Non-Associated Flow Plasticity

2022 Fall Semester

Jeong Whan Yoon

Professor of Mechanical Engineering
KAIST

Publications for Non-AFR of Metals

• **Spitzig, W.A., Richmond, O**., 1984. “The effect of pressure on the flow stress of metals” Acta Metall. 32, 457–463.

• **Stoughton, T.B**., 2002. “A non-associated flow rule for sheet metal forming”. *Int. J. Plasticity* 18, 687–714.

•**T.B. Stoughton and J.W. Yoon**, “A pressure-sensitive yield criterion under a non-associated flow rule for sheet metal forming”, *Int. J. of Plasticity*, Vol. 20, No.9, pp. 705-731 (2004).

• **T.B. Stoughton and J.W. Yoon**, “Review of Drucker’s Postulate and the Issue of Plastic Stability”, *Int. J. Plasticity,* Vol.22, pp.391-433 (2006).

• **T.B. Stoughton, J.W. Yoon**, “On the existence of indeterminate solutions to the equations of motion under non-associated flow”, *Int. J. Plasticity,* Vol.24, pp.583-613(2008).

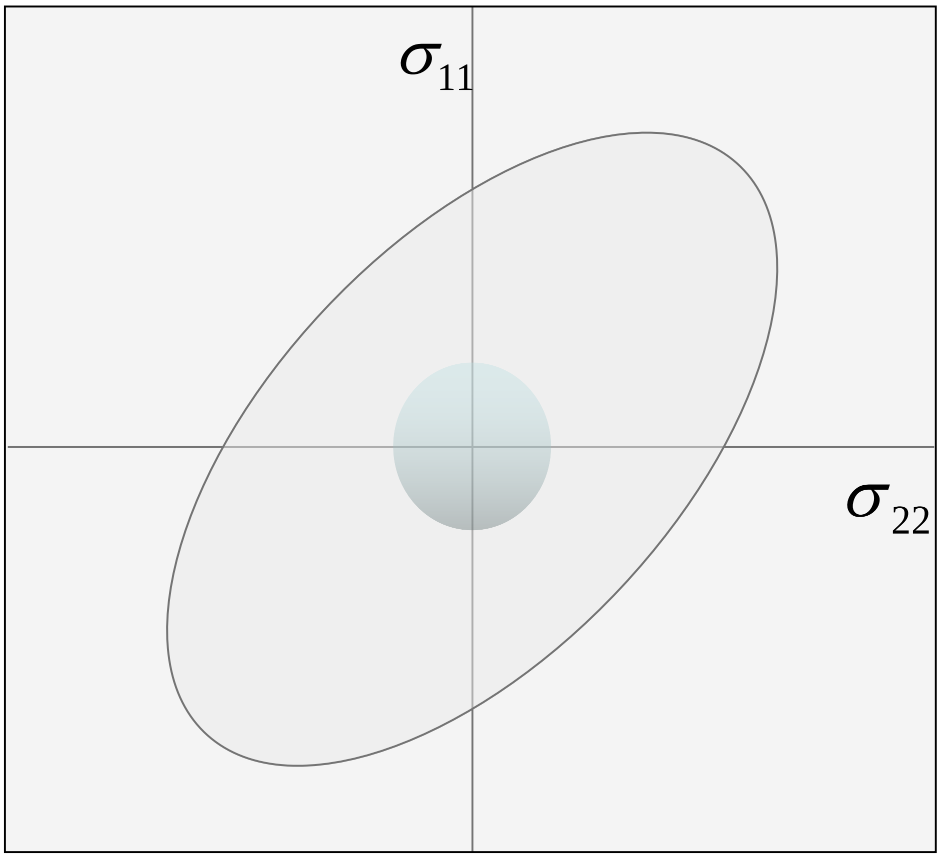
• **T.B. Stoughton, J.W. Yoon**, “Anisotropic hardening and non-associated flow in proportional loading of sheet metals”, *Int. J. of Plasticity*, Vol.25, pp.1777-1818 (2009).

• **A. Taherizadeh, D.E. Green, A. Ghaei, J.W. Yoon**, “A Non-associated constitutive model with mixed iso-kinematic hardening for finite element simulation of sheet metal forming”, *Int. J. of Plasticity, 25,* 288-309 (2010). Cited “1”

• **A. Taherizadeh, D. E. Green, J.W. Yoon**, “Evaluation of advanced anisotropic models with mixed hardening for general associated and non-associated flow metal plasticity”, *International Journal of Plasticity*, *In Press, Corrected Proof*, *Available online 19 May 2011*

Classical Elastoplasticity Theory

$$E_{ij}^{(e)} = C_{ijkl}^{-1}\sigma_{kl}$$



At low stresses the deformation is pure elastic, where the relationship between the (elastic) strain and the applied stress state is assumed to be linear.

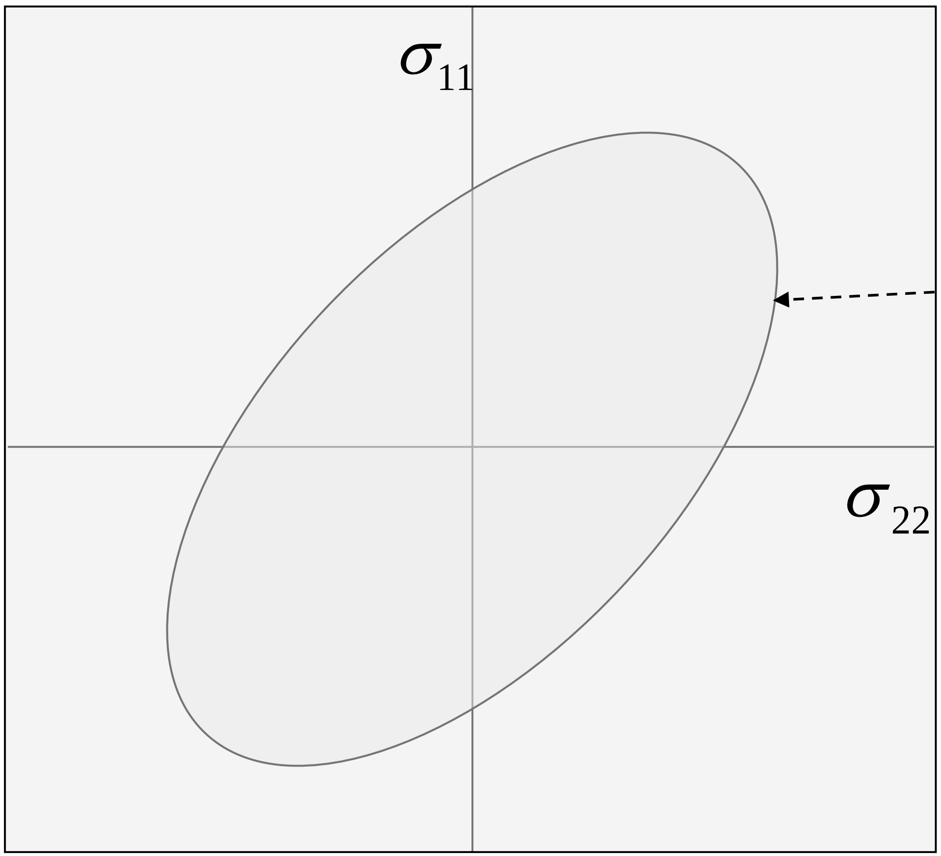
$$E_{11}^{(e)} = \frac{1}{E}\sigma^{11} - \frac{\nu}{E}\sigma^{22}$$

$$E_{22}^{(e)} = \frac{1}{E}\sigma^{22} - \frac{\nu}{E}\sigma^{11}$$

$$E_{12}^{(e)} = \frac{1+\nu}{E}\sigma^{12}$$

Classical Elastoplasticity Theory

$$E_{ij}^{(e)} = C_{ijkl}^{-1}\sigma_{kl}$$



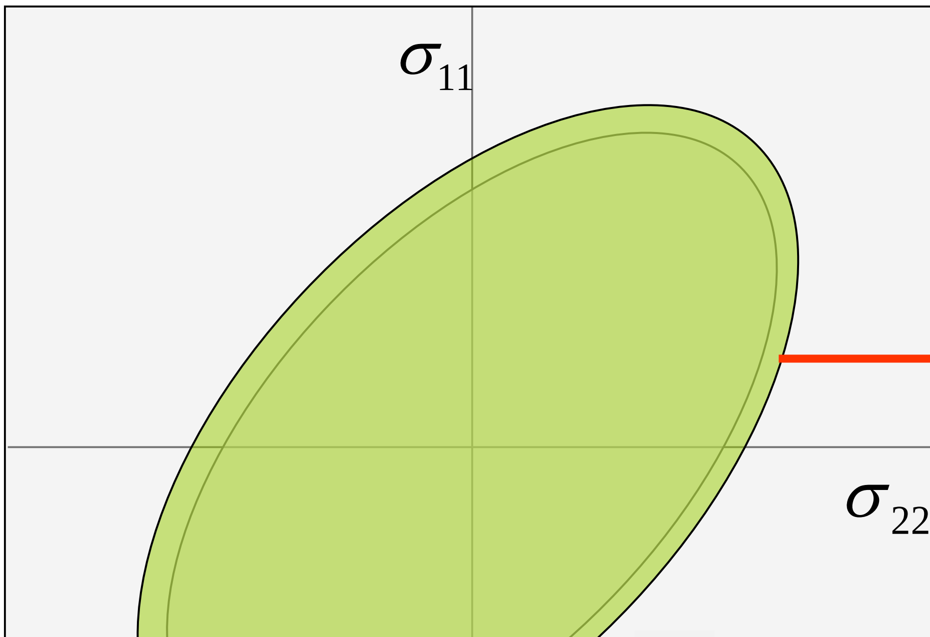
Deformation is *pure elastic* as long as the magnitude of the yield function is below a critical value called the yield stress.

$$\bar{\sigma}_y < \sigma_Y$$

$$\bar{\sigma}_y = \bar{\sigma}_y\left(\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31}\right)$$

Classical Elastoplasticity Theory

For materials with positive work hardening the magnitude of the yield stress increases with plastic strain.

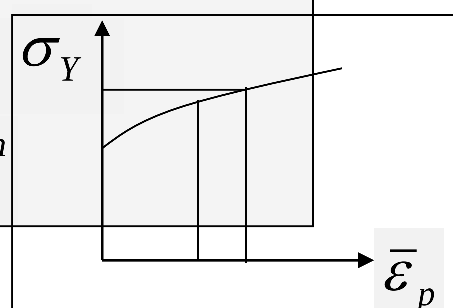


$$\bar{\sigma}_y = \bar{\sigma}_y(\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31})$$

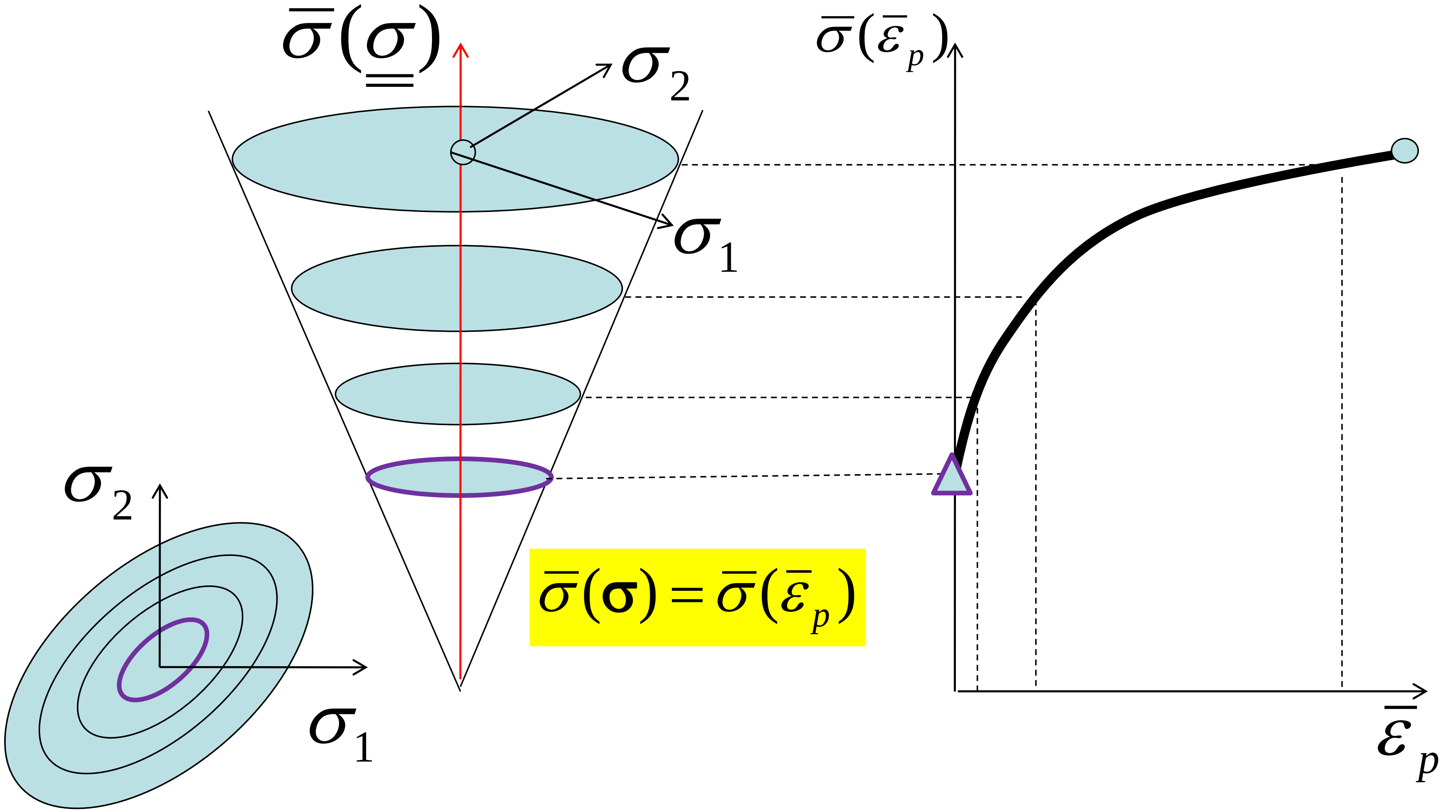
$$\bar{\sigma}_y = \sigma_Y$$

Power Law Example

$$\sigma_Y = \sigma_Y(\bar{\varepsilon}_p) = K\left(\varepsilon_o + \bar{\varepsilon}_p\right)^n$$

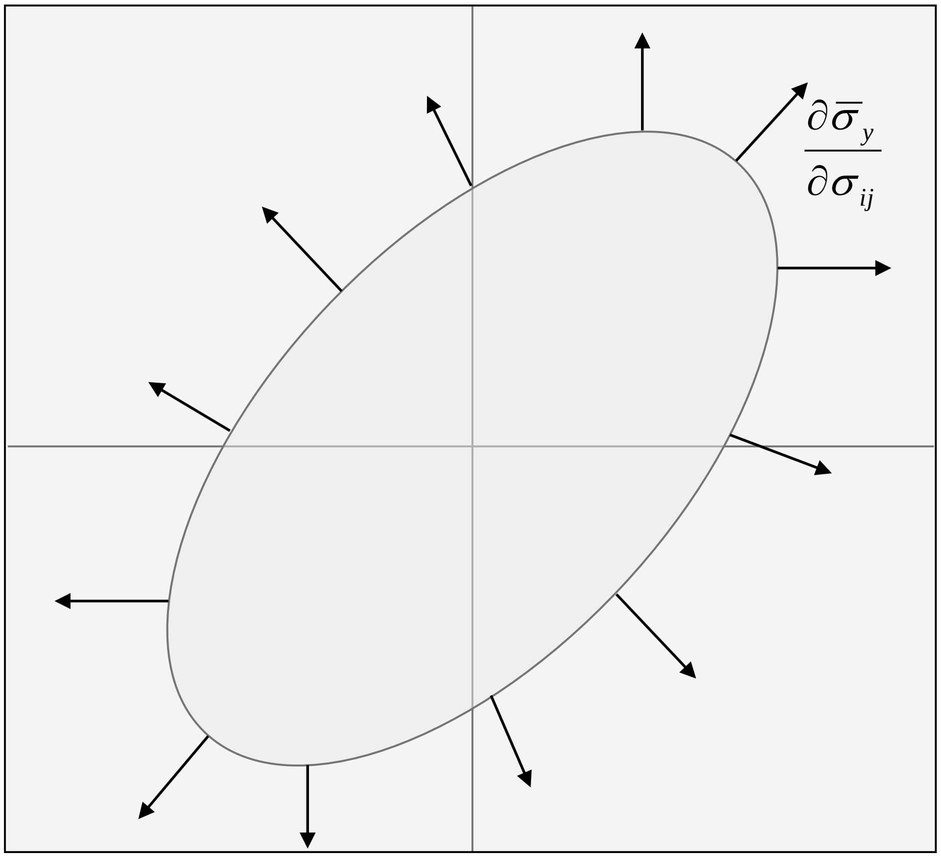


Yield Function and Hardening in View of Potential



Elastoplasticity Theory Based on Associated Flow Rule

$$E_{ij}^{(e)} = C_{ijkl}^{-1}\sigma_{kl}$$



The plastic strain rate is parallel to the gradient of the yield function.

$$\dot{E}_{ij}^{(p)} = \dot{\bar{\varepsilon}}_p \frac{\partial \bar{\sigma}_y}{\partial \sigma_{ij}}$$

$$\bar{\sigma}_y = \bar{\sigma}_y\left(\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31}\right)$$

$$\sigma_Y = \sigma_Y\left(\bar{\varepsilon}_p\right)$$

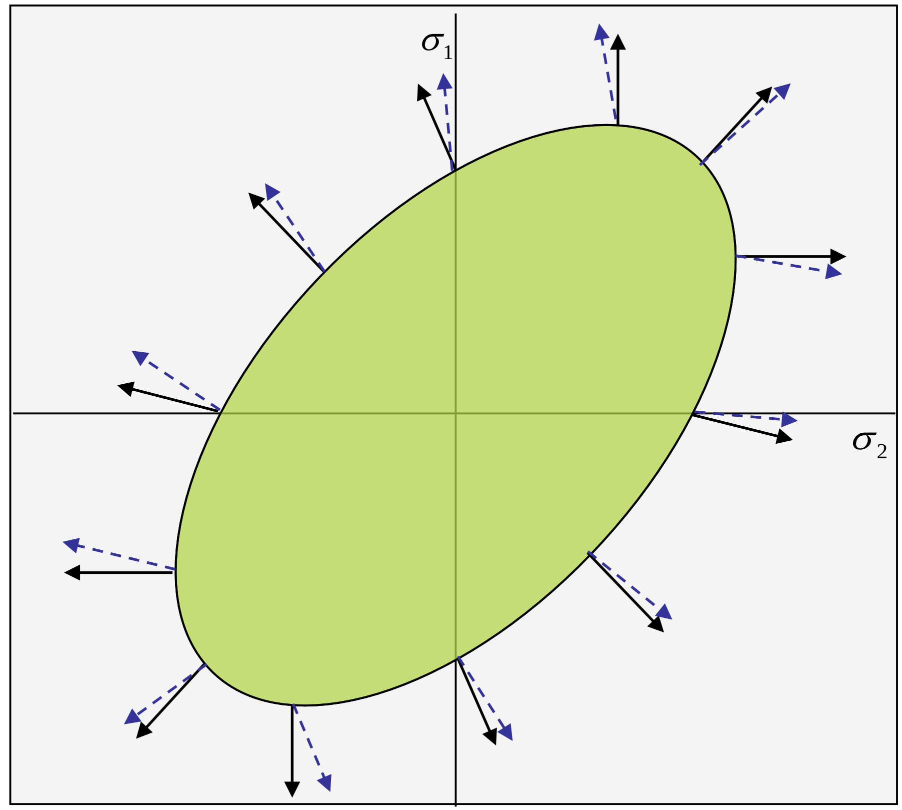
$$\bar{\sigma}_y = \sigma_Y$$

Non-Associated Flow Rule

$$\bar{\sigma}_y\left(\sigma_{ij}\right) = f_y\left(\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31}\right)$$

$$\dot{E}_{ij}^{(p)} = \dot{\bar{\varepsilon}}_p \frac{\partial \bar{\sigma}_p}{\partial \sigma_{ij}}$$

$$\bar{\sigma}_p\left(\sigma_{ij}\right) = f_p\left(\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31}\right)$$



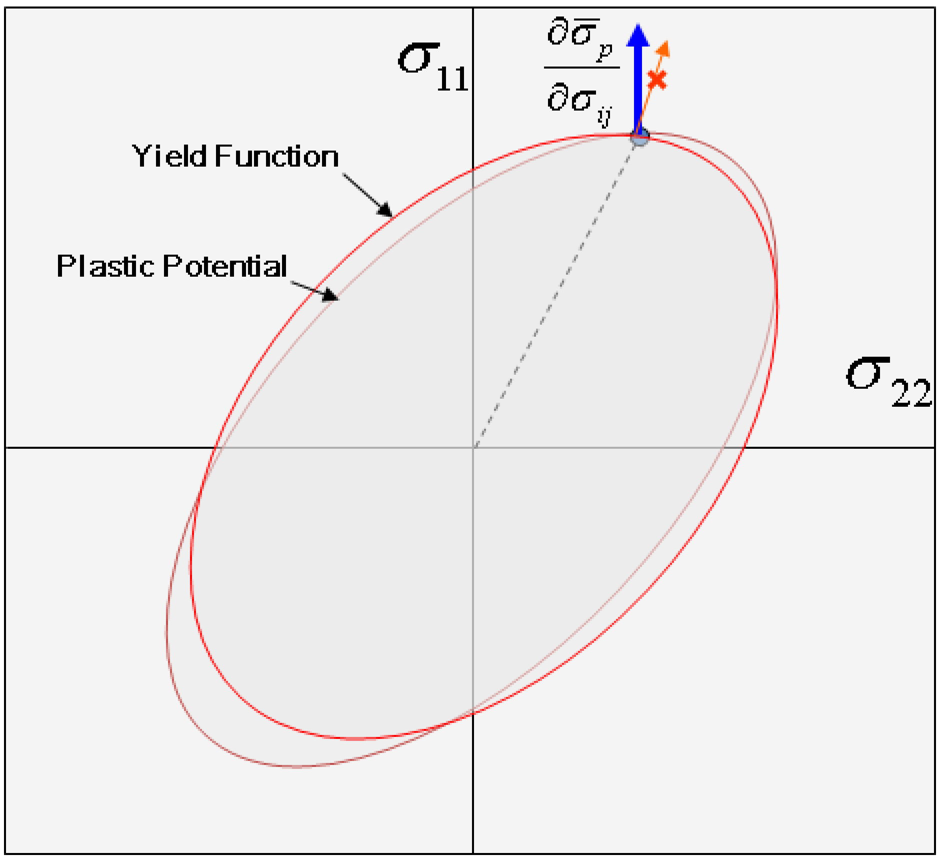
non-AFR Elastoplasticity Theory

$$E_{ij}^{(e)} = C_{ijkl}^{-1}\sigma_{kl}$$

The AFR is a special case of this general model.

$$\bar{\sigma}_p = \bar{\sigma}_y$$

(Non-AFR)



$$\dot{E}_{ij}^{(p)} = \dot{\bar{\varepsilon}}_p \frac{\partial \bar{\sigma}_p}{\partial \sigma_{ij}}$$

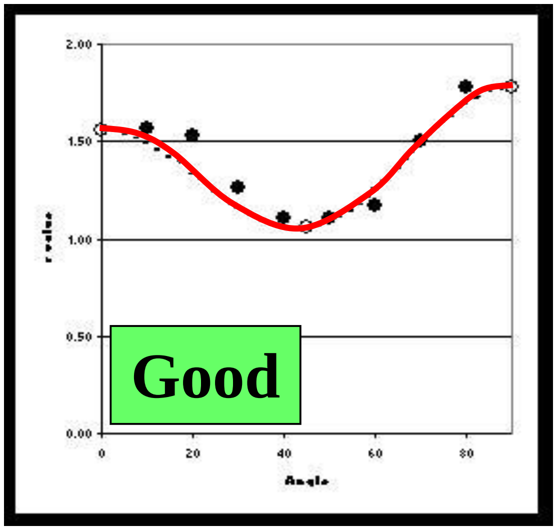
$$\bar{\sigma}_p = \bar{\sigma}_p\left(\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31}\right)$$

$$\bar{\sigma}_y = \bar{\sigma}_y\left(\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31}\right)$$

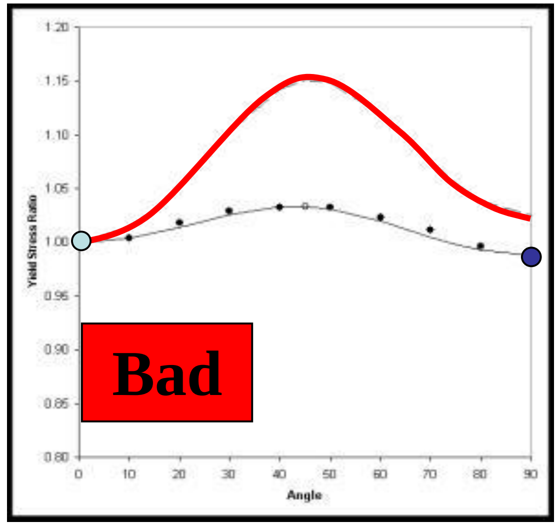
$$\sigma_Y = \sigma_Y\left(\bar{\varepsilon}_p\right)$$

$$\bar{\sigma}_y = \sigma_Y$$

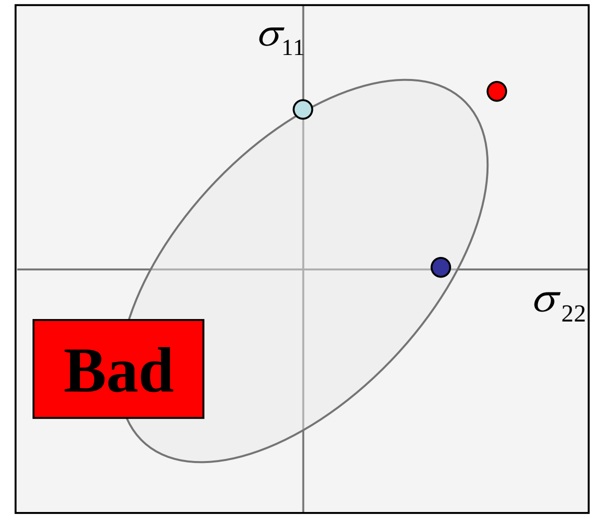
Hill's Function Based on Associated Flow Rule



R Value Distribution



Uniaxial Yield Distribution



Bi-axial Yield

$$\bar{\sigma}_y = \sqrt{\sigma_{11}{}^2 + \frac{F+H}{G+H}\sigma_{22}{}^2 - 2\frac{H}{G+H}\sigma_{11}\sigma_{22} + 2\frac{N}{G+H}\sigma_{12}\sigma_{21}}$$

$$\dot{E}_{ij}^{(p)} = \dot{\bar{\varepsilon}}_p \frac{\partial\bar{\sigma}_y}{\partial\sigma_{ij}}$$

Yield function & potential

$$2\bar{\sigma}^2 = (G+H)\sigma_{xx}^2 + (F+H)\sigma_{yy}^2 - 2H\sigma_{xx}\sigma_{yy} + 2N\sigma_{xy}^2 = 2\bar{\sigma}^2$$

$$\bar{\sigma}_y = \sqrt{\sigma_{11}{}^2 + \frac{F+H}{G+H}\sigma_{22}{}^2 - 2\frac{H}{G+H}\sigma_{11}\sigma_{22} + 2\frac{N}{G+H}\sigma_{12}\sigma_{21}}$$

$$F = (\bar{\sigma}/\sigma_{90})^2 + (\bar{\sigma}/\sigma_b)^2 - (\bar{\sigma}/\sigma_0)^2$$

$$G = (\bar{\sigma}/\sigma_b)^2 + (\bar{\sigma}/\sigma_0)^2 - (\bar{\sigma}/\sigma_{90})^2$$

$$H = (\bar{\sigma}/\sigma_0)^2 + (\bar{\sigma}/\sigma_{90})^2 - (\bar{\sigma}/\sigma_b)^2$$

$$N = 4(\bar{\sigma}/\sigma_{45})^2 - (\bar{\sigma}/\sigma_b)^2$$

$$F = \frac{r_o}{r_{90}(1+r_o)}2\left(\frac{\bar{\sigma}}{\sigma_o}\right)^2$$

$$G = \frac{1}{1+r_o}2\left(\frac{\bar{\sigma}}{\sigma_o}\right)^2$$

$$H = \frac{r_o}{1+r_o}2\left(\frac{\bar{\sigma}}{\sigma_o}\right)^2$$

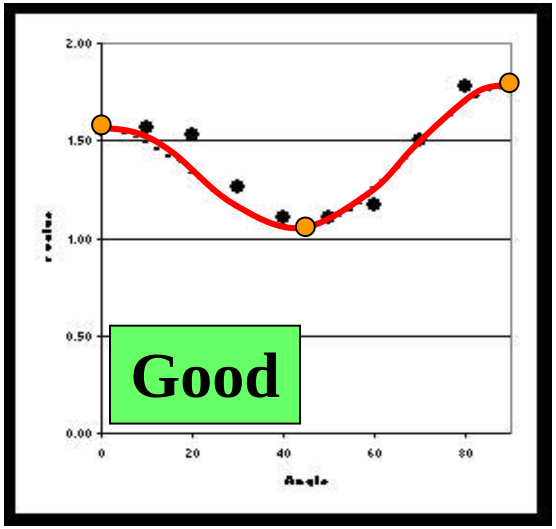
$$N = \frac{(r_o + r_{90})(2r_{45}+1)}{2r_{90}(1+r_o)}2\left(\frac{\bar{\sigma}}{\sigma_o}\right)^2$$

$$\bar{\sigma} = \sigma_0$$

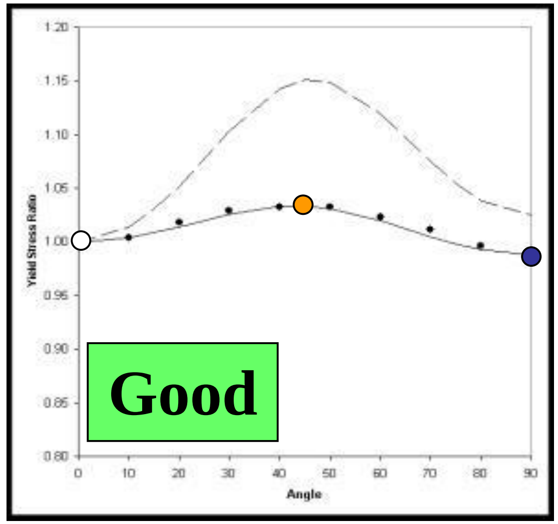
Yield function: $$\bar{\sigma}_y = \sqrt{\sigma_{11}^2 + \left(\frac{\sigma_0}{\sigma_{90}}\right)^2\sigma_{22}^2 - \left(1 + \left(\frac{\sigma_0}{\sigma_{90}}\right)^2 - \left(\frac{\sigma_0}{\sigma_{EB}}\right)^2\right)\sigma_{11}\sigma_{22} + \left(\left(\frac{2\sigma_0}{\sigma_{45}}\right)^2 - \left(\frac{\sigma_0}{\sigma_{EB}}\right)^2\right)\sigma_{12}^2}$$

Potential: $$\bar{\sigma}_p = \sqrt{\sigma_{11}{}^2 + \frac{1+1/r_{90}}{1+1/r_0}\sigma_{22}{}^2 - 2\frac{r_0}{1+r_0}\sigma_{11}\sigma_{22} + (1/r_0 + 1/r_{90})\frac{1+2r_{45}}{1+1/r_0}\sigma_{12}\sigma_{21}}$$

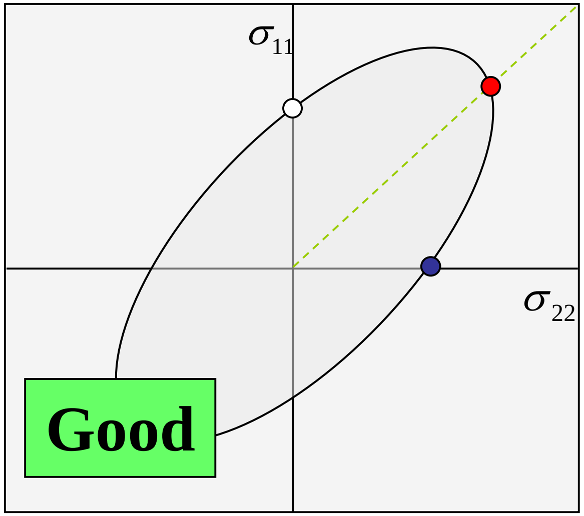
Hill's Function Based on Non-Associated Flow Rule



R Value Distribution



Uniaxial Yield Distribution



Bi-axial Yield

$$\bar{\sigma}_y=\sqrt{\sigma_{11}^2+\left(\frac{\sigma_0}{\sigma_{90}}\right)^2\sigma_{22}^2-\left(1+\left(\frac{\sigma_0}{\sigma_{90}}\right)^2-\left(\frac{\sigma_0}{\sigma_{EB}}\right)^2\right)\sigma_{11}\sigma_{22}+\left(\left(\frac{2\sigma_0}{\sigma_{45}}\right)^2-\left(\frac{\sigma_0}{\sigma_{EB}}\right)^2\right)\sigma_{12}^2}$$

$$\boxed{\dot{E}_{ij}^{(p)}=\dot{\bar{\varepsilon}}_p\frac{\partial\bar{\sigma}_p}{\partial\sigma_{ij}}}$$

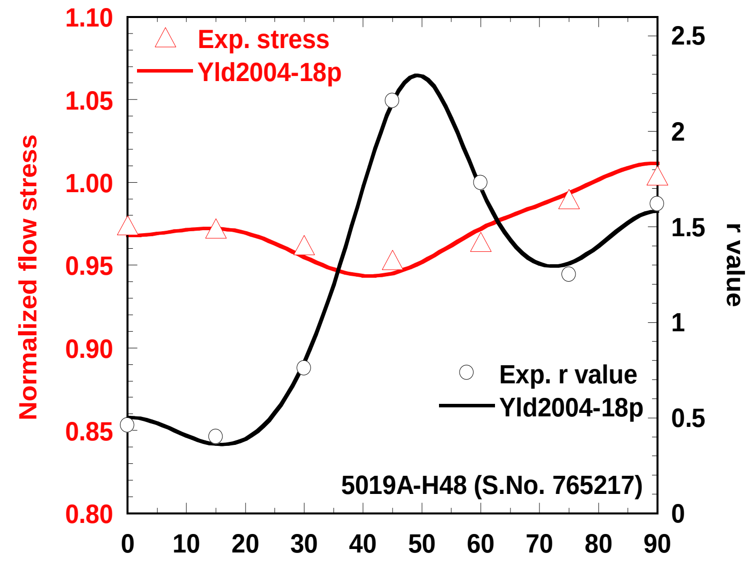
$$\bar{\sigma}_p=\sqrt{\sigma_{11}{}^2+\frac{1+1/r_{90}}{1+1/r_0}\sigma_{22}{}^2-2\frac{r_0}{1+r_0}\sigma_{11}\sigma_{22}+\left(1/r_0+1/r_{90}\right)\frac{1+2r_{45}}{1+1/r_0}\sigma_{12}\sigma_{21}}$$

Upgrade of Barlat's model (Yld2000 → Yld2004) with Non-AFR

(Yoon, Stoughton, Dick, 2007 : NUMIFORM2007)

AFR (Yld2004-18p) →

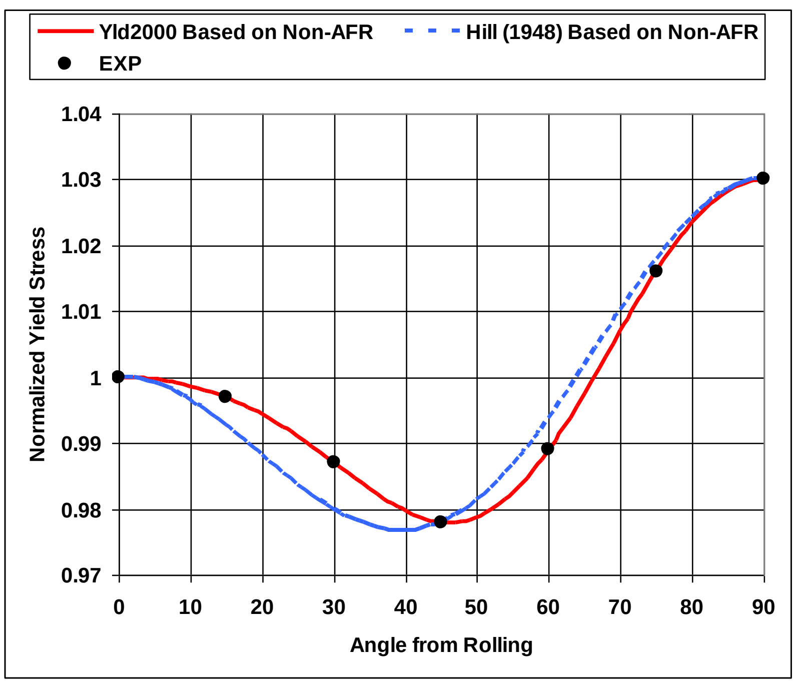
$$\bar{\sigma}_y = \bar{\sigma}_y(\sigma_b, \sigma_0, \sigma_{15}, \sigma_{30}, \sigma_{45}, \sigma_{60}, \sigma_{75}, \sigma_{90}; \; r_b,\ r_0,\ r_{15},\ r_{30},\ r_{45},\ r_{60},\ r_{75},\ r_{90})$$

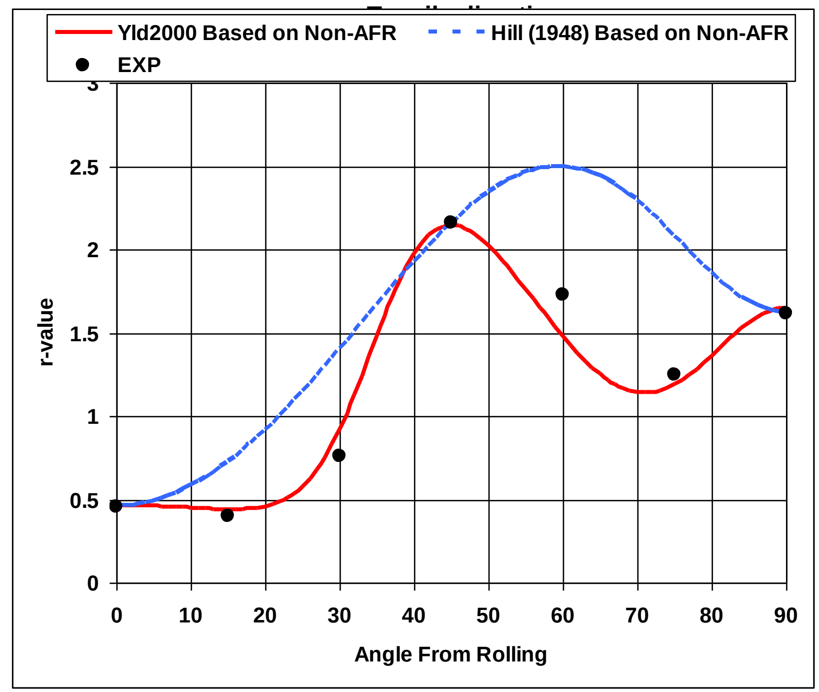


Non-AFR (Yld2000-2d)

$$\bar{\sigma}_y = \bar{\sigma}_y(\sigma_b, \sigma_0, \sigma_{15}, \sigma_{30}, \sigma_{45}, \sigma_{60}, \sigma_{75}, \sigma_{90})$$

$$\bar{\sigma}_p = \bar{\sigma}_p(r_b,\ r_0,\ r_{15},\ r_{30},\ r_{45},\ r_{60},\ r_{75},\ r_{90})$$





Derivation of Constitutive Law

$$\bar{\sigma}_y = \sqrt{\sigma_{11}^2 + \left(\frac{\sigma_0}{\sigma_{90}}\right)^2 \sigma_{22}^2 - \left(1 + \left(\frac{\sigma_0}{\sigma_{90}}\right)^2 - \left(\frac{\sigma_0}{\sigma_{EB}}\right)^2\right)\sigma_{11}\sigma_{22} + \left(\left(\frac{2\sigma_0}{\sigma_{45}}\right)^2 - \left(\frac{\sigma_0}{\sigma_{EB}}\right)^2\right)\sigma_{12}^2}$$

The yield function defines the *magnitude* of the plastic strain

$$\dot{E}_{ij}^{(p)} = \dot{\bar{\varepsilon}}_p \frac{\partial \bar{\sigma}_p}{\partial \sigma_{ij}}$$

The gradient of the plastic potential defines only the *direction* of the plastic strain tensor

This is introduced by imposing the stress strain relation as a constraint during plastic flow

$$\bar{\sigma}_y = \sigma_Y(\bar{\varepsilon}_p)$$

$$\frac{\partial \bar{\sigma}_y}{\partial \sigma_{ij}} \dot{\sigma}_{ij} = \frac{d\sigma_Y}{d\bar{\varepsilon}_p} \dot{\bar{\varepsilon}}_p$$

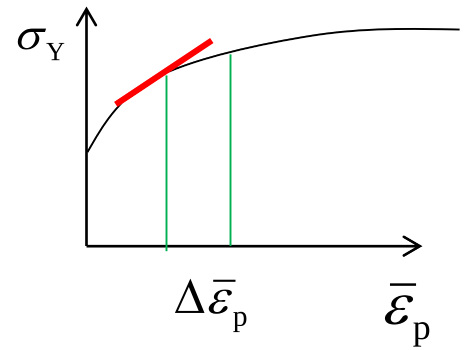
$$\dot{\bar{\varepsilon}}_p = \frac{1}{\dfrac{d\sigma_Y}{d\bar{\varepsilon}_p}} \frac{\partial \bar{\sigma}_y}{\partial \sigma_{ij}} \dot{\sigma}_{ij} > 0$$

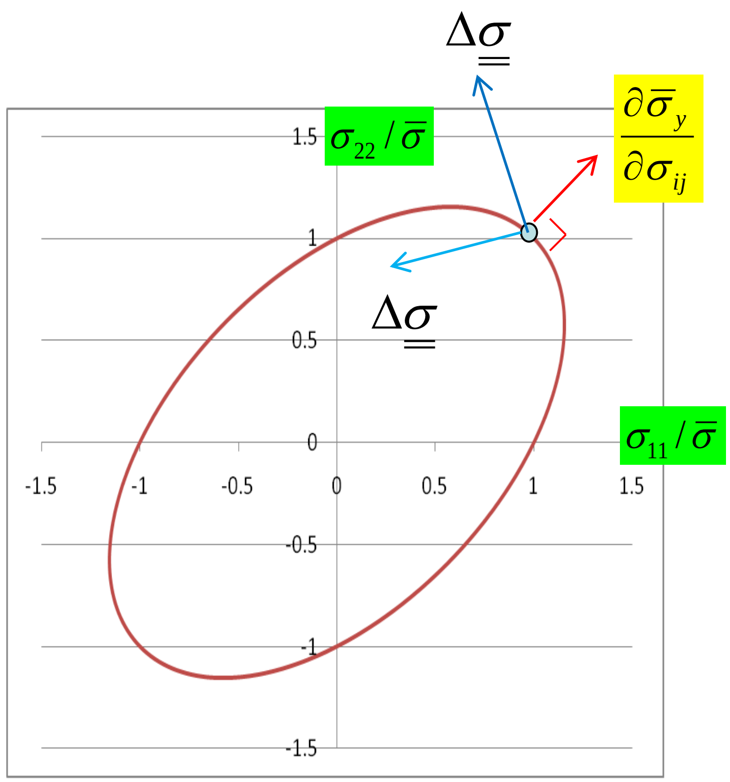
$$\bar{\sigma}_p = \sqrt{\sigma_{11}{}^2 + \frac{1+1/r_{90}}{1+1/r_0}\sigma_{22}{}^2 - 2\frac{r_0}{1+r_0}\sigma_{11}\sigma_{22} + \left(1/r_0 + 1/r_{90}\right)\frac{1+2r_{45}}{1+1/r_0}\sigma_{12}\sigma_{21}}$$

Effective Plastic Strain Rate

$$\dot{\bar{\varepsilon}}_p = \frac{1}{\dfrac{d\sigma_Y}{d\bar{\varepsilon}_p}} \frac{\partial \bar{\sigma}_y}{\partial \sigma_{ij}} \dot{\sigma}_{ij} = \frac{1}{\dfrac{d\sigma_Y}{d\bar{\varepsilon}_p}} \dot{\bar{\sigma}}_y > 0$$

$$where\ \frac{\mathrm{d}\sigma_{\mathrm{Y}}}{\mathrm{d}\bar{\varepsilon}_{\mathrm{p}}} > 0$$





$$\dot{\bar{\sigma}}_y = \frac{\partial \bar{\sigma}_y}{\partial \sigma_{ij}} \dot{\sigma}_{ij}$$

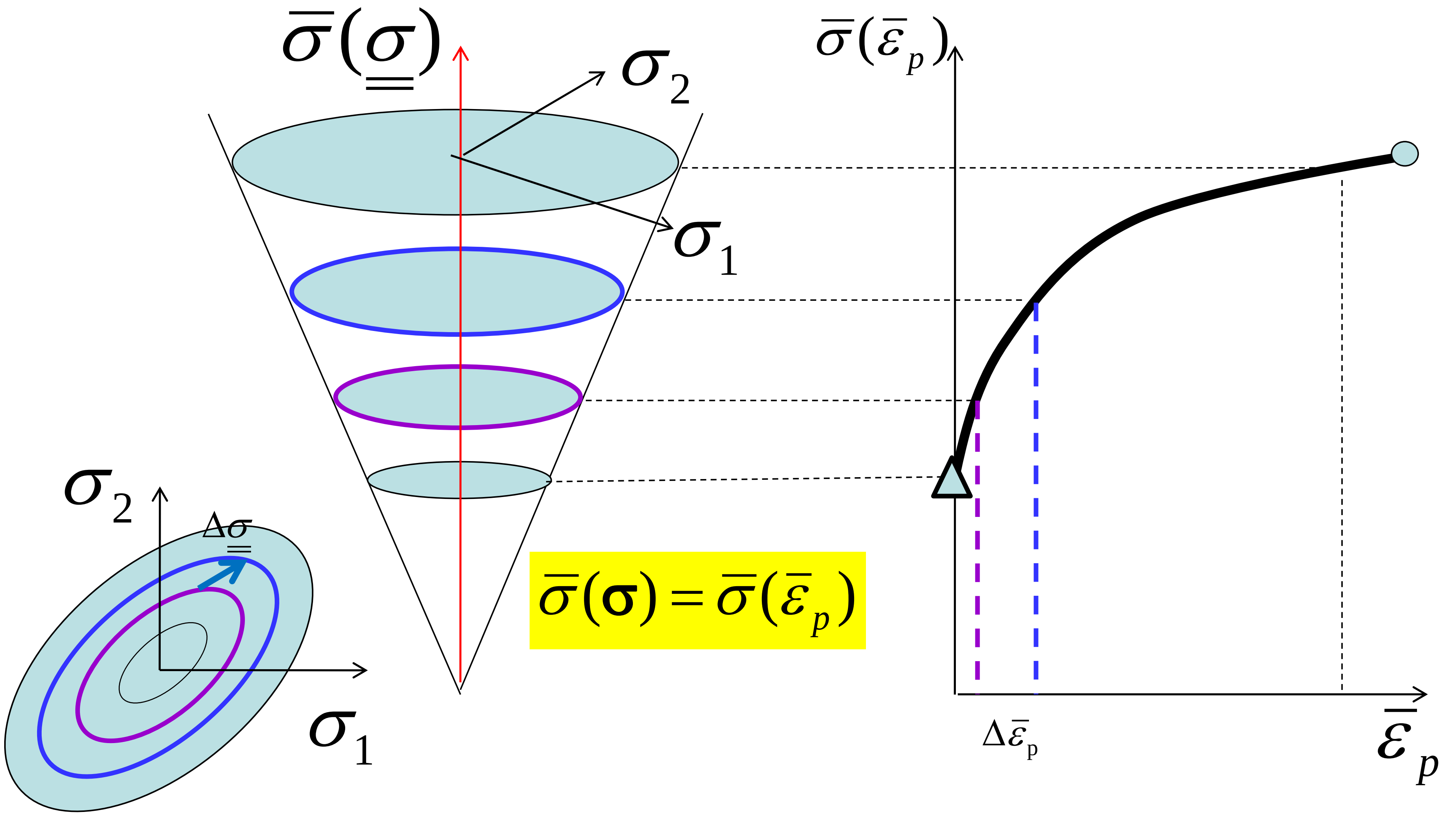
or

$$\Delta\bar{\sigma}_y = \frac{\partial \bar{\sigma}_y}{\partial \sigma_{ij}} \Delta\sigma_{ij} = \frac{\partial \bar{\sigma}_y}{\partial \underline{\underline{\sigma}}} \cdot \Delta\underline{\underline{\sigma}}$$

$$if\ \Delta\bar{\sigma}_y > 0, \Delta\bar{\varepsilon}_{\mathrm{p}} (or\ \dot{\bar{\varepsilon}}_{\mathrm{p}}) > 0$$

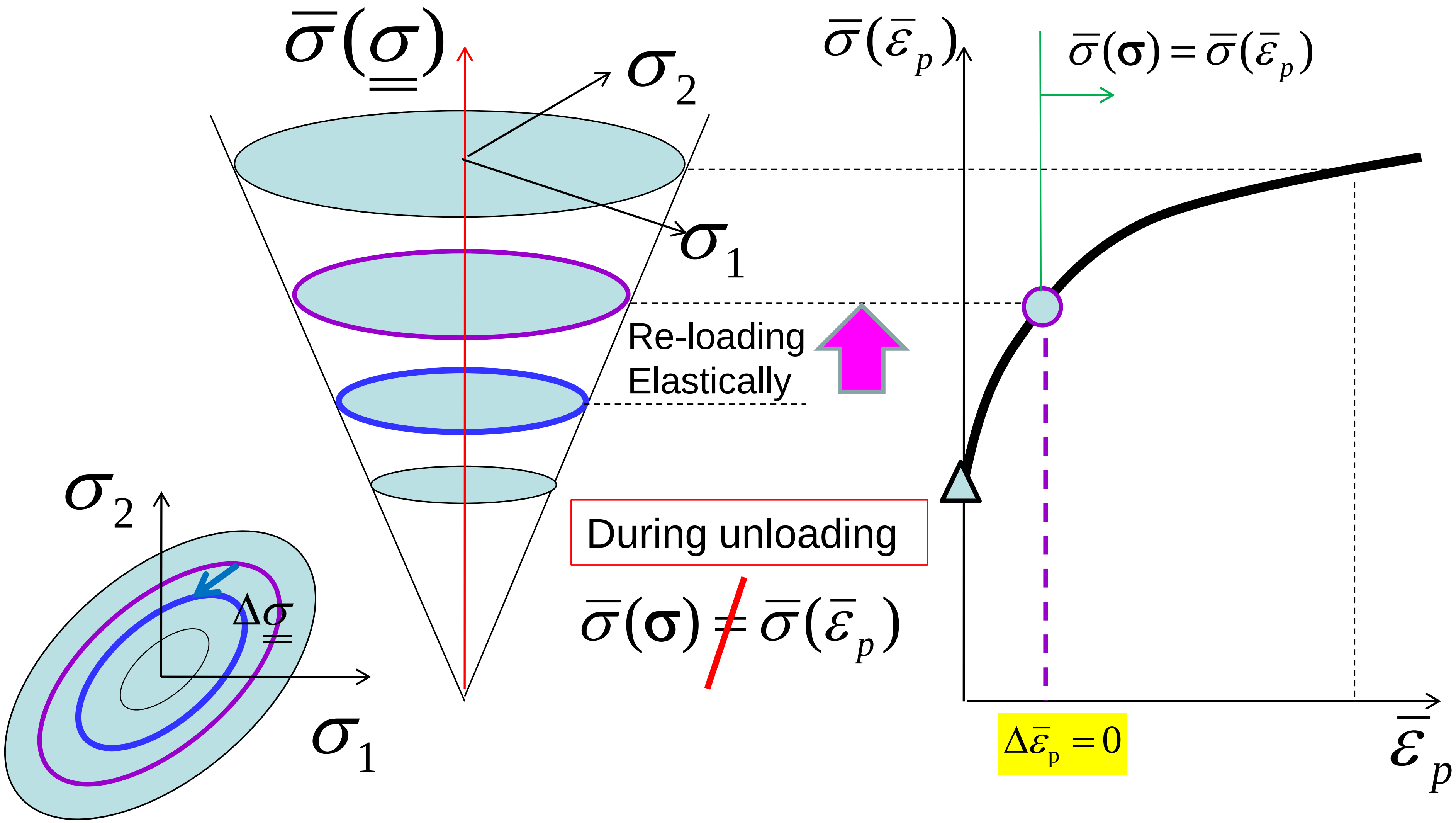
$$if\ \Delta\bar{\sigma}_y \le 0, \Delta\bar{\varepsilon}_{\mathrm{p}} (or\ \dot{\bar{\varepsilon}}_{\mathrm{p}}) = 0$$

Effective Plastic Strain Rate (Loading)



$$\bar{\sigma}(\boldsymbol{\sigma}) = \bar{\sigma}(\bar{\varepsilon}_p)$$

Effective Plastic Strain Rate (Unloading)



Derivation of Constitutive Law

$$\left.\begin{array}{ll}\bar{\sigma}_y(\sigma_{ij}) < \sigma_Y(\bar{\varepsilon}_p) & pure-elastic \\ \bar{\sigma}_y(\sigma_{ij}) = \sigma_Y(\bar{\varepsilon}_p) \quad and \quad \dot{\bar{\sigma}}_y(\sigma_{ij}) \le 0 & pure-elastic\end{array}\right\} H_{ep} = 0$$

$$\bar{\sigma}_y(\sigma_{ij}) = \sigma_Y(\bar{\varepsilon}_p) \quad and \quad \dot{\bar{\sigma}}_y(\sigma_{ij}) > 0 \quad elastic-plastic \qquad H_{ep} = 1$$

$$\sigma_Y = K(\varepsilon_O + \bar{\varepsilon}_p)^n \qquad \sigma_Y = A - B\exp(-C\bar{\varepsilon}_p) \qquad H_{ep} = 1$$

e-p

$$\frac{\partial \bar{\sigma}_y}{\partial \sigma_{ij}} \dot{\sigma}_{ij} = \frac{d\bar{\sigma}_Y}{d\bar{\varepsilon}_p} \dot{\bar{\varepsilon}}_p$$

$$\dot{\sigma}_{ij} = C_{ijkl} \dot{E}_{kl}^{(e)}$$

$$\begin{pmatrix} \dot{\sigma}_{11} \\ \dot{\sigma}_{22} \\ \dot{\sigma}_{12} \end{pmatrix} = \frac{E}{1-\nu^2} \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu) \end{pmatrix} \begin{pmatrix} \dot{E}_{11}^{(e)} \\ \dot{E}_{22}^{(e)} \\ 2\dot{E}_{12}^{(e)} \end{pmatrix}$$

Construction of the strain rate components

$$\bar{\sigma}_y = \sigma_Y(\bar{\varepsilon}_p)$$

$$H_{ep} = 1 \qquad \frac{\partial \bar{\sigma}_y}{\partial \sigma_{ij}} \dot{\sigma}_{ij} = \frac{d\sigma_Y}{d\bar{\varepsilon}_p} \dot{\bar{\varepsilon}}_p$$

$$\dot{E}_{\alpha\beta}^{(p)} = \dot{\bar{\varepsilon}}_p \frac{\partial \bar{\sigma}_p}{\partial \sigma_{\alpha\beta}}$$

$$H_{ep} = 0,1 \quad \Rightarrow \quad \dot{E}_{ij}^{(p)} = \frac{H_{ep}}{\frac{d\bar{\sigma}_Y}{d\bar{\varepsilon}_p}} \frac{\partial \bar{\sigma}_p}{\partial \sigma_{ij}} \frac{\partial \bar{\sigma}_y}{\partial \sigma_{kl}} \dot{\sigma}_{kl}$$

$$\dot{\sigma}_{ij} = C^{ijkl} \dot{E}_{kl}^{(e)}$$

$$H_{ep} = 0,1 \quad \Rightarrow \quad \dot{E}_{ij}^{(e)} = C_{ijkl}^{-1} \dot{\sigma}_{kl}$$

$$\dot{E}_{ij} = \left(C_{ijkl}^{-1} + \frac{H_{ep}}{\frac{d\bar{\sigma}_Y}{d\bar{\varepsilon}_p}} \frac{\partial \bar{\sigma}_p}{\partial \sigma_{ij}} \frac{\partial \bar{\sigma}_y}{\partial \sigma_{kl}} \right) \dot{\sigma}_{kl}$$

Incremental Elastoplasticity Theory

Constitutive law is invertible…

For a given $\sigma_{ij} \quad \bar{\varepsilon}_p \quad \dot{\sigma}_{ij}$

if $\bar{\sigma}_y < \sigma_Y$

or $\bar{\sigma}_y = \sigma_Y$ **and** $\dfrac{\partial \bar{\sigma}_y}{\partial \sigma_{kl}} \dot{\sigma}_{kl} \le 0$ **then** $H_{ep} = 0$

else (otherwise)

($\bar{\sigma}_y = \sigma_Y$ **and** $\dfrac{\partial \bar{\sigma}_y}{\partial \sigma_{kl}} \dot{\sigma}_{kl} > 0$ **)** $H_{ep} = 1$

Model Constraints

$$\frac{d\sigma_Y}{d\bar{\varepsilon}_p} > 0$$

$$\frac{d\sigma_Y}{d\bar{\varepsilon}_p} + \frac{\partial \bar{\sigma}_p}{\partial \sigma_{\alpha\beta}} C_{\alpha\beta\eta\rho} \frac{\partial \bar{\sigma}_y}{\partial \sigma_{\eta\rho}} > 0$$

$$\dot{E}_{ij} = \dot{E}_{ij}^{(e)} + H_{ep} \dot{E}_{ij}^{(p)}$$
$$= \left(C_{ijkl}^{-1} + H_{ep} \left(\frac{d\sigma_Y}{d\bar{\varepsilon}_p} \right)^{-1} \frac{\partial \bar{\sigma}_p}{\partial \sigma_{ij}} \frac{\partial \bar{\sigma}_y}{\partial \sigma_{kl}} \right) \dot{\sigma}_{kl}$$

$$\dot{\bar{\varepsilon}}_p = \left(\frac{H_{ep}}{\dfrac{d\sigma_Y}{d\bar{\varepsilon}_p} + \dfrac{\partial \bar{\sigma}_p}{\partial \sigma_{\alpha\beta}} C_{\alpha\beta\eta\rho} \dfrac{\partial \bar{\sigma}_y}{\partial \sigma_{\eta\rho}}} \right) \left(\frac{\partial \bar{\sigma}_y}{\partial \sigma_{ij}} C_{ijkl} \dot{E}_{kl} \right) > 0$$

$$\dot{\sigma}_{kl} = \left(C_{klij} - \frac{H_{ep}}{\dfrac{d\sigma_Y}{d\bar{\varepsilon}_p} + \dfrac{\partial \bar{\sigma}_p}{\partial \sigma_{\alpha\beta}} C_{\alpha\beta\eta\rho} \dfrac{\partial \bar{\sigma}_y}{\partial \sigma_{\eta\rho}}} C_{klab} \frac{\partial \bar{\sigma}_p}{\partial \sigma_{ab}} \frac{\partial \bar{\sigma}_y}{\partial \sigma_{cd}} C_{cdij} \right) \dot{E}_{ij}$$

$$d\underline{\sigma} = \underline{\underline{C}}\, d\underline{\varepsilon}^{e} = \underline{\underline{C}}\,(d\underline{\varepsilon} - d\underline{\varepsilon}^{p})$$

$$= \underline{\underline{C}}\,(d\underline{\varepsilon} - d\bar{\varepsilon}_p \frac{\partial \bar{\sigma}_p}{\partial \underline{\sigma}})$$

$$\bar{\sigma}_y(\underline{\sigma}) = \sigma_Y(\bar{\varepsilon}_p) \Rightarrow \frac{\partial \bar{\sigma}_y}{\partial \underline{\sigma}} d\underline{\sigma} = \frac{\partial \sigma_Y}{\partial \bar{\varepsilon}_p} d\bar{\varepsilon}_p$$

$$d\underline{\sigma} = \underline{\underline{C}}\,(d\underline{\varepsilon} - d\bar{\varepsilon}_p \frac{\partial \bar{\sigma}_p}{\partial \underline{\sigma}})$$

$$d\bar{\varepsilon}_p = ?$$

$$d\underline{\sigma} = \underline{\underline{C}}\, d\underline{\varepsilon}^{e} = \underline{\underline{C}}\,(d\underline{\varepsilon} - d\underline{\varepsilon}^{p})$$

$$= \underline{\underline{C}}\,\left(d\underline{\varepsilon} - d\bar{\varepsilon}_p \frac{\partial \bar{\sigma}_p}{\partial \underline{\sigma}}\right)$$

$$\bar{\sigma}_y(\underline{\sigma}) = \sigma_Y(\bar{\varepsilon}_p) \quad \Longrightarrow \quad \frac{\partial \bar{\sigma}_y}{\partial \underline{\sigma}}\, d\underline{\sigma} = \frac{\partial \sigma_Y}{\partial \bar{\varepsilon}_p}\, d\bar{\varepsilon}_p$$

$$\frac{\partial \bar{\sigma}_y}{\partial \underline{\sigma}}\left(\underline{\underline{C}}\, d\underline{\varepsilon} - d\bar{\varepsilon}_p \underline{\underline{C}}\, \frac{\partial \bar{\sigma}_p}{\partial \underline{\sigma}}\right) = \frac{\partial \sigma_Y}{\partial \bar{\varepsilon}_p}\, d\bar{\varepsilon}_p$$

$$d\bar{\varepsilon}_p = \frac{\dfrac{\partial \bar{\sigma}_y}{\partial \underline{\sigma}}\, \underline{\underline{C}}\, d\underline{\varepsilon}}{\dfrac{\partial \sigma_Y}{\partial \bar{\varepsilon}_p} + \dfrac{\partial \bar{\sigma}_y}{\partial \underline{\sigma}}\, \underline{\underline{C}}\, \dfrac{\partial \bar{\sigma}_p}{\partial \underline{\sigma}}} \left(or\ d\bar{\varepsilon}_p = \frac{\left[\dfrac{\partial \bar{\sigma}_y}{\partial \underline{\sigma}}\right]^T \left[\underline{\underline{C}}\, d\underline{\varepsilon}\right]}{\dfrac{\partial \sigma_Y}{\partial \bar{\varepsilon}_p} + \left[\dfrac{\partial \bar{\sigma}_y}{\partial \underline{\sigma}}\right]^T \left[\underline{\underline{C}}\right] \left[\dfrac{\partial \bar{\sigma}_p}{\partial \underline{\sigma}}\right]} \right)$$

$$d\bar{\varepsilon}_p = \frac{\frac{\partial \bar{\sigma}_y}{\partial \underline{\sigma}} \underline{\underline{C}} d\underline{\varepsilon}}{\frac{\partial \sigma_Y}{\partial \bar{\varepsilon}_p} + \frac{\partial \bar{\sigma}_y}{\partial \underline{\sigma}} \underline{\underline{C}} \frac{\partial \bar{\sigma}_p}{\partial \underline{\sigma}}}$$

$$d\underline{\sigma} = \underline{\underline{C}}\left(d\underline{\varepsilon} - \frac{\partial \bar{\sigma}_p}{\partial \underline{\sigma}} d\bar{\varepsilon}_p\right)$$

$$d\underline{\sigma} = ?$$

$$d\bar{\varepsilon}_p = \frac{\dfrac{\partial \bar{\sigma}_y}{\partial \underline{\sigma}} \underline{\underline{C}} d\underline{\varepsilon}}{\dfrac{\partial \sigma_Y}{\partial \bar{\varepsilon}_p} + \dfrac{\partial \bar{\sigma}_y}{\partial \underline{\sigma}} \underline{\underline{C}} \dfrac{\partial \bar{\sigma}_p}{\partial \underline{\sigma}}}$$

$$d\underline{\sigma} = \underline{\underline{C}}\left(d\underline{\varepsilon} - \frac{\partial \bar{\sigma}_p}{\partial \underline{\sigma}} d\bar{\varepsilon}_p\right)$$

$$d\underline{\sigma} = \underline{\underline{C}}\left(d\underline{\varepsilon} - \frac{\dfrac{\partial \bar{\sigma}_p}{\partial \underline{\sigma}} \dfrac{\partial \bar{\sigma}_y}{\partial \underline{\sigma}} \underline{\underline{C}}}{\dfrac{\partial \sigma_Y}{\partial \bar{\varepsilon}_p} + \dfrac{\partial \bar{\sigma}_y}{\partial \underline{\sigma}} \underline{\underline{C}} \dfrac{\partial \bar{\sigma}_p}{\partial \underline{\sigma}}} d\underline{\varepsilon} \right) = \left(\underline{\underline{C}} - \frac{\underline{\underline{C}} \dfrac{\partial \bar{\sigma}_p}{\partial \underline{\sigma}} \otimes \underline{\underline{C}} \dfrac{\partial \bar{\sigma}_y}{\partial \underline{\sigma}}}{\dfrac{\partial \sigma_Y}{\partial \bar{\varepsilon}_p} + \dfrac{\partial \bar{\sigma}_y}{\partial \underline{\sigma}} \underline{\underline{C}} \dfrac{\partial \bar{\sigma}_p}{\partial \underline{\sigma}}} \right) d\underline{\varepsilon}$$

Incremental Elastoplasticity Theory

Plastic Work Rate Must Always Be Positive

For a given $\sigma_{ij} \quad \bar{\varepsilon}_p \quad \dot{\sigma}_{ij}$

if $\bar{\sigma}_y < \sigma_Y$

or $\bar{\sigma}_y = \sigma_Y$ **and** $\dfrac{\partial \bar{\sigma}_y}{\partial \sigma_{ij}} \dot{\sigma}_{ij} \le 0$ **then** $H_{ep} = 0$

otherwise $\dfrac{\partial \bar{\sigma}_y}{\partial \sigma_{ij}} \dot{\sigma}_{ij} > 0 \qquad H_{ep} = 1$

$$\dot{E}_{ij} = \dot{E}_{ij}^{(e)} + H_{ep} \dot{E}_{ij}^{(p)}$$

$$= \left(C_{ijkl}^{-1} + H_{ep} \left(\frac{d\sigma_Y}{d\bar{\varepsilon}_p} \right)^{-1} \frac{\partial \bar{\sigma}_p}{\partial \sigma_{ij}} \frac{\partial \bar{\sigma}_y}{\partial \sigma_{kl}} \right) \dot{\sigma}_{kl}$$

Model Constraints

$$\frac{d\sigma_Y}{d\bar{\varepsilon}_p} > 0$$

$$\frac{d\sigma_Y}{d\bar{\varepsilon}_p} + \frac{\partial \bar{\sigma}_p}{\partial \sigma_{\alpha\beta}} C_{\alpha\beta\eta\rho} \frac{\partial \bar{\sigma}_y}{\partial \sigma_{\eta\rho}} > 0$$

$$\frac{\partial \bar{\sigma}_p}{\partial \sigma_{ij}} \sigma_{ij} > 0$$

$$\dot{w}^{(p)} = \sigma_{ij} \dot{E}_{ij}^{(p)} = \frac{H_{ep}}{\frac{d\sigma_Y}{d\bar{\varepsilon}_p}} \left(\frac{\partial \bar{\sigma}_p}{\partial \sigma_{ij}} \sigma_{ij} \right) \left(\frac{\partial \bar{\sigma}_y}{\partial \sigma_{kl}} \dot{\sigma}_{kl} \right) > 0$$

Positive

Proof that Drucker's Postulate is equivalent to the AFR

Drucker's Postulate:

$$\dot{\sigma}^{ij}\dot{\varepsilon}_{ij}^{(p)} = \dot{\boldsymbol{\sigma}}\cdot\dot{\boldsymbol{\varepsilon}}^{(p)} (or\ \Delta\boldsymbol{\sigma}\cdot\Delta\boldsymbol{\varepsilon}^{(p)}) > 0$$

Suppose $\bar{\sigma}_p \neq \bar{\sigma}_y$ **then, we can always define a stress rate** $\dot{\sigma}_{ij}$ **such that**

$$\dot{\sigma}_{ij}\frac{\partial\bar{\sigma}_y}{\partial\sigma_{ij}} > 0 \ \textbf{and}\ \dot{\sigma}_{ij}\frac{\partial\bar{\sigma}_p}{\partial\sigma_{ij}} < 0 \quad .$$

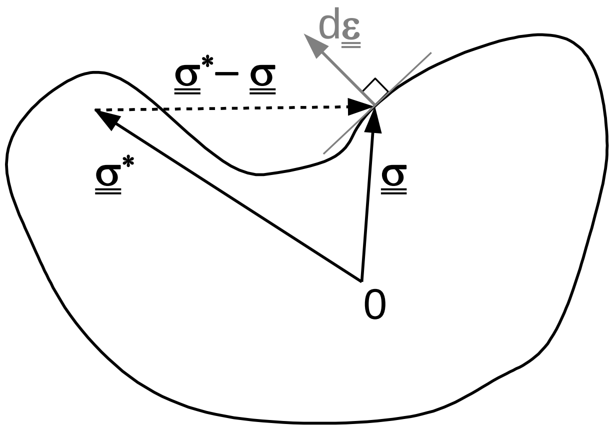
Since,

$$\dot{\sigma}^{ij}\dot{\varepsilon}_{ij}^{(p)} = \frac{1}{\dfrac{d\bar{\sigma}_Y}{d\bar{\varepsilon}_p}}\left(\dot{\sigma}_{ij}\frac{\partial\bar{\sigma}_y}{\partial\sigma_{ij}}\right)\left(\dot{\sigma}_{ij}\frac{\partial\bar{\sigma}_p}{\partial\sigma_{ij}}\right)$$

it follows that if $\bar{\sigma}_p \neq \bar{\sigma}_y$ **then, Drucker's Postulate is violated:** $\dot{\sigma}^{ij}\dot{\varepsilon}_{ij}^{(p)} = \Delta\boldsymbol{\sigma}\cdot\Delta\boldsymbol{\varepsilon}^{(p)} < 0$

Conversely, if $\bar{\sigma}_p = \bar{\sigma}_y$ **(AFR) then**

$$\dot{\sigma}^{ij}\dot{\varepsilon}_{ij}^{(p)} = \frac{1}{\dfrac{d\bar{\sigma}_Y}{d\bar{\varepsilon}_p}}\left(\dot{\sigma}_{ij}\frac{\partial\bar{\sigma}_y}{\partial\sigma_{ij}}\right)^2 > 0$$



$$\dot{E}_{ij}^{(p)} = \frac{H_{ep}}{\dfrac{d\bar{\sigma}_Y}{d\bar{\varepsilon}_p}}\frac{\partial\bar{\sigma}_p}{\partial\sigma_{ij}}\frac{\partial\bar{\sigma}_y}{\partial\sigma_{kl}}\dot{\sigma}_{kl}$$

2nd Order Plastic Work Under Proportional Loading

$$\left.\begin{aligned}\sigma_{ij} &= \sigma_s(t) n_{ij} \\ \dot{\sigma}_{ij} &= \dot{\sigma}_s(t) n_{ij}\end{aligned}\right\} \quad \frac{\sigma_{ij}}{\sigma_s} = n_{ij} = \frac{\dot{\sigma}_{ij}}{\dot{\sigma}_s} \quad \Big\} \qquad \dot{\sigma}_{ij} = \frac{\dot{\sigma}_s}{\sigma_s}\sigma_{ij}$$

$$\dot{\sigma}^{ij}\dot{E}_{ij}^{(p)} = \frac{1}{\dfrac{d\bar{\sigma}_Y}{d\bar{\varepsilon}_p}}\left(\frac{\dot{\sigma}_s}{\sigma_s}\right)^2\left(\frac{\partial\bar{\sigma}_y}{\partial\sigma_{ij}}\sigma_{ij}\right)\left(\frac{\partial\bar{\sigma}_p}{\partial\sigma_{ij}}\sigma_{ij}\right) > 0$$

even if $\dot{\sigma}_s(t) < 0$

Consequently, YPE is not permitted by this non-AFR model.

Model Constraints

$$\frac{d\sigma_Y}{d\bar{\varepsilon}_p} > 0$$

$$\frac{d\sigma_Y}{d\bar{\varepsilon}_p} + \frac{\partial\bar{\sigma}_p}{\partial\sigma_{\alpha\beta}} C_{\alpha\beta\eta\rho} \frac{\partial\bar{\sigma}_y}{\partial\sigma_{\eta\rho}} > 0$$

$$\frac{\partial\bar{\sigma}_y}{\partial\sigma_{ij}}\sigma_{ij} > 0 \qquad \frac{\partial\bar{\sigma}_p}{\partial\sigma_{ij}}\sigma_{ij} > 0$$

$$\dot{\sigma}^{ij}\dot{E}_{ij}^{(p)} = \frac{1}{\dfrac{d\bar{\sigma}_Y}{d\bar{\varepsilon}_p}}\left(\dot{\sigma}_{ij}\frac{\partial\bar{\sigma}_y}{\partial\sigma_{ij}}\right)\left(\dot{\sigma}_{ij}\frac{\partial\bar{\sigma}_p}{\partial\sigma_{ij}}\right)$$

Incremental Elastoplasticity Theory

For FEM analysis of stable metal forming processes, we require material models that will not allow yield point elongation (YPE).

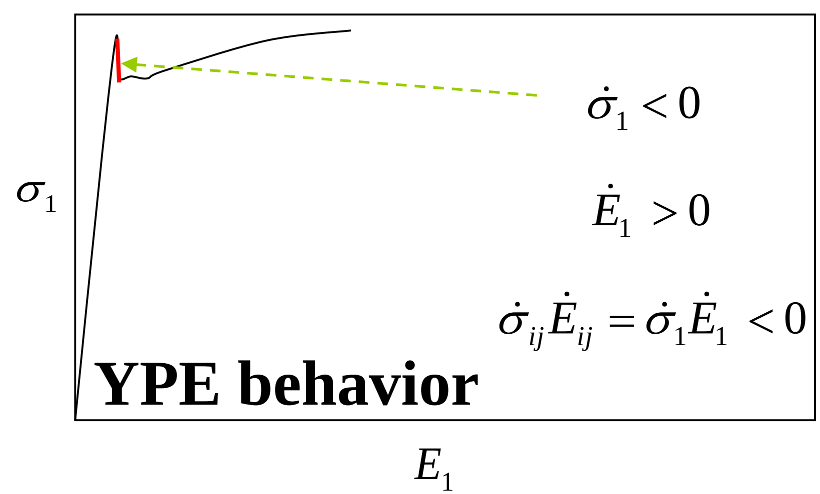
Model Constraints
$$\frac{d\sigma_Y}{d\bar{\varepsilon}_p} > 0$$
$$\frac{d\sigma_Y}{d\bar{\varepsilon}_p} + \frac{\partial\bar{\sigma}_p}{\partial\sigma_{\alpha\beta}} C_{\alpha\beta\eta\rho} \frac{\partial\bar{\sigma}_y}{\partial\sigma_{\eta\rho}} > 0$$
$$\left(\frac{\partial\bar{\sigma}_y}{\partial\sigma_{ij}}\sigma_{ij} > 0 \qquad \frac{\partial\bar{\sigma}_p}{\partial\sigma_{ij}}\sigma_{ij} > 0 \right)$$

AFR (for all loadings)

$$\dot{\sigma}^{ij}\dot{E}_{ij}^{(p)} = \frac{1}{\frac{d\bar{\sigma}_Y}{d\bar{\varepsilon}_p}} \left(\dot{\sigma}_{ij} \frac{\partial\bar{\sigma}_y}{\partial\sigma_{ij}} \right)^2 > 0$$

Non-AFR (for proportional loading)

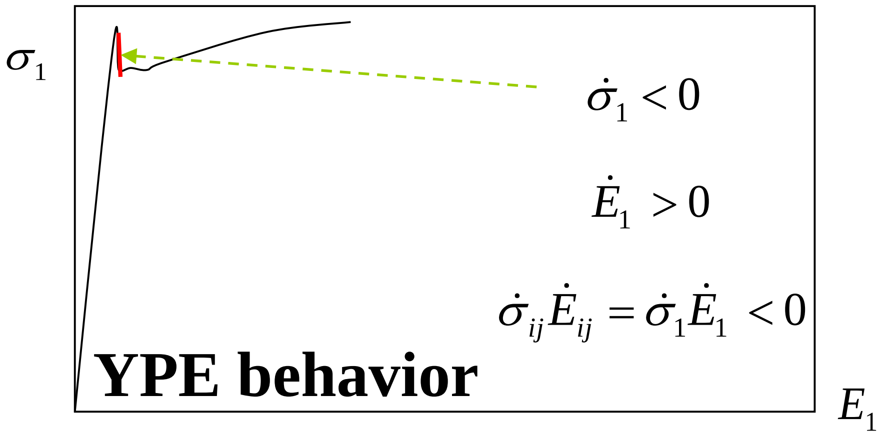
$$\dot{\sigma}^{ij}\dot{E}_{ij}^{(p)} = \frac{1}{\frac{d\bar{\sigma}_Y}{d\bar{\varepsilon}_p}} \left(\frac{\dot{\sigma}_s}{\sigma_s} \right)^2 \left(\frac{\partial\bar{\sigma}_y}{\partial\sigma_{ij}}\sigma_{ij} \right) \left(\frac{\partial\bar{\sigma}_p}{\partial\sigma_{ij}}\sigma_{ij} \right) > 0$$



Yield Point Elongation

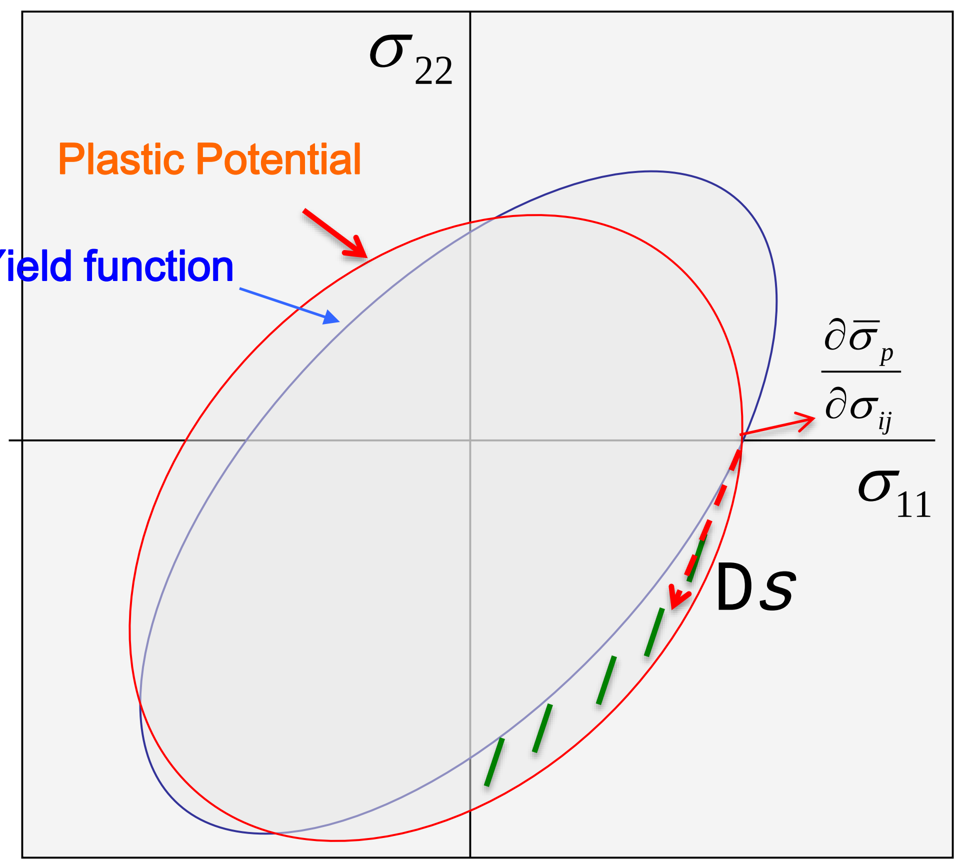
2nd order plastic work rate :

$$\dot{\sigma}^{ij}\dot{E}_{ij}^{(p)} = \frac{1}{\frac{d\bar{\sigma}_Y}{d\bar{\varepsilon}_p}}\left(\frac{\partial\bar{\sigma}_y}{\partial\sigma_{ij}}\dot{\sigma}_{ij}\right)\left(\frac{\partial\bar{\sigma}_p}{\partial\sigma_{ij}}\dot{\sigma}_{ij}\right)$$



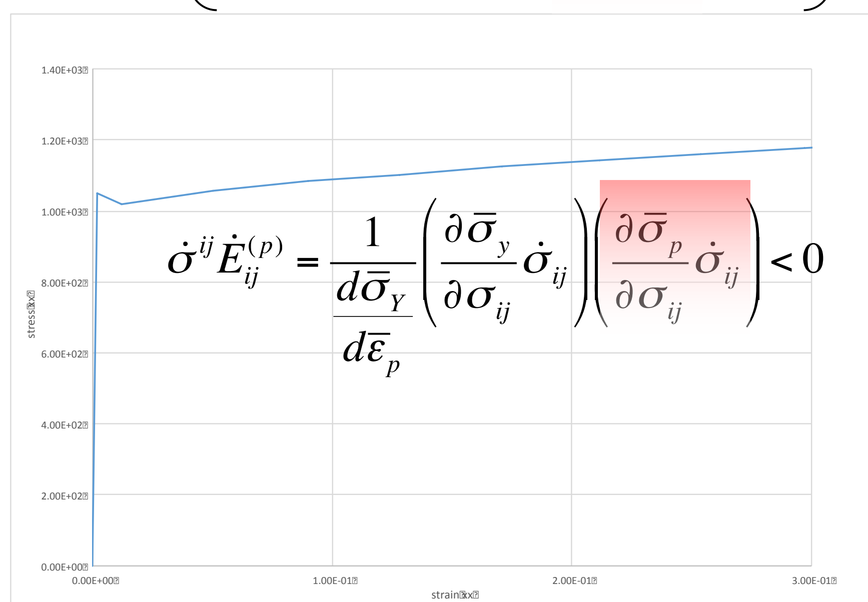
(Associated Flow Rule)

$$\dot{\sigma}^{ij}\dot{E}_{ij}^{(p)} = \frac{1}{\frac{d\bar{\sigma}_Y}{d\bar{\varepsilon}_p}}\left(\dot{\sigma}_{ij}\frac{\partial\bar{\sigma}_y}{\partial\sigma_{ij}}\right)^2 > 0$$



(Non-Associated Flow Rule)

$$\left(\frac{\partial\bar{\sigma}_y}{\partial\sigma_{ij}}\dot{\sigma}_{ij} > 0 \qquad \frac{\partial\bar{\sigma}_p}{\partial\sigma_{ij}}\dot{\sigma}_{ij} < 0\right)$$



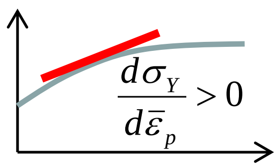
Stability Conditions for Non-AFR

(Stoughton & Yoon, 2006; IJP)

Model Constraints

$$\frac{d\sigma_Y}{d\bar{\varepsilon}_p} > 0$$

This avoids a singularity for the Stress Rate BC



$$\dot{\sigma}_{kl} = \left(C^{klij} - \frac{H_{ep}}{\frac{d\sigma_Y}{d\bar{\varepsilon}_p} + \frac{\partial\bar{\sigma}_p}{\partial\sigma_{\alpha\beta}} C_{\alpha\beta\eta\rho} \frac{\partial\bar{\sigma}_y}{\partial\sigma_{\eta\rho}}} C_{klab} \frac{\partial\bar{\sigma}_p}{\partial\sigma_{ab}} \frac{\partial\bar{\sigma}_y}{\partial\sigma_{cd}} C_{cdij} \right) \dot{E}_{ij}$$

$$\frac{d\sigma_Y}{d\bar{\varepsilon}_p} + \frac{\partial\bar{\sigma}_p}{\partial\sigma_{\alpha\beta}} C_{\alpha\beta\eta\rho} \frac{\partial\bar{\sigma}_y}{\partial\sigma_{\eta\rho}} > 0$$

This avoids a singularity and ensures that the effective plastic strain rate is positive

$$\dot{\bar{\varepsilon}}_p = \left(\frac{H_{ep}}{\frac{d\sigma_Y}{d\bar{\varepsilon}_p} + \frac{\partial\bar{\sigma}_p}{\partial\sigma_{\alpha\beta}} C_{\alpha\beta\eta\rho} \frac{\partial\bar{\sigma}_y}{\partial\sigma_{\eta\rho}}} \right) \left(\frac{\partial\bar{\sigma}_y}{\partial\sigma_{ij}} C_{ijkl} \dot{E}_{kl} \right) > 0$$

$$\frac{\partial\bar{\sigma}_p}{\partial\sigma_{ij}} \sigma_{ij} > 0$$

This ensures that the plastic work rate is positive

$$\dot{w}^{(p)} = \frac{\partial\bar{\sigma}_p}{\partial\sigma_{ij}} \sigma_{ij} \dot{\bar{\varepsilon}}_p > 0$$

$$\frac{\partial\bar{\sigma}_y}{\partial\sigma_{ij}} \sigma_{ij} > 0$$

This ensures that the second order plastic work rate is positive for all proportional loading

$$\dot{\sigma}^{ij} \dot{E}^{(p)}_{ij} = \frac{1}{\frac{d\bar{\sigma}_Y}{d\bar{\varepsilon}_p}} \left(\frac{\partial\bar{\sigma}_y}{\partial\sigma_{ij}} \dot{\sigma}_{ij} \right) \left(\frac{\partial\bar{\sigma}_p}{\partial\sigma_{ij}} \dot{\sigma}_{ij} \right) > 0$$

Notes on Constraints

Strain Rate BC for a non-hardening metal

For a given $\sigma_{ij} \quad \bar{\varepsilon}_p \quad \dot{E}_{ij}$

if $\bar{\sigma}_y < \sigma_Y$

or $\bar{\sigma}_y = \sigma_Y$ **and** $\frac{\partial \bar{\sigma}_y}{\partial \sigma_{ij}} C^{ijkl} \dot{E}_{kl} \leq 0$ **then** $H_{ep} = 0$

otherwise $H_{ep} = 1$

These two constraints are automatically satisfied if the functions are convex.

These constraints show that convexity is only sufficient for stability, i.e., convexity is not necessary under Non-AFR.

Model Constraints

$$\frac{d\sigma_Y}{d\bar{\varepsilon}_p} > 0$$

$$\frac{d\sigma_Y}{d\bar{\varepsilon}_p} + \frac{\partial \bar{\sigma}_p}{\partial \sigma_{\alpha\beta}} C_{\alpha\beta\eta\rho} \frac{\partial \bar{\sigma}_y}{\partial \sigma_{\eta\rho}} > 0$$

$$\frac{\partial \bar{\sigma}_p}{\partial \sigma_{ij}} \sigma_{ij} > 0 \qquad \frac{\partial \bar{\sigma}_y}{\partial \sigma_{ij}} \sigma_{ij} > 0$$

$$\dot{E}_{ij}^{(e)} = C_{ijkl}^{-1} \dot{\sigma}_{kl} \qquad \dot{E}_{ij}^{(p)} = \dot{\bar{\varepsilon}}_p \frac{\partial \bar{\sigma}_p}{\partial \sigma_{ij}}$$

$$\dot{w}^{(p)} = \frac{\partial \bar{\sigma}_p}{\partial \sigma_{ij}} \sigma_{ij} \dot{\bar{\varepsilon}}_p > 0$$

$$\dot{\sigma}^{ij} \dot{E}_{ij}^{(p)} = \left(\dot{\sigma}_{ij} \frac{\partial \bar{\sigma}_p}{\partial \sigma_{ij}} \right) \dot{\bar{\varepsilon}}_p$$

$$\dot{\bar{\varepsilon}}_p = \left(\frac{H_{ep}}{\frac{d\sigma_Y}{d\bar{\varepsilon}_p} + \frac{\partial \bar{\sigma}_p}{\partial \sigma_{\alpha\beta}} C_{\alpha\beta\eta\rho} \frac{\partial \bar{\sigma}_y}{\partial \sigma_{\eta\rho}}} \right) \left(\frac{\partial \bar{\sigma}_y}{\partial \sigma_{ij}} C_{ijkl} \dot{E}_{kl} \right) > 0$$

Is it possible to combine Hill and Barlat Models under Non-Associated Flow ?

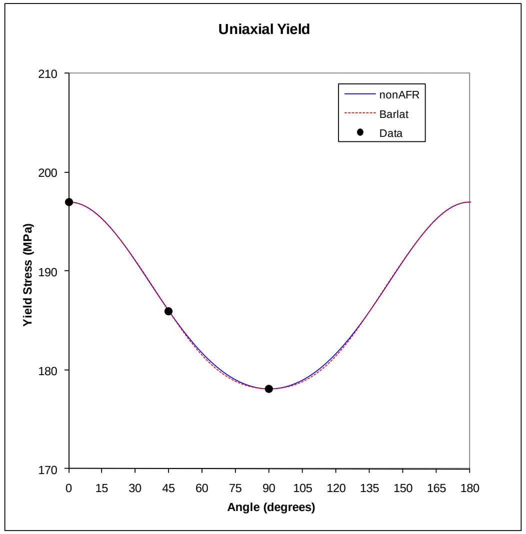
Hill's Model

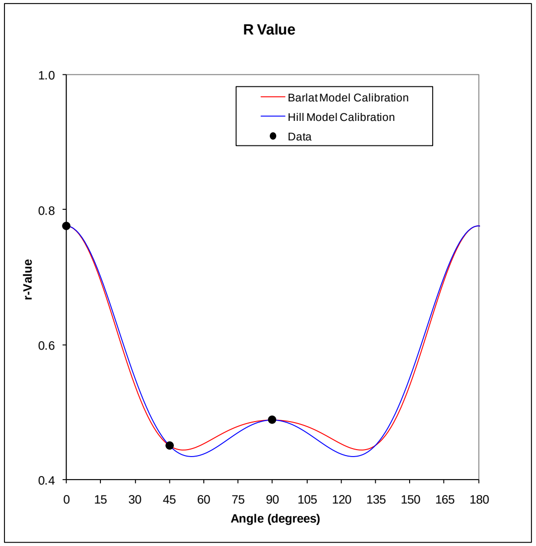
$$\bar{\sigma}^2 = F_p(\sigma_{22}-\sigma_{33})^2 + G_p(\sigma_{33}-\sigma_{11})^2 + H_p(\sigma_{11}-\sigma_{22})^2 + 2L_p\sigma_{23}^2 + 2M_p\sigma_{31}^2 + 2N_p\sigma_{12}^2$$

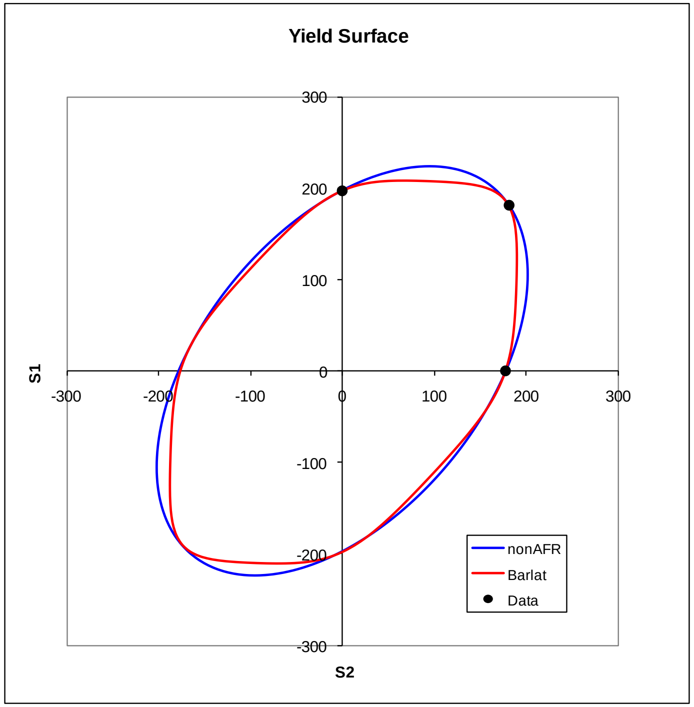
$$+ \left(\dot{E}_{ij}^{(p)} = \dot{\bar{\varepsilon}}_p \frac{\partial \bar{\sigma}_p}{\partial \sigma_{ij}} \right)$$

Barlat's Model

$$2\bar{\sigma}^a = \left|\tilde{S}_1 - \tilde{S}_2\right|^a + \left|\tilde{S}_2 - \tilde{S}_3\right|^a + \left|\tilde{S}_3 - \tilde{S}_1\right|^a$$





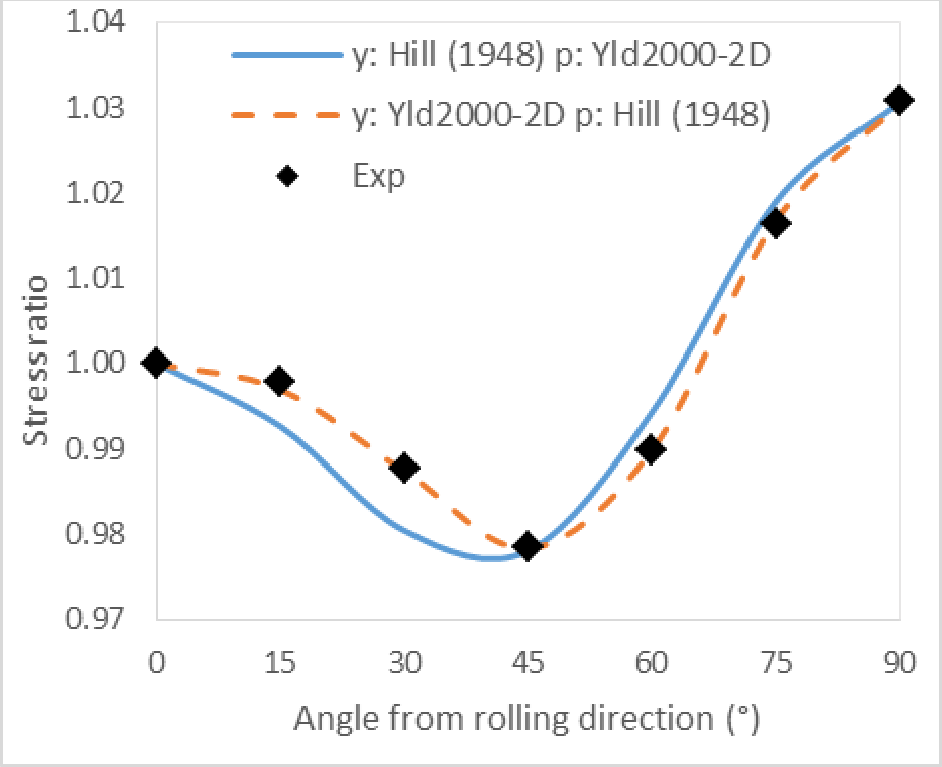


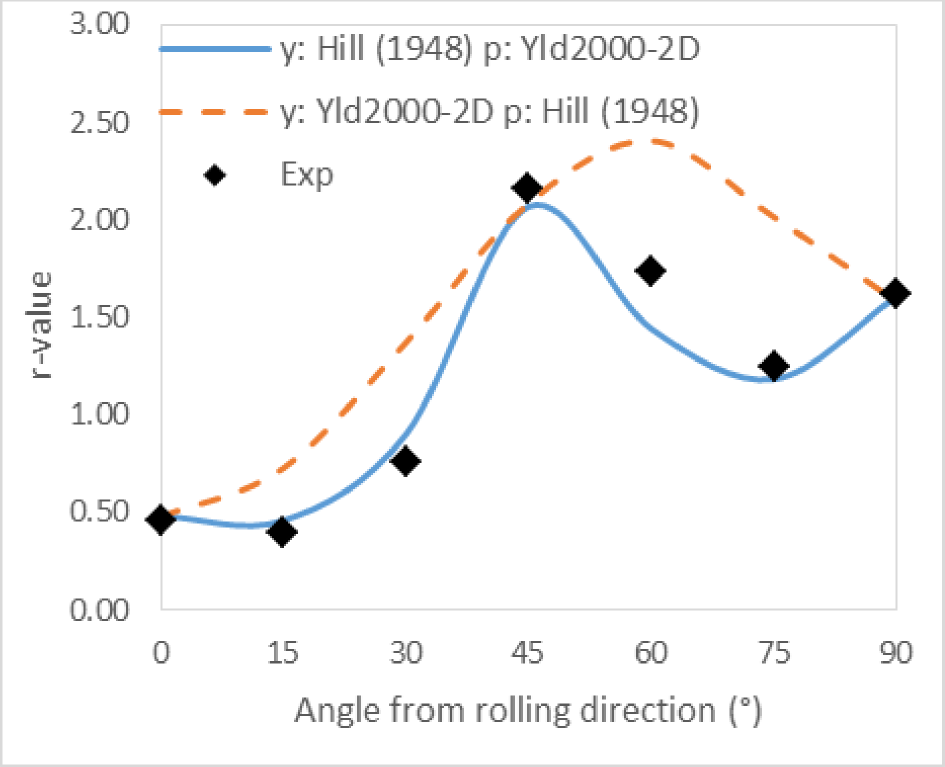
Stress and r-value directionalities from non-AFR

(Hill and Barlat Combination)

Case-1 : Barlat Yld2000 (Yld) + Barlat Yld2000 (Potential)
Case-2 : Hill (Yld) + Barlat Yld2000 (Potential)

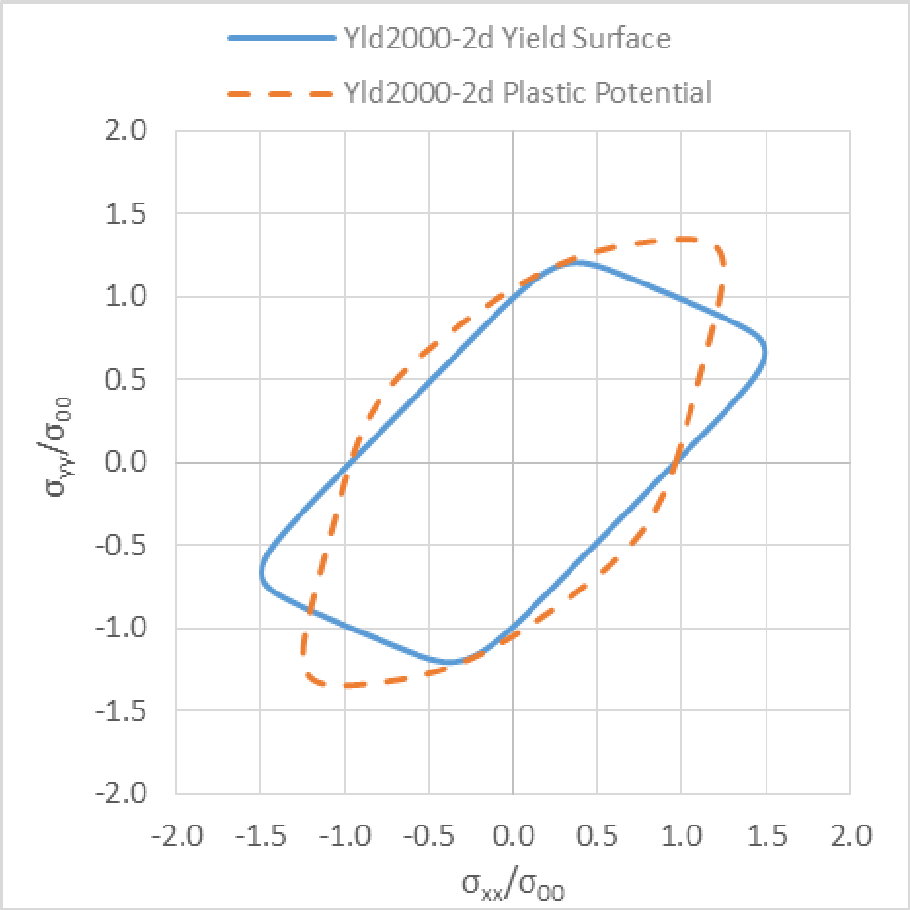
(AA 5019)



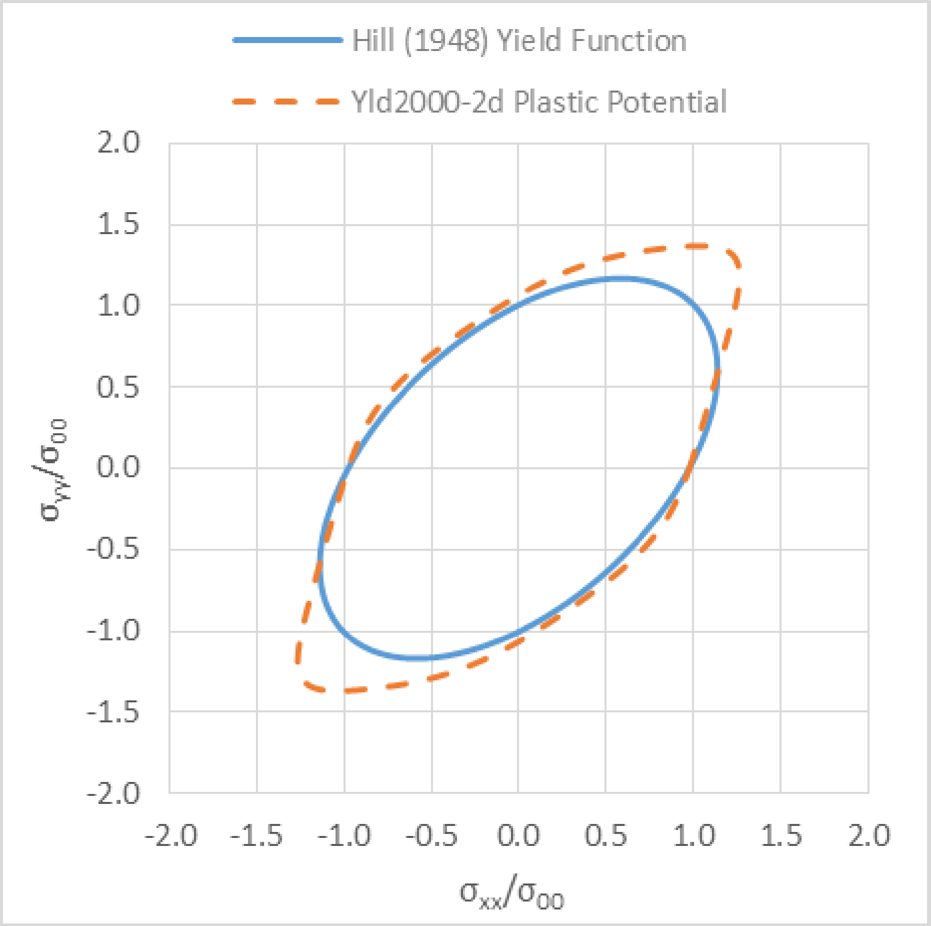


Yield function and Potential used for non-AFR (AA 5019)

Duplication of Yld2004 :
Yld2000 (Yld) + Yld2000 (Potential)

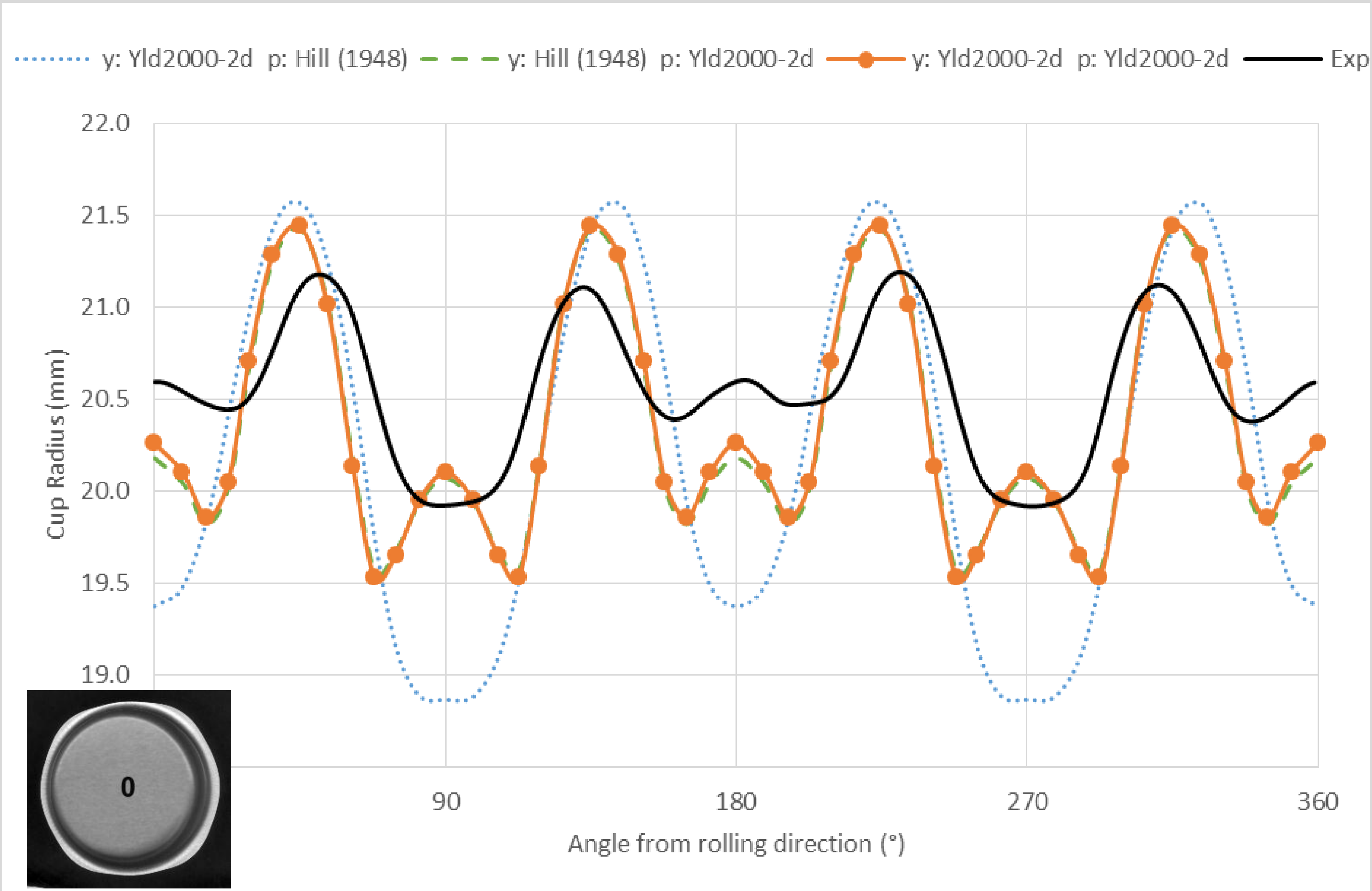


Hill and Barlat Combination :
Hill (Yld) +Yld2000 (Potential)



Earing Profile with a Combination of Hill and Barlat functions

(Paulino & Yoon, 2015; SRI)

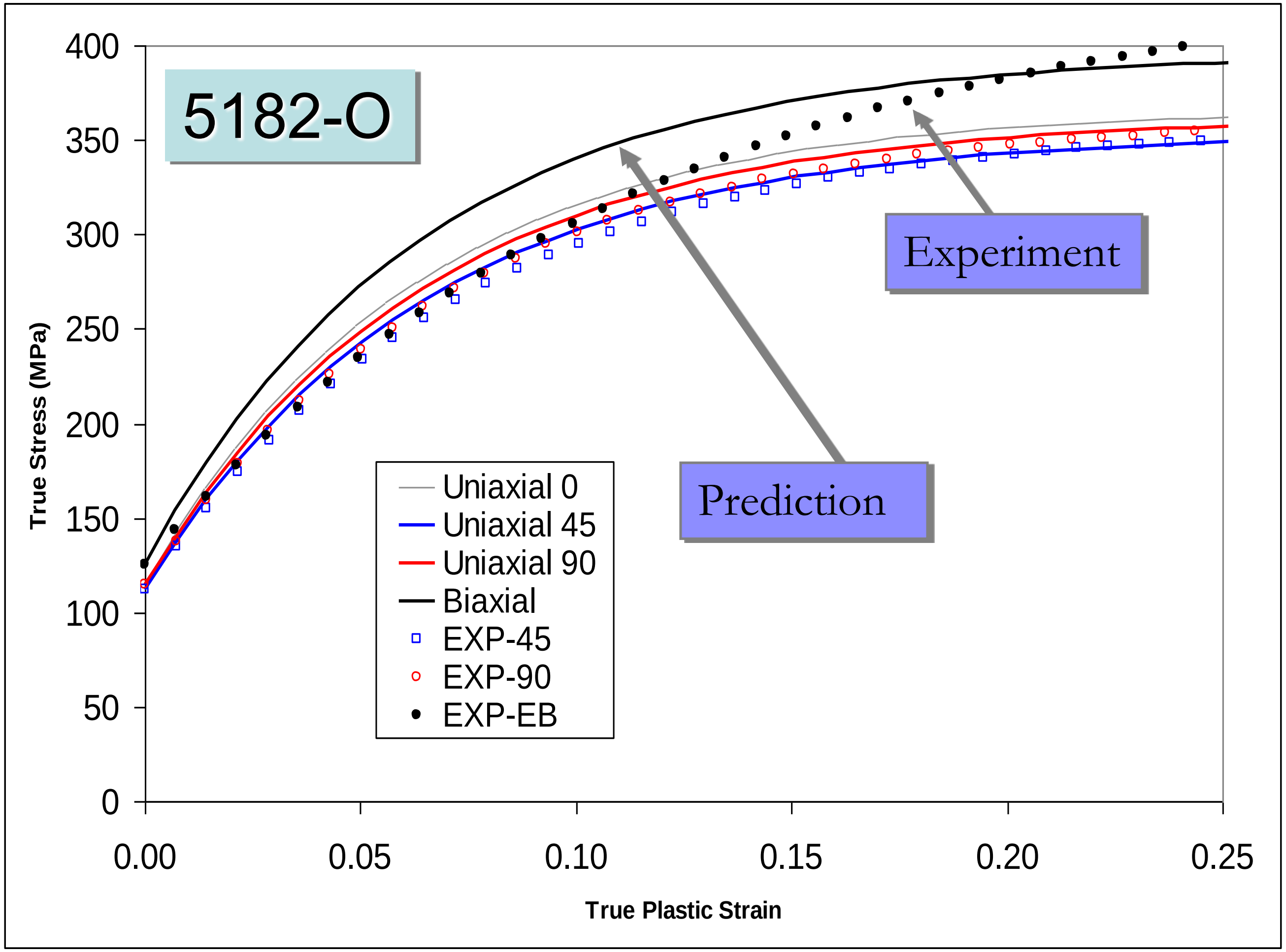


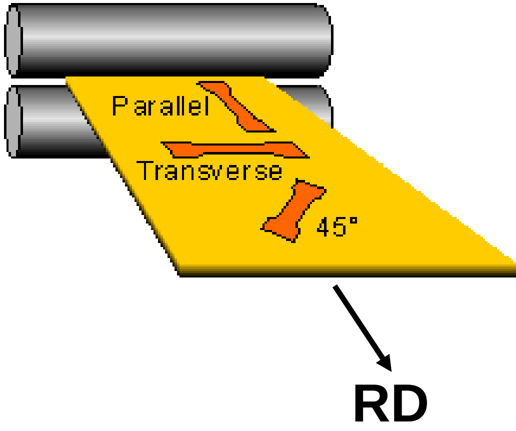
AFR : Yld2004 **CPU time : (1)**

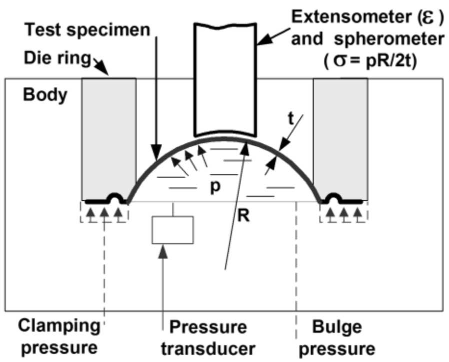
= Non-AFR : Yld2000 (Yld) + Yld2000 (Potential) = **CPU time : (0.5)**

Hill (Yld) + Yld2000 (Potential) **CPU time : (0.30)**

Anisotropic hardening







The Hardening Law is Critical for Springback

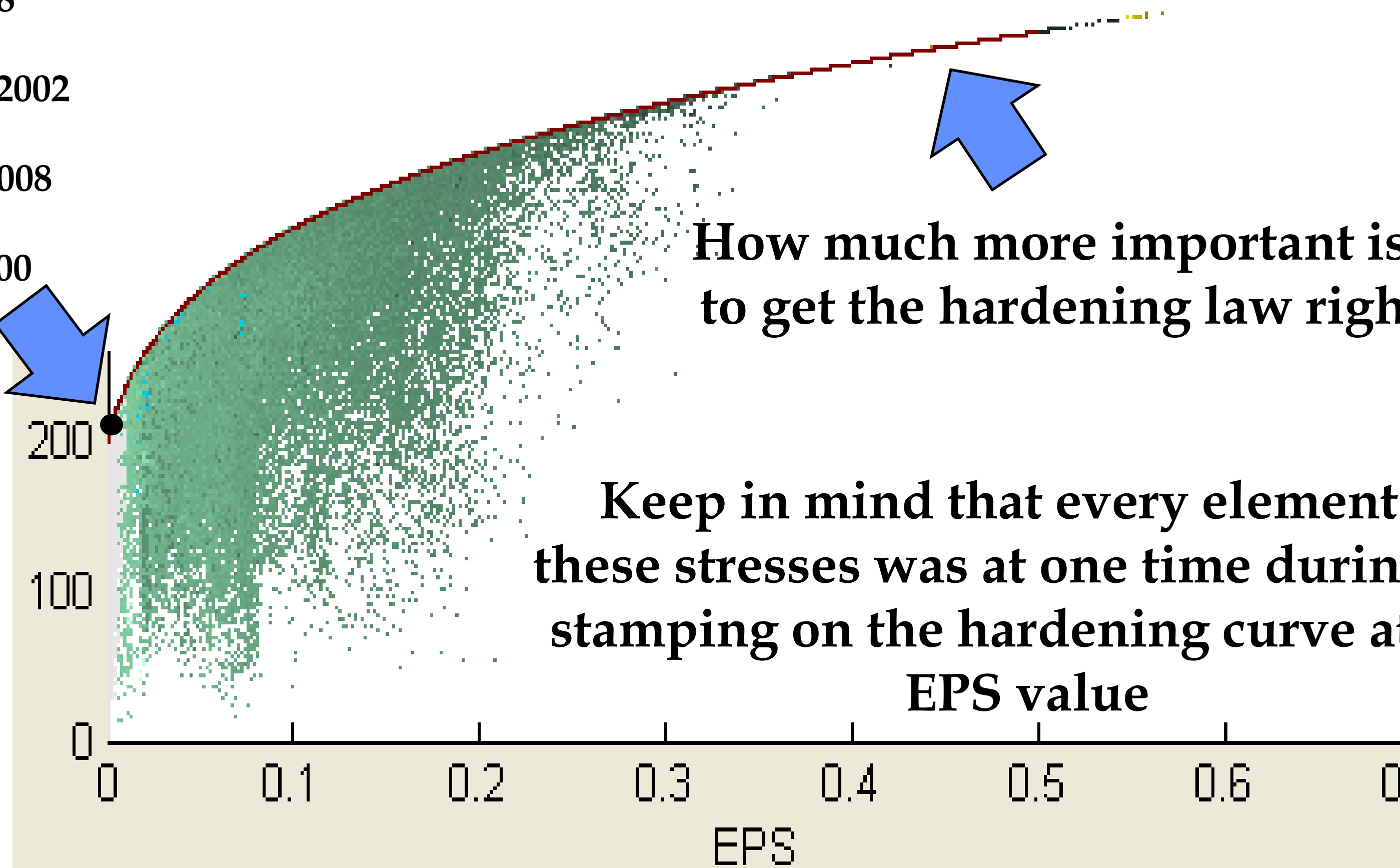
Von Mises

Hill 1948

Stoughton 2002

Cvitanic 2008

Barlat 2000



How much more important is it to get the hardening law right?

Keep in mind that every element at these stresses was at one time during the stamping on the hardening curve at the EPS value

Anisotropic Hardening Model Based On Non-Associated Flow

(Stoughton & Yoon [2009], Int. J. Plasticity)

Yield Function (Stress Anisotropy)

$$\hat{f}\left(\vec{\sigma},\bar{\varepsilon}_p\right)^2 = \frac{\sigma_{11}\left(\sigma_{11}-\sigma_{22}\right)}{\sigma_0^2\left(\bar{\varepsilon}_p\right)} + \frac{\sigma_{22}\left(\sigma_{22}-\sigma_{11}\right)}{\sigma_{90}^2\left(\bar{\varepsilon}_p\right)} + \frac{\sigma_{11}\sigma_{22}-\sigma_{12}^2}{\sigma_{EB}^2\left(\bar{\varepsilon}_p\right)} + \left(\frac{2\sigma_{12}}{\sigma_{45}\left(\bar{\varepsilon}_p\right)}\right)^2$$

Plastic Potential (Strain Anisotropy)

$$g(\vec{\sigma}) = \left(1+\frac{1}{r_0}\right)\sigma_{11}^2 + \left(1+\frac{1}{r_{90}}\right)\sigma_{22}^2 - 2\sigma_{11}\sigma_{22} + \left(\frac{1}{r_0}+\frac{1}{r_{90}}\right)\left(1+2r_{45}\right)\sigma_{12}^2$$

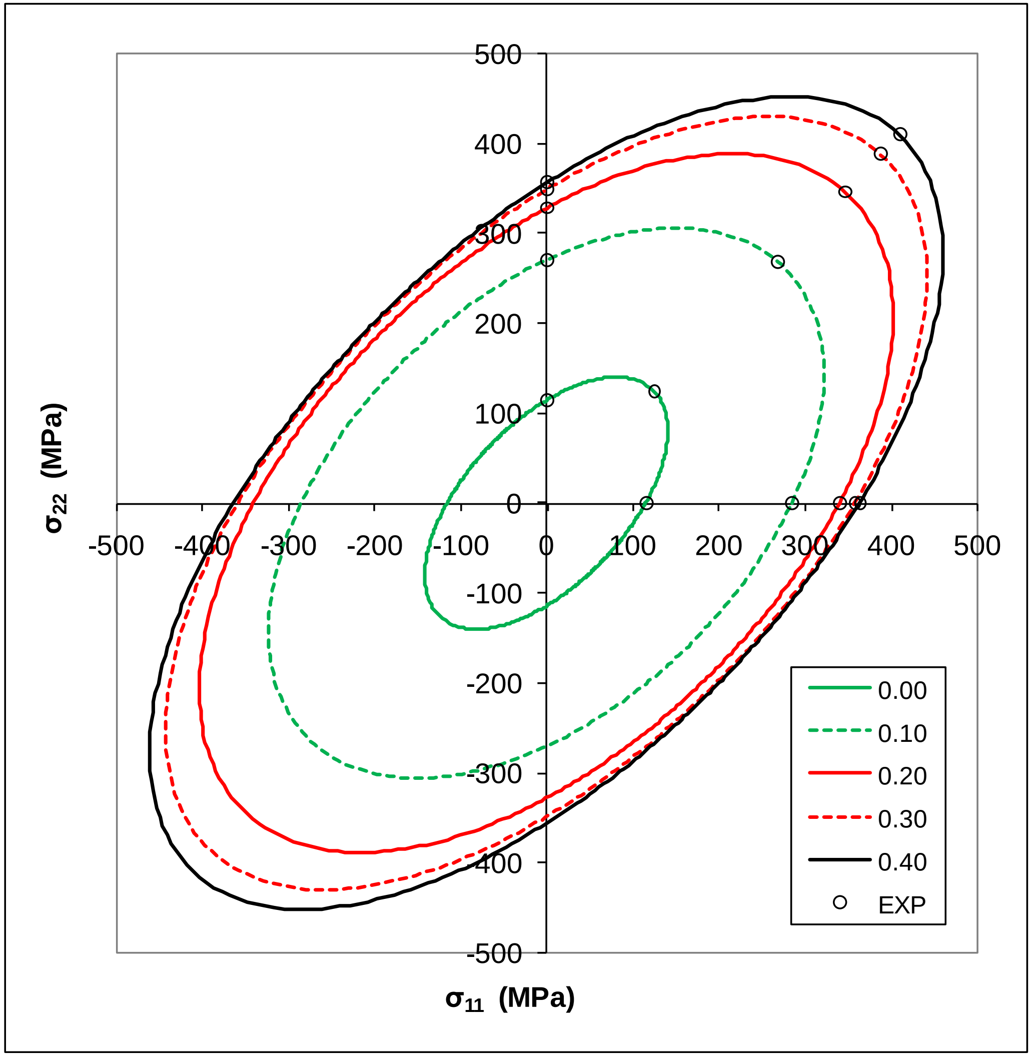
$$\dot{E}_{ij}^{(p)} = \dot{\bar{\varepsilon}}_p \frac{\partial g(\vec{\sigma})}{\partial \sigma^{ij}}$$

Advantages of this Approach

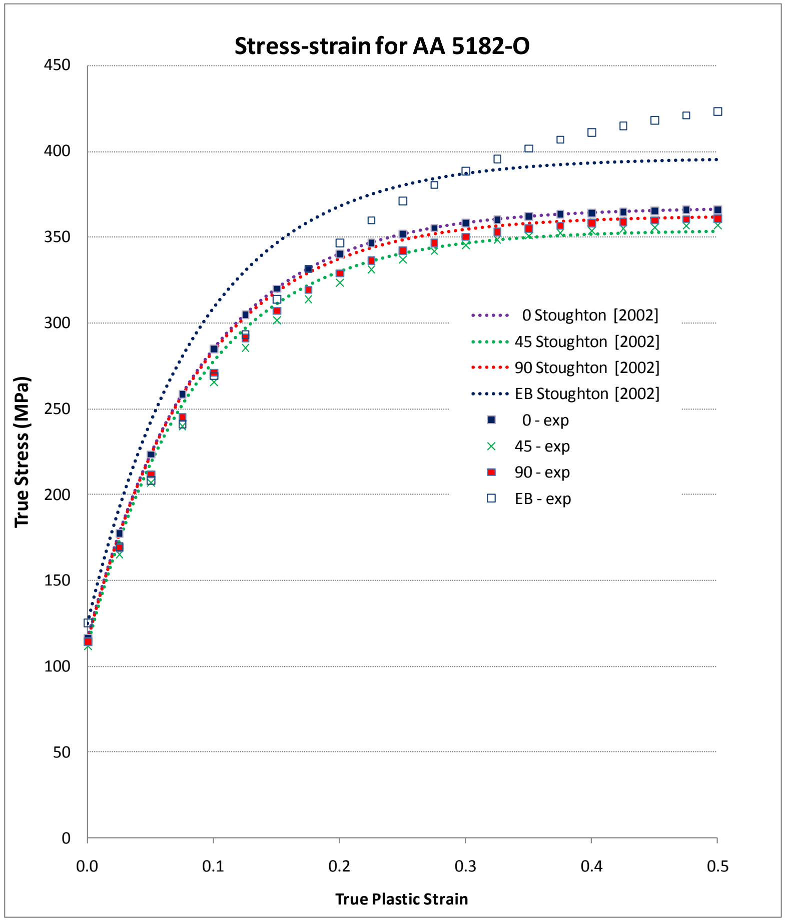
Exact Match of the stress-strain response in uniaxial tension at 0, 45, and 90 degrees

Exact Match of the stress-strain response in equal-biaxial tension

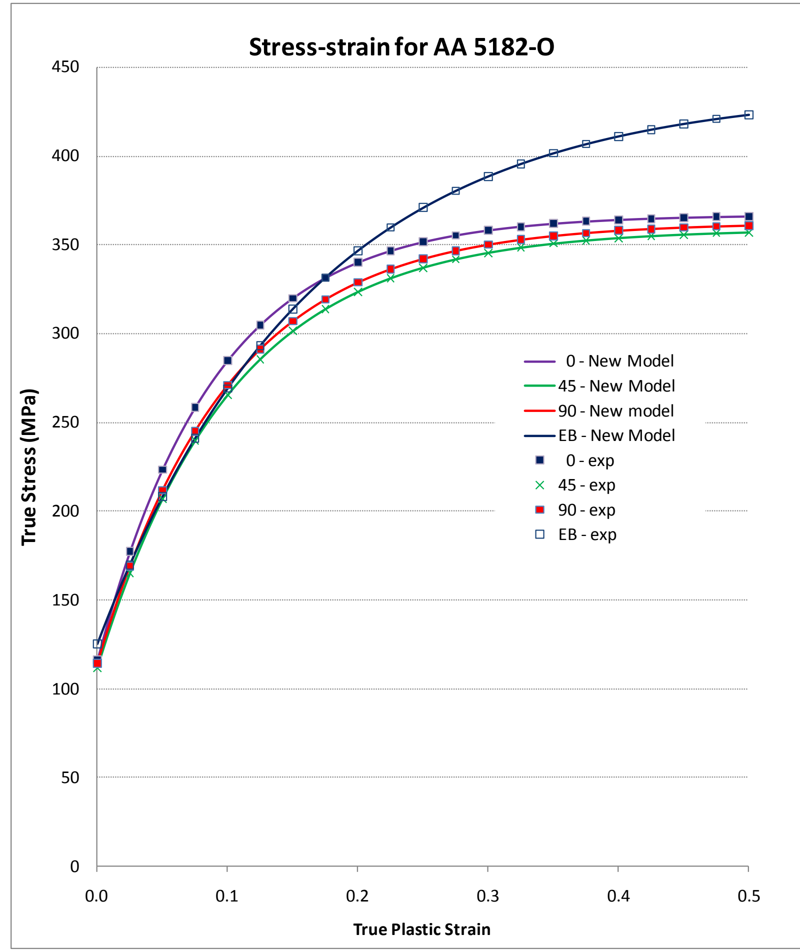
Exact Match to Constant Anisotropic R Values at 0, 45, 90 degrees
(due to non-AFR)



Example: AA 5182-O

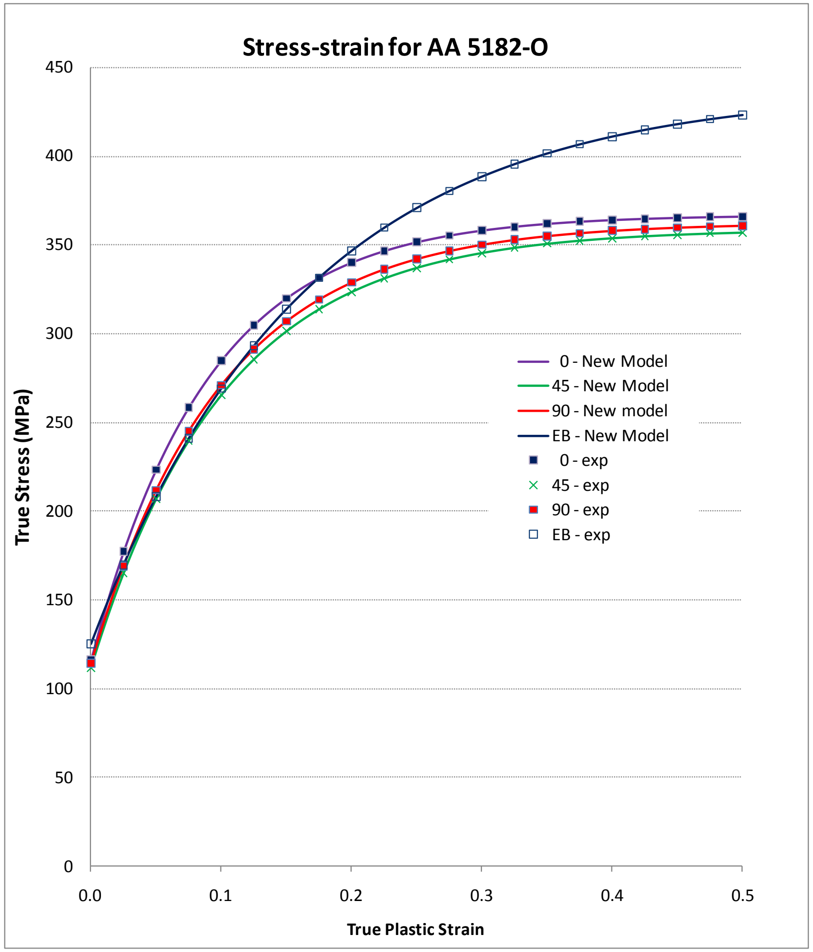


Best that can be done with I.H

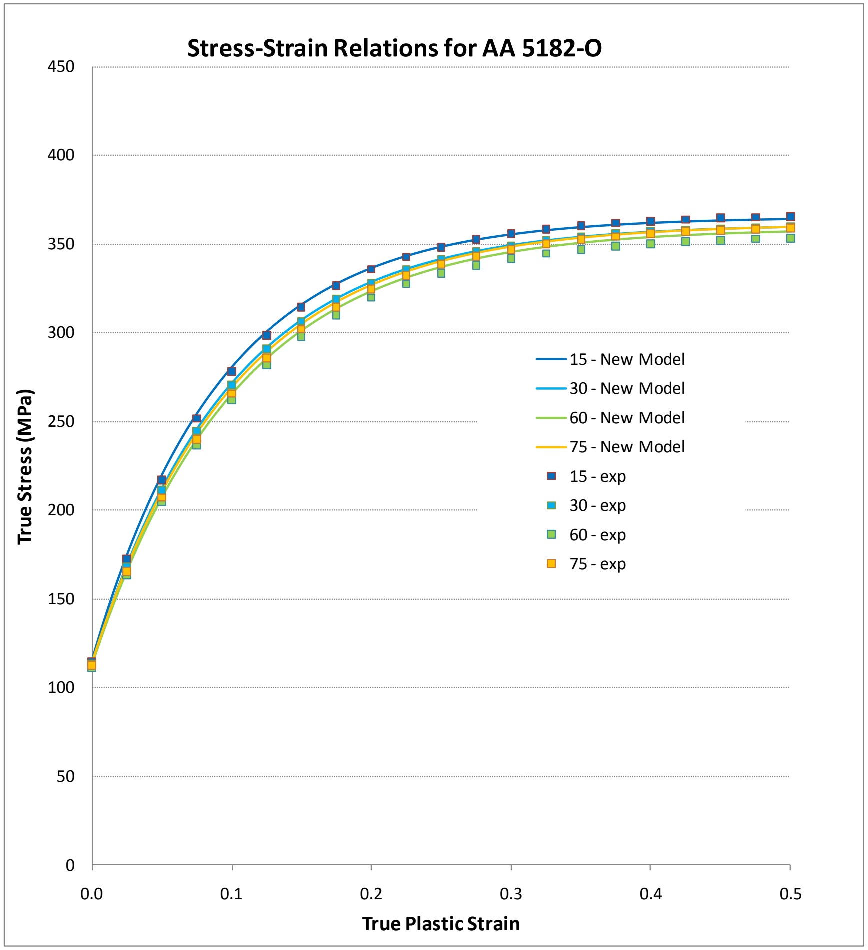


Exact Match with New Model

True Predictions in Uniaxial Tension at 15, 30, 60, and 75 degrees

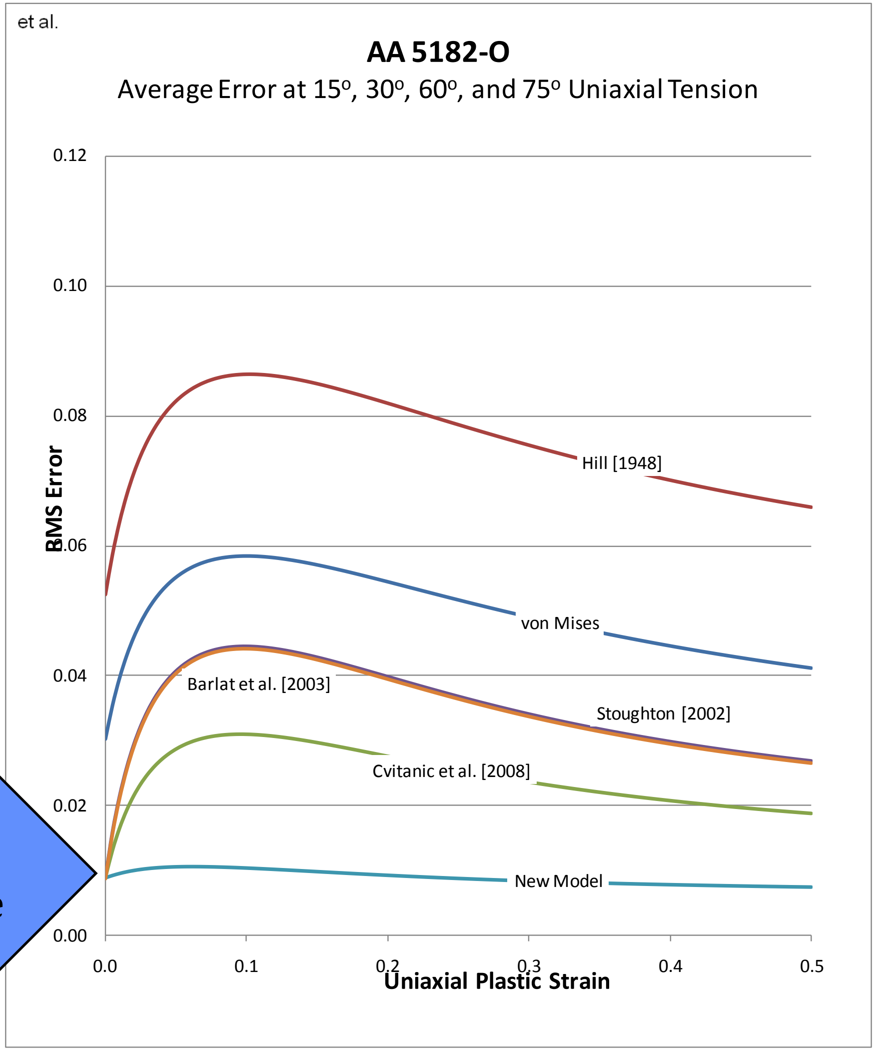


Exact Match but not Predicted



Nearly Exact Match

Comparison of New Model with Selected Models Based on Isotropic Hardening



$\Delta_{predict}(\varepsilon)$

Due to Match of Initial Yield Surface

Interesting points:

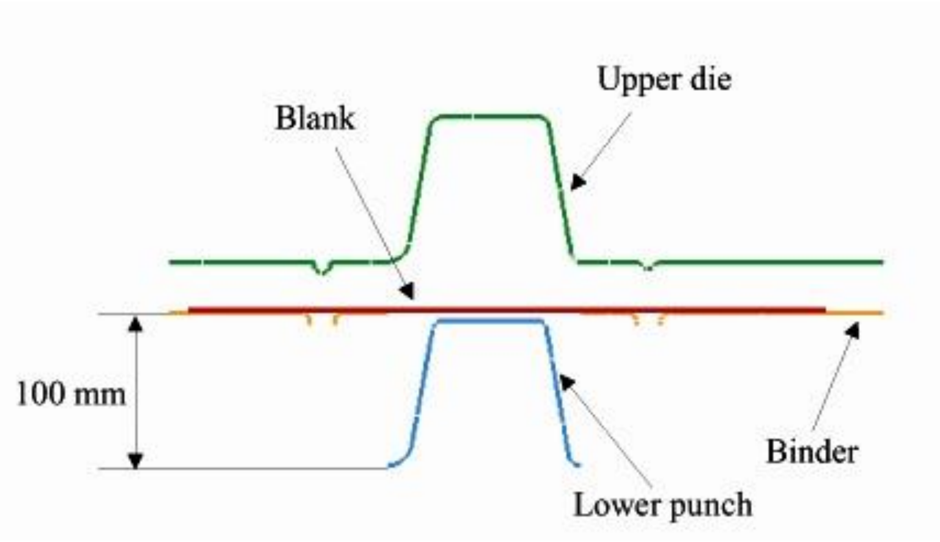
For this alloy in these directions

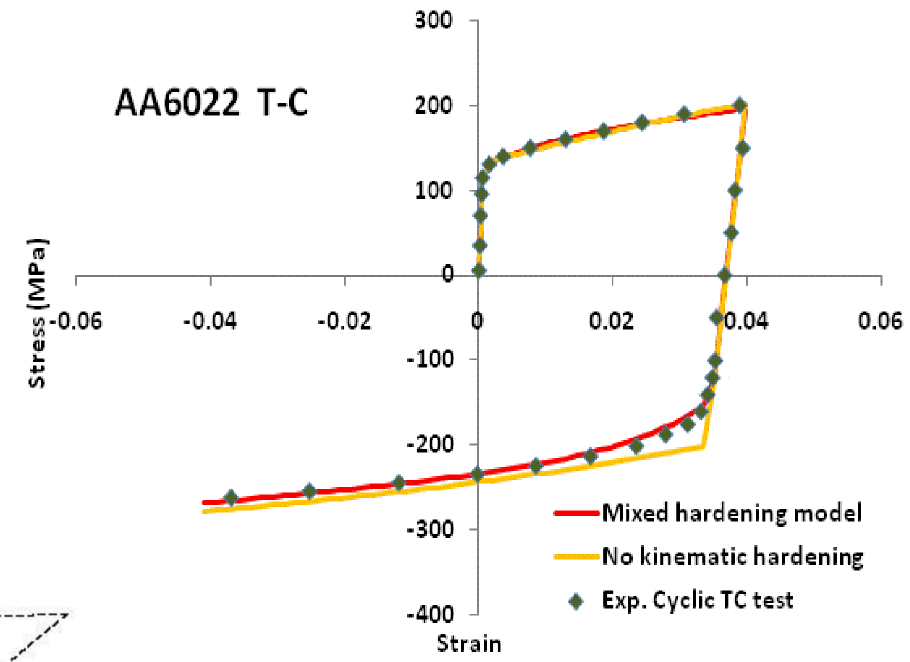
Von Mises is better than Hill

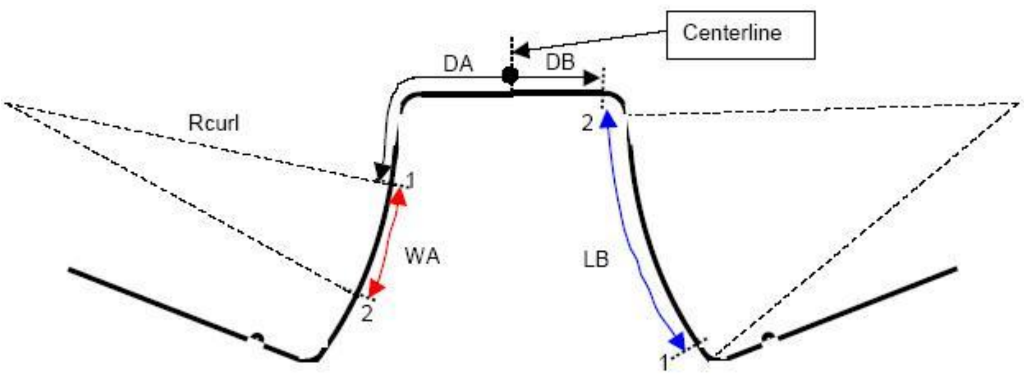
Cvitanic is the best I.H. model

The RMS error of the new model as well as the initial errors of four other models are at the 1% level consistent with uncertainty of the stress measurement.

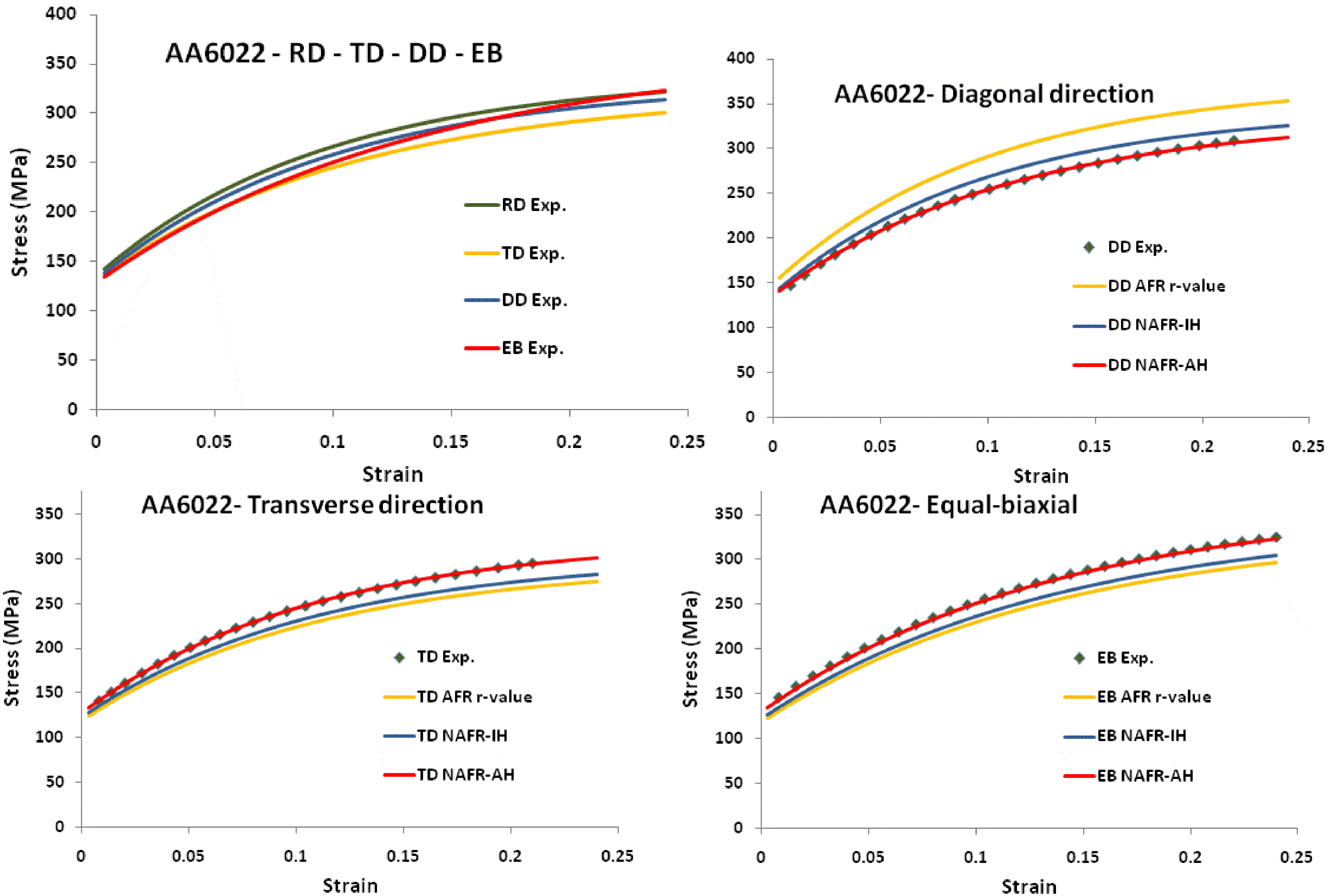
Numisheet2005 Draw-bead Benchmark Spring-Back with Anisotropic Hardening (AA 6022-T4)



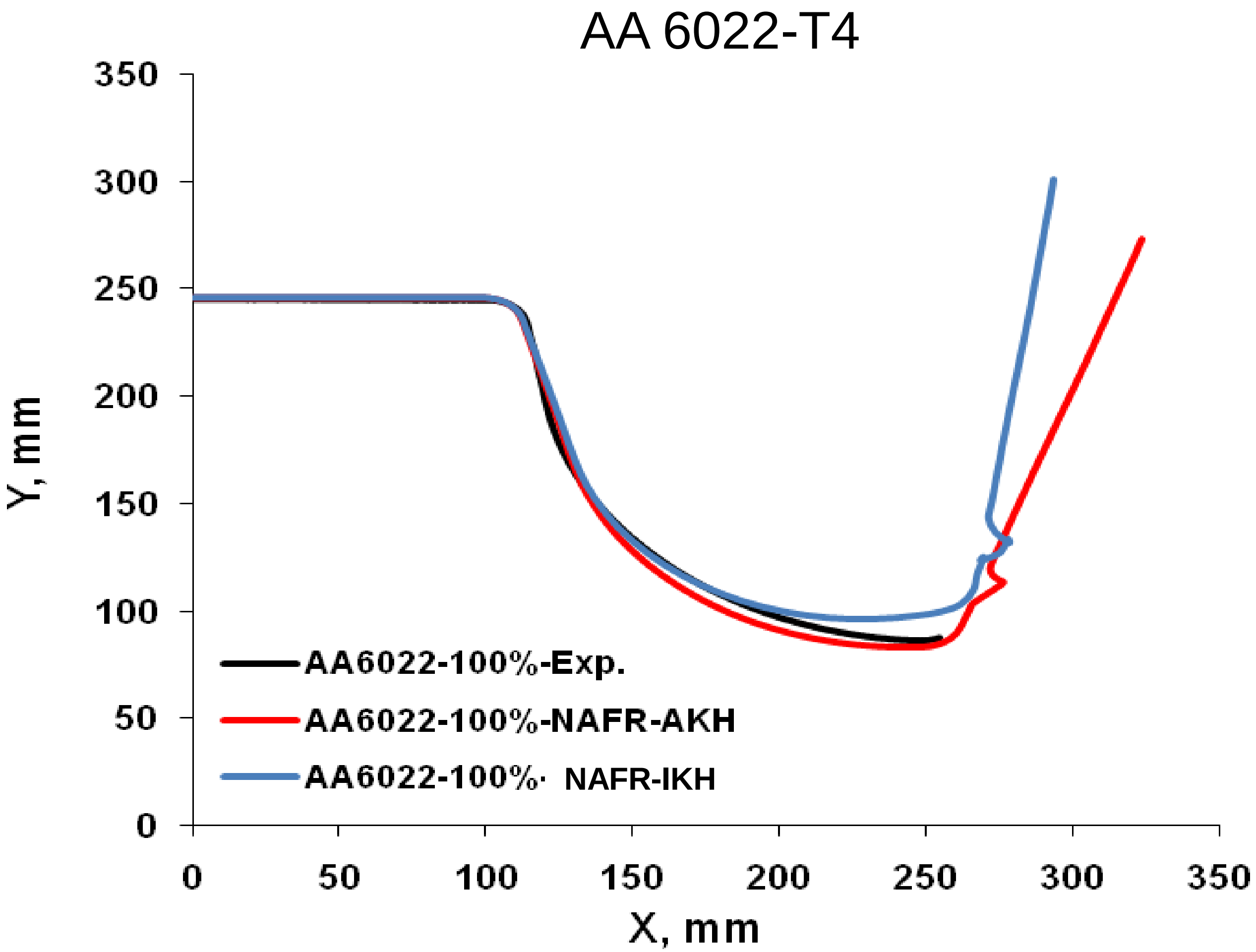




AA 6022-T4 Directional Hardening Behaviours



Spring-back results (Numisheet2005 benchmark)



(AKH-Anisotropic Kinematic Hardening shows a better match than IKH-isotropic kinematic hardening for the curvature prediction)

Coupled Quadratic and Non-quadratic (CQN) Directional Hardening Model (Lee-Stoughton-Yoon, IJP, 2017)

$$f_{Coup}\left(\boldsymbol{\sigma},\bar{\varepsilon}^{P}\right)=\left[\ f_{Quad}\left(\boldsymbol{\sigma},\bar{\varepsilon}^{P}\right)\cdot\ f_{Nonquad}\left(\boldsymbol{\sigma}\right)\right]^{\frac{1}{n+2}}$$

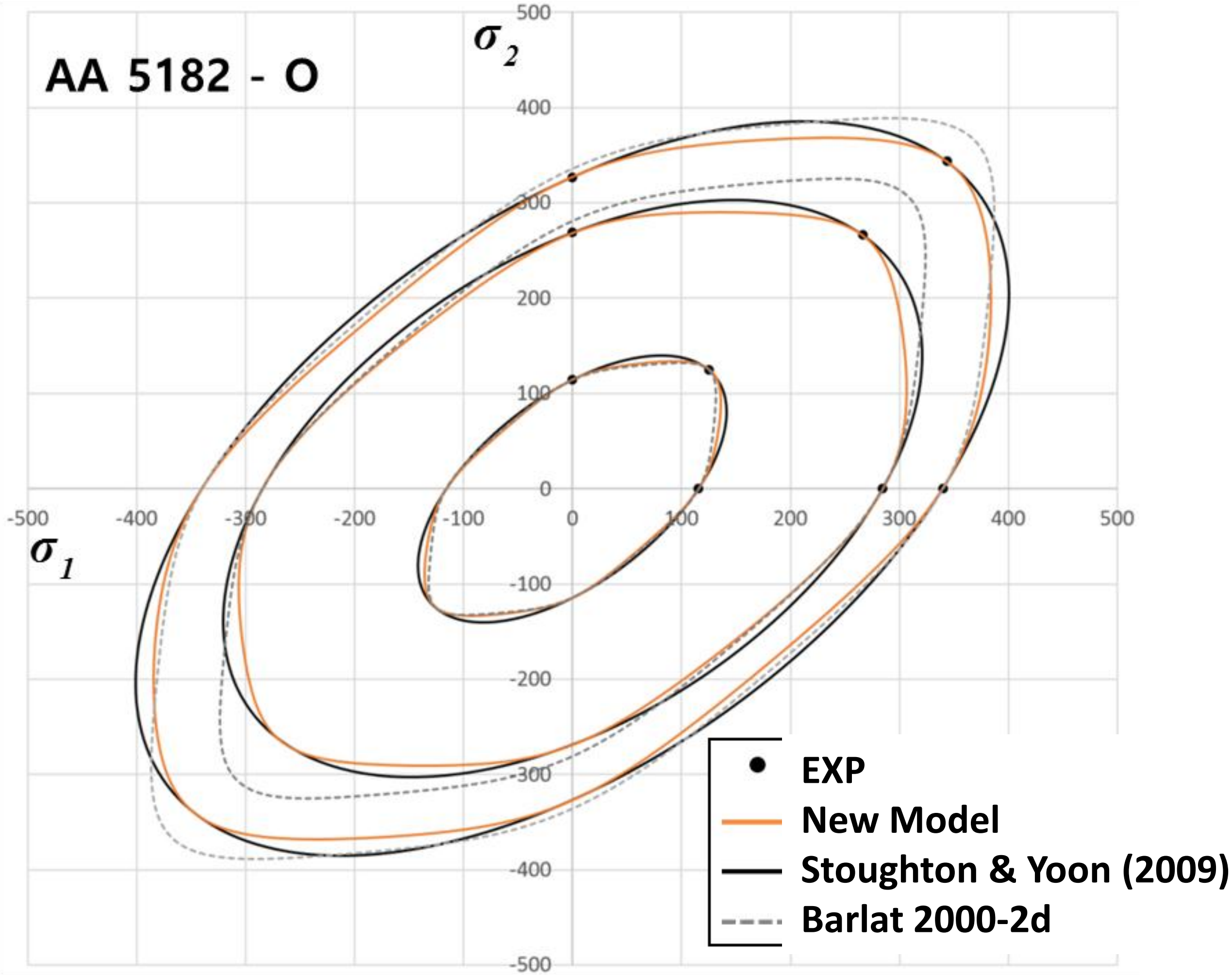
where

$$f_{Quad}\left(\boldsymbol{\sigma},\bar{\varepsilon}^{P}\right)=\left(\sigma_{11}-\frac{\bar{\sigma}_{0}^{n+2}\left(\bar{\varepsilon}^{P}\right)}{\bar{\sigma}_{90}^{n+2}\left(\bar{\varepsilon}^{P}\right)}\sigma_{22}\right)\left(\sigma_{11}-\sigma_{22}\right)+\frac{\bar{\sigma}_{0}^{n+2}\left(\bar{\varepsilon}^{P}\right)}{\bar{\sigma}_{EB}^{n+2}\left(\bar{\varepsilon}^{P}\right)}\left(\sigma_{11}\sigma_{22}-\sigma_{12}\sigma_{12}\right)+\frac{\bar{\sigma}_{0}^{n+2}\left(\bar{\varepsilon}^{P}\right)}{\bar{\sigma}_{45}^{n+2}\left(\bar{\varepsilon}^{P}\right)}4\sigma_{12}\sigma_{12}$$

and

$$f_{Nonquad}\left(\boldsymbol{\sigma}\right)=\frac{1}{2}\left|\sigma_{\mathrm{I}}\right|^{n}+\frac{1}{2}\left|\sigma_{\mathrm{II}}\right|^{n}+\frac{1}{2}\left|\sigma_{\mathrm{I}}-\sigma_{\mathrm{II}}\right|^{n}$$

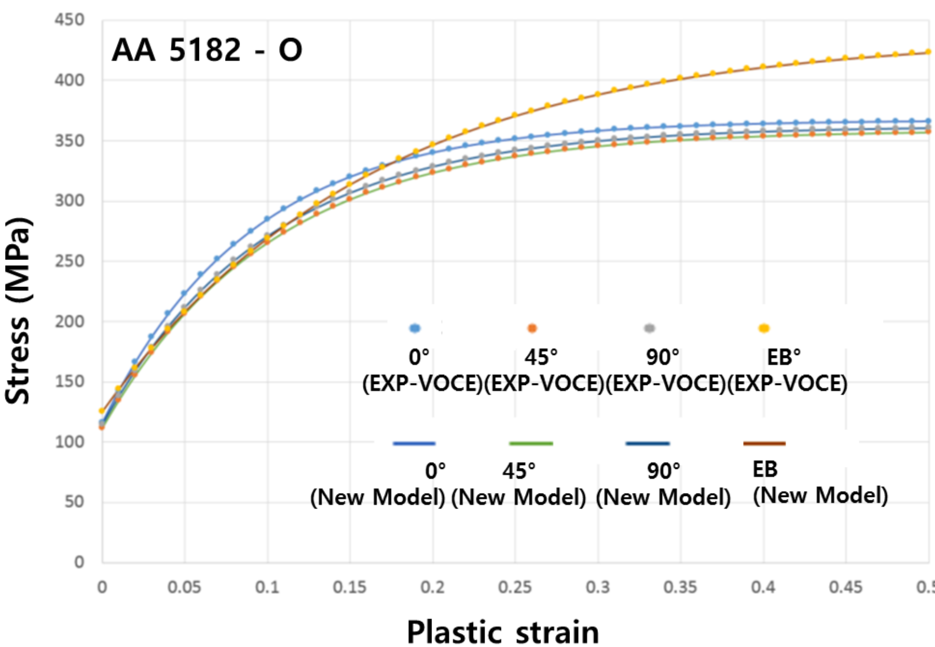
Coupled Quadratic and Non-quadratic (CQN) Directional Hardening Model (Lee-Stoughton-Yoon, IJP, 2017)



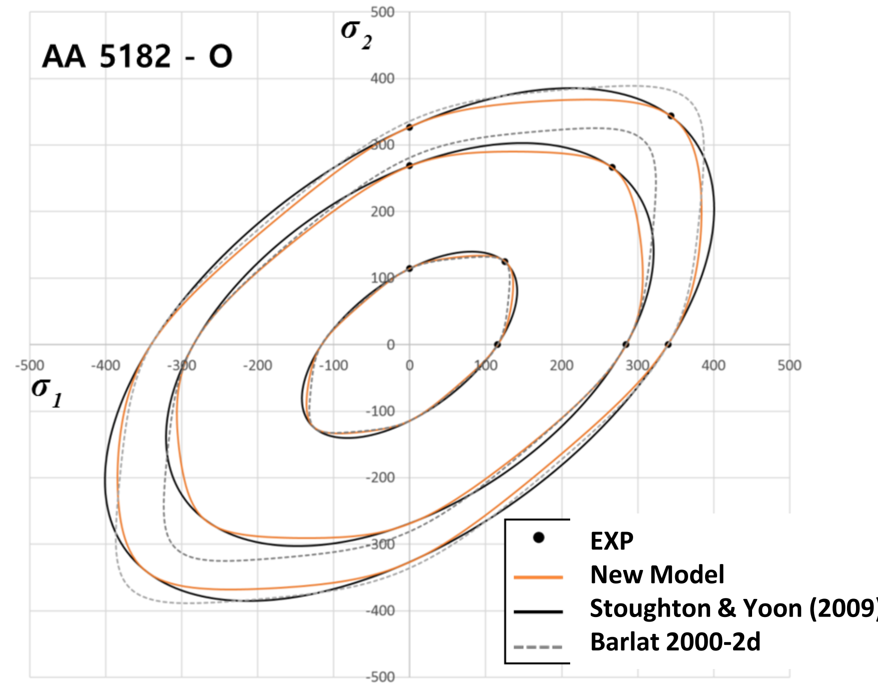
Coupled Quadratic and Non-quadratic (CQN) Directional Hardening Model (Lee-Stoughton-Yoon, IJP, 2017)

$$f_{Coup}\left(\boldsymbol{\sigma},\bar{\varepsilon}^{P}\right)=\left[\,f_{Quad}\left(\boldsymbol{\sigma},\bar{\varepsilon}^{P}\right)\cdot\,f_{Nonquad}\left(\boldsymbol{\sigma}\right)\right]^{\frac{1}{n+2}}$$

Flow stress – New model



Yield surface – New model



Evolution of Anisotropy/Asymmetry-induced Distortional Yielding (Park-Stoughton-Yoon, IJP, 2019)

- Strain
- Strain-rate
- Temperature
- Asymmetry

$$\Phi(\sigma, \bar{\varepsilon}^p, \dot{\bar{\varepsilon}}^p, T) = \underbrace{\left[\frac{\sigma_{11}(\sigma_{11}-\sigma_{22})}{\sigma_0^2(\bar{\varepsilon}^p, \dot{\bar{\varepsilon}}^p, T)} + \frac{\sigma_{22}(\sigma_{22}-\sigma_{11})}{\sigma_{90}^2(\bar{\varepsilon}^p, \dot{\bar{\varepsilon}}^p, T)} + \frac{4\sigma_{12}^2}{\sigma_{45}^2(\bar{\varepsilon}^p, \dot{\bar{\varepsilon}}^p, T)} + \frac{\sigma_{11}\sigma_{22}-\sigma_{12}^2}{\sigma_{EB}^2(\bar{\varepsilon}^p, \dot{\bar{\varepsilon}}^p, T)}\right]^{\frac{1}{2}}}_{\textit{Stoughton and Yoon (2009) Model}} \cdot [X]^k \cdot e^{K(\sigma, \bar{\varepsilon}^p, \dot{\bar{\varepsilon}}^p, T)I(\eta)}$$

Scaling Function

✓ *Control the flatness of a yield surface by the exponent of k*

Asymmetry Function

Evolution of Anisotropy/Asymmetry-induced Distortional Yielding (Park-Stoughton-Yoon, IJP, 2019)

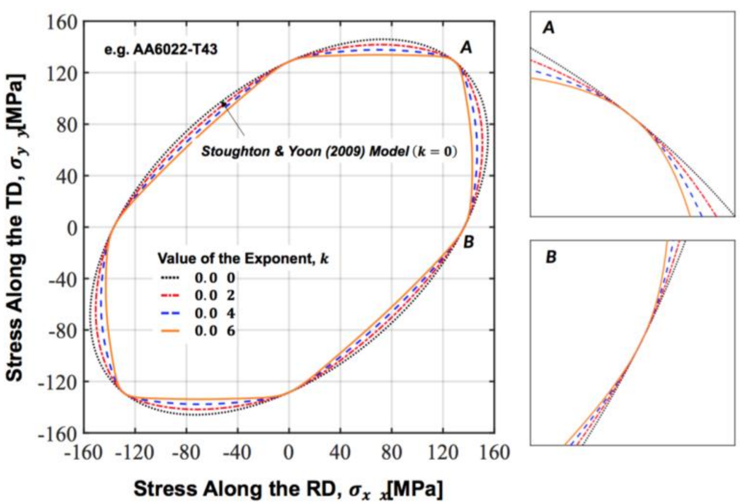
$$\Phi(\sigma, \bar{\varepsilon}^p, \dot{\bar{\varepsilon}}^p, T) = \underbrace{\left[\frac{\sigma_{11}(\sigma_{11}-\sigma_{22})}{\sigma_0^2(\bar{\varepsilon}^p, \dot{\bar{\varepsilon}}^p, T)} + \frac{\sigma_{22}(\sigma_{22}-\sigma_{11})}{\sigma_{90}^2(\bar{\varepsilon}^p, \dot{\bar{\varepsilon}}^p, T)} + \frac{4\sigma_{12}^2}{\sigma_{45}^2(\bar{\varepsilon}^p, \dot{\bar{\varepsilon}}^p, T)} + \frac{\sigma_{11}\sigma_{22}-\sigma_{12}^2}{\sigma_{EB}^2(\bar{\varepsilon}^p, \dot{\bar{\varepsilon}}^p, T)}\right]^{\frac{1}{2}}}_{\text{Stoughton and Yoon (2009) Model}} \cdot [X]^k \cdot e^{K(\sigma, \bar{\varepsilon}^p, \dot{\bar{\varepsilon}}^p, T) I(\eta)}$$

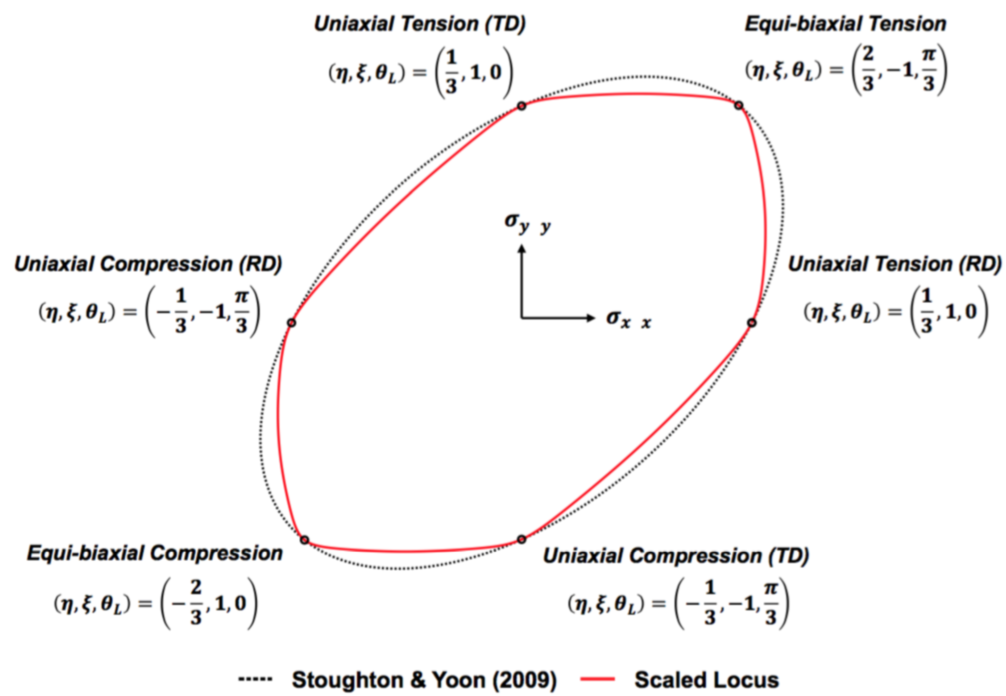
$$X = (C_0 - 1)[D(\theta_L)]^4 - 2(C_0 - 1)[D(\theta_L)]^2 + C_0$$

Scaling Function

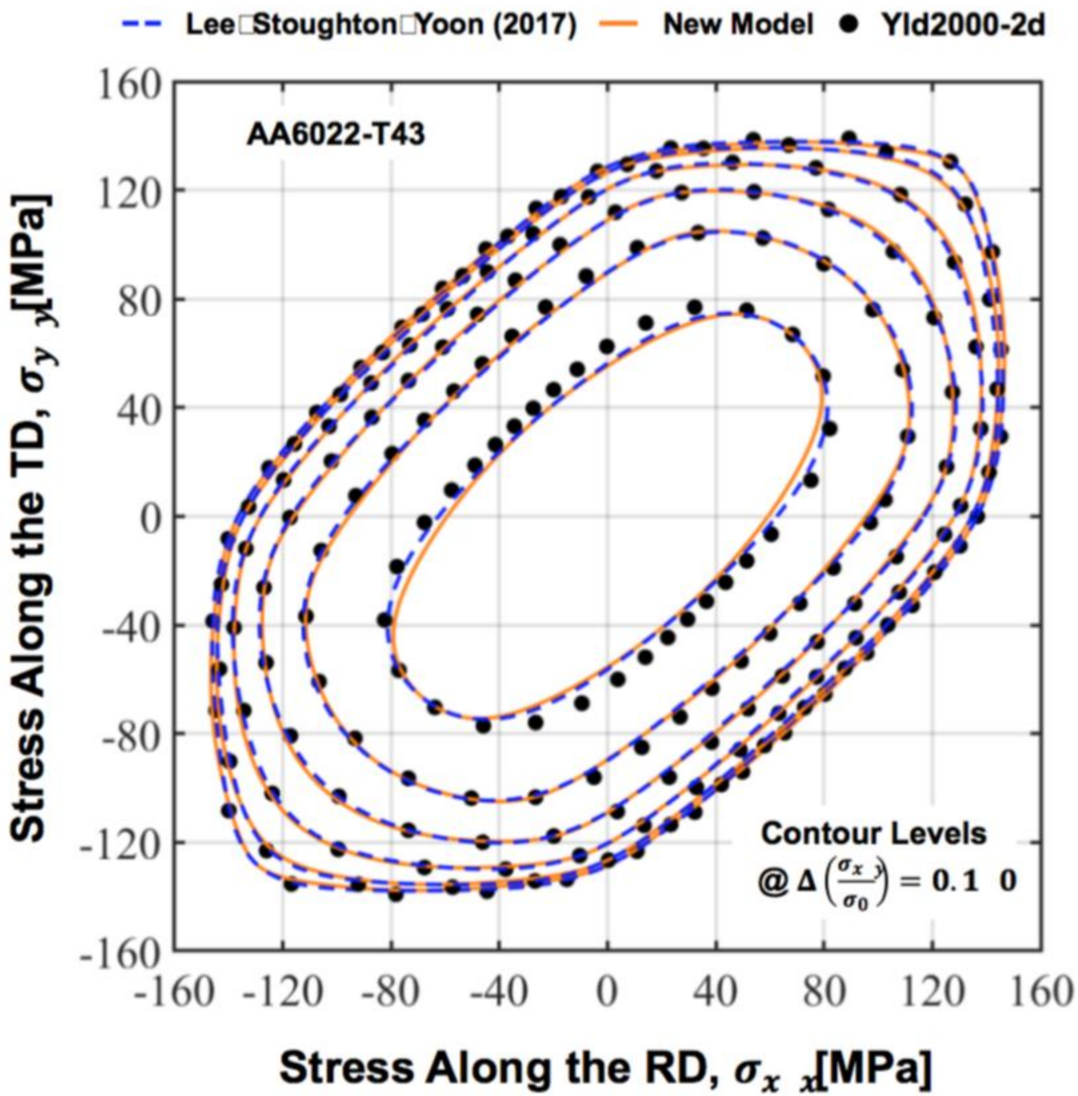
- ✓ ***Control the flatness of a yield surface by the exponent of k***

where $D(\theta_L) = \frac{2}{\sqrt{3}} \sin\left(\theta_L + \frac{\pi}{3}\right)$ (Khan and Liu (2012))





Prediction of Yield Loci



Evolution of Anisotropy/Asymmetry-induced Distortional Yielding (Park-Stoughton-Yoon, IJP, 2019)

$$\Phi\left(\sigma, \bar{\varepsilon}^p, \dot{\bar{\varepsilon}}^p, T\right) = \underbrace{\left[\frac{\sigma_{11}(\sigma_{11}-\sigma_{22})}{\sigma_0^2\left(\bar{\varepsilon}^p, \dot{\bar{\varepsilon}}^p, T\right)} + \frac{\sigma_{22}(\sigma_{22}-\sigma_{11})}{\sigma_{90}^2\left(\bar{\varepsilon}^p, \dot{\bar{\varepsilon}}^p, T\right)} + \frac{4\sigma_{12}^2}{\sigma_{45}^2\left(\bar{\varepsilon}^p, \dot{\bar{\varepsilon}}^p, T\right)} + \frac{\sigma_{11}\sigma_{22}-\sigma_{12}^2}{\sigma_{EB}^2\left(\bar{\varepsilon}^p, \dot{\bar{\varepsilon}}^p, T\right)}\right]^{\frac{1}{2}}} \cdot [X]^k \cdot e^{K\left(\sigma, \bar{\varepsilon}^p, \dot{\bar{\varepsilon}}^p, T\right) I(\eta)}$$

Stoughton and Yoon (2009) Model

$$X = (C_0 - 1)[D(\theta_L)]^4 - 2(C_0 - 1)[D(\theta_L)]^2 + C_0$$

Scaling Function

where $D(\theta_L) = \frac{2}{\sqrt{3}} \sin\left(\theta_L + \frac{\pi}{3}\right)$ (Khan and Liu (2012))

✓ *Control the flatness of a yield surface by the exponent of k*

$$I(\eta) = \begin{cases} 0 & \text{for } \eta = \frac{1}{3} \text{ or } \frac{2}{3} \\ -1 & \text{for } \eta = -\frac{1}{3} \text{ or } -\frac{2}{3} \end{cases}$$

Asymmetry Function

✓ *Strength Differential Effect*

$$k_i = \ln\left|\frac{\sigma_i^C\left(\bar{\varepsilon}^p, \dot{\bar{\varepsilon}}^p, T\right)}{\sigma_i^T\left(\bar{\varepsilon}^p, \dot{\bar{\varepsilon}}^p, T\right)}\right|$$

where $K\left(\sigma, \bar{\varepsilon}^p, \dot{\bar{\varepsilon}}^p, T\right) = \frac{1}{\bar{\sigma}_{vM}^2(\sigma)}\left[k_0\sigma_{11}(\sigma_{11}-\sigma_{22}) + k_{90}\sigma_{22}(\sigma_{22}-\sigma_{11}) + 4k_{45}\sigma_{12}^2 + k_{EB}\left(\sigma_{11}\sigma_{22}-\sigma_{12}^2\right)\right]$

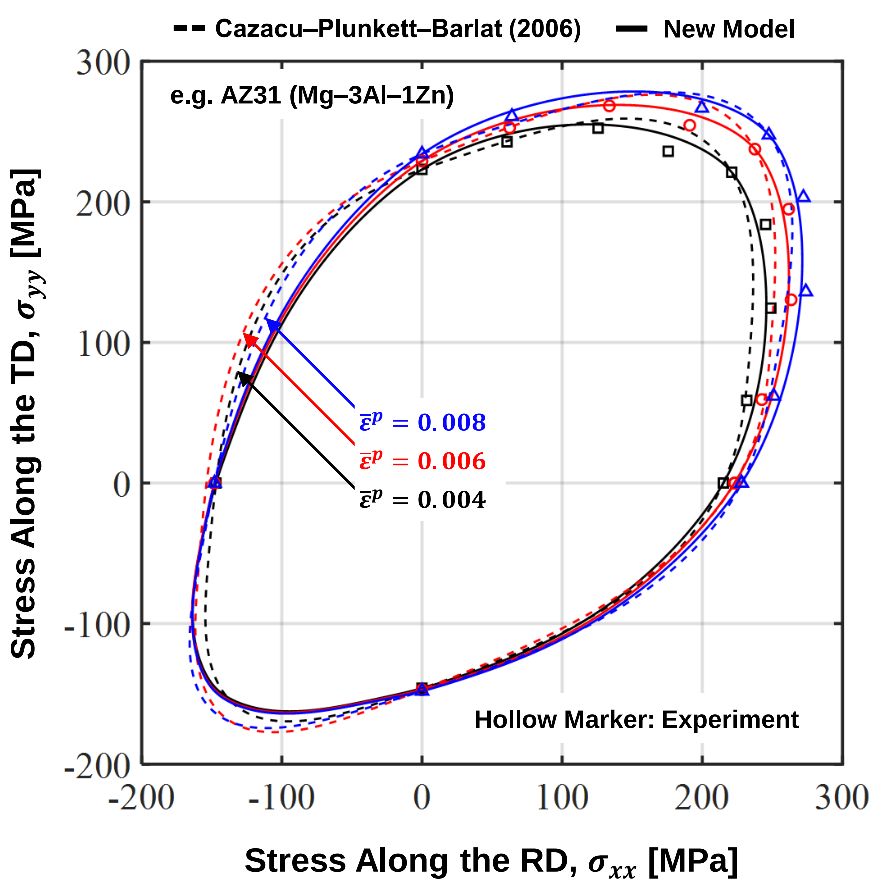
$$e^{K\left(\sigma, \bar{\varepsilon}^p, \dot{\bar{\varepsilon}}^p, T\right) I(\eta)} = \left[1 + \underbrace{K\left(\sigma, \bar{\varepsilon}^p, \dot{\bar{\varepsilon}}^p, T\right) I(\eta) + \frac{\left(K\left(\sigma, \bar{\varepsilon}^p, \dot{\bar{\varepsilon}}^p, T\right) I(\eta)\right)^2}{2!} + \frac{\left(K\left(\sigma, \bar{\varepsilon}^p, \dot{\bar{\varepsilon}}^p, T\right) I(\eta)\right)^3}{3!} + \cdots}\right]$$

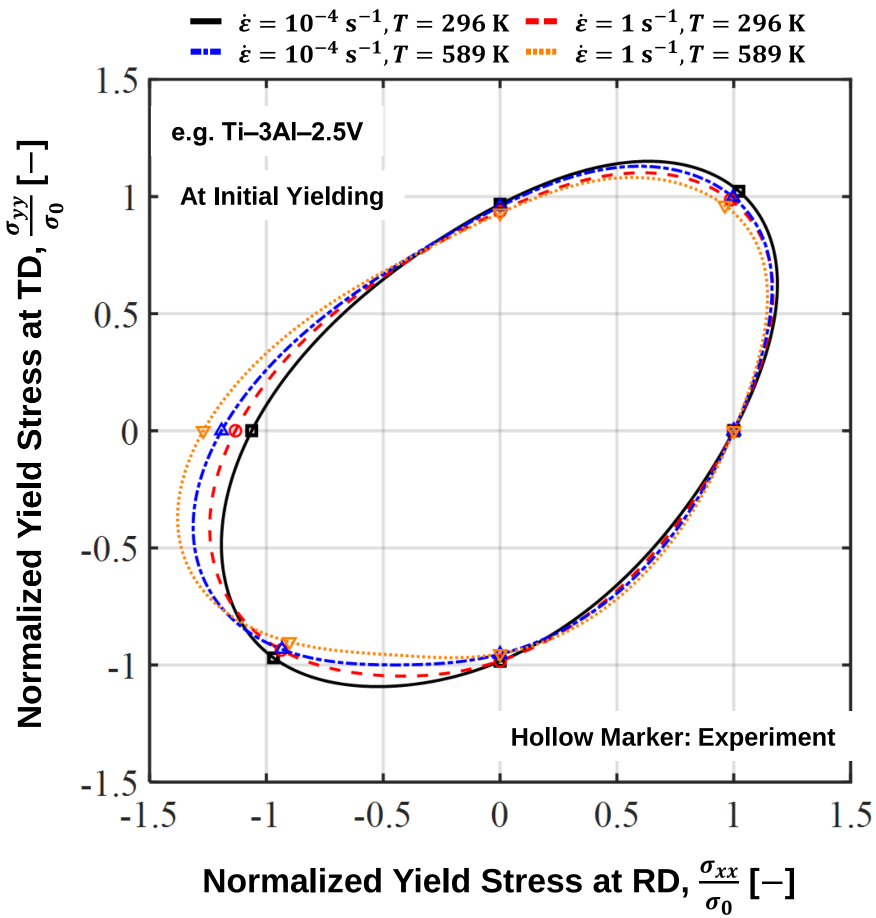
Taylor Expansion

Anisotropic Strength Differential Effect

Application to Various Metallic Materials

Evolution of Anisotropy/Asymmetry-induced Distortional Yielding





In considering the evolution of a distortional yield surface with respect to $\bar{\varepsilon}^p$, $\dot{\bar{\varepsilon}}^p$, and T

- Using neither any interpolation nor optimization techniques for the calibration of the yield surface

Asymmetry in Yield Function

Spitzig and Richmond (1984) showed that yielding *does* depend on the hydrostatic pressure even for Metals.

$$f\left(\sigma_{ij}\right) = \sqrt{3J_2} + aI_1 = \sigma_{\text{Mises}} + aI_1$$

Stoughton & Yoon (2004) proposed a pressure sensitive non-AFR Model

$$\bar{\sigma}_y = \underbrace{\sqrt{\lambda_1\sigma_{11}^2 + \lambda_2\sigma_{22}^2 - 2\nu_y\sigma_{11}\sigma_{22} + 2\rho_y\sigma_{12}^2}}_{\text{(Tensile)}}\underbrace{\left(1 + \alpha_1\sigma_{11} + \alpha_2\sigma_{22}\right)}_{\text{(Compressive)}}$$

Cazacu and Barlat (2004) restored the asymmetry of the third invariant to consider SD effect for pressure insensitive metals

$$f\left(\sigma_{ij}\right) = J_2^3 - cJ_3^2$$

Asymmetric yield function with dependence on three stress invariants (Yoon et al., IJP, 2013)

$$f\left(\sigma_{ij}\right) = bI_1 + \left(J_2'^{3/2} - J_3''\right)^{1/3}$$

where

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$$

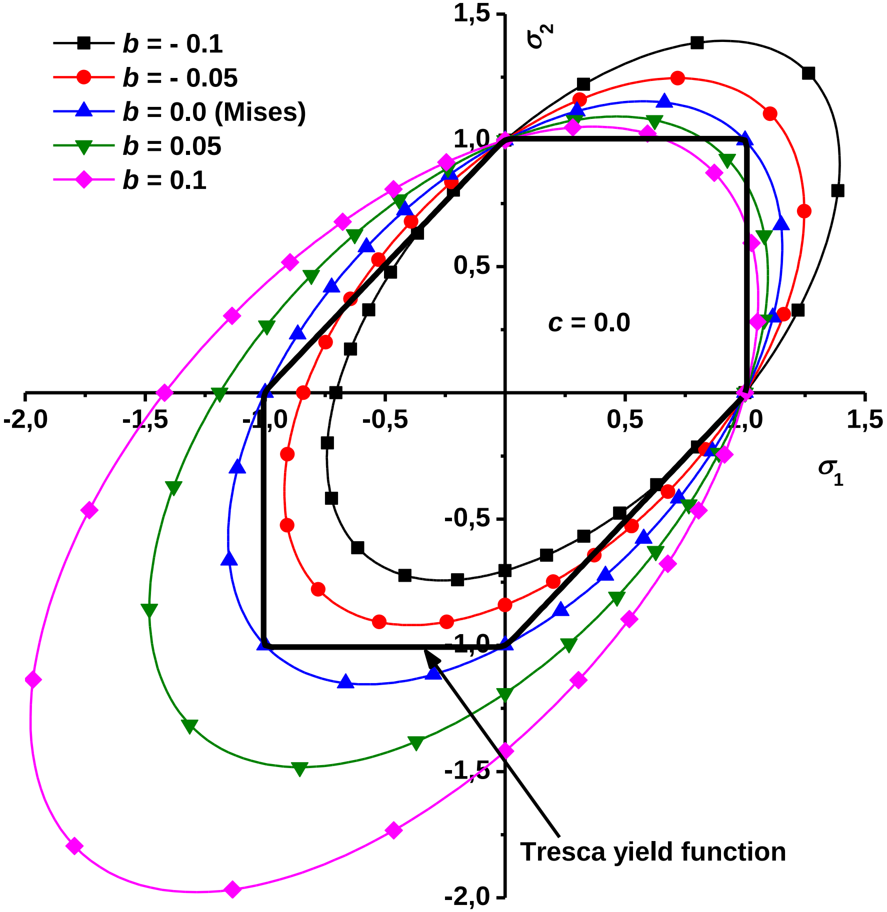
$$J_2 = \frac{1}{2}\boldsymbol{s}' : \boldsymbol{s}' \qquad J_3 = \det(\boldsymbol{s}'')$$

$$\mathbf{s}' = \mathbf{L}'\boldsymbol{\sigma}, \ \mathbf{s}'' = \mathbf{L}''\boldsymbol{\sigma}$$

$$\mathbf{L}' = \begin{bmatrix} (c_2' + c_3')/3 & -c_3'/3 & -c_2'/3 & 0 & 0 & 0 \\ -c_3'/3 & (c_3' + c_1')/3 & -c_1'/3 & 0 & 0 & 0 \\ -c_2'/3 & -c_1'/3 & (c_1' + c_2')/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_4' & 0 & 0 \\ 0 & 0 & 0 & 0 & c_5' & 0 \\ 0 & 0 & 0 & 0 & 0 & c_6' \end{bmatrix} \quad \mathbf{L}'' = \begin{bmatrix} (c_2'' + c_3'')/3 & -c_3''/3 & -c_2''/3 & 0 & 0 & 0 \\ -c_3''/3 & (c_3'' + c_1'')/3 & -c_1'/3 & 0 & 0 & 0 \\ -c_2''/3 & -c_1''/3 & (c_1'' + c_2'')/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_4'' & 0 & 0 \\ 0 & 0 & 0 & 0 & c_5'' & 0 \\ 0 & 0 & 0 & 0 & 0 & c_6'' \end{bmatrix}$$

Effect of b

$$f\left(\sigma_{ij}\right) = bI_1 + \left(J_2'^{3/2} - J_3''\right)^{1/3}$$



Asymmetric yield function under Non-Associated Flow

- Application to **AA 2090-T3** with a reduced form without pressure sensitivity

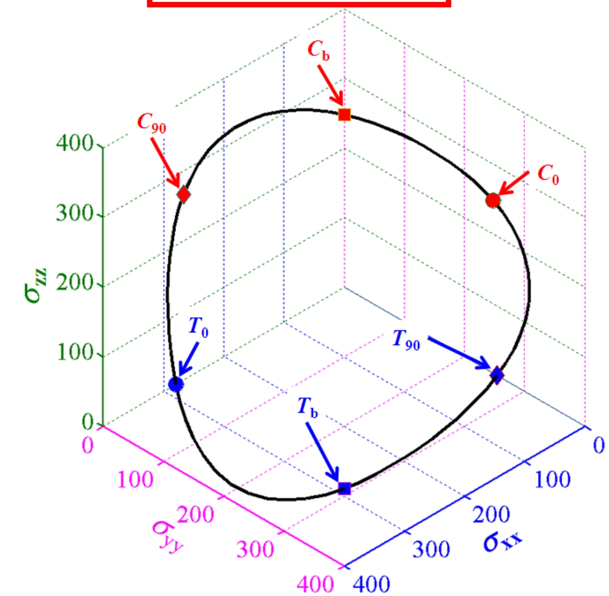
(Yield function)

$$f\left(s_{ij}\right) = bI_1 + \left({J_2'}^{3/2} - J_3''\right)^{1/3}$$

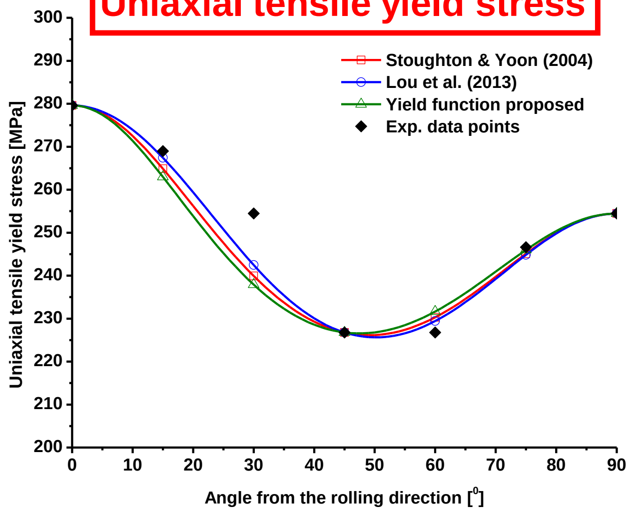
(Plastic potential)

$$g(\vec{\sigma}) = \left(1+\frac{1}{r_0}\right)\sigma_{11}^2 + \left(1+\frac{1}{r_{90}}\right)\sigma_{22}^2 - 2\sigma_{11}\sigma_{22} + \left(\frac{1}{r_0}+\frac{1}{r_{90}}\right)(1+2r_{45})\sigma_{12}^2$$

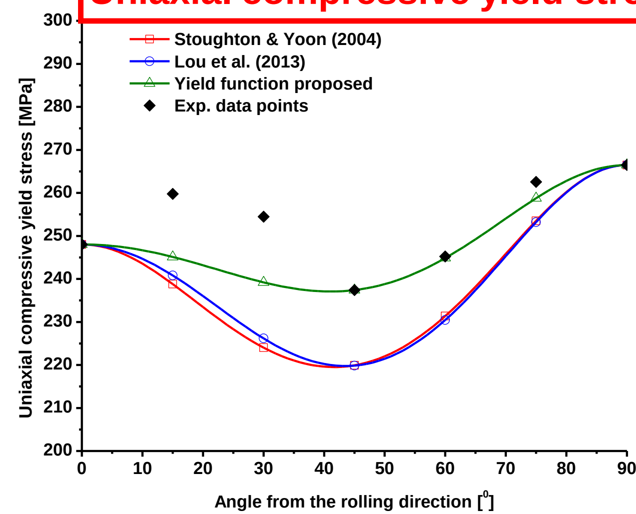
3d yield surface



Uniaxial tensile yield stress



Uniaxial compressive yield stress

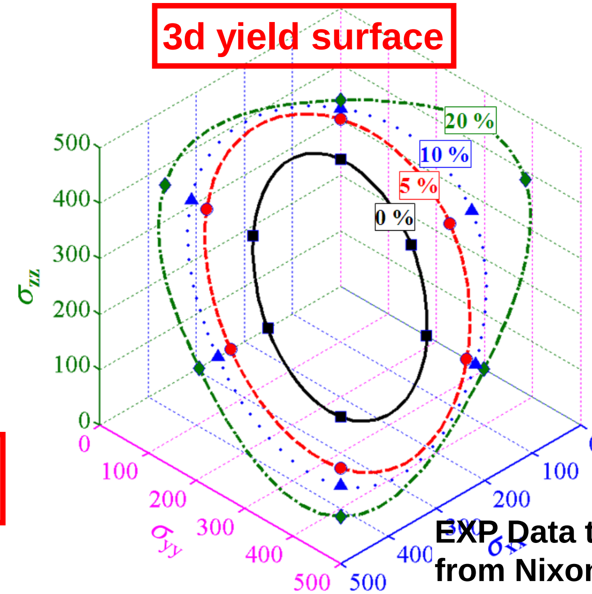


Asymmetric yield function with dependence on three stress invariants

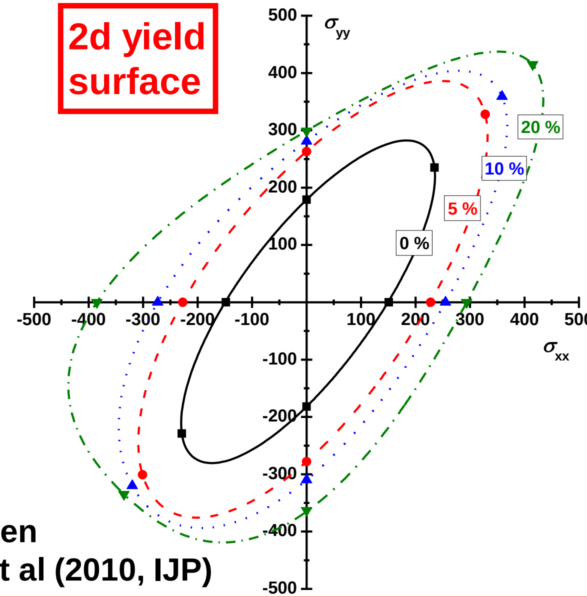
■ Application to high-purity α-titanium and AZ31

high-purity α-titanium

3d yield surface



2d yield surface

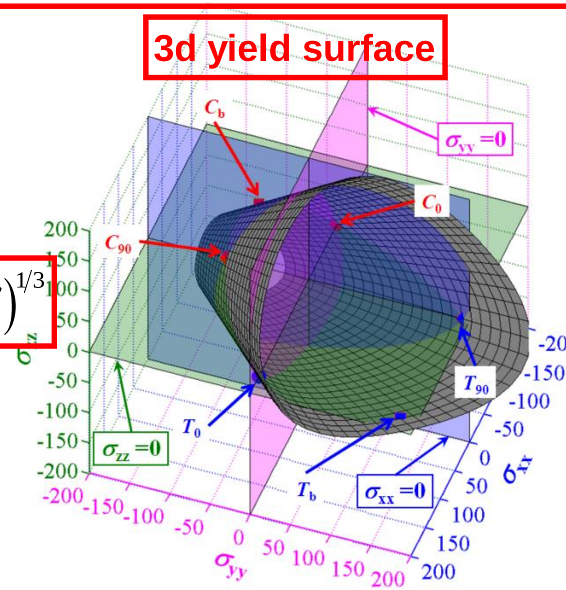


$$f\left(\sigma_{ij}\right)=\left(J_2'^{3/2}-J_3''\right)^{1/3}$$

EXP Data taken from Nixon et al (2010, IJP)

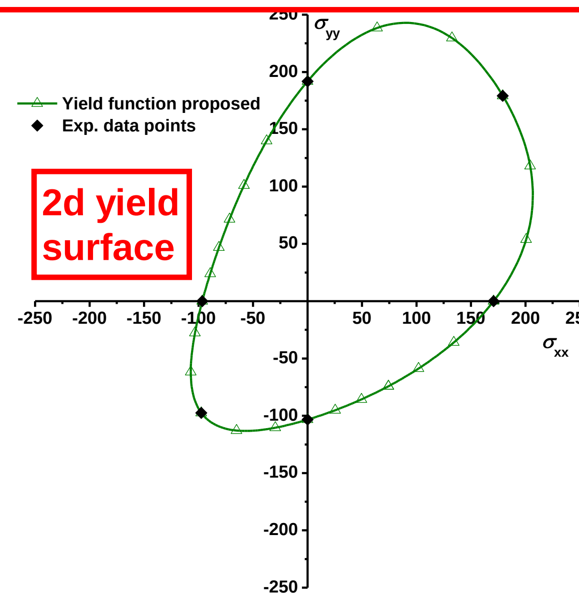
Mg AZ31

3d yield surface



$$f\left(\sigma_{ij}\right)=bI_1+\left(J_2'^{3/2}-J_3''\right)^{1/3}$$

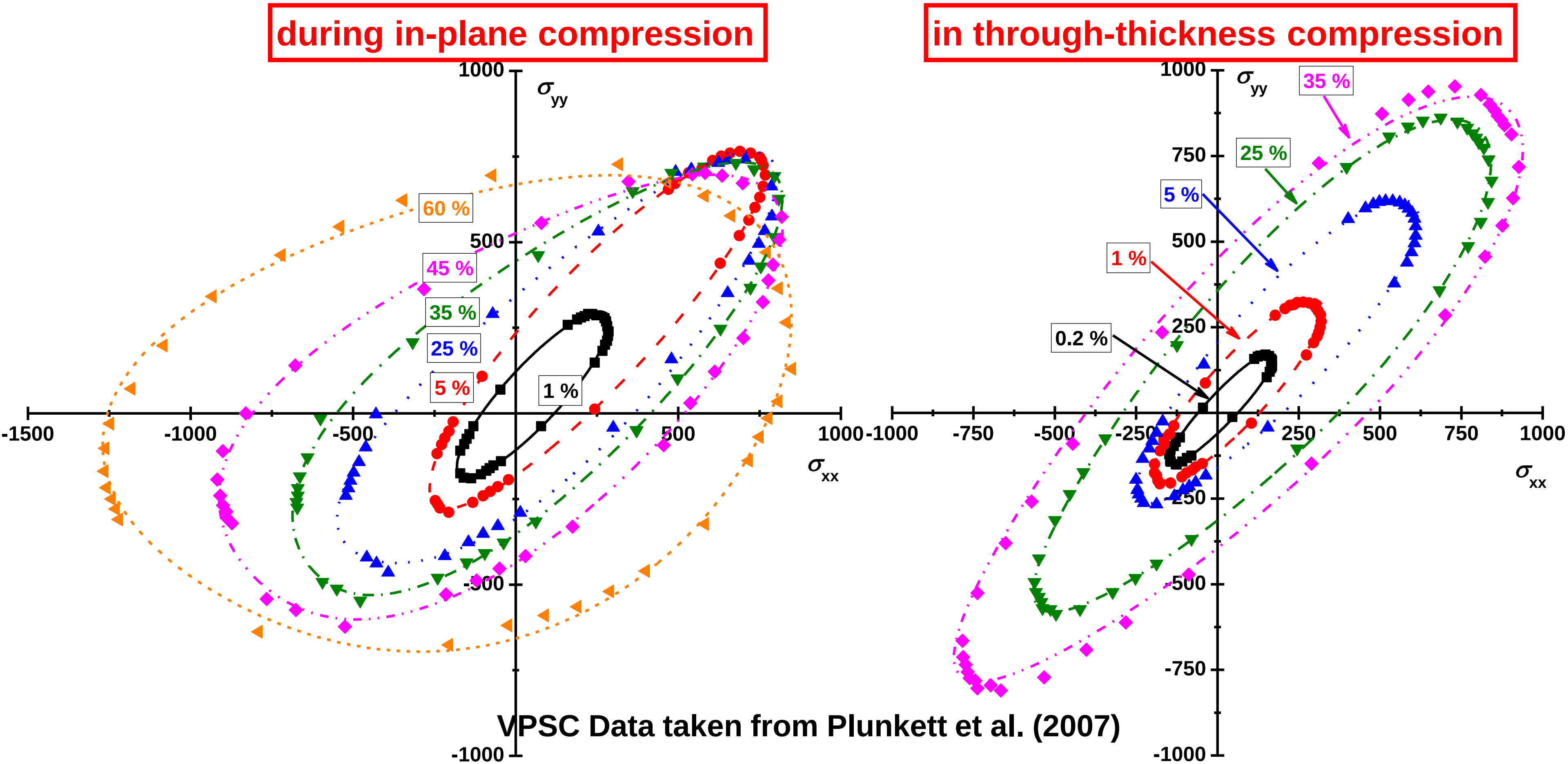
2d yield surface



Asymmetric yield function with dependence on three stress invariants

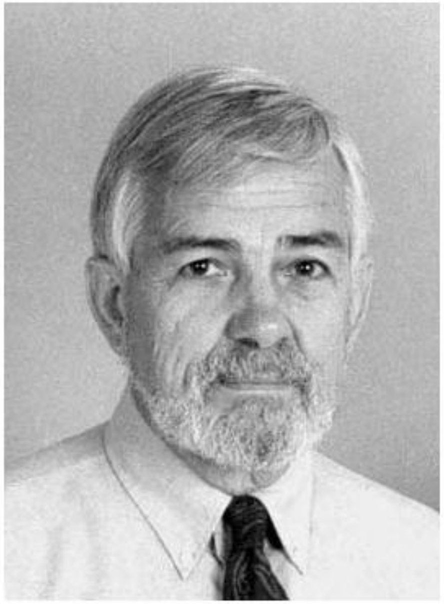
- **Application to yield surface evolution a zirconium clock-rolled plate for various levels of pre-strain**

Reduced form with isotropic pressure sensitivity: $f\left(\sigma_{ij}\right) = bI_1 + \left(J_2'^{3/2} - J_3''\right)^{1/3}$



VPSC Data taken from Plunkett et al. (2007)

Owen Richmond (1928-2001)



Dr. Owen Richmond, 1928–2001.

PhD (1958) :
Pennsylvania State University with a thesis entitled ‘‘A Hyperbolic Theory of Plasticity.”

1983 – 1998 : Alcoa Corporate Director and Fellow

1993 : The highest distinction, the Outstanding Engineering Alumni from Pennsylvania State University

1994 : ASME Nadai Award

1997 : Member of the US National Academy of Engineering

Publications : 120 articles from Continuum to Nano Mechanics.

Owen was interested in the general concept of design from the beginning of his career, when he worked on streamline dies for extruded products. One of Owen’s achievements was the theory of ideal forming for sheet and bulk materials. This theory, which is based on mechanics principles and material plasticity models, provides a method of designing products along with the associated tools and processes while using minimum deformation energy. Today, the use of computer software to design manufactured products is well accepted and a growing trend. The collective his works can be found in the book, ‘‘The Integration of Material, Process, and Product Design.’’ This book describes Owen’s vision for collaboration and integration of diverse disciplines in order to achieve a common goal.

Thomas STOUGHTON

General Motors Technical Fellow
GM R&D Center
Warren, MI, USA 48090-9055

Education:

9/1972-6/1976 California Institute of Technology, Bachelor of Science, Physics
9/1976-5/1981 Massachusetts Institute of Technology, Ph.D degree in High Energy Physics

Professional Experience:

6/1974-7/1976 Research Assistant, Caltech, Solar Radiation Lab
9/1975-6/1976 Teaching Assistant, Caltech, Physics Department
7/1976-6/1981 Research Associate, MIT, Lab for Nuclear Science
6/1981-3/1983 Post-doc, MIT, Lab for Nuclear Science
3/1983-present GM Research Fellow, General Motors Research Labs

COMMITTEE MEMBERSHIPS and ACTIVITIES

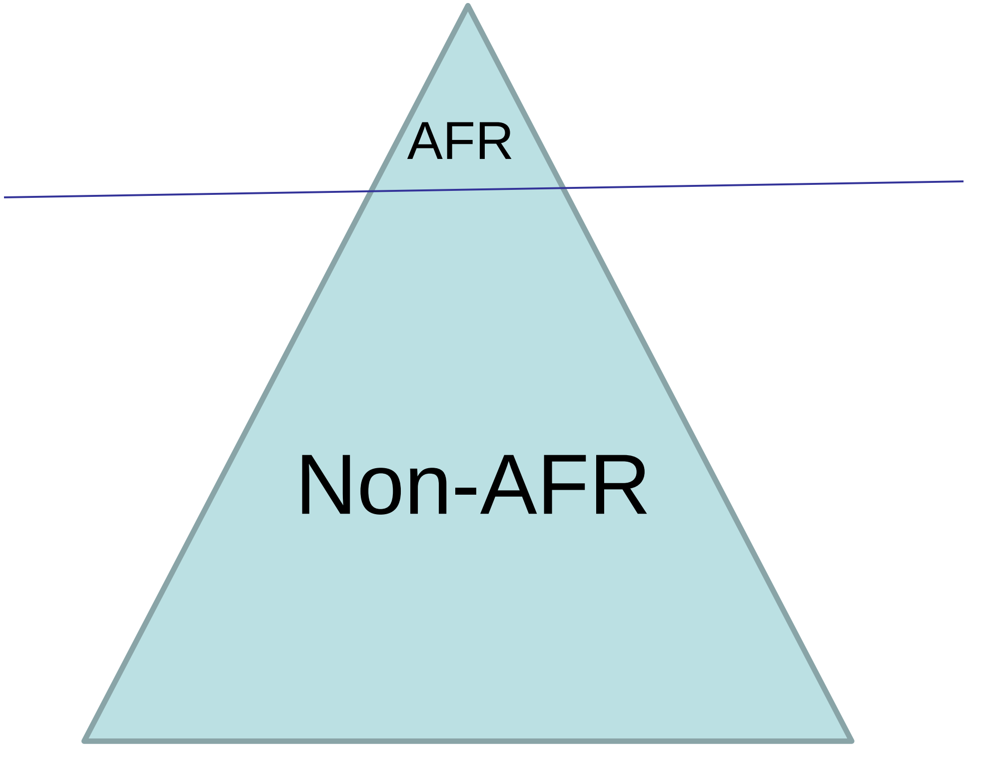
NORTH AMERICAN DEEP DRAWING RESEARCH GROUP (Vice-President)	2008-present
NUMISHEET CONFERENCES (Member of Steering Committee)	2005-present
INTERNATIONAL JOURNAL OF PLASTICITY (Editorial Board)	2006-present

HONORS

Graduated with Honors, Caltech
McCuen Award, 1995

PUBLICATIONS

Dr. Stoughton has published over 100 internal research reports and is an expert in metal forming technology at GM. He has published over 100 papers in journals and conference proceedings in the field of metal forming and FEA.



by M. Ortiz (Caltech, USA)
at WCCM2008