PROBLEM SHEET 4

1 (a) $$\langle f, g \rangle = \int_0^1 \frac{f(x)\,g(x)}{\sqrt{x}}\,dx$$

$f, g \in C([0,1])$

(i) $\dfrac{f(x)\,g(x)}{\sqrt{x}} \in C((0,1])$

(ii) $|f(x)\,g(x)| \leq M$ on $[0,1]$

(iii) $\left| \dfrac{f(x)\,g(x)}{\sqrt{x}} \right| \leq \dfrac{M}{\sqrt{x}}$

(iv) $\displaystyle\int_0^1 \frac{M}{\sqrt{x}}\,dx$ is convergent

$\Longrightarrow$ $\langle f, g \rangle$ is well-defined

(b)

(i) $\langle \cdot , \cdot \rangle$ is symmetric

(ii) $\langle f , f \rangle \geq 0$ for any f

(iii) if $\langle f , f \rangle = 0$, then $h(x) = \dfrac{[f(x)]^2}{\sqrt{x}}$

defined on $(0,1]$ vanishes on the interval since its integral is zero. So, f must be a zero function.

f is continuous, hence $f(0) = 0$, and we conclude that $f = 0$ over $[0,1]$

Conclusion: $\langle \cdot , \cdot \rangle$ is an inner product.

(c) (i) $\varphi_0(x) = a_0$; $\displaystyle\int_0^1 \frac{\varphi_0(x)^2}{\sqrt{x}}\,dx = 1$

$$\Rightarrow 2a_0^2 = 1 \quad \Rightarrow \varphi_0(x) = \frac{1}{\sqrt{2}}$$

(ii) $\varphi_1(x) = a_1 x + b_1$

$$\int_0^1 \frac{\varphi_0(x)\,\varphi_1(x)}{\sqrt{x}} = 0 \; ; \quad \int_0^1 \frac{\varphi_1(x)^2}{\sqrt{x}}\,dx = 1$$

$$\frac{2}{3} a_1 + 2 b_1 = 0 \implies b_1 = -\frac{a_1}{3}$$

$$\frac{2}{5} a_1 + \frac{4}{3} a_1 b_1 + 2 b_1^2 = 1 \implies \frac{8}{45} a_1^2 = 1$$

$$\implies \varphi_1(x) = \frac{1}{2}\sqrt{\frac{5}{2}}\,(3x - 1)$$

(iii) $\varphi_2(x) = a_2 x^2 + b_2 x + c_2$

There is no easy way out here :

$$\langle \varphi_0, \varphi_2 \rangle = 0 \quad \& \quad \langle \varphi_1, \varphi_2 \rangle = 0$$

$$\implies b_2 = -\frac{6}{7} a_2 \quad ; \quad c_2 = \frac{3}{35} a_2$$

$$\implies \varphi_2(x) = \frac{a_2}{35}\left(35x^2 - 30x + 3\right)$$

From $\langle \varphi_2, \varphi_2 \rangle = 1 \implies a_2 = \frac{3}{16}\sqrt{2}$

Interesting trick :

$$\varphi_2(x)^2 = \frac{a_2^2}{35}\left(35x^2 - 30x + 3\right)\left(x^2 + \lambda_1 \varphi_1(x) + \lambda_0 \varphi_0(x)\right)$$

$\implies$ no need to compute the inner products with $\varphi_1(x)$ and $\varphi_0(x)$

(d) The roots of $\varphi_2(x)$ are

$$x_1 = \frac{15 - 2\sqrt{30}}{35}, \qquad x_2 = \frac{15 + 2\sqrt{30}}{35}$$

We find the weights : Lagrange interpolation, $l_i(x)$

$$w_i = \langle 1, l_i(x) \rangle$$

Quadrature rule : $J(f) = w_1 f(x_1) + w_2 f(x_2)$

$$R(f) = \left| \int_0^1 \frac{f(x)}{\sqrt{x}}\, dx - J(f) \right|$$

$w_1 \simeq 1.304$ See MATLAB Companion.

$w_2 \simeq 0.696$

(e) Error $|R(f)| \leq c_2 M_4(f)$,

where $M_4(f) = \max\limits_{\xi \in [0,1]} |f^{(4)}(\xi)|$

and

$$c_2 = \frac{1}{4!} \int_0^1 \frac{\left[(x-x_1)(x-x_2)\right]^2}{\sqrt{x}}\, dx$$

$$\simeq 4.8 \cdot 10^{-4}$$

Here one has to be careful with indeces : x_1, x_2 vs x_0, x_1, x_2 What is n here ?

② $I = [-1, 1]$

(a) If $(x,y,z) \in K$, then $x^2 \le a^2$

or $-a \le x \le a$ or $x \in aI$;

similarly for y, z

(b)

$$\mathrm{vol}_K = \mathrm{vol}_C \frac{N_B}{N} \longrightarrow \text{points inside}$$

$$\mathrm{vol}_C = 2a\,2b\,2c = 8abc$$

Let $(x,y,z) = (au, bv, cw) \in \mathbb{R}^3$.

Then $(x,y,z) \in C \iff (u,v,w) \in C_B$

and $(x,y,z) \in K \iff (u,v,w) \in B$

The result then follows, since

$$(x,y,z) \in K \cap C \iff (u,v,w) \in B \cap C_B$$