7

DAMAGE MECHANICS

L'endommagement, comme le diable, invisible mais redoutable.

The phenomenon of damage, described from a physical point of view in Chapter 1, and then presented schematically in Chapter 3, represents surface discontinuities in the form of microcracks, or volume discontinuities in the form of cavities. It, therefore, involves a rheological process quite different from deformation, although the initial causes of the two phenomena are identical: movement and accumulation of dislocations in metals, modification of intermolecular bonds in organic materials, microdecohesion in minerals. Damage is marked by pronounced irreversibility; the traditional thermomechanical treatments can only partially remove the defects caused by it. Macroscopic fracture has been studied for a long time. Around 1500, Leonardo da Vinci was already preoccupied with the characterization of fracture by means of mechanical variables. A number of failure criteria, i.e., functions of components of stress or strain, characterizing the fracture of the volume element have been proposed (e.g., by Coulomb, Rankine, Tresca, von Mises, Mohr, Caquot). However, it is only quite recently that concern has been directed towards modelling the progressive deterioration of matter preceding the macroscopic fracture. The development of damage mechanics began in 1958. In that year, Kachanov published the first paper devoted to a continuous damage variable, conceived within the framework, limited indeed, of creep failure of metals under uniaxial loads. This concept was taken up again in the seventies, mainly in France (Lemaitre & Chaboche), Sweden (Hult), England (Leckie), Japan (Murakami) and extended to ductile fracture and fatigue fracture. It has been generalized to the multiaxial isotropic case within the framework of the thermodynamics of irreversible processes described in Chapter 2; anisotropic damage remains to be 'finished' in the eighties!

7.1 Domain of validity and use

First of all, it should be stated what is meant by the beginning and the end of the process of damage; in other words what is the scale of the phenomena under consideration? A material is said to be free of any damage if it is devoid of cracks and cavities at the microscopic scale, or from a more pragmatic point of view, if its deformation behaviour is that of the material formed under the best conditions. Let us recall what was said in Chapter 2: the initial state of a material cannot be objectively defined. Usually, it is the state starting from which the history of the loads is known. The final damage state is that of the fracture of the volume element, i.e., the existence of a macroscopic crack of the size of the representative volume element as estimated in Chapter 3: 0.1–1 mm for metals or polymers, of the order of 1 cm for wood, and 10 cm for concrete. Beyond this, is the domain of crack mechanics (Chapter 8).

The theory of damage therefore, describes the evolution of the phenomena between the virgin state and macroscopic crack initiation. This evolution, which is not always easily distinguishable from the deformation phenomena which usually accompany it, is due to several mechanisms which have been classified in Chapter 3:

> ductile plastic damage accompanying large plastic deformations of metals at ambient as well as average temperatures;
> brittle viscoplastic (or creep) damage, a function of time which, for metals at average and high temperatures, corresponds to intergranular decohesions accompanying viscoplastic strains;
> fatigue (or microplastic) damage, caused by stress repetitions and identified as a function of the number of cycles;
> macrobrittle damage produced by monotonic loads without appreciable irreversible deformations, as in the case of concrete for example.

Other phenomena can be considered as damage: the oxidation process, corrosion, irradiation; but there thermomechanical modelling is yet to be done!

It is therefore clear that the theory of damage is concerned with all materials at low as well as high temperatures under any kind of load. It is from the nature of the evolution models that we can represent these different phenomena that can accumulate or interact with each other. Knowing the stress and strain history for a given volume element of a structure, the damage laws provide, by integration with respect to time, the

damage evolution in the element up to the point of macroscopic crack initiation. Thus, the theory furnishes the time or the number of cycles corresponding to the initiation of such a crack at the most stressed point of the structure. This is the modern principle of analysing the resistance of structures used at the design level and in the verification and control of the service conditions of structures. This also provides a better means of optimizing the process of metal forming by plastic deformations, to avoid or reduce manufacturing defects. And, in any case, it is the means of knowing *a priori* the modifications of the product's mechanical properties which result from the forming operations.

7.2 Phenomenological aspects

The definition of a mechanical damage variable itself presents a difficult problem. There is nothing (almost nothing!) that macroscopically distinguishes a highly damaged volume element and a virgin one. It therefore becomes necessary to imagine internal variables which are representative of the deteriorated state of the matter. There are several possibilities for this choice depending on the school of thought and the types of damage measurement envisaged:

measurements at the scale of microstructure (density of micro-cracks or cavities) lead to microscopic models that can be 'integrated' over the macroscopic volume element with the help of mathematical homogenization techniques. We may thus obtain the properties of the damaged volume element; but from these it is difficult to define a macroscopic damage variable and a law of its evolution which is easy to use in continuum mechanics analysis;

global physical measurements (density, resistivity etc.) require the definition of a global model to convert them into properties which characterize mechanical resistance;

another type of damage evaluation is linked to the remaining life-time, but this concept does not directly lead to a damage constitutive law;

global mechanical measurements (of modification of elastic, plastic, or viscoplastic properties) are easier to interpret in terms of damage variables using the concept of effective stress introduced by Rabotnov. This is the approach we will be following.

7.2.1 *Damage variable*

Definition

Consider a damaged solid in which an element of finite volume has been isolated, of a sufficiently large size with respect to the inhomogeneities of the medium, and imagine that this element has been grossly enlarged (Fig. 7.1).

Let S be the area of a section of the volume element identified by its normal $\vec{n}$. On this section, cracks and cavities which constitute the damage leave traces of different forms. Let $\tilde{S}$ be the effective area of resistance ($\tilde{S} < S$) taking account of the area of these traces, stress concentrations in the neighbourhood of geometric discontinuities, and the interactions between the neighbouring defects. Let S_D be the difference:

$$S_D = S - \tilde{S}.$$

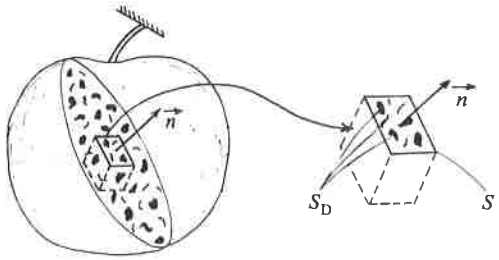
S_D is the total area of the defect traces corrected for stress concentrations and interactions. We will see in Section 7.2.2 that the concept of effective stress associated with the hypothesis of strain – equivalence enables us to avoid the calculation of S_D (or $\tilde{S}$) which would be extremely difficult to do because of the lack of knowledge of the precise geometry of the defects and because of the doubts one might have regarding the applicability of continuum mechanics on this scale.

By definition:

- $S_D/S = D_n$

is the mechanical measure of local damage relative to the direction $\vec{n}$. From a physical point of view, the damage variable is therefore the relative (or corrected) area of cracks and cavities cut by the plane normal to the direction $\vec{n}$. From a mathematical point of view, as S tends to 0, the variable

Fig. 7.1. Damaged element.



D_n is the (corrected) surface density of the discontinuities of the matter in the plane normal to $\vec{n}$.

$D_n = 0$ corresponds to the nondamaged or virgin state,
$D_n = 1$ corresponds to the breaking of the volume element into two parts along a plane normal to $\vec{n}$,
$0 \leqslant D_n < 1$ characterizes the damaged state.

In the general case of anisotropic damage consisting of cracks and cavities with preferred orientations, the value of the scalar variable D_n depends on the orientation of the normal. It will be seen in Section 7.3.1 that the corresponding intrinsic variable can be represented by a second or a fourth order tensor.

Hypothesis of isotropy

Isotropic damage consists of cracks and cavities with an orientation distributed uniformly in all directions. In this case, the variable does not depend on the orientation $\vec{n}$ and the damaged state is completely characterized by the scalar D. In this section we will limit ourselves to the case of isotropic damage:

$$D_n = D \qquad \forall \vec{n}.$$

7.2.2 *Effective stress*

Definition

The introduction of a damage variable which represents a surface density of discontinuities in the material leads directly to the concept of effective stress, i.e., to the stress calculated over the section which effectively resists the forces.

In the uniaxial case, if F is the applied force on a section of the representative volume element, $\sigma = F/S$ is the usual stress satisfying the equilibrium equation. In the presence of isotropic damage, D, the effective area of resistance is:

$$\tilde{S} = S - S_D = S(1 - D)$$

and by definition the effective stress $\tilde{\sigma}$ is taken to be:

$$\tilde{\sigma} = \sigma S/\tilde{S} \quad \text{or} \quad \tilde{\sigma} = \sigma/(1 - D).$$

Evidently $\tilde{\sigma} \geqslant \sigma$

$\tilde{\sigma} = \sigma$ for a virgin material,

$\tilde{\sigma} \to \infty$ at the moment of fracture.

In the case of multiaxial isotropic damage, the ratio $S/\tilde{S}$ does not depend on the orientation of the normal and the operator $(1 - D)$ can be applied to all the components. We will therefore write for the effective stress $\tilde{\boldsymbol{\sigma}}$:

- $\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma}/(1 - D)$.

Principle of strain-equivalence

We assume that the deformation behaviour of the material is only affected by damage in the form of effective stress:

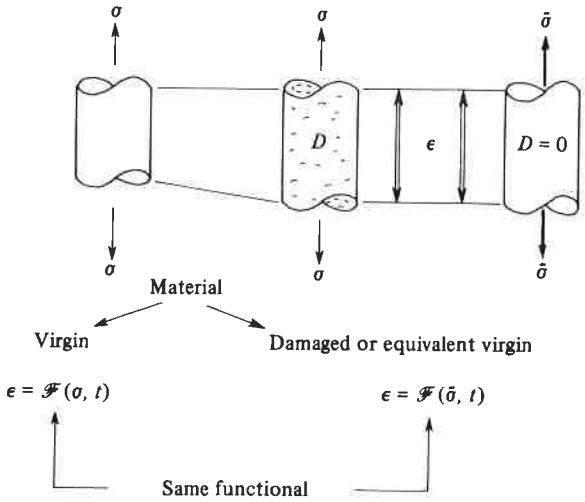
> any deformation behaviour, whether uniaxial or multiaxial, of a damaged material is represented by the constitutive laws of the virgin material in which the usual stress is replaced by the effective stress (Fig. 7.2).

For example, the uniaxial linear elastic law of a damaged material is written as:

$$\varepsilon_e = \frac{\tilde{\sigma}}{E} = \frac{\sigma}{(1 - D)E}$$

where E is the Young's modulus. This constitutes a nonrigorous hypothesis

Fig. 7.2. Effective stress and equivalence in strain.



which assumes that all the different behaviours (elasticity, plasticity, viscoplasticity) are affected in the same way by the surface density of the damage defects. However, its simplicity allows the establishment of a coherent and efficient formalism.

Critical damage at fracture

By applying the concept of effective stress at the instant of fracture by interatomic decohesion, we define the critical value of damage D_c, as that corresponding to the occurrence of this phenomenon.

If $\tilde{\sigma}_u$ is the uniaxial stress at fracture by decohesion and σ_u is the usual ultimate fracture stress, we have:

$$\tilde{\sigma}_u = \sigma_u/(1 - D_c)$$

or

$$D_c = 1 - (\sigma_u/\tilde{\sigma}_u).$$

The physics of solids shows that $\tilde{\sigma}_u$ is of the order of $E/50$–$E/20$; for common materials $\tilde{\sigma}_u$ is of the order of $E/100 - E/250$, and D_c is therefore of the order of 0.5–0.9. This allows us to neglect $(1 - D_c)^x$ (with $x \gg 1$), a term which often appears in calculations, in comparison to 1. Multiaxial criteria are studied in Section 7.2.5.

7.2.3 *Measurement of damage*

Damage is not directly accessible to measurement. Its quantitative evaluation, as for any physical value, is linked to the definition of the variable chosen to represent the phenomenon. Having chosen a definition based on the concept of effective stress, associated to the hypothesis of strain-equivalence, the measurements which result naturally from it, are essentially linked to the coupling between deformation and damage, i.e., to the modification of the mechanical properties caused by damage. These measurements are those of the properties of elasticity, plasticity, viscoplasticity, to which we also add the measurement of variation in resistivity.

Variation of the modulus of elasticity

Static method

Recall the uniaxial elastic damage law, which was mentioned in the previous Section, and which will be justified on thermodynamical grounds

in Section 7.3.2:

$$\tilde{\sigma} = \sigma/(1 - D) = E\varepsilon_e$$

or

$$\sigma = E(1 - D)\varepsilon_e$$

where E is Young's modulus, i.e., the elasticity modulus of the material free from any damage; $E(1 - D) = \tilde{E}$ can then be interpreted as the elastic modulus of the damaged material. If Young's modulus E is known, any measurement of elastic stiffness can be used to determine the damage by $D = 1 - \sigma/(E\varepsilon_e)$ and with $\sigma = \tilde{E}\varepsilon_e$, we find that:

- $D = 1 - \tilde{E}/E.$

Although very simple in principle, this measurement is rather tricky to perform for the following reasons:

any measurement of the modulus of elasticity requires precise measurement of very small strains, as we have seen in Chapter 4;
damage is usually very localized, which requires that the measurements be made on a very small base of the order of 0.5–5 mm;
at the required level of the precision, the best straight line in the (σ, ε) graph representing an elastic loading or unloading is difficult to define.

In view of the above reasons, the following procedure is recommended:

(i) use of specimens with a weakened central section so as to localize the damage there; an example is given in Fig. 7.3;

Fig. 7.3. Method of measuring damage.

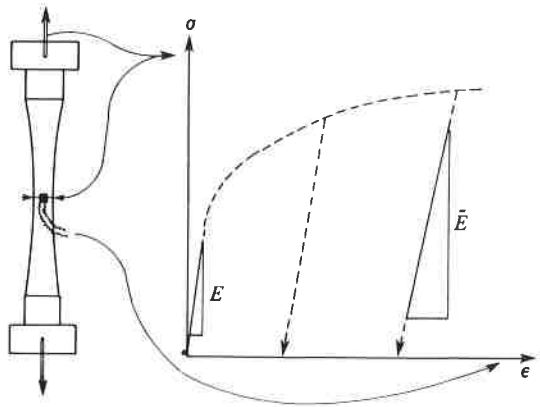
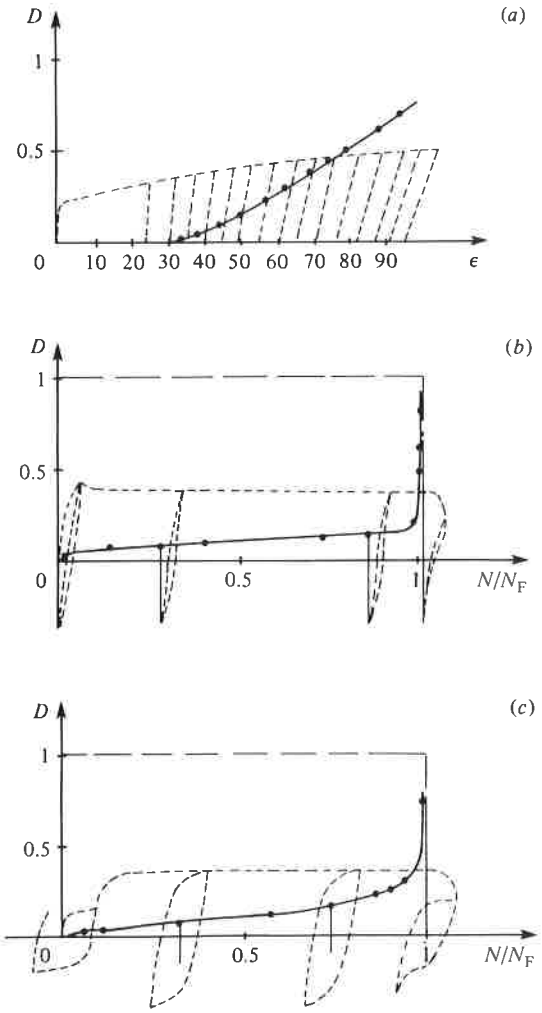


Fig. 7.4. Examples of evolution of damage: (*a*) ductile plastic damage: copper Cu/Al $T = 20\,°C$, $\dot{\varepsilon} \approx 10^{-4}\,s^{-1}$; (*b*) fatigue damage: 316L stainless steel $T = 20\,°C$

$$\varepsilon = \begin{cases} +0.7 \times 10^{-2} \\ 0 \end{cases} \quad N_F(\text{fracture}) = 70\,450 \text{ cycles;}$$

(*c*) cyclic creep damage: 316L stainless steel, $T = 550\,°C$

$$\varepsilon = \begin{cases} +10^{-2} \\ -10^{-2} \end{cases} \quad N_c(\text{fracture}) = 218 \text{ cycles.}$$



(ii) measurement of strain with small gauges: 0.5 mm × 0.5 mm when the temperature allows it, or by displacement transducers attached to as small a measurement base as possible, when the temperature is above 200 °C;

(iii) evaluation of the elasticity modulus during elastic unloading by avoiding zones of stronger nonlinearities: this is achieved by performing the evaluation in a range of stress defined as follows (see Fig. 7.3):

$$\sigma_{Max} < \sigma < \sigma_{min}$$
$$\sigma_{Max} = 0.85\,\sigma(\varepsilon)$$
$$\sigma_{min} = 0.15\,\sigma(\varepsilon).$$

If these precautions are taken, damage can be evaluated with a relative precision of the order of ± 5%, and the method can easily be applied for any kind of damage. Fig. 7.4 shows, as examples, evolution of damage in ductile plasticity, cyclic creep, and fatigue obtained by this technique.

Ultrasonic dynamic method

The same principle may be used in dynamic situations, particularly with regard to propagation of ultrasonic waves. A measurement of the speed or propagation time of plane waves in a specimen or a cylinder of damaged material leads to the determination of the elasticity modulus $\tilde{E}$ of the damaged material and of the damage by $D = 1 - \tilde{E}/E$. Expressed in terms of the velocity of longitudinal waves $\tilde{v}_L$ and transverse waves $\tilde{v}_T$ (see Section 4.1.3):

$$\tilde{E} = \rho \tilde{v}_T^2 \frac{3\tilde{v}_L^2 - 4\tilde{v}_T^2}{\tilde{v}_L^2 - \tilde{v}_T^2}.$$

In the framework of the isotropic damage hypothesis (constant Poisson's coefficient) and neglecting the variation in ρ (which does not result in a relative error in D of more than 5%), we have:

$$v_L^2 = \frac{E}{\rho}\frac{1-\nu}{(1+\nu)(1-2\nu)} \quad \text{and} \quad \tilde{v}_L^2 = \frac{\tilde{E}}{\rho}\frac{1-\nu}{(1+\nu)(1-2\nu)}$$

and we obtain:

- $D = 1 - \tilde{v}_L^2/v_L^2$.

An example of damage measurement using the ultrasonic technique, applied to a concrete specimen loaded in compression, is given in Fig. 7.5.

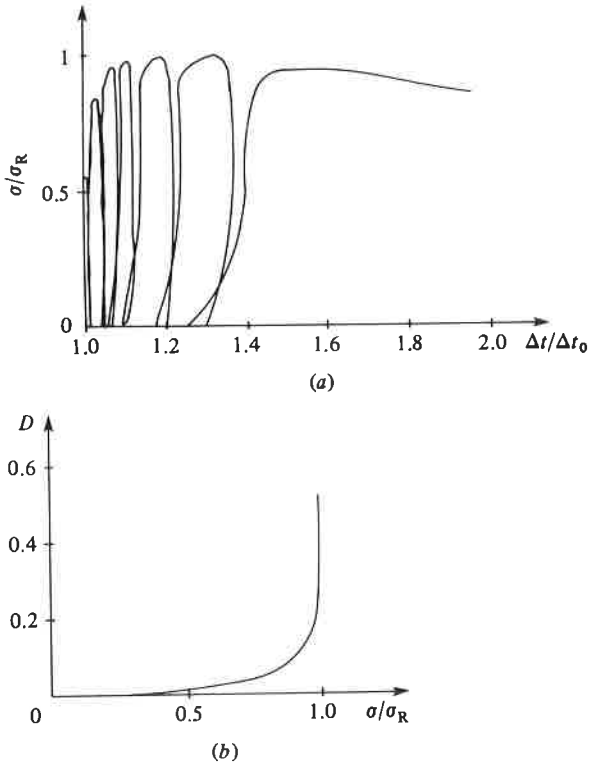
Variation of plasticity characteristics

Monotonic hardening characteristics

This method is of interest, especially for the characterization of ductile plastic damage as a complement of the method of measurement by means of the variation of the elasticity modulus. Ductile plastic damage accompanies large plastic strains in metals and in polymers and only becomes important as the necking condition is approached. In hardening tests, it is apparent through a drop in the stress commencing from the point of instability defined by $d\sigma/d\varepsilon = 0$. This drop in stress is therefore due to a combination of the geometric effect (necking) and the damage effect (reduction of the effective area $\tilde{S} = S - S_D$).

The hardening law of a nondamaged material is expressed by the

Fig. 7.5. Measurement of damage in concrete by an ultrasonic technique (after S. Benounich): (*a*) evolution of the traverse time of the wave $\Delta t/\Delta t_0 = v_L/\tilde{v}_L$ as a function of the stress expressed as a fraction of the fracture stress; (*b*) evolution of the damage.



equation introduced in Section 2.1 and written here for large strains ($\varepsilon_{pv} = \ln(1 + \varepsilon_p)$):

$$\varepsilon_{pv} = \left\langle \frac{\sigma_v - \sigma_Y}{K_Y} \right\rangle^{M_Y}.$$

As soon as the damage becomes perceptible ($\varepsilon_v > \varepsilon_v^*$), the hypothesis of strain equivalence associated with the concept of effective stress can be invoked to write:

$$\varepsilon_{pv} = \frac{1}{K_Y^{M_Y}} \left\langle \frac{\sigma_v}{1 - D} - \sigma_Y \right\rangle^{M_Y}.$$

This expression can be used to measure D indirectly from the $(\sigma_v, \varepsilon_v)$ graph as soon as K_Y and M_Y are known (these latter quantities can be determined from the same graph for $\varepsilon_v < \varepsilon_v^*$):

$$D = 1 - \frac{\sigma_v}{K_Y \varepsilon_{pv}^{1/M_Y} + \sigma_Y}.$$

Cyclic hardening characteristics

The method discussed here is particularly suited to the measurement of low-cycle fatigue damage. The basic cyclic plasticity law, described in Chapter 5, is used in the form of a relation between the maximum plastic strain at the stabilized cycle (when it exists) and the maximum stress:

$$\varepsilon_{pMax}^* = \left[\frac{\sigma_{Max}^*}{K_c} \right]^{M_c}.$$

We generally assume that the fatigue damage is negligible in the stabilized cycle. The further evolution of the maximum plastic strain in a stress controlled test ($\sigma_{Max} = \sigma_{Max}^*$) is then attributed to damage. The hypothesis of strain-equivalence associated with the concept of effective stress leads to writing for each cycle:

$$\varepsilon_{pMax} = \left[\frac{\sigma_{Max}^*}{K_c(1 - D)} \right]^{M_c}$$

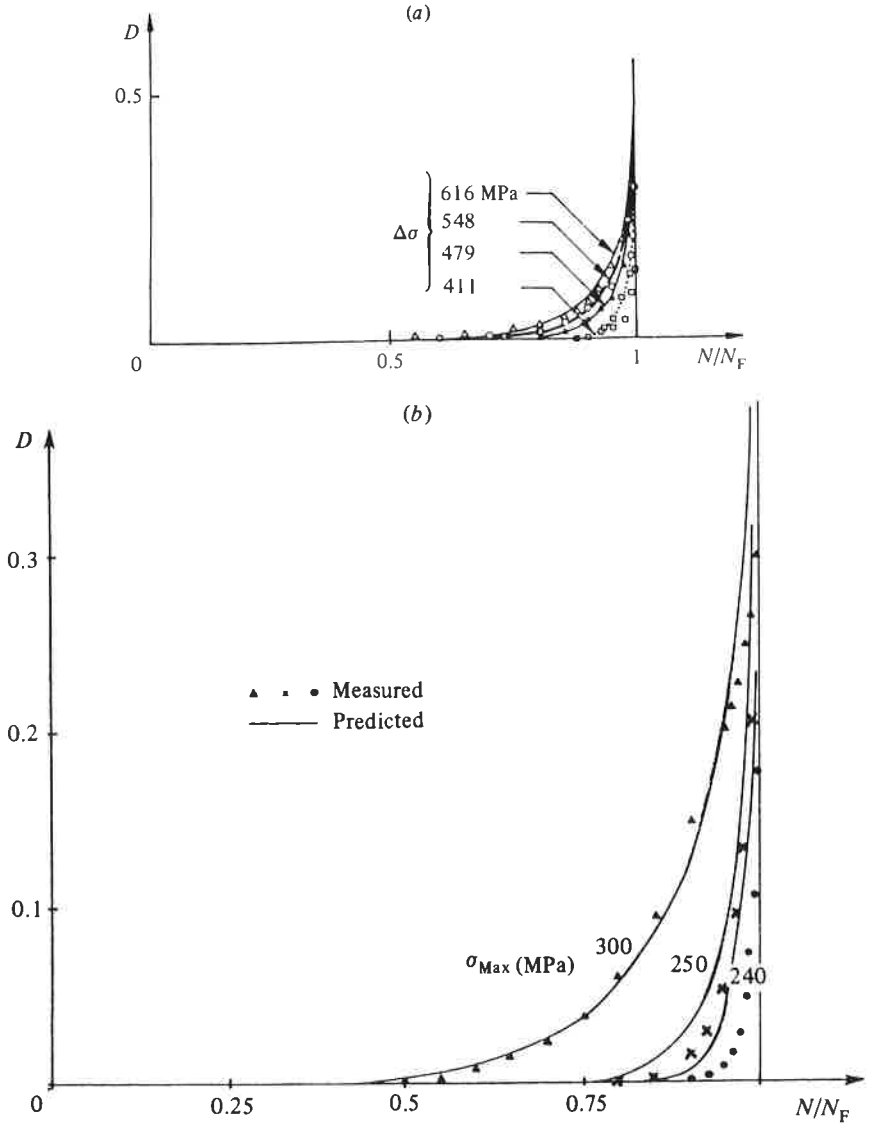
and the damage is obtained by combining the two relations:

$$D = 1 - \left(\frac{\varepsilon_{pMax}^*}{\varepsilon_{pMax}} \right)^{1/M_c}.$$

The relative precision of this method is of the order of 10–20%, the major difficulty being in the evaluation of the stabilized state. Fig. 7.6 shows the case of two alloys subject to fatigue at 5 Hz. The continuous curves correspond to the model of Section 7.4.3.

In contrast, under controlled plastic strain ($\varepsilon_{pMax} = \varepsilon^*_{pMax}$) loading, the

Fig. 7.6. Damage evolution in pure fatigue: (*a*) IN 100 at 1000 °C; (*b*) 316L at 20 °C.



same method leads to:

$$D = 1 - \sigma_{Max}/\sigma^*_{Max}.$$

For a controlled total strain, the interpretation is more complex since in such a case both the elasticity and the cyclic laws are affected by the damage.

For a fatigue test under asymmetric loading (repeated tension, for example), we could use the same method but include the progressive ratchetting effect with a constitutive equation of the form (N being the number of cycles):

$$\frac{\delta\varepsilon_{pMax}}{\delta N} = \left[\frac{\sigma_{Max}}{(1-D)K_r}\right]^{M_r}.$$

Finally, note that this method is applicable only in the regime of low-cycle fatigue (measurable plastic strains) for materials not subjected to cyclic hardening (or softening) for most of their lives. Moreover, the fatigue damage that can thus be measured is the one that occurs in the last stage of the life of the volume element when microcracks have already been initiated (stage I) and are spreading on a microscopic scale (stage II).

Variation of viscoplastic characteristics

This method demonstrates creep damage, which in metal corresponds to the nucleation and growth processes of mainly intercrystalline microcracks. This damage increases as a function of time under constant or slowly increasing loads. This phenomenon becomes more pronounced as the temperature rises.

In a uniaxial creep test at constant stress, it manifests itself, especially during tertiary creep, by an increase in the creep velocity which becomes very high as rupture is approached. In fact in creep tests at constant load, tertiary creep also results from a reduction in the cross-sectional area of the speciman. Fig. 6.3 shows an example of this quite well known phenomenon.

As for ductile plastic damage, we may use the coupling between damage and strain behaviour (here viscoplastic) to identify the damage. By assuming the damage to be zero or negligible in primary creep, we may account for secondary creep by Norton's law (see Chapter 6):

$$\dot{\varepsilon}^*_p = (\sigma_v/\lambda^*)^{N^*}$$

and for tertiary creep by the concept of effective stress:

$$\dot{\varepsilon}_p = [\sigma_v/(1-D)\lambda^*]^{N^*}.$$

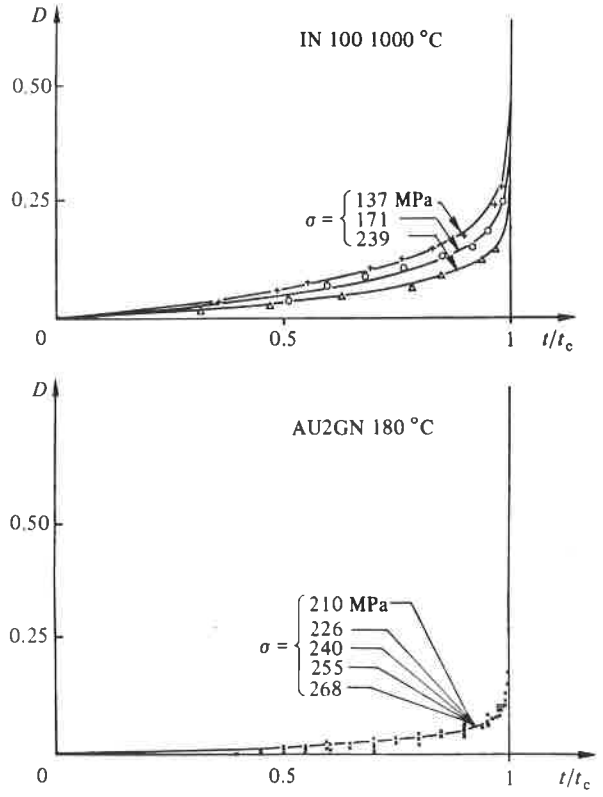
Knowing the viscosity exponent N^*, determined when $D = 0$, the damage can be expressed as a function of the secondary creep rate $\dot{\varepsilon}_p^*$ and the tertiary creep rate $\dot{\varepsilon}_p$ measured as functions of time on the creep curve. From the two relations above, it follows that when σ_v is constant:

$$D = 1 - (\dot{\varepsilon}_p^*/\dot{\varepsilon}_p)^{1/N^*}.$$

This method of evaluating damage can be used to plot the graph of $D(t/t_c)$ in tertiary creep as shown by Fig. 7.7 (where t_c is the rupture time). In practice, the test is performed at constant load and it is necessary to take into account the variation of σ_v. Noting that ε^* is the strain at the beginning of the secondary creep, we have:

$$D = 1 - \frac{1+\varepsilon}{1+\varepsilon^*}\left(\frac{\dot{\varepsilon}_p^*}{\dot{\varepsilon}_p}\right)^{1/N^*}.$$

Fig. 7.7. Creep damage evolution for the alloys IN 100 and AU2GN. The continuous curves correspond to the model of Section 7.4.2.



Variation of electrical resistance

The concept of effective stress is replaced here by the concept of effective current density:

$$\tilde{i} = i/(1 - D).$$

Consider an elementary tube of current (Fig. 7.8) and assume that this volume element is subjected to a simple tension in the direction of the current. The influence of the damage D is assumed to be distributed uniformly in the electrical resistance. The problem is complicated by the influence of deformations on the resistance. For an apparently constant current, the potential differences for the virgin material and the damaged one are expressed (with self-evident notation) by:

$$dv = \rho \frac{dl}{dS} di, \qquad dv' = \rho' \frac{dl'}{dS'} d\tilde{i}.$$

We assume that the influence of damage is entirely represented by the density of the effective current $d\tilde{i}$ and that the modifications of resistivity ρ, dl and dS are dependent only on the geometric changes resulting from elastic and plastic strains. In simple tension, the axial and transverse strains are:

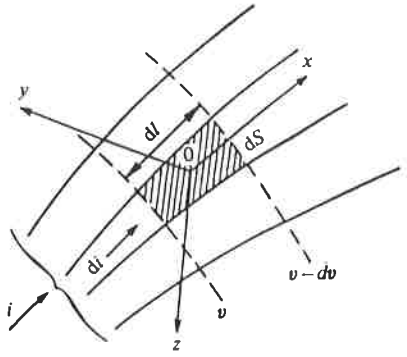
$$\varepsilon_e + \varepsilon_p \quad \text{and} \quad -\nu\varepsilon_e - \tfrac{1}{2}\varepsilon_p$$

which can be used to write:

$$dl' = dl(1 + \varepsilon_e + \varepsilon_p), \qquad dS' = dS(1 - 2\nu\varepsilon_e - \varepsilon_p).$$

Traditionally the strain is considered to influence the resistivity only through the geometric effect of the variation in volume (Bridgman's law).

Fig. 7.8. Elementary volume.



We therefore write:

$$\rho' = \rho(1 + C_1\varepsilon_e - 2\nu C_1\varepsilon_e).$$

In the framework of small strains, these assumptions lead to the relative variation of potential:

$$\frac{\Delta dv}{dv} = \frac{dv' - dv}{dv} = \frac{K_e\varepsilon_e + K_p\varepsilon_p + D}{1 - D}$$

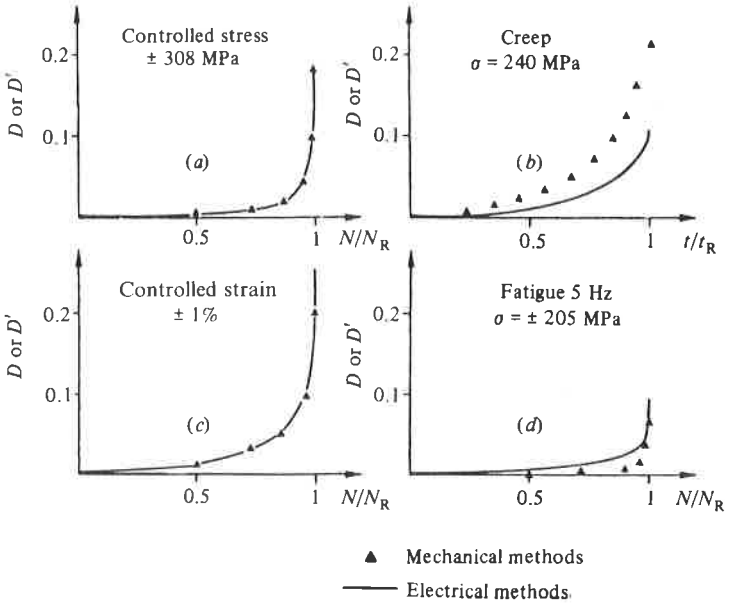
where

$$K_e = 1 + 2\nu + C_1(1 - 2\nu), \quad K_p = 2.$$

The damage is therefore obtained from measurements of the relative variation in electrical potential, taking into account the presence of strains:

$$D = \frac{\Delta dv/dv - K_e\varepsilon_e - K_p\varepsilon_p}{1 + \Delta dv/dv}.$$

Fig. 7.9. Comparison between measurements from electrical and mechanical methods on the 316 steel at 20 °C ((*a*), (*c*)), and the IN 100 alloy at 1000 °C ((*b*), (*d*)). Mechanical measurements: drop in modulus of elasticity ((*a*), (*c*)), increase in creep rate ((*b*)), increase in maximum plastic strain ((*d*)).



Tests performed on tension–compression specimens under very different conditions (stainless steel and aluminium alloy at 20 °C, refractory alloy at high temperature) show that the damage values D obtained by the electrical method are close to those from mechanical measurements. Fig. 7.9 shows some results obtained in fatigue and in creep. It is seen that in pure creep, where the damage manifests itself by the appearance of cavities, electrical measurements give lower values than mechanical ones. The comparison between the two types of measurement is more favourable in the case of pure fatigue, where the physical process consists in the initiation of microcracks, and for this case the electrical method is much more sensitive. However, on the whole, the agreement between the methods is good in view of the large uncertainty of this type of measurement which relies on the difference between two neighbouring mechanical or electrical states.

7.2.4 *Elementary damage laws*

We describe here some uniaxial phenomenological equations relative to three types of damage (ductile, creep and fatigue). It will be interesting to note the quality/complexity ratio.

Ductile plastic damage equation, linear in terms of strain

From a microscopic point of view this consists in the nucleation, growth and coalescence of cavities induced by large plastic strains. We have seen in the Section 7.2.3 how the variable D is measured from variations in the elasticity modulus or the monotonic hardening characteristics.

For many metallic materials subjected to a uniaxial monotonically increasing load, the damage D varies linearly with the strain (as in Fig. 7.4(*a*)). A simple law, well in accord with the experimental results, is therefore:

$$D = D_c\left\langle \frac{\varepsilon_v - \varepsilon_{vD}}{\varepsilon_{vR} - \varepsilon_{vD}} \right\rangle,$$

where ε_{vD} is the true strain at the damage threshold, before which the damage is zero or negligible, and ε_{vR} is the true strain at fracture at which the damage is equal to its critical value D_c. This empirical equation contains three coefficients which have been identified for some materials, their values are listed in Table 7.1. The equation will be justified by the formalism of thermodynamics in Section 7.4.1.

Table 7.1. *Values of the coefficients of ductile plastic damage*

Material	T(°C)	ε_{vD}	ε_{vR}	D_c
99.9% copper	20	0.35	1.04	0.85
2024 alloy	20	0.03	0.25	0.23
E 24 steel	20	0.50	0.88	0.17
XC 38 steel	20	0	0.56	0.22
30 CD 4 steel	20	0.02	0.37	0.24
INCO 718 alloy	20	0.02	0.29	0.24

Table 7.2. *Values of the creep damage coefficients*

Material	T(°C)	A_0(h MPa)	r	k	A
AU2GN alloy	180	317	28	43	322
Magnesium alloy	260	162	7.5	5.4	156
304 steel	593	520	7	15	575
304 steel	650	555	5.6	15	650
316 L steel	600	1160	6.5	5	1120
L605 alloy	980	102	7.5		
UDIMET 560 alloy	980	190	5		
UDIMET 700 alloy	900	662	6.3		
UDIMET 700 alloy	1000	520	4.8		
IN 100 alloy	800	1010	10	15	1050
IN 100 alloy	900	768	6.3	15	870
IN 100 alloy	1000	433	5.2	15	520
IN 100 alloy	1100	244	3.6	15	345
0.4 Mn–0.12 C steel	450	1660	3.5		
321 Laminated steel	500	2915	3.9		
310 Laminated steel	550	1766	4		
	700	675	3.5		

Kachanov's creep damage law

This is concerned with brittle viscoplastic damage which was mentioned in Section 7.2.3 and an example of which is shown in Fig. 7.7.

In 1958, Kachanov proposed the modelling of this type of evolution by:

$$\dot{D} = \left[\frac{\sigma_v}{A_0(1-D)}\right]^r,$$

where A_0 and r are the two characteristic creep damage coefficients for the material. This expression yields not zero but very low values for primary and secondary creep. A number of identifications have since shown the

correctness of this modelling, at least for cases of sufficiently simple load histories. Table 7.2 gives the value of A_0 and r for some materials.

The time to rupture t_c, in a creep test under constant true stress, is obtained from the solution of the differential equation of the model for a damage value equal to the critical value, $D = D_c$. With the initial condition $D = 0$ at $t = 0$:

$$t_c = \frac{1-(1-D_c)^{r+1}}{r+1}\left(\frac{\sigma_v}{A_0}\right)^{-r}.$$

For usual values of r we may neglect $(1-D_c)^{r+1}$ in comparison to 1, and express the damage evolution in a particularly simple form by integrating the differential equation between 0 and D:

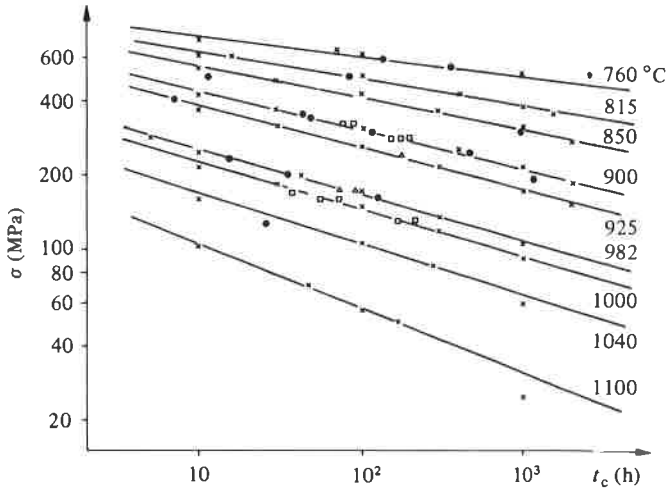
$$D = 1-\left(1-\frac{t}{t_c}\right)^{1/(r+1)} \quad \text{with} \quad t_c = \frac{1}{r+1}\left(\frac{\sigma_v}{A_0}\right)^{-r}.$$

Fig. 7.10 shows that the power relation between the stress and the time to rupture is verified satisfactorily by experiment and that the coefficients A_0 and r are strongly temperature dependent.

Fatigue damage equations

Fatigue damage in metals corresponds to nucleation and growth of microcracks, generally intracrystalline, under the action of cyclic loading,

Fig. 7.10. Time to reach creep rupture as a function of stress and temperature: IN 100 alloy (uncoated).



until the initiation of a macroscopic crack. This phenomenon can occur at low stress levels, lower than the conventional elastic limit. We thus distinguish:

> fatigue with a large number of cycles which almost certainly involves elastic deformations and which corresponds to a number of cycles to failure higher than approximately 50 000; beyond 10^6–10^7 or even 10^9 cycles, depending on the intended field of application, the life-time is generally considered to be unlimited, i.e., when the loads are lower than a conventional fatigue limit;
>
> fatigue with a low number of cycles (or low-cycle fatigue) which corresponds to less than 50 000 cycles in which plastic deformations occur; at high temperatures the fatigue phenomenon is usually coupled with creep damage and can be isolated only by performing tests at a sufficiently high frequency ($f > 5$–10 Hz).

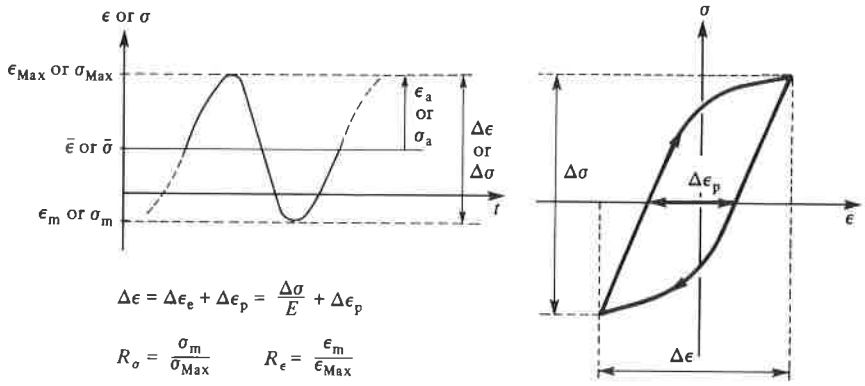
The classical laws are expressed as functions of quantities defined with respect to a cycle, see Fig. 7.11:

> the stress amplitude $\Delta\sigma/2$ or strain amplitude $\Delta\varepsilon/2$,
>
> the mean value (over a cycle) of stress $\bar{\sigma}$ or strain $\bar{\varepsilon}$,
>
> the ratio of the minimum stress (or strain) to the maximum stress (or strain) $R_\sigma = \sigma_m/\sigma_{Max}$ (or $R_\varepsilon = \varepsilon_m/\varepsilon_{Max}$).

Linear and nonlinear accumulation

Cumulative effects, whether linear or nonlinear, are of great importance in fatigue. We have seen in Chapter 3 (Section 3.2.3) that the rule of linear

Fig. 7.11. Definition of cyclic quantities for the characterization of fatigue.



accumulation is in fact a property of any linear or nonlinear differential equation with separable variables.

The Palmgreen – Miner linear rule is based on the assumption that damage is accumulated additively when it is defined by the associated life ratio N_i/N_{F_i} where N_i is the number of cycles applied under a given load for which the number of cycles to fracture (under periodic conditions) would be N_{F_i}. The fracture criterion is:

$$\sum_i (N_i/N_{F_i}) = 1.$$

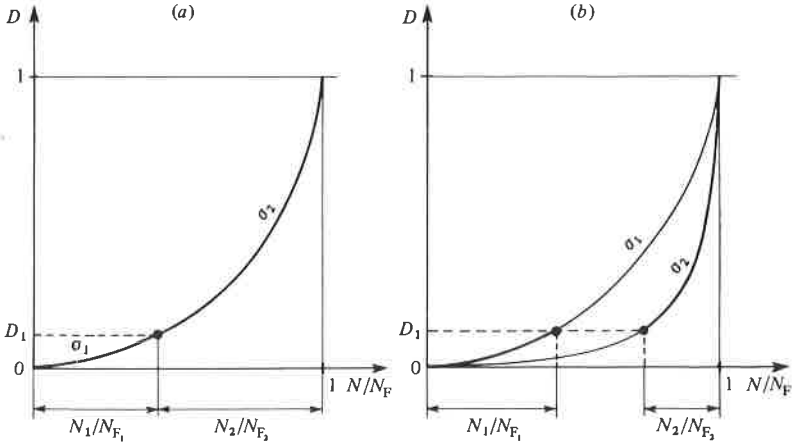
Therefore, in periodic tests, damage evolution is considered to be linear in that:

$$D = N/N_F.$$

In fact, the linear accumulation rule can be applied even to a damage which evolves nonlinearly. For this it is sufficient that a one-to-one relationship between D and N/N_F exists, or even that the damage evolution curve be a unique function (independent of the applied cycle) of the life ratio N/N_F. For a test at two stress levels, the evolution is as shown schematically in Fig. 7.12(*a*).

There are, therefore, two ways of defining a damage incremental law incorporating the linear accumulation rule. Following well-established rules this increment is introduced per cycle by:

Fig. 7.12. (*a*) Damage with nonlinear evolution but linear accumulation. (*b*) Nonlinear accumulation.



using simply an equation of the form:

$$\delta D = \delta N / N_F(...)$$

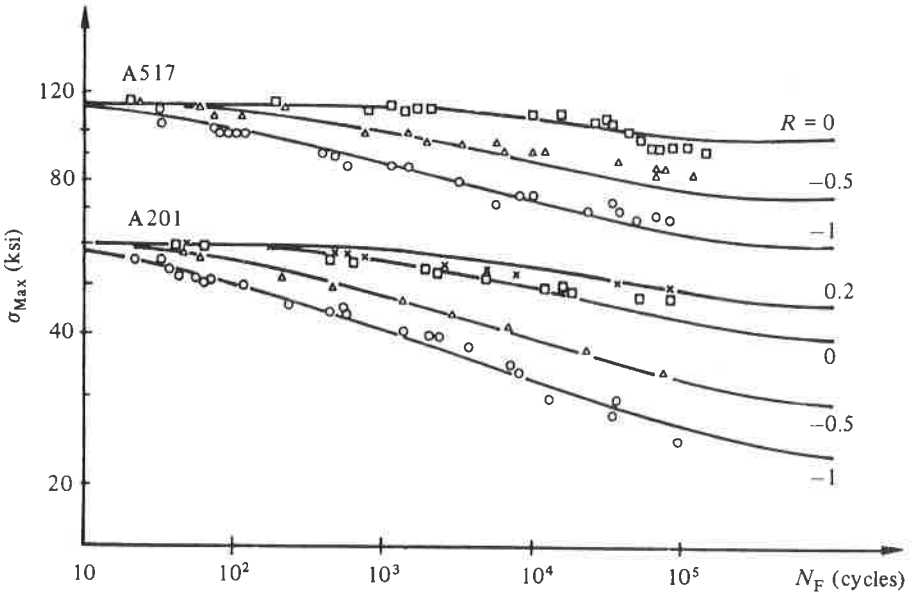
where N_F is the number of cycles to failure defined by the chosen parametric relation;
employing a nonlinear evolution of damage which, however, is a unique function of N/N_F, for example:

$$\delta D = \frac{(1-D)^{-k}}{k+1} \frac{\delta N}{N_F(...)}.$$

We recall (cf. Chapter 3) that such an equation, in which the loading and the damage D are separable variables, always leads to linear accumulation like that shown in Fig. 7.12(*a*).

In contrast, if the damage evolution curve, as a function of the life ratio N/N_F, depends on the applied loading we have the effect of nonlinear accumulation as shown in Fig. 7.12(*b*). D_1 represents the state of internal damage at the end of the first level σ_1. Evolution at the second level σ_2 continues from the same state, and it is clear that the sum of the life ratios is less than 1. From the point of view of the damage law, this nonlinearity

Fig. 7.13. Woehler curves for the A201 and A517 steels at 20 °C.



always corresponds to the case where the variables which represent the load and the damage variable D are not separable.

The Palmgreen–Miner linear accumulation law gives good results only for loads for which there is little variation in the amplitude and mean value of the stress. More sophisticated models for loads with complex histories will be described in Section 7.4.3.

Woehler–Miner law

This rule essentially applies to the fatigue of metals under periodic loading. Woehler curves express the experimental results of uniaxial fatigue tests. They represent the relations between the number of cycles to failure, the value of the maximum stress σ_{Max} and the mean stress $\bar{\sigma}$ (or the ratio $R_\sigma = \sigma_m/\sigma_{Max}$). A typical example is given in Fig. 7.13.

In general, each curve exhibits two limits: one, when σ_{Max} tends towards the static fracture stress σ_u (fracture in a quarter of the cycle), and the other, when σ_{Max} tends towards the fatigue limit σ_l. The influence of the mean stress on the fatigue limit is satisfactorily represented by Goodman's linear relation:

$$\sigma_l(\bar{\sigma}) = \bar{\sigma} + \sigma_{l_0}(1 - \bar{\sigma}/\sigma_u)$$

where σ_{l_0} is the fatigue limit at $\bar{\sigma} = 0$, or in a modified form by:

$$\sigma_l(\bar{\sigma}) = \bar{\sigma} + \sigma_{l_0}(1 - b\bar{\sigma})$$

in which b is a characteristic coefficient of the material and σ_l is the fatigue limit expressed in terms of maximum stress. Fig. 7.14 shows that this relation is applicable for a large range.

One way of writing a damage law which expresses these experimental results is to assume that the damage per cycle is a function of the maximum and the mean values of the stress:

$$\delta D/\delta N = f(\sigma_{Max}, \bar{\sigma}).$$

In order to recover, after integration, one of the many forms proposed to represent the Woehler curves, we let:

$$\frac{\delta D}{\delta N} = \frac{\sigma_{Max} - \sigma_l(\bar{\sigma})}{\sigma_u - \sigma_{Max}} \left(\frac{\sigma_{Max} - \bar{\sigma}}{B(\bar{\sigma})} \right)^\beta$$

with:

$$\sigma_l(\bar{\sigma}) = \bar{\sigma} + \sigma_{l_0}(1 - b\bar{\sigma})$$
$$B(\bar{\sigma}) = B_0(1 - b\bar{\sigma}).$$

The number of cycles to failure is obtained by an obvious integration, with the conditions:

$$N = 0 \rightarrow D = 0$$
$$N = N_F \rightarrow D = 1$$

so that:

$$N_F = \frac{\sigma_u - \sigma_{Max}}{\sigma_{Max} - \sigma_1(\bar{\sigma})} \left(\frac{\sigma_{Max} - \bar{\sigma}}{B(\bar{\sigma})} \right)^{-\beta}.$$

By identifying this expression with Woehler's curves, one can determine (possibly as functions of temperature) the coefficients σ_u, σ_{1_0}, b, B_0 and β for each material. Some examples of the values of these coefficients are given in Table 7.3. Fig. 7.13 illustrates the application of this expression to two steels (the coefficients a and M_0 correspond to a generalization to be studied in Section 7.4.3).

For periodic loading (σ_{Max} = constant, σ_m = constant), damage evolves as a linear function of the number of cycles:

$$D = \frac{\sigma_{Max} - \sigma_1(\bar{\sigma})}{\sigma_u - \sigma_{Max}} \left(\frac{\sigma_{Max} - \bar{\sigma}}{B(\bar{\sigma})} \right)^{\beta} N = \frac{N}{N_F}.$$

This property is, however, very poorly verified by experiment, as we shall see in Section 7.4.3.

Coffin–Manson law

In low-cycle fatigue, plastic strains become dominant. To include certain time or frequency related effects in such a case, we may assume that the

Fig. 7.14. Effect of the mean stress on fatigue limit in tension–compression.

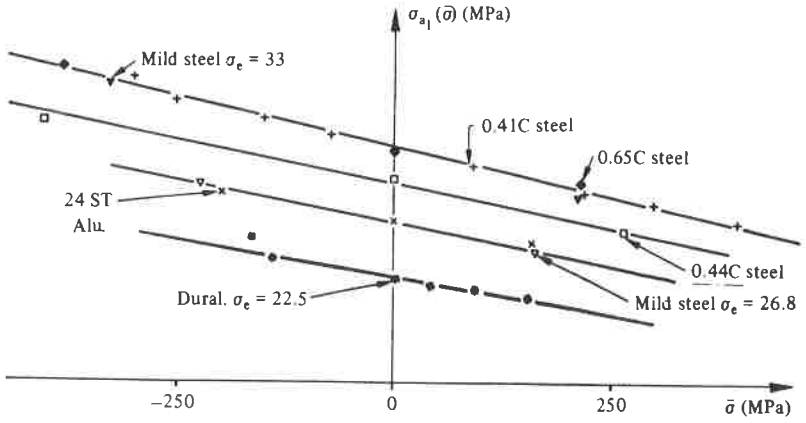


Table 7.3. *Coefficients of the fatigue damage law (in MPa units)*

Material	T(°C)	σ_1	σ_u	β	B_0	b	a	M_0
IN 100	700	300	950	2.6	6560		0.120	4750
IN 100	900	240	715	3	4430		0.072	2925
IN 100	950	180	600	4.8	1637	0.0013	0.108	1485
IN 100	1000	140	470	6	880		0.138	875
IN 100	1110	70	250	4.7	645		0.162	635
INCO 718	550	600	1150	10.3	1586		0.015	1335
A 201	20	150	412	6	932	0.0070		
A 517	20	400	790	7.4	1945	0.0038	0.710	1945
C 35	20	255	458	4	5434			
316 L	20	222	760	5	1213		0.900	1700
Maraging	20	310	2028	4	5820			
AISI 4130 S	20	240	890	5.5	1590			
AISI 4130 H	20	292	1420	5	2460			
AISI 304	20	240	950	5.8	2000			
52100 steel	20	360	2005	3.3	6320			
Cu 5456 H311	20	76	400	4.7	832			
AISI 304	593	150	700	3	4207	0.005	0.100	3100
AISI 304	650	140	580	3.7	1728		0.200	1700
316 L (ICL 167)	600	200	650	5.5	1144		0.200	1200

damage per cycle is a power function of the plastic strain range:

$$\delta D/\delta N = f(\Delta\varepsilon_p) = (\Delta\varepsilon_p/C_1)^{\gamma_1}.$$

The integration of this relation for periodic loading where the cycle is assumed to be stabilized, from the first to the last cycle, gives the Coffin–Manson relation:

$$N_F = (\Delta\varepsilon_p/C_1)^{-\gamma_1}$$

which corresponds to the conditions:

$$N = 0 \rightarrow D = 0$$
$$N = N_F \rightarrow D = 1.$$

In the low stress range where $\Delta\varepsilon_p$ is small, we may complement this relation by a damage law expressed as a function of the stress amplitude $\Delta\sigma$:

$$\delta D/\delta N = (\Delta\sigma/C_2)^{\gamma_2}$$

or, by integrating with the same initial and final conditions as before:

$$N_F = (\Delta\sigma/C_2)^{-\gamma_2}.$$

By using the linear elasticity law $\Delta\sigma = E\Delta\varepsilon_e$, solving for $\Delta\varepsilon_e$, and then

Table 7.4. *Coefficients of the Coffin–Manson relations*

Material	T(°C)	γ_1	C_1	γ_2	C_2
TA6V	350	1.818	0.38	20	1450
INCO 718	550	1.615	0.407	18.9	2580
IN 100	700	2	0.244	6.5	3168
IN 100	900	2	0.037	7.1	2459
IN 100	1000	2	0.031	8.75	1268
IN 100	1110	2	0.02	7.8	750
AISI 304	20			10.5	2120
AISI 304	430	2.2	0.189		
Maraging, hardened	20	1.22	1.634	11.1	2940
35 CD 4 Air Cooled steel	20	1.54	0.662	8.3	3565
X 20 GM_o V 12 1	350	1.83	0.356	13.5	1660
316 ICL 167 CN	20	2.07	0.34	5.7	3280
IMI 550	20	1.3	2.63	13.3	2940
COTAC 744	1000	1.32	0.408	11.1	1560
AISI 4130 soft	20			9.6	1925
AISI 4130 hard	20			9.2	2805
AISI 52100	20			6.5	5100
ALU 5456 H 311	20			7.2	1140
UDIMET 700	20	1.67	0.407	8.3	4830

adding $\Delta\varepsilon_p$ as determined from the Coffin–Manson relation, we obtain:

$$\Delta\varepsilon_e + \Delta\varepsilon_p = (C_2/E)N_F^{-1/\gamma_2} + C_1 N_F^{-1/\gamma_1}.$$

The coefficients C_1, γ_1, C_2, γ_2 depend on the material and the temperature. Table 7.4 and Fig. 7.15 give a few examples. This relation is interesting because after a large number of tests on many materials it has been possible to write it in the following form called 'universal slopes':

$$\Delta\varepsilon = \Delta\varepsilon_e + \Delta\varepsilon_p = 3.5\frac{\sigma_u}{E}N_F^{-0.12} + D_u^{0.6}N_F^{-0.6}$$

where σ_u is the ultimate static fracture stress, E the Young's modulus, and D_u the ductility which is expressed as a function of the reduction of the area RA at necking: $D_u = -\ln(1 - RA)$. Fig. 7.15(*b*) illustrates the case of the UDIMET 700 alloy at room temperature.

The Coffin–Manson relation itself also contains the rule of linear accumulation and can therefore be used for cases of varying amplitude. Nevertheless, its range of application is limited to low variations in strain range and to the quite low temperatures for which the creep damage is small.

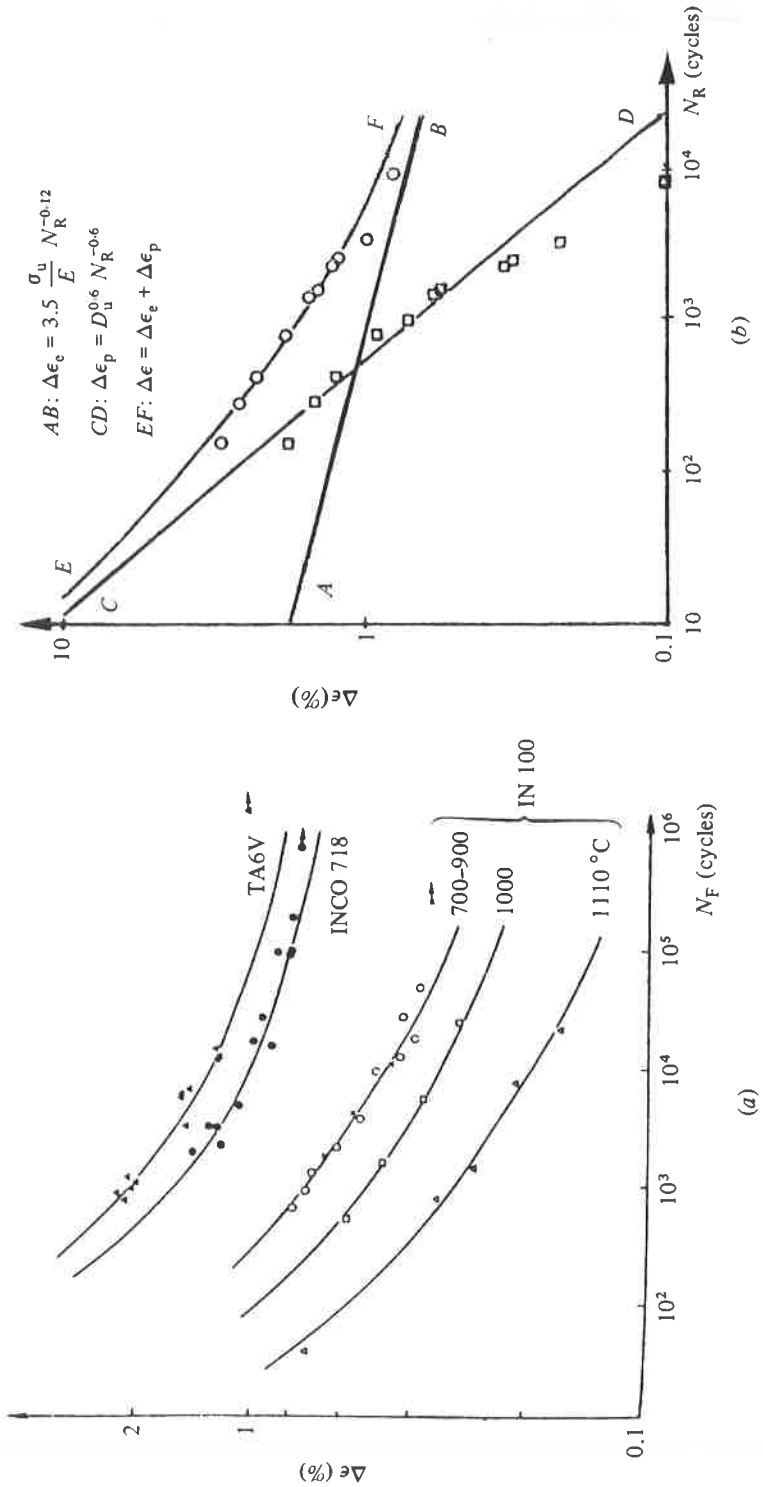


Fig. 7.15. (*a*) Coffin–Manson relations for three materials: TA6V at 350 °C, INCO 718 at 550 °C, IN 100 from 700 to 1110 °C. (*b*) Application of the rule of universal slopes to the UDIMET 700 alloy at 20 °C.

Energy criterion

An interesting parameter is Δw the energy dissipated per cycle, i.e., the area of the stress–strain hysteresis loop. This can be determined approximately from knowledge of the cyclic hardening coefficient by:

$$\Delta w = \frac{M_c - 1}{M_c + 1} \Delta\sigma \Delta\varepsilon_p.$$

The total dissipation energy up to failure is, unfortunately, not a constant intrinsic to the material; it has been observed to depend on the number of cycles to failure, generally as a power function (see, for example, Fig. 7.16):

$$w_F = N_F \Delta w = \alpha N_F^{-\delta}.$$

The laws of plasticity and their formulation within the thermodynamic framework can be used to replace Δw by the stored energy of the volume element (the rest being dissipated as heat). In the case of time-independent plasticity with isotropic hardening and superposed kinematic hardening with a yield function

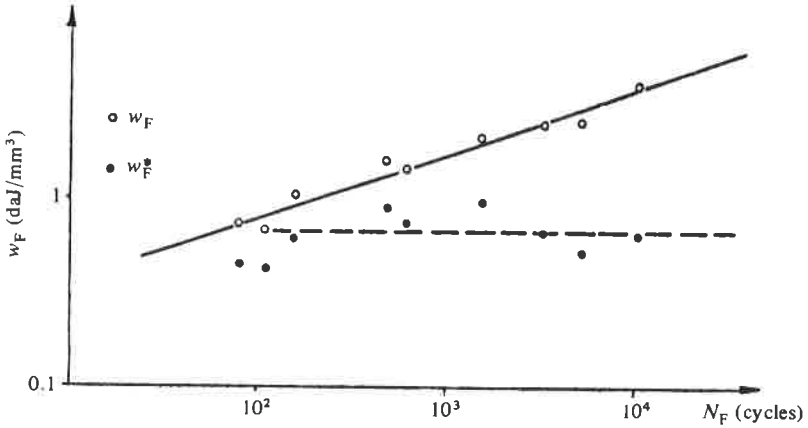
$$f = J(\boldsymbol{\sigma} - \mathbf{X}) - R - k$$

where J is the second invariant of the deviator of $\boldsymbol{\sigma} - \mathbf{X}$, the power dissipated as heat can be written as (see Chapter 5):

$$\Phi = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}_p - \mathbf{X} : \dot{\boldsymbol{\alpha}} - R\dot{p} = \dot{p}[J(\boldsymbol{\sigma} - \mathbf{X}) - R] = \dot{p}(f + k).$$

When there is no flow we have $\dot{p} = 0$. On the other hand, during flow we

Fig. 7.16. Dissipated energy of plastic deformation until failure, and its active part; 316 stainless steel at room temperature.



have $f = 0$, and the dissipated power is:

$$\Phi = k\dot{p} = k(\tfrac{2}{3}\dot{\boldsymbol{\varepsilon}}_p : \dot{\boldsymbol{\varepsilon}}_p)^{1/2}.$$

During a tension–compression cycle, the active energy stored in the volume element is therefore:

$$\Delta w^* = \Delta w - \Delta\Phi = \Delta w - 2k\Delta\varepsilon_p = \left(\frac{M_c - 1}{M_c + 1}\Delta\sigma - 2k\right)\Delta\varepsilon_p$$

where k represents the initial yield stress when

$$\mathbf{X}(0) = \varepsilon_p(0) = R(0) = 0.$$

The interesting feature of this parameter Δw^* is illustrated in Fig. 7.16; the energy to failure given by:

$$w_F^* = w_F - 2kN_F\Delta\varepsilon_p$$

is approximately constant and may be considered intrinsic to the material (the value $k = 220$ MPa used for this calculation corresponds to the limit of elasticity at 0.02% plastic strain for the 316 steel).

Strain range partitioning method

This method was developed to describe high temperature fatigue taking into account the effects of time. The basic assumption is that a complex cycle can always be decomposed into four parts corresponding to the four damage relations which can be determined independently.

Two types of damage can occur depending on whether the load generates plastic or creep deformation. A tensile load produces damage of a different type than a compressive load, and these two can be further distinguished to bring the total to four types, corresponding to the four basic cycles of Fig. 7.17.

In alternate tension–compression fatigue cycles, there is always:

plastic damage in both tension and compression:

D_{pp}, $\Delta\varepsilon_{pp}$;

plastic damage in tension, creep damage in compression:

D_{pc}, $\Delta\varepsilon_{pc}$;

creep damage in tension, plastic damage in compression:

D_{cp}, $\Delta\varepsilon_{cp}$;

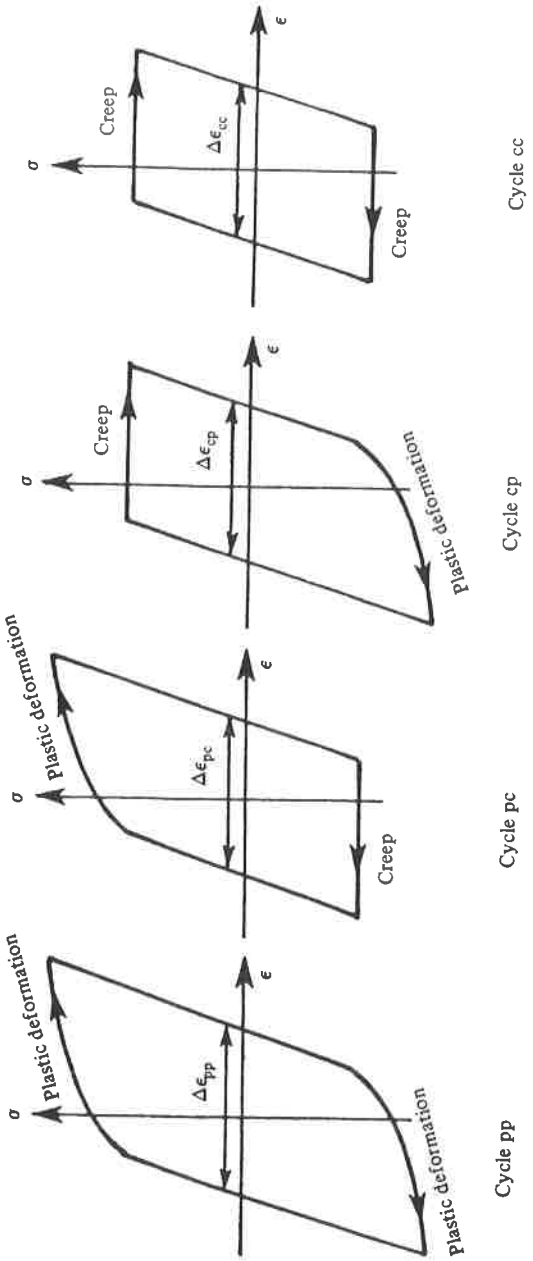


Fig. 7.17. The four basic cycles of strain range partitioning.

creep damage in both tension and compression:

D_{cc}, $\Delta\varepsilon_{cc}$.

Each damage is modelled by a differential relation with respect to the number of cycles. The form generally used reduces to assuming for each kind of damage:

$$\frac{\delta D_{ij}}{\delta N} = \left(\frac{\Delta\varepsilon_{in}}{C_{ij}}\right)^{\gamma_{ij}} \frac{\Delta\varepsilon_{ij}}{\Delta\varepsilon_{in}}.$$

Assuming that these damages are additive:

$$\frac{\delta D}{\delta N} = \left(\frac{\Delta\varepsilon_{in}}{C_{pp}}\right)^{\gamma_{pp}} \frac{\Delta\varepsilon_{pp}}{\Delta\varepsilon_{in}} + \left(\frac{\Delta\varepsilon_{in}}{C_{pc}}\right)^{\gamma_{pc}} \frac{\Delta\varepsilon_{pc}}{\Delta\varepsilon_{in}} + \left(\frac{\Delta\varepsilon_{in}}{C_{cp}}\right)^{\gamma_{cp}} \frac{\Delta\varepsilon_{cp}}{\Delta\varepsilon_{in}} + \left(\frac{\Delta\varepsilon_{in}}{C_{cc}}\right)^{\gamma_{cc}} \frac{\Delta\varepsilon_{cc}}{\Delta\varepsilon_{in}}.$$

In each of the four basic cases, the total inelastic strain $\Delta\varepsilon_{in}$ is equal to the strain $\Delta\varepsilon_{ij}$:

$$\Delta\varepsilon_{ij}/\Delta\varepsilon_{in} = 1.$$

The damage law is integrated assuming that in each case the loading is periodic with a constant strain range: $\Delta\varepsilon_{ij} = \Delta\varepsilon_{in}$ = constant. Let N_{ij} be the solution of $\delta D_{ij}/\delta N = (\Delta\varepsilon_{in}/C_{ij})^{\gamma_{ij}}$ subject to the conditions

$$N = 0 \rightarrow D_{ij} = 0$$
$$N = N_{ij} \rightarrow D_{ij} = 1:$$
$$N_{pp} = (\Delta\varepsilon_{pp}/C_{pp})^{-\gamma_{pp}}$$
$$N_{pc} = (\Delta\varepsilon_{pc}/C_{pc})^{-\gamma_{pc}}$$
$$N_{cp} = (\Delta\varepsilon_{cp}/C_{cp})^{-\gamma_{cp}}$$
$$N_{cc} = (\Delta\varepsilon_{cc}/C_{cc})^{-\gamma_{cc}}.$$

These relations can be used to express, in a particularly simple manner, the equation of total damage under periodic loads:

$$\frac{\delta D}{\delta N} = \frac{1}{N_{pp}}\frac{\Delta\varepsilon_{pp}}{\Delta\varepsilon_{in}} + \frac{1}{N_{pc}}\frac{\Delta\varepsilon_{pc}}{\Delta\varepsilon_{in}} + \frac{1}{N_{cp}}\frac{\Delta\varepsilon_{cp}}{\Delta\varepsilon_{in}} + \frac{1}{N_{cc}}\frac{\Delta\varepsilon_{cc}}{\Delta\varepsilon_{in}}$$

and for

$$D = 1 \rightarrow N = N_R$$

$$\frac{1}{N_R} = \frac{1}{N_{pp(\Delta\varepsilon_{in})}}\frac{\Delta\varepsilon_{pp}}{\Delta\varepsilon_{in}} + \frac{1}{N_{pc(\Delta\varepsilon_{in})}}\frac{\Delta\varepsilon_{pc}}{\Delta\varepsilon_{in}} + \frac{1}{N_{cp(\Delta\varepsilon_{in})}}\frac{\Delta\varepsilon_{cp}}{\Delta\varepsilon_{in}} + \frac{1}{N_{cc(\Delta\varepsilon_{in})}}\frac{\Delta\varepsilon_{cc}}{\Delta\varepsilon_{in}}.$$

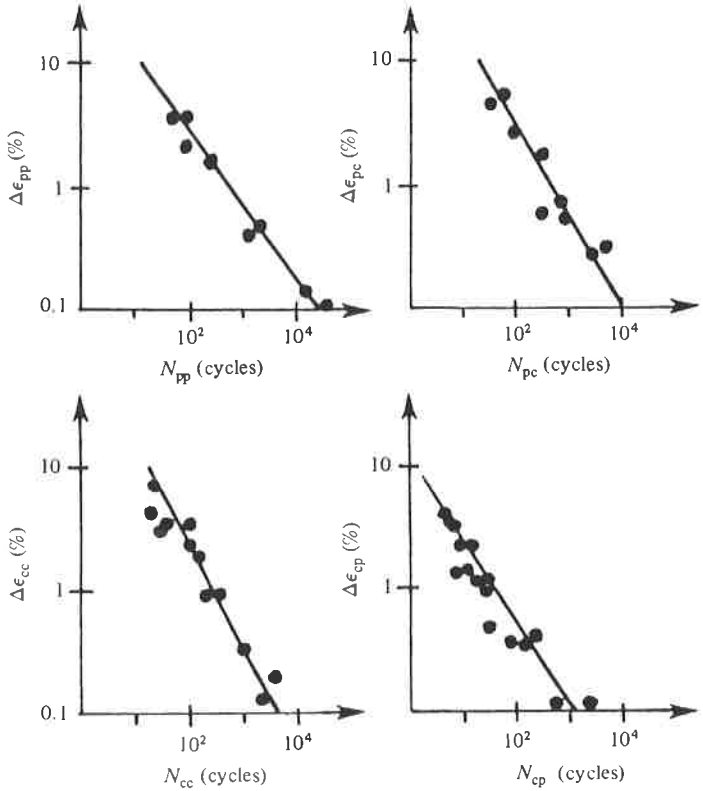
This is the relation of the strain range partitioning method. Fig. 7.18 illustrates the shape of the four curves $N_{ij}(\Delta\varepsilon_{ij})$ for the particular case of 316 stainless steel.

Use of the partitioning method requires knowledge of the four basic curves and of the strain components $\Delta\varepsilon_{pp}$, $\Delta\varepsilon_{pc}$, $\Delta\varepsilon_{cp}$, $\Delta\varepsilon_{cc}$; $\Delta\varepsilon_{in}$ is the total inelastic strain range. The numbers N_{pp}, N_{pc}, N_{cp}, N_{cc} are the abscissae of the points of intersection of a straight line of ordinate $\Delta\varepsilon_{in}$ with the four basic curves (Fig. 7.19(*a*)).

The decomposition of a real cycle into 'pp, pc, cp, cc' basic cycles is very difficult because in some cases it assumes a subjective character. Consider the real cycle *ABCDEF* of Fig. 7.19(*b*):

> *AB* is a 'plastic' loading in tension, assumed to be rapid;
> *BC* is a 'creep' part;

Fig. 7.18. The four basic curves of the strain range partitioning method for the 316 stainless steel.



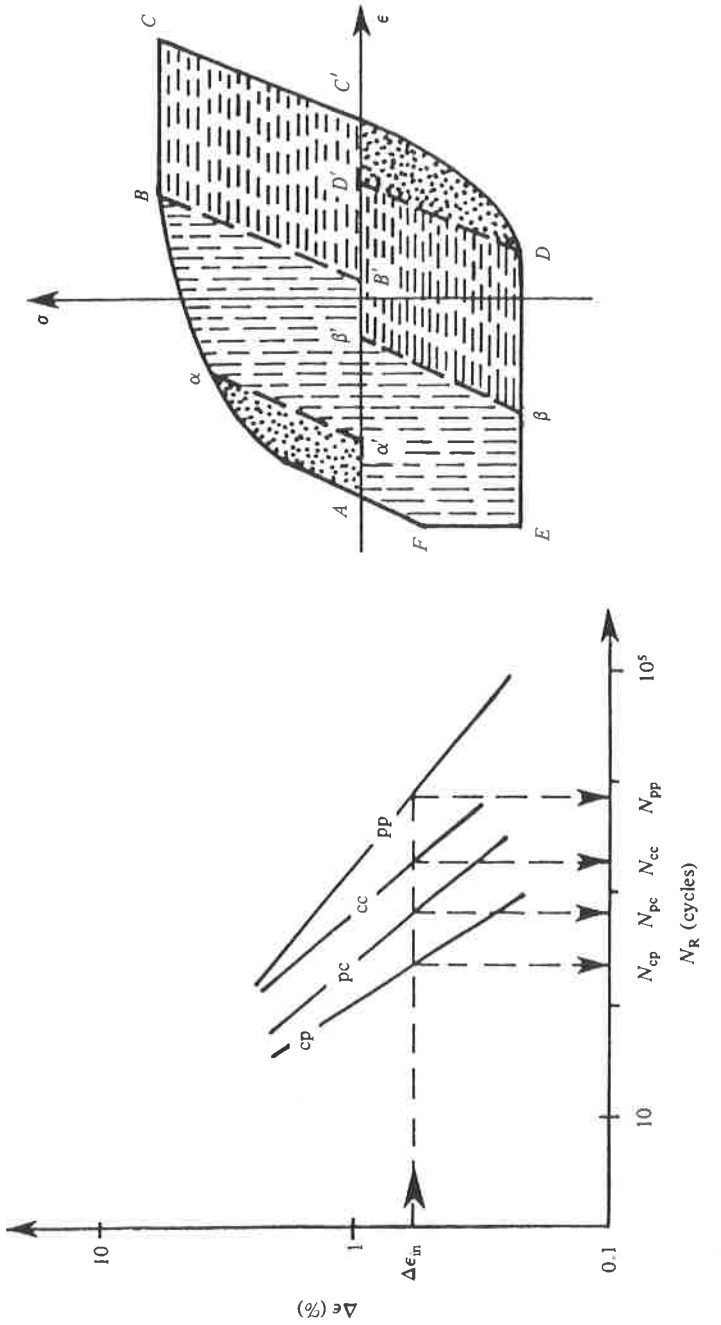


Fig. 7.19. (*a*) Obtaining the basic number of cycles to failure. (*b*) Partition of the actual cycle.

CD is an unloading and a plastic loading in compression, assumed to be rapid;
DE is a creep under compression;
EF is a relaxation in compression;
FA is an elastic loading.

(1) We search for the smallest plastic part, say CD. We associate to it a part which corresponds to a plastic strain of equal magnitude but of opposite sign, say $A\alpha$; we thus obtain the pp cycle with:

$$\Delta\varepsilon_{pp} = A\alpha' = D'C'.$$

(2) We search the smallest creep part, say BC. We associate to it a part which corresponds to a creep strain of equal magnitude but of opposite sign, say $D\beta$; we thus obtain the cc cycle with:

$$\Delta\varepsilon_{cc} = B'C' = \beta' D'.$$

(3) There remains the 'plastic' portion αB, the creep portion βE, and the relaxation portion EF. Relaxation and creep are governed by the same (viscoplastic) damage law.

The portion αB corresponds to a strain $\Delta\varepsilon_p = \alpha' B'$.
The portion βEF corresponds to a strain $\Delta\varepsilon_c = A\beta'$.

The partition is done in a way which equalizes these two strains. We may then associate these two portions to a pc cycle

$$\Delta\varepsilon_{pc} = \alpha' B' = A\beta'.$$

(4) In this particular case, the cp cycle does not exist: $\Delta\varepsilon_{cp} = 0$. We, of course, verify the fact that:

$$\sum_{ij} \Delta\varepsilon_{ij} = \Delta\varepsilon_{in} = AC'.$$

This method gives a unique decomposition for the cycle of Fig. 7.18(b), but it is conceivable that it could lead to different interpretations for cycles in which the transient loadings are so slow that there is creep (under variable load). In this case, which is quite common in practice, $\Delta\varepsilon_{pp}$ must be determined by a tension–compression cyclic test performed at a sufficiently high frequency (5–10 Hz) and with the same stress range. Another case where the partition is difficult is when the temperature changes during the cycle. The strain range partitioning method is a simple one which can be used for predictive purposes provided it is applied to cases which represent interpolations between the four basic curves and not extrapolations. Cases

of varying temperatures, very small plastic strains, ratchetting effects, generalizations to multiaxial loading pose difficult, even unsolvable problems.

7.2.5 *Multiaxial damage criteria*

In Section 7.2.4 we introduced damage thresholds such as the stress or strain below which the damage is zero or negligible. This concept can be generalized to three dimensions, in the same manner as the plasticity criteria. Similarly, for time-dependent damage, we can use surfaces similar to the equipotentials introduced for viscoplasticity in Chapter 6.

In one dimension, the damage threshold (in terms of stress) defines the range of resistance of the material:

$$-\sigma_D < \sigma < \sigma_D \rightarrow \dot{D} = 0.$$

When $|\sigma|$ is higher than the threshold σ_D, there is damage.

In three dimensions, this concept is generalized by a damage threshold (yield) surface:

$$f_D(\boldsymbol{\sigma}, D) = 0.$$

When $f_D < 0$, no damage results but when $f_D \geqslant 0$, damage occurs. The following sub sections give some examples of such surfaces described in terms of the stress tensor invariants. This damage is assumed here to be isotropic. In some cases we will also employ a law in terms of strain and the limiting surface will then be expressed by:

$$f_D(\boldsymbol{\varepsilon}, D) = 0.$$

The elastic energy density release rate criterion

The von Mises plasticity criterion is concerned with the shear energy, since slip is the main mechanism of plastic deformation. Damage is equally sensitive to shear energy but also to the volumetric deformation energy since the growth of cavities and cracks is very sensitive to hydrostatic stress. To formulate an isotropic criterion, we may therefore postulate that the mechanism of damage is governed by the total elastic strain energy: distortion energy + volumetric energy:

$$w^e = w_d^e + w_H^e$$

$$w_e = \int_0^{\varepsilon^e} \boldsymbol{\sigma} : d\boldsymbol{\varepsilon}^e = \int_0^{\varepsilon^{e'}} \boldsymbol{\sigma}' : d\boldsymbol{\varepsilon}^{e'} + 3\int_0^{\varepsilon_H^e} \sigma_H \, d\varepsilon_H$$

$\boldsymbol{\sigma}'$ and $\boldsymbol{\varepsilon}^{e'}$ are, as before, the deviators of $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}^e$; σ_H and ε_H are the hydrostatic stress and strain respectively.

By introducing the linear elasticity law coupled with damage through the concept of effective stress:

$$\boldsymbol{\varepsilon}^{e'} = \frac{1+\nu}{E}\frac{\boldsymbol{\sigma}'}{1-D} \qquad \varepsilon_H = \frac{1-2\nu}{E}\frac{\sigma_H}{1-D}$$

and by assuming that the damage does not vary within the elastic domain, we obtain:

$$w_e = \frac{1}{2}\left(\frac{1+\nu}{E}\frac{\boldsymbol{\sigma}':\boldsymbol{\sigma}'}{1-D} + 3\frac{1-2\nu}{E}\frac{\sigma_H^2}{1-D}\right)$$

or, with the equivalent von Mises stress $\sigma_{eq} = (\frac{3}{2}\boldsymbol{\sigma}':\boldsymbol{\sigma}')^{1/2}$:

$$w_e = \frac{1}{2E(1-D)}[\tfrac{2}{3}(1+\nu)\sigma_{eq}^2 + 3(1-2\nu)\sigma_H^2].$$

Similar to the equivalent stress in plasticity, we define the equivalent damage stress σ^* by stating that this energy in a multiaxial state is equal to that in an equivalent uniaxial state defined by σ^*, for which ($\sigma_{eq} = \sigma^*$, $\sigma_H = \frac{1}{3}\sigma^*$):

$$\frac{\sigma^{*2}}{2E(1-D)} = \frac{1}{2E(1-D)}[\tfrac{2}{3}(1+\nu)\sigma_{eq}^2 + 3(1-2\nu)\sigma_H^2]$$

or

$$\bullet \qquad \sigma^* = \sigma_{eq}[\tfrac{2}{3}(1+\nu) + 3(1-2\nu)(\sigma_H/\sigma_{eq})^2]^{1/2}.$$

The ratio σ_H/σ_{eq} in this relation expresses the triaxiality of the state of stress.

We shall see in Section 7.3.2 in the discussion on the thermodynamic formulation of damage that this equivalent stress σ^* is related to Y, the dual variable of D, in a simple way (such that $Y\dot{D}$ represents the dissipation); Y represents, if we do not consider the sign, the elastic energy release rate during damage growth at constant temperature and stress:

$$\sigma^* = (1-D)(-2EY)^{1/2} \quad \text{with} \quad Y = -\frac{1}{2}\frac{\mathrm{d}w_e}{\mathrm{d}D}\bigg|_{\sigma,T}$$

from which ensues the name the elastic energy density release rate criterion.

Fig. 7.20 gives the graphical representation of this criterion for (*a*) two-dimensional biaxial loading, and (*b*) uniaxial loading–torsion.

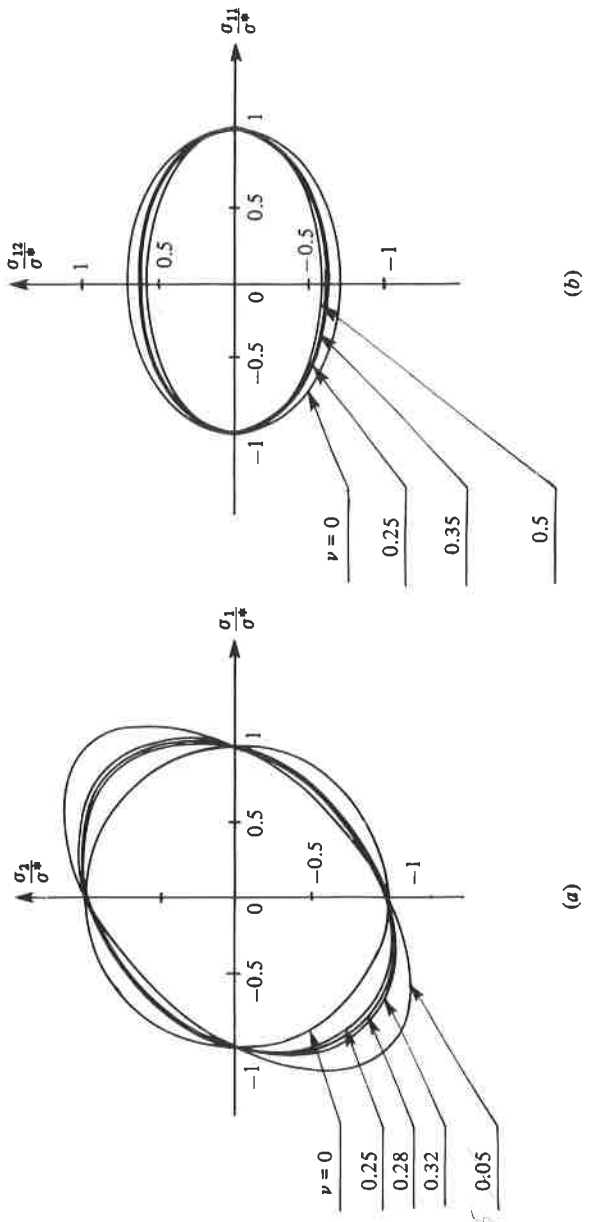


Fig. 7.20. Representation of the criterion of elastic energy density release rate: (*a*) two-dimensional biaxial loading and (*b*) uniaxial loading–torsion (after D. Baptiste).

Three invariants criterion

The elastic energy density release rate criterion is very simple and is formally justified by thermodynamics; no material coefficient, other than Poisson's ratio, appears in this criterion. However, in certain cases, it can be at fault, particularly, for creep damage where it gives the same answer in tension and in compression.

Within the framework of isotropy, a more general, but more complex, form consists in expressing the damage criterion in terms of the three basic invariants of the stress tensor:

$$\sigma_H = \tfrac{1}{3} J_1(\boldsymbol{\sigma}) = \tfrac{1}{3}\mathrm{Tr}(\boldsymbol{\sigma})$$

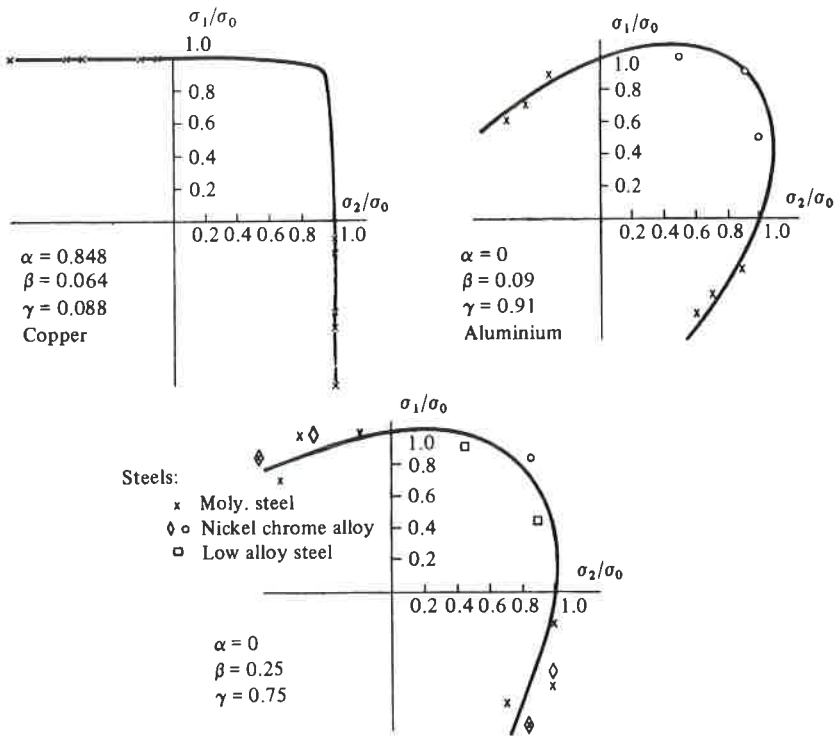
$$\sigma_{eq} = J_2(\boldsymbol{\sigma}) = [\tfrac{3}{2}\mathrm{Tr}(\boldsymbol{\sigma}'^2)]^{1/2}$$

$$J_3(\boldsymbol{\sigma}) = [\tfrac{9}{2}\mathrm{Tr}(\boldsymbol{\sigma}'^3)]^{1/3}.$$

Instead of $J_3(\boldsymbol{\sigma})$ it is more practical to choose:

$$J_0(\boldsymbol{\sigma}) = \mathrm{Sup}\,(\sigma_i).$$

Fig. 7.21. Isochronous curves for creep rupture times (after Hayhurst).



If a linear combination is chosen with α and β as phenomenological coefficients, then:

- $$\chi(\boldsymbol{\sigma}) = \alpha J_0(\boldsymbol{\sigma}) + \beta J_1(\boldsymbol{\sigma}) + (1 - \alpha - \beta) J_2(\boldsymbol{\sigma}).$$

For a uniaxial state we regain $\chi(\boldsymbol{\sigma}) = \sigma$.

Note that the three invariants correspond to the three basic mechanisms of nucleation and growth of defects, and that any isotropic surface in stress space can be described by this expression. $\chi(\boldsymbol{\sigma})$ may be negative for certain stress states (and certain values of α and β): these states then do not result in any damage growth.

Fig. 7.21 illustrates the possibilities of describing isochronic creep surfaces (loci of stress states that require the same time to failure in creep). Fig. 7.22 gives the shapes of these surfaces for different choices of the coefficients α and β.

A particularly interesting case is that where $\alpha = 0$:

$$\sigma^*_\beta = \beta J_1(\boldsymbol{\sigma}) + (1 - \beta) J_2(\boldsymbol{\sigma})$$

or

- $$\sigma^*_\beta = \sigma_{eq}[(1 - \beta) + 3\beta\sigma_H/\sigma_{eq}].$$

Here β acts as sensitivity coefficient of the material to the stress triaxiality. Fig. 7.22(*a*) shows graphs related to this equivalent stress.

Nonsymmetric criterion in terms of strain

This criterion has been developed for concrete, which has the property of being considerably more resistant to damage in compression than in tension. Microscopic observations reveal that microcracks always have a preferential orientation normal to the direction of the maximum principal extension. This suggests the idea of introducing a criterion dependent only upon the positive part of the principal strains.

Denoting the three principal strains by ε_1, ε_2 and ε_3, this criterion is expressed by:

- $$\varepsilon^* = (\langle\varepsilon_1\rangle^2 + \langle\varepsilon_2\rangle^2 + \langle\varepsilon_3\rangle^2)^{1/2}$$

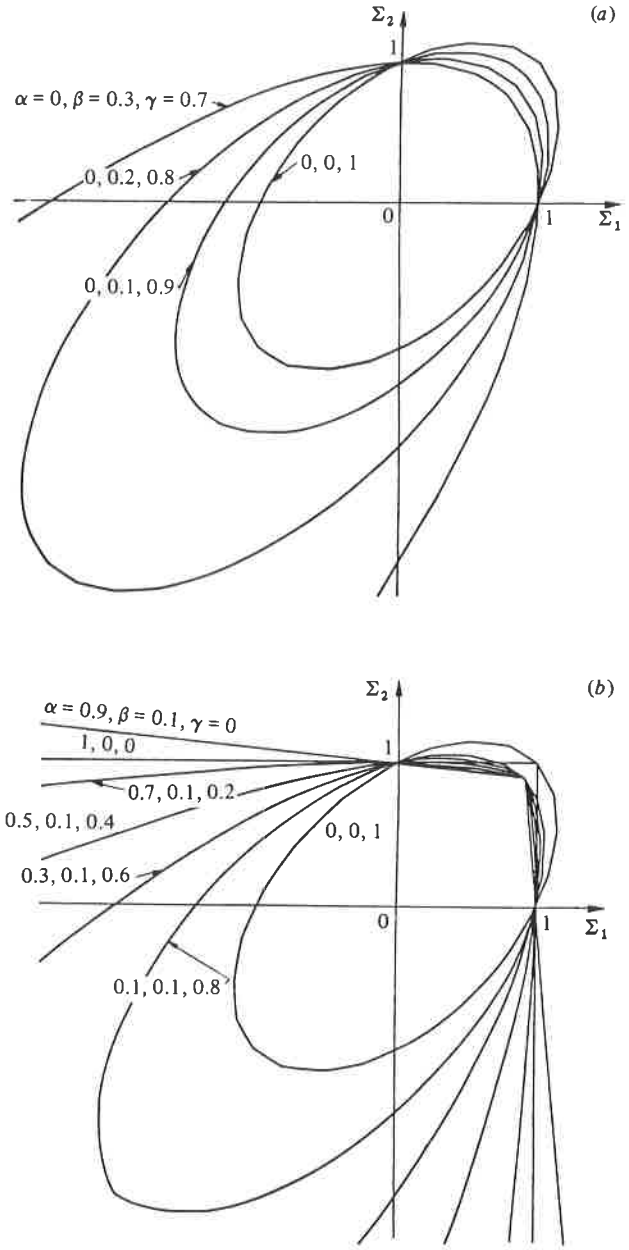
with

$$\langle\varepsilon_i\rangle = \varepsilon_i \quad \text{if} \quad \varepsilon_i > 0 \quad \text{for extension,}$$

$$\langle\varepsilon_i\rangle = 0 \quad \text{if} \quad \varepsilon_i \leqslant 0 \quad \text{for compression.}$$

Fig. 7.23 indicates the form of such a criterion compared to the

Fig. 7.22. Isochronous curves ($\Sigma_i = \sigma_i/\sigma_0$): (*a*) the influence of the hydrostatic stress ($\alpha = 0$); (*b*) the influence of the maximum principal stress ($\beta = 0.1$).



experimental results for biaxial tension, the stress is normalized to -1 for the failure stress in compression.

Fatigue limit criteria

The case of fatigue is more complex for two reasons:

> the loading is represented schematically by introducing the concept of a cycle: the parameters introduced are then defined globally for a cycle (amplitude, maximum value, mean value, etc.), the fatigue limit criteria expressed in terms of amplitude also depend on the average value during the cycle (see Section 7.2.4).

The concept of a uniaxial fatigue limit is therefore generalized by introducing at least two loading parameters. Numerous experiments have shown that the mean shear stress has no influence on the fatigue limit in either tension or torsion (Fig. 7.24).

On the other hand, the mean tensile stress influences the fatigue limit in tension (Fig. 7.14) and in torsion (Fig. 7.25) in a linear fashion. The two parameters which may be used to translate these effects are:

Fig. 7.23. Graphical representation of the strain criterion for concrete (after J. Mazars).

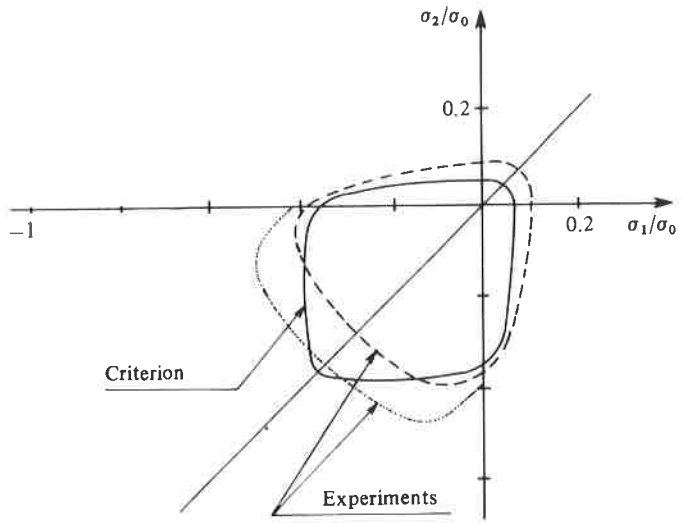


Fig. 7.24. Effect of mean torsion on the fatigue limit: (*a*) in torsion, and (*b*) in tension-compression. (τ_{a_1} = fatigue limit amplitude in torsion, $\bar{\tau}$ = mean stress in torsion, τ_{Max} = maximum stress in torsion; τ_e = elastic limit in torsion, σ_{a_1} = fatigue limit amplitude in bending.)

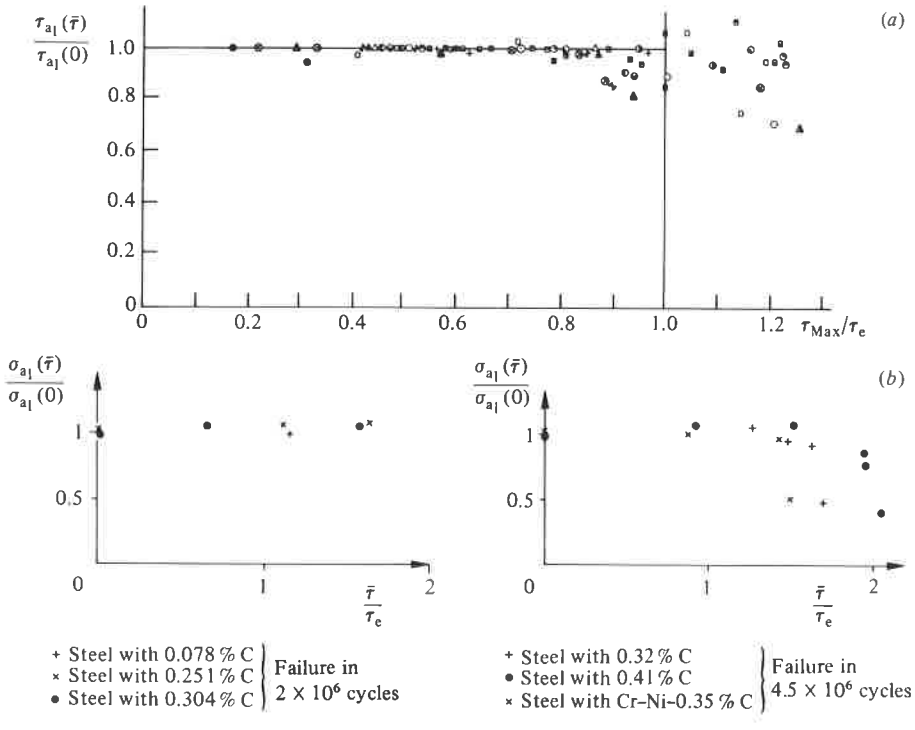
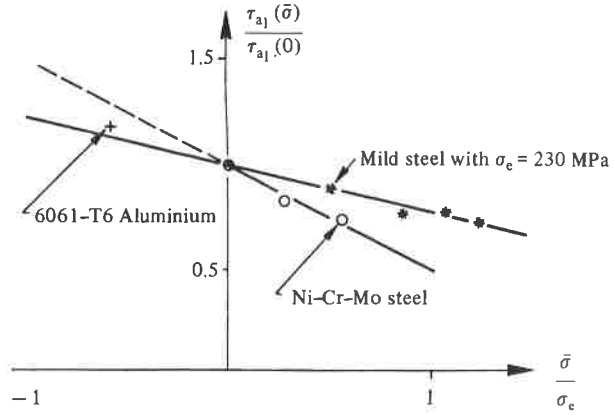


Fig. 7.25. Influence of mean tension on the fatigue limit in torsion.

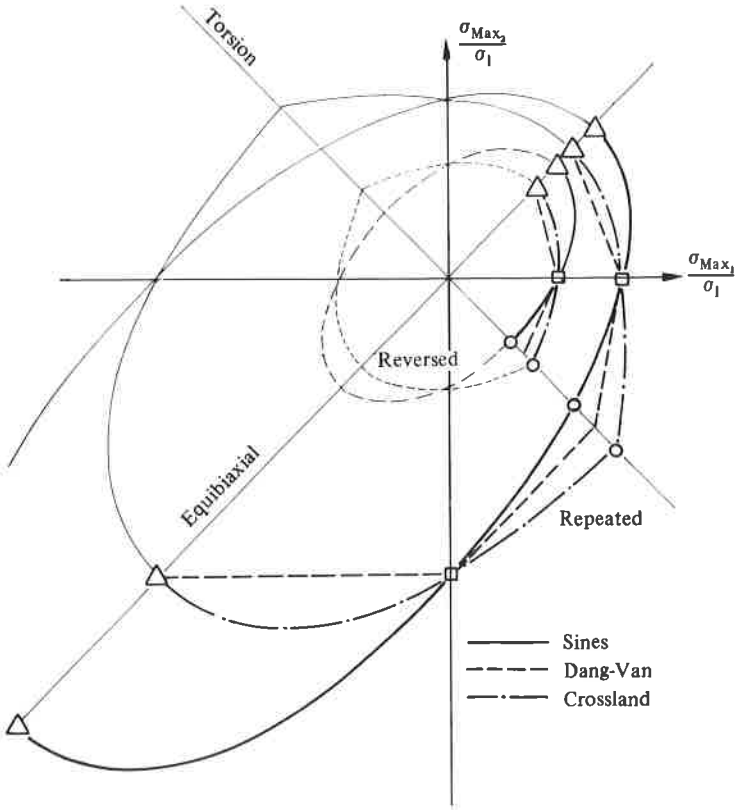


an octahedral shear stress amplitude, defined in proportional loading by:

$$A_{II} = \tfrac{1}{2}[\tfrac{3}{2}(\sigma'_{ij_{Max}} - \sigma'_{ij_m})(\sigma'_{ij_{Max}} - \sigma'_{ij_m})]^{1/2}$$
$$= \{\tfrac{1}{2}[(a_1 - a_2)^2 + (a_2 - a_3)^2 + (a_3 - a_1)^2]\}^{1/2}$$

where σ'_{Max} and σ'_m are the maximum and minimum values of each component of the deviator (during the cycle) and a_1, a_2, a_3 represent the principal stress amplitudes $a_i = \Delta\sigma_i/2$.
the mean hydrostatic pressure during the cycle $\bar{\sigma}_H = \frac{1}{3}$ mean $J_1(\boldsymbol{\sigma})$ in the Sines criterion, or the maximum hydrostatic pressure $\sigma_{H_{Max}} = \frac{1}{3}\text{Max}\, J_1(\boldsymbol{\sigma})$ as in the Crossland criterion.

Fig. 7.26. Criteria of Sines, Crossland, Dang-Van in proportional biaxial loading, reversed or repeated.



These two fatigue limit criteria can be written as:

Sines criterion:

- $A_{II} = \sigma_{I_0}(1 - 3b\bar{\sigma}_H)$;

Crossland criterion:

- $A_{II} = \sigma_{I_0}\dfrac{1 - 3b\sigma_{II_{Max}}}{1 - b\sigma_{I_1}}$.

In the first case, under alternate proportional loading, the fatigue limit surface in the plane $\sigma_3 = 0$ corresponds to the von Mises ellipse (Fig. 7.26): the Crossland criterion corresponds to a portion of an ellipse with a different eccentricity and centre.

Depending on the material, one or the other of the two criteria is the more

Fig. 7.27. Reversed tension–torsion: Sines and Crossland criteria for (*a*) 'silal' cast iron and (*b*) Cr–Va steel.

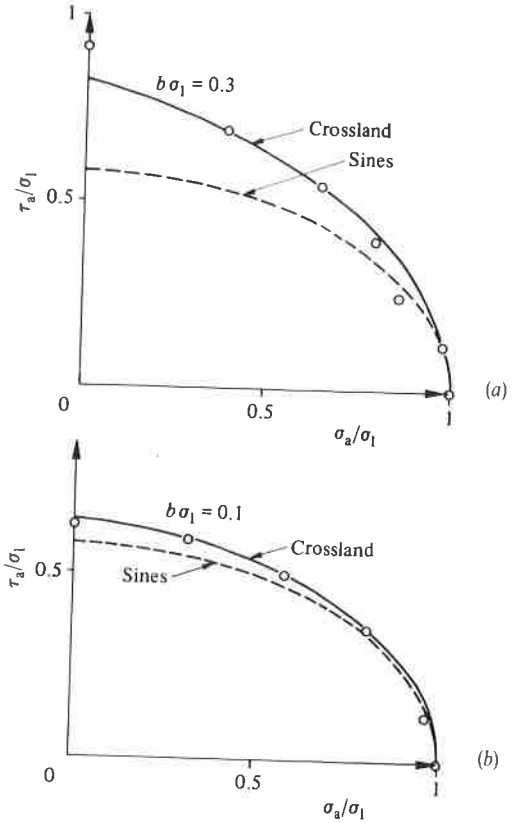


Table 7.5. *Fatigue limits in terms of the amplitude of σ_{a_1} for particular proportional loadings (σ_I means $\sigma_{I_0} = \sigma_I(0)$)*

Loading		Sines	Crossland
Reversed tension	$\sigma_2 = 0$	σ_I	
Repeated tension		$\sigma_I/(1 + b\sigma_I)$	
Repeated compression		$\sigma_I/(1 - b\sigma_I)$	
Reversed equibiaxial tension	$\sigma_2 = \sigma_1$	σ_I	$\sigma_I/(1 + b\sigma_I)$
Repeated equibiaxial tension		$\sigma_I/(1 + 2b\sigma_I)$	$\sigma_I/(1 + 3b\sigma_I)$
Repeated equibiaxial compression		$\sigma_I/(1 - 2b\sigma_I)$	$\sigma_I/(1 - b\sigma_I)$
Repeated and reversed torsion	$\sigma_2 = -\sigma_1$	$\sigma_I/\sqrt{3}$	$\sigma_I/\sqrt{3}(1 - b\sigma_I)$

Sines: $(\sigma_{a_1}^2 + \sigma_{a_2}^2 - \sigma_{a_1}\sigma_{a_2})^{1/2} = \sigma_I[1 - b(\bar{\sigma}_1 + \bar{\sigma}_2)]$

Crossland: $(\sigma_{a_1}^2 + \sigma_{a_2}^2 - \sigma_{a_1}\sigma_{a_2})^{1/2} = \dfrac{\sigma_I}{1 - b\sigma_I}[1 - b\,\mathrm{Max}\,(\bar{\sigma}_1 + \bar{\sigma}_2 + \sigma_{a_1} + \sigma_{a_2})]$

realistic (Fig. 7.27). Table 7.5 gives different expressions for the fatigue limit in terms of amplitude $\sigma_{a_1} = \Delta\sigma_1/2$ for different types of loading. The fatigue limit surface depends on the mean stress during the cycle. Fig. 7.28 shows, for example, the cases of reversed and repeated loadings.

Under nonproportional loadings, these criteria are always applicable but the definition of equivalent amplitude must be replaced by an intrinsic formula:

$$A_{II} = \tfrac{1}{2}\underset{t_0}{\mathrm{Max}}\,\underset{t}{\mathrm{Max}}\,J_2(\boldsymbol{\sigma}(t) - \boldsymbol{\sigma}(t_0))$$

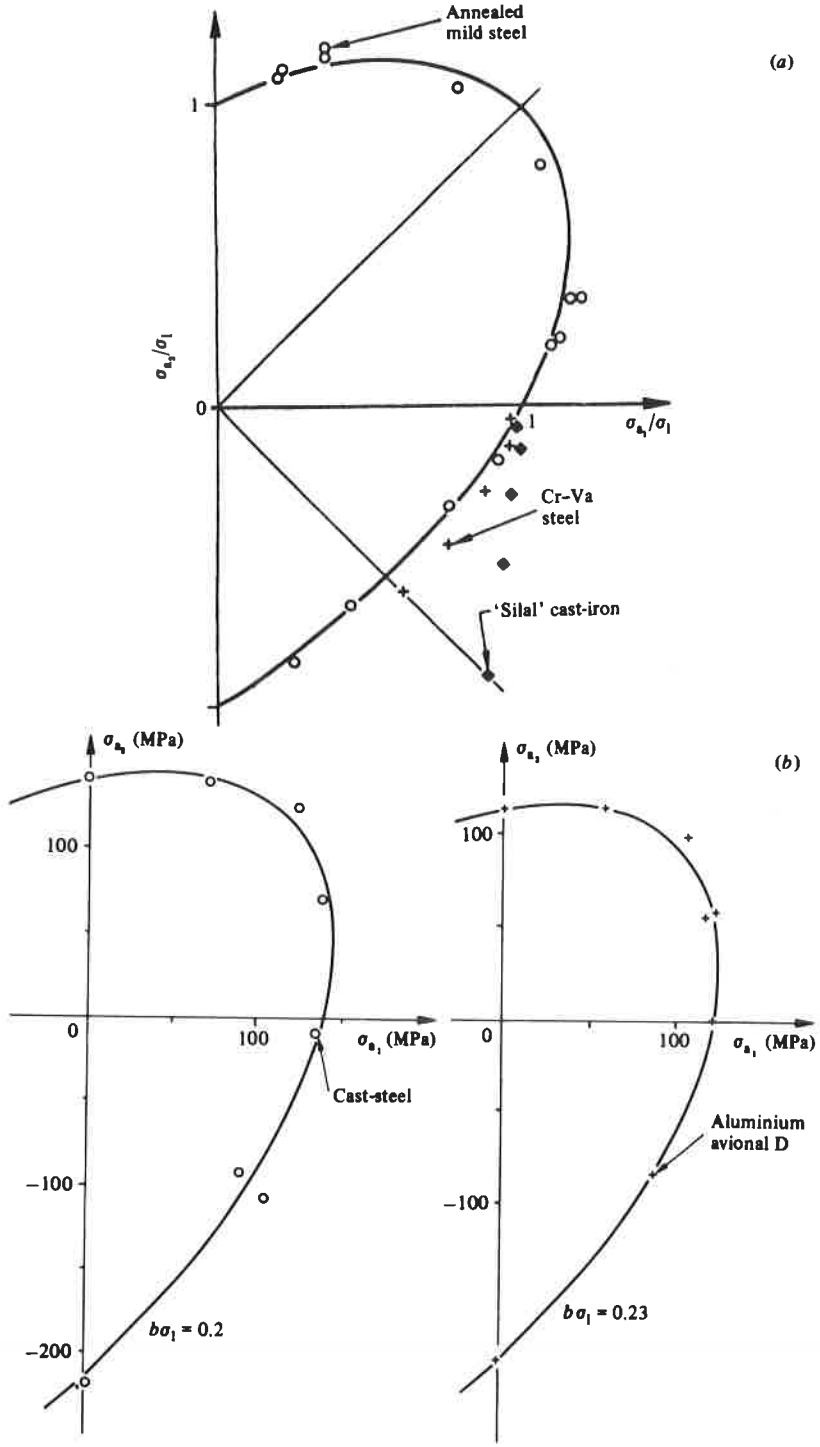
where J_2 denotes the second invariant of the deviator. This generalization is illustrated in Fig. 7.29(*a*) in the deviatoric plane for any loading.

An intermediate criterion, with a stronger physical basis, has been introduced by Dang-Van. In order to define the amplitude parameter, he considers the plane which suffers the maximum shear amplitude during the cycle. On this plane, represented by its normal $\vec{n}$, we consider at each instant the difference $\vec{\tau}(t) - \vec{\tau}$ where $\vec{\tau}(t)$ is the shear $\boldsymbol{\sigma}\cdot\vec{n} - (\boldsymbol{\sigma}\cdot\vec{n}\cdot\vec{n})\vec{n}$ and where $\vec{\tau}$ represents the mean shear stress on this plane during the cycle. Usually, $\vec{\tau}$ is defined by the centre of the smallest circle containing the path of $\vec{\tau}(t)$ in the plane. The radius of this circle is taken to be the shear amplitude (Fig. 7.29(*b*)):

$$\Delta\tau/2 = \tfrac{1}{2}\underset{t}{\mathrm{Max}}\,\underset{t_0}{\mathrm{Max}}\,\|\vec{\tau}(t) - \vec{\tau}(t_0)\|$$

where $\|\ \|$ denotes the magnitude. The quantity $\vec{\tau}(t) - \vec{\tau}$ is combined with the hydrostatic pressure $\sigma_H(t)$ to define the critical instant and the critical

Fig. 7.28. Sines criterion in (*a*) reversed or (*b*) repeated loading, (σ_I means $\sigma_{I0} = \sigma_I(0)$).

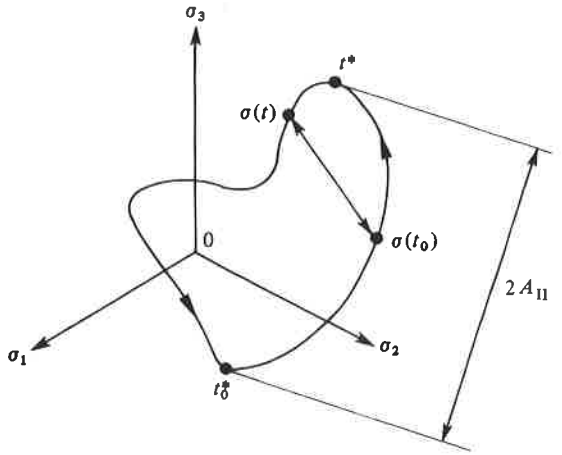


parameter. The criterion is then expressed as:

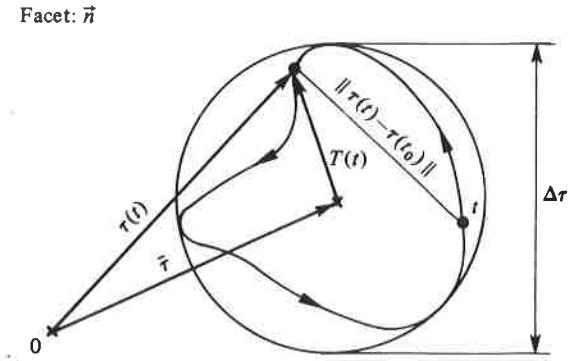
$$\underset{\vec{n}}{\text{Max}}\ \underset{t}{\text{Max}}\left[\ \|\vec{\tau}(t)-\vec{\tau}\| + \frac{1}{2}\frac{3b\sigma_{I_0}}{1-b\sigma_{I_0}}\sigma_H(t)\right] = \frac{\sigma_{I_0}}{2(1-b\sigma_{I_0})}.$$

Fig. 7.30 illustrates the recentring and the determination of the critical instant in the shear–pressure diagram (for the case where shear has a fixed

Fig. 7.29. Determination of the amplitude of the cycle in the general case: (*a*) Sines and Crossland criteria; (*b*) Dang-Van criterion.



(*a*)



(*b*)

direction). The essential difference with the two previous criteria is that the quantities appearing in the criterion (shear and pressure) are defined at the same instant (point A). In contrast, in the Crossland criterion, these quantities are defined at different instants (A' and C). We therefore have a more accurate description of the real process. Fig. 7.31 shows different cases of proportional loading after recentring the cycle. In this case, the critical plane is really the plane that minimizes the shear amplitude and we can write the criterion in the simplified form:

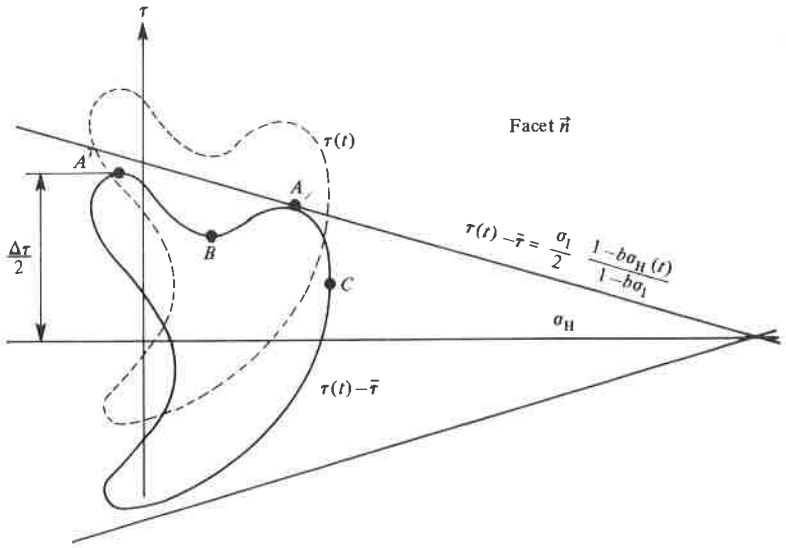
$$\frac{\Delta\tau}{2} = \frac{\sigma_{1_0}}{2(1 - b\sigma_{1_0})}(1 - 3b\sigma_{H_{Max}}).$$

In torsional proportional loading, the Dang-Van criterion gives results which are between those given by the criterion of Sines and that of Crossland:

$$\tau_a = \sigma_a = \frac{\sigma_{1_0}}{2(1 - b\sigma_{1_0})}.$$

For other proportional loadings (tension–compression, reversed and repeated equibiaxial), it is identical to the Crossland criterion as can be seen from Fig. 7.26.

Fig. 7.30. Application of the Dang-Van criterion when the shear has a fixed direction during the cycle. A: critical instant. B and C: instants corresponding to the choice of the hydrostatic pressure in the criteria of Sines and Crossland (σ_1 means $\sigma_1(0)$).



Note that this criterion implicitly gives the direction of defects (critical planes) but that, in the general case, it is much more complex to use because it involves three maximizations: twice as a function of t and t_0 at each plane, and then as a function of the direction of the normal $\vec{n}$.

Instead of using the shear amplitude $\Delta\tau/2$, we may also use the range of octahedral shear:

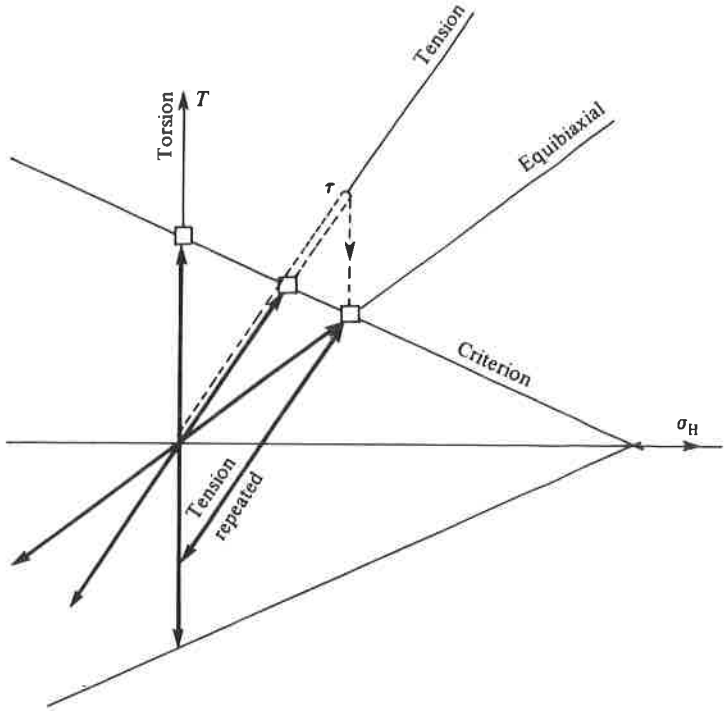
$$\Delta J_2 = \operatorname*{Max}_{t_1}\left[\operatorname*{Max}_{t_2} J_2(\boldsymbol{\sigma}(t_1) - \boldsymbol{\sigma}(t_2))\right].$$

This maximum being reached at a time t_i, the equivalent shear stress is defined by:

$$J_2^*(t) = J_2(\boldsymbol{\sigma}(t) - \boldsymbol{\sigma}(t_i)) - \Delta J_2/2.$$

It is then sufficient to replace $(\vec{\tau}(t), \sigma_H(t))$ by $(J_2^*(t)\, \sigma_H(t))$ in the criterion and we obtain an expression, equivalent to the Crossland criterion in proportional loading, which is simpler to use than the Dang-Van criterion.

Fig. 7.31. Application of the Dang-Van criterion for particular proportional loadings.



7.3 Thermodynamic formulation

7.3.1 *Multiaxial representation of damage*

Within the isotropic framework, the damage variable is a scalar D which, for the purpose of defining the effective stress $\tilde{\boldsymbol{\sigma}}$, affects all components of the stress tensor $\boldsymbol{\sigma}$ identically:

$$\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma}/(1 - D).$$

In fact, the surface microcracks and the volume cavities that make up damage usually have preferred orientations and thus constitute anisotropic damage. For example, microcracks in metals have a tendency to propagate in a direction perpendicular to that of the maximum stress. The damage operator, which applied to the usual stress tensor gives the effective stress tensor, is then tensorial and of an order which differs depending on the adopted theoretical scheme: a second order tensor, if a stress vector is defined on a damaged oriented surface, but this concept does not necessarily satisfy the second principle of thermodynamics; a fourth order tensor if the damage operator is defined on the basis of the elastic potential as we show below; eighth order and higher tensors are also possible!

If only geometrical effects of cavities and cracks are taken into account, we introduce, for each element of area, represented by its normal $\vec{n}$, an area reduction $\Omega(\vec{n})$, and the state of damage is characterized by a second order symmetric tensor expressed in principal axes representation by:

$$\boldsymbol{\Omega} = \sum_{j=1}^{3} \Omega_j \cdot \vec{n}_j \otimes \vec{n}_j.$$

The net stress tensor (referred to the noncorrected resistant sections), symmetrized according to Murakami and Ohno can be deduced from it by:

$$\tilde{\boldsymbol{\sigma}} = \tfrac{1}{2}[\boldsymbol{\sigma}\cdot(\mathbf{1} - \boldsymbol{\Omega})^{-1} + (\mathbf{1} - \boldsymbol{\Omega})^{-1}\cdot\boldsymbol{\sigma}].$$

In fact this effective stress cannot be used directly in the constitutive equations because the behaviour of the damaged material depends not only upon the area reduction caused by cracks and cavities, but also upon their arrangement and the effects of their interactions. It is necessary to introduce another effective stress derived from the Cauchy stress by a fourth order tensor. This tensor also characterizes the damage, but it is preferable to introduce it directly from a theory based on the concept of strain equivalence.

Definition of effective stress by strain equivalence

This theory, introduced by Chaboche and Lemaitre, is applicable to materials whose initial behaviour is undefined (elastic or plastic, isotropic or anisotropic). In practice, it is sufficient to examine the case of isothermal elasticity to define the effective stress tensor. Let, therefore

$$\boldsymbol{\sigma} = \mathbf{a}:\boldsymbol{\varepsilon}^{e}$$

be the constitutive equation of the virgin material, and

$$\tilde{\boldsymbol{\sigma}} = \mathbf{a}:\boldsymbol{\varepsilon}^{e}$$

be the constitutive equation of the damaged material, $\mathbf{a}$ being the fourth order elasticity tensor. The strain energy of the damaged material can be defined directly by using an elasticity tensor $\tilde{\mathbf{a}}$ different from $\mathbf{a}$ but possessing the same thermodynamic symmetry properties:

$$\tilde{a}_{ijkl} = \tilde{a}_{ijlk} = \tilde{a}_{jikl} = \tilde{a}_{klij}$$
$$\psi_e = \tfrac{1}{2}\tilde{\mathbf{a}}:\boldsymbol{\varepsilon}^{e}:\boldsymbol{\varepsilon}^{e}.$$

The constitutive equation of the damaged material is then deduced by:

$$\boldsymbol{\sigma} = \partial\psi_e/\partial\boldsymbol{\varepsilon}^{e} = \tilde{\mathbf{a}}:\boldsymbol{\varepsilon}^{e}.$$

According to the strain equivalence principle (Section 7.2.2), $\tilde{\boldsymbol{\sigma}}$ is that stress tensor which, when applied to the virgin material, produces the same strain tensor as that obtained by applying the tensor $\boldsymbol{\sigma}$ to the damaged material. The combination of the two laws easily leads to:

$$\tilde{\boldsymbol{\sigma}} = (\mathbf{a}:\tilde{\mathbf{a}}^{-1}):\boldsymbol{\sigma} = \boldsymbol{\Delta}:\boldsymbol{\sigma}$$

where $\boldsymbol{\sigma}$ and $\tilde{\boldsymbol{\sigma}}$ are second order tensors: it follows that the damage operator $\boldsymbol{\Delta} = \mathbf{a}:\tilde{\mathbf{a}}^{-1}$ is a fourth order tensor whose coefficients can be determined from the clasticity matrices of the virgin and damaged materials.

We can obtain the same result by using the mathematical technique of homogenization, at least, for the idealized cases of volume elements damaged by ellipsoidal cavities or periodically distributed parallel cracks. The elasticity matrix of the damaged material is expressible as:

$$\tilde{a}_{ijkl} = \left(\frac{\tilde{V}}{V}\delta_{ir}\delta_{js} - \frac{1}{V}\int_{\tilde{V}} b_{ijrs}\,\mathrm{d}V\right)a_{rskl}$$

where b_{ijrs} represents the matrix of the stress-concentration coefficients. V is the apparent volume of the element and $\tilde{V}$ is the solid volume (equal to the apparent volume minus the volume of voids). We then introduce $\mathbf{D}$, a 4th

order operator with components:

$$D_{ijrs} = \left(1 - \frac{\tilde{V}}{V}\right)\delta_{ir}\delta_{js} + \frac{1}{V}\int_{\tilde{V}} b_{ijrs}\,\mathrm{d}V$$

such that:

$$(\mathbf{1} - \mathbf{D})^{-1} = \mathbf{\Delta}.$$

This operator is not symmetric, for example $D_{1122} \neq D_{2211}$. We can write:

$$\tilde{\mathbf{a}} = (\mathbf{1} - \mathbf{D}):\mathbf{a}$$
$$\tilde{\boldsymbol{\sigma}} = (\mathbf{1} - \mathbf{D})^{-1}:\boldsymbol{\sigma}.$$

We will stop here in defining the three-dimensional damage variable **D** which can represent the most general states of anisotropic damage. It should, however, be noted that an energy equivalence, instead of the strain equivalence, allows us to introduce a second order damage tensor (still written as **Ω**) and an effective stress of the form:

$$(\mathbf{1} - \mathbf{\Omega})^{-1/2}\cdot\boldsymbol{\sigma}\cdot(\mathbf{1} - \mathbf{\Omega})^{-1/2}$$

but this theory suffers from the disadvantage that it cannot be identified with any general elastic anisotropy.

The two problems of damage multiaxiality

It is important to distinguish correctly between two aspects of damage which often cause confusion:

> the first is linked to the multiaxiality of the stress state in the form of the damage criterion mentioned in Section 7.2.5;
> the second is related to the isotropy or anisotropy of the current state of damage and its evolution.

To illustrate the first aspect, we may imagine different isotropic damages induced by states of stress with different directions. For example, in the same material, a tensile loading σ may produce an isotropic damage $D = 0.2$ (which is the same for all material directions), but a compressive loading $-\sigma$ may result in an isotropic damage $D = 0.02$. In other words, the directions of the stress state that results in damage should not be confused with those linked to the material. These latter ones define the directions of anisotropy of the current state of damage which will be explained below.

The second aspect may be illustrated by the case of a radial loading (where the principal directions of the stress tensor remain constant) which

produces an anisotropic damage with isotropic evolution; for example the case of a tensile loading which produces:

> $D = D(t)$ in the material direction parallel to that of the tension;
> $D = \alpha D(t)$ in other directions, for example with $\alpha = 0.1$ in a perpendicular direction.

The properties of isotropy and anisotropy can be applied independently to the current state and to the damage growth.

For radial (or proportional) loading, the principal directions of the damage tensor coincide with those of the stress tensor. A simplified form of the damage evolution equation is then:

$$\dot{\mathbf{D}} = \mathbf{Q}^*\dot{D}$$

where $\mathbf{Q}^*$ is a tensor dependent on the material, and possibly on the temperature. All nonlinearities of the evolution phenomenon are assumed to be incorporated in the scalar $\dot{D}$, the rate of damage, representative of the quantity of defects per unit volume, independent of their orientation.

In the general case, the evolution law depends on the principal directions of the effective stress tensor:

$$\dot{\mathbf{D}} = \mathbf{Q}(\boldsymbol{\sigma})\dot{D}.$$

The relation between $\mathbf{Q}(\boldsymbol{\sigma})$ and $\mathbf{Q}^*$ involves rotations to change from the principal system to the reference system.

In what follows, we will limit ourselves to the very simple case of isotropic damage, or to the more general case studied above in which the anisotropy of the damage evolution does not depend upon any particular instant during the life of the volume element.

7.3.2 *Theory of isotropic damage*

Thermodynamic potential

We will only consider damageable materials exhibiting elastic, elastoplastic or elastoviscoplastic behaviour (the viscoelastic case would require a slightly different formal treatment). The damage variable is the scalar D which will be considered as a state variable amenable to the thermodynamic representation described in Chapter 2. With the notations of the preceding section, this amounts to treating the damage as a spherical tensor $\mathbf{1}D$.

The free energy thermodynamic potential is therefore a convex function

of all the state variables, and, in particular, of the damage

$$\Psi = \Psi(\boldsymbol{\varepsilon}^e, T, D, V_k)$$

where V_k denotes the internal variables such as the hardening variables.

As in elastoplasticity, we may adopt the hypothesis of decoupling between hardening and other effects, represented by the variable V_k, and the effects of elasticity associated with the damage:

$$\Psi = \Psi_e(\boldsymbol{\varepsilon}^e, T, D) + \Psi_p(T, V_k)$$

where Ψ_e is the thermoelastic potential of a damageable material. We have seen in Chapter 4 that the linear thermoelasticity can be derived from a quadratic potential in $\boldsymbol{\varepsilon}^e$ and T. Remaining within this framework, the fundamental hypothesis of the effective stress and the definition chosen for D (see Section 7.3.1) require that Ψ_e depends linearly on D:

$$\rho\Psi_e = \tfrac{1}{2}(1-D)\mathbf{a}:(\boldsymbol{\varepsilon}^e - \mathbf{k}\Delta T):(\boldsymbol{\varepsilon}^e - \mathbf{k}\Delta T) + C\Delta T^2$$

where $\mathbf{k}\Delta T$ represents the thermal dilatation.

The specification of this thermodynamic potential furnishes the thermoelastic law of the damaged material:

$$\boldsymbol{\sigma} = \rho\,\partial\Psi_e/\partial\boldsymbol{\varepsilon}^e = (1-D)\mathbf{a}:(\boldsymbol{\varepsilon}^e - \mathbf{k}\Delta T) = \tilde{\mathbf{a}}:(\boldsymbol{\varepsilon}^e - \mathbf{k}\Delta T)$$

which is indeed of the form:

$$\tilde{\boldsymbol{\sigma}} = \frac{\boldsymbol{\sigma}}{1-D} = \mathbf{a}:(\boldsymbol{\varepsilon}^e - \mathbf{k}\Delta T).$$

The variable associated with the damage variable is the scalar:

$$Y = \rho\,\partial\Psi_e/\partial D = -\tfrac{1}{2}\mathbf{a}:(\boldsymbol{\varepsilon}^e - \mathbf{k}\Delta T):(\boldsymbol{\varepsilon}^e - \mathbf{k}\Delta T).$$

It is interesting to note that this quantity can be identified, except for the sign, with one-half of the variation in elastic energy generated by a damage variation at constant stress and temperature, $\mathrm{d}w_e/\mathrm{d}D|_{\sigma,T}$. To show this, let us calculate: $\mathrm{d}w_e = \boldsymbol{\sigma}:\mathrm{d}\boldsymbol{\varepsilon}^e$ by obtaining $\mathrm{d}\boldsymbol{\varepsilon}^e$ from the expression $\mathrm{d}\boldsymbol{\sigma} = 0$ deduced from the thermoelasticity law with $\Delta T = 0$:

$$\mathrm{d}\sigma|_{T=\text{constant}} = (1-D)\mathbf{a}:\mathrm{d}\boldsymbol{\varepsilon}^e - \mathbf{a}:\boldsymbol{\varepsilon}^e\,\mathrm{d}D = 0$$

$$\mathrm{d}w_e = \boldsymbol{\sigma}:\mathrm{d}\boldsymbol{\varepsilon}^e = \boldsymbol{\sigma}:\boldsymbol{\varepsilon}^e\,\mathrm{d}D/(1-D) = \mathbf{a}:\boldsymbol{\varepsilon}^e:\boldsymbol{\varepsilon}^e\,\mathrm{d}D$$

$$\mathrm{d}w_e/\mathrm{d}D|_{\sigma,T} = \mathbf{a}:\boldsymbol{\varepsilon}^e:\boldsymbol{\varepsilon}^e = a_{ijkl}\varepsilon^e_{ij}\varepsilon^e_{kl}$$

and indeed, as expected, we find the stated result:

$$\bullet \qquad -Y = \tfrac{1}{2}(\mathrm{d}w_e/\mathrm{d}D)|_{\sigma,T}.$$

$-Y$ is called the elastic energy release rate.

Thus, for isotropic damage and when the material is elastically isotropic, there exists a simple relation between the variable associated with the damage and the equivalent damage stress introduced in Section 7.2.5.

$$\bullet \qquad -Y = \frac{\sigma^{*2}}{2E(1-D)^2} = \frac{1}{2E}\tilde{\sigma}^{*2}.$$

This relation justifies the elastic energy density release rate criterion by giving a thermodynamic meaning to the damage equivalent stress: the variables $-Y$ and $\tilde{\sigma}^*$ are equivalent variables.

Dissipation potential

Under the hypothesis of decoupling between mechanical and thermal dissipations, the second principle of thermodynamics requires the mechanical dissipation to be positive:

$$\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}^p - Y\dot{D} - A_k\dot{V}_k \geqslant 0.$$

The phenomenon of plastic flow can occur without damage; similarly the damage phenomenon can occur without noticeable macroscopic flow; we must therefore have separately:

$$\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}^p - A_k\dot{V}_k \geqslant 0 \qquad -Y\dot{D} \geqslant 0$$

$-Y$ being a positive definite quadratic form, the second principle requires that $\dot{D} \geqslant 0$. The damage defined in this way can therefore only increase or remain constant.

In accordance with the phenomenological thermodynamic method presented in Chapter 2, the damage growth law can be derived from a dissipation potential whose existence is postulated:

$$\varphi(\dot{\boldsymbol{\varepsilon}}^p, \dot{V}_k, \dot{D}, \vec{q}; \boldsymbol{\varepsilon}^e, T, V_k, D).$$

This convex function with a scalar value is a function of all the flux variables, with observable and internal variables acting as parameters.

Using the Legendre–Fenchel transformation, we may construct another equivalent dual potential so as to be able to express $\dot{D}$ as a function of Y rather than the opposite

$$\varphi^*(\boldsymbol{\sigma}, A_k, Y, \vec{g}; \boldsymbol{\varepsilon}^e, T, V_k, D).$$

The generalized normality law that results from this is expressed by:

$$\dot{D} = -\partial\varphi^*/\partial Y.$$

Using the hypothesis of thermomechanical decoupling φ^* can be expressed

as a sum of two functions, one which depends only on $\vec{g} = \overrightarrow{\text{grad}}\ T$, and the other which depends on all the other variables, so that $\dot{D}$ does not depend on $\vec{g}$.

The balance of the dissipated power can be illustrated by a tensile stress cycle, shown schematically in Fig. 7.32, in which the increase in damage is assumed to occur at a constant maximum stress. $0A$ corresponds to the plastic flow during loading; AB represents the plastic flow accompanying the damage process; BC represents the increase in elastic strain produced by the increase in damage. The curve $0A'B'$ represents the evolution of the hardening variable A_k related to the plastic flow $\delta\varepsilon_p$. The total energy dissipated during the cycle is the area $0ABCD0$. The energy dissipated plastically corresponding to the integration of $\sigma d\varepsilon_p$ is the area $0ABD0$; from this we subtract (1), the energy blocked in dislocations, the integral of $A_k d V_k$, and obtain the energy dissipated as heat during the flow. The energy dissipated by the process of decohesion (eventually as heat) corresponds to the integral of $\Phi_D = -Y\dot{D}$. This is represented by (3).

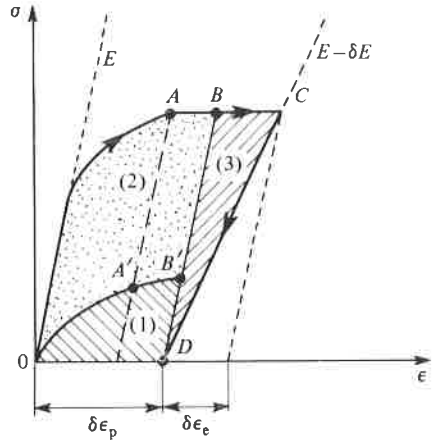
In effect, we have:

$$-Y\,dD = \frac{1}{2}\frac{dw_e}{dD}\bigg|_\sigma dD = \frac{1}{2}\sigma\frac{d\varepsilon_e}{dD}\bigg|_\sigma dD = \frac{1}{2}\sigma\,d\varepsilon_e.$$

Fracture criterion for the volume element

From what we have just said, an energetic definition of the criterion of macrocrack initiation is justified. The variable $-Y$, the elastic energy

Fig. 7.32. Balance of dissipated energies during plastic flow and damage.



release rate, represents the elastic energy of the equivalent virgin material (with ΔT taken here as zero):

$$-Y = \tfrac{1}{2}\mathbf{a}:\boldsymbol{\varepsilon}^e:\boldsymbol{\varepsilon}^e.$$

The criterion consists in postulating that the initiation of a macroscopic crack corresponds to a critical value of this energy, intrinsic to every material:

$$-Y = |Y| = Y_c \leftrightarrow \text{fracture}.$$

This critical energy is identified with the decohesion energy of the material which may be evaluated by performing a uniaxial tension test

$$-Y = \tfrac{1}{2}E\varepsilon_e^2$$

and by replacing ε_e by its expression taken from the elasticity law of the damaged material:

$$\varepsilon_e = \frac{\sigma}{(1-D)E}$$

$$|Y| = \frac{1}{2}\frac{\sigma^2}{(1-D)^2E}$$

and, for the fracture conditions:

$$\left.\begin{aligned} |Y| &= Y_c \\ \sigma &= \sigma_u \\ D &= D_c \end{aligned}\right\} \qquad Y_c = \frac{1}{2}\frac{\sigma_u^2}{E(1-D_c)^2}$$

or

$$D_c = 1 - \sigma_u/(2EY_c)^{1/2},$$

a criterion which is identified with the critical damage at failure calculated in Section 7.2.2:

$$D_c = 1 - \sigma_u/\tilde{\sigma}_u, \qquad \tilde{\sigma}_u = (2EY_c)^{1/2}.$$

7.3.3 *A nonisotropic damage theory*

In order to simplify the notation, we limit ourselves here to the isothermal case, $\Delta T = 0$. The theory consists in using the damage tensor $\mathbf{D}$, introduced in Section 7.3.1, conjointly with a scalar measure D which will be taken as the trace of the tensor $\mathbf{D}$. These two variables furnish respectively the orientation of defects and their density. The anisotropy of the growth law is

assumed to depend only on the material and on the principal stress directions.

Thermodynamic potential

The specific free energy depends on the tensor $\mathbf{D}$. As usual we assume decoupling between the hardening effects and the law of elasticity:

$$\Psi = \Psi_e(\boldsymbol{\varepsilon}^e, T, \mathbf{D}) + \Psi_p(T, V_k)$$

where Ψ_e depends linearly on the damage:

$$\rho\Psi_e = \tfrac{1}{2}(\mathbf{1} - \mathbf{D}):\mathbf{a}:\boldsymbol{\varepsilon}^e:\boldsymbol{\varepsilon}^e.$$

The elasticity law is:

$$\boldsymbol{\sigma} = \rho\,\partial\Psi/\partial\boldsymbol{\varepsilon}^e = (\mathbf{1} - \mathbf{D}):\mathbf{a}:\boldsymbol{\varepsilon}^e = \tilde{\mathbf{a}}:\boldsymbol{\varepsilon}^e$$

and the effective stress is:

$$\tilde{\boldsymbol{\sigma}} = (\mathbf{1} - \mathbf{D})^{-1}:\boldsymbol{\sigma}.$$

The variable associated with damage is expressed by:

$$\mathbf{Y} = \rho\,\partial\Psi_e/\partial\mathbf{D} = -\tfrac{1}{2}(\mathbf{a}:\boldsymbol{\varepsilon}^e)\boldsymbol{\varepsilon}^e \qquad \text{or} \qquad Y_{ijkl} = -\tfrac{1}{2}a_{ijrs}\varepsilon^e_{rs}\varepsilon^e_{kl}.$$

If we introduce the trace of $\mathbf{D}$ as a scalar measure of damage:

$$D = c\,\mathrm{Tr}(\mathbf{D}) = c\mathbf{D}::\mathbf{1}$$

where the symbol : : denotes the tensorial product contracted on four indices (the coefficient c will be specified later), we may consider that Ψ_e also depends on D and obtain:

$$Y = \mathrm{Tr}(\mathbf{Y}) = -\tfrac{1}{2}\mathbf{a}:\boldsymbol{\varepsilon}^e:\boldsymbol{\varepsilon}^e = \rho\frac{\partial\Psi_e}{\partial\mathbf{D}}::\mathbf{1} = \rho c\frac{\partial\Psi_e}{\partial D}.$$

We can still write these variables as functions of effective stress:

$$-\mathbf{Y} = \tfrac{1}{2}\tilde{\boldsymbol{\sigma}}\boldsymbol{\varepsilon}^e = \tfrac{1}{2}\tilde{\boldsymbol{\sigma}}\mathbf{a}^{-1}:\tilde{\boldsymbol{\sigma}}$$

$$-Y = \tfrac{1}{2}\tilde{\boldsymbol{\sigma}}:\boldsymbol{\varepsilon}^e = \tfrac{1}{2}\mathbf{a}^{-1}:\tilde{\boldsymbol{\sigma}}:\tilde{\boldsymbol{\sigma}}.$$

Dissipation potential

Using the same assumptions as before, we consider a dissipation potential such as:

$$\varphi^*(\boldsymbol{\sigma}, A_k, \mathbf{Y}; \boldsymbol{\varepsilon}^e, T, V_k, \mathbf{D}).$$

The generalized normality law resulting from this is expressed by:

$$\dot{\boldsymbol{\varepsilon}}^p = \partial\varphi^*/\partial\boldsymbol{\sigma}, \quad \dot{V}_k = -\partial\varphi^*/\partial A_k, \quad \dot{\mathbf{D}} = -\partial\varphi^*/\partial\mathbf{Y}.$$

For simplification, we assume that the dissipations due to the deformation process (and hardening) and the damage process are independent of each other. The potential φ^* can then be decomposed as:

$$\varphi^* = \varphi^*_p(\boldsymbol{\sigma}, A_k; V_k, T) + \varphi^*_D(\mathbf{Y}; \boldsymbol{\varepsilon}^e, T, \mathbf{D}).$$

Again, for simplification, the damage potential is assumed to depend linearly on $\mathbf{Y}$:

$$\varphi^*_D = -F(\boldsymbol{\varepsilon}^e, T, D)\mathbf{Q}::\mathbf{Y}$$

where $\mathbf{Q}$ is a fourth order tensor defining the anisotropy of the damage growth law. This anisotropy expressed with reference to the principal axes of the effective stress $\tilde{\boldsymbol{\sigma}}$, is thus taken as constant, the whole nonlinearity of the damage process being contained in the function F. The normality law gives:

$$\dot{\mathbf{D}} = \mathbf{Q}F(\boldsymbol{\varepsilon}^e, T, \mathbf{D}).$$

The growth of the scalar D is obtained by assuming that $D = c\,\mathrm{Tr}(\mathbf{D})$ and $Y = \mathrm{Tr}(\mathbf{Y})$, with $c = 1/\mathrm{Tr}(\mathbf{Q})$. We find:

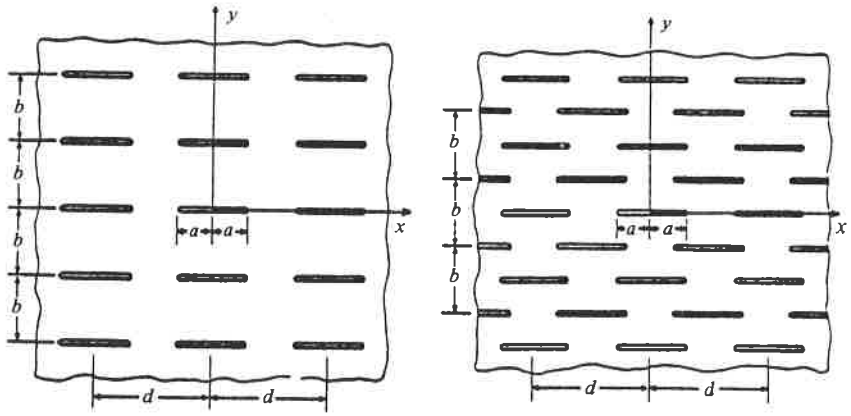
$$\dot{D} = c\,\mathrm{Tr}(\dot{\mathbf{D}}) = -c\,\mathrm{Tr}(\partial\varphi^*/\partial\mathbf{Y}) = -c(\partial\varphi^*/\partial Y) = F(\boldsymbol{\varepsilon}^e, T, \mathbf{D}).$$

These equations can also be written as functions of the effective stress with the help of the elasticity law. We will later use a law of the form:

$$\dot{D} = G(\tilde{\boldsymbol{\sigma}}, T, D)$$

$$\dot{\mathbf{D}} = \mathbf{Q}\dot{D}.$$

Fig. 7.33. Examples of microcracks in two-dimensional media calculated by the homogenization technique: (*a*) rectangular arrangement; (*b*) staggered arrangement.



Definition of the nonisotropic relation

The inconvenience of a general theory is often linked to the difficulty of the identification of the material parameters. In order to limit the number of degrees of freedom as far as possible, we may, for the definition of the operator $\mathbf{Q}$, use elementary linear elasticity solutions obtained for particular arrangements of defects. Thus the case of all parallel cracks (Fig. 7.33) may be homogenized and interpreted by a damage tensor $\mathbf{D} = \mathbf{1} - \tilde{\mathbf{a}}:\mathbf{a}^{-1}$ with the matrix representation:

$$D = \begin{bmatrix} D_1 & 0 & 0 & & & \\ \frac{\nu}{1-\nu} D_1 & 0 & 0 & & & \\ \frac{\nu}{1-\nu} D_1 & 0 & 0 & & & \\ & & & 0 & & \\ & & & & D_5 & \\ & & & & & D_5 \end{bmatrix}$$

The symmetry of the stress tensor limits it to a 6×6 matrix. Direction 1 is perpendicular to the plane of the cracks. The effective stress tensor is obtained as in Section 7.3.1:

$$\tilde{\boldsymbol{\sigma}} = (\mathbf{1} - \mathbf{D})^{-1}:\boldsymbol{\sigma}$$

which written in a matrix form is:

$$\begin{Bmatrix} \tilde{\sigma}_{11} \\ \tilde{\sigma}_{22} \\ \tilde{\sigma}_{33} \\ \tilde{\sigma}_{23} \\ \tilde{\sigma}_{31} \\ \tilde{\sigma}_{12} \end{Bmatrix} = \begin{bmatrix} \frac{1}{1-D_1} & 0 & 0 & & & \\ \frac{\nu}{1-\nu}\frac{D_1}{1-D_1} & 1 & 0 & & & \\ \frac{\nu}{1-\nu}\frac{D_1}{1-D_1} & 0 & 1 & & & \\ & & & 1 & & \\ & & & & \frac{1}{1-D_5} & \\ & & & & & \frac{1}{1-D_5} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{Bmatrix}.$$

Study of this particular case shows that it can be described by two

damage variables D_1 and D_5. If we admit a law of defect growth perpendicular to the maximum principal stress, it is sufficient to make the following choice: $\mathbf{Q} = \boldsymbol{\Gamma}$, with:

$$\boldsymbol{\Gamma} = \begin{bmatrix} 1 & 0 & 0 & & & \\ \frac{\nu}{1-\nu} & 0 & 0 & & & \\ \frac{\nu}{1-\nu} & 0 & 0 & & & \\ & & & 0 & & \\ & & & & \xi & \\ & & & & & \xi \end{bmatrix}.$$

In fact, each material is sensitive to a greater or lesser degree to this anisotropy effect. A linear combination with the isotropic case then offers the simplest way of describing possible cases with only two material-dependent coefficients (plus Poisson's ratio):

$$\mathbf{Q} = (1-\gamma)\boldsymbol{\Gamma} + \gamma\mathbf{1}.$$

The two characteristic coefficients in the above relation are ξ in the operator $\boldsymbol{\Gamma}$, and the coefficient γ which can be used to represent all the degrees of anisotropy: $\gamma = 1$ if the material is isotropic, and $\gamma = 0$ in the completely anisotropic case.

It is possible to verify that the results obtained for the isotropic damage law remain valid; the second principle requires that:

$$\dot{D} \geqslant 0.$$

The power dissipated during the process of damage is written as:

$$\Phi_D = -\mathbf{Y}::\dot{\mathbf{D}} = -\mathbf{Y}::\mathbf{Q}\dot{D} = -[(1-\gamma)\mathbf{Y}::\boldsymbol{\Gamma} + \gamma Y]\dot{D}.$$

The second term in the square brackets is always negative since $-Y$ is a positive definite form in thermoelastic strains. The quantity $\mathbf{Y}::\boldsymbol{\Gamma}$ is also negative as can be checked by replacing $\mathbf{Y}$ with its expression as a function of strains.

Anisotropy effectively develops only when the loading is nonproportional. In this case the anisotropy tensor depends on the principal direction of $\tilde{\boldsymbol{\sigma}}$. This is schematically illustrated in Fig. 7.34 for a simple example for which $\gamma = 0$. Under the action of a first tension load (I), the damage has grown and the defects are arranged in planes perpendicular to the

maximum principal stress. We have:

$$\mathbf{D}_1 = \mathbf{Q}D_1.$$

If an equibiaxial tension is now applied ($\sigma_1 = \sigma_2$, $\sigma_3 = 0$), the maximum principal effective stress remains such that $\tilde{\sigma}_1 > \tilde{\sigma}_2$. The defects continue to grow in the same plane (II). If a tension is now applied in direction 2 ($\sigma_1 = \sigma_3 = 0$), $\tilde{\sigma}_2$ becomes the maximum principal effective stress and further growth of defects takes place in planes perpendicular to it. We will therefore write (III):

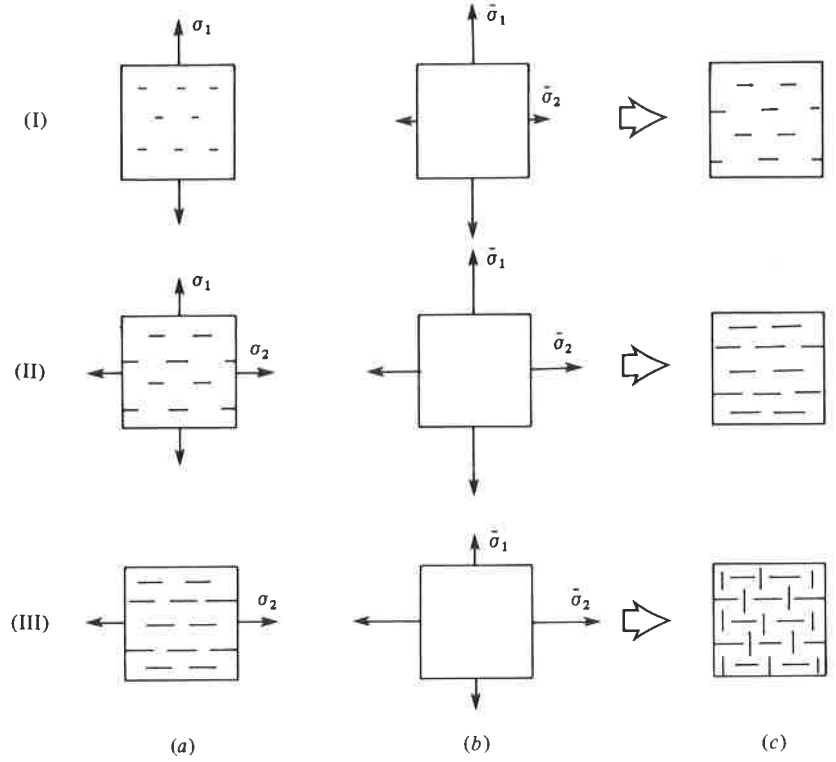
$$\mathbf{D}_2 = \mathbf{D}_1 + \mathbf{Q}_2\mathbf{D}_2 = \mathbf{Q}D_1 + \mathbf{V}{:}\mathbf{Q}{:}\mathbf{V}^{\mathrm{T}}D_2$$

with

$$\mathbf{V} = \mathbf{R} \otimes \mathbf{R}$$

where $\mathbf{R}$ denotes the rotation operator between axes 1 and 2. After a certain time there will be superposition of the two perpendicular systems of defects; when $D_1 = D_2$ the anisotropy will be much less, directions 1 and 2 exhibiting the same behaviour with only small variations for the intermediate directions. This illustrates quite well the point that anisotropy is a characteristic of the growth law and not of the current state of damage. Note that the choice of the expression $\dot{\mathbf{D}} = \mathbf{Q}\mathbf{D}$ implies that in simple tension (direction 1) the first component of $\mathbf{D}$ is always equal to D.

Fig. 7.34. Schematic representation of anisotropic damage under complex loading: (*a*) applied loading; (*b*) corresponding effective stresses; (*c*) resulting damage from this loading.



7.4 Particular models

The isotropic models described below assume that there is no coupling between damage and other dissipative phenomena. The evolution law of the variable D does not depend on other internal variables; this prevents taking into account some phenomena such as the increase in the fatigue lifetime produced by a prestrain. In order to account for this particular phenomenon, it is sufficient to introduce hardening variables in the damage law. This is possible with the formalism described in Section 7.3, but it is a complex process!

7.4.1 *Ductile plastic damage*

Ductile plastic damage accompanies large plastic deformations and, like plasticity, it is rate independent and does not involve time explicitly. If, on the other hand, we consider only isotropic damage and hardening, the only internal variable which appears, besides damage, is the accumulated plastic strain defined (see Chapter 5) by:

$$\dot{p} = (\tfrac{2}{3}\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}{:}\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}})^{1/2}.$$

The two damage models described above are derived from a dissipation potential formulated with

$$f(\boldsymbol{\sigma}, R) \quad \text{and} \quad \varphi_{\mathrm{D}}^{*}(Y; T, D).$$

Coupling between plasticity and damage is studied in detail in Section 7.5.2. Here, let us use the loading function in the form:

$$f = f(\boldsymbol{\sigma}, R, D) = \sigma_{\mathrm{eq}}/(1 - D) - R - k \leqslant 0.$$

The damage potential is chosen as a power function of $-Y$:

$$\varphi_{\mathrm{D}}^{*} = \frac{S_0}{s_0 + 1}\frac{1}{1 - D}\left(-\frac{Y}{S_0}\right)^{s_0 + 1}$$

where s_0 and S_0 are characteristic coefficients for each material and depend on temperature. By applying the general method we get:

$$\dot{\varepsilon}^p = \dot{\lambda}\frac{\partial f}{\partial \boldsymbol{\sigma}} = \frac{3}{2}\frac{\dot{\lambda}}{1-D}\frac{\boldsymbol{\sigma}'}{\sigma_{eq}}, \quad \dot{p} = \left(\tfrac{2}{3}\dot{\boldsymbol{\varepsilon}}^p : \dot{\boldsymbol{\varepsilon}}^p\right)^{1/2} = \frac{\dot{\lambda}}{1-D},$$

$$\dot{D} = -\dot{\lambda}\frac{\partial \varphi_D^*}{\partial Y} = \frac{\dot{\lambda}}{1-D}\left(-\frac{Y}{S_0}\right)^{s_0},$$

$$\dot{D} = \left(-\frac{Y}{S_0}\right)^{s_0}\dot{p}.$$

Uniaxial stress model

The associated variable Y and the equivalent effective stress $\tilde{\sigma}^*$ are equivalent variables, as has been shown in Section 7.3.2:

$$-Y = (1/2E)\tilde{\boldsymbol{\sigma}}^{*2}.$$

Moreover, with a monotonically increasing uniaxial load, which is the assumption for the validity of this model, $\tilde{\sigma}^*$ is identified with the effective stress $\tilde{\sigma}$ and $\dot{p}$ with the magnitude of the plastic strain rate $|\dot{\varepsilon}_p|$. We may therefore write:

$$\dot{D} = (\tilde{\sigma}^2/2ES_0)^{s_0}|\dot{\varepsilon}_p|.$$

In order to obtain the model suggested by phenomenological considerations, it is necessary to replace $\dot{\varepsilon}_p$ by its expression in the hardening law introduced in Section 5.4.2 and written here in terms of the effective stress and with $\sigma_Y = 0$:

$$\varepsilon_p = \left(\frac{\tilde{\sigma}}{K}\right)^M \quad \text{or} \quad \dot{\varepsilon}_p = \frac{M}{K}\left(\frac{\tilde{\sigma}}{K}\right)^{M-1}\dot{\tilde{\sigma}}.$$

After an obvious change in the notation of the constants:

$$\dot{D} = \left(\frac{\tilde{\sigma}}{S_1}\right)^{s_1}\frac{\dot{\tilde{\sigma}}}{S_1} \quad \text{or} \quad \mathrm{d}D = \left(\frac{\tilde{\sigma}}{S_1}\right)^{s_1}\frac{\mathrm{d}\tilde{\sigma}}{S_1}.$$

A closely related form, closer to the experimental results because it introduces a damage threshold σ_D, is the following:

- $$\mathrm{d}D = \left\langle\frac{\sigma-\sigma_D}{(1-D)S}\right\rangle^s\frac{\mathrm{d}\sigma}{S}.$$

The corresponding potential φ^* does not have an explicit form. The three

characteristic coefficients of ductile plastic damage, which are possibly dependent on temperature, are determined through identification with the damage measurements conducted, for example, with the help of the method of elastic modulus variation described in Section 7.2.3.

Integration of this differential equation gives the value of the damage for any value of stress. With the initial condition $\sigma = \sigma_D \rightarrow D = 0$, we find:

$$D = 1 - \left(1 - \left\langle\frac{\sigma-\sigma_D}{S}\right\rangle^{s+1}\right)^{1/s+1}.$$

The final ductile fracture occurs when:

$$D = D_c \quad \text{or} \quad \sigma = \sigma_R.$$

Identification of the characteristic coefficients of the ductile damage can be performed by measuring variations of the elasticity modulus during the hardening tests. Two examples are:

Copper Cu/Al, $T = 20\,^\circ\mathrm{C}$: $\sigma_D = 330\,\mathrm{MPa}$, $S = 445\,\mathrm{MPa}$, $s = 0.70$, $D_c = 0.85$;
AU4G – T_4 alloy, $T = 20\,^\circ\mathrm{C}$: $\sigma_D = 400\,\mathrm{MPa}$, $S = 580\,\mathrm{MPa}$, $s = 0.58$, $D_c = 0.23$.

Multiaxial strain model

Starting from the same dissipation potential, we can develop a multiaxial model within the framework of the hypotheses of isotropic damage and isotropic hardening which is valid for any loading:

$$-\frac{\partial \varphi_D^*}{\partial Y} = \left(\frac{-Y}{S_0}\right)^{s_0}\dot{p}.$$

We may replace $-Y$ by its expression in terms of equivalent von Mises stress σ_{eq} and the hydrostatic stress σ_H (Sections 7.2.5 and 7.3.2):

$$-Y = \frac{\tilde{\sigma}^{*2}}{2E} = \frac{\sigma_{eq}^2}{2E(1-D)^2}\left[\frac{2}{3}(1+\nu) + 3(1-2\nu)\left(\frac{\sigma_H}{\sigma_{eq}}\right)^2\right].$$

The ductile damage occurs only when the strain-hardening is saturated ($R = R_\infty = \text{constant}$) and the material is then considered perfectly plastic. The expression of the plastic criterion $\sigma_{eq}/(1-D) - R - k = 0$ shows that

$$\frac{\sigma_{eq}}{1-D} = \tilde{\sigma}_{eq} = \text{constant} = K.$$

Then

$$\dot{D} = (K^2/2ES_0)^{s_0}[\tfrac{2}{3}(1+\nu) + 3(1-2\nu)(\sigma_H/\sigma_{eq})^2]^{s_0}\dot{p}.$$

Evolution of the damage as a function of p is obtained by integrating this differential equation with the initial condition $p \leqslant p_D \rightarrow D = 0$, where p_D is the damage threshold in terms of strain.

If we restrict ourselves to radial loading for which the triaxiality ratio σ_H/σ_{eq} is constant, we obtain:

$$D = (K^2/2ES_0)^{s_0}[\tfrac{2}{3}(1+\nu) + 3(1-2\nu)(\sigma_H/\sigma_{eq})^2]^{s_0}\langle p - p_D\rangle.$$

We can simplify this expression by introducing the condition of fracture

$$p = p_R \rightarrow D = D_c$$

where p_R is the accumulated strain at fracture. Since D_c is a material constant, p_R depends on the triaxiality ratio σ_H/σ_{eq}

$$D_c = (K^2/2ES_0)^{s_0}[\tfrac{2}{3}(1+\nu) + 3(1-2\nu)(\sigma_H/\sigma_{eq})^2]^{s_0}\langle p_R - p_D\rangle.$$

The expression for D then may be written as:

$$D = D_c\left\langle \frac{p - p_0}{p_R - p_D}\right\rangle.$$

This expression generalizes to the multiaxial empirical model of Section 7.2.4 with the difference that p_R and p_D are functions of the triaxiality ratio. From the expression for D_c we get:

$$p_R - p_D = D_c\left(\frac{K^2}{2ES_0}\left[\frac{2}{3}(1+\nu) + 3(1-2\nu)\left(\frac{\sigma_H}{\sigma_{eq}}\right)^2\right]\right)^{-s_0}.$$

Assuming that the stress triaxiality affects the damage threshold p_D and the fracture strain p_R in the same manner, the ratio p_D/p_R is a constant for each material equal to its uniaxial value $\varepsilon_D/\varepsilon_R$ where no distinction is made between total strains and plastic strains. The model involves only two constants D_c and $\varepsilon_D/\varepsilon_R$, and a function p_R which can also be expressed as a function of the uniaxial fracture strain ε_R corresponding to a triaxiality ratio $\sigma_H/\sigma_{eq} = \frac{1}{3}$:

$$p_R\left(\frac{1}{3}\right) = \varepsilon_R = \left(\frac{2ES_0}{K^2}\right)^{s_0}\frac{D_c}{1 - \varepsilon_D/\varepsilon_R}$$

$$p_R\left(\frac{\sigma_H}{\sigma_{eq}}\right) = \varepsilon_R\left[\frac{2}{3}(1+\nu) + 3(1-2\nu)\left(\frac{\sigma_H}{\sigma_{eq}}\right)^2\right]^{-s_0}.$$

If the numerical values given by this expression are compared with those obtained by applying the cavity growth models of MacClintock, Rice and Tracey, we find that $s_0 = 1$ constitutes the best identification, also in agreement with the uniaxial stress model.

In summary, the multiaxial ductile plastic damage strain model is the following:

In differential form:

$$\dot{D} = \frac{D_c}{\varepsilon_R - \varepsilon_D}\left[\frac{2}{3}(1+\nu) + 3(1-2\nu)\left(\frac{\sigma_H}{\sigma_{eq}}\right)^2\right]\dot{p}.$$

In integrated form:

$$D \approx \frac{D_c}{\varepsilon_R - \varepsilon_D}\left\{p\left[\frac{2}{3}(1+\nu) + 3(1-2\nu)\left(\frac{\sigma_H}{\sigma_{eq}}\right)^2\right] - \varepsilon_D\right\}.$$

The three constants, intrinsic to each material, are determined by uniaxial tests and are those given in Table 7.1.

7.4.2 *Creep Damage*

The creep damage law commonly used is that described in Section 7.2.4. The relations developed below correspond to generalizations for complex loadings:

variable stress,
multiaxial proportional loading,
multiaxial nonproportional loading.

Nonlinear accumulation model

A form which is an improvement over Kachanov's law is obtained by introducing an extra coefficient k as suggested by Rabotnov:

$$\dot{D} = (\sigma/A)^r(1-D)^{-k}.$$

It always includes the rule of linear accumulation, and in general k is larger than r: the damage rate is influenced more strongly by the degree of damage than the global mechanical behaviour is. Integration of this law, for a constant stress, gives:

$$D = 1 - \left(1 - \frac{t}{t_c}\right)^{1/(k+1)} \quad \text{with} \quad t_c = \frac{1}{k+1}\left(\frac{\sigma}{A}\right)^{-r}.$$

Table 7.2 gives the values of coefficients r, k, $A = A_0[(k+1)/(r+1)]^{1/r}$. Fig. 7.7 shows the correlation with the damage measurements made on two materials. It also shows that the curves can depend on the stress level (the case of the IN 100 alloy) which is consistent with the existence of nonlinear accumulation effects and leads to the following generalization:

- $$\dot{D} = (\sigma/A)^r (1-D)^{-k(\sigma)}.$$

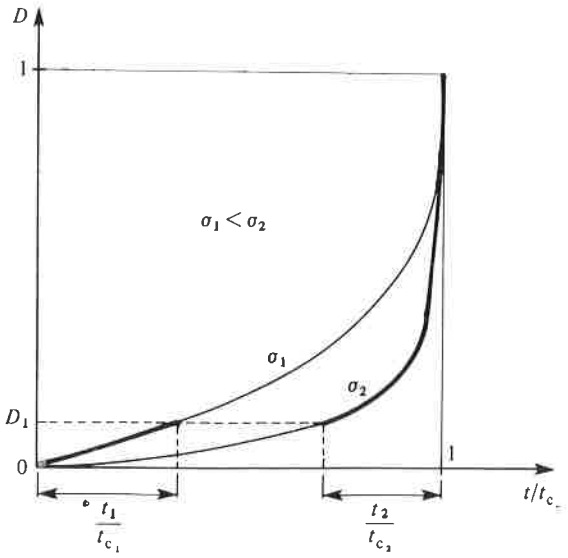
Since the exponent k depends on the stress level, the two variables σ and D are no longer separable. No theoretical justification exists for the introduction of the function $k(\sigma)$ except that it provides the simplest means of representing nonlinear accumulation.

Discrete values of $k(\sigma)$ are determined by identification with the damage growth measured during creep under constant stress. In fact, the evolution of D as a function of t/t_c, obtained by integration from 0 to D, depends on σ:

$$D = 1 - \left(1 - \frac{t}{t_c}\right)^{1/[k(\sigma)+1]}, \quad t_c = \frac{1}{k(\sigma)+1}\left(\frac{\sigma}{A}\right)^{-r}.$$

The function $k(\sigma)$ is an increasing one, because damage growth is faster for lower stress (Figs. 7.7 and 7.35). This is confirmed by the results of tests at

Fig. 7.35. Schematic diagram illustrating nonlinear accumulation in creep tests at two levels of stress.



two levels of stress. The equation represents the effects of nonlinear accumulation quite well; by integrating from 0 to D_1, at the first level of stress, and then from D_1 to 1 at the second level, we find (Fig. 7.36) that:

$$\frac{t_2}{t_{c_2}} = \left(1 - \frac{t_1}{t_{c_1}}\right)^{[k(\sigma_2)+1]/[k(\sigma_1)+1]}.$$

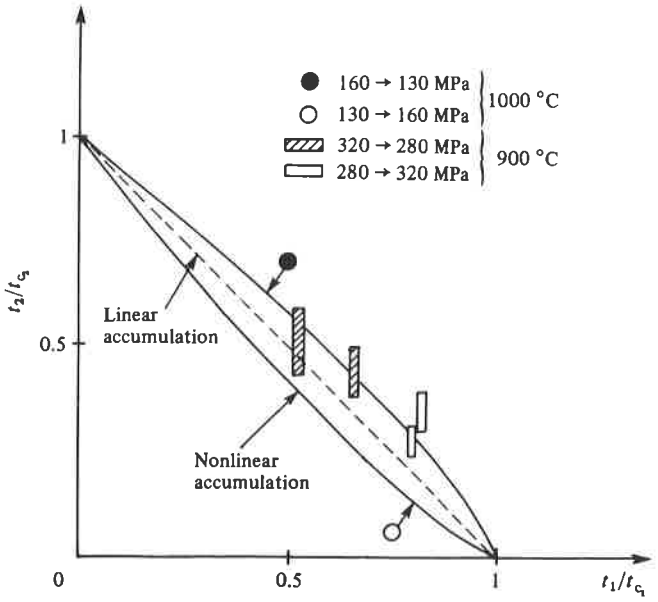
The above discussion applies to creep under tension; few studies have been made regarding creep damage in compression: fracture is never observed as it is preceded by instability of geometric origin (buckling). However, there is nothing to prove that this damage does not exist or that compression does not influence damage in further tension.

Multiaxial isotropic model

The domain of validity of this model is restricted to loading which can be considered to be almost proportional. If such is not the case, it may become necessary to consider the effects of damage anisotropy such as those mentioned in the next subsection.

The writing of a multiaxial law therefore requires only the use of an equivalent stress in the sense of creep damage. Depending on the material it

Fig. 7.36. Example of prediction by the damage model for the IN 100 refractory alloy.



will be possible to use one of the criteria described in Section 7.2.5 and consequently to describe isochronous surfaces of failure in stress space (surfaces linking the states of stress which result in creep rupture in the same time):

the elastic energy density release rate criterion:

$$\sigma^* = \sigma_{eq}[\tfrac{2}{3}(1+\nu) + 3(1-2\nu)(\sigma_H/\sigma_{eq})^2]^{1/2},$$

the shear–dilatation sensitivity coefficient criterion:

$$\sigma_\beta^* = (1-\beta)\sigma_{eq} + 3\beta\sigma_H,$$

the more general criterion involving a combination of the three basic invariants:

$$\chi(\boldsymbol{\sigma}) = \alpha J_0(\boldsymbol{\sigma}) + \beta J_1(\boldsymbol{\sigma}) + (1-\alpha-\beta)J_2(\boldsymbol{\sigma})$$

where $J_1(\boldsymbol{\sigma}) = 3\sigma_H$, $J_2(\boldsymbol{\sigma}) = \sigma_{eq}$, and $J_0(\boldsymbol{\sigma})$ denotes the maximum principal stress.

In the isotropic case, the damage variable $\mathbf{D}$ is reduced to a scalar

$$\mathbf{D} = \mathbf{1}D$$

and the effective stress is expressed by:

$$\tilde{\boldsymbol{\sigma}} = \frac{\boldsymbol{\sigma}}{1-D}.$$

The multiaxial law of damage evolution is expressed as follows:

$$\dot{D} = \left\langle \frac{\chi(\tilde{\boldsymbol{\sigma}})}{A} \right\rangle^r (1-D)^{r-k}.$$

If $r = k$, we obtain Kachanov's law. For simple tension, we have again exactly the same expressions as those obtained in the previous section. If the exponent depends on the stress, the following expression can be chosen:

$$\dot{D} = \left\langle \frac{\chi(\tilde{\boldsymbol{\sigma}})}{A} \right\rangle^r (1-D)^{r-k\langle\chi(\boldsymbol{\sigma})\rangle}.$$

Note that this can also be written as:

$$\dot{D} = \left\langle \frac{\chi(\boldsymbol{\sigma})}{A} \right\rangle^r (1-D)^{-k\langle\chi(\boldsymbol{\sigma})\rangle} \quad \text{or} \quad \dot{D} = \left\langle \frac{\chi(\tilde{\boldsymbol{\sigma}})}{A} \right\rangle^{k\langle\chi(\boldsymbol{\sigma})\rangle} \left[\frac{\chi(\boldsymbol{\sigma})}{A} \right]^{r-k\langle\chi(\boldsymbol{\sigma})\rangle}.$$

As usual the symbol $\langle\chi\rangle$ denotes the positive part of χ. In fact $\chi(\boldsymbol{\sigma})$ may be negative for certain stress states depending on the values of the coefficients α and β. The damage rate is then zero, and there is no isochronous surface passing through these points.

Anisotropic damage model

In accordance with the theory introduced in Section 7.3.3, a simple way of incorporating damage anisotropy (already produced) in the damage law (in further evolution) consists in admitting a multiplicative decomposition of the damage rate tensor:

$$\dot{\mathbf{D}} = \mathbf{Q}\dot{D} = [(1-\gamma)\boldsymbol{\Gamma} + \gamma\mathbf{1}]\dot{D}.$$

The anisotropic part is contained in the tensor $\mathbf{Q}$. It plays a part, it should be recalled, only in cases of nonproportional loading. The effect of anisotropy is thus considered fixed in the current principal directions of the effective stress tensor $\tilde{\boldsymbol{\sigma}}$ (see Section 7.3.3). The nonlinearity of the damage evolution process is entirely contained in the equation governing D. We may choose an equation for this similar to that used in the isotropic case:

$$\dot{D} = \left\langle \frac{\chi^*(\tilde{\boldsymbol{\sigma}}, D)}{A} \right\rangle^{k\langle\chi(\boldsymbol{\sigma})\rangle} \left(\frac{\chi(\boldsymbol{\sigma})}{A} \right)^{r-k\langle\chi(\boldsymbol{\sigma})\rangle}$$

where the equivalent effective stress depends on D:

$$\chi^*(\tilde{\boldsymbol{\sigma}}, D) = \alpha J_0(\tilde{\boldsymbol{\sigma}}) + \frac{\beta}{1+2A} J_1(\tilde{\boldsymbol{\sigma}}) + \frac{1-\alpha-\beta}{1-A} J_2(\tilde{\boldsymbol{\sigma}})$$

with

$$A = \frac{\nu}{1-\nu}\frac{(1-\gamma)D}{1-\gamma D}.$$

This change in the expression of the equivalent effective stress is introduced in order to recover Kachanov's law in the simple tension case. Indeed, the first component of the tensor $\mathbf{D}$ is equal to D and we then have:

$$\tilde{\sigma}_1 = \frac{\sigma_1}{1-D},$$

$$\tilde{\sigma}_2 = \frac{\nu(1-\gamma)}{1-\nu}\frac{D}{1-\gamma D}\frac{\sigma_1}{1-D} = \tilde{\sigma}_3.$$

The multiaxial anisotropic model is therefore completely identified with the uniaxial model and the coefficients ξ and γ which fix the anisotropy (in addition to ν). In the isotropic case ($\gamma = 1$) we recover $\tilde{\sigma}_2 = \tilde{\sigma}_3 = 0$ and the expressions of the previous subsection.

This formulation therefore includes the two limiting cases (isotropic and purely anisotropic) and also the intermediate ones (by linear combination). It applies to any nonproportional loading and can be used to represent the effects of damage anisotropy on the damage law and on material behaviour (through the effective stress established in Section 7.3.1).

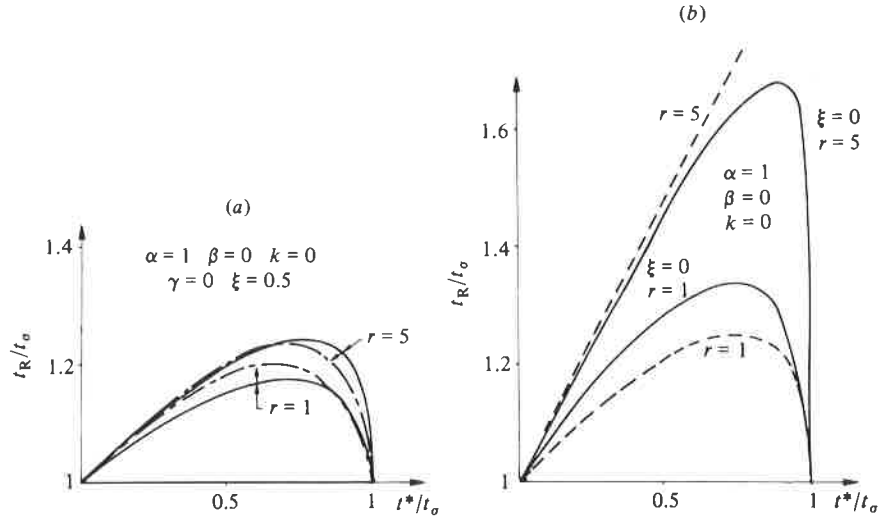
It has been applied to the case of creep under tension followed by creep under torsion. Fig. 7.37 shows a few examples of predictions of the total rupture time (in terms of time to failure under tension alone) as function of the time spent under tension. This is repeated for several values of the coefficients, and the results of Murakami's and Kachanov's theories are presented for comparison.

7.4.3 *Fatigue damage*

Definition of damage per cycle

The general theory of Section 7.3 uses time as a reference variable, and the increments of damages are written as functions of the increments of time, and of strain or stress. In the case of fatigue, we usually employ the concept of the loading cycle to evaluate the evolution of damage and to measure the fatigue life-time. The equations then depend on the load through globally

Fig. 7.37. Calculation of the total rupture time for creep under tension and then torsion; (*a*) $\xi = 0.5$; (*b*) $\xi = 0$. [Chain line – Murakami's theory, dashed line – Kachanov's theory.]



defined quantities over a cycle, such as amplitude, maximum value, mean value.

Therefore, we only have to imagine a formal integration, during each cycle, of an evolution law similar to that used for ductile plastic damage. The growth equation of fatigue damage is therefore taken in the form:

$$\delta D = f(\cdots)\delta N$$

in which the variables on which the function f depends are:

the state variables at the beginning of the cycle (temperature, damage D, and the hardening variables),
the maximum and the mean values over a cycle of the parameters which define the loading (stress, strain or plastic strain).

A particular problem occurs in representing the influence of the temperature when this changes during the cycle. This case will be studied later in this section.

Fatigue damage equations, which obey the rule of linear accumulation, have already been discussed as basic laws (Section 7.2.4). In contrast, the equation studied below allows us to describe the effects of nonlinear accumulation in the case of nonperiodic cyclic loads.

Uniaxial model

This model is derived from the macroscopic definition of damage with reference to two types of evaluation:

(1) damage evaluation in terms of the remaining life-time;
(2) damage evaluation by using the concept of effective stress.

We first examine the construction of a model based on the remaining life-time. Fatigue tests at two stress levels ($\Delta\sigma_1$ during N_1 cycles followed by $\Delta\sigma_2$ during N_2 cycles with $N_1 + N_2 = N_R$, the number of cycles to failure) demonstrate the nonlinearity of the damage accumulation and furnish an indirect measure of the current state of damage from knowledge of the remaining life-time N_2/N_{F_2}. These measurements are sufficient to prove that the damage evolution curves as functions of the life ratio N/N_F depend on the load level, as illustrated in Fig. 7.38(*a*): the remaining life at the second level N_2/N_F is then different from $1 - N_1/N_{F_1}$, the value given by the linear accumulation rule.

A simple way to introduce such effects in the damage growth equation consists in rendering the load and damage variables nonseparable. For

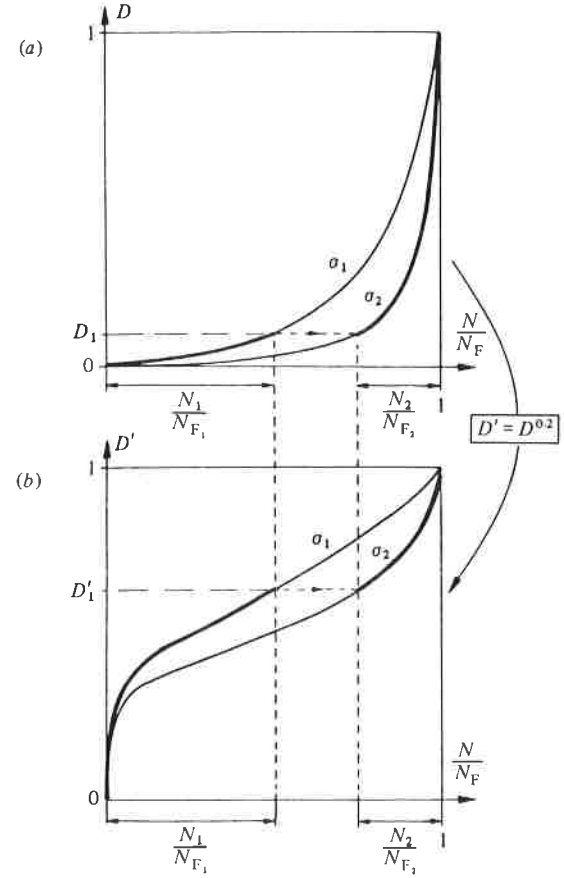
example, we may take:

$$\delta D = D^{\alpha(\sigma_{Max},\bar{\sigma})} \cdot \left(\frac{\sigma_{Max} - \bar{\sigma}}{C(\bar{\sigma})}\right)^{\beta} \delta N.$$

The exponent α depends on the loading $(\sigma_{Max}, \bar{\sigma})$, which results in nonseparability. Note that an equation similar to that used for creep damage is not suitable because of a reversal of the cumulative effect: in fatigue, the life-time at the second level, when the first load is higher, is shorter than predicted by the linear rule. Integrating from $D = 0$ to $D = 1$, the above equation gives the number of cycles to failure:

$$N_F = \frac{1}{1 - \alpha(\sigma_{Max}, \bar{\sigma})} \left(\frac{\sigma_{Max} - \bar{\sigma}}{C(\bar{\sigma})}\right)^{-\beta}.$$

Fig. 7.38. Nonlinear accumulation in fatigue and partial indetermination of the damage measured in terms of the remaining life-time.



D evolves as a function of N/N_F and, in a two-level test, the life-time at the second level is given by:

$$\frac{N_2}{N_{F_2}} = 1 - \left(\frac{N_1}{N_{F_1}}\right)^{\eta} \qquad \text{with} \qquad \eta = \frac{1 - \alpha_2}{1 - \alpha_1} = \frac{1 - \alpha(\sigma_{Max_2}, \bar{\sigma}_2)}{1 - \alpha(\sigma_{Max_1}, \bar{\sigma}_1)}.$$

This formulation permits a qualitative description of most of the results of tests conducted at different levels on a number of materials. The evaluation of D in terms of the remaining life is, however, not sufficient to totally fix its value at each instant. A look at Fig. 7.38 is enough to be convinced of this; a simple variable mapping in fact shows that the life-time can be identical for any loading while the damage value is different. In other words the concept of remaining life-time provides only a relative evaluation of damage.

In order to avoid this indetermination, the second type of evaluation is used. The concept of effective stress applied to fatigue provides an indirect measure as we have seen in Section 7.2.3. The measured evolutions are extremely nonlinear as can be seen in Fig. 7.6. With this concept, damage can really be measured only in the last part of the life-time, when microscopic initiations have already occurred (this is the phase of micropropagation of defects).

In order to combine this evaluation with the one corresponding to the 'remaining life-time', it is sufficient to make a change of variable by replacing D in the previous equation by:

$$1 - (1 - D)^{\beta + 1}.$$

The differential law can then be written as:

$$\bullet \qquad \delta D = [1 - (1 - D)^{\beta + 1}]^{\alpha(\sigma_{Max},\bar{\sigma})} \left[\frac{\sigma_{Max} - \bar{\sigma}}{M(\bar{\sigma})(1 - D)}\right]^{\beta} \delta N.$$

The form is more complex, but its properties are identical to the properties of the previous equation, except for the current value of damage. The number of cycles to failure, obtained by integrating the above is:

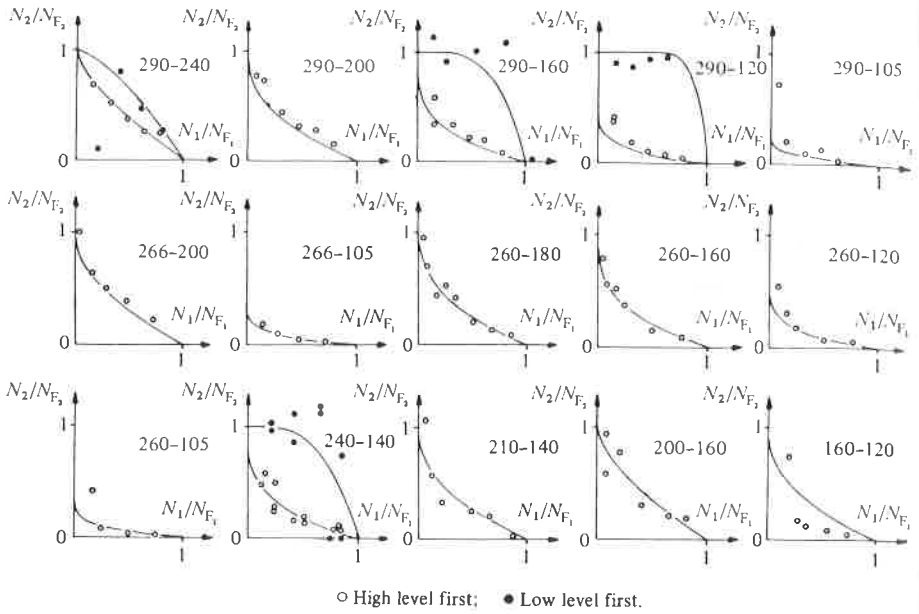
$$N_F = \frac{1}{(\beta + 1)[1 - \alpha(\sigma_{Max}, \bar{\sigma})]} \left(\frac{\sigma_{Max} - \bar{\sigma}}{M(\bar{\sigma})}\right)^{-\beta}$$

and we find that $M(\bar{\sigma}) = C(\bar{\sigma})(\beta + 1)^{1/\beta}$. The damage, expressed as a function of N/N_F is:

$$D = 1 - \left[1 - \left(\frac{N}{N_F}\right)^{1/(1-\alpha)}\right]^{1/(\beta+1)}.$$

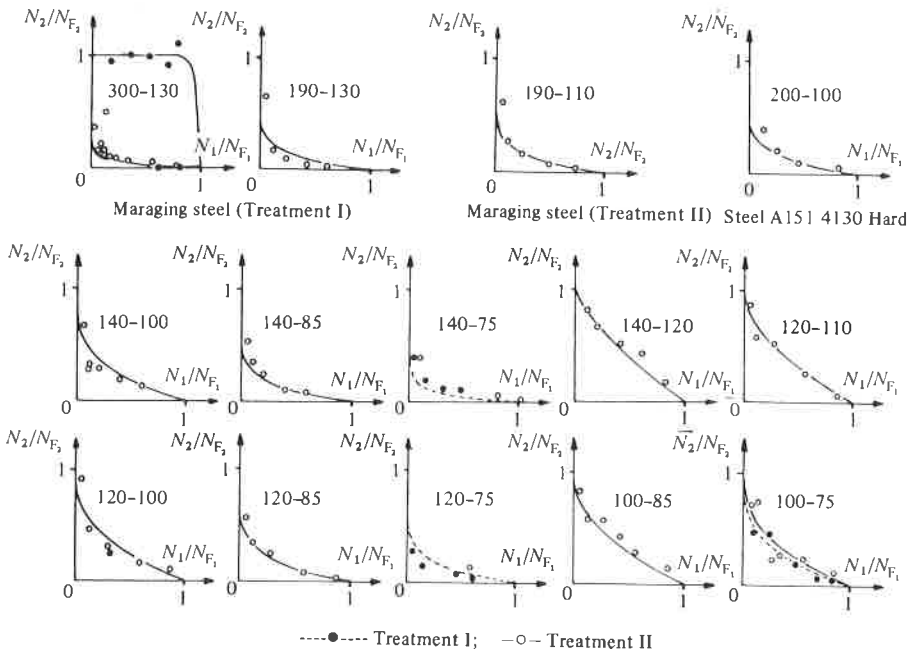
Fig. 7.39. Predictions of two-level fatigue tests on (*a*) maraging steel at room temperature (amplitudes in ksi: treatment I) (*b*) AISI 4130 soft steel and maraging steel (Treatment I & II) at room temperature and (*c*) IN 100 alloy at high temperatures.

(*a*)



○ High level first; ● Low level first.

(*b*)



Maraging steel (Treatment I) Maraging steel (Treatment II) Steel A151 4130 Hard

----●---- Treatment I; —○— Treatment II

The calculated curves of Fig. 7.6 show that this expression is in good agreement with experimental results. The functions α and M still have to be chosen to represent at the same time the fatigue limit and the static fracture (in one cycle) and the effects of nonlinear accumulation:

- $$\alpha(\sigma_{Max}, \bar{\sigma}) = 1 - a\left\langle \frac{\sigma_{Max} - \sigma_l(\bar{\sigma})}{\sigma_u - \sigma_{Max}} \right\rangle,$$
- $$\sigma_l(\bar{\sigma}) = \sigma_{l_0} + (1 - b\sigma_{l_0})\bar{\sigma},$$
- $$M(\bar{\sigma}) = M_0(1 - b\bar{\sigma})$$

where σ_{l_0} is the fatigue limit at zero mean stress and σ_u is the ultimate tensile stress. The choice of a linear relation to express the influence of $\bar{\sigma}$ is justified by experimental observations on the fatigue limit. Examples of correlation with Woehler's curves have been shown in Fig. 7.13 (the coefficients β and M_0 are determined from these experimental curves).

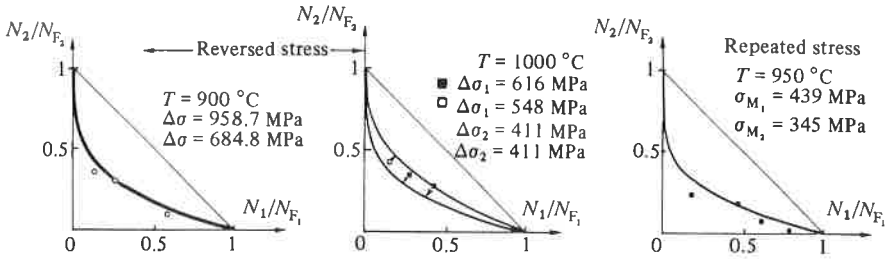
The coefficient a can be identified only from the damage measurements mentioned above (Fig. 7.6), but the choice of the value of a is important only when fatigue damage is combined with another type of damage (creep for example). In pure fatigue, all loadings can be described by an arbitrary value of a. The relation between B_0 and M_0 of Table 7.3 is: $M_0 = B_0[a(\beta + 1)]^{1/\beta}$.

The tests at two stress levels are described by the same relations as before. The remaining life-time at the second level is:

$$\frac{N_2}{N_{F_2}} = 1 - \left(\frac{N_1}{N_{F_1}}\right)^{\eta} \quad \text{with} \quad \eta = \frac{1 - \alpha_2}{1 - \alpha_1} = \frac{\sigma_{Max_2} - \sigma_l(\bar{\sigma}_2)}{\sigma_{Max_1} - \sigma_l(\bar{\sigma}_1)} \frac{\sigma_u - \sigma_{Max_1}}{\sigma_u - \sigma_{Max_2}}.$$

Experimental results on a number of materials show correct correlations at room temperature and at high temperatures (Fig. 7.39). These nonlinear cumulative effects are not very different from those described with the equations used by Subramanyan and by Manson. The same relation

(*c*)



governs N_2/N_{F_2}, with

$$\eta = \frac{\sigma_{Max_2} - \sigma_l}{\sigma_{Max_1} - \sigma_l} \quad \text{and} \quad \eta = \left(\frac{N_{F_1}}{N_{F_2}}\right)^{0.45}.$$

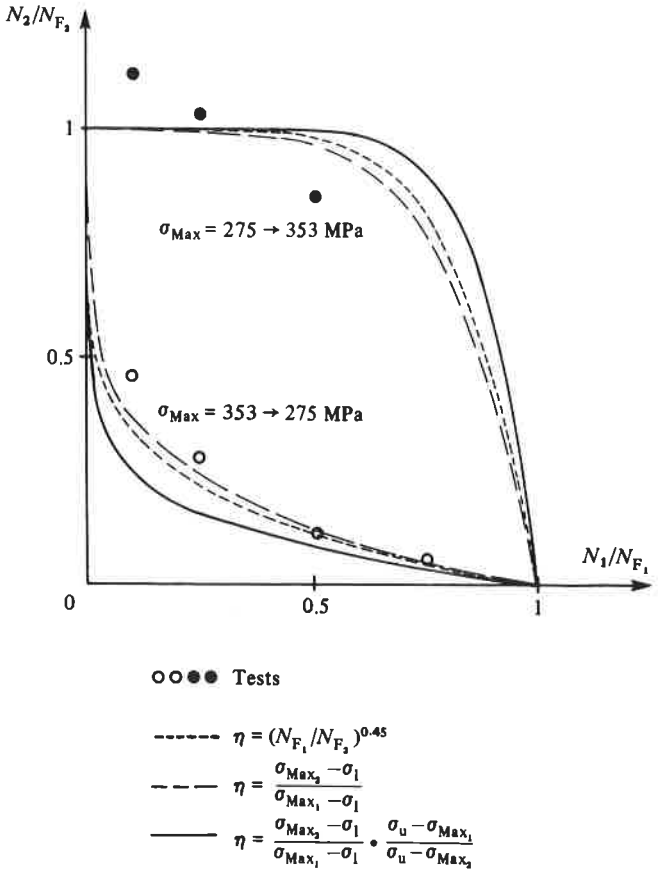
respectively (Fig. 7.40, and Table 7.3).

Extension to the multiaxial case

The principle of generalization of the preceding model to the multiaxial case consists in the simultaneous use of:

the uniaxial law for correctly describing the damage evolution in tension and, later, the nonlinear accumulative effects,

Fig. 7.40. Comparison of three exponents for nonlinear accumulation of fatigue damage. Case of C 35 steel (stress ranges in M Pa).



the multiaxial fatigue limit criteria as a guide in the choice of significant invariants for use in the law. Here, we choose the Sines criterion, but an analogous formulation is possible with the Crossland or the Dang-Van criterion (see Section 7.2.5),

the anisotropy of the damage law, with a method similar to that used in creep.

In the uniaxial case, it is the function α which is used to introduce the fatigue limit with the term $\sigma_{Max} - \sigma_l(\bar{\sigma})$ that can also be written in the form $\sigma_a - \sigma_{al}(\bar{\sigma})$. In the multiaxial case, a stress cycle corresponding to an unlimited life-time is such that:

$$A_{II} \leqslant A_{II}^*(\bar{\sigma}_H) = \sigma_l(1 - 3b\bar{\sigma}_H)$$

with

$$A_{II} = \tfrac{1}{2} \underset{t_0}{\text{Max}} \underset{t}{\text{Max}} J_2(\boldsymbol{\sigma}(t) - \boldsymbol{\sigma}(t_0)).$$

Similarly, static fracture is described by the von Mises criterion (for example) by replacing $\sigma_u - \sigma_{Max}$ in the function α by $\sigma_u - \sigma_{eqMax}$ where σ_{eqMax} is the maximum value of the second invariant of $\boldsymbol{\sigma}$ during the course of the cycle:

$$\sigma_{eq} = J_2(\boldsymbol{\sigma}) = \frac{1}{\sqrt{2}}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]^{1/2}.$$

The functions α and M are therefore expressed by:

$$\alpha(A_{II}, \bar{\sigma}_H, \sigma_{eqMax}) = 1 - a\left\langle \frac{A_{II} - A_{II}^*(\bar{\sigma}_H)}{\sigma_u - \sigma_{eqMax}} \right\rangle$$

$$M(\bar{\sigma}_H) = \sigma_{l_0}(1 - 3b\bar{\sigma}_H).$$

In the uniaxial damage law, the amplitude of effective stress is $(\sigma_{Max} - \bar{\sigma})/(1 - D)$. In the multiaxial case we have instead:

$$\tilde{A}_{II} = \tfrac{1}{2} \underset{t_0}{\text{Max}} \underset{t}{\text{Max}} J_2(\tilde{\boldsymbol{\sigma}}(t) - \tilde{\boldsymbol{\sigma}}(t_0)).$$

The general isotropic damage law is then written as:

$$\delta D = [1 - (1 - D)^{\beta+1}]^{\alpha(A_{II}, \bar{\sigma}_H, \sigma_{eqMax})} \left[\frac{\tilde{A}_{II}}{M(\bar{\sigma}_H)}\right]^{\beta} \delta N.$$

It should be noted that for an isotropic damage theory:

$$\tilde{A}_{II} = A_{II}/(1 - D).$$

The anisotropy of the fatigue damage growth law is introduced by

expressing the rate of growth of the tensor $\mathbf{D}$ by:

$$\dot{\mathbf{D}} = \mathbf{Q}_F \dot{D}$$

where $\mathbf{Q}_F$ depends on the material and is defined with respect to the principal axes of the effective stress tensor $\tilde{\boldsymbol{\sigma}}$ at the instant considered most damaging in the cycle. There is not enough research or experimental results (in 1984 !) that could enable us to establish and validate completely such an anisotropic theory.

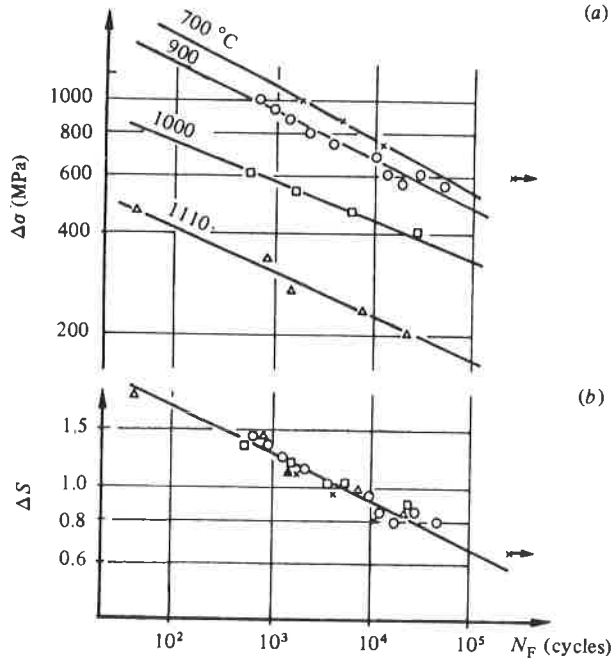
Case of variable temperature during the cycle

When the temperature changes slowly from one cycle to another, or from one group of cycles to another, it is sufficient to consider each cycle as isothermal, and take account of the temperature in the model coefficients by treating them as variables dependent on the number of cycles.

On the other hand, if the temperature varies greatly during each cycle, the problem is more tricky. It can be solved by introducing:

(i) A reduced stress: $S = \sigma/\sigma_u(T)$ where $\sigma_u(T)$ is the ultimate tensile stress

Fig. 7.41. (*a*) The influence of temperature on Woehler's curves of the IN 100 alloy. (*b*) Accounting for temperature by the reduced stress.



which depends on the temperature. Fig. 7.41 illustrates the case of the IN 100 alloy and shows quite well that a large part of the influence of the temperature is accounted for by this parameter. Note that a law expressed in terms of the plastic strain range (Coffin–Manson law described in Section 7.2.4) is also not much influenced by the temperature.

(ii) A mean equivalent temperature, as proposed by Taira, by linearly accumulating the temperature effects during a cycle. To define this equivalent temperature, we use a simplified (linearly accumulative) isothermally determined law such as:

$$\delta D = \frac{\delta N}{N_F(S_{Max}, \bar{S}, T)} = \frac{(S_{Max} - \bar{S})^{\beta(T)}}{A(\bar{S}, T)} \delta N$$

where S_{Max} and $\bar{S}$ are the maximum and mean values of S during the cycle. We then define the damage growth per cycle as:

$$\frac{\delta D}{\delta N} = \int_{cycle} \frac{\beta(T)}{2} \frac{\langle S - \bar{S} \rangle^{\beta(T)-1}}{A(\bar{S}, T)} \mathrm{d}S$$

where T depends on time, and therefore on S. For the isothermal case we recover:

$$\delta D/\delta N = 1/N_F(S_{Max}, \bar{S}, T).$$

When the temperature varies, we obtain the equivalent temperature T^*, i.e., the temperature of an isothermal loading producing the same damage:

$$\frac{1}{N_F(S_{Max}, \bar{S}, T^*)} = \frac{1}{2} \int_{\bar{S}}^{S_{Max}} \frac{\beta(T(S))\,\mathrm{d}S}{(S - \bar{S}) N_F(S, \bar{S}, T(S))}.$$

In practice, S_{Max} and $\bar{S}$ are known, as are the evolutions of S and T during the cycle. They can then be used to calculate the right hand side of the above equation. Assuming N_F to be a monotonic function of T, we can define the equivalent temperature T^*. It is this temperature which must be then used to define the coefficients of the damage law.

7.4.4 *Interaction effects of fatigue and creep damage*

At elevated temperatures, the processes of fatigue damage (due to cyclic loading) and creep damage (linked to the loading duration) can interact. The physical phenomena which occur at the microstructural scale are complex and the influence of the environment is not negligible. We limit ourselves here to a macroscopic description of these interaction effects through the damage growth models described in the preceding sections.

Accumulation of damages of different natures

The creep and fatigue damage models defined in Sections 7.4.2 and 7.4.3 consider separately two types of processes:

creep damage, D_c, characterized by intercrystalline defects, evolves as a function of time elapsed under stress; in the uniaxial case:

$$dD_c = f_c(\sigma, T, D_c, \ldots)\,dt,$$

fatigue damage D_F, for which the defects are generally initiated on the surface and progress across the crystals (transcrystalline); these evolve as functions of the loading cycles; in the uniaxial case:

$$\delta D_F = f_F(\sigma_{Max}, \bar{\sigma}, T, D_F, \ldots)\delta N.$$

When the two processes are present simultaneously, the interaction effects can be represented macroscopically by introducing the couplings:

$$dD_c = f_c(\sigma, T, D_c, D_F, \ldots)\,dt$$
$$\delta D_F = f_F(\sigma_{Max}, \bar{\sigma}, T, D_F, D_c, \ldots)\,\delta N.$$

How should D_F be introduced in the first equation and D_c in the second? In order to simplify and make possible a prediction from pure creep and pure fatigue, we are led to assume that the damage D_c and D_F have additive macroscopic effects:

$$dD_c = f_c(\sigma, T, D_c + D_F, \ldots)\,dt$$
$$\delta D_F = f_F(\sigma_{Max}, \bar{\sigma}, T, D_F + D_c, \ldots)\delta N$$

so that there is only one damage variable and its evolution can be described by:

$$dD = dD_c + dD_F = f_c(\sigma, T, D, \ldots)\,dt + f_F(\sigma_{Max}, \bar{\sigma}, T, D, \ldots)\,dN.$$

It is this form of the equation which will be used below.

Rule of linear accumulation and interaction

This rule was developed first by Robinson and then by Taira. It consists in adding together the creep and pure fatigue damages defined by the linear relation:

$$\bullet \qquad dD = \frac{dt}{t_c(\sigma, T)} + \frac{dN}{N_F(\Delta\varepsilon, T)},$$

where $t_c(\sigma, T)$ is the time to rupture in creep under constant stress σ and

temperature T and N_F is the number of cycles to failure in pure fatigue for a periodic loading of strain range $\Delta\varepsilon$ (in this method one uses $\Delta\varepsilon$ rather than the stress parameters).

The preceding relation when integrated over a cycle gives:

$$\delta D = \left[\int_0^{\Delta t} \frac{dt}{t_c(\sigma, T)} + \frac{1}{N_F(\Delta\varepsilon, T)}\right]\delta N$$

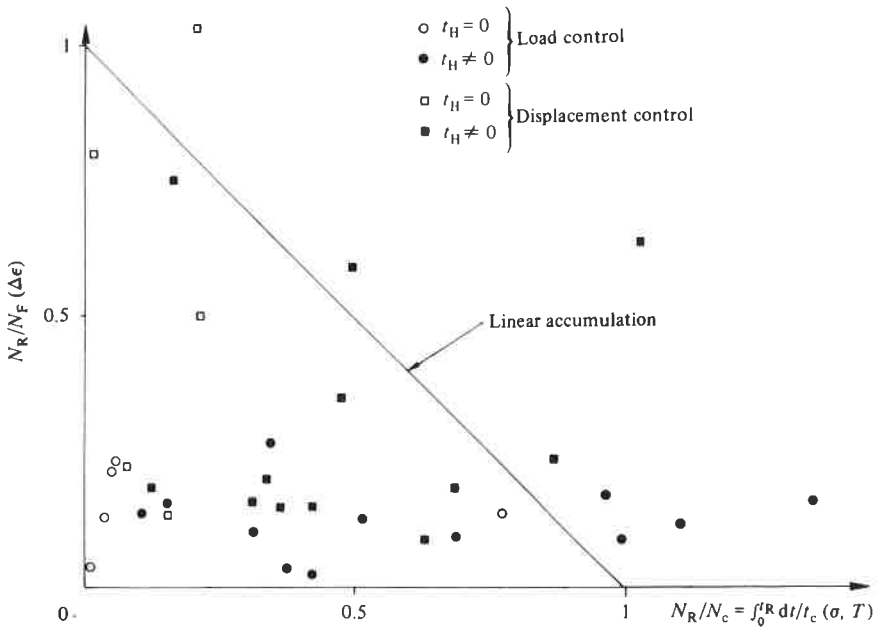
where Δt is the period of the cycle. When the load is periodic, the number of cycles to failure under combined fatigue–creep is found by integration to be:

$$\frac{1}{N_R} = \int_0^{\Delta t} \frac{dt}{t_c(\sigma, T)} + \frac{1}{N_F(\Delta\varepsilon, T)}.$$

If the temperature varies during the cycle, T in N_F is replaced by the equivalent temperature T^* defined in Section 7.4.3.

Although satisfactory in some cases, this method proves ineffective for many materials, in particular for the IN 100 alloy as shown in Fig. 7.42. This figure also shows the experimental values of the life ratios in fatigue

Fig. 7.42. Nonlinear interaction of fatigue and creep damages in the IN 100 alloy. Fatigue criterion expressed in terms of $\Delta\varepsilon$.



and creep. It should be noted that for materials which exhibit significant ductility in creep, it is necessary to perform geometric corrections; the creep tests must be interpreted in terms of the true stress as these data are then applied to cyclic tests in which the strains always remain small.

Model of nonlinear accumulation and interaction

The method presented here is derived directly from the damage equations with nonlinear accumulation:

for creep, Kachanov's law is used in which the characteristic coefficients are identified by pure creep tests;
for fatigue, the model presented in Section 7.4.3 is used in which damage evolution is highly nonlinear in accordance with the damage measurements.

These two models are coupled by the assumption of additivity of fatigue and creep damages. This enables us to take account of the following physical effects:

the presence of creep damage accelerates the growth of fatigue microcracks (the grain boundaries close to the tip of the crack are less resistant);
the presence of fatigue cracks increases the rate of growth of cavities by virtue of stress concentration effects.

In compression, the creep damage growth rate can be lower than in tension. This effect is incorporated by replacing σ by:

$$\chi(\sigma) = \langle c\sigma + (1-c)|\sigma| \rangle.$$

For $c = 0$, we have the symmetry property (compression identical to tension). In contrast, for $c = 1$ there is no creep damage in compression, since we then have $\chi = 0$. This is the application of the general criteria of Sections 7.2.5 and 7.4.2 to tension–compression.

Moreover, in the interest of simplicity, the exponent of Rabotnov's law is assumed to be constant. With these assumptions, the general evolution law under conditions of fatigue–creep interaction is written as:

$$\mathrm{d}D = \left[\frac{\chi(\sigma)}{A}\right]^r (1-D)^{-k}\,\mathrm{d}t + [1-(1-D)^{\beta+1}]^{\alpha(\sigma_{\mathrm{Max}},\bar{\sigma})}\left[\frac{\sigma_{\mathrm{Max}}-\bar{\sigma}}{M(\bar{\sigma})(1-D)}\right]^{\beta}\mathrm{d}N.$$

For life-times which are, at least, in the range of hundreds of cycles, as is the case during service, the damage variation during each isolated cycle

may be neglected. We may thus simply integrate the effects of creep damage during a cycle and include them at the end of this cycle. Accordingly we obtain:

$$\delta D = \left\{(1-D)^{-k}\int_0^{\Delta t}\left(\frac{\chi(\sigma)}{A}\right)^r \mathrm{d}t + [1-(1-D)^{\beta+1}]^{\alpha(\sigma_{\mathrm{Max}},\bar{\sigma})}\left[\frac{\sigma_{\mathrm{Max}}-\bar{\sigma}}{M(\bar{\sigma})(1-D)}\right]^{\beta}\right\}\delta N$$

where Δt is the period of the cycle.

Denoting by N_F the number of cycles to failure for fatigue damage alone and by N_c the number for creep damage alone, we have:

$$\frac{1}{N_c} = \int_0^{\Delta t}\frac{\mathrm{d}t}{t_c(\chi(\sigma),T)} = (k+1)\int_0^{\Delta t}\left(\frac{\chi(\sigma)}{A}\right)^r \mathrm{d}t$$

$$\frac{1}{N_F} = (\beta+1)[1-\alpha(\sigma_{\mathrm{Max}},\bar{\sigma})]\left[\frac{\sigma_{\mathrm{Max}}-\bar{\sigma}}{M(\bar{\sigma})}\right]^{\beta}.$$

By substituting these expressions into the preceding equation, we are led to the following relation:

$$\delta D = \left\{\frac{(1-D)^{-k}}{(k+1)N_c} + \frac{[1-(1-D)^{\beta+1}]^{\alpha(\sigma_{\mathrm{Max}},\bar{\sigma})}}{(\beta+1)N_F[1-\alpha(\sigma_{\mathrm{Max}},\bar{\sigma})](1-D)^{\beta}}\right\}\delta N.$$

The number of cycles to failure N_R is found by integrating for D from 0 to 1, and is expressed by:

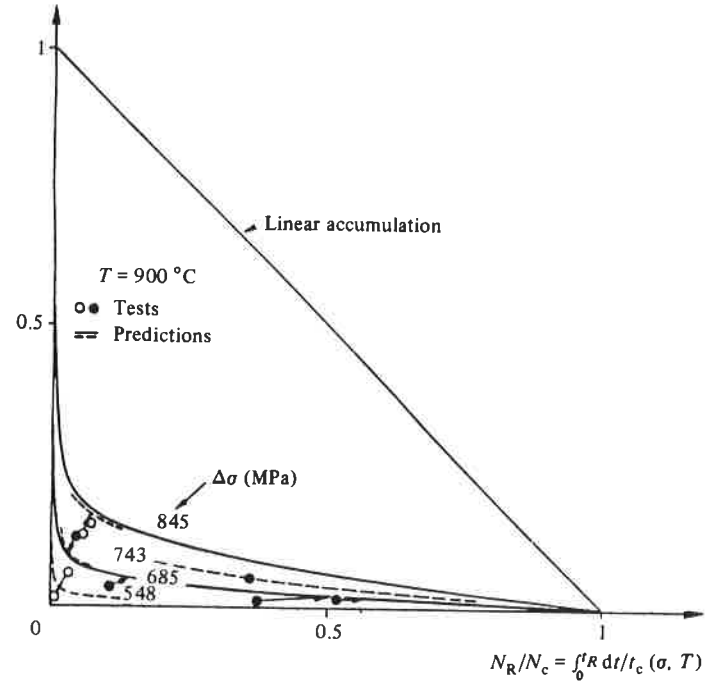
$$\frac{N_R}{N_F} = \int_0^1\left(\frac{N_F}{N_c}\frac{(1-D)^{-k}}{k+1} + \frac{[1-(1-D)^{\beta+1}]^{\alpha(\sigma_{\mathrm{Max}},\bar{\sigma})}}{(\beta+1)[1-\alpha(\sigma_{\mathrm{Max}},\bar{\sigma})](1-D)^{\beta}}\right)^{-1}\mathrm{d}D.$$

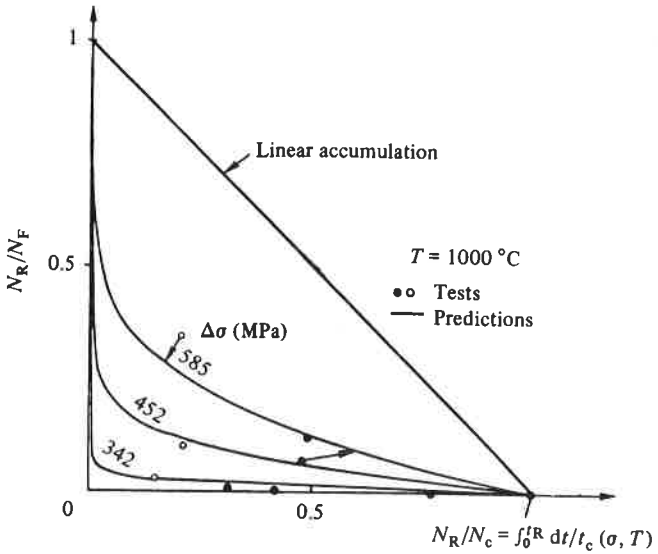
The calculation of N_R, the number of cycles to theoretical fracture is performed in two steps:

calculation of N_c and N_F, the numbers of cycles to failure in creep alone and fatigue alone respectively. In tests under displacement control, evolution of the stress cycle is taken into account until stable conditions are reached;
knowing the maximum and the mean stress, N_F/N_c, and the coefficients r, k, β, we can easily calculate N_R/N_F by performing the above integration which gives the value of N_R.

Comparison between predictions and experimental results is made on interaction diagrams; Fig. 7.43 gives an example of these for stress-

Fig. 7.43. Prediction of the nonlinear interaction between fatigue and creep damages for the IN 100 alloy (fatigue criterion in terms of stress).



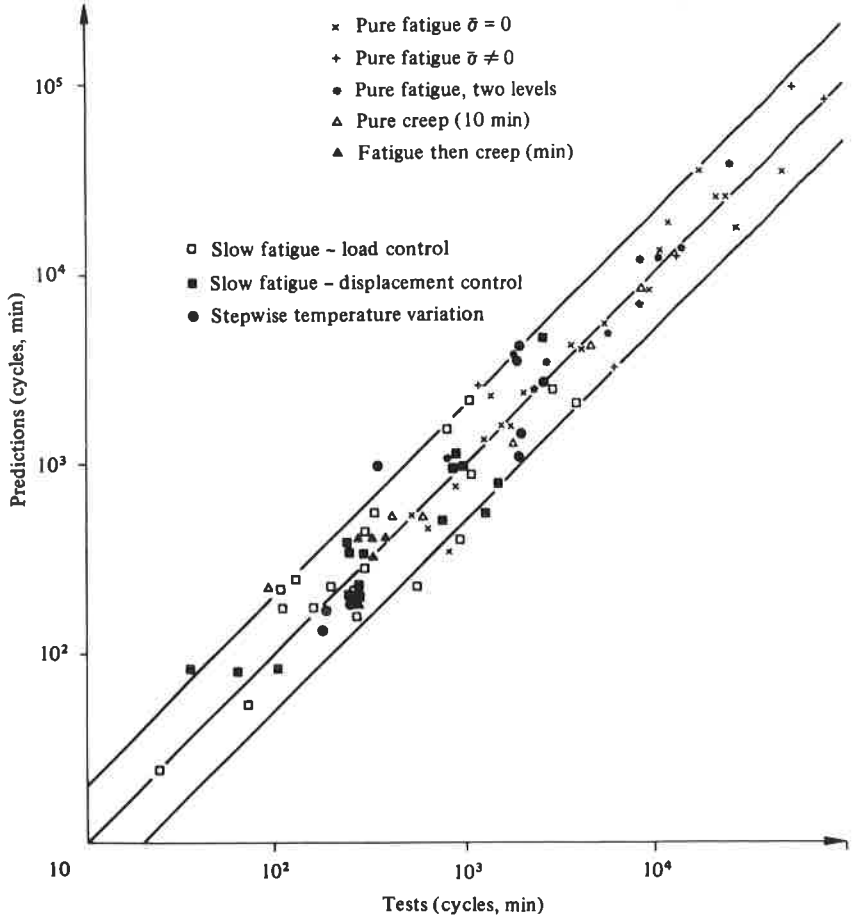


controlled tests. The experimental points correspond to the measured values of N_R divided by the calculated values of N_c and N_F.

For IN 100 and for many other materials, the nonlinearity of the interaction increases as the stress amplitude decreases, as predicted correctly by the theory, particularly when comparing to the prediction by the rule of linear accumulation.

Fig. 7.44 sums up the set of comparisons between predictions and the results of reference tests (pure fatigue and pure creep), stress and strain-controlled viscoplastic fatigue tests, viscoplastic tests at two temperature levels, pure fatigue tests at two stress levels, creep tests after fatigue (see

Fig. 7.44. Comparisons of predictions versus tests on the basis of number of cycles or time to failure for the IN 100 alloy.



below), and thermomechanical fatigue tests. The differences vary within a factor of 2, which constitutes a good verification of the model as the rule of linear accumulation can lead to errors higher than a factor of 10.

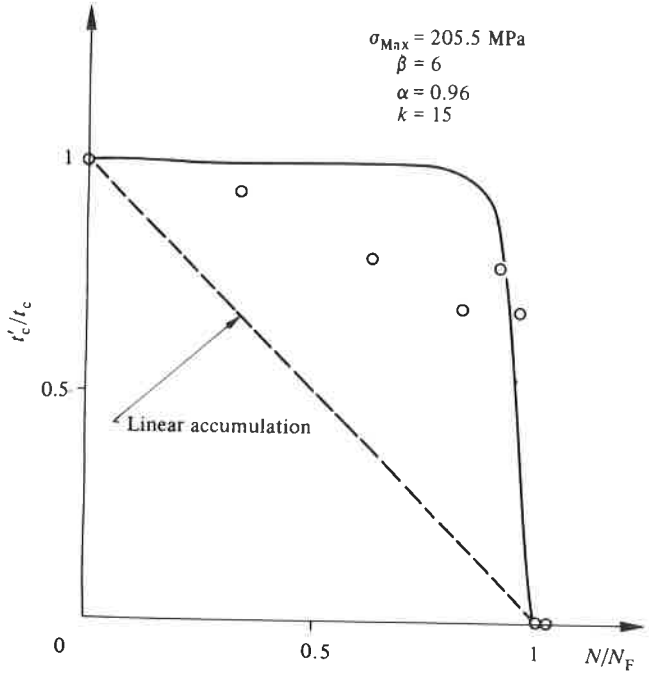
The validity of this modelling derives from some well-known facts for this type of refractory alloy:

creep damage evolves much faster than fatigue damage (relative to nominal life-time): this difference between the rates of evolution is correctly described by the chosen models;
there is no cyclic consolidation effect: the effect of hardening on the damage evolution is negligible.

The first remark implies the existence of test conditions for which the effect of nonlinear accumulation is observed in the opposite sense, i.e., in a sense beneficial with respect to the rule of linear accumulation. This represents tests started in fatigue for N_1 cycles followed by creep for a duration t_c until rupture; we must then have:

$$\frac{N_1}{N_F}+\frac{t'_c}{t_c}>1$$

Fig. 7.45. Creep rupture after pure fatigue: predictions and tests on IN 100 alloy at 1000 °C.



where N_F and t_c are respectively the number of cycles to failure in fatigue and the time to rupture in creep, corresponding to the applied stress.

Such tests have been conducted on the IN 100 alloy at 1000 °C and the predicted effect is indeed observed as can be seen from Fig. 7.45. The fatigue and creep damage models can be used for a more accurate prediction. In fatigue, after N_1 cycles, we have:

$$[1-(1-D_1)^{\beta+1}]^{1-\alpha}=N_1/N_F$$

where α denotes the value of the function $\alpha(\sigma_{Max},\bar{\sigma})$ corresponding to the maximum applied stress ($\bar{\sigma}=0$). The integration of the creep damage law between D_1 and 1 leads to:

$$(1-D_1)^{k+1}=t'_c/t_c$$

where k is the value of $k(\sigma)$. Hence we can deduce the relation predicting the time to rupture in creep as a function of the length of fatigue precycling:

$$\frac{t'_c}{t_c}=\left[1-\left(\frac{N_1}{N_F}\right)^{1/(1-\alpha)}\right]^{(k+1)/(\beta+1)}.$$

This relation, used with $\beta=6$, $\alpha=0.96$, $k=15$, which correspond to the loading under consideration, satisfactorily reproduces the experimental results considering the relatively significant dispersion among the creep rupture tests (Fig. 7.10).

7.5 Deformation and damage coupling

By deformation and damage coupling, we refer to the fact that the solution of a problem in terms of stresses and strains at the limit states depends on the state of damage of the structure under consideration. Generally, as we have seen, damage reduces the stiffness and strength of materials. For a given state of stresses, the larger are the strains the higher is the damage, and hence we recognize the importance of 'coupled' calculations for evolution problems in which stresses, strains and damage are calculated simultaneously.

The concept of effective stress, associated to the principle of strain equivalence, enables us to write the laws of coupled behaviour very simply since it is sufficient to replace the usual stress $\boldsymbol{\sigma}$ by the effective stress $\tilde{\boldsymbol{\sigma}}$.

We introduce here some basic elements of the general approach and apply them to a few examples of constitutive equations. For simplicity, we assume that the damage is isotropic.

7.5.1 *Elasticity coupled with damage*

This is important for materials for which plasticity effects are not very significant. A typical example is concrete in which nonlinear deformations, which are sometimes irreversible, are produced by the process of decohesion. This process may be analysed using a damage theory.

The theoretical study of this case issues directly from the choice of the effective stress. For isotropic elasticity and isotropic damage in the initial conditions, the elasticity law of the damaged material is written as:

$$\boldsymbol{\varepsilon} = \mathbf{A}:\tilde{\boldsymbol{\sigma}}$$

or

$$\boldsymbol{\varepsilon} = \frac{1+\nu}{E}\frac{\boldsymbol{\sigma}}{1-D} - \frac{\nu}{E}\frac{\mathrm{Tr}(\boldsymbol{\sigma})}{1-D}\mathbf{1}.$$

This relation must be coupled with the chosen damage law.

Let us take a very schematic example, specialized to uniaxial tension, by assuming that D evolves with (elastic) deformations and introducing a variable threshold ε_D:

$$\mathrm{d}D = \begin{cases} (\varepsilon/\varepsilon_0)^{s^*} & \text{when } \varepsilon = \varepsilon_D \quad \text{and} \quad \mathrm{d}\varepsilon = \mathrm{d}\varepsilon_D > 0 \\ 0 & \text{when } \varepsilon < \varepsilon_D \quad \text{or} \quad \mathrm{d}\varepsilon < 0. \end{cases}$$

By taking the initial conditions to be $D = \varepsilon_D = 0$, and integrating up to fracture, $D = 1$, we obtain:

$$D = (\varepsilon/\varepsilon_R)^{s^*+1} \qquad \sigma = E\varepsilon[1 - (\varepsilon/\varepsilon_R)^{s^*+1}]$$

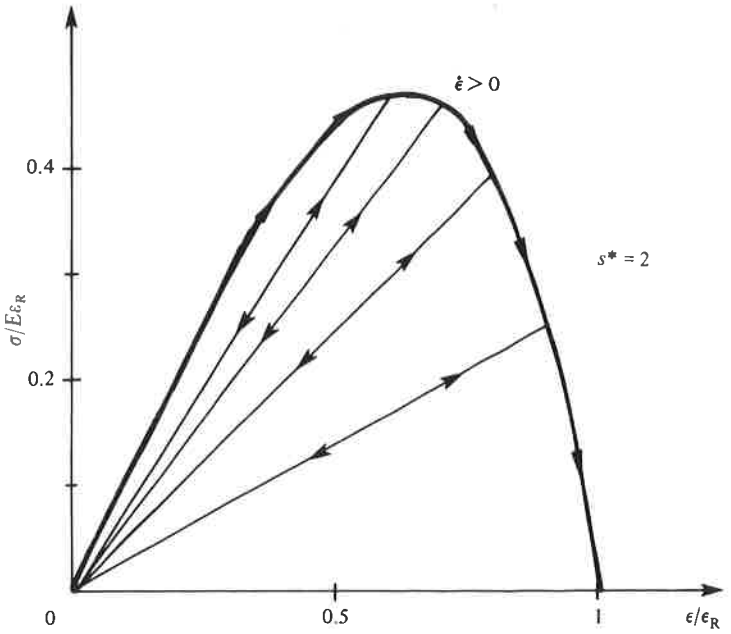
where $\varepsilon_R = [(s^*+1)\varepsilon_0^{s^*}]^{1/(s^*+1)}$ is the fracture strain. Fig. 7.46(*a*) illustrates the stress evolution obtained from such a theory ($s^* = 2$) of elasticity coupled with damage. During an eventual unloading–loading we again find linear elastic behaviour but with a lower elasticity modulus. The results obtained are similar to the experimental observations as shown in Fig. 7.46(*b*).

7.5.2 *Plasticity coupled with damage*

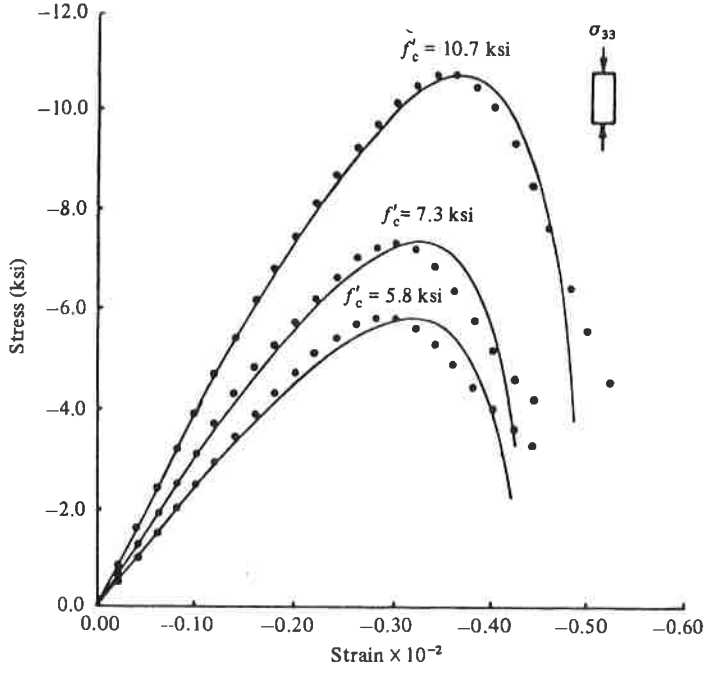
General formulation

In order to model the behaviour of a damaged and hardened plastic material, the strain equivalence principle is once more applied. The stress $\boldsymbol{\sigma}$ is replaced by the effective stress $\tilde{\boldsymbol{\sigma}}$ in the dissipation potential, all other variables remain unchanged (see Chapter 5). The associated variable of

Fig. 7.46. (*a*) Stress–strain curve for a theory of elasticity coupled with damage. (*b*) Example of concrete in compression (after D. Krajcinovic).



(*a*)



(*b*)

isotropic strain hardening remains R while the corresponding internal variable changes and is now denoted as r.

$$\boldsymbol{\sigma} \to \tilde{\boldsymbol{\sigma}} = \frac{\boldsymbol{\sigma}}{1-D}$$
$$\mathbf{X} \to \mathbf{X}$$
$$R \to R$$

Thus, the elastic domain is expressed by:

$$f = J_2(\tilde{\sigma} - \mathbf{X}) - R - k = J_2\left(\frac{\boldsymbol{\sigma}}{1-D} - \mathbf{X}\right) - R - k \leqslant 0.$$

The flow law and the evolutions of internal and damage variables are obtained by the law of normal dissipativity, as in the general case.

The dissipation potential φ^* is decomposed into:

$$f(\tilde{\boldsymbol{\sigma}}, \mathbf{X}, R) \quad \text{and} \quad \varphi_D^*(Y; \boldsymbol{\varepsilon}^e, \boldsymbol{\alpha}, r, D),$$

the first term corresponds to plastic dissipation, the second to dissipation by damage, assumed to be decoupled from the first.

The generalized normality rule provides the flow law and the evolutions of the internal variables:

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}(\partial f/\partial \boldsymbol{\sigma})$$
$$\dot{\boldsymbol{\alpha}} = -\dot{\lambda}(\partial f/\partial \mathbf{X})$$
$$\dot{r} = -\dot{\lambda}(\partial f/\partial R).$$
$$\dot{D} = -\dot{\lambda}(\partial \varphi_D^*/\partial Y).$$

The plastic multiplier $\dot{\lambda}$ is then determined by the plastic flow consistency condition: $\dot{f} = f = 0$ (see Chapter 5). Only the case of isotropic hardening will be studied below.

Coupling between isotropic ductile plastic damage and isotropic hardening

Assuming that the material obeys the von Mises criterion, the elastic domain is now expressed by $f = \sigma_{eq}/(1-D) - R - k \leqslant 0$; φ_D^* depends only on Y and D. We, therefore, have:

$$\dot{\boldsymbol{\varepsilon}}^p = \frac{3}{2}\frac{\dot{\lambda}}{1-D}\frac{\tilde{\boldsymbol{\sigma}}'}{\tilde{\sigma}_{eq}} = \frac{3}{2}\frac{\dot{\lambda}}{1-D}\frac{\boldsymbol{\sigma}'}{\sigma_{eq}}$$
$$\dot{r} = \dot{\lambda} = (\tfrac{2}{3}\dot{\boldsymbol{\varepsilon}}^p : \dot{\boldsymbol{\varepsilon}}^p)^{1/2}(1-D) = \dot{p}(1-D)$$

which represents the corrected accumulated plastic strain rate by virtue of irreversible strain due to damage, assuming that the given hardening law of the material without damage is:

$$R = \rho \partial \Psi(r)/\partial r = R(r).$$

$\dot{\lambda}$ or $\dot{r}$ is expressed by the consistency condition:

$$\dot{f} = \frac{\partial f}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} + \frac{\partial f}{\partial R}\dot{R} + \frac{\partial f}{\partial D}\dot{D} = 0$$

or

$$\frac{3}{2}\frac{\boldsymbol{\sigma}' : \dot{\boldsymbol{\sigma}}}{\sigma_{eq}(1-D)} - R'(r)\dot{r} + \frac{\sigma_{eq}}{(1-D)^2}\dot{D} = 0.$$

Then, with:

$$\dot{r} = \dot{\lambda}, \quad \frac{\sigma_{eq}}{1-D} - R = k \quad \text{and} \quad \dot{D} = -\frac{\partial \varphi_D^*}{\partial Y}\dot{\lambda}$$

we get:

$$\dot{\lambda} = \frac{\dot{\sigma}_{eq}}{(1-D)R'(r) + [k + R(r)]\partial \varphi_D^*/\partial Y}.$$

The loading–unloading criterion is expressed by:

$$\dot{\lambda} = 0 \quad \text{if} \quad \dot{\sigma}_{eq} \leqslant 0$$
$$\dot{\lambda} > 0 \quad \text{if} \quad \dot{\sigma}_{eq} > 0.$$

Moreover, there is flow only if $f = 0$. Denoting the hardening modulus by $h(r, D, Y)$ we have:

$$\dot{p} = \frac{\dot{\lambda}}{1-D} = H(f)\frac{\langle \dot{\sigma}_{eq} \rangle}{h(r, D, Y)}$$

with:

$$h(r, D, Y) = (1-D)^2 R'(r) + (1-D)[k + R(r)]\partial \varphi_D^*/\partial Y.$$

Since $\partial \varphi_D^*/\partial Y$ is negative, this hardening modulus may be negative. In fact it is positive when the strain is small and $R'(r)$ is large, and it becomes negative at the instability point:

$$h(r, D, Y) = 0.$$

In this case, the above description is no longer applicable and we must use the loading–unloading condition with $\dot{\tilde{\sigma}}_{eq}$.

Thus, the Prandtl–Reuss equations in which the hardening law appears

in the form of a power function

$$R(p) = K_Y p^{1/M_Y} = \sigma_{eq} - \sigma_Y$$

can now be written

- $$d\varepsilon = d\varepsilon^e + d\varepsilon^p$$
- $$d\varepsilon^e = \frac{1+\nu}{E}\frac{d\boldsymbol{\sigma}}{1-D} - \frac{\nu}{E}\frac{d(\mathrm{Tr}(\boldsymbol{\sigma}))}{1-D}\mathbf{1}$$
- $$d\varepsilon^p = \tfrac{3}{2}H(f)\frac{\langle d\sigma_{eq}\rangle}{\dfrac{K_Y(1-D)^2}{M_Y}\left\langle \dfrac{\sigma_{eq}}{K_Y(1-D)} - \dfrac{\sigma_Y}{K_Y}\right\rangle^{1-M_Y} + \sigma_{eq}\dfrac{\partial\varphi_D^*}{\partial Y}}\frac{\boldsymbol{\sigma}'}{\sigma_{eq}}.$$

To these, the corresponding damage law should be adjoined (see Section 7.4.1):

$$\dot{D} = -\frac{\partial\varphi_D^*}{\partial Y}\dot{\lambda} = -\frac{\partial\varphi_D^*}{\partial Y}\dot{p}(1-D)$$

or

- $$dD = \frac{D_c}{\varepsilon_R - \varepsilon_D}\left[\tfrac{2}{3}(1+\nu) + 3(1-2\nu)\left(\frac{\sigma_H}{\sigma_{eq}}\right)^2\right]dp$$

and

$$\frac{\partial\varphi^*}{\partial Y} = -\frac{D_c}{(\varepsilon_R - \varepsilon_D)(1-D)}\left[\tfrac{2}{3}(1+\nu) + 3(1-2\nu)\left(\frac{\sigma_H}{\sigma_{eq}}\right)^2\right].$$

7.5.3 *Viscoplasticity coupled with damage*

Assuming that the attentive [*sic* !] reader has understood the method presented in Sections 7.5.1 and 7.5.2 well, we immediately write the coupled isotropic viscoplasticity law studied in Section 6.4.2:

- $$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^e + \dot{\boldsymbol{\varepsilon}}^p,$$
- $$\dot{\boldsymbol{\varepsilon}}^e = \frac{1+\nu}{E}\frac{\dot{\boldsymbol{\sigma}}}{1-D} - \frac{\nu}{E}\frac{\mathrm{Tr}(\dot{\boldsymbol{\sigma}})}{1-D}\mathbf{1},$$
- $$\dot{\boldsymbol{\varepsilon}}^p = \tfrac{3}{2}\dot{p}\frac{\boldsymbol{\sigma}'}{\sigma_{eq}}, \quad \dot{r} = \dot{p}(1-D),$$
- $$\dot{r} = \frac{1}{1-D}\left[\frac{\sigma_{eq}}{(1-D)Kr^{1/M}}\right]^N.$$

As with plasticity, the relation $\dot{r} = \dot{p}(1-D)$ comes from the choice of a

viscoplastic potential of the form:

$$\varphi^* = \Omega = \frac{K}{N+1}\left\langle\frac{J_2(\tilde{\boldsymbol{\sigma}}) - R + h'(r)}{K}\right\rangle^{N+1} r^{-N/M}$$

and from the normality rule:

$$\dot{\boldsymbol{\varepsilon}}^p = \partial\Omega/\partial\boldsymbol{\sigma}.$$

Only creep damage can occur simultaneously with viscoplasticity:

- $$\dot{D} = \langle\chi(\boldsymbol{\sigma})/A\rangle^r(1-D)^{-k\langle\chi(\sigma)\rangle}.$$

In applications, we may consider that the damage is neglected during primary creep and that strain-hardening is saturated during tertiary creep. For example, in tension:

- during primary creep:

$$\varepsilon_p < \varepsilon_p^*, \quad D = 0 \rightarrow \qquad \varepsilon_p = \left[\frac{N+M}{M}\left(\frac{\sigma}{K}\right)^N t\right]^{M/(N+M)},$$

- during tertiary creep: $\varepsilon_p \geqslant \varepsilon_p^*$, $Kr^{1/M}$ is replaced by $K\varepsilon_p^{*1/M}$ first, the damage equation then gives

$$D = 1 - \left(1 - \frac{t}{t_c}\right)^{1/(k+1)} \qquad \text{with} \qquad t_c = \frac{1}{k+1}\left(\frac{\sigma}{A}\right)^{-r}.$$

Then, the constitutive equation is written as:

$$d\varepsilon_p = \left(1 - \frac{t}{t_c}\right)^{-(N+1)/(k+1)}\left(\frac{\sigma}{K}\right)^N dt.$$

After integrating for σ = constant, we find:

$$\varepsilon_p = \varepsilon_p^* + (\varepsilon_{pR} - \varepsilon_p^*)\left[1 - \left(\frac{1 - t/t_c}{1 - t^*/t_c}\right)^{(k-N)/(k+1)}\right]$$

$$\varepsilon_{pR} = \varepsilon_p^* + \frac{k-1}{k-N}\left(\frac{\sigma}{K\varepsilon_p^{*1/M}}\right)^N t_c\left(1 - \frac{t^*}{t_c}\right)^{(k-N)/(k+1)}$$

with

$$t^* = \frac{M}{N+M}\left(\frac{\sigma}{K}\right)^{-N}\varepsilon_p^{*(N+M)/M}$$

where the exponents are usually ordered as follows:

$$r \leqslant N \leqslant k.$$

Hence, it follows that the rupture strain is a decreasing function of the stress applied during creep. This is generally well confirmed by experiment. Fig. 7.47 shows good agreement for tertiary creep as well as for the rupture strain for IN 100 superalloy.

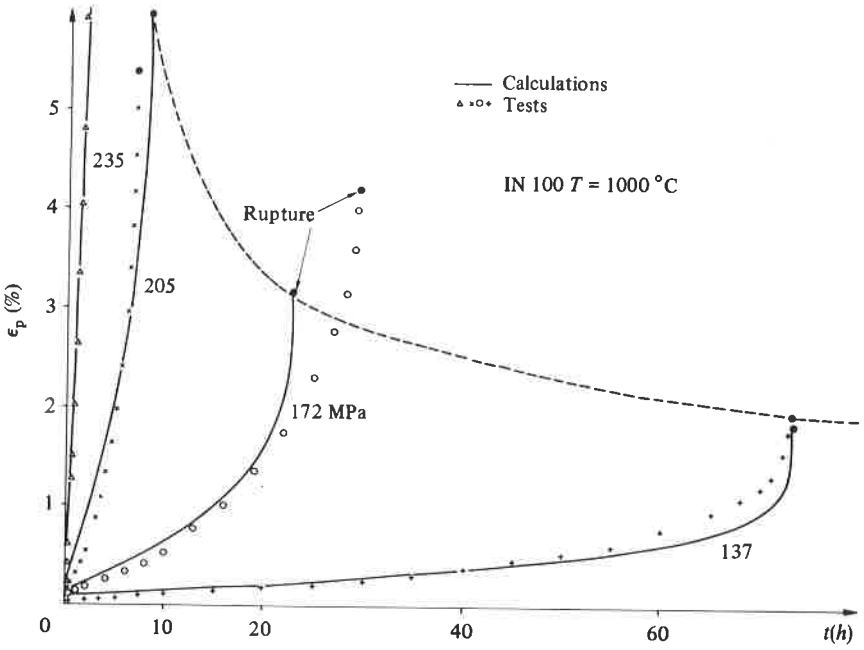
7.6 Prediction of crack initiation in structures

In this section, we present some basic elements on the use of damage models for predicting crack initiation. The use of these models to foresee the growth of cracks within the framework of the local approach will be discussed in Section 8.7.

7.6.1 *Initial damage*

The first problem that arises in the integration of differential damage models is that of the initial conditions. The processes of cold-working, forming, machining of mechanical parts, etc. can leave an initial damage, denoted by D_0, which must be taken into account in fracture prediction.

Fig. 7.47. Prediction of tertiary creep and rupture strain for the IN 100 superalloy at 1000 °C.



To write $D = D_0$ at $t = 0$ or $N = 0$ is easy enough! But to determine the value of the initial damage D_0 is not so. Ultrasonic methods can provide some indications, but, usually, it is necessary to resort to an assessment based on indirect information.

7.6.2 *Calculation of damage at critical points*

Classical method without coupling

In a number of applications, we may consider the influence of damage on the deformation behaviour to be weak and assume decoupling. In fatigue, for example, in the restricted deformation range, this is an acceptable assumption: the coupling effect does not come into play, as we have seen (in Section 7.2.3), until very late when one or several microcracks have already been initiated. In any case this is the simplest assumption, and is commonly made in most applications.

Calculation of fracture initiation is then performed in two steps:

(1) Calculation of the stress–strain response in the structure by the finite element method (see Chapters 4, 5 and 6 for a brief description of this method in elasticity, plasticity and viscoplasticity). Under enforced loading, assumed to be known, we have to solve a boundary value problem, which is made complex in many applications by the effects of plastic or viscoplastic flows. For cyclic loadings, we try to obtain the solution at the so-called stabilized cycle, that is to say for a constitutive equation of stable cyclic behaviour (the effects of deconsolidation or of cyclic hardening are not accounted for in this case), and for stabilized stress redistributions. To meet this last requirement, calculations have to be done successively for many cycles in order to take account of the stress redistribution in the structure.

(2) Knowing the stresses and strains at any point of the structure, we are able to determine the critical point or points (where there is a danger of a crack initiating) and a life-time of the structure, i.e., the time or the number of cycles corresponding to crack initiation. This step requires the use of an initiation criterion which may be derived from the damage evolution law. Generally it is sufficient to use a direct relation between the stresses and/or strains, calculated in the stabilized regime, and a time or a number of cycles to failure. In fatigue, for example, we will use:

$$N_R = N_F(\sigma_{Max}, \bar{\sigma}, T).$$

An additional problem arises when the loads are neither constant nor periodic. For certain applications, it is necessary to take into consideration the real service loads, which are usually idealized; but this involves consideration of the effects of accumulated damage. Staying within the context of a noncoupled theory, the structure is analysed independently for each loading, and the adopted rule is used to sum the damages which are obtained independently for each loading. For example, for the rule of linear accumulation in fatigue under a loading programme of repeated blocks (see Fig. 7.48), we have:

$$n_R \sum_{i=1}^{m} (N_i/N_{Fi}) = 1$$

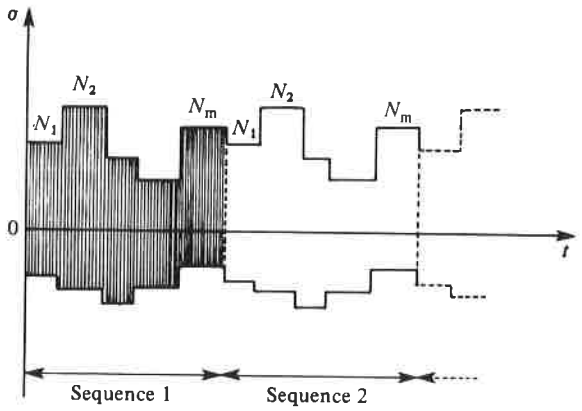
where n_R is the number of sequences before initiation, N_i the number of cycles in each block and N_{F_i} is the number of cycles to failure corresponding to the periodic stresses and/or strains equal to those calculated for the ith loading condition.

Example of crack initiation calculation in a turbine blade

This involves the blades of a high pressure turbine of an aeroplane engine cooled by internal circulation of compressed air through ducts or cavities (Fig. 7.49). The material is the Nickel based IN 100 superalloy. It is protected against oxydization by an aluminizing process. The main loads are those from the start–stop cycles of the engine:

forces of centrifugal origin,

Fig. 7.48. Block loading programmes.



cyclic temperature changes producing thermal stresses responsible for 'thermal fatigue'.

The method described above for predicting crack initiation is applied as follows:

use of a cyclic viscoplasticity law with nonlinear kinematic hardening (Section 6.4.3) in the stabilized regime;

Fig. 7.49. Turbine blade and temperature fields.

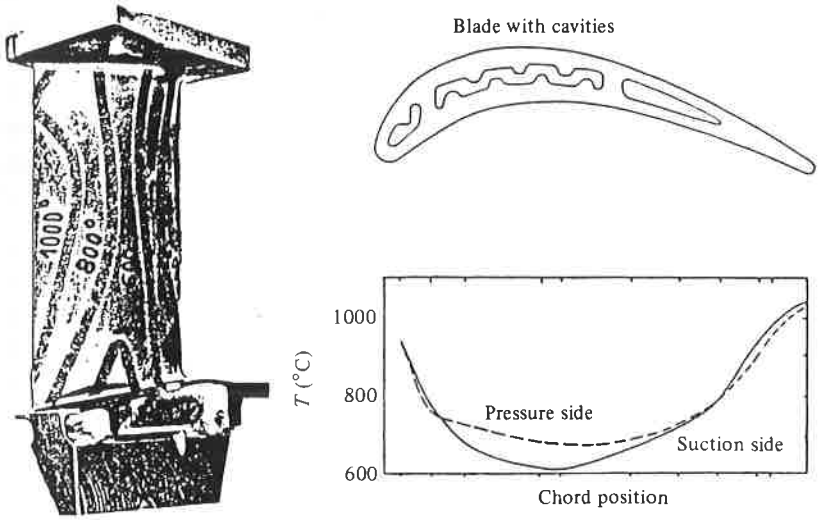
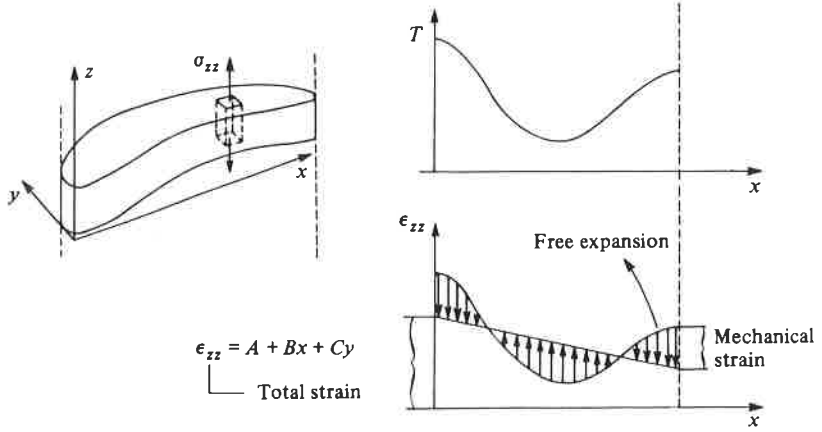


Fig. 7.50. Schematic illustration of the plane cross-section method.



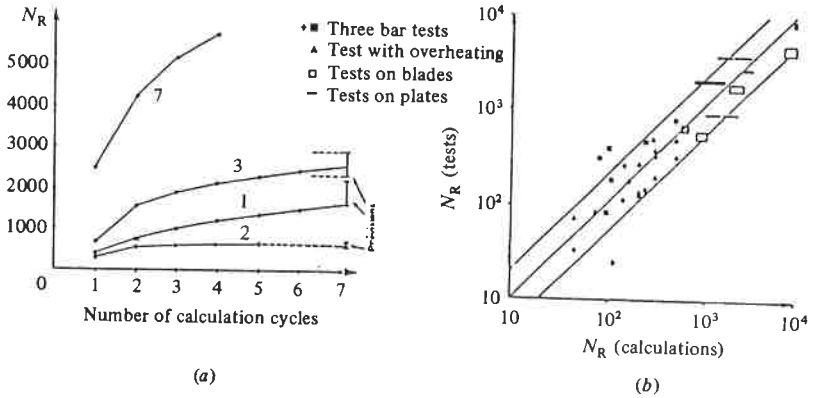
calculation of the stresses and strains by a method of plane cross-sections and uniaxial stresses, permitting three degrees of freedom (one of translation and two of bending). Fig. 7.50 shows schematically how the free expansion of the elements is inhibited, thus resulting in thermal stresses and elastoviscoplastic strains. The effects of stress redistribution due to viscoplasticity and successive cycles are taken into account by a step-by-step calculation as indicated in Section 6.5.2;

calculation of the number of cycles to crack initiation at each point, the critical initiation point obviously being that which corresponds to the minimum number of cycles to failure. These calculations are done with stresses and strains obtained at the first, second,..., sixth or seventh cycle in order to extrapolate the number of cycles at initiation, taking into account the stress redistribution at each cycle (Fig. 7.51). The nonlinear fatigue–creep interaction effects are incorporated by the models described in Section 7.4.4.

The predictions have been confirmed by experimental observations in practical laboratory tests on real blades under loadings close to those experienced by engines in service. Similar results have been obtained for plates under thermal gradients and for different types of anisothermal tension–compression tests, all with the same choice of coefficients (Fig. 7.51(*b*)).

This method is commonly used at the Société SNECMA in the final dimensioning phase of the turbine blades, with the help of the program

Fig. 7.51. (*a*) Prediction of number of cycles to crack initiation as function of the number of calculation cycles. (*b*) Comparison with tests.



(*a*) (*b*)

'CALIFAT' (CALcul d'Initiation des Fissures dans les Aubes de Turbine). The finite element method allows its generalization for use in other critical parts of the engines (discs, combustion chambers) and in other industrial sectors (nuclear industry, automobile industry, etc.).

Taking account of deformation–damage coupling

This effect can be important for certain materials or certain types of loading insofar as the calculated stresses and strains are modified by the introduction of such a coupling. Two examples are briefly described.

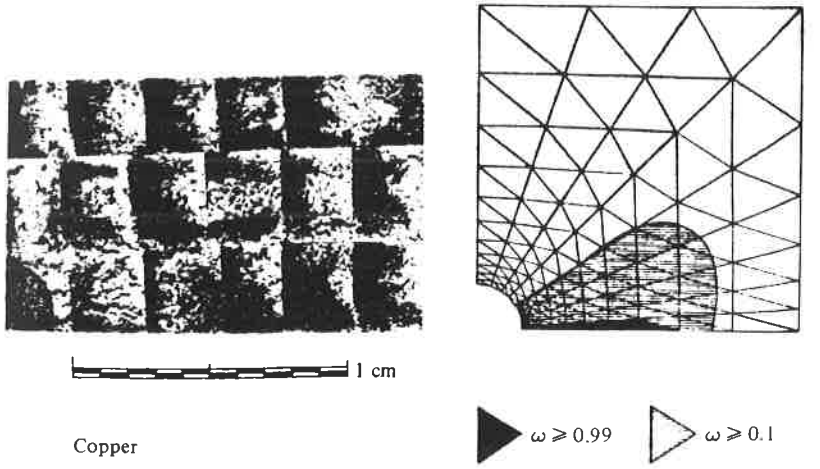
Creep damage

The influence of creep damage on viscoplastic behaviour is important. It becomes apparent, as we have seen, in tertiary creep. Even if a structure is loaded by specified forces, the influence of tertiary creep can result in an additional stress redistribution.

Such a calculation is not easy, even under a constant load. Fig. 7.52 shows the example of a notched plate with the calculation performed in plane stress by taking this coupling into consideration. The equations are those of perfect viscoplasticity (without hardening) and of isotropic damage specialized by:

$$\dot{\varepsilon}^p = \frac{3}{2}\left[\frac{\sigma_{eq}}{\lambda^*(1-D)}\right]^{N^*}\frac{\boldsymbol{\sigma}'}{\sigma_{eq}}, \qquad \dot{D} = \left[\frac{\sigma_{eq}}{A(1-D)}\right]^r.$$

Fig. 7.52. Damaged zone in a notched plate (after D. Hayhurst).

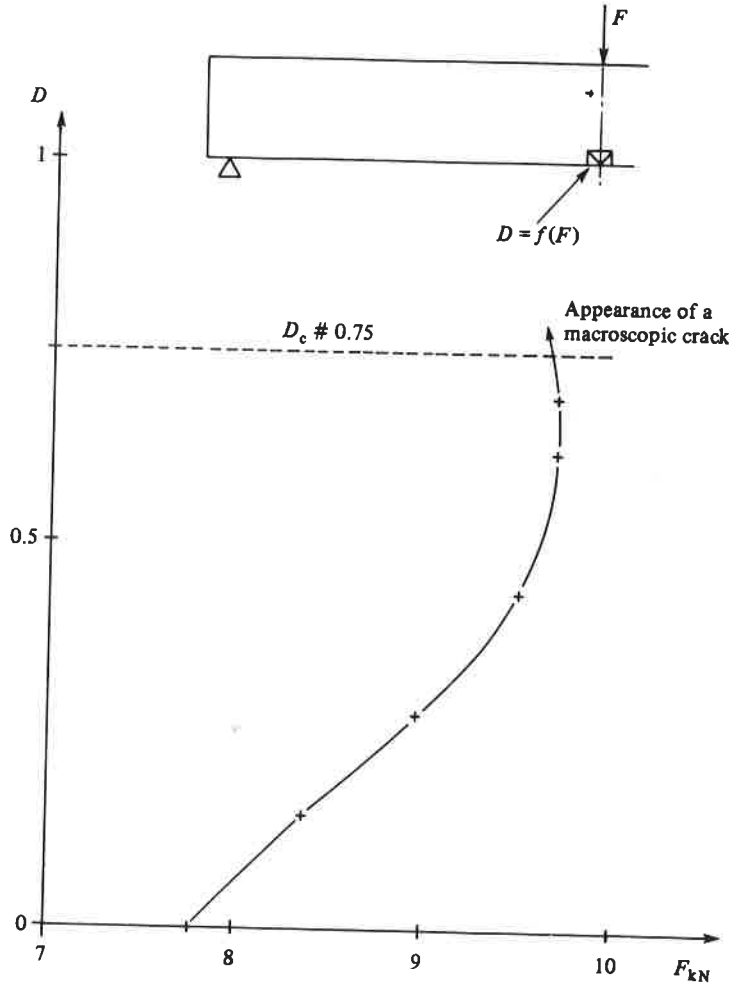


The calculation is done by the finite element method applied to viscoplasticity (see Chapter 6). The stiffness matrix of the most loaded elements is periodically updated to take into account the reduction of elastic moduli. The equations are integrated step-by-step in time.

Damage in concrete

Concrete is a material whose nonlinear behaviour in tension can be represented by coupling between the linear elastic law and a deformation

Fig. 7.53. Evolution of damage at the most critical point of a concrete beam loaded in bending (after J. Mazars).



induced damage. In Section 7.5.1 we have seen a simple but realistic model for describing its uniaxial behaviour. A multiaxial criterion in terms of strain is added to differentiate tension from compression. As an example, we mention the criterion described in Section 7.2.5 which contains the invariant ε^* derived from the positive parts of the principal strains:

$$\varepsilon^* = (\langle\varepsilon_1\rangle^2 + \langle\varepsilon_2^2\rangle + \langle\varepsilon_3^2\rangle)^{1/2}.$$

By assuming isotropic damage, we write:

$$\dot{D} = KH(f)\varepsilon^{*s^*}\langle\dot{\varepsilon}^*\rangle$$

where f is the damage criterion:

$$f = \varepsilon^* - \varepsilon_D \leqslant 0.$$

$H(f)$ is the Heaviside function as used in plasticity.

Structural analysis is performed by the finite element method. The damage equation is integrated step-by-step; the elastic stiffness of each finite element is modified at each step to take into account the effect of damage on the elasticity law:

$$\boldsymbol{\varepsilon} = \frac{1+\nu}{E}\frac{\boldsymbol{\sigma}}{1-D} - \frac{\nu}{E}\frac{\mathrm{Tr}(\boldsymbol{\sigma})}{1-D}\mathbf{1}.$$

We note that it is sufficient to introduce the elasticity modulus of the damaged material $\tilde{E} = E(1-D)$.

Fig. 7.53 shows an example of results obtained for a concrete beam subjected to bending. The damage law, integrated for monotonic loading, is here of the type:

$$D = 1 - \frac{\varepsilon_{D_0}(1-A)}{\varepsilon^*} - \frac{A}{\exp(B(\varepsilon^* - \varepsilon_{D_0}))}$$

where for the concrete under consideration, $E = 30\,000$ MPa $\nu = 0.2$; $\varepsilon_{D_0} = 10^{-4}$; $A = 0.8$; $B = 2 \times 10^4$.

The experiments performed in corresponding conditions have confirmed the location of the damage zones as well as the damage level for a given load (ultrasonic method of measurement). The prediction of crack propagation considered as the growth of a completely damaged zone is described in Section 8.7.

Bibliography

Kachanov L. M. *Introduction to continuum damage mechanics.* Martinus Nijhoff, Dordricht (1986).

Krajcinovic D. & Lemaitre J. *Continuum damage mechanics – theory and applications.* CISM course. Springer-Verlag, Berlin (1987).
Chaboche J. L. *Description thermodynamique et phénoménologique de la viscoplasticité cyclique avec endommagement.* Thèse d'Etat, Université P. et M. Curie (Paris VI) (1978).
Cazaud R., Janssen C., Pomey G. & Rabbe P. *La fatigue des métaux.* Dunod, Paris (1977).
Bathias C., Bailon J. P. *La fatigue des matériaux et des structures.* Maloine, Paris (1980).
Osgood C. C. *Fatigue design.* Pergamon Press, New York (1982).
Bazant Z. P. *Advanced topics in inelasticity and failure of concrete.* Cement Och. Betoninstituted, Stockholm (1979).
Wilshire B. & Owen D. R. J. *Engineering approaches of engineering design.* Pineridge Press, Swansea (1983).

8

CRACK MECHANICS

Une fissure instable est une crique qui craque!

All the previous chapters have treated the behaviour of a volume element by considering it as a continuous medium, represented by its state variables. The theory of cracking phenomena or fracture mechanics describes the behaviour of solids or structures with macroscopic geometric discontinuities at the scale of the structure. These discontinuities are line discontinuities in two-dimensional media (such as plates and shells) and surface discontinuities in three-dimensional media. The theory of damage can be used to predict the onset of a macroscopic crack. The theory of cracking phenomena, on the other hand, can be used to predict the evolution of the crack up to a complete failure of the structure.

In structural analysis these discontinuities must be taken into consideration as they modify the stress, strain and displacement fields on such a scale that the assumption of a homogeneous medium would no longer be meaningful. The theory of damage can also be used to study the evolution of cracks, as shown in Section 8.6.4, but more global methods allow consideration of these problems in a more synthetic and simplified way at least for quite simple structures and loads.

As early as 1920, Griffith showed that the failure of a brittle elastic medium could be characterized by a variable, later called the energy release rate, whose critical value, independent of the geometry of the structure, was a characteristic of the material. This approach, called the global approach, was not generalized and formally constructed starting from the thermodynamics of irreversible processes until the seventies (as evidenced by the works of Son, Lemaitre and Chaboche). It showed that in all cases the essential phenomena occur in the vicinity of the crack front and that it is possible to study the macroscopically cracked medium with the help of intrinsic variables. This is due to the high stress concentration present at the