

- Ward I. M. *Mechanical properties of solid polymers*. Wiley Interscience, New York (1971).
- Kausch H. H. *Polymer fracture*. Springer Verlag, Berlin (1978).
- Neville A. M. *Properties of concrete*. J. Wiley, New York (1973).
- Baron J. & Sautgrey R. *Le béton hydraulique*. Presses de l'ENPC, Paris (1982).
- Kollman F. F. P. & Cote W. A., *Principles of wood science and technology*. Springer Verlag, Berlin (1968).

## 2

## *ELEMENTS OF CONTINUUM MECHANICS AND THERMODYNAMICS*

*A chaque phénomène sa variable  
A chaque variable sa loi d'évolution*

To model the physical phenomena of deformation and fracture, described briefly in Chapter 1, a method based upon general principles which govern the variables representative of the state of the material medium is needed. The objective of this chapter is to present in a condensed form all the basic concepts that will be used in the following chapters. Two types of modelling are necessary: one, the so-called kinematical or mechanical modelling is concerned with the motions and forces in the continuum, and the second, the so-called phenomenological or physical modelling introduces the variables characterizing the phenomena under study.

The presentation given here is the result of the work of Germain as presented in a post graduate course and subsequently incorporated by one of the authors (Lemaitre) in his course, and also contained in *Cours de mécanique des milieux continus* cited in the bibliography of this chapter. This reference contains the details necessary for a deeper understanding of the concepts. Here, only the essential results are given, and the mathematical derivations have been largely omitted. The notation used is almost identical to that in the work referred to above which should help the reader who is anxious to go deeper into the subject.

The framework of mechanics presented here is based on the principle of virtual power. Although the basic idea was presented by D'Alembert as long ago as 1750, it is only, with the development of the variational methods of functional analysis in around 1970 (Duvaut, Lions, Nayroles), that systematic use of this principle has been made. The choice of a particular virtual movement for a given medium leads naturally to consistent definitions of stresses and strains, and to equations of equilibrium with corresponding boundary conditions.

The state of a continuum material depends, in general, on the whole history of its mechanical variables, and the modelling of its behaviour may be based on hereditary or integral laws. In order to obtain a formalism which is directly accessible to the methods of functional analysis, we prefer to adopt the approach of the thermodynamics of irreversible processes by introducing state variables. This approach was initiated first by chemists, and was applied to continuum mechanics by Eckart and Biot around 1950. The thermodynamic potential allows us to define associated variables from observable variables and internal variables chosen for the study of the phenomenon. This then naturally leads to the state laws. The pseudo-potential of dissipation furnishes the complementary laws of evolution for the variables which describe irreversible processes according to the Moreau (1970) formalism.

It should be mentioned, however, that the elementary thermodynamics presented here will be used without consideration of thermal or true dynamic effects for most of the phenomena studied. Nevertheless, this thermodynamic framework will be very useful to guide and limit the possible choices in phenomenological modelling.

## 2.1 Statement of the principle of virtual power

### 2.1.1 Motion and virtual power

In order to obtain a schematic representation of the forces involved in the phenomenon under study, it is convenient to imagine fictitious or virtual motions and to analyse the resulting work or power. For example:

to analyse the gravity forces acting on a car, we can imagine that it is being lifted (virtual motion from bottom to top);

to analyse the friction forces involved when the car skids on the road, we can imagine that the car is being pulled along with its brakes on (horizontal virtual motion);

to analyse the rigidity of the suspension, we can imagine that the body of the car is being moved with respect to the wheels (relative virtual motion of one point with respect to another).

More generally, a virtual motion of a material medium with respect to a frame of reference is defined at any instant by a velocity vector field dependent on the material point ( $M$ ):

$$\vec{v}(M).$$

By a suitable choice of  $\vec{v}(M)$  the mechanical phenomenon can be represented more or less precisely.

The virtual power of a system of forces in a given virtual motion is, by definition, a linear continuous function of the scalar value of  $\vec{v}(M)$  equal to the work done per unit time in the considered phenomenon:

$$\hat{P}(\vec{v}(M)).$$

### 2.1.2 Frames of reference and material derivatives

Motion may be described in terms of either Eulerian variables, or, alternatively, in terms of Lagrangian variables.

The Eulerian variables are the current time  $t$  and the coordinates of the current position of the material point  $M$ ; these variables identify the current configuration. The velocity of the point is expressed by:

$$\vec{v}(M) = \vec{v}(x_1(t), x_2(t), x_3(t), t).$$

The Lagrangian description is based on the current time  $t$  and the coordinates of the initial position  $M_0$  of the material point  $M$ . This identification is done with respect to the initial configuration and the velocity is expressed by:

$$\vec{v}(M) = \vec{v}(x_1^0, x_2^0, x_3^0, t).$$

The concept of a derivative following the particle, called the material derivative, is applicable to any quantity defined on a set whose motion is being followed.

In terms of Lagrangian variables, the material derivative is identical to the partial derivative with respect to time. For example, the acceleration of a point  $M$  is expressible as:

$$\vec{\gamma} = d\vec{v}/dt = \partial\vec{v}/\partial t.$$

In contrast, in terms of Eulerian variables, there is no such identity because the current coordinates of the point  $M$  depend on time. One then has:

$$\vec{\gamma} = \frac{d\vec{v}}{dt} = \frac{\partial\vec{v}}{\partial t} + \frac{\partial\vec{v}}{\partial\vec{x}} \frac{\partial\vec{x}}{\partial t} = \frac{\partial\vec{v}}{\partial t} + \vec{v} \cdot \overrightarrow{\text{grad}} \vec{v}$$

or in index notation,

$$\gamma_i = \frac{dv_i}{dt} = \frac{\partial v_i}{\partial t} + v_{i,j}v_j$$

where we have used the Einstein summation convention:

$$v_{i,j}v_j = v_{i,1}v_1 + v_{i,2}v_2 + v_{i,3}v_3.$$

### 2.1.3 Principle of virtual power (Germain, 1972)

It is possible in an isolated material medium to distinguish between the external forces acting on the medium and the internal forces which represent the bonds existing between all possible parts of the medium.

#### Axiom of objectivity

The virtual power of internal forces in any rigid movement of the medium is zero.

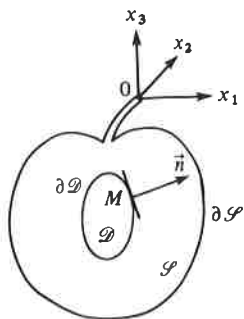
#### Axiom of equilibrium (static or dynamic)

For any material identified in an absolute frame of reference, at every instant and for any virtual movement, the virtual power of the acceleration quantities (i.e., inertia forces)  $\hat{P}_{(a)}$  is equal to the sum of the virtual power of internal forces  $\hat{P}_{(i)}$  and of external forces  $\hat{P}_{(x)}$

$$\bullet \quad \hat{P}_{(i)} + \hat{P}_{(x)} = \hat{P}_{(a)}.$$

These two axioms embody the fundamental law of dynamics.

Fig. 2.1. Isolated material medium.



## 2.2 Method of virtual power

### 2.2.1 Rates of strain and stress

The application of the principle of virtual power to an isolated region for a particular virtual motion and a choice of the linear forms which appear in the expression for virtual power lead directly to the mechanical equations of continua.

Let there be a region  $\mathcal{D}$  with boundary  $\partial\mathcal{D}$  in the interior of volume  $\mathcal{S}$  with boundary  $\partial\mathcal{S}$ , and let  $\vec{n}$  be the outward normal at a point  $M$  of  $\partial\mathcal{D}$  in an orthogonal frame of reference  $(0, x_1, x_2, x_3)$  (see Fig. 2.1).

#### Choice of a virtual motion, theory of first gradient

The first choice consists in defining the space of virtual velocities. Here a simple vector field of displacement rate  $\vec{v}(M)$  is sufficient as long as micro-rotations are not taken into account (micropolar media excluded). The second choice is concerned with the terms to be included in the virtual power calculations; a single term in  $\vec{v}(M)$  is insufficient to describe the behaviour of deformable bodies. The simplest idea consists in adding to it a field of velocity gradients,  $\overrightarrow{\text{grad}} \vec{v}$  or  $\hat{v}_{i,j}$ . Thus, for the theory of first gradient, the virtual motion is given by:

$$\vec{v}(M) \text{ and } \overrightarrow{\text{grad}} \vec{v}.$$

By decomposition of  $\overrightarrow{\text{grad}} \vec{v}$  into its symmetric and antisymmetric parts, one obtains the rate of deformation tensor  $\hat{D}$  (of second order) and the rate of rotation tensor  $\hat{\Omega}$ , as follows (the symbol  $()^T$  denotes a transpose):

$$\overrightarrow{\text{grad}} \vec{v} = \frac{1}{2}[\overrightarrow{\text{grad}} \vec{v} + (\overrightarrow{\text{grad}} \vec{v})^T] + \frac{1}{2}[\overrightarrow{\text{grad}} \vec{v} - (\overrightarrow{\text{grad}} \vec{v})^T]$$

$$\hat{\Omega} = \frac{1}{2}[\overrightarrow{\text{grad}} \vec{v} - (\overrightarrow{\text{grad}} \vec{v})^T]$$

$$\hat{D} = \frac{1}{2}[\overrightarrow{\text{grad}} \vec{v} + (\overrightarrow{\text{grad}} \vec{v})^T]$$

or

$$\hat{D}_{ij} = \frac{1}{2}(\hat{v}_{i,j} + \hat{v}_{j,i}).$$

#### Virtual power of internal forces

The virtual power of internal forces is defined by the integral over the whole domain  $\mathcal{D}$  of a volume density which is supposed to contain *a priori* three terms in  $\vec{v}$ ,  $\hat{D}$  and  $\hat{\Omega}$  associated respectively to a vector  $\vec{f}^*$ , and two second order tensors,  $\sigma$  which is symmetric, and  $\Gamma$  which is anti-

symmetric:

$$\hat{P}_{(i)} = - \int_{\mathcal{Q}} (\bar{f}^* \cdot \bar{v} + \boldsymbol{\sigma} : \hat{\mathbf{D}} + \boldsymbol{\Gamma} : \hat{\boldsymbol{\Omega}}) dV$$

where the symbol  $:$  denotes the tensorial product contracted on two indices and the minus sign is in accordance with the convention used in thermodynamics.

The first axiom of the principle of virtual power requires that for a rigid motion of the solid:

$$\begin{aligned} &\text{in translation } \bar{v} \neq 0, \mathbf{D} = 0, \boldsymbol{\Omega} = 0; \\ &\text{in rotation } \bar{v} = 0, \mathbf{D} = 0, \boldsymbol{\Omega} \neq 0 \end{aligned}$$

the power  $\hat{P}_{(i)}$  is zero.

Application of the fundamental lemma of the physics of continuum media to any function  $f(M)$  defined and continuous in  $\mathcal{D}$  gives

$$\int_{\mathcal{Q}} f(M) dV = 0 \quad \forall \mathcal{D} \text{ in } \mathcal{S} \Rightarrow f(M) = 0 \text{ in } \mathcal{D},$$

which leads to  $\bar{f}^* \cdot \bar{v} = 0$  and  $\boldsymbol{\Gamma} : \boldsymbol{\Omega} = 0$  regardless of  $\bar{v}$  and  $\boldsymbol{\Omega}$  so that

$$\bar{f}^* = 0, \quad \boldsymbol{\Gamma} = 0$$

and

$$\hat{P}_{(i)} = - \int_{\mathcal{Q}} \boldsymbol{\sigma} : \hat{\mathbf{D}} dV.$$

It will be seen that the tensor  $\boldsymbol{\sigma}$  introduced above is the Cauchy stress tensor. The first axiom of virtual power and the choice of virtual motion according to the theory of first gradient are together equivalent to the hypothesis generally made to introduce the stress tensor. In this hypothesis, it is supposed that the internal forces can be represented schematically by a surface density of cohesive forces  $\bar{T}$  which represent action at very short distance. By enlarging upon this hypothesis, it is possible to define a stress vector dependent on the material point  $M$  under consideration, on time  $t$ , and linearly on the normal  $\bar{n}$  to  $\partial\mathcal{D}$  at  $M$ :

$$\bar{T}(M, t, \bar{n}).$$

The stress tensor  $\boldsymbol{\sigma}$  can be shown to be given by:

$$\bar{T}(M, t, \bar{n}) = \boldsymbol{\sigma}(M, t) \cdot \bar{n}.$$

The tensor is symmetric and the components  $\sigma_{ij}$  of its matrix representation

in the frame of reference  $(0, x_1, x_2, x_3)$  are given by

$$T_i = \sigma_{ij} n_j.$$

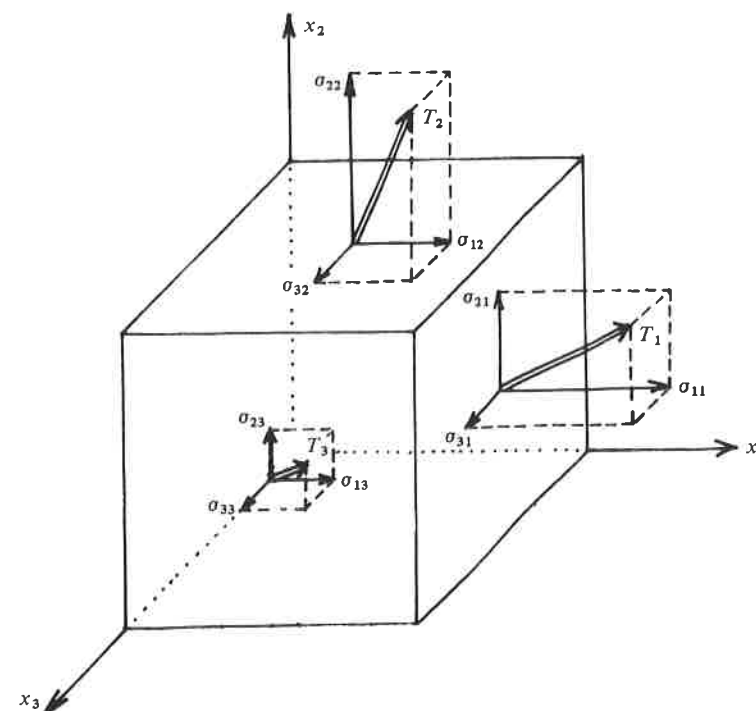
These components are shown schematically in Fig. 2.2. On each face, the stress vector is decomposed into one normal component and two shear components.

The method of virtual power enables us to bring more precision to the hypotheses. Moreover, it allows us, if necessary, to construct theories based on other hypotheses: for example, the introduction of a rotation field in the space of virtual velocities for a micropolar medium, the introduction of a second gradient in the axioms of virtual power to describe the deformations more precisely.

### Virtual power of external forces

The external forces consist of:

Fig. 2.2. Components of the stress vector and of the representative matrix of the stress tensor (after Germain-Muller).



the forces exerted at a distance by systems external to  $\mathcal{S}$ , supposed to be defined by a volume density of force  $\vec{f}$  (not of a tensor or of a couple);

the contact forces schematically represented by a surface density  $\vec{T}$  (Cauchy's stress hypothesis)

$$\hat{P}_{(x)} = \int_{\mathcal{Q}} \vec{f} \cdot \vec{v} \, dV + \int_{\partial\mathcal{Q}} \vec{T} \cdot \vec{v} \, dS.$$

We may add a term involving  $\overrightarrow{\text{grad}} \vec{v}$ , but this can rigorously be shown to be zero.

### Virtual work of acceleration quantities

If  $\vec{\gamma}$  is the acceleration vector of a particle  $M$ , and  $\rho$  is the mass density (the values for some materials are given in Table 2.4), the power of the acceleration quantities is expressed by:

$$\hat{P}_{(a)} = \int_{\mathcal{Q}} \vec{\gamma} \cdot \vec{v} \rho \, dV.$$

### 2.2.2 Equations of equilibrium

By applying the axiom of equilibrium of the principle of virtual power to the region  $\mathcal{Q}$  we obtain the following equilibrium equations:

$$\hat{P}_{(l)} + \hat{P}_{(x)} = \hat{P}_{(a)} \quad \forall \vec{v}$$

$$\bullet \quad - \int_{\mathcal{Q}} \boldsymbol{\sigma} : \hat{\mathbf{D}} \, dV + \int_{\mathcal{Q}} \vec{f} \cdot \vec{v} \, dV + \int_{\partial\mathcal{Q}} \vec{T} \cdot \vec{v} \, dS = \int_{\mathcal{Q}} \vec{\gamma} \cdot \vec{v} \rho \, dV$$

or

$$- \int_{\mathcal{Q}} \sigma_{ij} \hat{D}_{ij} \, dV + \int_{\mathcal{Q}} f_i \hat{v}_i \, dV + \int_{\partial\mathcal{Q}} T_i \hat{v}_i \, dS = \int_{\mathcal{Q}} \gamma_i \hat{v}_i \rho \, dV.$$

In order to take advantage of the fact that this identity holds for arbitrary virtual motions  $\vec{v}$ , it is essential to have  $\vec{v}$  itself in the first term. This is achieved by applying the divergence theorem (or integration by parts) as follows:

$$\begin{aligned} - \int_{\mathcal{Q}} \boldsymbol{\sigma} : \hat{\mathbf{D}} \, dV &= - \int_{\mathcal{Q}} \boldsymbol{\sigma} : \overrightarrow{\text{grad}} \vec{v} \, dV \\ &= - \int_{\partial\mathcal{Q}} \boldsymbol{\sigma} \cdot \vec{v} \cdot \vec{n} \, dS + \int_{\mathcal{Q}} \text{div} \boldsymbol{\sigma} \cdot \vec{v} \, dV \end{aligned}$$

where  $\vec{n}$  is the outward unit normal at the boundary  $\partial\mathcal{Q}$  of the region  $\mathcal{Q}$ . Then, the equation corresponding to the second axiom, may be written as:

$$- \int_{\partial\mathcal{Q}} \boldsymbol{\sigma} \cdot \vec{v} \cdot \vec{n} \, dS + \int_{\mathcal{Q}} \text{div} \boldsymbol{\sigma} \cdot \vec{v} \, dV + \int_{\mathcal{Q}} \vec{f} \cdot \vec{v} \, dV + \int_{\partial\mathcal{Q}} \vec{T} \cdot \vec{v} \, dS = \int_{\mathcal{Q}} \vec{\gamma} \cdot \vec{v} \rho \, dV$$

or

$$\int_{\mathcal{Q}} (\text{div} \boldsymbol{\sigma} + \vec{f} - \rho \vec{\gamma}) \cdot \vec{v} \, dV + \int_{\partial\mathcal{Q}} (\vec{T} - \boldsymbol{\sigma} \cdot \vec{n}) \cdot \vec{v} \, dS = 0 \quad \forall \vec{v}.$$

This identity, in accordance with the fundamental lemma, can be satisfied for any field  $\vec{v}$  if and only if:

$$\bullet \quad \text{div} \boldsymbol{\sigma} + \vec{f} - \rho \vec{\gamma} = 0 \quad \text{in } \mathcal{Q}$$

or

$$\sigma_{ij,j} + f_i - \rho \gamma_i = 0 \quad \text{in } \mathcal{Q},$$

and

$$\vec{T} = \boldsymbol{\sigma} \cdot \vec{n} \quad \text{or} \quad T_i = \sigma_{ij} n_j \quad \text{on } \partial\mathcal{Q}.$$

The first equation in its two forms expresses the local static or dynamic equilibrium. The second one defines the stress vector as the surface density of the forces introduced. It shows that  $\boldsymbol{\sigma}$  is really the Cauchy stress tensor: a second order symmetric tensor. It also yields the boundary conditions on forces if the axiom of equilibrium of virtual power is applied to the whole region  $\mathcal{S}$  under consideration.

If  $\vec{T}^d$  represents the density of the applied forces on the boundary  $\partial\mathcal{S}$  of the region  $\mathcal{S}$ , the same argument as the preceding one leads to:

$$\vec{T}^d = \boldsymbol{\sigma} \cdot \vec{n}$$

$$\text{or } T_i^d = \sigma_{ij} n_j \quad \text{on } \partial\mathcal{S}.$$

### 2.2.3 Strains and displacements

The method of virtual power introduces in a natural way the kinematical variables: the rate of displacement  $\vec{v}$  and the rate of deformation  $\mathbf{D}$ , defined at every instant  $t$ . Beyond this description of motion, the study of a solid continuum requires the characterization of its current state of deformation with respect to a reference configuration which may be chosen to be the initial configuration, or the relaxed configuration if elastic and inelastic phenomena are present simultaneously. It is then necessary to introduce the displacement vector and the strain tensor for finite deformations, or for infinitesimal deformations when these quantities are small.

### Assumptions of small strains and small displacements

In the case of infinitesimal deformations, the distinction between the Eulerian and Lagrangian variables can be ignored. We may then neglect second order terms in the analysis and pass from the latter to the former set of variables.

Let  $\vec{u}$  then be the displacement vector defined by:

$$\vec{u} = \int_0^t \vec{v} dt \quad \text{or} \quad \vec{v} = \dot{\vec{u}},$$

where  $t = 0$  corresponds to the reference configuration. Let  $\epsilon$  be the strain tensor defined by:

$$\epsilon = \int_0^t \mathbf{D} dt \quad \text{or} \quad \mathbf{D} = \dot{\epsilon}.$$

Invoking the definition of the tensor  $\mathbf{D}$ , we may then write:

$$\epsilon = \int_0^t \frac{1}{2} [\overrightarrow{\text{grad}} \vec{v} + (\overrightarrow{\text{grad}} \vec{v})^T] dt.$$

The assumption of small deformations results in the equivalence of the Eulerian and Lagrangian variables, and thus the material derivative is a partial derivative and the operations of gradient and integration become commutative:

$$\int_0^t \overrightarrow{\text{grad}} \vec{v} dt = \overrightarrow{\text{grad}} \left( \int_0^t \vec{v} dt \right) = \overrightarrow{\text{grad}} \vec{u}$$

whence

$$\epsilon = \frac{1}{2} [\overrightarrow{\text{grad}} \vec{u} + (\overrightarrow{\text{grad}} \vec{u})^T].$$

The strain tensor  $\epsilon$  defined from the deformation rate tensor  $\mathbf{D}$  is therefore the symmetric part of the gradient of displacement  $\vec{u}$  defined from the velocity  $\vec{v}$ .

The equations of continuum mechanics can be summarized in the case of small strains and small displacements by the following equations:

$$\bullet \quad \text{div } \sigma + \vec{f} = \rho \vec{\gamma}$$

$$\text{or } \sigma_{ij,j} + f_i = \rho \ddot{u}_i \text{ and}$$

$$\bullet \quad \epsilon = \frac{1}{2} [\overrightarrow{\text{grad}} \vec{u} + (\overrightarrow{\text{grad}} \vec{u})^T]$$

$$\text{or } \epsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}).$$

To solve a problem we need to add to the above equations:

the boundary conditions, generally given in terms of tractions,  $\sigma \cdot \vec{n} = \vec{T}^d$ , on part  $\partial \mathcal{S}_F$  of the surface, and in terms of displacements,  $\vec{u} = \vec{u}^d$ , on the complementary part  $\partial \mathcal{S}_u$  of the surface.

the constitutive laws which characterize the physics of the medium in terms of the relations between stress and strain; these will be studied in Chapters 4, 5 and 6.

In practice, the assumption of small strains can be applied as long as the strain modulus remains smaller than the order of magnitude of the precision with which the computations are to be done, i.e. when:

$$|\epsilon| < (2-5) \times 10^{-2}.$$

This will be the case for most of the topics treated in this book.

### Finite deformations with large strains and large displacements

When the assumption of small strains and small displacements is untenable it is necessary to introduce new concepts regarding the geometry of deformation, which are described in detail in the works cited in the references of this chapter; here, they are only summarized.

Let  $M$  be a material point in the current configuration of the body, at a time  $t$ , represented by Lagrangian variables with respect to the initial configuration  $M_0$  at  $t_0$  (Fig. 2.3):

$$M = M(M_0, t_0, t)$$

and let  $\mathcal{F}$  be the transformation which carries  $M_0$  to  $M$  by a vector  $\vec{x}$ :

$$\vec{x}(M) = \mathcal{F}(\vec{x}(M_0), t_0, t).$$

The deformation may be expressed by means of the linear tangential transformation which expresses the transformation in the vicinity of  $M_0$  and  $M$ :

$$d\vec{x}(M) = \mathbf{F} \cdot d\vec{x}(M_0)$$

where  $\mathbf{F}$  is the deformation gradient tensor of  $\mathcal{F}$

$$\mathbf{F} = \partial \mathcal{F} / \partial \vec{x}(M_0).$$

If we chose the reference state as the current configuration in  $M$ , we are led to characterize the deformation by the Almansi–Euler strain tensor. If we choose the reference state to be the initial configuration in  $M_0$ , we may then

introduce the Green-Lagrange tensor  $\Delta$  by means of the following definition:

$$\Delta = \frac{1}{2}(\mathbf{F}^T \cdot \mathbf{F} - \mathbf{1})$$

where a dot denotes the contracted product of the tensors on one index, and where  $\mathbf{1}$  is the unit second order tensor.

An important aspect of large deformations, which concerns the constitutive laws for elastoplastic (or viscoelastic) behaviour, is linked to the separation of (total) strain into elastic (reversible) and inelastic (irreversible) strains. The rule of partition used for small strain  $\epsilon = \epsilon^e + \epsilon^p$  (see Section 4.1) can only be generalized for large deformations through the introduction of a relaxed intermediate configuration. Fig. 2.3 gives a schematic representation of the decomposition of  $\mathbf{F}$  into an inelastic deformation  $\mathbf{P}$  between the initial and intermediate configuration, and an elastic deformation  $\mathbf{E}$  between the intermediate configuration and the current one:

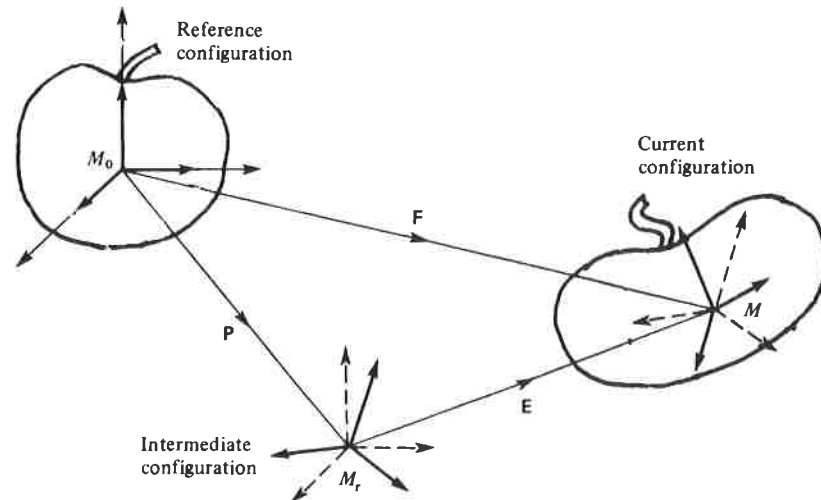
$$\mathbf{F} = \mathbf{E} \cdot \mathbf{P}.$$

The rate of deformation is defined with respect to the current configuration:

$$\mathbf{D} = \frac{1}{2}[\overrightarrow{\text{grad}} \vec{v} + (\overrightarrow{\text{grad}} \vec{v})^T]$$

where  $\vec{v}$  is the velocity vector of the material point. It can be decomposed in

Fig. 2.3. Initial ( $M_0$ ), current ( $M$ ) and relaxed ( $M_r$ ) configurations in finite deformation.



the additive fashion:

$$\mathbf{D} = \mathbf{D}^e + \mathbf{D}^p.$$

In contrast, the total strain tensor  $\Delta$  cannot be as simply decomposed as the sum of an elastic strain tensor (defined with respect to the relaxed configuration):

$$\Delta^e = \frac{1}{2}(\mathbf{E}^T \mathbf{E} - \mathbf{1})$$

and an inelastic strain tensor (defined with respect to the initial configuration):

$$\Delta^p = \frac{1}{2}(\mathbf{P}^T \mathbf{P} - \mathbf{1}).$$

The geometrical aspects related to large deformations will not be discussed in this book. However, it should be remembered that the laws developed here can be extended to large deformations, but only at the cost of a very complicated presentation. Table 2.1 indicates schematically the correspondences which exist between the variables and the equations in terms of the velocity vector in the current configuration and in terms of small and large strains. The great similarity between the first and the second columns should be noted: the quantities in the first column are effectively the derivatives with respect to time of those in the second one. As indicated in the table, the treatment of large deformations often requires the use of stress tensors different from the Cauchy stress tensor  $\sigma$  defined with respect to the current configuration. The equations of motion (and of equilibrium) given in the preceding paragraph in terms of Eulerian variables (current configuration) remain valid. They can, however, be replaced by those in Lagrangian variables, by introducing the second Piola-Kirchhoff stress tensor and using the initial mass density.

In the one-dimensional case, we often use the true strain  $\epsilon_v$  and the true stress  $\sigma_v$ , defined as Eulerian variables in the current configuration. Assuming homogeneous strain, the true strain can be expressed as:

$$d\epsilon_v = dl/l = dx(M)/x(M), \quad \text{or} \quad \epsilon_v = \int_{x_0(M_0)}^{x(M)} (dx/x),$$

and by introducing the expression for small strains with respect to the initial configuration  $\epsilon = (x - x_0)/x_0$ , we obtain:

$$\bullet \quad \epsilon_v = \ln(l/l_0) = \ln(x/x_0) = \ln(1 + \epsilon).$$

The Cauchy stress  $\sigma_v$  used until now is the true stress. We may only consider it as identical to the nominal stress  $\sigma = F/S_0$  (which happens to be the first

Table 2.1. Strain and displacement variables

Description in terms of velocity	Description in terms of small deformations	Description in terms of finite deformations
Eulerian variables $M = M(x, t)$	Eulerian variables = Lagrangian variables	Lagrangian variables $M = M(M_0, t_0, t)$
Current configuration	Current configuration = initial configuration	Initial configuration
Velocity $\vec{v}$	Displacement $\vec{u}$	Deformation $x(M) = \mathcal{F}(x(M_0), t_0, t)$
Velocity gradient $\overrightarrow{\text{grad}} \vec{v}$	Displacement gradient $\overrightarrow{\text{grad}} \vec{u}$	(Linear tangent) Deformation gradient $\mathbf{F} = \partial \mathcal{F} / \partial x(M_0)$
Deformation rate tensor $\mathbf{D} = \frac{1}{2}[\overrightarrow{\text{grad}} \vec{v} + (\overrightarrow{\text{grad}} \vec{v})^T]$	Strain tensor $\boldsymbol{\varepsilon} = \frac{1}{2}[\overrightarrow{\text{grad}} \vec{u} + (\overrightarrow{\text{grad}} \vec{u})^T]$	Green-Lagrange tensor $\boldsymbol{\Delta} = \frac{1}{2}(\mathbf{F}^T \mathbf{F} - \mathbf{1})$
Elastic strain rate tensor $\mathbf{D}^e$	Elastic strain tensor $\boldsymbol{\varepsilon}^e$	Elastic deformation $\mathbf{E}$ with reference to a relaxed configuration Elastic strain tensor $\boldsymbol{\Delta}^e = \frac{1}{2}(\mathbf{E}^T \mathbf{E} - \mathbf{1})$
Inelastic strain rate tensor $\mathbf{D}^p$	Inelastic strain tensor $\boldsymbol{\varepsilon}^p$	Inelastic deformation $\mathbf{P}$ Inelastic strain tensor $\boldsymbol{\Delta}^p = \frac{1}{2}(\mathbf{P}^T \mathbf{P} - \mathbf{1})$
Partition $\mathbf{D} = \mathbf{D}^e + \mathbf{D}^p$	Partition $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p$	Decomposition $\mathbf{F} = \mathbf{E} \cdot \mathbf{P}$ $\boldsymbol{\Delta} \neq \boldsymbol{\Delta}^e + \boldsymbol{\Delta}^p$
	$\dot{\boldsymbol{\varepsilon}} = \mathbf{D}$ , $\dot{\boldsymbol{\varepsilon}}^e = \mathbf{D}^e$ , $\dot{\boldsymbol{\varepsilon}}^p = \mathbf{D}^p$	$\dot{\boldsymbol{\Delta}} \neq \mathbf{D}$ , $\dot{\boldsymbol{\Delta}}^e \neq \mathbf{D}^e$ , $\dot{\boldsymbol{\Delta}}^p \neq \mathbf{D}^p$
Cauchy stress tensor $\boldsymbol{\sigma}$	Cauchy stress tensor $\boldsymbol{\sigma}$	First Piola-Kirchhoff stress tensor $\mathbf{S} = \det(\mathbf{F}) \boldsymbol{\sigma} \cdot \mathbf{F}^{T-1}$ Second Piola-Kirchhoff stress tensor $\mathbf{S}^* = \mathbf{F}^{-1} \cdot \mathbf{S}$

Piola-Kirchhoff stress) in the small strain case. Otherwise, it is necessary to take into account the change of the cross-sectional area:

$$S = S_0(1 - \nu^* \varepsilon)^2 \approx S_0(1 - 2\nu^* \varepsilon)$$

where  $\nu^*$  is the coefficient of elastic or elastoplastic contraction. As a first approximation, we may take  $\nu^* = \frac{1}{2}$  for elastoplastic cases, and write:

$$\bullet \quad \sigma_v = \frac{F}{S_0(1 - \nu^* \varepsilon)^2} \approx \sigma(1 + \varepsilon) \simeq \sigma e^{\varepsilon_v}$$

## 2.2.4 Tensorial representation: invariants

In writing three-dimensional constitutive laws, the assumption of isotropic behaviour leads to the use of invariants of the stress tensor  $\boldsymbol{\sigma}$  and the strain tensor  $\boldsymbol{\varepsilon}$  and also of their deviators defined by:

$$\begin{aligned} \boldsymbol{\sigma}' &= \boldsymbol{\sigma} - \frac{1}{3} \text{Tr}(\boldsymbol{\sigma}) \mathbf{1} & \text{or} & & \sigma'_{ij} &= \sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij} \\ \boldsymbol{\varepsilon}' &= \boldsymbol{\varepsilon} - \frac{1}{3} \text{Tr}(\boldsymbol{\varepsilon}) \mathbf{1} & \text{or} & & \varepsilon'_{ij} &= \varepsilon_{ij} - \frac{1}{3} \varepsilon_{kk} \delta_{ij} \end{aligned}$$

These second order tensors possess three basic invariants defined by three independent scalar functions. Mathematically, we may define the three invariants as the three coefficients of the following cubic equation in  $x$ :

$$\det(\boldsymbol{\sigma} - x \mathbf{1}) = 0.$$

This equation is called the characteristic equation. The three invariants are:

$$\begin{aligned} \Sigma_I &= \text{Tr}(\boldsymbol{\sigma}) = \sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33} \\ \Sigma_{II} &= \frac{1}{2}[(\text{Tr}(\boldsymbol{\sigma}))^2 - \text{Tr}(\boldsymbol{\sigma}^2)] = \frac{1}{2}(\sigma_{ii}^2 - \sigma_{ij} \sigma_{ij}) \\ &= \sigma_{11} \sigma_{22} + \sigma_{22} \sigma_{33} + \sigma_{33} \sigma_{11} - \sigma_{23}^2 - \sigma_{31}^2 - \sigma_{12}^2 \\ \Sigma_{III} &= \det(\boldsymbol{\sigma}). \end{aligned}$$

In practice, we may prefer to use the following invariants:

$$\begin{aligned} \sigma_I &= \Sigma_I = \text{Tr}(\boldsymbol{\sigma}) \\ \sigma_{II} &= \frac{1}{2} \text{Tr}(\boldsymbol{\sigma}^2) = \frac{1}{2} \sigma_{ij} \sigma_{ij} \\ \sigma_{III} &= \frac{1}{3} \text{Tr}(\boldsymbol{\sigma}^3) = \frac{1}{3} \sigma_{ij} \sigma_{jk} \sigma_{ki}. \end{aligned}$$

In the same way, we can define the invariants of the strain tensor:

$$\begin{aligned} \varepsilon_I &= \text{Tr}(\boldsymbol{\varepsilon}) = \varepsilon_{ii} \\ \varepsilon_{II} &= \frac{1}{2} \text{Tr}(\boldsymbol{\varepsilon}^2) = \frac{1}{2} \varepsilon_{ij} \varepsilon_{ij} \\ \varepsilon_{III} &= \frac{1}{3} \text{Tr}(\boldsymbol{\varepsilon}^3) = \frac{1}{3} \varepsilon_{ij} \varepsilon_{jk} \varepsilon_{ki}. \end{aligned}$$

In plasticity, we often use the invariants of the deviatoric stress and deviatoric plastic strain, defined in the same fashion as above:  $s_{II}$ ,  $s_{III}$ ,  $e_{pII}$ ,  $e_{pIII}$  (since  $s_I = e_{pI} = 0$ ). For convenience we will use functions of these invariants which represent concrete, physical quantities. It is practical to choose homogeneous invariants that can be identified in the simple tension case with the first component of the tensor. We thus define:

For stresses:  
the hydrostatic stress

$$J_1(\boldsymbol{\sigma}) = \sigma_1 = 3\sigma_H$$



the equivalent stress in the sense of von Mises

$$J_2(\boldsymbol{\sigma}) = (3s_{II})^{1/2} = \sigma_{eq}$$

the third homogeneous invariant of a uniaxial state of stress

$$J_3(\boldsymbol{\sigma}) = (\frac{27}{2}s_{III})^{1/3}.$$

For strains:

the volumetric strain

$$J_1(\boldsymbol{\varepsilon}) = \varepsilon_I = 3\varepsilon_H$$

the equivalent inelastic strain in the sense of von Mises

$$I_2(\boldsymbol{\varepsilon}^p) = (\frac{4}{3}e_{pII})^{1/2} = \varepsilon_{peq}.$$

The invariants defined in this way are sufficient for writing constitutive laws of isotropic materials. In the anisotropic case, the tensorial arguments, characteristic of the anisotropy must also be used. For example, let  $\mathbf{f}$  be an anisotropic function relating the stress tensor  $\boldsymbol{\sigma}$  to the strain tensor  $\boldsymbol{\varepsilon}$  for a general anisotropic behaviour:

$$\boldsymbol{\varepsilon} = \mathbf{f}(\boldsymbol{\sigma}).$$

If  $\vec{a}_1, \vec{a}_2, \vec{a}_3$  are the base vectors of an orthonormal set of axes in a reference configuration, we may write:

$$\boldsymbol{\varepsilon} = \mathbf{f}(\boldsymbol{\sigma}) = \mathbf{g}(\boldsymbol{\sigma}, \vec{a}_1, \vec{a}_2, \vec{a}_3).$$

To each rotation  $\vec{q}\vec{a}_1, \vec{q}\vec{a}_2, \vec{q}\vec{a}_3$  of the reference axes, we may associate the transformations:

$$\vec{q} \cdot \boldsymbol{\sigma} \cdot \vec{q}^T \quad \text{and} \quad \vec{q} \cdot \boldsymbol{\varepsilon} \cdot \vec{q}^T$$

which shows that  $\mathbf{g}$  is an isotropic function which can then be written as:

$$\boldsymbol{\varepsilon} = C_p \mathbf{G}_p.$$

$\mathbf{G}_p$  are the generating tensors. They number 21 in the general case, but can be expressed as linear functions of a smaller number of independent generators.  $C_p$  are coefficients which are functions of the 21 invariants associated with the tensor  $\boldsymbol{\sigma}$  and with the three vectors  $\vec{a}_1, \vec{a}_2, \vec{a}_3$ . These invariant coefficients can also be expressed as functions of a smaller number of independent invariants.

## 2.3 Fundamental statements of thermodynamics

### 2.3.1 Conservation laws; first principle

The equilibrium equation established in the preceding section can be interpreted as a form of the law of conservation of momentum:

$$(d/dt) \int_{\mathcal{Q}} \rho \vec{v} dV - \int_{\partial \mathcal{Q}} \vec{T} dS = \int_{\mathcal{Q}} \vec{f} dV$$

as it can be deduced by applying the divergence theorem with:

$$\vec{T} = \boldsymbol{\sigma} \cdot \vec{n} \quad \text{and} \quad d\vec{v}/dt = \vec{\gamma}$$

(where  $d/dt$  denotes the material derivative).

In the same way, the symmetry of the stress tensor,  $\sigma_{ij} = \sigma_{ji}$ , which results directly from the principle of virtual power, can be considered as a consequence of the equation of dynamic moment balance.

The second law of conservation is that of conservation of mass, which with the notation introduced, can be expressed as:

$$(d/dt) \int_{\mathcal{Q}} \rho dV = 0.$$

#### First principle of thermodynamics

This constitutes the third important law of conservation: the conservation of energy. Let us again consider a domain  $\mathcal{Q}$  with a boundary  $\partial \mathcal{Q}$  in the interior of the material medium  $\mathcal{S}$ . Let  $E$  be its internal energy, and  $e$  the specific internal energy:

$$E = \int_{\mathcal{Q}} \rho e dV.$$

Let  $K$  be its kinetic energy:

$$K = \frac{1}{2} \int_{\mathcal{Q}} \rho \vec{v} \cdot \vec{v} dV.$$

Let  $Q$  be the rate at which heat is received by the region  $\mathcal{Q}$ ; it consists of two terms, the heat generated within the volume  $\mathcal{Q}$  by the external agencies (inductive heating, for example), and the heat received by conduction through the boundary  $\partial \mathcal{Q}$  of  $\mathcal{Q}$ :

$$Q = \int_{\mathcal{Q}} r dV - \int_{\partial \mathcal{Q}} \vec{q} \cdot \vec{n} dS$$

where  $r$  is the volumetric density of the internal heat production,  $\vec{q}$  is the heat flux vector and  $\vec{n}$  is the outward unit normal to  $\partial\mathcal{D}$ . Let  $P_{(x)}$  be the actual power of the external forces:

$$P_{(x)} = \int_{\mathcal{D}} \vec{f} \cdot \vec{v} \, dV + \int_{\partial\mathcal{D}} \vec{T} \cdot \vec{v} \, dS.$$

The first principle of thermodynamics is expressed by:

$$\bullet \quad (d/dt)(E + K) = P_{(x)} + Q \quad \forall \mathcal{D}$$

or

$$(d/dt) \int_{\mathcal{D}} \rho(e + \frac{1}{2} \vec{v} \cdot \vec{v}) \, dV = \int_{\mathcal{D}} (\vec{f} \cdot \vec{v} + r) \, dV + \int_{\partial\mathcal{D}} (\vec{T} \cdot \vec{u} - \vec{q} \cdot \vec{n}) \, dS.$$

From this it is possible to derive a local expression which involves only the power of internal forces and the heat received. To do so, we write the second axiom of the principle of virtual power by treating the real motion as the virtual one:

$$P_{(x)} = P_{(a)} - P_{(i)}$$

Now, noting that

$$P_{(a)} = \int_{\mathcal{D}} \rho \vec{\gamma} \cdot \vec{v} \, dV = \frac{d}{dt} \frac{1}{2} \int_{\mathcal{D}} \rho \vec{v} \cdot \vec{v} \, dV = \frac{dK}{dt}$$

we obtain

$$\frac{dE}{dt} + \frac{dK}{dt} = \frac{dK}{dt} - P_{(i)} + Q.$$

Since

$$Q = \int_{\mathcal{D}} r \, dV - \int_{\partial\mathcal{D}} \vec{q} \cdot \vec{n} \, dS = \int_{\mathcal{D}} r \, dV - \int_{\mathcal{D}} \operatorname{div} \vec{q} \, dV$$

we have

$$(d/dt) \int_{\mathcal{D}} \rho e \, dV = \int_{\mathcal{D}} \boldsymbol{\sigma} : \mathbf{D} \, dV + \int_{\mathcal{D}} r \, dV - \int_{\mathcal{D}} \operatorname{div} \vec{q} \, dV.$$

This identity is valid for any region  $\mathcal{D}$ . Hence, according to the fundamental lemma it implies that:

$$\rho(de/dt) = \boldsymbol{\sigma} : \mathbf{D} + r - \operatorname{div} \vec{q}$$

or, under the assumption of small strains:

$$\bullet \quad \rho \dot{e} = \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} + r - \operatorname{div} \vec{q}$$

or

$$\rho \dot{e} = \sigma_{ij} \dot{\epsilon}_{ij} + r - q_{i,i}.$$

### 2.3.2 Entropy; second principle

In addition to the internal energy and the rate of heating it is necessary to introduce two more new variables: temperature and entropy. We assume that it is possible to represent the temperature by a scalar field of positive values defined at each instant  $t$  and at all points of the domain  $\mathcal{D}$  under study :  $T(M, t)$ . Entropy expresses a variation of energy associated with a variation in the temperature. It is defined for a domain  $\mathcal{D}$  by means of specific entropy  $s$  per unit mass:

$$S = \int_{\mathcal{D}} \rho s \, dV.$$

#### The second principle

The second principle postulates that the rate of entropy production is always greater than or equal to the rate of heating divided by the temperature:

$$\bullet \quad \frac{dS}{dt} \geq \int_{\mathcal{D}} \frac{r}{T} \, dV - \int_{\partial\mathcal{D}} \frac{\vec{q} \cdot \vec{n}}{T} \, dS \quad \forall \mathcal{D}$$

where  $d/dt$  denotes the material derivative. Use of the divergence theorem leads to

$$\int_{\mathcal{D}} \left( \rho \frac{ds}{dt} + \operatorname{div} \frac{\vec{q}}{T} - \frac{r}{T} \right) dV \geq 0.$$

This inequality is valid for any region  $\mathcal{D}$  of the body and implies the following local form of the irreversibility of the entropy production rate:

$$\rho \frac{ds}{dt} + \operatorname{div} \frac{\vec{q}}{T} - \frac{r}{T} \geq 0.$$

The fundamental inequality containing the first and second principles is obtained by replacing  $r$  with the expression resulting from the equation of conservation of energy:

$$\rho \frac{ds}{dt} + \operatorname{div} \frac{\vec{q}}{T} - \frac{1}{T} \left( \rho \frac{de}{dt} - \boldsymbol{\sigma} : \mathbf{D} + \operatorname{div} \vec{q} \right) \geq 0.$$

Noting that:

$$\operatorname{div} \frac{\vec{q}}{T} = \frac{\operatorname{div} \vec{q}}{T} - \frac{\vec{q} \cdot \overrightarrow{\operatorname{grad}} T}{T^2}$$

and multiplying by  $T > 0$ , we obtain:

$$\rho \left( T \frac{ds}{dt} - \frac{de}{dt} \right) + \sigma : \mathbf{D} - \vec{q} \cdot \frac{\overrightarrow{\operatorname{grad}} T}{T} \geq 0.$$

The Clausius–Duhem inequality is obtained by introducing a new variable, the specific free energy  $\Psi$  defined by:

$$\Psi = e - Ts.$$

Differentiating this, we obtain:

$$\frac{d\Psi}{dt} = \frac{de}{dt} - T \frac{ds}{dt} - s \frac{dT}{dt} \quad \text{or} \quad T \frac{ds}{dt} - \frac{de}{dt} = - \left( \frac{d\Psi}{dt} + s \frac{dT}{dt} \right)$$

which, when substituted in the first term of the fundamental inequality, yields:

$$\sigma : \mathbf{D} - \rho \left( \frac{d\Psi}{dt} + s \frac{dT}{dt} \right) - \vec{q} \cdot \frac{\overrightarrow{\operatorname{grad}} T}{T} \geq 0.$$

For small perturbations, the above may be written as:

$$\bullet \quad \sigma : \dot{\mathbf{e}} - \rho(\dot{\Psi} + s\dot{T}) - \vec{q} \cdot \frac{\overrightarrow{\operatorname{grad}} T}{T} \geq 0$$

or

$$\sigma_{ij} \dot{\epsilon}_{ij} - \rho(\dot{\Psi} + s\dot{T}) - q_i \frac{T_{,i}}{T} \geq 0.$$

## 2.4 Method of local state

In order to avoid any confusion in the reader's mind, it is time to summarize the variables used to describe the thermomechanical behaviour of solids. The concept of thermodynamic potential will clarify everything! However, before giving its definition, a choice must be made with regard to the nature of the variables. In this choice lie both the weakness and the richness of the method of phenomenological thermodynamics: weakness because the choice is partly subjective and results in different models depending on the

inclination of the authors, and richness because it allows the formulation of the theories to be adopted to the study of one or more phenomena, either coupled or uncoupled, depending on the intended use.

### 2.4.1 State variables

The method of local state postulates that the thermodynamic state of a material medium at a given point and instant is completely defined by the knowledge of the values of a certain number of variables at that instant, which depend only upon the point considered. Since the time derivatives of these variables are not involved in the definition of the state, this hypothesis implies that any evolution can be considered as a succession of equilibrium states. Therefore, ultrarapid phenomena for which the time scales of the evolutions are of the same order as the relaxation time for a return to thermodynamic equilibrium (atomic vibrations) are excluded from this theory's field of application. Physical phenomena can be described with a precision which depends on the choice of the nature and the number of state variables. The processes defined in this way will be thermodynamically admissible if, at any instant of the evolution, the Clausius–Duhem inequality is satisfied. The state variables, also called thermodynamic or independent variables, are the observable variables and the internal variables.

#### Observable variables

The formalism of continuum mechanics and thermodynamics as developed above requires the existence of a certain number of state variables; these are the observable variables:

the temperature  $T$

the total strain  $\mathbf{\epsilon}$  (assuming small strains).

We limit ourselves to the two observable variables as they are the only ones which occur in elasticity, viscoelasticity, plasticity, viscoplasticity, damage and fracture phenomena. For reversible (or elastic) phenomena, at every instant of time, the state depends uniquely on these variables. For example, the reversible power is defined with the help of the associated stress  $\sigma$  as:

$$\Phi_e = \sigma : \dot{\mathbf{e}}.$$

## Internal variables

For dissipative phenomena, the current state also depends on the past history which is represented, in the method of local state, by the values at each instant of other variables called internal variables.

Plasticity and viscoplasticity require the introduction of the plastic (or viscoplastic) strain as a variable. For small strains, the plastic strain  $\epsilon^p$  is the permanent strain associated with the relaxed configuration. This configuration is obtained by 'elastic unloading', leading to the additive strain decomposition:

$$\epsilon = \epsilon^p + \epsilon^e.$$

The two internal variables related to the above decomposition may formally be defined as: the plastic strain  $\epsilon^p$ , and the thermoelastic strain  $\epsilon^e$  (including, as well, the possibility of thermal dilatation).

Other phenomena such as hardening, damage, fracture, require the introduction of other internal variables of a less obvious nature. These represent the internal state of matter (density of dislocations, crystalline microstructure, configuration of microcracks and cavities, etc.) and there are no means of measuring them by direct observation. They do not appear explicitly either in the conservation laws or in the statement of the second principle of thermodynamics. They are called internal variables, but in fact, they are state variables which will be treated as observable ones.

There is no objective way of choosing the nature of the internal variables best suited to the study of a phenomenon. The choice is dictated by experience, physical feeling and very often by the type of application. They will be defined in the different chapters as the need arises. For their general study, they will be denoted by  $V_1, V_2, \dots, V_k, \dots$ ;  $V_k$  representing either a scalar or a tensorial variable.

## 2.4.2 Thermodynamic potential, state laws

Once the state variables have been defined, we postulate the existence of a thermodynamic potential from which the state laws can be derived. Without entering into the details, let us say that the specification of a function with a scalar value, concave with respect to  $T$ , and convex with respect to other variables, allows us to satisfy *a priori* the conditions of thermodynamic stability imposed by the inequalities that can be derived from the second principle. It is possible to work in an equivalent way with different potentials. Here we choose the free specific energy potential  $\Psi$ ,

which depends on observable state variables and internal variables:

$$\Psi = \Psi(\epsilon, T, \epsilon^e, \epsilon^p, V_k).$$

In elastoplasticity (or viscoplasticity) the strains appear only in the form of their additive decomposition  $\epsilon - \epsilon^p = \epsilon^e$ , so that:

$$\Psi = \Psi((\epsilon - \epsilon^p), T, V_k) = \Psi(\epsilon^e, T, V_k)$$

which shows that:

$$\partial\Psi/\partial\epsilon^e = \partial\Psi/\partial\epsilon = -\partial\Psi/\partial\epsilon^p.$$

We now use the Clausius–Duhem inequality with:

$$\dot{\Psi} = \frac{\partial\Psi}{\partial\epsilon^e}:\dot{\epsilon}^e + \frac{\partial\Psi}{\partial T}\dot{T} + \frac{\partial\Psi}{\partial V_k}\dot{V}_k.$$

to obtain:

$$\left(\sigma - \rho \frac{\partial\Psi}{\partial\epsilon^e}\right):\dot{\epsilon}^e + \sigma:\dot{\epsilon}^p - \rho\left(s + \frac{\partial\Psi}{\partial T}\right)\dot{T} - \rho \frac{\partial\Psi}{\partial V_k}\dot{V}_k - \frac{\bar{q}}{T} \cdot \overrightarrow{\text{grad}} T \geq 0.$$

A classical hypothesis permits us to cancel some terms in this inequality independently. We may imagine, first of all, an elastic deformation taking place at constant ( $\dot{T}=0$ ) and uniform ( $\overrightarrow{\text{grad}} T=0$ ) temperature which alters neither the plastic strain ( $\dot{\epsilon}^p=0$ ) nor the internal variables ( $\dot{V}_k=0$ ). For this to happen, it is necessary to consider that the elastic deformations can occur at a time scale higher than those which would question the validity of the hypothesis of local state, and lower than those of dissipative phenomena. Since the Clausius–Duhem inequality holds regardless of any particular  $\dot{\epsilon}^e$ , it necessarily follows that:

$$\sigma - \rho(\partial\Psi/\partial\epsilon^e) = 0.$$

Assuming this equality to hold, we now imagine a thermal deformation in which  $\dot{\epsilon}^p=0$ ,  $\dot{V}_k=0$ ,  $\overrightarrow{\text{grad}} T=0$ . Then, since  $T$  is arbitrary, it follows that:

$$s + \partial\Psi/\partial T = 0.$$

These expressions define the thermoelastic laws:

- $\sigma = \rho(\partial\Psi/\partial\epsilon^e),$
- $s = -\partial\Psi/\partial T.$

We note that:

$$\sigma = \rho(\partial\Psi/\partial\epsilon^e) = \rho(\partial\Psi/\partial\epsilon) = -\rho(\partial\Psi/\partial\epsilon^p)$$

which shows that the stress is a variable associated with the elastic strain, with the total strain, and with the plastic strain (with a minus sign).

In an analogous manner, we define the thermodynamic forces associated with the internal variables by:

$$A_k = \rho(\partial\Psi/\partial V_k).$$

These relations constitute the state laws:

the entropy  $s$  and the stress tensor  $\sigma$  having been defined elsewhere, the specification of the thermodynamic potential  $\Psi(\epsilon^e, T, V_k)$  furnishes the coupled or uncoupled theories of thermo-elasticity;

in contrast, the variables  $A_k$  associated with the internal variables, which have not yet been introduced, are defined by the specification of the thermodynamic potential  $\Psi(\dots, V_k)$ ;

$s$ ,  $\sigma$  and  $A_1, A_2, \dots, A_k$  constitute the associated variables. The vector formed by these variables is the gradient of the function  $\Psi$  in the space of the variables  $T, \epsilon^e, V_k$ . This vector is normal to the surface  $\Psi = \text{constant}$ .

The associated variables form a set of normal variables in duality with the observable and internal state variables. Table 2.2 summarizes the set of variables introduced in this way.

#### 2.4.3 Dissipation, complementary laws

As we have seen, the thermodynamic potential allows us to write relations between observable state variables and associated variables. However, for internal variables it allows only the definition of their associated variables. In order to describe the dissipation process, mainly the evolution of the internal variables, a complementary formalism is needed. This is precisely the objective of the dissipation potentials.

##### *Intrinsic dissipation, thermal dissipation*

Taking into account the state laws and putting  $\vec{g} = \overrightarrow{\text{grad}} T$ , the Clausius-Duhem inequality can be reduced to express the fact that dissipation is necessarily positive:

$$\bullet \quad \Phi = \sigma : \dot{\epsilon}^p - A_k \dot{V}_k - \vec{g} \cdot \vec{q}/T \geq 0.$$

We note that  $\Phi$  is a sum of the products of the force variables or dual

Table 2.2. *Thermodynamic variables*

State variables		
Observable	Internal	Associated variables
$\epsilon$		$\sigma$
$T$		$s$
	$\epsilon^e$	$\sigma$
	$\epsilon^p$	$-\sigma$
	$V_k$	$A_k$

variables  $\sigma, A_k, \vec{g}$  with the respective flux variables  $\dot{\epsilon}^p, -\dot{V}_k, -\vec{q}/T$ . The sum of the first two terms:

$$\Phi_1 = \sigma : \dot{\epsilon}^p - A_k \dot{V}_k$$

is called the intrinsic dissipation (or mechanical dissipation). It consists of plastic dissipation plus the dissipation associated with the evolution of the other internal variables; it is generally dissipated by the volume element in the form of heat. The last term:

$$\Phi_2 = -\vec{g} \cdot \frac{\vec{q}}{T} = -\frac{\vec{q}}{T} \cdot \overrightarrow{\text{grad}} T$$

is the thermal dissipation due to the conduction of heat.

##### *Dissipation potential*

In order to define the complementary laws related to the dissipation process, we postulate the existence of a dissipation potential (or pseudo-potential) expressed as a continuous and convex scalar valued function of the flux variables, wherein the state variables may appear as parameters:

$$\varphi(\dot{\epsilon}^p, \dot{V}_k, \vec{q}/T).$$

This potential is a positive convex function with a zero value at the origin of the space of the flux variables,  $\dot{\epsilon}^p, \dot{V}_k, \vec{q}/T$ . The complementary laws are then expressed by the normality property (or normal dissipativity):

$$\sigma = \frac{\partial \varphi}{\partial \dot{\epsilon}^p} \quad A_k = -\frac{\partial \varphi}{\partial \dot{V}_k} \quad \vec{g} = -\frac{\partial \varphi}{\partial (\vec{q}/T)}.$$

The thermodynamic forces are the components of the vector  $\overrightarrow{\text{grad}} \varphi$  normal to the  $\varphi = \text{constant}$  surfaces in the space of the flux variables.

In fact, the complementary laws are more easily expressed in the form of the evolution laws of flux variables as functions of dual variables. The Legendre–Fenchel transformation enables us to define the corresponding potential  $\varphi^*(\sigma, A_k, \bar{g})$ , the dual of  $\varphi$  with respect to the variables  $\dot{\epsilon}^p$ ,  $\dot{V}_k$  and  $\bar{q}/T$ . By definition:

$$\varphi^*(\sigma, A_k, \bar{g}) = \sup_{(\dot{\epsilon}^p, \dot{V}_k, \bar{q}/T)} ((\sigma : \dot{\epsilon}^p - A_k \dot{V}_k - \bar{g} \cdot \bar{q}/T) - \varphi(\dot{\epsilon}^p, \dot{V}_k, \bar{q}/T)).$$

The transformation, written in a slightly clumsy form, is illustrated graphically in Fig. 2.4 in which only one variable has been retained.

It can be shown that, if the function  $\varphi^*$  is differentiable, the normality property is preserved for the variables  $\dot{\epsilon}^p$ ,  $-\dot{V}_k$ ,  $-\bar{q}/T$ , and the complementary laws of evolution can then be written as:

- $\dot{\epsilon}^p = \partial \varphi^* / \partial \sigma$ ,
- $-\dot{V}_k = \partial \varphi^* / \partial A_k$ ,
- $-\frac{\bar{q}}{T} = \partial \varphi^* / \partial \bar{g}$ .

Let us note once more the properties that the potentials  $\varphi$  and  $\varphi^*$  must possess for the automatic satisfaction of the second principle of thermodynamics: they must be nonnegative, convex functions, zero at the origin: ( $\sigma = A_k = g = 0$ ). Later, we will generally use the potential  $\varphi^*$  and the rela-

Fig. 2.4. Construction of the graph of a potential  $\varphi^*(\sigma)$ , dual of  $\varphi(\dot{\epsilon}^p)$ , by the Legendre–Fenchel transformation.

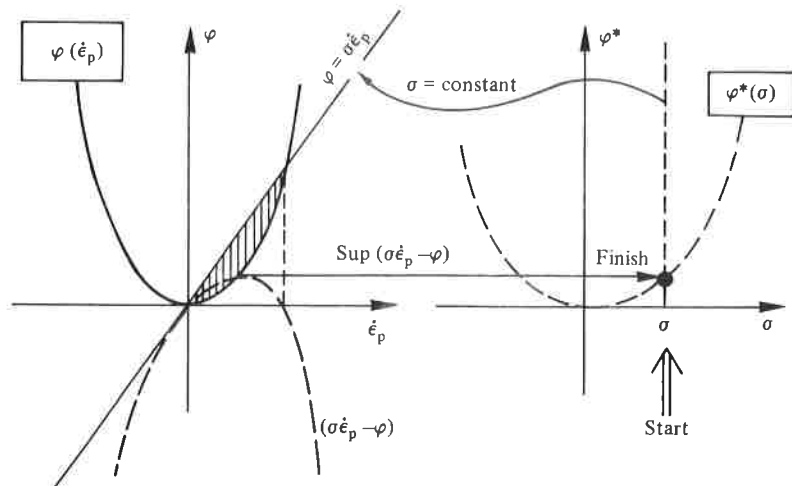


Table 2.3. Dissipation variables

Flux variables	Dual variables
$\dot{\epsilon}^p$	$\sigma$
$-\dot{V}_k$	$A_k$
$-\bar{q}/T$	$\bar{g} = \text{grad } T$

tions expressing the evolution of the flux variables. It should be noted that the normality rule is sufficient to ensure the satisfaction of the second principle of thermodynamics, but it is not a necessary condition. This rule applies to generalized standard materials. A standard material is defined as that for which only the first of the above three rules,  $\dot{\epsilon}^p = \partial \varphi^* / \partial \sigma$ , applies. This first relation yields the plasticity or viscoplasticity laws. The second equation expresses the evolution laws of the internal variables, and the third one leads to the Fourier law of thermostatics. Table 2.3 provides a summary of the dissipation variables.

The whole problem of modelling a phenomenon lies in the determination of the analytical expressions for the thermodynamic potential  $\Psi$  and for the dissipation potential  $\varphi$  or its dual  $\varphi^*$ , and their identification in characteristic experiments. In fact the values of  $\varphi$  or  $\varphi^*$  are almost impossible to measure as they represent an energy usually dissipated as heat. The flux variables and the dual variables are quite easy to measure and it is on their values that the modelling and identification are based. The complementary laws of evolution are therefore directly identified but the dissipation potential is used as a guideline for writing their analytical expression.

It should be noted that one can generalize the dissipation potentials by including the state variables themselves as parameters. The above development is not modified at all. The dissipation potentials are then written as:

$$\begin{aligned} &\varphi(\dot{\epsilon}^p, \dot{V}_k, \bar{q}/T; \epsilon^e, T, V_k) \\ &\varphi^*(\underbrace{\sigma, A_k, \bar{g}}_{\text{variable}}, \underbrace{\epsilon^e, T, V_k}_{\text{parameters}}). \end{aligned}$$

#### Onsager's symmetry relations

A first simplification consists in assuming that the function  $\varphi^*$  is of a positive-definite quadratic form in terms of the dual variables. Then denoting the flux variables by  $\dot{V}_\alpha$  and the dual variables by  $A_\alpha$  the potential

$\varphi^*$  may be written

$$\varphi^* = \frac{1}{2} C_{\alpha\beta}(\mathbf{\epsilon}^e, T, V_k) A_\alpha A_\beta.$$

Consequently, under such circumstances, every complementary law of evolution is linear with respect to the corresponding dual variables:

$$\dot{V}_\alpha = C_{\alpha\beta} A_\beta.$$

The matrix  $C_{\alpha\beta}$  is symmetric. This property is known as Onsager's symmetry relation.

### Decoupling of intrinsic and thermal dissipation

A second simplification consists in assuming a decoupling of intrinsic and thermal dissipations. This does not mean that the corresponding physical mechanisms are decoupled. This assumption amounts to considering the dissipation potential as the sum of two terms, one dependent on the dual variables  $\sigma, A_k$ , and the other on the variable  $\bar{g}$ :

$$\varphi^* = \varphi_1^*(\sigma, A_k) + \varphi_2^*(\bar{g})$$

and the second principle of thermodynamics is satisfied by the following inequalities respectively:

$$\Phi_1 = \sigma : \dot{\mathbf{\epsilon}}^p - A_k \dot{V}_k = \sigma : \frac{\partial \varphi_1^*}{\partial \sigma} + A_k \frac{\partial \varphi_1^*}{\partial A_k} \geq 0$$

$$\Phi_2 = -\bar{g} \cdot \frac{\vec{q}}{T} = \bar{g} \cdot \frac{\partial \varphi_2^*}{\partial \bar{g}} \geq 0.$$

Note that, since  $\varphi^*$  is convex in  $\sigma$  and  $A_k$  and passes through the origin, the first of the above inequalities is automatically satisfied. We then have:

$$\sigma : \frac{\partial \varphi_1^*}{\partial \sigma} + A_k \frac{\partial \varphi_1^*}{\partial A_k} \geq \varphi_1^* \geq 0.$$

### The phenomena of instantaneous dissipation

When the behaviour is independent of the velocities, the function  $\varphi(\dot{\mathbf{\epsilon}}^p, \dot{V}_k)$  is a positive, homogeneous function of degree 1 and its dual function  $\varphi^*$  is nondifferentiable. By extension, we write that  $\dot{\mathbf{\epsilon}}^p$  belongs to the subdifferential of  $\varphi^*$  defined by:

$$\partial \varphi_{(\sigma_0)}^* = \{ \dot{\mathbf{\epsilon}}^p / \varphi_{(\sigma_0)}^* \geq \varphi_{(\sigma_0)}^* + \dot{\mathbf{\epsilon}}^p : (\sigma - \sigma_0), \forall \sigma \}.$$

In addition, we take the convex function of the criterion  $f(\sigma, A_k)$  whose convex,  $f = 0$ , has  $\varphi^*$  as an indicator function.

$$\varphi^* = 0 \text{ if } f < 0 \rightarrow \dot{\mathbf{\epsilon}}^p = 0$$

$$\varphi^* = +\infty \text{ if } f = 0 \rightarrow \dot{\mathbf{\epsilon}}^p \neq 0.$$

A proof, not given here, allows us to assert that it is equivalent to write:

$$\dot{\mathbf{\epsilon}}^p \in \partial \varphi_{(\sigma)}^* \text{ and } \dot{\mathbf{\epsilon}}^p = \frac{\partial F}{\partial \sigma} \dot{\lambda} \text{ if } \begin{cases} f = 0 \\ \dot{f} = 0 \end{cases}$$

where  $F$  is a potential function equal to  $f$  in the case of 'associated' theories and  $\dot{\lambda}$  is a multiplier determined by the consistency condition  $\dot{f} = 0$ .

The equations describing normality have to be replaced by:

$$\dot{\mathbf{\epsilon}}^p = \dot{\lambda} (\partial F / \partial \sigma), \quad -\dot{V}_k = \dot{\lambda} (\partial F / \partial A_k)$$

or

$$\bullet \quad \dot{\mathbf{\epsilon}}^p = \dot{\lambda} \partial f / \partial \sigma, \quad -V_k = \dot{\lambda} \partial f / \partial A_k$$

## 2.5 Elements of heat

### 2.5.1 Fourier's law

The law of heat diffusion, or Fourier's law, expresses a linear relation between the heat flux vector  $\vec{q}$  and its dual variable  $\bar{g}$ . This is a direct consequence of the two simplifications introduced regarding dissipation potentials. In fact, we let:

$$\varphi_2^* = \frac{1}{2} \mathbf{C} \cdot \bar{g} \cdot \bar{g}$$

and

$$-\frac{\vec{q}}{T} = \frac{\partial \varphi_2^*}{\partial \bar{g}} = \mathbf{C} \cdot \bar{g} = \mathbf{C} \cdot \overrightarrow{\text{grad}} T.$$

If we now make the hypothesis that diffusion properties are isotropic for the material under consideration, then the tensor  $\mathbf{C}$  is reduced to a scalar tensor. Moreover, this scalar is considered to vary inversely with respect to the temperature so that it is possible to write:

$$\bullet \quad \vec{q} = -k \overrightarrow{\text{grad}} T$$

or

$$q_i = -kT, i$$

Table 2.4. Order of magnitudes of the thermodynamic properties of some materials

	Melting temperature $T_M$ °C	Density $\rho$ kg m <sup>-3</sup>	Thermal conductivity $k$ W m <sup>-1</sup> °C <sup>-1</sup>	Specific heat $C_e$ J kg <sup>-1</sup> °C <sup>-1</sup>
Aluminium alloys	600	2800	115 at 20 °C 150 at 200 °C	900 at 20 °C
Steels	1500	7800	46 at 20 °C 29 at 900 °C	460 at 20 °C 625 at 900 °C
Brass	900	8600	128 at 400 °C	450 at 400 °C
Polymers	200	1300	—	1500 at 29 °C
Concrete	2200	2000	1.2 at 20 °C	880 at 20 °C
Wood	—	500	0.15 at 20 °C	2000 at 20 °C

where  $k$  is the coefficient of thermal conductivity, a characteristic property of the material (see Table 2.4 for values pertinent to some particular materials).

### 2.5.2 Heat equation

Let us return to the equation of the conservation of energy (the first principle of thermodynamics):

$$\rho \dot{e} = \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} + r - \operatorname{div} \vec{q}$$

and replace  $\rho \dot{e}$  by the expression derived from  $e = \Psi + Ts$ :

$$\rho \dot{e} = \rho \dot{\Psi} + \rho T \dot{s} + \rho \dot{T}s$$

and  $\dot{\Psi}$  by its expression as a function of the state variables, so that:

$$\rho \dot{e} = \rho \left( \frac{1}{\rho} \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} - s \dot{T} + \frac{1}{\rho} A_k \dot{V}_k \right) + \rho T \dot{s} + \rho \dot{T}s$$

and we obtain:

$$\boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} + A_k \dot{V}_k + \rho T \dot{s} = \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} + r - \operatorname{div} \vec{q}.$$

Now, with  $s = (-\partial \Psi / \partial T)(\boldsymbol{\epsilon}^e, T, V_k)$ , we may express  $\dot{s}$  by:

$$\begin{aligned} \dot{s} &= -\frac{\partial^2 \Psi}{\partial \boldsymbol{\epsilon}^e \partial T} : \dot{\boldsymbol{\epsilon}}^e - \frac{\partial^2 \Psi}{\partial T^2} \dot{T} - \frac{\partial^2 \Psi}{\partial V_k \partial T} \dot{V}_k \\ &= -\frac{1}{\rho} \frac{\partial \boldsymbol{\sigma}}{\partial T} : \dot{\boldsymbol{\epsilon}}^e + \frac{\partial s}{\partial T} \dot{T} - \frac{1}{\rho} \frac{\partial A_k}{\partial T} \dot{V}_k. \end{aligned}$$

By introducing the specific heat defined by:

$$C = T \partial s / \partial T$$

and taking into account Fourier's law  $\vec{q} = -k \overrightarrow{\operatorname{grad}} T$  or

$$\operatorname{div} \vec{q} = -k \operatorname{div} (\overrightarrow{\operatorname{grad}} T) = -k \Delta T$$

we obtain, using  $\dot{\boldsymbol{\epsilon}}^p = \dot{\boldsymbol{\epsilon}} - \dot{\boldsymbol{\epsilon}}^e$ ,

$$\bullet \quad k \Delta T = \rho C \dot{T} - \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}^p + A_k \dot{V}_k - r - T \left( \frac{\partial \boldsymbol{\sigma}}{\partial T} : \dot{\boldsymbol{\epsilon}}^e + \frac{\partial A_k}{\partial T} \dot{V}_k \right)$$

which is the complete heat equation and where  $\Delta$  denotes the Laplacian operator.

### Heat propagation

The classical heat equation corresponds to a process:

- without variation in inelastic strains,  $\boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}^p = 0$ ;
- without variation in internal variables,  $A_k \dot{V}_k = 0$ ;
- without internal generation of heat created by the external sources,  $r = 0$ ;
- without thermomechanical coupling,  $(\partial \boldsymbol{\sigma} / \partial T) : \dot{\boldsymbol{\epsilon}}^e = 0$ ,
- $(\partial A_k / \partial T) \dot{V}_k = 0$ .

The values of these last two terms are actually negligible in most applications. The specific heat is then the specific heat at constant strain:

$$C = C_e = T(\partial s / \partial T)_e.$$

(Table 2.4 gives some specific values of  $C_e$ .) Under such conditions we may write:

$$\bullet \quad k \Delta T = \rho C_e \dot{T}.$$

### Adiabatic overheating

The complete heat equation also allows us to find the rise in temperature of a medium subjected to mechanical dissipation. To do this it is sufficient to solve:

$$\rho C \dot{T} = \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}}^p - A_k \dot{V}_k + r + k \Delta T + T \left( \frac{\partial \boldsymbol{\sigma}}{\partial T} : \dot{\boldsymbol{\epsilon}}^e + \frac{\partial A_k}{\partial T} \dot{V}_k \right).$$

The classical equation used to calculate the overheating of metallic



materials, for example during the forming process, corresponds to:

$k\Delta T = 0$ : adiabatic evolution

$r = 0$ : no internal heat production generated by external sources

$(\partial\sigma/\partial T)\dot{\epsilon}^e = 0$ ,  $(\partial A_k/\partial T)\dot{V}_k = 0$ : no thermomechanical coupling.

$A_k\dot{V}_k$  represents the nonrecoverable energy stored in the material. For metals, this is the energy of the field of the residual microstresses which accompany the increase in the dislocation density. It represents only 5–10% of the term  $\sigma:\dot{\epsilon}^p$  and is often negligible:

$$A_k\dot{V}_k \approx 0.$$

The equation of adiabatic overheating is given by:

$$\bullet \quad \rho C_\epsilon \dot{T} = \sigma:\dot{\epsilon}^p.$$

### Bibliography

- Germain P. & Muller P. *Introduction à la mécanique des milieux continus*. Masson, Paris (1980).
- Germain P. *Cours de mécanique des milieux continus*. Masson, Paris (1973).
- Valid R. *La mécanique des milieux continus et le calcul des structures*. Eyrolles, Paris (1977).
- Bamberger Y. *Mécanique de l'ingénieur, II: Milieux déformables*. Hermann (1981).
- Nemat-Nasser S. *Mechanics today*. Vol. I, II, III, IV, V. Pergamon Press, New York (1972–1980).
- Sanchez-Palencia E. *Non homogeneous media and vibration theory*. Springer Verlag, Berlin (1980).
- Truesdell C. *The elements of continuum mechanics*. Springer Verlag, Berlin (1966).
- Eringen A. C. *Mechanics of continua*. J. Wiley, New York (1967).
- Kestin J. & Rice J. R. *A critical review of thermodynamics*. Stuart Ed. Mono book (1970).

## 3

### IDENTIFICATION AND RHEOLOGICAL CLASSIFICATION OF REAL SOLIDS

*L'expérience (d'un Laboratoire), c'est l'ensemble des erreurs qu'on ne recommencera plus.*

Continuum mechanics and thermodynamics (Chapter 2) constitute the basic theoretical tools for the formulation of the physical phenomena of deformation and fracture. For fundamental and practical reasons, we model each broad class of phenomena separately. The aim of this chapter is to differentiate from a qualitative point of view and identify the most common types of material behaviour. The phenomenological method used is based on observed experimental results. We, therefore, present some basic elements on the types of tests, the machines and the modern measurement techniques likely to be used. Progress in electronics, automatic controls, digital measurements, and more recently in microprocessors has resulted in a radical transformation, especially during the 1970s, of the laboratories engaged in characterization of materials. We no longer have to be content with approximate measurements of a few quantities; we are now in a position to measure the evolution of any mathematically well defined variable precisely. The identification of complex models has thus become possible, but it requires numerical methods for the identification of nonlinear processes which still belong to the domain of 'heuristic' techniques.

The resulting schematic classification allows us to associate, *a priori*, to each material, a theory of only the dominant phenomena in a limited domain of state variables. Beyond its fundamental interest, this offers a guide for the choice of material in the design stage and helps to simplify the estimation of the resistance of a structure under service loads. The method used is that of rheology initiated by Bingham around 1930, with decisive developments taking place during the 1950s.