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School of Engineering

MEC-E6007 Mechanical Testing of Materials

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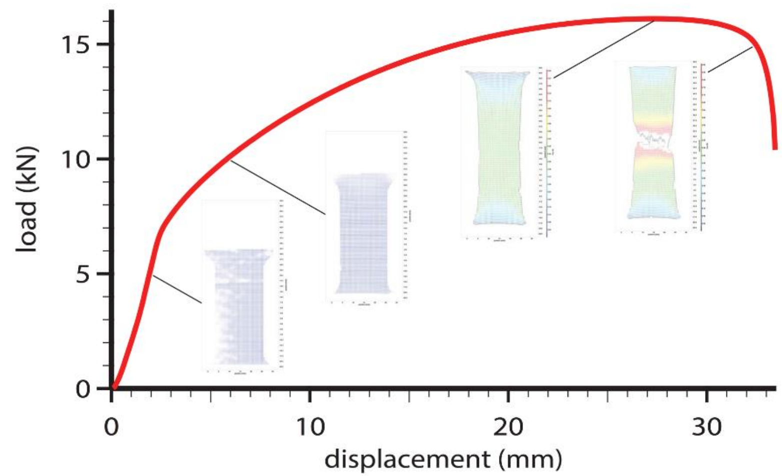
Course Content: *learning from breaking things*

- Load
  - *loadframes, actuators, and grips*
  - *quasi-static, dynamic, and cyclic loading*
- Measure
  - *measurement of force, displacement, and strain*
  - *digital image correlation and other full-field measurement techniques*
- Analyse
  - *selected special challenges in mechanical testing (ask for yours!)*
  - *introduction to inverse problem methodologies in experimental mechanics*



case study

summative assessment of the intended learning outcomes

- *You can treat writing this as the inverse problem to grading it.*

choose a question to be answered with a mechanical test

- provide context situating the question
- conclude how confident you would be to perform this test yourself
  - *or what help you would need*
- could be standardized, from a journal paper, from your research, or hypothetical
  - *cite your sources!*

describe the test

- specimen preparation
- qualitative and quantitative characterization of the specimen before and after the test
- physical quantities to be measured
  - *including measurement methods*
- instruments used
  - *criteria those must satisfy*
- relevant safety precautions
- how to analyse test results
- reasons the test results might not be valid
  - *how to detect that when it is the case*

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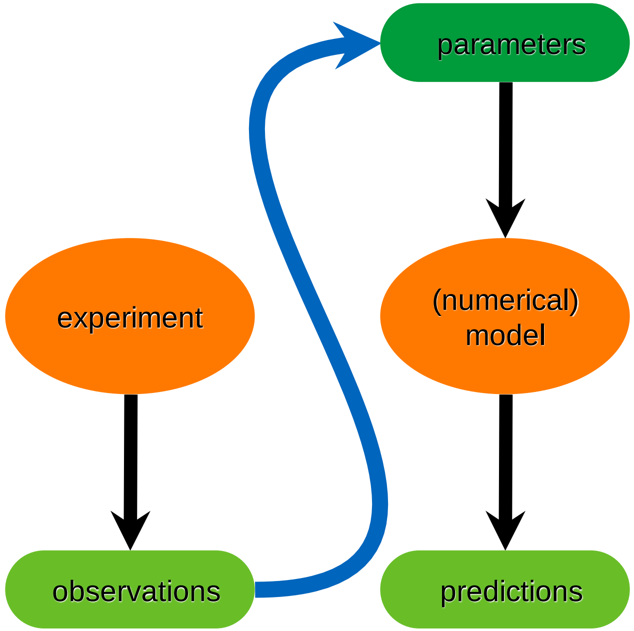
Inverse Problem Methodologies

Inverse Problems:

Mixed Numerical Experimental Techniques

determine model parameters from observed data

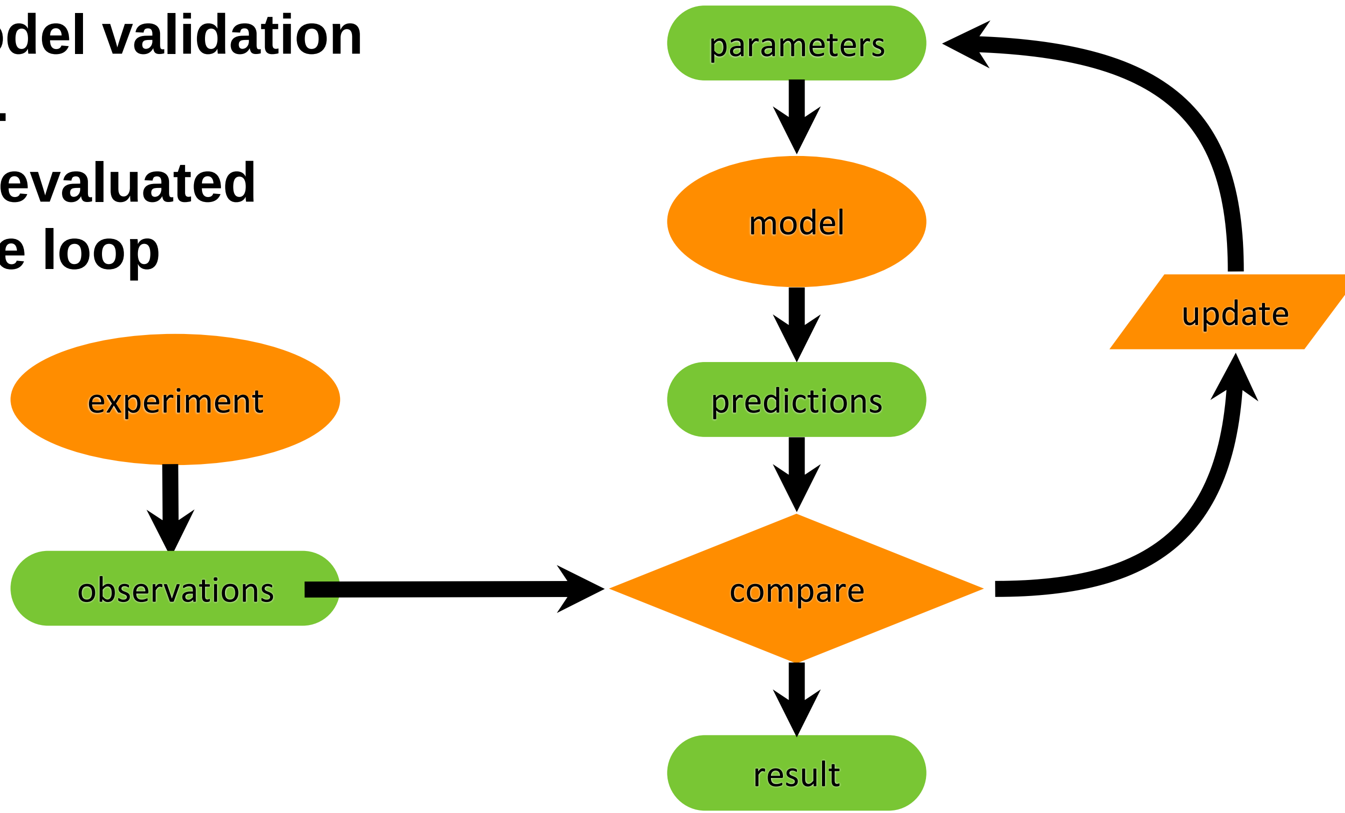
- forward problem predicts observations for given model parameters
  - *iterative solution to find model parameters that agree with observations*
- inverse problem is often ill-posed
  - *regularisation, preferably using a priori knowledge about actual experiment*



iterative solutions

model updating, parameter identification, model validation and verification…

forward problem evaluated many times inside loop



difficulty of inverse problems

Hadamard conditions:

- a problem is well-posed if
  - *a solution exists*
  - *the solution is unique*
  - *the solution depends continuously on the data*
- inverse problems are often ill-posed

regularisation modifies the problem statement to make it more well-posed

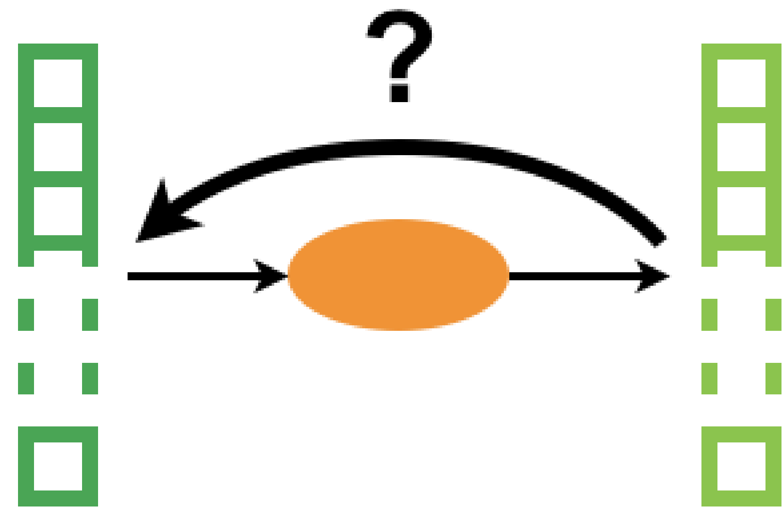
extensive literature and active research in mathematics

Ill-posed problems: Hadamard conditions

a problem is well-posed if

- a solution exists
- the solution is unique
- the solution is stable, i.e., it depends continuously on the data

inverse problems are often ill-posed



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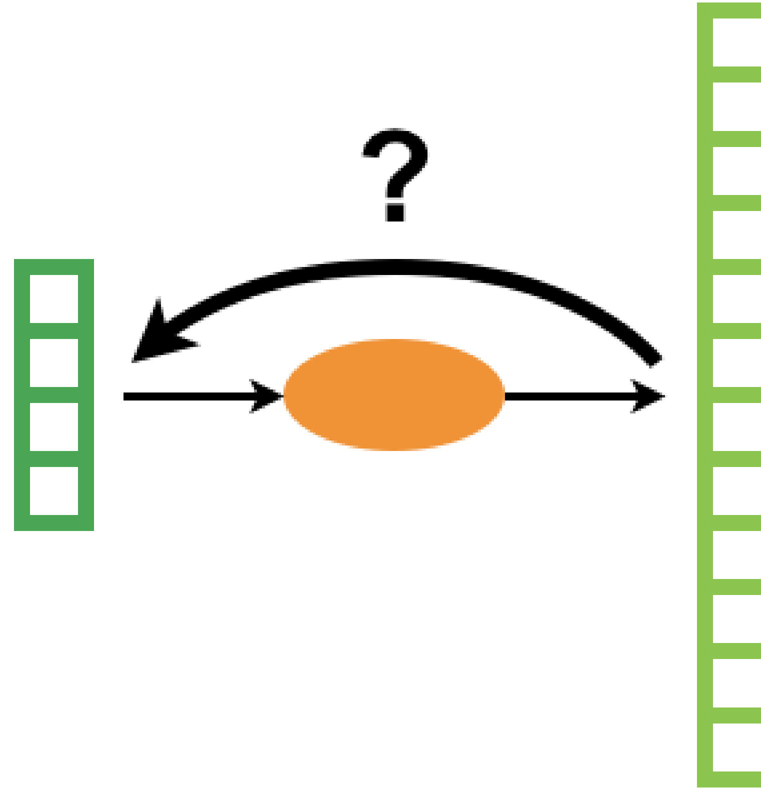

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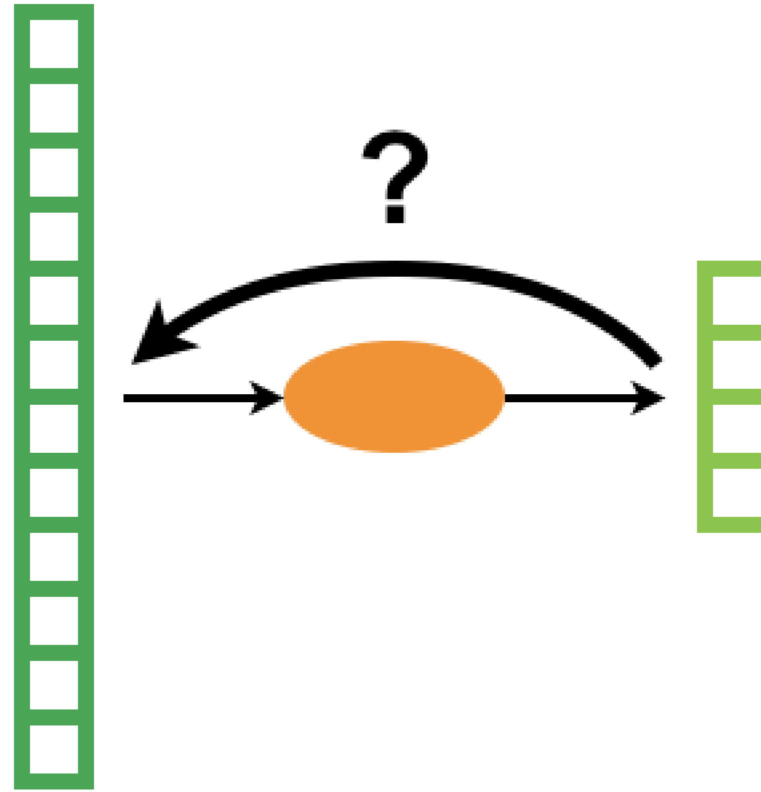

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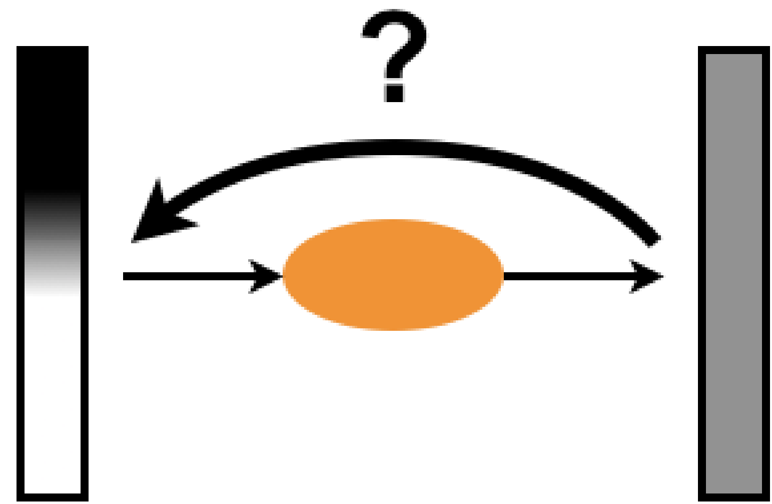

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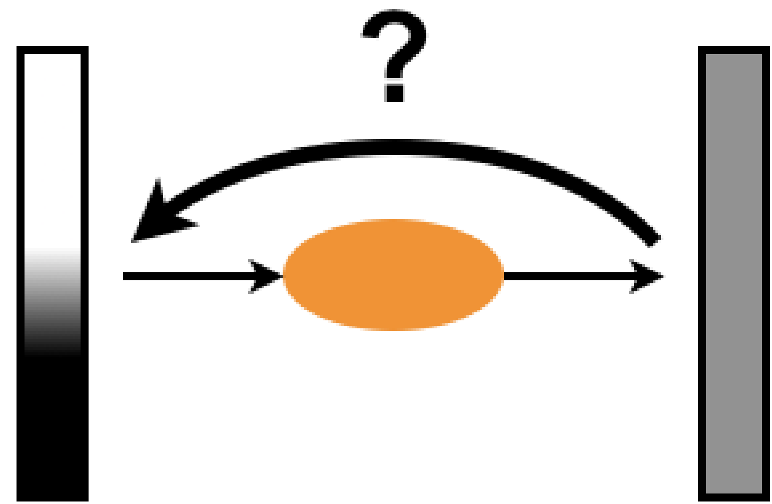

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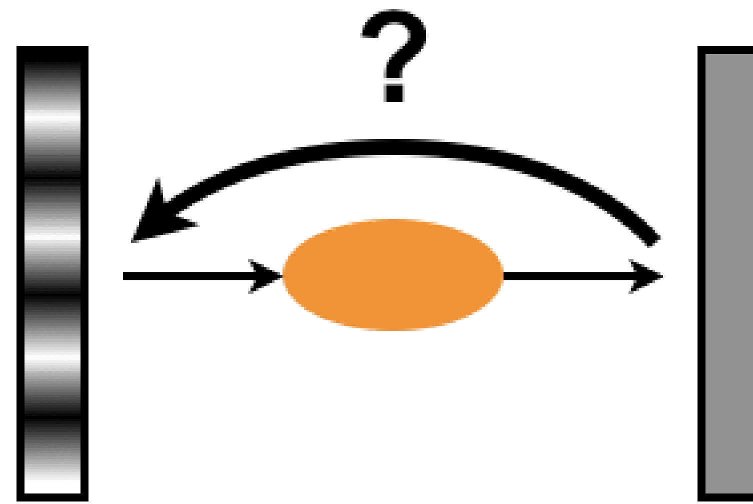


regularization

modify an ill-posed problem to make it well-posed

- explicitly modify equation, e.g., by adding regularization term
- implicitly modify problem statement or solution search space

define a new problem that is “close” to original problem

- recover original problem in limit where regularization parameter → zero

codify prior information about solution

- e.g., look for smoothly varying fields
- extreme case: vary simulation parameters to reproduce observations

Inverse Problems
in Experimental Mechanics

parameter identification

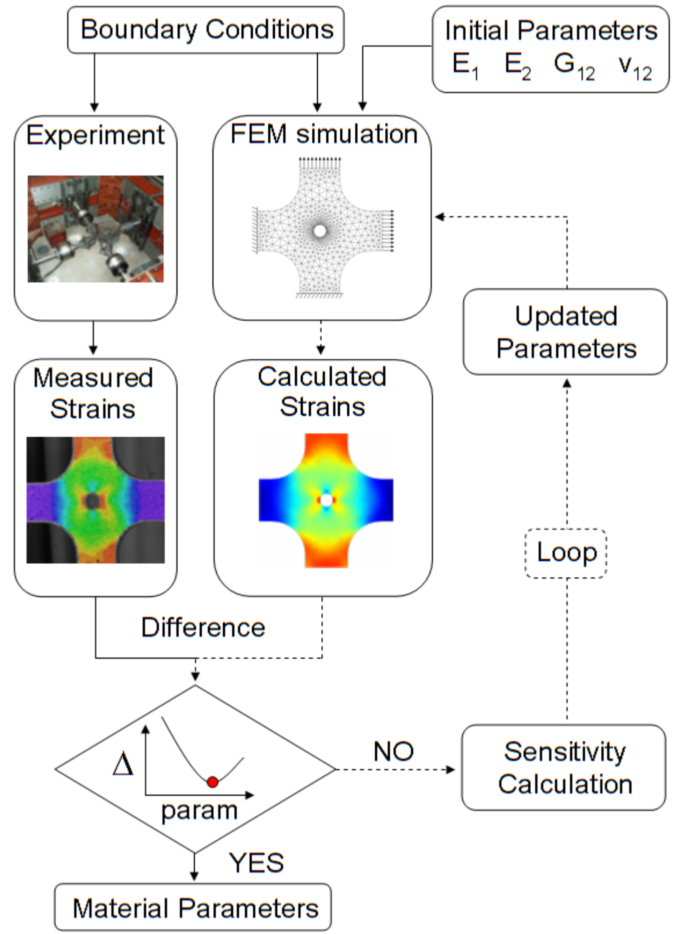
- over-determined
- ill-posed only with non-linearities

full-field measurements

- under-determined
- excessive regularization causes artifacts

forward problems solved by finite element models

- computationally intensive
- shape functions act as regularization
- finer mesh requires more computation and gives less regularization



identification of mechanical properties

simple tests with uniformly loaded gauge section

- inverse problem is trivial
- not always possible experimentally
- lots of tests when many parameters are needed

inverse problem and test with complex loading

- make use of computation power and full-field measurements
- sensitivity depends on test design (e.g., specimen shape)
  - *can test actual components of products in relevant load cases*
- coupled measurements of different parameters

inverse problem methodologies

for identification of mechanical properties

finite element model updating

- iterative solution comparing measured strain or displacement fields with finite element model results to update model parameters

equilibrium gap method

- if material model is wrong, then stresses calculated from measured strains are not in equilibrium, so iterate parameters in model to fix that

integrated DIC

- use finite element model parameters as parametrization of admissible displacement fields in DIC calculation

virtual fields method

- analytically calculate sensitivities of model parameters to measured displacements
- directly write system of equations relating measurements to sought parameters

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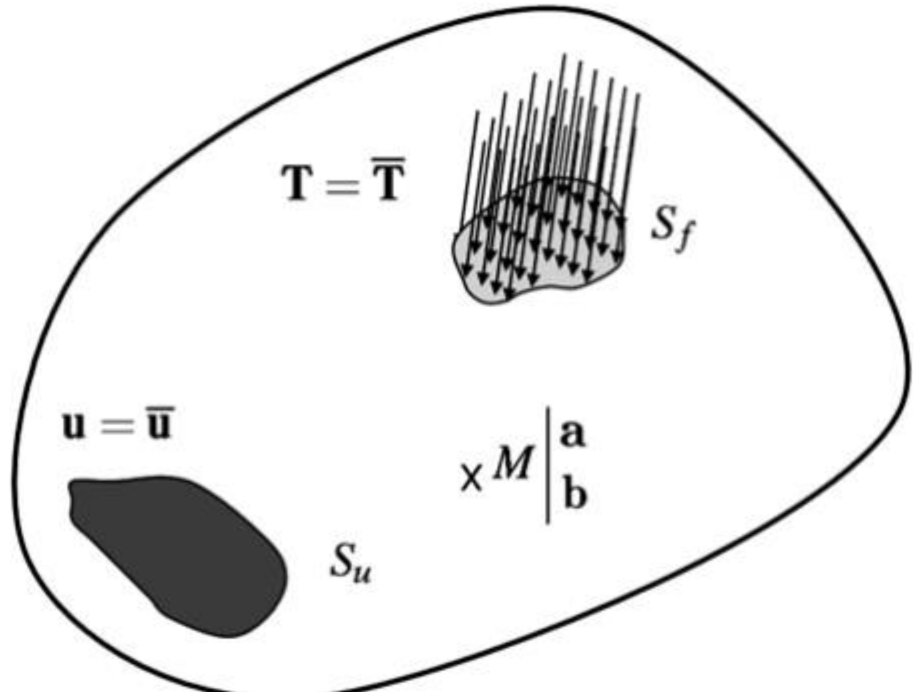
Virtual Fields Method

mechanics of deformable solids

- solid of any shape,
  subjected to mechanical load
  - *displacement field* $\mathbf{u}$
  - *strain field* $\boldsymbol{\varepsilon}$
  - *stress field* $\boldsymbol{\sigma}$
- body forces
  - *acceleration* $\mathbf{a}$ *in dynamic problems*
  - *other body forces* $\mathbf{b}$ *(e.g. gravity)*

- external surface $S = S_f \cup S_u$
- tractions $\mathbf{T}$ specified on S_f
  - *free surface has zero traction*
- displacements $\mathbf{u}$ specified on S_u



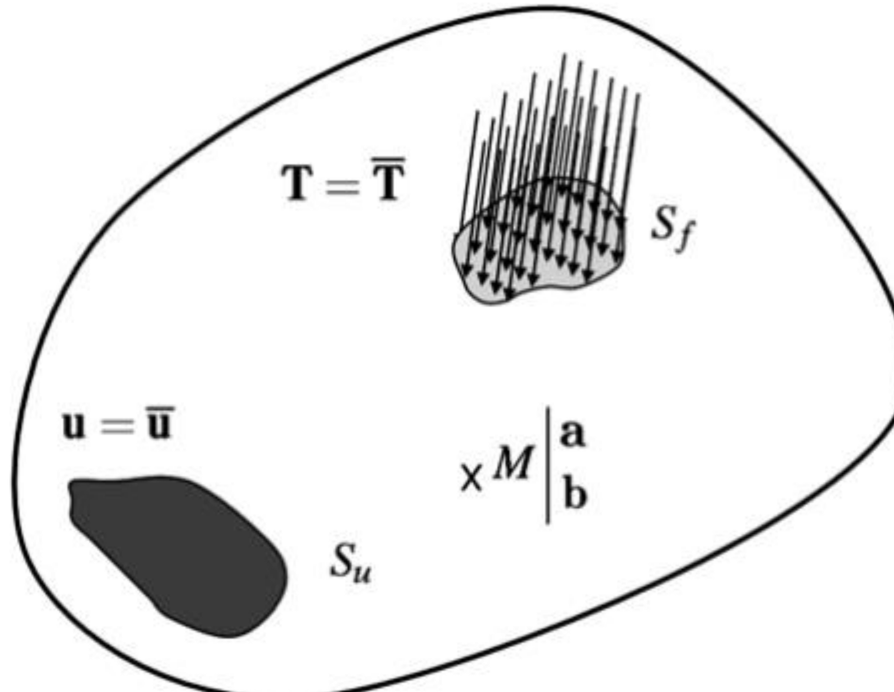
3 types of problems
in mechanics of deformable solids

solid of any shape,
subjected to mechanical load

1. find fields **u**, **ε** and **σ** for known boundary conditions and constitutive behaviour
   - *classical FEM problem*

2. identify constitutive behaviour from measured deformations

3. determine reaction forces



principle of virtual work

variational formulation of theoretical mechanics

- “weak form”
  - *weaker constraints on solution*
- multiply local equilibrium equation with test function and integrate

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \mathbf{b} = \rho \mathbf{a}$$

- for vector-valued test function $\mathbf{u}^*$

$$\int_V (\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}) \cdot \mathbf{u}^* dV + \int_V \mathbf{b} \cdot \mathbf{u}^* dV = \int_V \rho \mathbf{a} \cdot \mathbf{u}^* dV$$

principle of virtual work

- integrate by parts to get div $\mathbf{u}^*$ instead of div $\boldsymbol{\sigma}$

$$\int_V \boldsymbol{\nabla} \cdot (\boldsymbol{\sigma} \cdot \mathbf{u}^*) dV = \int_V (\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}) \cdot \mathbf{u}^* dV + \int_V \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} \mathbf{u}^* dV$$

- and use Gauss theorem to convert to surface integral of traction $\mathbf{T}$

$$\int_V \boldsymbol{\nabla} \cdot (\boldsymbol{\sigma} \cdot \mathbf{u}^*) dV = \oint_S \mathbf{n} \cdot \boldsymbol{\sigma} \cdot \mathbf{u}^* dS = \oint_S \mathbf{T} \cdot \mathbf{u}^* dS$$

- so that

$$\int_V (\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}) \cdot \mathbf{u}^* dV = \oint_S \mathbf{T} \cdot \mathbf{u}^* dS - \int_V \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} \mathbf{u}^* dV$$

principle of virtual work

$$\int_V (\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}) \cdot \mathbf{u}^* dV = \oint_S \mathbf{T} \cdot \mathbf{u}^* dS - \int_V \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} \mathbf{u}^* dV$$

- for symmetric $\boldsymbol{\sigma}$ and kinematically admissible $\mathbf{u}^*$ simplifies to

$$\int_V (\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}) \cdot \mathbf{u}^* dV = \oint_{S_f} \mathbf{T} \cdot \mathbf{u}^* dS - \int_V \boldsymbol{\sigma} \cdot (\boldsymbol{\nabla}^{\mathrm{sym}} \mathbf{u}^*) dV$$

- which yields

$$\oint_{S_f} \mathbf{T} \cdot \mathbf{u}^* dS - \int_V \boldsymbol{\sigma} \cdot (\boldsymbol{\nabla}^{\mathrm{sym}} \mathbf{u}^*) dV + \int_V \mathbf{b} \cdot \mathbf{u}^* dV = \int_V \rho \mathbf{a} \cdot \mathbf{u}^* dV$$

principle of virtual work

weak form of mechanical equilibrium equation in deformable solid

- applied load $$\oint_{S_f} \mathbf{T} \cdot \mathbf{u}^* dS$$
- reaction forces $$+ \oint_{S_u} \mathbf{T} \cdot \mathbf{u}^* dS$$
- internal stresses $$- \int_V \boldsymbol{\sigma} \cdot (\boldsymbol{\nabla}^{\text{sym}} \mathbf{u}^*) dV$$
- body forces $$+ \int_V \mathbf{b} \cdot \mathbf{u}^* dV$$
- acceleration $$= \int_V \rho \mathbf{a} \cdot \mathbf{u}^* dV \qquad \forall \mathbf{u}^*$$

remarks

- A continuous body is at equilibrium if the virtual work of all forces acting on the body is null in any kinematically admissible virtual displacement.
- If $\mathbf{u}^*$ is in meters, then the virtual work is in Joules.
- The test functions $\mathbf{u}^*$ are arbitrary weight functions, not related to the actual displacement field $\mathbf{u}$ and independent of stress field $\boldsymbol{\sigma}$.
  - *Expanding* $\mathbf{u}^*$ *in FEM shape functions is exact, not an approximation.*
  - *The mesh used for these shape functions does not have to conform to the object.*
- The reaction force term disappears for kinematically admissible $\mathbf{u}^*$.
- If $\mathbf{u}^*$ is discontinuous, then tractions along the discontinuity must be introduced to apply Gauss theorem.
- The Galerkin method and FEM can be derived from this.

linear elastic constitutive equations

$$\boldsymbol{\sigma} = \mathbf{Q}\boldsymbol{\epsilon}$$

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} & Q_{15} & Q_{16} \\ Q_{12} & Q_{22} & Q_{23} & Q_{24} & Q_{25} & Q_{26} \\ Q_{13} & Q_{23} & Q_{33} & Q_{34} & Q_{35} & Q_{36} \\ Q_{14} & Q_{24} & Q_{34} & Q_{44} & Q_{45} & Q_{46} \\ Q_{15} & Q_{25} & Q_{35} & Q_{45} & Q_{55} & Q_{56} \\ Q_{16} & Q_{26} & Q_{36} & Q_{46} & Q_{56} & Q_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{Bmatrix}$$

linear elastic constitutive equations

- orthotropic $\boldsymbol{\sigma} = \mathbf{Q}\boldsymbol{\epsilon}$

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & Q_{23} & 0 & 0 & 0 \\ Q_{13} & Q_{23} & Q_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{Bmatrix}$$

linear elastic constitutive equations

- cubic

$$\boldsymbol{\sigma} = \mathbf{Q}\boldsymbol{\epsilon}$$

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{12} & Q_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & Q_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{Bmatrix}$$

$$Q_{11} = \frac{E}{1-\nu^2} \qquad Q_{12} = \frac{\nu E}{1-\nu^2} \qquad Q_{66} = G$$

linear elastic constitutive equations

- isotropic $\boldsymbol{\sigma} = \mathbf{Q}\boldsymbol{\epsilon}$

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{12} & Q_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{Q_{11}-Q_{12}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{Q_{11}-Q_{12}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{Q_{11}-Q_{12}}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{Bmatrix}$$

$$Q_{11} = \frac{E}{1-\nu^2} \qquad Q_{12} = \frac{\nu E}{1-\nu^2} \qquad Q_{66} = G = \frac{E}{2(1+\nu)}$$

linear virtual fields method

- substitute linear elastic constitutive equations
- apply principle of virtual work

for simplicity, consider static in-plane cubic case without body forces

$$\oint_{S_f} \mathbf{T} \cdot \mathbf{u}^* dS - \int_V \boldsymbol{\sigma} \cdot (\boldsymbol{\nabla}^{\text{sym}} \mathbf{u}^*) dV + \cancel{\int_V \mathbf{b} \cdot \mathbf{u}^* dV} = \cancel{\int_V \rho \mathbf{a} \cdot \mathbf{u}^* dV}$$

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{11} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_6 \end{Bmatrix}$$

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for simplicity, consider static in-plane cubic case without body forces

$$
\begin{aligned}
&\int_V \boldsymbol{\sigma} \cdot (\boldsymbol{\nabla}^{\mathrm{sym}} \mathbf{u}^*) dV \\
&= \int_V (Q_{11}\epsilon_1\epsilon_1^* + Q_{11}\epsilon_2\epsilon_2^* + Q_{12}(\epsilon_1\epsilon_2^* + \epsilon_2\epsilon_1^*) + Q_{66}\epsilon_6\epsilon_6^*) dV \\
&= \oint_{S_f} \mathbf{T} \cdot \mathbf{u}^* dS
\end{aligned}
$$

linear virtual fields method

- substitute linear elastic constitutive equations
- apply principle of virtual work

for simplicity, consider static in-plane cubic case without body forces

$$\left(\int_V (\epsilon_1\epsilon_1^* + \epsilon_2\epsilon_2^*)dV\right) Q_{11} + \left(\int_V (\epsilon_1\epsilon_2^* + \epsilon_2\epsilon_1^*)dV\right) Q_{12} + \left(\int_V (\epsilon_6\epsilon_6^*)dV\right) Q_{66}$$
$$= \oint_{S_f} \mathbf{T} \cdot \mathbf{u}^* dS$$

linear virtual fields method

- substitute linear elastic constitutive equations
- apply principle of virtual work
- for as many virtual fields as independent coefficients

$$\left(\int_V (\epsilon_1\epsilon_1^{*(1)} + \epsilon_2\epsilon_2^{*(1)})dV\right) Q_{11} + \left(\int_V (\epsilon_1\epsilon_2^{*(1)} + \epsilon_2\epsilon_1^{*(1)})dV\right) Q_{12} + \left(\int_V (\epsilon_6\epsilon_6^{*(1)})dV\right) Q_{66} = \oint_{S_f} \mathbf{T}\cdot\mathbf{u}^{*(1)}dS$$

$$\left(\int_V (\epsilon_1\epsilon_1^{*(2)} + \epsilon_2\epsilon_2^{*(2)})dV\right) Q_{11} + \left(\int_V (\epsilon_1\epsilon_2^{*(2)} + \epsilon_2\epsilon_1^{*(2)})dV\right) Q_{12} + \left(\int_V (\epsilon_6\epsilon_6^{*(2)})dV\right) Q_{66} = \oint_{S_f} \mathbf{T}\cdot\mathbf{u}^{*(2)}dS$$

$$\left(\int_V (\epsilon_1\epsilon_1^{*(3)} + \epsilon_2\epsilon_2^{*(3)})dV\right) Q_{11} + \left(\int_V (\epsilon_1\epsilon_2^{*(3)} + \epsilon_2\epsilon_1^{*(3)})dV\right) Q_{12} + \left(\int_V (\epsilon_6\epsilon_6^{*(3)})dV\right) Q_{66} = \oint_{S_f} \mathbf{T}\cdot\mathbf{u}^{*(3)}dS$$

linear virtual fields method

- substitute linear elastic constitutive equations
- apply principle of virtual work
- for as many virtual fields as independent coefficients

$$\begin{bmatrix} \int_V (\epsilon_1\epsilon_1^{*(1)} + \epsilon_2\epsilon_2^{*(1)})dV & \int_V (\epsilon_1\epsilon_2^{*(1)} + \epsilon_2\epsilon_1^{*(1)})dV & \int_V (\epsilon_6\epsilon_6^{*(1)})dV \\ \int_V (\epsilon_1\epsilon_1^{*(2)} + \epsilon_2\epsilon_2^{*(2)})dV & \int_V (\epsilon_1\epsilon_2^{*(2)} + \epsilon_2\epsilon_1^{*(2)})dV & \int_V (\epsilon_6\epsilon_6^{*(2)})dV \\ \int_V (\epsilon_1\epsilon_1^{*(3)} + \epsilon_2\epsilon_2^{*(3)})dV & \int_V (\epsilon_1\epsilon_2^{*(3)} + \epsilon_2\epsilon_1^{*(3)})dV & \int_V (\epsilon_6\epsilon_6^{*(3)})dV \end{bmatrix} \begin{Bmatrix} Q_{11} \\ Q_{12} \\ Q_{66} \end{Bmatrix} = \begin{Bmatrix} \oint_{S_f} \mathbf{T}\cdot\mathbf{u}^{*(1)}dS \\ \oint_{S_f} \mathbf{T}\cdot\mathbf{u}^{*(2)}dS \\ \oint_{S_f} \mathbf{T}\cdot\mathbf{u}^{*(3)}dS \end{Bmatrix}$$

linear virtual fields method

- substitute linear elastic constitutive equations
- apply principle of virtual work
- for as many virtual fields as independent coefficients

$$\mathbf{A} \cdot \mathbf{Q} = \mathbf{B}$$

$$\mathbf{A} = \begin{bmatrix} \int_V (\epsilon_1 \epsilon_1^{*\,(1)} + \epsilon_2 \epsilon_2^{*\,(1)}) dV & \int_V (\epsilon_1 \epsilon_2^{*\,(1)} + \epsilon_2 \epsilon_1^{*\,(1)}) dV & \int_V (\epsilon_6 \epsilon_6^{*\,(1)}) dV \\ \int_V (\epsilon_1 \epsilon_1^{*\,(2)} + \epsilon_2 \epsilon_2^{*\,(2)}) dV & \int_V (\epsilon_1 \epsilon_2^{*\,(2)} + \epsilon_2 \epsilon_1^{*\,(2)}) dV & \int_V (\epsilon_6 \epsilon_6^{*\,(2)}) dV \\ \int_V (\epsilon_1 \epsilon_1^{*\,(3)} + \epsilon_2 \epsilon_2^{*\,(3)}) dV & \int_V (\epsilon_1 \epsilon_2^{*\,(3)} + \epsilon_2 \epsilon_1^{*\,(3)}) dV & \int_V (\epsilon_6 \epsilon_6^{*\,(3)}) dV \end{bmatrix}$$

$$\mathbf{Q} = \begin{Bmatrix} Q_{11} \\ Q_{12} \\ Q_{66} \end{Bmatrix} \qquad \mathbf{B} = \begin{Bmatrix} \oint_{S_f} \mathbf{T} \cdot \mathbf{u}^{*\,(1)} dS \\ \oint_{S_f} \mathbf{T} \cdot \mathbf{u}^{*\,(2)} dS \\ \oint_{S_f} \mathbf{T} \cdot \mathbf{u}^{*\,(3)} dS \end{Bmatrix}$$

special virtual fields

- The equation $\mathbf{A} \cdot \mathbf{Q} = \mathbf{B}$ still depends on the choice of virtual fields.
  - *valid for any choice of virtual fields*
  - *solvable for any linearly independent choice of virtual fields*
  - *numerically stable if condition number of* $\mathbf{A}$ *is close to 1*
- It is possible to choose the virtual fields such that $\mathbf{A}$ is perfectly conditioned and trivially solvable by making $\mathbf{A} = \mathbf{I}$ the identity matrix.
  - $\mathbf{A} = \mathbf{I}$ *is a set of linear equations*
  - *as many equations as elements of* $\mathbf{A}$
- It is possible to assess *a priori* how sensitive to noise the calculation is.
  - *coefficients depend on material coefficients* $\mathbf{Q}$
  - *optimized virtual fields minimize noise sensitivity*
  - *iteratively solving material coefficients and optimizing converges quickly*