Stress Invariants

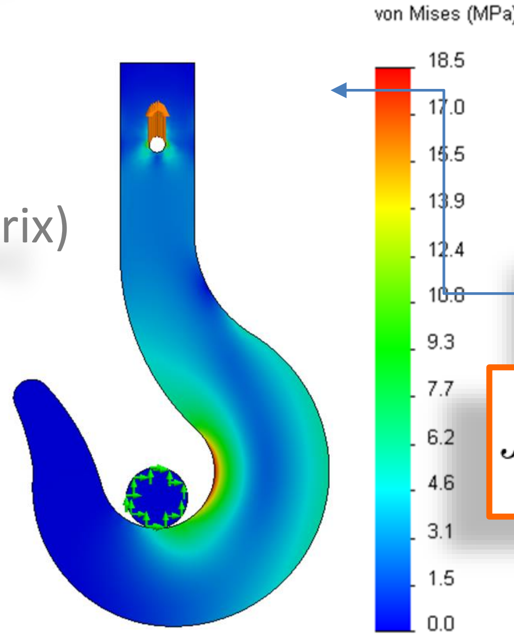
$$\sigma = -p\mathrm{I} + \mathbf{s}$$

Stress tensor (or matrix)

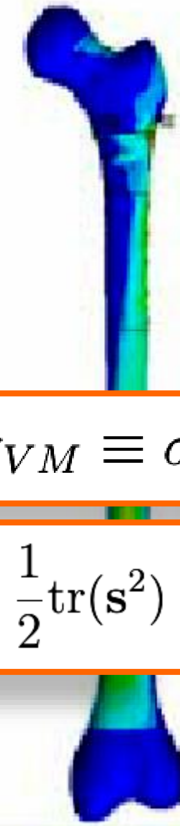
$$(\boldsymbol{\sigma} - \lambda \mathbf{I}) \cdot \hat{\mathbf{n}} = \mathbf{0}$$

Principal stresses

Principal directions



Ref. Srinivasan, Sowmianarayanan *et al.* (2018). *Finite Element Analysis of Proximal Femur Nail for Subtrochanteric Fractured Femur.*

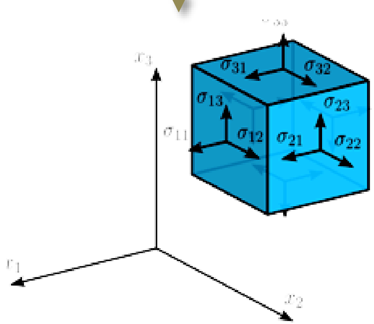


$$\boxed{\sigma_{VM} \equiv \sigma_e = \sqrt{3J_2}}$$

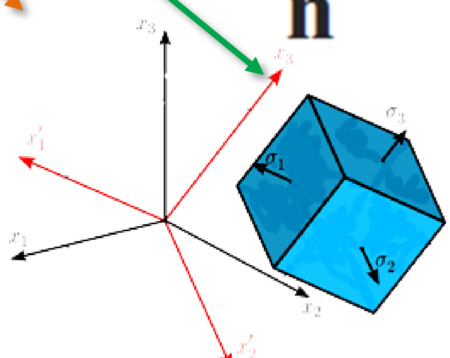
$$\boxed{J_2 = \frac{1}{2}\mathrm{tr}(\mathbf{s}^2)}$$

Maximum Principal Stress on Intact femur

Intact Femur

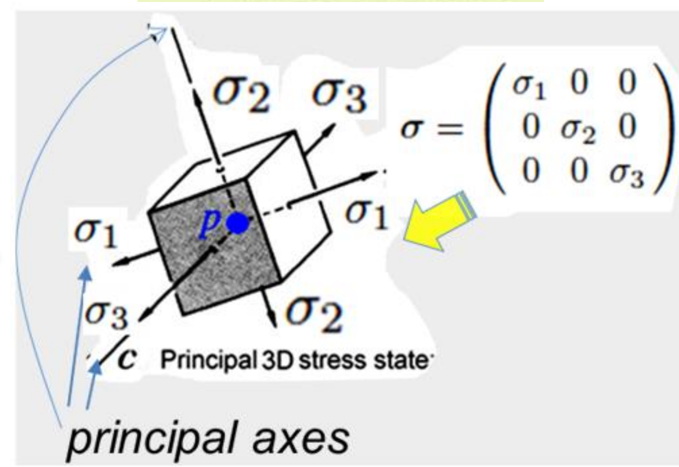


$\boldsymbol{\sigma}$



$\hat{\mathbf{n}}$

λ



Teacher
Djebar BAROUDI, Dr.

19.4.2022

Elements of solid mechanics – Stress Invariants

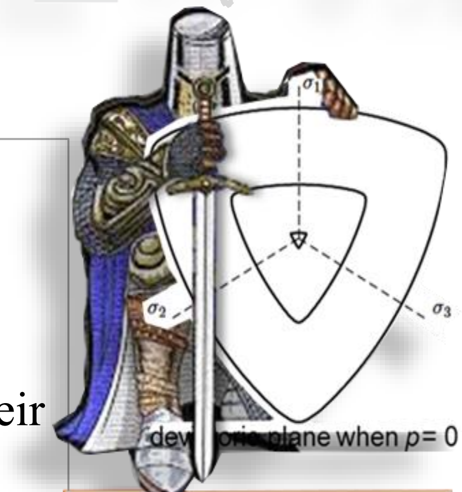
Limited to the necessary minimum to follow this course

Recall

de-σtress

We summarize necessary basic concepts for solid mechanics

- We use primarily uses **Cartesian coordinates**
- In parallel also the tensor notation is used; this makes it easy to read: vectors and tensors are represented either by their components or by their symbols



tensor A = relation which operates on a vector v and produces a vector so u = A(v) $t = \sigma \cdot n$

– in cartesian coordinate system, a tensor can be represented by its components which form a Matrix.

$$A\frac{J_2}{\sigma_c} + \Lambda\sqrt{J_2} + BI_1 - \sigma_c = 0,$$

failure criterion (concrete)

Ex.) Drucker-Prager **yield criterion**

$$\sqrt{J_2} = A + B\, I_1$$

Ottosen **criterion**

$$A\frac{J_2}{\sigma_c} + \Lambda\sqrt{J_2} + BI_1 - \sigma_c = 0,$$

What are these symbols J2 and I1 standing for?

Readings:

Chapter 1 from: D. Gross and T. Seelig, *Fracture Mechanics: With an Introduction to Micromechanics,*

or

Reddy; Chapters 3 *(kinematics of continua) and 4 (stress measures)*

... or read from any other source ...
https://en.wikipedia.org/wiki/Cauchy_stress_tensor

Principal Stresses and Principal Planes

Why?

- Determination of **maximum normal** and **shear stresses** at a point is of considerable interest
- Many *failure criteria* are expressed using **stress**/strain **Invariants** (= scalars)
- The general laws of physics are <u>independent</u> coordinate systems
So, they have **<u>Invariance</u>** & have symmetry properties

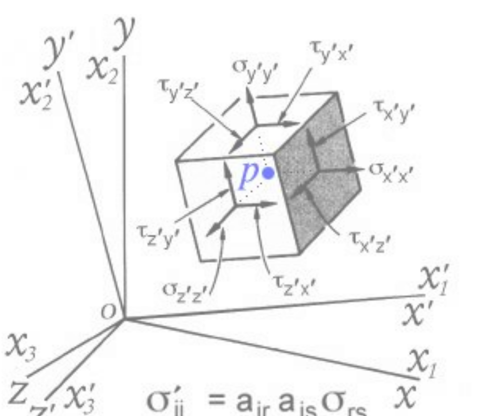
Stress tensor (or matrix)

$$(\boldsymbol{\sigma} - \lambda \mathbf{I}) \cdot \hat{\mathbf{n}} = 0$$

principal stresses **principal directions**

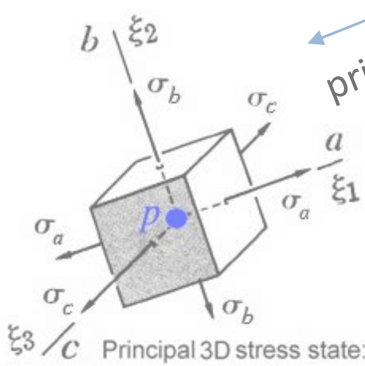
The principal axes: solely **normal stresses** and **no shear stresses** appear in sections perpendicular to these axes

principal axes



$\sigma'_{ij} = a_{ir}\, a_{js}\, \sigma_{rs}$

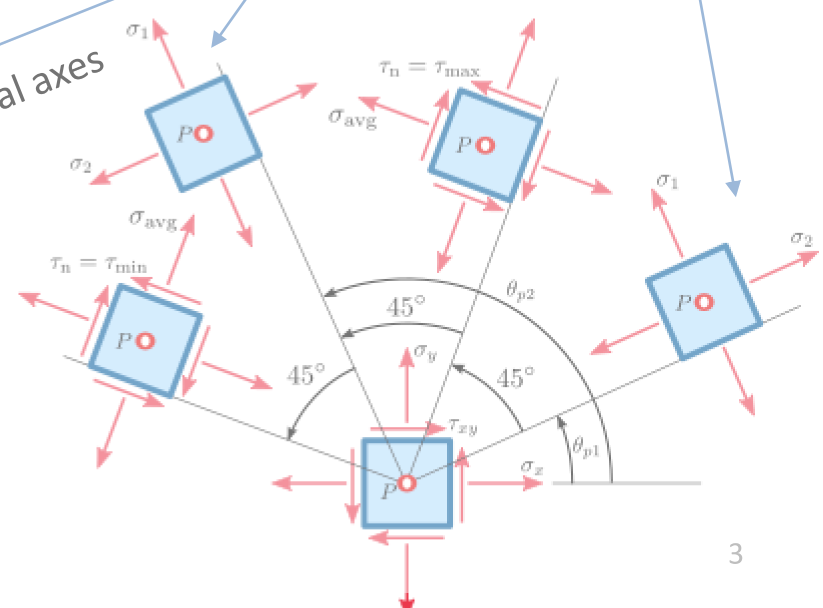
General rotational stress transformation using tensor notation.



Principal 3D stress state: Special rotational orientation

$\xi_i = x_i' = a_{ir}\, x_r$

where all shear stresses go to zero



Example: the speed of light is an invariant

Invariants

Mechanics

$$\mathbf{A} = \boldsymbol{\sigma}$$

- 2nd-order symmetric tensors (matrices) have always three groups independent invariants which, for our purposes, in mechanics, are defined as

$$\begin{aligned} I_1^A &= \operatorname{tr}\mathbf{A}, \\ J_2^A &= \tfrac{1}{2}\operatorname{tr}(\mathbf{A}^{\mathrm{d}})^2, \\ J_3^A &= \tfrac{1}{3}\operatorname{tr}(\mathbf{A}^{\mathrm{d}})^3, \end{aligned}$$

$$\mathbf{A} = \boldsymbol{\sigma}$$

$$\begin{aligned} \operatorname{tr}\mathbf{A} &= A_{kk} \\ \mathbf{A}^{\mathrm{d}} &= \mathbf{A} - \tfrac{1}{3}(\operatorname{tr}\mathbf{A})\mathbf{I} \end{aligned}$$

The stress deviator $\mathbf{S} = \boldsymbol{\sigma}^{\mathrm{d}}$

$$\begin{aligned} \mathbf{S} &= \mathbf{A}^{\mathrm{d}} = \boldsymbol{\sigma}^{\mathrm{d}} \equiv \mathbf{s} \\ \mathbf{T} &= (\mathbf{S}^{\mathrm{d}})^2, \end{aligned}$$

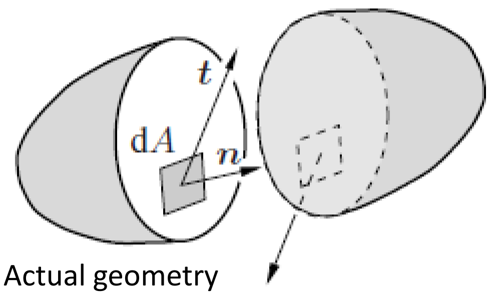
- In plasticity, one needs to compute gradients of functions of invariants with respect to the stress components , for instance, the gradient for an arbitrary isotropic yield function (Isotropic plasticity theory) is as

$$\frac{\partial f(I_1, J_2, J_3)}{\partial \boldsymbol{\sigma}} = \frac{\partial f}{\partial I_1}\mathbf{I} + \frac{\partial f}{\partial J_2}\mathbf{S} + \frac{\partial f}{\partial J_3}\mathbf{T}$$, where Hill tensor $\mathbf{T} = (\mathbf{S}^2)^{\mathrm{d}}$;

Stress

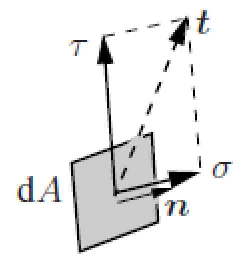
$$\begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix}$$

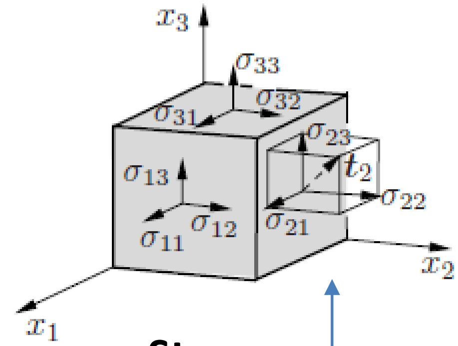
Stress vector:



Actual geometry

$$t = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} = \frac{\mathrm{d}F}{\mathrm{d}A}$$



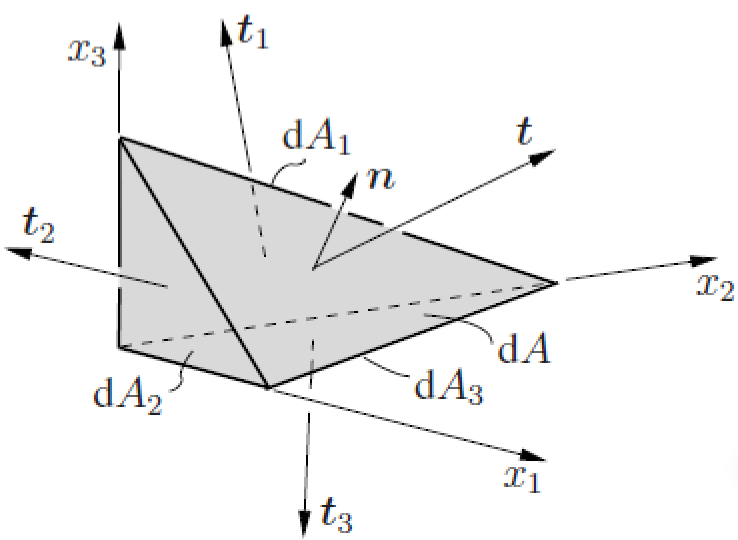


Stress components

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

stress components
in Cartesian rectangular
coordinates

σ - **Cauchy**'*s stress tensor*
(2nd order symm. tensor)



Stress state – Cauchy's tetrahedron

Cauchy's stress theorem:

$$t = \boldsymbol{\sigma} \cdot \boldsymbol{n}$$

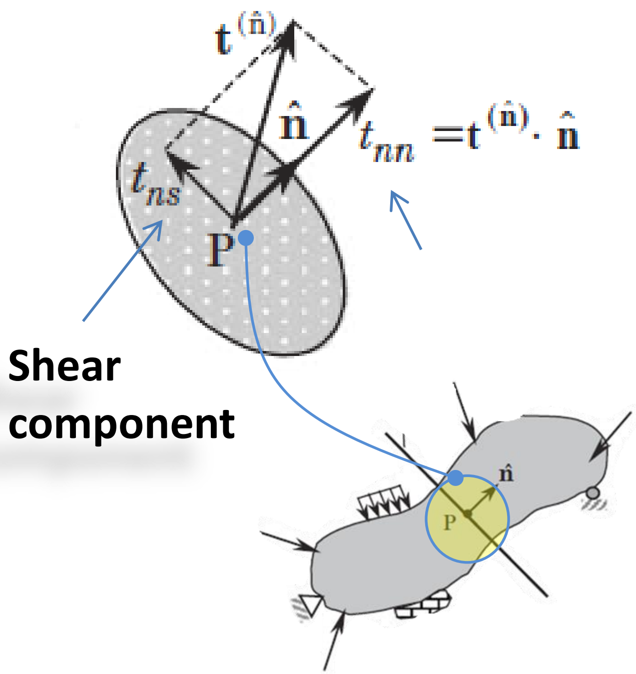
$$\vec{t}(\vec{x},\vec{n},t) = \vec{n} \cdot \boldsymbol{\sigma}(\vec{x},t) \leftrightarrow \mathbf{t} = \boldsymbol{\sigma}^{\mathrm{T}}\mathbf{n}$$

Shear- and normal components of the stress vector

Cauchy's stress theorem:

$$\mathbf{t}(\hat{\mathbf{n}}) = \hat{\mathbf{n}} \cdot \boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{T}} \cdot \hat{\mathbf{n}}$$

$$\underline{\mathrm{t}}(\underline{\mathrm{n}}) = \underline{\underline{\sigma}}^{\mathrm{T}}\,\underline{\mathrm{n}} \leftrightarrow \mathrm{t}_i = \sigma_{ji} n_j$$



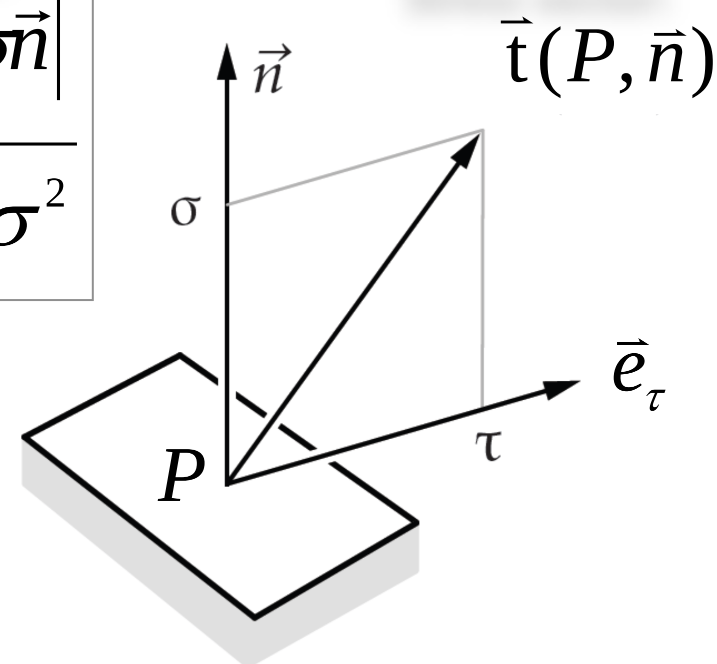
Stress vector: $\boxed{\vec{\mathrm{t}}(P,\vec{n}) = \sigma\vec{n} + \vec{\tau}}$

Normal stress: $\boxed{\sigma = \vec{\mathrm{t}} \cdot \vec{n}}$

Shear stress amplitude:

$$\boxed{\tau = |\vec{\tau}| = |\vec{\mathrm{t}} - \sigma\vec{n}| = \sqrt{|\vec{\mathrm{t}}|^2 - \sigma^2}}$$

Stress vector:

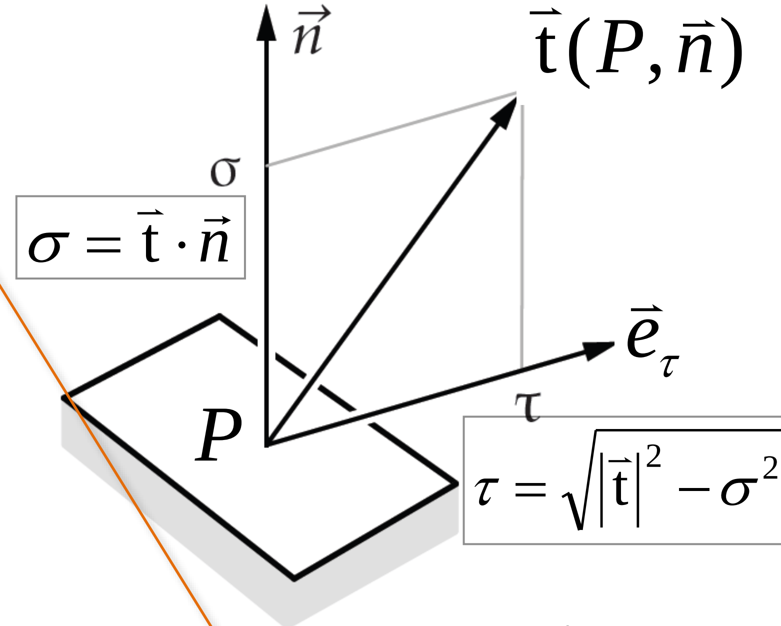


Mohr's stress representation in plane - (τ,σ)

$$\vec{t}(P,\vec{n}) = \sigma\vec{n} + \vec{\tau}$$

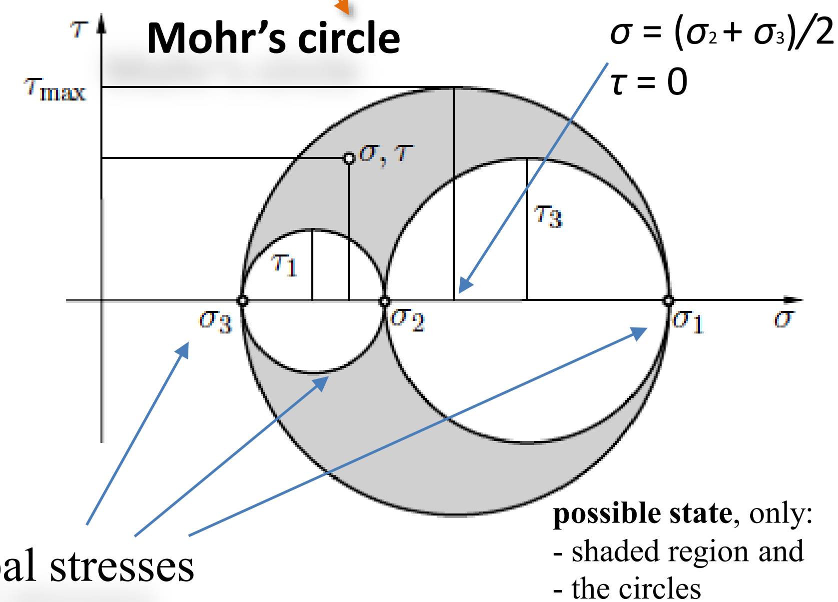
$$x^2 + y^2 = R^2$$

$$\tau^2 + \sigma^2 = \left|\vec{t}(P,\vec{n})\right|^2 \qquad \left\|\vec{n}\right\| = 1$$



Mohr's circle – *the equation* for points (τ,σ)
with the material point ***P*** is fixed and
the directions of the plan-section
across ***P*** vary through the outer normal ***n***
director-cosines.

- a graphical visualization of the stress state by *Mohr's circles* (1835-1918)
- a representation of **normal stresses σ** and corresponding **shear stresses τ** as points in a σ-τ--diagram for all possible cross sections through the material point P and for all directions



$$\underline{t}(\underline{n}) = \underline{\underline{\sigma}}^T\underline{n} \leftrightarrow t_i = \sigma_{ji} n_j$$

Principal Stresses and Principal Planes

Why?

- Determination of **maximum normal stresses and shear stresses** at a point is of **considerable interest**

Why?

- Many *failure criteria* are expressed using **stress**/strain **Invariants**

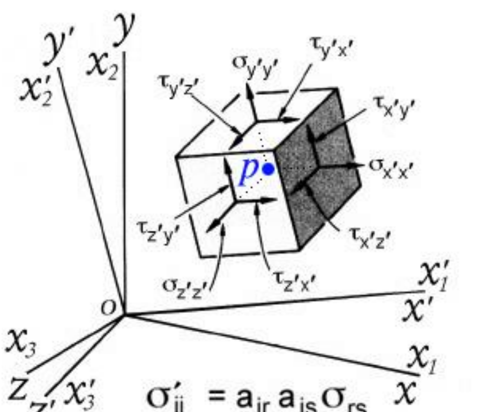
Stress tensor (or matrix)

$$(\boldsymbol{\sigma} - \lambda \mathbf{I}) \cdot \hat{\mathbf{n}} = 0$$

principal stresses **principal directions**

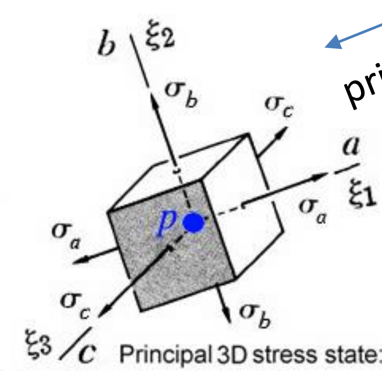
The principal axes: solely **normal stresses** and **no shear stresses** appear in sections perpendicular to these axes

principal axes



$\sigma'_{ij} = a_{ir}\, a_{js}\, \sigma_{rs}$

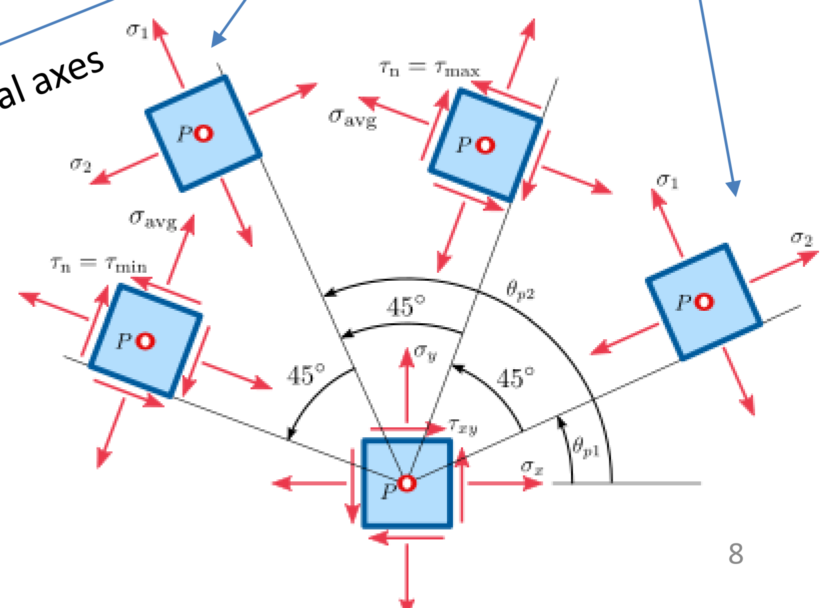
General rotational stress transformation using tensor notation.



Principal 3D stress state: Special rotational orientation

$\xi_i = x'_i = a_{ir}\, x_r$

where all shear stresses go to zero



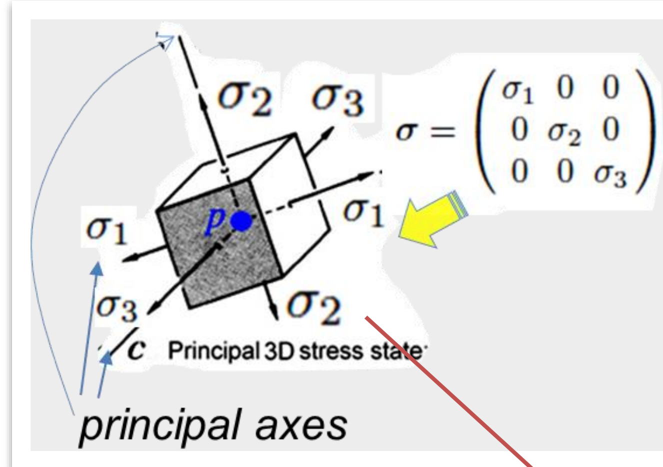
Principal Stresses and Principal Planes

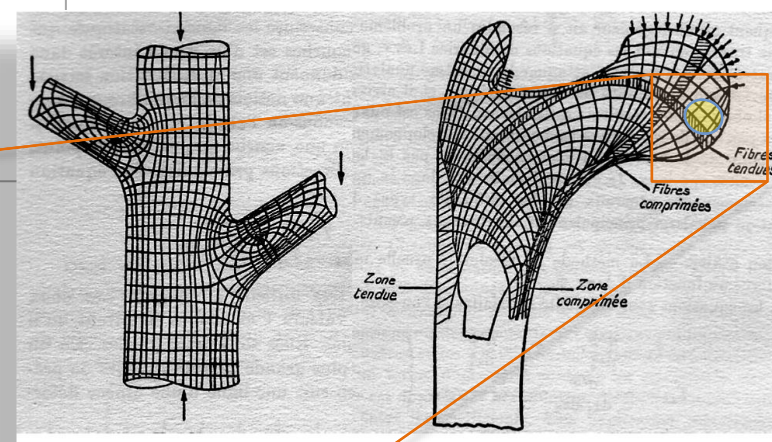
The same holds for strains

For **any arbitrary state of stress**, we can find a **set of orthogonal planes** on **which only normal stresses act** and **the shearing stresses are zero**

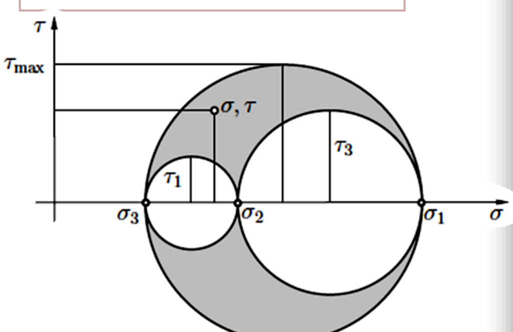
- Called Principal Planes and the normal stresses acting on these planes are Principal Stresses denoted as
- They are ordered such that : $\boxed{\sigma_1 > \sigma_2 > \sigma_3}$
- **Maximum shear stresses** are in planes forming **an angle of 45 deg. with Principal Planes**

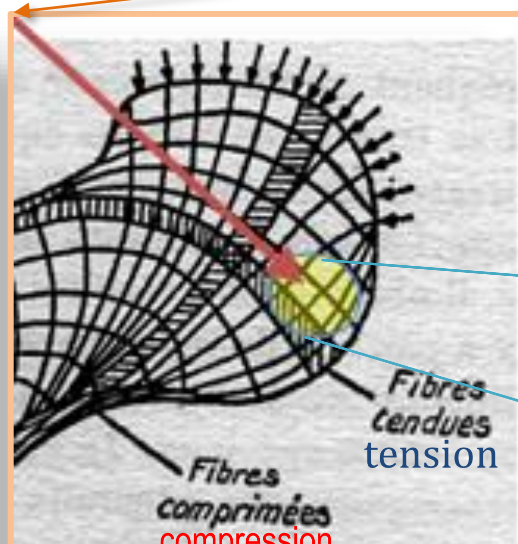
$$\boxed{\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2}}$$

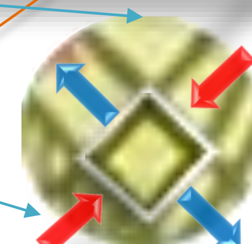




Isostatiques des neuds du bois et d'une tête de fémur







Stress Invariants 1(2)

Most tensors used in engineering are symmetric 3×3. For this case the invariants can be calculated as:

$$\mathrm{I}_A = \mathrm{tr}(\mathbf{A})$$
$$\mathrm{II}_A = \frac{1}{2}\left((\mathrm{tr}\mathbf{A})^2 - \mathrm{tr}(\mathbf{AA})\right)$$
$$\mathrm{III}_A = \det(\mathbf{A})$$

Stress tensor (or matrix)

Invariants

$$\lambda = \sigma$$

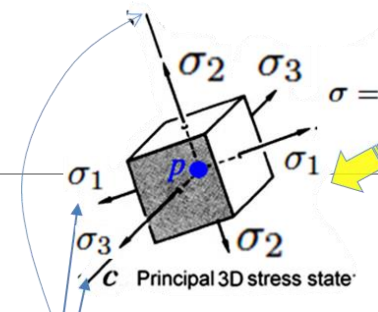
$$(\boldsymbol{\sigma} - \lambda \mathbf{I}) \cdot \hat{\mathbf{n}} = \mathbf{0} \quad \Rightarrow \quad \sigma^3 - I_\sigma \sigma^2 - II_\sigma \sigma - III_\sigma = 0$$

principal directions

principal stresses

Solutions give the principal stresses $\sigma_1, \sigma_2, \sigma_3$
Each principal stress corresponds to a principal direction

$$\sigma = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$



principal axes

Also denoted as: I_1, I_2, I_3

I_1
$$I_\sigma = \sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33} \; ,$$

I_2
$$II_\sigma = (\sigma_{ij}\sigma_{ij} - \sigma_{ii}\sigma_{jj})/2$$
$$= -(\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11}) + \sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2 \; ,$$

I_3
$$III_\sigma = \det(\sigma_{ij}) = \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{vmatrix} \; .$$

The same formula are valid for strain invariants too

Stress Invariants in terms of principle stresses

$$(\boldsymbol{\sigma} - \lambda \mathbf{I}) \cdot \hat{\mathbf{n}} = \mathbf{0} \quad \Rightarrow \quad \sigma^3 - I_\sigma \sigma^2 - II_\sigma \sigma - III_\sigma = 0$$

principal directions

principal stresses

Solutions give the principal stresses $\sigma_1, \sigma_2, \sigma_3$
Each principal stress corresponds to a principal direction

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$

Also denoted as: I_1, I_2, I_3

$$I_1 \equiv \mathrm{tr}(\boldsymbol{\sigma}) = \sigma_1 + \sigma_2 + \sigma_3$$

$$I_2 \equiv II_\sigma = \frac{1}{2}\left[\mathrm{tr}(\boldsymbol{\sigma})^2 - \mathrm{tr}(\boldsymbol{\sigma}^2)\right] \quad \Rightarrow$$

$$I_3 \equiv III_\sigma = \det(\boldsymbol{\sigma})$$

$$\begin{aligned} I_1 &= \sigma_1 + \sigma_2 + \sigma_3 \\ I_2 &= \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \\ I_3 &= \sigma_1\sigma_2\sigma_3 \end{aligned}$$

Most tensors used in engineering are symmetric 3×3.
For this case the invariants can be calculated as:

$$\mathbf{I}_A = \mathrm{tr}(\mathbf{A})$$

$$\mathbf{II}_A = \frac{1}{2}\left((\mathrm{tr}\mathbf{A})^2 - \mathrm{tr}(\mathbf{AA})\right)$$

$$\mathbf{III}_A = \det(\mathbf{A})$$

Invariant - Invariantti

Deviatoric stress

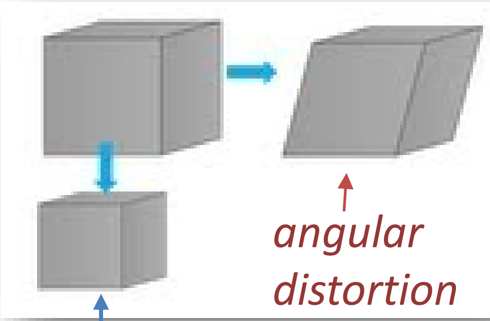
Jännitysdeviaattori

$$
\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix} + \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix}
$$

- additive decomposition of stress tensor – very useful

$$
\sigma_{ij} = \frac{\sigma_{kk}}{3}\,\delta_{ij} + s_{ij} \qquad \text{or} \qquad \boldsymbol{\sigma} = \sigma_m\, \boldsymbol{I} + \boldsymbol{s}\ .
$$

Why splitting Volumetric and deviatoric (shearing)**?**

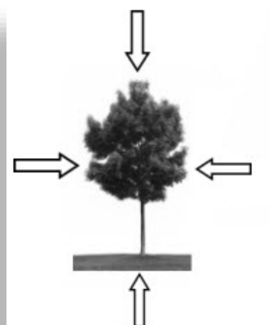


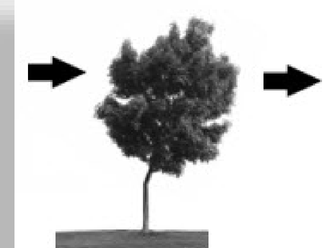
volume change

No volume change

Mean normal stress $-p \equiv \sigma_m$ or also called ***hydrostatic stress state***

Deviatoric stress $\mathbf{s}$ characterizes deviation of the stress state from a hydrostatic state

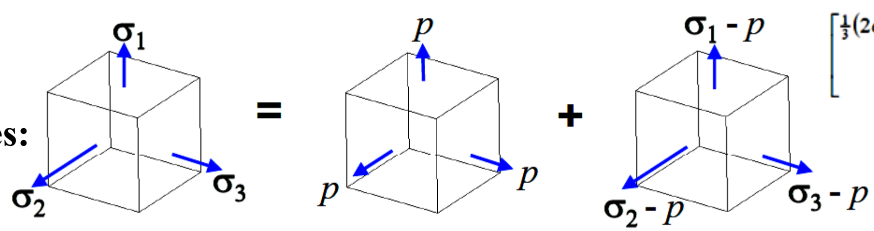




$$
\boldsymbol{\sigma} \quad = \quad -p\mathbf{I} \quad + \quad \mathbf{s}
$$

$$
\sigma_m = \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3}
$$

Expressed in principle stresses:



$$
\begin{bmatrix} \frac{1}{3}(2\sigma_{11} - \sigma_{22} - \sigma_{33}) & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \frac{1}{3}(2\sigma_{22} - \sigma_{11} - \sigma_{33}) & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \frac{1}{3}(2\sigma_{33} - \sigma_{11} - \sigma_{22}) \end{bmatrix}
$$

: $\sigma_1 \neq \sigma_2 \neq \sigma_3$)

hydrostatic stress
volume change only

deviatoric stress - *jännitysdeviaattori*
angular distortion only

Stress Invariants 2(2)

Also denoted as: J_1, J_2, J_3

$$J_1 = \mathrm{tr}(\mathbf{s}) = 0$$
$$J_2 = \frac{1}{2}\mathrm{tr}(\mathbf{s}^2)$$
$$J_3 = \det(\mathbf{s}) = \frac{1}{3}\mathrm{tr}(\mathbf{s}^3)$$

Invariants of the deviatoric Stress:

deviatoric Stress

$$\sigma = -p\mathrm{I} + \boxed{\mathbf{s}}$$

J_1 $\quad I_s = 0\,,$

$\frac{1}{2}\mathrm{tr}(\boldsymbol{s}^2)$

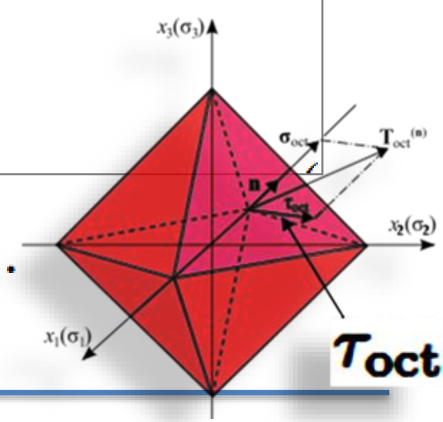
J_2 $\quad II_s = \frac{1}{2}s_{ij}s_{ij} = \frac{1}{6}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right]$

$$= \frac{1}{6}\left[(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2\right] + \sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2$$

J_3 $\quad III_s = \frac{1}{3}s_{ij}s_{jk}s_{ki}$

$\frac{1}{3}\mathrm{tr}(\boldsymbol{s}^3)$

$$\tau_{oct} = \sqrt{\frac{2}{3}J_2} \longleftrightarrow II_s = \frac{3}{2}\tau_{\mathrm{oct}}^2.$$

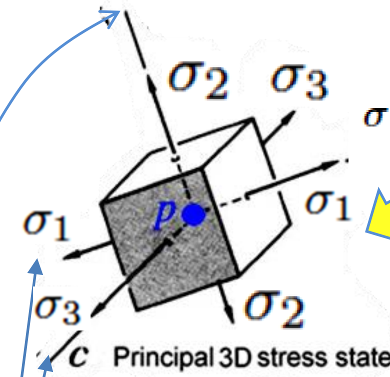


- For instance, the *equivalent stress* (von Mises stress) – *vertailujännitys* - commonly used in solid mechanics is .... = $\sigma_e = \sqrt{3J_2}$ (which is up to a constant coefficient the maximum shear stress on the octahedral plane)

$$\sigma_e \propto \tau_{oct}$$

Maximum shear stress

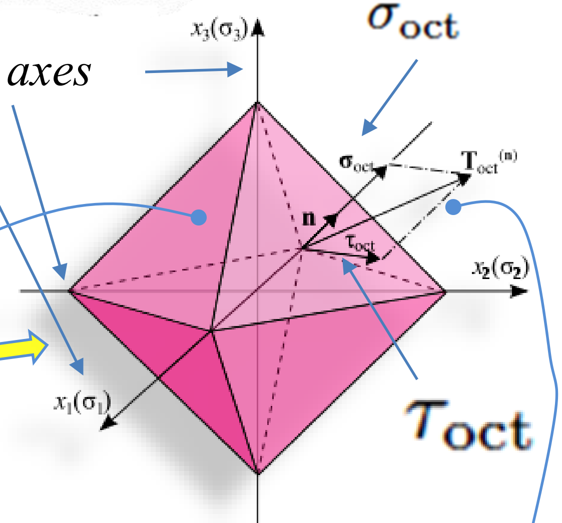
- ***Extreme shear*** stresses appear in sections with normal is perpendicular to one *principal axis* and forms ***angles of 45 deg.*** with the remaining two axes



$$\sigma = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$

principal axes

direction cosines = $|1/\sqrt{3}|$



Octahedral Stresses:

- Normal and shear stress in cross sections whose normal forms an equal angle with all 3 principal axes:

Mean normal stress $-p \equiv \sigma_m$ or also called ***hydrostatic stress state***

$$\begin{cases} \sigma_{oct} = \dfrac{\sigma_1 + \sigma_2 + \sigma_3}{3} = \dfrac{\sigma_{ii}}{3} = \dfrac{I_\sigma}{3}\,, \\ \tau_{oct} = \dfrac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}\,. \end{cases}$$

$$\boldsymbol{n} \equiv \boldsymbol{n}_{okt} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

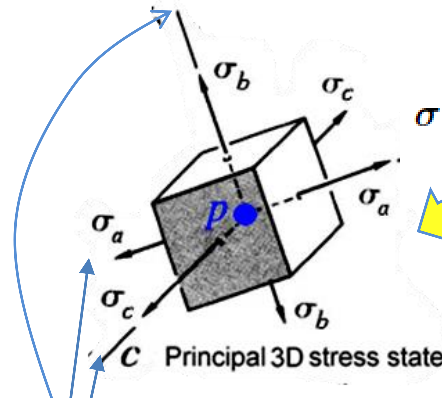
$$\tau_{oct} = \sqrt{\frac{2}{3} J_2}$$

$$\sigma_n = \boldsymbol{t} \cdot \boldsymbol{n} = t_i n_i = \sigma_{ij} n_i n_j .$$

Octa = eight = 8; we have 8 such planes

Maximum shear stress:

- ***Extreme shear*** stresses appear in sections with normal is perpendicular to one *principal axis* and forms ***angles of 45 deg***. with the remaining two axes



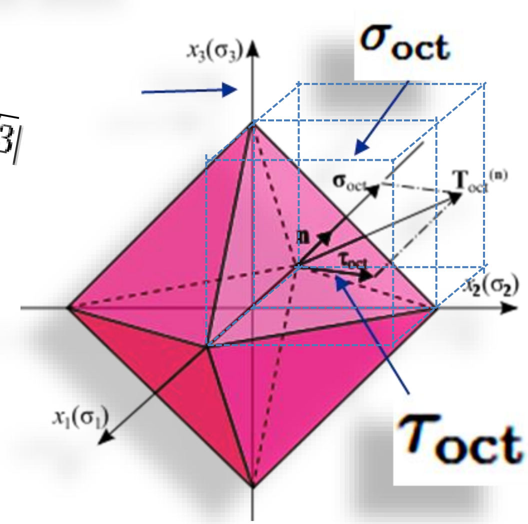
$$\sigma = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$

Octahedral stresses:

direction cosines = $|1/\sqrt{3}|$

- Normal and shear stress in cross sections whose normal forms an equal angle with all 3 principal axes:

$$\begin{cases} \sigma_{\text{oct}} = \dfrac{\sigma_1+\sigma_2+\sigma_3}{3} = \dfrac{\sigma_{ii}}{3} = \dfrac{I_\sigma}{3}\,, \\ \tau_{\text{oct}} = \dfrac{1}{3}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}\,. \end{cases}$$



Can be related to some type of weighted mean ~'max. shear stress'

$\tau_{oct} = \sqrt{2/3}\cdot\tau_y, \quad \tau_y = \sigma_y/\sqrt{3}$ von-Mises yield condition: $\sqrt{J_2} = \tau_y$

$$n \equiv n_{\text{okt}} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

Exercise: show these results:

$$\sigma_{\text{oct}} = \frac{I_\sigma}{3} \qquad \tau_{oct} = \sqrt{\frac{2}{3}J_2}$$

Hints:

$$\sigma_n = \boldsymbol{t}\cdot\boldsymbol{n} = t_i n_i = \sigma_{ij} n_i n_j.$$

Invariants in general

$$\mathbf{A} = \boldsymbol{\sigma}$$

- 2nd-order symmetric tensors (matrices) have always three groups independent invariants which, for our purposes, in mechanics, are defined as

$$I_1^A = \operatorname{tr}\mathbf{A},$$
$$J_2^A = \tfrac{1}{2}\operatorname{tr}(\mathbf{A}^{\mathrm{d}})^2,$$
$$J_3^A = \tfrac{1}{3}\operatorname{tr}(\mathbf{A}^{\mathrm{d}})^3,$$

$$\mathbf{A} = \boldsymbol{\sigma}$$

The stress deviator $\mathbf{S} = \boldsymbol{\sigma}^{\mathrm{d}}$

$$\mathbf{S} = \mathbf{A}^{\mathrm{d}} = \boldsymbol{\sigma}^{\mathrm{d}} \equiv \mathbf{s}$$
$$\mathbf{T} = (\mathbf{S}^{\mathrm{d}})^2,$$

$$\operatorname{tr}\mathbf{A} = A_{kk}$$
$$\mathbf{A}^{\mathrm{d}} = \mathbf{A} - \tfrac{1}{3}(\operatorname{tr}\mathbf{A})\mathbf{I}$$

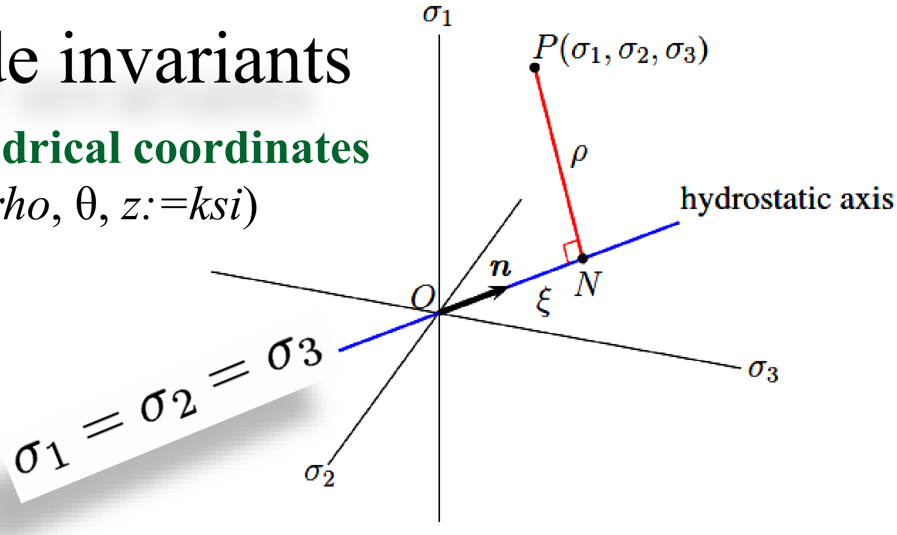
- In plasticity, one needs to compute gradients of functions of invariants, for instance, the gradient for an arbitrary isotropic yield function (Isotropic plasticity theory) is as

$$\frac{\partial f(I_1, J_2, J_3)}{\partial \boldsymbol{\sigma}} = \frac{\partial f}{\partial I_1}\mathbf{I} + \frac{\partial f}{\partial J_2}\mathbf{S} + \frac{\partial f}{\partial J_3}\mathbf{T}$$

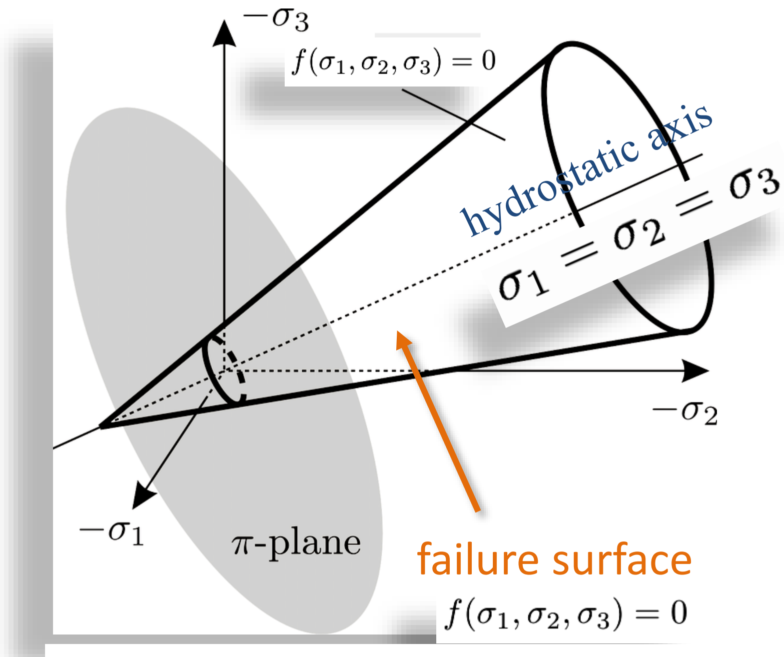
, where Hill tensor $\mathbf{T} = (\mathbf{S}^2)^{\mathrm{d}}$;

Lode invariants

cylindrical coordinates
(*r:=rho*, θ, *z:=ksi*)



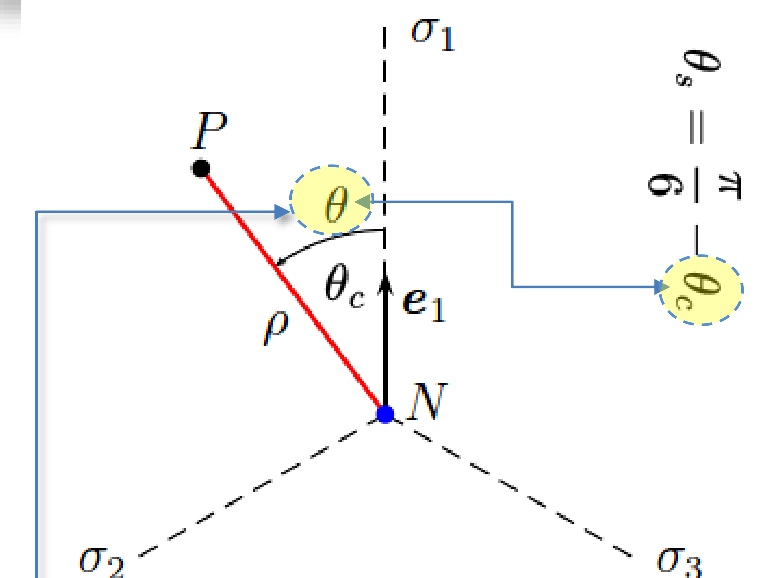
Principal stress space.



- **Lode** invariants is an other alternative invariant triplet that is more useful than principal stresses for geometrical visualization of isotropic yield surfaces and is the **cylindrical coordinates** (r,θ,z) **representation** of the above representation, where the z-coordinate points along the hydrostatic axis

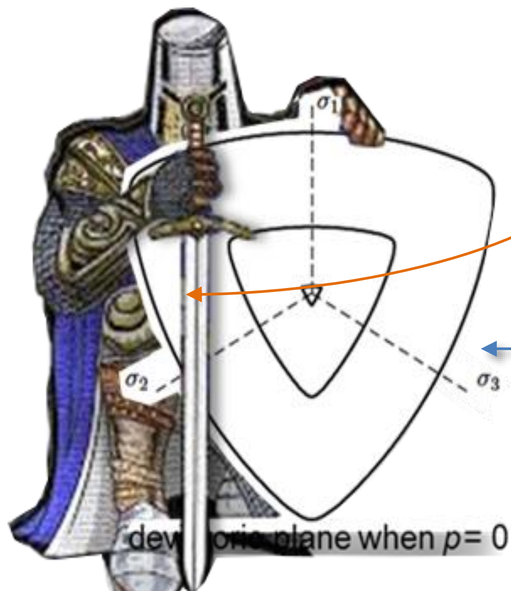
$$r = \sqrt{2J_2}, \qquad \sigma_1 = \sigma_2 = \sigma_3 \qquad z = \frac{I_1}{\sqrt{3}}$$

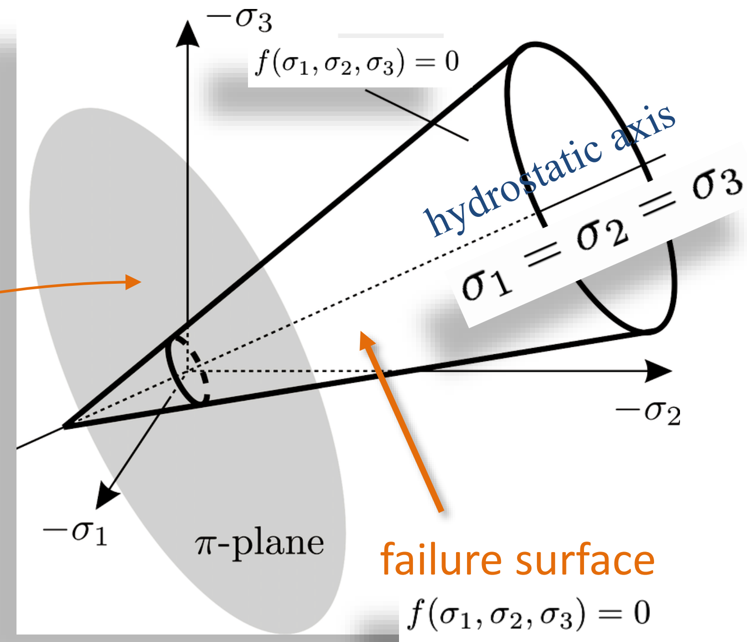
$$\sin(3\theta_s) = -\sin(3\bar{\theta}_s) = \cos(3\theta_c) = \frac{J_3}{2}\left(\frac{3}{J_2}\right)^{3/2}$$



Deviatoric plane. The projections of the principal stress axes are shown with dashed line (*ref*. Reijo's lecture Notes)

$$f(\sigma_1, \sigma_2, \sigma_3) = 0$$





Failure criteria for concrete:

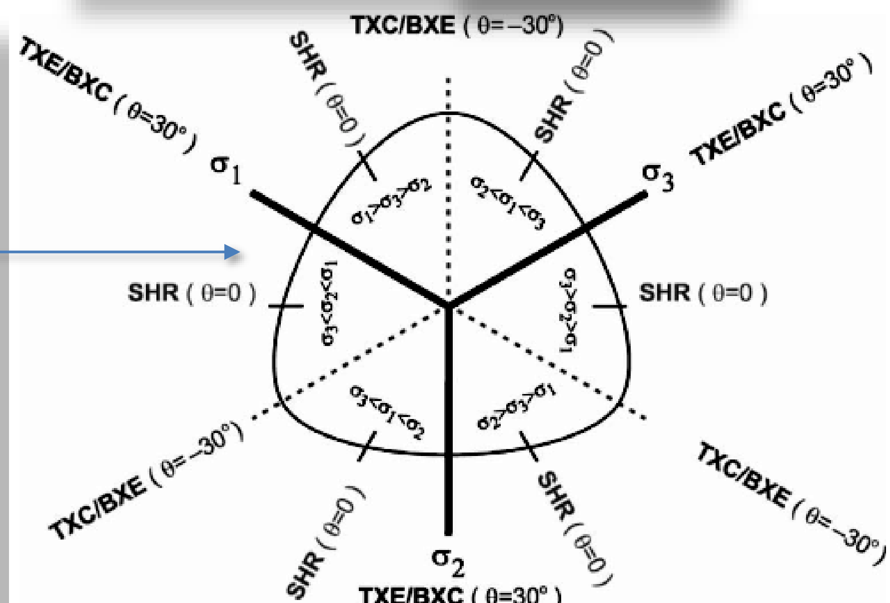
$$A\frac{J_2}{\sigma_c} + \Lambda\sqrt{J_2} + BI_1 - \sigma_c = 0,$$

$$r = \sqrt{2J_2},$$

$$\sigma_1 = \sigma_2 = \sigma_3$$

$$z = \frac{I_1}{\sqrt{3}}$$

$$\sin(3\theta_s) = -\sin(3\bar{\theta}_s) = \cos(3\theta_c) = \frac{J_3}{2}\left(\frac{3}{J_2}\right)^{3/2}$$

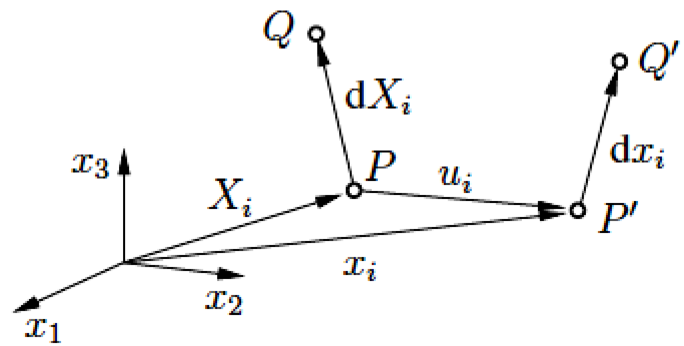


Periodicity of the Lode angle in an octahedral plane

Ref for this figure above: Brannon, R. M. (2007). Elements of Phenomenological Plasticity: geometrical insight, computational algorithms, and applications in shock physics. Shock Wave Science and Technology Reference Library: Solids I, Springer-New York. 2: pp. 189-274

Strain

Deformation



$$u_i = u_i(X_j, t) \; , \qquad x_i = x_i(X_j, t)$$

displacement vector

Spatial coordinates

Material coordinates

$$\left.\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}x_k \mathrm{d}x_k = \frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j}\,\mathrm{d}X_i \mathrm{d}X_j \; , \\ \mathrm{d}S^2 &= \mathrm{d}X_k \mathrm{d}X_k = \mathrm{d}X_i \mathrm{d}X_j \, \delta_{ij} \; . \end{aligned}\right\} \Rightarrow \quad \mathrm{d}s^2 - \mathrm{d}S^2 = 2\,E_{ij}\,\mathrm{d}X_i \mathrm{d}X_j$$

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right)$$

Green's strain tensor

Infinitesimal or Engineering strain tensor:

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \; . \longrightarrow \boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix}$$

$$(\partial u_i / \partial X_j \ll 1)$$

symmetric

Strain invariants

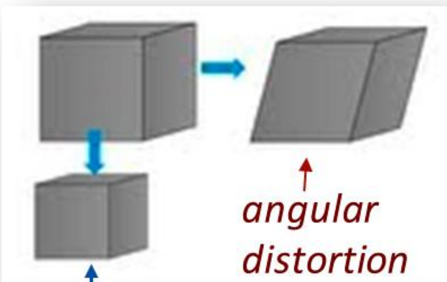
Same formula as for the stress tensor:

$$I_\varepsilon,\ II_\varepsilon,\ III_\varepsilon$$

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})\ . \qquad \boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix}$$

$(\partial u_i / \partial X_j \ll 1)$

Engineering strains



volume change   No volume change

$$\boxed{I_\varepsilon = \varepsilon_V = \varepsilon_{kk} = \varepsilon_1 + \varepsilon_2 + \varepsilon_3\ .}$$

Decomposition: *Volumetric strain* (relative volume change)

$$\boxed{\varepsilon_{ij} = \frac{\varepsilon_{kk}}{3}\,\delta_{ij} + e_{ij} \qquad \text{or} \qquad \boldsymbol{\varepsilon} = \frac{\varepsilon_V}{3}\boldsymbol{I} + \boldsymbol{e}}$$

Distortion = Deviator

Most **tensors used in engineering** are symmetric 3×3.
For this case the invariants can be calculated as:

$$\mathbf{I}_A = \mathrm{tr}(\mathbf{A})$$

$$\mathbf{II}_A = \frac{1}{2}\left((\mathrm{tr}\mathbf{A})^2 - \mathrm{tr}(\mathbf{AA})\right)$$

$$\mathbf{III}_A = \det(\mathbf{A})$$

Second Invariant
of the deviator:

$$II_e = \frac{1}{2}\,e_{ij}e_{ij} = \frac{1}{6}\left[(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2\right]\ .$$