Aalto University

MS-C1350 Partial differential equations
Chapter 1 – Introduction

Riikka Korte

Department of Mathematics and Systems Analysis
Aalto University
riikka.korte@aalto.fi

September 3, 2024

Official learning outcomes

- how to solve partial differential equations using separation of variables and Fourier techniques
- the physical interpretations of the equations
- the equations in the spherical and cylindrical coordinates
- the role of the fundamental solutions
- the maximum and comparison principles
- the variational forms of the equations.

Goal: to have more detailed learning goals biweekly this fall (work in progress –> also already published parts might be edited during the fall)

What we will do in this course

- We study three prototype equations: Laplace, heat and wave equation.
- First, we learn how to solve PDEs using Fourier series and Fourier transform.
- In the second part, we study each of the three equations one-by-one in more detail.
- We learn techniques that are useful in the study of also more general equations.

Lecture 1

- What is a PDE?
- What is a solution to PDE?
- Well posed problem.
- Function spaces C, C_0, C^kÂ and C^∞. Difference of $C((0,1))$ and $C([0,1])$?
- Introduction to Fourier series:
    - periodic functions,
    - L^p (and especially L^2) functions,
    - (complex) inner product, orthonormal basis and representing an element of a vector space using a basis.

Introduction – What is a PDE?

Introduction – What is a PDE?

- PDE = an equation involving an unknown function of two or more variables and its partial derivatives.

$$\text{Example: } \sum_{i,j=1}^{n} A_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} = f.$$

Introduction – What is a PDE?

- PDE = an equation involving an unknown function of two or more variables and its partial derivatives.

$$\text{Example: } \sum_{i,j=1}^{n} A_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} = f.$$

- PDEs are used to model a huge number of different phenomena e.g. in physics and engineering.

Introduction – What is a PDE?

- PDE = an equation involving an unknown function of two or more variables and its partial derivatives.

$$\text{Example: } \sum_{i,j=1}^{n} A_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} = f.$$

- PDEs are used to model a huge number of different phenomena e.g. in physics and engineering.
- For most PDEs, there is no explicit formula for the solution.
  - (numerical) estimates for solutions
  - qualitative properties

Introduction – What is a PDE?

- PDE = an equation involving an unknown function of two or more variables and its partial derivatives.

$$\text{Example: } \sum_{i,j=1}^{n} A_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} = f.$$

- PDEs are used to model a huge number of different phenomena e.g. in physics and engineering.
- For most PDEs, there is no explicit formula for the solution.
  - (numerical) estimates for solutions
  - qualitative properties
- There does not exist a general method for studying every PDE. Some methods have turned out to be more useful than others.

Introduction – What is a PDE?

- PDE = an equation involving an unknown function of two or more variables and its partial derivatives.

$$\text{Example: } \sum_{i,j=1}^{n} A_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} = f.$$

- PDEs are used to model a huge number of different phenomena e.g. in physics and engineering.
- For most PDEs, there is no explicit formula for the solution.
  - (numerical) estimates for solutions
  - qualitative properties
- There does not exist a general method for studying every PDE. Some methods have turned out to be more useful than others.
- In this course, we study special cases, in which explicit solutions and representation formulas are available, but focus on features that are present also in more general situations.

Introduction – What is a PDE?

- PDE = an equation involving an unknown function of two or more variables and its partial derivatives.

$$\text{Example: } \sum_{i,j=1}^{n} A_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} = f.$$

- PDEs are used to model a huge number of different phenomena e.g. in physics and engineering.
- For most PDEs, there is no explicit formula for the solution.
  - (numerical) estimates for solutions
  - qualitative properties
- There does not exist a general method for studying every PDE. Some methods have turned out to be more useful than others.
- In this course, we study special cases, in which explicit solutions and representation formulas are available, but focus on features that are present also in more general situations.
- Qualitative aspects are also important in numerical solutions of PDE. Without existence, uniqueness and stability, numerical methods may give inaccurate or completely wrong solutions.

Introduction – types of equations

1. Laplace equation (elliptic)

$$\Delta u = f, \quad u = u(x), \quad \Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2},$$

Introduction – types of equations

1. Laplace equation (elliptic)

$$\Delta u = f, \quad u = u(x), \quad \Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2},$$

2. Heat equation (parabolic)

$$\frac{\partial u}{\partial t} - \Delta u = f, \quad u = u(x,t),$$

Introduction – types of equations

1. Laplace equation (elliptic)

$$\Delta u = f, \quad u = u(x), \quad \Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2},$$

2. Heat equation (parabolic)

$$\frac{\partial u}{\partial t} - \Delta u = f, \quad u = u(x,t),$$

3. Wave equation (hyperbolic)

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = f \quad u = u(x,t).$$

Introduction – types of equations

1. Laplace equation (elliptic)

$$\Delta u = f, \quad u = u(x), \quad \Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2},$$

2. Heat equation (parabolic)

$$\frac{\partial u}{\partial t} - \Delta u = f, \quad u = u(x,t),$$

3. Wave equation (hyperbolic)

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = f \quad u = u(x,t).$$

There are also PDEs that are in none of the above classes.

Well posed problem

By finding a solution to a PDE we may mean

- Finding an explicit representation formulas for solutions, or
- deducing general properties that hold true for all solutions.

Well posed problem

By finding a solution to a PDE we may mean

- Finding an explicit representation formulas for solutions, or
- deducing general properties that hold true for all solutions.

A PDE problem is well posed, if

1. **EXISTENCE**: the problem has a solution,
2. **UNIQUENESS**: there exists only one solution and
3. **STABILITY**: the solution depends continuously on the data given in the problem.

Introduction – what does it mean to be a solution

- Classical solution: all partial derivatives, which appear in the PDE, exist and are continuous.
  $\Longrightarrow$ We can verify by direct computation that a function solves the PDE.

Introduction – what does it mean to be a solution

- Classical solution: all partial derivatives, which appear in the PDE, exist and are continuous.
  $\Longrightarrow$ We can verify by direct computation that a function solves the PDE.
- Weak solution: less regularity, e.g. saw tooth wave.

Many PDEs do not have any classical solutions, but some type of weak solutions might be physically meaningful.

MS-C1350 Partial differential equations
Chapters 2.1–2.2 – Periodic functions, L^p-functions and inner product

Riikka Korte

Department of Mathematics and Systems Analysis
Aalto University
riikka.korte@aalto.fi

September 3, 2024

2 Fourier series (intro)

- Fourier series is a series representation of a function as trigonometric polynomials.
- https://www.geogebra.org/m/fdaYrVCH

2 Fourier series (intro)

- Fourier series is a series representation of a function as trigonometric polynomials.
- `https://www.geogebra.org/m/fdaYrVCH`
- Euler's formula:

$$e^{ikt} = \cos(kt) + i \sin kt, \quad e^{-ikt} = \cos(kt) - i \sin kt$$

$\Longrightarrow$ We can use exponential function instead of trigonometric functions.

2 Fourier series (intro)

- Fourier series is a series representation of a function as trigonometric polynomials.
- https://www.geogebra.org/m/fdaYrVCH
- Euler's formula:

$$e^{ikt} = \cos(kt) + i\sin kt, \quad e^{-ikt} = \cos(kt) - i\sin kt$$

  $\Longrightarrow$ We can use exponential function instead of trigonometric functions.
- Compare to Taylor series – both express a function as infinite sum of elementary functions

2 Fourier series (intro)

- Fourier series is a series representation of a function as trigonometric polynomials.
- https://www.geogebra.org/m/fdaYrVCH
- Euler's formula:

$$e^{ikt} = \cos(kt) + i\sin kt, \quad e^{-ikt} = \cos(kt) - i\sin kt$$

  $\Longrightarrow$ We can use exponential function instead of trigonometric functions.
- Compare to Taylor series – both express a function as infinite sum of elementary functions
- Fourier series gives the best square approximation of the function.

2 Fourier series (intro)

- Fourier series is a series representation of a function as trigonometric polynomials.
- `https://www.geogebra.org/m/fdaYrVCH`
- Euler's formula:

$$e^{ikt} = \cos(kt) + i \sin kt, \quad e^{-ikt} = \cos(kt) - i \sin kt$$

  $\Longrightarrow$ We can use exponential function instead of trigonometric functions.
- Compare to Taylor series – both express a function as infinite sum of elementary functions
- Fourier series gives the best square approximation of the function.
- Fourier series converges pointwise if the function is smooth enough.

2 Fourier series (intro)

- Fourier series is a series representation of a function as trigonometric polynomials.
- https://www.geogebra.org/m/fdaYrVCH
- Euler's formula:

$$e^{ikt} = \cos(kt) + i \sin kt, \quad e^{-ikt} = \cos(kt) - i \sin kt$$

  $\Longrightarrow$ We can use exponential function instead of trigonometric functions.
- Compare to Taylor series – both express a function as infinite sum of elementary functions
- Fourier series gives the best square approximation of the function.
- Fourier series converges pointwise if the function is smooth enough.
- Many PDE problems can be solved using using Fourier series and convolutions.

2 Fourier series (intro)

- Fourier series is a series representation of a function as trigonometric polynomials.
- `https://www.geogebra.org/m/fdaYrVCH`
- Euler's formula:

$$e^{ikt} = \cos(kt) + i \sin kt, \quad e^{-ikt} = \cos(kt) - i \sin kt$$

  $\Longrightarrow$ We can use exponential function instead of trigonometric functions.
- Compare to Taylor series – both express a function as infinite sum of elementary functions
- Fourier series gives the best square approximation of the function.
- Fourier series converges pointwise if the function is smooth enough.
- Many PDE problems can be solved using using Fourier series and convolutions.
- We need following concepts to be able to do Fourier theory:
  - periodic functions
  - L^p-functions, in particular L^2-functions
  - Innerproduct spaces

Periodic functions (Chapter 2.1)

- We say that a function $f : \mathbb{R} \to \mathbb{C}$ is 2π-periodic if for every $t \in \mathbb{R}$ we have

$$f(t + 2\pi) = f(t). \tag{1}$$

Periodic functions (Chapter 2.1)

- We say that a function $f : \mathbb{R} \to \mathbb{C}$ is 2π-periodic if for every $t \in \mathbb{R}$ we have

$$f(t + 2\pi) = f(t). \tag{1}$$

- More generally, a function $f : \mathbb{R} \to \mathbb{C}$ is called T-periodic , $T \in \mathbb{R}$, $T \neq 0$, if

$$f(t + T) = f(t) \tag{2}$$

for every $t \in \mathbb{R}$.

Periodic functions (Chapter 2.1)

- We say that a function $f : \mathbb{R} \to \mathbb{C}$ is 2π-periodic if for every $t \in \mathbb{R}$ we have

(1) $$f(t + 2\pi) = f(t).$$

- More generally, a function $f : \mathbb{R} \to \mathbb{C}$ is called T-periodic , $T \in \mathbb{R}$, $T \neq 0$, if

(2) $$f(t + T) = f(t)$$

  for every $t \in \mathbb{R}$.
- Every function $f : [a, b] \to \mathbb{C}$, $a, b \in \mathbb{R}$, can be extended to a periodic function on the whole $\mathbb{R}$.

Periodic functions (Chapter 2.1)

- We say that a function $f:\mathbb{R}\to\mathbb{C}$ is 2π-periodic if for every $t\in\mathbb{R}$ we have

$$f(t+2\pi)=f(t). \tag{1}$$

- More generally, a function $f:\mathbb{R}\to\mathbb{C}$ is called T-periodic , $T\in\mathbb{R}$, $T\neq 0$, if

$$f(t+T)=f(t) \tag{2}$$

  for every $t\in\mathbb{R}$.
- Every function $f:[a,b]\to\mathbb{C}$, $a,b\in\mathbb{R}$, can be extended to a periodic function on the whole $\mathbb{R}$.
- Fourier series are defined only for periodic functions, but this is not a serious restriction, because we can extend the functions periodically.

L^p space (Chapter 2.2)

Solutions to PDEs are not always smooth. Therefore we use L^p spaces ($p \geq 1$). Especially L^1 and L^2 are important for Fourier series.

L^p space (Chapter 2.2)

Solutions to PDEs are not always smooth. Therefore we use L^p spaces ($p \geq 1$). Especially L^1 and L^2 are important for Fourier series.

Definition

Let $1 \leq p < \infty$. A function $f : [-\pi, \pi] \to \mathbb{C}$ belongs to $L^p([-\pi, \pi])$, if

$$\|f\|_{L^p([-\pi,\pi])} = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^p \, dt \right)^{\frac{1}{p}} < \infty.$$

The number $\|f\|_{L^p([-\pi,\pi])}$ is called the L^p-norm of f.

L^p space (Chapter 2.2)

Solutions to PDEs are not always smooth. Therefore we use L^p spaces ($p \geq 1$). Especially L^1 and L^2 are important for Fourier series.

Definition

Let $1 \leq p < \infty$. A function $f : [-\pi, \pi] \to \mathbb{C}$ belongs to $L^p([-\pi, \pi])$, if

$$\|f\|_{L^p([-\pi,\pi])} = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^p \, dt \right)^{\frac{1}{p}} < \infty.$$

The number $\|f\|_{L^p([-\pi,\pi])}$ is called the L^p-norm of f.

- L^p functions need not be smooth or even continuous.

L^p space (Chapter 2.2)

Solutions to PDEs are not always smooth. Therefore we use L^p spaces ($p \geq 1$). Especially L^1 and L^2 are important for Fourier series.

Definition

Let $1 \leq p < \infty$. A function $f : [-\pi, \pi] \to \mathbb{C}$ belongs to $L^p([-\pi, \pi])$, if

$$\|f\|_{L^p([-\pi,\pi])} = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^p \, dt \right)^{\frac{1}{p}} < \infty.$$

The number $\|f\|_{L^p([-\pi,\pi])}$ is called the L^p-norm of f.

- L^p functions need not be smooth or even continuous.
- $C([-\pi, \pi]) \subset L^2([-\pi, \pi])$ BUT $C((-\pi, \pi)) \not\subseteq L^2([-\pi, \pi])$

L^p space (Chapter 2.2)

Solutions to PDEs are not always smooth. Therefore we use L^p spaces ($p \geq 1$). Especially L^1 and L^2 are important for Fourier series.

Definition

Let $1 \leq p < \infty$. A function $f : [-\pi, \pi] \to \mathbb{C}$ belongs to $L^p([-\pi, \pi])$, if

$$\|f\|_{L^p([-\pi,\pi])} = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^p \, dt \right)^{\frac{1}{p}} < \infty.$$

The number $\|f\|_{L^p([-\pi,\pi])}$ is called the L^p-norm of f.

- L^p functions need not be smooth or even continuous.
- $C([-\pi, \pi]) \subset L^2([-\pi, \pi])$ BUT $C((-\pi, \pi)) \not\subset L^2([-\pi, \pi])$
- L^p functions can have singularities.

(Complex) Inner product

- Inner product space is a vector space V with a map: $< \cdot, \cdot >: V \times V \to \mathbb{C}$ that satisfies ($\forall\, x, y, z \in V$, $a \in \mathbb{C}$)
  - Linearity
$$\langle ax, y\rangle = a\langle x, y\rangle, \qquad \langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle,$$
  - conjugate symmetry
$$\langle x, y\rangle = \overline{\langle y, x\rangle},$$
  - positivedefinitess
$$\langle x, x\rangle > 0 \text{ if } x \neq 0.$$

(Complex) Inner product

- Inner product space is a vector space V with a map: $< \cdot, \cdot >: V \times V \to \mathbb{C}$ that satisfies ($\forall\ x, y, z \in V$, $a \in \mathbb{C}$)
  - Linearity
    $$\langle ax, y\rangle = a\langle x, y\rangle, \qquad \langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle,$$
  - conjugate symmetry
    $$\langle x, y\rangle = \overline{\langle y, x\rangle},$$
  - positivedefinitess
    $$\langle x, x\rangle > 0 \text{ if } x \neq 0.$$
- There are both finite dimensional vector spaces (e.g $\mathbb{R}^n, \mathbb{C}^n$) and infinite dimensional vector spaces (e.g function spaces).

(Complex) Inner product

- Inner product space is a vector space V with a map:
  $< \cdot, \cdot >: V \times V \to \mathbb{C}$ that satisfies ($\forall\, x, y, z \in V$, $a \in \mathbb{C}$)
  - Linearity
  $$\langle ax, y\rangle = a\langle x, y\rangle, \qquad \langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle,$$
  - conjugate symmetry
  $$\langle x, y\rangle = \overline{\langle y, x\rangle},$$
  - positivedefinitess
  $$\langle x, x\rangle > 0 \text{ if } x \neq 0.$$
- There are both finite dimensional vector spaces (e.g $\mathbb{R}^n, \mathbb{C}^n$) and infinite dimensional vector spaces (e.g function spaces).
- If $\{e_i\}_{i=-\infty}^{\infty}$ is an orthonormalisation basis for V then
$$f = \sum_{i=-\infty}^{\infty} \langle f, e_i\rangle e_i \quad \text{for every } f \in V.$$

Complex vector space $L^2([-\pi, \pi])$

- Addition and multiplication

$$(f+g)(t) = f(t) + g(t), \quad (\alpha f)(t) = \alpha f(t), \quad \alpha \in \mathbb{C}.$$

Complex vector space $L^2([-\pi, \pi])$

- Addition and multiplication

$$(f+g)(t) = f(t) + g(t), \quad (\alpha f)(t) = \alpha f(t), \quad \alpha \in \mathbb{C}.$$

- Inner product:

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \overline{g(t)} \, dt$$

Here $\overline{z} = x - iy \in \mathbb{C}$ is the complex conjugate of $z = x + iy$, where $x, y \in \mathbb{R}$ and i is the imaginary unit.

Complex vector space $L^2([-\pi, \pi])$

- Addition and multiplication

$$(f+g)(t) = f(t) + g(t), \quad (\alpha f)(t) = \alpha f(t), \quad \alpha \in \mathbb{C}.$$

- Inner product:

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\overline{g(t)}\, dt$$

  Here $\overline{z} = x - iy \in \mathbb{C}$ is the complex conjugate of $z = x + iy$, where $x, y \in \mathbb{R}$ and i is the imaginary unit.

- Norm given by the inner product:

$$\|f\|_{L^2([-\pi,\pi])} = \langle f, f \rangle^{1/2}$$

Important example of an inner product

- Let $e_j : [-\pi, \pi] \to \mathbb{C}$,

$$e_j(t) = e^{ijt} = \cos(jt) + i\sin(jt), \quad j \in \mathbb{Z} \quad \text{(Euler's formula).}$$

Important example of an inner product

- Let $e_j : [-\pi, \pi] \to \mathbb{C}$,

$$e_j(t) = e^{ijt} = \cos(jt) + i\sin(jt), \quad j \in \mathbb{Z} \quad \text{(Euler's formula)}.$$

- Then $e_j \in C([-\pi,\pi]) \subset L^2([-\pi,\pi])$ with

$$\|e_j\|_{L^2([-\pi,\pi])} = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{|e^{ijt}|^2}_{=1} \, dt \right)^{\frac{1}{2}} = 1, \quad j = 1, 2, \ldots.$$

Important example of an inner product

- Let $e_j : [-\pi, \pi] \to \mathbb{C}$,

$$e_j(t) = e^{ijt} = \cos(jt) + i\sin(jt), \quad j \in \mathbb{Z} \quad \text{(Euler's formula)}.$$

- Then $e_j \in C([-\pi,\pi]) \subset L^2([-\pi,\pi])$ with

$$\|e_j\|_{L^2([-\pi,\pi])} = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{|e^{ijt}|^2}_{=1} \, dt \right)^{\frac{1}{2}} = 1, \quad j = 1, 2, \ldots.$$

- The inner product of two such functions is

$$\langle e_j, e_k \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ijt} \overline{e^{ikt}} \, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(j-k)t} \, dt = 0, \quad \text{if } j \neq k$$

Important example of an inner product

- Let $e_j : [-\pi, \pi] \to \mathbb{C}$,

$$e_j(t) = e^{ijt} = \cos(jt) + i\sin(jt), \quad j \in \mathbb{Z} \quad \text{(Euler's formula)}.$$

- Then $e_j \in C([-\pi, \pi]) \subset L^2([-\pi, \pi])$ with

$$\|e_j\|_{L^2([-\pi,\pi])} = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{|e^{ijt}|^2}_{=1} \, dt \right)^{\frac{1}{2}} = 1, \quad j = 1, 2, \ldots.$$

- The inner product of two such functions is

$$\langle e_j, e_k \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ijt} \overline{e^{ikt}} \, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(j-k)t} \, dt = 0, \quad \text{if } j \neq k$$

- Thus the set $\{e_j\}_{j \in \mathbb{Z}}$ is an orthonormal set in $L^2([-\pi, \pi])$

$$\langle e_j, e_k \rangle = \delta_{jk} = \begin{cases} 0, & j \neq k, \\ 1, & j = k. \end{cases}$$

Important example of an inner product

- Let $e_j : [-\pi, \pi] \to \mathbb{C}$,

$$e_j(t) = e^{ijt} = \cos(jt) + i\sin(jt), \quad j \in \mathbb{Z} \quad \text{(Euler's formula)}.$$

- Then $e_j \in C([-\pi, \pi]) \subset L^2([-\pi, \pi])$ with

$$\|e_j\|_{L^2([-\pi,\pi])} = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{|e^{ijt}|^2}_{=1} \, dt \right)^{\frac{1}{2}} = 1, \quad j = 1, 2, \ldots .$$

- The inner product of two such functions is

$$\langle e_j, e_k \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ijt} \overline{e^{ikt}} \, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(j-k)t} \, dt = 0, \quad \text{if } j \neq k$$

- Thus the set $\{e_j\}_{j \in \mathbb{Z}}$ is an orthonormal set in $L^2([-\pi, \pi])$

$$\langle e_j, e_k \rangle = \delta_{jk} = \begin{cases} 0, & j \neq k, \\ 1, & j = k. \end{cases}$$

- It is actually also an orthonormal basis (which is not so easy to prove).

MS-C1350 Partial differential equations
Chapter 2.3 Fourier series

Riikka Korte

Department of Mathematics and Systems Analysis
Aalto University
riikka.korte@aalto.fi

September 3, 2024

Fourier series

- Let $e_j = e^{ijt} \in L^1([-\pi, \pi])$.

Fourier series

- Let $e_j = e^{ijt} \in L^1([-\pi, \pi])$.
- Recall that $\{e_j\}_{j=-\infty}^{\infty}$ form an orthonormal set with the inner product given in previous slides.

Fourier series

- Let $e_j = e^{ijt} \in L^1([-\pi, \pi])$.
- Recall that $\{e_j\}_{j=-\infty}^{\infty}$ form an orthonormal set with the inner product given in previous slides.
- Let $f \in L^1([-\pi, \pi])$. The nth partial sum of a Fourier series is

$$S_n f(t) = \sum_{j=-n}^{n} \widehat{f}(j) e^{ijt}, \quad n = 0, 1, 2, \ldots,$$

where

$$\widehat{f}(j) = \langle f, e_j \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ijt}\, dt, \quad j \in \mathbb{Z},$$

is the jth Fourier coefficient of f.

Fourier series

- Let $e_j = e^{ijt} \in L^1([-\pi,\pi])$.
- Recall that $\{e_j\}_{j=-\infty}^{\infty}$ form an orthonormal set with the inner product given in previous slides.
- Let $f \in L^1([-\pi,\pi])$. The nth partial sum of a Fourier series is

$$S_n f(t) = \sum_{j=-n}^{n} \widehat{f}(j) e^{ijt}, \quad n = 0, 1, 2, \dots,$$

  where

$$\widehat{f}(j) = \langle f, e_j \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ijt} \, dt, \quad j \in \mathbb{Z},$$

  is the jth Fourier coefficient of f.
- The Fourier series of f is the limit of the partial sums $S_n f$ as $n \to \infty$, provided the limit exists in some reasonable sense. In this case we may write

$$f(t) = \lim_{n\to\infty} S_n f(t) = \lim_{n\to\infty} \sum_{j=-n}^{n} \widehat{f}(j) e^{ijt} = \sum_{j=-\infty}^{\infty} \widehat{f}(j) e^{ijt}.$$

Remarks about Fourier series

- Definition makes sense if $f \in L^1([-\pi, \pi])$, but we will see that $L^2([-\pi, \pi])$ is needed to understand the Fourier coefficients and the convergence of the series.

Remarks about Fourier series

- Definition makes sense if $f \in L^1([-\pi,\pi])$, but we will see that $L^2([-\pi,\pi])$ is needed to understand the Fourier coefficients and the convergence of the series.
- We always consider symmetric Fourier series (sum from $-n$ to n).

Remarks about Fourier series

- Definition makes sense if $f \in L^1([-\pi, \pi])$, but we will see that $L^2([-\pi, \pi])$ is needed to understand the Fourier coefficients and the convergence of the series.
- We always consider symmetric Fourier series (sum from $-n$ to n).
- It corresponds to having trigonometric functions $\sin(t), \cos(t), \sin(2t), \cos(2t), \ldots, \sin(nt), \cos(nt)$.

Remarks about Fourier series

- Definition makes sense if $f \in L^1([-\pi, \pi])$, but we will see that $L^2([-\pi, \pi])$ is needed to understand the Fourier coefficients and the convergence of the series.
- We always consider symmetric Fourier series (sum from $-n$ to n).
- It corresponds to having trigonometric functions $\sin(t), \cos(t), \sin(2t), \cos(2t), \dots, \sin(nt), \cos(nt)$.
- e_j, $j \in \mathbb{Z}$ is always 2π-periodic. Therefore $S_n f$ is always 2π-periodic. Therefore we can only approximate 2π-periodic functions. (And by change of variables, other periodic functions.)

MS-C1350 Partial differential equations
Ch 2.4 Best square approximation

Riikka Korte

Department of Mathematics and Systems Analysis
Aalto University
riikka.korte@aalto.fi

September 10, 2024

Lecture 2 – topics

- Fourier series
  - Definition
  - Best square approximation
  - Fourier series on different intervals
  - Real form Fourier series (i.e. with $\sin \cos$)
  - Differentiation of Fourier series
  - Dirichlet kernel
- Convolution
- Function spaces $C^k(I)$ and $L^2(I)$, where I is open or closed interval.

The best square approximation

It is good to consider the Fourier series in terms of projections. $S_n f$ is the projection of $f \in L^2([-\pi,\pi])$ to a subspace spanned by $\{e_j\}_{j=-n}^{n}$:

$$S_n f(t) = \sum_{j=-n}^{n} \langle f, e_j \rangle e_j(t) = \sum_{j=-n}^{n} \widehat{f}(j) e^{ijt}, \quad n = 0, 1, 2, \ldots,$$

where

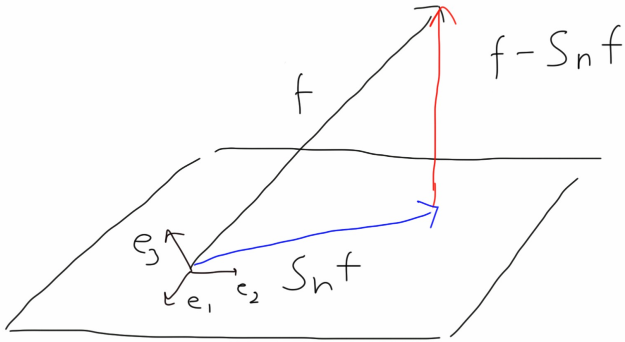
$$\widehat{f}(j) = \langle f, e_j \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ijt}\, dt.$$

This works for all orthonormal bases of vector spaces.

Orthogonality

$f - S_n f$ is orthogonal to the subspace spanned by $\{e_j\}_{j=-n}^{n}$ i.e.

$$\langle f - S_n f, e_j \rangle = 0 \quad \text{for every } j = -n, \dots, n.$$



Orthogonality

By orthogonality of $S_n f$ and $f - S_n f$, we have

$$
\begin{aligned}
\|f\|^2_{L^2([-\pi,\pi])} &= \|f - S_n f + S_n f\|^2_{L^2([-\pi,\pi])} \\
&= \langle (f - S_n f) + S_n f, (f - S_n f) + S_n f \rangle \\
&= \langle f - S_n f, f - S_n f \rangle + \underbrace{\langle f - S_n f, S_n f \rangle}_{=0} \\
&\qquad + \underbrace{\langle S_n f, f - S_n f \rangle}_{=0} + \langle S_n f, S_n f \rangle \\
&= \|f - S_n f\|^2_{L^2([-\pi,\pi])} + \|S_n f\|^2_{L^2([-\pi,\pi])}.
\end{aligned} \tag{1}
$$

Orthogonality

By orthogonality of $S_n f$ and $f - S_n f$, we have

$$
\begin{aligned}
\|f\|^2_{L^2([-\pi,\pi])} &= \|f - S_n f + S_n f\|^2_{L^2([-\pi,\pi])} \\
&= \langle (f - S_n f) + S_n f, (f - S_n f) + S_n f \rangle \\
&= \langle f - S_n f, f - S_n f \rangle + \underbrace{\langle f - S_n f, S_n f \rangle}_{=0} \\
&\qquad + \underbrace{\langle S_n f, f - S_n f \rangle}_{=0} + \langle S_n f, S_n f \rangle \\
&= \|f - S_n f\|^2_{L^2([-\pi,\pi])} + \|S_n f\|^2_{L^2([-\pi,\pi])}.
\end{aligned} \tag{1}
$$

Orthogonality

By orthogonality of S_nf and $f - S_nf$, we have

$$
\begin{aligned}
\|f\|^2_{L^2([-\pi,\pi])} &= \|f - S_nf + S_nf\|^2_{L^2([-\pi,\pi])} \\
&= \langle (f - S_nf) + S_nf, (f - S_nf) + S_nf \rangle \\
&= \langle f - S_nf, f - S_nf \rangle + \underbrace{\langle f - S_nf, S_nf \rangle}_{=0} \\
&\quad + \underbrace{\langle S_nf, f - S_nf \rangle}_{=0} + \langle S_nf, S_nf \rangle \\
&= \|f - S_nf\|^2_{L^2([-\pi,\pi])} + \|S_nf\|^2_{L^2([-\pi,\pi])}.
\end{aligned}
\tag{1}
$$

Orthogonality

By orthogonality of $S_n f$ and $f - S_n f$, we have

$$
\begin{aligned}
\|f\|^2_{L^2([-\pi,\pi])} &= \|f - S_n f + S_n f\|^2_{L^2([-\pi,\pi])} \\
&= \langle (f - S_n f) + S_n f, (f - S_n f) + S_n f \rangle \\
&= \langle f - S_n f, f - S_n f \rangle + \underbrace{\langle f - S_n f, S_n f \rangle}_{=0} \\
&\quad + \underbrace{\langle S_n f, f - S_n f \rangle}_{=0} + \langle S_n f, S_n f \rangle \\
&= \|f - S_n f\|^2_{L^2([-\pi,\pi])} + \|S_n f\|^2_{L^2([-\pi,\pi])}.
\end{aligned} \tag{1}
$$

Orthogonality

By orthogonality of $S_n f$ and $f - S_n f$, we have

$$
\begin{aligned}
\|f\|^2_{L^2([-\pi,\pi])} &= \|f - S_n f + S_n f\|^2_{L^2([-\pi,\pi])} \\
&= \langle (f - S_n f) + S_n f, (f - S_n f) + S_n f \rangle \\
&= \langle f - S_n f, f - S_n f \rangle + \underbrace{\langle f - S_n f, S_n f \rangle}_{=0} \\
&\qquad + \underbrace{\langle S_n f, f - S_n f \rangle}_{=0} + \langle S_n f, S_n f \rangle \\
&= \|f - S_n f\|^2_{L^2([-\pi,\pi])} + \|S_n f\|^2_{L^2([-\pi,\pi])}.
\end{aligned} \tag{1}
$$

This is Pythagorean theorem in $L^2([-\pi,\pi])$.

Orthogonality

By orthogonality of $S_n f$ and $f - S_n f$, we have

$$
\begin{aligned}
\|f\|^2_{L^2([-\pi,\pi])} &= \|f - S_n f + S_n f\|^2_{L^2([-\pi,\pi])} \\
&= \langle (f - S_n f) + S_n f, (f - S_n f) + S_n f \rangle \\
&= \langle f - S_n f, f - S_n f \rangle + \underbrace{\langle f - S_n f, S_n f \rangle}_{=0} \\
&\quad + \underbrace{\langle S_n f, f - S_n f \rangle}_{=0} + \langle S_n f, S_n f \rangle \\
&= \|f - S_n f\|^2_{L^2([-\pi,\pi])} + \|S_n f\|^2_{L^2([-\pi,\pi])}.
\end{aligned} \tag{1}
$$

This is Pythagorean theorem in $L^2([-\pi,\pi])$. Moreover, by orthogonality of $\{e_j\}_{j=-n}^{n}$

$$
\begin{aligned}
\|f\|^2_{L^2([-\pi,\pi])} &= \|f - S_n f\|^2_{L^2([-\pi,\pi])} + \|S_n f\|^2_{L^2([-\pi,\pi])} \\
&= \|f - S_n f\|^2_{L^2([-\pi,\pi])} + \sum_{j=-n}^{n} |\widehat{f}(j)|^2.
\end{aligned} \tag{2}
$$

Parseval's identity

- $$\|f\|^2_{L^2([-\pi,\pi])} = \|f - S_n f\|^2_{L^2([-\pi,\pi])} + \sum_{j=-n}^{n} |\widehat{f}(j)|^2.$$

  implies

  $$\|f\|^2_{L^2([-\pi,\pi])} \geq \sum_{j=-n}^{n} |\widehat{f}(j)|^2.$$

Parseval's identity

$$\|f\|^2_{L^2([-\pi,\pi])} = \sum_{j=-n}^{n} |\widehat{f}(j)|^2.$$

- Equality holds if and only if

$$\lim_{n\to\infty} \|f - S_n f\|^2_{L^2([-\pi,\pi])} = 0.$$

This is Parseval's identity and it holds for all $f \in L^2([-\pi, \pi])$.

Parseval's identity

$$\|f\|^2_{L^2([-\pi,\pi])} = \sum_{j=-n}^{n} |\widehat{f}(j)|^2.$$

- Equality holds if and only if

$$\lim_{n\to\infty} \|f - S_n f\|^2_{L^2([-\pi,\pi])} = 0.$$

  This is Parseval's identity and it holds for all $f \in L^2([-\pi,\pi])$.
- This implies that if $f \in L^2([-\pi,\pi])$, then

$$\hat{f}(j) \to 0, \quad |j| \to \infty.$$

  The same holds for all $f \in L^1([-\pi,\pi])$ (Riemann-Lebesgue lemma).

Best square approximation

Theorem

If $f \in L^2([-\pi, \pi])$, *then*

$$\|f - S_n f\|_{L^2([-\pi,\pi])} \leq \left\| f - \sum_{j=-n}^{n} a_j e_j \right\|_{L^2([-\pi,\pi])}$$

for every $a_j \in \mathbb{C}$, $j = -n, \ldots, n$.

Best square approximation

Theorem

If $f \in L^2([-\pi, \pi])$*, then*

$$\|f - S_n f\|_{L^2([-\pi,\pi])} \leq \left\| f - \sum_{j=-n}^{n} a_j e_j \right\|_{L^2([-\pi,\pi])}$$

for every $a_j \in \mathbb{C}$*,* $j = -n, \ldots, n$*.*

Theorem

Let $f \in L^2([-\pi, \pi])$*. Then*

$$\lim_{n \to \infty} \|f - S_n f\|_{L^2([-\pi,\pi])} = 0.$$

Best square approximation

Theorem

If $f \in L^2([-\pi,\pi])$*, then*

$$\|f - S_n f\|_{L^2([-\pi,\pi])} \leq \left\| f - \sum_{j=-n}^{n} a_j e_j \right\|_{L^2([-\pi,\pi])}$$

for every $a_j \in \mathbb{C}$*,* $j = -n, \ldots, n$*.*

Theorem

Let $f \in L^2([-\pi,\pi])$*. Then*

$$\lim_{n\to\infty} \|f - S_n f\|_{L^2([-\pi,\pi])} = 0.$$

Warning! This does not imply pointwise convergence in each point!

Best square approximation

- This implies that $\{e_j\}_{n\in\mathbb{Z}}$ is an orthonormal bases for $L^2([-\pi,\pi])$ in the sense that for every function $f \in L^2([-\pi,\pi])$

$$\lim_{n\to\infty} \left\| \sum_{j=-n}^{n} \widehat{f}(j)e_j - f \right\|_{L^2([-\pi,\pi])} = 0$$

for every $f \in L^2([-\pi,\pi])$. This means that

$$f = \lim_{n\to\infty} \sum_{j=-n}^{n} \widehat{f}(j)e_j = \sum_{j=-\infty}^{\infty} \widehat{f}(j)e_j$$

in $L^2([-\pi,\pi])$.

MS-C1350 Partial differential equations
Ch 2.5-2.10 Fourier series

Riikka Korte

Department of Mathematics and Systems Analysis
Aalto University
riikka.korte@aalto.fi

September 10, 2024

2.5 Other intervals

The nth partial sum of a Fourier series of f on $[a,b]$ is

$$S_n f(t) = \sum_{j=-n}^{n} \langle f, e_j \rangle e_j = \sum_{j=-n}^{n} \widehat{f}(j) e^{\frac{2\pi i j t}{b-a}}, \quad n = 0, 1, 2, \ldots,$$

where the Fourier coefficients are

$$\widehat{f}(j) = \frac{1}{b-a} \int_a^b f(t) e^{\frac{-2\pi i j t}{b-a}}\, dt, \quad j \in \mathbb{Z}.$$

This follows from a change of variables.
Thus everything works in the same way as with the interval $[-\pi, \pi]$, but the formulas get more messy when the length of the interval is not the same as the period of trigonometric functions.

2.6 Real form of Fourier series

Let $f \in L^1([-\pi, \pi))$. The nth partial sum of a Fourier series can be written as

$$S_n f(t) = \frac{a_0}{2} + \sum_{j=1}^{n} (a_j \cos(jt) + b_j \sin(jt)),$$

where

$$a_j = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(jt)\, dt, \quad j = 0, 1, 2, \dots$$

and

$$b_j = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(jt)\, dt, \quad j = 1, 2, \dots.$$

This is called the real form of the Fourier series of f. The coefficients a_j are called the Fourier cosine coefficients of f and b_j are called the Fourier sine coefficients of f. The corresponding series are called the Fourier cosine and sine series of f correspondingly.

2.6 Real form of Fourier series

- Recall that

$$e^{ijt} = \cos(jt) + i\sin(jt), \quad e^{-ijt} = \cos(jt) - i\sin(jt).$$

- If f is real, then also a_j, b_j are real whereas $\hat{f}(j)$'s are not usually all real.

2.6 Real form of Fourier series

- Recall that

$$e^{ijt} = \cos(jt) + i\sin(jt), \quad e^{-ijt} = \cos(jt) - i\sin(jt).$$

- If f is real, then also a_j, b_j are real whereas $\hat{f}(j)$'s are not usually all real.
- Even function: $f(-t) = f(t)$. Odd function: $f(-t) = -f(t)$.

2.6 Real form of Fourier series

- Recall that

$$e^{ijt} = \cos(jt) + i\sin(jt), \quad e^{-ijt} = \cos(jt) - i\sin(jt).$$

- If f is real, then also a_j, b_j are real whereas $\hat{f}(j)$'s are not usually all real.
- Even function: $f(-t) = f(t)$. Odd function: $f(-t) = -f(t)$.
- If f is even, then $b_j = 0$ for all j.
- If f is odd, then $a_j = 0$ for all j.

2.6 Real form of Fourier series

- Recall that

$$e^{ijt} = \cos(jt) + i\sin(jt), \quad e^{-ijt} = \cos(jt) - i\sin(jt).$$

- If f is real, then also a_j, b_j are real whereas $\hat{f}(j)$'s are not usually all real.
- Even function: $f(-t) = f(t)$. Odd function: $f(-t) = -f(t)$.
- If f is even, then $b_j = 0$ for all j.
- If f is odd, then $a_j = 0$ for all j.
- Let $f : [0, L] \to \mathbb{R}$.

2.6 Real form of Fourier series

- Recall that

$$e^{ijt} = \cos(jt) + i\sin(jt), \quad e^{-ijt} = \cos(jt) - i\sin(jt).$$

- If f is real, then also a_j, b_j are real whereas $\hat{f}(j)$'s are not usually all real.
- Even function: $f(-t) = f(t)$. Odd function: $f(-t) = -f(t)$.
- If f is even, then $b_j = 0$ for all j.
- If f is odd, then $a_j = 0$ for all j.
- Let $f : [0, L] \to \mathbb{R}$.
- Even $2L$-periodic extension: $f(-t) = f(t)$. Then the series contains only $\cos$-terms.

2.6 Real form of Fourier series

- Recall that

$$e^{ijt} = \cos(jt) + i\sin(jt), \quad e^{-ijt} = \cos(jt) - i\sin(jt).$$

- If f is real, then also a_j, b_j are real whereas $\hat{f}(j)$'s are not usually all real.
- Even function: $f(-t) = f(t)$. Odd function: $f(-t) = -f(t)$.
- If f is even, then $b_j = 0$ for all j.
- If f is odd, then $a_j = 0$ for all j.
- Let $f : [0, L] \to \mathbb{R}$.
- Even $2L$-periodic extension: $f(-t) = f(t)$. Then the series contains only $\cos$-terms.
- Odd $2L$-periodic extension: $f(-t) = -f(t)$. Then the series contains only $\sin$-terms.

2.6 Real form of Fourier series

- Recall that

$$e^{ijt} = \cos(jt) + i\sin(jt), \quad e^{-ijt} = \cos(jt) - i\sin(jt).$$

- If f is real, then also a_j, b_j are real whereas $\hat{f}(j)$'s are not usually all real.
- Even function: $f(-t) = f(t)$. Odd function: $f(-t) = -f(t)$.
- If f is even, then $b_j = 0$ for all j.
- If f is odd, then $a_j = 0$ for all j.
- Let $f : [0, L] \to \mathbb{R}$.
- Even $2L$-periodic extension: $f(-t) = f(t)$. Then the series contains only $\cos$-terms.
- Odd $2L$-periodic extension: $f(-t) = -f(t)$. Then the series contains only $\sin$-terms.
- This does not violate the uniqueness of Fourier series as the functions are different on $[-L, 0]$.

2.7 Differentiation and Fourier series

- The smoother the function, the faster the Fourier coefficients decay.

2.7 Differentiation and Fourier series

- The smoother the function, the faster the Fourier coefficients decay.
- Fourier transform of a derivative: Let $f \in C^1(\mathbb{R})$ be 2π-periodic. Then

$$\widehat{f'}(j) = ij\hat{f}(j), \quad j \in \mathbb{Z}.$$

2.7 Differentiation and Fourier series

- The smoother the function, the faster the Fourier coefficients decay.
- Fourier transform of a derivative: Let $f \in C^1(\mathbb{R})$ be 2π-periodic. Then

$$\widehat{f'}(j) = ij\hat{f}(j), \quad j \in \mathbb{Z}.$$

- Reason: Fourier series can be differentiated termwise:

$$f(t) = \sum_{j=-\infty}^{\infty} \widehat{f}(j)e^{ijt} \Longrightarrow f'(t) = \sum_{j=-\infty}^{\infty} ij\widehat{f}(j)e^{ijt}.$$

2.8 Dirichlet kernel

- Dirichlet kernel is a useful way of expressing $S_n f$.

$$S_n f(t) = \sum_{j=-n}^{n} \widehat{f}(j) e^{ijt}$$

2.8 Dirichlet kernel

- Dirichlet kernel is a useful way of expressing $S_n f$.

$$S_n f(t) = \sum_{j=-n}^{n} \widehat{f}(j) e^{ijt} = \sum_{j=-n}^{n} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) e^{-ijs}\, ds \right) e^{ijt}$$

2.8 Dirichlet kernel

- Dirichlet kernel is a useful way of expressing $S_n f$.

$$S_n f(t) = \sum_{j=-n}^{n} \widehat{f}(j) e^{ijt} = \sum_{j=-n}^{n} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) e^{-ijs} \, ds \right) e^{ijt}$$

$$= \frac{1}{2\pi} \sum_{j=-n}^{n} \int_{-\pi}^{\pi} f(s) e^{ij(t-s)} \, ds$$

2.8 Dirichlet kernel

- Dirichlet kernel is a useful way of expressing $S_n f$.

$$S_n f(t) = \sum_{j=-n}^{n} \widehat{f}(j) e^{ijt} = \sum_{j=-n}^{n} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) e^{-ijs}\, ds \right) e^{ijt}$$

$$= \frac{1}{2\pi} \sum_{j=-n}^{n} \int_{-\pi}^{\pi} f(s) e^{ij(t-s)}\, ds = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) \left(\sum_{j=-n}^{n} e^{ij(t-s)} \right)$$

2.8 Dirichlet kernel

- Dirichlet kernel is a useful way of expressing $S_n f$.

$$S_n f(t) = \sum_{j=-n}^{n} \widehat{f}(j) e^{ijt} = \sum_{j=-n}^{n} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) e^{-ijs} \, ds \right) e^{ijt}$$

$$= \frac{1}{2\pi} \sum_{j=-n}^{n} \int_{-\pi}^{\pi} f(s) e^{ij(t-s)} \, ds = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) \left(\sum_{j=-n}^{n} e^{ij(t-s)} \right)$$

- Now we define the Dirichlet kernel as

$$D_n(t) = \sum_{j=-n}^{n} e^{ijt}, \quad n = 0, 1, 2, \ldots .$$

2.8 Dirichlet kernel

- Dirichlet kernel is a useful way of expressing $S_n f$.

$$S_n f(t) = \sum_{j=-n}^{n} \widehat{f}(j) e^{ijt} = \sum_{j=-n}^{n} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) e^{-ijs}\, ds \right) e^{ijt}$$

$$= \tfrac{1}{2\pi} \sum_{j=-n}^{n} \int_{-\pi}^{\pi} f(s) e^{ij(t-s)}\, ds = \tfrac{1}{2\pi} \int_{-\pi}^{\pi} f(s) \left(\sum_{j=-n}^{n} e^{ij(t-s)} \right)$$

- Now we define the Dirichlet kernel as

$$D_n(t) = \sum_{j=-n}^{n} e^{ijt}, \quad n = 0, 1, 2, \ldots .$$

- With this definition we have the formula

$$S_n f(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) D_n(t-s)\, ds, \quad t \in [-\pi, \pi], \quad n = 0, 1, 2, \ldots .$$

Properties of the Dirichlet kernel

- $$D_n f(t) = \sum_{j=-n}^{n} e^{ijt}$$

Properties of the Dirichlet kernel

- $$D_n f(t) = \sum_{j=-n}^{n} e^{ijt} = \frac{\sin\left(\left(n+\frac{1}{2}\right)t\right)}{\sin\left(\frac{1}{2}t\right)}, \quad t \neq 0, \quad n = 0, 1, 2, \ldots.$$

- The Dirichlet kernel is a 2π-periodic function.

Properties of the Dirichlet kernel

- $$D_n f(t) = \sum_{j=-n}^{n} e^{ijt} = \frac{\sin\left(\left(n+\frac{1}{2}\right)t\right)}{\sin\left(\frac{1}{2}t\right)}, \quad t \neq 0, \quad n = 0, 1, 2, \ldots.$$
- The Dirichlet kernel is a 2π-periodic function.
- $D_n(0) = 2n+1$ and $D_n(\pi) = (-1)^n$, $n = 0, 1, 2, \ldots$.

Properties of the Dirichlet kernel

- $$D_n f(t) = \sum_{j=-n}^{n} e^{ijt} = \frac{\sin\left(\left(n+\frac{1}{2}\right)t\right)}{\sin\left(\frac{1}{2}t\right)}, \quad t \neq 0, \quad n = 0, 1, 2, \dots.$$
- The Dirichlet kernel is a 2π-periodic function.
- $D_n(0) = 2n+1$ and $D_n(\pi) = (-1)^n$, $n = 0, 1, 2, \dots$.
- The Dirichlet formula can be written as a convolution

$$S_n f(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s) D_n(t-s)\, ds = (D_n * f)(t), \quad t \in [-\pi, \pi].$$

Convolution

- Convolution is an important notation in Fourier analysis and PDEs.

Convolution

- Convolution is an important notation in Fourier analysis and PDEs.
- Let $f, g \in C(\mathbb{R})$ be 2π-periodic.
  The convolution of f and g on $[-\pi, \pi]$ is

$$(f * g)(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t-s)g(s)\,ds = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s)g(t-s)\,ds.$$

Convolution

- Convolution is an important notation in Fourier analysis and PDEs.
- Let $f, g \in C(\mathbb{R})$ be 2π-periodic.
  The convolution of f and g on $[-\pi, \pi]$ is

$$(f * g)(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t-s)g(s)\,ds = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s)g(t-s)\,ds.$$

- If $g = 1$ then $f * g$ is constant.
  Moral: Convolutions can be considered as weighted averages of the functions.

Convolution

- Convolution is an important notation in Fourier analysis and PDEs.
- Let $f, g \in C(\mathbb{R})$ be 2π-periodic.
  The convolution of f and g on $[-\pi, \pi]$ is

$$(f * g)(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t-s)g(s)\, ds = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s)g(t-s)\, ds.$$

- If $g = 1$ then $f * g$ is constant.
  Moral: Convolutions can be considered as weighted averages of the functions.
- $f * g$ is also 2π-periodic.

Convolution

- Convolution is an important notation in Fourier analysis and PDEs.
- Let $f, g \in C(\mathbb{R})$ be 2π-periodic.
  The convolution of f and g on $[-\pi, \pi]$ is

$$(f * g)(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t-s)g(s)\,ds = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s)g(t-s)\,ds.$$

- If $g = 1$ then $f * g$ is constant.
  Moral: Convolutions can be considered as weighted averages of the functions.
- $f * g$ is also 2π-periodic.
- $f * g = g * f$ and $f * (g * h) = (f * g) * h$.

Convolution

- Convolution is an important notation in Fourier analysis and PDEs.
- Let $f, g \in C(\mathbb{R})$ be 2π-periodic.
  The convolution of f and g on $[-\pi, \pi]$ is

$$(f * g)(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t-s)g(s)\,ds = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s)g(t-s)\,ds.$$

- If $g = 1$ then $f * g$ is constant.
  Moral: Convolutions can be considered as weighted averages of the functions.
- $f * g$ is also 2π-periodic.
- $f * g = g * f$ and $f * (g * h) = (f * g) * h$.
- $\widehat{(f * g)}(j) = \widehat{f}(j)\widehat{g}(j)$, $j \in \mathbb{Z}$.

Pointwise convergence

Theorem

Let $f \in C([-\pi, \pi])$ be a 2π-periodic function which is differentiable at some point $t_0 \in [-\pi, \pi]$. Then

$$\lim_{n\to\infty} S_n f(t_0) = f(t_0).$$

Pointwise convergence

Theorem

Let $f \in C([-\pi, \pi])$ be a 2π-periodic function which is differentiable at some point $t_0 \in [-\pi, \pi]$. Then

$$\lim_{n\to\infty} S_n f(t_0) = f(t_0).$$

- This shows that the convergence of the Fourier series depends only on the behaviour of the function in an arbitrarily small neighbourhood of the point.

Pointwise convergence

Theorem

Let $f \in C([-\pi,\pi])$ be a 2π-periodic function which is differentiable at some point $t_0 \in [-\pi,\pi]$. Then

$$\lim_{n\to\infty} S_n f(t_0) = f(t_0).$$

- This shows that the convergence of the Fourier series depends only on the behaviour of the function in an arbitrarily small neighbourhood of the point.
- There exists a function $f \in L^1([-\pi,\pi])$ whose Fourier series diverges at every point.

Pointwise convergence

Theorem

Let $f \in C([-\pi, \pi])$ be a 2π-periodic function which is differentiable at some point $t_0 \in [-\pi, \pi]$. Then

$$\lim_{n \to \infty} S_n f(t_0) = f(t_0).$$

- This shows that the convergence of the Fourier series depends only on the behaviour of the function in an arbitrarily small neighbourhood of the point.
- There exists a function $f \in L^1([-\pi, \pi])$ whose Fourier series diverges at every point.
- There are also continuous functions whose Fourier series diverge in a dense set.

MS-C1350 Partial differential equations
Chapter 2.11
Laplace equation in the unit disc

Riikka Korte

Department of Mathematics and Systems Analysis
Aalto University
riikka.korte@aalto.fi

September 17, 2024

Lecture 3

Today we see how to solve Dirichlet problem in the unit disc (Chapter 2.11). Needed techniques include:

- Moving to polar coordinates
- Separation of variables
- Solving equations we obtain by separation of variables.
- Using Fourier series to find the correct Fourier series solution.

In Chapter 2.12, the heat equation in 1D is solved in a similar way. That example is not discussed in the lecture (but there is an old lecture video about it).

Laplace equation in the unit disc

- Laplace equation:

$$\Delta u = 0$$

- Laplace equation models heat distribution when the system has reached thermal equilibrium.
- Appears in many other places as well.

Dirichlet problem

- Consider 2-dimensional unit disc in $\mathbb{R}^2$:

$$\Omega = \{(x, y) \in \mathbb{R}^2 : (x^2 + y^2)^{\frac{1}{2}} < 1\}$$

Dirichlet problem

- Consider 2-dimensional unit disc in $\mathbb{R}^2$:

$$\Omega = \{(x, y) \in \mathbb{R}^2 : (x^2 + y^2)^{\frac{1}{2}} < 1\}$$

- Let $g \in C(\partial\Omega)$ be a continuous function on the boundary

$$\partial\Omega = \{(x, y) \in \mathbb{R}^2 : (x^2 + y^2)^{\frac{1}{2}} = 1\}.$$

Dirichlet problem

- Consider 2-dimensional unit disc in $\mathbb{R}^2$:

$$\Omega = \{(x,y) \in \mathbb{R}^2 : (x^2+y^2)^{\frac{1}{2}} < 1\}$$

- Let $g \in C(\partial\Omega)$ be a continuous function on the boundary

$$\partial\Omega = \{(x,y) \in \mathbb{R}^2 : (x^2+y^2)^{\frac{1}{2}} = 1\}.$$

- The problem is to find $u \in C^2(\Omega) \cap C(\overline{\Omega})$ such that

$$\begin{cases} \Delta u(x,y) = \dfrac{\partial^2 u}{\partial x^2}(x,y) + \dfrac{\partial^2 u}{\partial y^2}(x,y) = 0, \quad (x,y) \in \Omega, \\ u(x,y) = g(x,y), \quad (x,y) \in \partial\Omega. \end{cases}$$

Solving the Dirichlet problem

- We will solve this problem using Fourier series together with technique called separation of variables.

Solving the Dirichlet problem

- We will solve this problem using Fourier series together with technique called separation of variables.
- This means that we (first forget about the boundary values and) look for solutions of the form

$$u(x,y) = A(x)B(y),$$

Solving the Dirichlet problem

- We will solve this problem using Fourier series together with technique called separation of variables.
- This means that we (first forget about the boundary values and) look for solutions of the form
$$u(x,y) = A(x)B(y),$$
- This would work well in rectangular areas, but for the disc, we switch to polar coordinates:
$$(x,y) = (r\cos\theta, r\sin\theta), \quad (x,y) \in \mathbb{R}^2, \quad 0 \le r < \infty, \quad -\pi \le \theta < \pi,$$
where $r^2 = x^2 + y^2$ and $\tan\theta = \frac{y}{x}$. In polar coordinates, we have
$$\Omega = \{(r,\theta) : 0 \le r < 1, \, -\pi \le \theta < \pi\}$$
and
$$\partial\Omega = \{(1,\theta) : -\pi \le \theta < \pi\}.$$

Solving the Dirichlet problem

- We will solve this problem using Fourier series together with technique called separation of variables.
- This means that we (first forget about the boundary values and) look for solutions of the form
$$u(x, y) = A(x)B(y),$$
- This would work well in rectangular areas, but for the disc, we switch to polar coordinates:
$$(x, y) = (r\cos\theta, r\sin\theta), \quad (x, y) \in \mathbb{R}^2, \quad 0 \le r < \infty, \quad -\pi \le \theta < \pi,$$
where $r^2 = x^2 + y^2$ and $\tan\theta = \frac{y}{x}$. In polar coordinates, we have
$$\Omega = \{(r, \theta) : 0 \le r < 1, -\pi \le \theta < \pi\}$$
and
$$\partial\Omega = \{(1, \theta) : -\pi \le \theta < \pi\}.$$
- Note that unit disc is a rectangular set in polar coordinates and this is compatible with separation of variables.

Laplace equation in polar coordinates

Lemma

The two-dimensional Laplace operator in polar coordinates is

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}, \quad 0 < r < \infty, \quad -\pi \leq \theta < \pi.$$

[You need to apply chain rule to prove this. Details are in lecture notes.]

Laplace equation in polar coordinates

Lemma

The two-dimensional Laplace operator in polar coordinates is

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}, \quad 0 < r < \infty, \quad -\pi \le \theta < \pi.$$

[You need to apply chain rule to prove this. Details are in lecture notes.]

Thus the Dirichlet problem assumes the following form:

$$\begin{cases} \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0, \quad 0 < r < 1, \quad -\pi \le \theta < \pi, \\ u(1,\theta) = g(\theta), \quad -\pi \le \theta < \pi, \end{cases}$$

for $u = u(r,\theta)$.

Solving the Dirichlet problem: Step 1

- **Separation of variables:** We look for a product solution (to $\Delta u = 0$) of the form

$$u(r, \theta) = A(\theta)B(r)$$

Solving the Dirichlet problem: Step 1

- **Separation of variables:** We look for a product solution (to $\Delta u = 0$) of the form

$$u(r,\theta) = A(\theta)B(r)$$

- We insert this into the PDE and write it in a form, where one side depends on r and the other on θ:

$$\frac{r^2 B''(r) + rB'(r)}{B(r)} = -\frac{A''(\theta)}{A(\theta)}$$

Solving the Dirichlet problem: Step 1

- **Separation of variables:** We look for a product solution (to $\Delta u = 0$) of the form

$$u(r,\theta) = A(\theta)B(r)$$

- We insert this into the PDE and write it in a form, where one side depends on r and the other on θ:

$$\frac{r^2 B''(r) + rB'(r)}{B(r)} = -\frac{A''(\theta)}{A(\theta)} = \lambda$$

- This can only happen, if both sides are equal to a constant, say equal to λ.

Solving the Dirichlet problem: Step 1

- **Separation of variables:** We look for a product solution (to $\Delta u = 0$) of the form

$$u(r,\theta) = A(\theta)B(r)$$

- We insert this into the PDE and write it in a form, where one side depends on r and the other on θ:

$$\frac{r^2 B''(r) + rB'(r)}{B(r)} = -\frac{A''(\theta)}{A(\theta)} = \lambda$$

- This can only happen, if both sides are equal to a constant, say equal to λ.
- The PDE has been reduced to a system of two ODEs:

$$\begin{cases} A''(\theta) + \lambda A(\theta) = 0, \\ r^2 B''(r) + rB'(r) - \lambda B(r) = 0. \end{cases}$$

Solving the Dirichlet problem: Step 2: Solving separated equations

- Take into account the boundary data:

$$\text{"}u(1,\theta) = A(\theta)B(1) = g(\theta).\text{"}$$

Solving the Dirichlet problem: Step 2: Solving separated equations

- Take into account the boundary data:

$$\text{"}u(1,\theta) = A(\theta)B(1) = g(\theta).\text{"}$$

- g is defined on the circle and it can be identiefied with a 2π-periodic function. Thus A must be 2π-periodic.

Solving the Dirichlet problem: Step 2: Solving separated equations

- Take into account the boundary data:

$$\text{"}u(1,\theta) = A(\theta)B(1) = g(\theta).\text{"}$$

- g is defined on the circle and it can be identiefied with a 2π-periodic function. Thus A must be 2π-periodic.
- For which λ do we have nontrivial solutions to the ODE:

$$A''(\theta) + \lambda A(\theta) = 0 \iff -A''(\theta) = \lambda A(\theta)?$$

Solving the Dirichlet problem: Step 2: Solving separated equations

- Take into account the boundary data:

$$"u(1,\theta) = A(\theta)B(1) = g(\theta)."$$

- g is defined on the circle and it can be identiefied with a 2π-periodic function. Thus A must be 2π-periodic.
- For which λ do we have nontrivial solutions to the ODE:

$$A''(\theta) + \lambda A(\theta) = 0 \Longleftrightarrow -A''(\theta) = \lambda A(\theta)?$$

- We are interested in eigenvalues λ and eigenfunctions A.

Solving the Dirichlet problem: Step 2: Solving separated equations

- Take into account the boundary data:

$$\text{"}u(1,\theta) = A(\theta)B(1) = g(\theta).\text{"}$$

- g is defined on the circle and it can be identiefied with a 2π-periodic function. Thus A must be 2π-periodic.
- For which λ do we have nontrivial solutions to the ODE:

$$A''(\theta) + \lambda A(\theta) = 0 \iff -A''(\theta) = \lambda A(\theta)?$$

- We are interested in eigenvalues λ and eigenfunctions A.
- There are no complex eigenvalues (proof in lecture notes).

Solving the Dirichlet problem: Step 2: Solving separated equations

- Take into account the boundary data:

$$\text{"}u(1,\theta) = A(\theta)B(1) = g(\theta).\text{"}$$

- g is defined on the circle and it can be identiefied with a 2π-periodic function. Thus A must be 2π-periodic.
- For which λ do we have nontrivial solutions to the ODE:

$$A''(\theta) + \lambda A(\theta) = 0 \iff -A''(\theta) = \lambda A(\theta)?$$

- We are interested in eigenvalues λ and eigenfunctions A.
- There are no complex eigenvalues (proof in lecture notes).
- **Case:** $\lambda \qquad < 0$**:**

Solving the Dirichlet problem: Step 2: Solving separated equations

- Take into account the boundary data:

$$\text{"}u(1,\theta) = A(\theta)B(1) = g(\theta).\text{"}$$

- g is defined on the circle and it can be identiefied with a 2π-periodic function. Thus A must be 2π-periodic.
- For which λ do we have nontrivial solutions to the ODE:

$$A''(\theta) + \lambda A(\theta) = 0 \iff -A''(\theta) = \lambda A(\theta)?$$

- We are interested in eigenvalues λ and eigenfunctions A.
- There are no complex eigenvalues (proof in lecture notes).
- **Case:** $\lambda = -\mu^2 < 0$**:**

Solving the Dirichlet problem: Step 2: Solving separated equations

- Take into account the boundary data:

$$"u(1,\theta) = A(\theta)B(1) = g(\theta)."$$

- g is defined on the circle and it can be identiefied with a 2π-periodic function. Thus A must be 2π-periodic.
- For which λ do we have nontrivial solutions to the ODE:

$$A''(\theta) + \lambda A(\theta) = 0 \iff -A''(\theta) = \lambda A(\theta)?$$

- We are interested in eigenvalues λ and eigenfunctions A.
- There are no complex eigenvalues (proof in lecture notes).
- **Case:** $\lambda = -\mu^2 < 0$**:**

$$A(\theta) = c_1 e^{\mu\theta} + c_2 e^{-\mu\theta}.$$

Solving the Dirichlet problem: Step 2: Solving separated equations

- Take into account the boundary data:

$$"u(1,\theta) = A(\theta)B(1) = g(\theta)."$$

- g is defined on the circle and it can be identiefied with a 2π-periodic function. Thus A must be 2π-periodic.
- For which λ do we have nontrivial solutions to the ODE:

$$A''(\theta) + \lambda A(\theta) = 0 \iff -A''(\theta) = \lambda A(\theta)?$$

- We are interested in eigenvalues λ and eigenfunctions A.
- There are no complex eigenvalues (proof in lecture notes).
- **Case:** $\lambda = -\mu^2 < 0$:

$$A(\theta) = c_1 e^{\mu\theta} + c_2 e^{-\mu\theta}.$$

The only periodic solution is $A = 0$. Then also $u = AB = 0$.

Solving the Dirichlet problem: Step 2: Solving separated equations

- **Case $\lambda = 0$:**

Solving the Dirichlet problem: Step 2: Solving separated equations

- **Case** $\lambda = 0$**:**
- The equation reduces to $A'' = 0$ with the general solution

$$A(\theta) = c_1\theta + c_2.$$

Solving the Dirichlet problem: Step 2: Solving separated equations

- **Case $\lambda = 0$:**
- The equation reduces to $A'' = 0$ with the general solution

$$A(\theta) = c_1\theta + c_2.$$

- The only periodic solution is $A(\theta) = c_2$.

Solving the Dirichlet problem: Step 2: Solving separated equations

- **Case $\lambda = 0$:**
- The equation reduces to $A'' = 0$ with the general solution

$$A(\theta) = c_1\theta + c_2.$$

- The only periodic solution is $A(\theta) = c_2$.
- In this case, the ODE for B becomes

$$rB'' + B' = 0.$$

Solving the Dirichlet problem: Step 2: Solving separated equations

- **Case $\lambda = 0$:**
- The equation reduces to $A'' = 0$ with the general solution

$$A(\theta) = c_1\theta + c_2.$$

- The only periodic solution is $A(\theta) = c_2$.
- In this case, the ODE for B becomes

$$rB'' + B' = 0.$$

- Its general solution is

$$B(r) = d_1 + d_2 \ln r.$$

Solving the Dirichlet problem: Step 2: Solving separated equations

- **Case $\lambda = 0$:**
- The equation reduces to $A'' = 0$ with the general solution

$$A(\theta) = c_1\theta + c_2.$$

- The only periodic solution is $A(\theta) = c_2$.
- In this case, the ODE for B becomes

$$rB'' + B' = 0.$$

- Its general solution is

$$B(r) = d_1 + d_2 \ln r.$$

- Thus we get:

$$u(r, \theta) = A(\theta)B(r) = c_2(d_1 + d_2 \ln r).$$

Solving the Dirichlet problem: Step 2: Solving separated equations

- **Case $\lambda = 0$:**
- The equation reduces to $A'' = 0$ with the general solution
$$A(\theta) = c_1\theta + c_2.$$
- The only periodic solution is $A(\theta) = c_2$.
- In this case, the ODE for B becomes
$$rB'' + B' = 0.$$
- Its general solution is
$$B(r) = d_1 + d_2 \ln r.$$
- Thus we get:
$$u(r,\theta) = A(\theta)B(r) = c_2(d_1 + d_2 \ln r).$$
- Solution becomes unbounded as $r \to 0$. This is against physical intuition and these solutions are excluded. (Not a solution at 0.)

Solving the Dirichlet problem: Step 2: Solving separated equations

Solving the Dirichlet problem: Step 2: Solving separated equations

- **The case $\lambda > 0$:**

Solving the Dirichlet problem: Step 2: Solving separated equations

- **The case $\lambda = \mu^2 > 0$:** $A'' + \mu^2 A = 0$

Solving the Dirichlet problem: Step 2: Solving separated equations

- **The case $\lambda = \mu^2 > 0$:** $A'' + \mu^2 A = 0$
- The general complex solution of this equation is

$$A(\theta) = c_1 e^{i\mu\theta} + c_2 e^{-i\mu\theta}.$$

Solving the Dirichlet problem: Step 2: Solving separated equations

- **The case $\lambda = \mu^2 > 0$:** $A'' + \mu^2 A = 0$
- The general complex solution of this equation is

$$A(\theta) = c_1 e^{i\mu\theta} + c_2 e^{-i\mu\theta}.$$

- This is 2π-periodic if $\mu \in \mathbb{Z}$. Thus $\lambda = \mu^2 = j^2$, $j \in Z \setminus \{0\}$.

Solving the Dirichlet problem: Step 2: Solving separated equations

- **The case** $\lambda = \mu^2 > 0$**:** $A'' + \mu^2 A = 0$
- The general complex solution of this equation is

$$A(\theta) = c_1 e^{i\mu\theta} + c_2 e^{-i\mu\theta}.$$

- This is 2π-periodic if $\mu \in \mathbb{Z}$. Thus $\lambda = \mu^2 = j^2$, $j \in Z \setminus \{0\}$.
- Now the ODE for B is Euler equation:

$$r^2 B'' + rB' - j^2 B = 0.$$

Solving the Dirichlet problem: Step 2: Solving separated equations

- **The case $\lambda = \mu^2 > 0$:** $A'' + \mu^2 A = 0$
- The general complex solution of this equation is

$$A(\theta) = c_1 e^{i\mu\theta} + c_2 e^{-i\mu\theta}.$$

- This is 2π-periodic if $\mu \in \mathbb{Z}$. Thus $\lambda = \mu^2 = j^2$, $j \in Z \setminus \{0\}$.
- Now the ODE for B is Euler equation:

$$r^2 B'' + rB' - j^2 B = 0.$$

- Its general solution is

$$B(r) = d_1 r^j + d_2 r^{-j}$$

Solving the Dirichlet problem: Step 2: Solving separated equations

- **The case $\lambda = \mu^2 > 0$:** $A'' + \mu^2 A = 0$
- The general complex solution of this equation is

$$A(\theta) = c_1 e^{i\mu\theta} + c_2 e^{-i\mu\theta}.$$

- This is 2π-periodic if $\mu \in \mathbb{Z}$. Thus $\lambda = \mu^2 = j^2$, $j \in Z \setminus \{0\}$.
- Now the ODE for B is Euler equation:

$$r^2 B'' + rB' - j^2 B = 0.$$

- Its general solution is

$$B(r) = d_1 r^j + d_2 r^{-j}$$

- The term with r^{-j}, $j > 0$ blows up at 0 and is therefore excluded.

Solving the Dirichlet problem: Step 2: Solving separated equations

- **The case** $\lambda = \mu^2 > 0$**:** $A'' + \mu^2 A = 0$
- The general complex solution of this equation is

$$A(\theta) = c_1 e^{i\mu\theta} + c_2 e^{-i\mu\theta}.$$

- This is 2π-periodic if $\mu \in \mathbb{Z}$. Thus $\lambda = \mu^2 = j^2$, $j \in Z \setminus \{0\}$.
- Now the ODE for B is Euler equation:

$$r^2 B'' + rB' - j^2 B = 0.$$

- Its general solution is

$$B(r) = d_1 r^j + d_2 r^{-j}$$

- The term with r^{-j}, $j > 0$ blows up at 0 and is therefore excluded.
- Thus $\lambda > 0$ gives relevant solutions of form

$$u(r, \theta) = A(\theta)B(r) = r^{|j|} e^{ij\theta}.$$

Solving the Dirichlet problem: Step 3: Fourier series solution for the whole problem

- Now we have found solutions

$$u_j(r,\theta) = A(\theta)B(r) = r^{|j|}e^{ij\theta}.$$

which solve the problem with boundary values

$$u_j(1,\theta) = e^{ij\theta}, \quad j \in \mathbb{Z}.$$

Solving the Dirichlet problem: Step 3: Fourier series solution for the whole problem

- Now we have found solutions

$$u_j(r,\theta) = A(\theta)B(r) = r^{|j|}e^{ij\theta}.$$

which solve the problem with boundary values

$$u_j(1,\theta) = e^{ij\theta}, \quad j \in \mathbb{Z}.$$

- Laplace operator is linear. Thus all linear combinations are also solutions:

$$u(r,\theta) = \sum_{j=-\infty}^{\infty} a_j r^{|j|} e^{ij\theta}, \quad 0 \le r < 1, \quad -\pi < \theta \le \pi.$$

Solving the Dirichlet problem: Step 3: Fourier series solution for the whole problem

- Now we have found solutions

$$u_j(r,\theta) = A(\theta)B(r) = r^{|j|}e^{ij\theta}.$$

which solve the problem with boundary values

$$u_j(1,\theta) = e^{ij\theta}, \quad j \in \mathbb{Z}.$$

- Laplace operator is linear. Thus all linear combinations are also solutions:

$$u(r,\theta) = \sum_{j=-\infty}^{\infty} a_j r^{|j|} e^{ij\theta}, \quad 0 \leq r < 1, \quad -\pi < \theta \leq \pi.$$

- This should be compatible with the boundary data when $r = 1$:

$$u(1,\theta) = \sum_{j=-\infty}^{\infty} a_j e^{ij\theta} = g(\theta).$$

Solving the Dirichlet problem: Step 3: Fourier series solution for the whole problem

- How to choose a_j's so that:

$$u(1,\theta) = \sum_{j=-\infty}^{\infty} a_j e^{ij\theta} = g(\theta)?$$

Solving the Dirichlet problem: Step 3: Fourier series solution for the whole problem

- How to choose a_j's so that:

$$u(1,\theta) = \sum_{j=-\infty}^{\infty} a_j e^{ij\theta} = g(\theta)?$$

- If $g \in L^2([-\pi,\pi])$ then we know that there is a solution if we interpret the equality in L^2-sense.

Solving the Dirichlet problem: Step 3: Fourier series solution for the whole problem

- How to choose a_j's so that:

$$u(1,\theta) = \sum_{j=-\infty}^{\infty} a_j e^{ij\theta} = g(\theta)?$$

- If $g \in L^2([-\pi,\pi])$ then we know that there is a solution if we interpret the equality in L^2-sense.
- If $g \in C^1([-\pi,\pi])$, then the Fourier series converges uniformly and

$$a_j = \hat{g}(j).$$

Things to think about:

1. Does the series solution we obtained converge?

Things to think about:

1. Does the series solution we obtained converge?
2. Is the function really a solution to our PDE?

Things to think about:

1. Does the series solution we obtained converge?
2. Is the function really a solution to our PDE?
3. Does the solution attain the correct boundary values?

Things to think about:

1. Does the series solution we obtained converge?
2. Is the function really a solution to our PDE?
3. Does the solution attain the correct boundary values?
4. Are there other solutions than the solution we found?

Things to think about:

1. Does the series solution we obtained converge?
2. Is the function really a solution to our PDE?
3. Does the solution attain the correct boundary values?
4. Are there other solutions than the solution we found?

On formal level (2) and (3) seem to be ok. On needs to be careful when switching the order of the limit and the infinite series. We will return later to this and to the question of uniqueness.

Step 4: Explicit representation formula

- We can plug in the formula for the Fourier coefficients $\hat{g}(j)$.

$$u(r,\theta) = \sum_{j=-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) e^{-ijt}\, dt \right) r^{|j|} e^{ij\theta}$$

Step 4: Explicit representation formula

- We can plug in the formula for the Fourier coefficients $\hat{g}(j)$.
- One has to be careful when changing the order of summation and integration.

$$u(r,\theta) = \sum_{j=-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) e^{-ijt}\, dt \right) r^{|j|} e^{ij\theta}$$

Step 4: Explicit representation formula

- We can plug in the formula for the Fourier coefficients $\hat{g}(j)$.
- One has to be careful when changing the order of summation and integration.

$$\begin{aligned} u(r,\theta) &= \sum_{j=-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) e^{-ijt}\, dt \right) r^{|j|} e^{ij\theta} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) \left(\sum_{j=-\infty}^{\infty} r^{|j|} e^{ij(\theta-t)} \right) dt \end{aligned}$$

Step 4: Explicit representation formula

- We can plug in the formula for the Fourier coefficients $\hat{g}(j)$.
- One has to be careful when changing the order of summation and integration.

$$\begin{aligned} u(r,\theta) &= \sum_{j=-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) e^{-ijt}\, dt \right) r^{|j|} e^{ij\theta} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) \left(\sum_{j=-\infty}^{\infty} r^{|j|} e^{ij(\theta - t)} \right) dt \\ &= (g * P_r)(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) P_r(\theta - t)\, dt, \end{aligned}$$

where we have the Poisson kernel:

$$P_r(\theta) = P(r,\theta) = \sum_{j=-\infty}^{\infty} r^{|j|} e^{ij\theta}.$$

Poisson kernel

$$P_r(\theta) = P(r,\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2}, \quad 0 \le r < 1, \quad -\pi \le \theta < \pi.$$

Some properties:

Poisson kernel

$$P_r(\theta) = P(r,\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2}, \quad 0 \le r < 1, \quad -\pi \le \theta < \pi.$$

Some properties:

- $P_r(\theta) \ge 0$

Poisson kernel

$$P_r(\theta) = P(r,\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2}, \quad 0 \le r < 1, \quad -\pi \le \theta < \pi.$$

Some properties:

- $P_r(\theta) \ge 0$
- $$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)\,d\theta = 1$$

Poisson kernel

$$P_r(\theta) = P(r,\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2}, \quad 0 \le r < 1, \quad -\pi \le \theta < \pi.$$

Some properties:

- $P_r(\theta) \ge 0$
- $$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)\,d\theta = 1$$
- Observe that

$$P(r,0) = \frac{1-r^2}{1-2r+r^2}, \quad 0 \le r < 1,$$

consequently $\lim_{r\to 1} P(r,0) = \infty$.
However $\lim_{r\to 1} P(r,\theta) = 0$ when $\theta \neq 0$.

Poisson kernel

$$P_r(\theta) = P(r,\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2}, \quad 0 \le r < 1, \quad -\pi \le \theta < \pi.$$

Some properties:

- $P_r(\theta) \ge 0$
- $$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)\,d\theta = 1$$
- Observe that
$$P(r,0) = \frac{1-r^2}{1-2r+r^2}, \quad 0 \le r < 1,$$
consequently $\lim_{r\to 1} P(r,0) = \infty$.
However $\lim_{r\to 1} P(r,\theta) = 0$ when $\theta \ne 0$.
- This formula does not work when $r = 1$!

MS-C1350 Partial differential equations
Chapter 2.12 and 2.13
Heat and weve equations in one-dimension

Riikka Korte

Department of Mathematics and Systems Analysis
Aalto University
riikka.korte@aalto.fi

September 24, 2024

Lecture 4

- We see how to solve wave equation in 1D.
  New things:
  - Now we use real form Fourier series. This problem could be solved with Fourier series with complex exponential functions equally

The heat equation in one-dimension (2.12)

- Suppose we have a ring of radius 1 centered at the origin.

The heat equation in one-dimension (2.12)

- Suppose we have a ring of radius 1 centered at the origin.
- Suppose that it is perfectly insulated.

The heat equation in one-dimension (2.12)

- Suppose we have a ring of radius 1 centered at the origin.
- Suppose that it is perfectly insulated.
- At time $t = 0$, the initia temperature is given by $g : [-\pi, \pi] \to \mathbb{R}$.

The heat equation in one-dimension (2.12)

- Suppose we have a ring of radius 1 centered at the origin.
- Suppose that it is perfectly insulated.
- At time $t = 0$, the initia temperature is given by $g : [-\pi, \pi] \to \mathbb{R}$.
- Diffusion of heat on the circle is modeled by the heat equation:

$$\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial \theta^2} = 0,$$

where a^2 is the thermal diffusivity, which depends on the material of the ring. We set $a^2 = 1$.

The heat equation in one-dimension (2.12)

- Suppose we have a ring of radius 1 centered at the origin.
- Suppose that it is perfectly insulated.
- At time $t = 0$, the initia temperature is given by $g : [-\pi, \pi] \to \mathbb{R}$.
- Diffusion of heat on the circle is modeled by the heat equation:

$$\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial \theta^2} = 0,$$

  where a^2 is the thermal diffusivity, which depends on the material of the ring. We set $a^2 = 1$.
- We are looking for a solution $u = u(\theta, t)$:

$$\begin{cases} \frac{\partial u}{\partial t}(\theta, t) - \frac{\partial^2 u}{\partial \theta^2}(\theta, t) = 0, \quad -\pi \le \theta < \pi, \quad t > 0, \\ u(\theta, 0) = g(\theta), \quad -\pi \le \theta < \pi. \end{cases}$$

  Periodic initial value problem.

Solving the heat equation

1. Separation of variables
2. Solution to the separated equations
3. Fourier series solution of the entire equation
4. Explicit representation formula

Solving the heat equation

1. Separation of variables
2. Solution to the separated equations
3. Fourier series solution of the entire equation
4. Explicit representation formula

Solving the heat equation: Separation of variables

- Write
$$u(\theta, t) = A(\theta)B(t)$$
- insert this into the heat equation:
$$A(\theta)B'(t) - A''(\theta)B(t) = 0 \Longleftrightarrow \frac{B'(t)}{B(t)} = \frac{A''(\theta)}{A(\theta)}.$$
- Thus
$$\begin{cases} A''(\theta) = \lambda A(\theta), \\ B'(t) = \lambda B(t). \end{cases}$$

Solving the heat equation: Solution to the separated equations

- The initial condition:

$$\text{"}u(\theta, 0) = A(\theta)B(0) = g(\theta)\text{"}$$

is a 2π-periodic function.

Solving the heat equation: Solution to the separated equations

- The initial condition:

$$"u(\theta, 0) = A(\theta)B(0) = g(\theta)"$$

  is a 2π-periodic function.
- We do a case study as with the Laplace equation:

Solving the heat equation: Solution to the separated equations

- The initial condition:

$$"u(\theta,0) = A(\theta)B(0) = g(\theta)"$$

  is a 2π-periodic function.
- We do a case study as with the Laplace equation:
- **Case** $\lambda = \mu^2 > 0$:

Solving the heat equation: Solution to the separated equations

- The initial condition:

$$"u(\theta, 0) = A(\theta)B(0) = g(\theta)"$$

  is a 2π-periodic function.
- We do a case study as with the Laplace equation:
- **Case** $\lambda = \mu^2 > 0$:
- The equation

$$A''(\theta) - \mu^2 A(\theta) = 0$$

  has the general solution

$$A(\theta) = c_1 e^{\mu\theta} + c_2 e^{-\mu\theta},$$

  which is periodic only if $c_1 = c_2 = 0$.

Solving the heat equation: Solution to the separated equations

- **Case** $\lambda = 0$: Then $A''(\theta) = 0$, which implies $A(\theta) = c_1\theta + c_2$.

Solving the heat equation: Solution to the separated equations

- **Case** $\lambda = 0$: Then $A''(\theta) = 0$, which implies $A(\theta) = c_1\theta + c_2$.
- The only periodic solution is $A(\theta) = c_2$.

Solving the heat equation: Solution to the separated equations

- **Case** $\lambda = 0$: Then $A''(\theta) = 0$, which implies $A(\theta) = c_1\theta + c_2$.
- The only periodic solution is $A(\theta) = c_2$.
- In this case we also get that $B(t) = c_3$ i.e. u is a constant.

Solving the heat equation: Solution to the separated equations

- **Case** $\lambda = -\mu^2 < 0$: The ODE becomes

$$A''(\theta) + \mu^2 A(\theta) = 0$$

Solving the heat equation: Solution to the separated equations

- **Case** $\lambda = -\mu^2 < 0$: The ODE becomes

$$A''(\theta) + \mu^2 A(\theta) = 0$$

- The general solution is

$$A(\theta) = c_1 e^{i\mu\theta} + c_2 e^{-i\mu\theta},$$

Solving the heat equation: Solution to the separated equations

- **Case** $\lambda = -\mu^2 < 0$: The ODE becomes

$$A''(\theta) + \mu^2 A(\theta) = 0$$

- The general solution is

$$A(\theta) = c_1 e^{i\mu\theta} + c_2 e^{-i\mu\theta},$$

- which is 2π-periodic if $\mu \in \mathbb{Z}$ i.e. $\lambda = -j^2$, $j \in \mathbb{Z}$.

Solving the heat equation: Solution to the separated equations

- **Case** $\lambda = -\mu^2 < 0$: The ODE becomes

$$A''(\theta) + \mu^2 A(\theta) = 0$$

- The general solution is

$$A(\theta) = c_1 e^{i\mu\theta} + c_2 e^{-i\mu\theta},$$

- which is 2π-periodic if $\mu \in \mathbb{Z}$ i.e. $\lambda = -j^2$, $j \in \mathbb{Z}$.
- For these values, the ODE for B is

$$B'(t) + j^2 B(t) = 0,$$

Solving the heat equation: Solution to the separated equations

- **Case** $\lambda = -\mu^2 < 0$: The ODE becomes

$$A''(\theta) + \mu^2 A(\theta) = 0$$

- The general solution is

$$A(\theta) = c_1 e^{i\mu\theta} + c_2 e^{-i\mu\theta},$$

- which is 2π-periodic if $\mu \in \mathbb{Z}$ i.e. $\lambda = -j^2$, $j \in \mathbb{Z}$.
- For these values, the ODE for B is

$$B'(t) + j^2 B(t) = 0,$$

- which has the general solution

$$B(t) = ce^{-j^2 t}.$$

Solving the heat equation: Solution to the separated equations

- **Case** $\lambda = -\mu^2 < 0$: The ODE becomes
$$A''(\theta) + \mu^2 A(\theta) = 0$$
- The general solution is
$$A(\theta) = c_1 e^{i\mu\theta} + c_2 e^{-i\mu\theta},$$
- which is 2π-periodic if $\mu \in \mathbb{Z}$ i.e. $\lambda = -j^2$, $j \in \mathbb{Z}$.
- For these values, the ODE for B is
$$B'(t) + j^2 B(t) = 0,$$
- which has the general solution
$$B(t) = ce^{-j^2 t}.$$
- Thus we have special solutions:
$$u(\theta, t) = e^{-j^2 t} e^{ij\theta}, \quad j \in \mathbb{Z}.$$

Solving the heat equation: Fourier series solution

- The heat equation is linear. Thus any linear combination fo the special solutions will give again a solution.

Solving the heat equation: Fourier series solution

- The heat equation is linear. Thus any linear combination fo the special solutions will give again a solution.
- Therefore we define

$$u(\theta, t) = \sum_{j=-\infty}^{\infty} a_j e^{-j^2 t} e^{ij\theta}, \quad -\pi \leq \theta < \pi, \quad t > 0,$$

and try to determine a_j.

Solving the heat equation: Fourier series solution

- The heat equation is linear. Thus any linear combination fo the special solutions will give again a solution.
- Therefore we define

$$u(\theta,t)=\sum_{j=-\infty}^{\infty} a_j e^{-j^2 t} e^{ij\theta}, \quad -\pi \le \theta < \pi, \quad t>0,$$

  and try to determine a_j.
- Initial condition $u(\theta,0)=g(\theta)$ gives

$$\sum_{j=-\infty}^{\infty} a_j e^{ij\theta} = g(\theta)$$

Solving the heat equation: Fourier series solution

- The heat equation is linear. Thus any linear combination fo the special solutions will give again a solution.
- Therefore we define

$$u(\theta,t)=\sum_{j=-\infty}^{\infty} a_j e^{-j^2 t} e^{ij\theta}, \quad -\pi \le \theta < \pi, \quad t>0,$$

  and try to determine a_j.
- Initial condition $u(\theta,0)=g(\theta)$ gives

$$\sum_{j=-\infty}^{\infty} a_j e^{ij\theta} = g(\theta)$$

- Thus a_j is the Fourier coefficient $\hat{g}(j)$ of the initial data and

$$u(\theta,t)=\sum_{j=-\infty}^{\infty} \widehat{g}(j) e^{-j^2 t} e^{ij\theta}.$$

Solving the heat equation: Fourier series solution

- The heat equation is linear. Thus any linear combination fo the special solutions will give again a solution.
- Therefore we define

$$u(\theta,t)=\sum_{j=-\infty}^{\infty} a_j e^{-j^2 t} e^{ij\theta}, \quad -\pi \le \theta < \pi, \quad t>0,$$

  and try to determine a_j.
- Initial condition $u(\theta,0)=g(\theta)$ gives

$$\sum_{j=-\infty}^{\infty} a_j e^{ij\theta} = g(\theta)$$

- Thus a_j is the Fourier coefficient $\hat{g}(j)$ of the initial data and

$$u(\theta,t)=\sum_{j=-\infty}^{\infty} \widehat{g}(j) e^{-j^2 t} e^{ij\theta}.$$

- This converges nicely.

Solving the heat equation: Explicit representation formula

- Next goal is to derive an integral representation for the solution.

Solving the heat equation: Explicit representation formula

- Next goal is to derive an integral representation for the solution.
- The idea is to use the definition of the Fourier coefficients, then switch the order of the limit and the integral:

$$u(\theta, t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(s) \left(\sum_{j=-\infty}^{\infty} e^{-j^2 t} e^{ij(\theta - s)} \right) ds.$$

Solving the heat equation: Explicit representation formula

- Next goal is to derive an integral representation for the solution.
- The idea is to use the definition of the Fourier coefficients, then switch the order of the limit and the integral:

$$u(\theta, t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(s) \left(\sum_{j=-\infty}^{\infty} e^{-j^2 t} e^{ij(\theta - s)} \right) ds.$$

- We define the heat kernel for the circle as

$$H_t(\theta) = \sum_{j=-\infty}^{\infty} e^{-j^2 t} e^{ij\theta}, \quad -\pi \leq \theta < \pi, \quad t > 0,$$

Solving the heat equation: Explicit representation formula

- Next goal is to derive an integral representation for the solution.
- The idea is to use the definition of the Fourier coefficients, then switch the order of the limit and the integral:

$$u(\theta, t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(s) \left(\sum_{j=-\infty}^{\infty} e^{-j^2 t} e^{ij(\theta - s)} \right) ds.$$

- We define the heat kernel for the circle as

$$H_t(\theta) = \sum_{j=-\infty}^{\infty} e^{-j^2 t} e^{ij\theta}, \quad -\pi \leq \theta < \pi, \quad t > 0,$$

- and we can write the solution as

$$u(\theta, t) = (g * H_t)(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(s) H_t(\theta - s)\, ds, \quad -\pi \leq \theta < \pi, t > 0.$$

Ch 2.13: Wave equation in one dimension

- We have now used essentially the same strategy for solving Laplace equation in the unit disc (Ch 2.11) and the heat equation in 1-dimensional ring.
  1. Separation of variables so that the initial and/or boundary values can be expressed with one variable.
  2. Solutions to the separated equations. Any linear combination solves the PDE inside the domain.
  3. Fourier series solution of the entire equation.
  4. Explicit representation formula.
- The same strategy works also for the (one-dimensional) wave equation.
- It is presented in the lecture notes (Ch 2.13) using real form Fourier coefficients.

Aalto University

MS-C1350 Partial differential equations
Chapter 2.14:
Approximations of the identity

Riikka Korte

Department of Mathematics and Systems Analysis
Aalto University
riikka.korte@aalto.fi

September 17, 2024

Lecture 4

- We solve heat equation in 1D.
  Topics:
  - Compare to the steps needed to solve previous examples (Laplace in unit disc, heat equation in 1D).
  - The use of real form Fourier series. (The problem could be solved also with complex Fourier series.)
  - Initial and boundary conditions for wave equation.
- Approximations of the identity – a class of convolutions that behave well.

Approximation of the identity (intro)

Consider the formulas:

$$S_n f(\theta) = (f * D_n)(\theta) = \sum_{j=-n}^{n} \widehat{f}(j) e^{ij\theta}, \quad f = \sum_{j=-\infty}^{\infty} \widehat{f}(j) e^{ij\theta},$$

$$P_r f(\theta) = (f * P_r)(\theta) = \sum_{j=-\infty}^{\infty} \widehat{f}(j) r^{|j|} e^{ij\theta}, \quad 0 < r < 1,$$

$$H_t f(\theta) = (f * H_t)(\theta) = \sum_{j=-\infty}^{\infty} \widehat{f}(j) e^{-j^2 t} e^{ij\theta}, \quad t > 0.$$

Approximation of the identity (intro)

Consider the formulas:

$$S_n f(\theta) = (f * D_n)(\theta) = \sum_{j=-n}^{n} \widehat{f}(j) e^{ij\theta}, \quad f = \sum_{j=-\infty}^{\infty} \widehat{f}(j) e^{ij\theta},$$

$$P_r f(\theta) = (f * P_r)(\theta) = \sum_{j=-\infty}^{\infty} \widehat{f}(j) r^{|j|} e^{ij\theta}, \quad 0 < r < 1,$$

$$H_t f(\theta) = (f * H_t)(\theta) = \sum_{j=-\infty}^{\infty} \widehat{f}(j) e^{-j^2 t} e^{ij\theta}, \quad t > 0.$$

- QUESTIONS:
  - Can we recover f from the partial sums of its Fourier series?
  - If f is continuous, do we have

$$\lim_{r \to 1} (P_r * f) = f \quad \text{and} \quad \lim_{t \to 0} (H_t * f) = f?$$

Approximation of the identity (intro)

Consider the formulas:

$$S_n f(\theta) = (f * D_n)(\theta) = \sum_{j=-n}^{n} \widehat{f}(j) e^{ij\theta}, \quad f = \sum_{j=-\infty}^{\infty} \widehat{f}(j) e^{ij\theta},$$

$$P_r f(\theta) = (f * P_r)(\theta) = \sum_{j=-\infty}^{\infty} \widehat{f}(j) r^{|j|} e^{ij\theta}, \quad 0 < r < 1,$$

$$H_t f(\theta) = (f * H_t)(\theta) = \sum_{j=-\infty}^{\infty} \widehat{f}(j) e^{-j^2 t} e^{ij\theta}, \quad t > 0.$$

- QUESTIONS:
  - Can we recover f from the partial sums of its Fourier series?
  - If f is continuous, do we have

$$\lim_{r \to 1} (P_r * f) = f \quad \text{and} \quad \lim_{t \to 0} (H_t * f) = f?$$

Approximation of the identity (intro)

Consider the formulas:

$$S_n f(\theta) = (f * D_n)(\theta) = \sum_{j=-n}^{n} \widehat{f}(j) e^{ij\theta}, \quad f = \sum_{j=-\infty}^{\infty} \widehat{f}(j) e^{ij\theta},$$

$$P_r f(\theta) = (f * P_r)(\theta) = \sum_{j=-\infty}^{\infty} \widehat{f}(j) r^{|j|} e^{ij\theta}, \quad 0 < r < 1,$$

$$H_t f(\theta) = (f * H_t)(\theta) = \sum_{j=-\infty}^{\infty} \widehat{f}(j) e^{-j^2 t} e^{ij\theta}, \quad t > 0.$$

- QUESTIONS:
  - Can we recover f from the partial sums of its Fourier series?
  - If f is continuous, do we have

$$\lim_{r \to 1} (P_r * f) = f \quad \text{and} \quad \lim_{t \to 0} (H_t * f) = f?$$

- These questions are related to the question in which sense the boundary or initial values are obtained.

Approximation of the identity

Definition

A family of kernels $\{K_\varepsilon\}_{\varepsilon>0}$ which are 2π-periodic functions is said to be a family of good kernels, if

Approximation of the identity

Definition

A family of kernels $\{K_\varepsilon\}_{\varepsilon>0}$ which are 2π-periodic functions is said to be a family of good kernels, if

1. For every $\varepsilon > 0$ we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_\varepsilon(x)\,dx = 1.$$

Approximation of the identity

Definition

A family of kernels $\{K_\varepsilon\}_{\varepsilon>0}$ which are 2π-periodic functions is said to be a family of good kernels, if

1. For every $\varepsilon > 0$ we have
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_\varepsilon(x)\,dx = 1.$$

2. There exists $M > 0$ such that for every $\varepsilon > 0$
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |K_\varepsilon(x)|\,dx \le M.$$

Approximation of the identity

Definition

A family of kernels $\{K_\varepsilon\}_{\varepsilon>0}$ which are 2π-periodic functions is said to be a family of good kernels, if

1. For every $\varepsilon > 0$ we have
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_\varepsilon(x)\,dx = 1.$$
2. There exists $M > 0$ such that for every $\varepsilon > 0$
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |K_\varepsilon(x)|\,dx \le M.$$
3. For every $\delta > 0$ we have
$$\lim_{\varepsilon\to 0}\int_{\delta<|x|\le\pi} |K_\varepsilon(x)|\,dx = 0.$$

Approximation of the identity

- $P_{1-\varepsilon}$ and H_t are good kernels.

Approximation of the identity

- $P_{1-\varepsilon}$ and H_t are good kernels.
- D_n is not.

Approximation of the identity

- $P_{1-\varepsilon}$ and H_t are good kernels.
- D_n is not.
- Good kernels are also called approximations of the identity.

Approximation of the identity

- $P_{1-\varepsilon}$ and H_t are good kernels.
- D_n is not.
- Good kernels are also called approximations of the identity.

Theorem

Let $\{K_\varepsilon\}_{\varepsilon>0}$ be a family of good kernels and $f : [-\pi, \pi] \to \mathbb{C}$ be a bounded 2π-periodic function. Then

$$\lim_{\varepsilon \to 0} (f * K_\varepsilon)(x) = f(x)$$

whenever f is continuous at x. If f is continuous on the whole interval $[-\pi, \pi]$, then the above limit is uniform.

MS-C1350 Partial differential equations
Chapter 3.1-3.8
Fourier transform and PDEs

Riikka Korte

Department of Mathematics and Systems Analysis
Aalto University
riikka.korte@aalto.fi

October 1, 2024

Lecture 5:

The topic of the lecture is Fourier transform. It is an analogous theory to Fourier series.

- $L^p(\mathbb{R}^n)$–space
- Formula for the Fourier transform and inverse transform
- Formulas for translation, modulation and differentiation with Fourier transform.
- Convolution on $\mathbb{R}^n$
- Good kernels on $\mathbb{R}^n$.

3 Fourier transform

- Our next topic is Fourier transform.
- It gives a method to solve PDEs in the higher dimensional case (and on unbounded intervals).
- Note that we used Fourier series to solve e.g. Laplace equation on plane ($n = 2$), but the Fourier series technique was only used to choose the correct coefficients so that the boundary values on one-dimensional boundary were correct.
- When we used Fourier series, the boundary/initial values were always one-dimensional and either periodic (boundary of a unit sphere / initial temperature on ring) or defined on a bounded interval with zero values at endpoints.
- Fourier transform has many properties analogous to the Fourier series.
- When using Fourier transform, the functions need not be periodic.
- Convolution is a central tool in this theory.

3 Fourier transform

- Our next topic is Fourier transform.
- It gives a method to solve PDEs in the higher dimensional case (and on unbounded intervals).
- Note that we used Fourier series to solve e.g. Laplace equation on plane ($n = 2$), but the Fourier series technique was only used to choose the correct coefficients so that the boundary values on one-dimensional boundary were correct.
- When we used Fourier series, the boundary/initial values were always one-dimensional and either periodic (boundary of a unit sphere / initial temperature on ring) or defined on a bounded interval with zero values at endpoints.
- Fourier transform has many properties analogous to the Fourier series.
- When using Fourier transform, the functions need not be periodic.
- Convolution is a central tool in this theory.

3 Fourier transform

- Our next topic is Fourier transform.
- It gives a method to solve PDEs in the higher dimensional case (and on unbounded intervals).
- Note that we used Fourier series to solve e.g. Laplace equation on plane ($n = 2$), but the Fourier series technique was only used to choose the correct coefficients so that the boundary values on one-dimensional boundary were correct.
- When we used Fourier series, the boundary/initial values were always one-dimensional and either periodic (boundary of a unit sphere / initial temperature on ring) or defined on a bounded interval with zero values at endpoints.
- Fourier transform has many properties analogous to the Fourier series.
- When using Fourier transform, the functions need not be periodic.
- Convolution is a central tool in this theory.

3 Fourier transform

- Our next topic is Fourier transform.
- It gives a method to solve PDEs in the higher dimensional case (and on unbounded intervals).
- Note that we used Fourier series to solve e.g. Laplace equation on plane ($n = 2$), but the Fourier series technique was only used to choose the correct coefficients so that the boundary values on one-dimensional boundary were correct.
- When we used Fourier series, the boundary/initial values were always one-dimensional and either periodic (boundary of a unit sphere / initial temperature on ring) or defined on a bounded interval with zero values at endpoints.
- Fourier transform has many properties analogous to the Fourier series.
- When using Fourier transform, the functions need not be periodic.
- Convolution is a central tool in this theory.

3 Fourier transform

- Our next topic is Fourier transform.
- It gives a method to solve PDEs in the higher dimensional case (and on unbounded intervals).
- Note that we used Fourier series to solve e.g. Laplace equation on plane ($n = 2$), but the Fourier series technique was only used to choose the correct coefficients so that the boundary values on one-dimensional boundary were correct.
- When we used Fourier series, the boundary/initial values were always one-dimensional and either periodic (boundary of a unit sphere / initial temperature on ring) or defined on a bounded interval with zero values at endpoints.
- Fourier transform has many properties analogous to the Fourier series.
- When using Fourier transform, the functions need not be periodic.
- Convolution is a central tool in this theory.

3 Fourier transform

- Our next topic is Fourier transform.
- It gives a method to solve PDEs in the higher dimensional case (and on unbounded intervals).
- Note that we used Fourier series to solve e.g. Laplace equation on plane ($n = 2$), but the Fourier series technique was only used to choose the correct coefficients so that the boundary values on one-dimensional boundary were correct.
- When we used Fourier series, the boundary/initial values were always one-dimensional and either periodic (boundary of a unit sphere / initial temperature on ring) or defined on a bounded interval with zero values at endpoints.
- Fourier transform has many properties analogous to the Fourier series.
- When using Fourier transform, the functions need not be periodic.
- Convolution is a central tool in this theory.

3 Fourier transform

- Our next topic is Fourier transform.
- It gives a method to solve PDEs in the higher dimensional case (and on unbounded intervals).
- Note that we used Fourier series to solve e.g. Laplace equation on plane ($n = 2$), but the Fourier series technique was only used to choose the correct coefficients so that the boundary values on one-dimensional boundary were correct.
- When we used Fourier series, the boundary/initial values were always one-dimensional and either periodic (boundary of a unit sphere / initial temperature on ring) or defined on a bounded interval with zero values at endpoints.
- Fourier transform has many properties analogous to the Fourier series.
- When using Fourier transform, the functions need not be periodic.
- Convolution is a central tool in this theory.

3.1 The L^p-space on $\mathbb{R}^n$

The space $L^p(\mathbb{R}^n)$ consists of functions $f : \mathbb{R}^n \to \mathbb{C}$ such that

$$\|f\|_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p} < \infty.$$

For $p = \infty$, we set

$$\|f\|_{L^p(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n} |f(x)|.$$

It can be shown that $L^p(\mathbb{R}^n)$ is a complete normed space with the norm defined above.

- Whether $f \in L^1(\mathbb{R}^n)$ depends only on $|f|$.
- If $f \in L^1(\mathbb{R}^n)$, then the integral of f is well-defined and finite.

3.1 The L^p-space on $\mathbb{R}^n$

The space $L^p(\mathbb{R}^n)$ consists of functions $f : \mathbb{R}^n \to \mathbb{C}$ such that

$$\|f\|_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p} < \infty.$$

For $p = \infty$, we set

$$\|f\|_{L^p(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n} |f(x)|.$$

It can be shown that $L^p(\mathbb{R}^n)$ is a complete normed space with the norm defined above.

- Whether $f \in L^1(\mathbb{R}^n)$ depends only on $|f|$.
- If $f \in L^1(\mathbb{R}^n)$, then the integral of f is well-defined and finite.

3.1 The L^p-space on $\mathbb{R}^n$

The space $L^p(\mathbb{R}^n)$ consists of functions $f : \mathbb{R}^n \to \mathbb{C}$ such that

$$\|f\|_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p} < \infty.$$

For $p = \infty$, we set

$$\|f\|_{L^p(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n} |f(x)|.$$

It can be shown that $L^p(\mathbb{R}^n)$ is a complete normed space with the norm defined above.

- Whether $f \in L^1(\mathbb{R}^n)$ depends only on $|f|$.
- If $f \in L^1(\mathbb{R}^n)$, then the integral of f is well-defined and finite.

3.1 The L^p-space on $\mathbb{R}^n$

The space $L^p(\mathbb{R}^n)$ consists of functions $f : \mathbb{R}^n \to \mathbb{C}$ such that

$$\|f\|_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p} < \infty.$$

For $p = \infty$, we set

$$\|f\|_{L^p(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n} |f(x)|.$$

It can be shown that $L^p(\mathbb{R}^n)$ is a complete normed space with the norm defined above.

- Whether $f \in L^1(\mathbb{R}^n)$ depends only on $|f|$.
- If $f \in L^1(\mathbb{R}^n)$, then the integral of f is well-defined and finite.

3.1 The L^p-space on $\mathbb{R}^n$

The space $L^p(\mathbb{R}^n)$ consists of functions $f : \mathbb{R}^n \to \mathbb{C}$ such that

$$\|f\|_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p} < \infty.$$

For $p = \infty$, we set

$$\|f\|_{L^p(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n} |f(x)|.$$

It can be shown that $L^p(\mathbb{R}^n)$ is a complete normed space with the norm defined above.

- Whether $f \in L^1(\mathbb{R}^n)$ depends only on $|f|$.
- If $f \in L^1(\mathbb{R}^n)$, then the integral of f is well-defined and finite.

3.2 Fourier transform

Let $f : \mathbb{R}^n \to \mathbb{C}$ be a complex valued integrable function, that is $f \in L^1(\mathbb{R}^n)$. The Fourier transform $\widehat{f} : \mathbb{R}^n \to \mathbb{C}$ of f is defined as

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\, dx,$$

where $x \cdot \xi = x_1\xi_1 + \cdots + x_n\xi_n$ is the standard inner product (dot product) in $\mathbb{R}^n$.

Compare the Fourier series and Fourier transform:

- Function: S: $f : \mathbb{R} \to \mathbb{C}$. T: $f : \mathbb{R}^n \to \mathbb{C}$
- Coefficients: S: $\hat{f}(k)$, $k \in \mathbb{Z} \subset \mathbb{R}$. T: $\hat{f}(\xi)$, $\xi \in \mathbb{R}^n$.
- The formulas for calculating Fourier series coefficients and Fourier transform are essentially the same.

3.2 Fourier transform

Let $f : \mathbb{R}^n \to \mathbb{C}$ be a complex valued integrable function, that is $f \in L^1(\mathbb{R}^n)$. The Fourier transform $\widehat{f} : \mathbb{R}^n \to \mathbb{C}$ of f is defined as

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\, dx,$$

where $x \cdot \xi = x_1\xi_1 + \cdots + x_n\xi_n$ is the standard inner product (dot product) in $\mathbb{R}^n$.

Compare the Fourier series and Fourier transform:

- Function: S: $f : \mathbb{R} \to \mathbb{C}$. T: $f : \mathbb{R}^n \to \mathbb{C}$
- Coefficients: S: $\hat{f}(k)$, $k \in \mathbb{Z} \subset \mathbb{R}$. T: $\hat{f}(\xi)$, $\xi \in \mathbb{R}^n$.
- The formulas for calculating Fourier series coefficients and Fourier transform are essentially the same.

3.2 Fourier transform

Let $f : \mathbb{R}^n \to \mathbb{C}$ be a complex valued integrable function, that is $f \in L^1(\mathbb{R}^n)$. The Fourier transform $\widehat{f} : \mathbb{R}^n \to \mathbb{C}$ of f is defined as

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\, dx,$$

where $x \cdot \xi = x_1\xi_1 + \cdots + x_n\xi_n$ is the standard inner product (dot product) in $\mathbb{R}^n$.

Compare the Fourier series and Fourier transform:

- Function: S: $f : \mathbb{R} \to \mathbb{C}$. T: $f : \mathbb{R}^n \to \mathbb{C}$
- Coefficients: S: $\hat{f}(k)$, $k \in \mathbb{Z} \subset \mathbb{R}$. T: $\hat{f}(\xi)$, $\xi \in \mathbb{R}^n$.
- The formulas for calculating Fourier series coefficients and Fourier transform are essentially the same.

3.2 Fourier transform

Let $f : \mathbb{R}^n \to \mathbb{C}$ be a complex valued integrable function, that is $f \in L^1(\mathbb{R}^n)$. The Fourier transform $\widehat{f} : \mathbb{R}^n \to \mathbb{C}$ of f is defined as

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\, dx,$$

where $x \cdot \xi = x_1\xi_1 + \cdots + x_n\xi_n$ is the standard inner product (dot product) in $\mathbb{R}^n$.

Compare the Fourier series and Fourier transform:

- Function: S: $f : \mathbb{R} \to \mathbb{C}$. T: $f : \mathbb{R}^n \to \mathbb{C}$
- Coefficients: S: $\hat{f}(k)$, $k \in \mathbb{Z} \subset \mathbb{R}$. T: $\hat{f}(\xi)$, $\xi \in \mathbb{R}^n$.
- The formulas for calculating Fourier series coefficients and Fourier transform are essentially the same.

3.2 Fourier transform

Let $f : \mathbb{R}^n \to \mathbb{C}$ be a complex valued integrable function, that is $f \in L^1(\mathbb{R}^n)$. The Fourier transform $\widehat{f} : \mathbb{R}^n \to \mathbb{C}$ of f is defined as

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\, dx,$$

where $x \cdot \xi = x_1\xi_1 + \cdots + x_n\xi_n$ is the standard inner product (dot product) in $\mathbb{R}^n$.

Compare the Fourier series and Fourier transform:

- Function: S: $f : \mathbb{R} \to \mathbb{C}$. T: $f : \mathbb{R}^n \to \mathbb{C}$
- Coefficients: S: $\hat{f}(k)$, $k \in \mathbb{Z} \subset \mathbb{R}$. T: $\hat{f}(\xi)$, $\xi \in \mathbb{R}^n$.
- The formulas for calculating Fourier series coefficients and Fourier transform are essentially the same.

3.2 Several definitions

There are several alternative definitions for the Fourier transform in the literature, for example,

$$\int_{\mathbb{R}^n} f(x)e^{ix\cdot\xi}\,dx, \quad \int_{\mathbb{R}^n} f(x)e^{-2\pi ix\cdot\xi}\,dx$$

$$\text{and} \quad (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}\,dx.$$

There is an analogous theory for these definitions.
The factor 2π (and minus sign) appears somewhere in each of these choices.

3.2 Several definitions

There are several alternative definitions for the Fourier transform in the literature, for example,

$$\int_{\mathbb{R}^n} f(x)e^{ix\cdot\xi}\,dx, \quad \int_{\mathbb{R}^n} f(x)e^{-2\pi ix\cdot\xi}\,dx$$

$$\text{and} \quad (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}\,dx.$$

There is an analogous theory for these definitions.
The factor 2π (and minus sign) appears somewhere in each of these choices.

3.2 Properties of the Fourier transform

Let $f, g \in L^1(\mathbb{R}^n)$.

1. (Linearity) $\widehat{af + bg}(\xi) = a\widehat{f}(\xi) + b\widehat{g}(\xi)$, $a, b \in \mathbb{C}$.
2. (Boundedness) For every $\xi \in \mathbb{R}^n$ we have

$$|\widehat{f}(\xi)| \le \int_{\mathbb{R}^n} |f(x)|\,dx = \|f\|_{L^1(\mathbb{R}^n)}.$$

   This implies $\|\widehat{f}\|_{L^\infty(\mathbb{R}^n)} \le \|f\|_{L^1(\mathbb{R}^n)}$.
3. $\widehat{f}(0) = \int_{\mathbb{R}^n} f(x)\,dx$.
4. (Continuity) If $f \in L^1(\mathbb{R}^n)$, then $\widehat{f}$ is continuous.
5. (Dilation) Let $f_a(x) = \frac{1}{a^n} f\left(\frac{x}{a}\right)$, $a > 0$. Then $\widehat{f_a}(\xi) = \widehat{f}(a\xi)$.
6. (Translation) For $y \in \mathbb{R}^n$ we have $f\widehat{(x + y)}(\xi) = e^{iy\cdot\xi}\widehat{f}(\xi)$.
7. (Modulation) For $\eta \in \mathbb{R}^n$ we have $\widehat{e^{ix\cdot\eta} f(x)}(\xi) = \widehat{f}(\xi - \eta)$.

3.2 Properties of the Fourier transform

Let $f, g \in L^1(\mathbb{R}^n)$.

1. (Linearity) $\widehat{af+bg}(\xi) = a\widehat{f}(\xi) + b\widehat{g}(\xi)$, $a, b \in \mathbb{C}$.
2. (Boundedness) For every $\xi \in \mathbb{R}^n$ we have

$$|\widehat{f}(\xi)| \le \int_{\mathbb{R}^n} |f(x)|\,dx = \|f\|_{L^1(\mathbb{R}^n)}.$$

   This implies $\|\widehat{f}\|_{L^\infty(\mathbb{R}^n)} \le \|f\|_{L^1(\mathbb{R}^n)}$.

3. $\widehat{f}(0) = \displaystyle\int_{\mathbb{R}^n} f(x)\,dx$.
4. (Continuity) If $f \in L^1(\mathbb{R}^n)$, then $\widehat{f}$ is continuous.
5. (Dilation) Let $f_a(x) = \frac{1}{a^n} f\left(\frac{x}{a}\right)$, $a > 0$. Then $\widehat{f_a}(\xi) = \widehat{f}(a\xi)$.
6. (Translation) For $y \in \mathbb{R}^n$ we have $\widehat{f(x+y)}(\xi) = e^{iy\cdot\xi}\widehat{f}(\xi)$.
7. (Modulation) For $\eta \in \mathbb{R}^n$ we have $\widehat{e^{ix\cdot\eta}f(x)}(\xi) = \widehat{f}(\xi-\eta)$.

3.2 Properties of the Fourier transform

Let $f, g \in L^1(\mathbb{R}^n)$.

1. (Linearity) $\widehat{af + bg}(\xi) = a\widehat{f}(\xi) + b\widehat{g}(\xi)$, $a, b \in \mathbb{C}$.
2. (Boundedness) For every $\xi \in \mathbb{R}^n$ we have

$$|\widehat{f}(\xi)| \leq \int_{\mathbb{R}^n} |f(x)|\, dx = \|f\|_{L^1(\mathbb{R}^n)}.$$

   This implies $\|\widehat{f}\|_{L^\infty(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)}$.

3. $\widehat{f}(0) = \displaystyle\int_{\mathbb{R}^n} f(x)\, dx$.
4. (Continuity) If $f \in L^1(\mathbb{R}^n)$, then $\widehat{f}$ is continuous.
5. (Dilation) Let $f_a(x) = \frac{1}{a^n} f\left(\frac{x}{a}\right)$, $a > 0$. Then $\widehat{f_a}(\xi) = \widehat{f}(a\xi)$.
6. (Translation) For $y \in \mathbb{R}^n$ we have $\widehat{f(x+y)}(\xi) = e^{iy\cdot\xi}\widehat{f}(\xi)$.
7. (Modulation) For $\eta \in \mathbb{R}^n$ we have $\widehat{e^{ix\cdot\eta} f(x)}(\xi) = \widehat{f}(\xi - \eta)$.

3.2 Properties of the Fourier transform

Let $f, g \in L^1(\mathbb{R}^n)$.

1. (Linearity) $\widehat{af + bg}(\xi) = a\widehat{f}(\xi) + b\widehat{g}(\xi)$, $a, b \in \mathbb{C}$.
2. (Boundedness) For every $\xi \in \mathbb{R}^n$ we have

$$|\widehat{f}(\xi)| \leq \int_{\mathbb{R}^n} |f(x)|\, dx = \|f\|_{L^1(\mathbb{R}^n)}.$$

   This implies $\|\widehat{f}\|_{L^\infty(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)}$.

3. $\widehat{f}(0) = \displaystyle\int_{\mathbb{R}^n} f(x)\, dx$.
4. (Continuity) If $f \in L^1(\mathbb{R}^n)$, then $\widehat{f}$ is continuous.
5. (Dilation) Let $f_a(x) = \frac{1}{a^n} f\left(\frac{x}{a}\right)$, $a > 0$. Then $\widehat{f_a}(\xi) = \widehat{f}(a\xi)$.
6. (Translation) For $y \in \mathbb{R}^n$ we have $\widehat{f(x+y)}(\xi) = e^{iy\cdot\xi}\widehat{f}(\xi)$.
7. (Modulation) For $\eta \in \mathbb{R}^n$ we have $\widehat{e^{ix\cdot\eta} f(x)}(\xi) = \widehat{f}(\xi - \eta)$.

3.2 Properties of the Fourier transform

Let $f, g \in L^1(\mathbb{R}^n)$.

1. (Linearity) $\widehat{af + bg}(\xi) = a\widehat{f}(\xi) + b\widehat{g}(\xi)$, $a, b \in \mathbb{C}$.
2. (Boundedness) For every $\xi \in \mathbb{R}^n$ we have

$$|\widehat{f}(\xi)| \leq \int_{\mathbb{R}^n} |f(x)|\,dx = \|f\|_{L^1(\mathbb{R}^n)}.$$

   This implies $\|\widehat{f}\|_{L^\infty(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)}$.

3. $\widehat{f}(0) = \displaystyle\int_{\mathbb{R}^n} f(x)\,dx$.
4. (Continuity) If $f \in L^1(\mathbb{R}^n)$, then $\widehat{f}$ is continuous.
5. (Dilation) Let $f_a(x) = \frac{1}{a^n} f\left(\frac{x}{a}\right)$, $a > 0$. Then $\widehat{f_a}(\xi) = \widehat{f}(a\xi)$.
6. (Translation) For $y \in \mathbb{R}^n$ we have $\widehat{f(x+y)}(\xi) = e^{iy\cdot\xi}\widehat{f}(\xi)$.
7. (Modulation) For $\eta \in \mathbb{R}^n$ we have $\widehat{e^{ix\cdot\eta} f(x)}(\xi) = \widehat{f}(\xi - \eta)$.

3.2 Properties of the Fourier transform

Let $f, g \in L^1(\mathbb{R}^n)$.

1. (Linearity) $\widehat{af + bg}(\xi) = a\widehat{f}(\xi) + b\widehat{g}(\xi)$, $a, b \in \mathbb{C}$.
2. (Boundedness) For every $\xi \in \mathbb{R}^n$ we have

$$|\widehat{f}(\xi)| \leq \int_{\mathbb{R}^n} |f(x)|\,dx = \|f\|_{L^1(\mathbb{R}^n)}.$$

   This implies $\|\widehat{f}\|_{L^\infty(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)}$.

3. $\widehat{f}(0) = \displaystyle\int_{\mathbb{R}^n} f(x)\,dx$.
4. (Continuity) If $f \in L^1(\mathbb{R}^n)$, then $\widehat{f}$ is continuous.
5. (Dilation) Let $f_a(x) = \frac{1}{a^n} f\left(\frac{x}{a}\right)$, $a > 0$. Then $\widehat{f_a}(\xi) = \widehat{f}(a\xi)$.
6. (Translation) For $y \in \mathbb{R}^n$ we have $f\widehat{(x + y)}(\xi) = e^{iy\cdot\xi}\widehat{f}(\xi)$.
7. (Modulation) For $\eta \in \mathbb{R}^n$ we have $\widehat{e^{ix\cdot\eta}f(x)}(\xi) = \widehat{f}(\xi - \eta)$.

3.3 Fourier transform and differentiation

The Fourier transform interacts with derivation very nicely. In many cases, Fourier transform transforms a PDE to an ODE.

Theorem

Let $f \in C^1(\mathbb{R}^n)$ and assume that $\frac{\partial f}{\partial x_j} \in L^1(\mathbb{R}^n)$, $j = 1, 2, \dots, n$. We also assume that $\lim_{|x|\to\infty} f(x) = 0$. Then

$$\widehat{\frac{\partial f}{\partial x_j}}(\xi) = i\xi_j \widehat{f}(\xi), \quad j = 1, 2, \dots, n.$$

- Compare to the formula for Fourier series:

$$\widehat{f'}(j) = ij\widehat{f}(j), \quad j \in \mathbb{Z}.$$

- When we differentiate the function, the the transform gets multiplied by ξ_j and thus $\hat{f}(\xi)$ decays slower as $|\xi| \to \infty$.
- The differentiation formula works e.g. if $f \in C_0^\infty(\mathbb{R}^n)$.

3.3 Fourier transform and differentiation

The Fourier transform interacts with derivation very nicely. In many cases, Fourier transform transforms a PDE to an ODE.

Theorem

Let $f \in C^1(\mathbb{R}^n)$ and assume that $\frac{\partial f}{\partial x_j} \in L^1(\mathbb{R}^n)$, $j = 1, 2, \ldots, n$. We also assume that $\lim_{|x|\to\infty} f(x) = 0$. Then

$$\widehat{\frac{\partial f}{\partial x_j}}(\xi) = i\xi_j \widehat{f}(\xi), \quad j = 1, 2, \ldots, n.$$

- Compare to the formula for Fourier series:

$$\widehat{f'}(j) = ij\widehat{f}(j), \quad j \in \mathbb{Z}.$$

- When we differentiate the function, the the transform gets multiplied by ξ_j and thus $\hat{f}(\xi)$ decays slower as $|\xi| \to \infty$.
- The differentiation formula works e.g. if $f \in C_0^\infty(\mathbb{R}^n)$.

3.3 Fourier transform and differentiation

The Fourier transform interacts with derivation very nicely. In many cases, Fourier transform transforms a PDE to an ODE.

Theorem

Let $f \in C^1(\mathbb{R}^n)$ and assume that $\frac{\partial f}{\partial x_j} \in L^1(\mathbb{R}^n)$, $j = 1, 2, \ldots, n$. We also assume that $\lim_{|x|\to\infty} f(x) = 0$. Then

$$\widehat{\frac{\partial f}{\partial x_j}}(\xi) = i\xi_j \widehat{f}(\xi), \quad j = 1, 2, \ldots, n.$$

- Compare to the formula for Fourier series:

$$\widehat{f'}(j) = ij\widehat{f}(j), \quad j \in \mathbb{Z}.$$

- When we differentiate the function, the the transform gets multiplied by ξ_j and thus $\hat{f}(\xi)$ decays slower as $|\xi| \to \infty$.
- The differentiation formula works e.g. if $f \in C_0^\infty(\mathbb{R}^n)$.

3.3 Fourier transform and differentiation

The Fourier transform interacts with derivation very nicely. In many cases, Fourier transform transforms a PDE to an ODE.

Theorem

Let $f \in C^1(\mathbb{R}^n)$ *and assume that* $\frac{\partial f}{\partial x_j} \in L^1(\mathbb{R}^n)$, $j = 1, 2, \ldots, n$. *We also assume that* $\lim_{|x|\to\infty} f(x) = 0$. *Then*

$$\widehat{\frac{\partial f}{\partial x_j}}(\xi) = i\xi_j \widehat{f}(\xi), \quad j = 1, 2, \ldots, n.$$

- Compare to the formula for Fourier series:
$$\widehat{f'}(j) = ij\widehat{f}(j), \quad j \in \mathbb{Z}.$$
- When we differentiate the function, the the transform gets multiplied by ξ_j and thus $\hat{f}(\xi)$ decays slower as $|\xi| \to \infty$.
- The differentiation formula works e.g. if $f \in C_0^\infty(\mathbb{R}^n)$.

3.3 Proof of the differentiation formula

$$
\begin{aligned}
\widehat{\frac{\partial f}{\partial x_j}}(\xi) &= \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_j}(x) e^{-ix\cdot\xi}\,dx \qquad \text{by definition of the Fourier transf.} \\
&= \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \frac{\partial f}{\partial x_j}(x) e^{-ix\cdot\xi}\,dx_j \right) dx_1 \,\dots\, dx_{j-1}\,dx_{j+1} \,\dots\, dx_n \\
&= \int_{\mathbb{R}^{n-1}} \left(\lim_{a\to\infty} \Big|_{x_j=-a}^{a} f(x) e^{-ix\cdot\xi} \right) dx_1 \,\dots\, dx_{j-1}\,dx_{j+1} \,\dots\, dx_n \\
&\quad - \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} f(x) \frac{\partial}{\partial x_j}(e^{-ix\cdot\xi})\,dx_j \right) dx_1 \,\dots\, dx_{j-1}\,dx_{j+1} \,\dots\, dx_n \\
&= -\int_{\mathbb{R}^n} f(x)(-i\xi_j) e^{-ix\cdot\xi}\,dx = i\xi_j \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\,dx = i\xi_j \widehat{f}(\xi).
\end{aligned}
$$

3.3 Proof of the differentiation formula

$$\widehat{\frac{\partial f}{\partial x_j}}(\xi) = \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_j}(x) e^{-ix\cdot\xi}\, dx \quad \text{by definition of the Fourier transf.}$$

$$= \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \frac{\partial f}{\partial x_j}(x) e^{-ix\cdot\xi}\, dx_j \right) dx_1 \ldots dx_{j-1}\, dx_{j+1} \ldots dx_n$$

$$= \int_{\mathbb{R}^{n-1}} \left(\lim_{a\to\infty} \Big|_{x_j=-a}^{a} f(x) e^{-ix\cdot\xi} \right) dx_1 \ldots dx_{j-1}\, dx_{j+1} \ldots dx_n$$

$$- \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} f(x) \frac{\partial}{\partial x_j}(e^{-ix\cdot\xi})\, dx_j \right) dx_1 \ldots dx_{j-1}\, dx_{j+1} \ldots dx_n$$

$$= -\int_{\mathbb{R}^n} f(x)(-i\xi_j) e^{-ix\cdot\xi}\, dx = i\xi_j \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\, dx = i\xi_j \widehat{f}(\xi).$$

3.3 Proof of the differentiation formula

$$
\begin{aligned}
\widehat{\frac{\partial f}{\partial x_j}}(\xi) &= \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_j}(x) e^{-ix\cdot\xi}\, dx \qquad \text{by definition of the Fourier transf.} \\
&= \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \frac{\partial f}{\partial x_j}(x) e^{-ix\cdot\xi}\, dx_j \right) dx_1 \,\dots\, dx_{j-1}\, dx_{j+1} \,\dots\, dx_n \\
&= \int_{\mathbb{R}^{n-1}} \left(\lim_{a\to\infty} \Big|_{x_j=-a}^{a} f(x) e^{-ix\cdot\xi} \right) dx_1 \,\dots\, dx_{j-1}\, dx_{j+1} \,\dots\, dx_n \\
&\quad - \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} f(x) \frac{\partial}{\partial x_j}(e^{-ix\cdot\xi})\, dx_j \right) dx_1 \,\dots\, dx_{j-1}\, dx_{j+1} \,\dots\, dx_n \\
&= -\int_{\mathbb{R}^n} f(x)(-i\xi_j) e^{-ix\cdot\xi}\, dx = i\xi_j \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\, dx = i\xi_j \widehat{f}(\xi).
\end{aligned}
$$

3.3 Proof of the differentiation formula

$$
\begin{aligned}
\widehat{\frac{\partial f}{\partial x_j}}(\xi) &= \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_j}(x) e^{-ix\cdot\xi}\, dx \qquad \text{by definition of the Fourier transf.} \\
&= \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \frac{\partial f}{\partial x_j}(x) e^{-ix\cdot\xi}\, dx_j \right) dx_1 \ldots dx_{j-1}\, dx_{j+1} \ldots dx_n \\
&= \int_{\mathbb{R}^{n-1}} \left(\lim_{a\to\infty} \Big|_{x_j=-a}^{a} f(x) e^{-ix\cdot\xi} \right) dx_1 \ldots dx_{j-1}\, dx_{j+1} \ldots dx_n \\
&\quad - \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} f(x) \frac{\partial}{\partial x_j}(e^{-ix\cdot\xi})\, dx_j \right) dx_1 \ldots dx_{j-1}\, dx_{j+1} \ldots dx_n \\
&= -\int_{\mathbb{R}^n} f(x)(-i\xi_j) e^{-ix\cdot\xi}\, dx = i\xi_j \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\, dx = i\xi_j \widehat{f}(\xi).
\end{aligned}
$$

3.3 Proof of the differentiation formula

$$
\begin{aligned}
\widehat{\frac{\partial f}{\partial x_j}}(\xi) &= \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_j}(x) e^{-ix\cdot\xi}\,dx \quad \text{by definition of the Fourier transf.} \\
&= \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \frac{\partial f}{\partial x_j}(x) e^{-ix\cdot\xi}\,dx_j \right) dx_1 \dots dx_{j-1}\,dx_{j+1} \dots dx_n \\
&= \int_{\mathbb{R}^{n-1}} \left(\lim_{a\to\infty} \Big|_{x_j=-a}^{a} f(x) e^{-ix\cdot\xi} \right) dx_1 \dots dx_{j-1}\,dx_{j+1} \dots dx_n \\
&\quad - \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} f(x) \frac{\partial}{\partial x_j}(e^{-ix\cdot\xi})\,dx_j \right) dx_1 \dots dx_{j-1}\,dx_{j+1} \dots dx_n \\
&= -\int_{\mathbb{R}^n} f(x)(-i\xi_j) e^{-ix\cdot\xi}\,dx = i\xi_j \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\,dx = i\xi_j \widehat{f}(\xi).
\end{aligned}
$$

3.3 Why Fourier transform is useful for PDEs? (1/3)

- A vector of the form $\alpha = (\alpha_1, \dots, \alpha_n)$, where each component α_j is a nonnegative integer, is called a multi-index of order $|\alpha| = \alpha_1 + \cdots + \alpha_n$.
- For a multi-index α, we define

$$D^\alpha u(x) = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}(x)$$

- Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ and denote $\lambda^\alpha = \lambda_1^{\alpha_1} \dots \lambda_n^{\alpha_n}$.
- The function $e_\lambda : \mathbb{R}^n \to \mathbb{C}$,

$$e_\lambda(x) = e^{x \cdot \lambda} = e^{x_1 \lambda_1 + \cdots + x_n \lambda_n},$$

belongs to $C^\infty(\mathbb{R}^n)$ and

$$D^\alpha e_\lambda = \lambda^\alpha e_\lambda.$$

3.3 Why Fourier transform is useful for PDEs? (1/3)

- A vector of the form $\alpha = (\alpha_1, \ldots, \alpha_n)$, where each component α_j is a nonnegative integer, is called a multi-index of order $|\alpha| = \alpha_1 + \cdots + \alpha_n$.
- For a multi-index α, we define

$$D^\alpha u(x) = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha^n}}(x)$$

- Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ and denote $\lambda^\alpha = \lambda_1^{\alpha_1} \ldots \lambda_n^{\alpha_n}$.
- The function $e_\lambda : \mathbb{R}^n \to \mathbb{C}$,

$$e_\lambda(x) = e^{x\cdot\lambda} = e^{x_1\lambda_1 + \cdots + x_n\lambda_n},$$

belongs to $C^\infty(\mathbb{R}^n)$ and

$$D^\alpha e_\lambda = \lambda^\alpha e_\lambda.$$

3.3 Why Fourier transform is useful for PDEs? (1/3)

- A vector of the form $\alpha = (\alpha_1, \dots, \alpha_n)$, where each component α_j is a nonnegative integer, is called a multi-index of order $|\alpha| = \alpha_1 + \cdots + \alpha_n$.
- For a multi-index α, we define

$$D^\alpha u(x) = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha^n}}(x)$$

- Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ and denote $\lambda^\alpha = \lambda_1^{\alpha_1} \dots \lambda_n^{\alpha_n}$.
- The function $e_\lambda : \mathbb{R}^n \to \mathbb{C}$,

$$e_\lambda(x) = e^{x \cdot \lambda} = e^{x_1\lambda_1 + \cdots + x_n\lambda_n},$$

belongs to $C^\infty(\mathbb{R}^n)$ and

$$D^\alpha e_\lambda = \lambda^\alpha e_\lambda.$$

3.3 Why Fourier transform is useful for PDEs? (1/3)

- A vector of the form $\alpha = (\alpha_1, \ldots, \alpha_n)$, where each component α_j is a nonnegative integer, is called a multi-index of order $|\alpha| = \alpha_1 + \cdots + \alpha_n$.
- For a multi-index α, we define

$$D^\alpha u(x) = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha^n}}(x)$$

- Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ and denote $\lambda^\alpha = \lambda_1^{\alpha_1} \ldots \lambda_n^{\alpha_n}$.
- The function $e_\lambda : \mathbb{R}^n \to \mathbb{C}$,

$$e_\lambda(x) = e^{x \cdot \lambda} = e^{x_1\lambda_1 + \cdots + x_n\lambda_n},$$

  belongs to $C^\infty(\mathbb{R}^n)$ and

$$D^\alpha e_\lambda = \lambda^\alpha e_\lambda.$$

3.3 Why Fourier transform is useful for PDEs? (2/3)

- Thus, for every linear PDE operator with constant coefficients

$$P = P(D) = \sum_{|\alpha| \le k} a_\alpha D^\alpha : C_0^\infty(\mathbb{R}^n) \to C_0^\infty(\mathbb{R}^n)$$

we have

$$P(D)e_\lambda = \sum_{|\alpha| \le k} a_\alpha \lambda^\alpha e_\lambda.$$

- In other words, e_λ is an eigenvector corresponding the eigenvalue

$$\sum_{|\alpha| \le k} a_\alpha \lambda^\alpha$$

for every operator $P(D)$.

3.3 Why Fourier transform is useful for PDEs? (2/3)

- Thus, for every linear PDE operator with constant coefficients

$$P = P(D) = \sum_{|\alpha| \le k} a_\alpha D^\alpha : C_0^\infty(\mathbb{R}^n) \to C_0^\infty(\mathbb{R}^n)$$

  we have

$$P(D)e_\lambda = \sum_{|\alpha| \le k} a_\alpha \lambda^\alpha e_\lambda.$$

- In other words, e_λ is an eigenvector corresponding the eigenvalue

$$\sum_{|\alpha| \le k} a_\alpha \lambda^\alpha$$

  for every operator $P(D)$.

3.3 Why Fourier transform is useful for PDEs? (3/3)

- Note that the PDE related to the operator P is

$$P(D)u = \sum_{|\alpha| \leq k} a_\alpha D^\alpha u = 0.$$

- The idea behind the Fourier transform is that the partial differential operator P can be better understood if the functions on which they act are represented as linear combinations of the eigenvectors

$$e_\lambda(x) = e^{x \cdot \lambda} = e^{x_1\lambda_1 + \cdots + x_n\lambda_n}.$$

3.3 Why Fourier transform is useful for PDEs? (3/3)

- Note that the PDE related to the operator P is

$$P(D)u = \sum_{|\alpha| \le k} a_\alpha D^\alpha u = 0.$$

- The idea behind the Fourier transform is that the partial differential operator P can be better understood if the functions on which they act are represented as linear combinations of the eigenvectors

$$e_\lambda(x) = e^{x \cdot \lambda} = e^{x_1 \lambda_1 + \cdots + x_n \lambda_n}.$$

3.3 Fourier transform and differentiation

- We have shown that

$$\widehat{\frac{\partial f}{\partial x_j}}(\xi) = i\xi_j \widehat{f}(\xi), \quad j = 1, 2, \ldots, n.$$

but what if we want to differentiate $\widehat{f}$?

Theorem

Suppose that $f \in L^1(\mathbb{R}^n)$ and $-ix_j f(x) \in L^1(\mathbb{R}^n)$. Then $\widehat{f}$ is differentiable and

$$\frac{\partial \widehat{f}}{\partial \xi_j}(\xi) = -i\widehat{x_j f(x)}(\xi), \quad j = 1, 2, \ldots, n.$$

3.3 Fourier transform and differentiation

- We have shown that

$$\widehat{\frac{\partial f}{\partial x_j}}(\xi) = i\xi_j \widehat{f}(\xi), \quad j = 1, 2, \ldots, n.$$

but what if we want to differentiate $\widehat{f}$?

Theorem

Suppose that $f \in L^1(\mathbb{R}^n)$ and $-ix_j f(x) \in L^1(\mathbb{R}^n)$. Then $\widehat{f}$ is differentiable and

$$\frac{\partial \widehat{f}}{\partial \xi_j}(\xi) = -i\widehat{x_j f(x)}(\xi), \quad j = 1, 2, \ldots, n.$$

3.4 Fourier transform of the Gaussian

- Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{-x^2}$. Then

$$f'(x) = -2xe^{-x^2} = -2xf(x) \in L^1(\mathbb{R})$$

- Now

$$\frac{\partial \widehat{f}}{\partial \xi}(\xi) = -\widehat{ixf(x)}(\xi) = -\frac{1}{2i}\widehat{f'}(\xi) = -\frac{1}{2i}i\xi\widehat{f}(\xi) = -\frac{\xi}{2}\widehat{f}(\xi).$$

- Thus $\widehat{f}$ satisfies the ODE

$$\frac{\partial \widehat{f}}{\partial \xi}(\xi) + \frac{\xi}{2}\widehat{f}(\xi) = 0 \iff \frac{\partial}{\partial \xi}\left(\widehat{f}(\xi)e^{\xi^2/4}\right) = 0 \iff \widehat{f}(\xi) = ce^{-\xi^2/4}$$

for some constant c. (And the correct c is $\pi^{n/2}$, read the details from the lecture notes)

3.4 Fourier transform of the Gaussian

- Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{-x^2}$. Then

$$f'(x) = -2xe^{-x^2} = -2xf(x) \in L^1(\mathbb{R})$$

- Now

$$\frac{\partial \widehat{f}}{\partial \xi}(\xi) = -\widehat{ixf(x)}(\xi) = -\frac{1}{2i}\widehat{f'}(\xi) = -\frac{1}{2i}i\xi\widehat{f}(\xi) = -\frac{\xi}{2}\widehat{f}(\xi).$$

- Thus $\widehat{f}$ satisfies the ODE

$$\frac{\partial \widehat{f}}{\partial \xi}(\xi)+\frac{\xi}{2}\widehat{f}(\xi) = 0 \iff \frac{\partial}{\partial \xi}\left(\widehat{f}(\xi)e^{\xi^2/4}\right) = 0 \iff \widehat{f}(\xi) = ce^{-\xi^2/4}$$

for some constant c. (And the correct c is $\pi^{n/2}$, read the details from the lecture notes)

3.4 Fourier transform of the Gaussian

- Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{-x^2}$. Then

$$f'(x) = -2xe^{-x^2} = -2xf(x) \in L^1(\mathbb{R})$$

- Now

$$\frac{\partial \widehat{f}}{\partial \xi}(\xi) = -\widehat{ixf(x)}(\xi) = -\frac{1}{2i}\widehat{f'}(\xi) = -\frac{1}{2i}i\xi\widehat{f}(\xi) = -\frac{\xi}{2}\widehat{f}(\xi).$$

- Thus $\widehat{f}$ satisfies the ODE

$$\frac{\partial \widehat{f}}{\partial \xi}(\xi) + \frac{\xi}{2}\widehat{f}(\xi) = 0 \iff \frac{\partial}{\partial \xi}\left(\widehat{f}(\xi)e^{\xi^2/4}\right) = 0 \iff \widehat{f}(\xi) = ce^{-\xi^2/4}$$

for some constant c. (And the correct c is $\pi^{n/2}$, read the details from the lecture notes)

3.5 Fourier inversion formula

- If we are given $\hat{f}$, can we determine f?
- The Fourier inversion theorem will state that under certain assumptions, we have

$$f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{ix\cdot\xi}\, d\xi.$$

- Compare this to how we get f from the Fourier series coefficients! (sum $\Rightarrow$ integral!)
- 2π appears now in a bit different place, but it is just a choice. It must appear somewhere.

3.5 Fourier inversion formula

- If we are given $\hat{f}$, can we determine f?
- The Fourier inversion theorem will state that under certain assumptions, we have

$$f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{ix\cdot\xi}\, d\xi.$$

- Compare this to how we get f from the Fourier series coefficients! (sum $\Rightarrow$ integral!)
- 2π appears now in a bit different place, but it is just a choice. It must appear somewhere.

3.5 Fourier inversion formula

- If we are given $\hat{f}$, can we determine f?
- The Fourier inversion theorem will state that under certain assumptions, we have

$$f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{ix\cdot\xi}\, d\xi.$$

- Compare this to how we get f from the Fourier series coefficients! (sum $\Rightarrow$ integral!)
- 2π appears now in a bit different place, but it is just a choice. It must appear somewhere.

3.5 Fourier inversion formula

- If we are given $\hat{f}$, can we determine f?
- The Fourier inversion theorem will state that under certain assumptions, we have

$$f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{ix\cdot\xi}\, d\xi.$$

- Compare this to how we get f from the Fourier series coefficients! (sum $\Rightarrow$ integral!)
- 2π appears now in a bit different place, but it is just a choice. It must appear somewhere.

3.5 Easy proof for inversion formula?

This is a deep result and it is instructive to see what happens if we try to prove directly by substituting the formula for $\widehat{f}(\xi)$ into the integral above. If we do this, we have

$$
\begin{aligned}
\int_{\mathbb{R}^n} \widehat{f}(\xi) e^{ix\cdot\xi}\, d\xi &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(y) e^{-iy\cdot\xi}\, dy \right) e^{ix\cdot\xi}\, d\xi \\
&= \int_{\mathbb{R}^n} f(y) \underbrace{\left(\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\, d\xi \right)}_{=?} dy.
\end{aligned}
$$

This does not work out, because the inner integral does not exist, that is, the function is not integrable. Thus we have to choose another approach.

3.5 Main steps in the correct proof:

- Step 1: Study the functions $K(x) = \pi^{-n/2} e^{-|x|^2}$ and $K_a(x) = a^{-n} K(ax)$. We know already what is $\hat{K}$ and $\hat{K}_a$!
- Step 1b: $\int_{\mathbb{R}^n} f(x) K_a(x)\, dx \to f(0)$ as $a \to 0$.
- Step 2: $\int_{\mathbb{R}^n} \widehat{f}(\xi) g(\xi)\, d\xi = \int_{\mathbb{R}^n} f(\xi) \widehat{g}(\xi)\, d\xi$
- Step 3: $f(0) = (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{f}(\xi)\, d\xi$ is the claim at $x = 0$
- Step 4: The claim for the other points follows from the translation formula i.e. if $F(y) = f(x+y)$, then

$$f(x) = F(0) = (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{F}(\xi)\, d\xi = (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{ix\cdot\xi}\, d\xi$$

3.5 Main steps in the correct proof:

- Step 1: Study the functions $K(x) = \pi^{-n/2} e^{-|x|^2}$ and $K_a(x) = a^{-n} K(ax)$. We know already what is $\hat{K}$ and $\hat{K}_a$!
- Step 1b: $\int_{\mathbb{R}^n} f(x) K_a(x)\, dx \to f(0)$ as $a \to 0$.
- Step 2: $\int_{\mathbb{R}^n} \widehat{f}(\xi) g(\xi)\, d\xi = \int_{\mathbb{R}^n} f(\xi) \widehat{g}(\xi)\, d\xi$
- Step 3: $f(0) = (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{f}(\xi)\, d\xi$ is the claim at $x = 0$
- Step 4: The claim for the other points follows from the translation formula i.e. if $F(y) = f(x+y)$, then

$$f(x) = F(0) = (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{F}(\xi)\, d\xi = (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{ix\cdot\xi}\, d\xi$$

3.5 Main steps in the correct proof:

- Step 1: Study the functions $K(x) = \pi^{-n/2}e^{-|x|^2}$ and $K_a(x) = a^{-n}K(ax)$. We know already what is $\hat{K}$ and $\hat{K}_a$!
- Step 1b: $\int_{\mathbb{R}^n} f(x)K_a(x)\,dx \to f(0)$ as $a \to 0$.
- Step 2: $\int_{\mathbb{R}^n} \widehat{f}(\xi)g(\xi)\,d\xi = \int_{\mathbb{R}^n} f(\xi)\widehat{g}(\xi)\,d\xi$
- Step 3: $f(0) = (2\pi)^{-n}\int_{\mathbb{R}^n} \widehat{f}(\xi)\,d\xi$ is the claim at $x = 0$
- Step 4: The claim for the other points follows from the translation formula i.e. if $F(y) = f(x+y)$, then

$$f(x) = F(0) = (2\pi)^{-n}\int_{\mathbb{R}^n} \widehat{F}(\xi)\,d\xi = (2\pi)^{-n}\int_{\mathbb{R}^n} \widehat{f}(\xi)e^{ix\cdot\xi}\,d\xi$$

3.5 Main steps in the correct proof:

- Step 1: Study the functions $K(x) = \pi^{-n/2}e^{-|x|^2}$ and $K_a(x) = a^{-n}K(ax)$. We know already what is $\hat{K}$ and $\hat{K}_a$!
- Step 1b: $\int_{\mathbb{R}^n} f(x)K_a(x)\,dx \to f(0)$ as $a \to 0$.
- Step 2: $\int_{\mathbb{R}^n} \widehat{f}(\xi)g(\xi)\,d\xi = \int_{\mathbb{R}^n} f(\xi)\widehat{g}(\xi)\,d\xi$
- Step 3: $f(0) = (2\pi)^{-n}\int_{\mathbb{R}^n} \widehat{f}(\xi)\,d\xi$ is the claim at $x = 0$
- Step 4: The claim for the other points follows from the translation formula i.e. if $F(y) = f(x+y)$, then

$$f(x) = F(0) = (2\pi)^{-n}\int_{\mathbb{R}^n} \widehat{F}(\xi)\,d\xi = (2\pi)^{-n}\int_{\mathbb{R}^n} \widehat{f}(\xi)e^{ix\cdot\xi}\,d\xi$$

3.5 Main steps in the correct proof:

- Step 1: Study the functions $K(x) = \pi^{-n/2} e^{-|x|^2}$ and $K_a(x) = a^{-n} K(ax)$. We know already what is $\hat{K}$ and $\hat{K}_a$!
- Step 1b: $\int_{\mathbb{R}^n} f(x) K_a(x)\, dx \to f(0)$ as $a \to 0$.
- Step 2: $\int_{\mathbb{R}^n} \widehat{f}(\xi) g(\xi)\, d\xi = \int_{\mathbb{R}^n} f(\xi) \widehat{g}(\xi)\, d\xi$
- Step 3: $f(0) = (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{f}(\xi)\, d\xi$ is the claim at $x = 0$
- Step 4: The claim for the other points follows from the translation formula i.e. if $F(y) = f(x+y)$, then

$$f(x) = F(0) = (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{F}(\xi)\, d\xi = (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{ix\cdot\xi}\, d\xi$$

3.6 Fourier transform and convolution (1/2)

Let $f, g : \mathbb{R}^n \to \mathbb{C}$. The convolution $f * g : \mathbb{R}^n \to \mathbb{C}$ is defined by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x - y)g(y)\, dy,$$

whenever this integral exists.

- The convolution on $\mathbb{R}^n$ has a similar role in representation formulas for solutions of PDEs as in the one-dimensional case.
- It is commutative $f * g = g * f$ and
- associative $(f * g) * h = f * (g * h)$.
- The integral exists e.g. when $f, g \in L^1(\mathbb{R}^n)$ and

$$\|f * g\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)} < \infty.$$

3.6 Fourier transform and convolution (1/2)

Let $f, g : \mathbb{R}^n \to \mathbb{C}$. The convolution $f * g : \mathbb{R}^n \to \mathbb{C}$ is defined by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy,$$

whenever this integral exists.

- The convolution on $\mathbb{R}^n$ has a similar role in representation formulas for solutions of PDEs as in the one-dimensional case.
- It is commutative $f * g = g * f$ and
- associative $(f * g) * h = f * (g * h)$.
- The integral exists e.g. when $f, g \in L^1(\mathbb{R}^n)$ and

$$\|f * g\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)} < \infty.$$

3.6 Fourier transform and convolution (1/2)

Let $f, g : \mathbb{R}^n \to \mathbb{C}$. The convolution $f * g : \mathbb{R}^n \to \mathbb{C}$ is defined by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x - y)g(y)\, dy,$$

whenever this integral exists.

- The convolution on $\mathbb{R}^n$ has a similar role in representation formulas for solutions of PDEs as in the one-dimensional case.
- It is commutative $f * g = g * f$ and
- associative $(f * g) * h = f * (g * h)$.
- The integral exists e.g. when $f, g \in L^1(\mathbb{R}^n)$ and

$$\|f * g\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)} < \infty.$$

3.6 Fourier transform and convolution (1/2)

Let $f, g : \mathbb{R}^n \to \mathbb{C}$. The convolution $f * g : \mathbb{R}^n \to \mathbb{C}$ is defined by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x - y)g(y)\, dy,$$

whenever this integral exists.

- The convolution on $\mathbb{R}^n$ has a similar role in representation formulas for solutions of PDEs as in the one-dimensional case.
- It is commutative $f * g = g * f$ and
- associative $(f * g) * h = f * (g * h)$.
- The integral exists e.g. when $f, g \in L^1(\mathbb{R}^n)$ and

$$\|f * g\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)} < \infty.$$

3.6 Fourier transform and convolution (1/2)

Let $f, g : \mathbb{R}^n \to \mathbb{C}$. The convolution $f * g : \mathbb{R}^n \to \mathbb{C}$ is defined by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy,$$

whenever this integral exists.

- The convolution on $\mathbb{R}^n$ has a similar role in representation formulas for solutions of PDEs as in the one-dimensional case.
- It is commutative $f * g = g * f$ and
- associative $(f * g) * h = f * (g * h)$.
- The integral exists e.g. when $f, g \in L^1(\mathbb{R}^n)$ and

$$\|f * g\|_{L^1(\mathbb{R}^n)} \le \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)} < \infty.$$

3.6 Fourier transform and convolution (2/2)

- Assume that $f, g \in L^1(\mathbb{R}^n)$. Then

$$\widehat{f * g}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi)$$

for every $\xi \in \mathbb{R}^n$.

- Assume that $f, g, \widehat{f}, \widehat{g} \in L^1(\mathbb{R}^n)$. Then

$$\widehat{fg}(\xi) = (2\pi)^{-n}(\widehat{f} * \widehat{g})(\xi)$$

for every $\xi \in \mathbb{R}^n$.

3.6 Fourier transform and convolution (2/2)

- Assume that $f, g \in L^1(\mathbb{R}^n)$. Then

$$\widehat{f * g}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi)$$

  for every $\xi \in \mathbb{R}^n$.
- Assume that $f, g, \widehat{f}, \widehat{g} \in L^1(\mathbb{R}^n)$. Then

$$\widehat{fg}(\xi) = (2\pi)^{-n}(\widehat{f} * \widehat{g})(\xi)$$

  for every $\xi \in \mathbb{R}^n$.

3.7 Plancherel's formula

$$\int_{\mathbb{R}^n} |f(y)|^2 \, dy = (2\pi)^{-n} \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 \, d\xi$$

- Plancherel's formula tells that L^2 norm of f and $\widehat{f}$ are essentially the same.
- The factor $(2\pi)^{-n}$ appears with our definition for the Fourier transform, but there are different scalings in the literature.
- This can be used to define the Fourier transform of L^2-functions, but this is out of scope of this course.
- Compare to the Fourier series version of Plancherel's theorem.

3.7 Plancherel's formula

$$\int_{\mathbb{R}^n} |f(y)|^2 \, dy = (2\pi)^{-n} \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 \, d\xi$$

- Plancherel's formula tells that L^2 norm of f and $\widehat{f}$ are essentially the same.
- The factor $(2\pi)^{-n}$ appears with our definition for the Fourier transform, but there are different scalings in the literature.
- This can be used to define the Fourier transform of L^2-functions, but this is out of scope of this course.
- Compare to the Fourier series version of Plancherel's theorem.

3.7 Plancherel's formula

$$\int_{\mathbb{R}^n} |f(y)|^2 \, dy = (2\pi)^{-n} \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 \, d\xi$$

- Plancherel's formula tells that L^2 norm of f and $\widehat{f}$ are essentially the same.
- The factor $(2\pi)^{-n}$ appears with our definition for the Fourier transform, but there are different scalings in the literature.
- This can be used to define the Fourier transform of L^2-functions, but this is out of scope of this course.
- Compare to the Fourier series version of Plancherel's theorem.

3.7 Plancherel's formula

$$\int_{\mathbb{R}^n} |f(y)|^2 \, dy = (2\pi)^{-n} \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 \, d\xi$$

- Plancherel's formula tells that L^2 norm of f and $\widehat{f}$ are essentially the same.
- The factor $(2\pi)^{-n}$ appears with our definition for the Fourier transform, but there are different scalings in the literature.
- This can be used to define the Fourier transform of L^2-functions, but this is out of scope of this course.
- Compare to the Fourier series version of Plancherel's theorem.

3.7 Plancherel's formula

$$\int_{\mathbb{R}^n} |f(y)|^2 \, dy = (2\pi)^{-n} \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 \, d\xi$$

- Plancherel's formula tells that L^2 norm of f and $\widehat{f}$ are essentially the same.
- The factor $(2\pi)^{-n}$ appears with our definition for the Fourier transform, but there are different scalings in the literature.
- This can be used to define the Fourier transform of L^2-functions, but this is out of scope of this course.
- Compare to the Fourier series version of Plancherel's theorem.

3.8 Approximations of identity (1/2)

Let $\{K_\varepsilon\}_{\varepsilon>0}$ be a family of functions $K_\varepsilon : \mathbb{R}^n \to \mathbb{C}$, with the following properties.

1. For every $\varepsilon > 0$ we have $\int_{\mathbb{R}^n} K_\varepsilon(x)\, dx = 1$.
2. There exists some constant $M > 0$ such that, for every $\varepsilon > 0$ we have
$$\int_{\mathbb{R}^n} |K_\varepsilon(x)|\, dx \leq M.$$
3. For every $\delta > 0$, we have
$$\lim_{\varepsilon \to 0} \int_{\{|x|>\delta\}} |K_\varepsilon(x)|\, dx = 0.$$

Then the family $\{K_\varepsilon\}_{\varepsilon>0}$ will be called a family of good kernels.

3.8 Approximations of identity (2/2)

Theorem

Let $f \in L^1(\mathbb{R}^n)$ *be bounded and continuous at* $x \in \mathbb{R}^n$ *and let* $\{K_\varepsilon\}_{\varepsilon>0}$ *be a family of good kernels. Then*

$$\lim_{\varepsilon \to 0} (K_\varepsilon * f)(x) = f(x).$$

If $f \in C_0(\mathbb{R}^n)$ *then* $K_\varepsilon * f \to f$ *uniformly as* $\varepsilon \to 0$*.*

- Approximations of the identity give an interpretation that the convolution $K_\varepsilon * f$ can be seen as a weighted integral average. The pointwise value of a function is replaced with an integral average, which converges to the value of the function as $\varepsilon \to 0$.
- Example 1: $K_\varepsilon(x) = \frac{1}{\omega_n \varepsilon^n} \chi_{\{|x|<\varepsilon\}}$. Then $(K_\varepsilon * f)(x)$ is the average of f in a ball $B(x, \varepsilon)$.
- Example 2: Heat kernel H_t which is related to the solutions to the heat equation in upper half plane.

3.8 Approximations of identity (2/2)

Theorem

Let $f \in L^1(\mathbb{R}^n)$ be bounded and continuous at $x \in \mathbb{R}^n$ and let $\{K_\varepsilon\}_{\varepsilon>0}$ be a family of good kernels. Then

$$\lim_{\varepsilon \to 0} (K_\varepsilon * f)(x) = f(x).$$

If $f \in C_0(\mathbb{R}^n)$ then $K_\varepsilon * f \to f$ uniformly as $\varepsilon \to 0$.

- Approximations of the identity give an interpretation that the convolution $K_\varepsilon * f$ can be seen as a weighted integral average. The pointwise value of a function is replaced with an integral average, which converges to the value of the function as $\varepsilon \to 0$.
- Example 1: $K_\varepsilon(x) = \frac{1}{\omega_n \varepsilon^n} \chi_{\{|x|<\varepsilon\}}$. Then $(K_\varepsilon * f)(x)$ is the average of f in a ball $B(x, \varepsilon)$.
- Example 2: Heat kernel H_t which is related to the solutions to the heat equation in upper half plane.

3.8 Approximations of identity (2/2)

Theorem

Let $f \in L^1(\mathbb{R}^n)$ be bounded and continuous at $x \in \mathbb{R}^n$ and let $\{K_\varepsilon\}_{\varepsilon>0}$ be a family of good kernels. Then

$$\lim_{\varepsilon \to 0} (K_\varepsilon * f)(x) = f(x).$$

If $f \in C_0(\mathbb{R}^n)$ then $K_\varepsilon * f \to f$ uniformly as $\varepsilon \to 0$.

- Approximations of the identity give an interpretation that the convolution $K_\varepsilon * f$ can be seen as a weighted integral average. The pointwise value of a function is replaced with an integral average, which converges to the value of the function as $\varepsilon \to 0$.
- Example 1: $K_\varepsilon(x) = \frac{1}{\omega_n \varepsilon^n} \chi_{\{|x|<\varepsilon\}}$. Then $(K_\varepsilon * f)(x)$ is the average of f in a ball $B(x, \varepsilon)$.
- Example 2: Heat kernel H_t which is related to the solutions to the heat equation in upper half plane.

3.8 Approximations of identity (2/2)

Theorem

Let $f \in L^1(\mathbb{R}^n)$ be bounded and continuous at $x \in \mathbb{R}^n$ and let $\{K_\varepsilon\}_{\varepsilon>0}$ be a family of good kernels. Then

$$\lim_{\varepsilon \to 0} (K_\varepsilon * f)(x) = f(x).$$

If $f \in C_0(\mathbb{R}^n)$ then $K_\varepsilon * f \to f$ uniformly as $\varepsilon \to 0$.

- Approximations of the identity give an interpretation that the convolution $K_\varepsilon * f$ can be seen as a weighted integral average. The pointwise value of a function is replaced with an integral average, which converges to the value of the function as $\varepsilon \to 0$.
- Example 1: $K_\varepsilon(x) = \frac{1}{\omega_n \varepsilon^n} \chi_{\{|x|<\varepsilon\}}$. Then $(K_\varepsilon * f)(x)$ is the average of f in a ball $B(x, \varepsilon)$.
- Example 2: Heat kernel H_t which is related to the solutions to the heat equation in upper half plane.

Aalto University

MS-C1350 Partial differential equations
Chapter 3.9-3.12
Laplace, heat and wave equation in the upper half space

Riikka Korte

Department of Mathematics and Systems Analysis
Aalto University
riikka.korte@aalto.fi

October 8, 2024

Lecture 6

Examples on how to use Fourier transform for solving

- Laplace equation in the upper half-space $\mathbb{R}^{n+1}_+$
- Heat equation in the upper half-space $\mathbb{R}^{n+1}_+$

Solving Laplace, heat and wave equation in upper half space using Fourier transform

Find a solution of an initial/boundary value problem in $\mathbb{R}^{n+1}_+$.

1. Take the Fourier transform of the PDE and of the initial conditions, with respect to the space variables:

$$(x, y) \quad \longleftrightarrow \quad (\xi, y) \quad \text{or} \quad (x, t) \quad \longleftrightarrow \quad (\xi, t)$$

2. This reduces the problem to an ODE.
3. The ODE is solved on the Fourier side.
4. The initial or boundary conditions are used to determine the free parameters.
5. The Fourier inversion formula gives the solution of the original problem.
6. The solution of the original problem is represented as a convolution of the data with the fundamental solution.
7. This gives a solution to the original problem and the initial or boundary values are attained by using approximations of the unity.

Solving Laplace, heat and wave equation in upper half space using Fourier transform

Find a solution of an initial/boundary value problem in $\mathbb{R}^{n+1}_+$.

1. Take the Fourier transform of the PDE and of the initial conditions, with respect to the space variables:

$$(x,y) \quad \longleftrightarrow \quad (\xi,y) \quad \text{or} \quad (x,t) \quad \longleftrightarrow \quad (\xi,t)$$

2. This reduces the problem to an ODE.
3. The ODE is solved on the Fourier side.
4. The initial or boundary conditions are used to determine the free parameters.
5. The Fourier inversion formula gives the solution of the original problem.
6. The solution of the original problem is represented as a convolution of the data with the fundamental solution.
7. This gives a solution to the original problem and the initial or boundary values are attained by using approximations of the unity.

Solving Laplace, heat and wave equation in upper half space using Fourier transform

Find a solution of an initial/boundary value problem in $\mathbb{R}^{n+1}_+$.

1. Take the Fourier transform of the PDE and of the initial conditions, with respect to the space variables:

$$(x, y) \quad \longleftrightarrow \quad (\xi, y) \quad \text{or} \quad (x, t) \quad \longleftrightarrow \quad (\xi, t)$$

2. This reduces the problem to an ODE.
3. The ODE is solved on the Fourier side.
4. The initial or boundary conditions are used to determine the free parameters.
5. The Fourier inversion formula gives the solution of the original problem.
6. The solution of the original problem is represented as a convolution of the data with the fundamental solution.
7. This gives a solution to the original problem and the initial or boundary values are attained by using approximations of the unity.

Solving Laplace, heat and wave equation in upper half space using Fourier transform

Find a solution of an initial/boundary value problem in $\mathbb{R}^{n+1}_+$.

1. Take the Fourier transform of the PDE and of the initial conditions, with respect to the space variables:

$$(x, y) \quad \longleftrightarrow \quad (\xi, y) \quad \text{or} \quad (x, t) \quad \longleftrightarrow \quad (\xi, t)$$

2. This reduces the problem to an ODE.
3. The ODE is solved on the Fourier side.
4. The initial or boundary conditions are used to determine the free parameters.
5. The Fourier inversion formula gives the solution of the original problem.
6. The solution of the original problem is represented as a convolution of the data with the fundamental solution.
7. This gives a solution to the original problem and the initial or boundary values are attained by using approximations of the unity.

Solving Laplace, heat and wave equation in upper half space using Fourier transform

Find a solution of an initial/boundary value problem in $\mathbb{R}^{n+1}_+$.

1. Take the Fourier transform of the PDE and of the initial conditions, with respect to the space variables:

$$(x, y) \quad \longleftrightarrow \quad (\xi, y) \quad \text{or} \quad (x, t) \quad \longleftrightarrow \quad (\xi, t)$$

2. This reduces the problem to an ODE.
3. The ODE is solved on the Fourier side.
4. The initial or boundary conditions are used to determine the free parameters.
5. The Fourier inversion formula gives the solution of the original problem.
6. The solution of the original problem is represented as a convolution of the data with the fundamental solution.
7. This gives a solution to the original problem and the initial or boundary values are attained by using approximations of the unity.

Solving Laplace, heat and wave equation in upper half space using Fourier transform

Find a solution of an initial/boundary value problem in $\mathbb{R}^{n+1}_+$.

1. Take the Fourier transform of the PDE and of the initial conditions, with respect to the space variables:

$$(x, y) \quad \longleftrightarrow \quad (\xi, y) \quad \text{or} \quad (x, t) \quad \longleftrightarrow \quad (\xi, t)$$

2. This reduces the problem to an ODE.
3. The ODE is solved on the Fourier side.
4. The initial or boundary conditions are used to determine the free parameters.
5. The Fourier inversion formula gives the solution of the original problem.
6. The solution of the original problem is represented as a convolution of the data with the fundamental solution.
7. This gives a solution to the original problem and the initial or boundary values are attained by using approximations of the unity.

Solving Laplace, heat and wave equation in upper half space using Fourier transform

Find a solution of an initial/boundary value problem in $\mathbb{R}^{n+1}_+$.

1. Take the Fourier transform of the PDE and of the initial conditions, with respect to the space variables:

$$(x, y) \quad \longleftrightarrow \quad (\xi, y) \quad \text{or} \quad (x, t) \quad \longleftrightarrow \quad (\xi, t)$$

2. This reduces the problem to an ODE.
3. The ODE is solved on the Fourier side.
4. The initial or boundary conditions are used to determine the free parameters.
5. The Fourier inversion formula gives the solution of the original problem.
6. The solution of the original problem is represented as a convolution of the data with the fundamental solution.
7. This gives a solution to the original problem and the initial or boundary values are attained by using approximations of the unity.

3.9 Laplace equation (1/8)

- we consider the Laplace equation in the upper half-space

$$\mathbb{R}^{n+1}_+ = \{(x,y) : x \in \mathbb{R}^n,\ y > 0\}.$$

- Suppose function g is continuous and bounded on the boundary $\partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times \{0\}$.

$$\begin{cases} \Delta u(x,y) = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0, & (x,y) \in \mathbb{R}^{n+1}_+, \\ u(x,0) = g(x), & x \in \mathbb{R}^n. \end{cases}$$

- Note: Laplacian with $n+1$ variables!
- Physically: Temperature on the boundary is g and the temperature is not changing in upper half space.
- In disc, 'periodicity' of the boundary reduced the number of solutions to the separated equations (countable number).

3.9 Laplace equation (1/8)

- we consider the Laplace equation in the upper half-space

$$\mathbb{R}^{n+1}_+ = \{(x, y) : x \in \mathbb{R}^n, \, y > 0\}.$$

- Suppose function g is continuous and bounded on the boundary $\partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times \{0\}$.

$$\begin{cases} \Delta u(x, y) = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = 0, \quad (x, y) \in \mathbb{R}^{n+1}_+, \\ u(x, 0) = g(x), \quad x \in \mathbb{R}^n. \end{cases}$$

- Note: Laplacian with $n + 1$ variables!
- Physically: Temperature on the boundary is g and the temperature is not changing in upper half space.
- In disc, 'periodicity' of the boundary reduced the number of solutions to the separated equations (countable number).

3.9 Laplace equation (1/8)

- we consider the Laplace equation in the upper half-space

$$\mathbb{R}^{n+1}_+ = \{(x,y) : x \in \mathbb{R}^n,\, y > 0\}.$$

- Suppose function g is continuous and bounded on the boundary $\partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times \{0\}$.

$$\begin{cases} \Delta u(x,y) = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0, & (x,y) \in \mathbb{R}^{n+1}_+, \\ u(x,0) = g(x), \quad x \in \mathbb{R}^n. \end{cases}$$

- Note: Laplacian with $n+1$ variables!
- Physically: Temperature on the boundary is g and the temperature is not changing in upper half space.
- In disc, 'periodicity' of the boundary reduced the number of solutions to the separated equations (countable number).

3.9 Laplace equation (1/8)

- we consider the Laplace equation in the upper half-space

$$\mathbb{R}^{n+1}_+ = \{(x,y) : x \in \mathbb{R}^n,\ y > 0\}.$$

- Suppose function g is continuous and bounded on the boundary $\partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times \{0\}$.

$$\begin{cases} \Delta u(x,y) = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0, & (x,y) \in \mathbb{R}^{n+1}_+, \\ u(x,0) = g(x), \quad x \in \mathbb{R}^n. \end{cases}$$

- Note: Laplacian with $n+1$ variables!
- Physically: Temperature on the boundary is g and the temperature is not changing in upper half space.
- In disc, 'periodicity' of the boundary reduced the number of solutions to the separated equations (countable number).

3.9 Laplace equation (1/8)

- we consider the Laplace equation in the upper half-space

$$\mathbb{R}^{n+1}_+ = \{(x,y) : x \in \mathbb{R}^n,\ y > 0\}.$$

- Suppose function g is continuous and bounded on the boundary $\partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times \{0\}$.

$$\begin{cases} \Delta u(x,y) = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0, \quad (x,y) \in \mathbb{R}^{n+1}_+, \\ u(x,0) = g(x), \quad x \in \mathbb{R}^n. \end{cases}$$

- Note: Laplacian with $n+1$ variables!
- Physically: Temperature on the boundary is g and the temperature is not changing in upper half space.
- In disc, 'periodicity' of the boundary reduced the number of solutions to the separated equations (countable number).

Laplace equation (2/8)

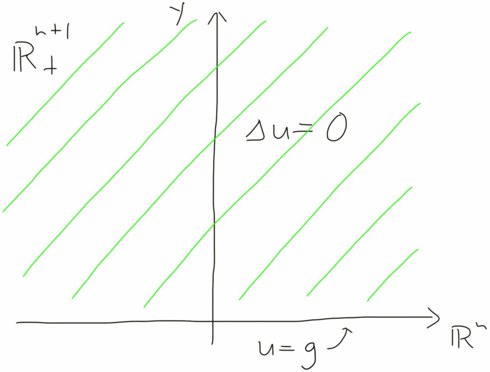


Figure: The Laplace equation in the upper half-space.

3.9 Laplace equation (3/8)

- **Step1: PDE on the Fourier side**
- Fix $y > 0$ and make the Fourier transform of $u(\cdot, y)$:

$$\widehat{u}(\xi, y) = \int_{\mathbb{R}^n} u(x,y)e^{-ix\cdot\xi}\,dx, \quad \xi \in \mathbb{R}^n, \quad y > 0.$$

- Differentiation becomes multiplication in Fourier transform:

$$\widehat{\frac{\partial^2 u}{\partial x_j^2}}(\xi, y) = i\xi_j \widehat{\frac{\partial u}{\partial x_j}}(\xi, y) = (i\xi_j)^2\widehat{u}(\xi, y) = -\xi_j^2\widehat{u}(\xi, y),\ j = 1, \ldots, n,$$

- As we did not make the transform with respect to y, $\partial^2/\partial y^2$ is not changed:

$$\begin{aligned}\widehat{\frac{\partial^2 u}{\partial y^2}}(\xi, y) &= \int_{\mathbb{R}^n} \frac{\partial^2 u}{\partial y^2}(x,y)e^{-ix\cdot\xi}\,dx \\ &= \frac{\partial^2}{\partial y^2}\left(\int_{\mathbb{R}^n} u(x,y)e^{-ix\cdot\xi}\,dx\right) = \frac{\partial^2 \widehat{u}}{\partial y^2}(\xi, y).\end{aligned}$$

3.9 Laplace equation (3/8)

- **Step1: PDE on the Fourier side**
- Fix $y > 0$ and make the Fourier transform of $u(\cdot, y)$:

$$\widehat{u}(\xi, y) = \int_{\mathbb{R}^n} u(x,y)e^{-ix\cdot\xi}\,dx, \quad \xi \in \mathbb{R}^n, \quad y > 0.$$

- Differentiation becomes multiplication in Fourier transform:

$$\widehat{\frac{\partial^2 u}{\partial x_j^2}}(\xi, y) = i\xi_j \widehat{\frac{\partial u}{\partial x_j}}(\xi, y) = (i\xi_j)^2 \widehat{u}(\xi, y) = -\xi_j^2 \widehat{u}(\xi, y),\ j = 1, \dots, n,$$

- As we did not make the transform with respect to y, $\partial^2/\partial y^2$ is not changed:

$$\begin{aligned}\widehat{\frac{\partial^2 u}{\partial y^2}}(\xi, y) &= \int_{\mathbb{R}^n} \frac{\partial^2 u}{\partial y^2}(x,y)e^{-ix\cdot\xi}\,dx \\ &= \frac{\partial^2}{\partial y^2}\left(\int_{\mathbb{R}^n} u(x,y)e^{-ix\cdot\xi}\,dx\right) = \frac{\partial^2 \widehat{u}}{\partial y^2}(\xi, y).\end{aligned}$$

3.9 Laplace equation (3/8)

- **Step1: PDE on the Fourier side**
- Fix $y > 0$ and make the Fourier transform of $u(\cdot, y)$:

$$\widehat{u}(\xi, y) = \int_{\mathbb{R}^n} u(x, y) e^{-ix\cdot\xi}\, dx, \quad \xi \in \mathbb{R}^n, \quad y > 0.$$

- Differentiation becomes multiplication in Fourier transform:

$$\widehat{\frac{\partial^2 u}{\partial x_j^2}}(\xi, y) = i\xi_j \widehat{\frac{\partial u}{\partial x_j}}(\xi, y) = (i\xi_j)^2 \widehat{u}(\xi, y) = -\xi_j^2 \widehat{u}(\xi, y),\ j = 1, \dots, n,$$

- As we did not make the transform with respect to y, $\partial^2/\partial y^2$ is not changed:

$$\begin{aligned}\widehat{\frac{\partial^2 u}{\partial y^2}}(\xi, y) &= \int_{\mathbb{R}^n} \frac{\partial^2 u}{\partial y^2}(x, y) e^{-ix\cdot\xi}\, dx \\ &= \frac{\partial^2}{\partial y^2}\left(\int_{\mathbb{R}^n} u(x, y) e^{-ix\cdot\xi}\, dx\right) = \frac{\partial^2 \widehat{u}}{\partial y^2}(\xi, y).\end{aligned}$$

3.9 Laplace equation (3/8)

- **Step1: PDE on the Fourier side**
- Fix $y > 0$ and make the Fourier transform of $u(\cdot, y)$:

$$\widehat{u}(\xi, y) = \int_{\mathbb{R}^n} u(x, y) e^{-ix\cdot\xi}\, dx, \quad \xi \in \mathbb{R}^n, \quad y > 0.$$

- Differentiation becomes multiplication in Fourier transform:

$$\widehat{\frac{\partial^2 u}{\partial x_j^2}}(\xi, y) = i\xi_j \widehat{\frac{\partial u}{\partial x_j}}(\xi, y) = (i\xi_j)^2 \widehat{u}(\xi, y) = -\xi_j^2 \widehat{u}(\xi, y),\ j = 1, \dots, n,$$

- As we did not make the transform with respect to y, $\partial^2/\partial y^2$ is not changed:

$$\begin{aligned}\widehat{\frac{\partial^2 u}{\partial y^2}}(\xi, y) &= \int_{\mathbb{R}^n} \frac{\partial^2 u}{\partial y^2}(x, y) e^{-ix\cdot\xi}\, dx \\ &= \frac{\partial^2}{\partial y^2}\left(\int_{\mathbb{R}^n} u(x, y) e^{-ix\cdot\xi}\, dx\right) = \frac{\partial^2 \widehat{u}}{\partial y^2}(\xi, y).\end{aligned}$$

3.9 Laplace equation (4/8)

- So now we have:

$$\widehat{\frac{\partial^2 u}{\partial x_j^2}}(\xi, y) = i\xi_j \widehat{\frac{\partial u}{\partial x_j}}(\xi, y) = (i\xi_j)^2 \widehat{u}(\xi, y) = -\xi_j^2 \widehat{u}(\xi, y),\ j = 1, \dots, n,$$

$$\text{and} \quad \widehat{\frac{\partial^2 u}{\partial y^2}}(\xi, y) = \frac{\partial^2 \widehat{u}}{\partial y^2}(\xi, y).$$

- It follows that

$$\begin{aligned} 0 &= \widehat{\Delta u}(\xi, y) \\ &= -\xi_1^2 \widehat{u}(\xi, y) - \dots - \xi_n^2 \widehat{u}(\xi, y) + \frac{\partial^2 \widehat{u}}{\partial y^2}(\xi, y) \\ &= -|\xi|^2 \widehat{u}(\xi, y) + \frac{\partial^2 \widehat{u}}{\partial y^2}(\xi, y). \end{aligned}$$

- Note that (under appropriate assumptions) $\Delta u = 0$ is equivalent to $\widehat{\Delta u} = 0$.

3.9 Laplace equation (4/8)

- So now we have:

$$\widehat{\frac{\partial^2 u}{\partial x_j^2}}(\xi, y) = i\xi_j \widehat{\frac{\partial u}{\partial x_j}}(\xi, y) = (i\xi_j)^2 \widehat{u}(\xi, y) = -\xi_j^2 \widehat{u}(\xi, y),\ j = 1, \dots, n,$$

$$\text{and} \quad \widehat{\frac{\partial^2 u}{\partial y^2}}(\xi, y) = \frac{\partial^2 \widehat{u}}{\partial y^2}(\xi, y).$$

- It follows that

$$\begin{aligned} 0 &= \widehat{\Delta u}(\xi, y) \\ &= -\xi_1^2 \widehat{u}(\xi, y) - \dots - \xi_n^2 \widehat{u}(\xi, y) + \frac{\partial^2 \widehat{u}}{\partial y^2}(\xi, y) \\ &= -|\xi|^2 \widehat{u}(\xi, y) + \frac{\partial^2 \widehat{u}}{\partial y^2}(\xi, y). \end{aligned}$$

- Note that (under appropriate assumptions) $\Delta u = 0$ is equivalent to $\widehat{\Delta u} = 0$.

3.9 Laplace equation (4/8)

- So now we have:

$$\widehat{\frac{\partial^2 u}{\partial x_j^2}}(\xi, y) = i\xi_j \widehat{\frac{\partial u}{\partial x_j}}(\xi, y) = (i\xi_j)^2 \widehat{u}(\xi, y) = -\xi_j^2 \widehat{u}(\xi, y),\ j = 1, \dots, n,$$

$$\text{and} \quad \widehat{\frac{\partial^2 u}{\partial y^2}}(\xi, y) = \frac{\partial^2 \widehat{u}}{\partial y^2}(\xi, y).$$

- It follows that

$$\begin{aligned} 0 &= \widehat{\Delta u}(\xi, y) \\ &= -\xi_1^2 \widehat{u}(\xi, y) - \dots - \xi_n^2 \widehat{u}(\xi, y) + \frac{\partial^2 \widehat{u}}{\partial y^2}(\xi, y) \\ &= -|\xi|^2 \widehat{u}(\xi, y) + \frac{\partial^2 \widehat{u}}{\partial y^2}(\xi, y). \end{aligned}$$

- Note that (under appropriate assumptions) $\Delta u = 0$ is equivalent to $\widehat{\Delta u} = 0$.

3.9 Laplace equation (5/8)

- **Step 2: Solution on the Fourier side**
- Fix ξ and solve

$$-|\xi|^2\widehat{u}(\xi,y) + \frac{\partial^2\widehat{u}}{\partial y^2}(\xi,y) = 0.$$

- The solution is

$$\widehat{u}(\xi,y) = c_1(\xi)e^{|\xi|y} + c_2(\xi)e^{-|\xi|y}.$$

- Note that the 'constants' c_1 and c_2 can now depend on ξ.
- We want to have physically relevant solutions and therefore we only keep the second term.
- Boundary condition on the Fourier side: $c_2(\xi) = \widehat{u}(\xi,0) = \widehat{g}(\xi)$.
- Thus we get

$$\widehat{u}(\xi,y) = \widehat{g}(\xi)e^{-|\xi|y}, \quad \xi \in \mathbb{R}^n, \quad y > 0.$$

3.9 Laplace equation (5/8)

- **Step 2: Solution on the Fourier side**
- Fix ξ and solve

$$-|\xi|^2\widehat{u}(\xi,y)+\frac{\partial^2\widehat{u}}{\partial y^2}(\xi,y)=0.$$

- The solution is

$$\widehat{u}(\xi,y)=c_1(\xi)e^{|\xi|y}+c_2(\xi)e^{-|\xi|y}.$$

- Note that the 'constants' c_1 and c_2 can now depend on ξ.
- We want to have physically relevant solutions and therefore we only keep the second term.
- Boundary condition on the Fourier side: $c_2(\xi)=\widehat{u}(\xi,0)=\widehat{g}(\xi)$.
- Thus we get

$$\widehat{u}(\xi,y)=\widehat{g}(\xi)e^{-|\xi|y},\quad \xi\in\mathbb{R}^n,\quad y>0.$$

3.9 Laplace equation (5/8)

- **Step 2: Solution on the Fourier side**
- Fix ξ and solve

$$-|\xi|^2\widehat{u}(\xi,y)+\frac{\partial^2\widehat{u}}{\partial y^2}(\xi,y)=0.$$

- The solution is

$$\widehat{u}(\xi,y)=c_1(\xi)e^{|\xi|y}+c_2(\xi)e^{-|\xi|y}.$$

- Note that the 'constants' c_1 and c_2 can now depend on ξ.
- We want to have physically relevant solutions and therefore we only keep the second term.
- Boundary condition on the Fourier side: $c_2(\xi)=\widehat{u}(\xi,0)=\widehat{g}(\xi)$.
- Thus we get

$$\widehat{u}(\xi,y)=\widehat{g}(\xi)e^{-|\xi|y},\quad \xi\in\mathbb{R}^n,\quad y>0.$$

3.9 Laplace equation (5/8)

- **Step 2: Solution on the Fourier side**
- Fix ξ and solve

$$-|\xi|^2\widehat{u}(\xi,y)+\frac{\partial^2\widehat{u}}{\partial y^2}(\xi,y)=0.$$

- The solution is

$$\widehat{u}(\xi,y)=c_1(\xi)e^{|\xi|y}+c_2(\xi)e^{-|\xi|y}.$$

- Note that the 'constants' c_1 and c_2 can now depend on ξ.
- We want to have physically relevant solutions and therefore we only keep the second term.
- Boundary condition on the Fourier side: $c_2(\xi)=\widehat{u}(\xi,0)=\widehat{g}(\xi)$.
- Thus we get

$$\widehat{u}(\xi,y)=\widehat{g}(\xi)e^{-|\xi|y},\quad \xi\in\mathbb{R}^n,\quad y>0.$$

3.9 Laplace equation (5/8)

- **Step 2: Solution on the Fourier side**
- Fix ξ and solve

$$-|\xi|^2\widehat{u}(\xi,y)+\frac{\partial^2\widehat{u}}{\partial y^2}(\xi,y)=0.$$

- The solution is

$$\widehat{u}(\xi,y)=c_1(\xi)e^{|\xi|y}+c_2(\xi)e^{-|\xi|y}.$$

- Note that the 'constants' c_1 and c_2 can now depend on ξ.
- We want to have physically relevant solutions and therefore we only keep the second term.
- Boundary condition on the Fourier side: $c_2(\xi)=\widehat{u}(\xi,0)=\widehat{g}(\xi)$.
- Thus we get

$$\widehat{u}(\xi,y)=\widehat{g}(\xi)e^{-|\xi|y},\quad \xi\in\mathbb{R}^n,\quad y>0.$$

3.9 Laplace equation (5/8)

- **Step 2: Solution on the Fourier side**
- Fix ξ and solve

$$-|\xi|^2\widehat{u}(\xi,y)+\frac{\partial^2\widehat{u}}{\partial y^2}(\xi,y)=0.$$

- The solution is

$$\widehat{u}(\xi,y)=c_1(\xi)e^{|\xi|y}+c_2(\xi)e^{-|\xi|y}.$$

- Note that the 'constants' c_1 and c_2 can now depend on ξ.
- We want to have physically relevant solutions and therefore we only keep the second term.
- Boundary condition on the Fourier side: $c_2(\xi)=\widehat{u}(\xi,0)=\widehat{g}(\xi)$.
- Thus we get

$$\widehat{u}(\xi,y)=\widehat{g}(\xi)e^{-|\xi|y},\quad \xi\in\mathbb{R}^n,\quad y>0.$$

3.9 Laplace equation (5/8)

- **Step 2: Solution on the Fourier side**
- Fix ξ and solve

$$-|\xi|^2\widehat{u}(\xi,y) + \frac{\partial^2\widehat{u}}{\partial y^2}(\xi,y) = 0.$$

- The solution is

$$\widehat{u}(\xi,y) = c_1(\xi)e^{|\xi|y} + c_2(\xi)e^{-|\xi|y}.$$

- Note that the 'constants' c_1 and c_2 can now depend on ξ.
- We want to have physically relevant solutions and therefore we only keep the second term.
- Boundary condition on the Fourier side: $c_2(\xi) = \widehat{u}(\xi,0) = \widehat{g}(\xi)$.
- Thus we get

$$\widehat{u}(\xi,y) = \widehat{g}(\xi)e^{-|\xi|y}, \quad \xi \in \mathbb{R}^n, \quad y > 0.$$

3.9 Laplace equation (6/8)

- **Step 3: Solution to the original problem**
- We use the Fourier inverse theorem:

$$
\begin{aligned}
u(x,y) &= (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{u}(\xi,y)e^{ix\cdot\xi}\,d\xi \\
&= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-|\xi|y}\widehat{g}(\xi)e^{ix\cdot\xi}\,d\xi \\
&= (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{P_y}(\xi)\widehat{g}(\xi)e^{ix\cdot\xi}\,d\xi,
\end{aligned}
$$

  where $\widehat{P_y}(\xi) = e^{-|\xi|y}$.
- The function $P_y(x)$ is called the Poisson kernel for the upper half-space.
- Note: At the moment we do not know what P_y is, only $\widehat{P_y}$!

3.9 Laplace equation (6/8)

- **Step 3: Solution to the original problem**
- We use the Fourier inverse theorem:

$$\begin{aligned} u(x,y) &= (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{u}(\xi,y) e^{ix\cdot\xi}\, d\xi \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-|\xi|y} \widehat{g}(\xi) e^{ix\cdot\xi}\, d\xi \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{P_y}(\xi) \widehat{g}(\xi) e^{ix\cdot\xi}\, d\xi, \end{aligned}$$

where $\widehat{P_y}(\xi) = e^{-|\xi|y}$.

- The function $P_y(x)$ is called the Poisson kernel for the upper half-space.
- Note: At the moment we do not know what P_y is, only $\widehat{P_y}$!

3.9 Laplace equation (6/8)

- **Step 3: Solution to the original problem**
- We use the Fourier inverse theorem:

$$\begin{aligned} u(x,y) &= (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{u}(\xi,y) e^{ix\cdot\xi}\, d\xi \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-|\xi|y} \widehat{g}(\xi) e^{ix\cdot\xi}\, d\xi \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{P_y}(\xi) \widehat{g}(\xi) e^{ix\cdot\xi}\, d\xi, \end{aligned}$$

  where $\widehat{P_y}(\xi) = e^{-|\xi|y}$.
- The function $P_y(x)$ is called the Poisson kernel for the upper half-space.
- Note: At the moment we do not know what P_y is, only $\widehat{P_y}$!

3.9 Laplace equation (6/8)

- **Step 3: Solution to the original problem**
- We use the Fourier inverse theorem:

$$
\begin{aligned}
u(x,y) &= (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{u}(\xi, y) e^{ix\cdot\xi}\, d\xi \\
&= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-|\xi|y} \widehat{g}(\xi) e^{ix\cdot\xi}\, d\xi \\
&= (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{P_y}(\xi) \widehat{g}(\xi) e^{ix\cdot\xi}\, d\xi,
\end{aligned}
$$

  where $\widehat{P_y}(\xi) = e^{-|\xi|y}$.
- The function $P_y(x)$ is called the Poisson kernel for the upper half-space.
- Note: At the moment we do not know what P_y is, only $\widehat{P}_y$!

3.9 Laplace equation (7/8)

- Now we see why this P_y-notation is useful:

$$\begin{aligned} u(x,y) &= (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{P_y}(\xi)\widehat{g}(\xi) e^{ix\cdot\xi}\, d\xi \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{P_y * g}(\xi) e^{ix\cdot\xi}\, d\xi \\ &= (P_y * g)(x) = \int_{\mathbb{R}^n} P_y(x-z) g(z)\, dz. \end{aligned}$$

3.9 Laplace equation (7/8)

- Now we see why this P_y-notation is useful:

$$
\begin{aligned}
u(x,y) &= (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{P_y}(\xi)\widehat{g}(\xi) e^{ix\cdot\xi}\, d\xi \\
&= (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{P_y * g}(\xi) e^{ix\cdot\xi}\, d\xi \\
&= (P_y * g)(x) = \int_{\mathbb{R}^n} P_y(x-z) g(z)\, dz.
\end{aligned}
$$

3.9 Laplace equation (7/8)

- Now we see why this P_y-notation is useful:

$$
\begin{aligned}
u(x,y) &= (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{P_y}(\xi)\widehat{g}(\xi)e^{ix\cdot\xi}\,d\xi \\
&= (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{P_y * g}(\xi)e^{ix\cdot\xi}\,d\xi \\
&= (P_y * g)(x) = \int_{\mathbb{R}^n} P_y(x-z)g(z)\,dz.
\end{aligned}
$$

3.9 Laplace equation (8/8)

- **Step 4: Explicit representation formula**
- $$P_y(x) = P(x,y) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \frac{y}{(|x|^2+y^2)^{\frac{n+1}{2}}}, \quad x \in \mathbb{R}^n, \quad y > 0.$$
- Here $\Gamma(\frac{n+1}{2})$ is a dimensional constant given by the Γ-function
$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x}\,dx.$$

3.9 Laplace equation (8/8)

- **Step 4: Explicit representation formula**
- $$P_y(x) = P(x,y) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \frac{y}{(|x|^2 + y^2)^{\frac{n+1}{2}}}, \quad x \in \mathbb{R}^n, \quad y > 0.$$
- Here $\Gamma(\frac{n+1}{2})$ is a dimensional constant given by the Γ-function
$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x}\, dx.$$

3.9 Laplace equation (8/8)

- **Step 4: Explicit representation formula**
- $$P_y(x) = P(x,y) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \frac{y}{(|x|^2 + y^2)^{\frac{n+1}{2}}}, \quad x \in \mathbb{R}^n, \quad y > 0.$$
- Here $\Gamma(\frac{n+1}{2})$ is a dimensional constant given by the Γ-function
$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x}\, dx.$$

3.9 Poisson kernel

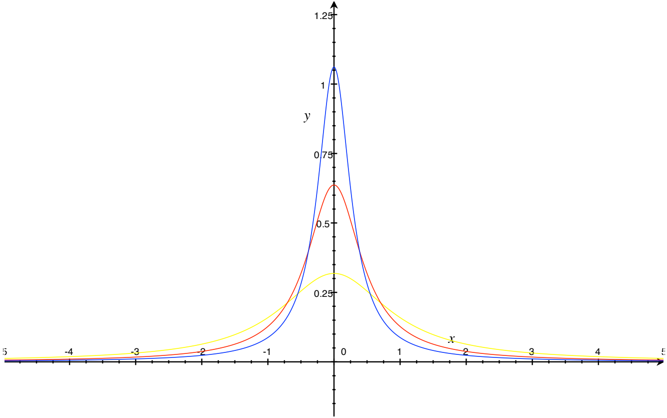


Figure: The graph of the Poisson kernel in dimension $n = 1$ for $y = 1$ (yellow), $y = 0.5$ (red) and $y = 0.3$ (blue).

3.9 Properties of the Poisson kernel

- $P_y(x) > 0$ for every $x \in \mathbb{R}^n$ and $y > 0$.
- For every $y > 0$ we have

$$\int_{\mathbb{R}^n} P_y(x)\,dx = \widehat{P_y}(0) = 1.$$

- The Poisson kernel $P(x,y) = P_y(x)$ is a solution of the Laplace equation in the upper half-space $\mathbb{R}^{n+1}_+$.
- It is called the fundamental solution in the upper half-space $\mathbb{R}^{n+1}_+$, since all other solutions can be represented as a convolution with it.

3.9 Properties of the Poisson kernel

- $P_y(x) > 0$ for every $x \in \mathbb{R}^n$ and $y > 0$.
- For every $y > 0$ we have

$$\int_{\mathbb{R}^n} P_y(x)\,dx = \widehat{P_y}(0) = 1.$$

- The Poisson kernel $P(x, y) = P_y(x)$ is a solution of the Laplace equation in the upper half-space $\mathbb{R}^{n+1}_+$.
- It is called the fundamental solution in the upper half-space $\mathbb{R}^{n+1}_+$, since all other solutions can be represented as a convolution with it.

3.9 Properties of the Poisson kernel

- $P_y(x) > 0$ for every $x \in \mathbb{R}^n$ and $y > 0$.
- For every $y > 0$ we have

$$\int_{\mathbb{R}^n} P_y(x)\,dx = \widehat{P_y}(0) = 1.$$

- The Poisson kernel $P(x, y) = P_y(x)$ is a solution of the Laplace equation in the upper half-space $\mathbb{R}^{n+1}_+$.
- It is called the fundamental solution in the upper half-space $\mathbb{R}^{n+1}_+$, since all other solutions can be represented as a convolution with it.

3.9 Properties of the Poisson kernel

- $P_y(x) > 0$ for every $x \in \mathbb{R}^n$ and $y > 0$.
- For every $y > 0$ we have

$$\int_{\mathbb{R}^n} P_y(x)\,dx = \widehat{P_y}(0) = 1.$$

- The Poisson kernel $P(x,y) = P_y(x)$ is a solution of the Laplace equation in the upper half-space $\mathbb{R}^{n+1}_+$.
- It is called the fundamental solution in the upper half-space $\mathbb{R}^{n+1}_+$, since all other solutions can be represented as a convolution with it.

3.9 Laplace equation

Theorem

Let $g \in C_0^\infty(\mathbb{R}^n)$. The solution to the Dirichlet problem is

$$u(x,y) = (g * P_y)(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \int_{\mathbb{R}^n} \frac{g(z)y}{(|x-z|^2+y^2)^{\frac{n+1}{2}}}\,dz.$$

The boundary condition is taken in the sense that

$$\lim_{y\to 0} u(x,y) = g(x) \quad \textit{for every} \quad x \in \mathbb{R}^n.$$

3.10 Heat equation in the upper half-space

- The general form of the heat equation is

$$u_t(x,t) - \Delta u(x,t) = 0,$$

where the Laplace operator is only in the x-variable

$$\Delta u(x,t) = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}(x,t).$$

- We consider the initial value problem

$$\begin{cases} u_t(x,t) - \Delta u(x,t) = 0, & x \in \mathbb{R}^n, \quad t > 0, \\ u(x,0) = g(x), & x \in \mathbb{R}^n. \end{cases}$$

- Cauchy problem for the heat equation.

3.10 Heat equation in the upper half-space

- The general form of the heat equation is

$$u_t(x,t) - \Delta u(x,t) = 0,$$

where the Laplace operator is only in the x-variable

$$\Delta u(x,t) = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}(x,t).$$

- We consider the initial value problem

$$\begin{cases} u_t(x,t) - \Delta u(x,t) = 0, & x \in \mathbb{R}^n, \quad t > 0, \\ u(x,0) = g(x), & x \in \mathbb{R}^n. \end{cases}$$

- Cauchy problem for the heat equation.

3.10 Heat equation in the upper half-space

- The general form of the heat equation is

$$u_t(x,t) - \Delta u(x,t) = 0,$$

where the Laplace operator is only in the x-variable

$$\Delta u(x,t) = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}(x,t).$$

- We consider the initial value problem

$$\begin{cases} u_t(x,t) - \Delta u(x,t) = 0, \quad x \in \mathbb{R}^n, \quad t > 0, \\ u(x,0) = g(x), \quad x \in \mathbb{R}^n. \end{cases}$$

- Cauchy problem for the heat equation.

Heat equation in the upper half-space

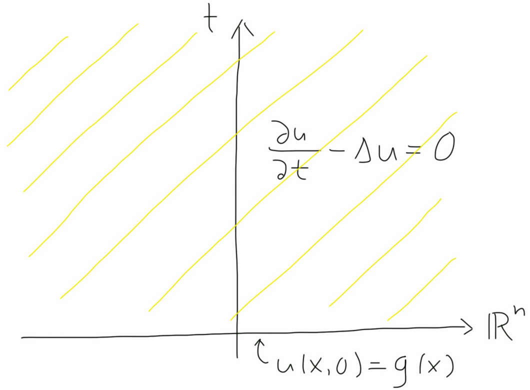


Figure: The heat equation in the upper half-space.

3.10 Heat equation in the upper half-space

- **Step 1: PDE on the Fourier side**
- Let $t > 0$ be fixed and denote by $\widehat{u}(\xi, t)$ the Fourier transform of $u(x, t)$ in the x-variable

$$\widehat{u}(\xi, t) = \int_{\mathbb{R}^n} u(x, t) e^{-ix\cdot\xi} dx.$$

- On the Fourier side, the heat equation becomes

$$0 = \widehat{u_t - \Delta u}(\xi, t) = \widehat{u_t}(\xi, t) - \widehat{\Delta u}(\xi, t) = \frac{\partial \widehat{u}}{\partial t}(\xi, t) + |\xi|^2 \widehat{u}(\xi, t).$$

3.10 Heat equation in the upper half-space

- **Step 1: PDE on the Fourier side**
- Let $t > 0$ be fixed and denote by $\widehat{u}(\xi, t)$ the Fourier transform of $u(x, t)$ in the x-variable

$$\widehat{u}(\xi, t) = \int_{\mathbb{R}^n} u(x, t) e^{-ix\cdot\xi} dx.$$

- On the Fourier side, the heat equation becomes

$$0 = \widehat{u_t - \Delta u}(\xi, t) = \widehat{u}_t(\xi, t) - \widehat{\Delta u}(\xi, t) = \frac{\partial \widehat{u}}{\partial t}(\xi, t) + |\xi|^2 \widehat{u}(\xi, t).$$

3.10 Heat equation in the upper half-space

- **Step 1: PDE on the Fourier side**
- Let $t > 0$ be fixed and denote by $\widehat{u}(\xi, t)$ the Fourier transform of $u(x, t)$ in the x-variable

$$\widehat{u}(\xi, t) = \int_{\mathbb{R}^n} u(x, t) e^{-ix\cdot\xi} dx.$$

- On the Fourier side, the heat equation becomes

$$0 = \widehat{u_t - \Delta u}(\xi, t) = \widehat{u_t}(\xi, t) - \widehat{\Delta u}(\xi, t) = \frac{\partial \widehat{u}}{\partial t}(\xi, t) + |\xi|^2 \widehat{u}(\xi, t).$$

3.10 Heat equation in the upper half-space

- The rest of the steps are very similar to the Laplace equation.
- Instead of the Poisson kernel, we obtain the heat kernel

$$H_t(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}. \quad x \in \mathbb{R}^n, \quad t > 0,$$

Theorem

Assume that $g \in C_0^\infty(\mathbb{R}^n)$*. The solution to Cauchy problem is*

$$u(x,t) = (H_t * g)(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy, \quad x \in \mathbb{R}^n, \quad t > 0.$$

The initial values are attained in the sense

$$\lim_{t \to 0} u(x,t) = g(x) \quad \textit{for every} \quad x \in \mathbb{R}^n.$$

3.10 Heat equation in the upper half-space

- The rest of the steps are very similar to the Laplace equation.
- Instead of the Poisson kernel, we obtain the heat kernel

$$H_t(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}. \quad x \in \mathbb{R}^n, \quad t > 0,$$

Theorem

Assume that $g \in C_0^\infty(\mathbb{R}^n)$. The solution to Cauchy problem is

$$u(x,t) = (H_t * g)(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy, \quad x \in \mathbb{R}^n, \quad t > 0.$$

The initial values are attained in the sense

$$\lim_{t\to 0} u(x,t) = g(x) \quad \textit{for every} \quad x \in \mathbb{R}^n.$$

3.10 Heat equation in the upper half-space

- The rest of the steps are very similar to the Laplace equation.
- Instead of the Poisson kernel, we obtain the heat kernel

$$H_t(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}. \quad x \in \mathbb{R}^n, \quad t > 0,$$

Theorem

Assume that $g \in C_0^\infty(\mathbb{R}^n)$. The solution to Cauchy problem is

$$u(x,t) = (H_t * g)(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy, \quad x \in \mathbb{R}^n, \quad t > 0.$$

The initial values are attained in the sense

$$\lim_{t \to 0} u(x,t) = g(x) \quad \textit{for every} \quad x \in \mathbb{R}^n.$$

3.11 Wave equation in the upper half-space

- The general form of the wave equation is

$$u_{tt}(x,t) - \Delta u(x,t) = 0,$$

where the Laplace operator is only in the x-variable

$$\Delta u(x,t) = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}(x,t).$$

- One can think of the wave equation as describing the displacement of a vibrating string (in dimension $n = 1$), a vibrating membrane (in dimension $n = 2$) or an elastic solid (dimension $n = 3$). In dimension $n = 3$ this equation also determines the behavior of electromagnetic waves in vacuum and the propagation of sound waves.

3.11 Wave equation in the upper half-space

- The general form of the wave equation is

$$u_{tt}(x,t) - \Delta u(x,t) = 0,$$

where the Laplace operator is only in the x-variable

$$\Delta u(x,t) = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}(x,t).$$

- One can think of the wave equation as describing the displacement of a vibrating string (in dimension $n = 1$), a vibrating membrane (in dimension $n = 2$) or an elastic solid (dimension $n = 3$). In dimension $n = 3$ this equation also determines the behavior of electromagnetic waves in vacuum and the propagation of sound waves.

3.11 Wave equation in the upper half-space

- The general goal is to find a solution of the Cauchy problem for the wave equation

$$\begin{cases} u_{tt} - \Delta u = 0, & x \in \mathbb{R}^n, \quad t > 0, \\ u(x,0) = g(x), & x \in \mathbb{R}^n, \\ u_t(x,0) = h(x), & x \in \mathbb{R}^n. \end{cases}$$

- **Step 1: PDE on the Fourier side.** Let $t > 0$ be fixed and denote by $\widehat{u}(\xi, t)$ the Fourier transform of $u(x,t)$ in the x-variable

$$\widehat{u}(\xi, t) = \int_{\mathbb{R}^n} u(x,t) e^{-ix\cdot\xi} \, dx.$$

3.11 Wave equation in the upper half-space

- The general goal is to find a solution of the Cauchy problem for the wave equation

$$\begin{cases} u_{tt} - \Delta u = 0, \quad x \in \mathbb{R}^n, \quad t > 0, \\ u(x,0) = g(x), \quad x \in \mathbb{R}^n, \\ u_t(x,0) = h(x), \quad x \in \mathbb{R}^n. \end{cases}$$

- **Step 1: PDE on the Fourier side.** Let $t > 0$ be fixed and denote by $\widehat{u}(\xi, t)$ the Fourier transform of $u(x,t)$ in the x-variable

$$\widehat{u}(\xi, t) = \int_{\mathbb{R}^n} u(x,t) e^{-ix\cdot\xi} dx.$$

3.11 Wave equation in the upper half-space

- The equation on the Fourier side:

$$0 = (\widehat{u_{tt} - \Delta} u)(\xi, t) = \widehat{u_{tt}}(\xi, t) - \widehat{\Delta u}(\xi, t) = \frac{\partial^2 \widehat{u}}{\partial t^2}(\xi, t) + |\xi|^2 \widehat{u}(\xi, t).$$

- The solution of this ODE for a fixed $\xi \in \mathbb{R}^n$ is

$$\widehat{u}(\xi, t) = c_1(\xi) \cos(|\xi| t) + c_2(\xi) \sin(|\xi| t)$$

for some functions $c_1(\xi)$ and $c_2(\xi)$.

- The initial conditions on the Fourier side:

$$\widehat{g}(\xi) = \widehat{u}(\xi, 0) = c_1(\xi) \quad \text{and} \quad \widehat{h}(\xi) = \widehat{u}_t(\xi, 0) = |\xi| c_2(\xi).$$

- Thus the solution on the Fourier side can be written in the form

$$\widehat{u}(\xi, t) = \widehat{g}(\xi) \cos(|\xi| t) + \widehat{h}(\xi) \frac{\sin(|\xi| t)}{|\xi|}.$$

3.11 Wave equation in the upper half-space

- The equation on the Fourier side:

$$0 = (\widehat{u_{tt} - \Delta u})(\xi,t) = \widehat{u_{tt}}(\xi,t) - \widehat{\Delta u}(\xi,t) = \frac{\partial^2 \widehat{u}}{\partial t^2}(\xi,t) + |\xi|^2 \widehat{u}(\xi,t).$$

- The solution of this ODE for a fixed $\xi \in \mathbb{R}^n$ is

$$\widehat{u}(\xi,t) = c_1(\xi)\cos(|\xi|t) + c_2(\xi)\sin(|\xi|t)$$

for some functions $c_1(\xi)$ and $c_2(\xi)$.

- The initial conditions on the Fourier side:

$$\widehat{g}(\xi) = \widehat{u}(\xi,0) = c_1(\xi) \quad \text{and} \quad \widehat{h}(\xi) = \widehat{u_t}(\xi,0) = |\xi| c_2(\xi).$$

- Thus the solution on the Fourier side can be written in the form

$$\widehat{u}(\xi,t) = \widehat{g}(\xi)\cos(|\xi|t) + \widehat{h}(\xi)\frac{\sin(|\xi|t)}{|\xi|}.$$

3.11 Wave equation in the upper half-space

- The equation on the Fourier side:

$$0 = (\widehat{u_{tt} - \Delta u})(\xi, t) = \widehat{u_{tt}}(\xi, t) - \widehat{\Delta u}(\xi, t) = \frac{\partial^2 \widehat{u}}{\partial t^2}(\xi, t) + |\xi|^2 \widehat{u}(\xi, t).$$

- The solution of this ODE for a fixed $\xi \in \mathbb{R}^n$ is

$$\widehat{u}(\xi, t) = c_1(\xi) \cos(|\xi| t) + c_2(\xi) \sin(|\xi| t)$$

for some functions $c_1(\xi)$ and $c_2(\xi)$.

- The initial conditions on the Fourier side:

$$\widehat{g}(\xi) = \widehat{u}(\xi, 0) = c_1(\xi) \quad \text{and} \quad \widehat{h}(\xi) = \widehat{u_t}(\xi, 0) = |\xi| c_2(\xi).$$

- Thus the solution on the Fourier side can be written in the form

$$\widehat{u}(\xi, t) = \widehat{g}(\xi) \cos(|\xi| t) + \widehat{h}(\xi) \frac{\sin(|\xi| t)}{|\xi|}.$$

3.11 Wave equation in the upper half-space

- The equation on the Fourier side:

$$0 = (\widehat{u_{tt} - \Delta u})(\xi, t) = \widehat{u_{tt}}(\xi, t) - \widehat{\Delta u}(\xi, t) = \frac{\partial^2 \widehat{u}}{\partial t^2}(\xi, t) + |\xi|^2 \widehat{u}(\xi, t).$$

- The solution of this ODE for a fixed $\xi \in \mathbb{R}^n$ is

$$\widehat{u}(\xi, t) = c_1(\xi) \cos(|\xi| t) + c_2(\xi) \sin(|\xi| t)$$

  for some functions $c_1(\xi)$ and $c_2(\xi)$.
- The initial conditions on the Fourier side:

$$\widehat{g}(\xi) = \widehat{u}(\xi, 0) = c_1(\xi) \quad \text{and} \quad \widehat{h}(\xi) = \widehat{u_t}(\xi, 0) = |\xi| c_2(\xi).$$

- Thus the solution on the Fourier side can be written in the form

$$\widehat{u}(\xi, t) = \widehat{g}(\xi) \cos(|\xi| t) + \widehat{h}(\xi) \frac{\sin(|\xi| t)}{|\xi|}.$$

3.11 Wave equation in the upper half-space

- The next step is to use the inverse transform in order to find u.
- Denote

$$\widehat{\Phi_t}(\xi) = \frac{\sin(|\xi|t)}{|\xi|} \quad \text{and} \quad \widehat{\Psi_t}(\xi) = \cos(|\xi|t) = \frac{\partial \widehat{\Phi_t}}{\partial t}(\xi).$$

- Then

$$\begin{aligned} u(x,t) &= (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\widehat{g}(\xi)\widehat{\Psi_t}(\xi) + \widehat{h}(\xi)\widehat{\Phi_t}(\xi) \right) e^{ix\cdot\xi}\, d\xi \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\widehat{g * \Psi_t}(\xi) + \widehat{h * \Phi_t}(\xi) \right) e^{ix\cdot\xi}\, d\xi \\ &= (g * \Psi_t)(x) + (h * \Phi_t)(x). \end{aligned}$$

- Determining Ψ_t and Φ_t and interpretation of this formula is a hard problem. We will return to it later.

3.11 Wave equation in the upper half-space

- The next step is to use the inverse transform in order to find u.
- Denote

$$\widehat{\Phi_t}(\xi) = \frac{\sin(|\xi|t)}{|\xi|} \quad \text{and} \quad \widehat{\Psi_t}(\xi) = \cos(|\xi|t) = \frac{\partial \widehat{\Phi_t}}{\partial t}(\xi).$$

- Then

$$\begin{aligned} u(x,t) &= (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\widehat{g}(\xi)\widehat{\Psi_t}(\xi) + \widehat{h}(\xi)\widehat{\Phi_t}(\xi) \right) e^{ix\cdot\xi}\, d\xi \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\widehat{g * \Psi_t}(\xi) + \widehat{h * \Phi_t}(\xi) \right) e^{ix\cdot\xi}\, d\xi \\ &= (g * \Psi_t)(x) + (h * \Phi_t)(x). \end{aligned}$$

- Determining Ψ_t and Φ_t and interpretation of this formula is a hard problem. We will return to it later.

3.11 Wave equation in the upper half-space

- The next step is to use the inverse transform in order to find u.
- Denote

$$\widehat{\Phi_t}(\xi) = \frac{\sin(|\xi|t)}{|\xi|} \quad \text{and} \quad \widehat{\Psi_t}(\xi) = \cos(|\xi|t) = \frac{\partial \widehat{\Phi_t}}{\partial t}(\xi).$$

- Then

$$\begin{aligned} u(x,t) &= (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\widehat{g}(\xi)\widehat{\Psi_t}(\xi) + \widehat{h}(\xi)\widehat{\Phi_t}(\xi) \right) e^{ix\cdot\xi}\, d\xi \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\widehat{g * \Psi_t}(\xi) + \widehat{h * \Phi_t}(\xi) \right) e^{ix\cdot\xi}\, d\xi \\ &= (g * \Psi_t)(x) + (h * \Phi_t)(x). \end{aligned}$$

- Determining Ψ_t and Φ_t and interpretation of this formula is a hard problem. We will return to it later.

3.11 Wave equation in the upper half-space

- The next step is to use the inverse transform in order to find u.
- Denote

$$\widehat{\Phi_t}(\xi) = \frac{\sin(|\xi|t)}{|\xi|} \quad \text{and} \quad \widehat{\Psi_t}(\xi) = \cos(|\xi|t) = \frac{\partial \widehat{\Phi_t}}{\partial t}(\xi).$$

- Then

$$\begin{aligned} u(x,t) &= (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\widehat{g}(\xi)\widehat{\Psi_t}(\xi) + \widehat{h}(\xi)\widehat{\Phi_t}(\xi) \right) e^{ix\cdot\xi}\, d\xi \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\widehat{g * \Psi_t}(\xi) + \widehat{h * \Phi_t}(\xi) \right) e^{ix\cdot\xi}\, d\xi \\ &= (g * \Psi_t)(x) + (h * \Phi_t)(x). \end{aligned}$$

- Determining Ψ_t and Φ_t and interpretation of this formula is a hard problem. We will return to it later.

MS-C1350 Partial differential equations
Chapter 4.1-4.5
Laplace equation

Riikka Korte

Department of Mathematics and Systems Analysis
Aalto University
riikka.korte@aalto.fi

October 22, 2024

Lecture 7

- Laplace equation and Poisson equation
- Harmonic function
- Gauss-Green theorem and Green's identities.
- Dirichlet and Neumann boundary value problems
- Uniqueness of solutions
- Compatibility condition for Neumann problems
- Fundamental solution and a solution to Poisson equation
- How to solve Poisson equation with correct boundary values.

4 Laplace equation (introduction)

- Now we concentrate on Laplace equation

$$\Delta u = 0$$

and the Poisson equation

$$-\Delta u = f.$$

- Boundary value problems for these equations appear frequently in natural sciences and engineering.
- Physically, solutions of the Poisson equation correspond to steady states for evolutions in time such as heat flow or wave motion, with f corresponding to external driving forces such as heat sources or wave generators.

4 Laplace equation (introduction)

- Now we concentrate on Laplace equation

$$\Delta u = 0$$

and the Poisson equation

$$-\Delta u = f.$$

- Boundary value problems for these equations appear frequently in natural sciences and engineering.
- Physically, solutions of the Poisson equation correspond to steady states for evolutions in time such as heat flow or wave motion, with f corresponding to external driving forces such as heat sources or wave generators.

4 Laplace equation (introduction)

- Now we concentrate on Laplace equation

$$\Delta u = 0$$

and the Poisson equation

$$-\Delta u = f.$$

- Boundary value problems for these equations appear frequently in natural sciences and engineering.
- Physically, solutions of the Poisson equation correspond to steady states for evolutions in time such as heat flow or wave motion, with f corresponding to external driving forces such as heat sources or wave generators.

4 Laplace equation (introduction)

- We will derive representation formulas and study general properties of solutions to the Laplace (and Poisson) equation.
- The topics include:
  - fundamental solutions
  - Green's functions
  - mean value property
  - Harnack's inequality, and
  - maximum principle.

Definition

A function $u \in C^2(\Omega)$, which satisfies $\Delta u = 0$ in Ω, is called a harmonic function in Ω.

4 Laplace equation (introduction)

- We will derive representation formulas and study general properties of solutions to the Laplace (and Poisson) equation.
- The topics include:
  - fundamental solutions
  - Green's functions
  - mean value property
  - Harnack's inequality, and
  - maximum principle.

Definition

A function $u \in C^2(\Omega)$, which satisfies $\Delta u = 0$ in Ω, is called a harmonic function in Ω.

4 Laplace equation (introduction)

- We will derive representation formulas and study general properties of solutions to the Laplace (and Poisson) equation.
- The topics include:
  - fundamental solutions
  - Green's functions
  - mean value property
  - Harnack's inequality, and
  - maximum principle.

Definition

A function $u \in C^2(\Omega)$, which satisfies $\Delta u = 0$ in Ω, is called a harmonic function in Ω.

4.1 Gauss-Green theorem

- We need certain integral formulas to be able to study the Laplacian.
- We assume that $\Omega \subset \mathbb{R}^n$ is bounded and open.
- We also assume that $\partial\Omega$ is smooth (i.e. it can be locally represented as a graph of a smooth function).
- Closure of Ω:
$$\overline{\Omega} = \Omega \cup \partial\Omega.$$
- We say that $u \in C^1(\overline{\Omega})$, if $u \in C^1(\Omega)$ is such that u and all partial derivatives $\frac{\partial u}{\partial x_j}$, $j = 1, \ldots, n$, can be extended continuously up to the boundary $\partial\Omega$.
- We start with Gauss-Green theorem, which is a generalization of
$$\int_a^b f'(t)dt = f(b) - f(a)$$
to higher dimensions.

4.1 Gauss-Green theorem

- We need certain integral formulas to be able to study the Laplacian.
- We assume that $\Omega \subset \mathbb{R}^n$ is bounded and open.
- We also assume that $\partial\Omega$ is smooth (i.e. it can be locally represented as a graph of a smooth function).
- Closure of Ω:
$$\overline{\Omega} = \Omega \cup \partial\Omega.$$
- We say that $u \in C^1(\overline{\Omega})$, if $u \in C^1(\Omega)$ is such that u and all partial derivatives $\frac{\partial u}{\partial x_j}$, $j = 1, \dots, n$, can be extended continuously up to the boundary $\partial\Omega$.
- We start with Gauss-Green theorem, which is a generalization of
$$\int_a^b f'(t)dt = f(b) - f(a)$$
to higher dimensions.

4.1 Gauss-Green theorem

- We need certain integral formulas to be able to study the Laplacian.
- We assume that $\Omega \subset \mathbb{R}^n$ is bounded and open.
- We also assume that $\partial\Omega$ is smooth (i.e. it can be locally represented as a graph of a smooth function).
- Closure of Ω:
$$\overline{\Omega} = \Omega \cup \partial\Omega.$$
- We say that $u \in C^1(\overline{\Omega})$, if $u \in C^1(\Omega)$ is such that u and all partial derivatives $\frac{\partial u}{\partial x_j}$, $j = 1, \ldots, n$, can be extended continuously up to the boundary $\partial\Omega$.
- We start with Gauss-Green theorem, which is a generalization of
$$\int_a^b f'(t)dt = f(b) - f(a)$$
to higher dimensions.

4.1 Gauss-Green theorem

- We need certain integral formulas to be able to study the Laplacian.
- We assume that $\Omega \subset \mathbb{R}^n$ is bounded and open.
- We also assume that $\partial\Omega$ is smooth (i.e. it can be locally represented as a graph of a smooth function).
- Closure of Ω:

$$\overline{\Omega} = \Omega \cup \partial\Omega.$$

- We say that $u \in C^1(\overline{\Omega})$, if $u \in C^1(\Omega)$ is such that u and all partial derivatives $\frac{\partial u}{\partial x_j}$, $j = 1, \ldots, n$, can be extended continuously up to the boundary $\partial\Omega$.
- We start with Gauss-Green theorem, which is a generalization of

$$\int_a^b f'(t)dt = f(b) - f(a)$$

to higher dimensions.

4.1 Gauss-Green theorem

- We need certain integral formulas to be able to study the Laplacian.
- We assume that $\Omega \subset \mathbb{R}^n$ is bounded and open.
- We also assume that $\partial\Omega$ is smooth (i.e. it can be locally represented as a graph of a smooth function).
- Closure of Ω:
$$\overline{\Omega} = \Omega \cup \partial\Omega.$$
- We say that $u \in C^1(\overline{\Omega})$, if $u \in C^1(\Omega)$ is such that u and all partial derivatives $\frac{\partial u}{\partial x_j}$, $j = 1, \ldots, n$, can be extended continuously up to the boundary $\partial\Omega$.
- We start with Gauss-Green theorem, which is a generalization of
$$\int_a^b f'(t)dt = f(b) - f(a)$$
to higher dimensions.

4.1 Gauss-Green theorem

- We need certain integral formulas to be able to study the Laplacian.
- We assume that $\Omega \subset \mathbb{R}^n$ is bounded and open.
- We also assume that $\partial\Omega$ is smooth (i.e. it can be locally represented as a graph of a smooth function).
- Closure of Ω:

$$\overline{\Omega} = \Omega \cup \partial\Omega.$$

- We say that $u \in C^1(\overline{\Omega})$, if $u \in C^1(\Omega)$ is such that u and all partial derivatives $\frac{\partial u}{\partial x_j}$, $j = 1, \ldots, n$, can be extended continuously up to the boundary $\partial\Omega$.
- We start with Gauss-Green theorem, which is a generalization of

$$\int_a^b f'(t)dt = f(b) - f(a)$$

to higher dimensions.

4.1 Gauss-Green theorem

Theorem (Gauss-Green theorem)

Assume that $u \in C^1(\overline{\Omega})$. Then

$$\int_\Omega \frac{\partial u}{\partial x_j}(x)\,dx = \int_{\partial\Omega} u(x)\nu_j(x)\,dS(x), \quad j = 1, \dots, n,$$

where dS denotes the surface measure on $\partial\Omega$. Here $\nu(x) = (\nu_1(x), \dots, \nu_n(x))$ is the outward pointing unit normal vector on $\partial\Omega$.

Or equivalently

Theorem (Divergence theorem)

$$\int_\Omega \text{div}\, F(x)\,dx = \int_{\partial\Omega} F(x) \cdot \nu(x)\,dS(x),$$

where $F = (F_1, \dots, F_n)$ is a vector field.

4.1 Gauss-Green theorem

Theorem (Gauss-Green theorem)

Assume that $u \in C^1(\overline{\Omega})$. *Then*

$$\int_\Omega \frac{\partial u}{\partial x_j}(x)\,dx = \int_{\partial\Omega} u(x)\nu_j(x)\,dS(x), \quad j = 1, \ldots, n,$$

where dS *denotes the surface measure on* $\partial\Omega$. *Here* $\nu(x) = (\nu_1(x), \ldots, \nu_n(x))$ *is the outward pointing unit normal vector on* $\partial\Omega$.

Or equivalently

Theorem (Divergence theorem)

$$\int_\Omega \textit{div}\, F(x)\,dx = \int_{\partial\Omega} F(x)\cdot\nu(x)\,dS(x),$$

where $F = (F_1, \ldots, F_n)$ *is a vector field.*

4.1 Gauss-Green theorem

Reason: Gauss-Green $\Rightarrow$ Divergence.

Recall, that

$$\operatorname{div} F(x) = \sum_{j=1}^{n} \frac{\partial F_j}{\partial x_j}(x)$$

is the divergence of F.

$$\begin{aligned}\int_\Omega \operatorname{div} F(x)\,dx &= \int_\Omega \sum_{j=1}^{n} \frac{\partial F_j}{\partial x_j}(x)\,dx = \sum_{j=1}^{n} \int_\Omega \frac{\partial F_j}{\partial x_j}(x)\,dx \\ &= \sum_{j=1}^{n} \int_{\partial\Omega} F_j(x)\nu_j(x)\,dS(x) = \int_{\partial\Omega} \sum_{j=1}^{n} F_j(x)\nu_j(x)\,dS(x) \\ &= \int_{\partial\Omega} F(x)\cdot\nu(x)\,dS(x).\end{aligned}$$

4.1 Gauss-Green theorem

Reason: Gauss-Green $\Rightarrow$ Divergence.

Recall, that

$$\operatorname{div} F(x) = \sum_{j=1}^{n} \frac{\partial F_j}{\partial x_j}(x)$$

is the divergence of F.

$$\int_\Omega \operatorname{div} F(x)\,dx = \int_\Omega \sum_{j=1}^{n} \frac{\partial F_j}{\partial x_j}(x)\,dx = \sum_{j=1}^{n} \int_\Omega \frac{\partial F_j}{\partial x_j}(x)\,dx$$

$$= \sum_{j=1}^{n} \int_{\partial\Omega} F_j(x)\nu_j(x)\,dS(x) = \int_{\partial\Omega} \sum_{j=1}^{n} F_j(x)\nu_j(x)\,dS(x)$$

$$= \int_{\partial\Omega} F(x)\cdot\nu(x)\,dS(x).$$

4.1 Gauss-Green theorem

Reason: Gauss-Green $\Rightarrow$ Divergence.

Recall, that

$$\operatorname{div} F(x) = \sum_{j=1}^{n} \frac{\partial F_j}{\partial x_j}(x)$$

is the divergence of F.

$$\begin{aligned}\int_\Omega \operatorname{div} F(x)\,dx &= \int_\Omega \sum_{j=1}^{n} \frac{\partial F_j}{\partial x_j}(x)\,dx = \sum_{j=1}^{n} \int_\Omega \frac{\partial F_j}{\partial x_j}(x)\,dx \\ &= \sum_{j=1}^{n} \int_{\partial\Omega} F_j(x)\nu_j(x)\,dS(x) = \int_{\partial\Omega} \sum_{j=1}^{n} F_j(x)\nu_j(x)\,dS(x) \\ &= \int_{\partial\Omega} F(x)\cdot\nu(x)\,dS(x).\end{aligned}$$

4.1 Gauss-Green theorem

Reason: Gauss-Green $\Rightarrow$ Divergence.

Recall, that

$$\operatorname{div} F(x) = \sum_{j=1}^{n} \frac{\partial F_j}{\partial x_j}(x)$$

is the divergence of F.

$$\begin{aligned}
\int_{\Omega} \operatorname{div} F(x)\,dx &= \int_{\Omega} \sum_{j=1}^{n} \frac{\partial F_j}{\partial x_j}(x)\,dx = \sum_{j=1}^{n} \int_{\Omega} \frac{\partial F_j}{\partial x_j}(x)\,dx \\
&= \sum_{j=1}^{n} \int_{\partial\Omega} F_j(x)\nu_j(x)\,dS(x) = \int_{\partial\Omega} \sum_{j=1}^{n} F_j(x)\nu_j(x)\,dS(x) \\
&= \int_{\partial\Omega} F(x)\cdot\nu(x)\,dS(x).
\end{aligned}$$

$\square$

4.2 Gauss-Green theorem

- The Gauss-Green theorem gives information about the divergence of a vector field inside the domain by its values on the boundary of the domain.
- More precisely, the integral of the divergence of a vector field over a domain is equal to the total flow through the boundary.
- This is useful in boundary value problems for PDEs.

Theorem (Integration by parts)

Assume that $u, v \in C^1(\overline{\Omega})$. *Then for* $j = 1, \ldots, n$,

$$\int_\Omega \frac{\partial u}{\partial x_j}(x)v(x)\,dx = -\int_\Omega \frac{\partial v}{\partial x_j}(x)u(x)\,dx + \int_{\partial\Omega} u(x)v(x)\nu_j(x)\,dS(x).$$

4.2 Gauss-Green theorem

- The Gauss-Green theorem gives information about the divergence of a vector field inside the domain by its values on the boundary of the domain.
- More precisely, the integral of the divergence of a vector field over a domain is equal to the total flow through the boundary.
- This is useful in boundary value problems for PDEs.

Theorem (Integration by parts)

Assume that $u, v \in C^1(\overline{\Omega})$. *Then for* $j = 1, \ldots, n$,

$$\int_\Omega \frac{\partial u}{\partial x_j}(x)v(x)\,dx = -\int_\Omega \frac{\partial v}{\partial x_j}(x)u(x)\,dx + \int_{\partial\Omega} u(x)v(x)\nu_j(x)\,dS(x).$$

4.2 Gauss-Green theorem

- The Gauss-Green theorem gives information about the divergence of a vector field inside the domain by its values on the boundary of the domain.
- More precisely, the integral of the divergence of a vector field over a domain is equal to the total flow through the boundary.
- This is useful in boundary value problems for PDEs.

Theorem (Integration by parts)

Assume that $u, v \in C^1(\overline{\Omega})$. *Then for* $j = 1, \dots, n$,

$$\int_\Omega \frac{\partial u}{\partial x_j}(x)v(x)\,dx = -\int_\Omega \frac{\partial v}{\partial x_j}(x)u(x)\,dx + \int_{\partial\Omega} u(x)v(x)\nu_j(x)\,dS(x).$$

4.2 Gauss-Green theorem

- The Gauss-Green theorem gives information about the divergence of a vector field inside the domain by its values on the boundary of the domain.
- More precisely, the integral of the divergence of a vector field over a domain is equal to the total flow through the boundary.
- This is useful in boundary value problems for PDEs.

Theorem (Integration by parts)

Assume that $u, v \in C^1(\overline{\Omega})$. *Then for* $j = 1, \ldots, n$,

$$\int_\Omega \frac{\partial u}{\partial x_j}(x) v(x)\, dx = -\int_\Omega \frac{\partial v}{\partial x_j}(x) u(x)\, dx + \int_{\partial\Omega} u(x) v(x) \nu_j(x)\, dS(x).$$

4.1 Green's identities

Theorem

1. Green's first identity:

$$\int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx = -\int_\Omega u(x)\Delta v(x)\, dx + \int_{\partial\Omega} \frac{\partial v}{\partial \nu}(x) u(x)\, dS(x),$$

2. Green's second identity:

$$\int_\Omega (u(x)\Delta v(x) - v(x)\Delta u(x))\, dx = \int_{\partial\Omega} \Big(u(x)\frac{\partial v}{\partial \nu}(x) - v(x)\frac{\partial u}{\partial \nu}(x)\Big)\, dS(x),$$

3. Green's third identity:

$$\int_\Omega \Delta u(x)\, dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu}(x)\, dS(x).$$

Check the proofs from the lecture notes. (1) follows from the integration by parts formula, (2) and (3) follows from (1).

4.1 Green's identities

Theorem

1. Green's first identity:

$$\int_{\Omega} \nabla u(x) \cdot \nabla v(x)\, dx = -\int_{\Omega} u(x)\Delta v(x)\, dx + \int_{\partial\Omega} \frac{\partial v}{\partial \nu}(x) u(x)\, dS(x),$$

2. Green's second identity:

$$\int_{\Omega} (u(x)\Delta v(x) - v(x)\Delta u(x))\, dx = \int_{\partial\Omega} \left(u(x)\frac{\partial v}{\partial \nu}(x) - v(x)\frac{\partial u}{\partial \nu}(x)\right) dS(x),$$

3. Green's third identity:

$$\int_{\Omega} \Delta u(x)\, dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu}(x)\, dS(x).$$

Check the proofs from the lecture notes. (1) follows from the integration by parts formula, (2) and (3) follows from (1).

4.1 Applications of Green's identities to harmonic functions (1/2)

- Suppose u is harmonic and apply Green's first identity

$$\int_\Omega \nabla u(x)\cdot\nabla v(x)\,dx = -\int_\Omega u(x)\Delta v(x)\,dx + \int_{\partial\Omega} \frac{\partial v}{\partial \nu}(x)u(x)\,dS(x),$$

with $v = u$.

- We obtain

$$\begin{aligned} 0 \le \int_\Omega |\nabla u(x)|^2\,dx &= \int_\Omega \nabla u(x)\cdot\nabla u(x)\,dx \\ &= \int_{\partial\Omega} \frac{\partial u}{\partial \nu}(x)u(x)\,dS(x) = \frac{1}{2}\int_{\partial\Omega} \frac{\partial u^2}{\partial \nu}(x)\,dS(x) \end{aligned}$$

4.1 Applications of Green's identities to harmonic functions (1/2)

- Suppose u is harmonic and apply Green's first identity

$$\int_\Omega \nabla u(x)\cdot\nabla v(x)\,dx = -\int_\Omega u(x)\Delta v(x)\,dx+\int_{\partial\Omega}\frac{\partial v}{\partial\nu}(x)u(x)\,dS(x),$$

  with $v = u$.

- We obtain

$$\begin{aligned} 0 \le \int_\Omega |\nabla u(x)|^2\,dx &= \int_\Omega \nabla u(x)\cdot\nabla u(x)\,dx \\ &= \int_{\partial\Omega}\frac{\partial u}{\partial\nu}(x)u(x)\,dS(x) = \frac{1}{2}\int_{\partial\Omega}\frac{\partial u^2}{\partial\nu}(x)\,dS(x) \end{aligned}$$

4.1 Applications of Green's identities to harmonic functions (1/2)

- Suppose u is harmonic and apply Green's first identity

$$\int_\Omega \nabla u(x)\cdot\nabla v(x)\,dx = -\int_\Omega u(x)\Delta v(x)\,dx + \int_{\partial\Omega} \frac{\partial v}{\partial \nu}(x)u(x)\,dS(x),$$

  with $v = u$.
- We obtain

$$\begin{aligned} 0 \le \int_\Omega |\nabla u(x)|^2\,dx &= \int_\Omega \nabla u(x)\cdot\nabla u(x)\,dx \\ &= \int_{\partial\Omega} \frac{\partial u}{\partial \nu}(x)u(x)\,dS(x) = \frac{1}{2}\int_{\partial\Omega} \frac{\partial u^2}{\partial \nu}(x)\,dS(x) \end{aligned}$$

4.1 Applications of Green's identities to harmonic functions (2/2)

- Consider Green's third identity:

$$\int_\Omega \Delta u(x)\,dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu}(x)\,dS(x).$$

- It tells that the integral of the Laplacian is equal to the total flow through the boundary.
- If u is harmonic in Ω, then

$$\int_{\partial V} \frac{\partial u}{\partial \nu}(x)\,dS(x) = 0$$

for every subdomain V.
- This means that the total flow is zero through the boundary of any subdomain V.
- Physically this means that there are not heat sources or electric charges in the domain.

4.1 Applications of Green's identities to harmonic functions (2/2)

- Consider Green's third identity:

$$\int_\Omega \Delta u(x)\,dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu}(x)\,dS(x).$$

- It tells that the integral of the Laplacian is equal to the total flow through the boundary.
- If u is harmonic in Ω, then

$$\int_{\partial V} \frac{\partial u}{\partial \nu}(x)\,dS(x) = 0$$

  for every subdomain V.
- This means that the total flow is zero through the boundary of any subdomain V.
- Physically this means that there are not heat sources or electric charges in the domain.

4.1 Applications of Green's identities to harmonic functions (2/2)

- Consider Green's third identity:

$$\int_{\Omega} \Delta u(x)\, dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu}(x)\, dS(x).$$

- It tells that the integral of the Laplacian is equal to the total flow through the boundary.
- If u is harmonic in Ω, then

$$\int_{\partial V} \frac{\partial u}{\partial \nu}(x)\, dS(x) = 0$$

  for every subdomain V.
- This means that the total flow is zero through the boundary of any subdomain V.
- Physically this means that there are not heat sources or electric charges in the domain.

4.1 Applications of Green's identities to harmonic functions (2/2)

- Consider Green's third identity:

$$\int_{\Omega} \Delta u(x)\,dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu}(x)\,dS(x).$$

- It tells that the integral of the Laplacian is equal to the total flow through the boundary.
- If u is harmonic in Ω, then

$$\int_{\partial V} \frac{\partial u}{\partial \nu}(x)\,dS(x) = 0$$

  for every subdomain V.
- This means that the total flow is zero through the boundary of any subdomain V.
- Physically this means that there are not heat sources or electric charges in the domain.

4.1 Applications of Green's identities to harmonic functions (2/2)

- Consider Green's third identity:

$$\int_{\Omega} \Delta u(x)\, dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu}(x)\, dS(x).$$

- It tells that the integral of the Laplacian is equal to the total flow through the boundary.
- If u is harmonic in Ω, then

$$\int_{\partial V} \frac{\partial u}{\partial \nu}(x)\, dS(x) = 0$$

  for every subdomain V.
- This means that the total flow is zero through the boundary of any subdomain V.
- Physically this means that there are not heat sources or electric charges in the domain.

4.2 PDEs and physics

- In a typical case, u is a function that denotes the density of some quantity in steady state.
- Examples: temperature, chemical concentration or electrostatic potential.
- The total flow through the boundary ∂V is zero

$$\int_{\partial V} F(x) \cdot \nu(x)\, dS(x) = 0,$$

where $F = (F_1, \ldots, F)$ is the flux density and ν is the unit outer normal of ∂V.
- By the Gauss-Green theorem we have

$$\int_V \operatorname{div} F(x)\, dx = \int_{\partial V} F(x) \cdot \nu(x)\, dS(x) = 0.$$

- Since this holds for every subdomain V of Ω, we have

$$\operatorname{div} F(x) = 0 \quad \text{for every} \quad x \in \Omega.$$

4.2 PDEs and physics

- In a typical case, u is a function that denotes the density of some quantity in steady state.
- Examples: temperature, chemical concentration or electrostatic potential.
- The total flow through the boundary ∂V is zero

$$\int_{\partial V} F(x) \cdot \nu(x)\, dS(x) = 0,$$

where $F = (F_1, \dots, F)$ is the flux density and ν is the unit outer normal of ∂V.
- By the Gauss-Green theorem we have

$$\int_V \operatorname{div} F(x)\, dx = \int_{\partial V} F(x) \cdot \nu(x)\, dS(x) = 0.$$

- Since this holds for every subdomain V of Ω, we have

$$\operatorname{div} F(x) = 0 \quad \text{for every} \quad x \in \Omega.$$

4.2 PDEs and physics

- In a typical case, u is a function that denotes the density of some quantity in steady state.
- Examples: temperature, chemical concentration or electrostatic potential.
- The total flow through the boundary ∂V is zero

$$\int_{\partial V} F(x) \cdot \nu(x)\, dS(x) = 0,$$

where $F = (F_1, \dots, F)$ is the flux density and ν is the unit outer normal of ∂V.
- By the Gauss-Green theorem we have

$$\int_V \operatorname{div} F(x)\, dx = \int_{\partial V} F(x) \cdot \nu(x)\, dS(x) = 0.$$

- Since this holds for every subdomain V of Ω, we have

$$\operatorname{div} F(x) = 0 \quad \text{for every} \quad x \in \Omega.$$

4.2 PDEs and physics

- In a typical case, u is a function that denotes the density of some quantity in steady state.
- Examples: temperature, chemical concentration or electrostatic potential.
- The total flow through the boundary ∂V is zero

$$\int_{\partial V} F(x) \cdot \nu(x) \, dS(x) = 0,$$

where $F = (F_1, \dots, F)$ is the flux density and ν is the unit outer normal of ∂V.
- By the Gauss-Green theorem we have

$$\int_V \operatorname{div} F(x) \, dx = \int_{\partial V} F(x) \cdot \nu(x) \, dS(x) = 0.$$

- Since this holds for every subdomain V of Ω, we have

$$\operatorname{div} F(x) = 0 \quad \text{for every} \quad x \in \Omega.$$

4.2 PDEs and physics

- In a typical case, u is a function that denotes the density of some quantity in steady state.
- Examples: temperature, chemical concentration or electrostatic potential.
- The total flow through the boundary ∂V is zero

$$\int_{\partial V} F(x) \cdot \nu(x)\, dS(x) = 0,$$

where $F = (F_1, \dots, F)$ is the flux density and ν is the unit outer normal of ∂V.
- By the Gauss-Green theorem we have

$$\int_V \operatorname{div} F(x)\, dx = \int_{\partial V} F(x) \cdot \nu(x)\, dS(x) = 0.$$

- Since this holds for every subdomain V of Ω, we have

$$\operatorname{div} F(x) = 0 \quad \text{for every} \quad x \in \Omega.$$

4.2 PDEs and physics

- It is physically reasonable to assume that the flux F is proportional to the gradient ∇u but in the opposite direction, since the flow is from regions of high temperature to regions of low temperature or high concentration to low concentration. Thus

$$F(x) = -a\nabla u(x), \quad a > 0.$$

- This gives

$$\operatorname{div} F(x) = -a \operatorname{div} \nabla u(x) = -a\Delta u(x) = 0 \quad \text{for every} \quad x \in \Omega,$$

which implies $\Delta u = 0$ in Ω.

4.2 PDEs and physics

- It is physically reasonable to assume that the flux F is proportional to the gradient ∇u but in the opposite direction, since the flow is from regions of high temperature to regions of low temperature or high concentration to low concentration. Thus

$$F(x) = -a\nabla u(x), \quad a > 0.$$

- This gives

$$\operatorname{div} F(x) = -a \operatorname{div} \nabla u(x) = -a\Delta u(x) = 0 \quad \text{for every} \quad x \in \Omega,$$

which implies $\Delta u = 0$ in Ω.

4.3 Boundary values and physics

- We consider two types of boundary conditions:
- **Dirichlet problem**

$$\begin{cases} \Delta u = 0 \quad \text{in} \quad \Omega, \\ u = g \quad \text{on} \quad \partial\Omega. \end{cases}$$

- Temperature: Boundary values g describe e.g. the temperature distribution on $\partial\Omega$.
- Electrostatistics: g specifies the values of the potential u on $\partial\Omega$, which induces the electric field $E = -\nabla u$ in Ω.

4.3 Boundary values and physics

- We consider two types of boundary conditions:
- **Dirichlet problem**

$$\begin{cases} \Delta u = 0 \quad \text{in} \quad \Omega, \\ u = g \quad \text{on} \quad \partial\Omega. \end{cases}$$

- Temperature: Boundary values g describe e.g. the temperature distribution on $\partial\Omega$.
- Electrostatistics: g specifies the values of the potential u on $\partial\Omega$, which induces the electric field $E = -\nabla u$ in Ω.

4.3 Boundary values and physics

- We consider two types of boundary conditions:
- **Dirichlet problem**

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ u = g & \text{on} \quad \partial\Omega. \end{cases}$$

- Temperature: Boundary values g describe e.g. the temperature distribution on $\partial\Omega$.
- Electrostatistics: g specifies the values of the potential u on $\partial\Omega$, which induces the electric field $E = -\nabla u$ in Ω.

4.3 Boundary values and physics

- We consider two types of boundary conditions:
- **Dirichlet problem**

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ u = g & \text{on} \quad \partial\Omega. \end{cases}$$

- Temperature: Boundary values g describe e.g. the temperature distribution on $\partial\Omega$.
- Electrostatistics: g specifies the values of the potential u on $\partial\Omega$, which induces the electric field $E = -\nabla u$ in Ω.

4.3 Boundary values and physics

- **Neumann problem**

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \nu} = h & \text{on} \quad \partial\Omega. \end{cases}$$

- Physically the Neumann problem describes the steady state temperature distribution in Ω when the heat flow through $\partial\Omega$ is given by the normal derivative $\frac{\partial u}{\partial \nu} = h$.
- For example, if the surface of the body $\partial\Omega$ is insulated, the function h in the Neumann boundary condition is zero.

4.3 Boundary values and physics

- **Neumann problem**

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \nu} = h & \text{on} \quad \partial\Omega. \end{cases}$$

- Physically the Neumann problem describes the steady state temperature distribution in Ω when the heat flow through $\partial\Omega$ is given by the normal derivative $\frac{\partial u}{\partial \nu} = h$.
- For example, if the surface of the body $\partial\Omega$ is insulated, the function h in the Neumann boundary condition is zero.

4.3 Boundary values and physics

- **Neumann problem**

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \nu} = h & \text{on} \quad \partial\Omega. \end{cases}$$

- Physically the Neumann problem describes the steady state temperature distribution in Ω when the heat flow through $\partial\Omega$ is given by the normal derivative $\frac{\partial u}{\partial \nu} = h$.
- For example, if the surface of the body $\partial\Omega$ is insulated, the function h in the Neumann boundary condition is zero.

4.3 Uniqueness of solutions for Dirichlet problem

- Step 1: If u is harmonic in Ω and $u = 0$ on $\partial\Omega$, then $u = 0$ in Ω.
- Proof: By Green's first identity

$$\int_\Omega |\nabla u(x)|^2\,dx = \int_\Omega \nabla u(x)\cdot\nabla u(x)\,dx$$
$$= -\int_\Omega u(x)\underbrace{\Delta u(x)}_{=0}\,dx + \int_{\partial\Omega}\frac{\partial u}{\partial\nu}(x)\underbrace{u(x)}_{=0}\,dS(x) = 0.$$

- This implies that $|\nabla u(x)| = 0$ and thus $u(x) = c$.
- Boundary condition implies that $c = 0$.
- **Step 2**: If u and v are harmonic (or satisfy Poisson equation with same f) and have the same boundary values g, then $w = u - v$ is a harmonic function with zero boundary values and thus $w = 0$ in Ω.

4.3 Uniqueness of solutions for Dirichlet problem

- Step 1: If u is harmonic in Ω and $u = 0$ on $\partial\Omega$, then $u = 0$ in Ω.
- Proof: By Green's first identity

$$\begin{aligned}\int_\Omega |\nabla u(x)|^2\,dx &= \int_\Omega \nabla u(x)\cdot\nabla u(x)\,dx \\ &= -\int_\Omega u(x)\underbrace{\Delta u(x)}_{=0}\,dx + \int_{\partial\Omega}\frac{\partial u}{\partial \nu}(x)\underbrace{u(x)}_{=0}\,dS(x) = 0.\end{aligned}$$

- This implies that $|\nabla u(x)| = 0$ and thus $u(x) = c$.
- Boundary condition implies that $c = 0$.
- **Step 2**: If u and v are harmonic (or satisfy Poisson equation with same f) and have the same boundary values g, then $w = u - v$ is a harmonic function with zero boundary values and thus $w = 0$ in Ω.

4.3 Uniqueness of solutions for Dirichlet problem

- Step 1: If u is harmonic in Ω and $u = 0$ on $\partial\Omega$, then $u = 0$ in Ω.
- Proof: By Green's first identity

$$\int_\Omega |\nabla u(x)|^2\,dx = \int_\Omega \nabla u(x)\cdot\nabla u(x)\,dx$$
$$= -\int_\Omega u(x)\underbrace{\Delta u(x)}_{=0}\,dx + \int_{\partial\Omega}\frac{\partial u}{\partial\nu}(x)\underbrace{u(x)}_{=0}\,dS(x) = 0.$$

- This implies that $|\nabla u(x)| = 0$ and thus $u(x) = c$.
- Boundary condition implies that $c = 0$.
- **Step 2**: If u and v are harmonic (or satisfy Poisson equation with same f) and have the same boundary values g, then $w = u - v$ is a harmonic function with zero boundary values and thus $w = 0$ in Ω.

4.3 Uniqueness of solutions for Dirichlet problem

- Step 1: If u is harmonic in Ω and $u = 0$ on $\partial\Omega$, then $u = 0$ in Ω.
- Proof: By Green's first identity

$$\int_\Omega |\nabla u(x)|^2\,dx = \int_\Omega \nabla u(x)\cdot\nabla u(x)\,dx$$
$$= -\int_\Omega u(x)\underbrace{\Delta u(x)}_{=0}\,dx + \int_{\partial\Omega}\frac{\partial u}{\partial\nu}(x)\underbrace{u(x)}_{=0}\,dS(x) = 0.$$

- This implies that $|\nabla u(x)| = 0$ and thus $u(x) = c$.
- Boundary condition implies that $c = 0$.
- **Step 2**: If u and v are harmonic (or satisfy Poisson equation with same f) and have the same boundary values g, then $w = u - v$ is a harmonic function with zero boundary values and thus $w = 0$ in Ω.

4.3 Uniqueness of solutions for Dirichlet problem

- Step 1: If u is harmonic in Ω and $u = 0$ on $\partial\Omega$, then $u = 0$ in Ω.
- Proof: By Green's first identity

$$\begin{aligned}\int_\Omega |\nabla u(x)|^2\,dx &= \int_\Omega \nabla u(x)\cdot\nabla u(x)\,dx \\ &= -\int_\Omega u(x)\underbrace{\Delta u(x)}_{=0}\,dx + \int_{\partial\Omega}\frac{\partial u}{\partial\nu}(x)\underbrace{u(x)}_{=0}\,dS(x) = 0.\end{aligned}$$

- This implies that $|\nabla u(x)| = 0$ and thus $u(x) = c$.
- Boundary condition implies that $c = 0$.
- **Step 2**: If u and v are harmonic (or satisfy Poisson equation with same f) and have the same boundary values g, then $w = u - v$ is a harmonic function with zero boundary values and thus $w = 0$ in Ω.

4.3 Uniqueness of solutions for Neumann problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \nu} = h & \text{on} \quad \partial\Omega. \end{cases}$$

- If u is a solution to the Neumann problem, then also $u(x) + c$ is a solution. $\Rightarrow$ **NOT UNIQUE**
- **BUT**: If u is a harmonic function in Ω and $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, then $u = c$ in Ω. Proof: By Green's 1st identity

$$\begin{aligned} \int_\Omega |\nabla u(x)|^2 \, dx &= \int_\Omega \nabla u(x) \cdot \nabla u(x) \, dx \\ &= -\int_\Omega u(x) \underbrace{\Delta u(x)}_{=0} \, dx + \int_{\partial\Omega} \underbrace{\frac{\partial u}{\partial \nu}(x)}_{=0} u(x) \, dS(x) = 0. \end{aligned}$$

- Thus $u = c$. But if Ω is not connected, then c might be different in each component of Ω.

4.3 Uniqueness of solutions for Neumann problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \nu} = h & \text{on} \quad \partial\Omega. \end{cases}$$

- If u is a solution to the Neumann problem, then also $u(x) + c$ is a solution. $\Rightarrow$ **NOT UNIQUE**
- **BUT**: If u is a harmonic function in Ω and $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, then $u = c$ in Ω. Proof: By Green's 1st identity

$$\int_\Omega |\nabla u(x)|^2 \, dx = \int_\Omega \nabla u(x) \cdot \nabla u(x) \, dx$$
$$= -\int_\Omega u(x) \underbrace{\Delta u(x)}_{=0} \, dx + \int_{\partial\Omega} \underbrace{\frac{\partial u}{\partial \nu}(x)}_{=0} u(x) \, dS(x) = 0.$$

- Thus $u = c$. But if Ω is not connected, then c might be different in each component of Ω.

4.3 Uniqueness of solutions for Neumann problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \nu} = h & \text{on} \quad \partial\Omega. \end{cases}$$

- If u is a solution to the Neumann problem, then also $u(x) + c$ is a solution. $\Rightarrow$ **NOT UNIQUE**
- **BUT**: If u is a harmonic function in Ω and $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, then $u = c$ in Ω. Proof: By Green's 1st identity

$$\begin{aligned} \int_\Omega |\nabla u(x)|^2 \, dx &= \int_\Omega \nabla u(x) \cdot \nabla u(x) \, dx \\ &= -\int_\Omega u(x) \underbrace{\Delta u(x)}_{=0} \, dx + \int_{\partial\Omega} \underbrace{\frac{\partial u}{\partial \nu}(x)}_{=0} u(x) \, dS(x) = 0. \end{aligned}$$

- Thus $u = c$. But if Ω is not connected, then c might be different in each component of Ω.

4.3 Uniqueness of solutions for Neumann problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \nu} = h & \text{on} \quad \partial\Omega. \end{cases}$$

- If u is a solution to the Neumann problem, then also $u(x) + c$ is a solution. $\Rightarrow$ **NOT UNIQUE**
- **BUT**: If u is a harmonic function in Ω and $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, then $u = c$ in Ω. Proof: By Green's 1st identity

$$\int_\Omega |\nabla u(x)|^2 \, dx = \int_\Omega \nabla u(x) \cdot \nabla u(x) \, dx$$

$$= -\int_\Omega u(x) \underbrace{\Delta u(x)}_{=0} \, dx + \int_{\partial\Omega} \underbrace{\frac{\partial u}{\partial \nu}(x)}_{=0} u(x) \, dS(x) = 0.$$

- Thus $u = c$. But if Ω is not connected, then c might be different in each component of Ω.

4.3 Uniqueness of solutions for Neumann problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \nu} = h & \text{on} \quad \partial\Omega. \end{cases}$$

- If u is a solution to the Neumann problem, then also $u(x) + c$ is a solution. $\Rightarrow$ **NOT UNIQUE**
- **BUT**: If u is a harmonic function in Ω and $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, then $u = c$ in Ω. Proof: By Green's 1st identity

$$\begin{aligned} \int_\Omega |\nabla u(x)|^2 \, dx &= \int_\Omega \nabla u(x) \cdot \nabla u(x) \, dx \\ &= -\int_\Omega u(x) \underbrace{\Delta u(x)}_{=0} \, dx + \int_{\partial\Omega} \underbrace{\frac{\partial u}{\partial \nu}(x)}_{=0} u(x) \, dS(x) = 0. \end{aligned}$$

- Thus $u = c$. But if Ω is not connected, then c might be different in each component of Ω.

4.3 Uniqueness of solutions for Neumann problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \nu} = h & \text{on} \quad \partial\Omega. \end{cases}$$

- If u is a solution to the Neumann problem, then also $u(x) + c$ is a solution. $\Rightarrow$ **NOT UNIQUE**
- **BUT**: If u is a harmonic function in Ω and $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, then $u = c$ in Ω. Proof: By Green's 1st identity

$$\begin{aligned} \int_\Omega |\nabla u(x)|^2 \, dx &= \int_\Omega \nabla u(x) \cdot \nabla u(x) \, dx \\ &= -\int_\Omega u(x) \underbrace{\Delta u(x)}_{=0} \, dx + \int_{\partial\Omega} \underbrace{\frac{\partial u}{\partial \nu}(x)}_{=0} u(x) \, dS(x) = 0. \end{aligned}$$

- Thus $u = c$. But if Ω is not connected, then c might be different in each component of Ω.

4.3 Compatibility condition for Neumann problems

- Green's third identity gives the following compatibility condition of the Neumann problem

$$0 = \int_\Omega \underbrace{\Delta u}_{=0} \, dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \, dS = \int_{\partial\Omega} h \, dS.$$

- Thus if the Neumann boundary condition is given by a function h such that

$$\int_{\partial\Omega} h \, dS \neq 0,$$

then there does not exist any solutions.

- Physically it means that the total heat flow though the boundary has to be zero.

4.3 Compatibility condition for Neumann problems

- Green's third identity gives the following compatibility condition of the Neumann problem

$$0 = \int_\Omega \underbrace{\Delta u}_{=0} \, dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \, dS = \int_{\partial\Omega} h \, dS.$$

- Thus if the Neumann boundary condition is given by a function h such that

$$\int_{\partial\Omega} h \, dS \neq 0,$$

then there does not exist any solutions.

- Physically it means that the total heat flow though the boundary has to be zero.

4.3 Compatibility condition for Neumann problems

- Green's third identity gives the following compatibility condition of the Neumann problem

$$0 = \int_\Omega \underbrace{\Delta u}_{=0} \, dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \, dS = \int_{\partial\Omega} h \, dS.$$

- Thus if the Neumann boundary condition is given by a function h such that

$$\int_{\partial\Omega} h \, dS \neq 0,$$

then there does not exist any solutions.

- Physically it means that the total heat flow though the boundary has to be zero.

4.3 Dirichlet and Neumann problems for Poisson equation

- We discuss Dirichlet and Neumann problems for the Poisson equation $-\Delta u = f$, but it is enough to consider boundary value problems, where either the equation is homogeneous ($\Delta u = 0$) or the boundary condition is homogeneous ($g = 0$ or $h = 0$).
- For example, to solve

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega, \\ u = g & \text{on} \quad \partial\Omega, \end{cases}$$

we may write $u = u_1 + u_2$ with

$$\begin{cases} -\Delta u_1 = f & \text{in} \quad \Omega, \\ u_1 = 0 & \text{on} \quad \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta u_2 = 0 & \text{in} \quad \Omega, \\ u_2 = g & \text{on} \quad \partial\Omega. \end{cases}$$

- Then $u = u_1 + u_2$ is the solution to the original problem.

4.3 Dirichlet and Neumann problems for Poisson equation

- We discuss Dirichlet and Neumann problems for the Poisson equation $-\Delta u = f$, but it is enough to consider boundary value problems, where either the equation is homogeneous ($\Delta u = 0$) or the boundary condition is homogeneous ($g = 0$ or $h = 0$).
- For example, to solve

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega, \\ u = g & \text{on} \quad \partial\Omega, \end{cases}$$

we may write $u = u_1 + u_2$ with

$$\begin{cases} -\Delta u_1 = f & \text{in} \quad \Omega, \\ u_1 = 0 & \text{on} \quad \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta u_2 = 0 & \text{in} \quad \Omega, \\ u_2 = g & \text{on} \quad \partial\Omega. \end{cases}$$

- Then $u = u_1 + u_2$ is the solution to the original problem.

4.3 Dirichlet and Neumann problems for Poisson equation

- We discuss Dirichlet and Neumann problems for the Poisson equation $-\Delta u = f$, but it is enough to consider boundary value problems, where either the equation is homogeneous ($\Delta u = 0$) or the boundary condition is homogeneous ($g = 0$ or $h = 0$).
- For example, to solve

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega, \\ u = g & \text{on} \quad \partial\Omega, \end{cases}$$

we may write $u = u_1 + u_2$ with

$$\begin{cases} -\Delta u_1 = f & \text{in} \quad \Omega, \\ u_1 = 0 & \text{on} \quad \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta u_2 = 0 & \text{in} \quad \Omega, \\ u_2 = g & \text{on} \quad \partial\Omega. \end{cases}$$

- Then $u = u_1 + u_2$ is the solution to the original problem.

4.4 Fundamental solution

- Fundamental solution u is a solution to a (linear) partial differential equation L is a function that satisfies:

$$Lu = \delta,$$

where δ is the Dirac delta function, which has a unit mass at 0.

- Now we are interested in the fundamental solution of the Laplace equation in the whole $\mathbb{R}^n$.
- As the equation is linear, any linear combination, or integral, of fundamental solution will be a solution to the Laplace (Poisson) equation as well.
- We will be able to represent all other solutions as integrals (or convolutions) with the fundamental solution.
- We look for a radial solution that has a singularity at the origin.

$$u(x) = v(|x|) = v(r(x)).$$

4.4 Fundamental solution

- Fundamental solution u is a solution to a (linear) partial differential equation L is a function that satisfies:

$$Lu = \delta,$$

where δ is the Dirac delta function, which has a unit mass at 0.

- Now we are interested in the fundamental solution of the Laplace equation in the whole $\mathbb{R}^n$.
- As the equation is linear, any linear combination, or integral, of fundamental solution will be a solution to the Laplace (Poisson) equation as well.
- We will be able to represent all other solutions as integrals (or convolutions) with the fundamental solution.
- We look for a radial solution that has a singularity at the origin.

$$u(x) = v(|x|) = v(r(x)).$$

4.4 Fundamental solution

- Fundamental solution u is a solution to a (linear) partial differential equation L is a function that satisfies:

$$Lu = \delta,$$

where δ is the Dirac delta function, which has a unit mass at 0.

- Now we are interested in the fundamental solution of the Laplace equation in the whole $\mathbb{R}^n$.
- As the equation is linear, any linear combination, or integral, of fundamental solution will be a solution to the Laplace (Poisson) equation as well.
- We will be able to represent all other solutions as integrals (or convolutions) with the fundamental solution.
- We look for a radial solution that has a singularity at the origin.

$$u(x) = v(|x|) = v(r(x)).$$

4.4 Fundamental solution

- Fundamental solution u is a solution to a (linear) partial differential equation L is a function that satisfies:

$$Lu = \delta,$$

  where δ is the Dirac delta function, which has a unit mass at 0.
- Now we are interested in the fundamental solution of the Laplace equation in the whole $\mathbb{R}^n$.
- As the equation is linear, any linear combination, or integral, of fundamental solution will be a solution to the Laplace (Poisson) equation as well.
- We will be able to represent all other solutions as integrals (or convolutions) with the fundamental solution.
- We look for a radial solution that has a singularity at the origin.

$$u(x) = v(|x|) = v(r(x)).$$

4.4 Fundamental solution

- Fundamental solution u is a solution to a (linear) partial differential equation L is a function that satisfies:

$$Lu = \delta,$$

  where δ is the Dirac delta function, which has a unit mass at 0.
- Now we are interested in the fundamental solution of the Laplace equation in the whole $\mathbb{R}^n$.
- As the equation is linear, any linear combination, or integral, of fundamental solution will be a solution to the Laplace (Poisson) equation as well.
- We will be able to represent all other solutions as integrals (or convolutions) with the fundamental solution.
- We look for a radial solution that has a singularity at the origin.

$$u(x) = v(|x|) = v(r(x)).$$

4.4 Fundamental solution

- **Step 1**: Find the Laplace equation for the radial functions using the chain rule.
- For a radial function

$$\Delta u(x) = 0, \quad x \neq 0 \quad \iff \quad v''(r) + \frac{n-1}{r} v'(r) = 0, \quad r > 0.$$

- Note: This is an ODE as it is allowed to depend only on one variable r.
- Solving this, we obtain

$$v(r) = \begin{cases} a \ln r + b, & n = 2, \\ \frac{c}{r^{n-2}} + d, & n \geq 3, \end{cases}$$

where a, b, c, d are constants.

4.4 Fundamental solution

- **Step 1**: Find the Laplace equation for the radial functions using the chain rule.
- For a radial function

$$\Delta u(x) = 0, \quad x \neq 0 \quad \Longleftrightarrow \quad v''(r) + \frac{n-1}{r} v'(r) = 0, \quad r > 0.$$

- Note: This is an ODE as it is allowed to depend only on one variable r.
- Solving this, we obtain

$$v(r) = \begin{cases} a \ln r + b, & n = 2, \\ \frac{c}{r^{n-2}} + d, & n \geq 3, \end{cases}$$

where a, b, c, d are constants.

4.4 Fundamental solution

- **Step 1**: Find the Laplace equation for the radial functions using the chain rule.
- For a radial function

$$\Delta u(x) = 0, \quad x \neq 0 \quad \Longleftrightarrow \quad v''(r) + \frac{n-1}{r} v'(r) = 0, \quad r > 0.$$

- Note: This is an ODE as it is allowed to depend only on one variable r.
- Solving this, we obtain

$$v(r) = \begin{cases} a \ln r + b, & n = 2, \\ \frac{c}{r^{n-2}} + d, & n \geq 3, \end{cases}$$

where a, b, c, d are constants.

4.4 Fundamental solution

- **Step 1**: Find the Laplace equation for the radial functions using the chain rule.
- For a radial function

$$\Delta u(x) = 0, \quad x \neq 0 \quad \iff \quad v''(r) + \frac{n-1}{r} v'(r) = 0, \quad r > 0.$$

- Note: This is an ODE as it is allowed to depend only on one variable r.
- Solving this, we obtain

$$v(r) = \begin{cases} a \ln r + b, & n = 2, \\ \frac{c}{r^{n-2}} + d, & n \geq 3, \end{cases}$$

where a, b, c, d are constants.

4.4 Fundamental solution

The function $\Phi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$,

$$\Phi(x) = \begin{cases} -\dfrac{1}{2\pi} \ln |x|, & n = 2, \\ \dfrac{1}{n(n-2)\alpha(n)} \dfrac{1}{|x|^{n-2}}, & n \geq 3, \end{cases}$$

is called the fundamental solution of the Laplace equation. Here we denote the volume of the unit ball in $\mathbb{R}^n$ by $\alpha(n) = |B(0,1)|$.

- Physically the fundamental solution is the potential induced by a unit point mass at $\overline{0}$. Constants are chosen so that

$$-\int_{\partial B(0,r)} \frac{\partial \Phi}{\partial \nu}(x)\, dS(x) = 1 \quad \text{for every} \quad r > 0,$$

- Φ is harmonic in $\mathbb{R}^n \setminus \{0\}$.

4.4 Fundamental solution

The function $\Phi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$,

$$\Phi(x) = \begin{cases} -\dfrac{1}{2\pi} \ln |x|, & n = 2, \\ \dfrac{1}{n(n-2)\alpha(n)} \dfrac{1}{|x|^{n-2}}, & n \geq 3, \end{cases}$$

is called the fundamental solution of the Laplace equation. Here we denote the volume of the unit ball in $\mathbb{R}^n$ by $\alpha(n) = |B(0,1)|$.

- Physically the fundamental solution is the potential induced by a unit point mass at $\overline{0}$. Constants are chosen so that

$$-\int_{\partial B(0,r)} \frac{\partial \Phi}{\partial \nu}(x)\, dS(x) = 1 \quad \text{for every} \quad r > 0,$$

- Φ is harmonic in $\mathbb{R}^n \setminus \{0\}$.

4.4 Fundamental solution

The function $\Phi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$,

$$\Phi(x) = \begin{cases} -\dfrac{1}{2\pi} \ln |x|, & n = 2, \\ \dfrac{1}{n(n-2)\alpha(n)} \dfrac{1}{|x|^{n-2}}, & n \geq 3, \end{cases}$$

is called the fundamental solution of the Laplace equation. Here we denote the volume of the unit ball in $\mathbb{R}^n$ by $\alpha(n) = |B(0,1)|$.

- Physically the fundamental solution is the potential induced by a unit point mass at $\overline{0}$. Constants are chosen so that

$$-\int_{\partial B(0,r)} \frac{\partial \Phi}{\partial \nu}(x)\, dS(x) = 1 \quad \text{for every} \quad r > 0,$$

- Φ is harmonic in $\mathbb{R}^n \setminus \{0\}$.

4.5 The Poisson equation

Theorem

Let $f \in C_0^\infty(\mathbb{R}^n)$ and define

$$u(x) = (f * \Phi)(x) = \int_{\mathbb{R}^n} f(y)\Phi(x-y)\,dy,$$

where Φ is the fundamental solution of the Laplace equation. Then $u \in C^2(\mathbb{R}^n)$ and $-\Delta u = f$ in $\mathbb{R}^n$.

- The problem does not have a unique solution, since we can add a function v with $\Delta v = 0$.
- Physically: f describes a charge density and u is the potential of the electric field induced by f.

4.5 The Poisson equation

Theorem

Let $f \in C_0^\infty(\mathbb{R}^n)$ and define

$$u(x) = (f * \Phi)(x) = \int_{\mathbb{R}^n} f(y)\Phi(x-y)\,dy,$$

where Φ is the fundamental solution of the Laplace equation. Then $u \in C^2(\mathbb{R}^n)$ and $-\Delta u = f$ in $\mathbb{R}^n$.

- The problem does not have a unique solution, since we can add a function v with $\Delta v = 0$.
- Physically: f describes a charge density and u is the potential of the electric field induced by f.

4.5 The Poisson equation

Theorem

Let $f \in C_0^\infty(\mathbb{R}^n)$ *and define*

$$u(x) = (f * \Phi)(x) = \int_{\mathbb{R}^n} f(y)\Phi(x-y)\,dy,$$

where Φ *is the fundamental solution of the Laplace equation. Then* $u \in C^2(\mathbb{R}^n)$ *and* $-\Delta u = f$ *in* $\mathbb{R}^n$*.*

- The problem does not have a unique solution, since we can add a function v with $\Delta v = 0$.
- Physically: f describes a charge density and u is the potential of the electric field induced by f.

4.5 The Poisson equation

- The theorem gives a solution u in the whole space without a specification of the boundary values.
- Consider an open and bounded set $\Omega \subset \mathbb{R}^n$.
- Let v be a solution of the Dirichlet problem

$$\begin{cases} \Delta v = 0 & \text{in} \quad \Omega, \\ v = -u & \text{on} \quad \partial\Omega. \end{cases}$$

Then $w = u + v$ is a solution to the problem

$$\begin{cases} -\Delta w = f & \text{in} \quad \Omega, \\ w = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

- This observation will be useful later.

4.5 The Poisson equation

- The theorem gives a solution u in the whole space without a specification of the boundary values.
- Consider an open and bounded set $\Omega \subset \mathbb{R}^n$.
- Let v be a solution of the Dirichlet problem

$$\begin{cases} \Delta v = 0 & \text{in} \quad \Omega, \\ v = -u & \text{on} \quad \partial\Omega. \end{cases}$$

Then $w = u + v$ is a solution to the problem

$$\begin{cases} -\Delta w = f & \text{in} \quad \Omega, \\ w = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

- This observation will be useful later.

4.5 The Poisson equation

- The theorem gives a solution u in the whole space without a specification of the boundary values.
- Consider an open and bounded set $\Omega \subset \mathbb{R}^n$.
- Let v be a solution of the Dirichlet problem

$$\begin{cases} \Delta v = 0 \quad \text{in} \quad \Omega, \\ v = -u \quad \text{on} \quad \partial\Omega. \end{cases}$$

  Then $w = u + v$ is a solution to the problem

$$\begin{cases} -\Delta w = f \quad \text{in} \quad \Omega, \\ w = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

- This observation will be useful later.

4.5 The Poisson equation

- The theorem gives a solution u in the whole space without a specification of the boundary values.
- Consider an open and bounded set $\Omega \subset \mathbb{R}^n$.
- Let v be a solution of the Dirichlet problem

$$\begin{cases} \Delta v = 0 \quad \text{in} \quad \Omega, \\ v = -u \quad \text{on} \quad \partial\Omega. \end{cases}$$

Then $w = u + v$ is a solution to the problem

$$\begin{cases} -\Delta w = f \quad \text{in} \quad \Omega, \\ w = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

- This observation will be useful later.

MS-C1350 Partial differential equations
Chapter 4.6-4.10

Riikka Korte

Department of Mathematics and Systems Analysis
Aalto University
riikka.korte@aalto.fi

October 29, 2024

Lecture 8

Laplace / Poisson equation continues

1. Green's function
   - What is Green's function in general domains?
   - Representing solutions using Green's function.
   - Green's function on upper half-space
   - Green's function for a ball.
2. Mean value property
3. Maximum principle and stability

Green's function (4.6) (intro)

- Goal: To use fundamental solution of the Laplace equation in $\mathbb{R}^n$ to solve a Dirichlet problem of the Poisson equation in $\Omega \subset \mathbb{R}^n$.
- Fundamental solution: Usually a solution to some PDE in the whole space except for one point, where it has a singularity corresponding to a unit mass.
- Green's function: A function that is a solution in Ω, has a unit mass singularity at one point $x \in \Omega$ and has zero boundary values at the boundary $\partial\Omega$.
- We only consider bounded domains $\Omega \subset \mathbb{R}^n$, but essentially fundamental solution is the same as the Green's function for $\mathbb{R}^n$.
- The term fundamental solution is used in various contexts for solutions that can be used to construct other solutions.
- The term Green's function can be used also when using other than zero boundary values.

Green's function (4.6) (intro)

- Goal: To use fundamental solution of the Laplace equation in $\mathbb{R}^n$ to solve a Dirichlet problem of the Poisson equation in $\Omega \subset \mathbb{R}^n$.
- Fundamental solution: Usually a solution to some PDE in the whole space except for one point, where it has a singularity corresponding to a unit mass.
- Green's function: A function that is a solution in Ω, has a unit mass singularity at one point $x \in \Omega$ and has zero boundary values at the boundary $\partial\Omega$.
- We only consider bounded domains $\Omega \subset \mathbb{R}^n$, but essentially fundamental solution is the same as the Green's function for $\mathbb{R}^n$.
- The term fundamental solution is used in various contexts for solutions that can be used to construct other solutions.
- The term Green's function can be used also when using other than zero boundary values.

Green's function (4.6) (intro)

- Goal: To use fundamental solution of the Laplace equation in $\mathbb{R}^n$ to solve a Dirichlet problem of the Poisson equation in $\Omega \subset \mathbb{R}^n$.
- Fundamental solution: Usually a solution to some PDE in the whole space except for one point, where it has a singularity corresponding to a unit mass.
- Green's function: A function that is a solution in Ω, has a unit mass singularity at one point $x \in \Omega$ and has zero boundary values at the boundary $\partial\Omega$.
- We only consider bounded domains $\Omega \subset \mathbb{R}^n$, but essentially fundamental solution is the same as the Green's function for $\mathbb{R}^n$.
- The term fundamental solution is used in various contexts for solutions that can be used to construct other solutions.
- The term Green's function can be used also when using other than zero boundary values.

Green's function (4.6) (intro)

- Goal: To use fundamental solution of the Laplace equation in $\mathbb{R}^n$ to solve a Dirichlet problem of the Poisson equation in $\Omega \subset \mathbb{R}^n$.
- Fundamental solution: Usually a solution to some PDE in the whole space except for one point, where it has a singularity corresponding to a unit mass.
- Green's function: A function that is a solution in Ω, has a unit mass singularity at one point $x \in \Omega$ and has zero boundary values at the boundary $\partial\Omega$.
- We only consider bounded domains $\Omega \subset \mathbb{R}^n$, but essentially fundamental solution is the same as the Green's function for $\mathbb{R}^n$.
- The term fundamental solution is used in various contexts for solutions that can be used to construct other solutions.
- The term Green's function can be used also when using other than zero boundary values.

Green's function (4.6) (intro)

- Goal: To use fundamental solution of the Laplace equation in $\mathbb{R}^n$ to solve a Dirichlet problem of the Poisson equation in $\Omega \subset \mathbb{R}^n$.
- Fundamental solution: Usually a solution to some PDE in the whole space except for one point, where it has a singularity corresponding to a unit mass.
- Green's function: A function that is a solution in Ω, has a unit mass singularity at one point $x \in \Omega$ and has zero boundary values at the boundary $\partial\Omega$.
- We only consider bounded domains $\Omega \subset \mathbb{R}^n$, but essentially fundamental solution is the same as the Green's function for $\mathbb{R}^n$.
- The term fundamental solution is used in various contexts for solutions that can be used to construct other solutions.
- The term Green's function can be used also when using other than zero boundary values.

Green's function (4.6) (intro)

- Goal: To use fundamental solution of the Laplace equation in $\mathbb{R}^n$ to solve a Dirichlet problem of the Poisson equation in $\Omega \subset \mathbb{R}^n$.
- Fundamental solution: Usually a solution to some PDE in the whole space except for one point, where it has a singularity corresponding to a unit mass.
- Green's function: A function that is a solution in Ω, has a unit mass singularity at one point $x \in \Omega$ and has zero boundary values at the boundary $\partial\Omega$.
- We only consider bounded domains $\Omega \subset \mathbb{R}^n$, but essentially fundamental solution is the same as the Green's function for $\mathbb{R}^n$.
- The term fundamental solution is used in various contexts for solutions that can be used to construct other solutions.
- The term Green's function can be used also when using other than zero boundary values.

Green's function (4.6) (intro 2)

- Let Ω be an open and bounded domain with smooth enough boundary.
- We consider the Dirichlet problem for the Poisson equation

$$\begin{cases} -\Delta u = f \quad \text{in} \quad \Omega, \\ u = g \quad \text{on} \quad \partial\Omega. \end{cases}$$

- Our goal is to derive a general representation formula for the solution of this problem using potential functions and so-called Green's function.

Green's function (4.6) (intro 2)

- Let Ω be an open and bounded domain with smooth enough boundary.
- We consider the Dirichlet problem for the Poisson equation

$$\begin{cases} -\Delta u = f \quad \text{in} \quad \Omega, \\ u = g \quad \text{on} \quad \partial\Omega. \end{cases}$$

- Our goal is to derive a general representation formula for the solution of this problem using potential functions and so-called Green's function.

Green's function (4.6) (intro 2)

- Let Ω be an open and bounded domain with smooth enough boundary.
- We consider the Dirichlet problem for the Poisson equation

$$\begin{cases} -\Delta u = f \quad \text{in} \quad \Omega, \\ u = g \quad \text{on} \quad \partial\Omega. \end{cases}$$

- Our goal is to derive a general representation formula for the solution of this problem using potential functions and so-called Green's function.

4.6 Delta distribution

Let us write δ_x for the generalized function (distribution) that has the property

$$\int_{\mathbb{R}^n} \phi(y)\delta_x(y)\,dy = \phi(x) \quad \text{for all} \quad \phi \in C_0^\infty(\mathbb{R}^n).$$

This generalized function is called Dirac's delta mass at x.

- Fundamental solution solves

$$\Delta\Phi = \delta_0$$

- Green's function for Laplace equation in Ω solves

$$\begin{cases} \Delta_y G(x,y) = \delta_x & y \in \Omega \\ G(x,y) = 0 & y \in \partial\Omega \end{cases}$$

- Formally $u = f * \Phi$ is a solution for $-\Delta u = f$, as

$$-\Delta u(x) = -\int_{\mathbb{R}^n} \Delta\Phi(y) f(x-y)\,dy = \int_{\mathbb{R}^n} \delta_0(y) f(x-y)\,dy = f(x).$$

4.6 Delta distribution

Let us write δ_x for the generalized function (distribution) that has the property

$$\int_{\mathbb{R}^n} \phi(y)\delta_x(y)\,dy = \phi(x) \quad \text{for all} \quad \phi \in C_0^\infty(\mathbb{R}^n).$$

This generalized function is called Dirac's delta mass at x.

- Fundamental solution solves

$$\Delta\Phi = \delta_0$$

- Green's function for Laplace equation in Ω solves

$$\begin{cases} \Delta_y G(x,y) = \delta_x & y \in \Omega \\ G(x,y) = 0 & y \in \partial\Omega \end{cases}$$

- Formally $u = f * \Phi$ is a solution for $-\Delta u = f$, as

$$-\Delta u(x) = -\int_{\mathbb{R}^n} \Delta\Phi(y) f(x-y)\,dy = \int_{\mathbb{R}^n} \delta_0(y) f(x-y)\,dy = f(x).$$

4.6 Delta distribution

Let us write δ_x for the generalized function (distribution) that has the property

$$\int_{\mathbb{R}^n} \phi(y)\delta_x(y)\,dy = \phi(x) \quad \text{for all} \quad \phi \in C_0^\infty(\mathbb{R}^n).$$

This generalized function is called Dirac's delta mass at x.

- Fundamental solution solves

$$\Delta\Phi = \delta_0$$

- Green's function for Laplace equation in Ω solves

$$\begin{cases} \Delta_y G(x,y) = \delta_x & y \in \Omega \\ G(x,y) = 0 & y \in \partial\Omega \end{cases}$$

- Formally $u = f * \Phi$ is a solution for $-\Delta u = f$, as

$$-\Delta u(x) = -\int_{\mathbb{R}^n} \Delta\Phi(y) f(x-y)\,dy = \int_{\mathbb{R}^n} \delta_0(y) f(x-y)\,dy = f(x).$$

4.6 Delta distribution

Let us write δ_x for the generalized function (distribution) that has the property

$$\int_{\mathbb{R}^n} \phi(y)\delta_x(y)\,dy = \phi(x) \quad \text{for all} \quad \phi \in C_0^\infty(\mathbb{R}^n).$$

This generalized function is called Dirac's delta mass at x.

- Fundamental solution solves

$$\Delta\Phi = \delta_0$$

- Green's function for Laplace equation in Ω solves

$$\begin{cases} \Delta_y G(x,y) = \delta_x & y \in \Omega \\ G(x,y) = 0 & y \in \partial\Omega \end{cases}$$

- Formally $u = f * \Phi$ is a solution for $-\Delta u = f$, as

$$-\Delta u(x) = -\int_{\mathbb{R}^n} \Delta\Phi(y) f(x-y)\,dy = \int_{\mathbb{R}^n} \delta_0(y) f(x-y)\,dy = f(x).$$

4.6 Delta distribution

Let us write δ_x for the generalized function (distribution) that has the property

$$\int_{\mathbb{R}^n} \phi(y)\delta_x(y)\,dy = \phi(x) \quad \text{for all} \quad \phi \in C_0^\infty(\mathbb{R}^n).$$

This generalized function is called Dirac's delta mass at x.

- Fundamental solution solves

$$\Delta\Phi = \delta_0$$

- Green's function for Laplace equation in Ω solves

$$\begin{cases} \Delta_y G(x,y) = \delta_x & y \in \Omega \\ G(x,y) = 0 & y \in \partial\Omega \end{cases}$$

- Formally $u = f * \Phi$ is a solution for $-\Delta u = f$, as

$$-\Delta u(x) = -\int_{\mathbb{R}^n} \Delta\Phi(y) f(x-y)\,dy = \int_{\mathbb{R}^n} \delta_0(y) f(x-y)\,dy = f(x).$$

4.6 Delta distribution

Let us write δ_x for the generalized function (distribution) that has the property

$$\int_{\mathbb{R}^n} \phi(y)\delta_x(y)\,dy = \phi(x) \quad \text{for all} \quad \phi \in C_0^\infty(\mathbb{R}^n).$$

This generalized function is called Dirac's delta mass at x.

- Fundamental solution solves

$$\Delta\Phi = \delta_0$$

- Green's function for Laplace equation in Ω solves

$$\begin{cases} \Delta_y G(x,y) = \delta_x & y \in \Omega \\ G(x,y) = 0 & y \in \partial\Omega \end{cases}$$

- Formally $u = f * \Phi$ is a solution for $-\Delta u = f$, as

$$-\Delta u(x) = -\int_{\mathbb{R}^n} \Delta\Phi(y) f(x-y)\,dy = \int_{\mathbb{R}^n} \delta_0(y) f(x-y)\,dy = f(x).$$

4.6 Delta distribution

Let us write δ_x for the generalized function (distribution) that has the property

$$\int_{\mathbb{R}^n} \phi(y)\delta_x(y)\,dy = \phi(x) \quad \text{for all} \quad \phi \in C_0^\infty(\mathbb{R}^n).$$

This generalized function is called Dirac's delta mass at x.

- Fundamental solution solves

$$\Delta\Phi = \delta_0$$

- Green's function for Laplace equation in Ω solves

$$\begin{cases} \Delta_y G(x,y) = \delta_x & y \in \Omega \\ G(x,y) = 0 & y \in \partial\Omega \end{cases}$$

- Formally $u = f * \Phi$ is a solution for $-\Delta u = f$, as

$$-\Delta u(x) = -\int_{\mathbb{R}^n} \Delta\Phi(y) f(x-y)\,dy = \int_{\mathbb{R}^n} \delta_0(y) f(x-y)\,dy = f(x).$$

4.6 Representation formula for all functions

Theorem

Assume that $\Omega \subset \mathbb{R}^n$ is a bounded open set with a smooth boundary and $u \in C^2(\overline{\Omega})$. Then

$$u(x) = \int_{\partial\Omega} \Phi(y-x) \frac{\partial u}{\partial \nu}(y)\, dS(y) - \int_{\partial\Omega} u(y) \frac{\partial \Phi}{\partial \nu}(y-x)\, dS(y) - \int_{\Omega} \Phi(y-x) \Delta u(y)\, dy,$$

for every $x \in \Omega$. Here ν denotes, as usual, the outward pointing unit normal vector of Ω.

- This representation formula works for all smooth enough functions, not only to solutions to some PDE's
- This allows us to determine u if we know Δu in Ω as well as u and its normal derivative $\partial u/\partial \nu$ on $\partial\Omega$.
- Laplace equation: We can only "choose" u or $\partial u/\partial \nu$ on $\partial\Omega$!
- Replace Φ with Green's function G: we get a similar formula without the problematic term.

4.6 Representation formula for all functions

Theorem

Assume that $\Omega \subset \mathbb{R}^n$ is a bounded open set with a smooth boundary and $u \in C^2(\overline{\Omega})$. Then

$$u(x) = \int_{\partial\Omega} \Phi(y-x)\frac{\partial u}{\partial \nu}(y)\,dS(y) - \int_{\partial\Omega} u(y)\frac{\partial \Phi}{\partial \nu}(y-x)\,dS(y) - \int_{\Omega} \Phi(y-x)\Delta u(y)\,dy,$$

for every $x \in \Omega$. Here ν denotes, as usual, the outward pointing unit normal vector of Ω.

- This representation formula works for all smooth enough functions, not only to solutions to some PDE's
- This allows us to determine u if we know Δu in Ω as well as u and its normal derivative $\partial u/\partial \nu$ on $\partial\Omega$.
- Laplace equation: We can only "choose" u or $\partial u/\partial \nu$ on $\partial\Omega$!
- Replace Φ with Green's function G: we get a similar formula without the problematic term.

4.6 Representation formula for all functions

Theorem

Assume that $\Omega \subset \mathbb{R}^n$ is a bounded open set with a smooth boundary and $u \in C^2(\overline{\Omega})$. Then

$$u(x) = \int_{\partial\Omega} \Phi(y-x)\frac{\partial u}{\partial \nu}(y)\,dS(y) - \int_{\partial\Omega} u(y)\frac{\partial \Phi}{\partial \nu}(y-x)\,dS(y) - \int_{\Omega} \Phi(y-x)\Delta u(y)\,dy,$$

for every $x \in \Omega$. Here ν denotes, as usual, the outward pointing unit normal vector of Ω.

- This representation formula works for all smooth enough functions, not only to solutions to some PDE's
- This allows us to determine u if we know Δu in Ω as well as u and its normal derivative $\partial u/\partial \nu$ on $\partial\Omega$.
- Laplace equation: We can only "choose" u or $\partial u/\partial \nu$ on $\partial\Omega$!
- Replace Φ with Green's function G: we get a similar formula without the problematic term.

4.6 Representation formula for all functions

Theorem

Assume that $\Omega \subset \mathbb{R}^n$ is a bounded open set with a smooth boundary and $u \in C^2(\overline{\Omega})$. Then

$$u(x) = \int_{\partial\Omega} \Phi(y-x)\frac{\partial u}{\partial \nu}(y)\,dS(y) - \int_{\partial\Omega} u(y)\frac{\partial \Phi}{\partial \nu}(y-x)\,dS(y) - \int_{\Omega} \Phi(y-x)\Delta u(y)\,dy,$$

for every $x \in \Omega$. Here ν denotes, as usual, the outward pointing unit normal vector of Ω.

- This representation formula works for all smooth enough functions, not only to solutions to some PDE's
- This allows us to determine u if we know Δu in Ω as well as u and its normal derivative $\partial u/\partial \nu$ on $\partial\Omega$.
- Laplace equation: We can only "choose" u or $\partial u/\partial \nu$ on $\partial\Omega$!
- Replace Φ with Green's function G: we get a similar formula without the problematic term.

4.6 Representation formula for all functions

Theorem

Assume that $\Omega \subset \mathbb{R}^n$ *is a bounded open set with a smooth boundary and* $u \in C^2(\overline{\Omega})$*. Then*

$$u(x) = \int_{\partial\Omega} \Phi(y-x)\frac{\partial u}{\partial \nu}(y)\,dS(y) - \int_{\partial\Omega} u(y)\frac{\partial \Phi}{\partial \nu}(y-x)\,dS(y) - \int_{\Omega} \Phi(y-x)\Delta u(y)\,dy,$$

for every $x \in \Omega$*. Here* ν *denotes, as usual, the outward pointing unit normal vector of* Ω*.*

- This representation formula works for all smooth enough functions, not only to solutions to some PDE's
- This allows us to determine u if we know Δu in Ω as well as u and its normal derivative $\partial u/\partial \nu$ on $\partial\Omega$.
- Laplace equation: We can only "choose" u or $\partial u/\partial \nu$ on $\partial\Omega$!
- Replace Φ with Green's function G: we get a similar formula without the problematic term.

4.6 Applications of the representation formula

- If $u \in C_0^2(\mathbb{R}^n)$, then 2 of the 3 terms in the previous theorem vanish and we get

$$u(x) = -\int_{\mathbb{R}^m} \Phi(y-x)\Delta_y u(y)\,dy.$$

- If $\Delta u = 0$, then

$$u(x) = \int_{\partial\Omega} \Phi(y-x)\frac{\partial u}{\partial \nu}(y)\,dS(y) - \int_{\partial\Omega} u(y)\frac{\partial \Phi}{\partial \nu}(y-x)\,dS(y)$$

- If we apply the previous formula to $u = 1$, we obtain

$$-\int_{\partial\Omega} \frac{\partial \Phi}{\partial \nu}(y-x)\,dS(y) = 1$$

for every $x \in \Omega$. This is related to the normalization of the fundamental solution.

4.6 Applications of the representation formula

- If $u \in C_0^2(\mathbb{R}^n)$, then 2 of the 3 terms in the previous theorem vanish and we get

$$u(x) = -\int_{\mathbb{R}^m} \Phi(y-x)\Delta_y u(y)\,dy.$$

- If $\Delta u = 0$, then

$$u(x) = \int_{\partial\Omega} \Phi(y-x)\frac{\partial u}{\partial \nu}(y)\,dS(y) - \int_{\partial\Omega} u(y)\frac{\partial \Phi}{\partial \nu}(y-x)\,dS(y)$$

- If we apply the previous formula to $u = 1$, we obtain

$$-\int_{\partial\Omega} \frac{\partial \Phi}{\partial \nu}(y-x)\,dS(y) = 1$$

for every $x \in \Omega$. This is related to the normalization of the fundamental solution.

4.6 Applications of the representation formula

- If $u \in C_0^2(\mathbb{R}^n)$, then 2 of the 3 terms in the previous theorem vanish and we get

$$u(x) = -\int_{\mathbb{R}^m} \Phi(y-x)\Delta_y u(y)\,dy.$$

- If $\Delta u = 0$, then

$$u(x) = \int_{\partial\Omega} \Phi(y-x)\frac{\partial u}{\partial \nu}(y)\,dS(y) - \int_{\partial\Omega} u(y)\frac{\partial \Phi}{\partial \nu}(y-x)\,dS(y)$$

- If we apply the previous formula to $u = 1$, we obtain

$$-\int_{\partial\Omega} \frac{\partial \Phi}{\partial \nu}(y-x)\,dS(y) = 1$$

for every $x \in \Omega$. This is related to the normalization of the fundamental solution.

4.6 Application: Poisson equation with Dirichlet boundary data

- Let us look at the representation problem in connection with the Dirichlet problem for Poisson equation:

$$u(x) = \int_{\partial\Omega} \Phi(y-x)\frac{\partial u}{\partial \nu}(y)\,dS(y) - \int_{\partial\Omega} u(y)\frac{\partial \Phi}{\partial \nu}(y-x)\,dS(y) - \int_{\Omega} \Phi(y-x)\Delta u(y)\,dy,$$

- We require that $\Delta u = f$ in Ω and $u = g$ on $\partial\Omega$, but $\partial u/\partial\nu$ is unknown.
- We solve this problem by adding a harmonic function to the fundamental solution. For a fixed $x \in \Omega$, let $\phi^x = \phi^x(y)$ be a corrector function, which is a solution the Dirichlet problem

$$\begin{cases} \Delta_y \phi^x(y) = 0, & y \in \Omega, \\ \phi^x(y) = \Phi(y-x), & y \in \partial\Omega. \end{cases}$$

- Interpretation: we solve the Dirichlet problem with boundary values given by the fundamental solution.

4.6 Application: Poisson equation with Dirichlet boundary data

- Let us look at the representation problem in connection with the Dirichlet problem for Poisson equation:

$$u(x) = \int_{\partial\Omega} \Phi(y-x)\frac{\partial u}{\partial \nu}(y)\,dS(y) - \int_{\partial\Omega} u(y)\frac{\partial \Phi}{\partial \nu}(y-x)\,dS(y)$$
$$- \int_{\Omega} \Phi(y-x)\Delta u(y)\,dy,$$

- We require that $\Delta u = f$ in Ω and $u = g$ on $\partial\Omega$, but $\partial u/\partial\nu$ is unknown.
- We solve this problem by adding a harmonic function to the fundamental solution. For a fixed $x \in \Omega$, let $\phi^x = \phi^x(y)$ be a corrector function, which is a solution the Dirichlet problem

$$\begin{cases} \Delta_y \phi^x(y) = 0, & y \in \Omega, \\ \phi^x(y) = \Phi(y-x), & y \in \partial\Omega. \end{cases}$$

- Interpretation: we solve the Dirichlet problem with boundary values given by the fundamental solution.

4.6 Application: Poisson equation with Dirichlet boundary data

- Let us look at the representation problem in connection with the Dirichlet problem for Poisson equation:

$$u(x) = \int_{\partial\Omega} \Phi(y-x)\frac{\partial u}{\partial \nu}(y)\, dS(y) - \int_{\partial\Omega} u(y)\frac{\partial \Phi}{\partial \nu}(y-x)\, dS(y) - \int_{\Omega} \Phi(y-x)\Delta u(y)\, dy,$$

- We require that $\Delta u = f$ in Ω and $u = g$ on $\partial\Omega$, but $\partial u/\partial \nu$ is unknown.
- We solve this problem by adding a harmonic function to the fundamental solution. For a fixed $x \in \Omega$, let $\phi^x = \phi^x(y)$ be a corrector function, which is a solution the Dirichlet problem

$$\begin{cases} \Delta_y \phi^x(y) = 0, & y \in \Omega, \\ \phi^x(y) = \Phi(y-x), & y \in \partial\Omega. \end{cases}$$

- Interpretation: we solve the Dirichlet problem with boundary values given by the fundamental solution.

4.6 Application: Poisson equation with Dirichlet boundary data

- Let us look at the representation problem in connection with the Dirichlet problem for Poisson equation:

$$u(x) = \int_{\partial\Omega} \Phi(y-x)\frac{\partial u}{\partial \nu}(y)\,dS(y) - \int_{\partial\Omega} u(y)\frac{\partial \Phi}{\partial \nu}(y-x)\,dS(y) \\ - \int_{\Omega} \Phi(y-x)\Delta u(y)\,dy,$$

- We require that $\Delta u = f$ in Ω and $u = g$ on $\partial\Omega$, but $\partial u/\partial\nu$ is unknown.
- We solve this problem by adding a harmonic function to the fundamental solution. For a fixed $x \in \Omega$, let $\phi^x = \phi^x(y)$ be a corrector function, which is a solution the Dirichlet problem

$$\begin{cases} \Delta_y \phi^x(y) = 0, & y \in \Omega, \\ \phi^x(y) = \Phi(y-x), & y \in \partial\Omega. \end{cases}$$

- Interpretation: we solve the Dirichlet problem with boundary values given by the fundamental solution.

4.6 Green's function

- The Green's function for Ω is

$$G(x,y) = \Phi(y-x) - \phi^x(y), \quad x,y \in \Omega, \quad x \neq y,$$

where $\phi^x = \phi^x(y)$ is a solution the Dirichlet problem

$$\begin{cases} \Delta_y \phi^x(y) = 0, & y \in \Omega, \\ \phi^x(y) = \Phi(y-x), & y \in \partial\Omega. \end{cases}$$

- Formally, G satisfies

$$\begin{cases} \Delta_y G(x,y) = \delta_x & y \in \Omega \\ G(x,y) = 0 & y \in \partial\Omega \end{cases}$$

4.6 Green's function

- By using the representation formula, Green's second identity and the definition of the Green's function, we finally (after quite long calculations) obtain that

$$u(x) = -\int_{\partial\Omega} u(y)\frac{\partial G}{\partial \nu}(x,y)\,dS(y) - \int_{\Omega} \Delta u(y) G(x,y)\,dy,$$

  where

$$\frac{\partial G}{\partial \nu}(x,y) = \nabla_y G(x,y)\cdot \nu(y).$$

- This holds for all $u \in C^2(\overline{\Omega})$. It allows us to determine u if we know Δu and the value of u on the boundary.

4.6 Green's function

- By using the representation formula, Green's second identity and the definition of the Green's function, we finally (after quite long calculations) obtain that

$$u(x) = -\int_{\partial\Omega} u(y)\frac{\partial G}{\partial \nu}(x,y)\,dS(y) - \int_{\Omega} \Delta u(y) G(x,y)\,dy,$$

where

$$\frac{\partial G}{\partial \nu}(x,y) = \nabla_y G(x,y)\cdot \nu(y).$$

- This holds for all $u \in C^2(\overline{\Omega})$. It allows us to determine u if we know Δu and the value of u on the boundary.

4.6 Poisson equation with Green's function

Theorem

Assume that u is a solution of the boundary value problem

$$\begin{cases} -\Delta u = f \quad \textit{in} \quad \Omega, \\ u = g \quad \textit{on} \quad \partial\Omega, \end{cases}$$

then

$$u(x) = -\int_{\partial\Omega} g(y)\frac{\partial G}{\partial \nu}(x,y)\,dS(y) + \int_{\Omega} f(y)G(x,y)\,dy,$$

where G is the Green's function for Ω.

4.7 Green's function for the upper half space

- Consider the upper half-space
  $\mathbb{R}^n_+ = \{(x_1, \dots, x_n) : x_n > 0\}$.
- This is not a bounded domain so the previous arguments do not apply directly.
- In order to construct the Green's function, for every $x \in \mathbb{R}^n_+$, we need to construct a corrector function ϕ^x such that

$$\begin{cases} \Delta \phi^x(y) = 0, & y \in \mathbb{R}^n_+, \\ \phi^x(y) = \Phi(y - x), & y \in \partial \mathbb{R}^n_+. \end{cases}$$

- The Green's function for $\mathbb{R}^n_+$ will then be

$$G(x, y) = \Phi(y - x) - \phi^x(y).$$

4.7 Green's function for the upper half space

- Consider the upper half-space $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) : x_n > 0\}$.
- This is not a bounded domain so the previous arguments do not apply directly.
- In order to construct the Green's function, for every $x \in \mathbb{R}^n_+$, we need to construct a corrector function ϕ^x such that

$$\begin{cases} \Delta\phi^x(y) = 0, & y \in \mathbb{R}^n_+, \\ \phi^x(y) = \Phi(y - x), & y \in \partial\mathbb{R}^n_+. \end{cases}$$

- The Green's function for $\mathbb{R}^n_+$ will then be

$$G(x, y) = \Phi(y - x) - \phi^x(y).$$

4.7 Green's function for the upper half space

- Consider the upper half-space $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) : x_n > 0\}$.
- This is not a bounded domain so the previous arguments do not apply directly.
- In order to construct the Green's function, for every $x \in \mathbb{R}^n_+$, we need to construct a corrector function ϕ^x such that

$$\begin{cases} \Delta\phi^x(y) = 0, & y \in \mathbb{R}^n_+, \\ \phi^x(y) = \Phi(y-x), & y \in \partial\mathbb{R}^n_+. \end{cases}$$

- The Green's function for $\mathbb{R}^n_+$ will then be

$$G(x,y) = \Phi(y-x) - \phi^x(y).$$

4.7 Green's function for the upper half space

- Consider the upper half-space $\mathbb{R}^n_+ = \{(x_1, \dots, x_n) : x_n > 0\}$.
- This is not a bounded domain so the previous arguments do not apply directly.
- In order to construct the Green's function, for every $x \in \mathbb{R}^n_+$, we need to construct a corrector function ϕ^x such that

$$\begin{cases} \Delta \phi^x(y) = 0, & y \in \mathbb{R}^n_+, \\ \phi^x(y) = \Phi(y - x), & y \in \partial \mathbb{R}^n_+. \end{cases}$$

- The Green's function for $\mathbb{R}^n_+$ will then be

$$G(x, y) = \Phi(y - x) - \phi^x(y).$$

4.7 Reflection principle

- We want to solve (for $x \in \mathbb{R}^n_+$).

$$\begin{cases} \Delta \phi^x(y) = 0, & y \in \mathbb{R}^n_+, \\ \phi^x(y) = \Phi(y-x), & y \in \partial\mathbb{R}^n_+. \end{cases}$$

- Let us reflect the vector x across the boundary $\partial\mathbb{R}^n_+$:

$$x^* = (x_1, \ldots, x_{n-1}, -x_n).$$

- $y \mapsto \Phi(y - x^*)$ is harmonic in the whole $\mathbb{R}^n_+$ as $x^* \in \mathbb{R}^n_-$.
- Moreover, $x = x^*$ on $\partial\mathbb{R}^n_+$ and thus

$$\Phi(y - x^*) = \Phi(y - x).$$

- Thus we can choose $\phi^x(y) = \Phi(y - x^*)$.
- And the Green function is $G(x, y) = \Phi(y - x) - \Phi(y - x^*)$

4.7 Reflection principle

- We want to solve (for $x \in \mathbb{R}^n_+$).

$$\begin{cases} \Delta \phi^x(y) = 0, & y \in \mathbb{R}^n_+, \\ \phi^x(y) = \Phi(y - x), & y \in \partial \mathbb{R}^n_+. \end{cases}$$

- Let us reflect the vector x across the boundary $\partial \mathbb{R}^n_+$:

$$x^* = (x_1, \dots, x_{n-1}, -x_n).$$

- $y \mapsto \Phi(y - x^*)$ is harmonic in the whole $\mathbb{R}^n_+$ as $x^* \in \mathbb{R}^n_-$.
- Moreover, $x = x^*$ on $\partial \mathbb{R}^n_+$ and thus

$$\Phi(y - x^*) = \Phi(y - x).$$

- Thus we can choose $\phi^x(y) = \Phi(y - x^*)$.
- And the Green function is $G(x, y) = \Phi(y - x) - \Phi(y - x^*)$

4.7 Reflection principle

- We want to solve (for $x \in \mathbb{R}^n_+$).

$$\begin{cases} \Delta \phi^x(y) = 0, & y \in \mathbb{R}^n_+, \\ \phi^x(y) = \Phi(y - x), & y \in \partial \mathbb{R}^n_+. \end{cases}$$

- Let us reflect the vector x across the boundary $\partial \mathbb{R}^n_+$:

$$x^* = (x_1, \dots, x_{n-1}, -x_n).$$

- $y \mapsto \Phi(y - x^*)$ is harmonic in the whole $\mathbb{R}^n_+$ as $x^* \in \mathbb{R}^n_-$.
- Moreover, $x = x^*$ on $\partial \mathbb{R}^n_+$ and thus

$$\Phi(y - x^*) = \Phi(y - x).$$

- Thus we can choose $\phi^x(y) = \Phi(y - x^*)$.
- And the Green function is $G(x, y) = \Phi(y - x) - \Phi(y - x^*)$

4.7 Reflection principle

- We want to solve (for $x \in \mathbb{R}^n_+$).

$$\begin{cases} \Delta \phi^x(y) = 0, & y \in \mathbb{R}^n_+, \\ \phi^x(y) = \Phi(y - x), & y \in \partial \mathbb{R}^n_+. \end{cases}$$

- Let us reflect the vector x across the boundary $\partial \mathbb{R}^n_+$:

$$x^* = (x_1, \ldots, x_{n-1}, -x_n).$$

- $y \mapsto \Phi(y - x^*)$ is harmonic in the whole $\mathbb{R}^n_+$ as $x^* \in \mathbb{R}^n_-$.
- Moreover, $x = x^*$ on $\partial \mathbb{R}^n_+$ and thus

$$\Phi(y - x^*) = \Phi(y - x).$$

- Thus we can choose $\phi^x(y) = \Phi(y - x^*)$.
- And the Green function is $G(x, y) = \Phi(y - x) - \Phi(y - x^*)$

4.7 Reflection principle

- We want to solve (for $x \in \mathbb{R}^n_+$).

$$\begin{cases} \Delta\phi^x(y) = 0, & y \in \mathbb{R}^n_+, \\ \phi^x(y) = \Phi(y-x), & y \in \partial\mathbb{R}^n_+. \end{cases}$$

- Let us reflect the vector x across the boundary $\partial\mathbb{R}^n_+$:

$$x^* = (x_1, \dots, x_{n-1}, -x_n).$$

- $y \mapsto \Phi(y - x^*)$ is harmonic in the whole $\mathbb{R}^n_+$ as $x^* \in \mathbb{R}^n_-$.
- Moreover, $x = x^*$ on $\partial\mathbb{R}^n_+$ and thus

$$\Phi(y - x^*) = \Phi(y - x).$$

- Thus we can choose $\phi^x(y) = \Phi(y - x^*)$.
- And the Green function is $G(x, y) = \Phi(y - x) - \Phi(y - x^*)$

4.7 Reflection principle

- We want to solve (for $x \in \mathbb{R}^n_+$).

$$\begin{cases} \Delta\phi^x(y) = 0, & y \in \mathbb{R}^n_+, \\ \phi^x(y) = \Phi(y-x), & y \in \partial\mathbb{R}^n_+. \end{cases}$$

- Let us reflect the vector x across the boundary $\partial\mathbb{R}^n_+$:

$$x^* = (x_1, \dots, x_{n-1}, -x_n).$$

- $y \mapsto \Phi(y - x^*)$ is harmonic in the whole $\mathbb{R}^n_+$ as $x^* \in \mathbb{R}^n_-$.
- Moreover, $x = x^*$ on $\partial\mathbb{R}^n_+$ and thus

$$\Phi(y - x^*) = \Phi(y - x).$$

- Thus we can choose $\phi^x(y) = \Phi(y - x^*)$.
- And the Green function is $G(x, y) = \Phi(y - x) - \Phi(y - x^*)$

4.7 Laplace equation in the upper half space

- We are considering the problem:

$$\begin{cases} \Delta u(y) = 0, & y \in \mathbb{R}^n_+, \\ u(y) = g(y), & y \in \partial\mathbb{R}^n_+. \end{cases}$$

- By inserting the Green's function to the representation formula, we obtain

$$u(x) = -\int_{\partial\mathbb{R}^n_+} \frac{\partial G}{\partial \nu}(x,y) g(y)\, dS(y).$$

- By deriving an explicit expression for $\frac{\partial G}{\partial \nu}(x,y)$, we get

$$u(x) = \frac{2x_n}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+} \frac{g(y)}{|x-y|^n}\, dy.$$

- We derived the same formula earlier with Fourier transform i.e. the solution is the convolution of the Poisson kernel and the boundary values.

4.7 Laplace equation in the upper half space

- We are considering the problem:

$$\begin{cases} \Delta u(y) = 0, & y \in \mathbb{R}^n_+, \\ u(y) = g(y), & y \in \partial\mathbb{R}^n_+. \end{cases}$$

- By inserting the Green's function to the representation formula, we obtain

$$u(x) = -\int_{\partial\mathbb{R}^n_+} \frac{\partial G}{\partial \nu}(x,y) g(y)\, dS(y).$$

- By deriving an explicit expression for $\frac{\partial G}{\partial \nu}(x,y)$, we get

$$u(x) = \frac{2x_n}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+} \frac{g(y)}{|x-y|^n}\, dy.$$

- We derived the same formula earlier with Fourier transform i.e. the solution is the convolution of the Poisson kernel and the boundary values.

4.7 Laplace equation in the upper half space

- We are considering the problem:

$$\begin{cases} \Delta u(y) = 0, & y \in \mathbb{R}^n_+, \\ u(y) = g(y), & y \in \partial\mathbb{R}^n_+. \end{cases}$$

- By inserting the Green's function to the representation formula, we obtain

$$u(x) = -\int_{\partial\mathbb{R}^n_+} \frac{\partial G}{\partial \nu}(x,y)g(y)\,dS(y).$$

- By deriving an explicit expression for $\frac{\partial G}{\partial \nu}(x,y)$, we get

$$u(x) = \frac{2x_n}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+} \frac{g(y)}{|x-y|^n}\,dy.$$

- We derived the same formula earlier with Fourier transform i.e. the solution is the convolution of the Poisson kernel and the boundary values.

4.7 Laplace equation in the upper half space

- We are considering the problem:

$$\begin{cases} \Delta u(y) = 0, & y \in \mathbb{R}^n_+, \\ u(y) = g(y), & y \in \partial\mathbb{R}^n_+. \end{cases}$$

- By inserting the Green's function to the representation formula, we obtain

$$u(x) = -\int_{\partial\mathbb{R}^n_+} \frac{\partial G}{\partial \nu}(x,y) g(y)\, dS(y).$$

- By deriving an explicit expression for $\frac{\partial G}{\partial \nu}(x,y)$, we get

$$u(x) = \frac{2x_n}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+} \frac{g(y)}{|x-y|^n}\, dy.$$

- We derived the same formula earlier with Fourier transform i.e. the solution is the convolution of the Poisson kernel and the boundary values.

4.8 Green's function for a ball

- We shall again use the method of reflection to construct a Green's function.
- We need to find the corrector function ϕ^x that satisfies

$$\begin{cases} \Delta \phi^x(y) = 0 & y \in B(0,1), \\ \phi^x(y) = \Phi(y - x), & y \in \partial B(0,1). \end{cases}$$

- As in the case of half space, we reflect the point across the boundary:

$$x^* = \frac{x}{|x|^2}.$$

- We set

$$\phi^x(y) = \Phi(|x|(y - x^*)) = \ldots = \frac{1}{|x|^{n-2}} \Phi(y - x^*).$$

- This is a harmonic function in $B(0,1)$ (with respect to y variable) as $x^* \notin B(0,1)$.
- Thus the Green's function for unit ball is

$$G(x,y) = \Phi(y - x) - \Phi(|x|(y - x^*)).$$

4.8 Green's function for a ball

- We shall again use the method of reflection to construct a Green's function.
- We need to find the corrector function ϕ^x that satisfies

$$\begin{cases} \Delta \phi^x(y) = 0 & y \in B(0,1), \\ \phi^x(y) = \Phi(y-x), & y \in \partial B(0,1). \end{cases}$$

- As in the case of half space, we reflect the point across the boundary:

$$x^* = \frac{x}{|x|^2}.$$

- We set

$$\phi^x(y) = \Phi(|x|(y-x^*)) = \ldots = \frac{1}{|x|^{n-2}} \Phi(y-x^*).$$

- This is a harmonic function in $B(0,1)$ (with respect to y variable) as $x^* \notin B(0,1)$.
- Thus the Green's function for unit ball is

$$G(x,y) = \Phi(y-x) - \Phi(|x|(y-x^*)).$$

4.8 Green's function for a ball

- We shall again use the method of reflection to construct a Green's function.
- We need to find the corrector function ϕ^x that satisfies

$$\begin{cases} \Delta\phi^x(y) = 0 & y \in B(0,1), \\ \phi^x(y) = \Phi(y-x), & y \in \partial B(0,1). \end{cases}$$

- As in the case of half space, we reflect the point across the boundary:

$$x^* = \frac{x}{|x|^2}.$$

- We set

$$\phi^x(y) = \Phi(|x|(y-x^*)) = \ldots = \frac{1}{|x|^{n-2}}\Phi(y-x^*).$$

- This is a harmonic function in $B(0,1)$ (with respect to y variable) as $x^* \notin B(0,1)$.
- Thus the Green's function for unit ball is

$$G(x,y) = \Phi(y-x) - \Phi(|x|(y-x^*)).$$

4.8 Green's function for a ball

- We shall again use the method of reflection to construct a Green's function.
- We need to find the corrector function ϕ^x that satisfies

$$\begin{cases} \Delta\phi^x(y) = 0 & y \in B(0,1), \\ \phi^x(y) = \Phi(y-x), & y \in \partial B(0,1). \end{cases}$$

- As in the case of half space, we reflect the point across the boundary:

$$x^* = \frac{x}{|x|^2}.$$

- We set

$$\phi^x(y) = \Phi(|x|(y-x^*)) = \ldots = \frac{1}{|x|^{n-2}}\Phi(y-x^*).$$

- This is a harmonic function in $B(0,1)$ (with respect to y variable) as $x^* \notin B(0,1)$.
- Thus the Green's function for unit ball is

$$G(x,y) = \Phi(y-x) - \Phi(|x|(y-x^*)).$$

4.8 Green's function for a ball

- We shall again use the method of reflection to construct a Green's function.
- We need to find the corrector function ϕ^x that satisfies

$$\begin{cases} \Delta\phi^x(y) = 0 & y \in B(0,1), \\ \phi^x(y) = \Phi(y-x), & y \in \partial B(0,1). \end{cases}$$

- As in the case of half space, we reflect the point across the boundary:

$$x^* = \frac{x}{|x|^2}.$$

- We set

$$\phi^x(y) = \Phi(|x|(y-x^*)) = \ldots = \frac{1}{|x|^{n-2}}\Phi(y-x^*).$$

- This is a harmonic function in $B(0,1)$ (with respect to y variable) as $x^* \notin B(0,1)$.
- Thus the Green's function for unit ball is

$$G(x,y) = \Phi(y-x) - \Phi(|x|(y-x^*)).$$

4.8 Green's function for a ball

- We shall again use the method of reflection to construct a Green's function.
- We need to find the corrector function ϕ^x that satisfies

$$\begin{cases} \Delta\phi^x(y) = 0 & y \in B(0,1), \\ \phi^x(y) = \Phi(y-x), & y \in \partial B(0,1). \end{cases}$$

- As in the case of half space, we reflect the point across the boundary:

$$x^* = \frac{x}{|x|^2}.$$

- We set

$$\phi^x(y) = \Phi(|x|(y-x^*)) = \ldots = \frac{1}{|x|^{n-2}}\Phi(y-x^*).$$

- This is a harmonic function in $B(0,1)$ (with respect to y variable) as $x^* \notin B(0,1)$.
- Thus the Green's function for unit ball is

$$G(x,y) = \Phi(y-x) - \Phi(|x|(y-x^*)).$$

4.9 Mean value formulas

- 1D-case: Laplace equation: $u'' = 0$. Harmonic functions:

$$u(x) = ax + b, \quad a, b \ \in \mathbb{R}.$$

- The value of these functions at the midpoint of the interval is the same as the arthmetic average of the values at the endpoints of the interval and the integral average over the whole interval.
- Higher dimensional case: Let $u \in C^2(\Omega)$ be harmonic in Ω. Then for every ball $B(x,r)$ such that $\overline{B(x,r)} \subset \Omega$ we have

$$u(x) = \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y)\,dy = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u(y)\,dS(y).$$

- This is a very important property for harmonic functions.

4.9 Mean value formulas

- 1D-case: Laplace equation: $u'' = 0$. Harmonic functions:

$$u(x) = ax + b, \quad a, b \ \in \mathbb{R}.$$

- The value of these functions at the midpoint of the interval is the same as the arthmetic average of the values at the endpoints of the interval and the integral average over the whole interval.
- Higher dimensional case: Let $u \in C^2(\Omega)$ be harmonic in Ω. Then for every ball $B(x,r)$ such that $\overline{B(x,r)} \subset \Omega$ we have

$$u(x) = \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y)\, dy = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u(y)\, dS(y).$$

- This is a very important property for harmonic functions.

4.9 Mean value formulas

- 1D-case: Laplace equation: $u'' = 0$. Harmonic functions:

$$u(x) = ax + b, \quad a, b \in \mathbb{R}.$$

- The value of these functions at the midpoint of the interval is the same as the arthmetic average of the values at the endpoints of the interval and the integral average over the whole interval.
- Higher dimensional case: Let $u \in C^2(\Omega)$ be harmonic in Ω. Then for every ball $B(x,r)$ such that $\overline{B(x,r)} \subset \Omega$ we have

$$u(x) = \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y)\, dy = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u(y)\, dS(y).$$

- This is a very important property for harmonic functions.

4.9 Mean value formulas

- 1D-case: Laplace equation: $u'' = 0$. Harmonic functions:

$$u(x) = ax + b, \quad a, b \ \in \mathbb{R}.$$

- The value of these functions at the midpoint of the interval is the same as the arthmetic average of the values at the endpoints of the interval and the integral average over the whole interval.
- Higher dimensional case: Let $u \in C^2(\Omega)$ be harmonic in Ω. Then for every ball $B(x, r)$ such that $\overline{B(x, r)} \subset \Omega$ we have

$$u(x) = \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y)\, dy = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u(y)\, dS(y).$$

- This is a very important property for harmonic functions.

4.9 Mean value property

- A converse result holds as well:

Theorem

If $u \in C(\Omega)$ satisfies

$$u(x) = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u(y)\, dS(y)$$

for all balls $\overline{B(x,r)} \subset \Omega$, then u is harmonic in Ω.

- Thus a continuous function u is harmonic if and only if for every point in the domain of definition the mean value property holds true for small enough balls centered at the point.

4.9 Mean value property

- A converse result holds as well:

Theorem

If $u \in C(\Omega)$ *satisfies*

$$u(x) = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u(y)\, dS(y)$$

for all balls $\overline{B(x,r)} \subset \Omega$*, then* u *is harmonic in* Ω*.*

- Thus a continuous function u is harmonic if and only if for every point in the domain of definition the mean value property holds true for small enough balls centered at the point.

4.10 Maximum principles

- Recall: A continuous function always attains its maximum and minimum values on a closed and bounded set.
- Maximum principle: Harmonic function attains its maximum and minimum at the boundary of the set.
- Physically: If the body is in thermal equilibrium, there cannot be internal hot or cold spots, since otherwise the heat energy would flow from hot to cold.
- If the maximum or minimum temperature is attained inside the body, then the temperature must be constant.
- Recall: An open set is connected if every pair of points can be connected by a piecewise linear path in the set.

4.10 Maximum principles

- Recall: A continuous function always attains its maximum and minimum values on a closed and bounded set.
- Maximum principle: Harmonic function attains its maximum and minimum at the boundary of the set.
- Physically: If the body is in thermal equilibrium, there cannot be internal hot or cold spots, since otherwise the heat energy would flow from hot to cold.
- If the maximum or minimum temperature is attained inside the body, then the temperature must be constant.
- Recall: An open set is connected if every pair of points can be connected by a piecewise linear path in the set.

4.10 Maximum principles

- Recall: A continuous function always attains its maximum and minimum values on a closed and bounded set.
- Maximum principle: Harmonic function attains its maximum and minimum at the boundary of the set.
- Physically: If the body is in thermal equilibrium, there cannot be internal hot or cold spots, since otherwise the heat energy would flow from hot to cold.
- If the maximum or minimum temperature is attained inside the body, then the temperature must be constant.
- Recall: An open set is connected if every pair of points can be connected by a piecewise linear path in the set.

4.10 Maximum principles

- Recall: A continuous function always attains its maximum and minimum values on a closed and bounded set.
- Maximum principle: Harmonic function attains its maximum and minimum at the boundary of the set.
- Physically: If the body is in thermal equilibrium, there cannot be internal hot or cold spots, since otherwise the heat energy would flow from hot to cold.
- If the maximum or minimum temperature is attained inside the body, then the temperature must be constant.
- Recall: An open set is connected if every pair of points can be connected by a piecewise linear path in the set.

4.10 Maximum principles

- Recall: A continuous function always attains its maximum and minimum values on a closed and bounded set.
- Maximum principle: Harmonic function attains its maximum and minimum at the boundary of the set.
- Physically: If the body is in thermal equilibrium, there cannot be internal hot or cold spots, since otherwise the heat energy would flow from hot to cold.
- If the maximum or minimum temperature is attained inside the body, then the temperature must be constant.
- Recall: An open set is connected if every pair of points can be connected by a piecewise linear path in the set.

4.10 Maximum principle

Theorem

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set and assume that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a harmonic function in Ω.

(1) *(Weak maximum principle) Then*

$$\max_{x \in \overline{\Omega}} u(x) = \max_{x \in \partial\Omega} u(x).$$

(2) *(Strong maximum principle) If Ω is a connected set and there exists $x_0 \in \Omega$ such that*

$$u(x_0) = \max_{x \in \overline{\Omega}} u(x),$$

then u is constant in Ω.

- By replacing u by $-u$, we get the minimum principles.

4.10 Maximum principle

Theorem

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set and assume that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a harmonic function in Ω.

(1) *(Weak maximum principle) Then*

$$\max_{x \in \overline{\Omega}} u(x) = \max_{x \in \partial\Omega} u(x).$$

(2) *(Strong maximum principle) If Ω is a connected set and there exists $x_0 \in \Omega$ such that*

$$u(x_0) = \max_{x \in \overline{\Omega}} u(x),$$

then u is constant in Ω.

- By replacing u by $-u$, we get the minimum principles.

4.10 Maximum principle

Theorem

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set and assume that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a harmonic function in Ω.

(1) *(Weak maximum principle) Then*

$$\max_{x \in \overline{\Omega}} u(x) = \max_{x \in \partial\Omega} u(x).$$

(2) *(Strong maximum principle) If Ω is a connected set and there exists $x_0 \in \Omega$ such that*

$$u(x_0) = \max_{x \in \overline{\Omega}} u(x),$$

then u is constant in Ω.

- By replacing u by $-u$, we get the minimum principles.

4.10 Maximum principle

Theorem

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set and assume that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a harmonic function in Ω.

(1) *(Weak maximum principle) Then*

$$\max_{x \in \overline{\Omega}} u(x) = \max_{x \in \partial\Omega} u(x).$$

(2) *(Strong maximum principle) If Ω is a connected set and there exists $x_0 \in \Omega$ such that*

$$u(x_0) = \max_{x \in \overline{\Omega}} u(x),$$

then u is constant in Ω.

- By replacing u by $-u$, we get the minimum principles.

4.10 Proof of strong maximum principle

- Suppose that there exists $x_0 \in \Omega$ such that
$$u(x_0) = \max_{x \in \overline{\Omega}} u(x) = M.$$
- For $B(x_0, r) \subset \Omega$, the mean value property implies that
$$M = u(x_0) = \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} u(y)dy \le \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} M dy = M.$$
- It follows that an equality holds throughout and thus
$$\int_{B(x_0,r)} (M - u(y))\, dy = 0.$$
- As $M - u(y) \geq 0$, we conclude that $u(y) = M$ for every $y \in B(x_0, r)$.
- As Ω is connected, we can reach any point in Ω with a finite chain of balls.

4.10 Proof of strong maximum principle

- Suppose that there exists $x_0 \in \Omega$ such that

$$u(x_0) = \max_{x \in \overline{\Omega}} u(x) = M.$$

- For $B(x_0, r) \subset \Omega$, the mean value property implies that

$$M = u(x_0) = \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} u(y)dy \leq \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} M dy = M.$$

- It follows that an equality holds throughout and thus

$$\int_{B(x_0,r)} (M - u(y))\, dy = 0.$$

- As $M - u(y) \geq 0$, we conclude that $u(y) = M$ for every $y \in B(x_0, r)$.
- As Ω is connected, we can reach any point in Ω with a finite chain of balls.

4.10 Proof of strong maximum principle

- Suppose that there exists $x_0 \in \Omega$ such that

$$u(x_0) = \max_{x \in \overline{\Omega}} u(x) = M.$$

- For $B(x_0, r) \subset \Omega$, the mean value property implies that

$$M = u(x_0) = \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} u(y)dy \leq \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} M dy = M.$$

- It follows that an equality holds throughout and thus

$$\int_{B(x_0,r)} (M - u(y))\, dy = 0.$$

- As $M - u(y) \geq 0$, we conclude that $u(y) = M$ for every $y \in B(x_0, r)$.
- As Ω is connected, we can reach any point in Ω with a finite chain of balls.

4.10 Proof of strong maximum principle

- Suppose that there exists $x_0 \in \Omega$ such that

$$u(x_0) = \max_{x \in \overline{\Omega}} u(x) = M.$$

- For $B(x_0, r) \subset \Omega$, the mean value property implies that

$$M = u(x_0) = \tfrac{1}{|B(x_0,r)|} \int_{B(x_0,r)} u(y)dy \leq \tfrac{1}{|B(x_0,r)|} \int_{B(x_0,r)} M dy = M.$$

- It follows that an equality holds throughout and thus

$$\int_{B(x_0,r)} (M - u(y))\, dy = 0.$$

- As $M - u(y) \geq 0$, we conclude that $u(y) = M$ for every $y \in B(x_0, r)$.
- As Ω is connected, we can reach any point in Ω with a finite chain of balls.

4.10 Proof of strong maximum principle

- Suppose that there exists $x_0 \in \Omega$ such that
$$u(x_0) = \max_{x \in \overline{\Omega}} u(x) = M.$$
- For $B(x_0, r) \subset \Omega$, the mean value property implies that
$$M = u(x_0) = \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} u(y)dy \leq \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} M dy = M.$$
- It follows that an equality holds throughout and thus
$$\int_{B(x_0,r)} (M - u(y))\, dy = 0.$$
- As $M - u(y) \geq 0$, we conclude that $u(y) = M$ for every $y \in B(x_0, r)$.
- As Ω is connected, we can reach any point in Ω with a finite chain of balls.

4.10 Consequences of the maximum principle

Let $\Omega \subset \mathbb{R}^n$ be open and bounded and assume that $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ are solutions to the Poisson equation

$$-\Delta u = f \quad \text{in } \Omega.$$

Theorem (Comparison principle)

If $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.

- Proof: $\Delta(u - v) = f - f = 0$. Thus $u - v$ is harmonic and satisfies the maximum principle.

Theorem (Stability)

If $|u - v| \leq \varepsilon$ on $\partial\Omega$, then $|u - v| \leq \varepsilon$ in Ω.

- Proof: $|u - v| \leq \varepsilon$ on $\partial\Omega$ means that $-\varepsilon \leq u - v \leq \varepsilon$ on $\partial\Omega$. By maximum and minimum principles $-\varepsilon \leq u - v \leq \varepsilon$ in Ω.

Theorem (Uniqueness)

The solution is unique.

4.10 Consequences of the maximum principle

Let $\Omega \subset \mathbb{R}^n$ be open and bounded and assume that $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ are solutions to the Poisson equation

$$-\Delta u = f \quad \text{in } \Omega.$$

Theorem (Comparison principle)

If $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.

- Proof: $\Delta(u - v) = f - f = 0$. Thus $u - v$ is harmonic and satisfies the maximum principle.

Theorem (Stability)

If $|u - v| \leq \varepsilon$ on $\partial\Omega$, then $|u - v| \leq \varepsilon$ in Ω.

- Proof: $|u - v| \leq \varepsilon$ on $\partial\Omega$ means that $-\varepsilon \leq u - v \leq \varepsilon$ on $\partial\Omega$. By maximum and minimum principles $-\varepsilon \leq u - v \leq \varepsilon$ in Ω.

Theorem (Uniqueness)

The solution is unique.

4.10 Consequences of the maximum principle

Let $\Omega \subset \mathbb{R}^n$ be open and bounded and assume that $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ are solutions to the Poisson equation

$$-\Delta u = f \quad \text{in } \Omega.$$

Theorem (Comparison principle)

If $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.

- Proof: $\Delta(u - v) = f - f = 0$. Thus $u - v$ is harmonic and satisfies the maximum principle.

Theorem (Stability)

If $|u - v| \leq \varepsilon$ on $\partial\Omega$, then $|u - v| \leq \varepsilon$ in Ω.

- Proof: $|u - v| \leq \varepsilon$ on $\partial\Omega$ means that $-\varepsilon \leq u - v \leq \varepsilon$ on $\partial\Omega$. By maximum and minimum principles $-\varepsilon \leq u - v \leq \varepsilon$ in Ω.

Theorem (Uniqueness)

The solution is unique.

4.10 Consequences of the maximum principle

Let $\Omega \subset \mathbb{R}^n$ be open and bounded and assume that $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ are solutions to the Poisson equation

$$-\Delta u = f \quad \text{in } \Omega.$$

Theorem (Comparison principle)

If $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.

- Proof: $\Delta(u - v) = f - f = 0$. Thus $u - v$ is harmonic and satisfies the maximum principle.

Theorem (Stability)

If $|u - v| \leq \varepsilon$ on $\partial\Omega$, then $|u - v| \leq \varepsilon$ in Ω.

- Proof: $|u - v| \leq \varepsilon$ on $\partial\Omega$ means that $-\varepsilon \leq u - v \leq \varepsilon$ on $\partial\Omega$. By maximum and minimum principles $-\varepsilon \leq u - v \leq \varepsilon$ in Ω.

Theorem (Uniqueness)

The solution is unique.

4.10 Consequences of the maximum principle

Let $\Omega \subset \mathbb{R}^n$ be open and bounded and assume that $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ are solutions to the Poisson equation

$$-\Delta u = f \quad \text{in } \Omega.$$

Theorem (Comparison principle)

If $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.

- Proof: $\Delta(u - v) = f - f = 0$. Thus $u - v$ is harmonic and satisfies the maximum principle.

Theorem (Stability)

If $|u - v| \leq \varepsilon$ on $\partial\Omega$, then $|u - v| \leq \varepsilon$ in Ω.

- Proof: $|u - v| \leq \varepsilon$ on $\partial\Omega$ means that $-\varepsilon \leq u - v \leq \varepsilon$ on $\partial\Omega$. By maximum and minimum principles $-\varepsilon \leq u - v \leq \varepsilon$ in Ω.

Theorem (Uniqueness)

The solution is unique.

4.10 Consequences of the maximum principle

Let $\Omega \subset \mathbb{R}^n$ be open and bounded and assume that $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ are solutions to the Poisson equation

$$-\Delta u = f \quad \text{in } \Omega.$$

Theorem (Comparison principle)

If $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.

- Proof: $\Delta(u - v) = f - f = 0$. Thus $u - v$ is harmonic and satisfies the maximum principle.

Theorem (Stability)

If $|u - v| \leq \varepsilon$ on $\partial\Omega$, then $|u - v| \leq \varepsilon$ in Ω.

- Proof: $|u - v| \leq \varepsilon$ on $\partial\Omega$ means that $-\varepsilon \leq u - v \leq \varepsilon$ on $\partial\Omega$. By maximum and minimum principles $-\varepsilon \leq u - v \leq \varepsilon$ in Ω.

Theorem (Uniqueness)

The solution is unique.

4.10 Stability and uniqueness in unbounded domains

- The assumption that Ω is bounded is essential for uniqueness.
- Consider, for example, the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \mathbb{R}^n_+, \\ u = 0 & \text{on} \quad \partial\mathbb{R}^n_+, \end{cases}$$

  in the unbounded open and connected unbounded set $\mathbb{R}^n_+$. Every function

$$u(x_1, \ldots, x_{n-1}, x_n) = a x_n,$$

  where $x_n > 0$ and $a \in \mathbb{R}$, is a solution to the problem.
- Without uniqueness, we cannot have stability results either.
- There are uniqueness results for unbounded domains for example with an extra condition that the solutions should approach to zero in a suitable sense when $|x| \to \infty$.

4.10 Stability and uniqueness in unbounded domains

- The assumption that Ω is bounded is essential for uniqueness.
- Consider, for example, the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \mathbb{R}^n_+, \\ u = 0 & \text{on} \quad \partial\mathbb{R}^n_+, \end{cases}$$

  in the unbounded open and connected unbounded set $\mathbb{R}^n_+$. Every function

$$u(x_1, \ldots, x_{n-1}, x_n) = a x_n,$$

  where $x_n > 0$ and $a \in \mathbb{R}$, is a solution to the problem.
- Without uniqueness, we cannot have stability results either.
- There are uniqueness results for unbounded domains for example with an extra condition that the solutions should approach to zero in a suitable sense when $|x| \to \infty$.

4.10 Stability and uniqueness in unbounded domains

- The assumption that Ω is bounded is essential for uniqueness.
- Consider, for example, the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \mathbb{R}^n_+, \\ u = 0 & \text{on} \quad \partial\mathbb{R}^n_+, \end{cases}$$

  in the unbounded open and connected unbounded set $\mathbb{R}^n_+$. Every function

$$u(x_1, \dots, x_{n-1}, x_n) = ax_n,$$

  where $x_n > 0$ and $a \in \mathbb{R}$, is a solution to the problem.
- Without uniqueness, we cannot have stability results either.
- There are uniqueness results for unbounded domains for example with an extra condition that the solutions should approach to zero in a suitable sense when $|x| \to \infty$.

4.10 Stability and uniqueness in unbounded domains

- The assumption that Ω is bounded is essential for uniqueness.
- Consider, for example, the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in} \quad \mathbb{R}^n_+, \\ u = 0 & \text{on} \quad \partial\mathbb{R}^n_+, \end{cases}$$

  in the unbounded open and connected unbounded set $\mathbb{R}^n_+$. Every function

$$u(x_1, \ldots, x_{n-1}, x_n) = ax_n,$$

  where $x_n > 0$ and $a \in \mathbb{R}$, is a solution to the problem.
- Without uniqueness, we cannot have stability results either.
- There are uniqueness results for unbounded domains for example with an extra condition that the solutions should approach to zero in a suitable sense when $|x| \to \infty$.

Aalto University

MS-C1350 Partial differential equations
Chapter 4.12-4.15
Harnack's inequality, energy methods, weak solutions, other coordinates

Riikka Korte

Department of Mathematics and Systems Analysis
Aalto University
riikka.korte@aalto.fi

November 6, 2024

Lecture 9

Laplace equation

- Harnack's inequality
- Energy methods
- Weak solutions
- Other coordinates

Heat equation

- Physical interpretation
- Fundamental solution
- Nonhomogeneous problem

4.11 Harnack's inequality

- For the Laplace equation, Harnack's inequality tells that if $u \geq 0$ is a harmonic function in Ω and $\overline{V} \subset \Omega$, then

$$\sup_{x \in V} u(x) \leq c \inf_{x \in V} u(x).$$

- Different PDE's have different Harnack-type inequalities.
- They can involve integral averages or waiting times (for time-dependent inequalities).

4.11 Harnack's inequality

- For the Laplace equation, Harnack's inequality tells that if $u \geq 0$ is a harmonic function in Ω and $\overline{V} \subset \Omega$, then

$$\sup_{x \in V} u(x) \leq c \inf_{x \in V} u(x).$$

- Different PDE's have different Harnack-type inequalities.
- They can involve integral averages or waiting times (for time-dependent inequalities).

4.11 Harnack's inequality

- For the Laplace equation, Harnack's inequality tells that if $u \geq 0$ is a harmonic function in Ω and $\overline{V} \subset \Omega$, then

$$\sup_{x \in V} u(x) \leq c \inf_{x \in V} u(x).$$

- Different PDE's have different Harnack-type inequalities.
- They can involve integral averages or waiting times (for time-dependent inequalities).

4.12 Energy methods

- Now we characterize the solution of the Dirichlet problem for the Poisson equation as a minimizer of an appropriate energy functional.
- The class of admissible functions for the Dirichlet problem:

$$\mathcal{A} = \{w \in C^2(\Omega) \cap C(\overline{\Omega}) : w = g \text{ on } \partial\Omega\}.$$

- The energy functional for the Poisson equation $\Delta u = f$:

$$I(w) = \int_\Omega \left(\frac{1}{2} |\nabla w|^2 - wf \right) dx.$$

4.12 Energy methods

- Now we characterize the solution of the Dirichlet problem for the Poisson equation as a minimizer of an appropriate energy functional.
- The class of admissible functions for the Dirichlet problem:

$$\mathcal{A} = \{w \in C^2(\Omega) \cap C(\overline{\Omega}) : w = g \text{ on } \partial\Omega\}.$$

- The energy functional for the Poisson equation $\Delta u = f$:

$$I(w) = \int_\Omega \left(\frac{1}{2} |\nabla w|^2 - wf \right) dx.$$

4.12 Energy methods

- Now we characterize the solution of the Dirichlet problem for the Poisson equation as a minimizer of an appropriate energy functional.
- The class of admissible functions for the Dirichlet problem:

$$\mathcal{A} = \{w \in C^2(\Omega) \cap C(\overline{\Omega}) : w = g \text{ on } \partial\Omega\}.$$

- The energy functional for the Poisson equation $\Delta u = f$:

$$I(w) = \int_\Omega \left(\frac{1}{2} |\nabla w|^2 - wf \right) dx.$$

4.12 Energy methods

Theorem (Dirichlet's principle)

Suppose that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ *solves*

$$\begin{cases} -\Delta u = f & \textit{in} \quad \Omega, \\ u = g & \textit{on} \quad \partial\Omega. \end{cases}$$

Then

$$I(u) = \min_{w \in \mathcal{A}} I(w).$$

Conversely, if $u \in \mathcal{A}$ *satisfies* $I(u) = \min_{w \in \mathcal{A}} I(w)$*, then* u *is a solution of the Dirichlet problem above for the Poisson equation.*

- The Poisson equation is said to be the Euler-Lagrange equation for the energy (or variational) integral above.
- A function is a solution to the Poisson equation if and only if it is a minimizer of the energy integral.
- Notice: Laplace involves 2^{nd} order derivatives, but minimization problem only 1^{st} order derivatives!

4.12 Energy methods

Theorem (Dirichlet's principle)

Suppose that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ *solves*

$$\begin{cases} -\Delta u = f \quad \textit{in} \quad \Omega, \\ u = g \quad \textit{on} \quad \partial\Omega. \end{cases}$$

Then

$$I(u) = \min_{w \in \mathcal{A}} I(w).$$

Conversely, if $u \in \mathcal{A}$ *satisfies* $I(u) = \min_{w \in \mathcal{A}} I(w)$*, then* u *is a solution of the Dirichlet problem above for the Poisson equation.*

- The Poisson equation is said to be the Euler-Lagrange equation for the energy (or variational) integral above.
- A function is a solution to the Poisson equation if and only if it is a minimizer of the energy integral.
- Notice: Laplace involves 2^{nd} order derivatives, but minimization problem only 1^{st} order derivatives!

4.12 Energy methods

Theorem (Dirichlet's principle)

Suppose that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ *solves*

$$\begin{cases} -\Delta u = f \quad \text{in} \quad \Omega, \\ u = g \quad \text{on} \quad \partial\Omega. \end{cases}$$

Then

$$I(u) = \min_{w \in \mathcal{A}} I(w).$$

Conversely, if $u \in \mathcal{A}$ *satisfies* $I(u) = \min_{w \in \mathcal{A}} I(w)$*, then* u *is a solution of the Dirichlet problem above for the Poisson equation.*

- The Poisson equation is said to be the Euler-Lagrange equation for the energy (or variational) integral above.
- A function is a solution to the Poisson equation if and only if it is a minimizer of the energy integral.
- Notice: Laplace involves 2^{nd} order derivatives, but minimization problem only 1^{st} order derivatives!

4.12 Energy methods

Theorem (Dirichlet's principle)

Suppose that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ *solves*

$$\begin{cases} -\Delta u = f & \textit{in} \quad \Omega, \\ u = g & \textit{on} \quad \partial\Omega. \end{cases}$$

Then

$$I(u) = \min_{w \in \mathcal{A}} I(w).$$

Conversely, if $u \in \mathcal{A}$ *satisfies* $I(u) = \min_{w \in \mathcal{A}} I(w)$*, then* u *is a solution of the Dirichlet problem above for the Poisson equation.*

- The Poisson equation is said to be the Euler-Lagrange equation for the energy (or variational) integral above.
- A function is a solution to the Poisson equation if and only if it is a minimizer of the energy integral.
- Notice: Laplace involves 2^{nd} order derivatives, but minimization problem only 1^{st} order derivatives!

4.14 Laplace equation in other coordinates

- In $\mathbb{R}^2$, we solved the Laplace equation on the disc by switching to polar coordinates.
- In $\mathbb{R}^3$, there are two coordinate systems that generalize polar coordinates in $\mathbb{R}^2$:
  - Cylindrical coordinates (r, θ, z): Use polar coordinates for (x, y), keep z.
  - Spherical coordinates (r, θ, ϕ): r distance to the origin, θ horizontal angle, ϕ angle between z-axes and the direction.

MS-C1350 Partial differential equations
Chapter 5.1-5.3
Heat equation – physical interpretation, fundamental solution and nonhomogeneous problem

Riikka Korte

Department of Mathematics and Systems Analysis
Aalto University
riikka.korte@aalto.fi

November 13, 2024

5.1 Heat equation

- Now we study the heat equation

$$u_t - \Delta u = 0$$

- and the nonhomogeneous heat equation

$$u_t - \Delta u = f$$

with appropriate initial and boundary conditions.

- The Laplace is taken with respect to the spatial variable x

$$\Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}.$$

- Let $\Omega \subset \mathbb{R}^n$ and $T > 0$. The problem is to find a function $u = u(x,t)$ such that it is a solution to the heat equation in $\Omega \times (0,T)$.

5.1 Heat equation

- Now we study the heat equation

$$u_t - \Delta u = 0$$

- and the nonhomogeneous heat equation

$$u_t - \Delta u = f$$

with appropriate initial and boundary conditions.

- The Laplace is taken with respect to the spatial variable x

$$\Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}.$$

- Let $\Omega \subset \mathbb{R}^n$ and $T > 0$. The problem is to find a function $u = u(x,t)$ such that it is a solution to the heat equation in $\Omega \times (0,T)$.

5.1 Heat equation

- Now we study the heat equation

$$u_t - \Delta u = 0$$

- and the nonhomogeneous heat equation

$$u_t - \Delta u = f$$

  with appropriate initial and boundary conditions.
- The Laplace is taken with respect to the spatial variable x

$$\Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}.$$

- Let $\Omega \subset \mathbb{R}^n$ and $T > 0$. The problem is to find a function $u = u(x,t)$ such that it is a solution to the heat equation in $\Omega \times (0,T)$.

5.1 Heat equation

- Now we study the heat equation

$$u_t - \Delta u = 0$$

- and the nonhomogeneous heat equation

$$u_t - \Delta u = f$$

  with appropriate initial and boundary conditions.
- The Laplace is taken with respect to the spatial variable x

$$\Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}.$$

- Let $\Omega \subset \mathbb{R}^n$ and $T > 0$. The problem is to find a function $u = u(x,t)$ such that it is a solution to the heat equation in $\Omega \times (0,T)$.

5.1 Heat equation

- Physically, a solution $u = u(x,t)$ of the heat equation represents the temperature of the body Ω at the point x and time t.
- Observe that any solution $v = v(x)$ of the Laplace equation induces a time independent solution $u = u(x,t) = v(x)$ of the heat equation
- This suggests that for every claim about solutions to the Laplace equation there should be a corresponding claim for the solutions of the heat equation.
- However, the dependence in time leads to new phenomena and challenges which are not visible in the stationary case.

5.1 Heat equation

- Physically, a solution $u = u(x,t)$ of the heat equation represents the temperature of the body Ω at the point x and time t.
- Observe that any solution $v = v(x)$ of the Laplace equation induces a time independent solution $u = u(x,t) = v(x)$ of the heat equation
- This suggests that for every claim about solutions to the Laplace equation there should be a corresponding claim for the solutions of the heat equation.
- However, the dependence in time leads to new phenomena and challenges which are not visible in the stationary case.

5.1 Heat equation

- Physically, a solution $u = u(x,t)$ of the heat equation represents the temperature of the body Ω at the point x and time t.
- Observe that any solution $v = v(x)$ of the Laplace equation induces a time independent solution $u = u(x,t) = v(x)$ of the heat equation
- This suggests that for every claim about solutions to the Laplace equation there should be a corresponding claim for the solutions of the heat equation.
- However, the dependence in time leads to new phenomena and challenges which are not visible in the stationary case.

5.1 Heat equation

- Physically, a solution $u = u(x,t)$ of the heat equation represents the temperature of the body Ω at the point x and time t.
- Observe that any solution $v = v(x)$ of the Laplace equation induces a time independent solution $u = u(x,t) = v(x)$ of the heat equation
- This suggests that for every claim about solutions to the Laplace equation there should be a corresponding claim for the solutions of the heat equation.
- However, the dependence in time leads to new phenomena and challenges which are not visible in the stationary case.

5.1 Initial and boundary conditions

- The appropriate initial condition is

$$u(x,0) = g(x), \quad x \in \Omega.$$

  This describes the initial temperature distribution at the time $t = 0$.
- In addition, we may have a Dirichlet type boundary condition

$$u(x,t) = g(x,t), \quad x \in \partial\Omega, \quad t > 0,$$

  which describes the temperature on the boundary,
- or a Neumann type boundary condition

$$\frac{\partial u}{\partial \nu}(x,t) = \nabla_x u(x,t) \cdot \nu(x) = h(x,t), \quad x \in \partial\Omega, \quad t > 0,$$

  which describes the heat flow through the boundary.
- More general equation with heat conductivity $a \in \mathbb{R}$: $u_t - a\Delta u = 0$.

5.1 Initial and boundary conditions

- The appropriate initial condition is

$$u(x,0) = g(x), \quad x \in \Omega.$$

This describes the initial temperature distribution at the time $t = 0$.

- In addition, we may have a Dirichlet type boundary condition

$$u(x,t) = g(x,t), \quad x \in \partial\Omega, \quad t > 0,$$

which describes the temperature on the boundary,

- or a Neumann type boundary condition

$$\frac{\partial u}{\partial \nu}(x,t) = \nabla_x u(x,t) \cdot \nu(x) = h(x,t), \quad x \in \partial\Omega, \quad t > 0,$$

which describes the heat flow through the boundary.

- More general equation with heat conductivity $a \in \mathbb{R}$: $u_t - a\Delta u = 0$.

5.1 Initial and boundary conditions

- The appropriate initial condition is

$$u(x,0) = g(x), \quad x \in \Omega.$$

  This describes the initial temperature distribution at the time $t = 0$.
- In addition, we may have a Dirichlet type boundary condition

$$u(x,t) = g(x,t), \quad x \in \partial\Omega, \quad t > 0,$$

  which describes the temperature on the boundary,
- or a Neumann type boundary condition

$$\frac{\partial u}{\partial \nu}(x,t) = \nabla_x u(x,t) \cdot \nu(x) = h(x,t), \quad x \in \partial\Omega, \quad t > 0,$$

  which describes the heat flow through the boundary.
- More general equation with heat conductivity $a \in \mathbb{R}$: $u_t - a\Delta u = 0$.

5.1 Initial and boundary conditions

- The appropriate initial condition is

$$u(x,0) = g(x), \quad x \in \Omega.$$

This describes the initial temperature distribution at the time $t = 0$.

- In addition, we may have a Dirichlet type boundary condition

$$u(x,t) = g(x,t), \quad x \in \partial\Omega, \quad t > 0,$$

which describes the temperature on the boundary,

- or a Neumann type boundary condition

$$\frac{\partial u}{\partial \nu}(x,t) = \nabla_x u(x,t) \cdot \nu(x) = h(x,t), \quad x \in \partial\Omega, \quad t > 0,$$

which describes the heat flow through the boundary.

- More general equation with heat conductivity $a \in \mathbb{R}$: $u_t - a\Delta u = 0$.

5.2 Fundamental solution

- We already derived (with Fourier techniques) a formula for the solution of the Cauchy problem

$$\begin{cases} u_t - \Delta u = 0 & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u = g & \text{on} \quad \mathbb{R}^n \times \{t=0\}, \end{cases}$$

  where $g \in C_0^\infty(\mathbb{R}^n)$.
- We obtained

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\,dy = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\,dy.$$

-

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}, & x \in \mathbb{R}^n, \quad t > 0, \\ 0, & x \in \mathbb{R}^n, \quad t \le 0, \end{cases}$$

  is the fundamental solution of the heat equation (or the heat kernel).

5.2 Fundamental solution

- We already derived (with Fourier techniques) a formula for the solution of the Cauchy problem

$$\begin{cases} u_t - \Delta u = 0 & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u = g & \text{on} \quad \mathbb{R}^n \times \{t=0\}, \end{cases}$$

  where $g \in C_0^\infty(\mathbb{R}^n)$.
- We obtained

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\, dy = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy.$$

-

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}, & x \in \mathbb{R}^n, \quad t > 0, \\ 0, \quad x \in \mathbb{R}^n, & t \le 0, \end{cases}$$

  is the fundamental solution of the heat equation (or the heat kernel).

5.2 Fundamental solution

- We already derived (with Fourier techniques) a formula for the solution of the Cauchy problem

$$\begin{cases} u_t - \Delta u = 0 \quad \text{in} \quad \mathbb{R}^n \times (0, \infty), \\ u = g \quad \text{on} \quad \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

where $g \in C_0^\infty(\mathbb{R}^n)$.

- We obtained

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\, dy = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy.$$

-

$$\Phi(x,t) = \begin{cases} \dfrac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}, \quad x \in \mathbb{R}^n, \quad t > 0, \\ 0, \quad x \in \mathbb{R}^n, \quad t \le 0, \end{cases}$$

is the fundamental solution of the heat equation (or the heat kernel).

5.2 Fundamental solution

- Φ is a solution to the heat equation in the upper half-space.
- Φ is unbounded in any neighbourhood of $(0, 0)$.
- We proved earlier that the solution attains the initial values g in the sense

$$\lim_{t \to 0} u(x,t) = g(x) \quad \text{for every} \quad x \in \mathbb{R}^n$$

  ("good kernels")
- This means that we have existence of a solution to the Cauchy problem.
- Uniqueness may fail in unbounded domains (we will return to this later).
- The solution is smooth, since the fundamental solution is smooth (in the upper half space) and convolution inherits smoothness.

5.2 Fundamental solution

- Φ is a solution to the heat equation in the upper half-space.
- Φ is unbounded in any neighbourhood of $(0, 0)$.
- We proved earlier that the solution attains the initial values g in the sense

$$\lim_{t \to 0} u(x, t) = g(x) \quad \text{for every} \quad x \in \mathbb{R}^n$$

  ("good kernels")
- This means that we have existence of a solution to the Cauchy problem.
- Uniqueness may fail in unbounded domains (we will return to this later).
- The solution is smooth, since the fundamental solution is smooth (in the upper half space) and convolution inherits smoothness.

5.2 Fundamental solution

- Φ is a solution to the heat equation in the upper half-space.
- Φ is unbounded in any neighbourhood of $(0,0)$.
- We proved earlier that the solution attains the initial values g in the sense

$$\lim_{t\to 0} u(x,t) = g(x) \quad \text{for every} \quad x \in \mathbb{R}^n$$

(“good kernels”)
- This means that we have existence of a solution to the Cauchy problem.
- Uniqueness may fail in unbounded domains (we will return to this later).
- The solution is smooth, since the fundamental solution is smooth (in the upper half space) and convolution inherits smoothness.

5.2 Fundamental solution

- Φ is a solution to the heat equation in the upper half-space.
- Φ is unbounded in any neighbourhood of $(0, 0)$.
- We proved earlier that the solution attains the initial values g in the sense

$$\lim_{t \to 0} u(x, t) = g(x) \quad \text{for every} \quad x \in \mathbb{R}^n$$

("good kernels")

- This means that we have existence of a solution to the Cauchy problem.
- Uniqueness may fail in unbounded domains (we will return to this later).
- The solution is smooth, since the fundamental solution is smooth (in the upper half space) and convolution inherits smoothness.

5.2 Fundamental solution

- Φ is a solution to the heat equation in the upper half-space.
- Φ is unbounded in any neighbourhood of $(0, 0)$.
- We proved earlier that the solution attains the initial values g in the sense

$$\lim_{t\to 0} u(x,t) = g(x) \quad \text{for every} \quad x \in \mathbb{R}^n$$

  ("good kernels")
- This means that we have existence of a solution to the Cauchy problem.
- Uniqueness may fail in unbounded domains (we will return to this later).
- The solution is smooth, since the fundamental solution is smooth (in the upper half space) and convolution inherits smoothness.

5.2 Fundamental solution

- Φ is a solution to the heat equation in the upper half-space.
- Φ is unbounded in any neighbourhood of $(0, 0)$.
- We proved earlier that the solution attains the initial values g in the sense

$$\lim_{t \to 0} u(x, t) = g(x) \quad \text{for every} \quad x \in \mathbb{R}^n$$

  ("good kernels")
- This means that we have existence of a solution to the Cauchy problem.
- Uniqueness may fail in unbounded domains (we will return to this later).
- The solution is smooth, since the fundamental solution is smooth (in the upper half space) and convolution inherits smoothness.

5.2 Fundamental solution

- Formally the fundamental solution solves the following PDE:
$$\begin{cases} \dfrac{\partial \Phi}{\partial t} - \Delta \Phi = 0 \quad \text{in} \quad \mathbb{R}^n \times (0, \infty), \\ \Phi = \delta_0 \quad \text{in} \quad \mathbb{R}^n \times \{t = 0\}, \end{cases}$$
where δ_0 is the Dirac measure in $\mathbb{R}^n$ giving unit mass to the point 0.

5.2 Remarks about heat equation

- The heat equation has infinite speed of propagation.
- For example, if initial values $g \in C(\mathbb{R}^n)$ are non-negative $g \geq 0$ and there is a point $y \in \mathbb{R}^n$ such that $g(y) > 0$, then $u > 0$ everywhere in the upper half-space.
- From the formula

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\, dy = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy.$$

we get the decay estimate

$$\sup_{x \in \mathbb{R}^n} |u(x,t)| \leq \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} |g(y)|\, dy, \quad t > 0.$$

- This means that the solution tends to zero as time goes to infinity.

5.2 Remarks about heat equation

- The heat equation has infinite speed of propagation.
- For example, if initial values $g \in C(\mathbb{R}^n)$ are non-negative $g \geq 0$ and there is a point $y \in \mathbb{R}^n$ such that $g(y) > 0$, then $u > 0$ everywhere in the upper half-space.
- From the formula

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\, dy = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy.$$

we get the decay estimate

$$\sup_{x \in \mathbb{R}^n} |u(x,t)| \leq \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} |g(y)|\, dy, \quad t > 0.$$

- This means that the solution tends to zero as time goes to infinity.

5.2 Remarks about heat equation

- The heat equation has infinite speed of propagation.
- For example, if initial values $g \in C(\mathbb{R}^n)$ are non-negative $g \geq 0$ and there is a point $y \in \mathbb{R}^n$ such that $g(y) > 0$, then $u > 0$ everywhere in the upper half-space.
- From the formula

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\, dy = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy.$$

we get the decay estimate

$$\sup_{x \in \mathbb{R}^n} |u(x,t)| \leq \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} |g(y)|\, dy, \quad t > 0.$$

- This means that the solution tends to zero as time goes to infinity.

5.2 Remarks about heat equation

- The heat equation has infinite speed of propagation.
- For example, if initial values $g \in C(\mathbb{R}^n)$ are non-negative $g \geq 0$ and there is a point $y \in \mathbb{R}^n$ such that $g(y) > 0$, then $u > 0$ everywhere in the upper half-space.
- From the formula

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)g(y)\,dy = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\,dy.$$

  we get the decay estimate

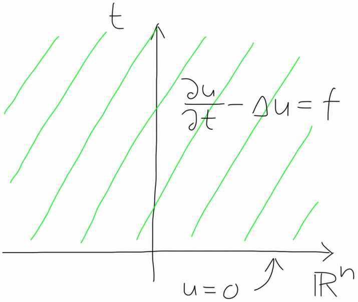
$$\sup_{x \in \mathbb{R}^n} |u(x,t)| \leq \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} |g(y)|\,dy, \quad t > 0.$$

- This means that the solution tends to zero as time goes to infinity.

5.3 Nonhomogeneous problem

- Consider the nonhomogeneous Cauchy problem

$$\begin{cases} u_t - \Delta u = f \quad \text{in} \quad \mathbb{R}^n \times (0, \infty), \\ u = 0 \quad \text{in} \quad \mathbb{R}^n \times \{t = 0\}. \end{cases}$$



5.3 Nonhomogeneous problem

- Consider the nonhomogeneous Cauchy problem

$$\begin{cases} u_t - \Delta u = f \quad \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u = 0 \quad \text{in} \quad \mathbb{R}^n \times \{t=0\}. \end{cases}$$

- From the formula (for the homogeneous problem)

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\, dy = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy.$$

- we obtain that for every fixed s, with $0 < s < t$, the function

$$u(x,t;s) = \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\, dy$$

solves the translated initial value problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t;s) - \Delta u(x,t;s) = 0, \quad x \in \mathbb{R}^n, \quad t > s, \\ u(x,s;s) = f(x,s), \quad x \in \mathbb{R}^n. \end{cases}$$

5.3 Nonhomogeneous problem

- Consider the nonhomogeneous Cauchy problem

$$\begin{cases} u_t - \Delta u = f & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u = 0 & \text{in} \quad \mathbb{R}^n \times \{t=0\}. \end{cases}$$

- From the formula (for the homogeneous problem)

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\,dy = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\,dy.$$

- we obtain that for every fixed s, with $0 < s < t$, the function

$$u(x,t;s) = \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\,dy$$

solves the translated initial value problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t;s) - \Delta u(x,t;s) = 0, & x \in \mathbb{R}^n, \quad t > s, \\ u(x,s;s) = f(x,s), & x \in \mathbb{R}^n. \end{cases}$$

5.3 Nonhomogeneous problem

- Consider the nonhomogeneous Cauchy problem

$$\begin{cases} u_t - \Delta u = f & \text{in} \quad \mathbb{R}^n \times (0, \infty), \\ u = 0 & \text{in} \quad \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

- From the formula (for the homogeneous problem)

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\, dy = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy.$$

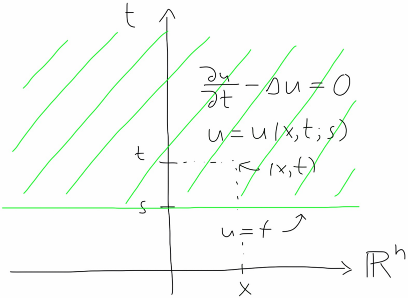
- we obtain that for every fixed s, with $0 < s < t$, the function

$$u(x,t;s) = \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s)\, dy$$

solves the translated initial value problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t;s) - \Delta u(x,t;s) = 0, \quad x \in \mathbb{R}^n, \quad t > s, \\ u(x,s;s) = f(x,s), \quad x \in \mathbb{R}^n. \end{cases}$$

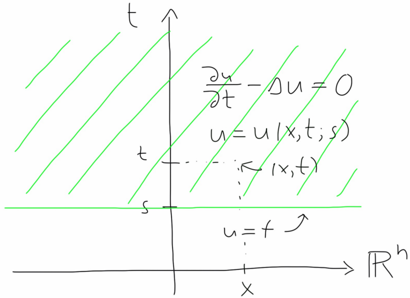
5.3 Duhamel's principle



- Duhamel's principle suggests that we can construct a solution to the nonhomogeneous problem by integrating solutions $u(x,t,;s)$ over $s \in (0,t)$ and have

$$u(x,t) = \int_0^t u(x,t;s)\,ds, \quad x \in \mathbb{R}^n, \quad t > 0.$$

5.3 Duhamel's principle



- Duhamel's principle suggests that we can construct a solution to the nonhomogeneous problem by integrating solutions $u(x,t,;s)$ over $s \in (0,t)$ and have

$$u(x,t) = \int_0^t u(x,t;s)\,ds, \quad x \in \mathbb{R}^n, \quad t > 0.$$

5.3 Duhamel's principle

- Duhamel's principle is a process of expressing the solution of a nonhomogeneous problem as an integral of the solutions to the homogeneous problem in the way that the source term is interpreted as the initial condition.
- It does not depend on the specific structure of the equation and it applies to other linear ODEs and PDEs as well.
- We can verify it by differentiating the function u defined by an integral.

$$u(x,t) = \int_0^t u(x,t;s)\,ds, \quad x \in \mathbb{R}^n, \quad t > 0.$$

- Recall that (Lemma 5.3)

$$\frac{\partial u}{\partial t}(x,t) = u(x,t;t) + \int_0^t \frac{\partial u}{\partial t}(x,t;s)\,ds.$$

5.3 Duhamel's principle

- Duhamel's principle is a process of expressing the solution of a nonhomogeneous problem as an integral of the solutions to the homogeneous problem in the way that the source term is interpreted as the initial condition.
- It does not depend on the specific structure of the equation and it applies to other linear ODEs and PDEs as well.
- We can verify it by differentiating the function u defined by an integral.

$$u(x,t) = \int_0^t u(x,t;s)\,ds, \quad x \in \mathbb{R}^n, \quad t > 0.$$

- Recall that (Lemma 5.3)

$$\frac{\partial u}{\partial t}(x,t) = u(x,t;t) + \int_0^t \frac{\partial u}{\partial t}(x,t;s)\,ds.$$

5.3 Duhamel's principle

- Duhamel's principle is a process of expressing the solution of a nonhomogeneous problem as an integral of the solutions to the homogeneous problem in the way that the source term is interpreted as the initial condition.
- It does not depend on the specific structure of the equation and it applies to other linear ODEs and PDEs as well.
- We can verify it by differentiating the function u defined by an integral.

$$u(x,t) = \int_0^t u(x,t;s)\,ds, \quad x \in \mathbb{R}^n, \quad t > 0.$$

- Recall that (Lemma 5.3)

$$\frac{\partial u}{\partial t}(x,t) = u(x,t;t) + \int_0^t \frac{\partial u}{\partial t}(x,t;s)\,ds.$$

5.3 Duhamel's principle

- Duhamel's principle is a process of expressing the solution of a nonhomogeneous problem as an integral of the solutions to the homogeneous problem in the way that the source term is interpreted as the initial condition.
- It does not depend on the specific structure of the equation and it applies to other linear ODEs and PDEs as well.
- We can verify it by differentiating the function u defined by an integral.

$$u(x,t) = \int_0^t u(x,t;s)\,ds, \quad x \in \mathbb{R}^n, \quad t > 0.$$

- Recall that (Lemma 5.3)

$$\frac{\partial u}{\partial t}(x,t) = u(x,t;t) + \int_0^t \frac{\partial u}{\partial t}(x,t;s)\,ds.$$

5.3 Duhamel's principle

Theorem

The solution of the nonhomogeneous Cauchy problem is

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\,dy\,ds$$
$$= \int_0^t \frac{1}{(4\pi(t-s))^{\frac{n}{2}}} \left(\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s)\,dy \right) ds.$$

- Formal proof:

$$\frac{\partial u}{\partial t}(x,t) = u(x,t;t) + \int_0^t \frac{\partial u}{\partial t}(x,t;s)\,ds$$
$$= f(x,t) + \int_0^t \Delta u(x,t;s)\,ds$$
$$= f(x,t) + \Delta \left(\int_0^t u(x,t;s)\,ds \right)$$
$$= f(x,t) + \Delta u(x,t).$$

5.3 Duhamel's principle

Theorem

The solution of the nonhomogeneous Cauchy problem is

$$
\begin{aligned}
u(x,t) &= \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\,dy\,ds \\
&= \int_0^t \frac{1}{(4\pi(t-s))^{\frac{n}{2}}} \left(\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s)\,dy \right) ds.
\end{aligned}
$$

- Formal proof:

$$
\begin{aligned}
\frac{\partial u}{\partial t}(x,t) &= u(x,t;t) + \int_0^t \frac{\partial u}{\partial t}(x,t;s)\,ds \\
&= f(x,t) + \int_0^t \Delta u(x,t;s)\,ds \\
&= f(x,t) + \Delta \left(\int_0^t u(x,t;s)\,ds \right) \\
&= f(x,t) + \Delta u(x,t).
\end{aligned}
$$

5.3 Duhamel's principle

Theorem

The solution of the nonhomogeneous Cauchy problem is

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s)\,dy\,ds$$
$$= \int_0^t \frac{1}{(4\pi(t-s))^{\frac{n}{2}}} \left(\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s)\,dy \right) ds.$$

- Formal proof:

$$\frac{\partial u}{\partial t}(x,t) = u(x,t;t) + \int_0^t \frac{\partial u}{\partial t}(x,t;s)\,ds$$
$$= f(x,t) + \int_0^t \Delta u(x,t;s)\,ds$$
$$= f(x,t) + \Delta \left(\int_0^t u(x,t;s)\,ds \right)$$
$$= f(x,t) + \Delta u(x,t).$$

5.3 Duhamel's principle

Theorem

The solution of the nonhomogeneous Cauchy problem is

$$\begin{aligned} u(x,t) &= \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\, dy\, ds \\ &= \int_0^t \frac{1}{(4\pi(t-s))^{\frac{n}{2}}} \left(\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s)\, dy \right) ds. \end{aligned}$$

- Formal proof:

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) &= u(x,t;t) + \int_0^t \frac{\partial u}{\partial t}(x,t;s)\, ds \\ &= f(x,t) + \int_0^t \Delta u(x,t;s)\, ds \\ &= f(x,t) + \Delta \left(\int_0^t u(x,t;s)\, ds \right) \\ &= f(x,t) + \Delta u(x,t). \end{aligned}$$

5.3 Duhamel's principle

Theorem

The solution of the nonhomogeneous Cauchy problem is

$$\begin{aligned} u(x,t) &= \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\,dy\,ds \\ &= \int_0^t \frac{1}{(4\pi(t-s))^{\frac{n}{2}}} \left(\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s)\,dy \right) ds. \end{aligned}$$

- Formal proof:

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) &= u(x,t;t) + \int_0^t \frac{\partial u}{\partial t}(x,t;s)\,ds \\ &= f(x,t) + \int_0^t \Delta u(x,t;s)\,ds \\ &= f(x,t) + \Delta \left(\int_0^t u(x,t;s)\,ds \right) \\ &= f(x,t) + \Delta u(x,t). \end{aligned}$$

5.3 Nonhomogeneous problem with general initial data

- To solve the general problem

$$\begin{cases} u_t - \Delta u = f & \text{in} \quad \mathbb{R}^n \times (0, \infty), \\ u = g & \text{in} \quad \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

  we can use the same approach as for the Laplace equation and write $u = u_1 + u_2$ with

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - \Delta u_1 = f & \text{in} \quad \mathbb{R}^n \times (0, \infty), \\ u_1 = 0 & \text{in} \quad \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

  and

$$\begin{cases} \dfrac{\partial u_2}{\partial t} - \Delta u_2 = 0 & \text{in} \quad \mathbb{R}^n \times (0, \infty), \\ u_2 = g & \text{in} \quad \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

Aalto University

MS-C1350 Partial differential equations Chapter 5.4-5.6: Heat equation – separation of variables, maximum principle and energy methods

Riikka Korte

Department of Mathematics and Systems Analysis
Aalto University
riikka.korte@aalto.fi

November 12, 2024

Lecture 10

We continue discussing the heat equation:

- Separation of variables (x and t) in bounded domains and eigenvalue problems for the Laplace equation.
- Maximum principles.
- Uniqueness.

5.4 Separation of variables for heat equation in $\mathbb{R}^n$

- We have now derived a solution to an initial value problem for the heat equation in the whole space $\mathbb{R}^n$.
- Now our goal is to derive corresponding solutions in a subdomain.
- Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with a smooth boundary.
- Consider the initial and boundary value problem

$$\begin{cases} u_t - \Delta u = 0 \quad \text{in} \quad \Omega \times (0, \infty), \\ u = 0 \quad \text{on} \quad \partial\Omega \times (0, \infty), \\ u = g \quad \text{on} \quad \Omega \times \{t = 0\}. \end{cases}$$

- We use separation of variables technique

5.4 Separation of variables for heat equation in $\mathbb{R}^n$

- We have now derived a solution to an initial value problem for the heat equation in the whole space $\mathbb{R}^n$.
- Now our goal is to derive corresponding solutions in a subdomain.
- Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with a smooth boundary.
- Consider the initial and boundary value problem

$$\begin{cases} u_t - \Delta u = 0 & \text{in} \quad \Omega \times (0, \infty), \\ u = 0 & \text{on} \quad \partial\Omega \times (0, \infty), \\ u = g & \text{on} \quad \Omega \times \{t = 0\}. \end{cases}$$

- We use separation of variables technique

5.4 Separation of variables for heat equation in $\mathbb{R}^n$

- We have now derived a solution to an initial value problem for the heat equation in the whole space $\mathbb{R}^n$.
- Now our goal is to derive corresponding solutions in a subdomain.
- Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with a smooth boundary.
- Consider the initial and boundary value problem

$$\begin{cases} u_t - \Delta u = 0 & \text{in} \quad \Omega \times (0, \infty), \\ u = 0 & \text{on} \quad \partial\Omega \times (0, \infty), \\ u = g & \text{on} \quad \Omega \times \{t = 0\}. \end{cases}$$

- We use separation of variables technique

5.4 Separation of variables for heat equation in $\mathbb{R}^n$

- We have now derived a solution to an initial value problem for the heat equation in the whole space $\mathbb{R}^n$.
- Now our goal is to derive corresponding solutions in a subdomain.
- Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with a smooth boundary.
- Consider the initial and boundary value problem

$$\begin{cases} u_t - \Delta u = 0 \quad \text{in} \quad \Omega \times (0, \infty), \\ u = 0 \quad \text{on} \quad \partial\Omega \times (0, \infty), \\ u = g \quad \text{on} \quad \Omega \times \{t = 0\}. \end{cases}$$

- We use separation of variables technique

5.4 Separation of variables for heat equation in $\mathbb{R}^n$

- We have now derived a solution to an initial value problem for the heat equation in the whole space $\mathbb{R}^n$.
- Now our goal is to derive corresponding solutions in a subdomain.
- Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with a smooth boundary.
- Consider the initial and boundary value problem

$$\begin{cases} u_t - \Delta u = 0 & \text{in} \quad \Omega \times (0, \infty), \\ u = 0 & \text{on} \quad \partial\Omega \times (0, \infty), \\ u = g & \text{on} \quad \Omega \times \{t = 0\}. \end{cases}$$

- We use separation of variables technique

5.4 Step 1: Separation of variables

- We separate variables and look for a solution in the form

$$u(x,t) = v(t)w(x), \quad x \in \Omega, \quad t > 0.$$

- Then

$$u_t(x,t) = v'(t)w(x) \quad \text{and} \quad \Delta u(x,t) = v(t)\Delta w(x)$$

- From the heat equation $u_t = \Delta u$, we get the condition

$$\frac{v'(t)}{v(t)} = \frac{\Delta w(x)}{w(x)} = -\lambda.$$

- As LHS depends only on t and RHS on x, both sides have to be the same constant $-\lambda$.
- We will soon see why it is convenient to use negative sign with λ.

5.4 Step 1: Separation of variables

- We separate variables and look for a solution in the form

$$u(x,t) = v(t)w(x), \quad x \in \Omega, \quad t > 0.$$

- Then

$$u_t(x,t) = v'(t)w(x) \quad \text{and} \quad \Delta u(x,t) = v(t)\Delta w(x)$$

- From the heat equation $u_t = \Delta u$, we get the condition

$$\frac{v'(t)}{v(t)} = \frac{\Delta w(x)}{w(x)} = -\lambda.$$

- As LHS depends only on t and RHS on x, both sides have to be the same constant $-\lambda$.
- We will soon see why it is convenient to use negative sign with λ.

5.4 Step 1: Separation of variables

- We separate variables and look for a solution in the form

$$u(x,t) = v(t)w(x), \quad x \in \Omega, \quad t > 0.$$

- Then

$$u_t(x,t) = v'(t)w(x) \quad \text{and} \quad \Delta u(x,t) = v(t)\Delta w(x)$$

- From the heat equation $u_t = \Delta u$, we get the condition

$$\frac{v'(t)}{v(t)} = \frac{\Delta w(x)}{w(x)} = -\lambda.$$

- As LHS depends only on t and RHS on x, both sides have to be the same constant $-\lambda$.
- We will soon see why it is convenient to use negative sign with λ.

5.4 Step 1: Separation of variables

- We separate variables and look for a solution in the form

$$u(x,t) = v(t)w(x), \quad x \in \Omega, \quad t > 0.$$

- Then

$$u_t(x,t) = v'(t)w(x) \quad \text{and} \quad \Delta u(x,t) = v(t)\Delta w(x)$$

- From the heat equation $u_t = \Delta u$, we get the condition

$$\frac{v'(t)}{v(t)} = \frac{\Delta w(x)}{w(x)} = -\lambda.$$

- As LHS depends only on t and RHS on x, both sides have to be the same constant $-\lambda$.
- We will soon see why it is convenient to use negative sign with λ.

5.4 Step 1: Separation of variables

- We separate variables and look for a solution in the form

$$u(x,t) = v(t)w(x), \quad x \in \Omega, \quad t > 0.$$

- Then

$$u_t(x,t) = v'(t)w(x) \quad \text{and} \quad \Delta u(x,t) = v(t)\Delta w(x)$$

- From the heat equation $u_t = \Delta u$, we get the condition

$$\frac{v'(t)}{v(t)} = \frac{\Delta w(x)}{w(x)} = -\lambda.$$

- As LHS depends only on t and RHS on x, both sides have to be the same constant $-\lambda$.
- We will soon see why it is convenient to use negative sign with λ.

5.4 Step 2: Solutions to the separated equations

$$\frac{v'(t)}{v(t)} = \frac{\Delta w(x)}{w(x)} = -\lambda.$$

- The general solution of $v' = -\lambda v$ is

$$v(t) = ce^{-\lambda t},$$

  where c is a constant.
- How about the other condition?

5.4 Step 2: Solutions to the separated equations

$$\frac{v'(t)}{v(t)} = \frac{\Delta w(x)}{w(x)} = -\lambda.$$

- The general solution of $v' = -\lambda v$ is

$$v(t) = ce^{-\lambda t},$$

  where c is a constant.
- How about the other condition?

5.4 Step 2: Solutions to the separated equations

- Consider then the other equation $-\Delta w = \lambda w$.
- We say that λ is an eigenvalue of the (negative) Laplacian in Ω, if there exists a solution w of the problem

$$\begin{cases} -\Delta w = \lambda w & \text{in} \quad \Omega, \\ w = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

so that w is not identically zero in Ω. The function w is the eigenfunction corresponding to the eigenvalue λ.

5.4 Step 2: Solutions to the separated equations

- Consider then the other equation $-\Delta w = \lambda w$.
- We say that λ is an eigenvalue of the (negative) Laplacian in Ω, if there exists a solution w of the problem

$$\begin{cases} -\Delta w = \lambda w \quad \text{in} \quad \Omega, \\ w = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

so that w is not identically zero in Ω. The function w is the eigenfunction corresponding to the eigenvalue λ.

5.4 Eigenfunctions

- The first Green's identity gives

$$
\begin{aligned}
0 \leq \int_\Omega |\nabla w|^2 \, dx &= \int_\Omega \nabla w \cdot \nabla w \, dx \\
&= -\int_\Omega w \Delta w \, dx + \int_{\partial\Omega} \frac{\partial w}{\partial \nu} \underbrace{w}_{=0} \, dS \\
&= \lambda \int_\Omega w^2 \, dx. \qquad (-\Delta w = \lambda w)
\end{aligned}
$$

- This implies that $\lambda \geq 0$. (In fact $\lambda > 0$ as $w = 0$ if $\lambda = 0$.)

5.4 Eigenfunctions

- The first Green's identity gives

$$
\begin{aligned}
0 \leq \int_\Omega |\nabla w|^2 \, dx &= \int_\Omega \nabla w \cdot \nabla w \, dx \\
&= -\int_\Omega w \Delta w \, dx + \int_{\partial\Omega} \frac{\partial w}{\partial \nu} \underbrace{w}_{=0} \, dS \\
&= \lambda \int_\Omega w^2 \, dx. \qquad (-\Delta w = \lambda w)
\end{aligned}
$$

- This implies that $\lambda \geq 0$. (In fact $\lambda > 0$ as $w = 0$ if $\lambda = 0$.)

5.4 Eigenfunctions

- The first Green's identity gives

$$
\begin{aligned}
0 \leq \int_\Omega |\nabla w|^2 \, dx &= \int_\Omega \nabla w \cdot \nabla w \, dx \\
&= -\int_\Omega w \Delta w \, dx + \int_{\partial\Omega} \frac{\partial w}{\partial \nu} \underbrace{w}_{=0} \, dS \\
&= \lambda \int_\Omega w^2 \, dx. \qquad (-\Delta w = \lambda w)
\end{aligned}
$$

- This implies that $\lambda \geq 0$. (In fact $\lambda > 0$ as $w = 0$ if $\lambda = 0$.)

5.4 Eigenfunctions

- The first Green's identity gives

$$
\begin{aligned}
0 \leq \int_\Omega |\nabla w|^2 \, dx &= \int_\Omega \nabla w \cdot \nabla w \, dx \\
&= -\int_\Omega w \Delta w \, dx + \int_{\partial\Omega} \frac{\partial w}{\partial \nu} \underbrace{w}_{=0} \, dS \\
&= \lambda \int_\Omega w^2 \, dx. \qquad (-\Delta w = \lambda w)
\end{aligned}
$$

- This implies that $\lambda \geq 0$. (In fact $\lambda > 0$ as $w = 0$ if $\lambda = 0$.)

5.4 Step 2: Solutions to the separated equations

- We conclude that

$$u(x,t) = ce^{-\lambda t} w(x)$$

is a solution to

$$\begin{cases} u_t - \Delta u = 0 \quad \text{in} \quad \Omega \times (0,\infty), \\ u = 0 \quad \text{on} \quad \partial\Omega \times (0,\infty), \end{cases}$$

with the initial condition $u(x,0) = cw(x)$ for $x \in \Omega$, whenever w is an eigenfunction with eigenvalue λ.

5.4 Step 3: Solution of the entire problem

- Let λ_j, $j = 1, 2, \ldots$ be eigenvalues and w_j, $j = 1, 2, \ldots$, the corresponding eigenfunctions.
- Then the linear combination

$$u(x,t) = \sum_{j=1}^{\infty} c_j e^{-\lambda_j t} w_j(x)$$

should be a solution with the initial condition

$$u(x,0) = \sum_{j=1}^{\infty} c_j w_j(x).$$

- If we can determine the coefficients c_j, $j = 1, 2, \ldots$, so that

$$\sum_{j=1}^{\infty} c_j w_j(x) = g(x),$$

then u is a solution of the original problem.

5.4 Step 3: Solution of the entire problem

- Let λ_j, $j = 1, 2, \dots$ be eigenvalues and w_j, $j = 1, 2, \dots,$ the corresponding eigenfunctions.
- Then the linear combination

$$u(x,t) = \sum_{j=1}^{\infty} c_j e^{-\lambda_j t} w_j(x)$$

should be a solution with the initial condition

$$u(x,0) = \sum_{j=1}^{\infty} c_j w_j(x).$$

- If we can determine the coefficients c_j, $j = 1, 2, \dots,$ so that

$$\sum_{j=1}^{\infty} c_j w_j(x) = g(x),$$

then u is a solution of the original problem.

5.4 Step 3: Solution of the entire problem

- Let λ_j, $j = 1, 2, \dots$ be eigenvalues and w_j, $j = 1, 2, \dots$, the corresponding eigenfunctions.
- Then the linear combination

$$u(x,t) = \sum_{j=1}^{\infty} c_j e^{-\lambda_j t} w_j(x)$$

  should be a solution with the initial condition

$$u(x,0) = \sum_{j=1}^{\infty} c_j w_j(x).$$

- If we can determine the coefficients c_j, $j = 1, 2, \dots$, so that

$$\sum_{j=1}^{\infty} c_j w_j(x) = g(x),$$

  then u is a solution of the original problem.

5.4 Step 3: Solution of the entire problem

- It is known that there is a countable number of eigenvalues $0 < \lambda_1 < \lambda_2 < \dots$.
- Moreover, the corresponding eigenfunctions $\{w_j\}_{j=1}^{\infty}$ can be chosen to be an orthonormal basis in $L^2(\Omega)$.
- This means that if $g \in L^2(\Omega)$, then

  $$c_j = \langle g, w_j \rangle = \int_\Omega g(y) w_j(y)\, dy, \quad j = 1, 2, \dots.$$

  Here $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $L^2(\Omega)$.
- The coefficients c_j, $j = 1, 2, \dots$, can be seen as the Fourier coefficients of $g \in L^2(\Omega)$ and the series

  $$g = \sum_{j=1}^{\infty} c_j w_j = \sum_{j=1}^{\infty} \langle g, w_j \rangle w_j$$

  converges in $L^2(\Omega)$,

5.4 Step 3: Solution of the entire problem

- It is known that there is a countable number of eigenvalues $0 < \lambda_1 < \lambda_2 < \dots$.
- Moreover, the corresponding eigenfunctions $\{w_j\}_{j=1}^{\infty}$can be chosen to be an orthonormal basis in $L^2(\Omega)$.
- This means that if $g \in L^2(\Omega)$, then

$$c_j = \langle g, w_j \rangle = \int_\Omega g(y) w_j(y)\, dy, \quad j = 1, 2, \dots .$$

Here $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $L^2(\Omega)$.
- The coefficients c_j, $j = 1, 2, \dots$, can be seen as the Fourier coefficients of $g \in L^2(\Omega)$ and the series

$$g = \sum_{j=1}^{\infty} c_j w_j = \sum_{j=1}^{\infty} \langle g, w_j \rangle w_j$$

converges in $L^2(\Omega)$,

5.4 Step 3: Solution of the entire problem

- It is known that there is a countable number of eigenvalues $0 < \lambda_1 < \lambda_2 < \dots$.
- Moreover, the corresponding eigenfunctions $\{w_j\}_{j=1}^{\infty}$ can be chosen to be an orthonormal basis in $L^2(\Omega)$.
- This means that if $g \in L^2(\Omega)$, then

$$c_j = \langle g, w_j \rangle = \int_\Omega g(y) w_j(y)\, dy, \quad j = 1, 2, \dots.$$

Here $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $L^2(\Omega)$.

- The coefficients c_j, $j = 1, 2, \dots$, can be seen as the Fourier coefficients of $g \in L^2(\Omega)$ and the series

$$g = \sum_{j=1}^{\infty} c_j w_j = \sum_{j=1}^{\infty} \langle g, w_j \rangle w_j$$

converges in $L^2(\Omega)$,

5.4 Step 3: Solution of the entire problem

- It is known that there is a countable number of eigenvalues $0 < \lambda_1 < \lambda_2 < \dots$.
- Moreover, the corresponding eigenfunctions $\{w_j\}_{j=1}^{\infty}$ can be chosen to be an orthonormal basis in $L^2(\Omega)$.
- This means that if $g \in L^2(\Omega)$, then

$$c_j = \langle g, w_j \rangle = \int_\Omega g(y) w_j(y)\, dy, \quad j = 1, 2, \dots .$$

  Here $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $L^2(\Omega)$.
- The coefficients c_j, $j = 1, 2, \dots$, can be seen as the Fourier coefficients of $g \in L^2(\Omega)$ and the series

$$g = \sum_{j=1}^{\infty} c_j w_j = \sum_{j=1}^{\infty} \langle g, w_j \rangle w_j$$

  converges in $L^2(\Omega)$,

5.4 Step 3: Solution of the entire problem

- Thus we have the representation formula

$$u(x,t) = \sum_{j=1}^{\infty} \langle g, w_j \rangle e^{-\lambda_j t} w_j(x)$$
$$= \int_\Omega K(x,y,t) g(y)\, dy,$$

where

$$K(x,y,t) = \sum_{j=1}^{\infty} e^{-\lambda_j t} w_j(x) w_j(y)$$

is the heat kernel in Ω.

- Note that this depends on whether we can find so many eigenfunctions that we can represent the initial value as an infinite linear combination of the eigenfunctions.
- We also have to check that the series converges and that the function u is really a solution to the original problem.

5.4 Step 3: Solution of the entire problem

- Thus we have the representation formula

$$
\begin{aligned}
u(x,t) &= \sum_{j=1}^{\infty} \langle g, w_j \rangle e^{-\lambda_j t} w_j(x) \\
&= \int_{\Omega} K(x,y,t) g(y)\, dy,
\end{aligned}
$$

  where

$$
K(x,y,t) = \sum_{j=1}^{\infty} e^{-\lambda_j t} w_j(x) w_j(y)
$$

  is the heat kernel in Ω.
- Note that this depends on whether we can find so many eigenfunctions that we can represent the initial value as an infinite linear combination of the eigenfunctions.
- We also have to check that the series converges and that the function u is really a solution to the original problem.

5.4 Step 3: Solution of the entire problem

- Thus we have the representation formula

$$u(x,t) = \sum_{j=1}^{\infty} \langle g, w_j \rangle e^{-\lambda_j t} w_j(x) = \int_\Omega K(x,y,t) g(y)\, dy,$$

  where

$$K(x,y,t) = \sum_{j=1}^{\infty} e^{-\lambda_j t} w_j(x) w_j(y)$$

  is the heat kernel in Ω.
- Note that this depends on whether we can find so many eigenfunctions that we can represent the initial value as an infinite linear combination of the eigenfunctions.
- We also have to check that the series converges and that the function u is really a solution to the original problem.

5.4 Asymptotical behaviour

- With time dependent problems, it is relevant to study the behaviour of solutions as $t \to \infty$.
- For this problem,

$$\|u(\cdot, t)\|_{L^2(\Omega)} \leq e^{-\lambda_1 t} \|g\|_{L^2(\Omega)}, \quad t > 0,$$

  where $\lambda_1 > 0$ is the first (and the smallest) eigenvalue of Laplacian.

5.5 Maximum principle

- As Laplace equation, also solutions to the heat equation satisfy a maximum principle.
- Let us introduce the space-time cylinder:

$$\Omega_T = \Omega \times (0, T),$$

where $\Omega \subset \mathbb{R}^n$ is open and bounded and $0 < T < \infty$.

- For Laplace equation, maximum is achieved on the boundary $\partial\Omega$.
- For heat equation, maximum is achieved on certain part of the boundary, which is called the parabolic boundary

$$\Gamma_T = (\partial\Omega \times [0, T]) \cup (\Omega \times \{t = 0\}).$$

5.5 Maximum principle

- As Laplace equation, also solutions to the heat equation satisfy a maximum principle.
- Let us introduce the space-time cylinder:

$$\Omega_T = \Omega \times (0, T),$$

  where $\Omega \subset \mathbb{R}^n$ is open and bounded and $0 < T < \infty$.
- For Laplace equation, maximum is achieved on the boundary $\partial\Omega$.
- For heat equation, maximum is achieved on certain part of the boundary, which is called the parabolic boundary

$$\Gamma_T = (\partial\Omega \times [0, T]) \cup (\Omega \times \{t = 0\}).$$

5.5 Maximum principle

- As Laplace equation, also solutions to the heat equation satisfy a maximum principle.
- Let us introduce the space-time cylinder:

$$\Omega_T = \Omega \times (0, T),$$

where $\Omega \subset \mathbb{R}^n$ is open and bounded and $0 < T < \infty$.
- For Laplace equation, maximum is achieved on the boundary $\partial\Omega$.
- For heat equation, maximum is achieved on certain part of the boundary, which is called the parabolic boundary

$$\Gamma_T = (\partial\Omega \times [0, T]) \cup (\Omega \times \{t = 0\}).$$

5.5 Maximum principle

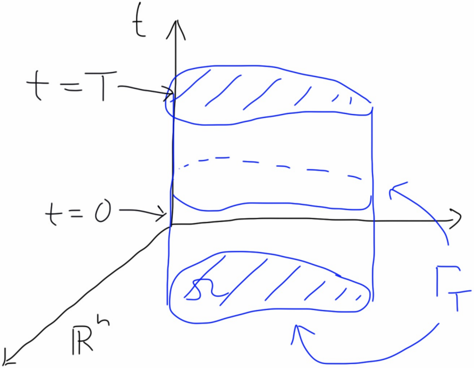
- As Laplace equation, also solutions to the heat equation satisfy a maximum principle.
- Let us introduce the space-time cylinder:

$$\Omega_T = \Omega \times (0, T),$$

where $\Omega \subset \mathbb{R}^n$ is open and bounded and $0 < T < \infty$.
- For Laplace equation, maximum is achieved on the boundary $\partial\Omega$.
- For heat equation, maximum is achieved on certain part of the boundary, which is called the parabolic boundary

$$\Gamma_T = (\partial\Omega \times [0, T]) \cup (\Omega \times \{t = 0\}).$$

5.5 Parabolic boundary



5.5 Weak maximum principle

Theorem

Let $\Omega \subset \mathbb{R}^n$ be open and bounded and assume that $u \in C^2(\Omega_T) \cap C(\overline{\Omega_T})$ is a solution to the heat equation in Ω_T. Then

$$\max_{(x,t)\in\overline{\Omega_T}} u(x,t) = \max_{(x,t)\in\Gamma_T} u(x,t).$$

- If u is a solution to the heat equation then also $-u$ is a solution.
- Thus, if we replace u by $-u$ we get the corresponding statement with $\min$ replacing $\max$.

5.5 Weak maximum principle

Theorem

Let $\Omega \subset \mathbb{R}^n$ be open and bounded and assume that $u \in C^2(\Omega_T) \cap C(\overline{\Omega_T})$ is a solution to the heat equation in Ω_T. Then

$$\max_{(x,t)\in\overline{\Omega_T}} u(x,t) = \max_{(x,t)\in\Gamma_T} u(x,t).$$

- If u is a solution to the heat equation then also $-u$ is a solution.
- Thus, if we replace u by $-u$ we get the corresponding statement with $\min$ replacing $\max$.

5.5 Weak maximum principle

Theorem

Let $\Omega \subset \mathbb{R}^n$ be open and bounded and assume that $u \in C^2(\Omega_T) \cap C(\overline{\Omega_T})$ is a solution to the heat equation in Ω_T. Then

$$\max_{(x,t)\in\overline{\Omega_T}} u(x,t) = \max_{(x,t)\in\Gamma_T} u(x,t).$$

- If u is a solution to the heat equation then also $-u$ is a solution.
- Thus, if we replace u by $-u$ we get the corresponding statement with $\min$ replacing $\max$.

5.5 Weak maximum principle – proof

- **Step 1**: Prove the maximum principle for
$$v(x,t) = u(x,t) - \varepsilon t.$$
- **Step 2**: Let $\varepsilon \searrow 0$ to conclude that it holds also for u.
- Proof of Step 1: Proof by contradiction: Suppose that the maximum is attained at $(x_0, t_0) \in \overline{\Omega_T} \setminus \Gamma_T = \Omega \times (0, T]$
- We have $v(x_0, t) \leq v(x_0, t_0)$ for all $t < t_0$. This implies that
$$\frac{\partial v}{\partial t}(x_0, t_0) \geq 0,$$
- As (x_0, t_0) is a maximum in x-direction, $\nabla v(x_0, t_0) = 0$ and
$$\frac{\partial^2 v}{\partial x_j^2}(x_0, t_0) \leq 0, \quad j = 1, \ldots, n,$$
- Thus
$$\frac{\partial v}{\partial t}(x_0, t_0) - \Delta v(x_0, t_0) \geq 0.$$

5.5 Weak maximum principle – proof

- **Step 1**: Prove the maximum principle for

$$v(x,t) = u(x,t) - \varepsilon t.$$

- **Step 2**: Let $\varepsilon \searrow 0$ to conclude that it holds also for u.
- Proof of Step 1: Proof by contradiction: Suppose that the maximum is attained at $(x_0, t_0) \in \overline{\Omega_T} \setminus \Gamma_T = \Omega \times (0, T]$
- We have $v(x_0, t) \le v(x_0, t_0)$ for all $t < t_0$. This implies that

$$\frac{\partial v}{\partial t}(x_0, t_0) \ge 0,$$

- As (x_0, t_0) is a maximum in x-direction, $\nabla v(x_0, t_0) = 0$ and

$$\frac{\partial^2 v}{\partial x_j^2}(x_0, t_0) \le 0, \quad j = 1, \dots, n,$$

- Thus

$$\frac{\partial v}{\partial t}(x_0, t_0) - \Delta v(x_0, t_0) \ge 0.$$

5.5 Weak maximum principle – proof

- **Step 1**: Prove the maximum principle for

$$v(x,t) = u(x,t) - \varepsilon t.$$

- **Step 2**: Let $\varepsilon \searrow 0$ to conclude that it holds also for u.
- Proof of Step 1: Proof by contradiction: Suppose that the maximum is attained at $(x_0, t_0) \in \overline{\Omega_T} \setminus \Gamma_T = \Omega \times (0, T]$
- We have $v(x_0, t) \leq v(x_0, t_0)$ for all $t < t_0$. This implies that

$$\frac{\partial v}{\partial t}(x_0, t_0) \geq 0,$$

- As (x_0, t_0) is a maximum in x-direction, $\nabla v(x_0, t_0) = 0$ and

$$\frac{\partial^2 v}{\partial x_j^2}(x_0, t_0) \leq 0, \quad j = 1, \ldots, n,$$

- Thus

$$\frac{\partial v}{\partial t}(x_0, t_0) - \Delta v(x_0, t_0) \geq 0.$$

5.5 Weak maximum principle – proof

- **Step 1**: Prove the maximum principle for
$$v(x,t) = u(x,t) - \varepsilon t.$$
- **Step 2**: Let $\varepsilon \searrow 0$ to conclude that it holds also for u.
- Proof of Step 1: Proof by contradiction: Suppose that the maximum is attained at $(x_0, t_0) \in \overline{\Omega_T} \setminus \Gamma_T = \Omega \times (0, T]$
- We have $v(x_0, t) \le v(x_0, t_0)$ for all $t < t_0$. This implies that
$$\frac{\partial v}{\partial t}(x_0, t_0) \ge 0,$$
- As (x_0, t_0) is a maximum in x-direction, $\nabla v(x_0, t_0) = 0$ and
$$\frac{\partial^2 v}{\partial x_j^2}(x_0, t_0) \le 0, \quad j = 1, \ldots, n,$$
- Thus
$$\frac{\partial v}{\partial t}(x_0, t_0) - \Delta v(x_0, t_0) \ge 0.$$

5.5 Weak maximum principle – proof

- So we obtained

$$\frac{\partial v}{\partial t}(x_0, t_0) - \Delta v(x_0, t_0) \geq 0.$$

- This is a contradiction with

$$\frac{\partial v}{\partial t} - \Delta v = u_t - \varepsilon - \Delta u = -\varepsilon < 0.$$

5.5 Weak maximum principle – proof

- So we obtained

$$\frac{\partial v}{\partial t}(x_0, t_0) - \Delta v(x_0, t_0) \geq 0.$$

- This is a contradiction with

$$\frac{\partial v}{\partial t} - \Delta v = u_t - \varepsilon - \Delta u = -\varepsilon < 0.$$

5.5 Uniqueness for bounded domains

Theorem (Uniqueness for bounded domains)

Assume that Ω_T is bounded, $g \in C(\Gamma_T)$ and $f \in C(\Omega_T)$. Then there exists at most one solution $u \in C^2(\Omega_T) \cap C(\overline{\Omega_T})$ of the initial and boundary value problem

$$\begin{cases} u_t - \Delta u = f \quad \textit{in} \quad \Omega_T, \\ u = g \quad \textit{on} \quad \Gamma_T. \end{cases}$$

- The initial and boundary values can be given only on the parabolic boundary Γ_T.
- The equation determines the values uniquely inside Ω_T and on the top $\Omega \times \{t = T\}$.

5.5 Uniqueness for bounded domains

Theorem (Uniqueness for bounded domains)

Assume that Ω_T *is bounded,* $g \in C(\Gamma_T)$ *and* $f \in C(\Omega_T)$*. Then there exists at most one solution* $u \in C^2(\Omega_T) \cap C(\overline{\Omega_T})$ *of the initial and boundary value problem*

$$\begin{cases} u_t - \Delta u = f & \textit{in} \quad \Omega_T, \\ u = g & \textit{on} \quad \Gamma_T. \end{cases}$$

- The initial and boundary values can be given only on the parabolic boundary Γ_T.
- The equation determines the values uniquely inside Ω_T and on the top $\Omega \times \{t = T\}$.

5.5 Uniqueness for bounded domains

Theorem (Uniqueness for bounded domains)

Assume that Ω_T is bounded, $g \in C(\Gamma_T)$ and $f \in C(\Omega_T)$. Then there exists at most one solution $u \in C^2(\Omega_T) \cap C(\overline{\Omega_T})$ of the initial and boundary value problem

$$\begin{cases} u_t - \Delta u = f \quad \textit{in} \quad \Omega_T, \\ u = g \quad \textit{on} \quad \Gamma_T. \end{cases}$$

- The initial and boundary values can be given only on the parabolic boundary Γ_T.
- The equation determines the values uniquely inside Ω_T and on the top $\Omega \times \{t = T\}$.

5.5 Uniqueness for unbounded domains?

- Consider the problem

$$\begin{cases} u_t - u_{xx} = 0 & \text{in} \quad \mathbb{R} \times (0, \infty), \\ u = 0 & \text{on} \quad \mathbb{R} \times \{t = 0\}. \end{cases}$$

- Now $\Omega = \mathbb{R}$ is unbounded.
- One solution to the problem is $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$, with

$$u(x,t) = \begin{cases} \displaystyle\sum_{j=0}^{\infty} \frac{x^{2j}}{(2j)!} \frac{\partial^j}{\partial t^j}(e^{-1/t^2}), & x \in \mathbb{R}, \quad t > 0, \\ 0, & t = 0. \end{cases}$$

- au is a solution to the same problem for every $a \in \mathbb{R}$.
$\implies$ No uniqueness!

5.5 Uniqueness for unbounded domains?

- Consider the problem

$$\begin{cases} u_t - u_{xx} = 0 & \text{in} \quad \mathbb{R} \times (0, \infty), \\ u = 0 & \text{on} \quad \mathbb{R} \times \{t = 0\}. \end{cases}$$

- Now $\Omega = \mathbb{R}$ is unbounded.
- One solution to the problem is $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$, with

$$u(x,t) = \begin{cases} \displaystyle\sum_{j=0}^{\infty} \frac{x^{2j}}{(2j)!} \frac{\partial^j}{\partial t^j} (e^{-1/t^2}), & x \in \mathbb{R}, \quad t > 0, \\ 0, & t = 0. \end{cases}$$

- au is a solution to the same problem for every $a \in \mathbb{R}$.
$\Longrightarrow$ No uniqueness!

5.5 Uniqueness for unbounded domains?

- Consider the problem

$$\begin{cases} u_t - u_{xx} = 0 \quad \text{in} \quad \mathbb{R} \times (0, \infty), \\ u = 0 \quad \text{on} \quad \mathbb{R} \times \{t = 0\}. \end{cases}$$

- Now $\Omega = \mathbb{R}$ is unbounded.
- One solution to the problem is $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$, with

$$u(x,t) = \begin{cases} \displaystyle\sum_{j=0}^{\infty} \frac{x^{2j}}{(2j)!} \frac{\partial^j}{\partial t^j} (e^{-1/t^2}), \qquad x \in \mathbb{R}, \quad t > 0, \\ 0, \quad t = 0. \end{cases}$$

- au is a solution to the same problem for every $a \in \mathbb{R}$.
  $\Longrightarrow$ No uniqueness!

5.5 Uniqueness for unbounded domains?

- Consider the problem

$$\begin{cases} u_t - u_{xx} = 0 \quad \text{in} \quad \mathbb{R} \times (0,\infty), \\ u = 0 \quad \text{on} \quad \mathbb{R} \times \{t = 0\}. \end{cases}$$

- Now $\Omega = \mathbb{R}$ is unbounded.
- One solution to the problem is $u : \mathbb{R} \times [0,\infty) \to \mathbb{R}$, with

$$u(x,t) = \begin{cases} \displaystyle\sum_{j=0}^{\infty} \frac{x^{2j}}{(2j)!} \frac{\partial^j}{\partial t^j}(e^{-1/t^2}), \qquad x \in \mathbb{R}, \quad t > 0, \\ 0, \quad t = 0. \end{cases}$$

- au is a solution to the same problem for every $a \in \mathbb{R}$.
  $\Longrightarrow$ No uniqueness!

5.5 Remarks about the maximum principle

- There is also a version of the strong maximum principle for the heat equation.
- Changing t to $-t$ does not preserve heat equation. Thus solutions forward and backward in time are different.
- Given an initial temperature, we may predict future temperatures, but we cannot in general determine the thermal status that generated that particular temperature distribution.
- The backward in time problem is illposed i.e. it is not solvable in general.
- The backward problem is not stable. (In the lecture notes there is an example of a sequence of solutions, where the initial values approach 0, but the solutions are unbounded close to the initial boundary.

5.5 Remarks about the maximum principle

- There is also a version of the strong maximum principle for the heat equation.
- Changing t to $-t$ does not preserve heat equation. Thus solutions forward and backward in time are different.
- Given an initial temperature, we may predict future temperatures, but we cannot in general determine the thermal status that generated that particular temperature distribution.
- The backward in time problem is illposed i.e. it is not solvable in general.
- The backward problem is not stable. (In the lecture notes there is an example of a sequence of solutions, where the initial values approach 0, but the solutions are unbounded close to the initial boundary.

5.5 Remarks about the maximum principle

- There is also a version of the strong maximum principle for the heat equation.
- Changing t to $-t$ does not preserve heat equation. Thus solutions forward and backward in time are different.
- Given an initial temperature, we may predict future temperatures, but we cannot in general determine the thermal status that generated that particular temperature distribution.
- The backward in time problem is illposed i.e. it is not solvable in general.
- The backward problem is not stable. (In the lecture notes there is an example of a sequence of solutions, where the initial values approach 0, but the solutions are unbounded close to the initial boundary.

5.5 Remarks about the maximum principle

- There is also a version of the strong maximum principle for the heat equation.
- Changing t to $-t$ does not preserve heat equation. Thus solutions forward and backward in time are different.
- Given an initial temperature, we may predict future temperatures, but we cannot in general determine the thermal status that generated that particular temperature distribution.
- The backward in time problem is illposed i.e. it is not solvable in general.
- The backward problem is not stable. (In the lecture notes there is an example of a sequence of solutions, where the initial values approach 0, but the solutions are unbounded close to the initial boundary.

5.5 Remarks about the maximum principle

- There is also a version of the strong maximum principle for the heat equation.
- Changing t to $-t$ does not preserve heat equation. Thus solutions forward and backward in time are different.
- Given an initial temperature, we may predict future temperatures, but we cannot in general determine the thermal status that generated that particular temperature distribution.
- The backward in time problem is illposed i.e. it is not solvable in general.
- The backward problem is not stable. (In the lecture notes there is an example of a sequence of solutions, where the initial values approach 0, but the solutions are unbounded close to the initial boundary.

5.6 Energy methods for the heat equation

- There is an analogous approach with energy / variational methods to the heat equation as there is for the Laplace equation.
- Let $\Omega \subset \mathbb{R}^n$ be open and bounded. We consider the initial and boundary value problem for the heat equation

$$\begin{cases} u_t - \Delta u = f \quad \text{in} \quad \Omega_T, \\ u = g \quad \text{on} \quad \Gamma_T. \end{cases}$$

Theorem

There exists at most one solution to the problem above.

5.6 Energy methods for the heat equation

- There is an analogous approach with energy / variational methods to the heat equation as there is for the Laplace equation.
- Let $\Omega \subset \mathbb{R}^n$ be open and bounded. We consider the initial and boundary value problem for the heat equation

$$\begin{cases} u_t - \Delta u = f \quad \text{in} \quad \Omega_T, \\ u = g \quad \text{on} \quad \Gamma_T. \end{cases}$$

Theorem

There exists at most one solution to the problem above.

5.6 Energy methods for the heat equation

- There is an analogous approach with energy / variational methods to the heat equation as there is for the Laplace equation.
- Let $\Omega \subset \mathbb{R}^n$ be open and bounded. We consider the initial and boundary value problem for the heat equation

$$\begin{cases} u_t - \Delta u = f \quad \text{in} \quad \Omega_T, \\ u = g \quad \text{on} \quad \Gamma_T. \end{cases}$$

Theorem

There exists at most one solution to the problem above.

5.6 Proof of uniqueness

- Let u, v be two solutions. Then $w = u - v$ satisfies

$$w_t - \Delta w = 0 \quad \text{in} \quad \Omega_T, \quad w = 0 \quad \text{on} \quad \Gamma_T.$$

- For $0 \le t \le T$ we define the energy

$$e(t) = \int_\Omega w(x,t)^2\,dx.$$

- Then

$$\begin{aligned} e'(t) &= \int_\Omega 2w\frac{\partial w}{\partial t}\,dx = \int_\Omega 2w\Delta w\,dx \quad (w \text{ satisfies the heat equation}) \\ &= 2\int_{\partial\Omega}\frac{\partial w}{\partial \nu}w\,dS - 2\int_\Omega |\nabla w|^2\,dx \quad \text{(Green's first identity)} \\ &= -2\int_\Omega |\nabla w|^2\,dx \le 0. \quad (w = 0 \text{ on } \partial\Omega) \end{aligned}$$

- Thus $0 \le e(t) \le e(0) = 0 \Longrightarrow e(t) = 0$ for every $t \Longrightarrow w = 0$.

5.6 Proof of uniqueness

- Let u, v be two solutions. Then $w = u - v$ satisfies

$$w_t - \Delta w = 0 \quad \text{in} \quad \Omega_T, \qquad w = 0 \quad \text{on} \quad \Gamma_T.$$

- For $0 \le t \le T$ we define the energy

$$e(t) = \int_\Omega w(x,t)^2 \, dx.$$

- Then

$$\begin{aligned} e'(t) &= \int_\Omega 2w \frac{\partial w}{\partial t} \, dx = \int_\Omega 2w \Delta w \, dx \quad (w \text{ satisfies the heat equation}) \\ &= 2 \int_{\partial\Omega} \frac{\partial w}{\partial \nu} w \, dS - 2 \int_\Omega |\nabla w|^2 \, dx \quad (\text{Green's first identity}) \\ &= -2 \int_\Omega |\nabla w|^2 \, dx \le 0. \quad (w = 0 \text{ on } \partial\Omega) \end{aligned}$$

- Thus $0 \le e(t) \le e(0) = 0 \Longrightarrow e(t) = 0$ for every $t \Longrightarrow w = 0$.

5.6 Proof of uniqueness

- Let u, v be two solutions. Then $w = u - v$ satisfies

$$w_t - \Delta w = 0 \quad \text{in} \quad \Omega_T, \qquad w = 0 \quad \text{on} \quad \Gamma_T.$$

- For $0 \leq t \leq T$ we define the energy

$$e(t) = \int_\Omega w(x,t)^2 \, dx.$$

- Then

$$\begin{aligned} e'(t) &= \int_\Omega 2w \frac{\partial w}{\partial t} \, dx = \int_\Omega 2w \Delta w \, dx \quad (w \text{ satisfies the heat equation}) \\ &= 2 \int_{\partial\Omega} \frac{\partial w}{\partial \nu} w \, dS - 2 \int_\Omega |\nabla w|^2 \, dx \quad (\text{Green's first identity}) \\ &= -2 \int_\Omega |\nabla w|^2 \, dx \leq 0. \quad (w = 0 \text{ on } \partial\Omega) \end{aligned}$$

- Thus $0 \leq e(t) \leq e(0) = 0 \Longrightarrow e(t) = 0$ for every $t \Longrightarrow w = 0$.

5.6 Proof of uniqueness

- Let u, v be two solutions. Then $w = u - v$ satisfies

$$w_t - \Delta w = 0 \quad \text{in} \quad \Omega_T, \qquad w = 0 \quad \text{on} \quad \Gamma_T.$$

- For $0 \leq t \leq T$ we define the energy

$$e(t) = \int_\Omega w(x,t)^2 \, dx.$$

- Then

$$e'(t) = \frac{\partial}{\partial t} \int_\Omega w(x,t)^2 \, dx = \int_\Omega 2w\Delta w \, dx \quad (w \text{ satisfies the heat equation})$$
$$= 2\int_{\partial\Omega} \frac{\partial w}{\partial \nu} w \, dS - 2\int_\Omega |\nabla w|^2 \, dx \quad \text{(Green's first identity)}$$
$$= -2\int_\Omega |\nabla w|^2 \, dx \leq 0. \quad (w = 0 \text{ on } \partial\Omega)$$

- Thus $0 \leq e(t) \leq e(0) = 0 \Longrightarrow e(t) = 0$ for every $t \Longrightarrow w = 0$.

5.6 Proof of uniqueness

- Let u, v be two solutions. Then $w = u - v$ satisfies

$$w_t - \Delta w = 0 \quad \text{in} \quad \Omega_T, \qquad w = 0 \quad \text{on} \quad \Gamma_T.$$

- For $0 \le t \le T$ we define the energy

$$e(t) = \int_\Omega w(x,t)^2 \, dx.$$

- Then

$$e'(t) = \int_\Omega \frac{\partial}{\partial t}(w(x,t)^2) \, dx = \int_\Omega 2w\Delta w \, dx \quad (w \text{ satisfies the heat equation})$$

$$= 2\int_{\partial\Omega} \frac{\partial w}{\partial \nu} w \, dS - 2\int_\Omega |\nabla w|^2 \, dx \quad \text{(Green's first identity)}$$

$$= -2\int_\Omega |\nabla w|^2 \, dx \le 0. \quad (w = 0 \text{ on } \partial\Omega)$$

- Thus $0 \le e(t) \le e(0) = 0 \Longrightarrow e(t) = 0$ for every $t \Longrightarrow w = 0$.

5.6 Proof of uniqueness

- Let u, v be two solutions. Then $w = u - v$ satisfies

$$w_t - \Delta w = 0 \quad \text{in} \quad \Omega_T, \qquad w = 0 \quad \text{on} \quad \Gamma_T.$$

- For $0 \leq t \leq T$ we define the energy

$$e(t) = \int_\Omega w(x,t)^2 \, dx.$$

- Then

$$\begin{aligned} e'(t) &= \int_\Omega 2w \frac{\partial w}{\partial t} \, dx = \int_\Omega 2w\Delta w \, dx \quad (w \text{ satisfies the heat equation}) \\ &= 2\int_{\partial\Omega} \frac{\partial w}{\partial \nu} w \, dS - 2\int_\Omega |\nabla w|^2 \, dx \quad \text{(Green's first identity)} \\ &= -2\int_\Omega |\nabla w|^2 \, dx \leq 0. \quad (w = 0 \text{ on } \partial\Omega) \end{aligned}$$

- Thus $0 \leq e(t) \leq e(0) = 0 \implies e(t) = 0$ for every $t \implies w = 0$.

5.6 Proof of uniqueness

- Let u, v be two solutions. Then $w = u - v$ satisfies

$$w_t - \Delta w = 0 \quad \text{in} \quad \Omega_T, \qquad w = 0 \quad \text{on} \quad \Gamma_T.$$

- For $0 \leq t \leq T$ we define the energy

$$e(t) = \int_\Omega w(x,t)^2 \, dx.$$

- Then

$$\begin{aligned} e'(t) &= \int_\Omega 2w \frac{\partial w}{\partial t} \, dx = \int_\Omega 2w \Delta w \, dx \quad (w \text{ satisfies the heat equation}) \\ &= 2 \int_{\partial\Omega} \frac{\partial w}{\partial \nu} w \, dS - 2 \int_\Omega |\nabla w|^2 \, dx \quad (\text{Green's first identity}) \\ &= -2 \int_\Omega |\nabla w|^2 \, dx \leq 0. \quad (w = 0 \text{ on } \partial\Omega) \end{aligned}$$

- Thus $0 \leq e(t) \leq e(0) = 0 \Longrightarrow e(t) = 0$ for every $t \Longrightarrow w = 0$.

5.6 Proof of uniqueness

- Let u, v be two solutions. Then $w = u - v$ satisfies

$$w_t - \Delta w = 0 \quad \text{in} \quad \Omega_T, \qquad w = 0 \quad \text{on} \quad \Gamma_T.$$

- For $0 \le t \le T$ we define the energy

$$e(t) = \int_\Omega w(x,t)^2 \, dx.$$

- Then

$$\begin{aligned} e'(t) &= \int_\Omega 2w \frac{\partial w}{\partial t} \, dx = \int_\Omega 2w \Delta w \, dx \quad (w \text{ satisfies the heat equation}) \\ &= 2 \int_{\partial\Omega} \frac{\partial w}{\partial \nu} w \, dS - 2 \int_\Omega |\nabla w|^2 \, dx \quad \text{(Green's first identity)} \\ &= -2 \int_\Omega |\nabla w|^2 \, dx \le 0. \quad (w = 0 \text{ on } \partial\Omega) \end{aligned}$$

- Thus $0 \le e(t) \le e(0) = 0 \implies e(t) = 0$ for every $t \implies w = 0$.

5.6 Proof of uniqueness

- Let u, v be two solutions. Then $w = u - v$ satisfies

$$w_t - \Delta w = 0 \quad \text{in} \quad \Omega_T, \qquad w = 0 \quad \text{on} \quad \Gamma_T.$$

- For $0 \leq t \leq T$ we define the energy

$$e(t) = \int_\Omega w(x,t)^2 \, dx.$$

- Then

$$\begin{aligned} e'(t) &= \int_\Omega 2w \frac{\partial w}{\partial t} \, dx = \int_\Omega 2w \Delta w \, dx \quad (w \text{ satisfies the heat equation}) \\ &= 2 \int_{\partial\Omega} \frac{\partial w}{\partial \nu} w \, dS - 2 \int_\Omega |\nabla w|^2 \, dx \quad \text{(Green's first identity)} \\ &= -2 \int_\Omega |\nabla w|^2 \, dx \leq 0. \quad (w = 0 \text{ on } \partial\Omega) \end{aligned}$$

- Thus $0 \leq e(t) \leq e(0) = 0 \Longrightarrow e(t) = 0$ for every $t \Longrightarrow w = 0$.

5.6 Proof of uniqueness

- Let u, v be two solutions. Then $w = u - v$ satisfies

$$w_t - \Delta w = 0 \quad \text{in} \quad \Omega_T, \qquad w = 0 \quad \text{on} \quad \Gamma_T.$$

- For $0 \le t \le T$ we define the energy

$$e(t) = \int_\Omega w(x,t)^2 \, dx.$$

- Then

$$\begin{aligned} e'(t) &= \int_\Omega 2w \frac{\partial w}{\partial t} \, dx = \int_\Omega 2w \Delta w \, dx \quad (w \text{ satisfies the heat equation}) \\ &= 2 \int_{\partial\Omega} \frac{\partial w}{\partial \nu} w \, dS - 2 \int_\Omega |\nabla w|^2 \, dx \quad \text{(Green's first identity)} \\ &= -2 \int_\Omega |\nabla w|^2 \, dx \le 0. \quad (w = 0 \text{ on } \partial\Omega) \end{aligned}$$

- Thus $0 \le e(t) \le e(0) = 0 \Longrightarrow e(t) = 0$ for every $t \Longrightarrow w = 0$.

MS-C1350 Partial differential equations Wave equation – Physical interpretation and dimension 1 and 3

Riikka Korte

Department of Mathematics and Systems Analysis
Aalto University
riikka.korte@aalto.fi

November 19, 2024

Lecture 11

Wave equation

- Properties of solutions are different for different dimensions.
- Dimension 1: d'Alembert's formula. Its physical interpretation.
- Dimension 3: Solution by using Euler-Poisson-Darboux equation.
- Domain of dependence and the range of influence (different for different dimensions!)

6 Wave equation – introduction

- We study the wave equation in all dimensions, but focus on physically relevant cases $n = 1, 2, 3$
- Properties of the solutions depend on the dimension.
- Interpretation:
  - $n = 1$: the displacement of a vibrating string
  - $n = 2$: a vibrating membrane
  - $n = 3$: an elastic solid, the behaviour of electromagnetic waves in vacuum and the propagation of sound waves.
- The $n-$dimensional wave equation is

$$u_{tt} - \Delta u = 0$$

and the nonhomogeneous wave equation is

$$u_{tt} - \Delta u = f.$$

Here Laplace operator is taken with respect to x:

$$\Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}.$$

6 Wave equation – introduction

- We study the wave equation in all dimensions, but focus on physically relevant cases $n = 1, 2, 3$
- Properties of the solutions depend on the dimension.
- Interpretation:
  - $n = 1$: the displacement of a vibrating string
  - $n = 2$: a vibrating membrane
  - $n = 3$: an elastic solid, the behaviour of electromagnetic waves in vacuum and the propagation of sound waves.
- The $n-$dimensional wave equation is

  $$u_{tt} - \Delta u = 0$$

  and the nonhomogeneous wave equation is

  $$u_{tt} - \Delta u = f.$$

  Here Laplace operator is taken with respect to x:

  $$\Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}.$$

6 Wave equation – introduction

- We study the wave equation in all dimensions, but focus on physically relevant cases $n = 1, 2, 3$
- Properties of the solutions depend on the dimension.
- Interpretation:
  - $n = 1$: the displacement of a vibrating string
  - $n = 2$: a vibrating membrane
  - $n = 3$: an elastic solid, the behaviour of electromagnetic waves in vacuum and the propagation of sound waves.
- The $n-$dimensional wave equation is

$$u_{tt} - \Delta u = 0$$

and the nonhomogeneous wave equation is

$$u_{tt} - \Delta u = f.$$

Here Laplace operator is taken with respect to x:

$$\Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}.$$

6 Wave equation – introduction

- We study the wave equation in all dimensions, but focus on physically relevant cases $n = 1, 2, 3$
- Properties of the solutions depend on the dimension.
- Interpretation:
  - $n = 1$: the displacement of a vibrating string
  - $n = 2$: a vibrating membrane
  - $n = 3$: an elastic solid, the behaviour of electromagnetic waves in vacuum and the propagation of sound waves.
- The $n-$dimensional wave equation is

$$u_{tt} - \Delta u = 0$$

and the nonhomogeneous wave equation is

$$u_{tt} - \Delta u = f.$$

Here Laplace operator is taken with respect to x:

$$\Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}.$$

6.1 Wave equation – physical interpretation

- $u = u(x,t)$: the displacement of the point x at the time t.
- If $V \subset \Omega$, the acceleration in V is

$$\frac{\partial^2}{\partial t^2} \int_V u(x,t)\,dx$$

- the net contact force:

$$-\int_{\partial V} F(x,t) \cdot \nu(x)\,dS(x),$$

- By the Gauss-Green theorem (and $F = ma$) we have

$$\int_V \operatorname{div}_x F(x,t)\,dx = \int_{\partial V} F(x,t) \cdot \nu(x)\,dS(x) = -\int_V \frac{\partial^2 u}{\partial t^2}(x,t)\,dx.$$

- This holds in every subdomain and thus we get

$$-c^2 \operatorname{div}_x \nabla u(x,t) = \operatorname{div}_x F(x,t) = -\frac{\partial^2 u}{\partial t^2}(x,t).$$

as it is physically reasonable to assume that the force F is proportional to the gradient ∇u.

6.1 Wave equation – physical interpretation

- $u = u(x,t)$: the displacement of the point x at the time t.
- If $V \subset \Omega$, the acceleration in V is

$$\frac{\partial^2}{\partial t^2} \int_V u(x,t)\,dx$$

- the net contact force:

$$-\int_{\partial V} F(x,t) \cdot \nu(x)\,dS(x),$$

- By the Gauss-Green theorem (and $F = ma$) we have

$$\int_V \operatorname{div}_x F(x,t)\,dx = \int_{\partial V} F(x,t) \cdot \nu(x)\,dS(x) = -\int_V \frac{\partial^2 u}{\partial t^2}(x,t)\,dx.$$

- This holds in every subdomain and thus we get

$$-c^2 \operatorname{div}_x \nabla u(x,t) = \operatorname{div}_x F(x,t) = -\frac{\partial^2 u}{\partial t^2}(x,t).$$

as it is physically reasonable to assume that the force F is proportional to the gradient ∇u.

6.1 Wave equation – physical interpretation

- $u = u(x,t)$: the displacement of the point x at the time t.
- If $V \subset \Omega$, the acceleration in V is

$$\frac{\partial^2}{\partial t^2} \int_V u(x,t)\,dx$$

- the net contact force:

$$-\int_{\partial V} F(x,t)\cdot \nu(x)\,dS(x),$$

- By the Gauss-Green theorem (and $F = ma$) we have

$$\int_V \operatorname{div}_x F(x,t)\,dx = \int_{\partial V} F(x,t)\cdot\nu(x)\,dS(x) = -\int_V \frac{\partial^2 u}{\partial t^2}(x,t)\,dx.$$

- This holds in every subdomain and thus we get

$$-c^2 \operatorname{div}_x \nabla u(x,t) = \operatorname{div}_x F(x,t) = -\frac{\partial^2 u}{\partial t^2}(x,t).$$

as it is physically reasonable to assume that the force F is proportional to the gradient ∇u.

6.1 Wave equation – physical interpretation

- $u = u(x,t)$: the displacement of the point x at the time t.
- If $V \subset \Omega$, the acceleration in V is

$$\frac{\partial^2}{\partial t^2} \int_V u(x,t)\,dx$$

- the net contact force:

$$-\int_{\partial V} F(x,t) \cdot \nu(x)\,dS(x),$$

- By the Gauss-Green theorem (and $F = ma$) we have

$$\int_V \operatorname{div}_x F(x,t)\,dx = \int_{\partial V} F(x,t) \cdot \nu(x)\,dS(x) = -\int_V \frac{\partial^2 u}{\partial t^2}(x,t)\,dx.$$

- This holds in every subdomain and thus we get

$$-c^2 \operatorname{div}_x \nabla u(x,t) = \operatorname{div}_x F(x,t) = -\frac{\partial^2 u}{\partial t^2}(x,t).$$

as it is physically reasonable to assume that the force F is proportional to the gradient ∇u.

6.1 Wave equation – physical interpretation

- $u = u(x,t)$: the displacement of the point x at the time t.
- If $V \subset \Omega$, the acceleration in V is

$$\frac{\partial^2}{\partial t^2} \int_V u(x,t)\,dx$$

- the net contact force:

$$-\int_{\partial V} F(x,t)\cdot \nu(x)\,dS(x),$$

- By the Gauss-Green theorem (and $F = ma$) we have

$$\int_V \operatorname{div}_x F(x,t)\,dx = \int_{\partial V} F(x,t)\cdot\nu(x)\,dS(x) = -\int_V \frac{\partial^2 u}{\partial t^2}(x,t)\,dx.$$

- This holds in every subdomain and thus we get

$$-c^2 \operatorname{div}_x \nabla u(x,t) = \operatorname{div}_x F(x,t) = -\frac{\partial^2 u}{\partial t^2}(x,t).$$

as it is physically reasonable to assume that the force F is proportional to the gradient ∇u.

6.1 Wave equation – physical interpretation

- $u = u(x,t)$: the displacement of the point x at the time t.
- If $V \subset \Omega$, the acceleration in V is

$$\frac{\partial^2}{\partial t^2} \int_V u(x,t)\,dx$$

- the net contact force:

$$-\int_{\partial V} F(x,t) \cdot \nu(x)\,dS(x),$$

- By the Gauss-Green theorem (and $F = ma$) we have

$$\int_V \text{div}_x\, F(x,t)\,dx = \int_{\partial V} F(x,t) \cdot \nu(x)\,dS(x) = -\int_V \frac{\partial^2 u}{\partial t^2}(x,t)\,dx.$$

- This holds in every subdomain and thus we get

$$-c^2 \text{div}_x\, \nabla u(x,t) = \text{div}_x\, F(x,t) = -\frac{\partial^2 u}{\partial t^2}(x,t).$$

as it is physically reasonable to assume that the force F is proportional to the gradient ∇u.

6.2 One-dimensional case

- We have showed using the Fourier transform that the solution of the Cauchy problem

$$\begin{cases} u_{tt} - \Delta u = 0 \quad \text{in} \quad \mathbb{R}^n \times (0, \infty), \\ u = g \quad \text{in} \quad \mathbb{R}^n \times \{t = 0\}, \\ \dfrac{\partial u}{\partial t} = h \quad \text{in} \quad \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

for the n-dimensional wave equation is

$$u(x,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\widehat{g}(\xi) \cos(|\xi| t) + \widehat{h}(\xi) \frac{\sin(|\xi| t)}{|\xi|} \right) e^{ix\cdot\xi} \, d\xi.$$

- To consider initial value problems for the wave equation in a bounded domain $\Omega_T = \Omega \times (0, \infty)$, we can use eigenfunction expansions as for the heat equation.

6.2 One-dimensional case

- We have showed using the Fourier transform that the solution of the Cauchy problem

$$\begin{cases} u_{tt} - \Delta u = 0 \quad \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u = g \quad \text{in} \quad \mathbb{R}^n \times \{t=0\}, \\ \dfrac{\partial u}{\partial t} = h \quad \text{in} \quad \mathbb{R}^n \times \{t=0\}. \end{cases}$$

  for the n-dimensional wave equation is

$$u(x,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\widehat{g}(\xi)\cos(|\xi|t) + \widehat{h}(\xi)\frac{\sin(|\xi|t)}{|\xi|} \right) e^{ix\cdot\xi}\, d\xi.$$

- To consider initial value problems for the wave equation in a bounded domain $\Omega_T = \Omega \times (0,\infty)$, we can use eigenfunction expansions as for the heat equation.

6.2 One-dimensional case

$$u(x,t) = (2\pi)^{-n}\int_{\mathbb{R}^n}\left(\widehat{g}(\xi)\cos(|\xi|t) + \widehat{h}(\xi)\frac{\sin(|\xi|t)}{|\xi|}\right)e^{ix\cdot\xi}\,d\xi.$$

- First term:

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\mathbb{R}}\widehat{g}(\xi)\cos(\xi t)e^{ix\xi}\,d\xi \\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\widehat{g}(\xi)\frac{1}{2}(e^{i\xi t}+e^{-i\xi t})e^{ix\xi}\,d\xi \\
&= \frac{1}{2}\frac{1}{2\pi}\int_{\mathbb{R}}\widehat{g(x+t)}(\xi)e^{ix\xi}\,d\xi + \frac{1}{2}\frac{1}{2\pi}\int_{\mathbb{R}}\widehat{g(x-t)}(\xi)e^{ix\xi}\,d\xi \\
&= \frac{1}{2}\big(g(x+t)+g(x-t)\big).
\end{aligned}$$

6.2 One-dimensional case

$$u(x,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\widehat{g}(\xi) \cos(|\xi| t) + \widehat{h}(\xi) \frac{\sin(|\xi| t)}{|\xi|} \right) e^{ix\cdot\xi} \, d\xi.$$

- First term:

$$\frac{1}{2\pi} \int_{\mathbb{R}} \widehat{g}(\xi) \cos(\xi t) e^{ix\xi} \, d\xi$$
$$= \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{g}(\xi) \frac{1}{2} (e^{i\xi t} + e^{-i\xi t}) e^{ix\xi} \, d\xi$$
$$= \frac{1}{2} \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{g(x+t)}(\xi) e^{ix\xi} \, d\xi + \frac{1}{2} \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{g(x-t)}(\xi) e^{ix\xi} \, d\xi$$
$$= \frac{1}{2} (g(x+t) + g(x-t)).$$

6.2 One-dimensional case

$$u(x,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\widehat{g}(\xi)\cos(|\xi|t) + \widehat{h}(\xi)\frac{\sin(|\xi|t)}{|\xi|} \right) e^{ix\cdot\xi}\, d\xi.$$

- First term:

$$\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{g}(\xi)\cos(\xi t)e^{ix\xi}\, d\xi$$
$$= \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{g}(\xi)\frac{1}{2}(e^{i\xi t} + e^{-i\xi t})e^{ix\xi}\, d\xi$$
$$= \frac{1}{2}\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{g(x+t)}(\xi)e^{ix\xi}\, d\xi + \frac{1}{2}\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{g(x-t)}(\xi)e^{ix\xi}\, d\xi$$
$$= \frac{1}{2}(g(x+t) + g(x-t)).$$

6.2 One-dimensional case

$$u(x,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\widehat{g}(\xi)\cos(|\xi|t) + \widehat{h}(\xi)\frac{\sin(|\xi|t)}{|\xi|} \right) e^{ix\cdot\xi}\, d\xi.$$

- First term:

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{g}(\xi)\cos(\xi t) e^{ix\xi}\, d\xi \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{g}(\xi)\frac{1}{2}(e^{i\xi t} + e^{-i\xi t}) e^{ix\xi}\, d\xi \\
&= \frac{1}{2}\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{g(x+t)}(\xi) e^{ix\xi}\, d\xi + \frac{1}{2}\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{g(x-t)}(\xi) e^{ix\xi}\, d\xi \\
&= \frac{1}{2}(g(x+t) + g(x-t)).
\end{aligned}$$

6.2 One-dimensional case

$$u(x,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\widehat{g}(\xi) \cos(|\xi| t) + \widehat{h}(\xi) \frac{\sin(|\xi| t)}{|\xi|} \right) e^{ix\cdot\xi} \, d\xi.$$

- First term:

$$\begin{aligned}
&\frac{1}{2\pi} \int_{\mathbb{R}} \widehat{g}(\xi) \cos(\xi t) e^{ix\xi} \, d\xi \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{g}(\xi) \frac{1}{2} (e^{i\xi t} + e^{-i\xi t}) e^{ix\xi} \, d\xi \\
&= \frac{1}{2} \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{g(x+t)}(\xi) e^{ix\xi} \, d\xi + \frac{1}{2} \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{g(x-t)}(\xi) e^{ix\xi} \, d\xi \\
&= \frac{1}{2}(g(x+t) + g(x-t)).
\end{aligned}$$

6.2 One-dimensional case

$$u(x,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\widehat{g}(\xi) \cos(|\xi| t) + \widehat{h}(\xi) \frac{\sin(|\xi| t)}{|\xi|} \right) e^{ix\cdot\xi} \, d\xi.$$

- First term:

$$\begin{aligned}
&\frac{1}{2\pi} \int_{\mathbb{R}} \widehat{g}(\xi) \cos(\xi t) e^{ix\xi} \, d\xi \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{g}(\xi) \frac{1}{2} (e^{i\xi t} + e^{-i\xi t}) e^{ix\xi} \, d\xi \\
&= \frac{1}{2} \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{g(x+t)}(\xi) e^{ix\xi} \, d\xi + \frac{1}{2} \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{g(x-t)}(\xi) e^{ix\xi} \, d\xi \\
&= \frac{1}{2} (g(x+t) + g(x-t)).
\end{aligned}$$

6.2 One-dimensional case

$$u(x,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\widehat{g}(\xi)\cos(\xi t) + \widehat{h}(\xi)\frac{\sin(\xi t)}{\xi} \right) e^{ix\cdot\xi}\, d\xi.$$

- Second term:

$$\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{h}(\xi)\frac{1}{2i\xi}(e^{i\xi t} - e^{-i\xi t})e^{ix\xi}\, d\xi$$

$$= \frac{1}{2}\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi t}\widehat{H}(\xi)e^{ix\xi}\, d\xi - \frac{1}{2}\frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\xi t}\widehat{H}(\xi)e^{ix\xi}\, d\xi$$

$$= \frac{1}{2}\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{H(x+t)}(\xi)e^{ix\xi}\, d\xi - \frac{1}{2}\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{H(x-t)}(\xi)e^{ix\xi}\, d\xi$$

$$= \frac{1}{2}(H(x+t) - H(x-t)),$$

where

$$\widehat{H}(\xi) = \frac{\widehat{h}(\xi)}{i\xi} \qquad \text{i.e. } H'(x) = h(x).$$

6.2 One-dimensional case

$$u(x,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\widehat{g}(\xi)\cos(\xi t) + \widehat{h}(\xi)\frac{\sin(\xi t)}{\xi} \right) e^{ix\cdot\xi}\, d\xi.$$

- Second term:

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{h}(\xi)\frac{1}{2i\xi}(e^{i\xi t} - e^{-i\xi t})e^{ix\xi}\, d\xi \\
&= \frac{1}{2}\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi t}\widehat{H}(\xi)e^{ix\xi}\, d\xi - \frac{1}{2}\frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\xi t}\widehat{H}(\xi)e^{ix\xi}\, d\xi \\
&= \frac{1}{2}\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{H(x+t)}(\xi)e^{ix\xi}\, d\xi - \frac{1}{2}\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{H(x-t)}(\xi)e^{ix\xi}\, d\xi \\
&= \frac{1}{2}(H(x+t) - H(x-t)),
\end{aligned}$$

where

$$\widehat{H}(\xi) = \frac{\widehat{h}(\xi)}{i\xi} \qquad \text{i.e. } H'(x) = h(x).$$

6.2 One-dimensional case

$$u(x,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\widehat{g}(\xi)\cos(\xi t) + \widehat{h}(\xi)\frac{\sin(\xi t)}{\xi} \right) e^{ix\cdot\xi}\, d\xi.$$

- Second term:

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{h}(\xi)\frac{1}{2i\xi}(e^{i\xi t} - e^{-i\xi t})e^{ix\xi}\, d\xi \\
&= \frac{1}{2}\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi t}\widehat{H}(\xi)e^{ix\xi}\, d\xi - \frac{1}{2}\frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\xi t}\widehat{H}(\xi)e^{ix\xi}\, d\xi \\
&= \frac{1}{2}\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{H(x+t)}(\xi)e^{ix\xi}\, d\xi - \frac{1}{2}\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{H(x-t)}(\xi)e^{ix\xi}\, d\xi \\
&= \frac{1}{2}(H(x+t) - H(x-t)),
\end{aligned}$$

where

$$\widehat{H}(\xi) = \frac{\widehat{h}(\xi)}{i\xi} \qquad \text{i.e. } H'(x) = h(x).$$

6.2 One-dimensional case

$$u(x,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\widehat{g}(\xi)\cos(\xi t) + \widehat{h}(\xi)\frac{\sin(\xi t)}{\xi} \right) e^{ix\cdot\xi}\, d\xi.$$

- Second term:

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{h}(\xi)\frac{1}{2i\xi}(e^{i\xi t} - e^{-i\xi t})e^{ix\xi}\, d\xi \\
&= \frac{1}{2}\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi t}\widehat{H}(\xi)e^{ix\xi}\, d\xi - \frac{1}{2}\frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\xi t}\widehat{H}(\xi)e^{ix\xi}\, d\xi \\
&= \frac{1}{2}\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{H(x+t)}(\xi)e^{ix\xi}\, d\xi - \frac{1}{2}\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{H(x-t)}(\xi)e^{ix\xi}\, d\xi \\
&= \frac{1}{2}(H(x+t) - H(x-t)),
\end{aligned}$$

where

$$\widehat{H}(\xi) = \frac{\widehat{h}(\xi)}{i\xi} \qquad \text{i.e. } H'(x) = h(x).$$

6.2 One-dimensional case

$$u(x,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\widehat{g}(\xi)\cos(\xi t) + \widehat{h}(\xi)\frac{\sin(\xi t)}{\xi} \right) e^{ix\cdot\xi}\, d\xi.$$

- Second term:

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{h}(\xi)\frac{1}{2i\xi}(e^{i\xi t} - e^{-i\xi t})e^{ix\xi}\, d\xi \\
&= \frac{1}{2}\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi t}\widehat{H}(\xi)e^{ix\xi}\, d\xi - \frac{1}{2}\frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\xi t}\widehat{H}(\xi)e^{ix\xi}\, d\xi \\
&= \frac{1}{2}\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{H(x+t)}(\xi)e^{ix\xi}\, d\xi - \frac{1}{2}\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{H(x-t)}(\xi)e^{ix\xi}\, d\xi \\
&= \frac{1}{2}(H(x+t) - H(x-t)),
\end{aligned}$$

where

$$\widehat{H}(\xi) = \frac{\widehat{h}(\xi)}{i\xi} \quad \text{i.e. } H'(x) = h(x).$$

6.2 One-dimensional case

- We conclude the d'Alembert's formula:

$$\begin{aligned} u(x,t) &= \frac{1}{2}(g(x+t)+g(x-t)) + \frac{1}{2}(H(x+t)-H(x-t)) \\ &= \frac{1}{2}(g(x+t)+g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy. \end{aligned}$$

- The converse holds as well.
- The solution is of the form

$$u(x,t) = F(x+t) + G(x-t)$$

- $F(x+t)$ is a wave travelling in time with speed 1.
  - $F(x)$ is the shape of the wave at $t=0$.
  - At time t the wave has moved to left with speed one.
- $G(x-t)$ is a wave traveling to right.

6.2 One-dimensional case

- We conclude the d'Alembert's formula:

$$\begin{aligned} u(x,t) &= \frac{1}{2}(g(x+t)+g(x-t)) + \frac{1}{2}(H(x+t)-H(x-t)) \\ &= \frac{1}{2}(g(x+t)+g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy. \end{aligned}$$

- The converse holds as well.
- The solution is of the form

$$u(x,t) = F(x+t) + G(x-t)$$

- $F(x+t)$ is a wave travelling in time with speed 1.
  - $F(x)$ is the shape of the wave at $t=0$.
  - At time t the wave has moved to left with speed one.
- $G(x-t)$ is a wave traveling to right.

6.2 One-dimensional case

- We conclude the d'Alembert's formula:

$$\begin{aligned} u(x,t) &= \frac{1}{2}(g(x+t)+g(x-t)) + \frac{1}{2}(H(x+t)-H(x-t)) \\ &= \frac{1}{2}(g(x+t)+g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy. \end{aligned}$$

- The converse holds as well.
- The solution is of the form

$$u(x,t) = F(x+t) + G(x-t)$$

- $F(x+t)$ is a wave travelling in time with speed 1.
  - $F(x)$ is the shape of the wave at $t=0$.
  - At time t the wave has moved to left with speed one.
- $G(x-t)$ is a wave traveling to right.

6.2 One-dimensional case

- We conclude the d'Alembert's formula:

$$u(x,t) = \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}(H(x+t) - H(x-t))$$
$$= \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy.$$

- The converse holds as well.
- The solution is of the form

$$u(x,t) = F(x+t) + G(x-t)$$

- $F(x+t)$ is a wave travelling in time with speed 1.
  - $F(x)$ is the shape of the wave at $t=0$.
  - At time t the wave has moved to left with speed one.
- $G(x-t)$ is a wave traveling to right.

6.2 One-dimensional case

- We conclude the d'Alembert's formula:

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}(g(x+t)+g(x-t)) + \frac{1}{2}(H(x+t)-H(x-t)) \\
&= \frac{1}{2}(g(x+t)+g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy.
\end{aligned}
$$

- The converse holds as well.
- The solution is of the form

$$
u(x,t) = F(x+t) + G(x-t)
$$

- $F(x+t)$ is a wave travelling in time with speed 1.
  - $F(x)$ is the shape of the wave at $t=0$.
  - At time t the wave has moved to left with speed one.
- $G(x-t)$ is a wave traveling to right.

6.2 One-dimensional case

- We conclude the d'Alembert's formula:

$$\begin{aligned} u(x,t) &= \frac{1}{2}(g(x+t)+g(x-t)) + \frac{1}{2}(H(x+t)-H(x-t)) \\ &= \frac{1}{2}(g(x+t)+g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy. \end{aligned}$$

- The converse holds as well.
- The solution is of the form

$$u(x,t) = F(x+t) + G(x-t)$$

- $F(x+t)$ is a wave travelling in time with speed 1.
  - $F(x)$ is the shape of the wave at $t=0$.
  - At time t the wave has moved to left with speed one.
- $G(x-t)$ is a wave traveling to right.

6.2 One-dimensional case

- We conclude the d'Alembert's formula:

$$u(x,t) = \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}(H(x+t) - H(x-t))$$
$$= \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy.$$

- The converse holds as well.
- The solution is of the form

$$u(x,t) = F(x+t) + G(x-t)$$

- $F(x+t)$ is a wave travelling in time with speed 1.
  - $F(x)$ is the shape of the wave at $t=0$.
  - At time t the wave has moved to left with speed one.
- $G(x-t)$ is a wave traveling to right.

6.2 Remarks about the one-dimensional case

$$u(x,t) = \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy.$$

- d'Alembert's formula gives uniqueness.
- d'Alembert's formula gives stability:
  - u_1 is a solution with the initial values g_1 and h_1.
  - u_2 is a solution with the initial values g_2 and h_2.
  - Then $v = u_1 - u_2$ is a solution with initial values $g_1 - g_2$ and $h_1 - h_2$
  - If $|g_1 - g_2| \le \varepsilon$ and $|h_1 - h_2| \le \varepsilon$, we obtain $|v(x,t)| \le (1+t)\varepsilon$.
- The wave equation does not smoothen the solution. If $g \in C^k(\mathbb{R})$ and $h \in C^{k-1}(\mathbb{R})$, then $u \in C^k(\mathbb{R} \times (0,\infty))$.
- d'Alembert's formula makes sense even for discontinuous g and h, when corresponding u is not differentiable and thus not a solution to the wave equation.

6.2 Remarks about the one-dimensional case

$$u(x,t) = \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy.$$

- d'Alembert's formula gives uniqueness.
- d'Alembert's formula gives stability:
  - u_1 is a solution with the initial values g_1 and h_1.
  - u_2 is a solution with the initial values g_2 and h_2.
  - Then $v = u_1 - u_2$ is a solution with initial values $g_1 - g_2$ and $h_1 - h_2$
  - If $|g_1 - g_2| \le \varepsilon$ and $|h_1 - h_2| \le \varepsilon$, we obtain $|v(x,t)| \le (1+t)\varepsilon$.
- The wave equation does not smoothen the solution. If $g \in C^k(\mathbb{R})$ and $h \in C^{k-1}(\mathbb{R})$, then $u \in C^k(\mathbb{R} \times (0,\infty))$.
- d'Alembert's formula makes sense even for discontinuous g and h, when corresponding u is not differentiable and thus not a solution to the wave equation.

6.2 Remarks about the one-dimensional case

$$u(x,t) = \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy.$$

- d'Alembert's formula gives uniqueness.
- d'Alembert's formula gives stability:
  - u_1 is a solution with the initial values g_1 and h_1.
  - u_2 is a solution with the initial values g_2 and h_2.
  - Then $v = u_1 - u_2$ is a solution with initial values $g_1 - g_2$ and $h_1 - h_2$
  - If $|g_1 - g_2| \leq \varepsilon$ and $|h_1 - h_2| \leq \varepsilon$, we obtain $|v(x,t)| \leq (1+t)\varepsilon$.
- The wave equation does not smoothen the solution. If $g \in C^k(\mathbb{R})$ and $h \in C^{k-1}(\mathbb{R})$, then $u \in C^k(\mathbb{R} \times (0,\infty))$.
- d'Alembert's formula makes sense even for discontinuous g and h, when corresponding u is not differentiable and thus not a solution to the wave equation.

6.2 Remarks about the one-dimensional case

$$u(x,t) = \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy.$$

- d'Alembert's formula gives uniqueness.
- d'Alembert's formula gives stability:
  - u_1 is a solution with the initial values g_1 and h_1.
  - u_2 is a solution with the initial values g_2 and h_2.
  - Then $v = u_1 - u_2$ is a solution with initial values $g_1 - g_2$ and $h_1 - h_2$
  - If $|g_1 - g_2| \leq \varepsilon$ and $|h_1 - h_2| \leq \varepsilon$, we obtain $|v(x,t)| \leq (1+t)\varepsilon$.
- The wave equation does not smoothen the solution. If $g \in C^k(\mathbb{R})$ and $h \in C^{k-1}(\mathbb{R})$, then $u \in C^k(\mathbb{R} \times (0,\infty))$.
- d'Alembert's formula makes sense even for discontinuous g and h, when corresponding u is not differentiable and thus not a solution to the wave equation.

6.2 Remarks about the one-dimensional case

$$u(x,t) = \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy.$$

- d'Alembert's formula gives uniqueness.
- d'Alembert's formula gives stability:
  - u_1 is a solution with the initial values g_1 and h_1.
  - u_2 is a solution with the initial values g_2 and h_2.
  - Then $v = u_1 - u_2$ is a solution with initial values $g_1 - g_2$ and $h_1 - h_2$
  - If $|g_1 - g_2| \leq \varepsilon$ and $|h_1 - h_2| \leq \varepsilon$, we obtain $|v(x,t)| \leq (1+t)\varepsilon$.
- The wave equation does not smoothen the solution. If $g \in C^k(\mathbb{R})$ and $h \in C^{k-1}(\mathbb{R})$, then $u \in C^k(\mathbb{R} \times (0,\infty))$.
- d'Alembert's formula makes sense even for discontinuous g and h, when corresponding u is not differentiable and thus not a solution to the wave equation.

6.2 Remarks about the one-dimensional case

$$u(x,t) = \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy.$$

- d'Alembert's formula gives uniqueness.
- d'Alembert's formula gives stability:
  - u_1 is a solution with the initial values g_1 and h_1.
  - u_2 is a solution with the initial values g_2 and h_2.
  - Then $v = u_1 - u_2$ is a solution with initial values $g_1 - g_2$ and $h_1 - h_2$
  - If $|g_1 - g_2| \le \varepsilon$ and $|h_1 - h_2| \le \varepsilon$, we obtain $|v(x,t)| \le (1+t)\varepsilon$.
- The wave equation does not smoothen the solution. If $g \in C^k(\mathbb{R})$ and $h \in C^{k-1}(\mathbb{R})$, then $u \in C^k(\mathbb{R} \times (0,\infty))$.
- d'Alembert's formula makes sense even for discontinuous g and h, when corresponding u is not differentiable and thus not a solution to the wave equation.

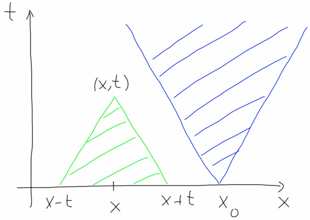
6.2 Remarks about the one-dimensional case

$$u(x,t) = \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy.$$

- d'Alembert's formula gives uniqueness.
- d'Alembert's formula gives stability:
  - u_1 is a solution with the initial values g_1 and h_1.
  - u_2 is a solution with the initial values g_2 and h_2.
  - Then $v = u_1 - u_2$ is a solution with initial values $g_1 - g_2$ and $h_1 - h_2$
  - If $|g_1 - g_2| \leq \varepsilon$ and $|h_1 - h_2| \leq \varepsilon$, we obtain $|v(x,t)| \leq (1+t)\varepsilon$.
- The wave equation does not smoothen the solution. If $g \in C^k(\mathbb{R})$ and $h \in C^{k-1}(\mathbb{R})$, then $u \in C^k(\mathbb{R} \times (0,\infty))$.
- d'Alembert's formula makes sense even for discontinuous g and h, when corresponding u is not differentiable and thus not a solution to the wave equation.

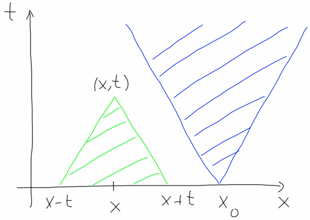
6.2 Remarks about the one-dimensional case

- The solution at the point (x,t) depends only on the values of g and h on the interval $[x-t, x+t]$. This is called the domain of dependence of (x,t).
- Conversely, for every $x_0 \in \mathbb{R}$, there is a conical region called the range of influence of x_0.
- Physically this means that the disturbances or signals propagate with a finite speed.



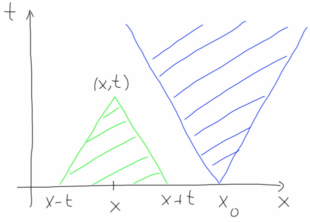
6.2 Remarks about the one-dimensional case

- The solution at the point (x,t) depends only on the values of g and h on the interval $[x-t, x+t]$. This is called the domain of dependence of (x,t).
- Conversely, for every $x_0 \in \mathbb{R}$, there is a conical region called the range of influence of x_0.
- Physically this means that the disturbances or signals propagate with a finite speed.



6.2 Remarks about the one-dimensional case

- The solution at the point (x, t) depends only on the values of g and h on the interval $[x - t, x + t]$. This is called the domain of dependence of (x, t).
- Conversely, for every $x_0 \in \mathbb{R}$, there is a conical region called the range of influence of x_0.
- Physically this means that the disturbances or signals propagate with a finite speed.



6.2 1D problem in $\mathbb{R}_+ \times \mathbb{R}_+$

- We have now a formula for the solution in 1D case where $x \in \mathbb{R}$ and $t \geq 0$.
- What if we want to solve wave equation in subdomain where $x > 0$ with extra boundary values $u(0,t) = 0$?

$$\begin{cases} u_{tt} - u_{xx} = 0 \quad \text{in} \quad \mathbb{R}_+ \times (0,\infty), \\ u = g, \quad u_t = h \quad \text{on} \quad \mathbb{R}_+ \times \{t = 0\}, \\ u = 0 \quad \text{on} \quad \{x = 0\} \times (0,\infty), \end{cases}$$

- We solve this with reflecting u over the boundary $x = 0$ with an odd reflection:

$$\widetilde{u}(x,t) = \begin{cases} u(x,t), & x \geq 0, \quad t \geq 0, \\ -u(-x,t), & x \leq 0, \quad t \geq 0, \end{cases}$$

- This will be needed later.

6.2 1D problem in $\mathbb{R}_+ \times \mathbb{R}_+$

- We have now a formula for the solution in 1D case where $x \in \mathbb{R}$ and $t \geq 0$.
- What if we want to solve wave equation in subdomain where $x > 0$ with extra boundary values $u(0,t) = 0$?

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in} \quad \mathbb{R}_+ \times (0,\infty), \\ u = g, \quad u_t = h & \text{on} \quad \mathbb{R}_+ \times \{t = 0\}, \\ u = 0 \quad \text{on} & \{x = 0\} \times (0,\infty), \end{cases}$$

- We solve this with reflecting u over the boundary $x = 0$ with an odd reflection:

$$\widetilde{u}(x,t) = \begin{cases} u(x,t), & x \geq 0, \quad t \geq 0, \\ -u(-x,t), & x \leq 0, \quad t \geq 0, \end{cases}$$

- This will be needed later.

6.2 1D problem in $\mathbb{R}_+ \times \mathbb{R}_+$

- We have now a formula for the solution in 1D case where $x \in \mathbb{R}$ and $t \geq 0$.
- What if we want to solve wave equation in subdomain where $x > 0$ with extra boundary values $u(0,t) = 0$?

$$\begin{cases} u_{tt} - u_{xx} = 0 \quad \text{in} \quad \mathbb{R}_+ \times (0, \infty), \\ u = g, \quad u_t = h \quad \text{on} \quad \mathbb{R}_+ \times \{t = 0\}, \\ u = 0 \quad \text{on} \quad \{x = 0\} \times (0, \infty), \end{cases}$$

- We solve this with reflecting u over the boundary $x = 0$ with an odd reflection:

$$\widetilde{u}(x,t) = \begin{cases} u(x,t), \quad x \geq 0, \quad t \geq 0, \\ -u(-x,t), \quad x \leq 0, \quad t \geq 0, \end{cases}$$

- This will be needed later.

6.2 1D problem in $\mathbb{R}_+ \times \mathbb{R}_+$

- We have now a formula for the solution in 1D case where $x \in \mathbb{R}$ and $t \geq 0$.
- What if we want to solve wave equation in subdomain where $x > 0$ with extra boundary values $u(0,t) = 0$?

$$\begin{cases} u_{tt} - u_{xx} = 0 \quad \text{in} \quad \mathbb{R}_+ \times (0,\infty), \\ u = g, \quad u_t = h \quad \text{on} \quad \mathbb{R}_+ \times \{t = 0\}, \\ u = 0 \quad \text{on} \quad \{x = 0\} \times (0,\infty), \end{cases}$$

- We solve this with reflecting u over the boundary $x = 0$ with an odd reflection:

$$\widetilde{u}(x,t) = \begin{cases} u(x,t), \quad x \geq 0, \quad t \geq 0, \\ -u(-x,t), \quad x \leq 0, \quad t \geq 0, \end{cases}$$

- This will be needed later.

6.3 The Euler-Poisson-Darboux equation

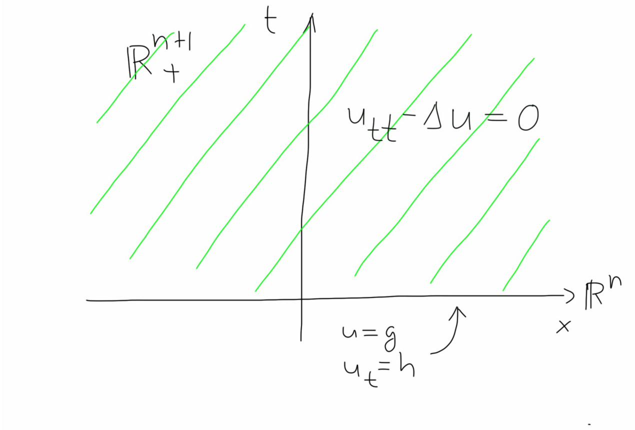


Figure: The Cauchy problem for the wave equation.

6.3 The Euler-Poisson-Darboux equation

- In higher dimensions, there is not as simple expression for the solutions as in $n = 1$ case.
- We shall use the method of spherical means:

$$U(x;r,t) = \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u(y,t)\,dS(y),$$

$$G(x;r) = \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} g(y)\,dS(y),$$

$$H(x;r) = \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} h(y)\,dS(y).$$

- Idea: replace pointwise values by integral averages over spheres.
- We recover the functions from the averages:

$$\lim_{r\to 0} U(x;r,t) = u(x,t)$$

6.3 The Euler-Poisson-Darboux equation

- In higher dimensions, there is not as simple expression for the solutions as in $n = 1$ case.
- We shall use the method of spherical means:

$$U(x;r,t) = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u(y,t)\, dS(y),$$

$$G(x;r) = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} g(y)\, dS(y),$$

$$H(x;r) = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} h(y)\, dS(y).$$

- Idea: replace pointwise values by integral averages over spheres.
- We recover the functions from the averages:

$$\lim_{r\to 0} U(x;r,t) = u(x,t)$$

6.3 The Euler-Poisson-Darboux equation

- In higher dimensions, there is not as simple expression for the solutions as in $n = 1$ case.
- We shall use the method of spherical means:

$$U(x;r,t) = \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u(y,t)\,dS(y),$$

$$G(x;r) = \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} g(y)\,dS(y),$$

$$H(x;r) = \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} h(y)\,dS(y).$$

- Idea: replace pointwise values by integral averages over spheres.
- We recover the functions from the averages:

$$\lim_{r\to 0} U(x;r,t) = u(x,t)$$

6.3 The Euler-Poisson-Darboux equation

- In higher dimensions, there is not as simple expression for the solutions as in $n = 1$ case.
- We shall use the method of spherical means:

$$U(x; r, t) = \frac{1}{|\partial B(x, r)|} \int_{\partial B(x,r)} u(y, t)\, dS(y),$$

$$G(x; r) = \frac{1}{|\partial B(x, r)|} \int_{\partial B(x,r)} g(y)\, dS(y),$$

$$H(x; r) = \frac{1}{|\partial B(x, r)|} \int_{\partial B(x,r)} h(y)\, dS(y).$$

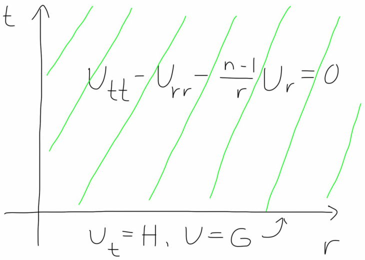
- Idea: replace pointwise values by integral averages over spheres.
- We recover the functions from the averages:

$$\lim_{r \to 0} U(x; r, t) = u(x, t)$$

Theorem (Euler-Poisson-Darboux equation)

Let u be a solution of the Cauchy problem. Then for every fixed $x \in \mathbb{R}^n$, we have

$$\begin{cases} U_{tt} - U_{rr} - \dfrac{n-1}{r} U_r = 0 & \textit{in} \quad \mathbb{R}_+ \times (0,\infty), \\ U = G, \quad U_t = H & \textit{in} \quad \mathbb{R}_+ \times \{t = 0\}. \end{cases}$$



Theorem (Euler-Poisson-Darboux equation)

Let u be a solution of the Cauchy problem. Then for every fixed $x \in \mathbb{R}^n$, we have

$$\begin{cases} U_{tt} - U_{rr} - \dfrac{n-1}{r} U_r = 0 & \textit{in} \quad \mathbb{R}_+ \times (0,\infty), \\ U = G, \quad U_t = H & \textit{in} \quad \mathbb{R}_+ \times \{t=0\}. \end{cases}$$

- Recognize the radial Laplace equation $\Delta u(x) = v''(r) + \frac{n-1}{r} v'(r)$ if $u(x) = v(|x|)$.
- Proof: Start as with the proof of the mean value property for Δu to obtain.

$$U_r(x;r,t) = \frac{1}{n\alpha(n)r^{n-1}} \int_{B(x,r)} \Delta u(y,t)dy$$

- Multiply by r^{n-1} and derivate with respect to r to obtain

$$\frac{\partial}{\partial r}(r^{n-1}U_r(x;r,t)) = \frac{r^{n-1}}{|\partial B(x,r)|} \int_{\partial B(x,r)} u_{tt}(y,t)\, dS(y) \quad = r^{n-1}U_{tt}(x;r,t),$$

Theorem (Euler-Poisson-Darboux equation)

Let u be a solution of the Cauchy problem. Then for every fixed $x \in \mathbb{R}^n$, we have

$$\begin{cases} U_{tt} - U_{rr} - \dfrac{n-1}{r} U_r = 0 & \textit{in} \quad \mathbb{R}_+ \times (0, \infty), \\ U = G, \quad U_t = H & \textit{in} \quad \mathbb{R}_+ \times \{t = 0\}. \end{cases}$$

- Recognize the radial Laplace equation $\Delta u(x) = v''(r) + \frac{n-1}{r} v'(r)$ if $u(x) = v(|x|)$.
- Proof: Start as with the proof of the mean value property for Δu to obtain.

$$U_r(x; r, t) = \frac{1}{n\alpha(n) r^{n-1}} \int_{B(x,r)} \Delta u(y, t) dy$$

- Multiply by r^{n-1} and derivate with respect to r to obtain

$$\frac{\partial}{\partial r}(r^{n-1} U_r(x; r, t)) = \frac{r^{n-1}}{|\partial B(x,r)|} \int_{\partial B(x,r)} u_{tt}(y, t)\, dS(y) \quad = r^{n-1} U_{tt}(x; r, t),$$

Theorem (Euler-Poisson-Darboux equation)

Let u be a solution of the Cauchy problem. Then for every fixed $x \in \mathbb{R}^n$, we have

$$\begin{cases} U_{tt} - U_{rr} - \dfrac{n-1}{r} U_r = 0 & \textit{in} \quad \mathbb{R}_+ \times (0, \infty), \\ U = G, \quad U_t = H & \textit{in} \quad \mathbb{R}_+ \times \{t = 0\}. \end{cases}$$

- Recognize the radial Laplace equation $\Delta u(x) = v''(r) + \frac{n-1}{r} v'(r)$ if $u(x) = v(|x|)$.
- Proof: Start as with the proof of the mean value property for Δu to obtain.

$$U_r(x; r, t) = \frac{1}{n\alpha(n) r^{n-1}} \int_{B(x,r)} \Delta u(y, t) dy$$

- Multiply by r^{n-1} and derivate with respect to r to obtain

$$\frac{\partial}{\partial r}(r^{n-1} U_r(x; r, t)) = \frac{r^{n-1}}{|\partial B(x,r)|} \int_{\partial B(x,r)} u_{tt}(y, t) \, dS(y) \quad = r^{n-1} U_{tt}(x; r, t),$$

Theorem (Euler-Poisson-Darboux equation)

Let u be a solution of the Cauchy problem. Then for every fixed $x \in \mathbb{R}^n$, we have

$$\begin{cases} U_{tt} - U_{rr} - \frac{n-1}{r} U_r = 0 \quad \textit{in} \quad \mathbb{R}_+ \times (0, \infty), \\ U = G, \quad U_t = H \quad \textit{in} \quad \mathbb{R}_+ \times \{t = 0\}. \end{cases}$$

- Recognize the radial Laplace equation $\Delta u(x) = v''(r) + \frac{n-1}{r} v'(r)$ if $u(x) = v(|x|)$.
- Proof: Start as with the proof of the mean value property for Δu to obtain.

$$U_r(x; r, t) = \frac{1}{n\alpha(n) r^{n-1}} \int_0^r \int_{\partial B(x,\rho)} \Delta u(y, t) dS(y) d\rho.$$

- Multiply by r^{n-1} and derivate with respect to r to obtain

$$\frac{\partial}{\partial r}(r^{n-1} U_r(x; r, t)) = \frac{r^{n-1}}{|\partial B(x,r)|} \int_{\partial B(x,r)} u_{tt}(y, t)\, dS(y) \quad = r^{n-1} U_{tt}(x; r, t),$$

Theorem (Euler-Poisson-Darboux equation)

Let u be a solution of the Cauchy problem. Then for every fixed $x \in \mathbb{R}^n$, we have

$$\begin{cases} U_{tt} - U_{rr} - \dfrac{n-1}{r} U_r = 0 & \textit{in} \quad \mathbb{R}_+ \times (0, \infty), \\ U = G, \quad U_t = H & \textit{in} \quad \mathbb{R}_+ \times \{t = 0\}. \end{cases}$$

- Recognize the radial Laplace equation $\Delta u(x) = v''(r) + \frac{n-1}{r} v'(r)$ if $u(x) = v(|x|)$.
- Proof: Start as with the proof of the mean value property for Δu to obtain.

$$U_r(x; r, t) = \frac{1}{n\alpha(n) r^{n-1}} \int_0^r \int_{\partial B(x,\rho)} u_{tt}(y, t) dS(y) d\rho.$$

- Multiply by r^{n-1} and derivate with respect to r to obtain

$$\frac{\partial}{\partial r}(r^{n-1} U_r(x; r, t)) = \frac{r^{n-1}}{|\partial B(x,r)|} \int_{\partial B(x,r)} u_{tt}(y, t) \, dS(y) \quad = r^{n-1} U_{tt}(x; r, t),$$

Theorem (Euler-Poisson-Darboux equation)

Let u be a solution of the Cauchy problem. Then for every fixed $x \in \mathbb{R}^n$, we have

$$\begin{cases} U_{tt} - U_{rr} - \dfrac{n-1}{r} U_r = 0 \quad \textit{in} \quad \mathbb{R}_+ \times (0,\infty), \\ U = G, \quad U_t = H \quad \textit{in} \quad \mathbb{R}_+ \times \{t = 0\}. \end{cases}$$

- Recognize the radial Laplace equation $\Delta u(x) = v''(r) + \frac{n-1}{r} v'(r)$ if $u(x) = v(|x|)$.
- Proof: Start as with the proof of the mean value property for Δu to obtain.

$$U_r(x;r,t) = \frac{1}{n\alpha(n)r^{n-1}} \int_0^r \int_{\partial B(x,\rho)} u_{tt}(y,t)dS(y)d\rho.$$

- Multiply by r^{n-1} and derivate with respect to r to obtain

$$\frac{\partial}{\partial r}(r^{n-1}U_r(x;r,t)) = \frac{r^{n-1}}{|\partial B(x,r)|} \int_{\partial B(x,r)} u_{tt}(y,t)\, dS(y) \quad = r^{n-1}U_{tt}(x;r,t),$$

6.4 Wave equation when $n = 3$

- Now we study the case $n = 3$.
- We continue using the integral averages U, G and H.
- and denote

$$\widetilde{U} = rU, \quad \widetilde{G} = rG \quad \text{and} \quad \widetilde{H} = rH.$$

- By the Euler-Darboux-Poisson equation

$$\begin{aligned}
\widetilde{U}_{tt} - \widetilde{U}_{rr} &= rU_{tt} - (U + rU_r)_r \\
&= rU_{tt} - (U_r + U_r + rU_{rr}) \\
&= rU_{tt} - 2U_r - rU_{rr} \\
&= r(U_{tt} - U_{rr} - \frac{2}{r}U_r) = 0
\end{aligned}$$

- This works only when $n - 1 = 2$!

6.4 Wave equation when $n = 3$

- Now we study the case $n = 3$.
- We continue using the integral averages U, G and H.
- and denote

$$\widetilde{U} = rU, \quad \widetilde{G} = rG \quad \text{and} \quad \widetilde{H} = rH.$$

- By the Euler-Darboux-Poisson equation

$$\begin{aligned}\widetilde{U}_{tt} - \widetilde{U}_{rr} &= rU_{tt} - (U + rU_r)_r \\ &= rU_{tt} - (U_r + U_r + rU_{rr}) \\ &= rU_{tt} - 2U_r - rU_{rr} \\ &= r(U_{tt} - U_{rr} - \frac{2}{r}U_r) = 0\end{aligned}$$

- This works only when $n - 1 = 2$!

6.4 Wave equation when $n = 3$

- Now we study the case $n = 3$.
- We continue using the integral averages U, G and H.
- and denote

$$\widetilde{U} = rU, \quad \widetilde{G} = rG \quad \text{and} \quad \widetilde{H} = rH.$$

- By the Euler-Darboux-Poisson equation

$$\begin{aligned}
\widetilde{U}_{tt} - \widetilde{U}_{rr} &= rU_{tt} - (U + rU_r)_r \\
&= rU_{tt} - (U_r + U_r + rU_{rr}) \\
&= rU_{tt} - 2U_r - rU_{rr} \\
&= r(U_{tt} - U_{rr} - \frac{2}{r}U_r) = 0
\end{aligned}$$

- This works only when $n - 1 = 2$!

6.4 Wave equation when $n = 3$

- Now we study the case $n = 3$.
- We continue using the integral averages U, G and H.
- and denote

$$\widetilde{U} = rU, \quad \widetilde{G} = rG \quad \text{and} \quad \widetilde{H} = rH.$$

- By the Euler-Darboux-Poisson equation

$$\begin{aligned}
\widetilde{U}_{tt} - \widetilde{U}_{rr} &= rU_{tt} - (U + rU_r)_r \\
&= rU_{tt} - (U_r + U_r + rU_{rr}) \\
&= rU_{tt} - 2U_r - rU_{rr} \\
&= r(U_{tt} - U_{rr} - \frac{2}{r}U_r) = 0
\end{aligned}$$

- This works only when $n - 1 = 2$!

6.4 Wave equation when $n = 3$

- Now we study the case $n = 3$.
- We continue using the integral averages U, G and H.
- and denote

$$\widetilde{U} = rU, \quad \widetilde{G} = rG \quad \text{and} \quad \widetilde{H} = rH.$$

- By the Euler-Darboux-Poisson equation

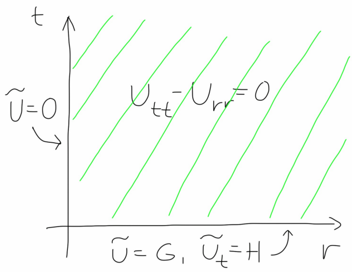
$$\begin{aligned}
\widetilde{U}_{tt} - \widetilde{U}_{rr} &= rU_{tt} - (U + rU_r)_r \\
&= rU_{tt} - (U_r + U_r + rU_{rr}) \\
&= rU_{tt} - 2U_r - rU_{rr} \\
&= r(U_{tt} - U_{rr} - \frac{n-1}{r}U_r) = 0
\end{aligned}$$

- This works only when $n - 1 = 2$!

6.4 Wave equation when $n = 3$

We have reduced the problem to

$$\begin{cases} \widetilde{U}_{tt} - \widetilde{U}_{rr} = 0 & \text{in} \quad \mathbb{R}_+ \times (0,\infty), \\ \widetilde{U} = \widetilde{G}, \quad \widetilde{U}_t = \widetilde{H} & \text{on} \quad \mathbb{R}_+ \times \{t = 0\}, \\ \widetilde{U} = 0 & \text{on} \quad \{r = 0\} \times (0,\infty). \end{cases}$$



6.4 Wave equation when $n = 3$

We have reduced the problem to

$$\begin{cases} \widetilde{U}_{tt} - \widetilde{U}_{rr} = 0 \quad \text{in} \quad \mathbb{R}_+ \times (0, \infty), \\ \widetilde{U} = \widetilde{G}, \quad \widetilde{U}_t = \widetilde{H} \quad \text{on} \quad \mathbb{R}_+ \times \{t = 0\}, \\ \widetilde{U} = 0 \quad \text{on} \quad \{r = 0\} \times (0, \infty). \end{cases}$$

- 3D wave equation is reduced to 1D wave equation, which can be solved with d'Alembert's formula.
- Notice that we are in the first quadrant i.e. need to use reflection principle.
- We recover the solution to the original problem by considering

$$u(x,t) = \lim_{r \to 0} U(x; r, t) = \lim_{r \to 0} \frac{rU(x; r, t)}{r} = \lim_{r \to 0} \frac{\widetilde{U}(x; r, t)}{r}.$$

6.4 Wave equation when $n = 3$

We have reduced the problem to

$$\begin{cases} \widetilde{U}_{tt} - \widetilde{U}_{rr} = 0 \quad \text{in} \quad \mathbb{R}_+ \times (0, \infty), \\ \widetilde{U} = \widetilde{G}, \quad \widetilde{U}_t = \widetilde{H} \quad \text{on} \quad \mathbb{R}_+ \times \{t = 0\}, \\ \widetilde{U} = 0 \quad \text{on} \quad \{r = 0\} \times (0, \infty). \end{cases}$$

- 3D wave equation is reduced to 1D wave equation, which can be solved with d'Alembert's formula.
- Notice that we are in the first quadrant i.e. need to use reflection principle.
- We recover the solution to the original problem by considering

$$u(x,t) = \lim_{r \to 0} U(x;r,t) = \lim_{r \to 0} \frac{rU(x;r,t)}{r} = \lim_{r \to 0} \frac{\widetilde{U}(x;r,t)}{r}.$$

6.4 Wave equation when $n = 3$

We have reduced the problem to

$$\begin{cases} \widetilde{U}_{tt} - \widetilde{U}_{rr} = 0 \quad \text{in} \quad \mathbb{R}_+ \times (0, \infty), \\ \widetilde{U} = \widetilde{G}, \quad \widetilde{U}_t = \widetilde{H} \quad \text{on} \quad \mathbb{R}_+ \times \{t = 0\}, \\ \widetilde{U} = 0 \quad \text{on} \quad \{r = 0\} \times (0, \infty). \end{cases}$$

- 3D wave equation is reduced to 1D wave equation, which can be solved with d'Alembert's formula.
- Notice that we are in the first quadrant i.e. need to use reflection principle.
- We recover the solution to the original problem by considering

$$u(x, t) = \lim_{r \to 0} U(x; r, t) = \lim_{r \to 0} \frac{rU(x; r, t)}{r} = \lim_{r \to 0} \frac{\widetilde{U}(x; r, t)}{r}.$$

6.4 Wave equation when $n = 3$

- Finally we obtain Kirchhoff's formula for the solution of the Cauchy problem for the three-dimensional wave equation.

$$
\begin{aligned}
u(x,t) &= \fint_{\partial B(x,t)} g(y)\, dS(y) + t\frac{\partial}{\partial t}\Big(\tfrac{1}{|\partial B(x,t)|}\int_{\partial B(x,t)} g(y)\, dS(y)\Big) \\
&\qquad + \tfrac{t}{|\partial B(x,t)|}\int_{\partial B(x,t)} h(y)\, dS(y) \\
&= \tfrac{1}{|\partial B(x,t)|}\int_{\partial B(x,t)} \left(th(y) + g(y) + \nabla g(y)\cdot(y-x)\right)\, dS(y).
\end{aligned}
$$

- To compute $u(x,t)$, we only need information on the data on the sphere $\partial B(x,t)$
- Similarly, the range of influence of a point $x_0 \in \mathbb{R}^3$ is the surface of the (light) cone

$$\{(x,t) \in \mathbb{R}^3 \times (0,\infty) : |x - x_0| = t\}.$$

6.4 Wave equation when $n = 3$

- Finally we obtain Kirchhoff's formula for the solution of the Cauchy problem for the three-dimensional wave equation.

$$
\begin{aligned}
u(x,t) &= \fint_{\partial B(x,t)} g(y)\,dS(y) + t\tfrac{\partial}{\partial t}\big(\tfrac{1}{|\partial B(x,t)|}\int_{\partial B(x,t)} g(y)\,dS(y)\big) \\
&\qquad + \tfrac{t}{|\partial B(x,t)|}\int_{\partial B(x,t)} h(y)\,dS(y) \\
&= \tfrac{1}{|\partial B(x,t)|}\int_{\partial B(x,t)} \left(th(y) + g(y) + \nabla g(y)\cdot(y-x)\right)\,dS(y).
\end{aligned}
$$

- To compute $u(x,t)$, we only need information on the data on the sphere $\partial B(x,t)$
- Similarly, the range of influence of a point $x_0 \in \mathbb{R}^3$ is the surface of the (light) cone

$$\{(x,t) \in \mathbb{R}^3 \times (0,\infty) : |x - x_0| = t\}.$$

6.4 Wave equation when $n = 3$

- Finally we obtain Kirchhoff's formula for the solution of the Cauchy problem for the three-dimensional wave equation.

$$
\begin{aligned}
u(x,t) &= \fint_{\partial B(x,t)} g(y)\, dS(y) + t\frac{\partial}{\partial t}\Big(\frac{1}{|\partial B(x,t)|}\int_{\partial B(x,t)} g(y)\, dS(y)\Big) \\
&\qquad + \frac{t}{|\partial B(x,t)|}\int_{\partial B(x,t)} h(y)\, dS(y) \\
&= \frac{1}{|\partial B(x,t)|}\int_{\partial B(x,t)} \left(th(y) + g(y) + \nabla g(y)\cdot(y-x)\right)\, dS(y).
\end{aligned}
$$

- To compute $u(x,t)$, we only need information on the data on the sphere $\partial B(x,t)$
- Similarly, the range of influence of a point $x_0 \in \mathbb{R}^3$ is the surface of the (light) cone

$$\{(x,t) \in \mathbb{R}^3 \times (0,\infty) : |x - x_0| = t\}.$$

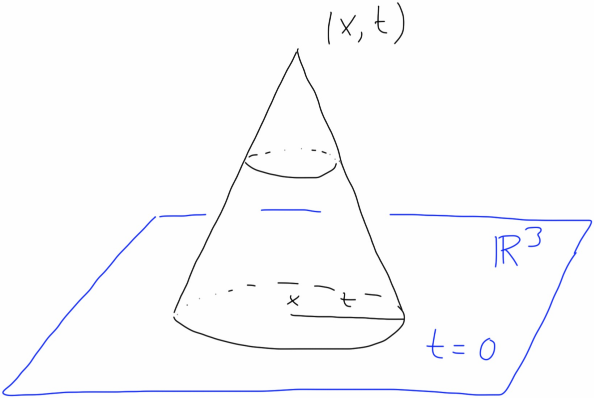


Figure: The domain of dependence in the three-dimensional case.

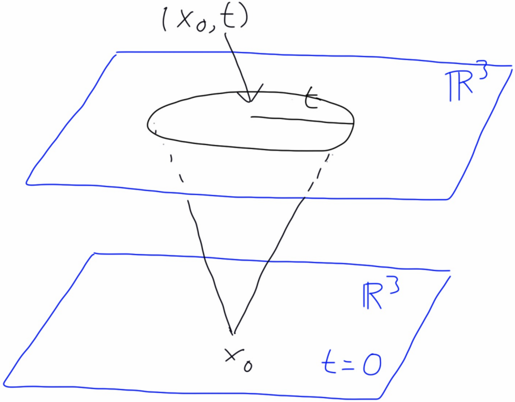


Figure: The range of influence in the three-dimensional case.

- 3D case: Information propagates at exactly unit speed, no faster and no slower! (Huygens' principle)
- 1D and 2D case: Slower is possible.
- Finite speed of propagation makes it possible to localize the process of solving initial value problems.
- Such a localization is not possible for boundary / initial value problems for Laplace and heat equation.
- in 3D case, the solution has less regularity than initial values:
- If $g \in C^2(\mathbb{R}^3)$ and $h \in C^1(\mathbb{R}^3)$, then Kirchhoff's formula gives only that u is C^1-function with respect to the space variable

- 3D case: Information propagates at exactly unit speed, no faster and no slower! (Huygens' principle)
- 1D and 2D case: Slower is possible.
- Finite speed of propagation makes it possible to localize the process of solving initial value problems.
- Such a localization is not possible for boundary / initial value problems for Laplace and heat equation.
- in 3D case, the solution has less regularity than initial values:
- If $g \in C^2(\mathbb{R}^3)$ and $h \in C^1(\mathbb{R}^3)$, then Kirchhoff's formula gives only that u is C^1-function with respect to the space variable

- 3D case: Information propagates at exactly unit speed, no faster and no slower! (Huygens' principle)
- 1D and 2D case: Slower is possible.
- Finite speed of propagation makes it possible to localize the process of solving initial value problems.
- Such a localization is not possible for boundary / initial value problems for Laplace and heat equation.
- in 3D case, the solution has less regularity than initial values:
- If $g \in C^2(\mathbb{R}^3)$ and $h \in C^1(\mathbb{R}^3)$, then Kirchhoff's formula gives only that u is C^1-function with respect to the space variable

- 3D case: Information propagates at exactly unit speed, no faster and no slower! (Huygens' principle)
- 1D and 2D case: Slower is possible.
- Finite speed of propagation makes it possible to localize the process of solving initial value problems.
- Such a localization is not possible for boundary / initial value problems for Laplace and heat equation.
- in 3D case, the solution has less regularity than initial values:
- If $g \in C^2(\mathbb{R}^3)$ and $h \in C^1(\mathbb{R}^3)$, then Kirchhoff's formula gives only that u is C^1-function with respect to the space variable

- 3D case: Information propagates at exactly unit speed, no faster and no slower! (Huygens' principle)
- 1D and 2D case: Slower is possible.
- Finite speed of propagation makes it possible to localize the process of solving initial value problems.
- Such a localization is not possible for boundary / initial value problems for Laplace and heat equation.
- in 3D case, the solution has less regularity than initial values:
- If $g \in C^2(\mathbb{R}^3)$ and $h \in C^1(\mathbb{R}^3)$, then Kirchhoff's formula gives only that u is C^1-function with respect to the space variable

MS-C1350 Partial differential equations Wave equation : dimension 2, nonhomogeneous problem and energy methods

Riikka Korte

Department of Mathematics and Systems Analysis
Aalto University
riikka.korte@aalto.fi

November 26, 2024

Lecture 12

- Wave equation
  - Wave equation in dimension 2
    (by ”descending” from dimension 3)
  - Non-homogeneous problem
  - Energy methods.
- The exam

6.5 Two-dimensional wave equation

- Now we consider the Cauchy problem for the wave equation when $n = 2$.
- We shall use Hadamard's method of descent.
- 2D problem = special case of 3D problem where x_3-variable does not appear.
- Let us write

$$\widetilde{u}(x_1, x_2, x_3, t) = u(x_1, x_2, t), \quad (x_1, x_2, x_3, t) \in \mathbb{R}^3 \times (0, \infty).$$

- Similarly

$$\widetilde{g}(x_1, x_2, x_3) = g(x_1, x_2) \quad \text{and} \quad \widetilde{h}(x_1, x_2, x_3) = h(x_1, x_2).$$

- If u satisfies 2D problem, $\widetilde{u}$ satisfies 3D problem:

$$\begin{cases} \widetilde{u}_{tt} - \Delta \widetilde{u} = 0 & \text{in} \quad \mathbb{R}^3 \times (0, \infty), \\ \widetilde{u} = \widetilde{g}, \quad \widetilde{u}_t = \widetilde{h} & \text{on} \quad \mathbb{R}^3 \times \{t = 0\}. \end{cases}$$

- Then we use Kirchhoff's formula for the 3D problem.

6.5 Two-dimensional wave equation

- Now we consider the Cauchy problem for the wave equation when $n = 2$.
- We shall use Hadamard's method of descent.
- 2D problem = special case of 3D problem where x_3-variable does not appear.
- Let us write

$$\widetilde{u}(x_1, x_2, x_3, t) = u(x_1, x_2, t), \quad (x_1, x_2, x_3, t) \in \mathbb{R}^3 \times (0, \infty).$$

- Similarly

$$\widetilde{g}(x_1, x_2, x_3) = g(x_1, x_2) \quad \text{and} \quad \widetilde{h}(x_1, x_2, x_3) = h(x_1, x_2).$$

- If u satisfies 2D problem, $\widetilde{u}$ satisfies 3D problem:

$$\begin{cases} \widetilde{u}_{tt} - \Delta \widetilde{u} = 0 & \text{in} \quad \mathbb{R}^3 \times (0, \infty), \\ \widetilde{u} = \widetilde{g}, \quad \widetilde{u}_t = \widetilde{h} & \text{on} \quad \mathbb{R}^3 \times \{t = 0\}. \end{cases}$$

- Then we use Kirchhoff's formula for the 3D problem.

6.5 Two-dimensional wave equation

- Now we consider the Cauchy problem for the wave equation when $n = 2$.
- We shall use Hadamard's method of descent.
- 2D problem = special case of 3D problem where x_3-variable does not appear.
- Let us write

$$\widetilde{u}(x_1, x_2, x_3, t) = u(x_1, x_2, t), \quad (x_1, x_2, x_3, t) \in \mathbb{R}^3 \times (0, \infty).$$

- Similarly

$$\widetilde{g}(x_1, x_2, x_3) = g(x_1, x_2) \quad \text{and} \quad \widetilde{h}(x_1, x_2, x_3) = h(x_1, x_2).$$

- If u satisfies 2D problem, $\widetilde{u}$ satisfies 3D problem:

$$\begin{cases} \widetilde{u}_{tt} - \Delta \widetilde{u} = 0 \quad \text{in} \quad \mathbb{R}^3 \times (0, \infty), \\ \widetilde{u} = \widetilde{g}, \quad \widetilde{u}_t = \widetilde{h} \quad \text{on} \quad \mathbb{R}^3 \times \{t = 0\}. \end{cases}$$

- Then we use Kirchhoff's formula for the 3D problem.

6.5 Two-dimensional wave equation

- Now we consider the Cauchy problem for the wave equation when $n = 2$.
- We shall use Hadamard's method of descent.
- 2D problem = special case of 3D problem where x_3-variable does not appear.
- Let us write

$$\widetilde{u}(x_1, x_2, x_3, t) = u(x_1, x_2, t), \quad (x_1, x_2, x_3, t) \in \mathbb{R}^3 \times (0, \infty).$$

- Similarly

$$\widetilde{g}(x_1, x_2, x_3) = g(x_1, x_2) \quad \text{and} \quad \widetilde{h}(x_1, x_2, x_3) = h(x_1, x_2).$$

- If u satisfies 2D problem, $\widetilde{u}$ satisfies 3D problem:

$$\begin{cases} \widetilde{u}_{tt} - \Delta \widetilde{u} = 0 & \text{in} \quad \mathbb{R}^3 \times (0, \infty), \\ \widetilde{u} = \widetilde{g}, \quad \widetilde{u}_t = \widetilde{h} & \text{on} \quad \mathbb{R}^3 \times \{t = 0\}. \end{cases}$$

- Then we use Kirchhoff's formula for the 3D problem.

6.5 Two-dimensional wave equation

- Now we consider the Cauchy problem for the wave equation when $n = 2$.
- We shall use Hadamard's method of descent.
- 2D problem = special case of 3D problem where x_3-variable does not appear.
- Let us write

$$\widetilde{u}(x_1, x_2, x_3, t) = u(x_1, x_2, t), \quad (x_1, x_2, x_3, t) \in \mathbb{R}^3 \times (0, \infty).$$

- Similarly

$$\widetilde{g}(x_1, x_2, x_3) = g(x_1, x_2) \quad \text{and} \quad \widetilde{h}(x_1, x_2, x_3) = h(x_1, x_2).$$

- If u satisfies 2D problem, $\widetilde{u}$ satisfies 3D problem:

$$\begin{cases} \widetilde{u}_{tt} - \Delta \widetilde{u} = 0 & \text{in} \quad \mathbb{R}^3 \times (0, \infty), \\ \widetilde{u} = \widetilde{g}, \quad \widetilde{u}_t = \widetilde{h} & \text{on} \quad \mathbb{R}^3 \times \{t = 0\}. \end{cases}$$

- Then we use Kirchhoff's formula for the 3D problem.

6.5 Two-dimensional wave equation

- Now we consider the Cauchy problem for the wave equation when $n = 2$.
- We shall use Hadamard's method of descent.
- 2D problem = special case of 3D problem where x_3-variable does not appear.
- Let us write

$$\widetilde{u}(x_1, x_2, x_3, t) = u(x_1, x_2, t), \quad (x_1, x_2, x_3, t) \in \mathbb{R}^3 \times (0, \infty).$$

- Similarly

$$\widetilde{g}(x_1, x_2, x_3) = g(x_1, x_2) \quad \text{and} \quad \widetilde{h}(x_1, x_2, x_3) = h(x_1, x_2).$$

- If u satisfies 2D problem, $\widetilde{u}$ satisfies 3D problem:

$$\begin{cases} \widetilde{u}_{tt} - \Delta \widetilde{u} = 0 \quad \text{in} \quad \mathbb{R}^3 \times (0, \infty), \\ \widetilde{u} = \widetilde{g}, \quad \widetilde{u}_t = \widetilde{h} \quad \text{on} \quad \mathbb{R}^3 \times \{t = 0\}. \end{cases}$$

- Then we use Kirchhoff's formula for the 3D problem.

6.5 Two-dimensional wave equation

- Now we consider the Cauchy problem for the wave equation when $n = 2$.
- We shall use Hadamard's method of descent.
- 2D problem = special case of 3D problem where x_3-variable does not appear.
- Let us write

$$\widetilde{u}(x_1, x_2, x_3, t) = u(x_1, x_2, t), \quad (x_1, x_2, x_3, t) \in \mathbb{R}^3 \times (0, \infty).$$

- Similarly

$$\widetilde{g}(x_1, x_2, x_3) = g(x_1, x_2) \quad \text{and} \quad \widetilde{h}(x_1, x_2, x_3) = h(x_1, x_2).$$

- If u satisfies 2D problem, $\widetilde{u}$ satisfies 3D problem:

$$\begin{cases} \widetilde{u}_{tt} - \Delta \widetilde{u} = 0 & \text{in} \quad \mathbb{R}^3 \times (0, \infty), \\ \widetilde{u} = \widetilde{g}, \quad \widetilde{u}_t = \widetilde{h} & \text{on} \quad \mathbb{R}^3 \times \{t = 0\}. \end{cases}$$

- Then we use Kirchhoff's formula for the 3D problem.

6.5 Two-dimensional wave equation

- Kirchhoff's formula for 3D problem gives that $u(x,t) = \widetilde{u}(\widetilde{x},t)$ is

$$\frac{\partial}{\partial t}\left(t\frac{1}{|\partial \widetilde{B}(\widetilde{x},t)|}\int_{\partial \widetilde{B}(\widetilde{x},t)} \widetilde{g}(\widetilde{y})\,dS(\widetilde{y})\right) + \frac{t}{|\partial \widetilde{B}(\widetilde{x},t)|}\int_{\partial \widetilde{B}(\widetilde{x},t)} \widetilde{h}(\widetilde{y})\,dS(\widetilde{y}).$$

- Integrals are over spheres $\partial B(\widetilde{x},t) \subset \mathbb{R}^3$. The calculations become complicated as projecting 3D spheres to 2D produces a 2D ball with a non-constant weight.
- As $n = 3$, $|\partial \widetilde{B}(\widetilde{x},t)| = 4\pi t^2$.
- Let $\widetilde{y} = (y, y_3)$ with $y \in \mathbb{R}^2$, $y_3 \in \mathbb{R}$.
- Integrals are over $\widetilde{y} \in \partial \widetilde{B}(\widetilde{x},t)$ i.e.

$$|\widetilde{y} - \widetilde{x}| = t \iff |y - x|^2 + y_3^2 = t^2 \iff y_3^2 = t^2 - |x - y|^2.$$

6.5 Two-dimensional wave equation

- Kirchhoff's formula for 3D problem gives that $u(x,t) = \widetilde{u}(\widetilde{x},t)$ is

$$\frac{\partial}{\partial t}\left(t\frac{1}{|\partial\widetilde{B}(\widetilde{x},t)|}\int_{\partial\widetilde{B}(\widetilde{x},t)}\widetilde{g}(\widetilde{y})\,dS(\widetilde{y})\right)+\frac{t}{|\partial\widetilde{B}(\widetilde{x},t)|}\int_{\partial\widetilde{B}(\widetilde{x},t)}\widetilde{h}(\widetilde{y})\,dS(\widetilde{y}).$$

- Integrals are over spheres $\partial B(\widetilde{x},t) \subset \mathbb{R}^3$. The calculations become complicated as projecting 3D spheres to 2D produces a 2D ball with a non-constant weight.
- As $n = 3$, $|\partial\widetilde{B}(\widetilde{x},t)| = 4\pi t^2$.
- Let $\widetilde{y} = (y, y_3)$ with $y \in \mathbb{R}^2$, $y_3 \in \mathbb{R}$.
- Integrals are over $\widetilde{y} \in \partial\widetilde{B}(\widetilde{x},t)$ i.e.

$$|\widetilde{y}-\widetilde{x}| = t \iff |y-x|^2 + y_3^2 = t^2 \iff y_3^2 = t^2 - |x-y|^2.$$

6.5 Two-dimensional wave equation

- Kirchhoff's formula for 3D problem gives that $u(x,t) = \widetilde{u}(\widetilde{x},t)$ is

$$\frac{\partial}{\partial t}\left(t\frac{1}{|\partial\widetilde{B}(\widetilde{x},t)|}\int_{\partial\widetilde{B}(\widetilde{x},t)}\widetilde{g}(\widetilde{y})\,dS(\widetilde{y})\right)+\frac{t}{|\partial\widetilde{B}(\widetilde{x},t)|}\int_{\partial\widetilde{B}(\widetilde{x},t)}\widetilde{h}(\widetilde{y})\,dS(\widetilde{y}).$$

- Integrals are over spheres $\partial B(\widetilde{x},t) \subset \mathbb{R}^3$. The calculations become complicated as projecting 3D spheres to 2D produces a 2D ball with a non-constant weight.
- As $n = 3$, $|\partial\widetilde{B}(\widetilde{x},t)| = 4\pi t^2$.
- Let $\widetilde{y} = (y, y_3)$ with $y \in \mathbb{R}^2$, $y_3 \in \mathbb{R}$.
- Integrals are over $\widetilde{y} \in \partial\widetilde{B}(\widetilde{x},t)$ i.e.

$$|\widetilde{y}-\widetilde{x}| = t \iff |y-x|^2 + y_3^2 = t^2 \iff y_3^2 = t^2 - |x-y|^2.$$

6.5 Two-dimensional wave equation

- Kirchhoff's formula for 3D problem gives that $u(x,t) = \widetilde{u}(\widetilde{x},t)$ is

$$\frac{\partial}{\partial t}\left(t\frac{1}{|\partial \widetilde{B}(\widetilde{x},t)|}\int_{\partial \widetilde{B}(\widetilde{x},t)} \widetilde{g}(\widetilde{y})\,dS(\widetilde{y})\right) + \frac{t}{|\partial \widetilde{B}(\widetilde{x},t)|}\int_{\partial \widetilde{B}(\widetilde{x},t)} \widetilde{h}(\widetilde{y})\,dS(\widetilde{y}).$$

- Integrals are over spheres $\partial B(\widetilde{x},t) \subset \mathbb{R}^3$. The calculations become complicated as projecting 3D spheres to 2D produces a 2D ball with a non-constant weight.
- As $n = 3$, $|\partial \widetilde{B}(\widetilde{x},t)| = 4\pi t^2$.
- Let $\widetilde{y} = (y, y_3)$ with $y \in \mathbb{R}^2$, $y_3 \in \mathbb{R}$.
- Integrals are over $\widetilde{y} \in \partial \widetilde{B}(\widetilde{x},t)$ i.e.

$$|\widetilde{y} - \widetilde{x}| = t \iff |y-x|^2 + y_3^2 = t^2 \iff y_3^2 = t^2 - |x-y|^2.$$

6.5 Two-dimensional wave equation

- Kirchhoff's formula for 3D problem gives that $u(x,t) = \widetilde{u}(\widetilde{x},t)$ is

$$\frac{\partial}{\partial t}\left(t\frac{1}{|\partial\widetilde{B}(\widetilde{x},t)|}\int_{\partial\widetilde{B}(\widetilde{x},t)}\widetilde{g}(\widetilde{y})\,dS(\widetilde{y})\right)+\frac{t}{|\partial\widetilde{B}(\widetilde{x},t)|}\int_{\partial\widetilde{B}(\widetilde{x},t)}\widetilde{h}(\widetilde{y})\,dS(\widetilde{y}).$$

- Integrals are over spheres $\partial B(\widetilde{x},t) \subset \mathbb{R}^3$. The calculations become complicated as projecting 3D spheres to 2D produces a 2D ball with a non-constant weight.
- As $n = 3$, $|\partial\widetilde{B}(\widetilde{x},t)| = 4\pi t^2$.
- Let $\widetilde{y} = (y, y_3)$ with $y \in \mathbb{R}^2$, $y_3 \in \mathbb{R}$.
- Integrals are over $\widetilde{y} \in \partial\widetilde{B}(\widetilde{x},t)$ i.e.

$$|\widetilde{y}-\widetilde{x}| = t \Longleftrightarrow |y-x|^2 + y_3^2 = t^2 \Longleftrightarrow y_3^2 = t^2 - |x-y|^2.$$

6.5 Integral over hemishpere

- Consider the upper hemisphere $\partial\widetilde{B}(\widetilde{x},t)^+$ of $\partial\widetilde{B}(\widetilde{x},t)$ in $\mathbb{R}^3$.
- Parametrization of the surface:

$$\widetilde{y} = (y_1, y_2, \gamma(y_1, y_2)), \quad \text{where} \quad \gamma(y_1, y_2) = (t^2 - |x-y|^2)^{1/2}.$$

- Integral of the surface using the parametrization $y = (y_1, y_2) \in \mathbb{R}^2$.

$$\int_{\partial\widetilde{B}(\widetilde{x},t)^+} F(\widetilde{y})\, dS(\widetilde{y}) = \int_{B(x,t)} F(y, \gamma(y))(1 + |\nabla\gamma(y)|^2)^{1/2}\, dy$$

- Lower hemisphere produces exactly the same integral.
- Direct calculation gives

$$|\nabla\gamma(y)| = \frac{|x-y|}{(t^2 - |x-y|^2)^{1/2}}$$

6.5 Integral over hemishpere

- Consider the upper hemisphere $\partial \widetilde{B}(\widetilde{x},t)^+$ of $\partial \widetilde{B}(\widetilde{x},t)$ in $\mathbb{R}^3$.
- Parametrization of the surface:

$$\widetilde{y} = (y_1, y_2, \gamma(y_1, y_2)), \quad \text{where} \quad \gamma(y_1, y_2) = (t^2 - |x-y|^2)^{1/2}.$$

- Integral of the surface using the parametrization $y = (y_1, y_2) \in \mathbb{R}^2$.

$$\int_{\partial \widetilde{B}(\widetilde{x},t)^+} F(\widetilde{y})\, dS(\widetilde{y}) = \int_{B(x,t)} F(y, \gamma(y))(1 + |\nabla \gamma(y)|^2)^{1/2}\, dy$$

- Lower hemisphere produces exactly the same integral.
- Direct calculation gives

$$|\nabla \gamma(y)| = \frac{|x-y|}{(t^2 - |x-y|^2)^{1/2}}$$

6.5 Integral over hemishpere

- Consider the upper hemisphere $\partial\widetilde{B}(\widetilde{x},t)^+$ of $\partial\widetilde{B}(\widetilde{x},t)$ in $\mathbb{R}^3$.
- Parametrization of the surface:

$$\widetilde{y} = (y_1, y_2, \gamma(y_1,y_2)), \quad \text{where} \quad \gamma(y_1,y_2) = (t^2 - |x-y|^2)^{1/2}.$$

- Integral of the surface using the parametrization $y = (y_1, y_2) \in \mathbb{R}^2$.

$$\int_{\partial\widetilde{B}(\widetilde{x},t)^+} F(\widetilde{y})\,dS(\widetilde{y}) = \int_{B(x,t)} F(y,\gamma(y))(1+|\nabla\gamma(y)|^2)^{1/2}\,dy$$

- Lower hemisphere produces exactly the same integral.
- Direct calculation gives

$$|\nabla\gamma(y)| = \frac{|x-y|}{(t^2-|x-y|^2)^{1/2}}$$

6.5 Integral over hemishpere

- Consider the upper hemisphere $\partial\widetilde{B}(\widetilde{x},t)^+$ of $\partial\widetilde{B}(\widetilde{x},t)$ in $\mathbb{R}^3$.
- Parametrization of the surface:

$$\widetilde{y} = (y_1, y_2, \gamma(y_1, y_2)), \quad \text{where} \quad \gamma(y_1, y_2) = (t^2 - |x-y|^2)^{1/2}.$$

- Integral of the surface using the parametrization $y = (y_1, y_2) \in \mathbb{R}^2$.

$$\int_{\partial\widetilde{B}(\widetilde{x},t)^+} F(\widetilde{y})\, dS(\widetilde{y}) = \int_{B(x,t)} F(y, \gamma(y))(1 + |\nabla\gamma(y)|^2)^{1/2}\, dy$$

- Lower hemisphere produces exactly the same integral.
- Direct calculation gives

$$|\nabla\gamma(y)| = \frac{|x-y|}{(t^2 - |x-y|^2)^{1/2}}$$

6.5 Integral over hemishpere

- Consider the upper hemisphere $\partial\widetilde{B}(\widetilde{x},t)^{+}$ of $\partial\widetilde{B}(\widetilde{x},t)$ in $\mathbb{R}^3$.
- Parametrization of the surface:

$$\widetilde{y} = (y_1, y_2, \gamma(y_1, y_2)), \quad \text{where} \quad \gamma(y_1, y_2) = (t^2 - |x-y|^2)^{1/2}.$$

- Integral of the surface using the parametrization $y = (y_1, y_2) \in \mathbb{R}^2$.

$$\int_{\partial\widetilde{B}(\widetilde{x},t)^{+}} F(\widetilde{y})\, dS(\widetilde{y}) = \int_{B(x,t)} F(y, \gamma(y))(1 + |\nabla\gamma(y)|^2)^{1/2}\, dy$$

- Lower hemisphere produces exactly the same integral.
- Direct calculation gives

$$|\nabla\gamma(y)| = \frac{|x-y|}{(t^2 - |x-y|^2)^{1/2}}$$

6.5 Two-dimensional wave equation

- Applying these calculations to Kirchhoff's formula produces

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}\frac{\partial}{\partial t}\left(t^2 \frac{1}{|B(x,t)|}\int_{B(x,t)} \frac{g(y)}{(t^2-|x-y|^2)^{1/2}}\,dy\right) \\
&\qquad + \frac{t^2}{2}\frac{1}{|B(x,t)|}\int_{B(x,t)} \frac{h(y)}{(t^2-|x-y|^2)^{1/2}}\,dy \\
&= I_1 + I_2.
\end{aligned}
$$

- Notice that now the solution at (x,t) depends on the initial data on the whole cone $|y-x| \le t$, not only its boundary.
- Calculation of the t-derivative becomes easier if we make a change of variables so that the integral will be over the ball $B(0,1)$ i.e. $y = x + tz$. Then $dy = t^2 dz$ as it is 2-dimensional variable.
- Notice also that $t^2/|B(x,t)|$ is a constant.

6.5 Two-dimensional wave equation

- Applying these calculations to Kirchhoff's formula produces

$$\begin{aligned} u(x,t) = \frac{1}{2}\frac{\partial}{\partial t}\left(t^2 \frac{1}{|B(x,t)|}\int_{B(x,t)} \frac{g(y)}{(t^2-|x-y|^2)^{1/2}}\,dy\right) \\ + \frac{t^2}{2}\frac{1}{|B(x,t)|}\int_{B(x,t)} \frac{h(y)}{(t^2-|x-y|^2)^{1/2}}\,dy \\ = I_1 + I_2. \end{aligned}$$

- Notice that now the solution at (x,t) depends on the initial data on the whole cone $|y-x| \le t$, not only its boundary.
- Calculation of the t-derivative becomes easier if we make a change of variables so that the integral will be over the ball $B(0,1)$ i.e. $y = x + tz$. Then $dy = t^2 dz$ as it is 2-dimensional variable.
- Notice also that $t^2/|B(x,t)|$ is a constant.

6.5 Two-dimensional wave equation

- Applying these calculations to Kirchhoff's formula produces

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}\frac{\partial}{\partial t}\left(t^2 \frac{1}{|B(x,t)|}\int_{B(x,t)} \frac{g(y)}{(t^2-|x-y|^2)^{1/2}}\,dy \right) \\
&\qquad\qquad + \frac{t^2}{2}\frac{1}{|B(x,t)|}\int_{B(x,t)} \frac{h(y)}{(t^2-|x-y|^2)^{1/2}}\,dy \\
&= I_1 + I_2 .
\end{aligned}
$$

- Notice that now the solution at (x,t) depends on the initial data on the whole cone $|y-x| \le t$, not only its boundary.
- Calculation of the t-derivative becomes easier if we make a change of variables so that the integral will be over the ball $B(0,1)$ i.e. $y = x + tz$. Then $dy = t^2 dz$ as it is 2-dimensional variable.
- Notice also that $t^2/|B(x,t)|$ is a constant.

6.5 Two-dimensional wave equation

- Finally we arrive at the Poisson's formula for the 2-dimensional wave equation:

$$u(x,t) = \frac{1}{2}\frac{1}{|B(x,t)|}\int_{B(x,t)} \frac{tg(y) + t^2 h(y) + t\nabla g(y)\cdot(y-x)}{(t^2 - |x-y|^2)^{1/2}}\,dy.$$

- Domain of dependence for (x,t) is the disk $B(x,t)$.
- Finite propagation speed, but no sharp signals for 2-dimensional waves (e.g. water waves).

6.5 Two-dimensional wave equation

- Finally we arrive at the Poisson's formula for the 2-dimensional wave equation:

$$u(x,t) = \frac{1}{2}\frac{1}{|B(x,t)|}\int_{B(x,t)} \frac{tg(y) + t^2 h(y) + t\nabla g(y)\cdot(y-x)}{(t^2 - |x-y|^2)^{1/2}}\,dy.$$

- Domain of dependence for (x,t) is the disk $B(x,t)$.
- Finite propagation speed, but no sharp signals for 2-dimensional waves (e.g. water waves).

6.5 Two-dimensional wave equation

- Finally we arrive at the Poisson's formula for the 2-dimensional wave equation:

$$u(x,t) = \frac{1}{2}\frac{1}{|B(x,t)|}\int_{B(x,t)} \frac{tg(y) + t^2 h(y) + t\nabla g(y)\cdot(y-x)}{(t^2 - |x-y|^2)^{1/2}}\,dy.$$

- Domain of dependence for (x,t) is the disk $B(x,t)$.
- Finite propagation speed, but no sharp signals for 2-dimensional waves (e.g. water waves).

6.6 Nonhomogeneous problem

- We study the nonhomogeneous Cauchy problem

$$\begin{cases} u_{tt} - \Delta u = f & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u = 0, \quad u_t = 0 & \text{in} \quad \mathbb{R}^n \times \{t=0\}. \end{cases}$$

- Goal is to apply Duhamel's principle as for the heat equation.
- Let the function $u(x,t;s)$ be the solution of the Cauchy problem

$$\begin{cases} u_{tt}(x,t;s) - \Delta u(x,t;s) = 0, \quad x \in \mathbb{R}^n, \quad t > s, \\ u(x,s;s) = 0, \quad u_t(x,s;s) = f(x,s), \quad x \in \mathbb{R}^n. \end{cases}$$

- Duhamel's principle suggests that

$$u(x,t) = \int_0^t u(x,t;s)\,ds, \quad x \in \mathbb{R}^n, \quad t > 0$$

is the solution to the original problem.

6.6 Nonhomogeneous problem

- We study the nonhomogeneous Cauchy problem

$$\begin{cases} u_{tt} - \Delta u = f & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u = 0, \quad u_t = 0 & \text{in} \quad \mathbb{R}^n \times \{t=0\}. \end{cases}$$

- Goal is to apply Duhamel's principle as for the heat equation.
- Let the function $u(x,t;s)$ be the solution of the Cauchy problem

$$\begin{cases} u_{tt}(x,t;s) - \Delta u(x,t;s) = 0, \quad x \in \mathbb{R}^n, \quad t > s, \\ u(x,s;s) = 0, \quad u_t(x,s;s) = f(x,s), \quad x \in \mathbb{R}^n. \end{cases}$$

- Duhamel's principle suggests that

$$u(x,t) = \int_0^t u(x,t;s)\,ds, \quad x \in \mathbb{R}^n, \quad t > 0$$

is the solution to the original problem.

6.6 Nonhomogeneous problem

- We study the nonhomogeneous Cauchy problem

$$\begin{cases} u_{tt} - \Delta u = f \quad \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u = 0, \quad u_t = 0 \quad \text{in} \quad \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

- Goal is to apply Duhamel's principle as for the heat equation.
- Let the function $u(x,t;s)$ be the solution of the Cauchy problem

$$\begin{cases} u_{tt}(x,t;s) - \Delta u(x,t;s) = 0, \quad x \in \mathbb{R}^n, \quad t > s, \\ u(x,s;s) = 0, \quad u_t(x,s;s) = f(x,s), \quad x \in \mathbb{R}^n. \end{cases}$$

- Duhamel's principle suggests that

$$u(x,t) = \int_0^t u(x,t;s)\,ds, \quad x \in \mathbb{R}^n, \quad t > 0$$

is the solution to the original problem.

6.6 Nonhomogeneous problem

- We study the nonhomogeneous Cauchy problem

$$\begin{cases} u_{tt} - \Delta u = f & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u = 0, \quad u_t = 0 & \text{in} \quad \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

- Goal is to apply Duhamel's principle as for the heat equation.
- Let the function $u(x, t; s)$ be the solution of the Cauchy problem

$$\begin{cases} u_{tt}(x,t;s) - \Delta u(x,t;s) = 0, & x \in \mathbb{R}^n, \quad t > s, \\ u(x,s;s) = 0, \quad u_t(x,s;s) = f(x,s), & x \in \mathbb{R}^n. \end{cases}$$

- Duhamel's principle suggests that

$$u(x,t) = \int_0^t u(x,t;s)\,ds, \quad x \in \mathbb{R}^n, \quad t > 0$$

is the solution to the original problem.

6.6 Nonhomogeneous problem

- Formal calculation suggests that Duhamel's principle is ok. We skip a rigorous proof.
- Nonhomogeneous problem with more general boundary values can be solved by summing up the solutions as with Laplace and heat equations.

6.6 Nonhomogeneous problem

- Formal calculation suggests that Duhamel's principle is ok. We skip a rigorous proof.
- Nonhomogeneous problem with more general boundary values can be solved by summing up the solutions as with Laplace and heat equations.

6.7 Energy methods

- Wave equation preserves the energy

$$e(t) = \frac{1}{2}\int_\Omega \left((u_t)^2 + |\nabla u|^2\right)\, dx, \quad 0 \le t \le T.$$

  if we have zero boundary values.

- Proof:

$$\begin{aligned}
e'(t) &= \frac{1}{2}\int_\Omega \frac{\partial}{\partial t}\left(\left(\frac{\partial u}{\partial t}\right)^2 + |\nabla u|^2\right) dx \\
&= \frac{1}{2}\int_\Omega \left(2\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} + 2\nabla u \cdot \frac{\partial}{\partial t}\nabla u\right) dx \\
&= \int_\Omega \left(\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} - \frac{\partial u}{\partial t}\Delta u\right) dx \qquad \text{(Green's first identity)} \\
&= \int_\Omega u_t\,(u_{tt} - \Delta u)\, dx = 0.
\end{aligned}$$

- This implies uniqueness.

6.7 Energy methods

- Wave equation preserves the energy

$$e(t) = \frac{1}{2} \int_{\Omega} \left((u_t)^2 + |\nabla u|^2 \right) \, dx, \quad 0 \le t \le T.$$

  if we have zero boundary values.

- Proof:

$$\begin{aligned} e'(t) &= \frac{1}{2} \int_{\Omega} \frac{\partial}{\partial t} \left(\left(\frac{\partial u}{\partial t} \right)^2 + |\nabla u|^2 \right) \, dx \\ &= \frac{1}{2} \int_{\Omega} \left(2 \frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial t^2} + 2 \nabla u \cdot \frac{\partial}{\partial t} \nabla u \right) \, dx \\ &= \int_{\Omega} \left(\frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial t^2} - \frac{\partial u}{\partial t} \Delta u \right) \, dx \qquad \text{(Green's first identity)} \\ &= \int_{\Omega} u_t \left(u_{tt} - \Delta u \right) \, dx = 0. \end{aligned}$$

- This implies uniqueness.

6.7 Energy methods

- Wave equation preserves the energy

$$e(t) = \frac{1}{2}\int_\Omega \left((u_t)^2 + |\nabla u|^2\right)\, dx, \quad 0 \leq t \leq T.$$

if we have zero boundary values.

- Proof:

$$\begin{aligned}
e'(t) &= \frac{1}{2}\int_\Omega \frac{\partial}{\partial t}\left(\left(\frac{\partial u}{\partial t}\right)^2 + |\nabla u|^2\right)\, dx \\
&= \frac{1}{2}\int_\Omega \left(2\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} + 2\nabla u \cdot \frac{\partial}{\partial t}\nabla u\right)\, dx \\
&= \int_\Omega \left(\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} - \frac{\partial u}{\partial t}\Delta u\right)\, dx \qquad \text{(Green's first identity)} \\
&= \int_\Omega u_t\left(u_{tt} - \Delta u\right)\, dx = 0.
\end{aligned}$$

- This implies uniqueness.