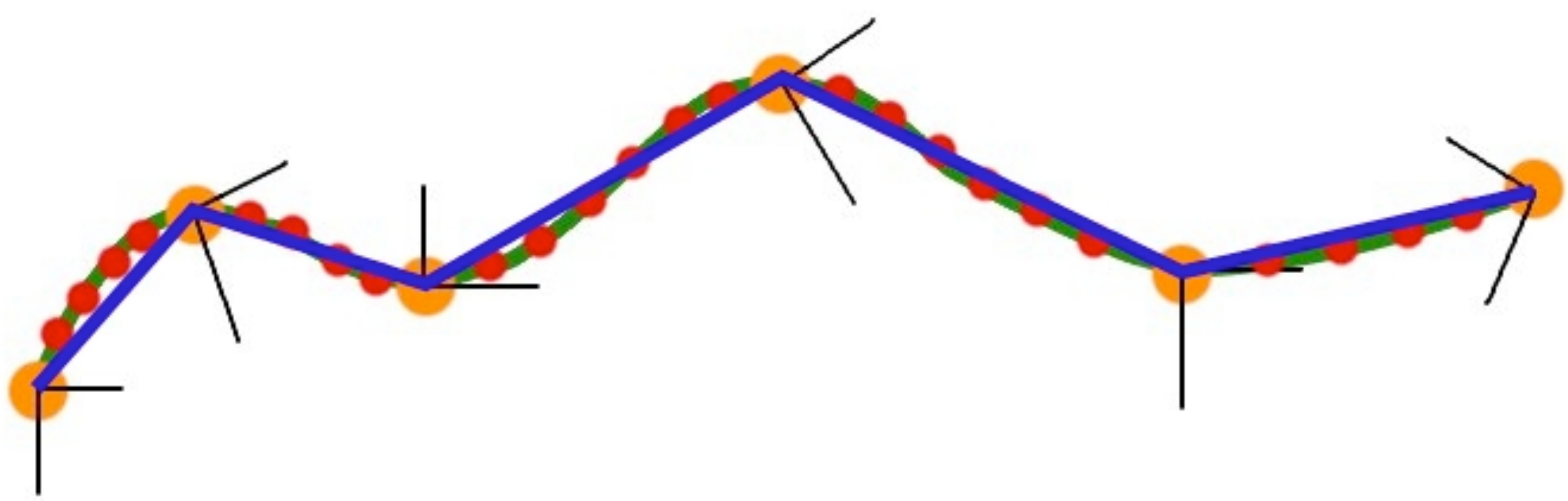
Curves and Splines

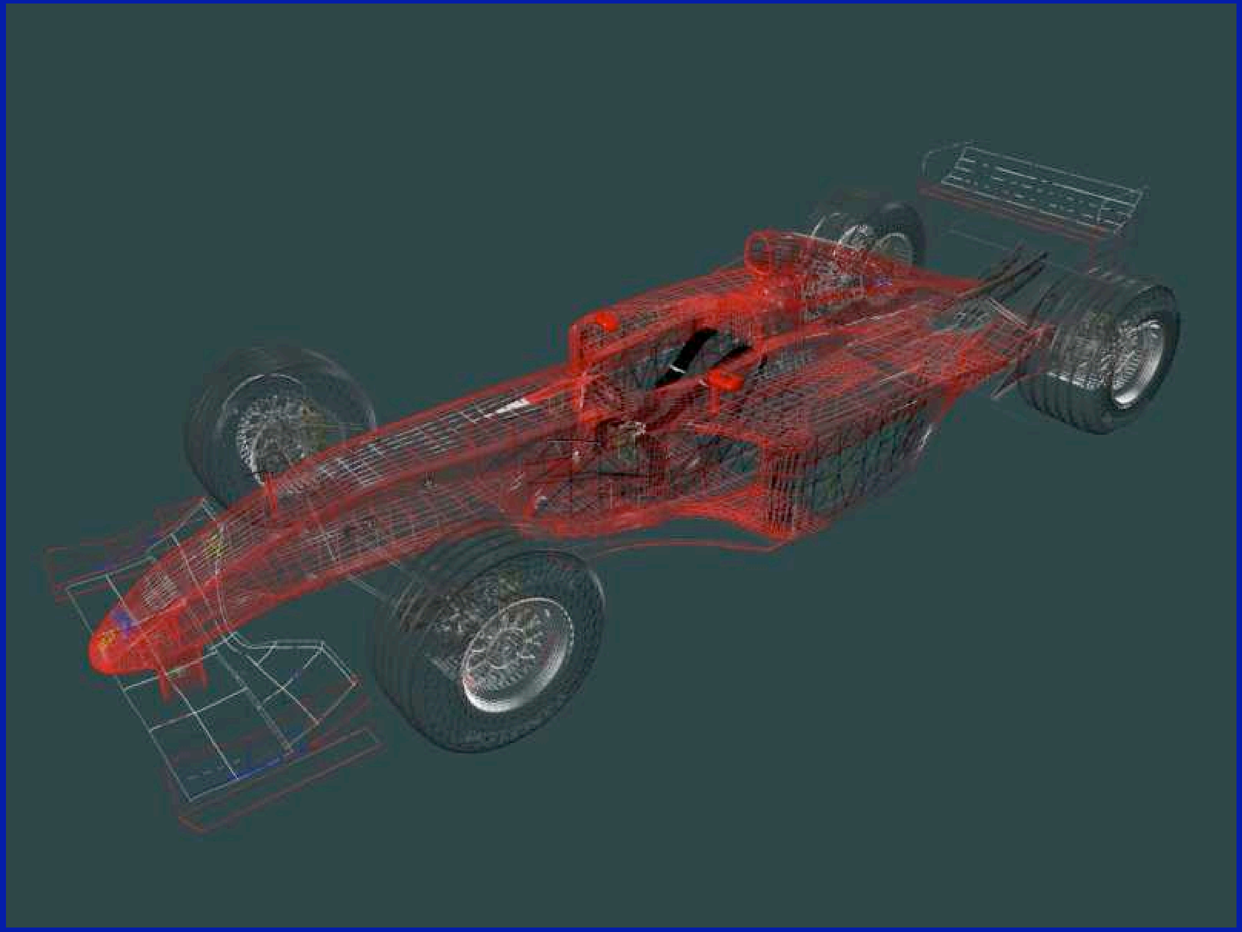


Outline

- Hermite Splines
- Catmull-Rom Splines
- Bezier Curves
- Higher Continuity: Natural and B-Splines
- Drawing Splines

Modeling Complex Shapes

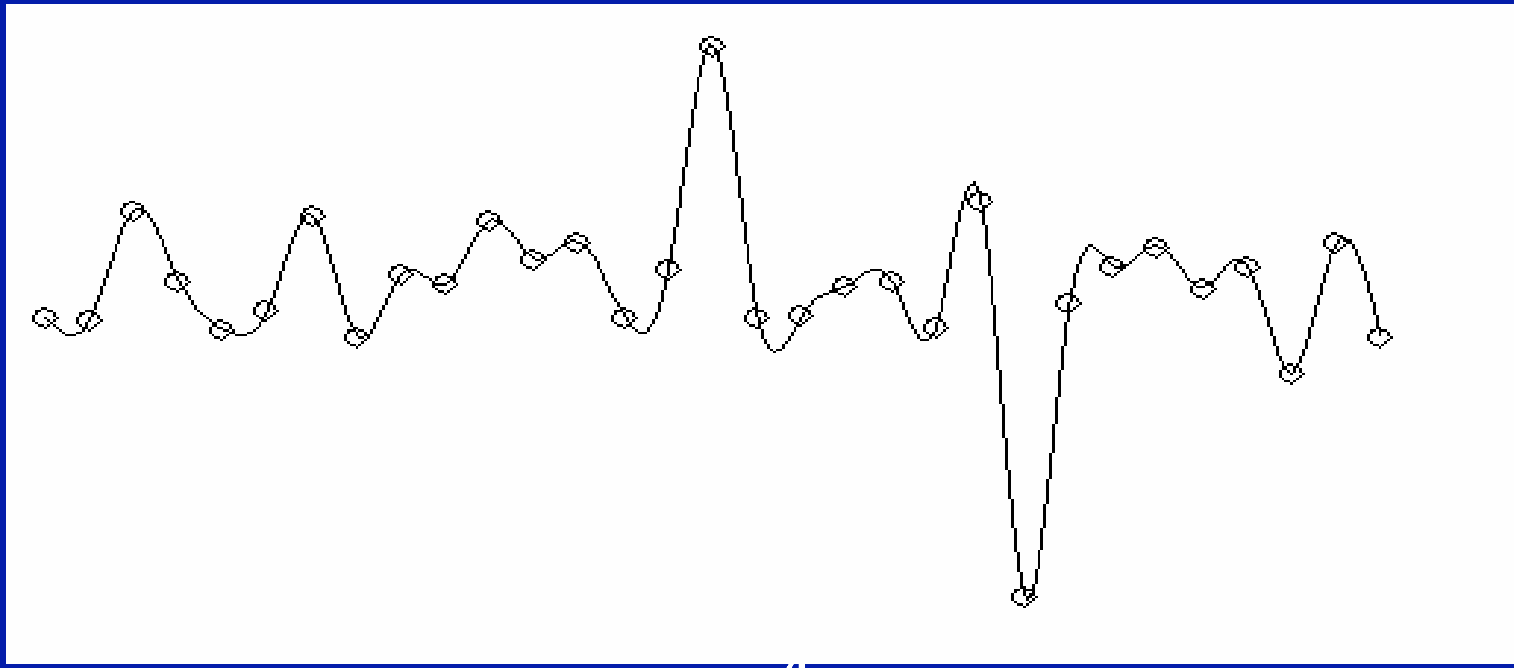
- We want to build models of very complicated objects
- An equation for a sphere is possible, but how about an equation for a telephone, or a face?
- Complexity is achieved using simple pieces
  - polygons, parametric curves and surfaces, or implicit curves and surfaces
  - This lecture: parametric curves



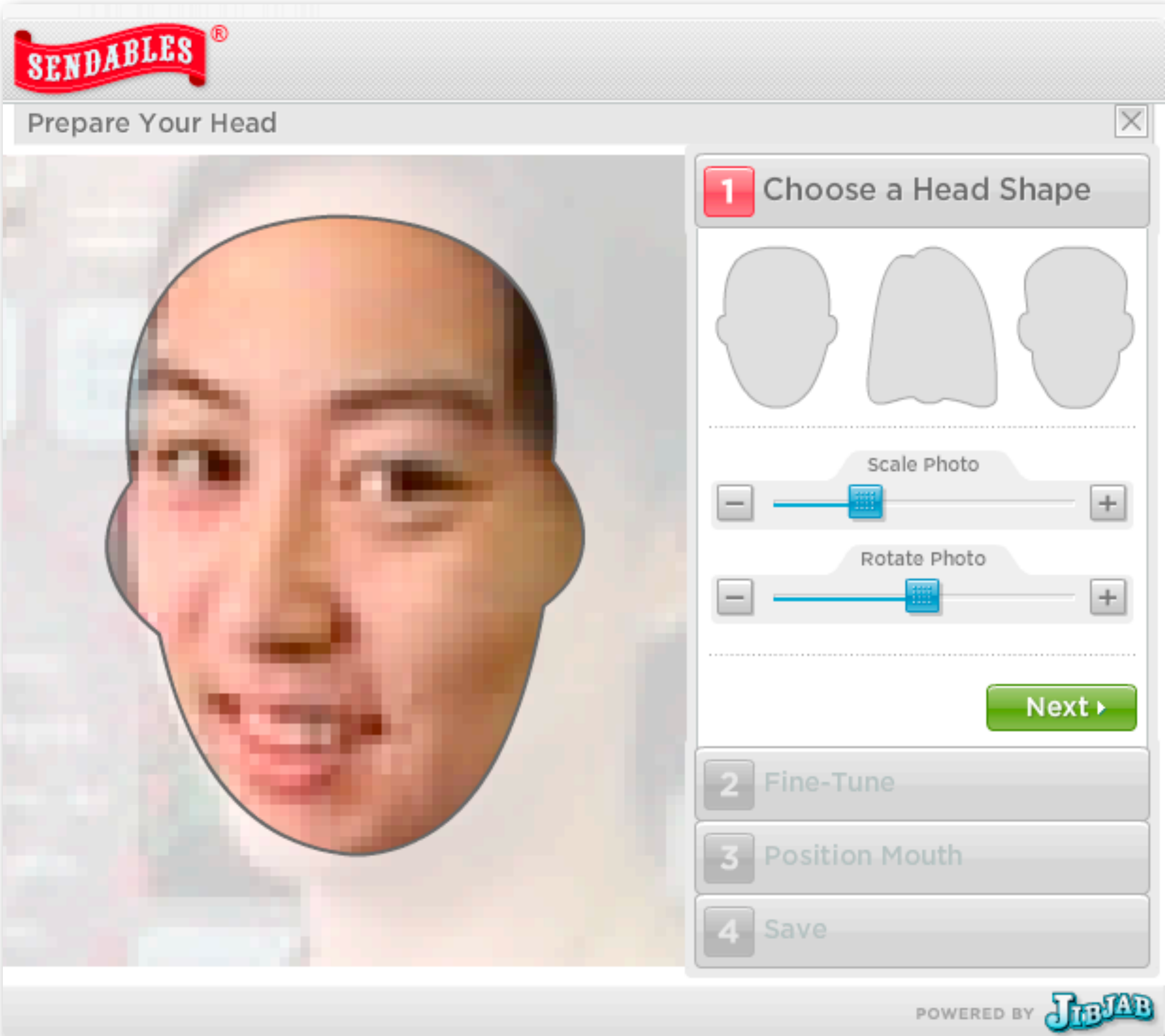
What Do We Need From Curves in Computer Graphics?

- Local control of shape (so that easy to build and modify)
- Stability
- Smoothness and continuity
- Ability to evaluate derivatives
- Ease of rendering

Demo



Curve Usage Demo

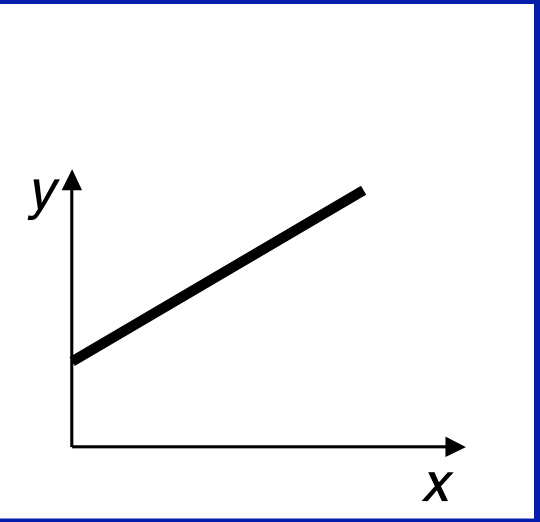


Curve Representations

- **Explicit: y = f(x)**

$$y = mx + b$$

  - Easy to generate points
  - Must be a function: big limitation—vertical lines?



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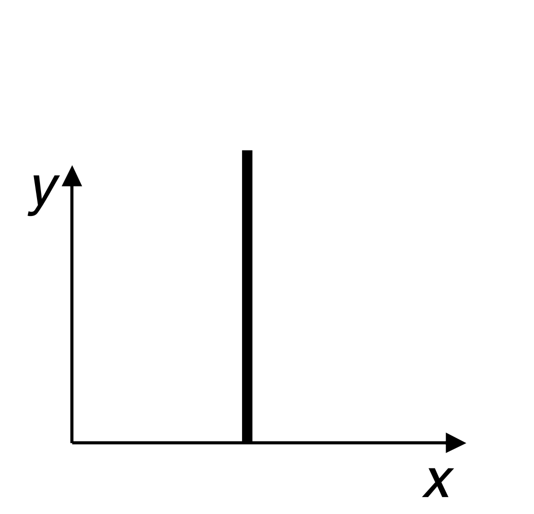
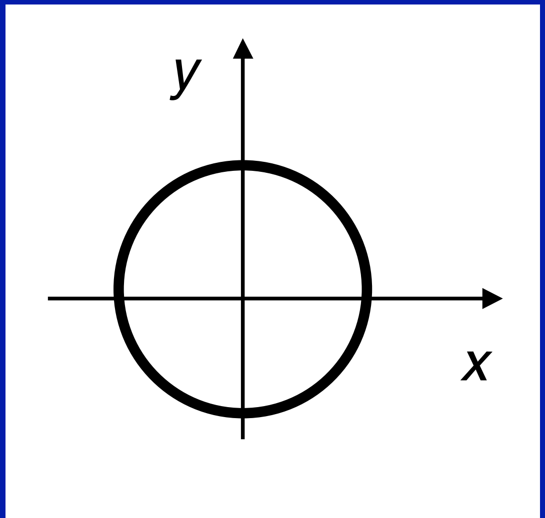


- **Implicit: f(x,y) = 0**

$$x^2 + y^2 - r^2 = 0$$

  +Easy to test if on the curve

  –Hard to generate points



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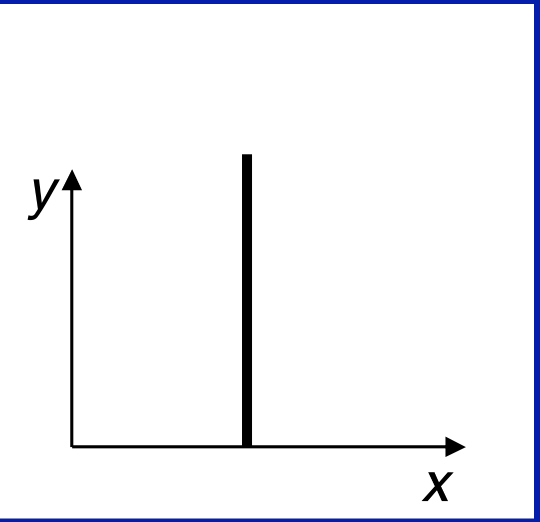

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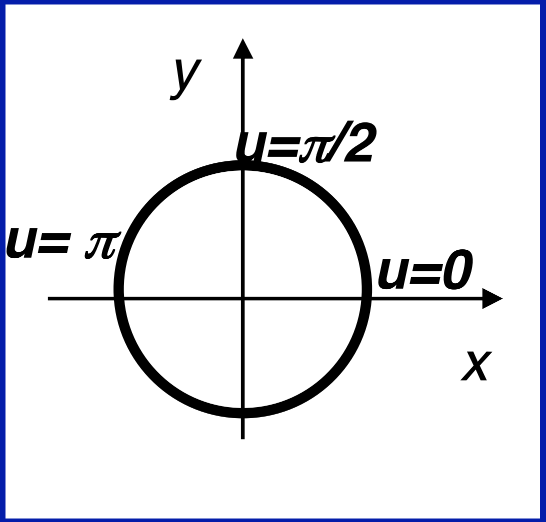
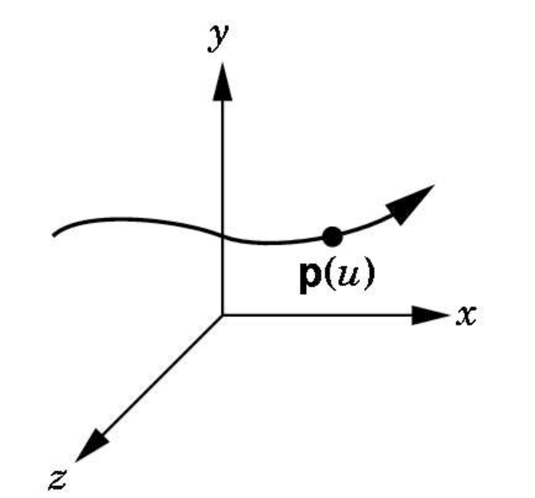


- **Parametric: (x,y) = (f(u), g(u))**

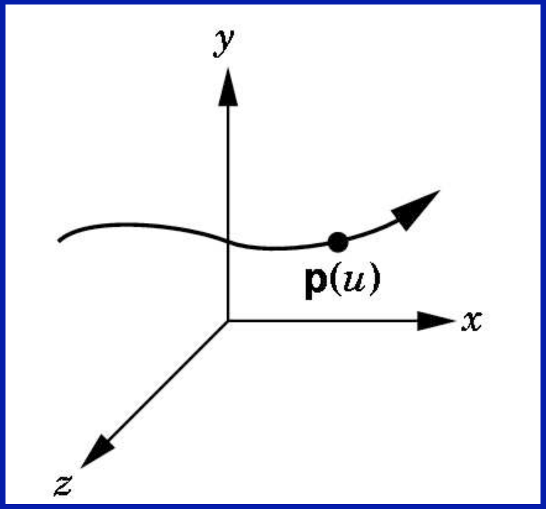
$$(x, y) = (\cos u, \sin u)$$

+Easy to generate points



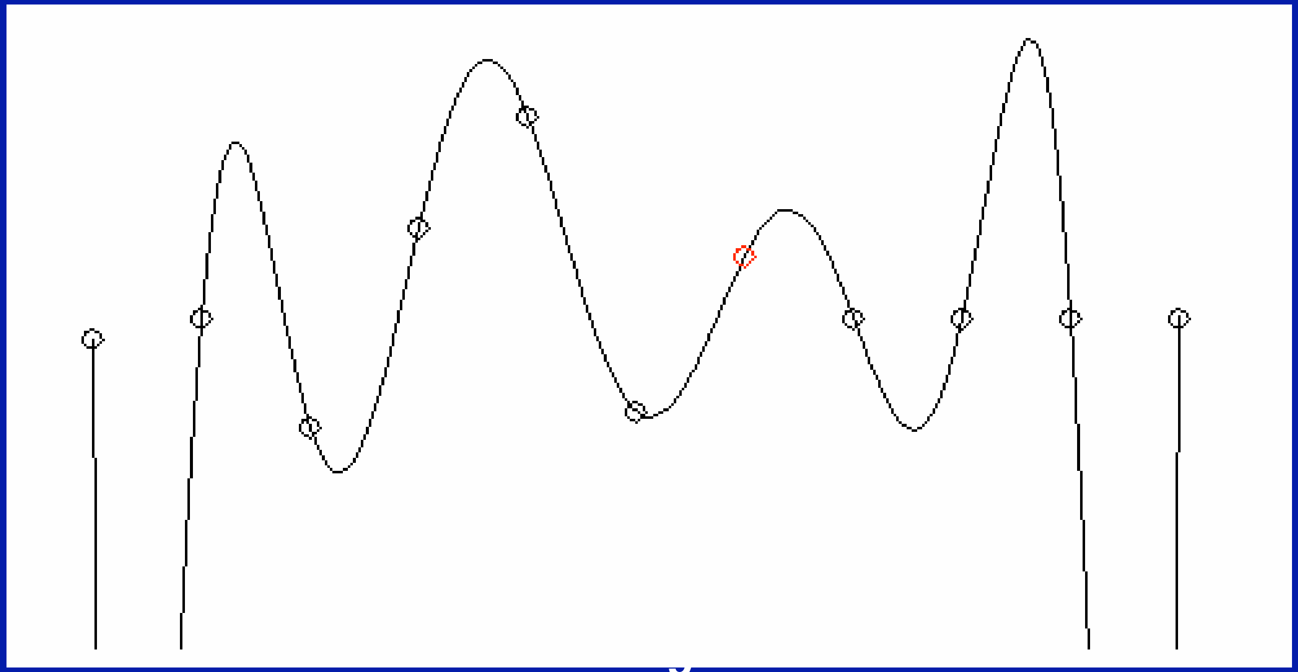
Parameterization of a Curve

- *Parameterization* of a curve: how a change in u moves you along a given curve in xyz space.

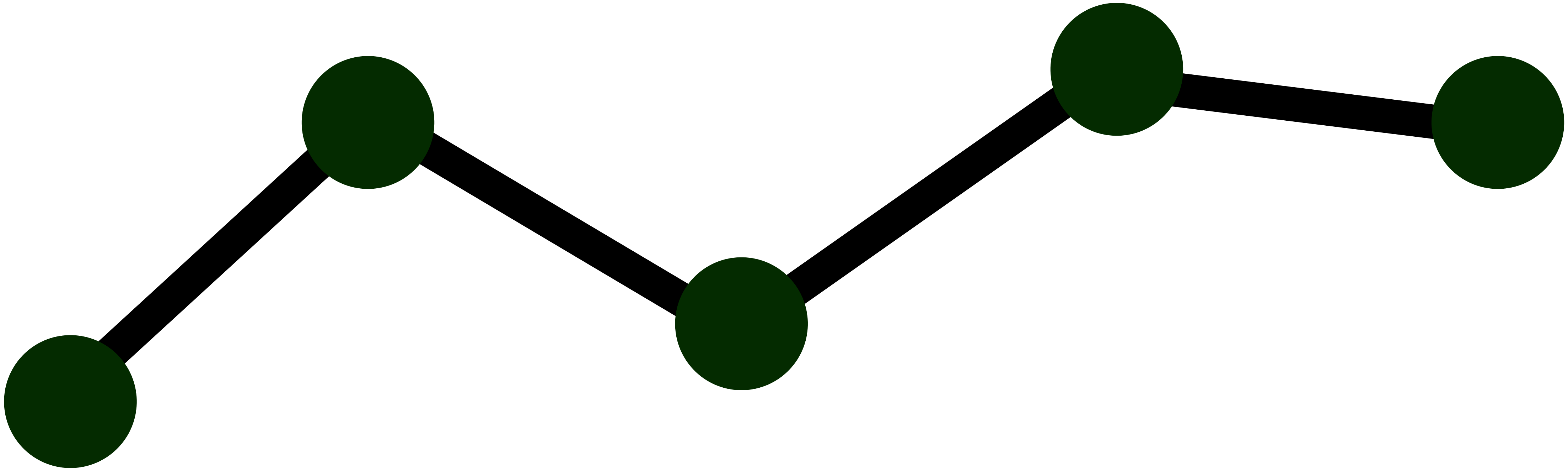


Polynomial Interpolation

- An *n*-th degree polynomial fits a curve to n+1 points
  - called Lagrange Interpolation
  - result is a curve that is too wiggly, change to any control point affects entire curve (nonlocal) – *this method is poor*
- We usually want the curve to be as smooth as possible
  - minimize the wiggles
  - high-degree polynomials are bad

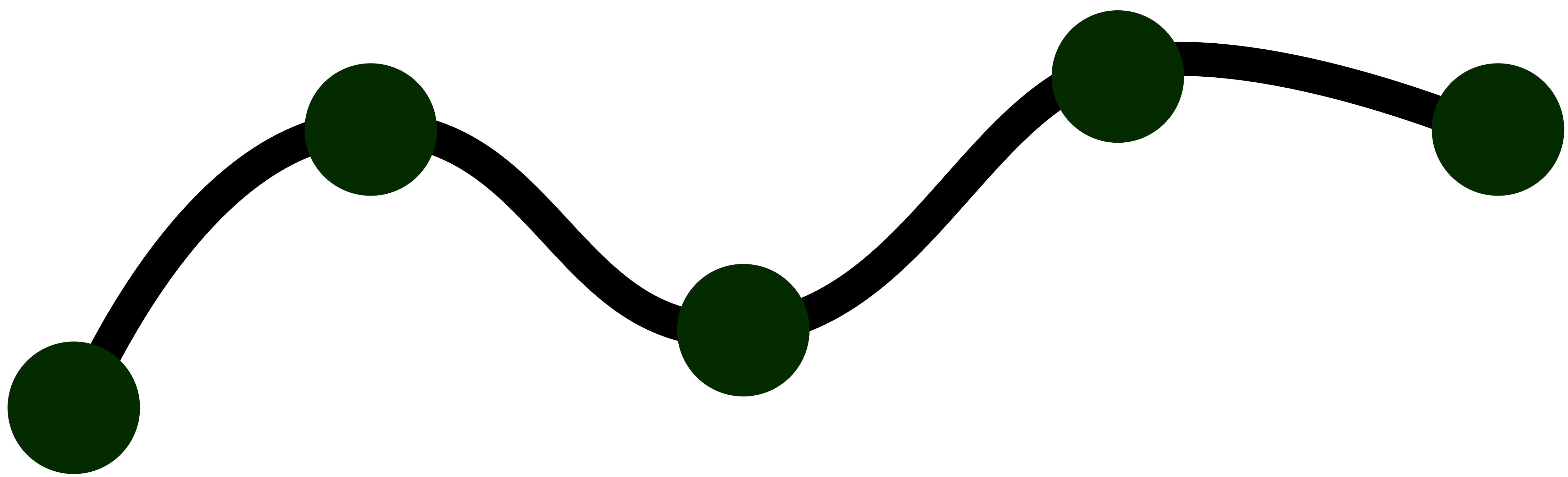


Linear Interpolation

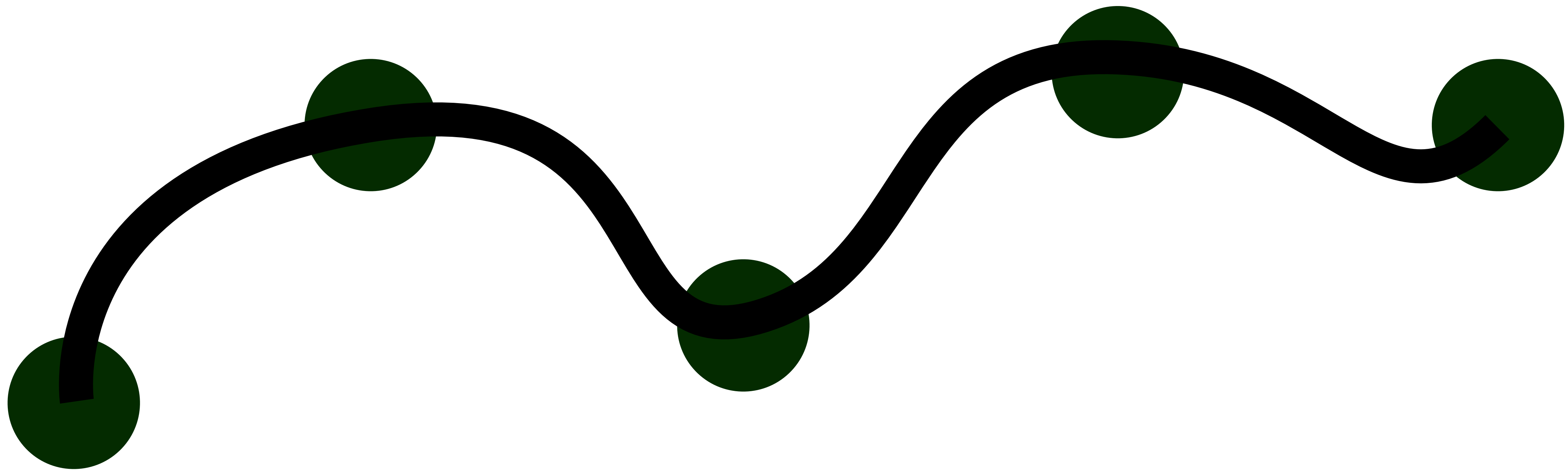


Chalkboard

Spline Interpolation

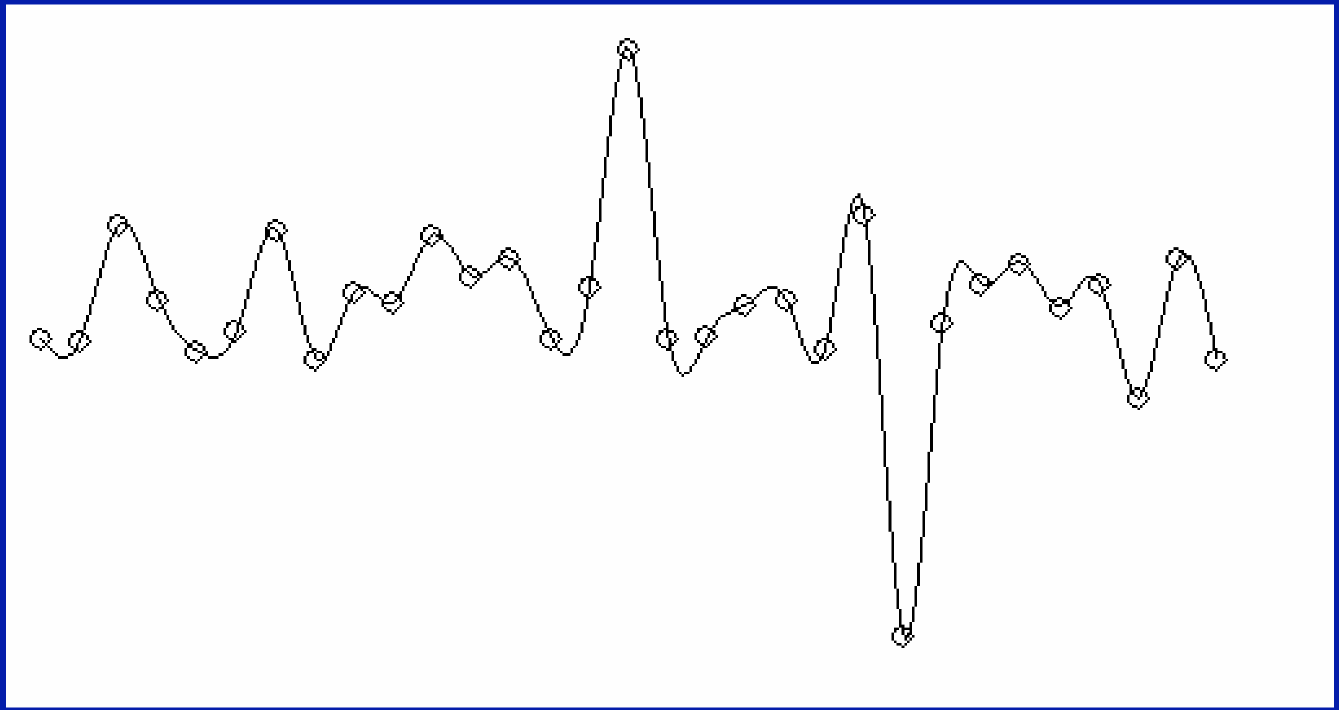


Spine Interpolation Demo



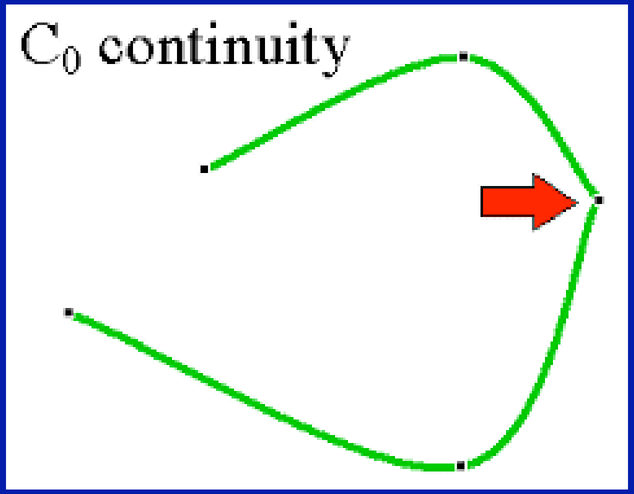
Splines: Piecewise Polynomials

- **A spline is a *piecewise polynomial* - many low degree polynomials are used to interpolate (pass through) the control points**
- ***Cubic piecewise* polynomials are the most common:**
  - **piecewise definition gives local control**

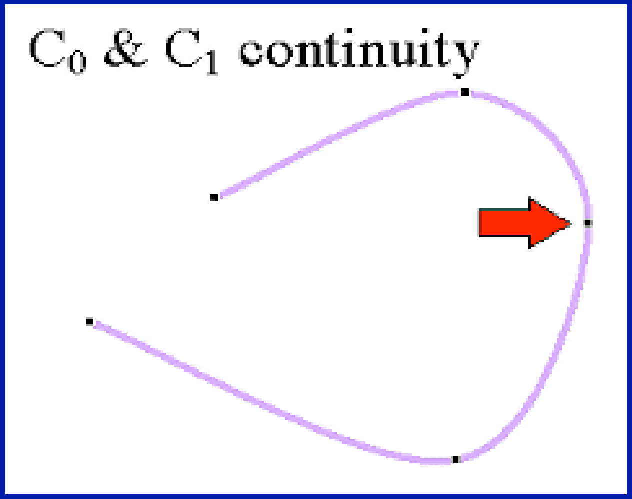


Piecewise Polynomials

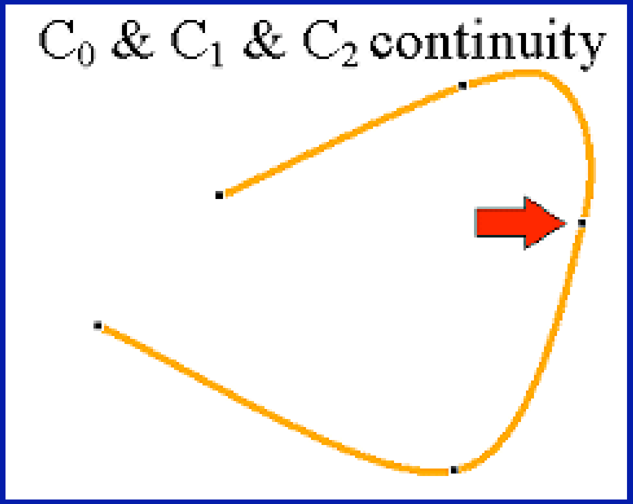
- Spline: lots of little polynomials pieced together
- Want to make sure they fit together nicely



Continuous in position



Continuous in position and tangent vector



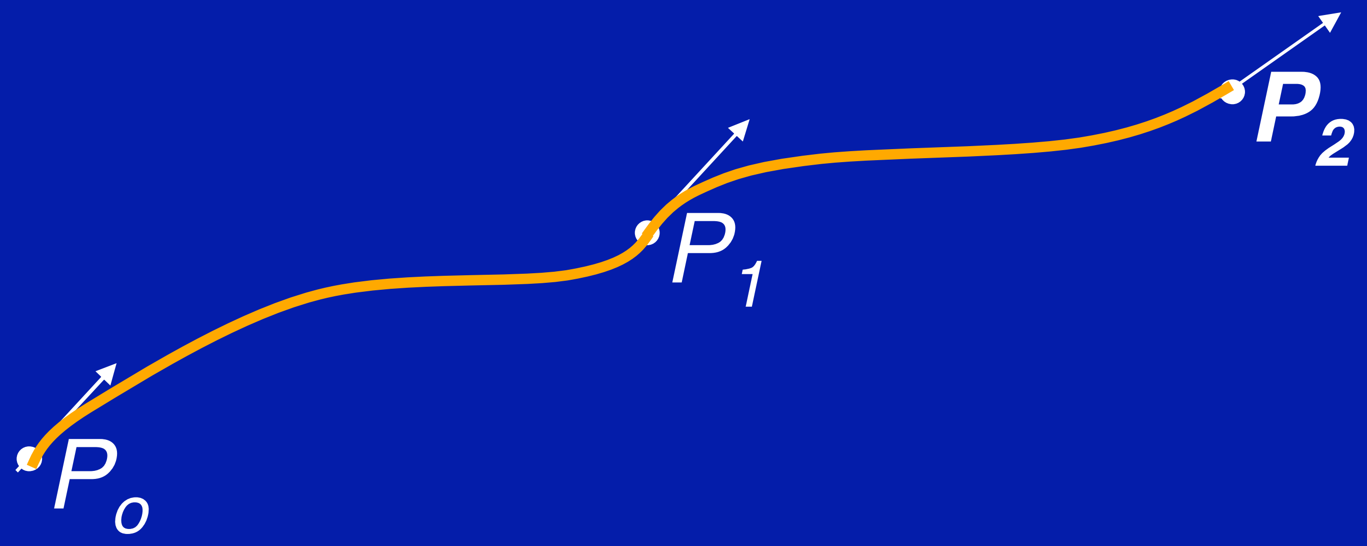
Continuous in position, tangent, and curvature

Splines

- Types of splines:
  - Hermite Splines
  - Catmull-Rom Splines
  - Bezier Splines
  - Natural Cubic Splines
  - B-Splines
  - NURBS

Hermite Curves

- **Cubic Hermite Splines**



That is, we want a way to specify the end points and the slope at the end points!

Splines

chalkboard

The Cubic Hermite Spline Equation

- Using some algebra, we obtain:

$$p(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \nabla p_1 \\ \nabla p_2 \end{bmatrix}$$

point that gets drawn

basis

control matrix (what the user gets to pick)

- This form typical for splines
  – basis matrix and meaning of control matrix change with the spline type

The Cubic Hermite Spline Equation

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control matrix (what the user gets to pick)

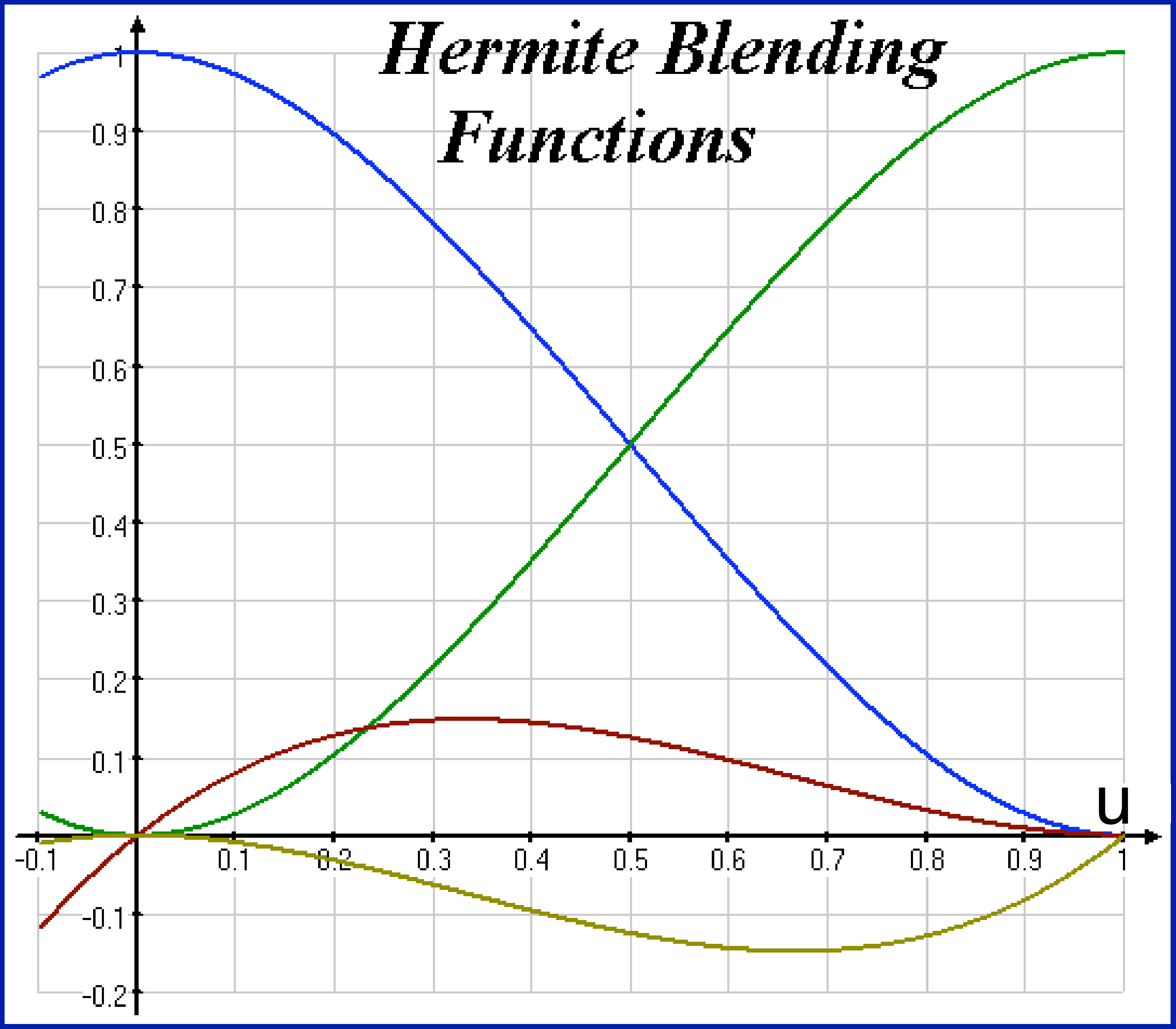
$$p(u) = \begin{bmatrix} 2u^3 - 3u^2 + 1 \\ -2u^3 + 3u^2 \\ u^3 - 2u^2 + u \\ u^3 - u^2 \end{bmatrix}^T \begin{bmatrix} p_1 \\ p_2 \\ \nabla p_1 \\ \nabla p_2 \end{bmatrix}$$

4 Basis Functions

Four Basis Functions for Hermite splines

$$p(u) = \begin{bmatrix} 2u^3 - 3u^2 + 1 \\ -2u^3 + 3u^2 \\ u^3 - 2u^2 + u \\ u^3 - u^2 \end{bmatrix}^T \begin{bmatrix} p_1 \\ p_2 \\ \nabla p_1 \\ \nabla p_2 \end{bmatrix}$$

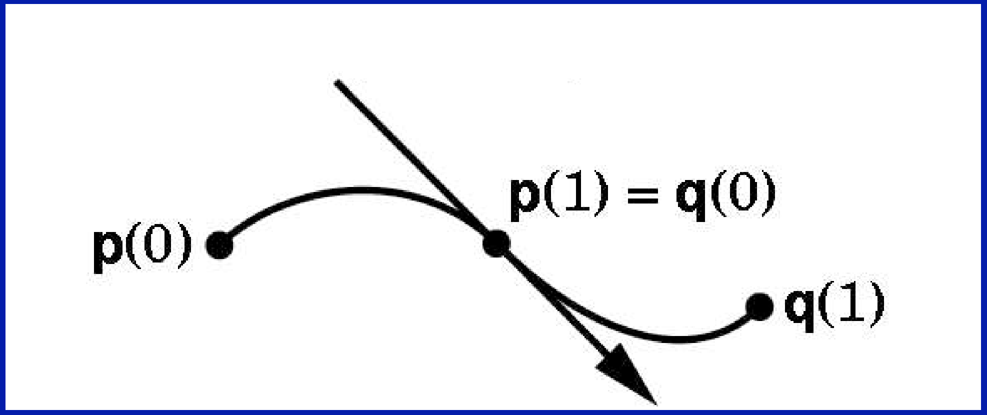
4 Basis Functions



Every cubic Hermite spline is a linear combination (blend) of these 4 functions

Piecing together Hermite Curves

- It's easy to make a multi-segment Hermite spline
  - each piece is specified by a cubic Hermite curve
  - just specify the position and tangent at each “joint”
  - the pieces fit together with matched positions and first derivatives
  - gives C1 continuity
- The points that the curve has to pass through are called *knots* or *knot points*



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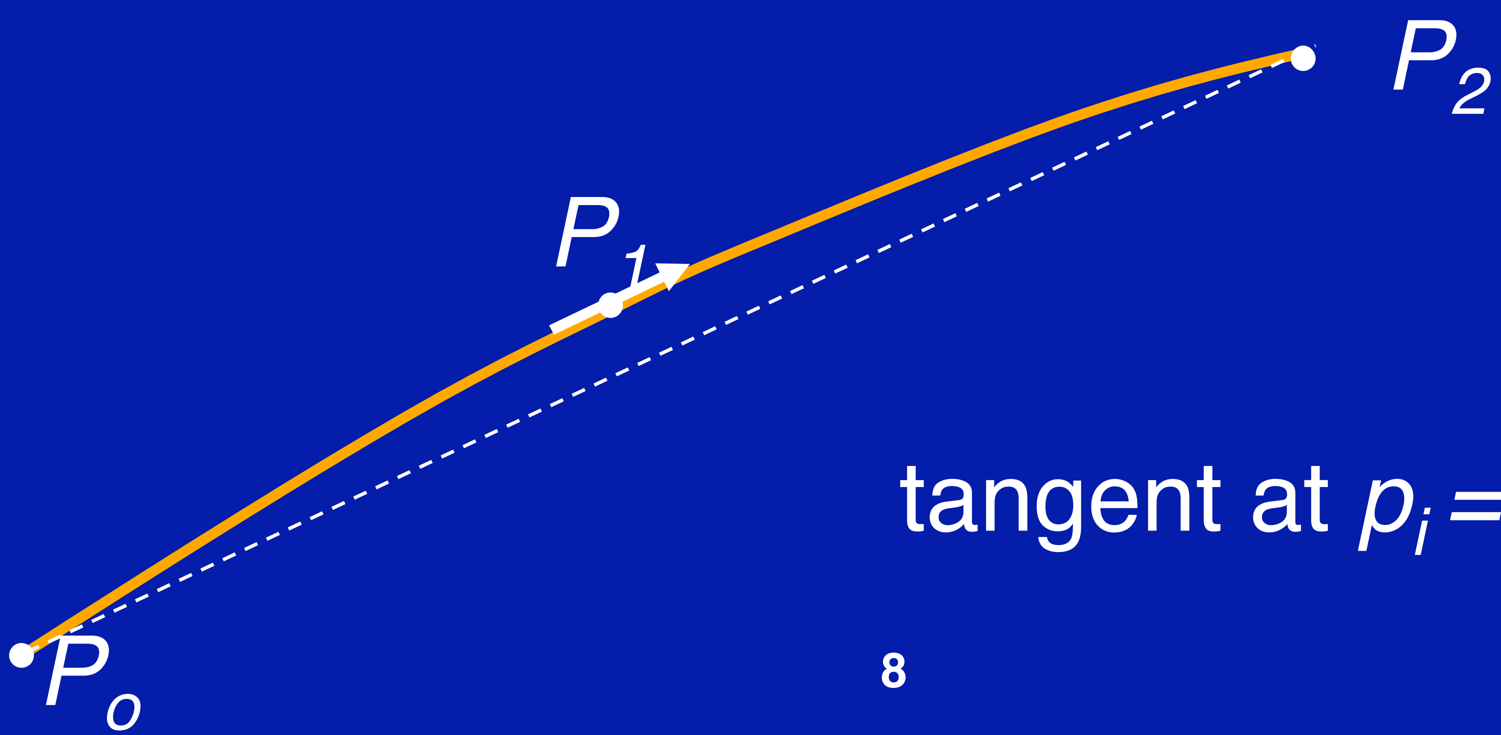
Problem with Hermite Splines?

- Must explicitly specify derivatives at each endpoint!
- How can we solve this?

Catmull-Rom Splines

- With Hermite splines, the designer must specify all the tangent vectors
- Catmull-Rom: an interpolating cubic spline with *built-in C^1 continuity*.



tangent at $p_i = s(p_{i+1} - p_{i-1})$

Catmull-Rom Splines

- **With Hermite splines, the designer must arrange for consecutive tangents to be collinear, to get C^1 continuity. This gets tedious.**
- **Catmull-Rom: an interpolating cubic spline with *built-in C^1 continuity*.**

chalkboard

Catmull-Rom Spline Matrix

$$p(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} -s & 2-s & s-2 & s \\ 2s & s-3 & 3-2s & -s \\ -s & 0 & s & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}$$

spline coefficients **CR basis** **control vector**

- **Derived similarly to Hermite**
- **Parameter s is typically set to s=1/2.**

Catmull-Rom Spline Matrix

$$\begin{bmatrix} x & y & z \end{bmatrix} = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} -s & 2-s & s-2 & s \\ 2s & s-3 & 3-2s & -s \\ -s & 0 & s & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{bmatrix}$$

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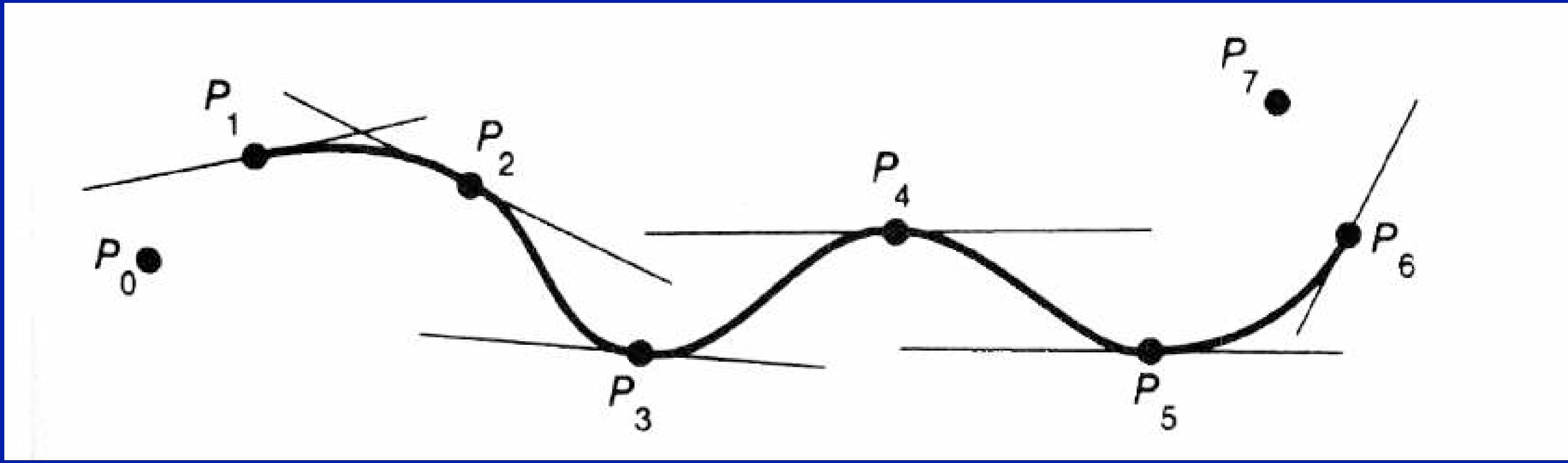

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- **Derived similarly to Hermite**
- **Parameter s is typically set to s=1/2.**

Cubic Curves in 3D

- **Three cubic polynomials, one for each coordinate**
  - $x(u) = a_x u^3 + b_x u^2 + c_x u + d_x$
  - $y(u) = a_y u^3 + b_y u^2 + c_y u + d_y$
  - $z(u) = a_z u^3 + b_z u^2 + c_z u + d_z$

- **In matrix notation**

$$\begin{bmatrix} x(u) & y(u) & z(u) \end{bmatrix} = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{bmatrix}$$

Catmull-Rom Spline Matrix in 3D

$$\begin{bmatrix} x(u) & y(u) & z(u) \end{bmatrix} = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \underbrace{\begin{bmatrix} -s & 2-s & s-2 & s \\ 2s & s-3 & 3-2s & -s \\ -s & 0 & s & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{\text{CR basis}} \underbrace{\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{bmatrix}}_{\text{control vector}}$$

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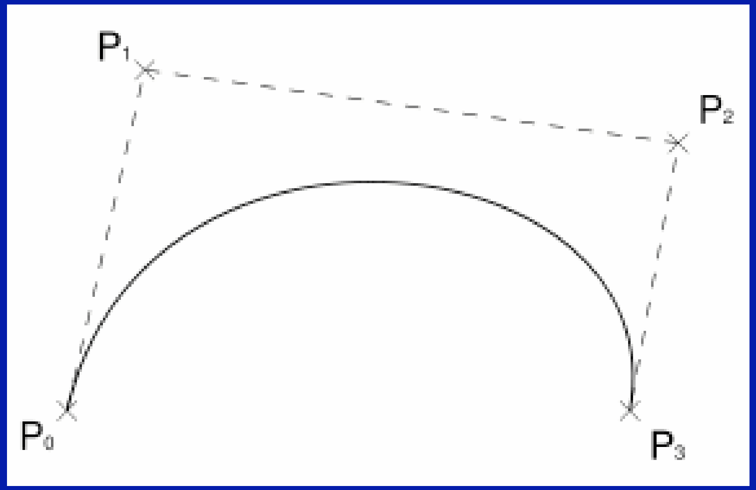
Problem with Catmull-Rom Splines?

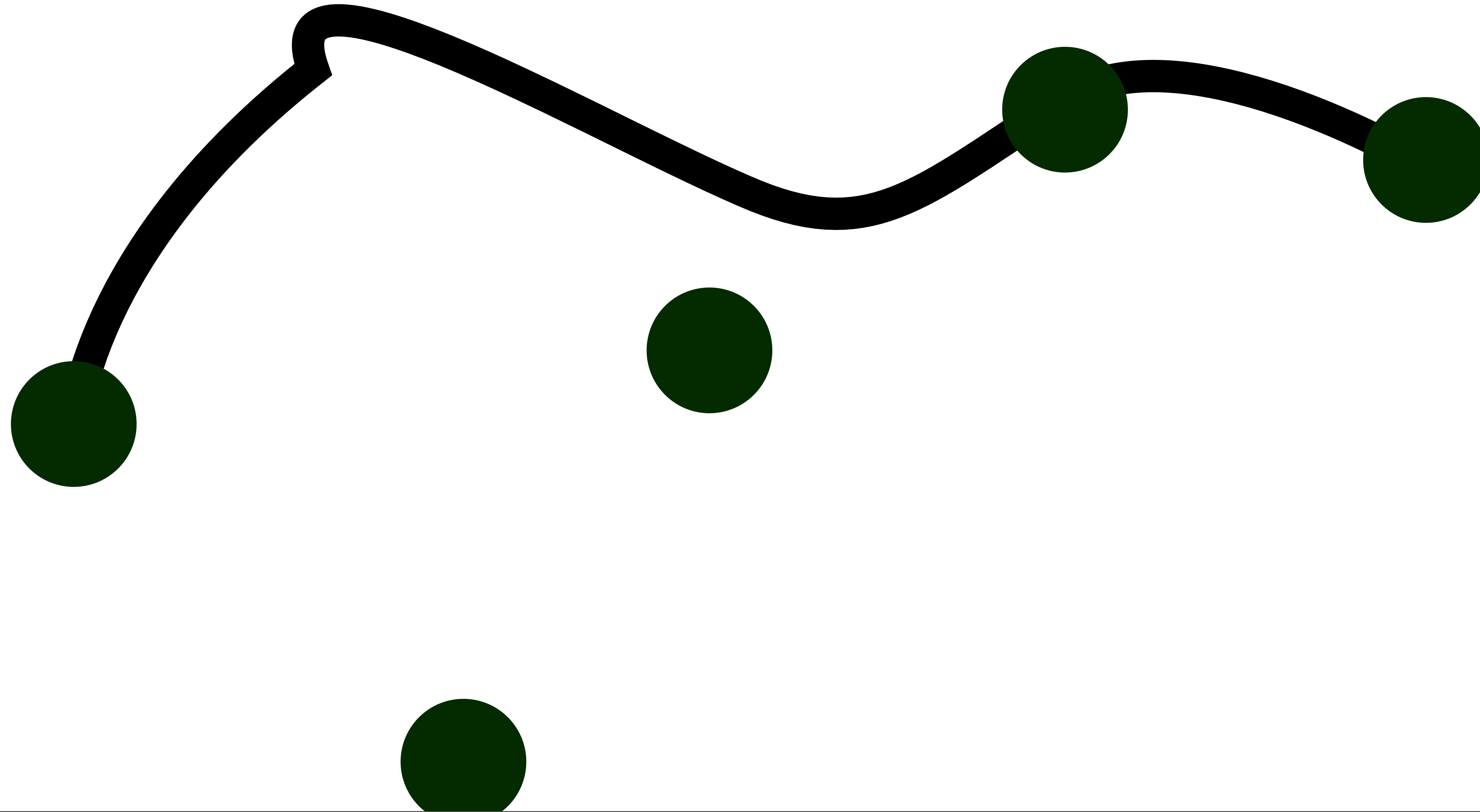
- *No* control of derivatives at endpoints!
- How can we solve this?
  - We want something intuitive.

Bezier Curves*

- Another variant of the same game
- Instead of endpoints and tangents, four control points
  - points P0 and P3 are on the curve: P(u=0) = P0, P(u=1) = P3
  - points P1 and P2 are off the curve
  - P'(u=0) = 3(P1-P0), P'(u=1) = 3(P3 – P2)
- Convex Hull property
  - curve contained within convex hull of control points
- Gives more control knobs (series of points) than Hermite
- Scale factor (3) is chosen to make "velocity" approximately constant



Bezier Spline Example



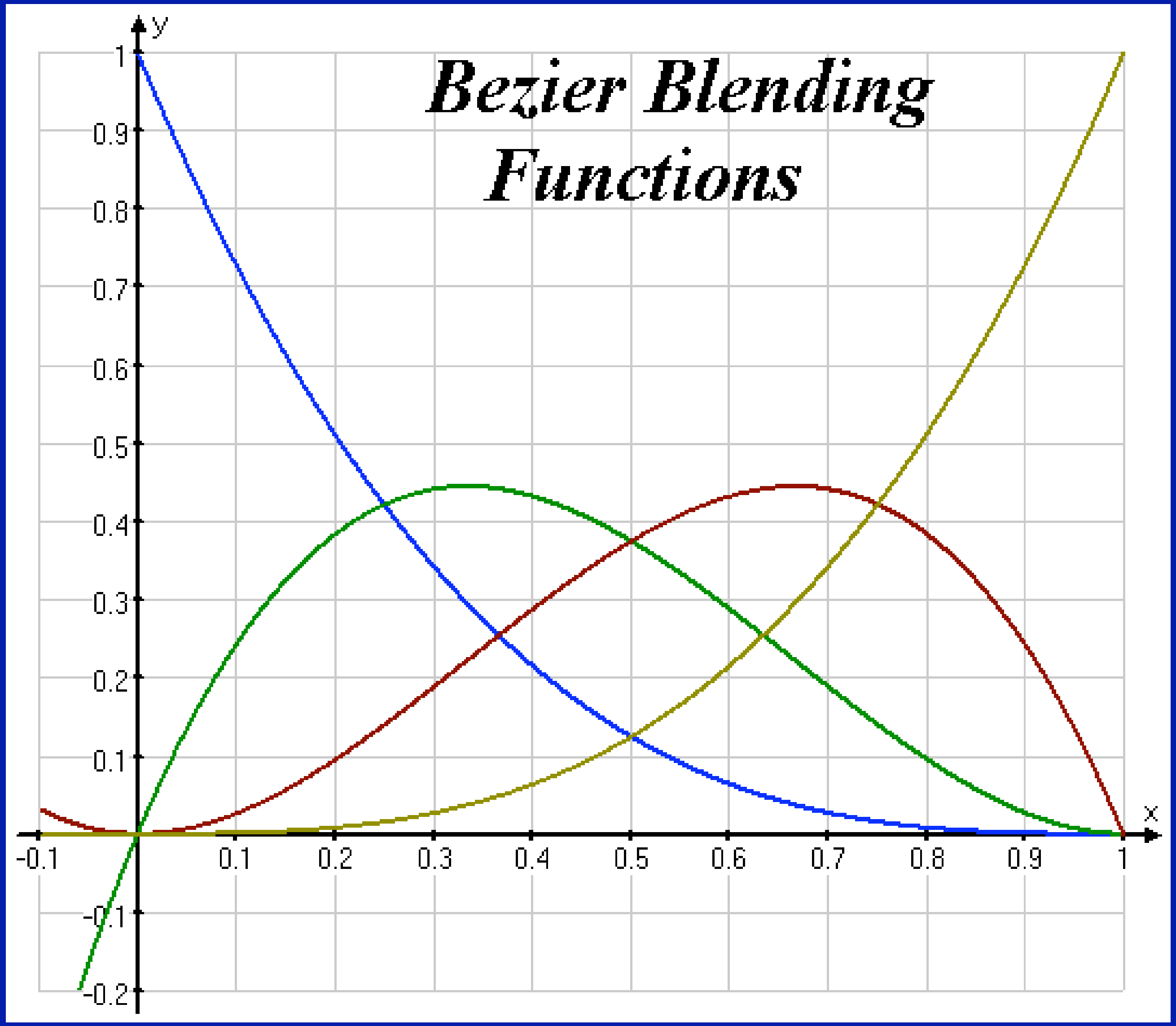
The Bezier Spline Matrix*

$$
\begin{bmatrix} x & y & z \end{bmatrix} = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{bmatrix}
$$

Bezier basis

Bezier control vector

Bezier Blending Functions*



$$p(t) = \begin{bmatrix} (1-t)^3 \\ 3t(1-t)^2 \\ 3t^2(1-t) \\ t^3 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}$$

- Also known as the order 4, degree 3 Bernstein polynomials
- Nonnegative, sum to 1
- The entire curve lies inside the polyhedron bounded by the control points

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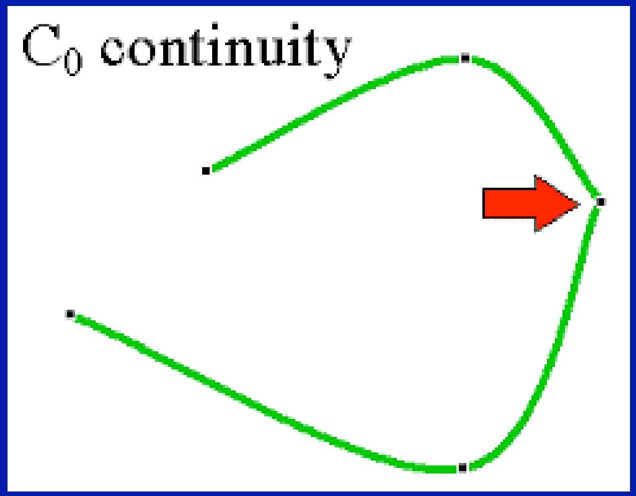
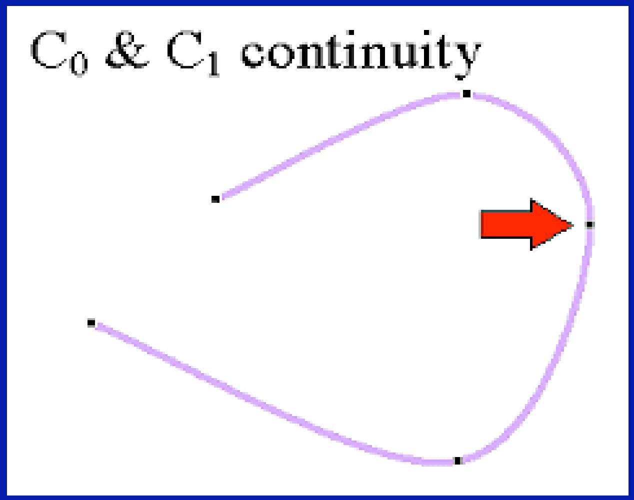
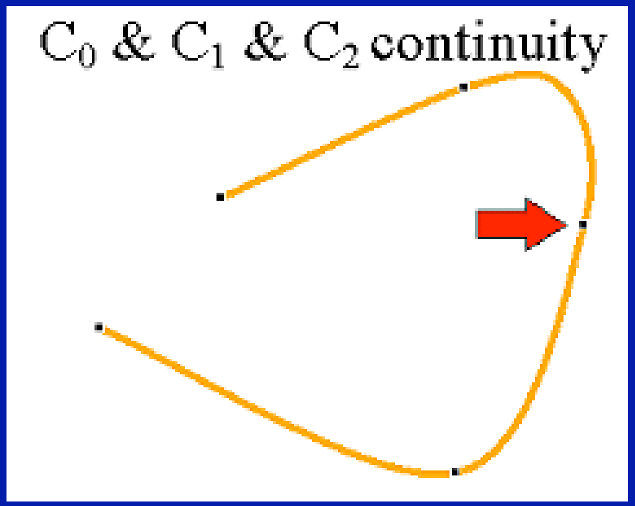
- Want to make sure they fit together nicely



Continuous in position



Continuous in position and tangent vector



Continuous in position, tangent, and curvature

Splines with More Continuity?

- How could we get C^2 continuity at control points?

- Possible answers:
  - Use higher degree polynomials

    degree 4 = quartic, degree 5 = quintic, … but these get computationally expensive, and sometimes wiggly
  - Give up local control   natural cubic splines

    A change to any control point affects the entire curve
  - Give up interpolation   cubic B-splines

    Curve goes near, but not through, the control points

Comparison of Basic Cubic Splines

Type	Local Control	Continuity	Interpolation
Hermite	YES	C1	YES
Bezier	YES	C1	YES
Catmull-Rom	YES	C1	YES
Natural	NO	C2	YES
B-Splines	YES	C2	NO

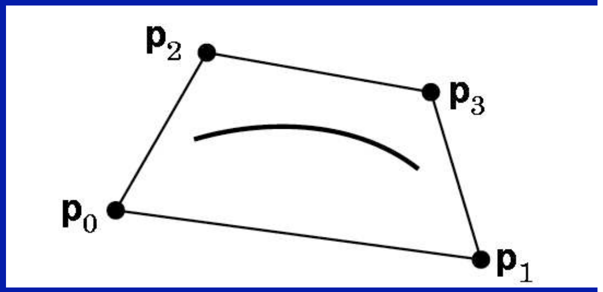
- Summary
  - Can't get C2, interpolation and local control with cubics

Natural Cubic Splines*

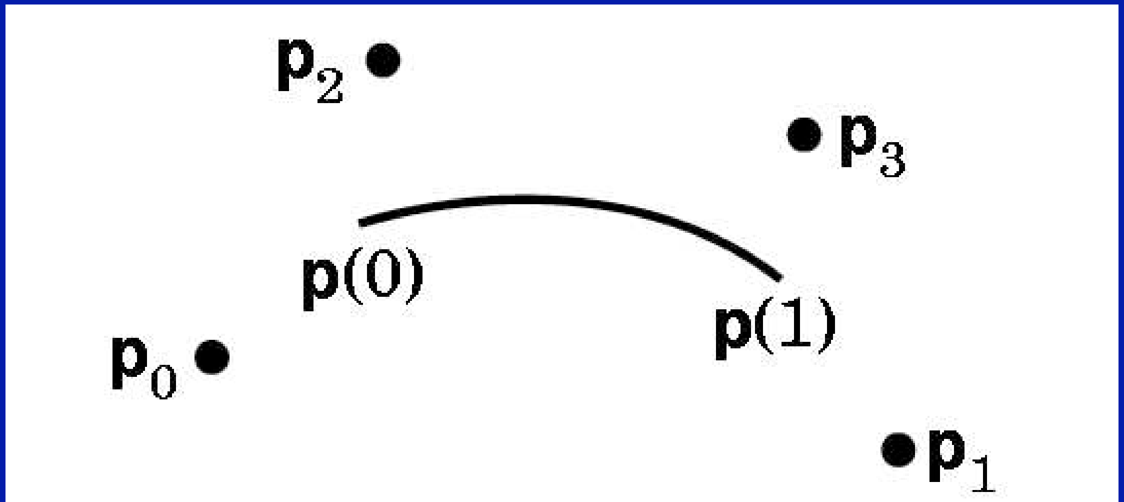
- **If you want 2nd derivatives at joints to match up, the resulting curves are called *natural cubic splines***
- **It's a simple computation to solve for the cubics' coefficients. (See *Numerical Recipes in C* book for code.)**
- **Finding all the right weights is a *global* calculation (solve tridiagonal linear system)**

B-Splines*

- **Give up interpolation**
  - **the curve passes *near* the control points**
  - **best generated with interactive placement (because it's hard to guess where the curve will go)**



- **Curve obeys the convex hull property**
- **C2 continuity and local control are good compensation for loss of interpolation**

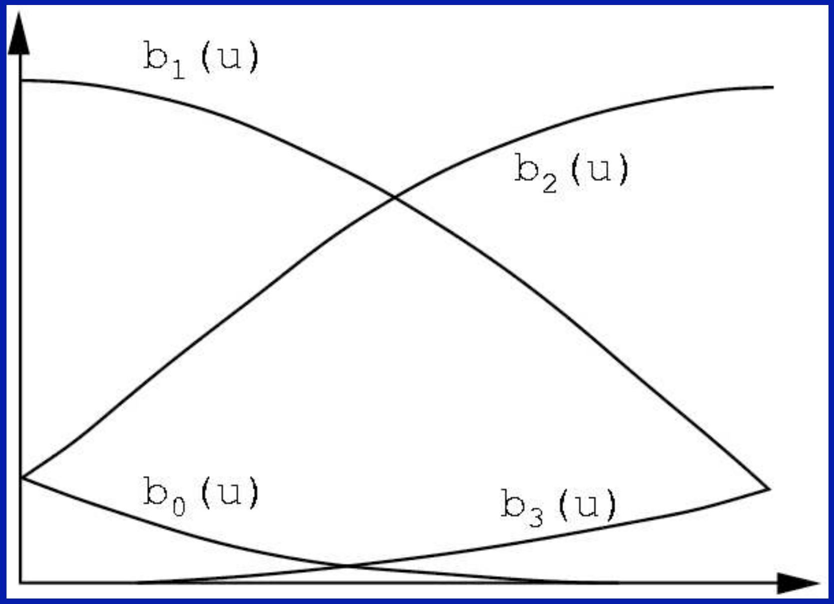


B-Spline Basis*

- We always need 3 more control points than spline pieces

$$M_{Bs} = \frac{1}{6}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}$$

$$G_{Bsi} = \begin{bmatrix} P_{i-3} \\ P_{i-2} \\ P_{i-1} \\ P_i \end{bmatrix}$$



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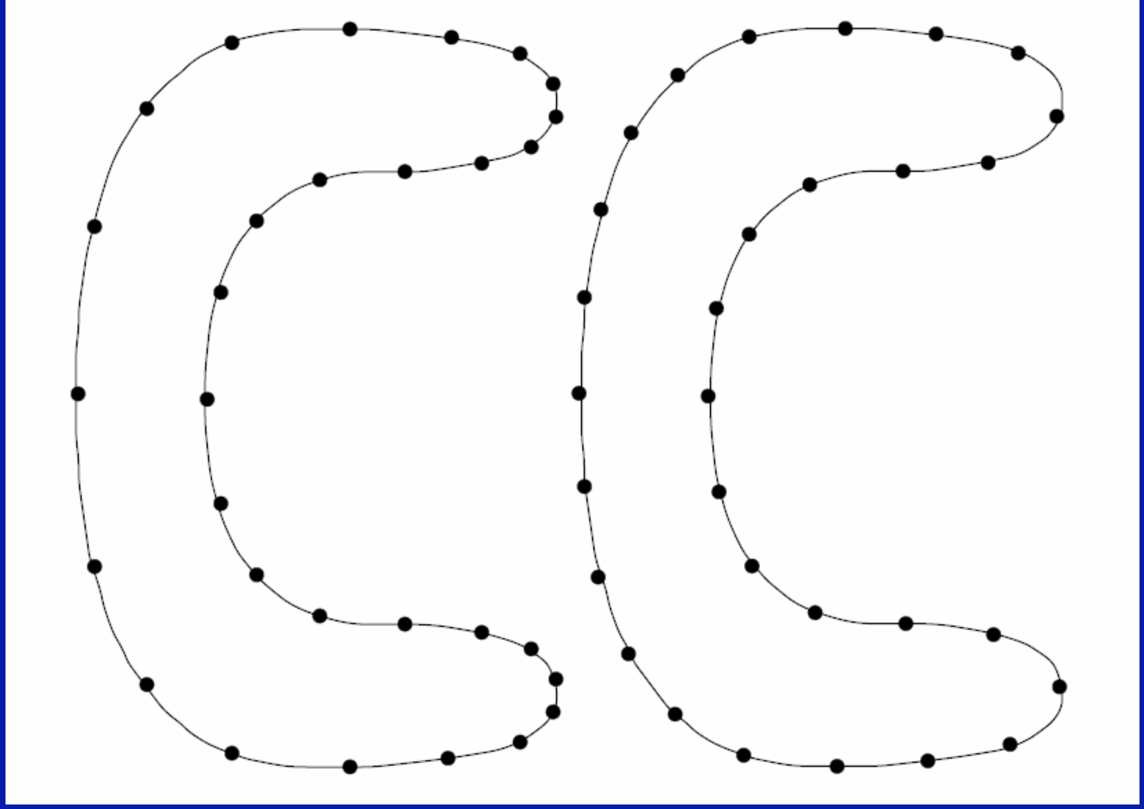
How to Draw Spline Curves

- **Basis matrix eqn. allows same code to draw any spline type**
- **<u>Method 1</u>: brute force**
  - **Calculate the coefficients**
  - **For each cubic segment, vary *u* from *0* to *1* (fixed step size)**
  - **Plug in *u* value, matrix multiply to compute position on curve**
  - **Draw line segment from last position to current position**

$$[x \quad y \quad z] = [u^3 \quad u^2 \quad u \quad 1] \underbrace{\begin{bmatrix} -s & 2-s & s-2 & s \\ 2s & s-3 & 3-2s & -s \\ -s & 0 & s & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{\text{CR basis}} \underbrace{\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{bmatrix}}_{\text{control vector}}$$

How to Draw Spline Curves

- What's wrong with this approach?
  - –Draws in even steps of u
  - –Even steps of u $\neq$ even steps of x
  - –Line length will vary over the curve
  - –Want to bound line length
    - »too long: curve looks jagged
    - »too short: curve is slow to draw

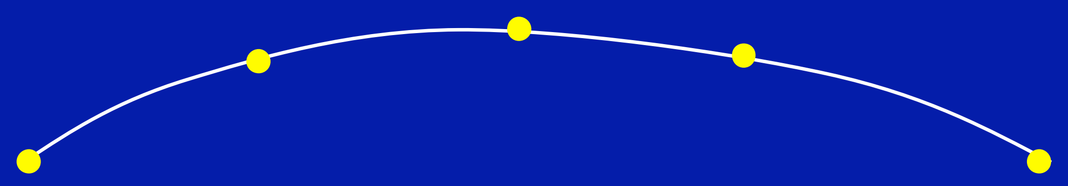


Drawing Splines, 2

- **Method 2: recursive subdivision - vary step size to draw short lines**

```
Subdivide(u0,u1,maxlinelength)
    umid = (u0 + u1)/2
    x0 = P(u0)
    x1 = P(u1)
    if |x1 - x0| > maxlinelength
       Subdivide(u0,umid,maxlinelength)
       Subdivide(umid,u1,maxlinelength)
    else drawline(x0,x1)
```

- **Variant on Method 2 - subdivide based on curvature**
  - **replace condition in "if" statement with straightness criterion**
  - **draws fewer lines in flatter regions of the curve**



In Summary...

- Summary:
  - piecewise cubic is generally sufficient
  - define conditions on the curves and their continuity

- Things to know:
  - basic curve properties (what are the conditions, controls, and properties for each spline type)
  - generic matrix formula for uniform cubic splines x(u) = uBG
  - given definition derive a basis matrix