CIV-E4080
Material Modelling in Civil Engineering D

Period V, 2024

Lecture 3. Plasticity: Basics

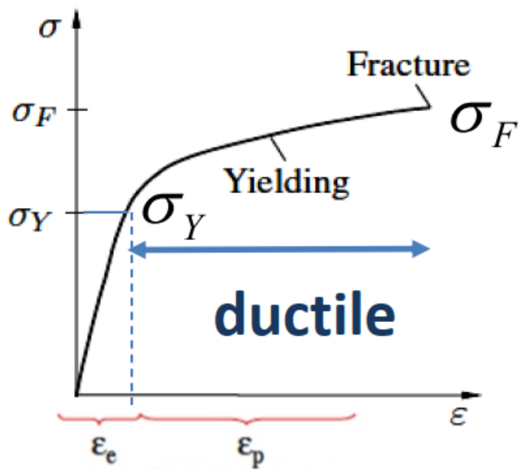
Contents

1. *Plastic deformation mechanisms*
2. *Experimental observations*
3. *Plasticity models*
4. *Yield criteria*
5. *Hardening rules*
6. *Software (Abaqus) demonstration*

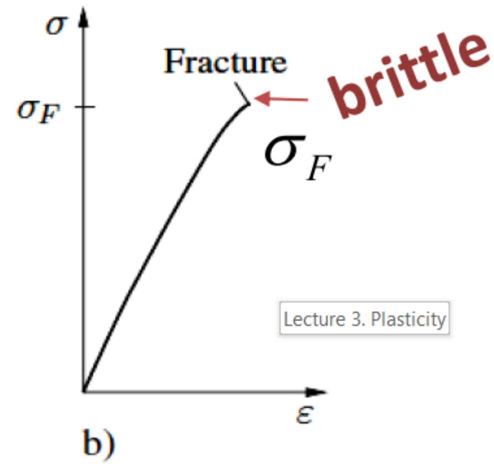
Lecture 3. Plasticity: Basics

Ductile vs Brittle material

ductile failure - *yield*



brittle failure - *fracture*



Ductile material: has a well defined **yield point** – ***ductile failure*** on yielding (**plasticity**)

Brittle material: has a no yield point & *sudden failure – brittle failure*

The ultimate stress at fracture will be reached only after sufficiently large inelastic deformations – *thus people have time to escape safely*

No significant inelastic deformations occur prior to Fracture – thus people doe not gat prior signs before failure

Lecture 3. Plasticity: Basics

Plasticity: theory of isothermal time-independent *irreversible deformations*

Domain of validity and use:

❖ **Metal and alloys**:

- mainly movement of dislocations:

The total strain (increment) can be partitioned into reversible (elastic) and irreversible (plastic)

- **Restrictions**: - **Low temperature** $\boxed{T < T_{melting} / 4}$
  - **Non-damaging loads**

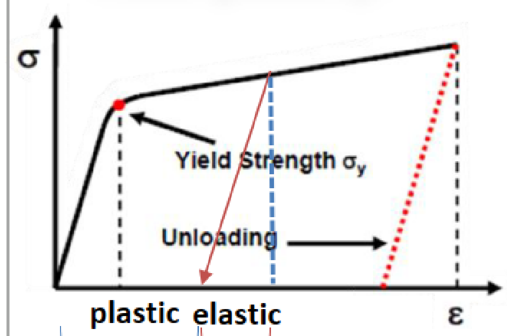
…otherwise → viscoplasticity, thermally activated creep
… further recrystallization…too complex

❖ **Soils:** Restrictions: occurrence of slip surfaces caused by instability

❖ **Polymers and Wood**: irreversible deformations are better accounted by **visco-elasticity** (however, for temperatures close to melting temperatures→ thermoplastics)

❖ **Concrete**: irreversible deformations are due mainly to micro-cracks therefore a *model coupling elasticity or plasticity* and *damage* may be preferable
(for longtime responce, creepresults in irrevesible deformations)

Elasto-plasticity:



$$\underbrace{\Delta\varepsilon}_{\text{total}} = \Delta\varepsilon^{(e)} + \Delta\varepsilon^{(p)}$$

Elastic increment
reversible

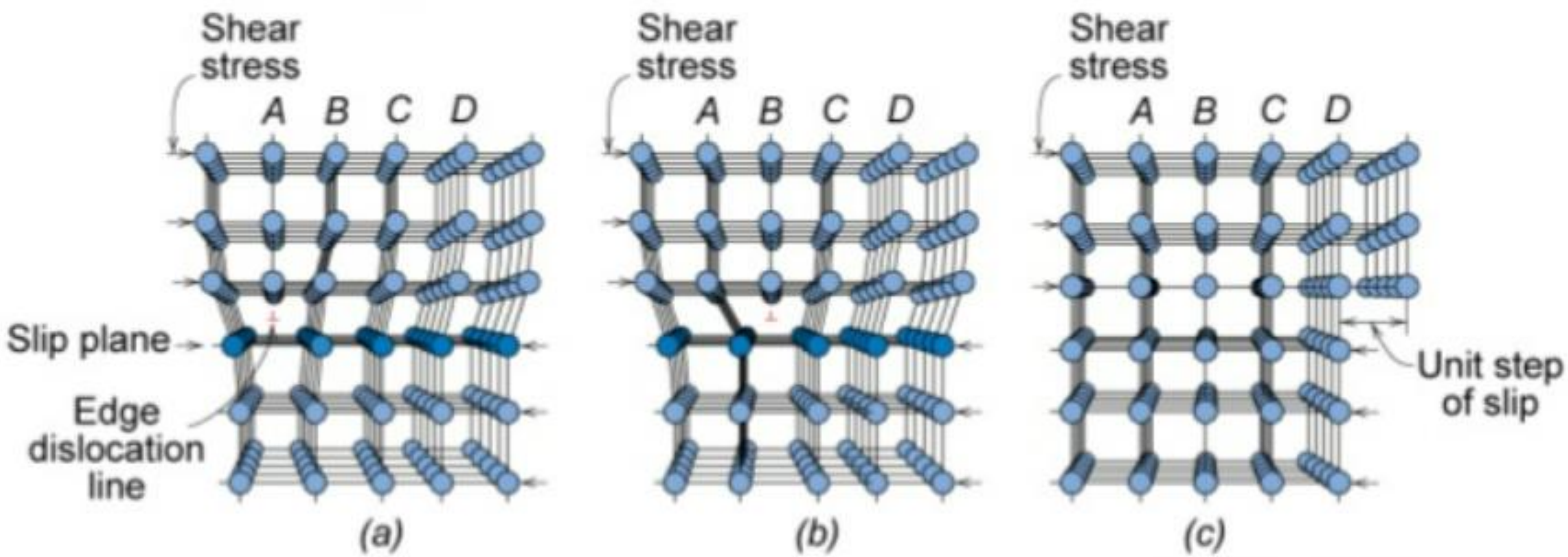
Plastic increment
irreversible

in soils plasticity or permanent shearing deformation occurs when constituting particles slide (& relatively rotate) over one another (*cohesive* and *frictional*)

Dislocation Motion

Dislocation motion & plastic deformation

- Metals - plastic deformation occurs by slip – an edge dislocation (extra half-plane of atoms) slides over adjacent plane half-planes of atoms.



- If dislocations can't move, plastic deformation doesn't occur!

Adapted from Fig. 7.1, *Callister & Rethwisch 8e.*

Chapter 7 - 4

Lecture 3. Plasticity: Basics

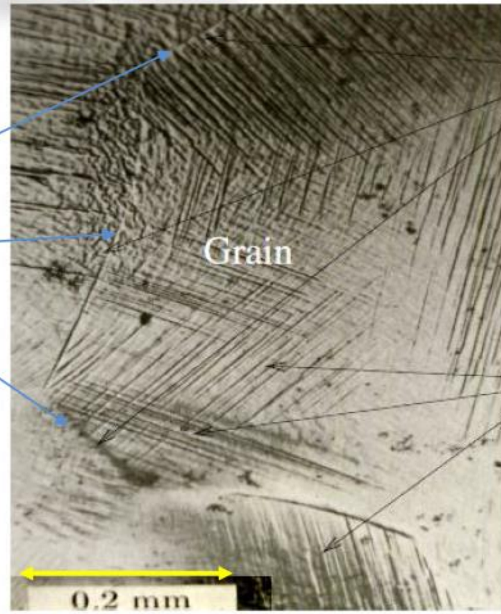
Illustration of slip planes in a grain of a metal (a monocristal)

Plastic Deformation of Polycrystalline Materials

Metal: Copper

G ~48 GPa

Grain boundaries



Joints de grain

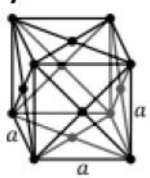
Grain joints

Note that we see <u>three</u> slip directions per grain

Trace de plans de glissement

Trace of slip movements along crystal planes

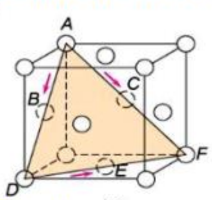
Copper: Crystal structure

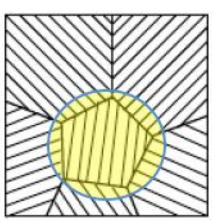


face-centered cubic (fcc)

Note that the slip orientations changes with the grain orientations

Cu: <u>three</u> slip-planes FCC {111} = Direction of easiest slippage





- **Slip plane** - **plane allowing easiest slippage**

Ref: *Rupture et Plasticité, lecture notes by* Pierre Suquet

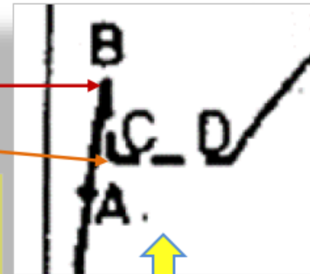
Dislocation := motion of atoms in the crystal structure rearranging themselves to have new neighbors

Lecture 3. Plasticity: Basics

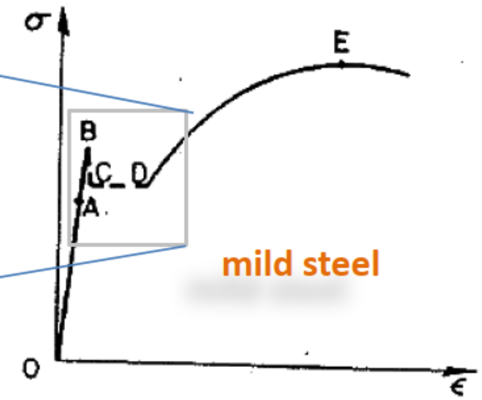
Monotonic loading

Upper yield stress

Lower yield stress



Behavior in the flat region CD is called *plastic flow*

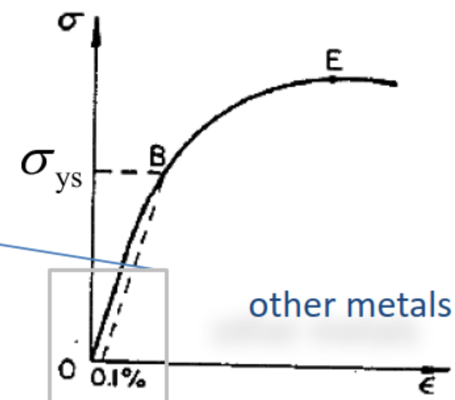


Stress-stress diagram

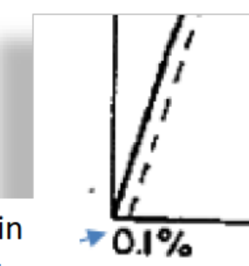
- For mild steel a well defined **flat plastic region** exists; so the ***yield stress*** *(yield point)* is well defined.
- For **most of metals**, such plastic plateau is **not well discernible** (visible); so the **yield strength** is **defined** by an ***offset of yield stress* corresponding** usually to a **certain amount of plastic strain** – 0.1%, 0.2% or what ever is relevant for the application.
- This apparent yield point is called *initial yield stress.*
- Above the yield point the behavior of the material is both elastic and plastic; elasto-plastic

offset yield stress

Initial yield stress



Stress-stress diagram



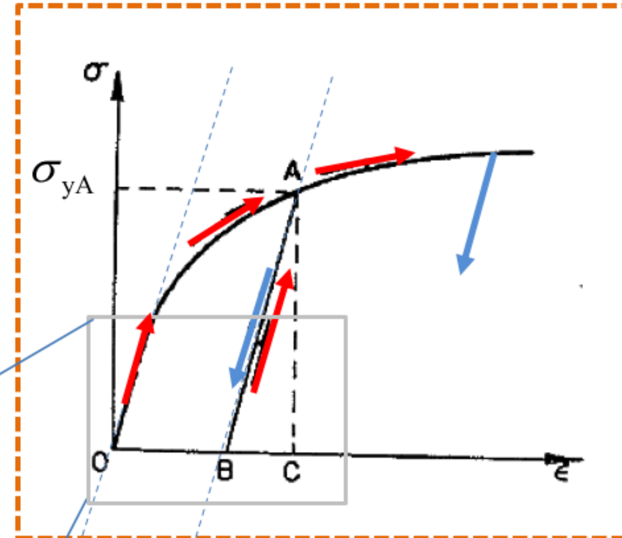
Permanent strain upon unloading

Lecture 3. Plasticity: Basics

Loading and unloading

Loading history: assume the test specimen is loaded monotonically to some specific value beyond the initial yield point and then then completely unloaded

- **Loading history dependence (or Load path dependence)**



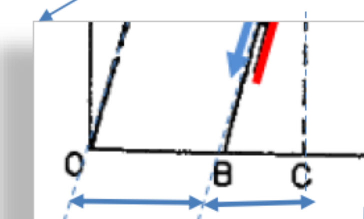
Parallels ; the unloading occurs elastically

σ_{yA} - subsequent yield stress

- In plasticity, because of path dependency of the behavior, the constitutive law for strain plastic is known only for strain increments

$$d\varepsilon = d\varepsilon_e + d\varepsilon_p$$

$$\Delta\varepsilon = \Delta\varepsilon_e + \Delta\varepsilon_p$$



Permanent strain OB

ε_p

Elastic strain BC

ε_e

$$\Delta\varepsilon = \Delta\varepsilon_e + \Delta\varepsilon_p$$

For many materials (metals), *after reaching the initial yield point*, the **stress-strain curve continues to rise** although with decreasing slope → This behavior is called **strain- or work hardening**

Lecture 3. Plasticity: Basics

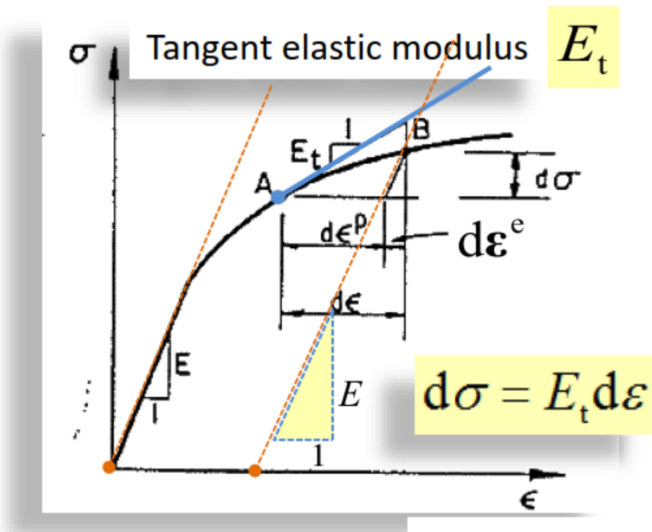
Tangent- and plastic modulus

In plasticity the material shows *path* or *history dependency* behavior; thus the *strain-stress* response of the material is *non-linear* by nature (elastic-plastic behavior),

$$\Downarrow$$

Stress-strain relations are **known** only in their **incremental** or equivalently **rate forms**.

$$\mathrm{d}\varepsilon = \mathrm{d}\varepsilon^{\mathrm{e}} + \mathrm{d}\varepsilon^{\mathrm{p}}$$



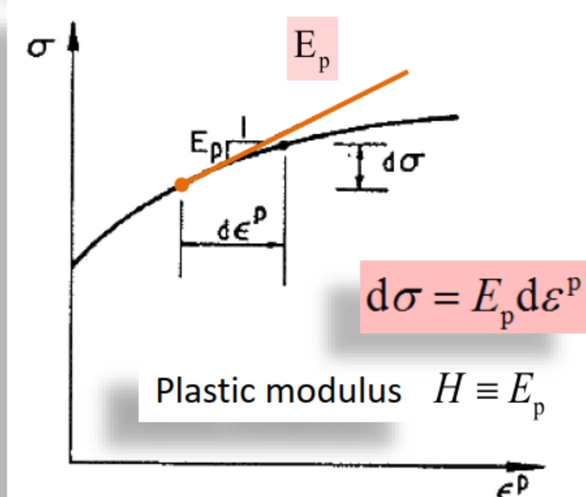
Tangent elastic modulus: E_{t}, $\quad \mathrm{d}\sigma = E_{\mathrm{t}}\mathrm{d}\varepsilon$

Plastic modulus: E_{p}, $\quad \mathrm{d}\sigma = E_{\mathrm{p}}\mathrm{d}\varepsilon^{\mathrm{p}}$

The relation between the three moduli:

$$\frac{1}{E_{\mathrm{t}}} = \frac{1}{E} + \frac{1}{E_{\mathrm{p}}}$$

Show that it is true!



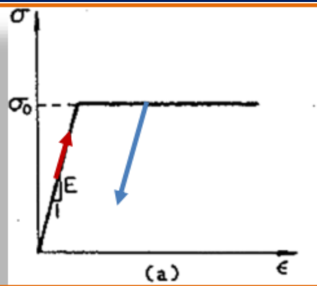
Lecture 3. Plasticity: Basics

Elastic-Perfectly Plastic Model

$$\varepsilon = \begin{cases} \dfrac{\sigma}{E}, & \sigma < \sigma_0 \\ \dfrac{\sigma}{E} + \lambda, & \sigma = \sigma_0 \end{cases}$$

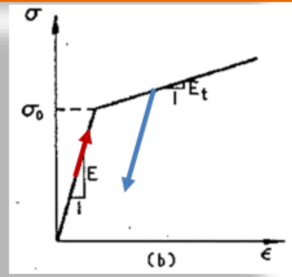
Total strain: $\varepsilon = \varepsilon^{e} + \varepsilon^{P}$

Strain increments: $d\varepsilon = d\varepsilon_e + d\varepsilon_p$



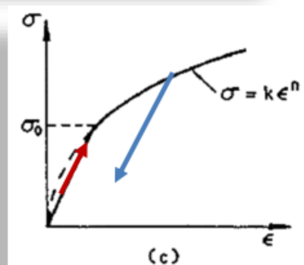
Elastic-Linear Work-hardening model

$$\varepsilon = \begin{cases} \dfrac{\sigma}{E}, & \sigma \le \sigma_0 \\ \dfrac{\sigma_0}{E} + \dfrac{1}{E_t}(\sigma - \sigma_0), & \sigma > \sigma_0 \end{cases}$$



Elastic-Exponential Hardening model

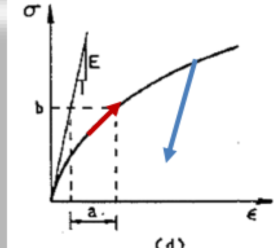
$$\sigma = \begin{cases} E\varepsilon, & \sigma \le \sigma_0 \\ k\varepsilon^{n}, & \sigma > \sigma_0 \end{cases}$$



Ramberg-Osgood model

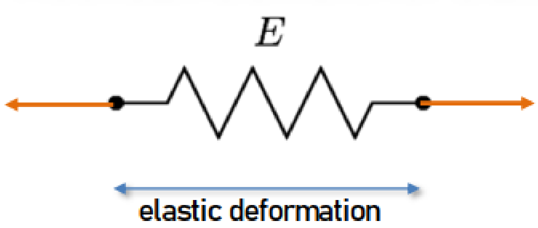
$$\varepsilon = \frac{\sigma}{E} + a\left(\frac{\sigma}{b}\right)^{n}$$ or more esthetically written $$\varepsilon = \frac{\sigma}{E} + K\left(\frac{\sigma}{E}\right)^{n}$$

$a, b, n, b, k, K, E_t, \lambda -$ describe the hardening behavior of the material.



Basic rheological models for plasticity

Hookean element for elasticity

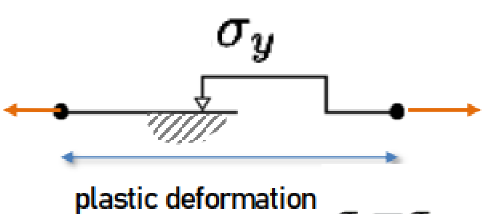


$$\sigma = E\varepsilon$$

Various plasticity and visco-plasticity material behaviours can be captured combining these basic rheological models

Friction element for plasticity

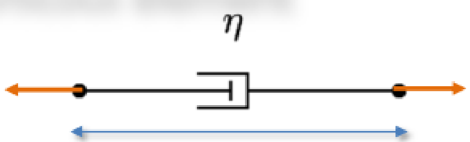
In the basic rheological friction model, the Coulomb's sliding friction coefficient μ corresponds to the threshold stress value at yield; the yield stress σ_y



$$\begin{cases} \dot{\varepsilon} = 0 & \text{if} \quad \sigma| < \sigma_y & \text{does not slide, the element is blocked} \\ \dot{\varepsilon} > 0 & \text{if} \quad \sigma| = \sigma_y & \text{slides, fully developped friction} \\ \dot{\varepsilon} < 0 & \text{if} \quad \sigma| = -\sigma_y & \text{slides, fully developped friction} \end{cases}$$

$$\sigma = \sigma_y \cdot \frac{\dot{\epsilon}}{|\dot{\epsilon}|}$$

Viscous element



$$\sigma = \eta\dot{\varepsilon}$$

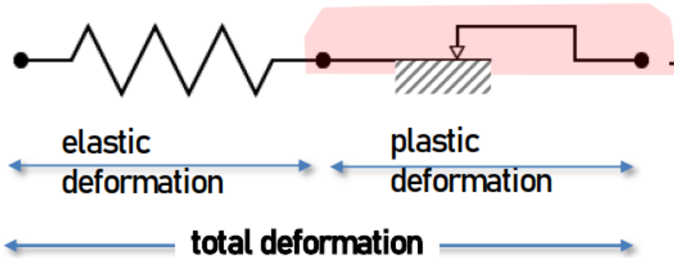
Lecture 3. Plasticity: Basics

Rheological models for plasticity & visco-plasticity

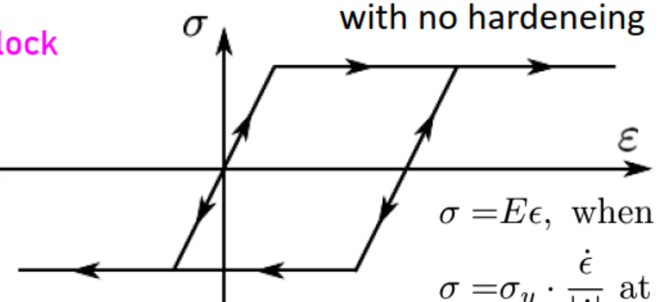
Simple rheological models

Rate independent Plasticity

friction element
or kidding threshold block
(= yield threshold)



with no hardeneing

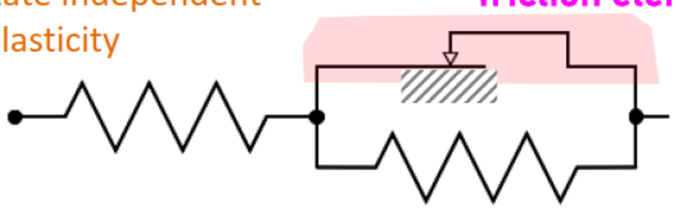


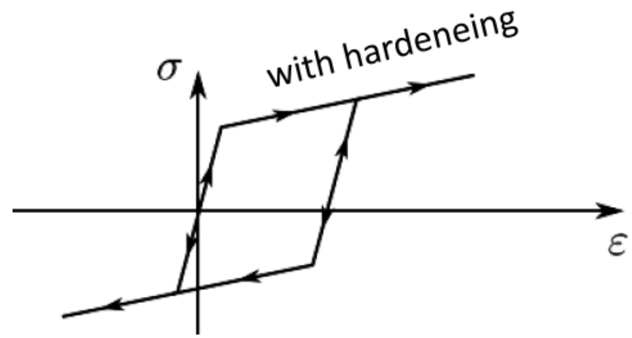
$$\sigma = E\epsilon, \text{ when } |\sigma| < \sigma_y$$

$$\sigma = \sigma_y \cdot \frac{\dot{\epsilon}}{|\dot{\epsilon}|} \text{ at yielding}$$

Rate independent Plasticity

friction element

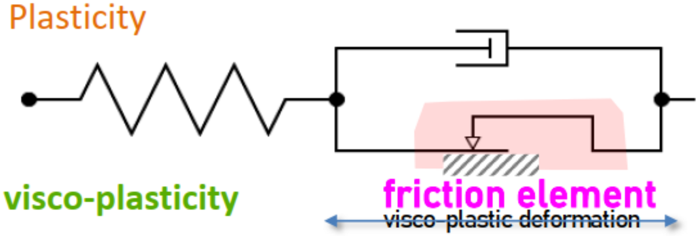




ductile metals...

Rate dependent Plasticity

bingham model



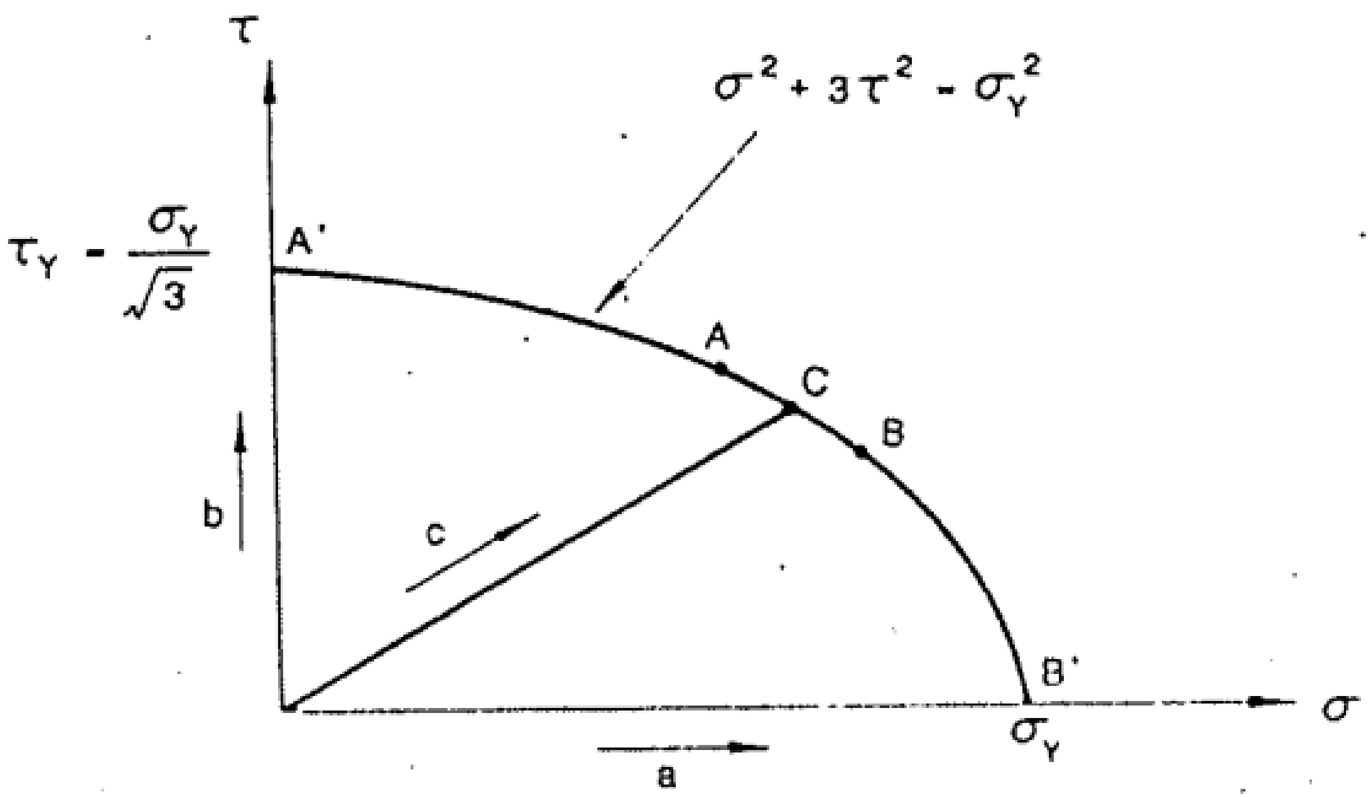
visco-plasticity

asphalts, some pastes, 'pâtes', teeth paste, ...

$$\sigma = \begin{cases} E\varepsilon, & |\sigma| < \mu \equiv \sigma_Y \\ \mu + \eta\dot{\varepsilon}, & |\sigma| \geq \mu > 0 \end{cases} \qquad \mu \equiv \sigma_Y$$

Lecture 3. Plasticity: Basics

Combined loading



Lecture 3. Plasticity: Basics

Von Mises and Tresca yield Criteria

- **Isotropic material**: the *yield criterion depends only on the stress invariants* or I_1, I_2, I_3 equivalently on invariants of deviatoric stresses and J_2, J_3 I_1

- Hydrostatic stress causes for many materials, *especially* ***metals***, a purely elastic volumetric strains (plastic volumetric strains = 0, **incompressible, zero dilatancy**) → I_1 does not influence yielding (for not very high pressures)

$$I_1 \equiv \mathrm{tr}(\boldsymbol{\sigma}) = \sigma_1 + \sigma_2 + \sigma_3$$

The plastic volume change is called "*dilatancy*"

Yield criteria - myötöehto tai -funktio

$$F(J_2, J_3) = 0$$

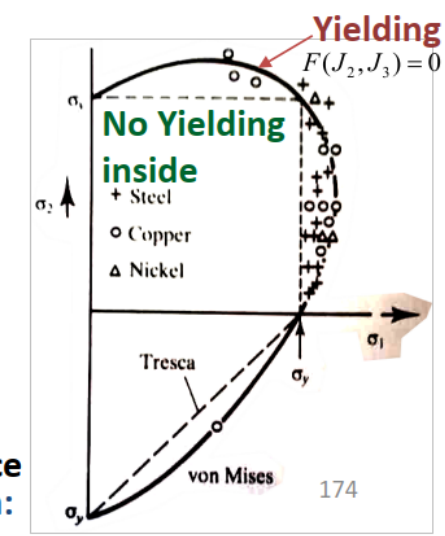
$$F(\sigma_{ij}) = 0$$

VON MISES yield criterion

TRESCA yield criterion

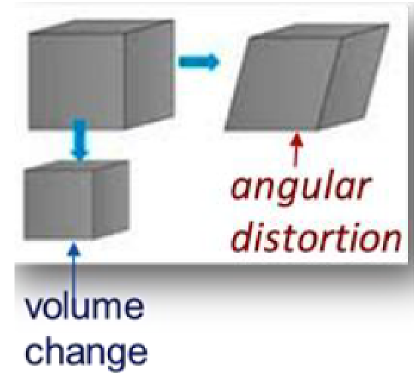
- Distortional energy – Von Mises **yield** condition
- Maximum shear stress – TRESCA **yield** condition
- Maximum Principal stress – RANKINE (1820–1872), LAMÉ (1795–1870) & NAVIER (1785–1836)
- Maximum Principal strain – SAINT-VENANT (1797–1886) & BACH (1889)
- Maximum strain energy – BELTRAMI (1835-1900)
- Mohr-Coulomb hypothesis ... for granular and geological materials
- Octahedral shear Stress theory or Drucker-Prager hypothesis - DRUCKER (1918-2001) & PRAGER (1903-1980) ... for granular and geological materials

Experimental evidence Von-Mises and Tresca:



Distortional Energy Criteria:

Yielding will occur when the ***distortion energy*** in a unit volume of the material equals the distortion energy in the same volume which is uniaxially stressed till yielding occurs



Von Mises Yield criteria: $F = J_2 - k^2 = 0 \;\Rightarrow$

$$\tau_Y \equiv \; k = \sigma_Y / \sqrt{3}$$

yield in uniaxial test

$$F = \frac{1}{6}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] - k^2 = 0 \;\Rightarrow$$

$$\Rightarrow \frac{1}{\sqrt{3}} \underbrace{\sqrt{\frac{1}{2}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]}}_{\equiv \sigma_e} = k \;= \frac{\sigma_Y}{\sqrt{3}}$$

von Mises equivalent stress

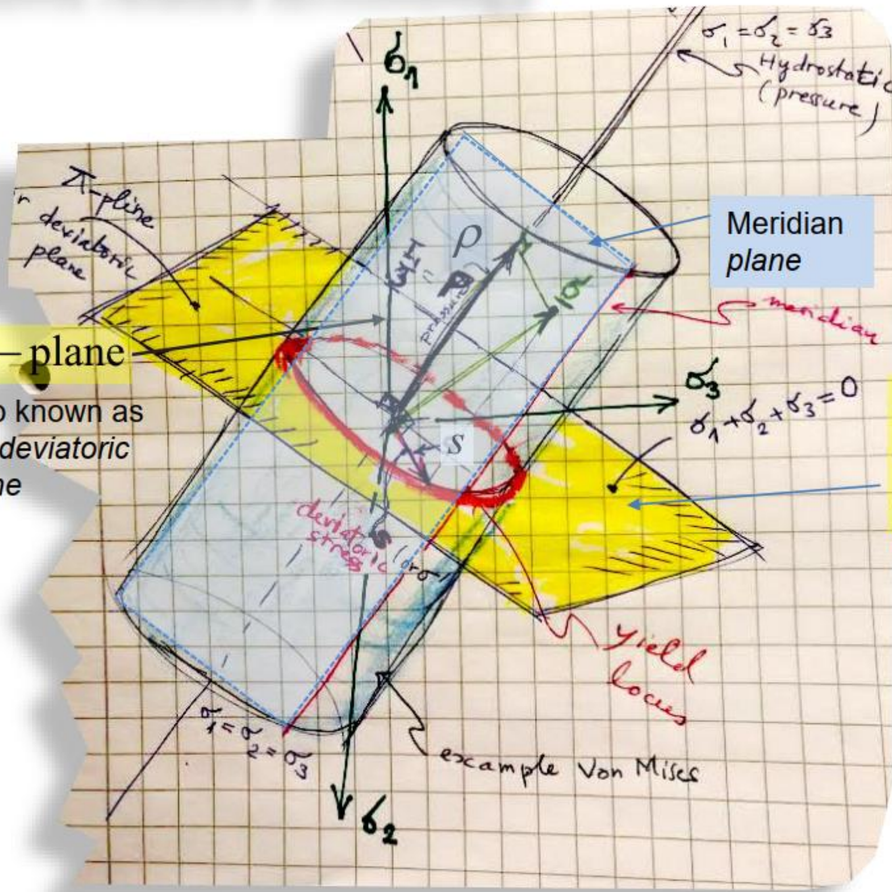
$$\Rightarrow \sigma_e = \sigma_Y$$ yielding criteria

The material yields when *von Mises equivalent stress* σ_e **exceeds** the uniaxial material yield strength

$$\sigma_e = \sigma_Y$$

Lecture 3. Plasticity: Basics

Graphical representation of the failure surface & some related terminology

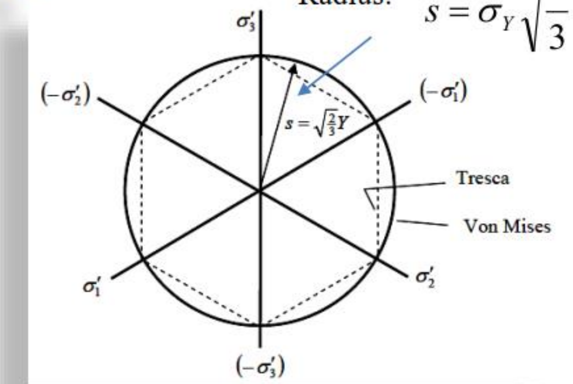


$\pi-$plane

Also known as the *deviatoric plane*

Yield surface in three dimensional stress-space

Radius: $s = \sigma_Y \sqrt{\frac{2}{3}}$



The Von Mises criterion in the *pi*-plane

$\pi-$plane

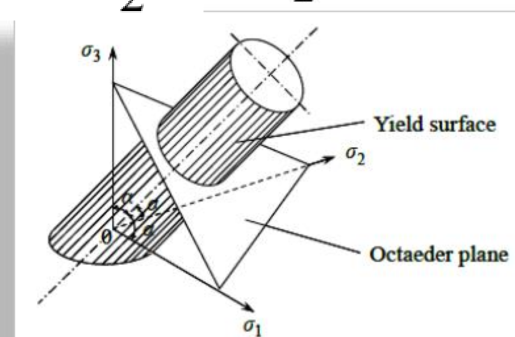
$\sigma_1 + \sigma_2 + \sigma_3 = 0$

$(I_1 = 0)$

perpendicular to the hydrostatic axis

$$\sigma_e = \sqrt{3J_2}$$

$$\sigma_e^2 = \frac{3}{2} s_{ij} s_{ij} = \frac{3}{2} \mathbf{s} : \mathbf{s} \equiv 3J_2$$



Tresca Yield Criterion – Maximum Shear Stress Criterion

Yield criterion: plastic flow (yield) occurs if the *maximum shear stress* re $\tau_{\max}$ s a certain critical value : k

Yield criteria - myötöehto tai -funktio

$$F = \tau_{\max} - k = 0,$$

$$\tau_{\max} = \frac{1}{2}\sigma_Y \Rightarrow \textbf{Yield}$$

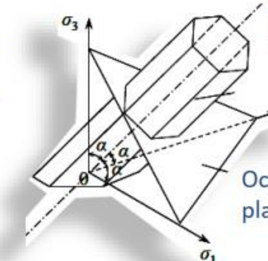
$k = \sigma_Y / 2 = \tau_Y$

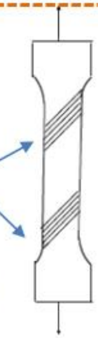
σ_Y - *yield* in *uniaxial* test maximum shear stresses $\Rightarrow \tau_{\max} = \max\left\{\frac{1}{2}|\sigma_1 - \cancel{\sigma_2}|\right\} = \boxed{\frac{1}{2}\sigma_Y = \tau_Y}$

$k = \sigma_Y / 2 = \tau_Y$

k - yield stress in pure shear according to Tresca yielding criterion

*At yield*ing in uniaxial tension test:





$\tau_{\max} = \tau_Y$

Lüders bands
Slip planes - dislocations

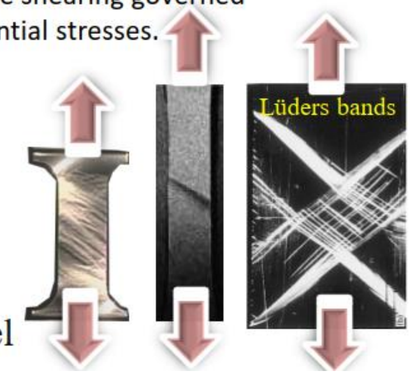
$$F = \max\left\{\frac{1}{2}|\sigma_1 - \sigma_2|, \frac{1}{2}|\sigma_2 - \sigma_3|, \frac{1}{2}|\sigma_3 - \sigma_1|\right\} - k$$

$$\textbf{Tresca:}\quad \tau_{\max} = \tfrac{1}{2}(\sigma_1 - \sigma_3) = \tau_Y$$

In metals, plastic deformation results from *slip*; inter-crystalline shearing governed by tangential stresses.

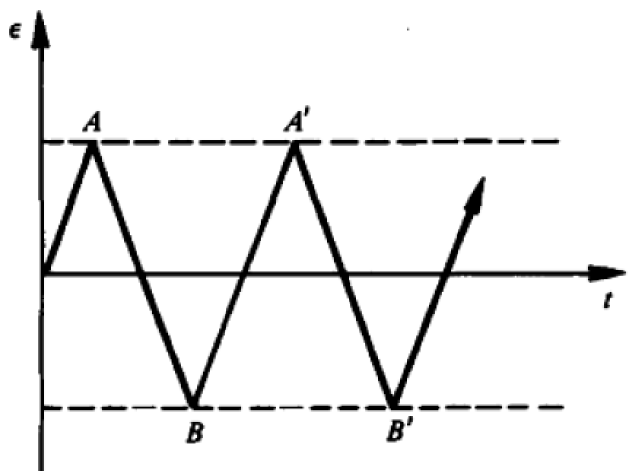
Note tha in Tresca criteria one of principle stresses does not enter the yield condition! here, max is σ3 – σ1 (in plane stress ok, where one principle stress component = 0)

$$F = \tau_{\max} - k < 0 \implies \text{elastic (no yieldin)}$$
$$F = \tau_{\max} - k = 0 \implies \text{onset of yielding}$$



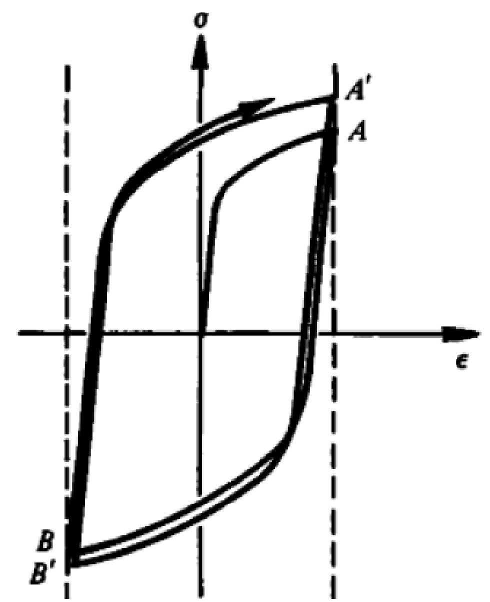
Cyclic loading

Cyclic test under prescribed starin history



Cyclic test under prescribed strain

ref: Chaboche *et al.*



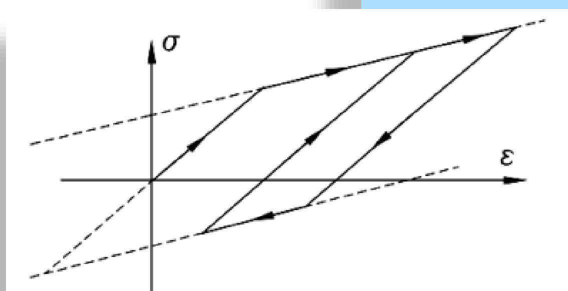
Steel

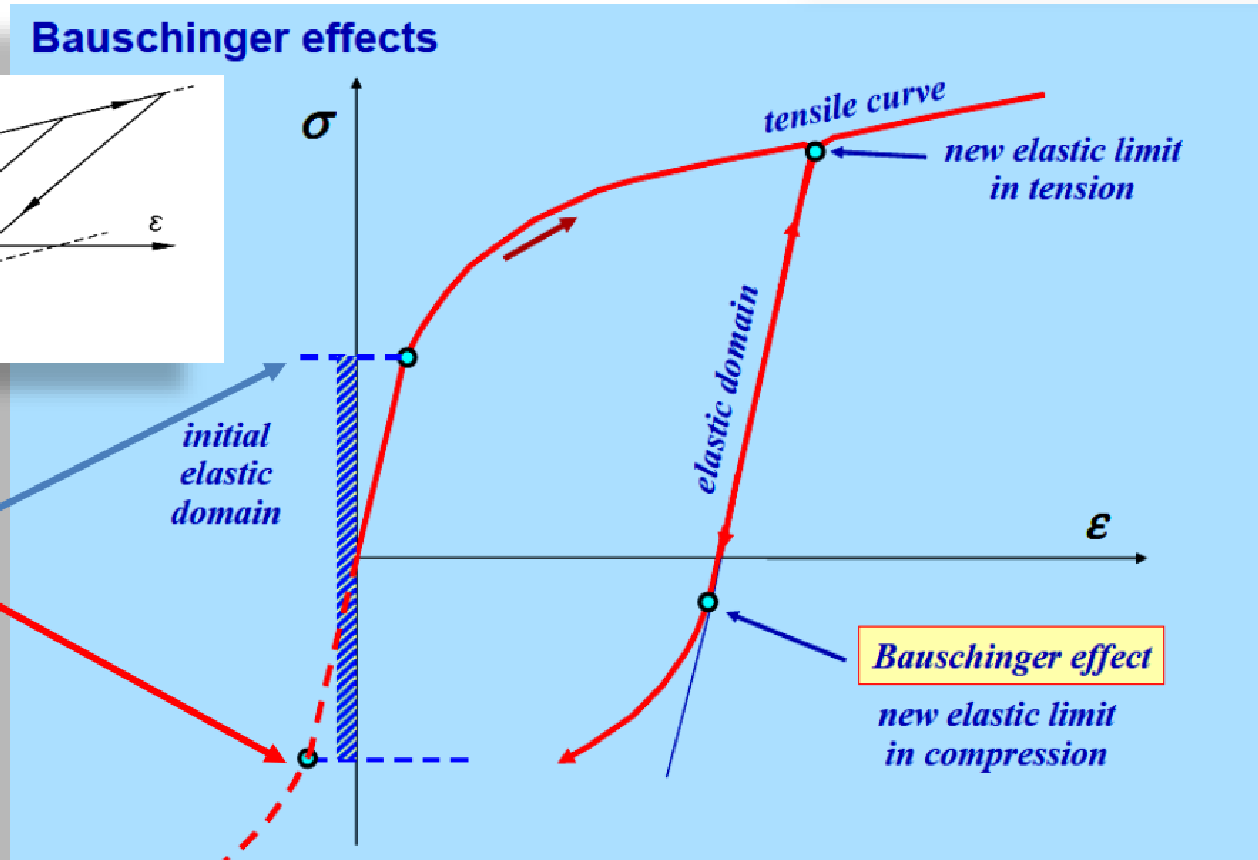
Lecture 3. Plasticity: Basics

Reversed loading

The yield stress in **compression** is **not the same** as the yield strength in **tension** after reloading the specimen (cyclic loading)

Such **Bauschinger** effect is present whenever there is a reversal loading (cyclic)





Initially, for a virgin metallic sample, the **compression** and **tension** yield strength are **the same**

Ref: This figure is based on Chaboche slide from: ATHENS – Course MP06 – 16 – 20 March 2009

Lecture 3. Plasticity: Basics

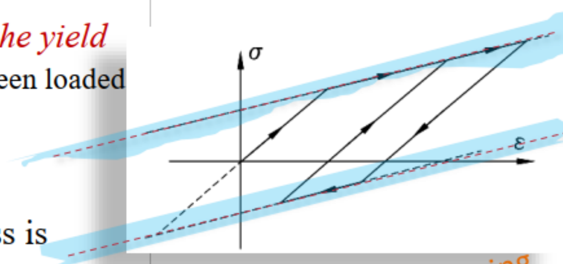
The Bauschinger Effect

Muokkaus, kylmämuovaus

Test: Load a virgin sample first in tension into the plastic range, and then unload it and continue to load it into compression beyond yield.

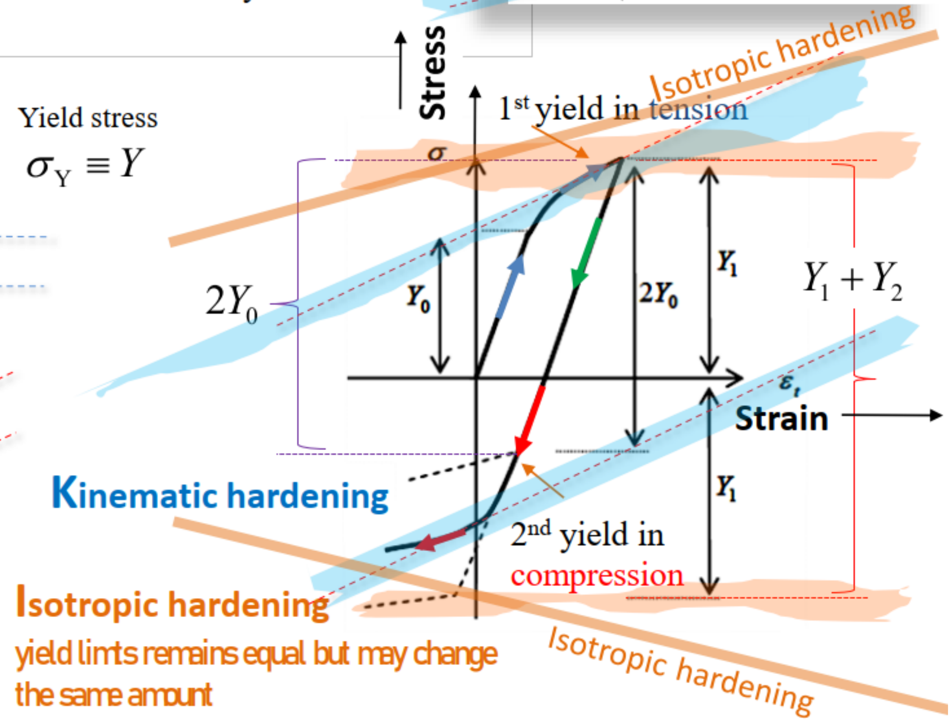
Observation: the *yield stress in compression is not the same as the yield strength in tension*, as it would have been if the specimen had not first been loaded in tension.

The yield point in compression will be significantly less than the corresponding yield stress in tension. This reduction in yield stress is known as the **Bauschinger effect**.



Two extreme cases used in plasticity models:

- **Isotropic hardening** (lujittuminen) model, in which the **yield stress** in **tension** and **compression are** maintained **equal**
- **Kinematic hardening**, in which the total elastic range is maintained **constant** throughout the deformation



Lecture 3. Plasticity: Basics

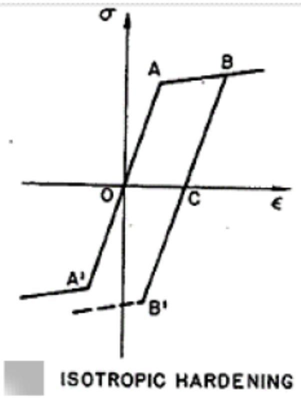
Hardening rules

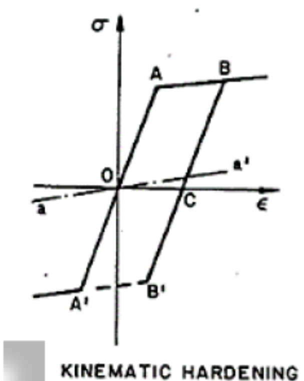
The phenomena of hardening: Yield stress increases with further increase of plastic strains

Hardening and its various stages is modelled by introducing some *internal variable* as for instance *a hardening parameter* $\kappa > 0$

and assumes, for instance, that the plastic modulus changes as a function of the hardening parameter

$$E_{\mathrm{p}} = E_{\mathrm{p}}(\kappa)$$





Choice of the hardening parameter:

$\kappa = W_{\mathrm{p}}$ or $\kappa = \varepsilon_{\mathrm{p}}$

plastic work

$$W_{\mathrm{p}} = \int \sigma_{ij} \mathrm{d}\varepsilon_{ij}^{p}$$

$W_{\mathrm{p}} > 0$

accumulated *effective or equivalent* plastic strain

$$\varepsilon_{\mathrm{p}} = \int \sqrt{\mathrm{d}\varepsilon_{ij}^{p} \mathrm{d}\varepsilon_{ij}^{p}} > 0$$

Effective plastic strain increment:

$$\left|\mathrm{d}\varepsilon_{\mathrm{p}}\right| = \sqrt{\mathrm{d}\varepsilon_{ij}^{p} \mathrm{d}\varepsilon_{ij}^{p}} \geq 0$$

Using such hardening parameter - the *lazy way* - is more practical than computing the plastic work

For a material, the functional relation $E_{\mathrm{p}} = E_{\mathrm{p}}(\kappa)$ can be determined from the uniaxial tensile stress-strain curve in terms of the above definition of the hardening parameter κ

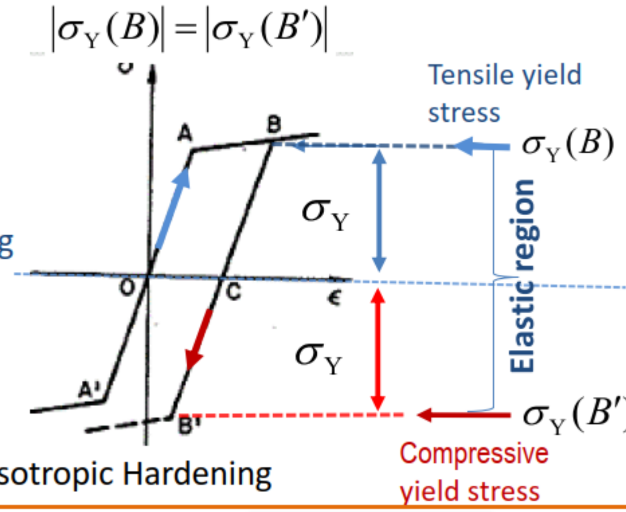
Hardening rules (models)

For a material under **reversal** or cyclic loading, the subsequent *yield stress* σ is usually determined usually by one of the three hardening rules below:

Isotropic Hardening rule:

- Assumes ***equal yield stress*** in tension and **compression**
  - thus the *Bauschinger* effect is *completely neglected*.

Hardening rule: $|\sigma| = |\sigma(\kappa)|, \quad \kappa > 0$



Isotropic Hardening

Kinematic Hardening rule:

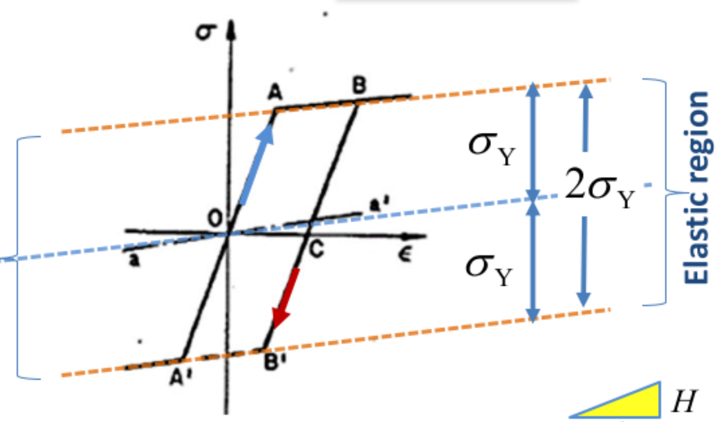
- The **elastic range remains unchanged** during hardening
  - So the *Bauschinger* effect is *accounted* to its full extent.

Hardening rule:

$$|\sigma - c(\kappa)| = \sigma_0,$$

shift of the yield stress, $\kappa > 0$

The center of the elastic region moves **(shifted)** along the straight line



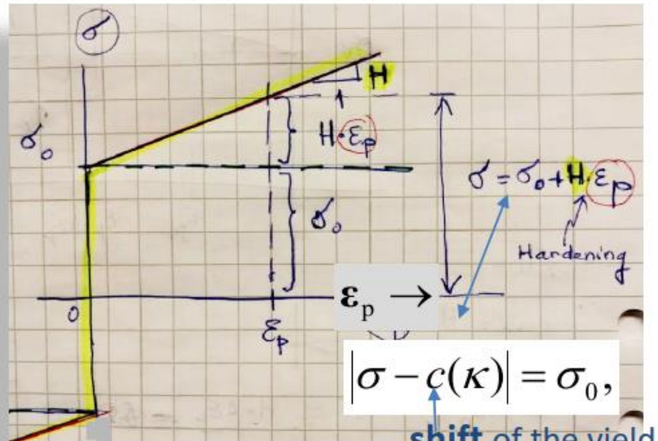
Linear Kinematic Hardening

Esim. $c(\kappa) \to H \cdot \varepsilon_p$

Lecture 3. Plasticity: Basics

Hardening rules (models)

Kinematic Hardening rule:

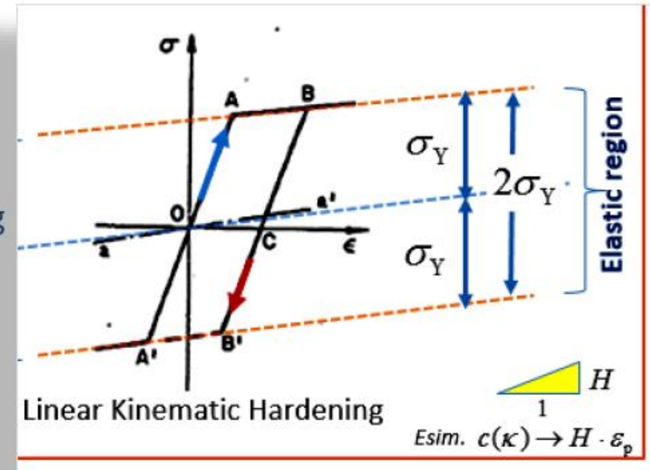


$$\left|\sigma - c(\kappa)\right| = \sigma_0,$$

shift of the yield stress

Loading

Reversal loading



Independent Hardening rule:

- Assumes ***yield stress*** in **tension** and **compression** are ***independent*** **(different)**

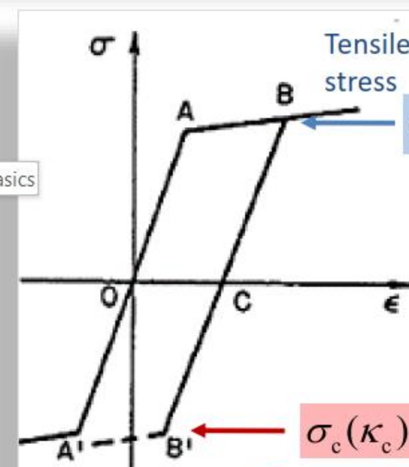
Hardening rule:

$$\sigma = \begin{cases} \sigma_t(\kappa_t), & \sigma > 0 \\ \sigma_c(\kappa_c), & \sigma < 0 \end{cases}$$

$\kappa_t > 0$ **tension**

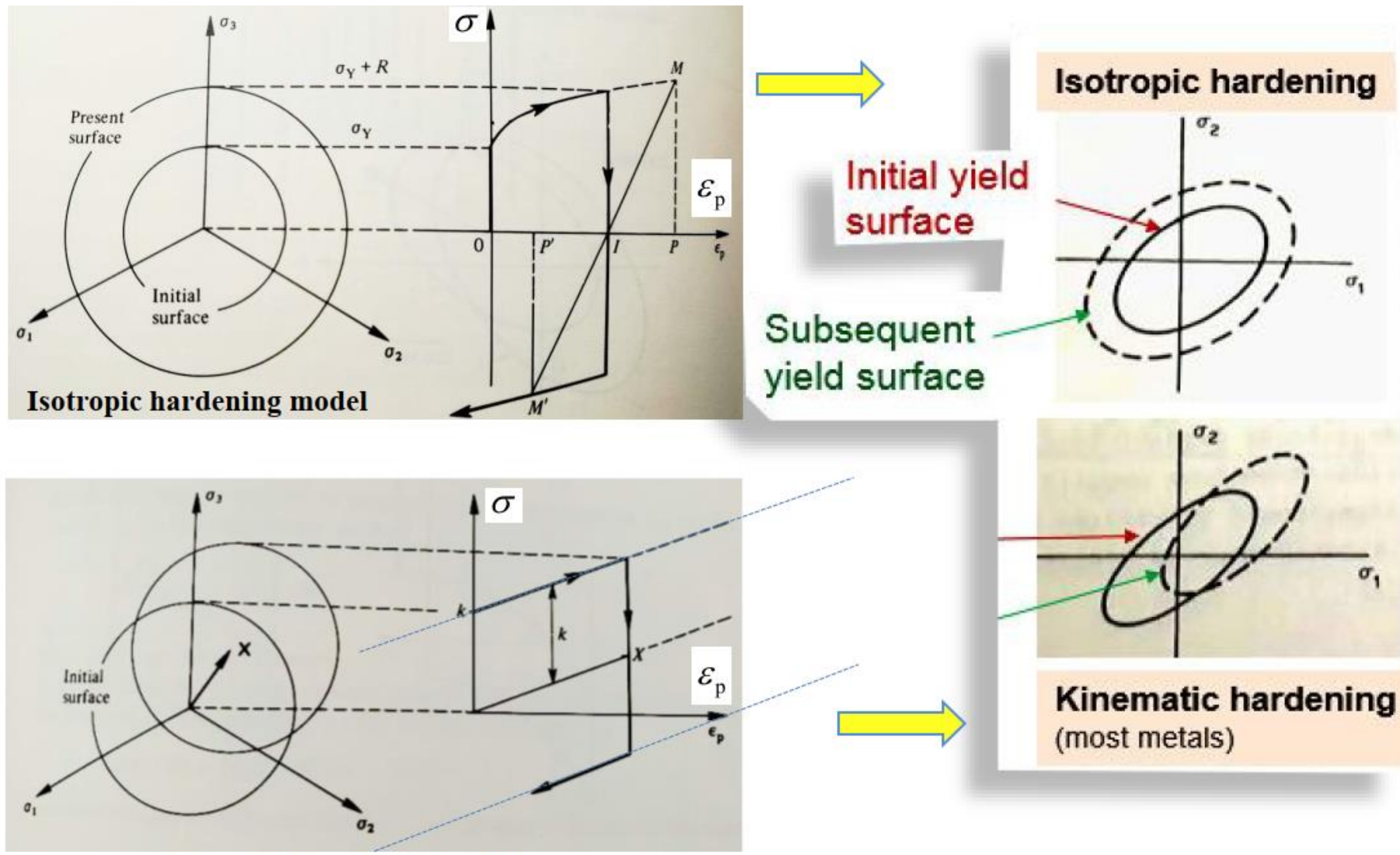
compression $\kappa_c > 0$

Lecture 3. Plasticity: Basics



Tensile yield stress $\sigma_t(\kappa_t)$

$\sigma_c(\kappa_c)$ Compressive yield stress



Kinematic hardening model

Some 1-D empirical models for work (also called strain-) hardening phenomena:

These models are relationship between the stress and the amount of plastic strain

$$\sigma = K\epsilon_p^n$$

$$\sigma = \sigma_y + K\epsilon_p^n \qquad n \sim 0.2 \text{ -… } 0.5$$

$$\sigma = \sigma_y + K(\epsilon_0 + \epsilon_p)^n$$

Lecture 3. Plasticity: Basics

Mixed hardening rule

A subsequent yield surface:

effective plastic strain

$$F(\sigma_{ij}, \varepsilon_{ij}^{p}; k(\varepsilon_{ij}^{p})) = F(\sigma_{ij} - \alpha_{ij}) - k^2(\bar{\varepsilon}^{p}) = 0,$$

***F* determines** the **shape**

Shift or (translation)

Kinematic hardening

Determines the **size** as function of cumulated plastic strain (**expansion**)

Isotropic hardening

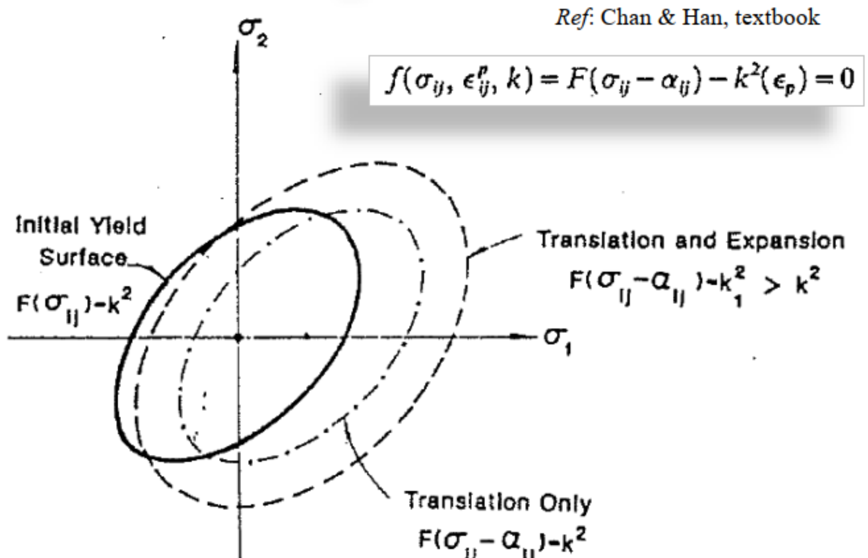
α_{ij}, k^2 - hardening parameters

Mixed hardening rule allows to model different degrees of Bauschinger effect by adjusting the two hardening parameters α_{ij} & k^2

Mixed hardening is a result of a combination of *kinematic* and *isotropic hardening* (Hodge, 1957)

Ref: Chan & Han, textbook

$$f(\sigma_{ij}, \epsilon_{ij}^{p}, k) = F(\sigma_{ij} - \alpha_{ij}) - k^2(\epsilon_p) = 0$$



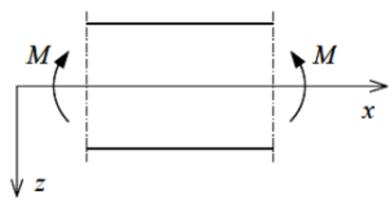
The elastic-plastic calculation

The moment-curvature relation for the symmetric rectangular cross-section

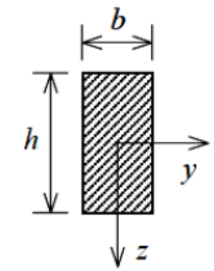
Task: Derive the relation between the bending moment $\boldsymbol{M}$ and the resulting curvature

$$\kappa(M)$$

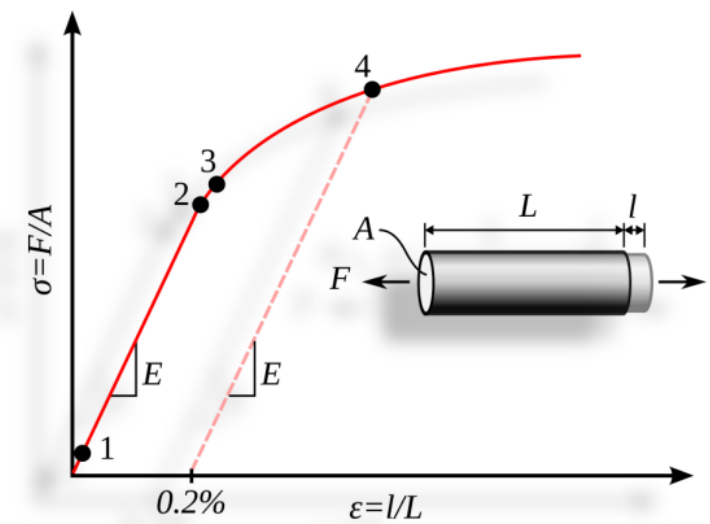
when the loading increases monotonically.



Pure bending of a beam

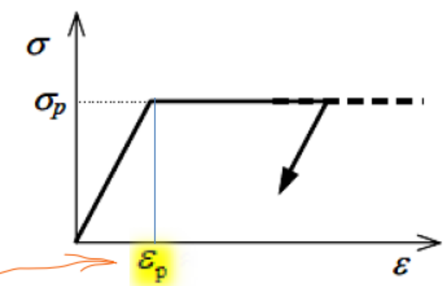


rectangular cross-section



2: Proportionality limit
3: Elastic limit
4: Offset yield

steel $\varepsilon_p \approx 0.2\%$



Elastic-perfectly plastic material behaviour (Idealised behaviour)

Idealised elastic-plastic constitutive model* describes simple mechanical behavior of metals

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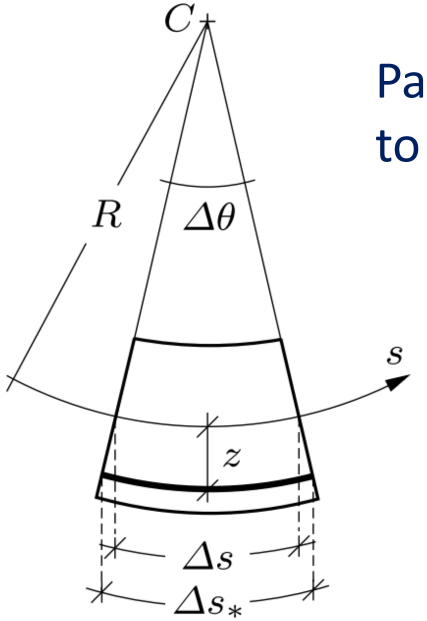
* Learn more details in *material modelling* course

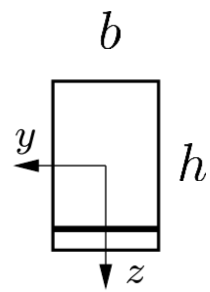
N.B. it is assumed that stability effects come after the profile reaches full plastic moment

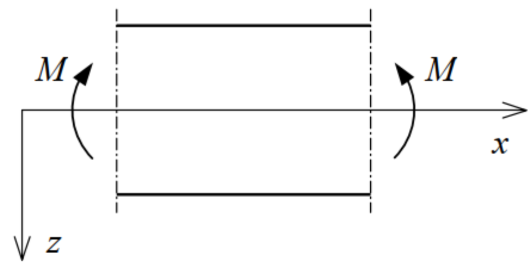
Lecture 3. Plasticity: Basics

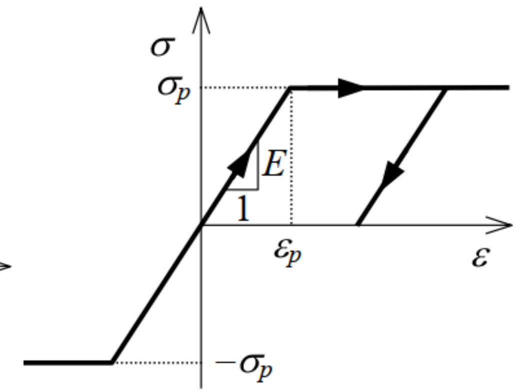
The moment-curvature relation for the rectangular cross-section



Part of a beam with rectangular cross section subjected to bending, with ideal-plastic material behaviour.







Normal strain: $\varepsilon = \dfrac{\Delta s_* - \Delta s}{\Delta s} = kz$

Normal stress: $\sigma = E\varepsilon = Ekz$

$$I = \int_A z^2 \mathrm{d}A = \frac{bh^3}{12}$$

Bending moment: $M = \displaystyle\int_A z\sigma \mathrm{d}A = k \int_A E z^2 \mathrm{d}A = EIk$ -- if *E* is constant

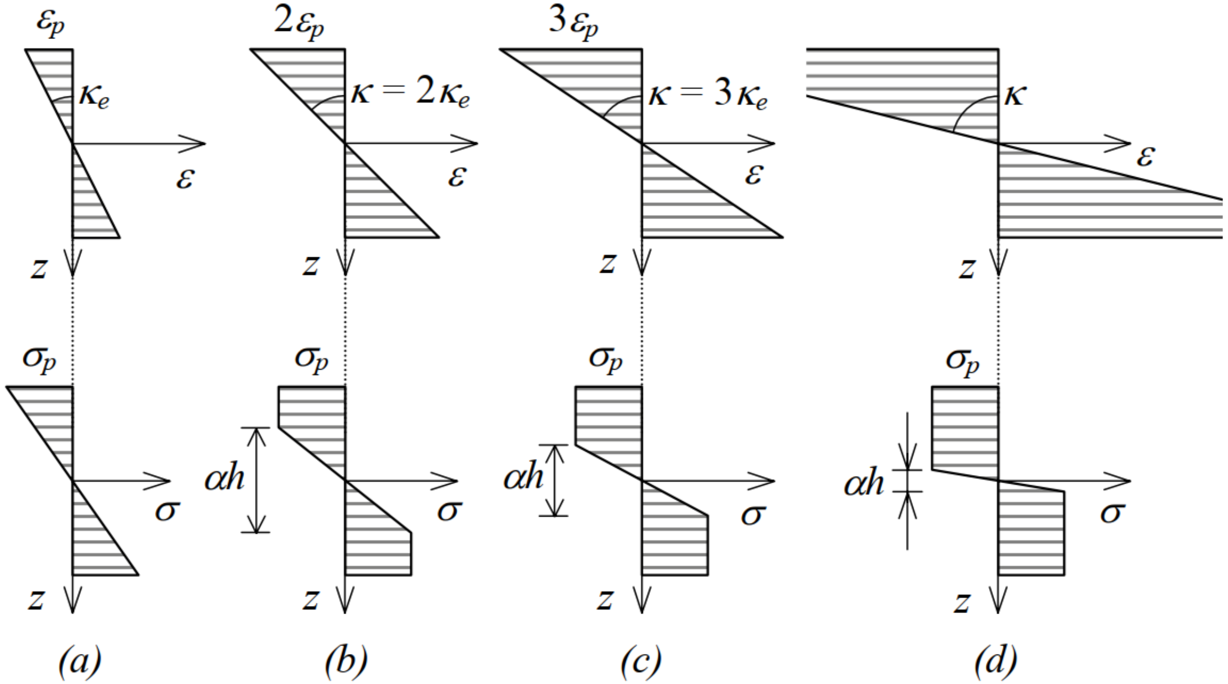
Lecture 3. Plasticity: Basics

The moment-curvature relation for the rectangular cross-section

This is valid until the yield stress σ_p is reached in the extreme fibres, i.e., until

$$M = M_e = \frac{1}{6} bh^2 \sigma_p \quad \text{and} \quad \kappa = \kappa_e = \frac{2}{h}\frac{\sigma_p}{E}$$

Strain and stress distributions in the cross-section:



$$\sigma = E\varepsilon \quad \text{for} \quad |\varepsilon| < \varepsilon_p$$

$$\sigma = \pm\sigma_p \quad \text{for} \quad |\varepsilon| \geq \varepsilon_p$$

or

$$\sigma = E\varepsilon \quad \text{for} \quad |z| < \frac{1}{2}\alpha h$$

$$\sigma = \pm\sigma_p \quad \text{for} \quad |z| \geq \frac{1}{2}\alpha h$$

The moment-curvature relation for the rectangular cross-section

Relation between M and κ in the elastic-plastic phase takes the form:

$$M = M_p\left[1-\frac{1}{3}\left(\frac{\kappa_e}{\kappa}\right)^2\right] \quad \text{for} \quad \kappa \geq \kappa_e, \quad \text{where} \quad M_p = \frac{1}{4}bh^2\sigma_p \text{ is the fully plastic moment.}$$

The moment-curvature diagram:

Slope continuity $\rightarrow \left(\frac{\partial M}{\partial \kappa}\right)_{\kappa=\kappa_e} = \left(\frac{2}{3}M_p\frac{\kappa_e^2}{\kappa^3}\right)_{\kappa=\kappa_e} = \frac{\frac{2}{3}M_p}{\kappa_e} = \frac{M_e}{\kappa_e}$