Introduction to Mechanics

A brief introduction into basic mechanics, aimed for Mechatronic students without Mechanical Engineering background

Markus Storsjö 2012

edited by Tapio Lantela 2015

Table of contents

1 Symbols

Table 1. Symbols used in this text

Term	Description	Unit
a	Acceleration	m/s^2
A	Area	m^2
g	Acceleration due to gravity	m/s^2
F	Force	kgm/s^2
J	Moment of inertia	kgm^2
m	Mass	kg
n	Rotational speed	(rev)/s, (rev)/min
n	Safety number	-
p	Pressure	Pa, N/m^2
P	Power	W, kgm^2/s^3
s	Distance	m
t	Time	s
T	Torque	Nm, kgm^2/s^2
U	Voltage	V
v	Linear velocity	m/s
W	Work	J, kgm^2/s^2
x	Position	m
z	Number of teeth	-
α	Angular acceleration	$(rad)/s^2$
α	Thermal expansion coefficient	1/°C
φ	Angular position	(rad)
η	Efficiency	-
μ	Friction coefficient	-
σ	Normal stress	Pa, N/m^2
τ	Shear stress	Pa, N/m^2
ω	Angular velocity	(rad)/s

2 Linear and rotational motion

2.1 Linear motion

Velocity is defined as the first time derivative of the position (1)

$$v = \frac{d}{dt}x = \dot{x} \tag{1}$$

Acceleration is the time derivative of velocity (2).

$$\boldsymbol{a} = \frac{\boldsymbol{d}}{\boldsymbol{dt}}\boldsymbol{v} = \dot{\boldsymbol{v}} = \ddot{\boldsymbol{x}} \tag{2}$$

In the case of constant acceleration, we can integrate equation (2) twice and get a convenient formula for the distance travelled during acceleration

$$x = \frac{1}{2}at^2 + v_0t + x_0 \tag{3}$$

2.2 Rotational motion

Angular position (φ) is measured in radians. The angle in radians is defined as the ratio between the circle bow and the radius (Figure 1). A whole revolution, 360°, equals therefore 2π radians. Observe, that because of its definition, an angle in radians is unitless ($\frac{m}{m} = 1$). The unit "rad" can still sometimes be written to make things clearer.

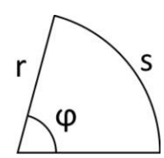


Figure 1. Angle, circle bow and radius

$$\varphi = \frac{s}{r} \tag{4}$$

Angular speed is defined as the time derivative of the angular position, and its unit is rad/s = 1/s.

$$\omega = \frac{d}{dt}\varphi = \dot{\varphi} \tag{5}$$

Angular acceleration is the time derivative of angular speed, and its unit is rad/s^2 = 1/s^2

$$\alpha = \frac{d}{dt}\omega = \dot{\omega} = \ddot{\varphi} \tag{6}$$

When we derive equation (4) we get the relation between rotational and linear motion

$$\omega = \frac{v}{r} \tag{7}$$

$$\alpha = \frac{a}{r} \tag{8}$$

Converting between radians and degrees is done according to

$$Angle\ in\ radians = \frac{2\pi}{360} Angle\ in\ degrees \tag{9}$$

Rotational speed is usually represented by the symbol n, and given as RPM (revolutions per minute) or revolutions per second. Since one revolution is 2π radians and one minute is 60 s, we get

$$Angular\ speed\ in \frac{rad}{s} = RPM * \frac{2\pi}{60} \tag{10}$$

The similarities between linear and rotational motion are gathered in Table 2. Formulas for linear motion can also be used for rotational motion, when changing the linear quantities to the corresponding rotational ones.

Table 2. Similarities between linear and rotational quantities

Linear motion	Rotational motion
Position x, distance s	Angle φ
Velocity v	Angular velocity ω
Acceleration a	Angular acceleration α
Force F	Torque T
Mass m	Moment of inertia J

2.2.1 Moment of inertia

The moment of inertia *J* of an object where all the mass *m* is located at a distance of *r* from the point of rotation can be calculated with equation (11).

$$J = mr^2 \tag{11}$$

For example the mass on a conveyor belt can be thought of as a single point mass located at the edge of the pulley rotating the conveyor.

The moment of inertia around the symmetry axis of a solid cylinder with radius *r* can be calculated with equation (12).

$$J = \frac{1}{2}mr^2 \tag{12}$$

In general, the moment of inertia is defined with respect to a specified rotation axis i.e. the same object has a different moment of inertia around a different rotation axis. The moment of inertia can be calculated for any object by summing the moment of inertia of all the point masses (integrating for continuous matter) in the object.

3 Statics

Newton's second law (13) gives the relation between force and acceleration.

$$\sum \bar{F} = m\bar{a} \tag{13}$$

In a static case the acceleration is always zero, which means that the sum of all forces also equals zero (14).

$$\sum \bar{F} = \bar{0} \tag{14}$$

Equation (15) is Newton's second law for rotational motion. T is the torque, J is the moment of inertia, and α is the angular acceleration.

$$\sum \bar{T} = J\bar{\alpha} \tag{15}$$

In a static case the angular acceleration is zero, which means that the sum of the torque is also zero (16).

$$\sum \bar{T} = \bar{0} \tag{16}$$

Equations (14) and (16) can be used for solving equilibrium equations. In a two-dimensional case we can write equation (14) for force balance in x-direction and y-direction separately, and apply equation (16) for rotation around any arbitrary point in the plane, giving us three equations, and making it possible to solve three unknown forces or torques.

Example: *A beam is loaded as seen in Figure 2. Determine the supporting forces.*

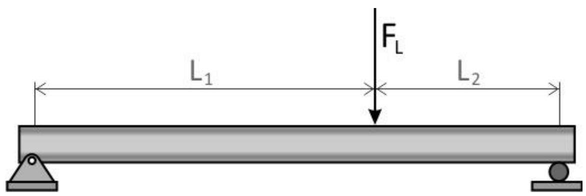


Figure 2. A loaded beam

The situation is simplified by drawing a free body diagram (Figure 3).

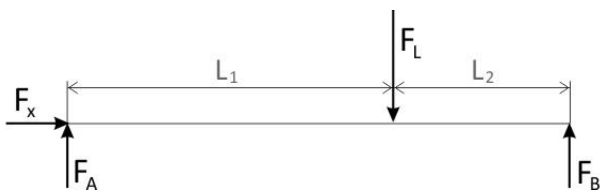


Figure 3. A free body diagram of a loaded beam

We then write the equilibrium equations in x-direction, y-direction, and for rotation around some point we choose.

- $\rightarrow \sum F_x = 0$: $\quad F_x = 0$
- $\uparrow \sum F_y = 0$: $\quad F_A + F_B - F_L = 0$
- $\sum T_A = 0$: $\quad F_B(L_1 + L_2) - F_L L_1 = 0$

From these three simultaneous equations we can then solve the three unknown forces, F_A, F_B and F_x.

4 Dynamics

In a dynamic case the acceleration is ≠ 0. Newton's second law is also most often the key to solving these cases, and it can be used for both linear motion (F=ma) and rotational motion (T=Jα). In the case of rotational motion, the torque refers to the torque applied to the center of gravity, and the moment of inertia is measured from the center of gravity. In some cases it could still be easier to examine the rotation around some other point than the center of gravity, which will be presented in later examples.

4.1 Differential equations

When the force acting on a body is not constant, writing Newton's second law as a differential equation, and solving it using mathematical tools, can be convenient. It can then be written in the form

$$\sum F(t) = m\ddot{x}(t) \tag{17}$$

In Figure 4 an oscillator is seen. Since the spring force is proportional to the displacement of the body, and the viscous force of the damper is proportional to the velocity, we can write the differential equation for its motion as

$$-kx - c\dot{x} = m\ddot{x} \tag{18}$$

Figure 4. An oscillator and its free body diagram

4.2 **Different points of view**

Newton's second law refers to the motion of the centre of gravity. Sometimes it is still seen that the motion is examined around some other point of rotation, as seen in the following examples.

4.2.1 The physically correct way

A back-wheel-drive car with the weight distribution 50/50 is accelerating with an acceleration a (Figure 5). What is the weight distribution during acceleration?

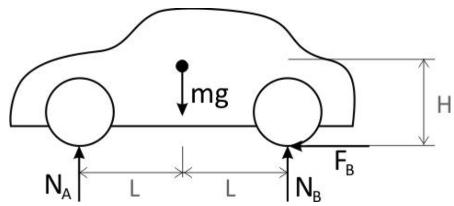


Figure 5. An accelerating car

We apply Newton's second law to the centre of gravity. The car has the acceleration a in x-direction, no acceleration in y-direction, and no angular acceleration. We get the following equations:

- $\leftarrow \sum F_x = ma$: $F_B = ma$
- $\uparrow \sum F_y = 0$: $N_A + N_B - mg = 0$
- $\sum T_G = 0$: $N_A L - N_B L + F_B H = 0$

We have now three simultaneous equations, from which we can solve the three unknown forces N_A, N_B and F_B.

4.2.2 "The engineer's way"

A back-wheel-drive car with the weight distribution 50/50 is accelerating with an acceleration a. What is the weight distribution during acceleration?

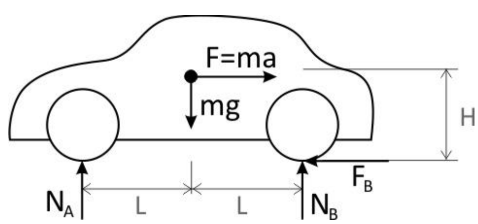


Figure 6. An accelerating car considered as a static problem

We treat the problem like a static problem, just adding the virtual inertial force F=ma acting on the center of gravity (Figure 6). We then write the equilibrium equations for forces in x- and y-directions, and for rotation around the back supporting point.

- $\leftarrow \sum F_x = 0$: $F_B - ma = 0$
- $\uparrow \sum F_y = 0$: $N_A + N_B - mg = 0$
- $\sum T_B = 0$: $N_A * 2L - mg * L + ma * H = 0$

We have now three simultaneous equations, from which we can solve the three unknown forces N_A, N_B and F_B.

This method is not as physically correct as the one in 4.2.1, but the benefit is that when we add the virtual inertial force, we can examine the rotation around any arbitrary point when writing the torque equation. We can therefore choose to examine the point where the rear wheel touches the ground, which feels natural, since it's the point that the car would rotate around if the acceleration was large enough.

5 Work and energy

Work is defined as the distance travelled multiplied by the force acting in the direction of the movement

$$W = \int \bar{F} \cdot d\bar{s} \tag{19}$$

which usually can be simplified as

$$W = Fs \tag{20}$$

where s and F are in the same direction. Work can be thought of as the energy being used.

Correspondingly for rotational motion we can write

$$W = T\varphi \tag{21}$$

Power is defined as how quickly the work is being done, or in other words, how fast the energy is changing.

$$P = \frac{d}{dt}W = \frac{d}{dt}E \tag{22}$$

and in the case of a constant force or torque we can derive equations (20) and (21) and get

$$P = Fv = T\omega \tag{23}$$

Kinetic energy is given by the formula

$$E_k = \frac{1}{2}mv^2 = \frac{1}{2}J\omega^2 \tag{24}$$

and potential energy by

$$E_p = mg\Delta h \tag{25}$$

where Δh is the height from the reference level.

The law of conservation of energy (26) gives the connection between kinetic and potential energy, but to apply it we need to know the amount of energy brought to the system and lost from the system.

$$E_{k1} + E_{p1} + W_{in} = E_{k2} + E_{p2} + W_{out} \tag{26}$$

Efficiency is defined as the ratio between the energy of the useful work that has been done, and the energy required for doing it. Correspondingly we can also get the efficiency as the ratio of the power of the work being done and the power needed for doing it.

$$\eta = \frac{E_{useful}}{E_{consumed}} = \frac{P_{useful}}{P_{consumed}} \tag{27}$$

The energy being lost when $\eta \neq 100\%$ is most often turned into heat. The power being lost as heat can be written as

$$P_{heat} = (1 - \eta) P_{consumed} \tag{28}$$

6 Mechanics

6.1 Gear ratio

Mechanical gearing is used to change the speed and torque.

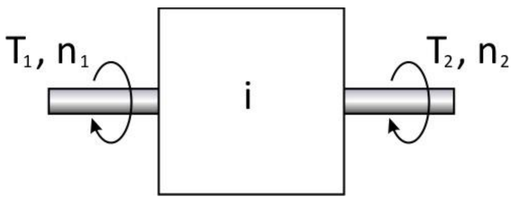


Figure 7. A gearbox

The gear ratio i is defined as

$$i = \frac{n_1}{n_2} = \frac{\omega_1}{\omega_2} \tag{29}$$

where n is the rotational speed in any unit (as long as n_1 and n_2 have the same unit!). In Mechanical Engineering most gearboxes are reduction gears, meaning that the speed is reduced, therefore i is usually a number greater than 1.

6.1.1 Different mechanisms

For a pair of two gears, the gear ratio is given by the ratio of the number of teeth.

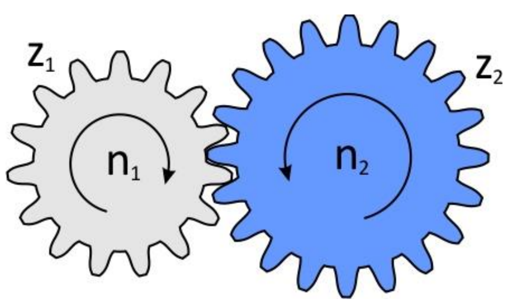


Figure 8. Gear pair

$$i = \frac{n_1}{n_2} = \frac{z_2}{z_1} \tag{30}$$

Correspondingly, in a belt drive (Figure 9) the gear ratio is given by the ratio of the pulley diameters.

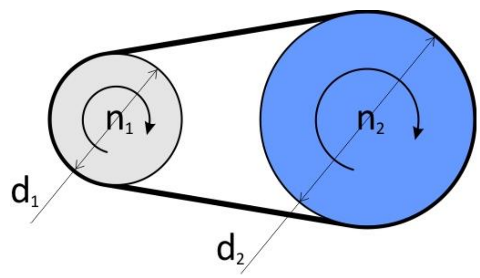


Figure 9. Belt drive

$$i = \frac{n_1}{n_2} = \frac{d_2}{d_1} \tag{31}$$

For a worm gear (Figure 10), the gear ratio is also given by equation (30), but in that case z_1 refers to the number of starts (spirals) of the worm.

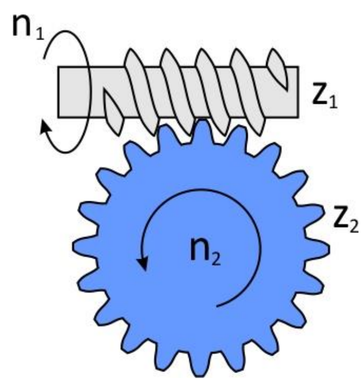


Figure 10. Worm gear

A screw (Figure 11) can be used for converting rotational motion into linear motion. The pitch of the screw refers to how far it moves per revolution. Therefore we get the following equations:

$$\Delta x = p * revolutions = p\frac{\Delta\varphi}{2\pi} \tag{32}$$

$$v = pn = p\frac{\omega}{2\pi} \tag{33}$$

where n has the unit rev/s.

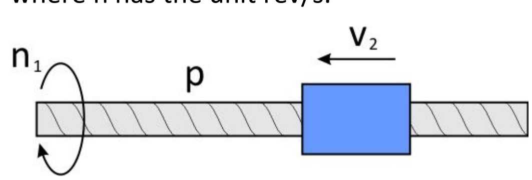


Figure 11. Screw drive

6.1.2 Gear efficiency

For an ideal gearing the input power and the output power are equal, $T_1\omega_1=T_2\omega_2$, which gives that

$$\frac{T_2}{T_1} = \frac{\omega_1}{\omega_2} = i \tag{34}$$

In real life gearing always contain some internal friction, which causes energy losses. We can then write

$$\eta = \frac{P_2}{P_1} = \frac{T_2\omega_2}{T_1\omega_1} \tag{35}$$

from which we can further get the outgoing torque

$$T_2 = \eta \frac{T_1\omega_1}{\omega_2} = \eta i T_1 \tag{36}$$

6.2 **Reduced mass**

Consider the following example: A motor with the moment of inertia J_1 is connected via a gearbox with the gear ratio i and efficiency η to a load with the moment of inertia J_2 (Figure 12). The motor is subject to a torque T_1. How do we get the acceleration for the system?

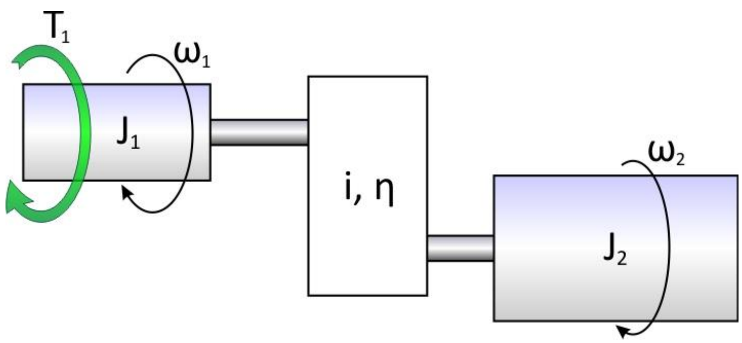


Figure 12. Two rotating loads connected though a gearbox

Since a torque is causing acceleration, Newton's second law should be applied. But it can only be applied for single bodies, and in this case there are two bodies connected to each other. We therefore need to reduce the system to a single body, as seen in Figure 13. The load and the gearbox are replaced by a load on the same shaft as the motor, giving the same inertia. The whole system spins with the same speed and acceleration.

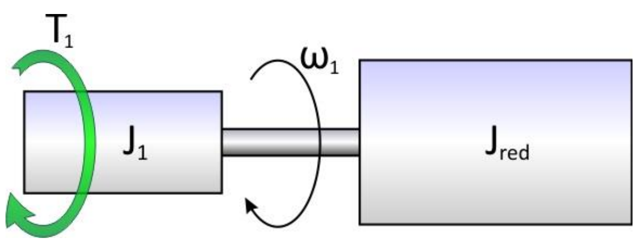


Figure 13. A reduced system with all loads on the same shaft

How do we get J_{red}? The kinetic energy of the load in the original system, and in the reduced system are

$$E_{k2}=\frac{1}{2}J_2{\omega_2}^2 \tag{37}$$

$$E_{k,red}=\frac{1}{2}J_{red}{\omega_1}^2 \tag{38}$$

In the reduced system, the kinetic energy should be the same as in the original system, when taking the efficiency into account

$$\eta E_{k,red}=E_{k2} \tag{39}$$

We then get

$$\eta\frac{1}{2}J_{red}{\omega_1}^2=\frac{1}{2}J_2{\omega_2}^2 \tag{40}$$

$$J_{red} = \frac{J_2}{\eta}\left(\frac{\omega_2}{\omega_1}\right)^2 = \frac{J_2}{\eta i^2} \tag{41}$$

When knowing the inertias in the single body system, solving the acceleration is trivial.

When mass *m* is driven linearly by a rotating lead/ball screw, the reduced (equivalent rotary) inertia of the mass on the axis of the screw can be calculated with equation (42), where *p* is the pitch of the ball screw (in m/rotation) and *η* is the efficiency of the screw.

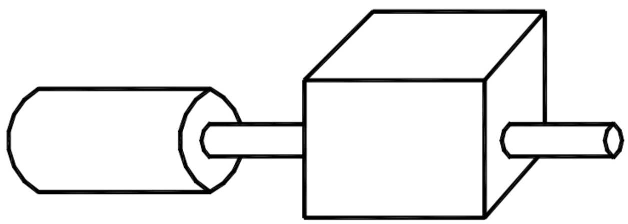


Figure 14 Mass driven by a leadscrew

$$J_{red} = \frac{mp^2}{4\pi^2\eta} \tag{42}$$

6.3 Reduced torque

Consider the following example (Figure 15): A motor with the moment of inertia J_1 is connected to a gearbox with the gear ratio i and efficiency η. The motor is accelerated with the torque T_1, and the output shaft of the gearbox is subject to a load torque T_2. How do we get the acceleration for the system?

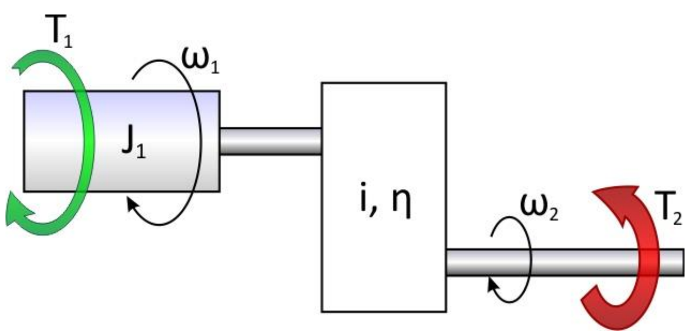


Figure 15. A load torque acting through a gearbox

We can't apply Newton's second law on the motor until we know all the torques acting on it. We therefore need to reduce the torque on the output shaft of the gearbox, to the motor shaft, as seen in Figure 16.

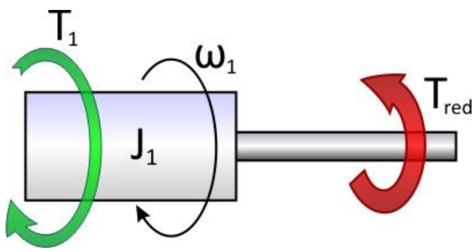


Figure 16. A reduced system with the load torque acting straight on the motor shaft

The power of the load torque and the reduced torque are:

$$P_2 = T_2 \omega_2 \tag{43}$$

$$P_{red} = T_{red} \omega_1 \tag{44}$$

In the reduced system, the power should be the same as in the original system, when taking the efficiency into account

$$\eta P_{red} = P_2 \tag{45}$$

We then get

$$\eta T_{red} \omega_1 = T_2 \omega_2 \tag{46}$$

$$T_{red} = \frac{T_2}{\eta} \frac{\omega_2}{\omega_1} = \frac{T_2}{\eta i} \tag{47}$$

When knowing all the torques acting on the single body in the reduced system, solving the acceleration is trivial.

7 Mechanics of materials

7.1 Stresses

Normal stress is defined as the ratio between the normal force acting on the surface and the area of the surface (Figure 17). The force is positive when acting outwards from the surface, meaning that a positive stress corresponds to tensile stress and negative stress is compressive.

$$\sigma = \frac{F}{A} \tag{48}$$

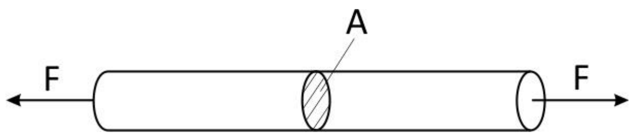


Figure 17. Normal stress

Shear stress is defined as the ratio between the force acting in the direction of the surface, and the area of the surface (Figure 18).

$$\tau = \frac{F}{A} \tag{49}$$

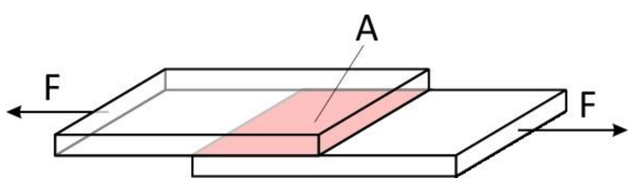


Figure 18. Shear stress

When a beam is bent (Figure 19), the material is in tension on one side, and in compression on the other side. The normal stress is at its maximum on the edge of the beam, and can be calculated from

$$\sigma = \frac{M_b}{W_b} \tag{50}$$

where M_b is the bending moment acting at the cross section, and W_b is the bending resistance.

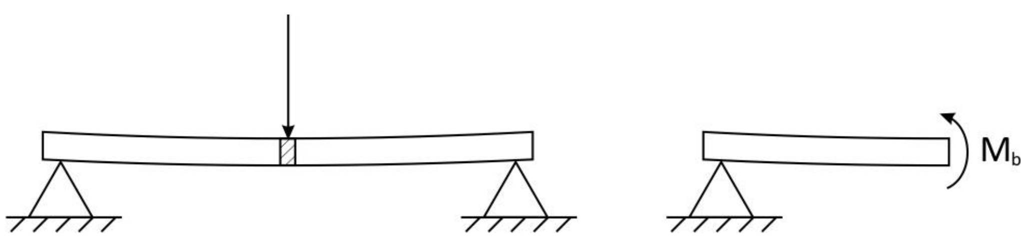


Figure 19. Bending stress

When a shaft is twisted (Figure 20), the shear stress is at its maximum at the surface of the shaft. The shear stress maximum is given from

$$\tau = \frac{T}{W_t} \tag{51}$$

where T is the torque acting on the shaft and W_t is the twisting resistance.

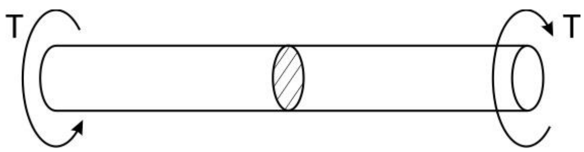
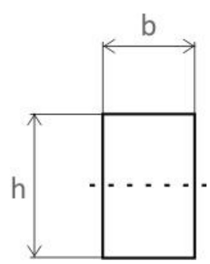
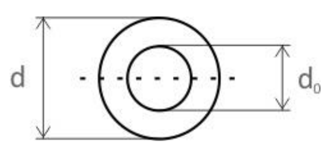


Figure 20. Twisting stress

The formulas for bending resistance and twisting resistance can be found in Table 3.

Table 3. Properties for different cross section shapes

Cross section shape	Bending and twisting resistance
	$W_b = \frac{1}{6}bh^2$ $W_t = \frac{2}{9}b^2h$
	$W_b = \frac{\pi}{32}\frac{d^4 - {d_0}^4}{d}$ $W_t = \frac{\pi}{16}\frac{d^4 - {d_0}^4}{d}$

A design criterion for mechanical structures is, that the maximum stress must not exceed the yield stress or fracture stress of the material in case. When designing a dynamically loaded structure, the maximum stress should not exceed the fatigue strength of the material.

The safety number, n, describes the ratio between the calculated strength, and the real stress acting in the structure. From this follows, that the safety number should always be greater than 1.

$$n = \frac{\sigma_{critical}}{\sigma_{load}} \tag{52}$$

7.2 Thermal expansion

Most materials expand due to heating. In the case of relatively small temperature changes the change can be thought of as nearly linear.

$$\Delta L = \alpha L_0 \Delta T \tag{53}$$

$$L = L_0 + \Delta L = L_0(1 + \alpha \Delta T) \tag{54}$$

where L_0 is the original length of the item, and α is the coefficient of length expansion.

8 Hydraulics

In hydraulics the pressure of a fluid is used for transferring power in a compact form. The basic formula for pressure is

$$p = \frac{F}{A} \tag{55}$$

8.1 Cylinders

In a hydraulic cylinder (Figure 21) the pressure on both sides of the piston causes a resultant force on the rod, given by

$$F = p_a A_a - p_b A_b \tag{56}$$

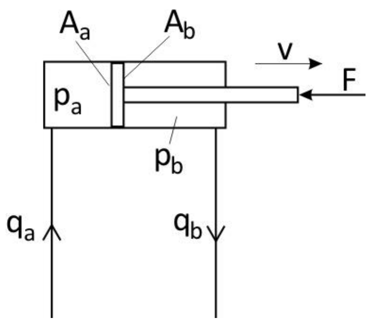


Figure 21. A hydraulic cylinder

The flow of the fluid is defined, as the volume of the fluid passing by a point during a given time, i.e.

$$q = \frac{\Delta V}{t} \tag{57}$$

In most calculations the fluid can be approximated to be incompressible. Therefore the velocity of the cylinder is depending on the incoming flow.

$$q = vA \tag{58}$$

Because of the different cross section areas in the chambers, the incoming flow will be different from the outgoing flow.

8.2 **Pumps and motors**

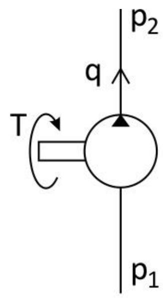


Figure 22. Hydraulic pump

In a pump (Figure 22) the incoming torque causes the pump to spin, which in turn creates a flow q through the pump. The volume being pumped when the pump spins one revolution is called the displacement of the pump, V_r. The correspondence between flow q and rotary speed n (rev/s) is

$$q = V_r n \tag{59}$$

The needed torque depends on the pressure difference over the pump according to

$$T = \frac{\Delta p V_r}{2\pi} \tag{60}$$

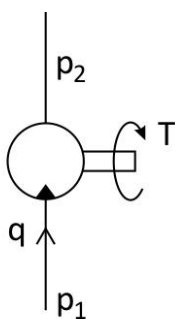


Figure 23. Hydraulic motor

A hydraulic motor, as seen in Figure 23, converts the flow into rotational motion. The equations for speed and torque are the same as for pumps.