MS-C1350 Partial differential equations
Chapter 2.3 Fourier series

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- Let $f \in L^1([-\pi, \pi])$. The nth partial sum of a Fourier series is

$$S_n f(t) = \sum_{j=-n}^{n} \widehat{f}(j) e^{ijt}, \quad n = 0, 1, 2, \dots,$$

where

$$\widehat{f}(j) = \langle f, e_j \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ijt}\, dt, \quad j \in \mathbb{Z},$$

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  is the jth Fourier coefficient of f.
- The Fourier series of f is the limit of the partial sums $S_n f$ as $n \to \infty$, provided the limit exists in some reasonable sense. In this case we may write

$$f(t) = \lim_{n\to\infty} S_n f(t) = \lim_{n\to\infty} \sum_{j=-n}^{n} \widehat{f}(j) e^{ijt} = \sum_{j=-\infty}^{\infty} \widehat{f}(j) e^{ijt}.$$

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- Definition makes sense if $f \in L^1([-\pi, \pi])$, but we will see that $L^2([-\pi, \pi])$ is needed to understand the Fourier coefficients and the convergence of the series.

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- We always consider symmetric Fourier series (sum from $-n$ to n).
- It corresponds to having trigonometric functions $\sin(t), \cos(t), \sin(2t), \cos(2t), \dots, \sin(nt), \cos(nt)$.
- e_j, $j \in \mathbb{Z}$ is always 2π-periodic. Therefore $S_n f$ is always 2π-periodic. Therefore we can only approximate 2π-periodic functions. (And by change of variables, other periodic functions.)