CIV-E4080
Material Modelling in Civil Engineering D

Period V, 2024

Lecture 6. Plasticity: Continuation

Contents

Lecture 6. Plasticity: Continuation

Ottosen (1977) developed a ***4-parameters failure criterion for concrete*** which reproduces the four key properties for 'concrete' and which is one of the best models [20, 21, 22, 23].

13111 AUGUST 1977 EM4

Good to know

JOURNAL OF THE ENGINEERING MECHANICS DIVISION

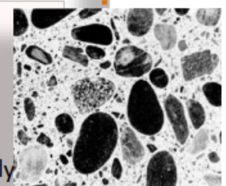
A FAILURE CRITERION FOR CONCRETE

By Niels Saabye Ottosen[1]

INTRODUCTION

At the present stage of computer programs development, the use of inadequate material models is often one of the limiting factors in structural analysis. This

Ottosen's 4-parameter failure criterion



The characteristic features for the failure surface:

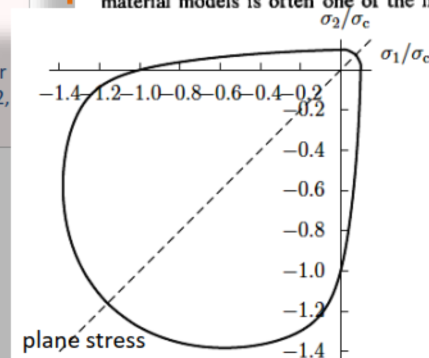
1. The uniaxial *tensile* strength is 5 – 10% of the uniaxial compressive strength
2. The shape of the failure curves on the meridian plane is slightly curved
3. Hydrostatic compression cannot cause failure
4. The shape of the failure locus on the deviatoric plane is triangular for small hydrostatic pressure and gets rounded with increasing hydrostatic pressure.

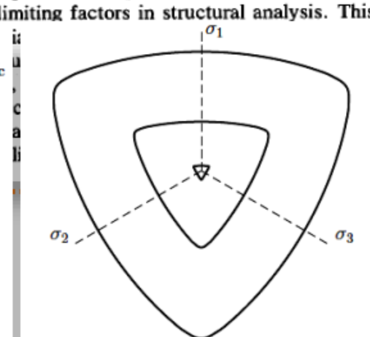
Ref. Saba Tahaei Yaghoubi, Reijo Kouhia, Juha Hartikainen and Kari Kolari. A continuum damage model based on Ottosen's four parameter failure criterion for concrete. *Rakenteiden Mekaniikka* (Journal of Structural Mechanics) Vol. 47, No 2, 2014, pp. 50 – 66

$$\boxed{A\frac{J_2}{\sigma_c} + \Lambda\sqrt{J_2} + BI_1 - \sigma_c = 0,}$$

$$\Lambda = \begin{cases} k_1 \cos[\frac{1}{3}\arccos(k_2\cos 3\theta)] & \text{if} \quad \cos 3\theta \geqslant 0 \\ k_1 \cos[\frac{1}{3}\pi - \frac{1}{3}\arccos(-k_2\cos 3\theta)] & \text{if} \quad \cos 3\theta \leqslant 0 \end{cases}.$$

$$\cos 3\theta = \frac{3\sqrt{3}}{2}\frac{J_3}{J_2^{3/2}}, \quad A \geqslant 0; \quad B \geqslant 0; \quad k_1 \geqslant 0; \quad 0 \leq k_2 \leq 1.$$





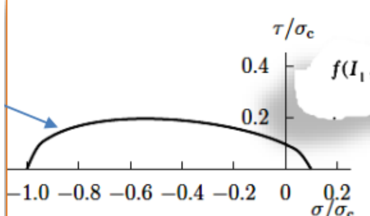
[20] N.S. Ottosen. A failure criterion for concrete. *Journal of the Engineering Mechanics, ASCE*, 103(EM4):527–535, August 1977.

... element analysis of concrete structures. Technical Re- ... Laboratory, DK-4000 Roskilde, Denmark, May 1980.

... a. Th... 2005.

[23] K. William and E.P. Warnke. Consti... crete. In *IABSE Proceedings*, volu... Structures Subjected to Triaxial Str...

$$f(I_1, J_2, \cos 3\theta) = A\frac{J_2}{\sigma_c^2} + \lambda\frac{\sqrt{J_2}}{\sigma_c} + B\frac{I_1}{\sigma_c} - 1 = 0\,.$$



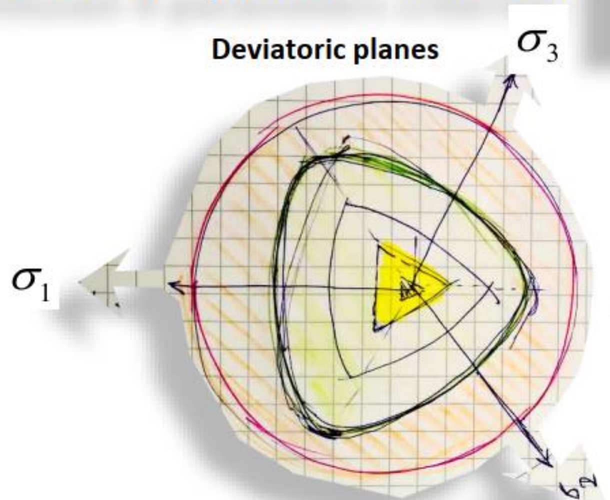
Recommended reading: **State-of-the-art report**, bulletin 45. 2008, task group 4.4. *fib* CEB-FIP – ***Practitioner's guide to finite element modelling of reinforced concrete structures***

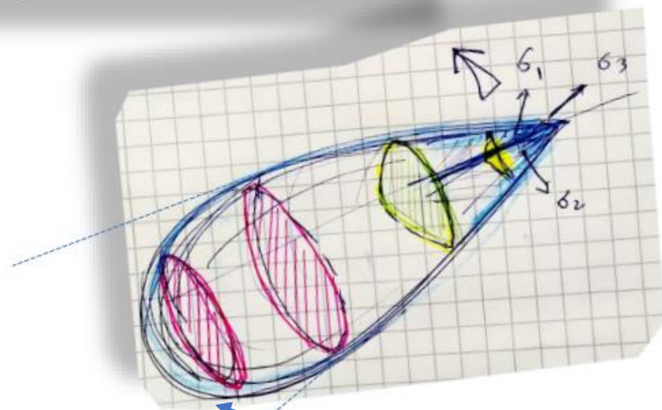
σ_c – uniaxial compression test, failure stress

Lecture 6. Plasticity: Continuation

Ottosen 4-parameters criterion: $A\frac{J_2}{\sigma_c}+\Lambda\sqrt{J_2}+BI_1-\sigma_c=0,$

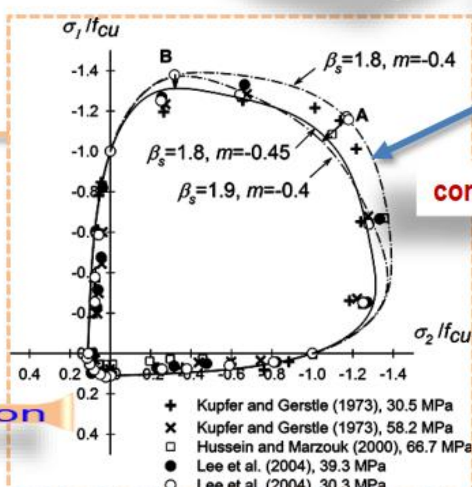
Deviatoric planes





+ cap model

NB the asymmetry in tension and compression

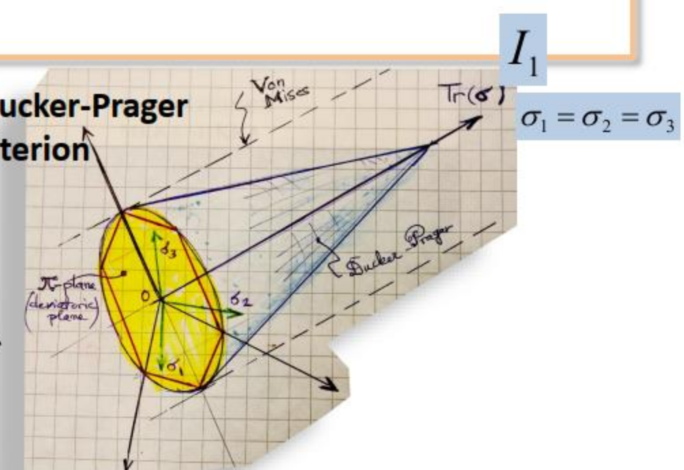


compression

tension

Concrete – plane stress

Drucker-Prager criterion

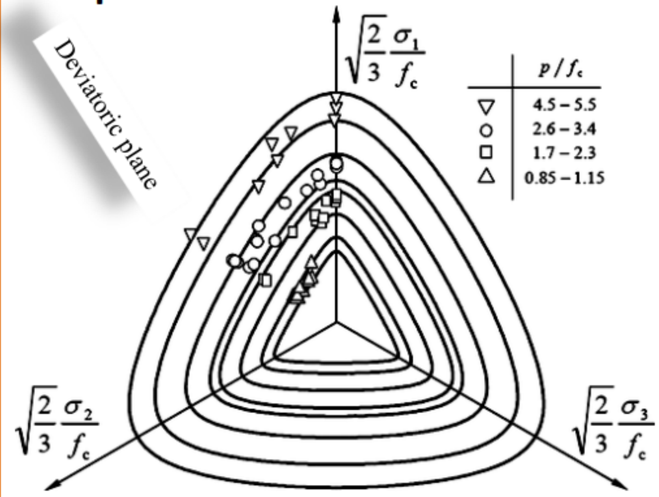


The DP-criterion does not see such asymmetry!

Lecture 6. Plasticity: Continuation

The idea in this slide is to show experimental evidence of the existence of a failure surface

Experimental validation

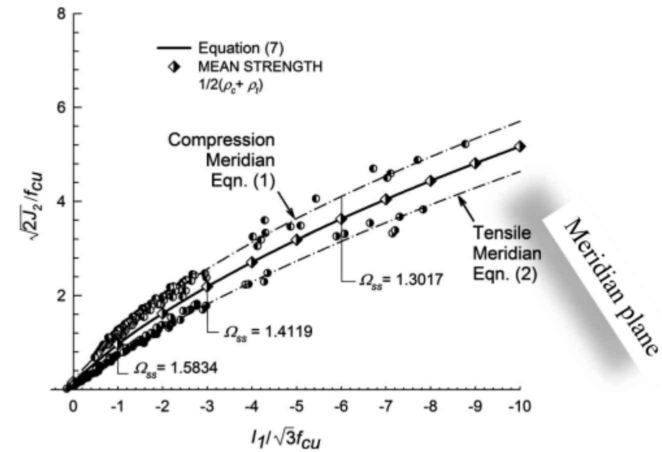


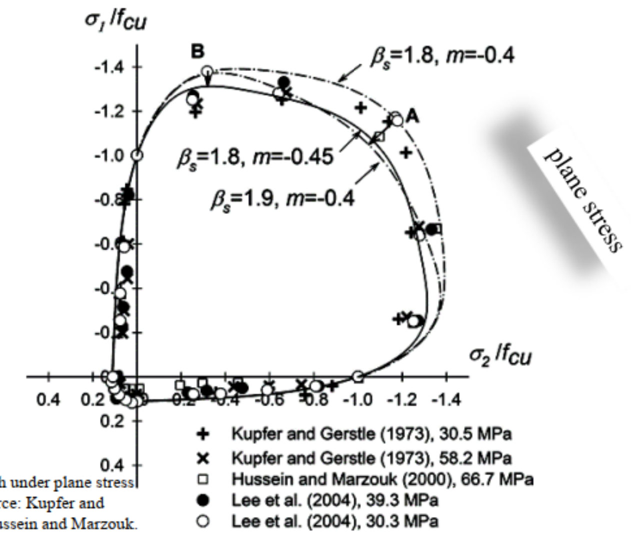
Comparison of the experimental results by Guo and Wang 1991 and the modeling results by using the proposed criterion p is held constant

Ref: *J. Eng. Mech.*, 2010, 136(1): 51-59

Nonlinear Unified Strength Criterion for Concrete under Three-Dimensional stress States. Xiu-Li Du; De-Chun Lu; Qiu-Ming Gong; and Mi Zhao

Mean concrete strength from compression and tension data





Concrete strength under plane stress Conditions. Source: Kupfer and Gerstle 1973; Hussein and Marzouk. 2000; Lee et al. 2004

Ref: Simple Single-Surface Failure Criterion for Concrete. Sean D. Hinchberger

J. Eng. Mech., 2009, 135(7): 729-732

Lecture 6. Plasticity: Continuation

Comparison of yield criteria

- von Mises: $\sqrt{3J_2} = \sigma_y, \quad \sigma_y = \sigma_c = \sigma_t$
- Drucker-Prager: $\sqrt{J_2} + \alpha I_1 - k = 0$
- Tresca: $\max\limits_{i \neq j}(\mid \sigma_i - \sigma_j \mid) = \sigma_y$
- Mohr-Coulomb: $\dfrac{m+1}{2} \max\limits_{i \neq j}(\mid \sigma_i - \sigma_j \mid + K(\sigma_i + \sigma_j)) = \sigma_c$

$$m = \frac{\sigma_c}{\sigma_t}, \quad K = \frac{m-1}{m+1}$$

- Ottosen: $A\dfrac{J_2}{\sigma_c} + \Lambda\sqrt{J_2} + BI_1 - \sigma_c = 0$

J_2, I_1 - stress invariants; $\sigma_c, \ \sigma_t$ - yield stresses in uniaxial compression, tension;

$\alpha, \ k, \ A, \ B, \ \Lambda$ - material parameters; $\sigma_1, \ \sigma_2, \ \sigma_3$ - principal stresses.

Lecture 6. Plasticity: Continuation

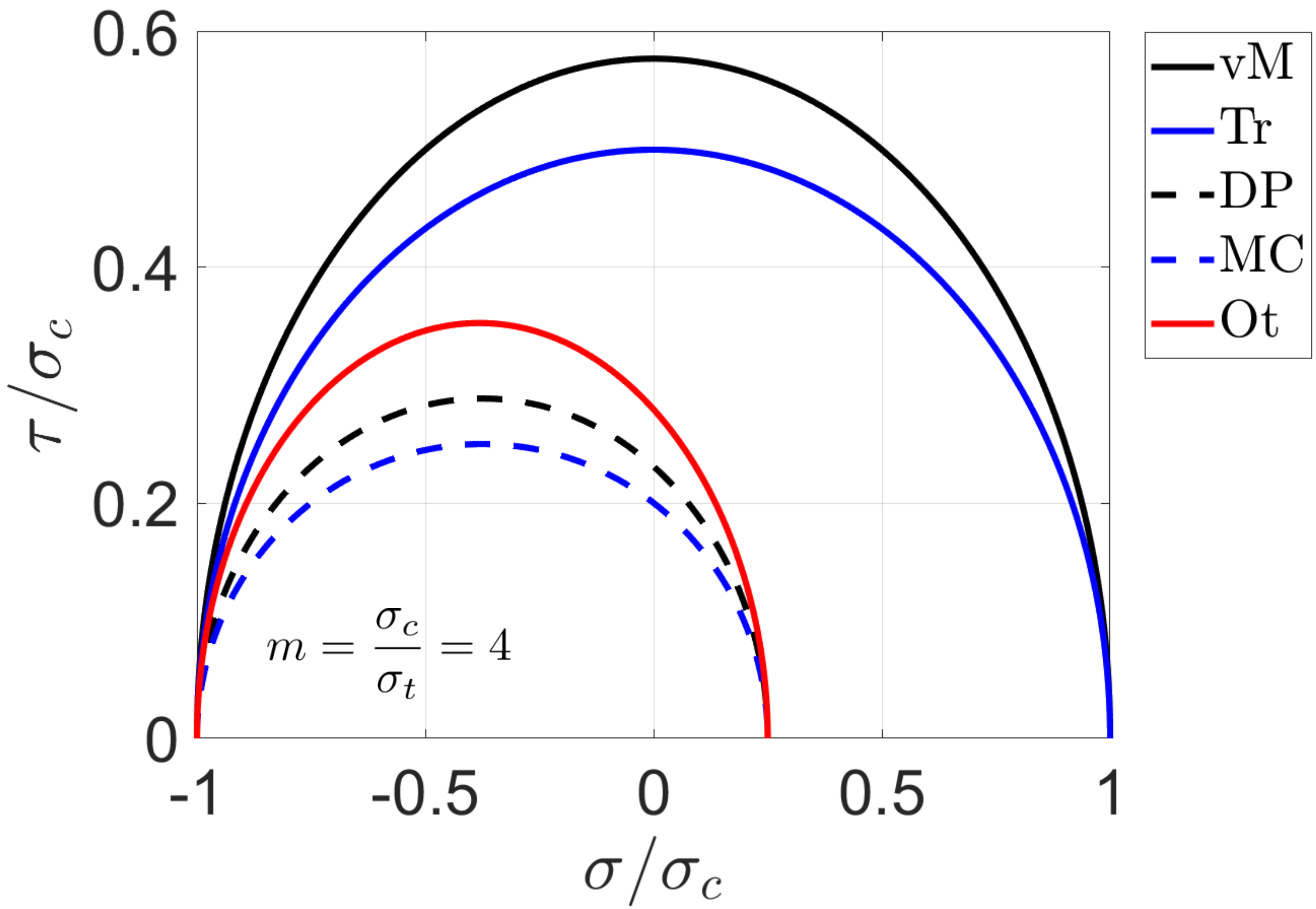
Comparison of yield criteria: loci in $\sigma - \tau$ plane

- von Mises: $$\sigma^2 + 3\tau^2 = \sigma_c^2$$
- Drucker-Prager: $$\frac{m+1}{2}\sqrt{\sigma^2 + 3\tau^2} + \frac{m-1}{2}\sigma = \sigma_c$$
- Tresca: $$\sigma^2 + 4\tau^2 = \sigma_c^2$$
- Mohr-Coulomb: $$\frac{m+1}{2}\sqrt{\sigma^2 + 4\tau^2} + \frac{m-1}{2}\sigma = \sigma_c$$ $$m = \frac{\sigma_c}{\sigma_t}$$
- Ottosen: $$\frac{A}{3\sigma_c}(\sigma^2 + 3\tau^2) + \frac{1}{\sqrt{3}}\sqrt{\sigma^2 + 3\tau^2} + B\sigma = \sigma_c$$

$$A = \frac{3m^2 + (3 - 2\sqrt{3})m}{m+1}, \quad B = \frac{3m^2 - \sqrt{3}m - 3 + \sqrt{3}}{3(m+1)}, \quad \Lambda = 1$$

Lecture 6. Plasticity: Continuation

Comparison of yield criteria: loci in $\sigma - \tau$ plane



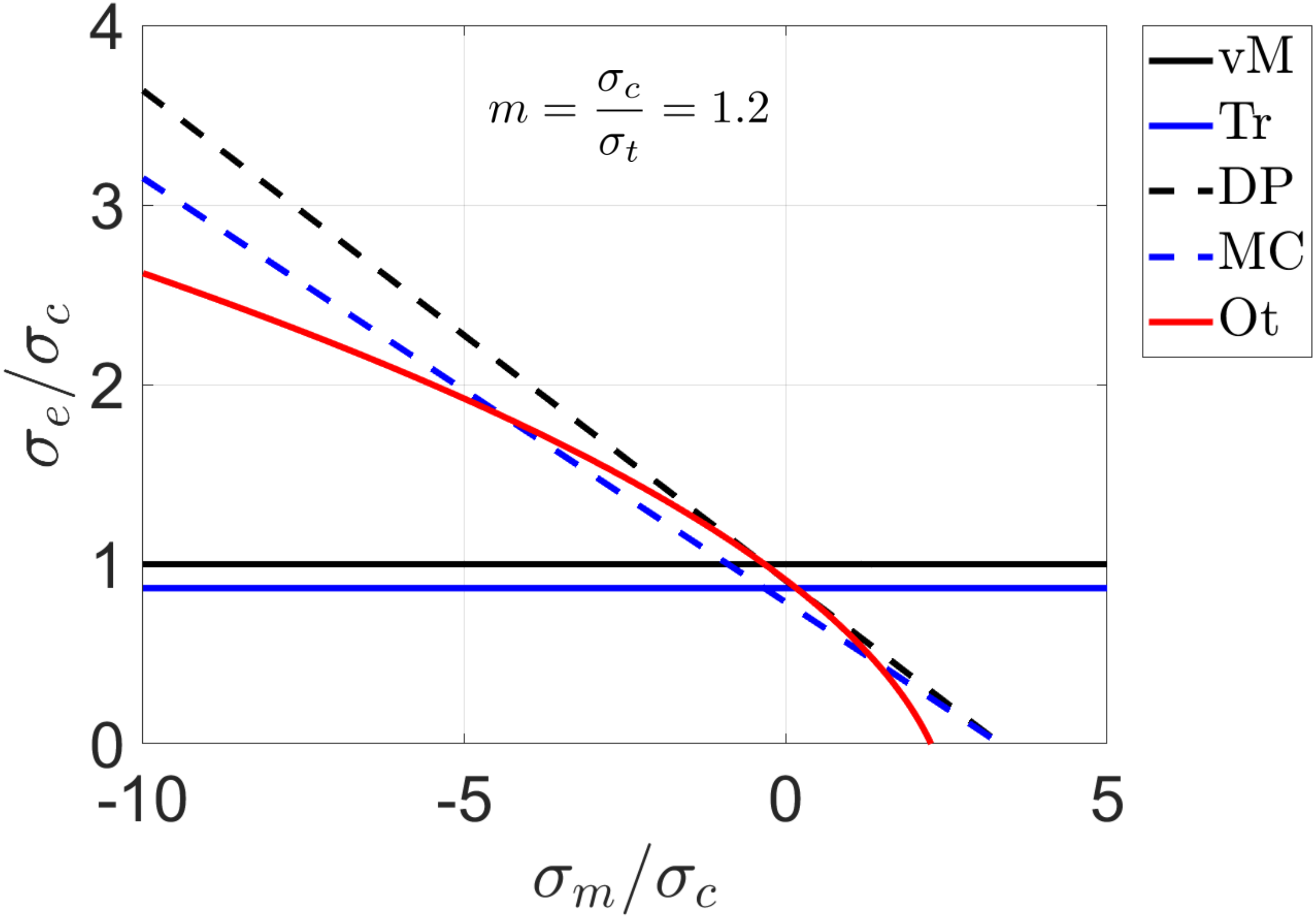
Lecture 6. Plasticity: Continuation

Comparison of yield criteria: loci in meridian plane

- von Mises: $$\frac{\sigma_e}{\sigma_c} = 1$$
- Drucker-Prager: $$\frac{m+1}{2}\frac{\sigma_e}{\sigma_c} + 3\frac{m-1}{2}\frac{\sigma_m}{\sigma_c} = 1$$
- Tresca: $$\frac{\sigma_e}{\sigma_c} = \frac{\sqrt{3}}{2}$$
- Mohr-Coulomb: $$\frac{m+1}{\sqrt{3}}\frac{\sigma_e}{\sigma_c} + 3\frac{m-1}{2}\frac{\sigma_m}{\sigma_c} = 1$$
- Ottosen: $$\frac{A}{3}\left(\frac{\sigma_e}{\sigma_c}\right)^2 + \frac{1}{\sqrt{3}}\frac{\sigma_e}{\sigma_c} + 3B\frac{\sigma_m}{\sigma_c} = 1$$

Lecture 6. Plasticity: Continuation

Comparison of yield criteria: loci in meridian plane



Comparison of yield criteria: loci in meridian plane

- Ottosen:

$$A\frac{J_2}{\sigma_c} + \Lambda\sqrt{J_2} + BI_1 - \sigma_c = 0$$

$$\Downarrow$$

$$\frac{A}{3}\left(\frac{\sigma_e}{\sigma_c}\right)^2 + \frac{1}{\sqrt{3}}\frac{\sigma_e}{\sigma_c} + 3B\frac{\sigma_m}{\sigma_c} - 1 = 0$$

- Ottosen + cap model:

$$A\frac{J_2}{\sigma_c} + \Lambda\sqrt{J_2} + F_c(BI_1 - \sigma_c) = 0$$

$$\Downarrow$$

$$\frac{A}{3}\left(\frac{\sigma_e}{\sigma_c}\right)^2 + \frac{1}{\sqrt{3}}\frac{\sigma_e}{\sigma_c} + F_c(3B\frac{\sigma_m}{\sigma_c} - 1) = 0$$

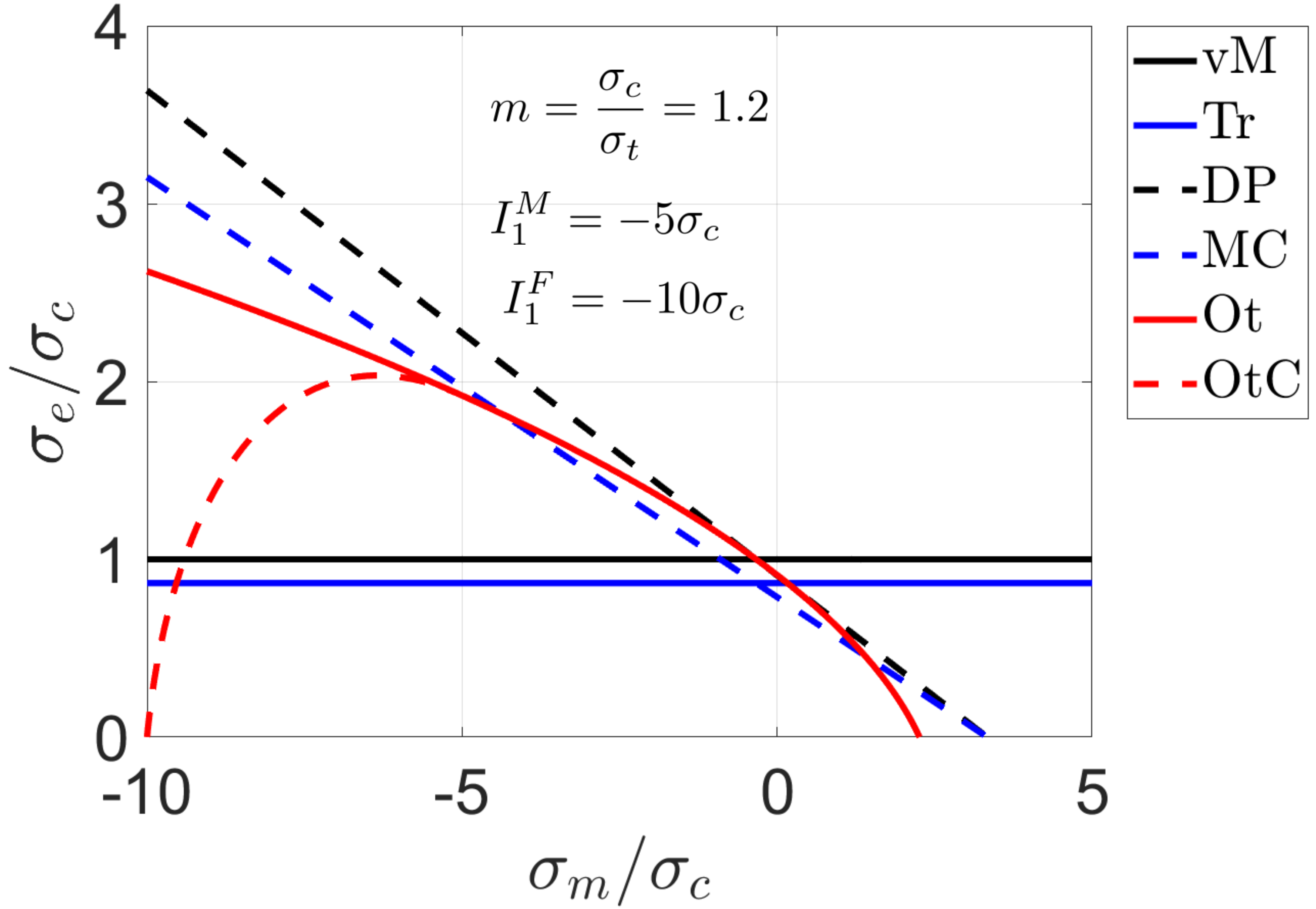
$$F_c = \begin{cases} 1, & I_1 \geq I_1^M \\ 1 - \left(\frac{I_1 - I_1^M}{I_1^F - I_1^M}\right)^2, & I_1 < I_1^M \end{cases}$$

I_1^F - corresponds to value of I_1 for which the yield surface intersects the hydrostatic axis

I_1^M - value of I_1 for which the yield curve in compression departs from the original curve

Lecture 6. Plasticity: Continuation

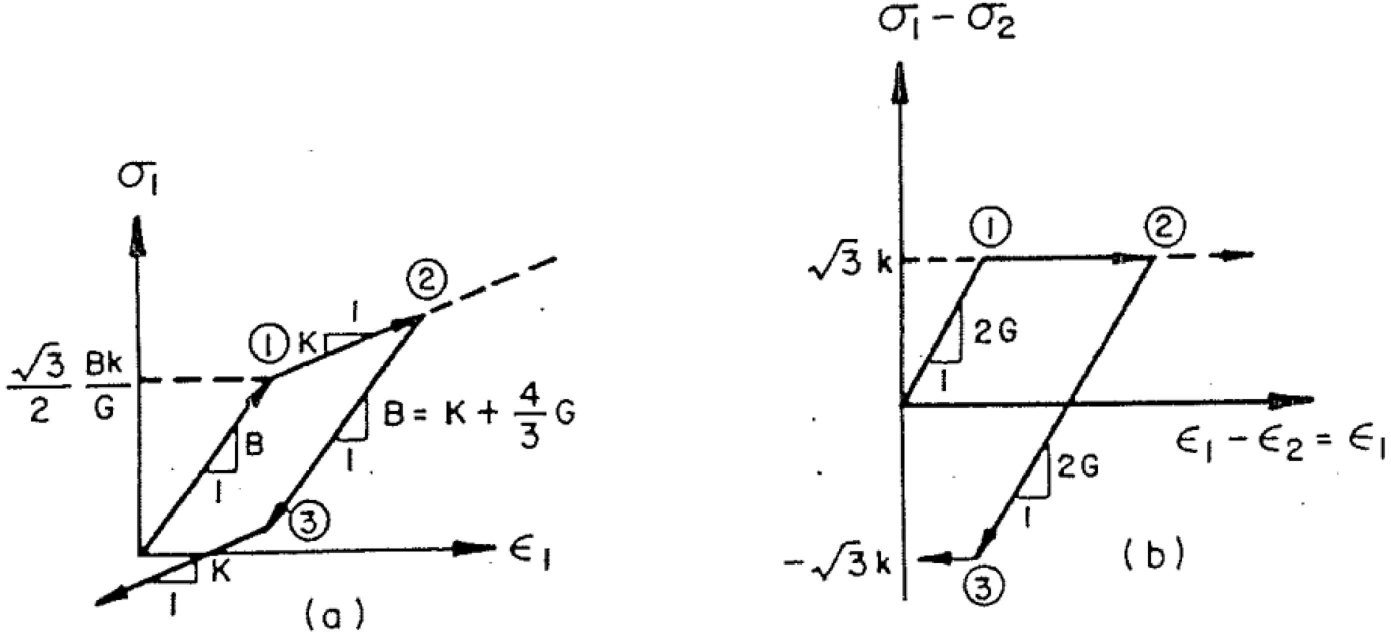
Comparison of yield criteria: loci in meridian plane



Lecture 6. Plasticity: Continuation

Uniaxial state-of-strain test

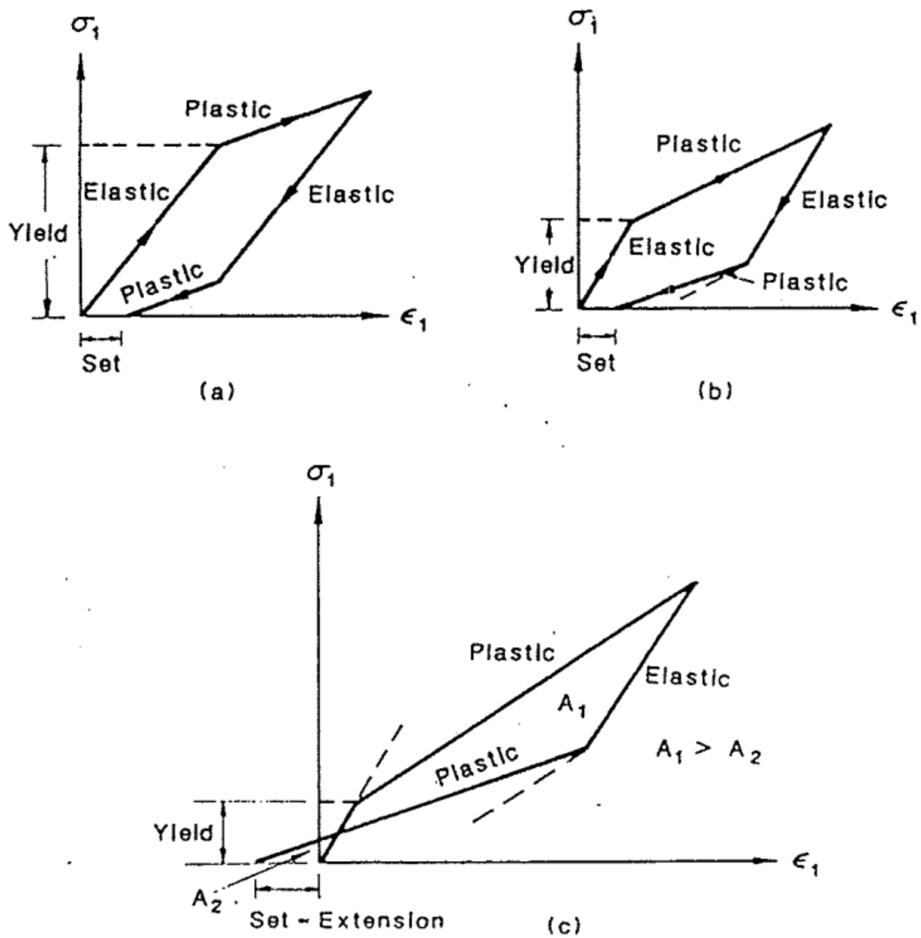
- von Mises: $\sqrt{J_2} = k$



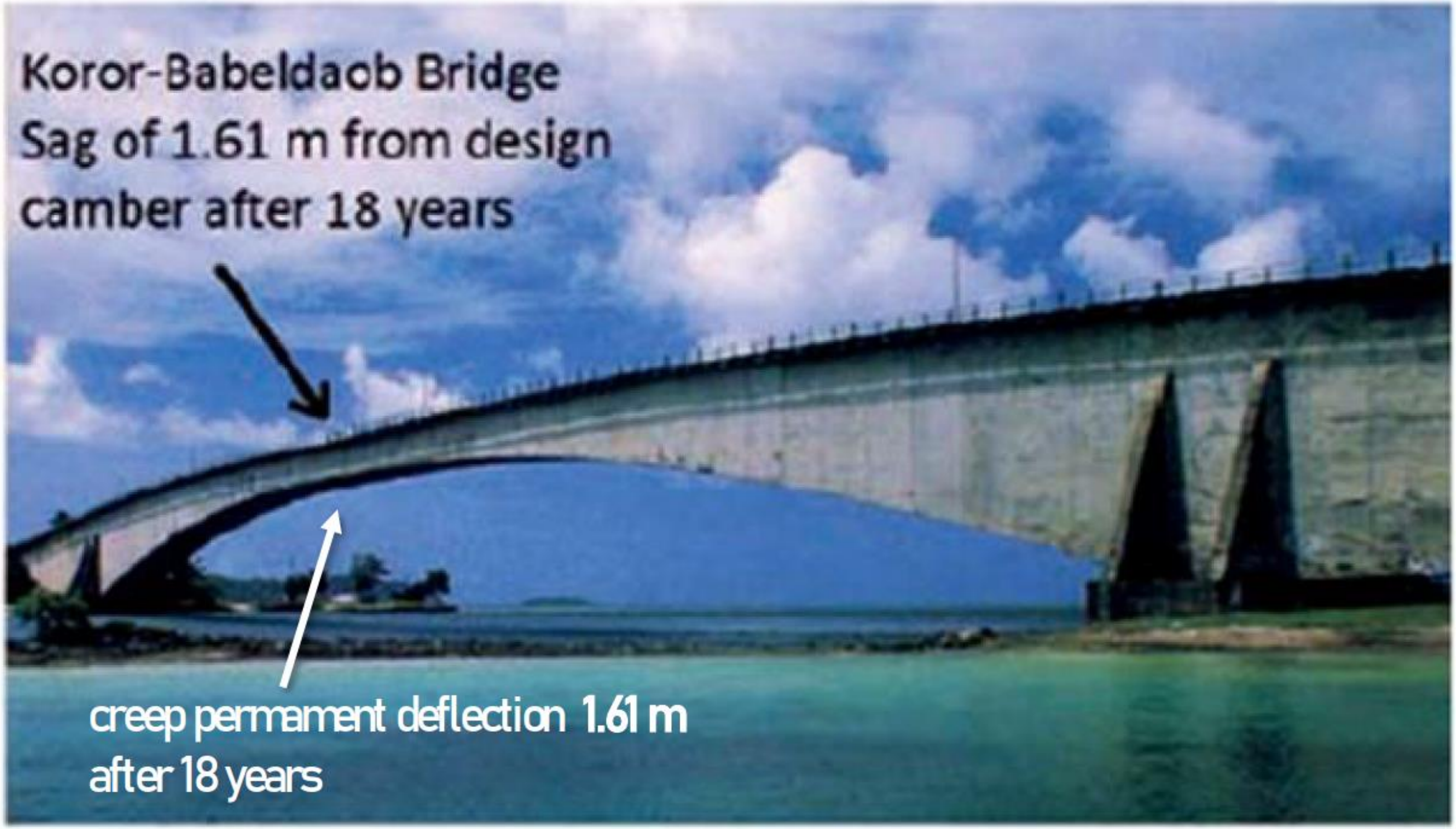
Lecture 6. Plasticity: Continuation

Uniaxial state-of-strain test

- von Mises: $\sqrt{J_2} = k$
- Drucker-Prager: $\sqrt{J_2} + \alpha I_1 - k = 0$



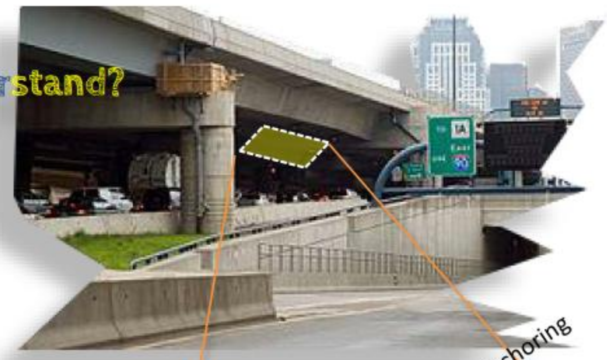
Lecture 7. Viscoelasticity



Lecture 7. Viscoelasticity

Viscoelastic material behaviour

Why viscoelasticity is it important to understand?
Why Hooke's Law is not sufficient here?

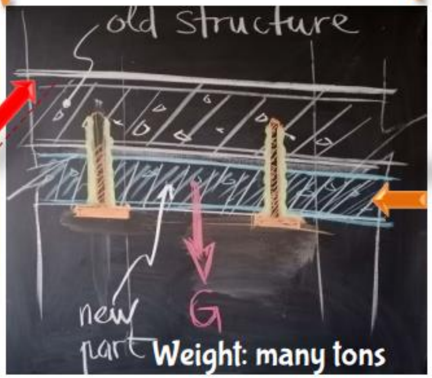


Accident: The **Big Dig ceiling collapse** occurred on July 10, 2006, when a concrete ceiling panel and debris weighing few tons (~6 x 12 m) fell in *Fort Point Channel Tunnel* (Boston). The panel fell on killing a car passenger and injuring the driver

other type anchoring

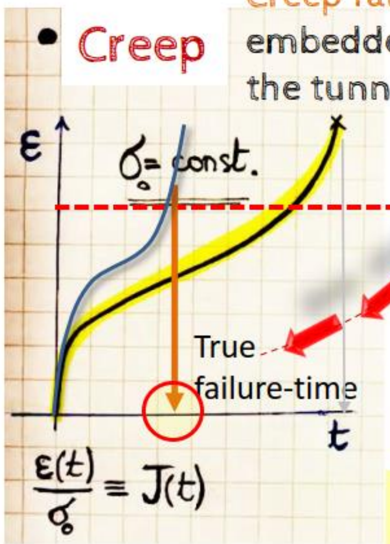
a **viscoelastic polymer**

Epoxy glue filling between concrete and bolt



added hanging concrete blocs

Creep failure of anchors embedded in epoxy in the tunnel's roof slab



Here, bolts pull out of the epoxy

- Hanging concrete ceiling blocks 24 tons/block
- Anchoring the concrete blocks with bolts embedded into the epoxy

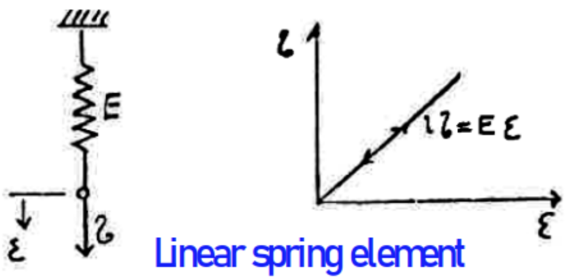
Over time, the bolts pulled out of the epoxy

The **epoxy** used was a **viscoelastic material** that **deforms over time when a force is applied (creep)**

Leading to **creep failure**

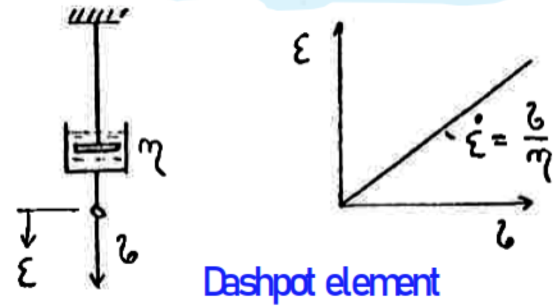
Basic rheological models

To help developing and better understanding *constitutive relations* of such various material behaviours simple mechanical sub-models and models are used to capture the key or *basic responses* of *fluid*- and *solid*-like behaviours. Such simple models are called *Rheological models*. Such basic rheological models can be combined to obtain a more complex (realistic) response of materials.



Linear spring element

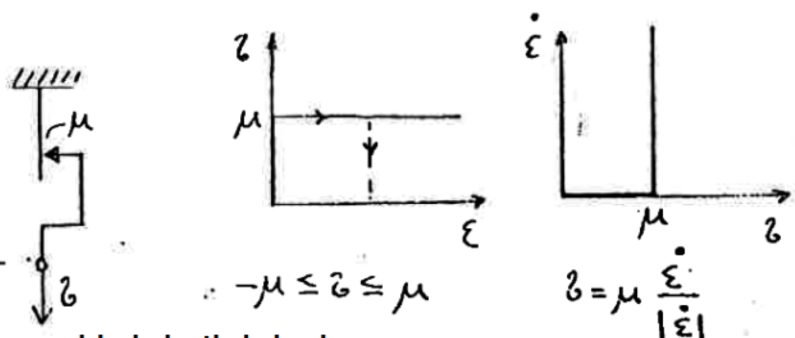
linear elastic behaviour (solid)
E = material elasticity modulus



Dashpot element

ideal viscous behaviour (Newton's fluid)
η = viscosity coefficient

Friction element



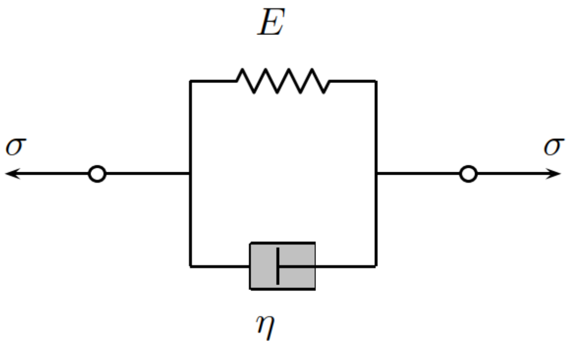
ideal plastic behaviour
μ = plastic friction coefficient or yield stress (threshold)

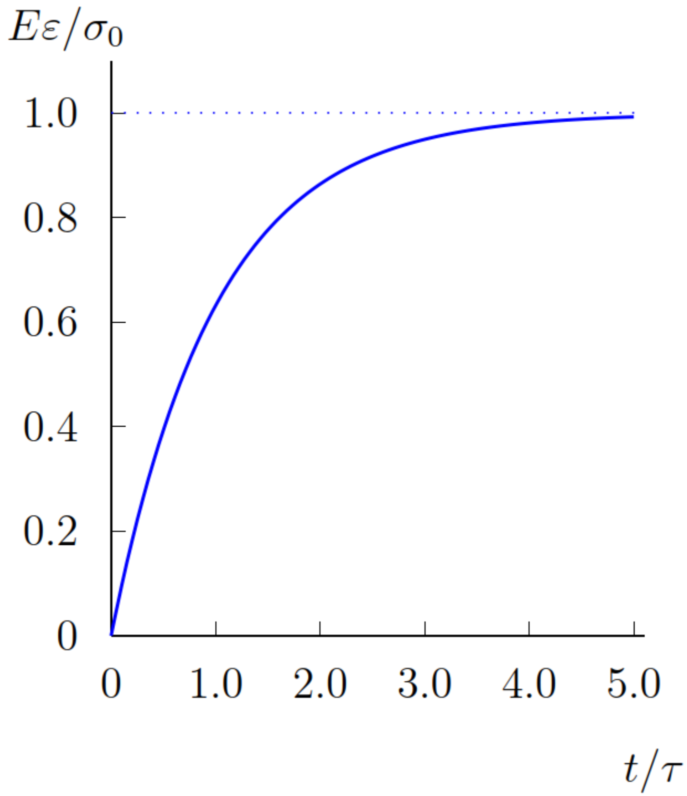
Lecture 7. Viscoelasticity

Contents

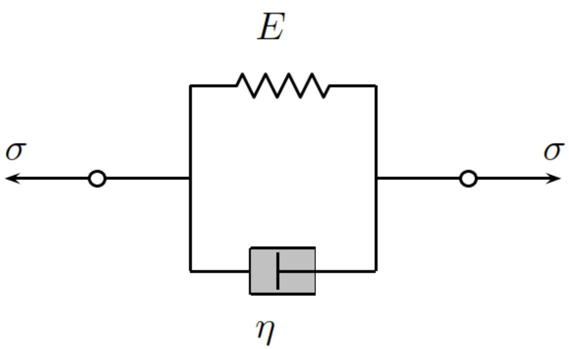
Kelvin-Voigt rheological model

- Constitutive equation: $$\sigma = E\varepsilon + \eta\dot{\varepsilon}$$
- Creep test: $$\varepsilon(t) = \frac{\sigma_0}{E}(1 - \mathrm{e}^{-Et/\eta})$$

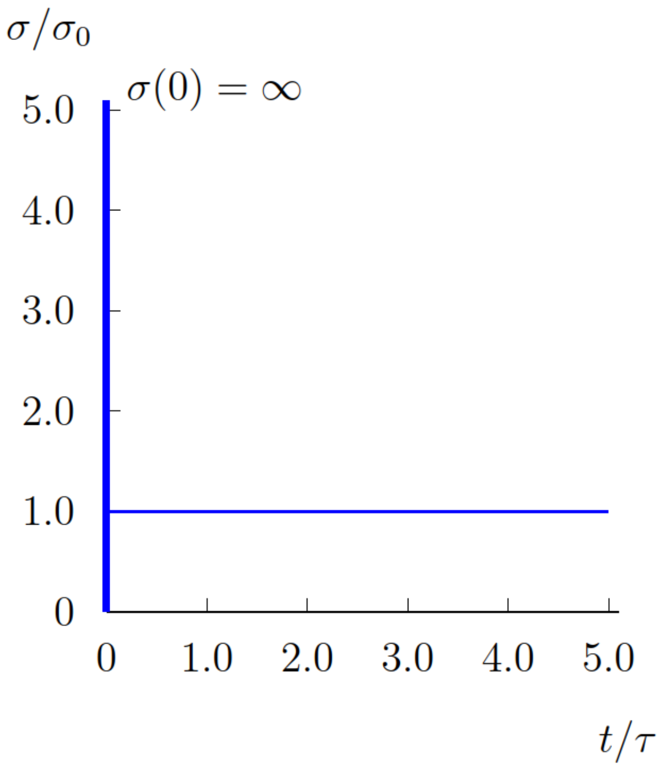




Kelvin-Voigt rheological model



- Constitutive equation: $\sigma = E\varepsilon + \eta\dot{\varepsilon}$
- Relaxation test: $\sigma(t) = E\varepsilon_0 H(t) + \eta\varepsilon_0 \delta(t)$



Heaviside step function:

$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

Dirac delta function:

$$\delta(t) = \begin{cases} +\infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

Lecture 7. Viscoelasticity

Kelvin-Voigt solid

- Stress decomposition: $\boldsymbol{\sigma} = \boldsymbol{\sigma}^e + \boldsymbol{\sigma}^v$
- Thermodynamic potential: $\rho\Psi = \frac{1}{2}\lambda(\mathrm{Tr}(\boldsymbol{\varepsilon}))^2 + \mu\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon}$
- Dissipation potential: $\varphi = \frac{1}{2}\lambda\theta_\lambda(\mathrm{Tr}(\dot{\boldsymbol{\varepsilon}}))^2 + \mu\theta_\mu\dot{\boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}}$
- Constitutive equations:

$$\boldsymbol{\sigma}^e = \rho\frac{\partial\Psi}{\partial\boldsymbol{\varepsilon}} = \lambda\mathrm{Tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}$$

$$\boldsymbol{\sigma}^v = \frac{\partial\varphi}{\partial\dot{\boldsymbol{\varepsilon}}} = \lambda\theta_\lambda\mathrm{Tr}(\dot{\boldsymbol{\varepsilon}})\boldsymbol{I} + 2\mu\theta_\mu\dot{\boldsymbol{\varepsilon}}$$

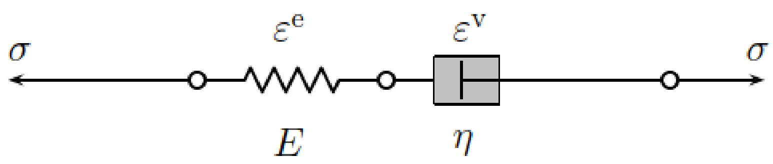
$$\boldsymbol{\sigma} = \lambda\big(\mathrm{Tr}(\boldsymbol{\varepsilon}) + \theta_\lambda\mathrm{Tr}(\dot{\boldsymbol{\varepsilon}})\big)\boldsymbol{I} + 2\mu\big(\boldsymbol{\varepsilon} + \theta_\mu\dot{\boldsymbol{\varepsilon}}\big)$$

$\lambda,\ \mu$ - Lamé parameters

$\theta_\lambda,\ \theta_\mu$ - characteristic retardation times

Lecture 7. Viscoelasticity

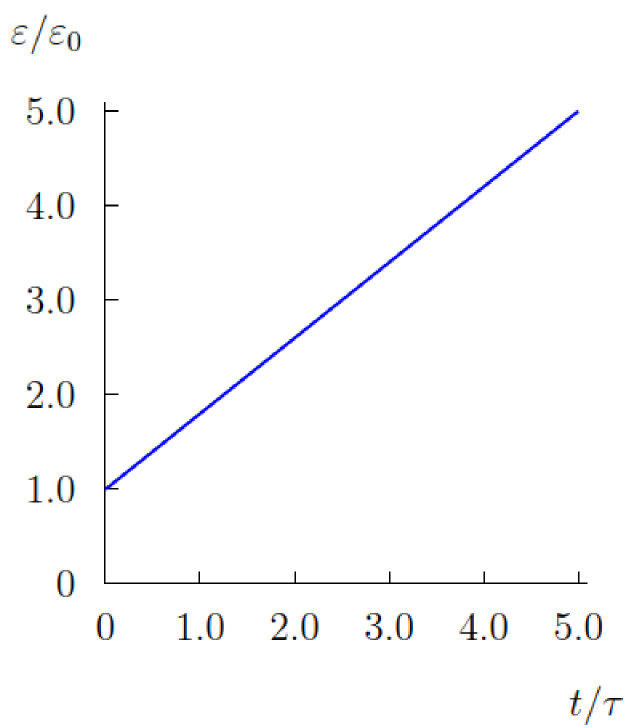
Maxwell rheological model



- Constitutive equation: $\dot{\sigma} + \frac{E}{\eta}\sigma = E\dot{\varepsilon}$

- Creep test:

$$\varepsilon(t) = \frac{\sigma_0}{E}\left(1 + \frac{t}{\tau}\right) = \sigma_0 J(t)$$



Relaxation time:

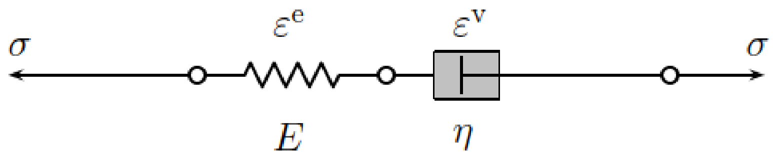
$$\tau = \frac{\eta}{E}$$

Creep compliance:

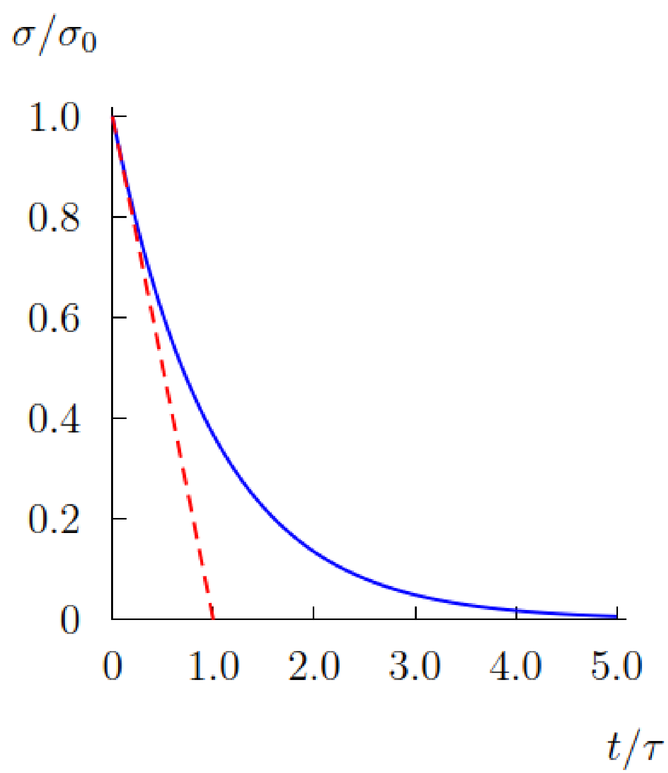
$$J(t) = \frac{1}{E}\left(1 + \frac{t}{\tau}\right)$$

Lecture 7. Viscoelasticity

Maxwell rheological model



- Constitutive equation: $\dot{\sigma} + \frac{E}{\eta}\sigma = E\dot{\varepsilon}$
- Relaxation test: $\sigma(t) = \sigma_0 \mathrm{e}^{-t/\tau} = \varepsilon_0 E \mathrm{e}^{-t/\tau} = \varepsilon_0 G(t)$



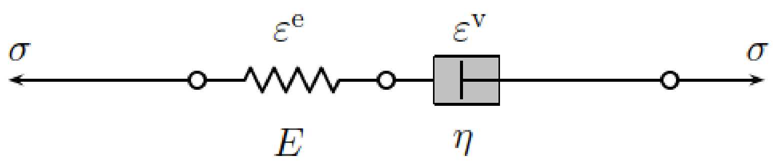
Relaxation time:

$$\tau = \frac{\eta}{E}$$

Relaxation modulus:

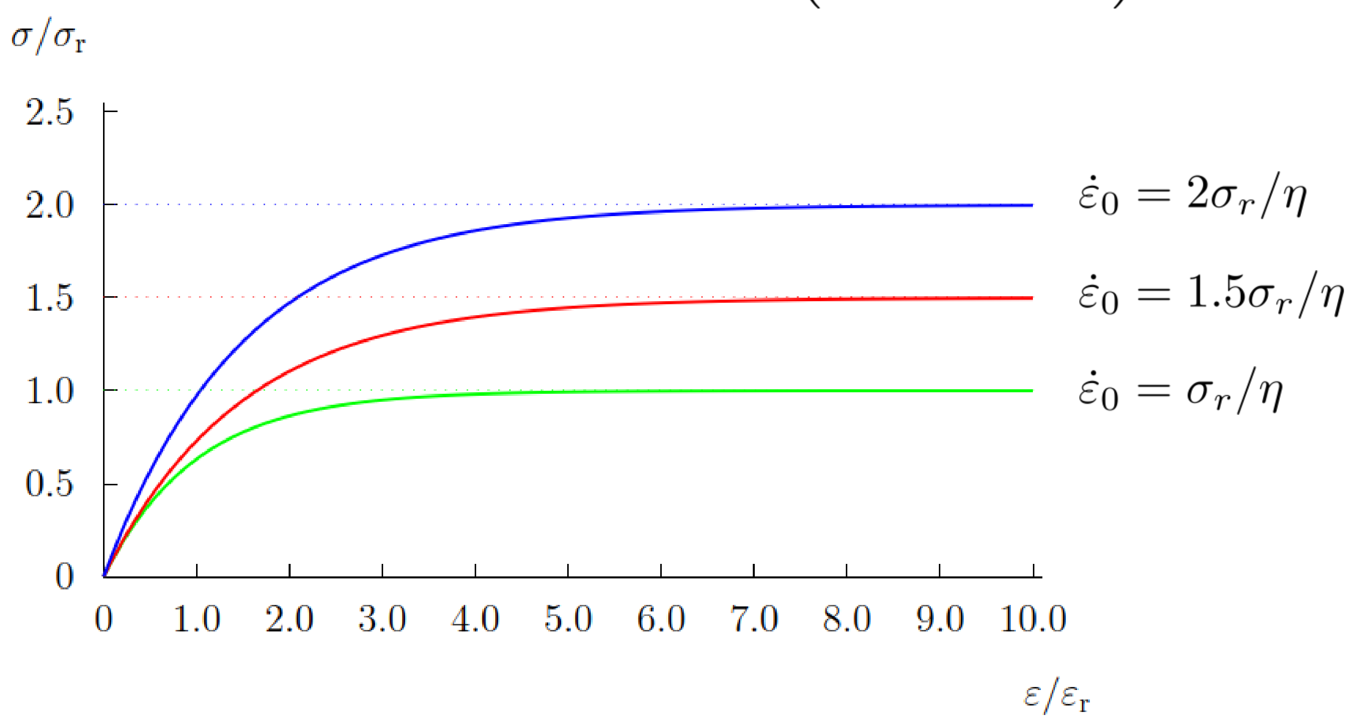
$$G(t) = E\mathrm{e}^{-t/\tau}$$

Maxwell rheological model



- Constitutive equation: $\dot{\sigma} + \frac{E}{\eta}\sigma = E\dot{\varepsilon}$
- Uniaxial tensile test: $\varepsilon(t) = \dot{\varepsilon}_0 t$

$$\sigma(\varepsilon) = \eta\dot{\varepsilon}_0\left(1 - \mathrm{e}^{-E\varepsilon/\eta\dot{\varepsilon}_0}\right)$$



Lecture 7. Viscoelasticity

Maxwell solid

- Strain decomposition: $$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^v$$
- Dual thermodynamic potential: $$\rho\Psi^* = \frac{1}{2}\Big(\frac{1+\nu}{E}\boldsymbol{\sigma} : \boldsymbol{\sigma} - \frac{\nu}{E}(\mathrm{Tr}(\boldsymbol{\sigma}))^2\Big)$$
- Dual dissipation potential: $$\varphi^* = \frac{1}{2}\Big(\frac{1+\nu}{E\tau_1}\boldsymbol{\sigma} : \boldsymbol{\sigma} - \frac{\nu}{E\tau_2}(\mathrm{Tr}(\boldsymbol{\sigma}))^2\Big)$$
- Constitutive equations:

$$\boldsymbol{\varepsilon}^e = \rho\frac{\partial\Psi^*}{\partial\boldsymbol{\sigma}} = \frac{1+\nu}{E}\boldsymbol{\sigma} - \frac{\nu}{E}\mathrm{Tr}(\boldsymbol{\sigma})\boldsymbol{I}$$

$$\dot{\boldsymbol{\varepsilon}}^v = \frac{\partial\varphi^*}{\partial\boldsymbol{\sigma}}$$

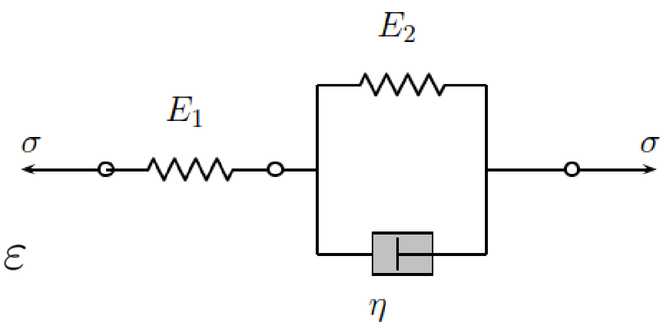
$$\dot{\boldsymbol{\varepsilon}} = \frac{1+\nu}{E}\Big(\dot{\boldsymbol{\sigma}} + \frac{\boldsymbol{\sigma}}{\tau_1}\Big) - \frac{\nu}{E}\Big(\mathrm{Tr}(\dot{\boldsymbol{\sigma}}) + \frac{\mathrm{Tr}(\boldsymbol{\sigma})}{\tau_2}\Big)\boldsymbol{I}$$

$E,\ \nu$ - elastic parameters

$\tau_1,\ \tau_2$ - parameters characterizing viscosity

Lecture 7. Viscoelasticity

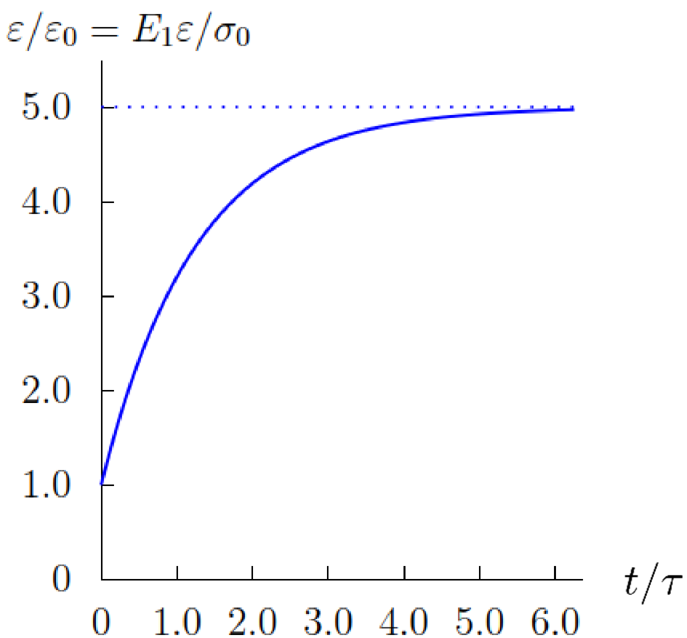
Linear standard (Zener) rheological model



- Constitutive equation:

$$\dot{\sigma} + \frac{E_1}{\eta}\left(1 + \frac{E_2}{E_1}\right)\sigma = E_1\dot{\varepsilon} + \frac{E_1 E_2}{\eta}\varepsilon$$

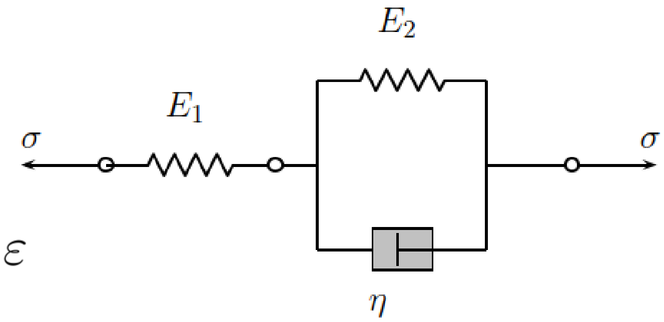
- Creep test: $$\varepsilon(t) = \frac{\sigma_0}{E_1}\left(1 + \frac{E_1}{E_2}\left(1 - \mathrm{e}^{-E_2 t/\eta}\right)\right) = \sigma_0 J(t)$$



Creep compliance:

$$J(t) = \frac{1}{E_1}\left(1 + \frac{E_1}{E_2}\left(1 - \mathrm{e}^{-E_2 t/\eta}\right)\right)$$

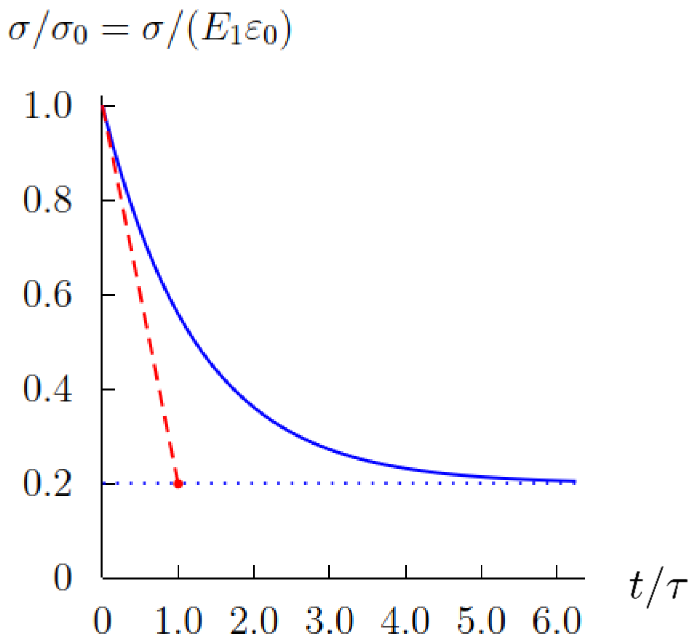
Linear standard (Zener) rheological model



- Constitutive equation:

$$\dot{\sigma} + \frac{E_1}{\eta}\left(1 + \frac{E_2}{E_1}\right)\sigma = E_1\dot{\varepsilon} + \frac{E_1E_2}{\eta}\varepsilon$$

- Relaxation test: $\sigma(t) = \dfrac{E_1\varepsilon_0}{E_1 + E_2}\left(E_1\mathrm{e}^{-t/\tau} + E_2\right) = \varepsilon_0 G(t)$



Relaxation time:

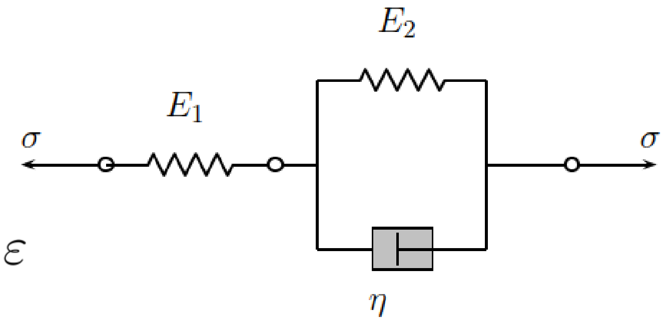
$$\tau = \frac{\eta}{E_1 + E_2}$$

Relaxation modulus :

$$G(t) = \frac{E_1}{E_1 + E_2}\left(E_1\mathrm{e}^{-t/\tau} + E_2\right)$$

Lecture 7. Viscoelasticity

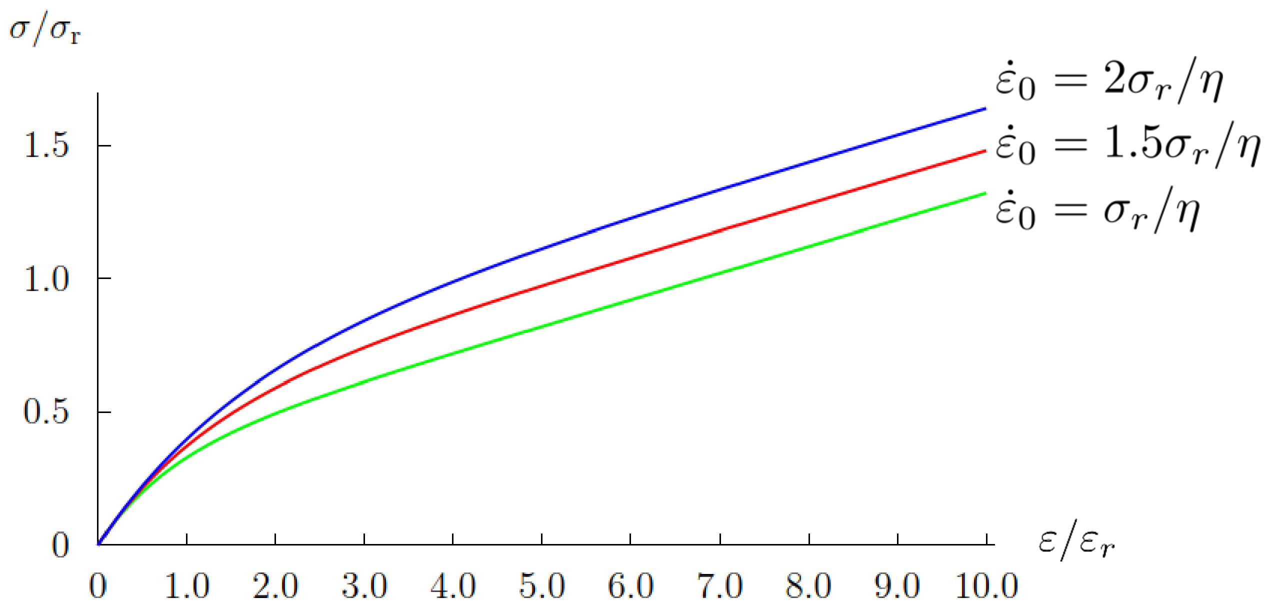
Linear standard (Zener) rheological model



- Constitutive equation:

$$\dot{\sigma} + \frac{E_1}{\eta}\left(1 + \frac{E_2}{E_1}\right)\sigma = E_1\dot{\varepsilon} + \frac{E_1 E_2}{\eta}\varepsilon$$

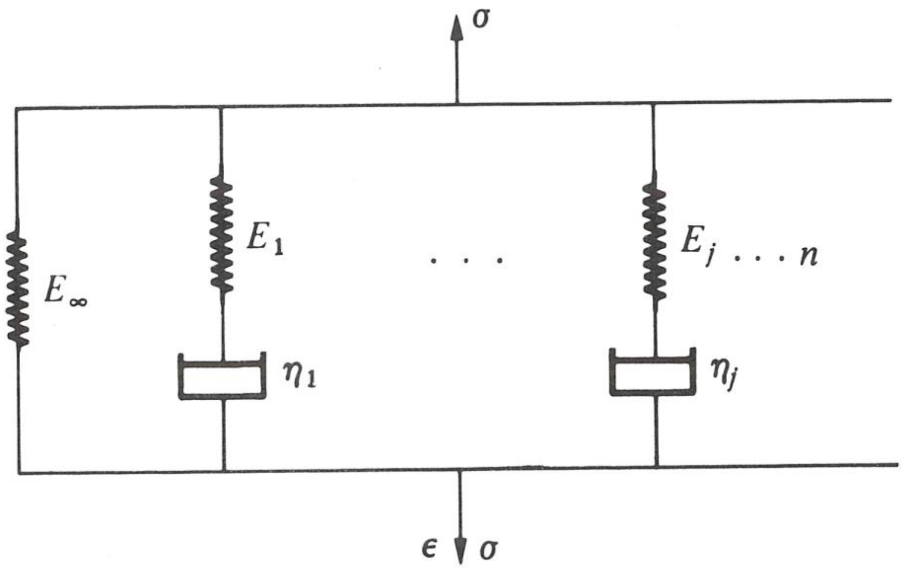
- Uniaxial tensile test: $\sigma(\varepsilon) = \frac{E_1^2}{(E_1+E_2)^2}\eta\dot{\varepsilon}_0\left(1 - \mathrm{e}^{-\varepsilon/\tau\dot{\varepsilon}_0}\right) + \frac{E_1 E_2}{E_1+E_2}\varepsilon$



Lecture 7. Viscoelasticity

Generalized Maxwell model

- Constitutive equations:



$$\sigma = \sigma_0 + \sum_{j=1}^{n} \sigma_j$$

$$\varepsilon = \frac{\sigma_0}{E_\infty}$$

$$\dot{\varepsilon} = \dot{\varepsilon}_j^e + \dot{\varepsilon}_j^v, \quad j = 1, \ldots, n$$

$$\dot{\varepsilon} = \frac{\dot{\sigma}_j}{E_j} + \frac{\sigma_j}{\eta_j}, \quad j = 1, \ldots, n \quad (\text{no sum on } j)$$

$$\sigma = E_\infty \varepsilon + \sum_{j=1}^{n} \eta_j \dot{\varepsilon}_j^v$$

$$\dot{\varepsilon}_j^v = \frac{E_j}{\eta_j}(\varepsilon - \varepsilon_j^v), \quad j = 1, \ldots, n \quad (\text{no sum on } j)$$

...

J. Lemaitre, J.-L. Chaboche, 1990.