CIV-E4080
Material Modelling
in Civil Engineering D

Period V, 2024

Lecture 3. Plasticity: Basics

Contents

Lecture 3. Plasticity: Basics

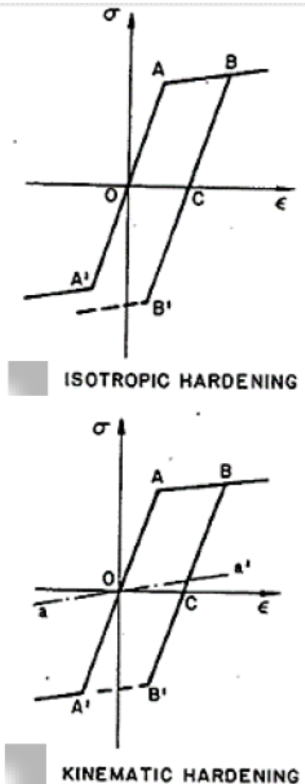
Hardening rules

The phenomena of hardening: Yield stress increases with further increase of plastic strains

Hardening and its various stages is modelled by introducing some *internal variable* as for instance *a hardening parameter* $\kappa > 0$

and assumes, for instance, that the plastic modulus changes as a function of the hardening parameter

$$E_\mathrm{p} = E_\mathrm{p}(\kappa)$$



Choice of the hardening parameter:

$\kappa = W_\mathrm{p}$ or $\kappa = \varepsilon_\mathrm{p}$

plastic work

$$W_\mathrm{p} = \int \sigma_{ij} \mathrm{d}\varepsilon_{ij}^p$$

$$W_\mathrm{p} > 0$$

accumulated *effective or equivalent* plastic strain

$$\varepsilon_\mathrm{p} = \int \sqrt{\mathrm{d}\varepsilon_{ij}^p \mathrm{d}\varepsilon_{ij}^p} > 0$$

Effective plastic strain increment:

$$\left|\mathrm{d}\varepsilon_\mathrm{p}\right| = \sqrt{\mathrm{d}\varepsilon_{ij}^p \mathrm{d}\varepsilon_{ij}^p} \geq 0$$

Using such hardening parameter - the *lazy way* - is more practical than computing the plastic work

For a material, the functional relation $E_\mathrm{p} = E_\mathrm{p}(\kappa)$ can be determined from the uniaxial tensile stress-strain curve in terms of the above definition of the hardening parameter κ

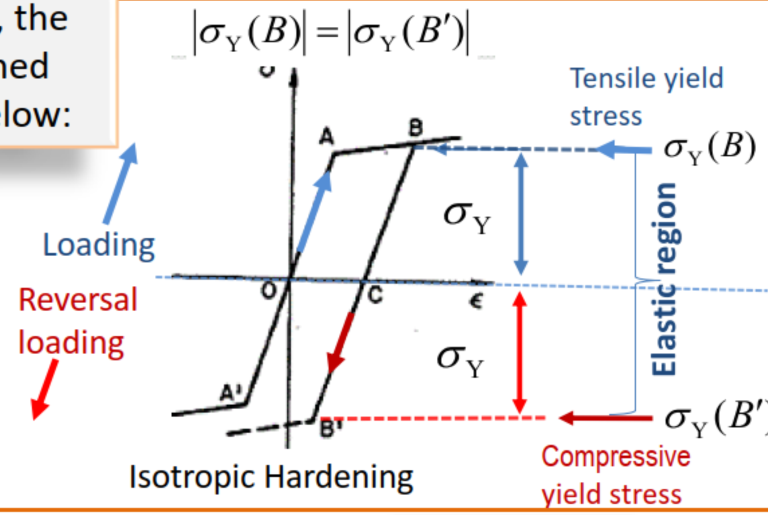
Hardening rules (models)

For a material under **reversal** or cyclic loading, the subsequent *yield stress* σ is usually determined usually by one of the three hardening rules below:

Isotropic Hardening rule:

- Assumes ***equal yield stress*** in tension and **compression**
  - thus the *Bauschinger* effect is *completely neglected*.

Hardening rule: $|\sigma| = |\sigma(\kappa)|, \quad \kappa > 0$



Isotropic Hardening

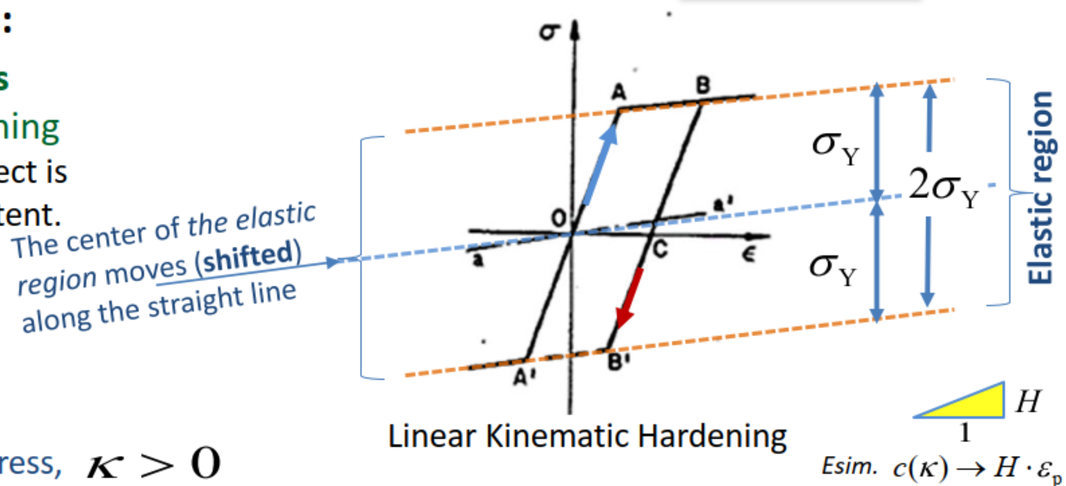
Kinematic Hardening rule:

- The **elastic range remains unchanged** during hardening
  - So the *Bauschinger* effect is *accounted* to its full extent.

Hardening rule:

$$|\sigma - c(\kappa)| = \sigma_0,$$

shift of the yield stress, $\kappa > 0$



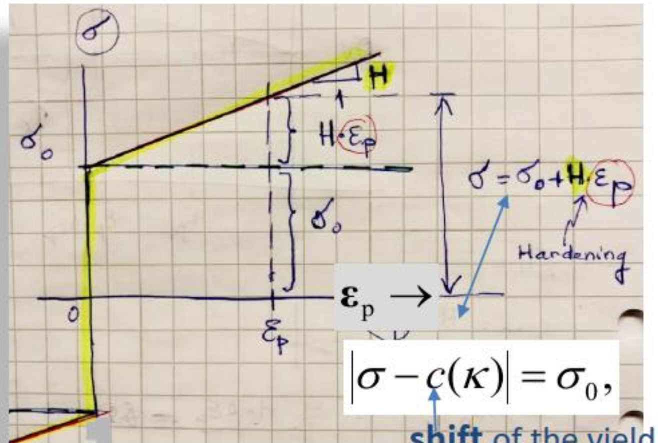
Linear Kinematic Hardening

Esim. $c(\kappa) \to H \cdot \varepsilon_p$

Lecture 3. Plasticity: Basics

Hardening rules (models)

Kinematic Hardening rule:

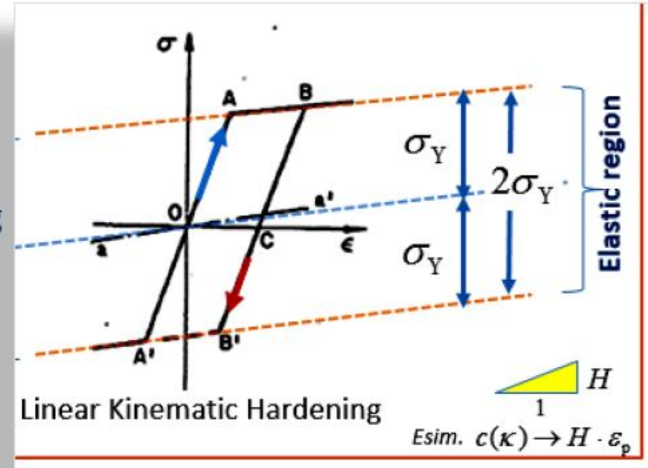


$$|\sigma - c(\kappa)| = \sigma_0,$$

shift of the yield stress

Loading

Reversal loading



Independent Hardening rule:

- Assumes ***yield stress*** in **tension** and **compression** are ***independent*** **(different)**

Hardening rule:

$$\sigma = \begin{cases} \sigma_t(\kappa_t), & \sigma > 0 \\ \sigma_c(\kappa_c), & \sigma < 0 \end{cases}$$

$\kappa_t > 0$ **tension**

compression $\kappa_c > 0$

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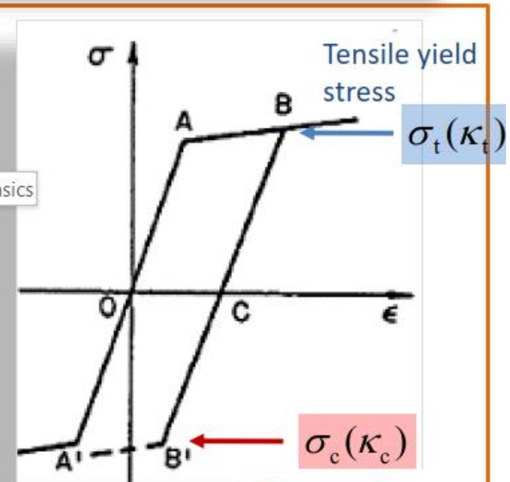
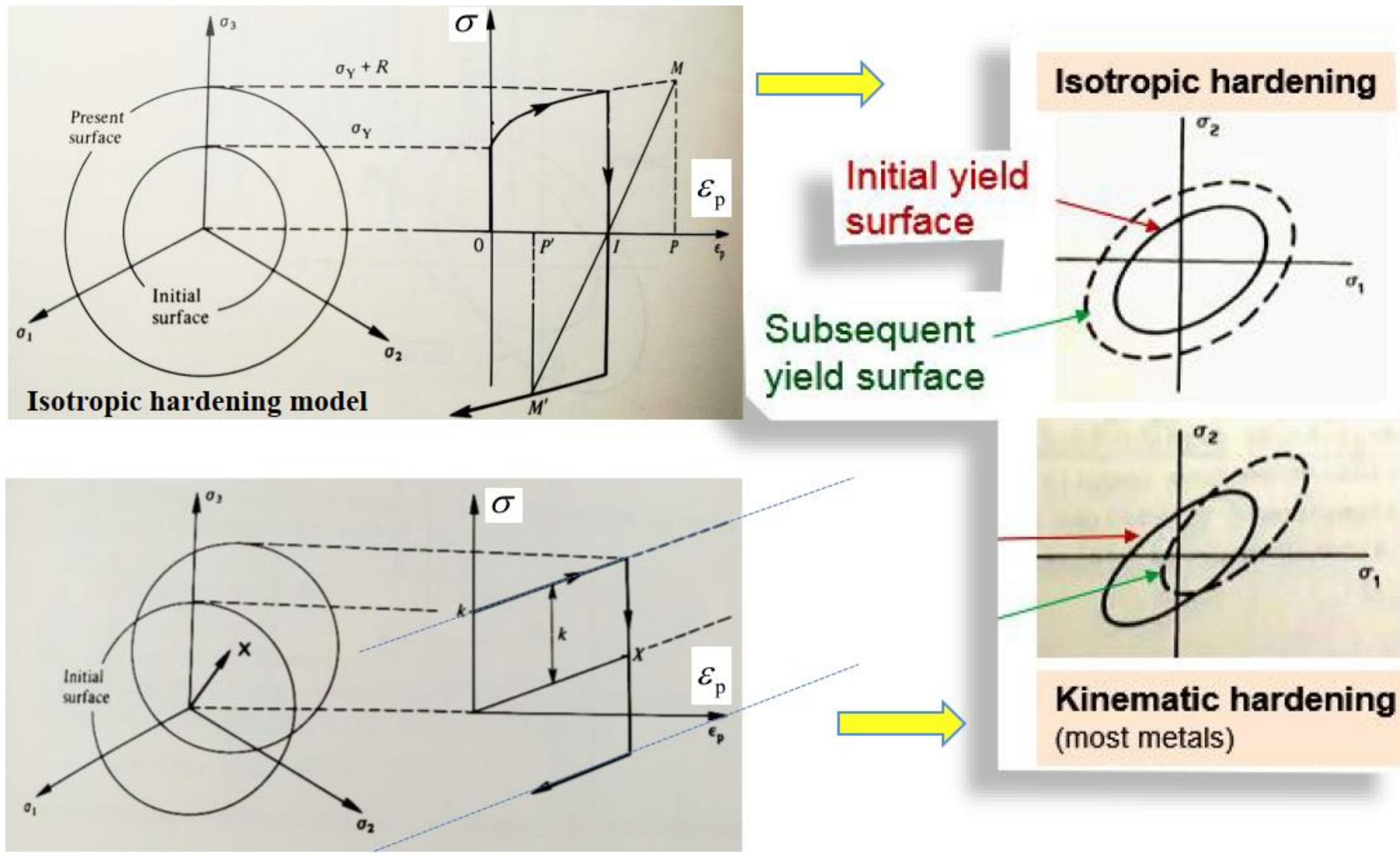


Lecture 3. Plasticity: Basics



Kinematic hardening model

Some 1-D empirical models for work (also called strain-) hardening phenomena:

These models are relationship between the stress and the amount of plastic strain

$$\sigma = K\epsilon_p^n$$

$$\sigma = \sigma_y + K\epsilon_p^n \qquad n \sim 0.2 - \dots\ 0.5$$

$$\sigma = \sigma_y + K(\epsilon_0 + \epsilon_p)^n$$

Lecture 3. Plasticity: Basics

Mixed hardening rule

A subsequent yield surface:

effective plastic strain

$$F(\sigma_{ij}, \varepsilon_{ij}^{p}; k(\varepsilon_{ij}^{p})) = F(\sigma_{ij} - \alpha_{ij}) - k^2(\bar{\varepsilon}^{p}) = 0,$$

***F* determines** the **shape**

Shift or (translation)

Kinematic hardening

Determines the **size** as function of cumulated plastic strain (**expansion**)

Isotropic hardening

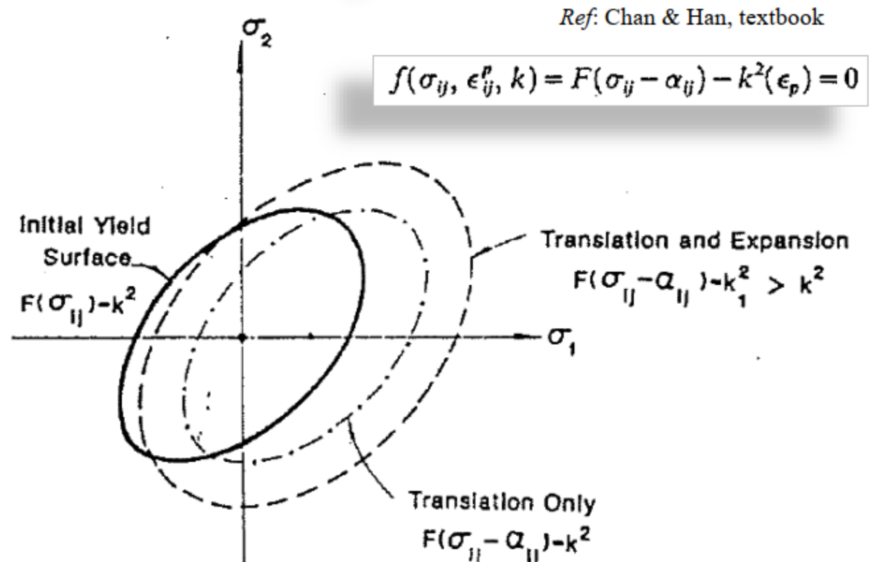
α_{ij}, k^2 - hardening parameters

Mixed hardening rule allows to model different degrees of Bauschinger effect by adjusting the two hardening parameters α_{ij} & k^2

Mixed hardening is a result of a combination of *kinematic* and *isotropic hardening* (Hodge, 1957)

Ref: Chan & Han, textbook

$$f(\sigma_{ij}, \epsilon_{ij}^{p}, k) = F(\sigma_{ij} - \alpha_{ij}) - k^2(\epsilon_p) = 0$$



Lecture 3. Plasticity: Basics

Anisotropic yield criteria

- For incompressible plastic materials, generalization of the von Mises criterion
- Cannot be expressed as functions of the stress invariants

$$\boldsymbol{s} : \boldsymbol{D} : \boldsymbol{s} = 1, \quad D_{ijkl} = D_{klij} = D_{jikl} = D_{ijlk}$$

Lecture 3. Plasticity: Basics

Anisotropic yield criteria

- For incompressible plastic materials, generalization of the von Mises criterion
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$$\boldsymbol{s} : \boldsymbol{D} : \boldsymbol{s} = 1, \quad D_{ijkl} = D_{klij} = D_{jikl} = D_{ijlk}$$

The Hill criterion

- Corresponds to a particular kind of anisotropy in which 3 planes of symmetry are conserved during hardening of the material
- Principal axes of anisotropy are intersections of these 3 planes
- The criterion is formulated w.r.t. these axes as the reference axes, say $(0, x_1, x_2, x_3)$.

$$\begin{aligned} D_{1111} &= F + H & D_{2222} &= F + G & D_{3333} &= G + H \\ D_{1122} &= -F & D_{2233} &= -G & D_{3311} &= -H \\ D_{1212} &= L/2 & D_{2323} &= M/2 & D_{3131} &= N/2 \end{aligned}$$

$$\begin{aligned} &F(\sigma_{11} - \sigma_{22})^2 + G(\sigma_{22} - \sigma_{33})^2 + H(\sigma_{33} - \sigma_{11})^2 \\ &+ 2L\sigma_{12}^2 + 2M\sigma_{23}^2 + 2N\sigma_{13}^2 = 1 \end{aligned}$$

The Hill criterion

- *F, G, H, L, M, N* are the 6 scalar parameters which characterize the state of anisotropic hardening
- Three simple tension and three simple shear experiments to determine the parameters:

1. Tensile yield stress σ_{s_1} in the direction $x_1 \rightarrow F + H = 1/\sigma_{s_1}^2$
2. Tensile yield stress σ_{s_2} in the direction $x_2 \rightarrow F + G = 1/\sigma_{s_2}^2$
3. Tensile yield stress σ_{s_3} in the direction $x_3 \rightarrow G + H = 1/\sigma_{s_3}^2$
4. Shear yield stress $\sigma_{s_{12}}$ in the plane $(0, x_1, x_2) \rightarrow 2L = 1/\sigma_{s_{12}}^2$
5. Shear yield stress $\sigma_{s_{23}}$ in the plane $(0, x_2, x_3) \rightarrow 2M = 1/\sigma_{s_{23}}^2$
6. Shear yield stress $\sigma_{s_{31}}$ in the plane $(0, x_3, x_1) \rightarrow 2N = 1/\sigma_{s_{31}}^2$

I reproduce some failure criteria communly used for wood. This pdf-material is from the presentation in this course (2020) given by Joonas Jaaranen (reproduced with his permission)

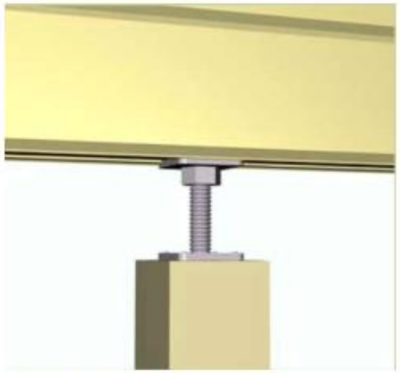
Failure criteria and material modelling of wood

Presentation on CIV-E4080 Material Modelling in Civil Engineering

Joonas Jaaranen, Aalto University

Motivation

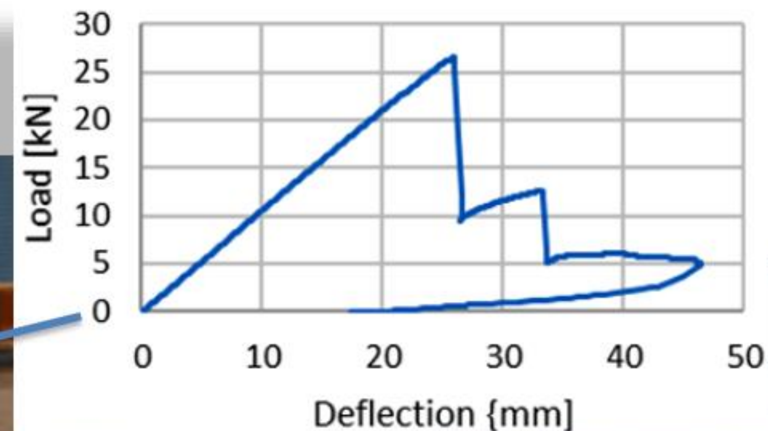
Details with complex behaviour

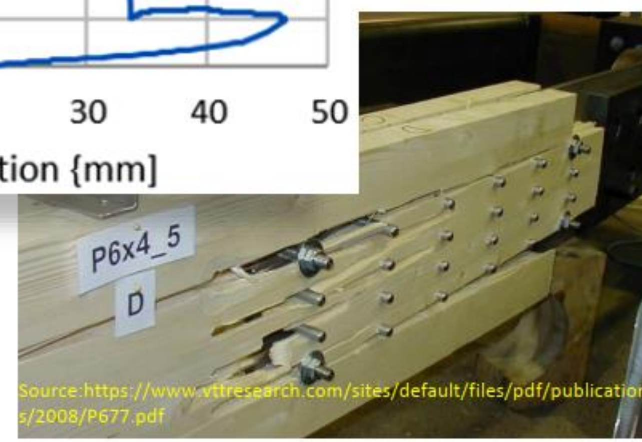


Source: RT 82-11168

Source: https://eurocodes.jrc.ec.europa.eu/doc/WS2008/EN1995_3_Dietsch.pdf

Civil Engineering Department Lab 17.10.2019





Source:https://www.vttresearch.com/sites/default/files/pdf/publications/2008/P677.pdf

Wood as material

- Naturally grown material
- Microscale: cellular structure
- Macroscale growth characteristics: annual rings, knots, spiral grain, compression wood,...

- Time- and moisture-dependent behaviour

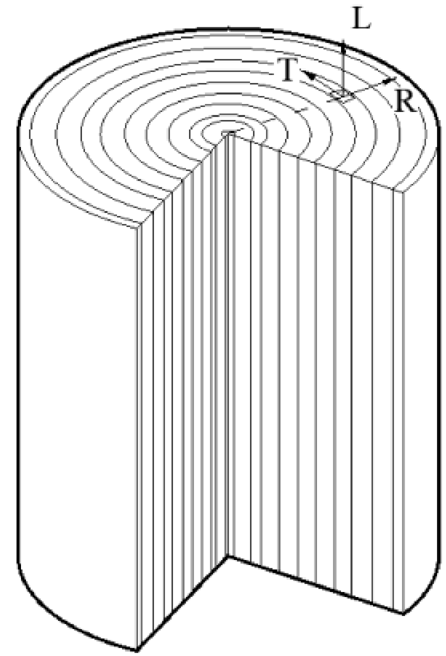
- 'Ductile' like compressive failure behaviour
- Brittle tensile failure

- Strongly orientated properties

- Different strength in tension and compression

- Properties depend on wood species, growth conditions, density, moisture content, stress-history,...

Wood as material

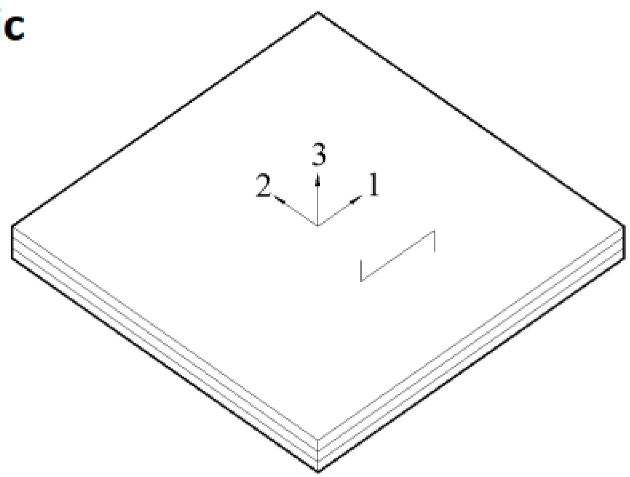


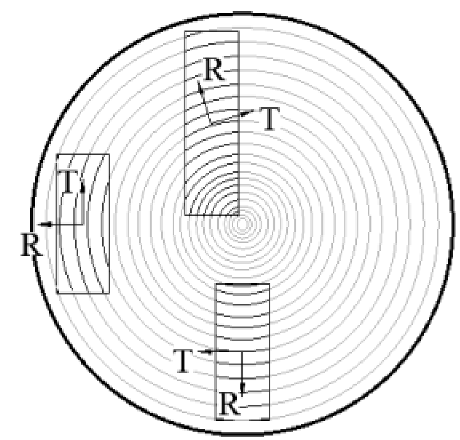
Common assumptions for modelling

- Homogeneity
- Orthotropy
- Grain orientation aligns with length-wise direction of the component

Choices of material coordinate system

- Cylindrical orthotrophic
- Cartesian orthotropic
- Transverse isotropic





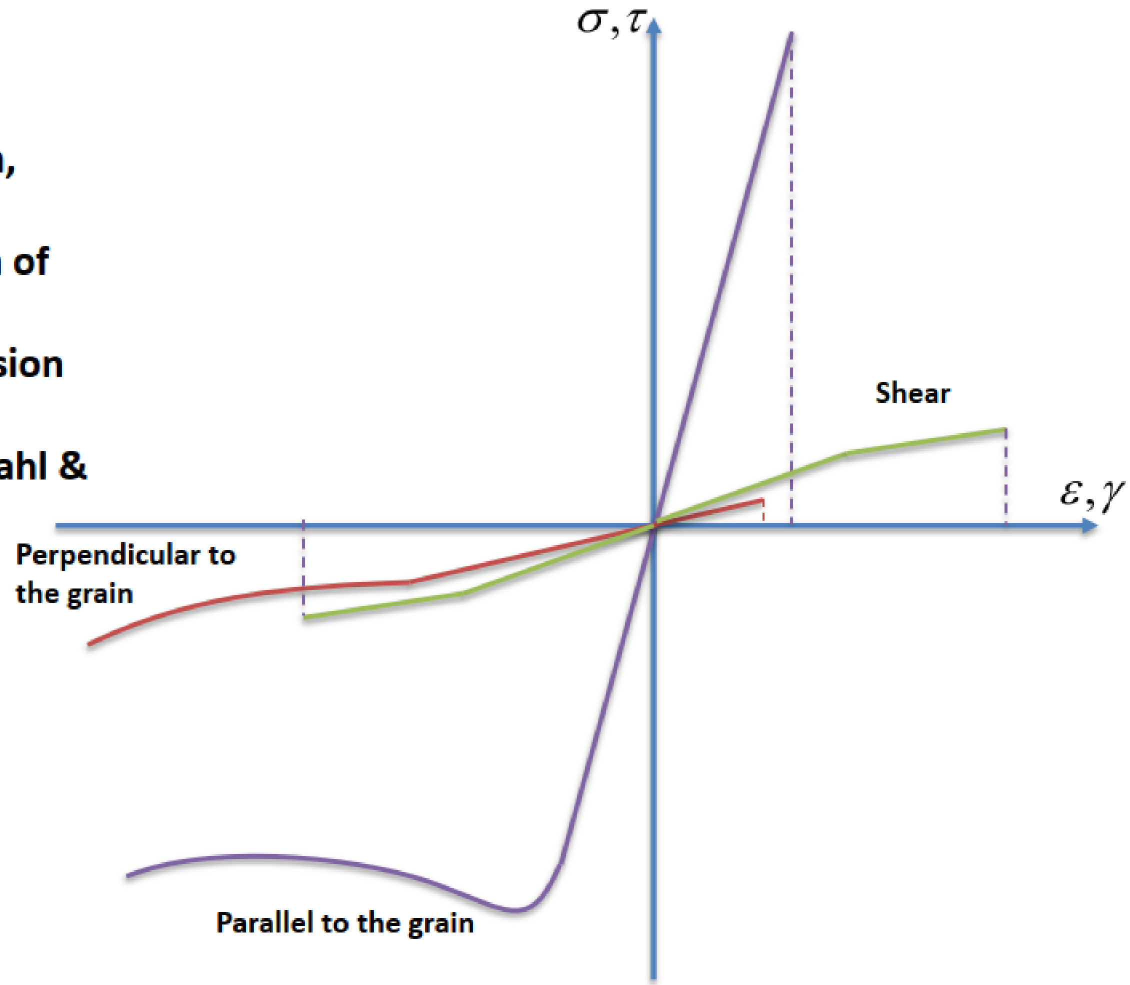
Wood as material

Material behaviour

- Different behaviour in tension, compression and shear
- Dependent on the orientation of the load
- Example references: compression (Holmberg et al. 1998, Widehammar 2004), shear (Dahl & Malo 2009)

$$\frac{f_{t,90}}{f_{t,0}} \approx \frac{1}{20} \dots \frac{1}{30}$$

$$\frac{f_{c,90}}{f_{c,0}} \approx \frac{1}{10}$$



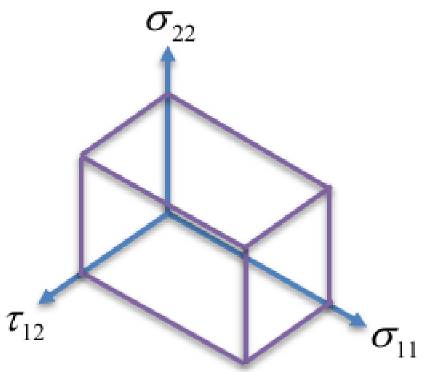
Common failure criteria

Basic criteria

- Common in practice
- Basis for many multi-surface criteria

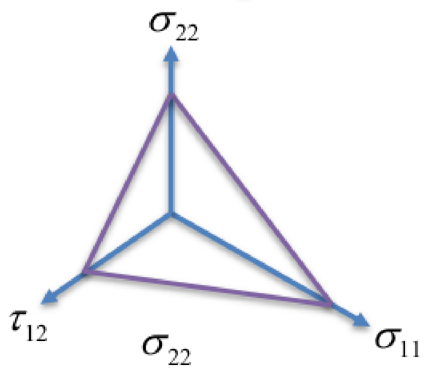
Maximum stress

$$\frac{\sigma_{11}}{f_1} = 1, \quad \frac{\sigma_{22}}{f_2} = 1, \quad \frac{\tau_{12}}{f_{12}} = 1$$



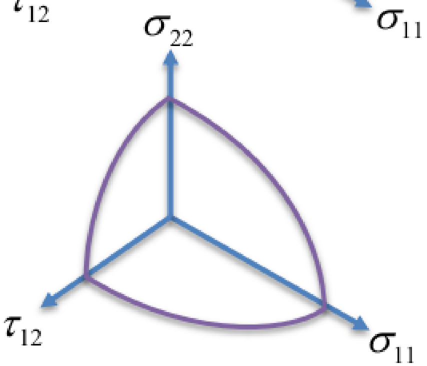
Linear interaction

$$\frac{\sigma_{11}}{f_1} + \frac{\sigma_{22}}{f_2} + \frac{\tau_{12}}{f_{12}} = 1$$



Quadratic interaction

$$\left(\frac{\sigma_{11}}{f_1}\right)^2 + \left(\frac{\sigma_{22}}{f_2}\right)^2 + \left(\frac{\tau_{12}}{f_{12}}\right)^2 = 1$$



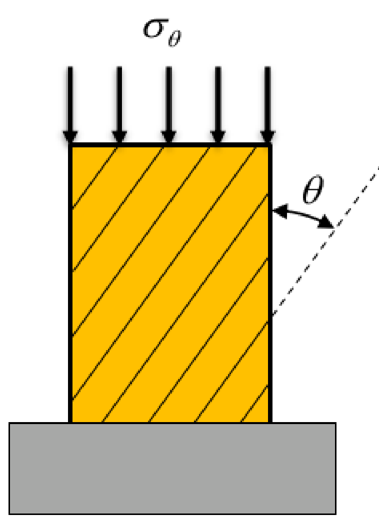
Common failure criteria

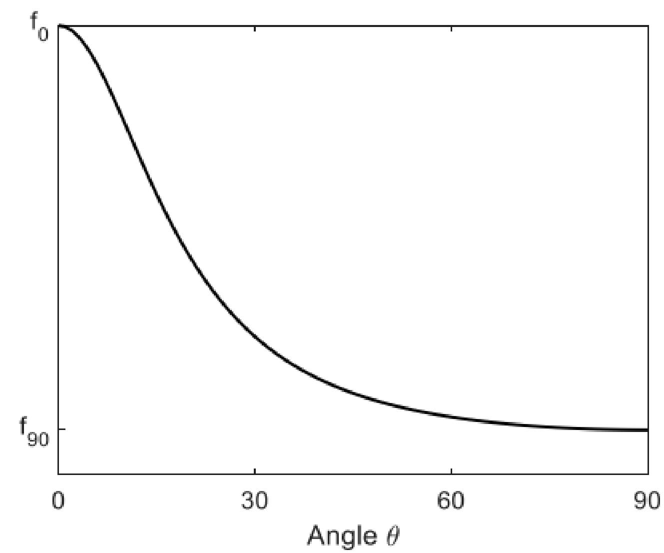
Hankinson equation (Hankinson, 1921)

- **Axial strength at angle to the grain**
- **Originally for compression**
- **Widely used for practical calculations**
- **For example EN 1995-1-1**

$$\boxed{\frac{\sigma_\theta}{f_\theta} = 1}$$

$$\boxed{f_\theta = \frac{f_0 f_{90}}{f_0 \sin^2\theta + f_{90}\cos^2\theta}}$$



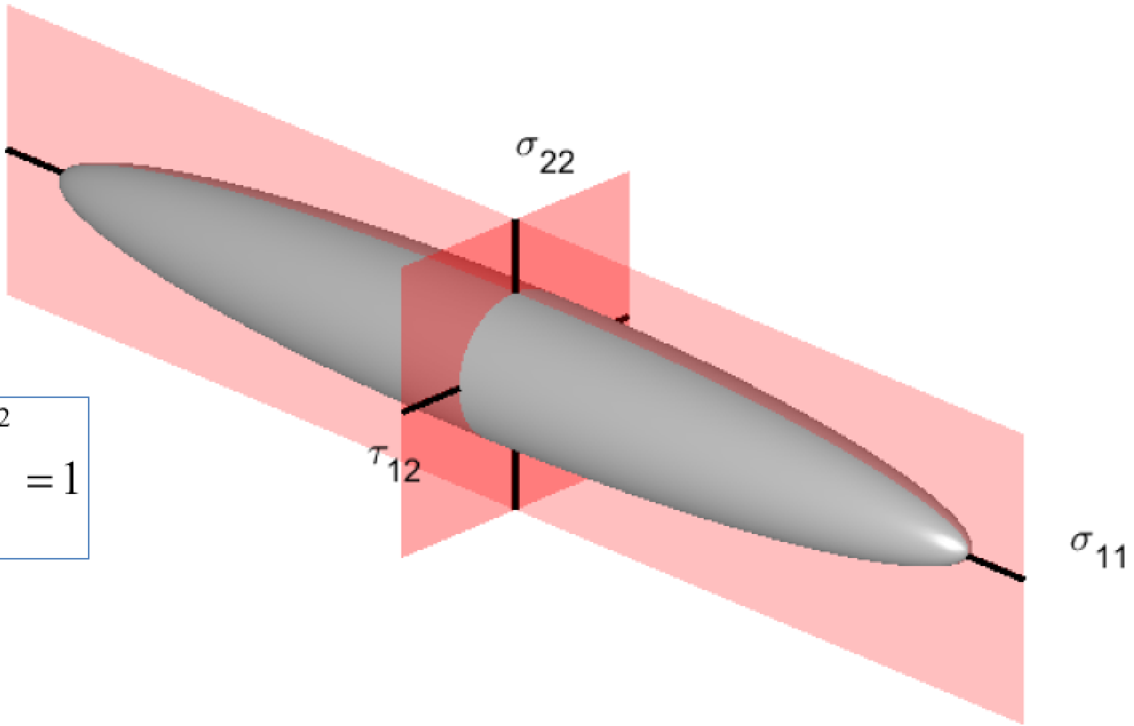


Common failure criteria

Tsai-Hill criterion (Azzi & Tsai 1965)

- **Accounts different orientation**
- **No distinction tension/compression**

$$\left(\frac{\sigma_{11}}{f_1}\right)^2 - \frac{\sigma_{11}\sigma_{22}}{f_1^2} + \left(\frac{\sigma_{22}}{f_2}\right)^2 + \left(\frac{\tau_{12}}{f_{12}}\right)^2 = 1$$



Common failure criteria

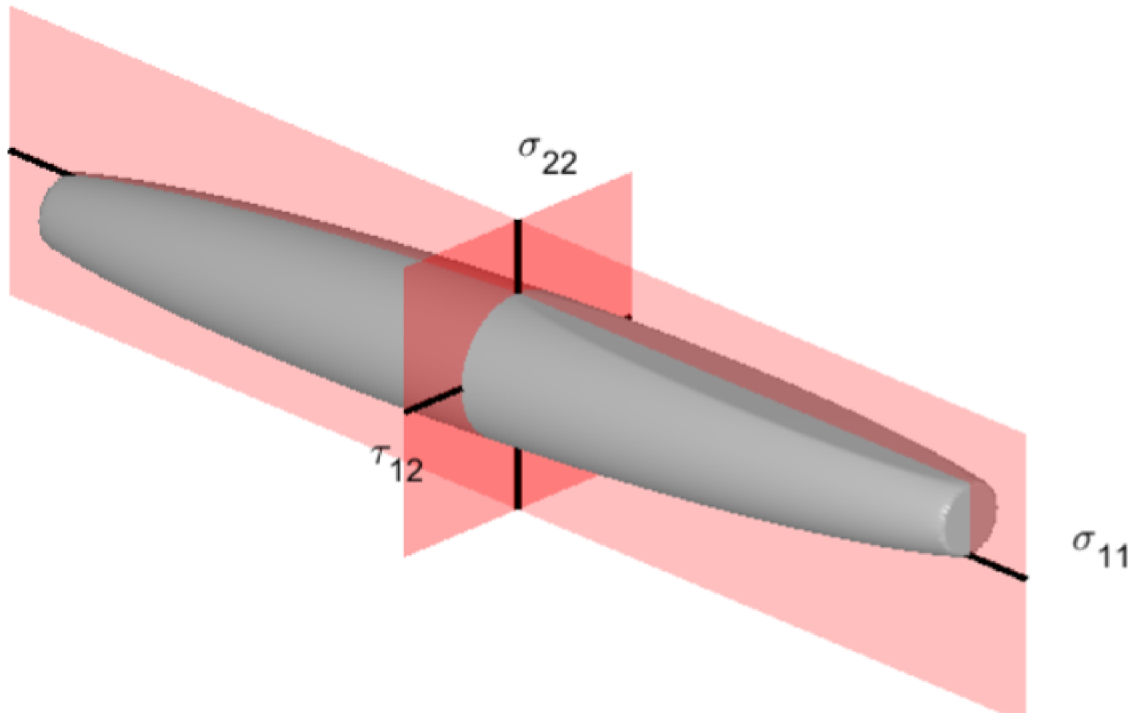
Norris criterion (Norris 1962)

- **Multi-surface approach**
- **No distinction tension/compression**

$$\left(\frac{\sigma_{11}}{f_1}\right)^2 - \frac{\sigma_{11}\sigma_{22}}{f_1 f_2} + \left(\frac{\sigma_{22}}{f_2}\right)^2 + \left(\frac{\tau_{12}}{f_{12}}\right)^2 = 1$$

$$\left(\frac{\sigma_{11}}{f_1}\right)^2 = 1$$

$$\left(\frac{\sigma_{22}}{f_2}\right)^2 = 1$$



Lecture 3. Plasticity: Basics

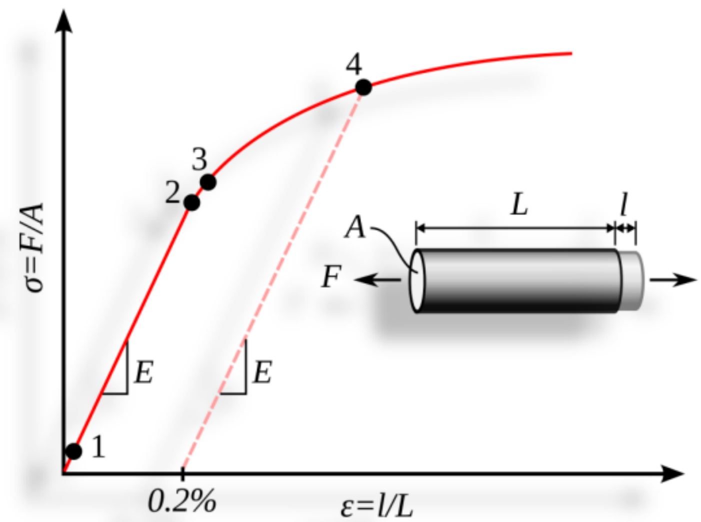
The elastic-plastic calculation

The moment-curvature relation for the symmetric rectangular cross-section

Task: Derive the relation between the bending moment $\boldsymbol{M}$ and the resulting curvature

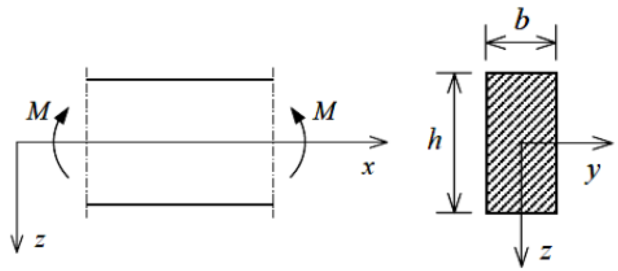
$$\kappa(M)$$

when the loading increases monotonically.



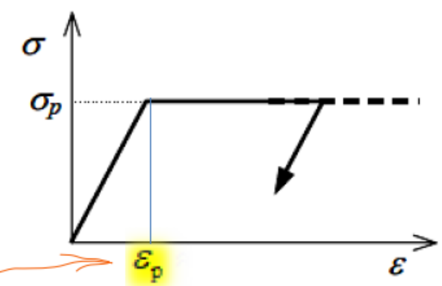
2: Proportionality limit
3: Elastic limit
4: Offset yield

steel $\varepsilon_p \approx 0.2\%$



Pure bending of a beam

rectangular cross-section



Elastic-perfectly plastic material behaviour (Idealised behaviour)

Idealised elastic-plastic constitutive model* describes simple mechanical behavior of metals

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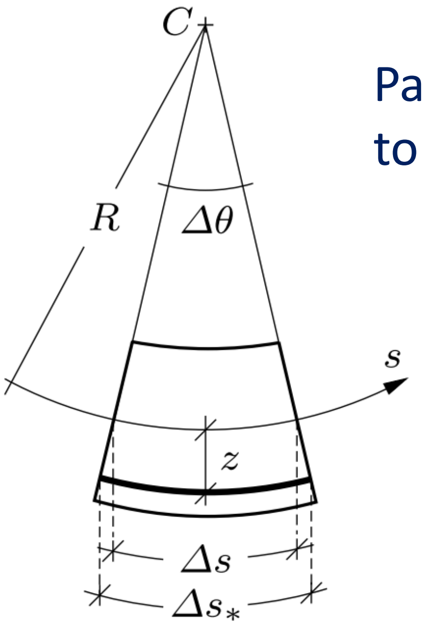
* Learn more details in *material modelling* course

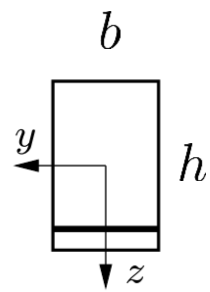
N.B. it is assumed that stability effects come after the profile reaches full plastic moment

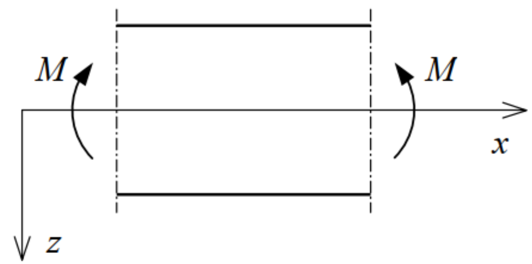
Lecture 3. Plasticity: Basics

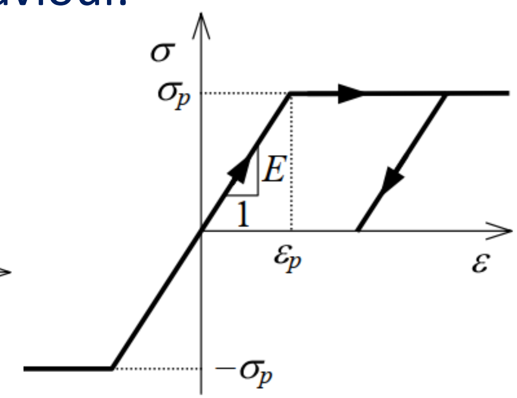
The moment-curvature relation for the rectangular cross-section

Part of a beam with rectangular cross section subjected to bending, with ideal-plastic material behaviour.









Normal strain: $\varepsilon = \dfrac{\Delta s_* - \Delta s}{\Delta s} = kz$

Normal stress: $\sigma = E\varepsilon = Ekz$

$$I = \int_A z^2 \mathrm{d}A = \frac{bh^3}{12}$$

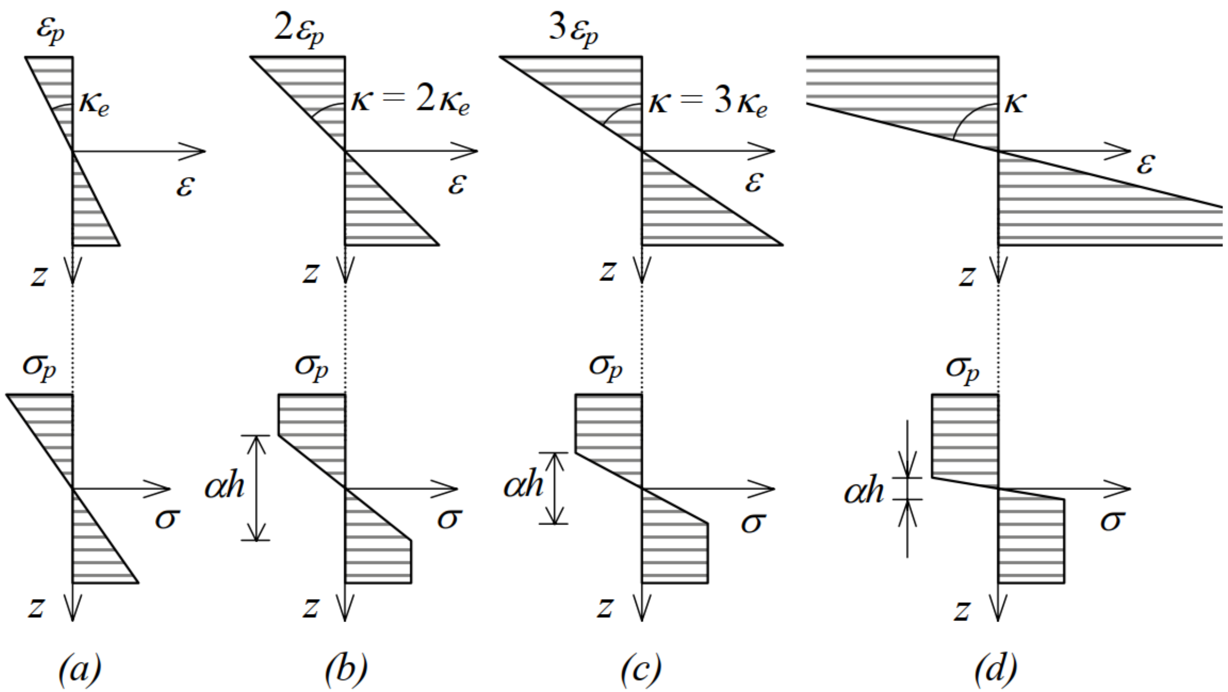
Bending moment: $M = \displaystyle\int_A z\sigma \mathrm{d}A = k \int_A E z^2 \mathrm{d}A = EIk$ -- if *E* is constant

The moment-curvature relation for the rectangular cross-section

This is valid until the yield stress σ_p is reached in the extreme fibres, i.e., until

$$M = M_e = \frac{1}{6} b h^2 \sigma_p \quad \text{and} \quad \kappa = \kappa_e = \frac{2}{h}\frac{\sigma_p}{E}$$

Strain and stress distributions in the cross-section:



$$\sigma = E\varepsilon \quad \text{for} \quad |\varepsilon| < \varepsilon_p$$

$$\sigma = \pm\sigma_p \quad \text{for} \quad |\varepsilon| \geq \varepsilon_p$$

or

$$\sigma = E\varepsilon \quad \text{for} \quad |z| < \frac{1}{2}\alpha h$$

$$\sigma = \pm\sigma_p \quad \text{for} \quad |z| \geq \frac{1}{2}\alpha h$$

Lecture 3. Plasticity: Basics

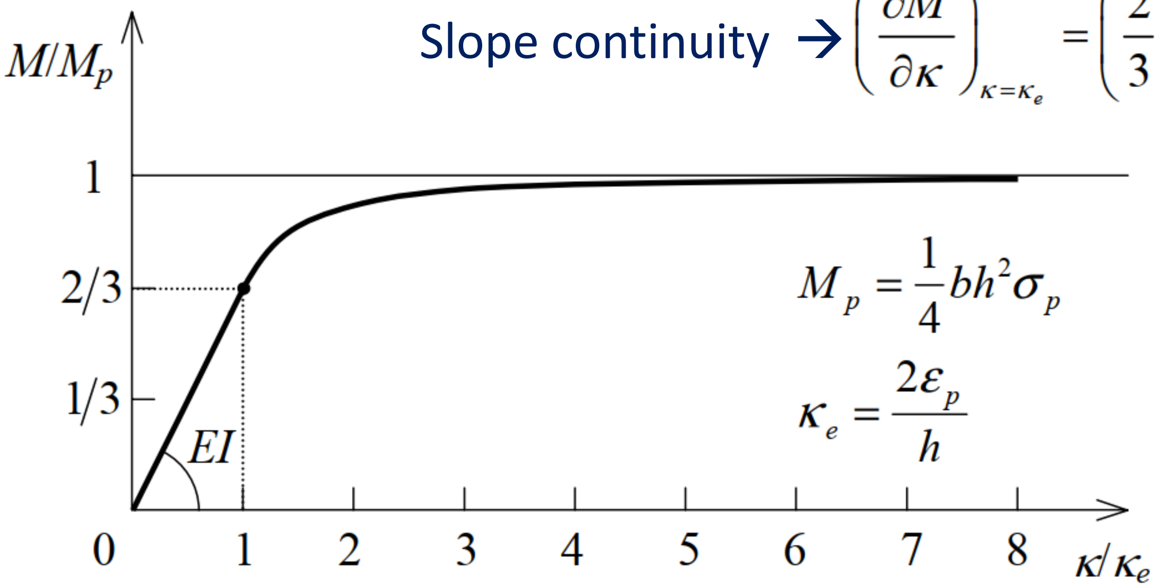
The moment-curvature relation for the rectangular cross-section

Relation between M and κ in the elastic-plastic phase takes the form:

$$M = M_p\left[1-\frac{1}{3}\left(\frac{\kappa_e}{\kappa}\right)^2\right] \quad \text{for} \quad \kappa \geq \kappa_e, \quad \text{where} \quad M_p = \frac{1}{4}bh^2\sigma_p \text{ is the fully plastic moment.}$$

The moment-curvature diagram:

Slope continuity → $\left(\frac{\partial M}{\partial \kappa}\right)_{\kappa=\kappa_e} = \left(\frac{2}{3}M_p\frac{\kappa_e^2}{\kappa^3}\right)_{\kappa=\kappa_e} = \frac{\frac{2}{3}M_p}{\kappa_e} = \frac{M_e}{\kappa_e}$



$$M_p = \frac{1}{4}bh^2\sigma_p$$

$$\kappa_e = \frac{2\varepsilon_p}{h}$$

Lecture 4. Plasticity: Associative flow rule

Contents

Lecture 4. Plasticity: Associative flow rule

The three ingredients of engineering plasticity theory

Firstly *three ingredients* is not related to *la haute cuisine* but is an imaged saying for the *three sub-models* needed in the constitutive model involved during plastic flow. These sub-models can also be combined. These sub-models will tell you, depending on the material, the actual stress state, the history of plastic deformations, will the considered material point yield or remain elastic? If it yields then what will be the resulting additional plastic increment. That's all. Now, the complexity arises from the fact that the yielding criteria, usually, evolves with subsequent plastic deformations, so it depends on the history. In other word, the mechanical problem is simply *materially non-linear*.

Here we mean time-independent[8] plasticity under isothermal processes. So, the temperature is constant and should not activate the creep; so $T \leq T_m/4$, where T_m being the melting temperature. When this is not the case, this results in thermally activated creep and leads to one type of time dependent plasticity.

- **Yielding criteria**: $f(\sigma_1, \sigma_2, \sigma_2, k; \kappa) = 0$

  The yield function can also expressed in terms of stress invariants I_1, J_2 and J_3 instead of principle stresses. The threshold stress $k = k(\kappa)$ can depend on some internal parameter κ, like cumulated effective plastic strain of plastic work, is yield stress σ_y in uni-axial tension or τ_y yield shear stress in pure shear test.

  Note that in simple words, the yield criterion given by $f = 0$ is in general, *non-linear*. This non-linearity results not only from the fact that for many materials the plasticity threshold k depends on history and evolves $k = k(\sigma, \kappa)$ but also, the internal parameters, here the plastic deformations, depend also on history even in time independent plasticity. For instance, in some metals, the Bauschinger effect, means that the yield thresholds are changed by *cold forming* due to plastic deformations in cyclic loading.

Lecture 4. Plasticity: Associative flow rule

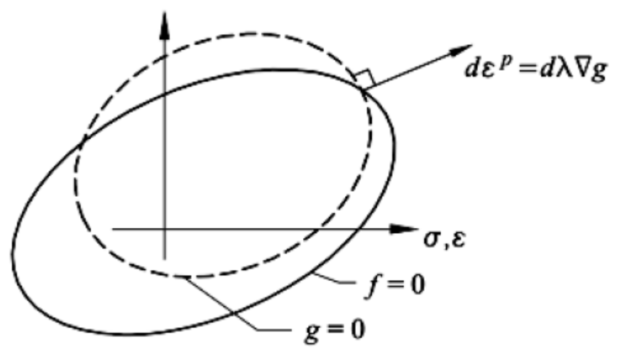
- **Flow rule**: The flow rule is just a model to determine the increment of subsequent plastic strains. Classically, it is defined to be proportional to gradient $\nabla_\sigma f \equiv \partial f / \partial \sigma_{ij}$ of a scalar plastic potential f or g. The proportionality coefficient $\mathrm{d}\lambda$ has to be determined by constraining the subsequent stress state resulting from this increment of plastic deformation to remain on the yield surface $f(\sigma) = 0$. This condition is know as *consistency* condition for the plastic flow.

$$\mathrm{d}\epsilon^{(\mathrm{p})} = \mathrm{d}\lambda \cdot \underbrace{\nabla_\sigma f}_{\equiv \mathbf{n}_f},$$

associative case = hydrostatic pressure insensitive

$$\mathrm{d}\epsilon^{(\mathrm{p})} = \mathrm{d}\lambda \cdot \underbrace{\nabla_\sigma g}_{\equiv \mathbf{n}_g},$$

non-associative case = hydrostatic pressure sensitive



- **Hardening rule** – defines how the yielding or loading surface evolves during the course of evolution of plastic flow.

Lecture 4. Plasticity: Associative flow rule

Plastic flow occurs only if the following 2 conditions are simultaneously satisfied

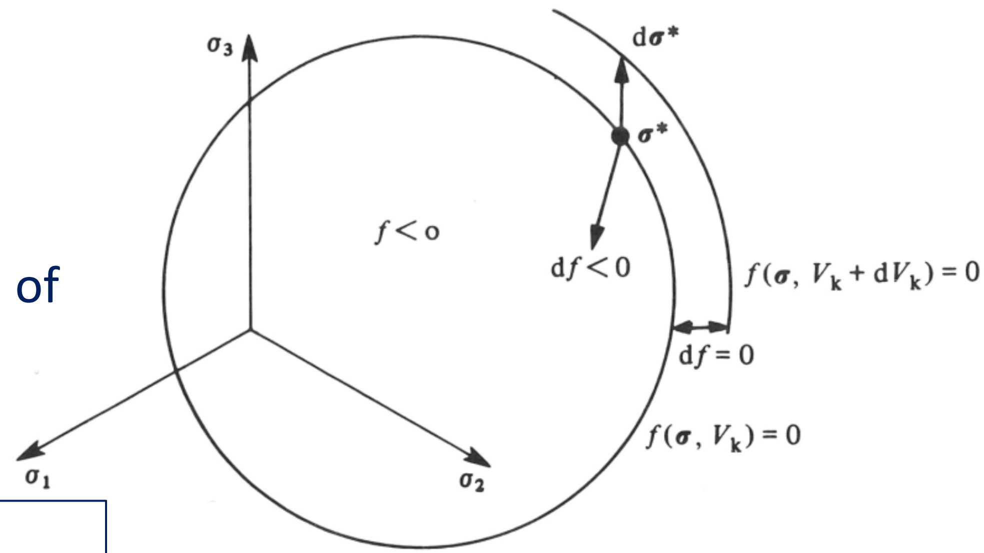
1. The representative point of the stress state is situated on the loading surface

$$f(\boldsymbol{\sigma}^*, V_k) = 0$$

2. The representative point of the stress state does not leave the loading surface ($f > 0$ is impossible). During the continuous flow, the consistency condition,

$$\mathrm{d}f(\boldsymbol{\sigma}^*) = \frac{\partial f}{\partial \boldsymbol{\sigma}} : \mathrm{d}\boldsymbol{\sigma}^* + \frac{\partial f}{\partial V_k}\mathrm{d}V_k = 0$$

must be satisfied.
This implies that the point representative of the stress $(\boldsymbol{\sigma}^* + \mathrm{d}\boldsymbol{\sigma}^*)$ remains on the loading surface.



- $f < 0$: → elastic behaviour
- $f = 0$ and $\mathrm{d}f = 0$: → plastic flow
- $f = 0$ and $\mathrm{d}f < 0$: → elastic unloading

Lecture 4. Plasticity: Associative flow rule

Flow rule (evolution of plastic strain)

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\gamma}\boldsymbol{r}(\boldsymbol{\sigma}, \xi)$$

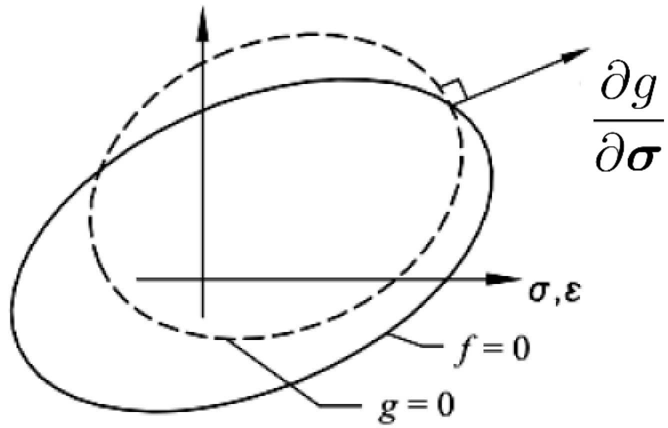
$\xi = (\boldsymbol{\alpha}, e_p)$ - plastic variables

- plastic consistency parameter γ: $\gamma > 0$ (plastic), $\gamma = 0$ (elastic)

Flow potential

$$\boldsymbol{r} = \frac{\partial g(\boldsymbol{\sigma}, \xi)}{\partial \boldsymbol{\sigma}} \quad \Rightarrow \quad \dot{\boldsymbol{\varepsilon}}^p = \dot{\gamma}\frac{\partial g(\boldsymbol{\sigma}, \xi)}{\partial \boldsymbol{\sigma}}$$

- plastic strain increases in the normal direction to the flow potential



Loading-unloading conditions

$$\dot{\gamma} \geq 0, \quad f \leq 0, \quad \dot{\gamma} f = 0.$$

Consistency condition

$$\dot{\gamma}\dot{f} = 0.$$

- $f < 0$: $\rightarrow$ elastic behaviour
- $f = 0$ and $\mathrm{d}f = 0$: $\rightarrow$ plastic flow
- $f = 0$ and $\mathrm{d}f < 0$: $\rightarrow$ elastic unloading

Lecture 4. Plasticity: Associative flow rule

Elasto-Plastic tangent operator

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^e + \dot{\boldsymbol{\varepsilon}}^p$$

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\gamma}\boldsymbol{r} = \dot{\gamma}\frac{\partial g(\boldsymbol{\sigma}, \xi)}{\partial \boldsymbol{\sigma}}$$

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{C}^{ep}\dot{\boldsymbol{\varepsilon}}$$

$$\boldsymbol{C}^{ep} = \boldsymbol{C} - \frac{\boldsymbol{C} : \boldsymbol{r}\frac{\partial f}{\partial \boldsymbol{\sigma}} : \boldsymbol{C}}{\frac{\partial f}{\partial \boldsymbol{\sigma}} : \boldsymbol{C} : \boldsymbol{r} - \frac{\partial f}{\partial \xi}h}$$

- elasto-plastic tangent operator

- the elasto-plastic constitutive relation defines the stress increment uniquely once the total strain increment and the current stress state is known.

Lectures 4 & 5. Plasticity

Contents

1. *Associative flow rule*
2. *Elasto-Plastic tangent operator*
3. ***Consistent tangent operator***
4. ***Pressure-dependent yield criteria***

Lecture 4. Plasticity: Associative flow rule

Elasto-Plastic tangent operator

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^e + \dot{\boldsymbol{\varepsilon}}^p$$

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\gamma}\boldsymbol{r} = \dot{\gamma}\frac{\partial g(\boldsymbol{\sigma}, \xi)}{\partial \boldsymbol{\sigma}}$$

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{C}^{ep}\dot{\boldsymbol{\varepsilon}}$$

$$\boldsymbol{C}^{ep} = \boldsymbol{C} - \frac{\boldsymbol{C} : \boldsymbol{r}\frac{\partial f}{\partial \boldsymbol{\sigma}} : \boldsymbol{C}}{\frac{\partial f}{\partial \boldsymbol{\sigma}} : \boldsymbol{C} : \boldsymbol{r} - \frac{\partial f}{\partial \xi} h}$$

- elasto-plastic tangent operator

- the elasto-plastic constitutive relation defines the stress increment uniquely once the total strain increment and the current stress state is known.

For linear combined hardening model

$$f(\boldsymbol{s}, \boldsymbol{\alpha}, e_p) = || \ \boldsymbol{s} - \boldsymbol{\alpha} \ || - \sqrt{\frac{2}{3}} k(e_p)$$

$$k(e_p) = \sigma_y^0 + (1 - \beta) H e_p$$

$$\dot{\boldsymbol{\alpha}} = \frac{2}{3}\beta H \dot{\boldsymbol{e}}^p$$

$$\boldsymbol{C}^{ep} = \boldsymbol{C} - \frac{4\mu^2}{2\mu + \frac{2}{3}H}\boldsymbol{N}\boldsymbol{N}$$

Lecture 4. Plasticity: Associative flow rule

Consistent (algorithmic) tangent operator

$$\boldsymbol{C}^{alg} = \frac{\partial \Delta \boldsymbol{\sigma}}{\partial \Delta \boldsymbol{\varepsilon}} \qquad \text{vs} \qquad \boldsymbol{C}^{ep} = \frac{\partial \dot{\boldsymbol{\sigma}}}{\partial \dot{\boldsymbol{\varepsilon}}}$$

For linear combined hardening model

$$f(\boldsymbol{s}, \boldsymbol{\alpha}, e_p) = \| \, \boldsymbol{s} - \boldsymbol{\alpha} \, \| - \sqrt{\frac{2}{3}} k(e_p)$$

$$k(e_p) = \sigma_y^0 + (1 - \beta) H e_p$$

$$\dot{\boldsymbol{\alpha}} = \frac{2}{3} \beta H \dot{\boldsymbol{e}}^p$$

$$\boldsymbol{C}^{ep} = \boldsymbol{C} - \frac{4\mu^2}{2\mu + \frac{2}{3}H} \boldsymbol{N}\boldsymbol{N}$$

$$\boldsymbol{C}^{alg} = \boldsymbol{C} - \frac{4\mu^2}{2\mu + \frac{2}{3}H} \boldsymbol{N}\boldsymbol{N} - \frac{4\mu^2 \Delta\gamma}{\| \, \boldsymbol{\eta}^{tr} \, \|} \left(\boldsymbol{I}_{dev} - \boldsymbol{N}\boldsymbol{N} \right)$$

Strength Hypotheses

Failure criteria **are functions in stress or strain space which separate "failed" states from "unfailed" states**

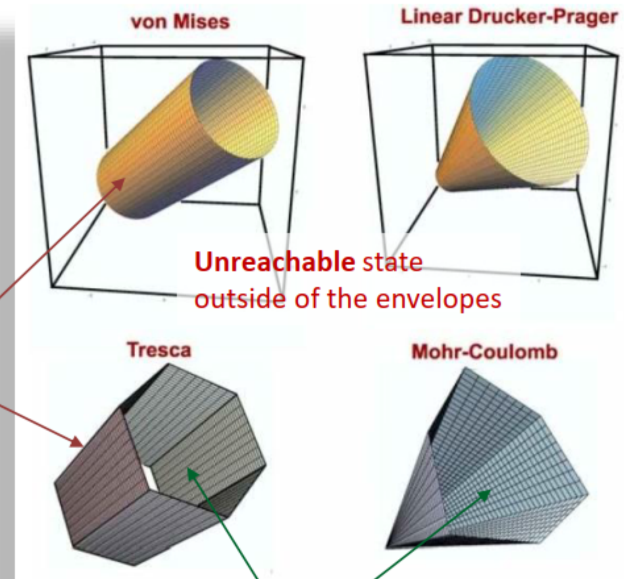
Failure criteria:

$$F(\sigma_{ij}) = 0$$

or

$$G(\varepsilon_{ij}) = 0$$

Failure envelope
= failure criteria
= failure function



Unfailed – inside the envelope
safe zone

The **failure condition**, as for *instance the* ***yield*** *or fracture condition*, **corresponds** to a ***failure surface*** in the 6-dimensional **stress space** or alternatively, **in the 3-dimensional** space of **principal stresses** or expressed as functions of stress or strain Invariants $I_1,\ J_2\ \&\ \cos(3\theta)$

$$f(\sigma_1, \sigma_2, \sigma_3;\ \kappa_1,\ \kappa_2,\ \ldots,\ \kappa_n) = 0$$

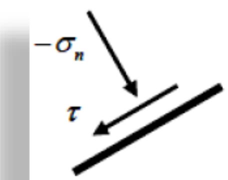
The basis of strength or failure hypothesis (for example *yield criteria*) are experiments conducted under specific and often more or less simple loading conditions

Lecture 5. Plasticity: Pressure-dependent

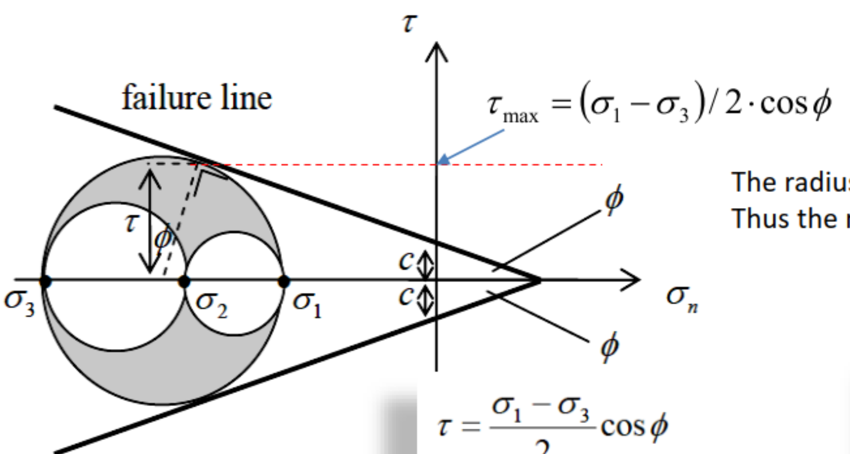
Mohr-Coulomb (failure) criterion is based on Coulomb's 1773 friction equation (Mohr 1900)

on failure one have: $\boxed{|\tau| = c - \sigma_n \underbrace{\tan\phi}_{\equiv \mu}}$

shear stress: τ; cohesion: c; normal stress: σ_n; angle of internal friction: ϕ; sliding friction coefficient: μ



Coulomb friction over a plane



$$\tau = \frac{\sigma_1 - \sigma_3}{2}\cos\phi$$

$$\sigma_n = \frac{\sigma_1 + \sigma_3}{2} + \frac{\sigma_1 - \sigma_3}{2}\sin\phi$$

Mohr-Coulomb failure criterion

The radius of the large Mohr circle is $(\sigma_1 - \sigma_3)/2$

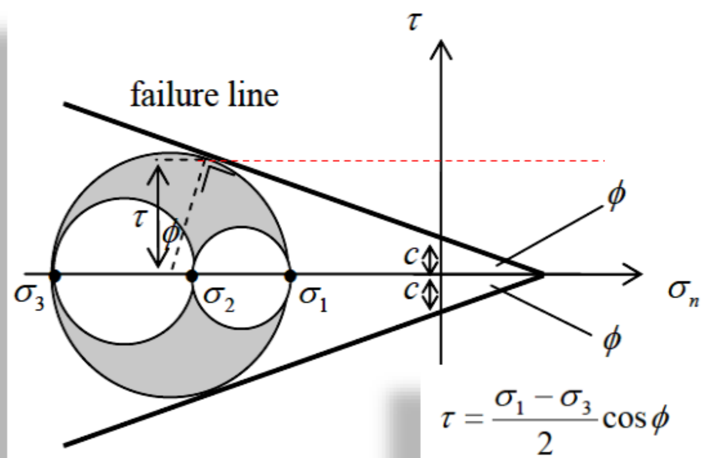
Thus the maximum shear stress is $\tau_{max} = (\sigma_1 - \sigma_3)/2 \cdot \cos\phi$

Mohr-Coulomb criterion:

$$(\sigma_1 - \sigma_3) = 2c\cos\phi - (\sigma_1 + \sigma_3)\sin\phi$$

$$\max_{i \neq j}\left[|\sigma_i - \sigma_j| + (\sigma_i + \sigma_j)\sin\phi\right] = 2c\cos\phi$$

Lecture 5. Plasticity: Pressure-dependent

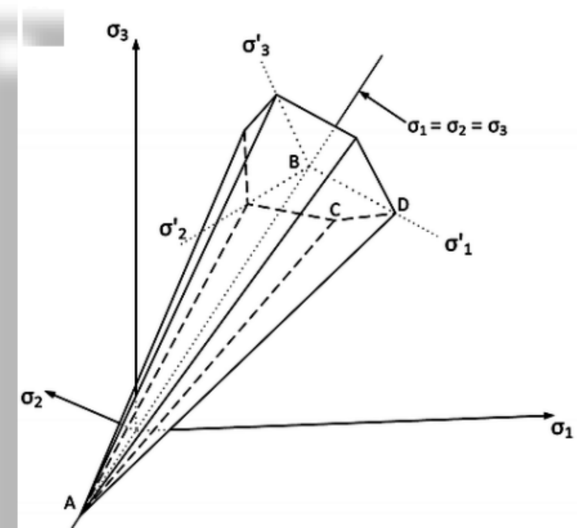


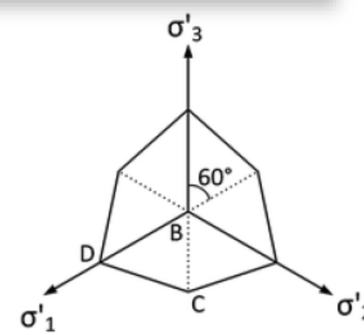
Mohr-Coulomb failure criterion

$$\tau = \frac{\sigma_1 - \sigma_3}{2}\cos\phi$$

$$\sigma_n = \frac{\sigma_1 + \sigma_3}{2} + \frac{\sigma_1 - \sigma_3}{2}\sin\phi$$

A hexagonal pyramid





Mohr-Coulomb criterion:

$$(\sigma_1 - \sigma_3) = 2c\cos\phi - (\sigma_1 + \sigma_3)\sin\phi$$

In terms of principle stresses:

$$\max_{i \neq j}\left[\left|\sigma_i - \sigma_j\right| + (\sigma_i + \sigma_j)\sin\phi\right] = 2\cos\phi$$

Family of Mohr-Coulomb criteria:

$$F(J_2, J_3) = c - \lambda I_1$$

Lecture 5. Plasticity: Pressure-dependent

Drucker-Prager criterion

$\boldsymbol{\sigma}' \equiv \mathbf{s} = \boldsymbol{\sigma} - p\mathbf{I} \leftrightarrow s_{ij} = \sigma_{ij} - p\delta_{ij}I_1 \rightarrow$ The deviatoric stress (shear)

$J_2 = \frac{1}{2}\mathbf{s}:\mathbf{s} \Rightarrow \sqrt{J_2} = \sqrt{\frac{1}{2}s_{ij}s_{ij}} \equiv \tau_e$ Equivalent stress

The model parameters in terms of *internal friction angle* ϕ and of *cohesion* intercept c of the material are:

$$\alpha = \frac{2\sin\phi}{\sqrt{3}(3-\sin\phi)}, \qquad k = \frac{6c\cdot\cos\phi}{\sqrt{3}(3-\sin\phi)}$$

Drucker-Prager (DP) criterion:

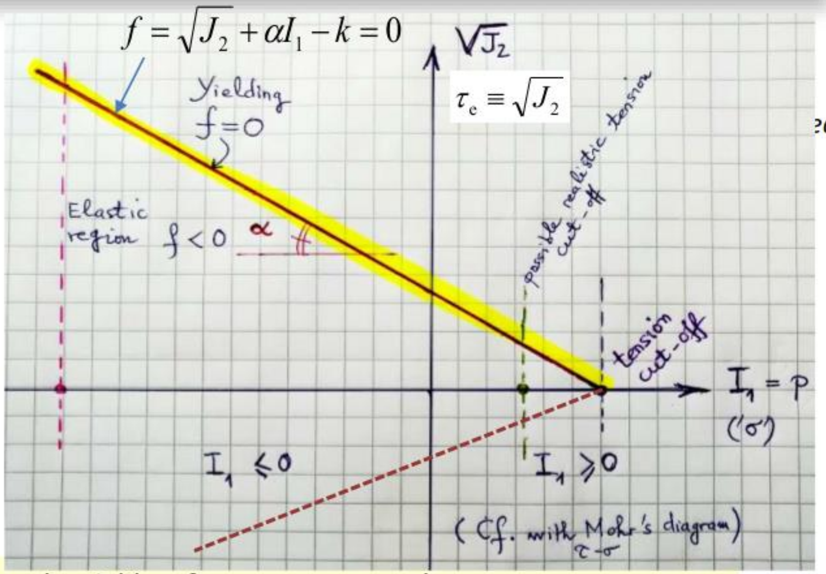
$$\boxed{f(I_1, J_2; k) = \sqrt{J_2} + \alpha I_1 - k = 0,}$$

Hydrostatic pressure:

$$I_1 = \frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma}) = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})$$

α – a friction-type material parameter; k – a threshold parameter

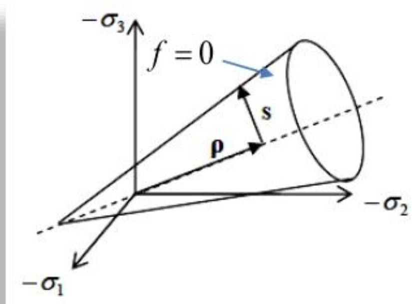
These material parameters can be expressed as '*cohesion*' and '*internal friction angle*' or they can be determined from triaxial tests graphs in $I_1, \sqrt{J_2}$ – space.



From the yield surface one can note that, as compared to the von Mises criterion (a 'tube'),

- there is *a limit for tensile mean stresses*
- the material *strengthening* by superimposing of *compressive mean pressure* (confinement)

The Drucker-Prager cone



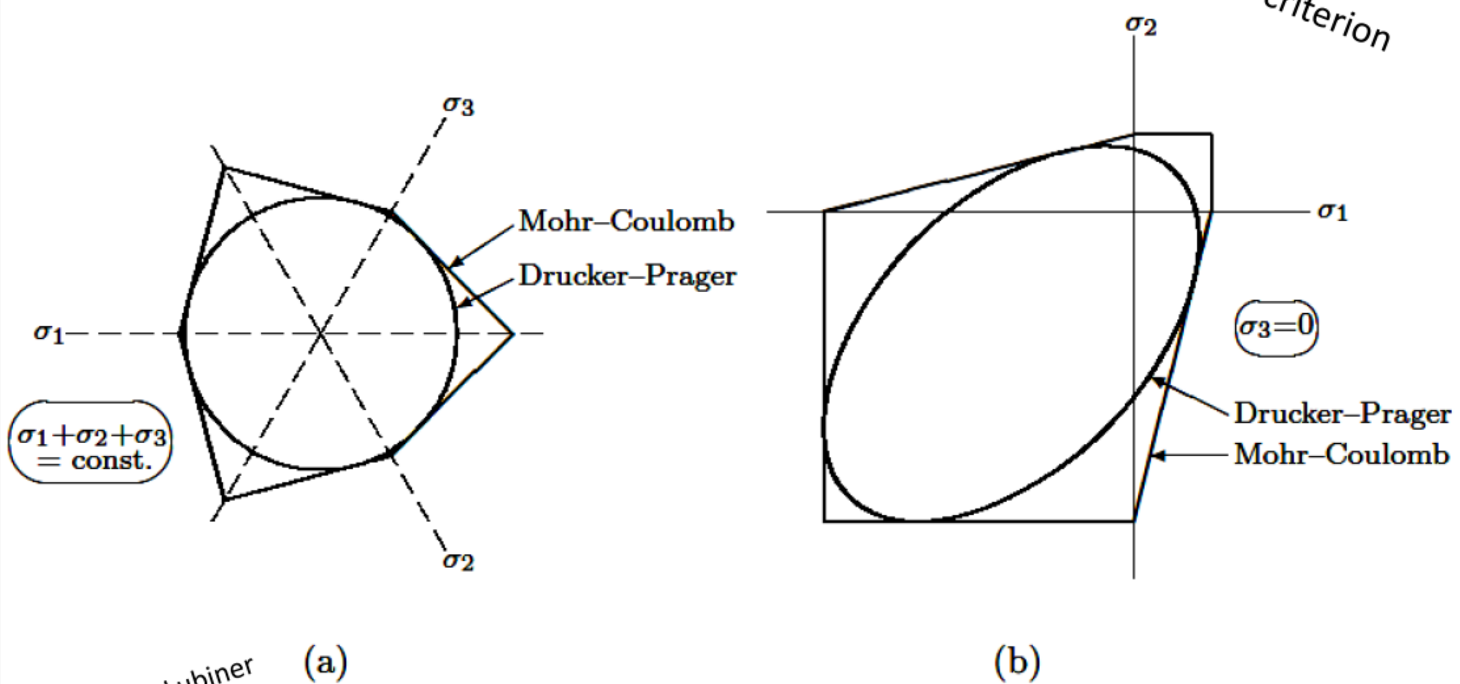
NB. Some experimental results on RC show indicate that **concrete** does not response linearly, as assumed in DP-model, in the space $\sqrt{J_2} - I_1$ when subjected to *severe hydrostatic pressures* (and probably also for very low pressures)

Lecture 5. Plasticity: Pressure-dependent

Comparison

Drucker-Prager criterion

Mohr-Coulomb yield criterion



Ref: Figures from Lubiner

Mohr–Coulomb and Drucker–Prager criteria: (a) plane parallel to π-plane; (b) plane stress.

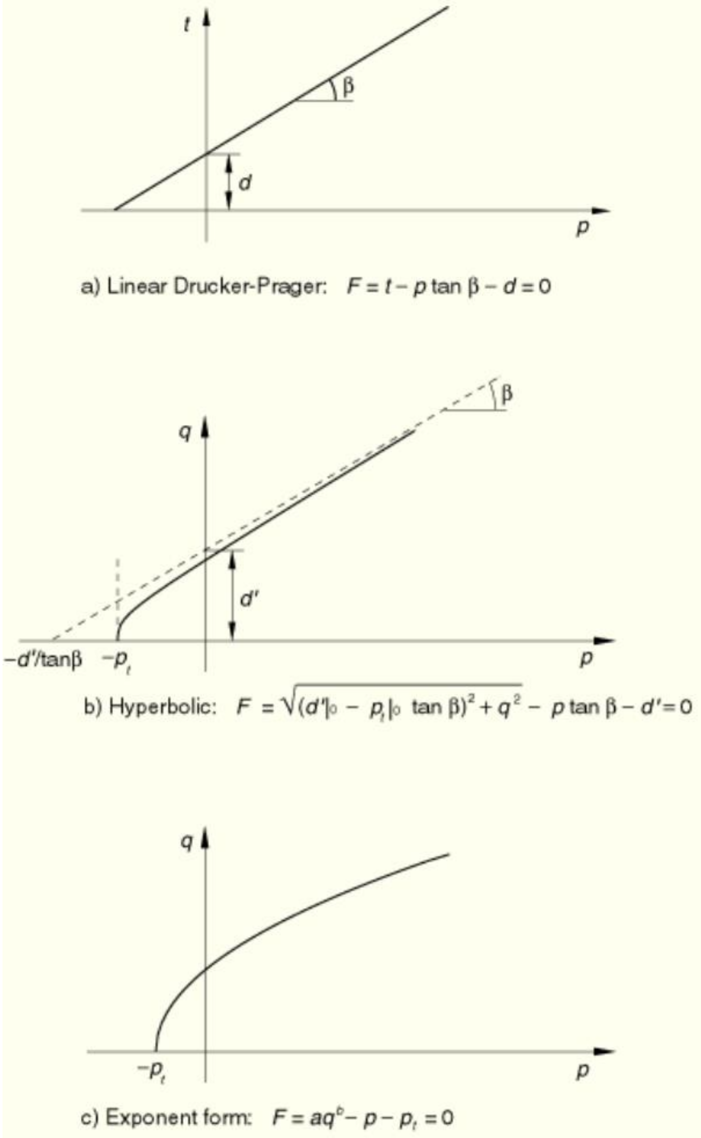
N.B. ***Good to know***: For Mohr-Coulomb and Drucker-Prager (failure, yield) surfaces applying a pure hydrostatic pressure will not affect yield since such surfaces are ***open***. However, many geomaterials, as for instance soils, a large enough hydrostatic pressure will induce permanent deformation. For such cases, a ***closed*** *(capped) yield surface* is more appropriate. Such capped yield surface can be the modified *Cam-Clay criterion*… to learn more of this subject, please join the courses with these topics of geotechnics and soil mechanics, with prof. Woiteck.

Cam-Clay criterion: $3J_2 = -\frac{1}{3}I_1 M^2\left(2p_c + \frac{1}{3}I_1\right)$

Lecture 5. Plasticity: Pressure-dependent

Non-associated flow rule

Yield surfaces in the meridional plane for extended Drucker-Prager models.



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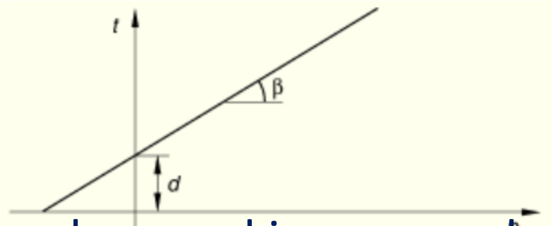
Lecture 5. Plasticity: Pressure-dependent

Non-associated flow rule



For linear Drucker-Prager model: $F = t - p\tan\beta - d = 0$

d - is the cohesion of the material

β - is the slope of the linear yield surface in the *p–t* stress plane and is commonly referred to as the friction angle of the material

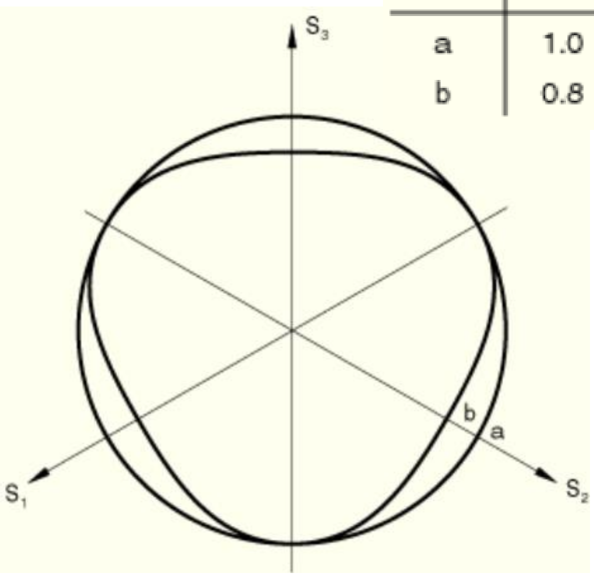
$$t = \frac{1}{2}q\left(1 + \frac{1}{K} - \left(1 - \frac{1}{K}\right)\left(\frac{r}{q}\right)^3\right)$$

$p = -\frac{1}{3}\mathrm{tr}\boldsymbol{\sigma}$ - is the pressure stress

$q = \sqrt{\frac{3}{2}\boldsymbol{s} : \boldsymbol{s}}$ - is the Mises equivalent stress

$r = \left(\frac{9}{2}\boldsymbol{s} \cdot \boldsymbol{s} : \boldsymbol{s}\right)^{\frac{1}{3}}$ - is the third invariant of deviatoric stress

Curve	K
a	1.0
b	0.8



K - is the ratio of the yield stress in triaxial tension to the yield stress in triaxial compression and, thus, controls the dependence of the yield surface on the value of the intermediate principal stress

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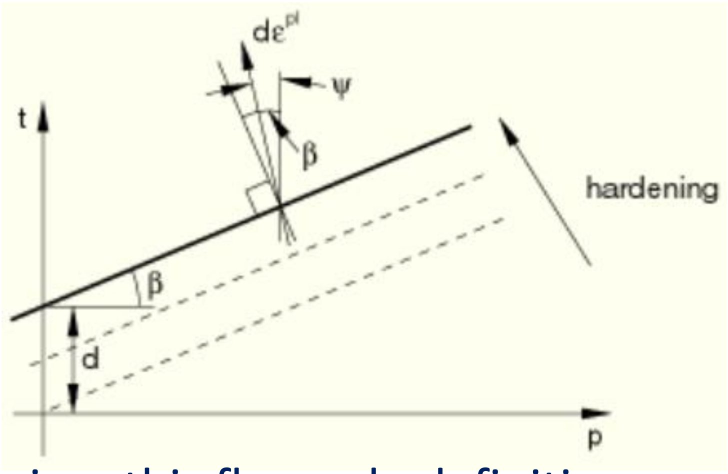
Lecture 5. Plasticity: Pressure-dependent

Non-associated flow rule



Plastic flow:

$G = t - p\tan\psi$ - is the flow potential

ψ - is the dilation angle in the *p–t* plane

In the case of hardening defined in uniaxial compression, this flow rule definition precludes dilation angles $\psi > 71.5°$.

For granular materials the linear model is normally used with non-associated flow in the *p–t* plane, in the sense that the flow is assumed to be normal to the yield surface in the dev-plane but at an angle ψ to the *t*-axis in the *p–t* plane, where usually $\psi < \beta$.

Associated flow results from setting $\psi = \beta$.

The original Drucker-Prager model is available by setting $\psi = \beta$ and $K = 1$.

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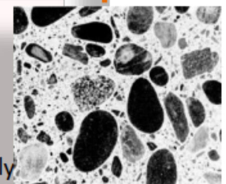
Lecture 5. Plasticity: Pressure-dependent

Ottosen (1977) developed a ***4-parameters failure criterion for concrete*** which reproduces the four key properties for 'concrete' and which is one of the best models [20, 21, 22, 23].

Ottosen's 4-parameter failure criterion



The characteristic features for the failure surface:
1. The uniaxial *tensile* strength is 5 – 10% of the uniaxial compressive strength
2. The shape of the failure curves on the meridian plane is slightly curved
3. Hydrostatic compression cannot cause failure
4. The shape of the failure locus on the deviatoric plane is triangular for small hydrostatic pressure and gets rounded with increasing hydrostatic pressure.

Ref. Saba Tahaei Yaghoubi, Reijo Kouhia, Juha Hartikainen and Kari Kolari. A continuum damage model based on Ottosen's four parameter failure criterion for concrete. *Rakenteiden Mekaniikka* (Journal of Structural Mechanics) Vol. 47, No 2, 2014, pp. 50 – 66

Good to know

13111 AUGUST 1977 EM4

JOURNAL OF THE ENGINEERING MECHANICS DIVISION

A Failure Criterion for Concrete

By Niels Saabye Ottosen[1]

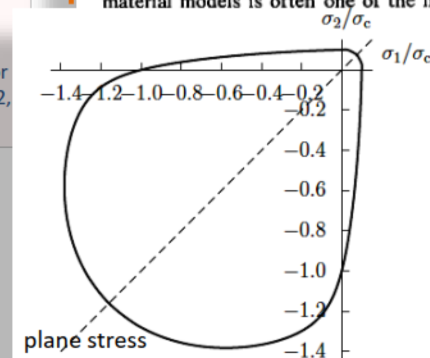
Introduction

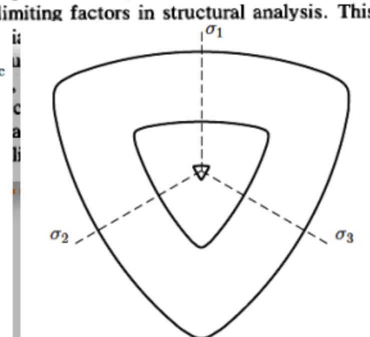
At the present stage of computer programs development, the use of inadequate material models is often one of the limiting factors in structural analysis. This

$$\boxed{A\frac{J_2}{\sigma_c}+\Lambda\sqrt{J_2}+BI_1-\sigma_c=0,}$$

$$\Lambda=\begin{cases} k_1\cos[\frac{1}{3}\arccos(k_2\cos 3\theta)] & \text{if} \quad \cos 3\theta\geqslant 0 \\ k_1\cos[\frac{1}{3}\pi-\frac{1}{3}\arccos(-k_2\cos 3\theta)] & \text{if} \quad \cos 3\theta\leq 0 \end{cases}.$$

$$\cos 3\theta=\frac{3\sqrt{3}}{2}\frac{J_3}{J_2^{3/2}},\quad A\geqslant 0;\quad B\geqslant 0;\quad k_1\geqslant 0;\quad 0\leq k_2\leq 1.$$





[20] N.S. Ottosen. A failure criterion for concrete. *Journal of the Engineering Mechanics, ASCE*, 103(EM4):527–535, August 1977.

...element analysis of concrete structures. Technical Re... Laboratory, DK-4000 Roskilde, Denmark, May 1980.

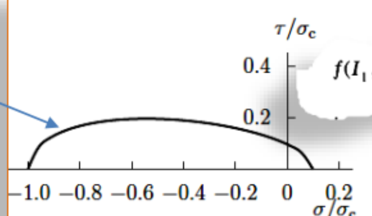
...a. Th... 2005.

[23] K. William and E.P. Warnke. Const... crete. In *IABSE Proceedings*, volu... Structures Subjected to Triaxial Str...

$$f(I_1, J_2, \cos 3\theta) = A\frac{J_2}{\sigma_c^2}+\lambda\frac{\sqrt{J_2}}{\sigma_c}+B\frac{I_1}{\sigma_c}-1=0\,.$$

Recommended reading: **State-of-the-art report**, bulletin 45. 2008, task group 4.4. *fib* CEB-FIP – ***Practitioner's guide to finite element modelling of reinforced concrete structures***

N.B. the graph is a curved curve and not a straight line as in Mohr-Coulomb or Drucker-Prager



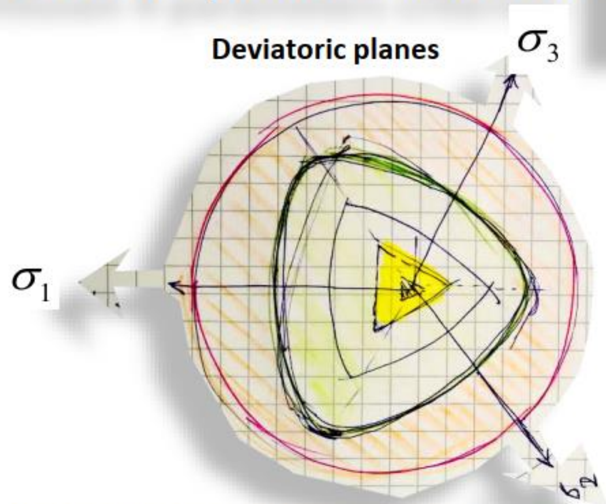
σ_c – uniaxial compression test, failure stress

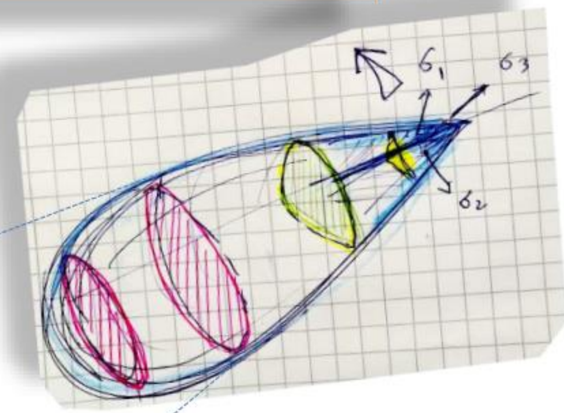
Lecture 5. Plasticity: Pressure-dependent

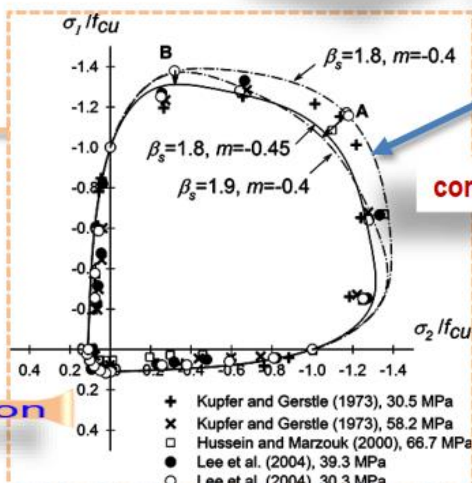
Ottosen 4-parameters criterion:

$$A\frac{J_2}{\sigma_c} + \Lambda\sqrt{J_2} + BI_1 - \sigma_c = 0,$$

Deviatoric planes







tension

Concrete – plane stress

NB the asymmetry in tension and **compression**

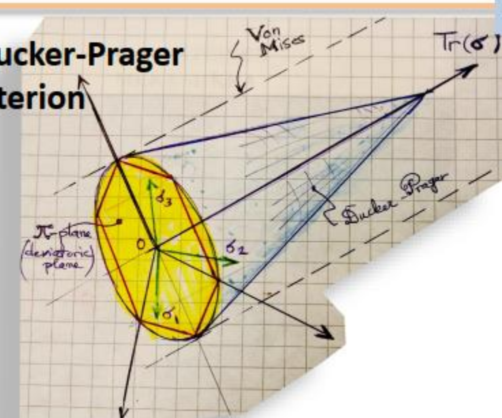
compression

The DP-criterion does not see such asymmetry!

Drucker-Prager criterion

I_1

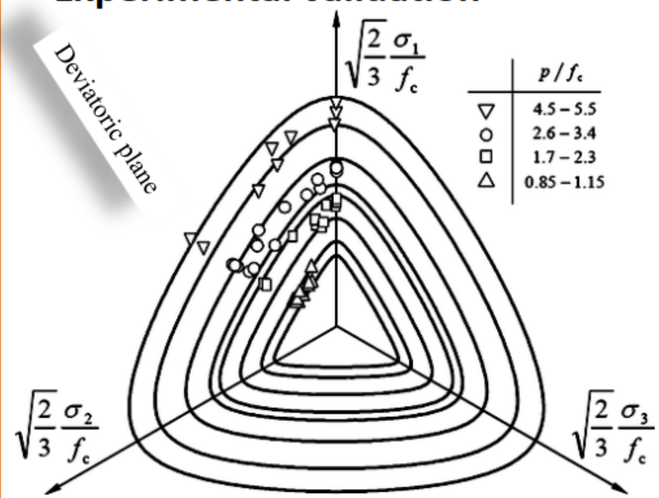
$\sigma_1 = \sigma_2 = \sigma_3$



Lecture 5. Plasticity: Pressure-dependent

The idea in this slide is to show experimental evidence of the existence of a failure surface

Experimental validation

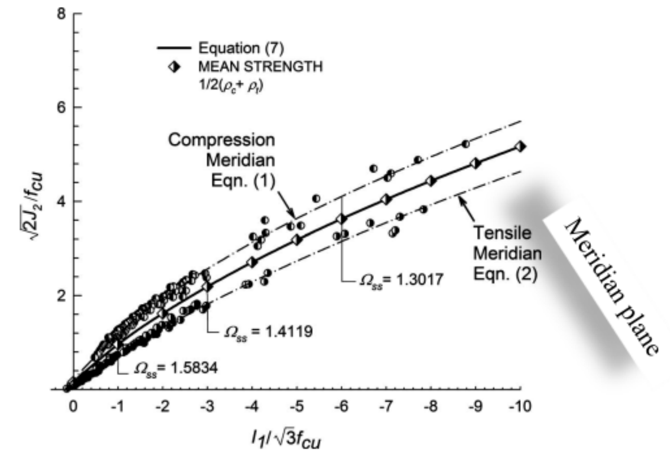


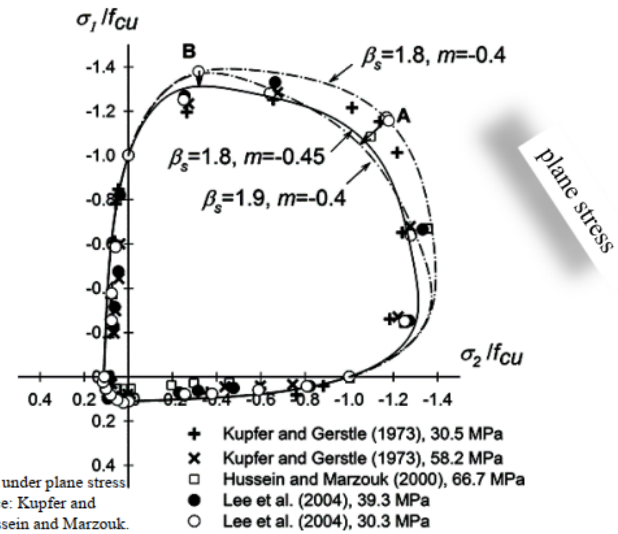
Comparison of the experimental results by Guo and Wang 1991 and the modeling results by using the proposed criterion *p* is held constant

Ref: *J. Eng. Mech.*, 2010, 136(1): 51-59

Nonlinear Unified Strength Criterion for Concrete under Three-Dimensional stress States. Xiu-Li Du; De-Chun Lu; Qiu-Ming Gong; and Mi Zhao

Mean concrete strength from compression and tension data





Concrete strength under plane stress Conditions. Source: Kupfer and Gerstle 1973; Hussein and Marzouk. 2000; Lee et al. 2004

Ref: Simple Single-Surface Failure Criterion for Concrete. Sean D. Hinchberger

J. Eng. Mech., 2009, 135(7): 729-732