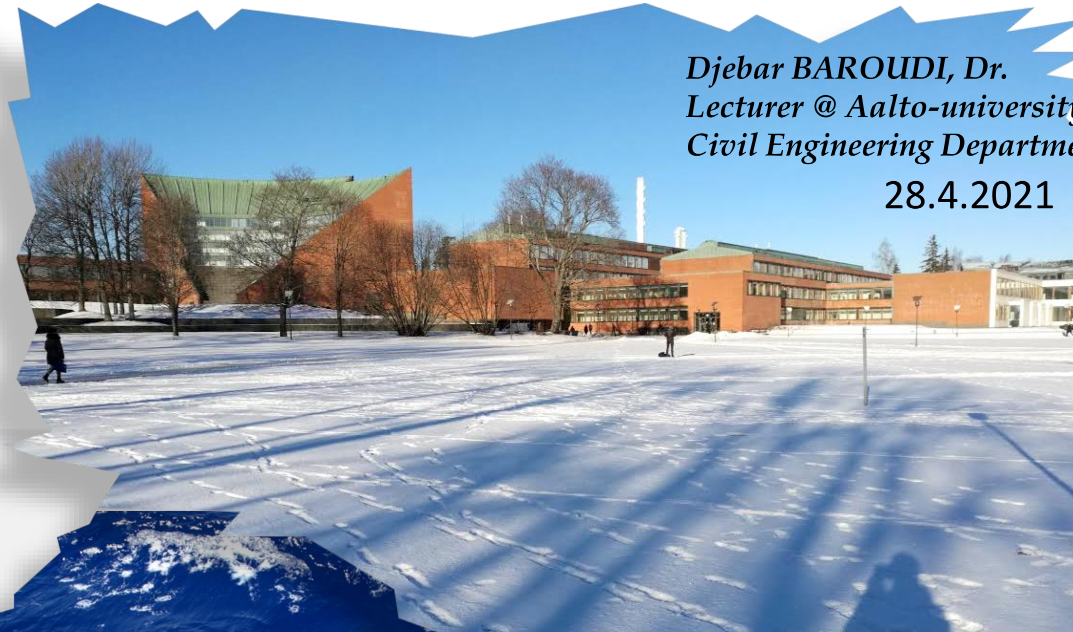
$$\mathcal{F}\left(\sigma, \dot{\sigma}, \varepsilon, \dot{\varepsilon}, T, \dot{T}\right) = 0$$

Slides #1- #35 introduction

Start: slide #35 or #66

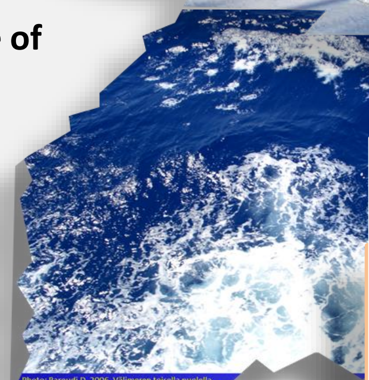


Djebar BAROUDI, Dr.
Lecturer @ Aalto-university
Civil Engineering Department
28.4.2021

Viscoelasticity

Content

- experimental observations: evidence of viscoelastic behavior
- stress relaxation at constant strain
- creep at constant stress
- strain-rate dependence
- constitutive models in the rate form:
  - Maxwell model
  - Kelvin-Voight model
  - Standard linear solid model
  - Burgers model
  - Generalized Maxwell model
  - Kelvin chain model



Reading: **Textbooks**

- Lemaitre and Chaboche – *Mechanics of Solid Materials*. **Chapter 4.3**
- Ottosen & Ristinmaa – *Introduction to time-dependent material behaviour.* **Chapter 14**

Experimental: Compressive responses of **balsa** wood at static, intermediate, and high strain rate

Lemaitre & Chaboche textbook as an **e-book**:
http://proquestcombo.safaribooksonline.com.libproxy.aalto.fi/book/physics/9781107384712

Version 3.5.2021

Content

Introduction

Why the constitutive equation is needed?
How Viscoelasticity does manifest itself?
What is meant by material aging?
Non-aging material
Experimental observation: typical mechanical responses of Viscoelastic materials
The characteristic time scales involved in viscoelastic response

Engineering Viscoelasticity

Examples of viscoelastic materials
Strain-rate dependence
Concepts and Definitions
Stress **relaxation** at constant strain
Creep at constant stress
Dynamic loading tests
Example of measurement of complex modulus

Constitutive models of linear viscoelasticity

Constitutive models in the rate form
(On constitutive models for Viscoplasticity)

- Maxwell model
- Kelvin-Voight model
- Standard linear solid model
- Generalized Maxwell model
- Kelvin chain model

3D-formulation of isotropic viscoelasticity
… ***continued***

Core content

…

Homework and examples

Elective Reading material

This material will be presented and commented during the last lecture on the topic, I hope we will have time for that

Constitutive models for viscoelasticity

Relation between Creep 'coefficient' and compliance function
Example of experimental evidence of creep in concrete

Reading: ***Textbook*** **-** Lemaitre and Chaboche – *Mechanics of Solid Materials*. *Chapter* 4.3

Course of **materials modelling** in other universities

TECHNISCHE UNIVERSITÄT DARMSTADT

Technische Universität Darmstadt

9.10.2016

13-02-0003-vl Werkstoffmechanik

Veranstaltungsdetails

Lehrende: Prof. Dr.-Ing. Michael Vormwald; Dipl.-Ing. Melanie Fiedler

Veranstaltungsart: Vorlesung

Orga-Einheit: FB13 Bau- und Umweltingenieurwissenschaften

Anzeige im Stundenplan: Werkstm. (V)

Fach:

Anrechenbar für:

Semesterwochenstunden: 3

Unterrichtssprache: Deutsch

Min. | Max. Teilnehmerzahl: - | -

Lehrinhalte:

- Klassifizierung der Phänomene des Deformations- und Festigke
- Lineare Elastizität
- Isotropie, Anisotropie (Orthotropie, transversale Isotropie)
- Elastoplastizität
- Idealplastizität, Isotrope und kinematische Verfestigung
- Viskoelastizität, Viskoplastizität
- Werkstoffgesetze für Stahl, Beton, Glas, Holz, Kunststoffe und Geomaterialien
- Numerische Umsetzung

Today subject

... for comparison of the courses contents at **two universities to show the relevance of such course content for CIV-engineers**

Aalto-university,
summer 2016

Content:

Djebar BAROUDI, PhD
Aalto-University

- **Elasticity – kimmoisuus tai elastisuus**
  (linear, hyper-elasticity, isotropy, anisotropy, orthotropy)
- **Viscoelasticity - viskoelastisuus**
- **Viscoplasticity** or creep – **viskoplastisuus ... viruminen**
- **Failure hypotheses - lujuushypoteesit**
- **Plasticity - plastisuus**
  associative, non-associative
- **Damage - vauriotuminen**
  damage-plasticity *ex.* Concrete Damage Plasticity, Model in Abaqus

If time...

Ref: thanks go to an exchange student for providing the course content-list above

http://www.werkstoffmechanik.tu-darmstadt.de/lehre_10/werkstoffmechanik_2/index.de.jsp

You can read also from

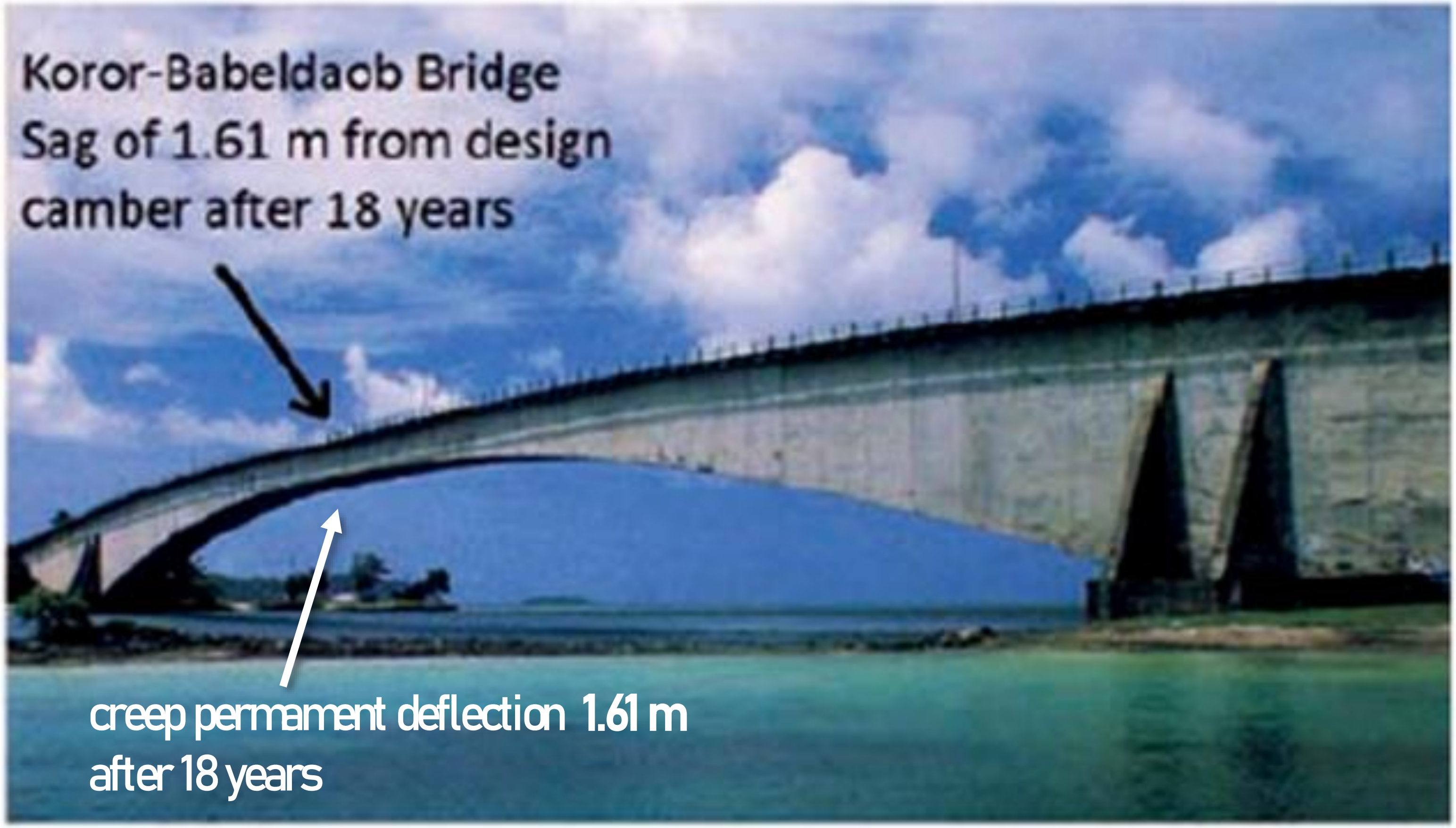
Prof. **Reijo Kouhia's lecture notes:**

Introduction to materials modelling **,** 103 pp.

These pdfs are in MyCourses

CONTENTS

Maximal Minimalism: Less than 20 pp.


Koror-Babeldaob Bridge
Sag of 1.61 m from design camber after 18 years
creep permament deflection 1.61 m after 18 years

Cement and Concrete Research 41 (2011) 1310–1319

ELSEVIER

Contents lists available at ScienceDirect

Cement and Concrete Research

journal homepage: http://ees.elsevier.com/CEMCON/default.asp

Long-term deformational simulation of PC bridges based on the thermo-hygro model of micro-pores in cementitious composites

Koichi Maekawa *, Nobuhiro Chijiwa, Tetsuya Ishida

Department of Civil Engineering, The University of Tokyo, Japan

Experimental data

The variation with time of the vertical deflection at the span center, measured by an optical measurement device since completion of construction in 1982.

Overview of Tsukiyono Bridge

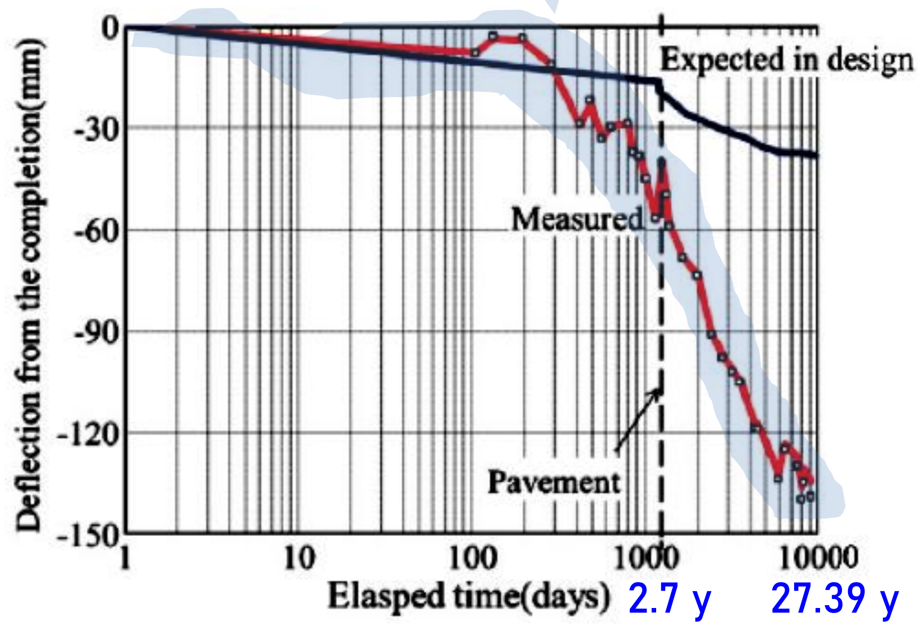
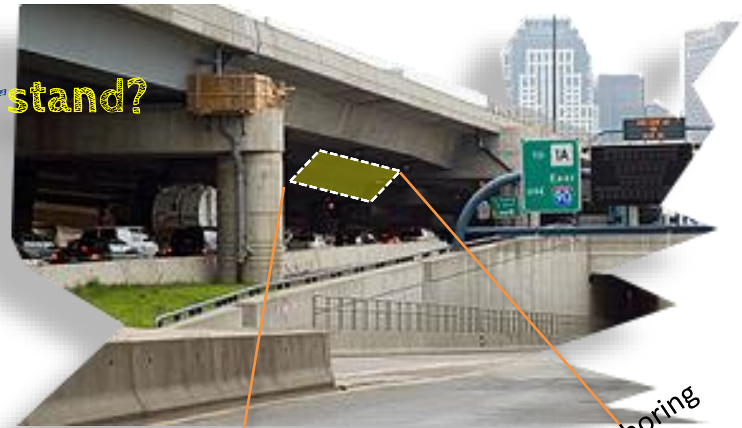


Fig. 3. Difference in predicted and measured values of deflection (the component caused by the overlaid pavement is taken out from the entire deflection).

Viscoelastic material behaviour

Why viscoelasticity is it important to understand?
Why Hooke's Law is not sufficient here?

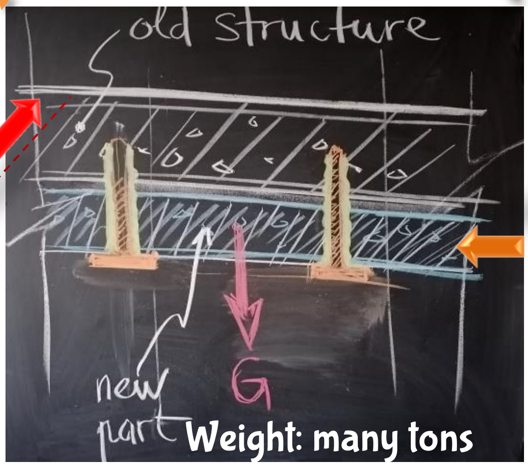


Accident: The **Big Dig ceiling collapse** occurred on July 10, 2006, when a concrete ceiling panel and debris weighing few tons (~6 x 12 m) fell in *Fort Point Channel Tunnel* (Boston). The panel fell on killing a car passenger and injuring the driver

other type anchoring

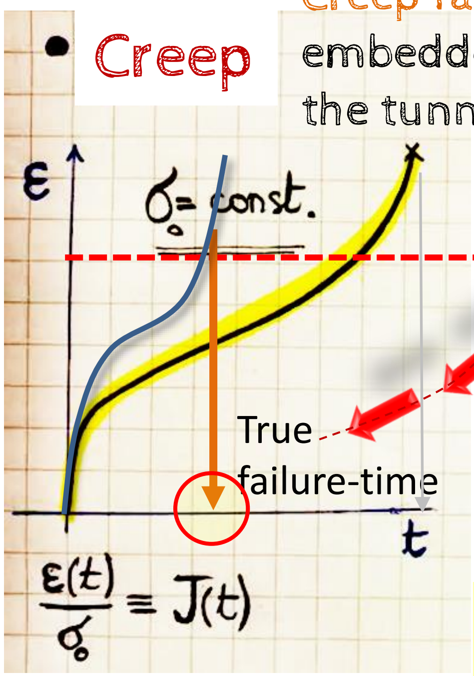
a **viscoelastic polymer**

Epoxy glue filling between concrete and bolt



added hanging concrete blocs

Creep failure of anchors embedded in epoxy in the tunnel's roof slab



Here, bolts pull out of the epoxy

- Hanging concrete ceiling blocks 24 tons/block
- Anchoring the concrete blocks with bolts embedded into the epoxy

Over time, the bolts pulled out of the epoxy

The **epoxy** used was a **viscoelastic material** that **deforms over time when a force is applied (creep)**

Leading to **creep failure**

Basics of viscoelasticity — Time and temperature dependent materials

In many previous courses, we were used to deal with the *elastic* mechanical response of solid materials to external excitations. Such elastic response is independent of time and te induced deformations are completely reversible. We treated linear ad non-linear elasticity. Such behaviour is shortly labelled *elastic*. In addition, we had also been introduced some basics of *engineering plasticity* when studying limit states of beam- and frame structures to determine *limit plastic loads*. In this cases, the mechanical response is termed *plastic*. In such behaviour we limited our-selves, because of the above application, to *time-independent plasticity*.

In this short course we will study material mechanical behaviours of solids that are *time* and *temperature* dependent. Such responses is called *thermo-visco-elasticity*. Time-dependent[2] materials can be regrouped in the following classes

- **Viscous materials** - flowing fluids, mechanical response can be essentially captured by a viscosity coefficient.
- **Visco-plastic materials** - the material starts to flow above some *stress threshold*[3] value, friction viscous flow (kikallinen virtaus).
- **Visco-elastic materials** - the material posses both of *viscous fluid* and *elastic solid* properties.
- **Visco-elasto-plastic materials** - in addition to the properties of visco-elasticty, *flow* or *yielding*[4] starts in the material above some stress threshold value. Often, this behaviour is also regrouped with *visco-elasticity* family.

In the above classification, we have assumed that only isothermal processes occur during loading and deformation (in solids) or flow (in solids and fluids). When the temperature changes during the loading, the material response is termed as *thermo-visco-elastic*.

A generic response $\boldsymbol{\sigma} = f(\epsilon, \dot{\epsilon}, \dot{\boldsymbol{\sigma}}, T, \dot{T}, t; \beta)$ $\quad$ β are some internal (hidden) parameters

Rheology is a branch of physics which study the motions (flows) and deformation (in solids) of materials resulting from various loading at the *phenomenological*[5] *description scale.*

Rheology characterizes flow and deformation of materials that exhibits a combination of *elastic*, *viscous* and *plastic* behaviour.

"Panta rhei" is a dictum that became the motto of the (American) Society of Rheology. The saying *"Everything flows and nothing stays"*[6], is said to belong to **Heraclitus** (535 BC – 475 BC).

Another saying *"The mountains flowed before the Lord"* (The Song of Deborah, Bible) is related to the definition of a non-dimensional *Deborah*-number

Deborah-number $\mathrm{De} = \dfrac{\tau_c}{\tau_p},$

τ_c a characteristic time-scale relaxation-time of processes resulting in creep (flow)

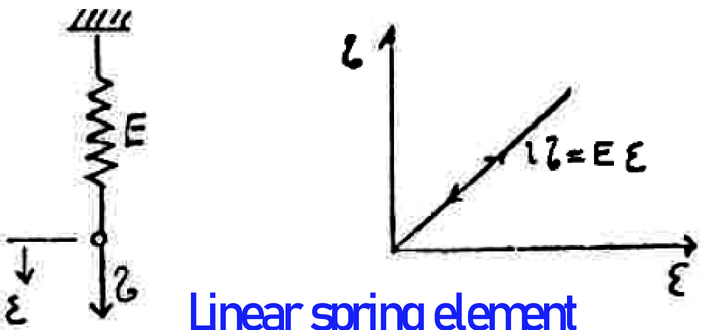
τ_p: observation time (or time-window)

$$\tau_c \gg \tau_p \implies \textbf{solid}$$

$$\tau_c \ll \tau_p \implies \textbf{fluid}$$

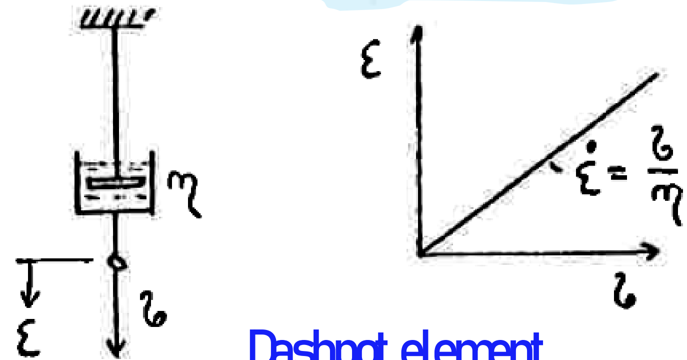
Basic rheological models

To help developing and better understanding *constitutive relations* of such various material behaviours simple mechanical sub-models and models are used to capture the key or *basic responses* of *fluid*- and *solid*-like behaviours. Such simple models are called *Rheological models*. Such basic rheological models can be combined to obtain a more complex (realistic) response of materials.



Linear spring element

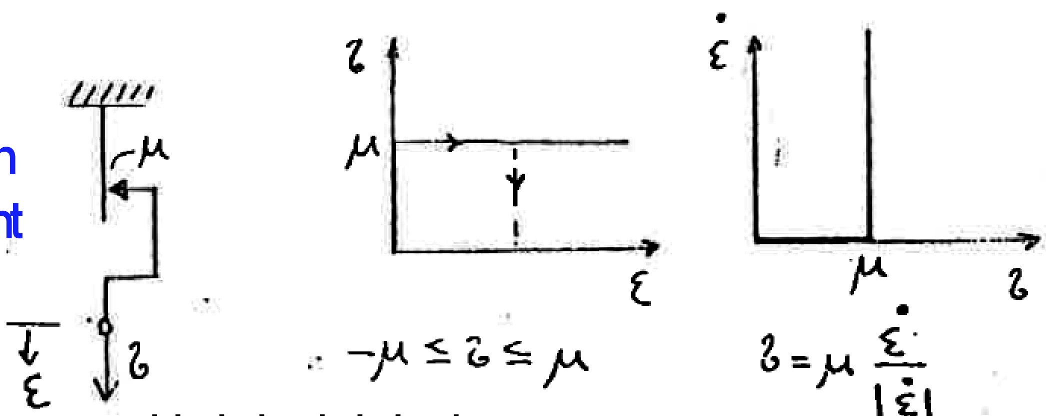
linear elastic behaviour (solid)
E = material elasticity modulus



Dashpot element

ideal viscous behaviour (Newton's fluid)
η = viscosity coefficient

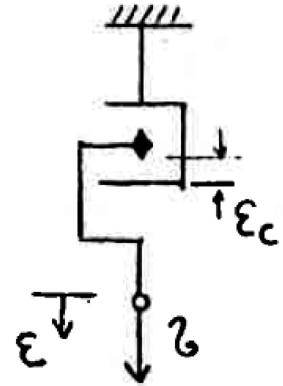
Friction element

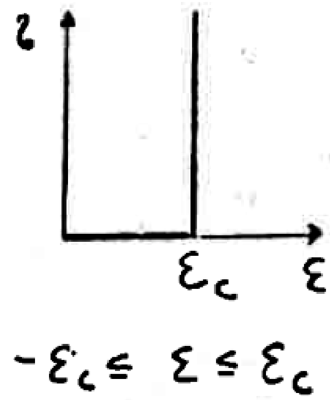


ideal plastic behaviour
μ = plastic friction coefficient or yield stress (threshold)

Basic rheological models

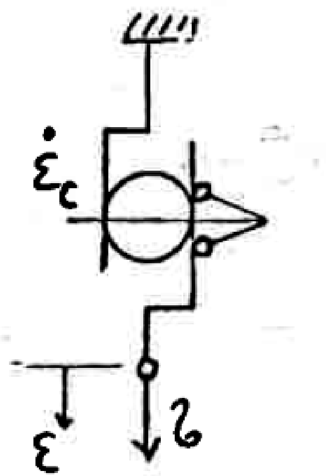
Strain limiting element

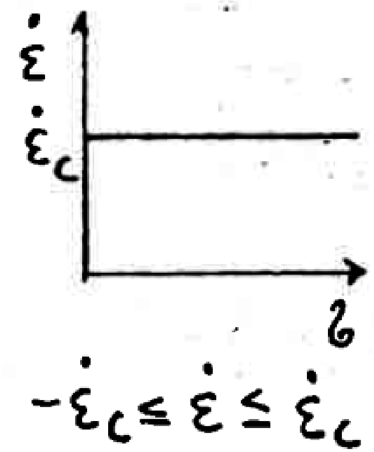




used to limit the deformations (strain locking materials)

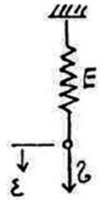
velocity (or deformation rate) limiting element





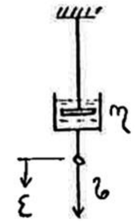
used to limit the velocity of deformations (strain rate locking materials)

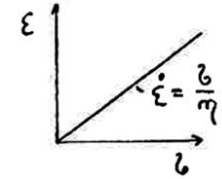
Basic rheological models



Linear spring element

linear elastic behaviour (**solid**)
E = material elasticity modulus

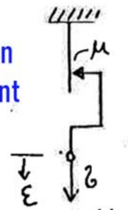


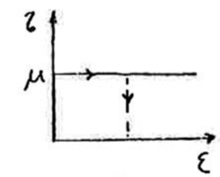


Dashpot element

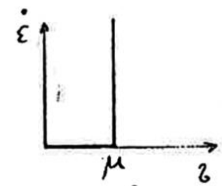
ideal viscous behaviour (Newton's **fluid**)
η = viscosity coefficient

Friction element





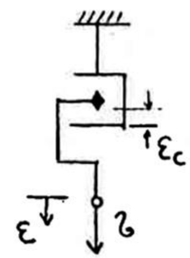
$-\mu \le \sigma \le \mu$

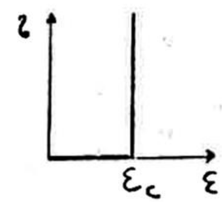


$\sigma = \mu \frac{\dot{\varepsilon}}{|\dot{\varepsilon}|}$

ideal plastic behaviour
μ = plastic friction coefficient or **yield stress** (threshold)

Strain limiting element

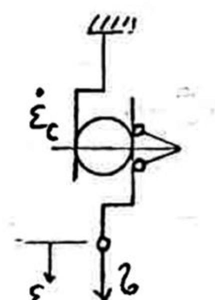


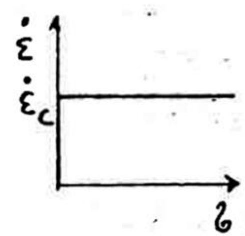


$-\varepsilon_c \le \varepsilon \le \varepsilon_c$

used to limit the deformations (strain locking materials)

velocity (or deformation rate) limiting element





$-\dot{\varepsilon}_c \le \dot{\varepsilon} \le \dot{\varepsilon}_c$

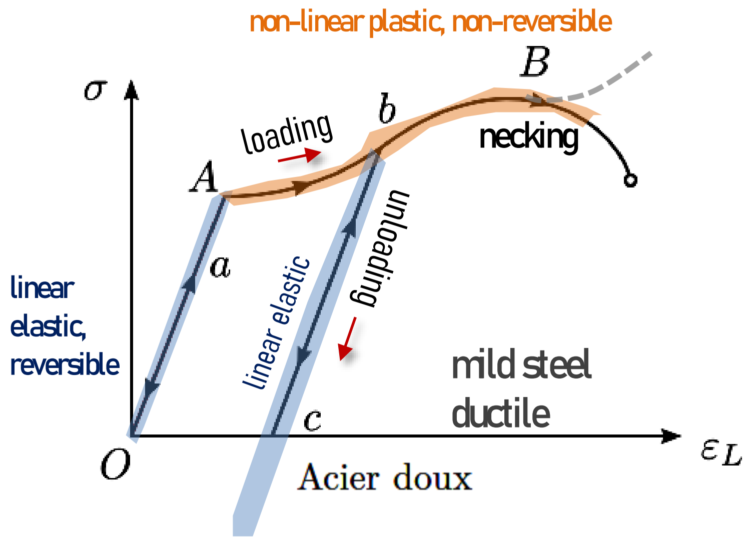
used to limit the velocity of deformations
(strain rate locking materials)

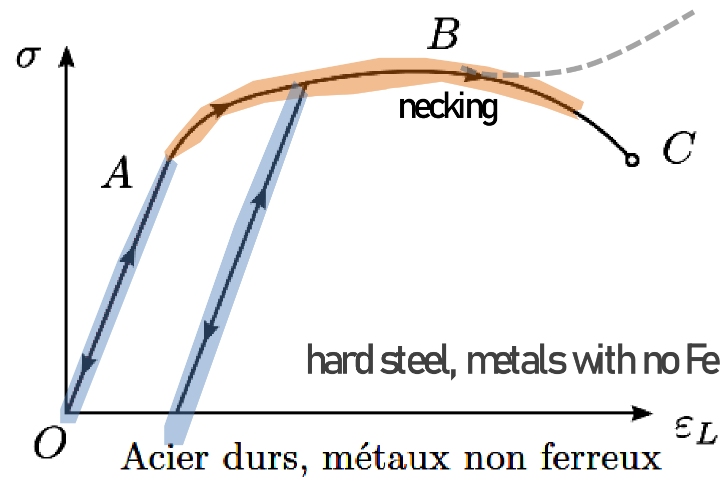
Overview from the first lecture:
= Recall

Some typical mechanical responses of solids

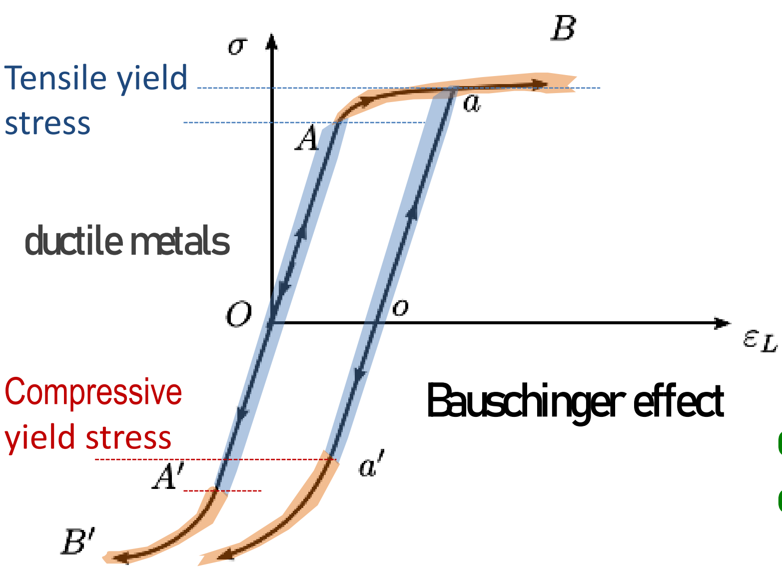
Diversity of material responses of solids – typical examples

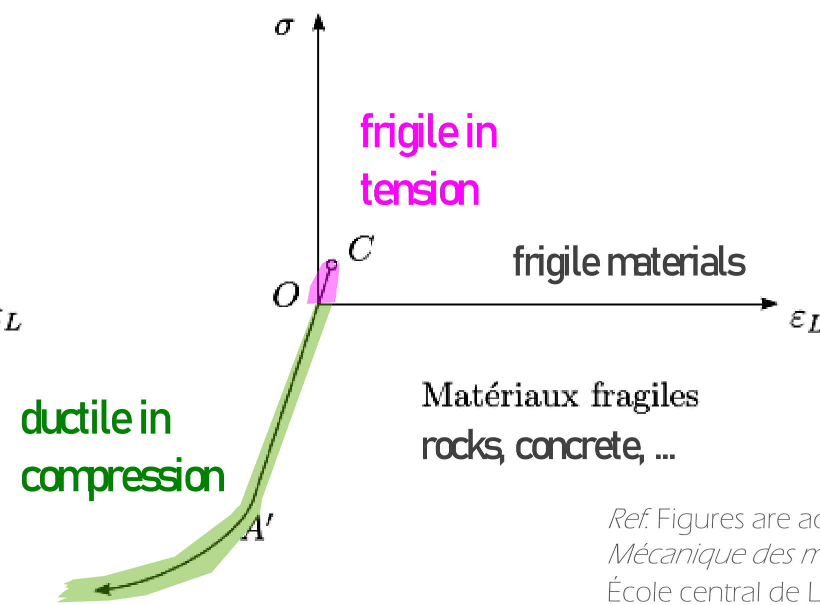
Metals at ambient temperature & quasi-satic loading





Traction test characterisation





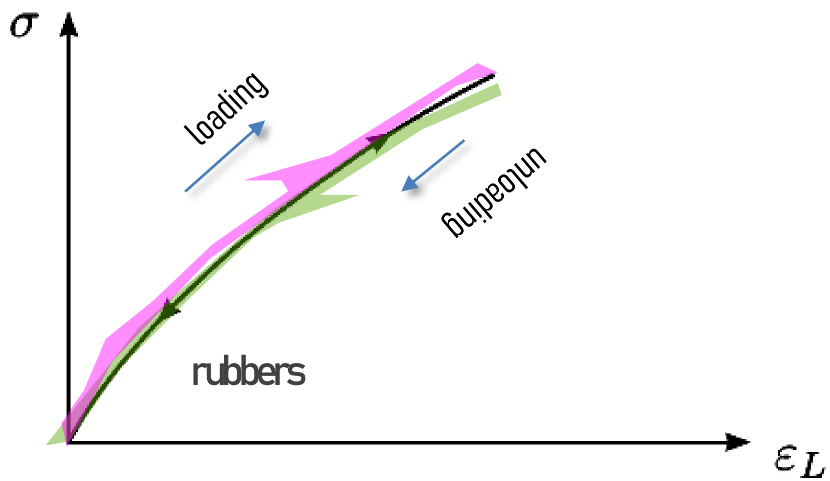
Traction test characterisation

Ref. Figures are adapted from: F. Sidoroff *Mécanique des milieux continus* École central de Lyon.

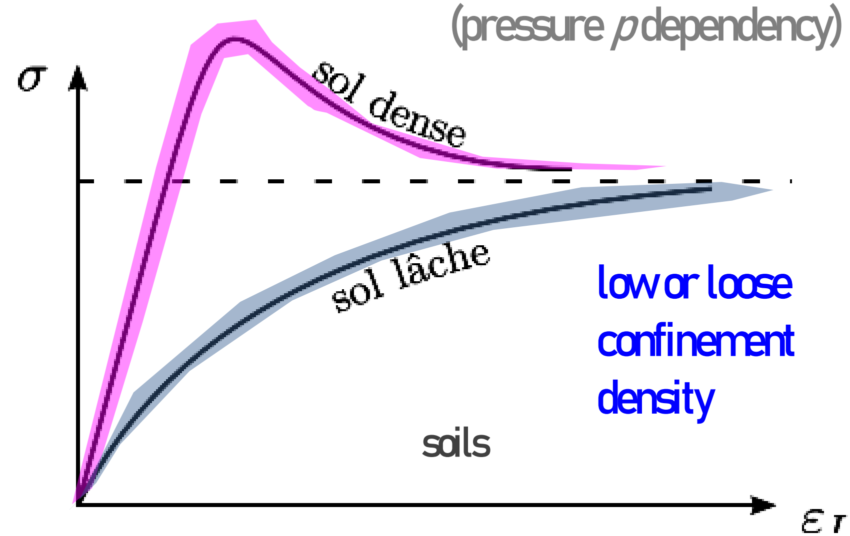
Diversity of material responses of solids – typical examples

At ambient temperature & quasi-satic loading

Traction/compression test characterisation



high confinement density (pressure *p* dependency)



low or loose confinement density

Traction/compression or shear-box test characterisation

- monotonic traction or shear-box tests can be used to characterise the mechanical material response
- mechanical response of materials like plastics, polymers, and metals at high temperatures cannot anymore be described by such traction tests because it crucially depends on deformation rate

visco-elastic
visco-plastic

- Needs for creep (viruminen) and stress relaxation (relaksaatio) tests

Ref. Figures are adapted from: F. Sidoroff *Mécanique des milieux continus* École central de Lyon.

Diversity of material responses of solids - typical examples

visco-elastic
visco-plastic

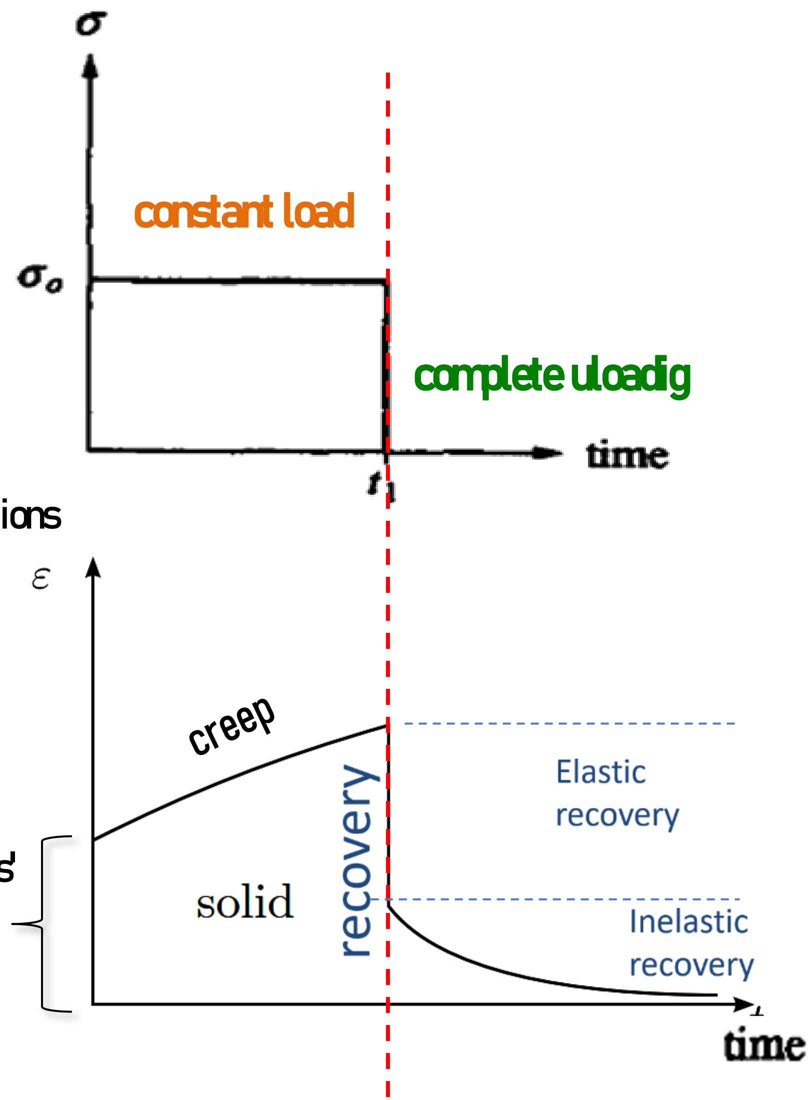
Material response is crucially rate dependent

Monotonic traction or shear tests makes no sense anymore for such behaviours

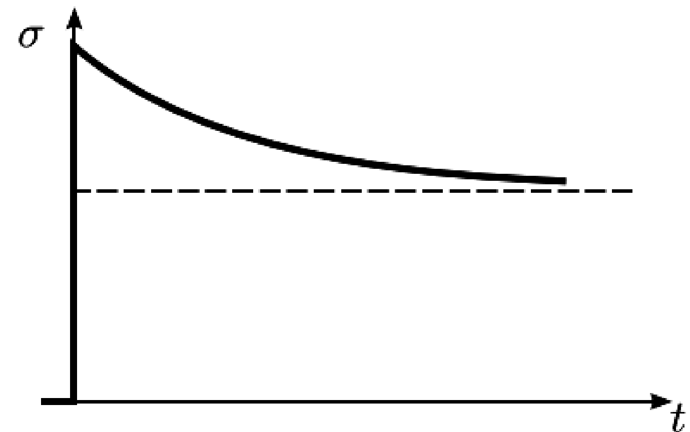
- creep tests
- reaxaation tests
- dynamic tests

for material characterisation

Creep tests
- constant load
- measure deformations



'instantaneous' elastic deformation



Relaxation tests
- constant deformation
- measure stresses

Diversity of material responses of solids - typical examples

visco-elastic
visco-plastic

Material response is crucially rate dependent

Monotonic traction or shear tests makes no sense anymore for such behaviours

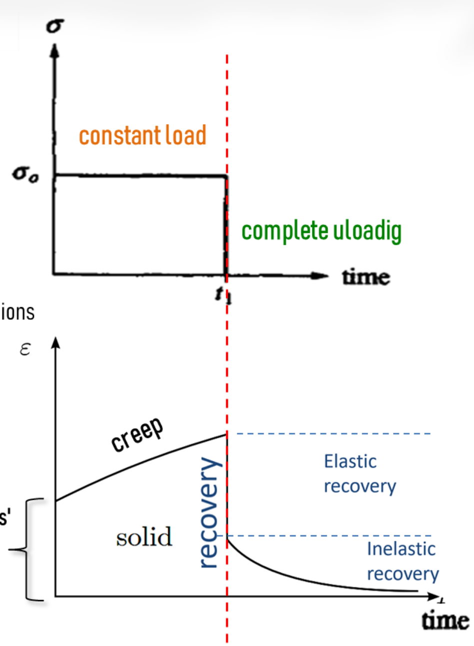
- **creep tests**
- **reaxation tests**
- **dynamic tests**

for material characterisation

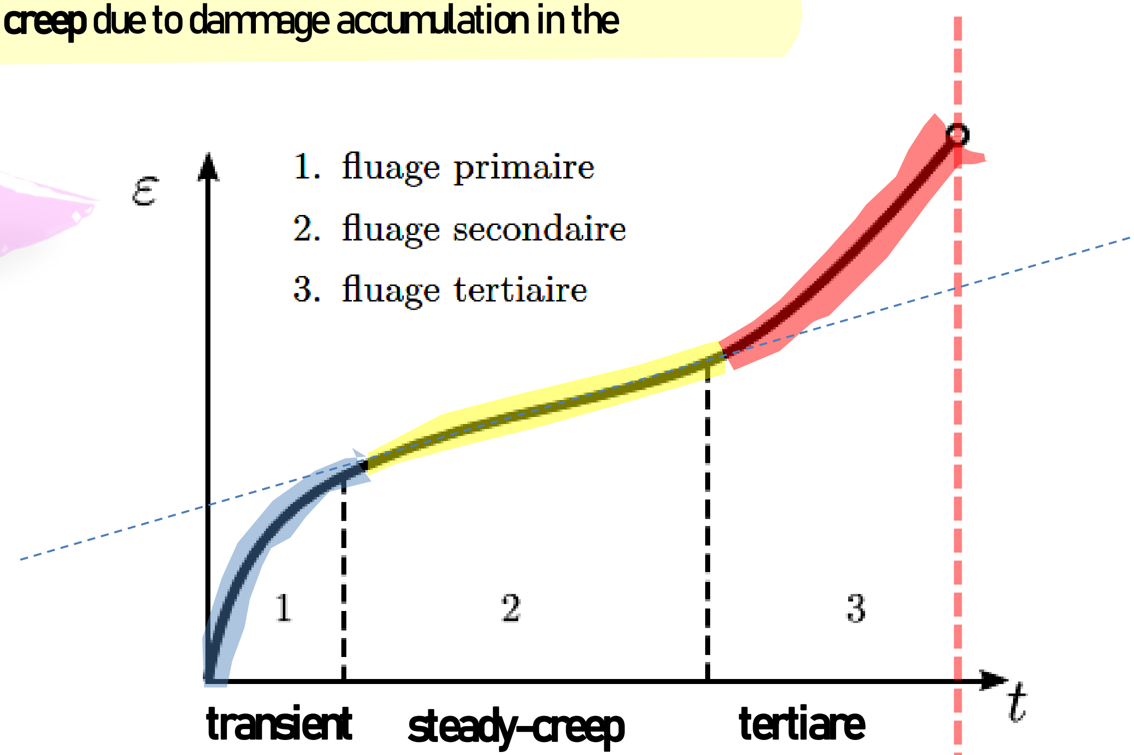
If we continue the creep test for lo-o-o-o-o-o-ong time then

- after the **transient (initial) creep [1]**, ...
- ... a **steady-creep [2]** period appears which can be followed ....
- ... by a **tertiare [3]** period with **accelerating creep** due to dammage accumulation in the material

lo-o-o-o-o-o-ong time loading



Creep tests
- constant load
- measure deformations



Creep test
- constant deformation
- measure stresses

Viscoelasticity in words:

Engineering Viscoelasticity

Viscoelastic materials have i) a time dependent response to constant loading as for instance, to force, temperature and strain, and ii) they exhibit also rate depend responses. iii) depending on the duration of the excitation (fast as in shock problems or slow as in long-term stability problems), the same material may exhibit some of solid or fluid behavior or both of them at the same time.

They have the ability to creep, recover partially or fully, undergo stress relaxation and absorb energy. They possess some of the combined mechanical properties of fluid and solids as related to the stress-strain response.

Engineering Viscoelasticity

Heraclitus 535 BC – 475 BC

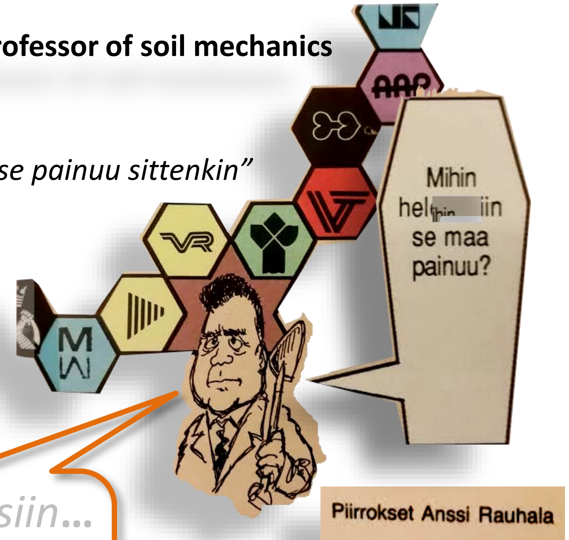
Ἡράκλειτος : ’*πάντα χωρεῖ* καὶ οὐδὲν μένει’ (dictum)

‘**Everything flows** and nothing stays’

a translation

TKK-professor of soil mechanics

”se painuu sittenkin”



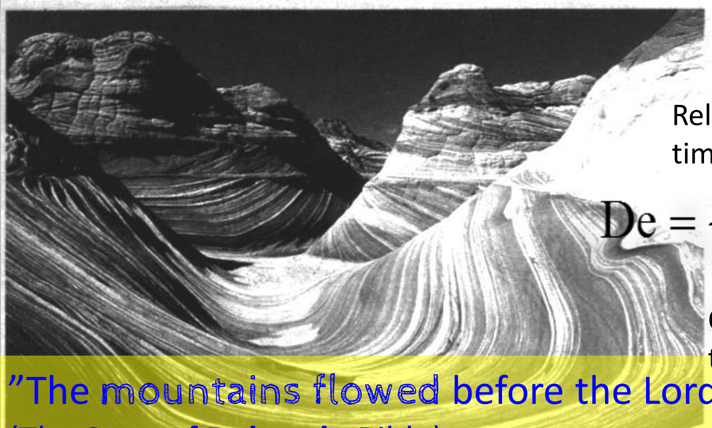
Mihin ... hitsiin...
maa painuu?

πάντα χωρεῖ

”panta rhei”, this dictum became...

... the **motto** of the (American) Society of Rheology

Journal of Rheology



$$\mathrm{De} = \frac{\tau_c}{\tau_p}$$

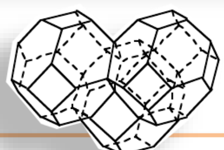
Relaxation time (τ_c)

Observation time (τ_p)

small: **fluid**

large: **solid**

”The mountains flowed before the Lord”
(The Song of **Deborah**, Bible)
דְּבוֹרָה



N.B. The creep response of materials results intimately from microstructural processes occurring inside the material during deformation. Despite this, the distribution of ‘defects’ and other type of faults in microstructure and macrostructure play a major role.

The continuum (or phenomenological) description of the creep process has been proven to very useful in solving engineering problems. On the other hand, such macroscopic approach is by definition incapable to open the black-box of the underlying processes needed to understand the basic mechanisms of creep. For this purpose, a microscopical approach is necessary.

Rheology
characterizes *flow* of materials that exhibits a combination of *elastic*, *viscous* and *plastic* behavior.

In **ABAQUS** and many other FE-software …

…time dependency is encoded through Prony-series

Prony series in the form of a scaled shear moduli

$$g(t) \equiv \frac{G(t)}{G_0} = 1 - \sum_{i=1}^{N} \frac{G_i}{G_0}\left(1 - e^{-t/\tau_i}\right)$$

SIMULIA

ABAQUS 6.14

THEORY GUIDE

In the numerical implementation (often a displacement formulation) the rate form of the **constitutive equation** is used. The incremental form for stress

Convolution product

sigma = conv(E, epsilon)

$$\boldsymbol{\sigma}(t) = \int_{t'=0}^{t} E(t-t')\dot{\boldsymbol{\varepsilon}}(t')\mathrm{d}t'$$

Relaxation modulus

3D-formulation:

$$\boldsymbol{\sigma}(t) = \int_{t'=0}^{t'=t} 2G(t-t')\dot{\mathbf{e}}_{\mathrm{dev}}(t')\mathrm{d}t' + \mathbf{I}\int_{t'=0}^{t} K(t-t')\dot{e}_{\mathrm{vol}}(t')\mathrm{d}t'$$

$$e_{\mathrm{ij}}^{\mathrm{dev}} = \varepsilon_{\mathrm{ij}} - \frac{1}{3}\varepsilon_{\mathrm{kk}}\delta_{\mathrm{ij}} \qquad e^{\mathrm{vol}} = \varepsilon_{\mathrm{kk}}\delta_{\mathrm{ij}},$$

Motivation: theory manuals are written in mathematical language that engineers should understand

Viscoelasticity

Download and install the free student version for doing the exposé-work

SIMULIA

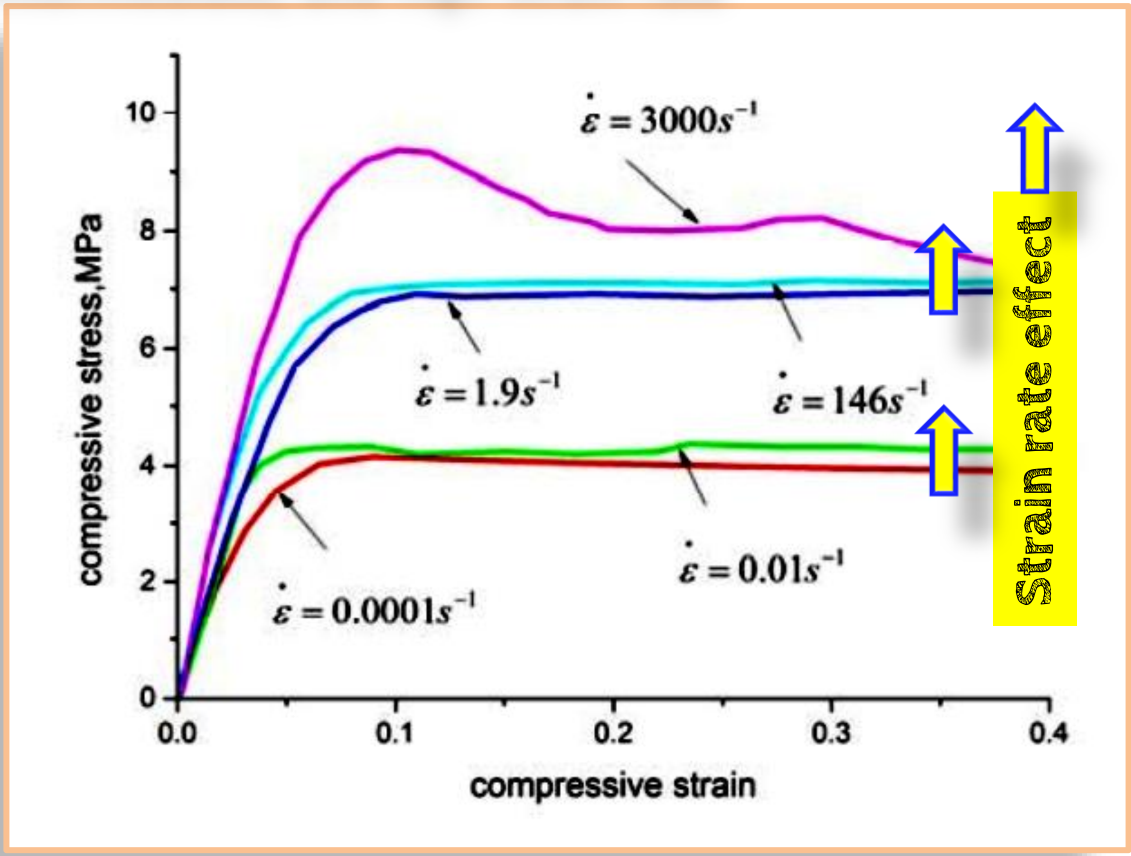
ABAQUS 6.14
THEORY GUIDE

DASSAULT SYSTEMES

https://academy.3ds.com/en/software/abaqus-student-edition

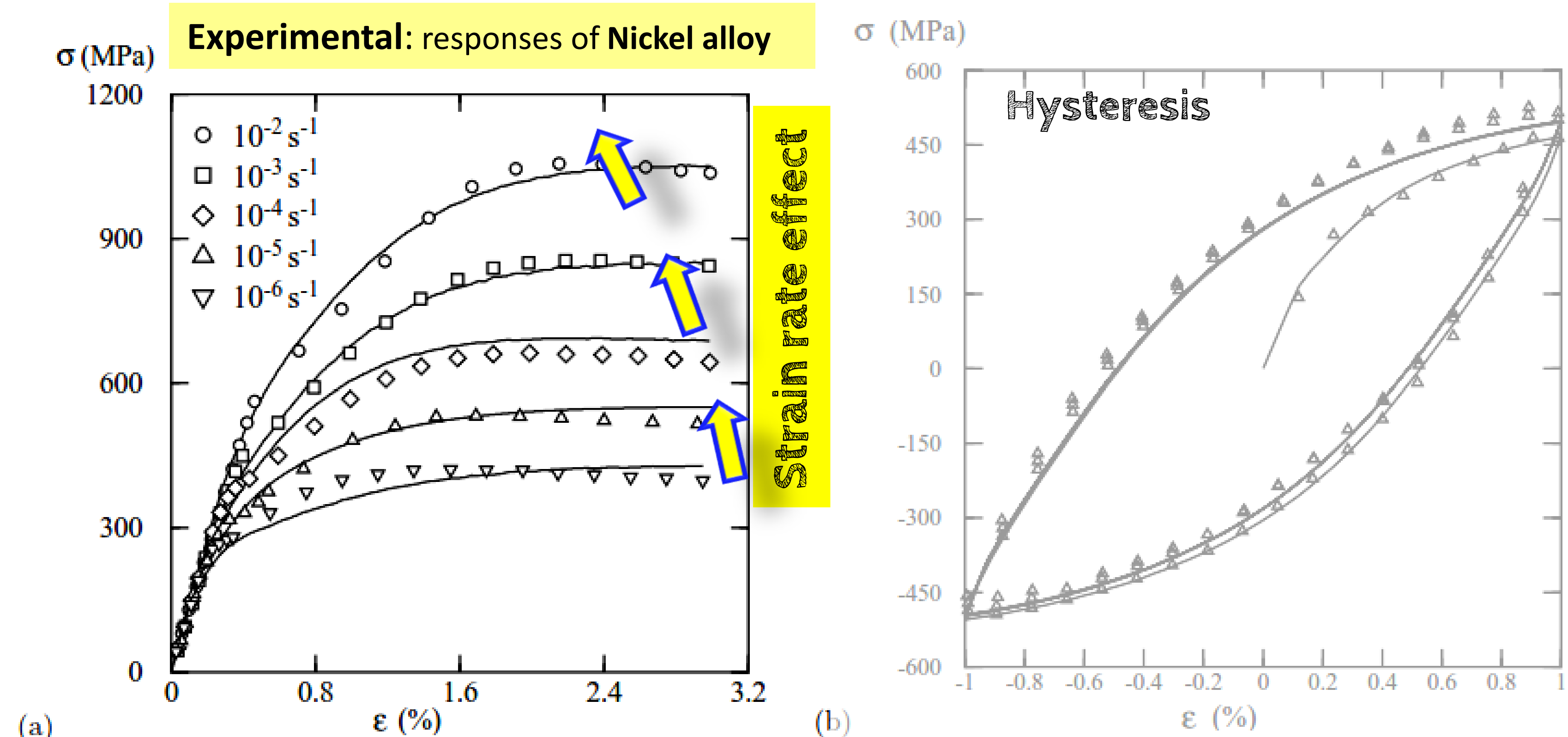
Experimental evidences

Compressive responses of **balsa wood at static, intermediate, and high strain rate**



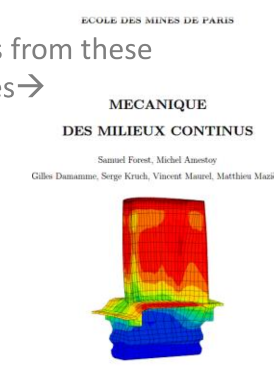
Visco-elasticity

Strain-rate dependence - experimental evidence



Comportement mécanique d'un superalliage à base de nickel pour aube de turbine : (a) comportement en traction à vitesse de déformation imposée, (b) comportement cyclique. Les résultats expérimentaux sont donnés par les symboles et les prévisions du modèle sont en trait continu.

This figure is from these lecture-notes→



Experimental evidences

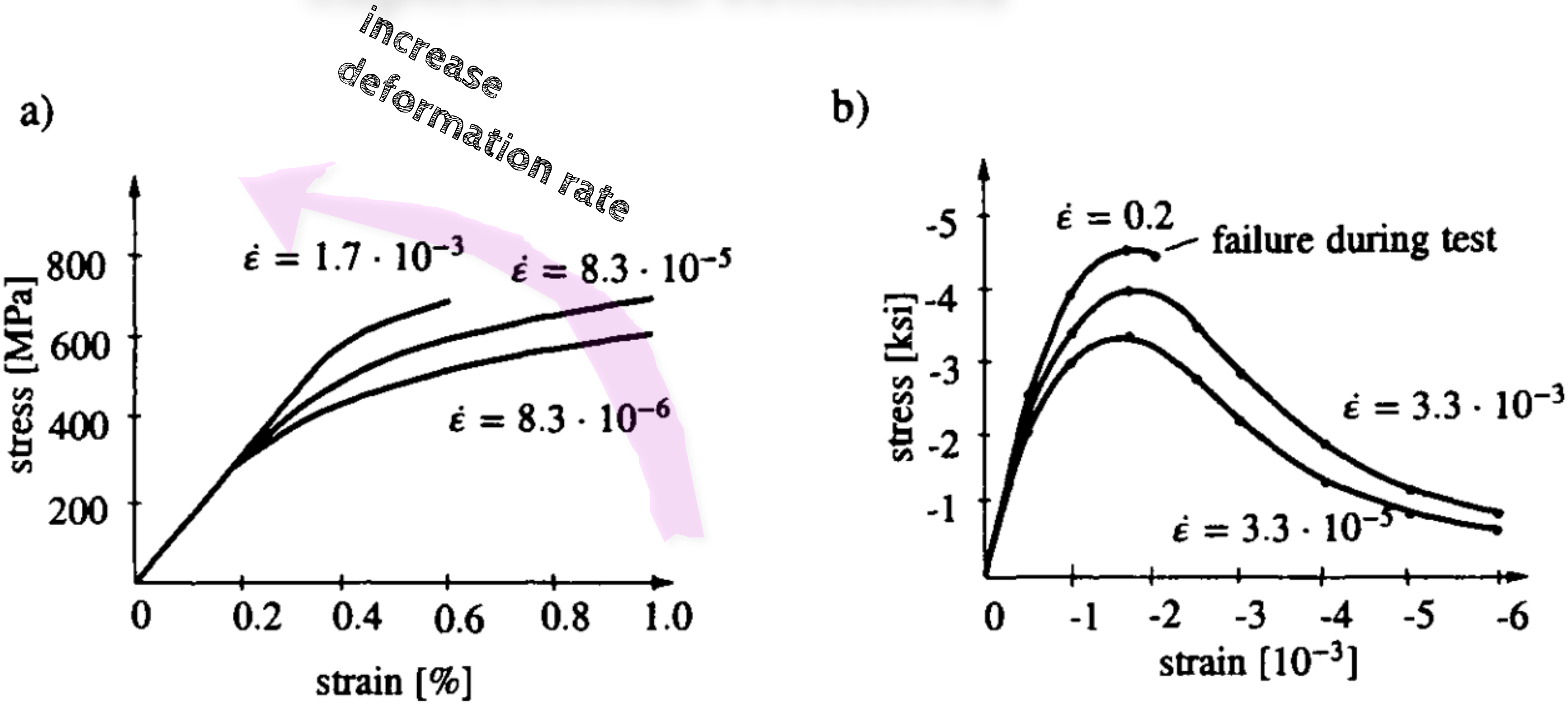


Figure 15.14: Constant strain-rate tests for uniaxial loading; a) nickel-based superalloy, B1900+Hf at 871°C , Chan *et al.* (1988), b) concrete in compression, Dilger *et al.* (1984).

THE MECHANICS OF CONSTITUTIVE MODELING

Niels Saabye Ottosen

Matti Ristinmaa

Division of Solid Mechanics,
Lund University, Sweden

About This **Reading Supporting Material**

- In addition to the pointed chapters from the course textbooks for each topic, I will provide an additional reading material together with weekly homework series (3-4 exercises/week).

- The Lecturer's written material is not a collection of lecture-slides but it is a supporting material to help and motivate students in their reading the chosen subject from the course textbooks & elsewhere. Lecture slides cannot capture the dynamics of the *in-vivo* lecture nor replace texbooks.

- The topics treated in this intensive course cannot be avoided by future structural engineers:
  - these topics will wait for them hidden, inside the FE-software you will use to perform structural analysis, in the black-boxes called material models.

Supporting Material in MyCourses
Content & enumeration of the pdf-material

0. INTRODUTION
1. ELASTICITY
2. VISCOELASTICITY (& some basics of creep)
3. PLASTICITY (& Failure Hypotheses)

CONST nweeks = 6

iw= 0;

Aalto University

BEGIN

iw = 3

END

Content:

Djebar BAROUDI, PhD.
Aalto-University

- **Elasticity – kimmoisuus tai elastisuus**
  (linear, hyper-elasticity, isotropy, anisotropy, orthotropy)
- **Viscoelasticity - viskoelastisuus**
- **Viscoplasticity** or creep – **viskoplastisuus … viruminen**
- **Failure hypotheses - lujuushypoteesit**
- **Plasticity - plastisuus**
  associative, non-associative
- **Damage - vauriotuminen**
  damage-plasticity *ex.* Concrete Damage Plasticity, Model in Abaqus

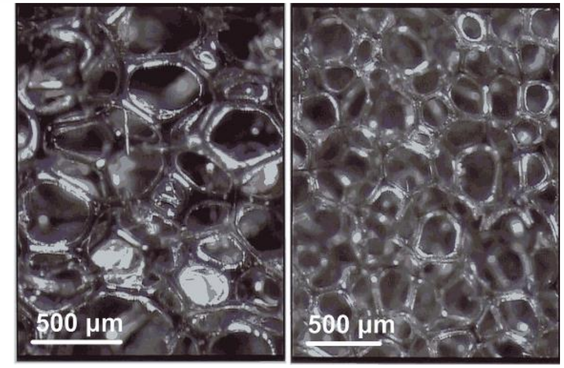
If time…

Introduction

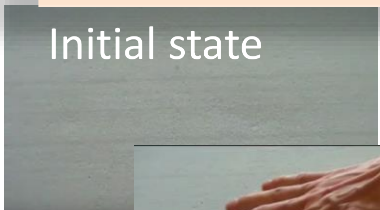
Viscoelasticity

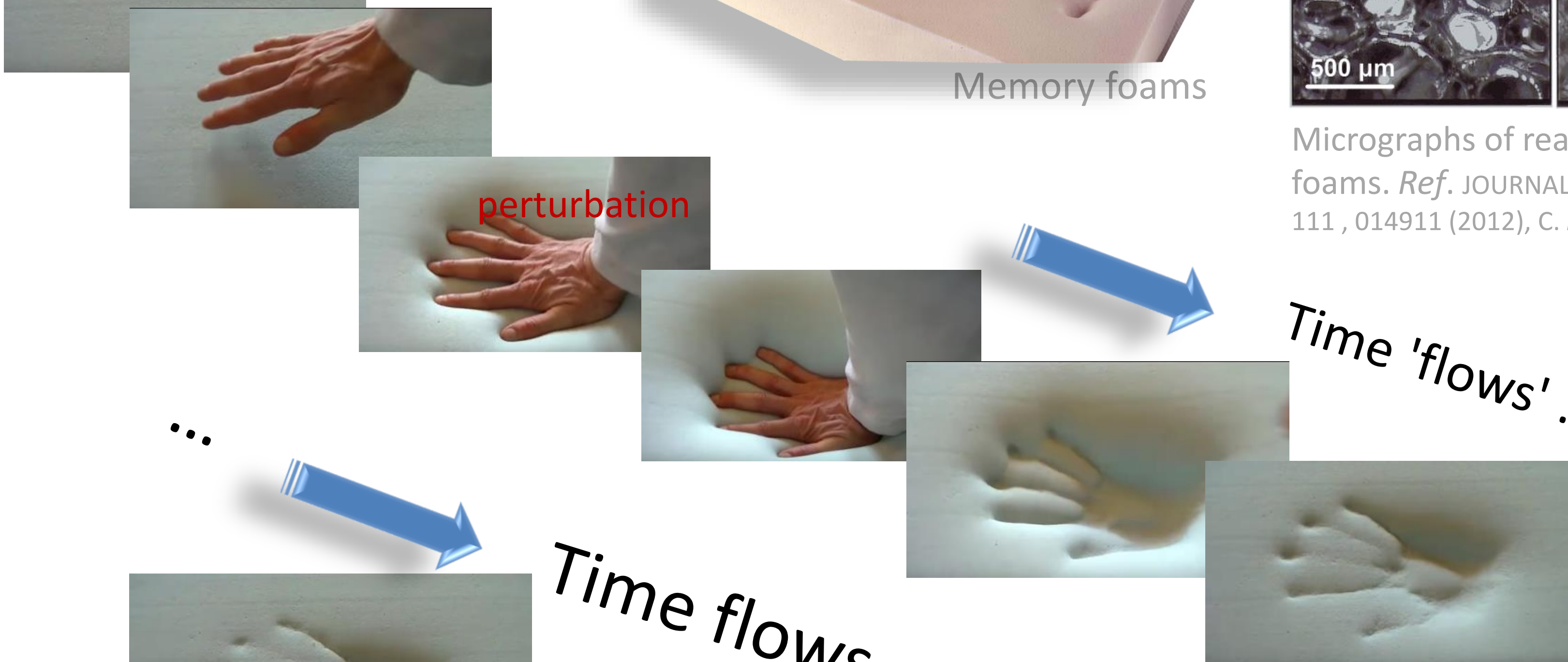
These are visco-elastic open-cell foams

Memory foams

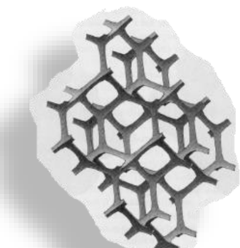


Micrographs of real open-cell foams. *Ref*. JOURNAL OF APPLIED PHYSICS 111 , 014911 (2012), C. *Perrot et al.*



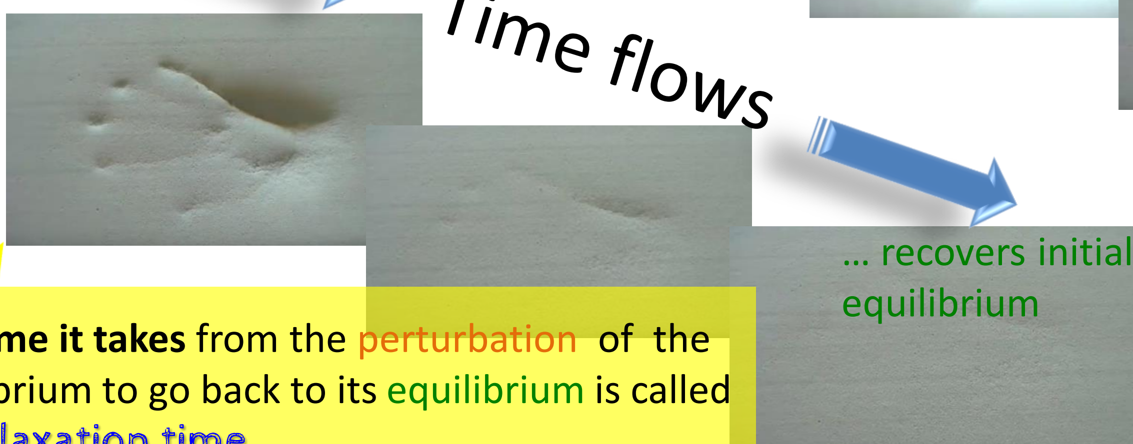


Time 'flows' ...



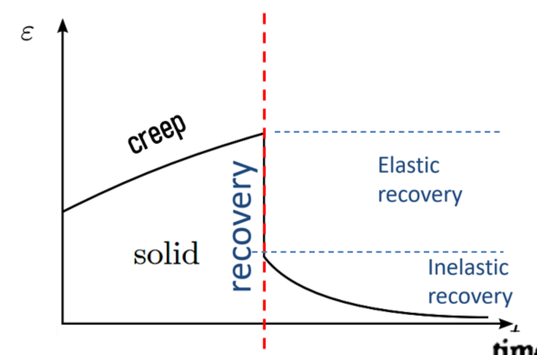
...

Time flows



... recovers initial equilibrium

The **time it takes** from the perturbation of the equilibrium to go back to its equilibrium is called the relaxation time



Visco-elasticity

In general, almost all the materials exhibit less or more several inelastic properties

- ✓ In general, stress in such materials depends on strain and the history of strain
- ✓ Such properties can be **modeled** by the theory of viscoelasticity

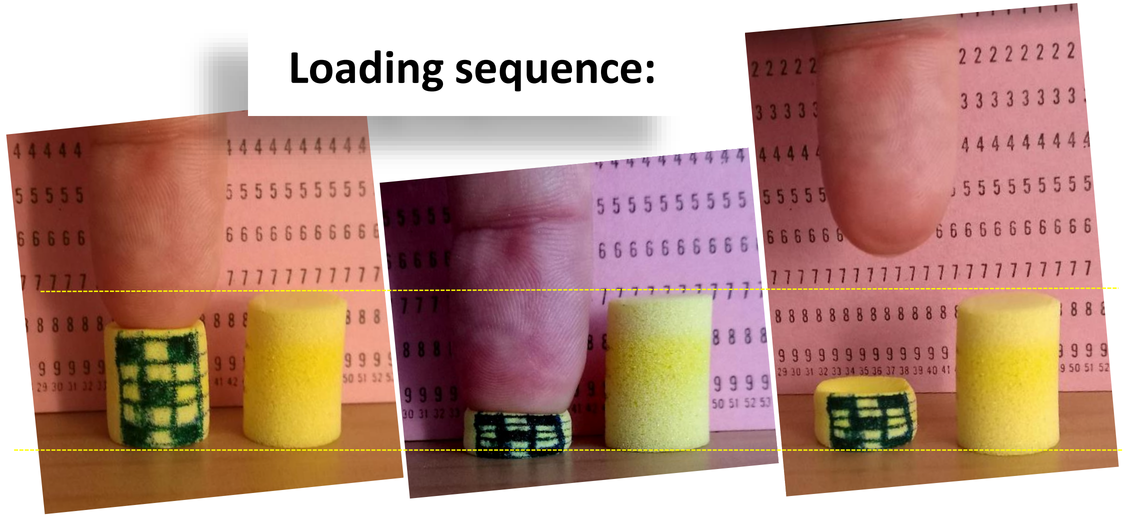
Viscoelasticity, in short:

Viscoelastic materials have i) a time dependent response to constant loading as for instance, to force, temperature and strain, and ii) they exhibit also rate depend responses. iii) depending on the duration of the excitation (fast as in shock problems or slow as in long-term stability problems), the same material may exhibit some of solid or fluid behavior or both of them at the same time.

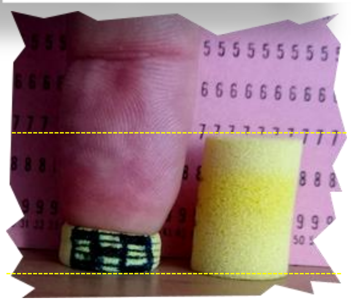
They have the ability to creep, recover partially or fully, undergo stress relaxation and absorb energy. They possess some of the combined mechanical properties of fluid and solids as related to the stress-strain response.

Viscoelasticity

Loading sequence:



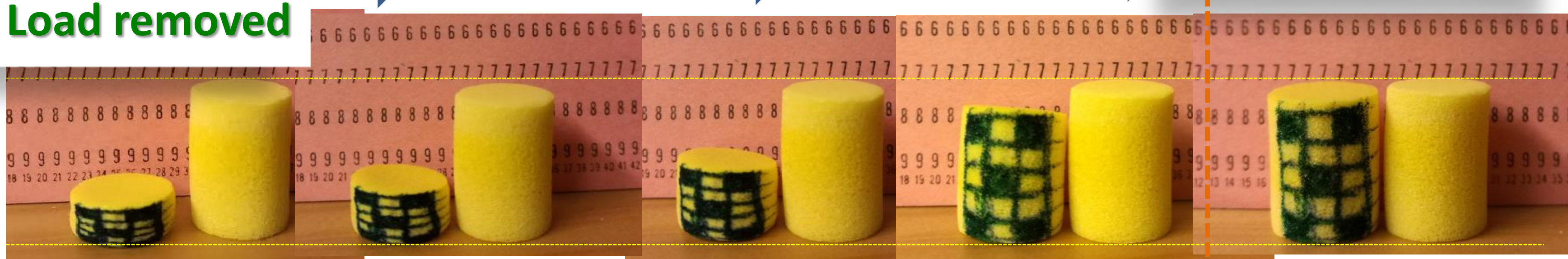
Initial loading



Load removed

Full **recovery***

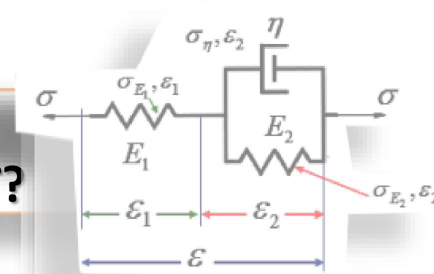
(Inverse phenomenon to creep)



t = 0 min

t = 40 min

NB. Yes, **time-dependent response ...** , but ... **SO WHAT?**



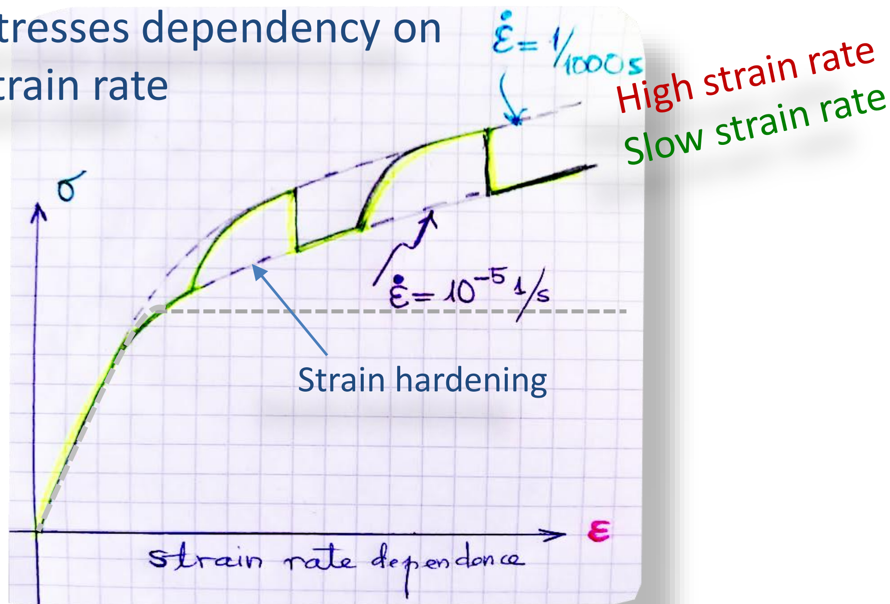
← *This is a question to the students*: **Why** this is important to know (=quantify) for an Engineer time to failure ...INSA-eau-cube plastiques - troués - effondrement déféré après stationnement prolongé-test-transient-seulement-failure-stress-pas-de-dimension-temporelle...

* In case of partial recovery for $t \to \infty$ - residual deformations remain after complete unloading (= Visco<u>plasticity</u>)

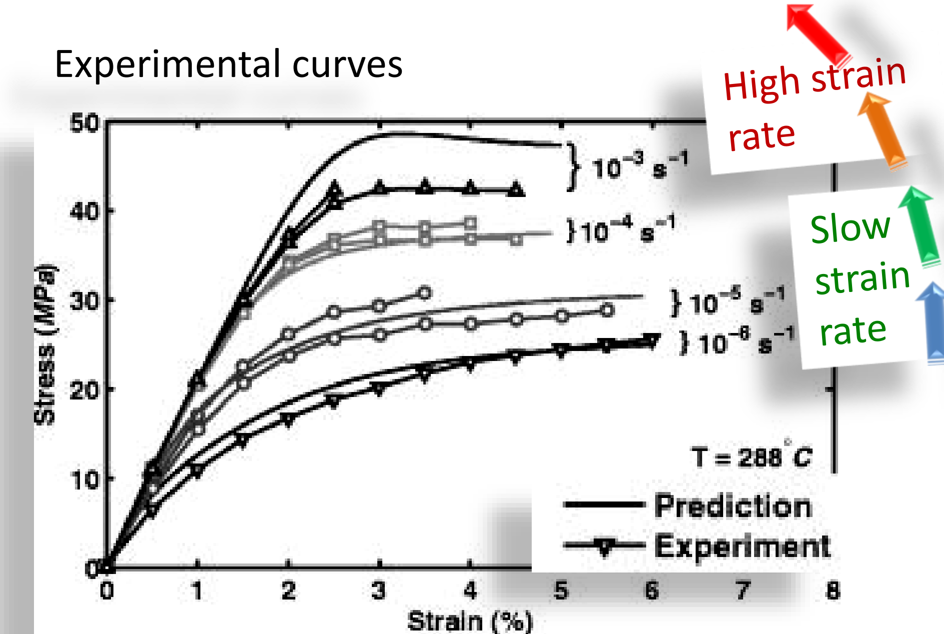
Viscoelastic material behaviour

Some Experimental Evidence

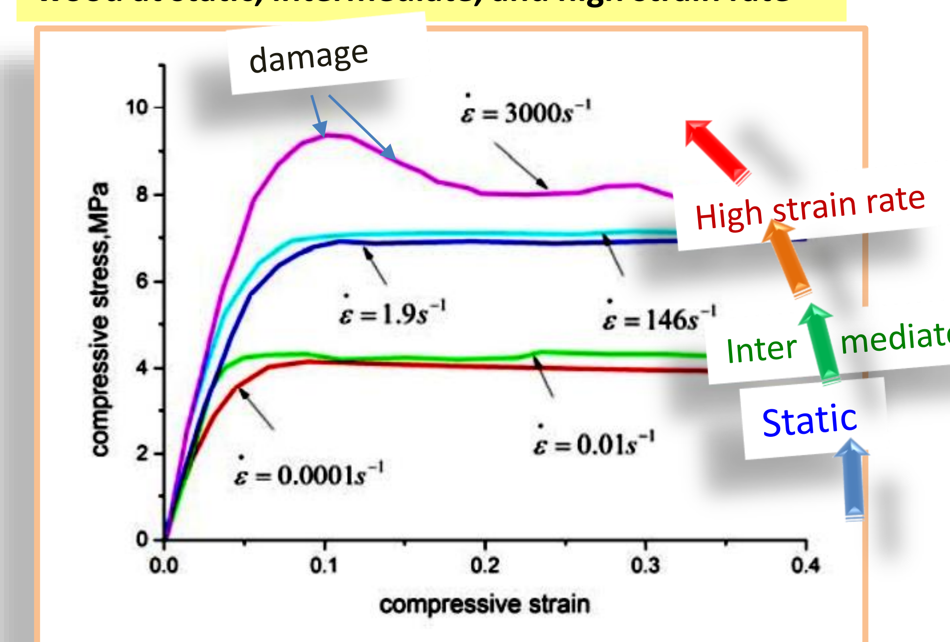
Stresses dependency on strain rate



Experimental curves



Experimental: Compressive responses of **balsa wood at static, intermediate, and high strain rate**



Stresses dependency on strain rate

Intoductionary examples

Concise reading:

https://en.wikipedia.org/wiki/Viscoelasticity

https://en.wikipedia.org/wiki/Viscoplasticity

https://en.wikipedia.org/wiki/Plasticity_(physics)

How Viscoelastic does manifest itself?

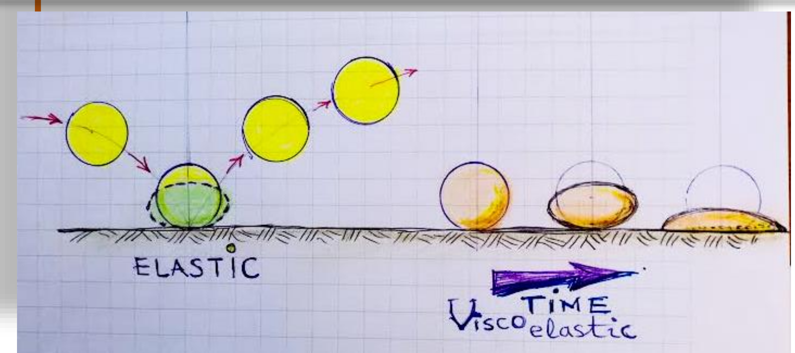
Reading

Here, the material 'aging' effects are assumed not occurring (they are assumed* extremely slow as regarded to the effects of the excitations or actions listed)

assumed*: if it is not the case, then should be accounted for … chemistry, opto-chemistery, …, etc.

Viscoelastic materials have

- *i*) a **time dependent response** to constant loading *as for instance*, to **force**, **temperature** and **strain**
- *ii*) they **exhibit also rate depend responses**
- *iii*) **depending on the duration of the excitation** (*fast* as in *shock* problems or *slow* as in *long-term* stability problems), **the same material may exhibit some of solid- or fluid-like behavior** or both of them at the same time.



- Such materials have the ability to **creep**, recover partially or fully, undergo **stress relaxation** and **absorb energy**.

They possess some of the **combined mechanical properties** of fluid and solids as related to the stress-strain response.

So, what is meant by material aging?

… **next slide** …

N.B. The material 'aging' effects are not addressed in this Master course.

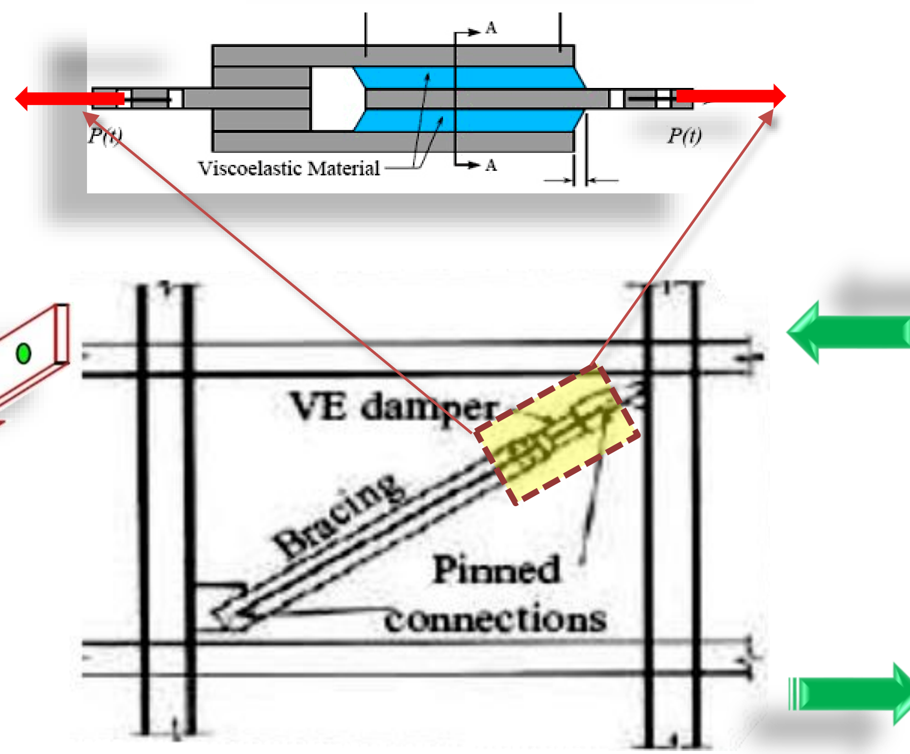
Some Examples of use of VE-materials in Structural Engineering

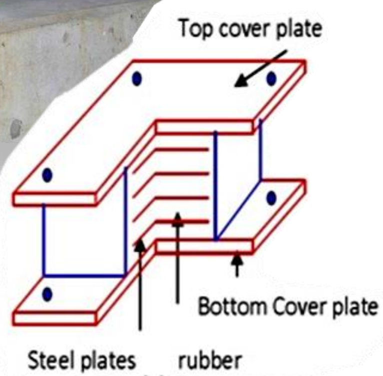
Damping & Dissipating …

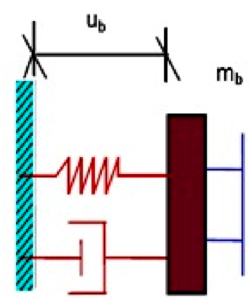
… unwanted vibrations, seismic loading, …

VISCO-ELASTIC DAMPERS

- *Visco-elastic (VE) dampers utilize high damping from VE materials to dissipate* energy through shear deformation.
  Such materials include rubber, polymers, and glassy substances.







Rubber layer as foundation support

Examples of viscoelastic materials

Viscoelasticity manifests itself through a *delayed response* of the material to an external action

- **The disks in the human spine (& most biomaterials)**

  The disks creep and get shorter under body weight with time and recover when lying down

  - this means that you are taller in the morning than in the evening
  - under **micro-gravity**, **astronauts gain up to 5 cm** in height

- **Skin tissue, soft tissues and cells**

  Pinch the skin at the back of the hand, hold a while and then release it; it takes some time for the skin to recover its original flat position

- **Wood, concrete, bitumen, soils, …**

  Creeping under gravity can take decades or centuries to be noticeable depending on materials

  Glacier motion or creep within the glacier some parts move forward faster than others or basal creep

  Concrete: the creep of concrete has its origin in the calcium silicate hydrates (C-S-H) of the hardened Portland cement past (cement gel) and happens at all stress levels …aging…relaxation

N.B. **Wood** is (for major part) a natural **thermoplastic**

Metals can creep significantly at high temperatures (high as regard to melting temperature) $\sim > 0.4T_m$

Materials which are elastic at room temperature often have significant viscoelastic properties when heated

Cf. Homologous temperature

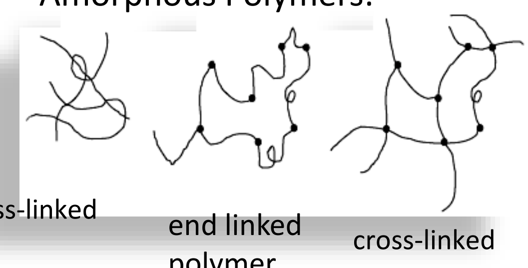
- **Polymers and polymer foams creep**

  Elastomers are highly viscoelastic → good impact absorbers

  amorphous polymers, semicrystalline polymers

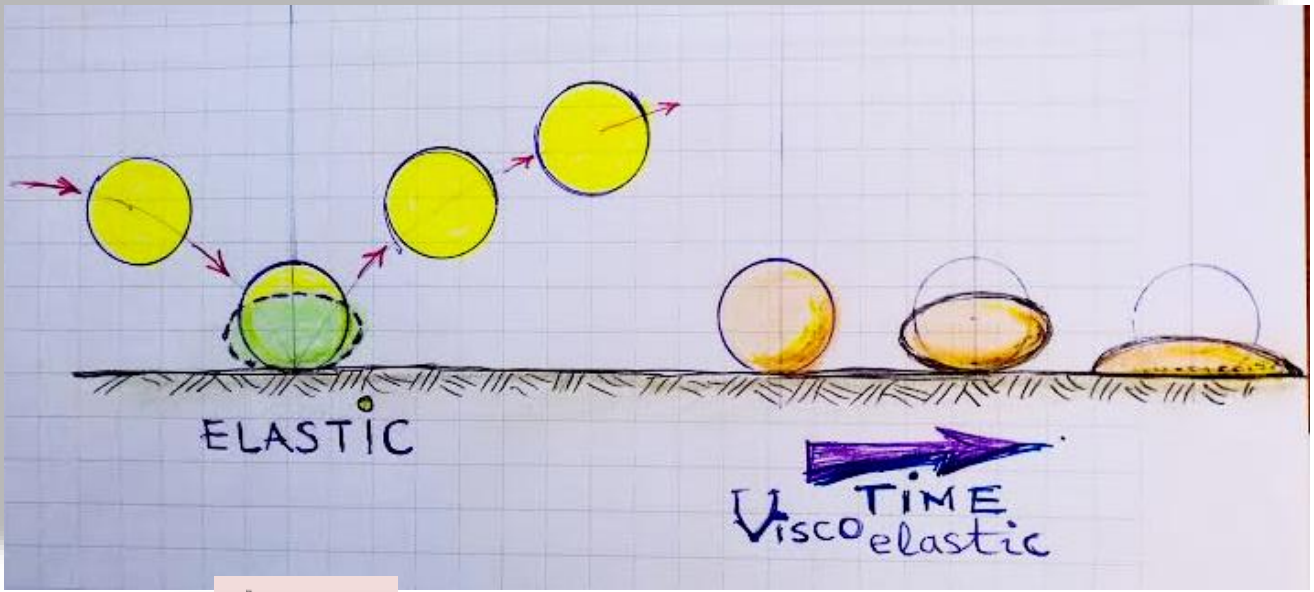
- **Ice** - polycristal

Amorphous Polymers:



N.B. for detailed various mechanisms of creep, please refer to your pervious courses on *materials* …

Illustration of Viscoelasticity



Short time

Long time

$$\mathrm{De} = \frac{\tau_c}{\tau_p}$$

Relaxation time (τ_c)

Observation time (τ_p)

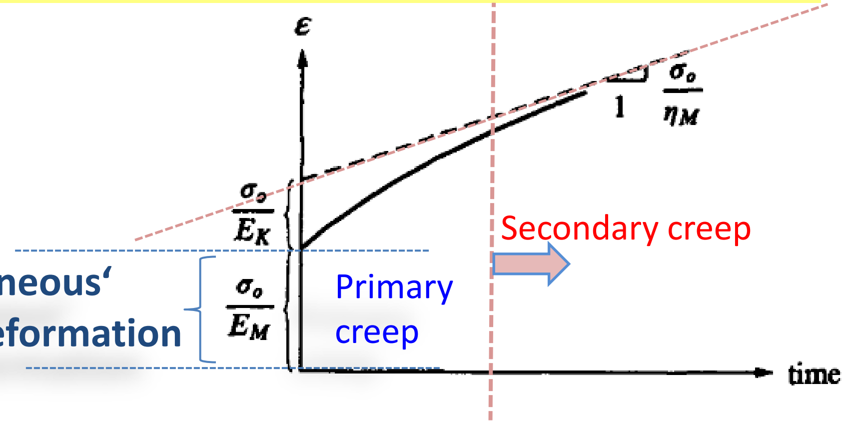
small: **fluid**

large: **solid**

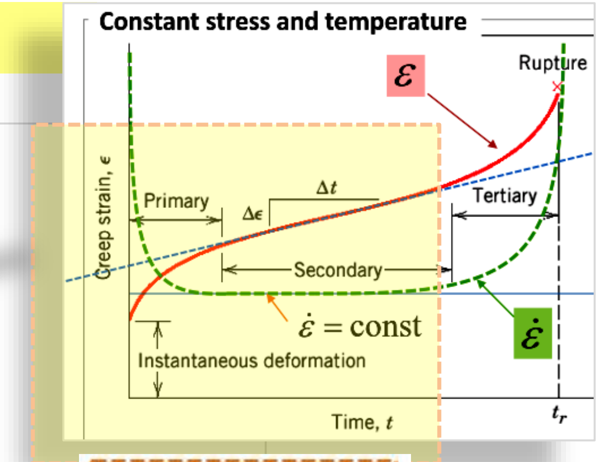
Creep Response:

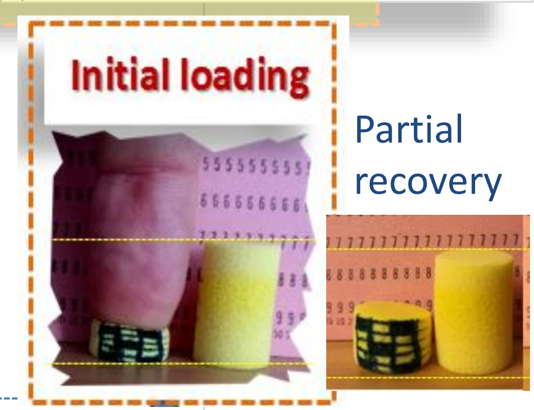
Apply a **constant stress** $\sigma(0)=\sigma_0$ $t=0$

... and determine (measure) the time dependent strain $\varepsilon(t)$

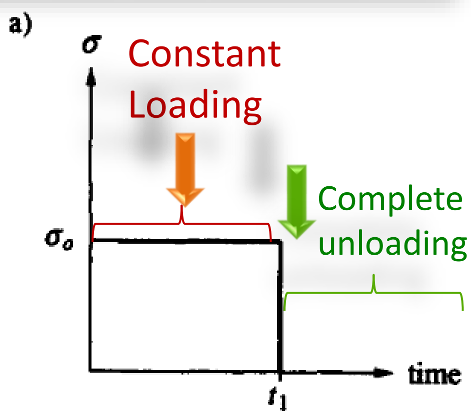


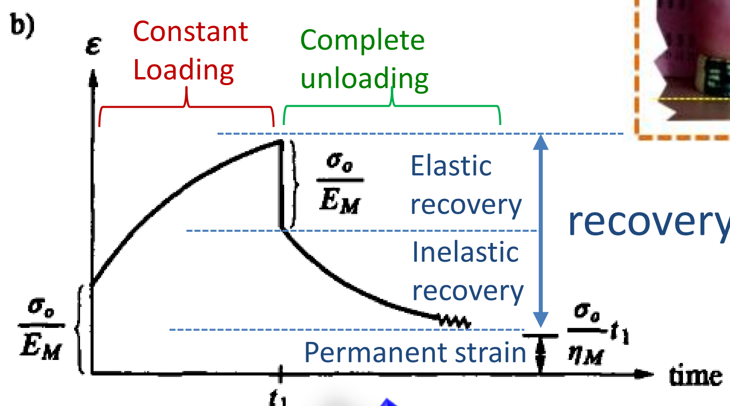
'Instantaneous' elastic deformation





Motivation - the first contact





A generic viscoelastic material response

NB. The term 'Instantaneous' usually used for elastic deformation may sound inappropriate since the elastic deformation takes time to reach its equilibrium. This time is characteristic of the elastic wave propagation in the medium and is finite since the velocity of the sound or the compression or acoustic waves in the media are finite. However, this characteristic time is much smaller in many orders of magnitudes than the characteristic time of relaxation related to creep deformation.

One of the scope of this lecture series is to present some classical models for such time –dependent behavior.

So, "let start from the beginning, go on and keep going till the end."

Generalities on Viscoelasticity

You can go directly to

Constitutive Models of Linear Viscoelasticity
(Slide #51)

Additional reading & Literature → Appendix 1

Viscoelasticity

Djebar BAROUDI, PhD.
Lecturer @ Aalto-university
Civil Engineering Department

22.1.2017

V i s c o e l a s t i c i t y

Content:

- experimental observations: evidence of viscoelastic behavior
- stress relaxation at constant strain
- creep at constant stress
- strain-rate dependence
- constitutive models in the rate form

…

Chinese new-year in Helsinki, 2017

Vi… rials have i) a time de… constant loading as… perature ar… ii)… end res… iii… on of the exci… problems or slo… ity proble… al may exhibit … d behavior or both … ime.

They h… creep, recover partially or fully, undergo stress relaxation and absorb energy. They possess some of the combined mechanical properties of fluid and solids as related to the stress-strain response.

Reading: ***Textbook* -** Lemaitre and Chaboche – *Mechanics of Solid Materials*. *Chapter* 4.3

How Viscoelastic does manifest itself?

Reading

Viscoelastic materials have

Here, the material 'aging' effects are assumed not occurring (they are assumed* extremely slow as regarded to the effects of the excitations or actions listed)

assumed*: if it is not the case, then should be accounted for … chemistry, opto-chemistery, …, etc.

- *i*) a **time dependent response** to constant loading *as for instance*, to **force**, **temperature** and **strain**
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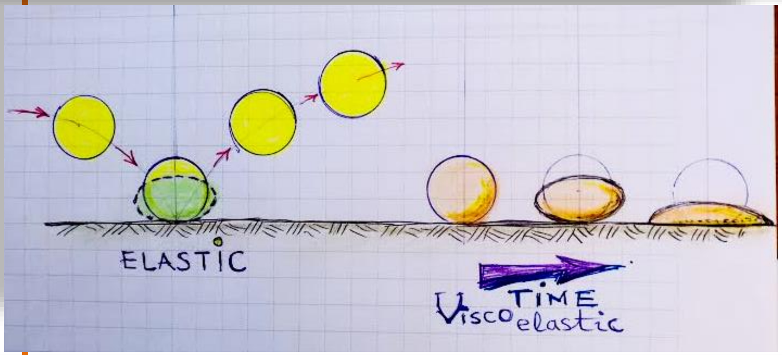


- Such materials have the ability to **creep**, recover partially or fully, undergo **stress relaxation** and **absorb energy**.

They possess some of the **combined mechanical properties** of fluid and solids as related to the stress-strain response.

So, what is meant by material aging?

… next slide …

N.B. The material 'aging' effects are not addressed in this Master course.

So, what is meant by material aging?

Reading

Material aging:

Observations shows that some of **physical properties** of the material may **evolve** and change **with time,** as for instance its mechanical properties **independently of the applied mechanical loading**.

Such changes may result from various action of phenomena such

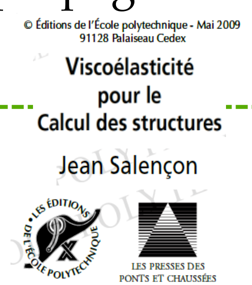
- Temperature
- Hygrometry (humidity)
- Light radiation (especially, ultraviolet UV)
- Ionizing radiation (radiation consisting of energetic subatomic particles, ions or atoms moving at high speeds and electromagnetic waves)
- Chemical reactions (*cf.* hydration reactions in concrete)
- Crystallization
- Fusion or melting
- Fault or damage propagation within the material
- …

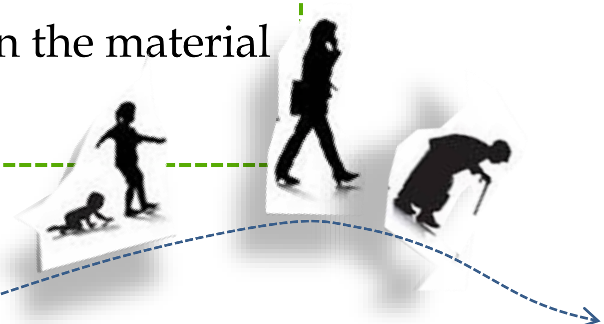
Material aging

Such general phenomena is called **material aging** and it has a negative connotation implying the *degradation of the mechanical properties* – this is true for the polymers, for instance.

Despite this, **material aging** can have also positive aspects as in the case of concrete, for instance. To get convinced, just see how its mechanical properties improve dramatically with time due to hardening reactions from 1, 7 and 28 days, for instance.

Ref: some of the material of following three slides is freely translated and edited from:





Continued

Non-aging material

Reading

Aging is a general property concerning all the materials starting at the beginning of their conception (or fabrication). However, this phenomena manifests itself more or less strongly dependently on the specific material and of the time-window of the history concerned.

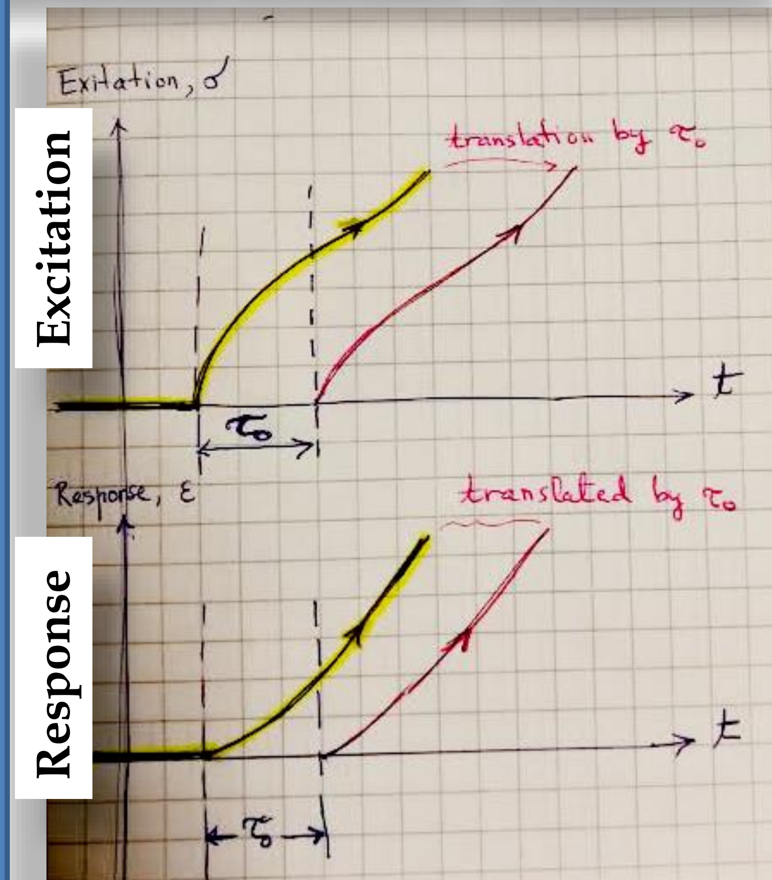
Thus, for a given material, **there exist often a significant period** of time, where its mechanical **properties become** somehow "**stabilized**" or insignificantly slowly evolves within this period.

Thus, **for a given material**, there **exist often a significant period** of time, **where its mechanical properties become** somehow "**stabilized**" or insignificantly slowly evolves with this period.

Such a material is called a **non-aging material**

Concrete and its hardening is a counter-example for this … aging occurs

For a non-aging material, one have translational invariance* of behavior



*This means that, given two histories of excitations (stresses) which are identical but shifted by an amount of time τ_0, one obtains two histories of responses (strains) which are identical but also shifted by the same amount τ_0.

Viscoelasticity

Content

- experimental observations: evidence of viscoelastic behavior
- stress relaxation at constant strain
- creep at constant stress
- strain-rate dependence
- **constitutive models in the rate form**:
  - Maxwell model
  - Kelvin-Voight model
  - Standard linear solid model
  - Generalized Maxwell model
  - Kelvin chain model

Viscoelasticity, in short:

Viscoelastic materials have i) a time dependent response to constant loading as for instance, to force, temperature and strain, and ii) they exhibit also rate depend responses. iii) depending on the duration of the excitation (fast as in shock problems or slow as in long-term stability problems), the same material may exhibit some of solid or fluid behavior or both of them at the same time.

They have the ability to creep, recover partially or fully, undergo stress relaxation and absorb energy. They possess some of the combined mechanical properties of fluid and solids as related to the stress-strain response.

Why the constitutive equation is needed*?

The structure of the problem

$$\mathcal{F}(\sigma,\dot{\sigma},\varepsilon,\dot{\varepsilon},T,\dot{T})=0$$

Example problem : think of the need of solving the *equation of motion* or the *quasi-static equilibrium* in order to **determine the field of displacements** within a body having **a rate dependent mechanical response**.

- **¿The unknown field of displacements:?**

$$\vec{u}(\vec{x},t)=\begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

- **The equation of motion** or also of equilibrium when $\rho\dot{\vec{u}}=\vec{0}$

$$\operatorname{div}\boldsymbol{\sigma}+\rho\vec{f}=\rho\dot{\vec{u}},\ \text{in}\ \Omega$$

- **The constitutive equation or the behavior law:** $\mathcal{F}(\sigma,\dot{\sigma},\varepsilon,\dot{\varepsilon},T,\dot{T})=0$

Response functional

- **The kinematic relation:** $\boldsymbol{\varepsilon}=\nabla^{\text{sym}}\begin{bmatrix} u \\ v \\ w \end{bmatrix}$

- **The boundary conditions:**

- **The initial condition at** $t=0,\ in\ \Omega$

The initial solution is given

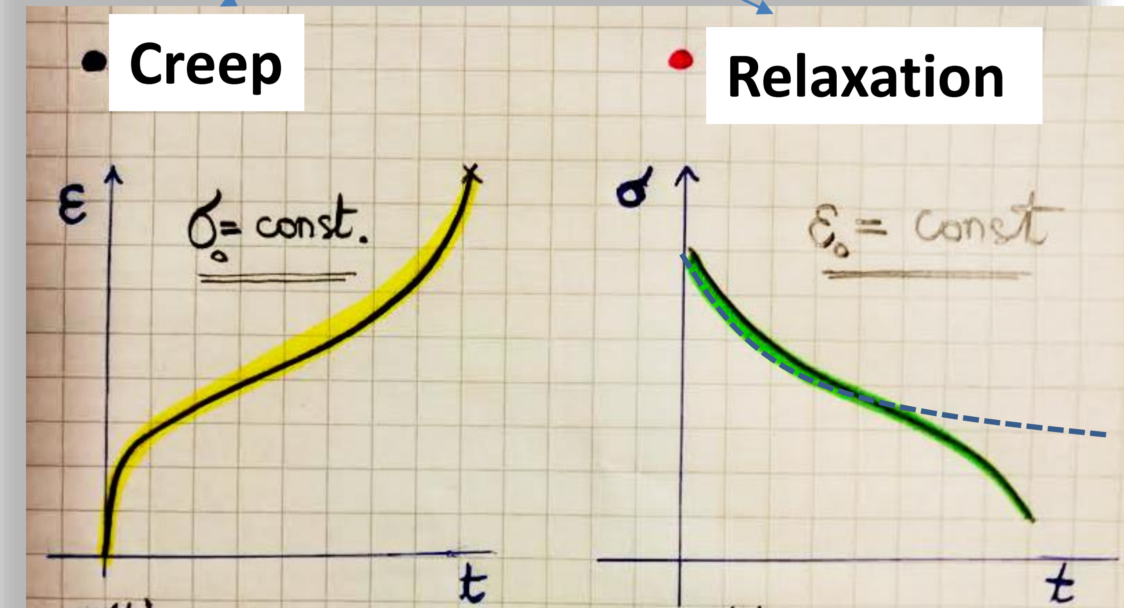
$$\vec{u}(\vec{x},t=0)=\vec{u}_0(\vec{x})$$

In this short course, explicit forms for $\mathcal{F}(\boldsymbol{\sigma},\dot{\boldsymbol{\sigma}},\boldsymbol{\epsilon},\dot{\boldsymbol{\epsilon}})=0$ will be derived for some classical viscoelastic mechanical response

* … to mathematically close the problem

Observation: typical *mechanical responses* of viscoelastic materials

At constant temperature *= isothermal viscoelasticity*



Creep function: $\dfrac{\varepsilon(t)}{\sigma_0} \equiv J(t)$ $= (1+\phi)\,/\,E$

$\dfrac{\sigma(t)}{\varepsilon_0} \equiv G(t)$ ← **Relaxation function**

$'E(t)'$

Viscoelastic materials have the ability to creep, recover partially or fully, undergo stress relaxation and absorb energy. They posse some of the combined mechanical properties of fluid and solids as related to the stress-strain response.

Relaxation - relaksaatio

Creep - viruminen

Some experimental observations

Time scales

- This **fluid** & **solid** character become relevant at *long times*

- The ***characteristic time scales*** involved in viscoelastic response are ***'very long'*** as compared to an '***instantaneous***' elastic response of the material with **characteristic time scale** being of order $\tau \approx \sqrt{\rho \ell^2 / E}$ where ℓ is a characteristic length of the system.

For instance, $\tau_p \rightarrow \sqrt{\rho \ell^2 / E}$

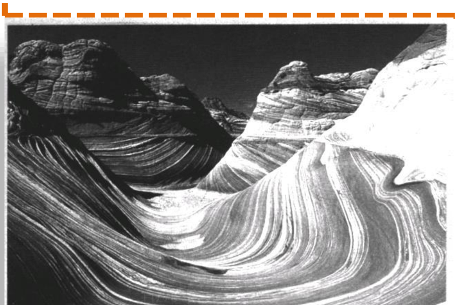
Speed of sound:

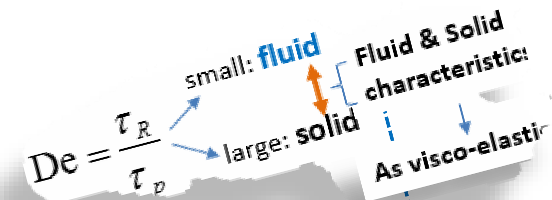
in solid: $c = \sqrt{E / \rho}$

in gas: $c = \sqrt{K_s / \rho}$

modulus of bulk elasticity

= velocity of propagation of an elastic wave





It is the magnitude of the **Deborah** number which distinguish between **solid** and fluid.

- If the relaxation time of the material is very small as compared to observation time, then you see the material **flowing:** $\tau_R << \tau_p$

- On the opposite, if the is relaxation time is much larger than the observation time then the material you are observing is a **solid:** $\tau_R >> \tau_p$

Deborah number: [1]

דְּבוֹרָה

$$\mathrm{De} = \frac{\tau_R}{\tau_p}$$

small: **fluid**

large: **solid**

Fluid & Solid characteristics

As visco-elastic

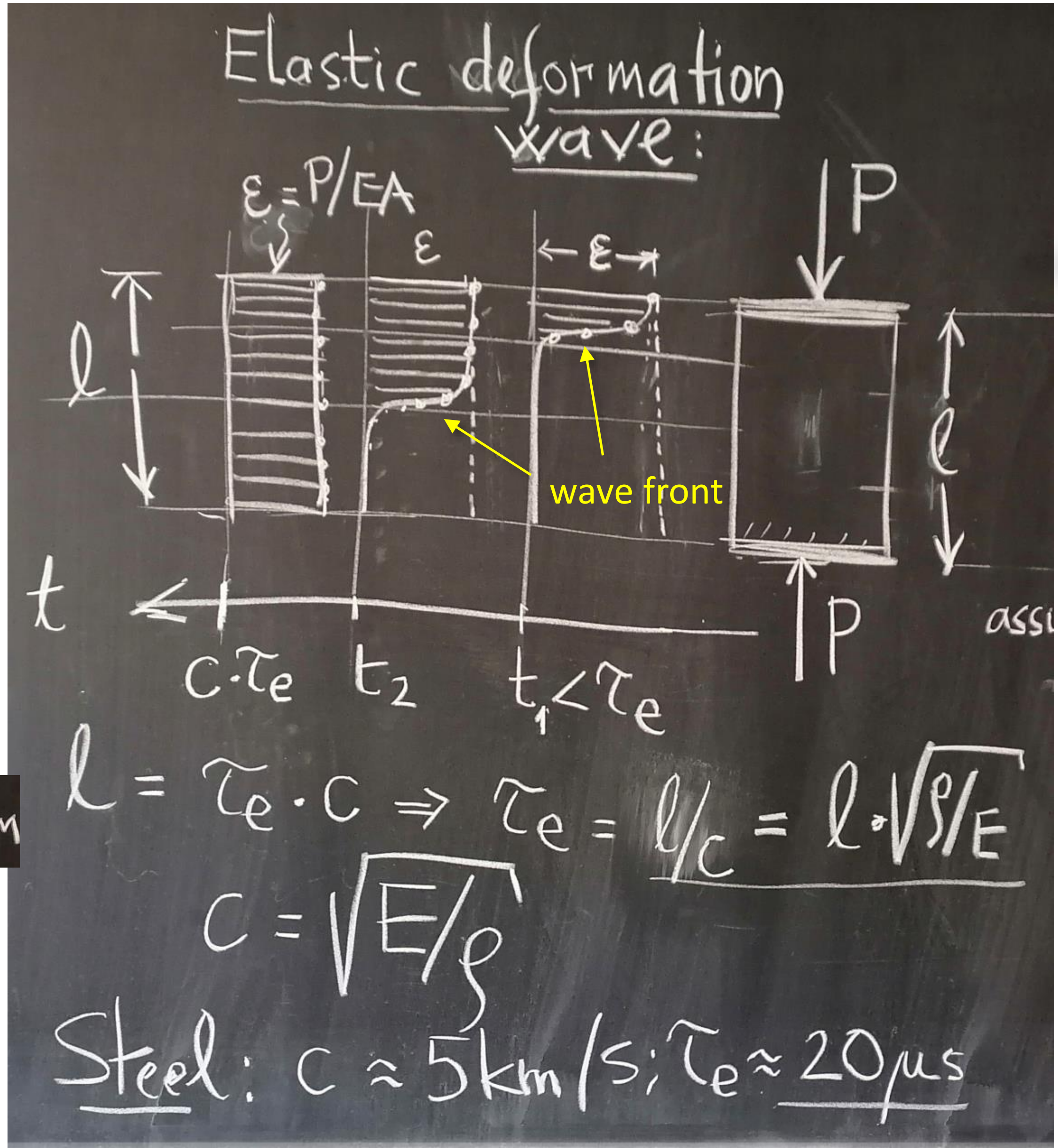
τ_R - stress relaxation time

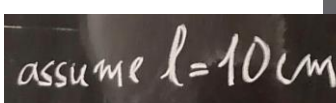
τ_p - observation time scale (characteristic time for the deformation process)

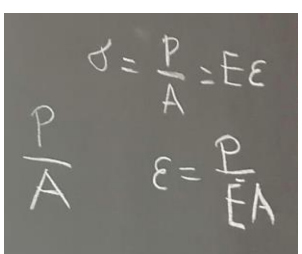
'The mountains **flowed** before the Lord'
(The Song of Deborah, Bible)

[1] **R J Poole** (2012), The Deborah and Weissenberg numbers. The British Society of Rheology, *Rheology Bulletin*. **53(2)** pp 32-39

Time scale for of elastic compression wave (= deformation)







Speed of sound:

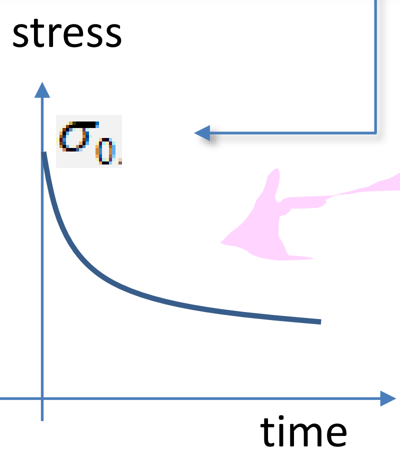
in solid: $c = \sqrt{E / \rho}$

'<u>instantaneous</u> elastic deformation' takes about 20 microseconds to reach the bottom of 10 cm thick steel specimen under compression

Some examples

$\exp(-t/\lambda)$

$\lambda \equiv \tau_R$ - stress relaxation characteristic time



Liquid	Viscosity η (Pa.s)	Relaxation time λ (s)	Modulus G (Pa)
Water	10^{-3}	10^{-12}	10^9
An Oil	0.1	10^{-9}	10^8
A polymer solution	1	0.1	10
A polymer melt	10^5	10	10^4
A glass	$> 10^{15}$	10^5	$> 10^{10}$

~ 11.6 days

Linear Viscoelastic properties of common liquids, values are typical order of magnitude approximations [1].

Speed of sound:

in solid: $c = \sqrt{E/\rho}$

1 . H.A. Barnes, *Handbook of Elementary Rheology* (University of Wales, Institute of Non-Newtonian Fluid Mechanics, Aberystwyth, 2000)

Two time scales

(Stress) **relaxation modulus** often contains terms like:

$\sigma(t) = \varepsilon_0 G(t;\varepsilon_0)$

$G(t;\varepsilon_0) \rightarrow \exp(-t/\tau_R) \longrightarrow$ τ_R – characteristic stress ***relaxation time***

It measures how quickly material relaxes

(Strain) **creep compliance** often contains terms like:

$\varepsilon(t) = \sigma_0 J(t;\sigma_0)$

$J(t;\sigma_0) \rightarrow \exp[-(E/\eta)\cdot t] \longrightarrow$ $\tau_C \equiv \eta / E$ – characteristic ***retardation time***

It measures how fast material creeps

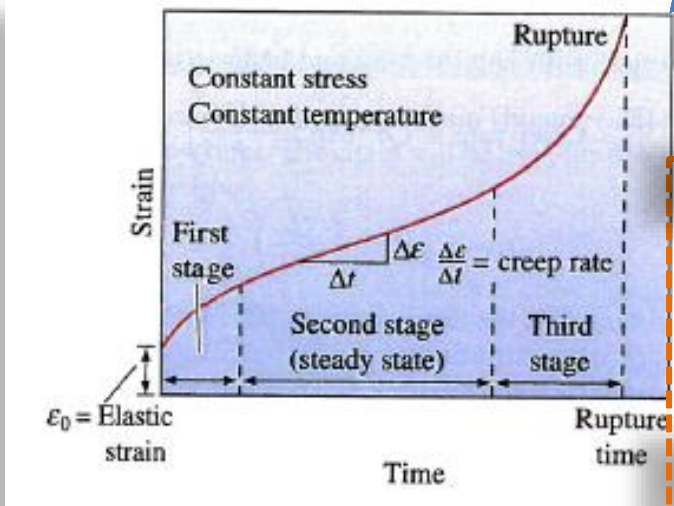
Some experimental observations

$$\dot{\varepsilon} = f(\sigma, T, t)$$

RECALL

Rate Dependent Deformation – Creep | viruminen

Stages of creep



Schematic typical creep curve showing strains as function of time under a constant stress and temperature levels

Creep test experimental setup

Lever arm creep arrangement

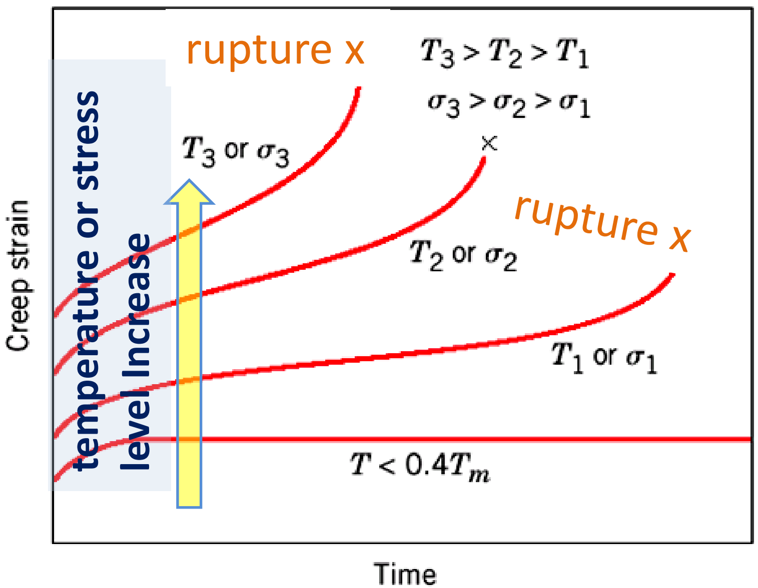
Steady state creep:

$$\dot{\varepsilon} = K_2 \sigma^n \exp\left(-\frac{Q_c}{RT}\right)$$

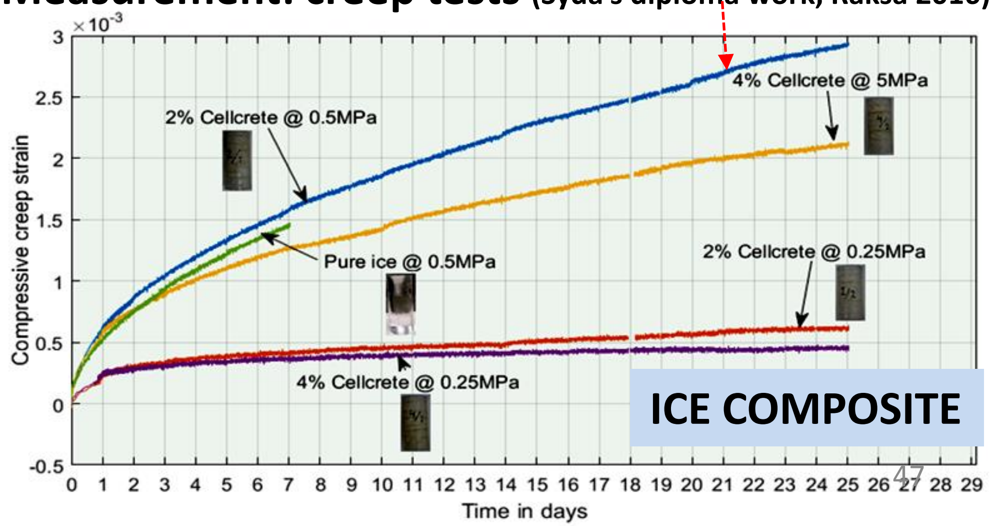
Q_c = activation energy for creep

K_2 and n are material constants

R - universal gas constant ~ 8.314 [J/(K.mol)]



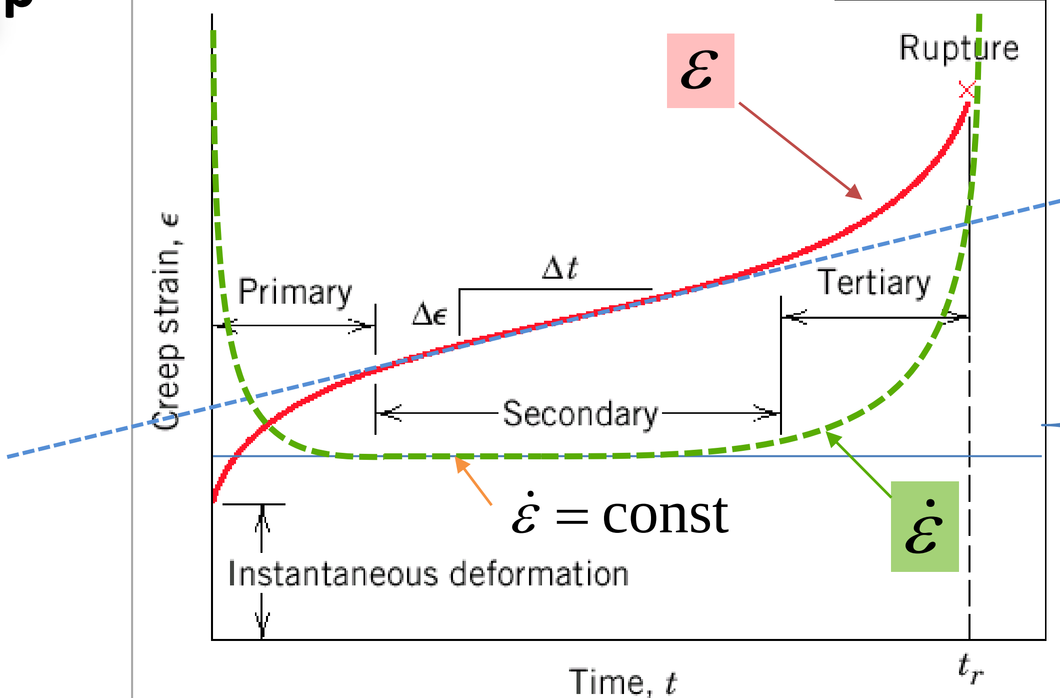
Measurement: creep tests (Syda's diploma work, Raksa 2016)



Stages of creep

Concepts & Definitions

Constant stress and temperature



Primary creep:

$$\varepsilon = A(\sigma)\cdot t^{1/\beta},$$
$$2 < \beta < 4$$

To avoids meaningless units write in this form:

$$\varepsilon = A(\sigma)\cdot\left[\frac{t}{t_{\mathrm{REF}}}\right]^{1/\beta},$$

$$\dot{\varepsilon} = K_2\left[\frac{\sigma}{\sigma_{\mathrm{Ref}}}\right]^{n}\exp\left(-\frac{Q_c}{RT}\right)$$

Steady creep rate:

$$\dot{\varepsilon}_s = \Delta\varepsilon / \Delta t$$

1. **Instantaneous deformation**, mainly elastic
2. **Primary/transient creep.** Slope of strain *vs*. time decreases *t*: work-hardening
3. **Secondary/steady-state creep**. Rate of straining constant: work-hardening and recovery.
4. **Tertiary**. Rapidly accelerating strain rate up to failure: formation of internal cracks, voids, grain boundary separation, necking*: accumulation of damage - rupture*

Steady state creep:

$$\dot{\varepsilon} = K_2\sigma^n\exp\left(-\frac{Q_c}{RT}\right)$$

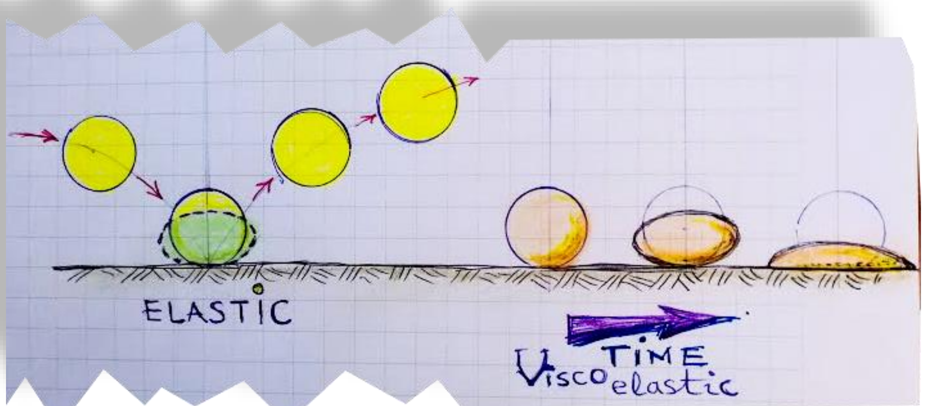
Q_c = activation energy for creep

K_2 and n are material constants

R - universal gas constant ~ 8.314 [J/(K.mol)]

Viscoelasticity –

Concepts & Definitions



'very short' loading times 'very long' loading times

$$\sigma = \sigma(\varepsilon, \dot{\varepsilon}, \text{strain history})$$

Homologous temperature - definition

Cf. Homologous temperature: θ

$$\theta = \frac{T}{T_m},$$ temperatures in Kelvins

In general, creep becomes important when approximately the homologous temperature $\theta \geq 0.4$.

In other terms, for such relative temperatures, there can be sufficient molecular mobility that entropic effects (disordering effects) should be considered. [weakening of intermolecular bonding]

Steady-state creep rate:

Creep depends on stress

$$\dot{\varepsilon} = K_2 \sigma^n \exp\left(-\frac{Q_c}{RT}\right)$$

Creep depends on temperature (thermally activated creep; important for metals, polymers and ice under 'high' temperatures or transient heating)

Q_c = activation energy for creep

K_2 and n are material constants

R - universal gas constant ~ 8.314 [J/(K.mol)]

n depends on mechanism of creep

N.B. the **homologous temperature** for ice, even at $-20\ ^oC$

$\theta = T / T_m = 0.93 \Rightarrow$ Creep deformation becomes very significant

theta = (-20+273)/(0+273)=0.93

Mechanical characteristics of viscoelastic materials

In the following section, we consider mechanical characteristics under constant temperature

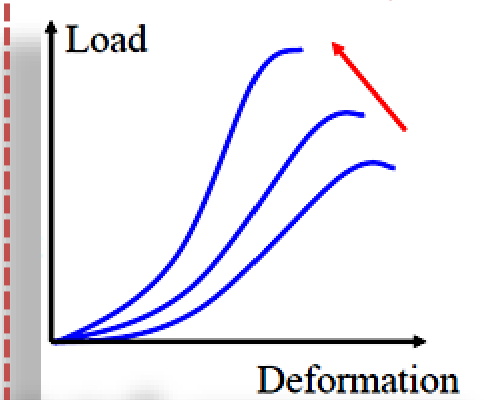
– **stress relaxation** at constant strain
– **creep** at constant stress
– **hysteresis** in loading and unloading cycle
– **strain-rate dependence**

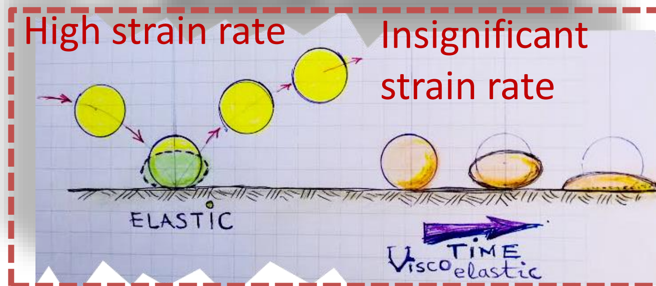
- Thus, in general, **stress** depends on **strain**, on **strain rate** and the **history of strain**

$$\rightarrow \sigma = \sigma(\varepsilon, \dot{\varepsilon}, \text{strain history})$$

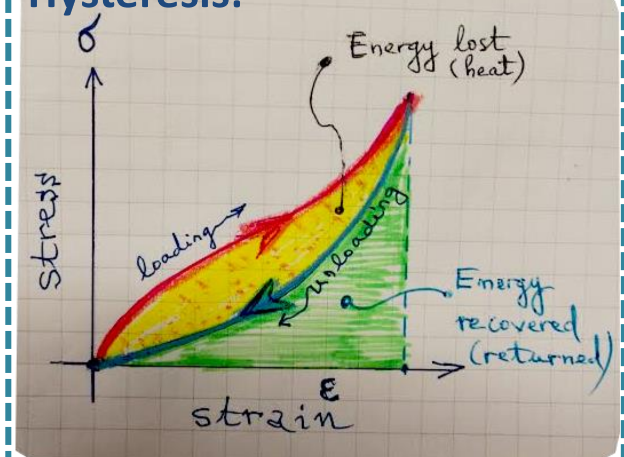
- The above mechanical behavior is generally modeled by the theory of ***viscoelasticity***

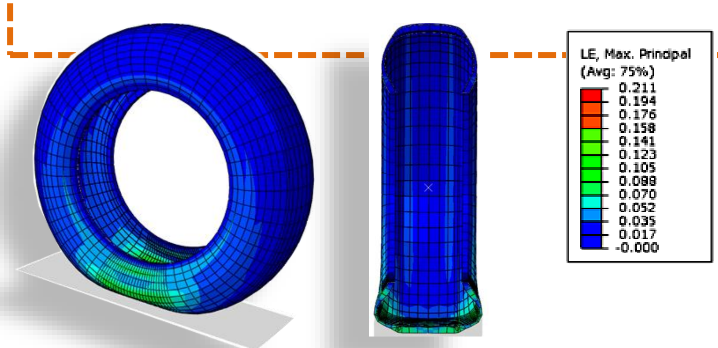
Strain rate increase leads to stiffer response





Hysteresis:





Phenomenology

In following sections, we consider mechanical characteristics under constant temperature (isothermal case)

These experiments allow

1) assessment of material viscoelastic response and

2) determination of the parameters of the relevant models (**parameter identification**)

Observations and experimentation

- **Engineers** often experimentally **characterize** (the mechanical behavior of) **materials** through simple **laboratory tests**
- For **mechanical characterization of viscoelastic materials**, uniaxial tensile tests similar to those used for elastic solids are ordinarily performed

  These **tensile tests** are **adapted** in order **to measure** the **time dependency** of the material response
- Engineers commonly **use** only **three type of tests**: creep, stress relaxation, and dynamic (sinusoidal) loading testing

Testing

Creep

Stress Relaxation

Dynamic or periodic **(sinusoidal)** loading **tests**

***Quasi-static* tests** (slow, inertia effects can be ignored)

Provide information on long-time response

Dynamic tests (quasi-static & rapid when inertia cannot be ignored) provide information as regard to short-time response supplementary to static creep and stress-relaxation tests

Phenomenology - Rheological Modeling

Aphorism: ***panta rhei*** 'everything flows'

We consider mechanical characteristics under constant temperature

- In general, **stress** depends on **strain**, on **strain rate** and the **history of strain**

$$\rightarrow \sigma = \sigma(\varepsilon, \dot{\varepsilon}, \text{strain history})$$

- The above mechanical behavior is generally modeled by the theory of **viscoelasticity** using rheological models

- **Mathematical models to determine** stress and strain or force and displacement interactions for viscoelastic materials (the mechanical response) are called: **Rheological Models**

- **Viscoelastic** response of a material reflects both **elastic** and **viscous** (fluid) behaviors and is modeled as a **combinations** of springs and dashpots respectively
- This fluid & solid character become relevant at *'long times'*

'flow'

ῥέω+ λογία

'study'

Rapid intro to rheology: https://en.wikipedia.org/wiki/Rheology (3.12.2016)

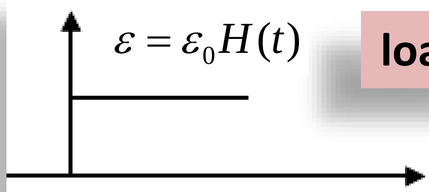
Distinction between viscoelastic liquid and viscoelastic solid materials …

… in their mechanical response:

RECALL

Qualitative responses to stress-relaxation:

Applied initial constant strain and time dependent stress is measured



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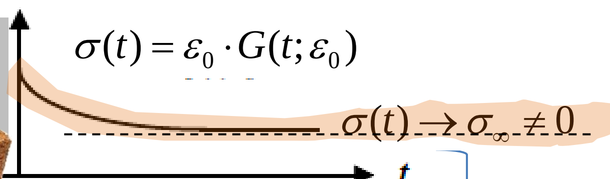
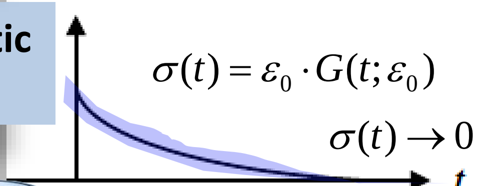
Viscoelastic solid



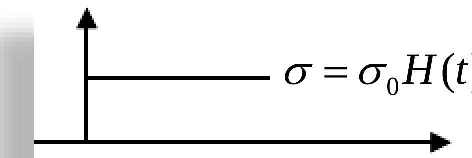
Viscoelastic fluid



Responses
What is the difference?

Qualitative responses to creep:

Applied initial constant stress and time dependent strain is measured



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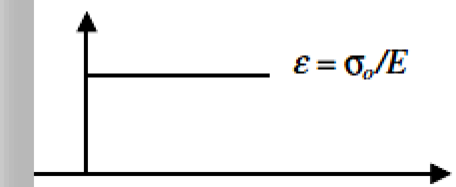
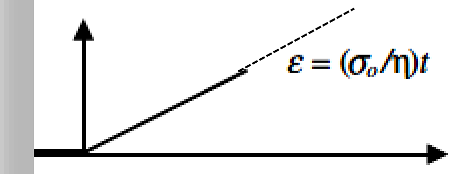
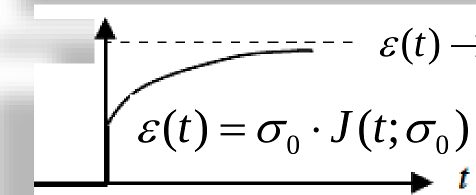


Elastic solid



Viscous fluid



Viscoelastic solid

$\varepsilon = \varepsilon_0 H(t)$: step-strain history

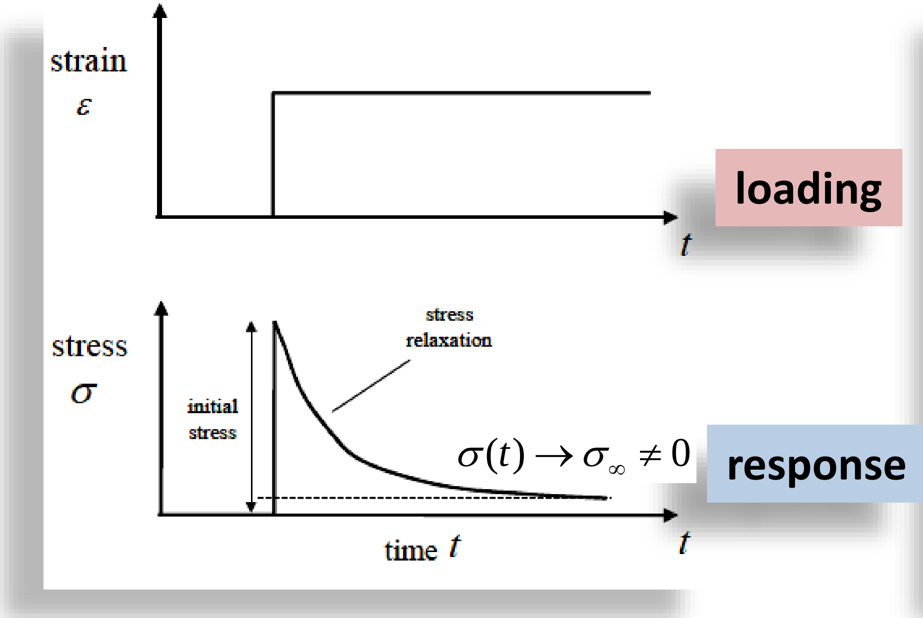
$H(t)$ – Heaviside function (unit step function)

$\sigma = \sigma_0 H(t)$: step-stress history

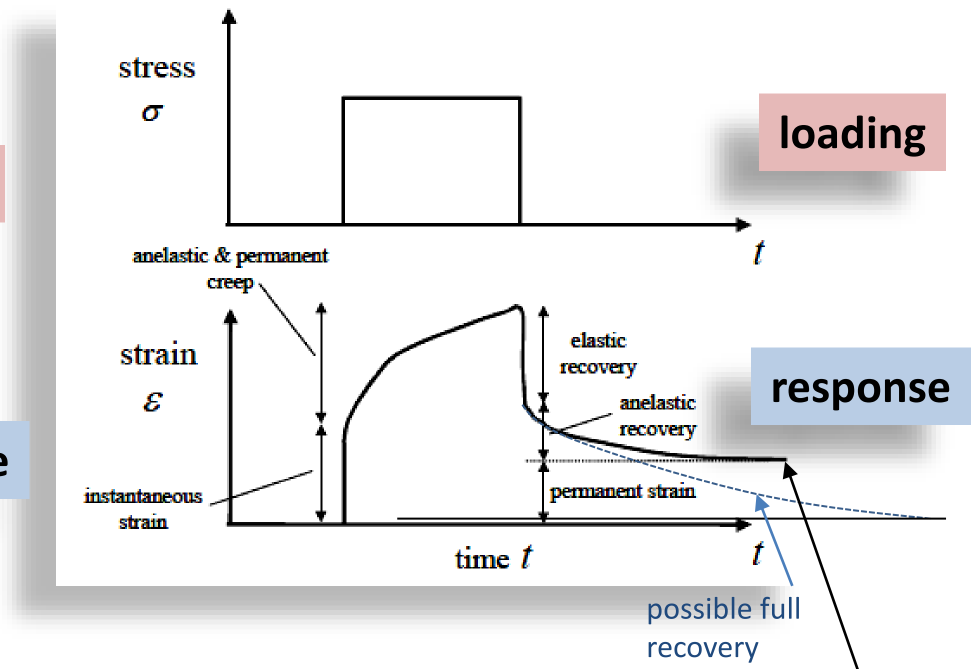
Observed behavior in real viscoelastic solid materials

RECALL

Typical response to a
Stress-relaxation test



Typical response to a **creep** recovery test



Permanent deformation if 'yield' stress is reached…(viscoplasticity)

Creep

$$C(t) \equiv J(t) = \frac{\varepsilon(t)}{\sigma_0}$$

For linear elastic material compliance:

$$J = \frac{\varepsilon}{\sigma} = \frac{1}{E} \equiv C$$

- **The creep test**: applying a steady uniaxial stress and σ_0 measuring the time dependent resulting strain $\varepsilon(t)$

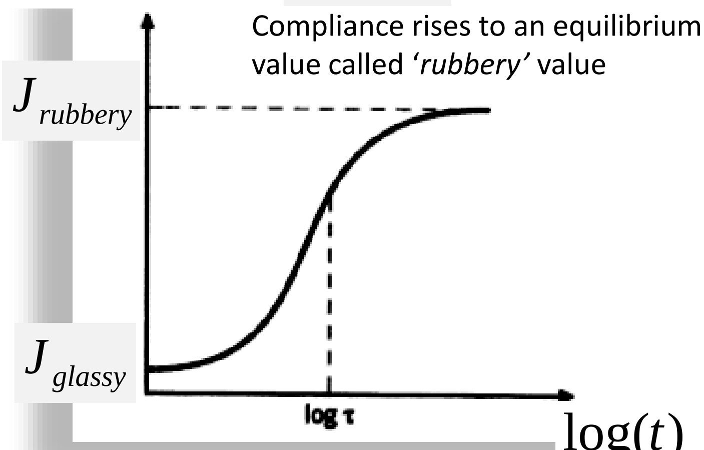
- **Creep compliance** (function) is the ratio of time-varying resulting strain from a constant stress to this stress

$$C(t) \equiv J(t) = \frac{\varepsilon(t)}{\sigma_0}$$

$$\Downarrow \quad \varepsilon(t) = \sigma_0 J(t;\sigma_0)$$

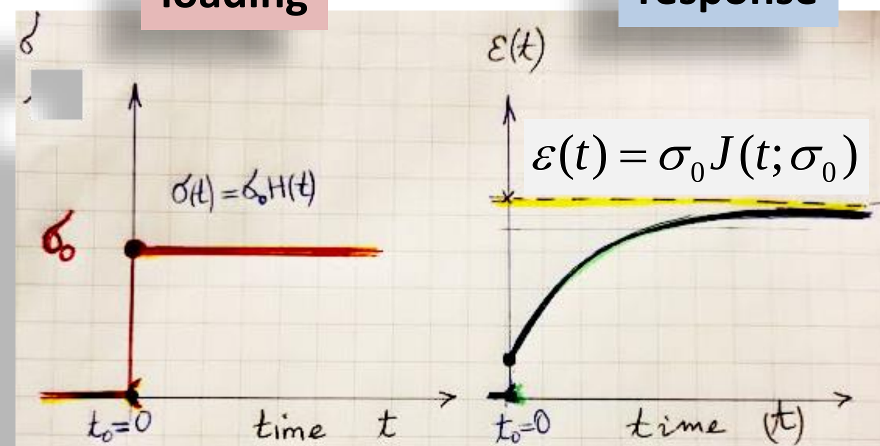
Compliance $J(t;\sigma_0)$

Compliance rises to an equilibrium value called *'rubbery'* value



INPUT → **OUTPUT**

loading **response**



Creep strain for a constant stress

Some notations: $J(t;\sigma_0) \leftrightarrow \phi(t;\sigma_0)$

Creep compliance | virumismoduuli tai -funktio

Creep

Relation between Creep 'coefficient' and compliance function

Creep deformation (mechanical strain): $\varepsilon(t,\tau) = \sigma \cdot \boxed{J(t,\tau)}$

Given in such form in many codes (**creep coefficient**)

$$\varepsilon(t,\tau) = \underset{\text{elastic}}{\varepsilon_{\mathrm{e}}} + \underset{\text{creep}}{\varepsilon_{\mathrm{cr}}} \equiv \varepsilon_{\mathrm{e}} + \boxed{\varphi(t,\tau)}\varepsilon_{\mathrm{e}}$$

$$\equiv \varepsilon_{\mathrm{e}} \cdot \left[1 + \varphi(t,\tau)\right]$$

strain at time *t* du to constant stress acting since age $\tau < t$

$$\Downarrow$$

$$\boxed{J(t,\tau)} = \frac{\varepsilon(t,\tau)}{\sigma} = \frac{\varepsilon_{\mathrm{e}} + \varepsilon_{\mathrm{cr}}}{\sigma} = \ldots$$

$$\ldots = \frac{\varepsilon_{\mathrm{e}}}{\sigma}\left[1 + \varphi(t,\tau)\right] = \boxed{\frac{1 + \varphi(t,\tau)}{E(\tau)}}$$

Elastic modulus characteristizing the instantaneous strain at age $\tau < t$

Creep coefficient

$$\varepsilon_{\mathrm{cr}} \equiv \varphi(t,\tau)\varepsilon_{\mathrm{e}}$$

Stress relaxation

Relaxation modulus:

$$E(t) \equiv G(t) = \frac{\sigma(t)}{\varepsilon_0}$$

- **The stress relaxation**: applying a steady strain ε_0 and measuring the time dependent resulting (relaxing) stress $\sigma(t)$

- **Relaxation modulus** (function) is the ratio of time-varying resulting stress from a constant steady strain

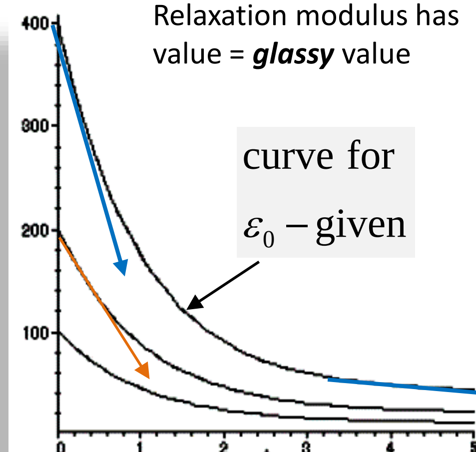
$$E(t) \equiv G(t) = \frac{\sigma(t)}{\varepsilon_0}$$

$$\Downarrow \quad \sigma(t) = \varepsilon_0 G(t;\varepsilon_0)$$

Relaxation modulus

Relaxation modulus has high initial value = ***glassy*** value

$G(t;\varepsilon_0)$

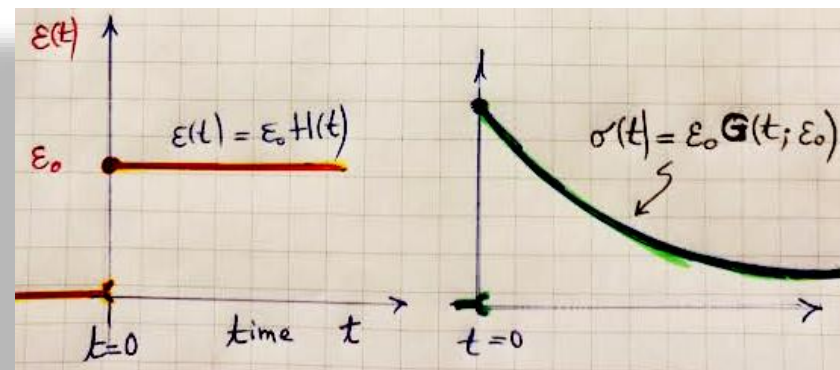


curve for ε_0 – given

time

INPUT → **OUTPUT**

loading **response**



Stress relaxation foe a steady strain

Some notations:

$$E_{rel}(t;\varepsilon_0) \leftrightarrow Y(t;\varepsilon_0) \leftrightarrow G(t;\varepsilon_0)$$

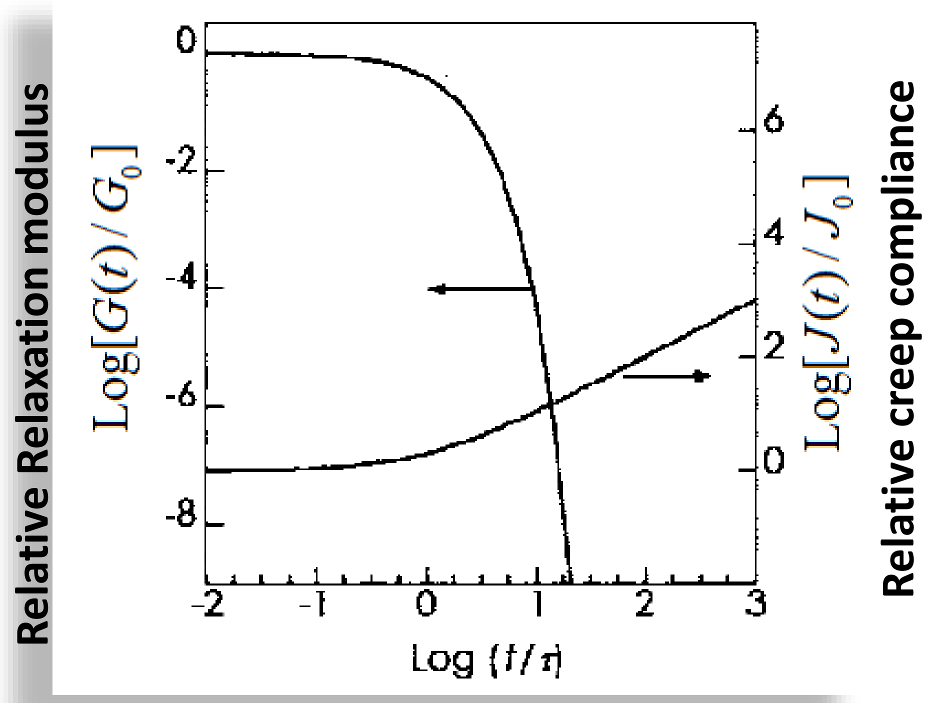
Relaxation modulus | relaksaatiomoduuli tai -funktio

Relationship between
Creep Compliance & Stress Relaxation Modulus

- the two curves have noticeably different shapes
- they would be 'mirror' *images if reciprocally related*

relaxation **creep**

$$G(t)=\frac{\sigma(t)}{\varepsilon_0}\neq\frac{1}{J(t)}=\frac{1}{\frac{\varepsilon(t)}{\sigma_0}}=\frac{\sigma_0}{\varepsilon(t)}$$



Relationship between creep compliance and stress relaxation modulus

Viscoelasticity - Complex Shear Modulus

Periodic response

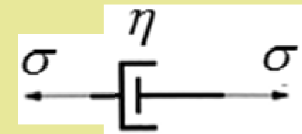
✓ Consider an **imposed periodic** (sinusoidal) **shear strain** on an **elastic** and on a **viscose** elements

- Imposed strain $$\boxed{\epsilon = \epsilon_0 \cdot \exp(i\omega t)}$$

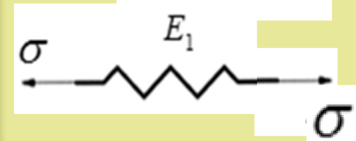
- Elastic **response**: $$\boxed{\sigma = E\epsilon = E\epsilon_0 \cdot \exp(i\omega t)}$$

- Viscous **response**: $$\boxed{\sigma = \eta\dot{\epsilon} = i\omega\eta\,\underbrace{\epsilon_0 \cdot \exp(i\omega t)}_{=\epsilon} = i\omega\eta \cdot \epsilon}$$

Newtonian dashpot



$\sigma = \eta\dot{\epsilon}$



$\sigma = E\epsilon$

Elastic spring

- In a general ViscoElastic solid: $\sigma = E\epsilon + \eta\dot{\epsilon}$

$$\boxed{\sigma = (E + i\omega\eta)\epsilon \equiv E^{*}\epsilon}$$

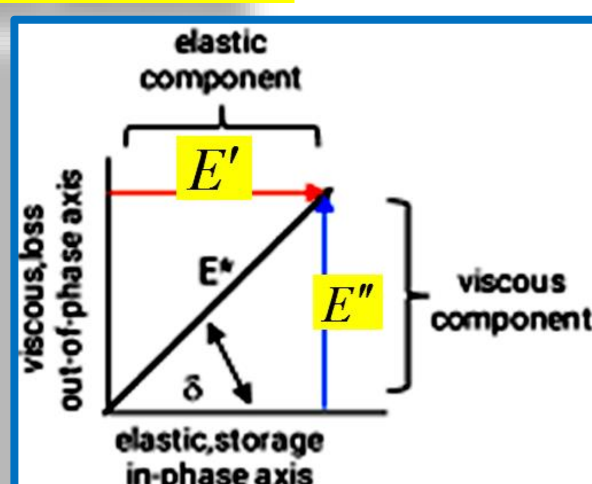
Complex Modulus

$$\boxed{E^{*} = (E + i\omega\eta) \equiv E' + iE''}$$

Loss modulus

Shear or (elastic modulus)

Storage modulus

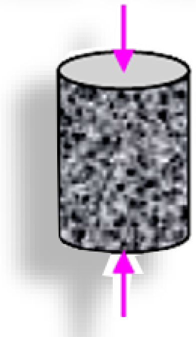


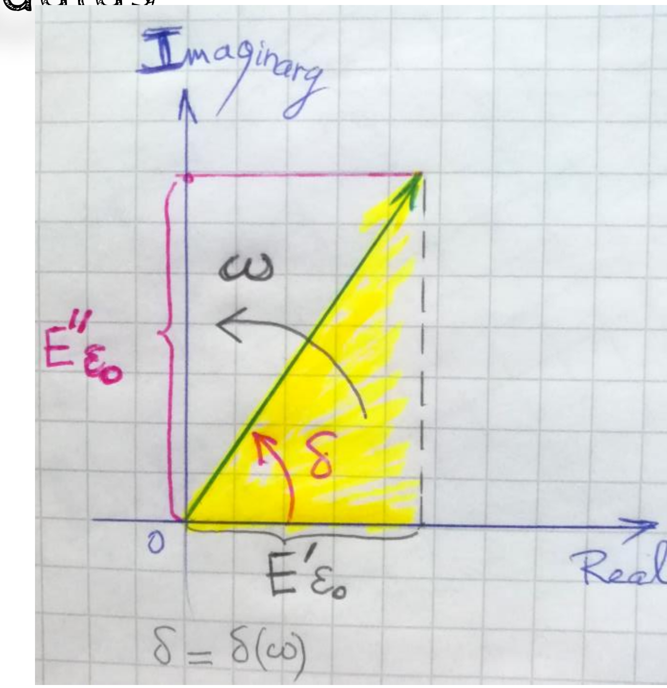
G^{*} : complex shear modulus

Viscoelasticity - Complex Shear Modulus

Periodic response

- ✓ Consider an **imposed periodic** (sinusoidal) **shear strain** on an **elastic** and on a **viscose** elements





$$\boxed{\sigma = (E + i\omega\eta)\epsilon \equiv E^*\epsilon}$$

$$\boxed{E^* = (E + i\omega\eta) \equiv \underbrace{E' + iE''}}$$

Complex Modulus

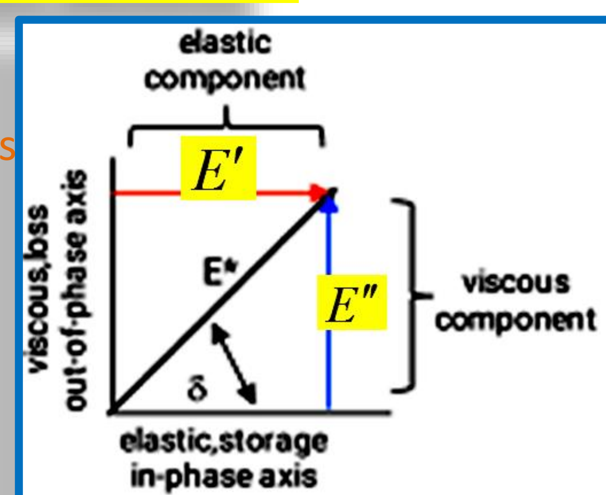
E^* or G^* : complex shear modulus

Loss tangent: $\tan\delta = E''/E'$

Loss modulus

Shear or (elastic modulus)

Storage modulus



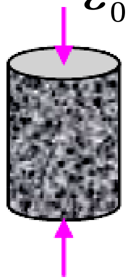
Elasticity (storage), **loss** moduli and **phase** shift angle depend on **frequency**

Dynamic (sinusoidal) loading tests

Dynamic modulus **Relaxation test**

Trigonometric formulation

$\varepsilon_0 \cos(\omega t)$



loading

sinusoidal strain (or stress)

$$\varepsilon(t) = \varepsilon_0 \cos(\omega t)$$

response

sinusoidal stress (or strain)

$$\sigma(t) = \sigma_0 \cos(\omega t - \delta), \quad \delta > 0 \qquad (1)$$

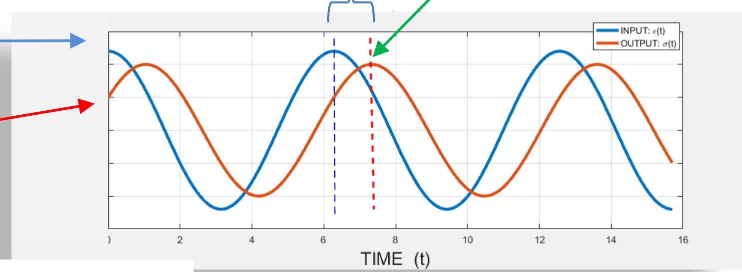
Lag by a phase shift or delay angle depends on frequency (delayed response)

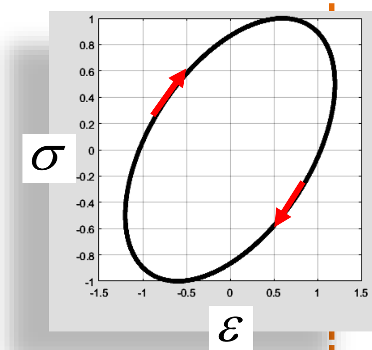
$0^{o} < \delta < 90^{o}$

$\delta / \omega =$

input $\varepsilon(t)$

output $\sigma(t)$





The complex relaxation **modulus:**

Complex formulation

$$E^* \equiv G^* = \sigma_0^* / \varepsilon_0^*$$

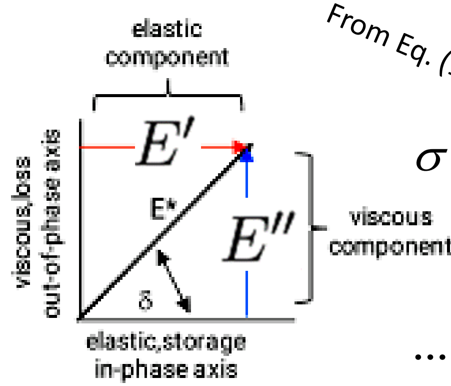
From Eq. (1)

$$\boxed{G^* = G' + iG''}$$

$$(E^* = E' + iE'')$$

$$\varepsilon(t) = \varepsilon_0^* e^{i\omega t}$$

$$\sigma(t) = \sigma_0^* e^{i\omega t}$$



Storage modulus: (elastic, reversible) $E' = \dfrac{\sigma_0}{\varepsilon_0}\cos\delta$

Loss modulus: (dissipative) $E'' = \dfrac{\sigma_0}{\varepsilon_0}\sin\delta$

$$\sigma(t) = \sigma_0\left[\cos(\omega t)\cos\delta + \sigma_0 \sin(\omega t)\sin\delta\right]$$

$$\Downarrow$$

$$\boxed{G(t;\varepsilon_0) = \sigma(t)/\varepsilon_0 = \ldots}$$

$$\ldots = \left[\frac{\sigma_0}{\varepsilon_0}\cos\delta\right]\cos(\omega t) + \left[\frac{\sigma_0}{\varepsilon_0}\sin\delta\right]\sin(\omega t)$$

The complex relaxation modulus - illustration

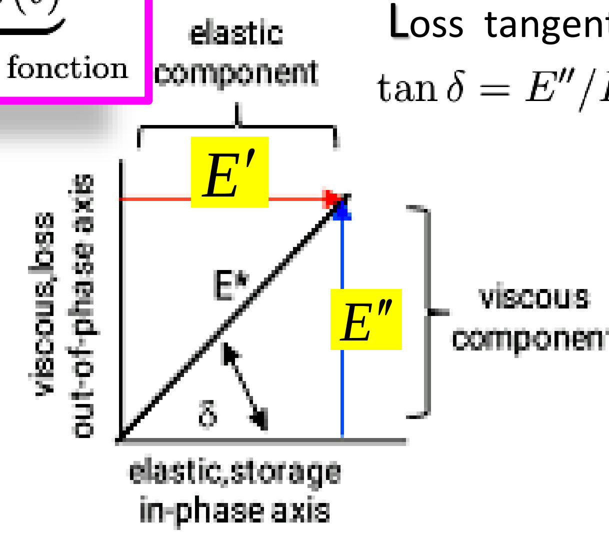
Useful to know

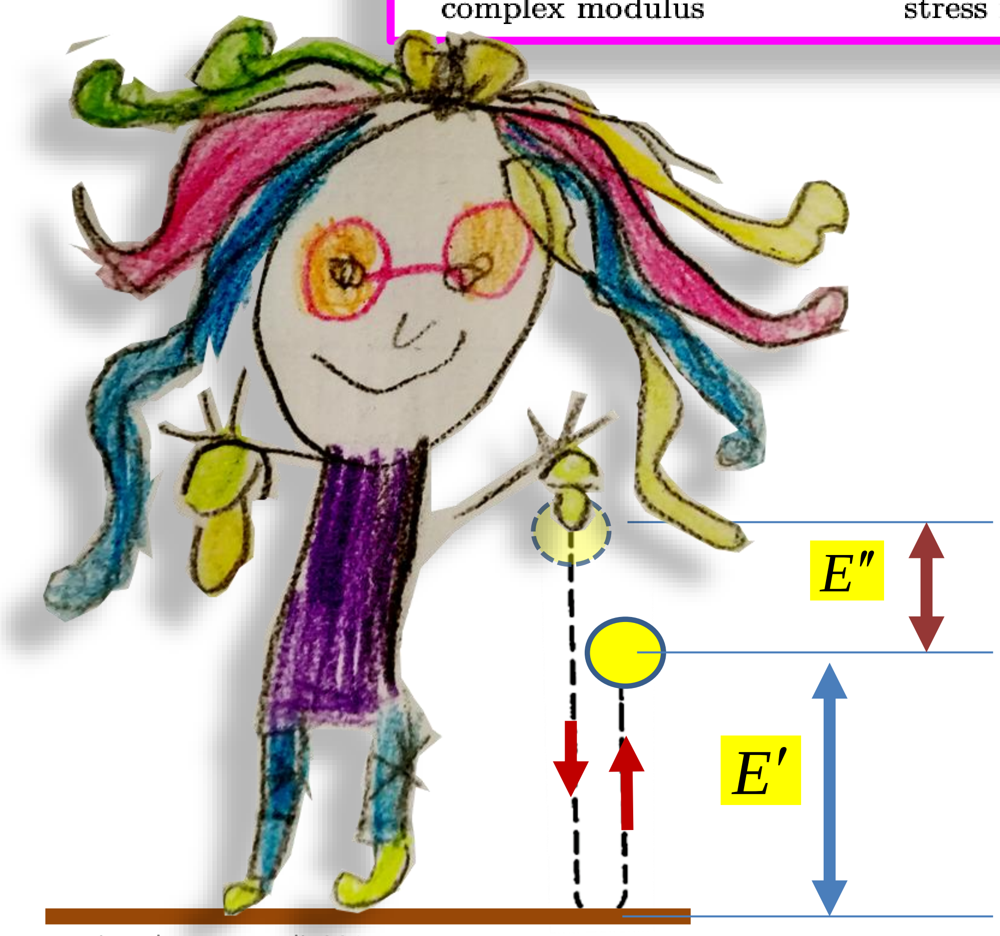
The complex modulus $G^*(\omega) \equiv E^*(\omega) = E' + \mathrm{i}E''$ is the frequency-domain representation of the stress relaxation function $G(t)$ of viscoelastic material. The ratio G''/G' is the ***loss factor*** (damping).

$$\underbrace{G^*(\omega) \equiv E^*(\omega)}_{\text{complex modulus}} \longleftrightarrow \underbrace{G(t) \equiv E(t)}_{\text{stress relaxation fonction}}$$

Loss tangent:

$\tan\delta = E''/E'$





Drawing: Ilona Baroudi, 2017.

E'' - ***Loss modulus*** (viscous, internal friction, irreversible, damping)

$$E'' = \frac{\sigma_0}{\varepsilon_0}\sin\delta$$

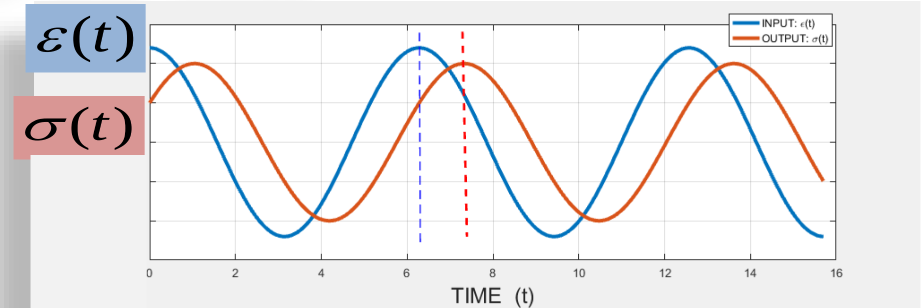
E' - ***Storage modulus*** (reversible, **ELASTIC**, restitution coefficient)

$$E' = \frac{\sigma_0}{\varepsilon_0}\cos\delta$$

The Physical meaning of the terms "Storage" and "Loss"

Consider the mechanical work done per loading cycle which is the strain energy:

$$\boxed{W = \oint \sigma \cdot d\varepsilon = \oint \sigma \cdot \dot{\varepsilon}\, dt}$$



where

$$\sigma(t) = \underbrace{\sigma_0' \cos(\omega t)}_{\text{in-phase stress}} + \underbrace{\mathbf{i}\sigma_0'' \sin(\omega t)}_{\text{out-of-phase stress}},$$

$$\varepsilon(t) = \varepsilon_0 \cos(\omega t) \Rightarrow \dot{\varepsilon} = -\varepsilon_0 \sin(\omega t)$$

The maximum energy stored by the in-phase components occurs at the quarter-cycle point:

$$W = \int_0^{\pi/2\omega} [\sigma_0' \cos(\omega t)][-\varepsilon_0 \sin(\omega t)] dt = $$

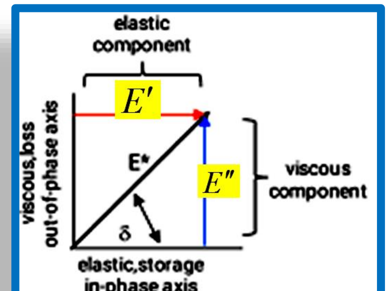
$$= -\frac{1}{2}\sigma_0'\varepsilon_0 = -\frac{1}{2}\sigma_0'\varepsilon_0 \cos\delta$$

Integrating separately the two components:

$$W = \underbrace{\int_0^{2\pi/\omega} [\sigma_0' \cos(\omega t)][-\varepsilon_0 \sin(\omega t)] dt}_{\equiv W_{rev} = 0} + \underbrace{\int_0^{2\pi/\omega} [\sigma_0'' \sin(\omega t)][-\varepsilon_0 \sin(\omega t)] dt}_{\equiv W_{dis} = \pi\sigma_0''\varepsilon_0 \neq 0}$$

strain energy associated with in-phase stress and strain is reversible (elastic); *i.e.* Energy which is **stored** in the material during a *loading cycle* can be **recovered** (restituted) without loss during *unloading*

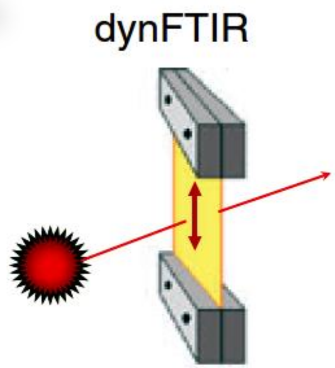
On the contrary, energy supplied to the material by the out-of-phase components is **converted irreversibly to heat** (dissipated, **lost**)



Relative dissipation = $W_{dis} / W_{rev} = 2\pi \tan\delta$

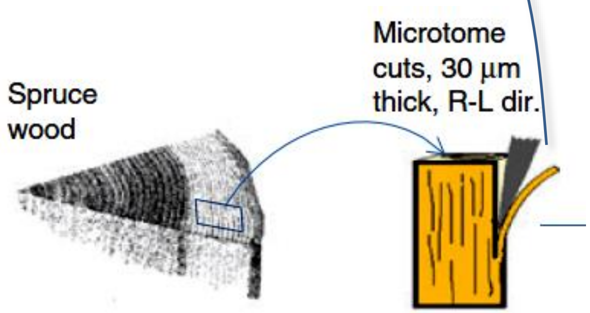
Loss tangent: $\tan\delta = E''/E'$

Example of Storage Modulus

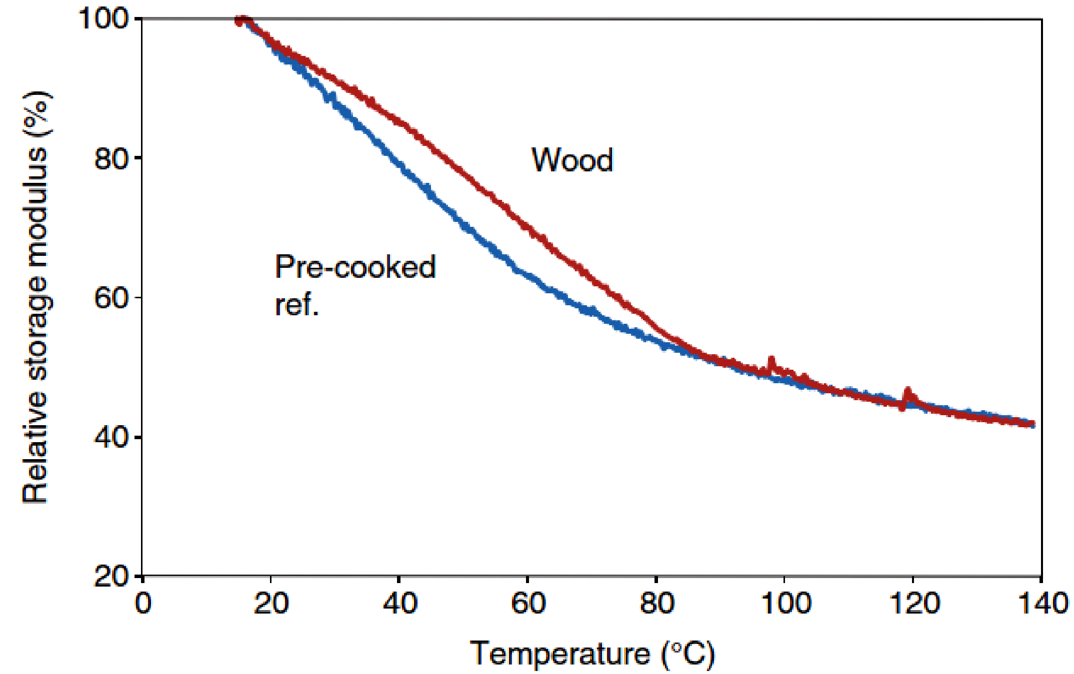


subjected to dynFTIR spectroscopy

Wood

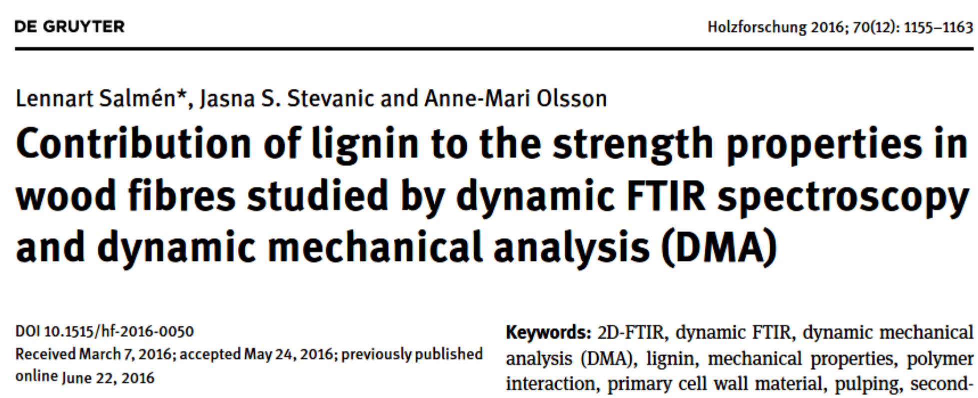


Storage modulus = Elastic modulus



Relative storage modulus as a function of temperature from dynamic mechanical measurements of spruce wood samples at 1 Hz and of the same wood impregnated with kraft-cooking liquor and heated to 120°C.

Reference:



DE GRUYTER Holzforschung 2016; 70(12): 1155–1163

Lennart Salmén*, Jasna S. Stevanic and Anne-Mari Olsson

Contribution of lignin to the strength properties in wood fibres studied by dynamic FTIR spectroscopy and dynamic mechanical analysis (DMA)

DOI 10.1515/hf-2016-0050
Received March 7, 2016; accepted May 24, 2016; previously published online June 22, 2016

Keywords: 2D-FTIR, dynamic FTIR, dynamic mechanical analysis (DMA), lignin, mechanical properties, polymer interaction, primary cell wall material, pulping, second-

Homework 5 - EXTRA mini-project in Visco-elasticity [EXTRA 15 points]

Good to do

Dynamic loading tests

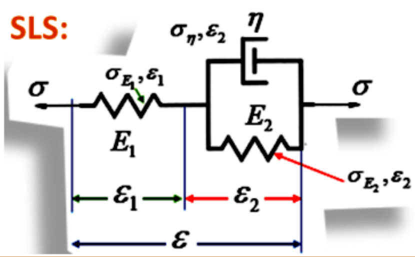
Key words Dynamic loading tests, dynamic modulus, relaxation test, storage modulus, loss modulus, lag or phase-shift angle, hystheresis, relative dissipation.

This mini-programming project is intended to help student to consolidate what he has learnt during the lectures about dynamic modulus and related consepts for collectivistic materials under dynamic loading. In addition to this, the student we recall or learn to do numerical-integration while solving the stress history from known strain history.

The time-integration of the bellow ODE (1) can be done either by using Matlab standard time-integration functions for stiff-ODEs or use student's own functions as the back-ward Euler or the mid-point integration scheme, or any other reverent method.

1 The Standard Linear Solid

Assume a Linear Standard Solid. Derive the ordinary differential equation below:



$$\boxed{\sigma + \tau\dot{\sigma} = G_\infty \epsilon + \tau G_0 \dot{\epsilon}} \quad (1)$$

$$G_\infty = \frac{E_1 E_2}{E_1 + E_2}, G_0 = E_1, \tau = \frac{\eta}{E_1 + E_2} \quad (2)$$

$$\dot{\sigma} = \frac{G_\infty}{\tau}\epsilon + G_0\dot{\epsilon} - \frac{\sigma}{\tau} \quad (3)$$

$$\dot{\sigma} = f(\epsilon, \sigma) \quad (4)$$

The differential equation (1) or (4) can be numerically integrated using appropriate in tial conditions for any known history of the deformations or of the stresses, respectivel

In order to determine some material properties of the visco-elastic material the cyclic strain history below is imposed and the corresponding stress history was recorded.

$$\epsilon(t) = \epsilon_0 \sin(\omega t), \quad (5)$$

where $\omega = 1$ (1/s), $\tau = 1$ (s) , $\epsilon_0 = 0.008, G_\infty = 550$ MPa, $G_0 = 1.5$ GPa. The initial conditions are $t = 0, \epsilon(0) = 0, \sigma(0) = 0$.

1. determine the time-series of the response in term of stress $\sigma(t)$ for the given periodic excitation $\epsilon(t)$
2. draw the graphs of $\sigma(t)$ and $\epsilon(t)$
3. draw the graph $\sigma - \epsilon$ for few cycles in order to observe the hysteresis loop
4. estimate from the delay (the lag) $\Delta t = \delta/\omega$ of the two time-serie; the excitation $\epsilon(t)$ and the response $\sigma(t)$ (scale adequately these graphs to draw them on the same axes to estimate, graphically Δt)
5. give an estimation for the storage E' and loss modulus E''
6. estimate the relative dissipation

Estimate from the delay (the lag) $\Delta t = \delta/\omega$ of the two time-serie; the exitation $\epsilon(t)$ and the response $\sigma(t)$ Give an estimation for the storage E' and loss modulus E''.

For the numerical time-integration you can use and edit the two m-scripts I put in MyCourses: `Main-SLS-distribute.m` and `Main-SLS-distribute.m`.

Some results obtained by eye-reading the graphs: Her follows an approximate partial solution $E' = \sigma_0/\epsilon_0 \cos\delta \approx 850$ MPa

Matlab Code

The m-scripts are self-explaining. The main program "Main-SLS-distribute" is calling the function "Standard-Linear-Model-SLS.m"

```
dot_EPSILON_TAB = diff(EPSILON_TAB) ./ diff(times);
dot_EPSILON_TAB = [dot_EPSILON_TAB 0];


[Time, Y] = ode23s('Standard_Linear_Model_SLS',time,[sigma_0
    t_0]);
Sigma = Y(:,1);
t     = Y(:,2);

epsilon_interp = interp1(times, epsilon, Time);

plot(Time/tau, Sigma/1e6);
xlabel('t/\tau')
ylabel('\sigma  (MPa)')
grid on
```

[Time, Y] = ode23s('Standard_Linear_Model_SLS',time,[sigma_0 t_0]);

Example of measurement of complex modulus

Good to know

$$|G^*| \equiv |E^*| \qquad \varphi \equiv \delta$$

$$|G^*| = \sigma_0 / \varepsilon_0$$

frequency

Dynamic Modulus

phase angle

f (1/sec)	-10ºC		4.4ºC		21.1ºC		37.8ºC		54.4ºC	
	$\lvert E^*\rvert$, GPa	φ(º)	$\lvert E^*\rvert$, GPa	φ(º)	$\lvert E^*\rvert$, GPa	φ(º)	$\lvert E^*\rvert$, GPa	φ(º)	$\lvert E^*\rvert$, GPa	φ(º)
25	12.571	3.9	9.981	7.0	5.575	14.2	2.972	22.9	1.206	28.6
10	12.802	3.9	9.366	6.5	4.974	13.7	2.492	22.3	0.878	21.2
5	12.506	3.8	8.981	7.5	4.376	14.5	2.062	22.8	0.614	26.7
1	11.628	4.9	7.856	8.6	3.322	17.4	1.237	23.8	0.306	25.8
0.5	11.224	5.3	7.381	8.6	2.914	18.5	0.978	25.7	0.234	26.0
0.1	10.285	6.1	6.227	11.6	1.991	20.3	0.634	25.2	0.146	28.7

Dynamic Moduli and Phase Angles for a specific asphalt-concrete mix (no tension, only compression was applied), *ref.* to [1] – PhD. thesis

[1] ***Ref***: Material data from: PhD-thesis:

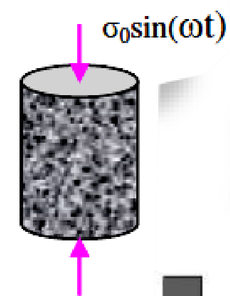
VISCOELASTIC FE MODELING OF ASPHALT PAVEMENTS AND ITS APPLICATION TO U.S. 30 PERPETUAL PAVEMENT

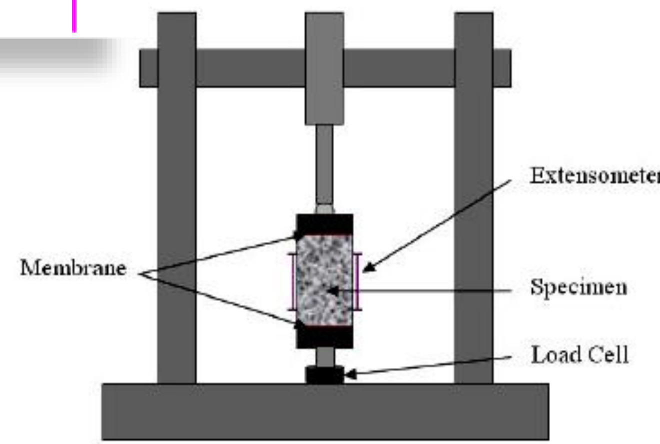
the Department of Civil Engineering and the Russ College of Engineering and Technology by

by

YUN LIAO

Doctor of Philosophy
Yun Liao
November 2007





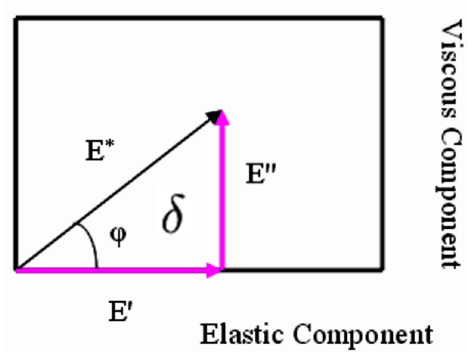
Schematic of dynamic modulus test

The complex (relaxation) **modulus**:

$$G^*(\omega) = G'(\omega) + iG''(\omega)$$

$$(E^* = E' + iE'')$$

Their exist methods to convert frequency domain representation to the time domain representation of the relaxation modulus, please ref. to literature.



$$\varphi = 2\pi \cdot \frac{t_{\mathrm{LAG}:\sigma\text{-}\varepsilon}}{t_{\sigma\text{-cycle}}}$$

δ

ω - angular frequency

Their exist methods to convert frequency domain representation to the time domain representation of the relaxation modulus, please *ref.* to literature for conversion methodology.

Good to know

Relaxation Moduli

t (sec)	-10 ºC E(t), GPa	4.4 ºC E(t), GPa	21.1 ºC E(t), GPa	37.8 ºC E(t), GPa	54.4 ºC E(t), GPa
0.04	12.498	9.598	5.014	2.399	0.842
0.1	12.646	8.968	4.457	2.015	0.659
0.2	12.295	8.548	3.893	1.656	0.445
1	11.291	7.388	2.894	0.988	0.229
2	10.842	6.916	2.520	0.761	0.176
10	9.816	5.734	1.796	0.498	0.110

Relaxation Moduli for a specific asphalt-concrete mix (no tension, only compression was applied), *ref.* to [1] – PhD. Thesis. /converted (see *ref.* [2] for the procedure) from dynamic moduli shown in the previous slide/

[2] Christensen, R. M. (1982). *Theory of Viscoelasticity*. 2nd Ed., Academic Press, New York.

Elasticity (storage), **loss** moduli and **phase** shift angle depend on **frequency**

Good to know

The complex (relaxation) **modulus:**

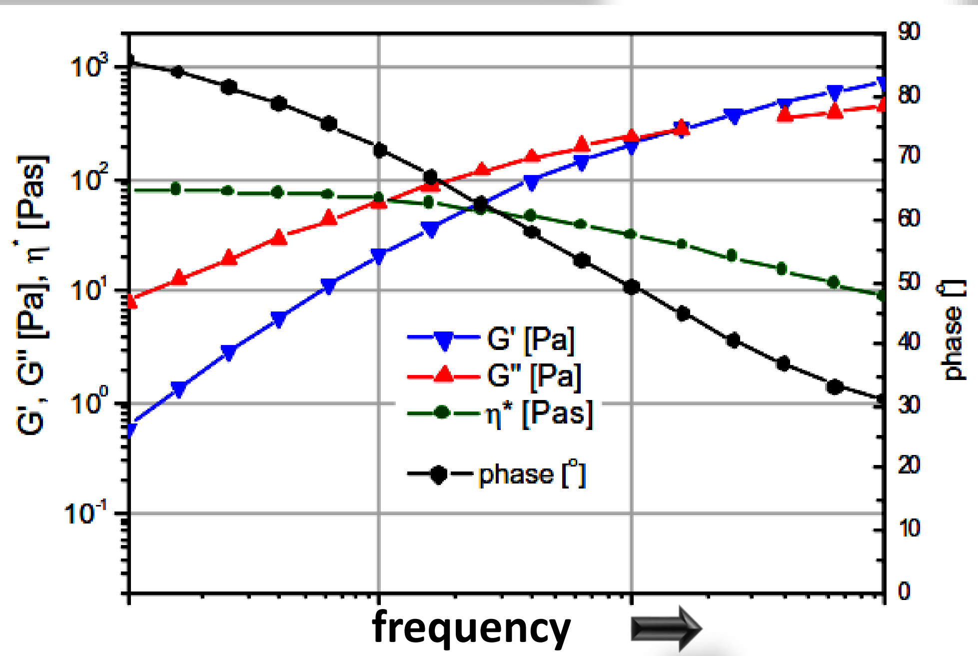
$$G^*(\omega) = G'(\omega) + iG''(\omega)$$

$$(E^* = E' + iE'')$$

Elasticity (storage), **loss** moduli and **phase** shift angle depend on **frequency**

E'' or *G''* - **Loss modulus**

E' or G' - **Storage modulus**



International Journal of Solids and Structures 47 (2010) 374–382

Contents lists available at ScienceDirect

International Journal of Solids and Structures

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Measurement method of complex viscoelastic material properties

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TA

AN004

Viscoelasticity and dynamic mechanical testing

A. Franck, TA Instruments Germany

Keywords: dynamic mechanical testing, viscoelasticity, Hookean body, Newtonian fluid, relaxation time, ,

VISCOELASTICITY

Most materials are not purely viscous and often show significant elastic behavior. Such materials are referred to as viscoelastic materials and the key parameter, the time determines whether viscous or elastic behavior prevails.

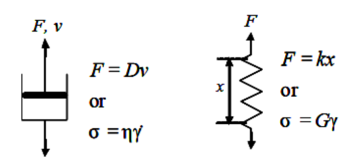
Therefore in a slow deformation or flow process the viscous behavior dominates, whereas in a short time process the material behaves predominately elastic. Whether a process is fast or slow depends on a characteristic internal material time.

A material will be perceived as a viscous liquid if the material time is very short in comparison to the time of the deformation process. For example, the material time of water is about 10-10 s and any deformation process must seem very long compared

MECHANICAL MODELS TO DESCRIBE VISCOELASTICITY

Viscoelastic materials can exhibit both viscous and elastic behavior. They can therefore be seen as a combination of both ideal types of materials: purely viscous fluids and ideally elastic solids.

The flow properties of a purely viscous material can be determined in a simple flow experiment. If



Maxwell model:

Dynamic response – example/homework?

$$\epsilon = \epsilon_0 \cos \omega t$$

$$\sigma = \sigma_0 \cos(\omega t + \delta)$$

mmon in the analysis of react
ıt to write the stress function
ain and whose imaginary part

$$\sigma^* = \sigma_0' \cos \omega t + i\,\sigma_0'' \sin \omega t$$

Γhe complex modulus c

$$E^* = \sigma_0^* / \epsilon_0^*$$

$$\sigma = \sigma_0^* e^{i\omega t}$$

$$\epsilon = \epsilon_0^* e^{i\omega t}$$

t of these is the "real," or "storage,"
strain:

$$E' = \sigma_0' / \epsilon_0$$

Maxwell material,

Multiplying by k and using $\tau = \eta/k$:

$$\boxed{k\dot{\epsilon} = \dot{\sigma} + \frac{1}{\tau}\sigma}$$

"imaginary," or "loss," modulus, define

$$E'' = \sigma_0'' / \epsilon_0$$

Example 3

In the case of the dynamic response, the time dependency of both the stress and the strain are of the form $\exp(i\omega t)$. All time derivatives will therefore contain the expression $(i\omega)\exp(i\omega t)$, so Eqn. 22 gives:

$$k\,(i\omega)\,\epsilon_0^* \exp(i\omega t) = \left(i\omega + \frac{1}{\tau_j}\right)\sigma_0^* \exp(i\omega t)$$

The complex modulus E^* is then

$$E^* = \frac{\sigma_0^*}{\epsilon_0^*} = \frac{k(i\omega)}{i\omega + \frac{1}{\tau_j}} = \frac{k(i\omega\tau)}{1 + i\omega\tau} \qquad (24)$$

This equation can be manipulated algebraically (multiply and divide by the complex conjugate of the denominator) to yield:

$$E^* = \frac{k\omega^2\tau^2}{1+\omega^2\tau^2} + i\frac{k\omega\tau}{1+\omega^2\tau^2} \qquad (25)$$

E'' or *G''* - **Loss modulus**

In Eq. 25, the real and imaginary components of the complex modulus are given explicitly; these are the "Debye" relations also important in circuit theory.

E' or G' - **Storage modulus**

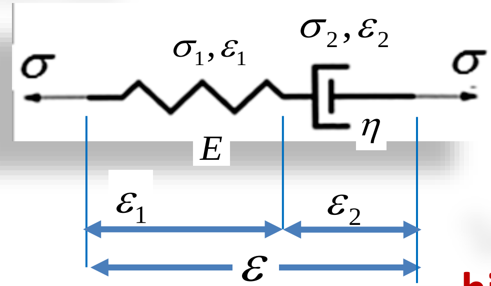
Maxwell model

Dynamic response

Maxwell material,

Input $\epsilon = \epsilon_0 \cos \omega t$

Output $\sigma = \sigma_0 \cos(\omega t + \delta)$



Relaxation time

$$\tau = \eta / E$$

$$\boxed{E\dot{\epsilon} = \dot{\sigma} + \frac{1}{\tau}\sigma}$$

$$E^* = \frac{E\omega^2\tau^2}{1+\omega^2\tau^2} + i\frac{E\omega\tau}{1+\omega^2\tau^2}$$

E′ E″

E′ or G′ - **Storage modulus**

E″ or *G″* - **Loss modulus**

'Loading' speed

high $\upsilon = \omega\tau >> 1$

E′ E″

$$\frac{k\omega^2\tau^2}{1+\omega^2\tau^2} >> \frac{k\omega\tau}{1+\omega^2\tau^2}$$

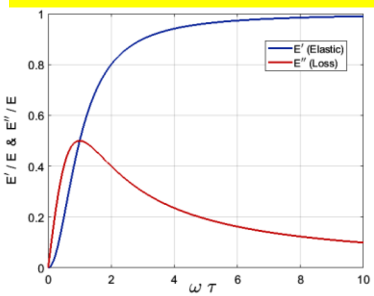
Behaves like an **Elastic Solid** E′

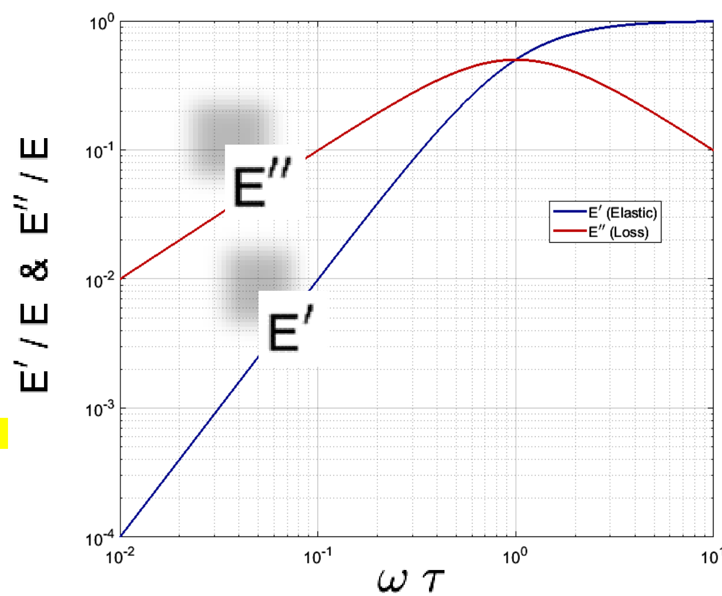
low $\upsilon = \omega\tau << 1$

$$\frac{k\omega^2\tau^2}{1+\omega^2\tau^2} << \frac{k\omega\tau}{1+\omega^2\tau^2}$$

E″

Behaves like a **Viscous Fluid**





Dynamic response

Input $\epsilon = \epsilon_0 \cos \omega t$

Output $\sigma = \sigma_0 \cos(\omega t + \delta)$

Journal of Sound and Vibration (1998) 214(1), 83–104

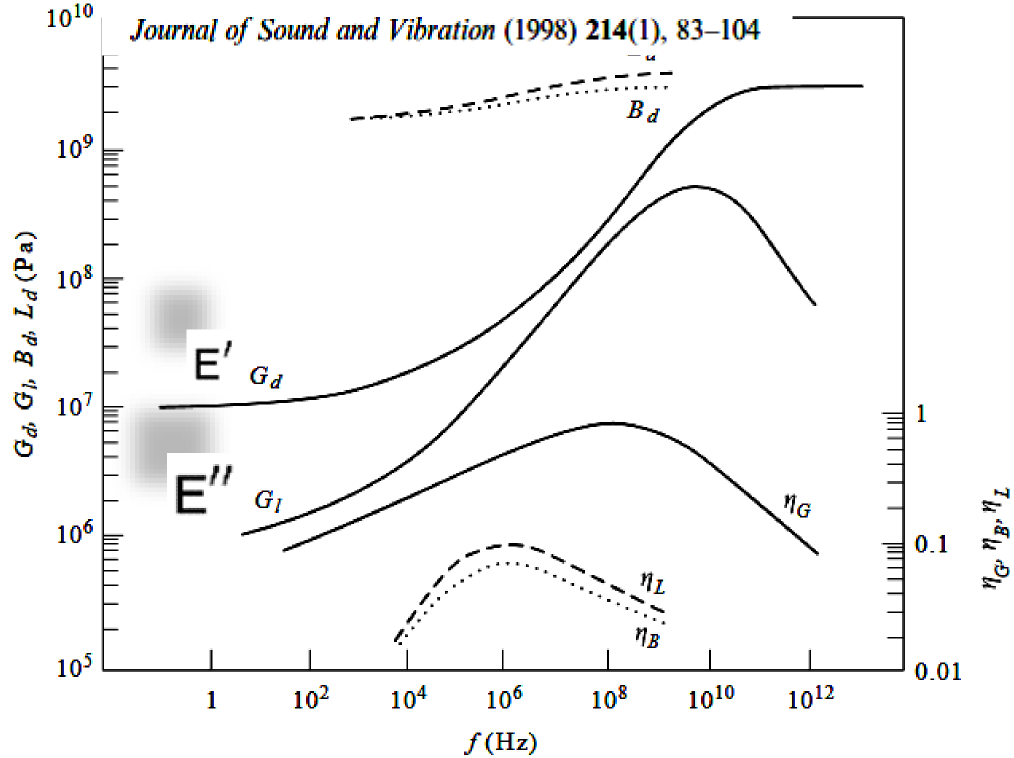
FREQUENCY DEPENDENCES OF COMPLEX MODULI AND COMPLEX POISSON'S RATIO OF REAL SOLID MATERIALS

T. Pritz

Behaves like an **Elastic Solid**

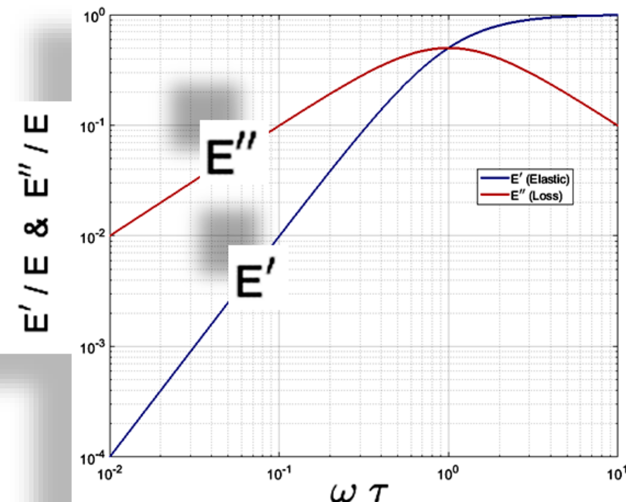
Behaves like a **Viscous Fluid**

Real material - Experimental



Frequency dependences of dynamic shear, bulk and longitudinal moduli, and the relevant loss functions of rubbery materials. ——, Shear dynamic properties of a natural rubber filled with carbon black at 0°C. (Data from Payne and Scott [20].) Bulk (. . .) and longitudinal (- - -) dynamic properties of a styrene-butadiene rubber at 20°C. (Data from Wada *et al*. [21]. The loss factors η_B and η_L have been calculated from the published values of the relevant loss and dynamic moduli.)

Maxwell model



E″ **Loss modulus**

E″ **or G′ - Storage modulus**

Mathematical Models for Linear Viscoelastic Response

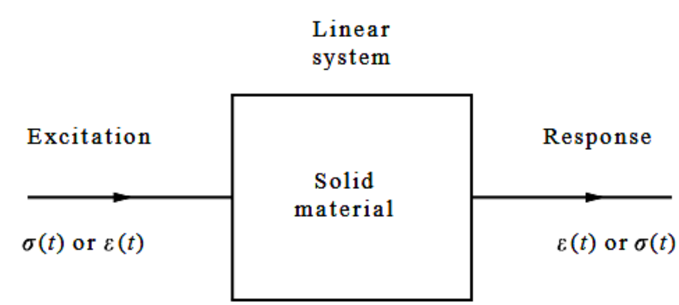
- In general, **stress** depends on **strain**, on **strain rate** and the **history of strain**

$$\rightarrow \sigma = \sigma(\varepsilon, \dot{\varepsilon}, \text{strain history})$$

- The above mechanical behavior is generally modeled by the theory of **viscoelasticity** using rheological models

- **Mathematical models to determine** stress and strain or force and displacement interactions for viscoelastic materials (the mechanical response) are called: **Rheological Models**

- **Viscoelastic** response of a material reflects both elastic and viscous (fluid) behaviors and is modeled as **combinations** of springs and dashpots respectively.
- This fluid & solid character become relevant at *'long times'*



Constitutive models for viscoelasticity

Good to know

Reading

Example: viscoelastic **creep** (we will come back to this example later may be in homework!)

- **Concrete is a viscoelastic material***

Creep: Apply constant stress

(compressive stress bellow yield stress)

$$\sigma = \sigma_0 H(t - t_0)$$

age of the concrete when loaded

creep deformation:

$$\boxed{\varepsilon(t,t_0) = \sigma(t_0) J(t,t_0),}$$

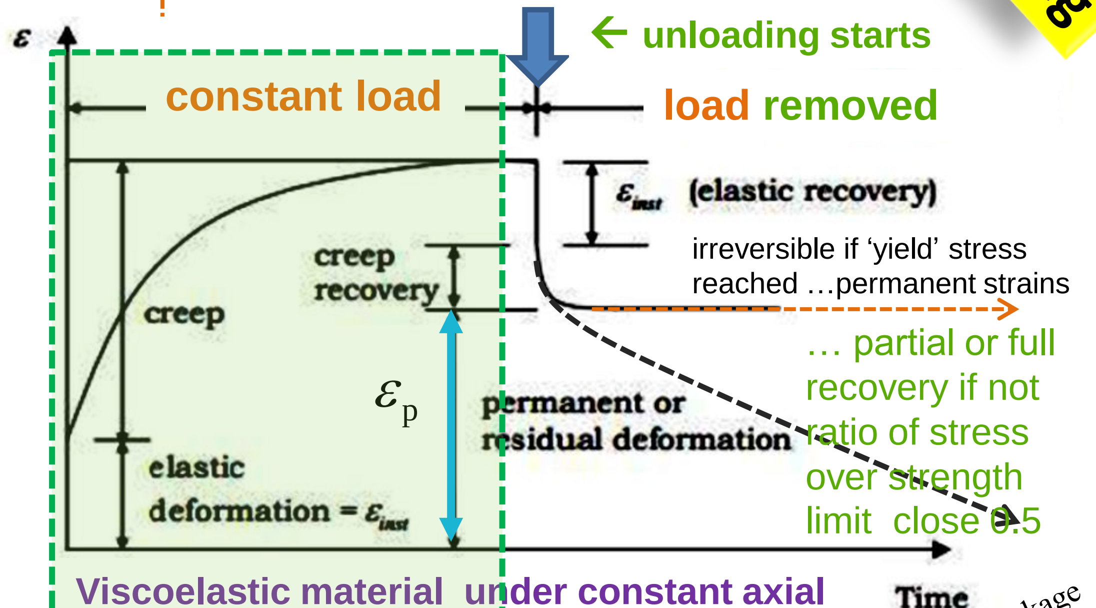
Generic Code-type of European Creep model:

$$J(t,t_0) = \frac{1}{E_c(t_0)} + \frac{\varphi(t,t_0)}{E_c}$$

creep function or creep compliance

Modulus of elasticity at age of 28 days in MPa

where for example: $\varphi(t,t_0) = \varphi_\infty \left[1 - \mathrm{e}^{-\beta(t-t_0)}\right]$



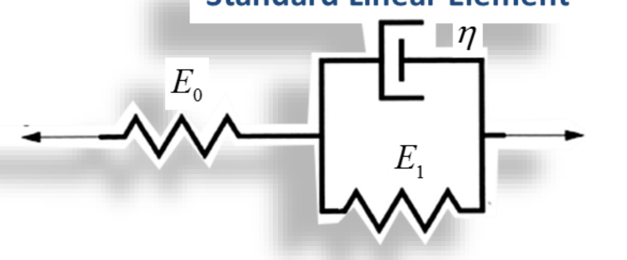
Viscoelastic material under constant axial compressive stress

Concrete shrinkage is not included here

* For creep linear response, it is assumed applied compressive stress is assumed bellow yield or ultimate stress σ_Y let say : $|\sigma| < \sigma_Y / 2$ at t_0

The **creep function** of concrete can be derived from extensive and lengthy experiments. Concrete **norms or codes** compile such experimental work and provides *creep functions* for a given class of concrete, sample size temperature and humidity conditions after specifying the type & age of the concrete at the loading instant, curing conditions,w/c, ….

Standard Linear Element



Relation between *Creep* 'coefficient' and *compliance* function

Reading

Creep deformation (mechanical strain): $\varepsilon(t,\tau) = \sigma \cdot \boxed{J(t,\tau)}$

$$\varepsilon(t,\tau) = \underbrace{\varepsilon_{\mathrm{e}}}_{\text{elastic}} + \underbrace{\varepsilon_{\mathrm{cr}}}_{\text{creep}} \equiv \varepsilon_{\mathrm{e}} + \varphi(t,\tau)\varepsilon_{\mathrm{e}} \equiv \varepsilon_{\mathrm{e}} \cdot [1 + \varphi(t,\tau)]$$

strain at time *t* du to constant stress acting since age $\tau < t$

$$\Downarrow$$

$$\boxed{J(t,\tau)} = \frac{\varepsilon(t,\tau)}{\sigma} = \frac{\varepsilon_{\mathrm{e}} + \varepsilon_{\mathrm{cr}}}{\sigma} = \ldots$$

$$\ldots = \frac{\varepsilon_{\mathrm{e}}}{\sigma}[1 + \varphi(t,\tau)] = \boxed{\frac{1 + \varphi(t,\tau)}{E(\tau)}}$$

Elastic modulus characteristizing the instantaneous strain at age $\tau < t$

Creep coefficient

$$\varepsilon_{\mathrm{cr}} \equiv \varphi(t,\tau)\varepsilon_{\mathrm{e}}$$

Superposition principle & Hereditary Integrals

$$\varepsilon(t) = \sigma_0 J(t, t_0)$$

As for linear elasticity, the *principle of* (Boltzmann's) *superposition* is also valid in linear viscoelasticity:

- For the three-stepped stress history:

$$\varepsilon(t) = \Delta\sigma_0 J(t-t_0) + \Delta\sigma_1 J(t-t_1) + \Delta\sigma_2 J(t-t_2)$$

$$= \sum_{i=0}^{2} \Delta\sigma_i J(t-t_i)$$

Creep compliance (function) $J(t) = \varepsilon(t)/\sigma_0$

$$\Downarrow$$

- Can be generalized to an arbitrary stress loading decomposed into an infinite number of infinitesimal step loadings:

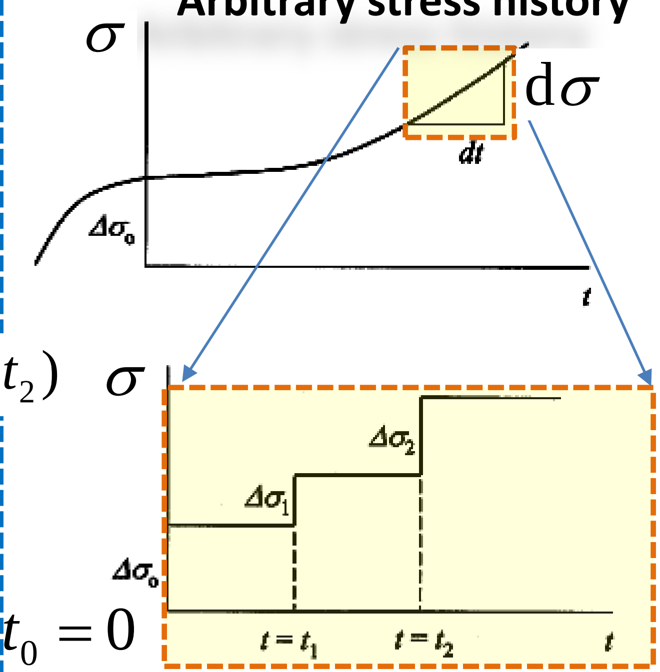
Hereditary Integral:

$$\varepsilon(t) = \int_{\tau=-\infty}^{\tau=t} J(t-\tau)\frac{\mathrm{d}\sigma(\tau)}{\mathrm{d}\tau}\mathrm{d}\tau$$

$$\varepsilon(t) = \sigma_0 J(t;\sigma_0)$$

A convolution integral

Arbitrary stress history



Applied stress history

If

$$\sigma = 0,\ t < 0, \qquad \sigma_0 = \sigma(0) \neq 0$$
$$\sigma \neq 0,\ t \geq 0,$$

$$\varepsilon(t) = \sigma_0 J(t) + \int_{\tau=0}^{\tau=t} J(t-\tau)\frac{\mathrm{d}\sigma(\tau)}{\mathrm{d}\tau}\mathrm{d}\tau$$

Linear viscoelastic deformation

Linear viscoelastic deformation can be described using the Boltzmann integral *representation*:

relaxation form:

$$\sigma(t) = \int_{-\infty}^{t} E(t-t')\dot{\varepsilon}(t')dt'$$

E(*t*) - **relaxation modulus**

(an other notation ***G***(*t*))

Retardation form:

$$\varepsilon(t) = \int_{-\infty}^{t} D(t-t')\dot{\sigma}(t')dt'$$

D(*t*) – **creep compliance**

(an other notation ***J***(*t*))

All the needed information concerning time-dependent behavior of the material is *encoded* in these two viscoelastic functions: the ***relaxation modulus*** *E*(*t*) or the ***creep compliance*** *D*(*t*).

$$E=\frac{1}{2}m\dot{v}^2+\frac{1}{2}kx^2$$

Cf.

$$+\frac{1}{2}m\omega^2x^2$$

Mechanical Models

The viscoelastic mechanical response of an elementary volume of the material – *the material point* - can be described by means of *mechanical** discrete *models* constructed using *elastic springs, viscous dashpots* and *friction elements*
(physical discretization)

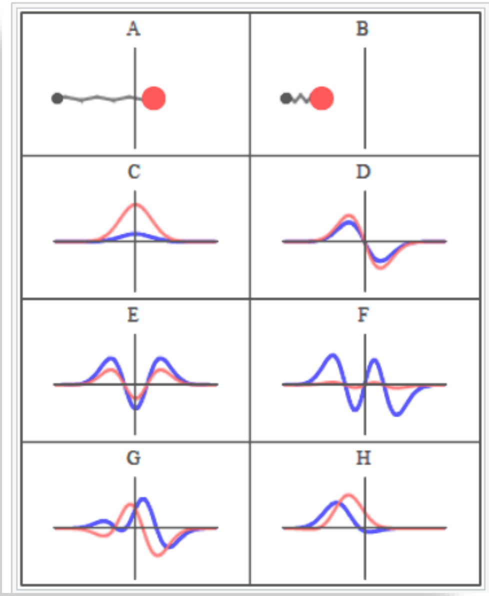
$$\frac{-\hbar^2}{2m}\frac{d^2\Psi(x)}{dx^2}+\frac{1}{2}m\omega^2x^2\Psi(x)=E\Psi(x)$$

Hamiltonian and energy eigenstates

The Hamiltonian of the particle is:

$$\hat{H}=\frac{\hat{p}^2}{2m}+\frac{1}{2}m\omega^2\hat{x}^2\,,$$

Model as 1-dimensional quantum harmonic oscillator (mass-spring mechanical system)



Ref: https://en.wikipedia.org/wiki/Quantum_harmonic_oscillator (21.1.2017)

$$\frac{-\hbar^2}{2m}\frac{d^2\Psi(x)}{dx^2}+\frac{1}{2}m\omega^2x^2\Psi(x)=E\Psi(x)$$

* *This type of mechanical modelling or **idealization** – physical discretization, through elastic springs, viscous dashpots, …, may truly sound a very barbarian and a rough approach, however, it is not,* cf. *quantum physics and* cf. *Einstein specific heat capacity model for solids, for instance, where such mechanical sub-models have been used , …. May be one of the most important idea in modeling is to match (well approximate) the energy content of the system under study by the energy of the model…*

Constitutive Models of Linear Viscoelasticity

Constitutive models of linear viscoelasticity

Basic rheological elements

Viscoelastic response of a material reflects both elastic and viscous behaviors and is modeled (rheology) as a combinations of springs and dashpots, respectively. Friction-elements can be added to model plasticity and plasticity-like behavior (yield-point, threshold value).

Example: toothpaste

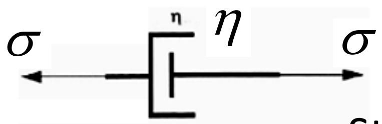
Spring (Hooke element)

Strain

$$\sigma = E \cdot \varepsilon$$

Elastic modulus

Viscous Damper (Newton element)



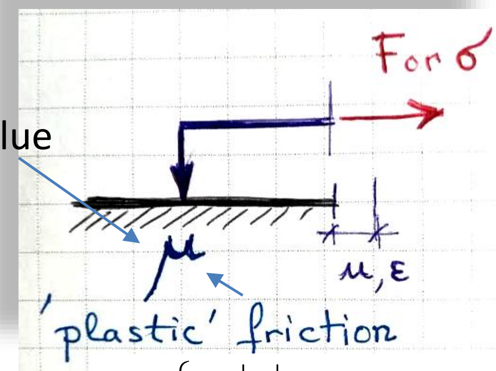
Strain rate

$$\sigma = \eta \cdot \dot{\varepsilon}$$

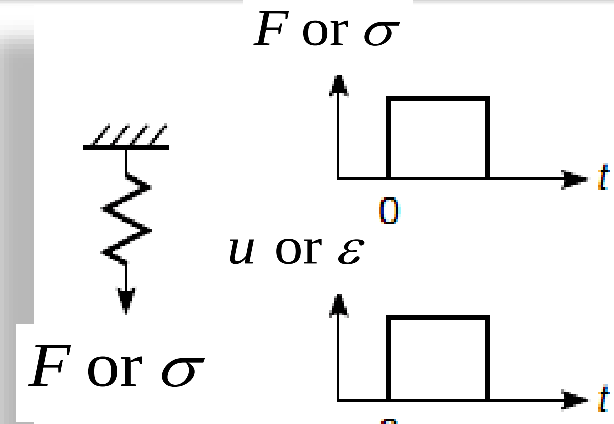
Viscosity coefficient

threshold value

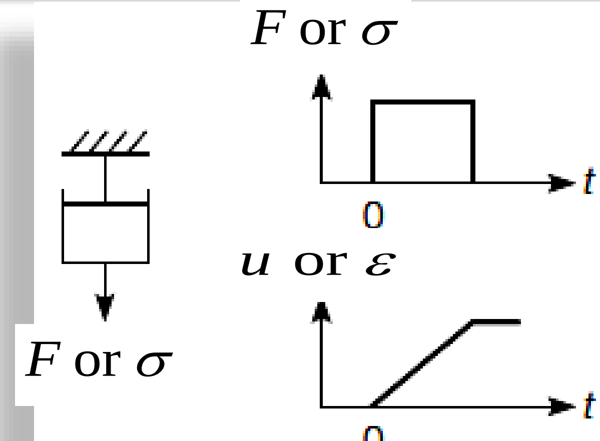
'Yield-point'



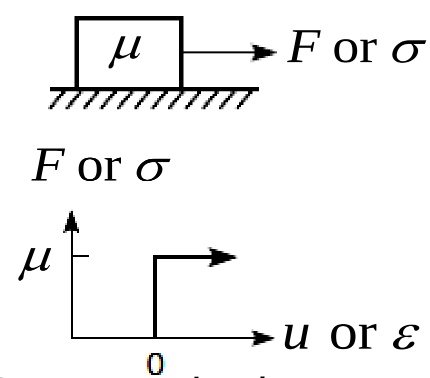
$$\sigma = \begin{cases} 0, & |\sigma| < \mu \\ \mu \cdot \mathrm{sign}(\sigma), & |\sigma| \geq \mu \end{cases}$$



Hookean element



Newton element



St. Venant's element

NB. The elastic modulus or the viscous elements **can follow non-linear laws**, *for instance*, like $\dot{\varepsilon} = A|\sigma|^{n-1}\sigma$

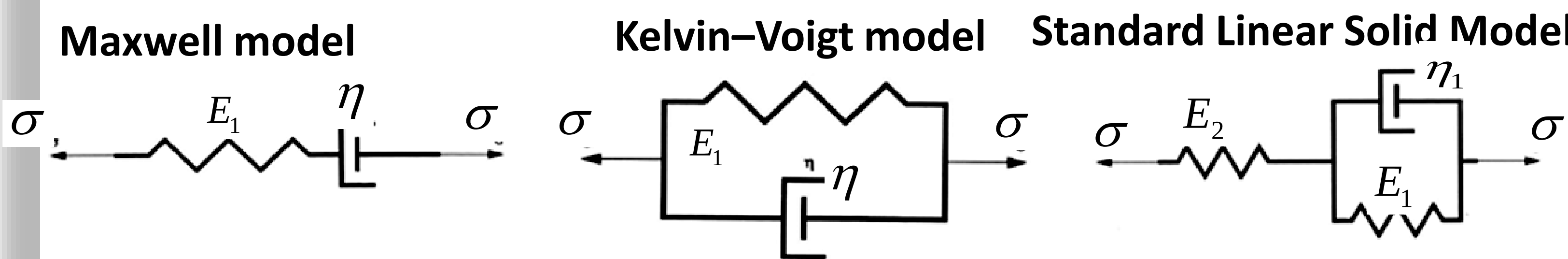
Constitutive models of linear viscoelasticity

Simple viscoelastic models

Three or four basic elements are commonly used:

- **They are combined in either series or parallel to form models that define complex material behaviors**

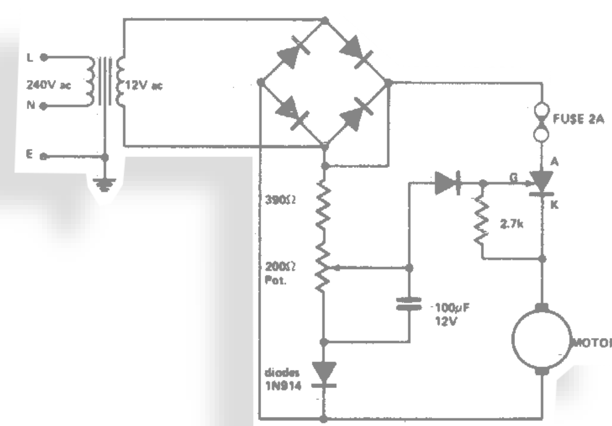
Stress depends on strain *and* strain-rate:



- Elastic stress depends on strain (spring)
- Viscous stress depends on strain-rate (dashpot)

  Each dashpot element corresponds to a relaxation mechanism
- Strains add **in series** to obtain the total strain (compatibility), stresses are equal (equilibrium)
- Stresses add **in parallel** (equilibrium) , strains are equal (compatibility)

A model of an Electric circuit

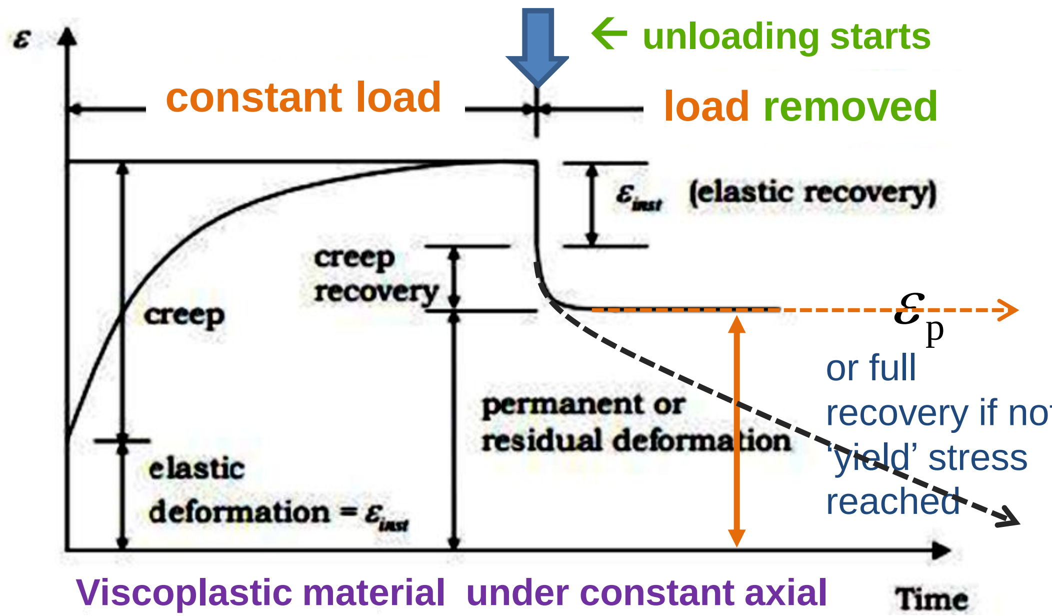


On constitutive models for viscoplasticity*

Good to know

Example of viscoplasticity

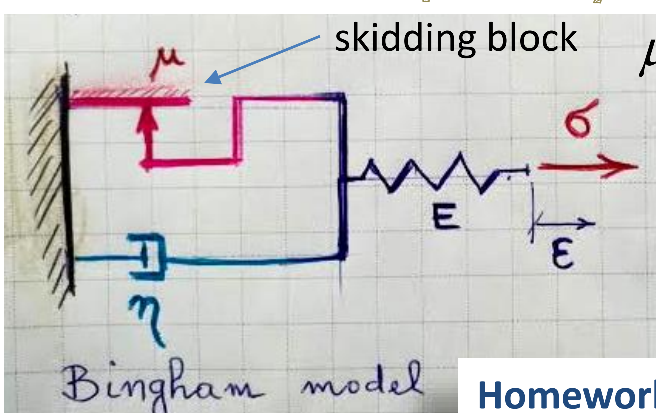
- The response is **viscoplastic** when the applied **stress is higher than the 'yield' stress** (or some other elastic threshold point) for viscoplastic materials



Viscoplastic material under constant axial compressive stress

Relevance of the Bingham model? Describes material behaving as a fluid at high shearing rate (or at high stress) and as a solid at low shearing rates (or at low stress), for instance 'tooth paste' (dentifrice) or mayonnaise …

Bingham model – is often used to model combined elastic viscoplasticity behavior



$\mu \equiv \sigma_Y$ – **Yield point** as 'static-friction'

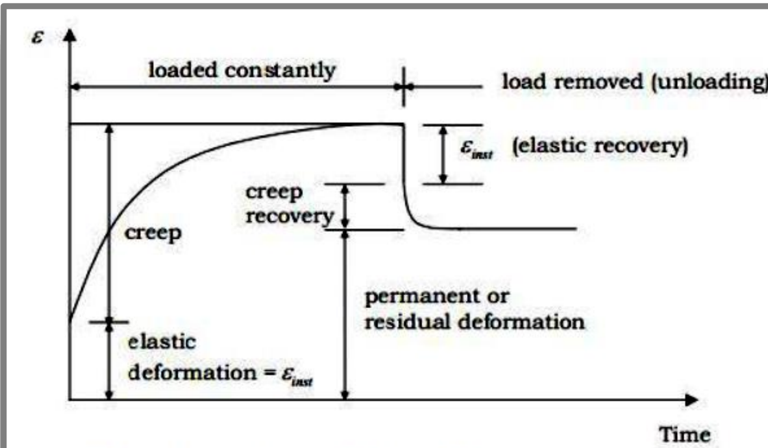
$$\sigma = \begin{cases} E\varepsilon, & |\sigma| < \mu \equiv \sigma_Y \\ \mu + \eta\dot{\varepsilon}, & |\sigma| \geq \mu > 0 \end{cases}$$

Homework? check that these are correct

* **Viscoplastic solids** exhibits **permanent deformation** after applying a load (as plastic solids) and continue to creep as function of time under the applied load (so, no equilibrium can be achieved)

Generality

- **Experimental (mechanical) viscoelastic behavior can be described well using different theoretical models as**
  - Maxwell model (*elementary*)
  - Kelvin-Voight model (*elementary*)
  - Standard linear solid model (*combined*)
  - Generalized Maxwell model (*combined*)
  - Kelvin chain model (*combined*)
  - …
- Each one of them has a specific field of applications



Concrete under constant axial compressive stress

Generalized Maxwell chain:

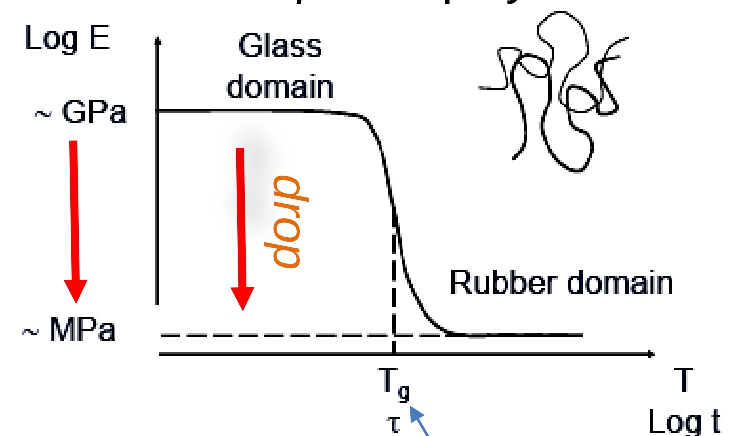
- suitable for modelling stress relaxation phenomena [1]

Kelvin Chain:

- adequate model for modelling Creep phenomena [1]

Good to know

Amorphous polymers



Glass-transition temperature (lower than melting temperature)

[1] *Computational Viscoelasticity*, S. Marques and G. Creus, Springer 2012

Deriving the constitutive models in the differential form

The idea for deriving the 'formula' of the constitutive law:

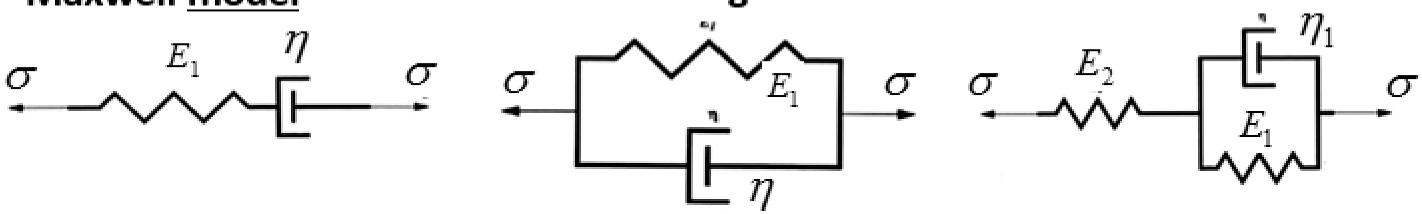
Simple viscoelastic models:

Three basic elements are commonly used:

- They are combined in either series or parallel to form models that define complex material behaviors

Stress depends on strain *and* strain-rate:

Maxwell model | **Kelvin–Voigt model** | **Standard Linear Solid Model**



Rheological models for visco-elasticity

- Elastic stress depends on strain (spring)
- Viscous stress depends on strain-rate (damper)

- Maxwell: Strains add in series, stresses are equal
- Kelvin: Stresses add in parallel, strains are equal
- SLS: Combination of Maxwell and Kelvin

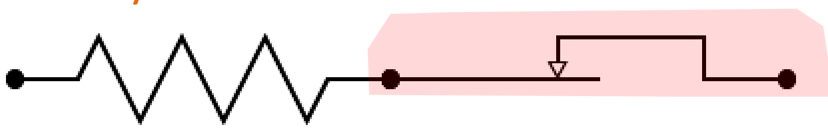
The idea

Combining equilibrium and compatibility equations

Rheological models for plasticity & visco-plasticity

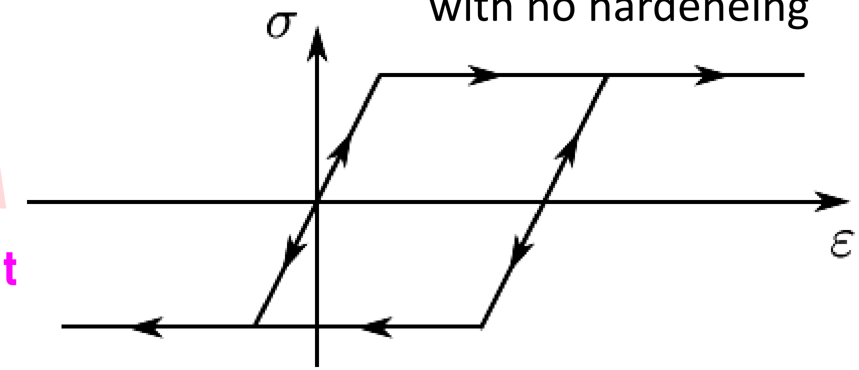
Simple rheological models

Rate independent
Plasticity



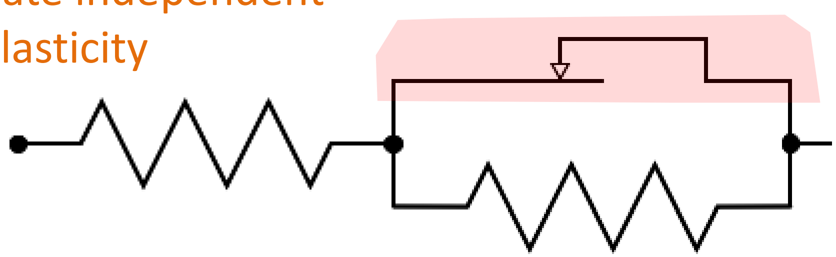
static friction element
(= yield threshold)

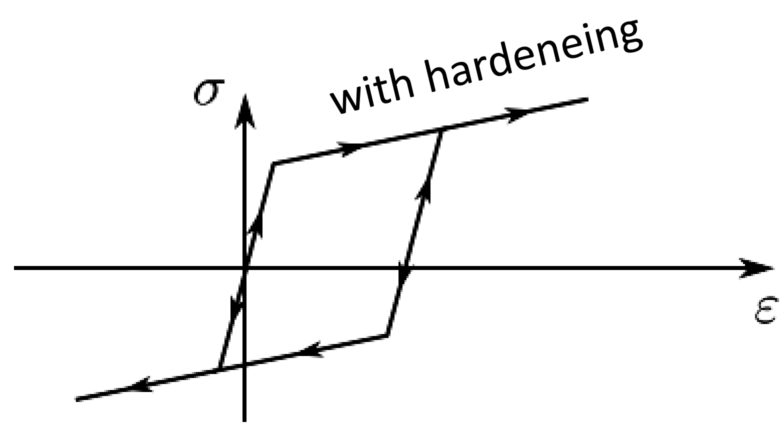
with no hardeneing



Rate independent
Plasticity

static friction element

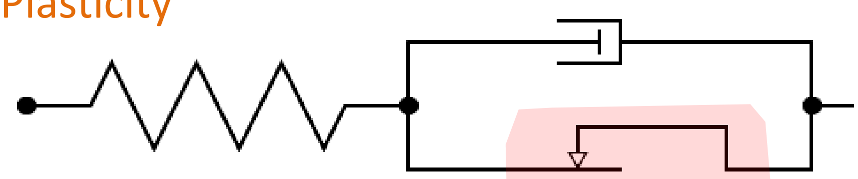




ductile metals...

Rate dependent
Plasticity

bingham model



visco-plasticity

friction element

asphalts, some 'pâtes', tooth paste, ...

Visco-elasticity

Constitutive models in the differential form*

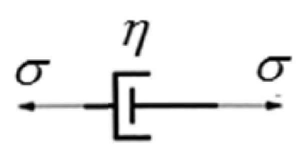
Spring and Dashpot Models

Creep Compliance and Relaxation Modulus

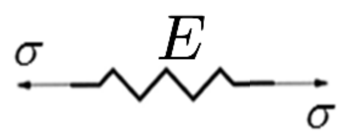
Viruma- ja Relaksaatiofunktiot

- linear material response is assumed in models
- each dashpot element corresponds to a relaxation mechanism
- each spring correspond to an elastic response mechanism

Newtonian dashpot



$$\sigma = \eta\dot{\epsilon}$$



$$\sigma = E\epsilon$$

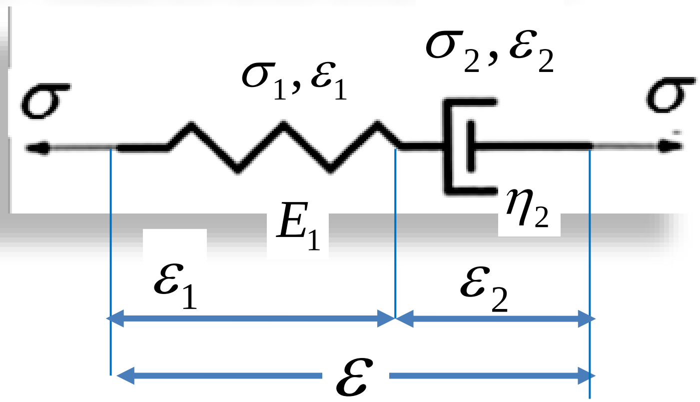
Hookean spring

*such rate or incremental formulation is also more adequate for numerical implementations

Deriving the constitutive models in the differential form

Applications: 'soft' solids as thermoplastic polymers near melting temperature and many metals close to their melting temperature, fresh concrete (when aging effects neglected)
/*ref*: Lemaitre & Chaboche in the course textbook/

Maxwell model



(viscous fluid, fluid-like)

$$\sigma + \frac{\eta_2}{E_1}\dot{\sigma} = \eta_2 \dot{\varepsilon}$$

Use Equilibrium and Compatibility (of strains) equations:

Compatibility:

$$\varepsilon_1 + \varepsilon_2 = \varepsilon \Rightarrow \dot{\varepsilon}_1 + \dot{\varepsilon}_2 = \dot{\varepsilon}$$

Equilibrium:

$$\sigma_1 = \sigma_2 = \sigma$$

$$\sigma_1 = E_1 \varepsilon_1 \qquad \sigma_2 = \eta_2 \dot{\varepsilon}_2$$

$$\varepsilon_1 = \frac{\sigma_1}{E_1} \Rightarrow \dot{\varepsilon}_1 = \frac{\dot{\sigma}_1}{E_1} \qquad \dot{\varepsilon}_2 = \frac{\sigma_2}{\eta_2}$$

Constitutive equation in the differential form:

$$\dot{\varepsilon} = \dot{\varepsilon}_1 + \dot{\varepsilon}_2 = \frac{\dot{\sigma}}{E_1} + \frac{\sigma}{\eta_2}$$

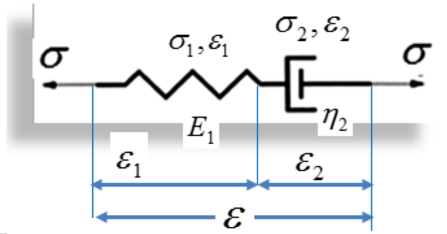
A linear first-order ordinary differential equation (ODE)

What are the
Creep response?
Relaxation response?

Maxwell model:

Creep Compliance and Relaxation Modulus

What are the
Creep response?
& Relaxation response?



Creep response: Apply a constant stress $\sigma(0) = \sigma_0$ $t = 0$

… and determine (measure) the time dependent strain : $\varepsilon(t)$

$$\boxed{\dot{\varepsilon} = \frac{\dot{\sigma}}{E_1} + \frac{\sigma}{\eta_2}} \leftrightarrow \frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = \frac{1}{E_1}\frac{\mathrm{d}\sigma}{\mathrm{d}t} + \frac{\sigma}{\eta_2} \qquad t > 0$$

$$\Rightarrow \mathrm{d}\varepsilon = \frac{1}{E_1}\mathrm{d}\sigma + \frac{\sigma}{\eta_2}\mathrm{d}t$$

integrate for constant stress

$$\Rightarrow \int_{\varepsilon(0)}^{\varepsilon(t)} \mathrm{d}\varepsilon = \frac{1}{E_1}\underbrace{\int_{\sigma(0)}^{\sigma(t)} \mathrm{d}\sigma}_{=0,\; t>0} + \frac{\sigma_0}{\eta_2}\int_0^t \mathrm{d}t$$

constant stress

$\Downarrow$

integration constant

Initial conditions: instantaneous elastic response only:

$$t = 0 \rightarrow \varepsilon(0) = \sigma_1 / E_1 = \sigma_0 / E_1$$

$$\boxed{\Rightarrow \varepsilon(t) = \varepsilon(0) + \frac{\sigma_0}{\eta_2} t} \qquad \boxed{\Rightarrow \varepsilon(t) = \frac{\sigma_0}{\eta_2} t + \frac{\sigma_0}{E_1}}$$

The *initial* instantaneous deformation is ***elastic*** (only the spring reacts)

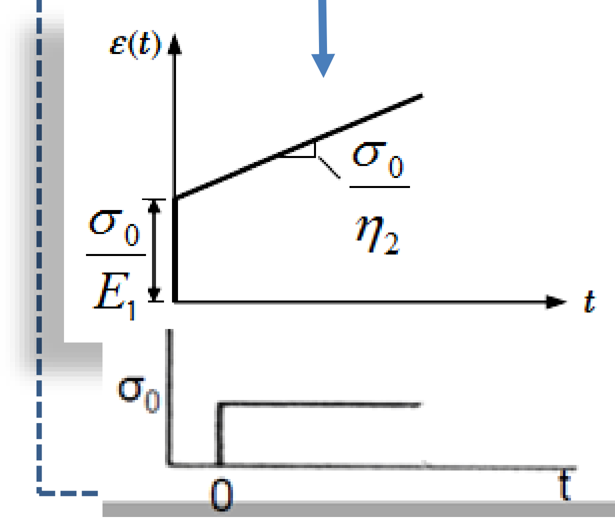
$$\Rightarrow \varepsilon(0) = \frac{\sigma_0}{E_1} = C$$

$$\boxed{\Rightarrow \varepsilon(t) = \sigma_0\left(\frac{1}{E_1} + \frac{t}{\eta_2}\right)}$$

$$\equiv J(t)$$

$$\boxed{= \sigma_0 J(t)}$$

Creep Compliance or -function



Maxwell model: Creep Compliance and **Relaxation Modulus**

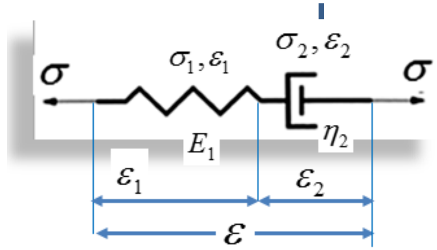
Relaxation response?

Apply a constant strain at $t=0$ $\varepsilon(0)=\varepsilon_0$

… and determine (measure) the time dependent stress : $\sigma(t)$

$\varepsilon(t>0)=\varepsilon_0=\text{constant} \Rightarrow$ For $t>0, \quad \dfrac{\mathrm{d}\varepsilon}{\mathrm{d}t}=0$

$$\Rightarrow \boxed{\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}=\frac{1}{E_1}\frac{\mathrm{d}\sigma}{\mathrm{d}t}+\frac{\sigma}{\eta_2}=0} \rightarrow \dots$$



$$\dots \rightarrow \frac{\mathrm{d}\sigma}{\sigma}=\frac{E_1}{\eta_2}\mathrm{d}t \qquad t>0$$

After integration one obtains:

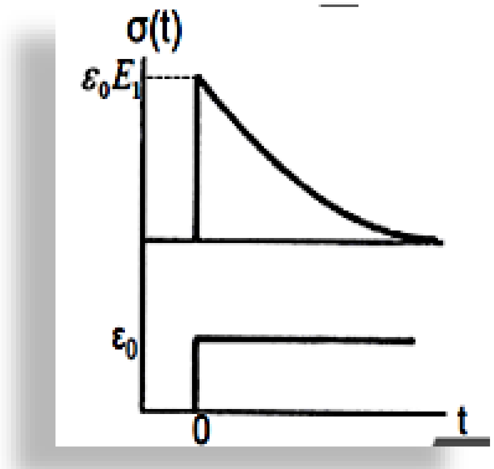
$$\int_{\sigma(0)}^{\sigma(t)}\frac{\mathrm{d}\sigma}{\sigma}=\int_0^t\frac{E_1}{\eta_2}\mathrm{d}t+C \quad \Rightarrow \sigma(t)=Ae^{-\frac{E_1}{\eta_2}t}$$

Initial condition: $t=0 \quad \Rightarrow \sigma(0)=A=E_1\varepsilon_0$

(elastic instantaneous strain)

$$\Rightarrow \sigma(t)=\varepsilon_0\cdot\boxed{E_1e^{-\frac{E_1}{\eta_2}t}}\equiv\varepsilon_0\cdot\boxed{G(t)}$$

Relaxation function:



Relaxation function:

$$\Rightarrow \sigma(t)=\varepsilon_0\cdot\underbrace{\boxed{E_1e^{-\frac{E_1}{\eta_2}t}}}$$

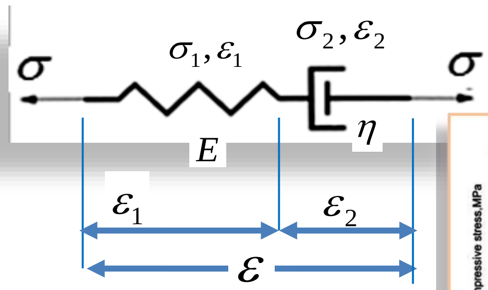
$$\equiv\varepsilon_0\cdot G(t)$$

Relaxation function:

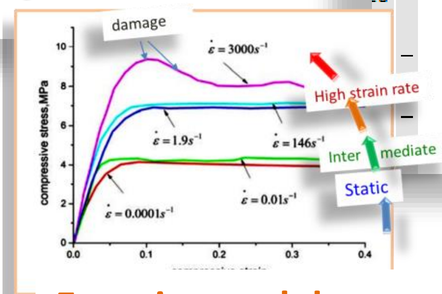
Definition: τ = η/E - relaxation time

Rate dependence? An example

Integrate using Matlab - odeset



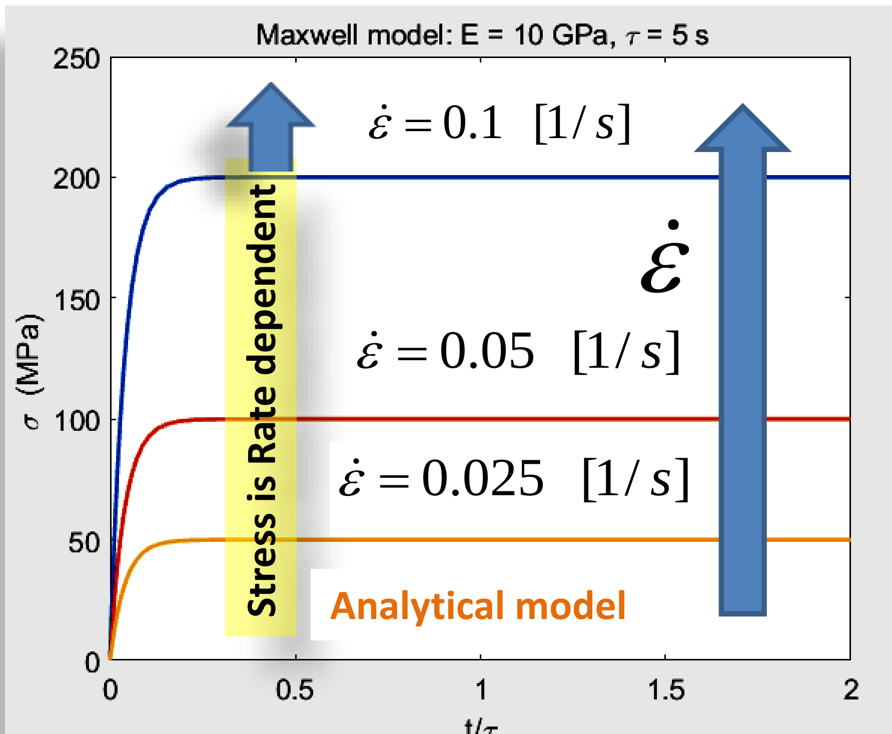
Maxwell model



Experimental data

```
global E tau dot_epsilon
E = 10e9    % E (Pa)
tau = 5; % (s) retardation time

sigma_0      = 0.0;      % alkujännitys (Pa);
dot_epsilon  = 0.1/4;    % vakiovenymänopeus (1/s)

%----
time           = [0 2*tau];     %  integrointiväli
[Time, Sigma] = ode23s('Maxwell_Model',time,[sigma_0]);

plot(Time/ta
xlabel('t/\t
ylabel('\sig
% ----------
```

```
% INPUT:
% ------
% t     = current time = scalar;
% Y     = is the temperature in Celsius deg.
%
% OUTPUT:
% -------
% DY    = is the initial value problem : right side f(Y(t);t)
%-----------------------------------------------------------------------
function [DY] = Maxwell_Model(t, Y)
global E tau dot_epsilon

%-----------------------------------
% the right side f(Y(t);t):   Begin
%-------------------------
Sigma         = Y;
eta           = E/tau; % viscosity coefficient
f_right_side  = - tau * Sigma + E * dot_epsilon ;

DY  =  f_right_side ;
%-----------------------------------
% the right side f(Y(t);t):   End
%-------------------------
```



Example: Integrate the linear ODE:

$$\frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta} = \dot{\varepsilon} = \dot{\varepsilon}_1 + \dot{\varepsilon}_2 \qquad \tau \equiv E/\eta$$

$$\varepsilon(t) = H(t) \cdot at, \quad a \in R_+^*$$

Approx. in Matlab

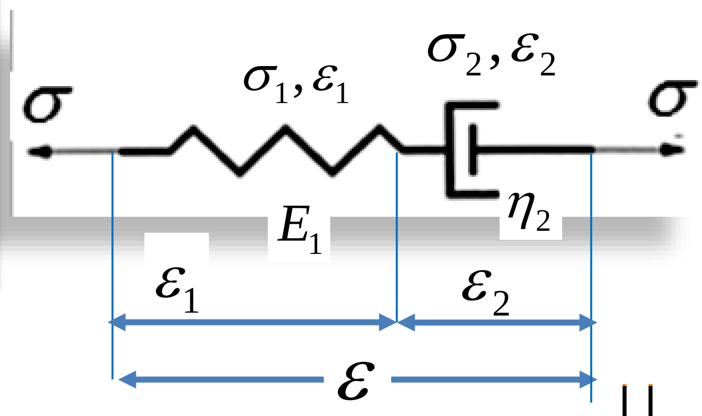
$$H(x) \approx \tfrac{1}{2} + \tfrac{1}{2}\tanh kx = \frac{1}{1+e^{-2kx}}, \qquad H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

Generalization of the uniaxial Maxwell model (3D case)

It appears trough **experimental experience** that **many materials have** an elastic volumetric **response** and the viscoelastic behaviour concentrates in the deviatoric (shearing) response. Thus, the **viscous strain** is often **incompressible** [Remember: earlier in your lectures we've seen arguments for the practical reasons behind splitting the general stress-strain Behavior into volumetric and deviatoric responses.]

Maxwell model

(viscous fluid, fluid-like)



Viscous strain:

$$\dot{\varepsilon}_{ij}^{v} = -\frac{\sigma_{kk}}{\eta_1}\delta_{ij} + \frac{\sigma_{ij}}{\eta_2}$$

Two viscosity coefficients: one for volumetric and the another for deviatoric responses

$$\underbrace{\dot{\varepsilon}_{kk}^{v} = \left(-\frac{3}{\eta_1} + \frac{1}{\eta_2}\right)\sigma_{kk} = 0}_{\text{viscous strain incompressibility}}, \quad \Rightarrow \eta_1 = 3\eta_2$$

Viscous strain

$$\dot{\varepsilon} = \dot{\varepsilon}_1 + \dot{\varepsilon}_2 = \frac{\dot{\sigma}}{E_1} + \frac{\sigma}{\eta_2}$$

Multi-axial Isotropic case:

Viscous strain

$$\dot{\varepsilon}_{ij} = \frac{1}{2G}\left(\dot{\sigma}_{ij} - \frac{\nu}{1+\nu}\dot{\sigma}_{kk}\delta_{ij}\right) - \frac{\sigma_{kk}}{\eta_1}\delta_{ij} + \frac{\sigma_{ij}}{\eta_2}$$

Ref: Ottosen *et al.*, Chapter 14.

Uniaxial case:

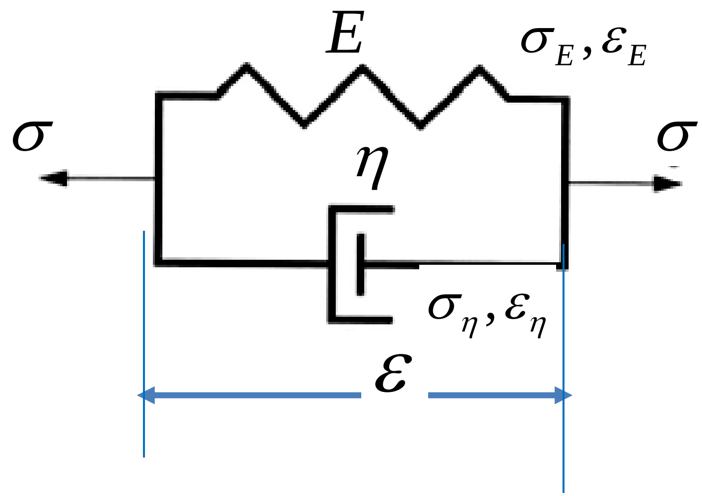
$$\dot{\varepsilon} = \dot{\varepsilon}_1 + \dot{\varepsilon}_2 = \frac{\dot{\sigma}}{E_1} + \frac{\sigma}{\eta_2}$$

Deriving the constitutive models in the differential form

Derive the constitutive models in the differential form for the Kelvin-Voigt model

Kelvin-Voigt element

(viscos & elastic)



Applications: organic polymers, rubber, wood when load is not to high /*ref*: Lemaitre & Chaboche in the course textbook/

$$\boxed{\sigma = E\varepsilon + \eta\dot{\varepsilon}}$$

Use equilibrium and compatibility (of strains) equations:

Compatibility: $$\boxed{\varepsilon_E = \varepsilon_\eta = \varepsilon}$$

Equilibrium: $$\boxed{\sigma_E + \sigma_\eta = \sigma}$$

$$\sigma_E = E\varepsilon_E = E\varepsilon \qquad \sigma_\eta = \eta\dot{\varepsilon}_\eta = \eta\dot{\varepsilon}$$

Constitutive equation in the differential form:

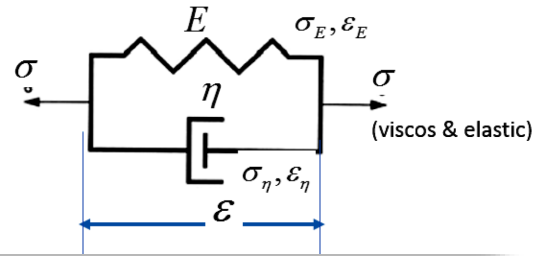
$$\boxed{\sigma = E\varepsilon + \eta\dot{\varepsilon}}$$

A linear first-order ordinary differential equation (ODE)

What are the
Creep response?
Relaxation response?

Kelvin-Voigt model: Creep Compliance and Relaxation Modulus

What is the **Creep response?**



Creep response: Apply a constant stress $\sigma(0) = \sigma_0$ $t = 0$

… and determine (measure) the time dependent strain : $\varepsilon(t)$

$$\Rightarrow \varepsilon(t) = \underbrace{\frac{\sigma_0}{E}[1 - e^{-\frac{E}{\eta}t}]}_{= \sigma_0 J(t)}$$

$$\boxed{\frac{E}{\eta}\varepsilon + \dot{\varepsilon} = \frac{\sigma_0}{\eta}} \leftrightarrow \begin{cases} \dfrac{d\varepsilon}{dt} + \dfrac{E}{\eta}\varepsilon = \dfrac{\sigma_0}{\eta}, \quad t > 0 & (1) \\ \text{with initial condition: } \varepsilon(0) = 0 & (2) \end{cases}$$

$$\rightarrow \varepsilon = \varepsilon_0 + \bar{\varepsilon}$$ $\leftarrow$ ε_0 homogeneous general solution + any special solution $\bar{\varepsilon}$

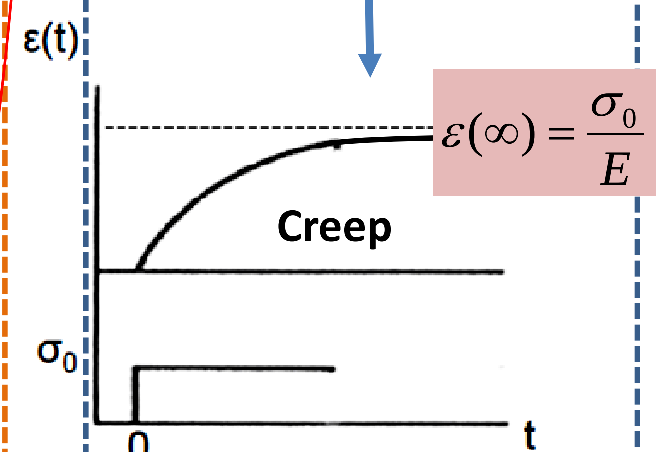
$$\begin{cases} \dfrac{d\varepsilon_0}{dt} + \dfrac{E}{\eta}\varepsilon_0 = 0 \Rightarrow \varepsilon_0(t) = C_1 e^{-\frac{E}{\eta}t} \\ \bar{\varepsilon} = C_2 \rightarrow \varepsilon = \varepsilon_0 + \bar{\varepsilon} = C_1 e^{-\frac{E}{\eta}t} + C_2 \end{cases}$$

$$(1) \Rightarrow -\frac{E}{\eta}C_1 e^{-\frac{E}{\eta}t} + \frac{E}{\eta}C_1 e^{-\frac{E}{\eta}t} + \frac{E}{\eta}C_2 = \frac{\sigma_0}{\eta} \Rightarrow \boxed{C_2 = \frac{\sigma_0}{E}}$$

Initial condition→ $\varepsilon(0) = 0 \Rightarrow C_1 = -\sigma_0 / E \Rightarrow \ldots$

$$\therefore J(t) = \frac{1}{E}[1 - e^{-\frac{E}{\eta}t}]$$

Creep Compliance or -function



Kelvin-Voigt model: Creep Compliance and Relaxation Modulus

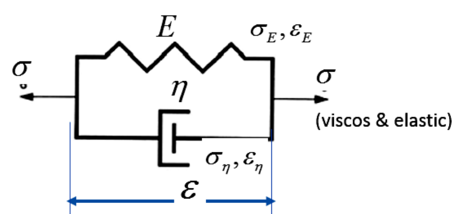
Relaxation response?

Apply a constant strain at $t=0$ $\varepsilon(0)=\varepsilon_0$

... and determine (measure) the time dependent stress : $\sigma(t)$

$\varepsilon(t>0)=\varepsilon_0=\text{constant} \Rightarrow$ For $t>0, \quad \frac{d\varepsilon}{dt}=0$

$$\boxed{\sigma = E\varepsilon + \eta\dot{\varepsilon}}$$



when strain applied instantaneously (over dt → 0)

$$t=0 \quad \Rightarrow \dot{\varepsilon} \to \infty \Rightarrow \quad \sigma(0) \to \infty$$

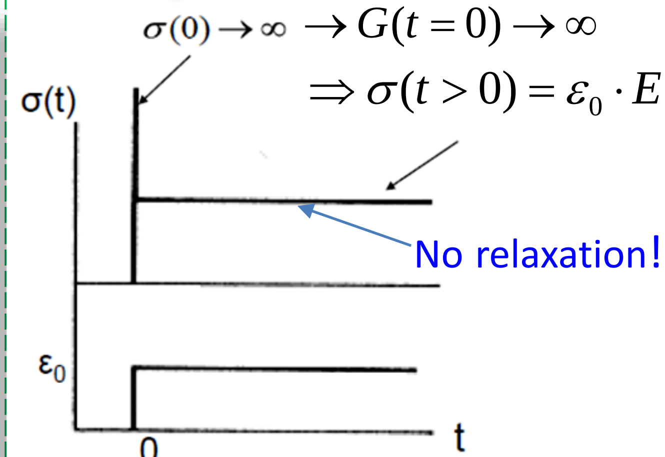
A 'non-physical' value for stress at *t* = 0 is obtained when strain applied instantaneously (infinitely stiff)

The stress becomes more physical when strain applied gradually over a small non-zero time-step $\Rightarrow \dot{\varepsilon} < \infty$

In practical situations, the elementary **rheological elements are combined** in order to quantitatively 'mimic' or model realistic macroscopically mechanical behaviour of materials as for instance the following **Standard Linear Solid...**

Relaxation function:

A non-physical value when strain applied instantaneously. Becomes more physical when strain applied gradually over a non-zero time-step



Relaxation function:

$$\Rightarrow \sigma(t>0) = \underbrace{\boxed{\varepsilon_0 \cdot E}}_{} \equiv \varepsilon_0 \cdot G(t)$$

'Relaxation function' for *t* >0, KV-model has no relaxation

Instantaneous elastic modulus: $G(t=0) \to \infty$

Rate dependence?

Kelvin–Voigt element

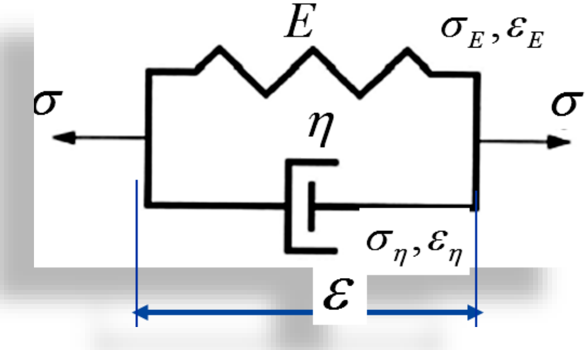
Rate dependence?

Obviously, yes

$$\sigma(t) = E\varepsilon(t) + \eta\dot{\varepsilon}(t)$$

Elastic part does not depend on strain rate

Viscous part depends on strain rate

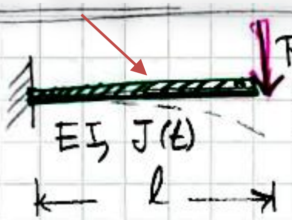


Creep displacements for a elastic-viscoelastic **beam**

Example: cantilever

BEAM: Kelvin–Voigt material

Ex :



$P(t) = P_0 H(t)$

$\delta(t) = ?$

Compute $\delta(t)$

In the elastic solution, replace $1/E$ by $J(t)$

$$\frac{1}{E} \longleftrightarrow J(t)$$

ex. Voigt Kelvin material

$J(t) = \frac{1}{k}(1 - e^{-t/\tau})$

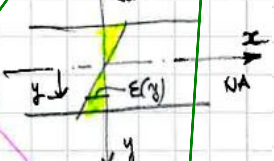
$\tau = \eta / k$, $(k = E)$

This is why we can do so

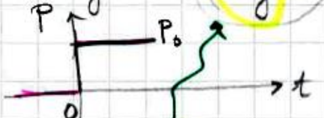
Creep : $\varepsilon(t) = \sigma_0 J(t)$ creep compliance

Creep! $K(x,y,t) = \frac{\varepsilon(x,y,t)}{y} = \frac{\sigma_0 J(t)}{y}$

curvature



$P(t) = P_0 H(t)$



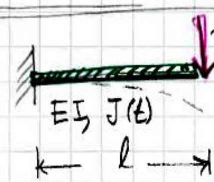
$$\frac{1}{E} \longleftrightarrow J(t)$$

when computing displacements

Elastic : $\varepsilon = \frac{\sigma_0}{E} \longleftrightarrow K_e = \frac{\varepsilon}{y} = \frac{\sigma_0}{E}\frac{1}{y}$

elastic curvature

Ex :



$P(t) = P_0 H(t)$

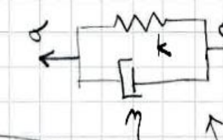
$\delta(t) = ?$

Compute $\delta(t)$

ex. Voigt Kelvin material

$J(t) = \frac{1}{k}(1 - e^{-t/\tau})$

$\tau = \eta / k$, $(k = E)$



- Elastic solution :

$1 \cdot \delta_e = \int_0^{\ell} K_e \cdot \bar{M}\, dx = \frac{1}{3}\frac{P\ell^3}{EI}$, with $K_e = \frac{Px}{EI}$, $\bar{M} = x$

- Viscoelastic solution $1/E \longleftrightarrow J(t)$ in

$$\delta(t) = \frac{1}{3}\frac{P\ell^3}{I} J(t) = \frac{P\ell^3}{3I}\frac{1}{k}(1 - e^{-t/\tau}), \quad \text{now } k = E$$

$$= \underbrace{\boxed{\frac{P\ell^3}{3EI}}}_{\delta_e}(1 - e^{-t/\tau}), \quad \tau = \eta/k = \eta/E$$

Q: $\delta(t)$, for Maxwell material?

Q: $\delta(t)$, for Maxwell material?

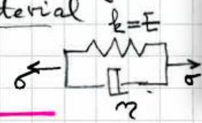
$$\delta_{Maxwell} = \frac{P\ell^3}{3EI}\left(1 + \frac{t}{\tau}\right)$$

Creep displacements for a elastic-viscoelastic **beam**

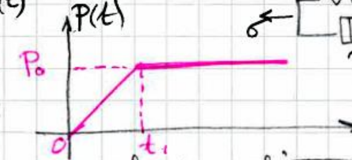
Example: cantilever

Ex 2.

• Kelvin–Voigt material



$P(t) = \begin{cases} P_0 \cdot \dfrac{t}{t_1}, & t \le t_1 \\ P_0, & t > t_1 \end{cases}$



loading history

Q : determine the viscoelastic tip-deflection δ(t)!

$1/E \longleftrightarrow J(t)$ creep compliance

• Elastic solution

$$\delta_e = \frac{1}{3}\frac{Pl^3}{EI} \qquad (1)$$

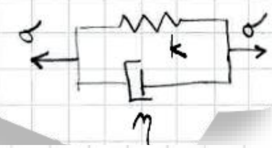
BEAM: Kelvin–Voigt material

In the elastic solution, replace $1/E$ by $J(t)$

ex. Voigt Kelvin material

$J(t) = \frac{1}{k}\left(1 - e^{-t/\tau}\right)$

$\tau = \eta/k, \ (k=E)$



• Visco-elastic tip-displacement :

$\delta(t) = \ ? \qquad 0 < t \le t_1$

$$\boxed{\varepsilon(t) = \int_0^{t'=t} J(t-t')\,\dot{\sigma}(t')\,dt'} \ ; \quad P(t) = \frac{P_0}{t_1}\,t$$

convolution integral $\forall\, t \ge 0$

$\dot{\sigma}(t) = \dfrac{\dot{P}(t)\cdot x}{W_e} = \dfrac{P_0 \cdot x}{t_1 W_e} = \sigma_0$

$$= \sigma_0 \cdot \frac{1}{t_1}\int_0^{t'=t} J(t-t')\,dt' \equiv \bar{J}(t)$$

$\therefore\ \dot{\sigma}(t) = \frac{1}{t_1}\cdot\sigma_0$

Now

$\therefore\ 1/E \longleftrightarrow \bar{J}(t)$ in (1) $\Longrightarrow$

$$\delta(t) = \frac{Pl^3}{3EI}\cdot\frac{1}{t_1}\cdot\int_0^{t'=t} J(t-t')\,dt', \quad t \in [0, t_1]$$

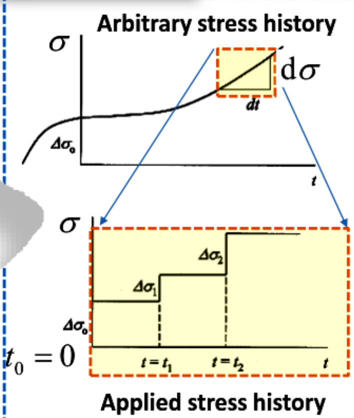
$$= \frac{Pl^3}{3EI}\cdot\frac{1}{t_1}\cdot\left[t - \frac{\eta}{E}\left(1 - e^{-t/\tau}\right)\right]$$

and

$$\delta(t) = \frac{Pl^3}{3EI}\cdot\frac{1}{t_1}\cdot\left[1 + \frac{\eta}{Et_1}\left(1 - e^{-t_1/\tau}\right)e^{-t/\tau}\right], \quad t \ge t_1$$

show this result!

$\varepsilon(t) = \sigma_0 J(t, t_0)$



If

$\sigma = 0, \ t < 0, \qquad \sigma_0 = \sigma(0) \ne 0$

$\sigma \ne 0, \ t \ge 0,$

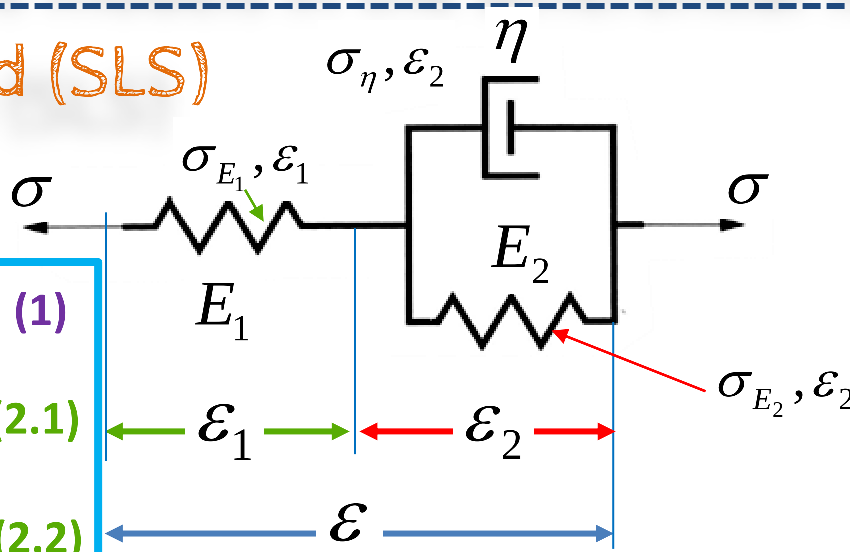
$$\varepsilon(t) = \sigma_0 J(t) + \int_{\tau=0}^{\tau=t} J(t-\tau)\frac{d\sigma(\tau)}{d\tau}\,d\tau$$

This is why we can do:

$$\frac{1}{E} \longleftrightarrow \bar{J}(t)$$

Deriving the constitutive models in the differential form

Standard Linear Solid (SLS)



Compatibility: $\varepsilon = \varepsilon_1 + \varepsilon_2$ **(1)**

Equilibrium:

$$\sigma_{E_1} = \sigma \quad \textbf{(2.1)}$$

$$\sigma_{E_2} + \sigma_\eta = \sigma \quad \textbf{(2.2)}$$

(2)

$$\sigma_{E_1} = E_1 \varepsilon_1$$

$$\sigma_{E_2} + \sigma_\eta = E_2 \varepsilon_2 + \eta \dot{\varepsilon}_2$$

$$\Longrightarrow$$

(2.1)-(2.2) $\Rightarrow$

$$\sigma - \sigma_{E_2} - \sigma_\eta = 0 \Rightarrow \sigma - E_2 \varepsilon_2 - \eta \dot{\varepsilon}_2 = 0$$

$$\Rightarrow \sigma - E_2(\varepsilon \overset{(1)}{-} \varepsilon_1) - \eta(\dot{\varepsilon} - \dot{\varepsilon}_1) = 0$$

$$\varepsilon_1 = \sigma_{E_1} / E_1 \qquad \dot{\varepsilon}_1 = \dot{\sigma}_{E_1} / E_1$$

$$\Longrightarrow$$

The constitutive equation:

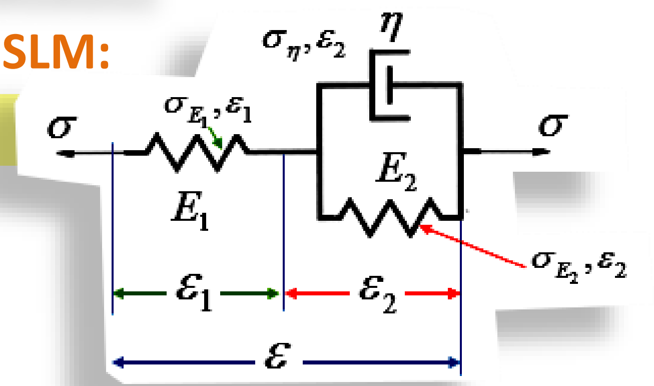
$$\dot{\varepsilon} + \frac{E_2}{\eta}\varepsilon = \frac{\dot{\sigma}}{E_1} + \frac{\sigma}{\eta}\left(1 + \frac{E_2}{E_1}\right)$$

A linear first-order ordinary differential equation (ODE)

Creep Compliance and Relaxation Modulus

What is the **Creep response?**

SLM:



Creep response: Apply a **constant stress** $\sigma(0) = \sigma_0$ $t = 0$

... and determine (measure) the time dependent strain : $\varepsilon(t)$

Initial condition: $\varepsilon(0) = \sigma_0 / E_1$

Initially, only he Hooke element reacts. The dash element is infinitely stiff at t = 0, since

$$\dot{\varepsilon}_2(0) \rightarrow \infty$$

$$\Downarrow$$

Homework

Solve the ODE:

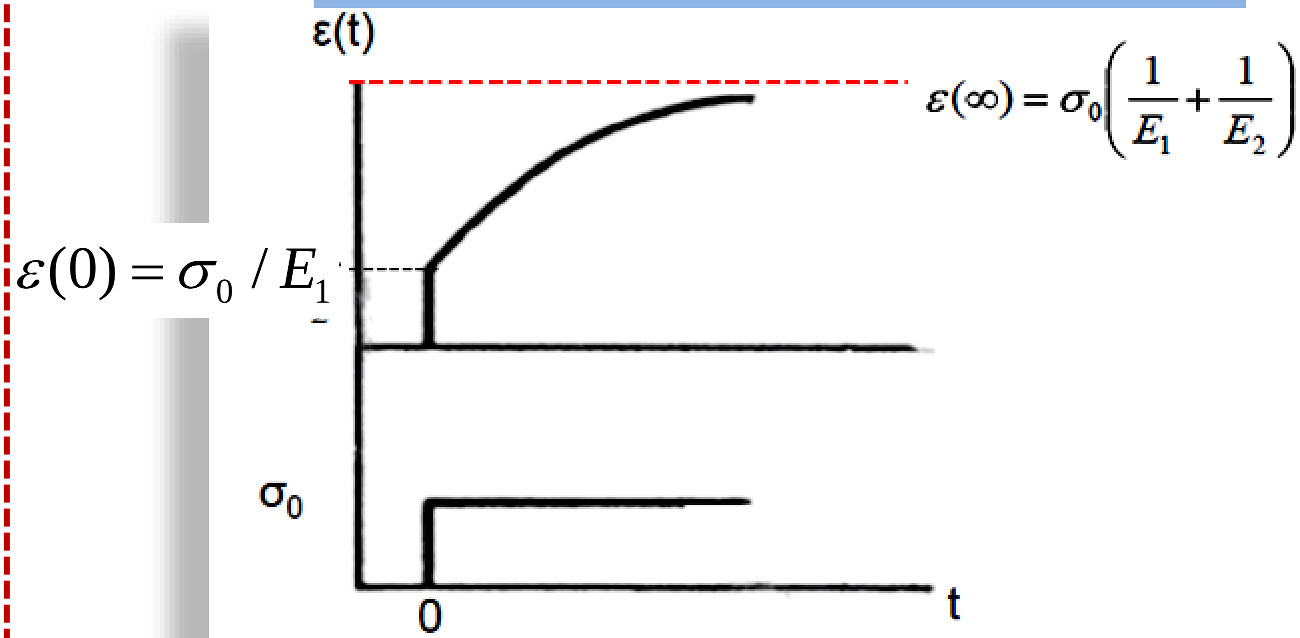
$$\dot{\varepsilon} + \frac{E_2}{\eta}\varepsilon = \frac{\dot{\sigma}}{E_1} + \frac{\sigma}{\eta}(1 + \frac{E_2}{E_1})$$

and show that the creep function is: $J(t)$

$$\left[\frac{1}{E_1} + \frac{1}{E_2}[1 - e^{-\frac{E_2}{\eta}t}]\right]$$

Creep response:

$$\varepsilon(t) = \sigma_0\left[\frac{1}{E_1} + \frac{1}{E_2}[1 - e^{-\frac{E_2}{\eta}t}]\right] \equiv \sigma_0 J(t)$$



Standard Linear Solid (SLS) : Creep Compliance and **Relaxation Modulus**

Relaxation response?

Apply a **constant strain at** $t=0$ $\quad \varepsilon(0)=\varepsilon_0$

... and determine (measure) the time dependent stress : $\sigma(t)$

$$\varepsilon(t>0)=\varepsilon_0=\text{constant} \;\Rightarrow\; \text{For}\quad t>0,\quad \mathrm{d}\varepsilon/\mathrm{d}t=0$$

$$\text{SLM} \Rightarrow \dot{\varepsilon}=\frac{\dot{\sigma}}{E_1}+\frac{\sigma}{\eta}\left(1+\frac{E_2}{E_1}\right)-\frac{E_2}{\eta}\varepsilon_0=0,\;\; t>0$$

This is a linear 1st order ODE ... having the solution:

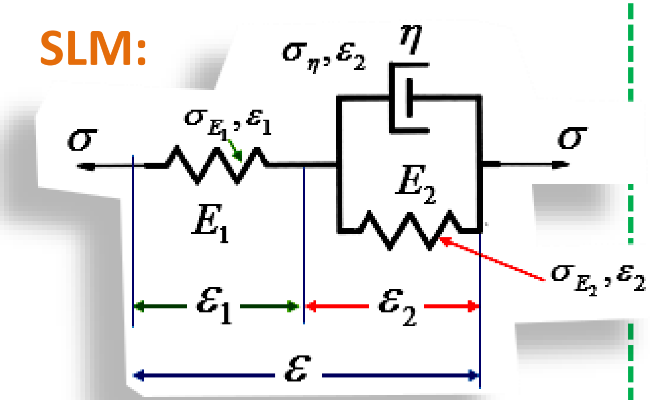
Relaxation solution:

$$\sigma(t)=\varepsilon_0\cdot\underbrace{\frac{E_1}{E_1+E_2}\left[E_2+E_1e^{-\frac{E_1+E_2}{\eta}t}\right]}_{\equiv\,\varepsilon_0\cdot G(t)}\equiv\varepsilon_0 G(t)$$

Relaxation function:

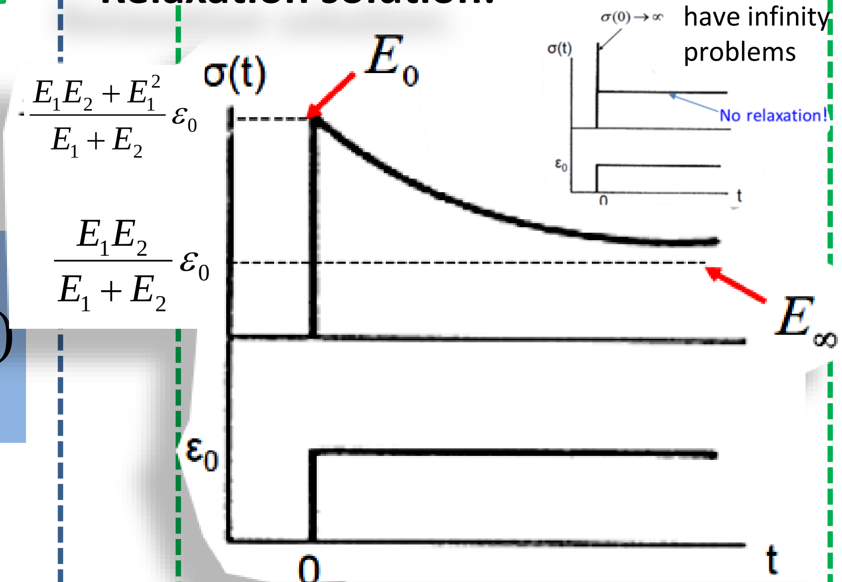
Relaxation function

SLM:



K-V model does not relaxe and have infinity problems

Relaxation solution:



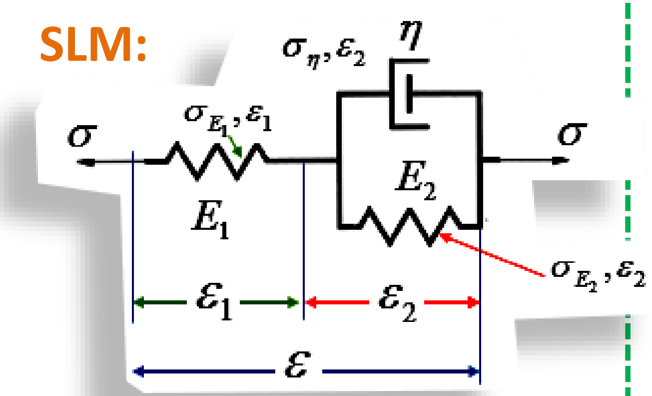
τ_R – relaxation time $\qquad \frac{1}{\tau_R}\equiv\frac{E_1+E_2}{\eta}$

Standard Linear Model :

Relaxation solution – some terminology

$$\sigma(t) = \varepsilon_0 \cdot \underbrace{\boxed{\frac{E_1}{E_1+E_2}\left[E_2 + E_1 e^{-\frac{E_1+E_2}{\eta}t}\right]}}_{\text{Relaxation Modulus}} \equiv \varepsilon_0 \cdot G(t)$$

SLM:



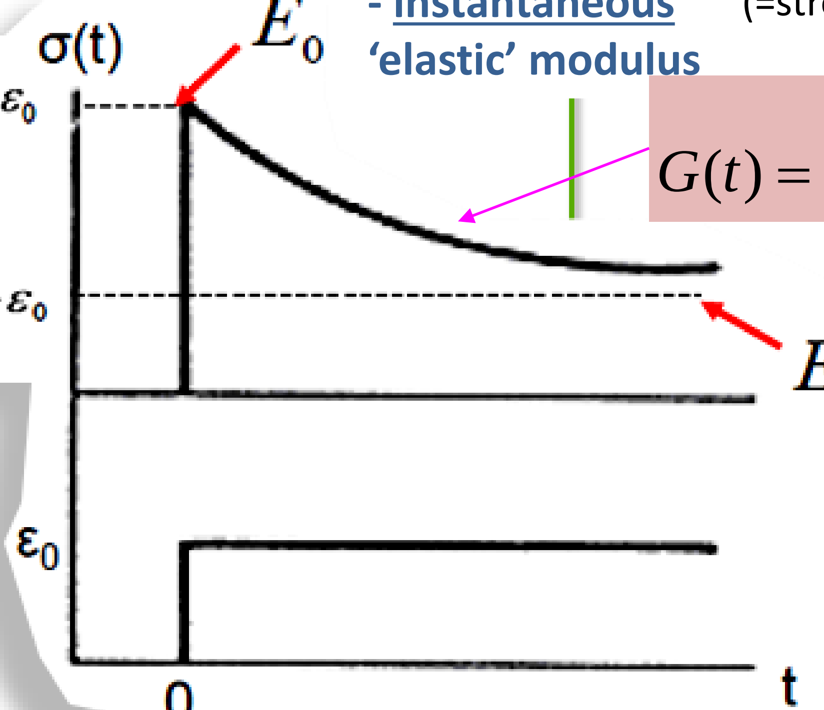
$$G(0) \equiv E_0 = \frac{E_2^2 + E_1E_2}{E_1+E_2}\varepsilon_0$$

E_0 - instantaneous 'elastic' modulus (=stress-strain ration at t = 0)

$$G(t) = E_\infty + \left[E_0 - E_\infty\right] e^{-\frac{E_1+E_2}{\eta}t}$$

$$G_\infty \equiv E_\infty = \frac{E_1E_2}{E_1+E_2}\varepsilon_0$$

E_∞ - asymptotic 'elastic' modulus (is = stress-strain ratio at $t \to \infty$)

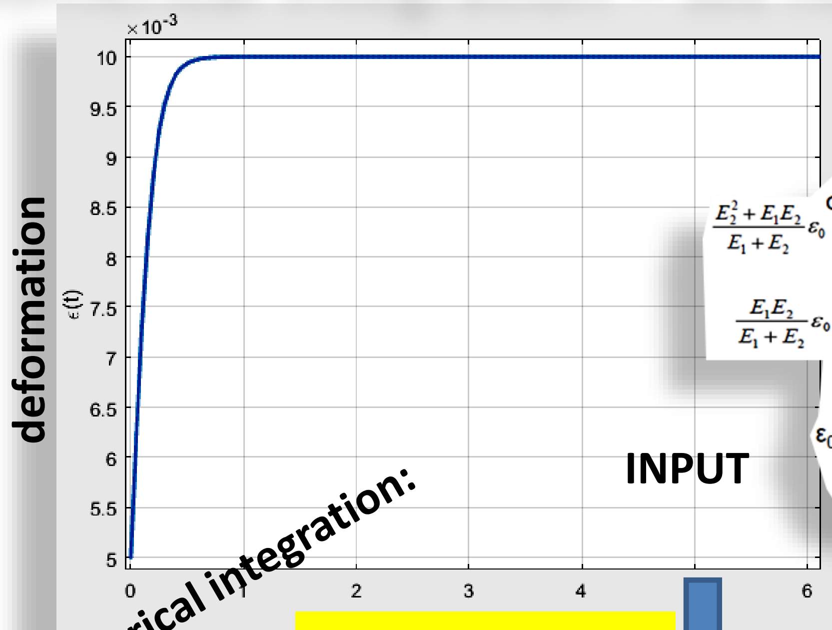


$$\frac{1}{\tau_R} \equiv \frac{E_1+E_2}{\eta}$$

relaxation time

Numerical integration – see next slide Matlab-code

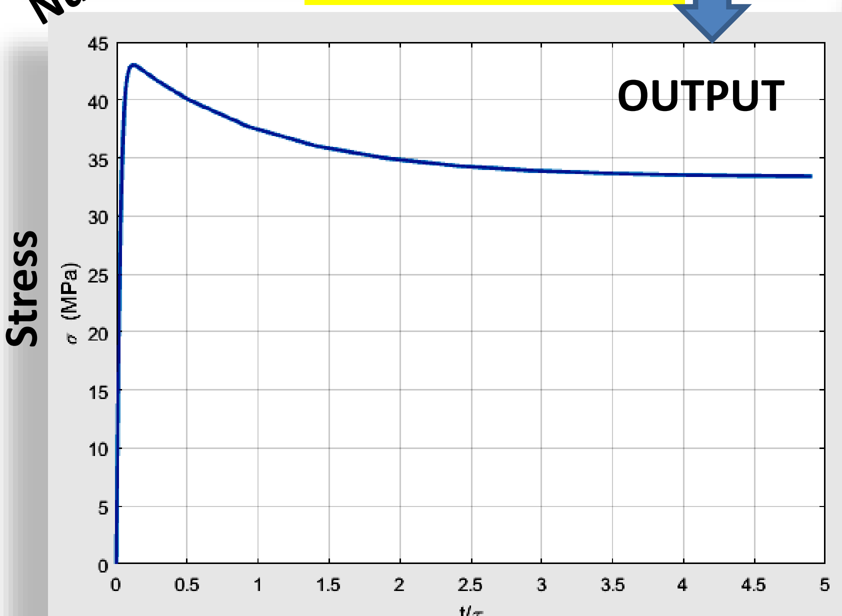
Standard Linear Solid (SLS)



INPUT

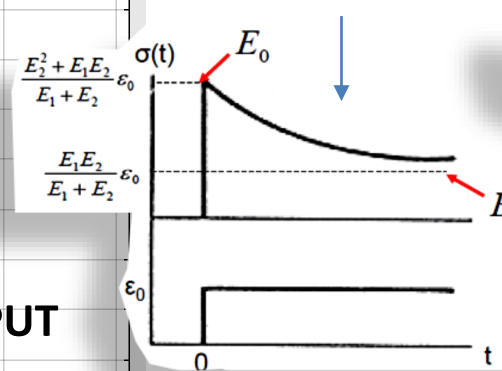
Numerical integration:

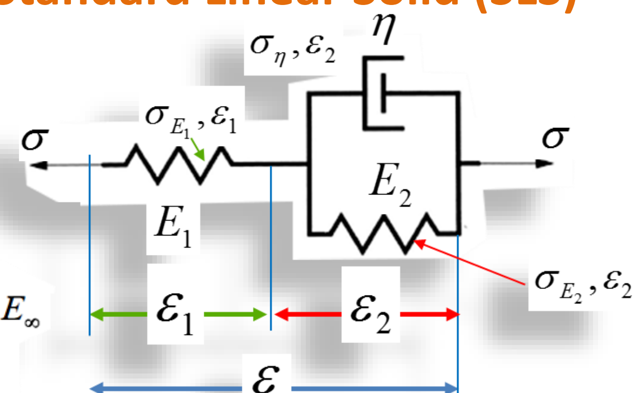
Stress Relaxation



OUTPUT

Theory:





The constitutive equation:

$$\dot{\varepsilon} + \frac{E_2}{\eta}\varepsilon = \frac{\dot{\sigma}}{E_1} + \frac{\sigma}{\eta}\left(1 + \frac{E_2}{E_1}\right)$$

```
t_TAB = ...;
NT      = length(time
k = 5;
H = 1 ./ ( 1 + exp(-2*
% epsilon history - given
epsilon_0         = 1e-2;
EPSILON_TAB       = epsilon_0 * H;
dot_EPSILON_TAB = diff(EPSILON_TAB) ./ diff(times);
dot_EPSILON_TAB = [dot_EPSILON_TAB dot_EPSILON_TAB(NT-1)];

[Time, Y] = ode23s('Standard_Linear_Model',time,[sigma_0 t_0]
Sigma = Y(:,1);
      = Y(:,2)
```

The ODE to numerically integrate …

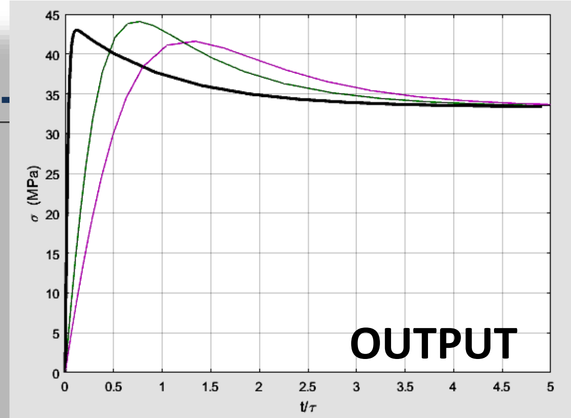
$$\frac{\dot{\sigma}}{E_1} = \dot{\varepsilon} + \frac{E_2}{\eta}\varepsilon - \frac{\sigma}{\eta}\left(1 + \frac{E_2}{E_1}\right)$$

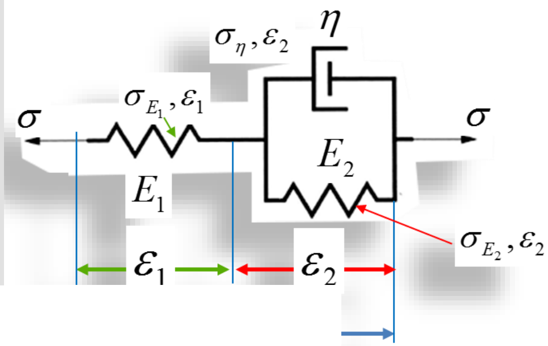
… given know deformation history

Numerical integration

Standard Linear Solid (SLS)

```
global E_1 E_2 eta
global t_TAB EPSILON_TAB dot_EPSILON_TAB

E_1 = 10e9    % E (Pa)
E_2 = E_1/2   % E (Pa)

tau = 5; % (s) retardation time
eta = tau * (E_1 + E_2);

E_0     = (E_2*E_2 + E_1 * E_1) / (E_1 + E_2)
E_infty = E_1 * E_1 / (E_1 + E_2)

sigma_0     = 0.0;      % alkujännitys (Pa);
t_0         = 0;        % t = 0; initial time
dot_epsilon = 0.1/4;    % vakiovenymänopeus (1/s)

%-- integration tiem
time          = [0 5*tau];     %  integrointiväli

% Heaviside ----
dt = tau/100;
times = 0:dt:max(time);
t_TAB = times;
NT     = length(times);
k = 5;
H = 1 ./ ( 1 + exp(-2* k * times));
% epsilon history - given
epsilon_0        = 1e-2;
EPSILON_TAB      = epsilon_0 * H;
dot_EPSILON_TAB = diff(EPSILON_TAB) ./ diff(times);
dot_EPSILON_TAB = [dot_EPSILON_TAB dot_EPSILON_TAB(NT-1)];

[Time, Y] = ode23s('Standard_Linear_Model',time,[sigma_0 t_0]
Sigma = Y(:,1);
t     = Y(:,2);
```





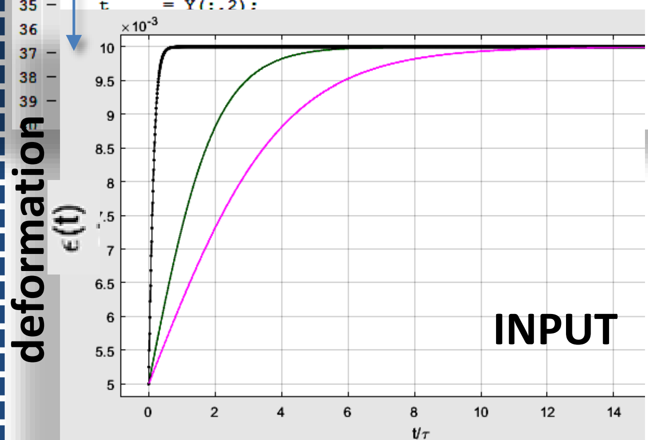
$$\sigma_{E_1} = E_1 \varepsilon_1$$

$$\sigma_{E_2} + \sigma_\eta = E_2 \varepsilon_2 + \eta \dot{\varepsilon}_2$$

```
% INPUT:
% ------
% t    = current time = scalar;
% Y    = is the  temperature in Celsius deg.
%
% OUTPUT:
% -------
% DY   = is the initial value problem : right side f(Y( . . .
%--------------------------------------------------------------------------
function [DY] = Standard_Linear_Model(t, Y)
global E_1 E_2 eta
global t_TAB EPSILON_TAB dot_EPSILON_TAB

%------------------------------------
% the right side f(Y(t);t):   Begin
%--------------------------
Sigma     = Y(1);
tim       = Y(2); % t := Y(2)

epsilon     = interp1(t_TAB, EPSILON_TAB, tim);
dot_epsilon = interp1(t_TAB, dot_EPSILON_TAB, tim);

f_right_side  = + E_1 * dot_epsilon + E_1 * E_2 / eta * epsilon + ...
                - E_1 * Sigma / eta * ( 1 + E_2 / E_1 ) ;

DY(1)   =  f_right_side ;
DY(2)   =  1.; %

DY = DY';
%------------------------------------
% the right side f(Y(t);t):   End
%--------------------------
```

The ODE to integrate:

$$\boxed{\frac{\dot{\sigma}}{E_1} = \dot{\varepsilon} + \frac{E_2}{\eta}\varepsilon - \frac{\sigma}{\eta}\left(1 + \frac{E_2}{E_1}\right)}$$




```
f_right_side  = + E_1 * dot_epsilon + E_1 * E_2 / eta * epsilon + ...
                - E_1 * Sigma / eta * ( 1 + E_2 / E_1 ) ;
```

```
global E_1 E_2 eta
global t_TAB EPSILON_TAB dot_EPSILON_TAB

E_1 = 10e9    % E (Pa)
E_2 = E_1/2   % E (Pa)

tau = 5; % (s) retardation time
eta = tau * (E_1 + E_2);

E_0     = (E_2 * E_2 + E_1 * E_1) / (E_1 + E_2)
E_infty = E_1 * E_1 / (E_1 + E_2)

sigma_0      = 0.0;       % alkujännitys (Pa);
t_0          = 0;         % t = 0; initial time

%-- integration tiem
time           = [0 5*tau];     %  integrointiväli

% Heaviside ----
dt = tau/500;
times = 0:dt:max(time);
t_TAB = times;
NT     = length(times);
k_0 = 5;
k = k_0  / 1;
H = 1 ./ ( 1 + exp(-2* k * times));
% epsilon history - given
epsilon_0       = 1e-2;
EPSILON_TAB     = epsilon_0 * H;
dot_EPSILON_TAB = diff(EPSILON_TAB) ./ diff(times);
dot_EPSILON_TAB = [dot_EPSILON_TAB dot_EPSILON_TAB(NT-1)]

[Time, Y] = ode23s('Standard_Linear_Model',time,[sigma_0 t_0]);
Sigma = Y(:,1);
t     = Y(:,2);

plot(Time/tau, Sigma/1e6);
xlabel('t/\tau')
ylabel('\sigma  (MPa)')
% --------------------------
```

```
% INPUT:
% ------
% t     = current time = scalar;
% Y     = is the  temperature in Celsius deg.
%
% OUTPUT:
% -------
% DY    = is the initial value problem : right side f(Y(t);t)
%----------------------------------------------------------------
-------------------
function [DY] = Standard_Linear_Model(t, Y)
global E_1 E_2 eta
global t_TAB EPSILON_TAB dot_EPSILON_TAB

%------------------------------------
% the right side f(Y(t);t):   Begin
%---------------------------
Sigma     = Y(1);
tim       = Y(2); % t := Y(2)

epsilon     = interp1(t_TAB, EPSILON_TAB, tim);
dot_epsilon = interp1(t_TAB, dot_EPSILON_TAB, tim);

f_right_side  = + E_1 * dot_epsilon + E_1 * E_2 / eta * epsilon
+ ...
                - E_1 * Sigma / eta * ( 1 + E_2 / E_1 ) ;

DY(1)   =  f_right_side ;
DY(2)   =  1.; %

DY = DY';
%------------------------------------
% the right side f(Y(t);t):   End
%---------------------------
```

Standard Linear Solid

Examp

$$\sigma + \tau\dot{\sigma} = G_\infty \epsilon + \tau G_0 \dot{\epsilon} \qquad (1.1)$$

$$G_\infty = \frac{E_1 E_2}{E_1 + E_2}, G_0 = E_1, \tau = \frac{\eta}{E_1 + E_2} \qquad (1.2)$$

$$\dot{\sigma} = \frac{G_\infty}{\tau}\epsilon + G_0\dot{\epsilon} - \frac{\sigma}{\tau} \qquad (1.3)$$

relaxation time

$$\dot{\sigma} = f(\epsilon, \sigma) \qquad (1.4)$$

The differential equation (1.1) or (1.4) can be numerically integrated using appropriate initial conditions for any known history of the deformations or of the stresses.

Given the imposed strain history below

$$\epsilon(t) = \epsilon_0 \sin(\omega t), \qquad (1.5)$$

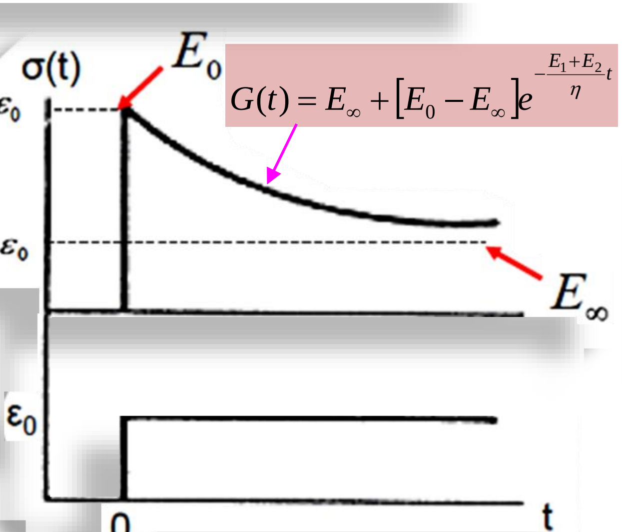
where $\omega = 1$ (1/s), $\tau = 1$ (s) , $\epsilon_0 = 0.008, G_\infty = 550$ MPa, $G_0 = 1.5$ GPa. The initial conditions are $t = 0, \epsilon(0) = 0, \sigma(0) = 0$.

Solve the stresses by integrating analytically and numerically. Draw $\sigma - \epsilon$ and the corresponding time-series for few cycles in order to observe the hysteresis loop.

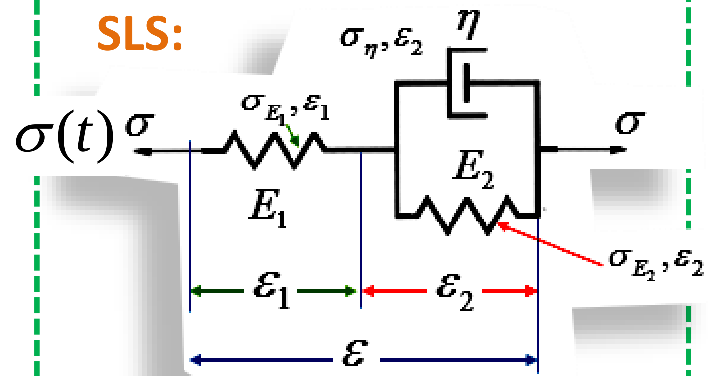
$$G(0) \equiv E_0 = \frac{E_2^2 + E_1 E_2}{E_1 + E_2}\varepsilon_0$$

$$G_\infty \equiv E_\infty = \frac{E_1 E_2}{E_1 + E_2}\varepsilon_0$$

$$G(t) = E_\infty + \left[E_0 - E_\infty\right] e^{-\frac{E_1 + E_2}{\eta}t}$$



SLS:



$$\dot{\sigma} = \frac{G_\infty}{\tau}\epsilon + G_0\dot{\epsilon} - \frac{\sigma}{\tau}$$

$$\dot{\sigma} = f(\epsilon, \sigma)$$

$$\sigma + \tau\dot{\sigma} = G_\infty \epsilon + \tau G_0 \dot{\epsilon}$$

$$G_\infty = \frac{E_1 E_2}{E_1 + E_2}, G_0 = E_1, \tau = \frac{\eta}{E_1 + E_2}$$

relaxation time

Standard Linear Solid

Example 2

$$\sigma + \tau\dot{\sigma} = G_\infty \epsilon + \tau G_0 \dot{\epsilon} \tag{1.1}$$

$$G_\infty = \frac{E_1 E_2}{E_1 + E_2}, G_0 = E_1, \tau = \frac{\eta}{E_1 + E_2} \tag{1.2}$$

$$\dot{\sigma} = \frac{G_\infty}{\tau}\epsilon + G_0\dot{\epsilon} - \frac{\sigma}{\tau} \tag{1.3}$$

$$\dot{\sigma} = f(\epsilon, \sigma)$$

relaxation time

The differential equation (1.1) or (1.4) can be numerically integrated using appropriate initial conditions for any known history of the deformations or of the stresses.

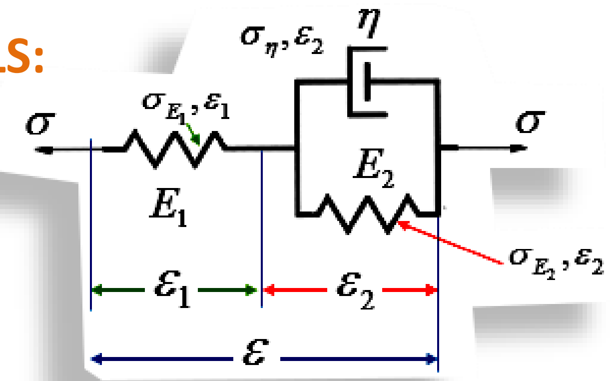
Given the imposed strain history below

$$\epsilon(t) = \epsilon_0 \sin(\omega t), \tag{1.5}$$

where $\omega = 1$ (1/s), $\tau = 1$ (s) , $\epsilon_0 = 0.008, G_\infty = 550$ MPa, $G_0 = 1.5$ GPa. The initial conditions are $t = 0, \epsilon(0) = 0, \sigma(0) = 0$.

Solve the stresses by integrating analytically and numerically. Draw $\sigma - \epsilon$ and the corresponding time-series for few cycles in order to observe the hysteresis loop.

SLS:

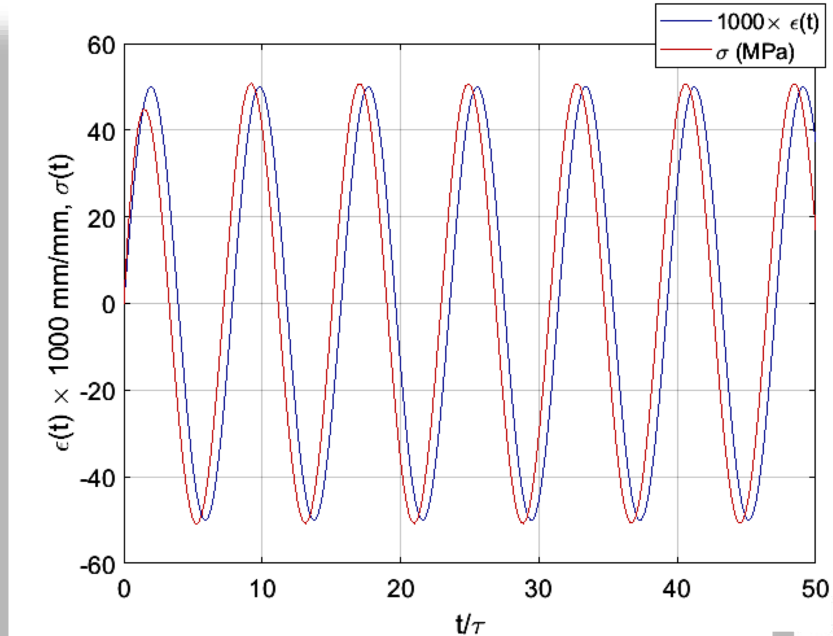


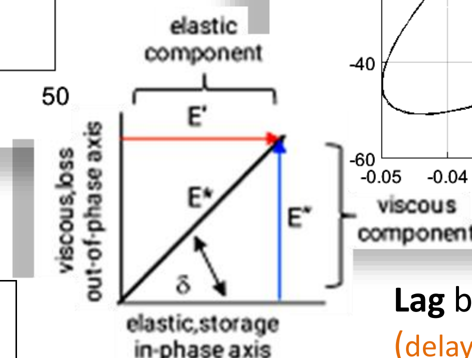
FILE NAVIGATE BREAKPOINTS RUN

```
% INPUT:
% ------
% t     = current time = scalar;
% Y     = is the  temperature in Celsius deg.
%
% OUTPUT:
% -------
% DY    = is the initial value problem : right side f(Y(t);t)
%--------------------------------------------------------------------
function [DY] = Standard_Linear_Model_SLS(t, Y)
global E_0 E_infty tau
global t_TAB EPSILON_TAB dot_EPSILON_TAB

%----------------------------------
% the right side f(Y(t);t):   Begin
%-------------------------

Sigma     = Y(1);
tim       = Y(2); % t := Y(2)

epsilon     = interp1(t_TAB, EPSILON_TAB, tim);
dot_epsilon = interp1(t_TAB, dot_EPSILON_TAB, tim);

f_right_side  = + E_0 * dot_epsilon + E_infty/tau * epsilon + ...
                - Sigma / tau;

DY(1)   =  f_right_side ;
DY(2)   =  1.; %

DY = DY';
%-------------------------
% the right side f(Y(t);t):
%-------------------------

```

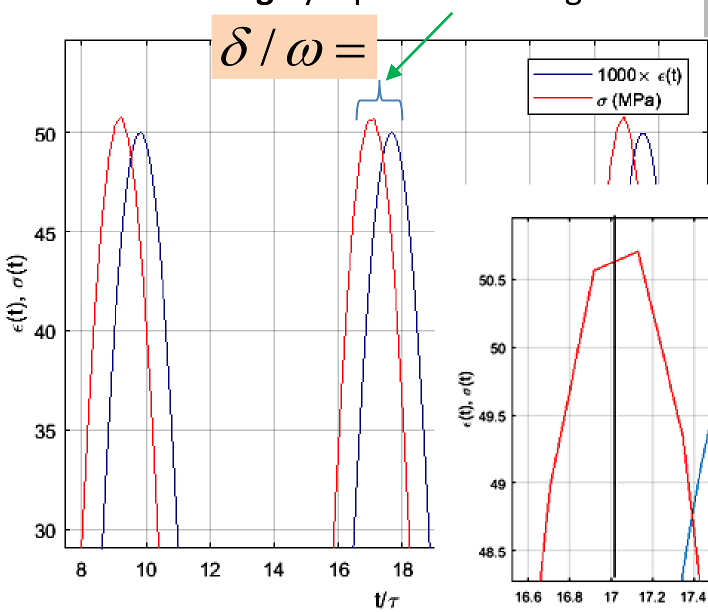
$$\dot{\sigma} = \frac{G_\infty}{\tau}\epsilon + G_0\dot{\epsilon} - \frac{\sigma}{\tau}$$

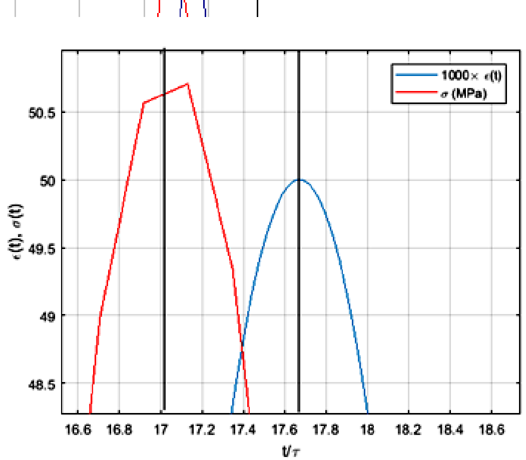
$$\dot{\sigma} = f(\epsilon, \sigma)$$





Lag by a phase shift angle





Lag by a phase shift angle

(delayed response)

delay time

Storage modulus: (dissipative) $E' = \dfrac{\sigma_0}{\varepsilon_0}\cos\delta$

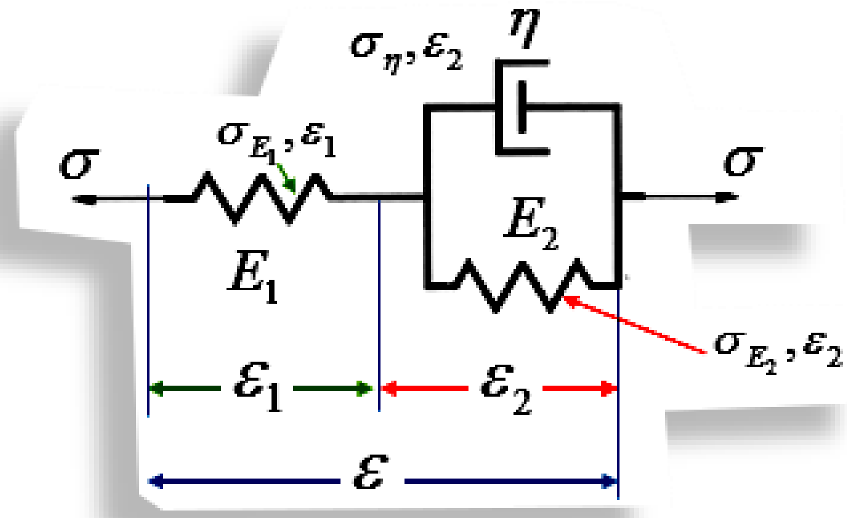
Loss modulus: (elastic, reversible) $E'' = \dfrac{\sigma_0}{\varepsilon_0}\sin\delta$

$$\delta/\omega = 0.7\tau \approx 0.7\text{s} \rightarrow \delta \approx 0.56 \ \sigma_0 \approx 50\text{MPa},$$
$$E' = \sigma_0/\epsilon_0 \cos\delta \approx 850 \text{ MPa and}$$
$$E'' = \sigma_0/\epsilon_0 \sin\delta \approx 530 \text{ MPa}$$
$$\phi = \arcsin(E''/E') \approx 39 \text{ degrees.}$$

Three-Parameter Models

Standard Linear Solid Model

$$\dot{\varepsilon} + \frac{E_2}{\eta}\varepsilon = \frac{\dot{\sigma}}{E_1} + \frac{\sigma}{\eta}\left(1 + \frac{E_2}{E_1}\right)$$



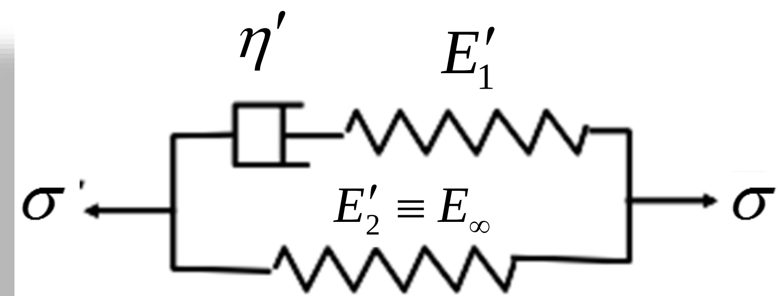
Good to know:

There is another possible configuration - called the Maxwell form of the SLS - for an **equivalent** Kelvin Standard Linear Solid Model for the one studied previously:

Homework/exercise: *derive the constitutive equation!*

$$f(\varepsilon, \dot{\varepsilon}, \sigma, \dot{\sigma}) = 0$$

The Maxwell form of the Kelvin Standard Linear Solid Model (SLS)



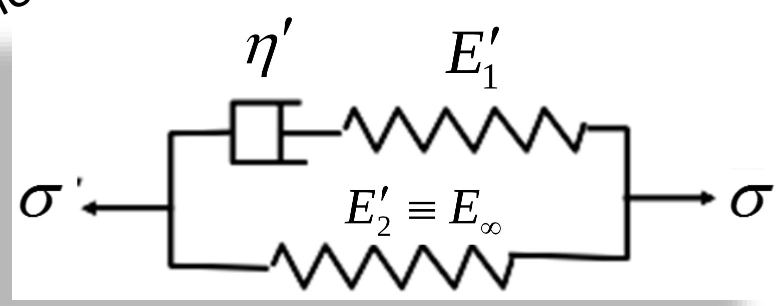
Three-Parameter Models

Standard Linear Solid Model (SLS)

Student: derive this constitutive equation!

$$f\left(\varepsilon, \dot{\varepsilon}, \sigma, \dot{\sigma}\right) = 0$$

Good to know



The Maxwell form of the **Standard Linear Solid** (SLS)

The constitutive equation:

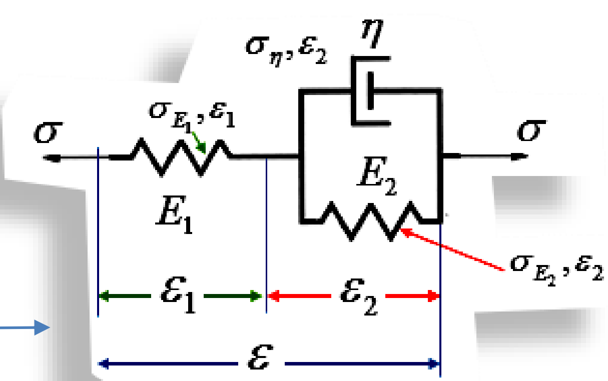
$$\dot{\sigma} + \frac{E_1'}{\eta'}\sigma = \left(E_1' + E_\infty'\right)\dot{\varepsilon} + \frac{E_1' E_\infty'}{\eta'}\varepsilon$$

A linear first-order ordinary differential equation (ODE)

Relaxation Modulus: $G(t) = E_\infty' + E_1' e^{-\frac{E_1'}{\eta'}t}$

This is another possible configuration - called the *Maxwell form of the SLS* - for an **equivalent** Kelvin Standard Linear Solid Model for the one *studied previously*.

Standard Linear Solid Model (SLS):



$$E_1 = E_1' + E_\infty'$$

$$E_2 = E_\infty' \cdot \left(1 + \frac{E_\infty'}{E_1'}\right)$$

$$\eta = \eta' \cdot \left(1 + \frac{E_\infty'}{E_1'}\right)^2$$

Numerical integration of the ODE-system (the constitutive equation)

Good to know:

In general, in a FE- Structural Analysis software, the constitutive equations (with visco-elasticity, visco-plasticity, plasticity and other kind of path-dependent stress-stress behavior) are known in rate or equivalently in incremental form. Therefore, these constitutive equations should be integrated numerically at each time step of the numerical simulation.

The student is encouraged to learn how to use the Matlab-ODE-suite* to numerically integrate any given constitutive law for any arbitrary excitation.

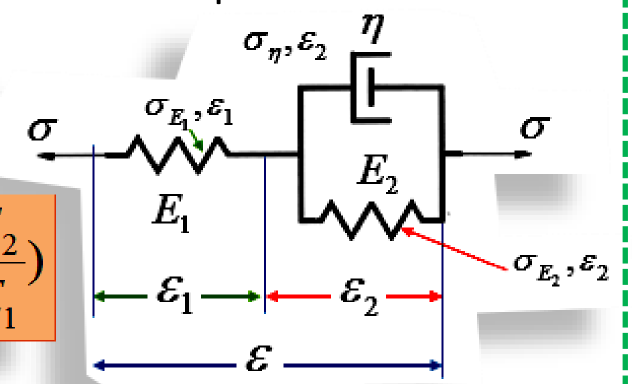
Exercise to do: Consider the stress relaxation problem for the SLS-model
Using the Matlab ODE-suite, solve numerically the relaxation problem
a) with constant initial condition: $\varepsilon(0) = 0.1$

b) harmonic strain input

using the parameters:

$$\eta = 1000, \;\; E_1 = 2, \;\; E_2 = 1.5$$

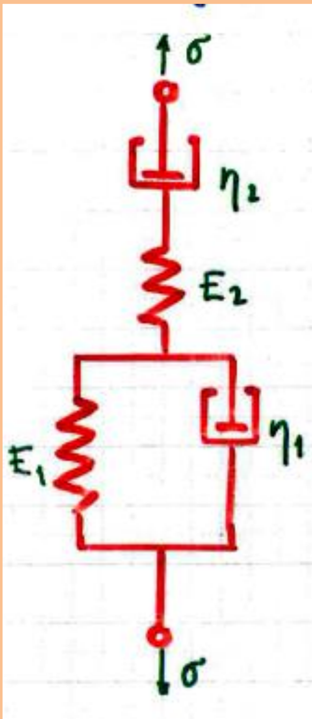
$$\boxed{\dot{\varepsilon} + \frac{E_2}{\eta}\varepsilon = \frac{\dot{\sigma}}{E_1} + \frac{\sigma}{\eta}\left(1 + \frac{E_2}{E_1}\right)}$$



*a general ODE-system numerical integration functions for stiff and non-stiff systems

Deriving the constitutive models in the differential form

Burgersin aine — Burgers model



Burgersin aineen malli saadaan kytkemällä sarjaan l. peräkkäin Maxwellin ja Voigt-in aineen mallit.

Näin saadaan

$$\sigma_1 = E_1\varepsilon_1 + \eta_1\dot{\varepsilon}_1$$

$$\eta_2\dot{\sigma}_2 + E_2\sigma_2 = E_2\eta_2\dot{\varepsilon}_2$$

Tasapaino- ja yhteensopivuusehdot ovat

$$\sigma_1 = \sigma_2 = \sigma \quad \text{(tsp. ehto)}$$

$$\varepsilon_1 + \varepsilon_2 = \varepsilon \quad \text{(yht. sop.)}$$

Tuloksena saadaan

$$\ddot{\varepsilon} + \frac{E_1}{\eta_1}\dot{\varepsilon} = \frac{1}{E_2}\ddot{\sigma} + \left(\frac{E_1+E_2}{E_2\eta_1} + \frac{1}{\eta_2}\right)\dot{\sigma} + \frac{E_1}{\eta_1\eta_2}\sigma$$

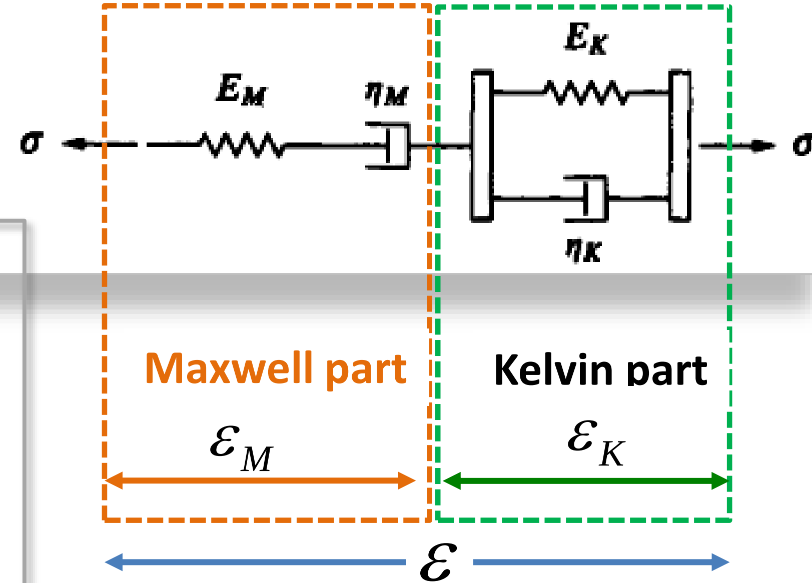
Such model represents well realistic viscoelastic behavior except for tertiary creep.
(*ref*. Ottosen *et al*.)

Deriving the constitutive models in the differential form

Exercise - derive the constitutive equation.

Such model represents well realistic viscoelastic behavior except for tertiary creep. (*ref*. Ottosen *et al.*)

Burgers model



Recall:

Maxwell model:

$$\dot{\varepsilon}_M = \frac{\dot{\sigma}}{E_M} + \frac{\sigma}{\eta_M} \qquad (1)$$

Kelvin model:

$$\sigma(t) = E\varepsilon(t) + \eta\dot{\varepsilon}(t)$$

Compatibility & Equilibrium

$\varepsilon = \varepsilon_M + \varepsilon_K$; $\boxed{\sigma = \sigma_M = \sigma_K}$

$$\sigma = E_K(\varepsilon - \varepsilon_M) + \eta_K(\dot{\varepsilon} - \dot{\varepsilon}_M) \qquad (2)$$

Differentiating Eq. (2) twice and inserting Eq. (1) one obtains:

Burgers model

$$\frac{\eta_K}{E_M}\ddot{\sigma} + \left(1 + \frac{\eta_K}{\eta_M} + \frac{E_K}{E_M}\right)\dot{\sigma} + \frac{E_K}{\eta_M}\sigma = \eta_K\ddot{\varepsilon} + E_K\dot{\varepsilon}$$

Creep compliance

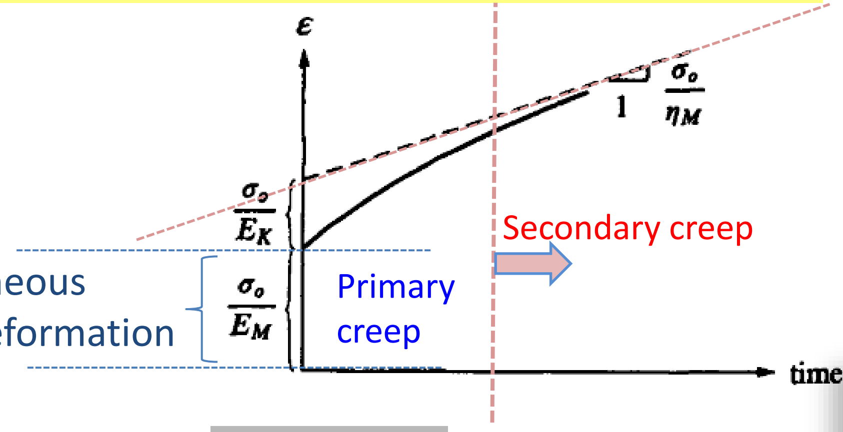
$$J(t) = \frac{1}{E_M} + \frac{t}{\eta_M} + \frac{1}{E_K}\left(1 - e^{-\frac{E_K}{\eta_K}t}\right) \qquad \textit{Burgers model}$$

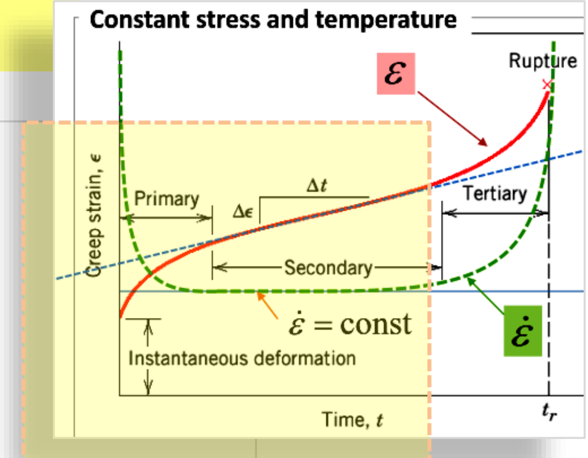
$$\varepsilon(t) = \sigma_0 J(t)$$

Creep response

Apply a **constant stress** $\sigma(0)=\sigma_0$ $\,.t=0$

… and determine (measure) the time dependent strain $\varepsilon(t)$



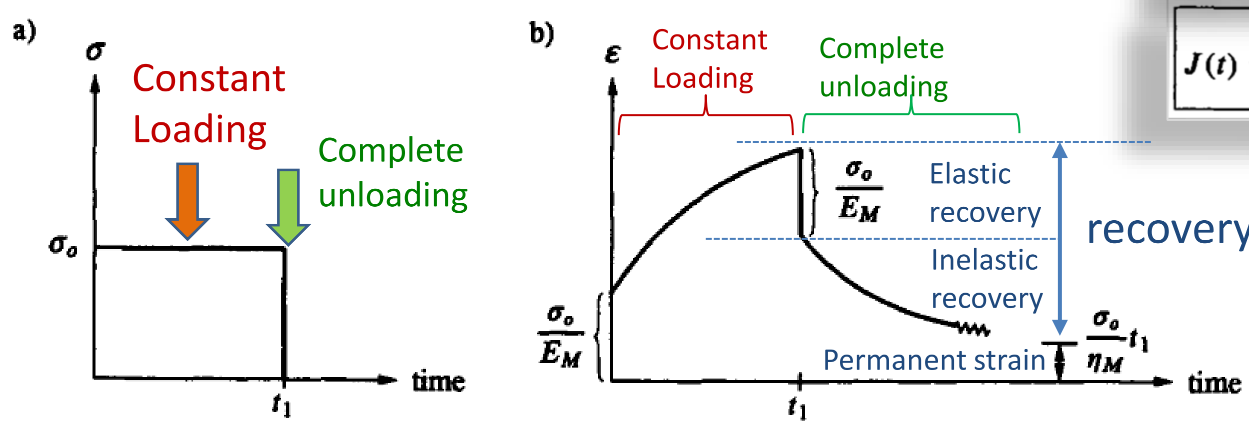


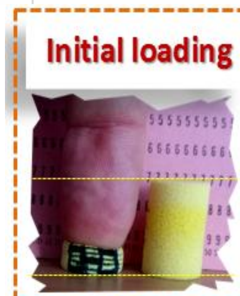
$$\frac{\eta_K}{E_M}\ddot{\sigma}+\left(1+\frac{\eta_K}{\eta_M}+\frac{E_K}{E_M}\right)\dot{\sigma}+\frac{E_K}{\eta_M}\sigma=\eta_K\ddot{\varepsilon}+E_K\dot{\varepsilon}$$

Response of Burgers model in creep test.

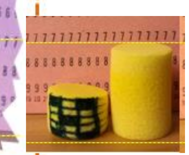
Creep compliance

$$J(t)=\frac{1}{E_M}+\frac{t}{\eta_M}+\frac{1}{E_K}\left(1-e^{-\frac{E_K}{\eta_K}t}\right)$$





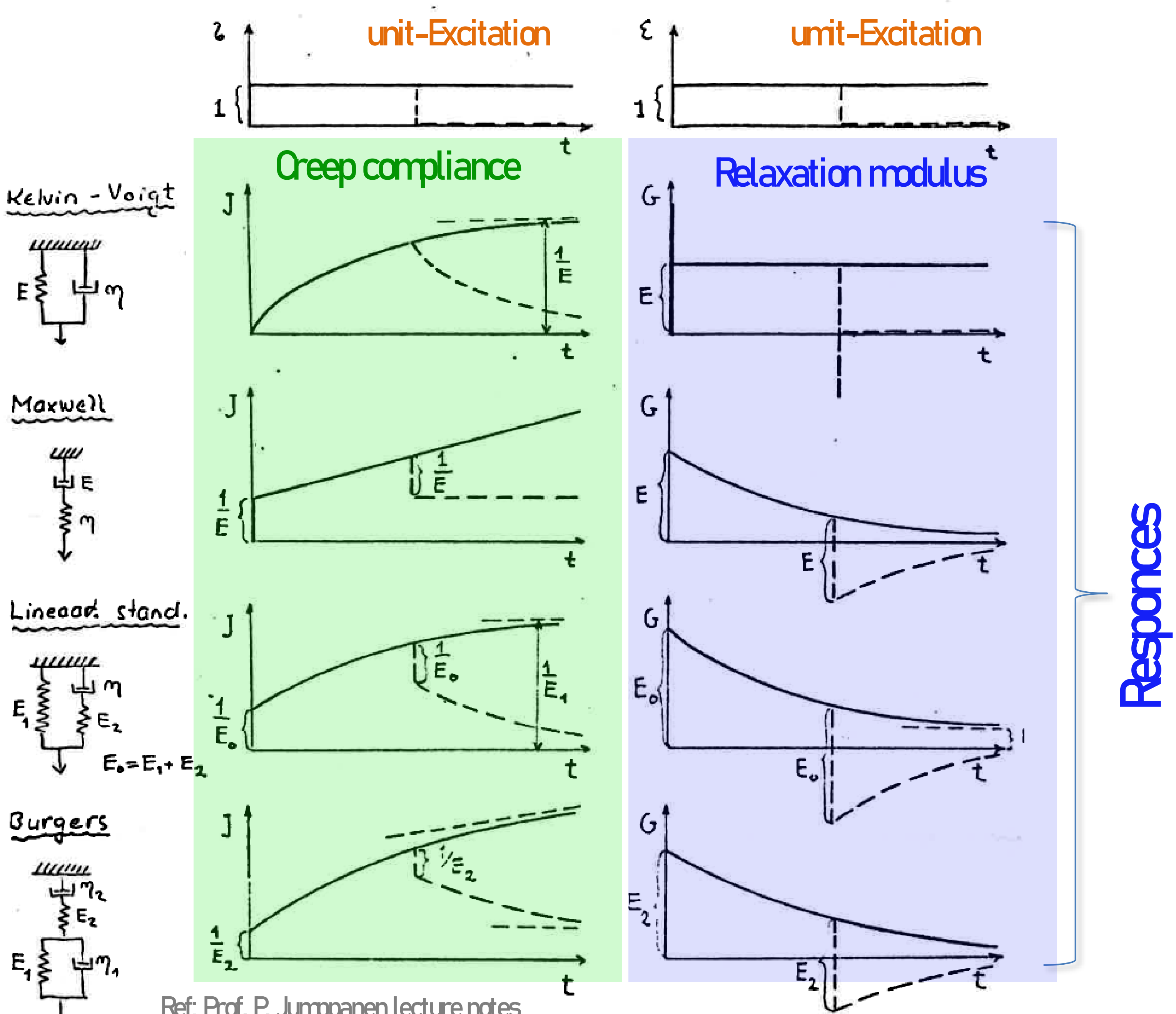
Partial
recovery



Response of Burgers model; partial recovery.

This model represents realistic viscoelastic behavior as primary, secondary and recovery well except for tertiary creep. (*ref*. Ottosen *et al*.)

Overview of responses of rheological model



Ref: Prof. P. Jumppanen lecture notes

Résumé

Virumisfunktiot:

Voigtin aine: $J(t) = \frac{1}{E}\left(1 - e^{-\frac{Et}{\eta}}\right) H(t)$

Maxwell'in aine: $J(t) = \left(\frac{1}{E} + \frac{t}{\eta}\right) H(t)$

Kelvin'in aine:

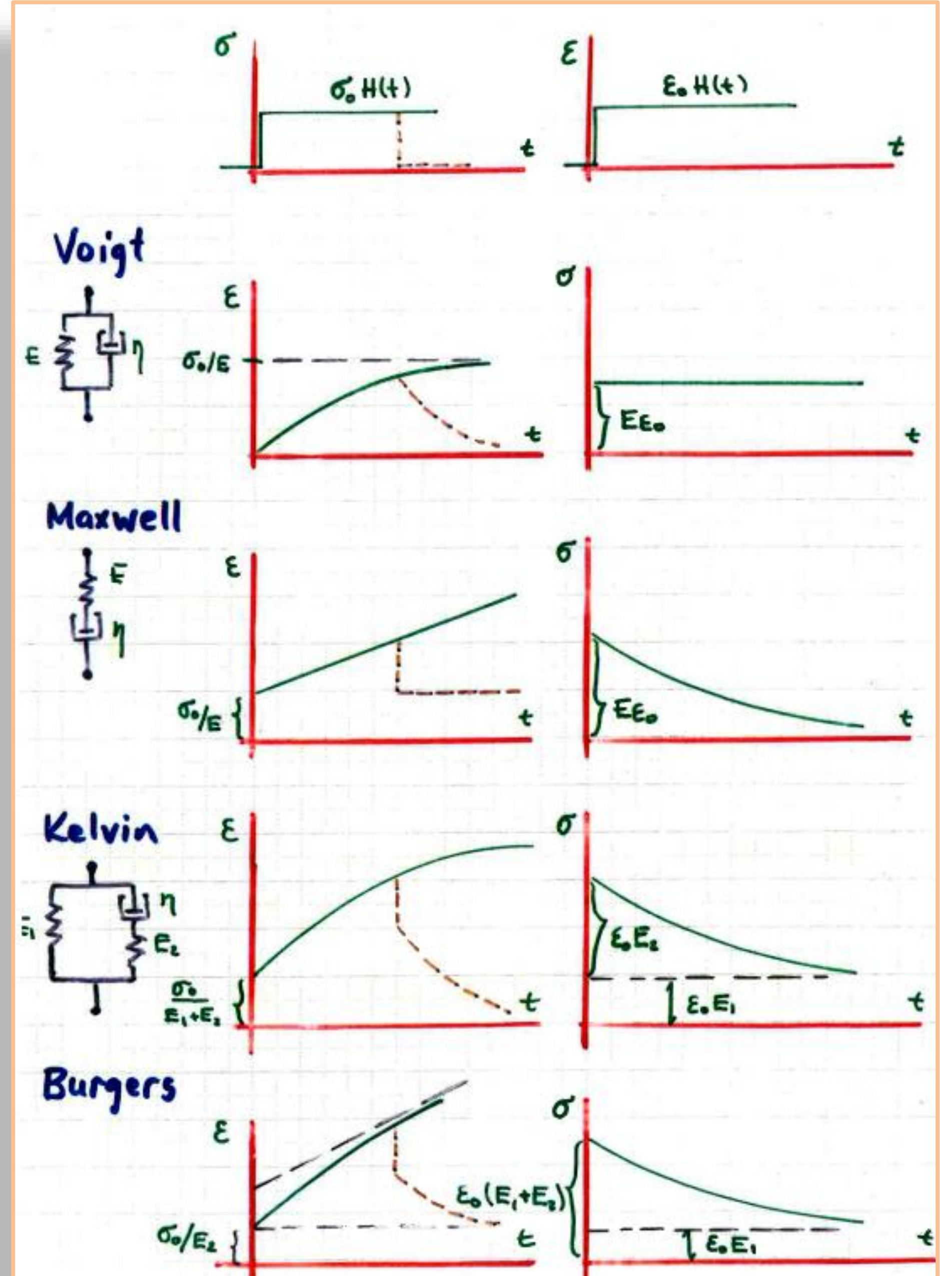
$$J(t) = \frac{1}{E_1}\left[1 - \frac{E_2}{E_1+E_2}\exp\left(-\frac{E_1 E_2 t}{(E_1+E_2)\eta}\right)\right] H(t)$$

Burgersin aine:

$$J(t) = \left[\frac{1}{E_1}\left(1 - e^{-\frac{E_1 t}{\eta}}\right) + \frac{1}{E_2} + \frac{t}{\eta_2}\right] H(t)$$

$$J(t) \propto \varepsilon(t)$$

Havaitaan, että mielivaltaisen sarjaan kytketyn mallin virumisfunktio on peruselementtien virumisfunktioiden summa.



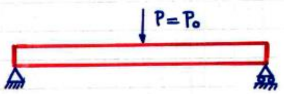
Ref: Prof. J. Paavola lecture slides

The Elastic-Viscoelastic analogy or The viscoelastic correspondence principle

The “viscoelastic correspondence principle” works by adapting an available elastic solution to make it applicable to viscoelastic materials as well, so making you save effort and time.

Elastis-viskoelastinen analogia

Tarkastellaan kimmoista palkin taivutusta.



Ratkaisuna saadaan $\begin{cases} M(x) = \frac{P}{2}x & (0 \le x \le \frac{l}{2}) \\ Q(x) = \frac{P}{2} \end{cases}$

$\Rightarrow \sigma(x) = \frac{M(x)}{I_z} \cdot y \Rightarrow \varepsilon(x) = \frac{M(x)}{EI_z} \cdot y$

Oletetaan seuraavaksi, että palkin materiaali on jotakin viskoelastista ainetta, ja että kuormituksena on pistekuorma $P = P_0 H(t)$

Näin saadaan

$M(x,t) = \frac{P_0}{2} \cdot x\, H(t)$

$Q(x,t) = \frac{P_0}{2} H(t) = \frac{dM(t,x)}{dx}$

Edelleen $\sigma(x,t) = \frac{M(x,t)}{I_z} y = \frac{P_0 x}{2I_z} y\, H(t)$

Havaitaan, että voimasuureisiin ei aineen "viskoelastisuudella" ole vaikutusta. Sen sijaan siirtymäsuureet (viruminen)

$\varepsilon(x,t) = \sigma_0 J(t) = \frac{P_0 x}{2I_z} y\, J(t)$

Nähdään, että kuormituksen ollessa muotoa $P_0 H(t)$ viskoelastinen ratkaisu saadaan kimmoisesta ratkaisusta korvaamalla $E \longrightarrow 1/J(t)$:llä.

Tarkastellaan seuraavaksi palkkia, johon aiheutetaan kimmoinen taipuma $w(x)$.

Palkissa syntyy käyristymä $\kappa = -w''$ ja venymä on

$\varepsilon = y\kappa = -y w''(x)$

ja jännitys $\sigma = Ey\kappa = -Ey w''(x)$

$M(x) = EI\kappa = -EI w''(x)$

Oletetaan palkin olevan jälleen viskoelastista ainetta, ja taipuman muotoa $w(x,t) = w(x) H(t)$

Tällöin venymä on $\varepsilon(x,t) = -y \frac{\partial^2 w(x)}{\partial x^2} H(t)$

Jännitys on (relaksaatio) $\sigma(x,t) = -y \frac{\partial^2 w(x)}{\partial x^2} G(t)$

Nähdään, että siirtymätilan (pakotetun) ollessa muotoa $w(x)\, H(t)$ viskoelastinen ratkaisu saadaan kimmoisesta korvaamalla $E \longrightarrow G(t)$:llä.

Linear viscoelastic models can be represented by a Volterra equation relating stress and strain:

$$\varepsilon(t) = \frac{\sigma(t)}{E_{\text{inst,creep}}} + \int_0^t K(t-t')\dot{\sigma}(t')dt' \qquad \sigma(t) = E_{\text{inst,relax}}\varepsilon(t) + \int_0^t F(t-t')\dot{\varepsilon}(t')dt'$$

Also called Boltzman Superposition Integrals

The Elastic-Viscoelastic analogy or The viscoelastic correspondence principle

Siirrytään jälleen tarkastelemaan virumistapausta, mutta otaksutaan, että kuormitus on jollakin muulla tavalla ajasta t riippuva funktio. Esimerkkinä edellä oleva palkki, mutta kuormana $P(t) = P_0 t H(t)$. Samainen korvausperiaate toimii edelleen, mutta ei aivan yhtä yksinkertaisesti, koska kysymyksessä ei ole puhdas viruminen määritelmänsä muodossa.

Ratkaisu voidaan suorittaa usealla tavalla. Käytetään alutsi integraalimuotoista esitystä.

$$\varepsilon(t) = \frac{1}{E}\sigma(t) + \int_0^t K(t-\tau)\,\sigma(\tau)\,d\tau$$

$$= \int_0^t J(t-\tau)\,\dot{\sigma}(\tau)\,d\tau$$

Esim. Maxwellin aine $K(t) = \frac{1}{\eta}$

$$J(t) = \frac{1}{E} + \frac{t}{\eta}$$

$$\Rightarrow \varepsilon(t) = \frac{M(x)\,t\,H(t)}{E I_z}\,y + \int_0^t \frac{1}{\eta}\,\tau\,\frac{M(x)}{I_z}\,y\,d\tau$$

$$\Rightarrow \varepsilon(t) = \frac{M(x)\,y}{I_z}\left(\frac{t}{E} + \frac{t^2}{2\eta}\right)H(t)$$

tai

$$\varepsilon(t) = \frac{M(x)\,y}{I_z}\int_0^t \left(\frac{1}{E} + \frac{\tau}{\eta}\right)d\tau$$

tai Laplace-muunnosta käyttämällä

$$\frac{\sigma}{\eta} + \frac{\dot{\sigma}}{E} = \dot{\varepsilon} \quad \div$$

$$\left(\frac{1}{\eta} + \frac{s}{E}\right)\tilde{\sigma} = s\,\tilde{\varepsilon} \qquad \left(\tilde{\sigma} = \frac{M}{I}\,y\,\frac{1}{s^2} = \frac{\sigma_0}{s^2}\right)$$

$$\Rightarrow \tilde{\varepsilon} = \left(\frac{1}{\eta}\,\frac{1}{s^3} + \frac{1}{E}\,\frac{1}{s^2}\right) \div \left(\frac{1}{\eta}\,\frac{t^2}{2} + \frac{1}{E}\,t\right)\sigma_0$$

tai vielä kimmoisesta ratkaisusta

$$\varepsilon = \frac{\sigma_0 t}{E} \div \sigma_0\left(\frac{1}{s^2}\,s\tilde{J}(s)\right) =$$

$$= \sigma_0 \cdot \frac{1}{s}\cdot\left(\frac{1}{E}\cdot\frac{1}{s} + \frac{1}{\eta}\,\frac{1}{s^2}\right) \div$$

$$\div\ \sigma_0\left(\frac{t}{E} + \frac{t^2}{2\eta}\right)H(t)$$

Muunnettaessa kimmoisesta ratkaisusta saadaan esim. muuntamalla yhtälö

$$\varepsilon(t) = \int_0^t J(t-\tau)\,\dot{\sigma}(\tau)\,d\tau$$

$$\tilde{\varepsilon} = \underbrace{\tilde{J}(s)\cdot s}_{1/\tilde{E}(s) = s\,\tilde{J}(s)}\,\tilde{\sigma}(s)$$

Relaksaatio tapaus

Esimerkiksi tuki painuu tai sauva (tahi jokin muu rakenne) saa ajasta riippuvan siirtymän. Tällöin voidaan ratkaisun perustana käyttää

$$\sigma(t) = E\,\varepsilon(t) + \int_0^t R(t-\tau)\,\varepsilon(\tau)\,d\tau$$

$$= \int_0^t G(t-\tau)\,\dot{\varepsilon}(\tau)\,d\tau$$

tai Laplace-muuntamalla

esim. Maxwell

$$\frac{\sigma}{\eta} + \frac{\dot{\sigma}}{E} = \dot{\varepsilon} \quad \div$$

$$\left(\frac{1}{\eta} + \frac{s}{E}\right)\tilde{\sigma}(s) = s\,\tilde{\varepsilon}(s) \Rightarrow$$

$$\tilde{\sigma}(s) = \frac{E\eta s}{E + \eta s}\,\underbrace{\tilde{\varepsilon}(s)}_{\text{tunnettu}}$$

Käytettäessä Laplace muunnosta kimmoisessa ratkaisusta tulee huomioida

$$\sigma(t) = \int_0^t G(t-\tau)\,\dot{\varepsilon}(\tau)\,d\tau \Rightarrow$$

$$\tilde{\sigma}(s) = \underbrace{\tilde{G}(s)\cdot s}_{\tilde{E}(s) = \tilde{G}(s)\cdot s}\cdot\,\tilde{\varepsilon}(s)$$

Edeltä $\tilde{E}(s) = \dfrac{1}{s\,\tilde{J}(s)} = \tilde{G}(s)\cdot s$

$$\Rightarrow \tilde{G}(s)\,\tilde{J}(s) = \frac{1}{s^2}$$

The viscoelastic correspondence principle

Mechanics problem = the structure, its materials, and its boundary conditions of traction and displacement

The result of Laplace transformation of a **mechanics problem** often has the consequence that none of the spatial aspects of its description is be altered: the problem will appear the same, at least spatially. The only alteration will be the time-dependent aspects in the material properties.

So, the Laplace-plane version of problem can then be interpreted as representing a stress analysis problem for an elastic body of the same shape as the viscoelastic body, in a way that a solution for an elastic body will apply, in the Laplace plane, to the corresponding viscoelastic body as well. Then by taking the inverse Laplace transform of theses equations when obtains the solution for the visco-elastic problem.

However, there is an exception to this correspondence: boundary conditions for traction or displacement may be altered spatially on transformation.

Good to know

viscoelastic analogy

$$\left.\begin{array}{c} E \\ \nu \end{array}\right\} \longrightarrow \left\{\begin{array}{l} G \longrightarrow \mathcal{G} \\ K \longrightarrow \mathcal{K} \end{array}\right.$$

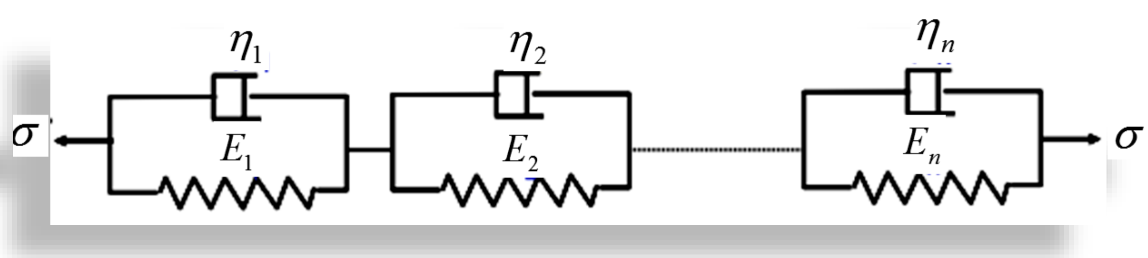
$$\underline{E \longrightarrow 1/J(t)}$$

$$\underline{E \longrightarrow G(t)}$$

Modelling complex experimental response

In order to account for a more complex experimental response* for some materials, the previously studied elementary elements **may be combined further into rheological chains as the *Generalized Voigt* and *Generalized Maxwell models***

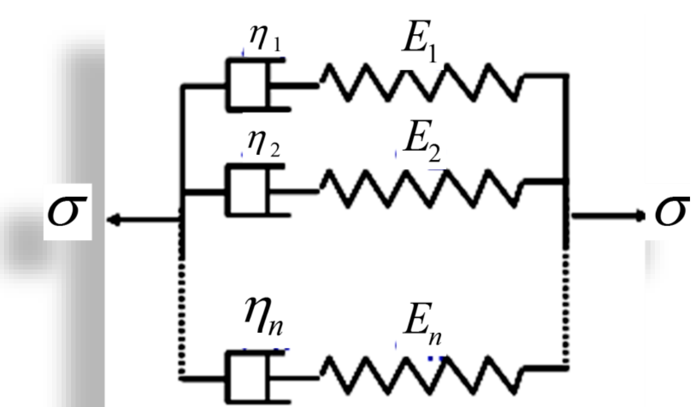
*Such combined models account better for the spectrum of different times scales involved in the relaxation due to presence of various lengths scales and other different physical mechanisms involved.



Generalized (Kelvin-) Voigt model

Describes the creep well.

Generalized Maxwell model



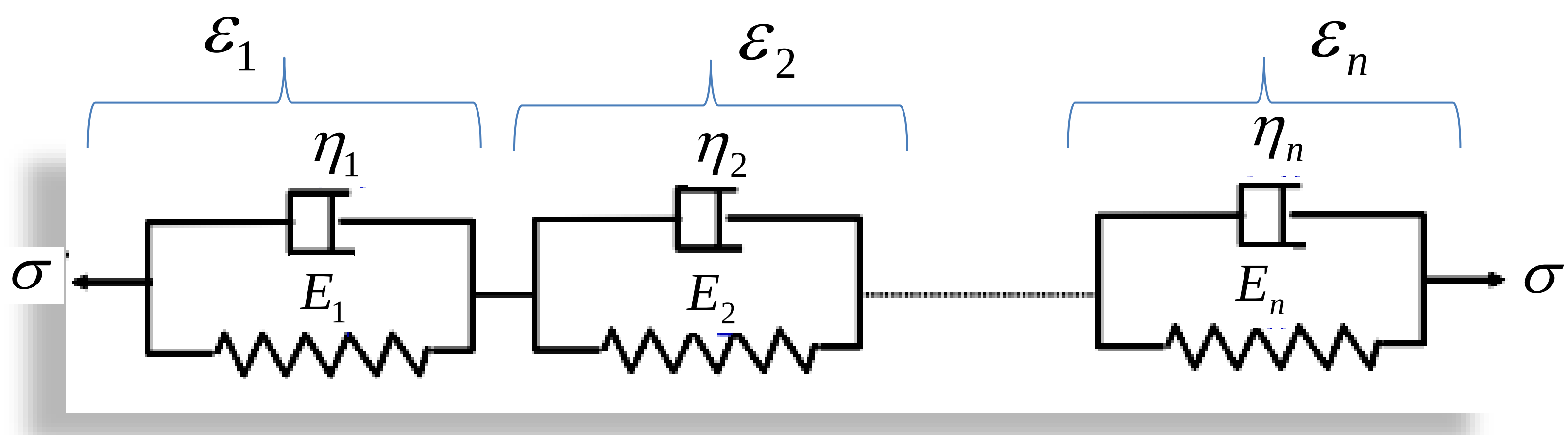
Describes stress relaxation well.

Kelvin (-Voigt) Chain:
- adequate model for modelling Creep phenomena [1]

Generalized Maxwell chain:
- suitable for modelling stress relaxation phenomena [1]

[1] *Computational Viscoelasticity*, S. Marques and G. Creus, Springer 2012

Generalized (Kelvin-) Voigt model



Creep response:

$$\varepsilon(t) = \sigma_0 H(t) \cdot \underbrace{\sum_{i=1}^{n} \frac{1}{E_i}\left(1 - e^{-\frac{E_i}{\eta_i}t}\right)}_{\text{Creep function}}, \quad t > 0$$

Total strain:

$$\varepsilon(t) = \sum_{i=1}^{n} \varepsilon_i$$

$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

The parameters of the model are obtained to match experimental creep data using adequate curve fitting techniques (+ some regularization methods)

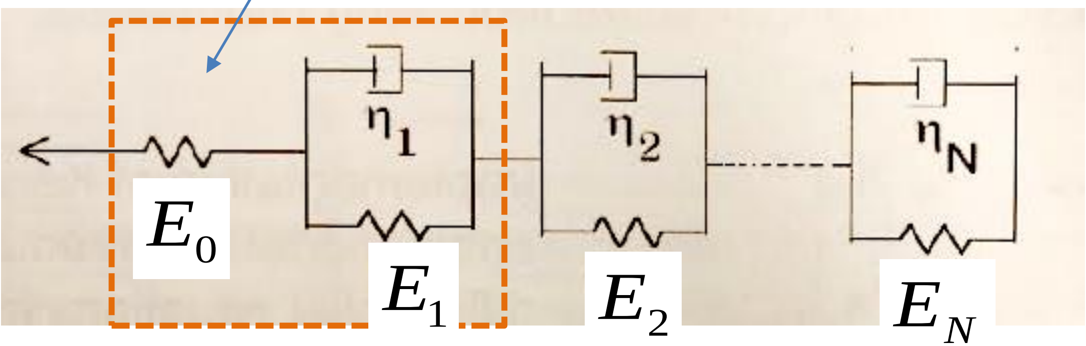
Generalized Kelvin chain

Useful to know

Sometimes, an additional spring is added into the chain for more realistic modeling

Note the additional elastic element! Why?

This corresponds to a Kelvin Standard Linear Solid Model
(another equivalent configuration of SLS)



Generalized Kelvin chain

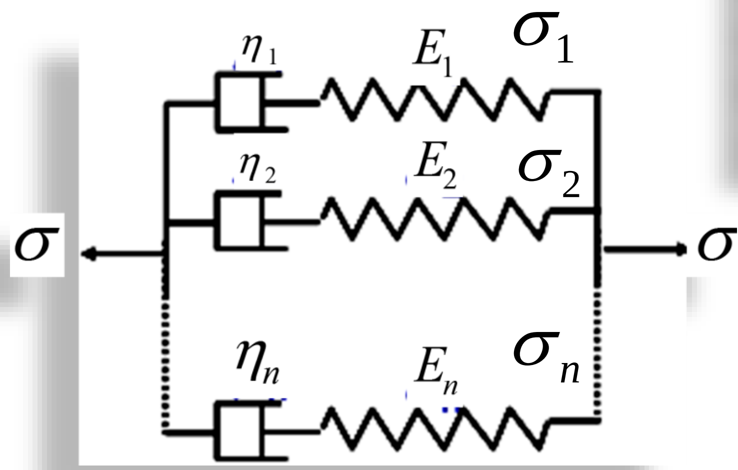
Creep function :

$$J(t) = \boxed{\frac{1}{E_0}} + \sum_{i=1}^{N} \frac{1}{E_i}\left(1 - e^{-\frac{E_i}{\eta_i}t}\right)$$

N.B. **The parameters of the model can be obtained from experimental creep data using adequate curve fitting techniques** (together with some regularization methods)

Generalized Maxwell model

N.B. **The parameters of the model can be obtained from experimental creep data using adequate curve fitting techniques** (together with some regularization methods)



Relaxation Function:

$$\boxed{\sigma(t) = \varepsilon_0 H(t) \cdot \sum_{i=1}^{n} E_i e^{-\frac{E_i}{\eta_i}t}}$$

Total stress:

$$\boxed{\sigma(t) = \sum_{i=1}^{n} \sigma_i}$$

H(*t*) - Heaviside step function, or the unit-step function*

$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

Generalized Maxwell model

An additional spring can be added into Maxwell chain

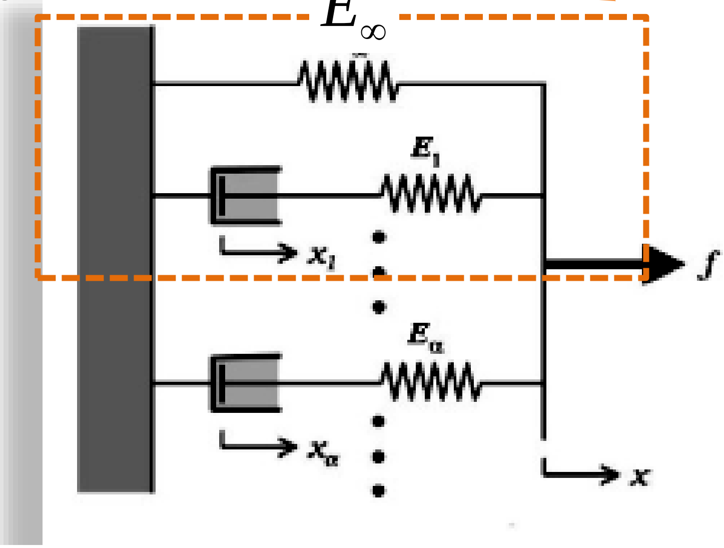
Useful to know

***N.B.* The parameters of the model can be obtained from experimental creep data using adequate curve fitting techniques** (together with some regularization methods)

Note the additional elastic element! Why?**

Long-term modulus E_∞

This corresponds to a Kelvin Standard Linear Solid Model (another equivalent configuration of SLS)



Generalized Maxwell rheological model

(known as *The Wiechert model* or Maxwell-Wiechert Model):

$$\sigma_i(t) = \varepsilon_0 \cdot \underbrace{[E_\infty + \overbrace{E_i \exp(-t/\tau_i)]}^{\text{Relaxation function for a single Maxwell element}}}_{\text{Relaxation function}}$$

The *Prony** series for **Relaxation Modulus:**

Instantaneous elastic modulus: E_0

Long-term modulus: E_∞

$$G(t) = E_\infty + \sum_{i=1}^{N} E_i e^{-t/\tau_i},$$

$$= E_0 - \sum_{i=1}^{N} E_i \left(1 - e^{-t/\tau_i}\right),$$

Relaxation times: $\tau_i \equiv \dfrac{E_i}{\eta_i}$

since $G(0) \equiv E_0 = E_\infty + \sum_{i=1}^{N} E_i$

***Numerical implementation:**

In many Finite Elements Codes (Abaqus, Ansys, …), viscoelasticity is implemented through ***Prony*** series.

The *shear* and *volumetric* responses are *separated*.

****Answer**: *to avoid the asymptotic 'elastic' modulus tends to zero for a **solid material***

Recall: Linear viscoelastic deformation or stress can be described using the Boltzmann integral representation:

Memory in form of a convolution product

relaxation form:

$$\sigma(t) = \int_{-\infty}^{t} E(t-t')\dot{\varepsilon}(t')dt'$$

$E(t)$ - **relaxation modulus**

(an other notation $G(t)$)

Retardation form:

$$\varepsilon(t) = \int_{-\infty}^{t} D(t-t')\dot{\sigma}(t')dt'$$

$D(t)$ – **creep compliance**

(an other notation $J(t)$)

3D-formulation of constitutive law:

time-dependent stress (viscous)

$$\boldsymbol{\sigma}(t) = \boxed{\int_{t'=0}^{t'=t} 2G(t-t')\dot{\mathbf{e}}_{\mathrm{dev}}(t')\mathrm{d}t' \quad e^{\mathrm{vol}} = \varepsilon_{\mathrm{kk}}\delta_{\mathrm{ij}},} + \boxed{\mathbf{I}\int_{t'=0}^{t} K(t-t')\dot{e}_{\mathrm{vol}}(t')\mathrm{d}t' \quad e_{\mathrm{ij}}^{\mathrm{dev}} = \varepsilon_{\mathrm{ij}} - \varepsilon_{\mathrm{kk}}/3\delta_{\mathrm{ij}}}$$

K, G - relaxation functions

Deviatoric response

Volumetric response

All the needed information concerning time-dependent behavior of the material is *encoded* in these two viscoelastic functions: the ***relaxation modulus*** $E(t)$ or the ***creep compliance*** $D(t)$.

3D-formulation - isotropic viscoelasticity

3D-formulation of constitutive law:

$$\boldsymbol{\sigma}(t) = \int_{t'=0}^{t'=t} 2G(t-t')\dot{\mathbf{e}}_{\mathrm{dev}}(t')\mathrm{d}t' + \mathbf{I}\int_{t'=0}^{t} K(t-t')\dot{e}_{\mathrm{vol}}(t')\mathrm{d}t'$$

$$e^{\mathrm{vol}} = \varepsilon_{\mathrm{kk}}\delta_{\mathrm{ij}}, \quad e^{\mathrm{dev}}_{\mathrm{ij}} = \varepsilon_{\mathrm{ij}} - \varepsilon_{\mathrm{kk}}/3\delta_{\mathrm{ij}}$$

K, G - relaxation functions

Example - Kelvin-Voigt solid

Uniaxial **viscoelasticity:**

$$\sigma = \sigma^{\mathrm{e}} + \sigma^{\mathrm{an}}$$

$$\sigma^{\mathrm{e}} = E\varepsilon \qquad \sigma^{\mathrm{ne}} = \eta\dot{\varepsilon}$$

$$\sigma = E\varepsilon + \eta\dot{\varepsilon}$$

Often, ***deviatoric*** (shear) and ***volumetric*** (bulk) components (strains, stress) are *separated* since their respective viscoelastic material responses can be different.

Multi-axial isotropic viscoelasticity:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{e}} + \boldsymbol{\sigma}^{\mathrm{an}}$$

$$\boldsymbol{\sigma}^{\mathrm{e}} = \lambda\mathrm{Tr}(\boldsymbol{\varepsilon})1 + 2\mu\boldsymbol{\varepsilon}$$

$$\boldsymbol{\sigma}^{\mathrm{an}} = \lambda\tau_\lambda\mathrm{Tr}(\dot{\boldsymbol{\varepsilon}})\mathbf{1} + 2\mu\tau_\mu\dot{\boldsymbol{\varepsilon}}$$

$\tau_\lambda, \tau_\mu -$ characteristic retardation times;

These **parameters** can be identified from uniaxial tensile test and shear test (torsion of tube, for instance). They may be $\tau_\lambda \neq \tau_\mu$ or $\tau_\lambda = \tau_\mu \equiv \tau$

$$\boxed{\sigma_{\mathrm{ij}} = \lambda\left(\varepsilon_{\mathrm{kk}} + \tau_\lambda\dot{\varepsilon}_{\mathrm{kk}}\right)\delta_{\mathrm{ij}} + 2\mu\left(\varepsilon_{\mathrm{ij}} + \tau_\mu\dot{\varepsilon}_{\mathrm{ij}}\right)}$$

$K_\infty, G_\infty -$ long-term *bulk* and shear *moduli*

$K_0, G_0 -$ instantaneous *bulk* and *shear* moduli

Lamé-parameters:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)},$$

$$\mu = G = \frac{E}{2(1+\nu)} = K - \frac{2}{3}G$$

Bulk modulus: $K = \lambda + 2\mu/3,$

3D-formulation of isotropic viscoelasticity

Example - Kelvin-Voigt solid

Uniaxial viscoelasticity:

$$\sigma = \sigma^{\mathrm{e}} + \sigma^{\mathrm{an}}$$

$$\sigma^{\mathrm{e}} = E\varepsilon \qquad \sigma^{\mathrm{ne}} = \eta\dot{\varepsilon}$$

$$\sigma = E\varepsilon + \eta\dot{\varepsilon}$$

Deviatoric (shear) and ***volumetric*** (bulk) components (strains, stress) are *separated* since their respective viscoelastic material responses may be different.

The material model parameters in **relaxation functions** should be identified from uniaxial tensile tests, shear tests and/or dynamic tests.

Shear test:

$$\sigma_{12} = 2\mu\left(\varepsilon_{12} + \tau_\mu \dot{\varepsilon}_{12}\right)$$

Uniaxial tensile test:

$$\sigma_{11} = \underbrace{\left[\lambda(1-2\nu) + 2\mu\right]}_{=E}\varepsilon_{11} + \ \ldots$$

$$\ldots\ + \underbrace{\left[\lambda(1-2\nu)\tau_\lambda + 2\mu\tau_\mu\right]}_{\equiv\eta}\dot{\varepsilon}_{11}$$

(for a Generalized Maxwell model)

Multi-axial isotropic viscoelasticity:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{e}} + \boldsymbol{\sigma}^{\mathrm{an}}$$

$$\boldsymbol{\sigma}^{\mathrm{e}} = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}$$

$$\boldsymbol{\sigma}^{\mathrm{an}} = \lambda\tau_\lambda \mathrm{Tr}(\dot{\boldsymbol{\varepsilon}})\mathbf{1} + 2\mu\tau_\mu\dot{\boldsymbol{\varepsilon}}$$

τ_λ, τ_μ – characteristic retardation times;

These **parameters** can be identified from uniaxial tensile test and shear test (torsion of tube, for instance). They may be $\tau_\lambda \neq \tau_\mu$ or $\tau_\lambda = \tau_\mu \equiv \tau$

$$\boxed{\sigma_{\mathrm{ij}} = \lambda\left(\varepsilon_{\mathrm{kk}} + \tau_\lambda\dot{\varepsilon}_{\mathrm{kk}}\right)\delta_{\mathrm{ij}} + 2\mu\left(\varepsilon_{\mathrm{ij}} + \tau_\mu\dot{\varepsilon}_{\mathrm{ij}}\right)}$$

The *relaxation functions* can be defined individually in terms of Prony series:

$$G_\mu(t) = G_\infty + \sum_{\mathrm{i}=1}^{\mathrm{N}_\mu} G_{\mathrm{i}} e^{-t/\tau_{\mathrm{i}}^{\mathrm{G}}}, \qquad \text{bulk}$$

$$G_\lambda(t) = K_\infty + \sum_{\mathrm{i}=1}^{\mathrm{N}_\lambda} K_{\mathrm{i}} e^{-t/\tau_{\mathrm{i}}^{\mathrm{K}}}, \qquad \text{shear}$$

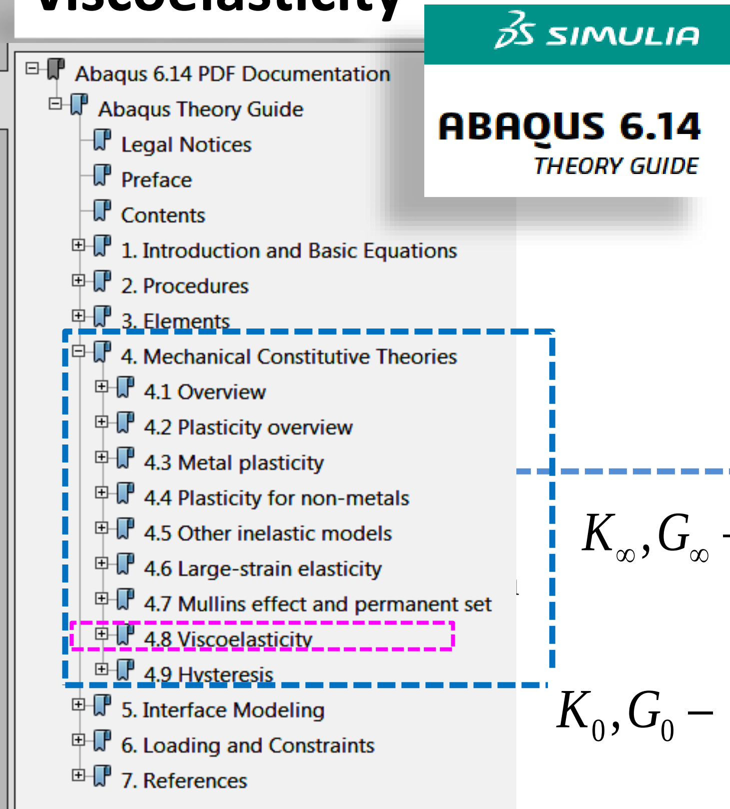
K_∞, G_∞ – long-term **bulk** and **shear** moduli

K_∞, G_∞ – long-term *bulk* and shear *moduli*

K_0, G_0 – instantaneous *bulk* and *shear* moduli

In many practical cases it can be assumed that and ı$\mathrm{N}_\lambda = 0$ $G_\lambda(t) \approx K_\infty$,

Example of 3D-formulation of isotropic viscoelasticity



Lamé-parameters:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)},$$

$$\mu = G = \frac{E}{2(1+\nu)} = K - \frac{2}{3}G$$

Bulk modulus:

$$K = \lambda + 2\mu/3,$$

Deviatoric (shear) and *volumetric* (bulk) components (strains, stress) are *separated* since their respective viscoelastic material responses may be different.

For instance, in Abaqus, the *parameters* – characterizing the viscoelastic response of the *material* - in these *Prony* series are inputted as material data.

K_∞, G_∞ – long-term *bulk* and shear *moduli*

K_0, G_0 – instantaneous *bulk* and *shear* moduli

$$G_\mu(t) = G_0 - \sum_{i=1}^{N_\mu} G_i\left(1 - e^{-t/\tau_i^G}\right),$$

$$G_\lambda(t) = K_0 - \sum_{i=1}^{N_\lambda} K_i\left(1 - e^{-t/\tau_i^K}\right),$$

or equivalently

(for a Generalized Maxwell model)

The relaxation functions can be defined individually in terms of Prony series:

$$G_\mu(t) = G_\infty + \sum_{i=1}^{N_\mu} G_i e^{-t/\tau_i^G},$$

$$G_\lambda(t) = K_\infty + \sum_{i=1}^{N_\lambda} K_i e^{-t/\tau_i^K},$$

K_∞, G_∞ – long-term bulk and shear moduli

In many practical cases it can be assumed that and $N_\lambda = 0$ $G_\lambda(t) \approx K_\infty$,

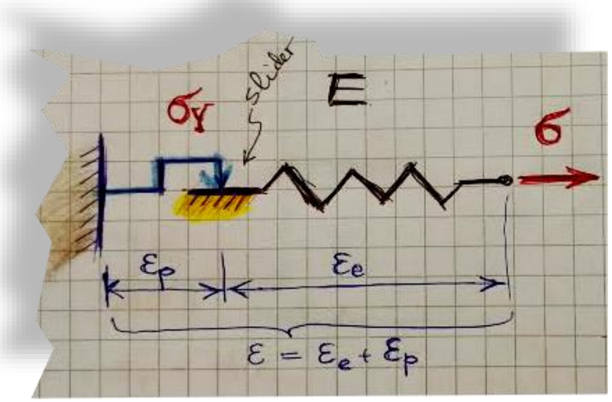
3D-formulation of constitutive law:

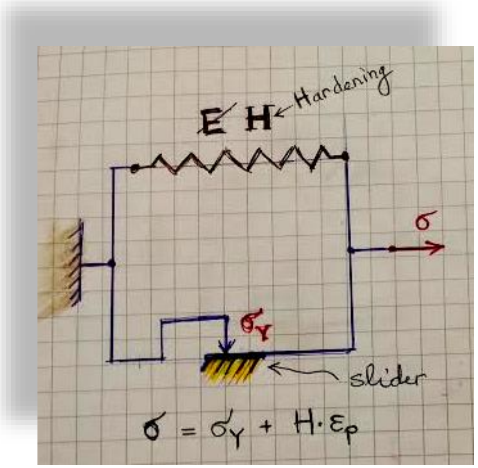
$$\boldsymbol{\sigma}(t) = \int_{t'=0}^{t'=t} 2G(t-t')\dot{\mathbf{e}}_{dev}(t')dt' + \mathbf{I}\int_{t'=0}^{t} K(t-t')\dot{e}_{vol}(t')dt'$$

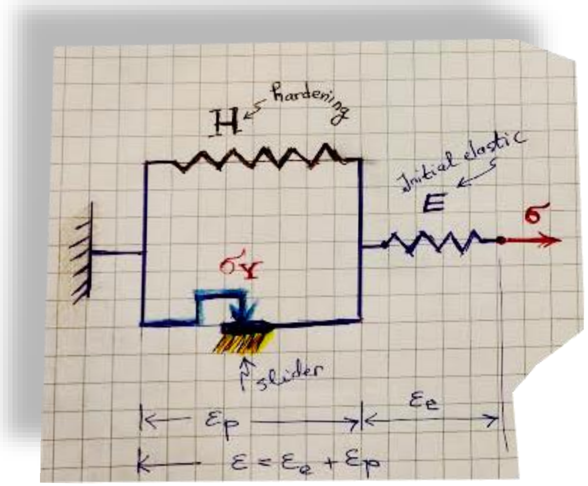
$$e^{vol} = \varepsilon_{kk}\delta_{ij}, \qquad e_{ij}^{dev} = \varepsilon_{ij} - \varepsilon_{kk}/3\delta_{ij}$$

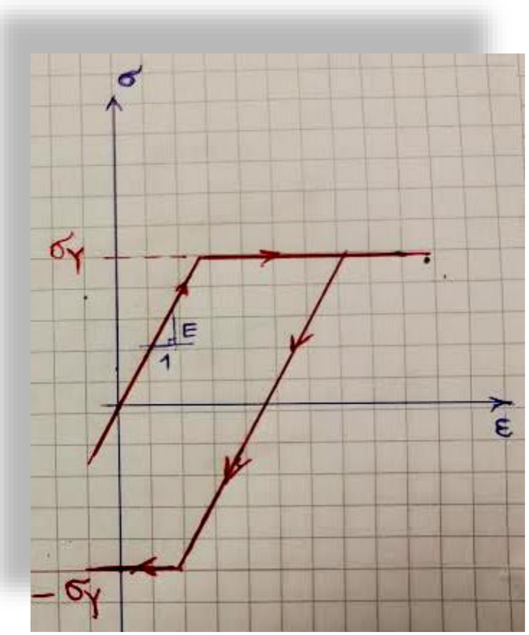
Simple rheological models for Rate-independent plasticity

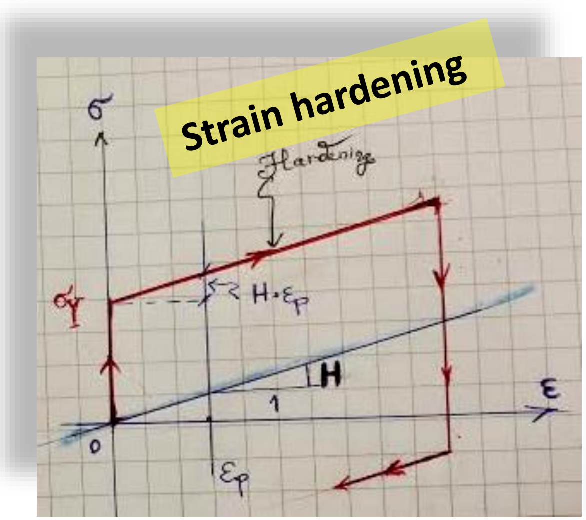
Some strain hardening models

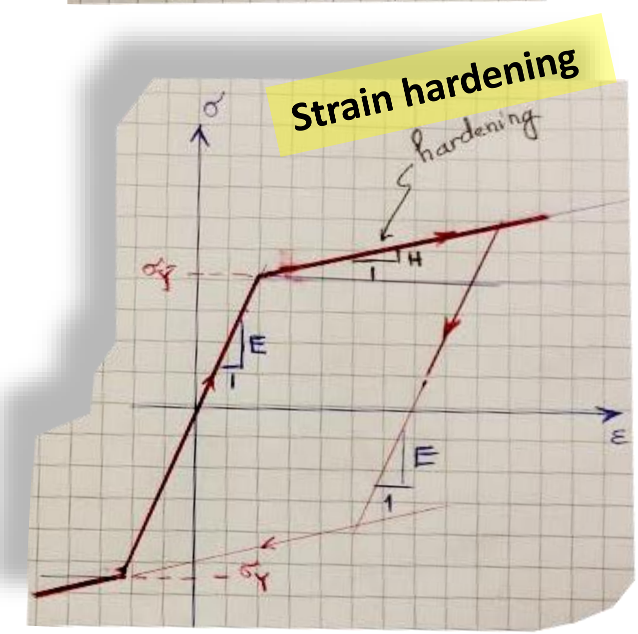










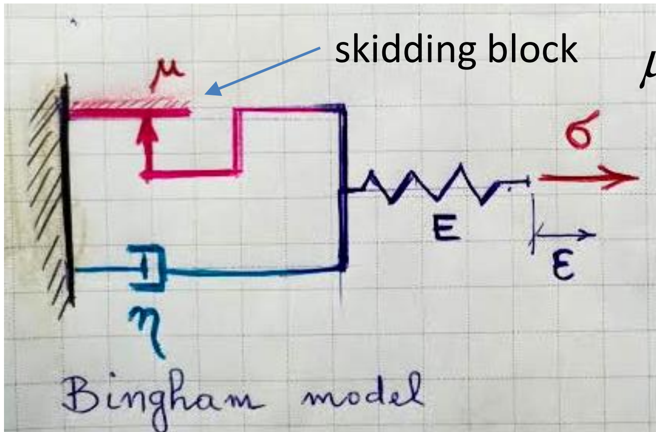


N.B. insert 'dashpots' to obtain rate-dependent plasticity as for Bingham model

Simple rheological models

for Rate-dependent plasticity

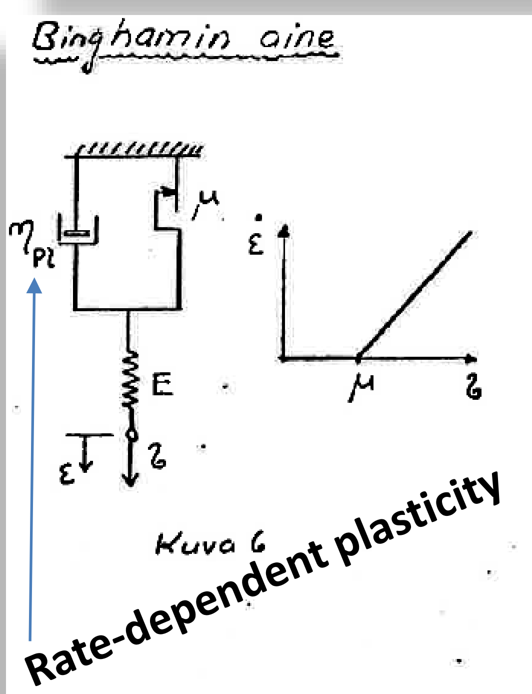
Bingham model – is often used to model combined elastic **visco-plasticity** behavior



$\mu \equiv \sigma_Y$ – **Yield point** as 'static-friction'

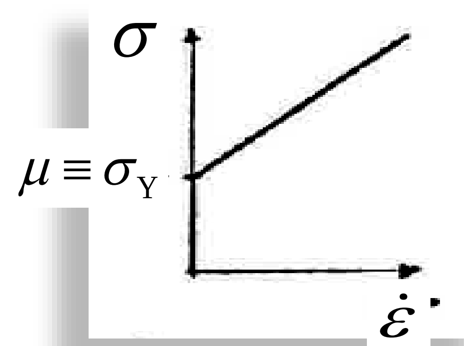
$$\sigma = \begin{cases} E\varepsilon, & |\sigma| < \mu \equiv \sigma_Y \\ \mu + \eta\dot{\varepsilon}, & |\sigma| \geq \mu > 0 \end{cases}$$

* *Viscoplastic solids* exhibits permanent deformation after applying a load (as plastic solids) and continue to creep as function of time under the applied load (so, no equilibrium can be achieved)



B-aine on yksinkertainen visko-elasto-plastinen aine, ns. plasto-dynamiikan perusaine. Mallin konst. yhtälö löydetään seur. yksinkertaisella tarkastelulla. Kun $\sigma < \mu$, toimii vain jousi. Kun $\sigma \geq \mu$, tapahtuu viskoosi virtaus voiman $\sigma - \mu$ vaikuttaessa nestesylinteriin. Konst. yhtälö voidaan siten kirj. muotoon

$$\left.\begin{array}{l} \sigma = E\varepsilon, \text{ kun } |\sigma| < |\mu| \\ \sigma - \mu = \eta_{Pl}\dot{\varepsilon}, \text{ kun } |\sigma| \geq |\mu| \end{array}\right\}$$



Constitutive Models
with viscosity

Overview

To be re-edited

- Newtonien (métaux liquides, thermodurcissables liquides, thermoplastiques peu visqueux) :

$$\tau = \eta\dot{\gamma}$$

- Newtonien généralisé (polymères, poudre avec liant) :

$$\tau = \eta(\dot{\gamma}, T)\dot{\gamma}$$

ce qui inclut le comportement pseudoplastique :

$$\tau = K(T)\,\dot{\gamma}^m$$

- Newtonien avec seuil (Bingham) (bétons) :

$$\tau = \tau_0 + K\dot{\gamma}$$

- Pseudoplastique avec seuil (Herschel-Bulkley) (bétons) :

$$\tau = \tau_0 + K\dot{\gamma}^m$$

- Viscoélastique linéaire (polymères, verres). Loi de Maxwell :

$$\sigma + \theta\frac{d\sigma}{dt} = \eta\dot{\varepsilon}$$

θ étant le temps de relaxation.

Ces lois sont utilisées dans la simulation numérique des procédés.

Useful to know

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Applied and Computational Mechanics 1 (2007) 175 - 184

Applied and Computational Mechanics

On the modeling of the linear viscoelastic behaviour of biological materials using Comsol Multiphysics

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Received 23 August 2007; received in revised form 4 October 2007

Abstract

The aim of this article is to simulate the viscoelastic behaviour during quasi-static equilibrium processes using the software *Comsol Multiphysics*. Two-dimensional constitutive equations of the three most classical rheological models (the *Kelvin-Voigt*, *Maxwell* and *Three-Elements* models) are developed and the mechanical equations for each model are rewritten to be compatible with the predefined Comsol format in the so-called *Coefficients mode*. Results of the simulations on creep and relaxation tests are compared with experimental data obtained on a sample of biological tissue.

Keywords: viscoelasticity, mechanical equilibrium, Comsol modeling

Useful to know

viscoelasticity, time-domain viscoelasticity, frequency-domain viscoelasticity, rheological model, generalized maxwell model, prony series expansion, asphalt concrete, complex modulus, flexible pavement design

MICHALCZYK Rafał[1]

IMPLEMENTATION OF GENERALIZED VISCOELASTIC MATERIAL MODEL IN ABAQUS CODE

An accurate description of behavior of bituminous mixes is necessary to adequately predict and evaluate the time dependent characteristics and the evolution of pavement distress. The generalized Maxwell model, currently considered to be one of the most suitable to characterize mechanical behavior of viscoelastic materials and widely used in many commercial FEM codes, is presented. A Prony series expansion implemented in ABAQUS code, is used to express the material's behavior. The theoretical background of the model is briefly discussed. Finally, the model was efficiently used for numerical simulation of stress relaxation experiment.

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2.2 Material characterization

Typically, two types of testing methods are used to characterize the viscoelastic properties of asphalt concrete: dynamic modulus testing and constant static testing. In the first case, the material is characterized by the complex module *(E*)* and the phase angle (f). Under constant static load, the viscoelastic behavior is characterized through the creep compliance or the relaxation modulus. In this research, dynamic modulus testing was chosen as the measure of the complex modulus of HMA materials. Values presented in Tab.1 were extracted from literature [3]. The material is characterized by the phase angle *(f)* and the complex modulus *(E*)* which can be expressed as storage modulus and loss modulus using the following relation (7)

$$E^* = E' \cos(\varphi) + E'' i \sin(\varphi)$$

Numerical implementation

In many modern Finite Elements Systems (Abaqus, Ansys), viscoelasticity is implemented through the use of *Prony* series.

The shear and volumetric responses are separated.

Ref: "00_implementation_abaqus_viscoielasticity.pdf" in MyCourses

A useful analytic approximation* of the Heaviside function

Useful to know

H(*t*) **- Heaviside step function, or the unit-step function*:**

$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

Or using the half-maximum convention:

$$H(t) = \begin{cases} 0, & t < 0, \\ \frac{1}{2}, & t = 0, \\ 1, & t > 0 \end{cases}$$

An analytic approximation:

$$H(x) \approx \frac{1}{2} + \frac{1}{2}\tanh kx = \frac{1}{1+e^{-2kx}},$$

k – A free parameter chosen adequately

$$H(x) = \lim_{k\to\infty} \frac{1}{2}(1+\tanh kx) = \lim_{k\to\infty} \frac{1}{1+e^{-2kx}}.$$

* Such smooth approximation can be used in numerical simulations to avoid loading 'brutally' with the step-function (avoids spurious numerical problems)

Homework and examples

Maxwell model: dynamic response – example/homework?

$\epsilon = \epsilon_0 \cos \omega t$

$\sigma = \sigma_0 \cos(\omega t + \delta)$

mmon in the analysis of react
ıt to write the stress function
ain and whose imaginary part

$\sigma^* = \sigma_0' \cos \omega t + i\, \sigma_0'' \sin \omega t$

Γhe complex modulus c

$E^* = \sigma_0^* / \epsilon_0^*$

$\sigma = \sigma_0^* e^{i\omega t}$

$\epsilon = \epsilon_0^* e^{i\omega t}$

t of these is the "real," or "storage,"
strain:

$E' = \sigma_0' / \epsilon_0$

"imaginary," or "loss," modulus, define

$E'' = \sigma_0'' / \epsilon_0$

Maxwell material,

Multiplying by k and using $\tau = \eta/k$:

$$\boxed{k\dot{\epsilon} = \dot{\sigma} + \frac{1}{\tau}\sigma}$$

Example 3

In the case of the dynamic response, the time dependency of both the stress and the strain are of the form $\exp(i\omega t)$. All time derivatives will therefore contain the expression $(i\omega)\exp(i\omega t)$, so Eqn. 22 gives:

$$k\,(i\omega)\,\epsilon_0^* \exp(i\omega t) = \left(i\omega + \frac{1}{\tau_j}\right)\sigma_0^* \exp(i\omega t)$$

The complex modulus E^* is then

$$E^* = \frac{\sigma_0^*}{\epsilon_0^*} = \frac{k(i\omega)}{i\omega + \frac{1}{\tau_j}} = \frac{k(i\omega\tau)}{1 + i\omega\tau} \tag{24}$$

This equation can be manipulated algebraically (multiply and divide by the complex conjugate of the denominator) to yield:

$$E^* = \frac{k\omega^2\tau^2}{1+\omega^2\tau^2} + i\frac{k\omega\tau}{1+\omega^2\tau^2} \tag{25}$$

In Eq. 25, the real and imaginary components of the complex modulus are given explicitly; these are the "Debye" relations also important in circuit theory.

Homework – first week

Exercise 0: A material can be described by a In a Kelvin-Voigt (KV) model. Consider **the creep response** of a such material for a constant stress σ_0 . Such response is modeled by the following expression:

$$\varepsilon(t) = \frac{\sigma_0}{E}[1 - e^{-t/\tau_c}], \qquad \tau_c \equiv \frac{\eta}{E} \quad \text{- retardation time}$$

η - viscosity of the dashpot

1) Show the result above for the creep response of the material.

2) **Consider the following creep test:** a material, having E = 600 MPa, is loaded initially with the constant stress σ_0 . The constitutive behavior (behavior law) of the material can described by Kelvin-Voigt (KV) model.

Half an hour (t = 30 min) after applying stress, the measured strain is 0.111
An hour (t = 90 min) later, the strain becomes 0.264.

Determine the strain after three hours of loading?
After what time the strain reached, initially, 0.001?

Homework – first week

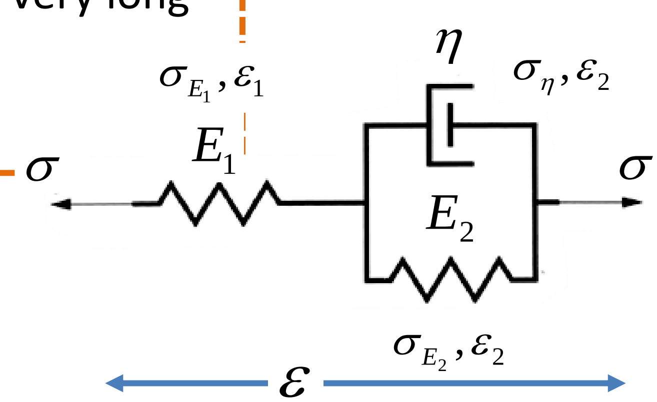
Exercise 1: Describe concisely the difference between a **Creep Test** and a **Stress Relaxation Test**.
Answer: refer to your lectures. Creep test has constant stress and relaxation test has constant strain…

Problem 1:
a) Consider the "standard linear solid" (SLS). **Derive the constitutive equation** relating the overall stress, stress rate, strain and strain rate. The model parameters of the system are E_1, E_2 and η

b) For the standard linear solid discussed earlier, determine the expression of the total strain ε(t) in terms of E_1, E_2 and η for the case of constant stress. What is the creep function?

c) As you may remember, *retardation time* is defined as $\tau = \eta/E$ [s]. Consider the following situation: immediately after applying stress, the strain is 0.002 (instantaneous strain), after 1000 seconds the strain grows to 0.004 and approaches asymptotically 0.006 after a very long time the strain.
Determine the retardation time τ?

Standard Linear Solid

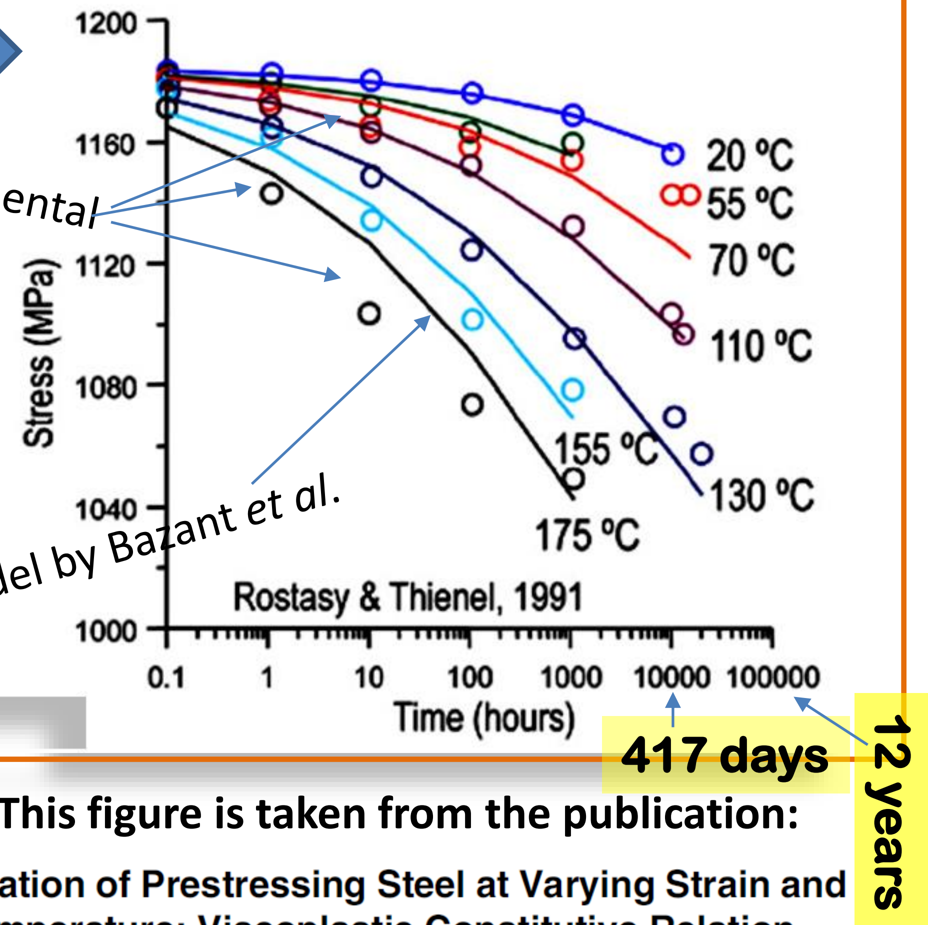


Stress Relaxation in presestressing steel

Introduction to a computation example

Experimental evidence of stress relaxation in steel at various temperatures – measurements shown by the small circles (o)

o - Rostasy and Thienel (1991) tests



NB. - relaxation increases with the rise of temperature

- the relaxation increases also with the prestressing level (stress level)

This figure is taken from the publication:

Relaxation of Prestressing Steel at Varying Strain and Temperature: Viscoplastic Constitutive Relation

Zdeněk P. Bažant, Hon.M.ASCE[1]; and Qiang Yu[2]

814 / JOURNAL OF ENGINEERING MECHANICS © ASCE / JULY

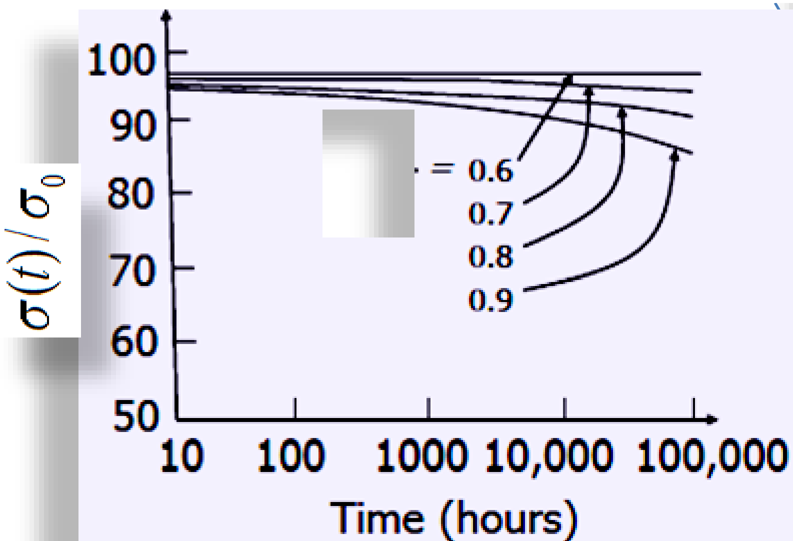
Examples or Homework

Restressing steel …*wire strands* … and *stress relaxation*

Some additional context:

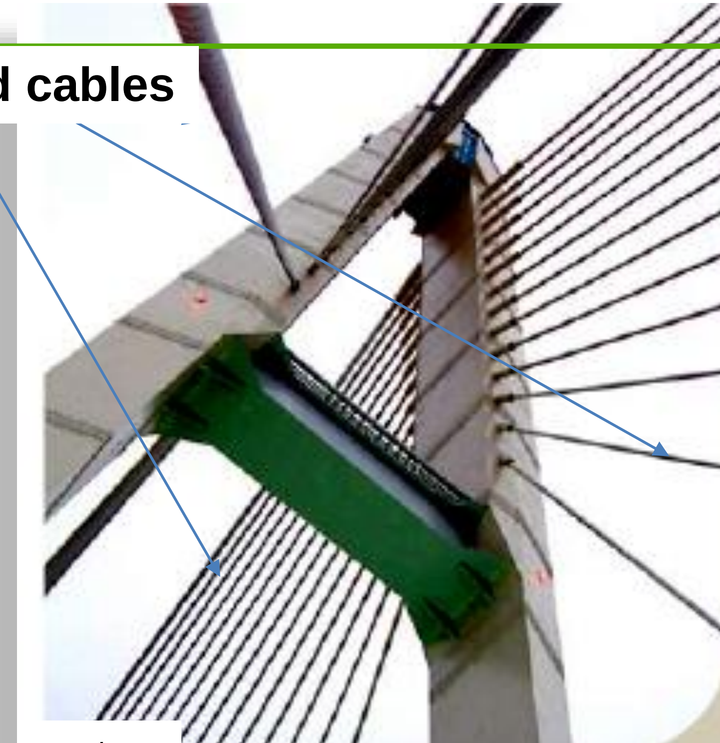
Prestressed cables

there is a significant relaxation loss when applied stress is more than 70% of the yield stress.



Relative variation of stress with time for various prestressing levels

σ_0/σ_Y



Maxwell model relaxation function – the simplest model:

τ_R *-the characteristic relaxation time of the material*

$$\sigma(t) = \varepsilon_0 \cdot E e^{-\frac{E}{\eta}t} = \sigma_0 e^{-\frac{E}{\eta}t} \equiv \sigma_0 e^{-t/\tau_R}$$

Recall: Maxwell chain or models are suitable for modelling stress relaxation phenomena

Examples or Homework:

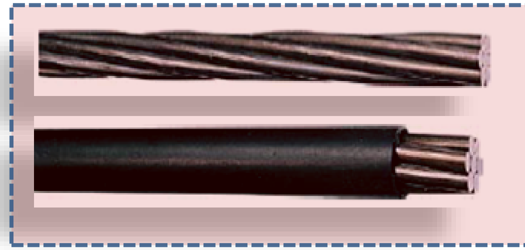
Restressing Steel …wire strands … and *stress relaxation*

Relaxation experiment for the material: (observation)

- After 2 weeks, a loss of 2 MPa is observed in the material of the cable while the initial stress was 100 MPa.

Problem:

- **Derive relaxation function** (modulus) – use simple *Maxwell model*
- Determine from relaxation experiment characteristic relaxation time
- What should be the initial pre-stress level in order to keep > 150 MPa over one year?
- Assume a constant operating temperature of ~20 °C



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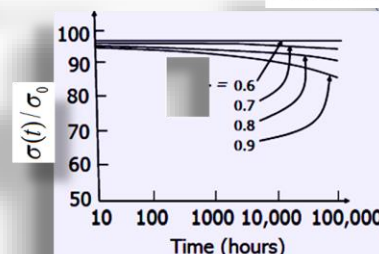
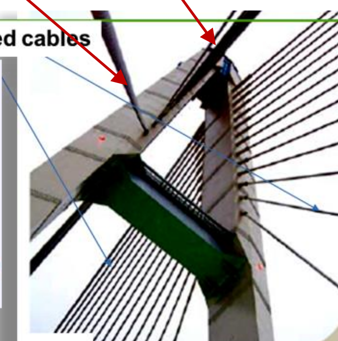


Relative variation of stress with time for various prestressing levels σ_0/σ_Y

Prestressed cables



Maxwell model relaxation function – the simplest model:

τ_R *-the characteristic relaxation time of the material*

$$\sigma(t) = \varepsilon_0 \cdot E e^{-\frac{E}{\eta}t} = \sigma_0 e^{-\frac{E}{\eta}t} \equiv \sigma_0 e^{-t/\tau_R}$$

Maxwell models are suitable for modelling stress relaxation phenomena

Jänneteräksen laskentalujuus

Y 1860 S 7

- 7 = lankojen lukumäärä (2/3/7)
- S = Punos (C = Lanka, H = Tanko)
- 1860 = f_{pk} vetolujuus [MPa]
- Y = Jänneteräs

Useful to know

Jänneteräksen kimmomoduulin arvona käytetään:
- •195 GPa punoksille
- •205 GPa langoille ja tangoille

In the exercise, I will round to 200 GPa

Nimike	f_{pk}
Y1770S7	1770
Y1860S7	1860

For a constant temperature ~20 oC

Relaksaatio

Relaksaatiossa jänneteräksen jännitys pienenee venymän pysyessä vakiona. Jänneteräkset on eurokoodissa luokiteltu kolmeen relaksaatioluokkaan.

Luokka 1	Langat ja punokset joilla tavanomainen relaksaatio
Luokka 2	Langat ja punokset joilla pieni relaksaatio
Luokka 3	Kuumavalssatut ja muokatut tangot

Jänneteräksen mitoituslaskelmissa käytetään arvoa ρ_{1000}. ρ_{1000} kuvaa relaksaatiohäviötä, joka tapahtuu ensimmäisen 1000 tunnin aikana +20°C lämpötilassa.

Relaksaatioluokka	ρ_{1000}
Luokka 1	8%
Luokka 2	2,5 %
Luokka 3	4 %

Suomessa käytettävät langat ja punokset kuuluvat

Relaksaatiohäviön arvo tietyllä ajanhetkellä voidaan laskea seuraavilla kaavoilla:

Luokka 1: $$\frac{\Delta\sigma_{pr}}{\sigma_{pi}} = 5{,}39\rho_{1000}e^{6{,}7\mu}\left(\frac{t}{1000}\right)^{0{,}75(1-\mu)}10^{-5}$$

Luokka 2: $$\frac{\Delta\sigma_{pr}}{\sigma_{pi}} = 0{,}66\rho_{1000}e^{9{,}1\mu}\left(\frac{t}{1000}\right)^{0{,}75(1-\mu)}10^{-5}$$

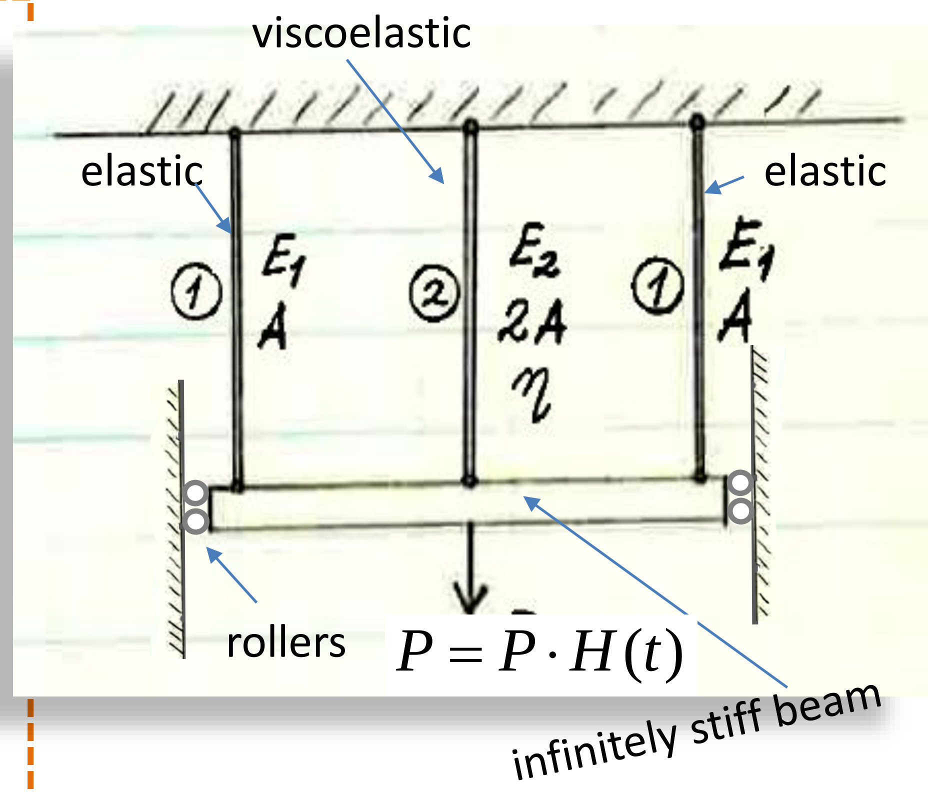
Luokka 3: $$\frac{\Delta\sigma_{pr}}{\sigma_{pi}} = 1{,}98\rho_{1000}e^{8\mu}\left(\frac{t}{1000}\right)^{0{,}75(1-\mu)}10^{-5}$$

missä
$\Delta\sigma_{pr}$= jännityksen relaksaatiohäviöiden itseisarvo
σ_{pi}= jänteeseen laukaistu vetojännitys vähennettynä jännittämisen aikaisilla välittömillä häviöillä
$\mu = \sigma_{pi}/ f_{pk}$
t = jännittämisen jälkeinen aika tunteina

Relaksaatiohäviöiden pitkäaikaisarvo vastaa eurokoodin mukaan betonin ikää 500 000h (n. 57vuotta)

Homework or example – first week

$A-$ Cross section area



Exercise:

Consider the mechanic system formed by the three vertical slender bars #1, #2 and #3 in tension. The horizontal beam is infinitely stiff and remains horizontal during the motion. The constitutive equations of the materials are given bellow.

Determine the time dependent member forces when loaded quasi-statically by a constant force $P = P \cdot H(t)$. (inertia terms are ignored.)

The material behavior (constitutive law)

Member #1: $\varepsilon = \sigma / E_1$ (Hooke element)

Member #2: $\dot{\varepsilon} = \dot{\sigma} / E_2 + \sigma / \eta$ (Maxwell element)

Constant initial loading:

$$P = P \cdot H(t)$$

$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

Homework – short RC column

Consider the short reinforced concrete column concentrically loaded by a constant force compressive $P = P \cdot H(t)$, where $H(t)$ Being the Heaviside unit-step function.

The material behavior (Constitutive laws)

Steel: considered *elastic* as compared to the concrete for the time durations considered here.

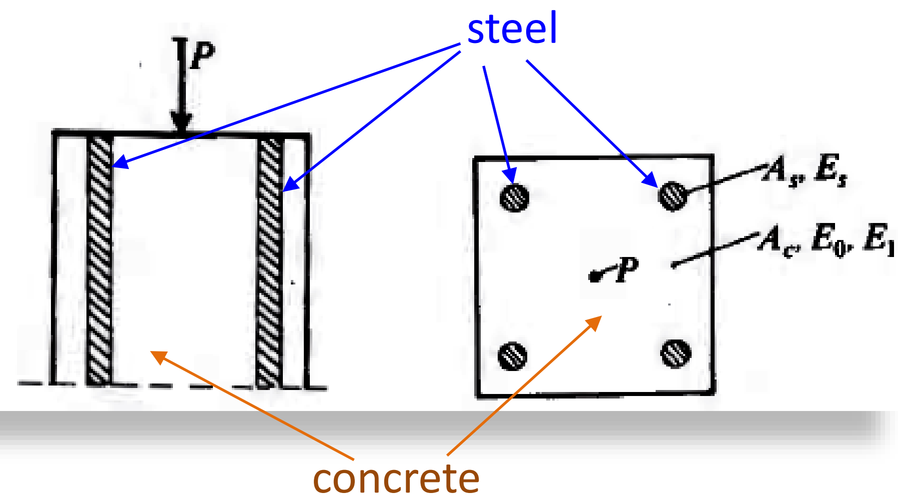
$$\sigma_s = E_s \varepsilon_s$$

Concrete: *viscoelastic* having obeying constitutive law of a Standard Linear Solid (SLS) in the form*

$$\dot{\sigma}_c + \frac{E_2}{\eta}\sigma_c = E_0 \dot{\varepsilon}_c + \frac{E_1 E_2}{\eta}\varepsilon_c$$

where the initial elastic modulus is

$$E_0 \equiv E_1 + E_2$$



Assume that the steel reinforcement is perfectly bonded to the concrete $\varepsilon_s = \varepsilon_c$.

Question: determine the stresses separately in the concrete σ_c and in the reinforcement σ_s .

Hint: compatibility and equilibrium… nothing new since almost 5000 y.

Cross-section area

$$A_c + A_s = A$$

$$A_s, \; A_c \approx A$$

Steel ratio:

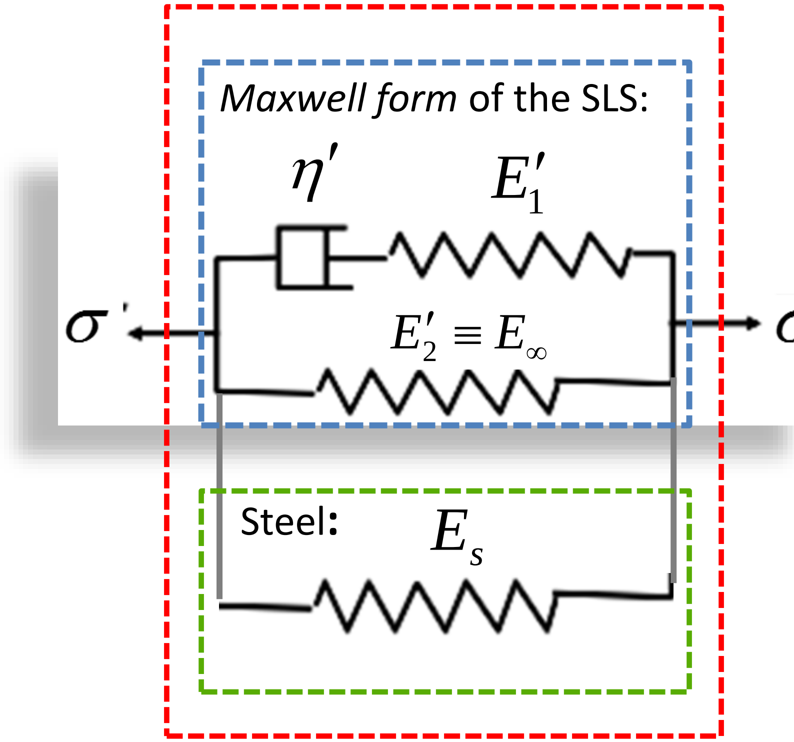
$$n \equiv A_s / A$$

Remark:

Concrete: *viscoelastic* having obeying constitutive law of a Standard Linear Solid (SLS) in the form* written for the equivalent *Maxwell form* of the SLS which is shown later as:

$$\dot{\sigma}_c + \frac{E_1'}{\eta'}\sigma_c = \underbrace{\left(E_1' + E_\infty'\right)}_{\equiv E_0}\dot{\varepsilon}_c + \frac{E_1' E_\infty'}{\eta'}\varepsilon_c$$

Steel: *elastic* $\sigma_s = E_s \varepsilon_s$



Composite: steel and concrete – reinforced concrete:

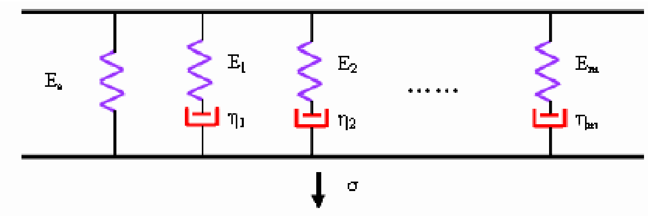
Good to know

Free homework: FE-simulations using COMSOL or other FE-software as Abaqus, Ansys, Lusas, etc.

Extra for those who wish: solve the pavement problem using Abaqus, Ansys or Lusas or any other FE-software

Assume a *isothermal* conditions for this problem.

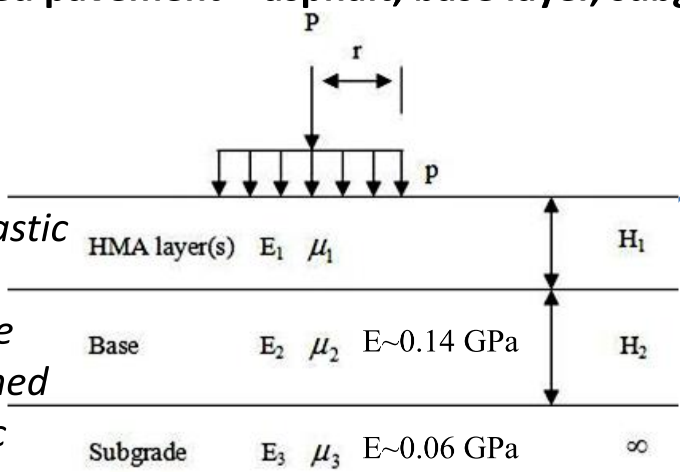
Generalized Maxwell model



Layered pavement – asphalt, base layer, subgrade soil

viscoelastic

Can be assumed elastic



$H_1 \approx 30-50$ mm

$H_2 \approx 150-450$ mm

Poisson's ratio ~0.40 for aggregates, and 0.48 for subgrade soils.

Layer	Calibrated Instantaneous Elastic modulus (GPa)	Measured Relaxation Modulus (GPa)
Base	0.1379$	0.069 – 0.276*$
Subgrade	0.0552$	0.0517$

$ These are resilient moduli.

Good to know

Ref: Material data from:

VISCOELASTIC FE MODELING OF ASPHALT PAVEMENTS AND ITS APPLICATION TO U.S. 30 PERPETUAL PAVEMENT

the Department of Civil Engineering and the Russ College of Engineering and Technology by

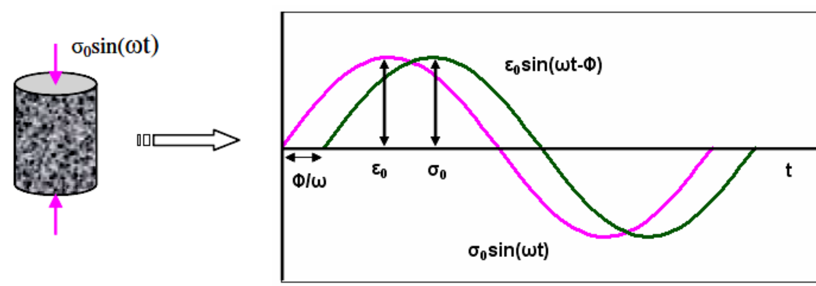
by

YUN LIAO

Doctor of Philosophy
Yun Liao
November 2007

Perform a short-term and a long-term analysis.
Assume reasonable thicknesses of the layers.
Determine stress and settlement in time.

Viscoelastic material parameters are calculated from complex modulus to incorporate into the FE model



Example of measurement of complex modulus

Good to know

[1] ***Ref***: Material data from: PhD-thesis:

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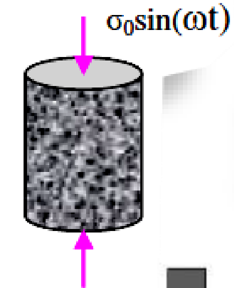
$$|G^*| \equiv |E^*| \qquad \varphi \equiv \delta$$

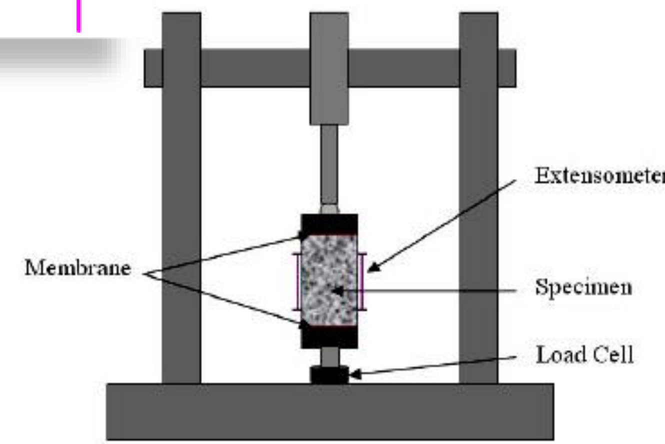
$$|G^*| = \sigma_0 / \varepsilon_0$$

frequency — Dynamic Modulus — phase angle

f (1/sec)	-10ºC		4.4ºC		21.1ºC		37.8ºC		54.4ºC	
	$\|E^*\|$, GPa	φ(º)	$\|E^*\|$, GPa	φ(º)	$\|E^*\|$, GPa	φ(º)	$\|E^*\|$, GPa	φ(º)	$\|E^*\|$, GPa	φ(º)
25	12.571	3.9	9.981	7.0	5.575	14.2	2.972	22.9	1.206	28.6
10	12.802	3.9	9.366	6.5	4.974	13.7	2.492	22.3	0.878	21.2
5	12.506	3.8	8.981	7.5	4.376	14.5	2.062	22.8	0.614	26.7
1	11.628	4.9	7.856	8.6	3.322	17.4	1.237	23.8	0.306	25.8
0.5	11.224	5.3	7.381	8.6	2.914	18.5	0.978	25.7	0.234	26.0
0.1	10.285	6.1	6.227	11.6	1.991	20.3	0.634	25.2	0.146	28.7

Dynamic Moduli and Phase Angles for a specific asphalt-concrete mix (no tension, only compression was applied), *ref.* to [1] – PhD. thesis





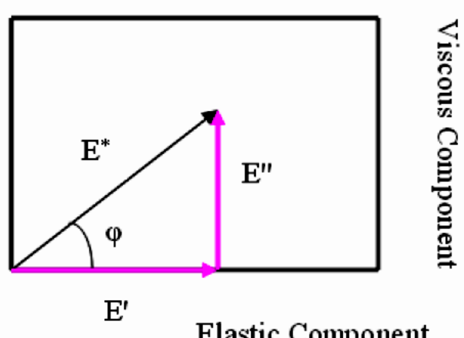
Schematic of dynamic modulus test

The complex (relaxation) **modulus**:

$$G^*(\omega) = G'(\omega) + iG''(\omega)$$

$$(E^* = E' + iE'')$$

Their exist methods to convert frequency domain representation to the time domain representation of the relaxation modulus, please ref. to literature.



$$\varphi = 2\pi \cdot \frac{t_{\text{LAG:}\sigma\text{-}\varepsilon}}{t_{\sigma\text{-cycle}}}$$

ω - angular frequency

Relaxation Moduli

$E(t)$

DATA:

t (sec)	-10 °C E(t), GPa	4.4 °C E(t), GPa	21.1 °C E(t), GPa	37.8 °C E(t), GPa	54.4 °C E(t), GPa
0.04	12.498	9.598	5.014	2.399	0.842
0.1	12.646	8.968	4.457	2.015	0.659
0.2	12.295	8.548	3.893	1.656	0.445
1	11.291	7.388	2.894	0.988	0.229
2	10.842	6.916	2.520	0.761	0.176
10	9.816	5.734	1.796	0.498	0.110

Relaxation Moduli for a specific asphalt-concrete mix (no tension, only compression was applied), *ref.* to [1] – PhD. Thesis. /converted from dynamic moduli shown in the previous slide/

$\nu \approx 0.35 \approx \text{const}$

Good to know

Lamé-parameters:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)},$$

$$\mu = G = \frac{E}{2(1+\nu)} = K - \frac{2}{3}G$$

Bulk modulus:

$$K = \lambda + 2\mu/3,$$

Time Dependency and Prony Series

In ABAQUS, time dependency (the relaxation functions) can be represented by using Prony series in the form of a scaled shear moduli as:

$$r = \left| g_{\text{MODEL}}(G_i, \tau_i) - g_{\text{DATA}} \right|$$

In general, at least 4-5 Prony series terms are needed

MODEL:

$$g(t) = 1 - \sum_{i=1}^{N} g_i \left(1 - e^{-t/\tau_i}\right),$$

where

instantaneous *shear* moduli

$$G(t) = G_0 - \sum_{i=1}^{N} G_i \left(1 - e^{-t/\tau_i}\right),$$

$$g(t) \equiv \frac{G(t)}{G_0} = 1 - \sum_{i=1}^{N} \frac{G_i}{G_0} \left(1 - e^{-t/\tau_i}\right)$$

Relaxation modulus

$$G(t) = \frac{E(t)}{2(1+\nu)} = K(t) - \frac{2}{3}G(t)$$

Shear modulus

G_0 - can be obtained from a separate test if needed.

Fit by the root-mean-square error or other more adequate method (data-versus model) in order to obtain the material parameters needed in the Prony series.

Relaxation Moduli

$E(t)$

DATA:

Good to know

t (sec)	-10 ºC E(t), GPa	4.4 ºC E(t), GPa	21.1 ºC E(t), GPa	37.8 ºC E(t), GPa	54.4 ºC E(t), GPa
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Relaxation Moduli for a specific asphalt-concrete mix (no tension, only compression was applied), *ref.* to [1] – PhD. Thesis. /converted from dynamic moduli shown in the previous slide/

The specific asphalt concrete mix

Constants of Prony Series:

Constant	SMA
g_1	0.2301
g_2	0.2847
g_3	0.2432
g_4	0.1566
g_5	7.03E-02
τ_1	9.66E-06
τ_2	2.37E-03
τ_3	1.58E-01
τ_4	8.479
τ_5	470.5

Time Dependency and Prony Series

In ABAQUS, time dependency (the relaxation functions) can be represented by using Prony series in the form of a scaled shear moduli as:

MODEL:

$$\boxed{g(t) = 1 - \sum_{i=1}^{N} g_i \left(1 - e^{-t/\tau_i}\right),}$$

$$g(t) \equiv \frac{G(t)}{G_0} = 1 - \sum_{i=1}^{N} \frac{G_i}{G_0}\left(1 - e^{-t/\tau_i}\right)$$

G_0 - is better obtained from a separate test if needed.

In this example, one may brutally approximate:

$$G_0 \approx G(t \to 0) = \frac{E(t \to 0)}{2(1+\nu)}$$

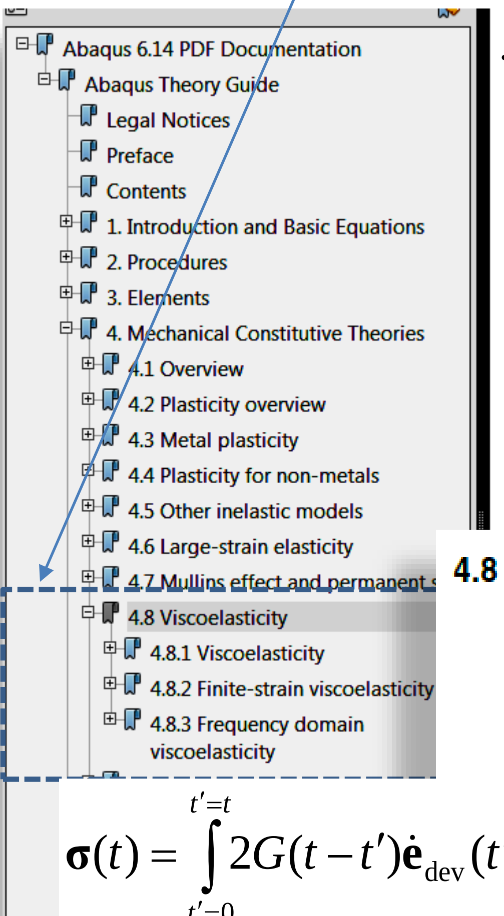
$$G_0 \approx \frac{5.014}{2(1+0.35)} = 1.86$$

Observed: $\nu \approx 0.35 \approx \text{const}$

Why use, in this example, the Relaxation Modulus?

Time Dependency and Prony Series

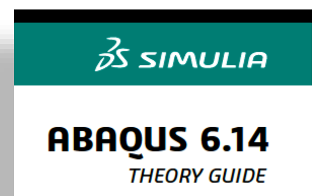
In ABAQUS, time dependency (the relaxation functions) can be represented by using Prony series in the form of a scaled shear moduli as:



$$\boxed{g(t) = 1 - \sum_{i=1}^{N} g_i \left(1 - e^{-t/\tau_i}\right),}$$

$$g(t) \equiv \frac{G(t)}{G_0} = 1 - \sum_{i=1}^{N} \frac{G_i}{G_0}\left(1 - e^{-t/\tau_i}\right)$$

Ref: ABAQUS



4.8 Viscoelasticity

- "Viscoelasticity," Section 4.8.1
- "Finite-strain viscoelasticity," Section 4.8.2
- "Frequency domain viscoelasticity," Section 4.8.3

$$\boldsymbol{\sigma}(t) = \int_{t'=0}^{t'=t} 2G(t-t')\dot{\mathbf{e}}_{\text{dev}}(t')\mathrm{d}t' + \mathbf{I}\int_{t'=0}^{t} K(t-t')\dot{e}_{\text{vol}}(t')\mathrm{d}t'$$

3D-formulation: $e^{\text{vol}} = \varepsilon_{kk}\delta_{ij}$, $\quad e^{\text{dev}}_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij}$

Answer:

In the numerical implementation (often a displacement formulation) the rate form of the **constitutive equation** is used. The incremental form for stress

$$\boldsymbol{\sigma}(t) = \int_{t'=0}^{t} E(t-t')\dot{\boldsymbol{\varepsilon}}(t')\mathrm{d}t'$$

Relaxation modulus

with **strain rate** assumed to be **constant** within the current time step, lead to a ***relaxation problem*** during the current k^{th}-time step . $\Delta t'_k$

This is why the relaxation moduli were experimentally determined in previous pavement example

In many practical cases it can be assumed that and $N_\lambda = 0$ $G_\lambda(t) \approx K_\infty,$

Abaqus and viscoelasticity formulation

SIMULIA

ABAQUS 6.14
THEORY GUIDE

4.8.1 VISCOELASTICITY

Products: Abaqus/Standard Abaqus/Explicit

The basic hereditary integral formulation for linear isotropic viscoelasticity is

$$\boldsymbol{\sigma}(t) = \int_0^t 2G(\tau - \tau')\dot{\mathbf{e}}\, dt' + \mathbf{I}\int_0^t K(\tau - \tau')\dot{\phi}\, dt'.$$

Here $\mathbf{e}$ and ϕ are the mechanical deviatoric and volumetric strains; K is the bulk modulus and G is the ... modulus, which are functions of the reduced time τ; and $\dot{}$ denotes differentiation with respect to t'.

The reduced time is relat

$$\boxed{\boldsymbol{\sigma}(t) = \int_0^t 2G(\tau - \tau')\dot{\mathbf{e}}\, dt' + \mathbf{I}\int_0^t K(\tau - \tau')\dot{\phi}\, dt'.}$$

and A_θ is the shift function. (Hence, if $A_\theta = 1$, $\tau = t$.) A commonly used s'

Creep strain:

$$\boxed{\mathbf{e}_i = \int_0^\tau \left(1 - e^{(\tau' - \tau)/\tau_i}\right)\frac{d\mathbf{e}}{d\tau'}\, d\tau'}$$

The relaxation functions $K(t)$ and $G(t)$ can be defined individually in terms of a series of expone known as the Prony series:

$$K(\tau) = K_\infty + \sum_{i=1}^{n_K} K_i e^{-\tau/\tau_i^K} \qquad G(\tau) = G_\infty + \sum_{i=1}^{n_G} G_i e^{-\tau/\tau_i^G},$$

Creep strain increment:

$$\begin{aligned}\Delta\mathbf{e}_i &= \left(1 - e^{-\Delta\tau/\tau_i}\right)\mathbf{e}^n + \left(e^{-\Delta\tau/\tau_i} - 1\right)\mathbf{e}_i^n + \left(\Delta\tau - \tau_i\left(1 - e^{-\Delta\tau/\tau_i}\right)\right)\frac{\Delta\mathbf{e}}{\Delta\tau} \\ &= \frac{\tau_i}{\Delta\tau}\left(\frac{\Delta\tau}{\tau_i} + e^{-\Delta\tau/\tau_i} - 1\right)\Delta\mathbf{e} + \left(1 - e^{-\Delta\tau/\tau_i}\right)\left(\mathbf{e}^n - \mathbf{e}_i^n\right).\end{aligned} \qquad (4.8.1\text{–}3)$$

Viscos (creep) strain is calculated in terms of the Prony series material parameters for relaxation

The Method proposed by Christensen (1982) [2]

[1] ***Ref***: Material data from: PhD-thesis:

VISCOELASTIC FE MODELING OF ASPHALT PAVEMENTS
AND ITS APPLICATION TO U.S. 30 PERPETUAL PAVEMENT

the Department of Civil Engineering
and the Russ College of Engineering and Technology by

by
YUN LIAO

Doctor of Philosophy
Yun Liao
November 2007

Conversion between Dynamic Modulus and Relaxation Modulus

The dynamic modulus, $|E^*|$, in the frequency domain can be converted to the relaxation modulus, $E(t)$, in the time domain. In what follows, the symbols of t and f for time (sec.) and frequency ($1/_{\text{sec.}}$), respectively, are used. An approximation method proposed by Schapery and Park (1999) was adopted in this research with the basic steps summarized as follows:

- Calculate the storage modulus, $E'(f)$, based on the dynamic modulus, $|E^*|$, and the phase angle, φ, by the relationship $E'(f) = |E^*| \cdot \cos(\varphi)$.
- Regress $E'(f)$ and compute the local log-log slope, n, at every measured frequency by the relationship $n = {d\log(E'(f))}/{d\log(f)}$.
- Calculate the adjustment function $\lambda' = \Gamma(1-n)\cdot\cos(n\pi/2)$, where $\Gamma(1-n)$ is the gamma function.
- Calculate $E(t) = {E'(f)}/{\lambda'}$, where $t = 1/f$.

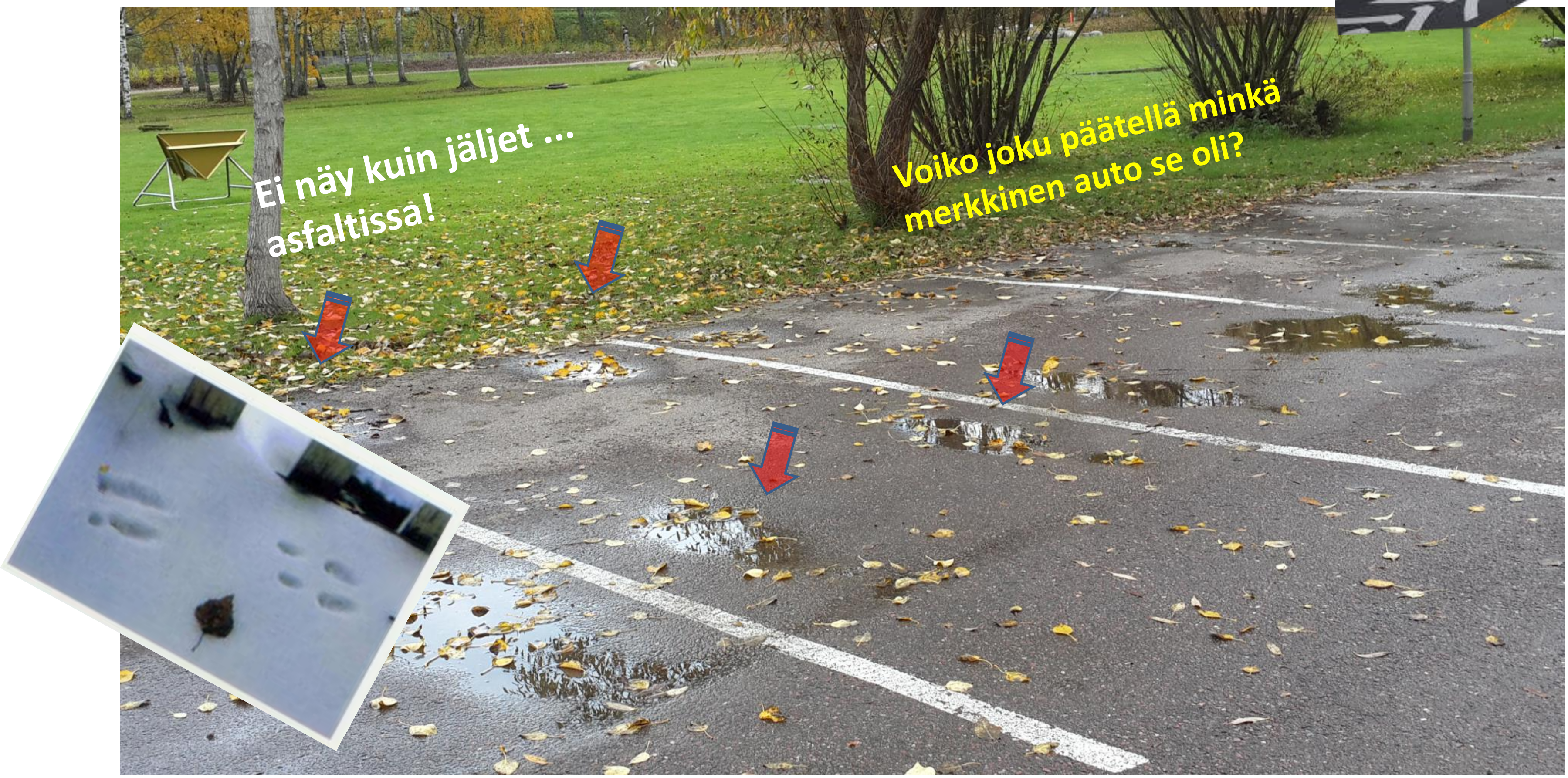
Good to know

[2] Christensen, R. M. (1982). *Theory of Viscoelasticity*. 2nd Ed., Academic Press, New York.

Visco-elasto-plasticity

Parkkipaikka raksan
takana, 12.10.2014
(kuva Dj. Baroudi)

Visco-elasto-plasticity



This material will be presented and commented during the last lecture on the topic, I hope we will have time for that.

Extra - elective

Elective Reading

Constitutive models for viscoelasticity

Example: viscoelastic **creep** (we will come back to this example later may be in homework!)

- **Concrete is a viscoelastic material***

Creep: Apply constant stress

(compressive stress bellow yield stress)

$$\sigma = \sigma_0 H(t - t_0)$$

age of the concrete when loaded

creep deformation:

$$\boxed{\varepsilon(t,t_0) = \sigma(t_0) J(t,t_0),}$$

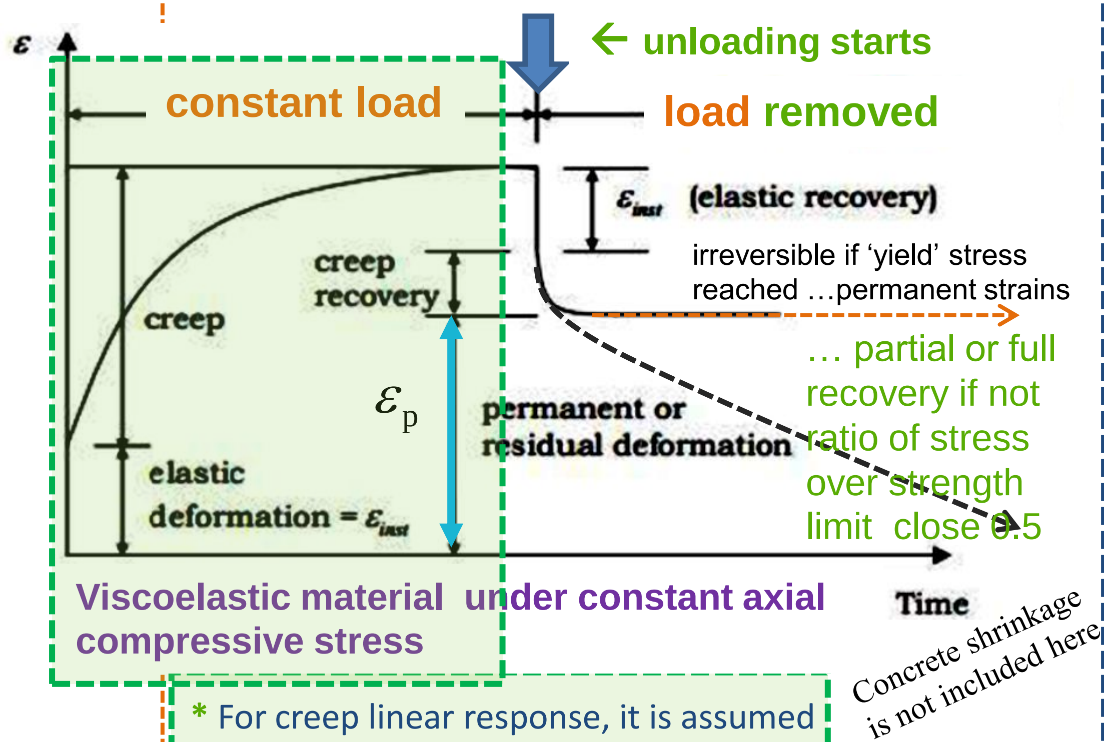
Generic Code-type of European Creep model:

$$J(t,t_0) = \frac{1}{E_c(t_0)} + \frac{\varphi(t,t_0)}{E_c}$$

creep function or creep compliance

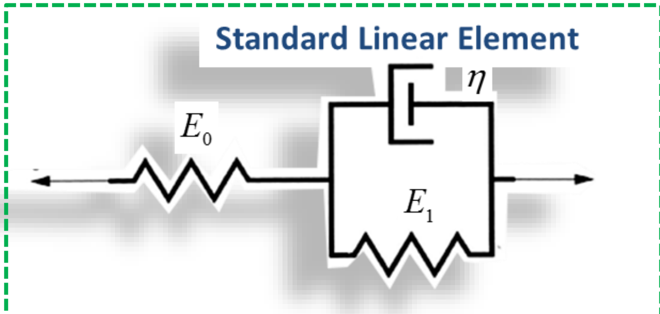
Modulus of elasticity at age of 28 days in MPa

where for example: $\varphi(t,t_0) = \varphi_\infty \left[1 - e^{-\beta(t-t_0)}\right]$



* For creep linear response, it is assumed applied compressive stress is assumed bellow yield or ultimate stress σ_Y let say : $|\sigma| < \sigma_Y / 2$ at t_0

The **creep function** of concrete can be derived from extensive and lengthy experiments. Concrete **norms or codes** compile such experimental work and provides *creep functions* for a given class of concrete, sample size temperature and humidity conditions after specifying the type & age of the concrete at the loading instant, curing conditions,w/c, ….



Relation between *Creep* 'coefficient' and *compliance* function

Creep deformation (mechanical strain): $\varepsilon(t,\tau) = \sigma \cdot \boxed{J(t,\tau)}$

$$\varepsilon(t,\tau) = \underbrace{\varepsilon_{\mathrm{e}}}_{\text{elastic}} + \underbrace{\varepsilon_{\mathrm{cr}}}_{\text{creep}} \equiv \varepsilon_{\mathrm{e}} + \varphi(t,\tau)\varepsilon_{\mathrm{e}}$$

$$\equiv \varepsilon_{\mathrm{e}} \cdot [1 + \varphi(t,\tau)]$$

strain at time *t* du to constant stress acting since age $\tau < t$

$$\Downarrow$$

$$\boxed{J(t,\tau)} = \frac{\varepsilon(t,\tau)}{\sigma} = \frac{\varepsilon_{\mathrm{e}} + \varepsilon_{\mathrm{cr}}}{\sigma} = \ldots$$

$$\ldots = \frac{\varepsilon_{\mathrm{e}}}{\sigma}[1 + \varphi(t,\tau)] = \boxed{\frac{1+\varphi(t,\tau)}{E(\tau)}}$$

Elastic modulus characteristizing the instantaneous strain at age $\tau < t$

Creep coefficient

$$\varepsilon_{\mathrm{cr}} \equiv \varphi(t,\tau)\varepsilon_{\mathrm{e}}$$

Creep governs strongly long–term behavior of concrete

ref ↗

ACI Structural Journal/November-December 2011

ACI STRUCTURAL JOURNAL TECHNICAL PAPER

Title no. 108-S72 Vol. 108, No. 6

Pervasiveness of Excessive Segmental Bridge Deflections: Wake-Up Call for Creep

by Zdeněk P. Bažant, Mija H. Hubler, and Qiang Yu

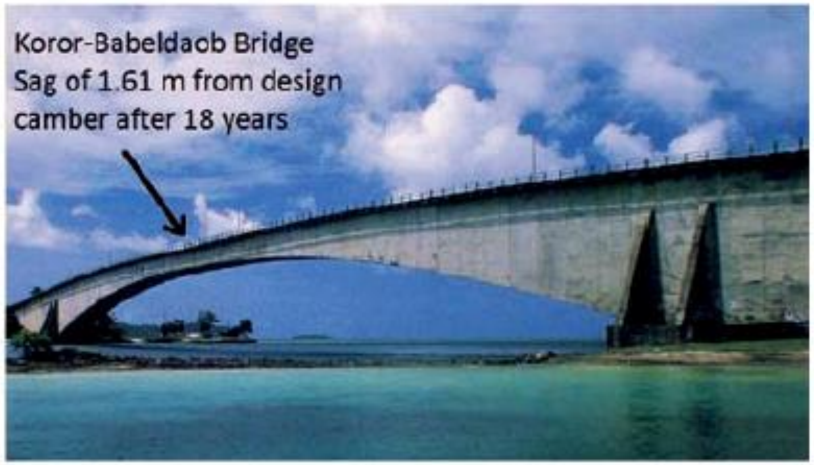


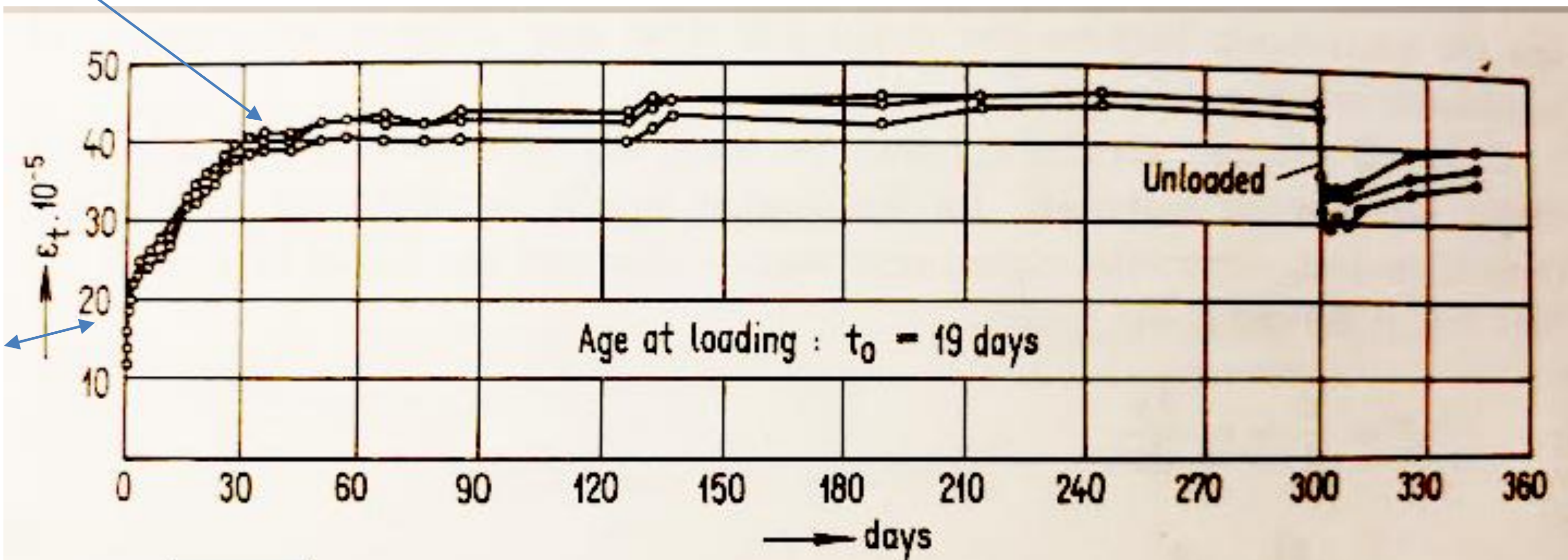
Fig. 1—(a) ACI authorized reprint from cover of ACI SP-194[4] *taken by Adam Neville before retrofit showing deck view of KB Bridge at midspan hinge before retrofit; (b) side view of KB Bridge before retrofit (main span: 241 m [790.7 ft] with two symmetric cantilevers consisting of 25 cast-in-place segments); and (c) collapse of KB Bridge on September 26, 1996—3 months after retrofit with remedial prestressing. (Note: 1 m = 3.28 ft.)*

Example of experimental evidence of creep in concrete

Total strain = elastic (instantaneous) + creep + shrinkage

Uniaxial test

instantaneous

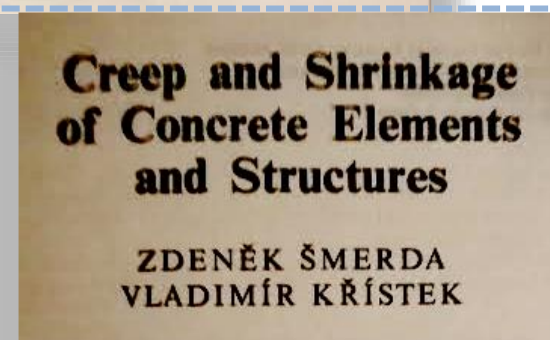


Total strain of concrete prisms 100/100/400 mm; strength = 35.30 MPa

$$\sigma_0 = 4.45 \text{ MPa}$$

$$\sigma_0 / f_c = 4.45 / 35.30$$

Ref →



Creep and Shrinkage of Concrete Elements and Structures

ZDENĚK ŠMERDA
VLADIMÍR KŘÍSTEK

1988

SNTL – PUBLISHERS OF TECHNICAL LITERATURE, PRAGUE

Constitutive models for viscoelasticity

Example: viscoelastic **creep** (we will come back to this example later may be in homework!)

- **Concrete is a** complex **viscoelastic material**

Creep: Apply constant stress

(stress bellow compressive strength)

$$\sigma = \sigma_0 H(t - t_0)$$

age of the concrete when loaded

creep deformation:

$$\boxed{\varepsilon(t,t_0) = \sigma(t_0) J(t,t_0),}$$

$$J(t,t_0) = \frac{1}{E_c(t_0)} + \frac{\varphi(t,t_0)}{E_c}$$

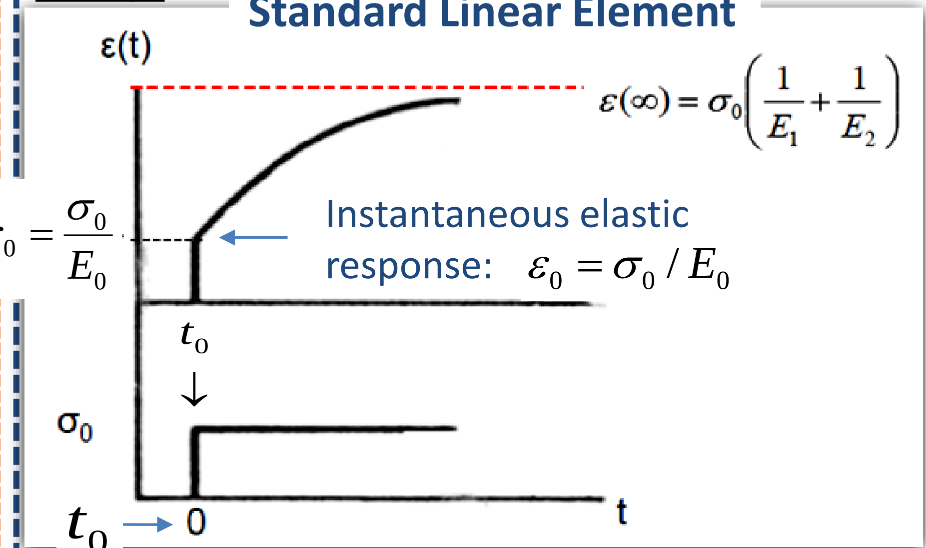
creep function or creep compliance

Modulus of elasticity at age of 28 days in MPa

where $\varphi(t,t_0) = \varphi_\infty \left[1 - e^{-\beta(t-t_0)}\right]$

compare

Creep:



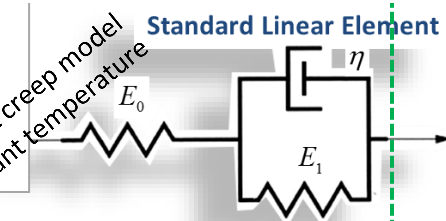
Standard Linear Element: $t_0 = 0$

$$\Rightarrow \varepsilon(t) = \sigma_0\left[\frac{1}{E_1}\left(1 - e^{-\frac{E_1}{\eta}t}\right) + \frac{1}{E_0}\right] = \sigma_0 J(t)$$

Creep coefficient can be also expressed in function of age of concrete, of time, relative humidity, water-to-cement ratio *etc.*...*ref* to norms and codes..

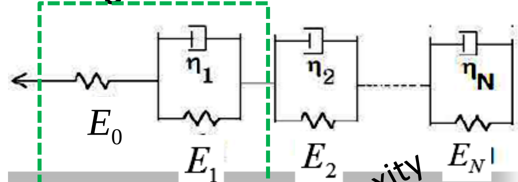
The **creep function *J*** for concrete can be derived from extensive and lengthy experiments. Concrete norms compile such experimental work and *estimates* for *creep function* for a given *class* of concrete, sample size, humidity and temperature conditions when specifying the age of the concrete at the loading instant

Standard Linear Element



Simplest creep model constant temperature

A generalised Kelvin model



More complexity

Example: Concrete is a complex 'viscoelastic' material

Creep governs strongly long–term behavior of concrete

Good to know

w/c = water/cement

Experiments:

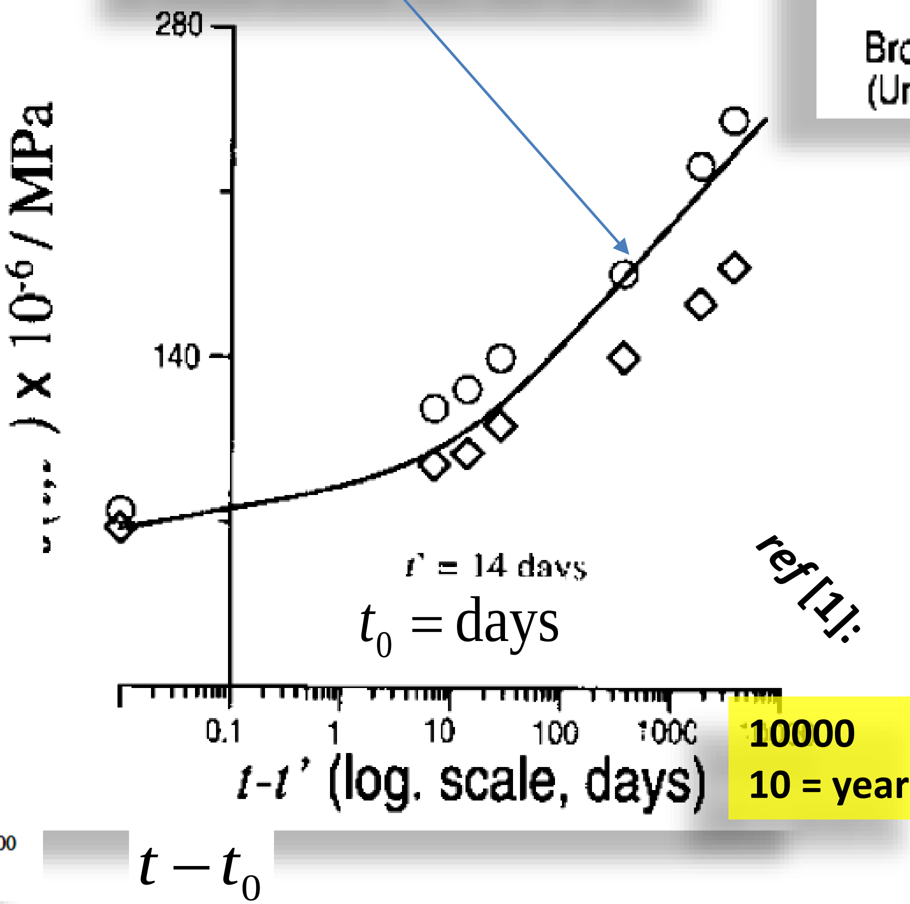
◇ w/c = 0.56

○ w/c = 0.67

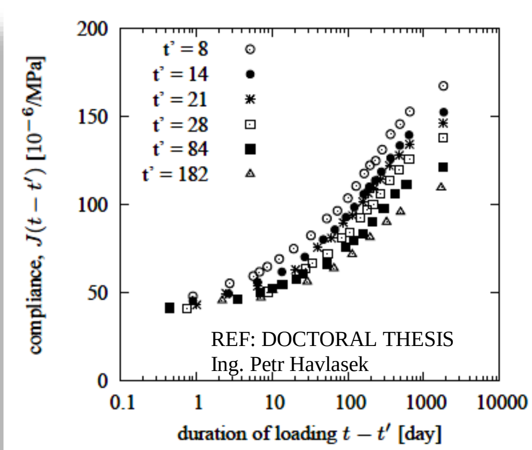
Brooks, 1984
(University of Leeds)

$$\varepsilon_{cr}(t,t_0)=\sigma(t_0)J(t,t_0),$$

A model (called 'B3', see ref [1])



10-year creep data by Brooks.



Good to know:
According to Bazant *et al.* and Bazant (1997 and 2010), the complex creep properties of hardened Portland cement concrete are caused by physical and chemical processes occurring in the nano-pores (or gel pores)

"Creep is most likely caused by separation, slip and restoration of bonds in the nanocrystaline hydrophilic C-H-S", [1]

[2] Brooks, J.J. 2005. 30-year creep and shrinkage of concrete. *Magazine of Concrete Research*, 57(9), 545-556.

[1]:

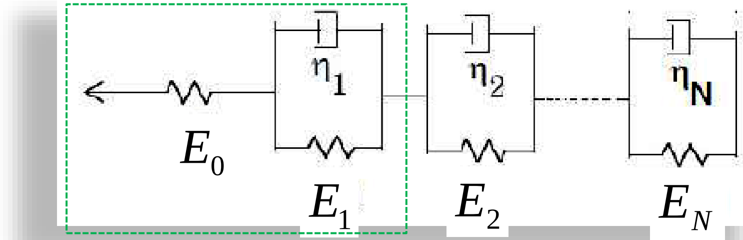
Bažant, Z.P., and Yu, Q. (2010). "Modeling of concrete creep and hygrothermal defor-mations, and computation of their structural effects." *Computational Modeling of Concrete structures* (plenary lecture, EURO-C Conf., Schladming/Rohrmoos, Austria, March 15). N. Bićanić et al., eds., Taylor & Francis, London, pp. 3–13.

Computational Modelling of Concrete Structures – Bićanić et al. (eds)
© 2010 Taylor & Francis Group, London, ISBN 978-0-415-58479-1

Kelvin Chain:

Favored model for modelling ***Creep*** phenomena, [1]

To capture more complexity of experimental response of the material one can use relevant combination of standard basic elements:



A generalised Kelvin (chain) model

It can be shown easily that: (equilibrium + compatibility)

$$J(t-t_0)=\frac{1}{E_0}+\sum_{i=1}^{n}\frac{1}{E_i}\left[1-e^{-\frac{t-t_0}{\tau_i}}\right]$$

$$\tau_i=\frac{\eta_i}{E_i}$$ - retardation time per branch (characteristic time)

The parameters of the model should be adjusted to model some experiments (*calibration*). After this step one can move to the *prediction* step.

Usually after simple hand- or analytical derivations of needed models or sub-models (*as substructures and constitutive models*) one moves, for structural analysis part, to use some efficient numerical software packages having the relevant material models (COMSOL see [2] for such eample, Lusas, Abaqus, Ansys, …)

[2] Modelling of Viscoelastic Phenomena in Concrete Structures

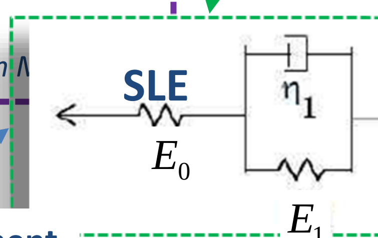
A.A. Pomarico*[1], G. Roselli[1], D. Caltabiano

Excerpt from the *Proceedings of the **2016** COMSOL Conference in* M

Example of a set of parameters … from **[2]**

Branch number	Elastic modulus E_i [MPa]	Retardation time τ_i [s]
0	7254	
1	25388	5E+02
2	36268	5E+03
3	29015	5E+04
4	18134	5E+05
5	10881	5E+06

Cylinder height 7cm, diameter7cm, 48 MPa characteristic compressive cylindrical strength, 50% environmental Humidity.



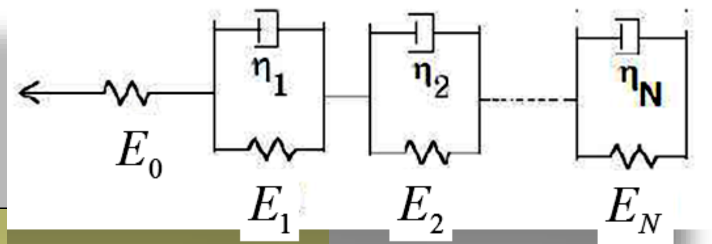
A smaller set of parameters may be sufficient depending on the complexity of experimental response of the material

Standard Linear Element

Good to know: *Generalized Maxwell chain* is suitable for modelling stress ***relaxation*** phenomena, [1]

[1] Computational Viscoelasticity, S. Marques and G. Creus, Springer 2012

Continued: Example of a set of parameters … from **[2]**

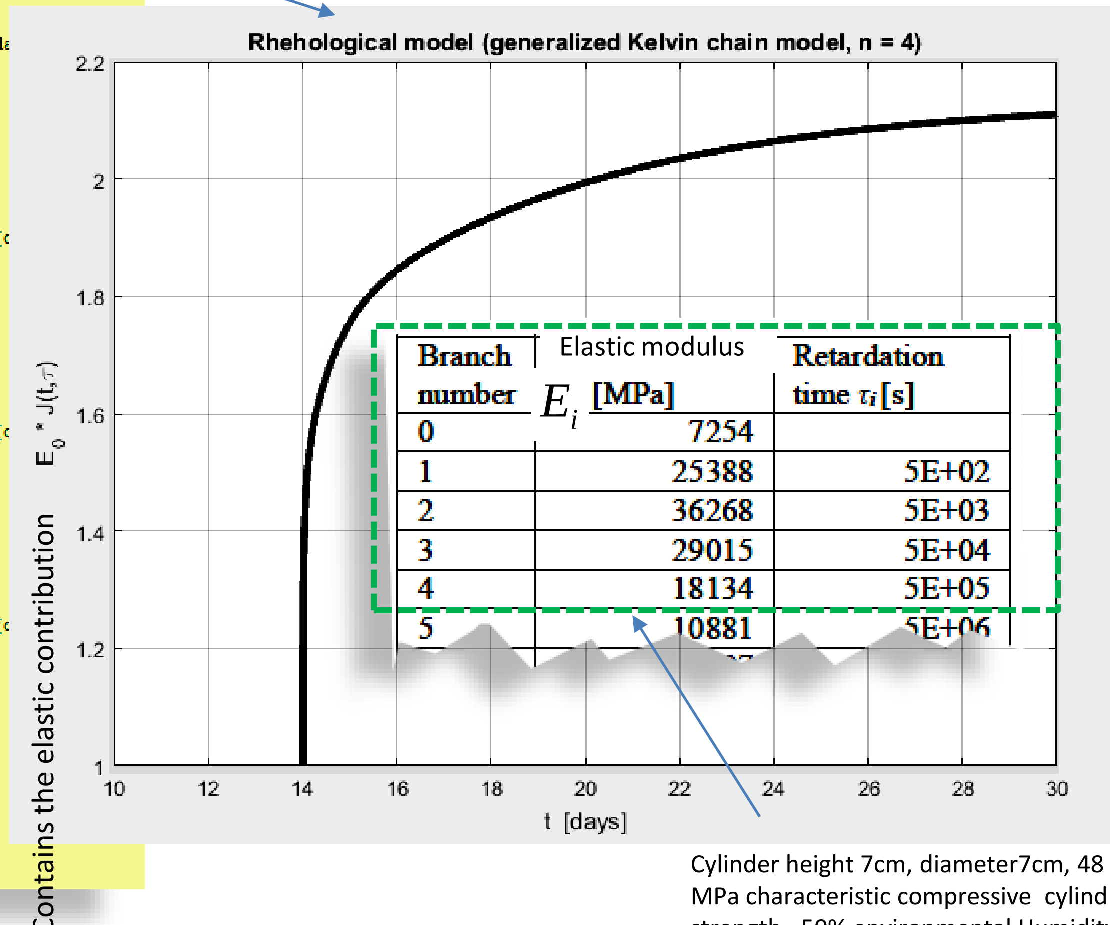


$$J(t-t_0)=\frac{1}{E_0}+\sum_{i=1}^{n}\frac{1}{E_i}\left[1-e^{-\frac{t-t_0}{\tau_i}}\right]$$

A generalised Kelvin (chain) model

```
%% t_s = 1:60*60:1e6;
t_s = 1:60*60:1e6;
%%ts    = t_day *24*60*60;
E_0 =  7250e6; % Pa
E_1 = 25388e6; % Pa
tau_1 = 500; % s
eta_1   = tau_1 * E_1;

J_t   = 1 - exp ( -E_1/eta_1 .* t_s ) ; % NB. now t [da
J_SLE = 1 / E_0 + 1/E_1 * J_t;
Fi_SLE = E_0 * J_SLE;

E_2 = 36268e6;
tau_2 = 5000;
eta_2   = tau_2 * E_2;
J_t2    = 1 - exp ( -E_2/eta_2 .* t_s ) ; % NB. now t [d
J_SLE2 = + 1/E_2 * J_t2;
Fi_SLE2 = E_0 * J_SLE2;

E_3   = 29015e6;
tau_3 = 50000;
eta_3   = tau_3 * E_3;
J_t3    = 1 - exp ( -E_3/eta_3 .* t_s ) ; % NB. now t [d
J_SLE3 = + 1/E_3 * J_t3;
Fi_SLE3 = E_0 * J_SLE3;

E_4   = 18134e6;
tau_4 = 5e5;
eta_4   = tau_4 * E_4;
J_t4    = 1 - exp ( -E_4/eta_4 .* t_s ) ; % NB. now t [d
J_SLE4 = + 1/E_4 * J_t4;
Fi_SLE4 = E_0 * J_SLE4;

Fi_SLE_tot = Fi_SLE + Fi_SLE2 + Fi_SLE3 + Fi_SLE4;
figure
plot(t0+t_s/3600, Fi_SLE_tot)
xlabel('t  [hours]')
grid on
ylabel('E_0 * J(t,\tau)')
```

A generalised Kelvin (chain) model



Branch number	Elastic modulus E_i [MPa]	Retardation time τ_i [s]
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4	18134	5E+05
5	10881	5E+06

Cylinder height 7cm, diameter7cm, 48 MPa characteristic compressive cylindrical strength, 50% environmental Humidity.

*Reference: **THE NEW MODEL FOR CREEP OF CONCRETE IN FIP MODEL CODE 2010**

F. Mola, Politecnico di Milano, Italy
L.M. Pellegrini, ECSD, Italy

http://cipremier.com/100037006

37th Conference on OUR WORLD IN CONCRETE & STRUCTURES: 29 - 31 August 2012 Singapore

Example* - THE MODELS IN THE EUROPEAN MODEL CODES

The *fib* Model Code 2010* - analytical form of the creep function

$$J(t,t') = \frac{1}{E_c(t')} + \frac{\varphi(t,t')}{E_{c,28}}$$

creep coefficient: $\varphi(t,t')$

$$J(t,t') = \frac{1}{E(t')} + \frac{1}{E_{28}} \cdot [\underbrace{\varphi_{bc}(t,t') + \varphi_{dc}(t,t')}_{\varphi(t,t')}]$$

basic creep

drying creep

for $|\sigma| \leq 0.4 \cdot f_{cm}(t_0)$

where

$$\varphi_{bc}(t,t') = \beta_{bc}(f_{cm}) \cdot \beta_{bc}(t,t')$$

$$\beta_{bc}(f_{cm}) = \frac{1.8}{(f_{cm})^{0.7}}$$

$$\beta_{bc}(t,t') = ln\left[\left(\frac{30}{t'_{adj}} + 0.035\right)^2 \cdot (t-t') + 1\right]$$

$$\varphi_{dc}(t,t') = \beta_{dc}(f_{cm}) \cdot \beta(RH) \cdot \beta_{dc}(t') \cdot \beta_{dc}(t,t')$$

$$\beta_{dc}(f_{cm}) = \frac{412}{(f_{cm})^{1.4}}$$

$$t'_{adj} = t_{0,T} \cdot \left[\frac{9}{2+t_{0,T}^{1.2}} + 1\right]^{\alpha} \geq 0.5 \text{ days}$$

$$\beta(RH) = \frac{1-RH/100}{\sqrt[3]{0.1 \cdot h/100}}$$

$$\beta_{dc}(t') = \frac{1}{0.1+(t'_{adj})^{0.2}}$$

$$f_{cm} = f_{ck} + 8 \text{ MPa}$$

$$\beta_{dc}(t,t') = \left[\frac{(t-t')}{\beta_h+(t-t')}\right]^{\gamma(t')}$$

$t = 0\ldots$ 50 years or 70 years

$$\gamma(t') = \frac{1}{2.3+\frac{3.5}{\sqrt{t'_{adj}}}}$$

$$\beta_h = 1.5 \cdot h + 250 \cdot \alpha_{f_{cm}} \leq 1500 \cdot \alpha_{f_{cm}}$$

$$\alpha_{f_{cm}} = \left(\frac{35}{f_{cm}}\right)^{0.5}$$

f_{cm}, RH, h respectively represent the mean compressive strength at 28 days MPa, the relative humidity in % and the notional thickness in mm.

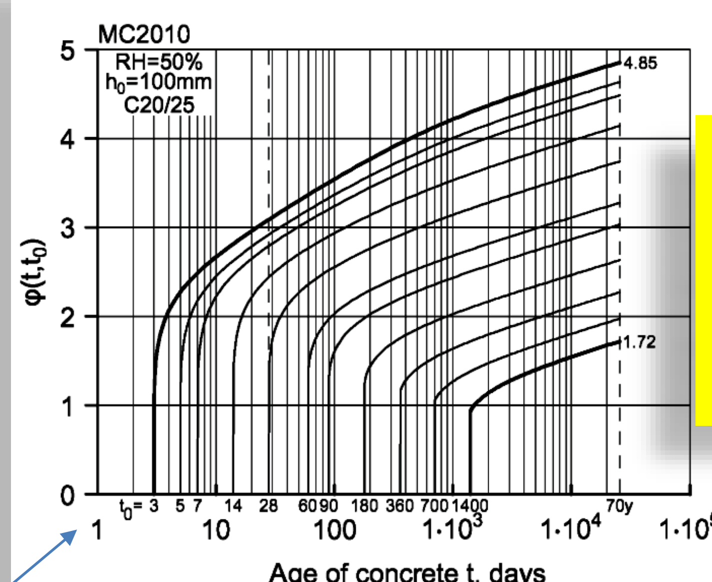
Elective reading -

In the interval $0.4 \cdot f_{cm}(t_0) < |\sigma| \leq 0.6 \cdot f_{cm}(t_0)$

$$\varphi_\sigma(t,t') = \varphi(t,t')e^{[1.5\cdot(k_\sigma - 0.4)]}$$

$$0.4 < k_\sigma \leq 0.6 \qquad k_\sigma = \frac{|\sigma|}{f_{cm}(t_0)}$$

Creep Coefficient $\varphi(t,t_0)$



Ref: Janusz Hołowaty. **Creep of Concrete in Contemporary Code-Type Models.** *Journal of Civil Engineering and Architecture* 9 (2015) 1025-1033.

Good to know:

International Federation for Structural concrete. 2013. ***fib Model Code for Concrete Structures 2010***. Berlin: Ernest & Son.

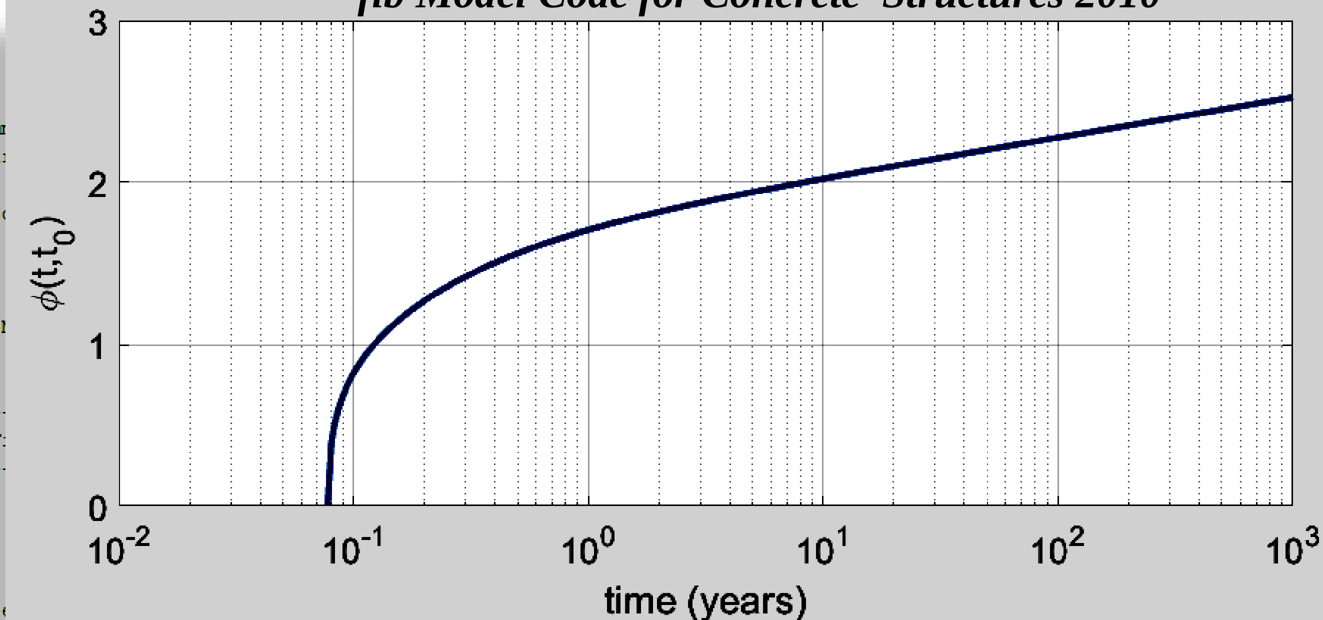
*more in design courses

```
% the creep function: The Concrete Model Code 2010 FIB
% now assume that |sigma| < 0.4 fcm(t0)
```

```
fck   = 48 % MPa
RH    = 50 % in (%) 50/100 %
fcm   = fck + 8
t0    = 14 % (days) loading time
b     = 200;  % square section bxb (mm
b_RH = 2*(2*b); % altiina RH:lle pii
A     = b * b;
h_RH = A / b_RH;    % RH:lle altiina (

dt     = 1; % (days)
t_100_years = 1000;
t_end = 12*30*t_100_years   % (days) 1
t = t0:dt:t_end;

% ---------------------------------
% Evaluating the creep coefficient F
% ---------------------------------

% 1. Basic creep
% -----------------
beta_bc_fcm = 1.8 / (fcm^0.7);
alpha = 1; % depends on strength case
t0_T  = t0; % assumed for example
t_adj_min = t0_T * (( 1 + 9/(2 + t0^1.2) ).^alpha );
t_adj = max(5, t_adj_min );  %  Temperature and cement type adjuste
beta_bc_t_tp = log( (30./t_adj)^2 * ( t - t0) + 1  );
%% t_adj = t0 + 1;  % ... KORJA KYSY JAARASELTa mika on t_adj????

Fi_bc = beta_bc_fcm .* beta_bc_t_tp; % basic creep

% 2. Dry creep
% --------------
beta_dc_fcm = 412 / (fcm^1.4);
beta_RH      = ( 1 - RH/100) / ((0.1*h_RH/100)^(1/3));
beta_dc_tp   = 1 / (0.1  + t_adj^0.2 );
gamma_tp     = 2.3 + 3.5 / sqrt(t_adj);
gamma_tp     = 1 / gamma_tp;
alpha_m      = sqrt(35/fcm);
beta_n       = min( 1.5 * h_RH + alpha_m, 1500 * alpha_m);
beta_dc_t_tp = ( ( t - t0) ./ (beta_n + (t - t0) ) ) .^ gamma_tp;
Fi_dc = beta_dc_fcm * beta_RH * beta_dc_tp .* beta_dc_t_tp;
% Overall creep (not including shrinkage or temperature effects)
```

creep_dat_normi_beton_48MPA.m

fib Model Code for Concrete Structures 2010



A concise overview of the creep of concrete can be found in:

https://en.wikipedia.org/wiki/Creep_and_shrinkage_of_concrete

(c) Effect of type of cement and curing temperature

The effect of the type of cement on the creep coefficient of concrete may be taken into account by modifying the age at loading t_0 to t_0 ...

$$t_{0,\,adj} = t_{0,T} \cdot \left[\frac{9}{2 + t_{0,T}^{1.2}} + 1 \right]^{\alpha} \geq 0.5 \text{ days} \qquad (5.1\text{-}73)$$

where:

$t_{0,T}$ is the age of concrete at loading in days adjusted according to Eq. (5.1-85);

α is a coefficient which depends on the type of cement:

$\alpha = -1$ for strength class 32.5 N;

$\alpha = 0$ for strength classes 32.5 R, 42.5 N;

$\alpha = 1$ for strength classes 42.5 R, 52.5 N, 52.5 R.

Origin of Creep in Concrete – theories & readings

Good to know

Engineers estimate creep using empirical models. However, long time predictions > 50 years, - are by nature extrapolations of the models - are often poor (**ref**. see next slide).

Till recently, the origin of Creep of Concrete was controversial and not satisfactory understood in the sense of finding the basic physical mechanisms. Many theories have been postulated. A consensus that the processes in the microstructure are the justification for creep. But what processes? Many theories exists, *ref*. literature.

AIP The Journal of Chemical Physics
Published Online: August 2016 Accepted: June 2016

More recently in 2016, a reported scientific work propose an experimentally **verified physical explanation** identifying, atomistic modeling & *measuring* microstructural processes responsible for creep (for the *basic* creep at fixed moisture content). The processes of *dissolution-precipitation* (**DP**) of the **C-S-H *hydrides acting at nanoscale in contact regions of C-S-H grains*** was identified as the origin of creep, and measurements showed that the **DP** and creep are *experimentally* highly correlating (Unlocking the Secrets of Creeping Concrete. **REF**: *The Journal of Chemical Physics* AIP Publishing 2016, see bellow) . The cement paste is hold together by a binding phase that tend to dissolve at high-stress regions, and re-precipitate at low-stress regions

1188 / JOURNAL OF ENGINEERING MECHANICS / NOVEMBER 1997

MICROPRESTRESS-SOLIDIFICATION THEORY FOR CONCRETE CREEP. I: AGING AND DRYING EFFECTS

By Zdeněk P. Bažant,[1] Fellow, ASCE, Anders Boe Hauggaard,[2] Sandeep Baweja,[3] and Franz-Josef Ulm[4]

ABSTRACT: A
concrete creep a
modeled by the
constituent (cem
is too short-live
flow term of the
crossing the mic
pressure of the h
caused by hydra
from viscous sh
the micropores (
term in the cree
Pickett effect (d
pressure, which
implementation,
this issue.

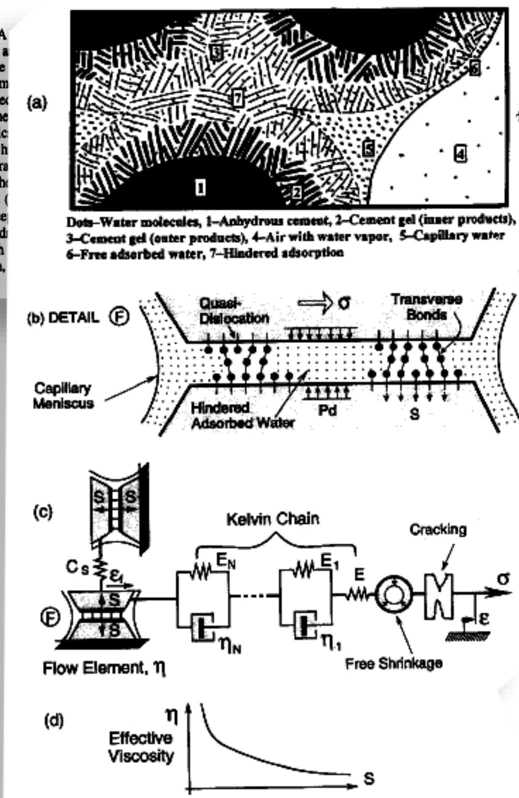


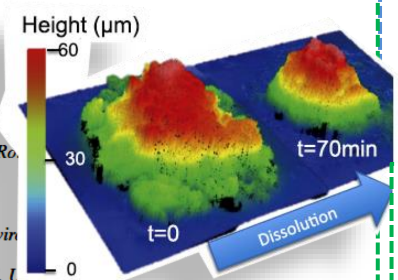
FIG. 1. (a) Idealized Microstructure of Hardened Cement Paste; (b) Micropore in Cement Gel, Disjoining Pressure p_d and Microprestress S; (c) Kelvin Chain Model with Flow Element and Microprestress Relaxation Mechanism; (d) Dependence of Flow Viscosity on Microprestress

THE JOURNAL OF CHEMICAL PHYSICS 145, 054701 (2016)

A dissolution-precipitation mechanism is at the origin of concrete creep in moist environments

Isabella Pignatelli,[1] Aditya Kumar,[2] Rouhollah Alizadeh,[3] Yann Le Pape,[4] Mathieu Bauchy,[5,a)] and Gaurav Sant[1,6,a)]
[1]*Laboratory for the Chemistry of Construction Materials (LC[2]), Department of Civil and Environm Engineering, University of California, Los Angeles, California 90095, USA*
[2]*Materials Science and Engineering Department, Missouri University of Science and Technology, Ro Missouri 65409, USA*
[3]*Giatec Scientific, Ottawa, Ontario K2H 9C4, Canada*
[4]*Oak Ridge National Laboratory, Oak Ridge, Tennessee 37831, USA*
[5]*Physics of AmoRphous and Inorganic Solids Laboratory (PARISlab), Department of Civil and Envir Engineering, University of California, Los Angeles, California 90095, USA*
[6]*California Nanosystems Institute (CNSI), University of California, Los Angeles, California 90095, U*

(Received 21 March 2016; accepted 23 June 2016; published online 2 August 2016)



Used: Vertical scanning interferometry → quantitative 3D-imaging of surfaces and interfaces, Micro-indentation creep tests, Molecular dynamics simulations of C–S–H creep.

JOURNAL OF CHEMICAL PHYSICS, 145(5)

Basic creep at: humidity ≥ 50%, but fixed moisture content

The Figure and the explanations are from the PhD. D dissertation:

ČESKÉ VYSOKÉ UČENÍ TECHNICKÉ V PRAZE
Fakulta stavební

Doktorský studijní program: **STAVEBNÍ INŽENÝRSTVÍ**
Studijní obor: **Fyzikální a materiálové inženýrství**

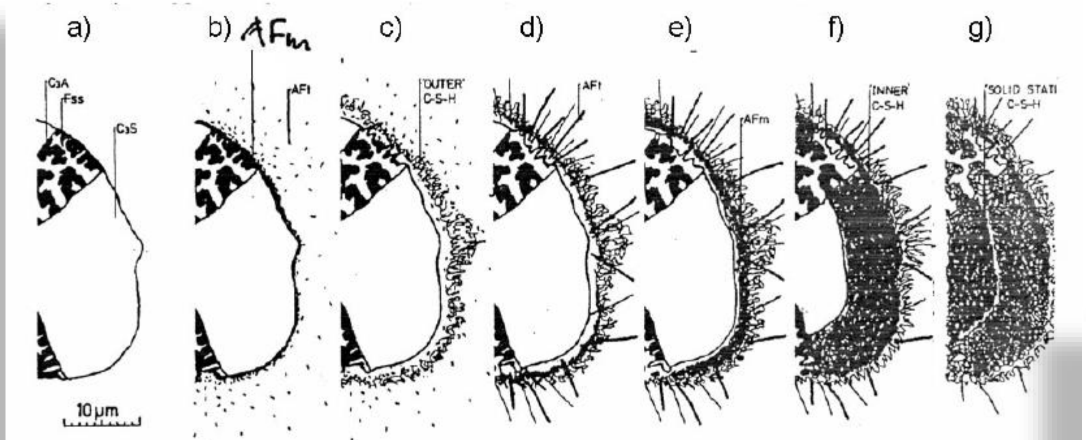
Mgr. Denis Davydov

VISCO-ELASTO-PLASTICKÉ VLASTNOSTI CEMENTOVÉ PASTY
VISCO-ELASTO-PLASTIC PROPERTIES OF CEMENT PASTE

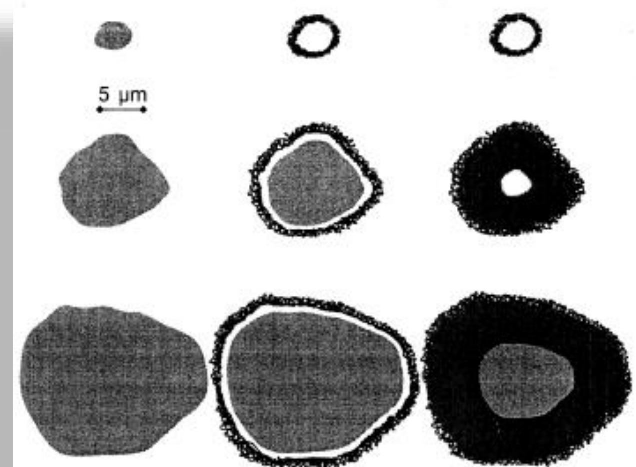
DISERTAČNÍ PRÁCE K ZÍSKÁNÍ AKADEMICKÉHO TITULU Ph.D.

Školitel: **Prof. Ing. Milan Jirásek, DrSc**

Praha, leden 2010



Summary of the microstructural development of a grain of cement. (a) Unhydrated section of polymineral grain (b) -10min. Some C_3A reacts with calcium sulfate in solution. Amorphous, aluminate-rich gel forms on the surface and short AFt $C_6A\bar{S}_3H_{32}$ rods nucleate at edge of gel and in solution. (c) -10h. Reaction of C_3S to produce "outer" product C-S-H on AFt rod network leaving $1\mu m$ between grain surface and hydrated shell. (d) – 18h. Secondary hydration of C_3A reacts with any AFt $C_6A\bar{S}_3H_{32}$ inside shell forming hexagonal plates of AFm. Continuing formation of "inner" product reduces separation of anhydrous grain and hydrated shell. (f) – 14 days. Sufficient "inner" C-S-H has formed to fill in the space between grain and shell. The "outer" C-S-H has become more fibrous. (g) – years. The remaining anhydrous material reacts to form additional "inner" product C-S-H. The ferrite phase appears to remain unreacted. (after [89])



Formation of separated hydration shells over the time according to the grain size. The middle column shows the situation after about one day, all grains are surrounded by a thin shell of C-S-H separated by around 1 μm from the underlying grain. The column on the right shows the situation in the mature paste. Small grains (top) hydrate completely in the first stage and remain as hollow shells of hydration product. Medium size grains (middle) for a thicker rim of C-S-H as the grain continues to hydrate, perhaps with a small hole at the center. Grains larger than about 15 μm hydrate to fill in the gap between shell and grain. (after [93])

[89] K.L. Scrivener. The microstructure of concrete. In J.P. Skalny, editor, *Materials Science of Concrete I*, American Cermamic Society, Westerville, OH, 1989.

[93] K.L. Scrivener. Quantitative study of portland cement hydration by X-ray diffraction. reitveld analysis and independent methods. Cement and Conrete Research, 34:1541–1547, 2004.

Cement and Concrete Research 41 (2011) 1310–1319

Contents lists available at ScienceDirect

Cement and Concrete Research

journal homepage: http://ees.elsevier.com/CEMCON/default.asp

Long-term deformational simulation of PC bridges based on the thermo-hygro model of micro-pores in cementitious composites

Koichi Maekawa *, Nobuhiro Chijiwa, Tetsuya Ishida

Department of Civil Engineering, The University of Tokyo, Japan

The variation with time of the vertical deflection at the span center, measured by an optical measurement device since completion of construction in 1982.

Overview of Tsukiyono Bridge

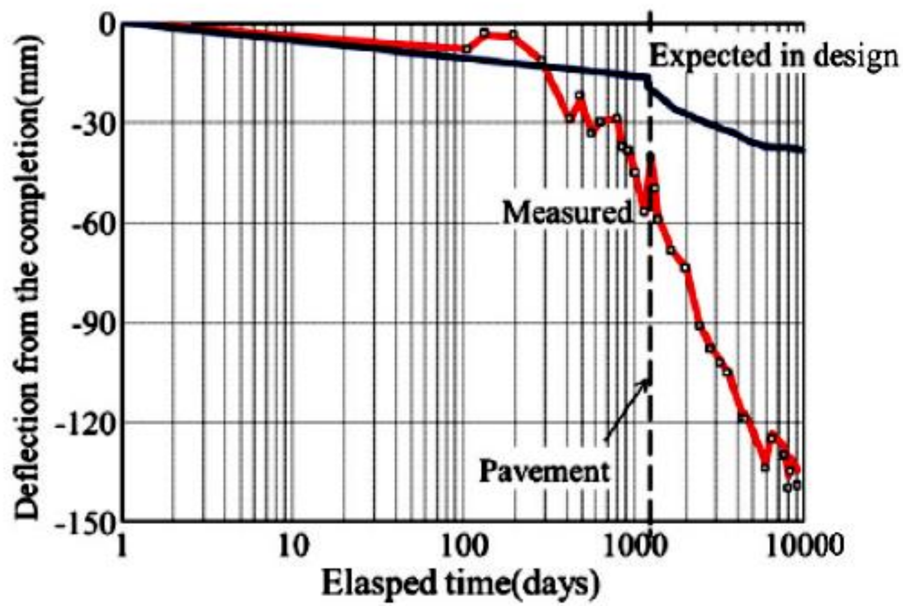


Fig. 3. Difference in predicted and measured values of deflection (the component caused by the overlaid pavement is taken out from the entire deflection).

END of Lectures

Elective reading:

LOGISTYKA - NAUKA

viscoelasticity, time-domain viscoelasticity, frequency-domain viscoelasticity, rheological model, generalized maxwell model, prony series expansion, asphalt concrete, complex modulus, flexible pavement design

MICHALCZYK Rafał[1]

IMPLEMENTATION OF GENERALIZED VISCOELASTIC MATERIAL MODEL IN ABAQUS CODE

An accurate description of behavior of bituminous mixes is necessary to adequately predict and evaluate the time dependent characteristics and the evolution of pavement distress. The generalized Maxwell model, currently considered to be one of the most suitable to characterize mechanical behavior of viscoelastic materials and widely used in many commercial FEM codes, is presented. A Prony series expansion implemented in ABAQUS code, is used to express the material's behavior. The theoretical background of the model is briefly discussed. Finally, the model was efficiently used for numerical simulation of stress relaxation experiment.

[1]Warsaw University of Technology, Faculty of Civil Engineering, POLAND;
Warsaw 00-637; Al. Armii Ludowej 16. Phone: +48 22 825-35-72, 234-65-43,
Fax: +48 22 825-89-46, E-mail: r.michalczyk@il.pw.edu.pl

Elective reading:

The elastic-viscoelastic analogy

Also known as: *The elastic-viscoelastic correspondence principle*

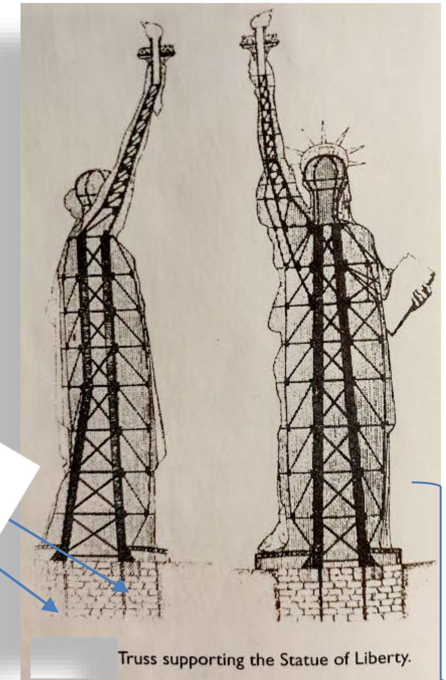
Elastis-viskoelastinen analogia

Appendix 1

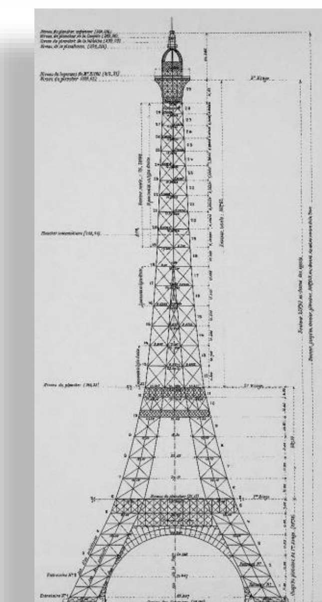
Additional reading
Literature
&
Miscellaneous

Q: what the term '*viscoelasticity*' or '*viskoelastisuus*' brings you to mind? *5min…*

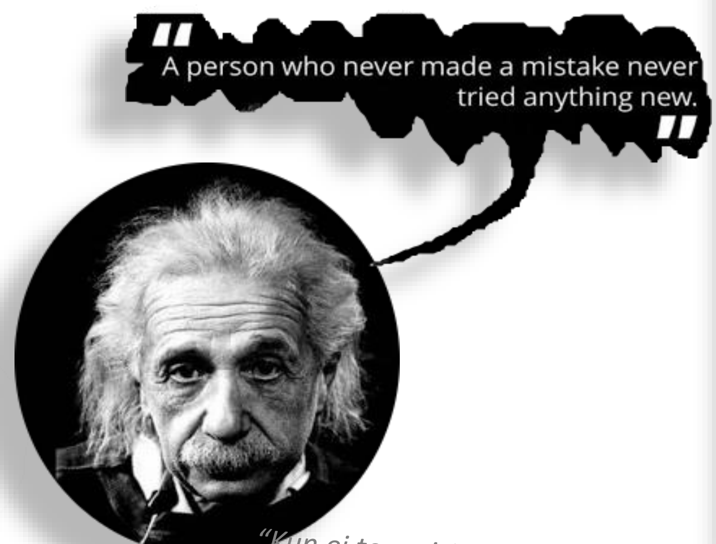
Stress relaxation in anchoring cables


Truss supporting the Statue of Liberty.

What is the relation?

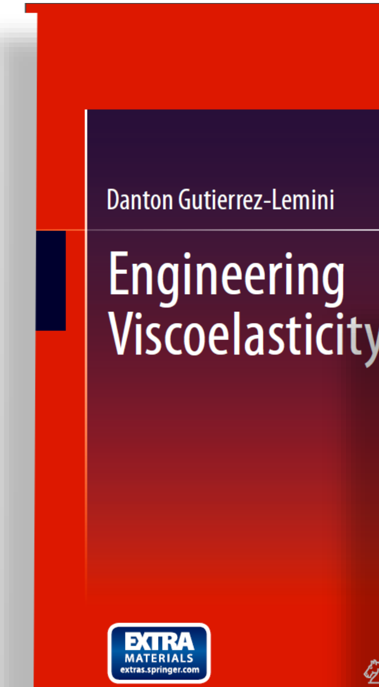


Good to know:
Structural Mechanics and Geotechnics - Vocabulary in Finnish, Swedish and English with equations and figures…



"Kun ei tee mitään, ei tee virkavirheitäkään (old saying)"

Not compulsory literature 1(2)



Reading: A concise course from MIT:

ENGINEERING VISCOELASTICITY

David Roylance
Department of Materials Science and Engineering
Massachusetts Institute of Technology
Cambridge, MA 02139

October 24, 2001

1Cours__MIT_visco.pdf

1 Introduction

This document is intended to outline an important aspect of the mechanical response of polymers and polymer-matrix composites: the field of *linear viscoelasticity*. The topics included here are aimed at providing an instructional introduction to this large and elegant subject, and should not be taken as a thorough or comprehensive treatment. The references appearing either as footnotes to the text or listed separately at the end of the notes should be consulted for more thorough coverage.

Viscoelastic response is often used as a probe in polymer science, since it is sensitive to the material's chemistry and microstructure. The concepts and techniques presented here are important for this purpose, but the principal objective of this document is to demonstrate how linear viscoelasticity can be incorporated into the general theory of mechanics of materials, so that structures containing viscoelastic components can be designed and analyzed.

While not all polymers are viscoelastic to any important practical extent, and even fewer are *linearly* viscoelastic[1], this theory provides a usable engineering approximation for many applications in polymer and composites engineering. Even in instances requiring more elaborate treatments, the linear viscoelastic theory is a useful starting point.

2 Molecular Mechanisms

When subjected to an applied stress, polymers may deform by either or both of two fundamentally different atomistic mechanisms. The lengths and angles of the chemical bonds connecting the atoms may distort, moving the atoms to new positions of greater internal energy. This is a small motion and occurs very quickly, requiring only $\approx 10^{-12}$ seconds.

If the polymer has sufficient molecular mobility, larger-scale rearrangements of the atoms may also be possible. For instance, the relatively facile rotation around backbone carbon-carbon single bonds can produce large changes in the conformation of the molecule. Depending on the mobility, a polymer molecule can extend itself in the direction of the applied stress, which decreases its conformational entropy (the molecule is less "disordered"). Elastomers — rubber — respond almost wholly by this entropic mechanism, with little distortion of their covalent bonds or change in their internal energy.

[1]For an overview of *nonlinear* viscoelastic theory, see for instance W.N. Findley et al., *Creep and Relaxation of Nonlinear Viscoelastic Materials*, Dover Publications, New York, 1989.

Have a look:

https://ocw.mit.edu/courses/

http://web.mit.edu/course/3/3.11/www/modules/visco.pdf (30.11.2016)

Not compulsory literature 1(2)

Have a look:

- CONTINUUM MECHANICS for ENGINEERS
- Chapter 01: Continuum Theory
- Chapter 02: Essential Mathematics
- Chapter 03: Stress Principles
- Chapter 04: Kinematics of Deformation and Motion
- Chapter 05: Fundamental Laws and Equations
- Chapter 06: Linear Elasticity
- Chapter 07: Classical Fluids
- Chapter 08: Nonlinear Elasticity
- Chapter 09: Linear Viscoelasticity
  - CONTINUUM MECHANICS for ENGINEERS
    - Table of Contents
    - Linear Viscoelasticity
      - 9.1 Introduction
      - 9.2 Viscoelastic Constitutive Equations in Linear Differential Operator Form
      - 9.3 One-Dimensional Theory, Mechanical Models
      - 9.4 Creep and Relaxation
      - 9.5 Superposition Principle, Hereditary Integrals
      - 9.6 Harmonic Loadings, Complex Modulus, and Complex Compliance
      - 9.7 Three-Dimensional Problems, The Correspondence Principle
      - References

CONTINUUM MECHANICS for ENGINEERS

Second Edition

Creep and Shrinkage of Concrete Elements and Structures

ZDENĚK ŠMERDA
VLADIMÍR KŘÍSTEK

1988

SNTL – PUBLISHERS OF TECHNICAL LITERATURE, PRAGUE

Some experimental observations

Examples of viscoelastic materials: stress – strain behavior

Three type of behavior of polymers:

1. **Brittle**
2. **Plastic**
3. **Elastomer**