are practically incompressible (for not very high pressures) when undergoing yield even for very large plastic strain changes.
- The only volumetric change being elastic.



Effective Stress σ_{eff} versus I_1 for 4330 steel

Therefore, **a metal** is assumed to have incompressible plastic strains

- *no influence of hydrostatic pressure on yielding*
- and *incompressibility for plastic strain*
- The only volumetric change being elastic

$$\sqrt{J_2} = \sqrt{\frac{1}{2} s_{ij} s_{ij}} \equiv \tau_e$$

May be can improve Von Mises yield criteria this way: $f(\sigma) = \sigma_e + \sqrt{3}\mu\sigma_m - \sigma_0 = 0$

Engineering Fracture Mechanics Vol. 44, No. 5, PP. 649-661, 1993

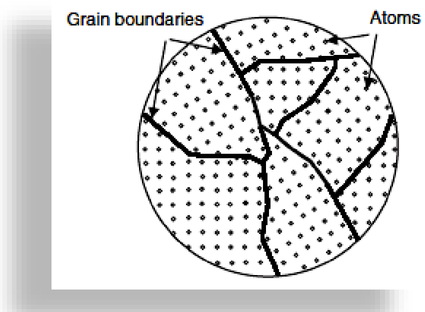
A pressure sensitive parameter

Influence of pressure on shear modulus:
Between 0 and 10 GPA, the shear modulus changed by about 2% for coiled steel and nickel (Bridgeman's experiments) / shear modulus *G* encodes the *influence of hydrostatic pressure on yielding* /

Effect of Hydrostatic Pressure

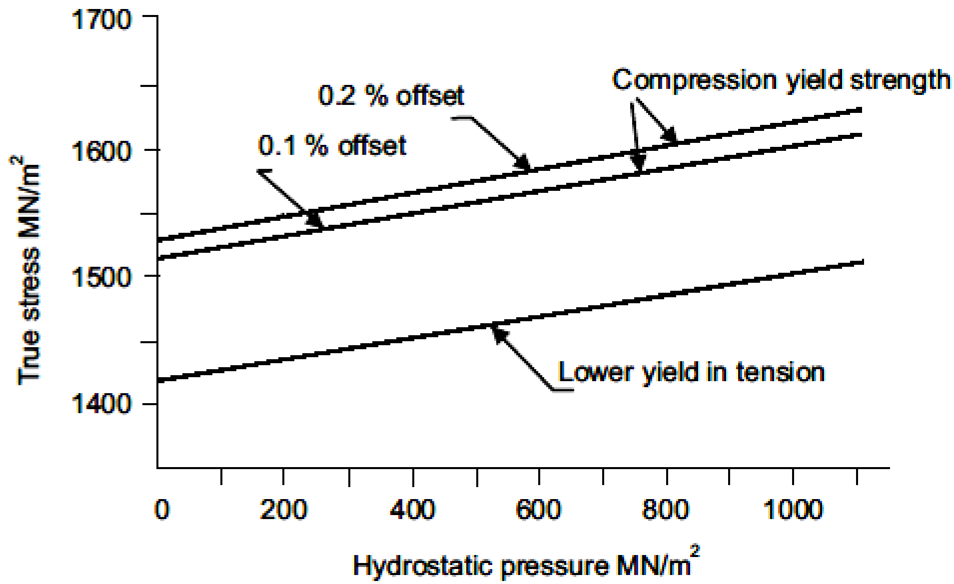
Good to know 3(3)

- Hydrostatic pressure may affect the properties of metals [3]



Grain structure

- Even brittle materials may be plastically deformed under suitable hydrostatic pressures



Effect of hydrostatic pressure on yield strength of quenched and tempered AISI 4330 steel, [3].

[3] Spitzig W.A., Sober R.J. and Richmond O.; *The Effect of Hydrostatic Pressure on the Deformation Behavior of Maraging and HY-80 Steels and its Implications for Plasticity Theory*; Metallurgical Transactions A, Vol. 7A, p. 1703, November 1976

NB. **In soils**, during **plastic deformation**, both **deviatoric** (shear) **plastic** increment and **volumetric plastic increments occurs** (plastic volume changes or dilatancy during plastic shearing).

Characteristic Plastic Behavior in simple tension & compression

Some basic plasticity properties – uniaxial test

in the macro-mechanical meaning

Engineering stress-strain curves:

$$\varepsilon = \varepsilon^{(e)} + \varepsilon^{(p)}$$

elastic *plastic*

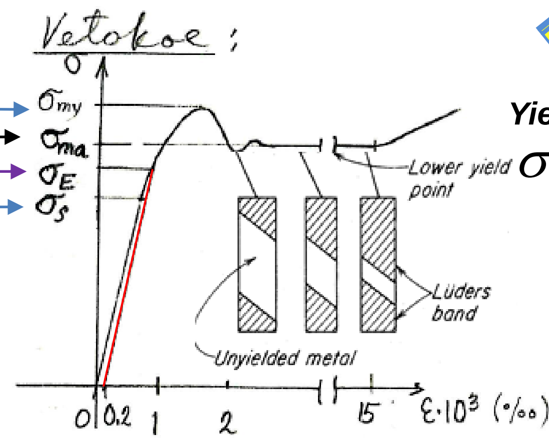
Uniaxial monotonic tension tests

$$\varepsilon_{n+1} = \varepsilon_n + \Delta\varepsilon_n$$

$$\Delta\varepsilon_n = \Delta\varepsilon_n^{(e)} + \Delta\varepsilon_n^{(p)}$$

σ_Y

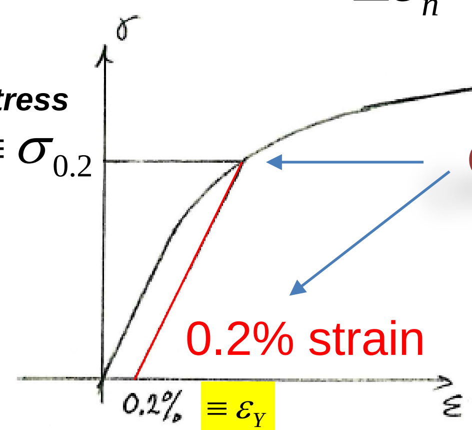
Yield stress:
Upper &
lower limits
Elastic limit
proportional limit



Kuumavalssattu niukka-hiilinen teräs

'Mild and low-carbon' steel (structural steel)

Yield stress $\sigma_Y \equiv \sigma_{0.2}$



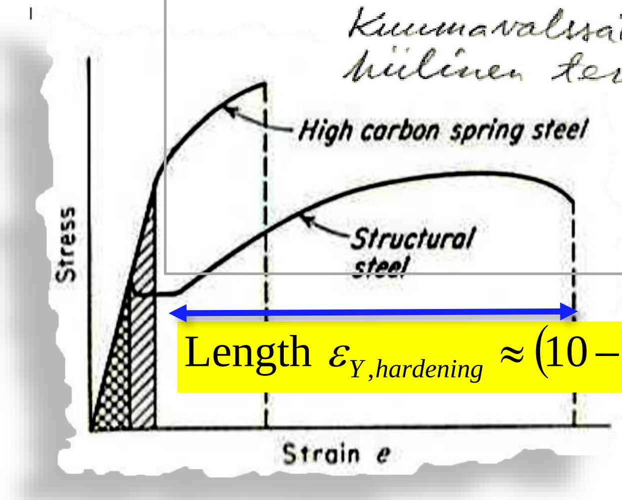
$\equiv \varepsilon_Y$

Offset **yield**

0.2% strain

Alumiiniseos

Aluminum alloy



$$\text{Length } \varepsilon_{Y,hardening} \approx (10-15)\cdot\varepsilon_Y$$

Ref: Manuscripts & drawing in this slide: Emeritus prof. M. Mikkola

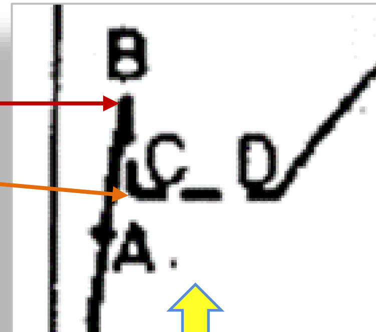
Teräksen ominaisuuksia: (kuumavalss. niukkahiilin.)

– kimmokerroin $E \approx 2{,}1 \cdot 10^6$ kp/cm² (206 000 MN/m²)
– myötöraja $\sigma_m \approx 2400 - 8000$ kp/cm² (jopa ylikin) (235–785 MN/m²)
– myötövenymä $\varepsilon_m \approx 1 - 4$ ‰
– lujenemisalue $\varepsilon_{ml} \approx (10-15) \times \varepsilon_m$
– murtovenymä $\delta \approx 250\, \varepsilon_m$

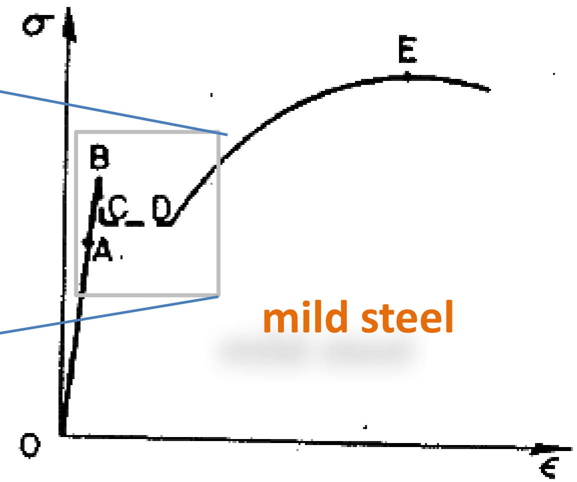
Monotonic loading

Upper yield stress

Lower yield stress



Behavior in the flat region CD is called *plastic flow*



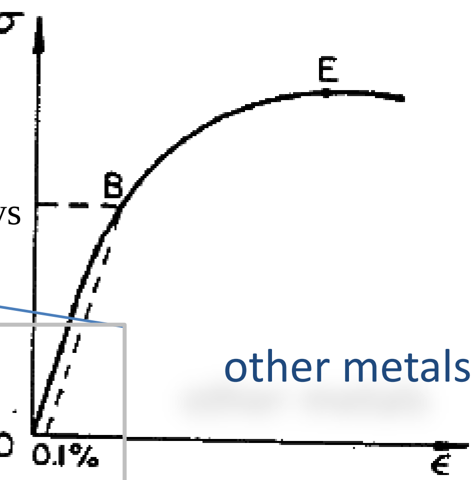
mild steel

Stress-stress diagram

- For mild steel a well defined **flat plastic region** exists; so the ***yield stress*** *(yield point)* is well defined.
- For **most of metals**, such plastic plateau is **not well discernible** (visible); so the **yield strength** is **defined** by an ***offset of yield stress* corresponding** usually to a **certain amount of plastic strain** – 0.1%, 0.2% or what ever is relevant for the application.
- This apparent yield point is called *initial yield stress.*
- Above the yield point the behavior of the material is both elastic and plastic; elasto-plastic

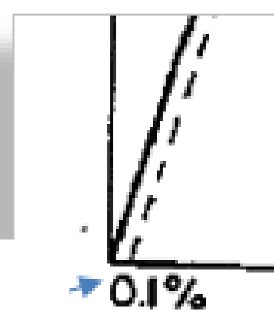
offset yield stress
Initial yield stress

σ_{ys}



other metals

Stress-stress diagram

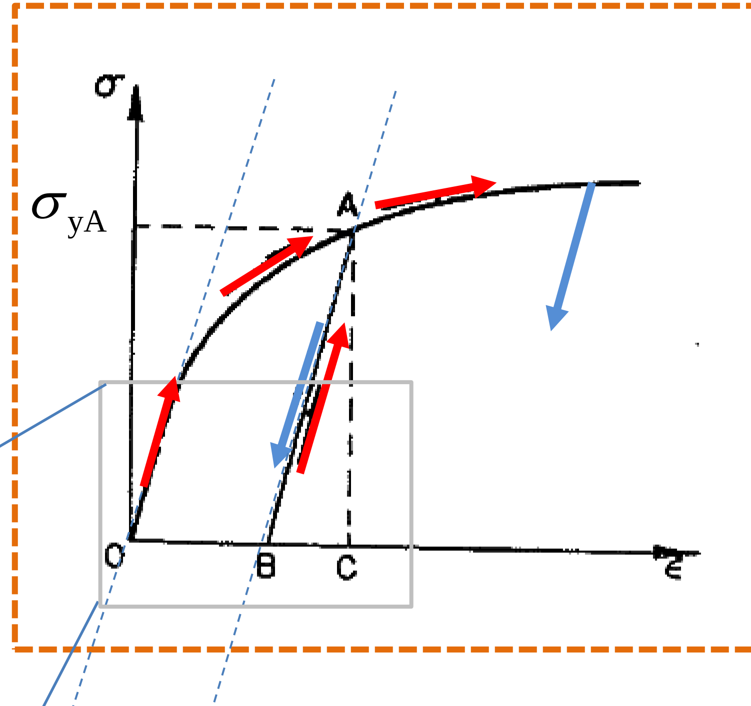


Permanent strain upon unloading

Loading and unloading

Loading history: assume the test specimen is loaded monotonically to some specific value beyond the initial yield point and then then completely unloaded

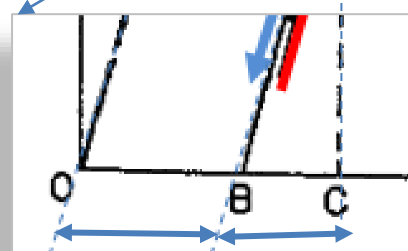
- **Loading history dependence (or Load path dependence)**



Parallels ; the unloading occurs elastically

σ_{yA} - subsequent yield stress

- In plasticity, because of path dependency of the behavior, the constitutive law for strain plastic is known only for strain increments



Permanent strain OB

ε_p

Elastic strain BC

ε_e

$$\Delta\varepsilon = \Delta\varepsilon_e + \Delta\varepsilon_p$$

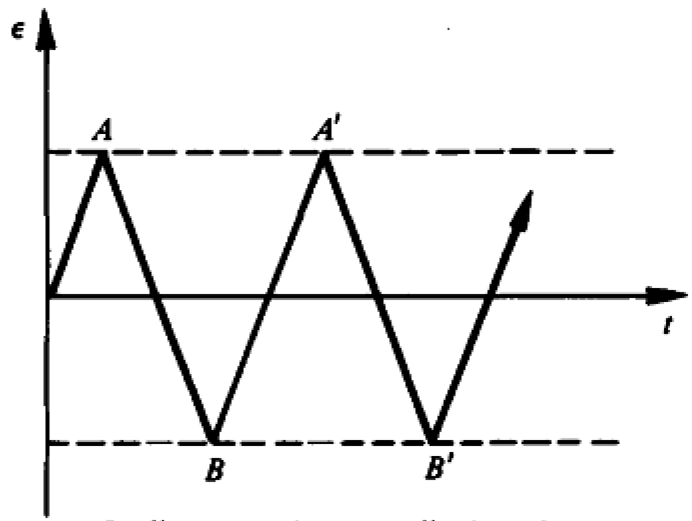
$$d\varepsilon = d\varepsilon_e + d\varepsilon_p$$

$$\Delta\varepsilon = \Delta\varepsilon_e + \Delta\varepsilon_p$$

For many materials (metals), *after reaching the initial yield point*, the **stress-strain curve continues to rise** although with decreasing slope → This behavior is called **strain- or work hardening**

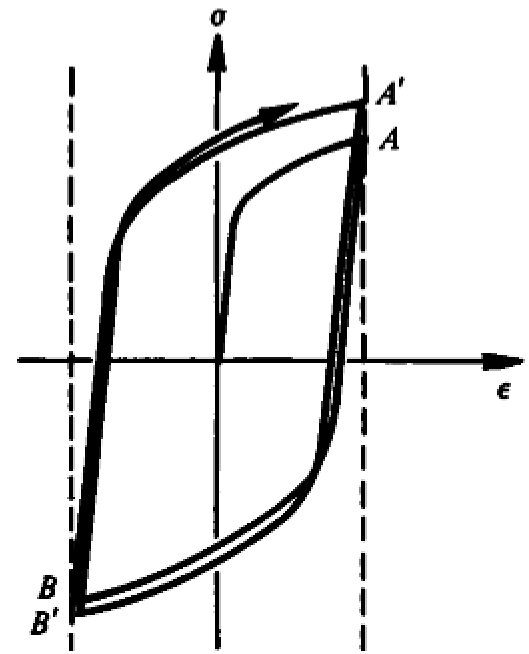
Cyclic loading

Cyclic test under prescribed starin history



Cyclic test under prescribed strain

ref: Chaboche *et al.*

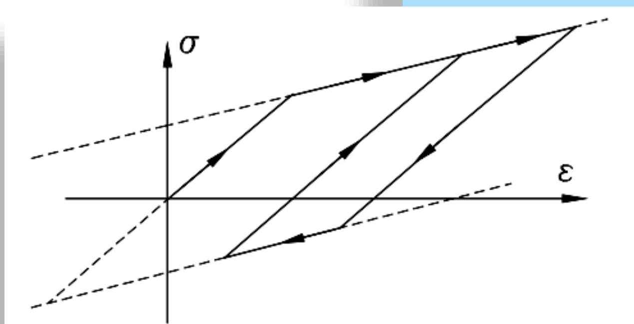


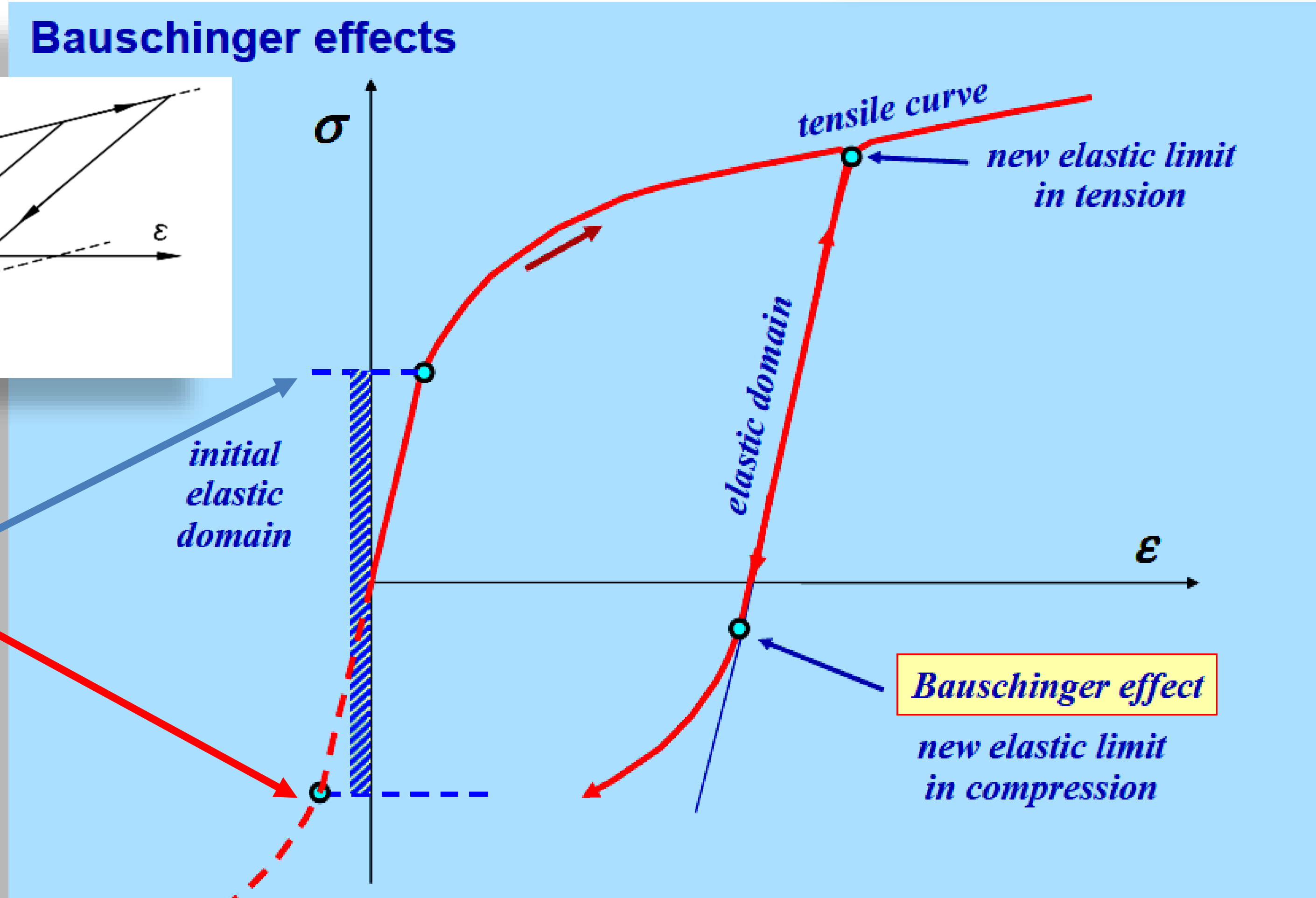
Steel

Reversed loading

The yield stress in **compression** is **not the same** as the yield strength in **tension** after reloading the specimen (cyclic loading)

Such **Bauschinger** effect is present whenever there is a reversal loading (cyclic)





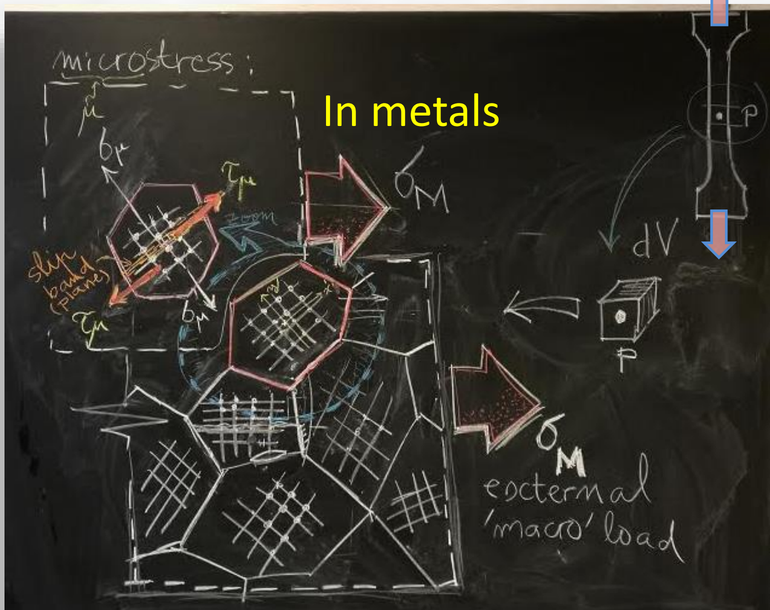
Initially, for a virgin metallic sample, the **compression** and **tension** yield strength are **the same**

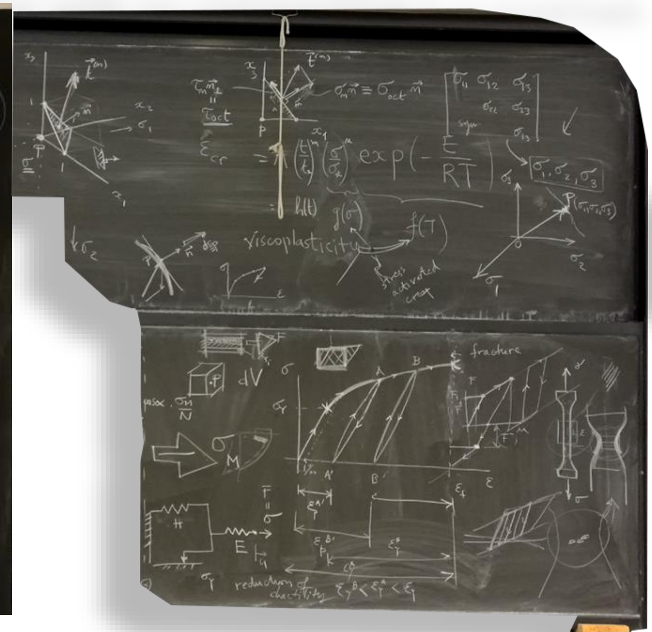
Ref: This figure is based on Chaboche slide from: ATHENS – Course MP06 – 16 – 20 March 2009

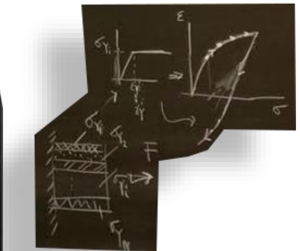
- Load path dependence

During the lecture held on 26.4.2017 on plasticity we discussed a lot about the subject below with interested students present.

I reproduce some of the trace of such discussion with no commenting & no sound track.

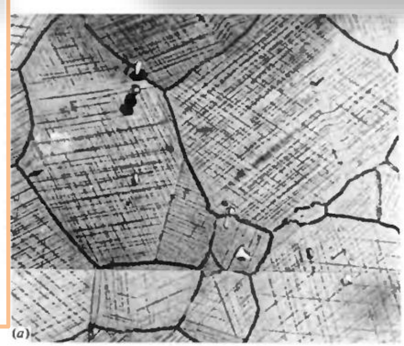






Stainless-steel at 315 °C. Diagonal lines are slip bands, i.e., regions with high degree of plastic slip (*ref.* Shanley, Strength of materials, 1957)

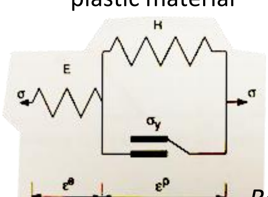
- *Bauschinger effect* is partly explained by the *presence residuals stresses* (in grains and grain boundaries) during reverse loading
- Strain-hardening results from the sequential activation of more and more different slip planes in different grains as the load increases
- Ductility (the value of maximum strain prior fracture) is reduced during loading-unloading cycles

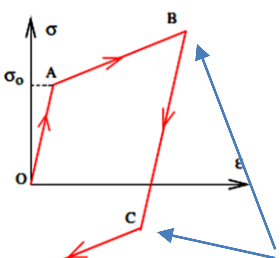


Slip bands in nickel based Wasploy.

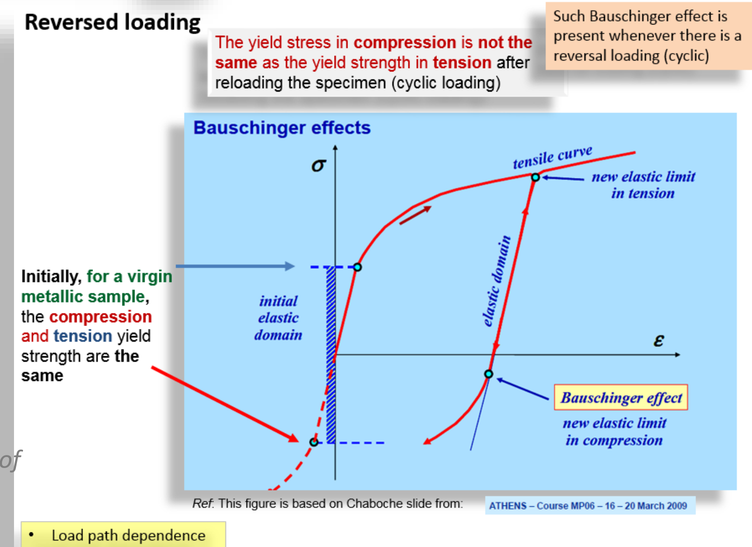
Ref. Lemaitre & Chaboche, *Mechanics of Solid materials, Cambridge Press University, 1990.*

Rheological model for Elastic-hardening plastic material

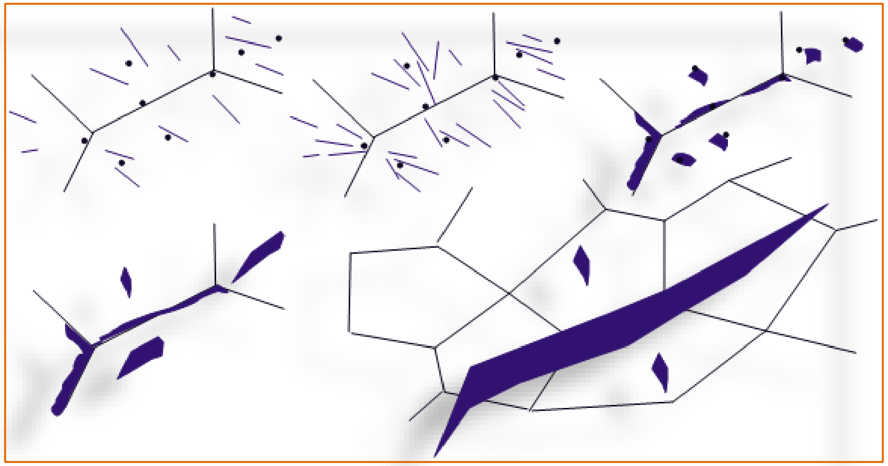




Residual stress in the spring H in reverse loading explains this difference



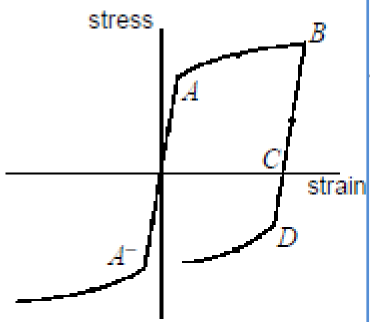
Voids growing and merging - coalescence in ductile material lead to failure



http://www.ltas-cm3.ulg.ac.be/FractureMechanics/overview_P3.html

Examples of some basic plasticity properties – uniaxial test

in the macro-mechanical meaning



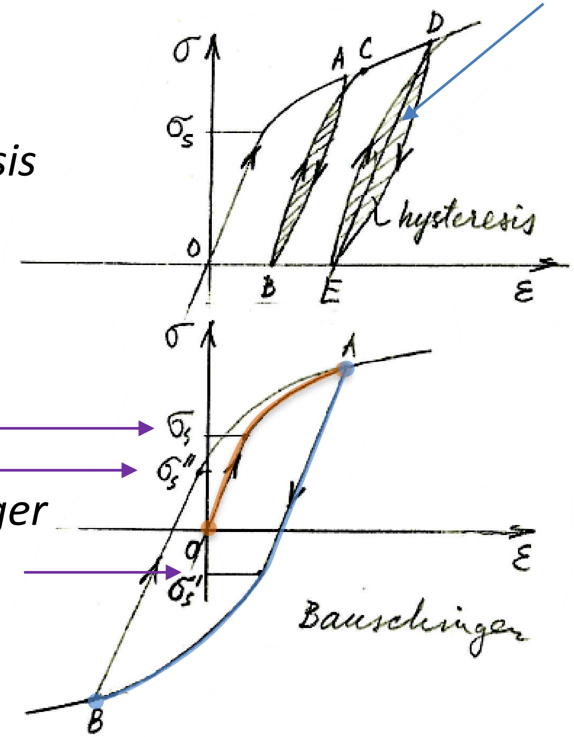
The Bauschinger effect
($D < B$)

cyclic loading

Uniaxial tests

Dissipated work

Hysteresis



For example, yield strength of a metal decreases when the direction of strain is changed

Bauschinger effect

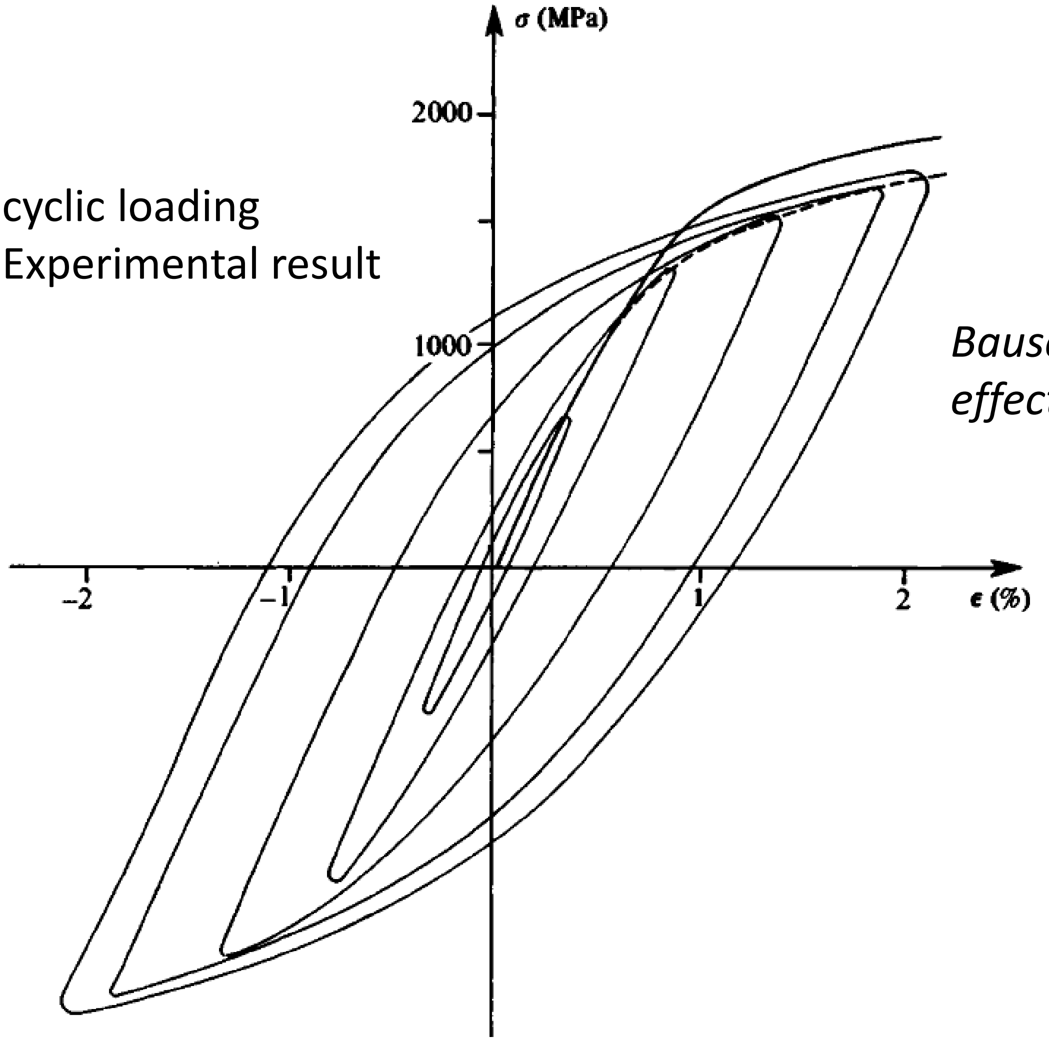
Effect: material's stress/strain characteristics *change* in *cyclic loading* as for instance yield stress (strength) or elasticity limit, …

In soils permanent shearing deformation occurs when constituting particles slide over one another

Examples of some basic plasticity properties – uniaxial test

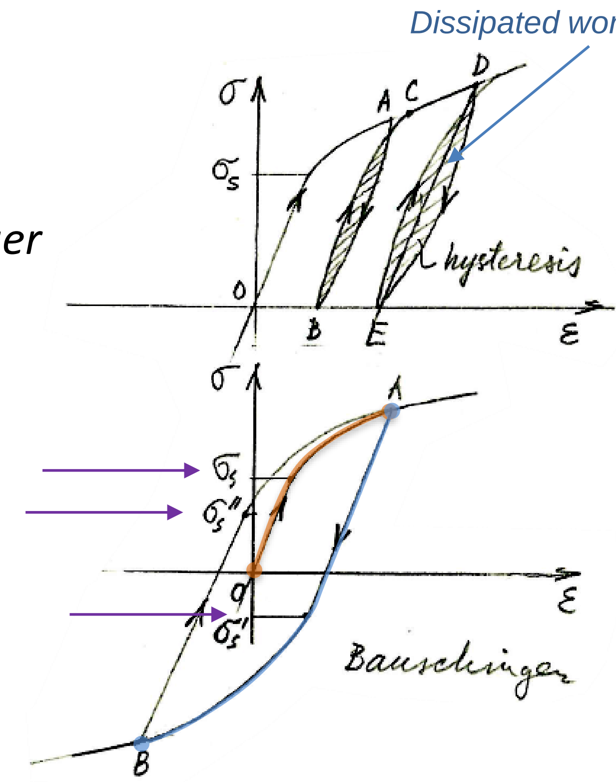
cyclic loading

cyclic loading
Experimental result



Bauschinger effect

Uniaxial tests



Cyclic hardening curve: 30 NCD 16 nickel–chromium steel (after Lieurade).

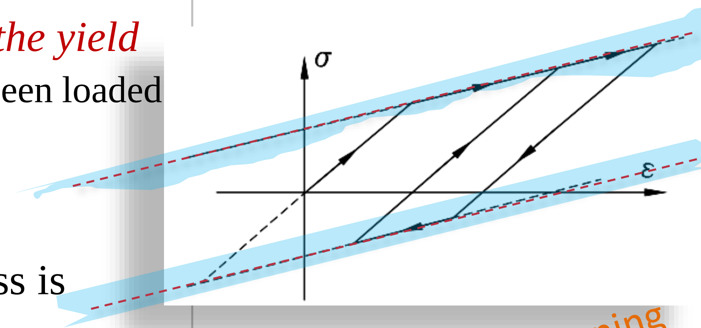
The Bauschinger Effect

Muokkaus, kylmämuovaus

Test: Load a virgin sample first in tension into the plastic range, and then unload it and continue to load it into compression beyond yield.

Observation: the *yield stress in compression is not the same as the yield strength in tension*, as it would have been if the specimen had not first been loaded in tension.

The yield point in compression will be significantly less than the corresponding yield stress in tension. This reduction in yield stress is known as the **Bauschinger effect**.

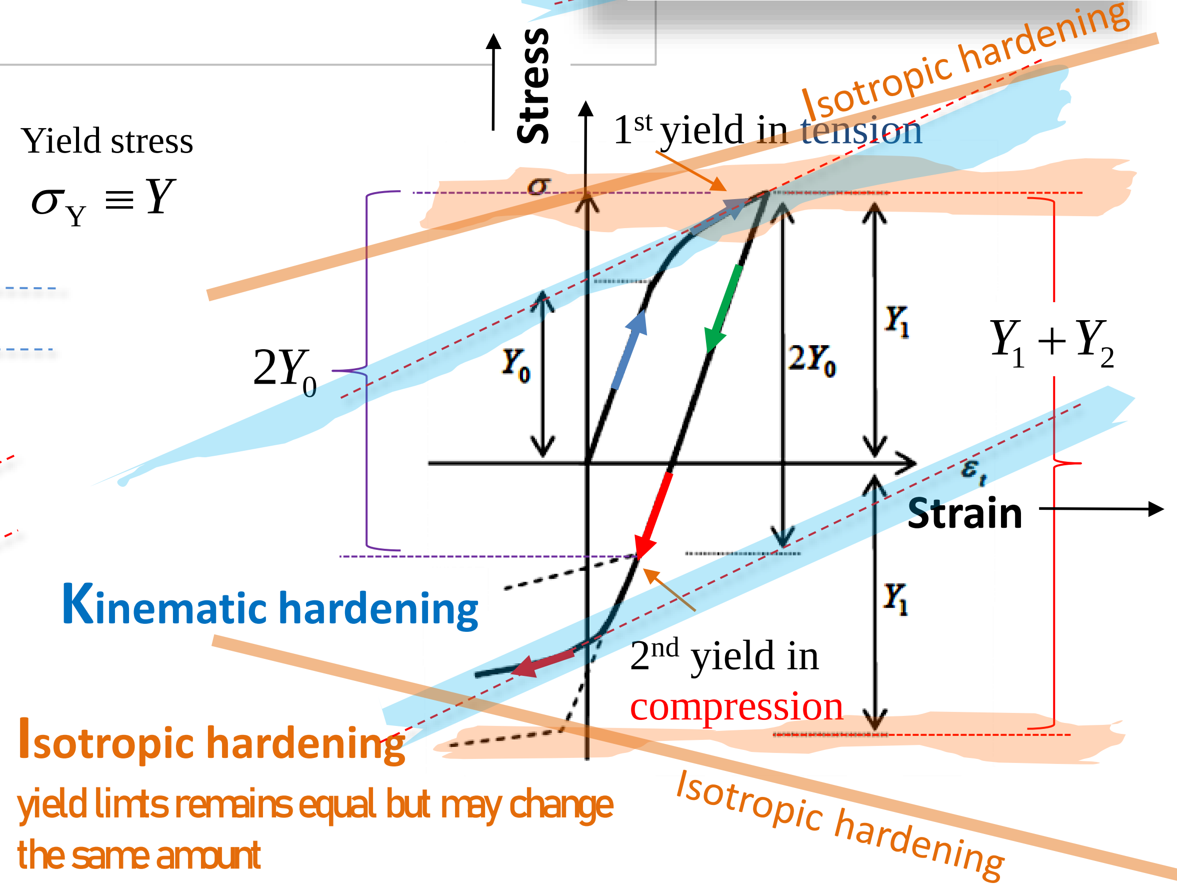


Two extreme cases used in plasticity models:

- **Isotropic hardening** (lujittuminen) model, in which the **yield stress** in **tension** and **compression are** maintained **equal**
- **Kinematic hardening**, in which the total elastic range is maintained **constant** throughout the deformation

Yield stress

$\sigma_Y \equiv Y$



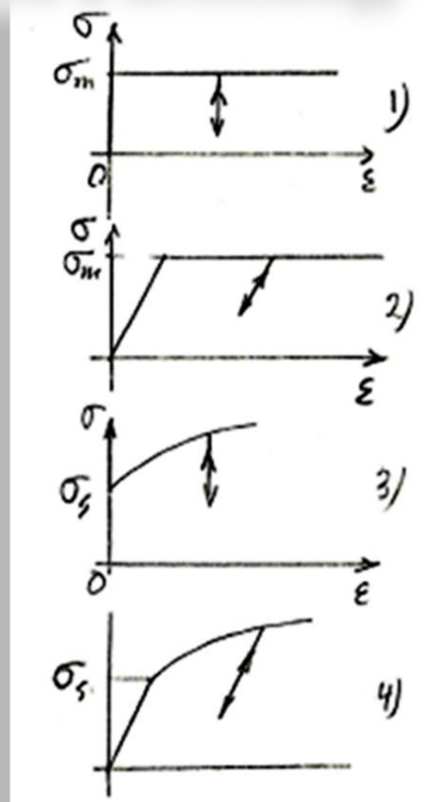
Idealization of elasto-plastic behaviour, some examples of 1D-models & terminology

Rigid Perfectly Plastic
Jäykkä ideaaliplastinen

Elasto-plastic perfect plasticity
Kimmoinen ideaaliplastinen

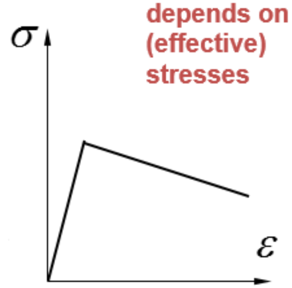
Rigid perfectly plastic with hardening
Jäykkä myötälujeneva

Elasto-plastic hardening
Kimmoinen myötälujeneva



Elasto-plastic softening
Kimmo-plastinen myötä-pehmenevä
Maa, soils,…

Yield limit depends on (effective) stresses



Elasto-plastic softening

For example,
1D-curve of *work-hardening* solids (without a sharply defined yield stress) can be approximated by:
Ramberg–Osgood formula

$$\varepsilon = \frac{\sigma}{E} + \alpha \frac{\sigma_R}{E}\left(\frac{\sigma}{\sigma_R}\right)^m ,$$

reference stress

dimensionless constants

To accommodate an elastic range with an initial **yield** → modified to:

$$\sigma = \begin{cases} E\varepsilon, & \varepsilon \le \dfrac{\sigma_E}{E}, \\ \sigma_E \left(\dfrac{E\varepsilon}{\sigma_E}\right)^n, & \varepsilon \ge \dfrac{\sigma_E}{E}. \end{cases}$$

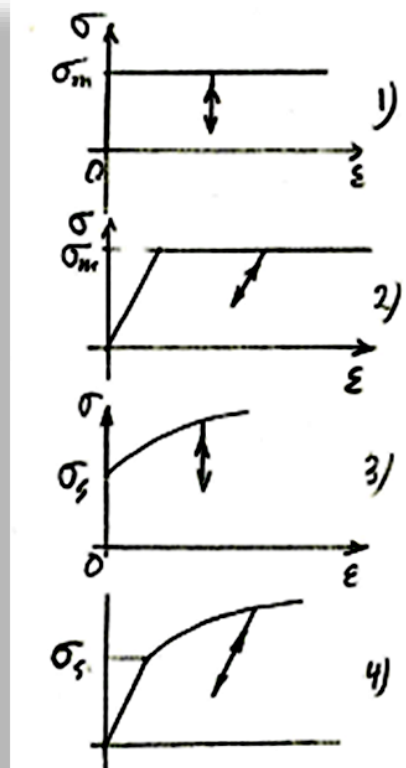
Examples of rheological models for plasticity

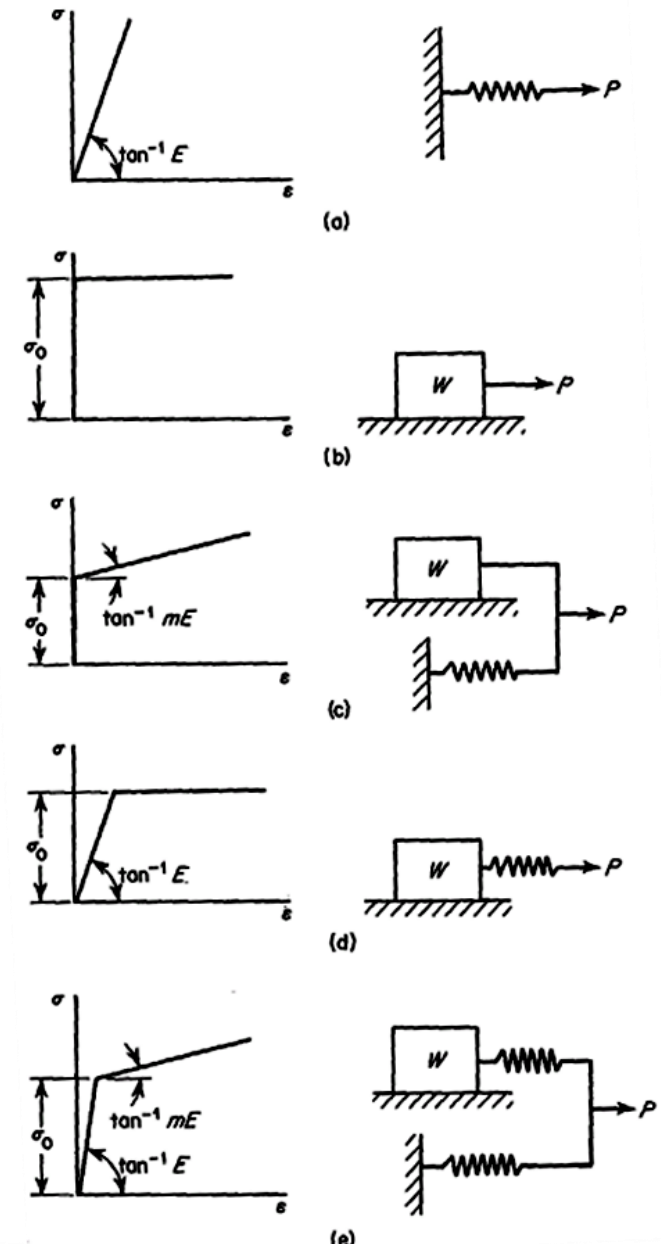
Rigid Perfectly Plastic
Jäykkä ideaaliplastinen

Elasto-plastic perfect plasticity
Kimmoinen ideaaliplastinen

Rigid perfectly plastic with hardening
Jäykkä myötälujeneva

Elasto-plastic hardening
Kimmoinen myötälujeneva





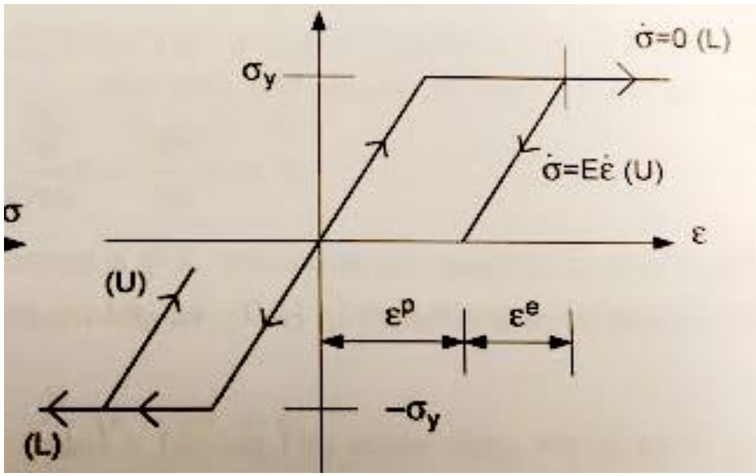
Idealized stress–strain curves: (a) perfectly elastic, brittle; (b) rigid, perfectly plastic; (c) rigid, linear strain hardening; (d) elastic, perfectly plastic; (e) elastic, linear strain hardening.

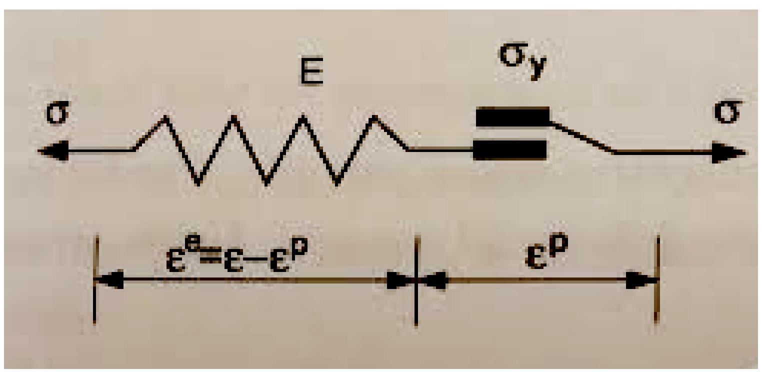
Examples of rheological models for two cases of plasticity

Good to know

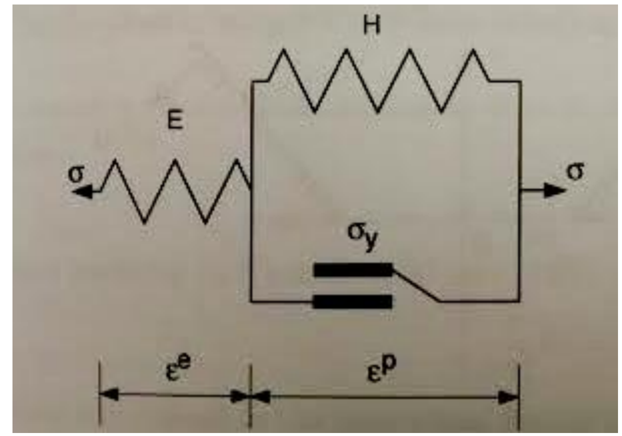
Free to do: Derive the constitutive laws and solve them then draw the strain-stress graphs for some cyclic loading history. You can integrate the ODE by hand or Matlab…
Construct one corresponding mechanical device (at least for the simplest case) and do the experiments by yourself. What do you observe? Feel the force. Is it conform to what is described by the models?

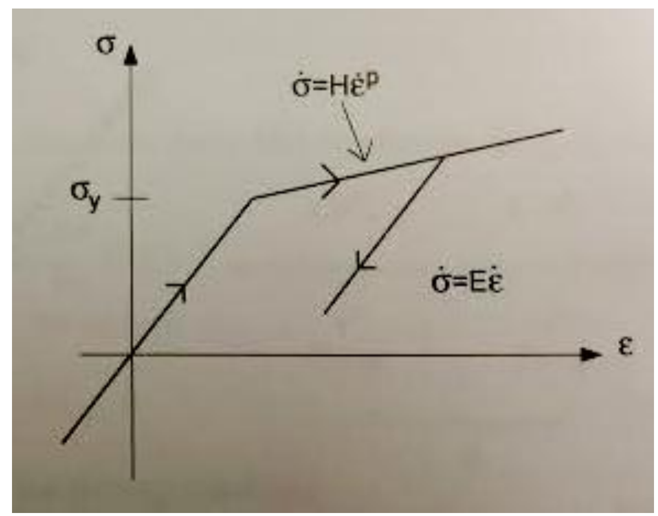
Rheological model form an Ela perfectly plastic material





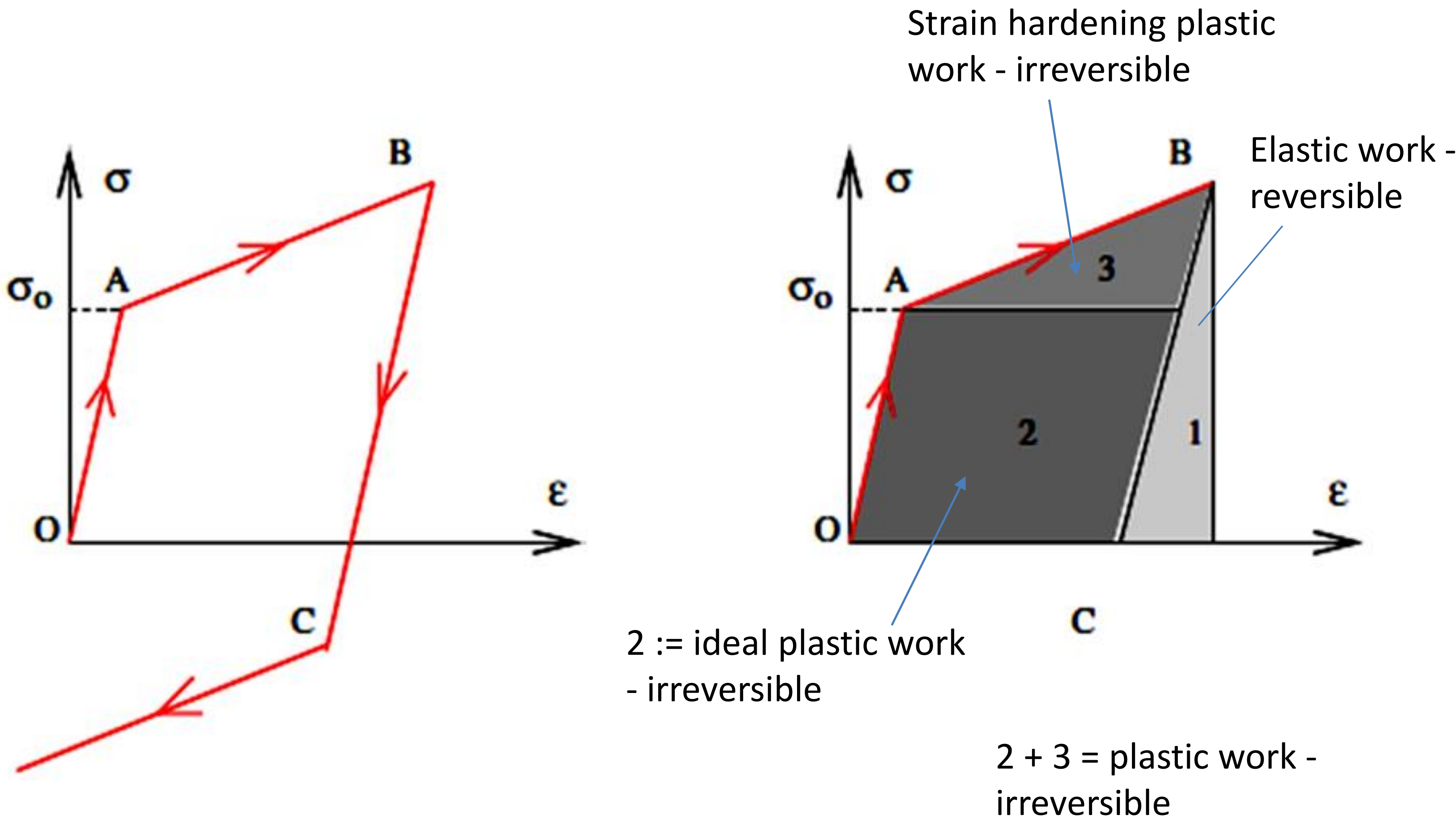
Rheological model form an Elastic-hardening plastic material





In this course, rate-independent plasticity is not treated through rheological models

Linear kinematic strain hardening – partition of work partition



Modelling of uniaxial behavior in plasticity – simplified models

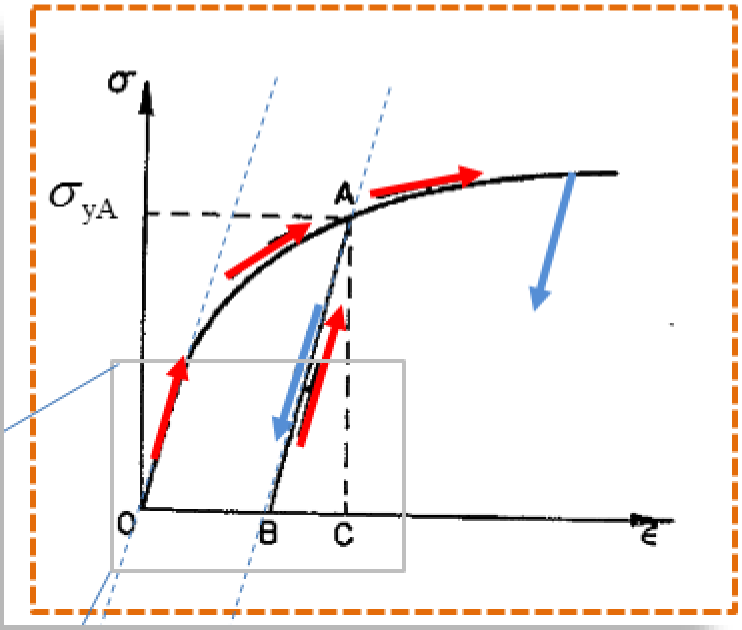
- Elastic-Perfectly Plastic Model
- Elastic-Linear Work-hardening model
- Elastic-Exponential Hardening model
- Ramberg-Osgood model

N.B.

In plasticity, *because of path dependency of the material behavior*, the constitutive law for the plastic strain is known only for *strain increments*

$$\mathrm{d}\varepsilon = \mathrm{d}\varepsilon_{\mathrm{e}} + \mathrm{d}\varepsilon_{\mathrm{p}}$$

Stress-strain relations are known only in their incremental or equivalently rate forms.

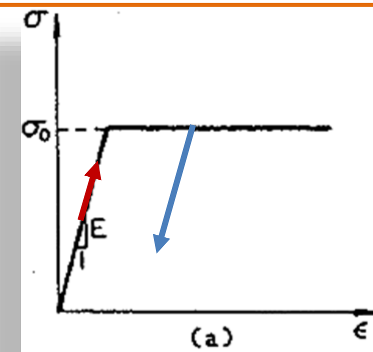


Elastic-Perfectly Plastic Model

$$\varepsilon = \begin{cases} \dfrac{\sigma}{E}, & \sigma < \sigma_0 \\ \dfrac{\sigma}{E} + \lambda, & \sigma = \sigma_0 \end{cases}$$

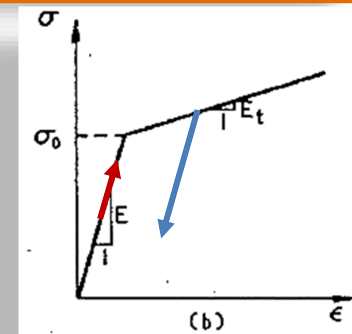
Total strain: $\varepsilon = \varepsilon^{\mathrm{e}} + \varepsilon^{\mathrm{P}}$

Strain increments: $\mathrm{d}\varepsilon = \mathrm{d}\varepsilon_{\mathrm{e}} + \mathrm{d}\varepsilon_{\mathrm{p}}$



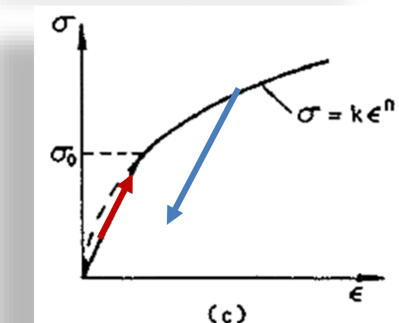
Elastic-Linear Work-hardening model

$$\varepsilon = \begin{cases} \dfrac{\sigma}{E}, & \sigma \le \sigma_0 \\ \dfrac{\sigma_0}{E} + \dfrac{1}{E_t}(\sigma - \sigma_0), & \sigma > \sigma_0 \end{cases}$$



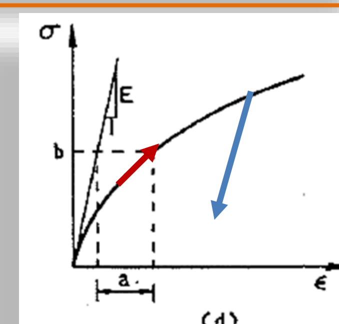
Elastic-Exponential Hardening model

$$\sigma = \begin{cases} E\varepsilon, & \sigma \le \sigma_0 \\ k\varepsilon^n, & \sigma > \sigma_0 \end{cases}$$



Ramberg-Osgood model

$$\varepsilon = \frac{\sigma}{E} + a\left(\frac{\sigma}{b}\right)^n$$ or more esthetically written $$\varepsilon = \frac{\sigma}{E} + K\left(\frac{\sigma}{E}\right)^n$$



$a, b, n, b, k, K, E_{\mathrm{t}}, \lambda -$ describe the hardening behavior of the material.

Tangent- and plastic modulus

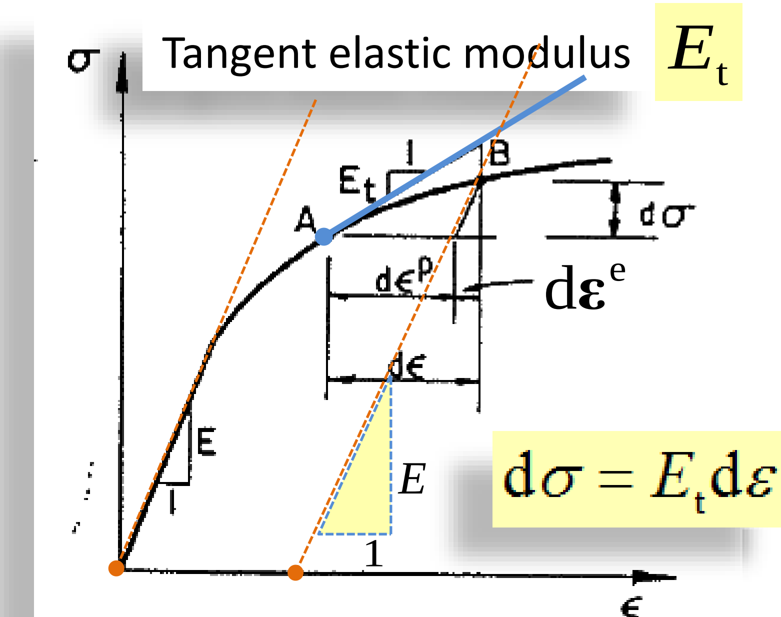
In plasticity the material shows *path* or *history dependency* behavior; thus the *strain-stress* response of the material is *non-linear* by nature (elastic-plastic behavior),

$$\Downarrow$$

Stress-strain relations are **known** only in their **incremental** or equivalently **rate forms**.

$$\mathrm{d}\varepsilon = \mathrm{d}\varepsilon^{\mathrm{e}} + \mathrm{d}\varepsilon^{\mathrm{p}}$$

$$\mathrm{d}\sigma = \mathrm{d}\sigma^{\mathrm{e}} + \mathrm{d}\sigma^{\mathrm{p}}$$



Tangent elastic modulus: E_{t}, $\quad \mathrm{d}\sigma = E_{\mathrm{t}}\mathrm{d}\varepsilon$

Plastic modulus: E_{p}, $\quad \mathrm{d}\sigma = E_{\mathrm{p}}\mathrm{d}\varepsilon^{\mathrm{p}}$

The relation between the three moduli:

$$\frac{1}{E_{\mathrm{t}}} = \frac{1}{E} + \frac{1}{E_{\mathrm{p}}}$$

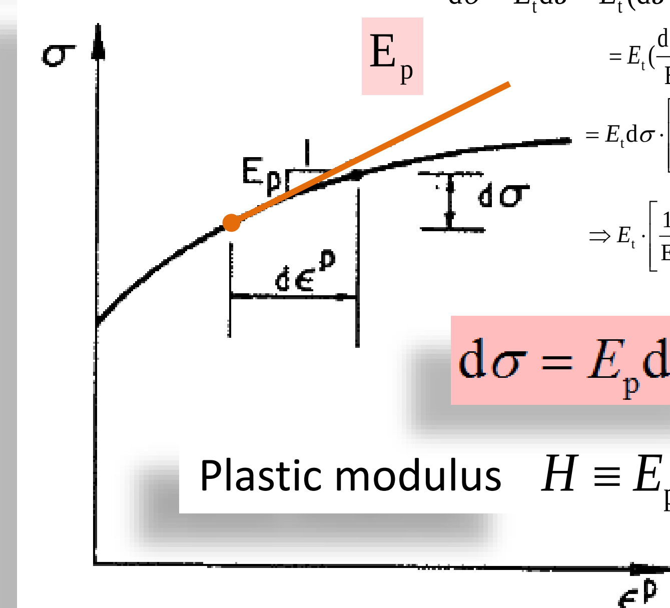
Show that it is true!

$$\mathrm{d}\sigma = E_{\mathrm{t}}\mathrm{d}\varepsilon = E_{\mathrm{t}}(\mathrm{d}\varepsilon^{\mathrm{e}} + \mathrm{d}\varepsilon^{\mathrm{p}})$$

$$= E_{\mathrm{t}}\left(\frac{\mathrm{d}\sigma}{\mathrm{E}} + \frac{\mathrm{d}\sigma}{E_{\mathrm{p}}}\right)$$

$$= E_{\mathrm{t}}\mathrm{d}\sigma \cdot \left[\frac{1}{\mathrm{E}} + \frac{1}{E_{\mathrm{p}}}\right]$$

$$\Rightarrow E_{\mathrm{t}} \cdot \left[\frac{1}{\mathrm{E}} + \frac{1}{E_{\mathrm{p}}}\right] = 1$$



Elasto-Plastic constitutive matrix

3D-example

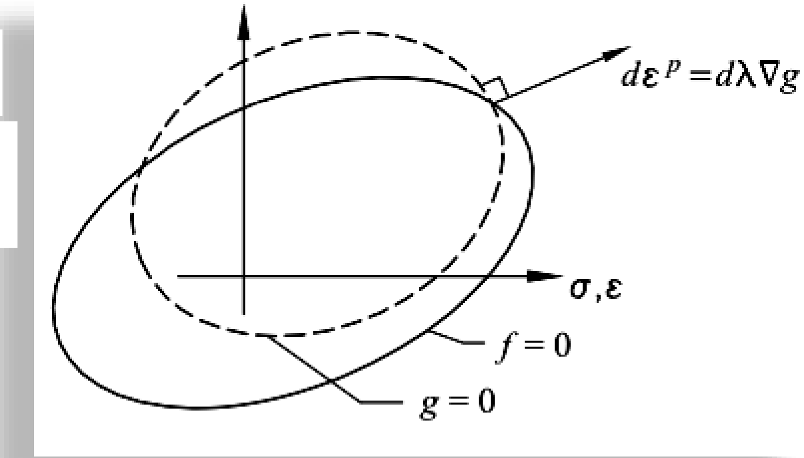
We will come back in details to these examples…

$$d\boldsymbol{\varepsilon} = d\boldsymbol{\varepsilon}^e + d\boldsymbol{\varepsilon}^p$$

$$d\boldsymbol{\varepsilon}^p = d\lambda \frac{\partial g}{\partial \boldsymbol{\sigma}} = d\lambda \nabla g$$

$$d\boldsymbol{\sigma} = \boldsymbol{D}^{ep} d\boldsymbol{\varepsilon}$$

the elasto-plastic constitutive matrix



The consistency condition leads to:

The **elasto-plastic** constitutive relation

No hardening:

$$d\boldsymbol{\sigma} = \left(\boldsymbol{D} - \frac{\boldsymbol{D}\frac{\partial g}{\partial \boldsymbol{\sigma}}\left(\frac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T \boldsymbol{D}}{\left(\frac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T \boldsymbol{D}\frac{\partial g}{\partial \boldsymbol{\sigma}}} \right) d\boldsymbol{\varepsilon}$$

the elastic constitutive matrix

1-D:

$$\mathrm{d}\sigma = E_t \mathrm{d}\varepsilon$$

With hardening:

$$d\boldsymbol{\sigma} = \left(\boldsymbol{D} - \frac{\boldsymbol{D}\frac{\partial g}{\partial \boldsymbol{\sigma}}\left(\frac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T \boldsymbol{D}}{H + \left(\frac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T \boldsymbol{D}\frac{\partial g}{\partial \boldsymbol{\sigma}}} \right) d\boldsymbol{\varepsilon}$$

Hardening parameter

The elasto-plastic constitutive relation defines the stress increment uniquely once the total strain increment and the current state of stress is known

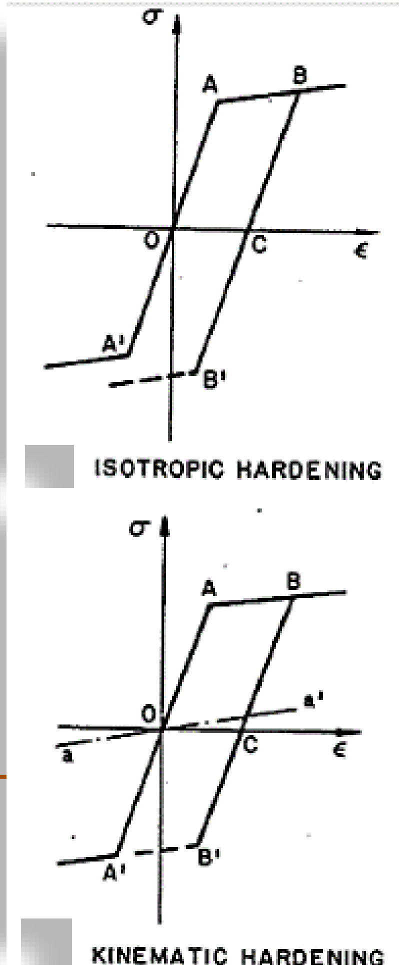
Hardening rules

The phenomena of hardening: Yield stress increases with further increase of plastic strains

Hardening and its various stages is modelled by introducing some *internal variable* as for instance *a hardening parameter* $\kappa > 0$

and assumes, for instance, that the plastic modulus changes as a function of the hardening parameter

$$E_{\mathrm{p}} = E_{\mathrm{p}}(\kappa)$$



Choice of the hardening parameter:

$\kappa = W_{\mathrm{p}}$ or $\kappa = \varepsilon_{\mathrm{p}}$

plastic work

$$W_{\mathrm{p}} = \int \sigma_{ij} \mathrm{d}\varepsilon_{ij}^{p}$$

$$W_{\mathrm{p}} > 0$$

accumulated *effective or equivalent* plastic strain

$$\varepsilon_{\mathrm{p}} = \int \sqrt{\mathrm{d}\varepsilon_{ij}^{p} \mathrm{d}\varepsilon_{ij}^{p}} > 0$$

Effective plastic strain increment:

$$\left|\mathrm{d}\varepsilon_{\mathrm{p}}\right| = \sqrt{\mathrm{d}\varepsilon_{ij}^{p} \mathrm{d}\varepsilon_{ij}^{p}} \geq 0$$

Using such hardening parameter - the *lazy way* - is more practical than computing the plastic work

For a material, the functional relation $E_{\mathrm{p}} = E_{\mathrm{p}}(\kappa)$ can be determined from the uniaxial tensile stress-strain curve in terms of the above definition of the hardening parameter κ

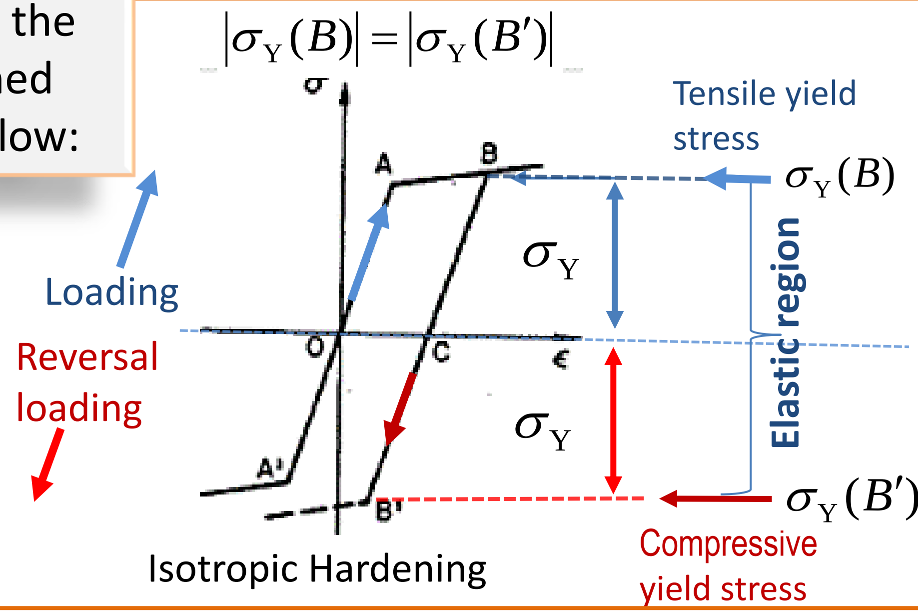
Hardening rules (models)

For a material under **reversal** or cyclic loading, the subsequent *yield stress* σ is usually determined usually by one of the three hardening rules below:

Isotropic Hardening rule:

- Assumes ***equal yield stress*** in tension and **compression**
  - ➢ thus the *Bauschinger* effect is *completely neglected*.

Hardening rule: $|\sigma| = |\sigma(\kappa)|, \quad \kappa > 0$



Isotropic Hardening

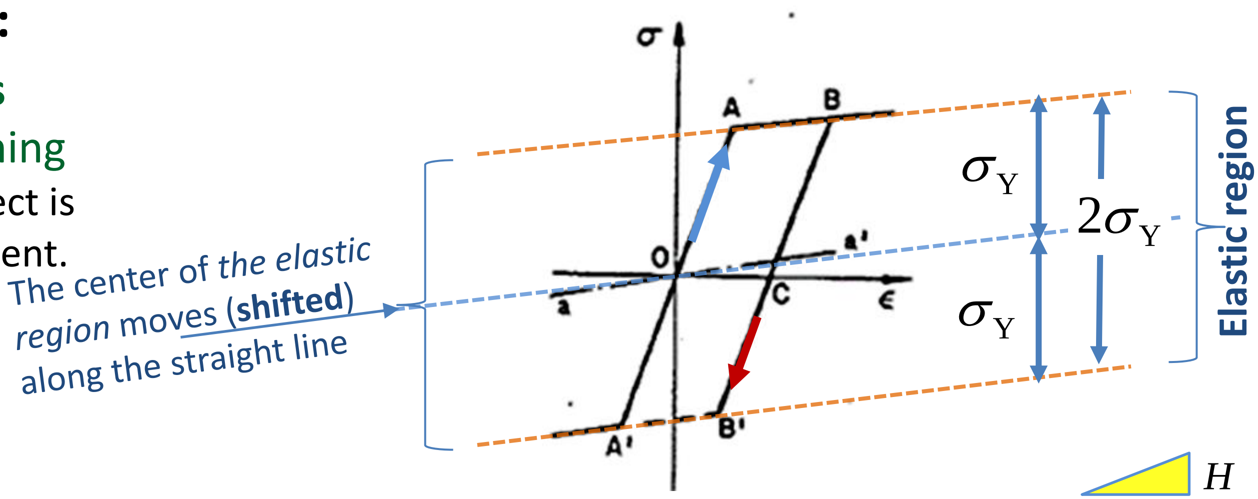
Kinematic Hardening rule:

- The **elastic range remains unchanged** during hardening
  - ➢ So the *Bauschinger* effect is *accounted* to its full extent.

Hardening rule:

$$|\sigma - c(\kappa)| = \sigma_0,$$

shift of the yield stress, $\kappa > 0$

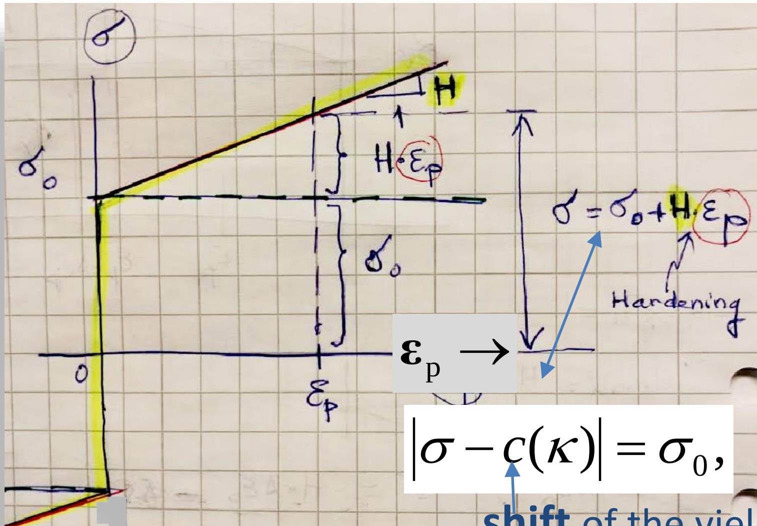


Linear Kinematic Hardening

Esim. $c(\kappa) \rightarrow H \cdot \varepsilon_p$

Hardening rules (models)

Kinematic Hardening rule:



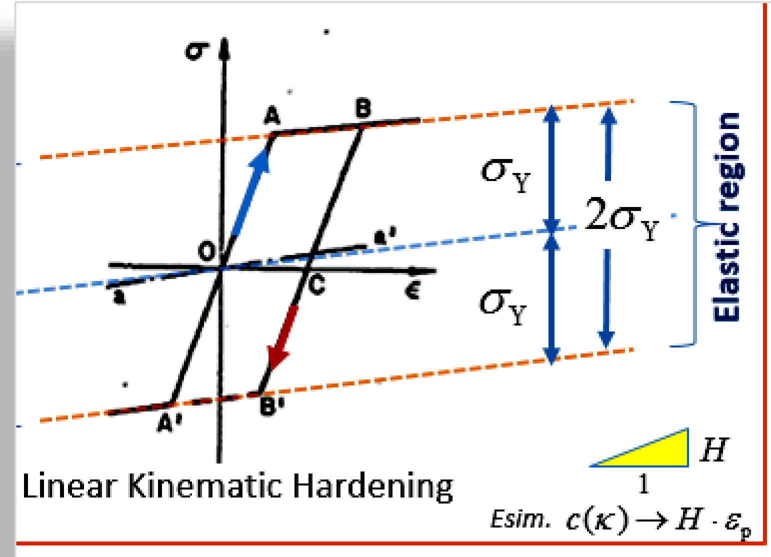
$\boldsymbol{\varepsilon}_{\mathrm{p}} \rightarrow$

$$\left|\sigma - c(\kappa)\right| = \sigma_0,$$

shift of the yield stress

Loading

Reversal loading



Independent Hardening rule:

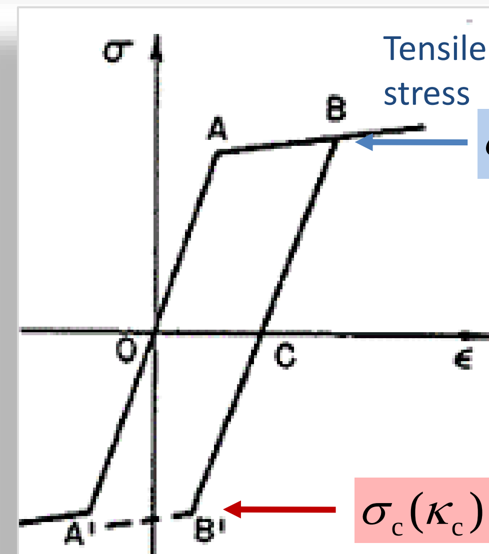
- Assumes ***yield stress*** in **tension** and **compression** are ***independent*** **(different)**

Hardening rule:

$$\sigma = \begin{cases} \sigma_t(\kappa_t), & \sigma > 0 \\ \sigma_c(\kappa_c), & \sigma < 0 \end{cases}$$

$\kappa_t > 0$ **tension**

compression $\kappa_c > 0$



Tensile yield stress $\sigma_t(\kappa_t)$

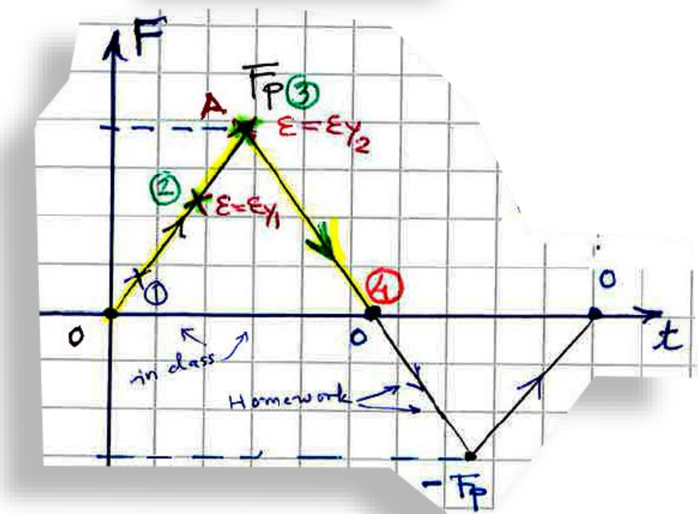
$\sigma_c(\kappa_c)$ Compressive yield stress

Elastic-plastic behaviour – Cyclic loading

Worked uniaxial **example**

This simple example demonstrates some fundamental aspects related to engineering plasticity as applied to structures such a *non-linear force-displacement relation*, *loading path (history) dependence*, *residual stresses* and *strains, residual displacements upon complete unloading, reloading,*… etc.

loading history



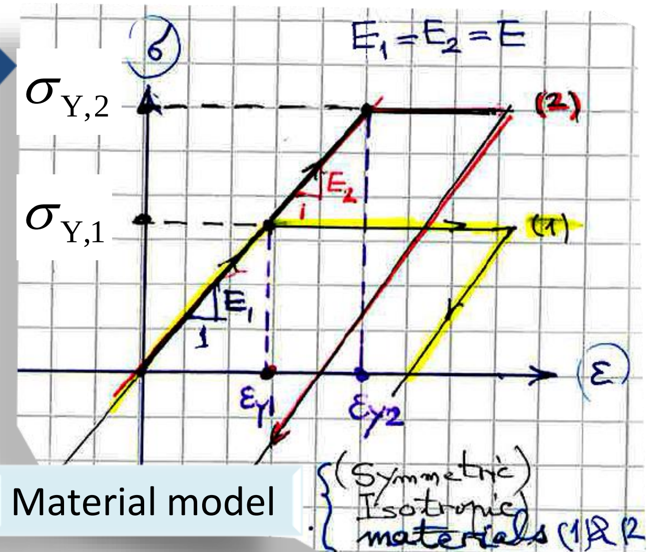
Example

Elastic-plastic behaviour – an example of uniaxial behaviour

Example

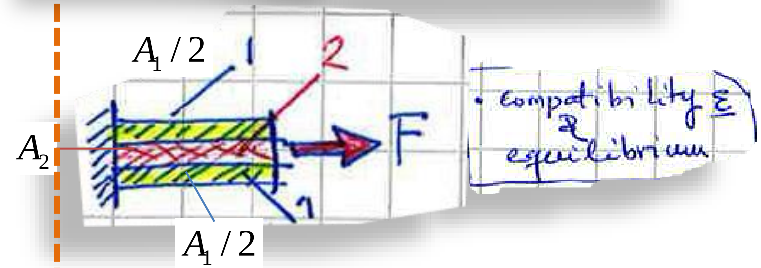
Loading history dependency, residual stresses and residual displacement upon complete unloading and Bauschinger effect (as an exercise to be continued by the student)

Introduction: It is well known that the mechanical behaviour of a polycrystal metallic material, as metals, can be analogous to a truss structure composed of many bares having elastic-plastic individual material behaviour.



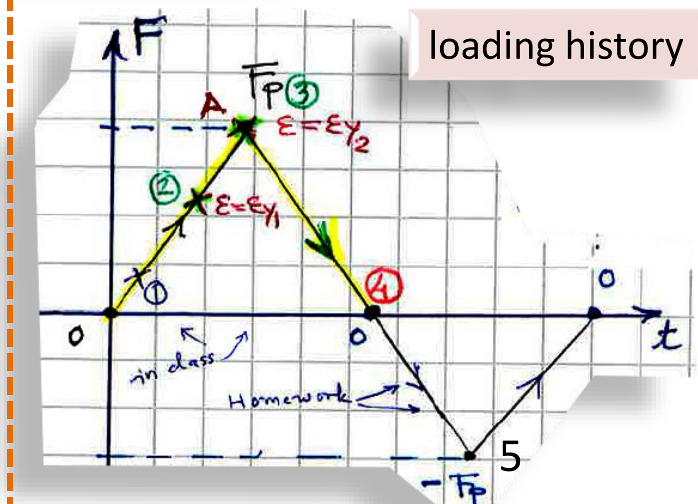
Material model

Consider a single such bare of length ℓ which is composed of two different metallic materials which are perfectly bond to each other. Both materials isotropic and are made of elastic-perfectly plastic materials. The cross-section of the bares are and $A_1 = A_2 = A$ modulus is $E_1 = E_2 = E$.



Both materials behave separately symmetrically in compression and in tension.

The composite bare is loaded according to the history shown on the figure (bottom). The applied load increases from $F = 0$ the plastic limit load $F = F_p$ and then decreases to zero (loading path **0**-1-2-**3**-**4**-**5**).



i) Determine the corresponding **force-displacement curve**
ii) Upon complete unloading, find the **residual stresses** and strains and **residual displacement**.
iii) Finish the force-displacement curve for the compression till point **5**.

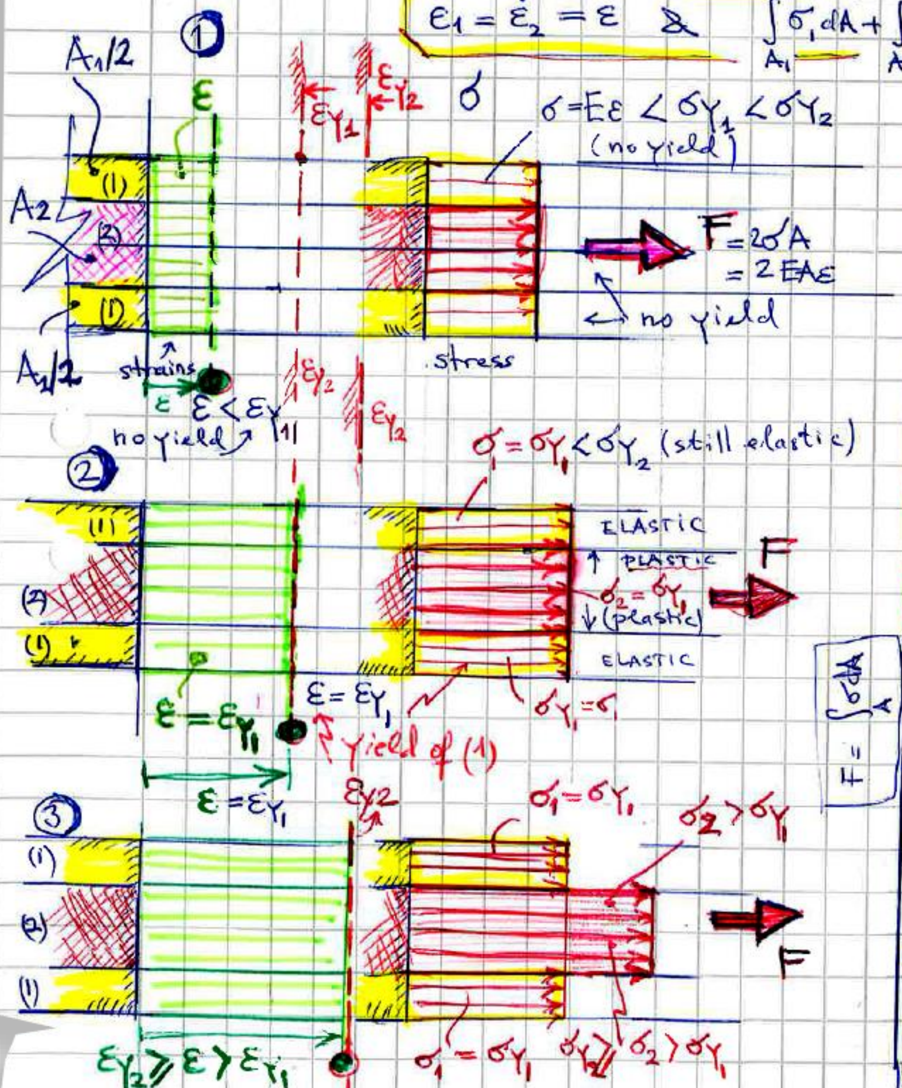
Solution 1(5)

Example

I • Loading from $F=0$ to F_p monotonically

(Hint: think that the ~~forcce~~ force F is imposed by a displacement control: increasing tip displacement monotically until the limit plastic (resistance) load F_p is reached)

$$\boxed{\varepsilon_1 = \varepsilon_2 = \varepsilon \quad \& \quad \int_{A_1} \sigma_1 dA + \int_{A_2} \sigma_2 dA = F} \quad (I) \quad \text{seed}$$



$\Delta\sigma_i = E_i \Delta\varepsilon_i$

(1) $\sigma_1 = E_1\varepsilon_1 = E\varepsilon < \sigma_{Y_1}$

$\sigma_2 = E_2\varepsilon_2 = E\varepsilon < \sigma_{Y_1}$

$F = \sigma_1 A_1 + \sigma_2 A_2$

$\Downarrow$

$\boxed{F} = E\varepsilon A + E\varepsilon A = 2EA\varepsilon$

$\boxed{u} = \varepsilon \cdot l = \boxed{\frac{\sigma}{E}} l = \frac{Fl}{2EA}$

(2) $\sigma_1 = E_1\varepsilon_1 = E_1\varepsilon_{Y_1}$

$\sigma_2 = E_2\varepsilon_2 = E_2\varepsilon_{Y_1}$

$\boxed{F} = \sigma_1 A_1 + \sigma_2 A_2 = E_1\varepsilon_{Y_1} A_1 + E_2\varepsilon_{Y_1} A_2 = EA\varepsilon_{Y_1} \cdot 2 = \boxed{2\sigma_{Y_1} A}$

$u = \varepsilon l = \varepsilon_{Y_1} l = \boxed{\frac{\sigma_{Y_1}}{E_1} l}$

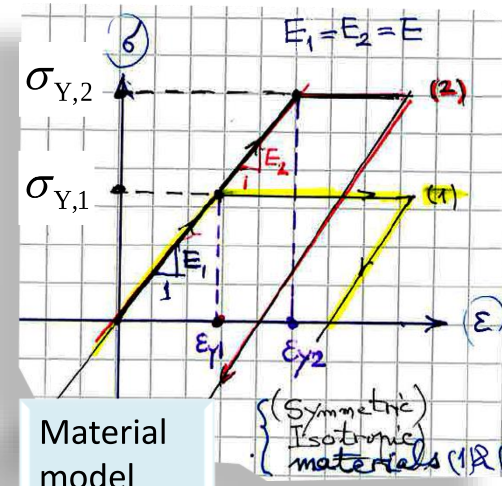
(3) $\sigma_1 = \sigma_{Y_1}$, $\sigma_{Y_1} < \sigma_2 \le \sigma_{Y_2}$; $\varepsilon_1 = \varepsilon_2 = \varepsilon$

$F = \sigma_Y A_1 + \sigma_2 A_2 = A(\sigma_{Y_1} + \sigma_2)$

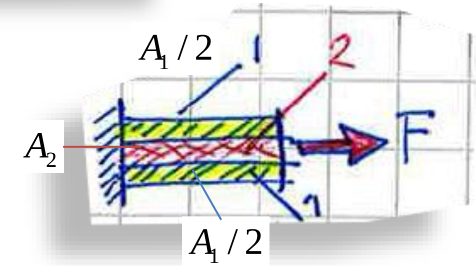
at yield $\Rightarrow F_p = \sigma_{Y_1} A_1 + \sigma_{Y_2} A_2$

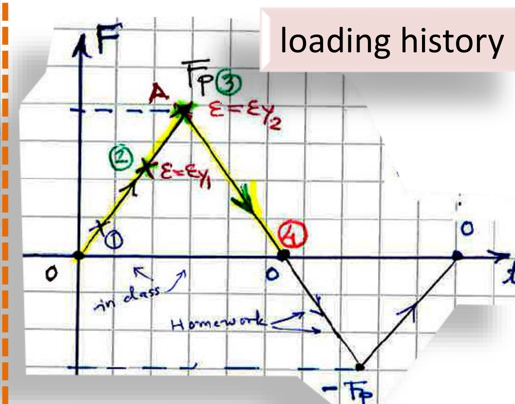
$\boxed{F_p = A(\sigma_{Y_1} + \sigma_{Y_2})}$

$u = \varepsilon \cdot l = \varepsilon_2 \cdot l = \frac{\sigma_{Y_2}}{E_2} \cdot l$



Material model





loading history

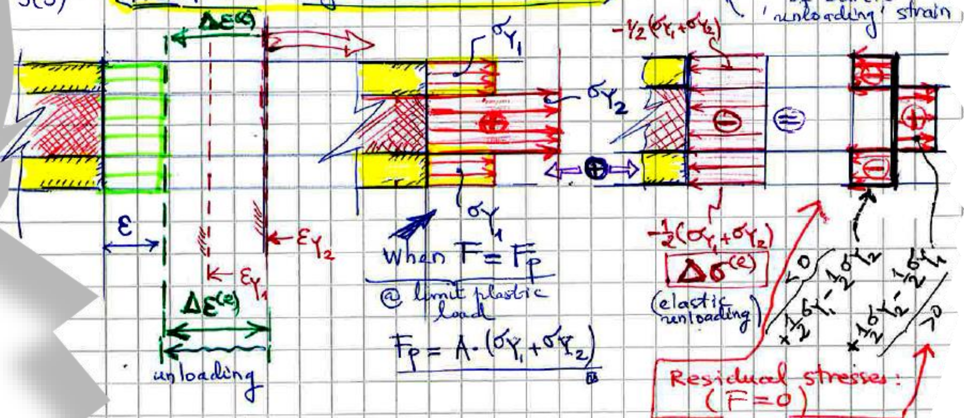
The 'seed': • compatibility ε & equilibrium

Solution 2(5)

3(3)

II • Unloading from $F = F_p$ to $F = 0$

N.B.: unloading occurs elastically : $\Delta\varepsilon^{(e)}$ the amount of elastic 'unloading' strain, unknown yet



Elastic unloading by an amount $\Delta\varepsilon^{(e)} = \Delta\varepsilon_i^{(e)}$, $i = 1,2$ such that the resulting stress distribution have a zero resultant when $F = 0$

New stress state: $\begin{cases} \boxed{\sigma_i = \sigma_{Y_i} + \Delta\sigma_i^{(e)}} \ (a), \ i = 1,2 \\ \Delta\sigma_i^{(e)} = E_i\,\Delta\varepsilon^{(e)} \end{cases}$ and $F = \int_A \sigma\, dA = 0 \Rightarrow \Delta\varepsilon^{(e)}$ will be solved

$$\left.\begin{aligned}\sigma_1 &= \sigma_{Y_1} + E_1\Delta\varepsilon^{(e)}\\ \sigma_2 &= \sigma_{Y_2} + E_2\Delta\varepsilon^{(e)}\end{aligned}\right\} \Rightarrow A_1\sigma_1 + A_2\sigma_2 = 0 \Rightarrow$$

$$A_1(\sigma_{Y_1} + E_1\Delta\varepsilon^{(e)}) + A_2(\sigma_{Y_2} + E_2\Delta\varepsilon^{(e)}) = 0$$

$$A_1\sigma_{Y_1} + A_2\sigma_{Y_2} = -(E_1A_1 + E_2A_2)\,\Delta\varepsilon^{(e)} \Rightarrow$$

$$\Delta\varepsilon^{(e)} = -\frac{A_1\sigma_{Y_1} + A_2\sigma_{Y_2}}{E_1A_1 + E_2A_2}; \quad \text{now } \begin{cases} E_1 = E_2 = E \\ \text{and} \\ A_1 = A_2 = A \end{cases}$$

$$\Rightarrow \boxed{E\Delta\varepsilon^{(e)} = +\Delta\sigma^{(e)} = -\frac{1}{2}(\sigma_{Y_1} + \sigma_{Y_2})} \ (b)$$

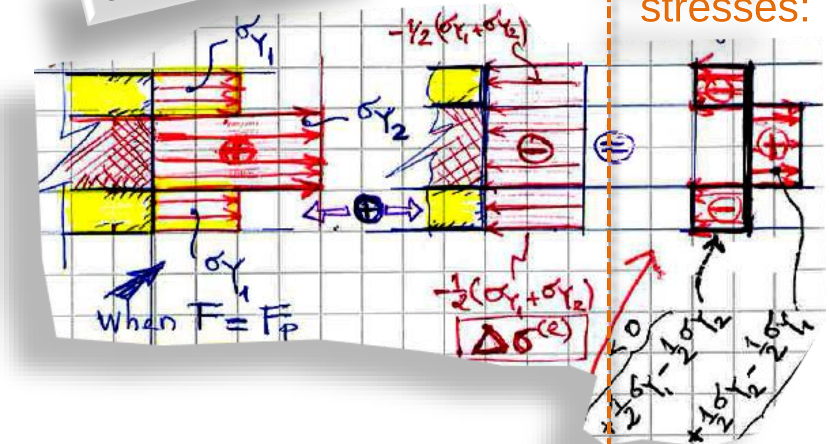
$\hookrightarrow$ therefore $\sigma_i = \underbrace{\sigma_{Y_i}}_{\text{stress distribution @ } F=F_p} + \underbrace{\Delta\sigma^{(e)}}_{\text{elastic stress distribution}}$

N.B. $\int_A \Delta\sigma^{(e)}\, dA = -\frac{1}{2}A(\sigma_{Y_1}+\sigma_{Y_2}) = -F_p$

stress distribution upon complete unloading $F=0$:

RESIDUAL STRESSES

$$\begin{aligned}\sigma_1 &= \sigma_{Y_1} - \tfrac{1}{2}\sigma_{Y_1} - \tfrac{1}{2}\sigma_{Y_2} = +\tfrac{1}{2}\sigma_{Y_1} - \tfrac{1}{2}\sigma_{Y_2} \ \ {}^{\geq}_{<}0\\ \sigma_2 &= \sigma_{Y_2} - \tfrac{1}{2}\sigma_{Y_1} - \tfrac{1}{2}\sigma_{Y_2} = +\tfrac{1}{2}\sigma_{Y_2} - \tfrac{1}{2}\sigma_{Y_1} \ \ {}^{\geq}_{<}0\end{aligned}$$



Residual stresses:

Residual stresses: $F = 0$

$$\sigma_1^{R} = (\sigma_{Y_1} - \sigma_{Y_2})/2 = -\sigma_2^{R}$$
$$\sigma_2^{R} = (\sigma_{Y_2} - \sigma_{Y_1})/2 = -\sigma_1^{R}$$

$$u(F=0) = u(F_p) + \Delta\varepsilon^{(e)}\cdot\ell$$
$$u(F=0) = \frac{\sigma_{Y_2}}{E}\cdot\ell - \frac{1}{2}\left(\frac{\sigma_{Y_1}+\sigma_{Y_2}}{E}\right)\cdot\ell = \frac{1}{2E}(\sigma_{Y_2}-\sigma_{Y_1})\cdot\ell = \frac{\ell}{2}(\sigma_{Y_2}-\sigma_{Y_1})/E$$

Residual tip displacement: $u(F=0) \equiv u^{R} = \dfrac{\sigma_{Y_2} - \sigma_{Y_1}}{2E}\,\ell$

Residual stresses: $\sigma_1^{\mathrm{R}} = \left(\sigma_{\mathrm{Y}_1} - \sigma_{\mathrm{Y}_2}\right)/\,2 = -\sigma_2^{\mathrm{R}}$

$F = 0$

$\sigma_2^{\mathrm{R}} = \left(\sigma_{\mathrm{Y}_2} - \sigma_{\mathrm{Y}_1}\right)/\,2 = -\sigma_1^{\mathrm{R}}$

$u(F=0) = u(F_p) + \Delta\varepsilon^{(e)} \cdot \ell$

$u(F=0) = \frac{\sigma_{Y_2}}{E} \cdot \ell - \frac{1}{2}\left(\frac{\sigma_{Y_1} + \sigma_{Y_2}}{E}\right) \cdot \ell$

$= \frac{1}{2E}\left(\sigma_{Y_2} - \sigma_{Y_1}\right) \cdot \ell \quad = \frac{\ell}{2}\left(\sigma_{Y_2} - \sigma_{Y_1}\right)/E$

Residual tip displacement: $u(F=0) \equiv u^{\mathrm{R}} = \dfrac{\sigma_{\mathrm{Y}_2} - \sigma_{\mathrm{Y}_1}}{2E}\,\ell$

Good to know: non-homogeneous residual strains lead to **residual stresses** upon full unloading.

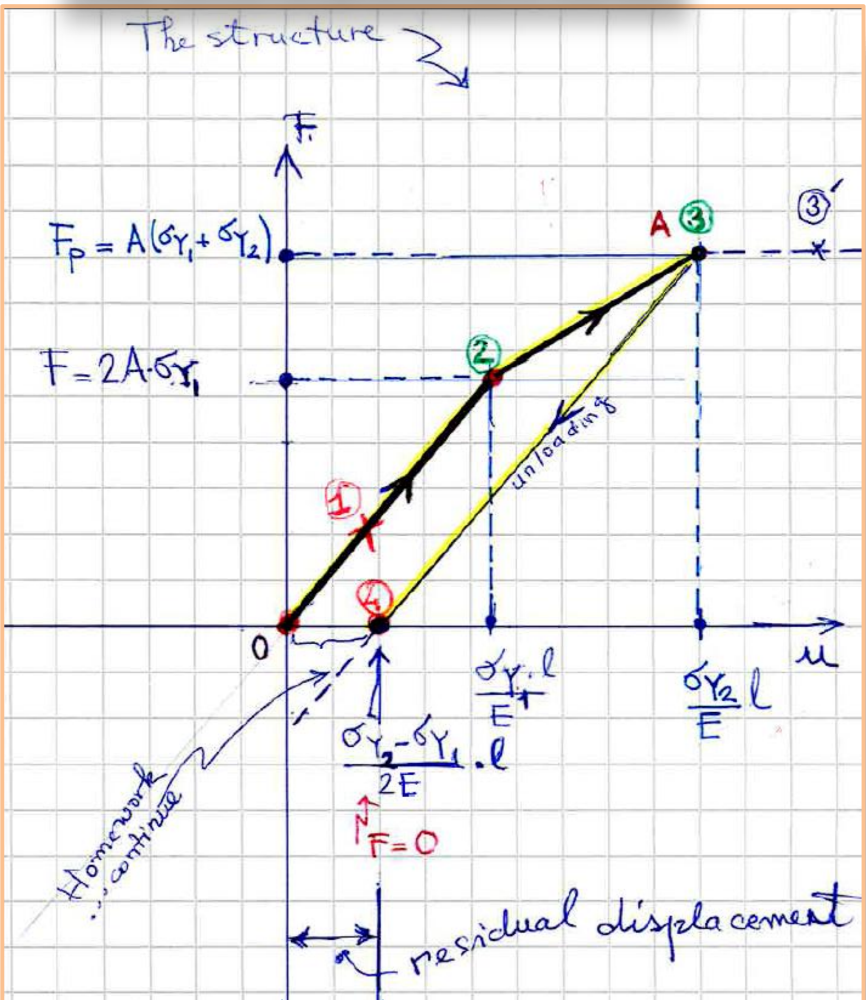
In order to decrease or reduce the these residual stresses heating can used: annealing or recovery temperature of $T \approx 0.3 T_m$ For metals is needed.

Melting temperature

→ stress relaxation by thermal annealing

Solution 3(5)

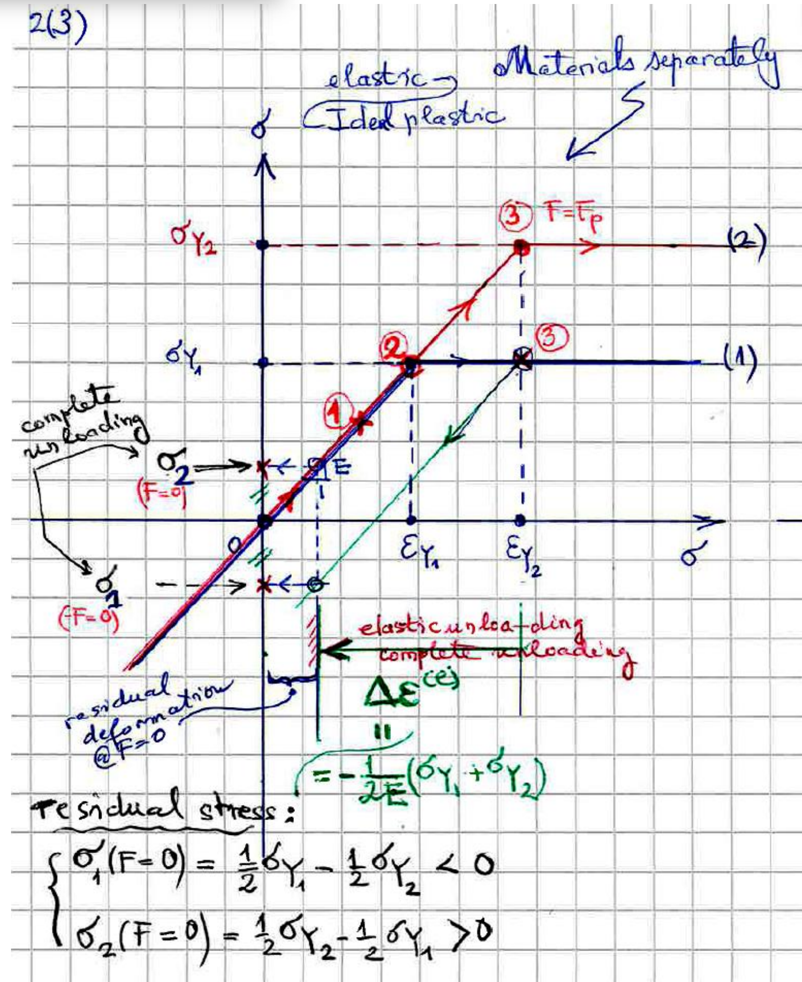
The force-displacement curve



Residual displacements upon complete unloading

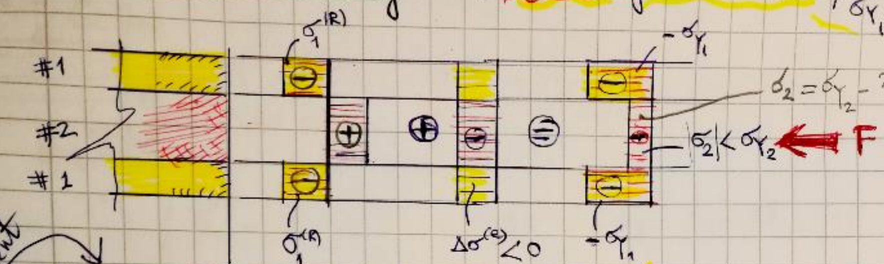
$F = 0$

Material model



Solution 4(5)

Compression re-loading: (Elastically until #1 yields) $\sigma_{Y_1} < \sigma_{Y_2}$



stress increment

- Find $\Delta\sigma^{(e)}$ such that $\sigma_1 = -\sigma_{Y_1}$ (yield occurs first in #1)

$\Rightarrow \Delta\sigma^{(e)} = ? \quad \Delta\varepsilon^{(e)} = ?$

$\Rightarrow \sigma_1 = \sigma_1^{(R)} + \Delta\sigma^{(e)} = -\sigma_{Y_1}$ $\quad \boxed{\sigma_1 = -\sigma_{Y_1}}$

residual stress from previous state

$\Rightarrow \Delta\sigma^{(e)} = -\sigma_{Y_1} - \sigma_1^{(R)}$ (assume $Y_2 > 2Y_1$)

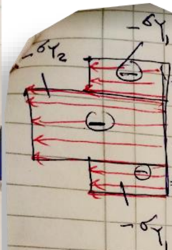
$\Rightarrow \Delta\sigma^{(e)} = -\sigma_{Y_1} - \frac{1}{2}\sigma_{Y_1} + \frac{1}{2}\sigma_{Y_2} = -\frac{3}{2}\sigma_{Y_1} + \frac{1}{2}\sigma_{Y_2} = \Delta\sigma^{(e)}$

$\therefore \sigma_2 = \sigma_2^{(R)} + \Delta\sigma^{(e)} = \frac{1}{2}\sigma_{Y_2} - \frac{1}{2}\sigma_{Y_1} - \frac{3}{2}\sigma_{Y_1} + \frac{1}{2}\sigma_{Y_2}$ elastic reloading

$= \sigma_{Y_2} - 2\sigma_{Y_1}$

$F' = F = \sigma_1 A_1 + \sigma_2 A_2 = -\sigma_{Y_1} A + \sigma_{Y_2} A - 2A\sigma_{Y_1} = A(\sigma_{Y_2} - 3\sigma_{Y_1})$

$\Rightarrow \frac{F}{A_1 + A_2} = \frac{F}{2A} = \frac{\sigma_{Y_2} - 3\sigma_{Y_1}}{2} < 0$ (equiv. stress)



$F' + \Delta F \rightarrow$ such that $\sigma_2 = -\sigma_{Y_2}$

$$\begin{cases} \sigma_2 = (\sigma_{Y_2} - 2\sigma_{Y_1}) + \Delta\sigma_2 = -\sigma_{Y_2} \ \text{(yields)} \quad (1) \\ \sigma_1 = -\sigma_{Y_1} = \text{const} \end{cases}$$

no increase of $\sigma_1 = -\sigma_{Y_1}$

$(1) \rightarrow \Delta\sigma_2 = -2\sigma_{Y_2} + 2\sigma_{Y_1} \Rightarrow \Delta F = A_2 \cdot \Delta\sigma_2 = A \cdot \Delta\sigma_2$

$\Delta F = A_2(-2\sigma_{Y_2} + 2\sigma_{Y_1})$ $\quad \begin{cases}\Delta\sigma_1 = 0 \\ \Delta\sigma_2\end{cases}$

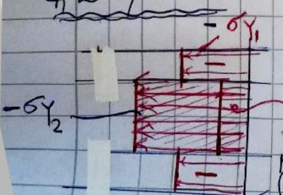
$\Downarrow$

compression limit plastic load (initial stress ≠ 0 residual stresses in the initial state)

$F^* = F'_p = F' + \Delta F = A(\underbrace{\sigma_{Y_2} - 3\sigma_{Y_1}}_{F'}) + A \cdot (-2\sigma_{Y_2} + 2\sigma_{Y_1})$ $\quad -\sigma_{Y_1}$

$= -A(\sigma_{Y_1} + \sigma_{Y_2})$

#2 yields: $F = F_5$ so what is the force (compressive) increment leading to yield #2 ⇒ giving the ~~tyield~~ limit plastic load $F_5 = F_p^{(5)}$?



Displacements: 4→7→5→6

Displacements: re-compression ~~#1 yields first~~ from 4 → 7 → 5 → 6

$\Delta\varepsilon^{(e)}_{4\to7} = \frac{\Delta\sigma^{4\to7}}{E} = +\frac{\sigma_{Y_2} - 3\sigma_{Y_1}}{2E}$

$\varepsilon_7 = \varepsilon_{\text{old at } 4} + \Delta\varepsilon^{(e)} = \left[\frac{\sigma_{Y_2} - \sigma_{Y_1}}{2E}\right]_{@4} + \frac{\sigma_{Y_2}}{2E} - \frac{3\sigma_{Y_1}}{2E}$

$= (\sigma_{Y_2} - 2\sigma_{Y_1})/E$ @ point 7

$\Delta\varepsilon^{(e)}_{7\to5} = \frac{\Delta\sigma^{5-6}}{E} = \frac{-2Y_2 + 2Y_1}{E}$

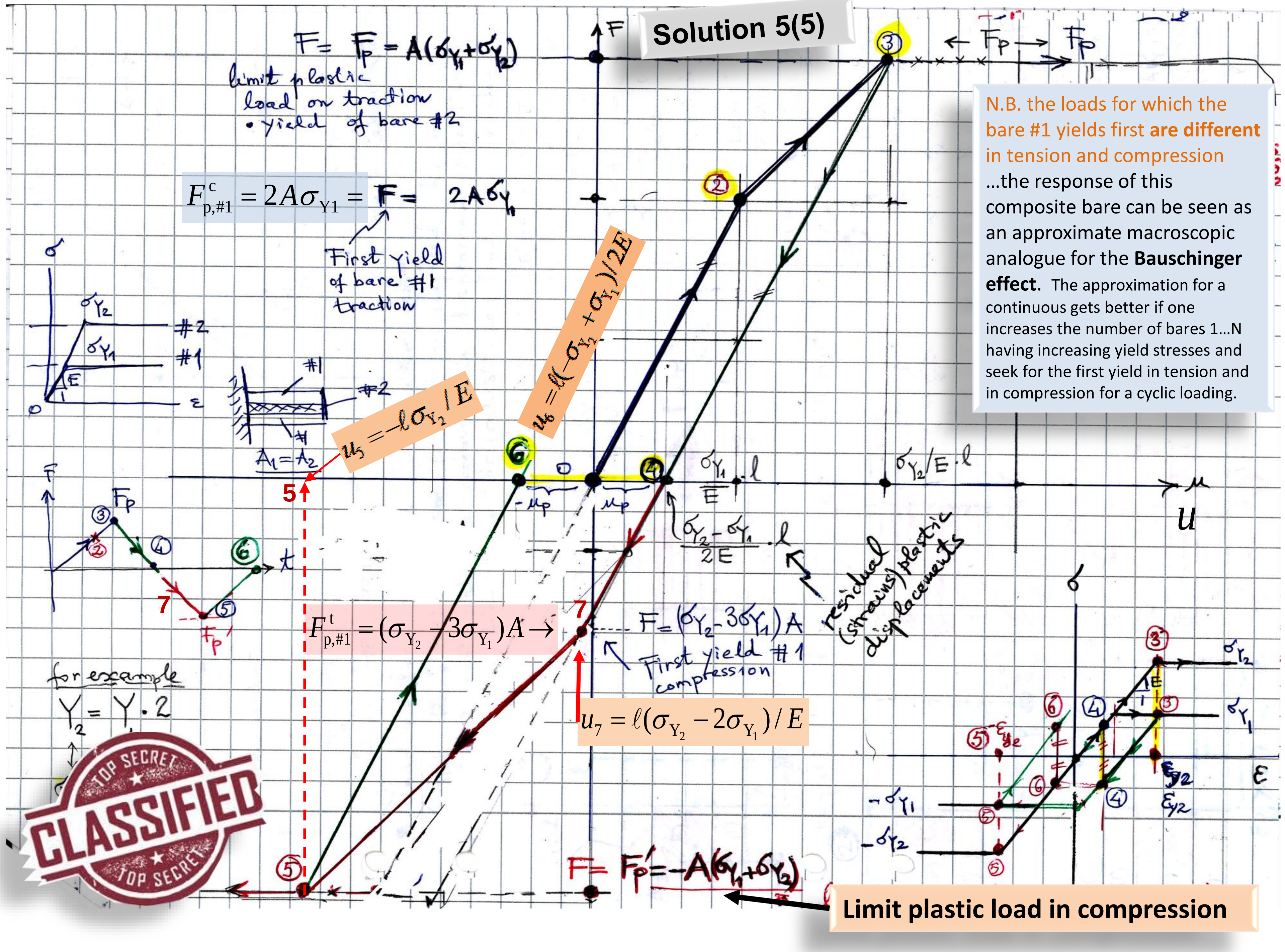
$\varepsilon^{(5)} = \varepsilon^{(7)} + \Delta\varepsilon^{(e)}_{7\to5} = -\frac{2Y_2}{E} + \frac{2Y_1}{E} + \frac{Y_2}{E} - \frac{2Y_1}{E}$

$= -Y_2/E$

5→6: $\Delta\sigma = +\frac{1}{2}(\sigma_{Y_1} + \sigma_{Y_2}) \Rightarrow \Delta\varepsilon^{(e)}_{5\to6} = \frac{1}{2E}(\sigma_{Y_1} + \sigma_{Y_2}) \Rightarrow$

$\varepsilon(5) \leftarrow \varepsilon_6 = -Y_2/E + \frac{1}{2}Y_1/E + \frac{1}{2}Y_2/E = \frac{-\sigma_{Y_2} + \sigma_{Y_1}}{2E}$

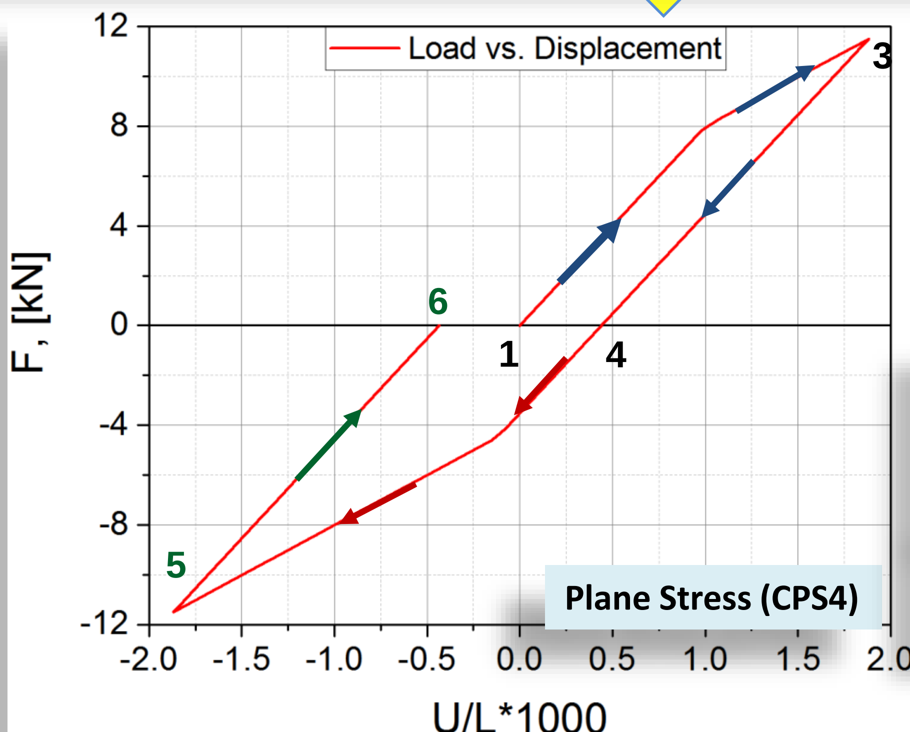
Solution 5(5)



N.B. the loads for which the bare #1 yields first **are different** in tension and compression

…the response of this composite bare can be seen as an approximate macroscopic analogue for the **Bauschinger effect**. The approximation for a continuous gets better if one increases the number of bares 1…N having increasing yield stresses and seek for the first yield in tension and in compression for a cyclic loading.

Solution 6(5): thanks to our doctoral student Serguei Khakalo who did the **Abaqus simulation**



00_elasto_elastic_bere_by_Serguei_El_Pl_Strip_01.inp

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** file created by  A N S A  Thu Feb 09 14:07:38 2017
**
** output from :
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** C:/WORK/Aalto/Abaqus/Djebar/2017_02_Elasto_Plastic_Strip/Unti
**
*HEADING

**
**
**
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        3,
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D STEP
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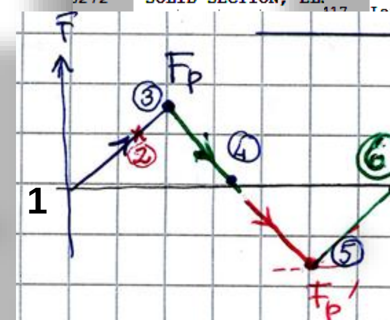


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*PLASTIC, HARDENING=KINEMATIC
          200.,           0.,            0.
          200.,           1.,            0.
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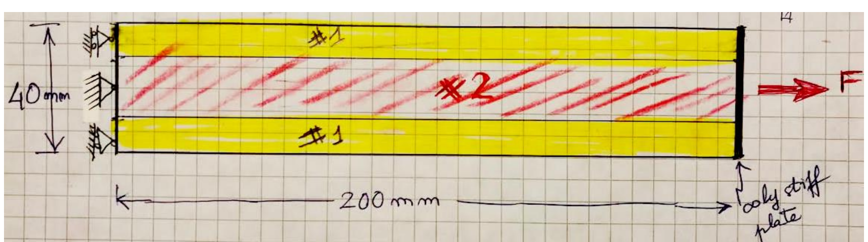
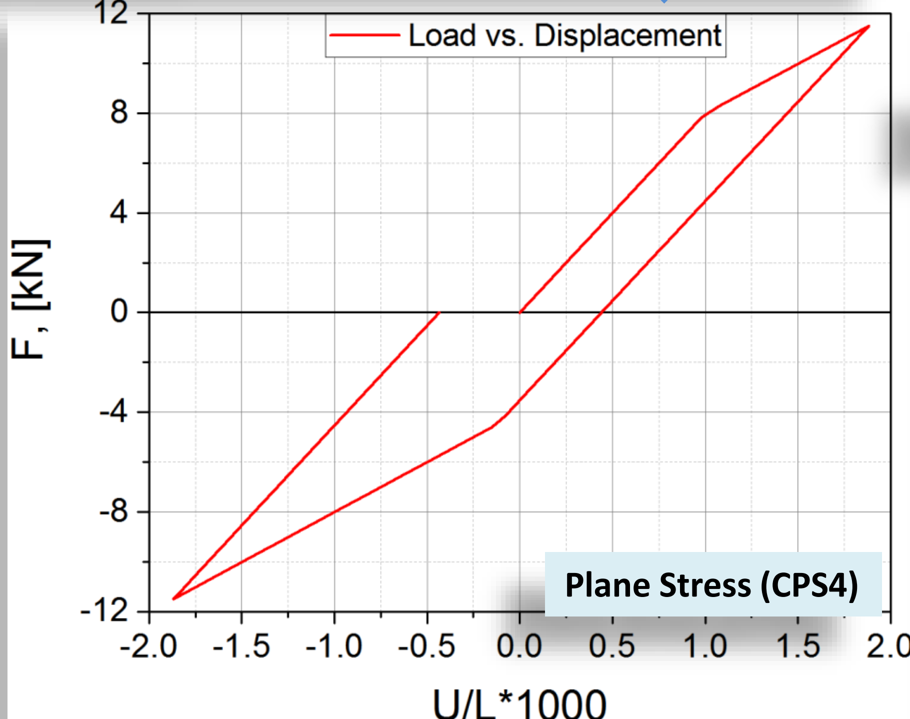
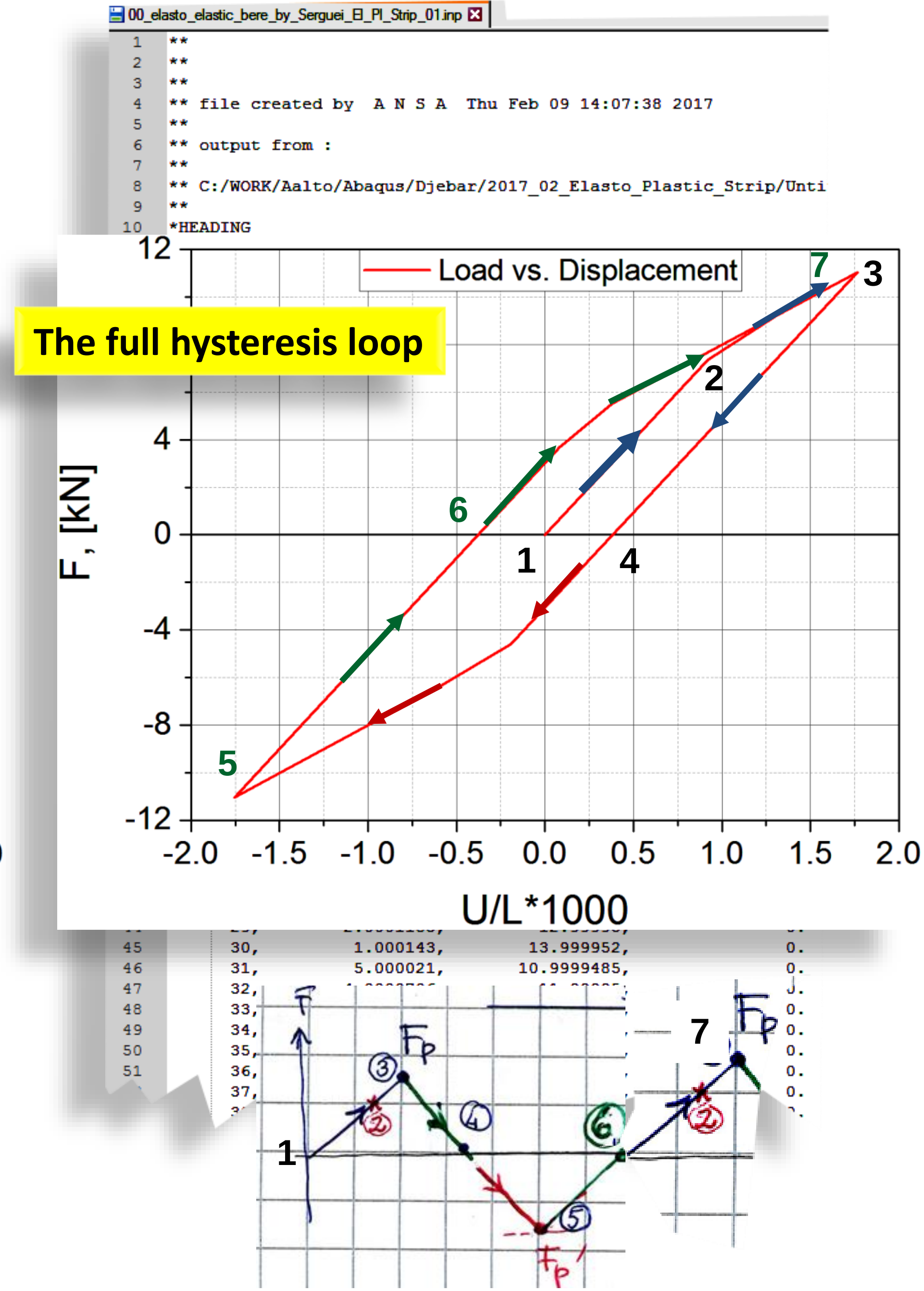
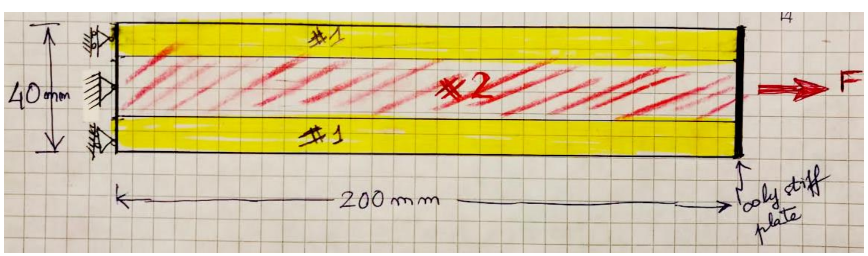


Abaqus input file:
00_ABAQUS_elasto_elastic_bere_by_Serguei_El_Pl_Strip_01.inp

Solution 6(5): thanks to our doctoral student Serguei Khakalo who did the **Abaqus simulation**







Abaqus input file:
00_ABAQUS_elasto_elastic_bere_by_Serguei_El_Pl_Strip_01.inp

Elastic-plastic behaviour – an example of uniaxial behaviour

Example

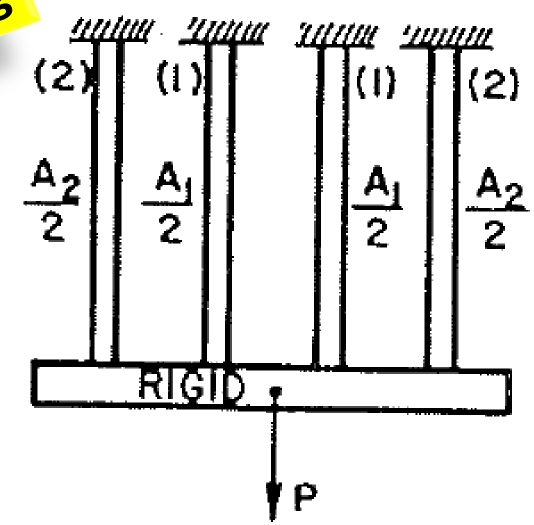
Self-reading

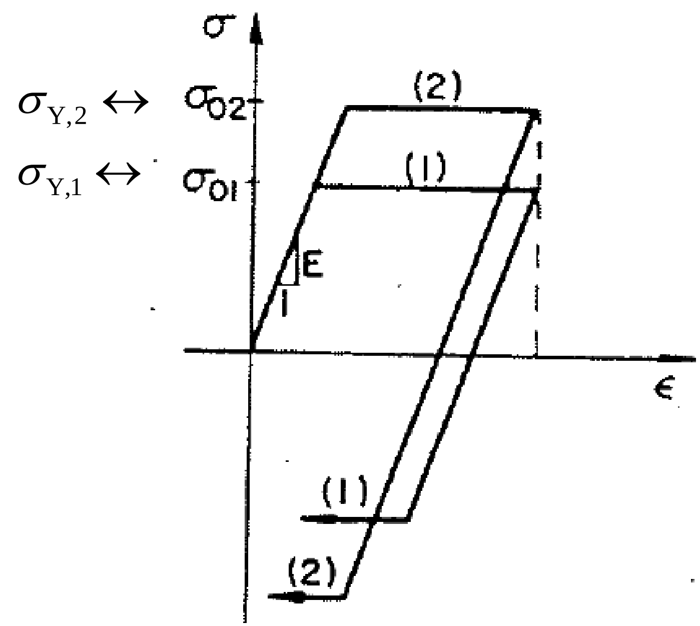
Loading history dependency and Bauschinger effect

EXAMPLE 1.1. The behavior of a polycrystal metallic material composed of many monocrystals is analogous to a truss structure composed of many individual bars. Therefore, it is possible to use a simple truss model to simulate the elastic–plastic behavior of metallic materials. In this example, an overlay truss structure shown in Fig. 1.7 is considered. The Bauschinger effect will be simulated by the model.

In Fig. 1.7, two pairs of bar elements in parallel carry the load P. The vertical bars are made of elastic–perfectly plastic materials with different yield strength. Discuss the loading, unloading, and reloading characteristics of this structural model.

Loading Behavior. With the load P increasing from zero, the first two significant stages occur when bars 1 yield followed by the yielding of bars





N.B. This simple example demonstrates some fundamental aspects related to engineering plasticity as applied to structures such a *non-linear force-displacement relation*, *loading path (history) dependence*, *residual stresses* and *strains, residual displacements upon complete unloading* … etc.

There will be a homework on this specific topic.

This worked example from reference below is meant for self-study.

[2] Chen & Han. ***Plasticity for Structural Engineers***

Elastic-plastic behaviour – an example of uniaxial behaviour

Solution

2. Noting that both materials have the same elastic modulus, the load at first yield is found as

$$P_a = \sigma_{01}A_1 + \sigma_{01}A_2 \tag{1.15}$$

The equivalent stress can be expressed as

$$\sigma_a = \frac{\sigma_{01}A_1 + \sigma_{01}A_2}{A_1 + A_2} = \sigma_{01} \tag{1.16}$$

The corresponding strain is

$$\epsilon_a = \frac{\sigma_{01}}{E} \tag{1.17}$$

At the yielding of bars 2, the load, the stress, and the strain may be expressed as

$$P_b = \sigma_{01}A_1 + \sigma_{02}A_2 \tag{1.18}$$

$$\sigma_b = \frac{\sigma_{01}A_1 + \sigma_{02}A_2}{A_1 + A_2} \tag{1.19}$$

$$\epsilon_b = \frac{\sigma_{02}}{E} \tag{1.20}$$

Unloading Behavior. After this, further elongation of the bars does not result in any increase in load. Therefore, the next significant occurrence will be an unloading phase. During unloading, the modulus is the same as the initial modulus E. Therefore, load P reaches zero when the strain has been reduced by an amount

$$\epsilon = \frac{\sigma_{01}A_1 + \sigma_{02}A_2}{E(A_1 + A_2)} \tag{1.21}$$

At this point, the stress in bars 1, σ_1, and that in bars 2, σ_2, are represented by

$$\sigma_1 = \sigma_{01} - E\epsilon = \frac{(\sigma_{01} - \sigma_{02})A_2}{A_1 + A_2} \tag{1.22}$$

$$\sigma_2 = \sigma_{02} - E\epsilon = \frac{(\sigma_{02} - \sigma_{01})A_1}{A_1 + A_2} \tag{1.23}$$

Because $\sigma_{02} > \sigma_{01}$, we have $\sigma_1 < 0$, $\sigma_2 > 0$, indicating that there exists a residual compression in the lower-yield-strength bars 1 and a residual tension in the higher-yield-strength bars 2 with the applied load being reduced to zero.

Solution:

Self-reading

$$P_a = \sigma_{01}A_1 + \sigma_{01}A_2$$

$$\sigma_a = \frac{\sigma_{01}A_1 + \sigma_{01}A_2}{A_1 + A_2} = \sigma_{01}$$

$$\epsilon_a = \frac{\sigma_{01}}{E}$$

$$P_b = \sigma_{01}A_1 + \sigma_{02}A_2$$

$$\sigma_b = \frac{\sigma_{01}A_1 + \sigma_{02}A_2}{A_1 + A_2}$$

$$\epsilon_b = \frac{\sigma_{02}}{E}$$

$$\epsilon = \frac{\sigma_{01}A_1 + \sigma_{02}A_2}{E(A_1 + A_2)}$$

$$\sigma_1 = \sigma_{01} - E\epsilon = \frac{(\sigma_{01} - \sigma_{02})A_2}{A_1 + A_2}$$

$$\sigma_2 = \sigma_{02} - E\epsilon = \frac{(\sigma_{02} - \sigma_{01})A_1}{A_1 + A_2}$$

Since we assume that the bar itself is elastic-perfectly plastic, compressive yield will occur in bars 1 when the strain has been reduced by an amount

$$\epsilon = 2\frac{\sigma_{01}}{E} \tag{1.24}$$

At this instance, the load in bars 1 is

$$P_1 = -\sigma_{01}A_1 \tag{1.25}$$

The load in bars 2 is

$$P_2 = (\sigma_{02} - 2\sigma_{01})A_2 \tag{1.26}$$

The equivalent stress is

$$\sigma_d = \frac{(\sigma_{02} - 2\sigma_{01})A_2 - \sigma_{01}A_1}{A_1 + A_2} \tag{1.27}$$

Noting that the reversed yield load in Eq. (1.27) is less in magnitude than the initial yield load in Eq. (1.16), it follows therefore that bars 1 yield much earlier than they would have on initial loading, because they are already in compression when $P = 0$ as indicated in Eq. (1.22).

Compressive yielding occurs in bars 2 when the strain has been reduced by the amount

$$\epsilon = 2\frac{\sigma_{02}}{E} \tag{1.28}$$

At this point, the load and the equivalent stress can be expressed as

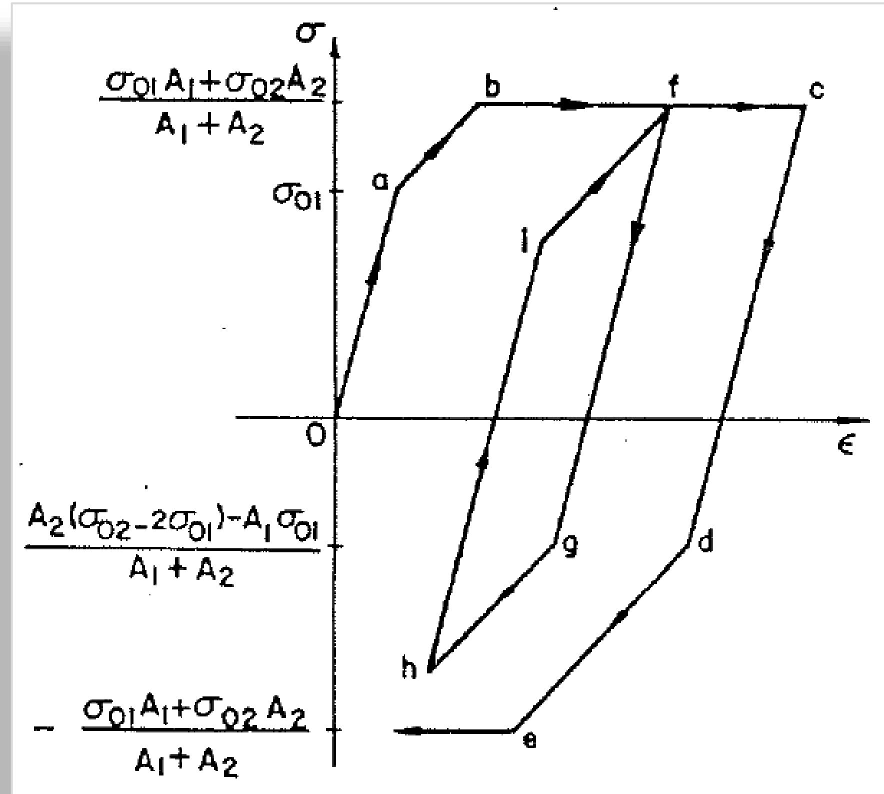
$$P_e = -\sigma_{01}A_1 - \sigma_{02}A_2 \tag{1.29}$$

$$\sigma_e = \frac{-\sigma_{01}A_1 - \sigma_{02}A_2}{A_1 + A_2} \tag{1.30}$$

The σ–ϵ curve for this loading path is shown in Fig. 1.8 by 0–a–b–c–d–e. If more bar elements with different yield strengths are included in the structural model, a more smooth σ–ϵ curve will be obtained.

Reloading Behavior. Consider now the case in which compressive loading is terminated at point h before compressive-plastic flow in bars 2 begins. Reloading of the structural system in tension will follow the linear response with the initial modulus E but bars 1 will eventually yield again in tension at point i and plastic flow begins at point f. The σ–ϵ curve for this loading cycle is shown in Fig. 1.8 by f–g–h–i.

This structural model with an assemblage of bars of different yield strengths may be considered, qualitatively, to represent a real specimen with slip planes of different strengths, and therefore explains why a real specimen generally exhibits the Bauschinger effect.



Loading and unloading characteristics of the overlay model.

Solution

Self-reading

Self-reading

Loading history dependency and **strain hardening effects**

An example – *reading assignment*

Elastic-plastic behaviour – an example of uniaxial behaviour

Example

Loading history dependency and **strain hardening effects**

EXAMPLE 1.2. The $\sigma-\epsilon$ response in simple tension for an elastic-linear hardening plastic material is approximated by the following expressions.

$$\sigma = \sigma_0 + m\epsilon^p \qquad \text{for } \sigma \geq \sigma_0$$

$$\epsilon^e = \frac{\sigma}{E}$$

where $\sigma_0 = 207$ MPa, $E = 207$ GPa, and $m = 25.9$ GPa. A material sample is first stretched to a total strain $\epsilon = 0.007$, is subsequently returned to its initial strain-free state ($\epsilon = 0$) by continued compressive stressing, and then is unloaded and reloaded in tension again to reach the same strain $\epsilon = 0.007$ (see Fig. 1.9). Sketch the stress-strain curve for the following hardening rules: (i) isotropic hardening, (ii) kinematic hardening, (iii) independent acting tensile and compressive hardening.

SOLUTION. According to the definition of the plastic modulus in Eq. (1.7), we have

$$E_p = \frac{d\sigma}{d\epsilon^p} = m = 25{,}900 \text{ MPa}$$

and the tangent modulus E_t is found from Eq. (1.9) as

$$E_t = \frac{1}{\dfrac{1}{E} + \dfrac{1}{E_p}} = \frac{1}{\dfrac{1}{207{,}000} + \dfrac{1}{25{,}900}} = 23{,}020 \text{ MPa}$$

which assumes the constant value 23,020 MPa for the linear hardening material.

With the material sample stretching, yielding occurs at the point with strain

$$\epsilon = \frac{\sigma_0}{E} = 0.001$$

Then, the sample is further stretched to point A with strain $\epsilon = 0.007$, at which the stress σ_A is found as

$$\begin{aligned} \sigma_A &= \sigma_0 + \Delta\sigma \\ &= \sigma_0 + E_t \Delta\epsilon \\ &= 207 + 23{,}020(0.007 - 0.001) = 345 \text{ MPa} \end{aligned}$$

$$d\sigma = E_t\, d\epsilon$$

The subsequent stresses are determined for the three hardening rules as follows:

(i) *Isotropic hardening case* [Fig. 1.9(i)]: During unloading and reversed loading in compression, the sample behaves elastically until it yields again in compression at point B. According to the isotropic hardening rule, we have

$$\sigma_B = -\sigma_A = -345 \text{ MPa}$$

$$\epsilon_B = \epsilon_A - 2\frac{\sigma_A}{E} = 0.007 - 2\left(\frac{345}{207{,}000}\right) = 0.00367$$

Now, the material sample is yielding until load reversal occurs at point C as $\epsilon = 0$.

reading assignment

$$\begin{aligned} \sigma_C &= \sigma_B + E_t \Delta\epsilon \\ &= -345 + 23{,}020(0 - 0.00367) = -429 \text{ MPa} \end{aligned}$$

Upon reversal of the straining, the material is elastic up to point D at which

$$\sigma_D = 429 \text{ MPa}$$

$$\epsilon_D = \epsilon_C + \frac{2\sigma_D}{E} = 0 + 2\left(\frac{429}{207{,}000}\right) = 0.004145$$

As the strain ϵ reaches 0.007 at point E, the stress is

$$\begin{aligned} \sigma_E &= \sigma_D + E_t \Delta\epsilon \\ &= 429 + 23{,}020(0.007 - 0.004145) \\ &= 495 \text{ MPa} \end{aligned}$$

Self-reading

(ii) *Kinematic hardening case* [Fig. 1.9(ii)]: The yield stress at point B i

$$\sigma_B = \sigma_A - 2\sigma_0 = 345 - 2(207) = -69 \text{ MPa}$$

$$\epsilon_B = \epsilon_A - 2\frac{\sigma_0}{E} = 0.007 - 2\left(\frac{207}{207{,}000}\right) = 0.005$$

At point C,

$$\sigma_C = \sigma_B + E_t \Delta\epsilon$$
$$= -69 + 23{,}020(0 - 0.005) = -184 \text{ MPa}$$

At point D, the sample yields again in tension at a stress

$$\sigma_D = \sigma_C + 2\sigma_0$$
$$= -184 + 2(207) = 230 \text{ MPa}$$

$$\epsilon_D = \epsilon_C + \frac{2\sigma_0}{E} = 0 + 2\left(\frac{207}{207{,}000}\right) = 0.002$$

At point E, the stress is

$$\sigma_E = \sigma_D + E_t \Delta\epsilon$$
$$= 230 + 23{,}020(0.007 - 0.002) = 345 \text{ MPa}$$

(iii) *Independent hardening case* [Fig. 1.9(iii)]: Because the material has not yielded in compression before, the yield stress at point B is

$$\sigma_B = -\sigma_0 = -207 \text{ MPa}$$

$$\epsilon_B = \epsilon_A - \frac{\sigma_A}{E} - \frac{\sigma_0}{E}$$
$$= 0.007 - \frac{345}{207{,}000} - \frac{207}{207{,}000} = 0.00433$$

At point C,

$$\sigma_C = \sigma_B + E_t \Delta\epsilon$$
$$= -207 + 23{,}020(0 - 0.00433) = -307 \text{ MPa}$$

At point D, the material yields again in tension at a stress equal to σ_A, i.e.,

$$\sigma_D = \sigma_A = 345 \text{ MPa}$$

$$\epsilon_D = \epsilon_C - \frac{\sigma_C}{E} + \frac{\sigma_D}{E} = 0 - \frac{(-307)}{207{,}000} + \frac{345}{207{,}000}$$
$$= 0.00315$$

At point E, the stress is

$$\sigma_E = \sigma_D + E_t \Delta\epsilon$$
$$= 345 + 23{,}020(0.007 - 0.00315) = 434 \text{ MPa}$$

The stress–strain curves for each of the three cases are shown in Fig. 1.9.

$$d\epsilon = d\epsilon^e + d\epsilon^p \quad (1.5)$$

The *stress increment* $d\sigma$ is related to the *strain increment* $d\epsilon$ by

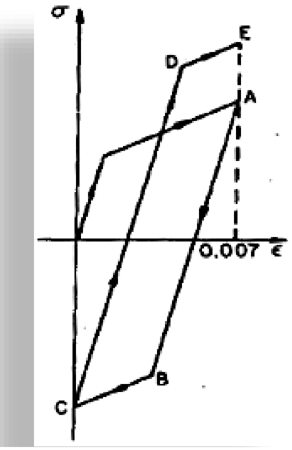
$$d\sigma = E_t\, d\epsilon \quad (1.6)$$

where E_t is the *tangent modulus* which is changing during plastic deformation. In the case of uniaxial loading, E_t is the current slope of the σ–ϵ curve (Fig. 1.5a). If we separate the plastic strain ϵ^p from the total strain ϵ, then the plastic strain increment $d\epsilon^p$ and the stress increment $d\sigma$ are related by

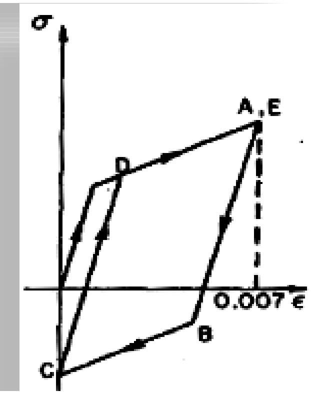
$$d\sigma = E_p\, d\epsilon^p \quad (1.7)$$

$$d\sigma = E_t\, d\epsilon$$

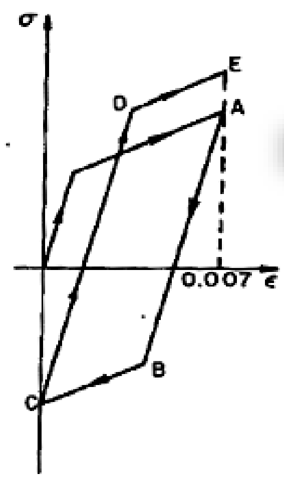
strain hardening



I) ISOTROPIC HARDENING



II) KINEMATIC HARDENING



III) INDEPENDENT HARDENING

Self-reading

Uniaxial Elastic-plastic behaviour

Homework #1, #2 and some other exercises

Uniaxial Elastic-plastic behaviour

Homework #1, #2 and some other exercises

Homework – plasticity, basic concepts

Homework #1.1

Hardening behavior and yield offset – uniaxial behavior in Plasticity

During a uniaxial (monotonic) stress testing of particular solid material the stress-strain curve was measured. It was found that this curve can be approximated by the non-linear **Ramberg-Osgood** model for 1-D case as

or equivalently

$$\varepsilon = \varepsilon^{\mathrm{e}} + \varepsilon^{\mathrm{P}} = \frac{\sigma}{E} + K'\left(\frac{\sigma}{\sigma_{0.2}}\right)^n .$$

$$\varepsilon = \varepsilon^{\mathrm{e}} + \varepsilon^{\mathrm{P}} = \frac{\sigma}{E} + \left(\frac{\sigma}{b}\right)^n, \quad \text{or equivalently} \quad \varepsilon = \varepsilon^{\mathrm{e}} + \varepsilon^{\mathrm{P}} = \frac{\sigma}{E} + K\left(\frac{\sigma}{E}\right)^n .$$

0. Make graphs for three or four different values of the exponent parameter with a fixed value of K or b

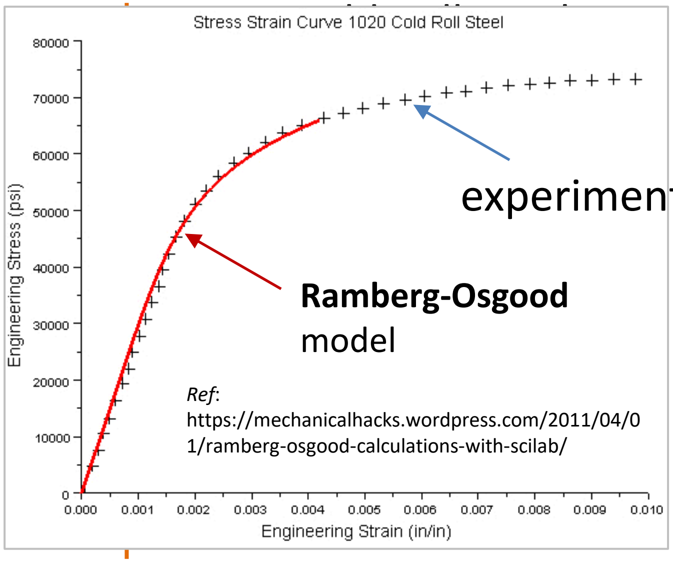
1. Determine the *tangent elastic modulus* E_{t} and the *plastic modulus* $H \equiv E_{\mathrm{p}}$ as functions of stress and strain. Show that $1/E_{\mathrm{t}} = 1/E + 1/E_{\mathrm{p}}$ holds.

2. Determine the plastic work W_{p} as function of stress and plastic strain. On the strain-stress graph show the elastic and the plastic work and also the hardening work partition.

3. Determine the yield stress. What is the value of the yield stress?

4. Assume $n = 5$ and determine the (offset tensile) stresses corresponding to values of 0.1 % and 0.2 % plastic strain (permanent strain). Plot your results on the graphs.

5. Assume n = 1, draw the stress-strain curve for loading followed b a complete unloading.



$n, b, K -$ describe the hardening behavior of the material.

Elastic-plastic behaviour – uniaxial behaviour

Homework #1.2

Consider a single such bare of length $2L$ which is composed of two different metallic materials which are perfectly bond to each other (see Figure). Both materials are isotropic and are made of **elastic-perfectly plastic** materials. The cross-section of the bares are $A_1 = A_2 = A$ and equal elasticity moduli $E_1 = E_2 = E$.

Both materials behave separately symmetrically in compression and in tension.

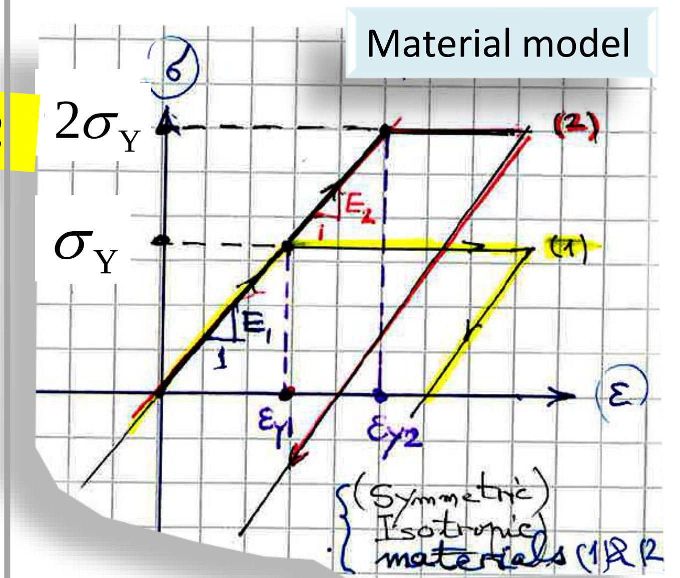
The composite bare is loaded according to the history shown on the figure (bottom). The applied load increases from $P = 0$ to the plastic limit load $P = P_{\mathrm{p}}$ and then decreases to zero (loading history 1-**3**-**4**-**5**-**6**).

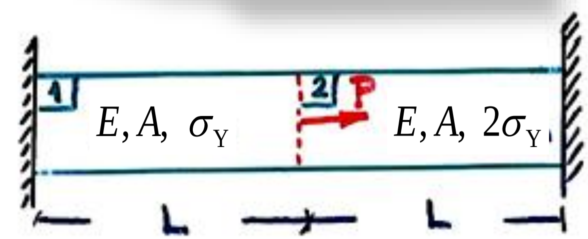
i) Determine the corresponding full **force-displacement curve**

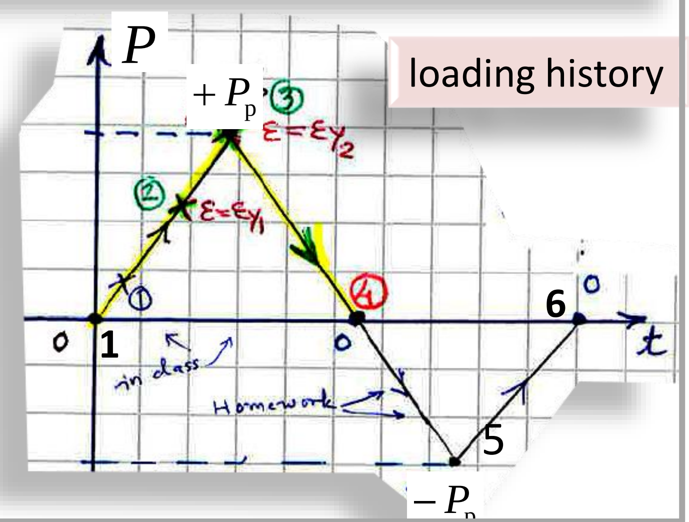
ii) Upon complete unloading (points 4 and then 6), find the **residual stresses,** strains and **residual displacement**.

iii) **Extra**: Choose, for basic properties steel having two different yield stresses and do the numerical simulation [plane stress] on **Abaqus** [free download student version] or equivalent…(non-linear FEA)

iv) Conclusions?

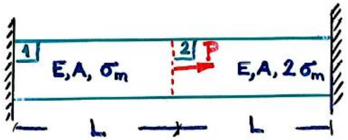






Solution

Esimerkki



Vaihe 1: Molemmat sauvat käyttäytyvät kimmoisesti.

$$|\sigma| = \frac{P}{2A} \le \sigma_m \Rightarrow P \le 2\sigma_m A$$

Vaihe 2: Tämän jälkeen kuormituksen lisäys jää vain oikeanpuoleisen osan kannettavaksi

$$|\sigma| = \sigma_m + \frac{\bar{P}}{A} \le 2\sigma_m \Rightarrow \bar{P} \le \sigma_m A$$

$$\underline{\underline{P_{kok} = P + \bar{P} = 3\sigma_m A}}$$

Vaihe 3: Kuorman poistaminen. Tällöin molemmat osat toimivat kimmoisesti. Jännityksen muutos

$$\Delta\sigma = \frac{P}{2A} = \frac{3\sigma_m A}{2A} = \frac{3}{2}\sigma_m$$

Toisin sanoen, osassa 1 jännitys on

$$\sigma_m - \frac{3}{2}\sigma_m = \underline{\underline{-\frac{1}{2}\sigma_m}}$$

ja osassa 2 vastaavasti

$$-2\sigma_m + \frac{3}{2}\sigma_m = \underline{\underline{-\frac{1}{2}\sigma_m}}$$

Siirtymähistoria:

Vaihe 1: Sauva 1 venyy ja sauva 2 kokoonpuristuu määrän

$$\delta = \frac{PL}{2EA} = \frac{2\sigma_m AL}{2EA} = \frac{\sigma_m L}{E}$$

Vaihe 2: Sauva 2 kokoonpuristuu kuorman vaikutuksesta, sauva 1 seuraa mukana

$$\bar{\delta} = \frac{\bar{P}L}{EA} = \frac{\sigma_m AL}{EA} = \frac{\sigma_m L}{E}$$

Sauvan 1 pituus $L_1 = L + \frac{2\sigma_m L}{E}$

Sauvan 2 pituus $L_2 = L - \frac{2\sigma_m L}{E}$

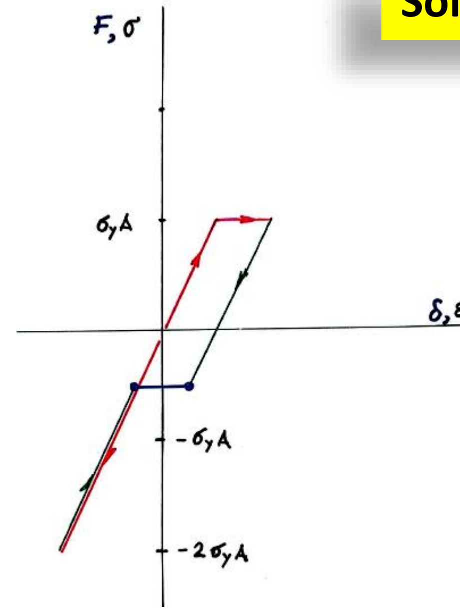
Vaihe 3: Kuorman poistaminen jälleen kimmoinen

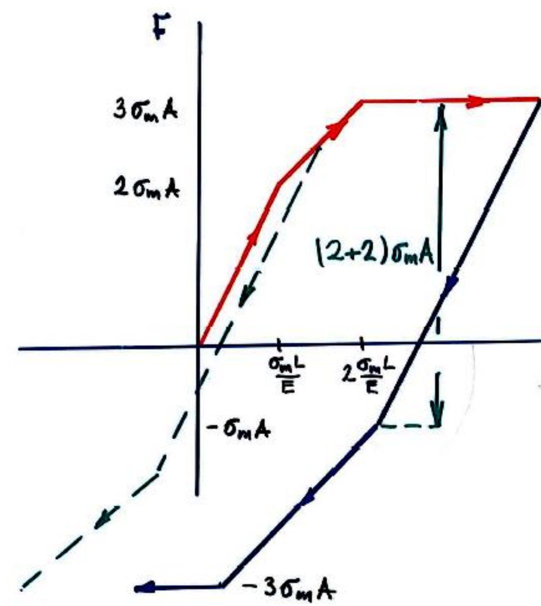
Sauva 1 $L_1 = L + \frac{2\sigma_m L}{E} - \frac{3\sigma_m AL}{2AE}$

$$= \underline{\underline{L + \frac{\sigma_m L}{2E}}}$$

Sauva 2 $L_2 = L - \frac{2\sigma_m L}{E} + \frac{3\sigma_m AL}{2AE}$

$$= \underline{\underline{L - \frac{\sigma_m L}{2E}}}$$





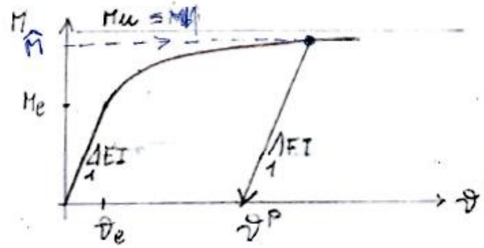
Solution by prof. J. Paavola
(very short)

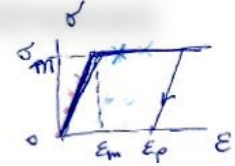
Engineering Plasticity

Some Solved examples

Homework/example-residual stresses

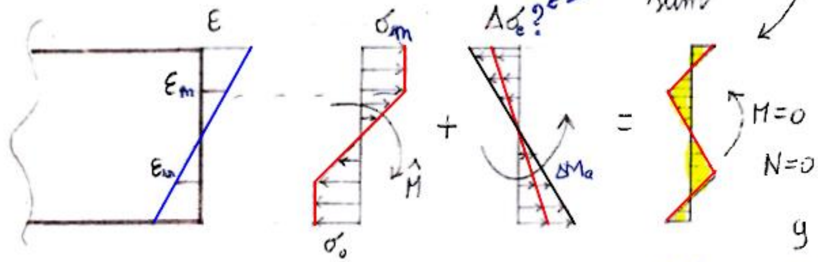
What happens if we release the load?





RESIDUAL STRESS DISTRIBUTION

Algebraic sum

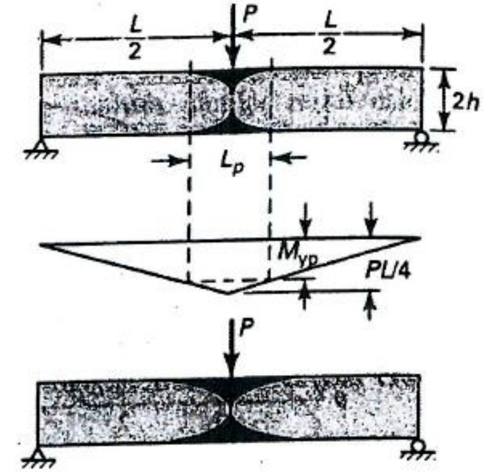


$\Delta N_e = -\int_A \Delta\sigma_e \, dA$

$\Delta\sigma_e$ - määräytyy ehdosta $\begin{cases} M = \hat{M} + \Delta M_e = 0 \\ N = \hat{N} + \Delta N_e = 0 \end{cases}$, $\hat{N}=0$

$\int_A \Delta\sigma_e y \, dA$

Bending of a rectangular beam: (a) plastic region; (b) moment diagram; (c) plastic hinge.



Self-reading

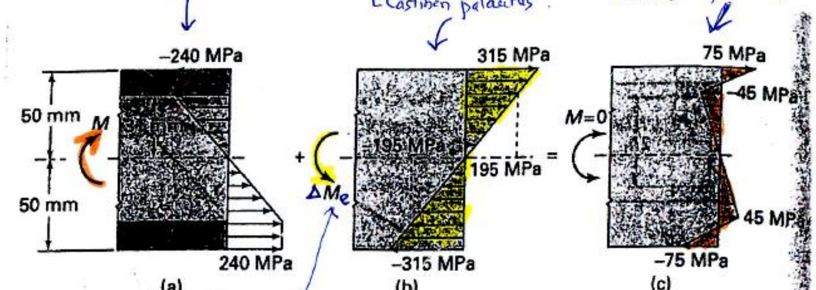
Tehtävä: Olkoon jännitystila (a) saavutettu taivuttamalla. Mitkä jännitykset jäävät kun poistuu ulkoinen kuormitus?

Kimmoplastinen tilanne

ulkoinen voima poistuu täysin

"Elastinen palautus"

Jäännösjännitykset



Example 12.7 (continued). Stress distribution in a rectangular beam: (a) elastic–plastic; (b) elastic rebound; (c) residual.

ΔM_e: tuntematon "elastinen palautustaivutusmomentti": ΔM_e määräytyy ehdoista: $\begin{cases} M + \Delta M_e = 0 \\ N + \Delta N_e = 0 \end{cases}$, jossa $\Delta M_e = \int \Delta\sigma_e y \, dA$, $\Delta N_e = \int_A \Delta\sigma_e \, dA$

Formulate more properly...if you'll have time ..

Lower Bound Plastic Collapse Theorem

The lower bound theorem provides a safe estimate of the collapse loads for a rigid plastic solid.

Statically admissible stress field…

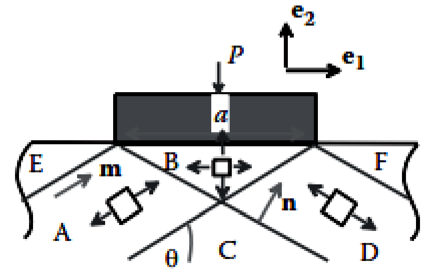
KESKEN, PAHASTI…

6.2.6.2 Example 2: Rigid Indenter in Contact with a Half-Space

We consider a flat indenter with width a that is pushed into the surface of a half-space by a force P. The stress state illustrated in Figure 6.31 will be used to obtain a lower bound to the collapse load in the solid. Note the following:

1. Regions C, E, and F are stress free.
2. The stress in regions A and D consists of a state of uniaxial stress, with direction parallel to the boundaries between AC (or AE) and CD (or DF), respectively. We will denote this stress by $\sigma^A \mathbf{m}\otimes\mathbf{m}$, where $\mathbf{m}$ is a unit vector parallel to the direction of the uniaxial stress.
3. The stress state in the triangular region B has principal directions of stress parallel to $\mathbf{e}_\alpha$. We will write this stress state as $\sigma_{11}^B \mathbf{e}_1 \otimes \mathbf{e}_1 + \sigma_{22}^B \mathbf{e}_2 \otimes \mathbf{e}_2$.

The stresses in each region must be chosen to satisfy equilibrium and to ensure that the stress is below yield everywhere. The stress is constant in each region, so equilibrium is satisfied locally. However, the stresses are discontinuous across AC, AB, etc. To satisfy



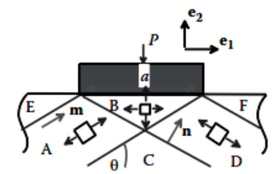


FIGURE 6.31 Statically admissible stress field for a rigid plastic solid indented by a flat punch.

equilibrium, equal and opposite tractions must act on the material surfaces adjacent to the discontinuity, which requires, for example, that $\sigma_{ij}^A n_j = \sigma_{ij}^B n_j$, where $\mathbf{n}$ is a unit vector normal to the boundary between A and B as indicated in Figure 6.31. We enforce this condition as follows:

1. Note that $\mathbf{m} = \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2 \qquad \mathbf{n} = \sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2$.
2. Equilibrium across the boundary between A and B requires

$$\sigma^A(\mathbf{m}\otimes\mathbf{m})\cdot\mathbf{n} = \left(\sigma_{11}^B \mathbf{e}_1 \otimes \mathbf{e}_1 + \sigma_{22}^B \mathbf{e}_2 \otimes \mathbf{e}_2\right)\cdot\mathbf{n}$$

$$\Rightarrow \sigma^A(\cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2)2\sin\theta\cos\theta = \sigma_{11}^B \mathbf{e}_1\sin\theta + \sigma_{22}^B \mathbf{e}_2\cos\theta$$

$$\Rightarrow \sigma_{11}^B = 2\sigma^A\cos^2\theta \qquad \sigma_{22}^B = 2\sigma^A\sin^2\theta.$$

3. We must now choose σ^A and θ to maximize the collapse load but ensure that the stresses do not exceed yield in regions A or B. Clearly, this requires $\sigma^A = -Y$, whereas θ must be chosen to ensure that $|\sigma_{22}^B - \sigma_{11}^B| < Y$. This requires $1/2 < \cos\theta < \sqrt{3}/2$. The largest value for θ maximizes the bound.
4. Finally, substituting for θ gives $\sigma_{22}^B = -3Y/2$. We see that the lower bound is $P = -3Ya/2$.

Aalto University

Self-reading

Lower and upper-Bound Plastic Collapse Theorem

Self-reading

SOIL MECHANICS AND PLASTIC ANALYSIS OR LIMIT DESIGN*

BY

D. C. DRUCKER AND W. PRAGER

Brown University

*Received Nov. 19, 1951. The results presented in this paper were obtained in the course of research sponsored by the Office of Naval Research under Contract N7onr-35801 with Brown University.

1. Introduction. Problems of soil mechanics involving stability of slopes, bearing capacity of foundation slabs and pressures on retaining walls are often treated as problems of plasticity. The soil is replaced by an idealized material which behaves elastically up to some state of stress at which slip or yielding occurs. The shear stress required for simple slip is often considered to depend upon the cohesion and linearly upon the normal pressure on the slip surface. In more complete plane investigations an extended Coulomb's rule is used,**

7. Simple discontinuous solutions. Any soil mass, in particular the slope shown in Fig. 3, may fail by rigid body "sliding" motion in two simple ways. As is well known, the surfaces of discontinuity, idealizations of discontinuity layers, are planes and circular cylinders for a Prandtl-Reuss material, $\alpha = 0$. When $\alpha \neq 0$, a plane discontinuity surface is still permissible but the circle is replaced by a logarithmic spiral which is at an

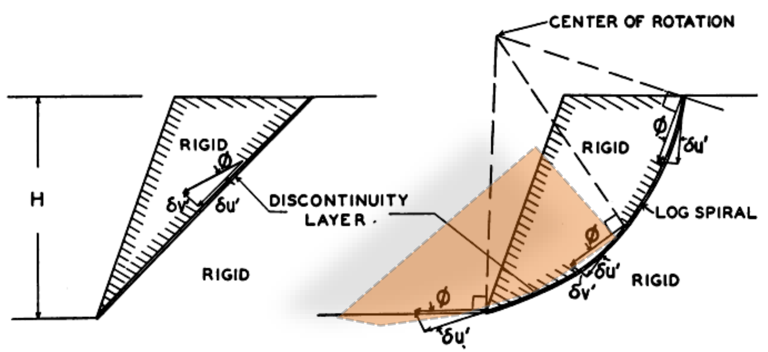


FIG. 3. Rigid body "slide" motions.

angle φ to the radius from the center of rotation. The circle is not a permissible surface for rigid body "sliding" because a discontinuity in tangential velocity requires a separation velocity. For a "slide" discontinuity the angle between the velocity vector and the discontinuity surface is φ, as is seen from Eq. (16).

8. Example: Critical height of vertical bank. The computation of the critical height H_c of a vertical bank, Fig. 4, will serve as an illustration of the procedure consistent with plasticity theory. No attempt will be made here to get an exact plasticity solution to this problem for even if it could be obtained it would not be helpful for more complicated slopes. Instead, the upper and lower bound techniques of limit design will be used.

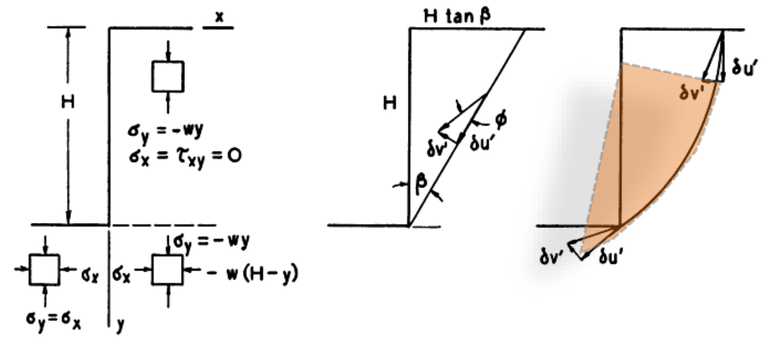


FIG. 4. (a) An equilibrium solution, (b) Plane, (c) Log. spiral compatible solutions.

Denoting the specific weight of the soil by w, Fig. 4a shows a discontinuous equilibrium solution in which the maximum shearing stress is $wH/2$ at the lower ground level, where the average normal stress is $-wH/2$. It should be remembered that the stress field need bear no resemblance to the actual state of stress. Equilibrium alone must be satisfied in addition to the yield condition (1) or (2) for H to be safe. From Eq. (1)

$$\frac{wH}{2} = c \cos \varphi + \frac{wH}{2} \sin \varphi$$

or a lower bound on the critical height H_c is given by

$$H_c \geq H = \frac{2c}{w} \frac{\cos \varphi}{1 - \sin \varphi} = \frac{2c}{w} \tan \left(45 + \frac{\varphi}{2}\right) \tag{22}$$

An upper bound on H_c may be found by taking a plane slide as a velocity pattern, Fig. 4b. Equating external rate of work to the dissipation Eq. (18) gives

$$\frac{wH^2 \tan \beta}{2} \frac{\delta u'}{\cos \varphi} \cos (\varphi + \beta) = \frac{cH}{\cos \beta} \delta u',$$

so that

$$\boxed{H_c \leq H = \frac{2c}{w} \frac{\cos \varphi}{\sin \beta \cos (\varphi + \beta)}}. \tag{23}$$

minimizing the right hand side gives

$$\boxed{\beta = 45 - \frac{\varphi}{2}} \tag{24}$$

and

$$\boxed{H_c \leq \frac{4c}{w} \tan \left(45 + \frac{\varphi}{2}\right)} \tag{25}$$

The usual theory furnishes the right-hand side of (25) as the critical height. However there is a factor of 2 between the upper bound (25) and the lower bound (22) for H_c.

The upper bound can be improved by considering a rotational discontinuity (logarithmic spiral, Fig. 4c) instead of the translational type.

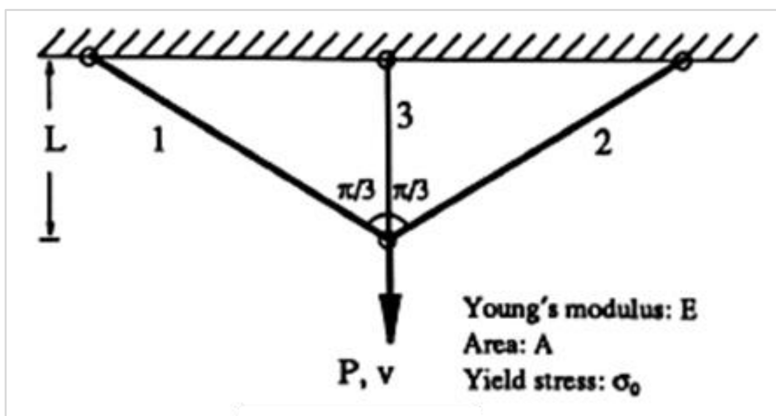
Solved problems 1(1)

Self-reading

Three-Bar Truss Problems

A three-bar truss subjected to a vertical load P is shown in ..... The bars are made of an elastic-perfectly plastic material with elastic modulus E and yield stress σ_0. All three bars have the same area A. The load P is first increased to the plastic limit P_p and then is unloaded to zero. Afterward, P is increased in the reversed direction until all three bars yield again in compression and then is unloaded again to zero.

a. Determine the elastic limit load P_e and the plastic limit load P_p in the initial loading.
b. Determine the elastic limit load P_e' and the plastic limit load P_p' in the reversed loading.
c. Determine the residual stresses and strains of the bars and the residual horizontal and vertical displacements at the load point at the end of the loading history.



Solved problems 1(2)

Self-reading

Solution: The basic equilibrium and compatibility equations are

$$\sigma_1 = \sigma_2, \quad \sigma_1 + \sigma_3 = \frac{P}{A}$$

$$\varepsilon_1 = \varepsilon_2, \quad 4\varepsilon_1 = \varepsilon_3$$

Using the elastic stress-strain relations, we obtain the elastic solution as

$$\sigma_1 = \sigma_2 = \frac{1}{5}\frac{P}{A}, \quad \sigma_3 = \frac{4}{5}\frac{P}{A}$$

Initial Loading

The elastic limit load is obtained by equating σ_3 to σ_0,

$$\frac{P_e}{A} = \frac{5}{4}\sigma_0$$

and at $P = P_e$, we have

$$\sigma_1 = \sigma_2 = \frac{1}{4}\sigma_0, \quad \sigma_3 = \sigma_0, \quad \varepsilon_1 = \varepsilon_2 = \frac{1}{4}\varepsilon_0, \quad \varepsilon_3 = \varepsilon_0$$

$$u = 0, \quad v = L\varepsilon_0$$

where u and v are the horizontal and vertical displacements respectively at the load point.

For $P > P_e$, $\Delta\sigma_3 = 0$. Thus, from the basic equations we obtain the plastic limit load as

$$\frac{P_p}{A} = 2\sigma_0$$

and at $P = P_p$, we have

$$\sigma_1 = \sigma_2 = \sigma_3 = \sigma_0, \quad \varepsilon_1 = \varepsilon_2 = \varepsilon_0, \quad \varepsilon_3 = 4\varepsilon_0$$

$$u = 0, \quad v = 4L\varepsilon_0$$

Unloading

When the load P is unloaded to zero, the incremental stresses are

$$\Delta\sigma_1 = \Delta\sigma_2 = -\frac{2}{5}\sigma_0$$

Thus, at $P = 0$, we obtain

$$\sigma_1 = \sigma_2 = \frac{3}{5}\sigma_0, \quad \sigma_3 = -\frac{3}{5}\sigma_0, \quad \varepsilon_1 = \varepsilon_2 = \frac{3}{5}\varepsilon_0, \quad \varepsilon_3 = \frac{12}{5}\varepsilon_0$$

$$u = 0, \quad v = \frac{12}{5}L\varepsilon$$

Reversed Loading

In the reversed loading, Bar 3 yields first again. We can easily obtain the elastic limit and plastic limit load in the reversed loading.

$$\frac{P_e'}{A} = -\frac{1}{2}\sigma_0, \quad \frac{P_p'}{A} = -2\sigma_0$$

and at $P = P_p'$, we have

$$\sigma_1 = \sigma_2 = \sigma_3 = -\sigma_0, \quad \varepsilon_1 = \varepsilon_2 = -\varepsilon_0, \quad \varepsilon_3 = -4\varepsilon_0$$

$$u = 0, \quad v = -4L\varepsilon_0$$

Unloading

Finally, when the load P is unloaded to zero again, we obtain the residual stresses and strains as

$$\sigma_1^* = \sigma_2^* = -\frac{3}{5}\sigma_0, \quad \sigma_3^* = \frac{3}{5}\sigma_0, \quad \varepsilon_1^* = \varepsilon_2^* = -\frac{3}{5}, \quad \varepsilon_3^* = -\frac{12}{5}\varepsilon_0$$

$$u = 0, \quad v = -\frac{12}{5}L\varepsilon_0$$

Solved problems 2(1)

Self-reading

Stepped and Tapped Bar Problems

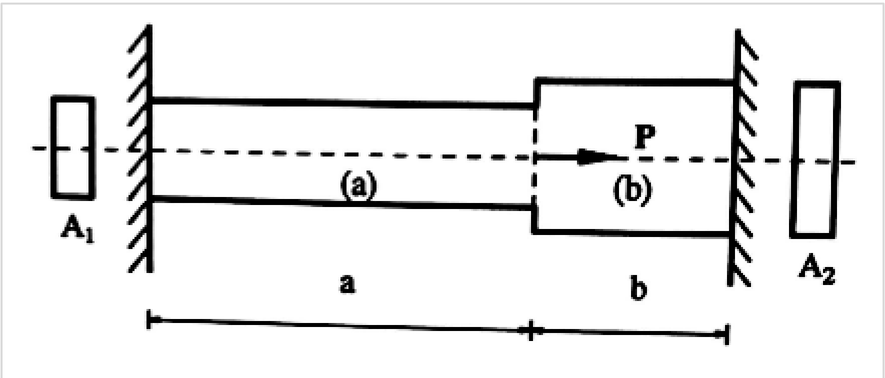
A fixed-end bar with two areas of cross-section is shown in Part a has an area A_1 and of length a, and Part b has an area A_2 and of length b. Assume $a > b$ and $A_2 / A_1 > 1$. The bar is made of an elastic-perfectly plastic material with a yield stress σ_y, and is subjected to an axial force P at the intersection of the two parts. For the given loading path: $P = 0 \rightarrow P_p \rightarrow 0 \rightarrow P_p'$, where P_p and P_p' are the plastic limit loads in the initial loading and in the reversed loading, respectively

a. Determine P_e and P_p in the initial loading;

b. Determine the residual stresses and residual strains when P is unloaded to zero;

c. Determine the elastic limit load P_e' and the plastic limit load P_p' in the reversed loading;

d. For the case $A_2 / A_1 = 2$ and $a = 2b$, plot curves for P vs. u, σ_a vs. P, and σ_b vs. P.



Solved problems 2(2)

Self-reading

Solution:

Basic Equations

The equilibrium and compatibility equations are

$$\sigma_a - \alpha\sigma_b = \frac{P}{A_1}, \qquad a\,\varepsilon_a + b\,\varepsilon_b = 0$$

where $\alpha = A_2 / A_1$.

Elastic Solution

Combining the above equations with the elastic stress-strain relation, the elastic solution for stresses is obtained as

$$\sigma_a = \frac{b}{b+\alpha a}\,\frac{P}{A_1}, \qquad \sigma_b = -\frac{a}{b+\alpha a}\,\frac{P}{A_1}$$

Initial Loading

Since $|\sigma_b| > |\sigma_a|$, the plastic flow starts in Part b at Point A
The elastic limit load is then obtained as

$$\frac{P_e}{A_1} = (\alpha + \frac{b}{a})\,\sigma_y$$

The plastic limit load at Point B is also easily obtained as

$$\frac{P_p}{A_1} = (1+\alpha)\,\sigma_y$$

Unloading

Using the elastic solution, the residual stresses and strains after the load P is unloaded to zero are obtained as (Point C)

$$\sigma_a^* = \frac{\alpha(a-b)}{b+\alpha a}\,\sigma_y > 0, \qquad \sigma_b^* = \frac{(a-b)}{b+\alpha a}\,\sigma_y > 0$$

$$\varepsilon_a^p = 0, \qquad \varepsilon_b^p = -(\frac{a}{b} - 1)\,\varepsilon_y$$

The plastic limit load at Point B is also easily obtained as

$$\frac{P_p}{A_1} = (1+\alpha)\,\sigma_y$$

Unloading

Using the elastic solution, the residual stresses and strains after the load P is unloaded to zero are obtained as (Point C)

$$\sigma_a^* = \frac{\alpha(a-b)}{b+\alpha a}\,\sigma_y > 0, \qquad \sigma_b^* = \frac{(a-b)}{b+\alpha a}\,\sigma_y > 0$$

$$\varepsilon_a^p = 0, \qquad \varepsilon_b^p = -(\frac{a}{b} - 1)\,\varepsilon_y$$

$$\varepsilon_a^* = \frac{\alpha(a-b)}{b+\alpha a}\,\varepsilon_y > 0, \qquad \varepsilon_b^* = -\frac{\alpha a(a-b)}{b(b+\alpha a)}\,\varepsilon_y$$

Reversed Loading

Using the elastic solution and the expressions of residual stresses, stresses during the reversed loading are expressed as

$$\sigma_a = \frac{\alpha(a-b)}{b+\alpha a}\,\sigma_y + \frac{b}{b+\alpha a}\,\frac{\Delta P}{A_1}$$

$$\sigma_b = \frac{(a-b)}{b+\alpha a}\,\sigma_y - \frac{a}{b+\alpha a}\,\frac{\Delta P}{A_1}$$

where ΔP is the incremental load. From the above expressions, we see that Part b yields first in the reversed loading. The elastic limit load and the plastic limit load in reversed loading are obtained as

$$\frac{P_e'}{A_1} = (1 - \alpha - 2\frac{b}{a})\,\sigma_y \qquad \text{(Point D)}$$

$$\frac{P_p'}{A_1} = -(1+\alpha)\,\sigma_y \qquad \text{(Point E)}$$

For $A_2/A_1 = 2$ and $a = 2b$, the P vs. u curve is plotted in Fig. S1.16a, and the σ_a, σ_b vs. P curves are plotted in Fig. S1.16b.

Plastic limit load and displacement-force relation

A cantilever made of elastic-ideal plastic material (steel) is loaded as shown in the figure. The cross-section is rectangular with thickness of half of its height. Assume that lateral buckling is prevented.

Determine the displacement-force relation for a load increasing from zero to the ultimate load and find the ultimate load.

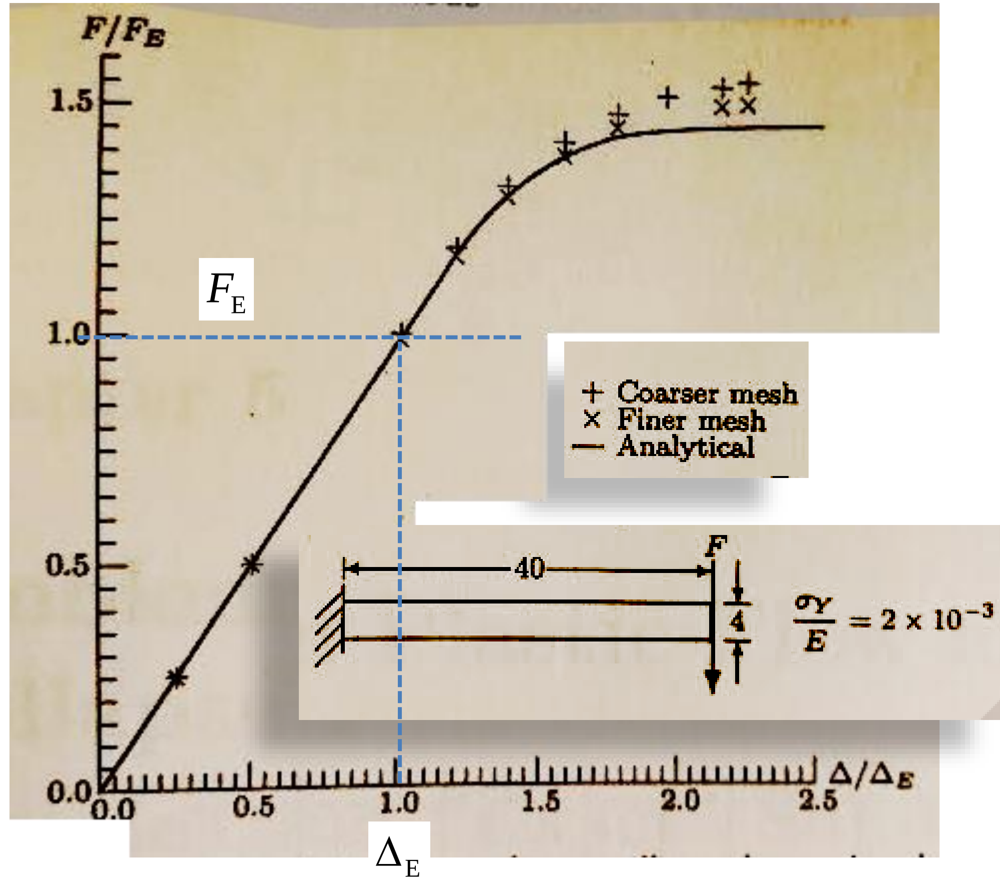
Further, unload completely and determine the residual tip displacement. Draw the full loading-unloading displacement diagram.

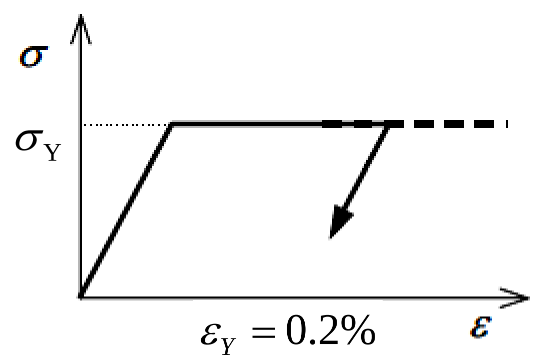
Determine the residual stress distribution

Solve the problem analytically** and using Abaqus.

Abaqus: Show the evolution of the regions of extend of the plastic deformation between loading and complete-unloading.

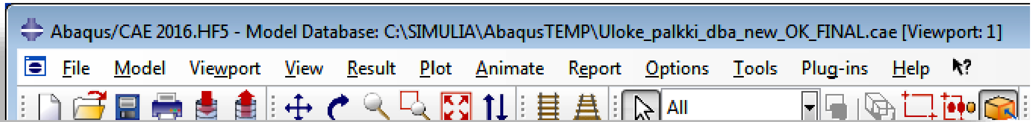
** at least the loading phase till plastic collapse was treated in details in your first master's course of structural mechanics (beams and frames)

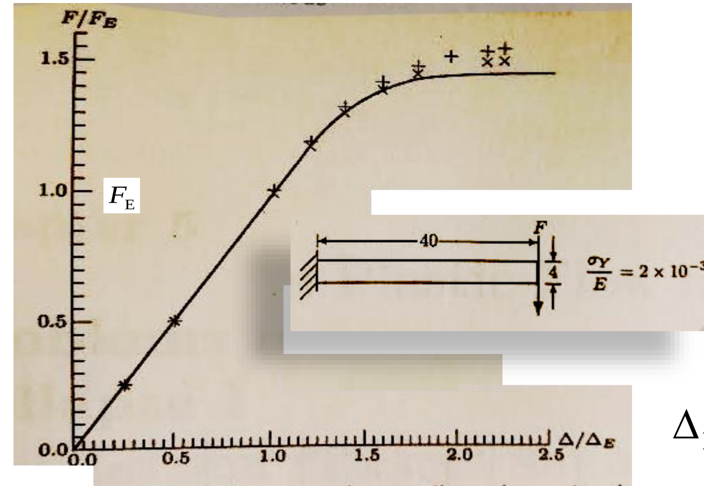




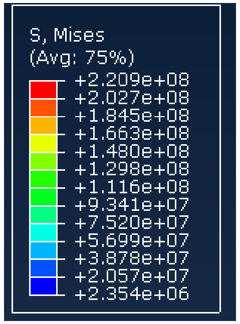
Elastic-perfectly plastic material behaviour symmetric for compression.

$h = 4$ cm
$b = 1$ cm
$L = 40$ cm





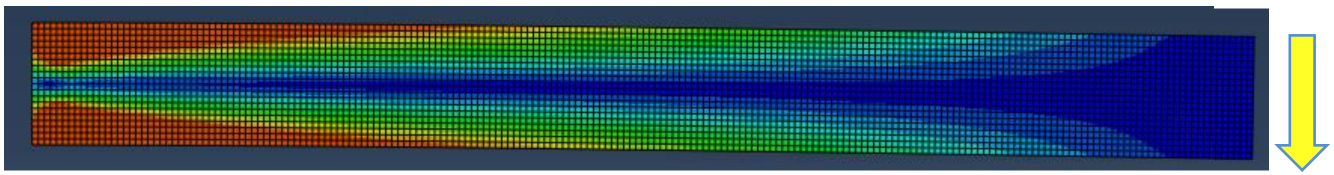
Δ_E

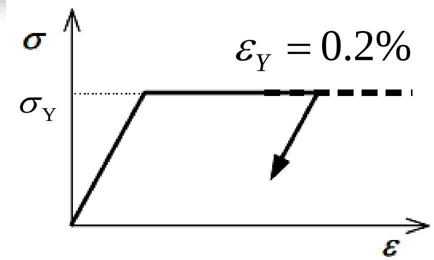


Plane stress

Load increment step 0. 7533 (of 1)

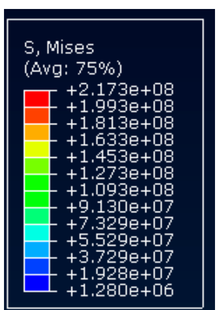
$F = -F_\mathrm{p}$

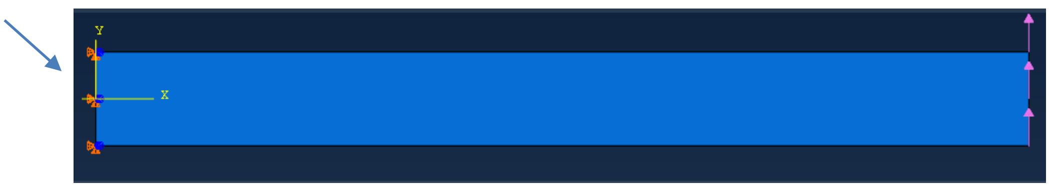




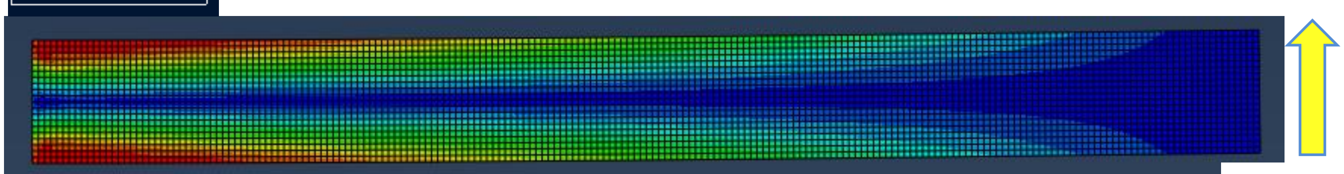
$L = 40$ cm $b = 1$ cm $h = 4$ cm

BC all dofs fixed

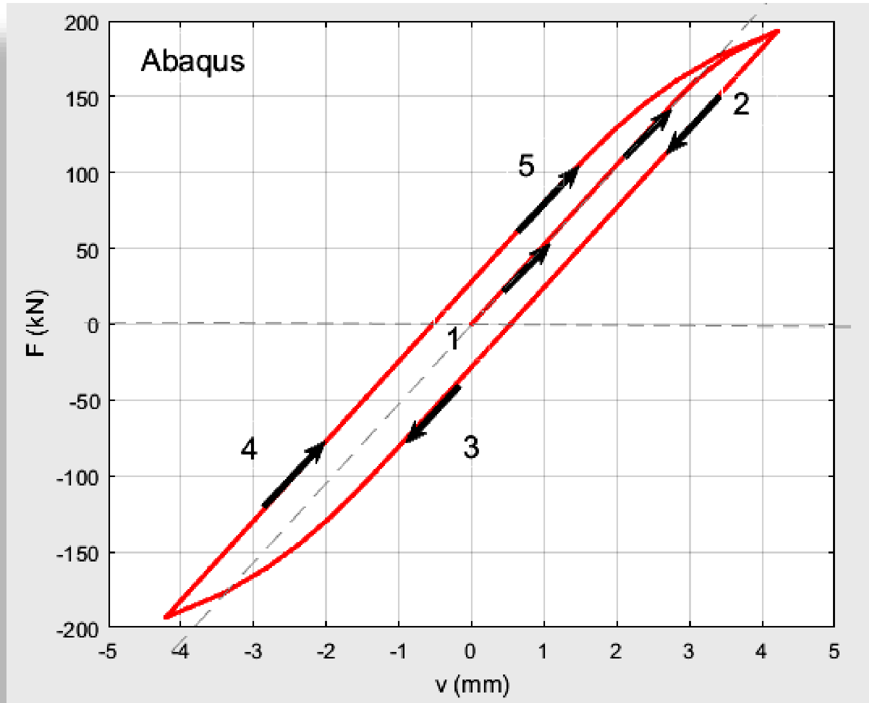




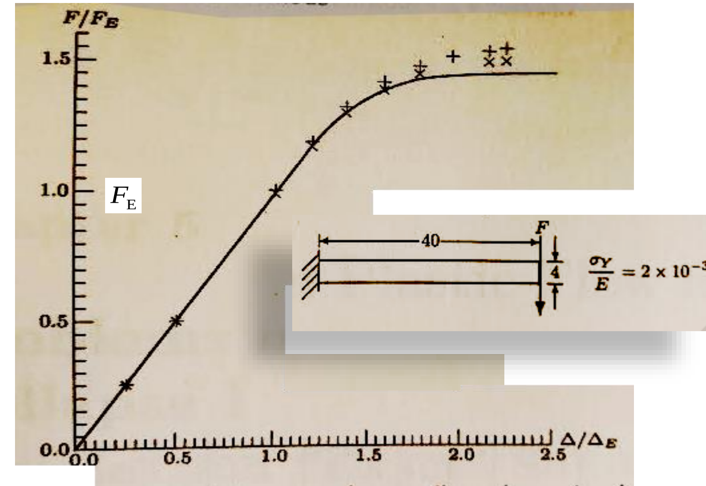
Load increment step 0.2875 (of 1)



$F = F_\mathrm{p}$



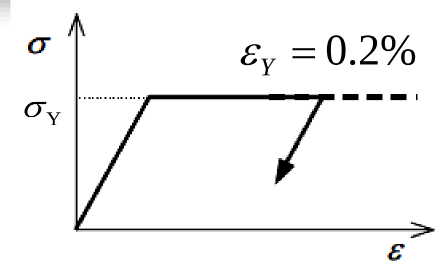
$F_{\mathrm{p}} = 200\mathrm{kN}$

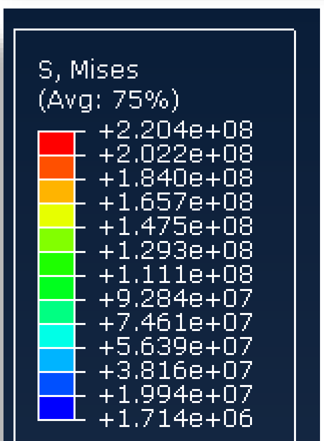


$L = 40 \text{ cm}$

$b = 1 \text{ cm}$

$h = 4 \text{ cm}$





Student: Compare to analytical solution. For the elasto-plastic phase we have

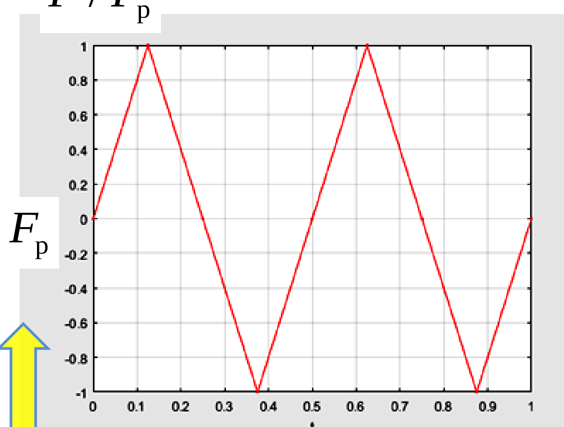
$$\Delta = \Delta_m \left(\frac{P_m}{P}\right)^2 \left[5 - \left(3 + \frac{P}{P_m}\right)\sqrt{3 - 2\frac{P}{P_m}}\right].$$

$v_{el.beam} \approx 2.5\mathrm{mm} \quad \Delta_{\mathrm{E}}$

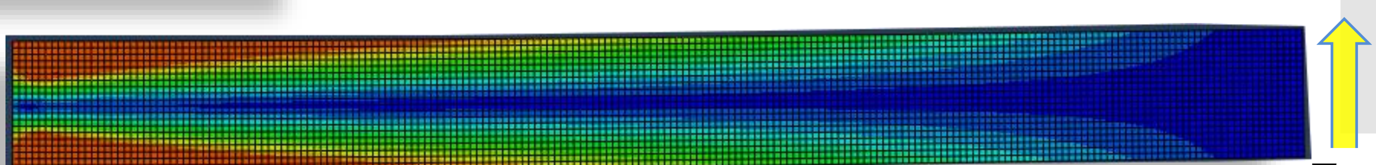
Load increment step 0.155 (of 1)

$F_{\mathrm{p}} = 200\mathrm{kN}$

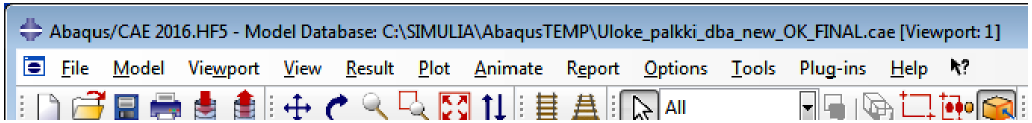
F / F_{p}



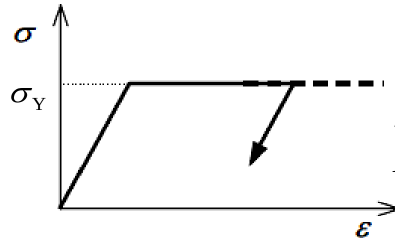
F / F_{p}



$F = F_{\mathrm{p}}$

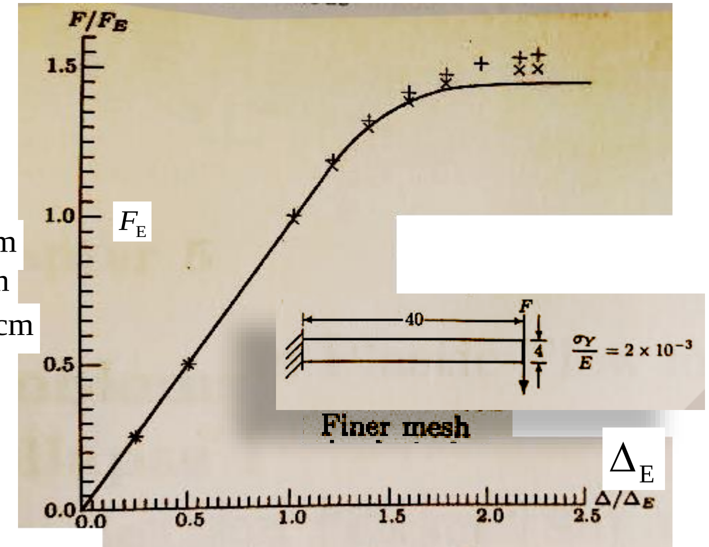


Uloke_palkki_dba_new_OK.cae



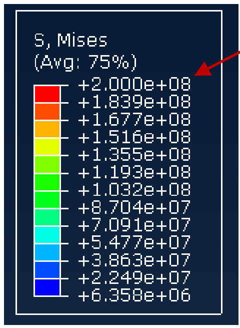
$h = 4$ cm
$b = 1$ cm
$L = 40$ cm

$\varepsilon_Y = 0.2\%$

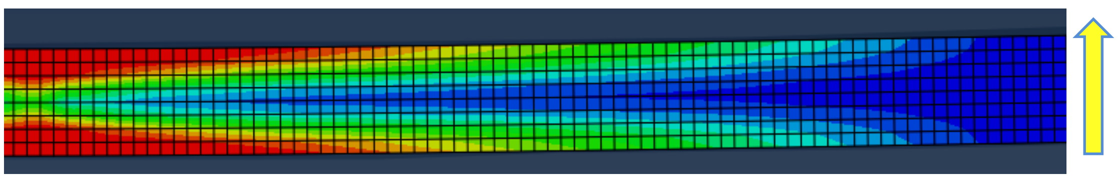


$t = 0.25$

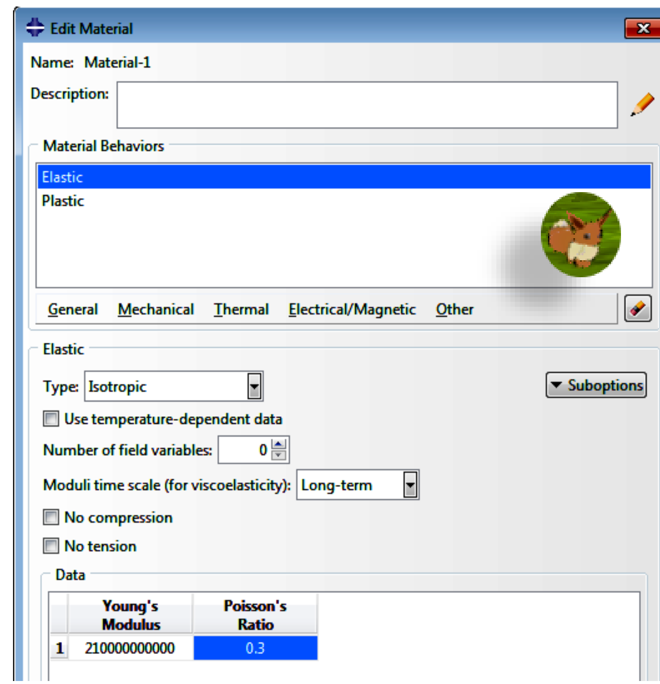
$\sigma_Y = 200$ MPa

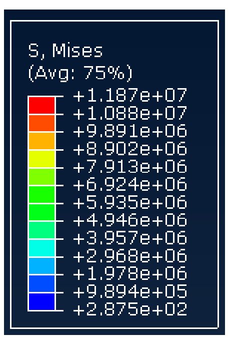


$t = 0.25, \quad F = F_p$

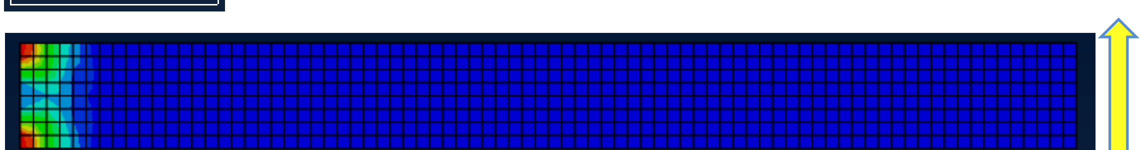


$F = F_p$



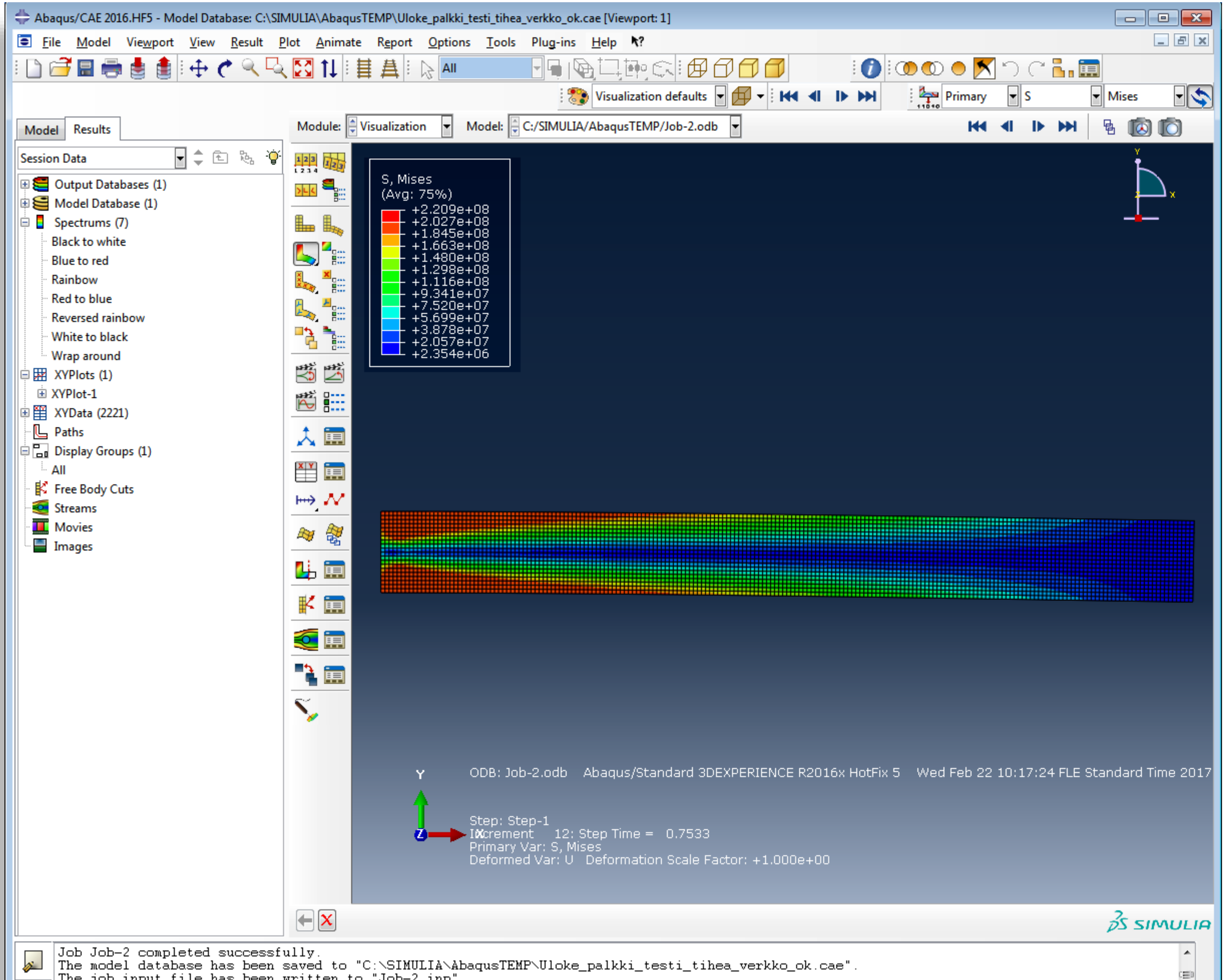


$t = 0.5, \quad F = 0$



Residual stresses on complete 1st unloading

$F = 0$


Abaqus/CAE 2016.HF5 - Model Database: C:\SIMULIA\AbaqusTEMP\Uloke_palkki_testi_tihea_verkko_ok.cae [Viewport: 1]
File Model Viewport View Result Plot Animate Report Options Tools Plug-ins Help
Visualization defaults
Primary S Mises
Module: Visualization Model: C:/SIMULIA/AbaqusTEMP/Job-2.odb
Model Results
Session Data
Output Databases (1)
Model Database (1)
Spectrums (7)
Black to white
Blue to red
Rainbow
Red to blue
Reversed rainbow
White to black
Wrap around
XYPlots (1)
XYPlot-1
XYData (2221)
Paths
Display Groups (1)
All
Free Body Cuts
Streams
Movies
Images
S, Mises
(Avg: 75%)
+2.209e+08
+2.027e+08
+1.845e+08
+1.663e+08
+1.480e+08
+1.298e+08
+1.116e+08
+9.341e+07
+7.520e+07
+5.699e+07
+3.878e+07
+2.057e+07
+2.354e+06
ODB: Job-2.odb Abaqus/Standard 3DEXPERIENCE R2016x HotFix 5 Wed Feb 22 10:17:24 FLE Standard Time 2017
Step: Step-1
Increment 12: Step Time = 0.7533
Primary Var: S, Mises
Deformed Var: U Deformation Scale Factor: +1.000e+00
SIMULIA
Job Job-2 completed successfully.
The model database has been saved to "C:\SIMULIA\AbaqusTEMP\Uloke_palkki_testi_tihea_verkko_ok.cae".
The job input file has been written to "Job-2.inp".

Engineering **Plasticity**

Classical theory - **fundamentals**

When **going beyond the** 'tiny' **domain of elasticity** of a material the **deformation** process **becomes** often **irreversible**.

There is many sources for irreversibility such as brittle fracture, fracture, **plasticity**, damage, aging, etc..

In previous courses you've come familiar with **elastic stability** of structures. Such phenomenon represents **a geometrical non-linearity** behavior.

In plasticity, we will **explore** the material response **far beyond** the **elastic range** and go…

In the present course, you will come familiar with another aspect of non-linearity since
Plasticity corresponds to **a material non-linearity**

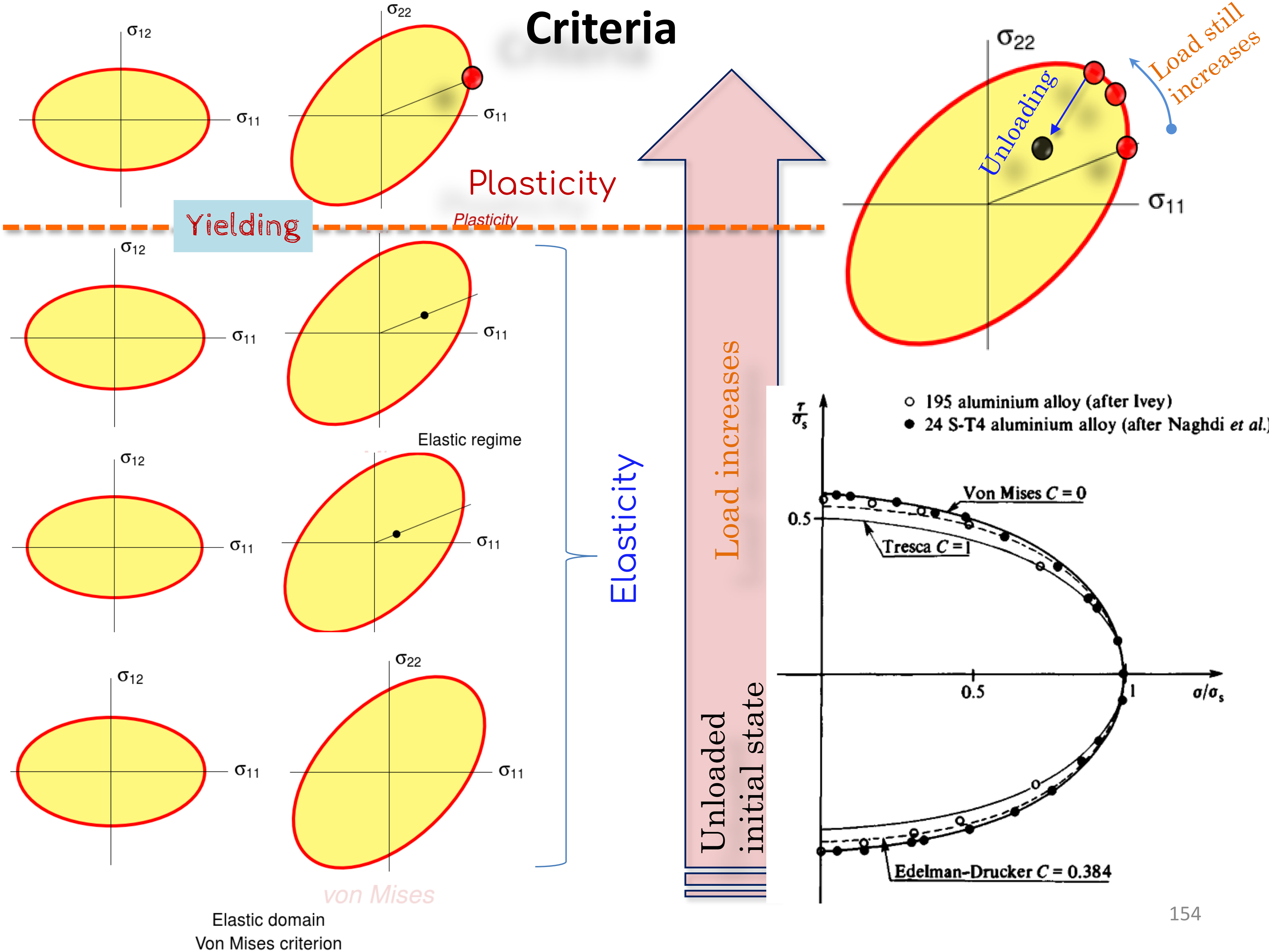
'tiny' **domain of elasticity**

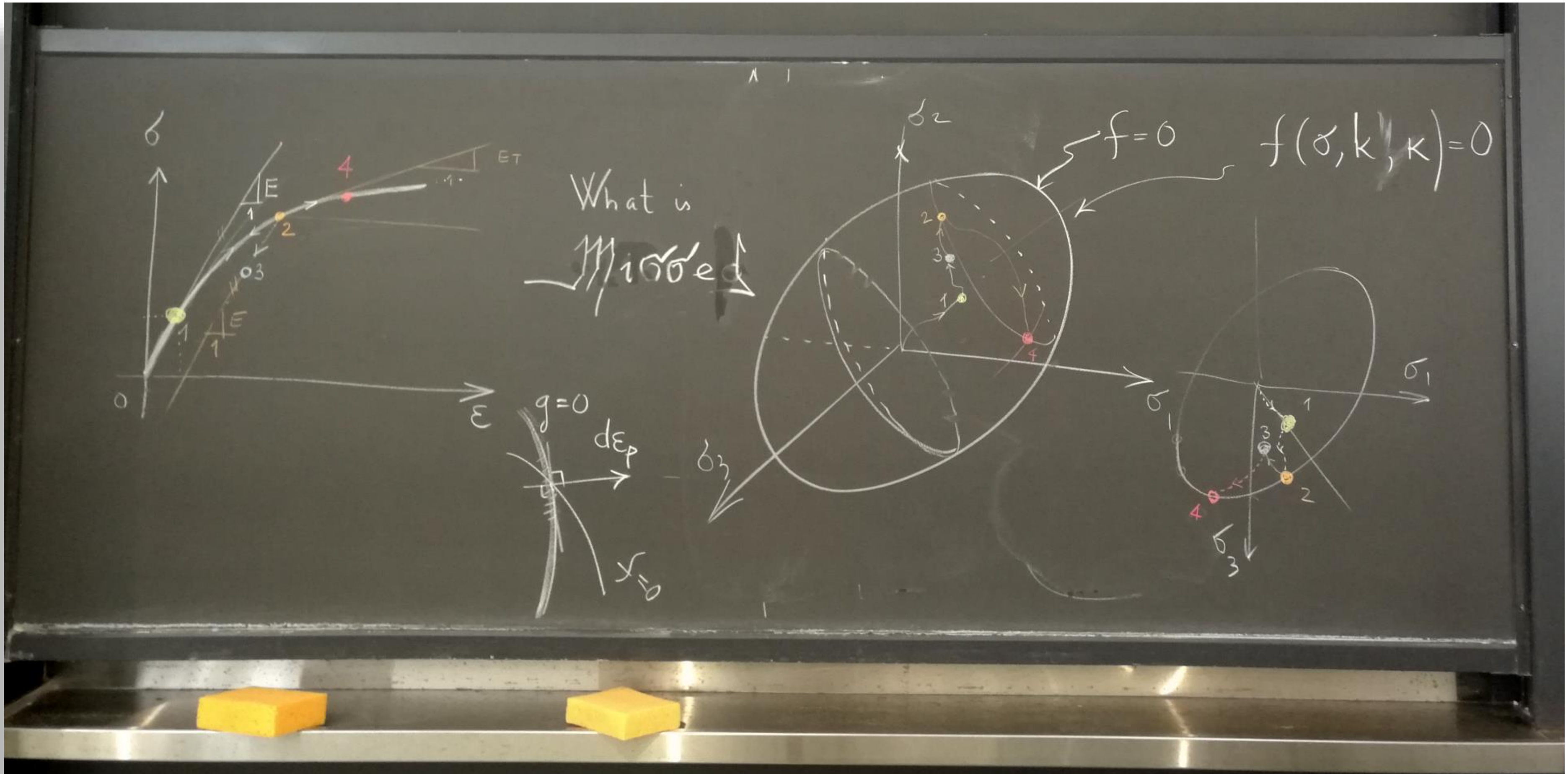
$$|\varepsilon_{\text{elastic}}| \leq 10^{-3}$$

Plastic **domain**
~10 – 20 x $|\varepsilon_{\text{elastic}}|$

GO … Where no man has gone before

Criteria


σ12
σ11
σ22
Plasticity
Yielding
Elastic regime
Elasticity
Load increases
Unloaded
initial state
Load still increases
Unloading
von Mises
Elastic domain
Von Mises criterion
τ/σs
σ/σs
0.5
1
○ 195 aluminium alloy (after Ivey)
● 24 S-T4 aluminium alloy (after Naghdi et al.)
Von Mises C = 0
Tresca C = 1
Edelman-Drucker C = 0.384

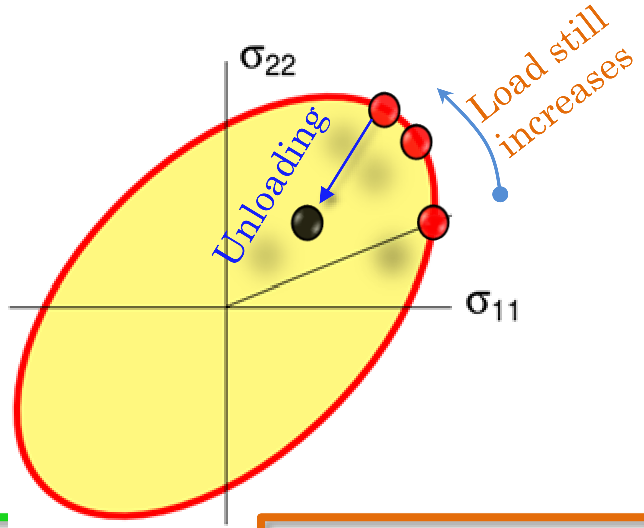


$$f(\sigma, k, \kappa) = 0$$

Plastic threshold

Hardening parameter

Plasticity in short



- $\epsilon = \epsilon^{\mathrm{e}} + \epsilon^{\mathrm{p}}$ Strain decomposition
- $f(\sigma, k, \kappa) = 0$ Yield criteria
- $\Delta\epsilon^{\mathrm{p}} = \dots$ Flow rule
- $\Delta\kappa = \dots$ Hardeningrule

How to compute?

$\Delta\epsilon^{\mathrm{p}}$

$\Delta\kappa$

$$f(\sigma, \underset{\text{Plastic threshold}}{k}, \underset{\text{Hardening parameter}}{\kappa}) = 0$$

$$f(\underbrace{\sigma_1, \sigma_2, \sigma_3}; \kappa_1, \kappa_2, \dots, \kappa_n) = 0$$

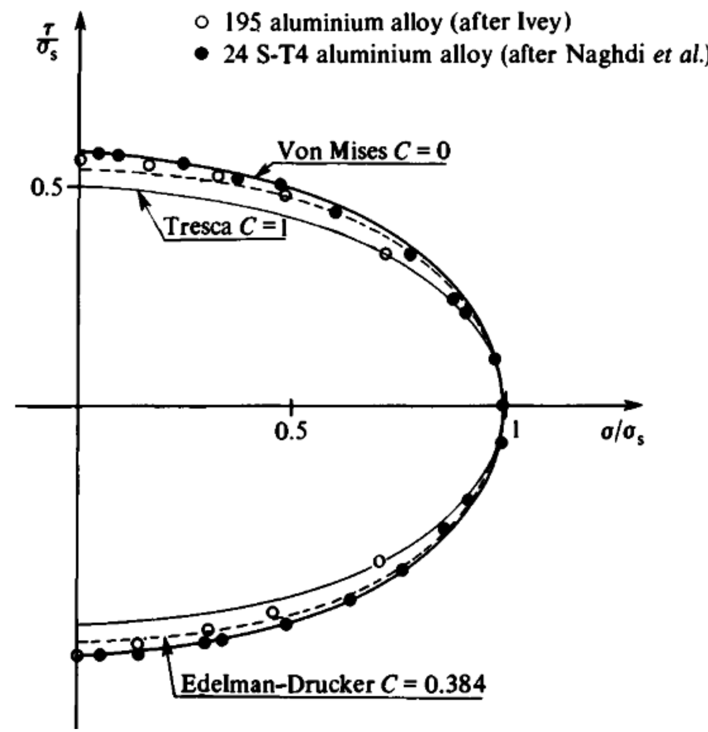
internal parameters

or in terms of stress invariants (usually)

$I_1,\ J_2\ \&\ \cos(3\theta)$

The three ingredients of engineering plasticity theory

- Yielding criteria myötöehto
- Flow rule myötösääntö
- Hardening rule lujittumisehto



$$f(\sigma_1, \sigma_2, \sigma_2, k; \kappa) = 0$$

or in terms of stress invariants (usually)

I_1, J_2 & $\cos(3\theta)$

or also expressed in terms of stress invariants I_1, J_2 and J_3 instead of principle stresses. The threshold stress $k = k(\kappa)$ can depend on some internal parameter κ, like cumulated effective plastic strain of plastic work, is yield stress σ_y in uni-axial tension or τ_y yield shear stress in pure shear test.

The three ingredients of engineering plasticity theory

- Yielding criteria myötöehto
- Flow rule myötösääntö
- Hardening rule lujittumisehto

$$f(\sigma_1, \sigma_2, \sigma_2, k; \kappa) = 0$$

or in terms of stress invariants (usually)

$$I_1,\ J_2\ \&\ \cos(3\theta)$$

Firstly *three ingredients* is not related to *la haute cuisine* but is an imaged saying for the *three sub-models* needed in the constitutive model involved during plastic flow. These sub-models can also be combined. These sub-models will tell you, depending on the material, the actual stress state, the history of plastic deformations, will the considered material point yield or remain elastic? If it yields then what will be the resulting additional plastic increment. That's all. Now, the complexity arises from the fact that the yielding criteria, usually, evolves with subsequent plastic deformations, so it depends on the history. In other word, the mechanical problem is simply *materially non-linear*.

[8] I have the convinced feeling that the complexity in model such plasticity, time-independent-plasticity, is the result of this restriction itself: the evolution problem is set as an non-evolution problem giving rise to inequality constraints in the formulation, for the process of plastic flow. I think, that this complexity should be simplified by re-giving the plastic flow it's time dimension and formulating it' as an evolutionary physical process which depends on time. Now probably, we will face an other difficulty rising from the the new time-scales introduced to the mechanical problem. These new time-scales may be order of magnitude shorter than the time-scales of the macroscopical processes related to observed deformations. However, this is worth to investigate. Introduce more physics, and please, time dependent physics.

Here we mean time-independent plasticity under isothermal processes. So, the temperature is constant and should not activate the creep; so $T \leq T_m/4$, where T_m being the melting temperature. When this is not the case, this results in thermally activated creep and leads to one type of time dependent plasticity.

The three ingredients of engineering plasticity theory

- **Yielding criteria**: $f(\sigma_1, \sigma_2, \sigma_2, k; \kappa) = 0$

  The yield function can also expressed in terms of stress invariants I_1, J_2 and J_3 instead of principle stresses. The threshold stress $k = k(\kappa)$ can depend on some internal parameter κ, like cumulated effective plastic strain of plastic work, is yield stress σ_y in uni-axial tension or τ_y yield shear stress in pure shear test.

  Note that in simple words, the yield criterion given by $f = 0$ is in general, *non-linear*. This non-linearity results not only from the fact that for many materials the plasticity threshold k depends on history and evolves $k = k(\sigma, \kappa)$ but also, the internal parameters, here the plastic deformations, depend also on history even in time independent plasticity. For instance, in some metals, the Bauschinger effect, means that the yield thresholds are changed by *cold forming* due to plastic deformations in cyclic loading.

- **Flow rule** $\mathrm{d}\epsilon_{ij}^{(\mathrm{p})} = \lambda \cdot \partial f / \partial \sigma_{ij}$ for associative response. This rule expresses normality of the plastic increment which is proportional to the gradient $\nabla_\sigma f$ in the principle stress space. For many metals normality expresses the incompressibility of the plastic flow (the volumetric part of the plastic deformation is zero). For non-associative case, like friction materials (concrete, soils,...) the *normality rule* does not hold and then $\mathrm{d}\epsilon_{ij}^{(\mathrm{p})} = \lambda \cdot \partial g / \partial \sigma_{ij}$. The function g has to be specified. Usually, the function g is 'close to the yield function f because the departure from normality is often not huge.

- **Hardening rule** – defines how the yielding or loading surface evolves during the course of evolution of plastic flow.

The Three Ingredients of the Classical Plasticity Theory

Total deformation increment $d\boldsymbol{\varepsilon} = d\boldsymbol{\varepsilon}^{(p)} + d\boldsymbol{\varepsilon}^{(e)}$

Plastic strain increment

Elastic strain increment

1. **Initial yield surface** – also called a *loading surface* - defines, in stress space, the stress level (*i.e.*, the combination of stress components) at which yielding occurs

2. **Hardening rule** – defines *how the yielding or loading surface evolves* during the course of plastic flow

3. **Flow rule** – defines the direction of the incremental plastic deformation *by postulating a plastic potential*. More precisely, defines an incremental plastic strain-stress relation.

a plastic potential

$$d\boldsymbol{\varepsilon}^{(p)} = d\lambda \frac{\partial g}{\partial \boldsymbol{\sigma}}$$

positive scalar

$$d\boldsymbol{\sigma} = \mathbf{D}^{ep} d\boldsymbol{\varepsilon}$$

Elastic-plastic material stiffness (non-linear)

The exact form should be derived using the *consistency condition* (see later)

Three ingredients of plasticity models

classical

$$\sigma = C : \varepsilon^e = C : (\varepsilon - \varepsilon^p)$$

$$\mathrm{d}\varepsilon = \mathrm{d}\varepsilon^e + \mathrm{d}\varepsilon^p$$

$$\dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p$$

$$\dot{\boldsymbol{\varepsilon}}^{(\mathrm{p})} = \dot{\lambda}\frac{\partial g}{\partial \boldsymbol{\sigma}}$$

g - **plastic potential**

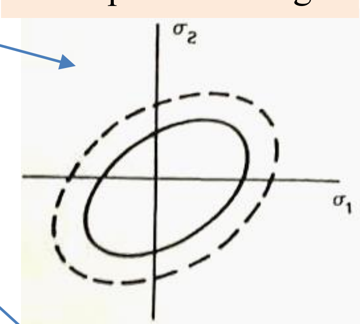
If stresses or strains exceed a certain limit, plastic ***yielding*** may be observed especially in metallic and ductile materials

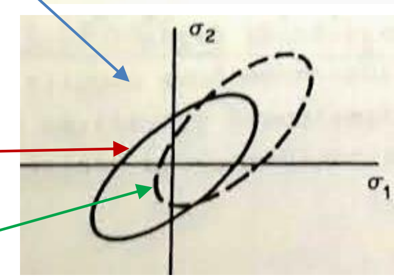
classical

Three ingredients of plasticity models: Threshold, plastic flow and hardening

1. **Yield criteria** - myötöehto tai -funktio $F(\sigma_{ij}) = 0$
2. **Flow rule** - myötösääntö $\mathrm{d}\varepsilon^p$
3. **Hardening rule** – myötölujenemissääntö
   The yield function may change due to plastic flow
   Work hardening, also known as **strain hardening** or **cold working**, is the **strengthening** of a metal by plastic deformation which leads to subsequent modifications of the yield surface

Isotropic hardening





Kinematic hardening
(most metals)

Initial yield surface

Subsequent yield surface after loading

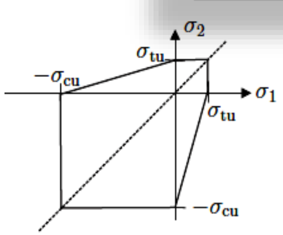
The yield surface remains constant in size and it moves

A general yield surface:

$$F\left(\sigma_{ij}, \varepsilon^{\mathrm{p}}_{ij}; k(\varepsilon^{\mathrm{p}})\right) = 0,$$

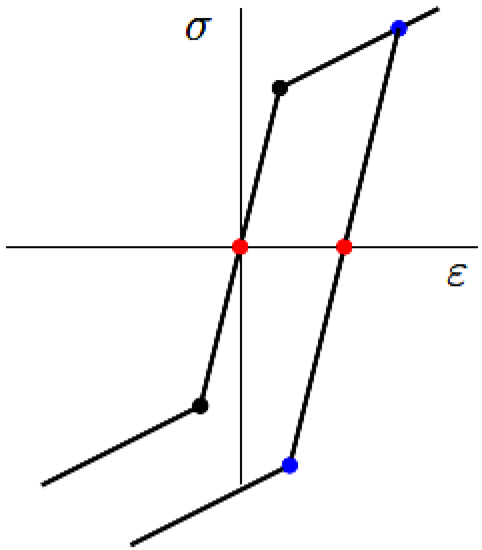
Effective plastic strain
An increasing function

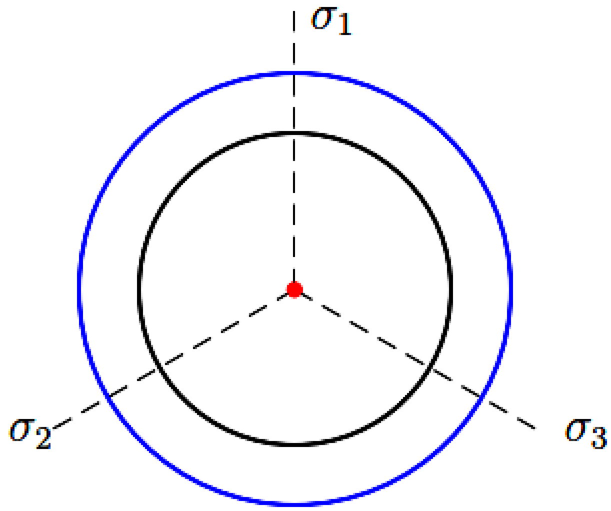
Threshold value
yield stress



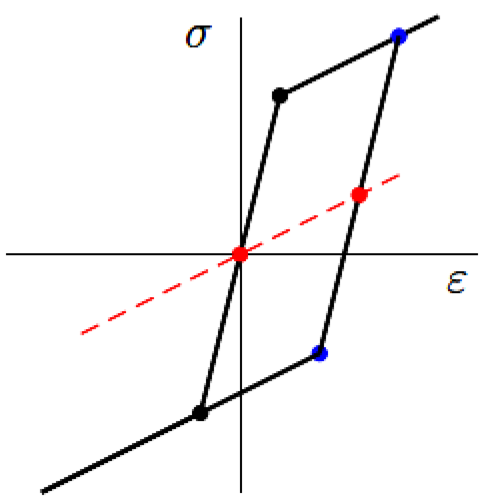
ex. Failure surface: **Mohr-Coulomb** for biaxial stress states

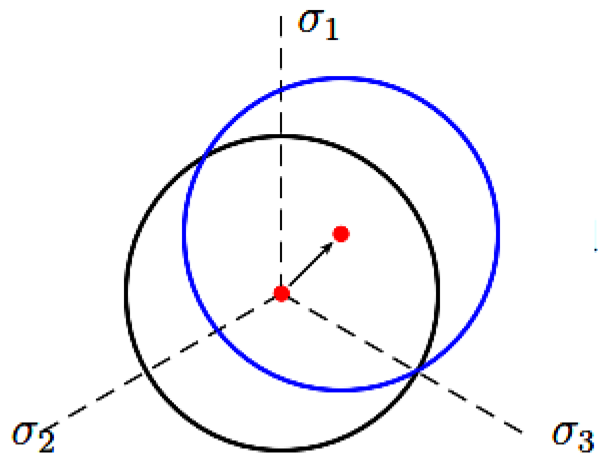
Linear isotropic (above) and kinematic hardening.





isotropic hardening.





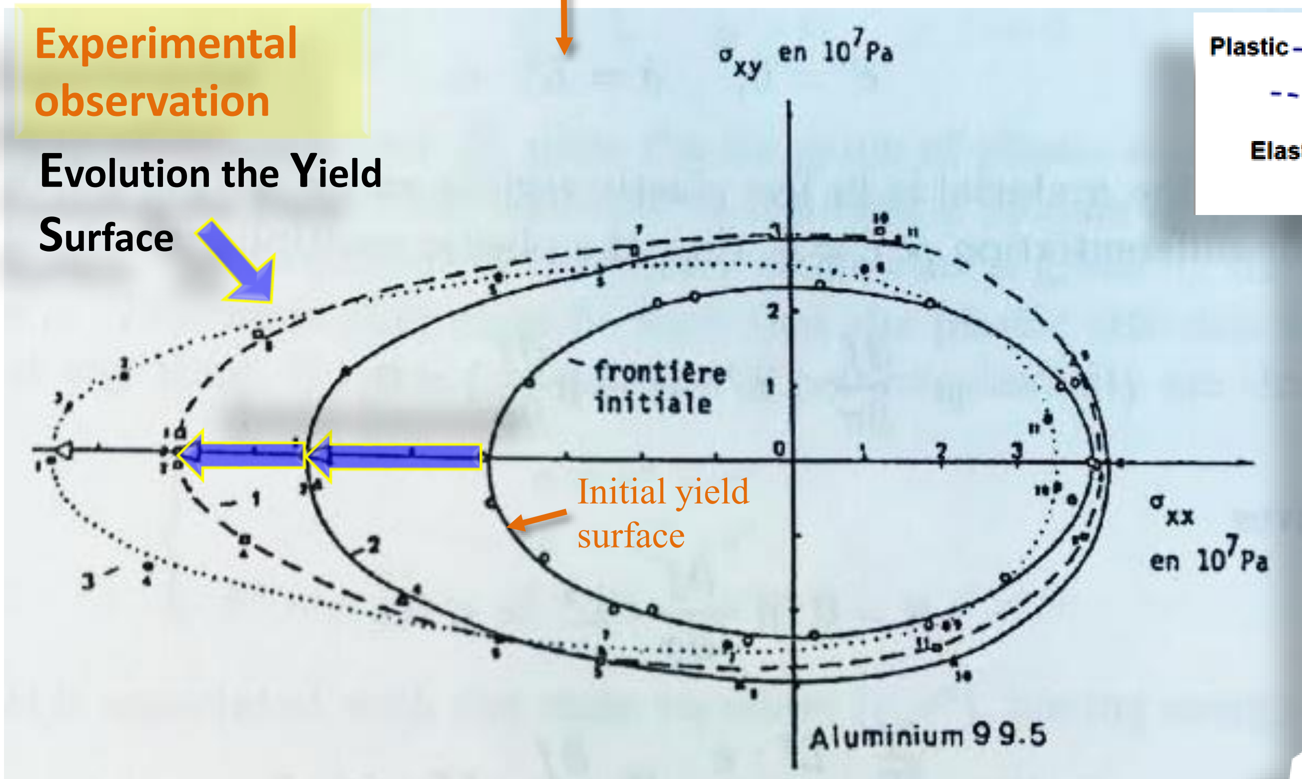
kinematic hardening.

Ref: Reijo Kouhia lecture notes (2020)

Experimental Evidence of Evolution the Yield Surface for metals (Aluminum)

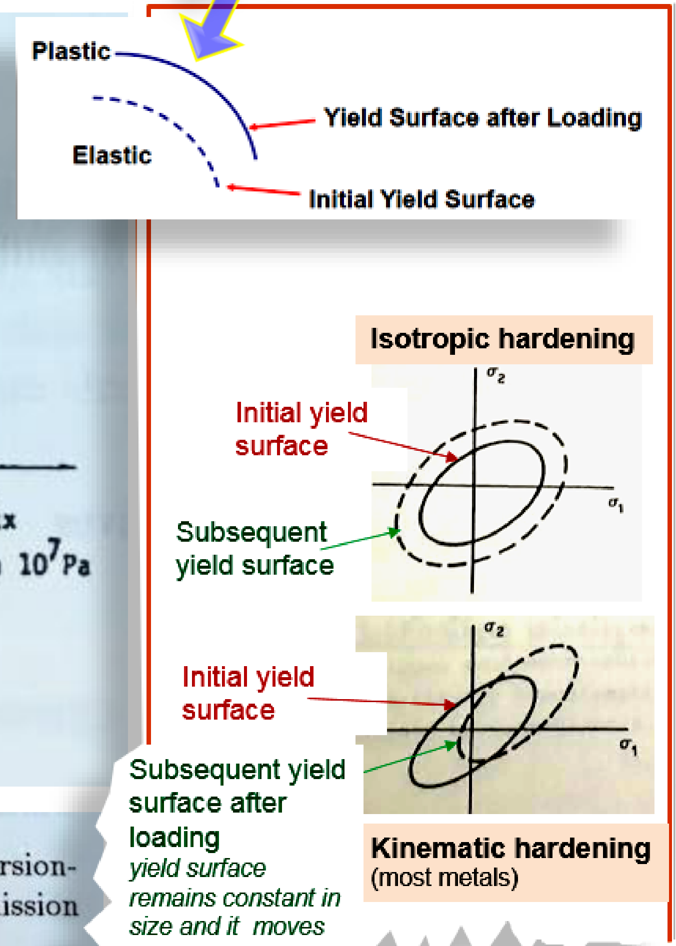
Experimental observation

Evolution the Yield Surface



Evolution of the elastic domain with the plastic strain in a torsion-traction experiment of a thin aluminum tube, after Bui [34]. Reproduced by permission of the author.

Evolution the Yield Surface



Reference: GA. Maugin: *The thermomechanics of plasticity and fracture.* Cambridge (book).

The experiments: Bui H.D. (1970), Evolution de la frontière du domaine élastique des métaux avec écrouissage plastique …. *Mém. Artillerie Franc., Sci. Tech. Armement*, **1**, 141-65.

Hardening rule – myötölujenemissääntö

The yield function F may change due to plastic flow

The hardening rule…

1. **describes** how the yield *surface changes* (evolves) due to plastic deformation
2. **determines** when the material *will yield again* in load continues or is reversed

A subsequent general yield surface:

$$F\left(\sigma_{ij}, \varepsilon_{ij}^{p}; k(\bar{\varepsilon}^{p})\right) = 0,$$

Threshold value
A yield stress

effective plastic strain
An increasing function … cumulated equivalent (deviatoric part) plastic strain

Given an initial *yield surface*, the hardening rule defines its modifications during plastic flow:

$$F(\sigma_{ij}, \varepsilon_{ij}^{p}; k(\varepsilon_{ij}^{p})) = F(\sigma_{ij}, \varepsilon_{ij}^{p}) - k^{2}(\bar{\varepsilon}^{p}) = 0,$$

General form of a hardening rule

F **determines** the **shape**

determines the **size**

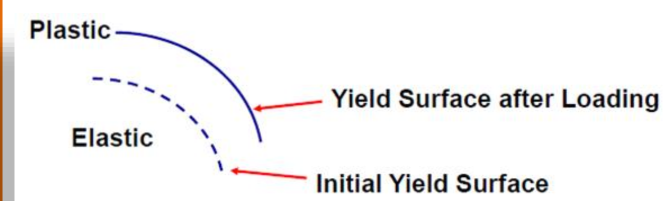
Isotropic hardening

- The yield surface expands uniformly in size in all directions with the plastic flow

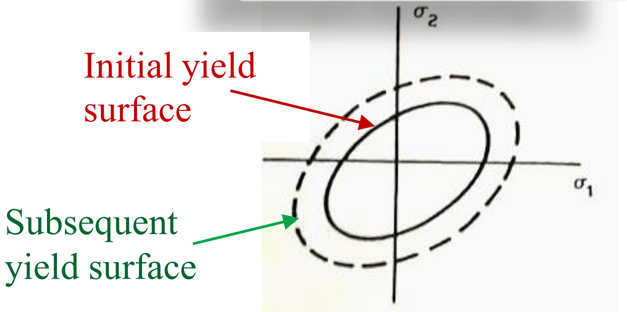
Kinematic hardening

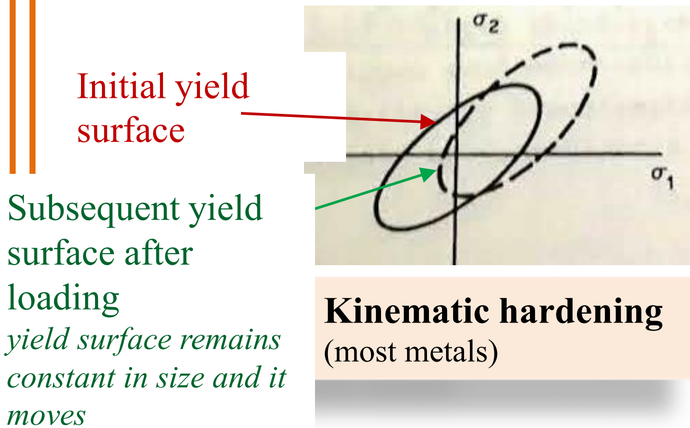
- The yield surface remains constant in size and moves in the direction of yielding

For instance, in the yield criterion $F(\sigma_{ij}; k(\kappa)) = 0$, the yield stress k may depend on a **kinematic** parameter as κ which depends for instance, on the plastic work wh $W_p = \int \boldsymbol{\sigma} : d\boldsymbol{\varepsilon}_p$ c $d\bar{\boldsymbol{\varepsilon}}_p = \sqrt{2/3 d\boldsymbol{\varepsilon}_p : d\boldsymbol{\varepsilon}_p}$ quivalent plastic strain $\bar{\boldsymbol{\varepsilon}}_p$

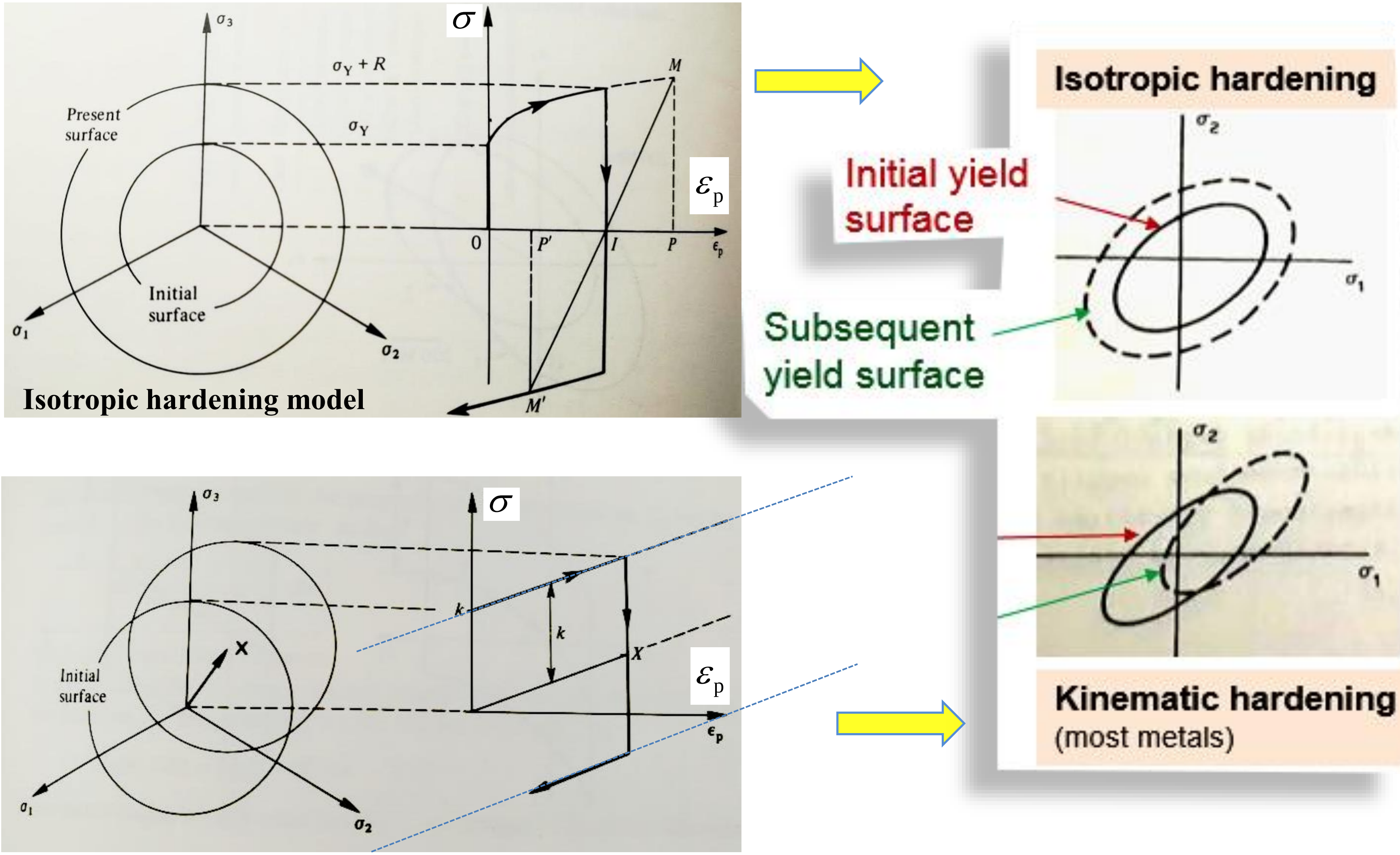


Isotropic hardening





Kinematic hardening
(most metals)



Kinematic hardening model

Some 1-D empirical models for work (also called strain-) hardening phenomena:

These models are relationship between the stress and the amount of plastic strain

$$\sigma = K\epsilon_p^n$$

$$\sigma = \sigma_y + K\epsilon_p^n$$

n ~0.2 -… 0.5

$$\sigma = \sigma_y + K(\epsilon_0 + \epsilon_p)^n$$

Mixed hardening rule

A subsequent yield surface:

effective plastic strain

$$F(\sigma_{ij},\varepsilon_{ij}^{p};k(\varepsilon_{ij}^{p})) = F(\sigma_{ij}-\alpha_{ij}) - k^{2}(\bar{\varepsilon}^{p}) = 0,$$

***F* determines** the **shape**

Shift or (translation)

Determines the **size** as function of cumulated plastic strain (**expansion**)

Kinematic hardening

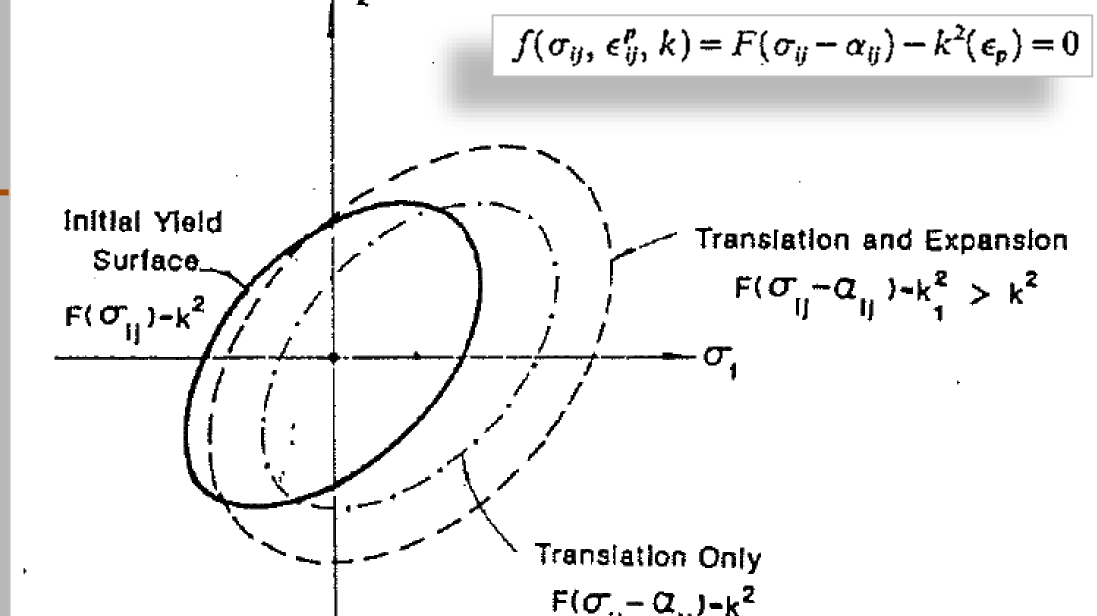
Isotropic hardening

α_{ij}, k^{2} - hardening parameters

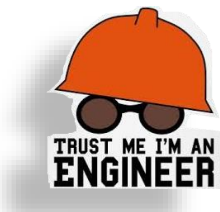
Mixed hardening rule allows to model different degrees of Bauschinger effect by adjusting the two hardening parameters α_{ij} & k^{2}

Mixed hardening is a result of a combination of *kinematic* and *isotropic hardening* (Hodge, 1957)

Ref: Chan & Han, textbook



Engineering Plasticity - Yield Criteria



Here we mean time-independent[8] plasticity under isothermal processes. So, the temperature is constant and should not activate the creep; so $T \leq T_m/4$, where T_m being the melting temperature. When this is not the case, this results in thermally activated creep and leads to one type of time dependent plasticity.

Note that in simple words, the yield criterion given by $f = 0$ is in general, *non-linear*. This non-linearity results not only from the fact that for many materials the plasticity threshold k depends on history and evolves $k = k(\sigma, \kappa)$ but also, the internal parameters, here the plastic deformations, depend also on history even in time independent plasticity. For instance, in some metals, the Bauschinger effect, means that the yield thresholds are changed by *cold forming* due to plastic deformations in cyclic loading.

Firstly *three ingredients* is not related to *la haute cuisine* but is an imaged saying for the *three sub-models* needed in the constitutive model involved during plastic flow. These sub-models can also be combined. These sub-models will tell you, depending on the material, the actual stress state, the history of plastic deformations, will the considered material point yield or remain elastic? If it yields then what will be the resulting additional plastic increment. That's all. Now, the complexity arises from the fact that the yielding criteria, usually, evolves with subsequent plastic deformations, so it depends on the history. In other word, the mechanical problem is simply *materially non-linear*.

The three ingredients of engineering plasticity theory

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- **Yielding criteria**: $f(\sigma_1, \sigma_2, \sigma_2, k; \kappa) = 0$

  The yield function can also expressed in terms of stress invariants I_1, J_2 and J_3 instead of principle stresses. The threshold stress $k = k(\kappa)$ can depend on some internal parameter κ, like cumulated effective plastic strain of plastic work, is yield stress σ_y in uni-axial tension or τ_y yield shear stress in pure shear test.

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- **Flow rule**: The flow rule is just a model to determine the increment of subsequent plastic strains. Classically, it is defined to be proportional to gradient $\nabla_\sigma f \equiv \partial f / \partial \sigma_{ij}$ of a scalar plastic potential f or g. The proportionality coefficient $\mathrm{d}\lambda$ has to be determined by constraining the subsequent stress state resulting from this increment of plastic deformation to remain on the yield surface $f(\sigma) = 0$. This condition is know as *consistency* condition for the plastic flow.

$$\mathrm{d}\epsilon^{(\mathrm{p})} = \mathrm{d}\lambda \cdot \underbrace{\nabla_\sigma f}_{\equiv \mathbf{n}_f}, \qquad \text{associative case} = \text{hydrostatic pressure insensitive}$$

$$\mathrm{d}\epsilon^{(\mathrm{p})} = \mathrm{d}\lambda \cdot \underbrace{\nabla_\sigma g}_{\equiv \mathbf{n}_g}, \qquad \text{non-associative case} = \text{hydrostatic pressure sensitive}$$

For hydrostatic pressure insensitive materials, like metals, the plastic potential is directly given by the flow rule $f(\sigma) = 0$. For such materials the relative volume change resulting from plastic deformations is zero. Mathematically, this is equivalent to say that the increment of plastic deformations $\mathrm{d}\epsilon^{(\mathrm{p})}$ is normal to the yield surface $f = 0$ as expressed by the relation $\mathbf{n}_f = \nabla_\sigma f$. This last relation is called *normal* flow rule. Normal means orthogonal. Now for hydrostatic pressure sensitive materials, like geo-materials, concrete, friction-materials, the material failure (='yield') is accompanied with a noticeable volume change due to rearrangements of the 'discrete' micro-structures. Mathematically speaking, the plastic strain increment is not any more *normal* to the yield function $f = 0$. There is a small deviation angle from it called *dilatation* or *dilatancy*. A new potential g, close to f, is constructed. Now the flow rule is again *normal* but expressed with respect to this new potential $g(\sigma) = 0$ as given by $\mathbf{n}_g = \nabla_\sigma g$. This was the story of the plastic flow.

- **Flow rule**: The flow rule is just a model to determine the increment of subsequent plastic strains. Classically, it is defined to be proportional to gradient $\nabla_\sigma f \equiv \partial f / \partial \sigma_{ij}$ of a scalar plastic potential f or g. The proportionality coefficient $\mathrm{d}\lambda$ has to be determined by constraining the subsequent stress state resulting from this increment of plastic deformation to remain on the yield surface $f(\sigma) = 0$. This condition is know as *consistency* condition for the plastic flow.

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associative case = hydrostatic pressure insensitive

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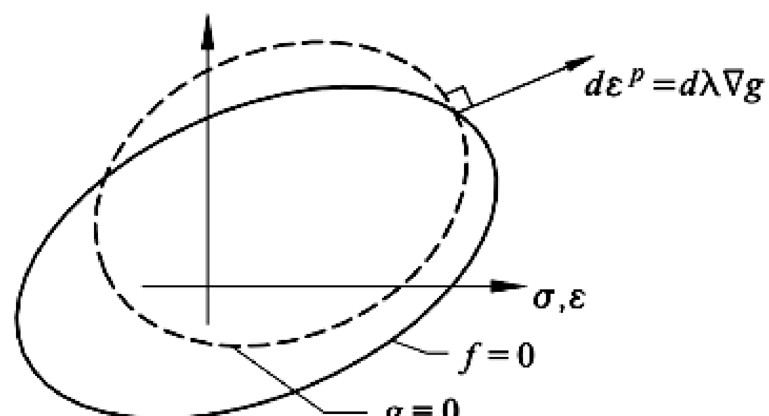
non-associative case = hydrostatic pressure sensitive



- **Hardening rule** – defines how the yielding or loading surface evolves during the course of evolution of plastic flow.

Engineering # Plasticity -

Yield Criteria

Pressure sensitive yield

For materials having **pressure dependent** yield as
Soils, sand, rocks, concrete, granular materials, powders,…

Pressure insensitive yield

For materials, as metals, for instance, plastic yield **independent** of the hydrostatic pressure (for moderate pressures)

Isotropic yield criteria also known as *failure hypotheses*:

- Maximum Principal Stress Theory (Rankine)
- Maximum Principal Strain Theory (St. Venant)

brittle materials

- **Maximum Shear Stress Theory** (Tresca)
- **Maximum Distortion Energy Theory** (von Mises)

- **Mohr-Coulomb yield criterion**
- **Drucker-Prager yield criterion**

into details

non-associative & associative flow laws

N.B. Depending on the specific application field, there exists an enormous amount of other three-dimensional failure or 'yield' criteria. They remain out of the scope of this course.

Pressure independent Yield criteria

- **Tresca yield criterion –** Maximum Shear Stress Theory
- **Von Mises yield criterion –** Maximum Distortion Energy Theory

Von Mises yield Criteria

$$\sqrt{J_2} = k \equiv \tau_Y \qquad = \text{yield stress of the material under pure shear}$$

It looks very simple **formula**

Von Mises Yield criteria:

$$\boxed{F = J_2 - k^2 = 0}$$

or equivalently $\sigma_e = \sqrt{3J_2} = \sigma_Y$

$$J_2 = k^2 \equiv \tau_Y^2$$

where *k* is the yield stress of the material in pure shear $k = \sigma_Y / \sqrt{3}$

yield in uniaxial test

$$k \equiv \tau_Y = \text{yield stress under pure shear}$$

... however, lets start from the beginning and have some fundaments and experimental observations on which the models are based and not only enumerating the various yield criteria

Von Mises and Tresca yield Criteria

- **Isotropic material**: the *yield criterion depends only on the stress invariants* or I_1, I_2, I_3 equivalently on invariants of deviatoric stresses and J_2, J_3 I_1

- Hydrostatic stress causes for many materials, *especially **metals***, a purely elastic volumetric strains (plastic volumetric strains = 0, **incompressible, zero dilatancy**) → I_1 does not influence yielding (for not very high pressures)

$$I_1 \equiv \mathrm{tr}(\boldsymbol{\sigma}) = \sigma_1 + \sigma_2 + \sigma_3$$

The plastic volume change is called "*dilatancy*"

Yield criteria -
myötöehto tai -funktio

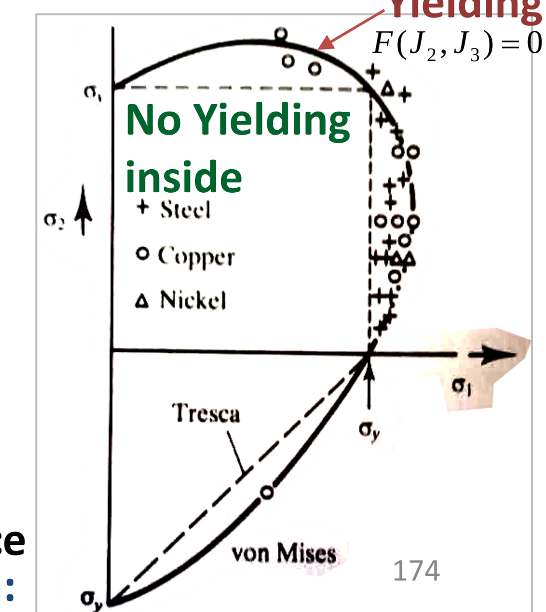
$$F(J_2, J_3) = 0$$

$$F(\sigma_{ij}) = 0$$

VON MISES yield criterion

TRESCA yield criterion

- Distortional energy – Von Mises **yield** condition
- Maximum shear stress – TRESCA **yield** condition
- Maximum Principal stress – RANKINE (1820–1872), LAMÉ (1795–1870) & NAVIER (1785–1836)
- Maximum Principal strain – SAINT-VENANT (1797–1886) & BACH (1889)
- Maximum strain energy – BELTRAMI (1835-1900)
- Mohr-Coulomb hypothesis ... for granular and geological materials
- Octahedral shear Stress theory or Drucker-Prager hypothesis - DRUCKER (1918-2001) & PRAGER (1903-1980) ... for granular and geological materials



Experimental evidence Von-Mises and Tresca:

Definitions

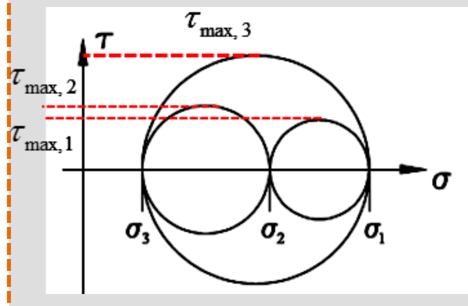
Vertailujännitys

Equivalent stress (effective stress or **von Mises Stress**):

It defines a uniaxial equivalent stress:
(Vertailujännitys)

$$\sigma_e^2 = \frac{3}{2} s_{ij} : s_{ij} = \frac{3}{2} \mathbf{s} : \mathbf{s} \equiv 3J_2$$

NB. Both $\tau_{max}(1)$ and $\sigma_e = 3/\sqrt{2} \cdot \tau_{oct}$ in (2) are shear stresses



$$\tau_{max} = \tfrac{1}{2}\left|\sigma_3 - \sigma_1\right| \quad (1)$$

$$J_2 \equiv II_s = \frac{1}{2} \mathbf{s} : \mathbf{s}, \qquad \boldsymbol{\sigma} - p\mathbf{I} = \mathbf{s}$$

deviator stress

Definition: A *three-dimensional* stress state $\boldsymbol{\sigma}$ or respectively $\mathbf{s}$ is *equivalent* to *one-dimensional* stress state σe,
(characterized by a material's **uniaxial** (1D) stress-strain curve $\sigma(\varepsilon)$),
if their **second invariants** J_2 **are equal**
(or equivalently, distortional elastic strain energies are equal)

$- p\mathbf{I}$

Yield, in metals or ductile materials, is *insensitive* to hydrostatic pressure *p*

3-D stress state: $\mathbf{s}$

1D- stress state:

$\sigma_1 = \sigma_e$, $\sigma_2 = \sigma_3 = 0$

$$J_2 = \frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$$

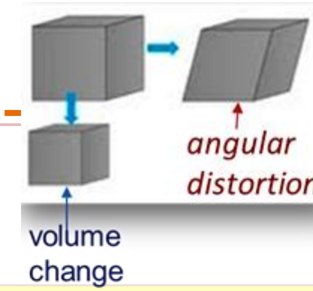
$$\sigma_e^2 = 3J_2 \Rightarrow \sigma_e = \sqrt{\frac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]} \quad (2)$$

$$\tau_{oct} = \sqrt{\frac{2}{3} J_2}$$

$$\tau_{oct} = \frac{\sqrt{2}}{3} \sigma_e$$

Effective stress or Von Mises Stress – tehokasjännitys, vertailujännitys

Von Mises yield Criterion (Distortional Energy Criteria)



- Failure or *yielding* occurs when equivalent stress $\sigma_e = \sqrt{3J_2}$ (Von Mises stress) reaches the *yield stress* k of the material $\sigma_e = \sigma_Y$

- Equivalently, *yielding* occurs when the (maximum) distortional elastic strain energy of density U_G reaches a critical value which is a property of the material

Yield criteria - myötöehto

$$F(\boldsymbol{\sigma};k) = J_2 - k^2 = 0$$

$$F = \frac{1}{6}[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2] - k^2 = 0$$

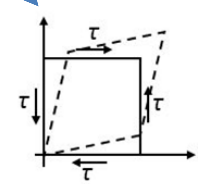
k - yield stress $\equiv \tau_Y$ in pure shear, $k = \sigma_Y / \sqrt{3}$

yield in **uniaxial test**

$$U_G = \frac{J_2}{2G} = \frac{k^2}{2G}$$

3D-stress state

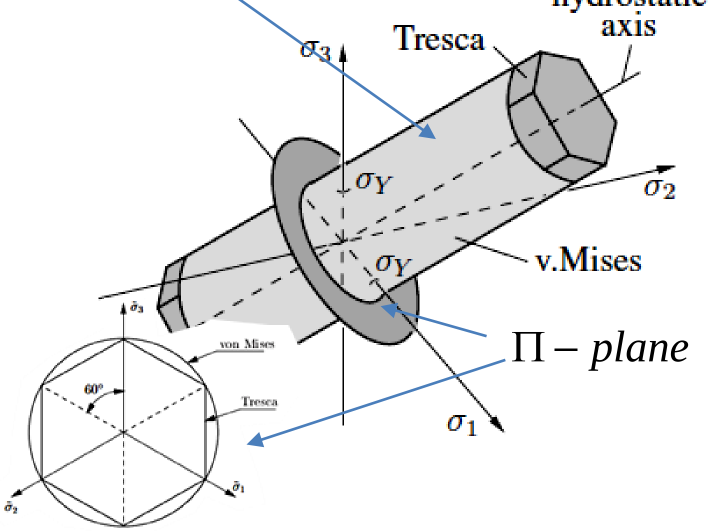
Uniaxial stress state

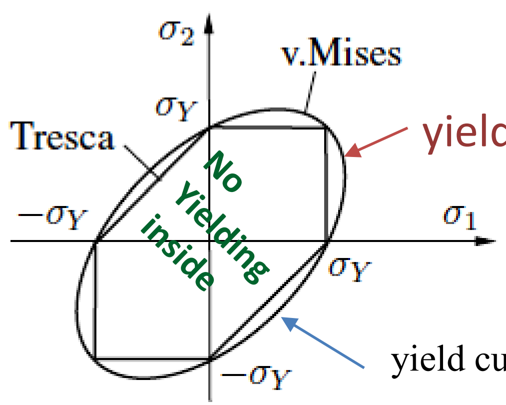


$$\boldsymbol{\sigma} = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{s}$$

$$\tau = \frac{1}{\sqrt{3}}\sigma_Y$$

cylindrical surface of radius of radius $\sqrt{2}k$ around *hydrostatic* axis $\sigma_1 = \sigma_2 = \sigma_3$





yield curve is an ellipse

Von-Mises criteria is also known as maximum octahedral shear stress criterion

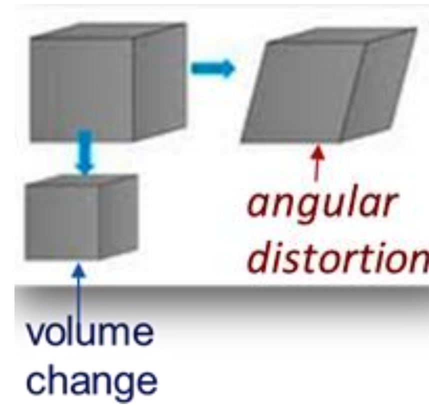
$$J_2 = \frac{3}{2}\tau^2_{oct}$$

octahedral shear stress

$$J_2 = \frac{1}{6}[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2]$$

Distortional Energy Criteria:

Yielding will occur when the ***distortion energy*** in a unit volume of the material equals the distortion energy in the same volume which is uniaxially stressed till yielding occurs



Von Mises Yield criteria: $F = J_2 - k^2 = 0 \;\Rightarrow$

$$\tau_Y \equiv k = \sigma_Y / \sqrt{3}$$

yield in uniaxial test

$$F = \frac{1}{6}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] - k^2 = 0 \;\Rightarrow$$

$$\Rightarrow \frac{1}{\sqrt{3}} \underbrace{\sqrt{\frac{1}{2}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]}}_{\equiv \sigma_e} = k = \frac{\sigma_Y}{\sqrt{3}}$$

von Mises equivalent stress

$$\Rightarrow \sigma_e = \sigma_Y$$ yielding criteria

The material yields when *von Mises equivalent stress* σ_e exceeds the uniaxial material yield strength

$$\sigma_e = \sigma_Y$$

Von Mises yield Criterion

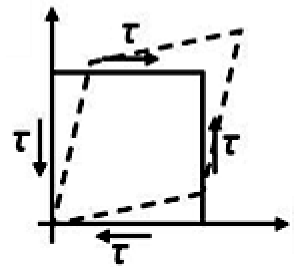
Yield criterion: $F(\boldsymbol{\sigma}, k) = J_2 - k^2 = 0$

$$F = \frac{1}{6}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] - k^2 = 0$$

k - yield stress $\equiv \tau_Y$ in pure shear, $k = \sigma_Y / \sqrt{3}$

yield in **uniaxial test**

$$\begin{cases} \sigma_e = \sqrt{3J_2} \\ J_2 = \frac{1}{6}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] \end{cases}$$



$$\boldsymbol{\sigma} = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{s}$$

$$\tau = \frac{1}{\sqrt{3}}\sigma_Y$$

Special cases of the **yield** condition:

1-D tensile case: $\sigma_1 = \sigma_Y, \sigma_2 = \sigma_3 = 0,$

$$F = \frac{1}{6}[\sigma_1^{\,2} + (-\sigma_1)^2] - k^2 = 0 \Rightarrow k^2 = \frac{1}{3}\sigma_Y^{\,2}$$

σ_Y $\quad \Rightarrow k = \sigma_Y / \sqrt{3} \approx 0.577\sigma_Y$

Effective stress or Von Mises Stress – tehokasjännitys, vertailujännitys:

$$\sigma_e^2 = \frac{3}{2} s_{ij} : s_{ij} = \frac{3}{2}\mathbf{s} : \mathbf{s} \equiv 3J_2$$

2-D plane stress state: **when** $\sigma_3 = 0$

Yield condition: $\sqrt{\sigma_1^2 + \sigma_1\sigma_2 + \sigma_2^2} = \sigma_Y$

Or $\sqrt{\sigma_{xx}^2 + 3\tau_{xy}^2} = \sigma_Y$

when $\sigma_{yy} = 0,$

For instance, for an *isotopically hardening* material, the yield criterion becomes $F(\sigma_{ij}, k) = 0$, the 'yield' or threshold parameter $k(\kappa)$ depends on some internal variable, *a kinematic variable* κ which depends *for ex.*, on the plastic work $W_p = \int \boldsymbol{\sigma} : d\boldsymbol{\varepsilon}_p'$ or other equivalent variables as the accumulated plastic deformations $\bar{\boldsymbol{\varepsilon}}_p = \int_0^{\boldsymbol{\varepsilon}_{ij}^p} d\boldsymbol{\varepsilon}_e^p$

$$d\boldsymbol{\varepsilon}_e^p = \sqrt{2/3 d\boldsymbol{\varepsilon}_p : d\boldsymbol{\varepsilon}_p}$$

Von Mises yield Criterion - Distortional Energy Criteria

$$F = J_2 - k^2 = 0$$ **Yield Function**

$$\Updownarrow$$

$$f = \frac{1}{2}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] - 3k^2 = 0$$

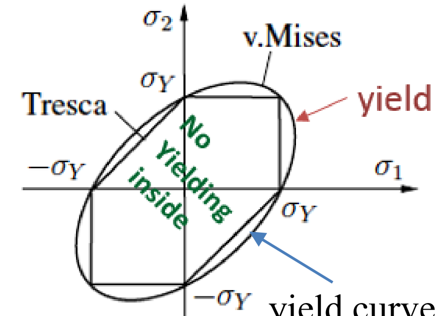
$$3k^2 = 3\sigma_Y^2 / 3 = \sigma_Y^2$$

Yield in **uniaxial test**

$$\underbrace{\frac{1}{2}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]}_{\equiv \sigma_e^2}$$

$$k = \sigma_Y / \sqrt{3} \equiv \tau_Y$$

Yield stress in **pure shear**

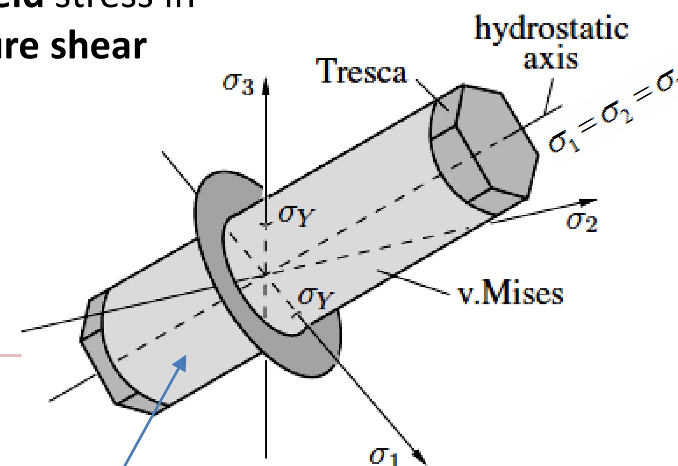


$$\Updownarrow$$

Yield Function in another equivalent form:

$$f = \frac{1}{2}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] - \sigma_Y^2 = 0$$

$$f = \sigma_e^2 - \sigma_Y^2 = 0$$



Definition of **equivalent Von Mises stress**: $\sigma_e = \sqrt{3J_2}$

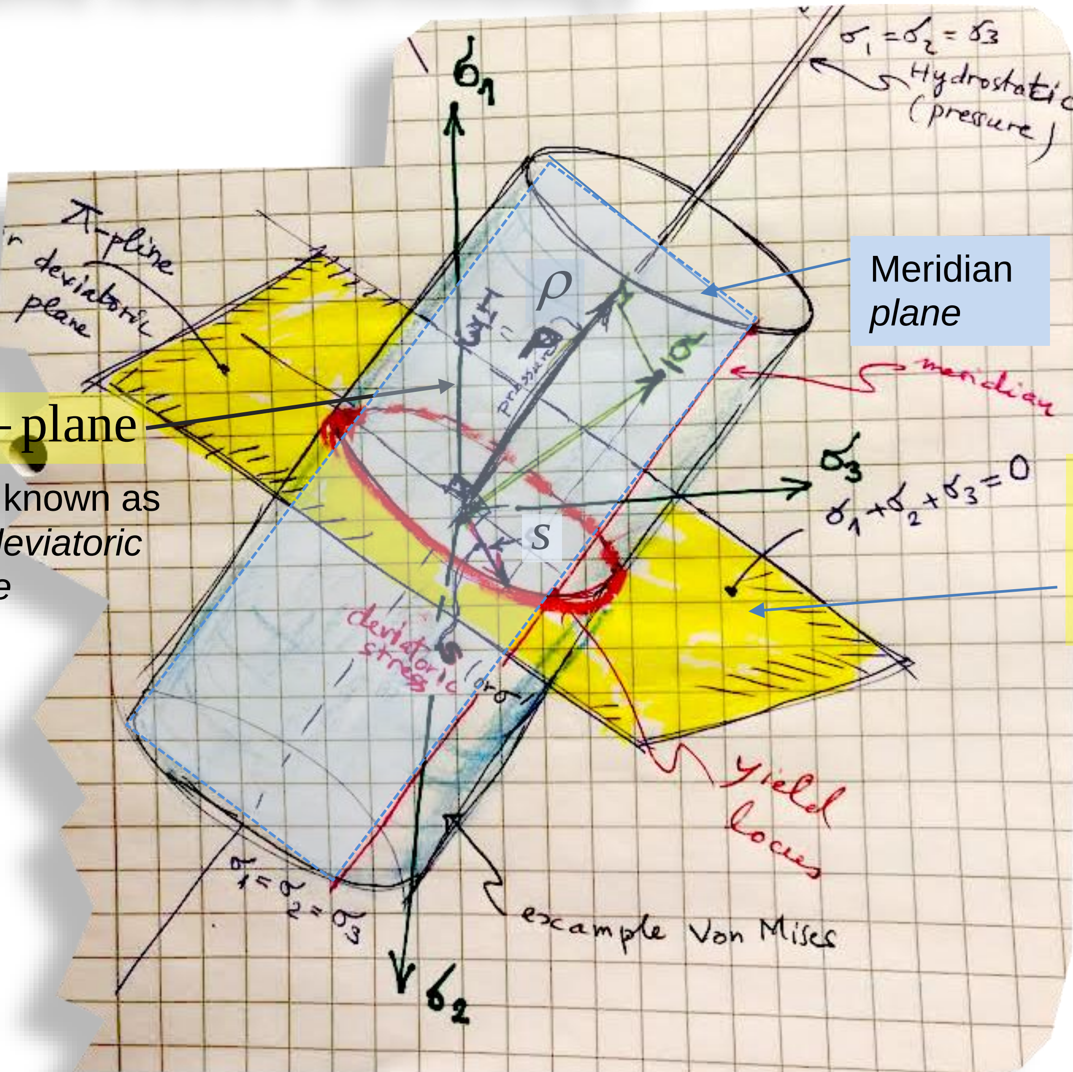
$$\sigma_e = \sqrt{\frac{1}{2}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]}$$

cylindrical surface of radius of radius $\sqrt{2}k$ around *hydrostatic* axis $\sigma_1 = \sigma_2 = \sigma_3$

$$f = \sigma_e^2 - \sigma_Y^2 = 0$$

← **Yield Function – a useful form**

Graphical representation of the failure surface & some related terminology

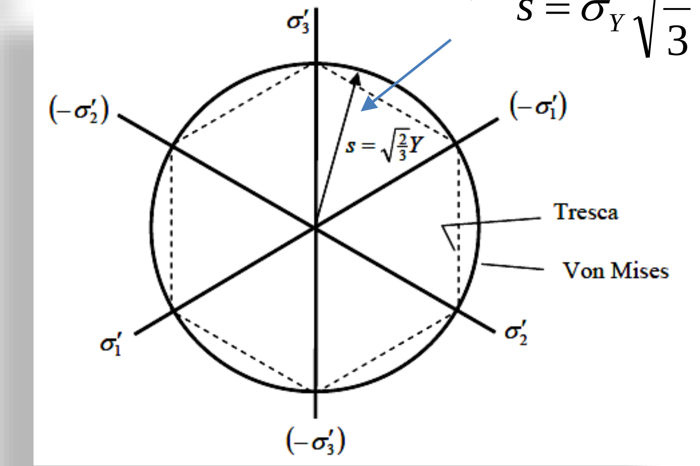


$\pi -$plane

Also known as the *deviatoric plane*

Yield surface in three dimensional stress-space

Radius: $s = \sigma_Y \sqrt{\frac{2}{3}}$



The Von Mises criterion in the *pi*-plane

$\pi -$plane

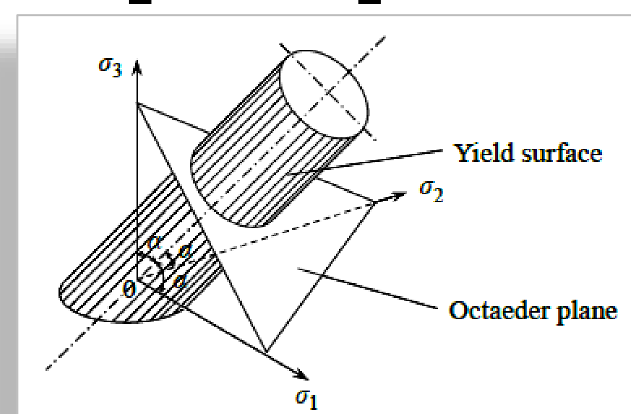
$\sigma_1 + \sigma_2 + \sigma_3 = 0$

$(I_1 = 0)$

perpendicular to the hydrostatic axis

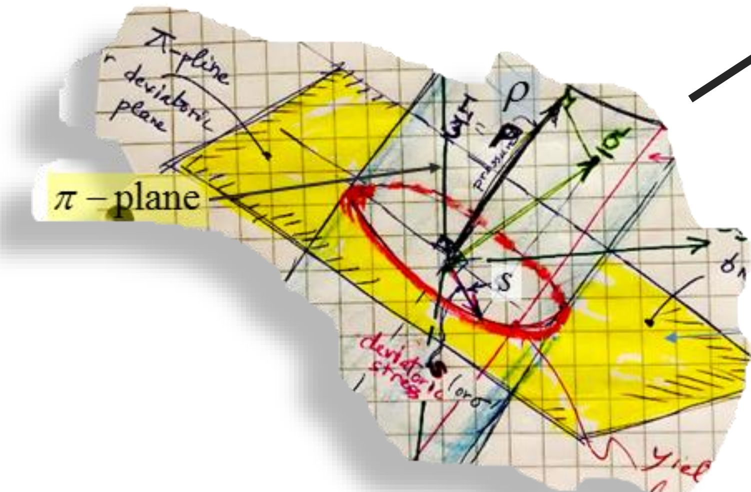
$$\sigma_e = \sqrt{3J_2}$$

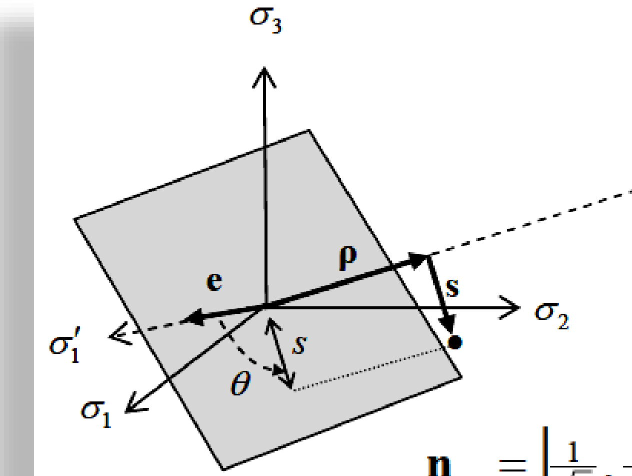
$$\sigma_e^2 = \frac{3}{2} s_{ij} : s_{ij} = \frac{3}{2} \mathbf{s} : \mathbf{s} \equiv 3J_2$$

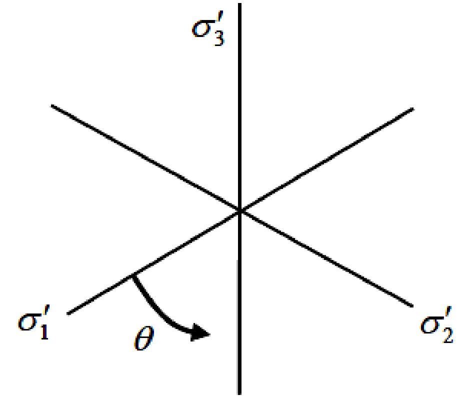


Haigh-Westergaard Stress Space [self-reading]

It is sometime more convenient to describe yield criteria with (ρ, s, θ) coordinates, than in terms of principal stresses.







$$\mathbf{n}_\rho = \left[\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\right],$$

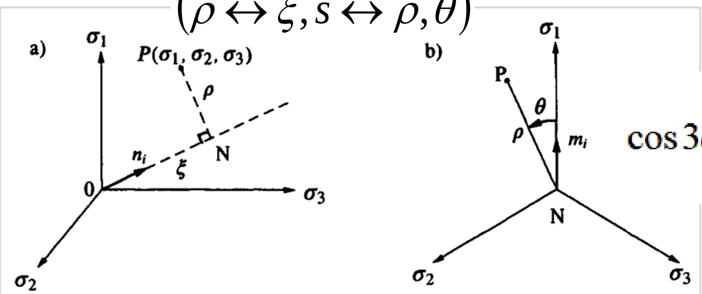
$$\rho = |\boldsymbol{\rho}| = \sqrt{3}\sigma_m = I_1/\sqrt{3}, \qquad s = |\mathbf{s}| = \sqrt{2J_2}$$

$$\boldsymbol{\rho} = (\sigma_m, \sigma_m, \sigma_m) \qquad \mathbf{s} = (s_1, s_2, s_3)$$

Haigh-Westergaard coordinates: (ρ, s, θ)

NB the different notation

$(\rho \leftrightarrow \xi, s \leftrightarrow \rho, \theta)$

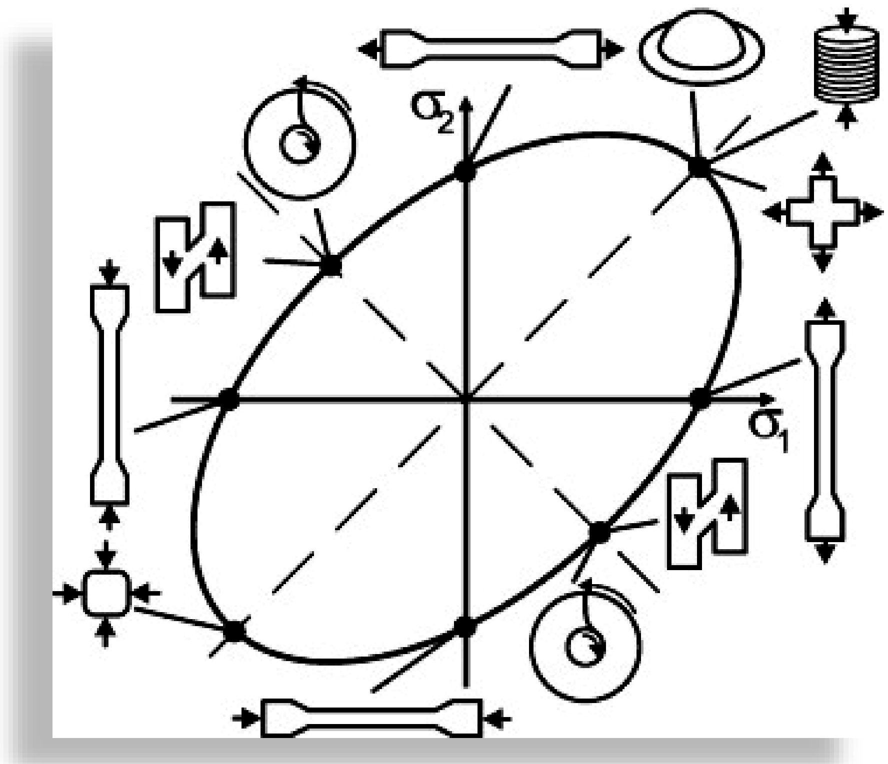


$$\cos 3\theta = \frac{3\sqrt{3}}{2}\frac{J_3}{J_2^{3/2}}$$

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix}\rho\\ \rho\\ \rho\end{bmatrix} + \sqrt{\frac{2}{3}}s\begin{bmatrix}\cos\theta\\ \cos(\theta - 2\pi/3)\\ \cos(\theta + 2\pi/3)\end{bmatrix} \qquad \begin{bmatrix}s_1\\ s_2\\ s_3\end{bmatrix} = \frac{2}{\sqrt{3}}\sqrt{J_2}\begin{bmatrix}\cos\theta\\ \cos(2\pi/3 - \theta)\\ \cos(2\pi/3 + \theta)\end{bmatrix}$$

Von Mises: $f(s) = \frac{1}{2}s^2 - k = 0$

Tresca $f(s,\theta) = \sqrt{2}s\sin\left(\theta + \frac{\pi}{3}\right) - Y = 0$


σ2
σ1

Von Mises yield Criterion – **yield surface**

Plane stress

$$\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 - \sigma_Y^2 = 0$$

$$F = \sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y + 3\tau_{xy}^2 - \sigma_Y^2 = 0$$

xy-stress components

Consider the section (intersection) plane of the yield surface with the plane formed by $\sigma_y = 0$. On this plane, the von Mises yield function will be:

$$F = \sigma_x^2 + 3\tau_{xy}^2 - \sigma_Y^2 = 0 \rightarrow \sqrt{\sigma_x^2 + 3\tau_{xy}^2} = \sigma_Y$$

$$\sigma_x^2 + 3\tau_{xy}^2 = \sigma_Y^2 \Rightarrow \left(\frac{\sigma_x / \sigma_Y}{1}\right)^2 + \left(\frac{\tau_{xy} / \sigma_Y}{1/\sqrt{3}}\right)^2 = 1$$

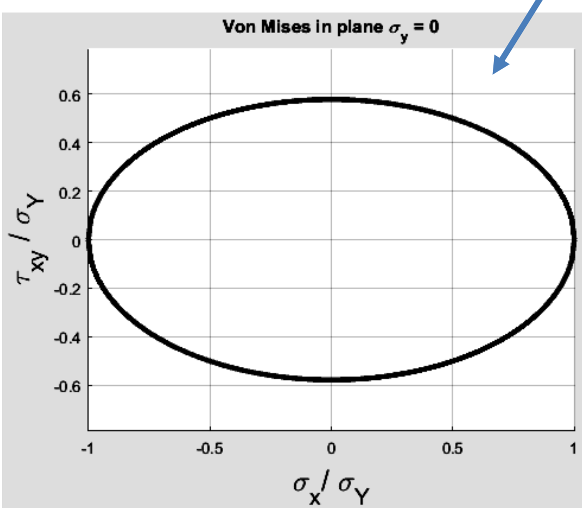
Ellipse

yield in *uniaxial* test

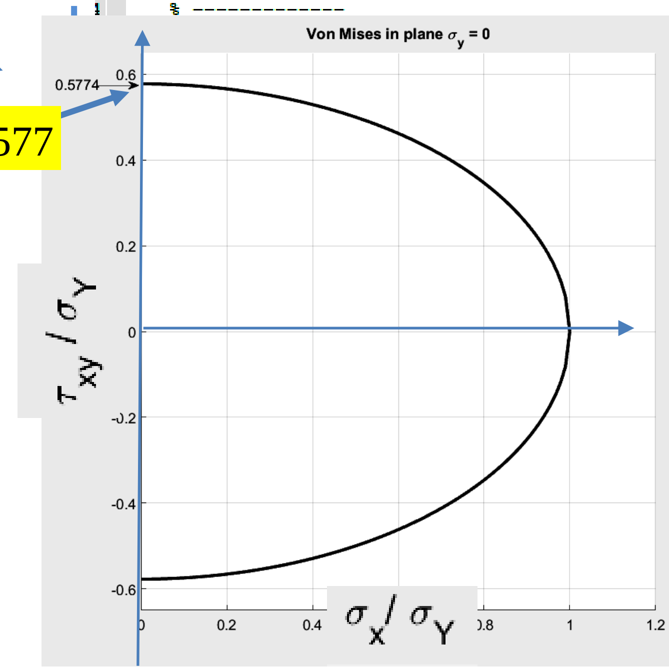
$$k = \sigma_Y / \sqrt{3} \equiv \tau_Y$$

yield stress in pure shear,

```
% Plane stress case:
% Von Mises in  plane sigma_y = 0
x  = 0:0.01:1;
y = sqrt( (1 - x.^2) ) / sqrt(3);
figure
plot( x, y, 'k-','linewidth','3');
hold on;
plot( x,-y, 'k-','linewidth','3')
plot(-x,-y, 'k-','linewidth','3')
plot(-x, y, 'k-','linewidth','3')
grid on; axis equal,
% -------------
```



$1/\sqrt{3} \approx 0.577$



Von Mises yield Criterion - Distortional Energy Criteria

Yield when (yield criteria):

3D-stress state Principal stresses	$\frac{1}{2}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] - \sigma_Y^2 = 0$ $\sigma_e^2 - \sigma_Y^2 = 0 \quad \Leftrightarrow \quad \tau_{oct}^2 = \frac{2}{3} J_2 = \frac{2}{3} k^2$
1D-uniaxial tension	$\sigma_1 = \sigma_Y, \; \sigma_2 = \sigma_3 = 0$ ← Yield when $k = \sigma_Y / \sqrt{3}$
Plane stress	$\sigma_1^2 + \sigma_2^2 - \sigma_1 \sigma_2 - \sigma_Y^2 = 0$ principle stresses $\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2 - \sigma_Y^2 = 0$ xy-stress components
3D-stress state General	$\sigma_e^2 = \frac{1}{2}[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2] + \ldots$ $\ldots + 3[\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2]$ $\quad \sigma_{23} \leftrightarrow \sigma_{yz}$

yield in *uniaxial* test

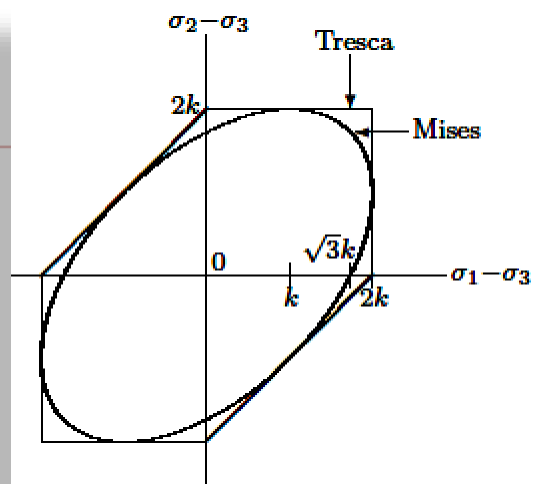
$$k = \sigma_Y / \sqrt{3}$$

k - yield stress in pure shear,

$$k = \sigma_Y / \sqrt{3} \equiv \tau_Y$$

Yield stress in pure shear

Yield stress in a uniaxial tension test



Von Mises stress: $\sigma_e = \sqrt{3J_2}$

$$\sigma_e = \sqrt{\frac{1}{2}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]}$$

$$f = \sigma_e^2 - \sigma_Y^2 = 0$$

← **Von Mises Yield Function**

Example – design of weld

Do you recognise Von Mises and related concepts as yield condition and equivalent stress?

$$\sigma_{eq} = \sqrt{\frac{(\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2 + 6 \cdot (\tau_{xy}{}^2 + \tau_{yz}{}^2 + \tau_{zx}{}^2)}{2}}$$

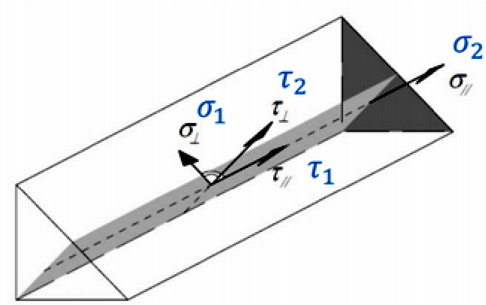
Applied to welds:

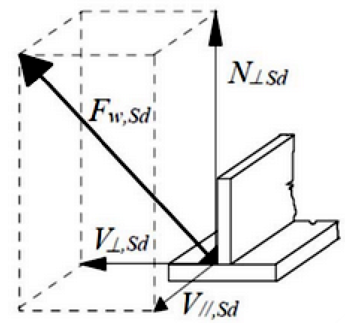
$$\sigma_2 \ll \tau_1$$

$$\sigma_{eq} = \sqrt{\sigma_1^2 + 3({\tau_1}^2 + {\tau_2}^2)} \leq \sigma_Y$$

Design code

$$\sigma_{eq} = \sqrt{\sigma_1^2 + 3({\tau_1}^2 + {\tau_2}^2)} \leq \frac{f_u}{\beta_w \cdot \gamma_{M2}}$$





Example

$$\tau_1 = \frac{F_{sd}}{2 \cdot a \cdot l} \qquad \tau_2 = 0 \qquad \sigma_1 = 0$$

$$\sigma_{eq} = \frac{\sqrt{3} \cdot F_{sd}}{2 \cdot a \cdot l} \leq \frac{f_u}{\beta_w \cdot \gamma_{M2}}$$

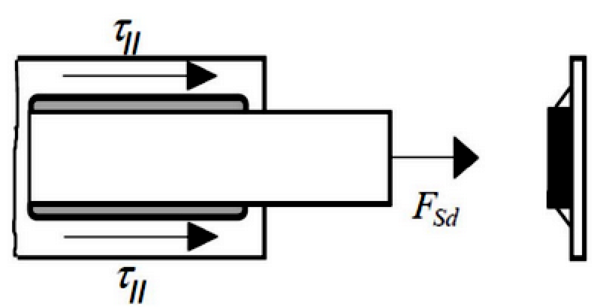


Illustration example provided by Wei Lu; go and take her courses …

Example – design of filled weld

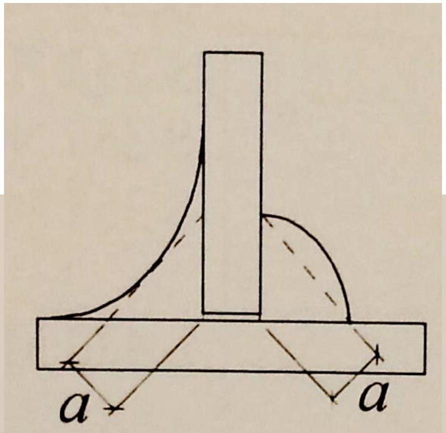
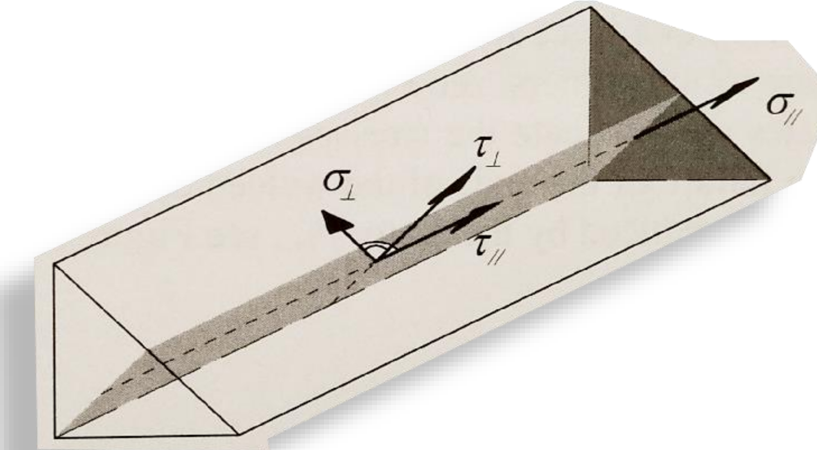


Figure 3.1 Definition of throat thickness a



Stress in critical plane of fillet weld

In the design procedure the internal force on the fillet weld is resolved into components paralle transverse to the critical plane of the weld throat, see Figure 3.2. A uniform stress distributi assumed on the critical throat section of the weld, leading to the following normal stresses and stresses:

- $\sigma_{\perp}$ the normal stress perpendicular to the critical plane of the throat,
- $\sigma_{//}$ the normal stress parallel to the axis of the weld, it should be neglected when calcul the design resistance of a fillet weld,
- $\tau_{\perp}$ the shear stress (in the critical plane of the throat) perpendicular to the weld axis,
- the shear stress (in the critical plane of the throat) parallel to the weld axis.

Do you recognise Von Mises?

2-D plane stress state:

$$\sqrt{\sigma_{xx}^2 + 3\tau_{xy}^2} = \sigma_Y$$

The resistance of the fillet weld will be sufficient if the

$$\sqrt{\sigma_{\perp}^2 + 3(\tau_{\perp} + \tau_{//})^2} \le \frac{f_u}{\beta_w \gamma_{Mw}}$$

and

$$\sigma_{\perp} \le \frac{f_u}{\gamma_{Mw}}.$$

The correlation factor β_w is summarised in Table 3.1.

prEN 1993-1-8 includes a simplified procedu the fillet weld per unit length independent of the direct

$$f_{vw.d} = \frac{f_u}{\sqrt{3}\,\beta_w \gamma_{Mw}}$$

$$\sqrt{\sigma_{\perp}^2 + 3(\tau_{\perp} + \tau_{//})^2} \le \frac{f_u}{\beta_w \gamma_{Mw}}$$

and $$\sigma_{\perp} \le \frac{f_u}{\gamma_{Mw}}.$$

Analysis using Abaqus Elastic-Plastic Steel K-joint

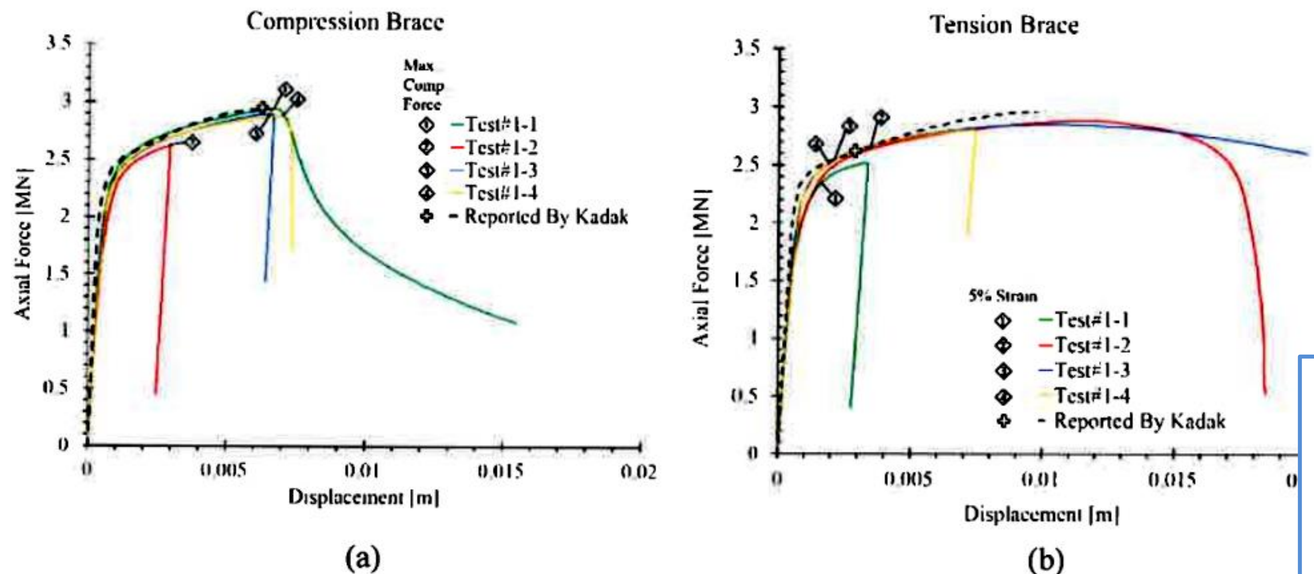


Figure D-1: Comparison of load-displacement curves of different loading conditions for specimens joint with 30 degrees brace inclination angle: a) Tension braces, b) Compression Braces

Example from Master Thesis by Pooya Saremi, 2016 at CIV-department@here

Elastic-Plastic Analysis of Steel K-joints

Flow rule associated with von Mises yield function (Structural Steel)

Von Mises stresses

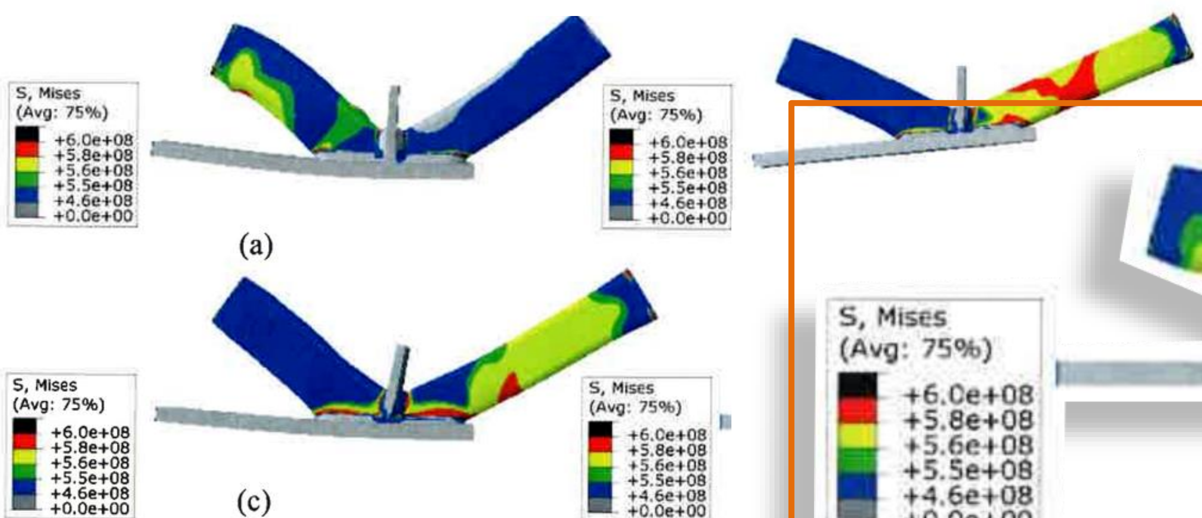
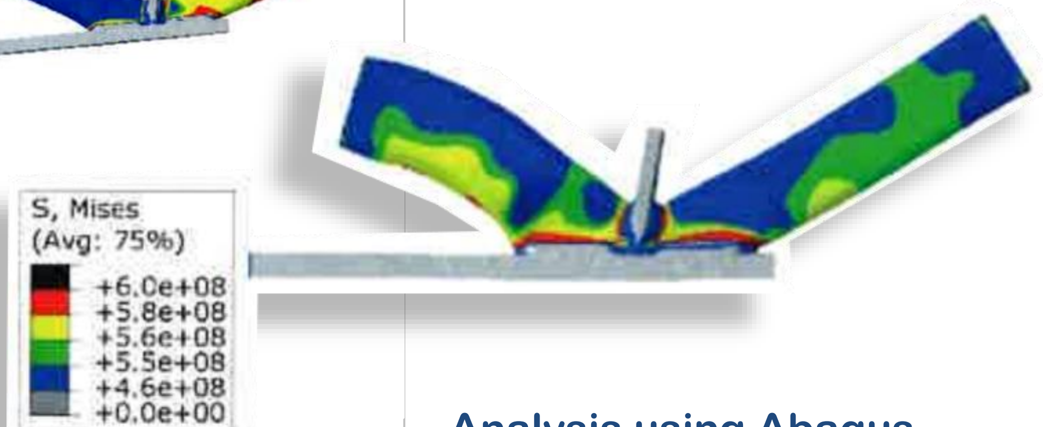


Figure D-2: The contoured, deformed shapes at peak compression load for: a) load type#1-1, b) load type#1-2, c) load type#1-3 and d) load type#1-4



Analysis using Abaqus Elastic-Plastic Steel K-joint

Tresca Yield Criterion – Maximum Shear Stress Criterion

Yield criterion: plastic flow (yield) occurs if the *maximum shear stress* τ_{max} reaches a certain critical value : k

k - yield stress in pure shear according to Tresca yielding criterion

Yield criteria - myötöehto tai -funktio

$$k = \sigma_Y / 2 = \tau_Y$$

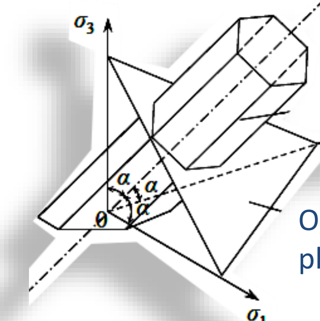
At yielding in uniaxial tension test:

σ_Y - *yield* in *uniaxial* test $\Rightarrow \tau_{max} = \max\left\{\frac{1}{2}|\sigma_1 - \cancel{\sigma_2}|\right\} = \boxed{\frac{1}{2}\sigma_Y = \tau_Y}$ maximum shear stresses

$$F = \tau_{max} - k = 0,$$

$$\tau_{max} = \frac{1}{2}\sigma_Y \Rightarrow \textbf{Yield}$$

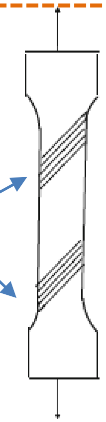
$$k = \sigma_Y / 2 = \tau_Y$$



Yield surface

Octahedral plane

$$\tau_{max} = \tau_Y$$



Lüders bands
Slip planes - dislocations

$$F = \max\left\{\frac{1}{2}|\sigma_1 - \sigma_2|, \frac{1}{2}|\sigma_2 - \sigma_3|, \frac{1}{2}|\sigma_3 - \sigma_1|\right\} - k$$

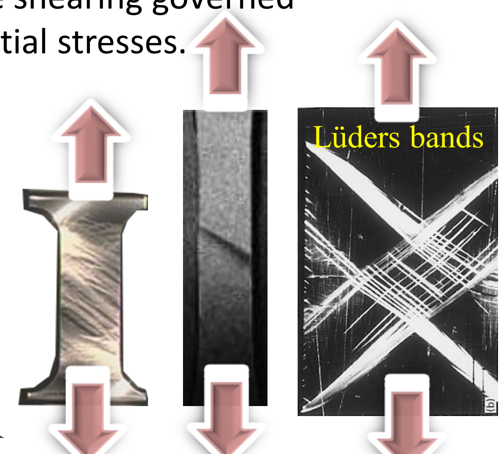
$$\textbf{Tresca:}\quad \tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3) = \tau_Y$$

In metals, plastic deformation results from *slip*; inter-crystalline shearing governed by tangential stresses.

Note tha in Tresca criteria one of principle stresses does not enter the yield condition! here, max is σ3 – σ1 (in plane stress ok, where one principle stress component = 0)

$$F = \tau_{max} - k < 0 \implies \text{elastic (no yieldin)}$$

$$F = \tau_{max} - k = 0 \implies \text{onset of yielding}$$



steel

Tresca Yield Criterion – Maximum Shear Stress Criterion

Yield criterion: plastic flow (yield) occurs if the *maximum shear stress* $\tau_{\max}$ reaches a certain *critical* value k

Yield criteria - myötöehto tai -funktio

$$F = \tau_{\max} - k = 0,$$

$$\tau_{\max} = \frac{1}{2}\sigma_Y \Rightarrow \textbf{Yield}$$

σ_Y – *yield* in *uniaxial* test

τ_Y – *yield* in *pure shear* test

$$k = \sigma_Y / 2 = \tau_Y$$

$$F = \max\left\{ \underbrace{\frac{1}{2}\left|\sigma_1 - \sigma_2\right|}, \underbrace{\frac{1}{2}\left|\sigma_2 - \sigma_3\right|}, \underbrace{\frac{1}{2}\left|\sigma_3 - \sigma_1\right|} \right\} - k$$

Tresca: $\tau_{\max} = \frac{1}{2}(\sigma_1 - \sigma_3) = \tau_{\mathrm{y}}$

Note that sigma_2 is not considered! This makes the criteria less accurate than the Von-Mises.

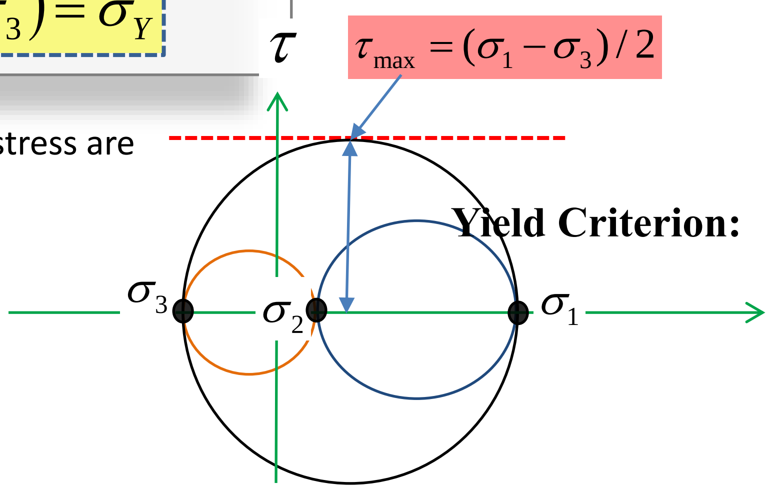
Example:

$$\tau_{\max} = (\sigma_1 - \sigma_3)/2 = k = \sigma_Y / 2$$

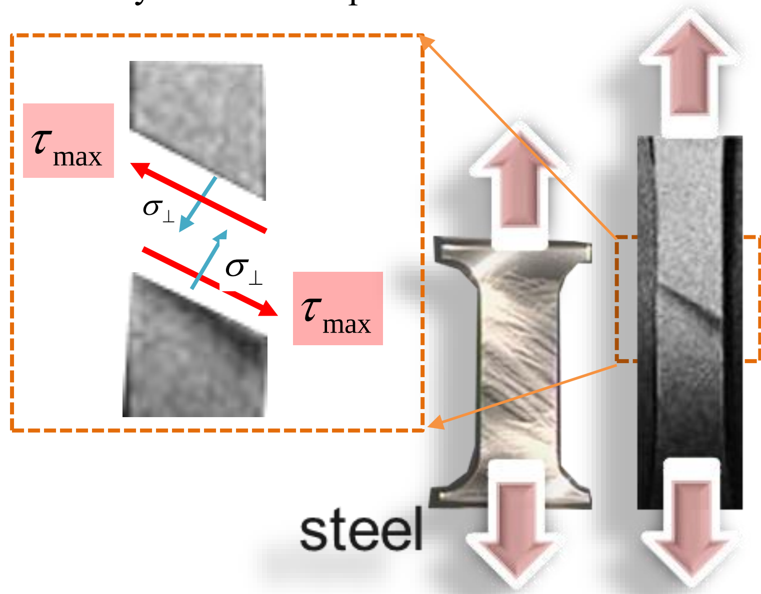
$$\boxed{\Rightarrow (\sigma_1 - \sigma_3) = \sigma_Y}$$

when principle stress are ordered:

$\sigma_1 \geq \sigma_2 \geq \sigma_3$



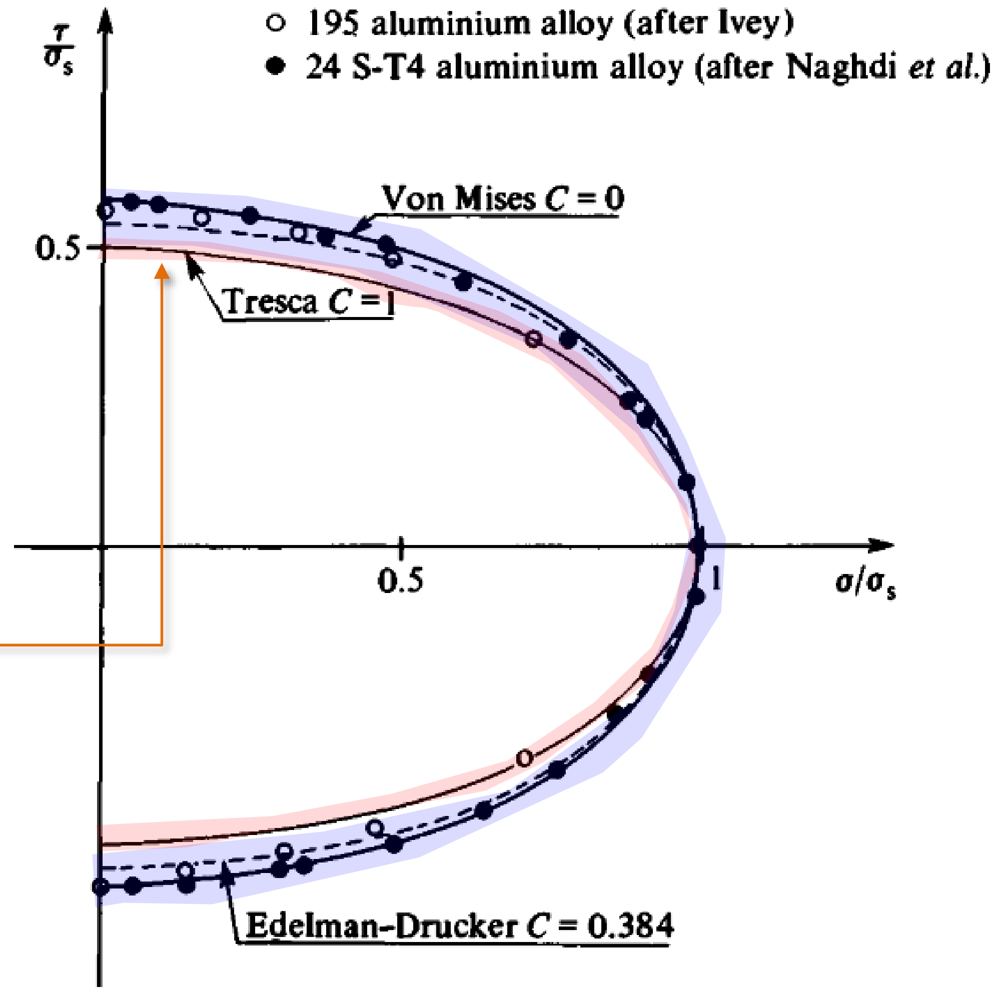
Yield: *When maximum shear stress* reaches yield stress in pure shear



$$k_{\text{Tresca}} = \sigma_Y / 2 = 0.5\sigma_Y$$

$$k_{\text{von Mises}} = \sigma_Y / \sqrt{3} \approx 0.577\sigma_Y$$

$$\textbf{Tresca: } \tau_{\max} = \frac{1}{2}(\sigma_1 - \sigma_3) = \tau_y$$



Tresca: $\tau_{\max} = \frac{1}{2}(\sigma_1 - \sigma_3) = \tau_y$. This criterion ignores σ_2, so it may be more suitable for plane stress cases having $\sigma_2 = 0$. Note that, for instance, the von-Mises criterion accounts for all the stress components σ_1, σ_2 and σ_3. Despite this remark, this model seems to give conservative results as compared to von-Mises.

PERGAMON
International Journal of Plasticity 19 (2003) 1297–1319
INTERNATIONAL JOURNAL OF Plasticity
www.elsevier.com/locate/ijplas

Plane stress yield function for aluminum alloy sheets—part 1: theory

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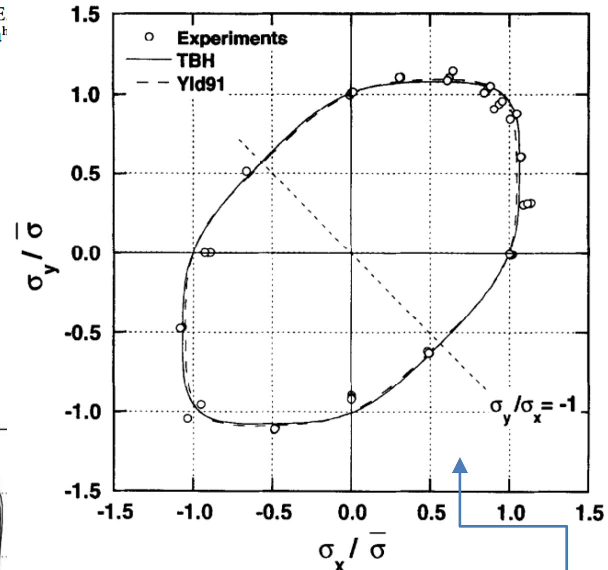
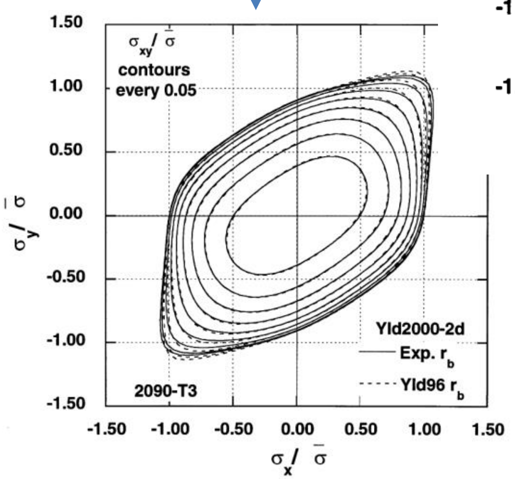
D.J. Lege[a], F. Pourboghrat[g], S.-H. Choi[f], E. Chu[h]

Tresca/Mises

Yield function development for aluminum alloy sheets





Tricomponent yield surface for AA2090-T3 predicted with Yld2000-2d (experimental r_b) and Yld2000-2d (Yld96 r_b).

Pergamon
J. Mech. Phys. Solids, Vol. 45, No. 11/12, pp. 1727–1763, 1997
Printed in Great Britain.
0022–5096/97 \$17.00 + 0.00
PII: S0022–5096(97)00034–3

YIELD FUNCTION DEVELOPMENT FOR ALUMINUM ALLOY SHEETS

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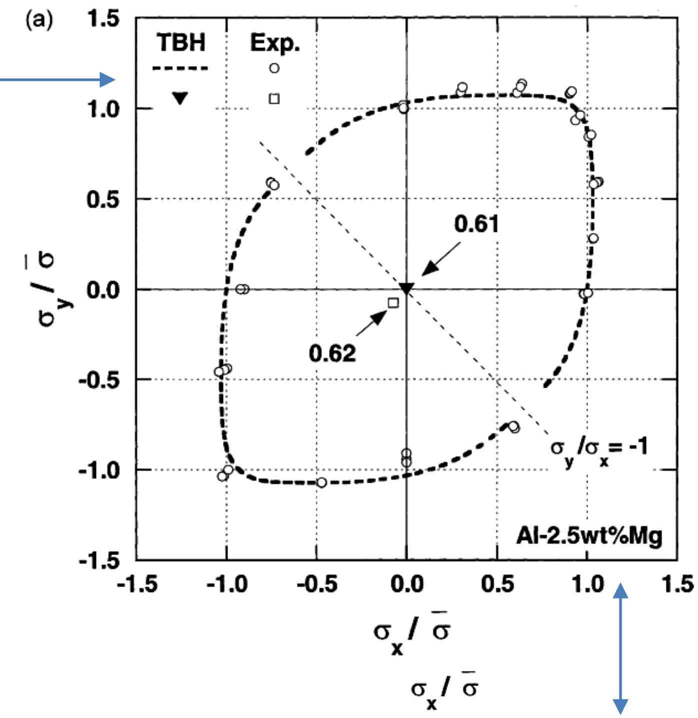
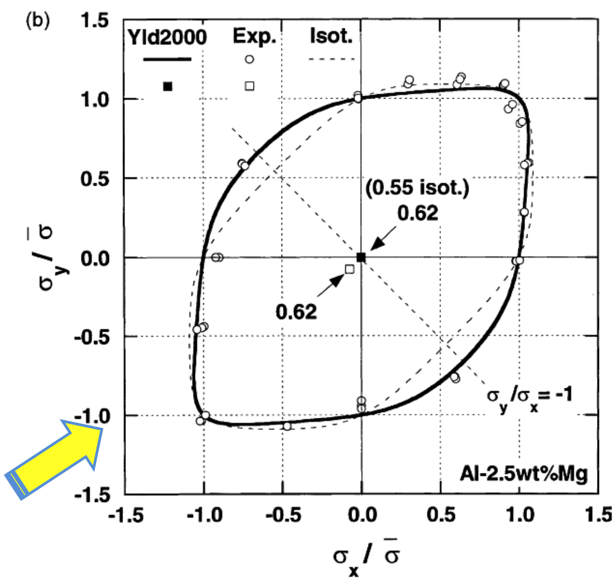
Y. HAYASHIDA,‡ D. J. LEGE,† K. MATSUI,‖ S. J. MURTHA,† S. HATTORI,‡
R. C. BECKER† and S. MAKOSEY†





. Normalized yield surface for an Al–2.5 wt.% Mg binary sheet sample. Open circles denote experimental planar locus ($\sigma_{xy} = 0$) and the open square denotes the maximum experimental simple shear stress ($\sigma_{xx} \approx \sigma_{yy} \approx 0$). (a) Dash line ($\sigma_{xy} = 0$) and solid triangle ($\sigma_{xx} \approx \sigma_{yy} \approx 0$) denote polycrystal TBH predictions. (b) The solid line (planar locus, $\sigma_{xy} = 0$) and full square (maximum simple shear stress, $\sigma_{xx} \approx \sigma_{yy} \approx 0$) denote Yld2000-2d predictions using $\sigma_0 = \sigma_{45} = \sigma_{90} = \sigma_b$ and $r_0 = 0.20$, $r_{45} = 0.28$, $r_{90} = 0.20$, $r_b = 1.0$. Dash line represents isotropic response (isot.).

Comparison between Von-Mises and Tresca criteria

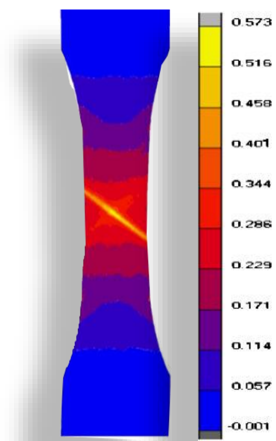
Tresca's criterion	Van-Mises criterion
1. Plastic flow occurs when the maximum shear stress reaches a particular value $\tau_{max} = k = ½\,(\sigma_1 - \sigma_3), \sigma_y = 2k$ when $\sigma_1 = \sigma_4$ and $\sigma_3 = 0$.	1. Plastic flow occurs when strain energy reaches a value equal to the strain energy at elastic point in uniaxial tension (σ_y) $\sigma_y = \sqrt{3}.k$.
2. Simple to use.	2. Not simple.
3. Less accurate as σ_2 is not considered except σ_1 and σ_3 are considered.	3. More accurate as principal stresses σ_1, σ_2 and σ_3 are considered.
4. It is more conservative.	4. Less conservative.

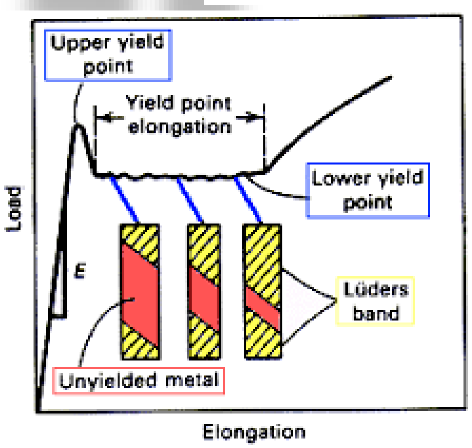
- Plastic deformation slip-bands in tensile stresses gives clear evidence in favor of **Tresca**' s yield condition saying that:

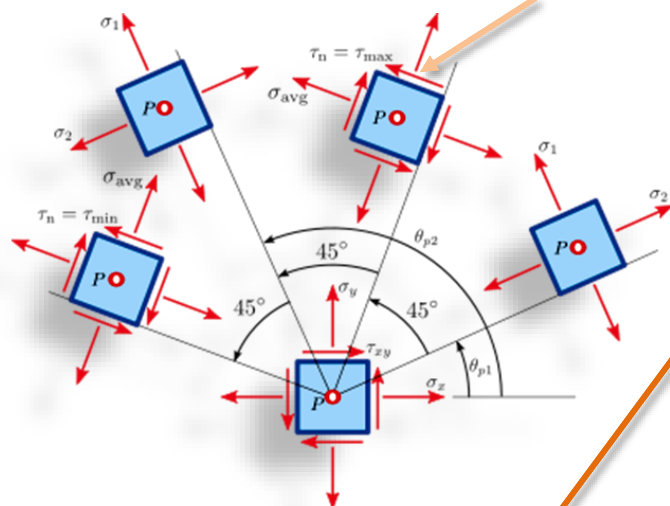
  yield occurs when the maximum shear stress equals the yield stress in shear

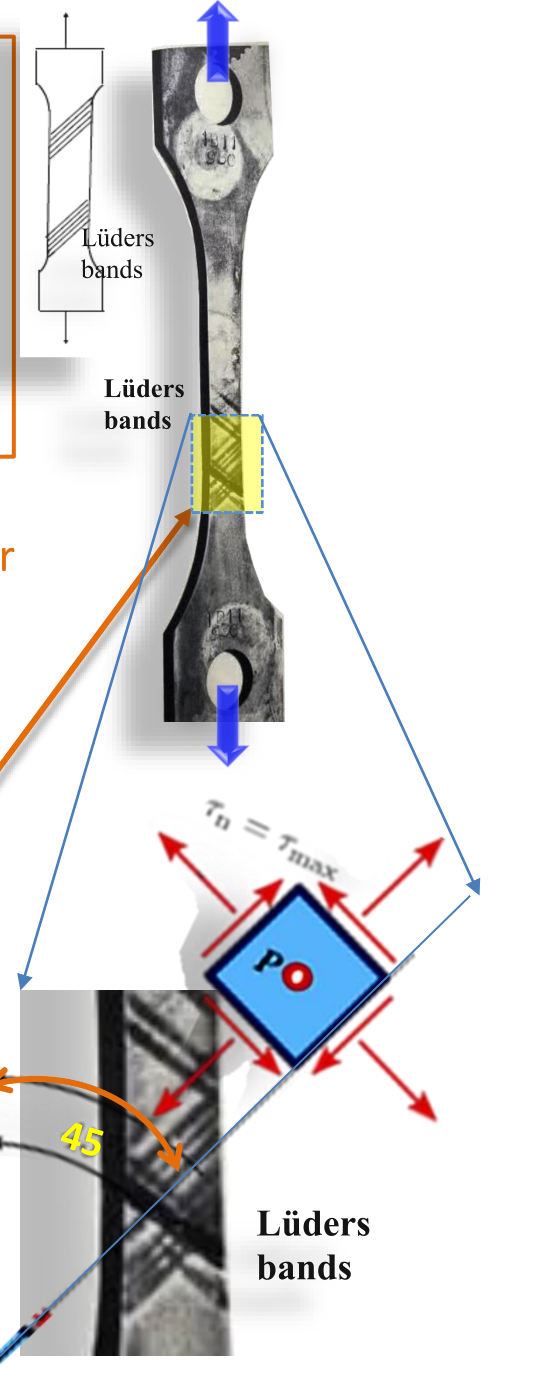
  Remember that maximum shear stresses makes an angle of 45 degree with the principle stresses

maximum shear









- *ex*. **Lüders bands** or slip-bands are localized bands of plastic deformation in tensile stresses and are common to low-carbon steels

Exercises – Example

Consider the following stress state (units in MPa): $\sigma_x = 200 \equiv \sigma_{11}$

$$\sigma_y = \sigma_z = 120$$

$$\tau_{xy} = 110, \quad \tau_{yz} = 0$$

Yield stress in uniaxial test: $\sigma_{\mathrm{Y}} = 240 \ \text{MPa}$

Question: how much the stress component τ_{xz} can be increased before yield occurs? *a*) according to von Mises and *b*) Tresca yield condition.

Let use the notations:

$$\tau_{xz} \equiv \alpha\sigma_0$$

$$\sigma_x = 200 \equiv \sigma_{11} = 2\sigma_0$$

$$\sigma_y = \sigma_z = 1.2\sigma_0$$

$$\tau_{xy} = 1.1\sigma_0, \quad \tau_{yz} = 0$$

$$\boldsymbol{\sigma} = \sigma_0 \begin{bmatrix} 2 & 1.1 & \alpha \\ & 1.2 & 0 \\ Symm. & & 1.2 \end{bmatrix}$$

HW b) do the same for Tresca yield

Tresca Yield Criteria:

$$F = \max\left\{\frac{1}{2}|\sigma_1 - \sigma_2|, \frac{1}{2}|\sigma_2 - \sigma_3|, \frac{1}{2}|\sigma_3 - \sigma_1|\right\} - k$$

Von Mises Yield Criteria: $F = \sigma_{\mathrm{e}}^2 - \sigma_Y^2 = 0$

$$F = \underbrace{\frac{1}{2}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]}_{\equiv \sigma_{\mathrm{e}}^2 = 3J_2} - \sigma_Y^2 = 0$$

… or in component form

$$\sigma_e^2 = \frac{1}{2}[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2] + \ldots$$

$$\ldots + 3[\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2] \quad \Downarrow$$

$$\sigma_e^2 = \frac{1}{2}\sigma_0^2 \cdot \underbrace{[(2-1.2)^2 + (1.2-1.2)^2 + (1.2-2)^2]}_{\equiv a} + \ldots$$

$$\ldots + 3\sigma_0^2[1.1^2 + 0 + \alpha^2] = \sigma_Y^2 \Rightarrow$$

$$3\alpha^2 = (\sigma_{\mathrm{Y}} / \sigma_0)^2 - 4.27 \rightarrow \left[(240/100)^2 - 4.27\right]/3 =$$

$$\alpha \approx 0.705$$

$$\Rightarrow \boxed{\tau_{xz} = \alpha\sigma_0} = 0.705 \cdot 100 \boxed{\approx 70 \ \text{MPa}}$$

Exercises – Example

Consider the following stress state (units in MPa): $\sigma_x = 200 \equiv \sigma_{11}$

$$\sigma_y = \sigma_z = 120$$

$$\tau_{xy} = 110, \quad \tau_{yz} = 0$$

Yield stress in uniaxial test: $\sigma_Y = 240 \ \text{MPa}$

Question: how much the stress component τ_{xz} can be increased before yield occurs? *a*) according to von Mises and *b*) Tresca yield condition.

Let use the notations:

$$\tau_{xz} \equiv \alpha\sigma_0$$

$$\sigma_x = 200 \equiv \sigma_{11} = 2\sigma_0$$

$$\sigma_y = \sigma_z = 1.2\sigma_0$$

$$\tau_{xy} = 1.1\sigma_0, \quad \tau_{yz} = 0$$

$$\boldsymbol{\sigma} = \sigma_0 \begin{bmatrix} 2 & 1.1 & \alpha \\ & 1.2 & 0 \\ \text{Symm.} & & 1.2 \end{bmatrix}$$

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} = \sigma_0 \cdot \begin{matrix} 0.2337 & 0 & 0 \\ 0 & 1.2000 & 0 \\ 0 & 0 & 2.9663 \end{matrix}$$

Von Mises Yield Criteria: $F = \sigma_e^2 - \sigma_Y^2 = 0$

$$F = \underbrace{\frac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]}_{\equiv \sigma_e^2 = 3J_2} - \sigma_Y^2 = 0$$

… or in component form

$$\sigma_e^2 = \frac{1}{2}\left[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2\right] + \ldots$$

$$\ldots + 3\left[\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2\right] \quad \Downarrow$$

$$\sigma_e^2 = \frac{1}{2}\sigma_0^2 \cdot \left[(2 - 1.2)^2 + (1.2 - 1.2)^2 + (1.2 - 2)^2\right] + \ldots$$

$$\ldots + 3\sigma_0^2\left[1.1^2 + 0 + \alpha^2\right] = \sigma_Y^2 \Rightarrow \boxed{\alpha \approx 0.705}$$

$$\Rightarrow \boxed{\tau_{xz} = \alpha\sigma_0} = 0.705 \cdot 100 \boxed{\approx 70 \ \text{MPa}}$$

$$\Rightarrow \sigma_e = 239.996 \approx 240 \ \text{MPa}$$

→ YIELD occurs

```
% check of the results
% ----------------------
sigma_0 = 100;
a =   0.7047;
sigma=[2   1.1  a;
       1.1 1.2  0;
       a    0  1.2];

s_e_0 = 0.5 * ( (sigma(1,1) - sigma(2,2))^2 + ...
              + (sigma(2,2) - sigma(3,3))^2 + ...
              + (sigma(3,3) - sigma(1,1))^2 ) + ...
        + 3* ( sigma(1,2)^2 + sigma(2,3)^2 + sigma(3,1)^2 )

SIGMA_EQUIV = sigma_0 * sqrt(s_e_0) % Von Mises stress
Y = 240;
```

SIGMA_EQUIV = 239.9960 MPa

Exercises – Example – HW – Tresca and von Mises

Consider a closed thin-walled steel cylindrical vessel having a diameter D with wall thickness t subjected to internal pressure p. The ends of the vessel are half-sphere in order to avoid bending state (boundary layers - reunahäiriö). The material is steel.

Yield stress in uniaxial test: $\sigma_0 = 225\ \text{MPa}$

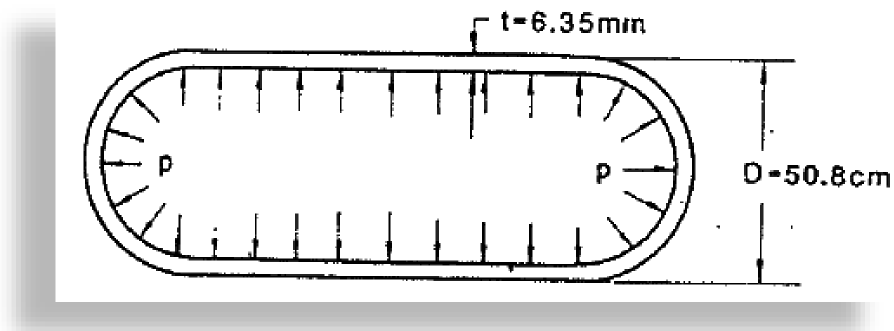
Question: how much the pressure can be increased before yield begins? *a*) according to von Mises and *b*) Tresca yield condition.

Hints: consider that the stress state is bi-axial (thin-walled cylindrical shell) where

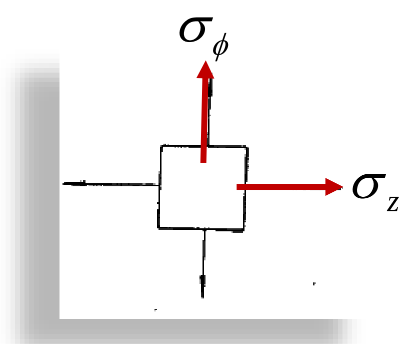
$$\sigma_\phi = \frac{pD}{2t}$$ - circumferential stress

$$\sigma_z = \frac{pD}{4t}$$ - axial stress

Hydrostatic pressure insensitive yield criteria



Assume a biaxial stress state, $t << 2R=D$



ex. 1-d tensile test and the hardening rule

Self-study

Determination of $d\sigma_f/d\epsilon_e^p$. Uniaxial tensile test

In a standard uniaxial tensile test, during plastic flow we have

$$\sigma_{11} \neq 0\,;\text{all other } \sigma_{ij} \equiv 0 \tag{16}$$

and

$$d\epsilon_{11}^p \neq 0\,; d\epsilon_{22} = d\epsilon_{33} = -\frac{1}{2}d\epsilon_{11}\,;\text{all other } d\epsilon_{ij}^p \equiv 0 \tag{17}$$

I.e.,

$$f = \sigma_e - \sigma_f = \sigma_{11} - \sigma_f = 0 \;\Rightarrow \sigma_f = \sigma_{11} \tag{18}$$

and

$$d\epsilon_e^p = \sqrt{\frac{2}{3}\left\{1 + 2\cdot\left(-\frac{1}{2}\right)^2\right\}}\cdot d\epsilon_{11}^p = d\epsilon_{11}^p \tag{19}$$

In the uniaxial tensile test, the function

$$\sigma_f = \phi(\epsilon_e^p) \tag{20}$$

will therefore simplify into

$$\sigma_{11} = \phi(\epsilon_{11}^p) \tag{21}$$

and the function can be found from the test results. Suppose, for instance, that the uniaxial test curve of Fig. 2 has been recorded. In the plastic region one then has

$$d\epsilon = d\epsilon^e + d\epsilon^p = \left(\frac{1}{E}+\frac{1}{\phi'}\right)d\sigma \Rightarrow \frac{d\sigma}{d\epsilon} = \frac{1}{\frac{1}{E}+\frac{1}{\phi'}} \;\Rightarrow \phi' = \frac{\frac{d\sigma}{d\epsilon}}{1-\frac{1}{E}\frac{d\sigma}{d\epsilon}}. \tag{22}$$

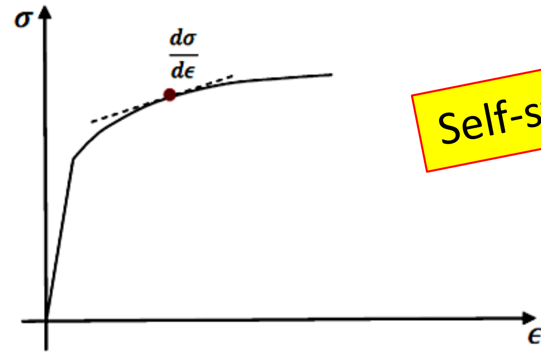


Fig.2 Uniaxial tensile test curve

Note, therefore, that the derivative $d\sigma/d\epsilon$ directly measured in the uniaxial test curve must be 'postprocessed' in order to find ϕ'!

In a simple case, this function may be linear, *i.e.,*

$$\sigma_f = \phi(\epsilon_e^p) = \sigma_s + c^{(i)}\epsilon_e^p \tag{23}$$

and

$$\frac{d\sigma_f}{d\epsilon_e^p} = c^{(i)} \tag{24}$$

The flow rule can then be simplified:

$$d\epsilon_{ij}^p = \frac{9}{4}\cdot\frac{s_{kl}d\sigma_{kl}}{c^{(i)}\sigma_e^2}\cdot s_{ij} \tag{25}$$

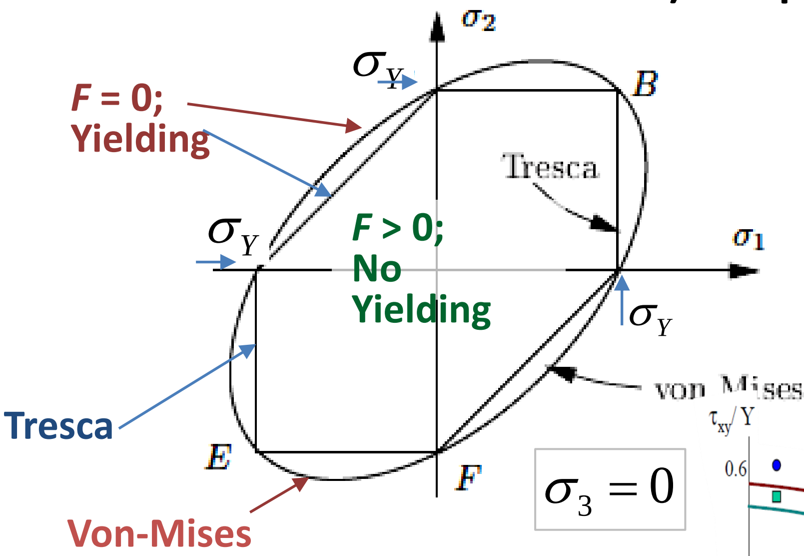
or

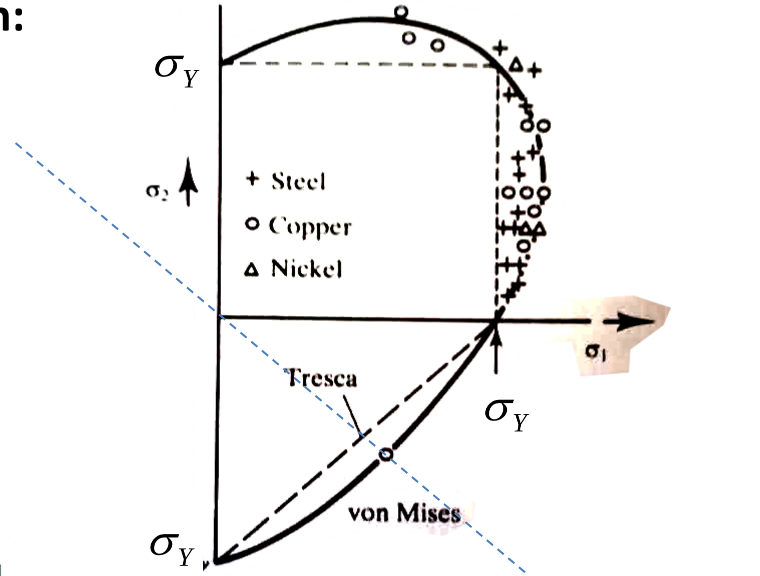
$$d\epsilon_{ij}^p = \frac{d\sigma_f}{c^{(i)}}\cdot\frac{3}{2}\frac{s_{ij}}{\sigma_e} = \frac{d\sigma_e}{c^{(i)}}\cdot\frac{3}{2}\frac{s_{ij}}{\sigma_e} \tag{26}$$

(the 2nd equality since during plastic flow, $\sigma_f = \sigma_e$). Eq. (26) is perhaps the most convenient version of the flow rule. Note, again, that a linear stiffness measured in a tensile test must be recalculated in order to find $c^{(i)}$ (see Eq. (22)!).

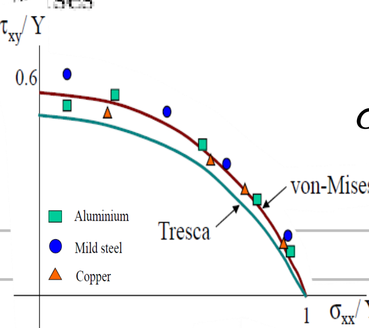
Failure envelopes:

a) comparison:





Experimental evidence
Von-Mises and Tresca



b) Tresca … conservative:

Von-Mises:

$$f = 1/2 \cdot [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] - 3\tau_Y{}^2 = 0$$

$$F = J_2 - \tau_Y{}^2 = 0$$

Tresca:

$$F = \max\left\{\frac{1}{2}|\sigma_1 - \sigma_2|, \frac{1}{2}|\sigma_2 - \sigma_3|, \frac{1}{2}|\sigma_3 - \sigma_1|\right\} - \tau_Y = 0$$

yield stress in pure shear

$F = \tau_{\max} - k < 0 \Rightarrow$	No yielding
$F = \tau_{\max} - k < 0 \Rightarrow$	Onset of yielding

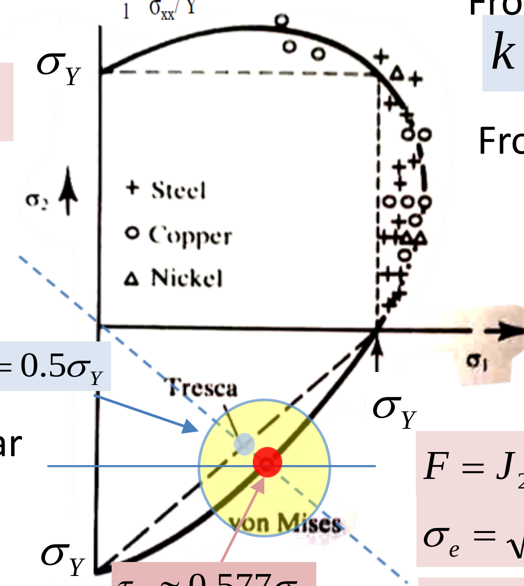
$$k \equiv \tau_Y$$

From uniaxial test:

$$k = \sigma_Y / 2 = \tau_Y$$

From pure shear test:

$$k = \sigma_Y / \sqrt{3}$$



$$\boldsymbol{\sigma} = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{s}$$

$\tau_Y = 0.5\sigma_Y$

$\tau_Y \approx 0.577\sigma_Y$

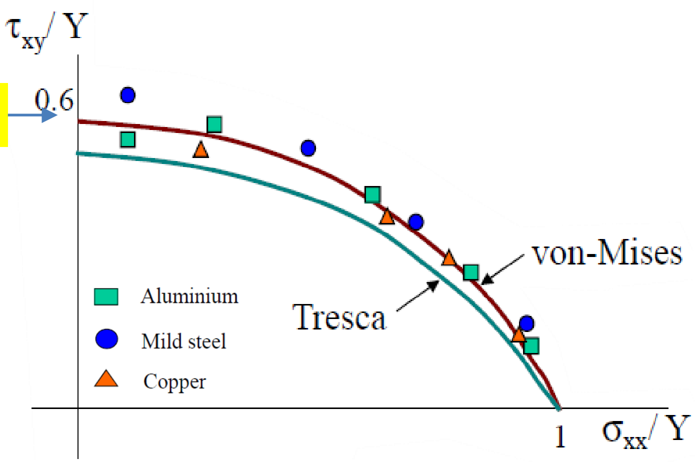
$$F = J_2 - \tau_Y{}^2 = 0 \Rightarrow \tau = \tau_Y$$

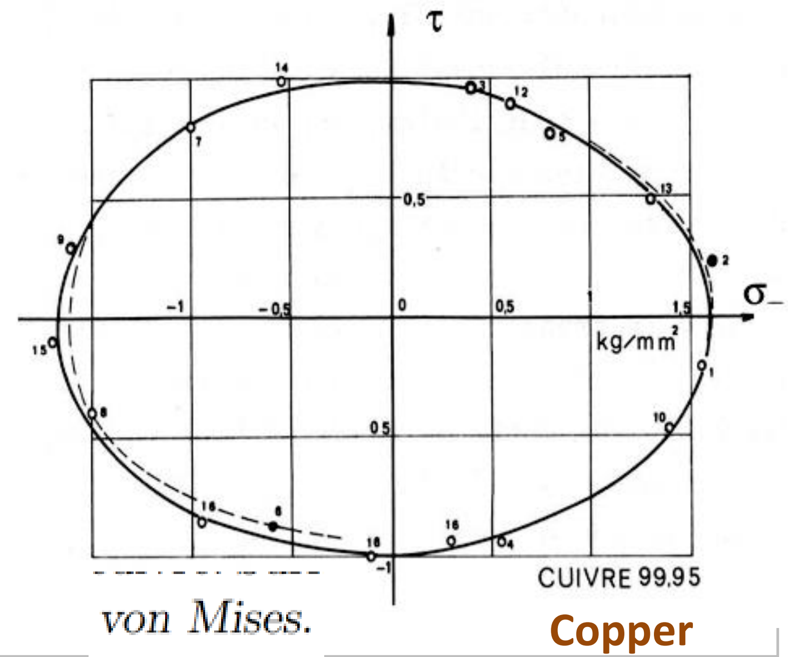
$$\sigma_e = \sqrt{3J_2} = \sqrt{3\tau_Y{}^2} = \sigma_Y$$

$$\Rightarrow \tau_Y = \sigma_Y / \sqrt{3} \approx 0.577\sigma_Y$$

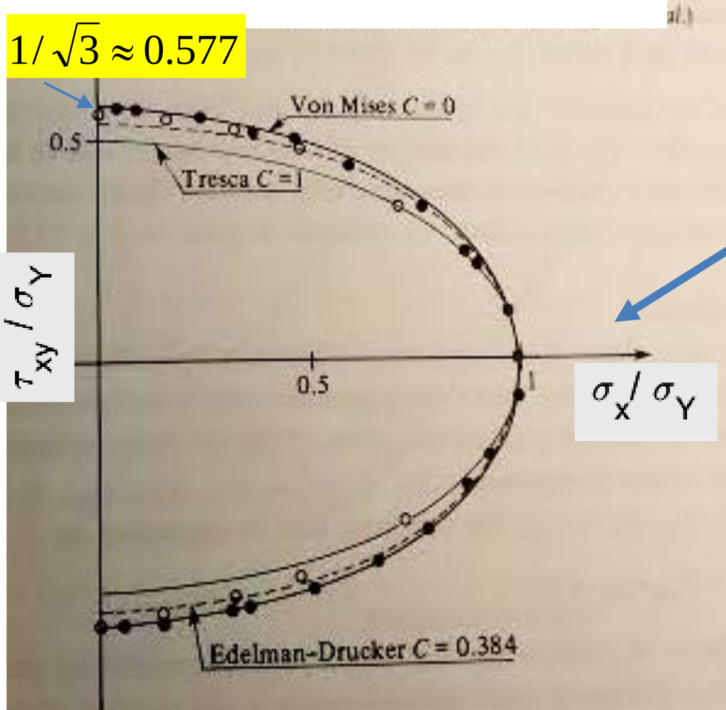
Experimentally validated hypothesis

Experimental Evidences Metals





o 195 aluminum alloy (after Ivey)
● 24 S-T4 aluminium alloy (after Naghdi *et al.)*



Ref: Lemaitre et al. , our textbook

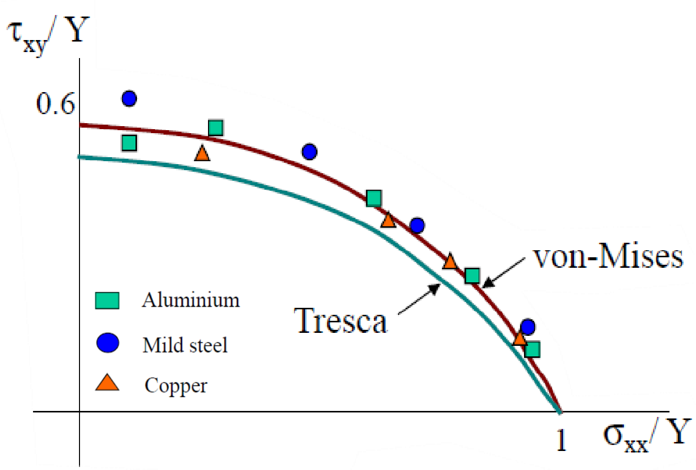
Experimental evidence
Von-Mises and Tresca

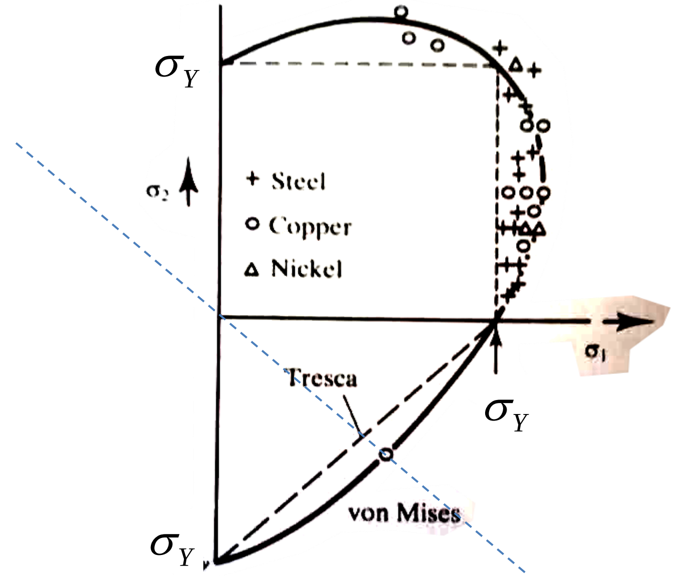
From these experimental results, it seems that 'von Mises' describe better than 'Tresca' the yield of metals (isotropic). **'Tresca is good too' despite its simplicity.**

Reason: The driving force for plastic flow in both criteria is *shearing*; maximum shear stress for Tresca and maximum distortion energy for von Mises, and distortion cannot be without shearing ***cqfd***.

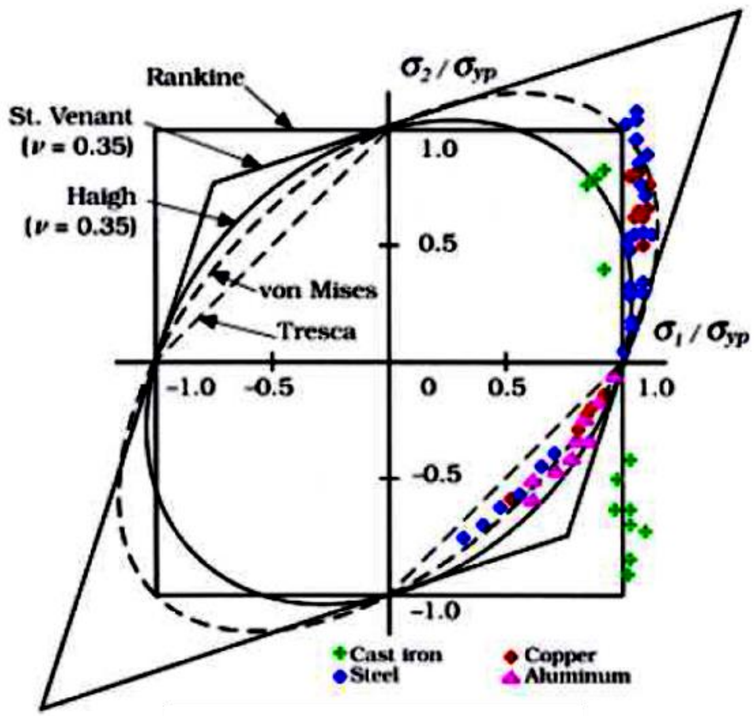
NB. **Anisotropic yield criteria**: In this case, the material have principle or preferred directions of anisotropy. Therefore, such information bout theses directions should appear in the yield criteria. Thus, such yield criteria cannot be described by stress invariant alone. Such criteria are for instance Hill criterion and Tsai criterion. (*ref*. Lemaitre et Chaboche, our course textbooks). Also, strain hardening introduces anisotropy. These aspects are not in the scope of this course, so *cf*. to literature.

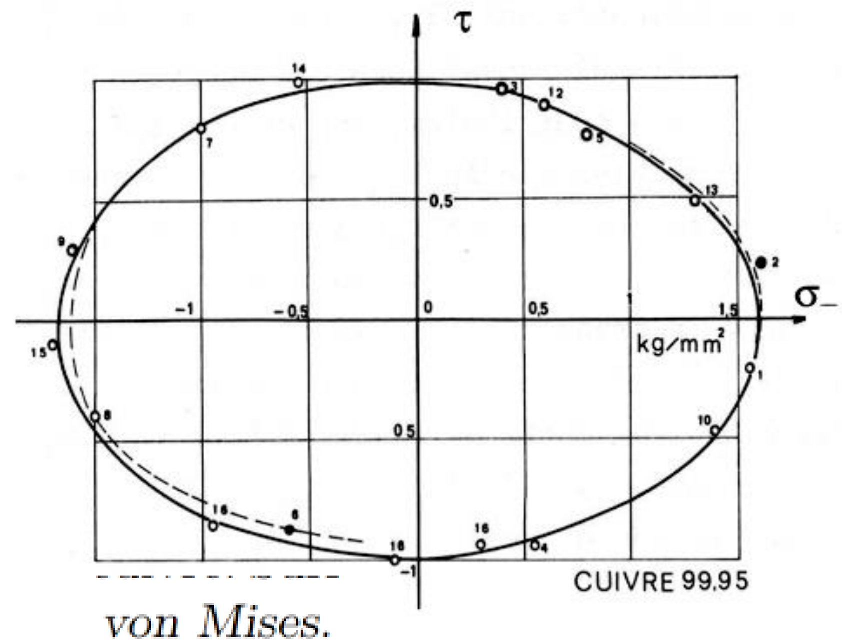
Experimental Evidences
Metals



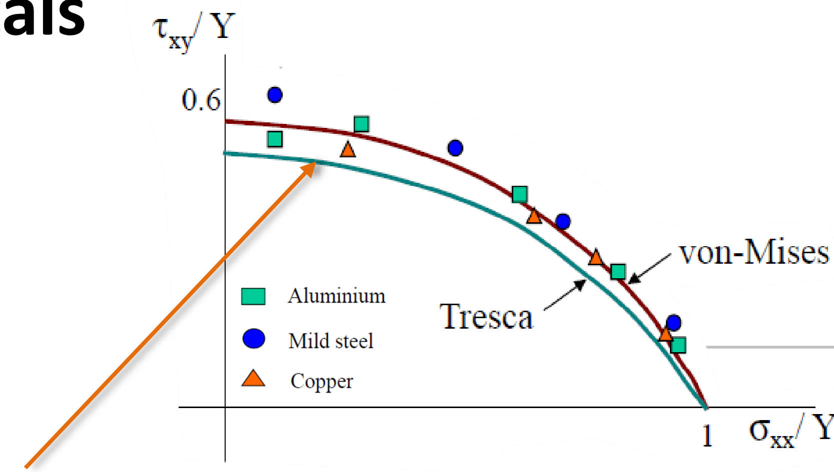


Experimental evidence
Von-Mises and Tresca

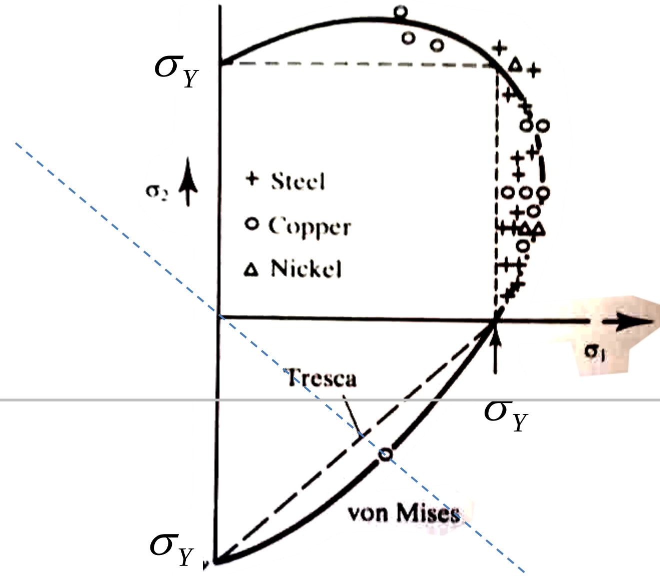




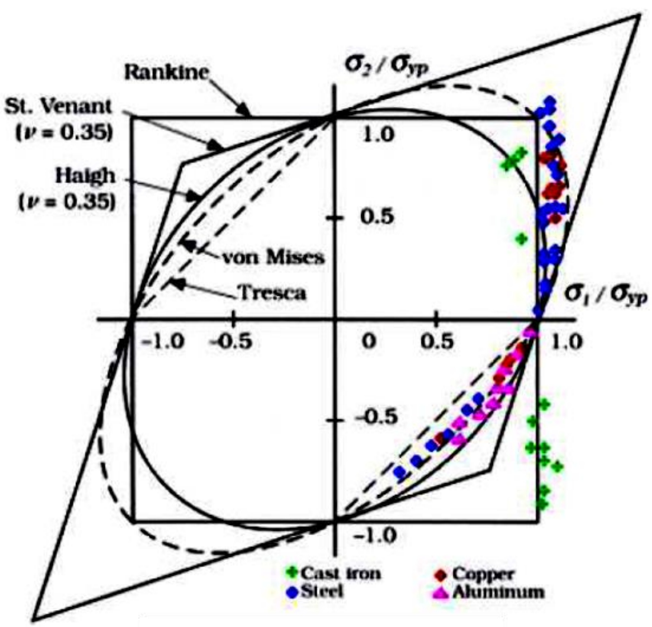
Experimental Evidences
Metals

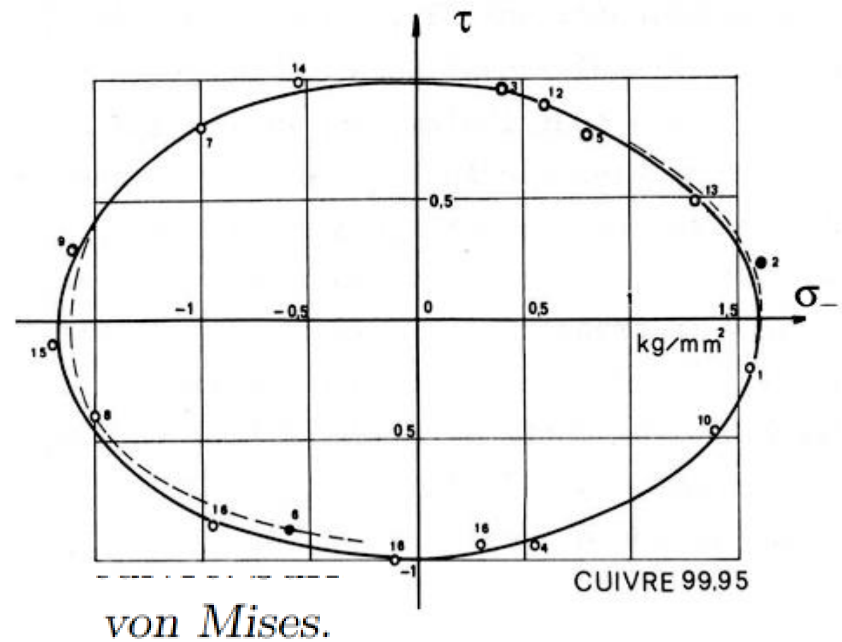


Challenge: show that Tresca yield locus, against direct intuition, is not a straight line but is a curved curve in this plane.

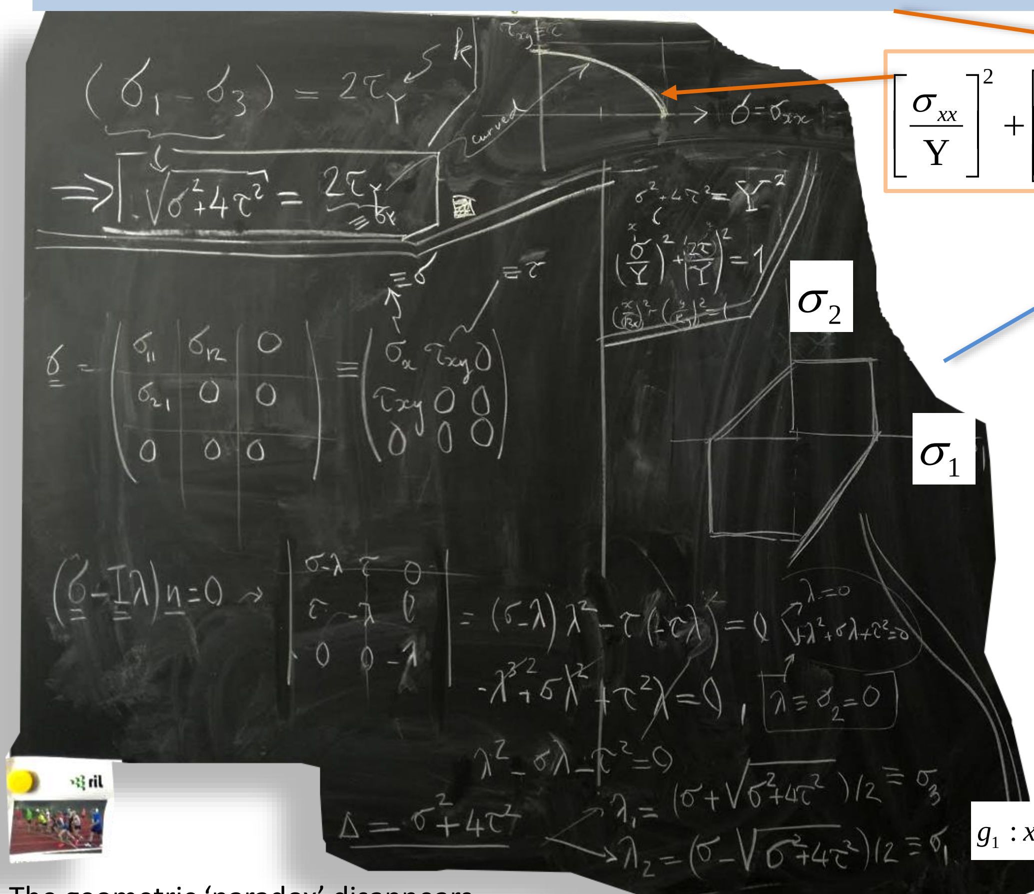


Experimental evidence
Von-Mises and Tresca





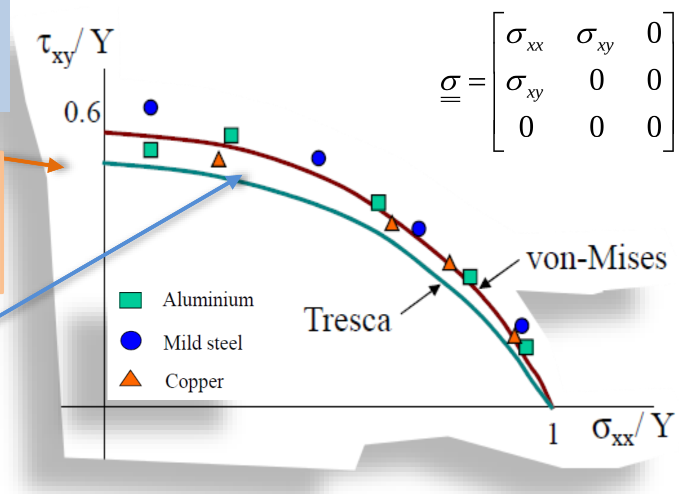
Challenge: show that Tresca yield locus, against a too rapid direct intuition, is not a straight line but is a *curved* curve in this plane.



$$\left[\frac{\sigma_{xx}}{\mathrm{Y}}\right]^2 + \left[\frac{\sigma_{xy}}{\mathrm{Y}/2}\right]^2 = 1$$

σ_2

σ_1



$$\sigma_{xx}^2 + 4\sigma_{xy}^2 = \mathrm{Y}^2 \equiv \sigma_{\mathrm{Y}}^2$$

$$\left[\frac{\sigma_{xx}}{\mathrm{Y}}\right]^2 + \left[\frac{\sigma_{xy}}{\mathrm{Y}/2}\right]^2 = 1$$

Is an ellipse when Tresca yield condition is expressed in the coordinates σ_{xx}, σ_{xy}

$g_1 : x = x' + \frac{1}{2}\sqrt{x' + 4y'^2}$

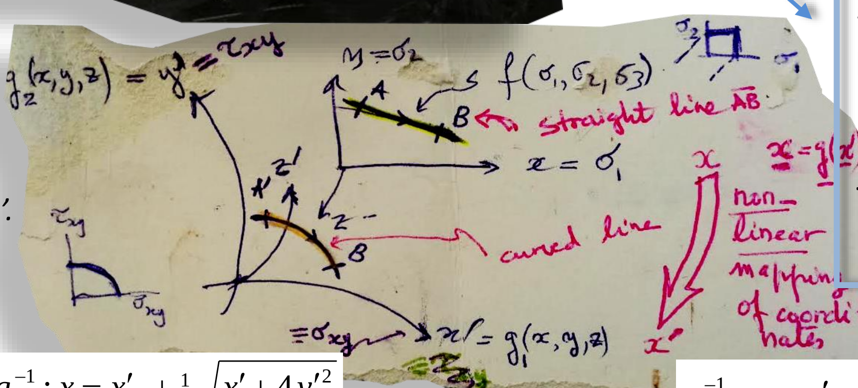
NB. when it is expressed in principle stresses coordinates

$$\sigma_1, \sigma_2 = 0, \sigma_3$$

then the yield locus is formed by straight lines

The geometric 'paradox' disappears when we write our geometrical mapping familiar notation from geometry and write the equation of a straight line - given in coordinates *x,y – in a new coordinate system x',y'.*

The mapping ***g***(x1,x2,x3) between the two coordinate systems is NOT LINEAR. It why the mapped straight segment AB is curved in the new coordinate system to A'B'

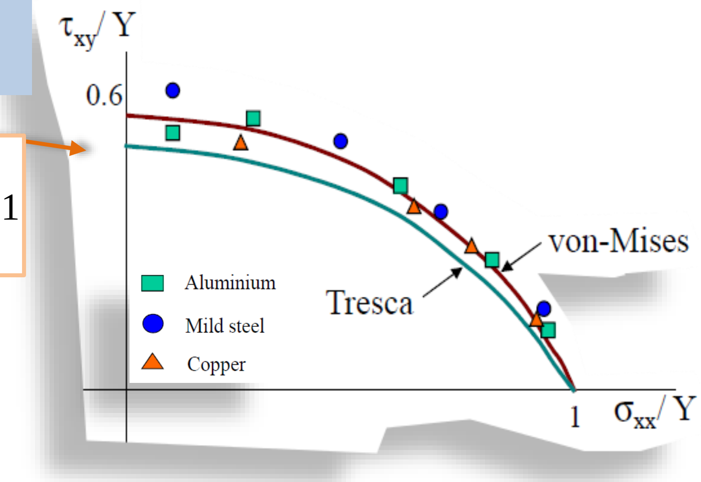


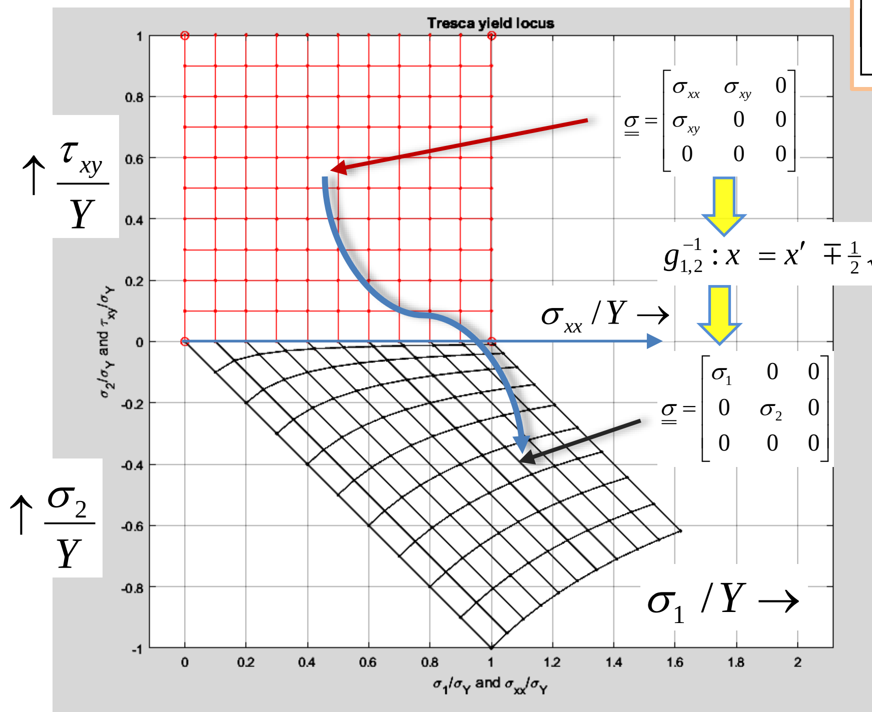
$g_1^{-1} : x = x' + \frac{1}{2}\sqrt{x' + 4y'^2}$

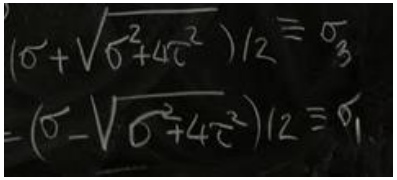
$g_2^{-1} : y = x' - \frac{1}{2}\sqrt{x' + 4y'^2}$

Challenge: show that Tresca yield locus, against a too rapid direct intuition, is not a straight line but is a *curved* curve in this plane.

$$\left[\frac{\sigma_{xx}}{Y}\right]^2+\left[\frac{\sigma_{xy}}{Y/2}\right]^2=1$$







$$\sigma_{xx}^2+4\sigma_{xy}^2=Y^2\equiv\sigma_Y^2$$

$$\left[\frac{\sigma_{xx}}{Y}\right]^2+\left[\frac{\sigma_{xy}}{Y/2}\right]^2=1$$

Is an ellipse when Tresca yield condition is expressed in the coordinates σ_{xx},σ_{xy}

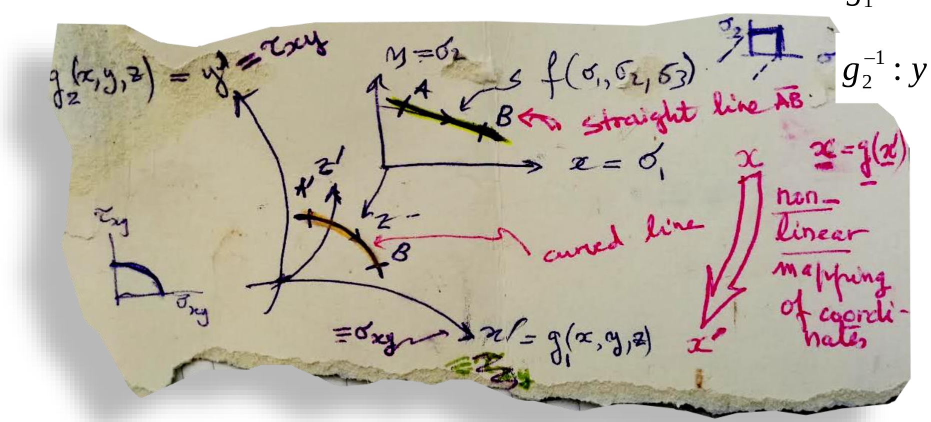
NB. when it is expressed in principle stresses coordinates

$$\sigma_1,\sigma_2=0,\sigma_3$$

then the yield locus is formed by straight lines

$$g_1^{-1}: x = x' + \tfrac{1}{2}\sqrt{x'+4y'^2}$$

$$g_2^{-1}: y = x' - \tfrac{1}{2}\sqrt{x'+4y'^2}$$



$$g_{1,2}^{-1}: x = x' \mp \tfrac{1}{2}\sqrt{x'+4y'^2}$$

Reading assignment:

Haigh-Westergaad Stress Space
chapter 2.1.9 - Chan & Han, *Plasticity for Structural Engineers*

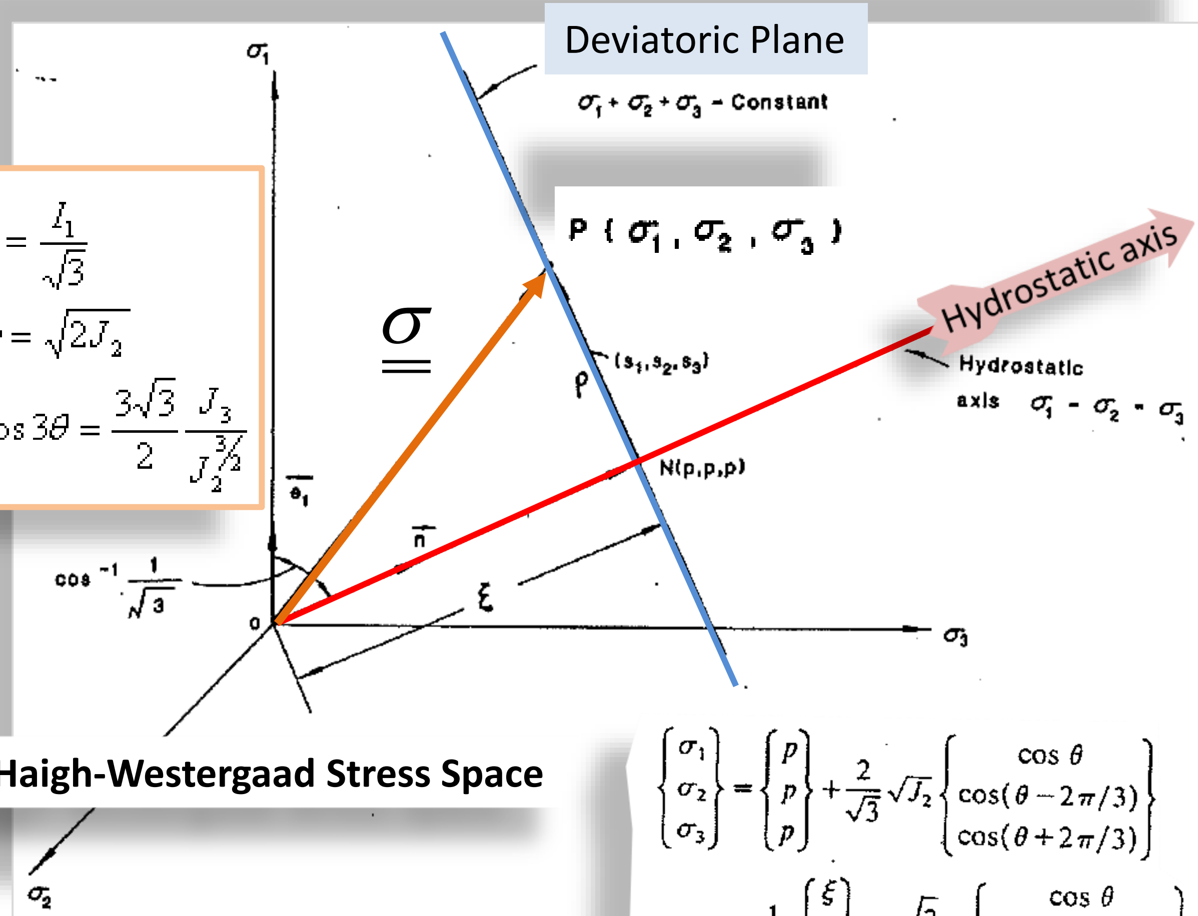
Haigh-Westergaad Stress Space

- a very useful and concepts clarifying 3D-geometric representation for stress-state and yield surfaces & in general, for studying plasticity

- The stress-space is chosen as the three principle stresses which play the role of coordinates for a representative point ***P*** of the stress state.

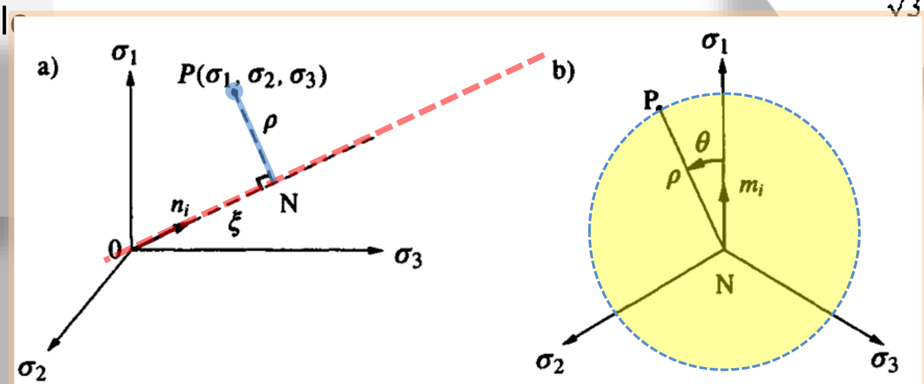
$$P(\underline{\underline{\sigma}}) = P(\sigma_1, \sigma_2, \sigma_3)$$

$$\begin{cases} \xi = \dfrac{I_1}{\sqrt{3}} \\ \rho = \sqrt{2J_2} \\ \cos 3\theta = \dfrac{3\sqrt{3}}{2}\dfrac{J_3}{J_2^{3/2}} \end{cases}$$



Haigh-Westergaad Stress Space

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{Bmatrix} = \begin{Bmatrix} p \\ p \\ p \end{Bmatrix} + \frac{2}{\sqrt{3}}\sqrt{J_2}\begin{Bmatrix} \cos\theta \\ \cos(\theta - 2\pi/3) \\ \cos(\theta + 2\pi/3) \end{Bmatrix} = \frac{1}{\sqrt{3}}\begin{Bmatrix} \xi \\ \xi \\ \xi \end{Bmatrix} + \sqrt{\frac{2}{3}}\rho\begin{Bmatrix} \cos\theta \\ \cos(\theta - 2\pi/3) \\ \cos(\theta + 2\pi/3) \end{Bmatrix}$$



Haigh-Westergaard coordinate system; deviatoric plane perpendicular

Pressure dependent Yield criteria

- **Mohr-Coulomb yield criterion**
- **Drucker-Prager yield criterion**

Drucker-Prager yield criterion

As stated earlier for metals and metal alloys – no plastic volume changes are shown during plastic flow for not very high hydrostatic pressures. Therefore the assumption of incompressibility during plastic deformation is realistic enough for structural engineers and the von Mises yield criteria describe well the plastic deformation in metals.

It is well known that granular and friction materials - as geomaterials; soils, rock and concrete* (multiaxial loading) – have a non-negligible plastic volume increase (volumetric dilatation, dilatancy) during 'plastic flow'(under shearing) and consequently, the von Mises yield criterion *fails* ... unless one modifies it in order to account for this change in plastic volume accompanying the yielding.

Drucker and Prager did exactly that and they formulated *a modified (or generalized*) von Mises criterion* by adding a term accounting for the hydrostatic pressure (the mean stress term).

* The Drucker–Prager criterion goes back to the *Mohr theory of rupture* in which failure (rupture) occurs on a plane in a body when the shear stress and normal stress (=pressure) on that plane achieve some defined critical combination. The Drucker–Prager failure criterion was established as a generalization of the Mohr–Coulomb criterion for soils.

From the yield surface one can note that, as compared to the von Mises criterion (a tube),

- there is *a limit for tensile* mean stresses
- the material is strengthened by superimposing of compressive mean pressure

Drucker-Prager criterion (1952)

$$\boxed{f(I_1, J_2; k) = \sqrt{J_2} + \alpha I_1 - k = 0,}$$

Hydrostatic pressure:

$$I_1 = \frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma}) = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})$$

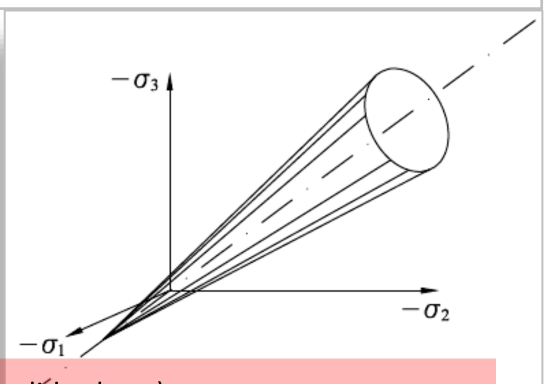
$\alpha -$ a friction-type material parameter

$k -$ a threshold parameter

These parameters can be expressed as 'cohesion' and 'internal friction angle' or they can be determined from triaxial tests drawing in I_1, J_2 –space.

$k_{VM} = \sigma_Y / \sqrt{3}$

Cf. Von Mises: $f(J_2; k) = \sqrt{J_2} - k_{VM} = 0,$



For concrete, **Ottosen's** 4-parameter failure criterion for concrete is more adequate, (presented few slides later)

Suggested reading: State-of-the-art report, bulletin 45. 2008, task group 4.4. Fib CEB-FIP – ***Practitioner's guide to finite element modelling of reinforced concrete structures***

Drucker-Prager criterion

$$\boldsymbol{\sigma}' \equiv \mathbf{s} = \boldsymbol{\sigma} - p\mathbf{I} \leftrightarrow s_{ij} = \sigma_{ij} - p\delta_{ij} I_1 \rightarrow$$ The deviatoric stress (shear)

$$J_2 = \frac{1}{2}\mathbf{s}:\mathbf{s} \Rightarrow \sqrt{J_2} = \sqrt{\frac{1}{2} s_{ij} s_{ij}} \equiv \tau_e$$ Equivalent stress

The model parameters in terms of *internal friction angle* ϕ and of *cohesion* intercept C of the material are:

$$\alpha = \frac{2\sin\phi}{\sqrt{3}(3-\sin\phi)}, \qquad k = \frac{6c\cdot\cos\phi}{\sqrt{3}(3-\sin\phi)}$$

Drucker-Prager (DP) criterion:

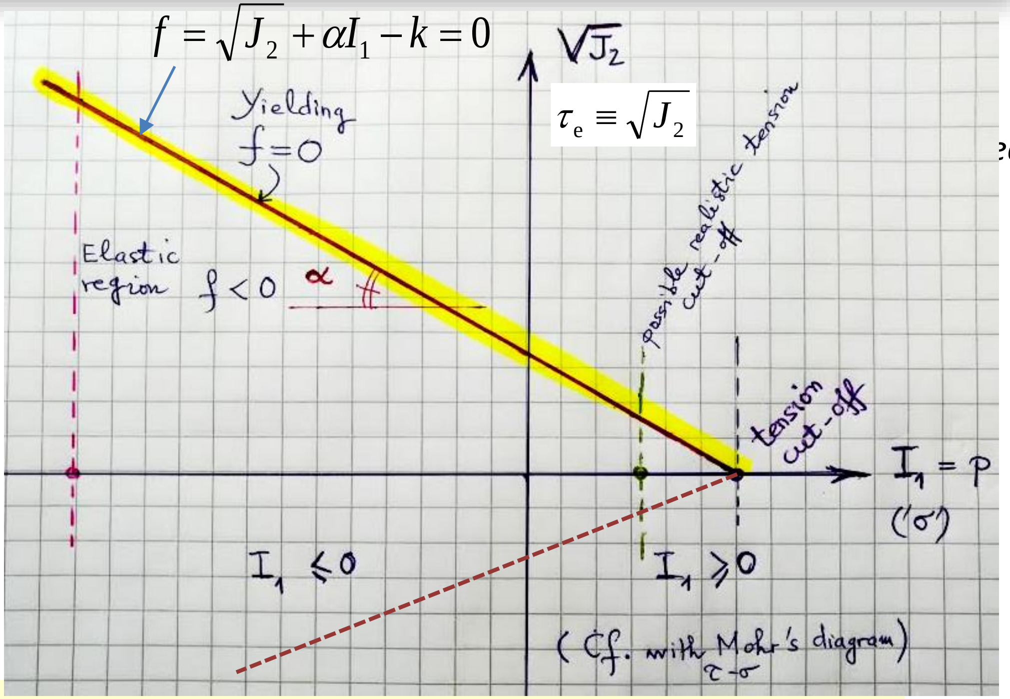
$$\boxed{f(I_1, J_2; k) = \sqrt{J_2} + \alpha I_1 - k = 0,}$$

Hydrostatic pressure:

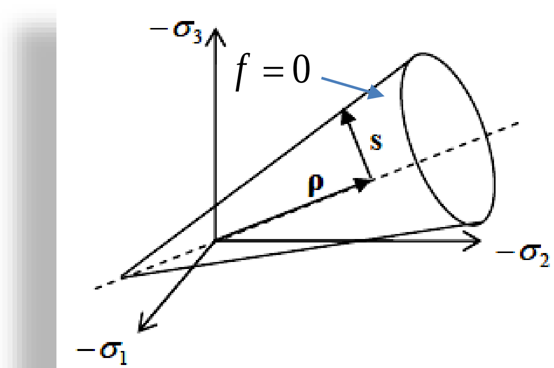
$$I_1 = \frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma}) = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})$$

$\alpha -$ a friction-type material parameter $\qquad k -$ a threshold parameter

These material parameters can be expressed as '*cohesion*' and '*internal friction angle*' or they can be determined from triaxial tests graphs in $I_1, \sqrt{J_2} -$ space.



The Drucker-Prager cone

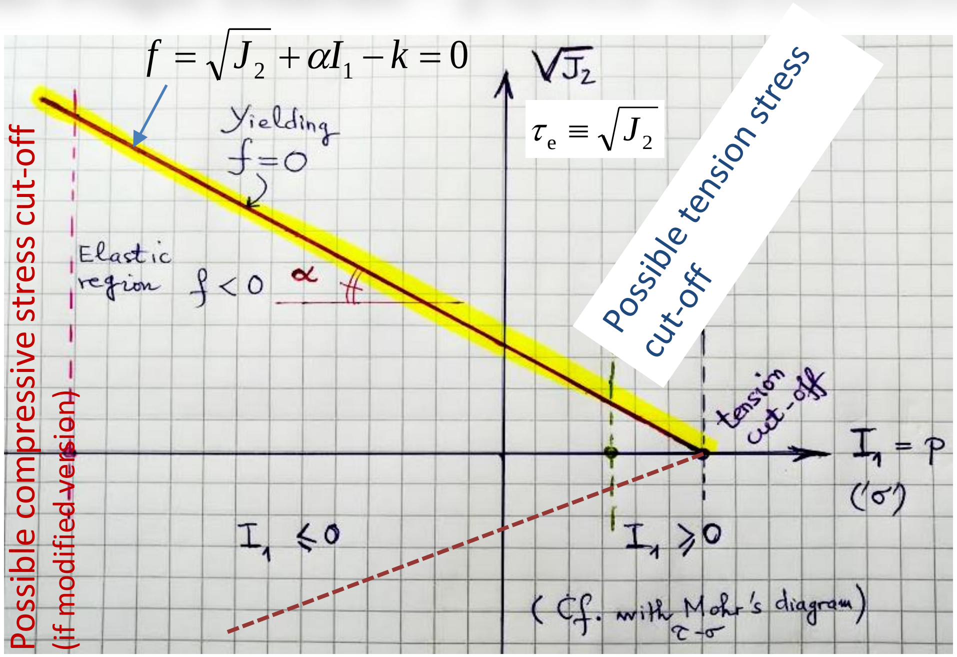


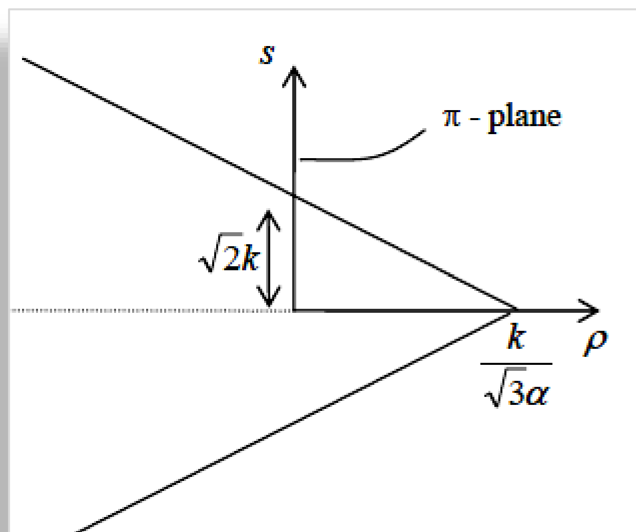
From the yield surface one can note that, as compared to the von Mises criterion (a 'tube'),

- there is *a limit for tensile mean stresses*
- the material *strengthening* by superimposing of *compressive mean pressure* (confinement)

NB. Some experimental results on RC show indicate that **concrete** does not response linearly, as assumed in DP-model, in the space $\sqrt{J_2} - I_1$ when subjected to *severe hydrostatic pressures* (and probably also for very low pressures)

Drucker-Prager criterion – graphical representation





The Drucker-Prager criteria in the Meridian Plane

The Drucker-Prager cone

$$\boxed{f(I_1, J_2; k) = \sqrt{J_2} + \alpha I_1 - k = 0,}$$

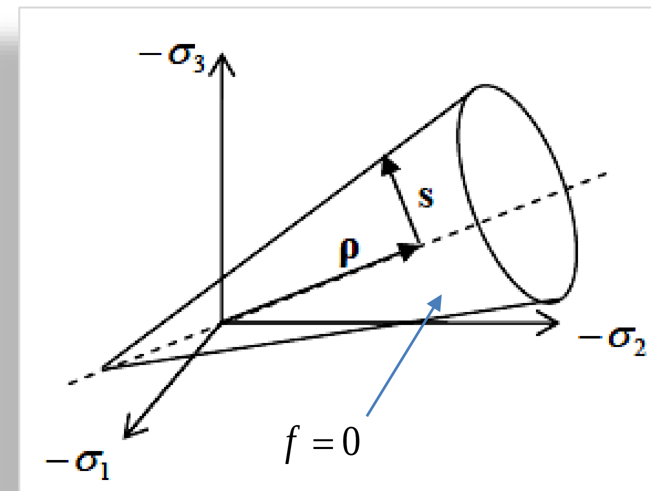
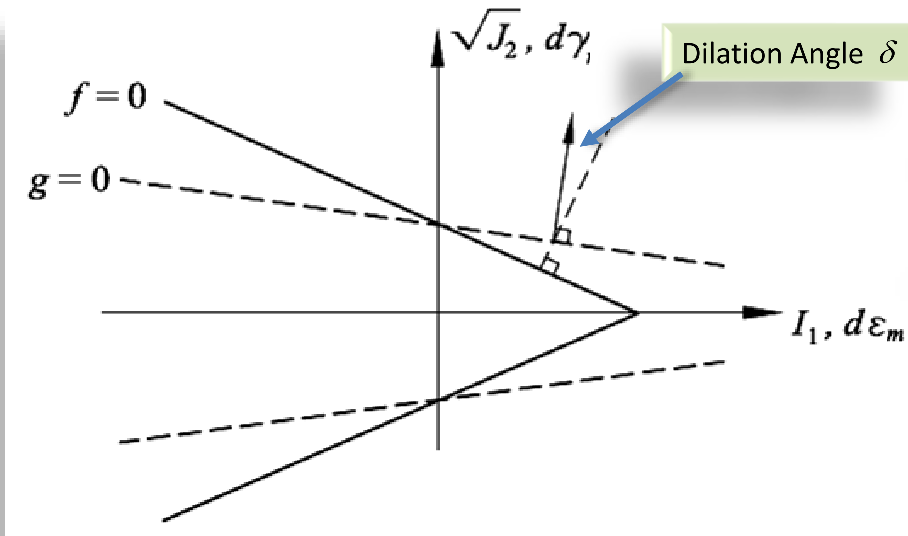


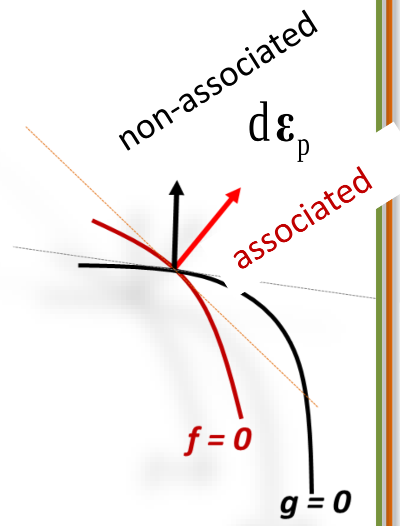
Illustration: **Non-associated plasticity, flow rule**

$$f(\boldsymbol{\sigma}) = \sqrt{J_2} + \alpha I_1 - k$$ **Drucker-Prager yield criterion**

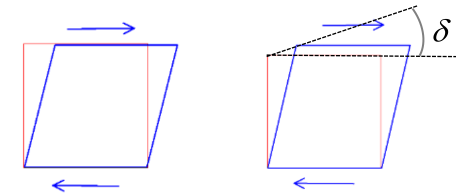
$$g(\boldsymbol{\sigma}) = \sqrt{J_2} + \beta I_1 - k$$ **Flow rule**



Drucker-Prager plasticity with nonassociated flow rule.



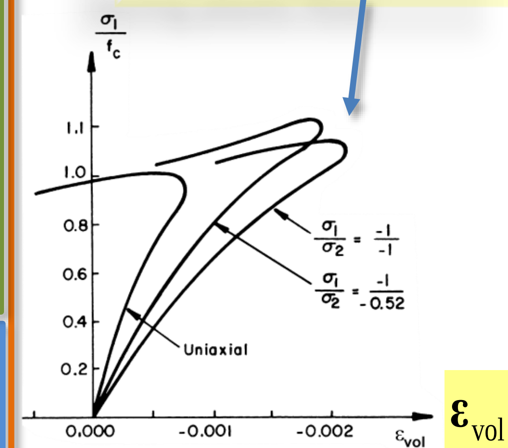
The angle of dilation controls an amount of plastic volumetric strain development during plastic flow



Zero dilatancy, associative
Incompressible plastic flow

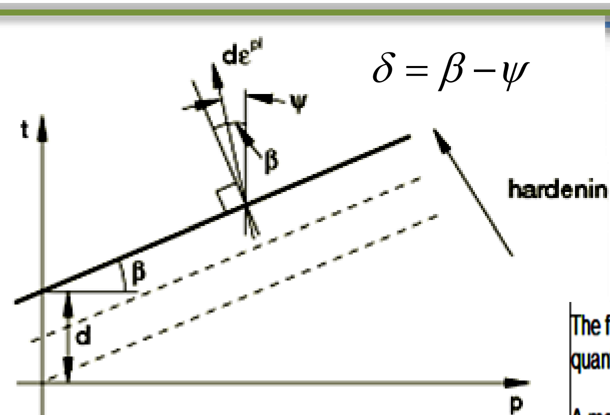
Dilatancy not null, non-associative, **compressible plastic flow**

Volumetric Strain grows during plastic flow



$\boldsymbol{\varepsilon}_{vol}$

Concrete Compressive Stress Versus Volumetric Strain (Data from Kupfer, 1969 as Presented by Chan and Han, 1988)



Schematic of hardening and flow for the linear model in the p–t plane.

$$\delta = \beta - \psi$$

$\beta \equiv \phi$ - internal friction angle

δ - dilancy angle

ABAQUS Version 6.6 Documentation

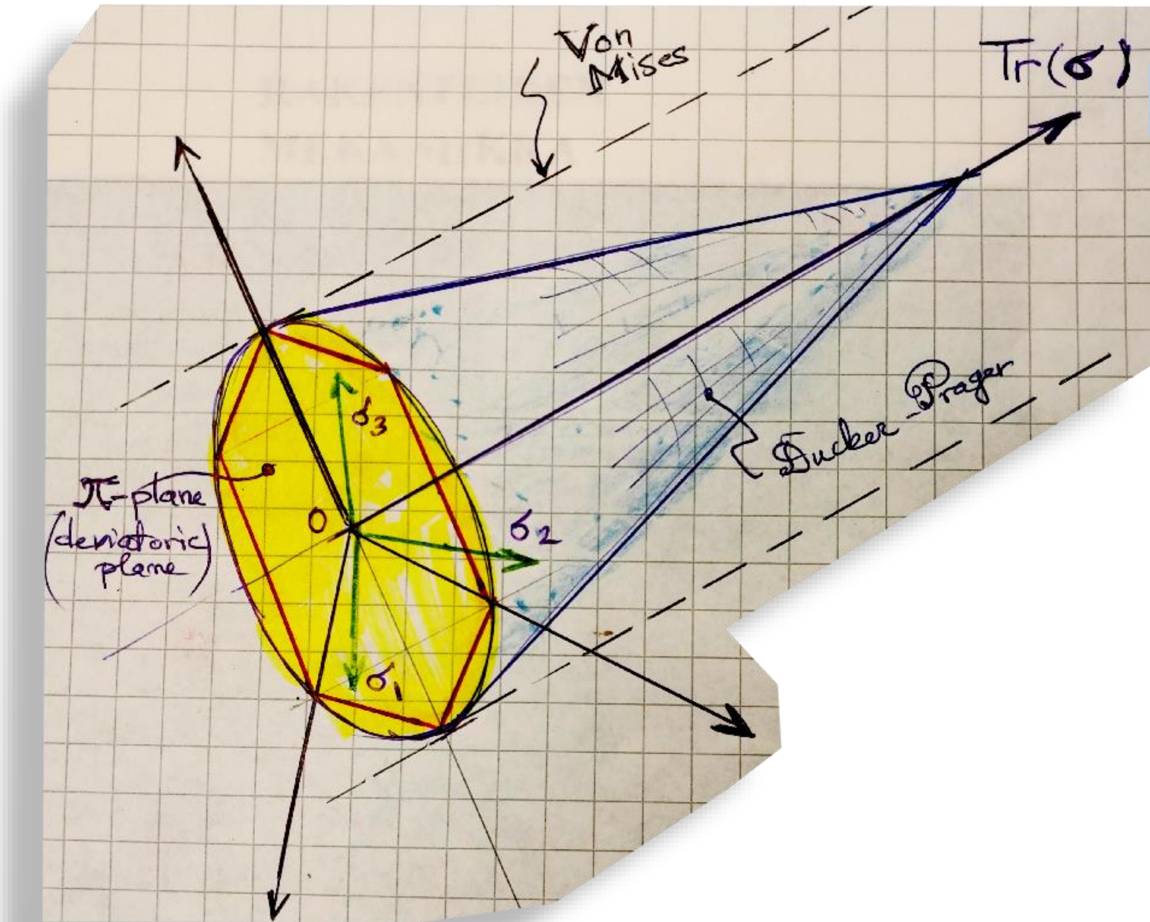
The formulation treats β as constant with respect to stress, although it is straightforward to extend the theory to provide for the functional dependence of β on quantities such as p.

A method for converting Mohr-Coulomb data (ϕ, the angle of Coulomb friction, and c, the cohesion) to appropriate values of β and d is described in the Abaqus Analysis User's Guide.

Drucker-Prager criterion

I_1

$\sigma_1 = \sigma_2 = \sigma_3$



$$f(I_1, J_2; k) = \sqrt{J_2} + \alpha I_1 - k = 0,$$

Equivalent stress: $\sqrt{J_2} = \sqrt{\frac{1}{2} s_{ij} s_{ij}} \equiv \tau_e$

Good to read: Extended Drucker-Prager models

Yield criteria

The yield criteria for this class of models are based on the shape of the yield surface in the meridional plane. In ABAQUS/Standard the yield surface can have a linear form, a hyperbolic form, or a general exponent form; in ABAQUS/Explicit only the linear form is available. These surfaces are illustrated in Figure 18.3.1–1.

In ABAQUS/Standard the *yield surface* can have a linear form, a hyperbolic form, or a general exponent form

NB the curved part of the curve in tension, often, corresponds better to observed behaviour.

Stress invariants

The yield stress sur

$$p = -\frac{1}{3}\mathrm{trace}(\boldsymbol{\sigma}),$$

and the Mises equivalent stress,

$$q = \sqrt{\frac{3}{2}(\mathbf{S} : \mathbf{S})},$$

where $\mathbf{S}$ is the stress deviator,

$$\mathbf{S} = \boldsymbol{\sigma} + p\mathbf{I}.$$

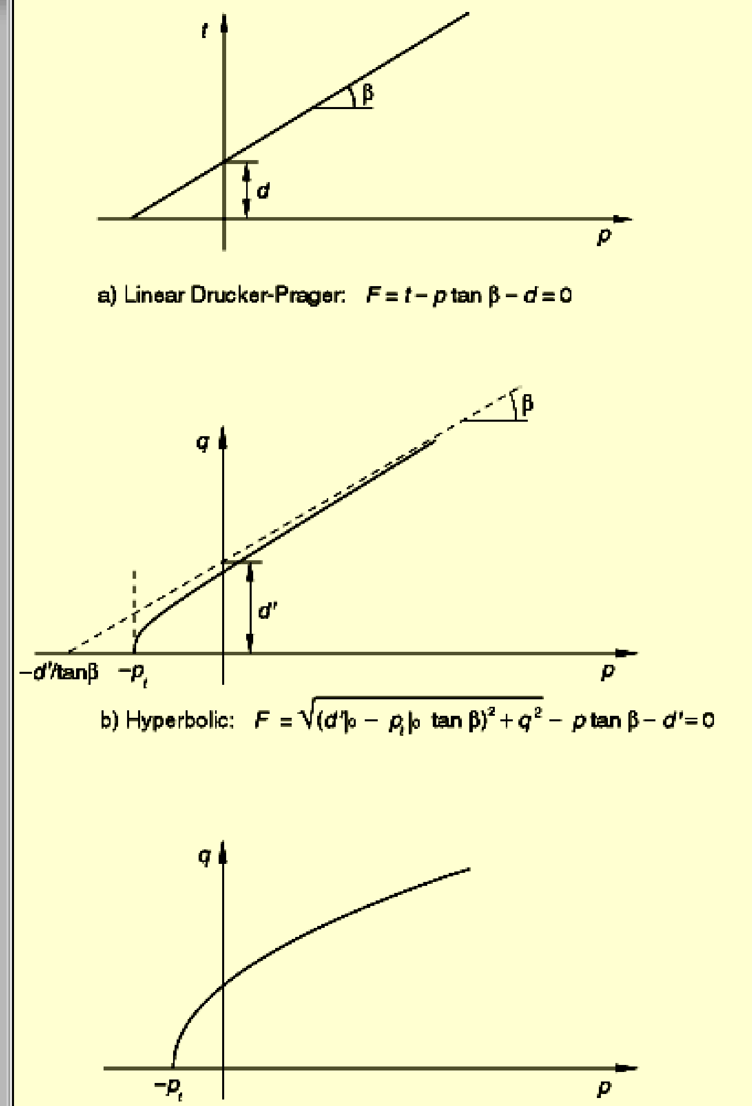
In addition, the linear model also uses the third invariant of deviatoric stress.

$$r = \left(\frac{9}{2}\mathbf{S}\cdot\mathbf{S} : \mathbf{S}\right)^{\frac{1}{3}}.$$

$$t = \frac{1}{2}q\left[1 + \frac{1}{K} - \left(1 - \frac{1}{K}\right)\left(\frac{r}{q}\right)^3\right].$$

$$0.778 \le K \le 1.0$$

Figure 18.3.1–1 Yield surfaces in the meridional plane.

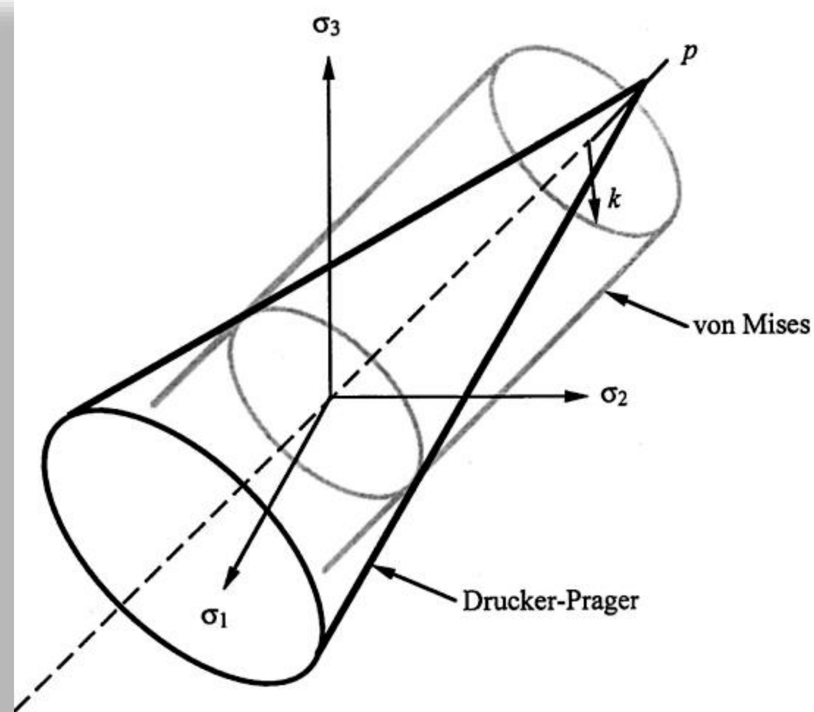


a) Linear Drucker-Prager: $F = t - p\tan\beta - d = 0$

b) Hyperbolic: $F = \sqrt{(d'|_0 - p_t|_0 \tan\beta)^2 + q^2} - p\tan\beta - d' = 0$

c) Exponent form: $F = aq^b - p - p_t = 0$

Summary - Drucker-Prager & von Mises criteria



von Mises and Drucker-Prager yield surfaces

Possible Abaqus

Simulation-

Slope Stability Failure HW

6.7.3 Numerical Examples

As an example, the plane-strain problem of a layer of material ($\phi = 20^\circ$ and $c = 10\ \text{lb/in}^2$) under a strip of uniform pressure p, as shown in Fig. 6.12, is analyzed by the finite-element method (Mizuno and Chen, 1980).

The Drucker-Prager criterion with different material constants is used. Bound-

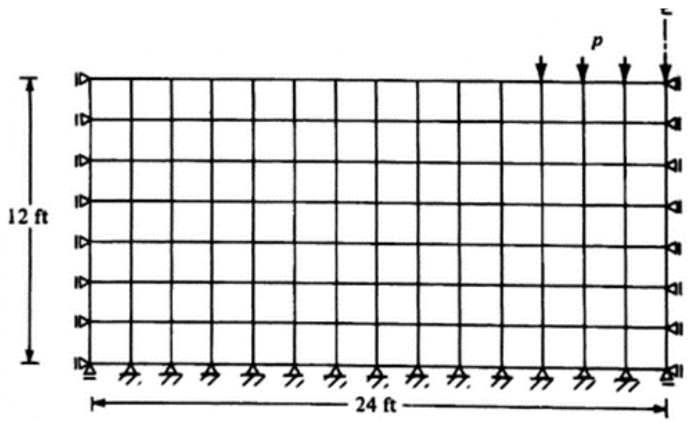
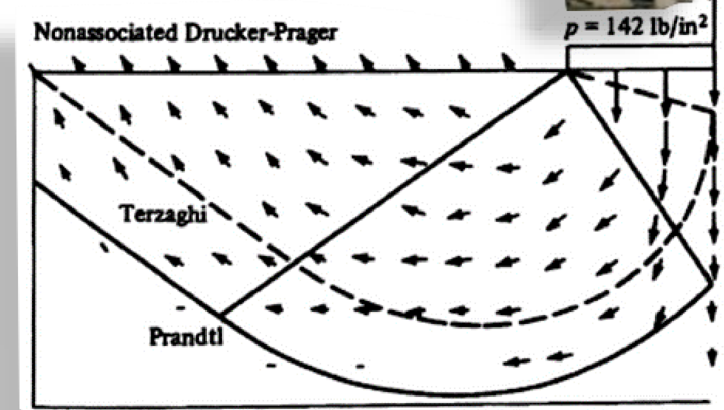


Figure 6.12 Plane-strain problem of a layer of material under a strip loading; $E = 30\ \text{kips/in}^2$, $\nu = 0.3$, $\phi = 20^\circ$, $c = 10\ \text{lb/in}^2$.

Figure 6.13 Velocity field at numerical collapse load: (*a*) associated flow rule and small deformation (158 lb/in²):, (*b*) nonassociated flow rule and small deformation (142 lb/in²).



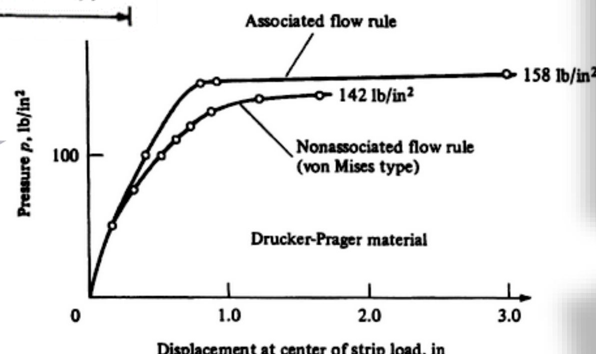
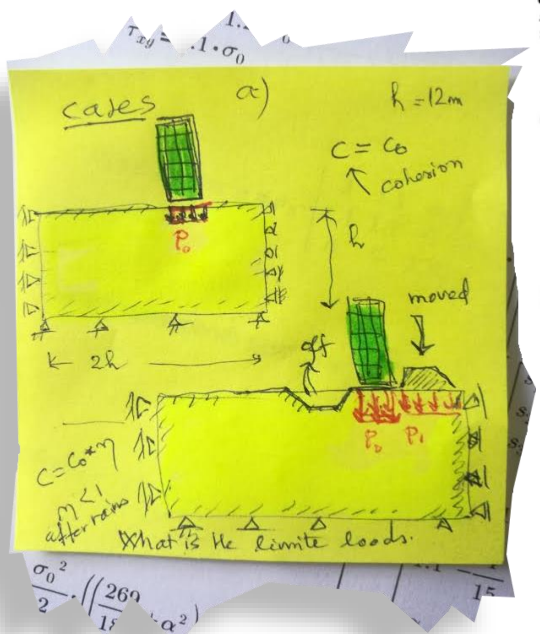
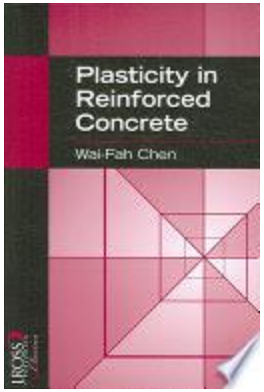


Figure 6.14 Comparison of analyses with an associated and nonassociated



Possible Comsol or Abaqus **Simulation** HW

Adapted from



Plasticity in Reinforced Concrete
By Wai-Fah Chen

6.7.3 Numerical Examples

As an example, the plane-strain problem of a layer of material ($\phi = 20^\circ$ and $c = 10\ \text{lb/in}^2$) under a strip of uniform pressure p, as shown in Fig. 6.12, is analyzed by the finite-element method (Mizuno and Chen, 1980).

The Drucker-Prager criterion with different material constants is used. Bound-

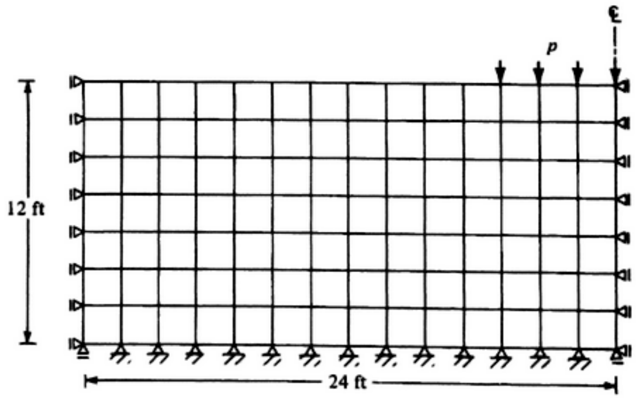
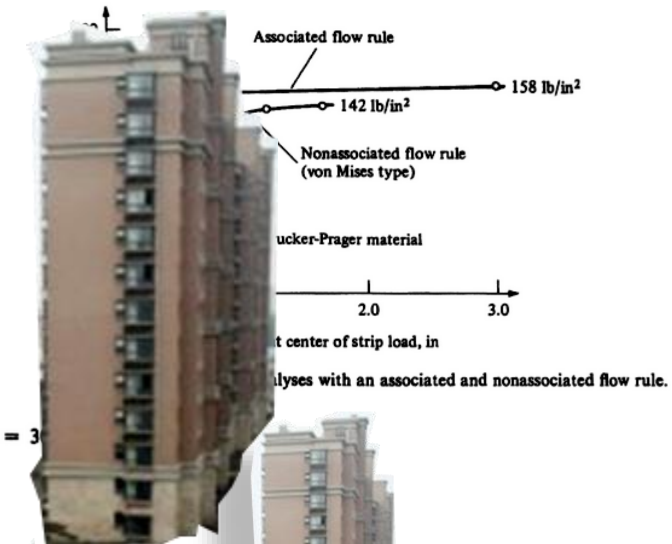
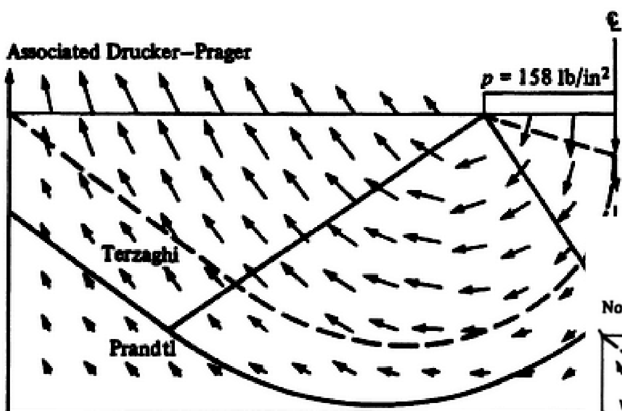
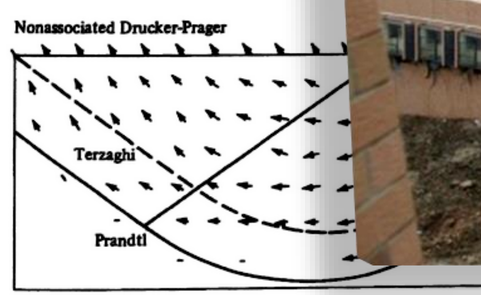


Figure 6.12 Plane-strain problem of a layer of material under a strip loading; E = 3
0.3, $\phi = 20^\circ$, $c = 10\ \text{lb/in}^2$.





(a)

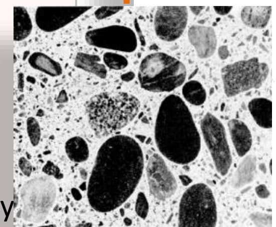


(b)

Figure 6.15 Velocity field at numerical collapse load: (*a*) associated flow rule and small deformation (158 lb/in²):, (*b*) nonassociated flow rule and small deformation (142 lb/in²).

Ottosen (1977) developed a ***4-parameters failure criterion for concrete*** which reproduces the four key properties for 'concrete' and which is one of the best models [20, 21, 22, 23].

Ottosen's 4-parameter failure criterion



The characteristic features for the failure surface:

1. The uniaxial *tensile* strength is 5 – 10% of the uniaxial compressive strength
2. The shape of the failure curves on the meridian plane is slightly curved
3. Hydrostatic compression cannot cause failure
4. The shape of the failure locus on the deviatoric plane is triangular for small hydrostatic pressure and gets rounded with increasing hydrostatic pressure.

Ref. Saba Tahaei Yaghoubi, Reijo Kouhia, Juha Hartikainen and Kari Kolari. A continuum damage model based on Ottosen's four parameter failure criterion for concrete. *Rakenteiden Mekaniikka* (Journal of Structural Mechanics) Vol. 47, No 2, 2014, pp. 50 – 66

Good to know

13111 AUGUST 1977 EM4

JOURNAL OF THE ENGINEERING MECHANICS DIVISION

A Failure Criterion for Concrete

By Niels Saabye Ottosen[1]

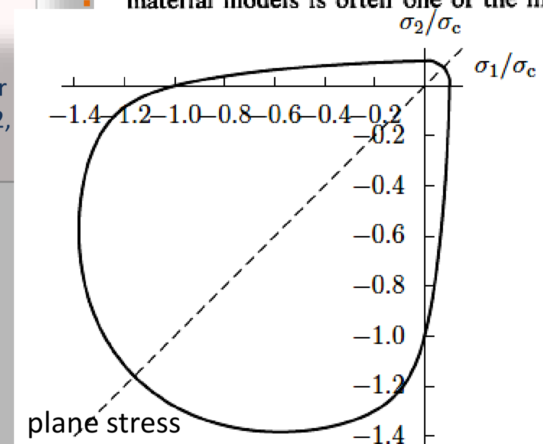
Introduction

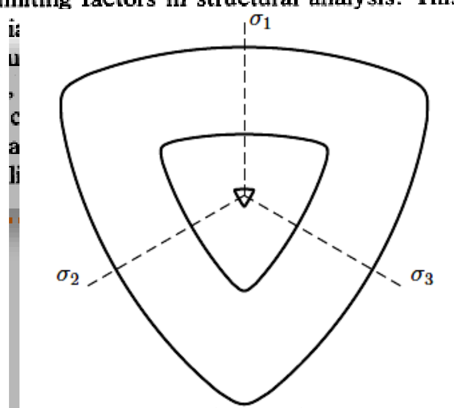
At the present stage of computer programs development, the use of inadequate material models is often one of the limiting factors in structural analysis. This

$$\boxed{A\frac{J_2}{\sigma_c}+\Lambda\sqrt{J_2}+BI_1-\sigma_c=0,}$$

$$\Lambda=\begin{cases}k_1\cos[\frac{1}{3}\arccos(k_2\cos 3\theta)] & \text{if}\quad \cos 3\theta\geqslant 0\\ k_1\cos[\frac{1}{3}\pi-\frac{1}{3}\arccos(-k_2\cos 3\theta)] & \text{if}\quad \cos 3\theta\leq 0\end{cases}.$$

$$\cos 3\theta=\frac{3\sqrt{3}}{2}\frac{J_3}{J_2^{3/2}},\quad A\geqslant 0;\quad B\geqslant 0;\quad k_1\geqslant 0;\quad 0\leq k_2\leq 1.$$





deviatoric plane when $p = 0$

[20] N.S. Ottosen. A failure criterion for concrete. *Journal of the Engineering Mechanics, ASCE*, 103(EM4):527–535, August 1977.

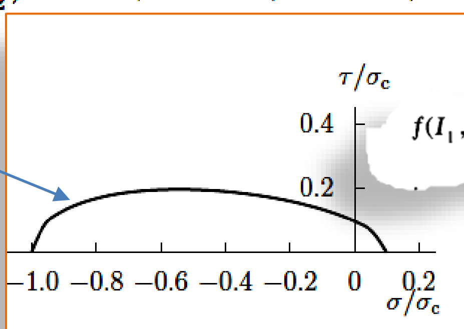
... element analysis of concrete structures. Technical Re... Laboratory, DK-4000 Roskilde, Denmark, May 1980.

... *The Mechanics of Constitutive Modeling*. Elsevier, 2005.

[23] K. Willam and E.P. Warnke. Constitutive model for the triaxial behaviour of concrete. In *IABSE Proceedings*, volume ... Structures Subjected to Triaxial Str... May 17-19, 1974.

$$f(I_1, J_2, \cos 3\theta) = A\frac{J_2}{\sigma_c^2}+\lambda\frac{\sqrt{J_2}}{\sigma_c}+B\frac{I_1}{\sigma_c}-1=0\,.$$

N.B. the graph is a curved curve and not a straight line as in Mohr-Coulomb or Drucker-Prager



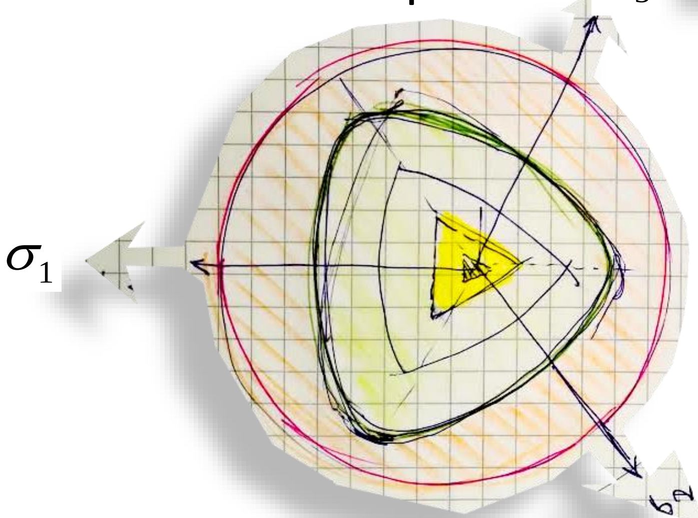
σ_c – uniaxial compression test, failure stress

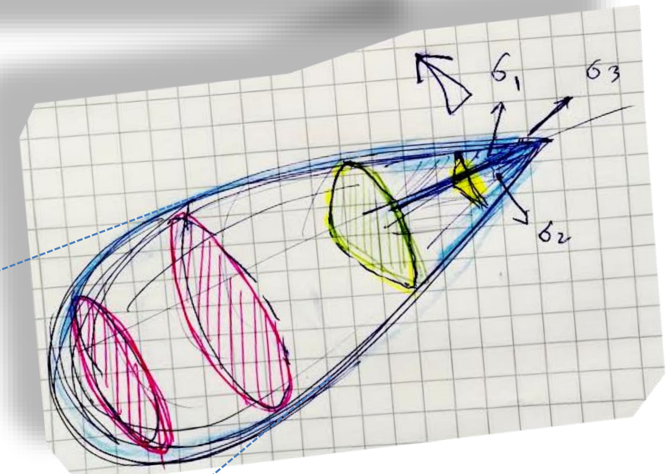
Recommended reading: **State-of-the-art report**, bulletin 45. 2008, task group 4.4. *fib* CEB-FIP – ***Practitioner's guide to finite element modelling of reinforced concrete structures***

Ottosen 4-parameters criterion:

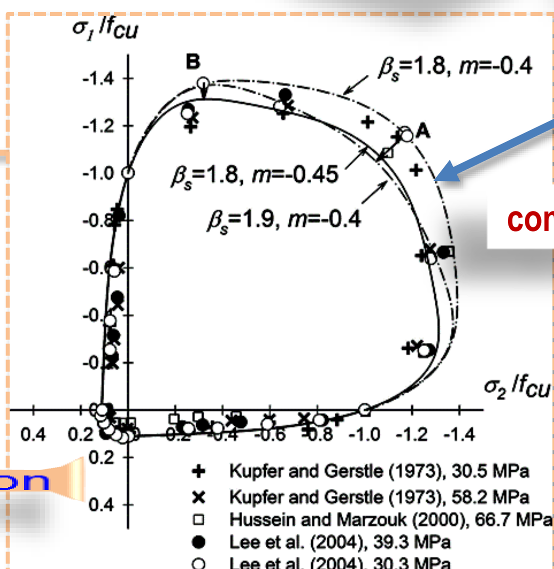
$$A\frac{J_2}{\sigma_c} + \Lambda\sqrt{J_2} + BI_1 - \sigma_c = 0,$$

Deviatoric planes





NB the asymmetry in tension and **compression**



compression

tension

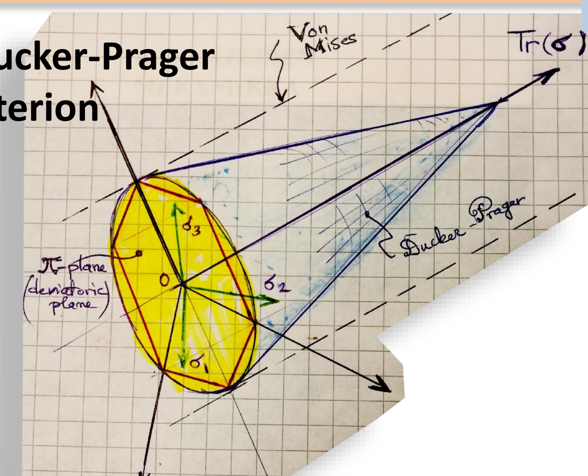
Concrete – plane stress

The DP-criterion does not see such asymmetry!

Drucker-Prager criterion

I_1

$\sigma_1 = \sigma_2 = \sigma_3$



The Ottosen 4-parameter General model

Ref: the Ottosen *et al.* textbook (reading elective assignment)

Good to know

4-parameter criterion was proposed by Ottosen (1977)

$$\boxed{A\frac{J_2}{\sigma_c^2}+\lambda\frac{\sqrt{J_2}}{\sigma_c}+B\frac{I_1}{\sigma_c}-1=0} \quad 8.62$$

$$\lambda=\begin{cases} K_1\cos[\frac{1}{3}\arccos(K_2\cos 3\theta)] & \text{if} \quad \cos 3\theta\geq 0 \\ K_1\cos[\frac{\pi}{3}-\frac{1}{3}\arccos(-K_2\cos 3\theta)] & \text{if} \quad \cos 3\theta\leq 0 \end{cases}$$

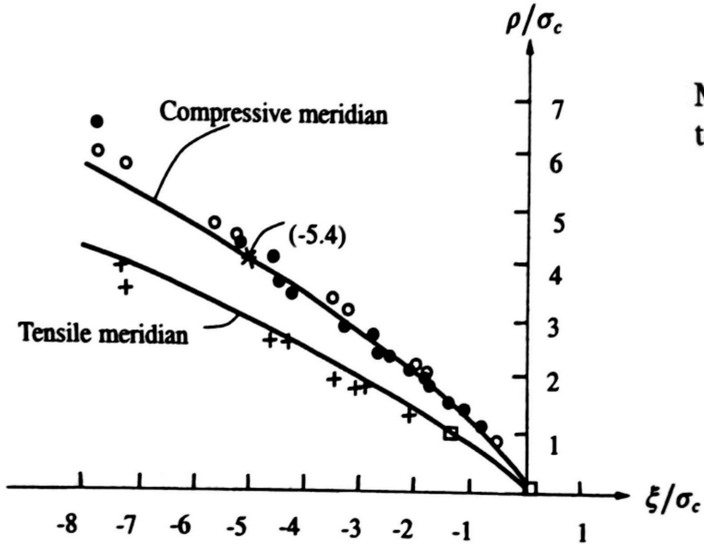
$$A\geq 0\,; \quad B\geq 0\,; \quad K_1\geq 0\,; \quad 0\leq K_2\leq 1$$

Moreover, (8.62) shows that the meridians are smooth, convex and curved and to obtain further insight we may solve (8.62) to obtain

Eq. 8.62 in the *meridian plane*

$$\frac{\sqrt{J_2}}{\sigma_c}=\frac{1}{2A}\left[-\lambda+\sqrt{\lambda^2-4A(B\frac{I_1}{\sigma_c}-1)}\right]$$

Experimental validation

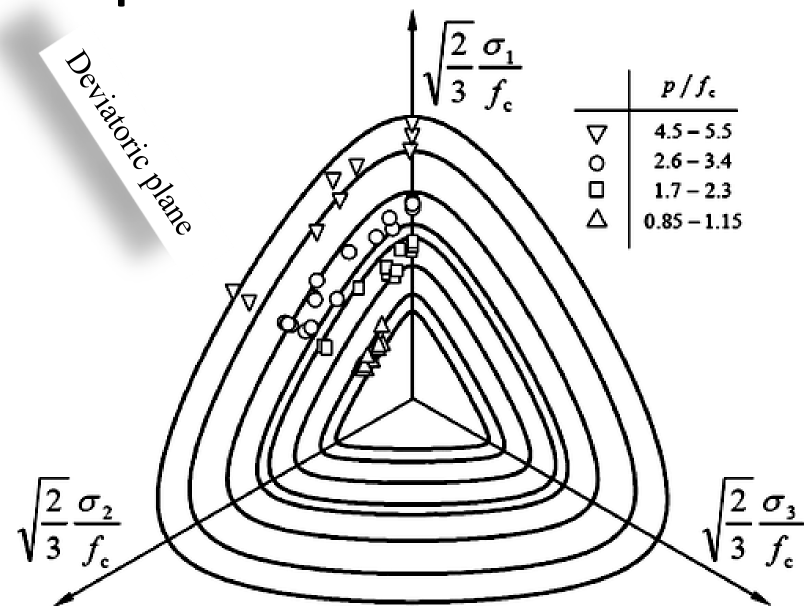


4-parameter criterion compared with experimental data in the meridian plane. Along compressive meridian: Balmer (1949)○, Richart *et al.* (1928)●; along tensile meridian: Richart *et al.* (1928)+, Kupfer *et al.* (1969)□. Moreover, $\sigma_t=0.10\sigma_c$ is assumed.

The idea in this slide is to show experimental evidence of the existence of a **failure surface**

Experimental validation

Deviatoric plane

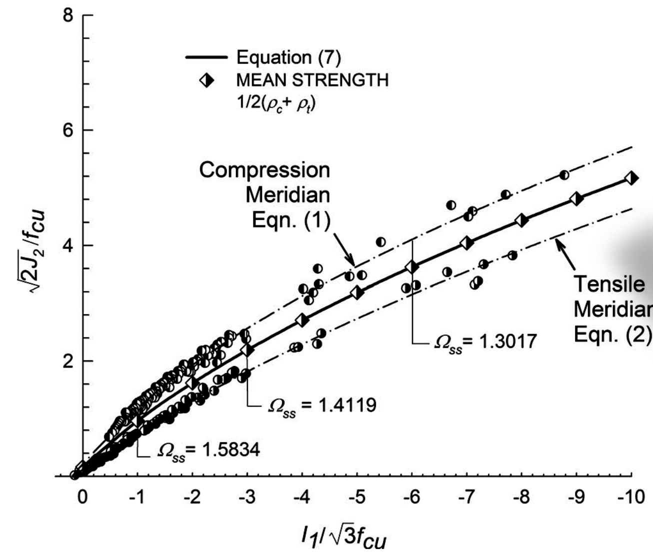


Comparison of the experimental results by Guo and Wang 1991 and the modeling results by using the proposed criterion p is held constant

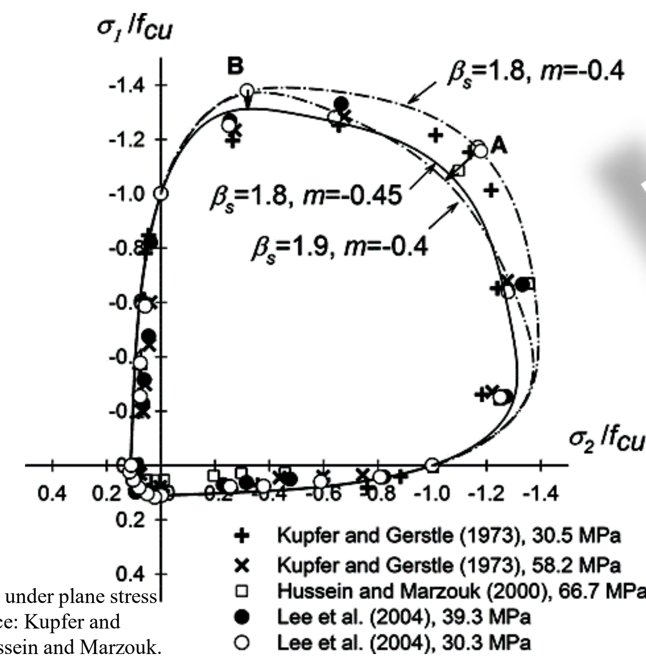
Ref: *J. Eng. Mech.*, 2010, 136(1): 51-59

Nonlinear Unified Strength Criterion for Concrete under Three-Dimensional stress States. Xiu-Li Du; De-Chun Lu; Qiu-Ming Gong; and Mi Zhao

Mean concrete strength from compression and tension data



Meridian plane



plane stress

Concrete strength under plane stress Conditions. Source: Kupfer and Gerstle 1973; Hussein and Marzouk. 2000; Lee et al. 2004

Ref: Simple Single-Surface Failure Criterion for Concrete. Sean D. Hinchberger

J. Eng. Mech., 2009, 135(7): 729-732

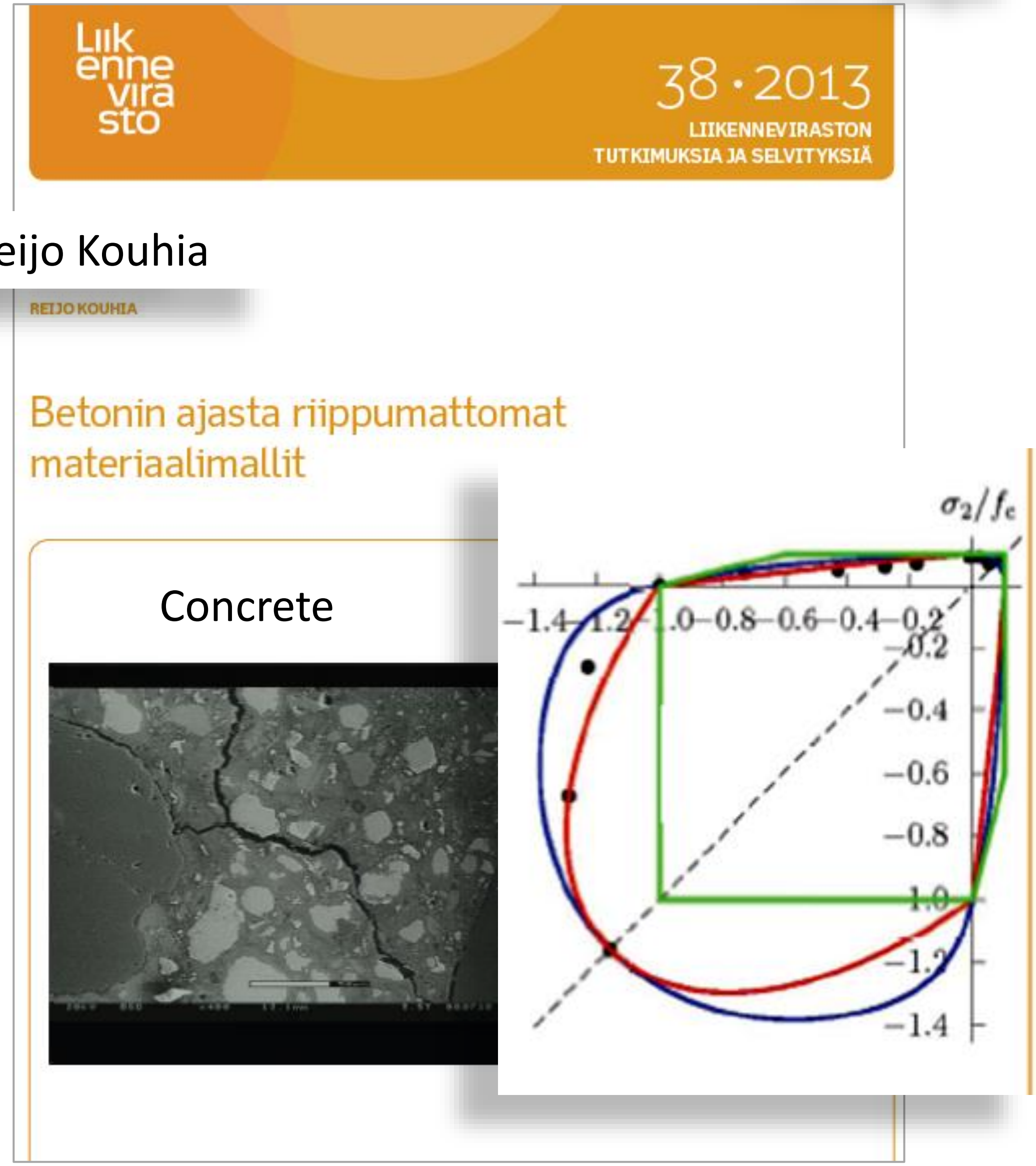
Concrete Material models (**in Finnish**)

Time-independent material models for concrete

Good to know

Recommended reading

By Prof. Reijo Kouhia



Recommended reading for future and actual engineers in structures:

An extensive detailed review on existing material models for concrete for time-independent behavior

Hoek–Brown failure criterion **Good to know**

The Hoek–Brown failure criterion is used widely in **mining engineering** design.

The **Hoek–Brown failure criterion** is an **empirical** stress failure criterion (surface) used in rock mechanics to predict the failure of rock.

Updated of the criterion (2002) exists and it includes improvements accounting for geological strength index (***GSI***), and may be also to include the effect of the missing stress component σ_2 . [*This criteria –* *Have two parameters only – it has some serious shortcomings (flaws) and it needs more improvements, work is going on.* *Cf.* with Ottosen 4-parameter model]

The basic idea of the Hoek–Brown criterion was to start with the properties *intact rock* and to add factors to reduce those properties because of the ex… k mass)

This criterion was conceived based on Hoek's experiences with brittle rock failure and his use of a parabolic Mohr envelope derived from Griffith's crack theory [1].

In its original form:

Failure criteria

In terms of principal effective stresses:

$$-\sigma_1 + \sigma_3 + \sqrt{A\sigma_3 + B^2} = 0$$

$\sigma_i, i = 1,2,3$ ***- effective stresses***

A, *B* - materials constants

$$A = A(GSI,...), \quad B = B(GSI,...)$$

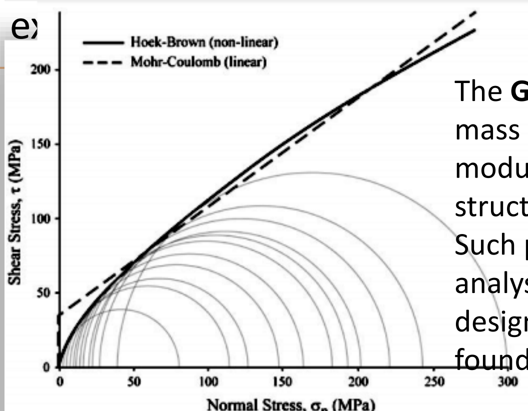
In terms of maximun shear stress and mean normal stresses:

$$\tau_m - \frac{1}{2}\sqrt{A(\sigma_m - \tau_m) + B^2} = 0$$

$$\tau_m = \frac{1}{2}(\sigma_1 - \sigma_3), \quad \sigma_m = \frac{1}{2}(\sigma_1 + \sigma_3)$$

σ_2 is assumed not 'affecting' the criterion

Good to know: attend courses in rock mechanics to know more …



The **GSI** characterizes for *ex*. rock-mass strength and deformation modulus (GSI-system describes rock structure and surface conditions). Such properties are needed in analysis in rock engineering for designing tunnels, slopes or foundations in rocks

Rock and rock mass

GSI …Decrease of interlocking of rock pieces

[1] Ref: Rock Mech Rock Eng (2012) 45:981–988
DOI 10.1007/s00603-012-0276-4

Ref: Catrin Edelbro, Slides of Licentiate Thesis , *Evaluation of rock mass strength criteria*

The GSI - Index

Good to know

Failure criteria

$$-\sigma_1+\sigma_3+\sqrt{A\sigma_3+B^2}=0$$
$$A=A(GSI,...),\; B=B(GSI,...)$$

Reference:

Evert Hoek

Rock mass properties for underground mines

Published in *Underground Mining Methods: Engineering Fundamentals and International Case Studies.* (Edited by W. A. Hustrulid and R. L. Bullock), Litleton, Colorado: Society for Mining, Metallurgy, and Exploration (SME) 2001

For intact rock:

Evert Hoek

$$\sigma_1=\sigma_3+\sigma_{ci}\sqrt{m_i\frac{\sigma_3}{\sigma_{ci}}+1}$$

$\sigma_i, i=1,2,3$ *- effective stresses*

σ_{ci} – *uniaxial compressive strength for intact rock elements*

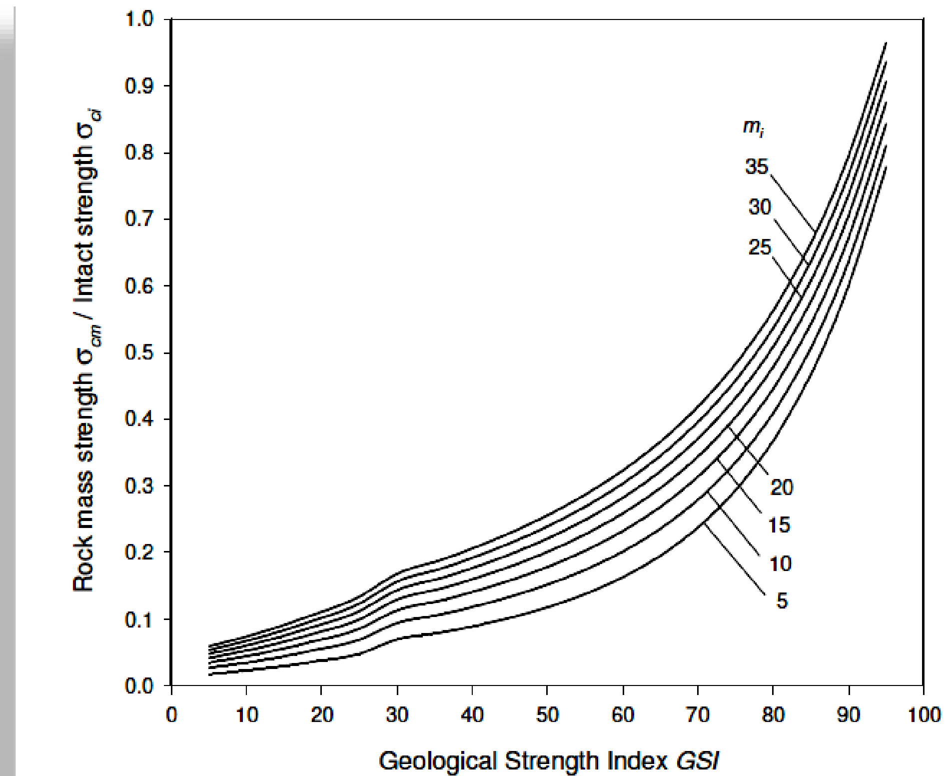
m_i – *Hoek-Brown constant for intact rock elements*

For joined rock mass:

$$\sigma_1=\sigma_3+\sigma_{ci}\sqrt{m_b\frac{\sigma_3}{\sigma_{ci}}+s}$$

Material constants for the rock mass

Example of *empiric relations*



Ex. Ratio of uniaxial compressive strength of rock mass to intact rock versus Geological Strength Index **GSI** (depth > 30 m)

$$m_b=m_i\exp[\frac{GSI-100}{28}]$$

$$s=\exp[\frac{GSI-100}{9}]$$

Abaqus – theory Manuel …to be able to read it, at least.

SIMULIA
ABAQUS 6.14
THEORY GUIDE

ISOTROPIC ELASTO-PLASTICITY

The elasticity can be written in volumetric and deviatoric components as follows.
Volumetric:

$$p = -K\varepsilon_{vol},$$

where

$$p = -\frac{1}{3}\,\mathrm{trace}(\boldsymbol{\sigma})$$

is the equivalent pressure stress.
Deviatoric:

$$\mathbf{S} = 2G\,\mathbf{e}^{el},$$

where $\mathbf{S}$ is the deviatoric stress,

$$\mathbf{S} = \boldsymbol{\sigma} + p\mathbf{I}.$$

The flow rule is

$$d\mathbf{e}^{pl} = d\bar{e}^{pl}\,\mathbf{n},$$

$$\mathbf{n} = \frac{3}{2}\frac{\mathbf{S}}{q},$$

$$q = \sqrt{\frac{3}{2}\mathbf{S}:\mathbf{S}},$$

) equivalent plastic strain rate.
uires that the material satisfy a uniaxial-str
dependent, this is the yield conditi

$$q = \sigma^0,$$

d is defined by the

$$f(\boldsymbol{\sigma}) = q,$$

$$q = \sqrt{\frac{3}{2}\mathbf{S}:\mathbf{S}},$$

ss:

$$\mathbf{S} = \boldsymbol{\sigma} - \frac{1}{3}\mathrm{trace}(\boldsymbol{\sigma})\mathbf{I} = \boldsymbol{\sigma} - \frac{1}{3}\mathbf{I}\mathbf{I}:(\boldsymbol{\sigma}).$$

circle in the plane normal to the hydrostatic axis in

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} = \frac{1}{q}\frac{3}{2}\mathbf{S},$$

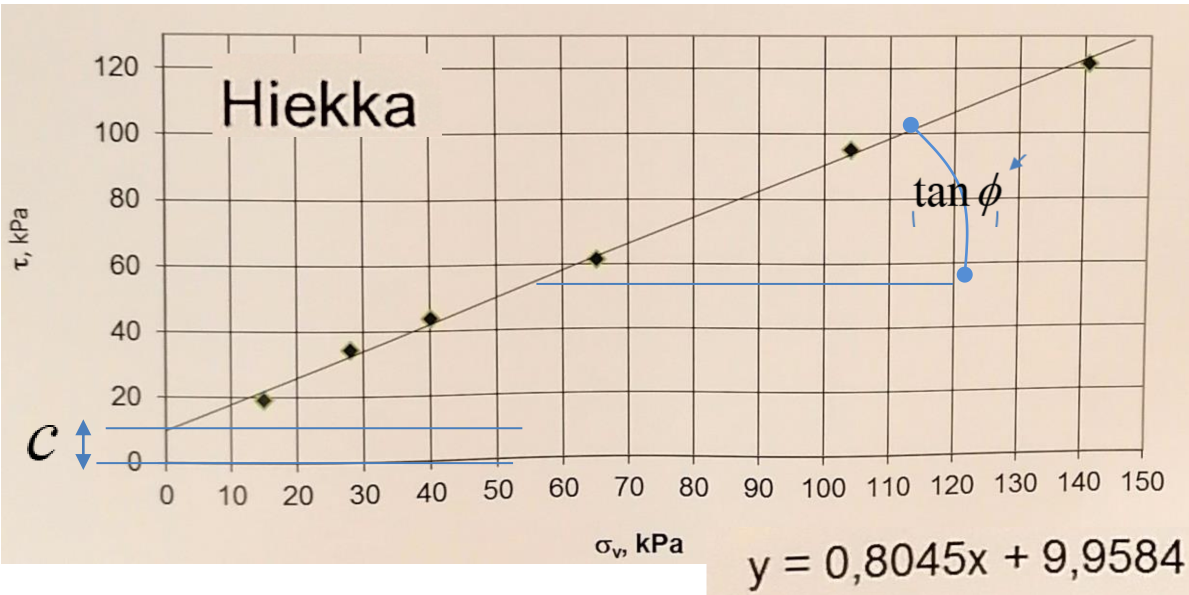
$$\frac{\partial^2 f}{\partial\boldsymbol{\sigma}\partial\boldsymbol{\sigma}} = \frac{1}{q}\left(\frac{3}{2}\mathfrak{I} - \frac{1}{2}\mathbf{I}\mathbf{I} - \frac{\partial f}{\partial\boldsymbol{\sigma}}\frac{\partial f}{\partial\boldsymbol{\sigma}}\right)$$

urth-order tensor.
unction … of the Mises function to allow

$$\sigma_e^2 = \frac{3}{2}s_{ij}:s_{ij} = \frac{3}{2}\mathbf{s}:\mathbf{s} \equiv 3J_2$$

Mohr-Coulomb yield criterion

Experimental result from a shear box with sand
(ref. 1st floor, soil mechanics lab, Raxa, 2021)



σ	τ
28,1	34,34
103,9	95,26
65,2	61,54
140,5	121,62
40,1	43,85
15	19,16

Kohde:	Hannan hiekka
Raekoko:	0,25-0,50 mm
k	1
A(0)	35,65 cm2
V	142,64
Ominaistilavuus :	1,65

shear stress — angle of internal friction

on failure one have: $$\boxed{|\tau| = c - \sigma_n \underbrace{\tan\phi}_{\equiv \mu}}$$

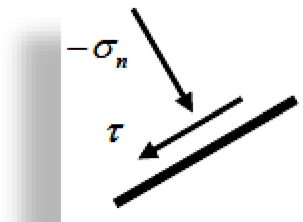
cohesion — normal stress — sliding friction coefficient

Mohr-Coulomb (failure) criterion is based on Coulomb's 1773 friction equation (Mohr 1900)

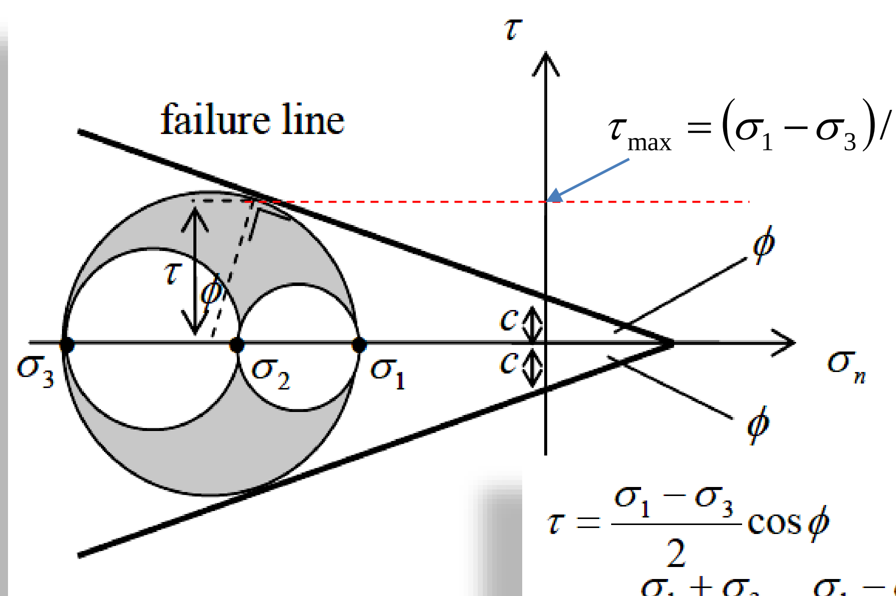
shear stress — angle of internal friction

on failure one have: $\boxed{|\tau| = c - \sigma_n \underbrace{\tan\phi}_{\equiv \mu}}$

cohesion — normal stress — sliding friction coefficient



Coulomb friction over a plane



Mohr-Coulomb failure criterion

The radius of the large Mohr circle is $(\sigma_1 - \sigma_3)/2$

Thus the maximum shear stress is $\tau_{\max} = (\sigma_1 - \sigma_3)/2 \cdot \cos\phi$

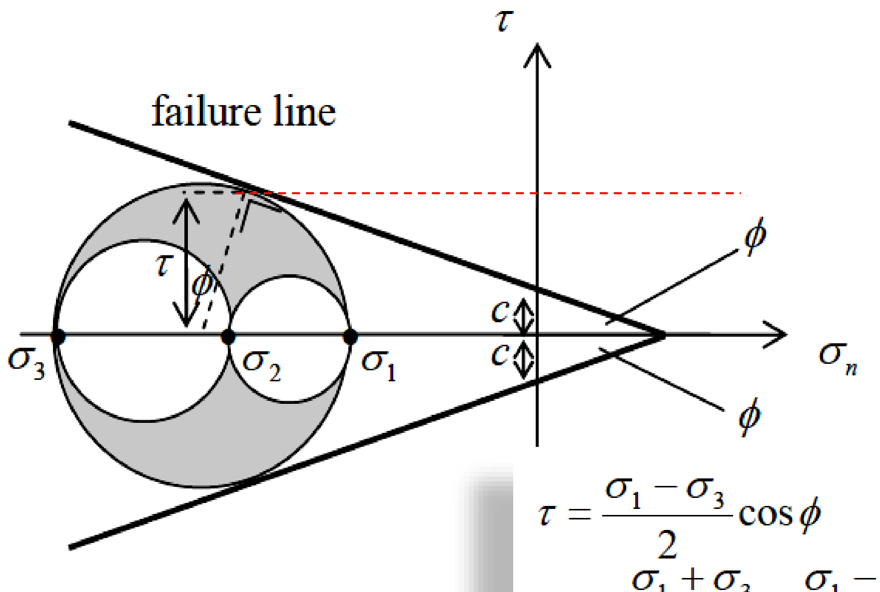
$$\tau = \frac{\sigma_1 - \sigma_3}{2}\cos\phi$$

$$\sigma_n = \frac{\sigma_1 + \sigma_3}{2} + \frac{\sigma_1 - \sigma_3}{2}\sin\phi$$

Mohr-Coulomb criterion:

$$(\sigma_1 - \sigma_3) = 2c\cos\phi - (\sigma_1 + \sigma_3)\sin\phi$$

$$\max_{i \neq j}\left[\left|\sigma_i - \sigma_j\right| + \left(\sigma_i + \sigma_j\right)\sin\phi\right] = 2\cos\phi$$

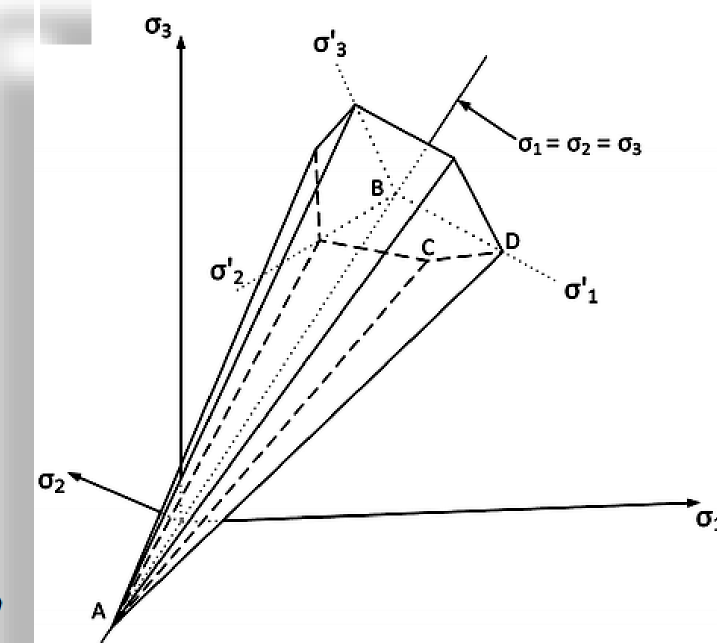


Mohr-Coulomb failure criterion

$$\tau = \frac{\sigma_1 - \sigma_3}{2}\cos\phi$$

$$\sigma_n = \frac{\sigma_1 + \sigma_3}{2} + \frac{\sigma_1 - \sigma_3}{2}\sin\phi$$

A hexagonal pyramid



Mohr-Coulomb criterion:

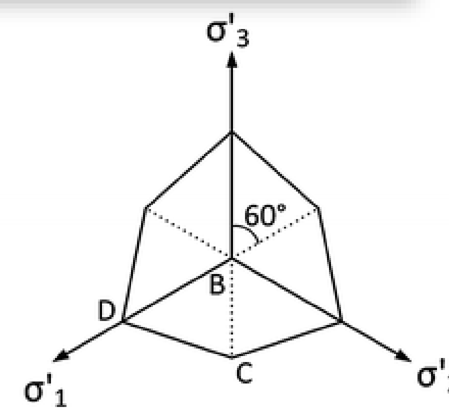
$$(\sigma_1 - \sigma_3) = 2c\cos\phi - (\sigma_1 + \sigma_3)\sin\phi$$

In terms of principle stresses:

$$\max_{i\neq j}\left[\left|\sigma_i - \sigma_j\right| + (\sigma_i + \sigma_j)\sin\phi\right] = 2\cos\phi$$

Family of Mohr-Coulomb criteria:

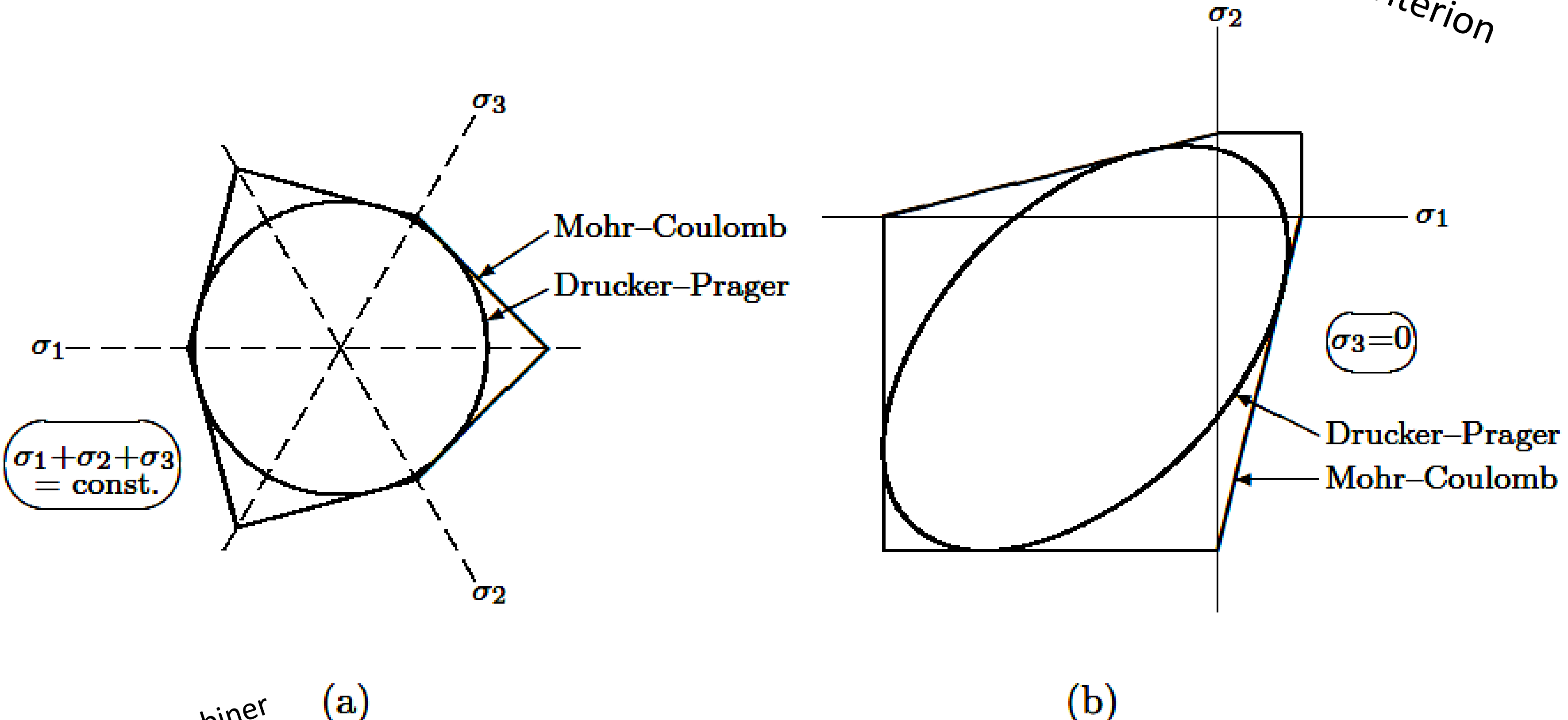
$$F(J_2, J_3) = c - \lambda I_1$$



Comparison

Drucker-Prager criterion

Mohr-Coulomb yield criterion



Ref: Figures from Lubiner

Mohr–Coulomb and Drucker–Prager criteria: (a) plane parallel to π-plane; (b) plane stress.

N.B. ***Good to know:*** For Mohr-Coulomb and Drucker-Prager (failure, yield) surfaces applying a pure hydrostatic pressure will not affect yield since such surfaces are ***open***. However, many geomaterials, as for instance soils, a large enough hydrostatic pressure will induce permanent deformation. For such cases, a ***closed*** *(capped) yield surface* is more appropriate. Such capped yield surface can be the modified *Cam-Clay criterion*… to learn more of this subject, please join the courses with these topics of geotechnics and soil mechanics, with prof. Woiteck.

Cam-Clay criterion: $3J_2 = -\frac{1}{3} I_1 M^2 \left(2p_c + \frac{1}{3} I_1\right)$

If you got interested →
join courses of
geotechnics …

Good to know

Material Behavior of Clay and Silt *in Otaniemi*

Yield Functions :

Cam-Clay

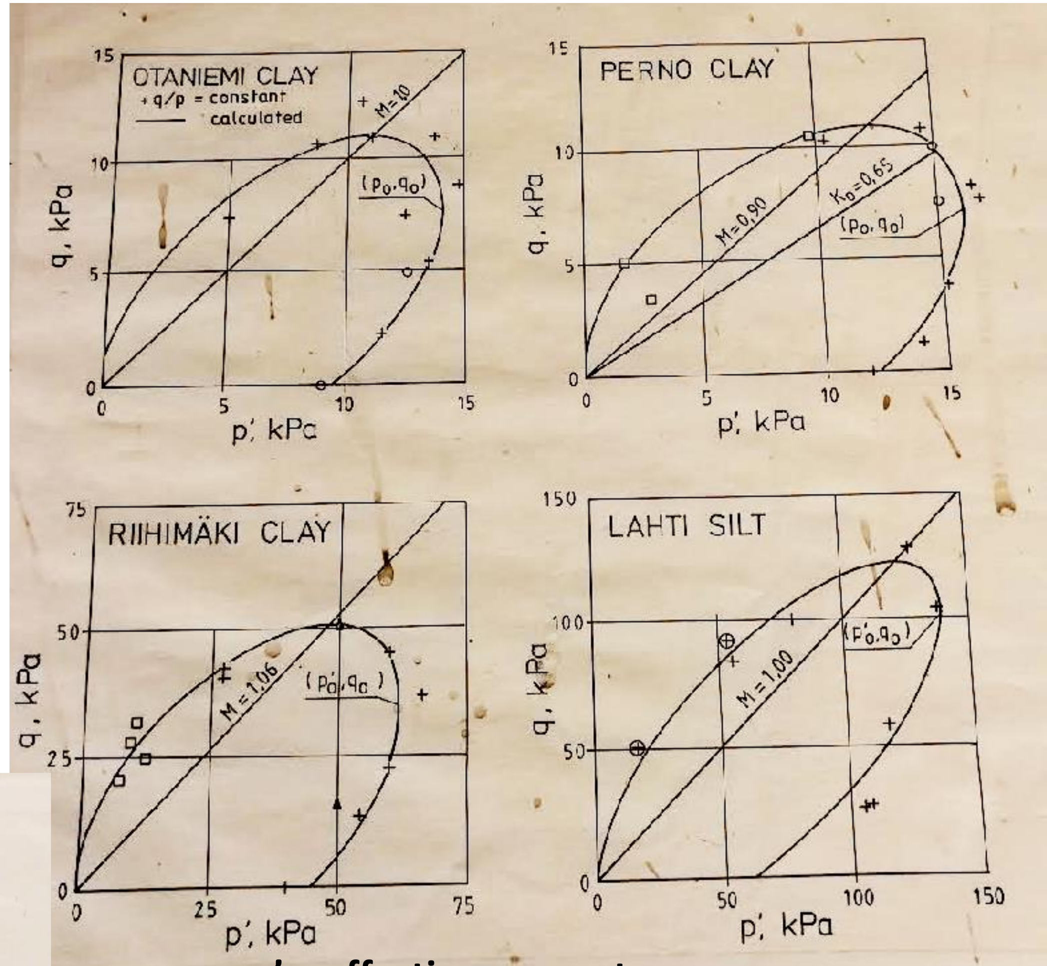
$$q + Mp' \ln\left(\frac{p'}{p'_o}\right) = 0$$

Modified Cam-Clay

$$\frac{q^2}{p'^2} + M^2\left(1 - \frac{p'_o}{p'}\right) = 0$$

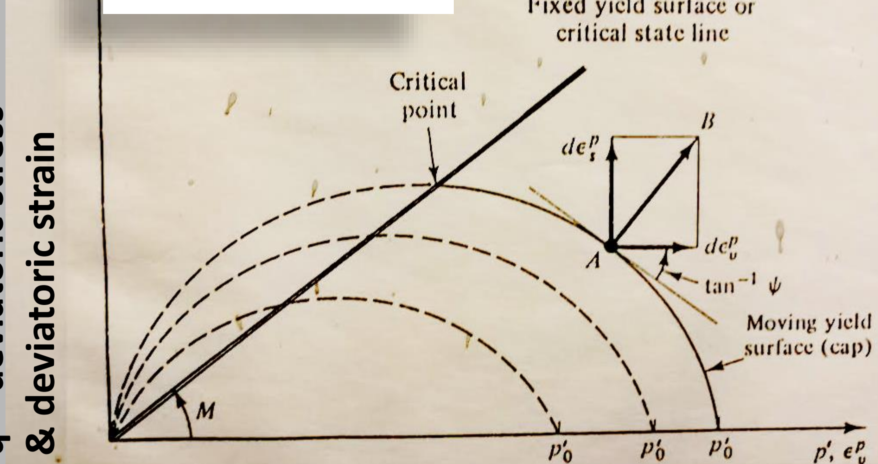
NB. In soils, *both incremental plastic volumetric strain* and *incremental plastic shear strain* components **exist**

q = deviatoric stress



p' = effective mean stress
(**tr**(sigma) - water pore pressure)

q = deviatoric stress & deviatoric strain



p' = effective mean stress & volumetric strain

The Cam-Clay model is assumes the soil isotropic, elasto-plastic, and it is not affected by creep (time-independent plasticity). **The yield surface** of the Cam clay model is described by a log arc (original Cam-Clay model) or an ellipse (modified Cam-Clay model).

Such model explain well the pressure-dependent soil strength and the volume change (contraction and dilation) of clayey soils during shear.

Ref: Photos by DBA from geotechnics lab ☺

Animation industry – example of use of Drucker-Prager elasto-plasticity model

Drucker-Prager Elastoplasticity for Sand Animation

Gergely Klár Theodore Gast Andre Pradhana Chuyuan Fu Craig Schroeder Chenfanfu Jiang Joseph Teran
University of California, Los Angeles

Pause: reading

..., we use Drucker-Prager elastoplasticity [Drucker and Prager 1952]. The elastic part of this relation is ex- ... F.

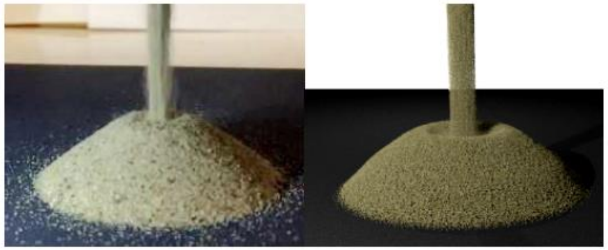
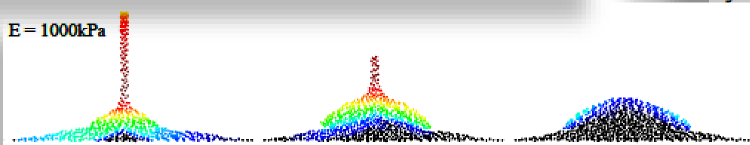


Figure 12: *Sand is poured from a spout into a pile in a lab (left) and with our method (right).*



Constitutive model. A consti... stress is needed. The text of B... ful background for elastoplast... Mast el al. [2013; 2014], we... [Drucker and Prager 1952]. T... pressed through the deformati...

For perfectly hyperelastic mat... fined through the potential en... the initial s... $\mathbf{F} = \mathbf{F}^E\mathbf{F}^P$ city, there wi... potential w... yond this state. In this case, th...

$$\sigma = \frac{\ }{\det}$$

$\psi(\mathbf{F}^E)$ is the elastic e... non-rigid $\mathbf{F}^E$ (see Section 6.3

Abstract

We simulate sand dynamics using an elastoplastic, continuum assumption. We demonstrate that the Drucker-Prager plastic flow model combined with a Hencky-strain-based hyperelasticity accurately recreates a wide range of visual sand phenomena with moderate computational expense. We use the Material Point Method (MPM) to discretize the governing equations for its natural treatment of contact, topological change and history dependent constitutive relations. The Drucker-Prager model naturally represents the frictional relation between shear and normal stresses through a yield stress criterion. We develop a stress projection algorithm used for enforcing this condition with a non-associative flow rule that works naturally with both implicit and explicit time integration. We demonstrate the efficacy of our approach on examples undergoing large deformation, collisions and topological changes necessary for producing modern visual effects.

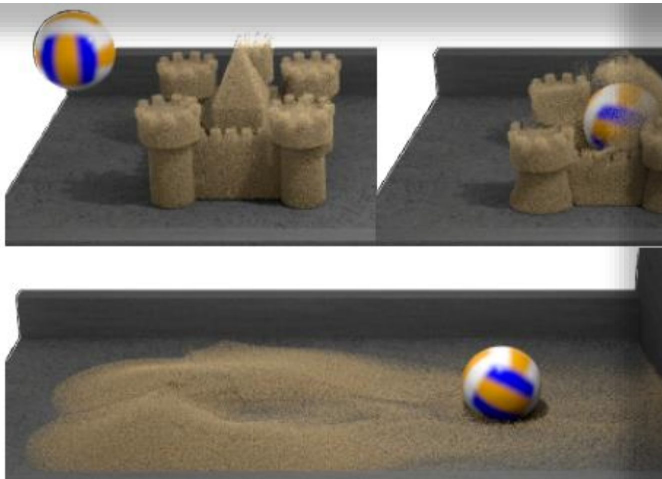


Figure 3: *A sand castle is hit with a deformable ball while ... The sand and ball are fully coupled in the simulation.*

With the Drucker-Prager model, frictional interactions between grains of sand can be expressed in the continuum via a relation between shear and normal stresses. Using a Coulomb friction model, shear stresses resisting sliding motions between grains can only be

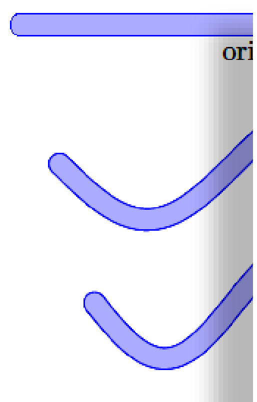


Figure 6: *Relationship between* $\mathbf{F}$, $\mathbf{F}^E$, *and* $\mathbf{F}^P$.

6.3 Constitutive model

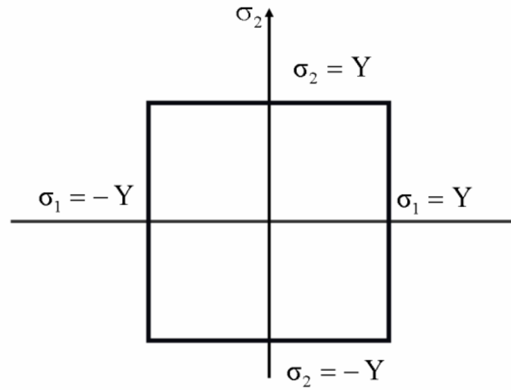
We adopt the energy density $\psi(\mathbf{F})$ from Mast et al. [2013]. This model uses the same energy density as St. Venant-Kirchhoff, but it replaces the left Cauchy Green strain with the Hencky strain $\frac{1}{2}\ln(\mathbf{F}\mathbf{F}^T)$. This makes a number of aspects of the Drucker-Prager plastic projection very simple (see the supplementary technical document [Klár et al. 2016]). The model is most conveniently written in terms of the singular value decomposition $\mathbf{F} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T$ as

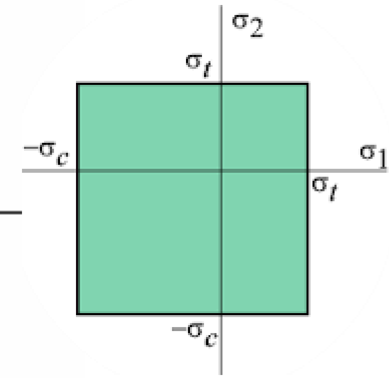
$$\psi(\mathbf{F}) = \mu \operatorname{tr}\left((\ln\boldsymbol{\Sigma})^2\right) + \frac{1}{2}\lambda(\operatorname{tr}(\ln\boldsymbol{\Sigma}))^2, \qquad (25)$$

Maximum Principle Stress Criteria (Rankine)

(Brittle) materials fails/ruptures/breaks when maximum principal stress in t $\sigma_{\max}$ specimen reaches some limit value (tensile k yield)

k - can be different in compression and tension





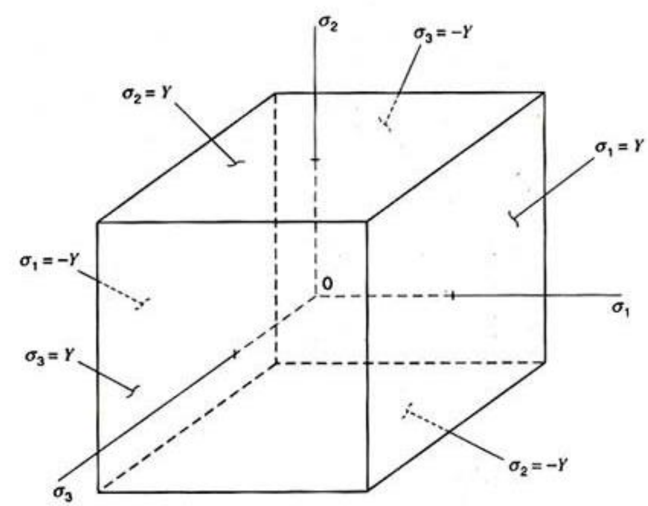
$$k \equiv Y \equiv \sigma_Y$$

Failure criteria:

$$F = \max\{|\sigma_1|, |\sigma_2|, |\sigma_3|\} - k = 0$$

$F < 0 \Rightarrow$ no yielding or no rupture

$F = 0 \Rightarrow$ onset of yielding or rupture



Applicable to brittle materials (mostly in tension)

The failure surface is called *tension-failure surface* or simply *tension cutoff.*

2.3.2. The Maximum-Tensile-Stress Criterion (Rankine)

The maximum-tensile-stress criterion of Rankine, dating from 1876, is generally accepted today to determine whether a tensile failure has occurred for a brittle material. According to this criterion, brittle failure takes place when the maximum principal stress at a point inside the material reaches a value equal to the tensile strength σ_0 as found in a simple tension test, regardless of the normal or shearing stresses that occur on other planes through this point. The equations for the failure surface defined by this criterion are

$$\sigma_1 = \sigma_0, \qquad \sigma_2 = \sigma_0, \qquad \sigma_3 = \sigma_0 \tag{2.168}$$

$$\sigma_1 = \sigma_0, \qquad \sigma_2 = \sigma_0, \qquad \sigma_3 = \sigma_0$$

which result in three planes perpendicular to the σ_1, σ_2, and σ_3 axes, respectively as shown in Fig. 2.20a. This surface will be referred to as the tension-failure surface or the simple *tension cutoff*. When the variables ξ, ρ, θ or I_1, J_2, θ are used, the failure surface can be fully described by the following equations within the range $0 \leq \theta \leq 60°$ using Eq. (2.123).

$$f(I_1, J_2, \theta) = 2\sqrt{3J_2}\cos\theta + I_1 - 3\sigma_0 = 0 \tag{2.169}$$

or identically

$$f(\xi, \rho, \theta) = \sqrt{2}\rho\cos\theta + \xi - \sqrt{3}\sigma_0 = 0 \tag{2.170}$$

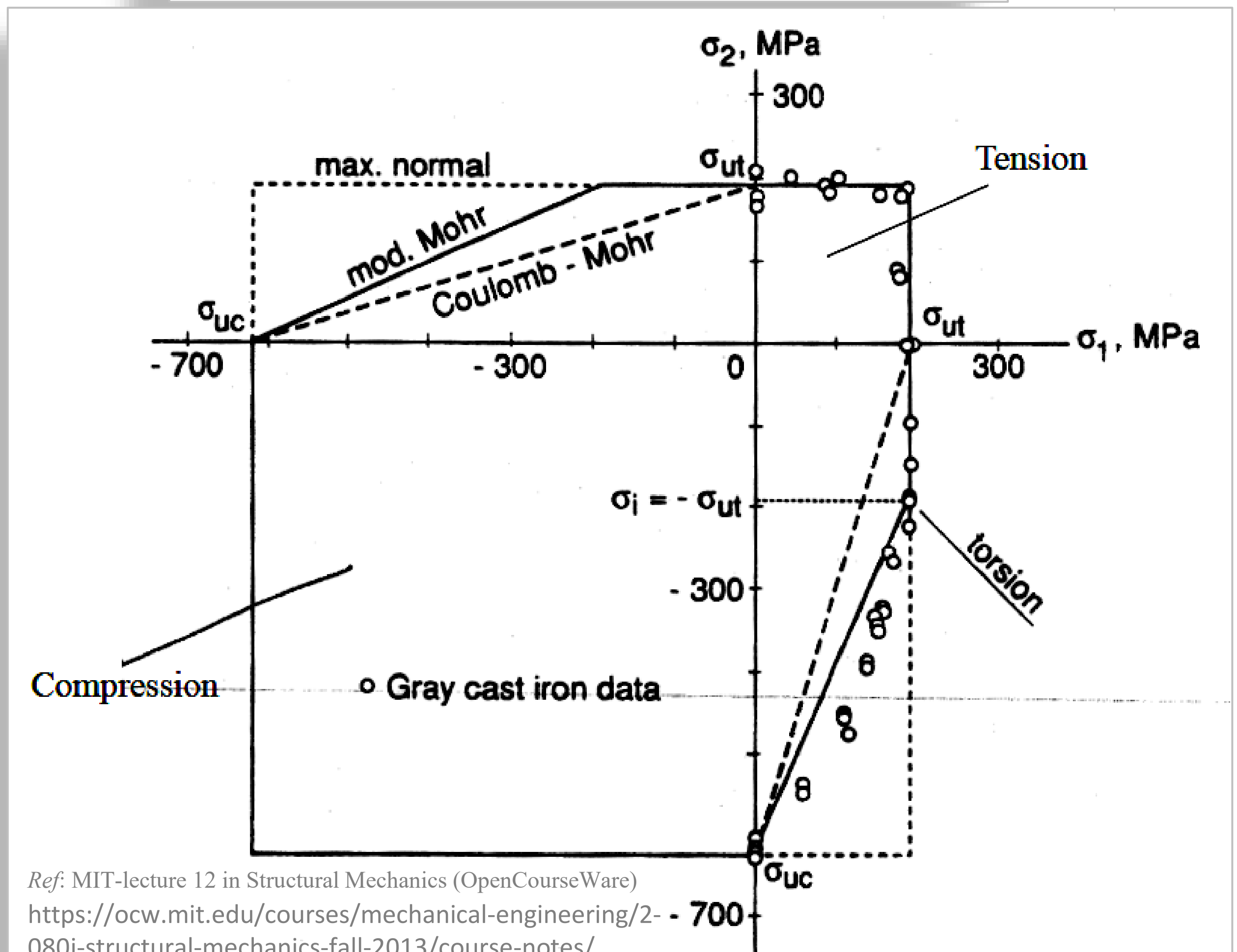
Figures (2.20b and c) show the cross-sectional shape on the π-plane ($\xi = 0$) and the tensile ($\theta = 0°$) and compressive ($\theta = 60°$) meridians of the failure surface.

As we know, some of the nonmetallic materials, such as concrete, rocks, and soils, have a good compressive strength. Under compression loading with confining pressure, this kind of material may even exhibit some ductile and shear failure behavior. Under tension loads, however, a brittle failure behavior with a very low tensile strength is generally observed. Hence, the Rankine criterion is sometimes combined with the Tresca or the von Mises criterion to approximate the failure behavior of such materials. The combined criteria are referred to as the Tresca or the von Mises criterion with a tension cutoff, and their graphical representations consist of two surfaces, corresponding to a combined behavior of shear failure in compression and tensile failure in tension. An example of such failure surfaces is shown in Fig. 2.21, in which the compressive strength is assumed three times as large as the tensile strength.

Experimental validation

Brittle or quasi-material as cast iron behave differently in tension than in compression (asymmetry). For such materials pressure or normal stress dependent criteria may be adequate. The maximum principal stress criteria is also adequate, *cf*. data bellow,

Biaxial fracture data of gray cast iron



Ref: MIT-lecture 12 in Structural Mechanics (OpenCourseWare)
https://ocw.mit.edu/courses/mechanical-engineering/2-080j-structural-mechanics-fall-2013/course-notes/

Anisotropic yield criteria

Anisotropic criteria

Hill's criterion :

orthotropic

$$f = \sqrt{\underline{\sigma} : \underline{\underline{H}} : \underline{\sigma}} - k(\ldots) \leq 0$$

$\underline{\underline{H}}$ = *4 rank tensor defined in the material principal directions*

$$\underline{\underline{H}} = \underline{\underline{I}}^d : \overline{\underline{\underline{H}}} : \underline{\underline{I}}^d$$

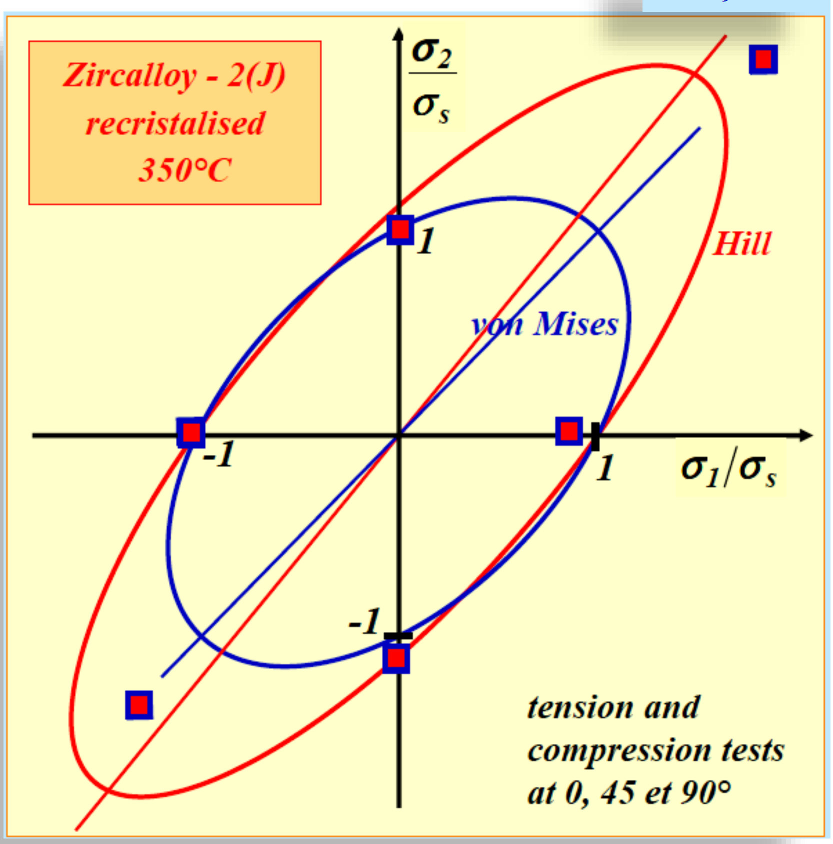
« isovolumic » case

$\underline{\underline{I}}^d$ = *deviatoric projector :*

$$I^d_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{1}{3}\delta_{ij}\delta_{kl}$$

Zirconium alloys (ZIRALLOY is a trademark by **Implant Manufacturing and Testing**) are alloys formed by zirconium or other metals, for instance few percent of tin, niobium, iron, chromium, nickel , etc. They are used in nuclear technology for their hardness, corrosion resistance, ductility and some other nuclear properties as regarded to low absorption of thermal neutrons, for instance. They have also biocompatibility properties and used for body implant. Etc.

Hill, 1951

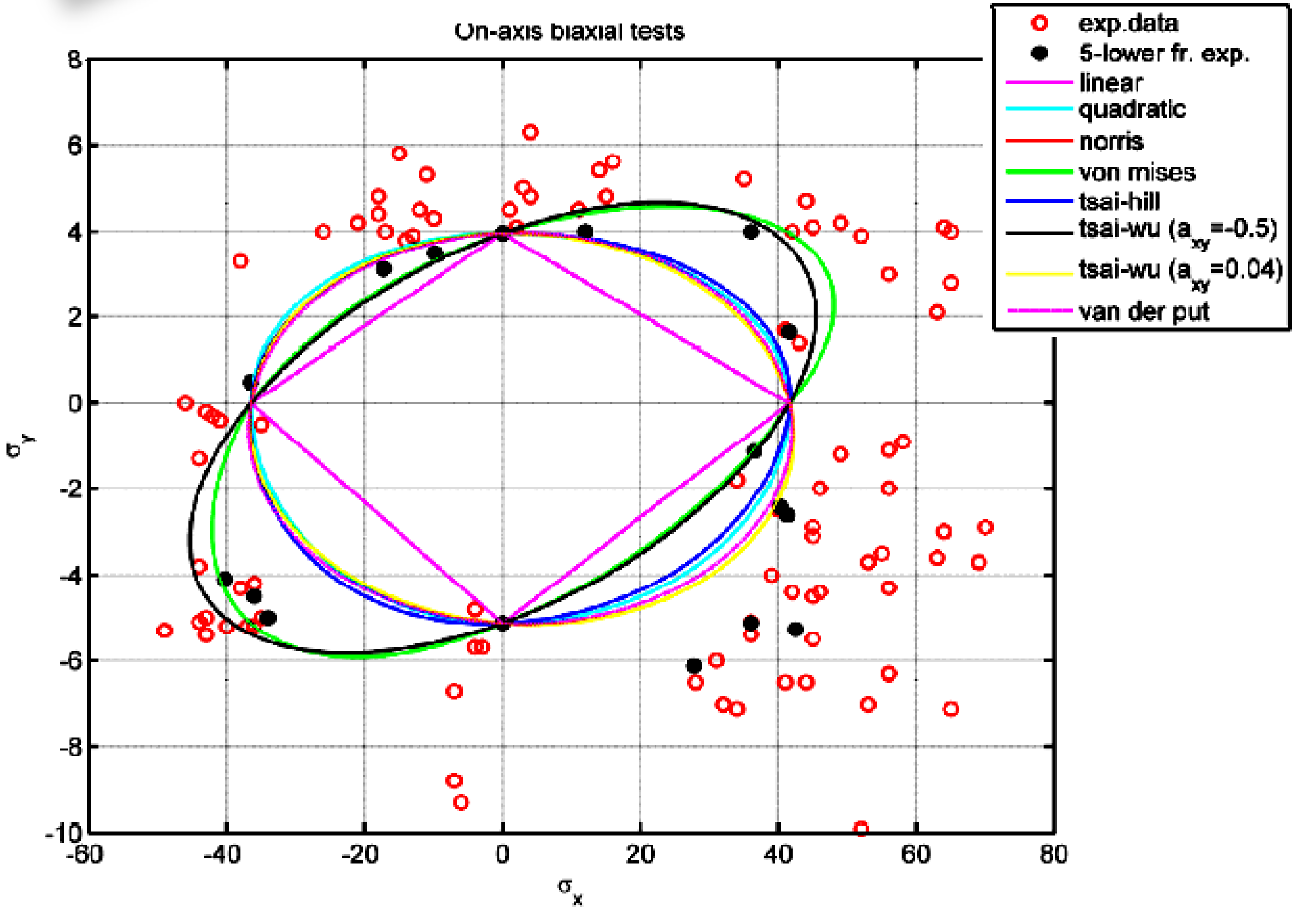


EVALUATION OF FAILURE CRITERIA IN WOOD MEMBERS

José M. Cabrero[1], Kifle G. Gebremedhin[2]

WCTE
WORLD CONFERENCE ON TIMBER ENGINEERING
2010

Example



Plot of the failure criteria with the on-axis experimental data from Eberhardsteiner (2002).

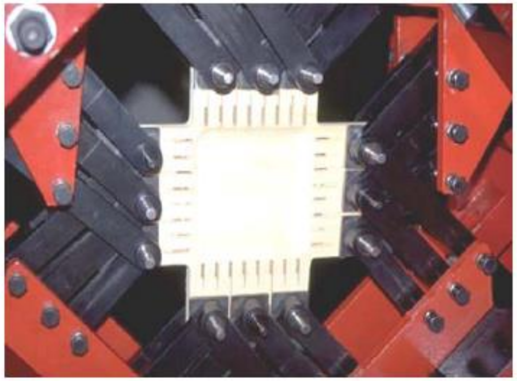
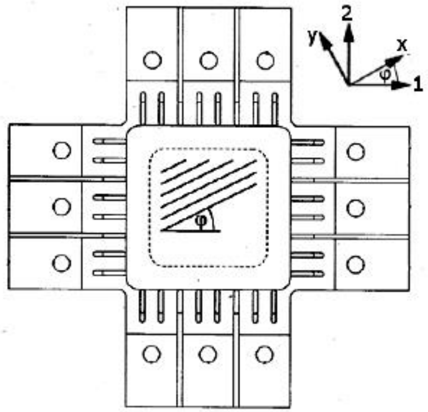
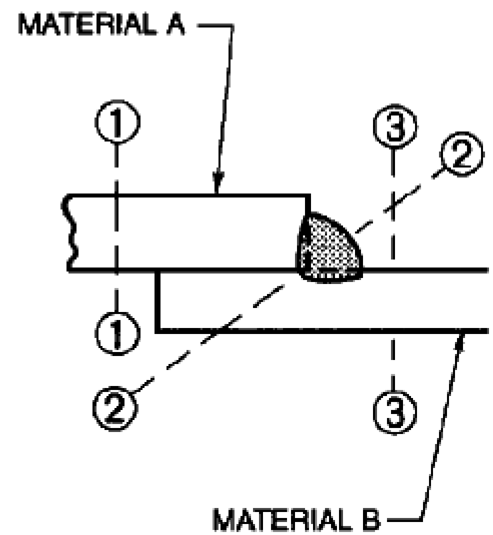


Figure 1. Experimental set up (taken from Eberhardsteiner, 2002).



2.1 Standards-Eurocodes

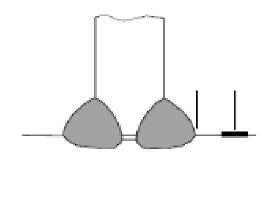
In Eurocodes, EN 1993-1-8 and EN 1993-1-12 are used in designing joints in high strength steels. Eurocode 3 part 1-12 extends the use of Eurocode 3 part1-8 in designing of joints for steel grades higher than S460 upto S700. It also allows the use of undermatched consumables and make the selection of filler materials against the strength of lower grade base material in the joint. Calculation of the design resistance of every component in a welded joint is recommended [1]. There are two methods to calculate the strength of fillet welded connections: the directional method and the simplified method [1]. In the directional method the carried forces are divided by the throat area and split into stress components parallel and transverse to the longitudinal axis of the weld and normal and transverse to the plane of its throat. The design resistance is calculated according to two conditions below;



$$\left[\sigma_{\perp}^{2}+3\left(\tau_{\perp}^{2}+\tau_{\parallel}^{2}\right)\right]^{0.5}\leq\frac{f_u}{\left(\beta_w\gamma_{M2}\right)} \tag{2.1}$$

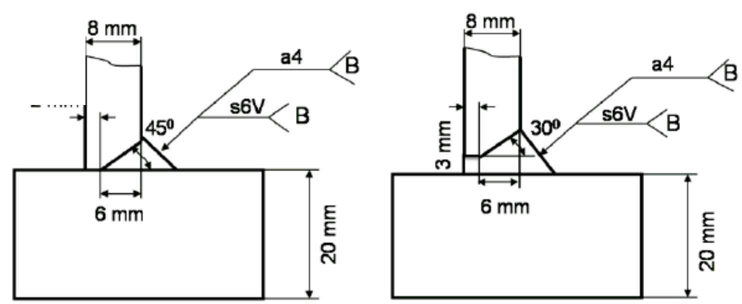
and

$$\sigma_{\perp}\leq\frac{0.9f_u}{\gamma_{M2}}. \tag{2.2}$$



where

fu - is the nominal ultimate tensile strength of the weaker part joined
β_w - is the appropriate correlation factor taken from Table 4.1 EN 1993 -1-8
$\sigma\perp$ - is the normal stress perpendicular to the throat
$\sigma_{\parallel}$ - is the normal stress parallel to the axis of the weld
$\tau\perp$ - is the shear stress (in the plane of the throat) perpendicular to the axis of the weld
$\tau_{\parallel}$ - is the shear stress (in the plane of the throat) parallel to the axis of the weld.



(a)75 % penetration (b)100 % penetration

I reproduce some failure criteria communly used for wood. This pdf-material is from the presentation in this course (2020) given by Joonas Jaaranen (reproduced with his permission)

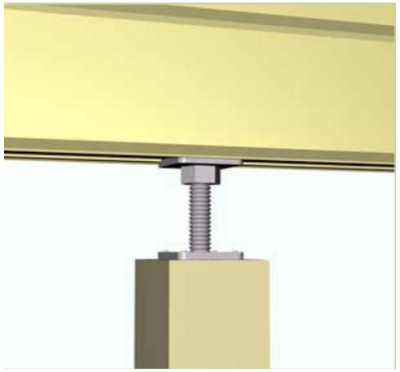
Failure criteria and material modelling of wood

Presentation on CIV-E4080 Material Modelling in Civil Engineering

Joonas Jaaranen, Aalto University

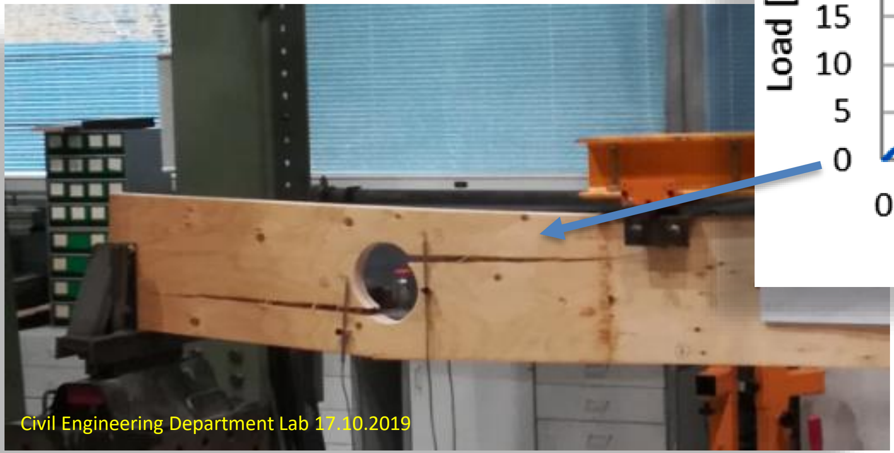
Motivation

Details with complex behaviour

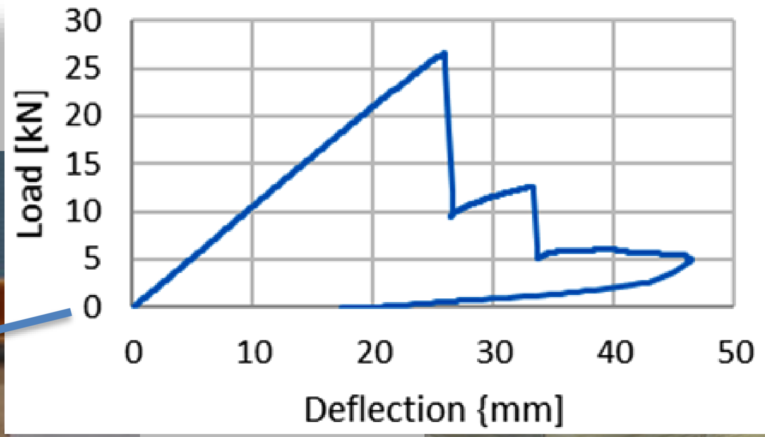


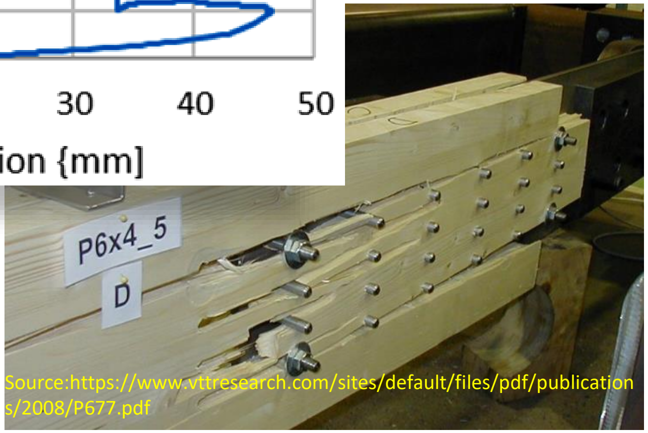
Source: RT 82-11168

Source: https://eurocodes.jrc.ec.europa.eu/doc/WS2008/EN1995_3_Dietsch.pdf



Civil Engineering Department Lab 17.10.2019





Source:https://www.vttresearch.com/sites/default/files/pdf/publications/2008/P677.pdf

Wood as material

- Naturally grown material
- Microscale: cellular structure
- Macroscale growth characteristics: annual rings, knots, spiral grain, compression wood,…

- Time- and moisture-dependent behaviour

- 'Ductile' like compressive failure behaviour
- Brittle tensile failure

- Strongly orientated properties

- Different strength in tension and compression

- Properties depend on wood species, growth conditions, density, moisture content, stress-history,…

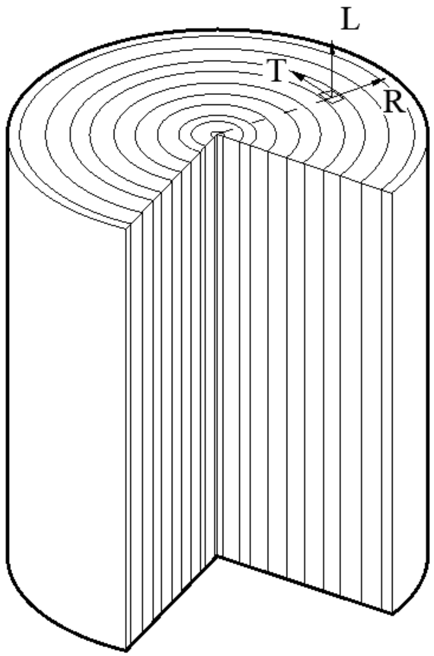
Wood as material

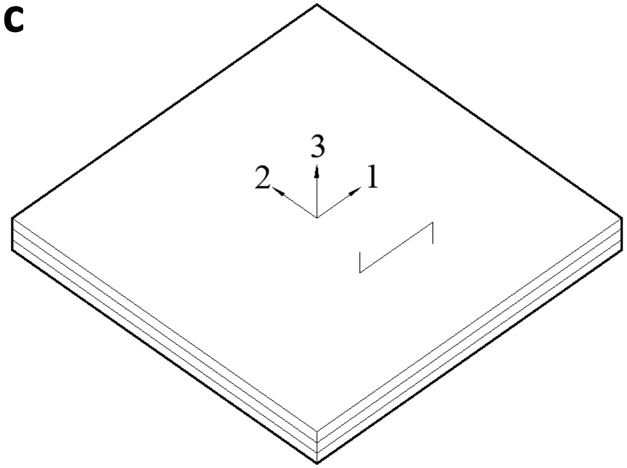
Common assumptions for modelling

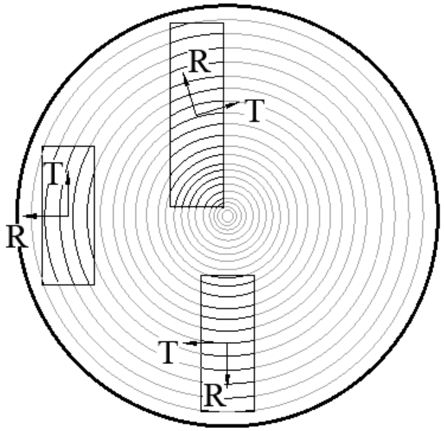
- **Homogeneity**
- **Orthotropy**
- **Grain orientation aligns with length-wise direction of the component**

Choices of material coordinate system

- **Cylindrical orthotrophic**
- **Cartesian orthotropic**
- **Transverse isotropic**







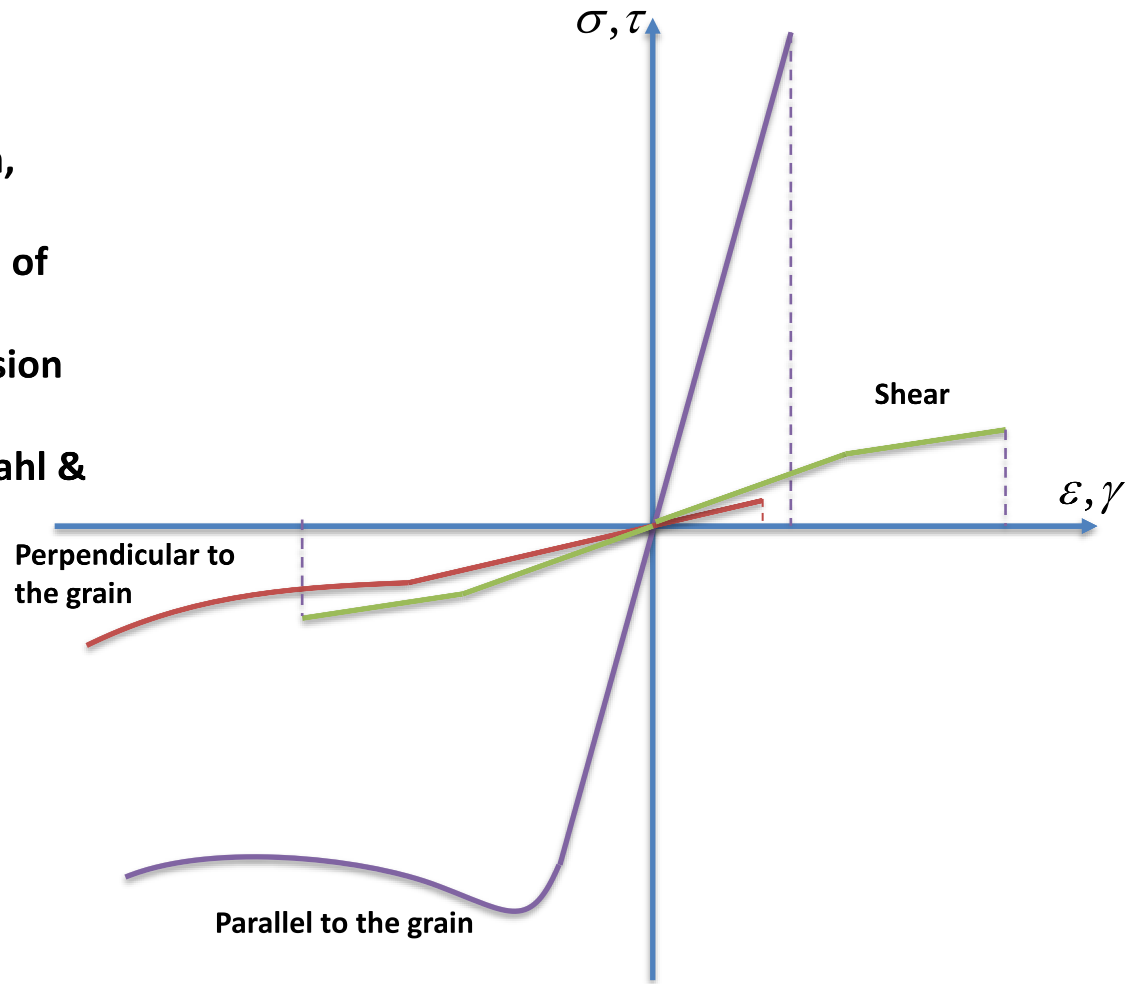
Wood as material

Material behaviour

- **Different behaviour in tension, compression and shear**
- **Dependent on the orientation of the load**
- **Example references: compression (Holmberg et al. 1998, Widehammar 2004), shear (Dahl & Malo 2009)**

$$\frac{f_{t,90}}{f_{t,0}} \approx \frac{1}{20} \dots \frac{1}{30}$$

$$\frac{f_{c,90}}{f_{c,0}} \approx \frac{1}{10}$$



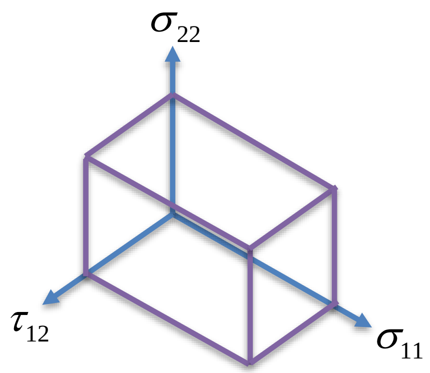
Common failure criteria

Basic criteria

- Common in practice
- Basis for many multi-surface criteria

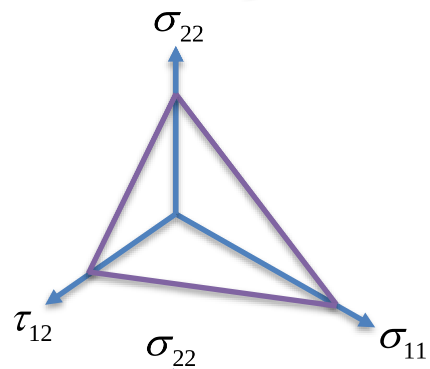
Maximum stress

$$\frac{\sigma_{11}}{f_1} = 1, \quad \frac{\sigma_{22}}{f_2} = 1, \quad \frac{\tau_{12}}{f_{12}} = 1$$



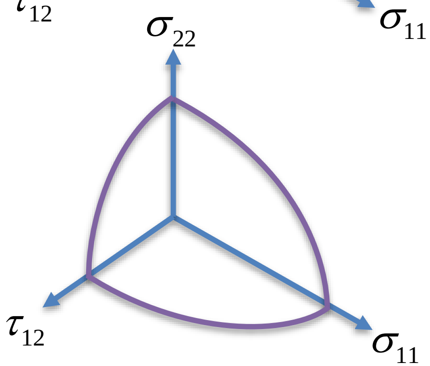
Linear interaction

$$\frac{\sigma_{11}}{f_1} + \frac{\sigma_{22}}{f_2} + \frac{\tau_{12}}{f_{12}} = 1$$



Quadratic interaction

$$\left(\frac{\sigma_{11}}{f_1}\right)^2 + \left(\frac{\sigma_{22}}{f_2}\right)^2 + \left(\frac{\tau_{12}}{f_{12}}\right)^2 = 1$$



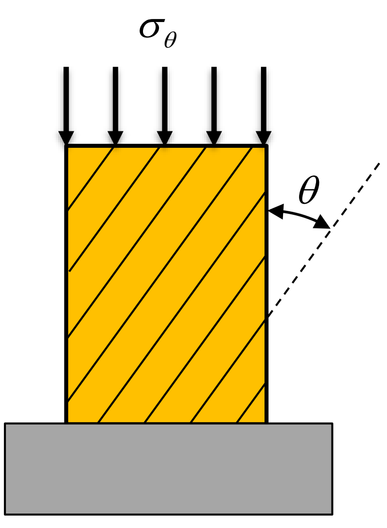
Common failure criteria

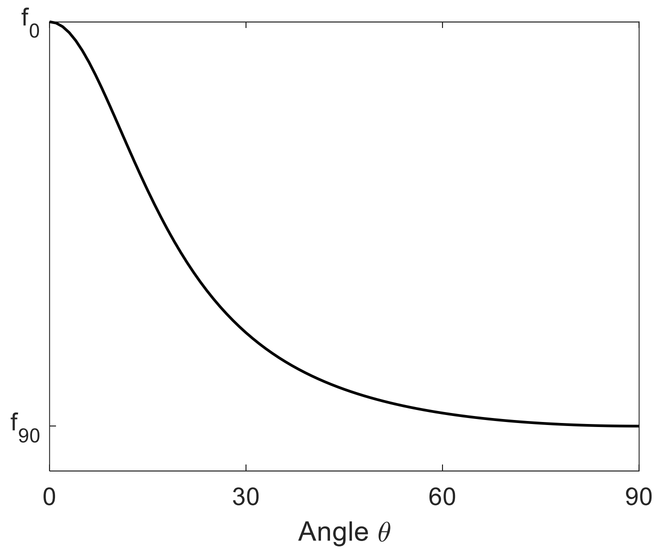
Hankinson equation (Hankinson, 1921)

- **Axial strength at angle to the grain**
- **Originally for compression**
- **Widely used for practical calculations**
- **For example EN 1995-1-1**

$$\frac{\sigma_\theta}{f_\theta} = 1$$

$$f_\theta = \frac{f_0 f_{90}}{f_0 \sin^2\theta + f_{90}\cos^2\theta}$$

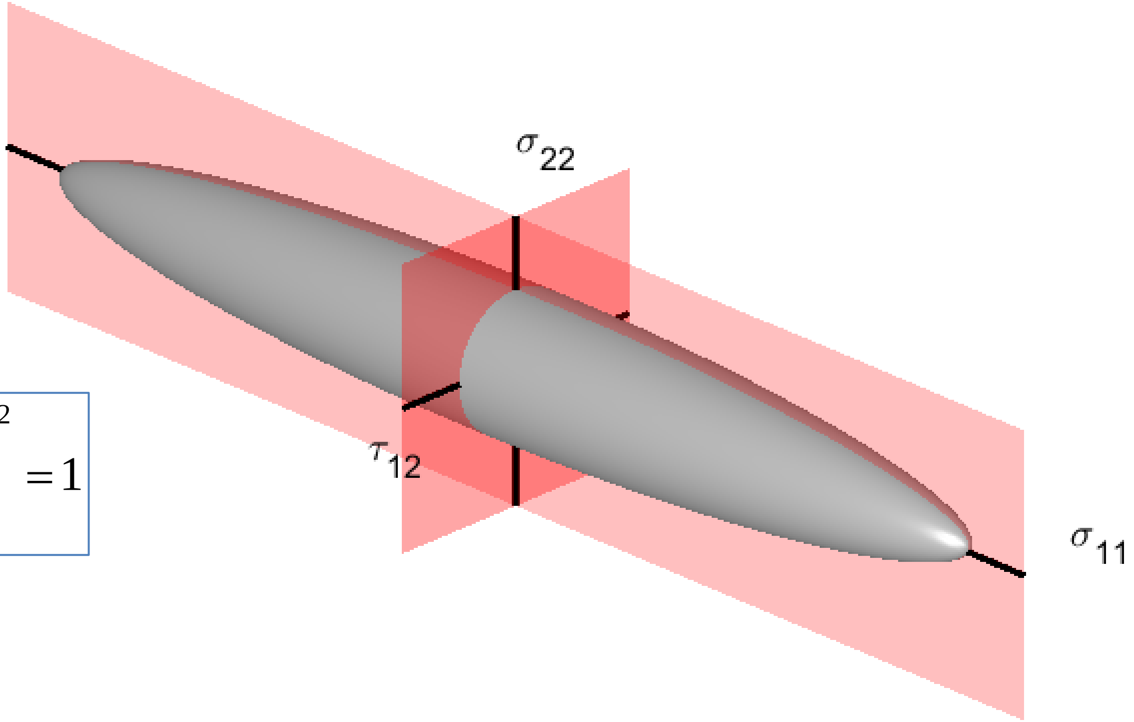




Common failure criteria

Tsai-Hill criterion (Azzi & Tsai 1965)

- **Accounts different orientation**
- **No distinction tension/compression**



$$\left(\frac{\sigma_{11}}{f_1}\right)^2 - \frac{\sigma_{11}\sigma_{22}}{f_1^{\,2}} + \left(\frac{\sigma_{22}}{f_2}\right)^2 + \left(\frac{\tau_{12}}{f_{12}}\right)^2 = 1$$

Common failure criteria

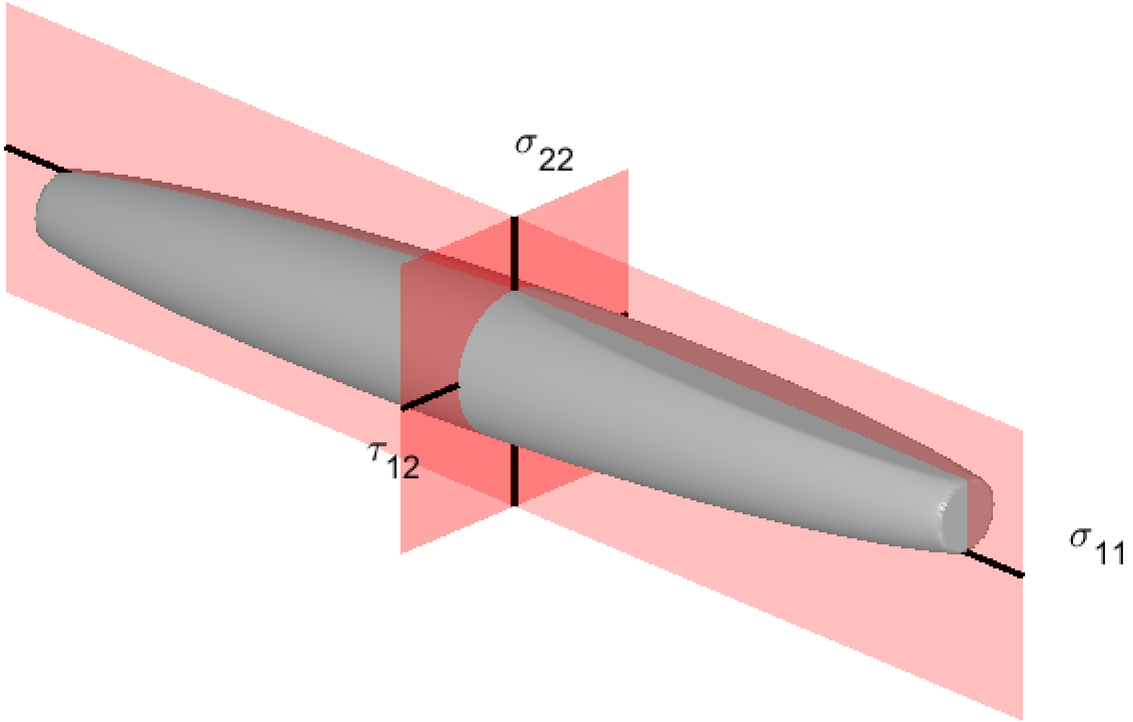
Norris criterion (Norris 1962)

- **Multi-surface approach**
- **No distinction tension/compression**

$$\left(\frac{\sigma_{11}}{f_1}\right)^2 - \frac{\sigma_{11}\sigma_{22}}{f_1 f_2} + \left(\frac{\sigma_{22}}{f_2}\right)^2 + \left(\frac{\tau_{12}}{f_{12}}\right)^2 = 1$$

$$\left(\frac{\sigma_{11}}{f_1}\right)^2 = 1$$

$$\left(\frac{\sigma_{22}}{f_2}\right)^2 = 1$$



More Failure theories – two of them

Maximum Principal strain SAINT-VENANT (1797–1886) & BACH (1889)

Maximum strain energy BELTRAMI (1835-1900)

For brittle materials

Maximum Principal Strain Theory

- ✓ Failure occurs when maximum value of applied strain exceeds the strain value corresponding to yield point of the material.
- ✓ If 'Y' is the yield stress then under uni-axial loading yield strain is defined as

$$\varepsilon_y = Y/E$$

- ✓ Maximum strain developed in the design component should be less than ε_y.
- ✓ Principal stresses σ_1, σ_2 & σ_3 corresponds to principal strains ε_1, ε_2 & ε_3.

$$\boxed{\varepsilon_1 = \frac{\sigma_1}{E} - \frac{\upsilon}{E}(\sigma_2 + \sigma_3)} \qquad \boxed{\varepsilon_2 = \frac{\sigma_2}{E} - \frac{\upsilon}{E}(\sigma_1 + \sigma_3)}$$

For brittle materials

Maximum Principal Stress Theory

- ✓ Maximum principal stress reaches tensile yield stress (Y).
- ✓ Estimate principal stresses σ_1, σ_2 & σ_3.
- ✓ Apply yield criteria:

$$f = \max\left(|\sigma_1|, |\sigma_2|, |\sigma_3|\right) - Y$$

$$\text{If, } f < 0 \quad \text{no yielding}$$

$$f = 0 \quad \text{onset of yielding}$$

$$f > 0 \quad \text{not defined}$$

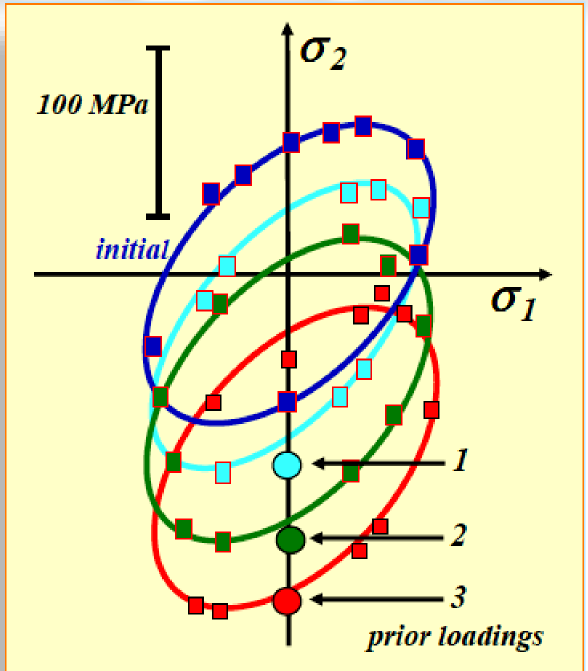
For brittle materials

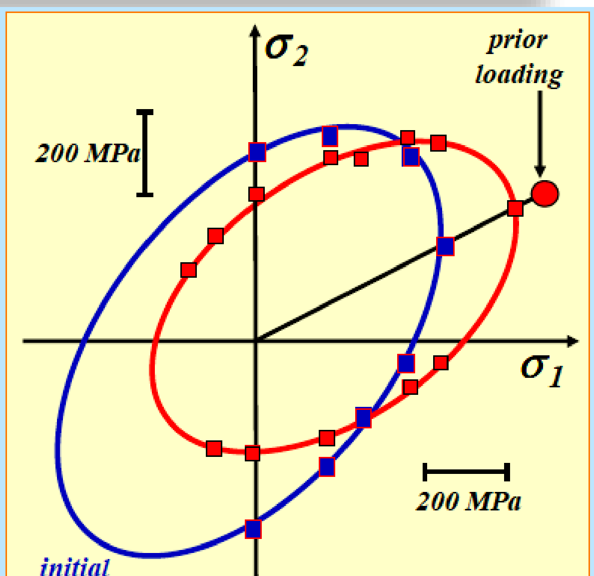
Hardening – notions & examples

1.2.2017, Otaniemi

Hardening – some examples

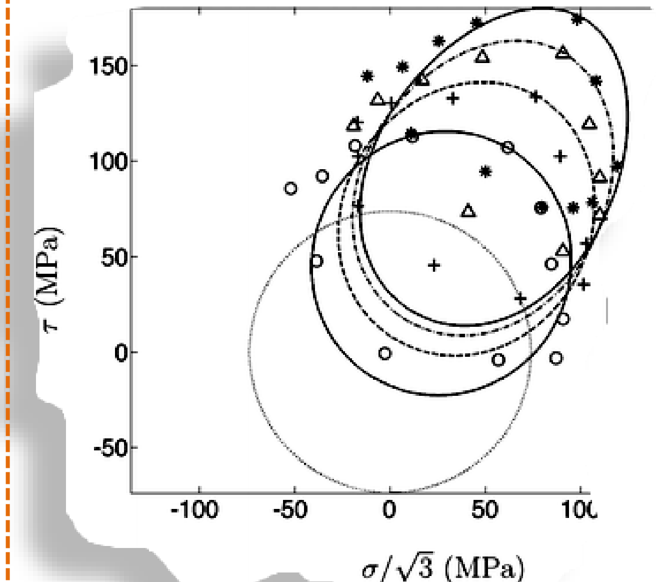
Kinematic hardening



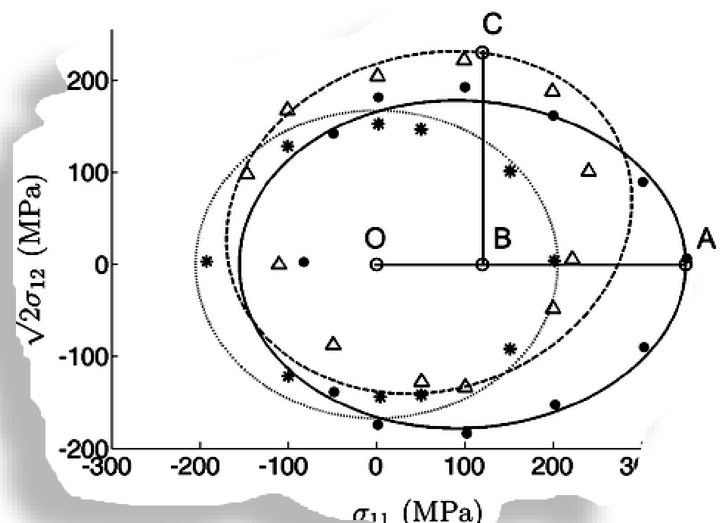


Ref: Moreton et al., 1984

Tension-compression and internal/external pressure



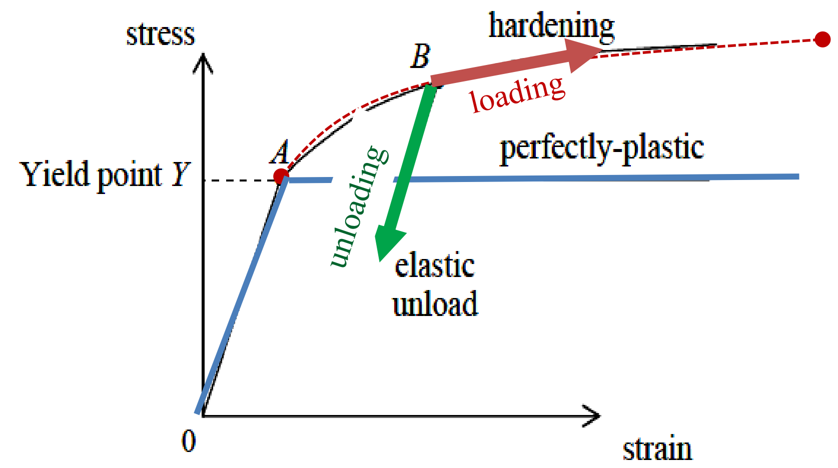
Directional Distortional Hardening in Metal Plasticity



Hardening - notion

Perfectly plastic: the yield surface remains unchanged

General case: the yield surface may change size, shape and position

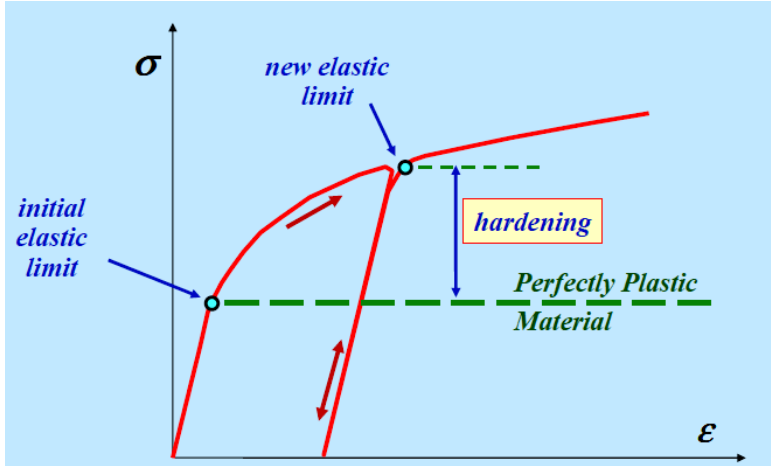


Uniaxial stress-strain curve (typical metal)

Hardening can be described by

$$f(\sigma_{ij}, K_k) = 0,$$

Hardening parameters (scalars, vectors, tensors), which *change* during plastic deformation and determine the *evolution of the yield surface*.
At first yield, the hardening parameters are zero; $f(\sigma_{ij}, 0) = f_0(\sigma_{ij})$

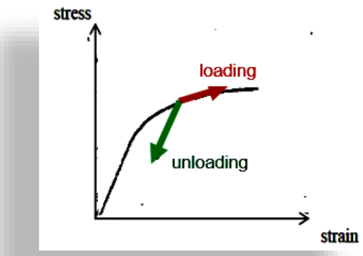


Ref: Chaboche slide from: **ATHENS – Course MP06 – 16 – 20 March 2009**

*Materials can also **strain soften**, for example soils

Hardening Rules

Isotropic hardening



Isotropic hardening: the yield surface remains the same shape but expands during loading increase

$$I_1 = (\sigma_{11} + \sigma_{22} + \sigma_{33})/3$$

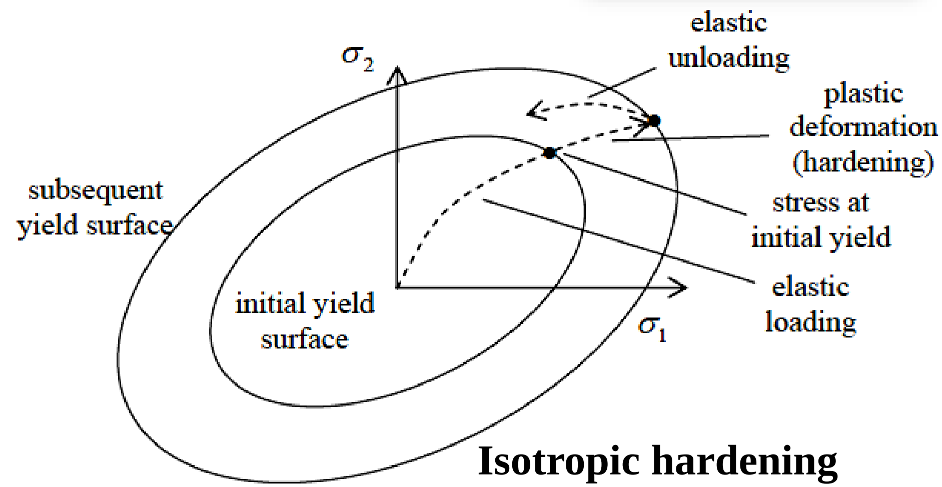
Can be modelled by

hardening parameter

$$f(\sigma_{ij}, K_k) = f_0(\sigma_{ij}) - \boxed{K} = 0,$$

The initial yield function specifies the shape its subsequent size changes as the *hardening parameter K* changes

not yet specified explicitly



Isotropic hardening

Examples of models:

- *Von Mises yield* surface:

initial yield surface: $f_0(\sigma_{ij}) = \sqrt{3J_2} - \sigma_Y$ cylinder with radius (in stress-space) $\sqrt{2/3}\sigma_Y$

subsequent yield surface: $f(\sigma_{ij}, K_k) = \sqrt{3J_2} - \sigma_Y \boxed{-K}$ radius expands for increasing loading $\sqrt{2/3}\cdot(\sigma_Y \boxed{+K})$

- *Drucker-Prager* criterion: initial : $f_0(\sigma_{ij}) = \sqrt{J_2} + \alpha I_1 - k = 0,$

uniaxial tension yield test→

$$I_1 = \sigma_Y \rightarrow \sqrt{J_2} = \sigma_Y / \sqrt{3}$$

$$\rightarrow k = (\alpha + 1/\sqrt{3})\cdot\sigma_Y$$

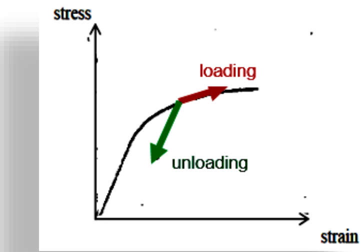
subsequent yield surface: $f(\sigma_{ij}) = \dfrac{1}{\alpha + 1/\sqrt{3}}\left(\sqrt{J_2} + \alpha I_1\right) - \sigma_Y - K = 0,$

Hardening Rules

Kinematic hardening

Kinematic hardening: the yield surface remains the same shape and size but moves during loading increase

$$I_1 = (\sigma_{11} + \sigma_{22} + \sigma_{33})/3$$



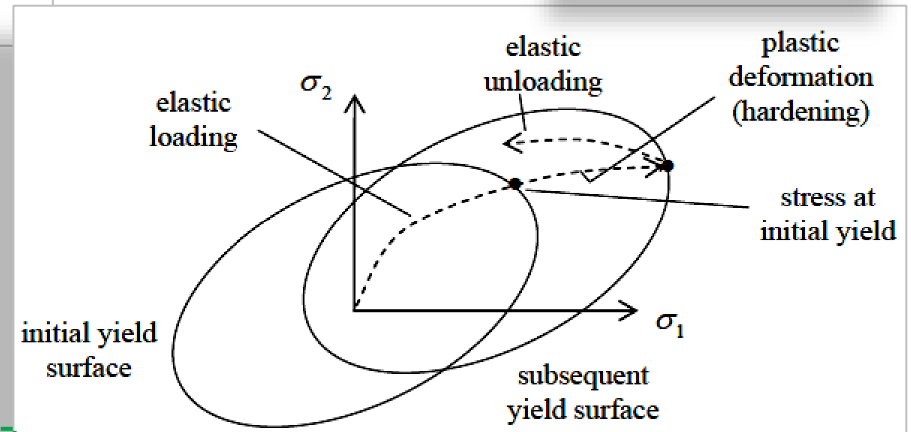
Can be modelled by

hardening parameter

$$f(\sigma_{ij}, K_k) = f_0(\sigma_{ij} + \boxed{\alpha_{ij}}) = 0,$$

The initial yield function specifies shape & size it is subsequently moved (shifted)

The hardening parameter is called *shift*- or *back-stress*. In practice, only the deviatoric stress is shifted $\boxed{(s_{ij} - \alpha'_{ij})}$



Kinematic hardening

Examples of models:

- *Von Mises yield* surface:
  initial yield surface: $$f_0(\sigma_{ij}) = \sqrt{\tfrac{3}{2} s_{ij} s_{ij}} - \sigma_Y = 0$$

α'_{ij} – deviatoric part of α_{ij}

subsequent yield surface: $$f(\sigma_{ij}, K_k) = \sqrt{\tfrac{3}{2}\left(s_{ij} - \boxed{\alpha'_{ij}}\right)\left(s_{ij} - \boxed{\alpha'_{ij}}\right)} - \sigma_Y = 0$$

- Based on experimental observations, an *evolution law,* should be specified, *to* determine how the hardening parameter(s) changes with deformation
- Mixed kinematic and isotropic hardening models can be combined

Mixed hardening rule

A subsequent yield surface:

effective plastic strain

$$F(\sigma_{ij}, \varepsilon_{ij}^{p}; k(\varepsilon_{ij}^{p})) = F(\sigma_{ij} - \alpha_{ij}) - k^{2}(\bar{\varepsilon}^{p}) = 0,$$

***F* determines** the **shape**

Shift or translation

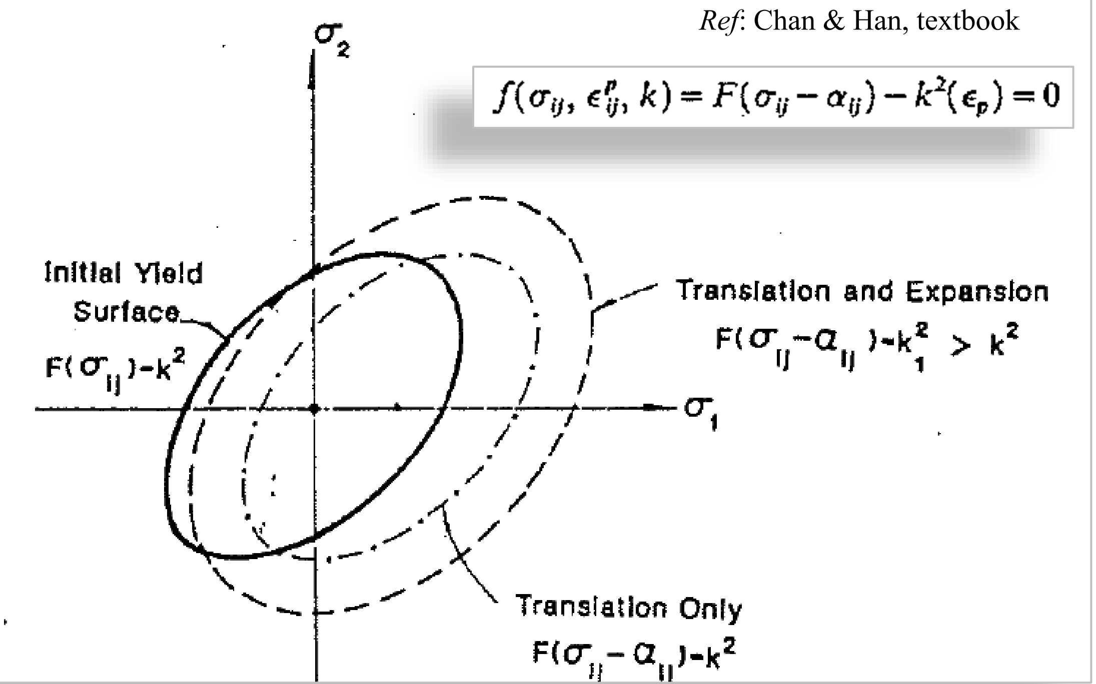
Determines the **size** as function of cumulated plastic strain (*expansion*)

α_{ij}, k^{2} - hardening parameters

Mixed hardening is a result of a combination of *kinematic* and *isotropic hardening* (Hodge, 1957)

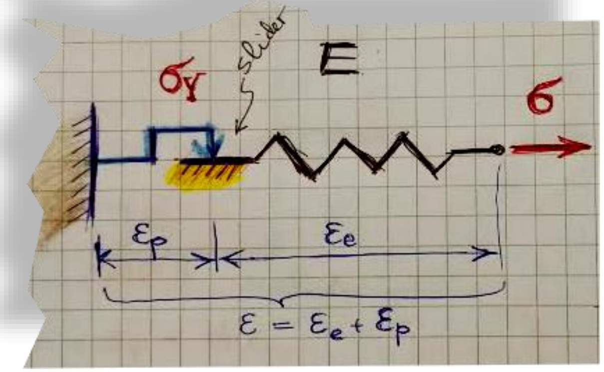
Mixed hardening rule allows to model different degrees of Bauschinger effect by adjusting the two hardening parameters α_{ij} & k^{2}

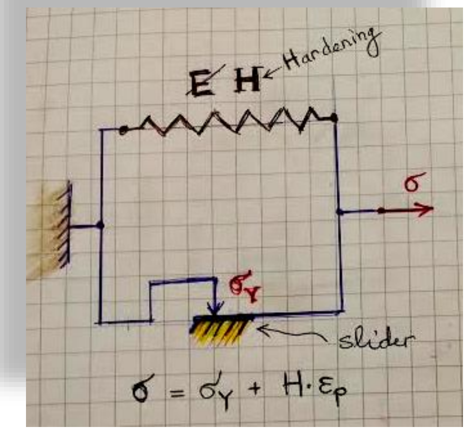
Ref: Chan & Han, textbook

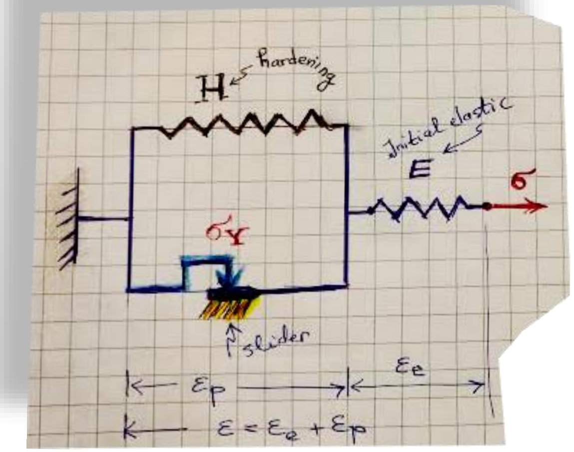


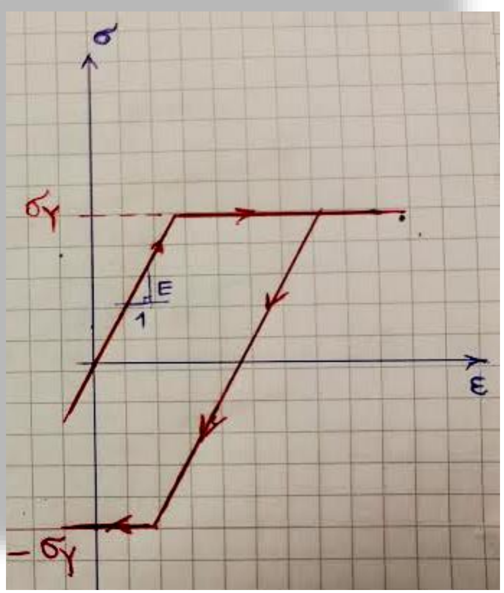
Simple rheological models for Rate-independent plasticity

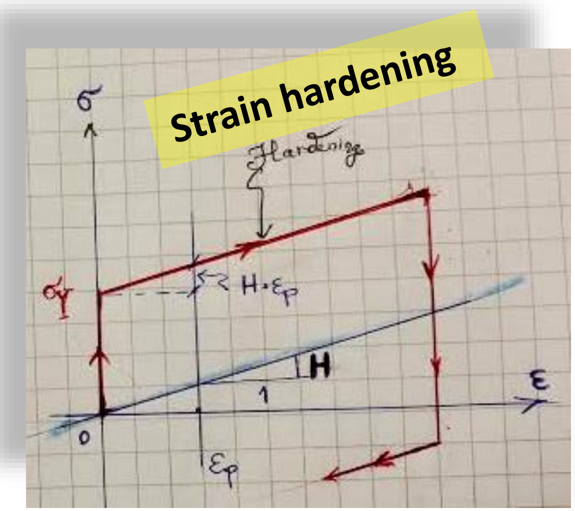
Some strain hardening models

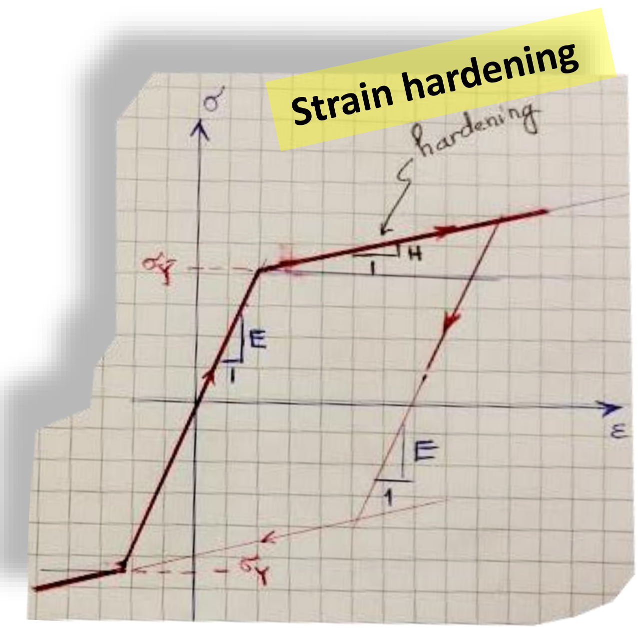












N.B. insert 'dashpots' to obtain rate-dependent plasticity as for Bingham model

Examples of hardening rules in Abaqus – how they looks like?

ABAQUS

Example 4: Kinematic Hardening Plasticity

- Plasticity:
  - Yield function:

$$\sqrt{\frac{3}{2}(S_{ij}-\alpha_{ij})(S_{ij}-\alpha_{ij})}-\sigma_y = 0 .$$

  - Equivalent plastic strain rate:

$$\dot{\bar{\varepsilon}}^{pl} = \sqrt{\frac{2}{3}\dot{\varepsilon}^{pl}_{ij}\dot{\varepsilon}^{pl}_{ij}}.$$

  - Plastic flow law:

$$\dot{\varepsilon}^{pl}_{ij} = \frac{3}{2}(S_{ij}-\alpha_{ij})\dot{\bar{\varepsilon}}^{pl}/\sigma_y .$$

  - Prager-Ziegler (linear) kinematic hardening:

$$\dot{\alpha}_{ij} = \frac{2}{3}h\dot{\varepsilon}^{pl}_{ij}.$$

ABAQUS

$$f(\sigma_{ij},K_k)=\sqrt{3J_2}-(\underbrace{\sigma_Y+K}_{\equiv\sigma_Y(\bar{\varepsilon}_{pl})})$$

Example 5: Isotropic Hardening Plasticity

$$\sqrt{\frac{3}{2}S_{ij}S_{ij}}-\sigma_y(\bar{\varepsilon}^{pl}) = 0 ,$$

$$\sqrt{\frac{3}{2}S_{ij}S_{ij}}-\sigma_y(\bar{\varepsilon}^{pl}) = 0 , \qquad S_{ij} = \sigma_{ij}-\frac{1}{3}\delta_{ij}\sigma_{kk}.$$

- Equivalent plastic strain:

$$\bar{\varepsilon}^{pl} = \int_0^t \dot{\bar{\varepsilon}}^{pl}dt, \qquad \dot{\bar{\varepsilon}}^{pl} = \sqrt{\frac{2}{3}\dot{\varepsilon}^{pl}_{ij}\dot{\varepsilon}^{pl}_{ij}}.$$

- Plastic flow law:

$$\dot{\varepsilon}^{pl}_{ij} = \frac{3}{2}\frac{S_{ij}}{\sigma_y}\dot{\bar{\varepsilon}}^{pl}.$$

$$\bar{\varepsilon}_{pl}\equiv\int_0^{\varepsilon^{pl}_{ij}}\sqrt{2/3\cdot d\varepsilon^{pl}_{ij}d\varepsilon^{pl}_{ij}}$$

This course notation:

$$f(\sigma_{ij},K_k)=\sqrt{\tfrac{3}{2}\left(s_{ij}-\boxed{\alpha'_{ij}}\right)\left(s_{ij}-\boxed{\alpha'_{ij}}\right)}-\sigma_Y=0$$

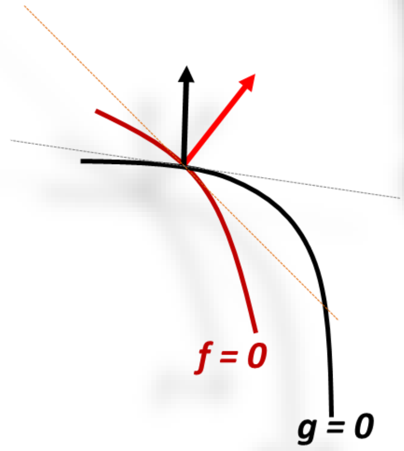
$$f(\sigma_{ij},K_k)=\sqrt{3J_2}-\sigma_Y\boxed{-K}$$

$$f(\sigma_{ij},K_k)=\sqrt{3J_2}-(\underbrace{\sigma_Y+K}_{\equiv\sigma_Y(\bar{\varepsilon}_{pl})})$$

$$\bar{\varepsilon}_{pl}\equiv\int_0^{\varepsilon^{pl}_{ij}}\sqrt{2/3\cdot d\varepsilon^{pl}_{ij}d\varepsilon^{pl}_{ij}}$$

Flow rules


Helsinki, January 2017
D. Baroudi



Reading: Chapter 5.3 *Formulation of general constitutive laws*
Lemaitre et Chaboche, our textbooks

Flow rule & Consistency condition

Plastic flow occurs under the simultaneous two conditions:

- 1. The representative point of the **stress** state $\boldsymbol{\sigma}^*$ is **located on *the loading*** (or **yield-**) ***surface*** (in stress-space representation), thus

$$f\left(\boldsymbol{\sigma}^*, V_k\right) = 0$$

internal variable related to the change of yielding surface during yielding (to account for hardening)

- 2. During the continuous plastic flow (or yielding), the representative point of the **stress state remains on the yield surface** $f > 0$ is impossible (in classical plasticity) and therefore $\mathrm{d}f = 0$

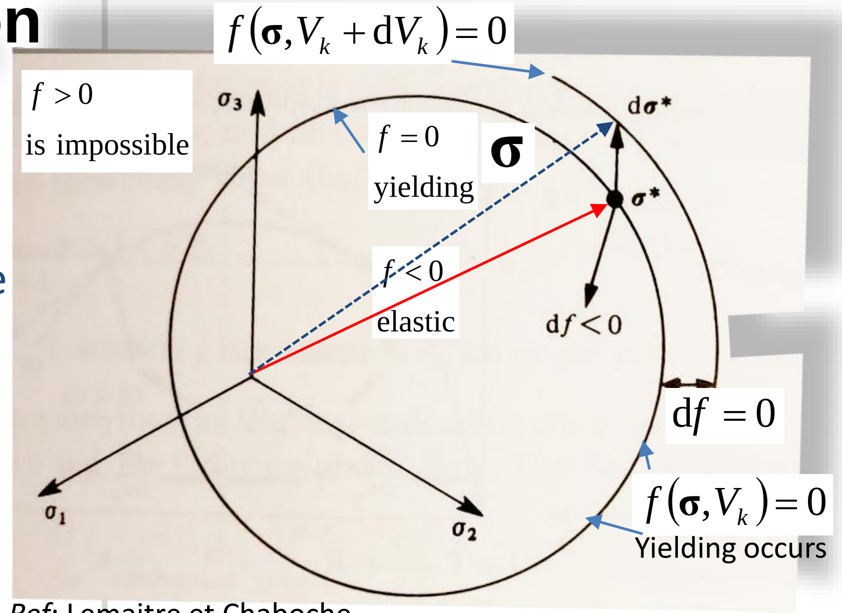
Consistency condition:

$$\mathrm{d}f\left(\boldsymbol{\sigma}^*\right) = \frac{\partial f}{\partial \boldsymbol{\sigma}} : \mathrm{d}\boldsymbol{\sigma}^* + \frac{\partial f}{\partial V_k}\mathrm{d}V_k = 0$$

$\Rightarrow (\boldsymbol{\sigma}^* + \mathrm{d}\boldsymbol{\sigma}^*)$ the new 'point' *P* remains on the yield surface

Unloading is allowed, the stress point moves to the interior of the surface. This corresponds to the condition: $\mathrm{d}f\left(\boldsymbol{\sigma}^*\right) < 0$

The behavior becomes **elastic** during unloading.



Ref: Lemaitre et Chaboche

Yield criterion for a positive hardening

Reading: Chapter 5.3 *Formulation of general constitutive laws*
Lemaitre et Chaboche, our textbooks

Summary:

$f < 0 \rightarrow$ Elastic behavior

$f = 0$ and $\mathrm{d}f = 0 \rightarrow$ Plastic behavior

$f = 0$ and $\mathrm{d}f < 0 \rightarrow$ Elastic unloading

Plastic strain

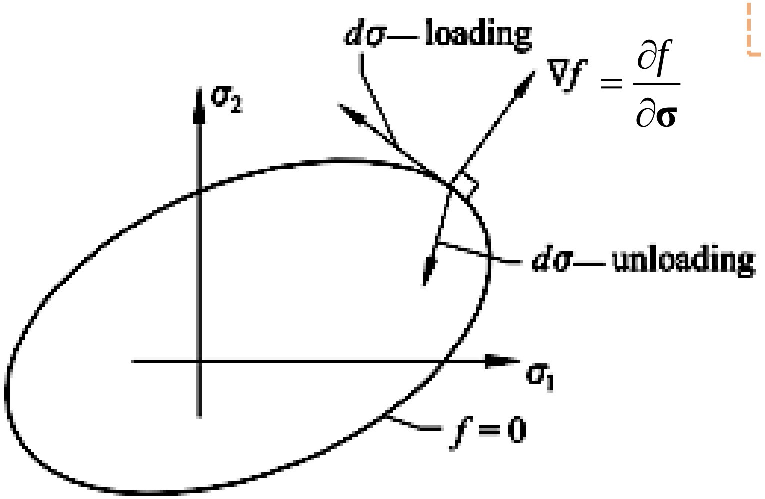
The total strain increment partition:

$$\mathrm{d}\boldsymbol{\varepsilon} = \mathrm{d}\boldsymbol{\varepsilon}^{\mathrm{e}} + \mathrm{d}\boldsymbol{\varepsilon}^{\mathrm{p}}$$

$f < 0 \rightarrow$ Elastic behavior

$f = 0$ and $\mathrm{d}f = 0 \rightarrow$ Plastic behavior

$f = 0$ and $\mathrm{d}f < 0 \rightarrow$ Elastic unloading



Plastic loading and unloading

$$\nabla f = \frac{\partial f}{\partial \underline{\boldsymbol{\sigma}}} = [\partial f / \partial \sigma_{11} \ldots \quad \partial f / \partial \sigma_{33}]^{\mathrm{T}}$$

Voigt's vector notation for stress

For an **associative flow*** rule: $\mathrm{d}\boldsymbol{\varepsilon}^{\mathrm{p}} = \boxed{\mathrm{d}\lambda} \cdot \dfrac{\partial f}{\partial \boldsymbol{\sigma}}$

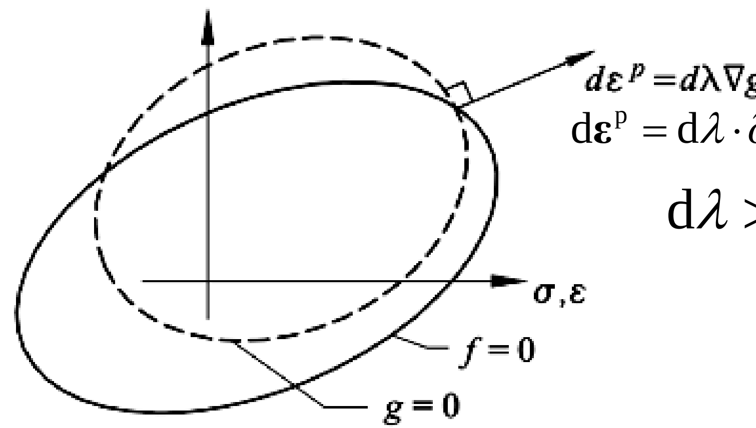
$g = f$

For **nonassociative flow**** rule: $\mathrm{d}\boldsymbol{\varepsilon}^{\mathrm{p}} = \boxed{\mathrm{d}\lambda} \cdot \dfrac{\partial g}{\partial \boldsymbol{\sigma}}$

$g \neq f$

A positive scalar (which determined by the consistency condition)

g : **A general *dissipation potential*** (convex) which provides the *laws of evolution of the plastic deformation and internal variables*. This potential is often called ***a plastic potential*** provides the direction of the plastic flow



$\mathrm{d}\boldsymbol{\varepsilon}^{\mathrm{p}} = \mathrm{d}\lambda \cdot \partial g / \partial \boldsymbol{\sigma}$

$\mathrm{d}\lambda > 0$

Plastic strain increment

In case of *associative flow* rule $\mathrm{d}\boldsymbol{\varepsilon}^{\mathrm{p}} = \mathrm{d}\lambda \cdot \partial f / \partial \boldsymbol{\sigma}$

* metals and metal alloys – no plastic volume changes during plastic flow (or negligible)

**describes well granular and friction materials as geomaterials, soils, concrete – have non negligible plastic volume change during plastic flow

Plastic strain and Plastic stress increments – graphical representation

During continued deformation *P* and *Q* describes curves representing the stress- and strain-paths which projections on the deviatoric plane $(\pi - \mathrm{plane})$ are shown in the figure.

The hydrostatic stress and strain components are perpendicular to Pi-plane of the figure thus all the strain and stress components shown on the deviatoric plane represent their respective deviatoric components.

The strain increment can be decomposed into an elastic part and a plastic part:

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}^{e}_{ij} + \dot{\varepsilon}^{p}_{ij}$$

$$d\varepsilon_{ij} = d\varepsilon^{e}_{ij} + d\varepsilon^{p}_{ij}$$

Rate or incremental formulations

Prandtl-Reuß equations:

(When plastic flow occurs)

$$d\varepsilon^{p}_{ij} = d\lambda \frac{\partial f}{\partial \sigma_{ij}} \qquad d\lambda > 0$$

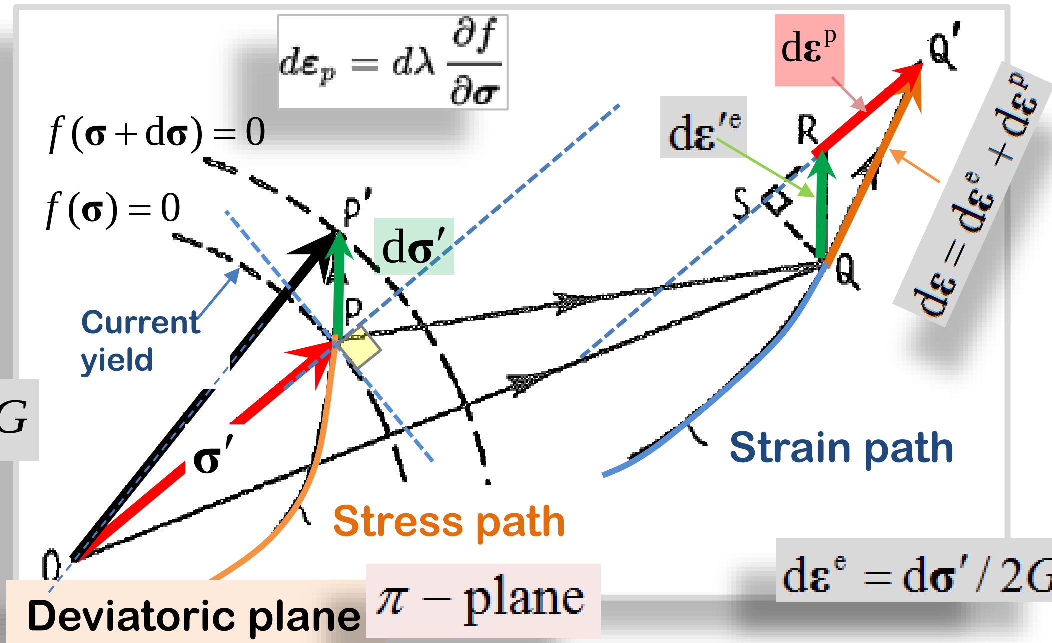
$$\dot{\varepsilon}^{p}_{ij} = \dot{\lambda} \frac{\partial f}{\partial \sigma_{ij}} \qquad \dot{\lambda} > 0$$

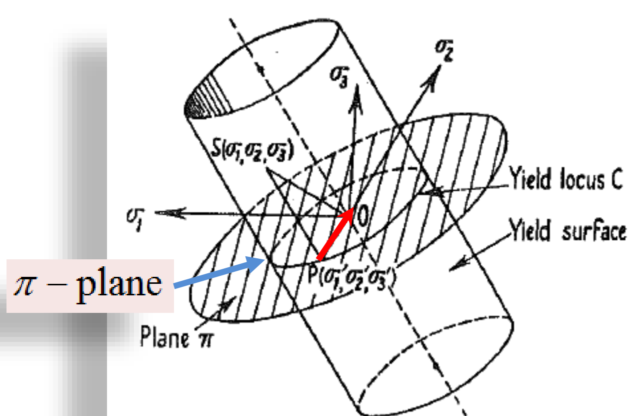
This graphical 2D-representation is due to Hill

QQ' – **total strain increment:** $d\boldsymbol{\varepsilon} = d\boldsymbol{\varepsilon}^{e} + d\boldsymbol{\varepsilon}^{p}$

PP' – **total stress increment** $d\boldsymbol{\sigma}'$

RQ – **elastic strain increment** $d\boldsymbol{\varepsilon}'^{e} = d\boldsymbol{\sigma}'/2G$





Representation of the Reuss relation for a plastic element of work-hardening metal when the principle axes of stress are fixed in the element. (think that the stress strain states are superposed and projected on the Pi-plane)

If you understand this graphical representation then you have understood already a lot about theory of plasticity.

Ref: textbook, p. 42, Hill

Principle of maximum plastic work
Normality rule
Consistency Condition
Associative and Non-associate Plasticity

Convexity of the criterion
Normality of the plastic flow

Reading: Chapter 4.6 & 5.4-5.7 – Chan & Han

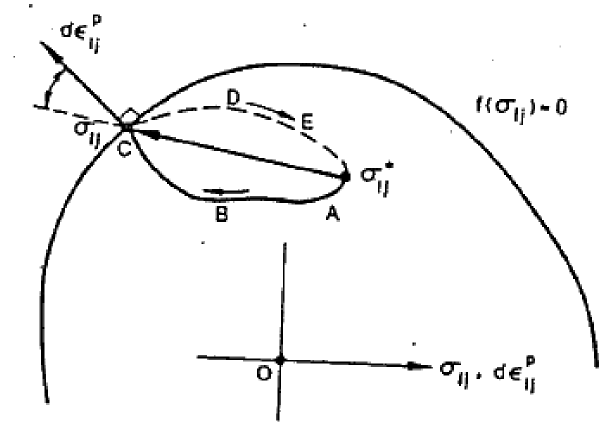
Principle of maximum plastic work

Work done on a loading-unloading cycle stress path from $\boldsymbol{\sigma}^*$ to $\boldsymbol{\sigma}$ and then unloading back to $\boldsymbol{\sigma}^*$ (occurs elastically) is dissipated as a plastic work increment and is positive for plastic deformation.

This is known as **Drucker's material stability postulate**: $\mathrm{d}\boldsymbol{\sigma} : \mathrm{d}\boldsymbol{\varepsilon}^{\mathrm{p}} \geq 0$

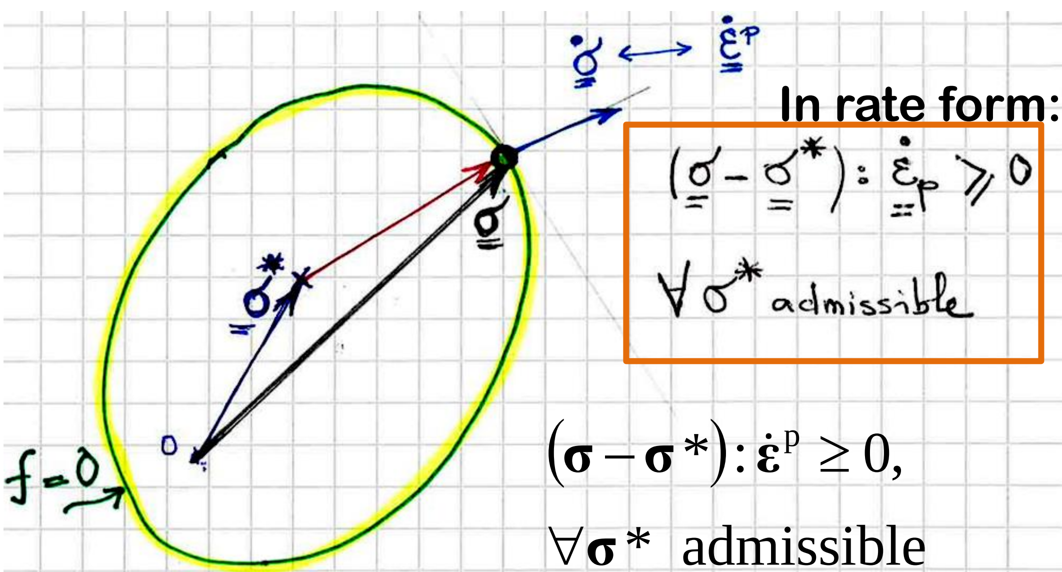
"The plastic work done in an infinitesimal stress cycle is always non-negative."

Applicable to hardening materials only

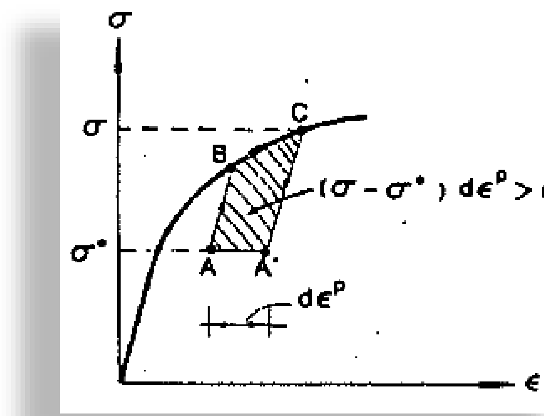


$$(\boldsymbol{\sigma} - \boldsymbol{\sigma}^*) : \mathrm{d}\boldsymbol{\varepsilon}^{\mathrm{p}} \geq 0, \quad \forall \boldsymbol{\sigma}^* \text{ admissible}$$

In rate form:



$$(\boldsymbol{\sigma} - \boldsymbol{\sigma}^*) : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} \geq 0, \quad \forall \boldsymbol{\sigma}^* \text{ admissible}$$

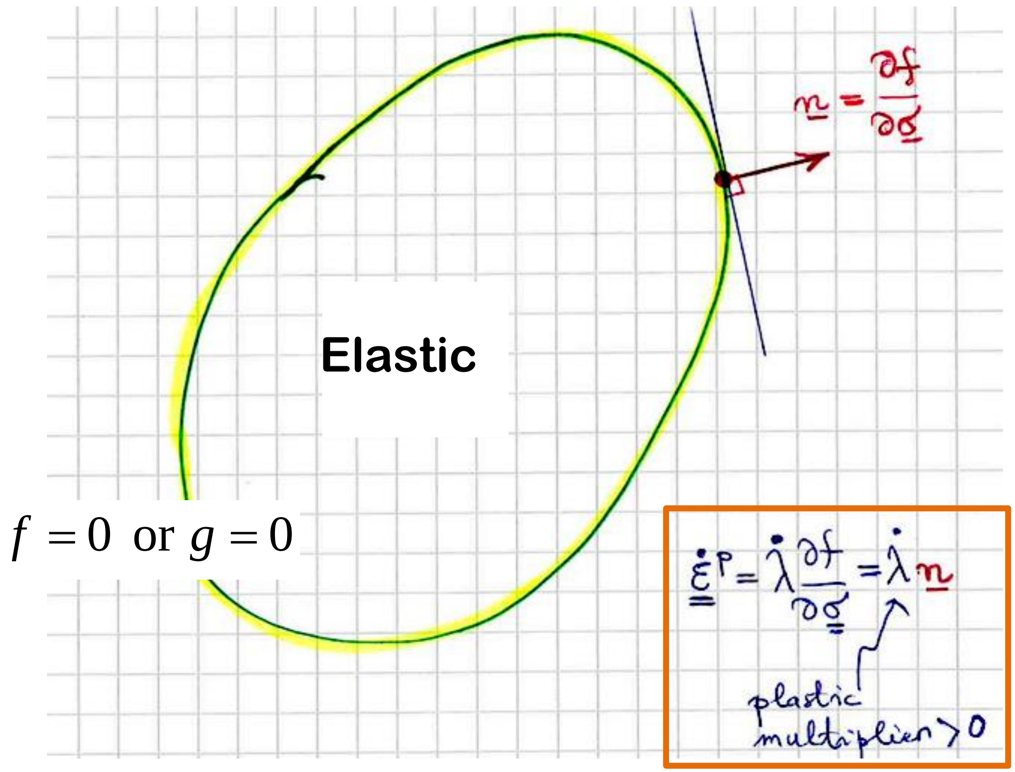


Some names: Drucker's (1951) and then later Ilyushin's (1961) postulates for *material stability*.

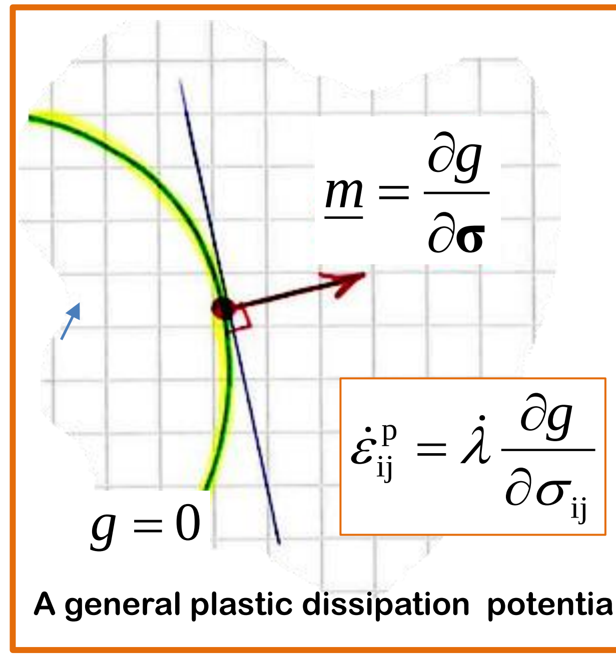
Normality rule

$f = 0 -$ Yield function

$g = 0 -$ A plastic dissipation potential which may be different from f



For a general plasticity with a general plastic dissipation potential not necessarily equal to the yield function:



$$\mathrm{d}\varepsilon_{ij}^{p} = \mathrm{d}\lambda \frac{\partial f}{\partial \sigma_{ij}}$$

$$\dot{\varepsilon}_{ij}^{p} = \dot{\lambda} \frac{\partial f}{\partial \sigma_{ij}}$$

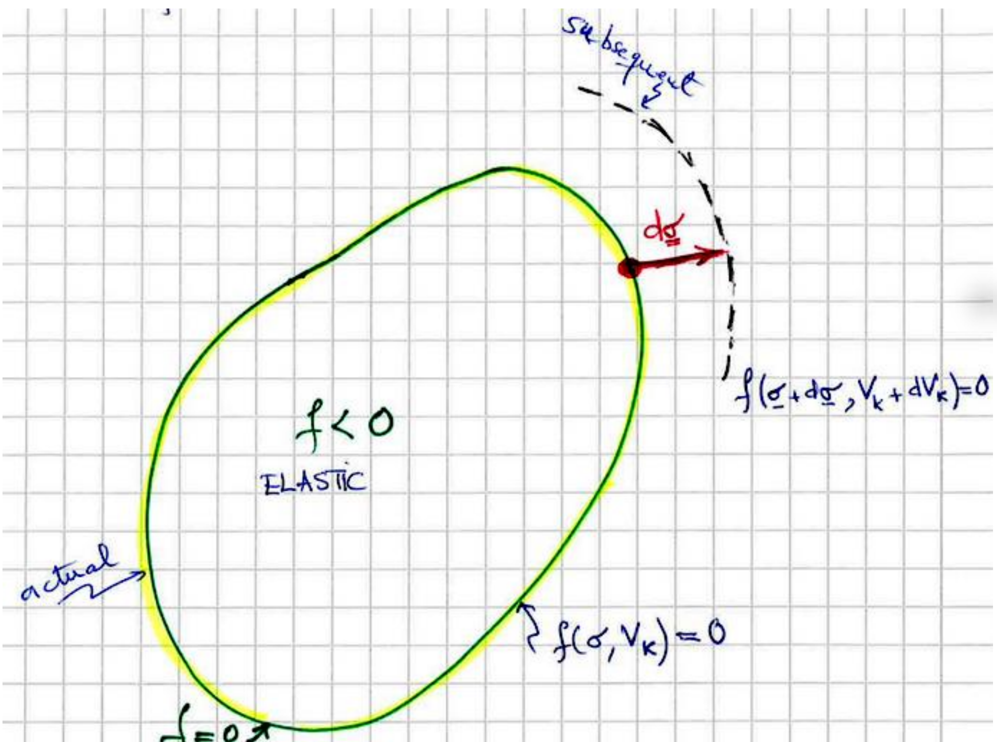
Normality rule implies

- convexity of the yield function
- Incompressibility of the plastic strain for hydrostatic pressure independent plasticity

Consistency condition

$\mathrm{d}\lambda$

The plastic multiplier $\dot{\lambda}$ is determined by combining the hardening rule and the consistency condition



Rate form:

$$\boxed{\dot{f} = \frac{\partial f}{\partial \sigma} : \dot{\sigma} + \frac{\partial f}{\partial V_k} \cdot \dot{V}_k = 0}$$

Prager consistency condition:

Incremental form:

$$\mathrm{d}f = \frac{\partial f}{\partial \boldsymbol{\sigma}} : \mathrm{d}\boldsymbol{\sigma} + \frac{\partial f}{\partial V_k} \mathrm{d}V_k = 0$$

Rate form:

$$\dot{f} = \frac{\partial f}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} + \frac{\partial f}{\partial V_k} \dot{V}_k = 0$$

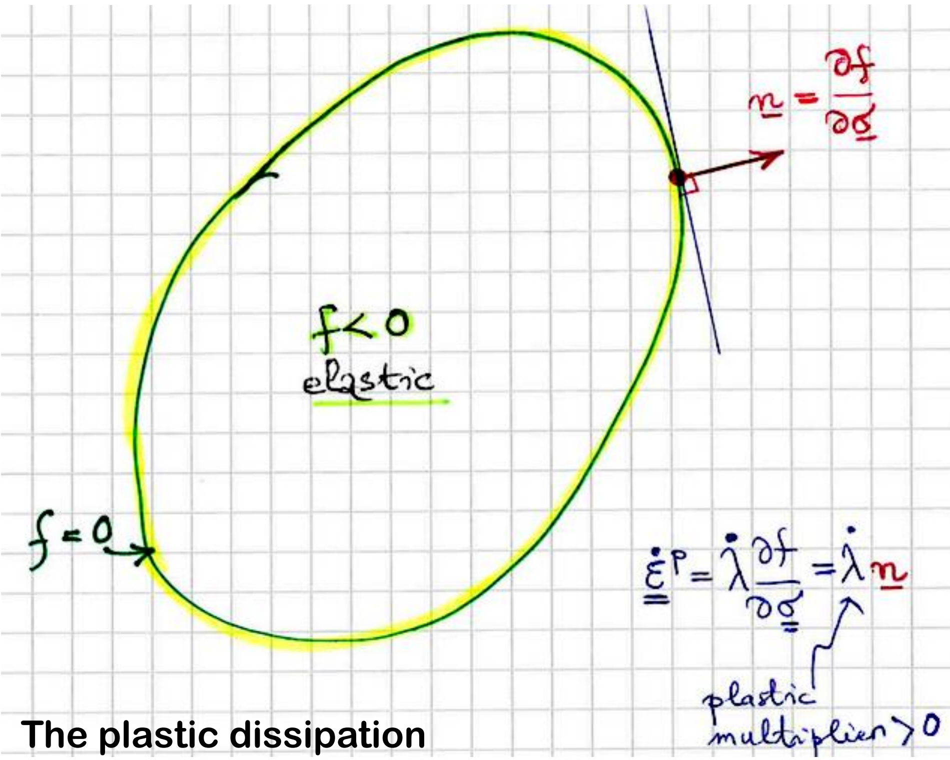
$$f\left(\boldsymbol{\sigma}^*, V_k\right) = 0$$

V_k is an internal (or hidden) variable related to the change of yielding surface during yielding (to account for hardening)

Associated and non-associated plasticity

Associative flow rule

$f = 0 -$ **Yield function**



The plastic dissipation potential is the same as the yield function $g = f$

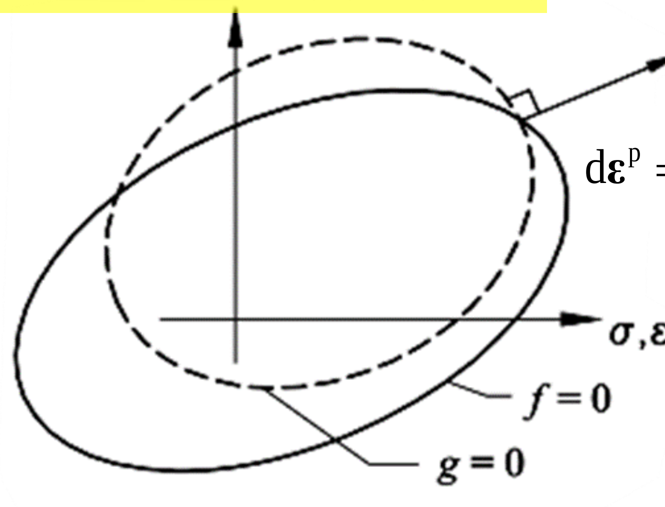
$$\boxed{\mathrm{d}\boldsymbol{\varepsilon}^{\mathrm{p}} = \mathrm{d}\lambda \frac{\partial f}{\partial \boldsymbol{\sigma}} = \mathrm{d}\lambda \cdot \mathbf{n}}$$

$$\boxed{\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \dot{\lambda} \frac{\partial f}{\partial \boldsymbol{\sigma}} = \dot{\lambda} \cdot \mathbf{n}}$$

$\mathrm{d}\lambda > 0$

$\dot{\lambda} > 0$

Non-associative flow rule



$$\mathbf{m} = \frac{\partial g}{\partial \boldsymbol{\sigma}}$$

$$\mathrm{d}\boldsymbol{\varepsilon}^{\mathrm{p}} = \mathrm{d}\lambda \frac{\partial g}{\partial \boldsymbol{\sigma}}$$

$g = 0:$ **A plastic dissipation potential which is different from the yield function** f

$$\boxed{\mathrm{d}\boldsymbol{\varepsilon}^{\mathrm{p}} = \mathrm{d}\lambda \frac{\partial g}{\partial \boldsymbol{\sigma}} = \mathrm{d}\lambda \cdot \mathbf{m}}$$

$$\boxed{\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \dot{\lambda} \frac{\partial g}{\partial \boldsymbol{\sigma}} = \dot{\lambda} \cdot \mathbf{m}}$$

兵马俑

The **Terracotta Army** in Xi'an, China

Clay samples – ***soils mechanics laboratory***
Civil Engineering Department
Aalto-University
Photo: DBA, February 2017

terra cocta (Latin)
means baked earth;
a **clay**-based ceramic.

Ref: Chinese soldiers: By Maros M r a z (Maros) - Own work, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=2704700

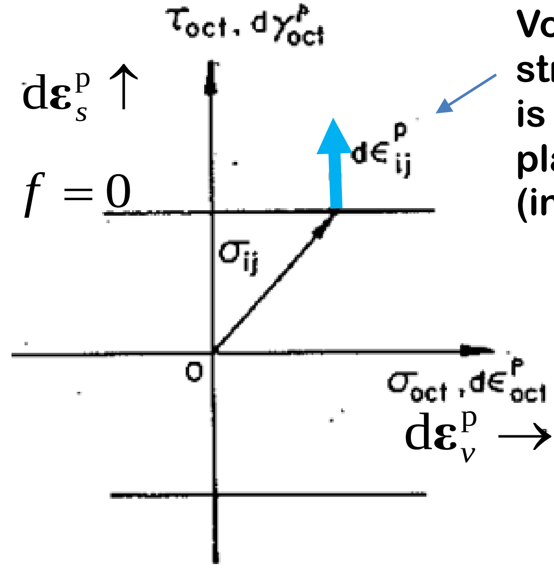
Modified Cam-Clays yield function:

$$f = q^2 - M^2[\,p'(p'_0 - p')] = 0$$

Some application examples of associated and non-associated plasticity

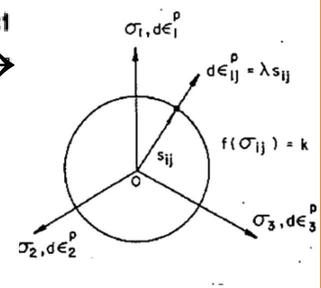
Examples

Flow rule associated with von Mises yield function (Metals)

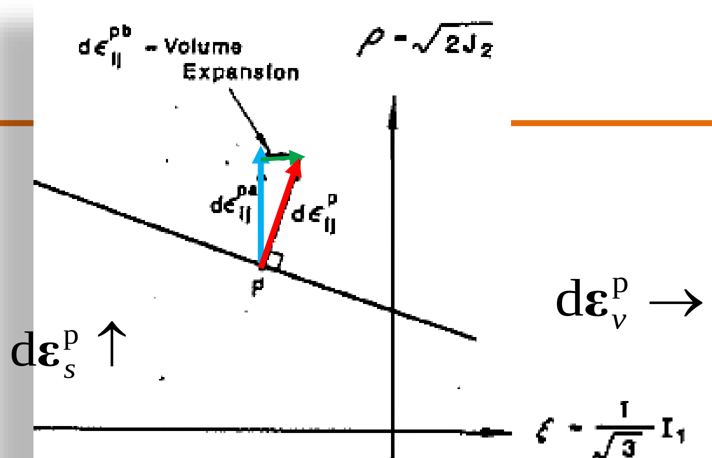


(a) Hydrostatic plane

Volumetric plastic strain component is null during plastic flow (incompressibility)



(b) Deviatoric plane



Plastic volume expansion associated with Drucker-Prager yield surface

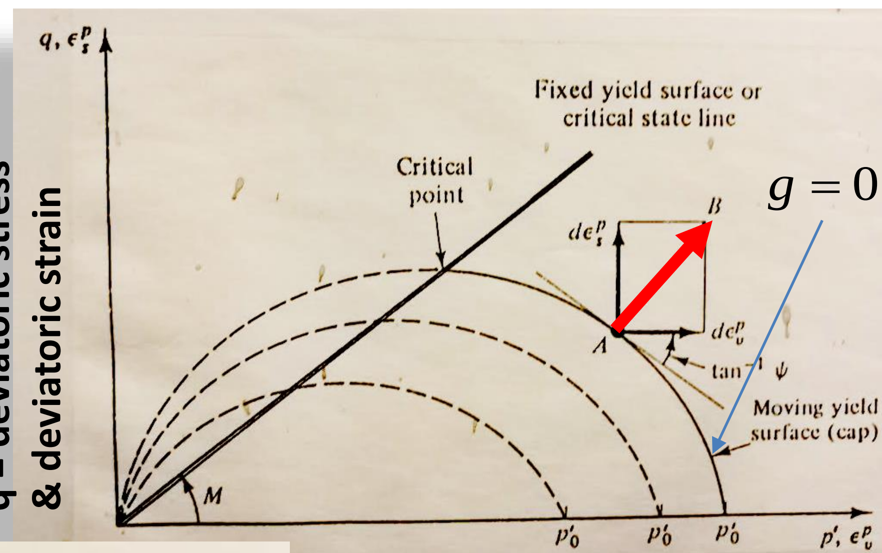
Flow rule associated with a non-associative plasticity (Soils)

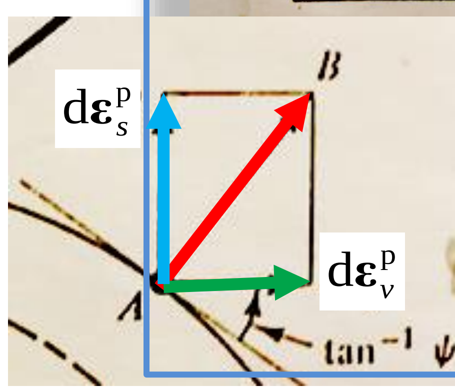
Modified Cam-Clays yield function: $f = q^2 - M^2[\,p'(p'_0 - p')] = 0$

p' = effective mean stress

Plastic strain increment has both a non-zero volumetric $d\varepsilon^p_v$ and deviator (shear) $d\varepsilon^p_s$ components in plastic flow (compressibility or dilatancy)

q = deviatoric stress & deviatoric strain





p' = effective mean stress & volumetric strain

Plastic deformation increment given by combining Consistency condition, Flow rule & Hardening rule

The general idea

This is in the rate form. To obtain incremental form

$$\dot{\lambda} \leftrightarrow \dot{\lambda} dt = d\lambda$$

Normality rule

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \dot{\lambda} \frac{\partial f}{\partial \boldsymbol{\sigma}} = \dot{\lambda} \cdot \mathbf{n}$$

Hardening rule (known behavior, modeled by an evolution low)

$$\dot{V}_k = \dot{\lambda} H_k(\boldsymbol{\sigma}, V_j)$$

V_k is an internal (or hidden) variable related to the change of yielding surface during yielding (to account for hardening)

Consistency condition

$$\dot{f} = \frac{\partial f}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} + \frac{\partial f}{\partial V_k} \dot{V}_k = \mathbf{n} : \dot{\boldsymbol{\sigma}} + \dot{\lambda} H_k \frac{\partial f}{\partial V_k} = 0$$

The plastic multiplier:

$$\Rightarrow \dot{\lambda} = \frac{\frac{\partial f}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}}}{\boxed{-H_k \frac{\partial f}{\partial V_k}}} = \frac{1}{h_{\mathrm{p}}} \frac{\partial f}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} = \frac{1}{h_{\mathrm{p}}} \mathbf{n} : \dot{\boldsymbol{\sigma}}$$

Hardening plastic modulus:

$$\boxed{h_{\mathrm{p}} \equiv -H_k \frac{\partial f}{\partial V_k}}$$

$$d\lambda > 0$$

$$\dot{\lambda} > 0$$

Loading-unloading condition and normality:

$$\boxed{\dot{\boldsymbol{\varepsilon}}_{\mathrm{p}} = \frac{1}{h_{\mathrm{p}}} H(f) \langle \mathbf{n} : \dot{\boldsymbol{\sigma}} \rangle \mathbf{n}}$$

$$d\boldsymbol{\varepsilon}_{\mathrm{p}} = \frac{1}{h_{\mathrm{p}}} H(f) \langle \mathbf{n} : d\boldsymbol{\sigma} \rangle \mathbf{n}$$

$$H(f) = \begin{cases} 1, & f \geq 0 \\ 0, & f < 0 \end{cases}$$

$$\langle x \rangle = x H(x)$$

Inserting the result above into Hooke's law will provide the *elastic-plastic tangent modulus* as shown in next slide

$$d\boldsymbol{\sigma} = \mathbf{C}^e : d\boldsymbol{\varepsilon}^e = \mathbf{C}^e : (d\boldsymbol{\varepsilon} - d\boldsymbol{\varepsilon}^{\mathrm{p}}) = \mathbf{C}^e : \left(d\boldsymbol{\varepsilon} - d\lambda \cdot \partial g / \partial \boldsymbol{\sigma}\right)$$

Incremental Stress-Strain Relationships

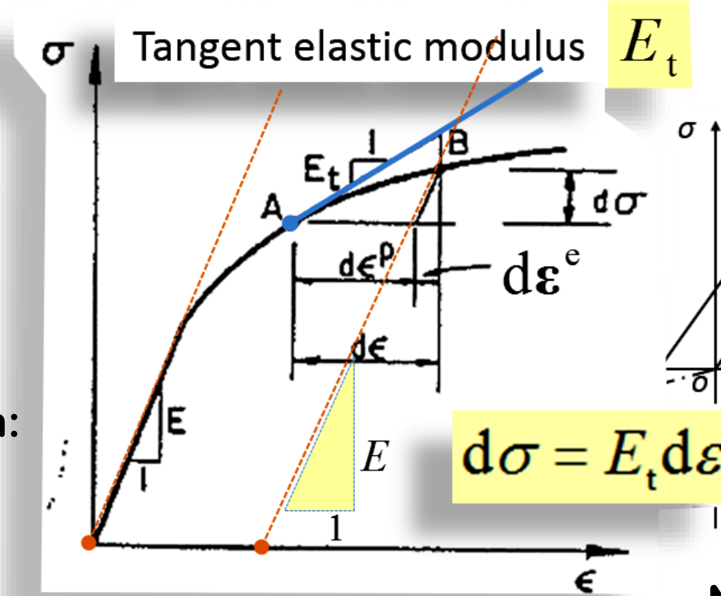
Reading: Chapter 5.7 – Chan & Han
Incremental Stress-Strain Relationships

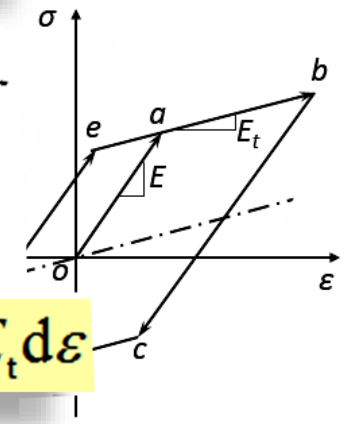
Elastic-Plastic stiffness tensor of tangent modulus

Case: general elastic-plastic work hardening material

You may remember for the 1-D case the idea behind the tangent modulus:

$$\mathrm{d}\sigma = E_{\mathrm{t}}\mathrm{d}\varepsilon$$





For the general 3D-case we seek for an equivalent incremental stress-strain relation:

$$\boxed{\mathrm{d}\boldsymbol{\sigma} = \mathbf{C}^{ep}\mathrm{d}\boldsymbol{\varepsilon} \leftrightarrow \mathrm{d}\sigma_{ij} = C_{ijkl}^{ep}\mathrm{d}\varepsilon_{kl}}$$

The elastic-plastic tangent modulus (tensor)
History dependent … needed in numerical algorithms

Notations:

$$\boxed{\mathbf{C}^{ep} \leftrightarrow \mathbf{D}^{ep}}$$

Isotropic hardening parameter

$$\bar{\varepsilon}_{\mathrm{p}} = C\sqrt{\mathrm{d}\boldsymbol{\varepsilon}^{\mathrm{p}} : \mathrm{d}\boldsymbol{\varepsilon}^{\mathrm{p}}}$$

Equivalent plastic strain

Yield surface: $f(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}^{\mathrm{p}}, k) = 0, \quad k = k(\bar{\varepsilon}_{\mathrm{p}})$

Plastic potential: $g(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}^{\mathrm{p}}, k) = 0,$ (if associated plasticity the $g = f$)

$$\boxed{C_{ijkl}^{ep} = C_{ijkl}^{e} - \frac{\dfrac{\partial f}{\partial \sigma_{mn}} C_{mnkl}^{e} C_{ijst}^{e} \dfrac{\partial g}{\partial \sigma_{st}}}{h + \dfrac{\partial f}{\partial \sigma_{pq}} C_{pquv}^{e} \dfrac{\partial g}{\partial \sigma_{uv}}}}$$

$$\mathrm{d}\boldsymbol{\varepsilon} = \mathrm{d}\boldsymbol{\varepsilon}^{\mathrm{e}} + \mathrm{d}\boldsymbol{\varepsilon}^{\mathrm{p}} \Rightarrow \quad \mathrm{d}\boldsymbol{\sigma} = \mathbf{C}^{e} : \mathrm{d}\boldsymbol{\varepsilon}^{\mathrm{e}} = \mathbf{C}^{e} : (\mathrm{d}\boldsymbol{\varepsilon} - \mathrm{d}\boldsymbol{\varepsilon}^{\mathrm{p}}) = \mathbf{C}^{e} : \left(\mathrm{d}\boldsymbol{\varepsilon} - \mathrm{d}\lambda \frac{\partial g}{\partial \boldsymbol{\sigma}}\right)$$

Hooke's law

Elastic modulus (tensor)

$$\mathrm{d}f = \frac{\partial f}{\partial \boldsymbol{\sigma}} : \mathrm{d}\boldsymbol{\sigma} + \frac{\partial f}{\partial \boldsymbol{\varepsilon}^{\mathrm{p}}} : \mathrm{d}\boldsymbol{\varepsilon}^{\mathrm{p}} + \frac{\partial f}{\partial k}\mathrm{d}k = 0$$

The constitutive relation is fully known once the *plastic multiplier* determined

Hardening plastic modulus (scalar function) /known once hardening rule specified/

As previously done, combining consistency condition and normality one obtains the *plastic multiplier* (for plastic flow)

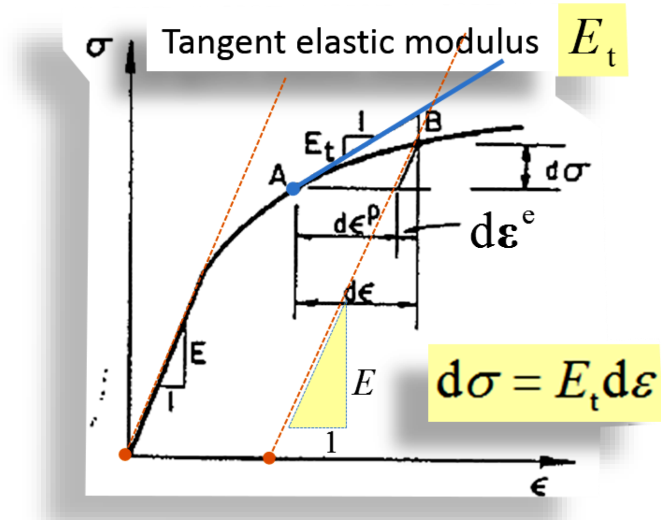
$h = 0$ – For elastic- ideal plastic case

Incremental Stress-Strain Relationships

Elastic-Plastic stiffness tensor of tangent modulus

Case: general elastic-plastic work hardening material

1-D the tangent modulus: $\mathrm{d}\sigma = E_{\mathrm{t}}\mathrm{d}\varepsilon$



3D-incremental stress-strain relation:

$$\mathbf{n} = \frac{\partial f}{\partial \boldsymbol{\sigma}} \qquad \mathbf{m} = \frac{\partial g}{\partial \boldsymbol{\sigma}}$$

$$\boxed{\mathrm{d}\boldsymbol{\sigma} = \mathbf{C}^{ep}\mathrm{d}\boldsymbol{\varepsilon} \leftrightarrow \mathrm{d}\sigma_{ij} = C^{ep}_{ijkl}\mathrm{d}\varepsilon_{kl}}$$

The elastic-plastic tangent modulus (tensor)

FEM: linearization of the weak form gives: $[K_e^T] = \int_{\Omega_e} [B]^T [D^{ep}][B]\, \mathrm{d}\Omega,$

$$\boxed{\mathbf{C}^{ep} \leftrightarrow \mathbf{D}^{ep}}$$

(if associated plasticity the $g = f$ *...symmetric positive definite, otherwise not symmetric and not necessarily positive definite*)

$$\boldsymbol{D}^{ep} = \boldsymbol{D} - \frac{\boldsymbol{D}\dfrac{\partial g}{\partial \boldsymbol{\sigma}}\left(\dfrac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T \boldsymbol{D}}{\left(\dfrac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T \boldsymbol{D}\,\dfrac{\partial g}{\partial \boldsymbol{\sigma}}}$$

$$\boxed{C^{ep}_{ijkl} = C^{e}_{ijkl} - \frac{\dfrac{\partial f}{\partial \sigma_{mn}} C^{e}_{mnkl} C^{e}_{ijst} \dfrac{\partial g}{\partial \sigma_{st}}}{h + \dfrac{\partial f}{\partial \sigma_{pq}} C^{e}_{pquv} \dfrac{\partial g}{\partial \sigma_{uv}}}}$$

matrix form (assumes Voight's notation for stress and stress tensors)

$$\boxed{\mathbf{C}^{ep} = \mathbf{C}^{e} - \frac{\mathbf{C}^{e}\mathbf{m}\mathbf{n}^{\mathrm{T}}\mathbf{C}^{e}}{h + \mathbf{n}^{\mathrm{T}}\mathbf{C}^{e}\mathbf{m}}}$$

Dyadic product $(\mathbf{a} \otimes \mathbf{b})_{ijkl} = a_{ij} b_{kl}$

$$(\mathbf{a} \otimes \mathbf{b}) \leftrightarrow \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}$$

Tensor form

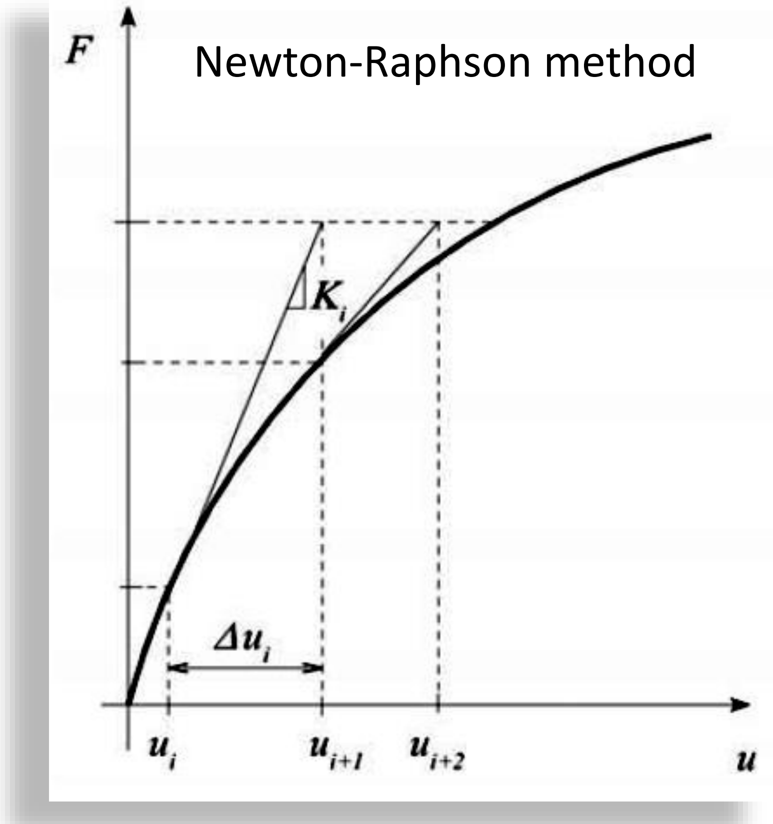
$$\boxed{\mathbf{C}^{ep} = \mathbf{C}^{e} - \frac{(\mathbf{C}^{e} : \mathbf{n}) \otimes (\mathbf{C}^{e} : \mathbf{m})}{h + \mathbf{n} : \mathbf{C}^{e} : \mathbf{m}}}$$

Double contraction:

$$\mathbf{A} : \mathbf{B} = \sum_{i,j} A_{ij} B_{ij}$$

Incremental Stress-Strain Relationships needed in non-linear FEM (plasticity)

Reading: Chapter 5.6 – Lemaitre et Chaboche
Elements of computational methods in elastoplasticity



FEM: linearization of the weak form gives:

$$[K^T_{n,i}]\{\Delta u_i\} = \{F^a_n\} - \{F^{nr}_{n,i}\},$$

$$[K^T_e] = \int_{\Omega_e} [B]^T [D^{ep}][B]\, d\Omega,$$

$$\boxed{d\boldsymbol{\sigma} = \mathbf{C}^{ep} d\boldsymbol{\varepsilon}}$$

$$\boxed{\mathbf{C}^{ep} \leftrightarrow \mathbf{D}^{ep}}$$

The elastic-plastic tangent modulus (tensor)

Incremental solution: In such case the resulting loading *{Fa}* is divided into substeps and in each of them is applied the Newton-Raphson procedure to solve displacements then strain and stress are computed…

$$\mathbf{K}\mathbf{u} = \mathbf{f}$$ Linear case

$$\mathbf{K}(\mathbf{u}) \cdot \Delta\mathbf{u} = \Delta\mathbf{f} + \Delta\mathbf{f}_{\sigma_0}$$ Elasto-plastic case (non-linear)

$$\Delta\mathbf{f}_{\sigma_0} = -\int_{\Omega} \mathbf{B}^{\mathrm{T}} \Delta\boldsymbol{\sigma}_0 d\Omega$$

$$\Delta\boldsymbol{\sigma}_0 = a\Delta\boldsymbol{\varepsilon}^{\mathrm{p}}$$

Treated as initial stress during the increment

Example of a flow rule

Reading: Chapter 5.3 *Formulation of general constitutive laws*
Lemaitre et Chaboche (our textbooks)

Otaniemi, 12.2.2017

Reading: **Chapter 5.3**
Formulation of general constitutive laws
Lemaitre et Chaboche, our textbooks

Flow rule for isotropic hardening – example 1(2)

Von Mises

$$\sigma_e = \sqrt{3J_2}$$

For metals, associative Flow rule is adequate

For **a *perfectly plastic*** material the yield criterion: $f(\sigma_{ij}) = 0,$

Von Mises: $f(\sigma_{ij}) = \sigma_e(\sigma_{ij}) - \sigma_Y = 0,$ where the equivalent stress $\sigma_e = \sqrt{3/2 s_{ij} s_{ij}},$

$$= \sqrt{3/2 s_{ij} s_{ij}} - \sigma_Y = 0,$$

For an ***isotropically hardening*** material, the yield criterion: $f\left(\sigma_{ij}, \kappa(\varepsilon_{ij}^p)\right) = 0,$

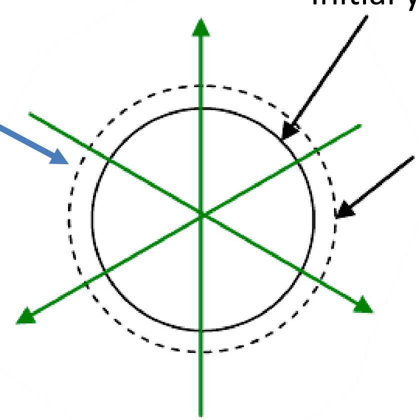
internal variable modeling the monotonic increase of the yielding surface (or of the yield stress)

In the von Mises case one can write:

$$f\left(\sigma_{ij}, \kappa(\varepsilon_{ij}^p)\right) = \sigma_e(\sigma_{ij}) - \sigma_f(\varepsilon_{ij}^p) = 0,$$

depends on plastic strain history

A scalar (instantaneous plastic yield $\sigma_Y(\varepsilon_{ij}^p)$ or flow stress) which increases monotonically with the accumulated plastic deformation.



$$f\left(\sigma_{ij}, \kappa(\varepsilon_{ij}^p)\right) = \sqrt{3/2 s_{ij} s_{ij}} - \sigma_f(\varepsilon_{ij}^p) = 0,$$

Frequently, the *isotropic hardening rule* is written as $\sigma_f(\varepsilon_{ii}^p) = \sigma_f(\varepsilon_e^p),$

where $\varepsilon_e^p \equiv \int_0^{\varepsilon_{ij}^p} \sqrt{2/3 \; d\varepsilon_{ij}^p d\varepsilon_{ij}^p}$, cumulated 'plastic equivalent strain' from $d\varepsilon_e^p \equiv \sqrt{2/3 \; d\varepsilon_{ij}^p d\varepsilon_{ij}^p}$

Flow rule for isotropic hardening – example 2(2)

Von Mises

Use of the consistency condition: $\leftrightarrow \mathrm{d}f(\boldsymbol{\sigma}^*) = \frac{\partial f}{\partial \boldsymbol{\sigma}} : \mathrm{d}\boldsymbol{\sigma}^* + \frac{\partial f}{\partial V_k}\mathrm{d}V_k = 0$

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} = \frac{1}{2}\frac{1}{\sqrt{3J_2}}3\frac{\partial J_2}{\partial \boldsymbol{\sigma}} = \frac{3}{2}\frac{1}{\sigma_Y}\mathbf{s}$$

$f = 0$ and $\mathrm{d}f = 0 \rightarrow$ Plastic behavior

$$f = \sigma_e(\sigma_{ij}) - \sigma_f(\varepsilon_e^p) = 0,$$

$$\begin{cases} \sigma_e \equiv \sqrt{3/2 s_{ij}s_{ij}}, \\ \sigma_e \equiv \sqrt{3J_2}, \end{cases}$$

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{3}{2}\frac{s_{ij}}{\sigma_e} \leftarrow \quad (3)$$

*This result can be shown easily; **homework**

$$\mathrm{d}f = \frac{\partial f}{\partial \sigma_{ij}}\mathrm{d}\sigma_{ij} + \frac{\partial f}{\partial \varepsilon_e^p}\mathrm{d}\varepsilon_e^p = 0$$

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} = \frac{1}{q}\frac{3}{2}\mathbf{s},$$

$$d\boldsymbol{\varepsilon}_p = d\lambda \frac{\partial f}{\partial \boldsymbol{\sigma}}$$

is an associated flow rule

$$\Rightarrow \frac{3}{2}\frac{s_{ij}}{\sigma_e}\mathrm{d}\sigma_{ij} + \frac{\partial f}{\partial \varepsilon_e^p}\mathrm{d}\varepsilon_e^p = 0 \rightarrow \mathrm{d}\varepsilon_e^p = \frac{3}{2}\left(\frac{s_{ij}\mathrm{d}\sigma_{ij}}{\sigma_e}\right) / \left(\frac{\partial f}{\partial \varepsilon_e^p}\right)$$

Flow rule: $\mathrm{d}\varepsilon_{ij}^p = \mathrm{d}\lambda \cdot \frac{\partial f}{\partial \sigma_{ij}}$ **(1)** $\leftrightarrow \dot{\varepsilon}_{ij}^p = \dot{\lambda} \cdot \frac{\partial f}{\partial \sigma_{ij}}$ (rate form)

$$\Rightarrow \quad \mathrm{d}\lambda = \frac{\frac{3}{2}\frac{s_{ij}}{\sigma_e}\mathrm{d}\sigma_{ij}}{\frac{\mathrm{d}\sigma_f}{\mathrm{d}\varepsilon_e^p}} \quad (4)$$

The problem is to determinate the value of the amplitude of the plastic increment $\mathrm{d}\lambda$

Using the definition $\varepsilon_e^p \equiv \int_0^{\varepsilon_{ij}^p} \sqrt{2/3\, \mathrm{d}\varepsilon_{ij}^p \mathrm{d}\varepsilon_{ij}^p},$

together with (1) one obtains the simple relation:

$$\mathrm{d}\varepsilon_e^p = \sqrt{\frac{2}{3}\mathrm{d}\lambda\frac{\partial f}{\partial \sigma_{ij}}\mathrm{d}\lambda\frac{\partial f}{\partial \sigma_{ij}}} = \quad \mathrm{d}\lambda\sqrt{\frac{3}{2}\frac{s_{ij}s_{ij}}{\sigma_e^2}} \quad (2)$$

Combining (1) & (3) & (4) results in:

$$\mathrm{d}\varepsilon_{ij}^p = \frac{3}{2}\frac{s_{ij}}{\sigma_e}\mathrm{d}\varepsilon_e^p$$

$$d\boldsymbol{\varepsilon}_p = d\lambda \frac{\partial f}{\partial \boldsymbol{\sigma}}$$

(Prandtl-Reuß equations)

Examples of hardening rules in Abaqus – how it looks like?

Classic Metal Plasticity

ABAQUS

Example 4: Kinematic Hardening Plasticity

- Plasticity:
  - Yield function:

$$\sqrt{\frac{3}{2}(S_{ij}-\alpha_{ij})(S_{ij}-\alpha_{ij})}-\sigma_y = 0 .$$

  - Equivalent plastic strain rate:

$$\dot{\bar{\varepsilon}}^{pl} = \sqrt{\frac{2}{3}\dot{\varepsilon}^{pl}_{ij}\dot{\varepsilon}^{pl}_{ij}}.$$

  - Plastic flow law:

$$\dot{\varepsilon}^{pl}_{ij} = \frac{3}{2}(S_{ij}-\alpha_{ij})\dot{\bar{\varepsilon}}^{pl}/\sigma_y .$$

  - Prager-Ziegler (linear) kinematic hardening:

$$\dot{\alpha}_{ij} = \frac{2}{3}h\dot{\varepsilon}^{pl}_{ij}.$$

ABAQUS

Example 5: Isotropic Hardening Plasticity

- Plasticity:
  - Yield function:

$$\sqrt{\frac{3}{2}S_{ij}S_{ij}}-\sigma_y(\bar{\varepsilon}^{pl}) = 0 ,$$

$$\sqrt{\frac{3}{2}S_{ij}S_{ij}}-\sigma_y(\bar{\varepsilon}^{pl}) = 0 , \qquad S_{ij} = \sigma_{ij}-\frac{1}{3}\delta_{ij}\sigma_{kk} .$$

  - Equivalent plastic strain:

$$\bar{\varepsilon}^{pl} = \int_0^t \dot{\bar{\varepsilon}}^{pl}dt, \qquad \dot{\bar{\varepsilon}}^{pl} = \sqrt{\frac{2}{3}\dot{\varepsilon}^{pl}_{ij}\dot{\varepsilon}^{pl}_{ij}}.$$

  - Plastic flow law:

$$\dot{\varepsilon}^{pl}_{ij} = \frac{3}{2}\frac{S_{ij}}{\sigma_y}\dot{\bar{\varepsilon}}^{pl}.$$

ABAQUS Version 6.6 Documentation

$$\bar{\varepsilon}^{pl} = \bar{\varepsilon}^{pl}|_0 + \int_0^t \sqrt{\tfrac{2}{3}\dot{\varepsilon}^{pl} : \dot{\varepsilon}^{pl}}\, dt$$

ldition to the standard output identifiers available in ABAQUS ("ABAQUS/Standard outpu… "ABAQUS/Explicit output variable identifiers," Section 4.2.2), the following variable has spe… tal plasticity models:

PEEQ

Equivalent plastic strain, $\bar{\varepsilon}^{pl} = \bar{\varepsilon}^{pl}|_0 + \int_0^t \sqrt{\frac{2}{3}\dot{\varepsilon}^{pl} : \dot{\varepsilon}^{pl}}\, dt$, where $\bar{\varepsilon}^{pl}|_0$ is the initial equivalent plastic strain (zero or user-specified; see "Initial conditions").

END OF this LECTURE SERIES

Recall

Appendix 1
Stress invariants (Recall)

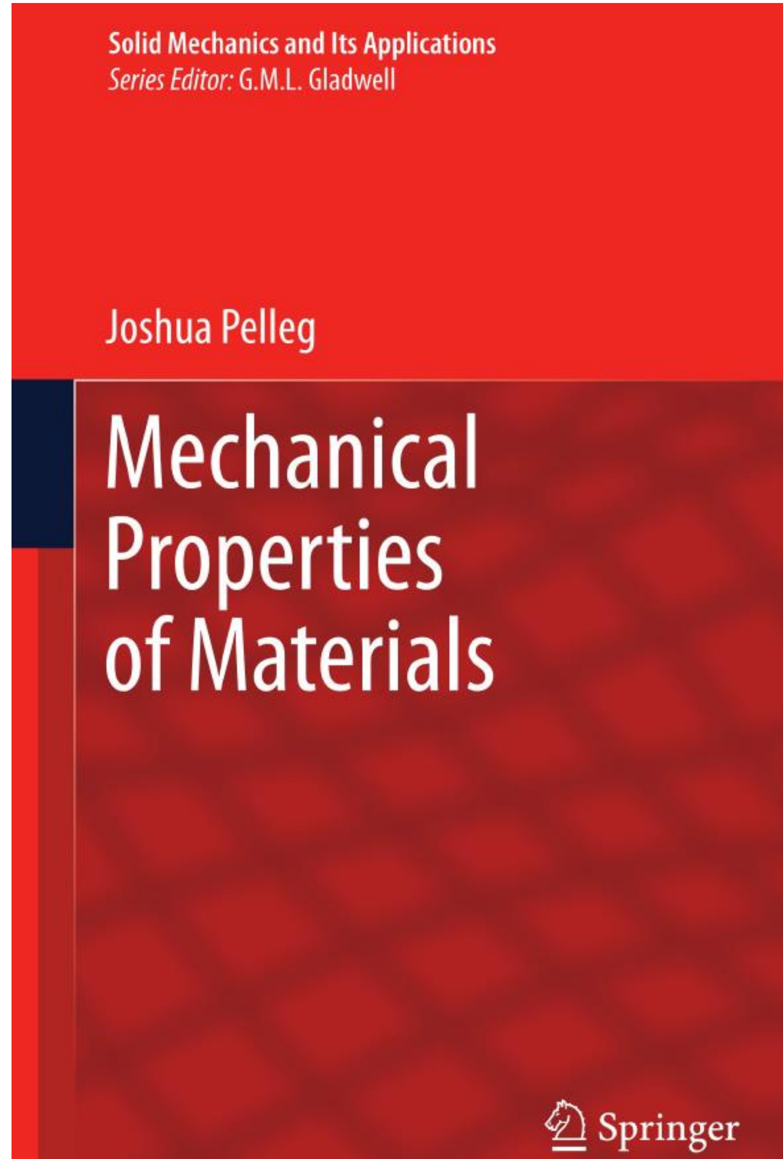
Recall

Elements of solid mechanics

Limited to the necessary minimum to follow this course

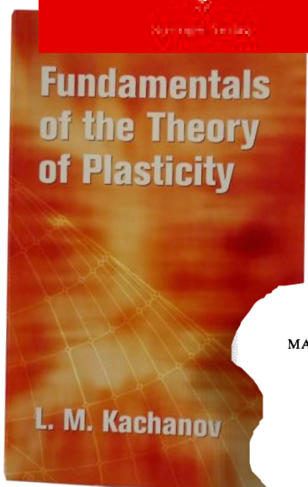
Not mandatory additional textbook – if you need to know about Mechanical Properties of Materials

Usually, our master students should have this type of knowledge (bachelor course)



Not mandatory additional classical textbooks and equivalent on plasticity

[Wai Fah Chen](), [D.J. Han]()
Springer New York, 6 Dec 2012 -
[Technology & Engineering]() - 606 pages



THE
MATHEMATICAL THEORY OF
PLASTICITY
BY
R. HILL
Formerly Professor of Mechanics of Solids, University of Cambridge

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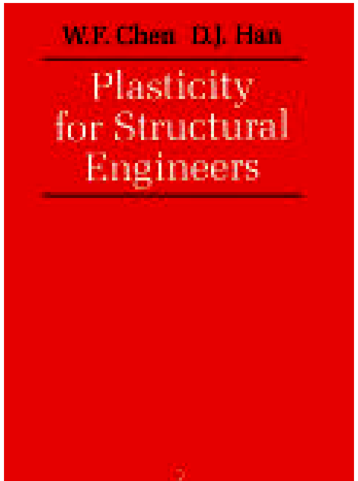
CLARENDON PRESS · OXFORD

PLASTICITY THEORY

Revised Edition (PDF)

Jacob Lubliner
University of California at Berkeley

[http://www.ewp.rpi.edu/hartford/~ernesto/F2008/MEF2/Z-Links/Papers/Lubliner.pdf]() (25.1.2017)



Wai Fah Chen, D.J. Han. ***Plasticity for Structural Engineers.***
Springer New York, 6 Dec 2012 - Technology & Engineering - 606 pages

Table of Contents

Plasticity for Structural Engineers is a practical work that provides engineers and students in structural engineering or structural mechanics with the background needed to make the transition from fundamental theory to computer implementation and engineering practice

Key Features

- Outlines the finite element implementation of the generalized stress-strain relations for the solution of practical steel and concrete structural problems
- Provides fully-worked examples, end-of-chapter problems, answers to selected problems, and clear illustrations and tables
- Features important constitutive equations for structural materials and applications to steel and concrete structures
- Includes examples of the most useful constitutive models and analytical methods

Raakeleita
Homework

Homework, exercises – plasticity – yield criteria …

2. Materiaalin yksiakselinen puristuslujuus on σ_c. Lisäksi se murtuu puristusmeridiaanilla jännitystilassa $\sigma_m = -3\sigma_c, \sigma_e = 5\sigma_c$, jossa $\sigma_m = \frac{1}{3}I_1 = \frac{1}{3}\mathrm{tr}\boldsymbol{\sigma}$ on keskimääräinen jännitys ja σ_e on von Misesin tehollinen jännitys $\sigma_e = \sqrt{3J_2}$. Määritä

   (a) Druckerin-Pragerin murtoehdon $\sqrt{3J_2} + \alpha I_1 - \beta = 0$ ja

   (b) Mohrin-Coulombin murtoehdon

materiaaliparametrit. Mohrin-Coulombin mutoehto voidaan ilmaista useilla parametrikombinaatioilla. Määritä ensin m ja f_c ja sieltä kitkakulma ϕ ja koheesio c.

Määritä myös minkä lujuusarvon kyseiset mallin antavat yksiakselisessa vedossa.

Piirrä puristusmeridiaani (σ_m, σ_e)-koordinaatistossa.

5) Drucker-Prager $\sqrt{3J_2} + \alpha I_1 - \beta = 0$

a) 1-aks. puristuslujuus σ_c

lisäksi tunnetaan murtotila, jossa $\sigma_m = -3\sigma_c$ ja $\sigma_e = 5\sigma_c$

$\sigma_e = \sqrt{3J_2}$, $I_1 = \mathrm{tr}\,\underline{\sigma} = 3\sigma_m$

1-aks. puristus $\sigma_{11} = -\sigma_c$

$$\underline{\sigma} = \begin{pmatrix} -\sigma_c & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \underline{s} = \begin{pmatrix} -\frac{2}{3}\sigma_c & & \\ & \frac{1}{3}\sigma_c & \\ & & \frac{1}{3}\sigma_c \end{pmatrix} \Rightarrow \sigma_e = \sqrt{3J_2} = \sigma_c$$

$$\Rightarrow \begin{cases} \sigma_c - \alpha\sigma_c - \beta = 0 \\ 5\sigma_c - 9\alpha\sigma_c - \beta = 0 \end{cases} \Rightarrow \begin{bmatrix} \sigma_c & 1 \\ 9\sigma_c & 1 \end{bmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \sigma_c \\ 5\sigma_c \end{pmatrix}$$

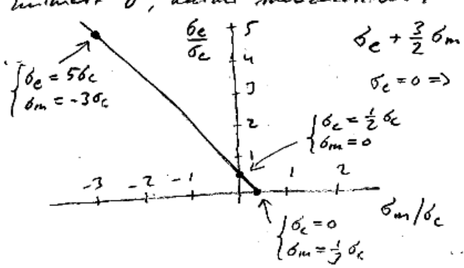
$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{\sigma_c - 9\sigma_c} \begin{pmatrix} 1 & -1 \\ -9\sigma_c & \sigma_c \end{pmatrix} \begin{pmatrix} \sigma_c \\ 5\sigma_c \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2}\sigma_c \end{pmatrix}$$

Murtoehto on siis $\sigma_e + \frac{1}{2}I_1 - \frac{1}{2}\sigma_c = 0$

b) 1-aks. veto $\underline{\sigma} = \begin{pmatrix} \sigma_t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \sigma_e = \sigma_t, \; I_1 = \sigma_t$

$\Rightarrow \frac{3}{2}\sigma_t = \frac{1}{2}\sigma_c \Rightarrow \sigma_t = \frac{1}{3}\sigma_c$

c) Koska murtoehto ei riipu J_3:sta tai Loden kulmasta θ, kaikki meridiaaniviivat yhtenevät



$\sigma_e + \frac{3}{2}\sigma_m - \frac{1}{2}\sigma_c = 0$

$\sigma_e = 0 \Rightarrow \sigma_m = \frac{1}{3}\sigma_c$

b) Mohrin-Coulombin murtoehto:

Puristusmeridiaanilla

$$\sigma_e + 3\frac{m-1}{m+2}\sigma_m - \frac{3}{m+2}\sigma_c = 0 \qquad f_c = \sigma_c$$

nyt $\sigma_e = 5\sigma_c$ ja $\sigma_m = -3\sigma_c$

$$\Rightarrow 5 - 9\frac{m-1}{m+2} - \frac{3}{m+2} = 0 \Rightarrow \frac{3m-2}{m+2} = \frac{5}{3}$$

$\Rightarrow m = 4$

$$\tan\phi = \frac{m-1}{2\sqrt{m}} = \frac{3}{4} \Rightarrow \phi = 36{,}9^\circ$$

$$c = \frac{\sigma_c}{2\sqrt{m}} = \frac{1}{4}\sigma_c$$

Vetomurtolujuus saadaan helpoimmin

$m\sigma_1 - \sigma_3 - \sigma_c = 0$ nyt $\sigma_3 = 0$

$$\sigma_t = \frac{\sigma_c}{m} = \frac{1}{4}\sigma_c$$

Puristusmeridiaani $\sigma_e + \frac{3}{2}\sigma_m - \frac{1}{2}\sigma_c = 0$

yhtyy Druckerin-Pragerin murtoehtoon!

Huom: DP murtoehdolla ∀ meridiaaniviivat yhtyvät. Näin ei ole MC-murtoehdossa, jossa vetomeridiaanilla on lauseke

$$\sigma_e + 3\frac{m-1}{2m+1}\sigma_m - \frac{3}{2m+1}\sigma_c = 0$$

eli tässä tapauksessa

$$\sigma_e + \sigma_m - \frac{1}{3}\sigma_c = 0.$$

Homework, exercises – plasticity – yield criteria …

2. Materiaalin yksiakselinen puristuslujuus on σ_c. Lisäksi se murtuu puristusmeridiaanilla jännitystilassa $\sigma_m = -3\sigma_c, \sigma_e = 5\sigma_c$, jossa $\sigma_m = \frac{1}{3}I_1 = \frac{1}{3}\mathrm{tr}\sigma$ on keskimääräinen jännitys ja σ_e on von Misesin tehollinen jännitys $\sigma_e = \sqrt{3J_2}$. Määritä

(a) Druckerin-Pragerin murtoehdon $\sqrt{3J_2} + \alpha I_1 - \beta = 0$ ja

(b) Mohrin-Coulombin murtoehdon

materiaaliparametrit. Mohrin-Coulombin mutoehto voidaan ilmaista useilla parametrikombinaatioilla. Määritä ensin m ja f_c ja sieltä kitkakulma ϕ ja koheesio c.

Määritä myös minkä lujuusarvon kyseiset mallin antavat yksiakselisessa vedossa.

5) Drucker-Prager $\sqrt{3J_2} + \alpha I_1 - \beta = 0$

a) 1-aks. puristuslujuus σ_c

Lisäksi tunnetaan murtotila, jossa $\sigma_m = -3\sigma_c$ ja $\sigma_e = 5\sigma_c$

$\sigma_e = \sqrt{3J_2}$, $I_1 = \mathrm{tr}\,\underline{\sigma} = 3\sigma_m$

1-aks. puristus $\sigma_{11} = -\sigma_c$

$$\underline{\sigma} = \begin{pmatrix} -\sigma_c & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \underline{s} = \begin{pmatrix} -\frac{2}{3}\sigma_c & & \\ & \frac{1}{3}\sigma_c & \\ & & \frac{1}{3}\sigma_c \end{pmatrix} \Rightarrow \sigma_e = \sqrt{3J_2} = \sigma_c$$

$$\Rightarrow \begin{cases} \sigma_c - \alpha\sigma_c - \beta = 0 \\ 5\sigma_c - 9\alpha\sigma_c - \beta = 0 \end{cases} \Rightarrow \begin{bmatrix} \sigma_c & 1 \\ 9\sigma_c & 1 \end{bmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \sigma_c \\ 5\sigma_c \end{pmatrix}$$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{\sigma_c - 9\sigma_c}\begin{pmatrix} 1 & -1 \\ -9\sigma_c & \sigma_c \end{pmatrix}\begin{pmatrix} \sigma_c \\ 5\sigma_c \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2}\sigma_c \end{pmatrix}$$

Murtoehto on siis $\sigma_e + \frac{1}{2}I_1 - \frac{1}{2}\sigma_c = 0$

b) 1-aks. veto $\underline{\sigma} = \begin{pmatrix} \sigma_t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \sigma_e = \sigma_t , \; I_1 = \sigma_t$

$$\Rightarrow \frac{3}{2}\sigma_t = \frac{1}{2}\sigma_c \Rightarrow \sigma_t = \frac{1}{3}\sigma_c$$

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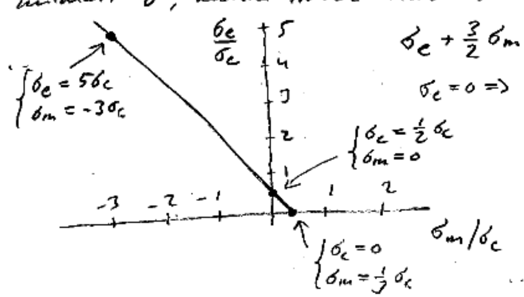

$\sigma_e + \frac{3}{2}\sigma_m - \frac{1}{2}\sigma_c = 0$

$\sigma_e = 0 \Rightarrow \sigma_m = \frac{1}{3}\sigma_c$

b) Mohrin-Coulombin murtoehto:

Puristusmeridiaanilla

$$\sigma_e + 3\frac{m-1}{m+2}\sigma_m - \frac{3}{m+2}\sigma_c = 0 \qquad f_c = \sigma_c$$

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Vetomurtolujuus saadaan helpoimmin

$$m\sigma_1 - \sigma_3 - \sigma_c = 0 \qquad \text{nyt } \sigma_3 = 0$$

$$\sigma_t = \frac{\sigma_c}{m} = \frac{1}{4}\sigma_c$$

Puristusmeridiaani $\sigma_e + \frac{3}{2}\sigma_m - \frac{1}{2}\sigma_c = 0$

yhtyy Druckerin-Pragerin murtoehtoon!

Huom: DP murtoehdolla $\forall$ meridiaaniviivat yhtyvät. Näin ei ole MC-murtoehdossa, jossa vetomeridiaanilla on lauseke

$$\sigma_e + 3\frac{m-1}{2m+1}\sigma_m - \frac{3}{2m+1}\sigma_c = 0$$

eli tässä tapauksessa

$$\sigma_e + \sigma_m - \frac{1}{3}\sigma_c = 0.$$

Rak-54.125 Plastisuus- ja viskoelastisuusteoria
Laskuharjoitus 17 23.3.1992

① Tarkastellaan tasomuodonmuutostilaa ja oletetaan, että pääjännitystä σ_3 vastaava venymä on nolla. Johda myötöehdot

a. Trescan mallille

b. Misesin mallille

c. Mohrin- Coulombin mallille.

Voidaanko käyttää kimmoteorian mukaista yhteyttä $\varepsilon_3 = (\sigma_3 - \nu\cdot(\sigma_1 + \sigma_2))/E = 0$ ja sijoittaa $\sigma_3 = \nu\cdot(\sigma_1 + \sigma_2)$ yleiseen myötöehtoon ?

② Ohutseinäistä ideaalikimmoista putkea vedetään, kunnes se myötää, siten sitä aletaan vääntää kuitenkin niin, että aksiaalinen siirtymä pysyy vakiona. Määritä putken liukukulma γ_{xy} leikkausjännityksen τ_{xy} funktiona.

③ Laske paksuseinäiseen putkeen jääviä jäännösjännityksiä, kun sisäpuolinen paine p poistetaan. (jatkoa tehtävälle 3 /laskuharjoitus 15). Miten jäännösjännitykset suhtautuvat uudelleen kuormitettaessa ?

Rak-54.125 1(2)
9.5

① Kimmoteorian yhteyttä $\varepsilon_3 = \frac{1}{E}(\sigma_3 - \nu(\sigma_1+\sigma_2)) = 0$ ei voi käyttää. Tämä johtaisi ristiriitaiseen tilanteeseen Trescan ja Mohr-Coulombin mallilla, jos ei olisi tilanne

$$\sigma_1 \geq \sigma_3 \geq \sigma_2 \quad \text{tai} \quad \sigma_1 \leq \sigma_3 \leq \sigma_2$$

Tällöin yhtyvät tasojännitystilaan. Jos σ_3 olisi suurin tai pienin pääjännitys murtotaso olisi vino $\sigma_1-\sigma_2$ -pintaa vastaan. Tämä rikkoo taso-muodonmuutostilan oletusta vastaan.

Tasomuodonmuutostilassa otaksutaan kimmoiset venymät häviävän pieniksi plastisiin verrattuina (oikeutettua murtotilaa tarkasteltaessa).

$\Rightarrow$ voidaan otaksua kokoonpuristumattomuus

$$\varepsilon_x + \varepsilon_y = 0 \Rightarrow \nu = 0{,}5$$

tällöin $\varepsilon_3 = \frac{1}{E}(\sigma_3 - \nu(\sigma_1+\sigma_2)) = 0$

$$\Rightarrow \sigma_3 = 0{,}5(\sigma_1+\sigma_2)$$

$$\Rightarrow \sigma_1 \leq \sigma_3 \leq \sigma_2 \quad \text{tai} \quad \sigma_1 \geq \sigma_3 \geq \sigma_2$$

a) Tresca

$$\sigma_m = |\sigma_1 - \sigma_2|$$

b) Mises

$$\sqrt{\tfrac{1}{2}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right]} = \sigma_m$$

sijoitetaan $\sigma_3 = 0{,}5(\sigma_1+\sigma_2)$

$$\Rightarrow \tfrac{1}{2}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2 - \tfrac{1}{2}(\sigma_1+\sigma_2))^2 + (\tfrac{1}{2}(\sigma_1+\sigma_2)-\sigma_1)^2\right] = \sigma_m^2$$

$$\Rightarrow \tfrac{1}{2}\left[(\sigma_2-\sigma_1)^2 + \tfrac{1}{4}(\sigma_2-\sigma_1)^2 + \tfrac{1}{4}(\sigma_2-\sigma_1)^2\right] = \sigma_m^2$$

$$\Rightarrow \tfrac{3}{4}(\sigma_1-\sigma_2)^2 = \sigma_m^2$$

c) Mohr-Coulomb

$$m\sigma_1 - \sigma_2 = f_c \quad , \sigma_1 \geq \sigma_3 \geq \sigma_2$$

$$m\sigma_2 - \sigma_1 = f_c \quad , \sigma_1 \leq \sigma_3 \leq \sigma_2$$

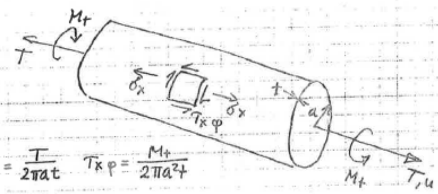
$$m = \frac{f_c}{f_t}$$

plasticity

7,54.125 PL & VE

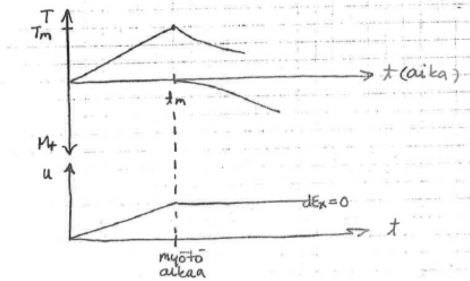
harj:

H 3,17



TSP $\sigma_x = \frac{T}{2\pi a t}$ $\tau_{x\varphi} = \frac{M_t}{2\pi a^2 t}$

Kuormitushistoria



Misesin myötöehto $\sigma_e - \sigma_m = 0$ (σ_m vakio)

$\sigma_e = \sqrt{\sigma_x^2 + 3\tau_{x\varphi}^2}$

Tilanne hetkestä t_m eteenpäin:

Kimmoiset venymän lisäykset (Hooken laki)

$$d\varepsilon_x^k = \frac{d(\sigma_x - \nu\sigma_\varphi)}{E} = \frac{d\sigma_x}{E} \quad (\sigma_\varphi = 0)$$

$$\left(d\varepsilon_\varphi^k = \frac{d(\sigma_\varphi - \nu\sigma_x)}{E} = -\nu\frac{d\sigma_x}{E} \right)$$

$$d\gamma_{x\varphi}^k = \frac{d\tau_{x\varphi}}{G} = \frac{2(1+\nu)}{E}d\tau_{x\varphi}$$

7,54.125 PL & VE harj:

H3,17 jatkuu

Plastiset venymän lisäykset (Prandtl-Reussin yht.)

$$d\varepsilon_x^p = d\lambda\frac{3(\sigma_x - p)}{2\sigma_e} = d\lambda\frac{\sigma_x}{\sigma_m}$$

$$\left(d\varepsilon_\varphi^p = d\lambda\frac{3(\sigma_\varphi - p)}{2\sigma_e} = -d\lambda\frac{\sigma_x}{2\sigma_m} \right)$$

$$d\gamma_{x\varphi}^p = d\lambda\frac{3\tau_{x\varphi}}{\sigma_e} = d\lambda\frac{3\tau_{x\varphi}}{\sigma_m}$$

paine $p = \frac{1}{3}(\sigma_x + \cancel{\sigma_\varphi}^{\,0} + \cancel{\sigma_r}^{\,0})$
$= \frac{1}{3}\sigma_x$,

myötö jatkuu $\Rightarrow \sigma_e = \sigma_m$

yhdistettynä $d\varepsilon_x = d\varepsilon_x^k + d\varepsilon_x^p$ jne.

Ehdosta $d\varepsilon_x = d\varepsilon_x^k + d\varepsilon_x^p = 0$ seuraa

$$d\lambda = -\frac{d\sigma_x}{\sigma_x}\frac{\sigma_m}{E}.$$

$$d\gamma_{x\varphi} = d\gamma_{x\varphi}^k + d\gamma_{x\varphi}^p = \frac{2(1+\nu)}{E}d\tau_{x\varphi} + d\lambda\frac{3\tau_{x\varphi}}{\sigma_m}$$

$$= \frac{2(1+\nu)}{E}d\tau_{x\varphi} - \frac{3\tau_{x\varphi}}{E}\frac{d\sigma_x}{\sigma_x}$$

Jännityksiä sitoo myötöehto $\sigma_e = \sigma_m$ (vakio)

$\Rightarrow \sigma_e^2 = \sigma_x^2 + 3\tau_{x\varphi}^2 = \sigma_m^2$ (vakio) $\Rightarrow \sigma_x^2 = \sigma_m^2 - 3\tau_{x\varphi}^2$

Kok. differentiaali $d(\sigma_e^2) = 2\sigma_x d\sigma_x + 6\tau_{x\varphi}d\tau_{x\varphi} = 0$

$$\Rightarrow d\sigma_x = -3\frac{\tau_{x\varphi}d\tau_{x\varphi}}{\sigma_x} \Rightarrow \frac{d\sigma_x}{\sigma_x} = 3\frac{\tau_{x\varphi}d\tau_{x\varphi}}{3\tau_{x\varphi}^2 - \sigma_m^2}$$

sij. $d\gamma_{x\varphi}$:n lausekkeehen

$$d\gamma_{x\varphi} = \frac{1}{E}\left\{2(1+\nu) - 9\frac{\tau_{x\varphi}^2}{3\tau_{x\varphi}^2 - \sigma_m^2}\right\}d\tau_{x\varphi}$$

integroidaan hetkestä t_m, jossa $\gamma_{x\varphi} = \tau_{x\varphi} = 0$

$$\Rightarrow \gamma_{x\varphi} = \int_0^{\tau_{x\varphi}} \frac{d\gamma_{x\varphi}}{d\tau_{x\varphi}}d\tau_{x\varphi} = \frac{1}{E}\left[(2\nu - 1)\tau_{x\varphi} + \frac{\sqrt{3}}{2}\ln\frac{\sigma_m + \sqrt{3}\tau_{x\varphi}}{\sigma_m - \sqrt{3}\tau_{x\varphi}}\right]$$

(Integrointi: $\int\frac{x^2}{ax^2+c}dx = \frac{x}{a} - \frac{c}{a}\int\frac{dx}{ax^2+c}$

ja $\int\frac{dx}{ax^2+c} = \frac{1}{2\sqrt{-ac}}\log\left|\frac{x\sqrt{a} - \sqrt{-c}}{x\sqrt{a} + \sqrt{-c}}\right|$ kun $a>0$, $c<0$

kts. esim. Pentikäinen)

Rak-54.125 Plastisuus- ja viskoelastisuusteoria

Laskuharjoitus 17 23.3.1992

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plasticity

Täysin kimmoinen

$$(1)\begin{cases}\sigma_r = \dfrac{pa^2}{b^2-a^2}\left[1-\left(\dfrac{b}{r}\right)^2\right]\\ \sigma_\varphi = \dfrac{pa^2}{b^2-a^2}\left[1+\left(\dfrac{b}{r}\right)^2\right]\\ \sigma_z = \dfrac{2pa^2}{b^2-a^2}\end{cases}$$

Osittain plastisoitunut

plastinen osa $a \le r \le c$

$$(2)\begin{cases}\sigma_r = \sigma_m\left[\ln\dfrac{r}{c} - \dfrac{1}{2}\left(1-\left(\dfrac{c}{b}\right)^2\right)\right]\\ \sigma_\varphi = \sigma_m\left[1+\ln\dfrac{r}{c} - \dfrac{1}{2}\left(1-\left(\dfrac{c}{b}\right)^2\right)\right]\end{cases}$$

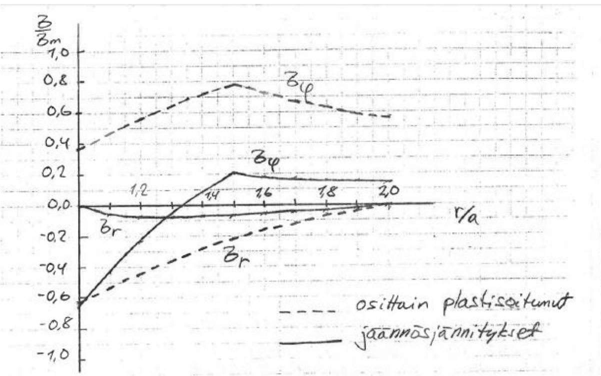
kimmoinen osa $c \le r \le b$

$$(3)\begin{cases}\sigma_r = \dfrac{\sigma_m}{2}\dfrac{c^2}{b^2-c^2}\left[1-\left(\dfrac{c}{b}\right)^2\right]\left[1-\left(\dfrac{b}{r}\right)^2\right]\\ \sigma_\varphi = \dfrac{\sigma_m}{2}\dfrac{c^2}{b^2-c^2}\left[1-\left(\dfrac{c}{b}\right)^2\right]\left[1+\left(\dfrac{b}{r}\right)^2\right]\end{cases}$$

Täysin kimmoinen paineella p

$$p = -\sigma_m\left[-\ln\frac{a}{c} + \frac{1}{2}\left(1-\left(\frac{c}{b}\right)^2\right)\right]$$

$$(4)\begin{cases}\sigma_r = -\dfrac{\sigma_m a^2}{b^2-a^2}\left[-\ln\dfrac{a}{c} + \dfrac{1}{2}\left(1-\left(\dfrac{c}{b}\right)^2\right)\right]\left[1-\left(\dfrac{b}{r}\right)^2\right]\\ \sigma_\varphi = -\dfrac{\sigma_m a^2}{b^2-a^2}\left[-\ln\dfrac{a}{c} + \dfrac{1}{2}\left(1-\left(\dfrac{c}{b}\right)^2\right)\right]\left[1+\left(\dfrac{b}{r}\right)^2\right]\end{cases}$$



Asetetaan putken mitoiksi: $b = 2a$ $c = \frac{3}{2}a$

Osittain plastisoituneen rakenteen jännitykset voidaan laskea

$a \le r \le c$

$$\sigma_r = \sigma_m\left[\ln\frac{2}{3}\frac{r}{a} - \frac{7}{32}\right]$$

$$\sigma_\varphi = \sigma_m\left[1+\ln\frac{2}{3}\frac{r}{a} - \frac{7}{32}\right]$$

r/a	σ_r/σ_m	σ_φ/σ_m
1,0	-0,624	+0,376
1,1	-0,529	0,471
1,2	-0,442	0,558
1,3	-0,362	0,638
1,4	-0,288	0,712
1,5	-0,219	0,781

$c \le r \le b$

$$\sigma_r = \sigma_m\left[\frac{9}{32} - \frac{9}{8}\left(\frac{a}{r}\right)^2\right]$$

$$\sigma_\varphi = \sigma_m\left[\frac{9}{32} + \frac{9}{8}\left(\frac{a}{r}\right)^2\right]$$

r/a	σ_r/σ_m	σ_φ/σ_m
1,5	-0,219	0,781
1,6	-0,158	0,721
1,7	-0,108	0,671
1,8	-0,066	0,628
1,9	-0,030	0,593
2,0	-0,000	0,563

ännösjännityl et saadaan vähentämällä istisoituneen tilan jännityksistä (yhtälöt 2 ja 3) taavan negatiivisen kuorman aiheuttama :aarinen jännitystila (yhtälöt 4)

Jäännösjännitysten käytännön merkitys:

A practical application of shakedown in thick tubes is the "autofrettage" process, which has been used for many years in the manufacture of gun barrels. It is clearly desirable that the inner bore of the barrel should retain its dimensional accuracy on repeated pressurization due to firing. By subjecting the barrel to an *overpressure* before the final surface machining is done, a residual stress system is set up in the barrel which ensures that the bore never goes into the plastic range subsequently, under normal conditions.

Murtumis- ja väsymistarkastelut ovat osoittautuneet tarpeellisiksi monien onnettomuuksien syitä selvitettäessä. Ensimmäinen suihkumoottorikäyttöinen matkustajakone Comet otettiin käyttöön vuonna 1952 yli 300 tuntia kestäneen koelento-ohjelman jälkeen. Neljä päivää tarkastusten jälkeen kone syöksyi Välimereen. Tutkimukset paljastivat, että rungon ikkuna-aukkojen kulmista oli lähtenyt liikkeelle väsymissäröjä. Runko oli paineistettu 18000 kertaa matkalentokorkeutta vastaavalla ylipaineella, jonka aiheuttama jännitys oli 40 % käytetyn alumiiniseoksen murtolujuudesta. Tätä kuormituskoetta ennen runkoa oli kuitenkin kuormitettu 30 kertaa 1.75...2.75 -kertaisella paineella. Tällainen ylikuormitus synnytti ilmeisesti kriittisiin kohtiin plastisten muodonmuutosten vuoksi niin suuren puristavan jäännösjännitystilan, että koerunko ei murtunut vaan antoi virheellisen kuvan rungon väsymiskestävyydestä todellisessa kuormitustilanteessa.

$a \le r \le c$

$$\sigma_r^j = \sigma_m\left[\ln\frac{r}{c} - \frac{1}{2}\left(1-\left(\frac{c}{b}\right)^2\right)\right] - \sigma_m\frac{a^2}{b^2-a^2}\left[-\ln\frac{a}{c} + \frac{1}{2}\left(1-\left(\frac{c}{b}\right)^2\right)\right]\left[1-\left(\frac{b}{r}\right)^2\right]$$

$$= \sigma_m\left[\ln\frac{2r}{3a} - \frac{1}{2}\left(1-\frac{9}{16}\right)\right] - \sigma_m\frac{1}{3}\left[-\ln\frac{2}{3} + \frac{1}{2}\left(1-\frac{9}{16}\right)\right]\left[1-4\left(\frac{a}{r}\right)^2\right]$$

$$= \sigma_m\left[-0{,}427 + \ln\frac{2r}{3a} + 0{,}832\left(\frac{a}{r}\right)^2\right]$$

$$\sigma_\varphi^j = \sigma_m\left[1+\ln\frac{r}{c} - \frac{1}{2}\left(1-\left(\frac{c}{b}\right)^2\right)\right] - \sigma_m\frac{a^2}{b^2-a^2}\left[-\ln\frac{a}{c} + \frac{1}{2}\left(1-\left(\frac{c}{b}\right)^2\right)\right]\left[1+\left(\frac{b}{r}\right)^2\right]$$

$$= \sigma_m\left[1+\ln\frac{2r}{3a} - \frac{1}{2}\left(1-\frac{9}{16}\right)\right] - \sigma_m\frac{1}{3}\left[-\ln\frac{2}{3} + \frac{1}{2}\left(1-\frac{9}{16}\right)\right]\left[1+4\left(\frac{a}{r}\right)^2\right]$$

$$= \sigma_m\left[0{,}573 + \ln\frac{2r}{3a} - 0{,}832\left(\frac{a}{r}\right)^2\right]$$

r/a	σ_r^j/σ_m	$\sigma_\varphi^j/\sigma_m$
1,0	0,000	-0,665
1,1	-0,049	-0,425
1,2	-0,072	-0,228
1,3	-0,077	-0,062
1,4	-0,071	+0,080
1,5	-0,057	+0,203

$c \le r \le b$

$$\sigma_r^j = \frac{\sigma_m}{2}\frac{c^2}{b^2-c^2}\left[1-\left(\frac{c}{b}\right)^2\right]\left[1-\left(\frac{b}{r}\right)^2\right] - \sigma_m\frac{a^2}{b^2-a^2}\left[-\ln\frac{a}{c} + \frac{1}{2}\left(1-\left(\frac{c}{b}\right)^2\right)\right]\left[1-\left(\frac{b}{r}\right)^2\right]$$

$$= \frac{\sigma_m}{2}\frac{\frac{9}{4}a^2}{4a^2-\frac{9}{4}a^2}\left[1-\frac{9}{16}\right]\left[1-4\left(\frac{a}{r}\right)^2\right] - \sigma_m\frac{a^2}{4a^2-a^2}\left[-\ln\frac{2a}{3a} + \frac{1}{2}\left(1-\frac{9}{16}\right)\right]\left[1-4\left(\frac{a}{r}\right)^2\right]$$

$$= \sigma_m\left[0{,}073 - 0{,}293\left(\frac{a}{r}\right)^2\right]$$

$$\sigma_\varphi^j = \frac{\sigma_m}{2}\frac{c^2}{b^2-c^2}\left[1-\left(\frac{c}{b}\right)^2\right]\left[1+\left(\frac{b}{r}\right)^2\right] - \sigma_m\frac{a^2}{b^2-a^2}\left[-\ln\frac{a}{c} + \frac{1}{2}\left(1-\left(\frac{c}{b}\right)^2\right)\right]\left[1+\left(\frac{b}{r}\right)^2\right]$$

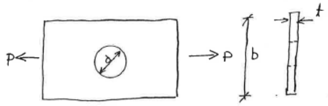
$$= \sigma_m\frac{9}{14}\cdot\frac{7}{16}\left[1+4\left(\frac{a}{r}\right)^2\right] - \sigma_m\frac{1}{3}\left[-\ln\frac{2}{3} + \frac{7}{32}\right]\left[1+4\left(\frac{a}{r}\right)^2\right]$$

$$= \sigma_m\left[0{,}073 + 0{,}293\left(\frac{a}{r}\right)^2\right]$$

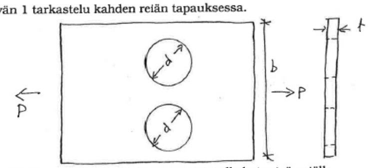
r/a	σ_r^j/σ_m	$\sigma_\varphi^j/\sigma_m$
1,5	-0,057	0,203
1,6	-0,041	0,188
1,7	-0,028	0,174
1,8	-0,017	0,164
1,9	-0,008	0,154
2,0	0,000	0,146

Rak-54.125 Plastisuus- ja viskoelastisuusteoria
Laskuharjoitus 18 25.3.1992

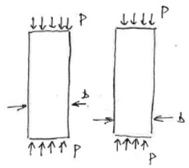
① Tarkastellaan vedettyä levyä, jossa on pyöreä reikä. Johda vetävälle rajakuormalle P staattisella menetelmällä alaraja ja kinemaattisella menetelmällä yläraja. Oletetaan aineen noudattavan Trescan myötöehtoa. Tarkastele sekä tasojännitystilaa (ohut levy) että tasomuodonmuutostilaa. Miten reiän muoto vaikuttaa ratkaisuun ?



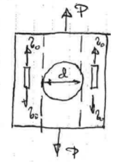
② Suorita tehtävän 1 tarkastelu kahden reiän tapauksessa.



③ Johda kinemaattisella menetelmällä yläraja puristetulle betonisärmiölle. Betonin otaksutaan noudattavan Mohrin- Coulombin myötöehtoa.



① a) TJT (TASOJÄNNITYSTILA) § TRESCA

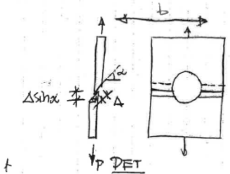


Alaraja

Oletetaan kuvan mukainen jännitysjakauma, joka täyttää tasapainoehdon

$\Rightarrow P^L = \tau_0 (b-d) t$

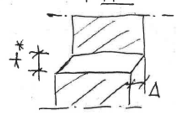
Yläraja



Oletetaan kuvan mukainen liukuma levyjen välillä.

Jos oletetaan liukupinnan muodostuvan DET mukaan t^* paksuisesta kerroksesta. Kerroksessa tapahtuva liukuma on

$$\gamma = \frac{\Delta}{t^*}$$



sisäiseksi työksi pinta-alan kohti saadaan

$$w_s = \tau_0 \gamma t^* = \tau_0 \Delta$$

$\Rightarrow$

Sisäinen työ koko liukupinnalla on

$$W_s = \tau_0 \Delta (b-d) \frac{t}{\cos\alpha}$$

Ulkoinen työ on

$$W_u = P \Delta \sin\alpha$$

$W_s = W_u$

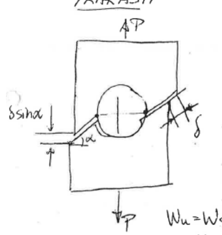
$$\Rightarrow P^u = \frac{\tau_0 (b-d) t}{\sin\alpha \cos\alpha} = \frac{2\tau_0 (b-d) t}{\sin 2\alpha}$$

$$P^u_{min} = 2\tau_0 (b-d) t = \sigma_0 (b-d) t \quad (\text{TRESCA})$$

b) TMT § TRESCA

ALARAJA
Sama kuin TJT:ssäkin $P^L = \sigma_0 (b-d) t$

YLÄRAJA



Oletetaan kuvan mukainen liukupinta. Liukupintojen pituus saa miniminsä kun liukupinnat lähtevät ympyrän ja sen keskiviivan leikkauspisteistä.

$$\Rightarrow W_s = \tau_0 \frac{(b-d)}{\cos\alpha} t \delta$$

Ulkoinen työ

$$W_u = P \delta \sin\alpha$$

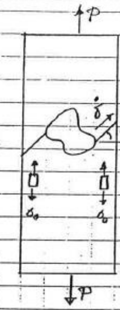
$W_u = W_s$

$$\Rightarrow P^u = \frac{\tau_0 (b-d) t \cancel{\delta}}{\cos\alpha \, \cancel{\delta} \sin\alpha}$$

$$P^u = \frac{2\tau_0 (b-d) t}{\sin 2\alpha} \quad \therefore \quad \sigma_0 = 2\tau_0 \ (\text{Tresca})$$

$$P^u_{min} = 2\tau_0 (b-d) t = \sigma_0 (b-d) t$$
$\alpha = 45°$

①
CHEN & HAN s. 486 P 8.1

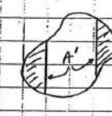


$$P^L = \sigma_0 A'$$

$$P^U \dot{\delta} \sin\alpha = k \frac{A'}{\cos\alpha} \dot{\delta}$$

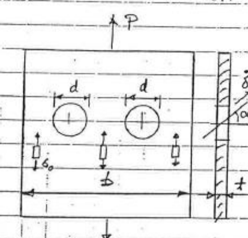
$$\Rightarrow \min_\alpha P^U = \min_\alpha \frac{2kA'}{\sin 2\alpha} = 2kA' = \sigma_0 A'$$

$$P^C = P^L = P^U = \sigma_0 A' \qquad \square$$



②
CHEN & HAN s. 487 P 8.2

(a) TJTK & TRESCA



$$P^L = \sigma_0 t (b-2d)$$

$$P^U \dot{\delta} \sin\alpha = k t (b-2d)/\cos\alpha \cdot \dot{\delta}$$

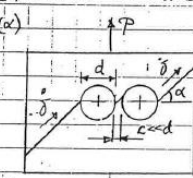
$$\min_\alpha P^U = \min \frac{2kt(b-2d)}{\sin 2\alpha} = \sigma_0 t (b-$$

$$= P^L \qquad \alpha = 45°$$

$$\Rightarrow P^C = \sigma_0 t (b-2d) \qquad \square$$

(b) TMTK & TRESCA

(α)



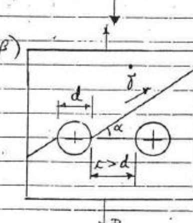
$$P^L = \sigma_0 t (b-2d)$$

$$P^U \dot{\delta} \sin\alpha = k t (b-2d)/\cos\alpha \cdot \dot{\delta}$$

$$\Rightarrow \min_\alpha P^U = \sigma_0 t (b-2d) \qquad \alpha = 45°$$

$$P^C = \sigma_0 t (b-2d)$$

(β)

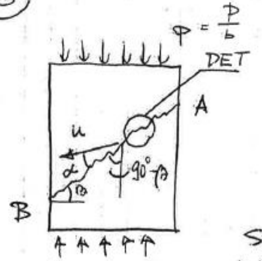


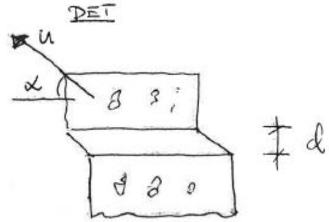
$$P^U \dot{\delta} \sin\alpha = k t (b-d)/\cos\alpha \cdot \dot{\delta}$$

$$\Rightarrow \min_\alpha P^U = \sigma_0 t (b-d) \qquad \alpha = 45°$$

③

$p = \frac{P}{b}$



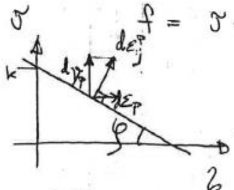


Särmiön ylempi osa liukuu matkan u alemman osan suhteen kulmassa α liukupintaan nähden. Liukupinnassa tapahtuu nyt sekä venymää että liukumaa.

$$\varepsilon = u\sin\alpha / d \qquad \gamma = u\cos\alpha / d \quad \Rightarrow \quad \frac{\varepsilon}{\gamma} = \tan\alpha \qquad (1)$$

Materiaali noudattaa Mohrin-Coulombin myötöehtoa

$$f = \sigma + \tau\tan\varphi - k = 0 \qquad (2)$$



Myötösäännön mukaan plastiset muodonmuutoslisäykset tapahtuvat kohtisuorassa myötöpintaa vastaan ts.

$$d\varepsilon^p_{ij} = d\lambda \frac{\partial f}{\partial \sigma_{ij}} \quad \Rightarrow \quad d\varepsilon_p = d\lambda \frac{\partial f}{\partial \tau} = \tan\varphi, \qquad d\gamma_p = d\lambda \frac{\partial f}{\partial \sigma} = 1$$

$$\Rightarrow \quad \frac{d\varepsilon_p}{d\gamma_p} = \tan\varphi \qquad (2)$$

Vertaamalla kaavoja (1) ja (2) havaitaan, että myötösäännön mukaan kulman $\alpha = \varphi$

$$\Rightarrow \quad \tau = \frac{k-\sigma}{\tan\varphi}$$

Sisäinen työ liukupinnalla:

$$W_s = \frac{b}{\cos\beta}(\sigma\gamma + \tau\varepsilon)\, d = \frac{b}{\cos\beta}\left(\sigma\frac{u\cos\varphi}{d} + \frac{(k-\sigma)}{\tan\varphi}\frac{u\sin\varphi}{d}\right) d = \frac{b}{\cos\beta}\left(\sigma\cos\varphi + (k-\sigma)\cos\varphi\right) u = \frac{bk}{\cos\beta}\cos\varphi \cdot u$$

$$W_u = P\cos(90° - \beta + \varphi) = Pu\sin(\beta - \varphi)$$

$$W_{sis} = W_{ulk} \Rightarrow \quad buk\frac{\cos\varphi}{\cos\beta} = Pu\sin(\beta-\varphi) \qquad (3)$$

$$\Rightarrow \quad k = \frac{P\sin(\beta-\varphi)}{b\cos\varphi}\cos\beta$$

$$\frac{dk}{d\beta} = \frac{P}{b\cos\varphi}\left\{\cos(\beta-\varphi)\cos\beta - \sin\beta\sin(\beta-\varphi)\right\} = 0$$

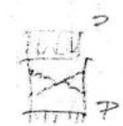
$$\Rightarrow \frac{P}{b\cos\varphi}\cos(\beta - \varphi + \beta) = 0 \quad \Rightarrow \cos(2\beta - \varphi) = 0$$

$$\Rightarrow \quad 2\beta - \varphi = \frac{\pi}{2} \quad \Rightarrow \quad \beta = \frac{\pi}{4} + \frac{\varphi}{2} \qquad (4)$$

$$(3) \Rightarrow \quad p = \frac{P}{b} = \frac{k\cos\varphi}{\sin(\beta-\varphi)\cos\beta} \qquad \text{sij } (4)$$

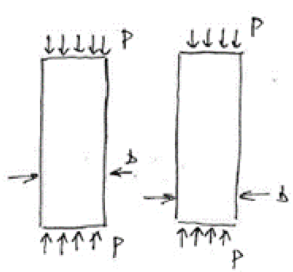
Tällöin $\sin(\beta-\varphi)\cos\beta = \sin(\frac{\pi}{4} - \frac{\varphi}{2})\cos(\frac{\pi}{4} + \frac{\varphi}{2}) = \frac{1}{2}(1 - \sin\varphi)$

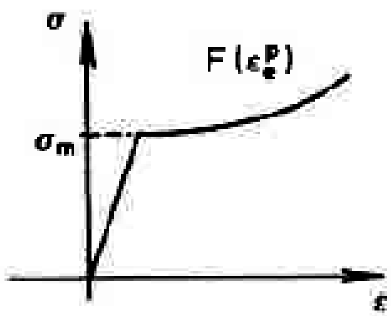
$$\Rightarrow (3) \Rightarrow \quad p = \frac{2k\cos\varphi}{1-\sin\varphi} = 2k\tan\left\{\frac{\pi}{4} + \frac{\varphi}{2}\right\}$$



sama voidaan [illegible] [illegible]

③ Johda kinemaattisella menetelmällä yläraja puristetulle betonisärmiölle. Betonin otaksutaan noudattavan Mohrin- Coulombin myötöehtoa.





H3.20

H3.20 Vedetyn sauvan materiaali on isotrooppisesti myötölujenevaa ja noudattaa Misesin myötöehtoa $f = \sigma_e - F(\epsilon_e^p)$, missä

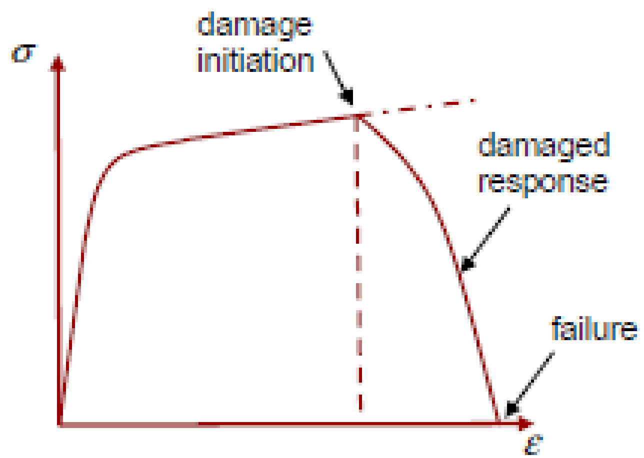
a) $F(\epsilon_e^p) = \begin{cases} \sigma_m + k\sqrt{\epsilon_e^p}, & d\epsilon_e^p > 0 \\ \sigma_m, & \text{muulloin} \end{cases}$

b) $F(\epsilon_e^p) = \begin{cases} \sigma_m + k(\epsilon_e^p)^{2/3}, & d\epsilon_e^p > 0 \\ \sigma_m, & \text{muulloin} \end{cases}$

Määritä sauvan jännitys-muodonmuutosriippuvuus. Kimmoiset muodonmuutokset otaksutaan merkityksettömän pieniksi.

Additional material

Ei jakeluun



Typical material response showing progressive damage

PT 1.2

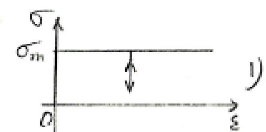
Muodonmuutos jaetaan kimmoiseen ja plastiseen osaan

$$\varepsilon = \varepsilon^e + \varepsilon^p \qquad (1)$$

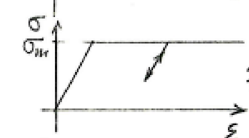
Kimmoinen osa on palautuva; plastinen osa on palautumaton eli irreversiibeli, ajasta riippumaton, selitetään ns. dislokaatioteorian avulla (ks. Hult: Hållfasthetslära s.209)

Käytettyjä idealisointeja

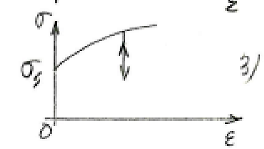
1) Jäykkä ideaaliplastinen



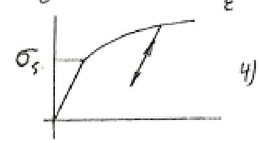
2) Kimmoinen ideaaliplastinen



3) Jäykkä myötölujeneva

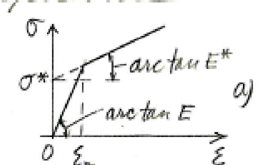


4) Kimmoinen myötölujeneva

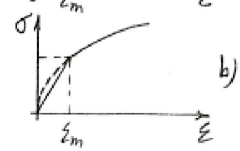


Analyyttisiä lausekkeita myötölujeneville malleille:

a) $$\sigma = \begin{cases} E\varepsilon, & -\varepsilon_m \le \varepsilon \le \varepsilon_m \\ \sigma^* \operatorname{sgn}\varepsilon + E^*\varepsilon, & |\varepsilon| \ge \varepsilon_m \end{cases}$$

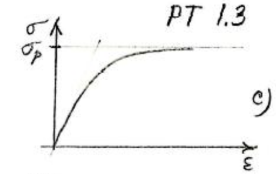


b) $$\sigma = \begin{cases} E\varepsilon, & |\varepsilon| \le \varepsilon_m \\ A|\varepsilon|^k \operatorname{sgn}\varepsilon, & (0 \le k \le 1), |\varepsilon| \ge \varepsilon_m \end{cases}$$



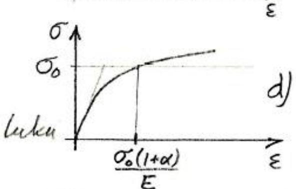
PT 1.3

c) $$\sigma = \sigma_p \tanh \frac{E\varepsilon}{\sigma_p}, \quad \forall \varepsilon$$



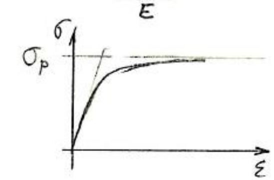
d) $$\varepsilon = \frac{\sigma}{E}\left[1 + \alpha\left|\frac{\sigma}{\sigma_0}\right|^{\beta-1}\right]$$

Ramberg-Osgood; $\alpha = 3/7$, n kokon. luku



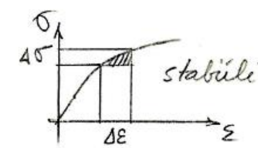
e) $$\varepsilon = \frac{\sigma}{E}\,\frac{1 - c\left|\frac{\sigma}{\sigma_p}\right|^n}{1 - \left|\frac{\sigma}{\sigma_p}\right|^n}$$

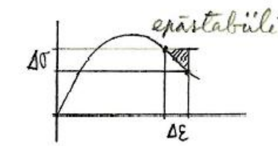
Ylinen: $0 \le c < 1$, n kokon. luku



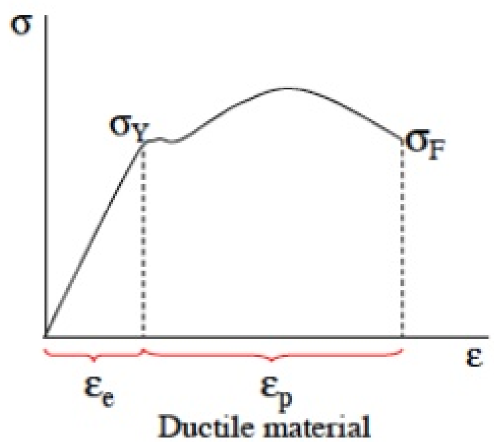
Stabiili – epästabiili

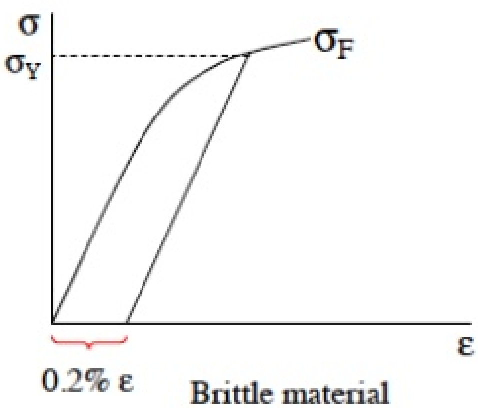
Ainetta sanotaan stabiiliksi tietyssä pisteessä, jos
$\Delta\sigma\,\Delta\varepsilon > 0$
muutoin epästabiiliksi.





Ductile vs. Brittle Material





- ✓ Ductile material : Well defined yield point– Failure on yielding.
- ✓ Brittle material : No yield point & sudden failure – Failure on failure load.

Joitakin esim. tehtäviä – plasticity

- Tresca
- Von Mises
- Mohr-Coulomb

plasticity

Rak-54.125 Plastisuus- ja viskoelastisuusteoria

Laskuharjoitus 17 23.3.1992

① Tarkastellaan tasomuodonmuutostilaa ja oletetaan, että pääjännitystä σ_3 vastaava venymä on nolla. Johda myötöehdot

a. Trescan mallille

b. Misesin mallille

c. Mohrin- Coulombin mallille.

Voidaanko käyttää kimmoteorian mukaista yhteyttä $\varepsilon_3 = (\sigma_3 - \nu\cdot(\sigma_1 + \sigma_2))/E = 0$ ja sijoittaa $\sigma_3 = \nu\cdot(\sigma_1 + \sigma_2)$ yleiseen myötöehtoon ?

② Ohutseinäistä ideaalikimmoista putkea vedetään, kunnes se myötää, siten sitä aletaan vääntää kuitenkin niin, että aksiaalinen siirtymä pysyy vakiona. Määritä putken liukukulma γ_{xy} leikkausjännityksen τ_{xy} funktiona.

③ Laske paksuseinäiseen putkeen jääviä jäännösjännityksiä, kun sisäpuolinen paine p poistetaan. (jatkoa tehtävälle 3 /laskuharjoitus 15). Miten jäännösjännitykset suhtautuvat uudelleen kuormitettaessa ?

Rak- 54.125 1(2) 9.5

① Kimmoteorian yhteyttä $\varepsilon_3 = \frac{1}{E}(\sigma_3 - \nu(\sigma_1+\sigma_2)) = 0$ ei voi käyttää. Tämä johtaisi ristiriitaiseen tilanteeseen Trescan ja Mohr-Coulombin mallilla, jos ei olisi tilanne

$$\sigma_1 \geq \sigma_3 \geq \sigma_2 \quad \text{tai} \quad \sigma_1 \leq \sigma_3 \leq \sigma_2$$

Tällöin yhtyvät tasojännitystilaan. Jos σ_3 olisi suurin tai pienin pääjännitys murtotaso olisi vino σ_1-σ_2 -pintaa vastaan. Tämä rikkoo taso- muodonmuutostilan oletusta vastaan.

Tasomuodonmuutostilassa otaksutaan kimmoiset venymät häviävän pieniksi plastisiin verrattuina (oikeutettua murtotilaa tarkasteltaessa).

$\Rightarrow$ voidaan otaksua kokoonpuristumattomuus

$$\varepsilon_x + \varepsilon_y = 0 \Rightarrow \nu = 0{,}5$$

tällöin $\varepsilon_3 = \frac{1}{E}(\sigma_3 - \nu(\sigma_1+\sigma_2)) = 0$

$$\Rightarrow \sigma_3 = 0{,}5(\sigma_1+\sigma_2)$$

$$\Rightarrow \sigma_1 \leq \sigma_3 \leq \sigma_2$$

tai

$$\sigma_1 \geq \sigma_3 \geq \sigma_2$$

a) Tresca

$$\sigma_m = |\sigma_1 - \sigma_2|$$

b) Mises

$$\sqrt{\tfrac{1}{2}[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2} = \sigma_m$$

sijoitetaan $\sigma_3 = 0{,}5(\sigma_1+\sigma_2)$

$$\Rightarrow \tfrac{1}{2}[(\sigma_1-\sigma_2)^2 + (\sigma_2 - \tfrac{1}{2}(\sigma_1+\sigma_2))^2 + (\tfrac{1}{2}(\sigma_1+\sigma_2)-\sigma_1)^2] = \sigma_m^2$$

$$\Rightarrow \tfrac{1}{2}[(\sigma_2-\sigma_1)^2 + \tfrac{1}{4}(\sigma_2-\sigma_1)^2 + \tfrac{1}{4}(\sigma_2-\sigma_1)^2] = \sigma_m^2$$

$$\Rightarrow \tfrac{3}{4}(\sigma_1-\sigma_2)^2 = \sigma_m^2$$

c) Mohr- Coulomb

$$m\sigma_1 - \sigma_2 = f_c \quad , \sigma_1 \geq \sigma_3 \geq \sigma_2$$

$$m\sigma_2 - \sigma_1 = f_c \quad , \sigma_1 \leq \sigma_3 \leq \sigma_2$$

$$m = \frac{f_c}{f_t}$$

plasticity

Rak-54.125 Plastisuus- ja viskoelastisuusteoria
Laskuharjoitus 17 23.3.1992

① Tarkastellaan tasomuodonmuutostilaa ja oletetaan, että pääjännitystä σ_3 vastaava venymä on nolla. Johda myötöehdot

a. Trescan mallille

b. Misesin mallille

c. Mohrin- Coulombin mallille.

Voidaanko käyttää kimmoteorian mukaista yhteyttä $\varepsilon_3 = (\sigma_3 - \nu\cdot(\sigma_1 + \sigma_2))/E = 0$ ja sijoittaa $\sigma_3 = \nu\cdot(\sigma_1 + \sigma_2)$ yleiseen myötöehtoon ?

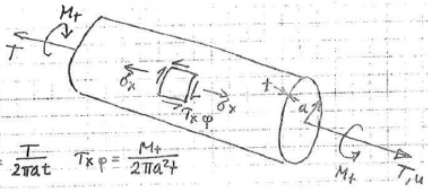
② Ohutseinäistä ideaalikimmoista putkea vedetään, kunnes se myötää, siten sitä aletaan vääntää kuitenkin niin, että aksiaalinen siirtymä pysyy vakiona. Määritä putken liukukulma γ_{xy} leikkausjännityksen τ_{xy} funktiona.

③ Laske paksuseinäiseen putkeen jääviä jäännösjännityksiä, kun sisäpuolinen paine p poistetaan. (jatkoa tehtävälle 3 /laskuharjoitus 15).
Miten jäännösjännitykset suhtautuvat uudelleen kuormitettaessa ?

7.54.125 PL & VE

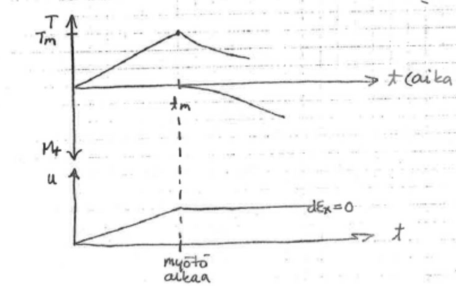
harj:

H 3,17



TSP $\sigma_x = \dfrac{T}{2\pi a t}$ $\tau_{x\varphi} = \dfrac{M_t}{2\pi a^2 t}$

Kuormitushistoria



Misesin myötöehto $\sigma_e - \sigma_m = 0$ (σ_m vakio)

$\sigma_e = \sqrt{\sigma_x^2 + 3\tau_{x\varphi}^2}$

Tilanne hetkestä t_m eteenpäin:

Kimmoiset venymän lisäykset (Hooken laki)

$$d\varepsilon_x^k = \frac{d(\sigma_x - \nu\sigma_\varphi)}{E} = \frac{d\sigma_x}{E} \qquad (\sigma_\varphi = 0)$$

$$\left(d\varepsilon_\varphi^k = \frac{d(\sigma_\varphi - \nu\sigma_x)}{E} = -\nu \frac{d\sigma_x}{E} \right)$$

$$d\gamma_{x\varphi}^k = \frac{d\tau_{x\varphi}}{G} = \frac{2(1+\nu)}{E} d\tau_{x\varphi}$$

7.54.125 PL & VE harj.

H3,17 jatkuu

Plastiset venymän lisäykset (Prandtl-Reussin yht.)

$$d\varepsilon_x^p = d\lambda \frac{3(\sigma_x - p)}{2\sigma_e} = d\lambda \frac{\sigma_x}{\sigma_m}$$

$$\left(d\varepsilon_\varphi^p = d\lambda \frac{3(\sigma_\varphi - p)}{2\sigma_e} = -d\lambda \frac{\sigma_x}{2\sigma_m} \right)$$

$$d\gamma_{x\varphi}^p = d\lambda \frac{3\tau_{x\varphi}}{\sigma_e} = d\lambda \frac{3\tau_{x\varphi}}{\sigma_m}$$

paine $p = \frac{1}{3}(\sigma_x + \overset{0}{\cancel{\sigma_\varphi}} + \overset{0}{\cancel{\sigma_r}})$
$= \frac{1}{3}\sigma_x$,
myötö jatkuu $\Rightarrow \sigma_e = \sigma_m$

yhdistettynä $d\varepsilon_x = d\varepsilon_x^k + d\varepsilon_x^p$ jne.

Ehdosta $d\varepsilon_x = d\varepsilon_x^k + d\varepsilon_x^p = 0$ seuraa

$$d\lambda = -\frac{d\sigma_x}{\sigma_x}\frac{\sigma_m}{E}.$$

$$d\gamma_{x\varphi} = d\gamma_{x\varphi}^k + d\gamma_{x\varphi}^p = \frac{2(1+\nu)}{E} d\tau_{x\varphi} + d\lambda \frac{3\tau_{x\varphi}}{\sigma_m}$$
$$= \frac{2(1+\nu)}{E} d\tau_{x\varphi} - \frac{3\tau_{x\varphi}}{E}\frac{d\sigma_x}{\sigma_x}$$

Jännityksiä sitoo myötöehto $\sigma_e = \sigma_m$ (vakio)

$\Rightarrow \sigma_e^2 = \sigma_x^2 + 3\tau_{x\varphi}^2 = \sigma_m^2$ (vakio) $\Rightarrow \sigma_x^2 = \sigma_m^2 - 3\tau_{x\varphi}^2$

Kok. differentiaali $d(\sigma_e^2) = 2\sigma_x d\sigma_x + 6\tau_{x\varphi} d\tau_{x\varphi} = 0$

$$\Rightarrow d\sigma_x = -3\frac{\tau_{x\varphi} d\tau_{x\varphi}}{\sigma_x} \Rightarrow \frac{d\sigma_x}{\sigma_x} = 3\frac{\tau_{x\varphi} d\tau_{x\varphi}}{3\tau_{x\varphi}^2 - \sigma_m^2}$$

sij. $d\gamma_{x\varphi}$:n lausekkeeseen

$$d\gamma_{x\varphi} = \frac{1}{E}\left\{ 2(1+\nu) - 9\frac{\tau_{x\varphi}^2}{3\tau_{x\varphi}^2 - \sigma_m^2} \right\} d\tau_{x\varphi}$$

integroidaan hetkestä t_m, jossa $\gamma_{x\varphi} = \tau_{x\varphi} = 0$

$$\Rightarrow \gamma_{x\varphi} = \int_0^{\tau_{x\varphi}} \frac{d\gamma_{x\varphi}}{d\tau_{x\varphi}} d\tau_{x\varphi} = \frac{1}{E}\left[(2\nu - 1)\tau_{x\varphi} + \frac{\sqrt{3}}{2} \ln \frac{\sigma_m + \sqrt{3}\tau_{x\varphi}}{\sigma_m - \sqrt{3}\tau_{x\varphi}} \right]$$

(Integrointi: $\int \frac{x^2}{ax^2 + c} dx = \frac{x}{a} - \frac{c}{a}\int \frac{dx}{ax^2 + c}$
ja $\int \frac{dx}{ax^2 + c} = \frac{1}{2\sqrt{-ac}} \log \left| \frac{x\sqrt{a} - \sqrt{-c}}{x\sqrt{a} + \sqrt{-c}} \right|$ kun $a > 0$, $c < 0$
kts. esim. Pentikäinen)

plasticity

Täysin kimmoinen

$$(1)\begin{cases}\sigma_r = \dfrac{pa^2}{b^2-a^2}\left[1-\left(\dfrac{b}{r}\right)^2\right]\\[2mm] \sigma_\varphi = \dfrac{pa^2}{b^2-a^2}\left[1+\left(\dfrac{b}{r}\right)^2\right]\\[2mm] \sigma_z = \dfrac{2pa^2}{b^2-a^2}\end{cases}$$

Osittain plastisoitunut

plastinen osa $a \le r \le c$

$$(2)\begin{cases}\sigma_r = \sigma_m\left[\ln\dfrac{r}{c} - \dfrac{1}{2}\left(1-\left(\dfrac{c}{b}\right)^2\right)\right]\\[2mm] \sigma_\varphi = \sigma_m\left[1+\ln\dfrac{r}{c} - \dfrac{1}{2}\left(1-\left(\dfrac{c}{b}\right)^2\right)\right]\end{cases}$$

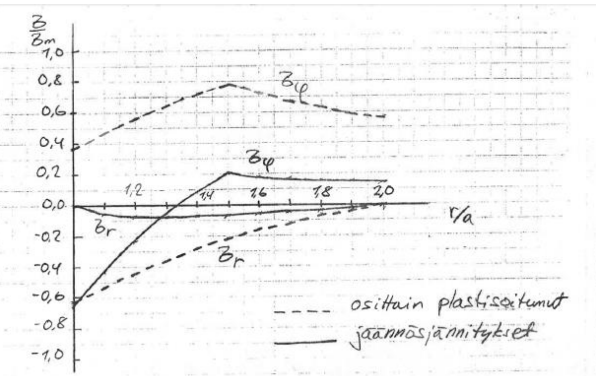
kimmoinen osa $c \le r \le b$

$$(3)\begin{cases}\sigma_r = \dfrac{\sigma_m}{2}\dfrac{c^2}{b^2-c^2}\left[1-\left(\dfrac{c}{b}\right)^2\right]\left[1-\left(\dfrac{b}{r}\right)^2\right]\\[2mm] \sigma_\varphi = \dfrac{\sigma_m}{2}\dfrac{c^2}{b^2-c^2}\left[1-\left(\dfrac{c}{b}\right)^2\right]\left[1+\left(\dfrac{b}{r}\right)^2\right]\end{cases}$$

Täysin kimmoinen paineella p

$$p = -\sigma_m\left[-\ln\frac{a}{c}+\frac{1}{2}\left(1-\left(\frac{c}{b}\right)^2\right)\right]$$

$$(4)\begin{cases}\sigma_r = -\dfrac{\sigma_m a^2}{b^2-a^2}\left[-\ln\dfrac{a}{c}+\dfrac{1}{2}\left(1-\left(\dfrac{c}{b}\right)^2\right)\right]\left[1-\left(\dfrac{b}{r}\right)^2\right]\\[2mm] \sigma_\varphi = -\dfrac{\sigma_m a^2}{b^2-a^2}\left[-\ln\dfrac{a}{c}+\dfrac{1}{2}\left(1-\left(\dfrac{c}{b}\right)^2\right)\right]\left[1+\left(\dfrac{b}{r}\right)^2\right]\end{cases}$$



Asetetaan putken mitoiksi: $b = 2a \quad c = \frac{3}{2}a$

Osittain plastisoituneen rakenteen jännitykset voidaan laskea

$a \le r \le c$

$$\sigma_r = \sigma_m\left[\ln\frac{2}{3}\frac{r}{a} - \frac{7}{32}\right]$$

$$\sigma_\varphi = \sigma_m\left[1+\ln\frac{2}{3}\frac{r}{a} - \frac{7}{32}\right]$$

r/a	σ_r/σ_m	σ_φ/σ_m
1,0	-0,624	+0,376
1,1	-0,529	0,471
1,2	-0,442	0,558
1,3	-0,362	0,638
1,4	-0,288	0,712
1,5	-0,219	0,781

$c \le r \le b$

$$\sigma_r = \sigma_m\left[\frac{9}{32} - \frac{9}{8}\left(\frac{a}{r}\right)^2\right]$$

$$\sigma_\varphi = \sigma_m\left[\frac{9}{32} + \frac{9}{8}\left(\frac{a}{r}\right)^2\right]$$

r/a	σ_r/σ_m	σ_φ/σ_m
1,5	-0,219	0,781
1,6	-0,158	0,721
1,7	-0,108	0,671
1,8	-0,066	0,628
1,9	-0,030	0,593
2,0	-0,000	0,563

ännösjännitykset saadaan vähentämällä
stisoituneen tilan jännityksistä (yhtälöt 2 ja 3)
taavan negatiivisen kuorman aiheuttama
aarinen jännitystila (yhtälöt 4)

Jäännösjännitysten käytännön merkitys:

A practical application of shakedown in thick tubes is the "autofrettage" process, which has been used for many years in the manufacture of gun barrels. It is clearly desirable that the inner bore of the barrel should retain its dimensional accuracy on repeated pressurization due to firing. By subjecting the barrel to an *overpressure* before the final surface machining is done, a residual stress system is set up in the barrel which ensures that the bore never goes into the plastic range subsequently, under normal conditions.

Murtumis- ja väsymistarkastelut ovat osoittautuneet tarpeellisiksi monien onnettomuuksien syitä selvitettäessä. Ensimmäinen suihkumoottorikäyttöinen matkustajakone Comet otettiin käyttöön vuonna 1952 yli 300 tuntia kestäneen koelento-ohjelman jälkeen. Neljä päivää tarkastusten jälkeen kone syöksyi Välimereen. Tutkimukset paljastivat, että rungon ikkuna-aukkojen kulmista oli lähtenyt liikkeelle väsymissäröjä. Runko oli paineistettu 18000 kertaa matkalentokorkeutta vastaavalla ylipaineella, jonka aiheuttama jännitys oli 40 % käytetyn alumiiniseoksen murtolujuudesta. Tätä kuormituskoetta ennen runkoa oli kuitenkin kuormitettu 30 kertaa 1.75...2.75 -kertaisella paineella. Tällainen ylikuormitus synnytti ilmeisesti kriittisiin kohtiin plastisten muodonmuutosten vuoksi niin suuren puristavan jäännösjännitystilan, että koerunko ei murtunut vaan antoi virheellisen kuvan rungon väsymiskestävyydestä todellisessa kuormitustilanteessa.

$a \le r \le c$

$$\sigma_r^j = \sigma_m\left[\ln\frac{r}{c} - \frac{1}{2}\left(1-\left(\frac{c}{b}\right)^2\right)\right] - \sigma_m\frac{a^2}{b^2-a^2}\left[-\ln\frac{a}{c}+\frac{1}{2}\left(1-\left(\frac{c}{b}\right)^2\right)\right]\left[1-\left(\frac{b}{r}\right)^2\right]$$

$$= \sigma_m\left[\ln\frac{2r}{3a} - \frac{1}{2}\left(1-\frac{9}{16}\right)\right] - \sigma_m\frac{1}{3}\left[-\ln\frac{2}{3}+\frac{1}{2}\left(1-\frac{9}{16}\right)\right]\left[1-4\left(\frac{a}{r}\right)^2\right]$$

$$= \sigma_m\left[-0{,}427 + \ln\frac{2r}{3a} + 0{,}832\left(\frac{a}{r}\right)^2\right]$$

$$\sigma_\varphi^j = \sigma_m\left[1+\ln\frac{r}{c} - \frac{1}{2}\left(1-\left(\frac{c}{b}\right)^2\right)\right] - \sigma_m\frac{a^2}{b^2-a^2}\left[-\ln\frac{a}{c}+\frac{1}{2}\left(1-\left(\frac{c}{b}\right)^2\right)\right]\left[1+\left(\frac{b}{r}\right)^2\right]$$

$$= \sigma_m\left[1+\ln\frac{2r}{3a} - \frac{1}{2}\left(1-\frac{9}{16}\right)\right] - \sigma_m\frac{1}{3}\left[-\ln\frac{2}{3}+\frac{1}{2}\left(1-\frac{9}{16}\right)\right]\left[1+4\left(\frac{a}{r}\right)^2\right]$$

$$= \sigma_m\left[0{,}573 + \ln\frac{2r}{3a} - 0{,}832\left(\frac{a}{r}\right)^2\right]$$

r/a	σ_r^j/σ_m	$\sigma_\varphi^j/\sigma_m$
1,0	0,000	-0,665
1,1	-0,049	-0,425
1,2	-0,072	-0,228
1,3	-0,077	-0,062
1,4	-0,071	+0,080
1,5	-0,057	+0,203

$c \le r \le b$

$$\sigma_r^j = \frac{\sigma_m}{2}\frac{c^2}{b^2-c^2}\left[1-\left(\frac{c}{b}\right)^2\right]\left[1-\left(\frac{b}{r}\right)^2\right] - \sigma_m\frac{a^2}{b^2-a^2}\left[-\ln\frac{a}{c}+\frac{1}{2}\left(1-\left(\frac{c}{b}\right)^2\right)\right]\left[1-\left(\frac{b}{r}\right)^2\right]$$

$$= \frac{\sigma_m}{2}\frac{\frac{9}{4}a^2}{4a^2-\frac{9}{4}a^2}\left[1-\frac{9}{16}\right]\left[1-4\left(\frac{a}{r}\right)^2\right] - \sigma_m\frac{a^2}{4a^2-a^2}\left[-\ln\frac{2a}{3a}+\frac{1}{2}\left(1-\frac{9}{16}\right)\right]\left[1-4\left(\frac{a}{r}\right)^2\right]$$

$$= \sigma_m\left[0{,}073 - 0{,}293\left(\frac{a}{r}\right)^2\right]$$

$$\sigma_\varphi^j = \frac{\sigma_m}{2}\frac{c^2}{b^2-c^2}\left[1-\left(\frac{c}{b}\right)^2\right]\left[1+\left(\frac{b}{r}\right)^2\right] - \sigma_m\frac{a^2}{b^2-a^2}\left[-\ln\frac{a}{c}+\frac{1}{2}\left(1-\left(\frac{c}{b}\right)^2\right)\right]\left[1+\left(\frac{b}{r}\right)^2\right]$$

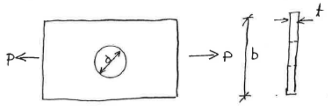
$$= \sigma_m\frac{9}{14}\cdot\frac{7}{16}\left[1+4\left(\frac{a}{r}\right)^2\right] - \sigma_m\frac{1}{3}\left[-\ln\frac{2}{3}+\frac{7}{32}\right]\left[1+4\left(\frac{a}{r}\right)^2\right]$$

$$= \sigma_m\left[0{,}073 + 0{,}293\left(\frac{a}{r}\right)^2\right]$$

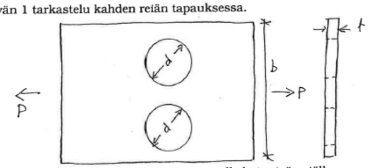
r/a	σ_r^j/σ_m	$\sigma_\varphi^j/\sigma_m$
1,5	-0,057	0,203
1,6	-0,041	0,188
1,7	-0,028	0,174
1,8	-0,017	0,164
1,9	-0,008	0,154
2,0	0,000	0,146

Rak-54.125 Plastisuus- ja viskoelastisuusteoria
Laskuharjoitus 18 25.3.1992

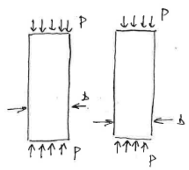
① Tarkastellaan vedettyä levyä, jossa on pyöreä reikä. Johda vetävälle rajakuormalle P staattisella menetelmällä alaraja ja kinemaattisella menetelmällä yläraja. Oletetaan aineen noudattavan Trescan myötöehtoa. Tarkastele sekä tasojännitystilaa (ohut levy) että tasomuodonmuutostilaa. Miten reiän muoto vaikuttaa ratkaisuun ?



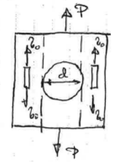
② Suorita tehtävän 1 tarkastelu kahden reiän tapauksessa.



③ Johda kinemaattisella menetelmällä yläraja puristetulle betonisärmiölle. Betonin otaksutaan noudattavan Mohrin- Coulombin myötöehtoa.



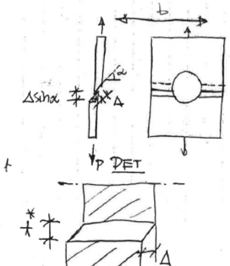
① a) TJT (TASOJÄNNITYSTILA) § TRESCA



Alaraja

Oletetaan kuvan mukainen jännitysjakauma, joka täyttää tasapainoehdon

$$\Rightarrow P^L = \tau_0 (b-d) t$$



Yläraja

Oletetaan kuvan mukainen liukuma levyjen välillä.

Jos oletetaan liukupinnan muodostuvan DET mukaan t^* paksuisesta kerroksesta. Kerroksessa tapahtuva liukuma on

$$\gamma = \frac{\Delta}{t^*}$$

sisäiseksi työksi pinta-alan kohti saadaan

$$w_s = \tau_0 \gamma t^* = \tau_0 \Delta$$

$\Rightarrow$

Sisäinen työ koko liukupinnalla on

$$W_S = \tau_0 \Delta (b-d) \frac{t}{\cos\alpha}$$

Ulkoinen työ on

$$W_u = P \Delta \sin\alpha$$

$W_S = W_u$

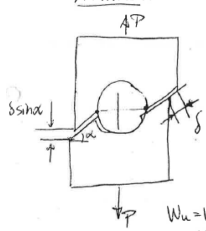
$$\Rightarrow P^u = \frac{\tau_0 (b-d) t}{\sin\alpha \cos\alpha} = \frac{2\tau_0 (b-d) t}{\sin 2\alpha}$$

$$P^u_{min} = 2\tau_0 (b-d) t = \sigma_0 (b-d) t \quad \text{(TRESCA)}$$

b) TMT § TRESCA

ALARAJA
Sama kuin TJT:ssäkin $P^L = \sigma_0 (b-d) t$

YLÄRAJA



Oletetaan kuvan mukainen liukupinta. Liukupintojen pituus saa miniminsä kun liukupinnat lähtevät ympyrän ja sen keskiviivan leikkauspisteistä.

$$\Rightarrow W_S = \tau_0 \frac{(b-d)}{\cos\alpha} t \delta$$

Ulkoinen työ

$$W_u = P \delta \sin\alpha$$

$W_u = W_S$

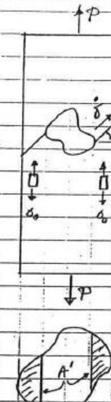
$$\Rightarrow P^u = \frac{\tau_0 (b-d) t \cancel{\delta}}{\cos\alpha \cancel{\delta} \sin\alpha}$$

$$P^u = \frac{2\tau_0 (b-d) t}{\sin 2\alpha} \quad \therefore \quad \sigma_0 = 2\tau_0 \text{ (Tresca)}$$

$$P^u_{min} = 2\tau_0 (b-d) t = \sigma_0 (b-d) t$$
$\alpha = 45°$

harjoitus — n:o 18 24

① CHEN & HAN s. 486 P8.1



$$P^L = \sigma_0 A'$$

$$P^U \delta \sin\alpha = k \frac{A'}{\cos\alpha} \delta$$

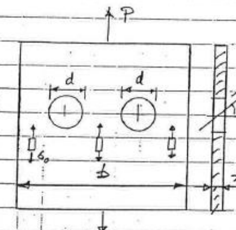
$$\Rightarrow \min_\alpha P^U = \min_\alpha \frac{2kA'}{\sin 2\alpha} = 2kA' = \sigma_0 A'$$

$$P^c = P^L = P^U = \sigma_0 A' \quad \square$$

25.3.1992 25

② CHEN & HAN s. 487 P8.2

(a) TJTK & TRESCA



$$P^L = \sigma_0 t (b-2d)$$

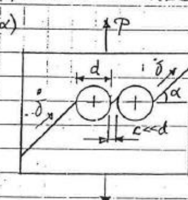
$$P^U \delta \sin\alpha = k t (b-2d)/\cos\alpha \cdot \delta$$

$$\min_\alpha P^U = \min \frac{2kt(b-2d)}{\sin 2\alpha} = \sigma_0 t (b-2d) = P^L \quad \alpha = 45°$$

$$\Rightarrow P^c = \sigma_0 t (b-2d) \quad \square$$

(b) TMTK & TRESCA

(α)



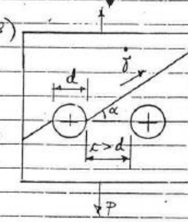
$$P^L = \sigma_0 t (b-2d)$$

$$P^U \delta \sin\alpha = k t (b-2d)/\cos\alpha \cdot \delta$$

$$\Rightarrow \min_\alpha P^U = \sigma_0 t (b-2d) \quad \alpha = 45°$$

$$P^c = \sigma_0 t (b-2d)$$

(β)

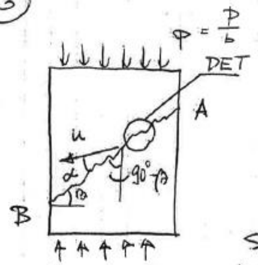


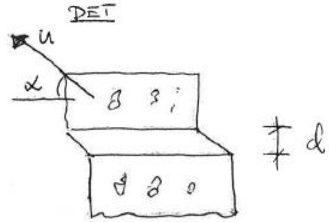
$$P^U \delta \sin\alpha = k t (b-d)/\cos\alpha \cdot \delta$$

$$\Rightarrow \min_\alpha P^U = \sigma_0 t (b-d) \quad \alpha = 45°$$

③

$p = \frac{P}{b}$



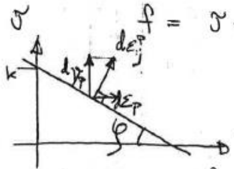


Särmiön ylempi osa liukuu matkan u alemman osan suhteen kulmassa α liukupintaan nähden. Liukupinnassa tapahtuu nyt sekä venymää että liukumaa.

$$\varepsilon = u \sin\alpha / d \quad \Rightarrow \quad \underline{\frac{\varepsilon}{\gamma} = \tan\alpha} \qquad (1)$$
$$\gamma = u\cos\alpha / d$$

Materiaali noudattaa Mohrin-Coulombin myötöehtoa

$$f = \sigma + \tau\tan\varphi - k = 0 \qquad (2)$$



Myötösäännön mukaan plastiset muodonmuutoslisäykset tapahtuvat kohtisuorassa myötöpintaa vastaan ts.

$$d\varepsilon^p_{ij} = d\lambda \frac{\partial f}{\partial \sigma_{ij}} \Rightarrow \quad d\varepsilon_p = d\lambda \frac{\partial f}{\partial \tau} = \tan\varphi, \qquad d\gamma_p = d\lambda \frac{\partial f}{\partial \sigma} = 1$$

$$\Rightarrow \underline{\underline{\frac{d\varepsilon_p}{d\gamma_p} = \tan\varphi}} \qquad (2)$$

Vertaamalla kaavoja (1) ja (2) havaitaan, että myötösäännön mukaan kulman $\underline{\alpha = \varphi}$

$$\Rightarrow \quad \tau = \frac{k-\sigma}{\tan\varphi}$$

Sisäinen työ liukupinnalla:

$$W_s = \frac{b}{\cos\beta}\left(\sigma\gamma + \tau\varepsilon\right) d$$
$$= \frac{b}{\cos\beta}\left(\sigma \frac{u\cos\varphi}{d} + \frac{(k-\sigma)}{\tan\varphi}\frac{u\sin\varphi}{d}\right) d =$$
$$= \frac{b}{\cos\beta}\left(\sigma\cos\varphi + (k-\sigma)\cos\varphi\right) u = \frac{bk}{\cos\beta}\cos\varphi \cdot u$$

$$W_u = P\cos(90^\circ - \beta + \varphi) = Pu\sin(\beta - \varphi)$$

$$W_{sis} = W_{ulk} \Rightarrow \quad buk\frac{\cos\varphi}{\cos\beta} = Pu\sin(\beta-\varphi) \qquad (3)$$

$$\Rightarrow \quad k = \frac{P\sin(\beta-\varphi)}{b\cos\varphi}\cos\beta$$

$$\frac{dk}{d\beta} = \frac{P}{b\cos\varphi}\left\{\cos(\beta-\varphi)\cos\beta - \sin\beta\sin(\beta-\varphi)\right\} = 0$$

$$\Rightarrow \frac{P}{b\cos\varphi}\cos(\beta - \varphi + \beta) = 0 \Rightarrow \cos(2\beta - \varphi) = 0$$

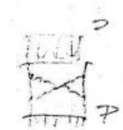
$$\Rightarrow 2\beta - \varphi = \frac{\pi}{2} \Rightarrow \underline{\underline{\beta = \frac{\pi}{4} + \frac{\varphi}{2}}} \qquad (4)$$

$$(3) \Rightarrow \quad p = \frac{P}{b} = \frac{k\cos\varphi}{\sin(\beta-\varphi)\cos\beta} \qquad \text{sij } (4)$$

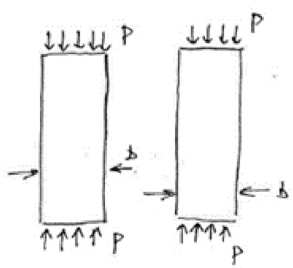
Tällöin $\sin(\beta-\varphi)\cos\beta = \sin\left(\frac{\pi}{4} - \varphi/2\right)\cos\left(\frac{\pi}{4} + \frac{\varphi}{2}\right) = \frac{1}{2}(1-\sin\varphi)$

$$\Rightarrow (3) \Rightarrow \quad p = \frac{2k\cos\varphi}{1-\sin\varphi} = \underline{\underline{2k\tan\left\{\frac{\pi}{4} + \frac{\varphi}{2}\right\}}}$$

sama voidaan [illegible] mekanismille

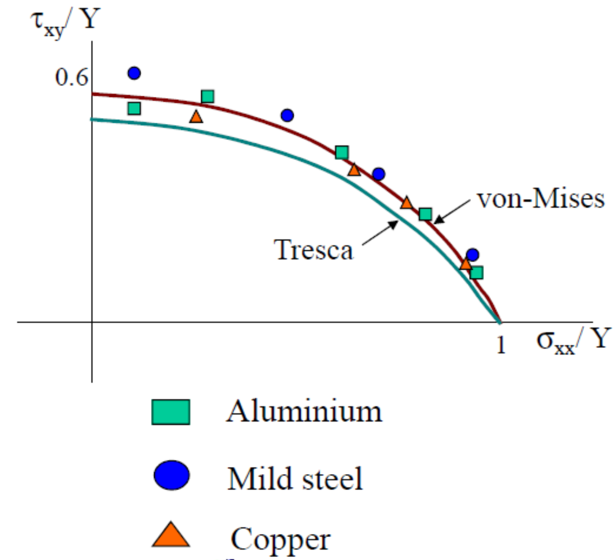


③ **Johda kinemaattisella menetelmällä yläraja puristetulle betonisärmiölle. Betonin otaksutaan noudattavan Mohrin- Coulombin myötöehtoa.**



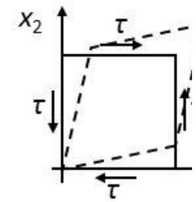
Plasticity - thanks n to prof. Woitek

- yield surface (yield locus)
  - why , how it is defined, what it defines
- plastic potential surface and flow rule
  - plastic potential surface (why we need it, what it defines)
  - what is flow rule, why we need it, differences between associated and non-associated flow rule
- hardening rule,
  - what is it, why we use it
- consistency conditions
  - why we need it, what we use it for
- elasto-plastic tangent matrix (what it is, why we need it)



Example

- Pure shear stress τ to yield



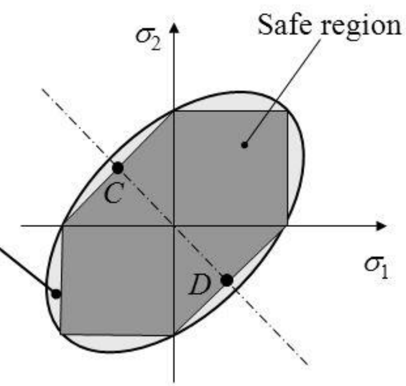
$$\boldsymbol{\sigma} = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{s}$$

$$\|\mathbf{s}\| = \sqrt{\mathbf{s} : \mathbf{s}} = \sqrt{\tau^2 + \tau^2} = \tau\sqrt{2}$$

- Yield surface:

$$\sqrt{2}\tau = \sqrt{\tfrac{2}{3}}\sigma_Y \quad \Rightarrow \quad \boxed{\tau = \frac{1}{\sqrt{3}}\sigma_Y}$$

- Failure in max. shear stress theory
  Safe in distortion energy theory
- Von Mises is more accurate, but
  Tresca is more conservative



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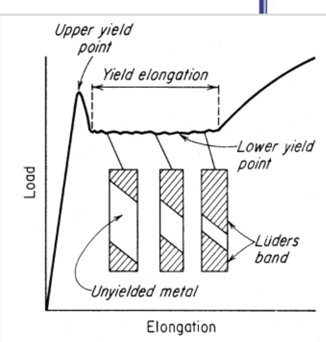
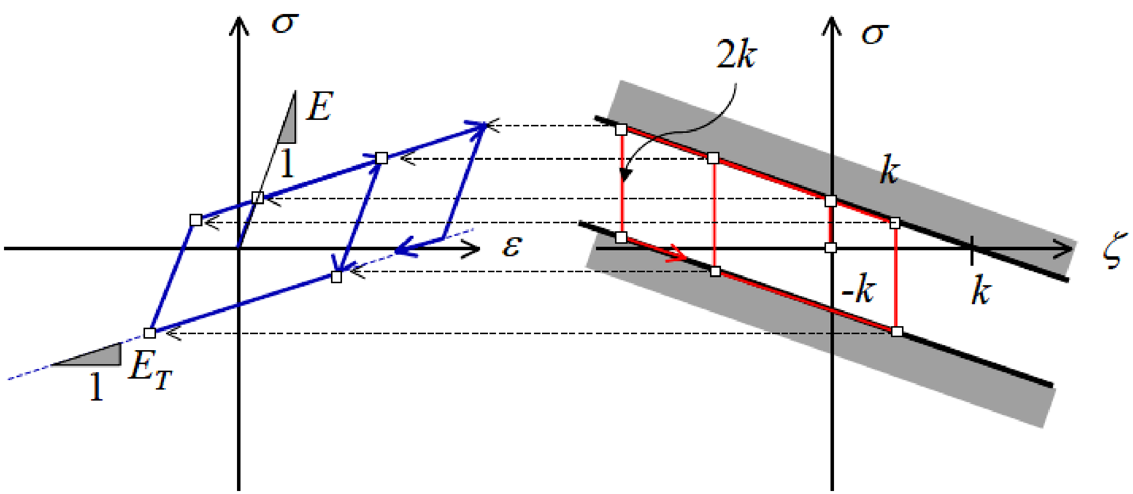


1D-Kinematic Hardening Model



1D →

Stress σ, Strain ε

Plastic Strain ε^p

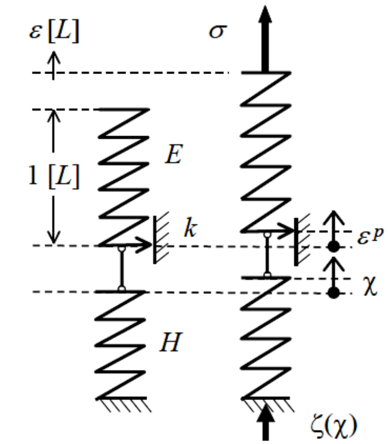
Hardening Variable χ

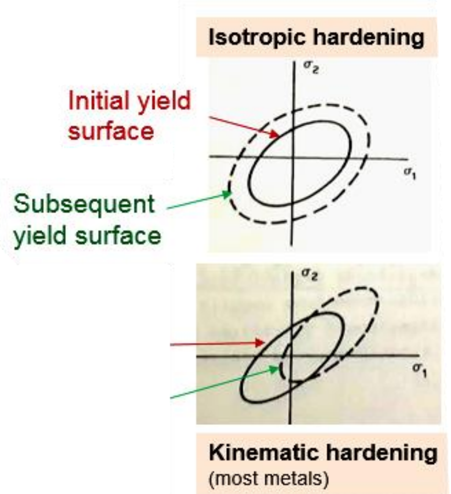
Hardening Force ζ

$f = |\sigma+\zeta| - k \leq 0$

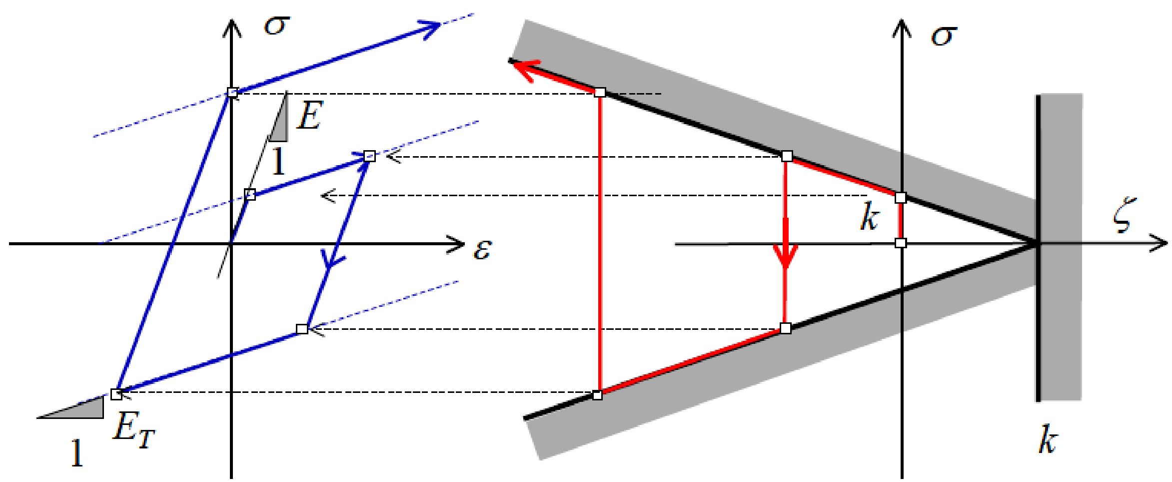
$\varphi dt = \sigma d\varepsilon - d\Psi \geq 0$

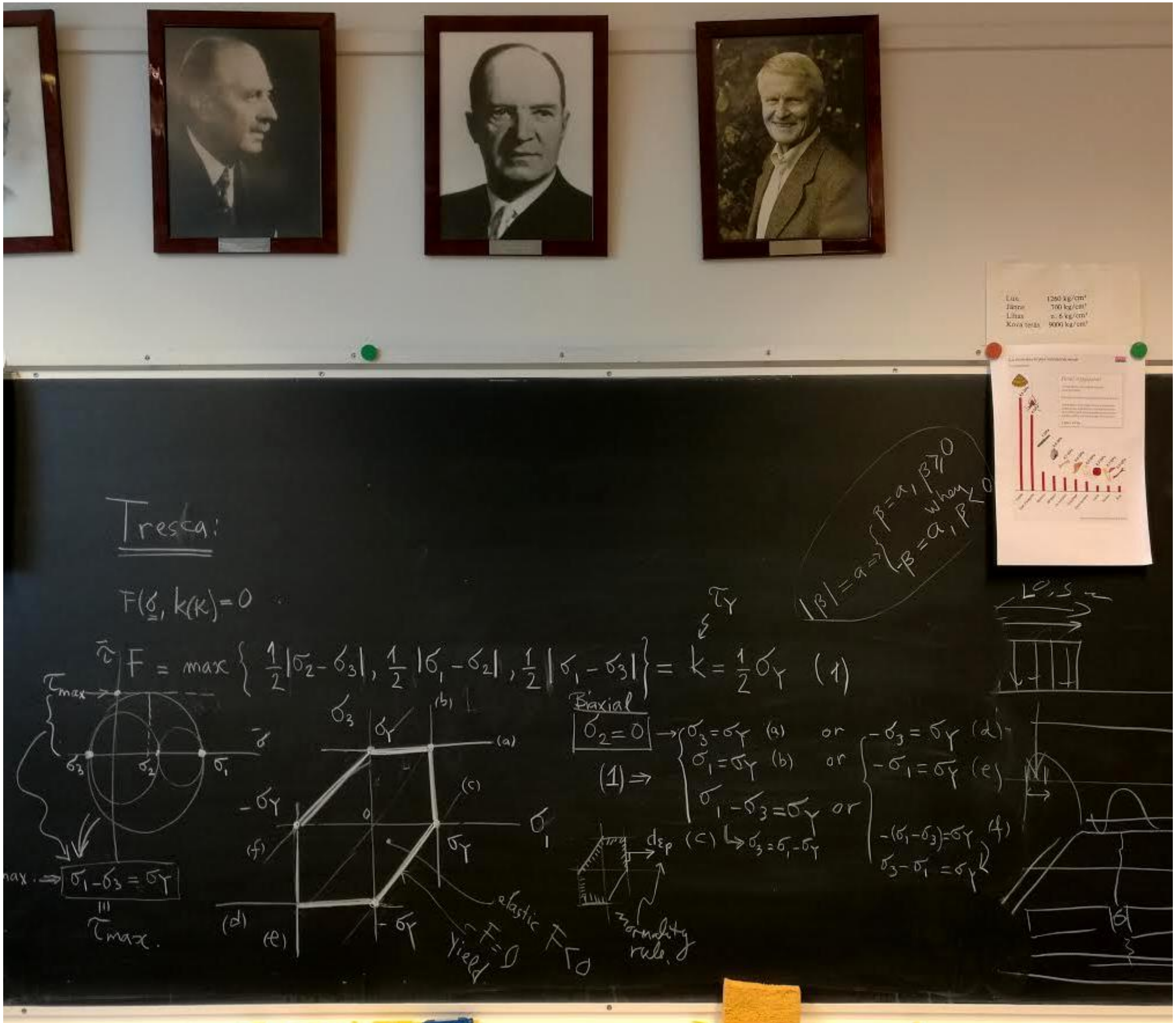
etc





1D-Isotropic Hardening Model




Tresca:
F(σ, k(κ)) = 0
F = max { ½|σ2 - σ3|, ½|σ1 - σ2|, ½|σ1 - σ3| } = k = ½σY (1)
Biaxial
σ2 = 0
(1) ⇒
σ3 = σY (a) or -σ3 = σY (d)
σ1 = σY (b) or -σ1 = σY (e)
σ1 - σ3 = σY or -(σ1-σ3) = σY (f)
σ3 = σ1 - σY
σ3 - σ1 = σY
σ1 - σ3 = σY
τmax
elastic F<0
F=0 Yield
normality rule.
dεp
|β| = α ⇒ β = α, β ≥ 0 when β = -α, β < 0

Semaine du 23 janvier

Poésie

Poésies de loups

23/1

par cœur
pour le 6
~~mars~~ FE
(au choix

Le loup vexé

Un loup sous la pluie,
Sous la pluie qui mouille,
Loup sans parapluie,
Pauvre loup gribouille.
Est-ce qu'un loup nage ?
Entre chien et loup,
Sous l'averse en rage,
Un hurluberloup ?
Un loup est vexé
Parce qu'on prétend
Que par mauvais temps
Un loup sous la pluie
Sent le chien mouillé.

Claude Roy

Le loup

Je suis poilu,
Fauve et dentu,
J'ai les yeux verts.
Mes crocs pointus
Me donnent l'air
Patibulaire.
Le vent qui siffle,
Moleste et gifle
Le promeneur,
Je le renifle
Et son odeur
Parle à mon cœur.
Sur l'autre rive
Qui donc arrive
A petits pas?
Hmmm! Je salive!
C'est mon repas
Qui vient là-bas!
Du bout du bois
Marche vers moi
Une gamine
Qui, je le vois,
Tantôt lambine,
Tantôt trottine.
Un chaperon
Tout rouge et rond
Bouge et palpite
D'un air fripon
Sur la petite
Chattemite...
Moi je me lèche
Et me pourlèche
Le bout du nez,
Je me dépêche
Pour accoster
Cette poupée.
Ah qu'il est doux
D'être le loup
De ces parages,
Le garde-fou
Des enfants sages
Du bois sauvage !

Pierre Gripari

kapa teknologia- tai ohjelmistoala. Tai ehkä päättäjien jakamaton huomio on keskittynyt viennin edistämiseen.

VIIME VUOSINA rakennusala ei ole juuri saanut innovaatioita rahoittavalta Tekesiltä varoja uusiin hankkeisiin. Myös koulutuspolitiikassa on tehty huonoja päätöksiä.

Tuntuu siltä, ettei Aalto-yliopistossa oteta rakennusalaa tosissaan. Päävastuu rakennusalan tarpeita vastaavan korkeakoulutuksen tarjoamisesta on siirtynyt melkein yksinomaan Tampereen teknilliselle yliopistolle. Osa lahjakkaista opiskelijoista valitseekin jonkin toisen koulutusalan, kun pääkaupunkiseudulla ei ole kunnollista alan korkeakoulutusta.

Aalto-yliopiston kannattaisi pyrkiä löytämään yhteinen sävel rakentajien kanssa koulutussisältöjen suunnittelussa ja esimerkiksi uusien tutkimus- ja kehityshankkeiden käynnistämisessä. Tästä olisi hyötyä kaikille osapuolille.

Huippututkimukseen kannattaa panostaa rakennusalallakin. Suomi on tässä selvästi esimerkiksi Ruotsia jäljessä.

VAHVAT SISÄMARKKINAT ovat Suomen etu, ja rakennusala on avainasemassa niiden luomisessa. Paikallisten rakennusalan yrittäjien saamat tuotot jäävät kiertämään paikallistalouteen ja luovat sinne vaurautta.

Rakennusalaa tukemalla olisi mahdollista nostaa työllisyysastetta

Rakennusalasta voisi kehittyä palveluviennin kivijalka.

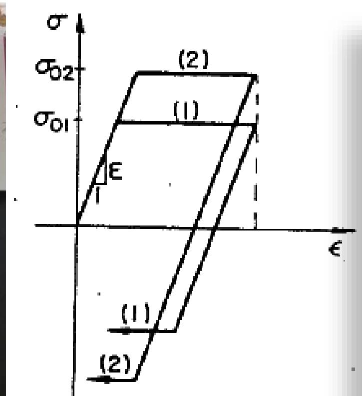
Kansainvälisille markkinoille päästäkseen cleantech-yritykset tarvitsevat ehdottomasti referenssejä kokeiluhankkeista. Onnistuneet kokeiluhankkeet taas todistaisivat rakennusalan yritysten osaamisesta.

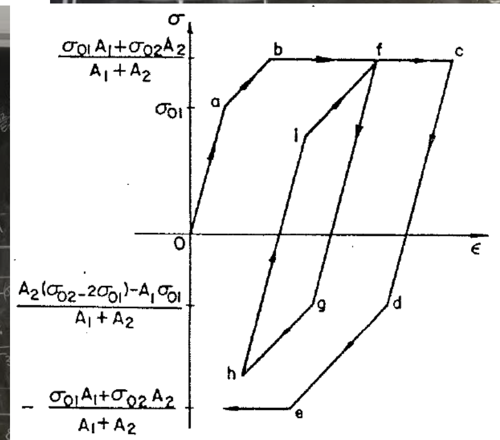
RUOTSISSA tällaisia jännittäviä hankkeita, joissa yhdistyvät monien alojen osaaminen, kehittää esimerkiksi Linköpingissä toimiva Plantagon-yhtiö. Kyseessä on ruotsalainen ratkaisu kaupungistumisen ja ruoan tuottamisen yhdistämiseen kestävällä tavalla.

Vastaavat rakennukset voivat yleistyä tulevaisuudessa, mikäli tällaiset kokeiluhankkeet osoittautuvat lupaaviksi tai menestyksellisiksi.

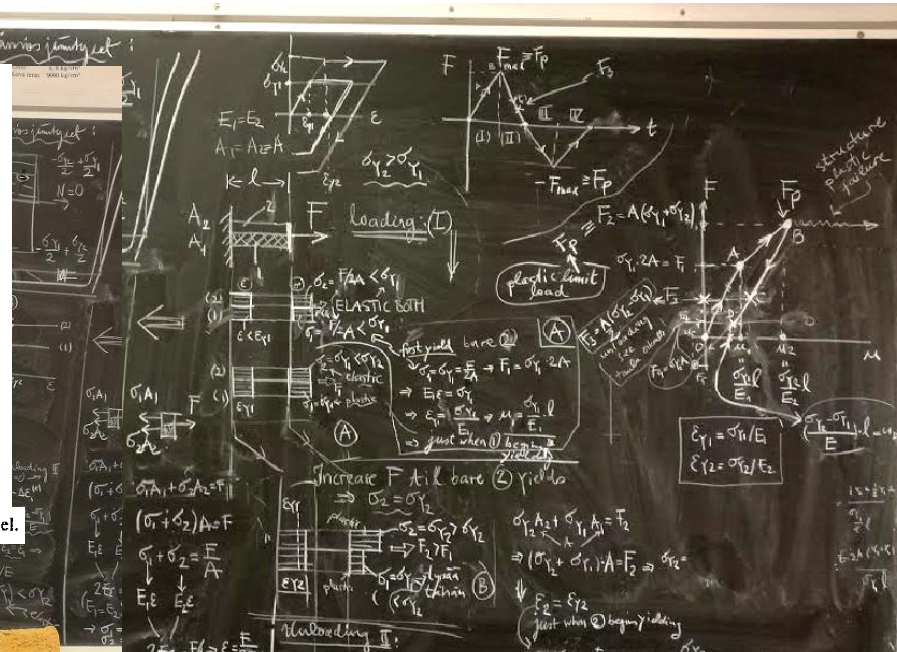
Juha Metsälä
Kirjoittaja on Pohjola Rakennuksen konsernijohtaja ja Rakennusteollisuus RT:n varapuheenjohtaja.

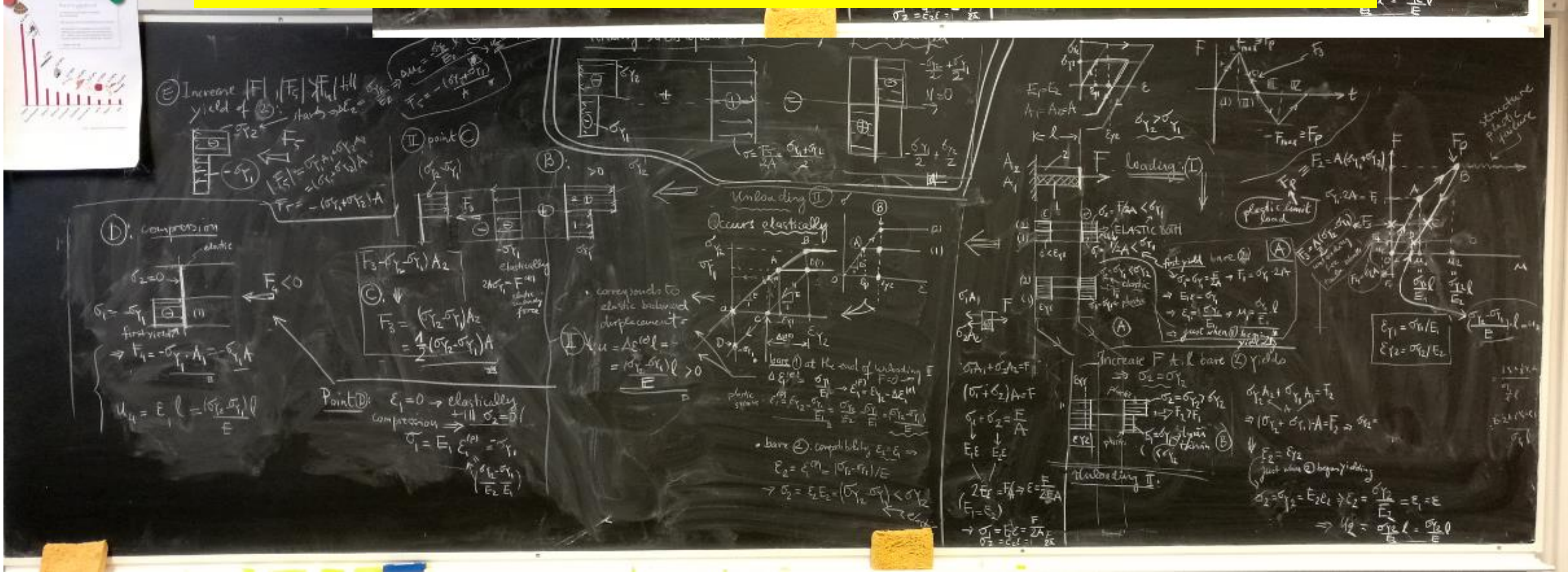
Simplified version of previous example





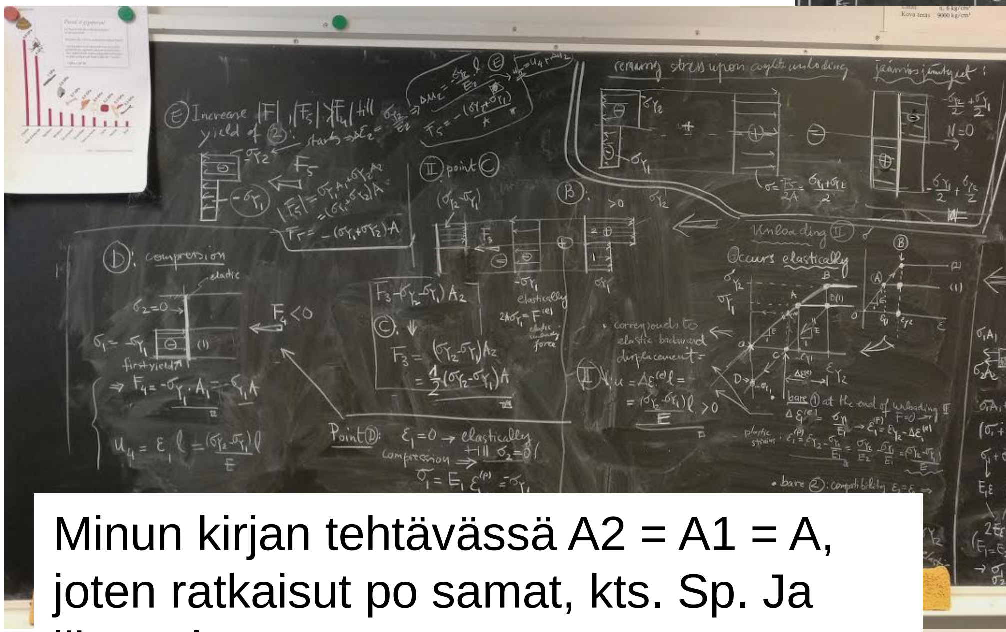
Loading and unloading characteristics of the overlay model.

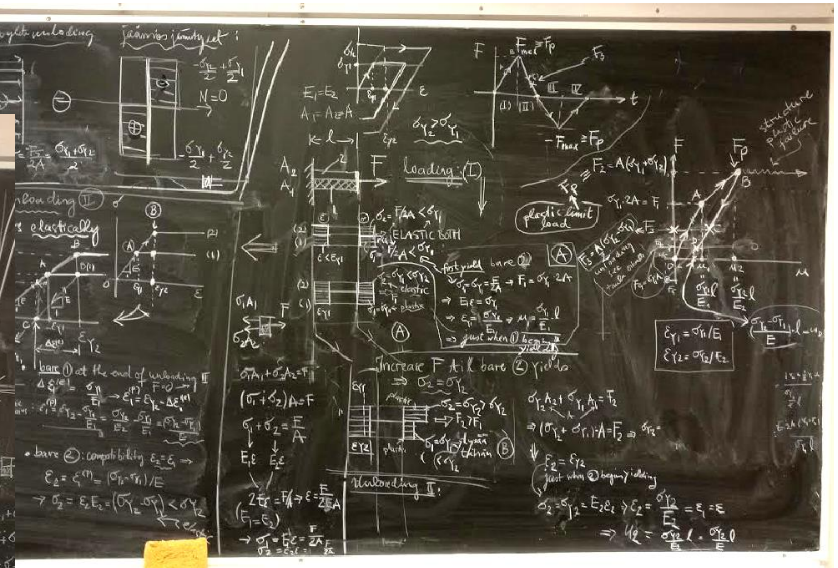




Ota symmetrinen matsku, eli 2 kertaa ympäri sama ja keskellä eri, saat taiv. Momentit nolla jäännösjännistyx

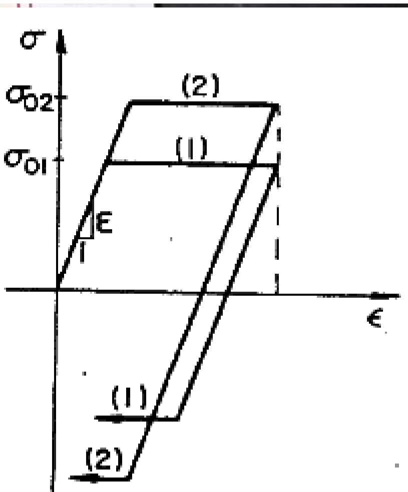
Simplified version of previous example

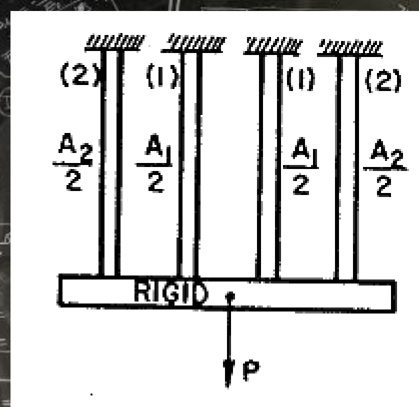


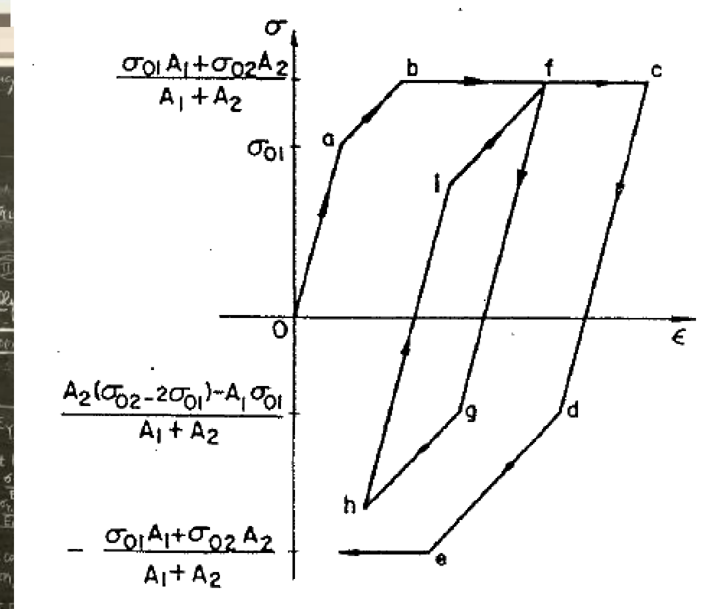


Minun kirjan tehtävässä A2 = A1 = A, joten ratkaisut po samat, kts. Sp. Ja liitutaulu

Ota symmetrinen matsku, eli 2 kertaa ympäri sama ja keskellä eri, saat taiv. Momentit nolla jäännösjännistyx

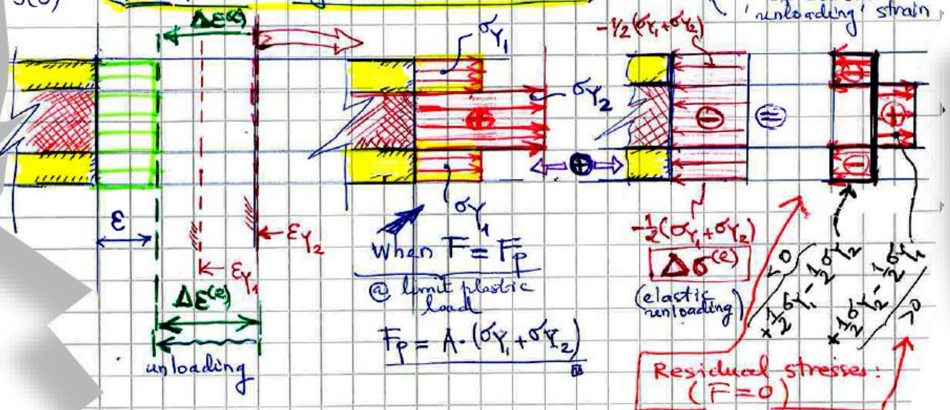


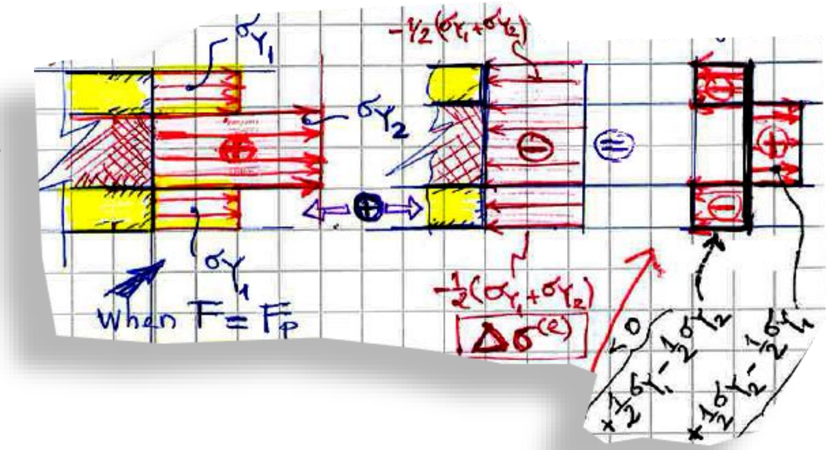




II • Unloading from $F = F_p$ to $F = 0$

3(3) N.B.: unloading occurs elastically : $\Delta\varepsilon^{(e)}$ — the amount of elastic 'unloading' strain (unknown yet)





Elastic unloading by an amount $\Delta\varepsilon^{(e)} = \Delta\varepsilon_i^{(e)}$, $i=1,2$ such that the resulting stress distribution have a zero resultant when $F=0$

New stress state:
$$\begin{cases} \boxed{\sigma_i = \sigma_{Y_i} + \Delta\sigma_i^{(e)}} \ (a), \quad i=1,2 \\ \Delta\sigma_i^{(e)} = E_i\,\Delta\varepsilon^{(e)} \end{cases} \quad \text{and } F = \int_A \sigma\, dA = 0 \Rightarrow \Delta\varepsilon^{(e)} \text{ will be solved}$$

$$\left.\begin{aligned} \sigma_1 &= \sigma_{Y_1} + E_1\Delta\varepsilon^{(e)} \\ \sigma_2 &= \sigma_{Y_2} + E_2\Delta\varepsilon^{(e)} \end{aligned}\right\} \Rightarrow A_1\sigma_1 + A_2\sigma_2 = 0 \Rightarrow$$

$$A_1(\sigma_{Y_1} + E_1\Delta\varepsilon^{(e)}) + A_2(\sigma_{Y_2} + E_2\Delta\varepsilon^{(e)}) = 0$$

$$A_1\sigma_{Y_1} + A_2\sigma_{Y_2} = -(E_1A_1 + E_2A_2)\,\Delta\varepsilon^{(e)} \Rightarrow$$

$$\Delta\varepsilon^{(e)} = -\frac{A_1\sigma_{Y_1} + A_2\sigma_{Y_2}}{E_1A_1 + E_2A_2}\ ; \quad \text{now } \begin{cases} E_1 = E_2 = E \\ \text{and} \\ A_1 = A_2 = A \end{cases}$$

$$\Rightarrow \boxed{E\Delta\varepsilon^{(e)} = +\Delta\sigma^{(e)} = -\frac{1}{2}(\sigma_{Y_1} + \sigma_{Y_2})} \ ^{(**)} \quad (b)$$

↳ therefore $\sigma_i = \underbrace{\sigma_{Y_i}}_{\text{stress distribution @ } F=F_p} + \underbrace{\Delta\sigma^{(e)}}_{\text{elastic stress distribution}}$

σ_i: stress distribution upon complete unloading $F=0$

N.B. $\int_A \Delta\sigma^{(e)}\, dA = -\frac{1}{2}A\cdot(\sigma_{Y_1}+\sigma_{Y_2}) \doteq -F_p$

$\therefore$

$\sigma_1 = \sigma_{Y_1} - \frac{1}{2}\sigma_{Y_1} - \frac{1}{2}\sigma_{Y_2} = +\frac{1}{2}\sigma_{Y_1} - \frac{1}{2}\sigma_{Y_2}$ (C)

N.B. Residual stresses:

• $u(F=0) = u(F_p) + \Delta\varepsilon^{(e)}\cdot \ell$

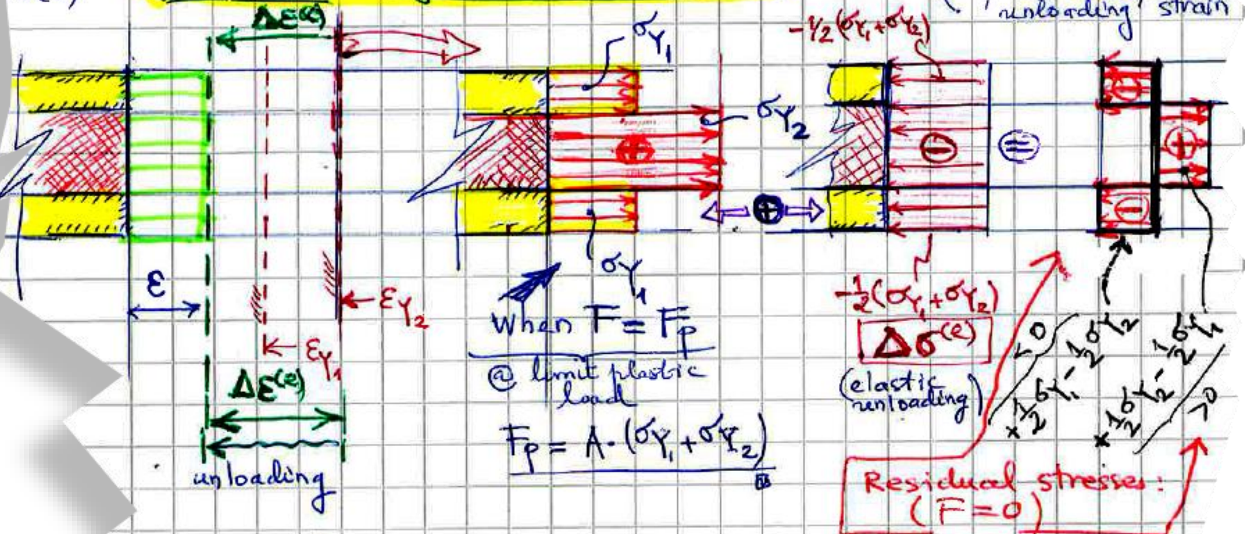
$u(F=0) = \frac{\sigma_{Y_2}}{E}\cdot\ell - \frac{1}{2}\left(\frac{\sigma_{Y_1}+\sigma_{Y_2}}{E}\right)\ell$

$= \frac{1}{2E}(\sigma_{Y_2} - \sigma_{Y_1})\cdot\ell$

3(3)

II • Unloading from $F = F_p$ to $F = 0$

N.B.: unloading occurs elastically : $\Delta\varepsilon^{(e)}$ the amount of elastic 'unloading' strain (unknown yet)



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and $F = \int_A \sigma\, dA = 0 \Rightarrow \Delta\varepsilon^{(e)}$ will be solved

$$\left.\begin{array}{l} \sigma_1 = \sigma_{Y_1} + E_1\Delta\varepsilon^{(e)} \\ \sigma_2 = \sigma_{Y_2} + E_2\Delta\varepsilon^{(e)} \end{array}\right\} \Rightarrow A_1\sigma_1 + A_2\sigma_2 = 0 \Rightarrow$$

$$A_1(\sigma_{Y_1} + E_1\Delta\varepsilon^{(e)}) + A_2(\sigma_{Y_2} + E_2\Delta\varepsilon^{(e)}) = 0$$

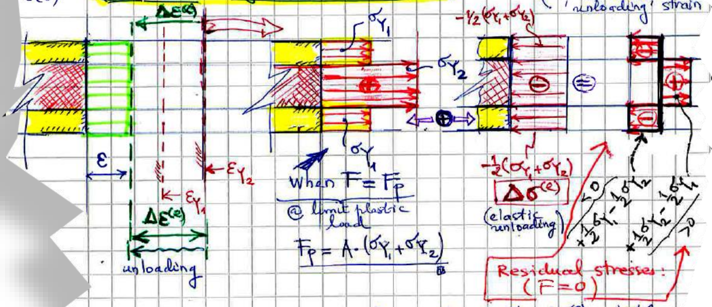
$$A_1\sigma_{Y_1} + A_2\sigma_{Y_2} = -(E_1A_1 + E_2A_2)\,\Delta\varepsilon^{(e)} \Rightarrow$$

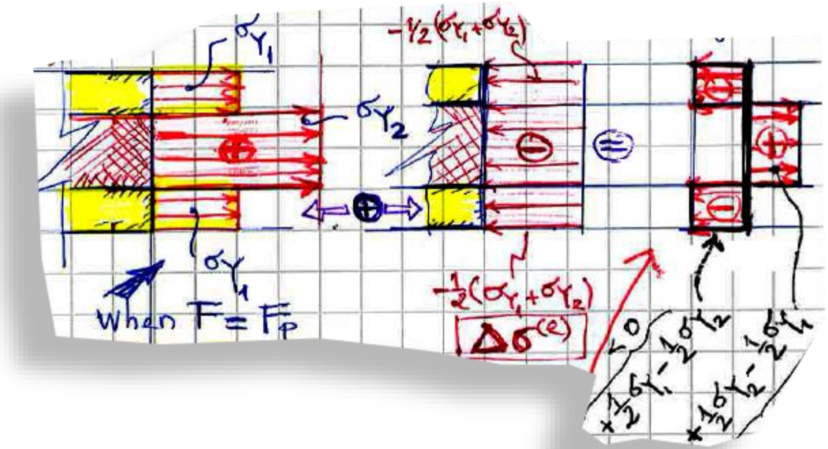
$A_1\sigma_{Y_1} + A_2\sigma_{Y_2}$. now $(E_1 = E_2 = E$

II • Unloading from $F=F_p$ to $F=0$

3(3)

N.B.: unloading occurs elastically : $\Delta\varepsilon^{(e)}$ — unknown yet the amount of elastic 'unloading' strain





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$$\left.\begin{aligned} \sigma_1 &= \sigma_{Y_1} + E_1\Delta\varepsilon^{(e)} \\ \sigma_2 &= \sigma_{Y_2} + E_2\Delta\varepsilon^{(e)} \end{aligned}\right\} \Rightarrow A_1\sigma_1 + A_2\sigma_2 = 0 \Rightarrow$$

$$A_1(\sigma_{Y_1} + E_1\Delta\varepsilon^{(e)}) + A_2(\sigma_{Y_2} + E_2\Delta\varepsilon^{(e)}) = 0$$

$$A_1\sigma_{Y_1} + A_2\sigma_{Y_2} = -(E_1A_1 + E_2A_2)\,\Delta\varepsilon^{(e)} \Rightarrow$$

$$\Delta\varepsilon^{(e)} = -\frac{A_1\sigma_{Y_1} + A_2\sigma_{Y_2}}{E_1A_1 + E_2A_2}; \quad \text{now} \begin{cases} E_1 = E_2 = E \\ \text{and} \\ A_1 = A_2 = A \end{cases}$$

$$\Rightarrow \boxed{E\Delta\varepsilon^{(e)} = +\Delta\sigma^{(e)} = -\frac{1}{2}(\sigma_{Y_1} + \sigma_{Y_2})} \ (b)$$

therefore $\sigma_i = \underbrace{\sigma_{Y_i}}_{\text{stress distribution @}F=F_p} + \underbrace{\Delta\sigma^{(e)}}_{\text{elastic stress distribution}}$

stress distribution upon complete unloading $F=0$

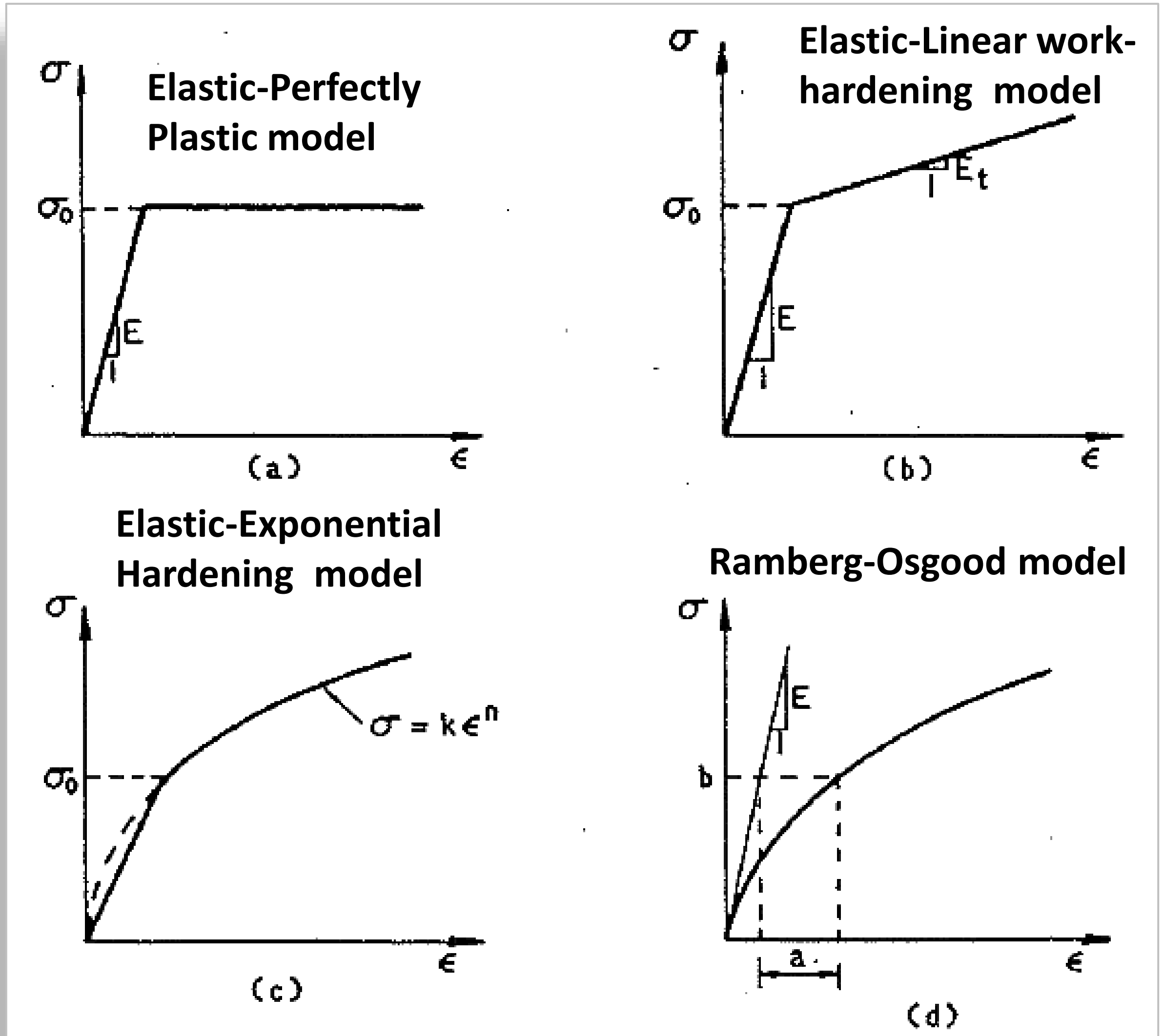
N.B. $\int_A \Delta\sigma^{(e)}\, dA = -\frac{1}{2}A\cdot(\sigma_{Y_1}+\sigma_{Y_2}) \equiv -F_p$

N.B. → RESIDUAL STRESSES:
$$\begin{aligned} \sigma_1 &= \sigma_{Y_1} - \tfrac{1}{2}\sigma_{Y_1} - \tfrac{1}{2}\sigma_{Y_2} = +\tfrac{1}{2}\sigma_{Y_1} - \tfrac{1}{2}\sigma_{Y_2} \neq 0 \\ \sigma_2 &= \sigma_{Y_2} - \tfrac{1}{2}\sigma_{Y_1} - \tfrac{1}{2}\sigma_{Y_2} = +\tfrac{1}{2}\sigma_{Y_2} - \tfrac{1}{2}\sigma_{Y_1} \neq 0 \end{aligned}$$

• $u(F=0) = u(F_p) + \Delta\varepsilon^{(e)}\cdot \ell$

$$u(F=0) = \frac{\sigma_{Y_2}}{E}\cdot\ell - \frac{1}{2}\left(\frac{\sigma_{Y_1}+\sigma_{Y_2}}{E}\right)\cdot\ell$$
$$= \frac{1}{2E}(\sigma_{Y_2} - \sigma_{Y_1})\cdot\ell = \frac{\ell}{2}(\sigma_{Y_2}-\sigma_{Y_1})/E$$

Modelling of uniaxial behavior in plasticity – simplified models



Idealized stress-strain curves

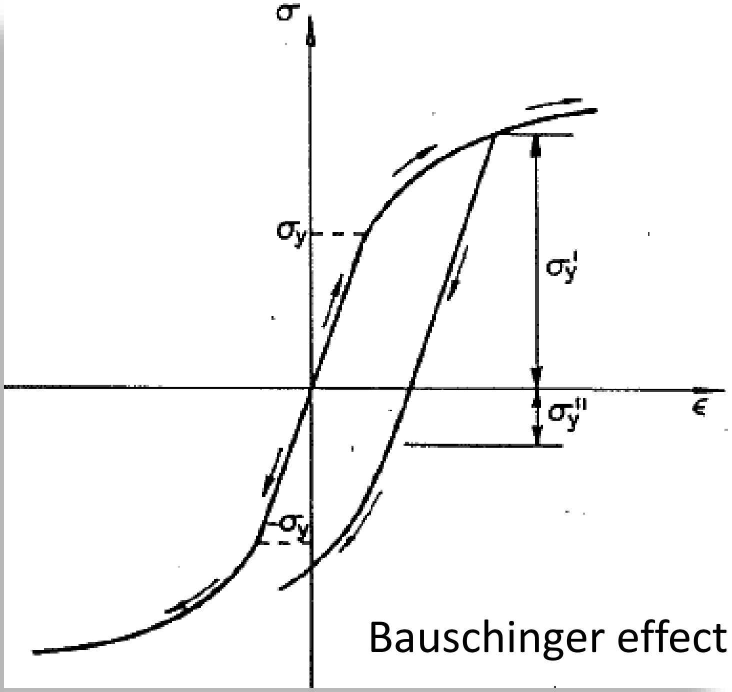
Reversed loading

Loading history: *Cf*. the graph →

- Load path dependence

Bauschinger effect – *see later the detailed description* (theses slides)

Such Bauschinger effect is present whenever there is a reversal loading (cyclic)



Bauschinger effect

Homework – plasticity

Hardening behavior and yield offset – uniaxial behavior in Plasticity

During a uniaxial (monotonic) stress testing of particular solid material the stress-strain curve was measured. It was found that this curve can be approximated by the non-linear **Ramberg-Osgood** model for 1-D case as

$$\varepsilon = \varepsilon^{\mathrm{e}} + \varepsilon^{\mathrm{P}} = \frac{\sigma}{E} + \left(\frac{\sigma}{b}\right)^{n}, \quad \text{or equivalently} \quad \varepsilon = \varepsilon^{\mathrm{e}} + \varepsilon^{\mathrm{P}} = \frac{\sigma}{E} + K\left(\frac{\sigma}{E}\right)^{n}.$$

0. Make graphs for three or four different values of the exponent parameter with a fixed value of K or b

1. Determine the *tangent elastic modulus* E_{t} and the *plastic modulus* $H \equiv E_{\mathrm{p}}$ as functions of stress and strain. Show that $1/E_{\mathrm{t}} = 1/E + 1/E_{\mathrm{p}}$ holds.

2. Determine the plastic work W_{p} as function of stress and plastic strain. On the strain-stress graph show the elastic and the plastic work and also the hardening work partition.

3. Determine the yield stress. What is the value of the yield stress?

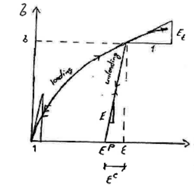
4. Assume $n = 5$ and determine the (offset tensile) stresses corresponding to values of 0.1 % and 0.2 % plastic strain (permanent strain). Plot your results on the graphs.

5. Assume n = 1, draw the stress-strain curve for loading followed b a complete unloading.

$n, b, K -$ describe the hardening behavior of the material.

Solution

3)



$\varepsilon = \varepsilon^e + \varepsilon^p$ (1)

$\varepsilon^e = \frac{\sigma}{E}$ (2)

$\varepsilon^p = \left(\frac{\sigma}{b}\right)^n$ (3)

a) $d\sigma = E_t\, d\varepsilon$; $E_t = \frac{d\sigma}{d\varepsilon} = \frac{1}{\frac{d\varepsilon}{d\sigma}}$ $\varepsilon = \frac{\sigma}{E} + \left(\frac{\sigma}{b}\right)^n \Rightarrow$

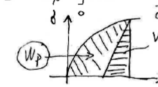
$\frac{d\varepsilon}{d\sigma} = \frac{1}{E} + \frac{n}{b}\left(\frac{\sigma}{b}\right)^{n-1} \Rightarrow E_t = \frac{1}{\frac{1}{E} + \frac{n}{b}\left(\frac{\sigma}{b}\right)^{n-1}}$

$d\sigma = E_p\, d\varepsilon^p \Rightarrow E_p = \frac{d\sigma}{d\varepsilon^p} = \frac{1}{\frac{d\varepsilon^p}{d\sigma}}$ $\varepsilon^p = \left(\frac{\sigma}{b}\right)^n \Rightarrow$

$\frac{d\varepsilon^p}{d\sigma} = \frac{n}{b}\left(\frac{\sigma}{b}\right)^{n-1} \Rightarrow E_p = \frac{1}{\frac{n}{b}\left(\frac{\sigma}{b}\right)^{n-1}}$

E_t ja E_p voidaan laskea ε^p:n avulla kun σ:n paikalle sijoitetaan kaavan (3) perusteella $\frac{\sigma}{b} = (\varepsilon^p)^{\frac{1}{n}}$.

b) $W_p = \int_0^{\varepsilon^p} \sigma\, d\varepsilon^p = b\int_0^{\varepsilon^p} (\varepsilon^p)^{\frac{1}{n}} d\varepsilon^p = \frac{nb}{n+1} \Big/_0^{\varepsilon^p} (\varepsilon^p)^{\frac{n+1}{n}} = \frac{nb}{n+1}(\varepsilon^p)^{\frac{n+1}{n}}$



$= \frac{nb}{n+1}\left(\frac{\sigma}{b}\right)^{n+1}$

Huom. ε^p ja W_p ovat vakioita unloading-käyrällä.

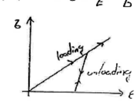
c) $W_p = \frac{nb}{n+1}(\varepsilon^p)^{\frac{n+1}{n}} \Rightarrow (\varepsilon^p)^{\frac{n+1}{n}} = \frac{n+1}{n}\frac{W_p}{b}$;

$\varepsilon^p = \left(\frac{n+1}{n}\frac{W_p}{b}\right)^{\frac{n}{n+1}}$. Sijoittamalla tämä σ:n ja E_p:n lausekkeisiin saadaan kysytyt tulokset.

d) "Initial yield stress" on se jännitystaso, jota saadaan kaavasta $\sigma = b(\varepsilon^p)^{1/n}$ sijoittamalla ε^p:n paikalle sovittu arvo (usein 0.1 % = 0.001) yleensä 0,2 % metallit

e) $n = 1 \Rightarrow \varepsilon = \frac{\sigma}{E} + \frac{\sigma}{b}$ loading: $\varepsilon = \left(\frac{1}{E} + \frac{1}{b}\right)\sigma$

unloading: $\varepsilon = \frac{\sigma}{E}$



f) sijoitetaan kaavaan $\sigma = b(\varepsilon^p)^{\frac{1}{n}}$

Huom kohta d) periaatteessa "initial yield stress" on Rambergin-Osgoodin mallilla 0, sillä plastisia venymiä syntyy heti, kun jännitystäkin. Mallia käytetään sellaisille aineille, joilla ei ole selvää kimmoista aluetta. Tällaisia metalleja ovat esim. ruostumaton teräs, kupari, jne. Tavallisella kuumavalssatulla teräksellä σ-ε noudattaa erittäin hyvin kimmo-ideaaliplastista mallia.

Incremental Stress-Strain Relationships

Reading: Chapter 5.7 – Chan & Han
Incremental Stress-Strain Relationships

Plastiset muodonmuutokset ovat myötöpinnan normaalin suuntaiset:

$$d\varepsilon_{ij}^{p} = d\lambda \frac{\partial f}{\partial \sigma_{ij}} \qquad (4)$$

Ideaaliplastiselle aineelle $d\lambda$ määräämätön vakio, joka on positiivinen plastisten muodonmuutosten tapahtuessa (seuraus myötöpinnan kuperuudesta). Myötö~~sääntöä~~ (4) sanotaan myötöehtoon f liittyväksi myötösäännöksi. (Myös: Misesin plastinen potentiaali) Kimmoiset muodonmuutokset tapahtuvat Hooken lain mukaan

$$d\varepsilon_{ij}^{e} = B_{ijkl}\, d\sigma_{kl} \qquad (5)$$

Kokonaismuodonmuutokset ovat

$$d\varepsilon_{ij} = d\varepsilon_{ij}^{e} + d\varepsilon_{ij}^{p} \qquad (6)$$

Yhtälöt (4), (5) ja (6) määrittelevät kimmoisen ideaaliplastisen aineen käyttäytymisen.

Kuormitus-palautuminen

Määritellään:

Kuormitus: $f(\sigma_{ij}) = 0 \,, \quad \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} = df = 0$

Palautuminen: $f(\sigma_{ij}) = 0 \,, \quad df < 0$

Plastisia muodonmuutoksia vain kuormitustilanteessa. Palautumisessa Hooken laki pätee.

Jännitysinkrementit voidaan lausua jännitystilan ja muodonmuutosinkrementtien avulla

$$d\sigma_{ij} = A_{ijkl}\, d\varepsilon_{kl} - \frac{A_{ijpq} A_{rskl} \frac{\partial f}{\partial \sigma_{pq}} \frac{\partial f}{\partial \sigma_{rs}}}{A_{pqrs} \frac{\partial f}{\partial \sigma_{pq}} \frac{\partial f}{\partial \sigma_{rs}}} d\varepsilon_{kl} \qquad (7)$$

Huom. Jos $\frac{\partial f}{\partial I_{\sigma}} \equiv 0$, niin plastinen tilavuudenmuutos $d\varepsilon_{kk}^{p} = 0$.

It can be shown that (for details ref. to Chapter 5.7 of Chan & Han)

2.4.2 Fundamentals of Classical Theory of Plasticity

There are three basic assumptions used in the development of the classical theory of plasticity: an initial yield surface, a hardening rule, and a flow rule. An initial yield surface in stress space defines the stress level at which plastic deformation begins. A hardening rule regulates the evolution of the subsequent loading surfaces during the course of plastic flow. A flow rule defines an incremental plastic stress-strain relationship using a plastic potential function. And the formulation of plasticity theory gives the following incremental stress-strain relation (Chen and Han, 1988).

$$d\sigma_{ij} = D_{ijkl}^{ep}\, d\varepsilon_{kl} \qquad (2.4.1)$$

where

$$D_{ijkl}^{ep} = D_{ijkl}^{e} - \frac{\frac{\partial f}{\partial \sigma_{mn}} D_{mnkl}^{e} D_{ijst}^{e} \frac{\partial g}{\partial \sigma_{st}}}{h + \frac{\partial f}{\partial \sigma_{pq}} D_{pquv}^{e} \frac{\partial g}{\partial \sigma_{uv}}} \qquad (2.4.2)$$

in which D_{ijkl}^{e}, D_{ijkl}^{ep} = elastic and elastic-plastic material tensors; f = a loading function (surface); g = a potential function; and h = a scalar function related to the hardening rule associated with a particular material. For an associated flow rule, f is equal to g while f≠g is assumed for a non-associated flow rule. This implies that in general the symmetr

$$d\sigma_{ij} = D_{ijkl}^{ep}\, d\varepsilon_{kl}$$

$$D_{ijkl}^{ep} = D_{ijkl}^{e} - \frac{\frac{\partial f}{\partial \sigma_{mn}} D_{mnkl}^{e} D_{ijst}^{e} \frac{\partial g}{\partial \sigma_{st}}}{h + \frac{\partial f}{\partial \sigma_{pq}} D_{pquv}^{e} \frac{\partial g}{\partial \sigma_{uv}}}$$

2.5 Incremental stress-strain relations

In computational elasto-plastic analysis one usually proceeds by applying a load increment which produces a displacement increment, and thus a total strain increment. The stress increment corresponding to this increment in total strain can be determined by a constitutive relation similar to the relation one has in elasticity, i.e.

$$d\boldsymbol{\sigma} = \boldsymbol{D}^{ep} d\boldsymbol{\varepsilon} \tag{38}$$

where $\boldsymbol{D}^{ep}$ is the elasto-plastic constitutive matrix. Such a relation was first derived and used in an finite element context by Zienkiewicz *et al.* [2]. The elasto-plastic constitutive matrix can be derived by considering the basic relations discussed in the foregoing.
The total strain increment is given as the sum of the elastic strain increment and the plastic strain increment

$$d\boldsymbol{\varepsilon} = d\boldsymbol{\varepsilon}^e + d\boldsymbol{\varepsilon}^p \tag{39}$$

Hooke's law gives the relation between stresses and elastic strains as

$$\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\varepsilon}^e = \boldsymbol{D}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) \tag{40}$$

or in rate form as

$$d\boldsymbol{\sigma} = \boldsymbol{D}(d\boldsymbol{\varepsilon} - d\boldsymbol{\varepsilon}^p) \tag{41}$$

where $\boldsymbol{D}$ is the elastic constitutive matrix. The plastic strain increment is determined by the flow rule (29) as

$$d\boldsymbol{\varepsilon}^p = d\lambda \frac{\partial g}{\partial \boldsymbol{\sigma}}$$

$$\boldsymbol{D}^{ep} = \boldsymbol{D} - \frac{\boldsymbol{D}\dfrac{\partial g}{\partial \boldsymbol{\sigma}}\left(\dfrac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T \boldsymbol{D}}{\left(\dfrac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T \boldsymbol{D}\dfrac{\partial g}{\partial \boldsymbol{\sigma}}}$$

$$f(\boldsymbol{\sigma} + d\boldsymbol{\sigma}) = f(\boldsymbol{\sigma}) + \nabla f^T d\boldsymbol{\sigma} = 0 \tag{24}$$

where

$$\nabla f = [\partial f/\partial \sigma_x, ..., \partial f/\partial \tau_{zx}]^T \tag{25}$$

is the normal to the yield surface and $d\boldsymbol{\sigma}$ is a stress increment, see Figure 9. Since $f(\boldsymbol{\sigma}) = 0$

$$df = \nabla f^T d\boldsymbol{\sigma} = 0 \tag{26}$$

In other words, (26) states that during plastic loading the change in stress, if any, occurs tangential to the yield surface. This is the so-called consistency condition which, as shall be shown later, is a key ingredient in the general theory.
In unloading the state of stress immediately becomes elastic which can be written as

$$df = \nabla f^T d\boldsymbol{\sigma} < 0 \tag{27}$$

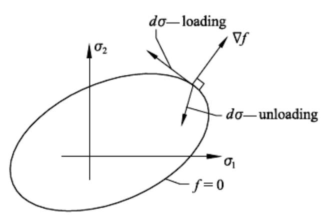


Figure 9: Plastic loading and unloading.

$$df = \nabla f^T d\boldsymbol{\sigma} = 0$$

where $\boldsymbol{D}$ is the elastic constitutive matrix. The plastic strain increment is determined by the flow rule (29) as

$$d\boldsymbol{\varepsilon}^p = d\lambda \frac{\partial g}{\partial \boldsymbol{\sigma}} \tag{42}$$

Thus, the stress increment is given by

$$d\boldsymbol{\sigma} = \boldsymbol{D}d\boldsymbol{\varepsilon} - d\lambda \boldsymbol{D}\frac{\partial g}{\partial \boldsymbol{\sigma}} \tag{43}$$

The expression for the stress increment is now substituted into the consistency condition (26)

$$\left(\frac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T \left(\boldsymbol{D}d\boldsymbol{\varepsilon} - d\lambda \boldsymbol{D}\frac{\partial g}{\partial \boldsymbol{\sigma}}\right) = 0 \tag{44}$$

Solving this equation for the scalar $d\lambda$ one obtains

$$d\lambda = \frac{\left(\dfrac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T \boldsymbol{D}\, d\boldsymbol{\varepsilon}}{\left(\dfrac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T \boldsymbol{D}\dfrac{\partial g}{\partial \boldsymbol{\sigma}}} \tag{45}$$

Finally, $d\lambda$ is substituted back into (43) to yield

$$d\boldsymbol{\sigma} = \boldsymbol{D}\left(d\boldsymbol{\varepsilon} - \frac{\left(\dfrac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T \boldsymbol{D}\, d\boldsymbol{\varepsilon}\, \dfrac{\partial g}{\partial \boldsymbol{\sigma}}}{\left(\dfrac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T \boldsymbol{D}\dfrac{\partial g}{\partial \boldsymbol{\sigma}}}\right) \tag{46}$$

nged to give the elasto-plastic constitutive relation

$$d\boldsymbol{\sigma} = \left(\boldsymbol{D} - \frac{\boldsymbol{D}\dfrac{\partial g}{\partial \boldsymbol{\sigma}}\left(\dfrac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T \boldsymbol{D}}{\left(\dfrac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T \boldsymbol{D}\dfrac{\partial g}{\partial \boldsymbol{\sigma}}}\right) d\boldsymbol{\varepsilon} \tag{47}$$

and thus, the elasto-plastic constitutive matrix introduced in (38) is

$$\boldsymbol{D}^{ep} = \boldsymbol{D} - \frac{\boldsymbol{D}\dfrac{\partial g}{\partial \boldsymbol{\sigma}}\left(\dfrac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T \boldsymbol{D}}{\left(\dfrac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T \boldsymbol{D}\dfrac{\partial g}{\partial \boldsymbol{\sigma}}} \tag{48}$$

The elasto-plastic constitutive relation defines the stress increment uniquely once the total strain increment and the current state of stress is known, whereas the a strain increment cannot be determined uniquely on the the basis of a stress increment, i.e. $\boldsymbol{D}^{ep}$ is singular. When used in finite element formulations (48) defines a nonlinear relation between stress and strain increments since the evaluation of the current stress must naturally be influenced by the magnitude of the stress increment. In this way the use of (48) leads to a classical type of finite element nonlinearity where the current state and an increment is known, but where the effect of the increment depends on the state that the increment gives rise to, i.e. an iterative procedure must be applied.

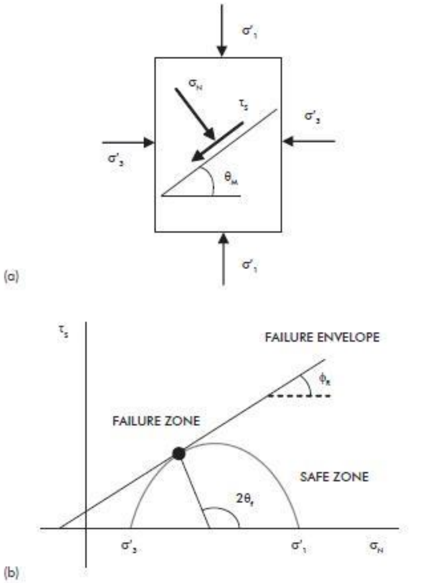
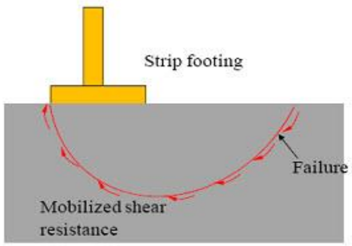
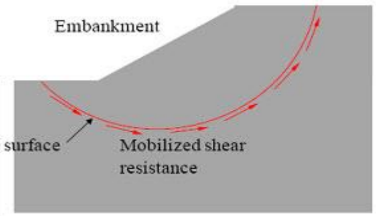
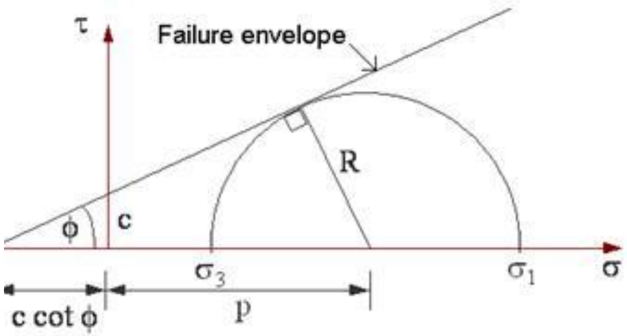
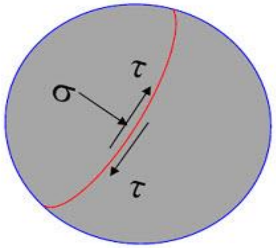


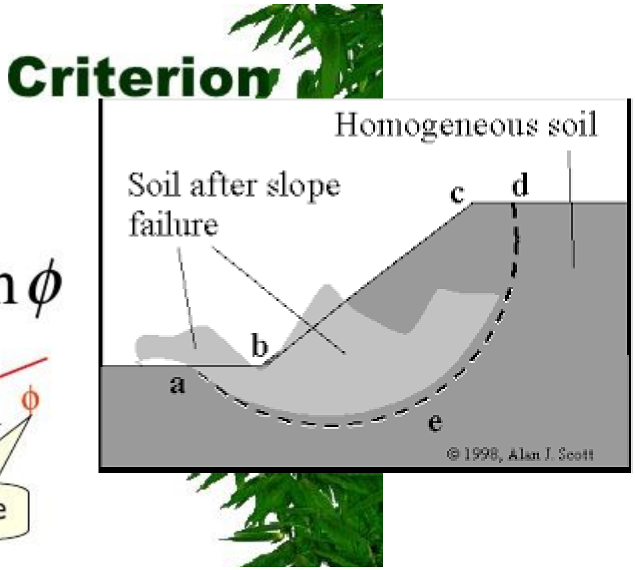
Figure 1 a) Shear failure under the Mohr Coulomb criterion (Osorio, 2003). b) Overlay of the Mohr circle with the failure envelope (Osorio, 2003)



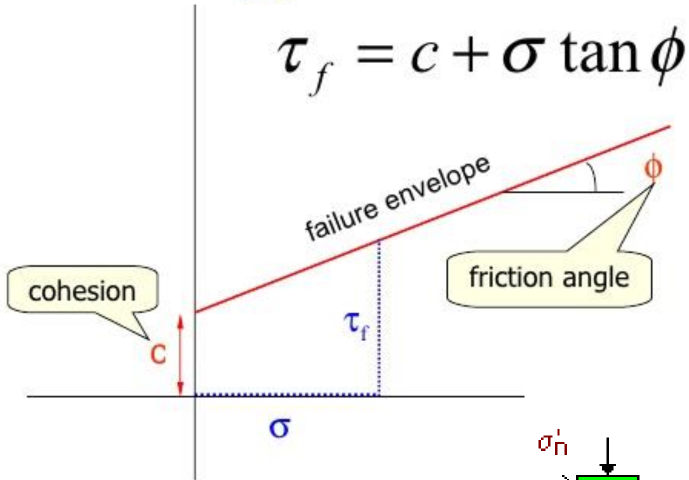








$$\tau_f = c + \sigma \tan\phi$$



τ_f is the maximum shear stress the
without failure, under normal stress

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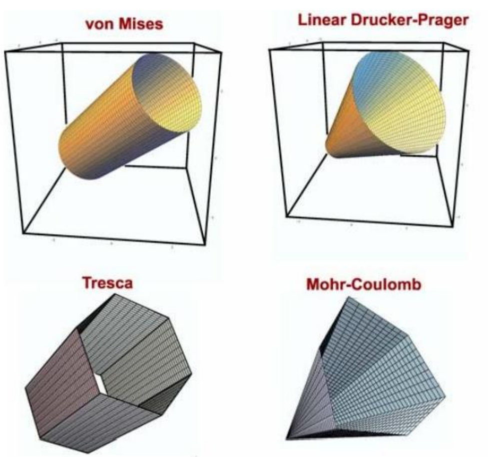
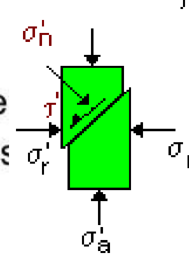
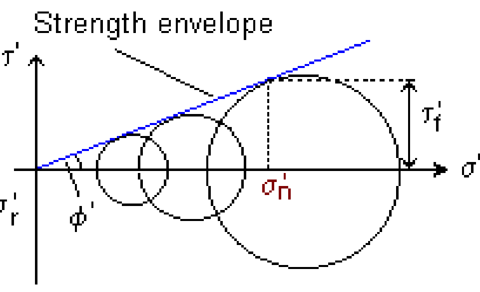






strength hypotheses

Aloita tästä "2.1 Basic concepts" ja sitten goto plasticity

Basic concepts

Plasticity

Yield criterion

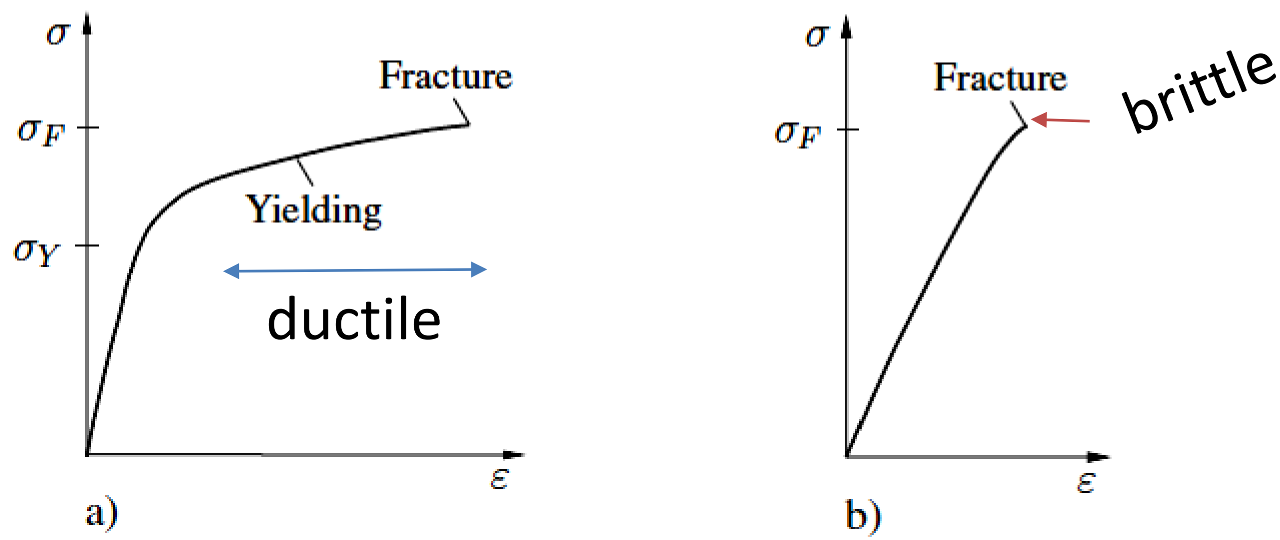
Incremental theory

Mohr–Coulomb theory

Strength hypotheses are intended to make a statement about the circumstances under which a material fails.

Their basis are experiments conducted under specific, mostly simple, loading conditions.

yield strength σY



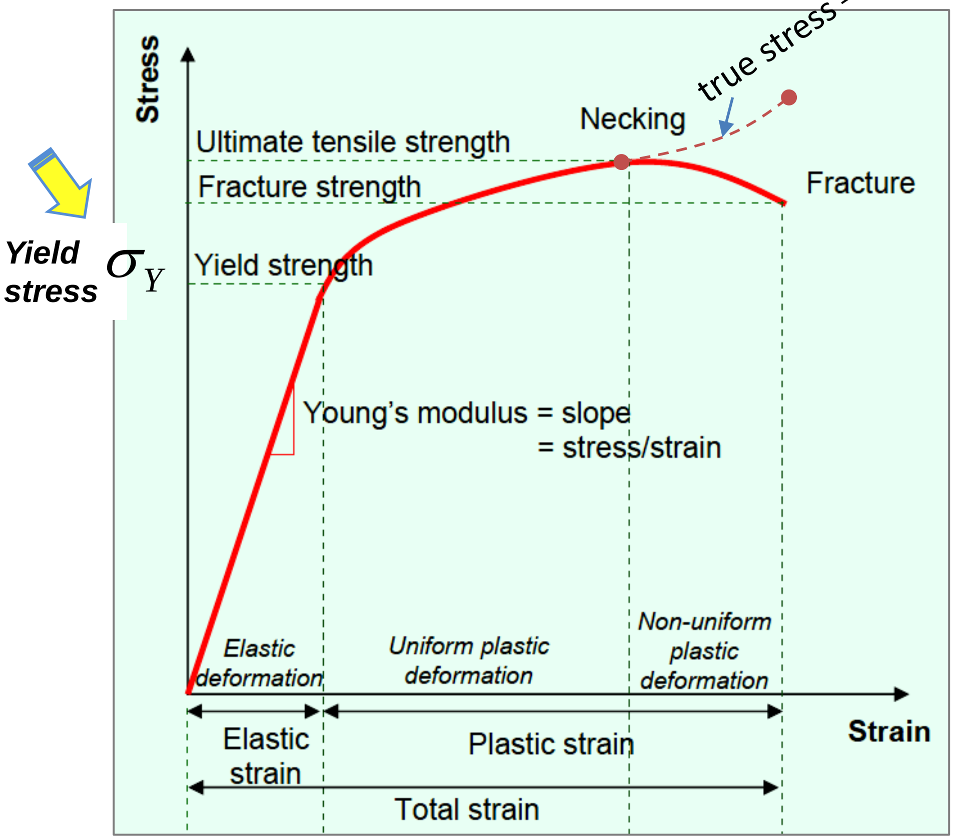
Mohr–Coulomb theory

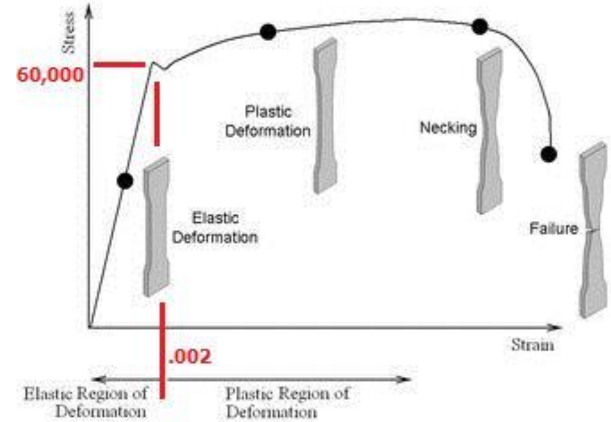
- In geotechnical engineering it is used to define shear strength of soils and rocks at different effective stresses
- In structural engineering it is used to determine failure load as well as the angle of fracture of a displacement fracture in concrete and similar materials

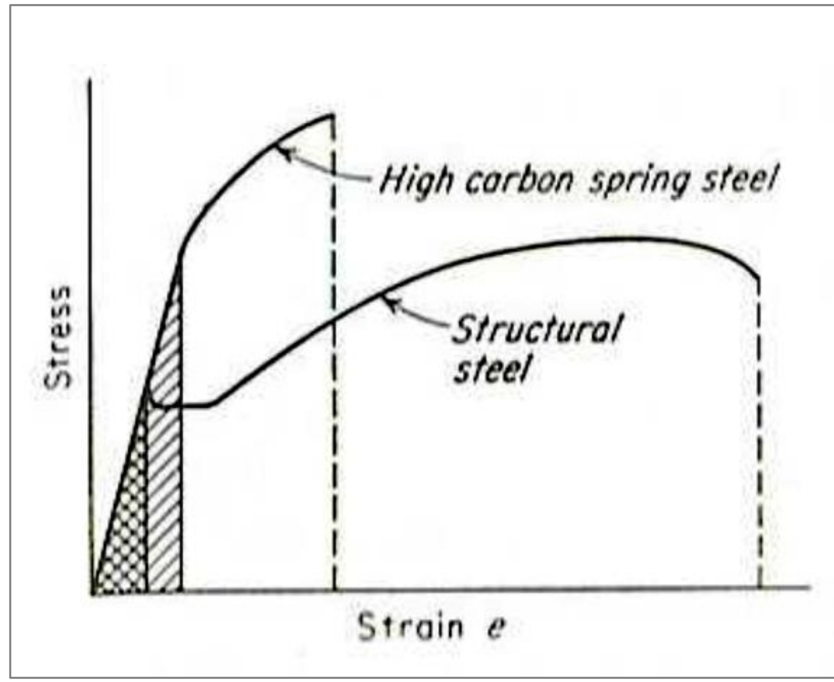
Examples plasticity properties:

uniaxial tension test - yksiaksiaalinen vetokoe

Some remarkable features and terminology







Concrete Material models (**in Finnish**)

Time-independent material models for concrete

Liik enne vira sto

38 · 2013

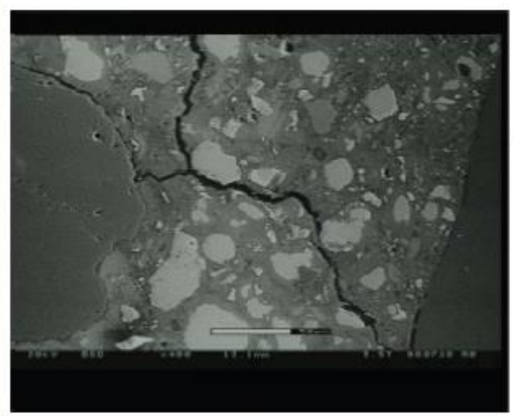
LIIKENNEVIRASTON TUTKIMUKSIA JA SELVITYKSIÄ

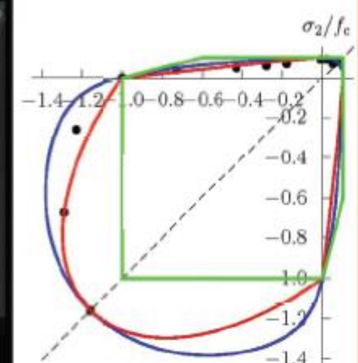
By Prof. Reijo Kouhia

REIJO KOUHIA

Betonin ajasta riippumattomat materiaalimallit

Concrete





Recommended reading for future and actual engineers in structures:

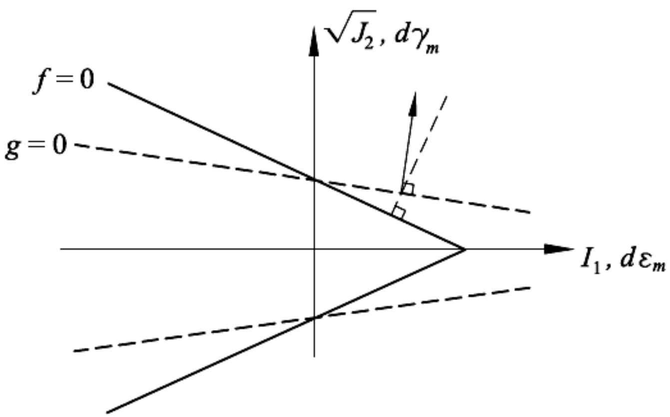
An extensive detailed review

ned by the Drucker-Prager criterion whereas the flow rule is c
:ion, e.g.

$$f(\boldsymbol{\sigma}) = \sqrt{J_2} + \alpha I_1 - k$$
$$g(\boldsymbol{\sigma}) = \sqrt{J_2} + \beta I_1 - k$$

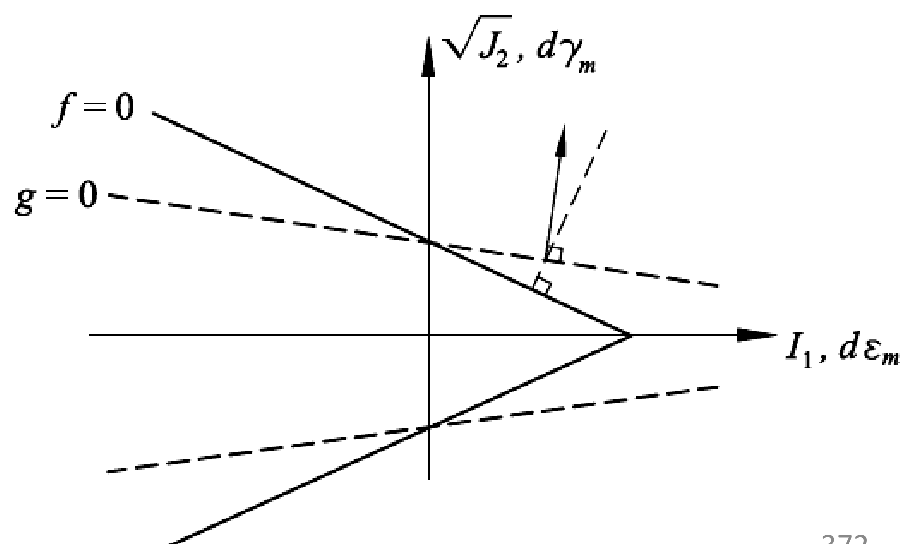
$$\beta < \alpha$$

be smaller than α. The situation is illustrated in Figure 11



:e 11: Drucker-Prager plasticity with nonassociated flow rule.

For soils, however, the volumetric dilatation predicted
by the associated Drucker-Prager flow rule is often somewhat larger than can be
verified experimentally. Therefore, a nonassociated flow rule can be used, i.e. the elastic
limit is still defined by the Drucker-Prager criterion whereas the flow rule is defined by
some other function, e.g.



Maximum Principal Stress Theory

✓ Maximum principal stress reaches tensile yield stress (Y).

✓ Estimate principal stresses σ_1, σ_2 & σ_3.

✓ Apply yield criteria:

$$f = \max\left(|\sigma_1|, |\sigma_2|, |\sigma_3|\right) - Y$$

$$\text{If, } f < 0 \quad \text{no yielding}$$

$$f = 0 \quad \text{onset of yielding}$$

$$f > 0 \quad \text{not defined}$$

Maximum Principal Strain Theory

✓ Failure occurs when maximum value of applied strain exceeds the strain value corresponding to yield point of the material.

✓ If 'Y' is the yield stress then under uni-axial loading yield strain is defined as

$$\varepsilon_y = Y/E$$

✓ Maximum strain developed in the design component should be less than ε_y.

✓ Principal stresses σ_1, σ_2 & σ_3 corresponds to principal strains ε_1, ε_2 & ε_3.

$$\boxed{\varepsilon_1 = \frac{\sigma_1}{E} - \frac{\upsilon}{E}(\sigma_2 + \sigma_3)} \qquad \boxed{\varepsilon_2 = \frac{\sigma_2}{E} - \frac{\upsilon}{E}(\sigma_1 + \sigma_3)}$$

Strain Energy Theory

✓ Failure at any point in a body is defined when the energy density in the body at the applied load equals the energy density corresponds to the elastic limit of the material.

✓ **Uni-axial loading :**

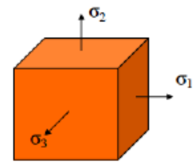
i. $\sigma = E\varepsilon$ (Hooke's Law)

ii. Strain energy density :

$$U = \int \sigma_{ij}\, d\varepsilon_{ij} \Rightarrow U = \int_0^{\varepsilon_y} \sigma\, d\varepsilon$$

$$\boxed{U = \frac{1}{2}\frac{Y^2}{E}}$$

✓ **Body subjected to principal stresses :**



$$U = \frac{1}{2}(\sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_3\varepsilon_3)$$

$$U = 1/2E\,[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3)]$$

✓ **For the onset of yielding :**

$$Y^2/2E = 1/2E\,[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3)]$$

✓ **Yield function**

$$f = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \upsilon(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) - Y^2$$

$$f = \sigma_e^2 - Y^2$$

$$\boxed{\text{Yielding} \Rightarrow f = 0, \ \text{safe } f < 0}$$

von Mises Criteria (Distortion Energy Criteria)

✓ Failure occurs when equivalent stress (von Mises stress) reaches the yield stress of the material.

✓ von –Mises yield criteria also suggests a failure or yielding when the elastic energy of distortion reaches a critical value. von Mises criteria is also known as maximum distortion energy criteria.

$$U_D = \frac{1}{12G}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$$

$$2Y^2 = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2$$

$$\boxed{f = \sigma_e^2 - Y^2}$$

Tresca Theory (Maximum Shear Stress Theory)

✓ Failure/yielding occurs when the maximum shear stress at a point equals the maximum shear stress at yield.

$$\tau_{\max} = K_T = \left|\frac{\sigma_1 - \sigma_2}{2}\right| \Rightarrow \tau_{\max} = \frac{Y}{2} = K_T$$

✓ Maximum shear stress less than 0.5 Y (No failure).

$$\boxed{\text{Shear stress yield} = 0.5\ (\text{Tensile stress yield})}$$

Tresca Theory (Cont...)

✓ **In terms of principal stresses σ_1, σ_2 & σ_3 .**

✓ **Maximum shear stresses.**

$$\max\left\{\left|\frac{\sigma_1 - \sigma_2}{2}\right|, \left|\frac{\sigma_2 - \sigma_3}{2}\right|, \left|\frac{\sigma_3 - \sigma_1}{2}\right|\right\}$$

✓ **Yield function :**

$$f = \max\left\{\left|\frac{\sigma_1 - \sigma_2}{2}\right|, \left|\frac{\sigma_2 - \sigma_3}{2}\right|, \left|\frac{\sigma_3 - \sigma_1}{2}\right|\right\} - K_T\left(= \frac{Y}{2}\right)$$

$$f < 0 \Rightarrow \text{No yielding}$$

$$f = 0 \Rightarrow \text{Onset of yielding}$$

- The von Mises yield criterion predicts that yielding will occur whenever the distortion energy in a unit volume equals the distortion energy in the same volume when uniaxially stressed to the yield strength.
  - From this theory, a scalar invariant (von Mises equivalent stress) is derived as:

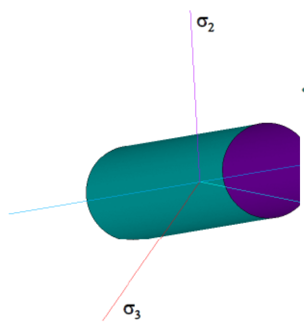
$$\sigma_e = \sqrt{\frac{1}{2}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right]}$$

- When von Mises equivalent stress exceeds the uniaxial material yield strength, general yielding will occur.
  - If plotted in 3D principal stress space, the von Mises yield surface is a cylinder.

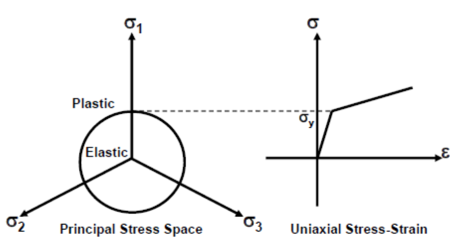
The cylinder is aligned with the axis $\sigma_1=\sigma_2=\sigma_3$.

Note that if the stress state is inside the cylinder, no yielding occurs. This means that if the material is under hydrostatic pressure ($\sigma_1=\sigma_2=\sigma_3$), no amount of hydrostatic pressure will cause yielding.

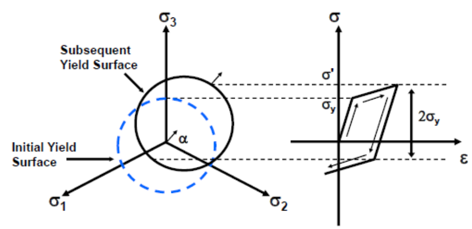
Another way to view this is that stresses which deviate from the axis ($\sigma_1=\sigma_2=\sigma_3$) contribute to the von Mises stress calculation.



- At the edge of the cylinder (circle), yielding will occu
- No stress state can exist *outside* of the cylinder.
- Instead, hardening rules will describe how the cylin with respect to yielding.

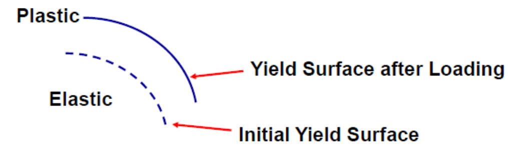


- The stress-strain behavior for linear kinematic hardening is illustrated below:

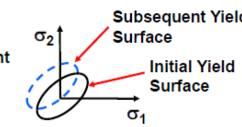


- Subsequent yield in compression is decreased by the amount that the yield stress in tension increased, so that a $2\sigma_y$ difference between the yields is always maintained. (This is known as the *Bauschinger effect*.)

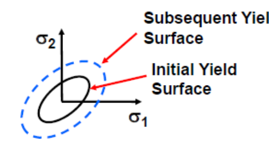
- The hardening rule describes how the yield surface changes (size, center, shape) as the result of plastic deformation.
- The hardening rule determines when the material will yield again if the loading is continued or reversed.
  - This is in contrast to elastic-perfectly-plastic materials which exhibit no hardening -- i.e., the yield surface remains fixed.



- There are two basic hardening rules to prescribe the modification of the yield surface:
  - *Kinematic* hardening.
    - The yield surface remains constant in size and translates in the direction of yielding.

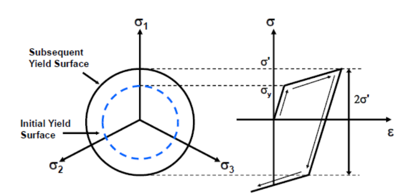


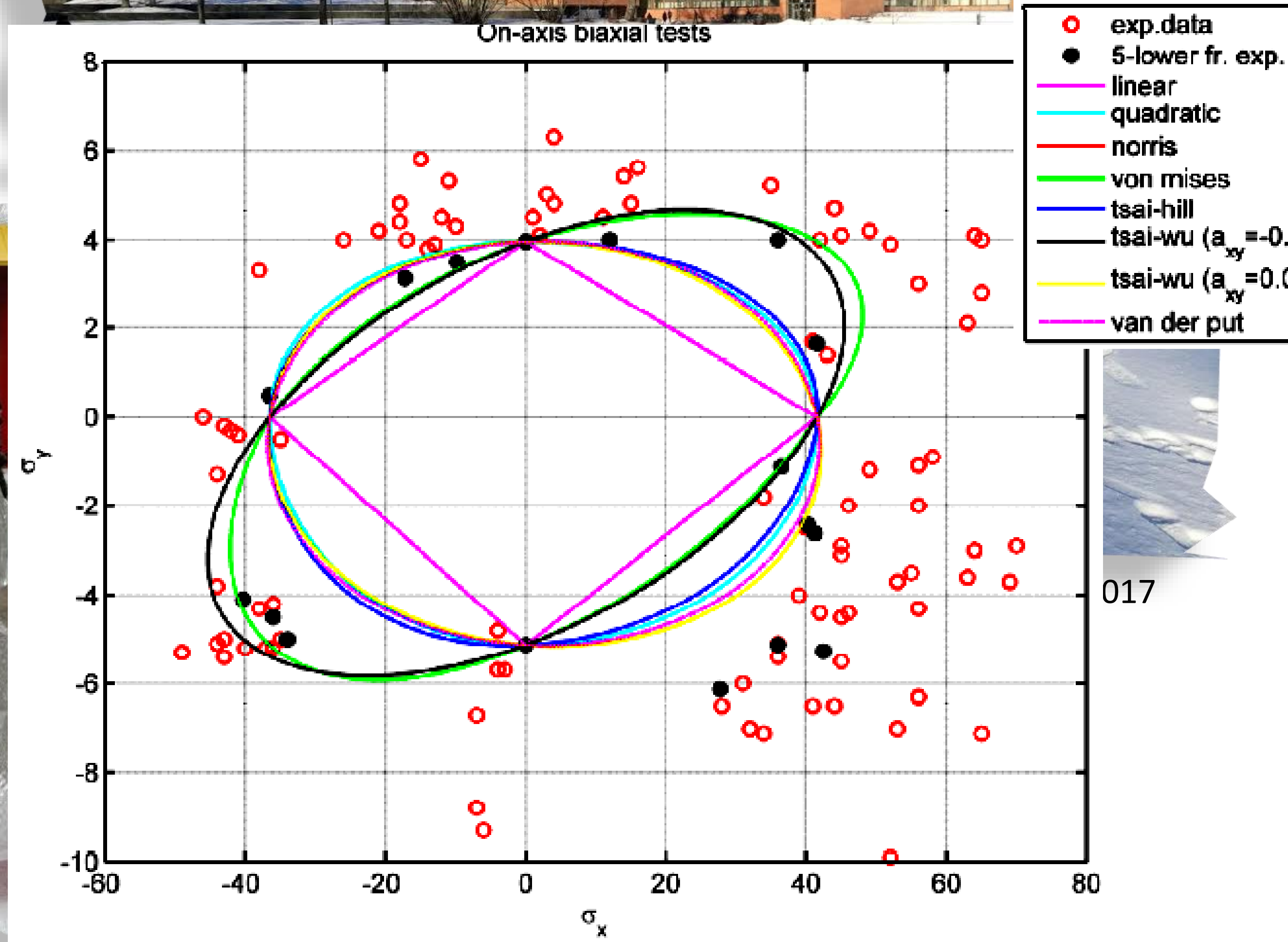
  - *Isotropic* hardening.
    - The yield surface expands uniformly in all directions with plastic flow.



metals exhibit kinematic hardening behavior for small
ı cyclic loading.

- Isotropic hardening states that the yield surface expands uniformly during plastic flow. The term 'isotropic' refers to the uniform dilatation of the yield surface and is different from an 'isotropic' yield criterion (i.e., material orientation).





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Chinese new-year in Helsinki, 20

Plot of the failure criteria with the on-axis experimental data from Eberhardsteiner (2002).

Homework, exercises – plasticity – yield criteria …

2. Materiaalin jännitys-venymä riippuvuutta yksiaksiaalisessa vedossa aproksimoidaan Ramberg-Osgood tyyppisellä kaavalla

$$\varepsilon = \varepsilon^{\mathrm{e}} + \varepsilon^{\mathrm{p}} = \frac{\sigma}{E} + \left(\frac{\sigma}{b}\right)^n .$$

(a) Määritä tangenttimoduuli E_{t} ja plastinen moduuli H sekä jännityksen, että plastisen venymän ε^{p} funktiona.

(b) Määritä plastinen työ W_{p} sekä jännityksen että plastisen venymän funktiona.

(c) Mikä on myötöjännityksen arvo?

(d) Mikäli $n = 5$, mikä on 0,1 % ja 0,2 % pysyvää venymää vastaava jännityksen arvo?

2)

$$\varepsilon = \varepsilon^e + \varepsilon^p = \frac{\sigma}{E} + \left(\frac{\sigma}{b}\right)^n$$

a)

$$\varepsilon^p = \left(\frac{\sigma}{b}\right)^n \qquad H = \frac{d\sigma}{d\varepsilon^p}$$

$$\sigma = b\,(\varepsilon^p)^{1/n} \qquad H = \frac{b}{n}(\varepsilon^p)^{\frac{1}{n}-1} = \frac{b}{n}(\varepsilon^p)^{\frac{1-n}{n}}$$

$$= \frac{b}{n}\left(\frac{\sigma}{b}\right)^{1-n}$$

$$E_t = \frac{d\sigma}{d\varepsilon}$$

$$d\varepsilon = \frac{1}{E}d\sigma + n\left(\frac{\sigma}{b}\right)^{n-1}\frac{1}{b}\,d\sigma$$

$$= \left(\frac{1}{E} + \frac{n}{b}\left(\frac{\sigma}{b}\right)^{n-1}\right)d\sigma$$

$$\frac{d\sigma}{d\varepsilon} = \frac{1}{\frac{1}{E} + \frac{n}{b}\left(\frac{\sigma}{b}\right)^{n-1}}$$

b)

$$W_p = \int_0^{\varepsilon^p} \sigma\, d\varepsilon^p = \int_0^{\varepsilon^p} b\,(\varepsilon^p)^{1/n}\, d\varepsilon^p = \frac{nb}{n+1}(\varepsilon^p)^{\frac{n+1}{n}}$$

$$= \frac{nb}{n+1}\left(\frac{\sigma}{b}\right)^{n+1}$$

c) riippuu määritelmästä yleensä 0,2% pysyvää venymää vastaava arvo valitaan myötöjännitysarvoksi. R-O-mallin plastinen muodonmuutos syntyy aina kun $\sigma \neq 0$.

$$\sigma_y = b\,(0{,}002)^{\frac{1}{n}}$$

d)

$$\sigma_{y0,1} = b\,(0{,}001)^{\frac{1}{5}}$$

$$\sigma_{y0,2} = b\,(0{,}002)^{\frac{1}{5}}$$

Näiden suhde on $\left(\frac{0{,}001}{0{,}002}\right)^{\frac{1}{5}} = \left(\frac{1}{2}\right)^{1/5} \simeq 0{,}87.$

Homework, exercises – plasticity – yield criteria …

3. Myötöehto

$$3J_2 + (\sigma_c - \sigma_t)I_1 - \sigma_c\sigma_t = 0$$

tunnetaan von Mises-Schleicher ehtona ja jossa σ_t ja σ_c ovat materiaalin yksiaksiaalisen jännitystilan veto- ja puristuslujuudet. Invarianttien I_1 ja J_2 lausekkeet ovat $I_1 = \sigma_{kk}$ ja $J_2 = \frac{1}{2}s_{ij}s_{ji}$, jossa s on deviatorinen jännitystensori.

(a) Määritä ja piirrä myötöehto tasojännitystilassa. Määritä myös biaksiaalinen puristuslujuus σ_{bc} kun $\sigma_1 = \sigma_2 = -\sigma_{bc}$.

(b) Määritä ja piirrä puristus- ja vetomeridiaanit meridiaanitasolla.

(c) Miten myötöehto eroaa Druckerin-Pragerin ehdosta?

3)

$$3J_2 + (\sigma_c - \sigma_t)I_1 - \sigma_c\sigma_t = 0$$

Tasojännitystila $\quad \underline{\sigma} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad$ pääjänn. koord.

$$I_1 = \sigma_1 + \sigma_2$$

$$\underline{s} = \underline{\sigma} - \frac{1}{3}I_1\underline{I} = \begin{pmatrix} \sigma_1 - \frac{1}{3}(\sigma_1+\sigma_2) & & \\ & \sigma_2 - \frac{1}{3}(\sigma_1+\sigma_2) & \\ & & -\frac{1}{3}(\sigma_1+\sigma_2) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{3}\sigma_1 - \frac{1}{3}\sigma_2 & & \\ & \frac{2}{3}\sigma_2 - \frac{1}{3}\sigma_1 & \\ & & -\frac{1}{3}(\sigma_1+\sigma_2) \end{pmatrix}$$

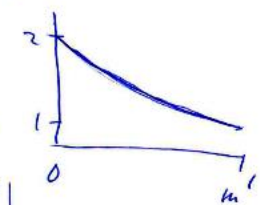
$$J_2 = \frac{1}{2}s_{ij}s_{ji} = \frac{1}{2}\left[\left(\frac{2}{3}\sigma_1 - \frac{1}{3}\sigma_2\right)^2 + \left(\frac{2}{3}\sigma_2 - \frac{1}{3}\sigma_1\right)^2 + \frac{1}{9}(\sigma_1+\sigma_2)^2\right]$$

$$= \frac{1}{2}\left[\frac{4}{9}\sigma_1^2 - \frac{4}{9}\sigma_1\sigma_2 + \frac{1}{9}\sigma_2^2 + \frac{4}{9}\sigma_2^2 - \frac{4}{9}\sigma_1\sigma_2 + \frac{1}{9}\sigma_1^2 + \frac{1}{9}\sigma_1^2 + \frac{1}{9}\sigma_2^2 + \frac{2}{9}\sigma_1\sigma_2\right]$$

$$= \frac{1}{2}\left(\frac{6}{9}\sigma_1^2 + \frac{6}{9}\sigma_2^2 - \frac{6}{9}\sigma_1\sigma_2\right) = \frac{1}{3}(\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2)$$

$$(\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2) + (\sigma_c - \sigma_t)(\sigma_1 + \sigma_2) - \sigma_c\sigma_t = 0$$

jos $\sigma_1 = \sigma_2 = -\sigma_{bc}$



$$\sigma_{bc}^2 - (\sigma_c - \sigma_t)2\sigma_{bc} - \sigma_c\sigma_t = 0$$

$$\sigma_{bc} = \frac{2(\sigma_c - \sigma_t) \mp \sqrt{4(\sigma_c - \sigma_t)^2 + 4\sigma_c\sigma_t}}{2}$$

$$= \frac{2(\sigma_c - \sigma_t)\left[1 \mp \sqrt{1 + \frac{\sigma_c\sigma_t}{(\sigma_c - \sigma_t)^2}}\right]}{2}$$

$$= (\sigma_c - \sigma_t)\left[1 + \sqrt{1 + \frac{\sigma_c\sigma_t}{(\sigma_c - \sigma_t)^2}}\right] \qquad \sigma_t = m\sigma_c$$

Homework, exercises – plasticity – yield criteria …

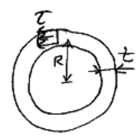
1. Ohutseinämäistä ympyräputkea käytetään usein plastisten ainemallin parametrien määrittämiseksi tehtävissä kokeissa. Oletetaan, että putken seinämän paksuus t on hyvin paljon pienempi kuin putken poikkileikkauksen säde R, eli $t/R \ll 1$. Putkea kuormitetaan vääntömomentilla T ja normaalivoimalla N.

   (a) Mikäli materiaali noudattaa von Misesin myötöehtoa $\sqrt{3J_2} = \sigma_y$, jossa σ_y on materiaalin myötöraja. Kirjoita myötöehto vääntömomentin T ja normaalivoiman N avulla. J_2 on jännitysmatrisiin deviaattorin toinen invariantti $J_2 = \frac{1}{2}\mathrm{tr}(s^2)$.

   (b) Mikä on Loden kulman θ arvo kun kuormituksena on pelkkä (i) vääntömomentti tai (ii) normaalivoima (huom: normaalivoiman merkki)?

   (c) Mikäli putkea rasittaa normaalivoima N, joka yksistään aiheuttaa jännitystilan, joka on puolet myötörajasta. Kuinka suuren putken sisäisen ylipaineen se kestää?

1)



Putken vääntömomentti

$T = 2\pi R t \tau \cdot R = 2\pi t R^2 \tau$

Normaalivoima

$N = 2\pi R t \sigma$

$$\Rightarrow \begin{cases} \tau = \dfrac{T}{2\pi t R^2} \\ \sigma = \dfrac{N}{2\pi t R} \end{cases} \qquad \underline{\sigma} = \begin{pmatrix} \sigma & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \sigma_m = \tfrac{1}{3}\sigma$$

$$\underline{s} = \begin{pmatrix} \frac{2}{3}\sigma & \tau & 0 \\ \tau & -\frac{1}{3}\sigma & 0 \\ 0 & 0 & -\frac{1}{3}\sigma \end{pmatrix}$$

$$J_2 = \tfrac{1}{2} s_{ij}s_{ji} = \tfrac{1}{2} s_{ij}s_{ij} = \tfrac{1}{2}\left(\tfrac{4}{9}\sigma^2 + \tfrac{1}{9}\sigma^2 + \tfrac{1}{9}\sigma^2 + 2\tau^2\right)$$
$$= \tfrac{1}{3}\sigma^2 + \tau^2$$

a) $\sqrt{3J_2} = \sqrt{\sigma^2 + 3\tau^2} = \sigma_y$

$$\Rightarrow \sqrt{\frac{N^2}{4\pi^2 t^2 R^2} + 3\frac{T^2}{4\pi^2 t^2 R^4}} = \sigma_y$$

$$\Rightarrow \sqrt{N^2 + 3\left(\frac{T}{R}\right)^2} = \sigma_y 2\pi t R = N_y$$

N_y on normaalivoima kun materiaali myötää, jos putkea rasittaa pelkkä normaalivoima.

b) $\theta = \frac{1}{3}\arccos \frac{3\sqrt{3}}{2}\frac{J_3}{J_2^{3/2}}$

pelkkä vääntömomentti $\Rightarrow$ pelkkä τ

$\Rightarrow J_2 = \tau^2, \ J_3 = 0 \Rightarrow \theta = 30°$

pelkkä normaalivoima $\Rightarrow$ pelkkä σ

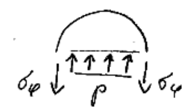
$\Rightarrow J_2 = \frac{1}{3}\sigma^2, \ J_3 = \frac{2}{27}\sigma^3 \Rightarrow$

$$\frac{3\sqrt{3}}{2}\cdot\frac{\frac{2}{27}\sigma^3}{\left(\frac{1}{3}\right)^{3/2}|\sigma^3|} = \frac{\sigma}{|\sigma|}$$

$\Rightarrow \theta = \frac{1}{3}\arccos 1$ jos $\sigma > 0 \Rightarrow \theta = 0°$

$\theta = \frac{1}{3}\arccos(-1)$ jos $\sigma < 0 \Rightarrow \theta = 60°$

c)



$p \cdot 2R = \sigma_\varphi \cdot 2t$

$\sigma_\varphi = p \cdot \frac{R}{t}$

$$\underline{\sigma} = \begin{pmatrix} \frac{1}{2}\sigma_y & 0 & 0 \\ 0 & p\frac{R}{t} & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \sigma_m = \tfrac{1}{3}\left(\tfrac{1}{2}\sigma_y + p\tfrac{R}{t}\right) = \tfrac{1}{6}\sigma_y + \tfrac{1}{3}p\tfrac{R}{t}$$

$$\underline{s} = \begin{pmatrix} \left(\frac{1}{2}-\frac{1}{6}\right)\sigma_y - \frac{1}{3}p\frac{R}{t} & & \\ & -\frac{1}{6}\sigma_y - \frac{1}{3}p\frac{R}{t} & \\ & & -\frac{1}{6}\sigma_y - \frac{1}{3}p\frac{R}{t} \end{pmatrix}$$

$\sqrt{3J_2} = \sqrt{\frac{3}{2}s_{ij}s_{ij}} = \sigma_y$ merkit. $p = \xi\sigma_y\frac{t}{R}$

$$\underline{s} = \begin{pmatrix} \frac{1}{3} - \frac{1}{3}\xi & & \\ & -\frac{1}{6} - \frac{1}{3}\xi & \\ & & -\frac{1}{6} - \frac{1}{3}\xi \end{pmatrix}\sigma_y$$

$$\sqrt{3J_2} = \sqrt{\tfrac{3}{2}\left[\tfrac{1}{9}(1-\xi)^2 + \tfrac{1}{36}(1-2\xi)^2 + \tfrac{1}{36}(1-2\xi)^2\right]}\ \sigma_y = \sigma_y$$

$$\Rightarrow \sqrt{\tfrac{1}{6}(1-\xi)^2 + \tfrac{1}{12}(1-2\xi)^2} = 1$$

$$\Rightarrow \tfrac{1}{12}\left[2 - 4\xi + 2\xi^2 + 1 - 4\xi + 4\xi^2\right] = 1$$

$$\Rightarrow 6\xi^2 - 8\xi - 9 = 0 \Rightarrow \xi = \tfrac{2}{3} \pm \sqrt{\tfrac{35}{18}}$$

Jos putkea rasittaa vetävä normaalivoima

$N = 2\pi t R \frac{1}{2}\sigma_y = \pi t R \sigma_y$

suurin sallittu ylipaine on

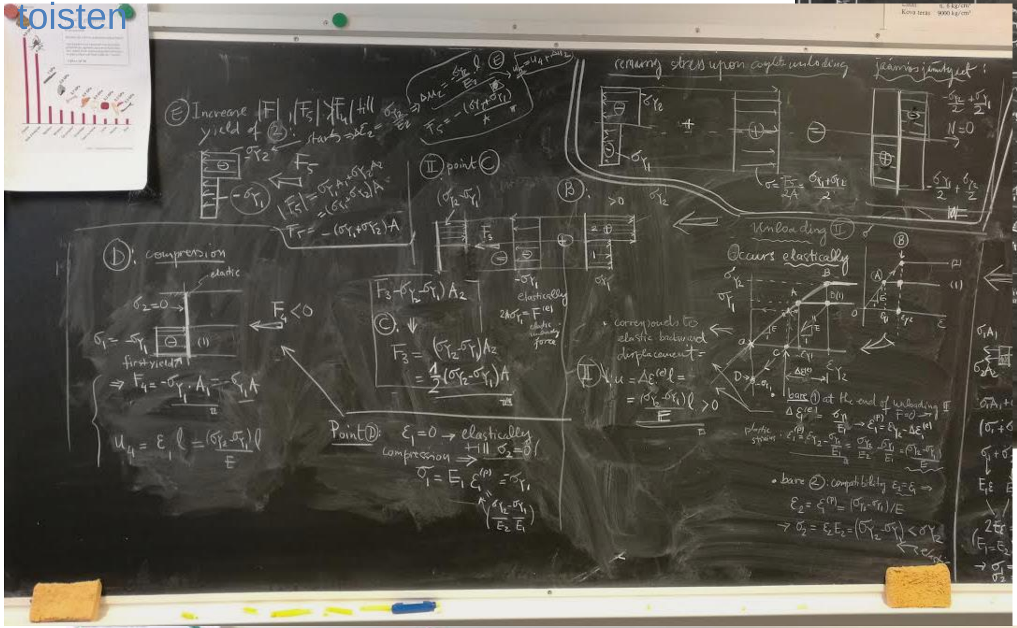
$p = \left(\frac{2}{3} + \sqrt{\frac{35}{18}}\right)\frac{t}{R}\sigma_y$

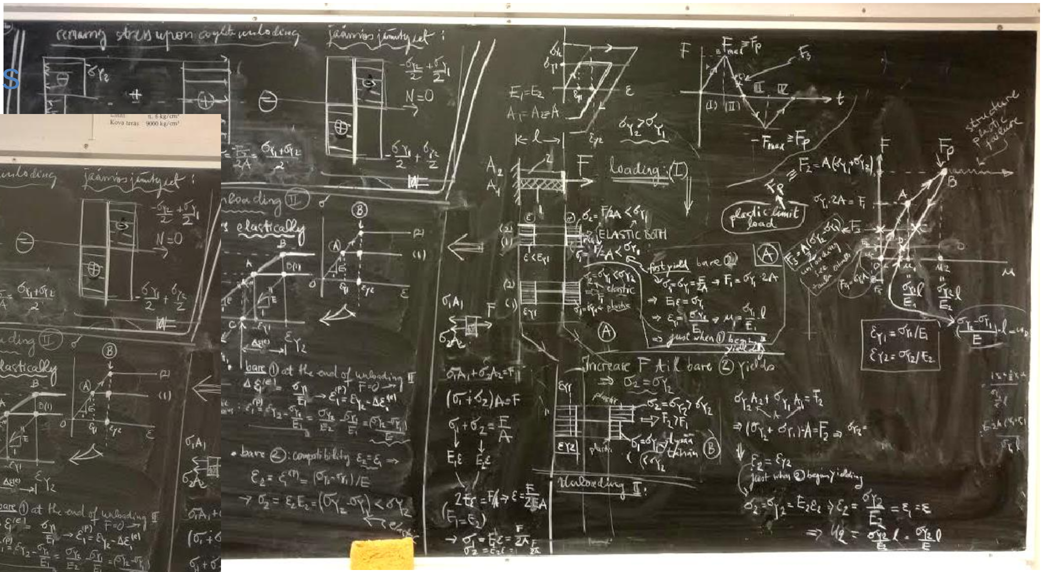
pienin alipaine vastaavasti

$p = \left(\frac{2}{3} - \sqrt{\frac{35}{18}}\right)\frac{t}{R}\sigma_y$

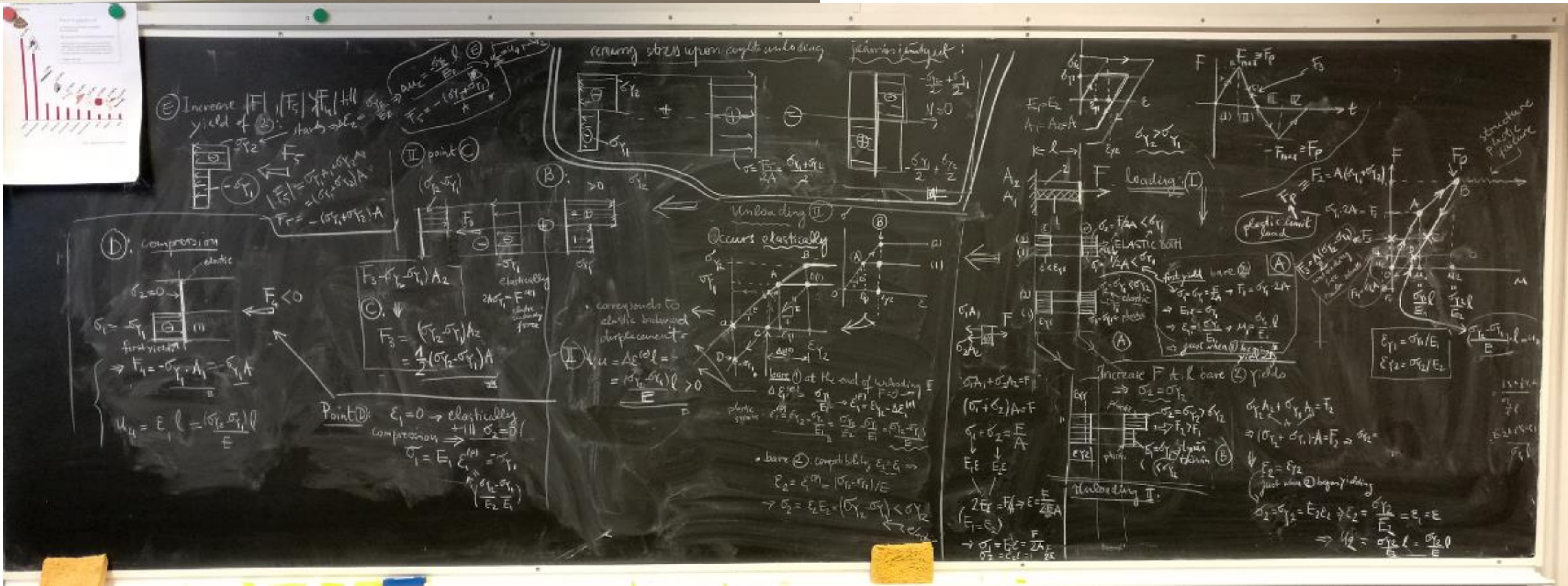
Mikäli putkea rasittaisi puristava normaalivoima, joka aiheuttaisi jännityksen $-\frac{1}{2}\sigma_y$, lasku menisi samaan tapaan.

Homework – exo 4? Ehkä joskus toisten





Ota symmetrinen matsku, eli 2 kertaa ympäri sama ja keskellä eri, saat taiv. Momentit nolla jäännösjännistyx



Example – design of filled weld

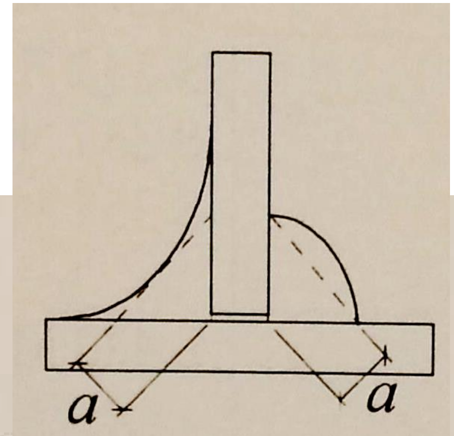
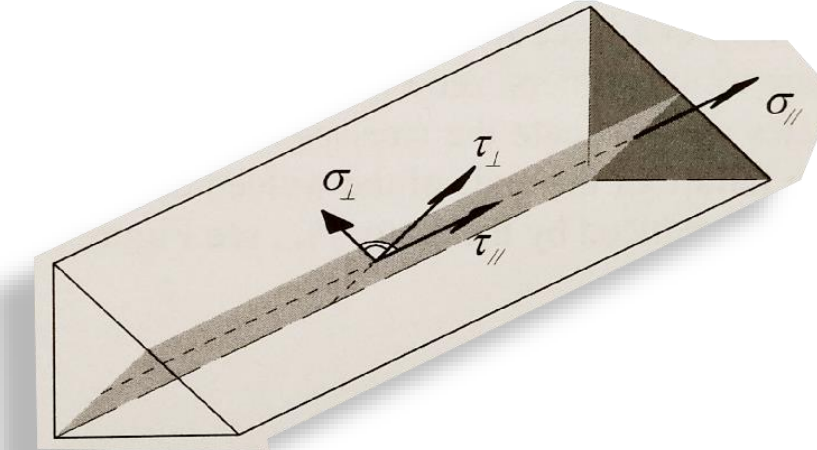


Figure 3.1 Definition of throat thickness a



Stress in critical plane of fillet weld

In the design procedure the internal force on the fillet weld is resolved into components paralle
transverse to the critical plane of the weld throat, see Figure 3.2. A uniform stress distributi
assumed on the critical throat section of the weld, leading to the following normal stresses and
stresses:

- $\sigma_\perp$ the normal stress perpendicular to the critical plane of the throat,
- $\sigma_{//}$ the normal stress parallel to the axis of the weld, it should be neglected when calcul the design resistance of a fillet weld,
- $\tau_\perp$ the shear stress (in the critical plane of the throat) perpendicular to the weld axis,
- $\tau_{//}$ the shear stress (in the critical plane of the throat) parallel to the weld axis.

2-D plane stress state:

$$\sqrt{\sigma_{xx}^2 + 3\tau_{xy}^2} = \sigma_Y$$

Do you recognise Von Mises?

$$\sqrt{\sigma_\perp^2 + 3(\tau_\perp + \tau_{//})^2} \le \frac{f_u}{\beta_w \, \gamma_{Mw}}$$

and

$$\sigma_\perp \le \frac{f_u}{\gamma_{Mw}}.$$

The resistance of the fillet weld will be sufficient if th

$$\sqrt{\sigma_\perp^2 + 3(\tau_\perp + \tau_{//})^2} \le \frac{f_u}{\beta_w \, \gamma_{Mw}}$$

and

$$\sigma_\perp \le \frac{f_u}{\gamma_{Mw}}.$$

The correlation factor β_w is summarised in Table 3.1.

prEN 1993-1-8 includes a simplified procedu
the fillet weld per unit length independent of the direct

$$f_{vw.d} = \frac{f_u}{\sqrt{3}\,\beta_w \, \gamma_{Mw}}$$

Examples of hardening rules in Abaqus – how it looks like?

Example 4: Kinematic Hardening Plasticity

- Plasticity:
  - Yield function:

$$\sqrt{\frac{3}{2}(S_{ij}-\alpha_{ij})(S_{ij}-\alpha_{ij})}-\sigma_y = 0 .$$

  - Equivalent plastic strain rate:

$$\dot{\bar{\varepsilon}}^{pl} = \sqrt{\frac{2}{3}\dot{\varepsilon}_{ij}^{pl}\dot{\varepsilon}_{ij}^{pl}}.$$

  - Plastic flow law:

$$\dot{\varepsilon}_{ij}^{pl} = \frac{3}{2}(S_{ij}-\alpha_{ij})\dot{\bar{\varepsilon}}^{pl}/\sigma_y .$$

  - Prager-Ziegler (linear) kinematic hardening:

$$\dot{\alpha}_{ij} = \frac{2}{3}h\dot{\varepsilon}_{ij}^{pl}.$$

Example 5: Isotropic Hardening Plasticity

- Plasticity:
  - Yield function:

$$\sqrt{\frac{3}{2}S_{ij}S_{ij}}-\sigma_y(\bar{\varepsilon}^{pl}) = 0 ,$$

$$\sqrt{\frac{3}{2}S_{ij}S_{ij}}-\sigma_y(\bar{\varepsilon}^{pl}) = 0 , \qquad S_{ij} = \sigma_{ij}-\frac{1}{3}\delta_{ij}\sigma_{kk} .$$

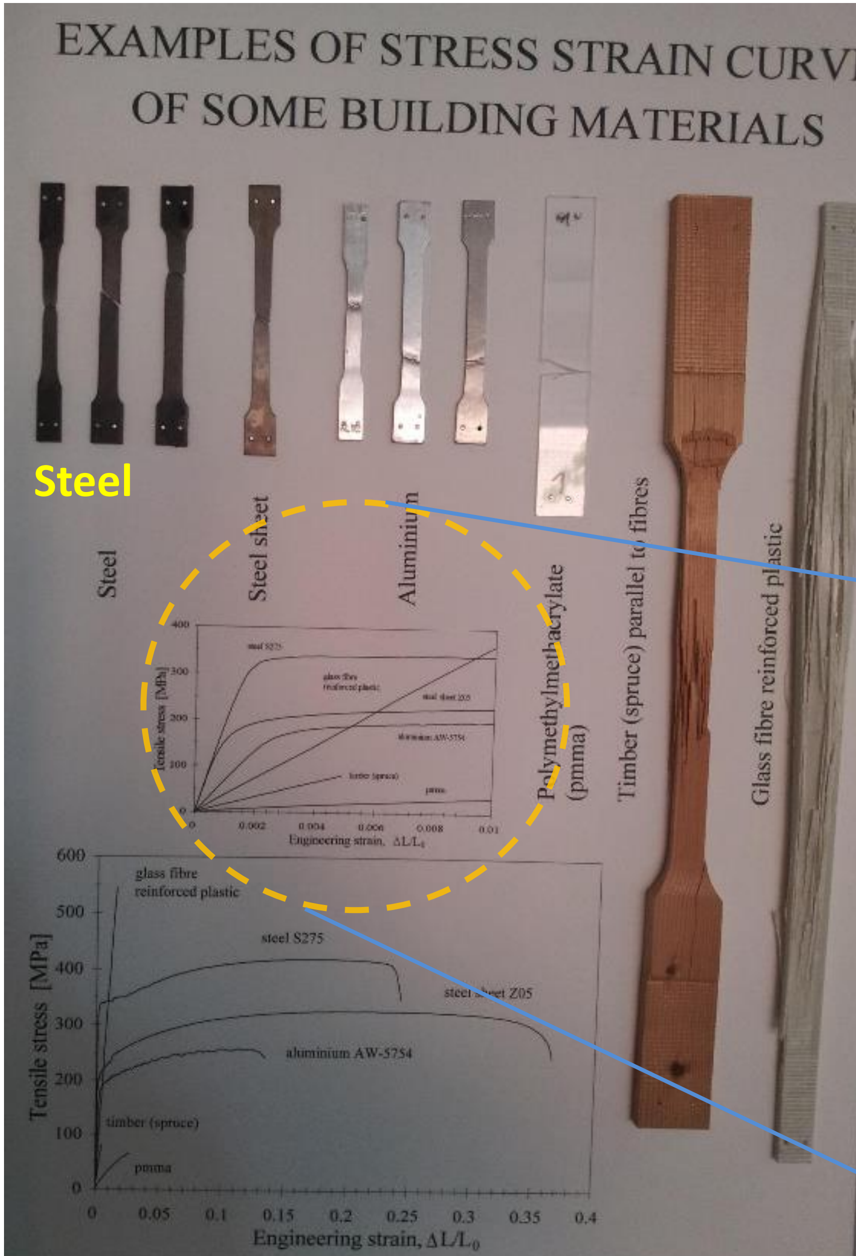
  - Equivalent plastic strain:

$$\bar{\varepsilon}^{pl} = \int_0^t \dot{\bar{\varepsilon}}^{pl}dt, \qquad \dot{\bar{\varepsilon}}^{pl} = \sqrt{\frac{2}{3}\dot{\varepsilon}_{ij}^{pl}\dot{\varepsilon}_{ij}^{pl}}.$$

  - Plastic flow law:

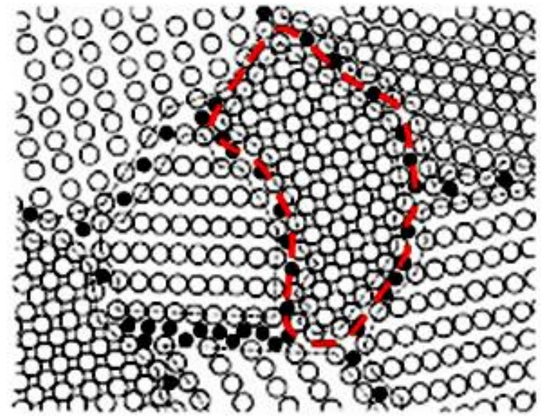
$$\dot{\varepsilon}_{ij}^{pl} = \frac{3}{2}\frac{S_{ij}}{\sigma_y}\dot{\bar{\varepsilon}}^{pl}.$$

A word about the tensile test

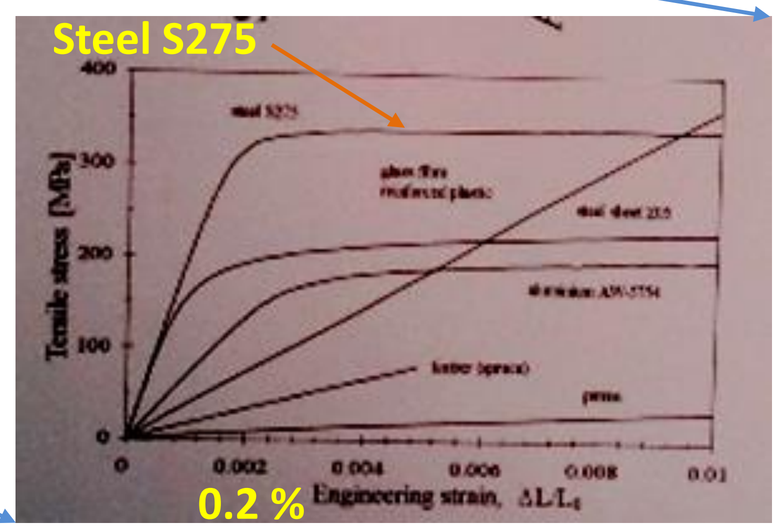


1 grain = 1 cristal + 1 orientation

Schematic view of a polycristal



H. Gleiter, Acta Mater. 48 (2000), p. 3



E3.2

Eräässä kontinuumin pisteessä vallitsee jännitystila, jossa jännityskomponentit suhtautuvat toisiinsa seuraavasti:

$$\tau_{zx}:\ \tau_{yz}:\ \tau_{xy}:\ \sigma_z:\ \sigma_y:\ \sigma_x = 1:\ 2:\ 3:\ 4:\ 5:\ 6$$

Millä jännityksen σ_x arvolla tapahtuu myötäminen a) Misesin, b) Trescan myötöehdon mukaan, kun aineen myötöraja on $\sigma_m = 240$ MPa?

a) Misesin myötöehto kolmidimensioisessa jännitystilassa on

$$\sigma_0^2 = \sigma_m^2/45 \quad \Rightarrow \quad \sigma_0 = 35.8 \text{ MPa},$$

josta edelleen ratkaisuksi $\sigma_x = 6\sigma_0 = 214.7$ MPa.

b) Trescan myötöehto kuuluu

$$\max\left\{ \tfrac{1}{2}\,|\sigma_1-\sigma_2|,\ \tfrac{1}{2}\,|\sigma_2-\sigma_3|,\ \tfrac{1}{2}\,|\sigma_3-\sigma_1| \right\} - \tau_m = 0$$

Pääjännitykset saadaan jännitysmatriisin ominaisarvoina ts. d

$$\det\begin{bmatrix} \sigma_x-\sigma & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y-\sigma & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z-\sigma \end{bmatrix} = 0,$$

mistä saadaan kolmannen asteen yhtälö

$$\sigma^3 - 15\,\sigma_0\sigma^2 + 60\,\sigma_0^2\sigma - 67\,\sigma_0^3 = 0$$

Yhtälön juuret ovat
$$\begin{cases} \sigma_1 = 1.921\ \sigma_0 \\ \sigma_2 = 3.730\ \sigma_0 \\ \sigma_3 = 9.349\ \sigma_0 \end{cases}$$

Trescan ehdosta tulee myötöjännitykseksi σ_0

$$\tfrac{1}{2}(\sigma_3-\sigma_1) - \tau_m = \tfrac{1}{2}(\sigma_3-\sigma_1) - \tfrac{1}{2}\sigma_m = 0 \quad \Rightarrow \quad \sigma_0 = 32.3 \text{ MPa},$$

mistä saadaan ratkaisuksi $\sigma_x = 6\sigma_0 = 193.9$ MPa.

Exercises – Example - HW

Erikoisaihe

Taipumien arviointi – plastisessa rajatilassa
Kehät

TAIPUMIEN ARVIOINTI **Plast.teoria 1996**

Aikaisemmin on tarkasteltu vain sauvarakenteen plastisen rajakuorman määrittämistä tai rajamitoitusta. Joissakin tapauksissa on tarpeen tutkia myös rakenteessa syntyviä siirtymiä. Jos rakenteen käyttökelpoisuus/toimintakelpoisuus riippuu siirtymien suuruudesta, on riittämätöntä tarkastella ainoastaan kuormankantokykyä vaan on otettava huomioon myös siirtymien arvot murtorajatilassa. Niiden tarkka määrittäminen on suhteellisen työlästä, varsinkin siinä tapauksessa että rakenteen geometrian muutokset siirtymien suuruuden takia on otettava huomioon. Tällainen geometrisesti epälineaarinen analyysi voidaan tehdä tietokoneella esim. elementtimenetelmään perustuen.

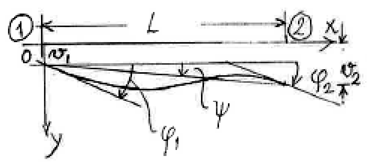
Seuraavassa esitettävä **likimääräinen menetelmä** perustuu otaksumiin:

- idealisoituun taivutusmomentti-käyristymä -riippuvuus pätee, jolloin plastisia muodonmuutoksia tapahtuu vain myötönivelissä ja sauvat ovat muutoin kimmoisia,
- kuormitus kasvaa monotonisesti yhden parametrin, kuormaparametrin, funktiona nollasta rajakuorman arvoon,
- sen jälkeen kun myötönivel on syntynyt, plastinen rotaatio siinä muuttuu koko ajan samaan suuntaan, ts. $\theta\dot{\theta} \geq 0$,
- siirtymät ovat niin pieniä, että rakenteen geometrian muutokset voidaan jättää huomioonottamatta.

Murtorajatilassa n kertaa staattisesti määräämättömässä rakenteessa on n+1 myötöniveltä. Taipuma lasketaan tilassa, jolloin viimeinen myötönivel syntyy, ts. siinä ei vielä ole tapahtunut plastista kiertymää. Nivelien syntymisjärjestystä ei yleensä tiedetä ennakolta, vaan on kokeiltava joitakin mahdollisia viimeisen nivelen kohtia ja määritettävä vastaava taipuma. Oikeaa viimeisen nivelen otaksumaa vastaa suurin taipuman arvo, ts. rakenne toimii plastisten nivelkiertymien suhteen mahdollisimman joustavasti. Tarkistus saadaan myös siitä, että nivelkiertymien ja nivelissä vallitsevien täysplastisten momenttien on oltava merkiltään samoja.

Kimmoisen sauvan muodonmuutokset

Kimmoisen tasajäykän sauvan sauvanpääkiertymät ovat (ks. RM1:n luennot):



$$\varphi_1 = 2\beta M_1 - \beta M_2 + \alpha_1^0 + \psi = 2\beta(M_1 - MK_1) - \beta(M_2 - MK_2) + \psi$$
$$\varphi_2 = -\beta M_1 + 2\beta M_2 + \alpha_2^0 + \psi = -\beta(M_1 - MK_1) + 2\beta(M_2 - MK_2) + \psi$$

Kaavoissa α_1^0 ja α_2^0 ovat kuormituksen aiheuttamat sauvanpääkiertymät, MK_1 ja MK_2 kuormituksen aiheuttamat kiinnitysmomentit ja ψ on sauvakiertymä $\psi = (v_2 - v_1)/L$. Sauvavakio β on $\beta = L/6EI$. Kiertymien ja momenttien positiivinen suunta on myötäpäivään.

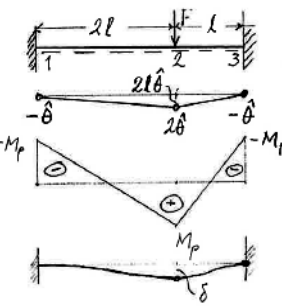
Myötönivelessä i syntyvä kiertymä lausutaan siitä oikealle ja vasemmalle olevien sauvanpääkiertymien avulla

$$\theta_i = -(\varphi_{i,oik} - \varphi_{i,vas})$$

Kimmoisessa sauvassa yllä oleva kiertymä on jatkuvuuden takia nolla samoin kuin myötönivelen juuri syntyessä, mutta muutoin se esittää plastista kiertymää. Edelleen on otettava huomioon, että sauvanpäämomentit myötönivelen oikealla ja vasemmalla puolella ovat

$$M_{i,oik} = \mathrm{sign}(\theta_i)M_p$$
$$M_{i,vas} = -\mathrm{sign}(\theta_i)M_p$$

Esimerkki 1. Määritä oheisen palkin taipuma δ pistekuorman kohdalla rajatilassa.



Plastinen rajakuorma:

$$(-M_p)(-\hat{\theta}) + M_p 3\hat{\theta} + (-M_p)(-2\hat{\theta}) = F2l\hat{\theta}$$

Saadaan $F_p = 3M_p/l$.

Kimmoiset sauvanpääkiertymät:

$$\varphi_{12} = -4\beta M_p + 2\beta M_p + \psi = -2\beta M_p + \psi$$
$$\varphi_{21} = 2\beta M_p - 4\beta M_p + \psi = -2\beta M_p + \psi$$
$$\varphi_{23} = 2\beta M_p - \beta M_p - 2\psi = \beta M_p - 2\psi$$
$$\varphi_{32} = -\beta M_p + 2\beta M_p - 2\psi = \beta M_p - 2\psi$$

Yhtälöissä on $\beta = l/6EI$ ja $\psi = \delta/2l$. Kuormitustermit ovat nollia, koska solmupisteiden välillä ei ole kuormaa.

Nivelkiertymät ovat:

$$\theta_1 = -\varphi_{12} = 2\beta M_p - \psi$$
$$\theta_2 = -(\varphi_{23} - \varphi_{21}) = -3\beta M_p + 3\psi$$
$$\theta_3 = \varphi_{32} = \beta M_p - 2\psi$$

Otaksutaan aluksi, että **nivel syntyy viimeiseksi solmuun 1**. Tällöin ehdosta $\theta_1 = 0$ saadaan $\psi = 2\beta M_p$. Nivelkiertymille θ_2 ja θ_3 seuraavat silloin arvot $\theta_2 = 3\beta M_p$ ja $\theta_3 = -3\beta M_p$, jotka ovat merkiltään oikeat. Taipuman arvoksi tulee $\delta = 4\beta M_p l = 2M_p l^2/3EI$.

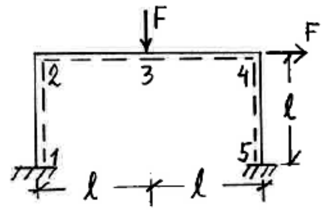
Seuraavaksi otaksutaan **viimeisen nivelen syntyvän solmuun 2.** Ehdosta $\theta_2 = 0$ seuraa $\psi = \beta M_p$ ja edelleen nivelkiertymät $\theta_1 = \beta M_p$ ja $\theta_3 = -\beta M_p$, joista edellinen on merkiltään väärä ts. ei vastaa murtomekanismin merkkiä. Taipuman arvo on $\delta = 2\beta M_p l = M_p l^2/3EI$, joka on vain puolet aikaisemmin saadusta.

Kolmas mahdollisuus on otaksua **viimeisen nivelen syntyvän solmuun 3.** Ehdosta $\theta_3 = 0$ seuraa $\psi = \beta M_p/2$ ja edelleen nivelkiertymät $\theta_1 = 3\beta M_p/2$ ja $\theta_2 = -3\beta M_p/2$, jotka ovat molemmat merkiltään vääriä verrattuna murtomekanismiin. Taipuman arvo on vain neljännes ensimmäisessä kohdassa saadusta.

Voidaan siis päätellä, että **nivel syntyy viimeiseksi solmuun 1** ja että solmun 2 taipuma rajatilassa on $\delta = 4\beta M_p l = 2M_p l^2/3EI$.

Esimerkki 2. Määritä oheisen kehän solmun 2 (tai 4) vaakasiirtymä δ rajatilassa.

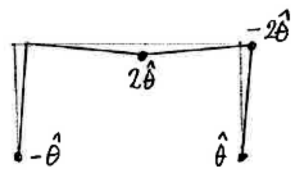
Plastinen rajakuorma:



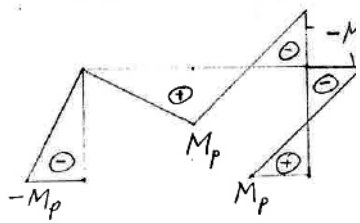
$$(-M_p)(-\hat{\theta})+M_p 2\hat{\theta}+(-M_p)(-2\hat{\theta})+M_p\hat{\theta}=2Fl\hat{\theta}$$

Saadaan $F_p=3M_p/l$. Taivutusmomentti M_2=0 ratkaistaan esim. palkkimekanismin 2-3-4 avulla.

Kimmoiset sauvanpääkiertymät:



$$\varphi_{12}=-2\beta M_p+\psi$$
$$\varphi_{21}=\beta M_p+\psi$$
$$\varphi_{23}=\beta M_p+\tilde{\psi}$$
$$\varphi_{32}=-2\beta M_p+\tilde{\psi}$$
$$\varphi_{34}=2\beta M_p-\beta M_p-\tilde{\psi}=\beta M_p-\tilde{\psi}$$
$$\varphi_{43}=-\beta M_p+2\beta M_p-\tilde{\psi}=\beta M_p-\tilde{\psi}$$
$$\varphi_{45}=-2\beta M_p+\beta M_p+\psi=-\beta M_p+\psi$$
$$\varphi_{54}=\beta M_p-2\beta M_p+\psi=-\beta M_p+\psi$$



Yhtälöissä on $\beta=l/6EI$, $\psi=\delta/l$ ja $\tilde{\psi}=v_3/l$. Kuormitustermit ovat nollia, koska solmupisteiden välillä ei ole kuormaa.

Nivelkiertymät ovat:

$$\theta_1=-\varphi_{12}=2\beta M_p-\psi$$
$$\theta_2=-(\varphi_{23}-\varphi_{21})=\psi-\tilde{\psi}=0$$
$$\theta_3=-(\varphi_{34}-\varphi_{32})=-3\beta M_p+2\tilde{\psi}$$
$$\theta_4=-(\varphi_{45}-\varphi_{43})=2\beta M_p-\psi-\tilde{\psi}$$
$$\theta_5=\varphi_{54}=-\beta M_p+\psi$$

Koska murtomekanismissa ei ole plastista niveltä solmussa 2, on $\tilde{\psi}=v_3/l=\psi=\delta/l$.

a) Otaksutaan, että **nivel syntyy viimeiseksi solmuun 1**. Tällöin $\theta_1=0$, koska sauva on jäykästi kiinnitetty 1:ssä. Tästä ehdosta saadaan $\psi=2\beta M_p$. Nivelkiertymille θ_3, θ_4 ja θ_5 seuraavat silloin arvot $\theta_3=\beta M_p$, $\theta_4=-2\beta M_p$ ja $\theta_5=\beta M_p$ jotka ovat merkiltään oikeat ts. vastaavat murtomekanismin merkkejä. Vaakasiirtymän arvoksi tulee $\delta=2\beta M_p l=M_p l^2/3EI$.

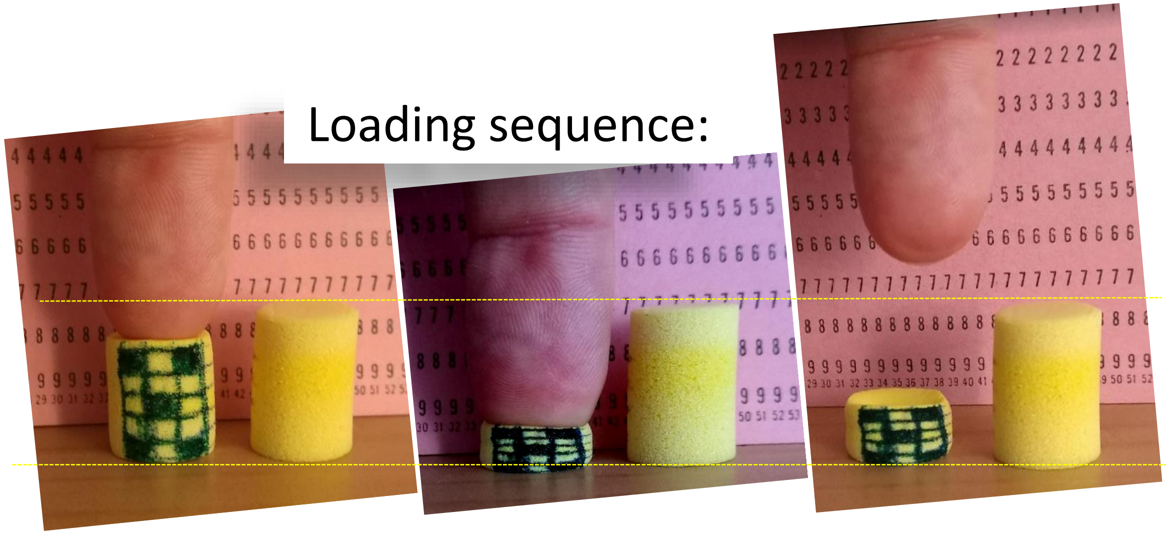
b) Seuraavaksi otaksutaan **viimeisen nivelen syntyvän solmuun 3.** Ehdosta $\theta_3=0$ seuraa $\psi=3\beta M_p/2$ ja edelleen nivelkiertymät $\theta_1>0$, $\theta_4<0$ ja $\theta_5>0$, joista ensimmäinen on merkiltään väärä ts. ei vastaa murtomekanismin merkkiä. Siirtymän arvo on kolme neljäsosaa aikaisemmin saadusta.

c) Otaksuttaessa **viimeisen nivelen syntyvän solmuun 4** saadaan ehdosta $\theta_4=0$ seuraa $\psi=\beta M_p$ ja edelleen nivelkiertymät $\theta_1=\beta M_p$ ja $\theta_3=-\beta M_p$, jotka ovat molemmat merkiltään vääriä verrattuna murtomekanismiin. Siirtymän arvo on vain puolet a)-kohdassa saadusta.

Viimeisen myötönivelen otaksuminen solmuun 5 tuottaa c)-kohdan mukaisen tuloksen. Voidaan siis päätellä, että **nivel syntyy viimeiseksi solmuun 1** ja että solmun 2 (ja 4) vaakasiirtymä rajatilassa on $\delta=2\beta M_p l=M_p l^2/3EI$.

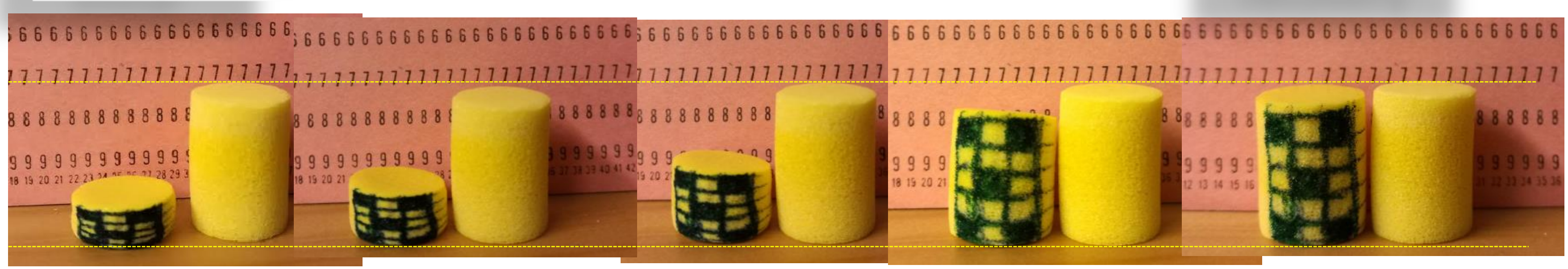
Viscoelasticity

Loading sequence:



Initial

Full recovery

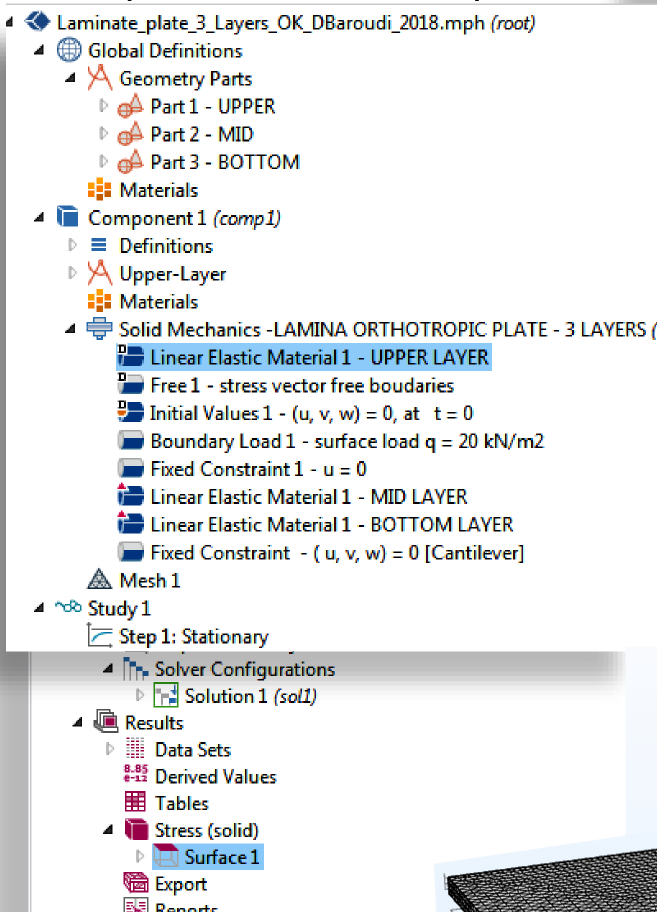


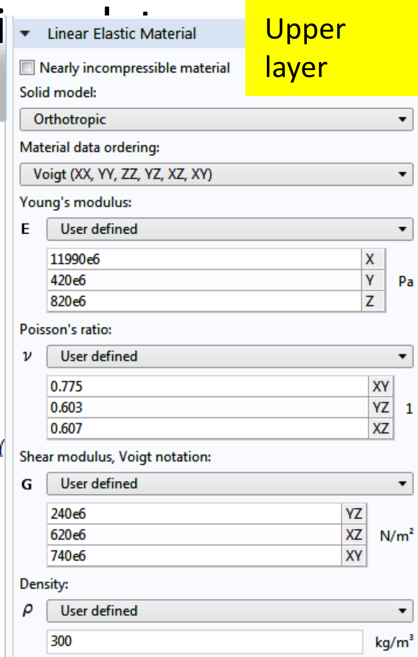
0min

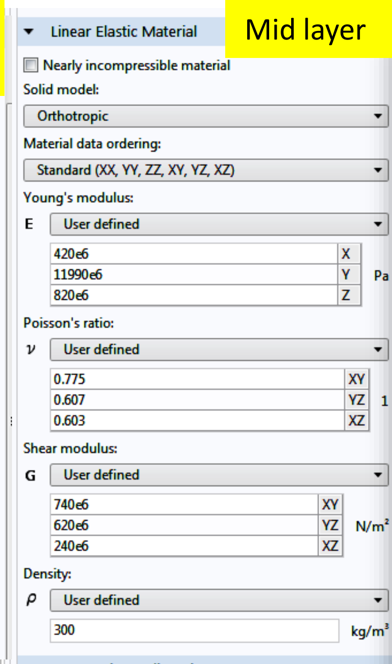
40 min

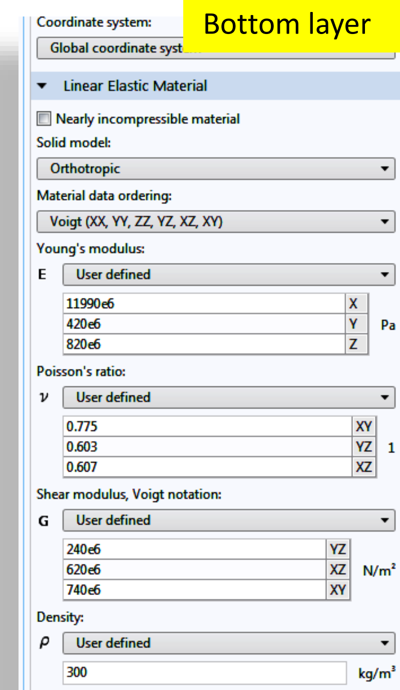
Multilayered orthotropic lami...

Example – A cantilever plate

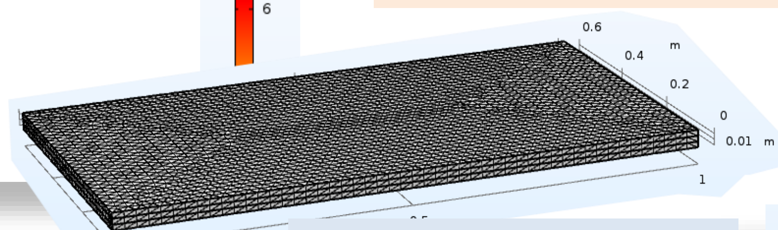


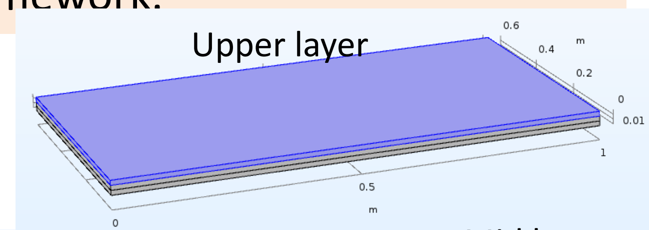


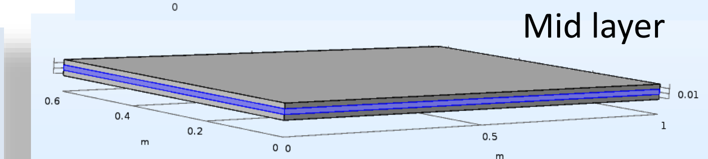




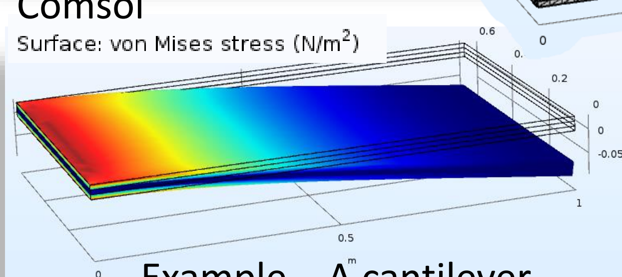
NB, here I did the FEA for a cantilever laminate plate, you should now update to the boundary conditions of the homework.

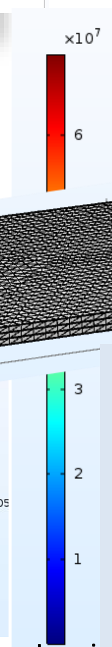






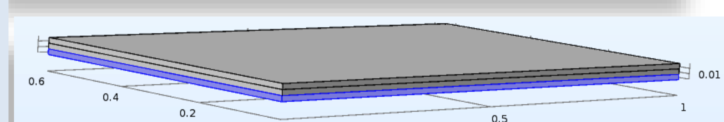
FEA Example using Comsol





Example – A cantilever

Here in FEA, I used 3D-solids elements. So I used 3D-elasticity and not plate theory. (It was a bit 'impossible' to have, in Comsol, layered plates bounded together! At least for me.)



Laminate_plate_3_Layers_OK_DBaroudi_2018.mph