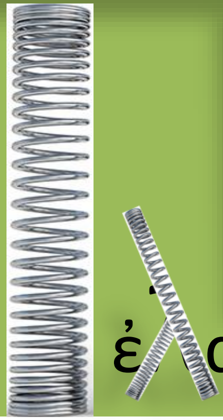
Dr. Djebar BAROUDI, lecturer



ἐλαστός

Readings ELASTICITY in Solids

- ***Lemaitre et al.*** - *Our course textbook* *Paragraphs 4.1 & 4.2* (linear elasticity)
- ***Ottosen et al.***, *chapters 4 and 5.* (elective reading)

Supporting Material (in MyCourses)
Content of the pdf-material

0. **INTRODUTION**
1. **ELASTICITY** (this lecture)
2. **VISCOELASTICITY** (& some basics of creep)
3. **PLASTICITY** (and a recall of *Failure Hypotheses*)

Elasticity? Visco-elasticity? Creep? Plasticity? Damage? Flow rule? Yield function?

$$\sigma = E \cdot \epsilon$$

ἐλαστός

ELASTICITY in Solids

Relaxation

Reading Supporting Material

Otaniemi, 25.4.2022

0. INTRODUTION
1. ELASTICITY >>Short version<<
2. VISCOELASTICITY (& some basics of creep)
3. PLASTICITY (& Failure Hypotheses)

Content

2nd week's lecture

Cauchy's equation of motion
(Newton's equation of motion for deformable bodies)

$$\nabla \cdot \sigma + \rho \mathbf{f} = \rho \frac{D\mathbf{v}}{Dt}$$

We need a **constitutive law** relating kinetics (stresses) with kinematics (deformation measure) to **account for specific material** response

$$\sigma = E \cdot \epsilon$$

Elasticity – kimmoisuus tai elastisuus

- Material Symmetries

Degrees of symmetry

- linear, hyper-elasticity, anisotropy, orthotropy, transverse isotropy, isotropy, homogenisation
- Thermodynamical framework

- **Viscoelasticity – viskoelastisuus** (& some basis for Viscoplasticity/creep)
- **Plasticity – plastisuus** (Engineering Plasticity)
  associative, non-associative
- **Damage** - vauriotuminen
  damage-plasticity *ex*. Concrete Damage Plasticity, Model in Abaqus

to **grasp** – to touch

com*pre***hand** - **käsi***ttää*
com**prehendere** (latin)

To take: <u>understand</u> various elastic response that many engineering materials exhibit

CONTENT

1. INTRODUTION
2. ELASTICITY
3. VISCOELASTICITY
4. PLASTICITY

Elasticity

Definitions

Thermodynamical framework

Elastic Solids

Isothermal Cauchy-elastic material

Green-Elastic or Hyper-elastic Materials

Examples of Non-Linear Elasticity

Hysteresis during loading and unloading

Equations of Elasticity

Minimum Master level

Material Symmetries

Degrees of symmetry

Linear Elasticity – Matrix Formulation

Anisotropy

Isotropy

Limits on Elastic Parameters Values

Orthotropy

Transversal isotropy

Limits on Elastic Parameters Values

Nonlinear isotropic Hooke formulation

Generalized Hooke's Law – Examples of problems

Orthotropic case – A worked example

Homogenisation

layered composite (transverse orthotropy)

Transformation of Stress and Strain Components

Example exercises for training

Nonlinear isotropic Hooke formulation

Some general aspects

Why splitting response to **volumetric** and **deviatoric** (shearing)?

Thermo-elasticity

Minimum level

Hyperelasticity

Rubber or rubber-like Elasticity

Terminology and some definitions

Thermodynamics of rubber – enthalpic and entropic forces

Some classical models

Neo-Hookean model

Mooney-Rivlin model

Yeoh model

Ogden model

Example of Rubber Elasticity In Abaqus

W. Gilbert's experiment

On thermodynamics of elastomers

Homework

Appendix 1

Stress invariants (Recall)

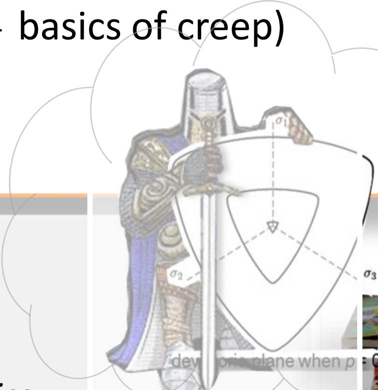
Appendix 2

On Thermodynamics of Rubber

Enthalpic and Entropic forces

Supporting Material in MyCourses

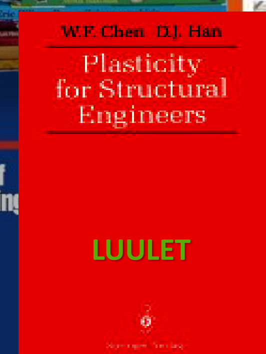
0. INTRODUTION
1. ELASTICITY
2. VISCOELASTICITY (+ basics of creep)
3. PLASTICITY
4. DAMAGE …

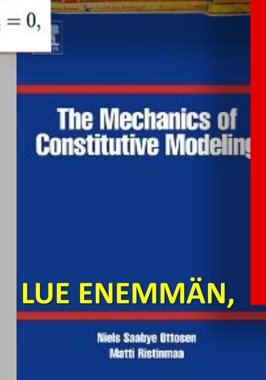


$$A\frac{J_2}{\sigma_c}+\Lambda\sqrt{J_2}+BI_1-\sigma_c=0,$$

Reading – Textbooks:

- **Lemaitre and Chaboche** – *Mechanics of Solid Materials*
- **Ottosen & Ristinmaa** – *The Mechanics of Constitutive Modeling*
- **W.F. Chen, D.J. Han** – *Plasticity for Structural Engineers* (only *chapters* 1-5)





Recommended elective textbooks

- **Kenneth Runesson** – *The Primer - Constitutive modelling of engineering materials.* This covers in details the entire scopes of our course.

- Prof. **Reijo Kouhia's lecture notes:**

1) ***Brief Introduction to continuum mechanics***. 100 pp
2) ***Introduction to materials modelling***

Erittäin ytimekkäitä
Very concise

- ***Plasticity Theory***. Jacob Lubliner
- ***Continuum Mechanics: Elasticity, Plasticity, Viscoelasticity*** Ellis H. Dill, November 10, 2006 by CRC Press

- http://solidmechanics.org/index.html

This electronic material summarizing: physical laws, mathematical models, and algorithms that are used to predict the response of materials and structures to mechanical or thermal loading.

https://www.appliedelementmethod.org/

Lemaitre & Chaboche textbook as an **e-book**:
http://proquestcombo.safaribooksonline.com.libproxy.aalto.fi/book/physics/9781107384712

A must have pdf

Ytimekäs

Concise and covers entirely our course

Prof. **Reijo Kouhia's lecture notes:**

Lecture notes of the course

Introduction to materials modelling

April 22, 2022

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These pdfs are in MyCourses

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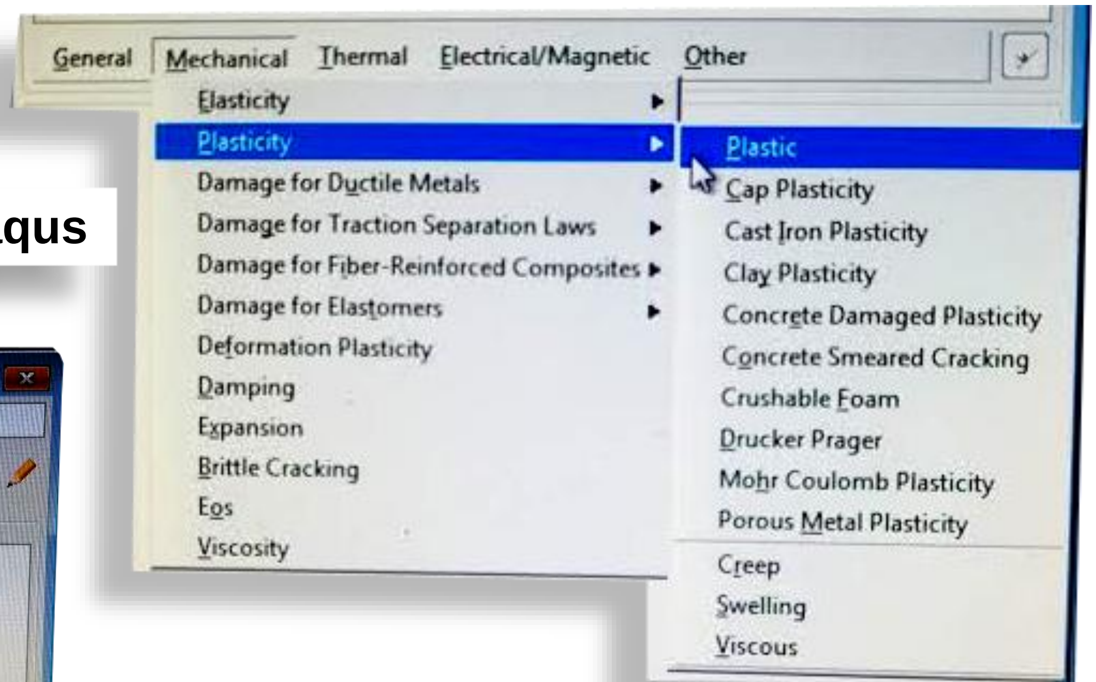
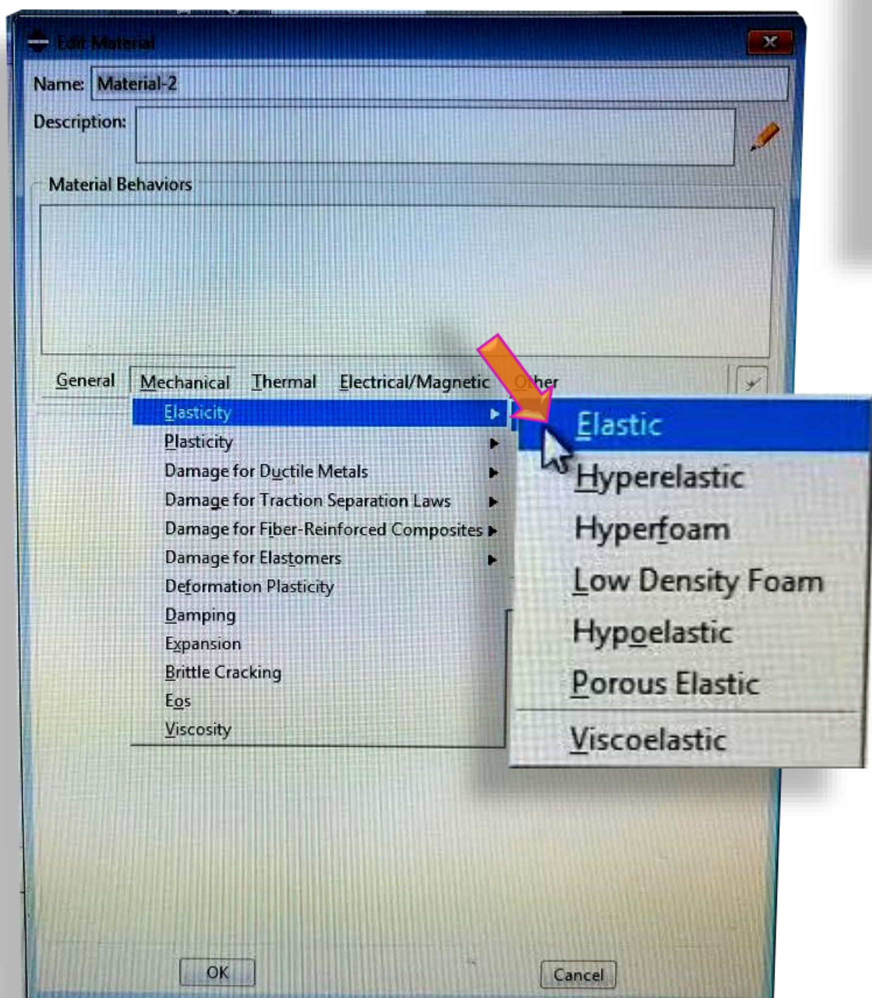
Why opening the *black-box* of Material Models?

Example - Material models in Abaqus





Terminology

Constitutive Equations
Material Behaviour Laws

Materiaalimallit
Konstitutiiviset yhtälöt

Readings

ELASTICITY in Solids

- **Lemaitre *et al.*-** *Our course textbook*
  Paragraphs 4.1 & 4.2 (linear elasticity)
  or
- **Ottosen et al.**, *chapters 4 and 5.*

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Content of the pdf-material

0. **INTRODUTION**
1. **ELASTICITY** (this lecture)
2. **VISCOELASTICITY** (& some basics of creep)
3. **PLASTICITY** (and a recall of *Failure Hypotheses*)

Elective textbooks

PRINCIPLES OF CONTINUUM MECHANICS
A Study of Conservation Principles with Applications

by

J. N. Reddy
Department of Mechanical Engineering
Texas A&M University
College Station, TX 77843-3123

6 CONSTITUTIVE EQUATIONS

6.1 Introduction
6.2 Elastic Solids
6.2.1 Introduction
6.2.2 Generalized Hooke's Law for Orthotropic Materials
6.2.3 Generalized Hooke's Law for Isotropic Materials

or

An Introduction to Continuum Mechanics, Second Edition

J. N. REDDY
Texas A & M University

CAMBRIDGE UNIVERSITY PRESS

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Must visit sites: http://solidmechanics.org/index.html

Summarizes physical laws, mathematical models, and algorithms that are used to predict the response of materials and structures to mechanical or thermal loading

Example of content:

3. Constitutive Equations: Relations between Stress and Strain

3.1 General Requirements for Constitutive Equations

3.2 Linear Elastic Material Behavior
3.2.1 Isotropic, linear elastic material behavior
3.2.2 Stress—strain relations for isotropic, linear elastic materials
3.2.3 Reduced stress-strain equations for plane deformation of isotropic solids
3.2.4 Representative values for density, and elastic constants of isotropic solids
3.2.5 Other elastic constants – bulk, shear and Lame modulus
3.2.6 Physical Interpretation of the elastic constants for isotropic materials
3.2.7 Strain energy density for isotropic solids
3.2.8 Stress-strain relation for a general anisotropic linear elastic material
3.2.9 Physical Interpretation of the anisotropic elastic constants.
3.2.10 Strain energy density for anisotropic, linear elastic solids
3.2.11 Basis change formulas for the anisotropic elastic constants
3.2.12 The effect of material symmetry on anisotropic stress-strain relations
3.2.13 Stress-strain relations for linear elastic orthotropic materials
3.2.14 Stress-strain relations for linear elastic transversely isotropic material
3.2.15 Values for elastic constants of transversely isotropic crystals
3.2.16 Linear elastic stress-strain relations for cubic materials
3.2.17 Representative values for elastic properties of cubic crystals

3.3 Hypoelasticity - elasticity with nonlinear stress-strain behavior

3.4 Generalized Hooke's law – elastic materials subjected to small stretches but large rotations

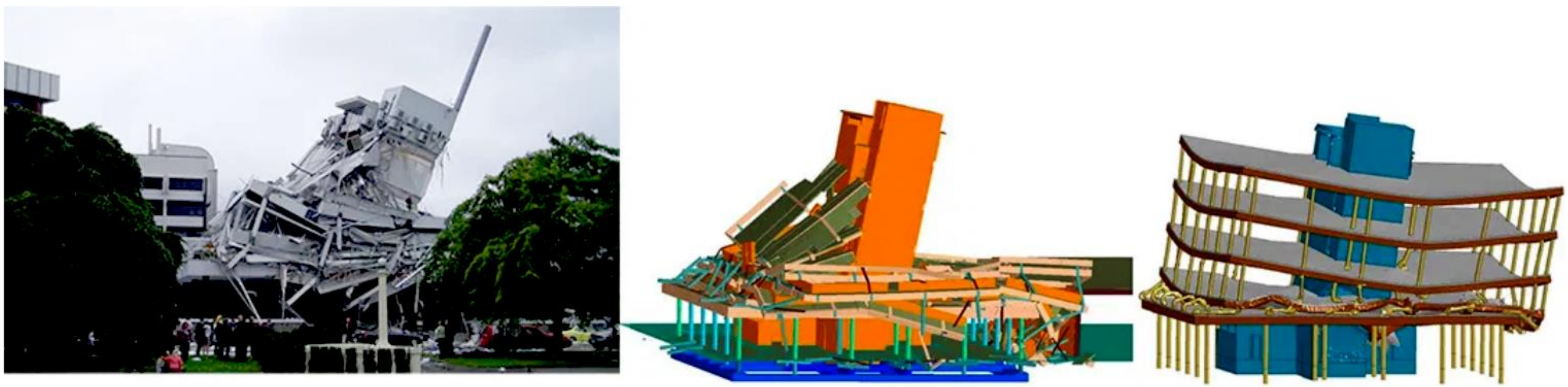
3.5 Hyperelasticity - time independent behavior of rubbers and foams subjected to large strains
3.5.1 Deformation measures used in finite elasticity
3.5.2 Stress measures used in finite elasticity
3.5.3 Calculating stress-strain relations from the strain energy density
3.5.4 A note on perfectly incompressible materials
3.5.5 Specific forms of the strain energy density
3.5.6 Calibrating nonlinear elasticity models
3.5.7 Representative values of material properties for rubbers and foams

Must visit sites: IF you want to know what happens after failure

Good to know:

https://www.appliedelementmethod.org/ (AEM)

- Numerical prediction of **progressive collapse of buildings due to extreme loading**
- Applied finite element AEM is a **ingenious com**bination (hybrid) of continuous finite element technology FEM and discrete elements DEM technology.
- FEM performs for regions with no discontinuities and DEM performs better when discontinuities and separation starts.



Pyne Gould Collapse AEM FEM

Reliability of collapse simulation – Comparing finite and applied element method at different levels

A Simulation TOOL: For structural engineering: **to test various designs and failure scenarios and to analyze (statistically) for progressive collapse** (jatkuva sortuma) **of buildings**

Engineers design to avoid progressive collapse or to keep and confine it remain localized.

A generic algorithm to start telling the complete story when you do not know exactly from what to start:

BEGIN

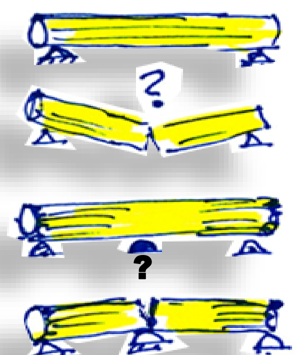
{

Elasticity

START,
GO on,
keep going …
… till the end
THEN
STOP

}

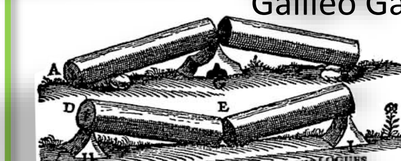
END



The question: How Things Break?


DISCORSI
E
DIMOSTRAZIONI
MATEMATICHE,
intorno à due nuoue scienze
Attenenti alla
MECANICA & I MOVIMENTI LOCALI,
del Signor
GALILEO GALILEI LINCEO,
Filosofo e Matematico primario del Sereniſſimo
Grand Duca di Toſcana.
Con una Appendice del centro di grauità d'alcuni Solidi.

Galileo Galilei



DIALOGUES
TWO NEW SCIENCES
BY
GALILEO GALILEI

Ref. Fig. 29 in:

Published 1638

IN LEIDA,
Appreſſo gli Elſevirii. M. D. C. XXXVIII.

The Law of Elasticity discovered by Robert Hooke in 1660

anagram ceiiinosssttuv

Constitutive Equations
or
Material **Behavior Laws**

Constitutive Equations

are **relations** between **stress** and **strain**

which are based on experiments

or more generally, are

relations between **internal forces** (kinetics) and corresponding **kinematics**.

Example of a full model as a conservation law

Field Equations of Linearized Isotropic Isothermal Elasticity

Equation of motion $\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{f} = \rho \dfrac{D\mathbf{v}}{Dt}$

Strain–displacement equations $\boldsymbol{\varepsilon} = \frac{1}{2}\left[\nabla \mathbf{u} + (\nabla \mathbf{u})^{\mathrm{T}}\right],$

Constitutive Equations $\sigma = \sigma(\varepsilon, \dot{\varepsilon}, T, \dot{T}, \beta, \dot{\beta})$

15 eqs. for 6 stresses, 6 strains, 3 displacements

They can be given either in

- **algebraic** form
- **integral** form
- **differential** form

A generic material model:

$$\sigma = \sigma(\varepsilon, \dot{\varepsilon}, T, \dot{T}, \underbrace{\beta, \dot{\beta}}_{\text{internal variables}})$$

$$\sigma = \sigma^{(c)} + \sigma^{(d)}$$

conservative dissipative

- **decide what physics is relevant to include in the Model**
- **validate experimentally**

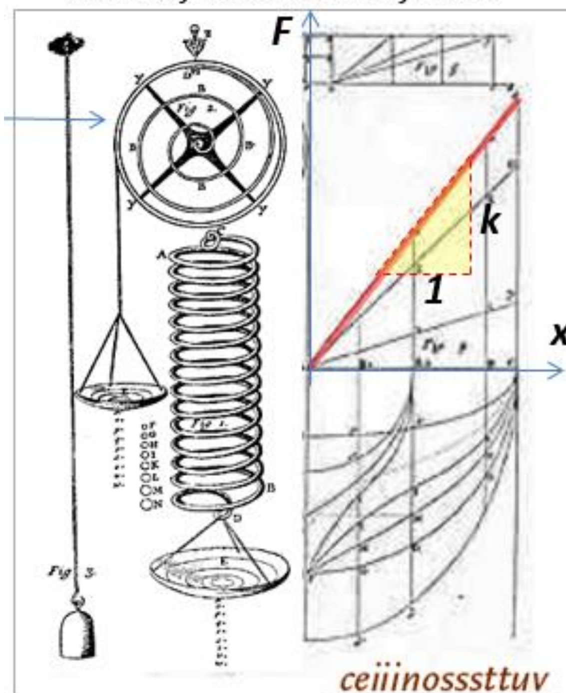
N.B. The material model is a sub-model needed to mathematically close the full model (the conservation laws). It represents a mathematical macroscopic description of sub- processes…

ELASTICITY
ἐ λ α σ τ ό ς
in Solids

All solids material possess a domain in the stress space within which a change in load results only in a change in elastic strains and upon full unloading, they recover their initial state (reversibility)

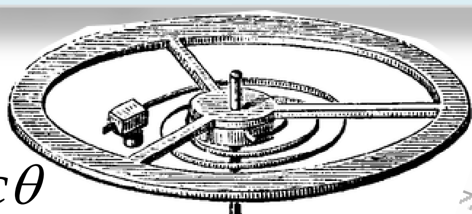
Hooke's law of springs

Kierre- ja lineaarinen jouset

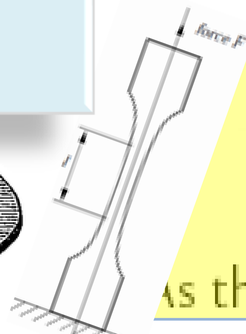


Hooke's law of springs (F = kx)

The knowledge was encoded into a secret code (anagram) ... so valuable is the knowledge (hiding)



$M = c\theta$



Ut tensio, sic vis

F = kx

"As the extension increases, so does the force."

$F = ku$

The Law of Elasticity discovered by **Robert Hooke in 1660**

Louis Cauchy (1789–1857) $\sigma = E \cdot \epsilon$

Hooke's law of springs

Proportionality between Cauchy and Hooke

The proportionality between elongation or change of angle and the load causing it, is known as *Hooke*'s law. R. Hooke (1635–1703) did not expressed it in the form we know actually, $\sigma = E \cdot \epsilon$ because the actual concept of stress came later in 1822, thanks to the Engineer[1] *A. Cauchy* (1789–1857).

[1]This is how Cauchy calls himself (I am an engineer).

$$\sigma = E \cdot \epsilon$$

1822

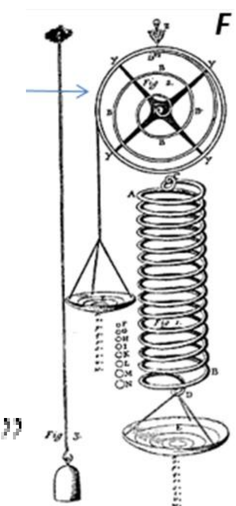
proportionality between elongation or change of angle and the load

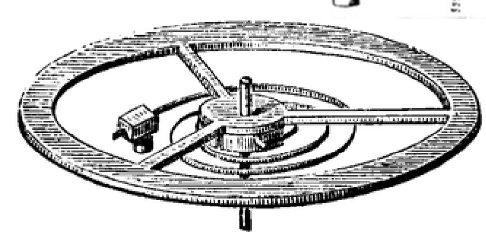
R. Hooke (1635–1703)

Ut tensio, sic vis

$$F = ku$$
$$M = c\theta$$

"As the extension increases, so does the force."





Hooke's law of springs

Proportionality between Cauchy and Hooke

The proportionality between elongation or change of angle and the load causing it, is known as *Hooke*'s law. R. Hooke (1635–1703) did not expressed it in the form we know actually, $\sigma = E \cdot \epsilon$ because the actual concept of stress came later in 1822, thanks to the Engineer[1] *A. Cauchy* (1789–1857).

[1] This is how Cauchy calls himself (I am an engineer).

R. Hooke (1635–1703)

$$\sigma = E \cdot \epsilon \quad 1822$$

anisotropy

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{pmatrix}$$

jäykkyysmatriisi

Stiffness matrix: matrix of stiffness coefficients (similar to Young's modulus in 1-D)

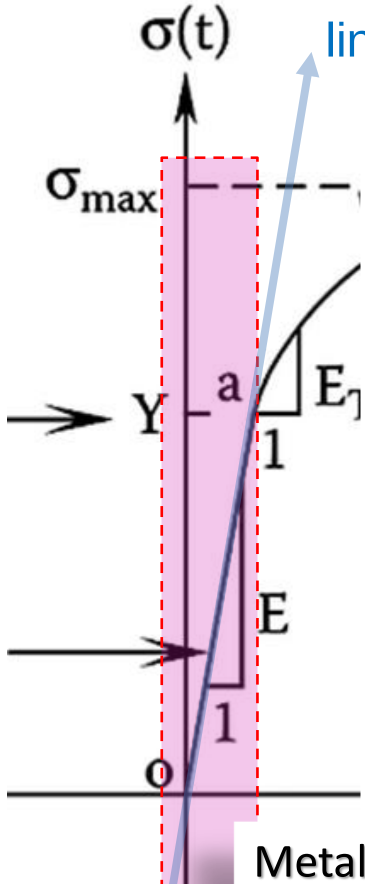
The inverse of the stiffness matrix is called **compliance** matrix joustomatriisi

About Hooke's Law and … what happens beyond

- Every material deforms linearly in a tiny region of deformation under monotonically increasing loading and 'normal' temperatures far bellow his melting point

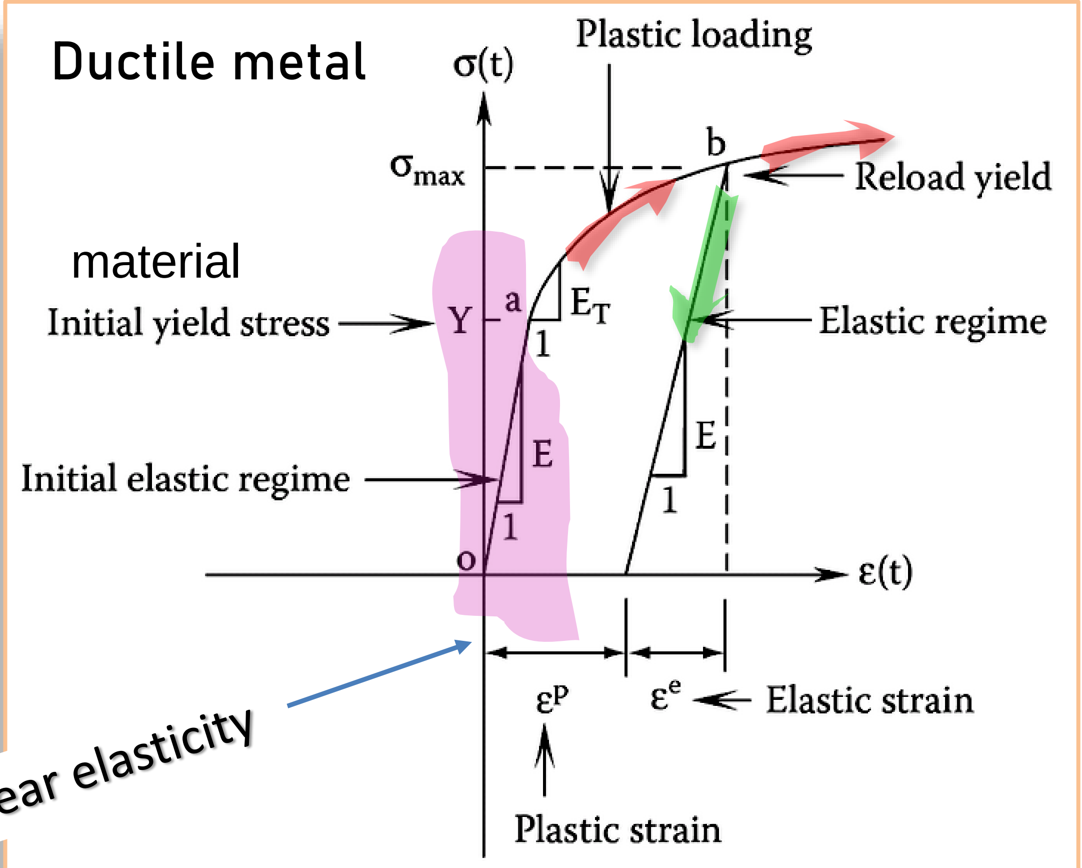
- Above this tiny region, all kind of **'material' non-linearities** start

This is what we will discover in this course



A tiny region of linear elastic deformation

Linear elasticity

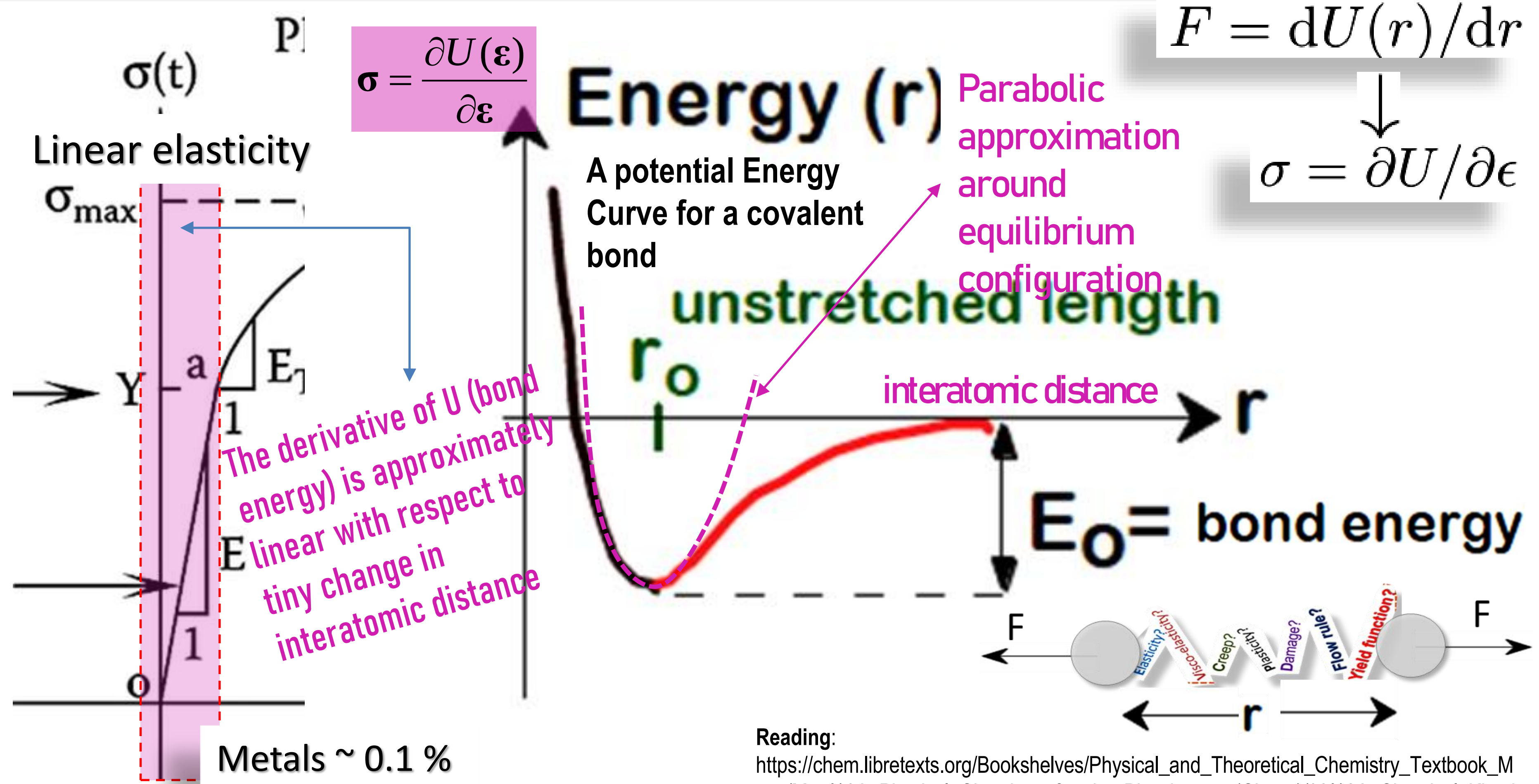


Tensile test for elastic-plastic material.

About Hooke's Law and … what happens beyond

- Every material deforms linearly in a tiny region of deformation under monotonically increasing loading and 'normal' temperatures far bellow his melting point

- Above this tiny region, all kind of **'material' non-linearities** start



A tiny region of linear elastic deformation

Reading: https://chem.libretexts.org/Bookshelves/Physical_and_Theoretical_Chemistry_Textbook_Maps/Map%3A_Physical_Chemistry_for_the_Biosciences_(Chang)/09%3A_Chemical_Kinetics/9.06%3A_Potential_Energy_Surfaces

Elastic Solids - Introduction

Concrete, biological tissues, rocks, soils ...
exhibit nonlinear mechanical behavior even under small deformations

Examples

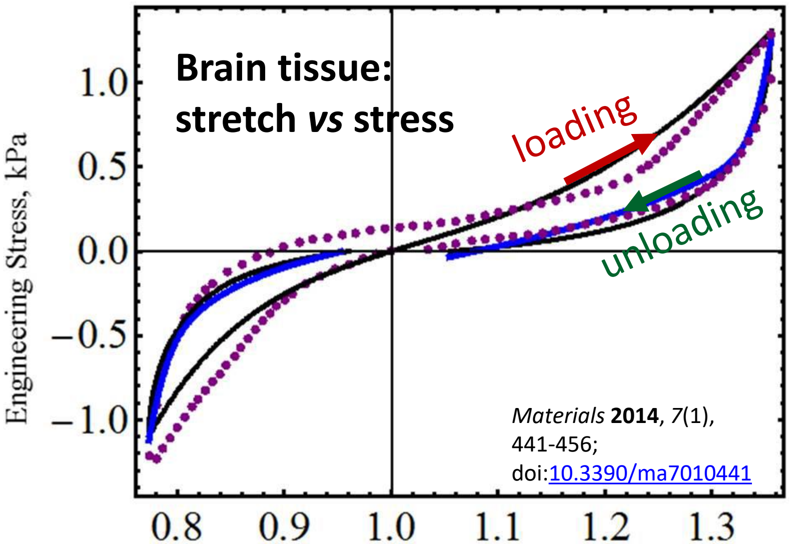
Terminology: $\boldsymbol{\epsilon} = \frac{1}{2}[\nabla \mathbf{u} + (\nabla \mathbf{u})^T]$

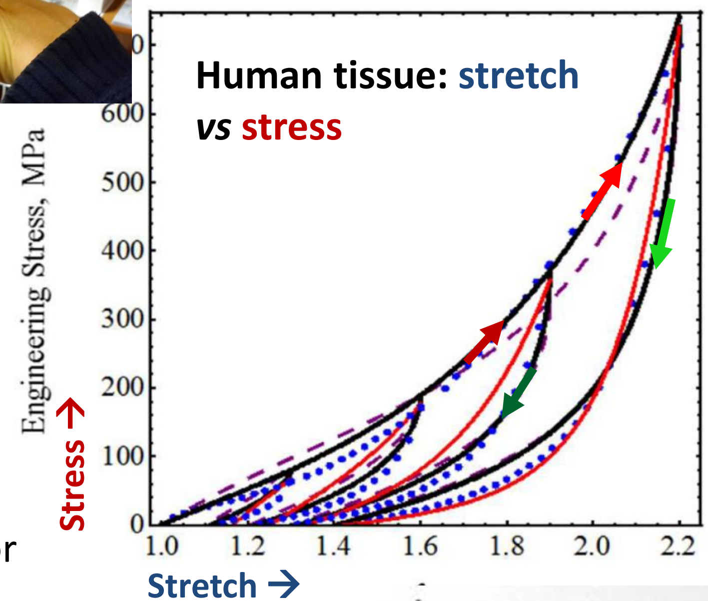
Linear: *small deformations* - $|\nabla \mathbf{u}| << 1$

Nonlinear: *large deformations* $- |\nabla \mathbf{u}| >> 1$
as Rubbers, polymer networks, foams, bio-solids, ...

Examples of Non-Linear Elasticity

Elastomers (rubbers), polymer networks, foams, bio-solids, soft-tissues, ...

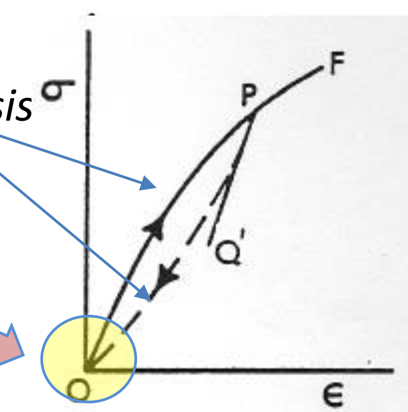




Materials **2014**, *7*(1), 441-456; doi:10.3390/ma7010441

Pseudo-elasticity: uses a different stress-strain relation for loading and unloading

NB. Stress may depend on the loading path – *elasticity with hysteresis*



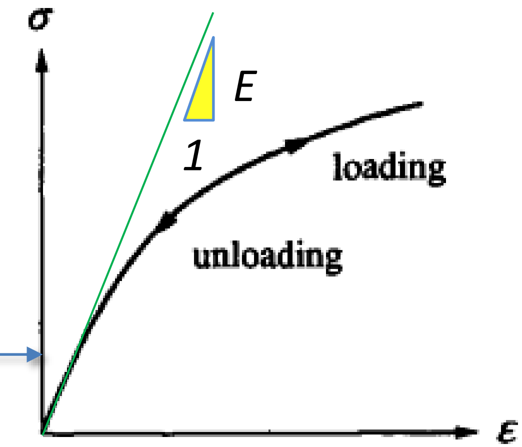
The body recovers its original form completely the forces causing deformation are removed
(under isothermal conditions)

Elasticity

Some definitions

On the thermodynamical framework (= phenomenologic

Elastic response is independent of the load history



Material	E in MPa	α_T in 1/°C
Steel	$2{,}1 \cdot 10^5$	$1{,}2 \cdot 10^{-5}$
Aluminium	$0{,}7 \cdot 10^5$	$2{,}3 \cdot 10^{-5}$
Concrete	$0{,}3 \cdot 10^5$	$1{,}0 \cdot 10^{-5}$
Wood (in fibre direction)	$0{,}7 \ldots 2{,}0 \cdot 10^4$	$2{,}2 \ldots 3{,}1 \cdot 10^{-5}$
Cast iron	$1{,}0 \cdot 10^5$	$0{,}9 \cdot 10^{-5}$
Copper	$1{,}2 \cdot 10^5$	$1{,}6 \cdot 10^{-5}$
Brass	$1{,}0 \cdot 10^5$	$1{,}8 \cdot 10^{-5}$

Elasticity

(for instance, in terms of σ, ϵ)

A material is *elastic* when its mechanical response to loading does not depend on the *history* of this loading f.

Therefore, an elastic material has no *memory*.

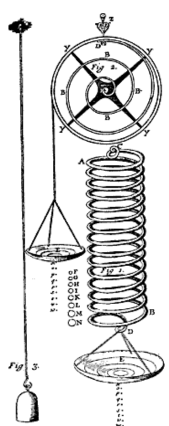
Such material deforms 'instantaneously' upon loading and maintains the same deformation states. Upon load removal, it returns to its original undeformed state 'instantaneously' (=with the speed of acoustic waves). so $\epsilon(t+\tau_c) = \sigma(t)/E$,

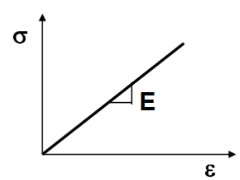
(elastic relaxation time)

Strain resulting from the application of stress at time t is activated only after a certain delay time

Note that σ and ϵ corresponds to stress and strain tensors matrices

Robert Hooke (1635 – 1703)
creative genius





$$\sigma = E \cdot \epsilon$$

ceiiinosssttuv

Ut tensio, sic vis

Linear aisotropic elasticity

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{pmatrix}$$

Elasticity

Definitions

- **Hypo-elasticity**
- **Hyper-elasticity** Green-elasticity
- **Cauchy-elasticity**

Elasticity 1(3)

- **Cauchy-elasticity**: most general form; the current stress state σ_{ij} depends entirely on the current deformations ($\boldsymbol{\epsilon}$ or $\mathbf{F}$) completely ϵ_{ij}. The relation between stress and strains is a *bijection* (=one-to-one relation) $\mathbf{F}$ is the deformation gradient $F_{ij} = \partial x_i / \partial X_j$.

$$\boldsymbol{\sigma} = \mathbf{f}(\boldsymbol{\epsilon}) \Longleftarrow \quad \Longrightarrow \boldsymbol{\epsilon} = \mathbf{g}(\sigma) = \mathbf{f}^{-1}(\boldsymbol{\sigma})$$

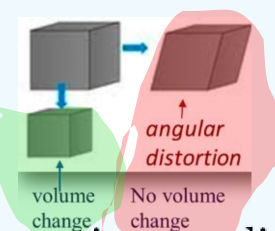
$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{F}) \qquad \boxed{\boldsymbol{\sigma} = \mathbf{f}(\boldsymbol{\epsilon})} \quad \Longleftrightarrow \quad \boxed{\boldsymbol{\epsilon} = \mathbf{g}(\sigma) = \mathbf{f}^{-1}(\boldsymbol{\sigma})}$$

During a deformation, a material point $P(\mathbf{X})$, occupying initially the position $\mathbf{X}$ moves to a new position $P(\boldsymbol{x})$. The deformation can be characterised by a one-to-one mapping $\boldsymbol{x} = \boldsymbol{\phi}(\boldsymbol{X})$. The *deformation gradient* is defined as $\mathbf{F} = \partial \boldsymbol{\phi} / \partial(\boldsymbol{X})$ and it relates the material infinitesimal vector $\mathrm{d}\boldsymbol{x} = \mathbf{F}\mathrm{d}\boldsymbol{X}$ In Cartesian notation $F_{ij} = \partial \phi_i / \partial X_j$.

Many, if not all, strain measures are defied through the deformation gradient. When needed the deformation gradient can be slitted into reversible and non-reversible part (for instance elastic and plastic part) $\mathbf{F} = \mathbf{F}^{(\mathrm{e})}\mathbf{F}^{(\mathrm{p})}$. The idea is to write a different constitutive model for the elastic and the plastic responses. For instance, the elastic response can be captured by giving the free energy (elastic potential) $\psi(\mathbf{F}^{(e)})$, for instance,

Lamé parameters

$$\psi(\mathbf{F}^{(\mathbf{e})}) = \underbrace{\mu \mathrm{tr}(\boldsymbol{\epsilon}^2)}_{\text{deviatoric strain}} + \underbrace{\frac{1}{2}\lambda[\mathrm{tr}(\boldsymbol{\epsilon})]^2}_{\text{volumetric strain}}$$



for isotropic case. The deformation measure $\boldsymbol{\epsilon}$ is defined in terms of the deformation gradient $\mathbf{F}^{(e)}$.

Elasticity

2(3)

- **Hyper-elasticity**: Hyper-elastic material can experience large strains without deforming plastically; they recover after unloading.

Such elastic behaviour (also called Green-elasticity) is best described within the framework of *thermodynamics* where the stress σ is obtained from as the gradient of an energy density *potential* $w(\epsilon)$ with respect to the generalised kinematics $\boldsymbol{\epsilon}$ as

$$\boldsymbol{\sigma}_{ij} = \partial w / \partial \boldsymbol{\epsilon}_{ij} \qquad w(\boldsymbol{\epsilon})$$

$$\sigma_{ij} = \partial w / \partial \epsilon_{ij}$$

In isothermal processes, the elastic potential $w(\boldsymbol{\epsilon})$ is simply the *strain energy density* $u(\boldsymbol{\epsilon})$ (J/m^3). In general $w = w(\mathbf{F}, T)$ where $\mathbf{F}$ being the deformation gradient ($F_{ij} = \partial x_i / \partial X_j$) and T the temperature.

We can also define a complementary elastic potential $w^*(\boldsymbol{\sigma})$ (called Legendre transform of w)

$$w^*(\boldsymbol{\sigma}) = \boldsymbol{\sigma} : \boldsymbol{\epsilon} - w(\boldsymbol{\epsilon})$$

which allows us to write

Both the elastic potential $w(\boldsymbol{\epsilon})$ and the strain energy density $u(\boldsymbol{\epsilon})$ can be expressed in terms of (strain) relevant invariants

$$I_1 = \mathrm{tr}(\epsilon)$$

$$I_2 = \frac{1}{2}\mathrm{tr}(\epsilon^2)$$

$$I_3 = \frac{1}{3}\mathrm{tr}(\epsilon^3)$$

or corresponding deviatoric invariants J_2 and J_3 while keeping I_1.

Elasticity 3(3)

- **Hypo-elasticity**: Fo such elastic materials, generally, the response is known only as a relations between teh strain and stress increments; $\Delta\boldsymbol{\epsilon}$ and $\Delta\boldsymbol{\sigma}$ can be addressed at a given time t. Naturally, the same relation holds for rates $\Delta\boldsymbol{\epsilon}/\Delta t$ and $\Delta\boldsymbol{\sigma}/\Delta t$ when making the time step Δt enough small.

The actual response of such materials is expressed in a *rate form*

$$\dot{\sigma} = f(\sigma, \dot{\epsilon}) \rightarrow \Delta\sigma = f(\sigma, \dot{\epsilon}) \cdot \Delta t$$

$$\dot{\sigma} = f(\sigma, \dot{\epsilon})$$

$$\Updownarrow$$

$$\Delta\sigma = f(\sigma, \dot{\epsilon}) \cdot \Delta t$$

when the incremental response is linear at time step Δt, the above relation can be simplified (Taylor expansion and keep the first terms) we get the *incrementally linear* response (or in its rate form)

$$\dot{\sigma} = \mathbf{C}(\sigma) \cdot \dot{\epsilon}$$

Elasticity

Resumé

- **Cauchy-elasticity**: most general form; the current stress state σ_{ij} depends entirely on the current deformations completely ϵ_{ij}. The relation between stress and strains is a *bijection* (one-to-one relation).

$$\sigma = \mathbf{f}(\epsilon) \Longleftarrow \quad \Longrightarrow \epsilon = \mathbf{g}(\sigma) = \mathbf{f}^{-1}(\sigma)$$

- **Hypo-elasticity**: The actual response of such materials is expressed in a *rate form*

$$\dot{\sigma} = f(\sigma, \dot{\epsilon}) \rightarrow \Delta\sigma = f(\sigma, \dot{\epsilon}) \cdot \Delta t \qquad \dot{\sigma} = \mathbf{C}(\sigma) \cdot \dot{\epsilon}$$

when the incremental response is linear at time step Δt, the above relation can be simplified (Taylor expansion and keep the first terms) we get the *incrementally linear* response (or in its rate form)

- **Hyper-elasticity**: Hyper-elastic material can experience large strains without deforming plastically; they recover after unloading.

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In isothermal processes, the elastic potential $w(\boldsymbol{\epsilon})$ is simply the *strain energy density* $u(\boldsymbol{\epsilon})$ $(\mathrm{J/m^3})$.
In general $w = w(\mathbf{F}, T)$ where $\mathbf{F}$ being the deformation gradient $(F_{ij} = \partial x_i / \partial X_j)$ and T the temperature.

The thermodynamics framework provides a powerful tool for developing constitutive models. Additionally, it naturally, imposes thermodynamic *constraints* through the entropy grow constraint that should obey each process in all closed systems.

Isothermal Cauchy-elastic material

Definition:

A material is said to be ***simple-elastic*** or ***Cauchy-elastic*** *if:*

- the body **recovers its original form** completely upon removal of the forces causing deformation (under isothermal conditions, otherwise, the temperature will be treated merely as a *parameter*)
- there is a ***one-to-one relationship*** between the state of stress and the state of strain in the current configuration **(Path independency)**

Cauchy-elasticity is the most general form of elasticity.

This relation is the only thing we know here about the material behavior

Cauchy-elasticity: $\boldsymbol{\sigma} = f(\mathbf{F})$

$\mathbf{F}$ is the *deformation gradient* tensor

$\Rightarrow$ The stress is not derivable from an elastic potential function (strain energy function, thermodynamic potential)

Cauchy stress σ does not depend on the path of deformation but depends only on the current state of deformation

$\Rightarrow$ **The strain energy function may depend on the loading history**

$$\mathbf{F} = \mathrm{Grad}\ \mathbf{x}(\mathbf{X}) \leftrightarrow F_{ij} = \frac{\partial x_i(\mathbf{X})}{\partial X_j}$$

$$\mathbf{x} = \mathbf{X} + \mathbf{u}$$

Small deformations: $\mathbf{E} \approx \mathbf{e} \approx \boldsymbol{\varepsilon} = \frac{1}{2}\left((\nabla \mathbf{u})^T + \nabla \mathbf{u}\right)$

Isothermal Cauchy-elastic material

Cauchy-elastic for <u>isotropic</u> material:

The most general representation of such elasticity can be on of the two relations:

A one-to-one relationships:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\varepsilon}) \leftrightarrow \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{\sigma})$$

Symmetric 2nd order tensors

Thanks to Cayley-Hamilton theorem

$$\boldsymbol{\varepsilon} = \alpha_0 \mathbf{1} + \alpha_1 \boldsymbol{\sigma} + \alpha_2 \boldsymbol{\sigma}^2$$

$$\alpha_i = \alpha_i(I_1, I_2, I_3)$$

Arbitrary coefficients α_i depend on stress invariants (objectivity)

$$\boldsymbol{\sigma} = \beta_0 \mathbf{1} + \beta_1 \boldsymbol{\varepsilon} + \beta_2 \boldsymbol{\varepsilon}^2$$

$$\beta_i = \beta_i(J_1, J_2, J_3)$$

Arbitrary coefficients β_i depend on strain invariants (objectivity)

$$I_1 = \mathrm{tr}(\epsilon)$$

$$I_2 = \frac{1}{2}\mathrm{tr}(\epsilon^2)$$

$$I_3 = \frac{1}{3}\mathrm{tr}(\epsilon^3)$$

Ref. to **Appendix 1** for stress and strain invariants

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3$$

$$I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$$

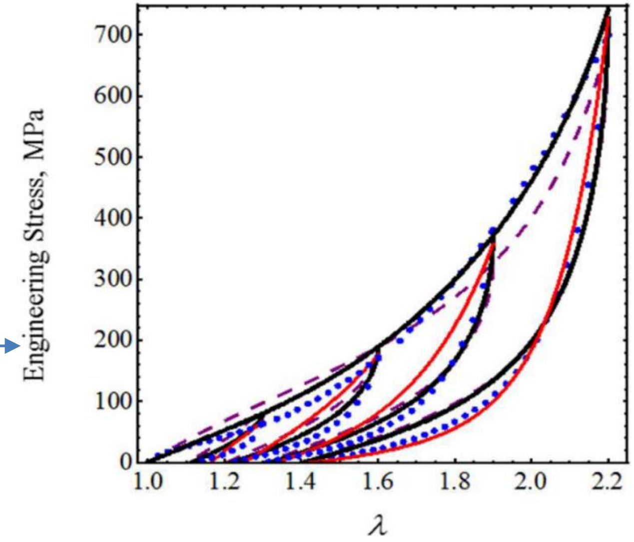
$$I_3 = \sigma_1\sigma_2\sigma_3$$

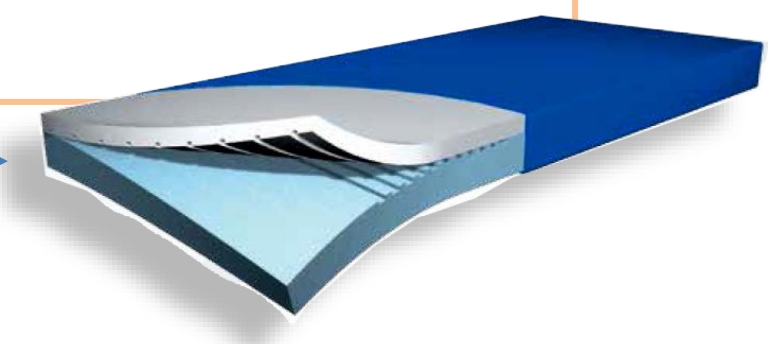
Hyperelastic Materials

They can experience large strains without deforming plastically; they recover after unloading

Examples of hyperelastic behavior in materials

- Rubbers
- Polymers
- Biological tissues
  ex. elastin, resilin and abductin
  - ➢ In such materials we have a combination of hyperelasticity and viscoelasticity
- Some metamaterials
- Hyperfoams





Green-Elastic or Hyperelastic Materials

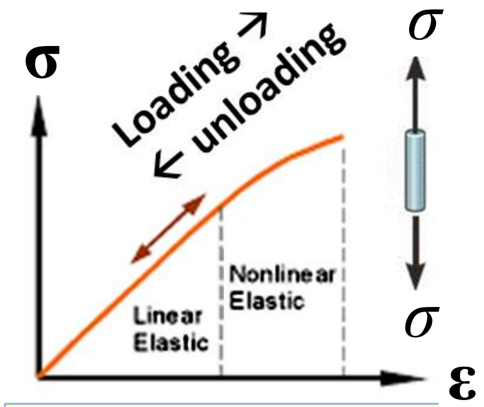
Definition:

A material is said to be ***hyper-elastic*** or **Green-Elastic** IF:

- **the stress is derivable** from an **elastic potential function** (strain energy function, **thermodynamic potential**)

$\Longrightarrow$ there exists a *Helmholtz free-energy density* ***potential*** ψ whose derivative with respect to a strain gives the corresponding stress

Hyperelasticity is a special case of Cauchy elasticity



Elastic response is independent of the load history

$\Updownarrow$

Conservative: the work done by the stresses producing the strains is stored as potential energy – the elastic energy which is a **reversible thermodynamical process**

In general, $\boxed{\psi = \psi(\mathbf{F}, T),}$ T is the absolute temperature

$\Longrightarrow$ $\partial\psi / \partial(\boldsymbol{\varepsilon}) = \boldsymbol{\sigma}$ and $\partial\psi / \partial T = heat\ \ flux$

In particular, $\boxed{\psi = \psi(\boldsymbol{\varepsilon}, T)}$

When ψ depends only on strains $\boldsymbol{\varepsilon}$ or on $\mathbf{F}$ then, it is called ***strain energy density function*** and denoted by U measured by unit mass.

$$\mathbf{F} = \mathrm{Grad}\ \mathbf{x}(\mathbf{X}) \leftrightarrow F_{ij} = \frac{\partial x_i(\mathbf{X})}{\partial X_j}$$

$$\mathbf{x} = \mathbf{X} + \mathbf{u} \qquad \mathbf{E} = \mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{1}$$

Definition: specific free energy $\psi = e - s\theta,$ $\theta-$ absolute temperature

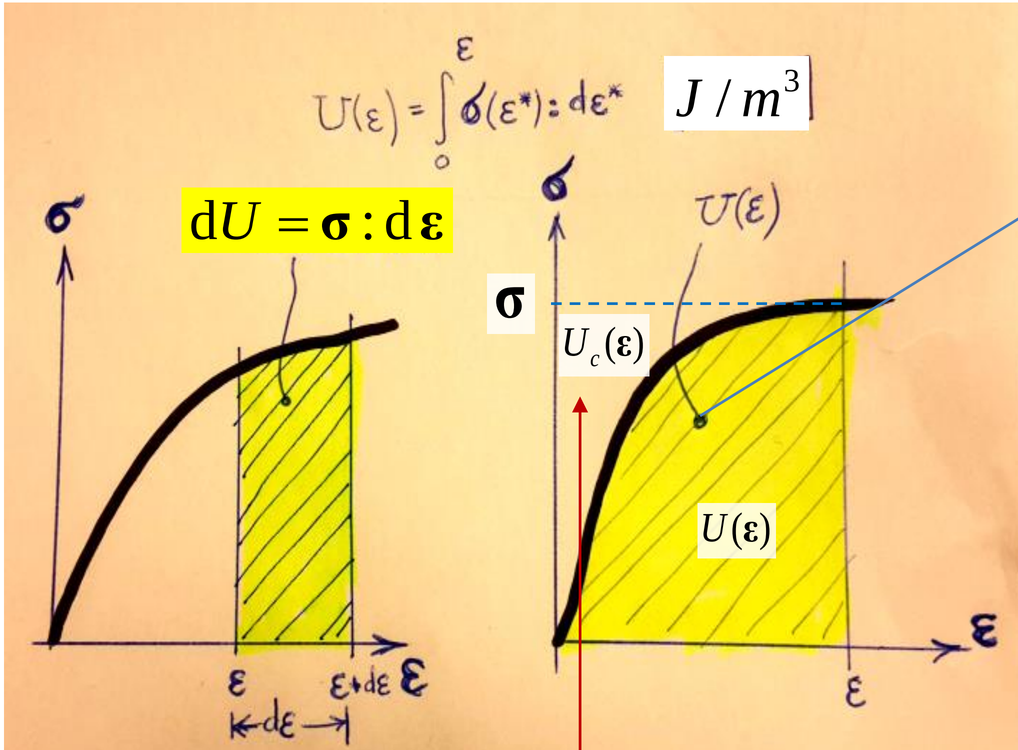
$e-$ specific internal energy, $s-$ specific entropy,

Small deformations: $\mathbf{E} \approx \mathbf{e} \approx \varepsilon = \frac{1}{2}\left((\nabla\mathbf{u})^T + \nabla\mathbf{u}\right)$

Green-Elastic or Hyper-elastic Materials

J/m^3 : energy density (work)

W/m^3 : power density (work rate)



$U(\boldsymbol{\varepsilon})$ ***strain energy density per unit volume***

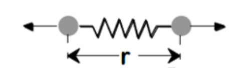
$$U(\boldsymbol{\varepsilon})=\int_0^{\boldsymbol{\varepsilon}}\boldsymbol{\sigma}(\boldsymbol{\varepsilon}^*):d\boldsymbol{\varepsilon}^*$$

$$\boldsymbol{\sigma}=\frac{\partial U(\boldsymbol{\varepsilon})}{\partial\boldsymbol{\varepsilon}}$$

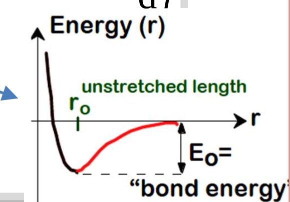
Cf.

Cf. Bond force between two atoms:

Bond length, r



$$f=\frac{dE(r)}{dr}$$



$U_c(\boldsymbol{\varepsilon})$ **Complementary energy**

$$U_c(\boldsymbol{\varepsilon})=\boldsymbol{\sigma}:\boldsymbol{\varepsilon}-U(\boldsymbol{\varepsilon})$$

In general, $\psi=\psi(\mathbf{F},T)$ T is the absolute temperature

$\Rightarrow$ $\partial\psi/\partial(\boldsymbol{\varepsilon})=\boldsymbol{\sigma}$ and $\partial\psi/\partial T=heat\ flux$

When ψ depends only on strains $\boldsymbol{\varepsilon}$ or on **F** then, it is called ***strain energy density function*** and denoted by U measured by unit mass.

$$dU=\boldsymbol{\sigma}:d\boldsymbol{\varepsilon}$$

Green-Elastic or Hyper-elastic Materials:

A material is **Hyper-Elastic** if there exist **strain energy density function** *U* per unit volume

$$U(\boldsymbol{\varepsilon}) = \int_0^{\boldsymbol{\varepsilon}} \boldsymbol{\sigma}(\boldsymbol{\varepsilon}^*) : \mathrm{d}\boldsymbol{\varepsilon}^*$$

Such that

$$\boxed{\boldsymbol{\sigma} = \frac{\partial U(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}}}$$

U - thermodynamic potential

Will be shown in the next slide

This relation gives the *constitutive law*

In a special case of *incompressible material* (rubber, …), the should be written in the form:

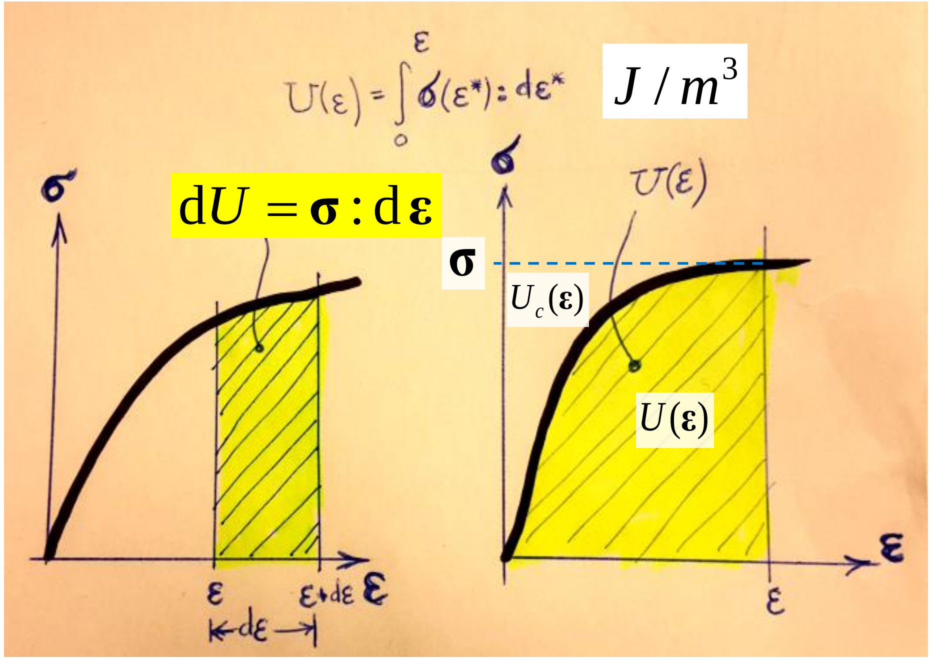
$$\boldsymbol{\sigma} = -p\mathbf{I} + \frac{\partial U(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}}$$

p is the Hydrostatic pressure

- Since the *material volume change is zero* - $\mathrm{Tr}(\boldsymbol{\varepsilon}) = \varepsilon_{kk} = 0$
- then corresponding hydrostatic stress $\sigma_{kk} \equiv -p/3$ is a constraint and cannot be given by a constitutive equation.

Hyper-elastic Materials

It is more convenient to work with energy than directly with forces. *Cf.* bond energy between atoms (or ions). The gradient of such energy potentials provides the forces.



$U(\boldsymbol{\varepsilon})$ ***strain energy density per unit volume***

$$U(\boldsymbol{\varepsilon}) = \int_0^{\boldsymbol{\varepsilon}} \boldsymbol{\sigma}(\boldsymbol{\varepsilon}^*) : \mathrm{d}\boldsymbol{\varepsilon}^*$$

$$\mathrm{d}U = \boldsymbol{\sigma} : \mathrm{d}\boldsymbol{\varepsilon} \qquad (1)$$

$$\boxed{\boldsymbol{\sigma} = \frac{\partial U(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}}}$$

We will show this result

We consider an isothermal process.

In general, strain energy depends both on the current strains and on the manner how these strains are achieved (history dependency)

$$\Rightarrow U = U(\varepsilon_{ij}, \text{load history})$$

hyper-elasticity is **independent** of load **history**

For **hyper-elastic** material, the elastic **response** is **independent** of the load **history**

$$\Rightarrow U = U(\varepsilon_{ij}) \quad \Rightarrow \mathrm{d}U = \sum_{ij} \frac{\partial U}{\partial \varepsilon_{ij}} \mathrm{d}\varepsilon_{ij}, \quad (2)$$

Subtracting Eq. (1) form Eq. (2) $\Rightarrow \mathrm{d}U - \mathrm{d}U = \sum_{ij} [\frac{\partial U}{\partial \varepsilon_{ij}} - \sigma_{ij}] \mathrm{d}\varepsilon_{ij} = 0 \boxed{\Rightarrow \sigma_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}}}$ **CQFD.**

Because the increments $\mathrm{d}\varepsilon_{ij}$ can be chosen arbitrary and independently of each other.

Example: linear elastic material in 1-D with a modulus of *E*

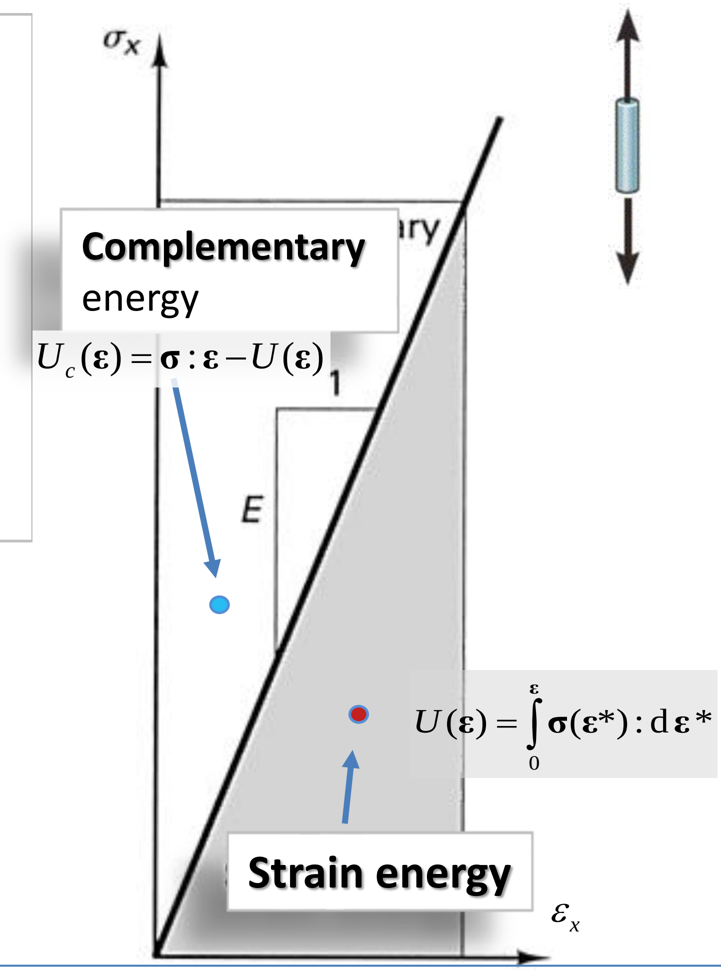
$$\boldsymbol{\sigma} = \mathbf{E} \cdot \boldsymbol{\epsilon}$$

$$\sigma_{xx} = E\varepsilon_{xx}$$

Strain energy density per unit volume:

$$U(\boldsymbol{\varepsilon}) = \int_0^{\boldsymbol{\varepsilon}} \boldsymbol{\sigma}(\boldsymbol{\varepsilon}^*) : \mathrm{d}\boldsymbol{\varepsilon}^* = \int_0^{\varepsilon_{xx}} \sigma(\varepsilon_{xx}^*)\,\mathrm{d}\varepsilon_{xx}^* = \frac{1}{2}E\varepsilon_{xx}^2$$

$$\boldsymbol{\sigma} = \frac{\partial U(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon_{xx}}\left(\frac{1}{2}E\varepsilon_{xx}^2\right) = E\varepsilon_{xx}$$



Remember: the area between the stress strain curve and the stress axis is called *complementary strain energy density U* :*

$$U^*(\boldsymbol{\sigma}) = \frac{1}{2E}\sigma_{xx}^2$$

Note: when the deformation reaches the maximum possible elastic strain, just before plasticity or other irreversible behavior begins, $U(\varepsilon_{max})$ is called Modulus of resilience

$$\boxed{U_{max} = \frac{1}{2}E\varepsilon_{max}^2}$$

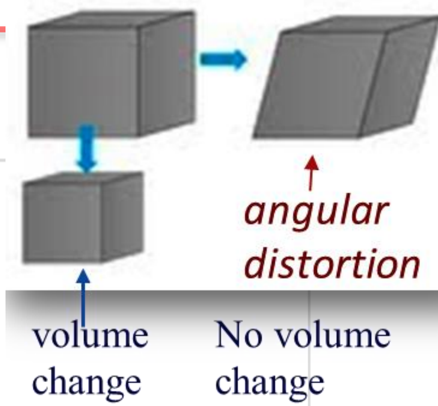
Maximum strain till failure (*ultimate strain*)

Definitions

Strain energy density:

(For isotropic case, otherwise even pure volumetric stress may lead to shape distortion)

$$U(\boldsymbol{\varepsilon}) = \tfrac{1}{2} K\, \mathrm{tr}(\boldsymbol{\varepsilon})^2 + \mu\, \mathbf{e} : \mathbf{e} = U_V + U_D$$



Pure Volumetric Strain energy density

Distortional (deviatoric) energy density

Bulk Modulus $K = \dfrac{E}{3(1-2\nu)}$

Shear Modulus $\mu = \dfrac{E}{2(1+\nu)} \equiv G$

Lame Modulus $\lambda = \dfrac{\nu E}{(1+\nu)(1-2\nu)}$

Why this split? **because material volumetric and distortional** (... ‘shearing’) **response are often different** (we will see this later and also in details when dealing with viscoelasticity and plasticity)

Cf. Appendix *Stress and strain invariants*

Volumetric strain: $\boldsymbol{\varepsilon}_v \equiv \mathrm{vol}(\boldsymbol{\varepsilon}) = \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\varepsilon})\mathbf{I} \leftrightarrow \delta_{ij}\varepsilon_{kk}/3$

$dV/V = \mathrm{tr}(\boldsymbol{\varepsilon}) \equiv \mathrm{v}$

Volume change

Deviatoric strain: $\mathbf{e} \equiv \mathrm{dev}(\boldsymbol{\varepsilon}) = \boldsymbol{\varepsilon} - \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\varepsilon}) \leftrightarrow e_{ij} = \varepsilon_{ij} - \varepsilon_{kk}/3$

Stress: $\boldsymbol{\sigma} = 3K\mathrm{vol}(\boldsymbol{\varepsilon}) + 2G\,\mathrm{dev}(\boldsymbol{\varepsilon})$

[Hooke's law for isotropic materials – coming soon back]

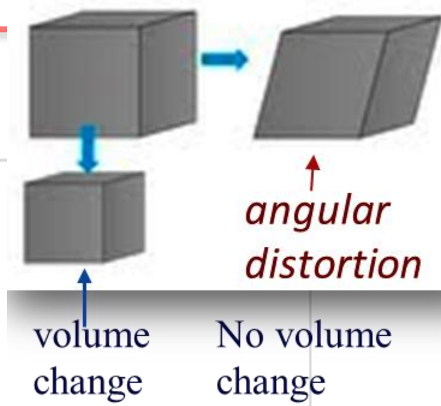
$2\mathbf{e} : \mathbf{e} \equiv \gamma^2$

the shear distortion

Definitions

Strain energy density:

(For isotropic case, otherwise even pure volumetric stress may lead to shape distortion)



$$U(\boldsymbol{\varepsilon}) = \tfrac{1}{2} K \,\mathrm{tr}(\boldsymbol{\varepsilon})^2 + \mu \mathbf{e} : \mathbf{e} = U_V + U_D$$

Pure Volumetric Strain energy density

Distortional (deviatoric) energy density

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Lame Modulus $\lambda = \dfrac{\nu E}{(1+\nu)(1-2\nu)}$

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} = K - \frac{2}{3}G = \frac{G(E-2G)}{(3G-E)},$$

$$\mu \equiv G = \frac{E}{2(1+\nu)} = \frac{\lambda(1-2\nu)}{2\nu} = \frac{3}{2}(K-\lambda),$$

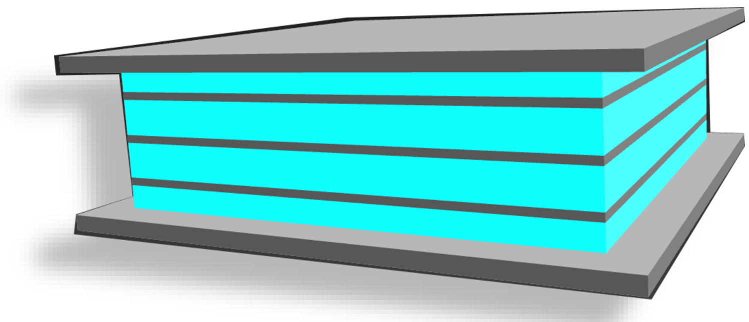
$$\nu = \frac{\lambda}{2(\lambda+\mu)} = \frac{\lambda}{(3K-\lambda)} = \frac{3K-2G}{2(3K+G)},$$

$$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} = \frac{\lambda(1+\nu)(1-2\nu)}{\nu} = \frac{9K(K-\lambda)}{3K-\lambda},$$

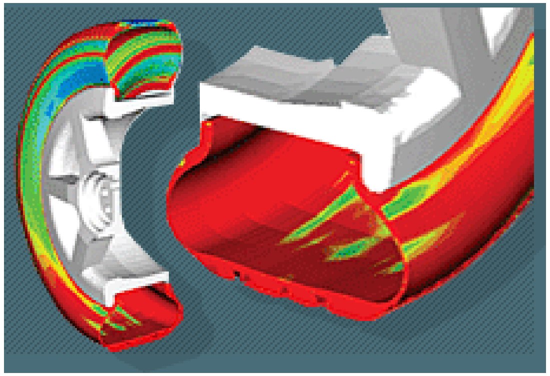
$$K = \lambda + \frac{2}{3}\mu = \frac{E}{3(1-2\nu)} = \frac{\lambda(1+\nu)}{3\nu} = \frac{GE}{3(3G-E)}.$$

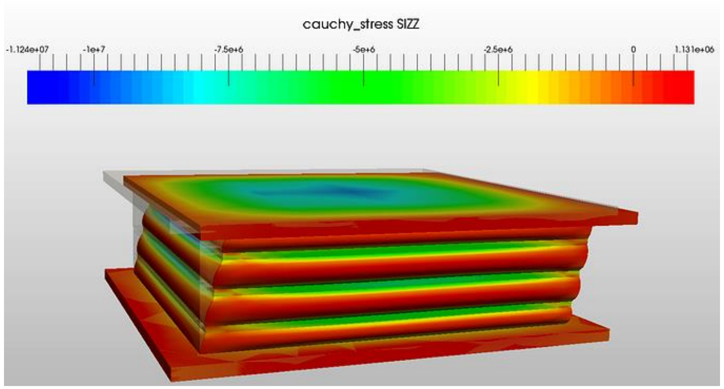
Ref: Reijo K. lecture notes

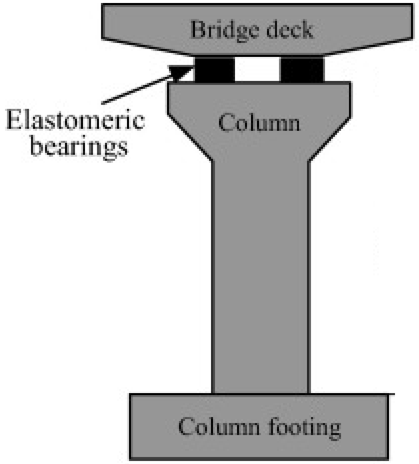
Hyperelasticity
some examples



Bridge elastomeric bearings



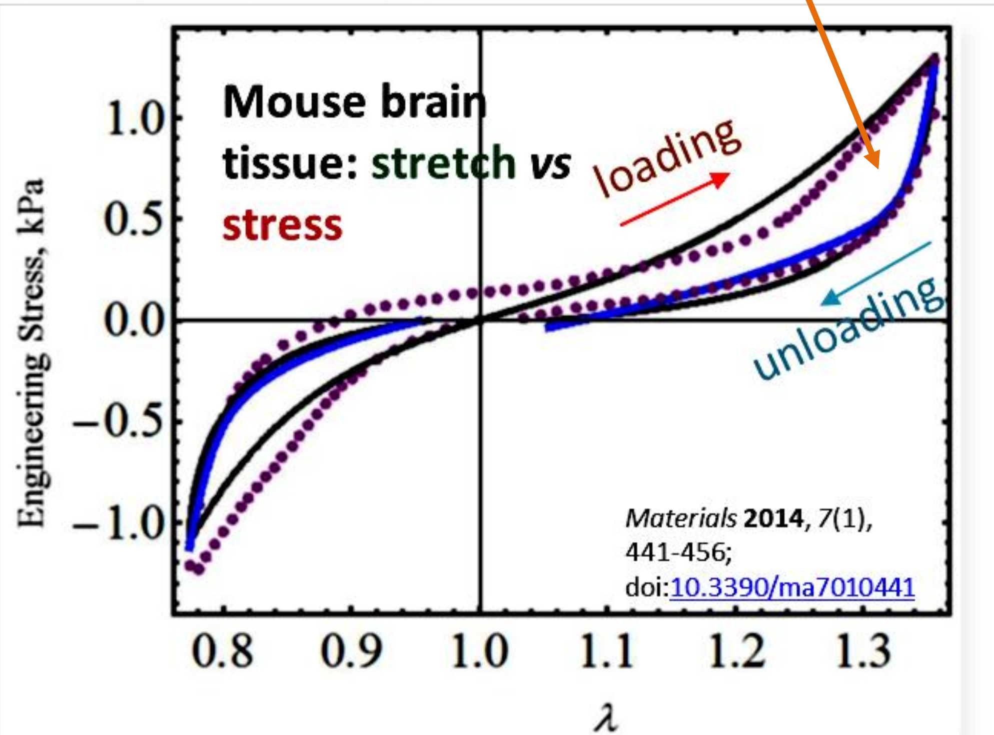




Examples of Non-Linear Elasticity

Elastomers (rubbers), polymer networks, foams, bio-solids, soft-tissues, …

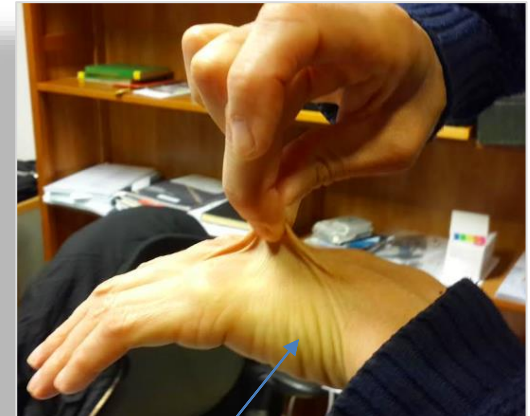
Elasticity with hysteresis



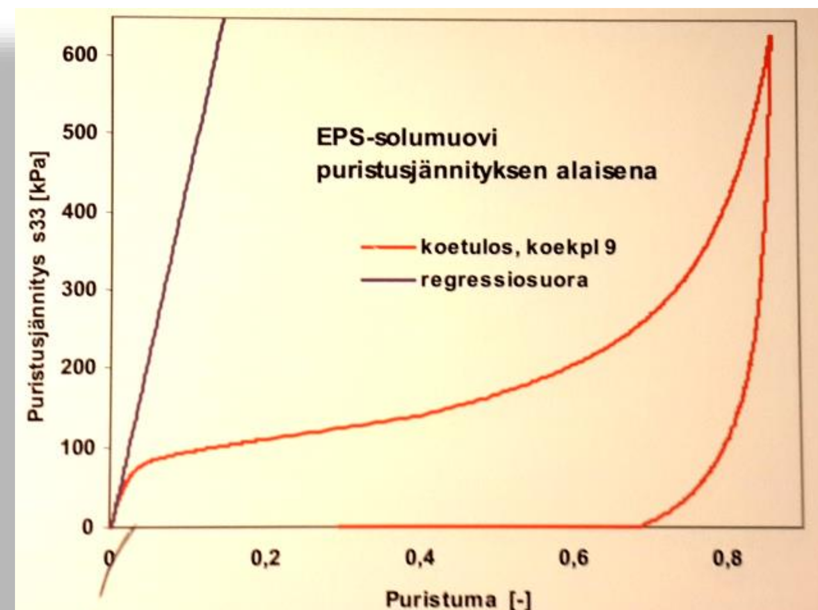
Pseudo-elasticity: uses a different stress-strain relation for loading and unloading

Elastic: The body recovers its original form completely the forces causing deformation are removed (recall)

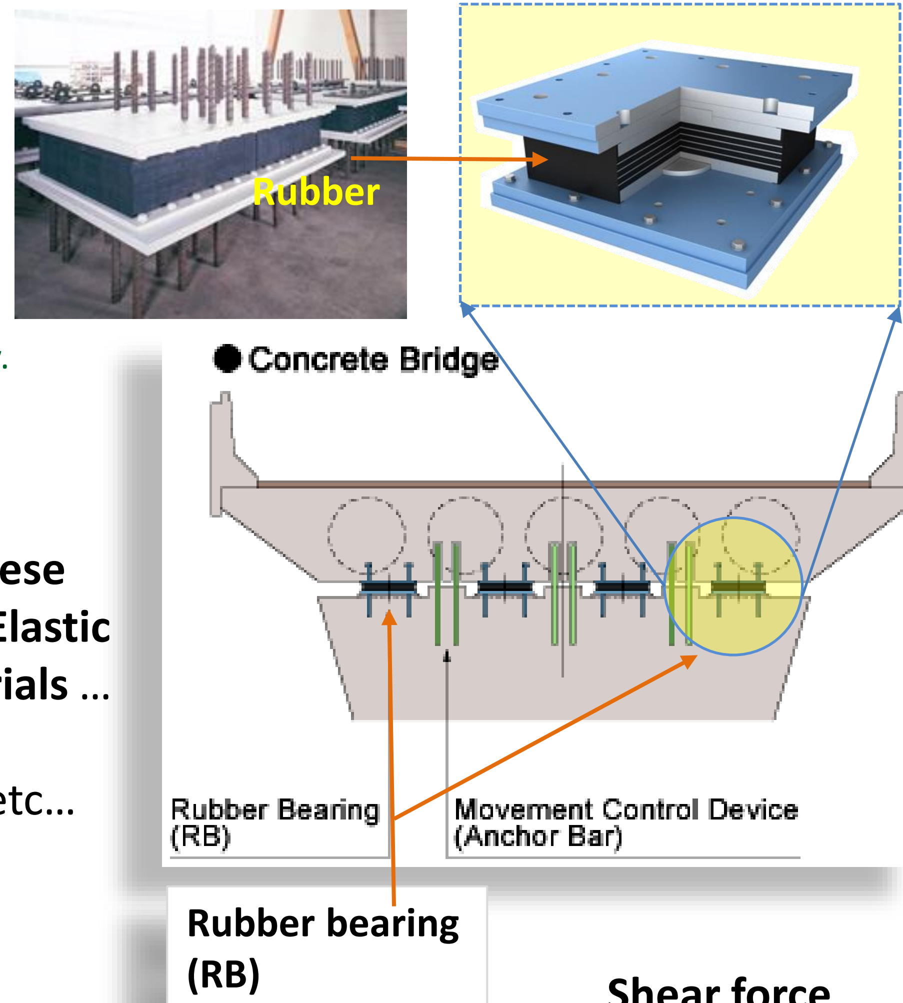
Elasticity with hysteresis



Good to know: Surface wrinkling can be explained by surface instability mechanism



CIV-application example
Hyperelasticity
Rubber



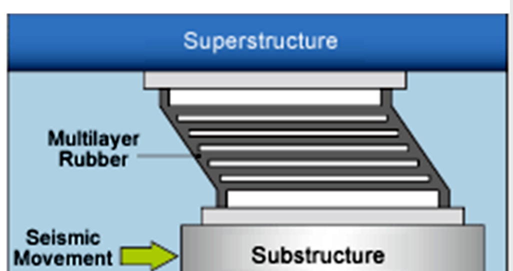
This is an important class of materials for an engineer to know. However, in this course, it is sufficient to know that such class exists and what are their key important thermomechanical response features must-know for an engineer.

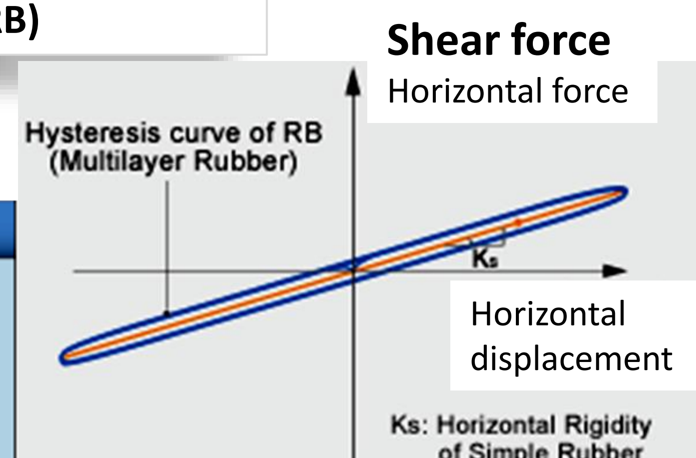
NB. In general, the response of a *polymer* dependent strongly on *temperature*, *strain history* and *loading rate*. Some aspects of such behavior will be detailed in next section treating of *viscoelasticity* and little bit when treating *rubber elasticity* in the current section.

Usually, these are Visco-Elastic (VE) materials … used for damping, etc…

In this section, we consider rubbery state only

It is known that that polymers have various regimes of mechanical response: **glassy**, **viscoelastic** and **rubbery**. These various regimes can be identified *via* dynamical loading.





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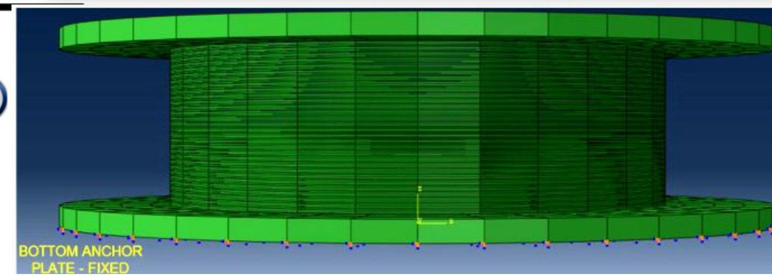
The Ogden strain energy density

$$U = \sum_{n=1}^{2} \frac{\mu_n}{\alpha_n}\left(\bar{\lambda}_1^{\alpha_n} + \bar{\lambda}_2^{\alpha_n} + \bar{\lambda}_3^{\alpha_n} - 3\right) + 4.5B\left(J^{\beta} - 1\right)^2$$

Numerical study on the response of steel-laminated elastomeric bearings subjected to variable axial loads and development of local tensile stresses

Konstantinos N. Kalfas [a], Stergios A. Mitoulis [b,*], Konstantinos Katakalos [c]



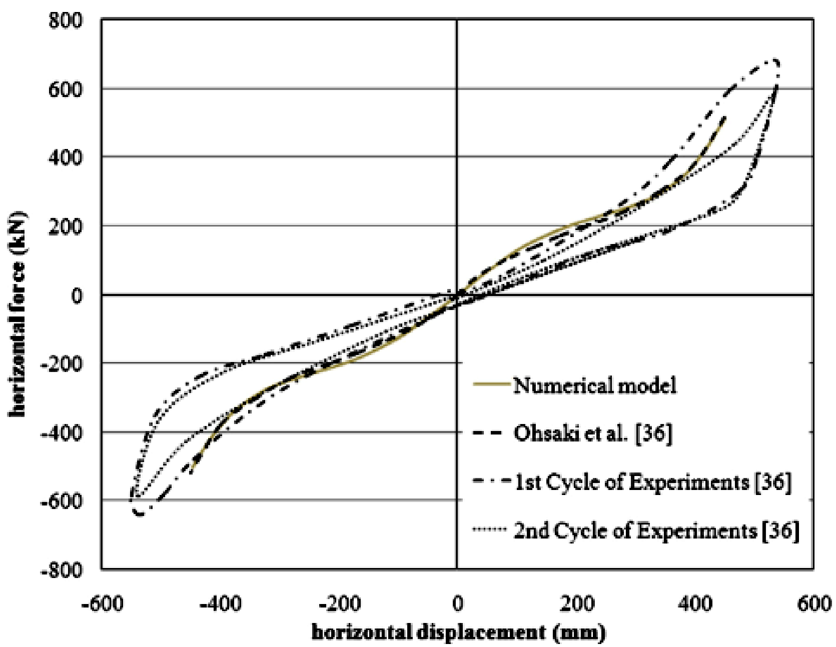


Fig. 2. Validation of the reference bearing model against the available numerical and experimental results.

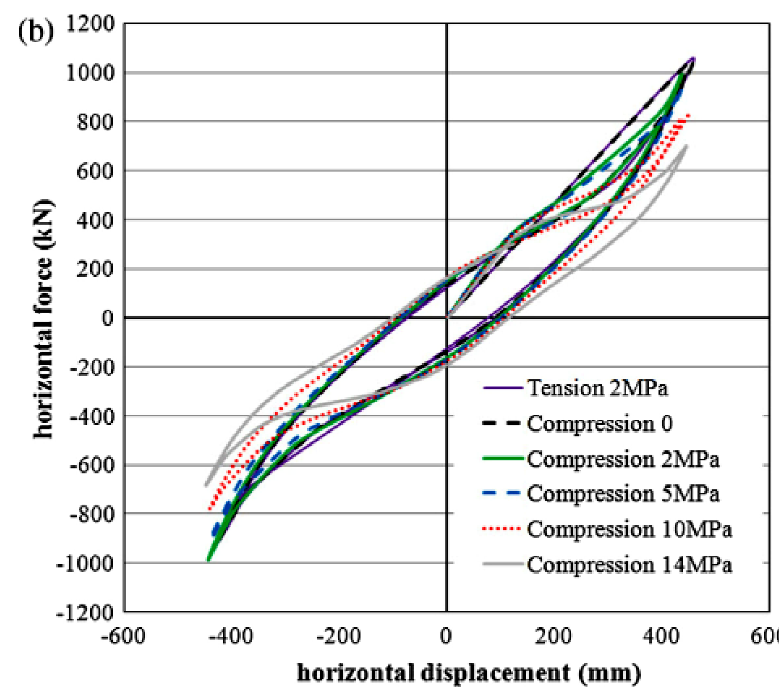


Fig. 3. Response of the reference steel-laminated elastomeric bearing for variable axial loads (shear strain 375%) for (a) one cycle of loading (NL_1–NL_6) and (b) two cycles of loading (NL_{10} - NL_{15}).

Material symmetries

- anisotropy
- orthotropy
- transverse isotropy
- isotropy

$$\begin{bmatrix}\sigma_{11}\\\sigma_{22}\\\sigma_{33}\\\sigma_{23}\\\sigma_{31}\\\sigma_{12}\end{bmatrix}=\begin{bmatrix}C_{11}&C_{11}&C_{13}&C_{14}&C_{15}&C_{16}\\C_{12}&C_{22}&C_{23}&C_{24}&C_{25}&C_{26}\\C_{13}&C_{23}&C_{33}&C_{34}&C_{35}&C_{36}\\C_{14}&C_{24}&C_{34}&C_{44}&C_{45}&C_{46}\\C_{15}&C_{25}&C_{35}&C_{45}&C_{55}&C_{56}\\C_{16}&C_{26}&C_{36}&C_{46}&C_{56}&C_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\\varepsilon_{22}\\\varepsilon_{33}\\\varepsilon_{23}\\\varepsilon_{31}\\\varepsilon_{12}\end{bmatrix}$$

21

$$\begin{bmatrix}\sigma_{11}\\\sigma_{22}\\\sigma_{33}\\\sigma_{23}\\\sigma_{31}\\\sigma_{12}\end{bmatrix}=\begin{bmatrix}A_1&B_{12}&B_{13}&0&0&0\\B_{12}&A_1&B_{23}&0&0&0\\B_{13}&B_{23}&A_1&0&0&0\\0&0&0&C_4&0&0\\0&0&0&0&C_5&0\\0&0&0&0&0&C_6\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\\varepsilon_{22}\\\varepsilon_{33}\\\varepsilon_{23}\\\varepsilon_{31}\\\varepsilon_{12}\end{bmatrix}$$

9

2

$$\begin{bmatrix}\sigma_{11}\\\sigma_{22}\\\sigma_{33}\\\sigma_{23}\\\sigma_{31}\\\sigma_{12}\end{bmatrix}=\begin{bmatrix}\lambda+2\mu&\lambda&0&0&0&0\\\lambda&\lambda+2\mu&\lambda&0&0&0\\0&\lambda&\lambda+2\mu&0&0&0\\0&0&0&2\mu&0&0\\0&0&0&0&2\mu&0\\0&0&0&0&0&2\mu\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\\varepsilon_{22}\\\varepsilon_{33}\\\varepsilon_{23}\\\varepsilon_{31}\\\varepsilon_{12}\end{bmatrix}$$

$$\begin{bmatrix}\sigma_{11}\\\sigma_{22}\\\sigma_{33}\\\sigma_{23}\\\sigma_{31}\\\sigma_{12}\end{bmatrix}=\begin{bmatrix}A&B&E&0&0&0\\B&A&E&0&0&0\\E&E&A&0&0&0\\0&0&0&C&0&0\\0&0&0&0&C&0\\0&0&0&0&0&A-B\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\\varepsilon_{22}\\\varepsilon_{33}\\\varepsilon_{23}\\\varepsilon_{31}\\\varepsilon_{12}\end{bmatrix}$$

5

A, B, C, E, λ and μ are elasticity constants

Such symmetries are naturally reflected in the material laws and their parameters. The number of needed material parameters to characterize such behaviour increases from isotropy to fully anisotropy.

Material Symmetries

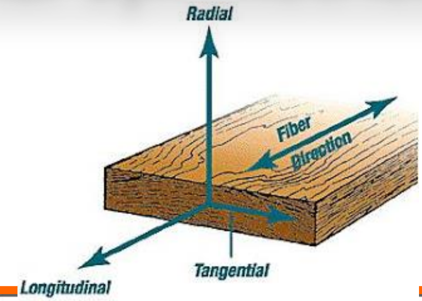
Examples

Hooke's Elasticity

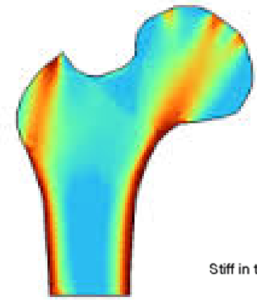
Courtesy by Luca Galuzzi.

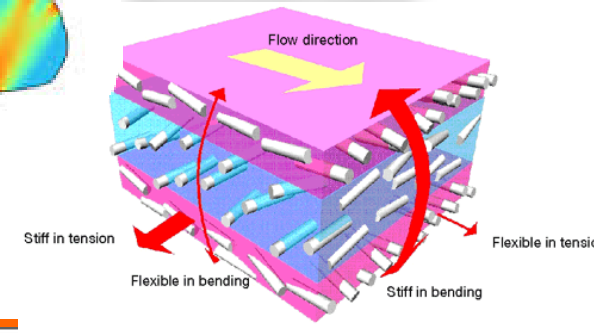
Stratified rock at Grand Canyon shows clearly transversely isotropic structure.

Example of Material Symmetries

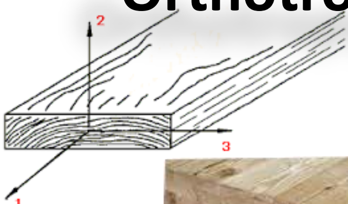


Anisotropy

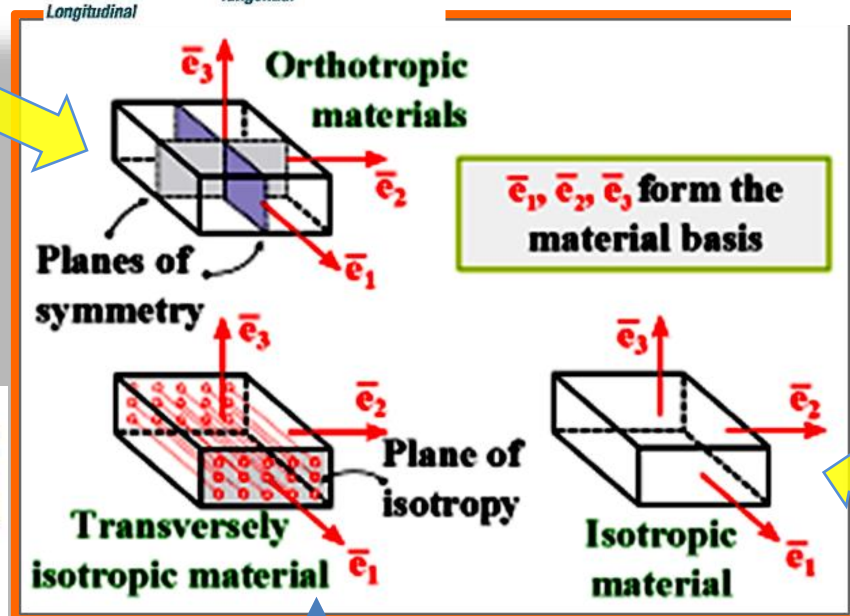




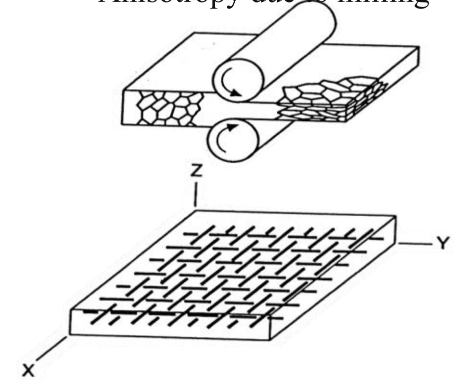
Orthotropy



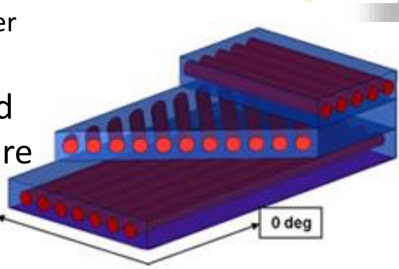
CLT
Cross
Laminated
Timber



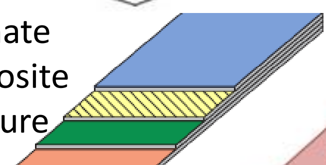
Anisotropy due to milling



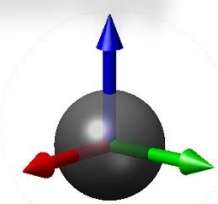
Layered structure

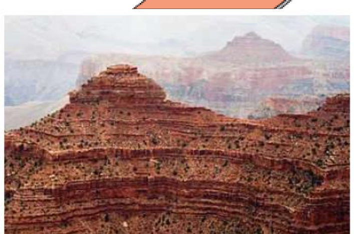


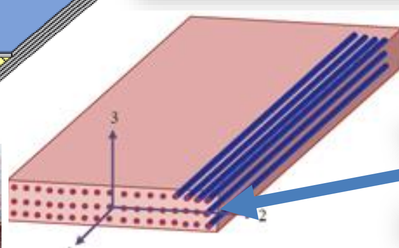
Laminate composite structure



Isotropy







$\vec{e}_1$

Transverse isotropy

Plane of isotropy is orthogonal to $\vec{e}_1$;
the axis of **transverse isotropy**

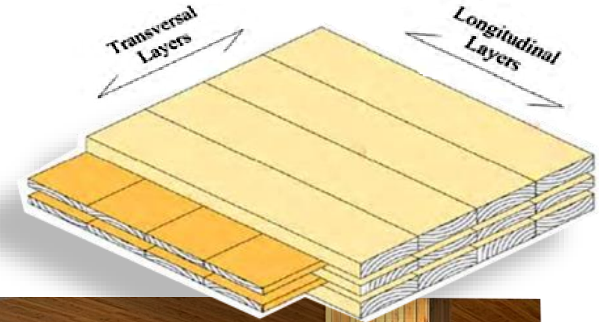
CLT, layered structure

Each layer is **orthotropic**

Example : CLT as a layered structure made from orthotropic layers of different orientations

Small project

Teacher
Djebar BAROUDI, PhD.



<u>Small project</u> [15 points (obligatory) + 15 points (bonus)]

ANALYSIS

Analytical

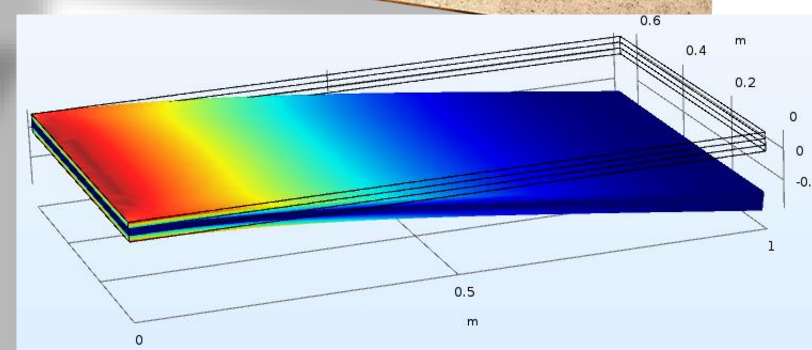
FEM

Elasticity

Elastic orthotropy
Bending of plates

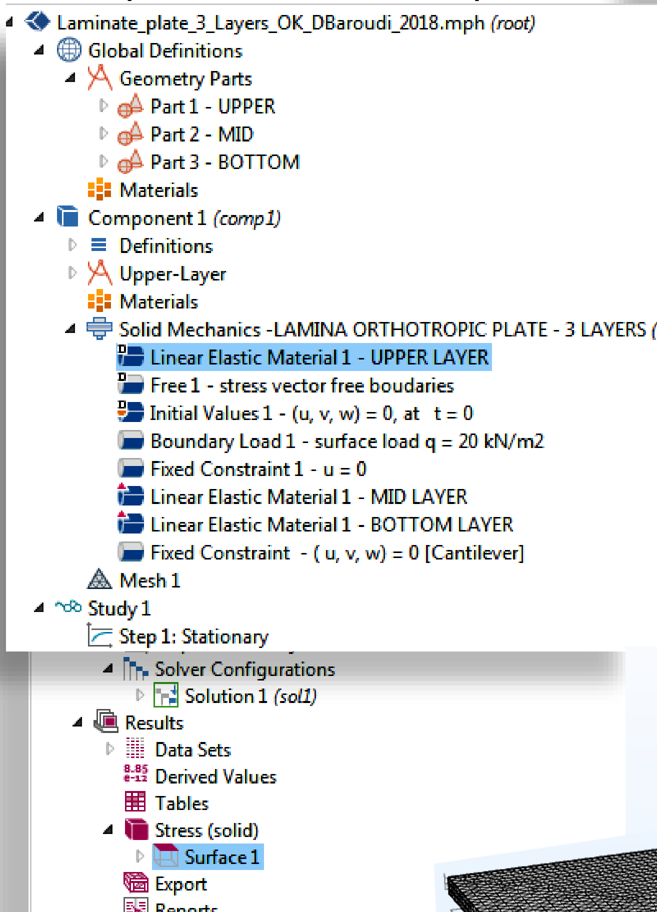
CLT, layered structure

Each layer is **orthotropic**

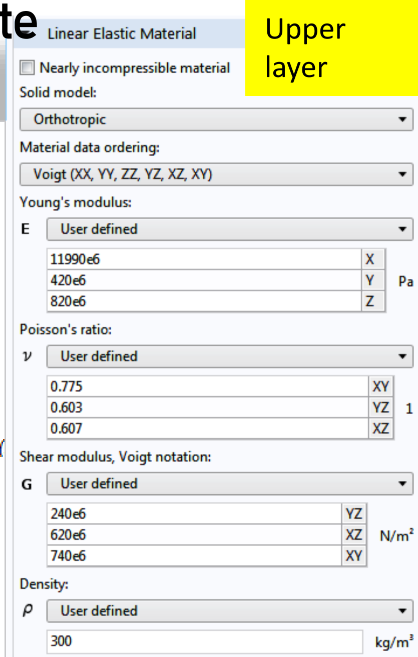


Multilayered orthotropic lamina plate

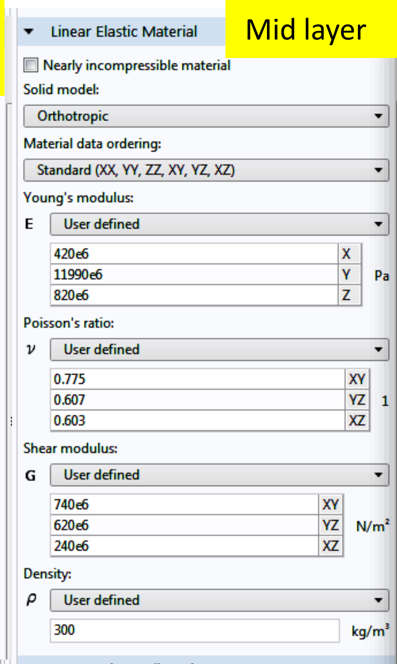
Example – A cantilever plate



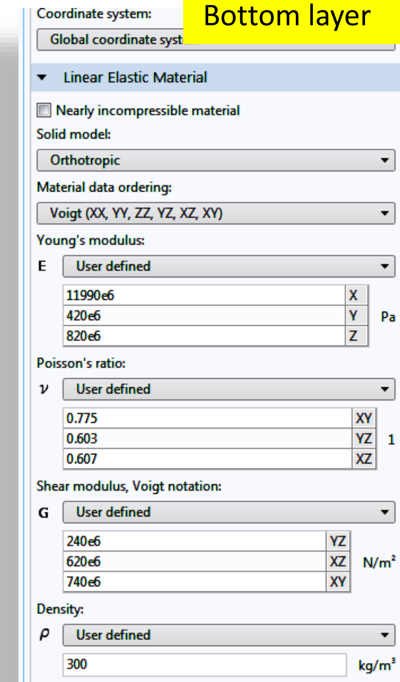
Upper layer



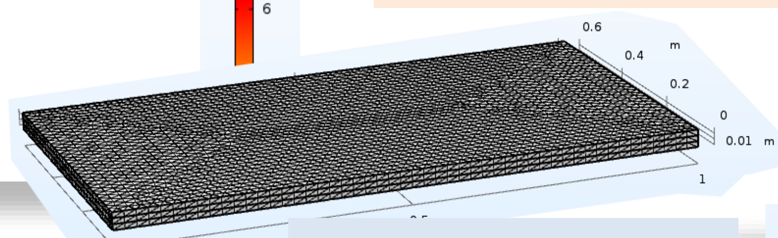
Mid layer



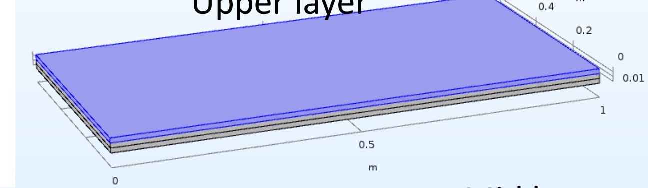
Bottom layer



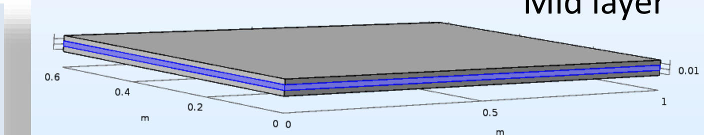
NB, here I did the FEA for a cantilever laminate plate, you should now update to the boundary conditions of the homework.



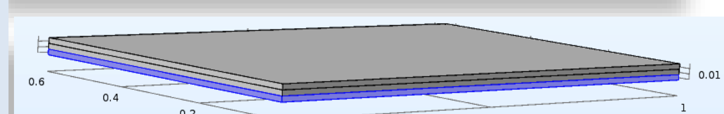
Upper layer



Mid layer

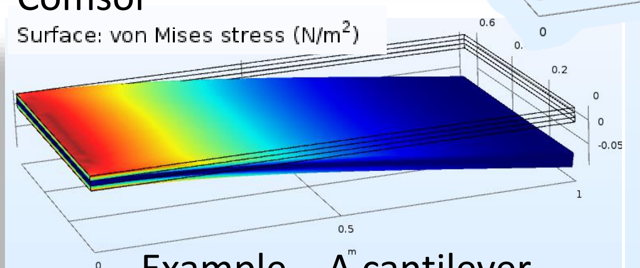


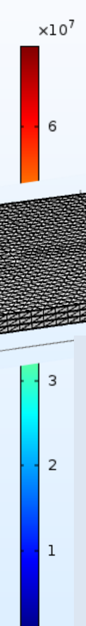
Bottom layer



FEA Example using Comsol

Surface: von Mises stress (N/m²)



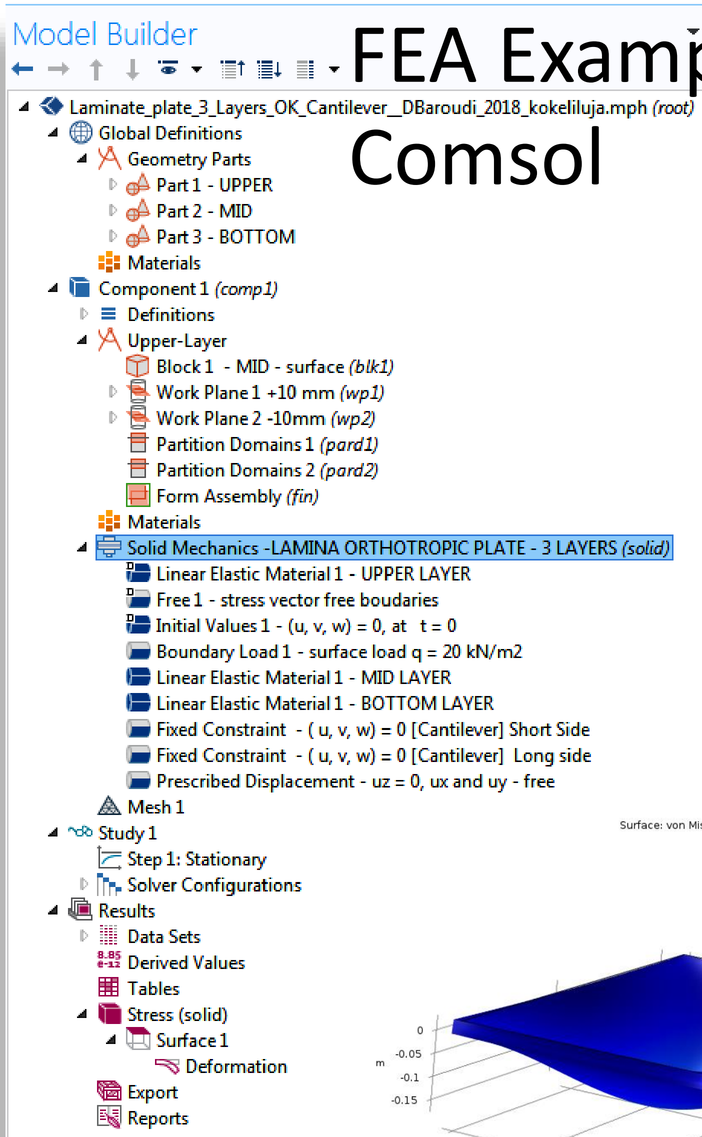


Example – A cantilever

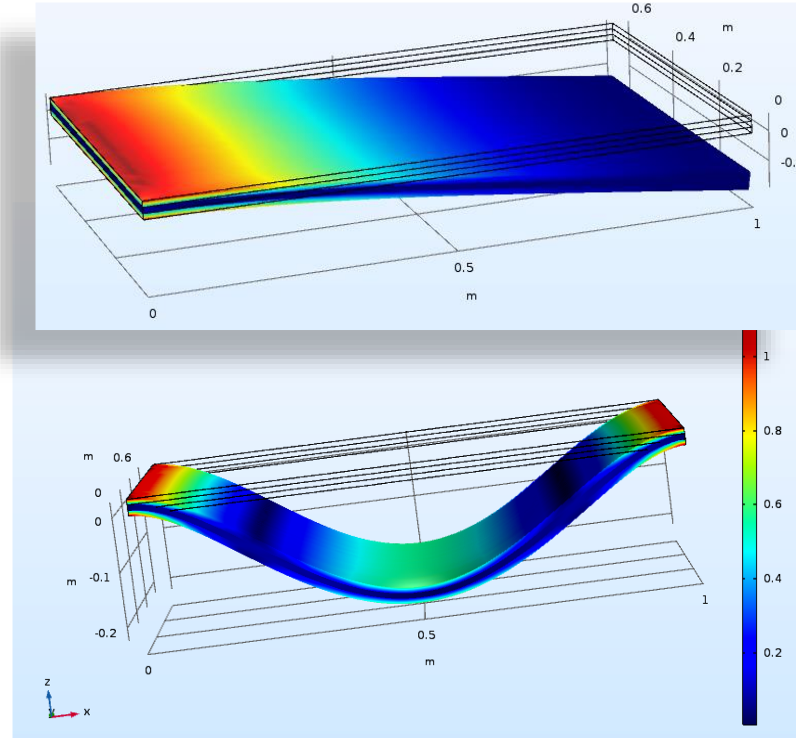
Here in FEA, [I used 3D-solids elements](). So I used 3D-elasticity and not plate theory. (It was a bit 'impossible' to have, in Comsol, layered plates bounded together! At least for me.)

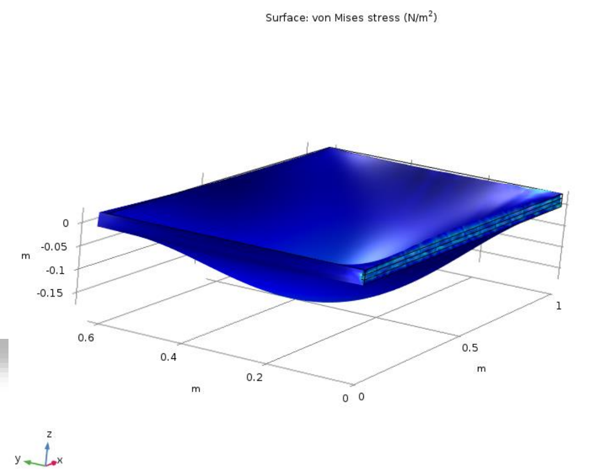
Laminate_plate_3_Layers_OK_DBaroudi_2018.mph

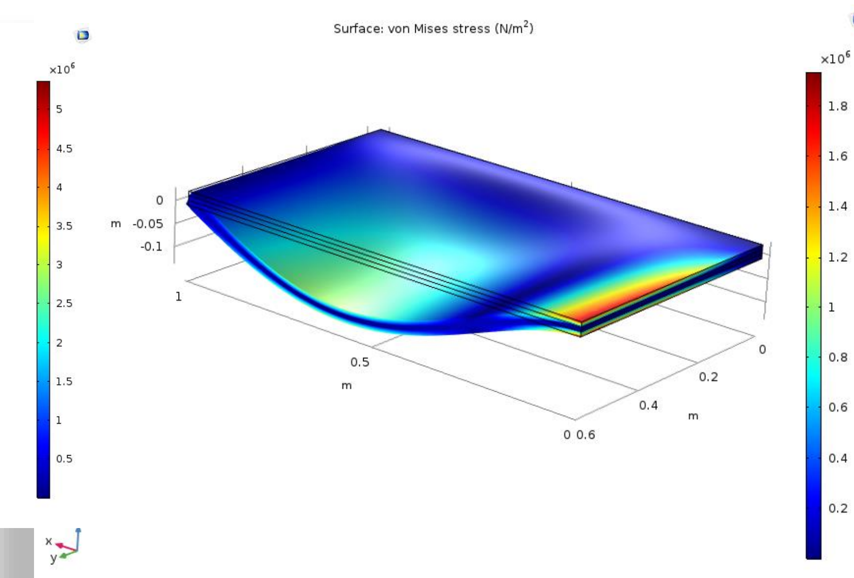
FEA Example using Comsol



Effects of varying boundary conditions







Importing *.inp files to Abaqus

after that, one can export it into *cae or whatever he wishes

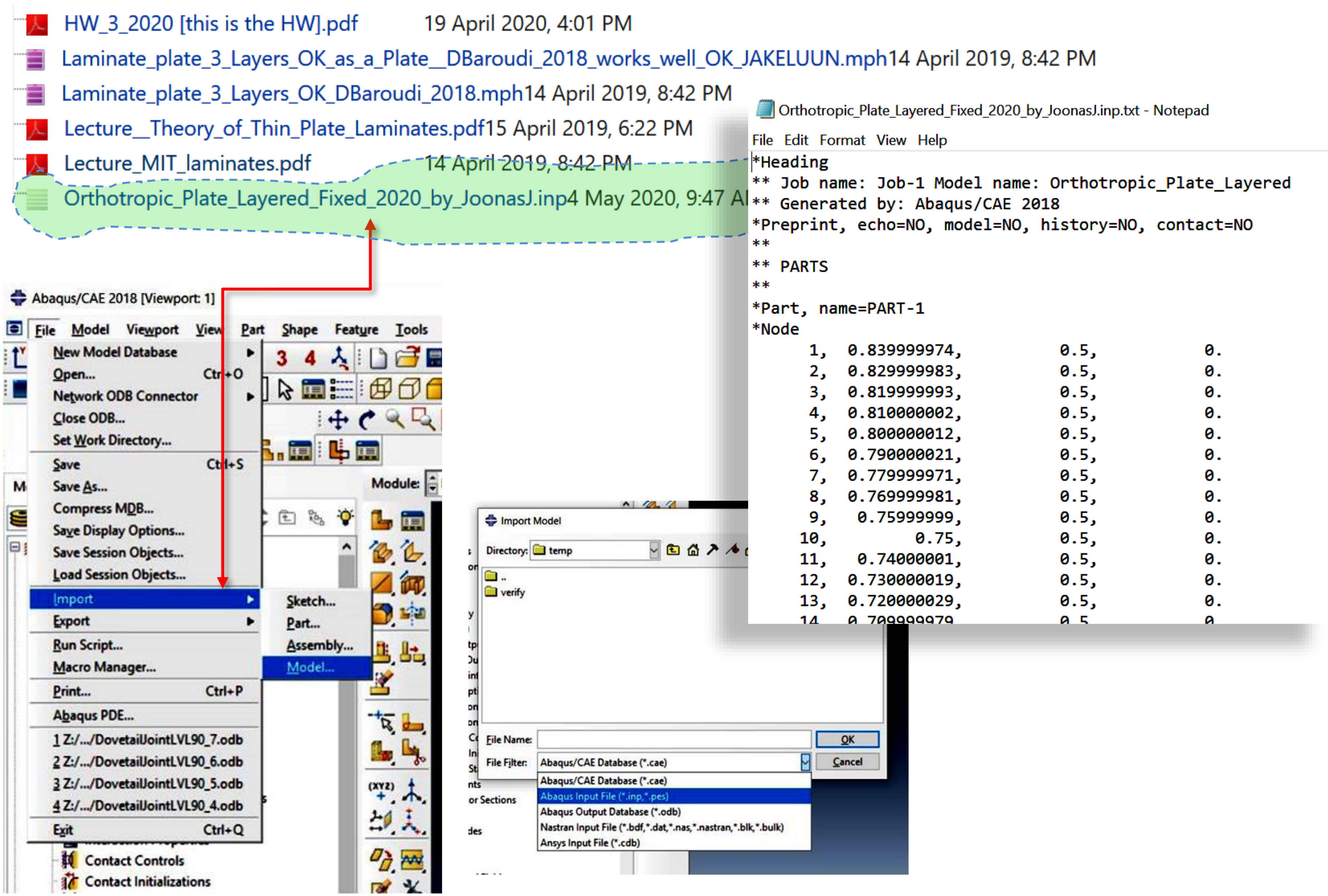
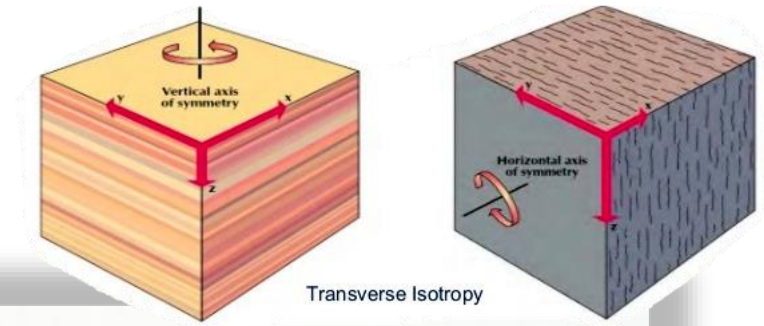
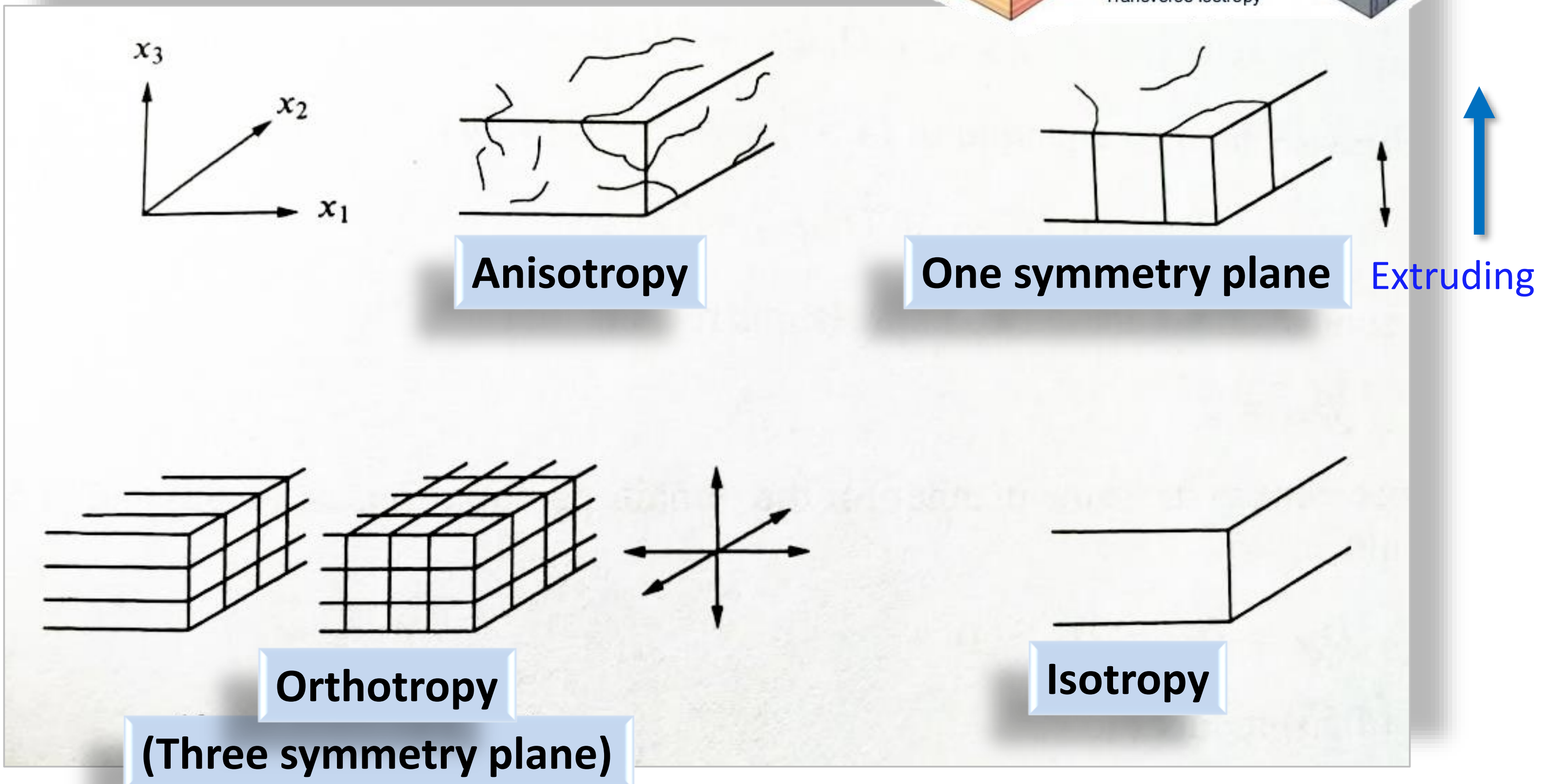


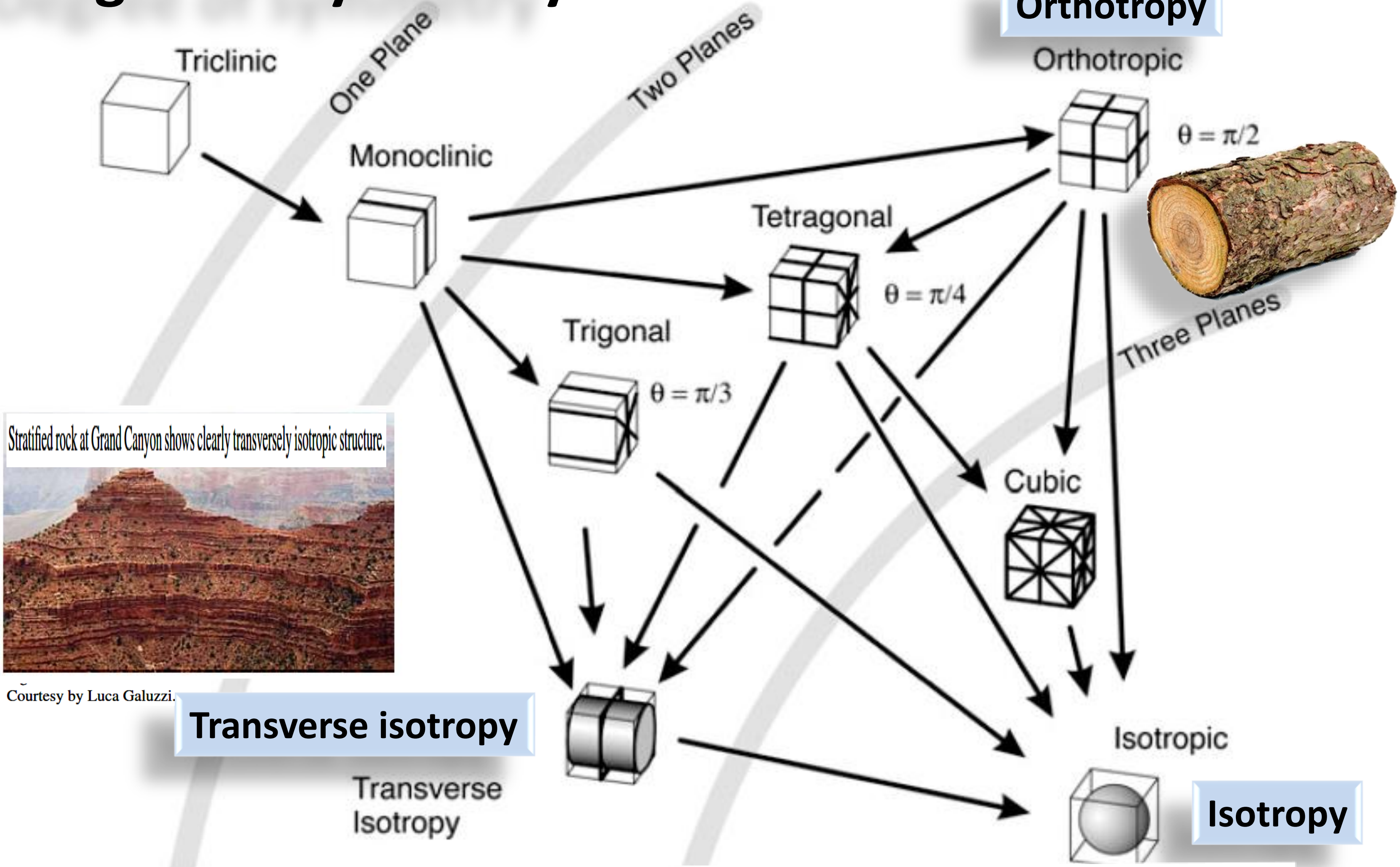
Illustration of increasing degree of symmetry





There is a link for RFEM, related to orthotropic plates: https://www.dlubal.com/en/support-and-learning/support/knowledge-base/001443

Degree of symmetry



Courtesy by Luca Galuzzi.

The hierarchical organization of the eight material symmetries of linear elasticity. The figure is organized such that the lower symmetries are at the upper left and as one moves down and across the table to the right one encounters crystal systems with greater and greater symmetry. From Chadwick et al. (2001)

Generalized Hooke's Law

Next case will be considered:

- **infinitesimal deformation** - $|\nabla \mathbf{u}| << 1$ $\qquad \mathbf{x} \approx \mathbf{X}$

- no distinction between various measures of stress and strain,
  - **σ** for the stress tensor and
  - **ε** for strain tensor

  the material description of solid mechanics will be used

- **Hookean solids**: the **relations** between stress and strain are **linear**

- the **linear constitutive model** for **infinitesimal deformations** is called: **G**eneralized **H**ooke's **L**aw

Linear elastic materials:

$$U_0 \equiv U(\boldsymbol{\varepsilon} - \mathbf{0}) = U(\mathbf{0}) + \mathbf{c} : \boldsymbol{\varepsilon} + \frac{1}{2} \hat{\mathbf{c}} : \boldsymbol{\varepsilon} : \boldsymbol{\varepsilon}$$

U_0 is a ***quadratic*** function of strains:

$$U_0 = C_0 + C_{ij}\varepsilon_{ij} + \frac{1}{2!}\hat{C}_{ijk\ell}\varepsilon_{ij}\varepsilon_{k\ell},$$

Elastic *stiffness* coefficients

In matrix notation

$$U_0 = U(\mathbf{0}) + \mathbf{c}^{\mathrm{T}}\boldsymbol{\varepsilon} + \frac{1}{2}\boldsymbol{\varepsilon}^{\mathrm{T}}\hat{\mathbf{C}}\boldsymbol{\varepsilon}$$

$$\rightarrow \boldsymbol{\sigma} = \frac{\partial U(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}}$$

The Engineering Problem

- Given, or measured **any multidimensional** (multiaxial) **strain state** you **want** to **convert** it **to** the **resulting stresses**, or vice versa.
- In order to do this, you need the **relation** which links the stresses to the **strains**:

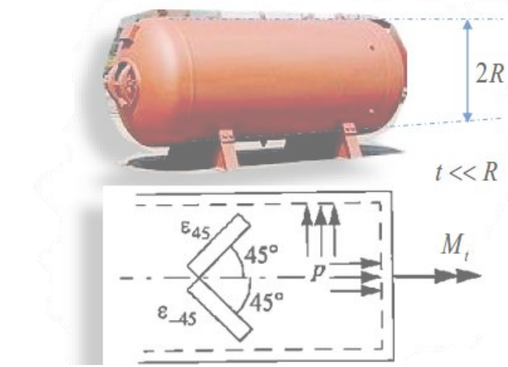
$$\sigma_{ij} = D_{ijkl}\, \varepsilon_{kl}\,, \quad \text{Tensor notations}$$

$$\boldsymbol{\sigma}_{6x1} = \mathbf{D}_{6x6}\boldsymbol{\varepsilon}_{6x1} \quad \text{Voigt's matrix notations}$$

Such example: **Homework, problem 4**

Problem 4: isotropy in 3D elasticity

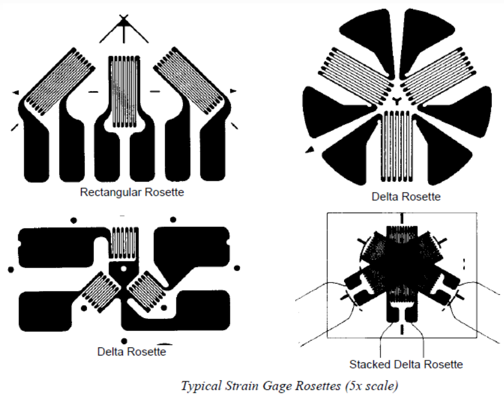
Composed stress state



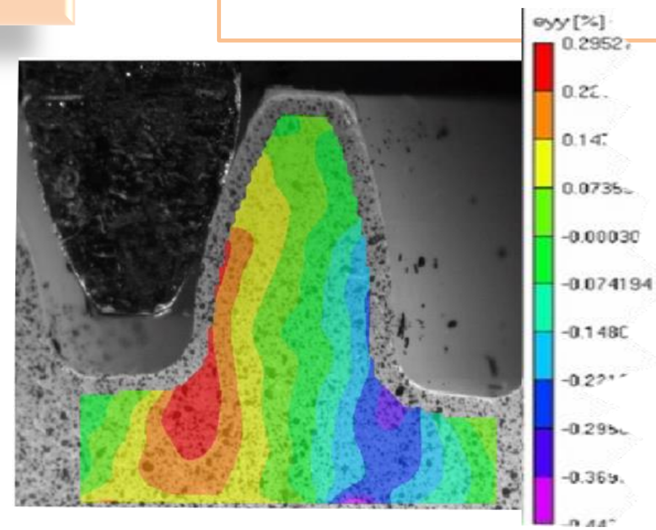
Q: measured strains determine the resulting stresses.

Strains can be measured using Strain Gages or any Digital Image Correlation (DIC) techniques (photogrammetry) and other means…

https://en.wikipedia.org/wiki/Digital_image_correlation



Typical Strain Gage Rosettes



Strain component ε_{yy} obtained by image correlation

Some classical such relations are shown in the following …

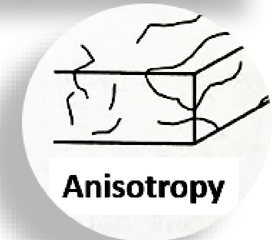
Linear Elasticity – Matrix Formulation

Anisotropy

Assume a Hooke's law (linear-elastic) Tensor notation

$$\sigma_{ij} = D_{ijkl}\,\varepsilon_{kl}\,, \qquad D_{ijkl} = D_{ijlk} -$$ symmetry property

Hooke
Linear-elastic



Can be rewritten using Voigt's matrix notation as $\boldsymbol{\sigma}_{6\text{x}1} = \mathbf{D}_{6\text{x}6}\boldsymbol{\varepsilon}_{6\text{x}1}$ where

$\boldsymbol{\sigma}^{\mathrm{T}} = \boldsymbol{\sigma}$
symmetry

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon} \leftrightarrow \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ D_{21} & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ D_{31} & D_{32} & D_{33} & D_{34} & D_{35} & D_{36} \\ D_{41} & D_{42} & D_{43} & D_{44} & D_{45} & D_{46} \\ D_{51} & D_{52} & D_{53} & D_{54} & D_{55} & D_{56} \\ D_{61} & D_{62} & D_{63} & D_{64} & D_{65} & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \end{bmatrix}$$

In general, **there are 21 independent elastic constants** (forth order tensor D_{ijkl} has 81 terms but reduces to **21** independent coefficients. This result follows from symmetries of strain and stress tensors and thermodynamical considerations – 1st law of thermodynamics)

$\mathbf{D} = \mathbf{D}^{\mathrm{T}} -$ stiffness matrix **symmetric**

$\mathbf{S} = \mathbf{C} = \mathbf{D}^{-1}$: compliance (or also flexibility) matrix

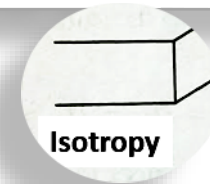
Various notations

Engineering shear strains:
$2\varepsilon_{12} \equiv \gamma_{12}$
$2\varepsilon_{23} \equiv \gamma_{23}$
$2\varepsilon_{31} \equiv \gamma_{31}$

Strain energy density: $U = \dfrac{1}{2}\boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{\sigma} = \dfrac{1}{2}\boldsymbol{\varepsilon}^{\mathrm{T}}\mathbf{D}\boldsymbol{\varepsilon} \geq 0, \quad \forall \boldsymbol{\varepsilon} \Rightarrow \exists \mathbf{D}^{-1} \equiv \mathbf{C}$

Generalized Hooke's Law

Isotropy



Linear isotropic material

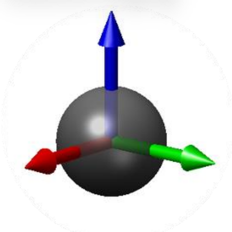
$$
\mathbf{D} = \frac{E}{(1+\nu)(1-2\nu)}
\begin{array}{c}
\begin{array}{cccccc} \varepsilon_{11} & \varepsilon_{22} & \varepsilon_{33} & 2\varepsilon_{12} & 2\varepsilon_{23} & 2\varepsilon_{31} \end{array} \\
\begin{bmatrix}
1-\nu & \nu & \nu & 0 & 0 & 0 \\
\nu & 1-\nu & \nu & 0 & 0 & 0 \\
\nu & \nu & 1-\nu & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu)
\end{bmatrix}
\end{array}
$$

There are only **two** (2) independent elastic constants

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon},$$

Elastic stiffness matrix

Isotropy



$$
\mathbf{S} = \frac{1}{E}
\begin{bmatrix}
1 & -\nu & -\nu & 0 & 0 & 0 \\
-\nu & 1 & -\nu & 0 & 0 & 0 \\
-\nu & -\nu & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\
0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\
0 & 0 & 0 & 0 & 0 & 2(1+\nu)
\end{bmatrix}
$$

Elastic compliance matrix

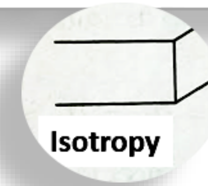
$$\boldsymbol{\varepsilon} = \mathbf{D}^{-1}\boldsymbol{\sigma} \equiv \mathbf{S}\boldsymbol{\sigma}$$

Material isotropy: the constitutive relation remains unchanged irrespective of the Cartesian coordinate system we use

Physically, this means that the material properties are the same in all the directions

Generalized Hooke's Law

Isotropy



Linear isotropic material

$$\boldsymbol{\varepsilon} = \mathbf{D}^{-1}\boldsymbol{\sigma} + \boldsymbol{\alpha}\Delta T$$

$$\equiv \mathbf{S}$$

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} + \alpha\Delta T\delta_{ij}.$$

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \cdot \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} + \alpha\Delta T \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

There are only two (2) independent elastic constants

$$E, \nu$$

$$\alpha$$

Linear thermo-elasticity

$$\boldsymbol{\sigma} = \mathbf{D}(\boldsymbol{\varepsilon} - \boldsymbol{\alpha}\Delta T),\qquad \sigma_{ij} = \frac{E}{1+\nu}\left\{\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij}\right\} - \frac{E\alpha\Delta T}{1-2\nu}\delta_{ij}.$$

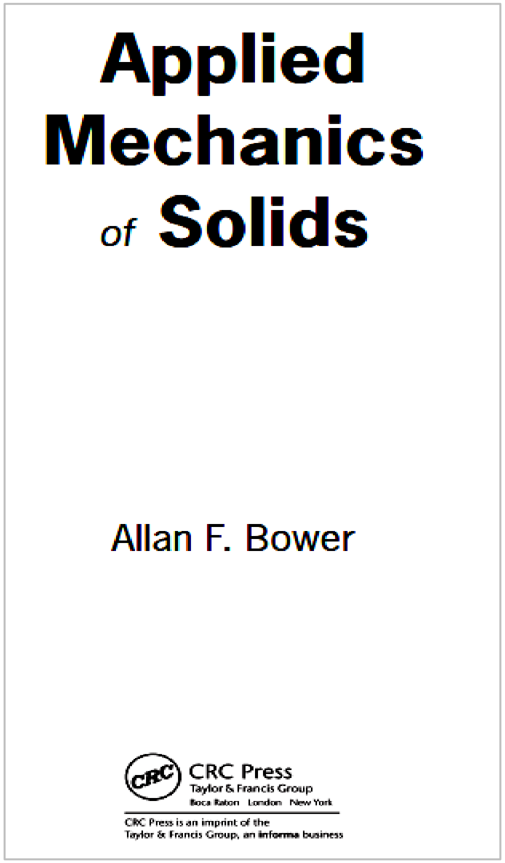
$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} - \frac{E\alpha\Delta T}{1-2\nu}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Elastic Constants for Isotropic Materials

Material	Mass Density ρ / Mgm^{-3}	Young's Modulus E / GNm^{-2}	Poisson's Ratio ν	Expansion Coefficient K^{-1}
Tungsten carbide	14–17	450–650	0.22	5×10^{-6}
Silicon carbide	2.5–3.2	450	0.22	4×10^{-6}
Tungsten	13.4	410	0.30	4×10^{-6}
Alumina	3.9	390	0.25	7×10^{-6}
Titanium carbide	4.9	380	0.19	13×10^{-6}
Silicon nitride	3.2	320–270	0.22	3×10^{-6}
Nickel	8.9	215	0.31	14×10^{-6}
CFRP	1.5–1.6	70–200	0.20	2×10^{-6}
Iron	7.9	196	0.30	13×10^{-6}
Low alloy steels	7.8	200–210	0.30	15×10^{-6}
Stainless steel	7.5–7.7	190–200	0.30	11×10^{-6}
Mild steel	7.8	196	0.30	15×10^{-6}
Copper	8.9	124	0.34	16×10^{-6}
Titanium	4.5	116	0.30	9×10^{-6}
Silicon	2.5–3.2	107	0.22	5×10^{-6}
Silica glass	2.6	94	0.16	0.5×10^{-6}
Aluminum and alloys	2.6–2.9	69–79	0.35	22×10^{-6}
Concrete	2.4–2.5	45–50	0.3	10×10^{-6}
GFRP	1.4–2.2	7–45	0.25	10×10^{-6}
Wood, parallel grain	0.4–0.8	9–16	0.2	40×10^{-6}
Polyimides	1.4	3–5	0.1–0.45	40×10^{-6}
Nylon	1.1 – 1.2	2–4	0.25	81×10^{-6}
PMMA	1.2	3.4	0.35–0.4	50×10^{-6}
Polycarbonate	1.2 – 1.3	2.6	0.36	65×10^{-6}
Natural rubbers	0.83–0.91	0.01–0.1	0.49	200×10^{-6}
PVC	1.3–1.6	0.003–0.01	0.41	70×10^{-6}

No differential in thermal elongations, thus no thermal stresses at the interface for reinforced concrete

← Reference

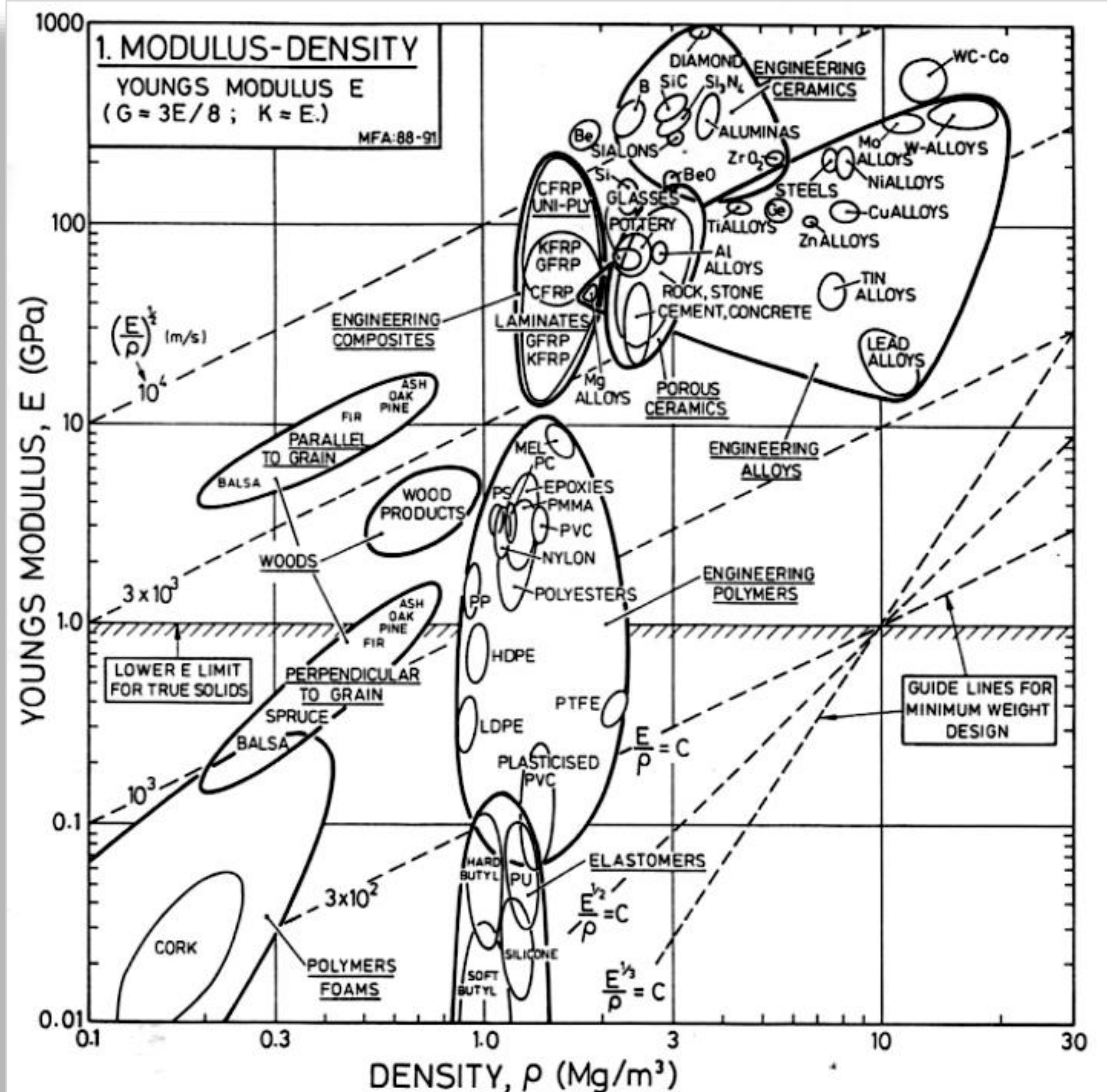


Nearly incompressible

$$\sigma_{ij} = \frac{E}{1+\nu}\left\{\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij}\right\} - \frac{E\alpha\Delta T}{1-2\nu}\delta_{ij}.$$

Troubles as $\nu \to 0.5$

$\Rightarrow \sigma_{kk} \to \infty$ → not physical!



Ref: *Material Selection in Mechanical Design*, M.F Ashby, Pergamon Press, Oxford, 1992

Generalized Hooke's Law – Isotropic case

Lamé notation

Linear isotropic material:

$$\boldsymbol{\sigma} = \lambda \,\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}$$

Lamé's coefficients

Bulk Modulus $K = \dfrac{E}{3(1-2\nu)}$

Shear Modulus $\mu = \dfrac{E}{2(1+\nu)} \equiv G$

Lame Modulus $\lambda = \dfrac{\nu E}{(1+\nu)(1-2\nu)}$

$$\boldsymbol{\varepsilon} = \frac{1+\nu}{E}\boldsymbol{\sigma} - \frac{\nu}{E}\mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1}$$

Material isotropy: the constitutive relation remains unchanged irrespective of the Cartesian coordinate system we use

Physically, this means that the material properties are the same in all the directions

Matrix notation:

Constitutive Model for Linear Elastic Isotropic Materials

Five elastic constants: $E \quad \nu \quad \lambda \quad G \quad K$

Only two of them are independent

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{13} \end{Bmatrix} = \begin{bmatrix} 1/E & -\nu/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & 1/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & -\nu/E & 1/E & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{Bmatrix}$$

Limits on Elastic Parameters Values

Isotropic materials

$$\text{Bulk Modulus } K = \frac{E}{3(1-2\nu)}$$

$$\text{Shear Modulus } \mu = \frac{E}{2(1+\nu)} \equiv G$$

$$\text{Lame Modulus } \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

$$\boldsymbol{\sigma} = \lambda \operatorname{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}$$

$$G = \mu$$

Material Class	Poisson's Ratio ν
Ceramics	0.2
Metals	0.3
Plastics	0.4
Rubber	0.5

Constitutive Model for Linear Elastic Isotropic Materials

Five elastic constants: $E \quad \nu \quad \lambda \quad G \quad K$

Only two of them are independent

For ordinary * Engineering Materials (isotropic)

$$\Longrightarrow \quad 0 < \nu < 1/2, \quad 0 < E, G, K < \infty$$

Nearly *incompressible*

Examples of the values of elastic parameters

Material	E (GPa)	ν
Rubber	0.003	~ 0.5
Timber	4	0.5
Concrete	10	0.2
Glass	72.1	0.23
Aluminum	70.3	0.345
Brass	103.5	0.33
Iron	211	0.29
Tungsten	400	0.27
Tungsten carbide	534.4	0.22

matériau	ν	E (MPa)	μ (MPa)	K (MPa)
acier	0,3	205 000	78 800	170 800
aluminium	0,33	70 000	26 300	68 600
verre ordinaire	0,22	60 000	24 600	35 700
béton	0,2	30 000	12 500	16 667
plomb	0,45	17 000	5 860	56 700
plexiglas (résine acrylique)	0,36	3 000	1 100	3 570
bakélite (polypropylène)	0,37	1 000	365	1 280
caoutchouc	0,5	2	0,67	∞

* For metamaterials (materials with substructures) the effective Poisson's ratio may be negative as well. **Materials** having negative Poisson's ratio are called **auxetic**

Good to know

by

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Ref:→

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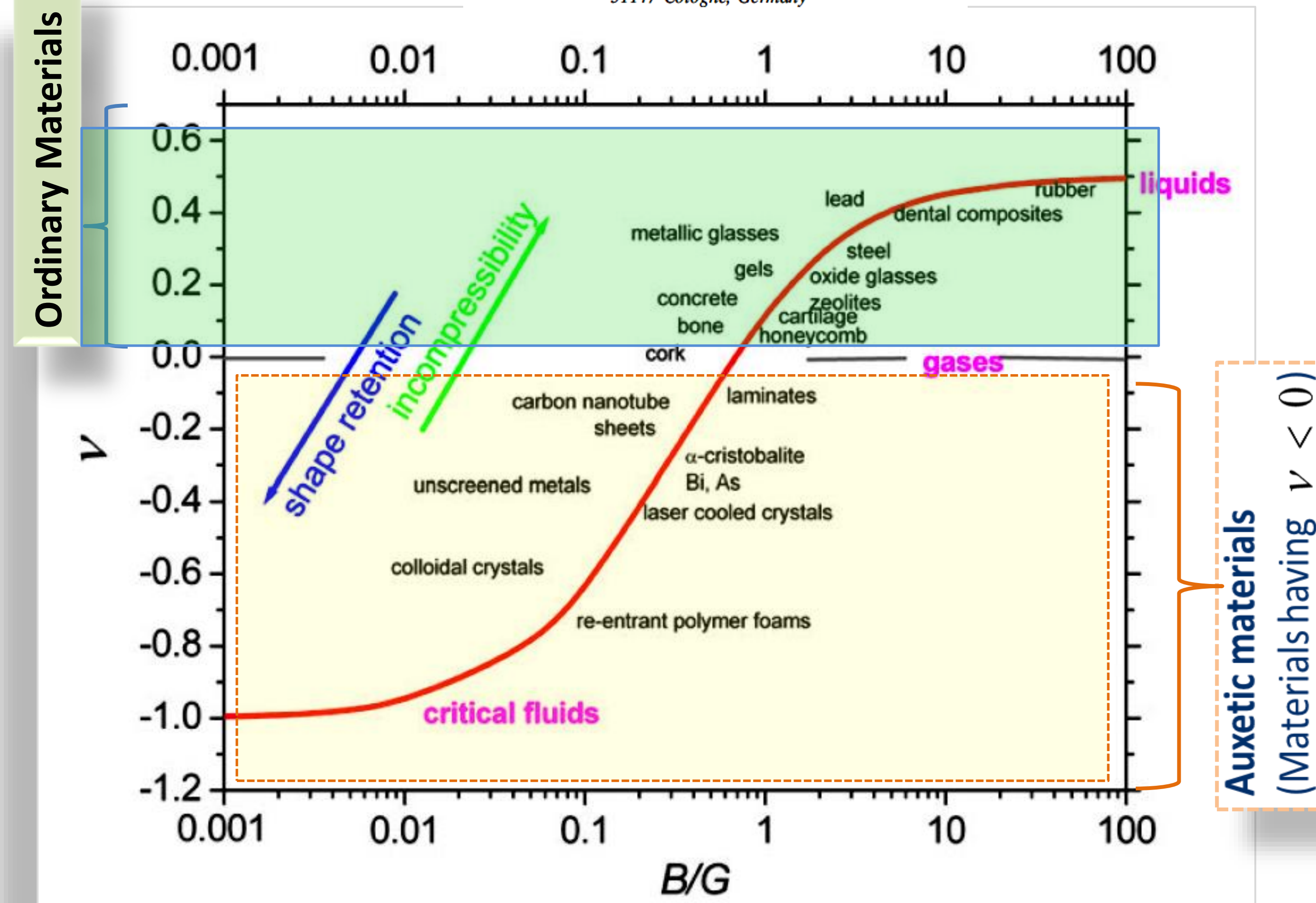
POISSON'S RATIO OVER TWO CENTURIES: CHALLENGING HYPOTHESES



Elastic moduli ***K*** bulk modulus (also called ***B***)

$$B \equiv K = \frac{E}{3(1-2\nu)}$$

G – modulus

Universal relationship between Poisson's ratio ν and the ratio of bulk to shear moduli B/G. All isotropic materials have ν values somewhere within the number window -1 to $\frac{1}{2}$. Materials with negative Poisson's ratio have been called auxetic. In going from negative to positive values, the shapes of materials are sacrificed for incompressibility.

Example of use of Hooke's law in experimental anaysis

Example 3.1 By using a strain gage rosette, the strains $\varepsilon_a = 12 \cdot 10^{-4}$, $\varepsilon_b = 2 \cdot 10^{-4}$ and $\varepsilon_c = -2 \cdot 10^{-4}$ have been measured in a steel sheet in the directions a, b and c (Fig. 3.5a).

Calculate the principal strains, the principal stresses and the principal directions.

Assuming linear elasticity

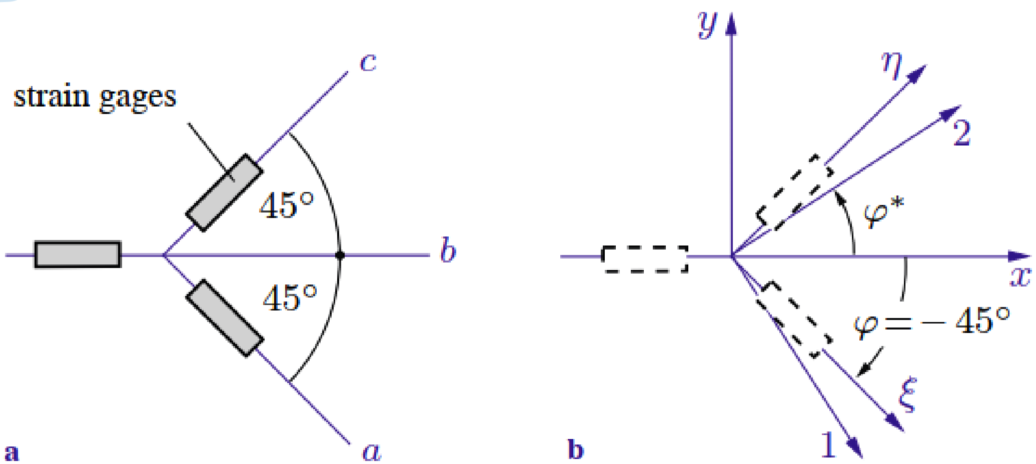
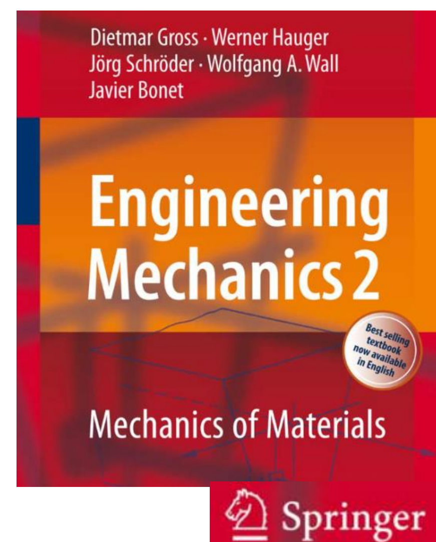


Fig. 3.5

Solution We introduce the two coordinate systems x, y and ξ, η according to Fig. 3.5b. Inserting the angle $\varphi = -45^\circ$ into the first and second transformation equation (3.3), we obtain

$$\varepsilon_\xi = \frac{1}{2}(\varepsilon_x + \varepsilon_y) - \frac{1}{2}\gamma_{xy}, \qquad \varepsilon_\eta = \frac{1}{2}(\varepsilon_x + \varepsilon_y) + \frac{1}{2}\gamma_{xy}\,.$$

This example is from this book



Addition and subtraction, respectively, yields

$$\varepsilon_\xi + \varepsilon_\eta = \varepsilon_x + \varepsilon_y, \qquad \varepsilon_\eta - \varepsilon_\xi = \gamma_{xy}.$$

With $\varepsilon_\xi = \varepsilon_a$, $\varepsilon_\eta = \varepsilon_c$ and $\varepsilon_x = \varepsilon_b$ we get

$$\varepsilon_y = \varepsilon_a + \varepsilon_c - \varepsilon_b = 8 \cdot 10^{-4}, \qquad \gamma_{xy} = \varepsilon_c - \varepsilon_a = -14 \cdot 10^{-4}.$$

The principal strains and principal directions are determined according to (3.5) and (3.4):

$$\varepsilon_{1,2} = (5 \pm \sqrt{9+49}) \cdot 10^{-4} \quad \rightarrow \quad \underline{\underline{\varepsilon_1 = 12.6 \cdot 10^{-4}}}, \ \underline{\underline{\varepsilon_2 = -2.6 \cdot 10^{-4}}},$$

$$\tan 2\varphi^* = \frac{-14}{2-8} = 2.33 \quad \rightarrow \quad \underline{\underline{\varphi^* = 33.4^\circ}}.$$

Introducing the angle φ^* into (3.3) shows that it is associated with the principal strain ε_2. The principal directions 1 and 2 are plotted in Fig. 3.5b.

Solving (3.13) for the stresses yields

$$\sigma_1 = \frac{E}{1-\nu^2}(\varepsilon_1 + \nu\,\varepsilon_2), \qquad \sigma_2 = \frac{E}{1-\nu^2}(\varepsilon_2 + \nu\,\varepsilon_1).$$

With $E = 2.1 \cdot 10^2$ GPa and $\nu = 0.3$ we obta

$$\underline{\underline{\sigma_1 = 273 \text{ MPa}}}, \qquad \underline{\underline{\sigma_2 = 27 \text{ MPa}}}.$$

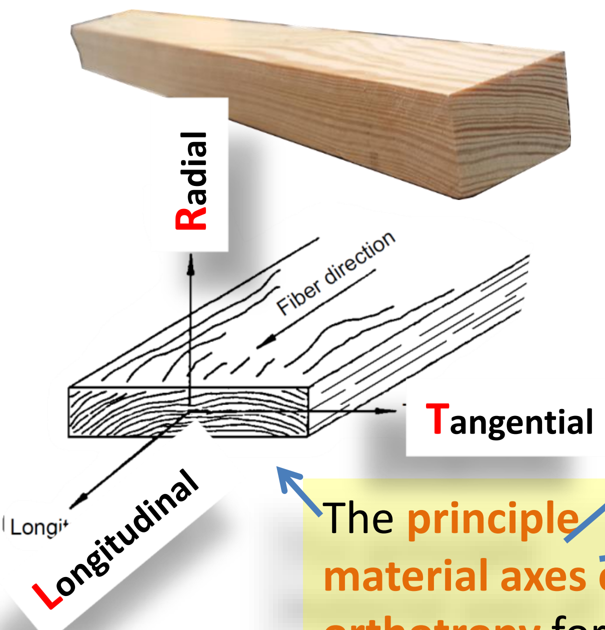
$$\begin{aligned}
\varepsilon_\xi &= \tfrac{1}{2}(\varepsilon_x + \varepsilon_y) + \tfrac{1}{2}(\varepsilon_x - \varepsilon_y)\cos 2\varphi + \tfrac{1}{2}\gamma_{xy}\sin 2\varphi, \\
\varepsilon_\eta &= \tfrac{1}{2}(\varepsilon_x + \varepsilon_y) - \tfrac{1}{2}(\varepsilon_x - \varepsilon_y)\cos 2\varphi - \tfrac{1}{2}\gamma_{xy}\sin 2\varphi, \qquad (3.3) \\
\tfrac{1}{2}\gamma_{\xi\eta} &= \qquad\qquad\quad - \tfrac{1}{2}(\varepsilon_x - \varepsilon_y)\sin 2\varphi + \tfrac{1}{2}\gamma_{xy}\cos 2\varphi.
\end{aligned}$$

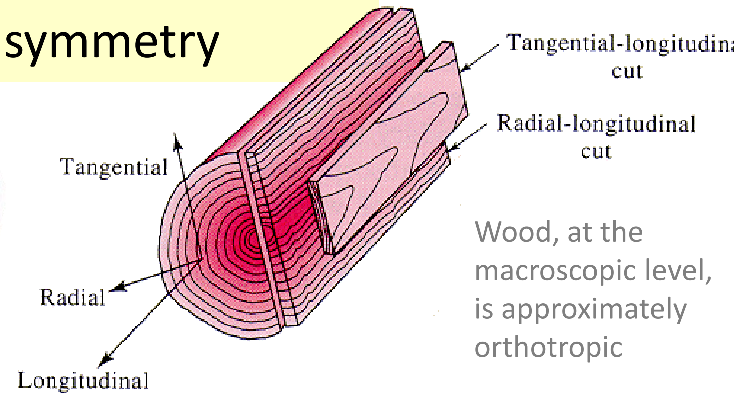
Generalized Hooke's Law

Orthotropy

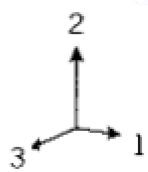
The material that has three orthogonal planes of symmetry

(at the macroscopic level, naturally)

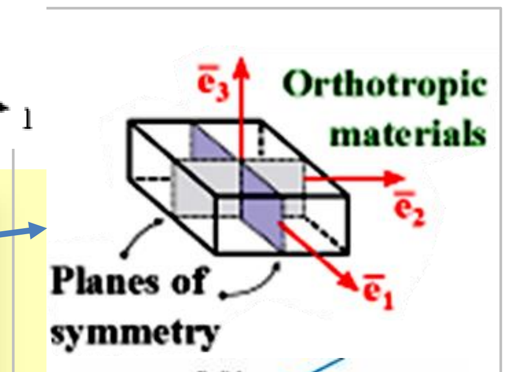




Wood, at the macroscopic level, is approximately orthotropic



The principle material axes of orthotropy forms the material basis: $\vec{e}_1, \vec{e}_2, \vec{e}_3$

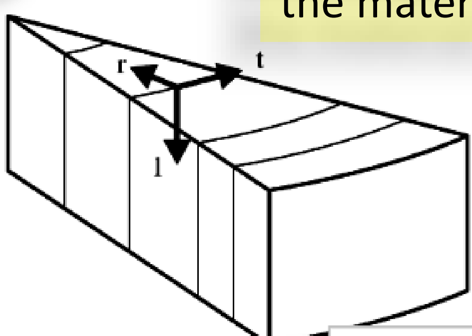


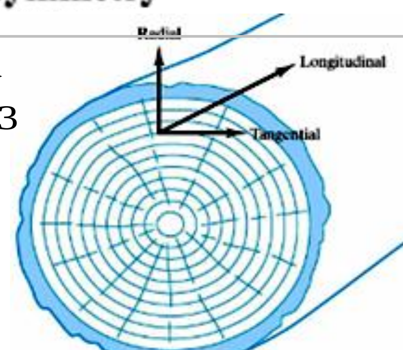
Examples: wood, laminated composite,…

$\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{D}}}} : \underline{\underline{\varepsilon}}$ - tensor notation

$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$ - matrix notation

Voigt's notation





Material orthotropy: three mutually orthogonal symmetry planes

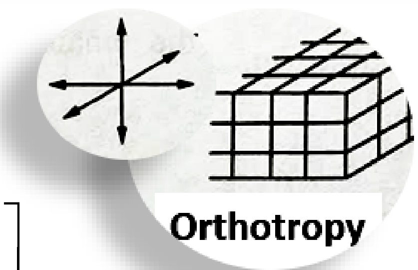
Generalized Hooke's Law

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$$

Orthotropy

There is **9** independent elastic constants

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & 0 & 0 & 0 \\ D_{21} & D_{22} & D_{23} & 0 & 0 & 0 \\ D_{31} & D_{32} & D_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \end{bmatrix}$$

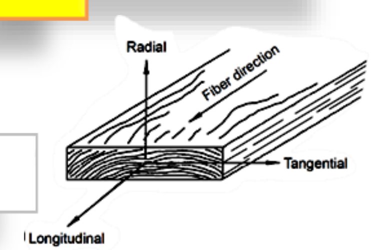


$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \end{bmatrix} = \begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & -\nu_{31}/E_3 & 0 & 0 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & -\nu_{32}/E_3 & 0 & 0 & 0 \\ -\nu_{13}/E_1 & -\nu_{23}/E_2 & 1/E_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_{23} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_{31} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix}$$

$$\boldsymbol{\varepsilon} = \mathbf{C}^{-1}\boldsymbol{\sigma}$$

$$\nu_{12}/E_1 = \nu_{21}/E_2$$
$$\nu_{31}/E_3 = \nu_{13}/E_1$$
$$\nu_{23}/E_2 = \nu_{32}/E_3$$

Since the **flexibility** matrix is **symmetric** then we have the **constraints**:



Material orthotropy: three mutually orthogonal symmetry planes

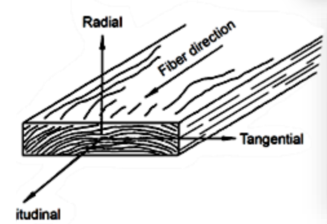
Example of a thermodynamical approach

$$\boldsymbol{\varepsilon} = \mathbf{C}^{-1}\boldsymbol{\sigma}$$

Orthotropy

Wooden structures

There is **9** independent elastic constants

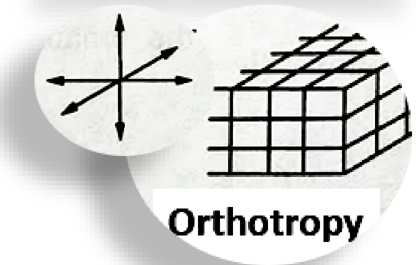


The elastic behaviour is modelled as St. Venant–Kirchhoff material with the strain-energy density function

$$\Psi = \frac{1}{2}\underline{E} : \underline{\underline{C}} : \underline{E}. \qquad \psi = \psi(\varepsilon, T) \qquad (4)$$

The elasticity tensor $\underline{\underline{C}}$ is defined by the Young's moduli E_r, E_t, E_l, the shear moduli G_{rt}, G_{tl}, G_{rl} as well as the Poisson's ratios υ_{rt}, υ_{tl} and υ_{rl}. Using the symmetry of the elasticity tensor, $\underline{\underline{C}}$ is defined in matrix notation as

$$\nu_{12}/E_1 = \nu_{21}/E_2$$
$$\nu_{31}/E_3 = \nu_{13}/E_1$$
$$\nu_{23}/E_2 = \nu_{32}/E_3$$



$$\underline{\underline{C}} = \begin{bmatrix} \frac{1}{E_r} & -\frac{\upsilon_{rt}}{E_t} & -\frac{\upsilon_{rl}}{E_l} & 0 & 0 & 0 \\ -\frac{\upsilon_{rt}}{E_t} & \frac{1}{E_t} & -\frac{\upsilon_{tl}}{E_l} & 0 & 0 & 0 \\ -\frac{\upsilon_{rl}}{E_l} & -\frac{\upsilon_{tl}}{E_l} & \frac{1}{E_l} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{rt}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{tl}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{rl}} \end{bmatrix}^{-1} \qquad (5)$$

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \end{bmatrix} = \begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & -\nu_{31}/E_3 & 0 & 0 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & -\nu_{32}/E_3 & 0 & 0 & 0 \\ -\nu_{13}/E_1 & -\nu_{23}/E_2 & 1/E_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_{23} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_{31} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix}$$

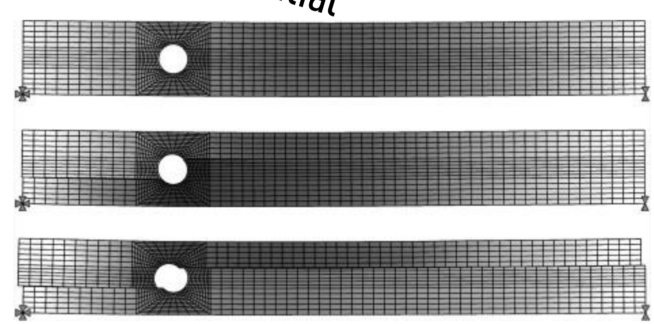
The dissipative or damage behavior should be modelled separately by postulating a dissipation potential

Eq. (5) is valid for the material coordinate system (cp. Fig. 2) and if

$$\frac{E_i}{E_j} = \frac{\upsilon_{ji}}{\upsilon_{ij}} \qquad \partial\psi/\partial(\varepsilon) = \boldsymbol{\sigma} \qquad (6)$$

is fulfilled. The stresses can be computed by

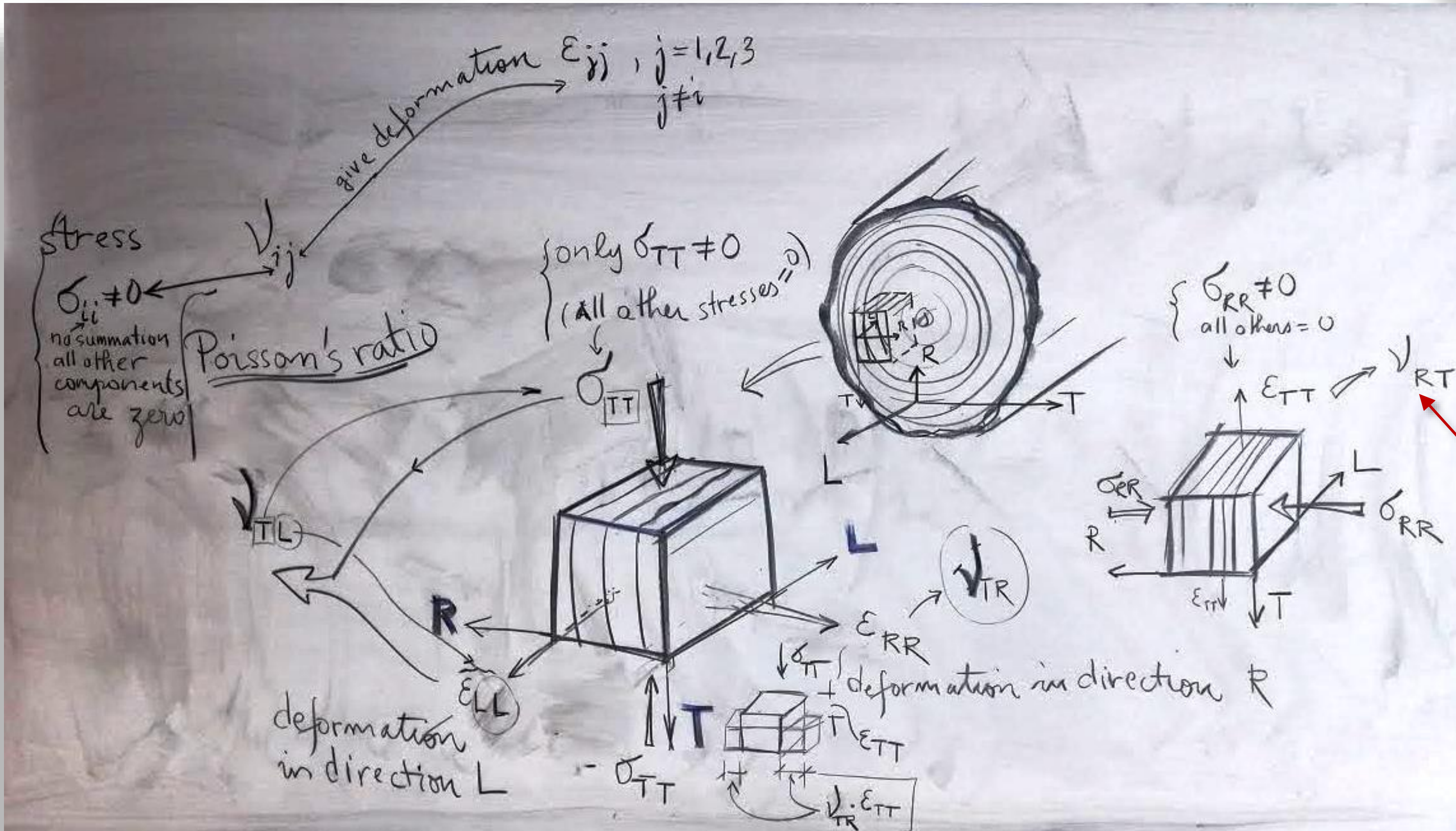
$$\underline{S} = \frac{\partial \Psi}{\partial \underline{E}} = \underline{\underline{C}} : \underline{E}. \qquad (7)$$

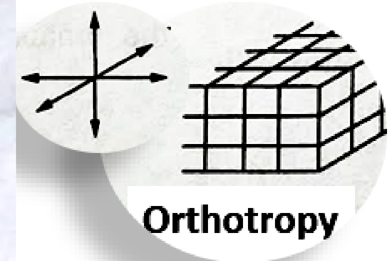


ref: Models for numerical failure analysis of wooden structures. Jörg Schmidt Michael Kaliske https://doi.org/10.1016/j.engstruct.2008.11.001

Poisson's ratios - notation

Orthotropy



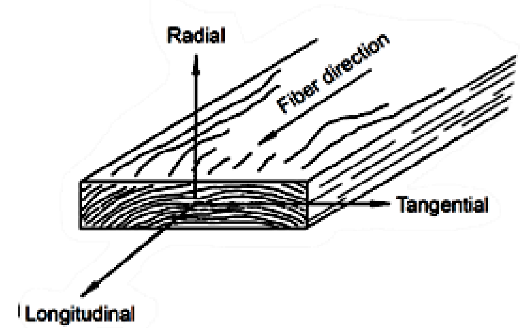


The first index (R) refers to the direction of the applied stress and the second index (T), to the direction of the lateral orthogonal direction of the induced deformation

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \end{bmatrix} = \begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & -\nu_{31}/E_3 & 0 & 0 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & -\nu_{32}/E_3 & 0 & 0 & 0 \\ -\nu_{13}/E_1 & -\nu_{23}/E_2 & 1/E_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_{23} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_{31} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix}$$

$$\boldsymbol{\varepsilon} = \mathbf{C}^{-1}\boldsymbol{\sigma}$$

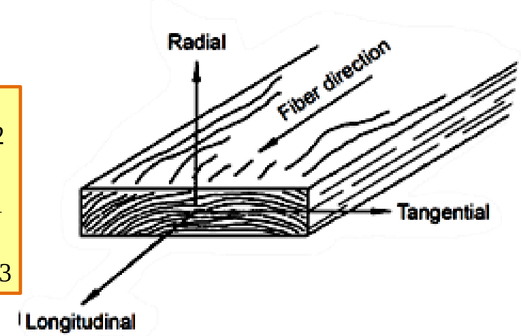
$$\nu_{12}/E_1 = \nu_{21}/E_2$$
$$\nu_{31}/E_3 = \nu_{13}/E_1$$
$$\nu_{23}/E_2 = \nu_{32}/E_3$$

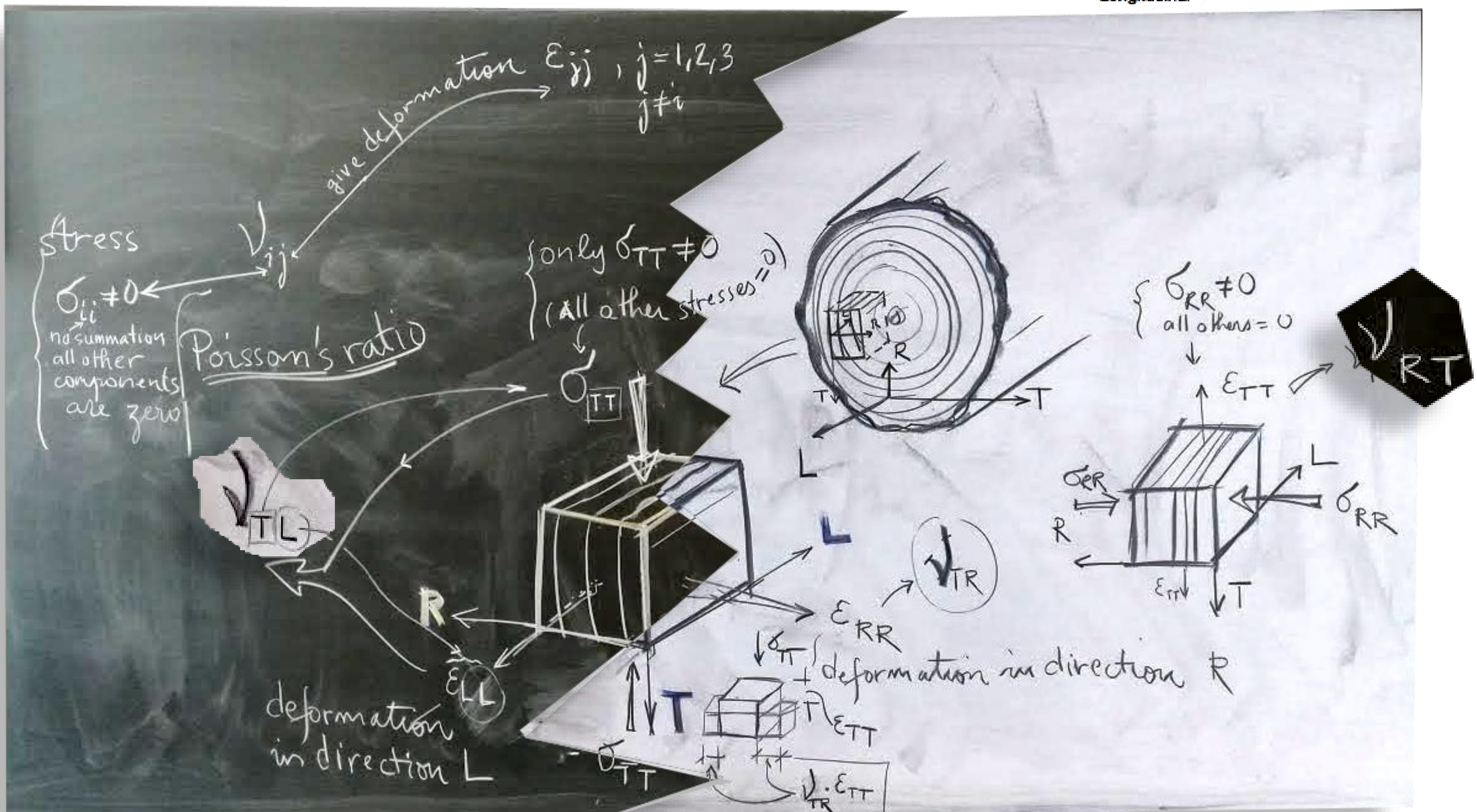


Material orthotropy: three mutually orthogonal symmetry plans

$$
\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \end{bmatrix}
=
\begin{bmatrix}
1/E_1 & -\nu_{21}/E_2 & -\nu_{31}/E_3 & 0 & 0 & 0 \\
-\nu_{12}/E_1 & 1/E_2 & -\nu_{32}/E_3 & 0 & 0 & 0 \\
-\nu_{13}/E_1 & -\nu_{23}/E_2 & 1/E_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 1/G_{12} & 0 & 0 \\
0 & 0 & 0 & 0 & 1/G_{23} & 0 \\
0 & 0 & 0 & 0 & 0 & 1/G_{31}
\end{bmatrix}
\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix}
$$

$$\nu_{12}/E_1 = \nu_{21}/E_2$$
$$\nu_{31}/E_3 = \nu_{13}/E_1$$
$$\nu_{23}/E_2 = \nu_{32}/E_3$$





Poisson's ratios - Notation

Inverting the **flexibility** matrix gives:

Orthotropy

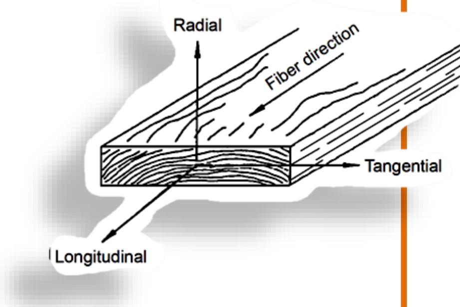
$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$$

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} \frac{1-\nu_{23}\nu_{32}}{E_2 E_3 \Delta} & \frac{\nu_{21}+\nu_{23}\nu_{31}}{E_2 E_3 \Delta} & \frac{\nu_{31}+\nu_{21}\nu_{32}}{E_2 E_3 \Delta} & 0 & 0 & 0 \\ & \frac{1-\nu_{13}\nu_{31}}{E_1 E_3 \Delta} & \frac{\nu_{32}+\nu_{12}\nu_{31}}{E_1 E_3 \Delta} & 0 & 0 & 0 \\ & & \frac{1-\nu_{12}\nu_{21}}{E_1 E_2 \Delta} & 0 & 0 & 0 \\ & & & G_{23} & 0 & 0 \\ & \text{symmetric} & & & G_{13} & 0 \\ & & & & & G_{12} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix}$$

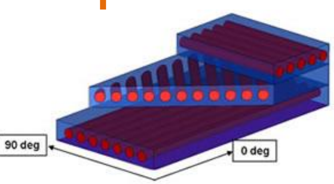
$$\Delta = \frac{\left(1-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}-\nu_{13}\nu_{31}-2\,\nu_{21}\nu_{32}\nu_{13}\right)}{\left(E_1 E_2 E_3\right)}$$

Reciprocal relations: $\frac{\nu_{21}}{E_2} = \frac{\nu_{12}}{E_1}, \quad \frac{\nu_{31}}{E_3} = \frac{\nu_{13}}{E_1}, \quad \frac{\nu_{23}}{E_2} = \frac{\nu_{32}}{E_3}$

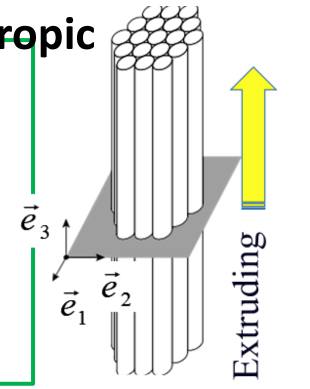
$$\frac{E_i}{E_j} = \frac{\nu_{ij}}{\nu_{ji}}$$



1, 2, 3 are the material principal directions of orthotropy



Transversely isotropic



To obtain: Do the following substitutions in the stiffness matrix $\mathbf{D}$ above

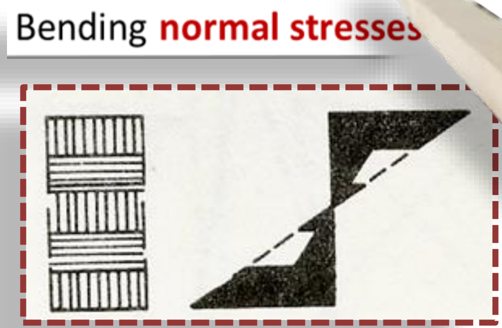
Transversely isotropic material: $E_2 = E_3,\ \nu_{12} = \nu_{13},\ G_{12} = G_{13}$ and $G_{23} = \dfrac{E_2}{2(1+\nu_{23})}$

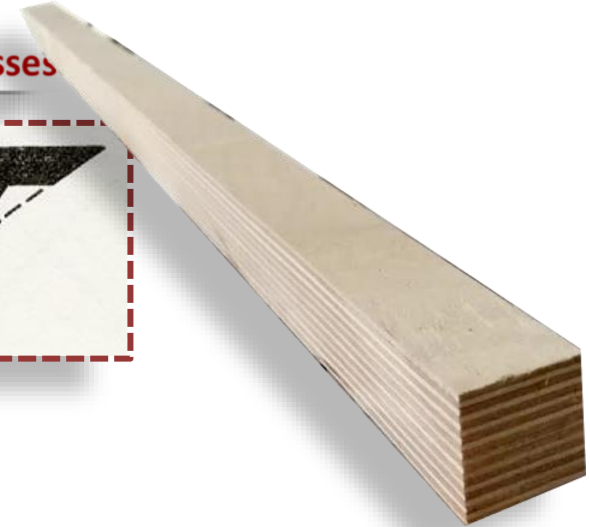
Isotropic material: $E = E_1 = E_2, \nu = \nu_{12} = \nu_{13} = \nu_{23}, G = G_{12} = G_{13}$ and $G = \dfrac{E}{2(1+\nu)}$

Homogenization

Laminated composite Beams

Determine the Effective Bending Rigidity of the composite beam or laminate





Why?

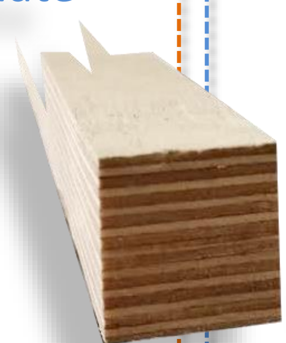
In order to use the conventional *beam theory* to do analysis by hand before feeding the FEA-software with a much more complex model ...and obtain a complex output, too

Sectional forces and strains and a stress distribution in laminates

Homogenization

Determine the Effective Bending Rigidity of the composite beam or laminate

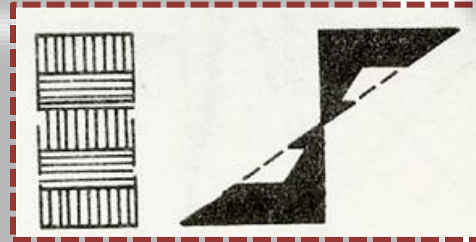
$$U(\boldsymbol{\varepsilon}) = \frac{1}{2}\int_V \boldsymbol{\sigma} : \boldsymbol{\varepsilon} \, \mathrm{d}V$$

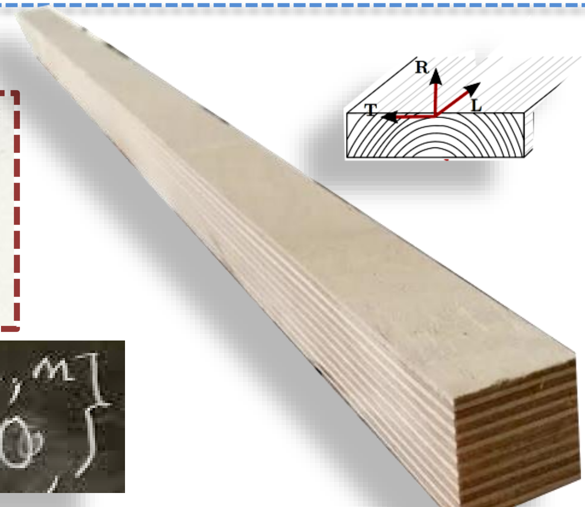


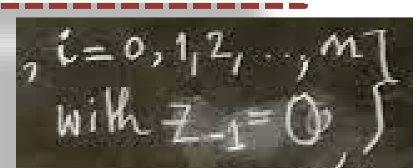
ἐλαστός

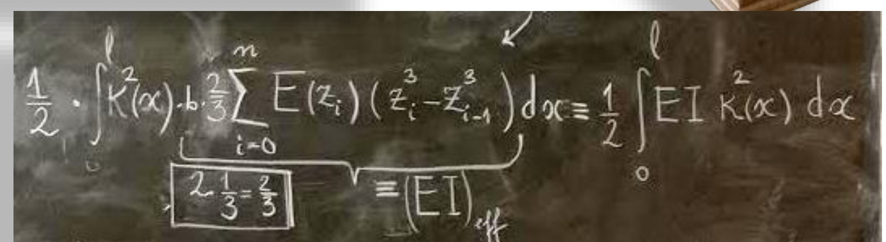
Elasticity

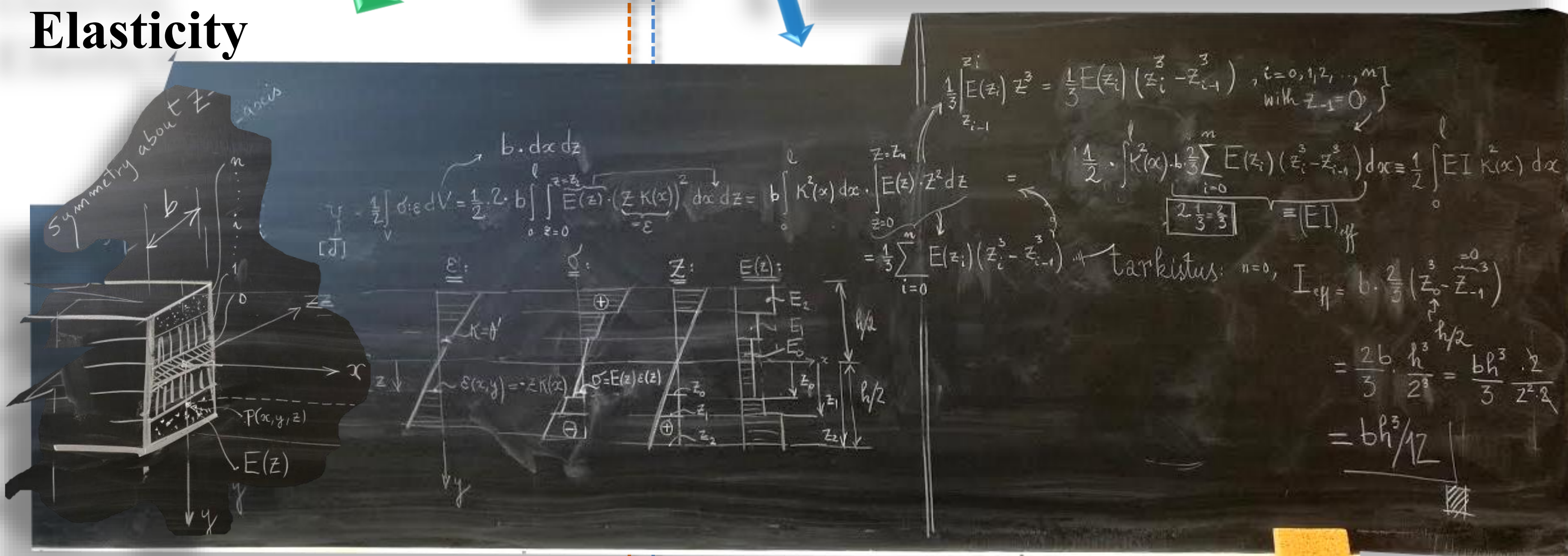
Bending **normal stresses**











Effective Bending Rigidity

$$U(\boldsymbol{\varepsilon}) = \frac{1}{2}\int_V \boldsymbol{\sigma} : \boldsymbol{\varepsilon}\,\mathrm{d}V$$

$$\frac{1}{2}\cdot\int_0^l K^2(x)\cdot \underbrace{b\cdot\frac{2}{3}\sum_{i=0}^{m} E(z_i)\left(z_i^3 - z_{i-1}^3\right)}_{\equiv (EI)_{eff}}\,dx \equiv \frac{1}{2}\int_0^l EI\,K^2(x)\,dx \qquad \boxed{2\cdot\frac{1}{3}=\frac{2}{3}}$$

$$, \; i = 0, 1, 2, \ldots, m, \text{ with } z_{-1} = 0$$

2.

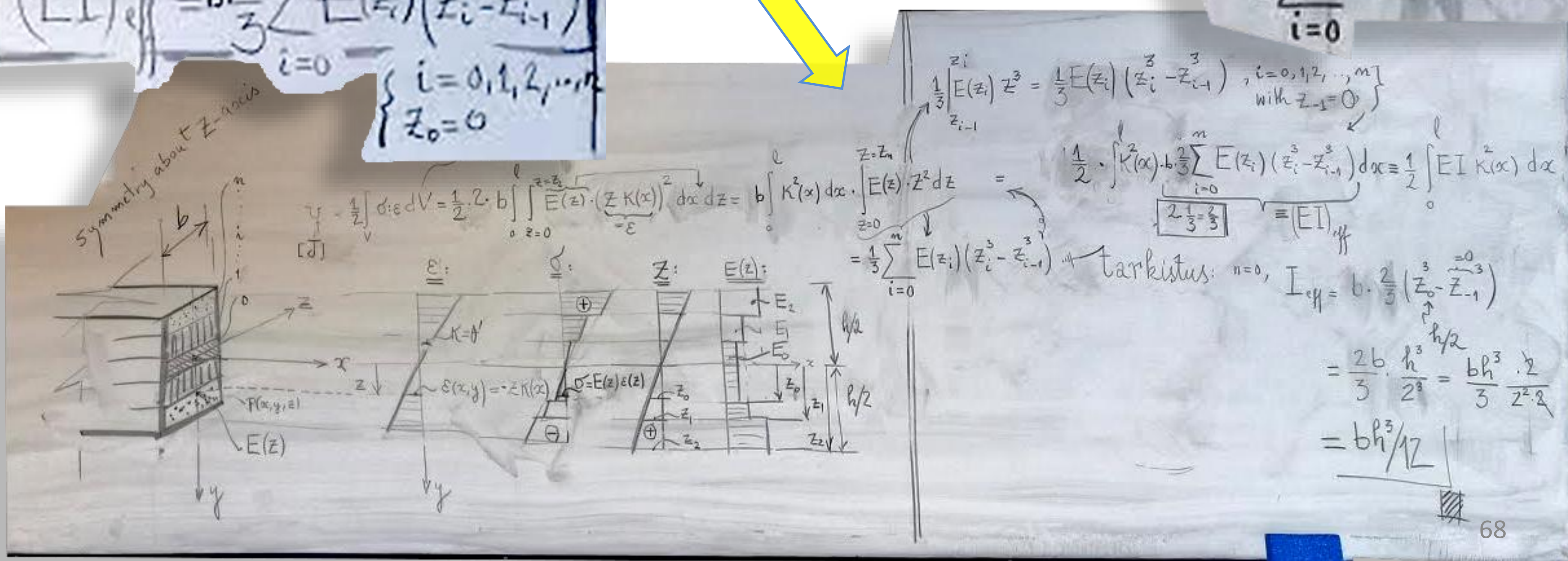
$$U(\boldsymbol{\varepsilon}) = \frac{1}{2}\int_V \boldsymbol{\sigma} : \boldsymbol{\varepsilon}\,\mathrm{d}V = \frac{1}{2}\cdot 2\cdot b\int_0^l\int_{z=0}^{z=z_n} E(z)\cdot(\underbrace{z\,K(x)}_{=\varepsilon})^2\,\underbrace{dx\,dz}_{b\cdot dx\,dz} = b\int_0^l K^2(x)\,dx\cdot\int_{z=0}^{z=z_n} E(z)\cdot z^2\,dz$$

$$= \frac{1}{3}\sum_{i=0}^{m} E(z_i)\left(z_i^3 - z_{i-1}^3\right)$$

3.

$$(EI)_{eff} = b\cdot\frac{2}{3}\sum_{i=0}^{m} E(z_i)\left(z_i^3 - z_{i-1}^3\right) \qquad \begin{cases} i = 0, 1, 2, \ldots, m \\ z_0 = 0 \end{cases}$$

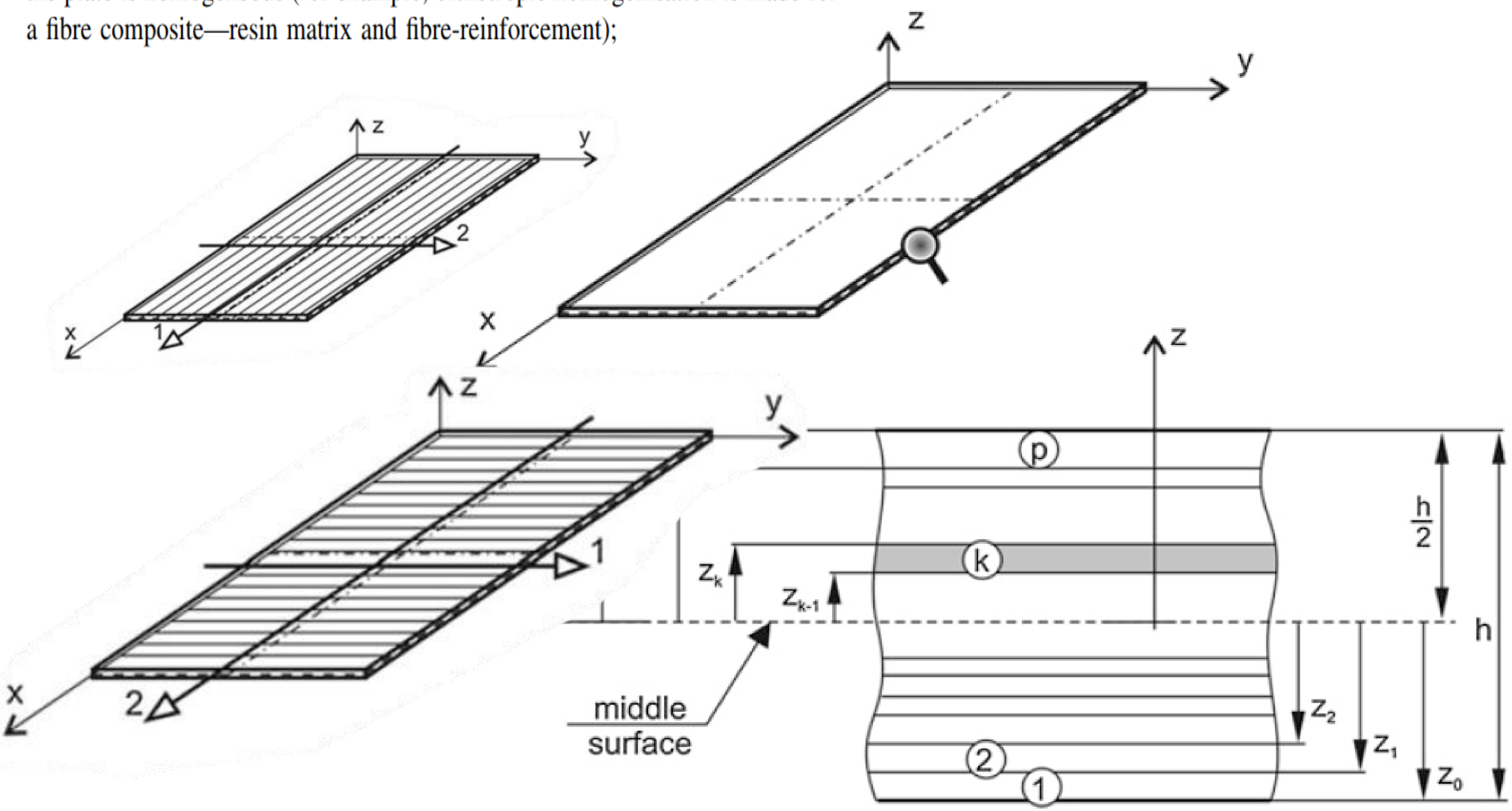
1.



Laminated composite thin Plates

Constitutive Equations for Laminates

- the plate is homogeneous (for example, orthotropic homogenisation is made for a fibre composite—resin matrix and fibre-reinforcement);



Assumed coordinate system for the layered plate

T. Kubiak, *Static and Dynamic Buckling of Thin-Walled Plate Structures*,
DOI: 10.1007/978-3-319-00654-3_2,

Laminated composite thin Plates

Constitutive Equations for Laminates

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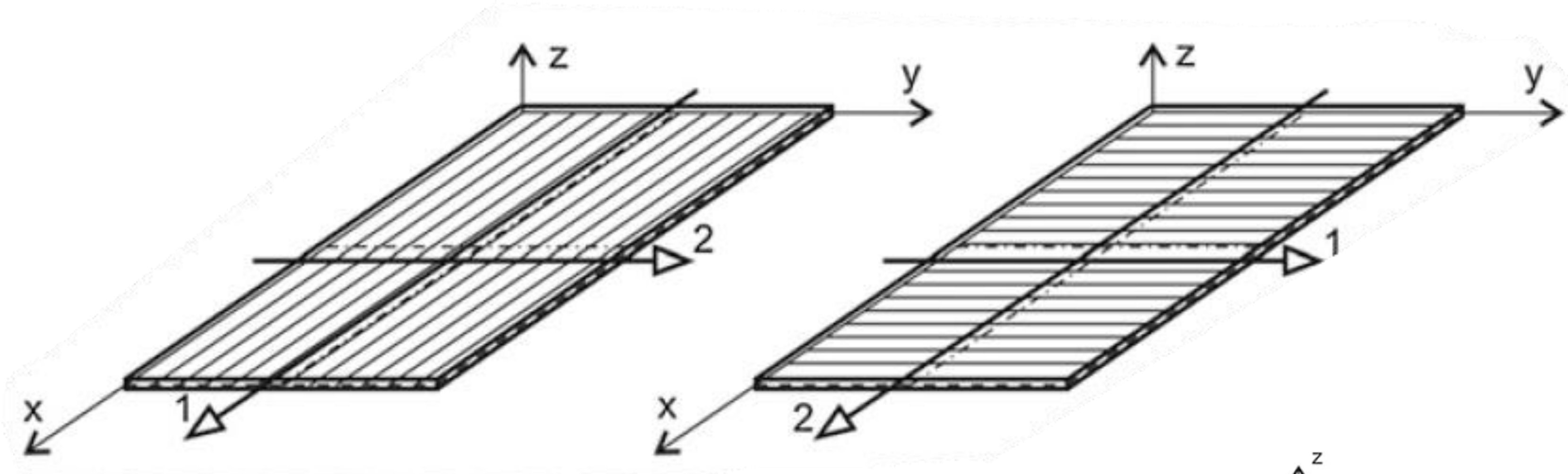
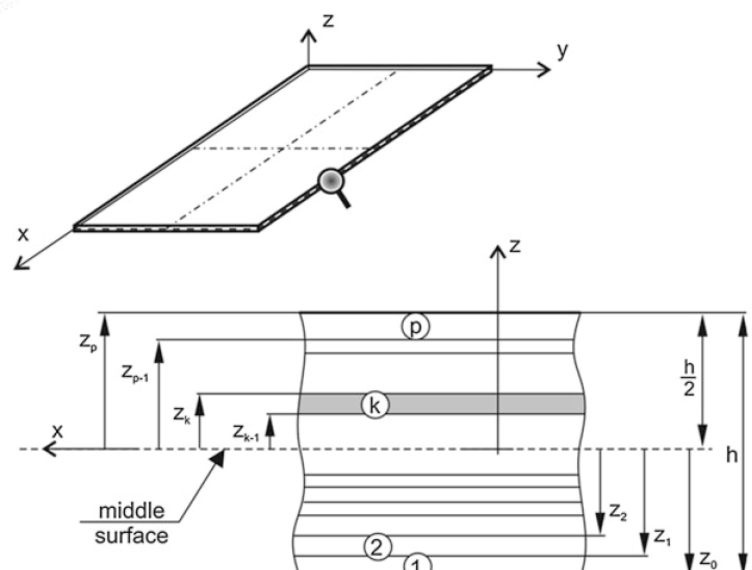


Principal axes of orthotropy for lamina



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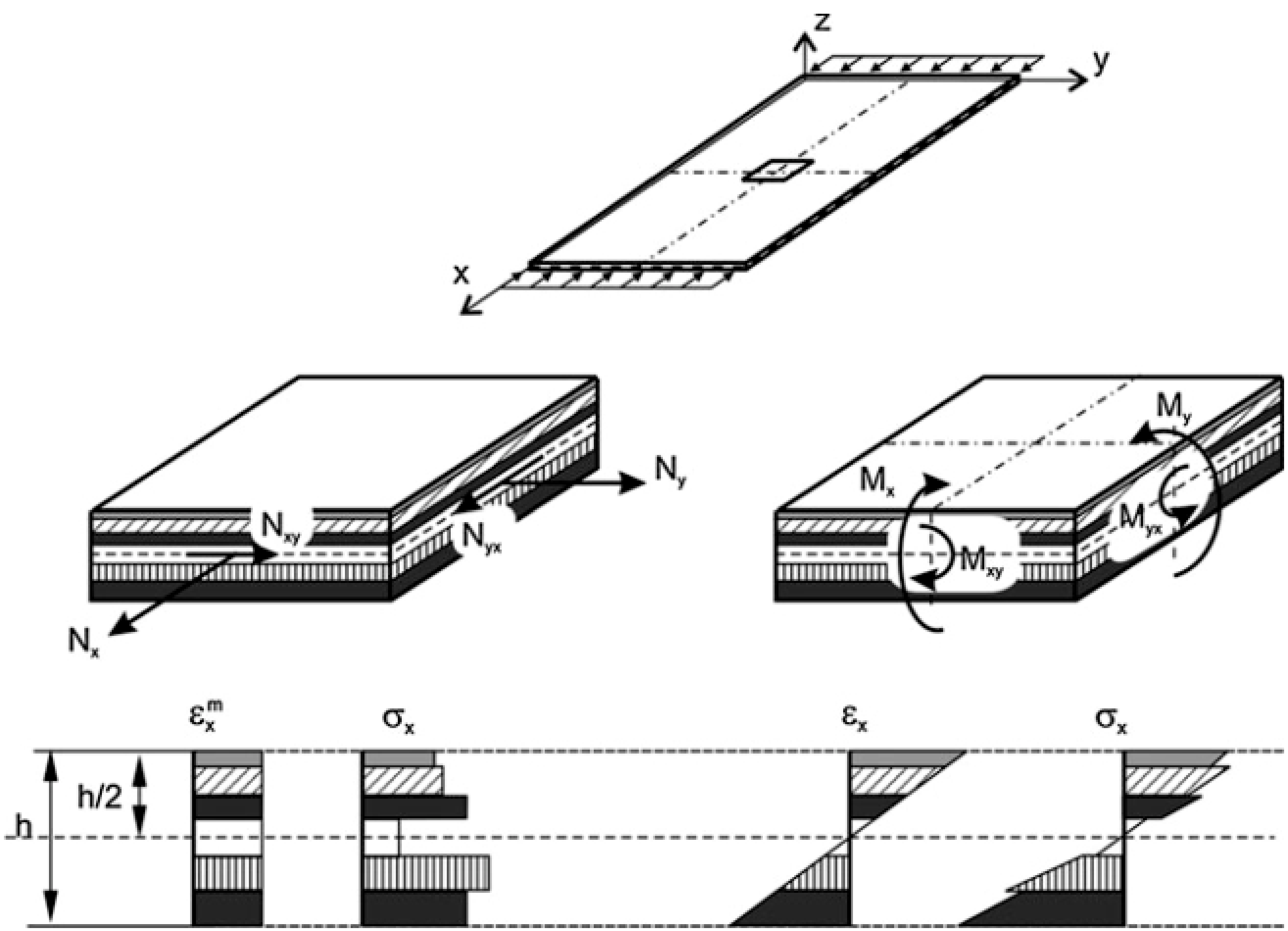
Laminated composite Plates

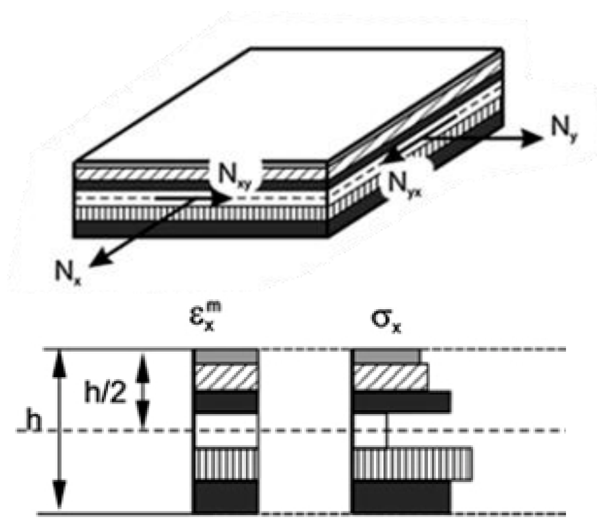
A short pdf can be found in MyCourses

Read from

T. Kubiak, *Static and Dynamic Buckling of Thin-Walled Plate Structures*,
DOI: 10.1007/978-3-319-00654-3_2,



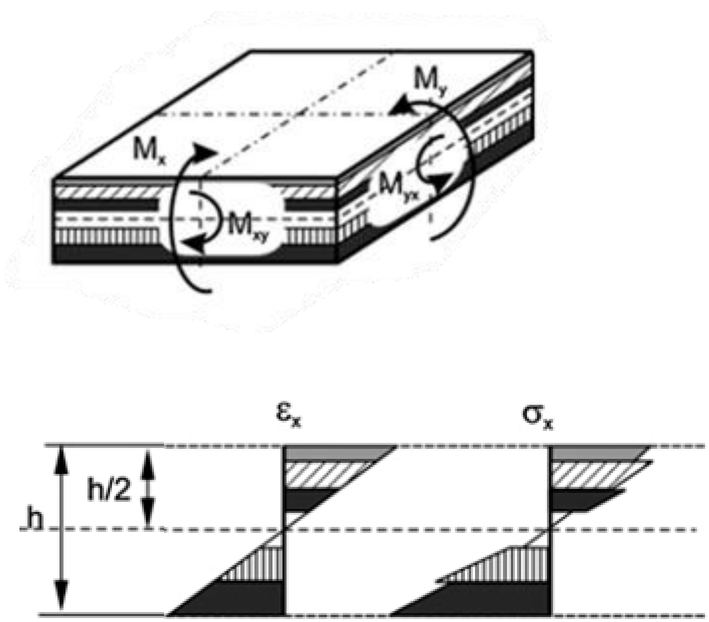
Sectional forces and strains and a stress distribution in laminates



Sectional forces and strains and a stress distribution in laminates

- for the orthotropic plate with the principal axes of orthotropy parallel to the plate (strip) edges:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{E_x}{1-\nu_{xy}\nu_{yx}} & \nu_{yx}\frac{E_x}{1-\nu_{xy}\nu_{yx}} & 0 \\ \nu_{xy}\frac{E_y}{1-\nu_{xy}\nu_{yx}} & \frac{E_y}{1-\nu_{xy}\nu_{yx}} & 0 \\ 0 & 0 & G_{xy} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix},$$



Sectional forces and strains and a stress distribution in laminates

- for the isotropic plate (wall of beam-columns):

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}.$$

In terms of stresses

Generalized Sectional Forces

for the i-th isotropic strip or wall of the beam-column are expressed by:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \frac{Eh}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^m \\ \varepsilon_y^m \\ \gamma_{xy}^m \end{Bmatrix},$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix},$$

where: $D = \frac{Eh^3}{12(1-\nu^2)}$

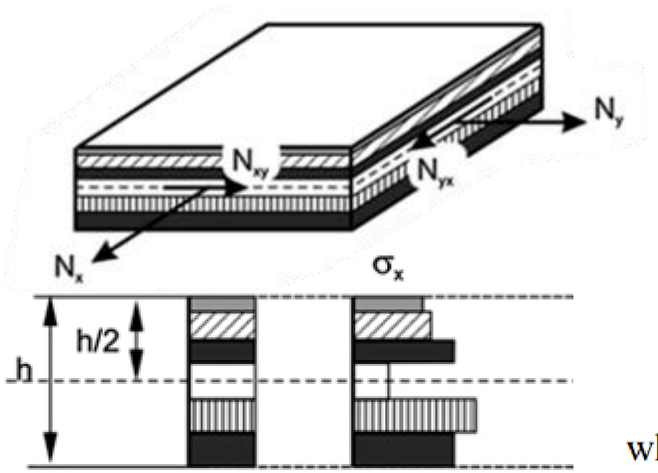
- for the i-th orthotropic strip or wall, they are:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \frac{h}{1-\nu_{xy}\nu_{yx}} \begin{bmatrix} E_x & \nu_{yx}E_x & 0 \\ \nu_{xy}E_y & E_y & 0 \\ 0 & 0 & (1-\nu_{xy}\nu_{yx})G_{xy} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^m \\ \varepsilon_y^m \\ \gamma_{xy}^m \end{Bmatrix},$$

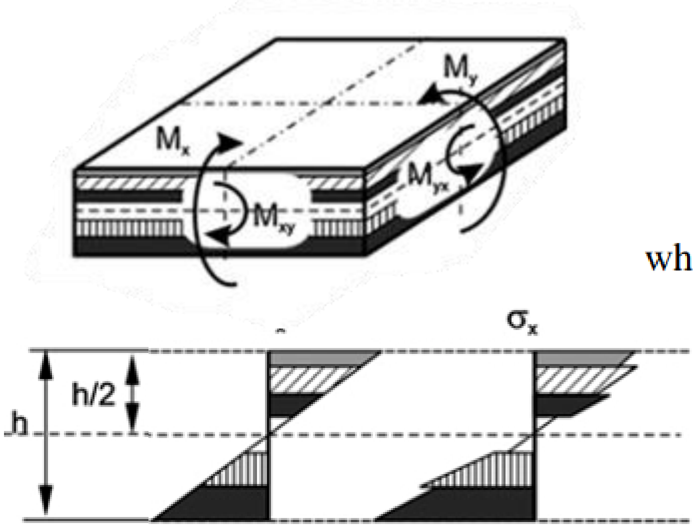
$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_x & \nu_{yx}D_x & 0 \\ \nu_{xy}D_y & D_y & 0 \\ 0 & 0 & D_{xy} \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix},$$

where: $D_x = \frac{E_x h^3}{12\left(1-\nu_{xy}\nu_{yx}\right)}$, $D_y = \frac{E_y h^3}{12\left(1-\nu_{xy}\nu_{yx}\right)}$, $D_{xy} = \frac{G_{xy}h^3}{6}$.

In terms of stress resultants



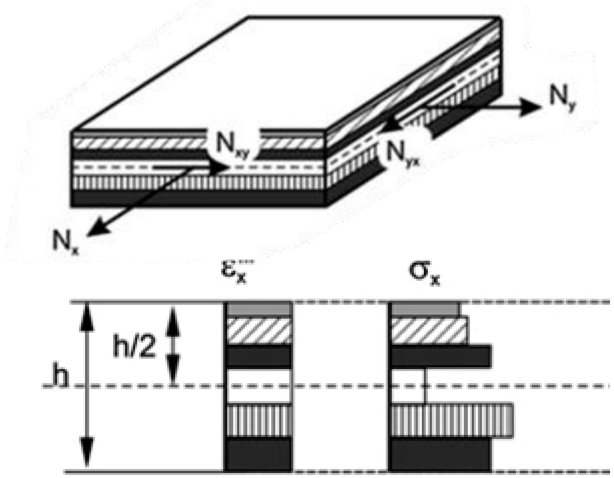
Sectional forces and strains and a stress distribution in laminates



Sectional forces and strains and a stress distribution in laminates

Constitutive Equations for Laminates

This is the final result of homogenisation

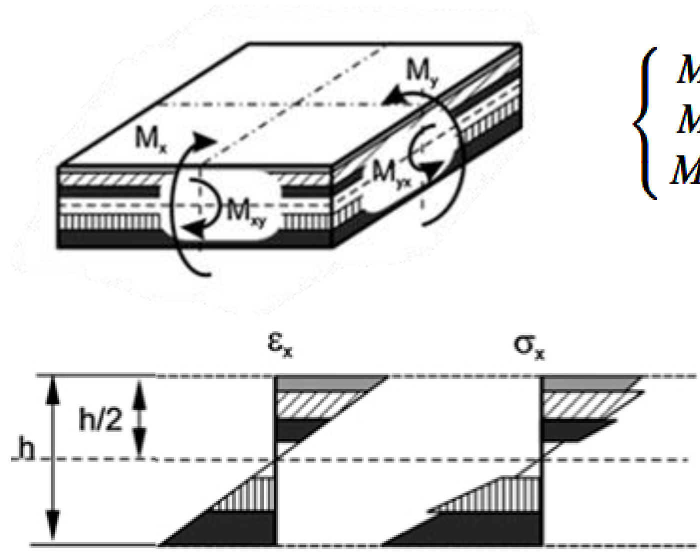


Sectional forces and strains and a stress distribution in laminates

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \sum_{k=1}^{p} \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k dz$$

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{21} & A_{22} & A_{26} & B_{21} & B_{22} & B_{26} \\ A_{61} & A_{62} & A_{66} & B_{61} & B_{62} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{21} & B_{22} & B_{26} & D_{21} & D_{22} & D_{26} \\ B_{61} & B_{62} & B_{66} & D_{61} & D_{62} & D_{66} \end{bmatrix}_i \begin{Bmatrix} \varepsilon_x^m \\ \varepsilon_y^m \\ \gamma_{xy}^m \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix},$$

$$\begin{Bmatrix} \{N\} \\ \{M\} \end{Bmatrix} = \begin{bmatrix} [A] & [B] \\ [B] & [D] \end{bmatrix} \begin{Bmatrix} \{\varepsilon^m\} \\ \{\kappa\} \end{Bmatrix},$$



Sectional forces and strains and a stress distribution in laminates

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \sum_{k=1}^{p} \int_{z_{k-1}}^{z_k} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k z\,dz$$

Qij := Elasticity coefficients

$$A_{pq} = \sum_{k=1}^{n} \left(\overline{Q}_{pq}\right)_k (z_k - z_{k-1}),$$

$$B_{pq} = \frac{1}{2} \sum_{k=1}^{n} \left(\overline{Q}_{pq}\right)_k (z_k^2 - z_{k-1}^2),$$

$$D_{pq} = \frac{1}{3} \sum_{k=1}^{n} \left(\overline{Q}_{pq}\right)_k (z_k^3 - z_{k-1}^3),$$

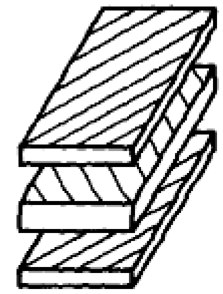
$$A_{pq} = A_{qp},\ B_{pq} = B_{qp},\ D_{pq} = D_{qp}.$$

Laminated composite Plates

The aim is to understand

✓ mechanics of fiber-reinforced laminated plates

- ❑ leading to a computational scheme that relates the in-plane **strain** and **curvature** of a laminate
- ❑ to the **resultant tractions** and **bending moments** imposed on it



A 3-ply symmetric laminate.

LAMINATED COMPOSITE PLATES

David Roylance
Department of Materials Science and Engineering
Massachusetts Institute of Technology
Cambridge, MA 02139

February 10, 2000

Refer to this pdf-reading material in MyCourses

Strain and curvature of a laminate:

$$\boldsymbol{\epsilon} = \left\{ \begin{matrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{matrix} \right\} = \left\{ \begin{matrix} u_{0,x} - z\, w_{0,xx} \\ v_{0,y} - z\, w_{0,yy} \\ (u_{0,y} + v_{0,x}) - 2z\, w_{0,xy} \end{matrix} \right\} = \boldsymbol{\epsilon}^0 + z\,\boldsymbol{\kappa}$$

$$\boldsymbol{\sigma} = \overline{\mathbf{D}}\boldsymbol{\epsilon} = \overline{\mathbf{D}}\boldsymbol{\epsilon}^0 + z\overline{\mathbf{D}}\boldsymbol{\kappa}$$

Stress resultants:

$$\mathbf{N} = \left\{ \begin{matrix} N_x \\ N_y \\ N_{xy} \end{matrix} \right\} \qquad \mathbf{M} = \left\{ \begin{matrix} M_x \\ M_y \\ M_{xy} \end{matrix} \right\}$$

$$\mathbf{N} = \int_{-h/2}^{+h/2} \boldsymbol{\sigma}\, dz = \sum_{k=1}^{N} \left(\overline{\mathbf{D}}\boldsymbol{\epsilon}^0 \int_{z_k}^{z_{k+1}} dz + \overline{\mathbf{D}}\boldsymbol{\kappa} \int_{z_k}^{z_{k+1}} z\, dz \right)$$

$$\mathbf{M} = \int_{-h/2}^{+h/2} \boldsymbol{\sigma} z\, dz = \mathcal{B}\boldsymbol{\epsilon}^0 + \mathcal{D}\boldsymbol{\kappa}$$

extensional stiffness matrix:

$$\mathcal{A} = \sum_{k=1}^{N} \overline{\mathbf{D}}(z_{k+1} - z_k)$$

coupling matrix:

$$\mathcal{B} = \frac{1}{2} \sum_{k=1}^{N} \overline{\mathbf{D}}(z_{k+1}^2 - z_k^2)$$

bending stiffness matrix:

$$\mathcal{D} = \frac{1}{3} \sum_{k=1}^{N} \overline{\mathbf{D}}(z_{k+1}^3 - z_k^3)$$

laminate stiffness matrix:

$$\left\{ \begin{matrix} \mathbf{N} \\ \mathbf{M} \end{matrix} \right\} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B} & \mathcal{D} \end{bmatrix} \left\{ \begin{matrix} \boldsymbol{\epsilon}^0 \\ \boldsymbol{\kappa} \end{matrix} \right\}$$

There will be a homework on this topic where you will derive and use such formulas

Energy based Homogenisation

HW-2

The strain energy

$$U = \frac{1}{2}\iiint_V \boldsymbol{\sigma}:\boldsymbol{\varepsilon}\, dV$$

Kirchhoff-Love plate theory

Multi-layred composite plate. Each layer (lamina) is orthotropic. **Determine effective rigidities of an equivalent orthotropic plate**

$$U = \frac{1}{2}\iint_A \left[D_x\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2D_{xy}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + D_y\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 4D_s\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 \right] dA.$$

Strain energy of orthotropic plate

In the composite case, planar stress-strain relationship in each layer is given as

$$\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{pmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\upsilon_{12}}{E_2} & 0 \\ -\frac{\upsilon_{21}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{pmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{pmatrix}$$

1 and 2 are the **principle** orthotropy **material directions for** a lamina (=layer)

inverting

$$\begin{pmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{pmatrix} = \begin{bmatrix} \frac{E_1}{1-\upsilon_{12}\upsilon_{21}} & \frac{\upsilon_{21}E_1}{1-\upsilon_{12}\upsilon_{21}} & 0 \\ \frac{\upsilon_{12}E_2}{1-\upsilon_{12}\upsilon_{21}} & \frac{E_2}{1-\upsilon_{12}\upsilon_{21}} & 0 \\ 0 & 0 & G_{12} \end{bmatrix} \begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{pmatrix}$$

$$\begin{pmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{pmatrix}(z) = -\begin{bmatrix} \frac{E_1}{1-\upsilon_{12}\upsilon_{21}} & \frac{\upsilon_{21}E_1}{1-\upsilon_{12}\upsilon_{21}} & 0 \\ \frac{\upsilon_{12}E_2}{1-\upsilon_{12}\upsilon_{21}} & \frac{E_2}{1-\upsilon_{12}\upsilon_{21}} & 0 \\ 0 & 0 & G_{12} \end{bmatrix} \begin{pmatrix}\frac{\partial^2 w}{\partial x^2}\\ \frac{\partial^2 w}{\partial y^2}\\ 2\frac{\partial^2 w}{\partial x \partial y}\end{pmatrix} z.$$

linear strain distribution

$$U = \frac{1}{2}\iiint_V \left[\frac{E_1}{1-\upsilon_{12}\upsilon_{21}}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \frac{\upsilon_{21}E_1+\upsilon_{12}E_2}{1-\upsilon_{12}\upsilon_{21}}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + \frac{E_2}{1-\upsilon_{12}\upsilon_{21}}\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 4G_{12}\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 \right] z^2 dV$$

steps: assume linear strains in the cross-section, integrate the strain energy layer by layer and identify the rigidities (bending and torsion) by comparing to the strain energy of orthotropic plate (eq. 1)

Energy based Homogenisation

Kirchhoff-Love plate theory

Solution distribute this version

Multi-layred composite plate. Each layer (lamina) is orthotropic. Determine effective rigidities of an equivalent orthotropic plate

The strain energy

$$U = \frac{1}{2}\iiint_V \boldsymbol{\sigma}:\boldsymbol{\varepsilon}\, dV$$

$$U = \frac{1}{2}\iint_A \left[D_x\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2D_{xy}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + D_y\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 4D_s\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 \right] dA.$$

Strain energy of orthotropic plate

$$U = \frac{1}{2}\iiint_V \left[\frac{E_1}{1-\upsilon_{12}\upsilon_{21}}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \frac{\upsilon_{21}E_1+\upsilon_{12}E_2}{1-\upsilon_{12}\upsilon_{21}}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + \frac{E_2}{1-\upsilon_{12}\upsilon_{21}}\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 4G_{12}\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 \right] z^2 dV$$

Comparing the strain energies of the orthotropic plate and composite plate, it may be seen that effective stiffnesses of the composite plate are given as

$$D_x = \frac{1}{3}\sum_{i=1}^{3}\frac{E_{1,i}}{1-\upsilon_{12,i}\quad}(z_i^3 - z_i^3\quad)$$

$$D_y = \frac{1}{3}\sum_{i=1}^{3}\frac{E_{2,i}}{1-\quad\upsilon_{21,i}}(\quad - z_{i-1}^3)$$

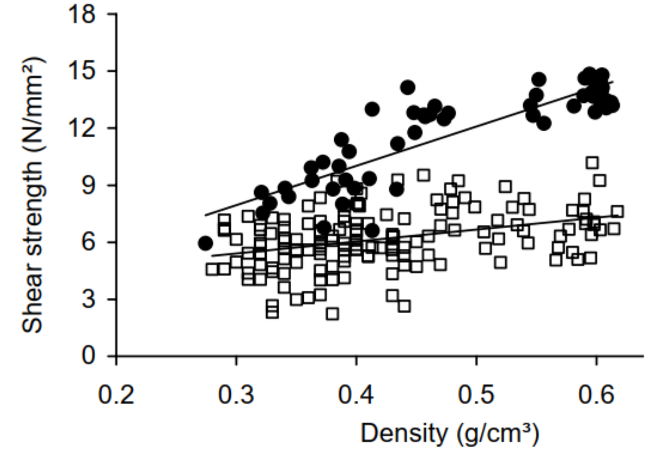
$$D_{xy} = \frac{1}{6}\sum_{i=1}^{3}\frac{\upsilon_{21,i}E\quad + \upsilon_{12,i}E_{2,i}}{1-\upsilon_1\quad\upsilon_{21,i}}(z_i^3 - z_{i-1}^3)$$

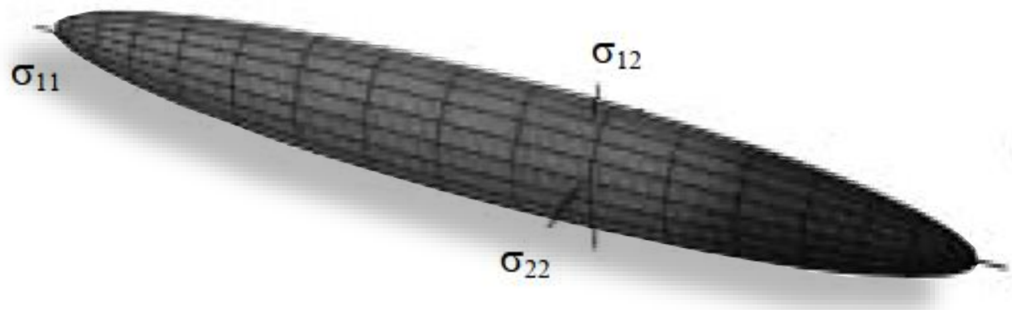
$$D_s = \frac{1}{3}\sum_{i=1}^{3} G_{1\ ,i}(\quad)$$

$$H = D_{xy} + 2D_s.$$

Wood failure criteria / more in lectures on plasticity/

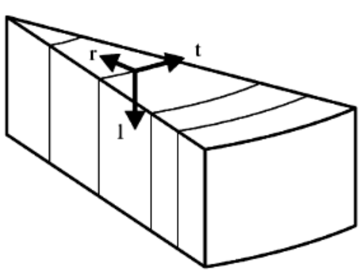
Here, an example of a phenomenological material **failure criterion** model, which is widely used for 'wood'-type orthotropic materials *having different strengths* as wood, for instance





– Fracture surface of the second order of Norway spruce (Eberhardsteiner 2002).

Failure occurs whenever we reach the failure surface



EFFECTS OF WOOD MACRO - AND MICRO-STRUCTURE ON SELECTED MECHANICAL PROPERTIES

Analyse der mechanischen Eigenschaften von Massivholz unter Berücksichtigung der Zellstruktur

Submitted by/eingereicht von:

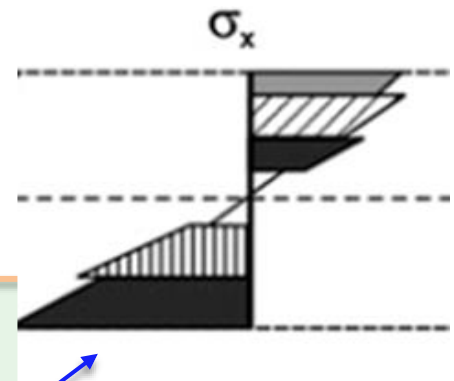
Univ.Ass. Dipl.-Ing.

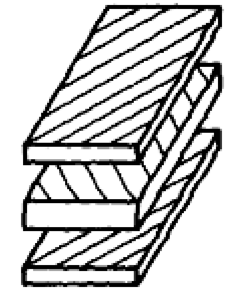
Ulrich Müller

2003

Tsai-Hill failure criterion

The **Tsai–Hill failure criterion** a phenomenological material failure model, which is widely used for anisotropic composite materials having different strengths in tension and compression





A 3-ply symmetric laminate.

https://en.wikipedia.org/wiki/Tsai-Hill_failure_criterion

Assessing the likelihood of a *ply failing:*
Individual *ply stresses* can be used in a commonly used failure criterion (**Tsai-Hill**)

The *Tsai-Hill* criterion

$$\left(\frac{\sigma_1}{\hat{\sigma}_1}\right)^2 - \frac{\sigma_1\sigma_2}{\hat{\sigma}_1^2} + \left(\frac{\sigma_2}{\hat{\sigma}_2}\right)^2 + \left(\frac{\tau_{12}}{\hat{\tau}_{12}}\right)^2 = 1$$

ply **tensile** strength parallel to fibre direction (0 deg.)

ply **tensile** strength transversal (90 deg.) fibre direction

Intra-laminar ply strength
(the in-plane **shear** allowable strength of the ply between the longitudinal and the transversal directions)

- Failure occurs whenever the left-hand-side of the criterion equals or exceeds unity
- The criteria is quite approximative

See: *Composite Airframe Structures.* Third Edition. (Michael Chun-Yung Niu (Author), Michael Niu (Author))

Tsai-Hill failure criterion

The **Tsai–Hill failure criterion** a phenomenological material failure model, which is widely used for anisotropic composite materials having different strengths in tension and compression

Assessing the likelihood of a *ply failing:*
Individual *ply stresses* can be used in a commonly used failure criterion (**Tsai-Hill**)

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$$\left(\frac{\sigma_1}{\hat{\sigma}_1}\right)^2 - \frac{\sigma_1\sigma_2}{\hat{\sigma}_1^2} + \left(\frac{\sigma_2}{\hat{\sigma}_2}\right)^2 + \left(\frac{\tau_{12}}{\hat{\tau}_{12}}\right)^2 = 1$$

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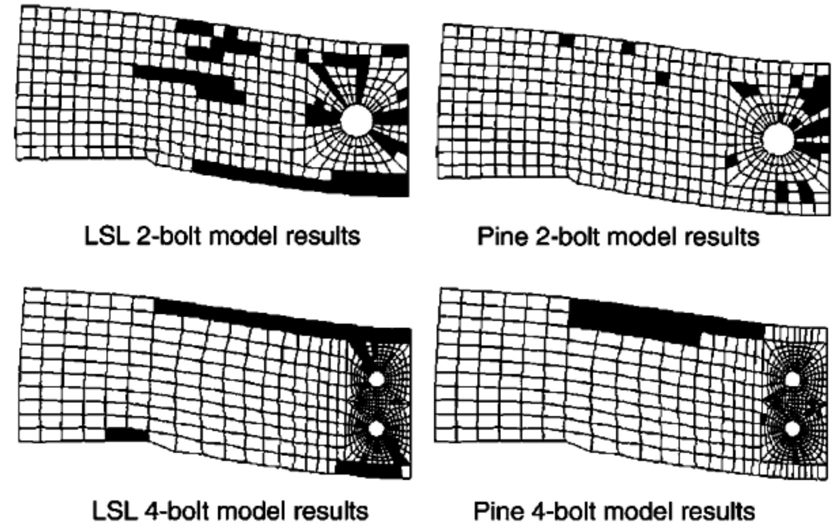
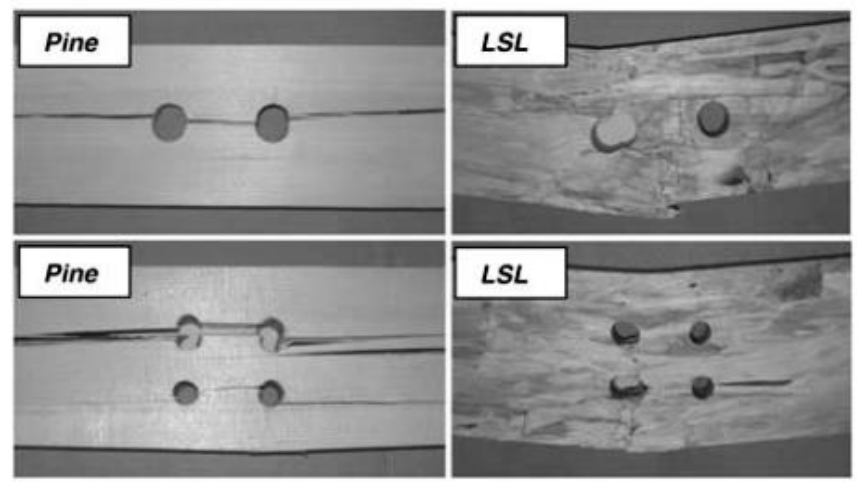


Typical deformed shapes of half-specimen finite-element models used with Tsai-Wu continuum failure theory. LSL, laminated strand lumber.



Failures in multiple bolt connections. LSL, laminated strand lumber.

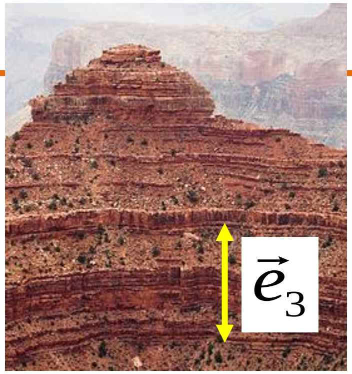
Holzforschung, Vol. 61, pp. 352–359, 2007 • Copyright © by Walter de Gruyter • Berlin • New York. DOI 10.1515/HF.2007.055

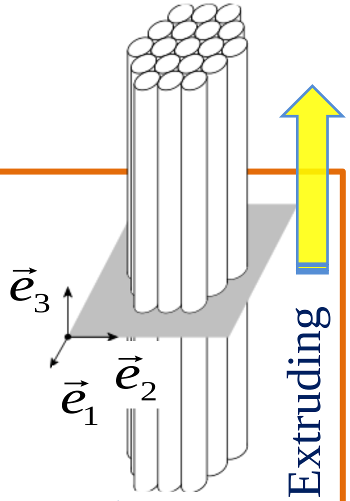
Failure mechanisms in wood-based materials: A review of discrete, continuum, and hybrid finite-element representations

Selected article from the 7th WCCM, Los Angeles, USA, July 16–22, 2006

Transversely Isotropic Material

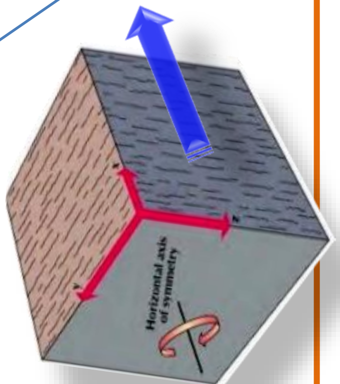
Transversal isotropy means that there is **one** *symmetry plane* (called the *plane of isotropy*)





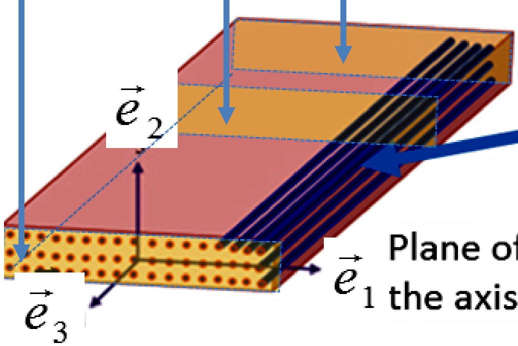
Assume the *axis of transverse isotropy* is $\vec{e}_3$, then the material is 'isotropic' in the plane orthogonal to $\vec{e}_3$ (the *plane of isotropy*)

Means that $\vec{e}_1 - \vec{e}_2$ is the symmetry plane



One symmetry plane

Example: a laminated composite obtained by bonding of layers formed from unidirectional fibers (unidirectional fiber composite lamina)



Transverse isotropy

Plane of isotropy is orthogonal to $\vec{e}_3$ the axis of transverse isotropy

Example: a composite reinforced by unidirectional fibers

Transversely isotropic material

$$\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{12}\\ 2\varepsilon_{23}\\ 2\varepsilon_{31}\end{bmatrix} = \begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & -\nu_{31}/E_3 & 0 & 0 & 0\\ -\nu_{12}/E_1 & 1/E_2 & -\nu_{32}/E_3 & 0 & 0 & 0\\ -\nu_{13}/E_1 & -\nu_{23}/E_2 & 1/E_3 & 0 & 0 & 0\\ 0 & 0 & 0 & 1/G_{12} & 0 & 0\\ 0 & 0 & 0 & 0 & 1/G_{23} & 0\\ 0 & 0 & 0 & 0 & 0 & 1/G_{31}\end{bmatrix}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{12}\\ \sigma_{23}\\ \sigma_{31}\end{bmatrix}$$

orthotropy

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon} \qquad \boldsymbol{\varepsilon} = \mathbf{C}\boldsymbol{\sigma}$$

Here $\vec{e}_3$ is the axis of symmetry.

one symmetry plane $\vec{e}_1 - \vec{e}_2$

$\vec{e}_3$ is the axis of symmetry (= *axis of transverse isotropy*).

$$\begin{Bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \varepsilon_{12}\\ \varepsilon_{23}\\ \varepsilon_{13}\end{Bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & -\frac{\nu_{13}}{E_1} & 0 & 0 & 0\\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_1} & -\frac{\nu_{13}}{E_1} & 0 & 0 & 0\\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{13}}{E_1} & \frac{1}{E_3} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{2G_{12}} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{2G_{13}} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2G_{13}}\end{bmatrix}}_{\mathbf{C} \equiv \mathbf{D}^{-1}} \begin{Bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{12}\\ \sigma_{23}\\ \sigma_{13}\end{Bmatrix}$$

With reference to the orthotropic case, we have the following equalities:

$E_1 = E_2$, same extensional moduli in directions 1 and 2;
$\nu_{13}/E_1 = \nu_{23}/E_2$, same contraction coefficient in directions 1 and 2 for a tension applied in direction 3;
$G_{13} = G_{23}$, same shear moduli for shear in planes perpendicular to axis 2 and axis 1;
$2G_{12} = E_1/(1 + \nu_{12})$, shear modulus around axis 3.

$$\frac{1}{2G_{12}} = \frac{1+\nu_{12}}{E_1}$$

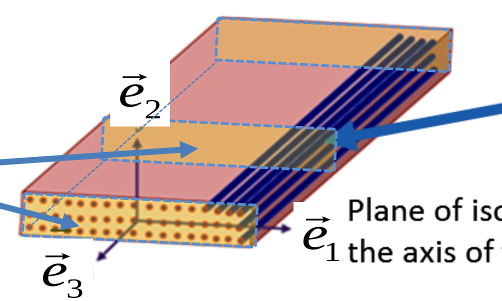
The five coefficients which characterize the material are:

two extensional moduli E_1 and E_3,
one shear modulus G_{13},
two coefficients of contraction ν_{12} and ν_{13}.

Remains only five (5) independent elastic coefficients

The material *stiffness matrix* is obtained by *inverting* the *compliance matrix* $\mathbf{C}$
(**Student**: do it as a homework, use symbolic algebra software)

Transversal isotropy: one symmetry plane $\vec{e}_1 - \vec{e}_2$



Transverse isotropy

Plane of isotropy is orthogonal to $\vec{e}_3$ the axis of transverse isotropy

Application example

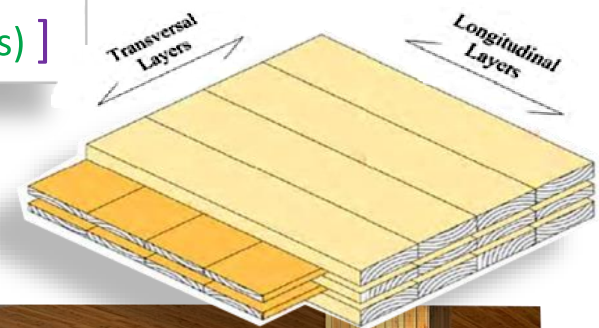
Teacher
Djebar BAROUDI, PhD.

Small project [15 points (obligatory) + 2x15 points (bonus)]

ANALYSIS

Analytical

FEM



Elasticity

Elastic orthotropy

Bending of plates

Reading: Chapter 7.2 from:

I put the necessary ‘equations’ at the end of this assignment

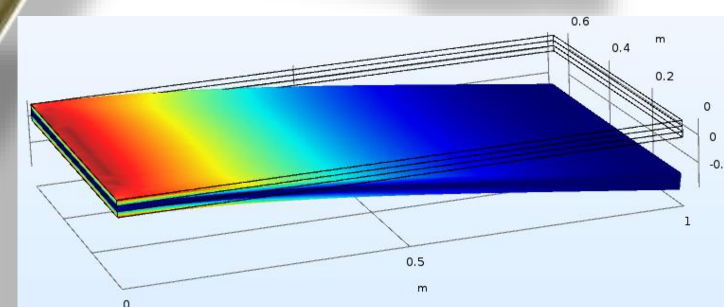
7.2 ORTHOTROPIC AND STIFFENED PLATES

Thin Plates and Shells

Theory, Analysis, and Applications

Eduard Ventsel
Theodor Krauthammer

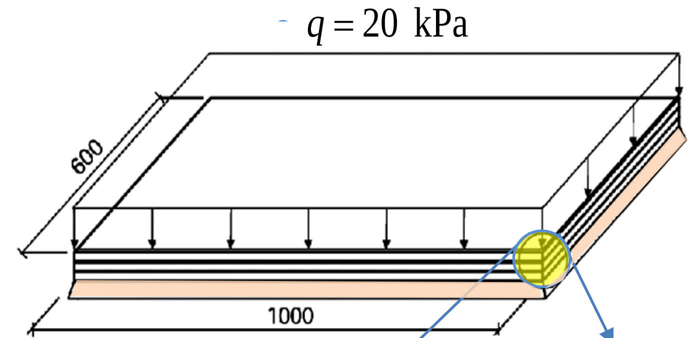
The Pennsylvania State University
University Park, Pennsylvania

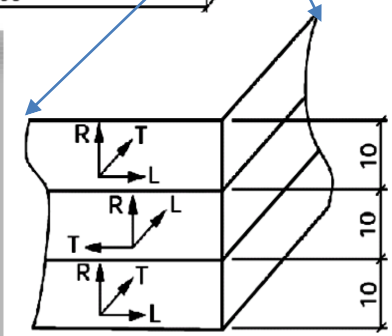


Problem 6: Orthotropic plate or laminate

Physical problem: Consider the laminate plate (Glue Laminated Timber, GLT) formed by three perfectly bonded layers having respective principle material directions L, T and R (*cf*. figure).

The plate is under a transversal loading. The plate can freely rotate along the four support lines (freely supported, hinged on all sides). The individual layers should be modeled as a linear elastic orthotropic material.





L – longitudinal
T – tangential
R – radial

Dimensions are in mm.

Problem [BONUS, elective 15 pnts]: Using Abaqus, Comsol (easy to use) or any other FE-software you need.

1. Determine the displacement along the horizontal center lines
2. The stress distributions at a section at the center and close to the supports.
3. Analyze your results

Problem [obligatory, 15 pnts]: You know well the thin plate theory of orthotropic plates (*cf.* textbooks, the pdf-version in your previous master course of *plate and shells*) and you are a clever and responsible engineer who wants somehow to check his FE-results. For this purpose you want to obtain an analytical (or semi-analytical) solution for comparison with FE-results. Do you have any idea how to proceed? If yes, then how? Do it.
Hint: one can find an *equivalent* one layer orthotropic plate having effective bending and torsional rigidities integrated from the 3-layer plate *in order to conserve strain energy (cf. the mentioned textbook). Other ways toward the solution exist and are allowed.*

Ref: this homework, except the bonus, was adapted from the course:
13-02-0003-vl Werkstoffmechanik
Technische Universität Darmstadt
Lehrende: Prof. Dr.-Ing. Michael Vormwald; Dipl.-Ing. Melanie Fiedler

Eine Platte aus Brettschichtholz wird allseitig gelenkig gelagert und mit eine konstanten Flächenlast q = 0.02 MPa belastet. Die einzelnen Schichten sollen als orthotroper Werkstoff modelliert werden. Die Werkstoffkonstanten in einem L-R-T-Koordinatensystem betragen:

E_L[MPa]	E_R[MPa]	E_T[MPa]	G_{LR}[MPa]	G_{LT}[MPa]	G_{RT}[MPa]	ν_{LT}	ν_{LR}	ν_{RT}
11990	820	420	620	740	240	0.7749	0.6071	0.6031

Some useful tables for wood

Table 4–2. Poisson's ratios for various species at approximately 12% moisture content

Species	μ_{LR}	μ_{LT}	μ_{RT}	μ_{TR}	μ_{RL}	μ_{TL}
Hardwoods						
Ash, white	0.371	0.440	0.684	0.360	0.059	0.051
Aspen, quaking	0.489	0.374	—	0.496	0.054	0.022
Balsa	0.229	0.488	0.665	0.231	0.018	0.009
Basswood	0.364	0.406	0.912	0.346	0.034	0.022
Birch, yellow	0.426	0.451	0.697	0.426	0.043	0.024
Cherry, black	0.392	0.428	0.695	0.282	0.086	0.048
Cottonwood, eastern	0.344	0.420	0.875	0.292	0.043	0.018
Mahogany, African	0.297	0.641	0.604	0.264	0.033	0.032
Mahogany, Honduras	0.314	0.533	0.600	0.326	0.033	0.034
Maple, sugar	0.424	0.476	0.774	0.349	0.065	0.037
Maple, red	0.434	0.509	0.762	0.354	0.063	0.044
Oak, red	0.350	0.448	0.560	0.292	0.064	0.033
Oak, white	0.369	0.428	0.618	0.300	0.074	0.036
Sweet gum	0.325	0.403	0.682	0.309	0.044	0.023
Walnut, black	0.495	0.632	0.718	0.378	0.052	0.035
Yellow-poplar	0.318	0.392	0.703	0.329	0.030	0.019
Softwoods						
Baldcypress	0.338	0.326	0.411	0.356	—	—
Cedar, northern white	0.337	0.340	0.458	0.345	—	—
Cedar, western red	0.378	0.296	0.484	0.403	—	—
Douglas-fir	0.292	0.449	0.390	0.374	0.036	0.029
Fir, subalpine	0.341	0.332	0.437	0.336	—	—
Hemlock, western	0.485	0.423	0.442	0.382	—	—
Larch, western	0.355	0.276	0.389	0.352	—	—
Pine						
Loblolly	0.328	0.292	0.382	0.362	—	—
Lodgepole	0.316	0.347	0.469	0.381	—	—
Longleaf	0.332	0.365	0.384	0.342	—	—
Pond	0.280	0.364	0.389	0.320	—	—
Ponderosa	0.337	0.400	0.426	0.359	—	—
Red	0.347	0.315	0.408	0.308	—	—
Slash	0.392	0.444	0.447	0.387	—	—
Sugar	0.356	0.349	0.428	0.358	—	—
Western white	0.329	0.344	0.410	0.334	—	—
Redwood	0.360	0.346	0.373	0.400	—	—
Spruce, Sitka	0.372	0.467	0.435	0.245	0.040	0.025
Spruce, Engelmann	0.422	0.462	0.530	0.255	0.083	0.058

Chapter 2

Structure of Wood

Regis B. Miller

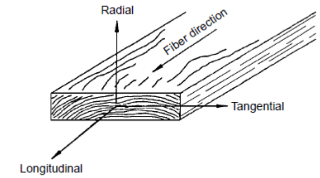


Figure 4–1. Three principal axes of wood with respect to grain direction and growth rings.

Table 4–1. Elastic ratios for various species at approximately 12% moisture content[a]

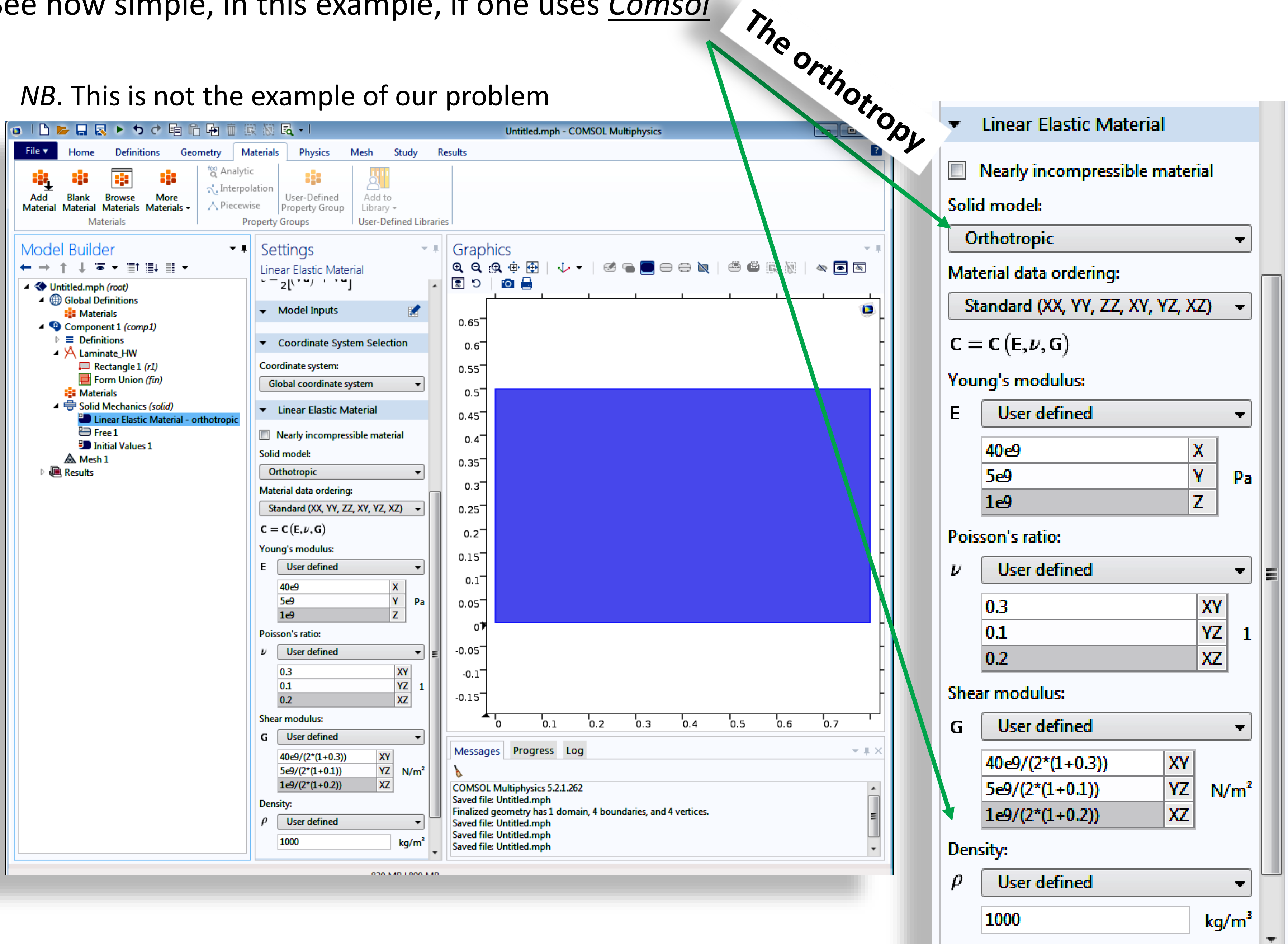
Species	E_T/E_L	E_R/E_L	G_{LR}/E_L	G_{LT}/E_L	G_{RT}/E_L
Hardwoods					
Ash, white	0.080	0.125	0.109	0.077	—
Balsa	0.015	0.046	0.054	0.037	0.005
Basswood	0.027	0.066	0.056	0.046	—
Birch, yellow	0.050	0.078	0.074	0.068	0.017
Cherry, black	0.086	0.197	0.147	0.097	—
Cottonwood, eastern	0.047	0.083	0.076	0.052	—
Mahogany, African	0.050	0.111	0.088	0.059	0.021
Mahogany, Honduras	0.064	0.107	0.066	0.086	0.028
Maple, sugar	0.065	0.132	0.111	0.063	—
Maple, red	0.067	0.140	0.133	0.074	—
Oak, red	0.082	0.154	0.089	0.081	—
Oak, white	0.072	0.163	0.086	—	—
Sweet gum	0.050	0.115	0.089	0.061	0.021
Walnut, black	0.056	0.106	0.085	0.062	0.021
Yellow-poplar	0.043	0.092	0.075	0.069	0.011
Softwoods					
Baldcypress	0.039	0.084	0.063	0.054	0.007
Cedar, northern white	0.081	0.183	0.210	0.187	0.015
Cedar, western red	0.055	0.081	0.087	0.086	0.005
Douglas-fir	0.050	0.068	0.064	0.078	0.007
Fir, subalpine	0.039	0.102	0.070	0.058	0.006
Hemlock, western	0.031	0.058	0.038	0.032	0.003
Larch, western	0.065	0.079	0.063	0.069	0.007
Pine					
Loblolly	0.078	0.113	0.082	0.081	0.013
Lodgepole	0.068	0.102	0.049	0.046	0.005
Longleaf	0.055	0.102	0.071	0.060	0.012
Pond	0.041	0.071	0.050	0.045	0.009
Ponderosa	0.083	0.122	0.138	0.115	0.017
Red	0.044	0.088	0.096	0.081	0.011
Slash	0.045	0.074	0.055	0.053	0.010
Sugar	0.087	0.131	0.124	0.113	0.019
Western white	0.038	0.078	0.052	0.048	0.005
Redwood	0.089	0.087	0.066	0.077	0.011
Spruce, Sitka	0.043	0.078	0.064	0.061	0.003
Spruce, Engelmann	0.059	0.128	0.124	0.120	0.010

[a] E_L may be approximated by increasing modulus of elasticity values in Table 4–3 by 10%.

$$\frac{\mu_{ij}}{E_i} = \frac{\mu_{ji}}{E_j}, \quad i \neq j \quad i,j = L,R,T$$

See how simple, in this example, if one uses <u>*Comsol*</u>

NB. This is not the example of our problem



Multilayered orthotropic lami[...]

Example – A cantilever plate

- Laminate_plate_3_Layers_OK_DBaroudi_2018.mph *(root)*
  - Global Definitions
    - Geometry Parts
      - Part 1 - UPPER
      - Part 2 - MID
      - Part 3 - BOTTOM
    - Materials
  - Component 1 *(comp1)*
    - Definitions
    - Upper-Layer
    - Materials
    - Solid Mechanics -LAMINA ORTHOTROPIC PLATE - 3 LAYERS (
      - Linear Elastic Material 1 - UPPER LAYER
      - Free 1 - stress vector free boudaries
      - Initial Values 1 - (u, v, w) = 0, at t = 0
      - Boundary Load 1 - surface load q = 20 kN/m2
      - Fixed Constraint 1 - u = 0
      - Linear Elastic Material 1 - MID LAYER
      - Linear Elastic Material 1 - BOTTOM LAYER
      - Fixed Constraint - (u, v, w) = 0 [Cantilever]
    - Mesh 1
  - Study 1
    - Step 1: Stationary
    - Solver Configurations
      - Solution 1 *(sol1)*
  - Results
    - Data Sets
    - Derived Values
    - Tables
    - Stress (solid)
      - Surface 1
    - Export
    - Reports

Upper layer

Linear Elastic Material

Nearly incompressible material

Solid model: Orthotropic

Material data ordering: Voigt (XX, YY, ZZ, YZ, XZ, XY)

Young's modulus: E User defined

11990e6	X	Pa
420e6	Y	
820e6	Z	

Poisson's ratio: ν User defined

0.775	XY	1
0.603	YZ	
0.607	XZ	

Shear modulus, Voigt notation: G User defined

240e6	YZ	N/m²
620e6	XZ	
740e6	XY	

Density: ρ User defined

300 kg/m³

Mid layer

Linear Elastic Material

Nearly incompressible material

Solid model: Orthotropic

Material data ordering: Standard (XX, YY, ZZ, XY, YZ, XZ)

Young's modulus: E User defined

420e6	X	Pa
11990e6	Y	
820e6	Z	

Poisson's ratio: ν User defined

0.775	XY	1
0.607	YZ	
0.603	XZ	

Shear modulus: G User defined

740e6	XY	N/m²
620e6	YZ	
240e6	XZ	

Density: ρ User defined

300 kg/m³

Bottom layer

Coordinate system: Global coordinate syst[...]

Linear Elastic Material

Nearly incompressible material

Solid model: Orthotropic

Material data ordering: Voigt (XX, YY, ZZ, YZ, XZ, XY)

Young's modulus: E User defined

11990e6	X	Pa
420e6	Y	
820e6	Z	

Poisson's ratio: ν User defined

0.775	XY	1
0.603	YZ	
0.607	XZ	

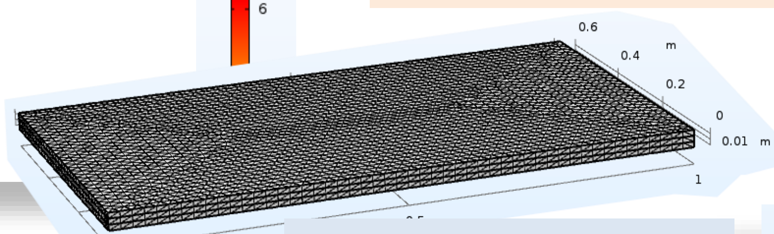
Shear modulus, Voigt notation: G User defined

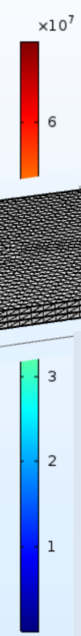
240e6	YZ	N/m²
620e6	XZ	
740e6	XY	

Density: ρ User defined

300 kg/m³

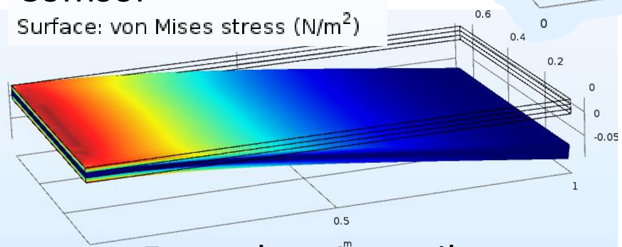
NB, here I did the FEA for a cantilever laminate plate, you should now update to the boundary conditions of the homework.





FEA Example using Comsol

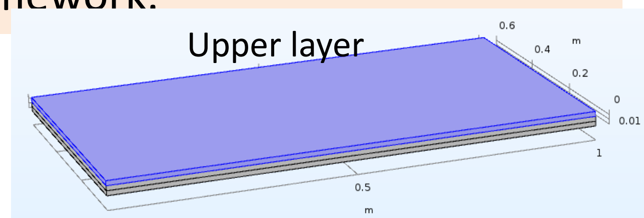
Surface: von Mises stress (N/m²)



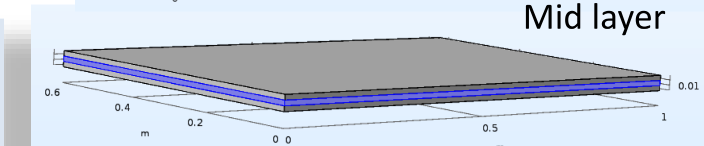
Example – A cantilever

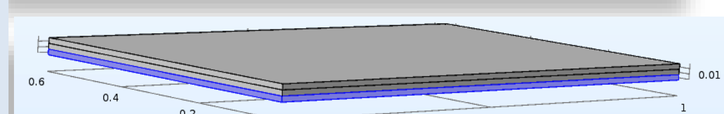
Here in FEA, I used 3D-solids elements. So I used 3D-elasticity and not plate theory. (It was a bit 'impossible' to have, in Comsol, layered plates bounded together! At least for me.)

Upper layer



Mid layer





Bottom layer

must obtain a new set of stress–strain relations that reflects the orthotropic properties of a material of the plate. Such a set of relations is shown below [3]:

$$\varepsilon_x = \frac{\sigma_x}{E_x} - \nu_y\frac{\sigma_y}{E_y}; \quad \varepsilon_y = \frac{\sigma_y}{E_y} - \nu_x\frac{\sigma_x}{E_x}; \quad \gamma_{xy} = \frac{\tau_{xy}}{G}, \tag{7.22}$$

where E_x, E_y, ν_x, ν_y, and G are assumed to be elastic constants of an orthotropic material, i.e., E_x, E_y, and ν_x, ν_y are the moduli of elasticity and Poisson's ratios in the x and y directions, respectively. They are independent of one another. G is the shear modulus, which is the same for both isotropic and orthotropic materials. It can be expressed in terms of E_x and E_y as follows:

$$G \approx \frac{\sqrt{E_x E_y}}{2\left(1 + \sqrt{\nu_x \nu_y}\right)}. \tag{7.23}$$

The following relationship exists between independent elastic constants introduced above:

$$\frac{\nu_x}{E_x} = \frac{\nu_y}{E_y}. \tag{7.24}$$

This equality directly results from Betti's reciprocal theorem. Solving Eqs (7.22) for the stress components and taking into account (7.24), we obtain

$$\sigma_x = \frac{E_x}{1-\nu_x\nu_y}(\varepsilon_x + \nu_y\varepsilon_y); \quad \sigma_y = \frac{E_y}{1-\nu_x\nu_y}(\varepsilon_y + \nu_x\varepsilon_x);$$
$$\tau_{xy} = G\gamma_{xy}. \tag{7.25}$$

The derivation of the governing differential equation of bending of an orthotropic plate is based on the general hypotheses introduced in Sec. 1.3. The strain-deflection relations (2.6) hold for orthotropic plates also. So, substituting the relations (2.6) into Eqs (7.25) gives the following:

$$\sigma_x = -\frac{E_x}{1-\nu_x\nu_y}\left(\frac{\partial^2 w}{\partial x^2} + \nu_y\frac{\partial^2 w}{\partial y^2}\right)z; \quad \sigma_y = -\frac{E_y}{1-\nu_x\nu_y}\left(\frac{\partial^2 w}{\partial y^2} + \nu_x\frac{\partial^2 w}{\partial x^2}\right)z;$$
$$\tau_{xy} = -2Gz\frac{\partial^2 w}{\partial x \partial y}. \tag{7.26}$$

Substituting the above into Eqs (2.11) and integrating over the plate thickness, yields the following bending and twisting moments deflection relations for orthotropic plates:

where D_x, D_y, D_{xy}, D_{yx}, and D_s are the flexural and torsional rigidities of an orthotropic plate, respectively, and are given as

$$D_x = \frac{E_x}{1-\nu_x\nu_y}\frac{h^3}{12}; \quad D_y = \frac{E_y}{1-\nu_x\nu_y}\frac{h^3}{12}; \quad D_{xy} = \frac{E_x\nu_y}{1-\nu_x\nu_y}\frac{h^3}{12};$$
$$D_{yx} = \frac{E_y\nu_x}{1-\nu_x\nu_y}\frac{h^3}{12}; \quad D_s = \frac{Gh^3}{12}. \tag{7.28}$$

In view of the expressions (7.24), one can conclude that $D_{xy} = D_{yx}$. The shear force expressions (2.22) become

$$Q_x = -\frac{\partial}{\partial x}\left(D_x\frac{\partial^2 w}{\partial x^2} + H\frac{\partial^2 w}{\partial y^2}\right); \quad Q_y = -\frac{\partial}{\partial y}\left(H\frac{\partial^2 w}{\partial x^2} + D_y\frac{\partial^2 w}{\partial y^2}\right), \tag{7.29}$$

where

$$H = D_{xy} + 2D_s. \tag{7.30}$$

The governing differential equation (2.24) for orthotropic plates becomes

$$D_x\frac{\partial^4 w}{\partial x^4} + 2H\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y\frac{\partial^4 w}{\partial y^4} = p(x, y). \tag{7.31}$$

We give below the expression for the potential energy of bending for orthotropic plates, which follows from Eqs (2.52) and (7.26):

$$U = \frac{1}{2}\iint_A \left[D_x\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2D_{xy}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + D_y\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 4D_s\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]\mathrm{d}A. \tag{7.32}$$

Thin Plates and Shells

Theory, Analysis, and Applications

Eduard Ventsel
Theodor Krauthammer

The Pennsylvania State University
University Park, Pennsylvania

$$U = \frac{1}{2}\iint_A \left[D_x\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2D_{xy}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + D_y\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 4D_s\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]\mathrm{d}A.$$

Partial solution by DI Jaaranen J. (2017)

b) Analytical modelling of the plate

Given plate can be homogenized to a plate with orthotropic properties by equation strain energies of the actual composite plate with strain energy of the orthotropic homogenous plate. Strain energy of orthotropic plate can be given as

$$U=\frac{1}{2}\iint_A\left[D_x\left(\frac{\partial^2 w}{\partial x^2}\right)^2+2D_{xy}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}+D_y\left(\frac{\partial^2 w}{\partial y^2}\right)^2+4D_s\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right]dA.$$

In the composite case, planar stress-strain relationship in each layer is given as

$$\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{pmatrix}=\begin{bmatrix}\frac{1}{E_1} & -\frac{\upsilon_{12}}{E_2} & 0\\ -\frac{\upsilon_{21}}{E_1} & \frac{1}{E_2} & 0\\ 0 & 0 & \frac{1}{G_{12}}\end{bmatrix}\begin{pmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{pmatrix}$$

The stresses are obtained by inverting the stiffness matrix, givin

$$\begin{pmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{pmatrix}=\begin{bmatrix}\frac{\cdots}{\cdots_{21}} & \frac{\upsilon_{21}E_1}{1-\upsilon_{12}\upsilon_{21}} & 0\\ \frac{\upsilon_{12}E_2}{1-\upsilon_{12}\upsilon_{21}} & \frac{E_2}{1-\upsilon_{12}\upsilon} & \\ 0 & 0 & \end{bmatrix}\begin{pmatrix}\varepsilon_1\\ \cdots\end{pmatrix}$$

Assuming linear strain distribution over the section, the stresses over the section

$$\begin{pmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{pmatrix}(z)=-\begin{bmatrix}\frac{\cdots}{1-\cdots} & \frac{\upsilon_{21}E_1}{1-\upsilon_{12}\upsilon_{21}} & 0\\ \frac{\upsilon_{12}E_{\cdots}}{1-\upsilon_{12}\upsilon_{21}} & \frac{E_2}{1-\upsilon_{12}\cdots} & \\ 0 & 0 & \cdots_2\end{bmatrix}\begin{pmatrix}\frac{\partial^2 w}{\partial x^2}\\ \frac{\partial^2 w}{\cdots}\\ 2\frac{\partial^{\cdots}}{\partial x\cdots}\end{pmatrix}z$$

The strain energy of the plate is given by

$$U=\frac{1}{2}\iiint_V \boldsymbol{\sigma}:\boldsymbol{\varepsilon}\,dV$$

and by substituting strains and stresses gives (in Kirchhoff-Love plate theory)

$$U=\frac{1}{2}\iiint_V\left[\frac{E_1}{1-\upsilon_{12}\upsilon_{21}}\left(\frac{\partial^2 w}{\partial x^2}\right)^2+\frac{\upsilon_{21}E_1}{1-\upsilon_{12}\upsilon_{21}}\cdots\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}+\frac{E_2}{1\cdots}\cdots\right.$$
$$\left.+4G_{12}\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right]z^2dV.$$

Due to symmetry in our case, the neutral plane is located in the mid-plane of the plate. Given that the distances of the layer surfaces from the mid-plane are given z_0, z_1, z_2 and z_3 and noting that the material properties are piecewise contantm one can write strain energy as

$$U=\frac{1}{2}\iint_A\left\{\left[\frac{1}{3}\sum_{i=1}^{3}\frac{E_{1,\cdots}}{1-\upsilon_{12}\cdots}\cdots-z_{i-1}^3)\right]\left(\frac{\partial^2 w}{\partial x^2}\right)^2\right.$$
$$+\left[\frac{1}{3}\sum_{i=1}^{3}\frac{\upsilon_{21,i}E_{1,i}+\upsilon_{12,i}E_{2,i}}{1-\upsilon_{12,i}\upsilon_{21,i}}(z\cdots_{i-1}^3)\right]\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}$$
$$+\left[\frac{1}{3}\sum_{i=1}^{3}\frac{E_{2,i}}{1-\upsilon_{12,i}\upsilon_{21,i}}(z_i^3-z_{i-1}^3)\right]\left(\frac{\partial^2 w}{\partial y^2}\right)^2$$
$$+4\left[\frac{1}{3}\sum_{i=1}^{3}G_{12,i}\,(z_i^3\cdots z_{i-}^3\cdots$$

Comparing the strain energies of the orthotropic plate and composite p… that effective stiffnesses of the composite plate are given as

$$D_x=\frac{1}{3}\sum_{i=1}^{3}\frac{E_{1,i}}{1-\upsilon_{12,i}\upsilon_{21,i}}(z_i^3-z_{i-1}^3)$$

$$D_{xy}=\frac{1}{6}\sum_{i=1}^{3}\frac{\upsilon_{21,i}\cdots{}_iE_{2,i}}{1-\cdots{}_{21,i}}(z_i^3-\cdots$$

$$D_y=\frac{1}{3}\sum_{i=1}^{3}\frac{E_{2,i}}{1-\upsilon_{12,i}\upsilon\cdots}(z_i^3-\cdots$$

$$D_s=\frac{1}{3}\sum_{i=1}^{3}G_{\cdots,i}\,(z\cdots$$

$$H=D_{xy}+\cdots$$

Substituting the given material properties E_L, E_T, G_{LT} and υ_{LT} for each layer as well as using t for the thickness of a layer gives

$$D_x=\frac{1}{12}\frac{t^3(26E_L+E_T)}{(1-\upsilon_{LT}\upsilon_{TL})}$$

$$D_{xy} = \frac{9\,t^3(\upsilon_T}{8\quad(1-}$$

$$D_y = \frac{1\;t^3(}{12\;(1-}$$

$$D_s = \frac{}{4}t^3G_{LT}$$

Deflection of simply supported uniformly loaded plate is

$$w = \frac{16p_0}{\pi^6}\sum_{m=1,3,\dots}^{\infty}\sum_{n=1,3,\dots}^{\infty}\frac{\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{mn\left[D_x(m^4/a^4)+2H(m^2n^2/a^2b^2)+D_y(n^4/b^4)\right]}$$

where a is the span in x-direction, b is the span in y-direction and p_0 is the unifrom load on the plate.

Calculating the deflection in the mid-span taking into account 3x3 terms gives w_{max} = 5.07 mm, which is equal to deflection obtained from the Abaqus model without the shear deformations (case b).

Equal deflection confirms that used Abaqus model is reliable in the case where transverse shear deformations are neglected. The stresses can be simply derived from

$$\begin{pmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{pmatrix}(z) = -\begin{bmatrix}\frac{E_1}{1-\upsilon_{12}\upsilon_{21}} & \frac{\upsilon_{21}E_1}{1-\upsilon_{12}\upsilon_{21}} & 0\\ \frac{\upsilon_{12}E_2}{1-\upsilon_{12}\upsilon_{21}} & \frac{E_2}{1-\upsilon_{12}\upsilon_{21}} & 0\\ 0 & 0 & G_{12}\end{bmatrix}\begin{pmatrix}\frac{\partial^2 w}{\partial x^2}\\ \frac{\partial^2 w}{\partial y^2}\\ 2\frac{\partial^2 w}{\partial x\partial y}\end{pmatrix}z.$$

where the curvature are obtained by differentation as

$$\frac{\partial^2 w}{\partial x^2} = -\frac{16p_0}{\pi^4 a^2}\sum_{m=1,3,\dots}^{\infty}\sum_{n=1,3,\dots}^{\infty}\frac{m\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{n\left[D_x(\quad 2H\left(\frac{m^2n^2}{a^2b^2}\right)+D_y\left(\frac{n^4}{b^4}\right)\right]},$$

$$\frac{\partial^2 w}{\partial y^2} = -\frac{16p_0}{\pi^4 b^2}\sum_{m=1,3,\dots}^{\infty}\sum_{n=1,3,\dots}^{\infty}\frac{\quad\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{m\left[D_x\left(\frac{m^4}{a^4}\right)+\quad\right)+D_y\left(\frac{n^4}{b^4}\right)\right]},$$

$$\frac{\partial^2 w}{\partial x\partial y} = \frac{16p_0}{\pi^4 ab}\sum_{m=1,3,\dots}^{\infty}\sum_{n=1,3,\dots}^{\infty}\frac{\cos\quad\frac{\quad\pi y}{b}}{\left[D_x\left(\frac{m^4}{a^4}\right)+2\quad+D_y\left(\frac{n^4}{b^4}\right)\right]}.$$

Finally the stresses from Abaqus and analytical model were plotten in the same figures. In the Figure 7 it may be seen that the stresses from the analytical model are almost equal to stresses obtained from the Abaqus model. In the Figure 8 stresses near the corner of the plate are

compared. There exists some differences between the Abaqus and analytical model. Partly the reason is that measuring the stresses from exact coordinate was not possible in Abaqus, and the stresses, which had large gradient in the region, were only approximations. Still the stresses in the corner are clearly comparable between the models and may be considered as …on of the used FE model.

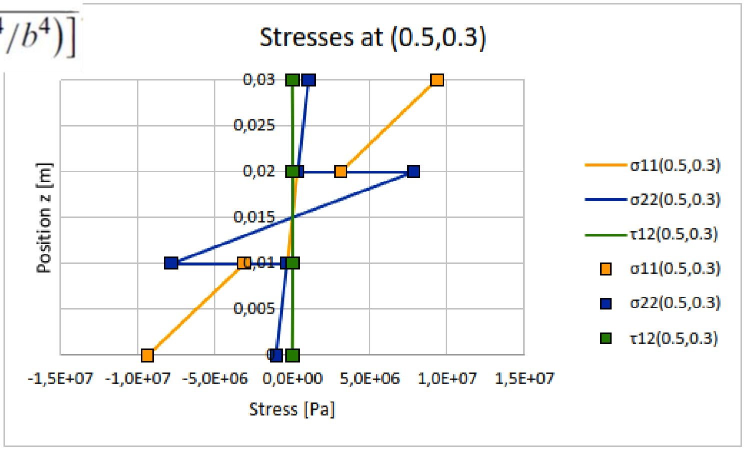


Figure 7. Stresses from the analytical model (squares) compared to Abaqus results (lines) on the mid-point of the plate.

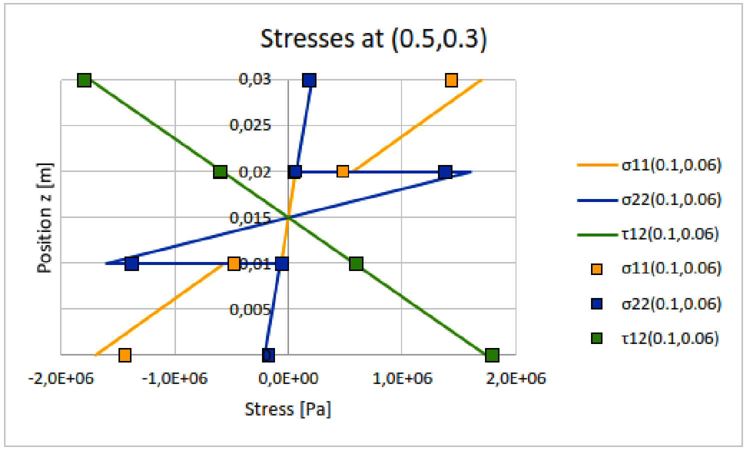


Figure 8. Stresses from the analytical model (squares) compared to Abaqus results (lines) near the corner of the plate.

Example solved in Abaqus (you can use other software if you wish)

Partial solution by DI Jaaranen J. (2017)
Reproduced with permission

a) Modelling the plate in Abaqus software

Model

The plate was modelled in Abaqus as a shell using 4-node shell elements (S4). The stiffness properties of the plate were given by defining composite layup for the plate, which Abaqus uses to calculate the stiffness of the plate for the calculation. All the layers were given same orthotropic material properties and the direction of the layers were defined in the composite layup. The material coordinate system was chosen so that; 1-axis refers to L-direction, 2-axis refers to T-direction and 3-axis refers to R-direction, which leads to properties below:

$E_1 = E_L = 11990$ MPa

$E_2 = E_T = 420$ MPa

$E_3 = E_R = 820$ MPa

$G_{12} = G_{LT} = 740$ MPa

$G_{13} = G_{LR} = 620$ MPa

$G_{23} = G_{RT} = 240$ MPa

$\nu_{12} = \nu_{LT} = 0.7749$

$\nu_{13} = \nu_{LR} = 0.6071$

$\nu_{23} = \nu_{TR} = \nu_{RT} E_T / E_R = 0.3089.$

The model is shown in the Figure 2. Top and bottom layers are orientated in x-direction and the middle layer in y-direction. Boundary conditions on all the edges are pinned ($u_1 = u_2 = u_3 = 0$). Material properties are shown in the Table 2. Abaqus takes transverse shear deformations automatically into account, so the analysis was done twice; in the case a) using given material properties and in the case b) using 1000x higher transverse shear moduli to neglect the effect of the transverse shear deformation.

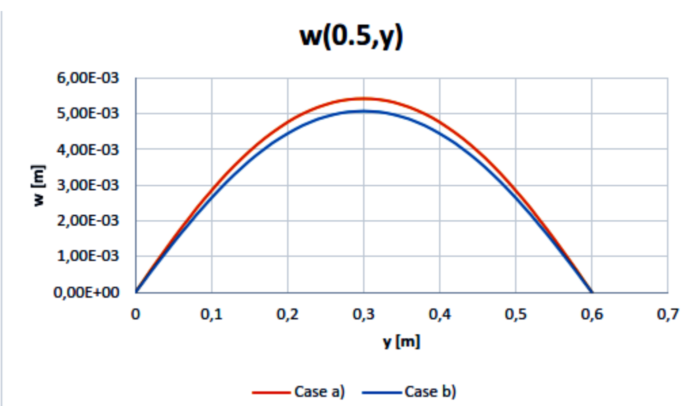


Figure 3. Deflection along the centerline in x-direction.

Table 2. Material properties in the analyses. In the case a) properties all the properties of the material as given is used and in the case b) transverse shear moduli are increased 1000x to neglect effects of the transverse shear deformation in the analysis.

	E_1 [MPa]	E_2 [MPa]	G_{12} [MPa]	G_{13} [MPa]	G_{23} [MPa]	ν_{12}
Case a)	11990	420	740	620	240	0.7749
Case b)	11990	420	740	620×10^3	240×10^3	0.7749

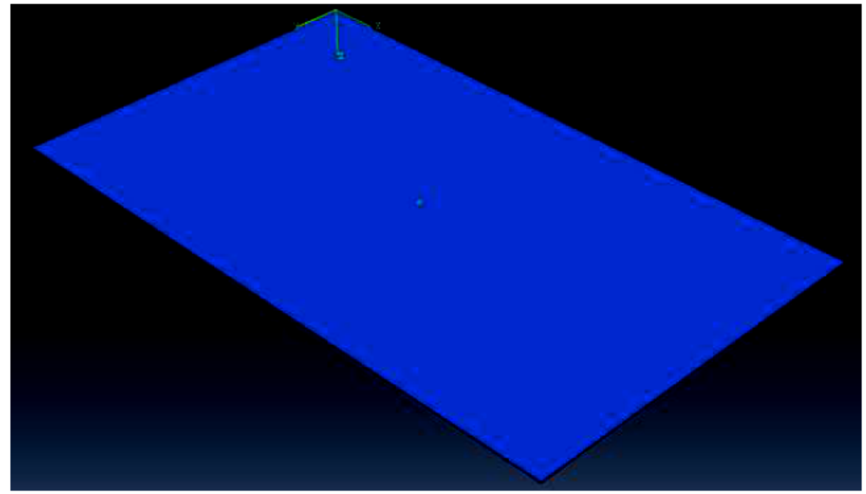


Figure 2. The model of the plate in Abaqus. All boundaries are pinned and uniform load $q = 20$ kN/m², acting downwards, is applied on the plate.

Results

Deflection of the centerlines of the plate along x- and y-directions are illustrated in the Figure 3 and Figure 4. The maximum deflection is found at the mid-point of the plate and it was in the case a) $w_{max} = 5.42$ mm and in the case b) $w_{max} = 5.07$ mm.

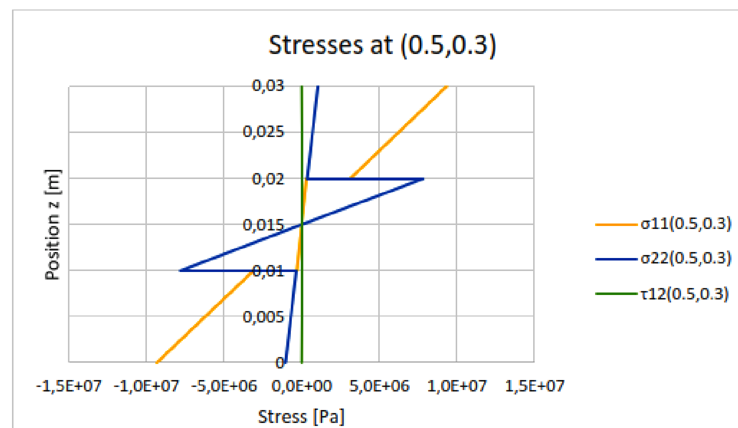


Figure 5. Stresses over the section at mid-point of the plate ($x = 0.5$ m, $y = 0.3$ m).

A pause

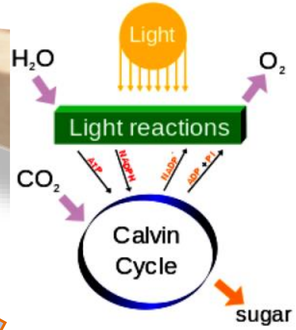
Let's look a bit to the microscopic level

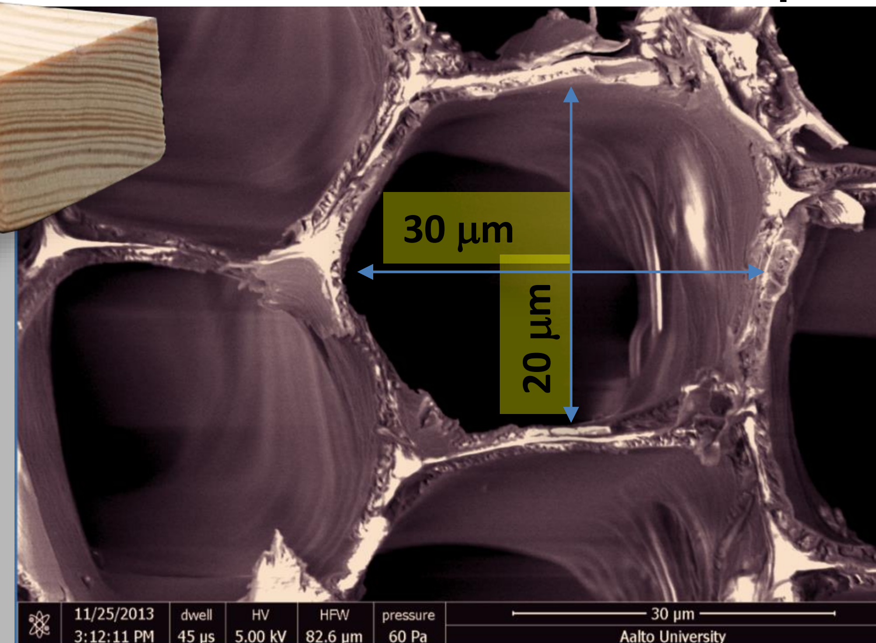
Did you knew that ...?

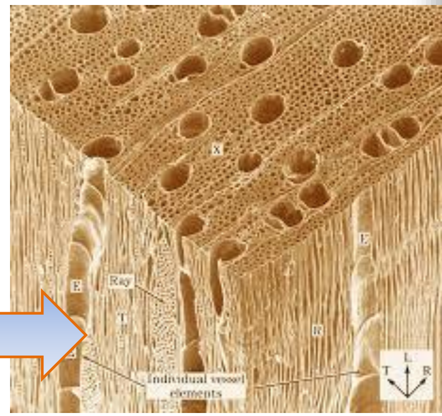
We are also made of natural polymers?

Wood is a natural cellular composite

Wood is a natural cellular composite

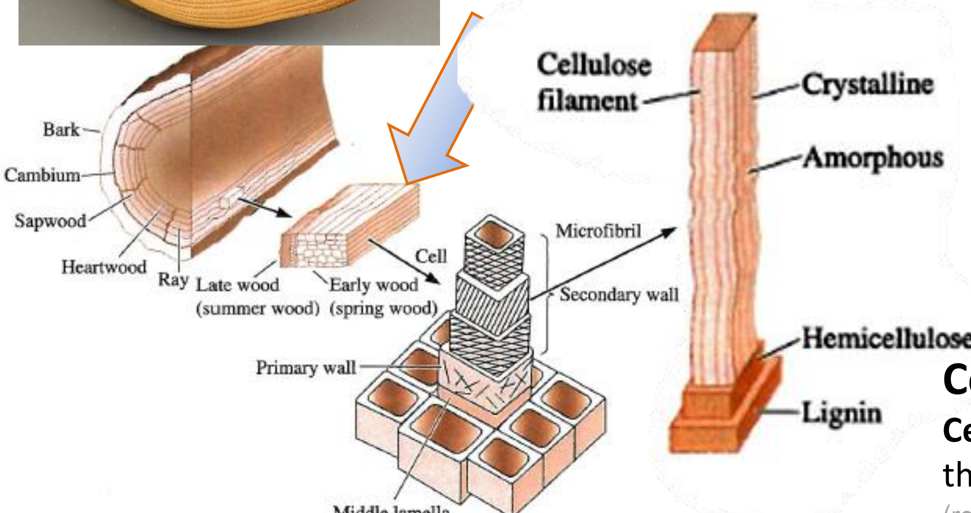


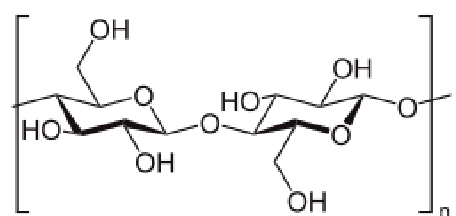




Wood: cellular microstructure – solurakenne

(SEM Photo prof. A. Cwirzen and Dr. D. Baroudi)





$(C_6H_{10}O_5)_n$

Cellulose content:
cotton fiber cotton 90%,
wood is 40–50%

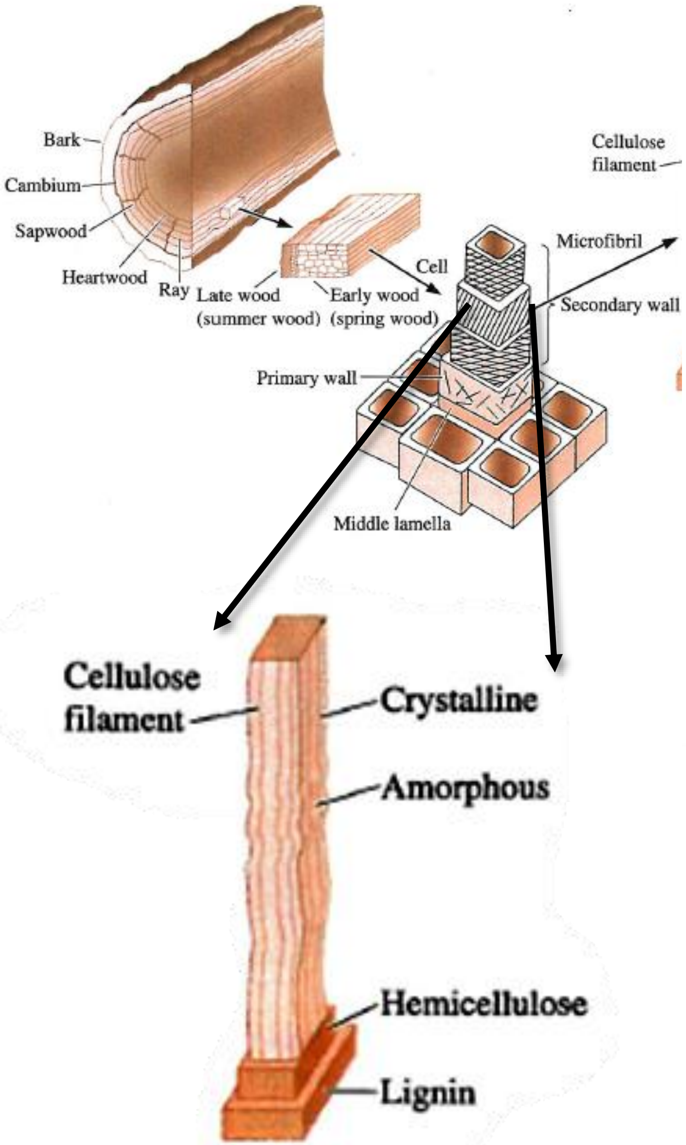
First produced thermoplastics from cellulose: Celluloid and cellophane

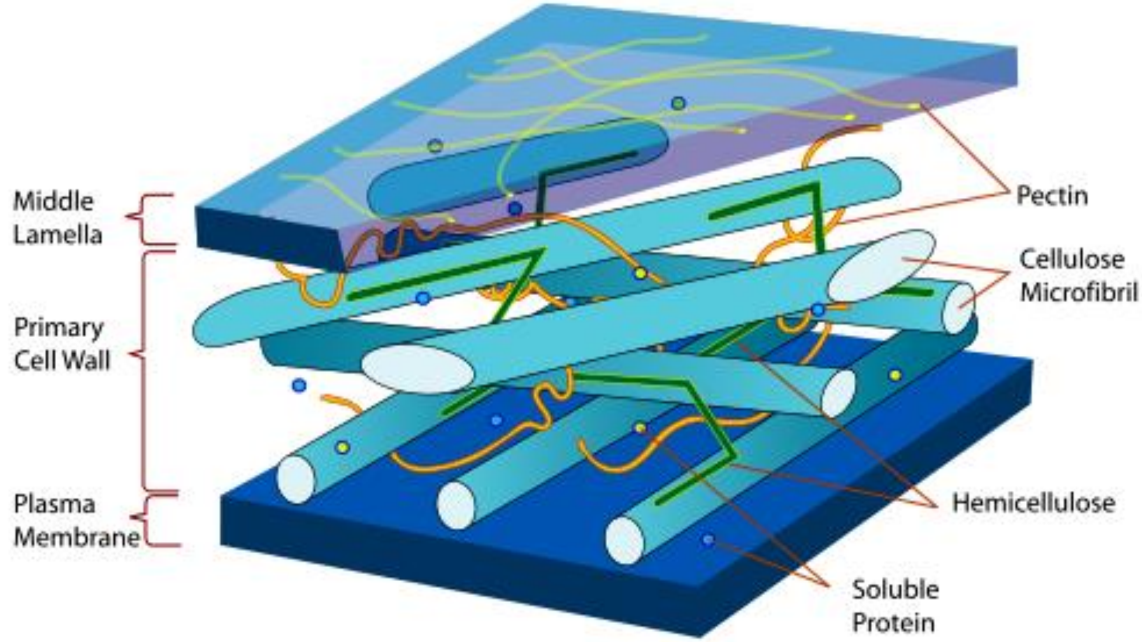
Cellulose is the most common natural polymer.

Cellulose consists of long, stretched out strands of glucose that plants fabricates by photosynthesis.

(recall that Proteins –thus, a large part of us - are also natural polymers too)

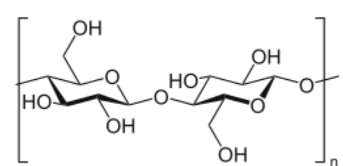
Arrangement of cellulose and other polysaccharides in a plant cell wall





- The highly complex structural organization of the wood polymers within the cell wall determines to a large extend the properties of the fibers and of the 'wood'
- Strength of solid wood results from strength of its fibers
- Cellulose is the main strength source
- Hemicellulose & lignin form as a matrix for the cellulose
- Lignin and hemicellulose closely associated

See also: Annals of Forest Science (2015) 72:679–684
Lennart Salmén : Wood morphology and properties from molecular perspectives
DOI 10.1007/s13595-014-0403-3



$(C_6H_{10}O_5)_n$

http://www.swst.org/wp/

(21.4.2019) Ref: https://en.wikipedia.org/wiki/Cellulose#/media/File:Plant_cell_wall_diagram-en.svg

The nature is not continuous...

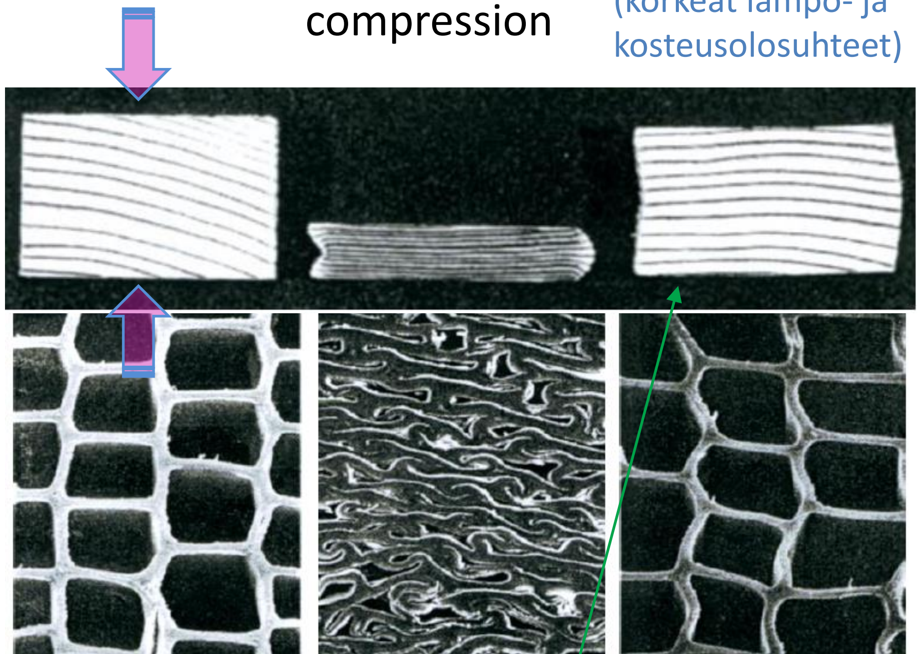
Wood - Puu

Compression

Result after compression

Hygrothermal *recovery*

Hygro-terminen palautuminen (korkeat lämpö- ja kosteusolosuhteet)



MACRO

Phenomenological scale

micro

Kurkistettu sisään - mikrorakenteeseen - mitä tapahtuu!

Opening the black box

Recovery is some kind of shape-memory of wood

shape-memory (muistimateriaali)→ smart material

References:

Research of Molecular-Topological Structure at Shape-Memory Effect of Wood

Galina Gorbacheva, Yuri Olkhov, Boris Ugolev, Serafim Belkovskiy

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Institute of Problems of Chemical Physics of the Russian Academy of Sciences
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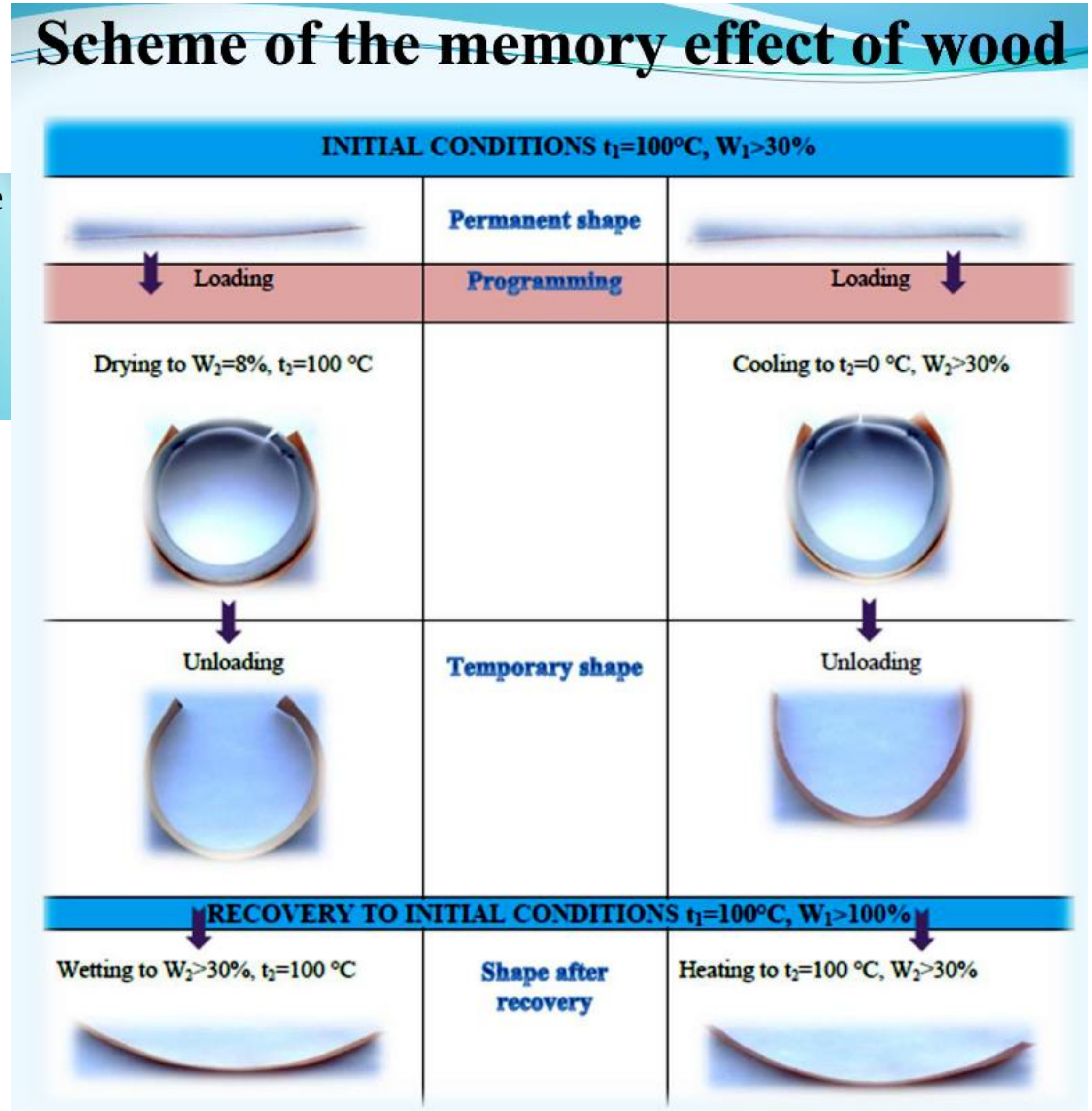
Wood has shape-memory (muistimateriaali)→ smart material



57th International Convention of Society of Wood Science and Technology June 23-27, 2014 - Zvolen, SLOVAKIA

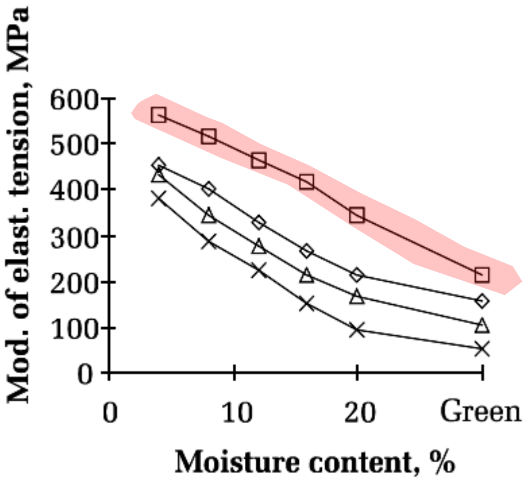
Moscow State Forest University
Institute of Problems of Chemical Physics of the Russian Academy of Sciences

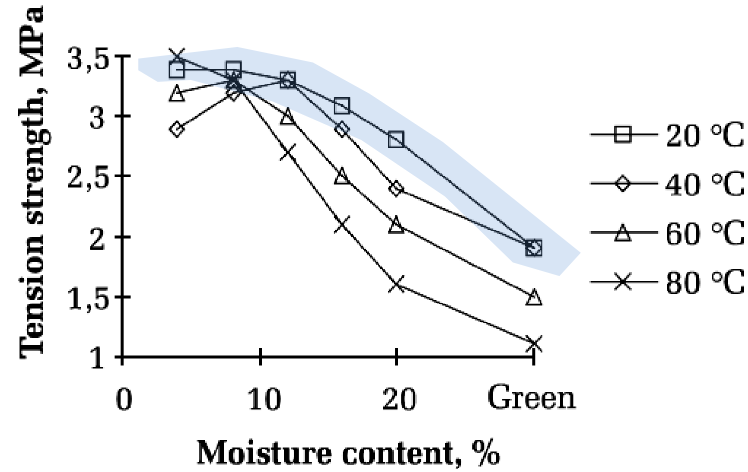
Strength and modulus of elasticity in tangential direction for pine

Ref: **Siimes** 1967
Pine ~490 kg/m3

Tangential direction perpendicular to grains most significant for cracking during drying

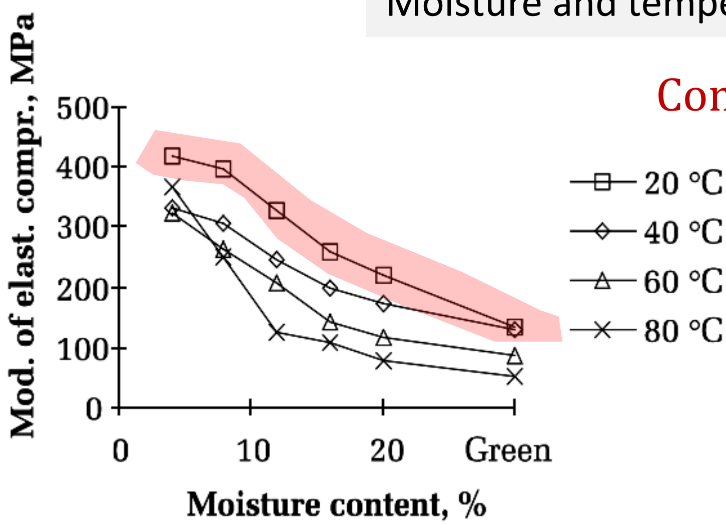
Tension

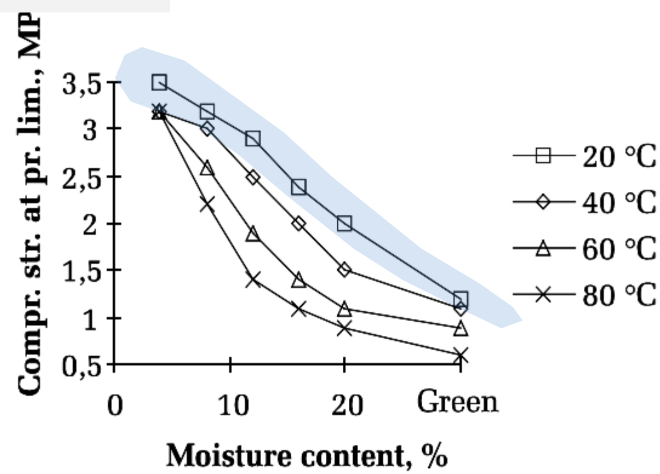




Moisture and temperature content in wood

Compression





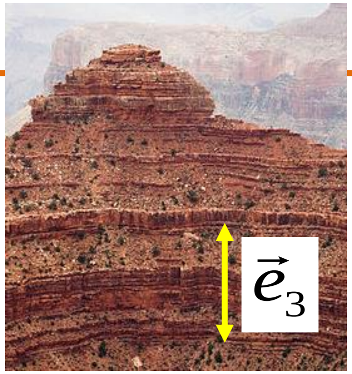
IMPORTANT for Structural Engineers

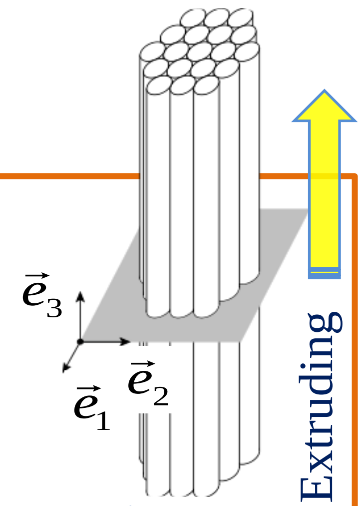
Ref: Siimes 19

Siimes, F. E. 1967. The effect of specific gravity, moisture content, temperature and heating time on the tension and compression strength and elasticity properties perpendicular to the grain of finnish pine spruce and birch wood and the significance of these factors on the checking of timber at kiln drying. The State Institute for Technical Research. Helsinki, Finland.

Transverse Isotropic Material

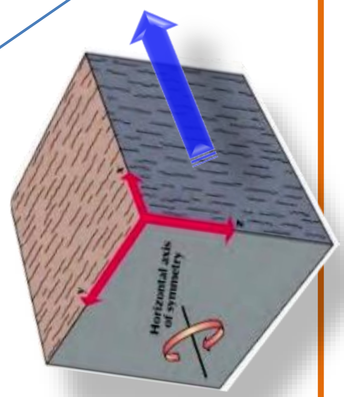
Transversal isotropy means that there is **one** *symmetry plane* (called the *plane of isotropy*)





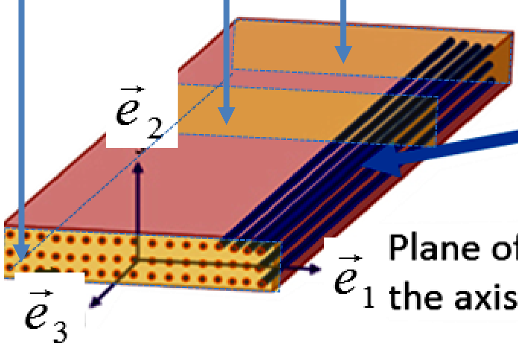
Assume the *axis of transverse isotropy* is $\vec{e}_3$, then the material is 'isotropic' in the plane orthogonal to $\vec{e}_3$ (the *plane of isotropy*)

Means that $\vec{e}_1 - \vec{e}_2$; the symmetry plane



One symmetry plane

Example: a laminated composite obtained by bonding of layers formed from unidirectional fibers (unidirectional fiber composite lamina)



Transverse isotropy

Plane of isotropy is orthogonal to $\vec{e}_3$ the axis of transverse isotropy

Example: a composite reinforced by unidirectional fibers

Transverse isotropic material

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon} \qquad \boldsymbol{\varepsilon} = \mathbf{C}\boldsymbol{\sigma}$$

one symmetry plane $\vec{e}_1 - \vec{e}_2$

Here $\vec{e}_3$ is the axis of symmetry.

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{13} \end{Bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & -\frac{\nu_{13}}{E_1} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_1} & -\frac{\nu_{13}}{E_1} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{13}}{E_1} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2G_{12}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2G_{13}} \end{bmatrix}}_{\mathbf{C} \equiv \mathbf{D}^{-1}} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{Bmatrix}$$

$\vec{e}_3$ is the axis of symmetry (= *axis of transverse isotropy*).

With reference to the orthotropic case, we have the following equalities:

$E_1 = E_2$, same extensional moduli in directions 1 and 2;
$\nu_{13}/E_1 = \nu_{23}/E_2$, same contraction coefficient in directions 1 and 2 for a tension applied in direction 3;
$G_{13} = G_{23}$, same shear moduli for shear in planes perpendicular to axis 2 and axis 1;
$2G_{12} = E_1/(1+\nu_{12})$, shear modulus around axis 3.

$$\frac{1}{2G_{12}} = \frac{1+\nu_{12}}{E_1}$$

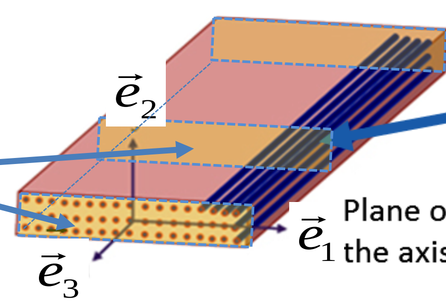
The five coefficients which characterize the material are:

two extensional moduli E_1 and E_3,
one shear modulus G_{13},
two coefficients of contraction ν_{12} and ν_{13}.

Remains only five (5) independent elastic coefficients

The material *stiffness matrix* is obtained by *inverting* the *compliance matrix* $\mathbf{C}$
(**Student**: do it as a homework, use symbolic algebra software)

Transversal isotropy: one symmetry plane $\vec{e}_1 - \vec{e}_2$



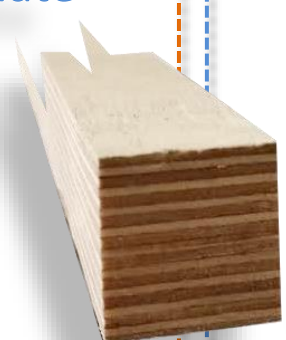
Transverse isotropy

Plane of isotropy is orthogonal to $\vec{e}_3$ the axis of transverse isotropy

Homogenization

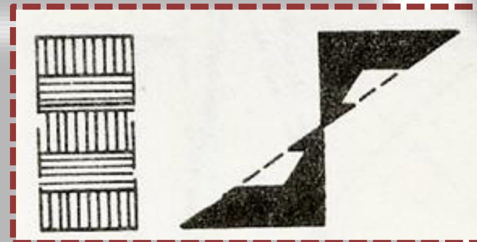
Determine the Effective Bending Rigidity of the composite beam or laminate

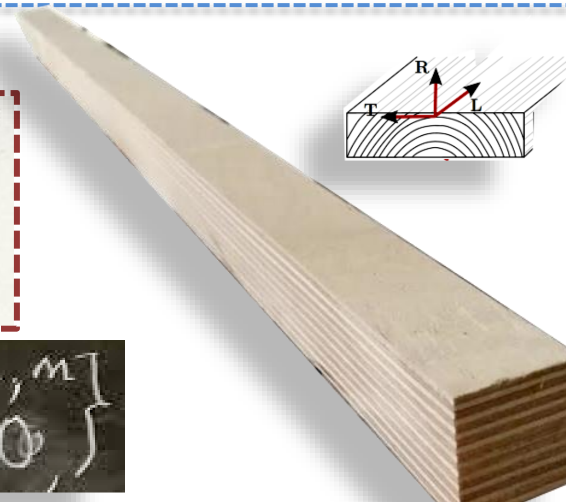
$$U(\boldsymbol{\varepsilon}) = \frac{1}{2}\int_V \boldsymbol{\sigma} : \boldsymbol{\varepsilon}\, \mathrm{d}V$$

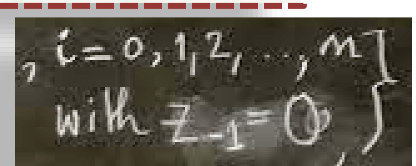


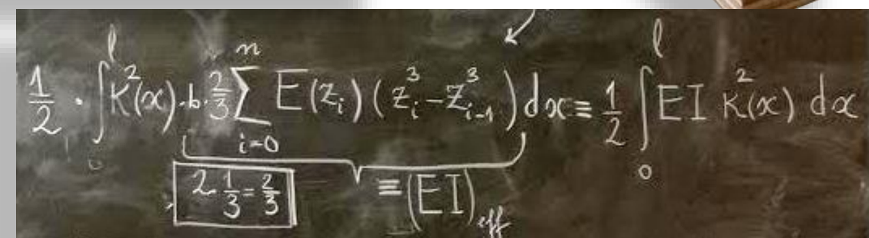
ἐλαστός
Elasticity

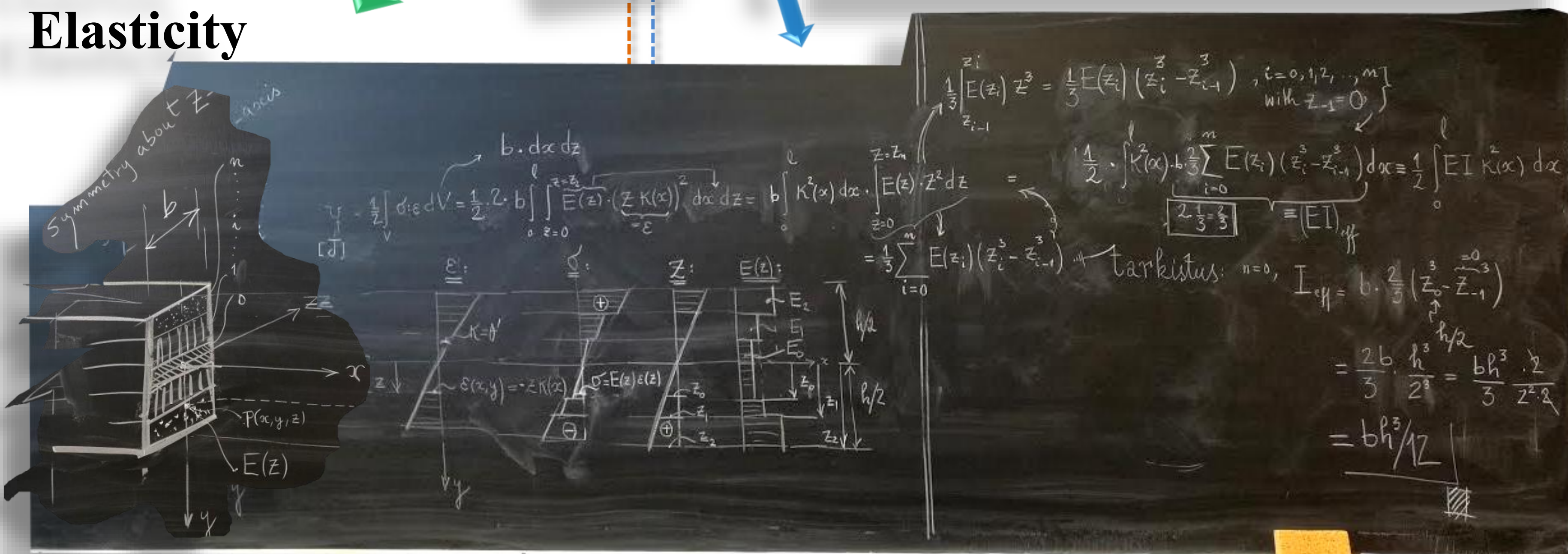
Bending **normal stresses**





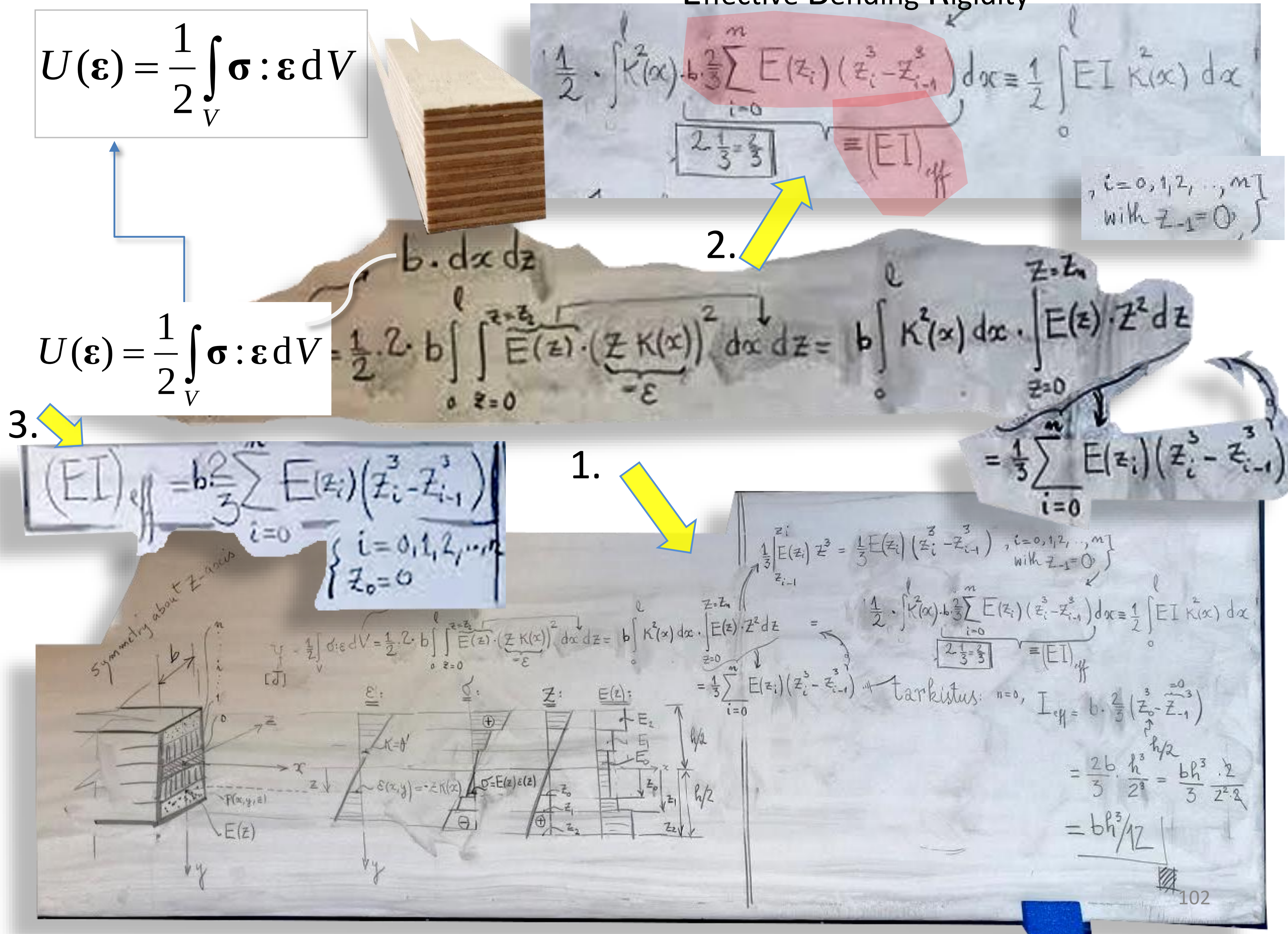






Effective Bending Rigidity

$$U(\boldsymbol{\varepsilon}) = \frac{1}{2}\int_V \boldsymbol{\sigma} : \boldsymbol{\varepsilon}\, \mathrm{d}V$$



$$U(\boldsymbol{\varepsilon}) = \frac{1}{2}\int_V \boldsymbol{\sigma} : \boldsymbol{\varepsilon}\, \mathrm{d}V = \frac{1}{2}\cdot 2\cdot b\int_0^{l}\int_{z=0}^{z=z_n} E(z)\cdot(\underbrace{z\,K(x)}_{=\varepsilon})^2\,dx\,dz = b\int_0^{l} K^2(x)\,dx\cdot\int_{z=0}^{z=z_n} E(z)\cdot z^2\,dz$$

$$= \frac{1}{3}\sum_{i=0}^{m} E(z_i)(z_i^3 - z_{i-1}^3)$$

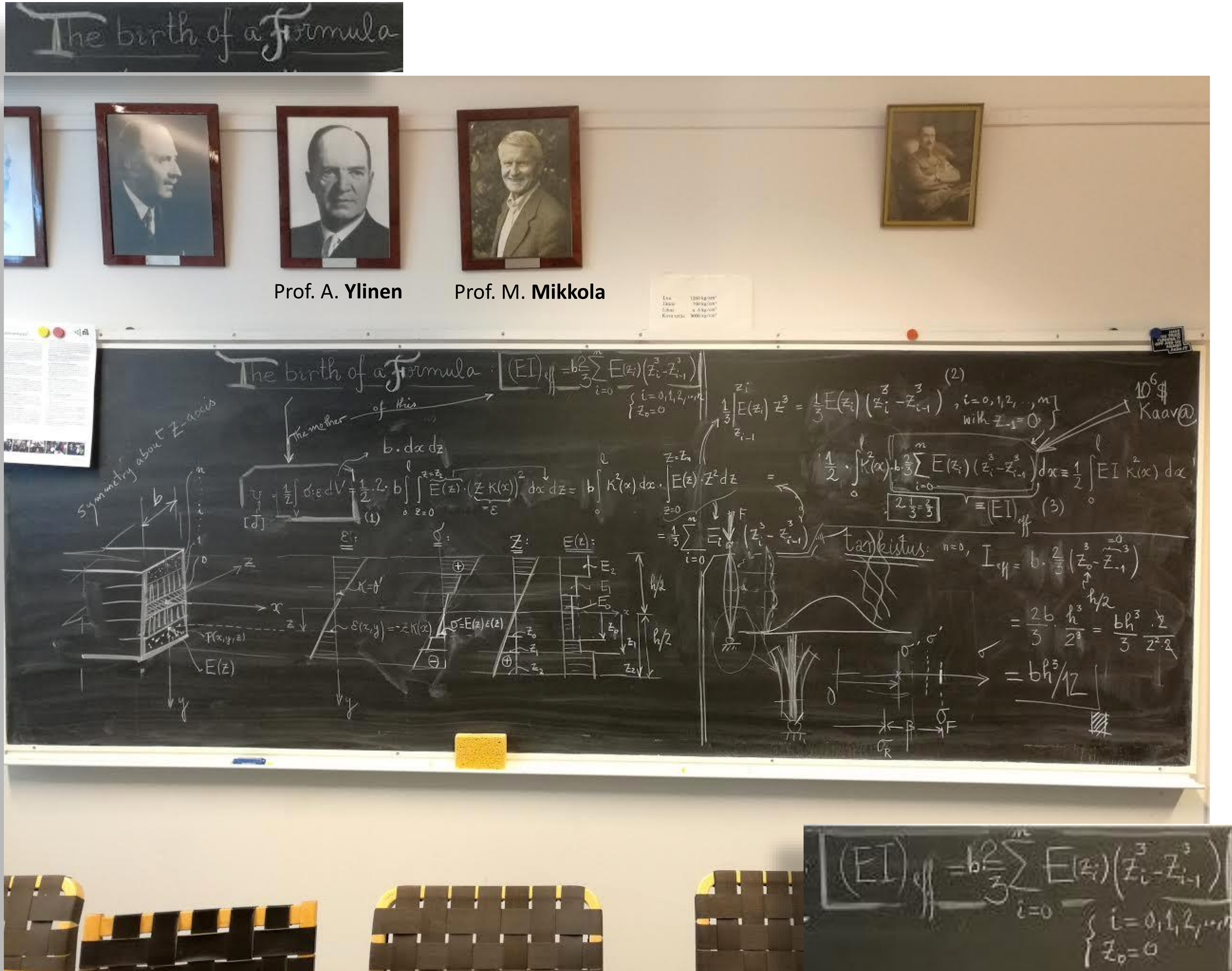
$$\frac{1}{3}\int_{z_{i-1}}^{z_i} E(z_i)\,z^3 = \frac{1}{3}E(z_i)(z_i^3 - z_{i-1}^3),\quad i=0,1,2,\ldots,m \text{ with } z_{-1}=0$$

$$\frac{1}{2}\cdot\int_0^{l} K^2(x)\cdot \underbrace{b\cdot\frac{2}{3}\sum_{i=0}^{m}E(z_i)(z_i^3 - z_{i-1}^3)}_{\equiv (EI)_{eff}}\,dx \equiv \frac{1}{2}\int_0^{l} EI\,K^2(x)\,dx \qquad \boxed{2\cdot\frac{1}{3}=\frac{2}{3}}$$

$$(EI)_{eff} = b\cdot\frac{2}{3}\sum_{i=0}^{m} E(z_i)(z_i^3 - z_{i-1}^3),\quad \begin{cases} i=0,1,2,\ldots,m \\ z_0 = 0\end{cases}$$

tarkistus: $n=0$, $I_{eff} = b\cdot\frac{2}{3}(z_0^3 - z_{-1}^3)$, $z_0 = h/2$, $z_{-1}=0$

$$= \frac{2b}{3}\cdot\frac{h^3}{2^3} = \frac{bh^3}{3}\cdot\frac{2}{2^2\cdot 2} = bh^3/12$$



Prof. A. **Ylinen** Prof. M. **Mikkola**

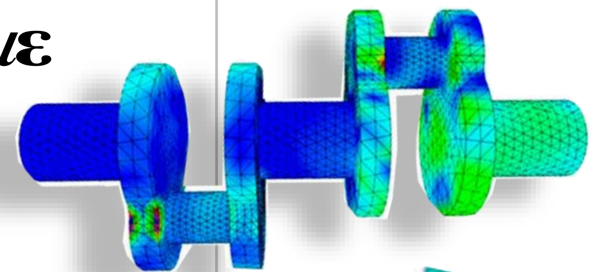
Examples of problems - generalized Hooke's Law

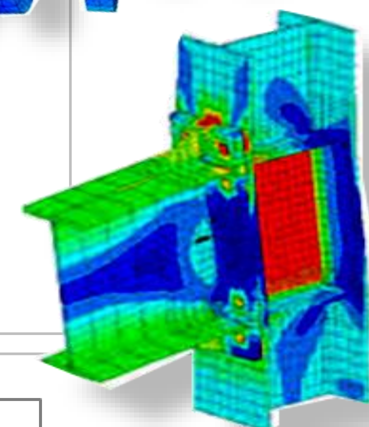
Linear isotropic material:

$$\boldsymbol{\sigma} = \lambda\,\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}$$

Lamé's coefficients

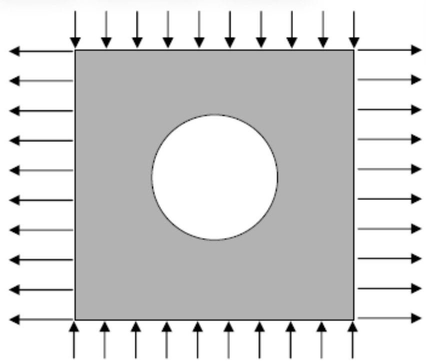
$$\boldsymbol{\varepsilon} = \frac{1+\nu}{E}\boldsymbol{\sigma} - \frac{\nu}{E}\mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1}$$

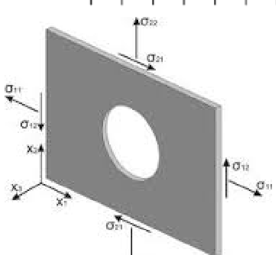




Problem: Find the strain and the stress distributions within the 3D solid

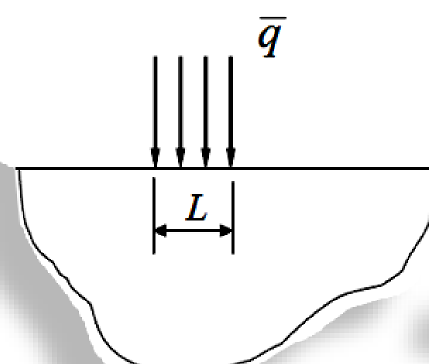
Plane stress state

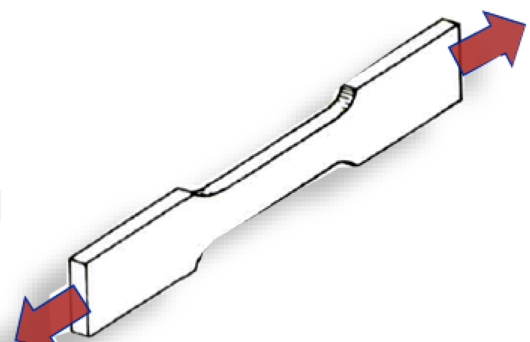


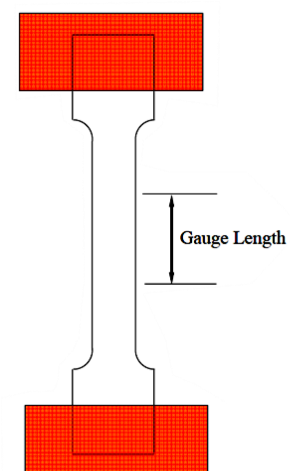


The Navier-Lame equations:

$$(\lambda + \mu)\ \mathrm{grad\ div}\ \mathbf{u} + \mu\nabla^2\mathbf{u} + \mathbf{F} = \rho\frac{\partial^2\mathbf{u}}{\partial t^2},$$







Generalized Hooke's Law – Examples of problems

2D

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$$

$$\boldsymbol{\varepsilon} = \mathbf{D}^{-1}\boldsymbol{\sigma} \equiv \mathbf{S}\boldsymbol{\sigma}$$

Linear isotropic material: $$\boldsymbol{\sigma} = \lambda\,\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}$$

$$\boldsymbol{\varepsilon} = \frac{1+\nu}{E}\boldsymbol{\sigma} - \frac{\nu}{E}\mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1}$$

Constitutive law

The Navier-Lame equations:

$$(\lambda+\mu)\ \mathrm{grad\ div}\ \mathbf{u} + \mu\nabla^2\mathbf{u} + \mathbf{F} = \rho\frac{\partial^2\mathbf{u}}{\partial t^2},$$

and initial & boundary conditions IBC

Plane stress: $\sigma_{33} = \sigma_{23} = \sigma_{13} = 0$

$\boldsymbol{\varepsilon} = \mathbf{D}^{-1}\boldsymbol{\sigma} \equiv \mathbf{S}\boldsymbol{\sigma}$ **reduces to:**

$$\sigma_{ij} = \lambda'\varepsilon_{kk} + 2\mu\varepsilon_{ij}$$

$$\lambda' = \nu E/(1-\nu^2)$$

$$G\left(\Delta u + \frac{1+\nu}{1-\nu}\frac{\partial e}{\partial x}\right) + f_x = 0,$$

$$G\left(\Delta v + \frac{1+\nu}{1-\nu}\frac{\partial e}{\partial y}\right) + f_y = 0,$$

$$e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y},$$

Component form:

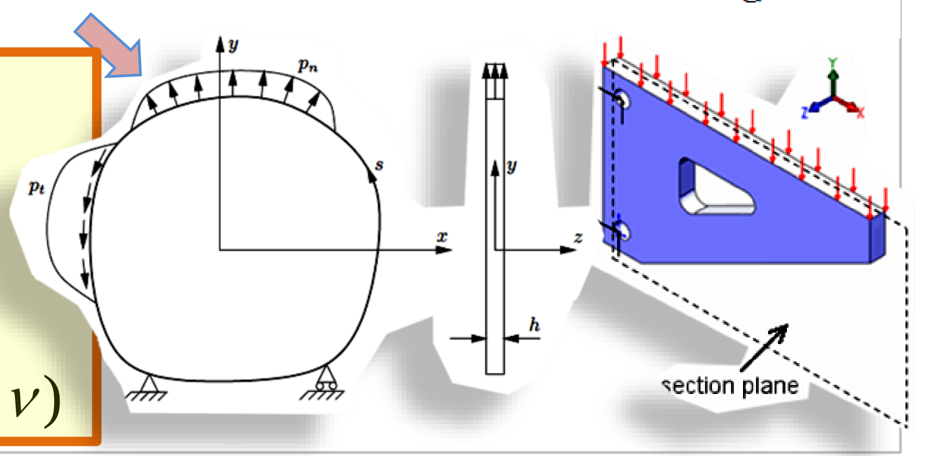
$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y), \qquad \varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y),$$

$$\sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x), \qquad \varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x),$$

$$\tau_{xy} = G\gamma_{xy}. \qquad \gamma_{xy} = \frac{1}{G}\tau_{xy},$$

$$\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{12}\end{bmatrix} = \begin{bmatrix}1/E & -\nu/E & 0\\ -\nu/E & 1/E & 0\\ 0 & 0 & (1+\nu)/E\end{bmatrix}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix}$$

$$\varepsilon_{13} = \varepsilon_{23} = 0 \qquad \varepsilon_{33} = -\nu(\varepsilon_{11}+\varepsilon_{22})/(1-\nu)$$



Generalized Hooke's Law – examples of problems

2D

Linear isotropic material: $\boldsymbol{\sigma} = \lambda\,\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}$

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$$

$$\boldsymbol{\varepsilon} = \mathbf{D}^{-1}\boldsymbol{\sigma} \equiv \mathbf{S}\boldsymbol{\sigma}$$

The Navier-Lame equations:

$$(\lambda+\mu)\ \mathrm{grad\ div}\ \mathbf{u} + \mu\nabla^2\mathbf{u} + \mathbf{F} = \rho\frac{\partial^2\mathbf{u}}{\partial t^2},$$

and initial & boundary conditions IBC

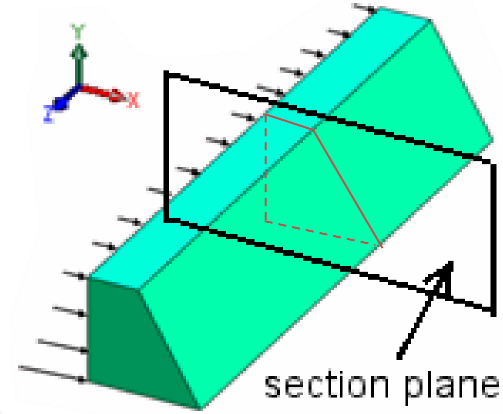
$$\boldsymbol{\varepsilon} = \frac{1+\nu}{E}\boldsymbol{\sigma} - \frac{\nu}{E}\mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1}$$

Bulk Modulus $K = \dfrac{E}{3(1-2\nu)}$

Shear Modulus $\mu = \dfrac{E}{2(1+\nu)} \equiv G$

Lame Modulus $\lambda = \dfrac{\nu E}{(1+\nu)(1-2\nu)}$

A dam



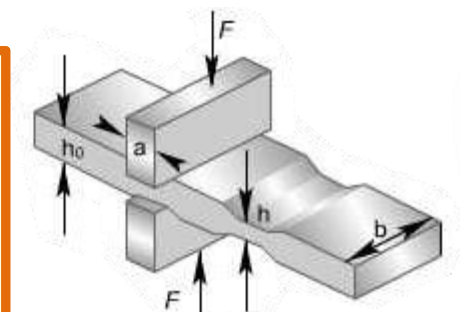
Plane strain: $\varepsilon_{33} = \varepsilon_{23} = \varepsilon_{13} = 0$

$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$ **reduces to:**

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix} = \begin{bmatrix}\lambda+2\mu & \lambda & 0\\ \lambda & \lambda+2\mu & 0\\ 0 & 0 & 2\mu\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{12}\end{bmatrix}$$

$$\sigma_{23} = \sigma_{13} = 0$$

$$\sigma_{33} = \nu(\sigma_{11}+\sigma_{22})$$

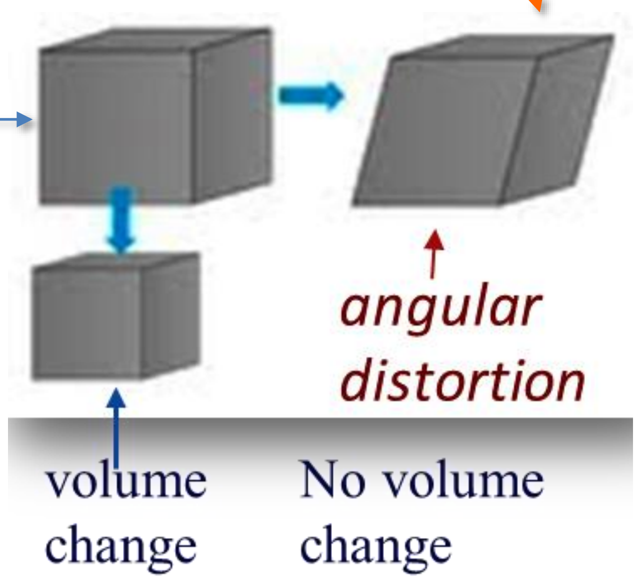


$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & \nu & 0 & 0 & 0\\ \nu & 1-\nu & \nu & 0 & 0 & 0\\ \nu & \nu & 1-\nu & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{23}\\ 2\varepsilon_{13}\\ 2\varepsilon_{12}\end{bmatrix}$$

Nonlinear Isotropic Hooke Formulation

Some general aspects

Why splitting Volumetric and deviatoric (shearing)**?**



Why split volumetric and deviatoric (shearing)?

- It is more convenient to work with scalar energy potentials (*Cf.* Bonding force or interatomic force e in solids … van der Waals, should be known to you) than directly with forces (vectors) or stresses (tensors)
- The forces are then derived from the gradients of such thermodynamic potentials

$$I_1 \equiv \mathrm{tr}(\boldsymbol{\sigma})$$

$$I_2 \equiv \frac{1}{2}\left[\mathrm{tr}(\boldsymbol{\sigma})^2 - \mathrm{tr}(\boldsymbol{\sigma}^2)\right]$$

$$J_2 = \frac{1}{2} s_{ij} s_{ij} \qquad \tau_{oct} = \sqrt{\frac{2}{3} J_2}$$

Maximum weighted shear stress

Introduction

Example:

$$U = \frac{1}{6} K I_1^2 + \frac{2n\sigma_0\varepsilon_0}{n+1}\left(\frac{I_2}{\varepsilon_0^2}\right)^{(n+1)/2n},$$

$K, \sigma_0, \varepsilon_0, n -$ material parameters

Strain energy density: characterized by $U = U(I_1, I_2, I_3)$ or equivalently by $U = U(I_1, J_2, J_3)$.

Most practical materials have

- **nonlinear** stress **behavior only** when subjected to **shear deformation** (= deviatoric behavior) ⇒ characterized by I_2 or J_2
- whereas **stress** response is **linear** with **volume changes** ⇒ characterized by I_1

Answer

Such behavior can be characterized by a strain energy density

$$\Rightarrow U = U_{\text{volumetric}} + U_{\text{deviatoric}} = \boxed{U(I_1, J_2)} \quad \leftrightarrow \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}_{\text{volumetric}} + \boldsymbol{\sigma}_{\text{deviatoric}}$$

Nonlinear isotropic Hooke formulation

A **nonlinear elasticity** theory that can be **expressed** in a form *similar* to **Hooke's law**:

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon} \quad \leftrightarrow \sigma_{ij} = D_{ijkl}\,\varepsilon_{kl}\,, \quad \textbf{(1)}$$

Now $\mathbf{D} \leftrightarrow D_{ijkl}$ depends on the amount of loading or stress

Hooke's law 'format' for nonlinear elasticity

This one reason why… $\Longrightarrow$

Generally, **materials respond differently** to **volume change** (*volumetric response*) and **shearing** or **shape distortion** (*deviatoric response*)

The strain and stress are **separated** into the **volumetric** and ***deviatoric*** parts

$$\sigma_{kk} \leftrightarrow \varepsilon_{kk} \qquad s_{ij} \leftrightarrow e_{ij}$$

$$e_{ij} \equiv \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij}$$

Constraining the constitutive relation to Hooke format (1) and some energy considerations (*Cf.* Ottosen *et al*. chap. 4.10), show that a) only I_1 & J_2 enter the formulation of the constitutive relation b) and this constitutive law have the form (2)

$$\sigma_{ij} = 3K\varepsilon_{kk}\delta_{ij} + 2Ge_{ij} \quad \textbf{(2)}$$

$$K = K(I_1, J_2), \quad G = G(I_1, J_2)$$

COUPLING

Volumetric behavior

$$I_1 = \sigma_{kk}$$

Deviatoric behavior

$$J_2 = \frac{1}{2}s_{kl}s_{lk} = \frac{1}{2}\mathbf{s}:\mathbf{s}$$

Cf. to **Linear elasticity:**

Bulk modulus: $K = \dfrac{E}{3(1-2\nu)}$

Shear modulus: $G = \dfrac{E}{2(1+\nu)}$

$$J_3 = \frac{1}{3}s_{kl}s_{lm}s_{mk}$$

$$s_{ij} \equiv \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$$

Nonlinear isotropic Hooke formulation

(*Cf*. Ottosen *et al*. chap. 4.10)

Strain energy

Complementary energy $U_c = \sigma_{ij}\varepsilon_{ij} - U(\varepsilon_{ij})$

$U_c = U_c(I_1, I_2, I_3)$ or equivalently

$U_c = U_c(I_1, J_2, J_3)$ $\quad I_1$ & J_2

Constraining the constitutive relation to Hooke format (1) requires (no quadratic terms) $\varepsilon_{ij} = \dfrac{\partial U_c(\sigma_{ij})}{\partial \sigma_{ij}}$

$$\partial U_c / \partial J_3 = 0 \Rightarrow \boxed{U_c = U_c(I_1, J_2)}$$

$$I_1 = \sigma_{kk}, \quad J_2 = \frac{1}{2}s_{kl}s_{lk}, \quad J_3 = \frac{1}{3}s_{kl}s_{lm}s_{mk}$$

$U_c = U_c(I_1, J_2) \Rightarrow \varepsilon_{ij} = \partial U_c / \partial \sigma_{ij} =$ chain rule

$$\Rightarrow \varepsilon_{ij} = \frac{\partial U_c}{\partial I_1}\delta_{ij} + \frac{\partial U_c}{\partial J_2}s_{ij}$$

Contracting indices

$$\varepsilon_{kk} = 3\underbrace{\frac{\partial U_c}{\partial I_1}}_{\frac{\sigma_{kk}}{9K}} \qquad e_{ij} = \underbrace{\frac{\partial U_c}{\partial J_2}}_{1/2G} s_{ij}$$

$$\sigma_{ij} = \frac{1}{3}\sigma_{kk}\delta_{ij} + s_{ij}$$

$$\sigma_{kk} \equiv -3p = 3K\varepsilon_{kk}$$

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon} \leftrightarrow \sigma_{ij} = D_{ijkl}\,\varepsilon_{kl}, \quad \textbf{(1)}$$

depends on the amount of loading

Hooke's law 'format' for nonlinear elasticity

The strain and stress are **separated** into the **volumetric** and ***deviatoric*** parts

$$\sigma_{kk} \leftrightarrow \varepsilon_{kk} \qquad s_{ij} \leftrightarrow e_{ij}$$

$$\sigma_{ij} = \frac{1}{3}\sigma_{kk}\delta_{ij} + s_{ij} \qquad e_{ij} \equiv \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij}$$

$$\sigma_{ij} = 3K\varepsilon_{kk}\delta_{ij} + 2Ge_{ij} \quad \textbf{(2)}$$

$$K = K(I_1, J_2), \quad G = G(I_1, J_2)$$

COUPLING

Volumetric behavior

$I_1 = \sigma_{kk}$

Deviatoric behavior

$$J_2 = \frac{1}{2}s_{kl}s_{lk} = \frac{1}{2}\mathbf{s}:\mathbf{s}$$

$$s_{ij} \equiv \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} \qquad J_3 = \frac{1}{3}s_{kl}s_{lm}s_{mk}$$

$K, G -$ are not arbitrary but have to fulfill an additional constraint, *ref*. to Ottosen *et al.*

Non-linear isotropic Hooke formulation

Metals and steel:

- ❑ Uncoupled **volumetric** and *deviatoric* Reponses
- **Linear volumetric** response
- **Nonlinear** *deviatoric* response

$$\Downarrow$$

Metals and steel:

$$K = const, \qquad G = G(J_2)$$

Fulfills also the constraint bellow

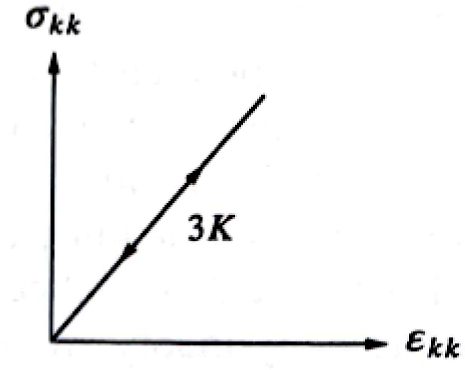
In general:

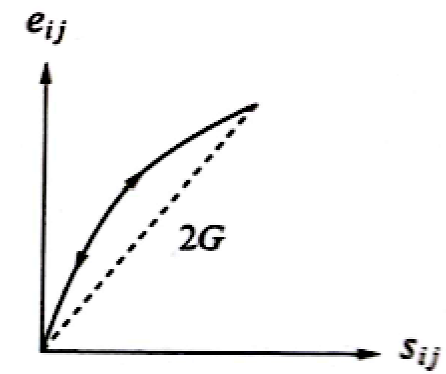
$$K = K(I_1, J_2), \qquad G = G(I_1, J_2)$$

$K,G-$ are not arbitrary but have to fulfill an additional constraint, *ref*. Ottosen *et al.*

$$\frac{\sigma_{kk}}{3}\frac{\partial}{\partial J_2}\left(\frac{1}{3K}\right) = \frac{\partial}{\partial I_1}\left(\frac{1}{2G}\right)$$

Metals and steel





(*Ref*. Ottosen *et al*. chap. 4.10)

Non-linear isotropic Hooke formulation

Rock, concrete & soil:

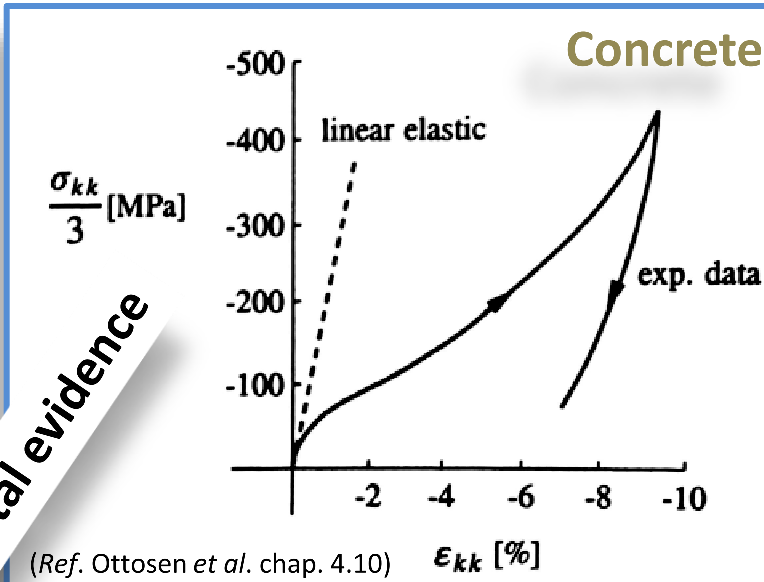
- ❑ Volumetric and *deviatoric* response are coupled
- • **Nonlinear** volumetric response
- • **Nonlinear** *deviatoric* response

$$\sigma_{ij} = \underbrace{3K\varepsilon_{kk}\delta_{ij}}_{\sigma_{kk}} + \underbrace{2Ge_{ij}}_{s_{ij}}$$

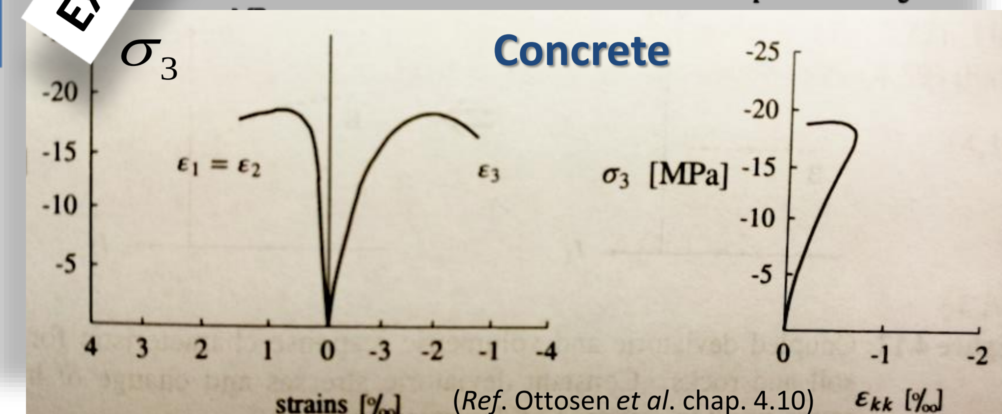
$$K = K(I_1, J_2), \qquad G = G(I_1, J_2)$$

Experimental evidence

Response to purely hydrostatic pressure



Hydrostatic compression of concrete ($\sigma_1 = \sigma_2 = \sigma_3 < 0$). Experimental data of Green and Swanson (1973); uniaxial compressive strength=48.5 MPa.



Uniaxial compression of concrete ($\sigma_1 = \sigma_2 = 0, \sigma_3 < 0$). Experimental data of Kupfer (1973); uniaxial compressive strength=18.7 MPa. a) stress-strain curves; b) development of volumetric strain ε_{ii}: first the volume decreases and then it increases.

Non-linear isotropic Hooke formulation

Rock, concrete & soil:

- ❑ Volumetric and *deviatoric* response are coupled
- **Nonlinear** volumetric response
- **Nonlinear** *deviatoric* response

$$\sigma_{ij} = \underbrace{3K\varepsilon_{kk}\delta_{ij}}_{\sigma_{kk}} + \underbrace{2Ge_{ij}}_{s_{ij}}$$

$$K = K(I_1, J_2), \qquad G = G(I_1, J_2)$$

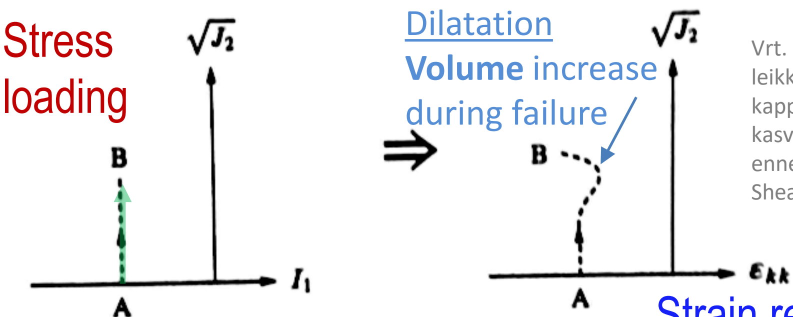
$$I_1 = \sigma_{kk}$$

$$J_2 = \frac{1}{2}s_{kl}s_{lk} = \frac{1}{2}\mathbf{s}:\mathbf{s}$$

$$s_{ij} \equiv \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$$

$$\sigma_{kk} = 3K(I_1, J_2)\varepsilon_{kk}$$

Const. varies



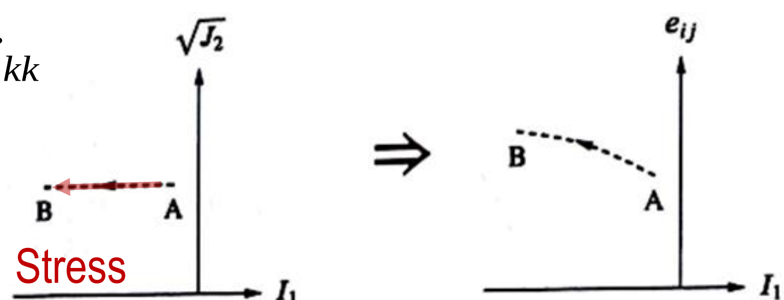
Vrt. Saven tai hiekan leikkauskoe, jossa kappaleen tilavuus kasvaa mäkimaallisesti ennen murtoa.
Shear box test

(*Ref*. Ottosen *et al*. chap. 4.10)

: Coupled deviatoric and volumetric response characteristic for concrete, soil and rocks. Constant hydrostatic stress and increasing deviatoric stresses result in a change of volumetric strain.

$$\sigma_{kk} = 3K(I_1, J_2)\varepsilon_{kk}$$

varies Const.



(*Ref*. Ottosen *et al*. chap. 4.10)

Coupled deviatoric and volumetric response characteristic for concrete, soil and rocks. Constant deviatoric stresses and change of hydrostatic stress result in a change of deviatoric strains.

Linear Thermo-Elasticity

Lineaarinen termo-elastisuus

- Lämpötilan vaikutuksesta materiaalit lämpö- laajenee (-kutistuu)

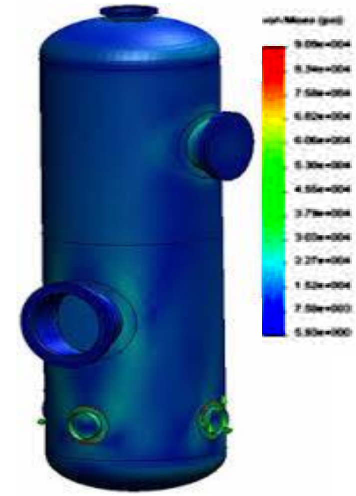
- Lämpötilasta johtuvat deformaatiot ovat verrannollisia lämpötilan muutoksee ΔT

$$\varepsilon_{ij}^{th} = \alpha_T \Delta T \delta_{ij}$$

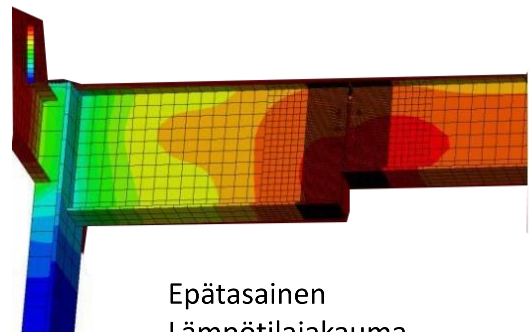
- **a) Tapaus vapaa**: kun rakenne on vapaa kinemaattisista esteistä $\Longrightarrow$ termisestä venymäkentästä ei aiheudu jännityksiä, mikäli venymäkenttä toteuttaa kompatibiliteetti-ehdot (tämä ehto vastaa ainakin sitä, että lämpötilajakauma on lineaarinen rakenteessa)
  - tasaisesti lämmitetty homogeeninen yhdestä pisteestä vapaasti roikkuva metallisauva lämpö laajenee ilman, että syntyy lämpöjännityksiä

- **b) Tapaus estetty**: sitä vastoin, rakenteessa kehittyy jännityskenttä jos .Not.a).
  - jos nyt pakotetaan *y.o.* sauvaa pitämään pituutensa vakiona lämmitessään, niin siinä syntyy lämpöjännityksiä
  - jos y.o. sauvaa, *esim.* lämmitetään epätasaisesti paksuussuunnassa (vaikkapa paraabelinen lämpötilajakauma), niin syntyy lämpöjännityksiä.

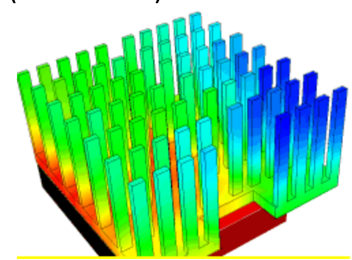
Epätasainen Lämpötilajakauma lasissa aiheutti lämpöjännityksiä, jotka aiheuttivat vaurioita (sen takia kuuma vettä miellummin hyvin ohkaseen lasiin)



Lämpöjännityksiä (Von-Mises)



Epätasainen Lämpötilajakauma



Epätasainen Lämpötilajakauma (jäähdytys-siili) - **radiator**

Estetty lämpölaajeneminen

Silta: liikunta-sauma, ettei tapahtuisi vaurioita

Lineaarinen termo-elastisuus

- Lämpötilan vaikutuksesta materiaalit lämpölaajenee (-kutistuu)
- Lämpötilasta johtuvat deformaatiot ovat verrannollisia läpötilan muutokseen ΔT

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  - tasaisesti lämmitetty homogeeninen yhdestä pisteestä vapaasti roikkuva metallisauva lämpölaajenee ilman, että syntyy lämpöjännityksiä
- **b) Tapaus estetty**: sitä vastoin, rakenteessa kehittyy jännityskenttä jos .Not.a).
  - jos nyt pakotetaan *y.o.* sauvaa pitämään pituutensa vakiona lämmitessään, niin siinä syyntyy lämpöjännityksiä
  - jos y.o. sauvaa, *esim*. lämmitetään epätasaisesti paksuussuunnassa (vaikkapa parabeelinen lämpötilajakauma), niin syntyy lämpöjännityksiä.

- Yleinen termo-mekaaninen rasitus: thermiset muodonmuutokset ynnäytyvät mekaanisiin muodonmuutoksiin.

$$\underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}}^{th} + \underline{\underline{\varepsilon}}^{m}$$

Lineaarisesti kimmoinen (hooke) isotrooppinen materiaali:

$$\varepsilon_{ij} = \underbrace{\frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}\sigma_{kk}}_{\text{Mekaaninen}} + \underbrace{\alpha_T \Delta T \delta_{ij}}_{\text{Terminen}}$$

kääntämällä

Termo-elastinen materiaalimalli:

$$\sigma_{ij} = \frac{E}{(1+\nu)(1-2\nu)}\left[\nu\delta_{ij}\varepsilon_{kk} + (1-2\nu)\varepsilon_{ij} - (1+\nu)\alpha\underbrace{(\theta-\theta_0)}_{\Delta T}\delta_{ij}\right]$$

Material	Linear coefficient α at 20 °C (10^{-6} K^{-1})
Aluminium	23
Steel	**12**
Concrete	**11**
Copper	17
Diamond	1
Ethanol	250
Glass	8.5
Gold	14
Iron	12
PVC	52
Quartz	0.33
Silver	18
Water	69

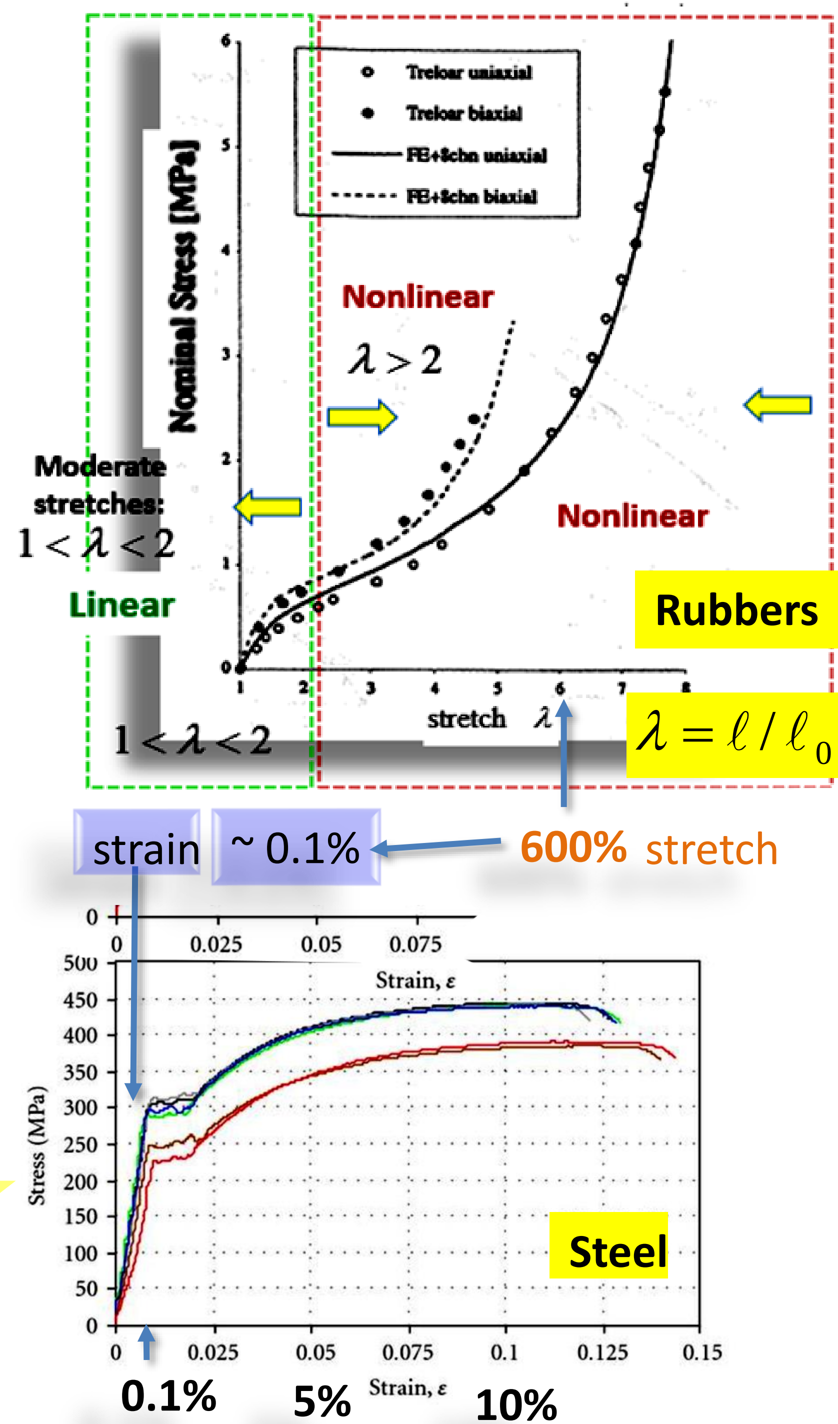
IMPORTANT

! Lähellä olevia arvoja ... **hyvä asia** teräsbetonirakenteissa. Mieti miksi?

Hyperelasticity

Application example:
Rubber or
rubber-like Elasticity

- Rubbers, artificial rubbers, elastomers and biological tissues are good example for studying **hyperelasticity** or equivalently non-linearly elasticity with high values of **stretch**ing **till 600%** far beyond small-strain elasticity with 0.1 – 1% of strain of common engineering materials.
- In addition, incompressibility of such materials, rubbers, can be easily accounted
- **For hyperplastic materials stress–strain relationship derives from a strain energy density function**



Hyper-elasticity

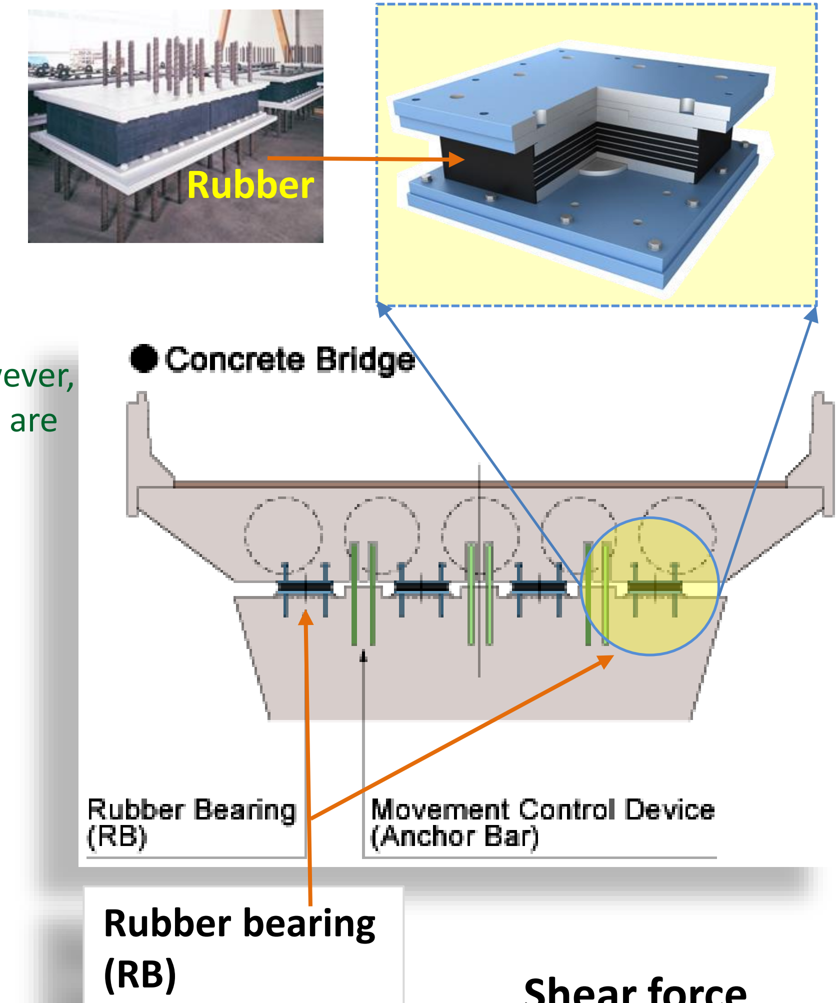
Rubber or rubber-like Elasticity

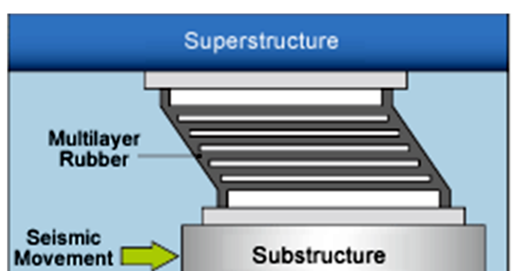
This is an important class of materials for an engineer to know. However, in this course, it is sufficient to know that such class exists and what are their key important mechanical response features to know for an engineer.

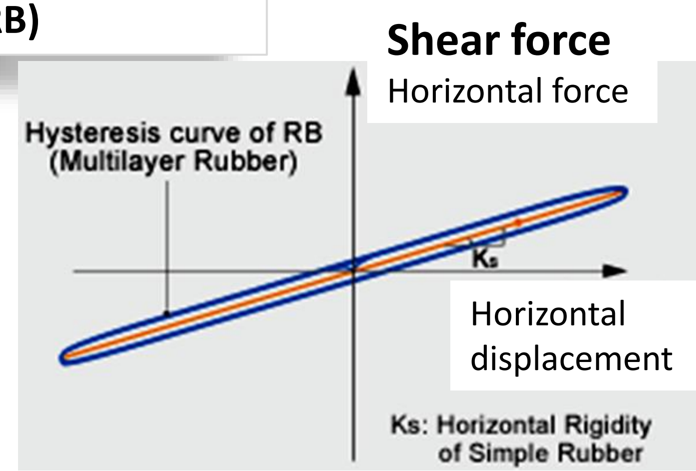
NB. In general, the response of a *polymer* dependent strongly on *temperature*, *strain history* and *loading rate*. Some aspects of such behavior will be detailed in next section treating of *viscoelasticity* and little bit when treating *rubber elasticity* in the current section.

In this section, we consider rubbery state only

It is known that that polymers have various regimes of mechanical response: **glassy**, **viscoelastic** and **rubbery**. These various regimes can be identified *via* dynamical loading.







On Thermodynamics of Rubber

Enthalpic (energetic) and Entropic forces

Cf. Appendix 2 for more details

Got interested? Read (elective) more from:

Original work
By Professor Treloar
is the guru…

A. N. GENT
The University of Akron
Akron, Ohio

1 Rubber Elasticity: Basic Concepts and Behavior

I. Introduction
II. Elasticity of a Single Molecule
III. Elasticity of a Three-Dimensional Network of Polymer Molecules
IV. Comparison with Experiment
V. Continuum Theory of Rubber Elasticity
VI. Second-Order Stresses
VII. Elastic Behavior Under Small Deformations
VIII. Some Unsolved Problems in Rubber Elasticity
Acknowledgments
References

Science and Technology of Rubber, Third Edition

The elasticity and related properties of rubbers

1973 Rep. Prog. Phys. 36 755

L R G TRELOAR

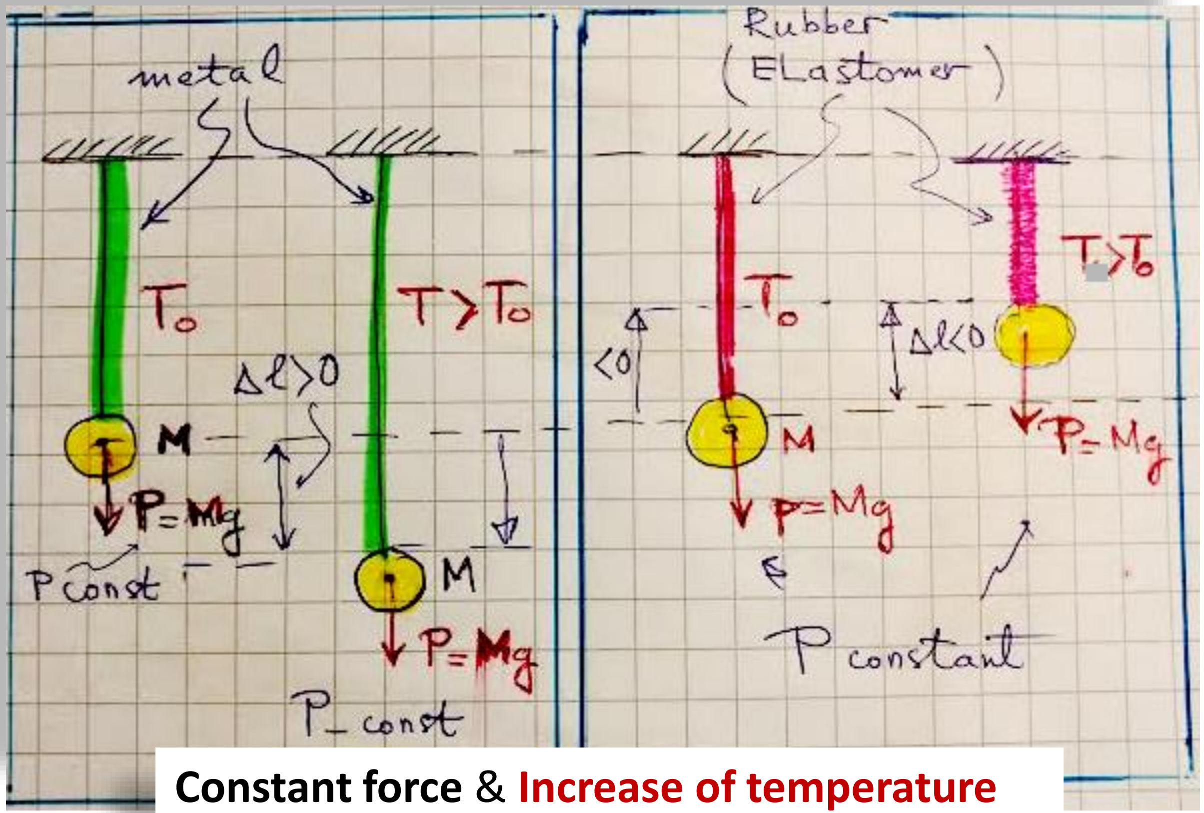
Department of Polymer and Fibre Science, University of Manchester Institute of Science and Technology, PO Box no 88, Sackville Street, Manchester M60 1QD, UK

W. Gilbert's experiment

Few word on elastomers

Metal in tension

Elastomer in tension

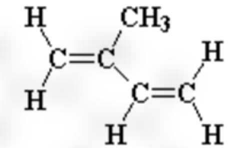


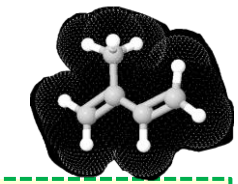
Constant force & **Increase of temperature**

Observation*: For a fixed force, the **stretch decreases** with temperature increase $\lambda = \ell / \ell_0$

On thermodynamics of elastomers

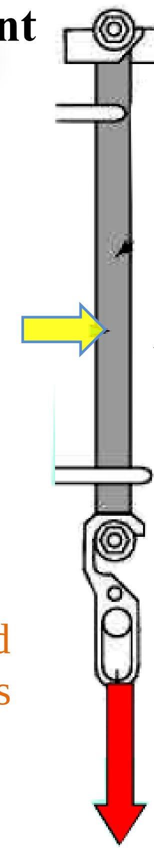
Isoprene





For details on *Thermodynamics of Rubber*, refer **to Appendix 2**

W. Gilbert's experiment



$\lambda_1 = \ell / \ell_0$

Elastomer
in extension

Constant force &
Increase of temperature

Observation*: For a fixed force, the stretch decreases with temperature increase

F

Experimental law:

$$\sigma \approx \frac{F}{S_0} \approx A\theta \cdot f(\lambda_i)$$

Statistical physics:

Elastic strain energy of the rubber chain assembly:

Boltzmann's constant

temperature

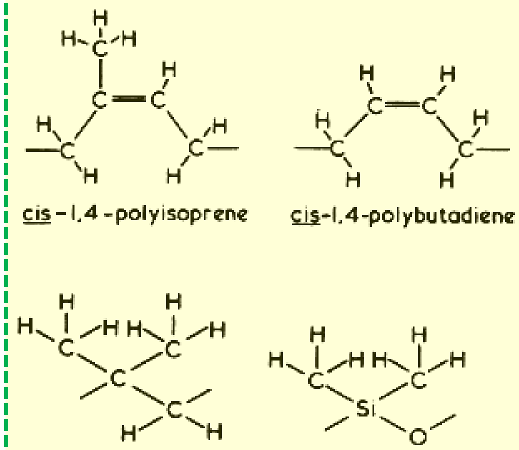
Principle stretches

$$W_G = \frac{1}{2} N k \theta (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)$$

Number of chains in the assembly

Statistical physics

Example of repeat units for some common elastomer molecules



cis-1,4-polyisoprene   cis-1,4-polybutadiene

poly(iso-butylene)   poly(dimethylsiloxane)

Ref: Rubber Elasticity: *Basic Concepts and Behavior*
A. N. Gent

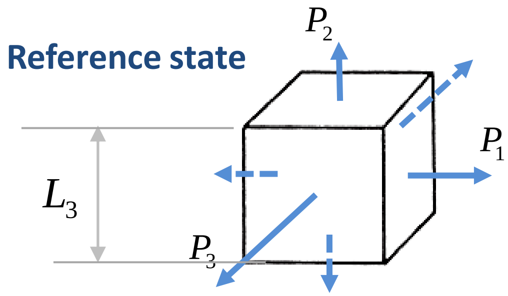
The glass transition temperature for usual rubbers is usually far bellow room temperatures.

Natural rubber: $T_g \sim -73\ ^{o}\mathrm{C}$

Range: $T_g \sim -20^{o}\mathrm{C} \div -130^{o}\mathrm{C}$

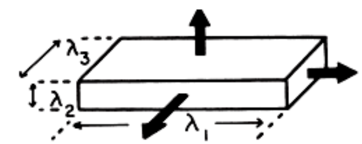
NB. * For normal range of temperatures well above the glass transition temperature and bellow those where chemical reaction begin to take part (thermal degradation etc.)

Terminology and some definitions



Consider a rectangular material block reference configuration: $L_1 \times L_2 \times L_3$

Apply forces P_i on its faces in this reference state: P_1, P_2, P_3



Principle stretches: $\lambda_i = \ell_i / L_i, \quad i = 1,2,3$

Actual length in the current state

Initial length in the reference state

Nominal stresses: $s_i \equiv \sigma_i^{(\mathrm{nom})} = \dfrac{P_i}{L_j L_k}, \quad j \neq i, k \neq i \qquad i = 1,2,3$

True stresses: $t_i \equiv \sigma_i^{(\mathrm{true})} = \dfrac{P_i}{\ell_j \ell_k}, \quad j \neq i, k \neq i$

True stresses:

$$\sigma_i^{(\mathrm{true})} = \frac{P_i}{\ell_j \ell_k}, \quad j \neq i, k \neq i$$

(They are not work-conjugate to stretches)

$$\frac{\sigma_i^{(\mathrm{nom})}}{\sigma_i^{(\mathrm{true})}} = \frac{\ell_j \ell_k}{L_j L_k} = \lambda_j \lambda_k$$

ΔF **Helmholtz free energy:** $\Psi = u - sT \qquad F = U - TS$

Absolute temperature

Internal energy

Entropy

A thermodynamic potential

$\Delta\psi$ Helmholtz specific free energy specific density: $\psi \equiv \dfrac{\Psi}{V}, \quad V = L_1 L_2 L_3$

If isothermal process

Work differential (by the external mechanical forces on elongations $\mathrm{d}\ell_i$):

$$\mathrm{d}W_{\mathrm{ext}} = \sum_i P_i \mathrm{d}\ell_i = \sum_i \sigma_i^{(\mathrm{nom})} \cdot \mathrm{d}\lambda_i \boxed{= \mathrm{d}\psi} \rightarrow$$

We see that nominal stress are work-conjugate to stretches

A thermodynamic potential

$$\psi = \psi(\lambda_1, \lambda_2, \lambda_3)$$

$$\Downarrow$$

$$\sigma_i^{(\mathrm{nom})} = \frac{\partial \psi(\lambda_1, \lambda_2, \lambda_3)}{\partial \lambda_i}$$

$$\sigma_i^{(\mathrm{true})} = \frac{\partial \psi(\lambda_1, \lambda_2, \lambda_3)}{\lambda_j \lambda_k \partial \lambda_i},$$

$$j \neq i, k \neq i, j \neq k$$

Terminology and some definitions

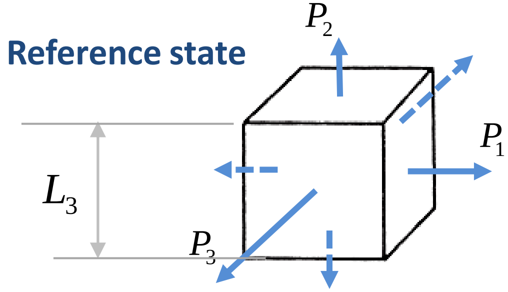
$$\mathrm{d}W_{ext} = \sum_{i=1}^{3} P_i \mathrm{d}\ell_i = \sum_i P_i L_i \mathrm{d}\lambda_i$$

$$= \sum_i \sigma_i \underbrace{L_j L_k}_{i\neq j,\, i\neq k} L_i \mathrm{d}\lambda_i = \sum_i \sigma_i V \mathrm{d}\lambda_i$$

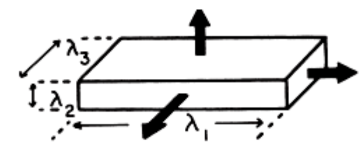
$$\Longrightarrow \psi = W \equiv \frac{W_{ext}}{V} = \sum_i \sigma_i \mathrm{d}\lambda_i,$$

σ_i is the nominal stress

Reference state



Actual state



True stresses:

$$\sigma_i^{(\mathrm{true})} = \frac{P_i}{\ell_j \ell_k}, \quad j \neq i, k \neq i$$

(They are not work-conjugate to stretches)

$$\frac{\sigma_i^{(\mathrm{nom})}}{\sigma_i^{(\mathrm{true})}} = \frac{\ell_j \ell_k}{L_j L_k} = \lambda_j \lambda_k$$

Absolute temperature

ΔF **Helmholtz free energy:** $\Psi = u - sT$ $\quad F = U - TS$

Internal energy

Entropy

A thermodynamic potential

$\Delta\psi$ Helmholtz specific free energy specific density: $\psi \equiv \dfrac{\Psi}{V}, \; V = L_1 L_2 L_3$

If isothermal process

Work differential (by the external mechanical forces on elongations $\mathrm{d}\ell_i$): $\mathrm{d}W_{\mathrm{ext}} = \sum_i P_i \mathrm{d}\ell_i = \sum_i \sigma_i^{(\mathrm{nom})} \cdot \mathrm{d}\lambda_i \boxed{= \mathrm{d}\psi} \rightarrow$

We see that nominal stress are work-conjugate to stretches

A thermodynamic potential

$$\psi = \psi(\lambda_1, \lambda_2, \lambda_3)$$

$$\Downarrow$$

$$\sigma_i^{(\mathrm{nom})} = \frac{\partial \psi(\lambda_1, \lambda_2, \lambda_3)}{\partial \lambda_i}$$

$$\sigma_i^{(\mathrm{true})} = \frac{\partial \psi(\lambda_1, \lambda_2, \lambda_3)}{\lambda_j \lambda_k \partial \lambda_i},$$

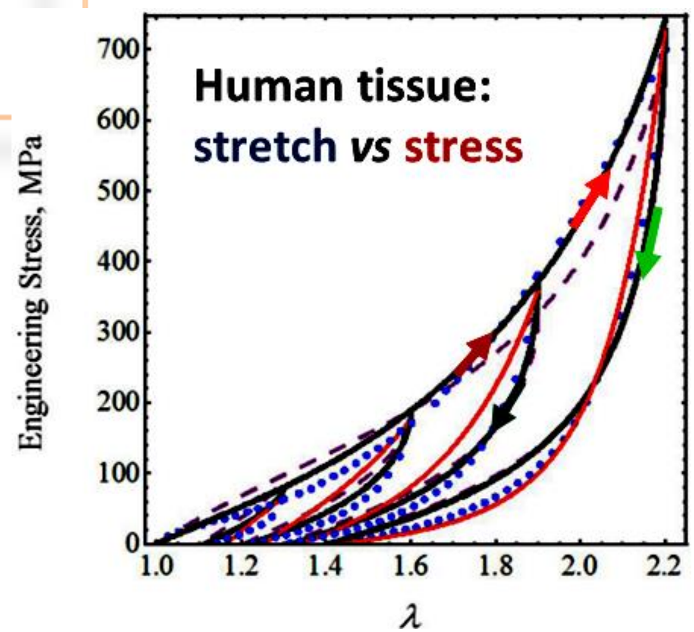
$$j \neq i, k \neq i, j \neq k$$

Hyperelasticity

Elastomers (rubbers) or elastomers-like material

Definitions and concepts
Deformation of rubbers
Principal stretches and their invariants
Overview of some classical models

For more details on *Thermodynamics of Rubber*, refer **to Appendix 2**



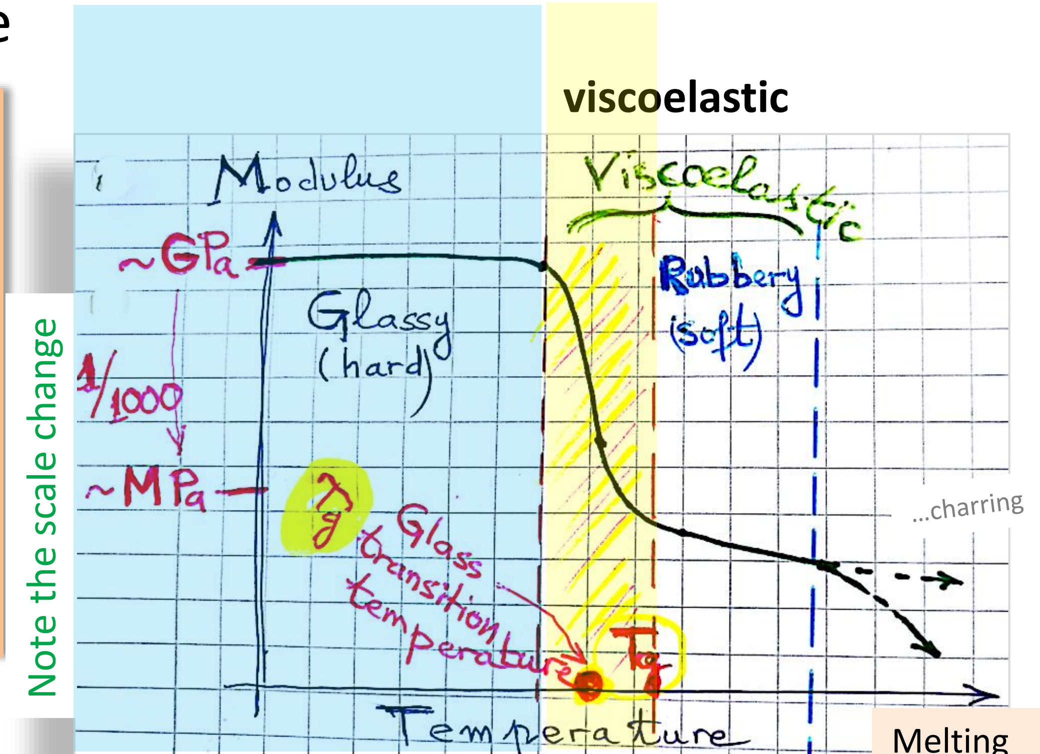
Materials **2014**, 7(1), 441-456; doi:10.3390/ma7010441

The glass transition temperature

The **glass transition** is a reversible transition in **amorphous materials** from a hard "glassy" state into a **viscous** or **rubbery** state with increase of temperature.

Ex. thermoplastics, glass

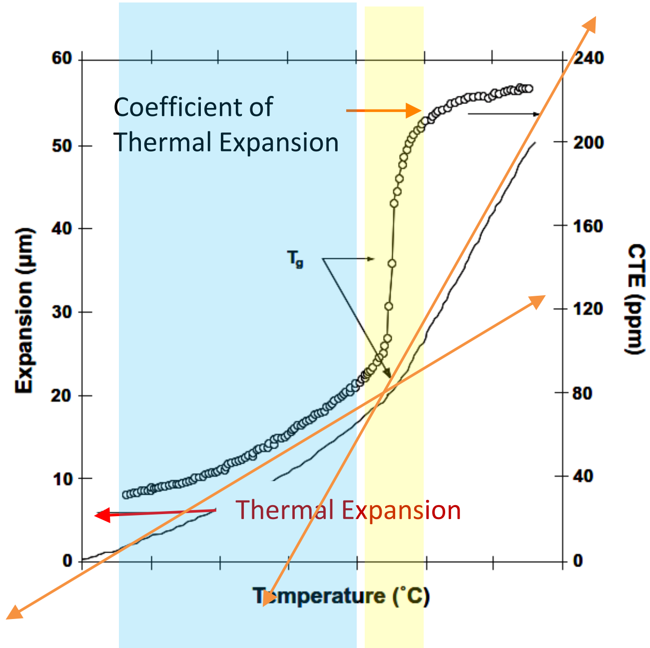
N.B! Glass transition is not a phase transition (*i.e*.., not a phase change).



$T_g < T_m$

Glass transition temperature (or zone)

Melting temperature



The **glass transition temperature** for **usual rubbers** is usually far bellow room temperatures

Natural rubber:

$$T_g \sim -20^{o}\,\mathrm{C} \;\div\; -130^{o}\,\mathrm{C}$$

Temperatures zone for normal use of rubbers (~room temperatures)

Kinematics of rubbers

Consider a rectangular material block reference configuration: $L_1 \times L_2 \times L_3$

Apply forces F_i on its faces in this reference state:

Principle stretches: $\lambda_i = \ell_i / L_i, \quad i = 1,2,3$

Actual length in the current state

Initial length in the reference state

Motion (deformation): $x_i = \lambda_i X_i, \quad i = 1,2,3$

Initial volume $\quad \mathrm{d}V_0 = L_1 L_2 L_3$

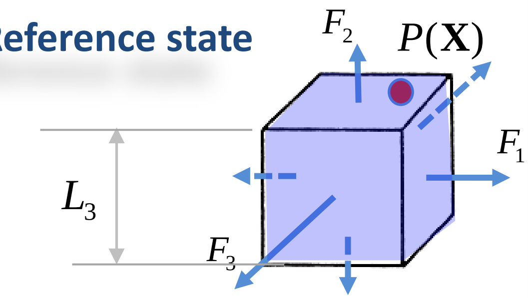
Deformed volume $\quad \mathrm{d}V = \ell_1 \ell_2 \ell_3 = \lambda_1 L_1 \cdot \lambda_2 L_2 \cdot \lambda_3 L_3 = \lambda_1 \lambda_2 \lambda_3 \cdot L_1 L_2 L_3 = \lambda_1 \lambda_2 \lambda_3 \mathrm{d}V_0$

Jacobian of the motion: $\quad J \equiv \mathrm{d}V / \mathrm{d}V_0 \neq 0, \qquad J(0) \equiv \mathrm{d}V(0) / \mathrm{d}V_0 = \mathrm{d}V_0 / \mathrm{d}V_0 = 1$

Material point in reference configuration: $P(X_1, X_2, X_3) \equiv P(\mathbf{X})$

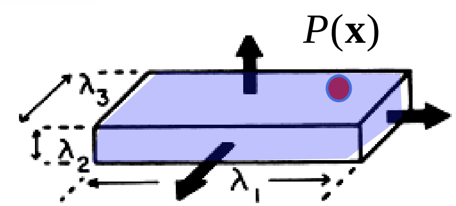
$X_i, \quad i = 1,2,3$

Reference state



Material point in deformed configuration: $P(x_1, x_2, x_3) \equiv P(\mathbf{x})$

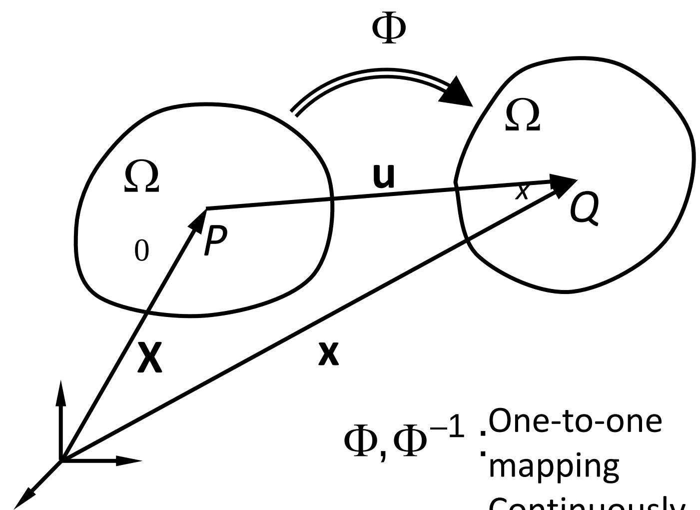
Actual state $\quad x_i, \quad i = 1,2,3$



Deformation and Mapping

(This is a **recall** from **continuum mechanics**)

1-1-**mapping**



- Initial domain Ω_0 is deformed to Ω_x
- **X**: material point in Ω_0 (reference configuration)
- $\boldsymbol{x}$ material point at location $\boldsymbol{x}$ in Ω_x (actual configuration)
- Material point **P** in Ω_0 is deformed to Q in Ω_x

displacement

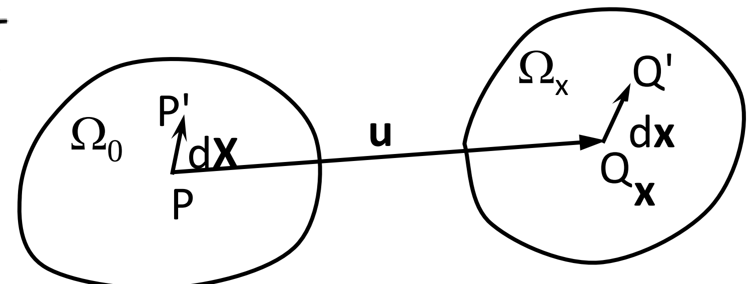
$$\boldsymbol{x} = \mathbf{X} + \mathbf{u}(\mathbf{X},t) \equiv \phi(\mathbf{X},t)$$

Φ, Φ^{-1} : One-to-one mapping Continuously differentiable

Deformation Gradient $\mathrm{d}\boldsymbol{x} = \mathbf{F}\mathrm{d}\boldsymbol{X}$

- Infinitesimal length $\mathrm{d}\boldsymbol{X}$ in Ω_0 deforms to $\mathrm{d}\boldsymbol{x}$ in Ω_x

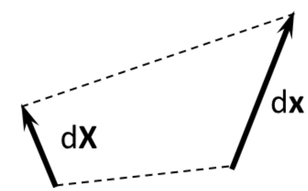
Since the mapping is continuously differentiable, one obtains



$$\mathrm{d}\boldsymbol{x} = \frac{\partial \phi(\mathbf{X},t)}{\partial \mathbf{X}} \mathrm{d}\mathbf{X} \equiv \mathbf{F}\mathrm{d}\mathbf{X}$$

Deformation gradient (2nd-order tensor)

Initially,
$\mathbf{u} = 0 \Rightarrow \mathbf{F} = \mathbf{1},$



$$F_{ij} = \partial \phi_i / \partial X_j \quad , \quad F_{ij} = \delta_{ij} + \frac{\partial \mathbf{u}(\mathbf{X},t)}{\partial X_j}, \qquad \mathbf{F} = \mathbf{1} + \nabla_X \mathbf{u}$$

Because one-to-one mapping: $\mathbf{dX} = \mathbf{F}^{-1}\mathbf{dx}$

$\det \mathbf{F} \equiv J > 0.$

F includes both deformation and rigid-body rotation

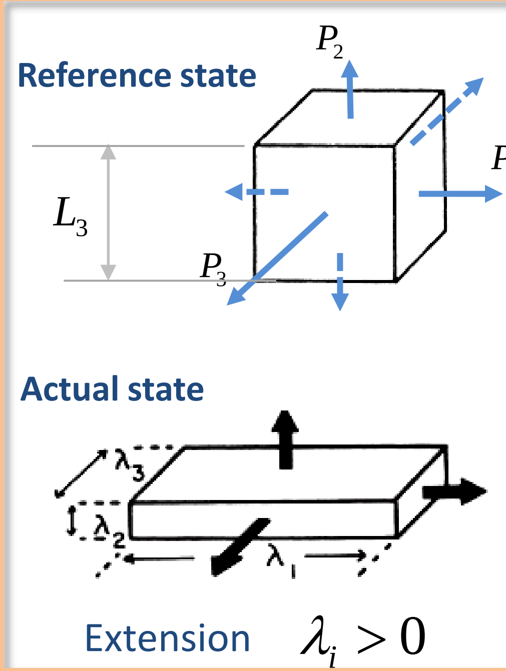
Example – uniform extension

- *Uniform extension* of a cube in all three directions

$$x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3$$

- Continuity requirement: $\det \mathbf{F} \equiv J < \infty,$
- Deformation gradient: $J \equiv \det \mathbf{F} = \lambda_1 \lambda_2 \lambda_3 > 0$

$$\mathbf{F} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$



- $\lambda_1 = \lambda_2 = \lambda_3 \equiv \lambda$: ***uniform expansion*** (dilatation) or contraction
- Volume change
  - initial volume: $dV_0 = dX_1 dX_2 dX_3$
  - deformed volume: $dV_x = dx_1 dx_2 dx_3 = \lambda_1 \lambda_2 \lambda_3 dX_1 dX_2 dX_3 = \lambda_1 \lambda_2 \lambda_3 dV_0$

$$dV = J dV_0 \Rightarrow dV / dV_0 = J = \det \mathbf{F}$$

Jacobian of the deformation mapping

Example – Uniaxial Tension

- Uniaxial tension of incompressible material $(\lambda_1 = \lambda > 1)$

$$x_1 = \lambda_1 X_1$$
$$x_2 = \lambda_2 X_2$$
$$x_3 = \lambda_3 X_3$$

- From incompressibility

$$\lambda_1 \lambda_2 \lambda_3 = 1 \quad \Rightarrow \quad \lambda_2 = \lambda_3 = \lambda^{-1/2}$$

- Deformation gradient and deformation tensor

$$F_{ij} = \delta_{ij} + \frac{\partial \mathbf{u}(\mathbf{X}, t)}{\partial X_j}$$
$$\mathbf{F} = \mathbf{1} + \nabla_{\mathbf{X}} \mathbf{u}$$

$$\mathbf{F} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1/2} & 0 \\ 0 & 0 & \lambda^{-1/2} \end{bmatrix} \qquad \boldsymbol{C} = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & \lambda^{-1} \end{bmatrix}$$

Right Cauchy-Green Deformation Tensor

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}} \mathbf{F}$$

- Green-Lagrange Strain: $\mathbf{E} = \frac{1}{2}(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{1}) = \mathbf{0}$

$$E_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i} \cdot u_{k,j})$$

Linear Small deformations

Nonlinear large Deformations component

$$\mathbf{E} = \frac{1}{2} \begin{bmatrix} \lambda^2 - 1 & 0 & 0 \\ 0 & \lambda^{-1} - 1 & 0 \\ 0 & 0 & \lambda^{-1} - 1 \end{bmatrix}$$

Engineering Strain:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \lambda - 1 & 0 & 0 \\ 0 & \lambda^{-1/2} - 1 & 0 \\ 0 & 0 & \lambda^{-1/2} - 1 \end{bmatrix}$$

Example – Uniaxial Tension

$$E_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i} + u_{i,k} \cdot u_{k,j})$$

Linear
Small deformations

Nonlinear
large Deformations component

G-L Strain (Green-Lagrange)

$$\mathbf{E} = \tfrac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{1}) = 0$$

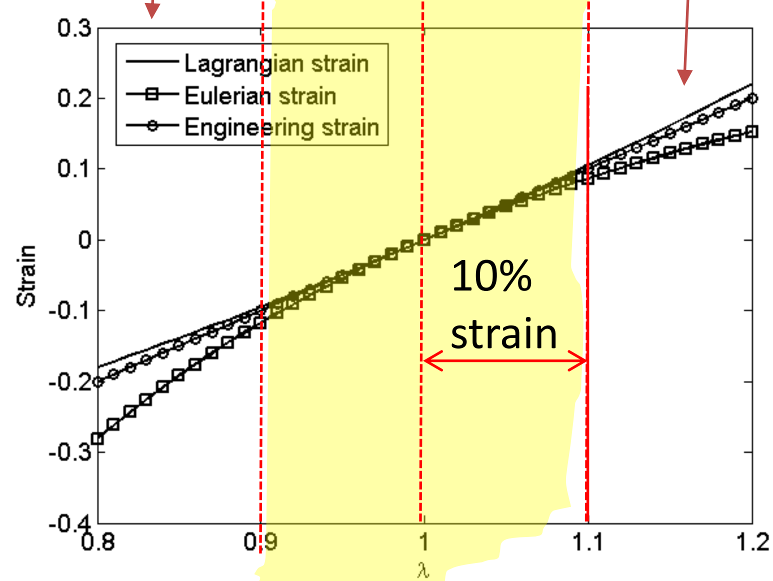
$$\mathbf{E} = \frac{1}{2}\begin{bmatrix} \lambda^2 - 1 & 0 & 0 \\ 0 & \lambda^{-1} - 1 & 0 \\ 0 & 0 & \lambda^{-1} - 1 \end{bmatrix}$$

Engineering Strain (small deformations)

$$\varepsilon = \begin{bmatrix} \lambda - 1 & 0 & 0 \\ 0 & \lambda^{-1/2} - 1 & 0 \\ 0 & 0 & \lambda^{-1/2} - 1 \end{bmatrix}$$

Nonlinear
large Deformations effect

Linear
~ domain of small deformations



$$E_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i} \cdot u_{k,j})$$

Linear
Small deformations

Nonlinear
large Deformations component

Strain invariants

- We consider isotropic materials
  - Material frame indifference: no matter what coordinate system is chosen, the response of the material is identical
  - The components of a deformation tensor depends on coordinate system
  - Three invariants of **C** are independent of coordinate system
- **Stretch Invariants**

$$I_1 = \mathrm{tr}(\mathbf{C}) = C_{11} + C_{22} + C_{33} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$

$$I_2 = \tfrac{1}{2}\left[(\mathrm{tr}\,\mathbf{C})^2 - \mathrm{tr}(\mathbf{C}^2)\right] = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2$$

$$I_3 = \det \mathbf{C} = \lambda_1^2\lambda_2^2\lambda_3^2$$

Right Cauchy-Green Deformation Tensor

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$$

  - In order to be material frame indifferent, constitutive laws (properties properties) must be expressed using invariants
  - For incompressibility, $I_3 = 1$

For the initial configuration one have: $\mathbf{u} = 0 \Rightarrow \mathbf{F} = \mathbf{1}, \quad \mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \mathbf{1} \Rightarrow I_1 = 3, I_2 = 3, I_3 = 1$

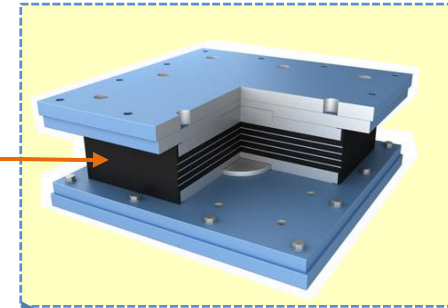
Hyperelasticity

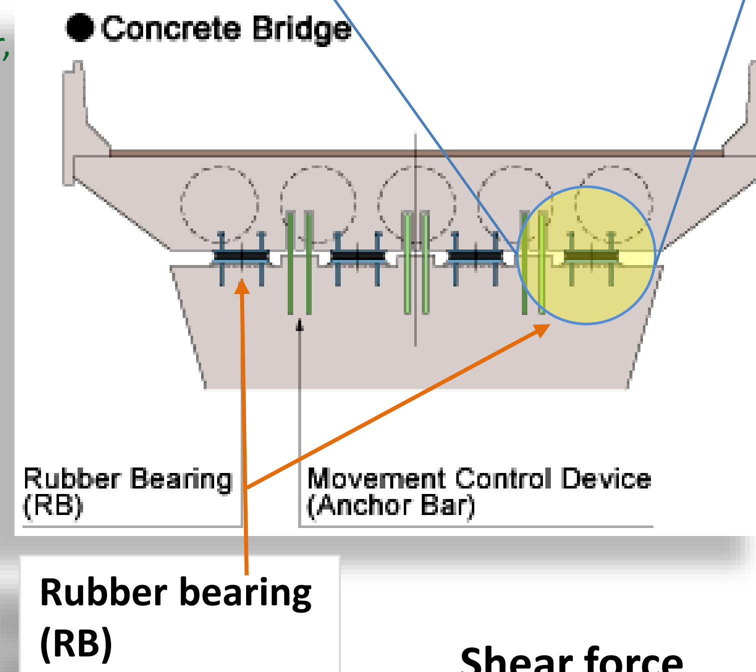
Rubber or

rubber-like Elasticity

Hyperelasticity

Rubber or rubber-like Elasticity



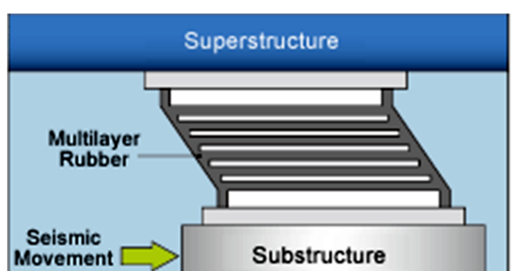


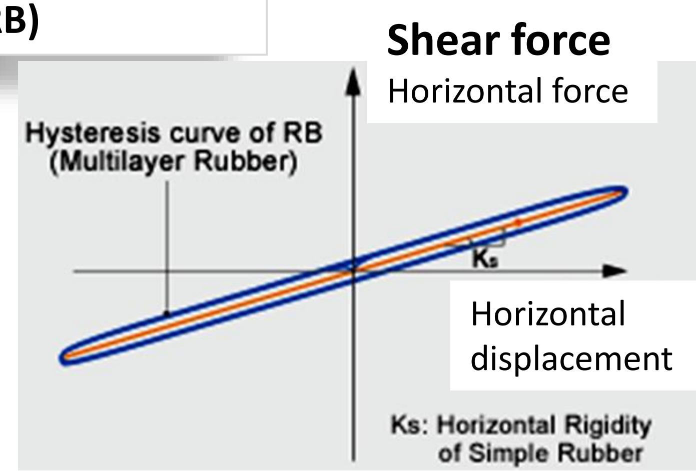
This is an important class of materials for an engineer to know. However, in this course, it is sufficient to know that such class exists and what are their key important mechanical response features to know for an engineer.

NB. In general, the response of a *polymer* dependent strongly on *temperature*, *strain history* and *loading rate*. Some aspects of such behavior will be detailed in next section treating of *viscoelasticity* and little bit when treating *rubber elasticity* in the current section.

In this section, we consider rubbery state only

It is known that that polymers have various regimes of mechanical response: **glassy**, **viscoelastic** and **rubbery**. These various regimes can be identified *via* dynamical loading.





On Thermodynamics of Rubber

Enthalpic (energetic) and Entropic forces

Got interested? Read (elective) more from:

A. N. GENT
The University of Akron
Akron, Ohio

1

Rubber Elasticity: Basic Concepts and Behavior

I. Introduction
II. Elasticity of a Single Molecule
III. Elasticity of a Three-Dimensional Network of Polymer Molecules
IV. Comparison with Experiment
V. Continuum Theory of Rubber Elasticity
VI. Second-Order Stresses
VII. Elastic Behavior Under Small Deformations
VIII. Some Unsolved Problems in Rubber Elasticity
Acknowledgments
References

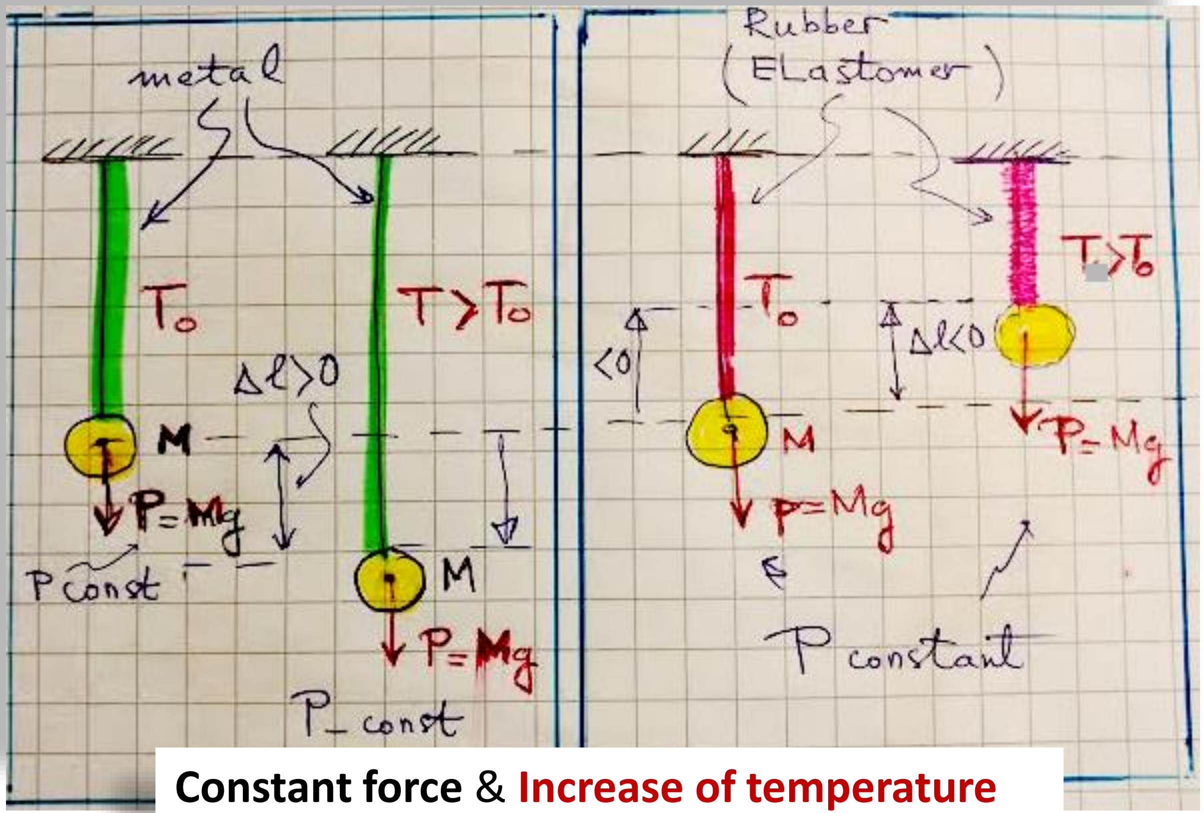
Science and Technology of Rubber, Third Edition

W. Gilbert's experiment

Few word on elastomers

Metal in tension

Elastomer in tension



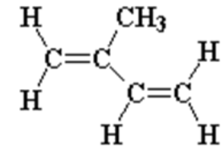
Constant force & **Increase of temperature**

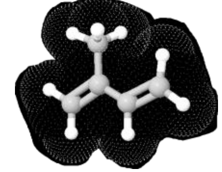
Observation*: For a fixed force, the **stretch decreases** with temperature increase

$$\lambda = \ell / \ell_0$$

On thermodynamics of elastomers

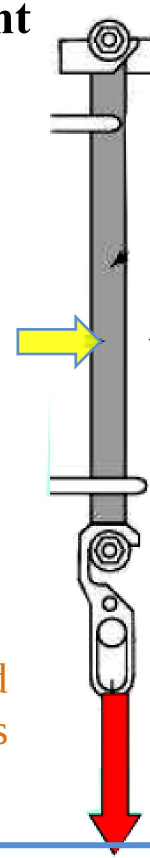
Isoprene





Good to know

W. Gilbert's experiment



$$\lambda_1 = \ell / \ell_0$$

Elastomer
in extension

Constant force &
Increase of temperature

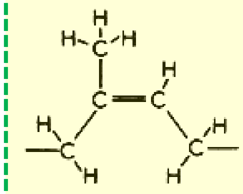
Observation*: For a fixed force, the stretch decreases with temperature increase

F

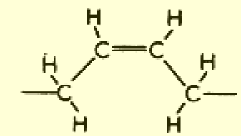
$$\boxed{\sigma \approx \frac{F}{S_0} \approx A\theta \cdot f(\lambda_i)}$$

temperature

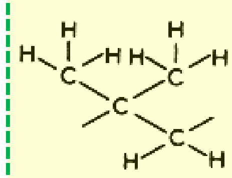
Example of repeat units for some common elastomer molecules



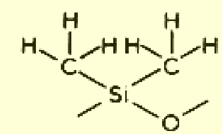
cis-1,4-polyisoprene



cis-1,4-polybutadiene



poly(iso-butylene)



poly(dimethylsiloxane)

Ref: Rubber Elasticity: *Basic Concepts and Behavior*
A. N. Gent

The glass transition temperature for usual rubbers is usually far bellow room temperatures.

Natural rubber: $T_g \sim -73\ ^{o}\mathrm{C}$

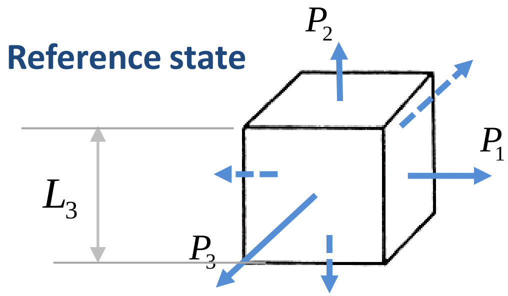
Range: $T_g \sim -20^{o}\mathrm{C} \div -130^{o}\mathrm{C}$

NB. * For normal range of temperatures well above the glass transition temperature and bellow those where chemical reaction begin to take part (thermal degradation etc.)

Terminology and some definitions

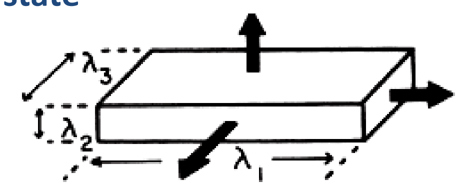
Consider a rectangular material block reference configuration: $L_1 \times L_2 \times L_3$

Reference state



Apply forces P_i on its faces in this reference state: P_1, P_2, P_3

Actual state



Principle stretches: $\lambda_i = \ell_i / L_i, \quad i = 1,2,3$

Actual length in the current state

Initial length in the reference state

Nominal stresses: $f_i \equiv \sigma_i = \dfrac{P_i}{L_j L_k}, \quad j \neq i, k \neq i$

$i = 1,2,3$

True stresses:

$$\sigma_i^{(\text{true})} = \frac{P_i}{\ell_j \ell_k}, \quad j \neq i, k \neq i$$

(They are not work-conjugate to stretches)

Absolute temperature

ΔF **Helmholtz free energy:** $\Psi = u - sT \qquad F = U - TS$

A thermodynamic potential

Internal energy

Entropy

A thermodynamic potential

$\Delta\psi$ Helmholtz free energy specific density: $\psi \equiv \dfrac{\Psi}{V}, \quad V = L_1 L_2 L_3$

If isothermal process

Work differential (by the external mechanical forces on elongations $\mathrm{d}\ell_i$):

$$\mathrm{d}W_{\text{ext}} = \sum_i P_i \mathrm{d}\ell_i = \sum_i \boxed{\sigma_i \cdot \mathrm{d}\lambda_i} = \mathrm{d}\psi \rightarrow$$

$$\boxed{\begin{gathered}\sigma_i = \frac{\partial \psi(\lambda_1, \lambda_2, \lambda_3)}{\partial \lambda_i} \\ \Uparrow \\ \psi = \psi(\lambda_1, \lambda_2, \lambda_3)\end{gathered}}$$

We see that nominal stress are work-conjugate to stretches

Thermodynamics of rubber – enthalpic (energetic) and entropic forces

The change of internal energy of a (closed) system can be achieved by:

(Static case – zero kinetic energy)

$$dU = TdS - pdV + fdL \quad \textbf{(1)}$$

Change in internal energy of a system

Heating of the system

Force

Changing the system volume

streching the system

Deforming the system

(This is from your previous courses of thermodynamics)

$$F \equiv \psi \qquad \sigma_i = \frac{\partial \psi}{\partial \lambda_i}$$

Helmholtz free energy: (state function)

Change of Helmholtz free energy

$$F = U - TS \Rightarrow dF = \underbrace{dU}_{= TdS - pdV + fdL \ (1)} - TdS - SdT$$

$$f = \frac{\partial \psi}{\partial L} = \frac{\partial F}{\partial L}$$

$$\Rightarrow dF = -SdT - pdV + \boxed{f} \cdot dL$$

← (total differential)

force

$$dF = \left(\frac{\partial F}{\partial T}\right)_{V,L} dT + \left(\frac{\partial F}{\partial V}\right)_{T,L} dV + \boxed{\left(\frac{\partial F}{\partial L}\right)_{T,V}} \cdot dL$$

(can de written in terms of a total differential)

Metals: (for typical cristalline solids) $|f_E| >> |f_S|$, $f_S = 0$

Ideal Polymer network: $|f_S| >> |f_E|$, $f_E = 0$

Applied **force:**

$$\Rightarrow f = \left(\frac{\partial F}{\partial L}\right)_{T,V} = \underbrace{\left(\frac{\partial U}{\partial L}\right)_{T,V}}_{\equiv f_E} \underbrace{- T\left(\frac{\partial S}{\partial L}\right)_{T,V}}_{\equiv f_S} \boxed{\equiv f_E + f_S} \quad \textbf{(2)}$$

$$f = \frac{\partial \psi}{\partial L} = \frac{\partial F}{\partial L}$$

Energetic contribution

Entropic contribution

Origin of the force:
Energetic and **entropic**

$$\boxed{f = f_E + f_S}$$

Hyper-elasticity - rubbers

Classical models

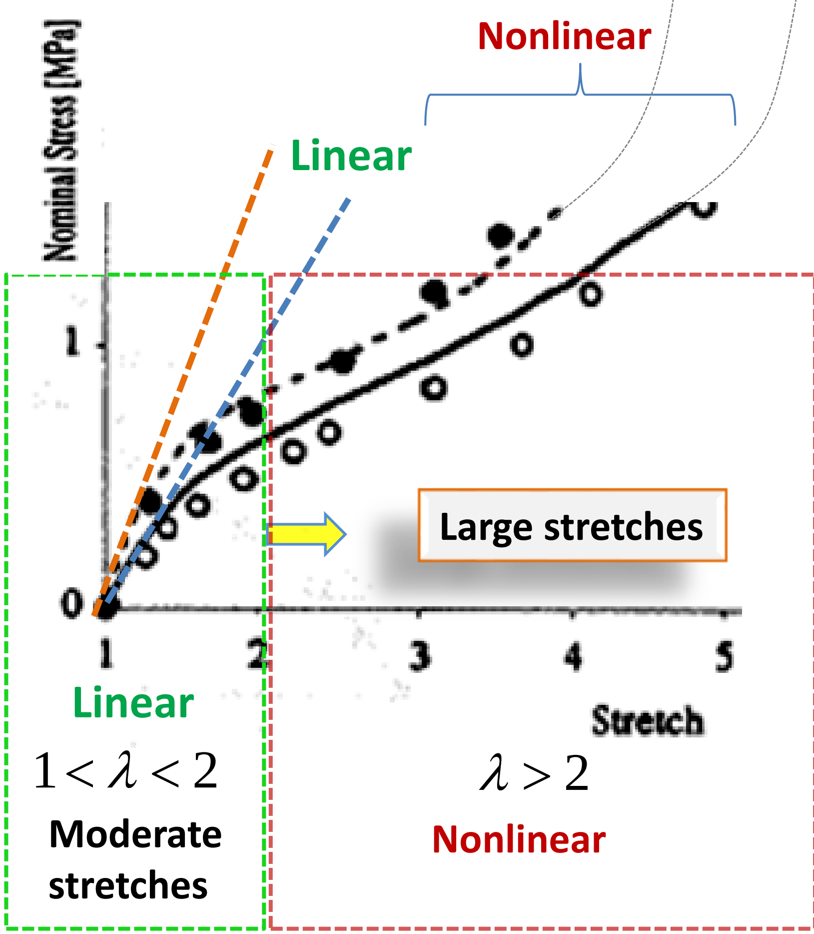
Hyperelasticity - rubbers

Observation: High degree of Nonlinearities

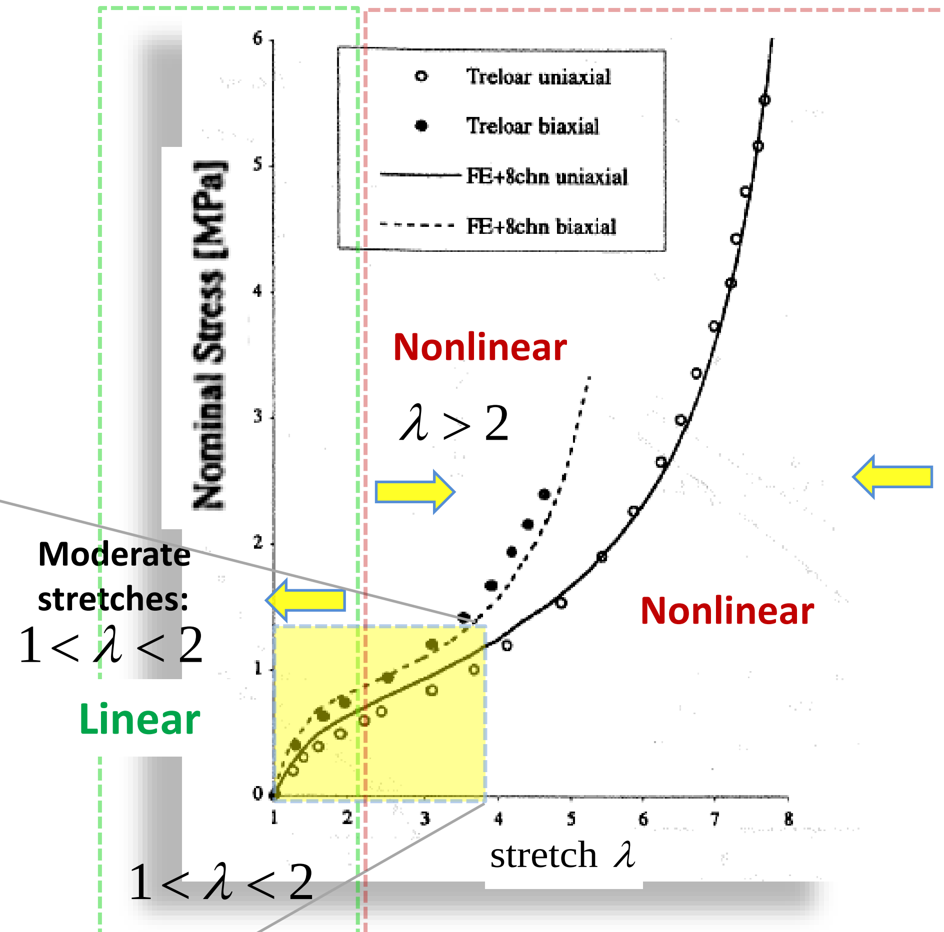
$\Downarrow$

There is a need for a non-linear elastic model

Stretch stiffening



Nonlinear: ***large deformations*** $|\nabla \mathbf{u}| >> 1$



Professor **L. R. G. Treloar** (1906 – 1985) was a leading person in the science of rubber and elasticity

Isotropic & isothermal cases

Overview: Some classical models

Isotropic & isothermal cases

The constitutive laws are given by postulating thermodynamic potentials (Elastic strain energy)

Elastomer is nearly incompressible. *However, most accurate models should incorporate compressibility into the constitutive law*

λ_i — Principle stretches

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$
$$I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_1^2\lambda_3^2$$
$$I_3 = \lambda_1^2\lambda_2^2\lambda_3^2$$

Elastic strain energy (proposed by **Rivlin**):

$$\psi = \sum_{i,j=0}^{\infty} C_{ij}(I_1-3)^i(I_2-3)^j$$

$C_{ij} \geq 0$ Material constants

$I_3 = 1$ Incompressibility

Initially:

$\mathbf{C}(0) = \mathbf{F}^T\mathbf{F} = \mathbf{1} \Downarrow$

$I_1 = 3, I_2 = 3, I_3 = 1$

Removes the initial constant level of energy

Constitutive laws are obtained by taking partial derivatives of the strain energy with respect to stretches:

$$\sigma_i = \frac{\partial \psi}{\partial \lambda_i} = \frac{\partial \psi}{\partial I_k}\frac{\partial I_k}{\partial \lambda_i}$$

Neo-Hookean model:

$$\psi = C_{10}(I_1-3)$$

Mooney-Rivlin model:

$$\psi = C_{10}(I_1-3) + C_{01}(I_2-3)$$

Yeoh model: $\psi = C_{10}(I_1-3) + C_{20}(I_1-3)^2 + C_{30}(I_1-3)^3$

Ogden model: $\psi = \sum_n \frac{\mu_n}{\alpha_n}(\lambda_1^{\alpha_n} + \lambda_2^{\alpha_n} + \lambda_3^{\alpha_n} - 3), \quad n \geq 3$ Gives reasonable fits to experiments

Nominal stress are work-conjugate to stretches (*Cf.* appendix)

Uniaxial example:

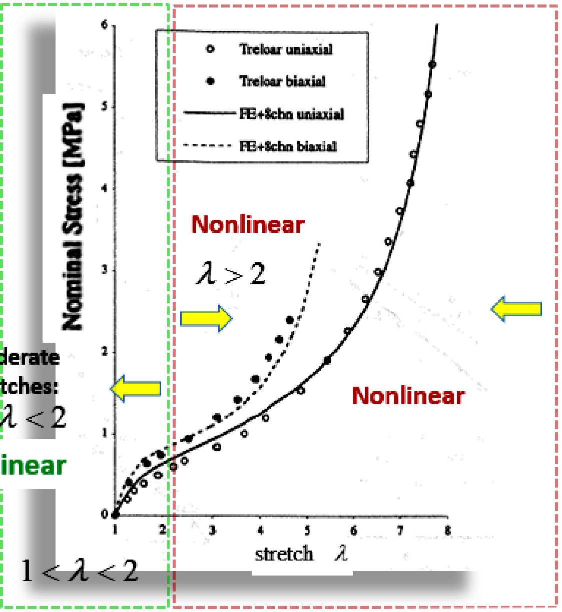
$$\sigma = 2C_{10}(\lambda - 1/\lambda^2)$$

Neo-Hookean

Nominal stress

$$\sigma = 2\left(C_{10} + \frac{C_{01}}{\lambda}\right)\left(\lambda - \frac{1}{\lambda^2}\right)$$

Mooney-Rivlin



The Drugger material stability criterion for the Hessian of strain energy positive restrains the coefficients Cij to be positive.

Strain energy density: $\psi = \psi(I_1, I_2) + \psi(J)$ = ***distortion*** + **dilatation**

$$\underbrace{\psi(I_1, I_2)}_{\text{Accounting for compressibility}}$$

$J = \sqrt{I_3}$

$$\begin{aligned} I_1 &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\ I_2 &= \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_1^2\lambda_3^2 \\ I_3 &= \lambda_1^2\lambda_2^2\lambda_3^2 \end{aligned}$$

Case 1)

Elastomer is *nearly* ***incompressible*** $I_3 \approx 1$ thus one stretch should be eliminated (dependent) since the volume is constant

$$I_3 = 1 \Rightarrow \quad \lambda_3^2 = \frac{1}{\lambda_1^2 \lambda_2^2} \Rightarrow \boxed{\psi = \psi(I_1, I_2)}$$

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \lambda_1^2 + \lambda_2^2 + \frac{1}{\lambda_1^2 \lambda_2^2}$$

Case 2)

$$\sigma_i = \frac{\partial \psi(I_1, I_2)}{\partial \lambda_i} = -p + \lambda_i \frac{\partial \psi}{\partial I_k} \frac{\partial I_k}{\partial \lambda_i}$$

for ***incompressibility*** cases (no sum over *i*) $J = \sqrt{I_3} = 1$

$$\sigma_i = \frac{\partial \psi(I_1, I_2)}{\partial \lambda_i} = \frac{\lambda_i}{J} \frac{\partial \psi}{\partial I_k} \frac{\partial I_k}{\partial \lambda_i} + \frac{\partial \psi}{\partial J}$$

for **compressibility** cases (no sum over *i*)

Volume ration (bulk): $J = \sqrt{I_3} < 1$

$J = \sqrt{I_3} = 1$

$$\boxed{\psi = \psi(I_1, I_2)}$$

$J = \sqrt{I_3} < 1$

$$\boxed{\psi = \psi(I_1, I_2) + \psi(J)}$$

$J = \sqrt{I_3}$

dilatation

N.B. The incompressibility $\rightarrow I_3 = 1$ means that the pressure (hydrostatic component p of the stress) cannot be derived from a constitutive relation but should be seen as a Lagrange multiplier for the incompressibility condition.

Cauchy stress: $\boxed{\sigma_i = -p + \lambda_i \frac{\partial \psi}{\partial \lambda_i}}$ (no sum on *i*)

Principle stresses

Principle stretches

Incompressibility

Case: Isotropy & isothermal

$$\begin{aligned} I_1 &= \lambda_1^2+\lambda_2^2+\lambda_3^2 \\ I_2 &= \lambda_1^2\lambda_2^2+\lambda_2^2\lambda_3^2+\lambda_1^2\lambda_3^2 \\ I_3 &= \lambda_1^2\lambda_2^2\lambda_3^2 \end{aligned}$$

Incompressibility condition: $J \equiv \sqrt{I_3} = \lambda_1\lambda_2\lambda_3 = 1 \Rightarrow \lambda_3 = 1/(\lambda_2\lambda_1)$ (1)

$\Downarrow$

Thus stretches **cannot vary independently**, since the above condition places a constraint among them. We choose λ_1 and λ_2 as the independent variables in the expression of the free energy: $\psi = \psi(\lambda_1, \lambda_2)$

Change in nominal strain energy density: $\mathrm{d}U = \sigma_1^{(\mathrm{nom})}\mathrm{d}\lambda_1 + \sigma_2^{(\mathrm{nom})}\mathrm{d}\lambda_2 + \sigma_3^{(\mathrm{nom})}\mathrm{d}\lambda_3 \equiv \mathrm{d}\psi$

(2)

The change expressed using true stresses: $\mathrm{d}U = \sigma_1\lambda_2\lambda_3\mathrm{d}\lambda_1 + \sigma_2\lambda_3\lambda_1\mathrm{d}\lambda_2 + \sigma_3\lambda_1\lambda_2\mathrm{d}\lambda_3 \equiv \mathrm{d}\psi$

$\sigma_i \equiv \sigma_i^{(\mathrm{true})}$

Inserting the constraint (1) $\mathrm{d}\lambda_3 = \lambda_1^{-2}\lambda_2^{-1}\mathrm{d}\lambda_1 + \lambda_1^{-1}\lambda_2^{-2}\mathrm{d}\lambda_2$ in Eq. (2b) gives:

$$\mathrm{d}U = \boxed{\frac{\sigma_1-\sigma_3}{\lambda_1}}\mathrm{d}\lambda_1 + \boxed{\frac{\sigma_2-\sigma_3}{\lambda_2}}\mathrm{d}\lambda_2$$

This is a total differential

$U = U(\lambda_1, \lambda_2) \Rightarrow$

$$\boxed{\mathrm{d}U = \frac{\partial U(\lambda_1,\lambda_2)}{\partial\lambda_1}\mathrm{d}\lambda_1 + \frac{\partial U(\lambda_1,\lambda_2)}{\partial\lambda_2}\mathrm{d}\lambda_2}$$

Stress-strain relation for incompressible materials

$$\psi \leftrightarrow U$$

$$\sigma_1 - \sigma_3 = \lambda_1\frac{\partial U(\lambda_1,\lambda_2)}{\partial\lambda_1}$$

$$\sigma_2 - \sigma_3 = \lambda_2\frac{\partial U(\lambda_1,\lambda_2)}{\partial\lambda_2}$$

and

$$\sqrt{I_3} = \lambda_1\lambda_2\lambda_3 = 1$$

- once all the stresses are prescribed, all the stretches are determined by the material model
- when all the stretches are prescribed, only the stress combinations $\sigma_1-\sigma_3$, $\sigma_2-\sigma_3$ are known because of incompressibility. The hydrostatic pressure acts as a reaction.

$\sigma_{\mathrm{i}}^{(\mathrm{true})} = \sigma_{\mathrm{i}}^{(\mathrm{eng})}\cdot\lambda_{\mathrm{j}}\lambda_{\mathrm{k}}$,

$j \neq i, k \neq i, j \neq k$

nominal = engineering

Example: Uniaxial case - how you did got this result?

$J = \sqrt{I_3}$

$$\sigma_2 = \sigma_3 = 0, \sigma_1 \equiv \sigma \rightarrow \quad \sigma^{(\text{true})} \equiv \sigma = 2C_{10}(\lambda^2 - 1/\lambda)$$

Neo- Hookean material

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$
$$I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_1^2\lambda_3^2$$
$$I_3 = \lambda_1^2\lambda_2^2\lambda_3^2$$

Neo- Hookean model: $\psi = C_{10}(I_1 - 3) = C_{10}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)$

$$(\lambda_1^2 + \lambda_2^2 - \tfrac{1}{\lambda_1^2\lambda_2^2} - 3)$$

$$\sigma_1 - 0 = \lambda_1 \frac{\partial \psi}{\partial \lambda_1} = \lambda_1 2C_{10}(\lambda_1 - \tfrac{\lambda_1}{\lambda_1^4\lambda_2^2}) = \boxed{2C_{10}(\lambda_1^2 - \tfrac{1}{\lambda_1^2\lambda_2^2})} \leftarrow \lambda_3^2 = \frac{1}{\lambda_1^2\lambda_2^2}$$

In simple extension: $\lambda_1 = \lambda, \lambda_2 = \lambda_3 = 1/\sqrt{\lambda},$

a single extension ratio with equal contractions in transverse directions

$$\sigma \equiv \sigma_1^{(\text{true})} = 2C_{10}(\lambda^2 - \frac{1}{\lambda}) \leftarrow \psi = C_{10}(\lambda^2 + 2/\lambda - 3)$$

nominal = engineering

$$\sigma_{\text{i}}^{(\text{true})} = \sigma_{\text{i}}^{(\text{eng})} / (\lambda_{\text{j}}\lambda_{\text{k}}),$$
$$j \neq i, k \neq i, j \neq k$$

$$\sigma = 2C_{10}(\lambda^2 - \frac{1}{\lambda}) \equiv \mu(\lambda^2 - \frac{1}{\lambda})$$

Shear modulus

Stress-strain relation for incompressible materials

True stresses

$$\sigma_1 - \sigma_3 = \lambda_1 \frac{\partial U(\lambda_1, \lambda_2)}{\partial \lambda_1}$$

$$\sigma_2 - \sigma_3 = \lambda_2 \frac{\partial U(\lambda_1, \lambda_2)}{\partial \lambda_2}$$

and

$$\sqrt{I_3} = \lambda_1\lambda_2\lambda_3 = 1$$

Have a look: Journal article you can find an extensive review of constitutive models

Arch Appl Mech **(2012)** 82:1183–1217
DOI 10.1007/s00419-012-0610-z
Paul Steinmann · Mokarram Hossain · Gunnar Possart
Hyperelastic models for rubber-like materials: consistent tangent operators and suitability for Treloar's data

See also:

The elasticity and related properties of rubbers

Rep. Prog. Phys. 1973 **36** 755–826

L R G TRELOAR

Department of Polymer and Fibre Science, University of Manchester Institute of Science and Technology, PO Box no 88, Sackville Street, Manchester M60 1QD, UK

Uniaxial case - interpretation of elasticity coefficient C_{10}

$J = \sqrt{I_3}$

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$
$$I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_1^2\lambda_3^2$$
$$I_3 = \lambda_1^2\lambda_2^2\lambda_3^2$$

$$\boxed{\sigma_2 = \sigma_3 = 0, \sigma_1 \equiv \sigma \rightarrow \quad \sigma^{(\mathrm{true})} \equiv \sigma = 2C_{10}(\lambda^2 - 1/\lambda)}$$

Neo- Hookean material

Neo- Hookean model: $\psi = C_{10}(I_1 - 3) = C_{10}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)$

$C_{10} = \mu / 2$

Shear modulus

Stress-strain relation **for incompressible materials**

True stresses

$$\sigma_1 - \sigma_3 = \lambda_1 \frac{\partial U(\lambda_1, \lambda_2)}{\partial \lambda_1}$$

$$\sigma_2 - \sigma_3 = \lambda_2 \frac{\partial U(\lambda_1, \lambda_2)}{\partial \lambda_2}$$

and

$$\sqrt{I_3} = \lambda_1\lambda_2\lambda_3 = 1$$

In simple extension:

$$\lambda_1 = \lambda, \lambda_2 = \lambda_3 = 1/\sqrt{\lambda},$$

$$\boxed{\sigma = 2C_{10}(\lambda^2 - \frac{1}{\lambda}) \equiv \mu(\lambda^2 - \frac{1}{\lambda})}$$

Shear modulus

nominal = engineering

$$\sigma_i^{(\mathrm{true})} = \sigma_i^{(\mathrm{eng})} \cdot \lambda_j \lambda_k,$$
$$j \neq i, k \neq i, j \neq k$$

$$\lambda = \ell / L = 1 + \varepsilon$$

For small engineering strains

$$\left.\begin{aligned} &\Rightarrow \lambda^2 = (1+\varepsilon)^2 \approx 1 + 2\varepsilon \\ &\Rightarrow \lambda^{-1} = (1+\varepsilon)^{-1} \approx 1 - \varepsilon \end{aligned}\right\} \boxed{\Rightarrow \sigma = \underbrace{3\mu}\varepsilon}$$

Young's modulus

Shear Modulus $\mu = \frac{E}{2(1+\nu)} \equiv G$

1/2

Statistical physics:

Elastic strain energy of the rubber chain assembly:

Boltzmann's constant

temperature

Principle stretches

$$\psi = \tfrac{1}{2} NkT(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)$$

Number of chains in the assembly

Statistical physics

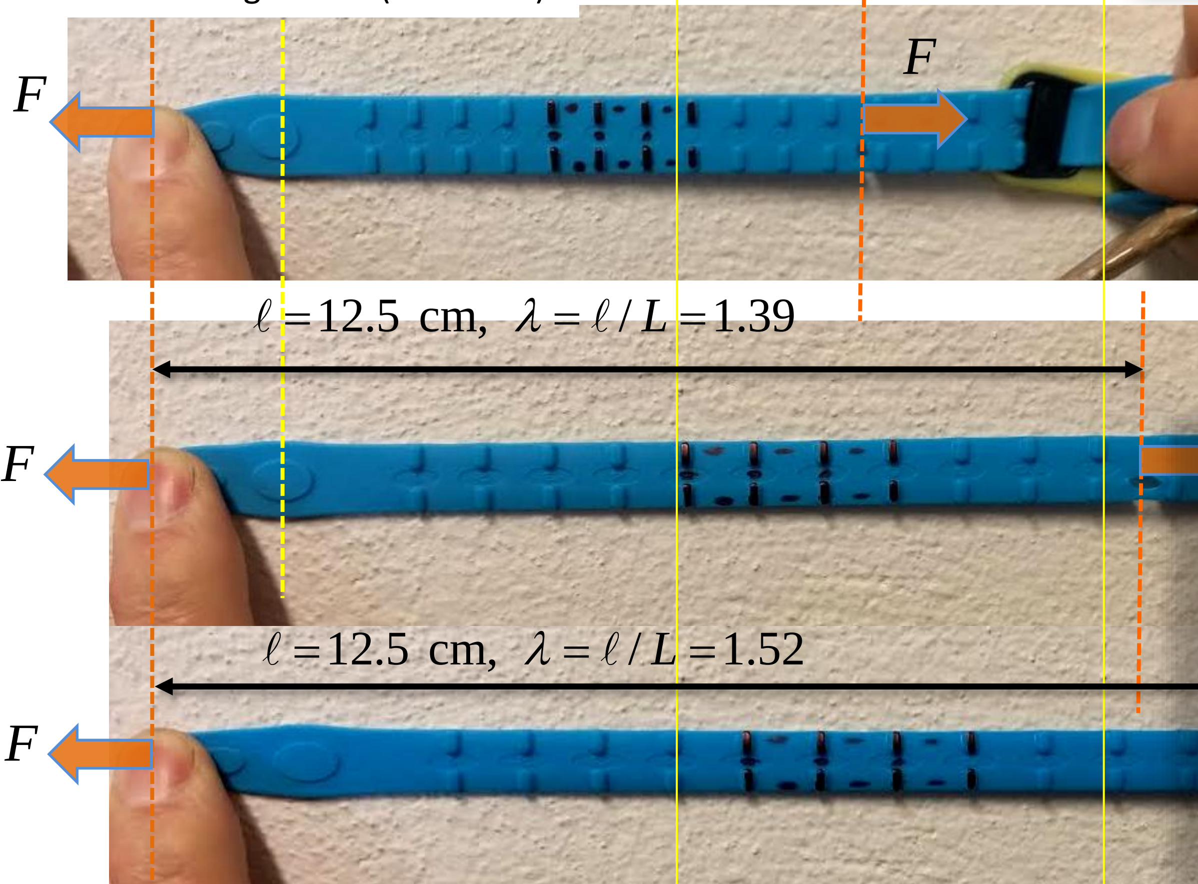
$$\boxed{\Rightarrow \mu = NkT}$$

N – number of polymer chains per unit volume

$$\text{Neo- Hookean}: \quad \sigma^{(\text{true})} = \mu(\lambda^2 - \frac{1}{\lambda})$$

$$\ell = L = 9\ \text{cm}, \quad \lambda = \ell / L = 1$$

Initial configuration (stressless)



Nominal = engineering

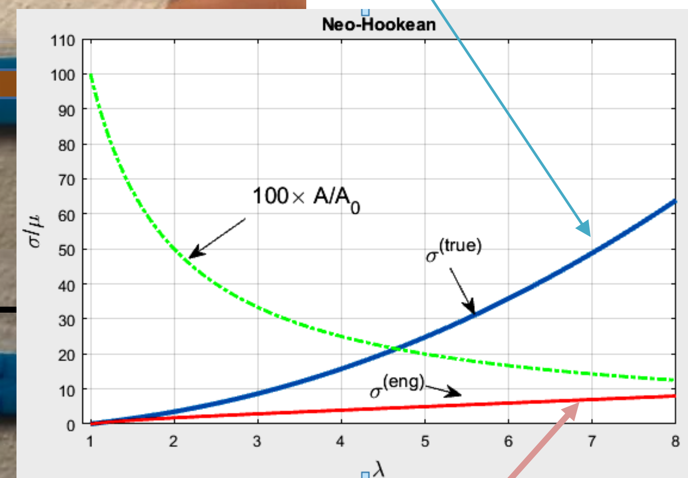
$$\sigma_{\mathrm{i}}^{(\text{true})} = \sigma_{\mathrm{i}}^{(\text{eng})} / (\lambda_{\mathrm{j}} \lambda_{\mathrm{k}}) = \frac{F}{A}$$

$$j \neq i, k \neq i, j \neq k$$

Actual cross-section area

In simple extension:

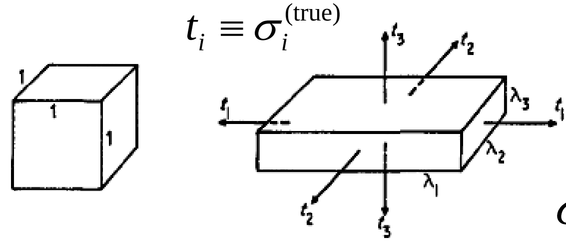
$$\lambda_1 = \lambda, \lambda_2 = \lambda_3 = 1/\sqrt{\lambda},$$



Rubber – swimming glass

$$A = \frac{A_0}{\lambda}$$

$$t_i \equiv \sigma_i^{(\text{true})}$$



$$\sigma^{(\text{eng})} = \underbrace{\lambda_2 \lambda_3}_{1/\lambda} \sigma^{(\text{true})} = \frac{1}{\lambda} \mu(\lambda^2 - \frac{1}{\lambda})$$

$1 < \lambda < 2$

Moderate stretches

$$\sigma_{\mathrm{i}}^{(\text{eng})} = \frac{F}{A_0}$$

$$A = \ell_2 \ell_3 = \lambda_2 \lambda_3 L_2 L_3 = \lambda_2 \lambda_3 A_0$$

Initial cross-section area

END

of

Lecture Slides

Additional material

- Appendices
- Homework

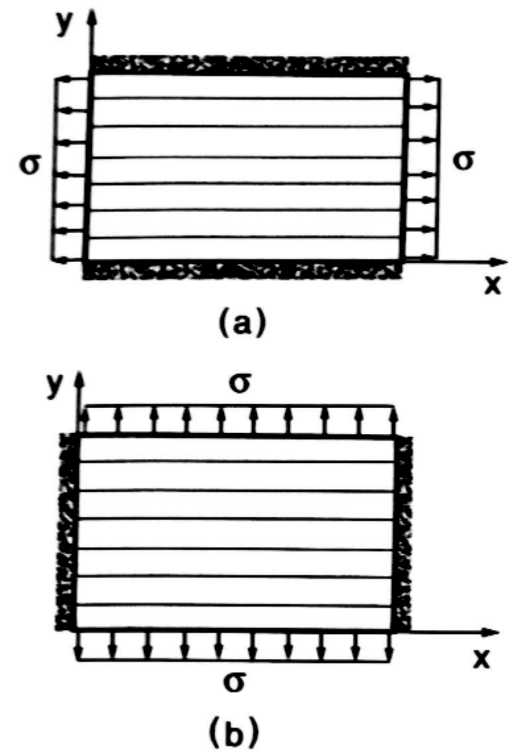
Orthotropic case – worked example

ESIMERKKI 1

Laminaattilevy (paksuus 0,10 mm) on tehty hiilikuituvahvisteisesta epoksista. Levy on tuettu kitkattomien jäykkien tukien väliin. Levyä kuormitetaan x-akselin suunnassa kuvan (a) ja y-akselin suunnassa kuvan (b) mukaisesti. Laske kummassakin tapauksessa tukipaine ja venymä vapaassa suunnassa.

$E_x = 230$ GPa, $E_y = 6{,}60$ GPa

$\nu_{xy} = 0{,}25$, $\sigma = 100$ MPa



Ref: T. Salmi and S. Virtanen, Materiaalien mekaniikka, 2008, Pressus Oy. (Finnish textbook)

RATKAISU

Resiprookkiyhtälöstä saadaan

$$\nu_{yx} = E_y \nu_{xy} / E_x = 6{,}60 \cdot 0{,}25 / 230$$

$$\Rightarrow \qquad \nu_{yx} \approx 0{,}00717$$

Orthotropic case – worked example

ESIMERKKI 1

Laminaattilevy (paksuus 0,10 mm) on tehty hiilikuituvahvisteisesta epoksista. Levy on tuettu kitkattomien jäykkien tukien väliin. Levyä kuormitetaan x-akselin suunnassa kuvan (a) ja y-akselin suunnassa kuvan (b) mukaisesti. Laske kummassakin tapauksessa tukipaine ja venymä vapaassa suunnassa.

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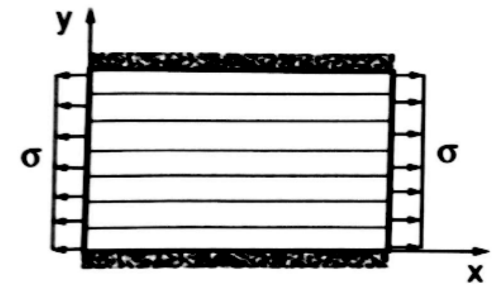
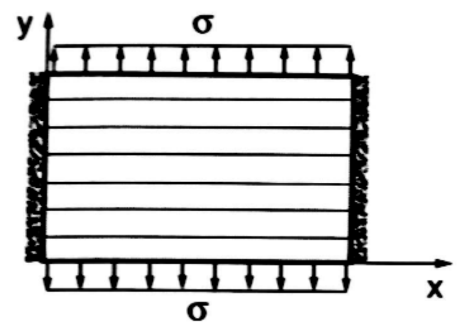


(a)



(b)

RATKAISU

Resiprookkiyhtälöstä saadaan

$$\nu_{yx} = E_y \nu_{xy} / E_x = 6{,}60 \cdot 0{,}25 / 230$$

$$\Rightarrow \qquad \nu_{yx} \approx 0{,}00717$$

Kuvan (a) tapauksessa venymä

$$\varepsilon_y = \frac{-\nu_{xy}}{E_x}\sigma_x + \frac{1}{E_y}\sigma_y = 0$$

$$\Rightarrow \qquad \sigma_y = \frac{E_y}{E_x}\nu_{xy}\,\sigma = \nu_{yx}\,\sigma = 0{,}00717 \cdot 100\,\text{MPa} \approx 0{,}72\,\text{MPa}$$

$$\Rightarrow \qquad \varepsilon_x = \frac{1}{E_x}\sigma - \frac{\nu_{yx}}{E_y}\sigma_y = \frac{100\,\text{MPa}}{230000\,\text{MPa}} - \frac{0{,}00717 \cdot 0{,}72\,\text{MPa}}{6600\,\text{MPa}} \approx 434\,\mu$$

Kuvan (b) tapauksessa venymä

$$\varepsilon_x = \frac{1}{E_x}\sigma_x - \frac{\nu_{yx}}{E_y}\sigma_y = 0$$

$$\Rightarrow \qquad \sigma_x = \frac{E_x}{E_y}\nu_{yx}\,\sigma = \nu_{xy}\,\sigma = 0{,}25 \cdot 100\,\text{MPa} \approx 25\,\text{MPa}$$

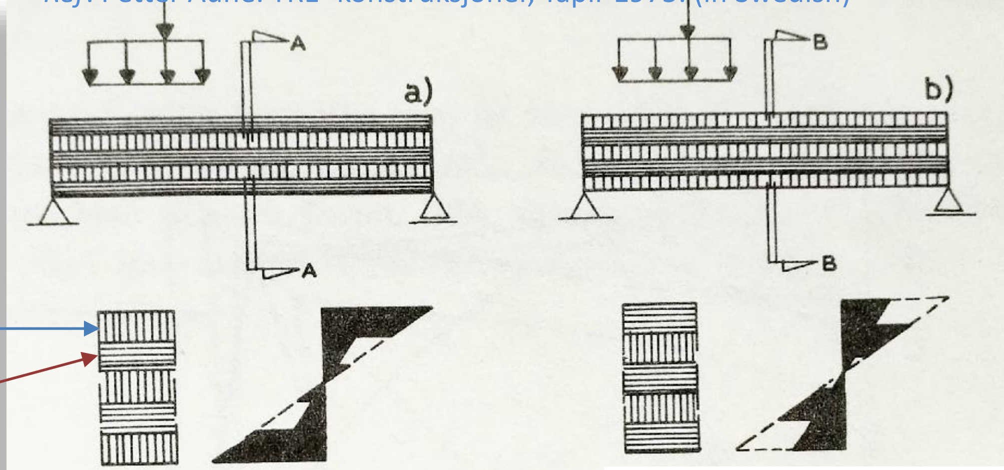
$$\Rightarrow \qquad \varepsilon_y = \frac{-\nu_{xy}}{E_x}\sigma_x + \frac{1}{E_y}\sigma = \frac{-0{,}25 \cdot 25\,\text{MPa}}{230000\,\text{MPa}} + \frac{100\,\text{MPa}}{6600\,\text{MPa}} \approx 15100\,\mu$$

Tukipaine on tapauksessa (b) $25 / 0{,}72 = 34{,}7$ -kertainen tapaukseen (a) verrattuna.

Orthotropic case - suggestion example/HW for students

Bending of a Glue Laminated Timber Beam

Ref: Petter Aune. TRE- konstruksjoner, Tapir 1975. (in Swedish)



$E_0 \equiv E_L$

$E_{90} \equiv E_T$

Bending **normal stresses**

Bending **normal stresses**

Wood:

$E_L >> E_T$

$E_L = nE_T, \quad n \approx 0.01 \div 0.1$

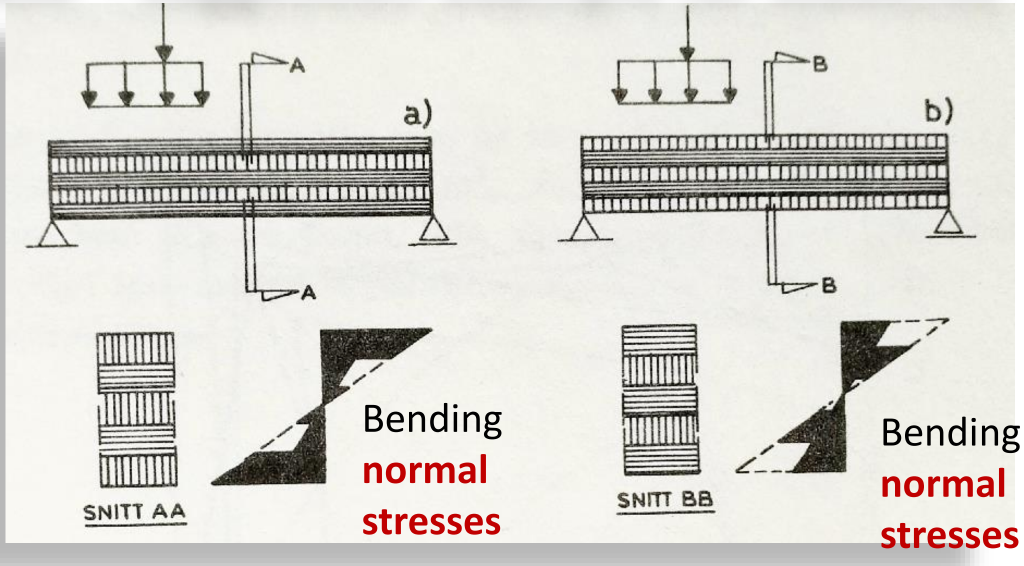
Q1: **Explain** qualitatively & **quantitatively**

- why these stresses are as they are?
- why this difference?

Q2: for a generic GL-beam & a constant distributed load, determine the deflection at the mid span for normal temperature and hygrometry. Account for both bending and shearing contribution for the deflection. For material properties refer to next Table (next-next slide)

Hints: 1. determine an effective bending stiffness ***B*** and shear stiffness ***S*** for the composite section *a*) and *b*) . 2. for picking the correct shear modulus values from tables, think in which material orthotropy plane the shear strain occurs.

Orthotropic case - suggestion example for students



Bending of a Glue Laminated Beam

Q1: **Explain** qualitatively **&** **quantitatively**

- why these stresses are as they are?
- why the difference

Q2: for a generic GL-beam & a constant distributed load, determine the deflection at the mid span for normal temperature and hygrometry. Account for both bending and shearing contribution for the deflection.

Hints: determine an effective bending stiffness ***B*** and shear stiffness ***S*** for the composite section a) and b)

Hyperelasticity

Rubber or

rubber-like Elasticity

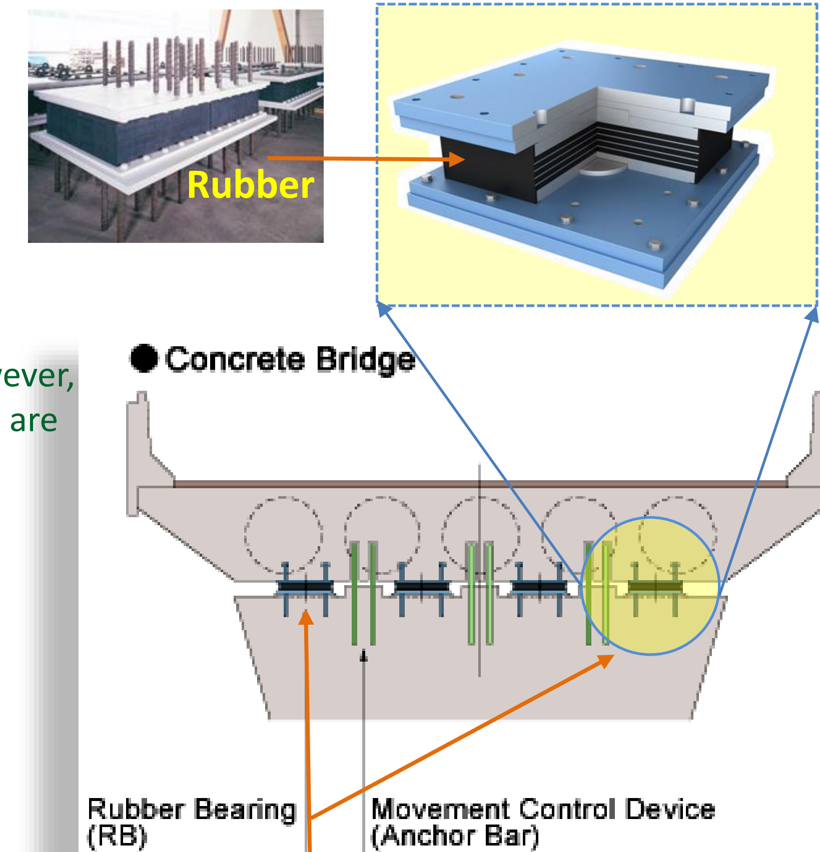
Hyperelasticity
Rubber or rubber-like Elasticity

This is an important class of materials for an engineer to know. However, in this course, it is sufficient to know that such class exists and what are their key important mechanical response features to know for an engineer.

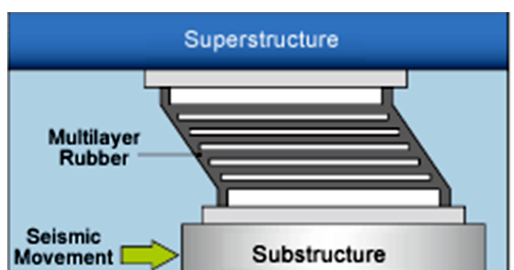
NB. In general, the response of a *polymer* dependent strongly on *temperature*, *strain history* and *loading rate*. Some aspects of such behavior will be detailed in next section treating of *viscoelasticity* and little bit when treating *rubber elasticity* in the current section.

In this section, we consider rubbery state only

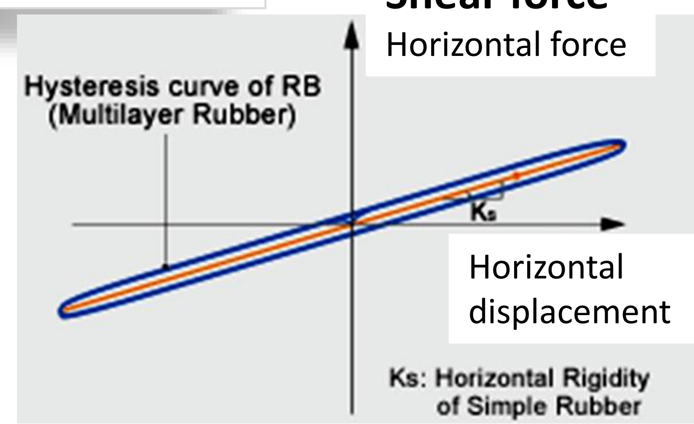
It is known that that polymers have various regimes of mechanical response: **glassy**, **viscoelastic** and **rubbery**. These various regimes can be identified *via* dynamical loading.



Rubber bearing (RB)



Shear force
Horizontal force



Horizontal displacement

On Thermodynamics of Rubber

Enthalpic (energetic) and Entropic forces

Cf. Appendix 2 for more details

Got interested? Read (elective) more from:

A. N. GENT
The University of Akron
Akron, Ohio

1

Rubber Elasticity: Basic Concepts and Behavior

I. Introduction
II. Elasticity of a Single Molecule
III. Elasticity of a Three-Dimensional Network of Polymer Molecules
IV. Comparison with Experiment
V. Continuum Theory of Rubber Elasticity
VI. Second-Order Stresses
VII. Elastic Behavior Under Small Deformations
VIII. Some Unsolved Problems in Rubber Elasticity
Acknowledgments
References

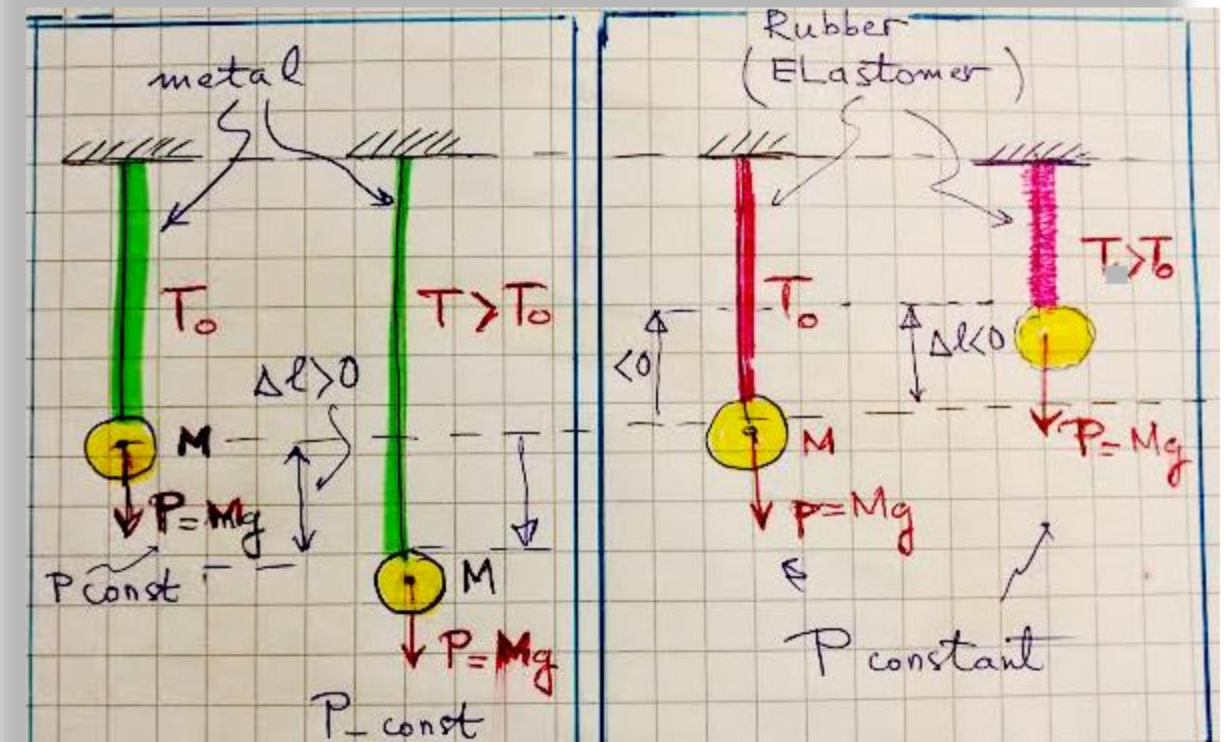
Science and Technology of Rubber, Third Edition

W. Gilbert's experiment

Few word on elastomers

Metal in tension

Elastomer in tension

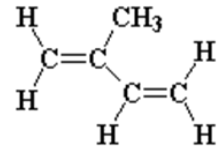


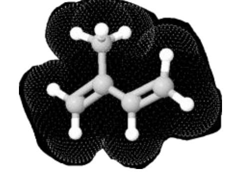
Constant force & **Increase of temperature**

Observation*: For a fixed force, the **stretch decreases** with temperature increase $\lambda = \ell / \ell_0$

On thermodynamics of elastomers

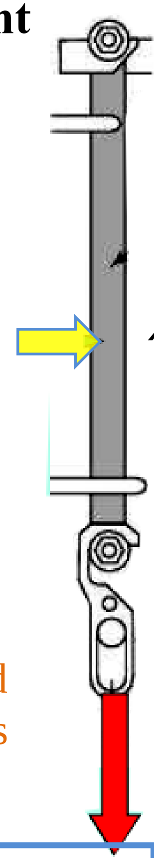
Isoprene





Good to know

W. Gilbert's experiment



Elastomer
in extension

$$\lambda_1 = \ell / \ell_0$$

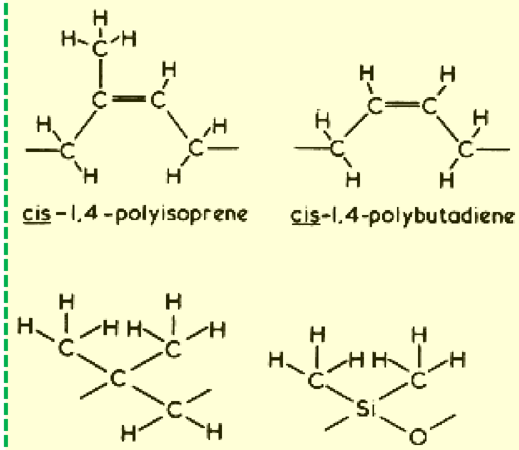
Constant force &
Increase of temperature

Observation*: For a fixed force, the stretch decreases with temperature increase

F

$$\sigma \approx \frac{F}{S_0} \approx \quad A\theta \cdot f(\lambda_i)$$

Example of repeat units for some common elastomer molecules



cis-1,4-polyisoprene   cis-1,4-polybutadiene

poly(iso-butylene)   poly(dimethylsiloxane)

Ref: Rubber Elasticity: *Basic Concepts and Behavior*
A. N. Gent

The glass transition temperature for usual rubbers is usually far bellow room temperatures.

Natural rubber: $T_g \sim -73\ ^{\circ}\mathrm{C}$

Range: $T_g \sim -20^{\circ}\mathrm{C} \div -130^{\circ}\mathrm{C}$

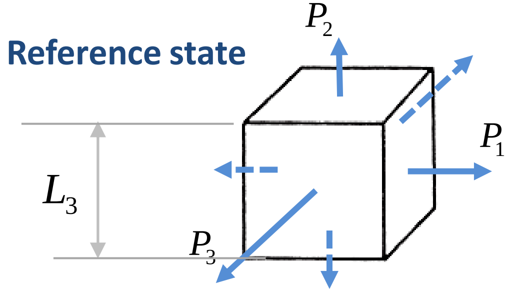
NB. * For normal range of temperatures well above the glass transition temperature and bellow those where chemical reaction begin to take part (thermal degradation etc.)

Terminology and some definitions

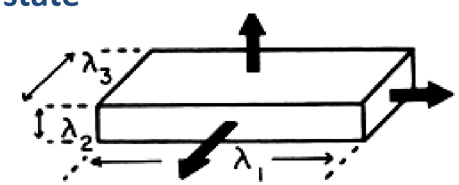
Consider a rectangular material block reference configuration: $L_1 \times L_2 \times L_3$

Apply forces P_i on its faces in this reference state: P_1, P_2, P_3

Reference state



Actual state



Principle stretches: $\lambda_i = \ell_i / L_i, \quad i = 1,2,3$

Actual length in the current state

Initial length in the reference state

Nominal stresses: $f_i \equiv \sigma_i = \dfrac{P_i}{L_j L_k}, \quad j \neq i, k \neq i$

$i = 1,2,3$

True stresses:

$\sigma_i^{(\text{true})} = \dfrac{P_i}{\ell_j \ell_k}, \quad j \neq i, k \neq i$

(They are not work-conjugate to stretches)

Absolute temperature

ΔF **Helmholtz free energy:** $\Psi = u - sT \qquad F = U - TS$

A thermodynamic potential

Internal energy

Entropy

A thermodynamic potential

$\Delta \psi$ Helmholtz free energy specific density: $\psi \equiv \dfrac{\Psi}{V}, \quad V = L_1 L_2 L_3$

If isothermal process

$$\boxed{\sigma_i = \frac{\partial \psi(\lambda_1, \lambda_2, \lambda_3)}{\partial \lambda_i} \\ \Uparrow \\ \psi = \psi(\lambda_1, \lambda_2, \lambda_3)}$$

Work differential (by the external mechanical forces on elongations $\mathrm{d}\ell_i$):

$$\mathrm{d}W_{\text{ext}} = \sum_i P_i \mathrm{d}\ell_i = \sum_i \boxed{\sigma_i \cdot \mathrm{d}\lambda_i} = \mathrm{d}\psi \rightarrow$$

We see that nominal stress Are work-conjugate to stretches

Thermodynamics of rubber – enthalpic (energetic) and entropic forces

The change of internal energy of a (closed) system can be achieved by:

(Static case – zero kinetic energy)

$$\mathrm{d}U = T\mathrm{d}S - p\mathrm{d}V + f\mathrm{d}L \quad \textbf{(1)}$$

Change in internal energy of a system

Heating of the system

Changing the system volume

Deforming the system

streching the system

Force

(This is from your previous courses of thermodynamics)

$$F \equiv \psi \qquad \sigma_i = \frac{\partial \psi}{\partial \lambda_i},$$

Helmholtz free energy: (state function)

$$F = U - TS \Rightarrow$$

Change of Helmholtz free energy

$$\mathrm{d}F = \underbrace{\mathrm{d}U}_{= T\mathrm{d}S - p\mathrm{d}V + f\mathrm{d}L \ (1)} - T\mathrm{d}S - S\mathrm{d}T$$

$$f = \frac{\partial \psi}{\partial L} = \frac{\partial F}{\partial L}$$

$$\Rightarrow \mathrm{d}F = -S\mathrm{d}T - p\mathrm{d}V + \boxed{f} \cdot \mathrm{d}L$$

← (total differential)

force

$$\mathrm{d}F = \left(\frac{\partial F}{\partial T}\right)_{V,L} \mathrm{d}T + \left(\frac{\partial F}{\partial V}\right)_{T,L} \mathrm{d}V + \boxed{\left(\frac{\partial F}{\partial L}\right)_{T,V}} \cdot \mathrm{d}L$$

(can de written in terms of a total differential)

Metals: $|f_E| >> |f_S|$

(for typical cristalline solids) $f_S = 0$

Ideal Polymer network: $|f_S| >> |f_E|$

$f_E = 0$

Applied **force:**

$$\Rightarrow f = \left(\frac{\partial F}{\partial L}\right)_{T,V} = \underbrace{\left(\frac{\partial U}{\partial L}\right)_{T,V}}_{\equiv f_E} \underbrace{- T\left(\frac{\partial S}{\partial L}\right)_{T,V}}_{\equiv f_S} \boxed{\equiv f_E + f_S} \quad \textbf{(2)}$$

$$f = \frac{\partial \psi}{\partial L} = \frac{\partial F}{\partial L}$$

Entropic contribution

Energetic contribution

Origin of the force:

Energetic and **entropic**

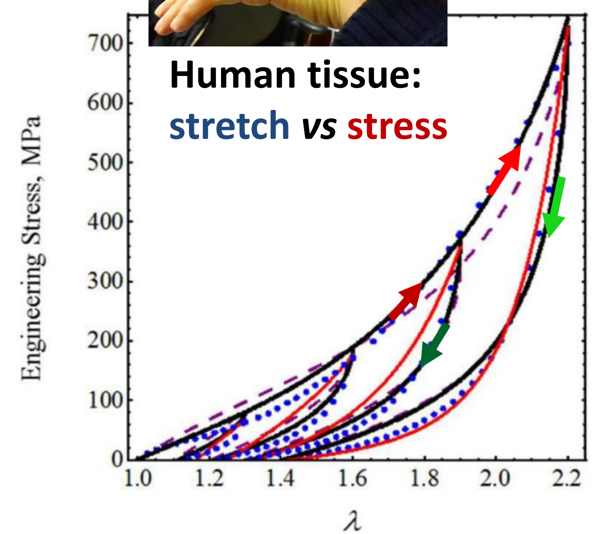
$$\boxed{f = f_E + f_S}$$

Hyper-elasticity

Elastomers (rubbers) or elastomers-like material

Overview: Some classical models

Hyperelasticity



Materials **2014**, 7(1), 441-456; doi:10.3390/ma7010441

Hyperelasticity

Elastomers (rubbers), polymer networks, foams, bio-solids, soft-tissues, …

Ref: L.R.G. **Treloar** , the elasticity and related properties of rubbers. *Rep. Prog. Phys.* 1973 (**36**) 755-826

Elastic strain energy (proposed by Rivlin):

$$\psi = \sum_{i,j=0}^{\infty} C_{ij}(I_1-3)^i(I_2-3)^j$$

Material constants

Continuum mechanics

incompressible $I_3 = 1$

Elastomer is nearly incompressible. $I_3 \approx 1$

However, most accurate models should incorporate compressibility into the constitutive law

$$\begin{aligned} I_1 &= \lambda_1^2+\lambda_2^2+\lambda_3^2 \\ I_2 &= \lambda_1^2\lambda_2^2+\lambda_2^2\lambda_3^2+\lambda_1^2\lambda_3^2 \\ I_3 &= \lambda_1^2\lambda_2^2\lambda_3^2 \end{aligned}$$

Neo-Hookean model:

$$\psi = C_{10}(I_1-3)$$

Nominal stress Are work-conjugate to stretches

uniaxial: *ex*.:

$$f \equiv \sigma = 2C_{10}(\lambda - 1/\lambda^2)$$

Neo-Hookean

Mooney-Rivlin model:

$$\psi = C_{10}(I_1-3)+C_{01}(I_2-3)$$

Nominal stress

Mooney-Rivlin

$$f \equiv \sigma = 2(C_{10}+\frac{C_{01}}{\lambda})(\lambda-\frac{1}{\lambda^2})$$

Yeoh model: $\psi = C_{10}(I_1-3)+C_{20}(I_1-3)^2+C_{30}(I_1-3)^3$

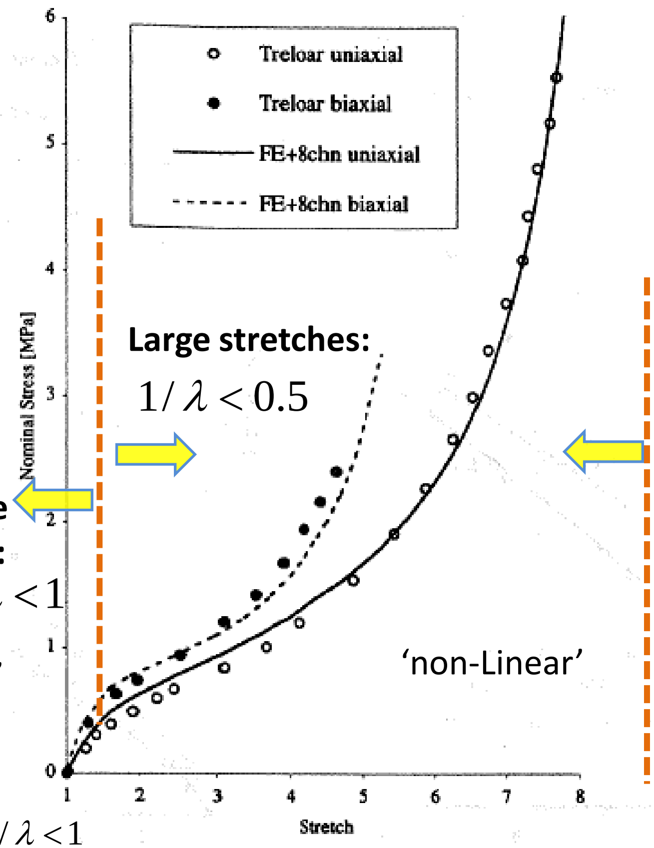
Gives reasonable fits to experiments

Ogden model: $\psi = \sum_n \frac{\mu_n}{\alpha_n}(\lambda_1^{\alpha_n}+\lambda_2^{\alpha_n}+\lambda_3^{\alpha_n}-3),\quad n \geq 3$

Constitutive laws is obtained by taking the partial derivatives of the strain energy with respect to stretches:

$$\sigma_i = \frac{\partial \psi}{\partial \lambda}$$

Nonlinear: ***large deformations*** $|\nabla \mathbf{u}| >> 1$



Large stretches: $1/\lambda < 0.5$

Moderate stretches: $0.5 < 1/\lambda < 1$

'Linear'

'non-Linear'

$1/\lambda < 1$

Elastic strain energy of the rubber chain assembly:

Boltzmann's constant

temperature

Principle stretches

$$W_G = \frac{1}{2}Nk\theta(\lambda_1^2+\lambda_2^2+\lambda_3^2-3)$$

Number of chains in the assembly

Statistical physics

$$f_i \equiv \sigma_i = \frac{\partial \psi}{\partial \lambda_i} = \frac{\partial \psi}{\partial I_k}\frac{\partial I_k}{\partial \lambda_i}$$

Professor **L. R. G. Treloar** (1906 – 1985) was a leading person in the science of rubber and elasticity

The Drugger material stability criterion for the Hessian of strain energy positive restrains the coefficients *Cij* to be positive.

Isotropic & isothermal cases

Strain energy density: $\psi = \psi(I_1, I_2) + \psi(J)$ = ***distortion*** **+ dilatation** $J = \sqrt{I_3}$

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$
$$I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_1^2\lambda_3^2$$
$$I_3 = \lambda_1^2\lambda_2^2\lambda_3^2$$

Uniaxial: example, how you did got this result? $\sigma = 2C_{10}(\lambda - 1/\lambda^2)$

$\sigma_2 = \sigma_3 = 0, \sigma_1 \equiv \sigma$

Neo-Hookean model: $\psi = C_{10}(I_1 - 3)$

Nominal stress: $\sigma_1 = \dfrac{\partial \psi}{\partial \lambda_1} = \dfrac{\partial \psi}{\partial I_k}\dfrac{\partial I_k}{\partial \lambda_1}$ In uniaxial case, the hydrostatical pressure can solved from the incompressibility constraint

Elastomer is nearly incompressible* $I_3 \approx 1$ thus one stretch should be eliminated (dependent) since the volume is constant

$$I_3 = 1 \Rightarrow \quad \lambda_3^2 = \frac{1}{\lambda_1^2\lambda_2^2} \Rightarrow \quad \psi = \psi(I_1, I_2)$$

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \lambda_1^2 + \lambda_2^2 + \frac{1}{\lambda_1^2\lambda_2^2}$$

$$\sigma_i = \frac{\partial \psi}{\partial \lambda_i} = \frac{\partial}{\partial \lambda_i} C_{10}(I_1 - 3) = C_{10}\frac{\partial I_1}{\partial \lambda_i}$$

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \lambda_1^2 + \lambda_2^2 + \frac{1}{\lambda_1^2\lambda_2^2}$$ ← The incompressibility constraint is accounted for

$$\frac{\partial I_1}{\partial \lambda_1} = 2\lambda_1 - \frac{1}{\lambda_2^2} \Rightarrow \sigma_1 = 2C_{10}(\lambda_1 - \frac{1}{\lambda_2^2})$$

If $\lambda_1 = \lambda_2 \to \sigma = 2C_{10}(\lambda - \dfrac{1}{\lambda^2})$

Uniaxial: example

$$\sigma_i = \frac{\partial \psi(I_1, I_2)}{\partial \lambda_i} = -p + \lambda_i \frac{\partial \psi}{\partial I_k}\frac{\partial I_k}{\partial \lambda_i}$$ for incompressibility cases (no sum over *i*) $J = \sqrt{I_3} = 1$

$$\sigma_i = \frac{\partial \psi(I_1, I_2)}{\partial \lambda_i} = \frac{\lambda_i}{J}\frac{\partial \psi}{\partial I_k}\frac{\partial I_k}{\partial \lambda_i} + \frac{\partial \psi}{\partial J}$$ for compressibility cases (no sum over *i*)

Volume ration (bulk): $J = \sqrt{I_3} < 1$

N.B. The incompressibility $\to I_3 = 1$ means that the pressure (hydrostatic component p of the stress) cannot be derived from a constitutive relation but should be seen as a Lagrange multiplier for the incompressibility condition.

Cauchy stress: $\boxed{\sigma_i = -p + \lambda_i \dfrac{\partial \psi}{\partial \lambda_i}}$ (no sum on *i*)

Principle stresses

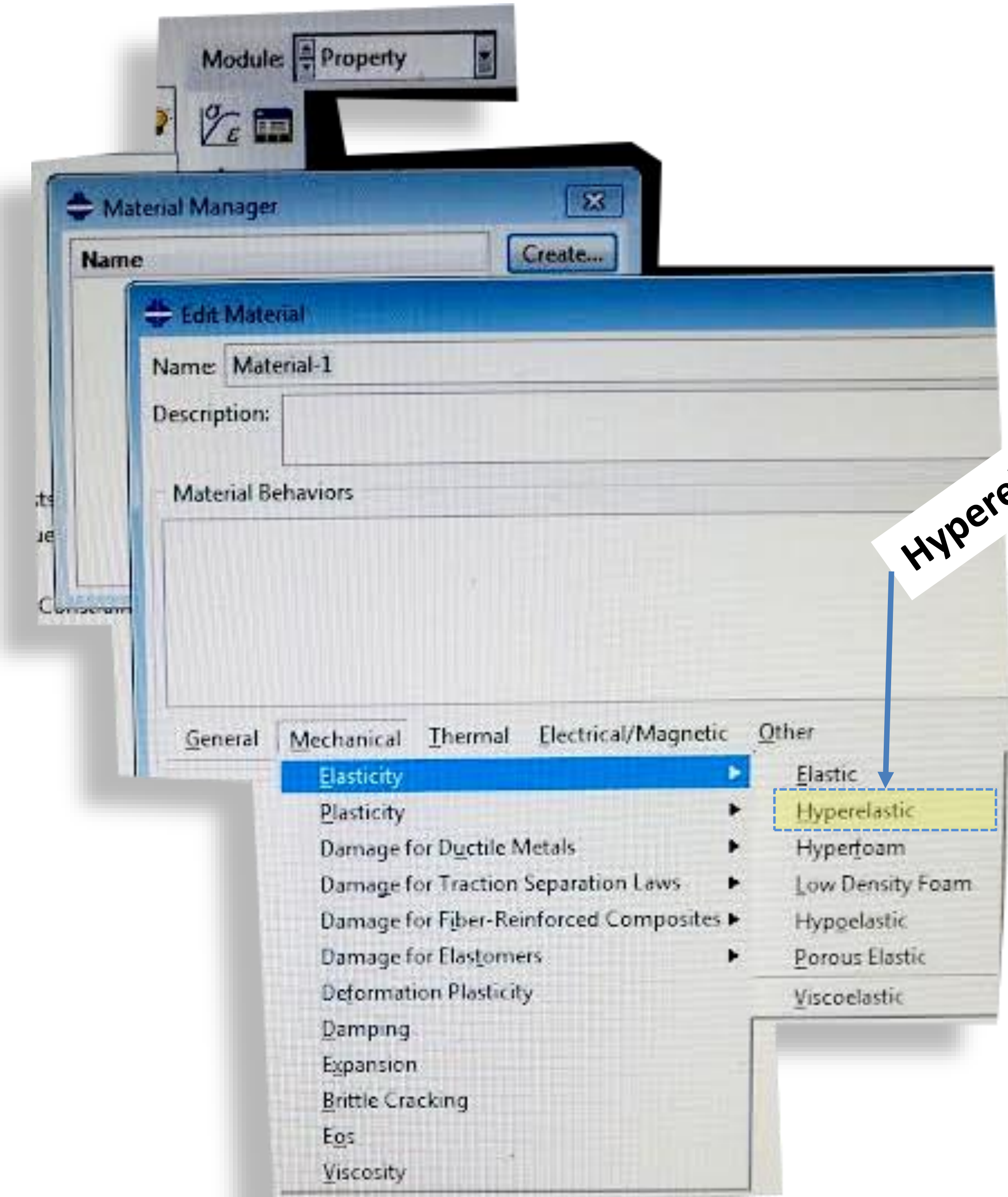
Principle stretches

Have a look: Journal article you can find an extensive review of constitutive models

Arch Appl Mech (**2012**) 82:1183–1217
DOI 10.1007/s00419-012-0610-z

Paul Steinmann · Mokarram Hossain · Gunnar Possart

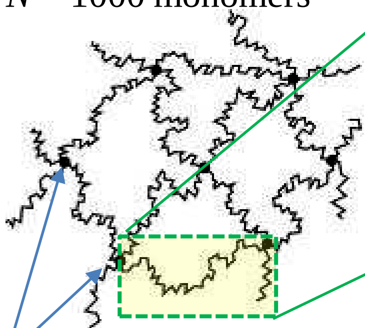
Hyperelastic models for rubber-like materials: consistent tangent operators and suitability for Treloar's data



Hyperelasticity

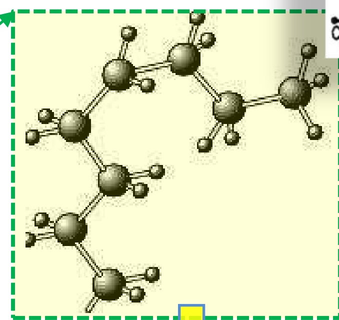
A chain network (assembly)

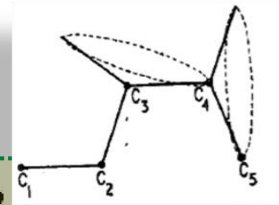
$N \sim 1000$ monomers



Nodes

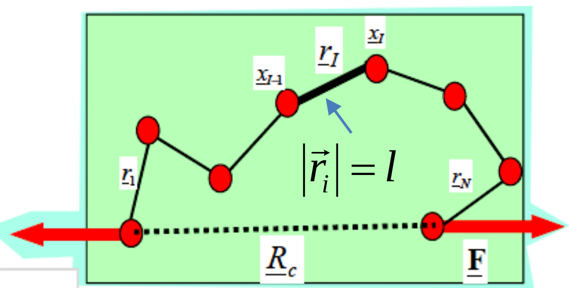
Isolated chain





Rotation about bonds in long-chain molecule take part more or less freely (Meyer *et al* 1932, 1935)

A model: isolated chain under extension



Configuration of nodes changes not the distance between them.

Isolated chain:

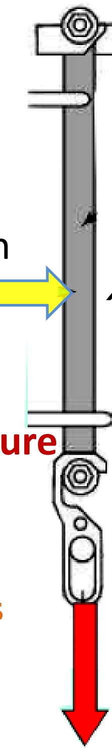
$$\langle \vec{F} \rangle \approx \frac{3k_B\theta}{Nl^2}\vec{R}_c$$

$$K_{eff} \propto \theta$$

Linear spring with stiffness proportional to temperature

https://en.wikipedia.org/wiki/Ideal_chain#Entropic_elasticity_of_an_ideal_chain

Gilbert's experiment



Elastomer in extension

$$\lambda_1 = \ell / \ell_0$$

Constant force &
Increase of temperature

$$F$$

Observation: For a fixed force, the stretch decreases with temperature increase (*NB*. For metals, the contrary behavior occurs)

$$\boxed{\frac{F}{S_0} \approx A\theta \cdot f(\lambda_i)}$$

Considerations from ***statistical physics*** lead to:

Elastic strain energy of the rubber chain assembly:

Boltzmann's constant

temperature

Principle stretches

$$\boxed{W_G = \frac{1}{2} N k \theta (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)}$$

Number of chains in the assembly

Constitutive equation:

$$\sigma_i = \frac{\partial \psi}{\partial \lambda_i} = \underbrace{Nk}_{\equiv A}\theta\lambda_i \equiv A\theta \cdot f(\lambda_i)$$

$$\rightarrow \frac{F}{S_0} \approx A\theta \cdot f(\lambda_i)$$

On Thermodynamics of Elastomers

Good to know

Such macroscopic mechanical behaviour *a-b-a* **is called**
Entropic elasticity

It is the thermal fluctuations that tend to bring a system (the chain assembly) toward its macroscopic state of maximum entropy. To maximize its entropy, an ideal chain 'reduces' the distance between its two free ends (bonds are not stretched nor torqued) (The polymer chains tend to remain random and form ball shape) . Resulting deformations are large.

It is known by experimental observations that the dependency of the internal energy on the inter-node distance (the black solid dots in the figure) is negligible, and moreover that entropic effect dominate the stress stretch-temperature behaviour.

In contrast, in metals for example, the elasticity results from stretching of **bonds** (change of strain energy) and one speaks of **enthalpic elasticity**. Usually, resulting deformations are relatively small.

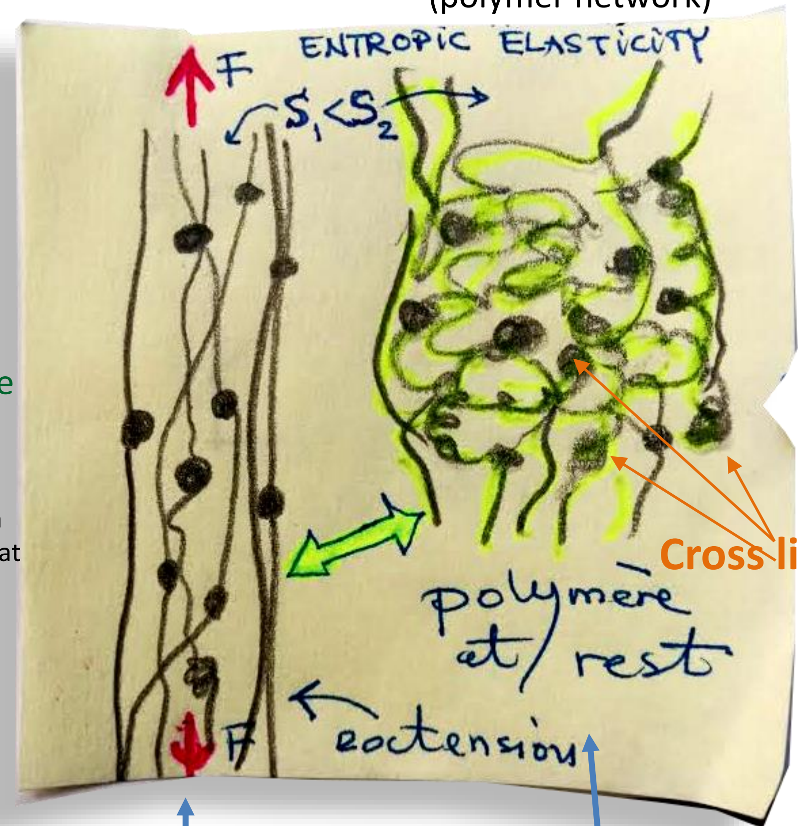
Cf. the ***pressure*** of an ideal gas in a containing box has an *entropic origin.*
(*ref*. to your thermodynamics lectures)

Definition: The system's tendency to increase its entropy results in an entropic force which is the result of action of some particular underlying microscopic forces

Isothermal stretching

Elastomer in stretch

Elastomer at rest
(polymer network)



b)

When stretched, the polymer network becomes mores ordered and thus the entropy decreases

a)

Entropy is maximized at rest - when the force is removed, the elastomer network regains its initial shape

Thermodynamics of rubber – enthalpic (energetic) and entropic forces

The change of internal energy of a (closed) system can be achieved by:

(Static case – zero kinetic energy)

Changing the system volume

$$\mathrm{d}U = T\mathrm{d}S - p\mathrm{d}V + f\mathrm{d}L \qquad \textbf{(1)}$$

Change in internal energy of a system

Heating of the system

Force

Deforming the system

(This is from your previous courses of thermodynamics)

$$F \equiv \psi \qquad \sigma_i = \frac{\partial \psi}{\partial \lambda_i},$$

Change of Helmholtz free energy

Helmholtz free energy:
(state function)

$$F = U - TS \Rightarrow \mathrm{d}F = \underbrace{\mathrm{d}U}_{= T\mathrm{d}S - p\mathrm{d}V + f\mathrm{d}L \ (1)} - T\mathrm{d}S - S\mathrm{d}T$$

$$f = \frac{\partial \psi}{\partial L} = \frac{\partial F}{\partial L}$$

$$\Rightarrow \mathrm{d}F = -S\mathrm{d}T - p\mathrm{d}V + \boxed{f} \cdot \mathrm{d}L$$

← (total differential)

force

$$\mathrm{d}F = \left(\frac{\partial F}{\partial T}\right)_{V,L} \mathrm{d}T + \left(\frac{\partial F}{\partial V}\right)_{T,L} \mathrm{d}V + \boxed{\left(\frac{\partial F}{\partial L}\right)_{T,V}} \cdot \mathrm{d}L$$

(can de written in terms of a total differential)

Metals: $|f_E| >> |f_S|$
(for typical cristalline solids) $f_S = 0$

Ideal Polymer network: $|f_S| >> |f_E|$
$f_E = 0$

Applied **force:**

$$\Rightarrow f = \left(\frac{\partial F}{\partial L}\right)_{T,V} = \underbrace{\left(\frac{\partial U}{\partial L}\right)_{T,V}}_{\equiv f_E} - T\underbrace{\left(\frac{\partial S}{\partial L}\right)_{T,V}}_{\equiv f_S} \boxed{\equiv f_E + f_S} \qquad \textbf{(2)}$$

$$f = \frac{\partial \psi}{\partial L} = \frac{\partial F}{\partial L}$$

Energetic contribution

Entropic contribution

Origin of the force:
Energetic and entropic

$$\boxed{f = f_E + f_S}$$

Thermo-elasticity

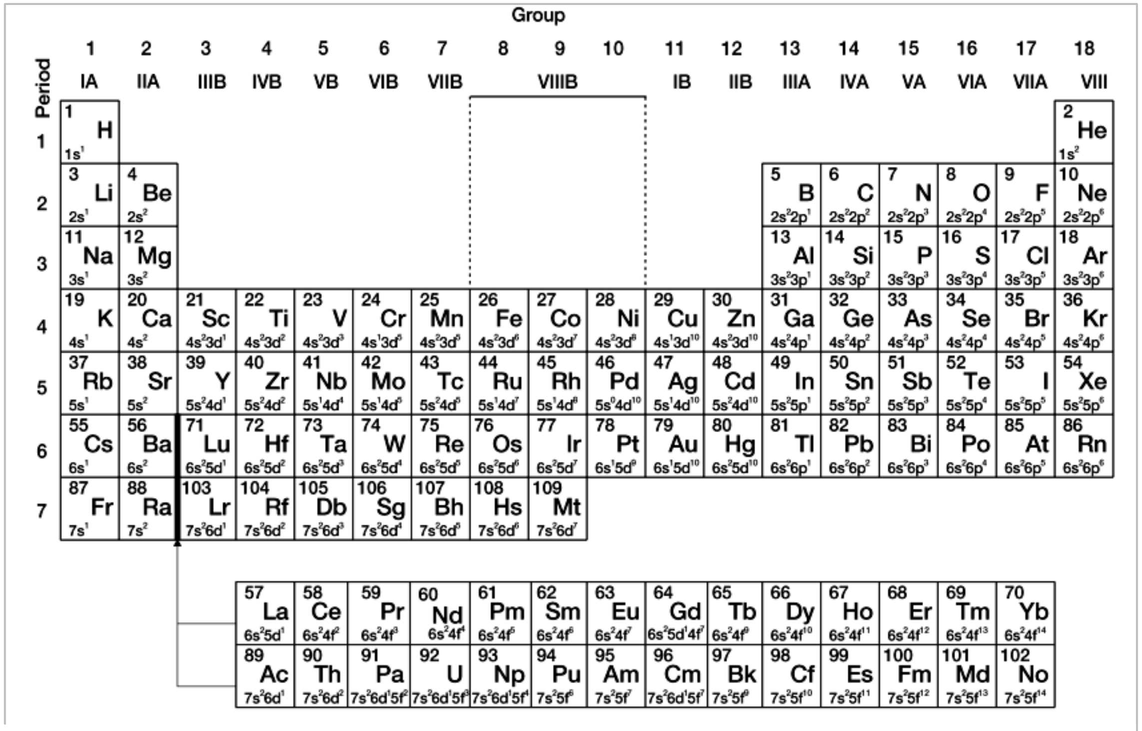
END OF LECTURE

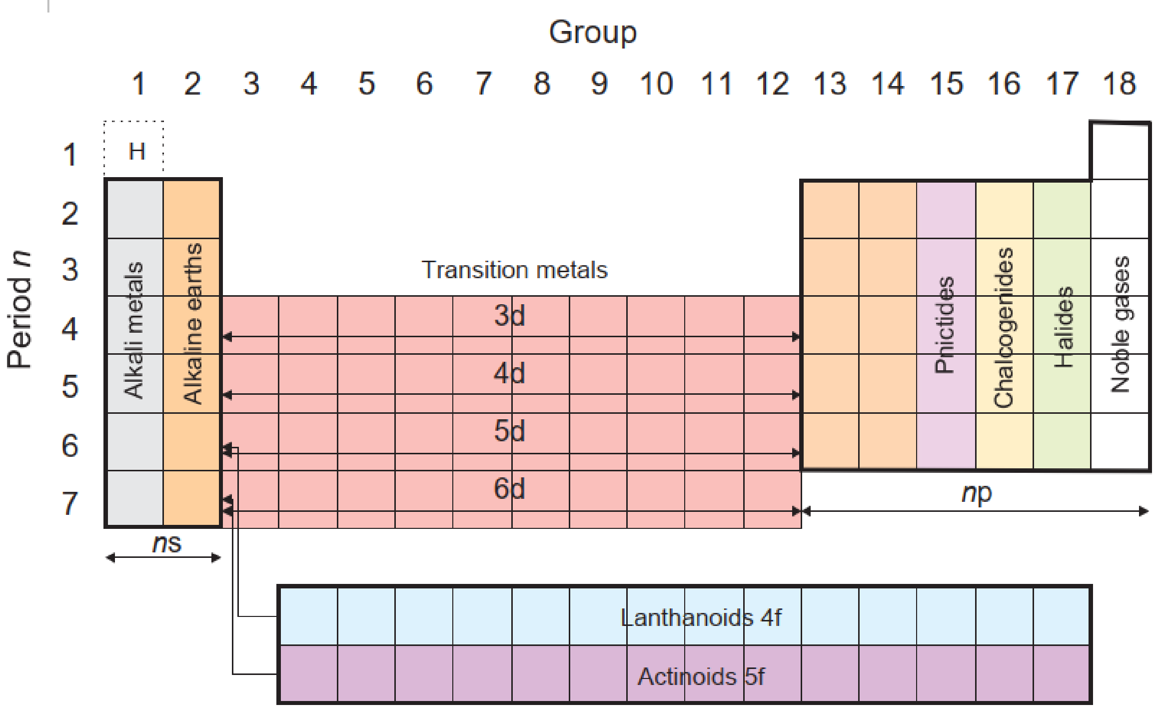
Homework

Skills! How to Acquire them?

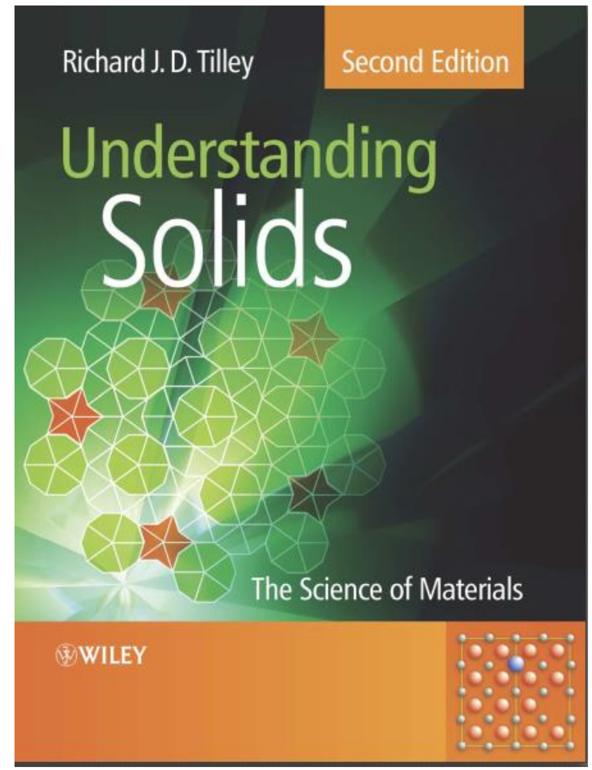
The secret: by **Training** you acquire skills… and by not training, you loose the skills you have…

Good to know:





The relationship between electronic configuration and the periodic table arrangement.



Homework # 2(2)

Elasticity

Content needed

Elasticity

Material Symmetries
Degree of symmetry

Linear Elasticity – Matrix Formulation
Anisotropy
Isotropy
Limits on Elastic Parameters Values
Orthotropy
Transversal isotropy
Limits on Elastic Parameters Values

…

Hyperelasticity

Rubber or rubber-like Elasticity

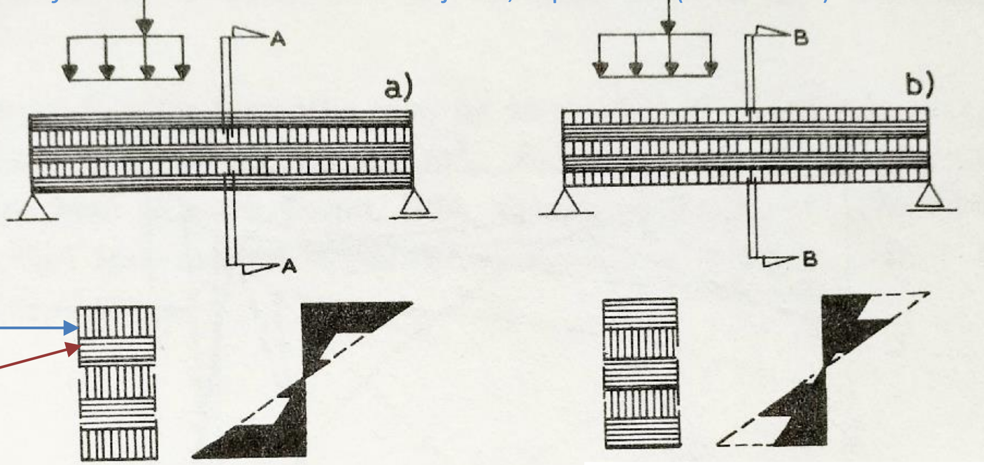
Problem 1: Orthotropy

<u>Elastic</u> Bending of a Glue Laminated Timber Beam

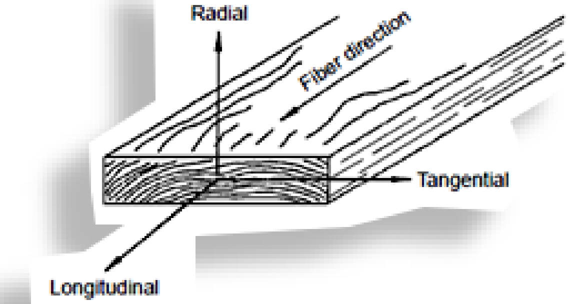
Ref: Petter Aune. TRE- konstruksjoner, Tapir 1975. (in Swedish)

$E_0 \equiv E_L$

$E_{90} \equiv E_T$



Bending **normal stresses**   Bending **normal stresses**



Three principal axes of wood with respect to grain direction and growth rings.

$$E_L >> E_T$$

$$E_L = nE_T\,,\quad n \approx 0.01 \div 0.1$$

Wood: cross section $h \times b$

Q1: **Explain** qualitatively **&** **quantitatively** (very concisely)

- why these stresses are as they are?
- why this difference?

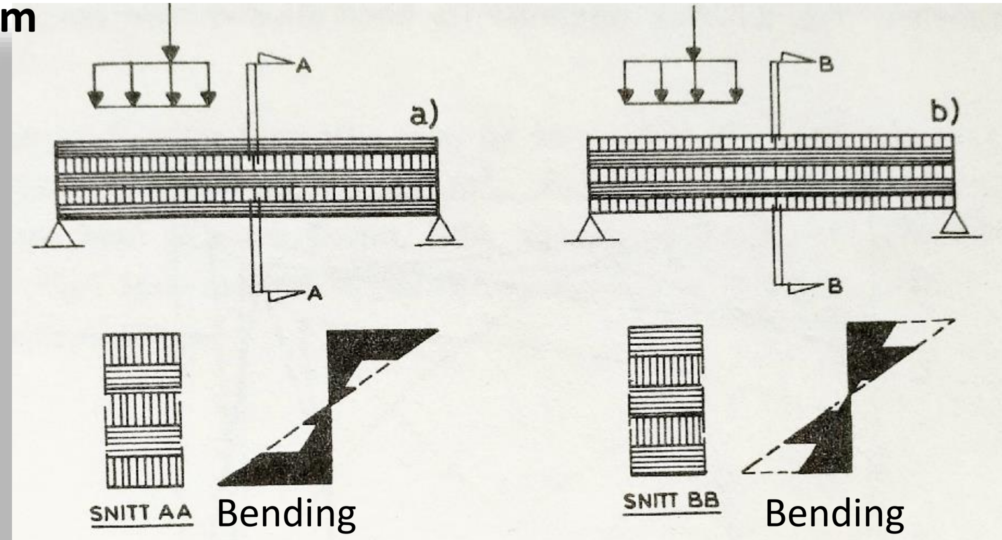
Q2: for a generic freely supported GL-beam & a constant distributed load, determine the elastic deflection at the mid span for normal temperature and hygrometry. Account for both bending and shearing contribution for the deflection. For material properties refer to next Table (next-next slide)

Hints: **1**: Determine an effective bending stiffness ***B*** and shear stiffness ***S*** for the composite section *a*) and *b*). **2**: For choosing the *correct shear modulus* values from tables, think in which material orthopy plane the corresponding shear strain component occurs.

$h \times b$

Orthotropy - solution

Elastic Bending of a Glue Laminated Beam



Bending **normal stresses**

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- why these stresses are as they are?
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Hints: determine an effective bending stiffness ***B*** and shear stiffness ***S*** for the composite section a) and b)

Problem 2: Linear isotropy in 3D elasticity (version 1)

Using Voigt's notation,

1. Write explicitly the stress-strain relation $\boldsymbol{\sigma} = \lambda\,\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon} \longleftrightarrow \sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$ in the matrix form $\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$ using the elasticity constants E, ν instead of the Lamé elastic constants.

What would be the expression, in matrix form, of the strain energy in terms of strains only?

What, not trivial result, can you conclude for the *material stiffness matrix* $\mathbf{D}$ from the sign of the strain energy?

2. Determine explicitly, <u>from answer of 1),</u> the three-dimensional *compliance matrix* given on the right for an isotropic linear elastic material.

Bonus: *what does mean physically speaking m*aterial isotropy? [1 pt]

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$$\boldsymbol{\varepsilon} = \mathbf{C}^{-1}\boldsymbol{\sigma}$$

Bonus: *what does mean physically speaking m*aterial isotropy? [1 pt]

Material isotropy: the constitutive relation remains unchanged irrespective of the Cartesian coordinate system we use

Physically, this means that the material properties are the same in all the directions

Problem 2: Linear isotropy in 3D elasticity (version 2)

Problem:

Using Voigt's notation, **derive** the three-dimensional **stiffness** (elastic) **matrix** starting from the compliance matrix given on the right for an isotropic linear elastic material. Write also the constitutive relation in tensor form.

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{13} \end{Bmatrix} = \begin{bmatrix} 1/E & -\nu/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & 1/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & -\nu/E & 1/E & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{Bmatrix}$$

Linear isotropic material:

$$\boldsymbol{\sigma} = \lambda \operatorname{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}$$

$$\sigma_{ij} = \lambda \varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$$

Problem 3: orthotropic elasticity in plane

Problem:

Consider a two-dimensional orthotropic material in plane stress state.

1. Determine the constitutive relation $\varepsilon = f(\sigma)$ in matrix form for 2-D plane stress. The coefficient matrix is the compliance matrix $\mathbf{C}$.

2. Invert the compliance matrix and deduce the stress strain relation in matrix form. $\mathbf{D} = ?$

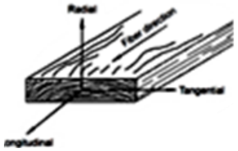
3. Derive the reciprocal relations between Poisson's coefficients ν_{ij} and elasticity modulus E_i

Hint: *from which* ***basic property****, are the reciprocity relations, derived*?

Reciprocal relations:

$$\nu_{12}/E_1 = \nu_{21}/E_2$$
$$\nu_{31}/E_3 = \nu_{13}/E_1$$
$$\nu_{23}/E_2 = \nu_{32}/E_3$$

1, 2, 3 are the material principal directions of orthotropy



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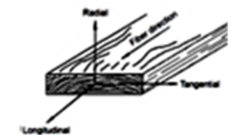
$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \end{bmatrix} = \begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & -\nu_{31}/E_3 & 0 & 0 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & -\nu_{32}/E_3 & 0 & 0 & 0 \\ -\nu_{13}/E_1 & -\nu_{23}/E_2 & 1/E_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_{23} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_{31} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix}$$

1, 2, 3 are the material principal directions of orthotropy



Hint: *from which **basic property**, are the reciprocity relations, derived?*

Reciprocal relations:

$$\boxed{\begin{aligned} \nu_{12}/E_1 &= \nu_{21}/E_2 \\ \nu_{31}/E_3 &= \nu_{13}/E_1 \\ \nu_{23}/E_2 &= \nu_{32}/E_3 \end{aligned}}$$

Problem 4: isotropy in 3D elasticity

Problem:

Consider a thin-walled cylindrical pressure wessel. The wall is very thin, $(t/R<1/10)$ $t \ll R$, as compared to the other dimensions as the radius and consequently you may assume that the stresses are uniform across the wall thickness t.
Here we have and $t = 2\ \text{mm}$ $D = 2R = 50\ \text{mm}$.

The internal (over-) pressure $p > 0$ uniform.
In addition to the gage pressure p torque moment is M_t applied at the ends of the wessel cylinder.

Two strain gages mutually perpendicular are perfectly glued on the external surface of the cylinder (as shown in the figure). The measured strains are

$$\varepsilon_{45} = 50\mu\text{m/m}$$

$$\varepsilon_{-45} = -20\mu\text{m/m}$$

The material is steel and considered *isotropic* and *linear elastic* since the stress state is such that no plastic flow yet occurs.

Question: Determine the torque moment M_t and the pressure p.

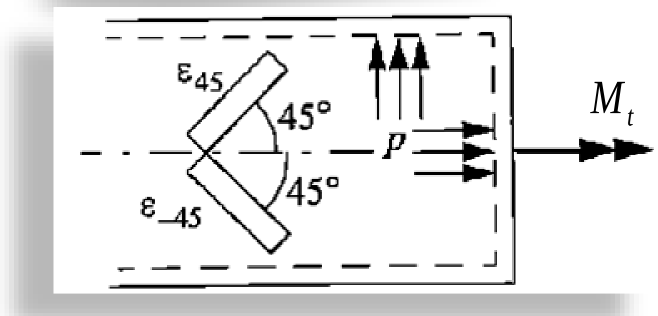
Hints:
Since the geometry and the loading are cylindrically symmetric then the stresses are independent of the angular coordinate of the cylindrically coordinate system.

Composed stress state

$(t/R<1/10)$

$2R$

$t \ll R$



$$p \equiv p_{inside} - p_{atmospheric} > 0$$

Oheisen kuvan mukaisen ohutseinämäisen suljetun putken sisällä vallitsee ylipaine p. Putken keskihalkaisija on $d = 50$ mm ja seinämän paksuus $t = 2$ mm. Putkea kuormittaa myös vääntömomentti M_v. Putken ulkopinnalta mitataan kahdella venymäliuskalla venymän arvot $\varepsilon_{45} = 0,00005$ ja $\varepsilon_{-45} = -0,00002$. Määritä paine p ja vääntömomentti M_v.

Problem 5 : Laminate plate

Problem:

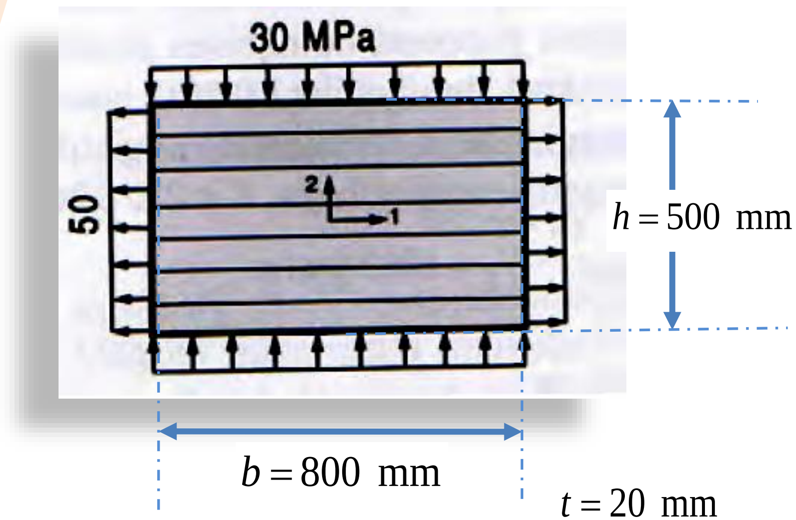
Physical problem: Consider the laminate plate with principle material direction 1 and 2. The fibers are aligned along direction 1. In the thickness direction of the plate we have traverse isotropy. The thickness is 20 mm. (Tehty hiilikuituvahvisteisesta epoksista)

The plate is under the stress state shown in the figure.

Question: Determine the length changes in both directions 1 and 2 and also in the thickness.

Bonus: compute the solution using *Abaqus* or *Comsol*

2-D stress state

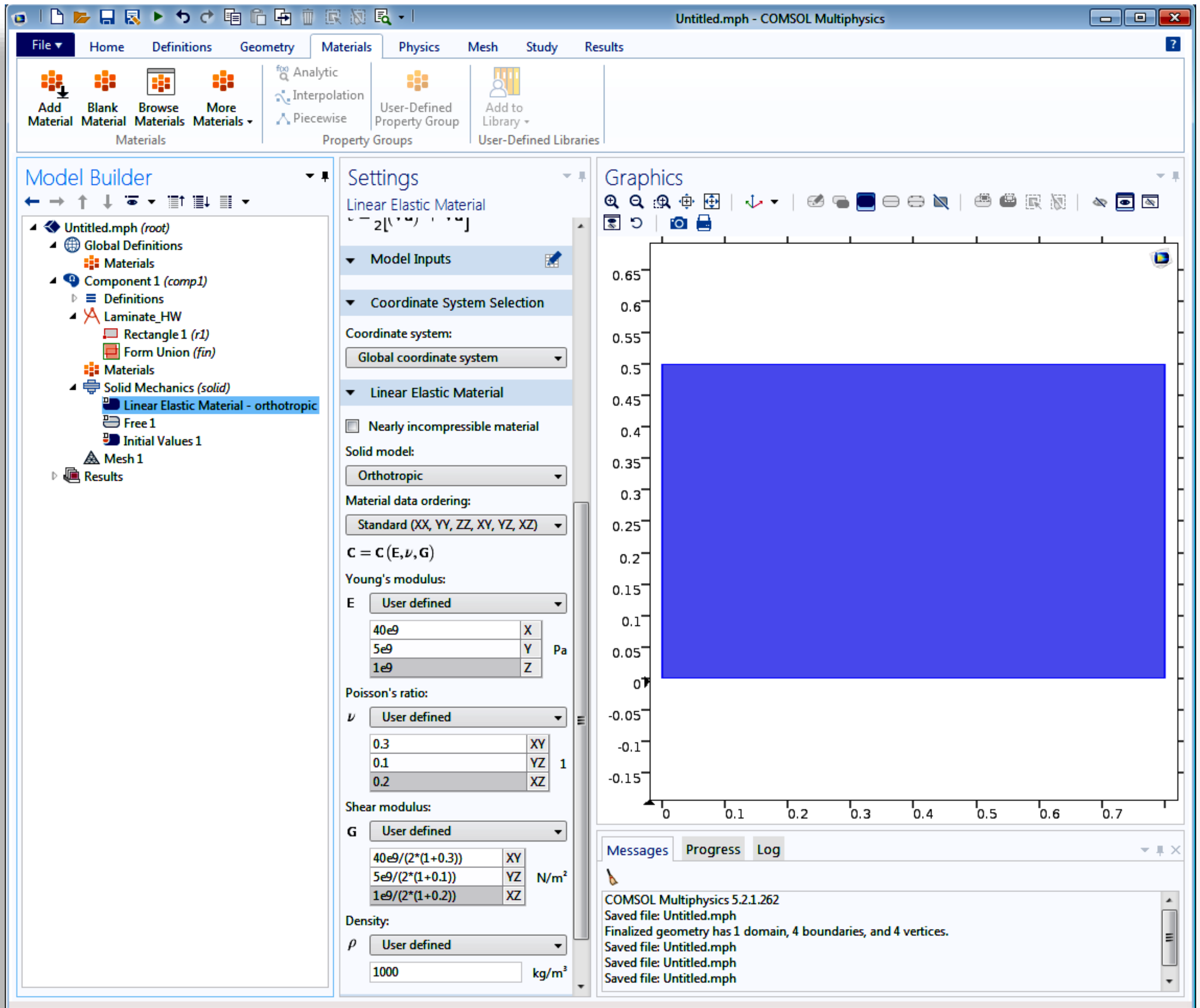


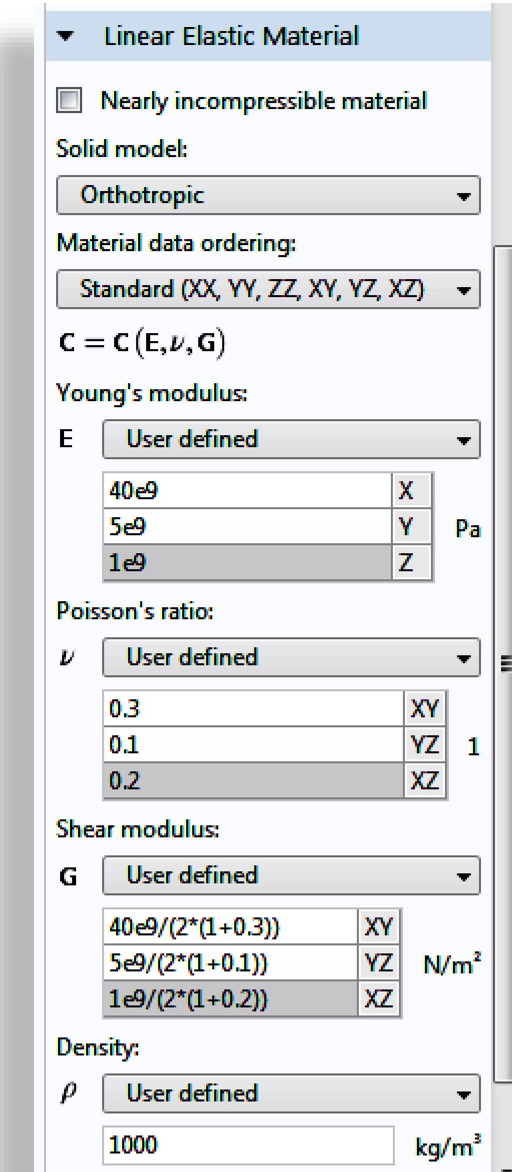
$$E_1 = 40 \text{ GPa}, \quad E_2 = 5 \text{ GPa}$$

$$\nu_{12} = 0.3, \quad \nu_{13} = 0.2, \quad \nu_{23} = 0.1,$$

Kuitulujitetun levyn paksuus on 20 mm, leveys kuitusuunnassa 1 on 800 mm ja leveys suunnassa 2 on 500 mm. Määritä levyn paksuuden ja leveyden muutokset. Materiaalikertoimet ovat $E_1 = 40$ GPa, $E_2 = 5$ GPa, ja $\nu_{12} = 0,3, \nu_{13} = 0,2$ ja $\nu_{23} = 0,1$.

Bonus: compute the solution using *Abaqus* or *Comsol*





Homework # 2(3)

Small project [15 points (obligatory) + 15 points (bonus)]

Elasticity

Elastic orthotropy

Bending of plates

Reading: Chapter 7.2 from:

I put the necessary 'equations' in the end slide

7.2 ORTHOTROPIC AND STIFFENED PLATES

Thin Plates and Shells

Theory, Analysis, and Applications

Eduard Ventsel
Theodor Krauthammer

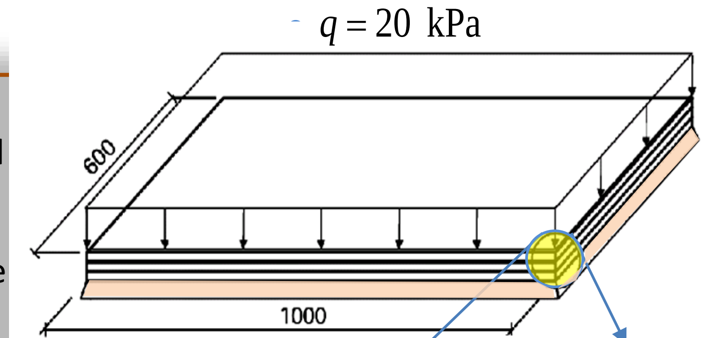
The Pennsylvania State University
University Park, Pennsylvania

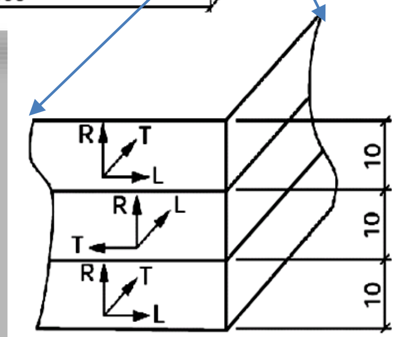
Due date: xx.xx.yyyy

Problem 6: Orthotropic plate or laminate

Physical problem: Consider the laminate plate (Glue Laminated Timber, GLT) formed by three perfectly bonded layers having respective principle material directions L, T and R (*cf*. figure).

The plate is under a transversal loading. The plate can freely rotate along the four support lines (freely supported, hinged on all sides). The individual layers should be modeled as a linear elastic orthotropic material.





L – longitudinal
T – tangential
R – radial

Dimensions are in mm.

Problem [BONUS, elective <u>15 pnts</u>]: Using Abaqus, <u>Comsol</u> (easy to use) or any other FE-software you need.

1. Determine the displacement along the horizontal center lines
2. The stress distributions at a section at the center and close to the supports.
3. Analyze your results

Problem [obligatory, <u>15 pnts</u>]: You know well the thin plate theory of orthotropic plates (*cf.* textbooks, the pdf-version in your previous master course of *plate and shells*) and you are a clever and responsible engineer who wants somehow to check his FE-results. For this purpose you want to obtain an analytical (or semi-analytical) solution for comparison with FE-results. Do you have any idea how to proceed? If yes, then how? Do it.
Hint: one can find an *equivalent* <u>one layer orthotropic plate</u> having effective bending and torsional rigidities integrated from the 3-layer plate *in order to conserve strain energy (cf. the mentioned textbook). Other ways toward the solution exist and are allowed.*

Ref: this homework, except the bonus, was adapted from the course:
13-02-0003-vl Werkstoffmechanik
Technische Universität Darmstadt
Lehrende: Prof. Dr.-Ing. Michael Vormwald; Dipl.-Ing. Melanie Fiedler

Eine Platte aus Brettschichtholz wird allseitig gelenkig gelagert und mit eine konstanten Flächenlast $q = 0.02$ MPa belastet. Die einzelnen Schichten sollen als orthotroper Werkstoff modelliert werden. Die Werkstoffkonstanten in einem L-R-T-Koordinatensystem betragen:

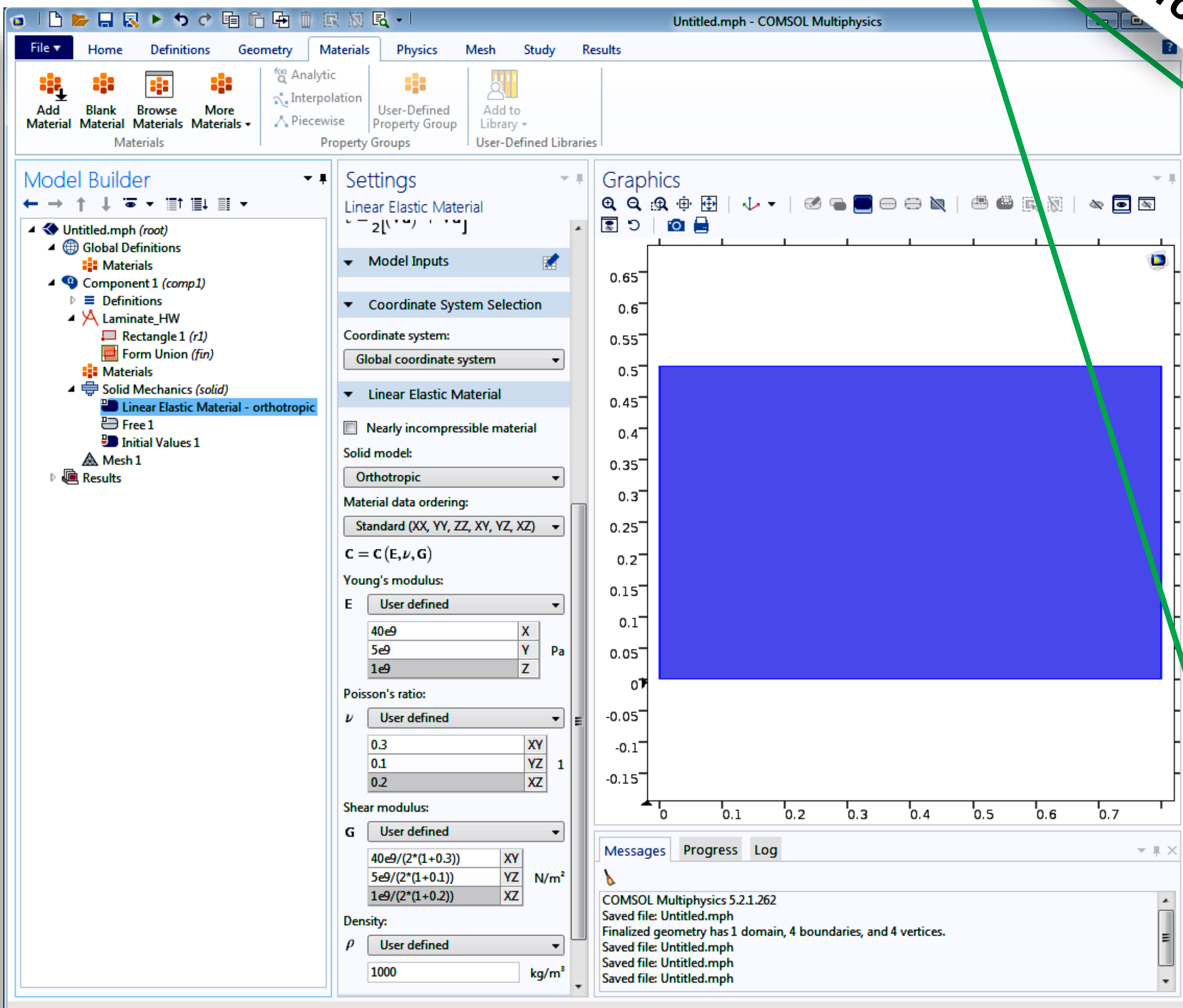
E_L[MPa]	E_R[MPa]	E_T[MPa]	G_{LR}[MPa]	G_{LT}[MPa]	G_{RT}[MPa]	ν_{LT}	ν_{LR}	ν_{RT}
11990	820	420	620	740	240	0.7749	0.6071	0.6031

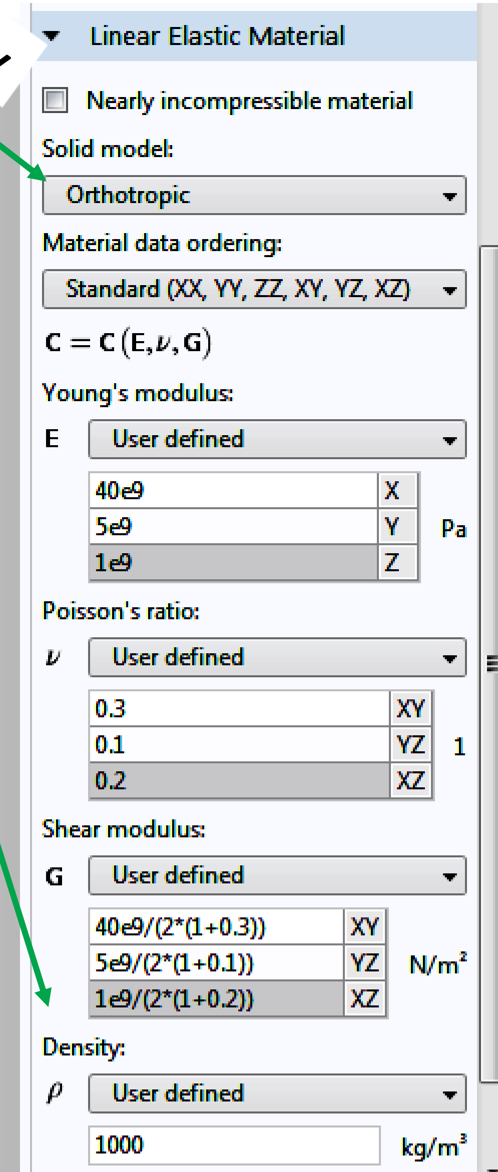
Due date: 30.4.2017

See how simple, in this example, if one uses *Comsol*

NB. This is not the example of our problem

The orthotropy





must obtain a new set of stress–strain relations that reflects the orthotropic properties of a material of the plate. Such a set of relations is shown below [3]:

$$\varepsilon_x = \frac{\sigma_x}{E_x} - \nu_y \frac{\sigma_y}{E_y}; \quad \varepsilon_y = \frac{\sigma_y}{E_y} - \nu_x \frac{\sigma_x}{E_x}; \quad \gamma_{xy} = \frac{\tau_{xy}}{G}, \tag{7.22}$$

where E_x, E_y, ν_x, ν_y, and G are assumed to be elastic constants of an orthotropic material, i.e., E_x, E_y, and ν_x, ν_y are the moduli of elasticity and Poisson's ratios in the x and y directions, respectively. They are independent of one another. G is the shear modulus, which is the same for both isotropic and orthotropic materials. It can be expressed in terms of E_x and E_y as follows:

$$G \approx \frac{\sqrt{E_x E_y}}{2\left(1 + \sqrt{\nu_x \nu_y}\right)}. \tag{7.23}$$

The following relationship exists between independent elastic constants introduced above:

$$\frac{\nu_x}{E_x} = \frac{\nu_y}{E_y}. \tag{7.24}$$

This equality directly results from Betti's reciprocal theorem. Solving Eqs (7.22) for the stress components and taking into account (7.24), we obtain

$$\begin{gathered} \sigma_x = \frac{E_x}{1 - \nu_x \nu_y}(\varepsilon_x + \nu_y \varepsilon_y); \quad \sigma_y = \frac{E_y}{1 - \nu_x \nu_y}(\varepsilon_y + \nu_x \varepsilon_x); \\ \tau_{xy} = G\gamma_{xy}. \end{gathered} \tag{7.25}$$

The derivation of the governing differential equation of bending of an orthotropic plate is based on the general hypotheses introduced in Sec. 1.3. The strain-deflection relations (2.6) hold for orthotropic plates also. So, substituting the relations (2.6) into Eqs (7.25) gives the following:

$$\begin{gathered} \sigma_x = -\frac{E_x}{1 - \nu_x \nu_y}\left(\frac{\partial^2 w}{\partial x^2} + \nu_y \frac{\partial^2 w}{\partial y^2}\right) z; \quad \sigma_y = -\frac{E_y}{1 - \nu_x \nu_y}\left(\frac{\partial^2 w}{\partial y^2} + \nu_x \frac{\partial^2 w}{\partial x^2}\right) z; \\ \tau_{xy} = -2Gz\frac{\partial^2 w}{\partial x \partial y}. \end{gathered} \tag{7.26}$$

Substituting the above into Eqs (2.11) and integrating over the plate thickness, yields the following bending and twisting moments deflection relations for orthotropic plates:

where D_x, D_y, D_{xy}, D_{yx}, and D_s are the flexural and torsional rigidities of an orthotropic plate, respectively, and are given as

$$\begin{gathered} D_x = \frac{E_x}{1 - \nu_x \nu_y}\frac{h^3}{12}; \quad D_y = \frac{E_y}{1 - \nu_x \nu_y}\frac{h^3}{12}; \quad D_{xy} = \frac{E_x \nu_y}{1 - \nu_x \nu_y}\frac{h^3}{12}; \\ D_{yx} = \frac{E_y \nu_x}{1 - \nu_x \nu_y}\frac{h^3}{12}; \quad D_s = \frac{Gh^3}{12}. \end{gathered} \tag{7.28}$$

In view of the expressions (7.24), one can conclude that $D_{xy} = D_{yx}$. The shear force expressions (2.22) become

$$Q_x = -\frac{\partial}{\partial x}\left(D_x \frac{\partial^2 w}{\partial x^2} + H\frac{\partial^2 w}{\partial y^2}\right); \quad Q_y = -\frac{\partial}{\partial y}\left(H\frac{\partial^2 w}{\partial x^2} + D_y \frac{\partial^2 w}{\partial y^2}\right), \tag{7.29}$$

where

$$H = D_{xy} + 2D_s. \tag{7.30}$$

The governing differential equation (2.24) for orthotropic plates becomes

$$D_x \frac{\partial^4 w}{\partial x^4} + 2H\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} = p(x, y). \tag{7.31}$$

We give below the expression for the potential energy of bending for orthotropic plates, which follows from Eqs (2.52) and (7.26):

$$U = \frac{1}{2}\iint_A \left[D_x\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2D_{xy}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + D_y\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 4D_s\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right] \mathrm{d}A. \tag{7.32}$$

Thin Plates and Shells

Theory, Analysis, and Applications

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$$U = \frac{1}{2}\iint_A \left[D_x\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2D_{xy}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + D_y\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 4D_s\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right] \mathrm{d}A.$$

Some useful tables for wood

Table 4–2. Poisson's ratios for various species at approximately 12% moisture content

Species	μ_{LR}	μ_{LT}	μ_{RT}	μ_{TR}	μ_{RL}	μ_{TL}
Hardwoods						
Ash, white	0.371	0.440	0.684	0.360	0.059	0.051
Aspen, quaking	0.489	0.374	—	0.496	0.054	0.022
Balsa	0.229	0.488	0.665	0.231	0.018	0.009
Basswood	0.364	0.406	0.912	0.346	0.034	0.022
Birch, yellow	0.426	0.451	0.697	0.426	0.043	0.024
Cherry, black	0.392	0.428	0.695	0.282	0.086	0.048
Cottonwood, eastern	0.344	0.420	0.875	0.292	0.043	0.018
Mahogany, African	0.297	0.641	0.604	0.264	0.033	0.032
Mahogany, Honduras	0.314	0.533	0.600	0.326	0.033	0.034
Maple, sugar	0.424	0.476	0.774	0.349	0.065	0.037
Maple, red	0.434	0.509	0.762	0.354	0.063	0.044
Oak, red	0.350	0.448	0.560	0.292	0.064	0.033
Oak, white	0.369	0.428	0.618	0.300	0.074	0.036
Sweet gum	0.325	0.403	0.682	0.309	0.044	0.023
Walnut, black	0.495	0.632	0.718	0.378	0.052	0.035
Yellow-poplar	0.318	0.392	0.703	0.329	0.030	0.019
Softwoods						
Baldcypress	0.338	0.326	0.411	0.356	—	—
Cedar, northern white	0.337	0.340	0.458	0.345	—	—
Cedar, western red	0.378	0.296	0.484	0.403	—	—
Douglas-fir	0.292	0.449	0.390	0.374	0.036	0.029
Fir, subalpine	0.341	0.332	0.437	0.336	—	—
Hemlock, western	0.485	0.423	0.442	0.382	—	—
Larch, western	0.355	0.276	0.389	0.352	—	—
Pine						
Loblolly	0.328	0.292	0.382	0.362	—	—
Lodgepole	0.316	0.347	0.469	0.381	—	—
Longleaf	0.332	0.365	0.384	0.342	—	—
Pond	0.280	0.364	0.389	0.320	—	—
Ponderosa	0.337	0.400	0.426	0.359	—	—
Red	0.347	0.315	0.408	0.308	—	—
Slash	0.392	0.444	0.447	0.387	—	—
Sugar	0.356	0.349	0.428	0.358	—	—
Western white	0.329	0.344	0.410	0.334	—	—
Redwood	0.360	0.346	0.373	0.400	—	—
Spruce, Sitka	0.372	0.467	0.435	0.245	0.040	0.025
Spruce, Engelmann	0.422	0.462	0.530	0.255	0.083	0.058

Chapter 2

Structure of Wood

Regis B. Miller

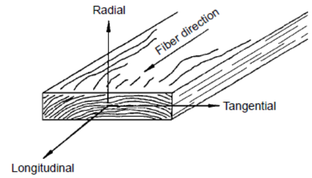


Figure 4–1. Three principal axes of wood with respect to grain direction and growth rings.

Table 4–1. Elastic ratios for various species at approximately 12% moisture content[a]

Species	E_T/E_L	E_R/E_L	G_{LR}/E_L	G_{LT}/E_L	G_{RT}/E_L
Hardwoods					
Ash, white	0.080	0.125	0.109	0.077	—
Balsa	0.015	0.046	0.054	0.037	0.005
Basswood	0.027	0.066	0.056	0.046	—
Birch, yellow	0.050	0.078	0.074	0.068	0.017
Cherry, black	0.086	0.197	0.147	0.097	—
Cottonwood, eastern	0.047	0.083	0.076	0.052	—
Mahogany, African	0.050	0.111	0.088	0.059	0.021
Mahogany, Honduras	0.064	0.107	0.066	0.086	0.028
Maple, sugar	0.065	0.132	0.111	0.063	—
Maple, red	0.067	0.140	0.133	0.074	—
Oak, red	0.082	0.154	0.089	0.081	—
Oak, white	0.072	0.163	0.086	—	—
Sweet gum	0.050	0.115	0.089	0.061	0.021
Walnut, black	0.056	0.106	0.085	0.062	0.021
Yellow-poplar	0.043	0.092	0.075	0.069	0.011
Softwoods					
Baldcypress	0.039	0.084	0.063	0.054	0.007
Cedar, northern white	0.081	0.183	0.210	0.187	0.015
Cedar, western red	0.055	0.081	0.087	0.086	0.005
Douglas-fir	0.050	0.068	0.064	0.078	0.007
Fir, subalpine	0.039	0.102	0.070	0.058	0.006
Hemlock, western	0.031	0.058	0.038	0.032	0.003
Larch, western	0.065	0.079	0.063	0.069	0.007
Pine					
Loblolly	0.078	0.113	0.082	0.081	0.013
Lodgepole	0.068	0.102	0.049	0.046	0.005
Longleaf	0.055	0.102	0.071	0.060	0.012
Pond	0.041	0.071	0.050	0.045	0.009
Ponderosa	0.083	0.122	0.138	0.115	0.017
Red	0.044	0.088	0.096	0.081	0.011
Slash	0.045	0.074	0.055	0.053	0.010
Sugar	0.087	0.131	0.124	0.113	0.019
Western white	0.038	0.078	0.052	0.048	0.005
Redwood	0.089	0.087	0.066	0.077	0.011
Spruce, Sitka	0.043	0.078	0.064	0.061	0.003
Spruce, Engelmann	0.059	0.128	0.124	0.120	0.010

[a] E_L may be approximated by increasing modulus of elasticity values in Table 4–3 by 10%.

$$\frac{\mu_{ij}}{E_i} = \frac{\mu_{ji}}{E_j}, \quad i \neq j \quad i,j = L,R,T$$

Hyper-elasticity

Problem 7: - hyper-elasticity

Physical : Consider an ideal rubber in a bi-axial stress state $\sigma_1 \equiv \sigma_x$ and $\sigma_2 \equiv \sigma_y$ at a constant temperature.

1) Starting with the expression for Helmholtz free energy derive the constitutive law for stresses *versus* stretches (extension ratio) that is find

$$\sigma_i \equiv \sigma_i(\lambda_1, \lambda_2), \; i = 1,2$$

For the ideal rubber, you can use a neo- Hokean model where the Gibbs free energy density is

$$\psi = C_{10}(I_1 - 3) \equiv \frac{1}{2} E_\theta (I_1 - 3)$$

with an effective elasticity coefficient E_θ

$$C_{10} = \frac{1}{2} N k_B \theta \equiv \frac{1}{2} E_\theta$$

Account first for incompressibility (deformation occurs at constant volume).

The invariant $I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$

Principle stretches

Partial answers: $\sigma_1 = E_\theta \left(\lambda_1 - \frac{1}{\lambda_1^3 \lambda_2^2} \right)$, $\sigma_2 = E_\theta \left(\lambda_2 - \frac{1}{\lambda_2^3 \lambda_1^2} \right)$

2) Assume that the elastic modulus is experimentally estimated as

$$E_\theta \approx 3.33 \text{ MPa}$$

Determine the stresses leading to the stretches

$$\lambda_1 = 2, \; \lambda_2 = 1/2$$

Model validity: by comparing with experimental results it is found that the used model over-estimates the stresses when $1.5 < \lambda_1 < 6$

What is the quality of your results for stresses?

Free reading:

Uniaxial case:

The stress σ is an engineering stresses

$$\lambda_1 = \lambda_2 \equiv \lambda \rightarrow \sigma = 2C_{10}(\lambda - \frac{1}{\lambda^2}) \equiv E_\theta(\lambda - \frac{1}{\lambda^2})$$

$$\sigma = N k_B \theta (\lambda - \frac{1}{\lambda^2}) \qquad \varepsilon_i = \lambda_i - 1$$

nominal = engineering

$$\sigma_{\mathrm{i}}^{(\mathrm{true})} = \sigma_{\mathrm{i}}^{(\mathrm{eng})} / (\lambda_{\mathrm{j}} \lambda_{\mathrm{k}}),$$

$$j \neq i, k \neq i, j \neq k$$

force

$$\sigma_{true} = \frac{F}{L_2 L_3} \qquad \sigma_{eng} = \frac{F}{L_{2,0} L_{3,0}}$$

actual cross-section

original cross-section

Example: Uniaxial case - how you did got this result?

$$\sigma_2 = \sigma_3 = 0, \sigma_1 \equiv \sigma \rightarrow \quad \sigma^{\text{(true)}} \equiv \sigma = 2C_{10}(\lambda^2 - 1/\lambda)$$

Neo- Hookean material

Neo- Hookean model: $\psi = C_{10}(I_1 - 3) = C_{10}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)$

$$(\lambda_1^2 + \lambda_2^2 - \tfrac{1}{\lambda_1^2\lambda_2^2} - 3)$$

$$\sigma_1 - 0 = \lambda_1 \frac{\partial \psi}{\partial \lambda_1} = \lambda_1 2C_{10}(\lambda_1 - \tfrac{\lambda_1}{\lambda_1^4\lambda_2^2}) = 2C_{10}(\lambda_1^2 - \tfrac{1}{\lambda_1^2\lambda_2^2})$$

$$\lambda_3^2 = \frac{1}{\lambda_1^2\lambda_2^2}$$

In simple extension: $\lambda_1 = \lambda, \lambda_2 = \lambda_3 = 1/\sqrt{\lambda},$

a single extension ratio with equal contractions in transverse directions

$$\sigma \equiv \sigma_1^{\text{(true)}} = 2C_{10}(\lambda^2 - \frac{1}{\lambda}) \quad \leftarrow \psi = C_{10}(\lambda^2 + 2/\lambda - 3)$$

$$J = \sqrt{I_3}$$

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$
$$I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_1^2\lambda_3^2$$
$$I_3 = \lambda_1^2\lambda_2^2\lambda_3^2$$

Stress-strain relation for incompressible materials

True stresses

$$\sigma_1 - \sigma_3 = \lambda_1 \frac{\partial U(\lambda_1, \lambda_2)}{\partial \lambda_1}$$

$$\sigma_2 - \sigma_3 = \lambda_2 \frac{\partial U(\lambda_1, \lambda_2)}{\partial \lambda_2}$$

and

$$\sqrt{I_3} = \lambda_1\lambda_2\lambda_3 = 1$$

Have a look: Journal article you can find an extensive review of constitutive models

***Arch Appl Mech* (2012)** 82:1183–1217

DOI 10.1007/s00419-012-0610-z

Paul Steinmann · Mokarram Hossain · Gunnar Possart

Hyperelastic models for rubber-like materials: consistent tangent operators and suitability for Treloar's data

See also:

nominal = engineering

The elasticity and related properties of rubbers

Rep. Prog. Phys. 1973 **36** 755–826

L R G TRELOAR

Department of Polymer and Fibre Science, University of Manchester Institute of Science and Technology, PO Box no 88, Sackville Street, Manchester M60 1QD, UK

End elasticity homework

BEGIN

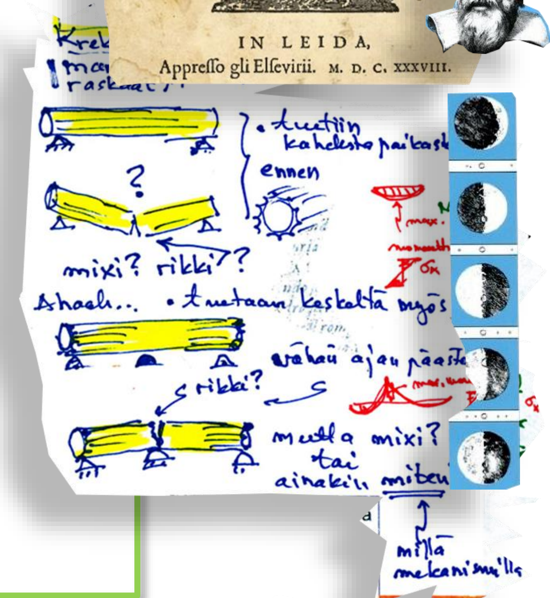
{

Elasticity

```
START,
  GO on,
  keep going …
  … till the end
      THEN
STOP
```

}

END



Ref. Fig. 29 in: