CIV-E4080
Material Modelling in Civil Engineering D

Period V, 2024

Practicalities

- Teacher (in charge): *Asst. Prof. Sergei Khakalo,* sergei.khakalo@aalto.fi
- Lecturer (invited): *Dr.-Ing. Aleksandr Morozov,* morozov@tu-berlin.de
- Assistants: *Ahmad Shahgordi,* ahmad.shahgordi@aalto.fi

  Dr. Viacheslav Balobanov, viacheslav.balobanov@vtt.fi

During 6 weeks:

- Lectures: Monday, 14.15–16.00

  Tuesday, 12.15–14.00
- Exercises: Wednesday, 14.15–16.00

  Thursday, 14.15–16.00

Practicalities

Workload

- ***Lectures***: 2h x (2 x week) = 4h/week contact teaching (**24h**)
- ***Guided exercises and homework solving***: 2h x (2 x week) contact teaching (1 homework/week) (**24h**)
- ***Self-studies*** in small groups or alone (not contact teaching): preparing, assimilating the course subjects and given reading assignments + doing the homework outside the guiding hours, at least 2h/day x 7 days x 6 weeks (**84 h**)
- Examination: (**3h**) - twice during the academic year

Practicalities

Passing the course

- A written EXAM should be passed successfully

  (Grades: 0-5. 0-fail, 1-5 pass)
- Course grade = Exam grade + possible upgrade
- IF collected HW points >= 2/3 of total HW points

  THEN course grade is upgraded by 1 point
- Exam on Wed 5.6.2024, at 13.00–16.00

Contents

- **Week 1:** Elasticity (linear, hyper-elasticity, isotropy, anisotropy, …)
- **Weeks 2, 3:** Plasticity (associative, non-associative)
- **Week 4:** Viscoelasticity
- **Week 5:** Viscoplasticity or creep (by Dr.-Ing. Morozov, TU Berlin)
- **Week 6:** Damage (damage coupled to plasticity)

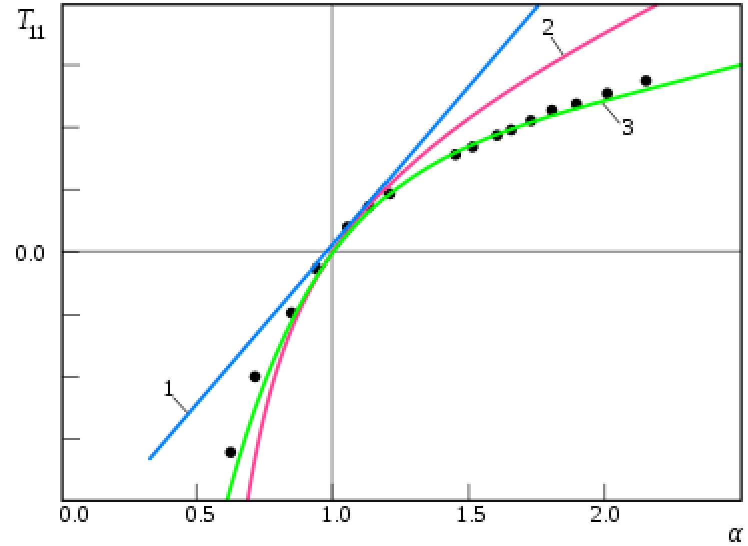
Contents

Week 1: Elasticity

Example: Isotropic material models

Linear

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda \mathrm{tr}(\boldsymbol{\varepsilon})\boldsymbol{I}$$



Non-linear

(e.g., incompressible Mooney-Rivlin)

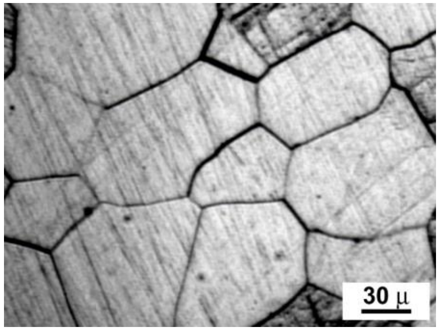
$$\boldsymbol{\sigma} = -p\,\mathit{1} + \lambda_1 \frac{\partial W}{\partial \lambda_1}\mathbf{n}_1 \otimes \mathbf{n}_1 + \lambda_2 \frac{\partial W}{\partial \lambda_2}\mathbf{n}_2 \otimes \mathbf{n}_2 + \lambda_3 \frac{\partial W}{\partial \lambda_3}\mathbf{n}_3 \otimes \mathbf{n}_3$$

$$W = C_1(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) + C_2(\lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 - 3)\,;\;\; \lambda_1\lambda_2\lambda_3 = 1$$

Contents

Week 1: Elasticity

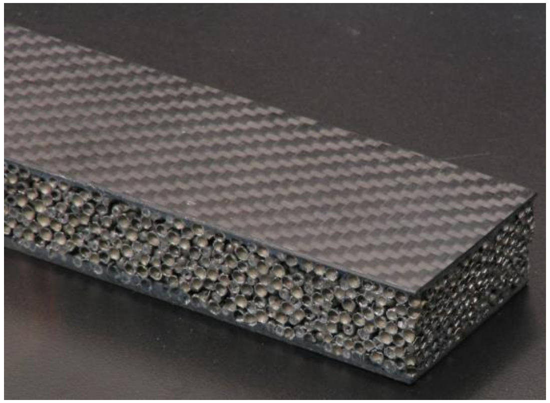
Example: Anisotropic material models



Linear

$\boldsymbol{\sigma} = \boldsymbol{C} : \boldsymbol{\varepsilon}$

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix}.$$



Contents

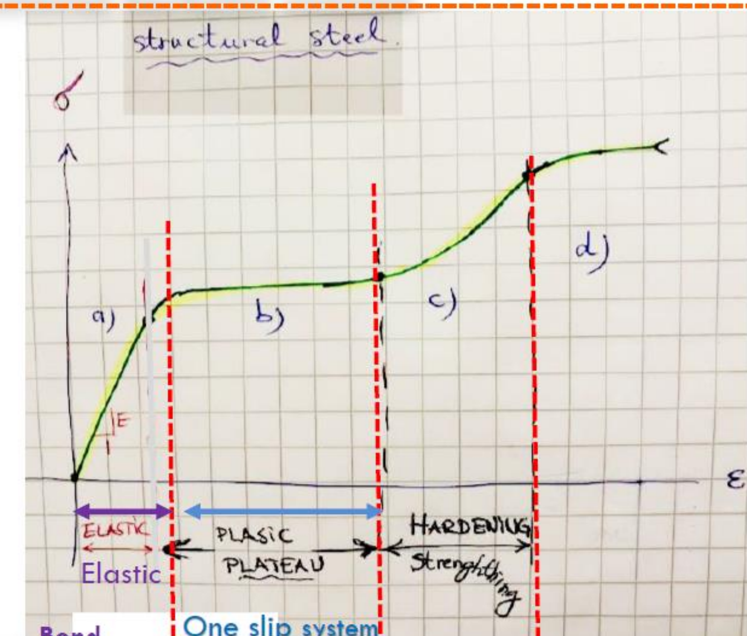
Weeks 2, 3: Plasticity

from Dr. Baroudi's lecture notes

Constitutive law

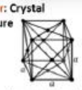
$$\boldsymbol{\sigma} = \boldsymbol{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{pl})$$

Should be known to you from earlier material bachelor courses



Bond stretching → Elastic

Copper: Crystal structure



face-centered cubic (fcc)

Cu: three slip-planes FCC {111}

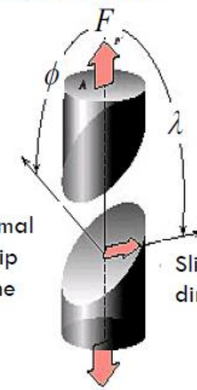
One slip system get activated

Slip on one slip system → '**easy** slip' or '**glide**' → plasticity plateau

Multiple slip systems get activated → hardening

Material fractures in some extreme orientations of grains (crystals)

Def: slip system = slip plane & slip direction



Stainless-steel at 315 ºC. Diagonal lines are slip bands, i.e., regions with high degree of plastic slip (ref. Shanley, Strength of materials, 1957)

Largest resolved shear stress (in a single crystal):

$$\tau_{\max} = \frac{F_s}{A_s} = \frac{F\cos\lambda}{A/\cos\phi} = \sigma[\cos\lambda\cos\phi]_{\max}$$

Schmidt factor

Therefore, the minimum stress to begin yielding (= slip) occurs when slip direction $\lambda = \phi = 45\,\mathrm{deg}$.

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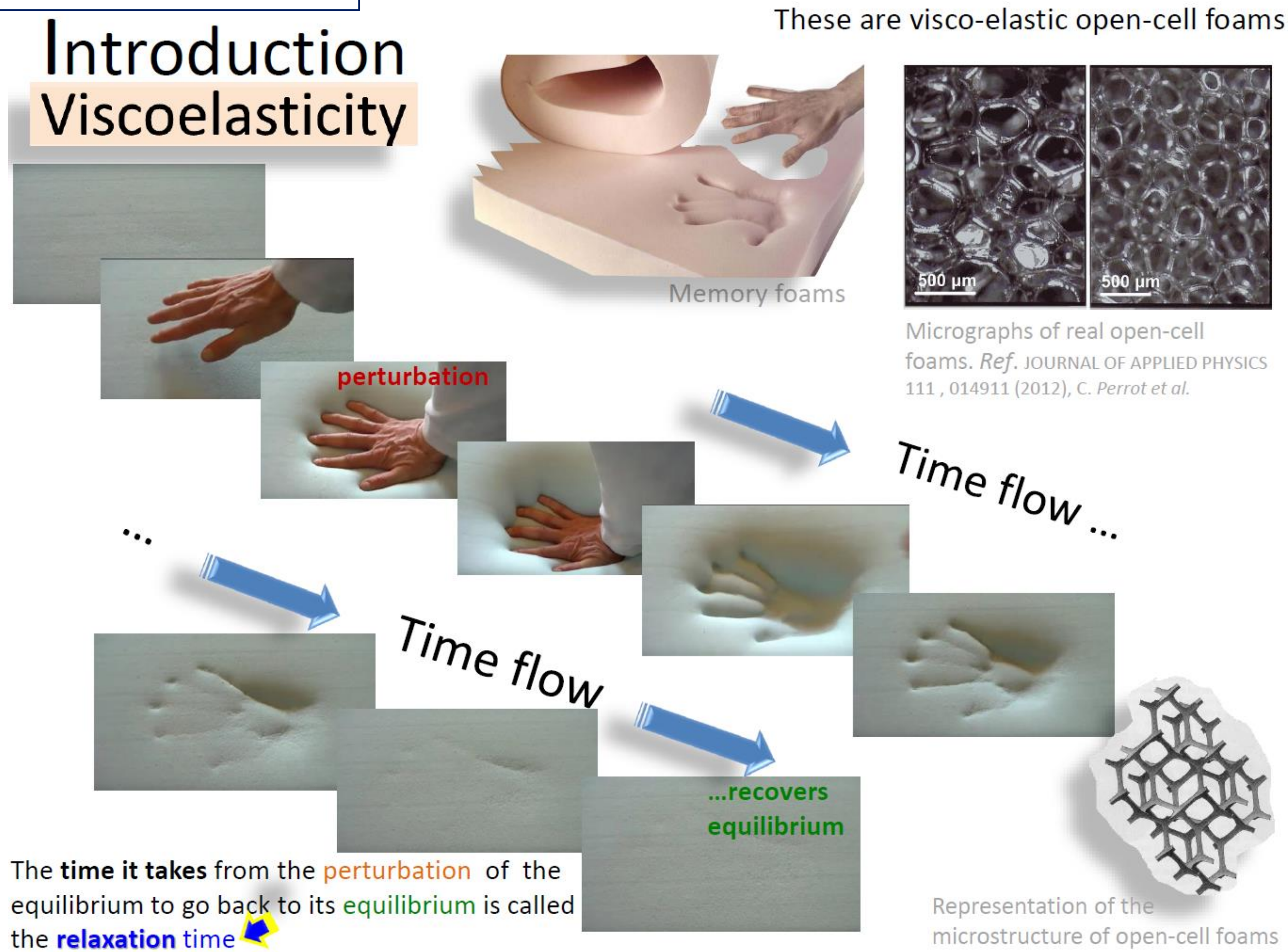
Week 4: Viscoelasticity

from Dr. Baroudi's lecture notes



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Week 5: Viscoplasticity (creep)

from Dr. Baroudi’s lecture notes

Parkkipaikka raksan takana, 12.10.2014
(kuva Dj. Baroudi)

Contents

Week 5: Viscoplasticity (creep)

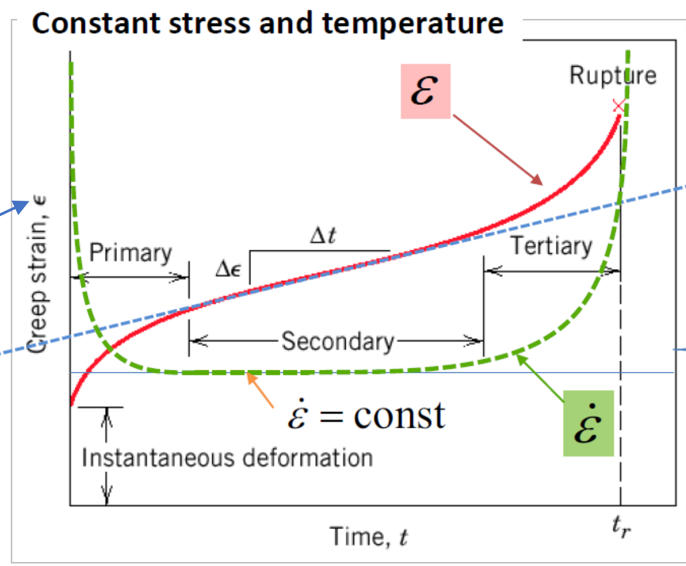
from Dr. Baroudi's lecture notes

Stages of creep

Concepts & Definitions

Constitutive law

$$\boldsymbol{\sigma} = \boldsymbol{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{cr})$$



Primary creep:

$$\varepsilon = A(\sigma) \cdot t^{1/\beta}, \quad 2 < \beta < 4$$

To avoids meaningless units write in this form:

$$\varepsilon = A(\sigma) \cdot \left[\frac{t}{t_{\text{REF}}}\right]^{1/\beta},$$

$$\dot{\varepsilon} = K_2 \left[\frac{\sigma}{\sigma_{\text{Ref}}}\right]^n \exp\left(-\frac{Q_c}{RT}\right)$$

Steady creep rate:

$$\dot{\varepsilon}_s = \Delta\varepsilon / \Delta t$$

1. **Instantaneous deformation**, mainly elastic
2. **Primary/transient creep.** Slope of strain *vs.* time decreases *t*: work-hardening
3. **Secondary/steady-state creep.** Rate of straining constant: work-hardening and recovery.
4. **Tertiary.** Rapidly accelerating strain rate up to failure: formation of internal cracks, voids, grain boundary separation, necking*: accumulation of damage - rupture*

Steady state creep:

$$\dot{\varepsilon} = K_2 \sigma^n \exp\left(-\frac{Q_c}{RT}\right)$$

Q_c = activation energy for creep
K_2 and n are material constants

Contents

Week 6: Damage (with plasticity)

Material degradation

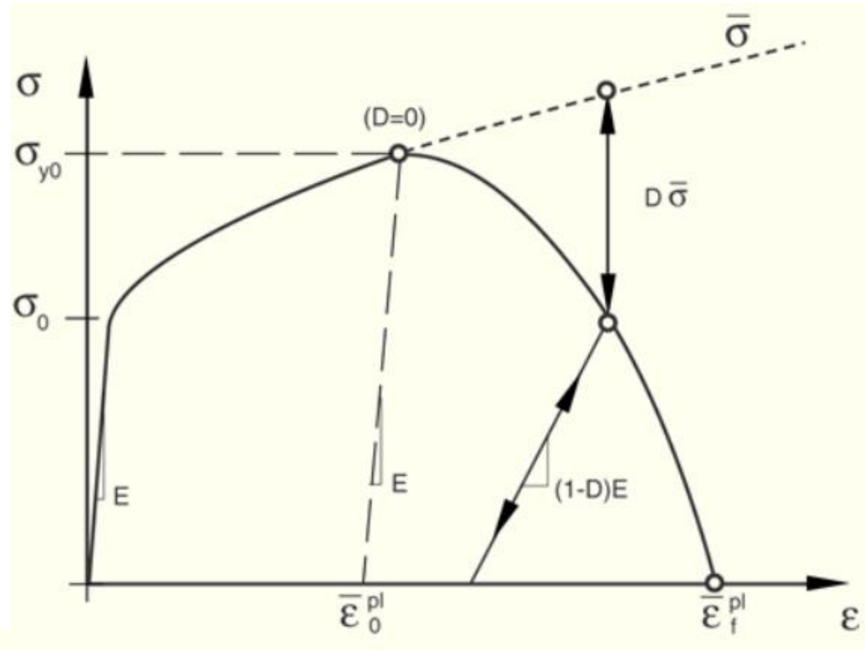
$$\boldsymbol{\sigma} = (1 - D)\bar{\boldsymbol{\sigma}}$$

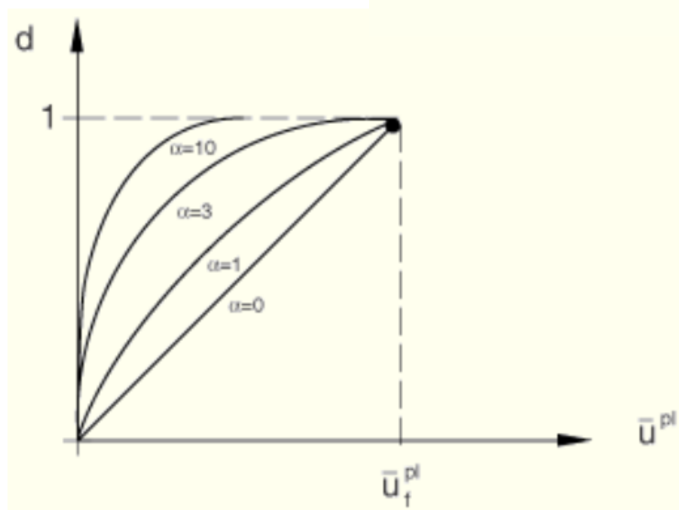
Damage initiation

$$\omega_D = \int \frac{d\bar{\varepsilon}^{pl}}{\bar{\varepsilon}_D^{pl}(\eta, \dot{\bar{\varepsilon}}^{pl})} = 1$$

Damage evolution

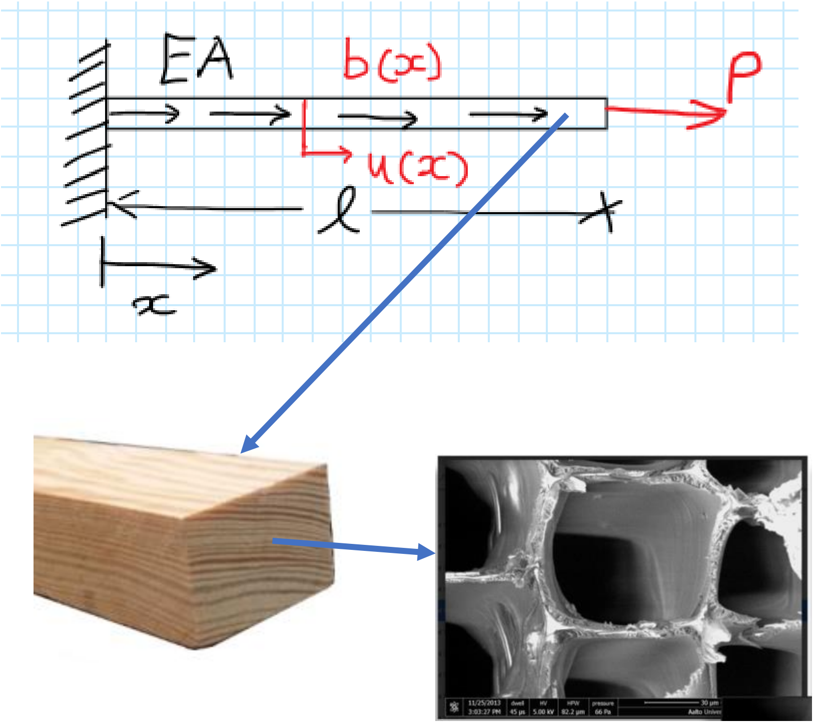
$$d = \frac{1 - e^{-\alpha(\bar{u}^{pl}/\bar{u}_f^{pl})}}{1 - e^{-\alpha}}$$





Lecture 1. Introduction

Example: 1D bar extension problem



$$-EAu'' = b$$

Main assumption:

A continuum is a body that can be continually sub-divided into infinitesimal elements with local material properties defined at any particular point. Properties of the bulk material can therefore be described by continuous functions.

Lecture 1. Introduction

Balance laws

$$\dot{\rho} + \rho(\nabla \cdot \boldsymbol{v}) = 0$$ - Balance of Mass

$$\rho\dot{\boldsymbol{v}} - \nabla \cdot \boldsymbol{\sigma} - \rho\boldsymbol{b} = 0$$ - Balance of Linear Momentum

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$$ - Balance of Angular Momentum

$$\rho\dot{e} - \boldsymbol{\sigma} : \nabla\boldsymbol{v} + \nabla \cdot \boldsymbol{q} - r = 0$$ - Balance of Energy

Clausius–Duhem inequality

$$\rho(T\dot{s} - \dot{e}) + \boldsymbol{\sigma} : \nabla\boldsymbol{v} - \frac{\boldsymbol{q} \cdot \nabla T}{T} \geq 0 \quad \longrightarrow \quad \boldsymbol{\sigma} : \nabla\boldsymbol{v} - \rho(\dot{\psi} + s\dot{T}) - \frac{\boldsymbol{q} \cdot \nabla T}{T} \geq 0$$

- We assume that it is possible to represent the temperature by a scalar field of positive values defined at each instant *t* and all material points of the volume *V*.
- Entropy expresses a variation of energy associated with a variation in the temperature.

Lecture 2. Elasticity

Linear elasticity

Orthotropy

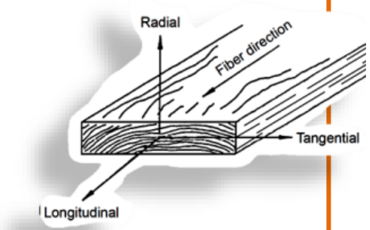
$$\boldsymbol{\sigma} = \boldsymbol{C} : \boldsymbol{\varepsilon}$$

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} \frac{1-\nu_{23}\nu_{32}}{E_2 E_3 \Delta} & \frac{\nu_{21}+\nu_{23}\nu_{31}}{E_2 E_3 \Delta} & \frac{\nu_{31}+\nu_{21}\nu_{32}}{E_2 E_3 \Delta} & 0 & 0 & 0 \\ & \frac{1-\nu_{13}\nu_{31}}{E_1 E_3 \Delta} & \frac{\nu_{32}+\nu_{12}\nu_{31}}{E_1 E_3 \Delta} & 0 & 0 & 0 \\ & & \frac{1-\nu_{12}\nu_{21}}{E_1 E_2 \Delta} & 0 & 0 & 0 \\ & & & G_{23} & 0 & 0 \\ & \text{symmetric} & & & G_{13} & 0 \\ & & & & & G_{12} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix}$$

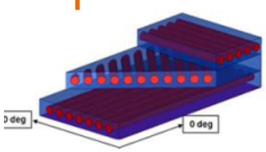
$$\Delta = \frac{(1-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}-\nu_{13}\nu_{31}-2\nu_{21}\nu_{32}\nu_{13})}{(E_1 E_2 E_3)}$$

Reciprocal relations: $\frac{\nu_{21}}{E_2} = \frac{\nu_{12}}{E_1}, \quad \frac{\nu_{31}}{E_3} = \frac{\nu_{13}}{E_1}, \quad \frac{\nu_{23}}{E_2} = \frac{\nu_{32}}{E_3}$

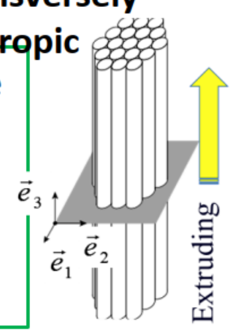
$$\frac{E_i}{E_j} = \frac{\nu_{ij}}{\nu_{ji}}$$



1, 2, 3 are the material principal directions of orthotropy



Transversely isotropic



To obtain: Do the following substitutions in the stiffness matrix $\boldsymbol{C}$ above

Transversely isotropic material: $E_2 = E_3,\ \nu_{12} = \nu_{13},\ G_{12} = G_{13}$ and $G_{23} = \frac{E_2}{2(1+\nu_{23})}$

Isotropic material: $E = E_1 = E_2, \nu = \nu_{12} = \nu_{13} = \nu_{23}, G = G_{12} = G_{13}$ and $G = \frac{E}{2(1+\nu)}$

Lecture 2. Elasticity

Thermoelasticity

Expansion joint in a road bridge used to avoid damage from thermal expansion.

Heat induced rail track buckling.

www.google.com

Lecture 2. Elasticity

Thermoelasticity

J. Lemaitre, J.-L. Chaboche. Mechanics of solid materials.

$$\rho\Psi = \frac{1}{2}\lambda(\mathrm{Tr}(\varepsilon))^2 + \mu\varepsilon : \varepsilon - (3\lambda + 2\mu)\alpha\theta\mathrm{Tr}(\varepsilon) - \frac{\rho C_\varepsilon}{2T_0}\theta^2 \text{ with } \theta = T - T_0,$$

where α and C_ε are two coefficients with a meaning given by the state laws derived from Ψ:

$$\boldsymbol{\sigma} = \rho(\partial\Psi/\partial\boldsymbol{\varepsilon}) = \lambda\,\mathrm{Tr}\,(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon} - (3\lambda + 2\mu)\alpha\theta\mathbf{1}$$

or

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} - (3\lambda + 2\mu)\alpha\theta\delta_{ij}$$

and

$$s = -\frac{\partial\Psi}{\partial T} = -\frac{\partial\Psi}{\partial\theta} = \frac{1}{\rho}(3\lambda + 2\mu)\alpha\,\mathrm{Tr}\,(\boldsymbol{\varepsilon}) + \frac{C_\varepsilon}{T_0}\theta.$$

This last equation allows us to calculate

$$(\partial s/\partial T)_{\varepsilon = \text{constant}} = (\partial s/\partial\theta)_{\varepsilon = \text{constant}} = C_\varepsilon/T_0.$$

which shows that $C_\varepsilon = T_0(\partial s/\partial T)$ is the specific heat at constant strain.

By inverting the first law of state or by employing the dual potential $\Psi^*(\boldsymbol{\sigma}, \theta)$, we may express strains as functions of stresses and temperature:

- $$\boldsymbol{\varepsilon} = \frac{1+\nu}{E}\boldsymbol{\sigma} - \frac{\nu}{E}\mathrm{Tr}\,(\boldsymbol{\sigma})\mathbf{1} + \alpha\theta\mathbf{1}$$

or

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} + \alpha\theta\delta_{ij}$$

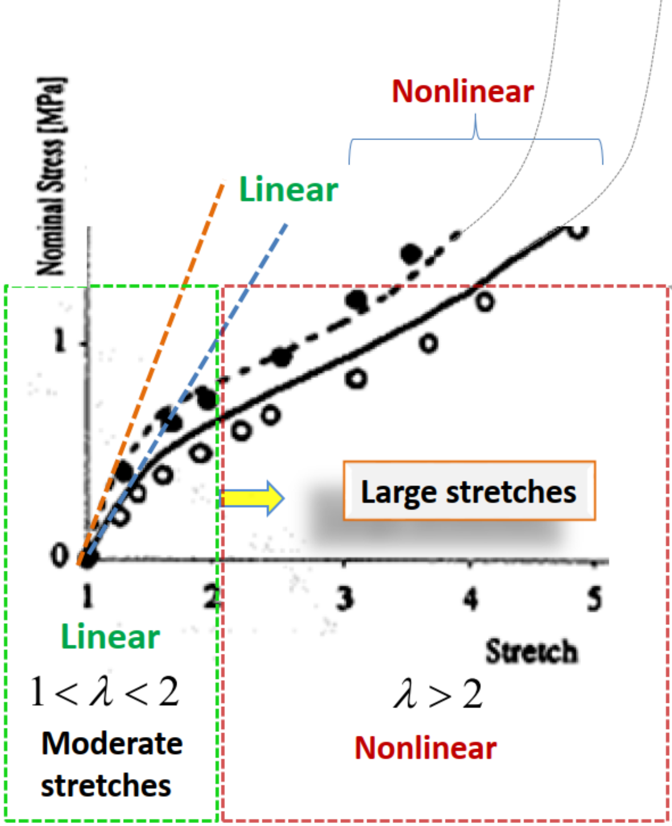
Hyperelasticity - rubbers

Observation: High degree of Nonlinearities

⇓

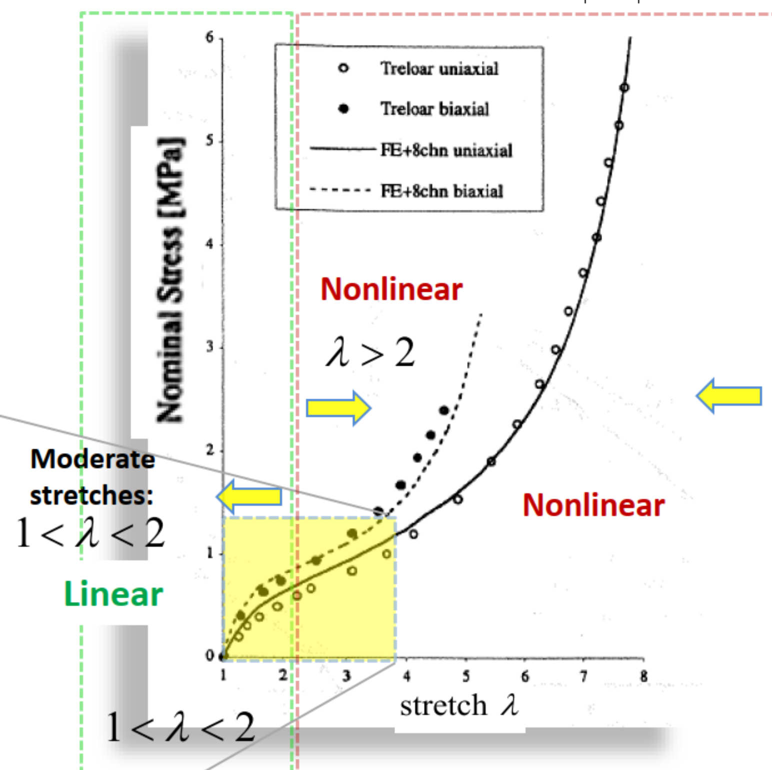
There is a need for a non-linear elastic model

Stretch stiffening



Ref: L.R.G. **Treloar**. **The elasticity and related properties of rubbers.** *Rep. Prog. Phys.* 1973 **(36)** 755-826

Nonlinear: *large deformations* $|\nabla \mathbf{u}| >> 1$



Professor **L. R. G. Treloar** (1906 – 1985) was a leading person in the science of rubber and elasticity

Isotropic & isothermal cases

Lecture 2. Elasticity

Kinematics of rubbers

Consider a rectangular material block reference configuration: $L_1 \times L_2 \times L_3$

Apply forces F_i on its faces in this reference state:

Principle stretches: $\lambda_i = \ell_i / L_i, \quad i = 1,2,3$

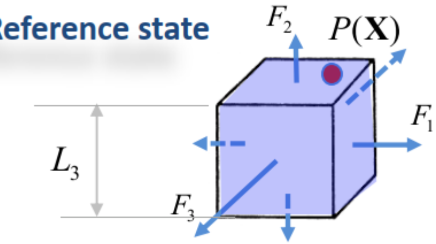
Actual length in the current state

Initial length in the reference state

Motion (deformation): $x_i = \lambda_i X_i, \quad i = 1,2,3$

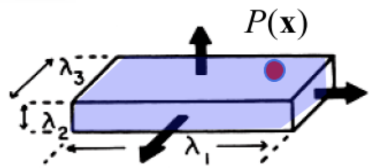
Material point in reference configuration: $P(X_1, X_2, X_3) \equiv P(\mathbf{X})$

$X_i, \quad i = 1,2,3$



Material point in deformed configuration: $P(x_1, x_2, x_3) \equiv P(\mathbf{x})$

Actual state $x_i, \quad i = 1,2,3$



Initial volume $\quad \mathrm{d}V_0 = L_1 L_2 L_3$

Deformed volume $\quad \mathrm{d}V = \ell_1 \ell_2 \ell_3 = \lambda_1 L_1 \cdot \lambda_2 L_2 \cdot \lambda_3 L_3 = \lambda_1 \lambda_2 \lambda_3 \cdot L_1 L_2 L_3 = \lambda_1 \lambda_2 \lambda_3 \mathrm{d}V_0$

Jacobian of the motion: $\quad J \equiv \mathrm{d}V / \mathrm{d}V_0 \neq 0, \qquad J(0) \equiv \mathrm{d}V(0) / \mathrm{d}V_0 = \mathrm{d}V_0 / \mathrm{d}V_0 = 1$

Strain invariants

- We consider isotropic materials
  - Material frame indifference: no matter what coordinate system is chosen, the response of the material is identical
  - The components of a deformation tensor depends on coordinate system
  - Three invariants of **C** are independent of coordinate system
- **Stretch Invariants**

$$I_1 = \mathrm{tr}(\boldsymbol{C}) = C_{11} + C_{22} + C_{33} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$

$$I_2 = \tfrac{1}{2}\left[(\mathrm{tr}\,\boldsymbol{C})^2 - \mathrm{tr}(\boldsymbol{C}^2) \right] = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2$$

$$I_3 = \det \boldsymbol{C} = \lambda_1^2\lambda_2^2\lambda_3^2$$

Right Cauchy-Green Deformation Tensor

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$$

  - In order to be material frame indifferent, constitutive laws (properties properties) must be expressed using invariants
  - For incompressibility, $I_3 = 1$

For the initial configuration one have: $\mathbf{u} = 0 \Rightarrow \mathbf{F} = \mathbf{1}, \quad \mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \mathbf{1} \Rightarrow I_1 = 3, I_2 = 3, I_3 = 1$

Overview: Some classical models

Isotropic & isothermal cases

The constitutive laws are given by postulating thermodynamic potentials (Elastic strain energy)

Elastomer is nearly incompressible. *However, most accurate models should incorporate compressibility into the constitutive law*

λ_i — Principle stretches

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$
$$I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_1^2\lambda_3^2$$
$$I_3 = \lambda_1^2\lambda_2^2\lambda_3^2$$

Elastic strain energy (proposed by **Rivlin**):

$$\psi = \sum_{i,j=0}^{\infty} C_{ij}(I_1-3)^i(I_2-3)^j$$

$C_{ij} \geq 0$ Material constants

$I_3 = 1$ Incompressibility

Initially:

$\mathbf{C}(0) = \mathbf{F}^T\mathbf{F} = \mathbf{1} \Downarrow$

$I_1 = 3, I_2 = 3, I_3 = 1$

Removes the initial constant level of energy

Constitutive laws are obtained by taking partial derivatives of the strain energy with respect to stretches:

$$\sigma_i = \frac{\partial \psi}{\partial \lambda_i} = \frac{\partial \psi}{\partial I_k}\frac{\partial I_k}{\partial \lambda_i}$$

Neo-Hookean model:

$$\psi = C_{10}(I_1-3)$$

Mooney-Rivlin model:

$$\psi = C_{10}(I_1-3) + C_{01}(I_2-3)$$

Yeoh model: $\psi = C_{10}(I_1-3) + C_{20}(I_1-3)^2 + C_{30}(I_1-3)^3$

Ogden model: $\psi = \sum_n \frac{\mu_n}{\alpha_n}(\lambda_1^{\alpha_n} + \lambda_2^{\alpha_n} + \lambda_3^{\alpha_n} - 3), \quad n \geq 3$ Gives reasonable fits to experiments

Nominal stress are work-conjugate to stretches (*Cf.* appendix)

Uniaxial example:

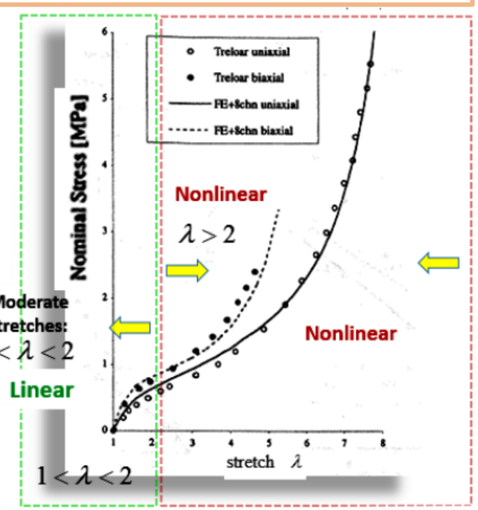
$$\sigma = 2C_{10}(\lambda - 1/\lambda^2)$$

Neo-Hookean

Nominal stress

$$\sigma = 2(C_{10} + \frac{C_{01}}{\lambda})(\lambda - \frac{1}{\lambda^2})$$

Mooney-Rivlin



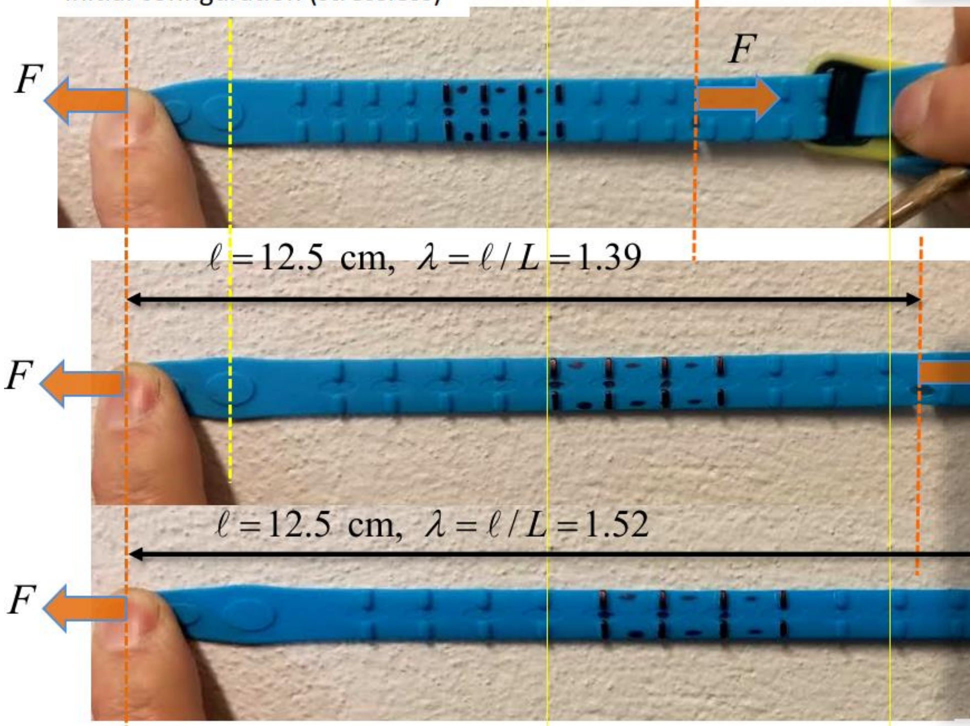
The Drugger material stability criterion for the Hessian of strain energy positive restrains the coefficients *C*ij to be positive.

Lecture 2. Elasticity

$$\text{Neo- Hookean}: \quad \sigma^{(\text{true})} = \mu\left(\lambda^2 - \frac{1}{\lambda}\right)$$

$\ell = L = 9$ cm, $\lambda = \ell / L = 1$

Initial configuration (stressless)



Nominal = engineering

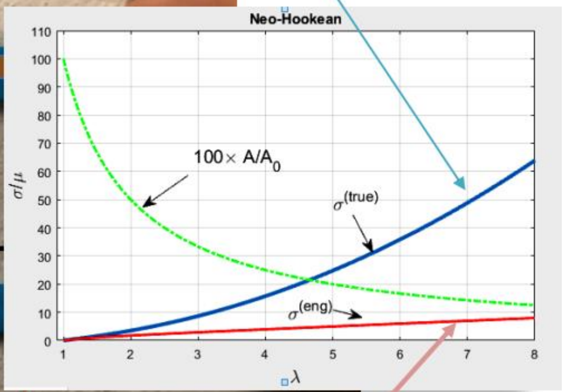
$$\sigma_i^{(\text{true})} = \sigma_i^{(\text{eng})} / (\lambda_j \lambda_k) = \frac{F}{A}$$

$j \neq i, k \neq i, j \neq k$

Actual cross-section area

In simple extension:

$$\lambda_1 = \lambda, \lambda_2 = \lambda_3 = 1/\sqrt{\lambda},$$



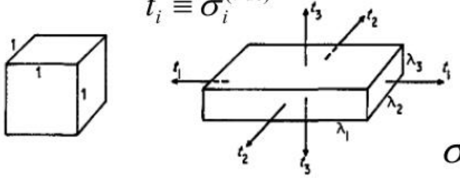
Rubber – swimming glass

$$A = \frac{A_0}{\lambda}$$

$1 < \lambda < 2$

Moderate stretches

$t_i \equiv \sigma_i^{(\text{true})}$



$$\sigma^{(\text{eng})} = \underbrace{\lambda_2 \lambda_3}_{1/\lambda} \sigma^{(\text{true})} = \frac{1}{\lambda} \mu\left(\lambda^2 - \frac{1}{\lambda}\right)$$

$$\sigma_i^{(\text{eng})} = \frac{F}{A_0}$$

$$A = \ell_2 \ell_3 = \lambda_2 \lambda_3 L_2 L_3 = \lambda_2 \lambda_3 A_0$$

Initial cross-section area