Aalto University

MS-C1350 Partial differential equations
Wave equation : dimension 2, nonhomogeneous problem and energy methods

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Lecture 12

- Wave equation
  - Wave equation in dimension 2
    (by ”descending” from dimension 3)
  - Non-homogeneous problem
  - Energy methods.
- The exam

6.5 Two-dimensional wave equation

- Now we consider the Cauchy problem for the wave equation when $n = 2$.
- We shall use Hadamard's method of descent.
- 2D problem = special case of 3D problem where x_3-variable does not appear.
- Let us write

$$\widetilde{u}(x_1, x_2, x_3, t) = u(x_1, x_2, t), \quad (x_1, x_2, x_3, t) \in \mathbb{R}^3 \times (0, \infty).$$

- Similarly

$$\widetilde{g}(x_1, x_2, x_3) = g(x_1, x_2) \quad \text{and} \quad \widetilde{h}(x_1, x_2, x_3) = h(x_1, x_2).$$

- If u satisfies 2D problem, $\widetilde{u}$ satisfies 3D problem:

$$\begin{cases} \widetilde{u}_{tt} - \Delta \widetilde{u} = 0 & \text{in} \quad \mathbb{R}^3 \times (0, \infty), \\ \widetilde{u} = \widetilde{g}, \quad \widetilde{u}_t = \widetilde{h} & \text{on} \quad \mathbb{R}^3 \times \{t = 0\}. \end{cases}$$

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- Kirchhoff's formula for 3D problem gives that $u(x,t) = \widetilde{u}(\widetilde{x},t)$ is

$$\frac{\partial}{\partial t}\left(t\frac{1}{|\partial\widetilde{B}(\widetilde{x},t)|}\int_{\partial\widetilde{B}(\widetilde{x},t)}\widetilde{g}(\widetilde{y})\,dS(\widetilde{y})\right)+\frac{t}{|\partial\widetilde{B}(\widetilde{x},t)|}\int_{\partial\widetilde{B}(\widetilde{x},t)}\widetilde{h}(\widetilde{y})\,dS(\widetilde{y}).$$

- Integrals are over spheres $\partial B(\widetilde{x},t) \subset \mathbb{R}^3$. The calculations become complicated as projecting 3D spheres to 2D produces a 2D ball with a non-constant weight.
- As $n = 3$, $|\partial\widetilde{B}(\widetilde{x},t)| = 4\pi t^2$.
- Let $\widetilde{y} = (y, y_3)$ with $y \in \mathbb{R}^2$, $y_3 \in \mathbb{R}$.
- Integrals are over $\widetilde{y} \in \partial\widetilde{B}(\widetilde{x},t)$ i.e.

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6.5 Integral over hemishpere

- Consider the upper hemisphere $\partial\widetilde{B}(\widetilde{x},t)^+$ of $\partial\widetilde{B}(\widetilde{x},t)$ in $\mathbb{R}^3$.
- Parametrization of the surface:

$$\widetilde{y} = (y_1, y_2, \gamma(y_1, y_2)), \quad \text{where} \quad \gamma(y_1, y_2) = (t^2 - |x-y|^2)^{1/2}.$$

- Integral of the surface using the parametrization $y = (y_1, y_2) \in \mathbb{R}^2$.

$$\int_{\partial\widetilde{B}(\widetilde{x},t)^+} F(\widetilde{y})\, dS(\widetilde{y}) = \int_{B(x,t)} F(y, \gamma(y))(1 + |\nabla\gamma(y)|^2)^{1/2}\, dy$$

- Lower hemisphere produces exactly the same integral.
- Direct calculation gives

$$|\nabla\gamma(y)| = \frac{|x-y|}{(t^2 - |x-y|^2)^{1/2}}$$

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- Applying these calculations to Kirchhoff's formula produces

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\begin{aligned}
u(x,t) &= \frac{1}{2}\frac{\partial}{\partial t}\left(t^2 \frac{1}{|B(x,t)|}\int_{B(x,t)} \frac{g(y)}{(t^2-|x-y|^2)^{1/2}}\,dy\right) \\
&\qquad + \frac{t^2}{2}\frac{1}{|B(x,t)|}\int_{B(x,t)} \frac{h(y)}{(t^2-|x-y|^2)^{1/2}}\,dy \\
&= I_1 + I_2.
\end{aligned}
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- Notice that now the solution at (x,t) depends on the initial data on the whole cone $|y-x| \leq t$, not only its boundary.
- Calculation of the t-derivative becomes easier if we make a change of variables so that the integral will be over the ball $B(0,1)$ i.e. $y = x + tz$. Then $dy = t^2 dz$ as it is 2-dimensional variable.
- Notice also that $t^2/|B(x,t)|$ is a constant.

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- Finally we arrive at the Poisson's formula for the 2-dimensional wave equation:

$$u(x,t) = \frac{1}{2}\frac{1}{|B(x,t)|}\int_{B(x,t)} \frac{tg(y) + t^2 h(y) + t\nabla g(y)\cdot(y-x)}{(t^2 - |x-y|^2)^{1/2}}\, dy.$$

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- Finite propagation speed, but no sharp signals for 2-dimensional waves (e.g. water waves).

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6.6 Nonhomogeneous problem

- We study the nonhomogeneous Cauchy problem

$$\begin{cases} u_{tt} - \Delta u = f & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u = 0, \quad u_t = 0 & \text{in} \quad \mathbb{R}^n \times \{t=0\}. \end{cases}$$

- Goal is to apply Duhamel's principle as for the heat equation.
- Let the function $u(x,t;s)$ be the solution of the Cauchy problem

$$\begin{cases} u_{tt}(x,t;s) - \Delta u(x,t;s) = 0, \quad x \in \mathbb{R}^n, \quad t > s, \\ u(x,s;s) = 0, \quad u_t(x,s;s) = f(x,s), \quad x \in \mathbb{R}^n. \end{cases}$$

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$$u(x,t) = \int_0^t u(x,t;s)\,ds, \quad x \in \mathbb{R}^n, \quad t > 0$$

is the solution to the original problem.

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- Goal is to apply Duhamel's principle as for the heat equation.
- Let the function $u(x, t; s)$ be the solution of the Cauchy problem

$$\begin{cases} u_{tt}(x, t; s) - \Delta u(x, t; s) = 0, \quad x \in \mathbb{R}^n, \quad t > s, \\ u(x, s; s) = 0, \quad u_t(x, s; s) = f(x, s), \quad x \in \mathbb{R}^n. \end{cases}$$

- Duhamel's principle suggests that

$$u(x, t) = \int_0^t u(x, t; s)\, ds, \quad x \in \mathbb{R}^n, \quad t > 0$$

is the solution to the original problem.

6.6 Nonhomogeneous problem

- Formal calculation suggests that Duhamel's principle is ok. We skip a rigorous proof.
- Nonhomogeneous problem with more general boundary values can be solved by summing up the solutions as with Laplace and heat equations.

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6.7 Energy methods

- Wave equation preserves the energy

$$e(t) = \frac{1}{2}\int_\Omega \left((u_t)^2 + |\nabla u|^2\right)\,dx, \quad 0 \le t \le T.$$

  if we have zero boundary values.

- Proof:

$$\begin{aligned} e'(t) &= \frac{1}{2}\int_\Omega \frac{\partial}{\partial t}\left(\left(\frac{\partial u}{\partial t}\right)^2 + |\nabla u|^2\right)\,dx \\ &= \frac{1}{2}\int_\Omega \left(2\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} + 2\nabla u \cdot \frac{\partial}{\partial t}\nabla u\right)\,dx \\ &= \int_\Omega \left(\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} - \frac{\partial u}{\partial t}\Delta u\right)\,dx \qquad \text{(Green's first identity)} \\ &= \int_\Omega u_t\,(u_{tt} - \Delta u)\,dx = 0. \end{aligned}$$

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