Aalto University

MS-C1350 Partial differential equations Wave equation – Physical interpretation and dimension 1 and 3

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November 19, 2024

Lecture 11

Wave equation

- Properties of solutions are different for different dimensions.
- Dimension 1: d'Alembert's formula. Its physical interpretation.
- Dimension 3: Solution by using Euler-Poisson-Darboux equation.
- Domain of dependence and the range of influence (different for different dimensions!)

6 Wave equation – introduction

- We study the wave equation in all dimensions, but focus on physically relevant cases $n = 1, 2, 3$
- Properties of the solutions depend on the dimension.
- Interpretation:
  - $n = 1$: the displacement of a vibrating string
  - $n = 2$: a vibrating membrane
  - $n = 3$: an elastic solid, the behaviour of electromagnetic waves in vacuum and the propagation of sound waves.
- The $n-$dimensional wave equation is

$$u_{tt} - \Delta u = 0$$

and the nonhomogeneous wave equation is

$$u_{tt} - \Delta u = f.$$

Here Laplace operator is taken with respect to x:

$$\Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}.$$

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6.1 Wave equation – physical interpretation

- $u = u(x,t)$: the displacement of the point x at the time t.
- If $V \subset \Omega$, the acceleration in V is
$$\frac{\partial^2}{\partial t^2} \int_V u(x,t)\,dx$$
- the net contact force:
$$-\int_{\partial V} F(x,t) \cdot \nu(x)\,dS(x),$$
- By the Gauss-Green theorem (and $F = ma$) we have
$$\int_V \operatorname{div}_x F(x,t)\,dx = \int_{\partial V} F(x,t) \cdot \nu(x)\,dS(x) = -\int_V \frac{\partial^2 u}{\partial t^2}(x,t)\,dx.$$
- This holds in every subdomain and thus we get
$$-c^2 \operatorname{div}_x \nabla u(x,t) = \operatorname{div}_x F(x,t) = -\frac{\partial^2 u}{\partial t^2}(x,t).$$

as it is physically reasonable to assume that the force F is proportional to the gradient ∇u.

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6.2 One-dimensional case

- We have showed using the Fourier transform that the solution of the Cauchy problem

$$\begin{cases} u_{tt} - \Delta u = 0 \quad \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u = g \quad \text{in} \quad \mathbb{R}^n \times \{t=0\}, \\ \dfrac{\partial u}{\partial t} = h \quad \text{in} \quad \mathbb{R}^n \times \{t=0\}. \end{cases}$$

  for the n-dimensional wave equation is

$$u(x,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\widehat{g}(\xi)\cos(|\xi|t) + \widehat{h}(\xi)\frac{\sin(|\xi|t)}{|\xi|} \right) e^{ix\cdot\xi}\, d\xi.$$

- To consider initial value problems for the wave equation in a bounded domain $\Omega_T = \Omega \times (0,\infty)$, we can use eigenfunction expansions as for the heat equation.

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- Second term:

$$\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{h}(\xi)\frac{1}{2i\xi}(e^{i\xi t} - e^{-i\xi t})e^{ix\xi}\, d\xi$$

$$= \frac{1}{2}\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi t}\widehat{H}(\xi)e^{ix\xi}\, d\xi - \frac{1}{2}\frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\xi t}\widehat{H}(\xi)e^{ix\xi}\, d\xi$$

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where

$$\widehat{H}(\xi) = \frac{\widehat{h}(\xi)}{i\xi} \qquad \text{i.e. } H'(x) = h(x).$$

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- We conclude the d'Alembert's formula:

$$\begin{aligned} u(x,t) &= \frac{1}{2}(g(x+t)+g(x-t)) + \frac{1}{2}(H(x+t)-H(x-t)) \\ &= \frac{1}{2}(g(x+t)+g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy. \end{aligned}$$

- The converse holds as well.
- The solution is of the form

$$u(x,t) = F(x+t) + G(x-t)$$

- $F(x+t)$ is a wave travelling in time with speed 1.
  - $F(x)$ is the shape of the wave at $t=0$.
  - At time t the wave has moved to left with speed one.
- $G(x-t)$ is a wave traveling to right.

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$$\begin{aligned} u(x,t) &= \frac{1}{2}(g(x+t)+g(x-t)) + \frac{1}{2}(H(x+t)-H(x-t)) \\ &= \frac{1}{2}(g(x+t)+g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy. \end{aligned}$$

- The converse holds as well.
- The solution is of the form

$$u(x,t) = F(x+t) + G(x-t)$$

- $F(x+t)$ is a wave travelling in time with speed 1.
  - $F(x)$ is the shape of the wave at $t=0$.
  - At time t the wave has moved to left with speed one.
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$$u(x,t) = \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy.$$

- d'Alembert's formula gives uniqueness.
- d'Alembert's formula gives stability:
  - u_1 is a solution with the initial values g_1 and h_1.
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  - Then $v = u_1 - u_2$ is a solution with initial values $g_1 - g_2$ and $h_1 - h_2$
  - If $|g_1 - g_2| \le \varepsilon$ and $|h_1 - h_2| \le \varepsilon$, we obtain $|v(x,t)| \le (1+t)\varepsilon$.
- The wave equation does not smoothen the solution. If $g \in C^k(\mathbb{R})$ and $h \in C^{k-1}(\mathbb{R})$, then $u \in C^k(\mathbb{R} \times (0,\infty))$.
- d'Alembert's formula makes sense even for discontinuous g and h, when corresponding u is not differentiable and thus not a solution to the wave equation.

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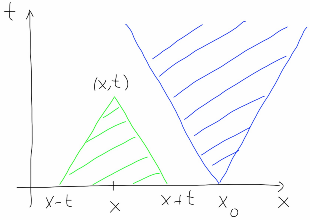
- The solution at the point (x, t) depends only on the values of g and h on the interval $[x - t, x + t]$. This is called the domain of dependence of (x, t).
- Conversely, for every $x_0 \in \mathbb{R}$, there is a conical region called the range of influence of x_0.
- Physically this means that the disturbances or signals propagate with a finite speed.



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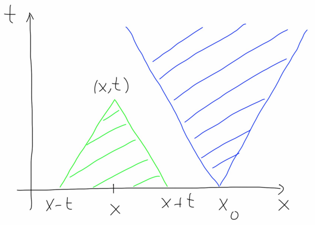

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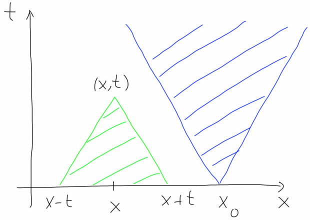

6.2 1D problem in $\mathbb{R}_+ \times \mathbb{R}_+$

- We have now a formula for the solution in 1D case where $x \in \mathbb{R}$ and $t \geq 0$.
- What if we want to solve wave equation in subdomain where $x > 0$ with extra boundary values $u(0,t) = 0$?

$$\begin{cases} u_{tt} - u_{xx} = 0 \quad \text{in} \quad \mathbb{R}_+ \times (0,\infty), \\ u = g, \quad u_t = h \quad \text{on} \quad \mathbb{R}_+ \times \{t = 0\}, \\ u = 0 \quad \text{on} \quad \{x = 0\} \times (0,\infty), \end{cases}$$

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6.3 The Euler-Poisson-Darboux equation

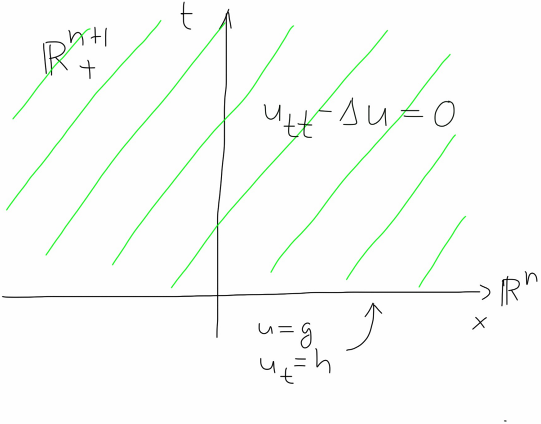

Figure: The Cauchy problem for the wave equation.

6.3 The Euler-Poisson-Darboux equation

- In higher dimensions, there is not as simple expression for the solutions as in $n = 1$ case.
- We shall use the method of spherical means:

$$U(x;r,t) = \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u(y,t)\,dS(y),$$
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- Idea: replace pointwise values by integral averages over spheres.
- We recover the functions from the averages:

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Theorem (Euler-Poisson-Darboux equation)

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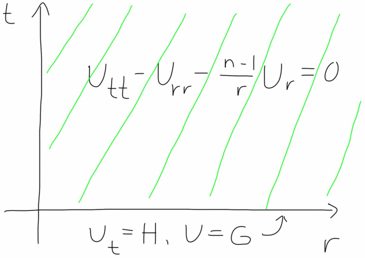

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- Recognize the radial Laplace equation $\Delta u(x) = v''(r) + \frac{n-1}{r} v'(r)$ if $u(x) = v(|x|)$.
- Proof: Start as with the proof of the mean value property for Δu to obtain.

$$U_r(x;r,t) = \frac{1}{n\alpha(n)r^{n-1}} \int_{B(x,r)} \Delta u(y,t)dy$$

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$$U_r(x;r,t) = \frac{1}{n\alpha(n) r^{n-1}} \int_0^r \int_{\partial B(x,\rho)} u_{tt}(y,t) dS(y) d\rho.$$

- Multiply by r^{n-1} and derivate with respect to r to obtain

$$\frac{\partial}{\partial r}(r^{n-1} U_r(x;r,t)) = \frac{r^{n-1}}{|\partial B(x,r)|} \int_{\partial B(x,r)} u_{tt}(y,t)\, dS(y) \quad = r^{n-1} U_{tt}(x;r,t),$$

6.4 Wave equation when $n = 3$

- Now we study the case $n = 3$.
- We continue using the integral averages U, G and H.
- and denote

$$\widetilde{U} = rU, \quad \widetilde{G} = rG \quad \text{and} \quad \widetilde{H} = rH.$$

- By the Euler-Darboux-Poisson equation

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\widetilde{U}_{tt} - \widetilde{U}_{rr} &= rU_{tt} - (U + rU_r)_r \\
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&= r(U_{tt} - U_{rr} - \frac{2}{r}U_r) = 0
\end{aligned}$$

- This works only when $n - 1 = 2$!

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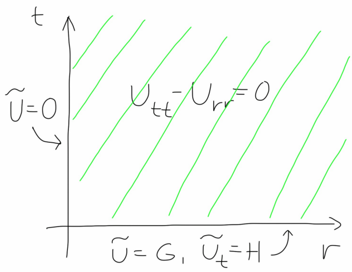

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- Notice that we are in the first quadrant i.e. need to use reflection principle.
- We recover the solution to the original problem by considering

$$u(x, t) = \lim_{r \to 0} U(x; r, t) = \lim_{r \to 0} \frac{rU(x; r, t)}{r} = \lim_{r \to 0} \frac{\widetilde{U}(x; r, t)}{r}.$$

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- Finally we obtain Kirchhoff's formula for the solution of the Cauchy problem for the three-dimensional wave equation.

$$u(x,t) = \fint_{\partial B(x,t)} g(y)\,dS(y) + t\tfrac{\partial}{\partial t}\big(\tfrac{1}{|\partial B(x,t)|}\int_{\partial B(x,t)} g(y)\,dS(y)\big) + \tfrac{t}{|\partial B(x,t)|}\int_{\partial B(x,t)} h(y)\,dS(y)$$

$$= \tfrac{1}{|\partial B(x,t)|}\int_{\partial B(x,t)} \left(th(y) + g(y) + \nabla g(y)\cdot(y-x)\right)\,dS(y).$$

- To compute $u(x,t)$, we only need information on the data on the sphere $\partial B(x,t)$
- Similarly, the range of influence of a point $x_0 \in \mathbb{R}^3$ is the surface of the (light) cone

$$\{(x,t) \in \mathbb{R}^3 \times (0,\infty) : |x - x_0| = t\}.$$

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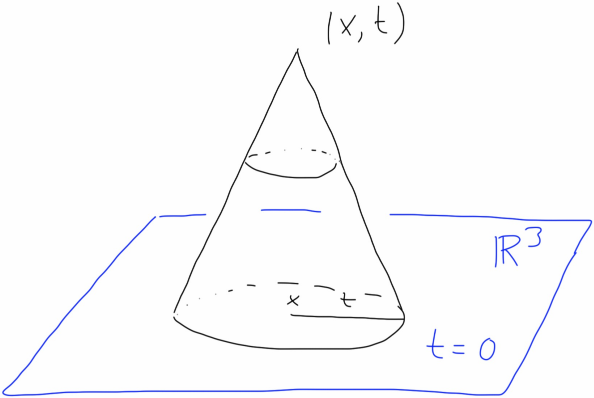

Figure: The domain of dependence in the three-dimensional case.

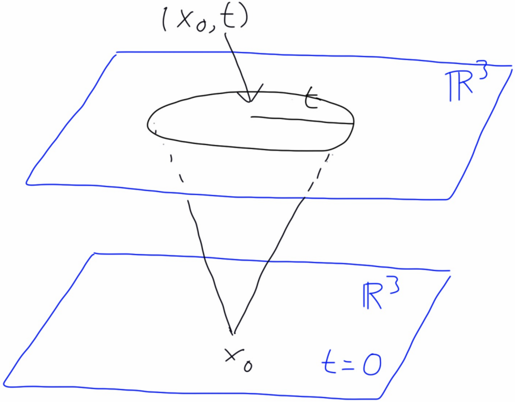

Figure: The range of influence in the three-dimensional case.

- 3D case: Information propagates at exactly unit speed, no faster and no slower! (Huygens' principle)
- 1D and 2D case: Slower is possible.
- Finite speed of propagation makes it possible to localize the process of solving initial value problems.
- Such a localization is not possible for boundary / initial value problems for Laplace and heat equation.
- in 3D case, the solution has less regularity than initial values:
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