ME 530 : Mechanics of Plastic Deformation

Introduction to Plasticity

2022 Fall Semester

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References

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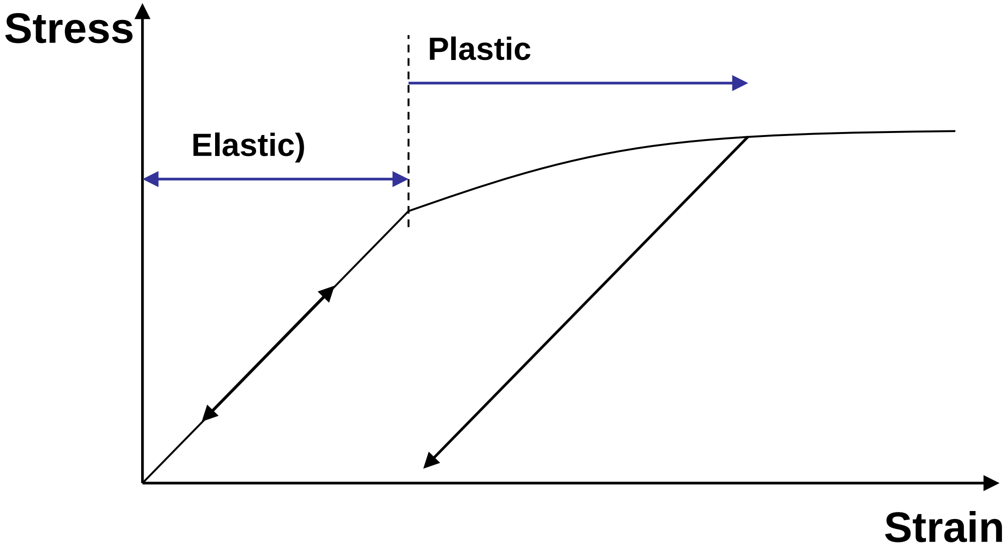
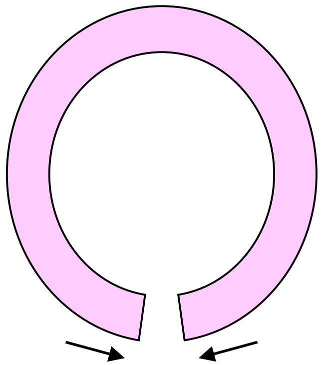
Plastic Deformation

Plastic Forming ?

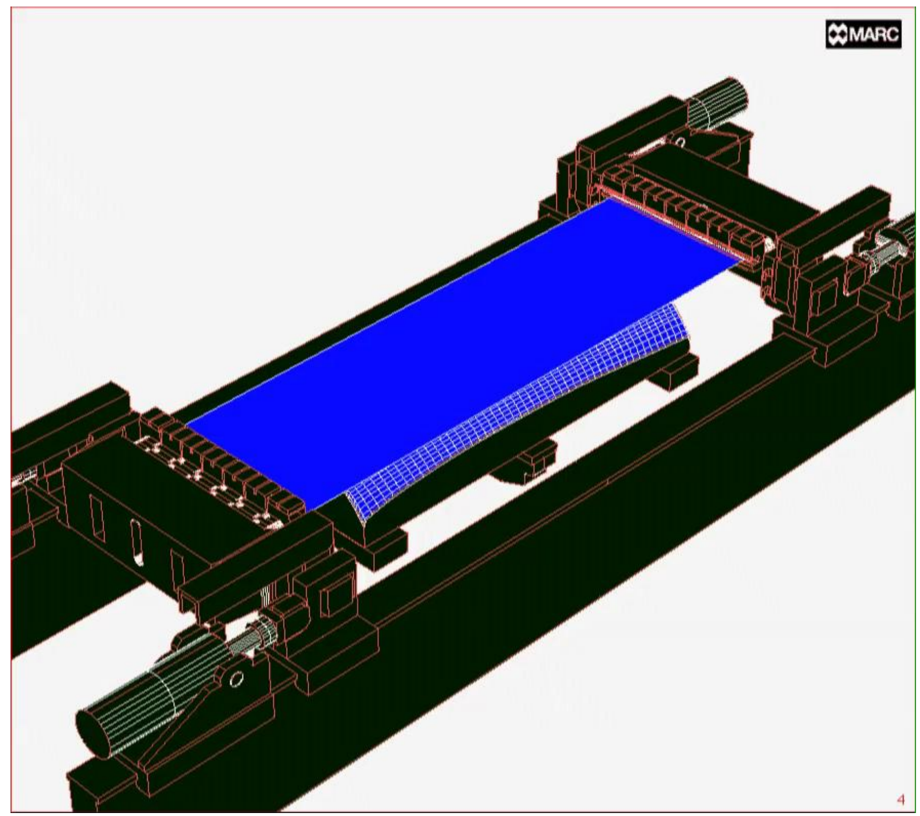
Plastic ⟺ **Elastic**



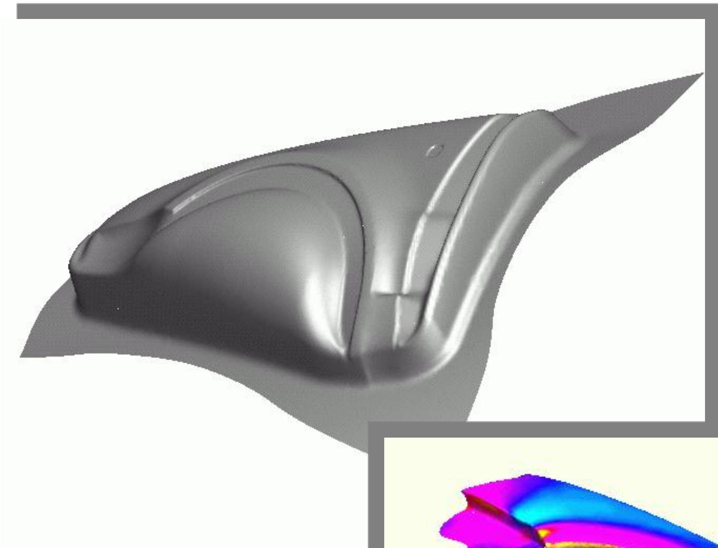


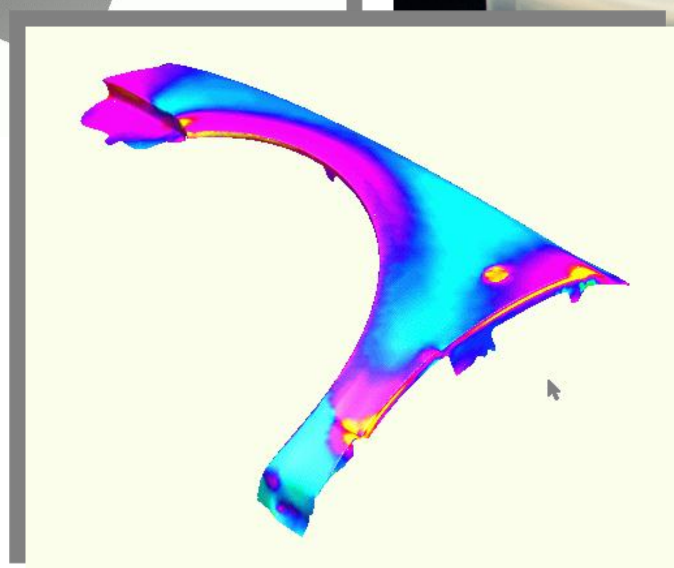
Example : Skin Panel Stretch Forming

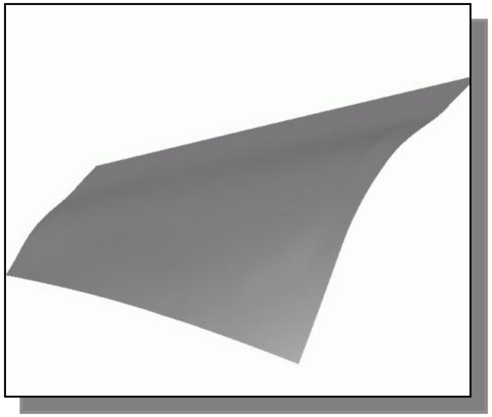
(Movie)

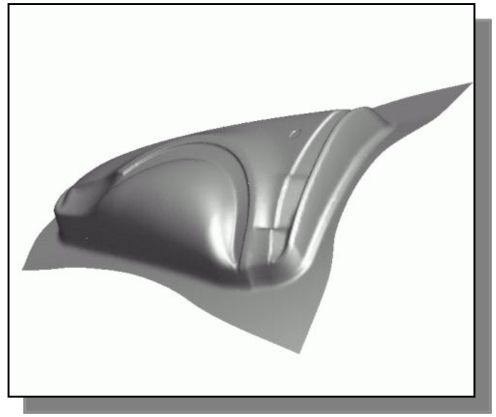


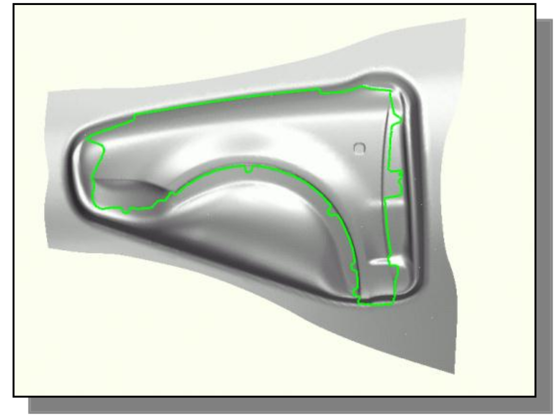
Sheet metal forming

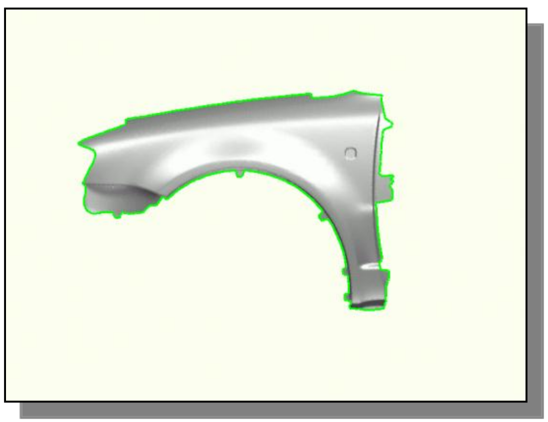




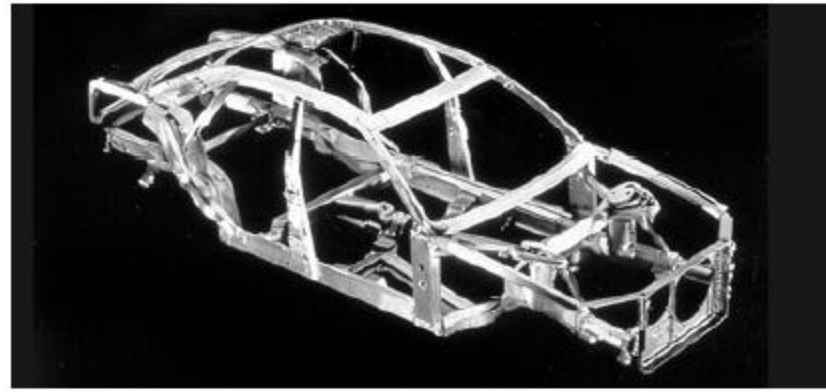




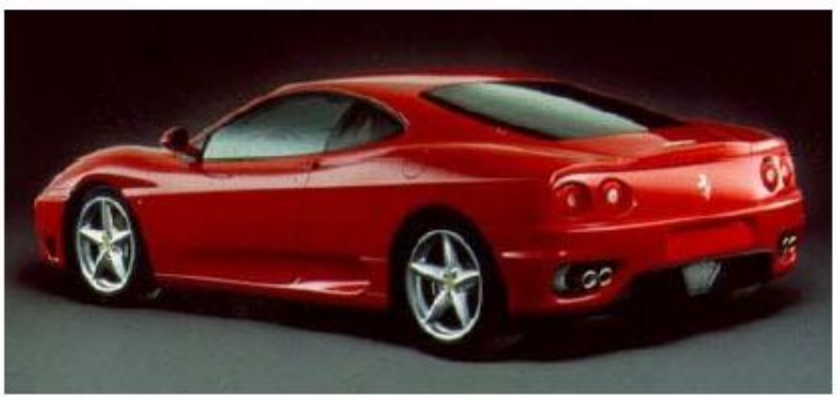


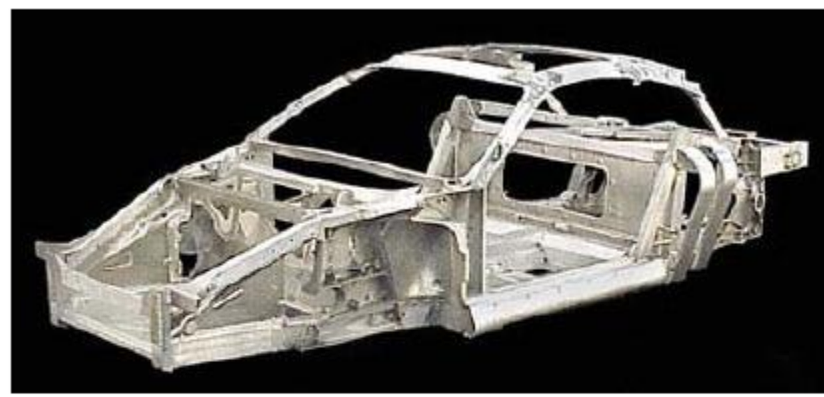


Automotive Applications for aluminum - Space Frames



Audi A8





Ferrari Modena

Blank & Draw - Cup

Commercial-cup making process

(1st Drawing) (2nd Drawing) (1st Ironing) (2nd Ironing) (3rd Ironing)

Trim

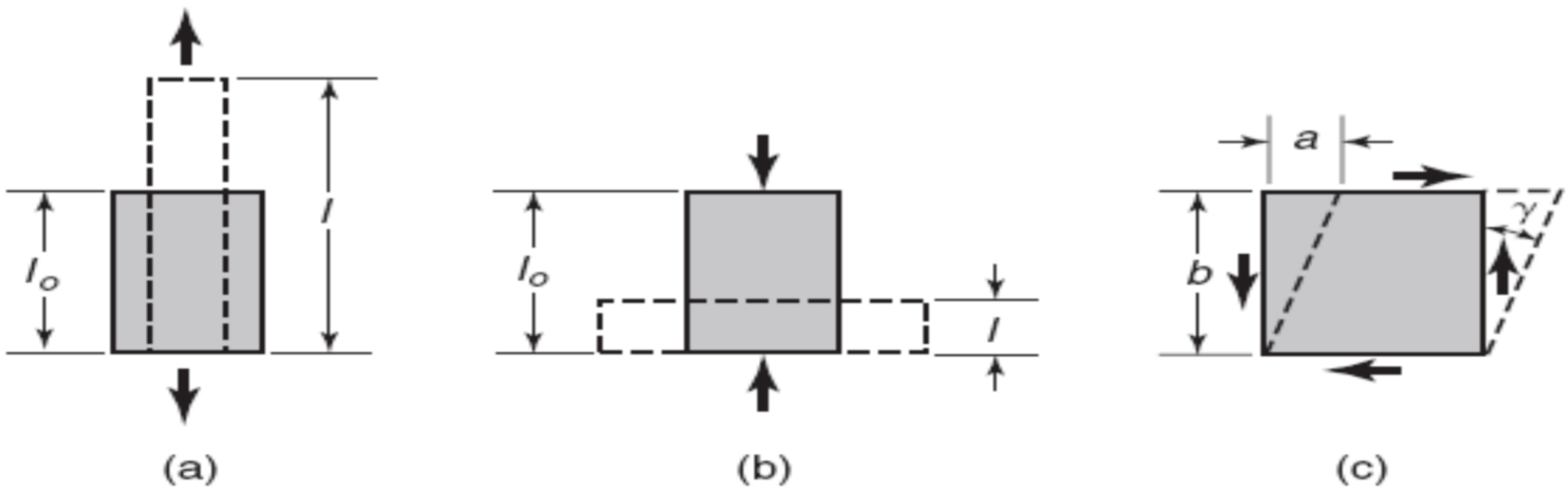
Clean & Decorate

Neck & Flange

ALCOA

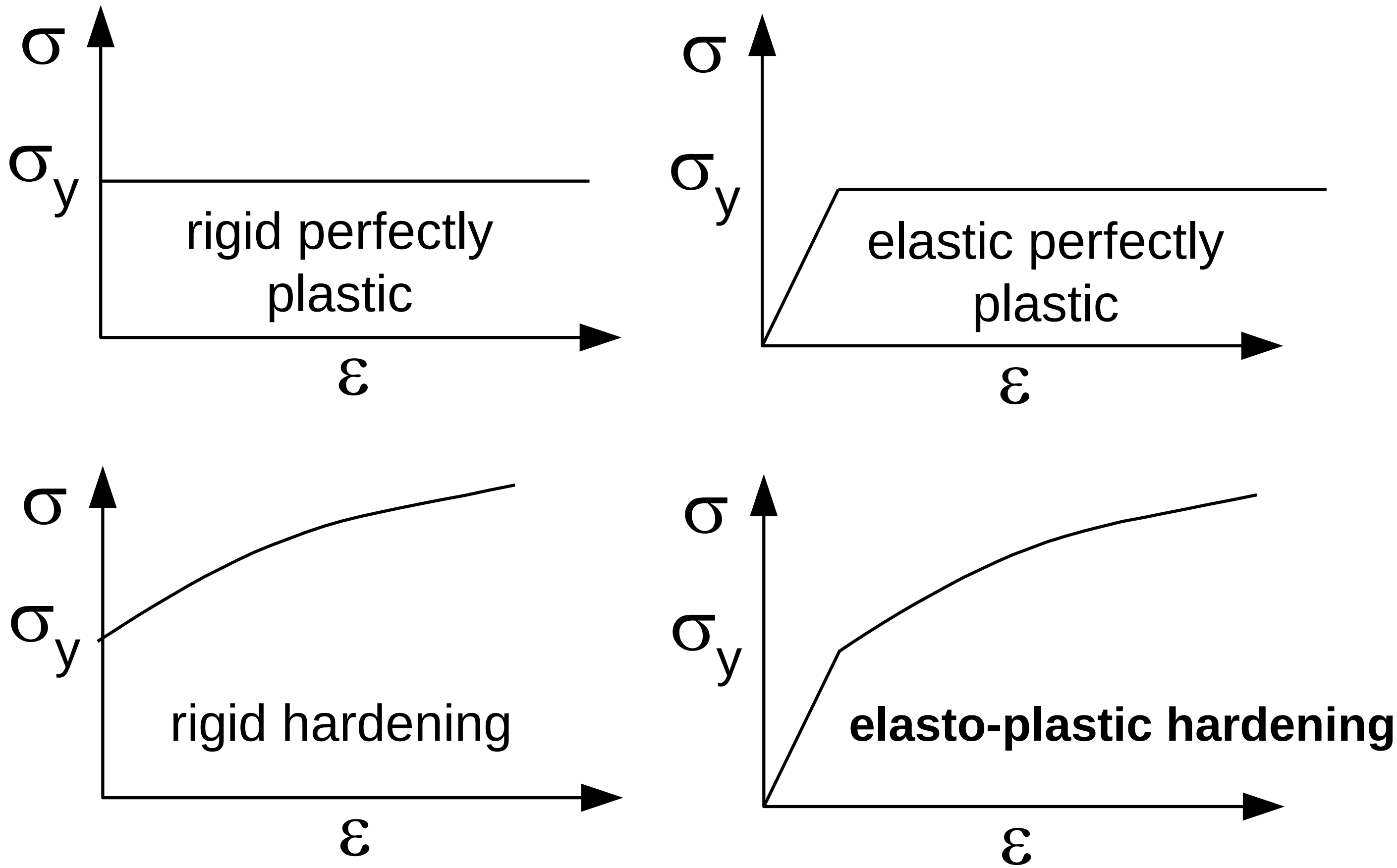
Deformation

- There are 3 types of strain: (a) tensile, (b) compressive and (c) shear.

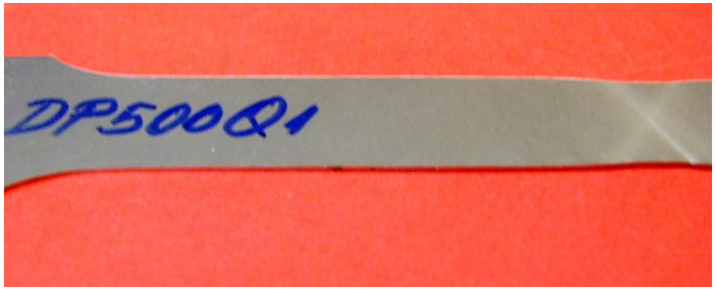


Tension test is the most common test for determining the strength-deformation characteristics of materials.

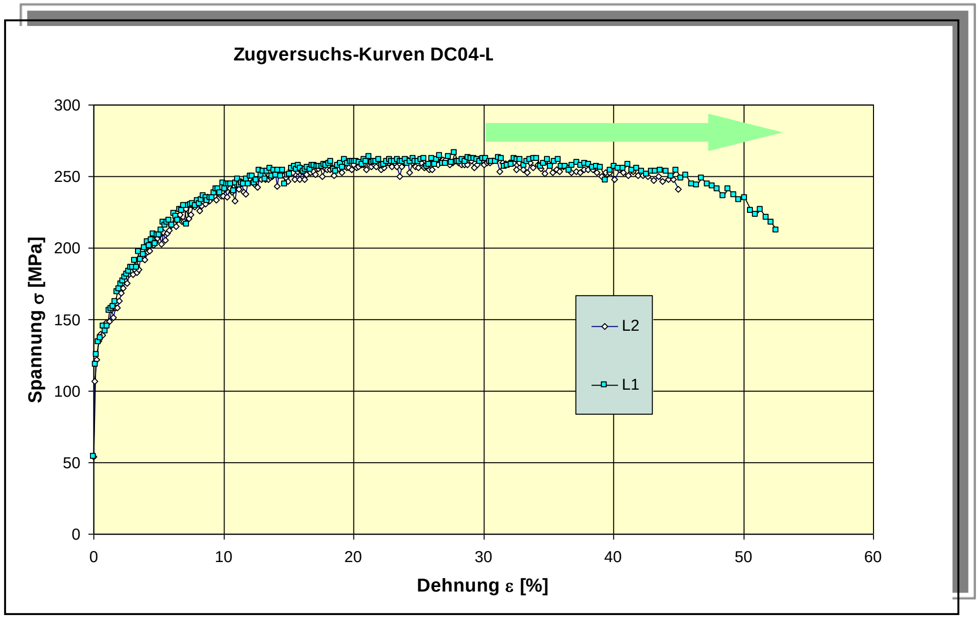
Time-independent plasticity (Ideal behavior)

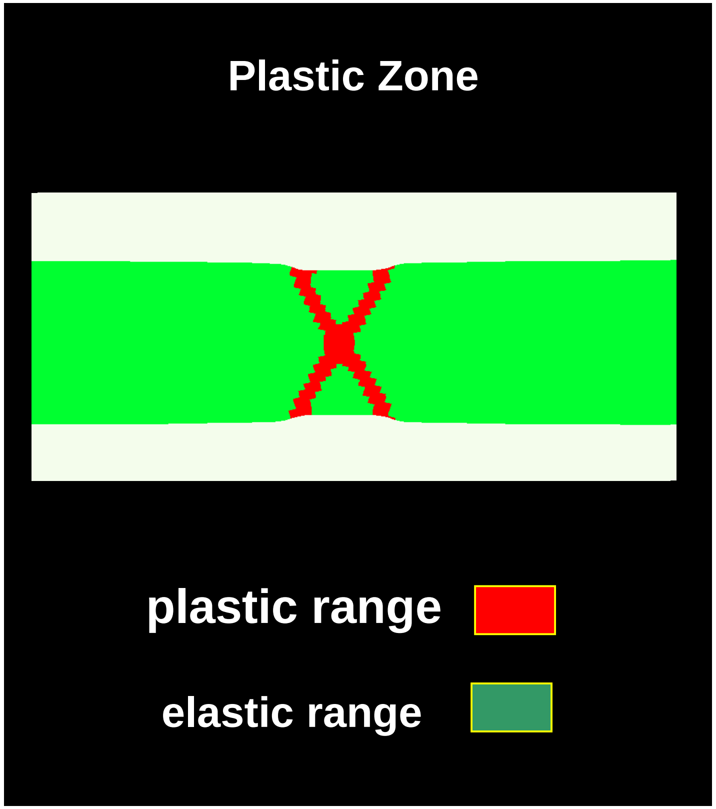


Tensile Test

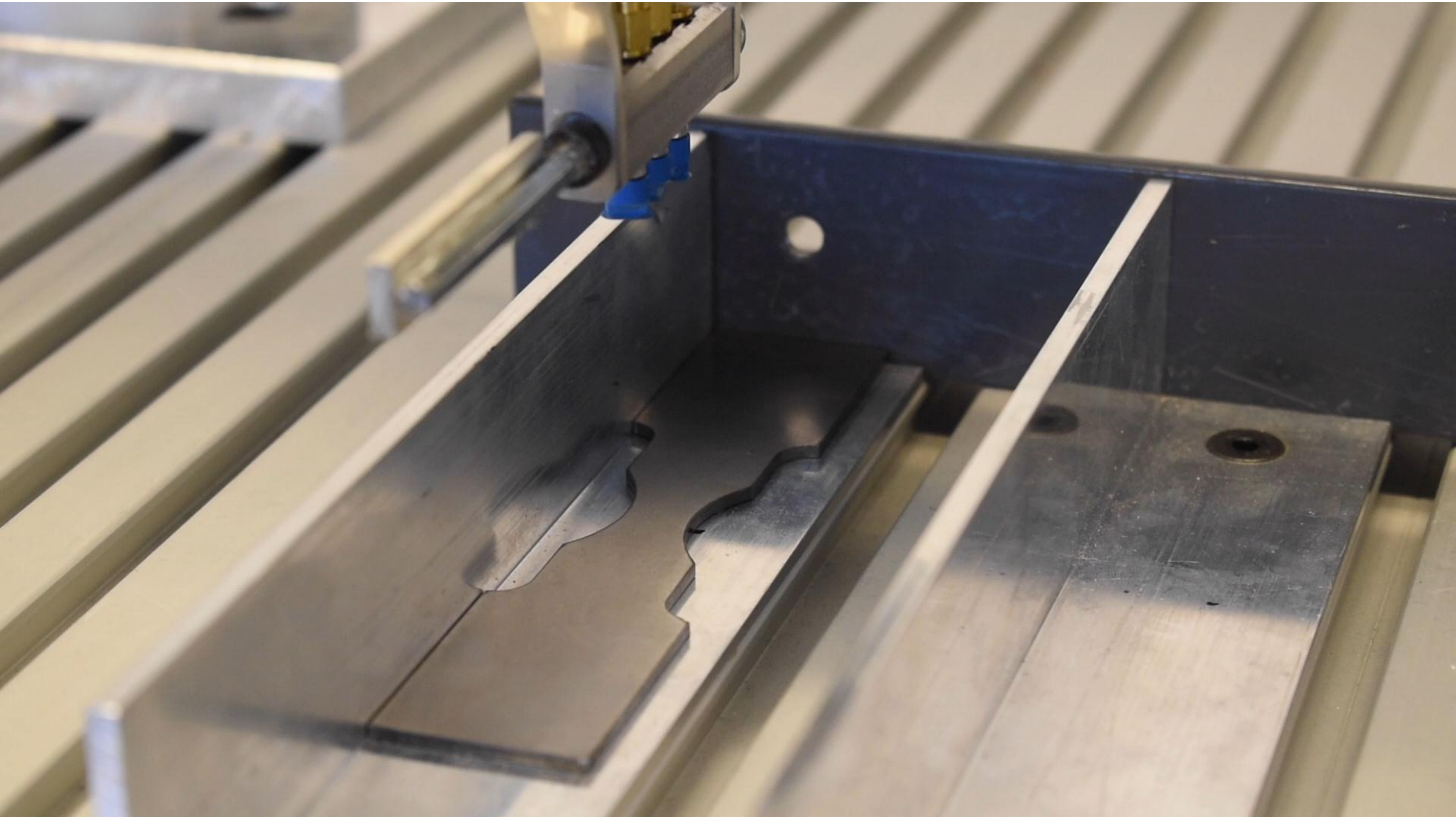


- **Localization of plastic Zone**

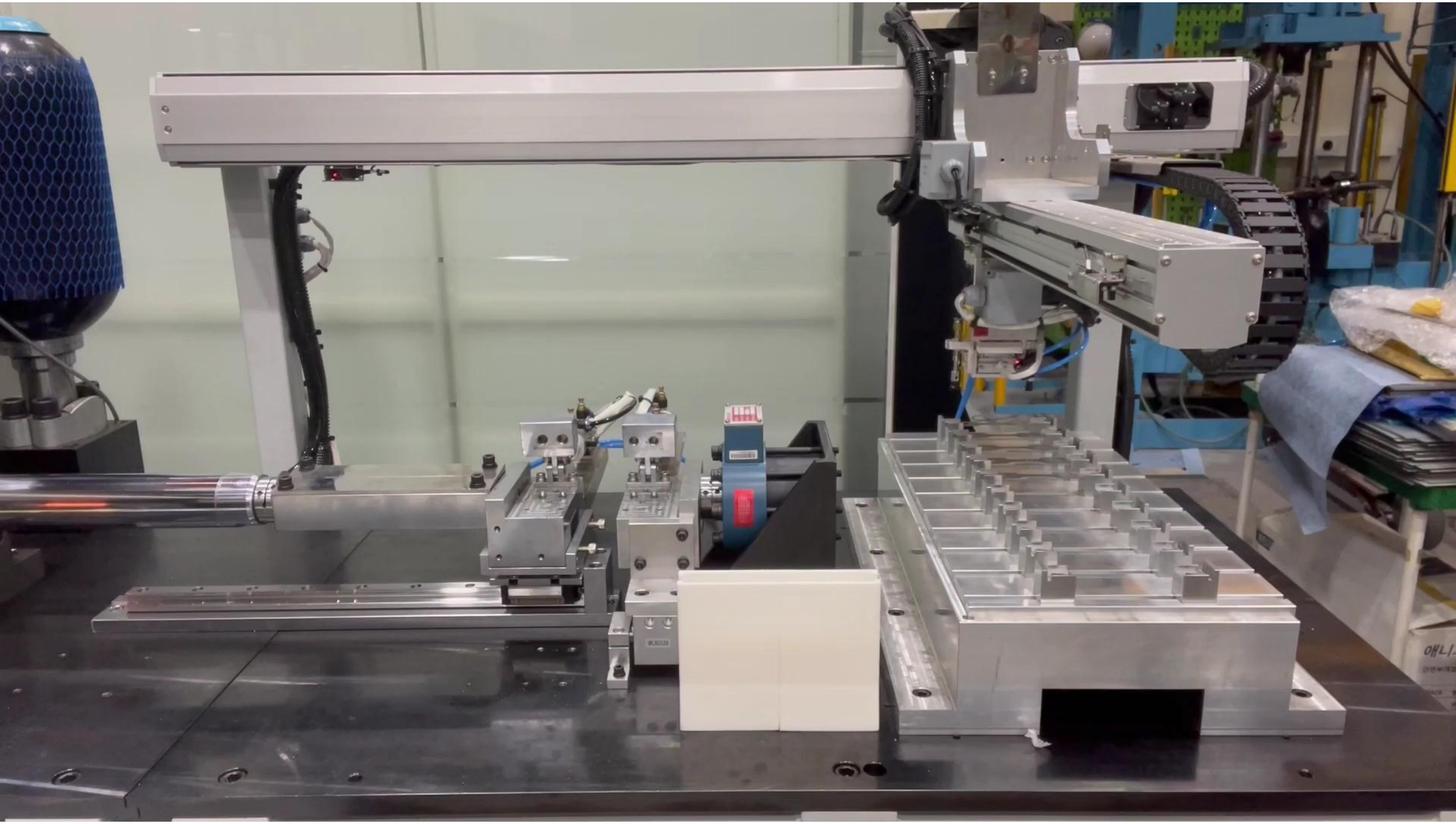




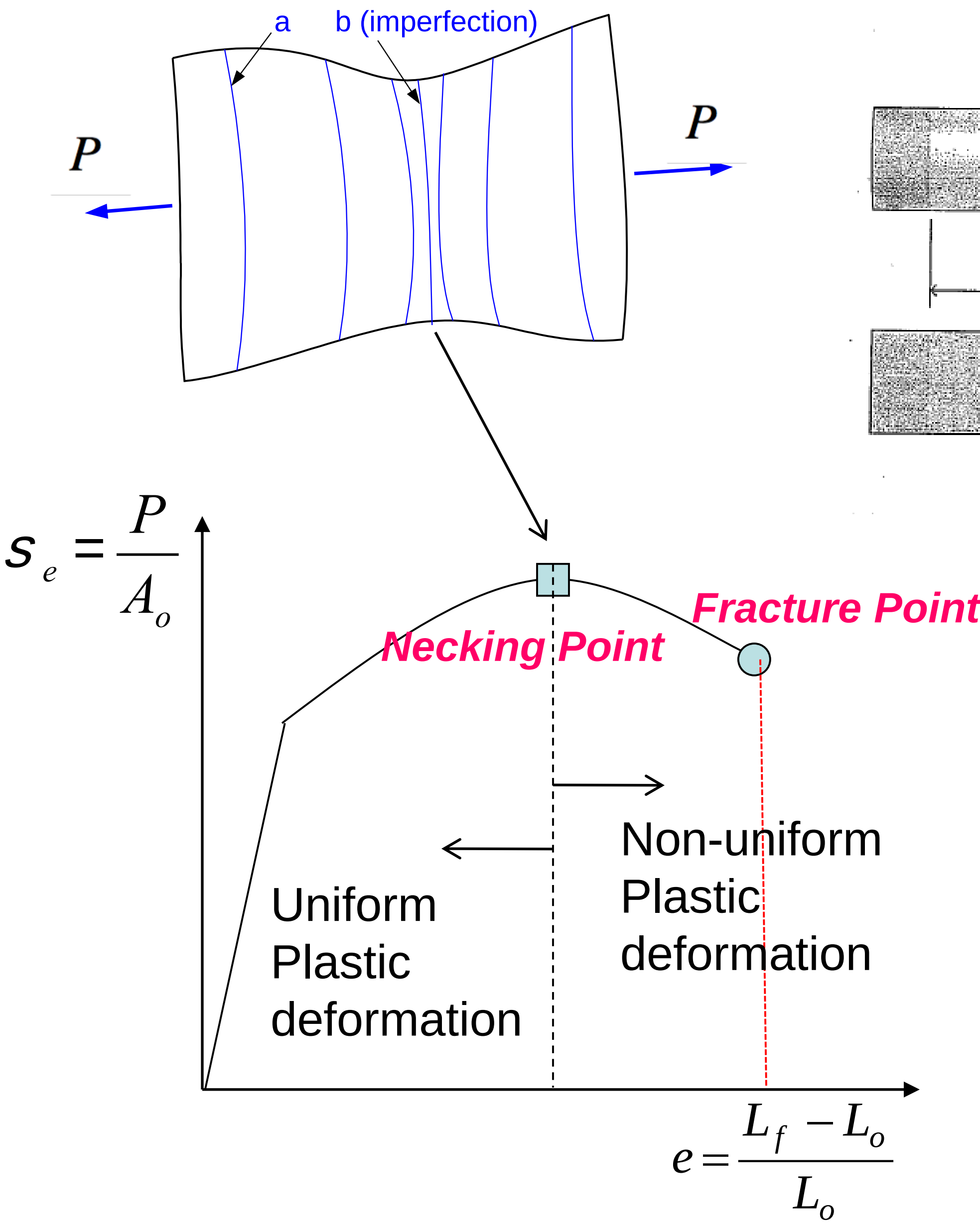
Tensile Test Stainless Steel Specimen

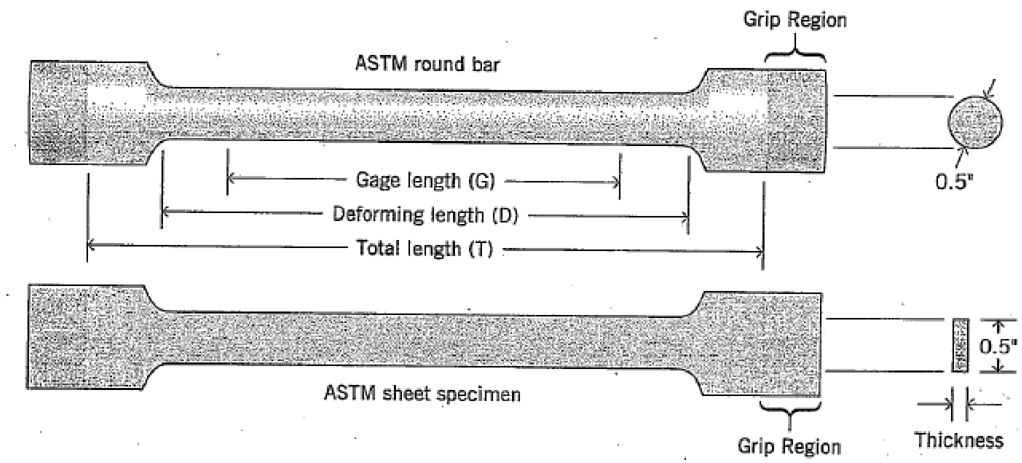


KAIST high strain-rate tensile testing machine



Engineering Strain & Stress





$$s_e = \frac{P}{A_o}$$

$$e = \frac{L_f - L_o}{L_o} = \frac{\Delta L}{L_o}$$

True Stress and True Strain

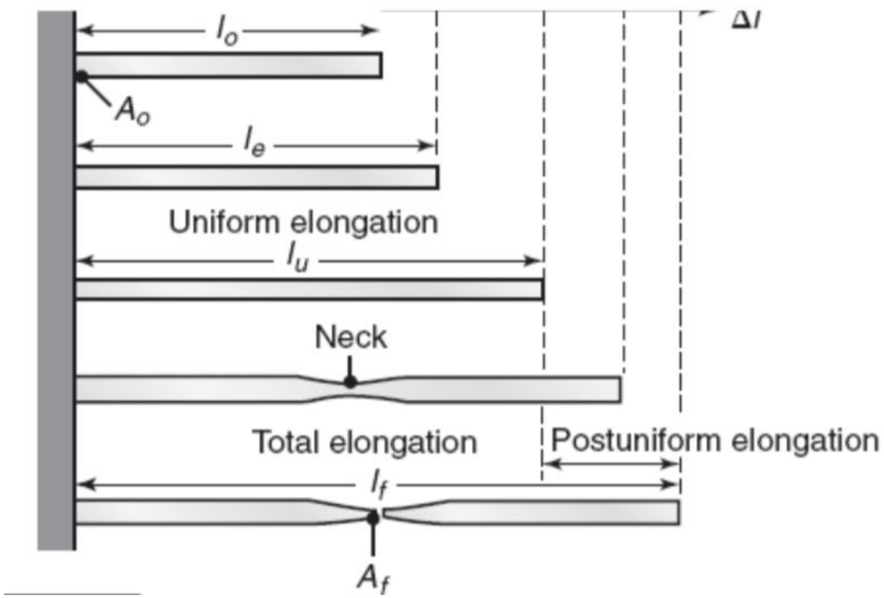
Similar to stress definition, true stress is defined as

$$\sigma = \frac{P}{A}$$

where A is the actual area supporting the load

True strain is defined as

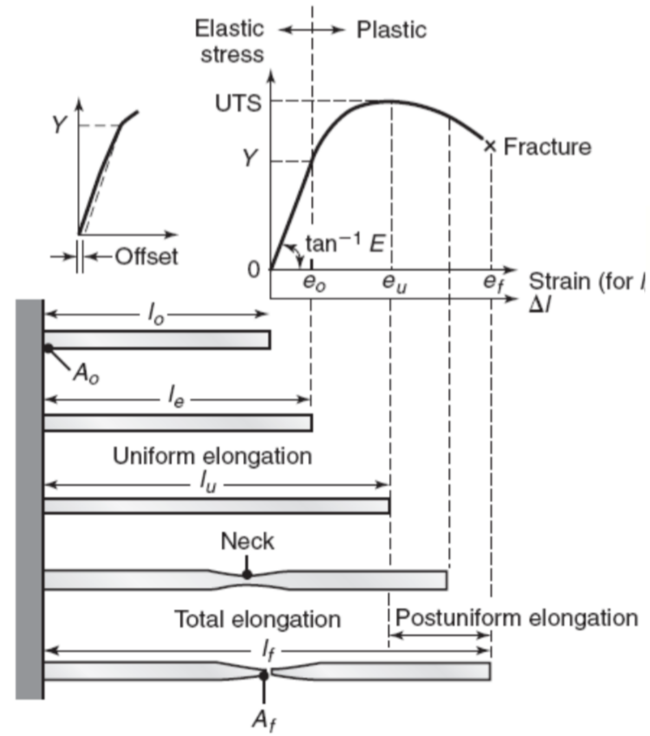
$$\varepsilon = \int_{l_0}^{l} \frac{dl}{l} = \ln\left(\frac{l}{l_0}\right)$$



True Stress and True Strain

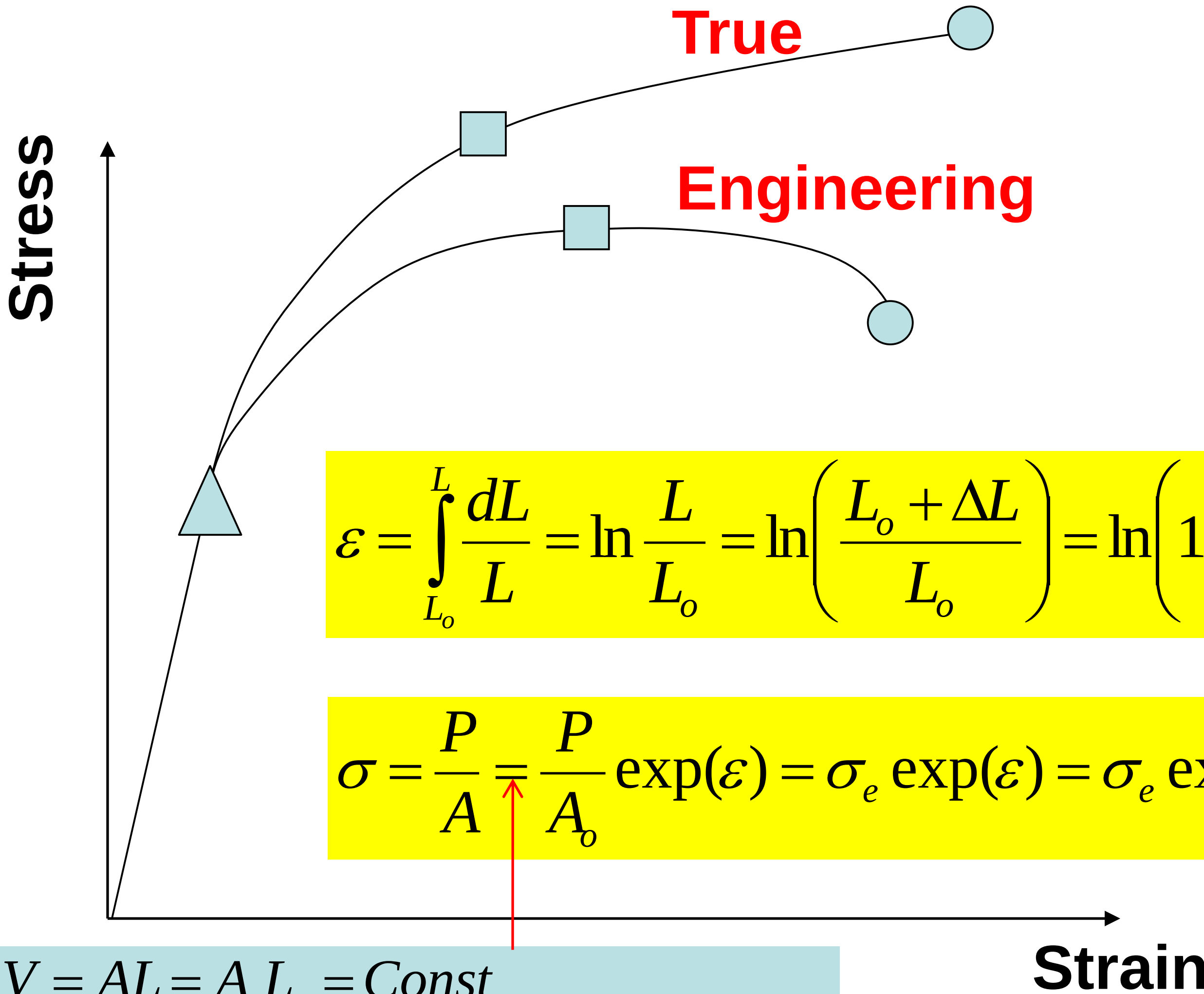
Since volume remains constant in the plastic region of the test (V=Vo), true strain can be expressed as

$$\varepsilon = \ln\left(\frac{l}{l_0}\right) = \ln\left(\frac{A_0}{A}\right)$$



- From above, the largest strain is at the narrowest region of the neck.

True Strain & Stress



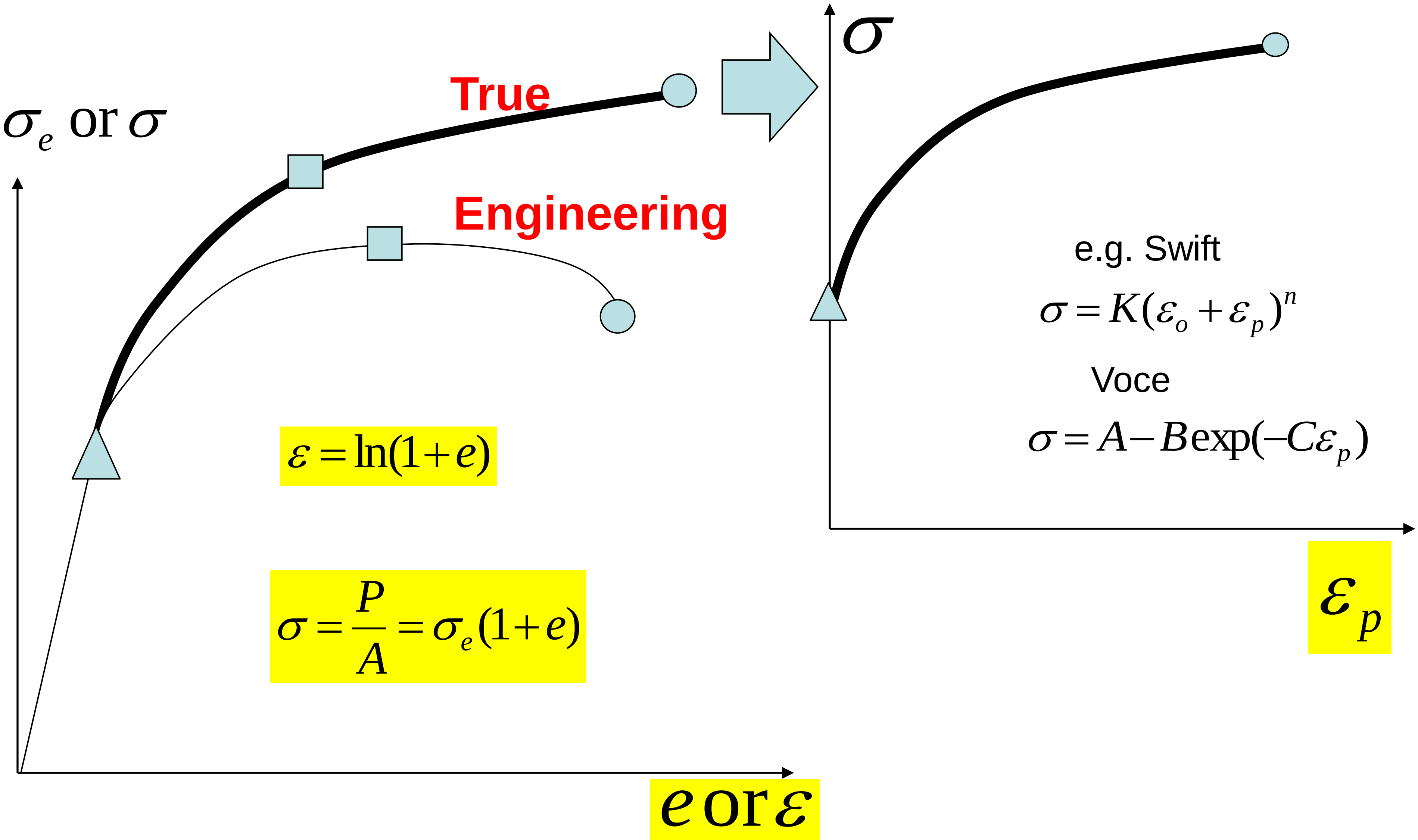
$$\varepsilon = \int_{L_o}^{L} \frac{dL}{L} = \ln\frac{L}{L_o} = \ln\left(\frac{L_o + \Delta L}{L_o}\right) = \ln\left(1 + \frac{\Delta L}{L_o}\right) = \ln(1+e)$$

$$\sigma = \frac{P}{A} = \frac{P}{A_o}\exp(\varepsilon) = \sigma_e \exp(\varepsilon) = \sigma_e \exp\big(\ln(1+e)\big) = \sigma_e (1+e)$$

$$V = AL = A_o L_o = Const$$

$$Then,\quad \frac{A_o}{A} = \frac{L}{L_o} = \exp(\varepsilon) \text{ or } \frac{1}{\mathrm{A}} = \frac{\exp(\varepsilon)}{\mathrm{A_o}}$$

True Strain & Stress

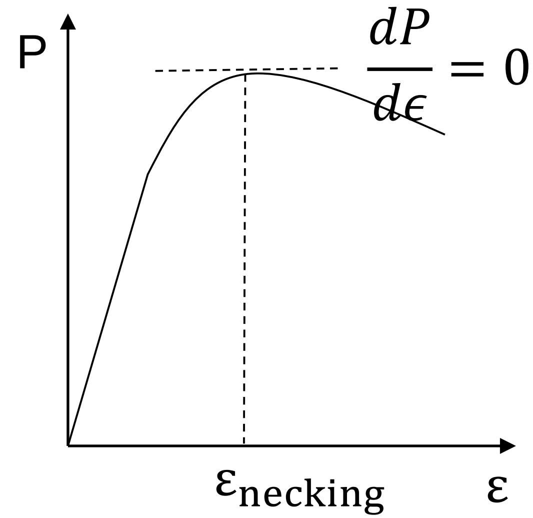


Instability in Tension

We have the following relationship:

$$\varepsilon = \ln\left(\frac{A_0}{A}\right), \quad A = A_0 e^{-\varepsilon}, \text{ and } \quad P = \sigma A = \sigma A_0 e^{-\varepsilon}$$

$$\frac{dP}{d\varepsilon} = \frac{d}{d\varepsilon}\left(\sigma A_0 e^{-\varepsilon}\right) = A_0\left(\frac{d\sigma}{d\varepsilon}\underline{e^{-\varepsilon}} - \sigma \underline{e^{-\varepsilon}}\right)$$

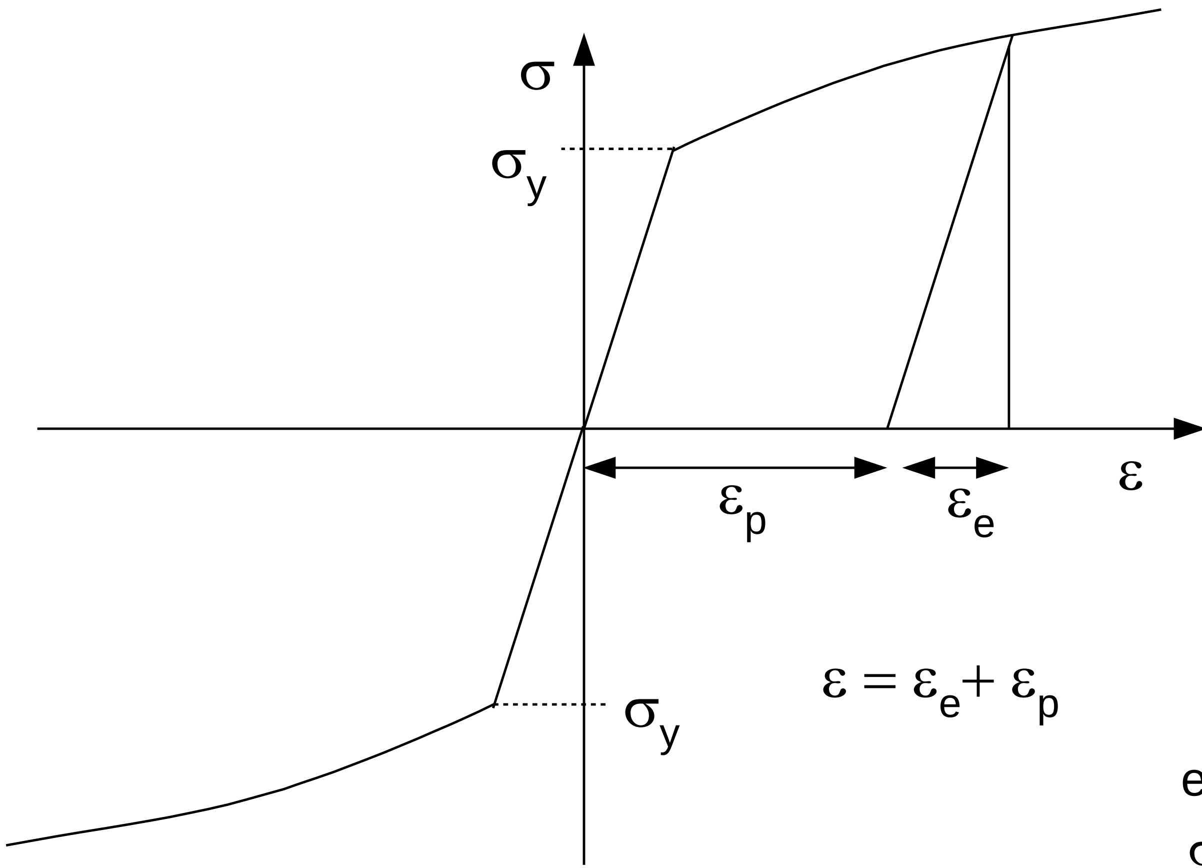


- Because *dP* = 0 at the UTS where necking begins,

$$\frac{d\sigma}{d\varepsilon} = \sigma$$

- However since $\sigma = K\varepsilon^n$, we have $\varepsilon_{necking} = n$

Main features of plasticity



$$\varepsilon = \varepsilon_e + \varepsilon_p$$

e.g. Swift

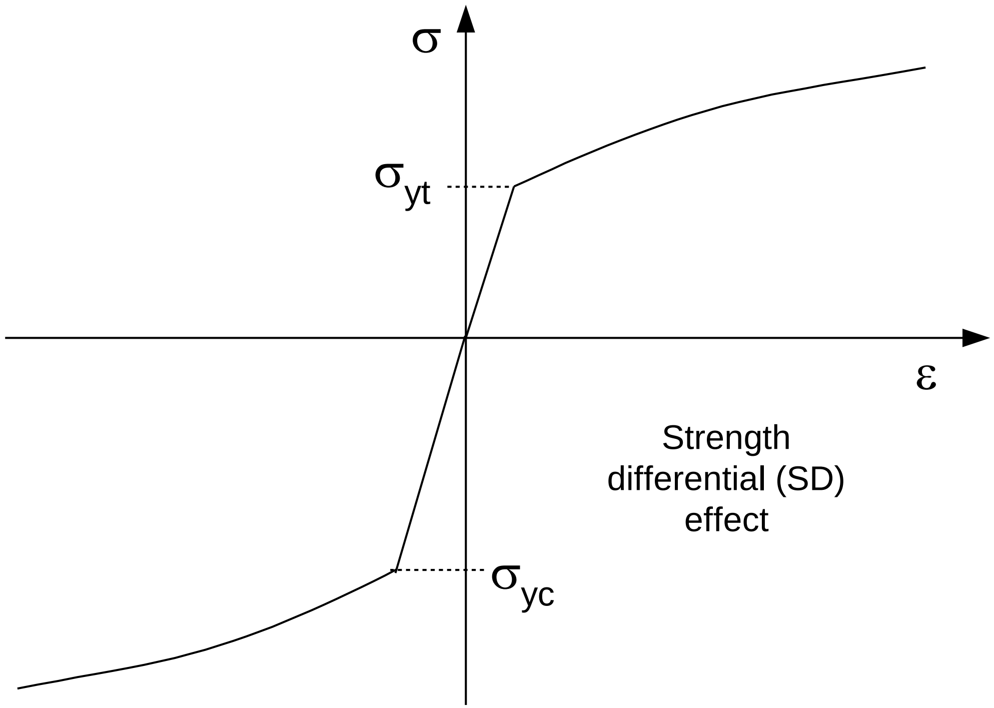
$$\sigma = K\left(\varepsilon_0 + \varepsilon^p\right)^n$$

Voce

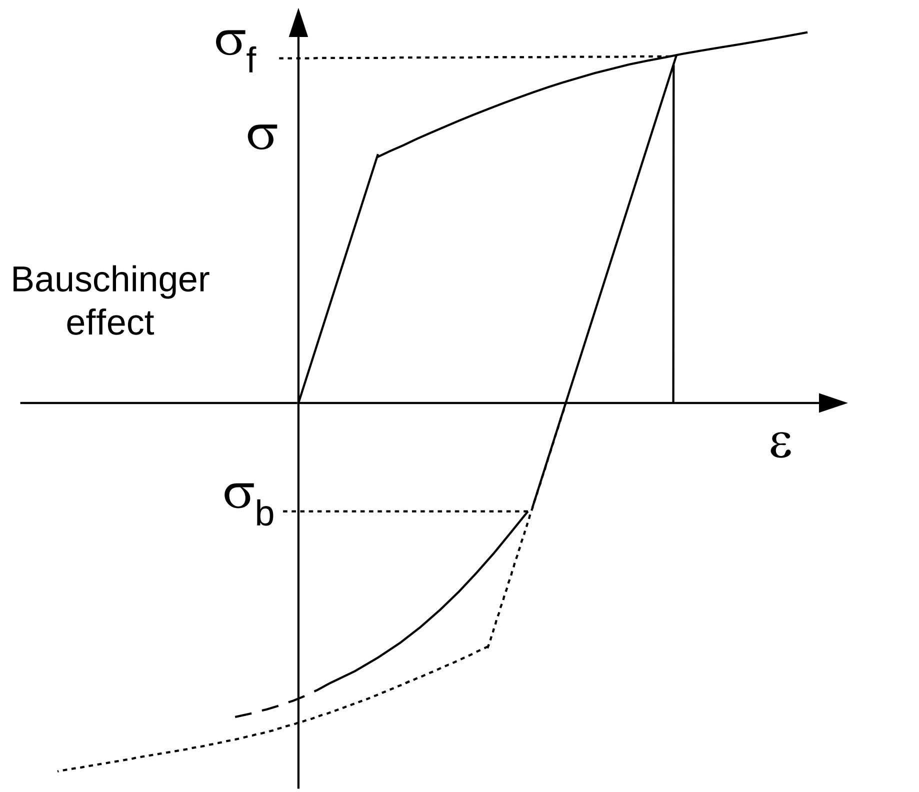
$$\sigma = A - B\exp\left(-C\varepsilon\right)$$

Other features

Strength differential (SD) effect

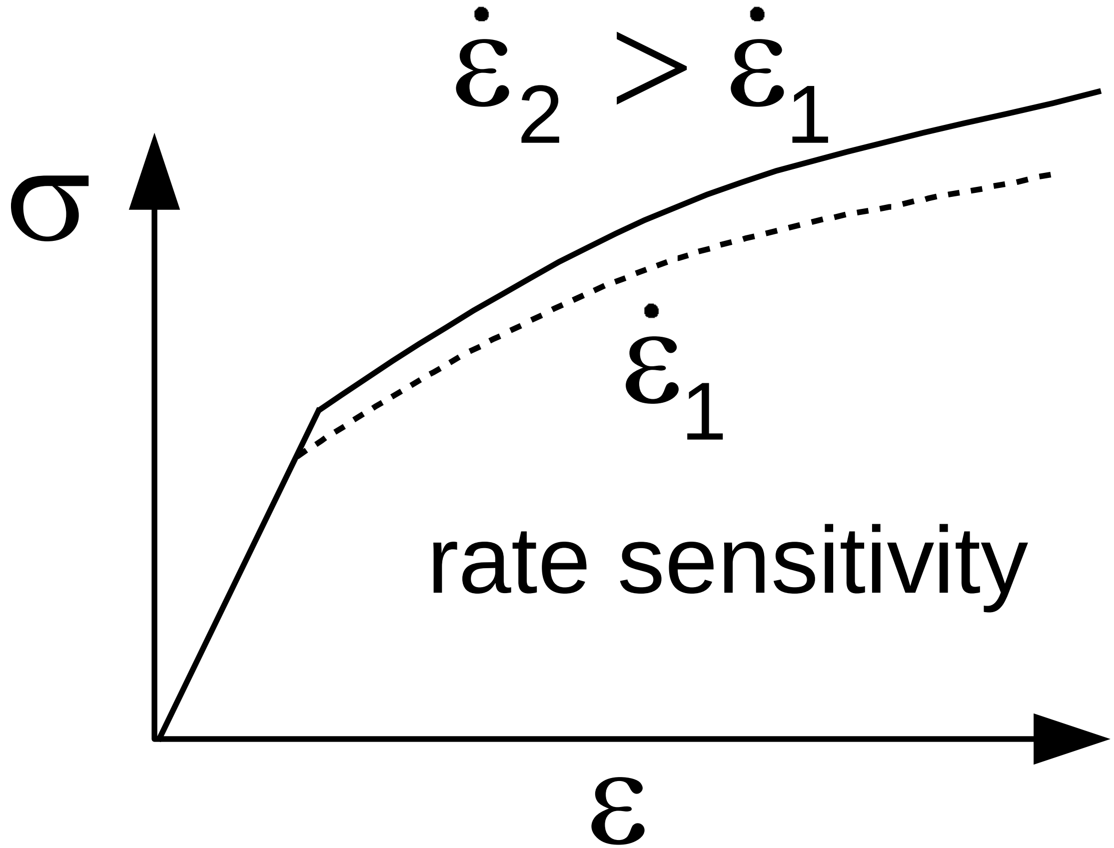


Bauschinger effect



Time-dependent behavior

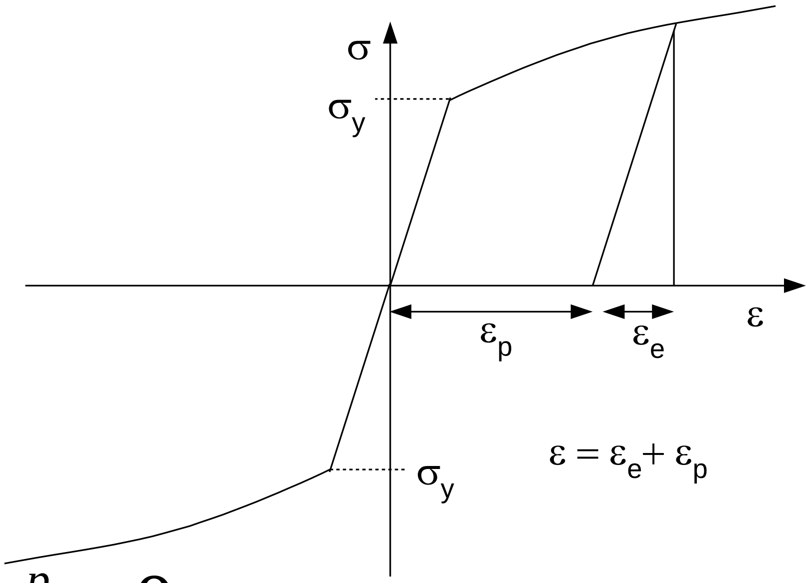
- Strain rate sensitivity (SRS)



Strain tensors

- Rate of deformation $\underline{\underline{\dot{\boldsymbol{\varepsilon}}}} \equiv \begin{bmatrix} \dot{\varepsilon}_{11} & \dot{\varepsilon}_{12} & \dot{\varepsilon}_{31} \\ \dot{\varepsilon}_{12} & \dot{\varepsilon}_{22} & \dot{\varepsilon}_{23} \\ \dot{\varepsilon}_{31} & \dot{\varepsilon}_{23} & \dot{\varepsilon}_{33} \end{bmatrix}$

- $d\underline{\underline{\boldsymbol{\varepsilon}}} = \underline{\underline{\dot{\boldsymbol{\varepsilon}}}}\, dt$



- $d\underline{\underline{\boldsymbol{\varepsilon}}} = d\underline{\underline{\boldsymbol{\varepsilon}}}^{e} + d\underline{\underline{\boldsymbol{\varepsilon}}}^{p}$

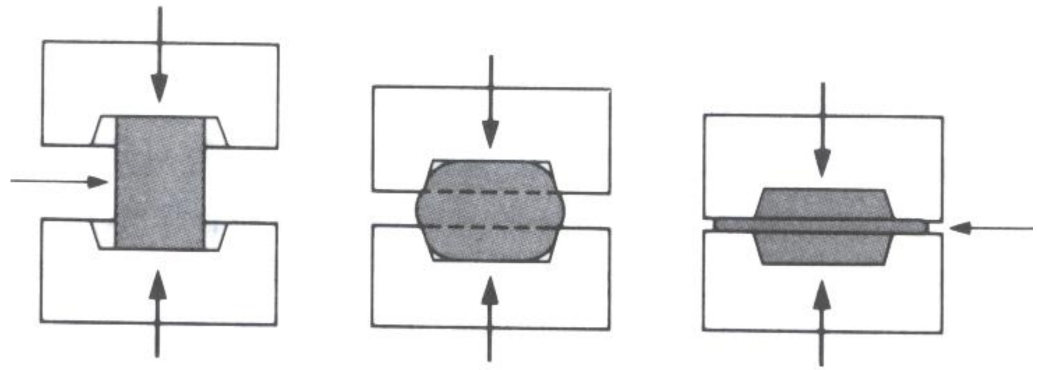
- $\varepsilon_{11}^{p} + \varepsilon_{22}^{p} + \varepsilon_{33}^{p} = \varepsilon_{1}^{p} + \varepsilon_{2}^{p} + \varepsilon_{3}^{p} = 0$

(Volume Constancy of Plastic Part)

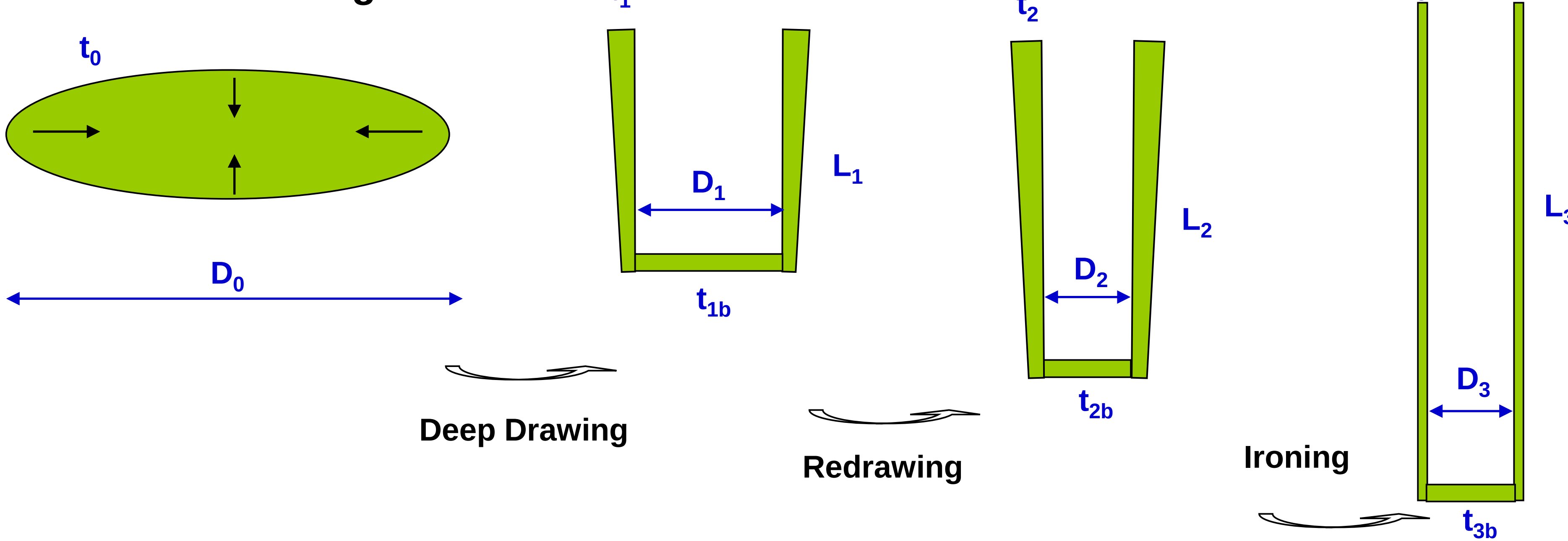
Volume Constancy in Plasticity

$$\varepsilon_{11}^{p}+\varepsilon_{22}^{p}+\varepsilon_{33}^{p}=0$$

Bulk Forming



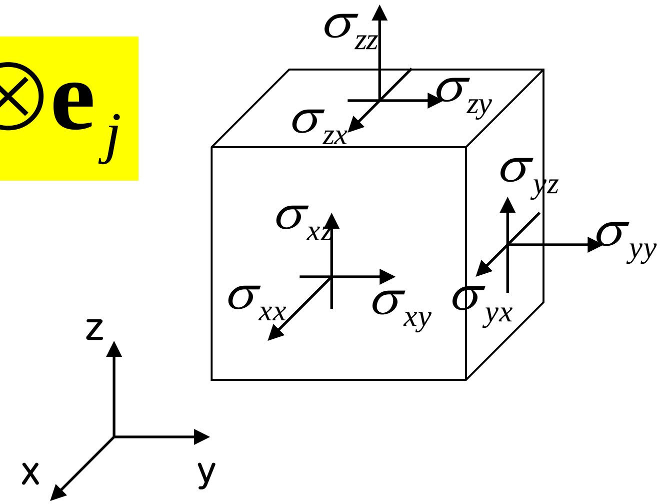
Sheet Forming



Stress Tensors

Cauchy Stress Tensor: $\boxed{\underline{\underline{\sigma}} = \boldsymbol{\sigma} = \sigma_{ij}\mathbf{e}_i \otimes \mathbf{e}_j}$

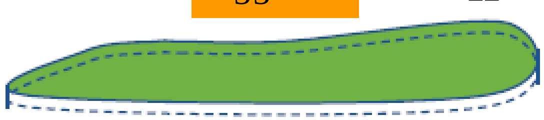
$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$



(Plane Stress)

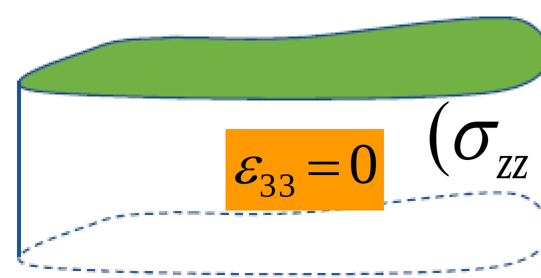
$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\boxed{\sigma_{33} = 0}$ $(\varepsilon_{zz} \neq 0)$



(Plane Strain)

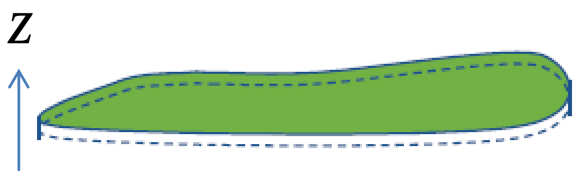
$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{21} & \varepsilon_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



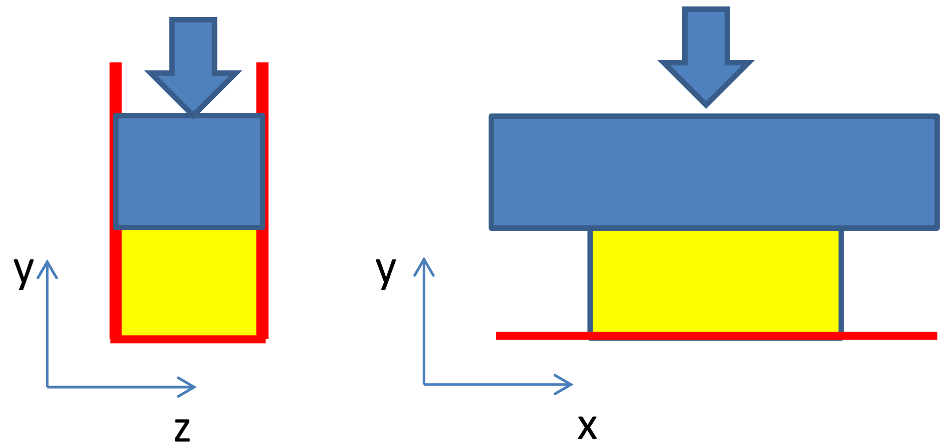
$\boxed{\varepsilon_{33} = 0}$ $(\sigma_{zz} \neq 0)$

Plane Stress $(\sigma_{zz} = 0)$

Plane Strain $(\varepsilon_{zz} = 0)$



$(\varepsilon_{zz} \neq 0)$



$(\sigma_{zz} \neq 0)$

Notation Remarks

Component Form

$$\sigma_{ij}=\begin{bmatrix}\sigma_{11} & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22} & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & \sigma_{33}\end{bmatrix}=\begin{bmatrix}\sigma_{xx} & \sigma_{xy} & \sigma_{xz}\\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz}\\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}\end{bmatrix}$$

$$=\begin{bmatrix}\sigma_{x} & \sigma_{xy} & \sigma_{xz}\\ \sigma_{yx} & \sigma_{y} & \sigma_{yz}\\ \sigma_{zx} & \sigma_{zy} & \sigma_{z}\end{bmatrix}=\begin{bmatrix}\sigma_{x} & \tau_{xy} & \tau_{xz}\\ \tau_{yx} & \sigma_{y} & \tau_{yz}\\ \tau_{zx} & \tau_{zy} & \sigma_{z}\end{bmatrix}$$

Principal Form

$$\sigma_{k}=\begin{bmatrix}\sigma_{1} & 0 & 0\\ 0 & \sigma_{2} & 0\\ 0 & 0 & \sigma_{3}\end{bmatrix}$$

Notation Remarks (Continued)

Symmetry

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ & \sigma_{22} & \sigma_{23} \\ SYM & & \sigma_{33} \end{bmatrix}$$

Vector form of Stress Tensor (with symmetry)

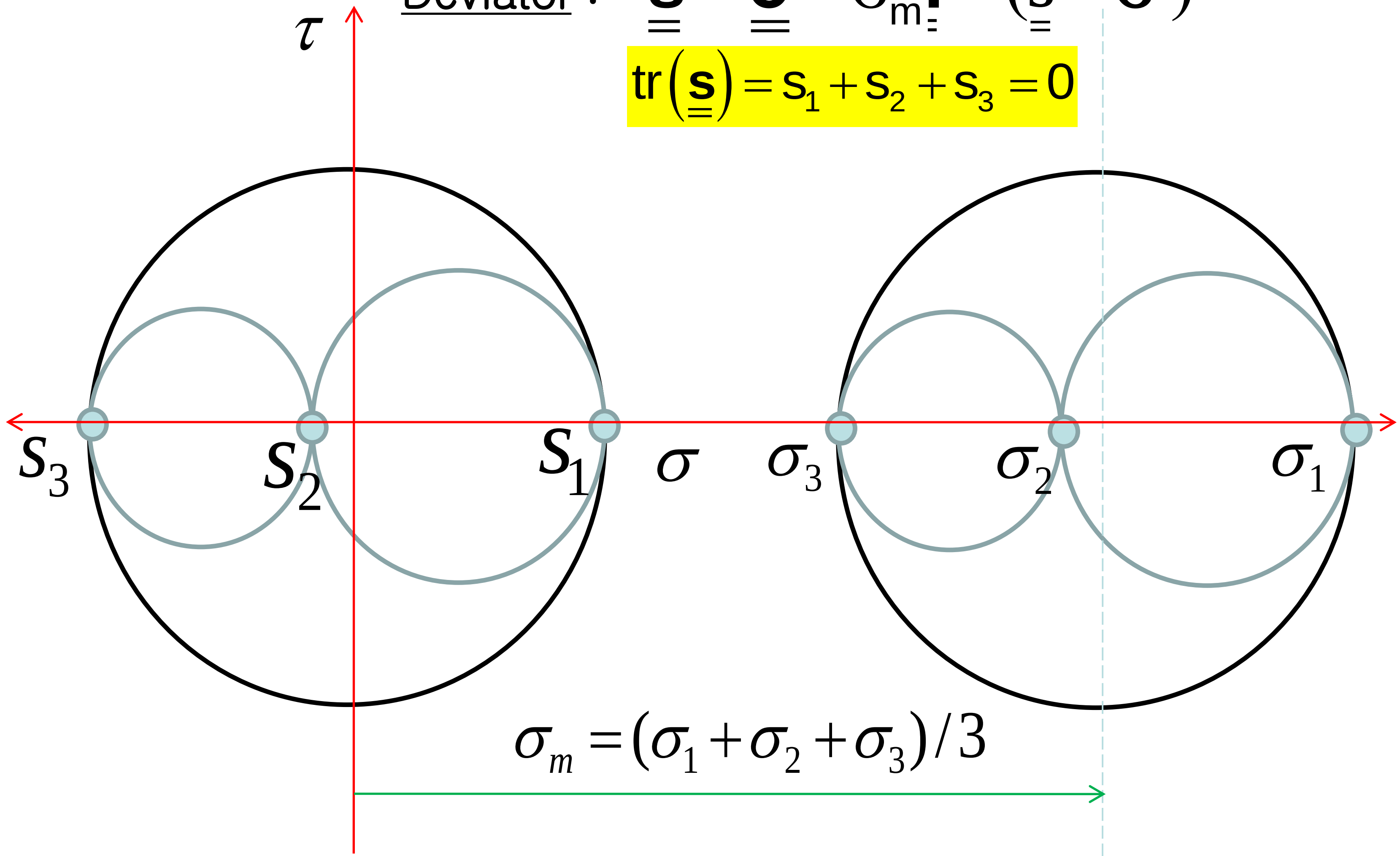
$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{22} & \sigma_{33} & \sigma_{12} & \sigma_{23} & \sigma_{13} \end{bmatrix}^T$$

Characteristics of Deviatoric Stress Tensor

$\underline{\underline{\mathbf{s}}}$ **is independent of pressure**

Deviator : $\underline{\underline{\mathbf{s}}} = \underline{\underline{\boldsymbol{\sigma}}} - \sigma_m \underline{\underline{\mathbf{I}}} \quad (\underline{\underline{\mathbf{s}}} = \boldsymbol{\sigma}')$

$\mathrm{tr}\left(\underline{\underline{\mathbf{s}}}\right) = s_1 + s_2 + s_3 = 0$



Example

- Split the following stress tensor into deviatoric and static parts and show that $tr(\underline{\underline{s}}) = s_{11} + s_{22} + s_{33} = 0$

$$\sigma_{ij} = \begin{bmatrix} 1.2 & 0.4 & 0 \\ 0.4 & 0.9 & -0.2 \\ 0 & -0.2 & 0.3 \end{bmatrix}$$

$$\sigma_m = (\sigma_{11} + \sigma_{22} + \sigma_{33})/3$$

$$\underline{\underline{\mathbf{s}}} = \underline{\underline{\boldsymbol{\sigma}}} - \sigma_m \underline{\underline{\mathbf{I}}}$$

$$\underline{\underline{\sigma}} = \underline{\underline{s}} + \sigma_m \underline{\underline{I}} \quad (\sigma_{ij} = s_{ij} + \delta_{ij}\sigma_m)$$

$$\sigma_{ij} = \begin{bmatrix} 1.2 & 0.4 & 0 \\ 0.4 & 0.9 & -0.2 \\ 0 & -0.2 & 0.3 \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix} + \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}$$

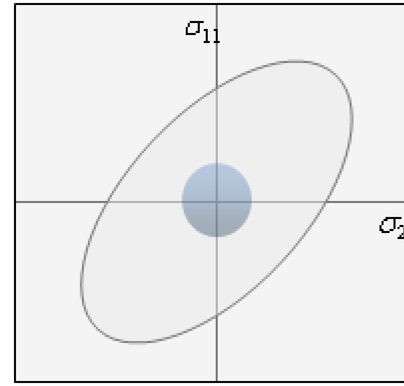
$$\underline{\underline{\sigma}} = \underline{\underline{s}} + \sigma_m \underline{\underline{I}}$$

$$\sigma_{ij} = \begin{bmatrix} 1.2 & 0.4 & 0 \\ 0.4 & 0.9 & -0.2 \\ 0 & -0.2 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.4 & 0.4 & 0 \\ 0.4 & 0.1 & -0.2 \\ 0 & -0.2 & -0.5 \end{bmatrix} + \begin{bmatrix} 0.8 & 0 & 0 \\ 0 & 0.8 & 0 \\ 0 & 0 & 0.8 \end{bmatrix}$$

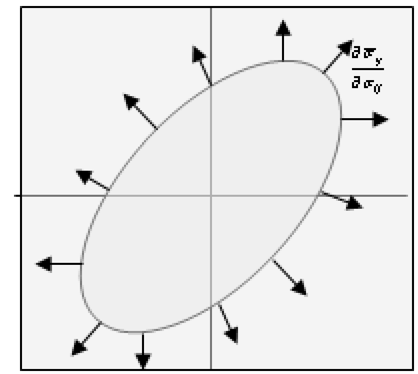
$$tr(\underline{\underline{s}}) = s_{11} + s_{22} + s_{33} = 0$$

Modeling plasticity

-Yield condition $$\bar{\sigma} = Y$$



- Flow rule $$\dot{\varepsilon}_{\alpha\beta}^{(p)} = \dot{\bar{\varepsilon}}_p \frac{\partial \bar{\sigma}}{\partial \sigma_{\alpha\beta}}$$



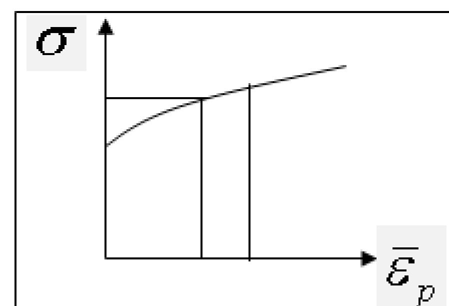
- Hardening law $$\sigma = \mathrm{h}\left(\varepsilon^{\mathrm{p}}\right)$$

e.g. Swift

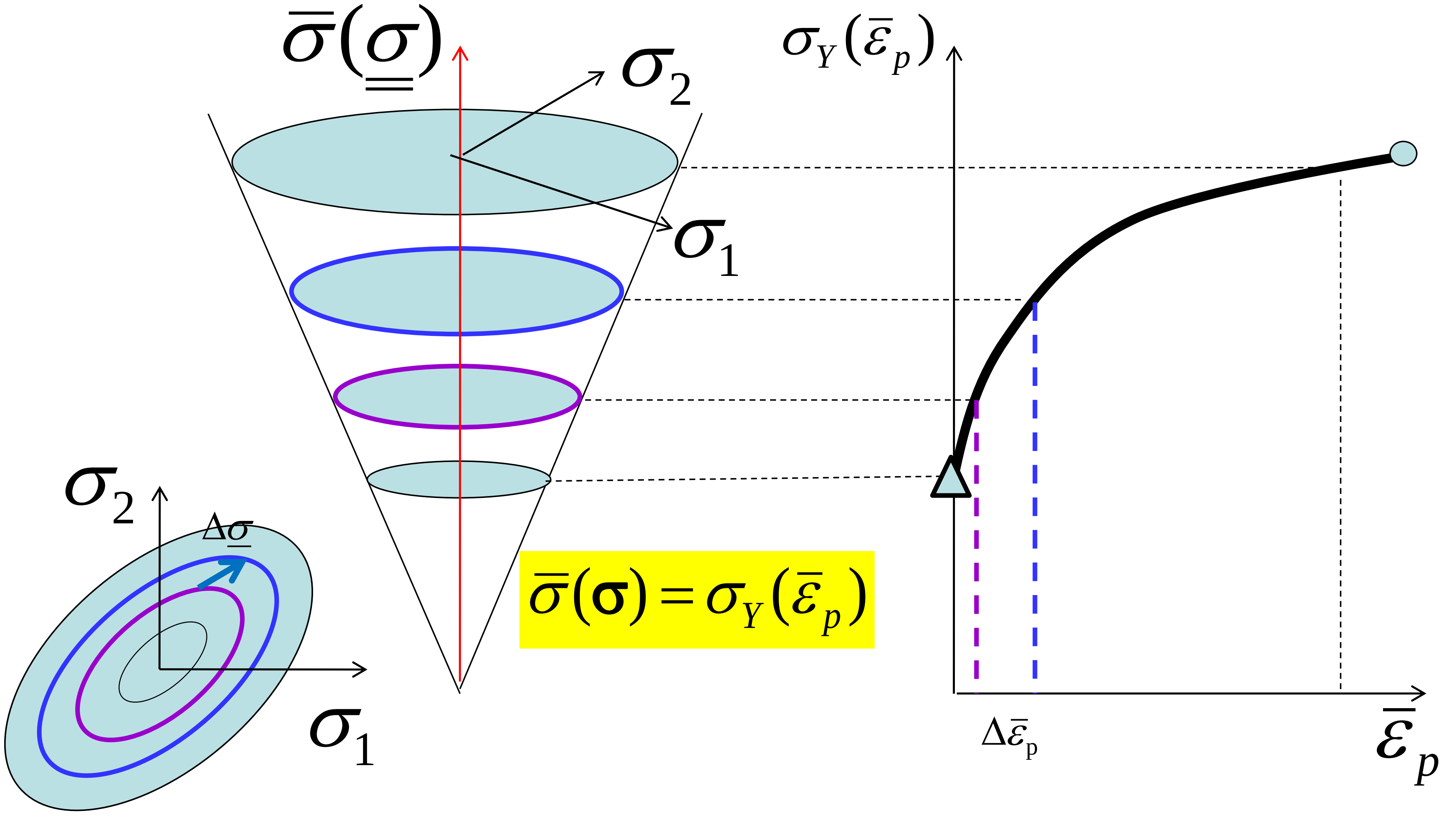
$$\sigma = K(\varepsilon_o + \varepsilon_p)^n$$

Voce

$$\sigma = A - B\exp(-C\varepsilon_p)$$

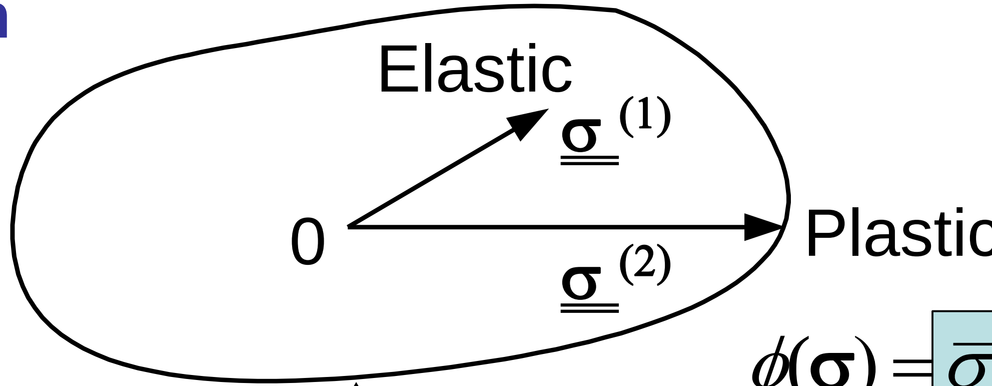


Modeling of Plasticity with Yield Function, Flow Rule and Hardening (Physical Meaning)



Yield Condition (1)

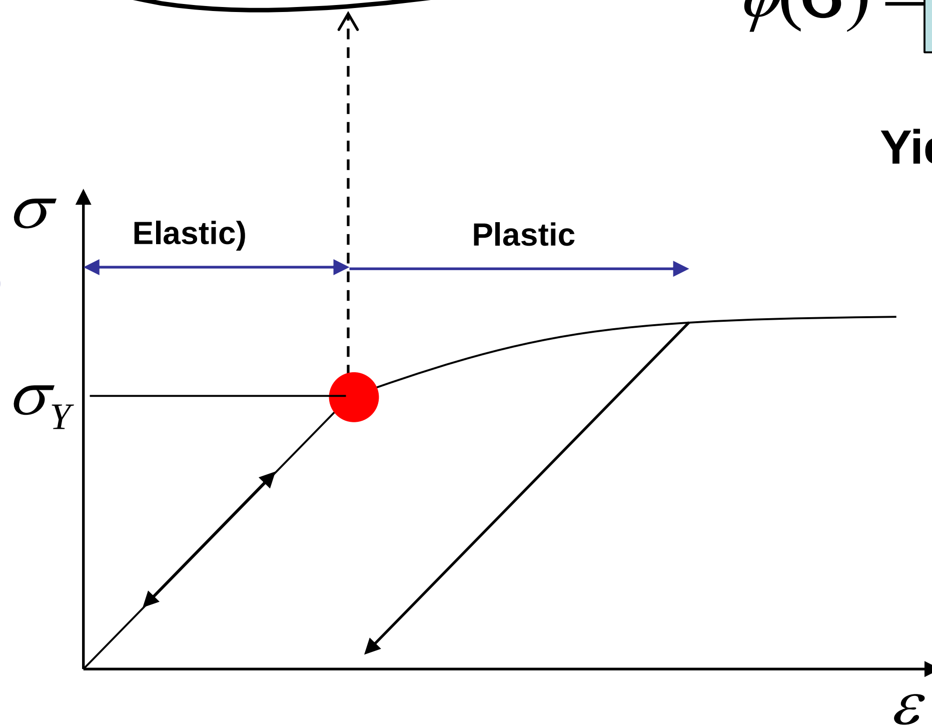
Yield Function



$$\phi(\boldsymbol{\sigma}) = \underbrace{\boxed{\bar{\sigma}(\boldsymbol{\sigma})}}_{\text{Yield Function}} - \sigma_Y = 0$$

Hardening

(True Stress-Strain Curve)

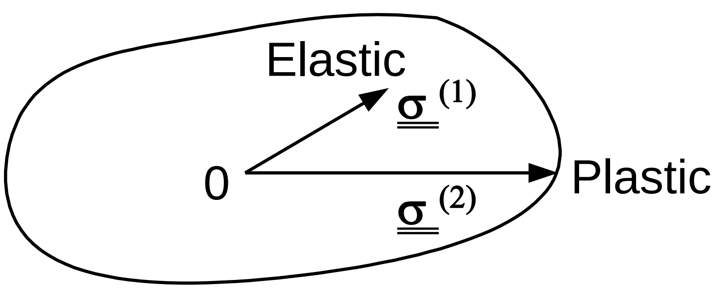


Yield condition (2)

- Uniaxial: $\phi(\sigma) = \sigma - \sigma_y = 0$

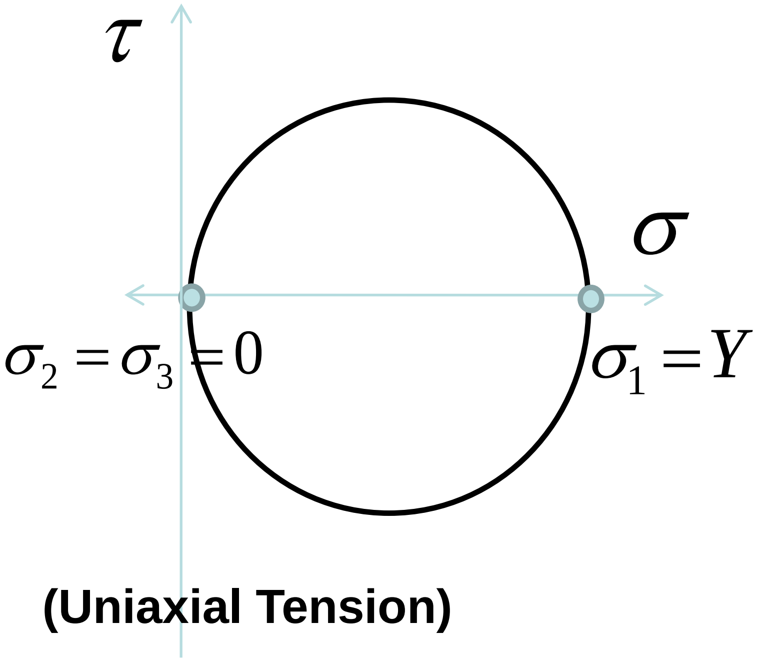
- Multiaxial: $\phi(\underline{\underline{\boldsymbol{\sigma}}}) = 0$ or
  $\phi(\underline{\underline{\mathbf{s}}}) = 0$ (pressure insensitive)

- Defines surface in multi-space

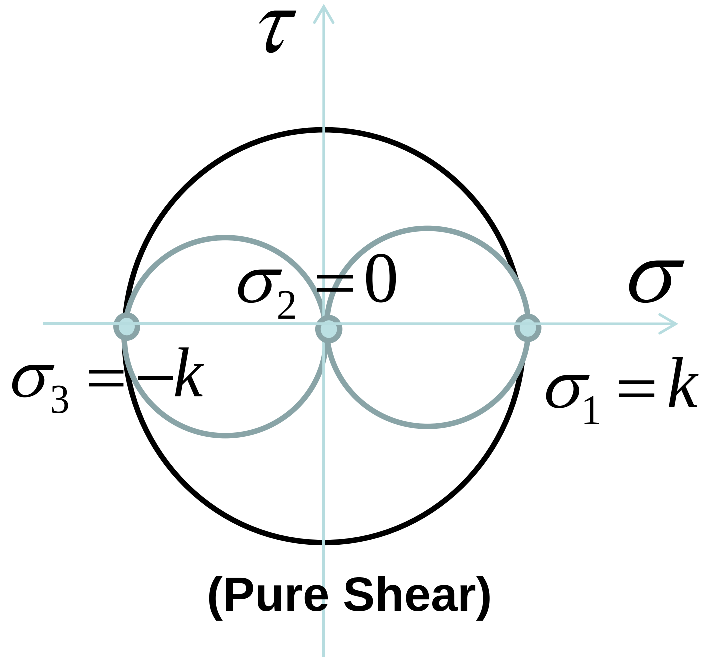


Tresca Criterion

$$\sigma_{max} - \sigma_{min} = C \text{ or } \sigma_1 - \sigma_3 = C \text{ if } \sigma_1 > \sigma_2 > \sigma_3$$



(Uniaxial Tension)



(Pure Shear)

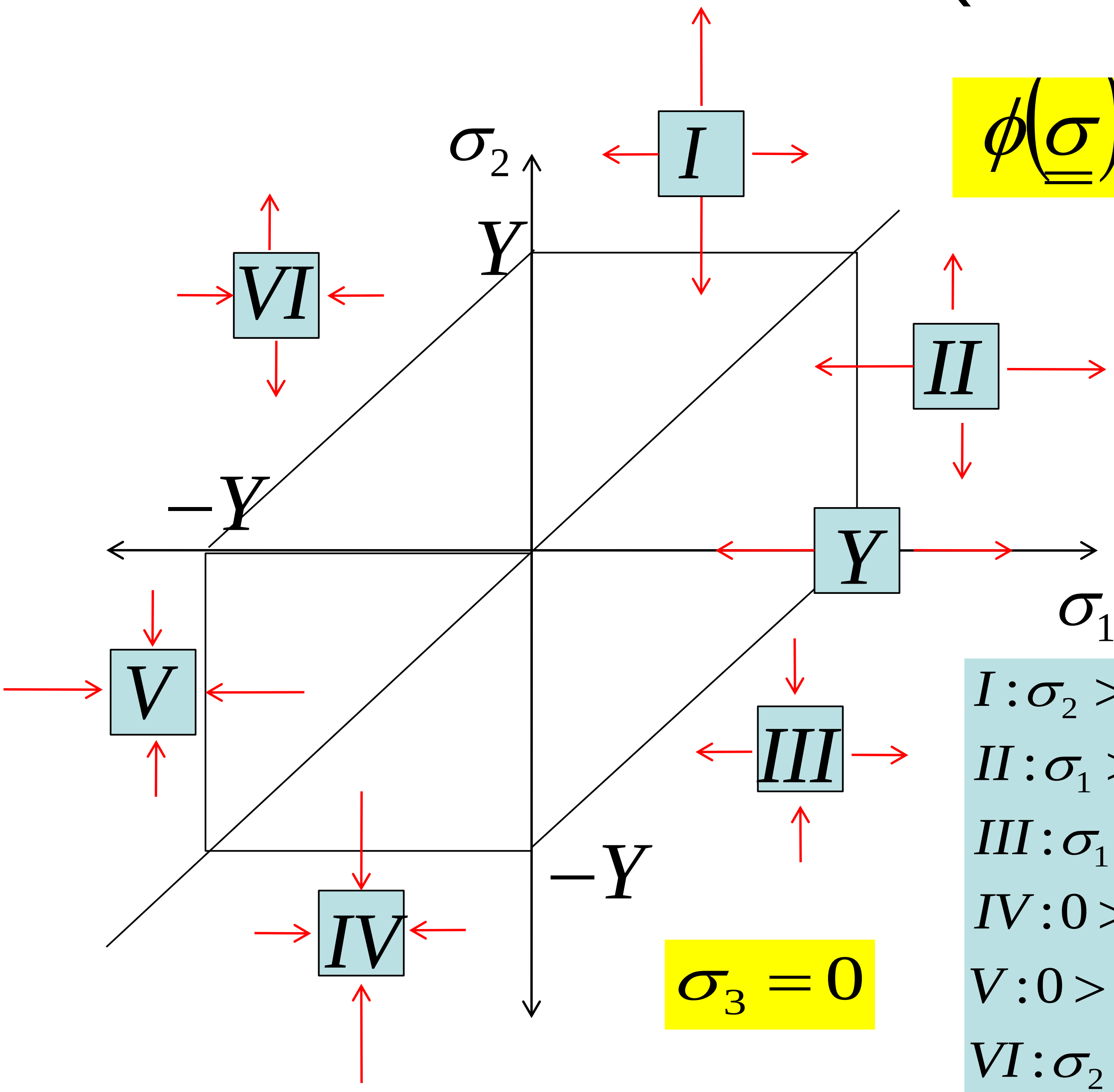
$$\text{Uniaxial Tension: } \sigma_1 = Y, \sigma_2 = \sigma_3 = 0; \; \sigma_1 - \sigma_3 = Y = C$$

$$\text{Pure Shear: } \sigma_1 = k, \sigma_2 = 0, \sigma_3 = -\sigma_1; \sigma_1 - \sigma_3 = 2k = C$$

$$\phi(\underline{\underline{\sigma}}) = \sigma_1 - \sigma_3 = Y = 2k \text{ if } \sigma_1 > \sigma_2 > \sigma_3$$

Tresca Yield Locus (Plane Stress)

$$\phi(\underline{\underline{\sigma}}) = \sigma_{\max} - \sigma_{\min} = Y$$



$$\sigma_3 = 0$$

$I: \sigma_2 > \sigma_1 > 0, \text{so } \sigma_2 = +Y$

$II: \sigma_1 > \sigma_2 > 0, \text{so } \sigma_1 = +Y$

$III: \sigma_1 > 0 > \sigma_2, \text{so } \sigma_1 - \sigma_2 = +Y$

$IV: 0 > \sigma_1 > \sigma_2, \text{so } \sigma_2 = -Y$

$V: 0 > \sigma_2 > \sigma_1, \text{so } \sigma_1 = -Y$

$VI: \sigma_2 > 0 > \sigma_1, \text{so } \sigma_2 - \sigma_1 = +Y$

Expression with Effective Stress

$$\bar{\sigma} = Y$$

Tresca : $\phi\left(\underline{\underline{\sigma}}\right) = \sigma_1 - \sigma_3 = Y \text{ if } \sigma_1 > \sigma_2 > \sigma_3$

$$\bar{\sigma}\left(\underline{\underline{\sigma}}\right) = \sigma_1 - \sigma_3$$

von Mises Criterion

$$\phi(\underline{\underline{\sigma}}) = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = C$$

$$Uniaxial\,\text{Tension:}\, \sigma_1 = Y, \sigma_2 = \sigma_3 = 0; \text{Then,} C = 2Y^2$$

$$Pure\,Shear: \sigma_1 = k, \sigma_2 = -k, \sigma_3 = 0; \text{Then,} C = 6k^2$$

Principal Form

$$\phi(\underline{\underline{\sigma}}) = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2Y^2 = 6k^2$$

General Component Form

$$\phi(\underline{\underline{\sigma}}) = (\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + 6(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{13}^2) = 2Y^2 = 6k^2$$

or

$$\phi(\underline{\underline{\sigma}}) = (\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xz})^2 + 6(\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{xz}^2) = 2Y^2 = 6k^2$$

Expression with Effective Stress

$$\bar{\sigma} = Y$$

von Mises:

$$\phi\left(\underline{\underline{\sigma}}\right) = (\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + 6(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2) = 2Y^2 = 6k^2$$

$$\bar{\sigma}\left(\underline{\underline{\sigma}}\right) = \sqrt{\frac{1}{2}\left[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + 6(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{13}^2)\right]}$$

(For plane stress)

$$\bar{\sigma}\left(\underline{\underline{\sigma}}\right) = \sqrt{(\sigma_{11})^2 + (\sigma_{22})^2 - \sigma_{11}\sigma_{22} + 3(\sigma_{12})^2}$$

Von Mises function in deviatoric space

$$\bar{\sigma}\left(\underline{\underline{\sigma}}\right)=\sqrt{\frac{1}{2}\left[(\sigma_{11}-\sigma_{22})^2+(\sigma_{22}-\sigma_{33})^2+(\sigma_{33}-\sigma_{11})^2+6(\sigma_{12}^2+\sigma_{23}^2+\sigma_{13}^2)\right]}$$

$$\underline{\underline{\mathbf{s}}}=\underline{\underline{\boldsymbol{\sigma}}}-\sigma_m\underline{\underline{\mathbf{I}}}$$

$$\sigma_{11}=S_{11}+\sigma_m$$

$$\sigma_{22}=S_{22}+\sigma_m$$

$$\sigma_{33}=S_{33}+\sigma_m$$

$$\sigma_{12}=S_{12}$$

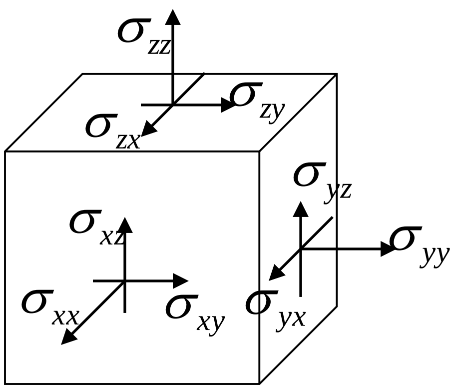
$$\sigma_{22}=S_{22}$$

$$\sigma_{33}=S_{33}$$

$$\bar{\sigma}\left(\underline{\underline{\sigma}}\right)=\sqrt{\frac{1}{2}\left[(S_{11}-S_{22})^2+(S_{22}-S_{33})^2+(S_{33}-S_{11})^2+6(S_{12}^2+S_{23}^2+S_{13}^2)\right]}$$

Yield Check with von Mises yield function ?

$$\bar{\sigma}(\underline{\underline{\sigma}}) = \sqrt{\frac{1}{2}\left[(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2 + 6(\sigma_{12}^2+\sigma_{23}^2+\sigma_{13}^2)\right]}$$



Example 1

$$\bar{\sigma}_Y = 100$$

$$\sigma_{ij} = \begin{bmatrix} 110 & 0 & 0 \\ 0 & 110 & 0 \\ 0 & 0 & 110 \end{bmatrix}$$

Yield ?

Example 2

$$\bar{\sigma}_Y = 100$$

$$\sigma_{ij} = \begin{bmatrix} 110 & 0 & 0 \\ 0 & 110 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Yield ?

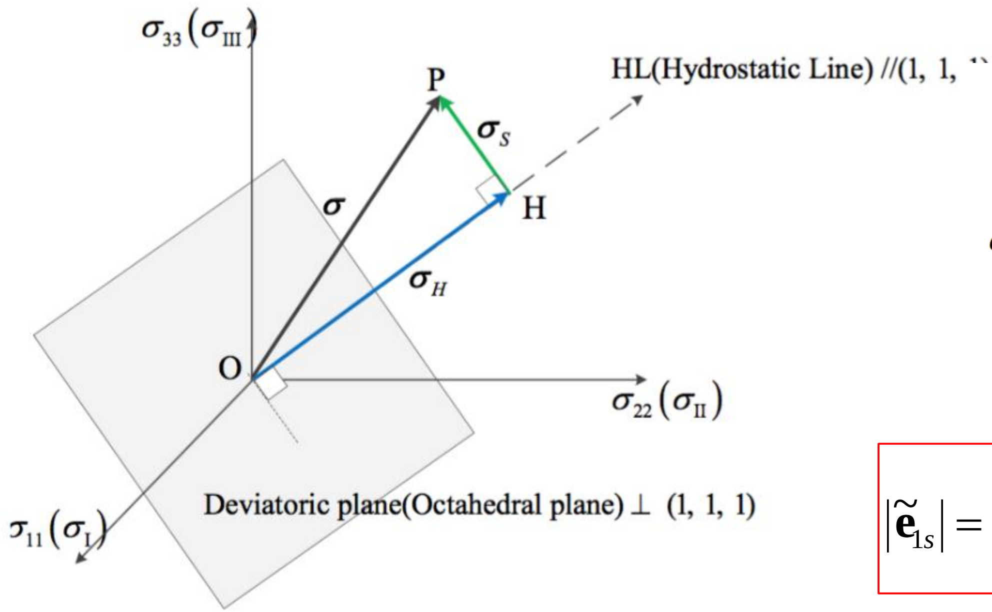
Example 3

$$\bar{\sigma}_Y = 100$$

$$\sigma_{ij} = \begin{bmatrix} 90 & 0 & 0 \\ 0 & -90 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

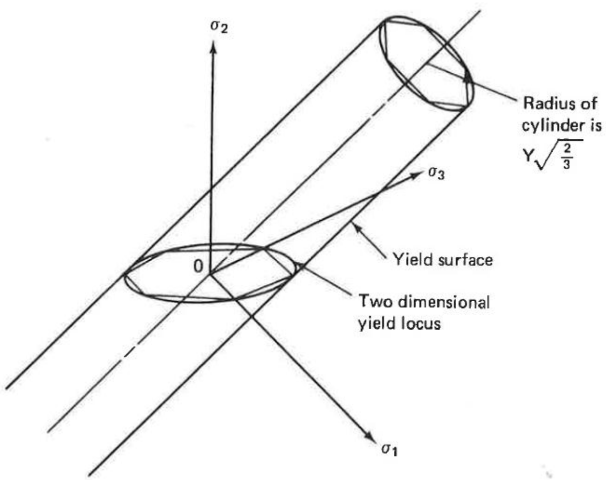
Yield ?

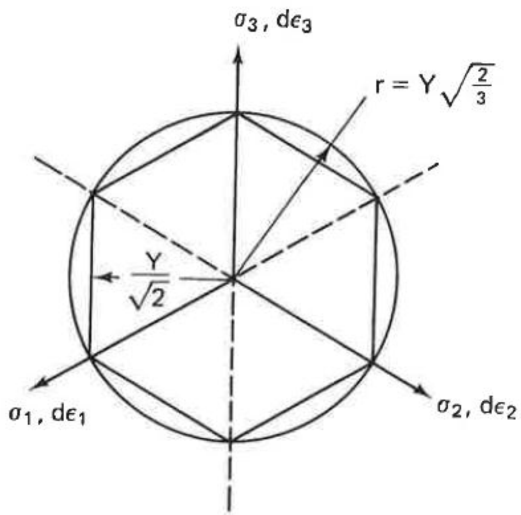
Hydrostatic and Deviatoric Planes

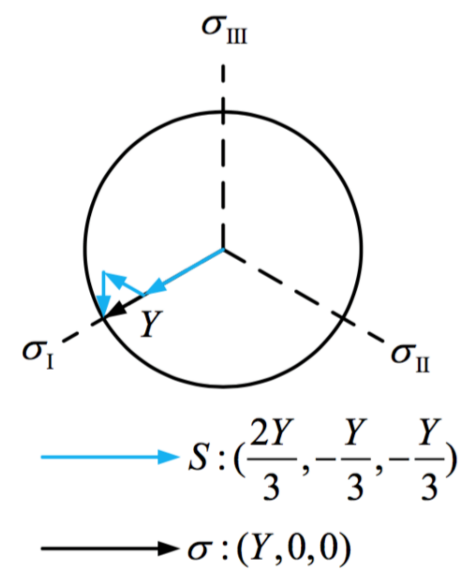


$$\tilde{e}_1 = \tilde{e}_{1S} + \tilde{e}_{1H} = \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}\frac{2}{3}\\ \frac{-1}{3}\\ \frac{-1}{3}\end{pmatrix} + \begin{pmatrix}\frac{1}{3}\\ \frac{1}{3}\\ \frac{1}{3}\end{pmatrix}$$

$$\boxed{\left|\tilde{\mathbf{e}}_{1S}\right| = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{-1}{3}\right)^2 + \left(\frac{-1}{3}\right)^2} = \sqrt{\frac{6}{9}} = \sqrt{\frac{2}{3}}}$$



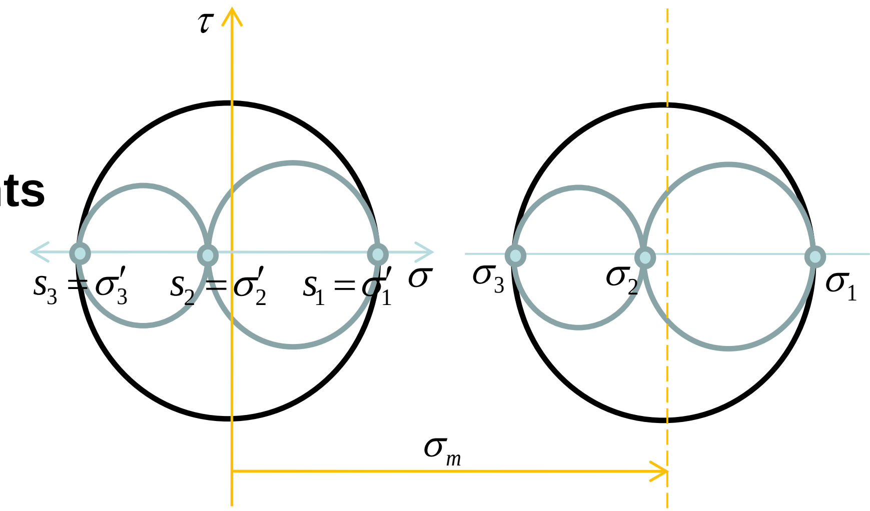




Invariants (Cauchy Stress)

$$P(\sigma_k) = \det\left(\underline{\underline{\boldsymbol{\sigma}}} - \sigma_k \underline{\underline{\mathbf{I}}}\right) = 0$$

Principal values $\sigma_1, \sigma_2, \sigma_3$ **are invariants**



$$\begin{vmatrix} \sigma_{11} - \sigma_p & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \sigma_p & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \sigma_p \end{vmatrix} = 0$$

$$\sigma_p^3 - J_1 \sigma_p^2 - J_2 \sigma_p - J_3 = 0$$

(Component form)

$$J_1 = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

$$J_2 = (\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{13}^2 - \sigma_{11}\sigma_{22} - \sigma_{22}\sigma_{33} - \sigma_{33}\sigma_{11})$$

$$J_3 = \sigma_{11}\sigma_{22}\sigma_{33} + 2\sigma_{12}\sigma_{23}\sigma_{13} - \sigma_{11}\sigma_{23}^2 - \sigma_{22}\sigma_{13}^2 - \sigma_{33}\sigma_{12}^2$$

(Principal form)

$$J_1 = \sigma_1 + \sigma_2 + \sigma_3$$

$$J_2 = -(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)$$

$$J_3 = \sigma_1\sigma_2\sigma_3$$

Invariants (Deviatoric Stress)

$$S_p^3 - J_1 S_p^2 - J_2 S_p - J_3 = 0$$

$$\begin{cases} J_1 = tr(\boldsymbol{S}) = S_{ii} = S_I + S_{II} + S_{III} = 0 \\ J_2 = \frac{1}{2}\left(S_{ij}S_{ji} - S_{ii}S_{jj}\right) = \frac{1}{2}S_{ij}S_{ij} = \frac{1}{2}(S_I^2 + S_{II}^2 + S_{III}^2) \\ J_3 = \det(\boldsymbol{S}) = \frac{1}{3}\left(S_{ij}S_{jk}S_{ki}\right) = S_I S_{II} S_{III} \end{cases}$$

$$\boxed{\left|\boldsymbol{S}^*\right| = \sqrt{2J_2}}$$

where $J_2 = \frac{1}{2}S_{ij}S_{ij} = \frac{1}{2}(S_{11}^2 + S_{22}^2 + S_{33}^2) + S_{12}^2 + S_{23}^2 + S_{31}^2$

$$= -(S_{11}S_{22} + S_{22}S_{33} + S_{33}S_{11}) + S_{12}^2 + S_{23}^2 + S_{31}^2$$

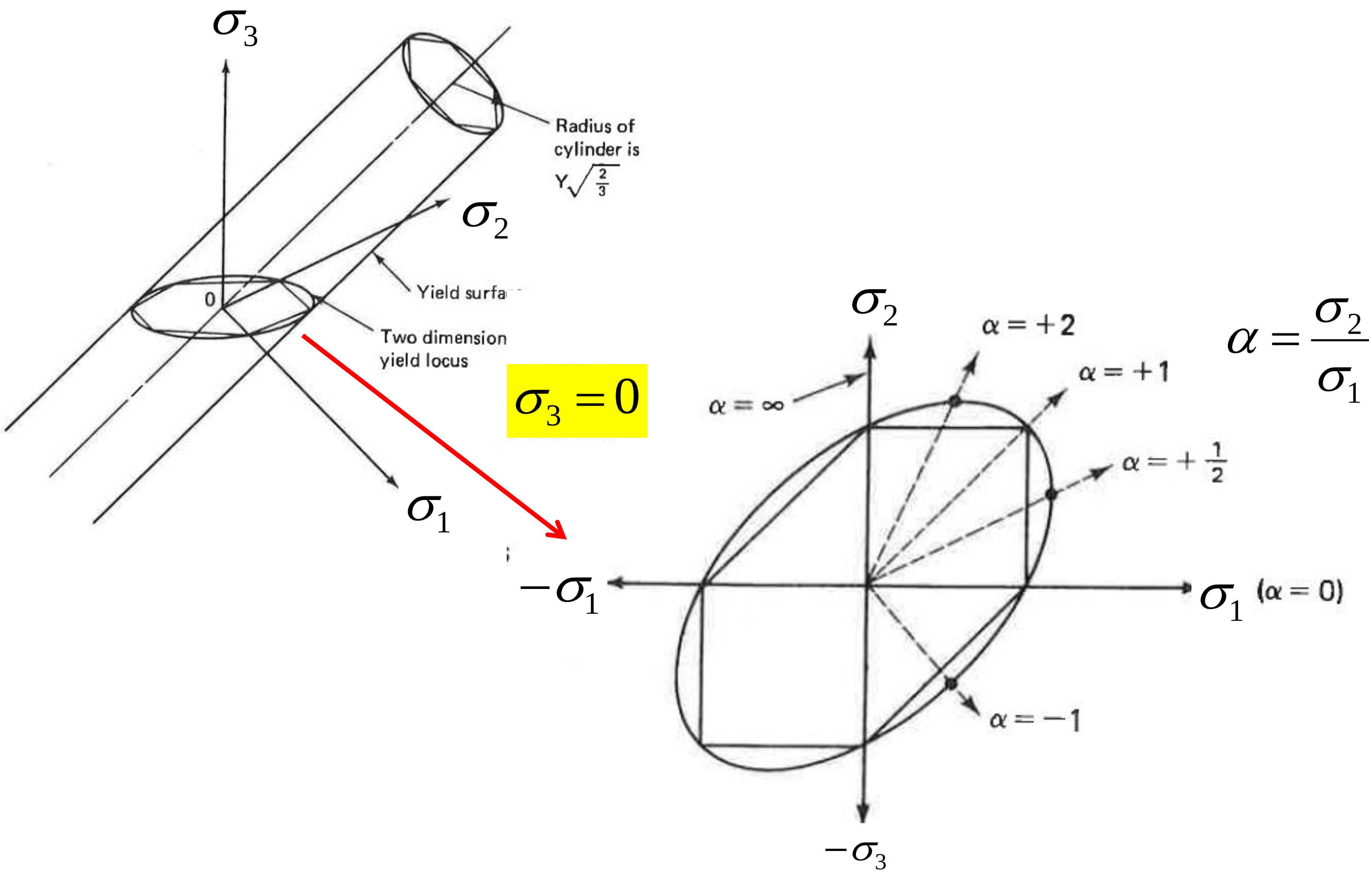
$$= \frac{1}{6}((S_{11} - S_{22})^2 + (S_{22} - S_{33})^2 + (S_{33} - S_{11})^2) + S_{12}^2 + S_{23}^2 + S_{31}^2$$

$$= \frac{1}{6}((\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2) + \sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2$$

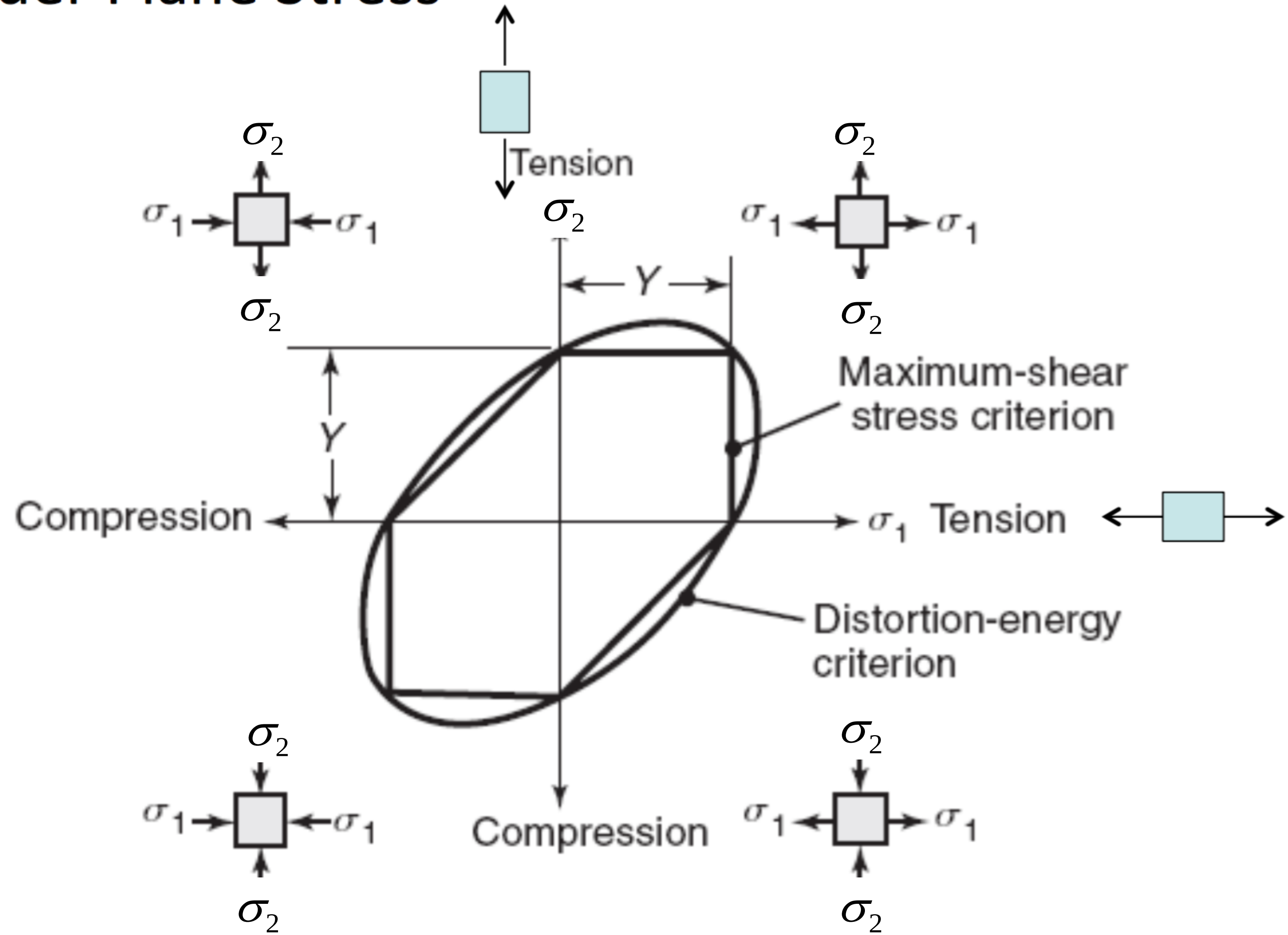
$$\bar{\sigma}_M\left(\underline{\underline{\sigma}}\right) = \sqrt{\frac{1}{2}\left[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + 6(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{13}^2)\right]}$$

$$\boxed{\bar{\sigma}_M\left(\underline{\underline{\sigma}}\right) = \sqrt{3J_2}}$$

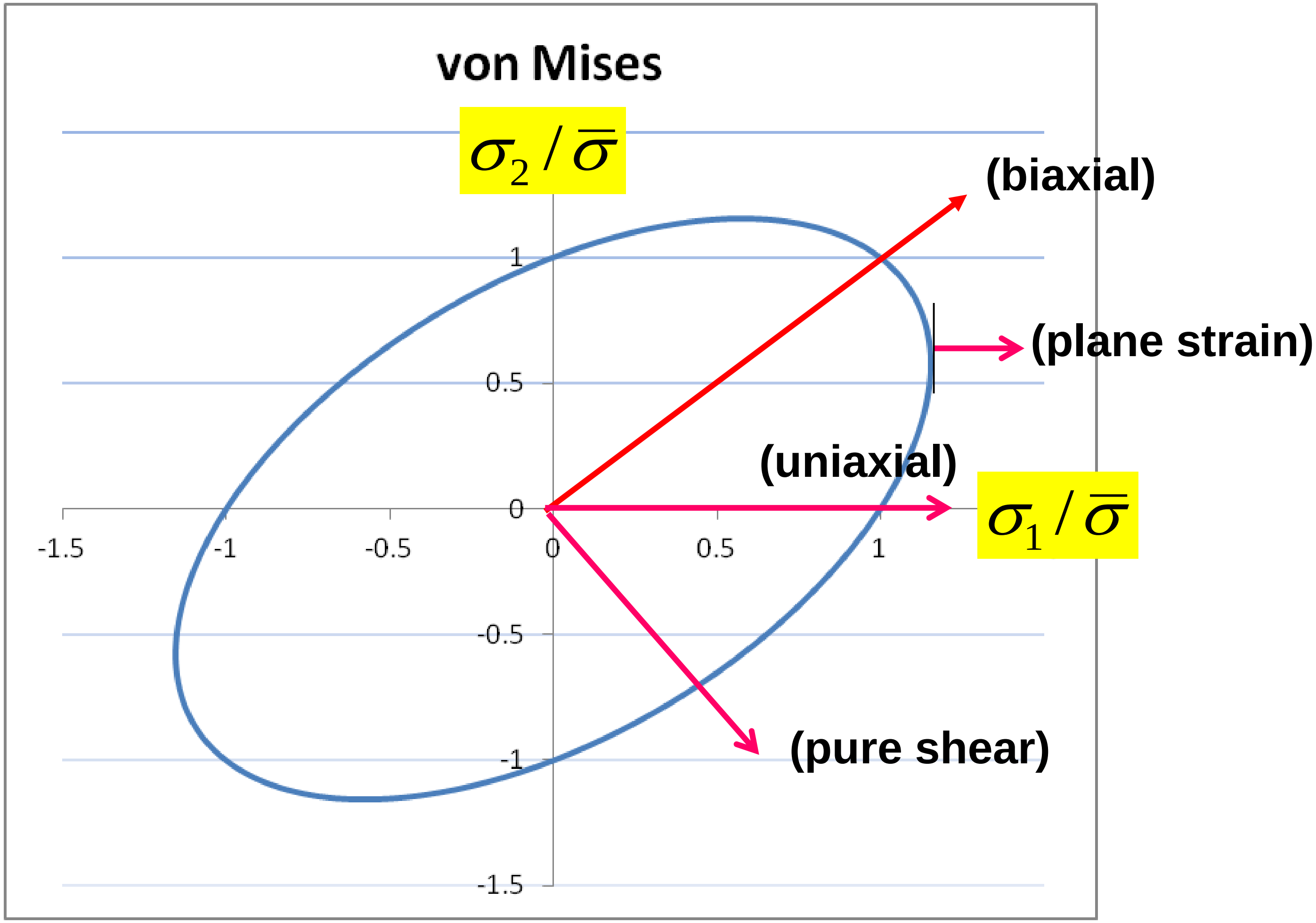
Tresca and von Mises yield surface projected to the plane : $\sigma_3 = 0$



Comparison between von Mises and Tresca under Plane Stress



von Mises Yield Surface under Plane Stress Condition

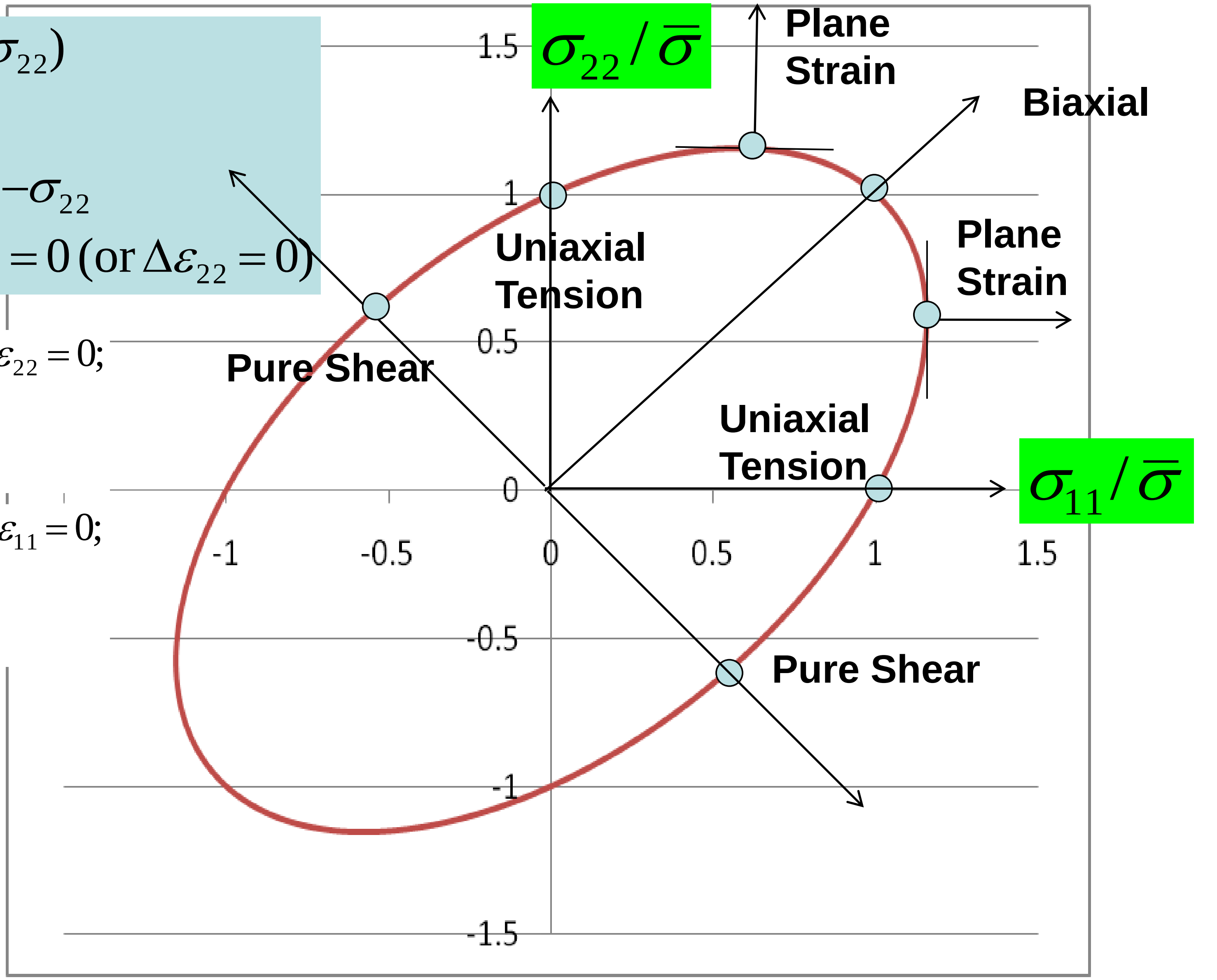


Typical Points at Yield Function (von Mises)

$$Uniaxial: \sigma_{11}\ (\mathrm{or}\ \sigma_{22})$$
$$Biaxial: \sigma_{11} = \sigma_{22}$$
$$PureShear: \sigma_{11} = -\sigma_{22}$$
$$PlaneStrain: \Delta\varepsilon_{11} = 0\ (\mathrm{or}\ \Delta\varepsilon_{22} = 0)$$

$$PlaneStrain\ for\ \Delta\varepsilon_{22} = 0;$$
$$\Delta\varepsilon_{11} = \Delta\bar{\varepsilon}_p \frac{\partial\bar{\sigma}}{\partial\sigma_{11}}$$

$$PlaneStrain\ for\ \Delta\varepsilon_{11} = 0;$$
$$\Delta\varepsilon_{22} = \Delta\bar{\varepsilon}_p \frac{\partial\bar{\sigma}}{\partial\sigma_{22}}$$

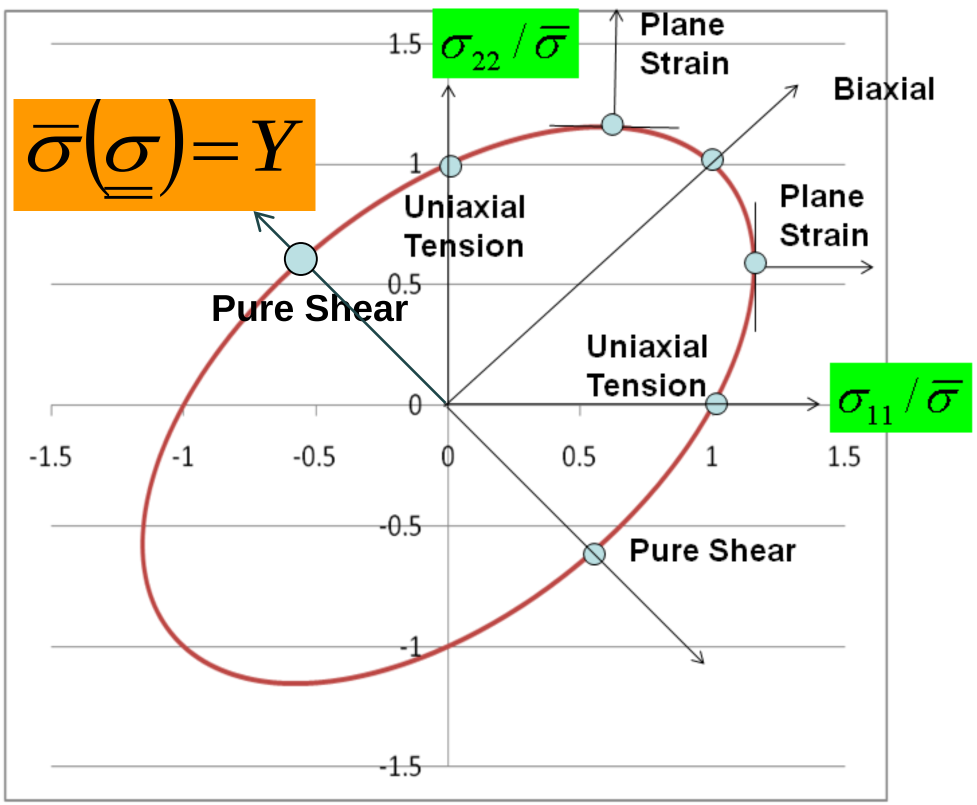


von Mises Yield Locus

$$\bar{\sigma}(\underline{\underline{\sigma}})=\sqrt{\frac{1}{2}\left[(\sigma_{11}-\sigma_{22})^2+(\sigma_{22}-\sigma_{33})^2+(\sigma_{33}-\sigma_{11})^2+6(\sigma_{12}^2+\sigma_{23}^2+\sigma_{13}^2)\right]}$$

$$\bar{\sigma}(\underline{\underline{\sigma}})=\sqrt{\sigma_{11}{}^2+\sigma_{22}{}^2-\sigma_{11}\sigma_{22}}$$

$$\bar{\sigma}(\underline{\underline{\sigma}})=Y$$



Uniaxial: $\sigma_{11}=Y$

Biaxial: $\sigma_{11}=\sigma_{22}=Y$

PureShear: $\sigma_{11}=-\sigma_{22}=0.577Y$

$$\downarrow$$

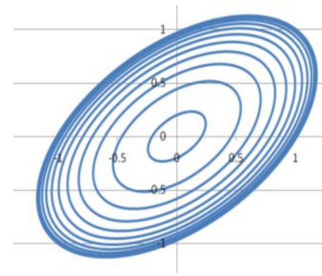
$$\bar{\sigma}(\underline{\underline{\sigma}})=\sqrt{\sigma_{11}{}^2+\sigma_{22}{}^2-\sigma_{11}\sigma_{22}}$$

$$\downarrow$$

$$\bar{\sigma}(\underline{\underline{\sigma}})=Y$$

How to draw von Mises yield surface in plane stress ?

Without Shear



$x = \sigma_{11}$

$y = \sigma_{22}$

$z = \sigma_{12}$

$$\bar{\sigma} = \sqrt{(\sigma_{11})^2 + (\sigma_{22})^2 - \sigma_{11}\sigma_{22} + 3(\sigma_{12})^2}$$

(No shear)

$$\bar{\sigma} = \sqrt{(\sigma_{11})^2 + (\sigma_{22})^2 - \sigma_{11}\sigma_{22}}$$

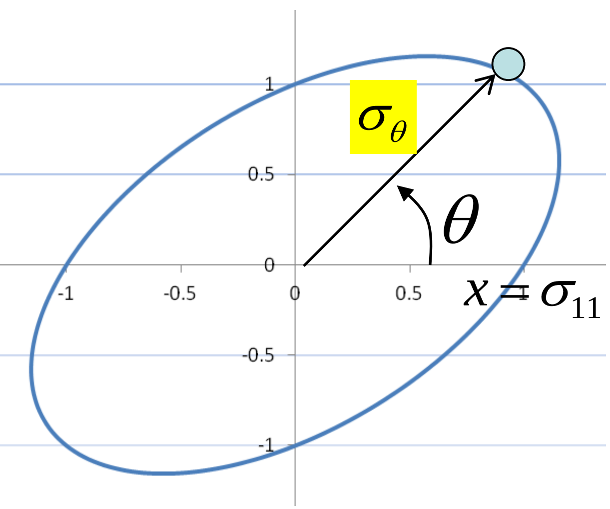
$$\boxed{\begin{aligned}\bar{\sigma} &= \sigma_\theta \sqrt{\cos^2\theta + \sin^2\theta - \cos\theta \sin\theta} \\ &= \sigma_\theta * fac\end{aligned}}$$

$$\boxed{\begin{aligned}\sigma_{11} &= \sigma_\theta \cos\theta \\ \sigma_{22} &= \sigma_\theta \sin\theta\end{aligned}}$$

where

$$\boxed{\begin{aligned} fac &= \sqrt{\cos^2\theta + \sin^2\theta - \cos\theta \sin\theta} \\ &= \bar{\sigma}(\sigma_1(=\cos\theta), \sigma_2(=\sin\theta))\end{aligned}}$$

$y = \sigma_{22}$



$x = \sigma_{11}$

$$x = \sigma_{11} = \frac{\bar{\sigma}}{fac}\cos\theta$$

$$y = \sigma_{22} = \frac{\bar{\sigma}}{fac}\sin\theta$$

$$\boxed{\sigma_\theta = \frac{\bar{\sigma}}{fac}}$$

How to draw von Mises yield surface in plane stress ?

With Shear

$$r=\sqrt{(\sigma_{11})^2+(\sigma_{22})^2+(\sigma_{12})^2}$$

$$\sigma_\theta = r\cos\beta$$

$$\sigma_{12} = r\sin\beta$$

Then, $\sigma_{11}=\sigma_\theta\cos\theta$

$\sigma_{22}=\sigma_\theta\sin\theta$

$\sigma_{12}=\sigma_\theta\tan\beta$

$$\bar{\sigma}=\sqrt{(\sigma_{11})^2+(\sigma_{22})^2-\sigma_{11}\sigma_{22}+3(\sigma_{12})^2}$$

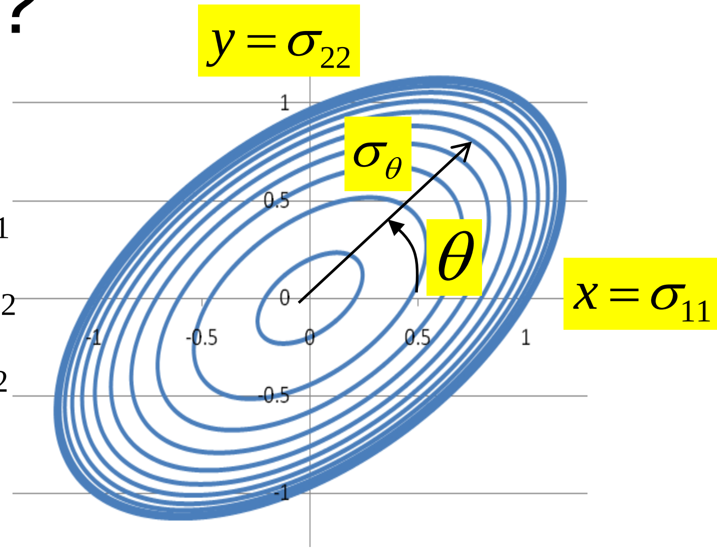
$$\sigma_\theta=\frac{\bar{\sigma}}{fac2}$$

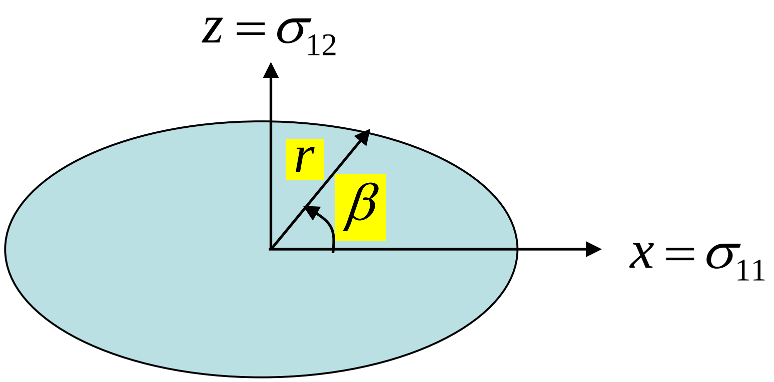
where $fac2=\sqrt{\cos^2\theta+\sin^2\theta-\cos\theta\sin\theta+3\tan^2\beta}$

$x=\sigma_{11}$

$y=\sigma_{22}$

$z=\sigma_{12}$





Hershey (1954)

- **Derived from crystal plasticity**

$$\phi\left(\underline{\underline{\mathbf{s}}}\right) = \left|s_1 - s_2\right|^a + \left|s_2 - s_3\right|^a + \left|s_3 - s_1\right|^a = 2\sigma_u^a$$

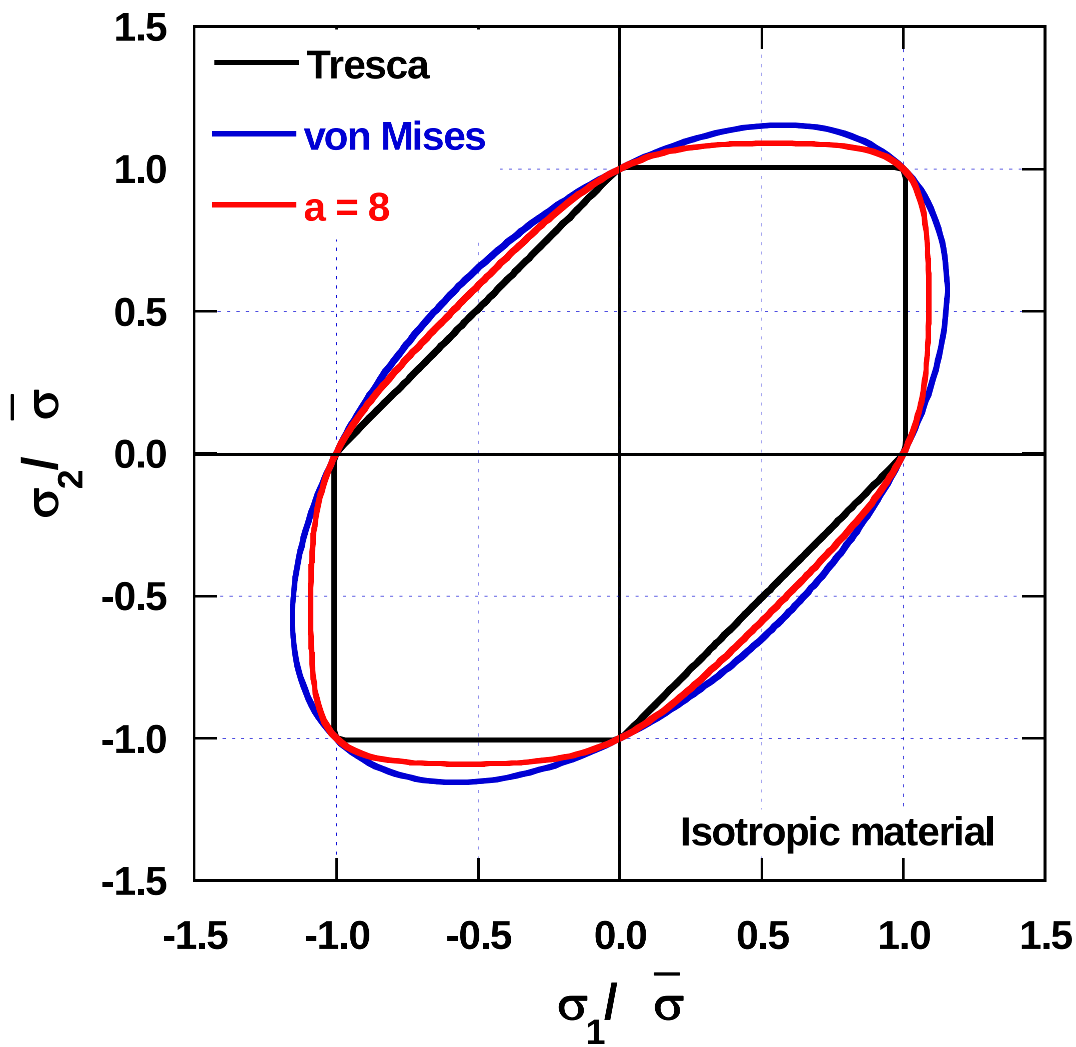
$a = \infty$ Tresca

$a = 2$ von Mises

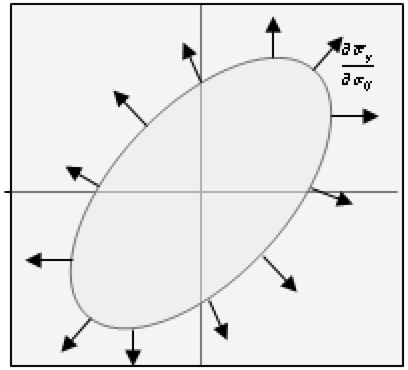
$a = 6$ for BCC materials

$a = 8$ for FCC materials

Isotropic Yield Loci

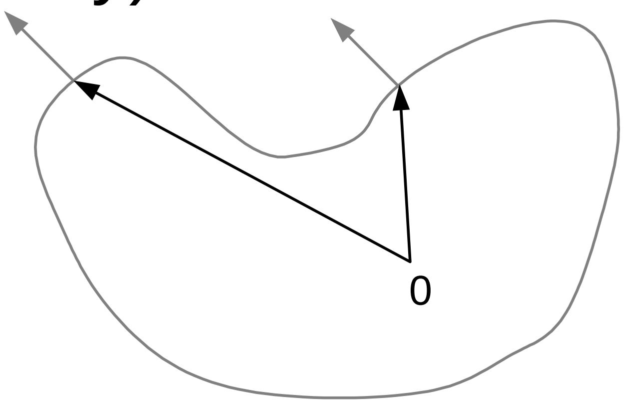


Flow Rule



Normality: $$\dot{\varepsilon}_{\alpha\beta}^{(p)} = \dot{\bar{\varepsilon}}_p \frac{\partial \bar{\sigma}}{\partial \sigma_{\alpha\beta}}$$

■ Unique Normal (because of convexity)



■ Convexity of a yield function

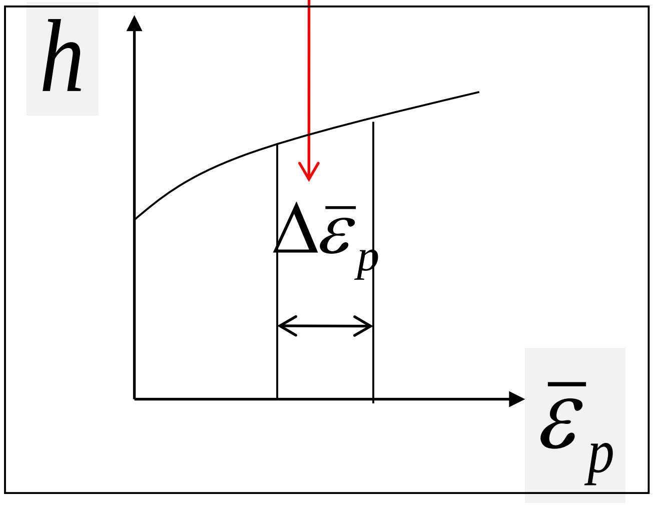
(Example of Violation)

Hessian matrix: $$[\mathbf{H}] = \frac{\partial^2 \phi}{\partial \underline{\underline{\boldsymbol{\sigma}}} \partial \underline{\underline{\boldsymbol{\sigma}}}}$$

must be positive definite

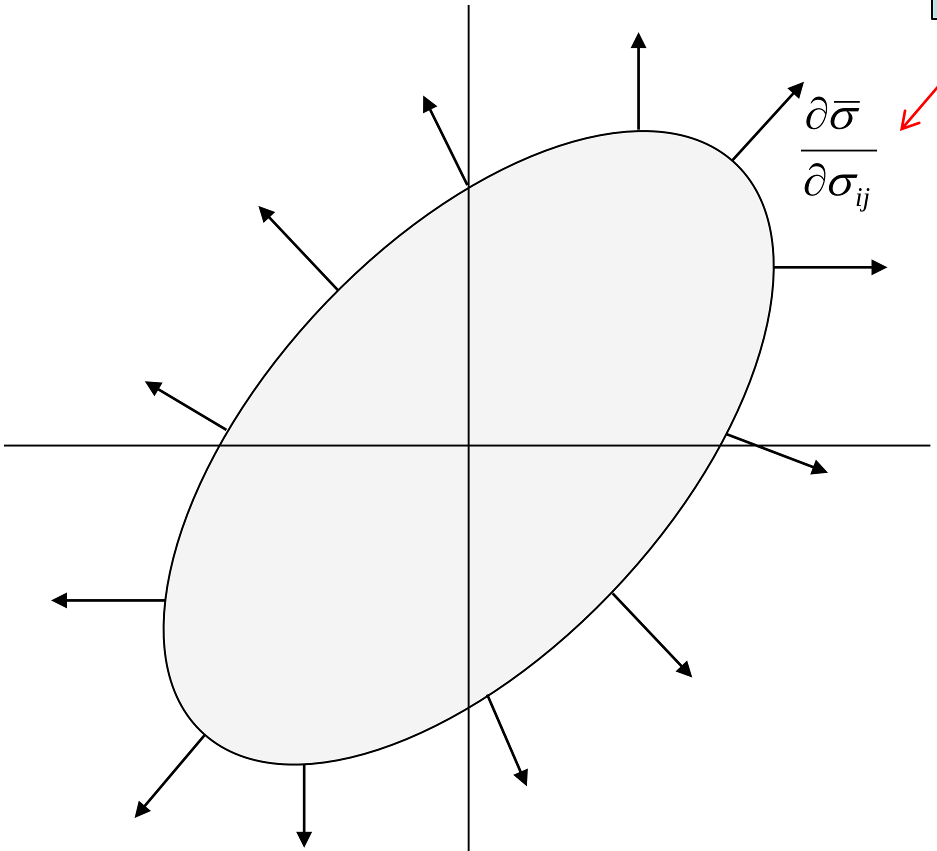
Magnitude of Plastic Flow : Controlled by Hardening

$$\Delta\varepsilon^{(p)}_{\alpha\beta} = \boxed{\Delta\bar{\varepsilon}_p} \frac{\partial\bar{\sigma}}{\partial\sigma_{\alpha\beta}}$$

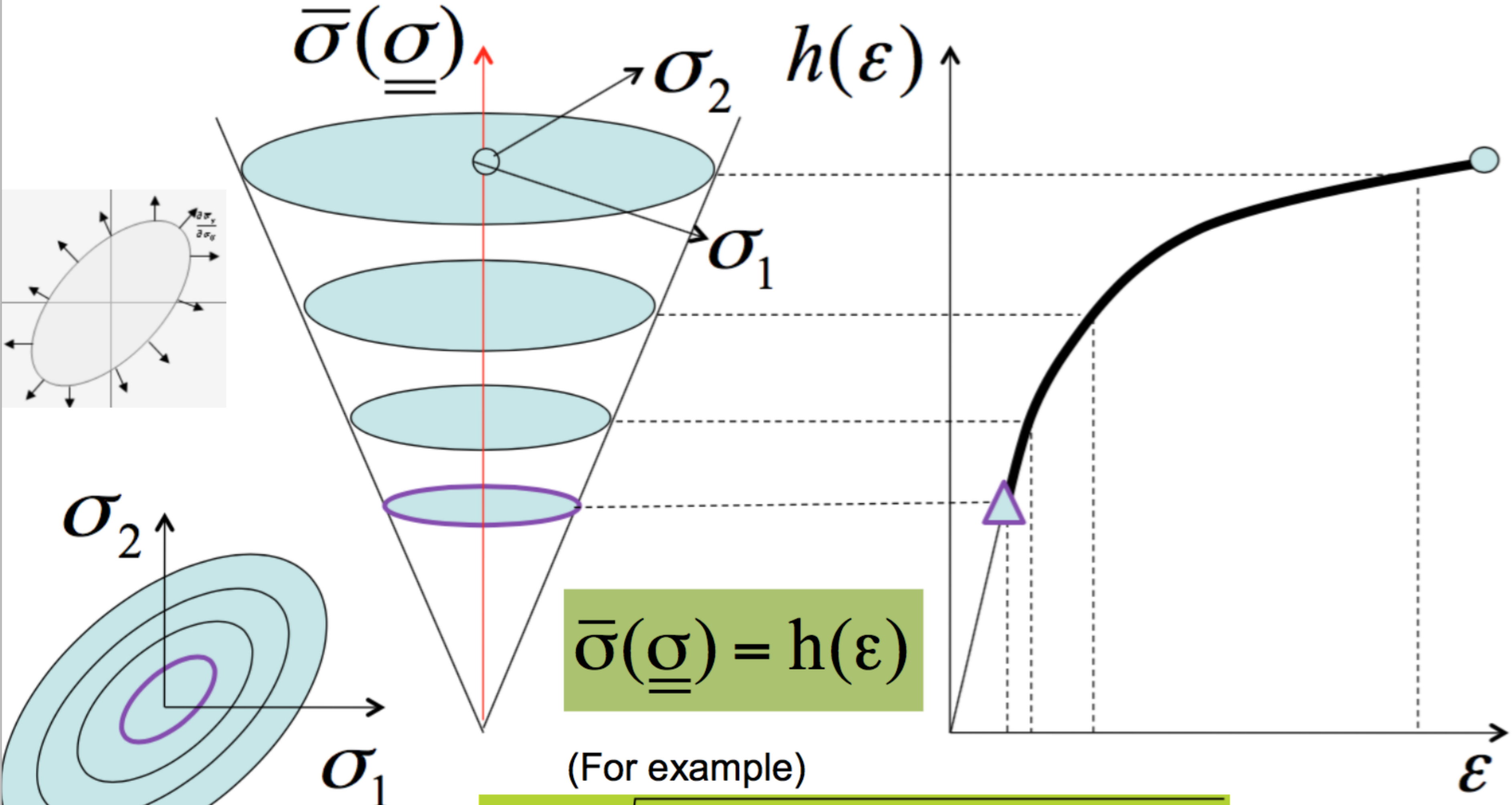


Direction of Plastic Flow : Normal to Yield Surface

$$\Delta\varepsilon_{\alpha\beta}^{(p)} = \Delta\bar{\varepsilon}_p \boxed{\frac{\partial\bar{\sigma}}{\partial\sigma_{\alpha\beta}}}$$



Yield Function and Hardening in View of Potential



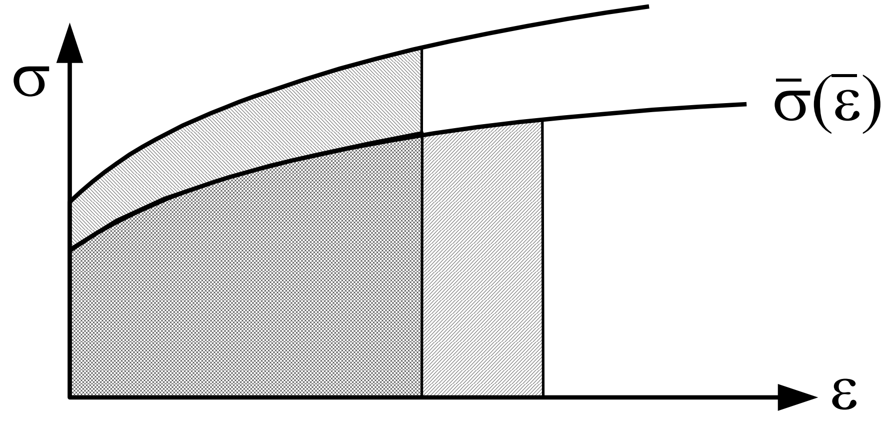
(For example)

$$\bar{\sigma} = Y = \sqrt{\frac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]}$$

$$h = \sigma = K\varepsilon^n$$

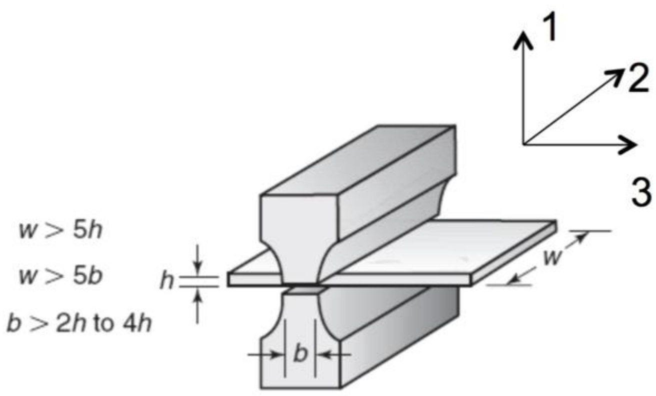
Plastic work

$$\sigma_{ij} d\varepsilon_{ij}^{p} = \sigma_{ij} d\bar{\varepsilon}^{\,p} \frac{\partial \bar{\sigma}}{\partial \sigma_{ij}} = \bar{\sigma} d\bar{\varepsilon}^{\,p}$$



Quiz

The compression test in the figure is conducted under the plane strain condition ($d\varepsilon_2 = 0$)

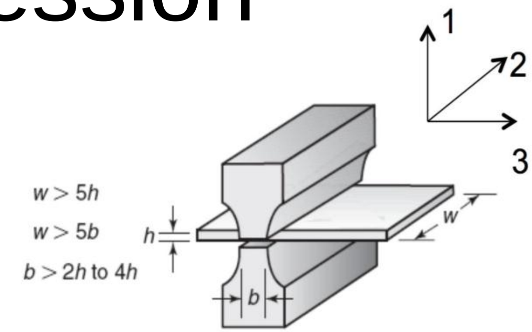


Answer if σ_3 is zero in the compression test

Under the above plane strain condition, which terms should be zero.

$$\bar{\sigma} d\bar{\varepsilon} = \sigma_1 d\varepsilon_1 + \sigma_2 d\varepsilon_2 + \sigma_3 d\varepsilon_3$$

Example: Channel Compression



- The distortion-energy criterion

$$(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2=2Y^2=2\bar{\sigma}^2$$

- When *plastic deformation,* the stress–strain relationships are obtained from **flow rules.**

$$\bar{\sigma}=Y=\sqrt{\frac{1}{2}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]}$$

Flow rule

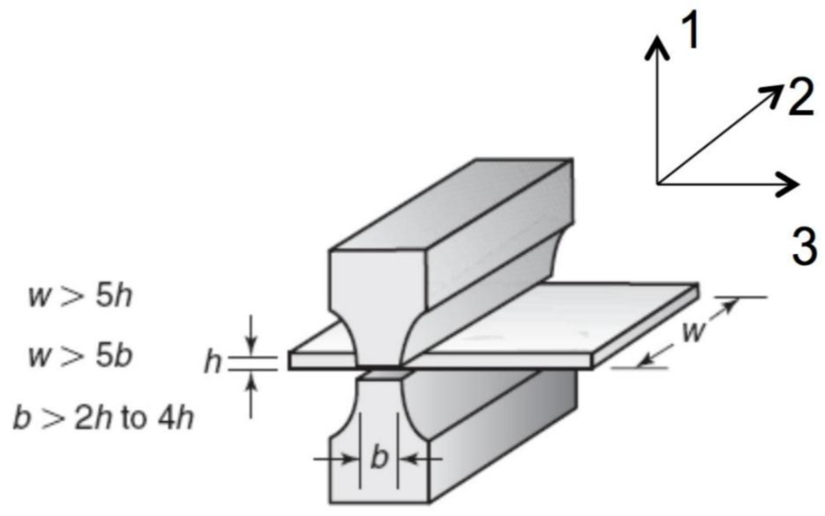
$$d\varepsilon_\alpha=d\bar{\varepsilon}\,\frac{\partial\bar{\sigma}}{\partial\sigma_\alpha}$$

$$d\varepsilon_1=\frac{d\bar{\varepsilon}}{\bar{\sigma}}\left[\sigma_1-\frac{1}{2}(\sigma_2+\sigma_3)\right]$$

$$d\varepsilon_2=\frac{d\bar{\varepsilon}}{\bar{\sigma}}\left[\sigma_2-\frac{1}{2}(\sigma_1+\sigma_3)\right]$$

$$d\varepsilon_3=\frac{d\bar{\varepsilon}}{\bar{\sigma}}\left[\sigma_3-\frac{1}{2}(\sigma_1+\sigma_2)\right]$$

Channel Compression (Plane Strain)

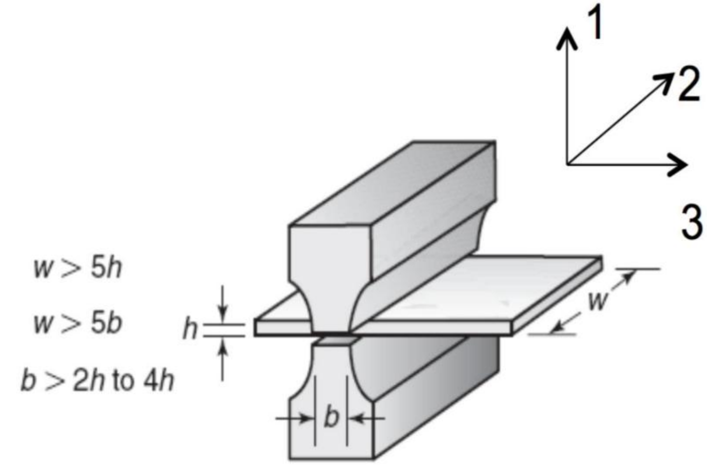


Under Plane Strain Condition

$$d\varepsilon_2 = \frac{d\bar{\varepsilon}}{\bar{\sigma}}\left[\sigma_2 - \frac{1}{2}(\sigma_1 + \sigma_3)\right] = 0 \Rightarrow \sigma_2 = \frac{\sigma_1 + \sigma_3}{2}$$

When $\sigma_3 = 0 \Rightarrow \sigma_2 = \frac{\sigma_1}{2}$

Channel Compression (Effective Plastic Strain)



With $\sigma_3 = 0$ and $\sigma_2 = \dfrac{\sigma_1}{2}$

$$\bar{\sigma} = \sqrt{\frac{1}{2}\left[\left(\sigma_1 - \frac{\sigma_1}{2}\right)^2 + (\sigma_1)^2 + \left(\frac{\sigma_1}{2}\right)^2\right]} = \frac{\sqrt{3}}{2}\sigma_1$$

$$\bar{\sigma}\,\mathrm{d}\bar{\varepsilon} = \sigma_1 \mathrm{d}\varepsilon_1 + \sigma_2 \cancelto{0}{d\varepsilon_2} + \cancelto{0}{\sigma_3} d\varepsilon_3 = \sigma_1 d\varepsilon_1$$

$$\mathrm{d}\bar{\varepsilon} = \frac{\sigma_1}{\bar{\sigma}}\mathrm{d}\varepsilon_1 = \frac{2}{\sqrt{3}}\,d\varepsilon_1$$