Numerical Analysis
MS-C1650
Tölle/Nyman
Exercises, Week 17, 2022

DATE[1]

On assignments: Submit homework to your assistant electronically via the course pages. MATLAB-assignments are submitted via MyCources.

EXERCISE 1 Let $\varphi(x) = \frac{1}{2}\left(x + \frac{3}{x}\right)$.

(a) Show that $x^* = \sqrt{3}$ is a fixed-point for φ. Are there other fixed-points?

(b) Write a MATLAB function `e=squareroot3(x,n)` that computes the errors $e_k := x_k - \sqrt{3}$ for $k = 0, 1, 2, \ldots, n$. Determine the order of the method. (Use `format short e`.)

(c) Show that the fixed-point method using ϕ converges to $\sqrt{3}$ for any $x_0 \in I = (\sqrt{3}, \infty)$.

(d) Show that the fixed-point method using ϕ converges to $\sqrt{3}$ for any $x_0 > 0$.

(e) Show that if $x_k > 0$ then

$$x_{k+1} - \sqrt{3} = \frac{1}{2x_k}(x_k - \sqrt{3})^2, \quad k = 0, 1, 2, \ldots, n.$$

Determine the order of the method.

(f) Show that the fixed-point method is in fact a Newton's method in disguise. Apply Newton to $f(x) = x^2 - 3$, $x > 0$.

EXERCISE 2 Suppose we want to approximate $f'(0)$ by $(f(h) - f(0))/h$, where $f(t) = e^t$. The following inequalities hold:

$$1 + 2^{-n} + 2^{-2n-1} < e^{2^{-n}} < 1 + 2^{-n} + 2^{-2n-1} + 2^{-3n-2},$$

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and

$$1 + 2^{-n-1} + 2^{-2n-3} < \frac{e^{2^{-n}} - 1}{h} < 1 + 2^{-n-1} + 2^{-2n-2}.$$

Let $x = e^h$, $y = 1$, and $h = 2^{-n}$. (a) Assuming the IEEE standard, estimate the accuracy of the difference approximation for different values of n. (b) Tabulate the values of $\mathrm{fl}((\mathrm{fl}(x) - y)/h)$, $\mathrm{fl}((x - y)/h)$, and $\mathrm{fl}(\mathrm{fl}(1 + h/2) + \mathrm{fl}(h^2/6))$, for $n \geq 25$. Comment on the accuracy of the series expansion.

EXERCISE 3 Show that Steffensen's method

$$x_{n+1} = x_n - \frac{f(x_n)}{g(x_n)}, \quad g(x_n) = \frac{f(x_n + f(x_n)) - f(x_n)}{f(x_n)}$$

is a second order method.

EXERCISE 4 Write and test a routine to compute $\arctan x$ for x in radians as follows. If $0 \leq x \leq 1.7 \times 10^{-9}$, set $\arctan x \approx x$. If $1.7 \times 10^{-9} < x \leq 2 \times 10^{-2}$, use the series approximation

$$\arctan x \approx x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}.$$

Otherwise, set $y = x$, $a = 0$, and $b = 1$ if $0 \leq x \leq 1$; set $y = 1/x$, $a = \pi/2$, and $b = -1$ if $1 < x$. Then set $c = \pi/16$ and $d = \tan c$ if $0 \leq y \leq \sqrt{2} - 1$; and $c = 3\pi/16$ and $d = \tan c$ if $\sqrt{2} - 1 < y \leq 1$. Compute $u = (y - d)/(1 + d\,y)$ and the approximation

$$\arctan u \approx u \left(\frac{135135 + 171962.46u^2 + 52490.4832u^4 + 2218.1u^6}{135135 + 217007.46u^2 + 97799.3033u^4 + 10721.3745u^6} \right)$$

Finally, set $\arctan x \approx a + b(c + \arctan u)$.
Test the accuracy of your routine. Report both absolute and relative errors. Is this a useful implementation?
Note: This algorithm uses telescoped rational and Gaussian continued fractions.