MS-C1350 Partial differential equations
Chapter 3.9-3.12
Laplace, heat and wave equation in the upper half space

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October 8, 2024

Lecture 6

Examples on how to use Fourier transform for solving

- Laplace equation in the upper half-space $\mathbb{R}^{n+1}_+$
- Heat equation in the upper half-space $\mathbb{R}^{n+1}_+$

Solving Laplace, heat and wave equation in upper half space using Fourier transform

Find a solution of an initial/boundary value problem in $\mathbb{R}^{n+1}_+$.

1. Take the Fourier transform of the PDE and of the initial conditions, with respect to the space variables:

$$(x, y) \quad \longleftrightarrow \quad (\xi, y) \quad \text{or} \quad (x, t) \quad \longleftrightarrow \quad (\xi, t)$$

2. This reduces the problem to an ODE.
3. The ODE is solved on the Fourier side.
4. The initial or boundary conditions are used to determine the free parameters.
5. The Fourier inversion formula gives the solution of the original problem.
6. The solution of the original problem is represented as a convolution of the data with the fundamental solution.
7. This gives a solution to the original problem and the initial or boundary values are attained by using approximations of the unity.

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3.9 Laplace equation (1/8)

- we consider the Laplace equation in the upper half-space

$$\mathbb{R}^{n+1}_+ = \{(x, y) : x \in \mathbb{R}^n, \, y > 0\}.$$

- Suppose function g is continuous and bounded on the boundary $\partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times \{0\}$.

$$\begin{cases} \Delta u(x,y) = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0, & (x,y) \in \mathbb{R}^{n+1}_+, \\ u(x,0) = g(x), & x \in \mathbb{R}^n. \end{cases}$$

- Note: Laplacian with $n+1$ variables!
- Physically: Temperature on the boundary is g and the temperature is not changing in upper half space.
- In disc, 'periodicity' of the boundary reduced the number of solutions to the separated equations (countable number).

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Laplace equation (2/8)

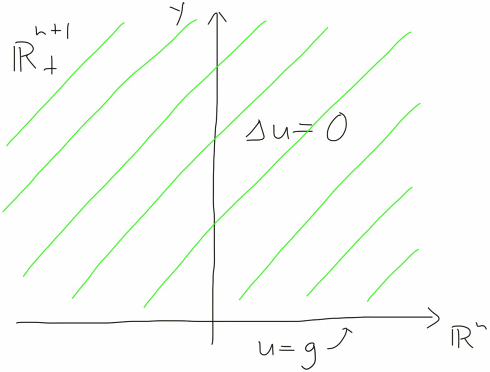

Figure: The Laplace equation in the upper half-space.

3.9 Laplace equation (3/8)

- **Step1: PDE on the Fourier side**
- Fix $y > 0$ and make the Fourier transform of $u(\cdot, y)$:

$$\widehat{u}(\xi, y) = \int_{\mathbb{R}^n} u(x, y) e^{-ix\cdot\xi}\, dx, \quad \xi \in \mathbb{R}^n, \quad y > 0.$$

- Differentiation becomes multiplication in Fourier transform:

$$\widehat{\frac{\partial^2 u}{\partial x_j^2}}(\xi, y) = i\xi_j \widehat{\frac{\partial u}{\partial x_j}}(\xi, y) = (i\xi_j)^2 \widehat{u}(\xi, y) = -\xi_j^2 \widehat{u}(\xi, y),\ j = 1, \ldots, n,$$

- As we did not make the transform with respect to y, $\partial^2/\partial y^2$ is not changed:

$$\begin{aligned}\widehat{\frac{\partial^2 u}{\partial y^2}}(\xi, y) &= \int_{\mathbb{R}^n} \frac{\partial^2 u}{\partial y^2}(x, y) e^{-ix\cdot\xi}\, dx \\ &= \frac{\partial^2}{\partial y^2}\left(\int_{\mathbb{R}^n} u(x, y) e^{-ix\cdot\xi}\, dx\right) = \frac{\partial^2 \widehat{u}}{\partial y^2}(\xi, y).\end{aligned}$$

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- So now we have:

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$$\text{and} \quad \widehat{\frac{\partial^2 u}{\partial y^2}}(\xi, y) = \frac{\partial^2 \widehat{u}}{\partial y^2}(\xi, y).$$

- It follows that

$$\begin{aligned} 0 &= \widehat{\Delta u}(\xi, y) \\ &= -\xi_1^2 \widehat{u}(\xi, y) - \dots - \xi_n^2 \widehat{u}(\xi, y) + \frac{\partial^2 \widehat{u}}{\partial y^2}(\xi, y) \\ &= -|\xi|^2 \widehat{u}(\xi, y) + \frac{\partial^2 \widehat{u}}{\partial y^2}(\xi, y). \end{aligned}$$

- Note that (under appropriate assumptions) $\Delta u = 0$ is equivalent to $\widehat{\Delta u} = 0$.

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3.9 Laplace equation (5/8)

- **Step 2: Solution on the Fourier side**
- Fix ξ and solve

$$-|\xi|^2\widehat{u}(\xi,y)+\frac{\partial^2\widehat{u}}{\partial y^2}(\xi,y)=0.$$

- The solution is

$$\widehat{u}(\xi,y)=c_1(\xi)e^{|\xi|y}+c_2(\xi)e^{-|\xi|y}.$$

- Note that the 'constants' c_1 and c_2 can now depend on ξ.
- We want to have physically relevant solutions and therefore we only keep the second term.
- Boundary condition on the Fourier side: $c_2(\xi)=\widehat{u}(\xi,0)=\widehat{g}(\xi)$.
- Thus we get

$$\widehat{u}(\xi,y)=\widehat{g}(\xi)e^{-|\xi|y},\quad \xi\in\mathbb{R}^n,\quad y>0.$$

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3.9 Laplace equation (6/8)

- **Step 3: Solution to the original problem**
- We use the Fourier inverse theorem:

$$
\begin{aligned}
u(x,y) &= (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{u}(\xi,y) e^{ix\cdot\xi}\, d\xi \\
&= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-|\xi|y} \widehat{g}(\xi) e^{ix\cdot\xi}\, d\xi \\
&= (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{P_y}(\xi) \widehat{g}(\xi) e^{ix\cdot\xi}\, d\xi,
\end{aligned}
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  where $\widehat{P_y}(\xi) = e^{-|\xi|y}$.
- The function $P_y(x)$ is called the Poisson kernel for the upper half-space.
- Note: At the moment we do not know what P_y is, only $\widehat{P_y}$!

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  where $\widehat{P_y}(\xi) = e^{-|\xi|y}$.
- The function $P_y(x)$ is called the Poisson kernel for the upper half-space.
- Note: At the moment we do not know what P_y is, only $\widehat{P}_y$!

3.9 Laplace equation (7/8)

- Now we see why this P_y-notation is useful:

$$\begin{aligned} u(x,y) &= (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{P_y}(\xi)\widehat{g}(\xi)e^{ix\cdot\xi}\,d\xi \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{P_y * g}(\xi)e^{ix\cdot\xi}\,d\xi \\ &= (P_y * g)(x) = \int_{\mathbb{R}^n} P_y(x-z)g(z)\,dz. \end{aligned}$$

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3.9 Laplace equation (8/8)

- **Step 4: Explicit representation formula**
- $$P_y(x) = P(x,y) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \frac{y}{(|x|^2+y^2)^{\frac{n+1}{2}}}, \quad x \in \mathbb{R}^n, \quad y > 0.$$
- Here $\Gamma(\frac{n+1}{2})$ is a dimensional constant given by the Γ-function
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3.9 Poisson kernel

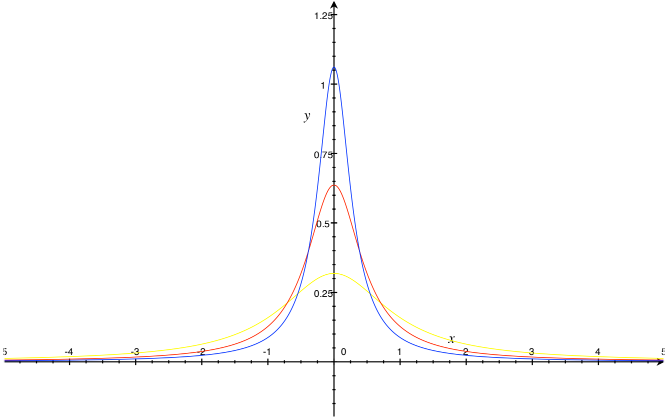

Figure: The graph of the Poisson kernel in dimension $n = 1$ for $y = 1$ (yellow), $y = 0.5$ (red) and $y = 0.3$ (blue).

3.9 Properties of the Poisson kernel

- $P_y(x) > 0$ for every $x \in \mathbb{R}^n$ and $y > 0$.
- For every $y > 0$ we have

$$\int_{\mathbb{R}^n} P_y(x)\, dx = \widehat{P_y}(0) = 1.$$

- The Poisson kernel $P(x, y) = P_y(x)$ is a solution of the Laplace equation in the upper half-space $\mathbb{R}^{n+1}_+$.
- It is called the fundamental solution in the upper half-space $\mathbb{R}^{n+1}_+$, since all other solutions can be represented as a convolution with it.

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3.9 Laplace equation

Theorem

Let $g \in C_0^\infty(\mathbb{R}^n)$. The solution to the Dirichlet problem is

$$u(x,y) = (g * P_y)(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \int_{\mathbb{R}^n} \frac{g(z)y}{(|x-z|^2 + y^2)^{\frac{n+1}{2}}}\, dz.$$

The boundary condition is taken in the sense that

$$\lim_{y\to 0} u(x,y) = g(x) \quad \textit{for every} \quad x \in \mathbb{R}^n.$$

3.10 Heat equation in the upper half-space

- The general form of the heat equation is

$$u_t(x,t) - \Delta u(x,t) = 0,$$

where the Laplace operator is only in the x-variable

$$\Delta u(x,t) = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}(x,t).$$

- We consider the initial value problem

$$\begin{cases} u_t(x,t) - \Delta u(x,t) = 0, & x \in \mathbb{R}^n, \quad t > 0, \\ u(x,0) = g(x), & x \in \mathbb{R}^n. \end{cases}$$

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Heat equation in the upper half-space

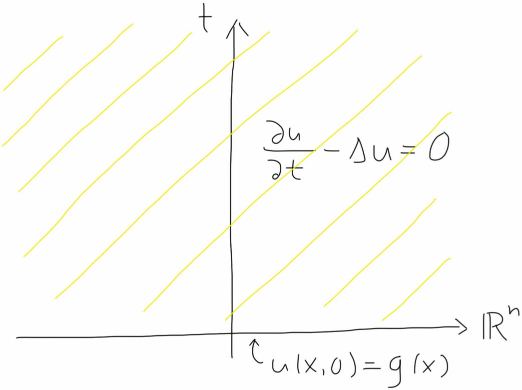

Figure: The heat equation in the upper half-space.

3.10 Heat equation in the upper half-space

- **Step 1: PDE on the Fourier side**
- Let $t > 0$ be fixed and denote by $\widehat{u}(\xi, t)$ the Fourier transform of $u(x, t)$ in the x-variable

$$\widehat{u}(\xi, t) = \int_{\mathbb{R}^n} u(x, t) e^{-ix\cdot\xi} dx.$$

- On the Fourier side, the heat equation becomes

$$0 = \widehat{u_t - \Delta u}(\xi, t) = \widehat{u}_t(\xi, t) - \widehat{\Delta u}(\xi, t) = \frac{\partial \widehat{u}}{\partial t}(\xi, t) + |\xi|^2 \widehat{u}(\xi, t).$$

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3.10 Heat equation in the upper half-space

- The rest of the steps are very similar to the Laplace equation.
- Instead of the Poisson kernel, we obtain the heat kernel

$$H_t(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}. \quad x \in \mathbb{R}^n, \quad t > 0,$$

Theorem

Assume that $g \in C_0^\infty(\mathbb{R}^n)$. The solution to Cauchy problem is

$$u(x,t) = (H_t * g)(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\,dy, \quad x \in \mathbb{R}^n, \quad t > 0.$$

The initial values are attained in the sense

$$\lim_{t \to 0} u(x,t) = g(x) \quad \text{for every} \quad x \in \mathbb{R}^n.$$

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- One can think of the wave equation as describing the displacement of a vibrating string (in dimension $n = 1$), a vibrating membrane (in dimension $n = 2$) or an elastic solid (dimension $n = 3$). In dimension $n = 3$ this equation also determines the behavior of electromagnetic waves in vacuum and the propagation of sound waves.

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3.11 Wave equation in the upper half-space

- The general goal is to find a solution of the Cauchy problem for the wave equation

$$\begin{cases} u_{tt} - \Delta u = 0, \quad x \in \mathbb{R}^n, \quad t > 0, \\ u(x,0) = g(x), \quad x \in \mathbb{R}^n, \\ u_t(x,0) = h(x), \quad x \in \mathbb{R}^n. \end{cases}$$

- **Step 1: PDE on the Fourier side.** Let $t > 0$ be fixed and denote by $\widehat{u}(\xi,t)$ the Fourier transform of $u(x,t)$ in the x-variable

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- The solution of this ODE for a fixed $\xi \in \mathbb{R}^n$ is

$$\widehat{u}(\xi, t) = c_1(\xi) \cos(|\xi| t) + c_2(\xi) \sin(|\xi| t)$$

for some functions $c_1(\xi)$ and $c_2(\xi)$.

- The initial conditions on the Fourier side:

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3.11 Wave equation in the upper half-space

- The next step is to use the inverse transform in order to find u.
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- Then

$$\begin{aligned} u(x,t) &= (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\widehat{g}(\xi)\widehat{\Psi_t}(\xi) + \widehat{h}(\xi)\widehat{\Phi_t}(\xi) \right) e^{ix\cdot\xi}\, d\xi \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\widehat{g * \Psi_t}(\xi) + \widehat{h * \Phi_t}(\xi) \right) e^{ix\cdot\xi}\, d\xi \\ &= (g * \Psi_t)(x) + (h * \Phi_t)(x). \end{aligned}$$

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