PROBLEM SHEET 2

1

a) Lagrange:

$$p(x) = \sum_{j=0}^{n} y_j \prod_{i \neq j} \frac{x - x_i}{x_j - x_i}$$

$$= \underbrace{\left(\prod_{i=0}^{n} (x - x_i)\right)}_{\varphi(x)} \sum_{j=0}^{n} y_j \frac{1}{x - x_j} \underbrace{\prod_{i \neq j} \frac{1}{x_j - x_i}}_{w_j}$$

$$= \varphi(x) \sum_{i=0}^{n} \frac{w_i}{x - x_i} y_i$$

(I just changed the summation index.)

b) Barycentric formula:

The famous observation is that $f(x) = 1$ (const) has the representation

$$1 = \sum_{i=0}^{n} L_i(x) = \varphi(x) \sum_{i=0}^{n} \frac{w_i}{x - x_i}$$

(All $y_i \equiv 1$ identically)

So, it follows that $\varphi(x) = 1 \Big/ \sum_{i=0}^{n} \frac{w_i}{x - x_i}$

c) Suppose the weights w_k^{m-1} for the points $x_0, x_1, \ldots, x_{m-1}$ have been computed and a new point x_m is added.

We get :

$$w_k^{m} = \frac{1}{x_k - x_m} w_k^{m-1} \quad , \quad k = 0, 1, \ldots, m-1$$

$$w_m^{m} = \prod_{j=0}^{m-1} \frac{1}{x_m - x_j}$$

Complexity is $\Theta(n)$. (Some textbooks have got this wrong!)

③

a) Since $\ell_p(x_i) = \delta_{pi}$ and $\bar{\ell}_q(y_j) = \delta_{qj}$

(δ is Kronecker's delta)

we obtain for all i, j

$$P(x_i, y_j) = \sum_p \sum_q z_{pq}\, \ell_p(x_i)\, \bar{\ell}_q(y_j) = z_{ij}$$

b) This looks complicated, but we can use the product form:

$$g_p(t) = \left[\sum_q \frac{\bar{w}_q}{(t - y_q)} z_{pq} \right] \Big/ \left[\sum_q \frac{\bar{w}_q}{t - y_q} \right]$$

for all p, $p = 0, \dots, m$.

Therefore, for all i, j

$$P(x_i, y_j) = \left[\sum_p \frac{w_p}{(x_i - x_p)} g_p(y_j) \right] \Big/ \left[\sum_p \frac{w_p}{x_i - x_p} \right]$$

$$= z_{ij}$$