1 Quadratic and Cubic Hermite Interpolation

Given interpolating values y_1, y_2 and derivatives s_1, s_2 at two nodes $a < b$, let $z \in (a, b)$ be a number called a knot. We seek a piecewise quadratic polynomial $g : [a, b] \to \mathbb{R}$ in the form

$$g(x) = \begin{cases} p_1(x), & \text{if } a \le x < z, \\ p_2(x), & \text{if } z < x \le b, \end{cases}$$

with $p_1, p_2 \in \mathbb{P}_2$ such that

$$g(a) = y_1,\ g'(a) = s_1,\ g(b) = y_2,\ g'(b) = s_2.$$

Moreover, we require

$$p_1(z) = p_2(z),\ p_1'(z) = p_2'(z),$$

which means that $g \in C^1[a, b]$. g is a unique quadratic Hermite interpolant.

EXERCISE 1

(a) Verify that $g : [0, 2] \to \mathbb{R}$ given by

$$g(x) = \begin{cases} 0, & \text{if } 0 \le x < 1, \\ 16(x-1)^2, & \text{if } 1 < x \le 2, \end{cases}$$

is a quadratic Hermite interpolant with a knot at $z = 1$ interpolating $f(x) = x^4$ at 0 and 2.

We have: a = 0, b = 0 and z = 1. We now test for the 6 conditions:

1) $g(a) = g(0) = 0,\ f(a) = f(0) = 0^4 = 0 \Rightarrow g(a) = f(a) = y_1$

2) $g(b) = g(2) = 16(2-1)^2 = 16,\ f(b) = f(2) = 2^4 = 16 \Rightarrow g(b) = f(b) = y_2$

$$g'(x) = \begin{cases} p_1' = 0 & ,\ 0 \le x < 1 \\ p_2' = 32(x-1), & 1 < x \le 2 \end{cases} \quad \text{and } f'(x) = 4x^3$$

3) $g'(a) = g'(0) = 0,\ f'(a) = f'(0) = 4 \times 0^3 = 0 \Rightarrow g'(a) = f'(a) = s_1$

4) $g'(b) = g'(2) = 32(2-1) = 32,\ f'(b) = f'(2) = 4 \times 2^3 = 32 \Rightarrow g'(b) = f'(b) = s_2$

5) $p_1(z) = p_1(1) = 0,\ p_2(z) = p_2(1) = 16(1-1)^2 = 0 \Rightarrow p_1(z) = p_2(z) = 0$

6) $p_1'(z) = p_1'(1) = 0,\ p_2'(z) = p_2'(1) = 32(1-1) = 0 \Rightarrow p_1'(z) = p_2'(z) = 0$

Since 6 conditions are satisfied, g(x): [0, 2] is a quadratic Hermite interpolant with a knot at z = 1 interpolating $f(x) = x^4$ at 0 and 2

(b) Find $g_3(x)$ the cubic Hermite interpolant interpolating the same data.

Let the cubic polynomial be $g_3(x) = ax^3 + bx^2 + cx + d \Rightarrow g_3'(x) = 3ax^2 + 2bx + c$

1) $g_3(0) = f(0) = 0 \Rightarrow d = 0$

2) $g_3(2) = f(2) = 16 \Rightarrow 8a + 4b + 2c + d = 16$

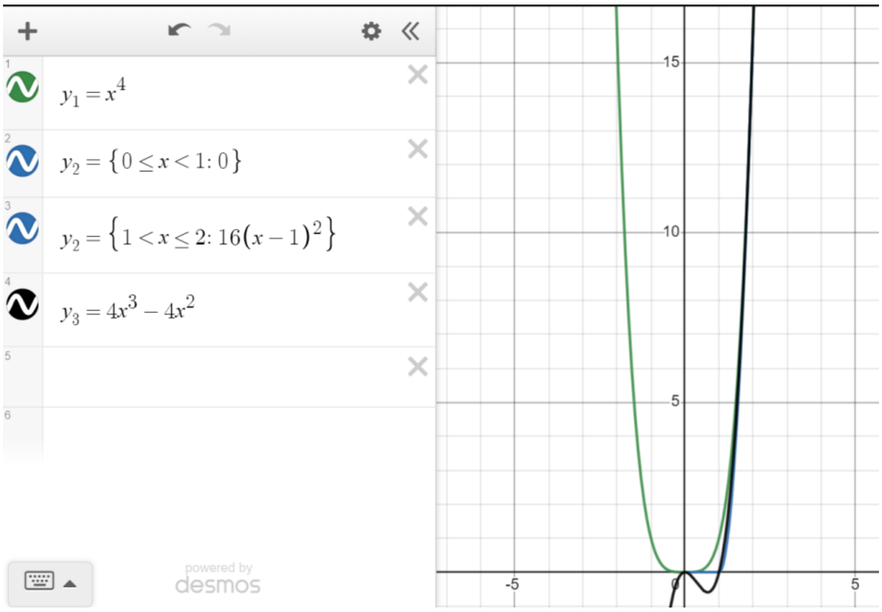
3) $g_3'(0) = f'(0) = 0 \Rightarrow c = 0$

4) $g_3'(2) = f'(2) = 32 \Rightarrow 12a + 4b = 32$

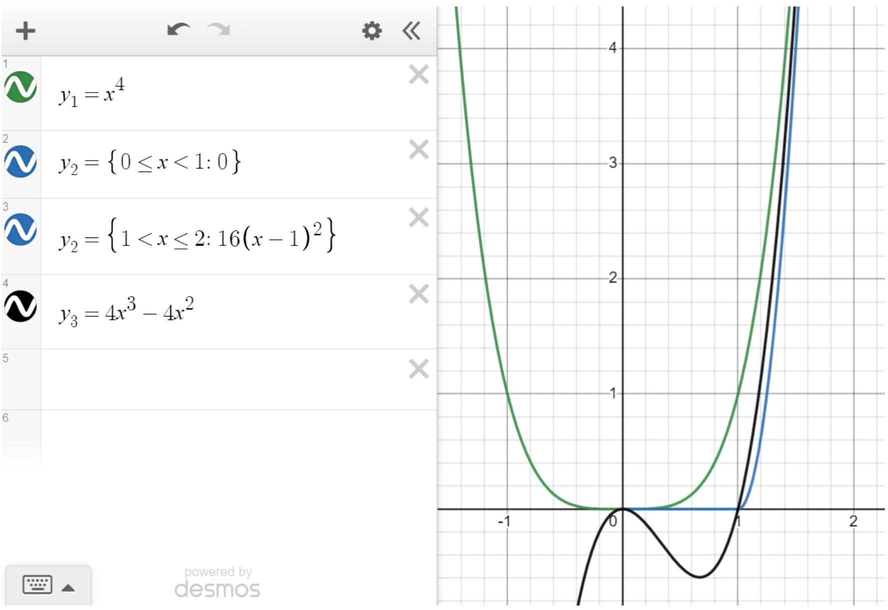
$\Rightarrow a = 4, b = -4, c = d = 0$

$\Rightarrow g_3(x) = 4x^3 - 4x^2$ (Answer)

(c) Plot f, g_2, and g_3 on the interval $[0, 2]$.



Where y1 is f, y2 is g2 and y3 is g3. Graph is made by Desmos. This is a closer look



2 Quadratic and Cubic Hermite Interpolation Using the Bernstein Basis

We recall that a polynomial $p \in \mathbb{P}_d$ is said to be in Bernstein form of degree d on an interval $[\alpha, \beta]$, with $h = \beta - \alpha > 0$, if

$$p(x) = \sum_{j=0}^{d} c_j B_j^d\left(\frac{x-\alpha}{h}\right), \text{ where } B_j^d(t) = \binom{d}{j} t^j (1-t)^{d-j},\ d \geq 0.$$

Moreover for $d \geq 2$ the values and derivatives at the endpoints are given by

$$p(\alpha) = c_0,\ p(\beta) = c_d,\ p'(\alpha) = \frac{d}{h}(c_1 - c_0),\ p'(\beta) = \frac{d}{h}(c_d - c_{d-1}).$$

EXERCISE 2

(a) Consider cubic Hermite interpolation on $[a, b]$. Compute first the general form of the coefficients c_j in the Bernstein form of degree 3 and then apply them to the setup of Exercise 1.

$$p(t) = \sum_{j=0}^{d} c_j B_j^d(t) = \sum_{j=0}^{3} c_j B_j^3(t)$$

$$= c_0 B_0^3(t) + c_1 B_1^3(t) + c_2 B_2^3(t) + c_3 B_3^3(t)$$

$$\Rightarrow p(t) = c_0 B_0^3(t) + c_1 B_1^3(t) + c_2 B_2^3(t) + c_3 B_3^3(t)$$

$$\Rightarrow p(t) = c_0\binom{3}{0} t^0 (1-t)^{3-0} + c_1\binom{3}{1} t^1 (1-t)^{3-1} + c_2\binom{3}{2} t^2 (1-t)^{3-2} + c_3\binom{3}{3} t^3 (1-t)^{3-3}$$

$$\Rightarrow p(t) = c_0\binom{3}{0} t^0 (1-t)^{3-0} + c_1\binom{3}{1} t^1 (1-t)^{3-1} + c_2\binom{3}{2} t^2 (1-t)^{3-2} + c_3\binom{3}{3} t^3 (1-t)^{3-3}$$

$$\Rightarrow p(t) = c_0 \times 1 \times t^0 (1-t)^3 + c_1 \times 3 \times t^1 (1-t)^2 + c_2 \times 3 \times t^2 (1-t)^1 + c_3 \times 1 \times t^3 (1-t)^0$$

$$\Rightarrow p(t) = c_0 (1-t)^3 + c_1 3t (1-t)^2 + c_2 3t^2 (1-t) + c_3 t^3$$

Plugging in t = (x – a)/h, we have:

$$p(x) = c_0\left(1 - \frac{x-\alpha}{h}\right)^3 + c_1 3\left(\frac{x-\alpha}{h}\right)\left(1 - \frac{x-\alpha}{h}\right)^2 + c_2 3\left(\frac{x-\alpha}{h}\right)^2\left(1 - \frac{x-\alpha}{h}\right) + c_3\left(\frac{x-\alpha}{h}\right)^3$$

Compute the general form of the coefficients:

$$p(\alpha) = c_0 (1-0)^3 + c_1 3(0)\left(1 - \frac{x-\alpha}{h}\right)^2 + c_2 3(0)^2 (1-0) + c_3 (0)^3 = c_0$$

$$p(\beta) = c_0 (1-1)^3 + c_1 3(1)(1-1)^2 + c_2 3(1)^2 (1-1) + c_3 (1)^3 = c_3$$

$$p'(\alpha) = \frac{3}{h}(c_1 - c_0),\ p'(\beta) = \frac{3}{h}(c_3 - c_2)$$

In exercise 1, we have a = 0, f(a) = 0, f'(a) = 0, b = 2, f(b) = 16, f'(b) = 32 and h = 2

=> $p(0) = c_0 = 0, p(2) = c_3 = 16$

=> $p'(\alpha) = \frac{3}{h}(c_1 - c_0) = \frac{3}{2}(c_1 - 0) = 0 => c_1 = 0$

=> $p'(\beta) = \frac{3}{h}(c_3 - c_2) = \frac{3}{2}(16 - c_2) = 32 => c_2 = -\frac{16}{3}$

Plugging into p(t), we have:

$$=> p(t) = -\frac{16}{3} \times 3t^2(1-t) + 16t^3 \text{ and } t = \frac{x-\alpha}{h} = \frac{x-0}{2} = \frac{x}{2}$$

$$=> p(x) = -\frac{16}{3} \times 3\left(\frac{x}{2}\right)^2\left(1-\frac{x}{2}\right) + 16\left(\frac{x}{2}\right)^3 = -16\frac{x^2}{4}\left(1-\frac{x}{2}\right) + 16\frac{x^3}{8}$$

$$=> p(x) = 4x^3 - 4x^2$$

We have get the same cubic Hermite interpolation like Exercise 1

(b) Compute the corresponding quadratic Hermite interpolant $g(x)$ above in Bernstein form. (Use notation c_{1j}, c_{2j}).

$$p(t) = \sum_{j=0}^{d} c_j B_j^d(t) = \sum_{j=0}^{2} c_j B_j^2(t)$$

$$= c_0 B_0^2(t) + c_1 B_1^2(t) + c_2 B_2^2(t)$$

$$=> p(t) = c_0 \binom{2}{0} t^0 (1-t)^{2-0} + c_1 \binom{2}{1} t^1 (1-t)^{2-1} + c_2 \binom{2}{2} t^2 (1-t)^{2-2}$$

$$=> p(t) = c_0 \times 1 \times t^0 (1-t)^2 + c_1 \times 2 \times t^1 (1-t)^1 + c_2 \times 1 \times t^2 (1-t)^0$$

$$=> p(t) = c_0 (1-t)^2 + c_1 2t(1-t) + c_2 t^2$$

For the first spline, we have a = 0, p1(a) = 0, p1'(a) = 0, z = 1, p1(z) = 0, p'(z) = 0 and h = 1

$$p_1(0) = c_{10} = 0, p_1(1) = c_{12}$$

$$p_1'(0) = \frac{d}{h}(c_{11} - c_{10}) = \frac{2}{1}(c_{11} - 0) = 0 => c_{11} = 0$$

$$p_1'(1) = \frac{d}{h}(c_{12} - c_{11}) = \frac{2}{1}(c_{12} - 0) = 2c_{12}$$

$$p_2(2) = c_{22} = 16, p_2(1) = c_{20}$$

$$p_2'(2) = \frac{d}{h}(c_{22} - c_{21}) = \frac{2}{1}(16 - c_{21}) = 32 => c_{21} = 0$$

$$p_2'(1) = \frac{d}{h}(c_{21} - c_{20}) = \frac{2}{1}(0 - c_{20}) = -2c_{20}$$

The final constraint, where z = 1, is given by

$p_1(z) = p_2(z)$ and $p_1'(z) = p_2'(z)$

$p_1(1) = c_{12}$ and $p_2(1) = c_{20} \Rightarrow c_{12} = c_{20}$

$p_1'(1) = 2c_{12}$ and $p_2'(1) = -2c_{20} \Rightarrow 2c_{12} = -2c_{20}$

From the system of equations => $c_{12} = c_{20} = 0$

=> The corresponding quadratic Hermite interpolant g(x) in Bernstein form is:

$$p_1(t) = c_{10}(1-t)^2 + c_{11}2t(1-t) + c_{12}t^2 = 0(1-t)^2 + 0\times 2t(1-t) + 0\times t^2 = 0$$

where t = $\dfrac{x-\alpha}{h} = \dfrac{x-0}{1} = x \Rightarrow p_1(x)=0,\ 0 \le x < 1$

$$p_2(t) = c_{20}(1-t)^2 + c_{21}2t(1-t) + c_{22}t^2 = 0\times(1-t)^2 + 0\times 2t(1-t) + 16\times t^2 = 16t^2$$

where t = $\dfrac{x-\alpha}{h} = \dfrac{x-1}{1} = x-1 \Rightarrow p_2(t) = 16(x-1)^2$ and $1 < x \le 2$

=> $c_{10} = c_{11} = c_{12} = 0,\ c_{20} = 0,\ c_{21} = 0,\ c_{22} = 16$

=> $g(x) = \begin{cases} p_1 = 0 & ,\ 0 \le x < 1 \\ p_2 = 16(x-1)^2, & 1 < x \le 2 \end{cases}$ as g2(x) the quadratic Hermite interpolation in Exercise 1

EXERCISE 3

(a) **Write a function quadhermite.m that computes the c_{1j}, c_{2j}, given arguments $a, b, z, y_1, y_2, s_1, s_2$.**

The Matlab code for quadhermite.m is

```
function hermiteQuadCoeff = quadhermite(a, b, z, y1, y2, s1, s2)
    c10 = y1;
    c22 = y2;
    d = 2;
    hSpline1 = z - a;
    hSpline2 = b - z;
    derivativeA = s1;
    derivativeB = s2;
    c11 = (derivativeA * hSpline1 / d) + c10;
    c21 = c22 - (derivativeB * hSpline2 / d);
    c120 = (hSpline1 * c21 + hSpline2 * c11)/(hSpline1 + hSpline2);
    hermiteQuadCoeff = [c10, c11, c120, c21, c22];
end
```

The result of the quadhermite function is:

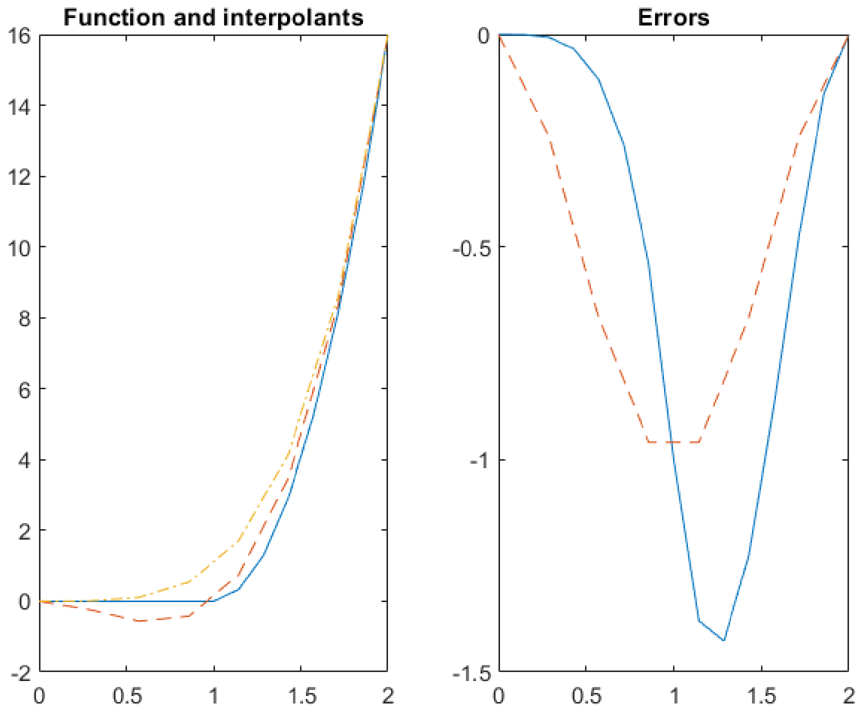
```
>> quadhermite(0,2,1,0,16,0,32)

ans =

     0     0     0     0    16
```

(b) **Let the knot $z = 1$. Plot first together f, g_3, g_2, where $f(x) = x^4$ on the interval $[0, 2]$. Second, plot together errors $f - g_2$ and $f - g_3$.**

The Matlab code for quadhermite_test.m is

```
vars = num2cell([0,2,1,0,16,0,32]);
[a,b,z,y1,y2,s1,s2] =  deal(vars{:});
C = quadhermite(a,b,z,y1,y2,s1,s2);
t = 0:1/n:1;
t2 = 1:1/n:2;
B10 = (1-t).^2;
B11 = 2*t.*(1-t);
B12 = t.^2;
B20 = (1-t).^2;
B21 = t*2.*(1-t);
B22 = t.^2;
x2 = [a:(z-a)/n:z,z:(b-z)/n:b];
g2 = [C(1).*B10 + C(2).*B11 + C(3).*B12, C(3).*B20 + C(4).*B21 +
C(5).*B22];
x3 = 2*t;
B30 = (1-t).^3;
B31 = 3*t.*((1-t).^2);
B32 = 3*(t.^2).*(1-t);
B33 = t.^3;
g3 = [0.*B30 + 0.*B31 - 16/3.*B32 + 16.*B33];
f2 = x2.^4;
f3 = x3.^4;
subplot(1,2,1)
plot(x2,g2,x3,g3,'--',x3,f3,'-.')
title('Function and interpolants')
subplot(1,2,2)
plot(x2,g2-f2,x3,g3-f3,'--')
title('Errors')
```

The function, interpolants and the errors are in the graph below:



3 Splines and Bezier Curves

EXERCISE 4 Draw one of your initials on (graph or grid) paper and design a font using either splines or Bezier curves. Implement your font with MATLAB.

The Matlab code for drawing_initials.m is

```
% My initials are N.X.B
% Drawing initial N
x = [1 1.5 2 2.5]; y = [1 3 1 3];
n = length(x);
t = 0:1:n-1;
tt = 0:.1:n-1;
xx = spline(t,x,tt); yy = spline(t,y,tt); hold on
plot(xx,yy','b','LineWidth',2), plot(x,y,'o'), grid on

% Drawing initial X
x = [3 3.5 4 4.5]; y = [1, 1.25, 2.75, 3];
n = length(x);
t = 0:1:n-1;
tt = 0:.1:n-1;
xx = spline(t,x,tt); yy = spline(t,y,tt); hold on
plot(xx,yy','g','LineWidth',2), plot(x,y,'o'), grid on

x = [3 3.5 4 4.5]; y = [3, 2.75, 1.25, 1];
n = length(x);
t = 0:1:n-1;
tt = 0:.1:n-1;
xx = spline(t,x,tt); yy = spline(t,y,tt); hold on
plot(xx,yy','g','LineWidth',2), plot(x,y,'o'), grid on

% Drawing initial B
x = [5 5.5 5]; y = [1, 2, 3];
n = length(x);
t = 0:1:n-1;
tt = 0:.1:n-1;
xx = spline(t,x,tt); yy = spline(t,y,tt); hold on
plot(xx,yy','r','LineWidth',2), plot(x,y,'o'), grid on

x = [5 6 6.5 5.5]; y = [1, 1, 1.5, 2];
n = length(x);
t = 0:1:n-1;
tt = 0:.1:n-1;
xx = spline(t,x,tt); yy = spline(t,y,tt); hold on
plot(xx,yy','r','LineWidth',2), plot(x,y,'o'), grid on

x = [5 6 6.5 5.5]; y = [3, 3, 2.5, 2];
n = length(x);
t = 0:1:n-1;
tt = 0:.1:n-1;
xx = spline(t,x,tt); yy = spline(t,y,tt); hold on
plot(xx,yy','r','LineWidth',2), plot(x,y,'o'), grid on
```

The drawing of my initials N.X.B is given using Spline (I can also use Bezier curves but it requires many control points so I use the default Matlab spline function for simplicity)