Aalto University

MS-C1350 Partial differential equations Chapter 5.4-5.6: Heat equation – separation of variables, maximum principle and energy methods

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Lecture 10

We continue discussing the heat equation:

- Separation of variables (x and t) in bounded domains and eigenvalue problems for the Laplace equation.
- Maximum principles.
- Uniqueness.

5.4 Separation of variables for heat equation in $\mathbb{R}^n$

- We have now derived a solution to an initial value problem for the heat equation in the whole space $\mathbb{R}^n$.
- Now our goal is to derive corresponding solutions in a subdomain.
- Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with a smooth boundary.
- Consider the initial and boundary value problem

$$\begin{cases} u_t - \Delta u = 0 \quad \text{in} \quad \Omega \times (0, \infty), \\ u = 0 \quad \text{on} \quad \partial\Omega \times (0, \infty), \\ u = g \quad \text{on} \quad \Omega \times \{t = 0\}. \end{cases}$$

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5.4 Step 1: Separation of variables

- We separate variables and look for a solution in the form

$$u(x,t) = v(t)w(x), \quad x \in \Omega, \quad t > 0.$$

- Then

$$u_t(x,t) = v'(t)w(x) \quad \text{and} \quad \Delta u(x,t) = v(t)\Delta w(x)$$

- From the heat equation $u_t = \Delta u$, we get the condition

$$\frac{v'(t)}{v(t)} = \frac{\Delta w(x)}{w(x)} = -\lambda.$$

- As LHS depends only on t and RHS on x, both sides have to be the same constant $-\lambda$.
- We will soon see why it is convenient to use negative sign with λ.

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5.4 Step 2: Solutions to the separated equations

$$\frac{v'(t)}{v(t)} = \frac{\Delta w(x)}{w(x)} = -\lambda.$$

- The general solution of $v' = -\lambda v$ is

$$v(t) = ce^{-\lambda t},$$

  where c is a constant.
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- Consider then the other equation $-\Delta w = \lambda w$.
- We say that λ is an eigenvalue of the (negative) Laplacian in Ω, if there exists a solution w of the problem

$$\begin{cases} -\Delta w = \lambda w \quad \text{in} \quad \Omega, \\ w = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

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5.4 Eigenfunctions

- The first Green's identity gives

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\begin{aligned}
0 \leq \int_\Omega |\nabla w|^2 \, dx &= \int_\Omega \nabla w \cdot \nabla w \, dx \\
&= -\int_\Omega w \Delta w \, dx + \int_{\partial\Omega} \frac{\partial w}{\partial \nu} \underbrace{w}_{=0} \, dS \\
&= \lambda \int_\Omega w^2 \, dx. \qquad (-\Delta w = \lambda w)
\end{aligned}
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- This implies that $\lambda \geq 0$. (In fact $\lambda > 0$ as $w = 0$ if $\lambda = 0$.)

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- We conclude that

$$u(x,t) = ce^{-\lambda t} w(x)$$

is a solution to

$$\begin{cases} u_t - \Delta u = 0 \quad \text{in} \quad \Omega \times (0,\infty), \\ u = 0 \quad \text{on} \quad \partial\Omega \times (0,\infty), \end{cases}$$

with the initial condition $u(x,0) = cw(x)$ for $x \in \Omega$, whenever w is an eigenfunction with eigenvalue λ.

5.4 Step 3: Solution of the entire problem

- Let λ_j, $j = 1, 2, \dots$ be eigenvalues and w_j, $j = 1, 2, \dots$, the corresponding eigenfunctions.
- Then the linear combination

$$u(x,t) = \sum_{j=1}^{\infty} c_j e^{-\lambda_j t} w_j(x)$$

should be a solution with the initial condition

$$u(x,0) = \sum_{j=1}^{\infty} c_j w_j(x).$$

- If we can determine the coefficients c_j, $j = 1, 2, \dots$, so that

$$\sum_{j=1}^{\infty} c_j w_j(x) = g(x),$$

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5.4 Step 3: Solution of the entire problem

- It is known that there is a countable number of eigenvalues $0 < \lambda_1 < \lambda_2 < \dots$.
- Moreover, the corresponding eigenfunctions $\{w_j\}_{j=1}^{\infty}$ can be chosen to be an orthonormal basis in $L^2(\Omega)$.
- This means that if $g \in L^2(\Omega)$, then

$$c_j = \langle g, w_j \rangle = \int_\Omega g(y) w_j(y)\, dy, \quad j = 1, 2, \dots .$$

  Here $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $L^2(\Omega)$.
- The coefficients c_j, $j = 1, 2, \dots$, can be seen as the Fourier coefficients of $g \in L^2(\Omega)$ and the series

$$g = \sum_{j=1}^{\infty} c_j w_j = \sum_{j=1}^{\infty} \langle g, w_j \rangle w_j$$

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- Thus we have the representation formula

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\begin{aligned}
u(x,t) &= \sum_{j=1}^{\infty} \langle g, w_j \rangle e^{-\lambda_j t} w_j(x) \\
&= \int_{\Omega} K(x,y,t) g(y)\, dy,
\end{aligned}
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where

$$
K(x,y,t) = \sum_{j=1}^{\infty} e^{-\lambda_j t} w_j(x) w_j(y)
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is the heat kernel in Ω.

- Note that this depends on whether we can find so many eigenfunctions that we can represent the initial value as an infinite linear combination of the eigenfunctions.
- We also have to check that the series converges and that the function u is really a solution to the original problem.

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5.4 Asymptotical behaviour

- With time dependent problems, it is relevant to study the behaviour of solutions as $t \to \infty$.
- For this problem,

$$\|u(\cdot,t)\|_{L^2(\Omega)} \leq e^{-\lambda_1 t}\|g\|_{L^2(\Omega)}, \quad t > 0,$$

where $\lambda_1 > 0$ is the first (and the smallest) eigenvalue of Laplacian.

5.5 Maximum principle

- As Laplace equation, also solutions to the heat equation satisfy a maximum principle.
- Let us introduce the space-time cylinder:

$$\Omega_T = \Omega \times (0, T),$$

where $\Omega \subset \mathbb{R}^n$ is open and bounded and $0 < T < \infty$.

- For Laplace equation, maximum is achieved on the boundary $\partial\Omega$.
- For heat equation, maximum is achieved on certain part of the boundary, which is called the parabolic boundary

$$\Gamma_T = (\partial\Omega \times [0, T]) \cup (\Omega \times \{t = 0\}).$$

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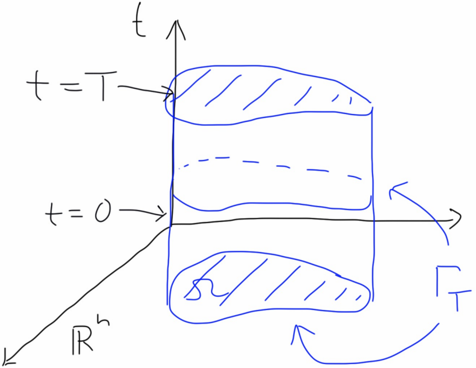
5.5 Parabolic boundary



5.5 Weak maximum principle

Theorem

Let $\Omega \subset \mathbb{R}^n$ be open and bounded and assume that $u \in C^2(\Omega_T) \cap C(\overline{\Omega_T})$ is a solution to the heat equation in Ω_T. Then

$$\max_{(x,t)\in\overline{\Omega_T}} u(x,t) = \max_{(x,t)\in\Gamma_T} u(x,t).$$

- If u is a solution to the heat equation then also $-u$ is a solution.
- Thus, if we replace u by $-u$ we get the corresponding statement with $\min$ replacing $\max$.

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5.5 Weak maximum principle – proof

- **Step 1**: Prove the maximum principle for
$$v(x,t) = u(x,t) - \varepsilon t.$$
- **Step 2**: Let $\varepsilon \searrow 0$ to conclude that it holds also for u.
- Proof of Step 1: Proof by contradiction: Suppose that the maximum is attained at $(x_0, t_0) \in \overline{\Omega_T} \setminus \Gamma_T = \Omega \times (0, T]$
- We have $v(x_0, t) \leq v(x_0, t_0)$ for all $t < t_0$. This implies that
$$\frac{\partial v}{\partial t}(x_0, t_0) \geq 0,$$
- As (x_0, t_0) is a maximum in x-direction, $\nabla v(x_0, t_0) = 0$ and
$$\frac{\partial^2 v}{\partial x_j^2}(x_0, t_0) \leq 0, \quad j = 1, \ldots, n,$$
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5.5 Uniqueness for bounded domains

Theorem (Uniqueness for bounded domains)

Assume that Ω_T is bounded, $g \in C(\Gamma_T)$ and $f \in C(\Omega_T)$. Then there exists at most one solution $u \in C^2(\Omega_T) \cap C(\overline{\Omega_T})$ of the initial and boundary value problem

$$\begin{cases} u_t - \Delta u = f & \text{in} \quad \Omega_T, \\ u = g & \text{on} \quad \Gamma_T. \end{cases}$$

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5.5 Uniqueness for unbounded domains?

- Consider the problem

$$\begin{cases} u_t - u_{xx} = 0 \quad \text{in} \quad \mathbb{R} \times (0, \infty), \\ u = 0 \quad \text{on} \quad \mathbb{R} \times \{t = 0\}. \end{cases}$$

- Now $\Omega = \mathbb{R}$ is unbounded.
- One solution to the problem is $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$, with

$$u(x,t) = \begin{cases} \displaystyle\sum_{j=0}^{\infty} \frac{x^{2j}}{(2j)!} \frac{\partial^j}{\partial t^j}(e^{-1/t^2}), & x \in \mathbb{R}, \quad t > 0, \\ 0, \quad t = 0. \end{cases}$$

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5.5 Remarks about the maximum principle

- There is also a version of the strong maximum principle for the heat equation.
- Changing t to $-t$ does not preserve heat equation. Thus solutions forward and backward in time are different.
- Given an initial temperature, we may predict future temperatures, but we cannot in general determine the thermal status that generated that particular temperature distribution.
- The backward in time problem is illposed i.e. it is not solvable in general.
- The backward problem is not stable. (In the lecture notes there is an example of a sequence of solutions, where the initial values approach 0, but the solutions are unbounded close to the initial boundary.

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5.6 Energy methods for the heat equation

- There is an analogous approach with energy / variational methods to the heat equation as there is for the Laplace equation.
- Let $\Omega \subset \mathbb{R}^n$ be open and bounded. We consider the initial and boundary value problem for the heat equation

$$\begin{cases} u_t - \Delta u = f \quad \text{in} \quad \Omega_T, \\ u = g \quad \text{on} \quad \Gamma_T. \end{cases}$$

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5.6 Proof of uniqueness

- Let u, v be two solutions. Then $w = u - v$ satisfies

$$w_t - \Delta w = 0 \quad \text{in} \quad \Omega_T, \quad w = 0 \quad \text{on} \quad \Gamma_T.$$

- For $0 \leq t \leq T$ we define the energy

$$e(t) = \int_\Omega w(x,t)^2 \, dx.$$

- Then

$$\begin{aligned} e'(t) &= \int_\Omega 2w \frac{\partial w}{\partial t} \, dx = \int_\Omega 2w \Delta w \, dx \quad (w \text{ satisfies the heat equation}) \\ &= 2 \int_{\partial\Omega} \frac{\partial w}{\partial \nu} w \, dS - 2 \int_\Omega |\nabla w|^2 \, dx \quad \text{(Green's first identity)} \\ &= -2 \int_\Omega |\nabla w|^2 \, dx \leq 0. \quad (w = 0 \text{ on } \partial\Omega) \end{aligned}$$

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