Table 3.2. *Approximate values of coefficients of friction*

Metal	Temperature °C	Lubrication	f	$\bar{m}$
Mild steel	20	Dry	0.25	0.5
Mild steel	950	Dry	0.45	0.3–1
Mild steel	20	Oil	0.03	0.05
Steel	20	Soaps	0.05	0.07
Aluminium	20	Oil	0.05	0.15
Copper	20	Dry	0.1	0.9
Copper	850	Graphite	0.25	0.2

contact of materials at high temperatures. Table 3.2 gives several characteristic values of $\bar{m}$.

Bibliography

Persoz B. *Introduction à l'étude de la rhéologie*. Dunod, Paris (1960).

Malvern L. E. *Introduction to the mechanics of a continuous medium*. Prentice Hall New Jersey (1969).

Mandel J. *Propriétés mécaniques des matériaux*. Eyrolles, Paris (1978).

Lieurade H. P. *La pratique des essais de fatigue*. P.Y.C. Edition, Paris (1982).

Broyden C. G., Fletcher R., Murray W., Powel M. J. D. & Swann W. M. *Numerical methods for unconstrained optimization*. Murray Ed. (1972).

Loveday M. S., Day M. F., Dyson B. F. *Measurement of high temperature mechanical properties of materials*. National Physical Laboratory. Her Majesty's Stationery Office, London (1982).

ASTM. *Metals Handbook*: Vols. 1–12. American Society for Metals, Philadelphia.

4

LINEAR ELASTICITY, THERMOELASTICITY AND VISCOELASTICITY

Une loi est un modèle qui n'est plus (et pas encore!) contesté.

Hooke (1676), Young (1807), Cauchy (1822) and Timoshenko (1934) have said almost everything about the linear-elastic behaviour of materials. In this field, we will therefore limit ourselves to giving a summary in the form of formulae. However, we will deal with anisotropic elasticity, which is so important for composite materials, and the identification of the coefficients. The term elasticity is taken here to mean the reversible deformations mentioned in Chapter 3. Neither thermal dissipation (thermoelasticity) nor mechanical dissipation (viscoelasticity) is excluded.

Thermoelasticity introduces several additional coefficients into the constitutive law including the dilatation coefficient, and permits the treatment of problems involving temperature variations, such as thermal stress analysis problems.

The theory of viscoelasticity was developed considerably with rheology during the 1960s. We will restrict ourselves in this chapter to linear viscoelasticity which is sufficient to deal with the mechanical behaviour of certain polymers.

These theories are continuum theories; the inhomogeneities are assumed to be small enough with respect to the size of the volume element that the results of experiments conducted on the latter are really the characteristics of average macroscopic behaviour. This is also true for composite materials and concrete whose behaviour can be more accurately described by homogenization techniques.

4.1   Elasticity

4.1.1   *Domain of validity and use*

All solid materials possess a domain in the stress space within which a load variation results only in a variation of elastic strains. As discussed in

Chapter 1, these elastic strains consist of reversible movements of atoms, molecules, or cells, corresponding to strains not exceeding $(0.2-0.5) \times 10^{-2}$ for metals, composites, concrete and wood. This fact, therefore, justifies the hypothesis of small strain elastic behaviour. The dissipation is zero.

The limit of the elastic domain, the exact study of which is presented in Chapter 5, depends on the temperature and on the previous loadings, experienced during the production process or in service as a mechanical part. The temperature, while conducive to irreversible movements, at the same time decreases the elastic domain until it becomes zero at the melting point. On the other hand, mechanical loads can considerably increase this domain as a result of the hardening phenomenon.

Thus, before applying the theory of linear elasticity, it is necessary to make sure that the order of magnitude of the stresses is indeed compatible with the elastic domain of the material under consideration as defined in Chapter 5. It may not always be possible to do so '*a priori*' but it can always be done '*a posteriori*'.

The hypothesis of elastic isotropy has been amply verified for metallic materials and a little less so for concrete. For composites and wood, however, it is necessary to use special anisotropic theories of elasticity.

4.1.2   *Formulation*

The method of local state presented in Chapter 2 is followed. The schematic representation of an elastic solid, as defined in Chapter 3, shows that the state variables to be used are temperature and elastic strain. If we restrict ourselves to isothermal deformations, only the elastic strain equal to the total strain appears as a state variable; the temperature acts as a parameter on which the elastic moduli depend. Since only small deformations are considered, there is no need to distinguish between the initial and final configurations of the body.

Thermodynamic potential

In order to obtain a linear theory, it is sufficient to choose as a convex thermodynamic potential a positive definite quadratic function in the components of the strain tensor.

Let

$$\Psi = (1/2\rho)\mathbf{a}:\varepsilon:\varepsilon$$

where the symbol : denotes the contraction of the tensorial product on two indices and where ρ is the density and $\mathbf{a}$ is a fourth order tensor whose components are the elastic moduli which may depend on the temperature.

By definition of the associated variable, the stress tensor $\boldsymbol{\sigma}$ is derived from the potential Ψ to give the law of state:

$$\boldsymbol{\sigma} = \rho(\partial\Psi/\partial\varepsilon) = \mathbf{a}:\varepsilon.$$

The tensor $\mathbf{a}$ satisfies certain symmetry properties. Since

$$\sigma_{ij} = \rho(\partial\Psi/\partial\varepsilon_{ij}) = a_{ijkl}\varepsilon_{kl} \quad \text{and} \quad \sigma_{kl} = a_{klij}\varepsilon_{ij}$$

we prove the following symmetry of the moduli by differentiating the above:

$$a_{ijkl} = \frac{\partial\sigma_{ij}}{\partial\varepsilon_{kl}} = \rho\frac{\partial^2\Psi}{\partial\varepsilon_{ij}\partial\varepsilon_{kl}} = \rho\frac{\partial^2\Psi}{\partial\varepsilon_{kl}\partial\varepsilon_{ij}} = \frac{\partial\sigma_{kl}}{\partial\varepsilon_{ij}} = a_{klij}.$$

The relation

● $\quad\quad\boldsymbol{\sigma} = \mathbf{a}:\varepsilon$

is the generalized Hooke's law for a linear, isotropic, elastic continuum which depends on 21 independent coefficients of the symmetric tensor of the elastic moduli. By inverting the above relations, or by defining a dual potential:

$$\Psi^* = (1/2\rho)\mathbf{A}:\boldsymbol{\sigma}:\boldsymbol{\sigma}$$

where $\mathbf{A}$ is a symmetric fourth order tensor of elastic compliances, we have:

$$\varepsilon = \mathbf{A}:\boldsymbol{\sigma}.$$

In the case of nonlinear elasticity, one chooses other forms of the potential function Ψ which, of course, possess the convexity property.

Linear isotropic elasticity

Isotropy and linearity require that the potential Ψ be a quadratic invariant of the strain tensor, i.e. a linear combination of the square of the first invariant $\varepsilon_I^2 = [\text{Tr}(\varepsilon)]^2$ and the second invariant $\varepsilon_{II}^2 = \frac{1}{2}\text{Tr}(\varepsilon^2)$:

$$\Psi = (1/2\rho)(\lambda\varepsilon_I^2 + 4\mu\varepsilon_{II})$$

and

● $\quad\quad\boldsymbol{\sigma} = \rho(\partial\Psi/\partial\varepsilon) = \lambda\,\text{Tr}(\varepsilon)\mathbf{1} + 2\mu\varepsilon$

where $\mathbf{1}$ is the unit tensor, or

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$$

where λ and μ are Lame's two coefficients. Employing the dual form:

$$\varepsilon = \mathbf{A}:\boldsymbol{\sigma}$$

we find that

● $\quad\quad\varepsilon = \dfrac{1+\nu}{E}\boldsymbol{\sigma} - \dfrac{\nu}{E}\text{Tr}(\boldsymbol{\sigma})\mathbf{1}$

or $\varepsilon_{ij} = \dfrac{1+v}{E}\sigma_{ij} - \dfrac{v}{E}\sigma_{kk}\delta_{ij}$

where E is Young's modulus and v is Poisson's ratio. Their values for different materials are given in the table in Section 4.1.4.

Alternatively, the relations between the spherical parts:

$$\sigma_H = \tfrac{1}{3}\mathrm{Tr}(\boldsymbol{\sigma})$$

$$\varepsilon_H = \tfrac{1}{3}\mathrm{Tr}(\boldsymbol{\varepsilon})$$

and the deviatoric parts:

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} - \sigma_H \mathbf{1}$$

$$\boldsymbol{\varepsilon}' = \boldsymbol{\varepsilon} - \varepsilon_H \mathbf{1}$$

are also useful, namely:

● $\varepsilon_H = \dfrac{1-2v}{E}\sigma_H,$

● $\boldsymbol{\varepsilon}' = \dfrac{1+v}{E}\boldsymbol{\sigma}'.$

Finally, we list some well-known relations between Lamé's coefficients, Young's modulus, Poisson's ratio, shear modulus and the bulk modulus in hydrostatic compression:

$$\lambda = \dfrac{vE}{(1+v)(1-2v)}, \quad \mu = \dfrac{E}{2(1+v)}$$

$$E = \mu\dfrac{3\lambda + 2\mu}{\lambda + \mu}, \qquad v = \dfrac{\lambda}{2(\lambda + \mu)}$$

$$G = \dfrac{E}{2(1+v)} = \mu, \quad K = \dfrac{E}{3(1-2v)} = \dfrac{3\lambda + 2\mu}{3}.$$

In the isotropic elasticity law derived from a quadratic potential, the only condition which must be satisfied by the coefficients (modulus E is assumed positive) is:

$$-1 < v < \tfrac{1}{2}.$$

This condition follows immediately from the general case of orthotropic elasticity (see Section 4.1.2).

Plane stress assumption
This is used, for example, for thin plates and is:

$$\sigma_{33} = \sigma_{13} = \sigma_{23} = 0.$$

Using $\sigma_{33} = 0$, the two-dimensional stress–strain relation can be easily obtained as:

$$\sigma_{ij} = \lambda'\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} \qquad (i \text{ and } j = 1, 2)$$

where

$$\lambda' = \dfrac{2\lambda\mu}{\lambda + 2\mu} = \dfrac{vE}{1 - v^2}.$$

The relation between $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ becomes:

$$\left\{\begin{array}{c} \varepsilon_{11} \\ \\ \varepsilon_{22} \\ \\ \varepsilon_{12} \end{array}\right\} = \left[\begin{array}{ccc} \dfrac{1}{E} & -\dfrac{v}{E} & 0 \\ -\dfrac{v}{E} & \dfrac{1}{E} & 0 \\ 0 & 0 & \dfrac{1+v}{E} \end{array}\right] \left\{\begin{array}{c} \sigma_{11} \\ \\ \sigma_{22} \\ \\ \sigma_{12} \end{array}\right\}$$

$$\varepsilon_{33} = -\dfrac{v}{1-v}(\varepsilon_{11} + \varepsilon_{22})$$

$$\varepsilon_{13} = 0$$

$$\varepsilon_{23} = 0.$$

Plane strain assumption
This is used for, example, for thick plates and is:

$$\varepsilon_{33} = \varepsilon_{13} = \varepsilon_{23} = 0$$

$$\left\{\begin{array}{c} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{array}\right\} = \left[\begin{array}{ccc} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & 2\mu \end{array}\right] \left\{\begin{array}{c} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{array}\right\}$$

$$\sigma_{33} = \dfrac{\lambda}{2(\lambda + \mu)}(\sigma_{11} + \sigma_{22}) = v(\sigma_{11} + \sigma_{22})$$

$$\sigma_{13} = 0$$

$$\sigma_{23} = 0.$$

Orthotropic elasticity

A medium is said to be orthotropic for a given property if this property is invariant with respect to a change in direction obtained by symmetry relative to two orthogonal planes (the symmetry with respect to the third

orthogonal plane then necessarily follows). In other words, if the elastic characteristics of specimens cut along the axes of symmetry with respect to a particular plane P_1 and along those with respect to another orthogonal plane P_2 are identical, then the material is orthotropic with P_1 and P_2 as the planes of orthotropy. The intersections of the three symmetry planes define the principal axes of orthotropy. This model provides a good represent-ation of the properties of not only unidirectional composites and wood, but also of laminated metallic products.

The compliance matrix of an orthotropic material contains only nine independent coefficients. Indeed it is possible to write the relations between stress and strain in a system of coordinates in which axis 1 forms an angle α or $-\alpha$ with one of the principal axes of orthotropy:

$$\{\varepsilon\}^{(\alpha)} = [A]^{(\alpha)}\{\sigma\}^{(\alpha)}$$
$$\{\varepsilon\}^{(-\alpha)} = [A]^{(-\alpha)}\{\sigma\}^{(-\alpha)}.$$

The matrices $[A]^{(\alpha)}$ and $[A]^{(-\alpha)}$ differ only by elementary rotations due to the change of axes. By definition, orthotropy means that $\{\varepsilon\}^{(\alpha)} = \{\varepsilon\}^{(-\alpha)}$ if identical stresses are applied along the α and $-\alpha$ sets of axes, i.e., $\{\sigma\}^{(\alpha)} = \{\sigma\}^{(-\alpha)}$. This remains true for arbitrary α and for the six independent components of strain. We, therefore, obtain six relations between the elastic compliance coefficients. Another six relations are obtained by similar considerations with respect to the second principal axis of orthotropy (those with respect to the third axis are automatically satisfied). Thus there remain $21 - 6 - 6 = 9$ independent coefficients.

With the principal axes of orthotropy as the reference axes, the elasticity law can be expressed in the following form:

$$
\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{31} \\ \varepsilon_{12} \end{Bmatrix} =
\begin{bmatrix}
\dfrac{1}{E_1} & -\dfrac{v_{12}}{E_1} & -\dfrac{v_{13}}{E_1} & 0 & 0 & 0 \\
-\dfrac{v_{21}}{E_2} & \dfrac{1}{E_2} & -\dfrac{v_{23}}{E_2} & 0 & 0 & 0 \\
-\dfrac{v_{31}}{E_3} & -\dfrac{v_{32}}{E_3} & \dfrac{1}{E_3} & 0 & 0 & 0 \\
0 & 0 & 0 & \dfrac{1}{2G_{23}} & 0 & 0 \\
0 & 0 & 0 & 0 & \dfrac{1}{2G_{31}} & 0 \\
0 & 0 & 0 & 0 & 0 & \dfrac{1}{2G_{12}}
\end{bmatrix}
\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{Bmatrix}
$$

with the symmetry conditions:

$$\frac{v_{12}}{E_1} = \frac{v_{21}}{E_2}, \quad \frac{v_{13}}{E_1} = \frac{v_{31}}{E_3}, \quad \frac{v_{23}}{E_2} = \frac{v_{32}}{E_3}.$$

The nine independent coefficients are:

three moduli of extension E_1, E_2, E_3 in the directions of orthotropy
three shear moduli G_{12}, G_{23}, G_{31}
three coefficients of contraction for example, v_{12}, v_{23}, v_{31}.

These coefficients must satisfy certain inequalities to ensure that the thermodynamic potential used in the derivation of the above laws is indeed a positive definite quadratic function:

$$\rho\Psi = \tfrac{1}{2}\varepsilon:\varepsilon = \tfrac{1}{2}\sigma:\varepsilon = \tfrac{1}{2}\sigma:\mathbf{A}:\sigma.$$

If we consider the extensional and shear moduli to be positive quantities, then Ψ will be a positive, definite quadratic function if the eigenvalues of the matrix of the contraction coefficients are all strictly positive. This condition requires that the following inequalities are satisfied:

$$1 - v_{12}v_{21} > 0, \quad 1 - v_{23}v_{31} > 0, \quad 1 - v_{13}v_{31} > 0$$
$$1 - v_{12}v_{23}v_{31} - v_{21}v_{13}v_{32} - v_{21}v_{12} - v_{31}v_{13} - v_{32}v_{23} > 0.$$

Examples of the values of orthotropic elasticity coefficients for some materials are given in the table in Section 4.1.4.

Transversely isotropic elasticity

An elastic medium is transversely isotropic if its elastic characteristics remain invariant for all pairs of directions symmetric with respect to an axis. Accordingly, if $\vec{x}_3$ is the axis of transverse isotropy then the material is 'isotropic' in planes normal to $\vec{x}_3$. A laminated composite obtained by the bonding of layers formed from unidirectional fibres and stacked in 'all' directions, can be considered as a transversely isotropic material. The same is true for alloys obtained from oriented solidification.

A proof, analogous to that for orthotropic material, shows that in the case of transverse isotropy only five elastic coefficients can be independent. More precisely, the property of isotropy perpendicular to the x_3 direction

implies that:

$$
\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{13} \end{Bmatrix}
=
\begin{bmatrix}
\dfrac{1}{E_1} & -\dfrac{v_{12}}{E_1} & -\dfrac{v_{13}}{E_1} & 0 & 0 & 0 \\[2mm]
-\dfrac{v_{12}}{E_1} & \dfrac{1}{E_1} & -\dfrac{v_{13}}{E_1} & 0 & 0 & 0 \\[2mm]
-\dfrac{v_{13}}{E_1} & -\dfrac{v_{13}}{E_1} & \dfrac{1}{E_3} & 0 & 0 & 0 \\[2mm]
0 & 0 & 0 & \dfrac{1+v_{12}}{E_1} & 0 & 0 \\[2mm]
0 & 0 & 0 & 0 & \dfrac{1}{2G_{13}} & 0 \\[2mm]
0 & 0 & 0 & 0 & 0 & \dfrac{1}{2G_{13}}
\end{bmatrix}
\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{Bmatrix}.
$$

With reference to the orthotropic case, we have the following equalities:

$E_1 = E_2$, same extensional moduli in directions 1 and 2;
$v_{13}/E_1 = v_{23}/E_2$, same contraction coefficient in directions 1 and 2 for a tension applied in direction 3;
$G_{13} = G_{23}$, same shear moduli for shear in planes perpendicular to axis 2 and axis 1;
$2G_{12} = E_1/(1 + v_{12})$, shear modulus around axis 3.

The five coefficients which characterize the material are:

two extensional moduli E_1 and E_3,
one shear modulus G_{13},
two coefficients of contraction v_{12} and v_{13}.

Values of these coefficients for some materials are listed in the table in Section 4.1.4.

4.1.3   *Identification*

Identification of the characteristic coefficients of elasticity consists in evaluating stiffnesses in tension–compression static tests, in vibration tests, or in wave propagation tests. These three broad types of tests do not yield exactly the same results for the same material because dynamic methods do not take into account certain viscous movements of atoms or molecules, and accordingly, give slightly higher values of the moduli.

Static methods

These methods are recommended for the calculation of the deformations and the static resistance of structures. Characterization of materials at room temperature or at temperatures below 200–300 °C is done by uniaxial tension or compression tests on standard specimens in which the axial strain ε_{11} and the transverse strains $\varepsilon_{22}, \varepsilon_{33}$ are measured by electric strain gauges.

Isotropic elastic case
The axial stress σ_{11} is determined by measuring the loading force on the specimen (see Chapter 3) and we obtain:

- $E = \sigma_{11}/\varepsilon_{11}$
- $v = -\varepsilon_{22}/\varepsilon_{11}$.

The relative accuracy generally obtained is of the order of 1%.

Orthotropic elastic case
The nine characteristic coefficients of orthotropic elasticity can be obtained by tension or compression tests on specimens cut out along the three directions of orthotropy and in three directions at $\pi/4$ (Fig. 4.1). To determine the coefficients from these tests, it is necessary to express by means of the orthotropic constitutive law, the relation between the normal stress $\sigma^{(n)}$ and the normal strain $\varepsilon^{(n)}$ in any direction $\vec{n}$ with respect to the axes of orthotropy (Fig. 4.1).

$$\vec{n} = \vec{x}_1 \sin \alpha \cos \Psi + \vec{x}_2 \sin \alpha \sin \Psi + \vec{x}_3 \cos \alpha.$$

A uniaxial stress σ^* in a reference system with $\vec{n}$ as a base vector has the

Fig. 4.1.   Directions for the characterization of elastic orthotropy.
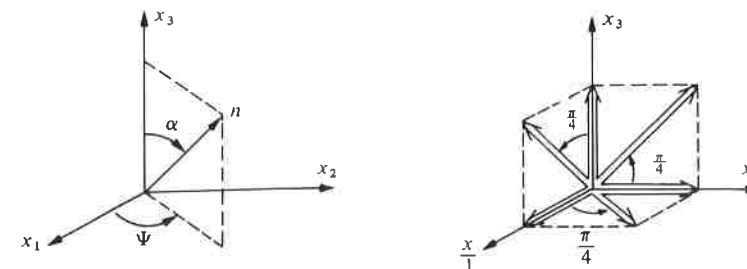
following components with respect to the orthotropy axes (x_1, x_2, x_3).

$$[\sigma] = \sigma^* \begin{bmatrix} \sin^2 \alpha \cos^2 \Psi & \sin^2 \alpha \sin \Psi \cos \Psi & \sin \alpha \cos \alpha \cos \Psi \\ \sin^2 \alpha \sin \Psi \cos \Psi & \sin^2 \alpha \sin^2 \Psi & \sin \alpha \cos \alpha \sin \Psi \\ \sin \alpha \cos \alpha \cos \Psi & \sin \alpha \cos \alpha \sin \Psi & \cos^2 \alpha \end{bmatrix}$$

The corresponding strain components are deduced from the orthotropic constitutive law:

$$\varepsilon = \mathbf{A} : \sigma.$$

We can then compute the following quantities depending on the measurements of $\sigma^{(n)}$, $\varepsilon_{11}^{(n)}$, $\varepsilon_{22}^{(n)}$, $\varepsilon_{33}^{(n)}$ for the six tension tests in the six particular directions:

$$\vec{n} = \vec{x}_1 \to E_1 = \frac{\sigma^{(n)}}{\varepsilon_{11}^{(n)}}, \quad v_{12} = -\frac{\varepsilon_{22}^{(n)}}{\varepsilon_{11}^{(n)}}, \quad v_{13} = -\frac{\varepsilon_{33}^{(n)}}{\varepsilon_{11}^{(n)}}$$

$$\vec{n} = \vec{x}_2 \to E_2 = \frac{\sigma^{(n)}}{\varepsilon_{22}^{(n)}}, \quad v_{21} = -\frac{\varepsilon_{11}^{(n)}}{\varepsilon_{22}^{(n)}}, \quad v_{23} = -\frac{\varepsilon_{33}^{(n)}}{\varepsilon_{22}^{(n)}}$$

$$\vec{n} = \vec{x}_3 \to E_3 = \frac{\sigma^{(n)}}{\varepsilon_{33}^{(n)}}, \quad v_{31} = -\frac{\varepsilon_{11}^{(n)}}{\varepsilon_{33}^{(n)}}, \quad v_{32} = -\frac{\varepsilon_{22}^{(n)}}{\varepsilon_{33}^{(n)}}$$

$$\vec{n} = \frac{\sqrt{2}}{2}(\vec{x}_1 + \vec{x}_2) \to \frac{1}{G_{12}} = 4\frac{\varepsilon^{(n)}}{\sigma^{(n)}} - \frac{1 - v_{21}}{E_2} - \frac{1 - v_{12}}{E_1}$$

$$\vec{n} = \frac{\sqrt{2}}{2}(\vec{x}_2 + \vec{x}_3) \to \frac{1}{G_{23}} = 4\frac{\varepsilon^{(n)}}{\sigma^{(n)}} - \frac{1 - v_{23}}{E_2} - \frac{1 - v_{32}}{E_3}$$

$$\vec{n} = \frac{\sqrt{2}}{2}(\vec{x}_3 + \vec{x}_1) \to \frac{1}{G_{31}} = 4\frac{\varepsilon^{(n)}}{\sigma^{(n)}} - \frac{1 - v_{13}}{E_1} - \frac{1 - v_{31}}{E_3}.$$

Tests in pure shear or in shear by the torsion of tubes allow interesting cross-checking. The relative precision obtained is of the order of 1–5%.

Dynamic methods

In the low or average frequency range (10–100 Hz), we may determine Young's modulus by measuring the first natural frequencies of flexural vibrations of a beam. For example, for a cantilever beam fixed at one end and free at the other, the first natural frequency is given by:

$$f = \frac{\omega}{2\pi} = \frac{3.5156}{2\pi L^2}\left(\frac{EI}{\rho S}\right)^{1/2}$$

from which

● $$E = \omega^2 \frac{\rho S L^4}{12.35961I}$$

where S is the (constant) cross-sectional area of the beam, L is its length, I is its flexural moment of inertia, ρ is the mass density of the material, and ω is the first natural frequency in radians obtained from the test measurements.

In the high frequency range ($> 20\,000$ Hz) the measurement of the period of traverse of ultrasonic waves can be used to determine the elastic constants. It can be shown that in a linear isotropic medium of elastic constants E and v and of density ρ, the rate of the propagation v_L of the longitudinal waves, and v_T that of the transverse waves, are given by:

$$v_L^2 = \frac{E}{\rho} \frac{1 - v}{(1 + v)(1 - 2v)}, \quad v_T^2 = \frac{E}{\rho} \frac{1}{2(1 + v)}$$

from which it follows that:

● $$E = \rho v_T^2 \frac{3v_L^2 - 4v_T^2}{v_L^2 - v_T^2}$$

● $$v = \frac{1}{2}\frac{v_L^2 - 2v_T^2}{v_L^2 - v_T^2}.$$

With ultrasonic techniques, it is possible to measure the time of traverse and therefore the rates with a relative precision of the order of 10^{-4} which yields a relative precision of 3×10^{-3} for Young's modulus E and Poisson's ratio v.

4.1.4   *Table of elastic properties of common materials*

Material	Temperature °C	Young's modulus MPa	Poisson's ratio
Aluminium alloy 2024	20	72 000	0.32
	200	66 000	0.325
	500	50 000	0.35
Titanium alloy Ti 4Al 4Mn	20	115 000	0.34
	200	103 400	
	315	95 000	
XC10 carbon steel	20	216 000	0.29
	200	205 000	0.30
	600	170 000	0.315
Grey cast iron	20	100 000	0.29

4.1.4 *Table of elastic properties of common materials*

Material	Temperature °C	Young's modulus MPa	Poisson's ratio
A316 stainless steel	20	196 000	0.3
	200	170 000	
	700	131 000	
A5 aluminimum	20	68 000	0.33
Bronze	180	61 000	
	20	130 000	0.34
Plexiglass	20	2 900	0.4
Araldite	20	3 000	0.4
Rubber	20	2	0.5
Unidirectional composites glass-epoxy (in long. direction)	20	19 000	0.30
Unidirectional composite carbon–epoxy (in long. direction)	20	87 600	0.32
Concrete	20	30 000	0.2
Granite	20	60 000	0.27
Wood: pinewood fibre direction	20	17 000	0.45
transverse direction	20	1 000	0.79

4.1.5 *Concepts of the finite element method*

We outline here the basis of the finite element method which is considered as a standard numerical method of solving field problems in elasticity. Other methods exist. In particular, those based on integral equation formulation, but we will not discuss them here.

Generalities

The objective of approximate numerical methods of structural analysis is to find in a class of 'reasonable' displacement (or stress) fields those which

minimize the potential energy $\mathscr{V}$ for kinematically admissible fields,

maximize the complementary potential energy $\mathscr{V}^*$ for statically admissible fields.

The finite element method consists in calculating the potential energy $\mathscr{V}$ or $\mathscr{V}^*$ as the sum of the energies of all the elements of the structure divided

into finite elements (these can be line, surface or volume elements depending on the schematic representation of the structure).

In each element the unknown fields are represented by a linear combination of functions of spatial coordinates. The coefficients of this combination depend on the displacements of the nodes which belong to the element (its vertices for example). These latter quantities are the ones that constitute the unknowns of the problem (nodal unknowns or degrees of freedom). In practice, the finite element method is equivalent to using the principle of virtual work as will be used later.

After imposing equality of nodal displacements at common nodes of elements, minimization of the potential energy with respect to these displacements leads to a system of linear algebraic equations. Thus, a problem of solving linear partial differential equations is replaced by that of solving a set of linear algebraic equations. Having solved the system, we can find the displacement, strains and stresses at points within the elements.

Formulation of the element properties

Numerous types of elements can be developed. A general methodology exists for the derivation of the characteristics of an element with any number of degrees of freedom. Our purpose here will only be to illustrate the principles of the finite element method in the simplest case, that of a plane triangular element with three nodes (Fig. 4.2).

The displacements of a point M identified by the coordinates x_1, x_2 are

Fig. 4.2. Three-node triangle.
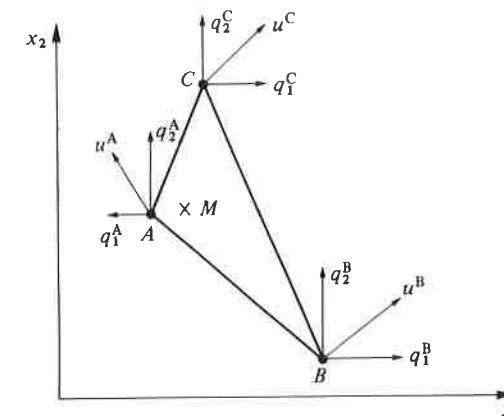
written in the form of a column vector:[†]

$$U = \begin{bmatrix} U_1(x_1, x_2) \\ U_2(x_1, x_2) \end{bmatrix}.$$

The degrees of freedom of the element are the six displacement components belonging to nodes A, B, C:

$$q = \begin{Bmatrix} q_1^A \\ q_2^A \\ q_1^B \\ q_2^B \\ q_1^C \\ q_2^C \end{Bmatrix}.$$

The unknown displacements u are represented by a combination of linear functions:

$$u = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} C_1 + C_2 x_1 + C_3 x_2 \\ C_4 + C_5 x_1 + C_6 x_2 \end{Bmatrix}.$$

Since at a node the displacement functions u must assume values identical to the nodal displacements q, the six coefficients $C_1, \ldots, C_6$ can be determined as linear functions of the six nodal displacements. The displacement function can then be expressed as:

$$u = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} N^{BC}(x_1, x_2)q_1^A + N^{CA}(x_1, x_2)q_1^B + N^{AB}(x_1, x_2)q_1^C \\ N^{BC}(x_1, x_2)q_2^A + N^{CA}(x_1, x_2)q_2^B + N^{AB}(x_1, x_2)q_2^C \end{Bmatrix}.$$

In the present case, the functions $N(x_1, x_2)$, called shape functions, are linear. In abridged notation, these relations may be expressed as:

$$u = Nq.$$

The strain components, expressed as a column vector, are:

$$\varepsilon = \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{Bmatrix} u_{1,1} \\ u_{2,2} \\ u_{1,2} + u_{2,1} \end{Bmatrix} = Bq$$

where B is a 3×6 matrix derived from N by differentiation. The factor 2 in the component $2\varepsilon_{12}$ is used in order to write the strain energy simply as $\sigma^T \varepsilon$

where the stress components are represented by the column vector;[†]

$$\sigma = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \frac{E}{1-v^2} \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} = a\varepsilon$$

in the case of isotropic elasticity and a state of plane stress.

The equilibrium of an element is expressed by invoking the principle of virtual work (see Chapter 2) with respect to an arbitrary kinematically admissible displacement field $u'(M)$ and the associated strain field $\varepsilon'(M)$:

$$\int_{\mathcal{D}} f^T u' \, dV + \int_{\partial \mathcal{D}} F^T u' \, dS - \int_{\mathcal{D}} \sigma^T \varepsilon' \, dV = 0 \qquad \forall u'$$

where f and F are respectively the externally applied body forces acting in $\mathcal{D}$ and the surface forces acting on $\partial \mathcal{D}$. Taking into account the relation between strains and nodal displacements, we may write:

$$\sigma^T \varepsilon' = \{a\varepsilon\}^T \varepsilon' = q^T B^T a B q'$$

where q' represents any choice for the values of the nodal displacements. By substituting the above into the principle of virtual work, we easily obtain:

$$\left(\int_{\mathcal{D}} f^T N \, dV + \int_{\partial \mathcal{D}} F^T N \, dS - q^T \int_{\mathcal{D}} B^T a B \, dV \right) q' = 0$$

or

$$[-q^T K + Q^T]q' = 0 \qquad \forall q'.$$

Since this equation is satisfied for arbitrary q', we conclude that:

$$Kq = Q$$

where K is the stiffness matrix of the element. It is symmetric since:

$$K = \int_{\mathcal{D}} B^T a B \, dV$$

and a is symmetric. Q is the column vector of generalized, externally

applied, nodal forces.

$$Q = \int_{\mathscr{D}} N^T f \, dV + \int_{\partial \mathscr{D}} N^T F \, dS.$$

If the external forces are known, the nodal displacements q are given by the solution of the set of linear equations $Kq = Q$. Knowing q we can determine within the element, the displacements $u = Nq$, the strains $\varepsilon = Bq$, and the stresses $\sigma = a\varepsilon$.

Assembly

To solve the problem of the whole structure (the calculation of q with respect to all elements) the virtual work of the whole structure has to be written by summing on all the elements. At the same time the condition of continuity of structure has to be imposed, which means that the displacements of a node common to several elements are the same for all these elements. Note that if the equality of nodal displacements ensures the equality of displacements at the interelement boundary, then the element is said to be a conforming one. In practice, this assembly operation consists in adding the stiffness contributions (and external forces) from each of the elements which meet at that node. More precisely, letting $^E q$ be the total degrees of freedom of all elements without considering their interconnections, and q be the total degrees of freedom when nodal compatibility is enforced, we may write:

$$^E q = Aq$$

where A is a rectangular matrix since the dimension of q is less than that of $^E q$. Similarly, the matrix $^E K$ and the column vector $^E Q$ obtained by simple juxtaposition of the individual element stiffness matrices and the force vectors are:

$$^E K = \begin{bmatrix} ^1 K & & & \\ & ^2 K & & \\ & & \cdot & \\ & & & \cdot \\ & & & & \cdot \end{bmatrix} \qquad ^E Q = \begin{Bmatrix} ^1 Q \\ ^2 Q \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix}$$

In terms of these matrices, the principle of virtual work can be written as:

$$[-\,^E q^T \, ^E K + \,^E Q^T]^E q' = 0$$

where for $^E q'$ to be kinematically admissible, it must satisfy the relation $^E q' = Aq'$. For the assembled equation this means that:

$$[-(Aq)^T \, ^E K + \,^E Q^T]Aq' = 0 \qquad \forall q'$$

which, in view of the arbitrariness of q', leads to:

$$- A^T \, ^E K Aq + A^T \, ^E Q = 0$$

which may be written as:

$$Kq = Q$$

where K is the assembled stiffness matrix $K = A^T \, ^E K A$, and Q and q are the column vectors which represent the externally applied nodal forces and nodal displacements respectively.

Solution of an elasticity problem by the finite element method thus requires the establishment of the stiffness matrix K and the load vector Q and then the numerical solution of the above set of linear equations. It is often necessary to divide a structure into a large number of elements with numerous degrees of freedom (from 1000 to 10 000 or even more). The matrix K can therefore be very large. However, it possesses a banded structure with nonzero terms around the main diagonal. Outside this band all the coefficients are zero which makes it easier to decompose K. The band width of the assembled stiffness matrix depends on the scheme used to number the nodes of the structures. A large difference in the node numbers of an element results in a large band width and the numbering scheme should preferably minimize this difference.

4.2   Thermoelasticity

Thermoelasticity involves one more variable than elasticity: the temperature. This induces the additional phenomenon of dilatation. The intrinsic dissipation is always zero but thermal dissipation does take place. The theory of thermoleasticity must be used for elastic materials whenever the loads involve variations in temperature. We limit ourselves here to aspects concerned with the constitutive law itself and with the determination of its characteristic coefficients as functions of temperature.

4.2.1   *Formulation*

Let T_0 be a reference temperature of a volume element of an elastic material and T its temperature at a subsequent instant so that the temperature

difference, $\theta = T - T_0$, is small with respect to T_0. Under such conditions the elastic properties and the density ρ can be considered to be constant.

The law of classical linear thermoelasticity is obtained by again selecting as the thermodynamic potential a positive definite quadratic form which, in addition to strain components, has temperature as one of the variables. Assuming isotropy of responses in both the deformation and dilatation phenomena, and taking cognizance of the classically used law, we are led to write:

$$\Psi = \frac{1}{\rho}\left[\tfrac{1}{2}(\lambda\varepsilon_I^2 + 4\mu\varepsilon_{II}) - (3\lambda + 2\mu)\alpha\theta\varepsilon_I\right] - \frac{C_\varepsilon}{2T_0}\theta^2$$

where α and C_ε are two coefficients with a meaning given by the state laws derived from Ψ:

$$\boldsymbol{\sigma} = \rho(\partial\Psi/\partial\boldsymbol{\varepsilon}) = \lambda\,\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon} - (3\lambda + 2\mu)\alpha\theta\mathbf{1}$$

or

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} - (3\lambda + 2\mu)\alpha\theta\delta_{ij}$$

and

$$s = -\frac{\partial\Psi}{\partial T} = -\frac{\partial\Psi}{\partial\theta} = \frac{1}{\rho}(3\lambda + 2\mu)\alpha\,\mathrm{Tr}(\boldsymbol{\varepsilon}) + \frac{C_\varepsilon}{T_0}\theta.$$

This last equation allows us to calculate

$$(\partial s/\partial T)_{\varepsilon = \text{constant}} = (\partial s/\partial\theta)_{\varepsilon = \text{constant}} = C_\varepsilon/T_0.$$

which shows that $C_\varepsilon = T_0(\partial s/\partial T)$ is the specific heat at constant strain.

By inverting the first law of state or by employing the dual potential $\Psi^*(\boldsymbol{\sigma}, \theta)$, we may express strains as functions of stresses and temperature:

$$\boldsymbol{\varepsilon} = \frac{1+\nu}{E}\boldsymbol{\sigma} - \frac{\nu}{E}\mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1} + \alpha\theta\mathbf{1}$$

or

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} + \alpha\theta\delta_{ij}$$

which constitutes the law of linear isotropic thermoelasticity. By considering a variation of temperature with zero stress we obtain:

$$\varepsilon_{ij} = \alpha\theta\delta_{ij}$$

which defines the isotropic dilatation coefficient α.

The coefficients λ, μ, α and C_ε are taken to be constant for small temperature variations ($\theta/T_0 \ll 1$). In practice, they vary with the temperature but 'weakly' as shown by Fig. 4.5.

The thermoelastic constitutive equation enables us to calculate the strains and stresses due to the thermal effect as soon as the temperature field is known. For the treatment of a complete thermoelastic problem in which the flux (and not the temperature itself) is given, it is necessary to use Fourier's law and the heat equation mentioned in Chapter 2.

4.2.2   *Identification*

The thermoelastic constitutive equation is written under the assumption of small temperature variations. If such is not the case, it is used step by step with the variable coefficients E, ν and α; this implies that their evolution as a function of temperature is known.

We are therefore confronted with a major experimental difficulty: the almost impossible task of measuring, with sufficient precision, the local deformation as soon as the temperature exceeds the limit for using electric resistance strain gauges. This limit varies from 200 to 300 °C. On the other hand, it is easy to obtain the dilatation coefficient by measuring the elongation of long wires heated in ovens; some values are given in Fig. 4.5.

For the elastic modulus we may also use rather long wires with an extensometer attached to the part of the wire outside the oven. However, this method raises the problem of the reliability of the fixture used for applying the forces (slipping, wrong definition of the useful length...). The two methods described below represent the means to obtain relatively good values from relatively short samples.

Stiffness comparison method

It is possible to overcome the problem of local measurement by employing two tension test specimens of the same material with identical heads but

Fig. 4.3.   Tension test specimens for the stiffness comparison method.
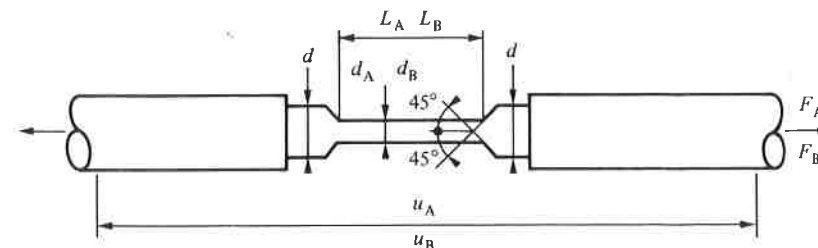
different useful lengths. The only measurements to be taken consist of recording the force F and the displacement u, which comprises the deformation of the useful part of the specimen, of its head, and of the part outside the heating zone and the heating device where the displacement transducer (or extensometer) is attached (Fig. 4.3). This latter part will be called the fixing-line part.

Let there be two samples A and B with dimensions as defined in Fig. 4.3. Let R_A and R_B be the measured stiffnesses depending on the geometry of the specimens and on their unknown Young's modulus E:

$$R_A = \delta F/\delta u_A, \quad R_B = \delta F/\delta u_B.$$

The stiffnesses of the useful parts of the two specimens are:

$$R_A^u = ES_A/L_A, \quad R_B^u = ES_B/L_B \quad \text{with } S = \pi d^2/4.$$

The second unknown is the stiffness R_m of the identical heads of the two specimens and of the fixing-line which is included in the measurement done with the extensometer. As a first approximation, it is permissible to consider that the measurement includes contributions from the conical grips (the difference between the two grips being very minimal). Moreover, the effects of stress concentration are considered negligible. We, therefore, obtain two equations in two unknowns:

$$\frac{1}{R_A} = \frac{1}{R_A^u} + \frac{1}{R_m}$$

$$\frac{1}{R_B} = \frac{1}{R_B^u} + \frac{1}{R_m}$$

which can easily be solved to obtain E:

$$\bullet \quad E = \frac{L_A/S_A - L_B/S_B}{1/R_A - 1/R_B}.$$

The precision of the method depends on the relative orders of magnitude of the different terms which appear in the expression for E. Generally $L_A/S_A = 2$–4 times L_B/S_B represents a good compromise between the numerical aspect of precision and the experimental aspect which requires the machine to load the specimen A without exceeding the elastic limit of the material. This method gives a relative precision of the order of $\pm 2\%$. The same principle used for torsion tests provides values of the shear modulus G, and when E is known, of Poisson's ratio, ν. Denoting the torsional stiffness per

unit angle of twist by R^* it can be shown that:

$$\bullet \quad G = \frac{32}{\pi} \left(\frac{L_A/d_A^4 - L_B/d_B^4}{1/R_A^* - 1/R_B^*} \right).$$

Thermal stress method

This method leads directly to the evolution of Young's modulus E and the dilatation coefficient α as functions of temperature, without displacement measurements. The load is compressive, which is of interest in view of the fact that in applications the hottest parts of the structure are usually loaded in compression. Only one specimen is used in this method of testing, but it requires a very stiff testing machine or even a special rig.

A cylindrical compression specimen ($L/d \approx 1.5$) in series with a dynamometer is placed between two thick plates fixed so as to permit almost zero relative displacement between them (Fig. 4.4). Heating of the specimen, by induction for example, induces in the sample a force measured by the dynamometer. Its expression given by the equations of thermoelasticity contains two unknowns E and α. A second piece of information is furnished by the existence of a preload at the reference temperature.

> The specimen with cross-sectional area S and length L is raised to a temperature T_1 which is the state of reference for strains.

Fig. 4.4.   Device for thermal compression without displacement (after Tichkiewitch).
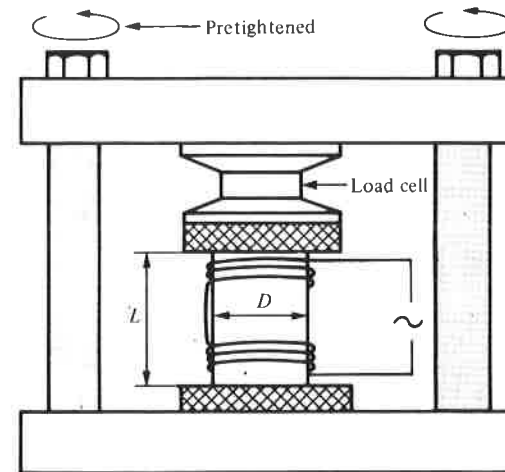
A preload F_1 is applied by the movement of one of the plates (indicated schematically in Fig. 4.4 by tightening of bolts). If E_1 is Young's modulus at the reference temperature T_1, the strain in the specimen is:

$$\varepsilon_1 = F_1/E_1 S.$$

The temperature of the specimen is then raised to T_2 without significant movement of the plates (since the stiffness of the clamps is very large). Denoting the stiffness of the dynamometer-mounted assembly by R_m, the strain in the specimen is given by:

$$\varepsilon_2 = \varepsilon_1 - (F_2 - F_1)/R_m L.$$

The linear thermoelastic law $\varepsilon = \sigma/E + \alpha\theta$ applied between the states $(T_1, F_1 = 0)$ and (T_2, F_2) gives:

$$\varepsilon_2 = \frac{F_2}{E_2 S} + \alpha_{21}(T_2 - T_1)$$

where E_2 and α_{21} denote the Young's modulus at temperature T_2 and the 'secant' coefficient of dilatation between the temperatures T_2 and T_1. Equating the two expressions for ε_2 we obtain the expression:

$$F_2 = F_1 \frac{1/E_1 S + 1/R_m L}{1/E_2 S + 1/R_m L} - \alpha_{21} \frac{T_2 - T_1}{1/E_2 S + 1/R_m L}$$

which shows that F_2 is linearly related to F_1 in the form:

$$F_2 = aF_1 + b.$$

By repeating the same operation several times with different values of preload F_1 but the same temperatures T_1 and T_2, constants a and b can be determined from the measurements of F_1 and F_2. We may then determine E_2 and α_{21} from:

$$E_2 = \frac{E_1 R_m La}{R_m L + E_1 S(1-a)}$$

$$\alpha_{21} = -\frac{b(E_2 S + R_m L)}{E_2 S R_m L(T_2 - T_1)}.$$

Knowing E_1, at a temperature T_1 (for example room temperature), a first experiment gives E_2 at T_2 and also α_{21}. Knowing E_2 at T_2, a

Fig. 4.5. Evolution of (a) the dilatation coefficient α, (b) Young's modulus E and (c) Poisson's ratio v, as functions of the temperature T (after Ben Cheikh).
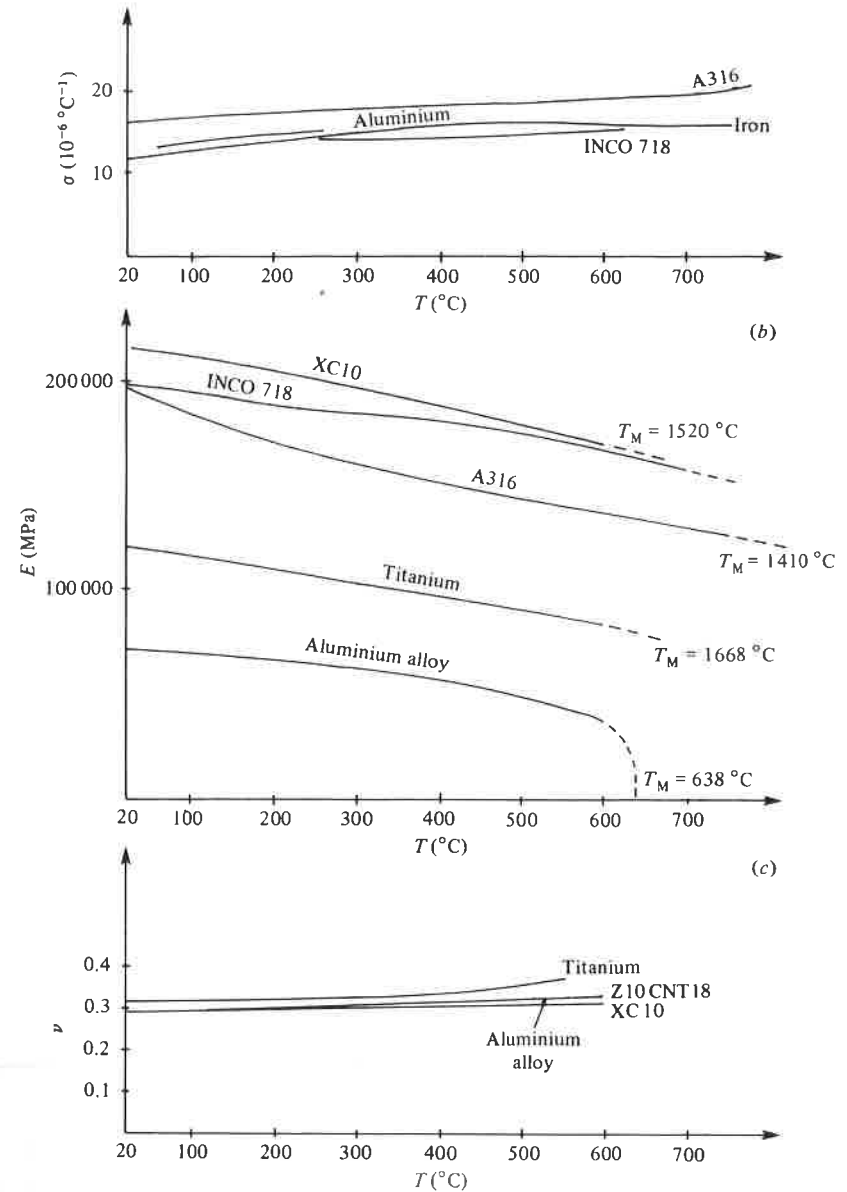
second experiment gives E_3 and α_{23} at T_3 and so on. The stiffness R_m of the assembly is first determined by using instead of the specimen a test bar with a much higher stiffness.

In following the above procedure, we can proceed in steps of 100 °C or 50 °C with a precision that is, of course, decreasing by cumulative effects, but which remains, however, in the order of 2% until over 1000 °C. It is, moreover, necessary to make sure that the elastic limit is not exceeded at the force F_2 for the temperature T_2 and that the effects of stress relaxation are negligible. The properties of A316 stainless steel given in Section 4.2.3 were obtained by this method.

4.2.3　*Thermoelastic properties of common materials*

Graphs showing the evolution of the dilatation coefficient α, Young's modulus E and Poisson's ratio v, as functions of the temperature T, for different materials are given in Fig. 4.5.

4.3　**Viscoelasticity**

4.3.1　*Domain of validity and use*

As discussed in Chapter 1, polymers, and, to a lesser extent, concrete and wood exhibit dissipative phenomena associated with elasticity that can be expressed globally by viscosity. Limiting ourselves to isothermal viscoelasticity we have the situation in which thermal dissipation is zero but intrinsic dissipation is not. The strain rate now becomes an independent, thermodynamic variable.

The theory of viscoelasticity can be used to deal with the reversible evolutions of viscoelastic solids as functions of time as defined in Chapter 3. In dynamic situations, this results in a very low damping (10^{-4}) at room temperature, but it can reach 10^{-2}–1 for polymers.

The Kelvin–Voigt and Maxwell models are derived here from a direct application of thermodynamics to materials with internal variables. For the particular case of linear viscoelasticity, the above functional approach is simpler to formulate and use. We should, however, note that these formulations are valid only for stable materials. The ageing of concrete and polymers requires special treatment which is not described here.

4.3.2　*Thermodynamic formulation*

Kelvin–Voigt solid

This model corresponds to the following choice of variables:

the observable variable is the total strain ε, its associated variable is the stress σ;
in order to define the reversible power and the dissipated power, it is necessary to divide the stress into an 'elastic' part σ^e and an anelastic one σ^{an}. This division corresponds to the analogical Kelvin–Voigt model of Chapter 3 with a spring and a damper in parallel. Accordingly, we have:

$$\sigma = \sigma^e + \sigma^{an}$$

where $\sigma^e:\dot{\varepsilon}$ is the reversible power per unit volume and $\sigma^{an}:\dot{\varepsilon}$ is the intrinsic dissipation per unit volume.

The linear isotropic theory is obtained by choosing the same thermodynamic potential as for linear isotropic elasticity:

$$\Psi = (1/2\rho)(\lambda \varepsilon_I^2 + 4\mu \varepsilon_{II}).$$

To express the anelastic stress σ^{an} (and not the strain rate $\dot{\varepsilon}$) in order to add it to the elastic stress σ^e, it is necessary to use the dissipation potential $\varphi(\dot{\varepsilon})$. For the same reasons as before, a positive definite quadratic form of strain rate components is chosen:

$$\varphi = \tfrac{1}{2}(\lambda \theta_\lambda \dot{\varepsilon}_I^2 + 4\mu \theta_\mu \dot{\varepsilon}_{II})$$

where $\dot{\varepsilon}_I$ and $\dot{\varepsilon}_{II}$ are the first and second principal invariants of the strain rate tensor $\dot{\varepsilon}$, and where θ_λ and θ_μ are two additional coefficients which characterize the viscosity of the material.

According to the method of local state described in Chapter 2, the state law is obtained as:

$$\sigma^e = \rho(\partial \Psi/\partial \varepsilon) = \lambda \operatorname{Tr}(\varepsilon)\mathbf{1} + 2\mu \varepsilon$$

and the complementary law as:

$$\sigma^{an} = \partial \varphi/\partial \dot{\varepsilon} = \lambda \theta_\lambda \operatorname{Tr}(\dot{\varepsilon})\mathbf{1} + 2\mu \theta_\mu \dot{\varepsilon}.$$

The mechanical Kelvin–Voigt model is then obtained by adding the two

stresses:

● $\quad \boldsymbol{\sigma} = \lambda(\mathrm{Tr}(\boldsymbol{\varepsilon}) + \theta_\lambda \mathrm{Tr}(\dot{\boldsymbol{\varepsilon}}))\mathbf{1} + 2\mu(\boldsymbol{\varepsilon} + \theta_\mu \dot{\boldsymbol{\varepsilon}})$

or

$$\sigma_{ij} = \lambda(\varepsilon_{kk} + \theta_\lambda \dot{\varepsilon}_{kk})\delta_{ij} + 2\mu(\varepsilon_{ij} + \theta_\mu \dot{\varepsilon}_{ij}).$$

θ_λ and θ_μ are the characteristic retardation times; they can be identified from the results of a uniaxial tensile test and shear test (torsion of tubes for example). The shear test is governed by the relation:

$$\sigma_{12} = 2\mu(\varepsilon_{12} + \theta_\mu \dot{\varepsilon}_{12}).$$

The response $\varepsilon_{12}(t)$ in a creep experiment in shear ($\sigma_{12} = $ constant), is given by the solution of the above differential equation. With the initial condition $\varepsilon = 0$ at $t = 0$, we find the solution as:

$$\varepsilon_{12} = \frac{\sigma_{12}}{2\mu}\left[1 - \exp\left(-\frac{t}{\theta_\mu}\right)\right]$$

which enables the identification of the two coefficients μ and θ_μ.

Remembering the definition of Poisson's ratio v, the tension test is governed by:

$$\sigma_{11} = [\lambda(1 - 2v) + 2\mu]\varepsilon_{11} + [\lambda(1 - 2v)\theta_\lambda + 2\mu\theta_\mu]\dot{\varepsilon}_{11}.$$

Letting:

$$E = \lambda(1 - 2v) + 2\mu$$

as in elasticity and

$$\eta = \lambda(1 - 2v)\theta_\lambda + 2\mu\theta_\mu,$$

the above relation can be written as:

$$\sigma_{11} = E\varepsilon_{11} + \eta\dot{\varepsilon}_{11}.$$

Hence, the response $\varepsilon_{11}(t)$ to a creep test in tension with $\sigma_{11} = $ constant (and the initial condition $\varepsilon = 0$ at $t = 0$) is:

$$\varepsilon_{11} = \frac{\sigma_{11}}{E}\left[1 - \exp\left(-\frac{E}{\eta}t\right)\right]$$

which, having determined μ and θ_μ, allows identification of the coefficients E and η, and then λ and θ_λ; v is measured from the lateral strain: $\varepsilon_{22} = -v\varepsilon_{11}$.

Maxwell body

The Maxwell model corresponds to the following choice of variables:

the observable variable is always the total strain $\boldsymbol{\varepsilon}$ which is associated to the stress $\boldsymbol{\sigma}$,

to define the reversible power and the dissipated power the strain is now divided into elastic and anelastic parts, so that:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^{an}$$

where $\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}^e$ defines the reversible power, and $\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}^{an}$ is the intrinsic dissipation.

Then, in accordance with the method of local state, the law of state derived from the thermodynamic potential $\Psi(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{an})$ is:

$$\boldsymbol{\sigma} = \rho(\partial\Psi/\partial\boldsymbol{\varepsilon}) = \rho(\partial\Psi/\partial\boldsymbol{\varepsilon}^e)$$

and the complementary law derived from the dual dissipation potential $\varphi^*(\boldsymbol{\sigma})$ is:

$$\dot{\boldsymbol{\varepsilon}}^{an} = \partial\varphi^*/\partial\boldsymbol{\sigma}.$$

In order to express the total strain rate as the sum of $\dot{\boldsymbol{\varepsilon}}^e$ and $\dot{\boldsymbol{\varepsilon}}^{an}$ it is better to work with the dual thermodynamic potential $\Psi^*(\boldsymbol{\sigma})$. Moreover, in order to obtain a linear theory, the same potential as for elasticity is chosen:

$$\Psi^* = \frac{1}{2\rho}\left\{\frac{1 + v}{E}\mathrm{Tr}(\boldsymbol{\sigma}^2) - \frac{v}{E}[\mathrm{Tr}(\boldsymbol{\sigma})]^2\right\}$$

with

$$\boldsymbol{\varepsilon}^e = \rho\frac{\partial\Psi^*}{\partial\boldsymbol{\sigma}} = \frac{1 + v}{E}\boldsymbol{\sigma} - \frac{v}{E}\mathrm{Tr}(\boldsymbol{\sigma})\mathbf{1}.$$

In the same manner, the dissipation potential φ is defined by a positive definite quadratic form which now includes two new coefficients τ_1 and τ_2 which are characteristic of the viscosity and which may be determined by uniaxial tension tests and shear tests:

$$\varphi = \frac{1}{2}\left(\frac{1 + v}{E\tau_1}\mathrm{Tr}(\boldsymbol{\sigma}^2) - \frac{v}{E\tau_2}[\mathrm{Tr}(\boldsymbol{\sigma})]^2\right).$$

By writing $\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^e + \dot{\boldsymbol{\varepsilon}}^{an} = \dot{\boldsymbol{\varepsilon}}^e + \partial\varphi^*/\partial\boldsymbol{\sigma}$ we obtain the following constitutive relation for the Maxwell body:

● $\quad \dot{\boldsymbol{\varepsilon}} = \frac{1 + v}{E}\left(\dot{\boldsymbol{\sigma}} + \frac{\boldsymbol{\sigma}}{\tau_1}\right) - \frac{v}{E}\left(\mathrm{Tr}(\dot{\boldsymbol{\sigma}}) + \frac{\mathrm{Tr}(\boldsymbol{\sigma})}{\tau_2}\right)\mathbf{1}$

or

$$\varepsilon_{ij} = \frac{1 + v}{E}\left(\dot{\sigma}_{ij} + \frac{\sigma_{ij}}{\tau_1}\right) - \frac{v}{E}\left(\dot{\sigma}_{kk} + \frac{\sigma_{kk}}{\tau_2}\right)\delta_{ij}.$$

Applied to the uniaxial case, which corresponds to the analogical model of a spring with a damper in series described in Chapter 3, the above relations give:

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \left(\frac{1+v}{E\tau_1} - \frac{v}{E\tau_2} \right)\sigma$$

or, by writing

$$\frac{1+v}{E\tau_1} - \frac{v}{E\tau_2} = \frac{1}{\eta}$$

we obtain

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta}.$$

This is a fluid model because the possibility of equilibrium at constant stress does not exist.

Generalized Maxwell model for solids

It is possible to obtain an equilibrium by adding a spring in parallel to the analogical Maxwell model. In the same way, it becomes possible to represent more complex behaviour of solids by adding other simple elements in parallel (Fig. 4.6). This also constitutes a generalization of the Kelvin–Voigt model.

The uniaxial constitutive law of such a model is expressed as:

$$\sigma = \sigma_0 + \sigma_1 + \cdots + \sigma_j + \cdots$$

$$\varepsilon = \sigma_0 / E_\infty$$

$$\dot{\varepsilon} = \frac{\dot{\sigma}_1}{E_1} + \frac{\sigma_1}{\eta_1}$$

$$\dot{\varepsilon} = \frac{\dot{\sigma}_j}{E_j} + \frac{\sigma_j}{\eta_j}.$$

It is also possible to introduce internal strains in the model:

$$\dot{\varepsilon} = \dot{\varepsilon}_j^e + \dot{\varepsilon}_j^{an}.$$

By virtue of equality of stress in the spring and in the damper of an

elementary model, the evolution equation for internal anelastic strains is obtained:

$$\dot{\varepsilon}_j^{an} = \frac{E_j}{\eta_j}(\varepsilon - \varepsilon_j^{an}).$$

The constitutive equation of the model may then be expressed in another form as:

$$\sigma = E_x \varepsilon + \sum_{j=1}^{n} \eta_j \dot{\varepsilon}_j^{an}$$

$$\dot{\varepsilon}_1^{an} = \frac{E_1}{\eta_1}(\varepsilon - \varepsilon_1^{an})$$

$$\dot{\varepsilon}_j^{an} = \frac{E_j}{\eta_j}(\varepsilon - \varepsilon_j^{an}).$$

The strains ε_j^{an} play the role of internal variables in the sense of thermodynamics ($\sum_{j=1}^{n} \sigma_j \dot{\varepsilon}_j^{an}$ is the intrinsic dissipation). The constitutive equation consists of a relation which contains internal variables and as many differential equations of evolution as the internal variables introduced. Note that this formulation can also be considered as a particular case of viscoplasticity (linear viscosity without yield limit) which will be studied in Chapter 6.

In order to represent complex viscoelastic behaviour, it is possible to generalize the association of springs and dampers in this way but soon the identification problem becomes unsolvable as it is necessary to determine at

Fig. 4.6.   Generalized Maxwell model.
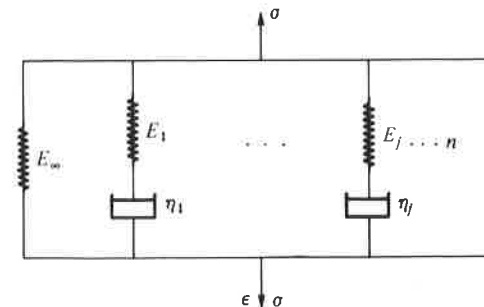
least as many coefficients as the number of elements introduced. The passage to an infinite number of elements or a continuum model is achieved by the concepts of creep or relaxation spectra. However, for linear viscoelasticity, the functional or hereditary approach offers undeniable advantages over that of discrete state variables.

4.3.3   *Functional formulation*

Creep function

Let us attempt to establish formally the response $\varepsilon(t)$ to a uniaxial load $\sigma(t)$ of a viscoelastic material under the following assumptions:

the strain $\varepsilon(t)$ is a functional of the whole history of the stress

$$\varepsilon(t) = \mathscr{F}(\sigma(\tau)), \qquad -\infty < \tau \leqslant t;$$

the material is nonageing and obeys the principle of objectivity; the functional $\mathscr{F}$ is linear.

Now consider a creep loading:

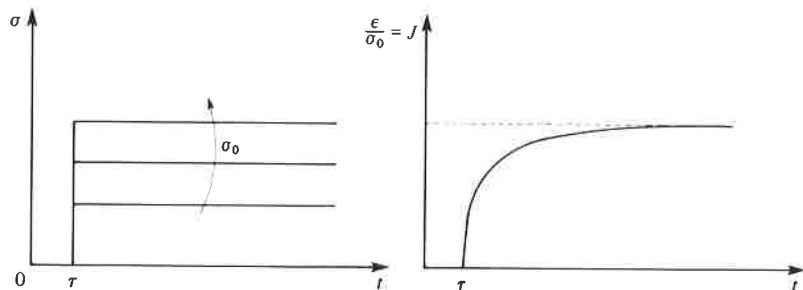$$\sigma = \sigma_0 H(t - \tau)$$

where σ_0 is a constant, τ a reference instant and H the Heaviside step function: $H = 0$ if $t - \tau < 0$ and $H = 1$ if $t - \tau \geqslant 0$.

In view of the assumptions made, the response in terms of strain is:

$$\varepsilon = J(t - \tau)\sigma_0.$$

By definition $J(t - \tau)\sigma_0$ is the creep function, a characteristic uniaxial viscoelastic property of the material (Fig. 4.7). It is independent of the initial

Fig. 4.7.   Creep function.

instant τ. Now consider a function represented by a series of steps (Fig. 4.8):

$$\sigma = \sum_{j=1}^{n} \Delta\sigma_j H(t - t_j).$$

The linearity or Boltzmann's principle of superposition can be used to write the strain response in the form:

$$\varepsilon(t) = \sum_{j=1}^{n} J(t - t_j)\Delta\sigma_j.$$

Finally, consider a load represented by a differentiable continuous or piecewise continuous function. Such a function can be considered as the limit of a series of step functions. The response $\varepsilon(t)$ can then be expressed in the form of a Riemann integral plus terms due to any discontinuities present in the loading.

$$\bullet \qquad \varepsilon(t) = \int_0^t J(t - \tau)\frac{d\sigma}{d\tau}(\tau)\,d\tau + \sum_{j=1}^{n} J(t - t_j)\Delta\sigma_j.$$

This expression can be written using the notation of a convolution product and the definition of a derivative in the sense of distribution, $D\sigma/D\tau$, for the discontinuities:

$$\varepsilon(t) = J \otimes \frac{D\sigma}{D\tau}.$$

Relaxation function

The dual formulation for seeking the response $\sigma(t)$ to a uniaxial strain loading $\varepsilon(t)$ leads, under the same assumptions as those for the creep

Fig. 4.8.   Creep response to a series of step loadings.
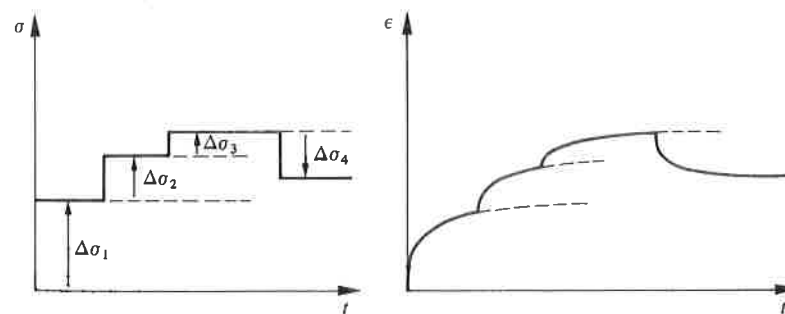
function, to

the definition of a relaxation function $R(t - \tau)$ as the response:

$$\sigma(t) = R(t - \tau)\varepsilon_0$$

to a step relaxation loading $\varepsilon = \varepsilon_0 H(t - \tau)$ (Fig. 4.9);
the response $\sigma(t)$ to any loading $\varepsilon(t)$:

● $$\sigma(t) = \int_0^t R(t - \tau)\frac{d\varepsilon}{d\tau}(\tau)\,d\tau + \sum_{j=1}^n R(t - t_j)\Delta\varepsilon_j$$

or

$$\sigma(t) = R \otimes \frac{D\varepsilon}{D\tau}.$$

For uniaxial loading, the material is characterized by the creep or by the relaxation function, one deducible from the other. For the creep test we have:

$$\sigma = \sigma_0 H \rightarrow \varepsilon = J\sigma_0$$

but

$$\sigma = \sigma_0 H = R \otimes \frac{d\varepsilon}{d\tau} = R \otimes \frac{dJ}{d\tau}\sigma_0.$$
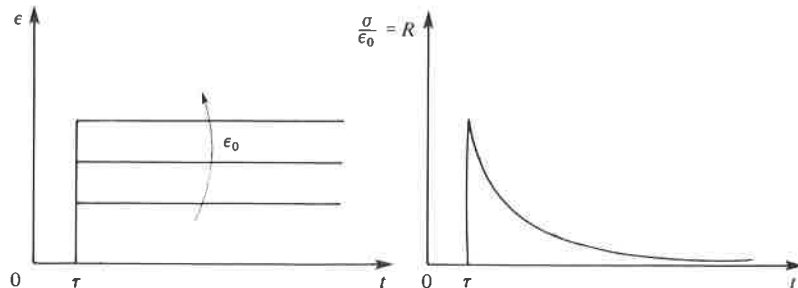
In the same way, for the relaxation test:

$$\varepsilon = \varepsilon_0 H = J \otimes \frac{dR}{d\tau}\varepsilon_0$$

from which we conclude that:

$$R \otimes \frac{dJ}{d\tau} = J \otimes \frac{dR}{d\tau} = H.$$

Fig. 4.9.   Relaxation function.

Generalized constitutive law

The assumptions made in connection with the definition of the creep function, when applied to the case of a general load lead us to define:

a creep function matrix $[J]$ such that:

$$\sigma_{ij} = \text{constant} = \sigma_{ij}^0 \rightarrow \varepsilon_{ij} = J_{ijkl}(t - \tau)\sigma_{kl}^0$$

a relaxation function matrix $[R]$ such that:

$$\varepsilon_{ij} = \text{constant} = \varepsilon_{ij}^0 \rightarrow \sigma_{ij} = R_{ijkl}(t - \tau)\varepsilon_{kl}^0$$

the responses to arbitrary loadings:

$$\boldsymbol{\sigma}(t) \quad \text{or} \quad \boldsymbol{\varepsilon}(t)$$

$$\varepsilon_{ij}(t) = J_{ijkl} \otimes \frac{D\sigma_{kl}}{D\tau} \quad \text{or} \quad \sigma_{ij}(t) = R_{ijkl} \otimes \frac{D\varepsilon_{kl}}{D\tau}.$$

If we assume isotropy of behaviour, then with the same reasoning as used for elasticity in Section 4.1.2 it is possible to reduce to two the number of independent functions which define the components of each of the matrices $[J]$ and $[R]$.

It is possible to choose:

two functions $I(\tau)$ and $K(\tau)$ which play a role equivalent to the coefficients $1/E$ and v/E in the uniaxial linear isotropic law:

● $$\boldsymbol{\varepsilon} = (I + K) \otimes \frac{D\boldsymbol{\sigma}}{D\tau} - K \otimes \frac{D[\text{Tr}(\boldsymbol{\sigma})]}{D\tau}\mathbf{1}$$

or, alternatively, two functions $L(\tau)$ and $M(\tau)$ which play a role equivalent to Lamé's coefficients:

● $$\boldsymbol{\sigma} = L \otimes \frac{D[\text{Tr}(\boldsymbol{\varepsilon})]}{D\tau}\mathbf{1} + 2M \otimes \frac{D\boldsymbol{\varepsilon}}{D\tau}.$$

The identification of functions I and K or L and M by simple tests is based upon creep or relaxation tests in tension and in shear.

Identification of functions I and K:

Creep test in simple tension, $\sigma_{11} = \text{constant}$

$$\varepsilon_{11} = [I(\tau) + K(\tau)]\sigma_{11} - K(\tau)\sigma_{11}$$
$$I(\tau) = \varepsilon_{11}(\tau)/\sigma_{11} = J(\tau).$$

Creep test in shear, $\sigma_{12} = $ constant

$$\varepsilon_{12} = [I(\tau) + K(\tau)]\sigma_{12}$$

$$K(\tau) = \varepsilon_{12}(\tau)/\sigma_{12} - I(\tau).$$

Identification of functions L and M:

Relaxation test in shear $\varepsilon_{12} = $ constant

$$\sigma_{12} = 2M(\tau)\varepsilon_{12}$$

$$M(\tau) = \sigma_{12}/2\varepsilon_{12}.$$

Relaxation test in simple tension $\varepsilon_{11} = $ constant

$$\sigma_{11} = L(\tau)\varepsilon_{11}(1 - 2\nu) + 2M(\tau)\varepsilon_{11}.$$

The contraction ratio ν can be expressed as a function of L and M by writing the relation between the zero component of stress ($\sigma_{22} = 0$) and the corresponding strain component ($\varepsilon_{22} = -\varepsilon_{11}$):

$$0 = L(\tau)\varepsilon_{11}(1 - 2\nu) - 2M\nu\varepsilon_{11}$$

$$\nu = L/2(L + M).$$

After measuring $\sigma_{11} = R(t)\varepsilon_{11}$, we obtain:

$$L(\tau) = \frac{M(R - 2M)}{3M - R}.$$

It is seen that the identification can be done with the same types of test as those used for the elasticity law, by taking into account the temporal evolution of the properties as given by the creep or relaxation functions.

4.3.4   *Viscoelastic properties of common materials*

Fig. 4.10(a) and (b) and 4.11(a) and (b) show graphs of creep and relaxation functions respectively for typical polymers identified as follows.

I   An amorphous polymer of low molecular weight (11 000), polyisobutylene at 25 °C: creep and relaxation functions in shear:

$$\varepsilon_{12} = J_c(t)\sigma_{12}^0, \quad \sigma_{12} = R(t)\varepsilon_{12}^0, \quad (A = -3).$$

II   An amorphous polymer of high molecular weight (300 000), a polyvinylacetate at 75 °C: creep relaxation functions in shear ($A = -1$).

III   An amorphous polymer of still higher molecular weight

(3 620 000), a methacrylate poly-*n*-octyl: creep and relaxation functions in shear ($A = 0$).

IV   An amorphous polymer of high molecular weight below its glass transition temperature, a polymethyl methacrylate at -22 °C: creep and relaxation functions in shear ($A = -7$).

Fig. 4.10. Creep functions for different, polymers (adapted from *Viscoelastic Properties of Polymers*, F. X. de Charentenay).
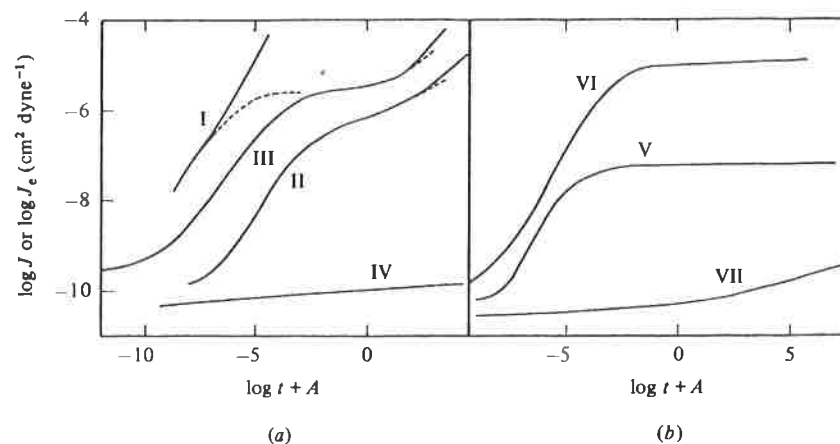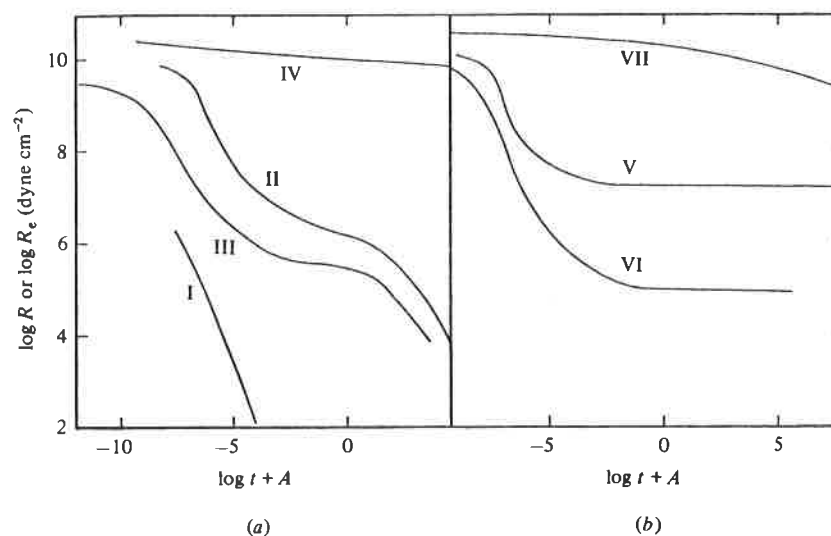

Fig. 4.11. Relaxation functions for different polymers (adapted from *Viscoelastic Properties of Polymers*, F. X. de Charentenay).

V   A weakly cross-linked amorphous polymer, a vulcanized rubber Hévéa at 25 °C: creep and relaxation functions in tension:

$$\varepsilon = J(t)\sigma^0, \quad \sigma = R(t)\varepsilon^0, \quad (A = 0).$$

VI   A diluted cross-linked gel at 10% of polyvinyl chloride in a dimethyl thianthrene at 25 °C: creep and relaxation functions in shear $(A = 0)$.

VII   A highly crystallized polymer, a linear polyethylene of density $0.965\, \mathrm{g\, ml^{-1}}$ at 20 °C: creep and relaxation functions in tension $(A = 2)$.

4.3.5   *Elements of viscoelastic analysis of structures*

The calculation of displacements, stresses and strains in linear viscoelastic structures can be reduced to solving classical elasticity problems by means of symbolic calculus: a linear viscoelasticity problem transformed by the Laplace–Carson transform is identical to the corresponding linear elasticity problem.

Laplace–Carson transform

For the purposes of viscoelasticity, it is sufficient to consider functions $f(t)$ of variable t, of real or complex values, piecewise indefinitely differentiable and identical to zero for $t \leqslant 0$.

Applied to any function $f(t)$, the Laplace–Carson transform gives a corresponding function $f^+(p)$ of variable p defined by:

$$f^+(p) = \mathscr{L}(f(t)) = p \int_0^\infty f(t)\exp(-pt)\,\mathrm{d}t.$$

The important property of the Laplace–Carson transform is that it transforms a convolution product into an ordinary product. We show this property here by using two functions ε and R:

$$\varepsilon^+ = p \int_0^\infty \varepsilon(\tau)\exp(-p\tau)\,\mathrm{d}\tau$$

$$R^+ = p \int_0^\infty R(t)\exp(-pt)\,\mathrm{d}t = p \int_\tau^\infty R(t-\tau)\exp(-p(t-\tau))\,\mathrm{d}t$$

By using the Heaviside function $H(t-\tau)$ we obtain:

$$R^+ \exp(-p\tau) = p \int_0^\infty R(t-\tau)H(t-\tau)\exp(-pt)\,\mathrm{d}t.$$

Since $\varepsilon(0) = 0$, we can, by using integration by parts, easily show that:

$$\varepsilon^+ = \int_0^\infty \frac{\mathrm{d}\varepsilon}{\mathrm{d}\tau}(\tau)\exp(-p\tau)\,\mathrm{d}\tau.$$

The product of the transform can then be written in the following form:

$$\varepsilon^+ R^+ = \int_0^\infty \frac{\mathrm{d}\varepsilon}{\mathrm{d}\tau} R^+(p)\exp(-p\tau)\,\mathrm{d}\tau$$

$$\varepsilon^+ R^+ = p \int_0^\infty \mathrm{d}\tau \int_0^\infty \frac{\mathrm{d}\varepsilon}{\mathrm{d}\tau} R(t-\tau)H(t-\tau)\exp(-pt)\,\mathrm{d}t$$

$$= p \int_0^\infty \left[\int_0^t \frac{\mathrm{d}\varepsilon}{\mathrm{d}\tau}(\tau)R(t-\tau)\,\mathrm{d}\tau \right]\exp(-pt)\,\mathrm{d}t = \left(\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} \otimes R \right)^+.$$

This proof applies also to piecewise continuous functions by writing the derivatives in the sense of distributions.

In general, we will therefore write:

$$\left(\frac{D\varepsilon}{Dt}(t) \otimes R(t) \right)^+ = \varepsilon^+(p)R^+(p)$$

or for the tensor σ_{ij}:

$$\left(\frac{D\sigma_{ij}}{Dt} \otimes A \right)^+ = \sigma_{ij}^+ A^+.$$

Table 4.1 lists some commonly used transforms.

Equations of viscoelastic problems

Constitutive equation

The property of the Laplace–Carson transform in relation to the convolution product can be used to write the linear viscoelastic constitutive equation in the form:

$$\sigma^+ = \lambda^+ \operatorname{Tr}(\varepsilon)^+ \mathbf{1} + 2\mu^+ \varepsilon^+$$

with

$$\lambda^+(p) = \mathscr{L}(L(t)), \quad \mu^+(p) = \mathscr{L}(M(t)).$$

These expressions are similar to the constitutive law for linear elasticity and this fact suggests an examination of all the equations of the problem transformed by the Laplace–Carson transform.

Knowns and unknowns

We express the knowns and unknowns in the transformed form by

Table 4.1. *Laplace–Carson transforms*

Function	Transform	Function	Transform
$f(t)$	$f^+(p)$	t	$1/p$
$C \cdot f(t)$	$C \cdot f^+(p)$	t^n	$n!/p^n$
$\dfrac{d}{dt}$	p	$\exp(-at)$	$p/(p+a)$
$H(t)$	1	$1-\exp(-at)$	$a/(p+a)$
$H(t-\tau)$	$\exp(-\tau p)$	$\cos \omega t$	$p^2/(p^2+\omega^2)$
$f(t-\tau)$	$\exp(-\tau p)f^+(p)$	$\sin \omega t$	$p\omega/(p^2+\omega^2)$
$If \qquad f(t)=0 \quad$ for $\quad t \leqslant 0$		$f(t)\exp(-at)$	$\dfrac{p}{p+a}f^+(a+p)$
$f'(t)$	$pf^+(p)$		
$\dfrac{df}{dt}(t)\otimes g(t)$	$f^+(p)g^+(p)$	$(-t)^m f(t)$	$p\dfrac{d^m}{dp^m}\left(\dfrac{f^+(p)}{p}\right)$

applying the Laplace–Carson transform to the case of a mixed boundary value problem. The transforms of the specified data are:

$\vec{f}^+$ in the solid $\mathscr{S}$,
$\vec{u}^+$ on the boundary ∂s_u,
$\vec{F}^+$ on the boundary ∂s_F.

The unknown fields are $\vec{u}^+(x)$, $\sigma^+(x)$, $\varepsilon^+(x)$.

Equations
In the case of quasi-static problems, the equations are:

$$\text{div}\,\sigma^+ + \vec{f}^+ = 0$$
$$\varepsilon^+ = \tfrac{1}{2}[\text{grad}\,\vec{u}^+ + (\text{grad}\,\vec{u}^+)^\mathsf{T}]$$
$$\sigma^+ = \lambda^+\,\text{Tr}(\varepsilon^+)\mathbf{1} + 2\mu^+\varepsilon^+$$
$$\sigma^+ \cdot \vec{n} = \vec{F}^+ \quad \text{on} \quad \partial S_F.$$

These equations are the same as those of a linear elastic problem if the boundary conditions do not vary with time (fixed ∂s_F and ∂s_u).

Solution method

The formal analogy between the transformed viscoelasticity problem and the elastic problem leads to the method of solution shown in a schematic way in Fig. 4.12. According to this method, a classical elasticity problem has to be solved first, followed by an inversion or deconvolution. The solutions in the transformed space are fields which depend on p and on coefficients defining the external loads.

The difficulty is focussed to inversion i.e. to the calculation of the original functions of time from the functions of variable p obtained by solving the elasticity problem. The calculations of the inverse are often tricky, particularly if the associated elasticity problem only has an approximate numerical solution. Nevertheless deconvolution codes now exist on computers.

Fig. 4.12. Flow chart for solving a problem of viscoelasticity
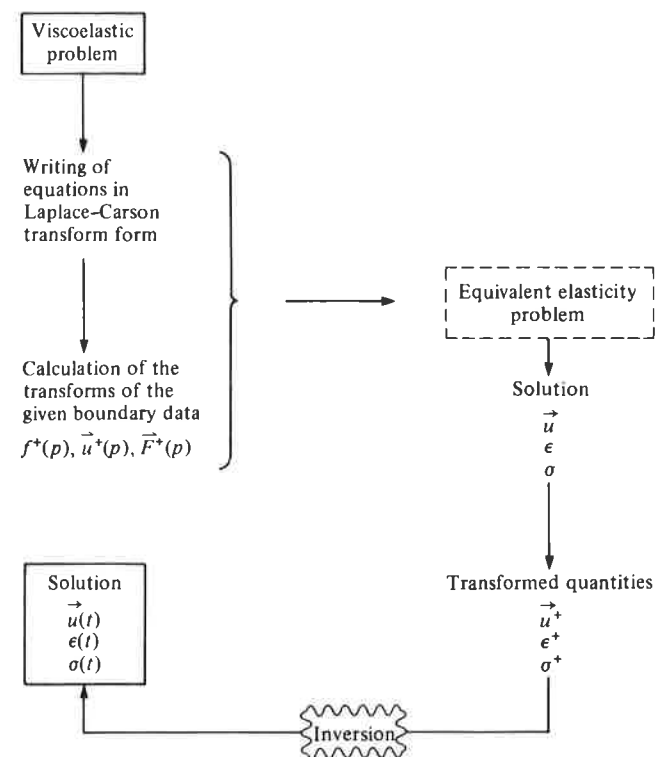
Bibliography

Solomon L. *Elasticité linéaire.* Masson, Paris (1966).

Schreiber E. & Anderson O. *Elastic constants and their measurements.* McGraw Hill, New York (1973).

Tsai S. W. & Thomas Han. *Introduction to composite materials.* Technomic (1980).

Massonet C. & Cescotto S. *Mécanique des matériaux.* Sciences et lettres, Liège (1982).

Courbon J. *Calcul des structures.* Dunod, Paris (1972).

Zienkiewicz O. C. The finite element method in engineering science, McGraw Hill, New York, 1st edn (1967)–3rd edn (1977).

Dhatt G. & Touzot G. *Une présentation de la méthode des éléments finis.* Maloine, Paris (1981).

Salençon J. *Viscoélasticité.* Presses de l'Ecole Nationale des Ponts et Chaussées, Paris (1983).

Findley W. N., Lai J. S. & Onaran K. *Creep and relaxation of non linear visco-elastic materials.* North Holland Publishing Company, Amsterdam (1976).

5

PLASTICITY

Un modèle devient une loi ... ou, sombre dans l'oubli!

This chapter deals with the phenomenological and mathematical modelling of plastic solids according to the schematic classification of Chapter 3: rigid, perfectly plastic solid; elastic, perfectly plastic solid; elastoplastic solid. It can be said that the first scientific work concerning plasticity goes back to Tresca's memoir in 1864 on the maximum shear stress criterion. Although the isotropic flow law was formulated as early as 1871 by St. Venant and Levy, its applications to structures had to wait until about 1950 when limit theorems were discovered. Since 1970 the availability of fast and large computers has led to application of the theory to practical problems. Proportional loading (where principal stresses do not rotate at any point of a structure) provides a large field of application for theories based on isotropic hardening.

Hardening, as we have seen in Chapter 1, is almost always anisotropic. This aspect of plasticity must be considered as soon as the loading is no longer proportional and especially under cyclic loads. Prager, around 1950, gave the first simple formulation of anisotropy, namely kinematic hardening, on which most of the present theories are based.

5.1   Domain of validity and use

The theory of plasticity is the mathematical theory of time-independent irreversible deformations; some comments on its physical nature were given in Chapter 1.

> For metals and alloys it involves mainly the movement of dislocations without the influence of viscous phenomena or the presence of decohesion which damages the matter. Its domain of validity is therefore restricted by the following two limitations: