PROBLEM SHEET 5

1 (a) $y(t) = \lambda e^{-150t} - t + 1/3$ is the general solution

Hence,

$$y_\varepsilon(t) = \varepsilon e^{-150t} - t + 1/3, \quad \varepsilon \geq 0$$

(b)

$$|y_0(t) - y_\varepsilon(t)| \leq |\varepsilon e^{-150t}| \leq |\varepsilon|, \quad t \geq 0$$

$$\| y_0(t) - y_\varepsilon(t) \|_\infty \leq |\varepsilon|$$

(c) $y_0(t)$ is the solution of the differential equation

Naturally

$$y_0(t+h) - y_0(t) = h y_0'(t)$$

By substitution:

$$y_0(t+h) = (1 - 150h) y_0(t) + h(49 - 150t)$$

(d) Uniform step size h

Euler's Method

$$u_{\varepsilon,0} = \varepsilon + 1/3$$

$$u_{\varepsilon,i+1} = u_{\varepsilon,i} + h f(t_i, u_{\varepsilon,i})$$

$$= (1 - 150h)\,u_{\varepsilon,i} + h\,(49 - 150\,t_i) \quad \circledast$$

(e) $$u^{\circledast}_{\varepsilon,i+1} - y^{\circledast}_0(t_{i+1}) = (1-150h)\left(u_{\varepsilon,i} - y_0(t_i)\right)$$

Initial step : $u_{\varepsilon,1} - y_0(0) = \varepsilon$

Geometric sequence : $u_{\varepsilon,i} - y_0(t_i) = (1-150h)^{i-1}\,\varepsilon$,

$$i = 1, \ldots, n+1$$

(f) $n = 50$, $\varepsilon = 0.01$

$$\Rightarrow \quad 150h = 3$$

$$\Rightarrow \quad \left| u_{\varepsilon,n+1} - y_0(1) \right| = \left| (-2)^{50} \cdot 0.01 \right|$$

$$\simeq 1.126 \cdot 10^{13}$$

(g) $$\max_{1 \le i \le n+1} \left| u_{\varepsilon,i} - y_0(t_i) \right| \le |\varepsilon|$$

$$-1 \le 1 - 150h \le 1 \quad ; \quad h = 1/n > 0$$

$$n \ge 75$$

(2) (a) The backward Euler gives

$$u_{\varepsilon,0} = \varepsilon + 1/3$$

$$u_{\varepsilon,i+1} = u_{\varepsilon,i} + h f(t_{i+1}, u_{\varepsilon,i+1})$$

$$= u_{\varepsilon,i} + h\left(49 - 150(t_i + h) - 150\, u_{\varepsilon,i+1}\right)$$

We get

$$\circledast \qquad (1 + 150h)\, u_{\varepsilon,i+1} = u_{\varepsilon,i} + h\left(49 - 150(t_i + h)\right)$$

(b) As before

$$y_0(t+h) = y_0(t) + h\, y_0'(t+h)$$

substituting and rearranging we get

$$\circledast \qquad (1 + 150h)\, y_0(t) = y_0(t) + h\left(49 - 150(t+h)\right)$$

Therefore, $\circledast - \circledast$ leads to

$$u_{\varepsilon,i} - y_0(t_i) = \frac{\varepsilon}{(1+150h)^{i-1}}, \quad i = 1, \dots, n+1$$

(c) This is a trick question, since for any n, we have

$$\max_{1 \le i \le n+1} | u_{\varepsilon,i} - y_0(t_i) | \le |\varepsilon| .$$