Aalto University

MS-C1350 Partial differential equations
Chapter 4.6-4.10

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Lecture 8

Laplace / Poisson equation continues

1. Green's function
   - What is Green's function in general domains?
   - Representing solutions using Green's function.
   - Green's function on upper half-space
   - Green's function for a ball.
2. Mean value property
3. Maximum principle and stability

Green's function (4.6) (intro)

- Goal: To use fundamental solution of the Laplace equation in $\mathbb{R}^n$ to solve a Dirichlet problem of the Poisson equation in $\Omega \subset \mathbb{R}^n$.
- Fundamental solution: Usually a solution to some PDE in the whole space except for one point, where it has a singularity corresponding to a unit mass.
- Green's function: A function that is a solution in Ω, has a unit mass singularity at one point $x \in \Omega$ and has zero boundary values at the boundary $\partial\Omega$.
- We only consider bounded domains $\Omega \subset \mathbb{R}^n$, but essentially fundamental solution is the same as the Green's function for $\mathbb{R}^n$.
- The term fundamental solution is used in various contexts for solutions that can be used to construct other solutions.
- The term Green's function can be used also when using other than zero boundary values.

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Green's function (4.6) (intro 2)

- Let Ω be an open and bounded domain with smooth enough boundary.
- We consider the Dirichlet problem for the Poisson equation

$$\begin{cases} -\Delta u = f \quad \text{in} \quad \Omega, \\ u = g \quad \text{on} \quad \partial\Omega. \end{cases}$$

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4.6 Delta distribution

Let us write δ_x for the generalized function (distribution) that has the property

$$\int_{\mathbb{R}^n} \phi(y)\delta_x(y)\,dy = \phi(x) \quad \text{for all} \quad \phi \in C_0^\infty(\mathbb{R}^n).$$

This generalized function is called Dirac's delta mass at x.

- Fundamental solution solves

$$\Delta\Phi = \delta_0$$

- Green's function for Laplace equation in Ω solves

$$\begin{cases} \Delta_y G(x,y) = \delta_x & y \in \Omega \\ G(x,y) = 0 & y \in \partial\Omega \end{cases}$$

- Formally $u = f * \Phi$ is a solution for $-\Delta u = f$, as

$$-\Delta u(x) = -\int_{\mathbb{R}^n} \Delta\Phi(y) f(x-y)\,dy = \int_{\mathbb{R}^n} \delta_0(y) f(x-y)\,dy = f(x).$$

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4.6 Representation formula for all functions

Theorem

Assume that $\Omega \subset \mathbb{R}^n$ is a bounded open set with a smooth boundary and $u \in C^2(\overline{\Omega})$. Then

$$u(x) = \int_{\partial\Omega} \Phi(y-x)\frac{\partial u}{\partial \nu}(y)\,dS(y) - \int_{\partial\Omega} u(y)\frac{\partial \Phi}{\partial \nu}(y-x)\,dS(y) - \int_{\Omega} \Phi(y-x)\Delta u(y)\,dy,$$

for every $x \in \Omega$. Here ν denotes, as usual, the outward pointing unit normal vector of Ω.

- This representation formula works for all smooth enough functions, not only to solutions to some PDE's
- This allows us to determine u if we know Δu in Ω as well as u and its normal derivative $\partial u/\partial \nu$ on $\partial\Omega$.
- Laplace equation: We can only "choose" u or $\partial u/\partial \nu$ on $\partial\Omega$!
- Replace Φ with Green's function G: we get a similar formula without the problematic term.

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4.6 Applications of the representation formula

- If $u \in C_0^2(\mathbb{R}^n)$, then 2 of the 3 terms in the previous theorem vanish and we get

$$u(x) = -\int_{\mathbb{R}^m} \Phi(y-x)\Delta_y u(y)\, dy.$$

- If $\Delta u = 0$, then

$$u(x) = \int_{\partial\Omega} \Phi(y-x)\frac{\partial u}{\partial \nu}(y)\, dS(y) - \int_{\partial\Omega} u(y)\frac{\partial \Phi}{\partial \nu}(y-x)\, dS(y)$$

- If we apply the previous formula to $u = 1$, we obtain

$$-\int_{\partial\Omega} \frac{\partial \Phi}{\partial \nu}(y-x)\, dS(y) = 1$$

for every $x \in \Omega$. This is related to the normalization of the fundamental solution.

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4.6 Application: Poisson equation with Dirichlet boundary data

- Let us look at the representation problem in connection with the Dirichlet problem for Poisson equation:

$$u(x) = \int_{\partial\Omega} \Phi(y-x)\frac{\partial u}{\partial \nu}(y)\,dS(y) - \int_{\partial\Omega} u(y)\frac{\partial \Phi}{\partial \nu}(y-x)\,dS(y) - \int_{\Omega} \Phi(y-x)\Delta u(y)\,dy,$$

- We require that $\Delta u = f$ in Ω and $u = g$ on $\partial\Omega$, but $\partial u/\partial \nu$ is unknown.
- We solve this problem by adding a harmonic function to the fundamental solution. For a fixed $x \in \Omega$, let $\phi^x = \phi^x(y)$ be a corrector function, which is a solution the Dirichlet problem

$$\begin{cases} \Delta_y \phi^x(y) = 0, & y \in \Omega, \\ \phi^x(y) = \Phi(y-x), & y \in \partial\Omega. \end{cases}$$

- Interpretation: we solve the Dirichlet problem with boundary values given by the fundamental solution.

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4.6 Green's function

- The Green's function for Ω is

$$G(x,y) = \Phi(y-x) - \phi^x(y), \quad x,y \in \Omega, \quad x \neq y,$$

where $\phi^x = \phi^x(y)$ is a solution the Dirichlet problem

$$\begin{cases} \Delta_y \phi^x(y) = 0, & y \in \Omega, \\ \phi^x(y) = \Phi(y-x), & y \in \partial\Omega. \end{cases}$$

- Formally, G satisfies

$$\begin{cases} \Delta_y G(x,y) = \delta_x & y \in \Omega \\ G(x,y) = 0 & y \in \partial\Omega \end{cases}$$

4.6 Green's function

- By using the representation formula, Green's second identity and the definition of the Green's function, we finally (after quite long calculations) obtain that

$$u(x) = -\int_{\partial\Omega} u(y)\frac{\partial G}{\partial \nu}(x,y)\,dS(y) - \int_{\Omega} \Delta u(y) G(x,y)\,dy,$$

  where

$$\frac{\partial G}{\partial \nu}(x,y) = \nabla_y G(x,y) \cdot \nu(y).$$

- This holds for all $u \in C^2(\overline{\Omega})$. It allows us to determine u if we know Δu and the value of u on the boundary.

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4.6 Poisson equation with Green's function

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then

$$u(x) = -\int_{\partial\Omega} g(y) \frac{\partial G}{\partial \nu}(x,y)\, dS(y) + \int_{\Omega} f(y) G(x,y)\, dy,$$

where G is the Green's function for Ω.

4.7 Green's function for the upper half space

- Consider the upper half-space $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) : x_n > 0\}$.
- This is not a bounded domain so the previous arguments do not apply directly.
- In order to construct the Green's function, for every $x \in \mathbb{R}^n_+$, we need to construct a corrector function ϕ^x such that

$$\begin{cases} \Delta \phi^x(y) = 0, & y \in \mathbb{R}^n_+, \\ \phi^x(y) = \Phi(y-x), & y \in \partial\mathbb{R}^n_+. \end{cases}$$

- The Green's function for $\mathbb{R}^n_+$ will then be

$$G(x,y) = \Phi(y-x) - \phi^x(y).$$

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$$\begin{cases} \Delta \phi^x(y) = 0, & y \in \mathbb{R}^n_+, \\ \phi^x(y) = \Phi(y - x), & y \in \partial \mathbb{R}^n_+. \end{cases}$$

- The Green's function for $\mathbb{R}^n_+$ will then be

$$G(x, y) = \Phi(y - x) - \phi^x(y).$$

4.7 Green's function for the upper half space

- Consider the upper half-space $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) : x_n > 0\}$.
- This is not a bounded domain so the previous arguments do not apply directly.
- In order to construct the Green's function, for every $x \in \mathbb{R}^n_+$, we need to construct a corrector function ϕ^x such that

$$\begin{cases} \Delta \phi^x(y) = 0, & y \in \mathbb{R}^n_+, \\ \phi^x(y) = \Phi(y - x), & y \in \partial \mathbb{R}^n_+. \end{cases}$$

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4.7 Reflection principle

- We want to solve (for $x \in \mathbb{R}^n_+$).

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- Let us reflect the vector x across the boundary $\partial \mathbb{R}^n_+$:

$$x^* = (x_1, \dots, x_{n-1}, -x_n).$$

- $y \mapsto \Phi(y - x^*)$ is harmonic in the whole $\mathbb{R}^n_+$ as $x^* \in \mathbb{R}^n_-$.
- Moreover, $x = x^*$ on $\partial \mathbb{R}^n_+$ and thus

$$\Phi(y - x^*) = \Phi(y - x).$$

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4.7 Laplace equation in the upper half space

- We are considering the problem:

$$\begin{cases} \Delta u(y) = 0, & y \in \mathbb{R}^n_+, \\ u(y) = g(y), & y \in \partial\mathbb{R}^n_+. \end{cases}$$

- By inserting the Green's function to the representation formula, we obtain

$$u(x) = -\int_{\partial\mathbb{R}^n_+} \frac{\partial G}{\partial \nu}(x,y) g(y)\, dS(y).$$

- By deriving an explicit expression for $\frac{\partial G}{\partial \nu}(x,y)$, we get

$$u(x) = \frac{2x_n}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+} \frac{g(y)}{|x-y|^n}\, dy.$$

- We derived the same formula earlier with Fourier transform i.e. the solution is the convolution of the Poisson kernel and the boundary values.

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4.8 Green's function for a ball

- We shall again use the method of reflection to construct a Green's function.
- We need to find the corrector function ϕ^x that satisfies

$$\begin{cases} \Delta\phi^x(y) = 0 & y \in B(0,1), \\ \phi^x(y) = \Phi(y-x), & y \in \partial B(0,1). \end{cases}$$

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4.9 Mean value formulas

- 1D-case: Laplace equation: $u'' = 0$. Harmonic functions:

$$u(x) = ax + b, \quad a, b \ \in \mathbb{R}.$$

- The value of these functions at the midpoint of the interval is the same as the arthmetic average of the values at the endpoints of the interval and the integral average over the whole interval.
- Higher dimensional case: Let $u \in C^2(\Omega)$ be harmonic in Ω. Then for every ball $B(x,r)$ such that $\overline{B(x,r)} \subset \Omega$ we have

$$u(x) = \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y)\, dy = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u(y)\, dS(y).$$

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4.9 Mean value property

- A converse result holds as well:

Theorem

If $u \in C(\Omega)$ satisfies

$$u(x) = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u(y)\, dS(y)$$

for all balls $\overline{B(x,r)} \subset \Omega$, then u is harmonic in Ω.

- Thus a continuous function u is harmonic if and only if for every point in the domain of definition the mean value property holds true for small enough balls centered at the point.

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4.10 Maximum principles

- Recall: A continuous function always attains its maximum and minimum values on a closed and bounded set.
- Maximum principle: Harmonic function attains its maximum and minimum at the boundary of the set.
- Physically: If the body is in thermal equilibrium, there cannot be internal hot or cold spots, since otherwise the heat energy would flow from hot to cold.
- If the maximum or minimum temperature is attained inside the body, then the temperature must be constant.
- Recall: An open set is connected if every pair of points can be connected by a piecewise linear path in the set.

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Theorem

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set and assume that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a harmonic function in Ω.

(1) *(Weak maximum principle) Then*

$$\max_{x \in \overline{\Omega}} u(x) = \max_{x \in \partial\Omega} u(x).$$

(2) *(Strong maximum principle) If Ω is a connected set and there exists $x_0 \in \Omega$ such that*

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- By replacing u by $-u$, we get the minimum principles.

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4.10 Proof of strong maximum principle

- Suppose that there exists $x_0 \in \Omega$ such that

$$u(x_0) = \max_{x \in \overline{\Omega}} u(x) = M.$$

- For $B(x_0, r) \subset \Omega$, the mean value property implies that

$$M = u(x_0) = \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} u(y)dy \leq \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} M dy = M.$$

- It follows that an equality holds throughout and thus

$$\int_{B(x_0,r)} (M - u(y))\, dy = 0.$$

- As $M - u(y) \geq 0$, we conclude that $u(y) = M$ for every $y \in B(x_0, r)$.
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- It follows that an equality holds throughout and thus

$$\int_{B(x_0,r)} (M - u(y))\, dy = 0.$$

- As $M - u(y) \geq 0$, we conclude that $u(y) = M$ for every $y \in B(x_0, r)$.
- As Ω is connected, we can reach any point in Ω with a finite chain of balls.

4.10 Proof of strong maximum principle

- Suppose that there exists $x_0 \in \Omega$ such that

$$u(x_0) = \max_{x \in \overline{\Omega}} u(x) = M.$$

- For $B(x_0, r) \subset \Omega$, the mean value property implies that

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4.10 Consequences of the maximum principle

Let $\Omega \subset \mathbb{R}^n$ be open and bounded and assume that $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ are solutions to the Poisson equation

$$-\Delta u = f \quad \text{in } \Omega.$$

Theorem (Comparison principle)

If $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.

- Proof: $\Delta(u - v) = f - f = 0$. Thus $u - v$ is harmonic and satisfies the maximum principle.

Theorem (Stability)

If $|u - v| \leq \varepsilon$ on $\partial\Omega$, then $|u - v| \leq \varepsilon$ in Ω.

- Proof: $|u - v| \leq \varepsilon$ on $\partial\Omega$ means that $-\varepsilon \leq u - v \leq \varepsilon$ on $\partial\Omega$. By maximum and minimum principles $-\varepsilon \leq u - v \leq \varepsilon$ in Ω.

Theorem (Uniqueness)

The solution is unique.

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4.10 Stability and uniqueness in unbounded domains

- The assumption that Ω is bounded is essential for uniqueness.
- Consider, for example, the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \mathbb{R}^n_+, \\ u = 0 & \text{on} \quad \partial\mathbb{R}^n_+, \end{cases}$$

  in the unbounded open and connected unbounded set $\mathbb{R}^n_+$. Every function

$$u(x_1, \dots, x_{n-1}, x_n) = ax_n,$$

  where $x_n > 0$ and $a \in \mathbb{R}$, is a solution to the problem.
- Without uniqueness, we cannot have stability results either.
- There are uniqueness results for unbounded domains for example with an extra condition that the solutions should approach to zero in a suitable sense when $|x| \to \infty$.

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