1 Euler's Method

For $m = 1$ let us consider the problem

$$(1) \qquad \begin{cases} y'(t) = -150y(t) + 49 - 150t, & t \in [0,1], \\ y(0) = 1/3 + \epsilon, \end{cases}$$

where $\epsilon \in \mathbb{R}$ is the error in the initial data.

EXERCISE 1

(a) Find the analytic solution y_ϵ.

Solving the linear ODE of this form:

$$\begin{cases} y_\varepsilon'(t) = -150y(t) + 49 - 150t, t \in [0,1] \\ y_\varepsilon(0) = \frac{1}{3} + \varepsilon, \text{ where } \varepsilon \in \mathbb{R} \text{ is the error in the initial data} \end{cases}$$

Linear ODE has the form: $\frac{dy}{dt} + P(t)y = Q(t)$

$\frac{dy}{dt} + 150y = 49 - 150t$ => $P(x) = 150,\ Q(x) = 49 - 150t$

Let $\frac{dy}{dt} = u\frac{dv}{dt} + v\frac{du}{dt}$ and $y = uv$. Substituting into the equations, we have:

$u\frac{dv}{dt} + v\frac{du}{dt} + 150uv = 49 - 150t$ => $u\frac{dv}{dt} + v\frac{du}{dt} + 150uv = 49 - 150t$

=> $u\frac{dv}{dt} + v\left(\frac{du}{dt} + 150u\right) = 49 - 150t$. Set $\frac{du}{dt} + 150u$ to 0:

$\frac{du}{dt} + 150u = 0 => \frac{du}{dt} = -150u => \frac{1}{u}\frac{du}{dt} = -150 => \int \frac{1}{u}\frac{du}{dt}dt = \int -150dt$

=> $\int \frac{1}{u}du = -150t => \ln(u) = -150t => u = e^{-150t}$

Substituting back to the equation:

$e^{-150t}\frac{dv}{dt} + 0 = 49 - 150t => \frac{dv}{dt} = \frac{49 - 150t}{e^{-150t}} => \int dv = \int \frac{49 - 150t}{e^{-150t}}dt$

=> $v = C - e^{-150t}t + \frac{e^{150t}}{3}$. We know both u and v. We have y = uv

=> $y = uv = e^{-150t}\left(C - e^{-150t}t + \frac{e^{150t}}{3}\right) = Ce^{-150t} - t + \frac{1}{3}$

Finally, from the condition:

$y_\varepsilon(0) = Ce^{-150\times 0} - 0 + \frac{1}{3} = C + \frac{1}{3} = \frac{1}{3} + \varepsilon => C = \varepsilon$

The formula of the analytic solution is thus:

$y_{\varepsilon}(t) = \varepsilon e^{-150t} - t + \frac{1}{3}$ (answer)

(b) Show that $\|y_0 - y_\epsilon\|_\infty \leq |\epsilon|$.

$$\|y_0 - y_\varepsilon\|_\infty = \max_{t\in[0,1]} |y_0(t) - y_\varepsilon(t)| = \max_{t\in[0,1]} \left| -t + \frac{1}{3} - \left(\varepsilon e^{-150t} - t + \frac{1}{3} \right) \right|$$

=> $\max_{t\in[0,1]} \left| -\varepsilon e^{-150t} \right| = \max_{t\in[0,1]} \left| \varepsilon e^{-150t} \right|$

We see that $-150t \in [-150, 0] => e^{-150t} \in [e^{-150}, 1] => e^{-150t} \leq 1, t \in [0,1]$

=> $\max_{t\in[0,1]} \left| \varepsilon e^{-150t} \right| \leq |\varepsilon|$. In other words, we have: $\|y_0 - y_\varepsilon\|_\infty \leq |\varepsilon|$ (proven)

(c) Let $h > 0$. If $t, t + h \in [0, 1]$, show that

$$y_0(t + h) = y_0(t) + h(-150y_0(t) + 49 - 150t).$$

Rearrange the formula as follows:

$y_0(t+h) - y_0(t) = h(-150y_0(t) + 49 - 150t)$

LHS: $y_0(t+h) - y_0(t) = -(t+h) + \frac{1}{3} - \left(-t + \frac{1}{3} \right) = -h$

RHS: $h(-150y_0(t) + 49 - 150t) = h\left(-150\left(-t + \frac{1}{3} \right) + 49 - 150t \right) = h(150t - 50 + 49 - 150t) = -h$

Both sides equal -h => $y_0(t+h) = y_0(t) + h(-150y_0(t) + 49 - 150t)$ (proven)

(d) Let $n \in \mathbb{N}$ with $n > 0$, $h = 1/n$, and $t_i = (i-1)h$, $i = 1, \ldots, n+1$. Compute the discrete solution $u_{\epsilon,i}$ for $i = 1, \ldots, n+1$ using Euler's Method.

The general formula of the Euler's Method:

$$y_{i+1} = y_i + hf(x_i, y_i)$$

where,

- y_{i+1} is the next estimated solution value;
- y_i is the current value;
- h is the interval between steps;
- $f(x_i, y_i)$ is the value of the derivative at the current (x_i, y_i) point.

Source: www.freecodecamp.org/news/eulers-method-explained-with-examples

Firstly:

$$\begin{cases} y_\varepsilon'(t) = -150y(t) + 49 - 150t, t \in [0,1] \\ y_\varepsilon(t) = \varepsilon e^{-150t} - t + \dfrac{1}{3} \\ y_\varepsilon(0) = \dfrac{1}{3} + \varepsilon \end{cases}$$

In this exercise, we have:

$u_{\varepsilon,i+1} = u_{\varepsilon,i} + hf(t_i, u_{\varepsilon,i})$, where $f(t_i, u_{\varepsilon,i}) = y_\varepsilon'(t_i) = -150y_\varepsilon(t_i) + 49 - 150t_i$

=> $u_{\varepsilon,i+1} = u_{\varepsilon,i} + h(-150y_\varepsilon(t_i) + 49 - 150t_i) = u_{\varepsilon,i} - 150y_\varepsilon(t_i)h + h(49 - 150t_i)$

The initial value is

$u_{\varepsilon,1} = y_\varepsilon(t_0) = y_\varepsilon(0) = \dfrac{1}{3} + \varepsilon$

=> $u_{\varepsilon,2} = u_{\varepsilon,1} - 150u_{\varepsilon,1}h + h(49 - 150t_1)$ => $u_{\varepsilon,2} = (1 - 150h)u_{\varepsilon,1} + h(49 - 150t_1)$

Next steps are:

$u_{\varepsilon,3} = (1 - 150h)u_{\varepsilon,2} + h(49 - 150t_2)$

$u_{\varepsilon,4} = (1 - 150h)u_{\varepsilon,3} + h(49 - 150t_3)$

and so on

The final general discrete solution by the Euler's method is:

$u_{\varepsilon,i+1} = (1 - 150h)u_{\varepsilon,i} + h(49 - 150t_i)$ (answer)

(e) Show that for $i = 1, \ldots, n$,

$$u_{\epsilon,i+1} - y_0(t_{i+1}) = (1 - 150h)(u_{\epsilon,i} - y_0(t_i))$$

and

$$u_{\epsilon,i} - y_0(t_i) = (1 - 150h)^{i-1}\epsilon$$

for $i = 1, \ldots, n+1$.

1) Prove that $u_{\varepsilon,i+1} - y_0(t_{i+1}) = (1 - 150h)(u_{\varepsilon,i} - y_0(t_i))$

From part (d), we can substitute $u_{\varepsilon,i+1} = (1 - 150h)u_{\varepsilon,i} + h(49 - 150t_i)$ into the equation:

$(1 - 150h)u_{\varepsilon,i} + h(49 - 150t_i) - y_0(t_{i+1}) = (1 - 150h)(u_{\varepsilon,i} - y_0(t_i))$

We can also see that $y_\varepsilon(t) = \varepsilon e^{-150t} - t + \dfrac{1}{3}$ => $y_0(t) = 0e^{-150t} - t + \dfrac{1}{3} = -t + \dfrac{1}{3}$

Thus, we can again substitute $y_0\left(t_{i+1}\right)=-t_{i+1}+\frac{1}{3}$ and $y_0\left(t_i\right)=-t_i+\frac{1}{3}$:

$$\left(1-150h\right)u_{\varepsilon,i}+h(49-150t_i)-\left(-t_{i+1}+\frac{1}{3}\right)=\left(1-150h\right)\left(u_{\varepsilon,i}-\left(-t_i+\frac{1}{3}\right)\right)$$

=> $\left(1-150h\right)u_{\varepsilon,i}+h(49-150t_i)+t_{i+1}-\frac{1}{3}=\left(1-150h\right)\left(u_{\varepsilon,i}+t_i-\frac{1}{3}\right)$

=> $\left(1-150h\right)u_{\varepsilon,i}+h(49-150t_i)+t_{i+1}-\frac{1}{3}=\left(1-150h\right)u_{\varepsilon,i}+\left(1-150h\right)\left(t_i-\frac{1}{3}\right)$

=> $h(49-150t_i)+t_{i+1}-\frac{1}{3}=\left(1-150h\right)\left(t_i-\frac{1}{3}\right)$

=> $49h-150ht_i+t_{i+1}-\frac{1}{3}=t_i-\frac{1}{3}-150ht_i+50h$ => $t_{i+1}=t_i+h$

We have: $t_i=\left(i-1\right)\mathrm{h}$ and $t_{i+1}=\left(i-1+1\right)h=ih$

=> $ih=\left(i-1\right)h+h$ => $ih=ih-h+h$ => $ih=ih$

We have LHS = RHS => The equation $u_{\varepsilon,i+1}-y_0\left(t_{i+1}\right)=\left(1-150h\right)\left(u_{\varepsilon,i}-y_0\left(t_i\right)\right)$ is proven

2) Prove that $u_{\varepsilon,i}-y_0\left(t_i\right)=\left(1-150h\right)^{i-1}\varepsilon$

We have already proven from part 1 that $u_{\varepsilon,i+1}-y_0\left(t_{i+1}\right)=\left(1-150h\right)\left(u_{\varepsilon,i}-y_0\left(t_i\right)\right)$

=> $u_{\varepsilon,i}-y_0\left(t_i\right)=\left(1-150h\right)\left(u_{\varepsilon,i-1}-y_0\left(t_{i-1}\right)\right)$ (1), decreasing by one step backward

Now, let's decrease by one step further

$u_{\varepsilon,i-1}-y_0\left(t_{i-1}\right)=\left(1-150h\right)\left(u_{\varepsilon,i-2}-y_0\left(t_{i-2}\right)\right)$. Replace this into the above equation (1), we have:

$$u_{\varepsilon,i}-y_0\left(t_i\right)=\left(1-150h\right)\left(\left(1-150h\right)\left(u_{\varepsilon,i-2}-y_0\left(t_{i-2}\right)\right)+y_0\left(t_{i-1}\right)-y_0\left(t_{i-1}\right)\right)$$

=> $u_{\varepsilon,i}-y_0\left(t_i\right)=\left(1-150h\right)^2\left(u_{\varepsilon,i-2}-y_0\left(t_{i-2}\right)\right)+\left(1-150h\right)\left(0\right)$

=> $u_{\varepsilon,i}-y_0\left(t_i\right)=\left(1-150h\right)^2\left(u_{\varepsilon,i-2}-y_0\left(t_{i-2}\right)\right)$

This will recurse back to (i-1) number of steps. At the base case:

$u_{\varepsilon,i}-y_0\left(t_i\right)=\left(1-150h\right)^{i-1}\left(u_{\varepsilon,1}-y_0\left(t_1\right)\right)=\left(1-150h\right)^{i-1}\left(\frac{1}{3}+\varepsilon-\left(-t_1+\frac{1}{3}\right)\right)$ and

$t_i=(i-1)h \Rightarrow t_1=(1-1)h=0$

=> $u_{\varepsilon,i}-y_0\left(t_i\right)=\left(1-150h\right)^{i-1}\varepsilon$ (proven)

(f) If $n = 50$ and $\epsilon = 0.01$, compute the error $u_{\epsilon,n+1} - y_0(1)$ at $t = 1$.

We need to write t = 1 in terms of i and h

$$t_i = (i-1)h = 1 \Rightarrow (i-1)\frac{1}{n} = 1 \Rightarrow (i-1)\frac{1}{50} = 1 \Rightarrow i = 51 = n+1$$

$$\Rightarrow u_{\varepsilon,n+1} - y_0(1) = u_{\varepsilon,n+1} - y_0(t_{n+1})$$

From part (e), we have: $u_{\varepsilon,i} - y_0(t_i) = (1-150h)^{i-1}\varepsilon$. The error can be calculated as follows:

$$\Rightarrow u_{\varepsilon,n+1} - y_0(t_{n+1}) = (1-150h)^n \varepsilon = \left(1-150\frac{1}{50}\right)^{50} \times 0.01 = 1.12589907 \times 10^{13} \text{ (answer)}$$

(g) Give a condition on n to obtain

$$\max_{i=1,\ldots,n+1} |u_{\epsilon,i} - y_0(t_i)| \le \epsilon.$$

From part (e), we can derive the formula as:

$\max\limits_{i=1,..,n+1}|u_{\varepsilon,i} - y_0(t_i)| = \max\limits_{i=1,..,n+1}\left|(1-150h)^{i-1}\varepsilon\right|$. At i = 1, we have $\left|(1-150h)^0\varepsilon\right| = \varepsilon \le \varepsilon$, satisfying the conditions. Now we only need to check the the other end where i = n + 1

$$\Rightarrow \max_{n\in\mathbb{N}}\left|\left(1-\frac{150}{n}\right)^n \varepsilon\right| \le \varepsilon \Rightarrow \max_{n\in\mathbb{N}}\left|\left(1-\frac{150}{n}\right)^n\right| \le 1$$

First we can find n where $\left(1-\frac{150}{n}\right)^n = 1 \Rightarrow 1-\frac{150}{n} = 1 \Rightarrow n = 75$

Analysis: if $n < 75 \Rightarrow \left|1-\frac{150}{n}\right|^n > 1$, violating the conditions

$n > 75 \Rightarrow \left|1-\frac{150}{n}\right|^n < 1$, satisfying the condition

=> The condition on n to obtain $\max\limits_{i=1,..,n+1}|u_{\varepsilon,i} - y_0(t_i)| \le \varepsilon$ is $n \ge 75$ (answer)

Exercise 2

(a) Show how $u_{\epsilon,i+1}$ for $1 \le i \le n$ can be computed from $u_{\epsilon,i}$ using the backward Euler method.

The forward Euler method is given by

$$u_{\varepsilon,i+1} = u_{\varepsilon,i} + hf(t_i, u_{\varepsilon,i}), \text{ where } f(t_i, u_{\varepsilon,i}) = y_\varepsilon'(t_i) = -150y_\varepsilon(t_i) + 49 - 150t_i$$

On the other hand, the backward Euler method is given by

$u_{\varepsilon,i+1} = u_{\varepsilon,i} + hf\left(t_{i+1}, u_{\varepsilon,i+1}\right)$, where $f\left(t_{i+1}, u_{\varepsilon,i+1}\right) = y_\varepsilon{}'\left(t_{i+1}\right) = -150 y_\varepsilon\left(t_{i+1}\right) + 49 - 150 t_{i+1}$

=> $u_{\varepsilon,i+1} = u_{\varepsilon,i} + h\left(-150 y_\varepsilon\left(t_{i+1}\right) + 49 - 150 t_{i+1}\right) = u_{\varepsilon,i} - 150 h y_\varepsilon\left(t_{i+1}\right) + 49h - 150 h t_{i+1}$

=> $u_{\varepsilon,i+1} = u_{\varepsilon,i} - 150 h y_\varepsilon\left(t_{i+1}\right) + 49h - 150h(t_i + h)$ and $y_\varepsilon\left(t_{i+1}\right)$ is approximated by $u_{\varepsilon,i+1}$

=> $u_{\varepsilon,i+1} = u_{\varepsilon,i} - 150 h u_{\varepsilon,i+1} + 49h - 150h(t_i + h)$

The general formula of the backward Euler method is therefore:

$u_{\varepsilon,i+1} = u_{\varepsilon,i} + 49h - 150h\left(t_i + h\right) - 150 h u_{\varepsilon,i+1}$ (answer)

(b) Show that for $i = 1, \ldots, n$

$$u_{\epsilon,i} - y_0(t_i) = \frac{1}{(1+150h)^{i-1}}\epsilon.$$

We can prove similarly in Exercise 1 - (e) that:

$u_{\varepsilon,i} - y_0\left(t_i\right) = \frac{1}{1+150h}\left(u_{\varepsilon,i+1} - y_0\left(t_{i+1}\right)\right)$ (1)

We move backward one step:

$u_{\varepsilon,i+1} - y_0\left(t_{i+1}\right) = \frac{1}{1+150h}\left(u_{\varepsilon,i+2} - y_0\left(t_{i+2}\right)\right)$ (2). Replacing this into (1)

=>

$$u_{\varepsilon,i} - y_0\left(t_i\right) = \frac{1}{1+150h}\left(\frac{1}{1+150h}\left(u_{\varepsilon,i+2} - y_0\left(t_{i+2}\right)\right) + y_0\left(t_{i+1}\right) - y_0\left(t_{i+1}\right)\right) = \frac{1}{\left(1+150h\right)^2}\left(u_{\varepsilon,i+2} - y_0\left(t_{i+2}\right)\right)$$

We can recurse this pattern up to the i = n. At the base case:

$u_{\varepsilon,i} - y_0\left(t_i\right) = \frac{1}{\left(1+150h\right)^{i-1}}\left(u_{\varepsilon,n+1} - y_0\left(t_{n+1}\right)\right)$ => $u_{\varepsilon,i} - y_0\left(t_i\right) = \frac{1}{\left(1+150h\right)^{i-1}}\varepsilon$ (proven)

(c) Give a condition on n to obtain

$$\max_{i=1,\ldots,n+1} |u_{\epsilon,i} - y_0(t_i)| \leq \epsilon.$$

Plugging in from (b), we have:

$$\max_{i=1,..,n+1}\left|u_{\varepsilon,i} - y_0\left(t_i\right)\right| = \max_{i=1,..,n+1}\left|\frac{1}{\left(1+150h\right)^{i-1}}\varepsilon\right| \Rightarrow \max_{n\in\mathbb{N}}\left|\left(\frac{1}{1+\frac{150}{n}}\right)^n \varepsilon\right| \leq \varepsilon \Rightarrow \max_{n\in\mathbb{N}}\left|\left(\frac{1}{1+\frac{150}{n}}\right)^n\right| \leq 1$$

This is the reverse case compared to Exercise 1 part (g). We only need to choose the range of n that is a difference between [75, infinity]

=> Condition on n is $n \leq 75$ and $n \geq 1$ (answer)

2 Heun's Method

Let us consider the same IVP.
EXERCISE 3

(a) Write a programme Heun.m that implements the Heun's method on an uniform partition.

The procedure for calculating the numerical solution to the initial value problem:

$$y'(t) = f(t, y(t)), \qquad y(t_0) = y_0,$$

by way of Heun's method, is to first calculate the intermediate value $\tilde{y}_{i+1}$ and then the final approximation y_{i+1} at the next integration point.

$$\tilde{y}_{i+1} = y_i + hf(t_i, y_i)$$
$$y_{i+1} = y_i + \frac{h}{2}[f(t_i, y_i) + f(t_{i+1}, \tilde{y}_{i+1})],$$

where h is the step size and $t_{i+1} = t_i + h$.

Source: https://en.wikipedia.org/wiki/Heun%27s_method

The Matlab code for heun.m is

```
function [t, u] = heun(func, range,initialvalue, partitions)
t = linspace(range(1),range(2),partitions + 1);
h = (range(2) - range(1))/partitions;
u = zeros([1, partitions + 1]);
u(1) = initialvalue;
for i=2:partitions+1
    temporary = u(i-1) + h*func(t(i-1),u(i-1));
    u(i) = u(i-1) + (h/2)*(func(t(i-1),u(i-1)) +
func(t(i),temporary));
end
```

The Matlab code for heuntest.m is

```
clc;
f1 = @(t,y) (-150)*y + 49 - 150*t;
[t,u] = heun(f1, [0,1], 1/3, 4);
disp("t is ")
disp(t)
disp("u is" )
disp(u)
```

Testing the function:

```
t is
         0    0.2500    0.5000    0.7500    1.0000

u is
    0.3333    0.0833   -0.1667   -0.4167   -0.6667
```

(b) **Write a program that plots the graphs of the approximation and of the exact solution and computes the error:** $\max_{i=1,\ldots,n+1} |u_i - y(t_i)|$, **where** $y(t)$ **is the exact solution. Test with** $n = 40, 73, 75, \ldots$ **and** $\epsilon = 0.01$.

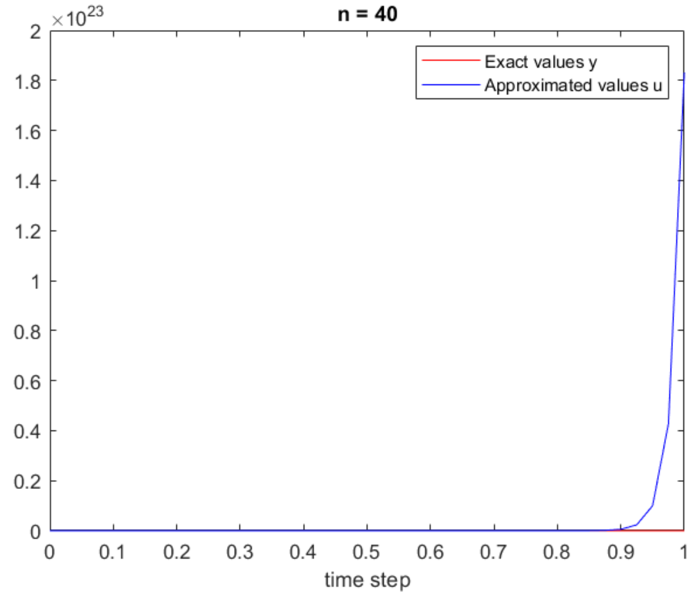
The Matlab code for heunplot.m is

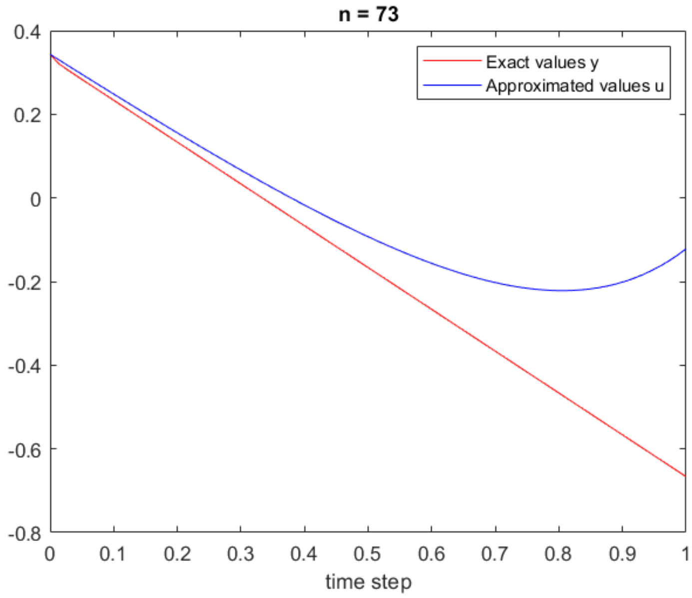
```
n = 40;
epsilon=0.01;
f1 = @(t,y) (-150)*y + 49 - 150*t;
figure(1)
[t,u]=heun(f1, [0, 1], 1/3 + epsilon, n);
y = epsilon * exp(-150.*t) - t + 1/3;
plot(t, y,'r')
hold on
plot(t, u,'b')
legend('Exact values y','Approximated values u')
title("n = 40")
xlabel("time step")
error40 = max(abs(u - y));
disp("Maximum error when n = 40: " + error40)

figure(2)
n = 73;
[t,u]=heun(f1, [0, 1], 1/3 + epsilon, n);
y = epsilon * exp(-150.*t) - t + 1/3;
plot(t, y,'r')
hold on
plot(t, u,'b')
legend('Exact values y','Approximated values u')
title("n = 73")
xlabel("time step")
error73 = max(abs(u - y));
disp("Maximum error when n = 73: " + error73)

figure(3)
n = 75;
[t,u]=heun(f1, [0, 1], 1/3 + epsilon, n);
y = epsilon * exp(-150.*t) - t + 1/3;
plot(t, y,'r')
hold on
plot(t, u,'b')
legend('Exact values y','Approximated values u')
title("n = 75")
xlabel("time step")
error75 = max(abs(u - y));
disp("Maximum error when n = 75: " + error75)
```

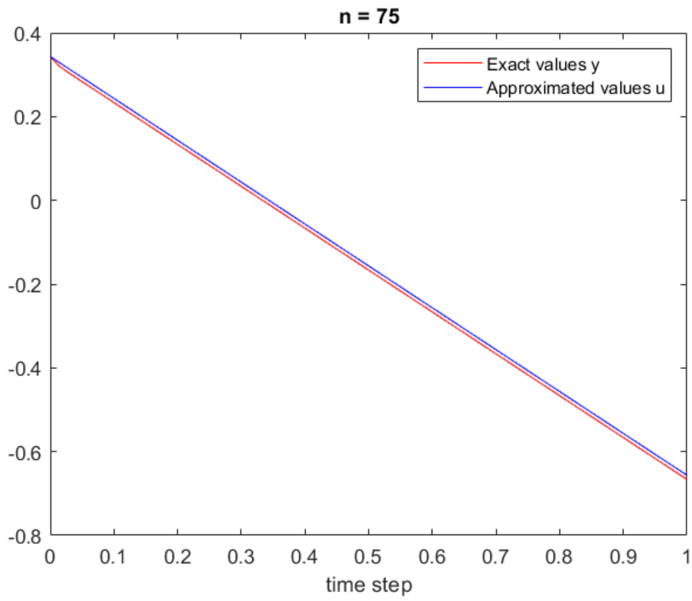
The Maximum errors for each case is:

```
Maximum error when n = 40: 1.831570135060366e+23
Maximum error when n = 73: 0.54493
Maximum error when n = 75: 0.01
```

Plotting the exact and approximated values for each case:







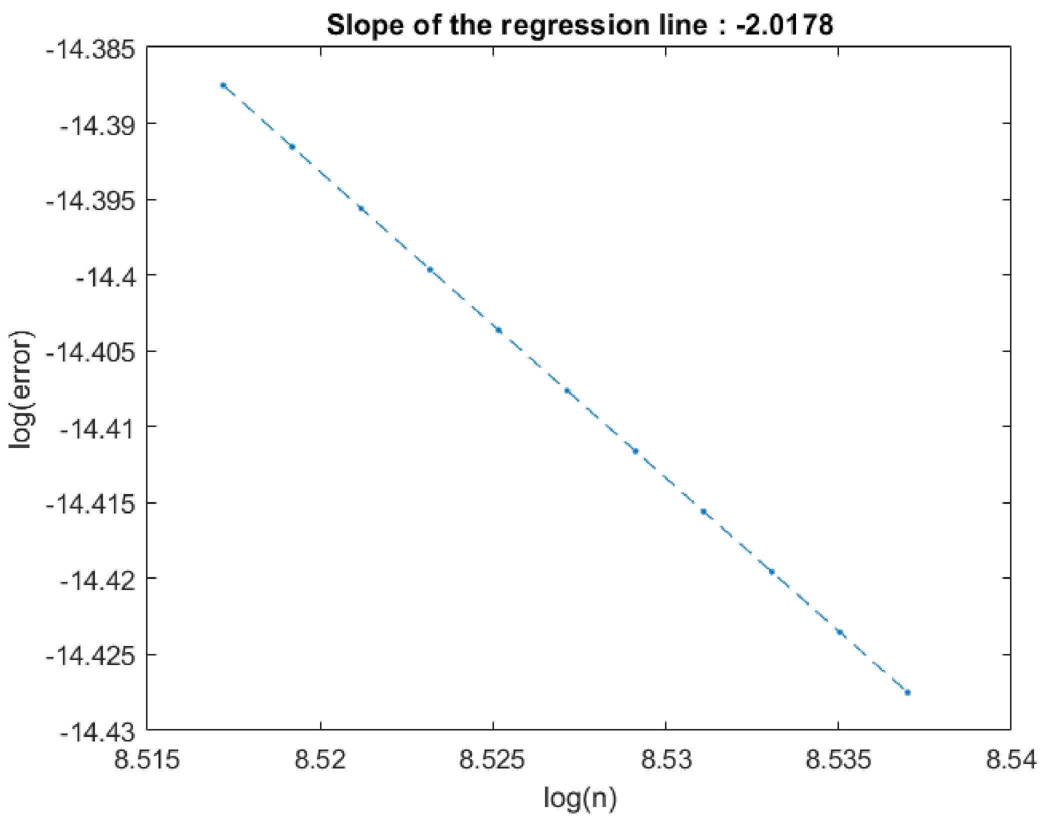
(c) Write a program Heunerror.m to study the error for different values of n. Beginning with an array arrn the program will compute the corresponding arrerror and plot $\ln(\text{arrerror})$ depending on $\ln(\text{arrn})$. What seems to be the order of the scheme?

Test with arrn=5000:10:5100 and $\epsilon = 0.01$.

The Matlab code for heunerror.m is

```
arrn = 5000:10:5100;
epsilon=0.01;
f1 = @(t,y) (-150)*y + 49 - 150*t;
for i=1:length(arrn)
n=arrn(i);
[t,u]=heun(f1, [0,1], 1/3 + epsilon, n);
y = epsilon * exp(-150.*t) - t + 1/3;
arrerror(i)=norm(u-y,inf);
end
plot(log(arrn),log(arrerror),'.--')
a=polyfit (log(arrn), log(arrerror) ,1);
title (['Slope of the regression line : ', num2str(a(1))])
xlabel('log(n)')
ylabel('log(error)')
```

Plotting the slope of the regression line:



The value of the errors are in the scale of 10e-7. The order of the convergence of the scheme seems to be 2, which is the absolute value of the slope of the regression line based on log(n) and log(error)

3 MATLAB

Let us assume that the RHS from above is implemented in f1.m.

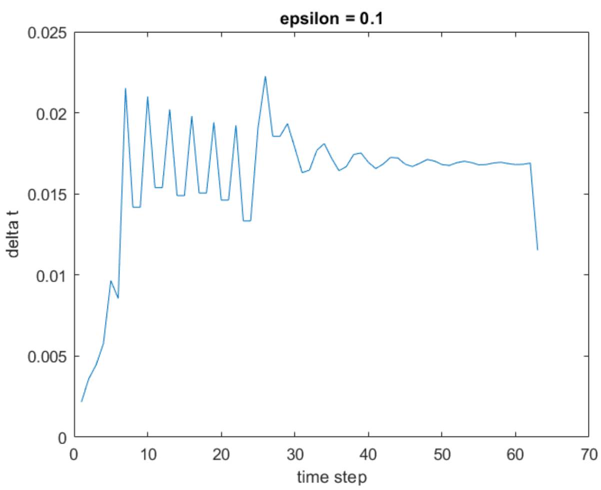
EXERCISE 4 Test the MATLAB tool ode23 with the initial data $u_1 = 1/3 + \epsilon$, and let $\epsilon = 0.1$ and 0.001.

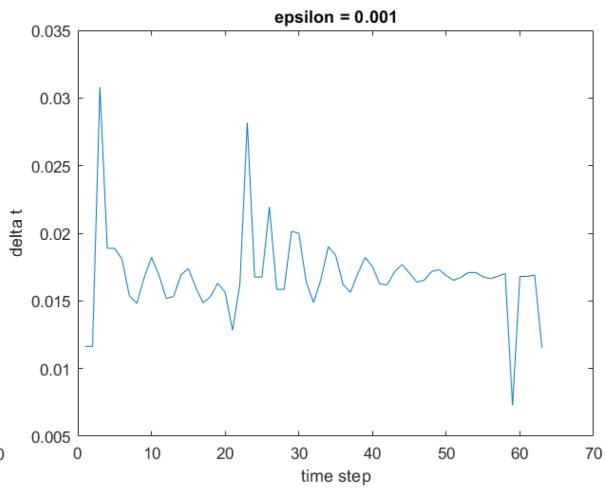
Test with [t,u] = ode23('f1',[0,1],1/3 + epsilon). Plot the vector Δt, with $\Delta t_i = t_{i+1} - t_i$ in the two cases.

The Matlab code for ode.m is

```
f1 = @(t,y) (-150)*y + 49 - 150*t;

figure(1)
epsilon=0.1;
[t,u] = ode23(f1,[0,1],1/3 + epsilon);
for i=1:length(t)-1
    delta_t(i) = t(i + 1) - t(i);
end
plot(delta_t)
title("epsilon = 0.1")
xlabel("time step")
ylabel("delta t")

figure(2)
epsilon=0.001;
[t,u] = ode23(f1,[0,1],1/3 + epsilon);
for i=1:length(t)-1
    delta_t(i) = t(i + 1) - t(i);
end
plot(delta_t)
plot(delta_t)
title("epsilon = 0.001")
xlabel("time step")
ylabel("delta t")
```

Plotting the delta_t for both cases of different errors:





The difference in timestep seems to fluctuate in the first half and starts to stay stable in the latter half. Finally, the difference drops at the end.