MS-C1350 Partial differential equations
Ch 2.5-2.10 Fourier series

Riikka Korte

Department of Mathematics and Systems Analysis
Aalto University
riikka.korte@aalto.fi

September 10, 2024

2.5 Other intervals

The nth partial sum of a Fourier series of f on $[a,b]$ is

$$S_n f(t) = \sum_{j=-n}^{n} \langle f, e_j \rangle e_j = \sum_{j=-n}^{n} \widehat{f}(j) e^{\frac{2\pi i j t}{b-a}}, \quad n = 0, 1, 2, \dots,$$

where the Fourier coefficients are

$$\widehat{f}(j) = \frac{1}{b-a} \int_a^b f(t) e^{\frac{-2\pi i j t}{b-a}} \, dt, \quad j \in \mathbb{Z}.$$

This follows from a change of variables.
Thus everything works in the same way as with the interval $[-\pi, \pi]$, but the formulas get more messy when the length of the interval is not the same as the period of trigonometric functions.

2.6 Real form of Fourier series

Let $f \in L^1([-\pi, \pi))$. The nth partial sum of a Fourier series can be written as

$$S_n f(t) = \frac{a_0}{2} + \sum_{j=1}^{n} (a_j \cos(jt) + b_j \sin(jt)),$$

where

$$a_j = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(jt)\, dt, \quad j = 0, 1, 2, \ldots$$

and

$$b_j = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(jt)\, dt, \quad j = 1, 2, \ldots.$$

This is called the real form of the Fourier series of f. The coefficients a_j are called the Fourier cosine coefficients of f and b_j are called the Fourier sine coefficients of f. The corresponding series are called the Fourier cosine and sine series of f correspondingly.

2.6 Real form of Fourier series

- Recall that

$$e^{ijt} = \cos(jt) + i\sin(jt), \quad e^{-ijt} = \cos(jt) - i\sin(jt).$$

- If f is real, then also a_j, b_j are real whereas $\hat{f}(j)$'s are not usually all real.

2.6 Real form of Fourier series

- Recall that

$$e^{ijt} = \cos(jt) + i\sin(jt), \quad e^{-ijt} = \cos(jt) - i\sin(jt).$$

- If f is real, then also a_j, b_j are real whereas $\hat{f}(j)$'s are not usually all real.
- Even function: $f(-t) = f(t)$. Odd function: $f(-t) = -f(t)$.

2.6 Real form of Fourier series

- Recall that

$$e^{ijt} = \cos(jt) + i\sin(jt), \quad e^{-ijt} = \cos(jt) - i\sin(jt).$$

- If f is real, then also a_j, b_j are real whereas $\hat{f}(j)$'s are not usually all real.
- Even function: $f(-t) = f(t)$. Odd function: $f(-t) = -f(t)$.
- If f is even, then $b_j = 0$ for all j.
- If f is odd, then $a_j = 0$ for all j.

2.6 Real form of Fourier series

- Recall that

$$e^{ijt} = \cos(jt) + i\sin(jt), \quad e^{-ijt} = \cos(jt) - i\sin(jt).$$

- If f is real, then also a_j, b_j are real whereas $\hat{f}(j)$'s are not usually all real.
- Even function: $f(-t) = f(t)$. Odd function: $f(-t) = -f(t)$.
- If f is even, then $b_j = 0$ for all j.
- If f is odd, then $a_j = 0$ for all j.
- Let $f : [0, L] \to \mathbb{R}$.

2.6 Real form of Fourier series

- Recall that

$$e^{ijt} = \cos(jt) + i\sin(jt), \quad e^{-ijt} = \cos(jt) - i\sin(jt).$$

- If f is real, then also a_j, b_j are real whereas $\hat{f}(j)$'s are not usually all real.
- Even function: $f(-t) = f(t)$. Odd function: $f(-t) = -f(t)$.
- If f is even, then $b_j = 0$ for all j.
- If f is odd, then $a_j = 0$ for all j.
- Let $f : [0, L] \to \mathbb{R}$.
- Even $2L$-periodic extension: $f(-t) = f(t)$. Then the series contains only $\cos$-terms.

2.6 Real form of Fourier series

- Recall that

$$e^{ijt} = \cos(jt) + i\sin(jt), \quad e^{-ijt} = \cos(jt) - i\sin(jt).$$

- If f is real, then also a_j, b_j are real whereas $\hat{f}(j)$'s are not usually all real.
- Even function: $f(-t) = f(t)$. Odd function: $f(-t) = -f(t)$.
- If f is even, then $b_j = 0$ for all j.
- If f is odd, then $a_j = 0$ for all j.
- Let $f : [0, L] \to \mathbb{R}$.
- Even $2L$-periodic extension: $f(-t) = f(t)$. Then the series contains only $\cos$-terms.
- Odd $2L$-periodic extension: $f(-t) = -f(t)$. Then the series contains only $\sin$-terms.

2.6 Real form of Fourier series

- Recall that

$$e^{ijt} = \cos(jt) + i\sin(jt), \quad e^{-ijt} = \cos(jt) - i\sin(jt).$$

- If f is real, then also a_j, b_j are real whereas $\hat{f}(j)$'s are not usually all real.
- Even function: $f(-t) = f(t)$. Odd function: $f(-t) = -f(t)$.
- If f is even, then $b_j = 0$ for all j.
- If f is odd, then $a_j = 0$ for all j.
- Let $f : [0, L] \to \mathbb{R}$.
- Even $2L$-periodic extension: $f(-t) = f(t)$. Then the series contains only $\cos$-terms.
- Odd $2L$-periodic extension: $f(-t) = -f(t)$. Then the series contains only $\sin$-terms.
- This does not violate the uniqueness of Fourier series as the functions are different on $[-L, 0]$.

2.7 Differentiation and Fourier series

- The smoother the function, the faster the Fourier coefficients decay.

2.7 Differentiation and Fourier series

- The smoother the function, the faster the Fourier coefficients decay.
- Fourier transform of a derivative: Let $f \in C^1(\mathbb{R})$ be 2π-periodic. Then

$$\widehat{f'}(j) = ij\hat{f}(j), \quad j \in \mathbb{Z}.$$

2.7 Differentiation and Fourier series

- The smoother the function, the faster the Fourier coefficients decay.
- Fourier transform of a derivative: Let $f \in C^1(\mathbb{R})$ be 2π-periodic. Then

$$\widehat{f'}(j) = ij\hat{f}(j), \quad j \in \mathbb{Z}.$$

- Reason: Fourier series can be differentiated termwise:

$$f(t) = \sum_{j=-\infty}^{\infty} \widehat{f}(j)e^{ijt} \Longrightarrow f'(t) = \sum_{j=-\infty}^{\infty} ij\widehat{f}(j)e^{ijt}.$$

2.8 Dirichlet kernel

- Dirichlet kernel is a useful way of expressing $S_n f$.

$$S_n f(t) = \sum_{j=-n}^{n} \widehat{f}(j) e^{ijt}$$

2.8 Dirichlet kernel

- Dirichlet kernel is a useful way of expressing $S_n f$.

$$S_n f(t) = \sum_{j=-n}^{n} \widehat{f}(j) e^{ijt} = \sum_{j=-n}^{n} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) e^{-ijs}\, ds \right) e^{ijt}$$

2.8 Dirichlet kernel

- Dirichlet kernel is a useful way of expressing $S_n f$.

$$S_n f(t) = \sum_{j=-n}^{n} \widehat{f}(j) e^{ijt} = \sum_{j=-n}^{n} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) e^{-ijs} \, ds \right) e^{ijt}$$

$$= \frac{1}{2\pi} \sum_{j=-n}^{n} \int_{-\pi}^{\pi} f(s) e^{ij(t-s)} \, ds$$

2.8 Dirichlet kernel

- Dirichlet kernel is a useful way of expressing $S_n f$.

$$S_n f(t) = \sum_{j=-n}^{n} \widehat{f}(j) e^{ijt} = \sum_{j=-n}^{n} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) e^{-ijs}\, ds \right) e^{ijt}$$

$$= \frac{1}{2\pi} \sum_{j=-n}^{n} \int_{-\pi}^{\pi} f(s) e^{ij(t-s)}\, ds = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) \left(\sum_{j=-n}^{n} e^{ij(t-s)} \right)$$

2.8 Dirichlet kernel

- Dirichlet kernel is a useful way of expressing $S_n f$.

$$S_n f(t) = \sum_{j=-n}^{n} \widehat{f}(j) e^{ijt} = \sum_{j=-n}^{n} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) e^{-ijs}\, ds \right) e^{ijt}$$

$$= \tfrac{1}{2\pi} \sum_{j=-n}^{n} \int_{-\pi}^{\pi} f(s) e^{ij(t-s)}\, ds = \tfrac{1}{2\pi} \int_{-\pi}^{\pi} f(s) \left(\sum_{j=-n}^{n} e^{ij(t-s)} \right)$$

- Now we define the Dirichlet kernel as

$$D_n(t) = \sum_{j=-n}^{n} e^{ijt}, \quad n = 0, 1, 2, \ldots .$$

2.8 Dirichlet kernel

- Dirichlet kernel is a useful way of expressing $S_n f$.

$$S_n f(t) = \sum_{j=-n}^{n} \widehat{f}(j) e^{ijt} = \sum_{j=-n}^{n} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) e^{-ijs}\, ds \right) e^{ijt}$$

$$= \frac{1}{2\pi} \sum_{j=-n}^{n} \int_{-\pi}^{\pi} f(s) e^{ij(t-s)}\, ds = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) \left(\sum_{j=-n}^{n} e^{ij(t-s)} \right)$$

- Now we define the Dirichlet kernel as

$$D_n(t) = \sum_{j=-n}^{n} e^{ijt}, \quad n = 0, 1, 2, \ldots.$$

- With this definition we have the formula

$$S_n f(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) D_n(t-s)\, ds, \quad t \in [-\pi, \pi], \quad n = 0, 1, 2, \ldots.$$

Properties of the Dirichlet kernel

- $$D_n f(t) = \sum_{j=-n}^{n} e^{ijt}$$

Properties of the Dirichlet kernel

- $$D_n f(t) = \sum_{j=-n}^{n} e^{ijt} = \frac{\sin\left(\left(n+\frac{1}{2}\right)t\right)}{\sin\left(\frac{1}{2}t\right)}, \quad t \neq 0, \quad n = 0, 1, 2, \dots .$$

- The Dirichlet kernel is a 2π-periodic function.

Properties of the Dirichlet kernel

- $$D_n f(t) = \sum_{j=-n}^{n} e^{ijt} = \frac{\sin\left(\left(n+\frac{1}{2}\right)t\right)}{\sin\left(\frac{1}{2}t\right)}, \quad t \neq 0, \quad n = 0, 1, 2, \ldots.$$
- The Dirichlet kernel is a 2π-periodic function.
- $D_n(0) = 2n+1$ and $D_n(\pi) = (-1)^n$, $n = 0, 1, 2, \ldots$.

Properties of the Dirichlet kernel

- $$D_n f(t) = \sum_{j=-n}^{n} e^{ijt} = \frac{\sin\left(\left(n+\frac{1}{2}\right)t\right)}{\sin\left(\frac{1}{2}t\right)}, \quad t \neq 0, \quad n = 0, 1, 2, \ldots .$$
- The Dirichlet kernel is a 2π-periodic function.
- $D_n(0) = 2n + 1$ and $D_n(\pi) = (-1)^n$, $n = 0, 1, 2, \ldots$.
- The Dirichlet formula can be written as a convolution

$$S_n f(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) D_n(t-s)\, ds = (D_n * f)(t), \quad t \in [-\pi, \pi].$$

Convolution

- Convolution is an important notation in Fourier analysis and PDEs.

Convolution

- Convolution is an important notation in Fourier analysis and PDEs.
- Let $f, g \in C(\mathbb{R})$ be 2π-periodic.
  The convolution of f and g on $[-\pi, \pi]$ is

$$(f * g)(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t-s)g(s)\,ds = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s)g(t-s)\,ds.$$

Convolution

- Convolution is an important notation in Fourier analysis and PDEs.
- Let $f, g \in C(\mathbb{R})$ be 2π-periodic.
  The convolution of f and g on $[-\pi, \pi]$ is

$$(f * g)(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t-s)g(s)\, ds = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s)g(t-s)\, ds.$$

- If $g = 1$ then $f * g$ is constant.
  Moral: Convolutions can be considered as weighted averages of the functions.

Convolution

- Convolution is an important notation in Fourier analysis and PDEs.
- Let $f, g \in C(\mathbb{R})$ be 2π-periodic.
  The convolution of f and g on $[-\pi, \pi]$ is

$$(f * g)(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t-s)g(s)\,ds = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)g(t-s)\,ds.$$

- If $g = 1$ then $f * g$ is constant.
  Moral: Convolutions can be considered as weighted averages of the functions.
- $f * g$ is also 2π-periodic.

Convolution

- Convolution is an important notation in Fourier analysis and PDEs.
- Let $f, g \in C(\mathbb{R})$ be 2π-periodic.
  The convolution of f and g on $[-\pi, \pi]$ is

$$(f * g)(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t-s)g(s)\, ds = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s)g(t-s)\, ds.$$

- If $g = 1$ then $f * g$ is constant.
  Moral: Convolutions can be considered as weighted averages of the functions.
- $f * g$ is also 2π-periodic.
- $f * g = g * f$ and $f * (g * h) = (f * g) * h$.

Convolution

- Convolution is an important notation in Fourier analysis and PDEs.
- Let $f, g \in C(\mathbb{R})$ be 2π-periodic.
  The convolution of f and g on $[-\pi, \pi]$ is

$$(f * g)(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t-s)g(s)\,ds = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s)g(t-s)\,ds.$$

- If $g = 1$ then $f * g$ is constant.
  Moral: Convolutions can be considered as weighted averages of the functions.
- $f * g$ is also 2π-periodic.
- $f * g = g * f$ and $f * (g * h) = (f * g) * h$.
- $\widehat{(f * g)}(j) = \widehat{f}(j)\widehat{g}(j)$, $j \in \mathbb{Z}$.

Pointwise convergence

Theorem

Let $f \in C([-\pi, \pi])$ be a 2π-periodic function which is differentiable at some point $t_0 \in [-\pi, \pi]$. Then

$$\lim_{n \to \infty} S_n f(t_0) = f(t_0).$$

Pointwise convergence

Theorem

Let $f \in C([-\pi, \pi])$ be a 2π-periodic function which is differentiable at some point $t_0 \in [-\pi, \pi]$. Then

$$\lim_{n\to\infty} S_n f(t_0) = f(t_0).$$

- This shows that the convergence of the Fourier series depends only on the behaviour of the function in an arbitrarily small neighbourhood of the point.

Pointwise convergence

Theorem

Let $f \in C([-\pi, \pi])$ be a 2π-periodic function which is differentiable at some point $t_0 \in [-\pi, \pi]$. Then

$$\lim_{n \to \infty} S_n f(t_0) = f(t_0).$$

- This shows that the convergence of the Fourier series depends only on the behaviour of the function in an arbitrarily small neighbourhood of the point.
- There exists a function $f \in L^1([-\pi, \pi])$ whose Fourier series diverges at every point.

Pointwise convergence

Theorem

Let $f \in C([-\pi, \pi])$ be a 2π-periodic function which is differentiable at some point $t_0 \in [-\pi, \pi]$. Then

$$\lim_{n\to\infty} S_n f(t_0) = f(t_0).$$

- This shows that the convergence of the Fourier series depends only on the behaviour of the function in an arbitrarily small neighbourhood of the point.
- There exists a function $f \in L^1([-\pi, \pi])$ whose Fourier series diverges at every point.
- There are also continuous functions whose Fourier series diverge in a dense set.