EXERCISE 1 $\qquad \varphi(x) = \frac{1}{2}\left(x + \frac{3}{x}\right)$

(a) $\varphi(\sqrt{3}) = \frac{1}{2}\left(\sqrt{3} + \frac{3}{\sqrt{3}}\right) = \sqrt{3}$

If $\frac{1}{2}\left(x + \frac{3}{x}\right) = x$ then $x^2 + 3 = 2x^2$ or $x^2 = 3$.

Therefore $-\sqrt{3}$ is also a fixed-point.

(c) The idea is to check if $\varphi(x)$ is increasing on $I = (\sqrt{3}, +\infty)$.

$\varphi'(x) = \frac{1}{2}\left(1 - \frac{3}{x^2}\right)$, so $0 < \varphi'(x) < 1$ on I.

If $x \in I$, $\varphi(x) > \varphi(\sqrt{3}) = \sqrt{3}$, $\varphi(x) \in I$, and the convergence results follows.

(d) Suppose $0 < x_0 < \sqrt{3}$.
Then

$$\frac{x_0}{2} + \frac{3}{2x_0} > \frac{1}{x_0} > \sqrt{3}.$$

therefore it remains to check $x_0 > \sqrt{3}$, but that was (c) already.

(e) $x_{k+1} - \sqrt{3} = \frac{1}{2}\left(x_k + \frac{3}{x_k}\right) - \sqrt{3} = \frac{1}{2x_k}\left(x_k^2 + 3 - 2\sqrt{3}x_k\right)$

$= \frac{1}{2x_k}\left(x_k - \sqrt{3}\right)^2$

Order is 2 since

$$\lim_{k\to\infty} \frac{e_{k+1}}{e_k^2} = \frac{1}{2\sqrt{3}}$$

(f) $\varphi(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^2-3}{2x}$

$= x - \frac{1}{2}x + \frac{3}{2}\frac{1}{x} = \frac{1}{2}\left(x + \frac{3}{x}\right)$

EXERCISE 2 $\qquad f(t) = e^t \; ; \; f'(0) \approx (f(h) - f(0))/h$

<u>Motivation</u>: If $x, y \in \mathbb{R}$ are nearly the same, then the computed value of $x-y$, namely $fl(fl(x) - fl(y))$ may have very few digits in common with the best we could hope for, namely $fl(x-y)$.

(a)

It is given that $x = e^h$, $y = 1$, $h = 2^{-n}$.

Notice, that
$$fl(fl(x) - fl(y)) = 0$$
$$fl(x-y) = h = 2^{-n}, \text{ for } n \geq 53.$$

We know that $f'(h) \xrightarrow[h \to 0]{} 1$, let us compute

$$fl\left(\frac{fl(fl(x) - fl(y))}{fl(h)}\right) = fl\left(\frac{fl(fl(x) - y)}{h}\right)$$

since y and h can be expressed exactly.

Using the inequality: $1 + 2^{-n} + 2^{-2n-1} < e^{2^{-n}}$

$$= \begin{cases} fl\left(\left(2^{-25} + 2^{-51}\right) / 2^{-25}\right), & n = 25 \\ fl\left(2^{-n} / 2^{-n}\right), & 26 \leq n \leq 52 \\ fl\left(0 / 2^{-n}\right), & n \geq 53 \end{cases}$$

$$= \begin{cases} fl\left(1 + 2^{-26}\right) \\ fl(1) \\ fl(0) \end{cases} = \begin{cases} 1 + 2^{-26}, & n = 25 \\ 1, & 26 \leq n \leq 52 \\ 0, & n \geq 53 \end{cases}$$

(b)

n	$fl((fl(x)-y)/h)$	$fl((x-y)/h)$	$fl(fl(1+h/2)+fl(h^2/6))$
25	$1+2^{-26}$	$1+2^{-26}+2^{-52}$	$1+2^{-26}+2^{-52}$
26	1	$1+2^{-27}$	$1+2^{-27}$
27	1	$1+2^{-28}$	$1+2^{-28}$
$\vdots$	$\vdots$		
51	1	$1+2^{-52}$	$1+2^{-52}$
52	1	1	1
53	0	1	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$

$$1+2^{-n-1}+2^{-2n-3} < \frac{e^{2^{-n}}-1}{2^{-n}}$$

$$< 1+2^{-n-1}+2^{-2n-2}$$

Notice:

$$fl(h) = h$$
$$fl(h/2) = h/2$$
$$fl(h^2) = h^2$$

The series expansion is remarkably good!

EXERCISE 3

Steffensen's Method

$$x_{n+1} = x_n - \frac{f(x_n)}{g(x_n)}$$

$$g(x_n) = \frac{f(x_n + f(x_n)) - f(x_n)}{f(x_n)}$$

Two function evaluations. Is it worth it?

Set $\beta_n = f(x_n)$

$$g(x_n) = \frac{f(x_n + \beta_n) - f(x_n)}{\beta_n}$$

nobody says you have to replace every instance

$$\stackrel{\text{Taylor}}{=} f'(x_n)\left(1 - \frac{1}{2} h_n f''(x_n) + \Theta(\beta_n^2)\right),$$

where $h_n = -\dfrac{f(x_n)}{f'(x_n)}$ from Newton's method.

Thus, the iteration becomes

$$x_{n+1} = x_n + h_n\left(1 + \frac{1}{2} h_n f''(x_n) + \Theta(\beta_n^2)\right)$$

We have shown before that for Newton's method :

$$x_{n+1} - x_* = \frac{1}{2} \frac{f''(\xi_n)}{f'(x_n)} (x_n - x_*)^2$$

(x_* is the root or exact value)

$$x_{n+1} = x_n \underbrace{- \frac{f(x_n)}{f'(x_n)}}_{h_n} , \quad \text{subtract } x_*$$

$$\Rightarrow h_n = -(x_n - x_*) + \frac{1}{2} \frac{f''(\xi_n)}{f'(x_n)} (x_n - x_*)^2$$

Substituting and taking the limit :

$$\frac{(x_{n+1} - x_*)}{(x_n - x_*)^2} \longrightarrow \frac{1}{2} \frac{f''(x_*)}{f'(x_*)} (1 + f'(x_*))$$

and indeed, we have a second order method!