MS-C1350 - Partial Differential Equations, Lecture, 3.9.2024-9.12.2024

This course space end date is set to 09.12.2024 **Search Courses: MS-C1350**

Learning goals

The official learning goals of the course is so study Laplace, heat and wave equation, and to learn to

- how to solve partial differential equations using separation of variables and Fourier techniques
- the physical interpretations of the equations
- the equations in the spherical and cylindrical coordinates
- the role of the fundamental solutions
- the maximum and comparison principles
- the variational forms of the equations.

During this fall, there is a plan to create more detailed list of learning goals in order to make it easier know where to focus on when studying. There will be a list of learning goals always for two weeks (weeks 1-2,3-4, etc). As this is work in progress, there may be changes in the learning goals of the earlier weeks during the fall after they are published.

In general, the focus is on understanding different phenomena related to these three equations and in being able to solve certain PDE problems. There is no need to learn lots of formulas by heart. If complicated formulas are needed in the exam, they are given in the exam papers.

Learning goals for weeks 1-2

(Lecture notes: Sections 1 and 2.1 - 2.10)

Topic	Level 1	Level 2
Well-posed problem	• What is a well-posed problem?	• Understanding why being well-posed is necessary in order to find correct physically relevant solutions.
Laplace, heat and wave equation	• Recognize the equations • Be able to name some phenomena that can be modelled by these equations. • Being able to do calculations with differential operators such as ∇, $\nabla\cdot$ and Δ that appear when working with these equations.	
Types of solutions		• What is a classical solution? • There exists also other types of solutions (e.g. weak solutions) that might not solve the PDE in every point, but that are still physically meaningful solutions.
Function spaces L^p, C, C^k and C^∞	• Know the definitions of these function spaces ($p = 1, 2, \infty$ are the most important L^p-spaces for this course)	• L^p functions are not defined at every point and they do not need to be continuous • Understand the difference between e.g. $C([0,1])$ and $C((0,1))$.
Periodic functions	• Definition • Period is not unique, but there is a minimal period	• Extending functions from an bounded interval into odd or even function.
Fourier series	• Functions can be represented as Fourier series • two options: sine/cosine function or complex exponential function • Calculating Fourier coefficients	• Partial sums of Fourier series gives the best approximation of function in L^2 with trigonometric polynomials. • Understand L^2 as an infinite dimensional vector space and the Fourier series as a representation of an L^2 function using a bases generated by trigonometric polynomials.
Change of variables	• Be able to find a suitable change of variable to transform a PDE (with possibly boundary and/or initial values) to another equivalent form.	

Learning goals for weeks 3-4

(Lecture notes: Sections 2.11-2.15)

Topic	Level 1	Level 2
Separation of variables	• Given a separation of variables, apply separation of variables technique to reduce the PDE into ODEs (or and ODE and simpler PDE). • (Solving the equations obtained by separation of variables, when they are linear first or second order ODEs with constant coefficient and when the form of the solutions is given.) • What is a special solution. • Solution with correct initial/boundary values can be obtained as a linear combination of special solutions. • How to use the boundary values to determine the correct coefficients for special solutions.	• How to choose a suitable coordinate system.
Cartesian, polar, spherical and cylindrical coordinates	• What are these coordinates? • Understand that Laplace operator can be expressed in each of these systems. (No need to learn the formulas by heart) • Apply separation of variables in each of these coordinates.	
Boundary and initial values	• What are Dirichlet and Neumann boundary values	
Initial and boundary values for the heat equation	• Initial and lateral boundary. • To have a unique solution, we need to know the initial temperature and to have a (Dirichlet or Neumann) boundary condition on lateral boundary. • We cannot set a boundary condition at the end time.	
Initial and boundary values for the wave equation	• In order to have a unique solution to the wave equation, we need to know both the initial shape ($u(x,0)$) and the initial velocity ($u_t(x,0)$) of the system. • In addition, we need a boundary condition (Dirichlet or Neumann) on the lateral boundary, if we are studying the problem in a bounded set.	
Kernels		• How solutions to the Dirichlet problem for the Laplace equation can be obtained using the Poisson kernel and convolution. • How heat kernel can be used to obtain solutions to the heat equation? • What is a good kernel / approximation of identity.

Learning goals for weeks 5-6

(Lecture notes: Section 3)

Topic	Level 1	Level 2
L^p -space	• Definition of L^p space on $\mathbb{R}^n$	
Fourier transform	• How to calculate Fourier transform and inverse transform. • What happens to differentiation and convolution in Fourier transform? (There is no need to remember the formulas in their exact form -- it is enough to know that translation becomes multiplication by exponential function with complex exponent, and that differentiation is essentially multiplication by ξ on the Fourier side etc.)	• What happens to translation, modulation and Gaussian function in Fourier transform?
Convolution	• Definition of convolution on $\mathbb{R}^n$ • Convolution becomes multiplication on the Fourier side and vice versa. • Understand the role of convolution and kernels in the solution formulas for many PDEs (e.g. Poisson and heat kernels).	• Basic properties of convolution (e.g. commutativity and associativity, smoothness)
Approximations of the identity / Good kernels		• Definition in $\mathbb{R}^n$ • Examples of good kernels • Pointwise convergence result for convolutions with good kernels.
Fourier transform and PDEs in upper-half space	• The strategy to solve PDEs using Fourier transform. • In particular, (a) with respect to which variables to apply Fourier transform in solving each of our equations, (b) how to make a Fourier transform of the equation, (c) "freezing" ξ when solving ODE.	• Understanding when to apply Fourier series and Fourier transform.
Fundamental solution.	• Fundamental solution means that all other solutions can be represented as convolutions with it. • For example, Poisson kernel and heat kernel are fundamental solutions.	

Learning goals for weeks 7-8

(Lecture notes: Sections 4.1-4.11)

Topic	Level 1	Level 2
Laplace and Poisson equation	• physical interpretation • Dirichlet and Neumann boundary conditions • Dividing problems to a "sum of two problems", where one is with zero source term and one with zero boundary values. • comparison principle	• When solutions are unique and proof(s) for uniqueness of solutions.
fundamental solution	• Fundamental solutions is a solution that corresponds to having a Delta mass at the origin • how to use fundamental solution to obtain other solutions • properties of the fundamental solution.	
Green's function		
harmonic functions	• definition • mean value properties • maximum principles (weak and strong)	• Harnack's inequality • proving maximum principle using mean value property and comparison principle, stability results and uniqueness using maximum principle.
Gauss-Green / Divergence theorem and Green's identities	• be able to apply the formulas in calculations	• understanding the physical interpretations
Green's function	• What is Green's function in a general domain?	• How to represent solutions using Green's function • The idea of using reflection principle to calculate Green's function in certain simple domains.

Learning goals for weeks 9-10

(Lecture notes: Sections 4.12-4.15 and 5)

Topic	Level 1	Level 2
Laplace equation	• Know that Laplace operator can be written also in polar (n=2), cylindrical and spherical coordinates (n=3). Ability to work with these coordinates (but no need to learn by heart the formulas!)	• Harnack's inequality for Laplace equation. • Minimization formulation for Laplace and Poisson equation. (Energy methods)
Heat equation in $\mathbb{R}^n$	• What kind of boundary conditions are needed to have a unique solution (initial values + Dirichlet or Neumann type condition on lateral boundary)? • Properties of the fundamental solutions. • Solving initial value problem using the fundamental solution and convolution. • Dividing a problem of solving non-homogenous problem with non-zero initial data into two problems.	• Duhamel's principle for solving non-homogeneous equation.
Heat equation in a bounded domain Ω_T	• (Weak) maximum principle (maximum is attained at the parabolic boundary) • Proving comparison principle, stability result and uniqueness starting from the maximum principle.	• Separation of variables (x and t) technique to transform a problem with zero lateral boundary values into an eigenvalue problem

Learning goals for weeks 11-12

(Lecture notes: Section 6: Wave equation)

Topic	Level 1	Level 2
Properties of solutions (wave equation)	• Finite speed of propagation • Wave equation does not "smoothen" the solution.	• The domain of dependence and the range of influence for dimension 1, 2 and 3.
1-dimensional wave equation	• using d'Alembert's formula for solving 1-dimensional wave equation	• Expressing the solution in form $u(x,t) = F(x+t) + G(x-t)$ and its physical interpretation. • Reflection technique for solving wave equation in half-space. • Deducing stability results from solutions formulas
3-dimensional wave equation	• Application of Kirchhoff's formula.	
2-dimensional wave equation		• Understanding the idea of "freezing" one variable to descent from dimension 3 to dimension 2.
Non-homogeneous wave equation		• The use of Duhamel's principle for solving non-homogeneous problem.