Aalto University

MS-C1350 Partial differential equations
Chapter 3.1-3.8
Fourier transform and PDEs

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Lecture 5:

The topic of the lecture is Fourier transform. It is an analogous theory to Fourier series.

- $L^p(\mathbb{R}^n)$–space
- Formula for the Fourier transform and inverse transform
- Formulas for translation, modulation and differentiation with Fourier transform.
- Convolution on $\mathbb{R}^n$
- Good kernels on $\mathbb{R}^n$.

3 Fourier transform

- Our next topic is Fourier transform.
- It gives a method to solve PDEs in the higher dimensional case (and on unbounded intervals).
- Note that we used Fourier series to solve e.g. Laplace equation on plane ($n = 2$), but the Fourier series technique was only used to choose the correct coefficients so that the boundary values on one-dimensional boundary were correct.
- When we used Fourier series, the boundary/initial values were always one-dimensional and either periodic (boundary of a unit sphere / initial temperature on ring) or defined on a bounded interval with zero values at endpoints.
- Fourier transform has many properties analogous to the Fourier series.
- When using Fourier transform, the functions need not be periodic.
- Convolution is a central tool in this theory.

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3.1 The L^p-space on $\mathbb{R}^n$

The space $L^p(\mathbb{R}^n)$ consists of functions $f : \mathbb{R}^n \to \mathbb{C}$ such that

$$\|f\|_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p} < \infty.$$

For $p = \infty$, we set

$$\|f\|_{L^p(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n} |f(x)|.$$

It can be shown that $L^p(\mathbb{R}^n)$ is a complete normed space with the norm defined above.

- Whether $f \in L^1(\mathbb{R}^n)$ depends only on $|f|$.
- If $f \in L^1(\mathbb{R}^n)$, then the integral of f is well-defined and finite.

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$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\, dx,$$

where $x \cdot \xi = x_1\xi_1 + \cdots + x_n\xi_n$ is the standard inner product (dot product) in $\mathbb{R}^n$.

Compare the Fourier series and Fourier transform:

- Function: S: $f : \mathbb{R} \to \mathbb{C}$. T: $f : \mathbb{R}^n \to \mathbb{C}$
- Coefficients: S: $\hat{f}(k)$, $k \in \mathbb{Z} \subset \mathbb{R}$. T: $\hat{f}(\xi)$, $\xi \in \mathbb{R}^n$.
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3.2 Several definitions

There are several alternative definitions for the Fourier transform in the literature, for example,

$$\int_{\mathbb{R}^n} f(x)e^{ix\cdot\xi}\,dx, \quad \int_{\mathbb{R}^n} f(x)e^{-2\pi ix\cdot\xi}\,dx$$

$$\text{and} \quad (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}\,dx.$$

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3.2 Properties of the Fourier transform

Let $f, g \in L^1(\mathbb{R}^n)$.

1. (Linearity) $\widehat{af + bg}(\xi) = a\widehat{f}(\xi) + b\widehat{g}(\xi)$, $a, b \in \mathbb{C}$.
2. (Boundedness) For every $\xi \in \mathbb{R}^n$ we have
$$|\widehat{f}(\xi)| \leq \int_{\mathbb{R}^n} |f(x)|\, dx = \|f\|_{L^1(\mathbb{R}^n)}.$$
This implies $\|\widehat{f}\|_{L^\infty(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)}$.
3. $\widehat{f}(0) = \displaystyle\int_{\mathbb{R}^n} f(x)\, dx$.
4. (Continuity) If $f \in L^1(\mathbb{R}^n)$, then $\widehat{f}$ is continuous.
5. (Dilation) Let $f_a(x) = \frac{1}{a^n} f\left(\frac{x}{a}\right)$, $a > 0$. Then $\widehat{f_a}(\xi) = \widehat{f}(a\xi)$.
6. (Translation) For $y \in \mathbb{R}^n$ we have $\widehat{f(x+y)}(\xi) = e^{iy\cdot\xi}\widehat{f}(\xi)$.
7. (Modulation) For $\eta \in \mathbb{R}^n$ we have $\widehat{e^{ix\cdot\eta} f(x)}(\xi) = \widehat{f}(\xi - \eta)$.

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3.3 Fourier transform and differentiation

The Fourier transform interacts with derivation very nicely. In many cases, Fourier transform transforms a PDE to an ODE.

Theorem

Let $f \in C^1(\mathbb{R}^n)$ and assume that $\frac{\partial f}{\partial x_j} \in L^1(\mathbb{R}^n)$, $j = 1, 2, \ldots, n$. We also assume that $\lim_{|x| \to \infty} f(x) = 0$. Then

$$\widehat{\frac{\partial f}{\partial x_j}}(\xi) = i\xi_j \widehat{f}(\xi), \quad j = 1, 2, \ldots, n.$$

- Compare to the formula for Fourier series:

$$\widehat{f'}(j) = ij\widehat{f}(j), \quad j \in \mathbb{Z}.$$

- When we differentiate the function, the the transform gets multiplied by ξ_j and thus $\hat{f}(\xi)$ decays slower as $|\xi| \to \infty$.
- The differentiation formula works e.g. if $f \in C_0^\infty(\mathbb{R}^n)$.

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\begin{aligned}
\widehat{\frac{\partial f}{\partial x_j}}(\xi) &= \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_j}(x) e^{-ix\cdot\xi}\, dx \qquad \text{by definition of the Fourier transf.} \\
&= \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \frac{\partial f}{\partial x_j}(x) e^{-ix\cdot\xi}\, dx_j \right) dx_1 \,\ldots\, dx_{j-1}\, dx_{j+1} \,\ldots\, dx_n \\
&= \int_{\mathbb{R}^{n-1}} \left(\lim_{a\to\infty} \Big|_{x_j=-a}^{a} f(x) e^{-ix\cdot\xi} \right) dx_1 \,\ldots\, dx_{j-1}\, dx_{j+1} \,\ldots\, dx_n \\
&\quad - \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} f(x) \frac{\partial}{\partial x_j}(e^{-ix\cdot\xi})\, dx_j \right) dx_1 \,\ldots\, dx_{j-1}\, dx_{j+1} \,\ldots\, dx_n \\
&= -\int_{\mathbb{R}^n} f(x)(-i\xi_j) e^{-ix\cdot\xi}\, dx = i\xi_j \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\, dx = i\xi_j \widehat{f}(\xi).
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&= \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \frac{\partial f}{\partial x_j}(x) e^{-ix\cdot\xi}\, dx_j \right) dx_1 \, \dots \, dx_{j-1}\, dx_{j+1} \, \dots \, dx_n \\
&= \int_{\mathbb{R}^{n-1}} \left(\lim_{a\to\infty} \Big|_{x_j=-a}^{a} f(x) e^{-ix\cdot\xi} \right) dx_1 \, \dots \, dx_{j-1}\, dx_{j+1} \, \dots \, dx_n \\
&\quad - \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} f(x) \frac{\partial}{\partial x_j}(e^{-ix\cdot\xi})\, dx_j \right) dx_1 \, \dots \, dx_{j-1}\, dx_{j+1} \, \dots \, dx_n \\
&= - \int_{\mathbb{R}^n} f(x)(-i\xi_j) e^{-ix\cdot\xi}\, dx = i\xi_j \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\, dx = i\xi_j \widehat{f}(\xi).
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3.3 Proof of the differentiation formula

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3.3 Why Fourier transform is useful for PDEs? (1/3)

- A vector of the form $\alpha = (\alpha_1, \dots, \alpha_n)$, where each component α_j is a nonnegative integer, is called a multi-index of order $|\alpha| = \alpha_1 + \dots + \alpha_n$.
- For a multi-index α, we define

$$D^\alpha u(x) = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha^n}}(x)$$

- Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ and denote $\lambda^\alpha = \lambda_1^{\alpha_1} \dots \lambda_n^{\alpha_n}$.
- The function $e_\lambda : \mathbb{R}^n \to \mathbb{C}$,

$$e_\lambda(x) = e^{x \cdot \lambda} = e^{x_1\lambda_1 + \dots + x_n\lambda_n},$$

belongs to $C^\infty(\mathbb{R}^n)$ and

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3.3 Why Fourier transform is useful for PDEs? (2/3)

- Thus, for every linear PDE operator with constant coefficients

$$P = P(D) = \sum_{|\alpha| \le k} a_\alpha D^\alpha : C_0^\infty(\mathbb{R}^n) \to C_0^\infty(\mathbb{R}^n)$$

we have

$$P(D)e_\lambda = \sum_{|\alpha| \le k} a_\alpha \lambda^\alpha e_\lambda.$$

- In other words, e_λ is an eigenvector corresponding the eigenvalue

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3.3 Why Fourier transform is useful for PDEs? (3/3)

- Note that the PDE related to the operator P is

$$P(D)u = \sum_{|\alpha|\le k} a_\alpha D^\alpha u = 0.$$

- The idea behind the Fourier transform is that the partial differential operator P can be better understood if the functions on which they act are represented as linear combinations of the eigenvectors

$$e_\lambda(x) = e^{x\cdot\lambda} = e^{x_1\lambda_1+\cdots+x_n\lambda_n}.$$

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3.3 Fourier transform and differentiation

- We have shown that

$$\widehat{\frac{\partial f}{\partial x_j}}(\xi) = i\xi_j \widehat{f}(\xi), \quad j = 1, 2, \dots, n.$$

but what if we want to differentiate $\widehat{f}$?

Theorem

Suppose that $f \in L^1(\mathbb{R}^n)$ and $-ix_j f(x) \in L^1(\mathbb{R}^n)$. Then $\widehat{f}$ is differentiable and

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3.4 Fourier transform of the Gaussian

- Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{-x^2}$. Then

$$f'(x) = -2xe^{-x^2} = -2xf(x) \in L^1(\mathbb{R})$$

- Now

$$\frac{\partial \widehat{f}}{\partial \xi}(\xi) = -\widehat{ixf(x)}(\xi) = -\frac{1}{2i}\widehat{f'}(\xi) = -\frac{1}{2i}i\xi\widehat{f}(\xi) = -\frac{\xi}{2}\widehat{f}(\xi).$$

- Thus $\widehat{f}$ satisfies the ODE

$$\frac{\partial \widehat{f}}{\partial \xi}(\xi) + \frac{\xi}{2}\widehat{f}(\xi) = 0 \iff \frac{\partial}{\partial \xi}\left(\widehat{f}(\xi)e^{\xi^2/4}\right) = 0 \iff \widehat{f}(\xi) = ce^{-\xi^2/4}$$

for some constant c. (And the correct c is $\pi^{n/2}$, read the details from the lecture notes)

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3.5 Fourier inversion formula

- If we are given $\hat{f}$, can we determine f?
- The Fourier inversion theorem will state that under certain assumptions, we have

$$f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{ix\cdot\xi} \, d\xi.$$

- Compare this to how we get f from the Fourier series coefficients! (sum $\Rightarrow$ integral!)
- 2π appears now in a bit different place, but it is just a choice. It must appear somewhere.

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3.5 Easy proof for inversion formula?

This is a deep result and it is instructive to see what happens if we try to prove directly by substituting the formula for $\widehat{f}(\xi)$ into the integral above. If we do this, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \widehat{f}(\xi)e^{ix\cdot\xi}\,d\xi &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} f(y)e^{-iy\cdot\xi}\,dy\right)e^{ix\cdot\xi}\,d\xi \\ &= \int_{\mathbb{R}^n} f(y)\underbrace{\left(\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\,d\xi\right)}_{=?}\,dy.\end{aligned}$$

This does not work out, because the inner integral does not exist, that is, the function is not integrable. Thus we have to choose another approach.

3.5 Main steps in the correct proof:

- Step 1: Study the functions $K(x) = \pi^{-n/2} e^{-|x|^2}$ and $K_a(x) = a^{-n} K(ax)$. We know already what is $\hat{K}$ and $\hat{K}_a$!
- Step 1b: $\int_{\mathbb{R}^n} f(x) K_a(x)\, dx \to f(0)$ as $a \to 0$.
- Step 2: $\int_{\mathbb{R}^n} \widehat{f}(\xi) g(\xi)\, d\xi = \int_{\mathbb{R}^n} f(\xi) \widehat{g}(\xi)\, d\xi$
- Step 3: $f(0) = (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{f}(\xi)\, d\xi$ is the claim at $x = 0$
- Step 4: The claim for the other points follows from the translation formula i.e. if $F(y) = f(x+y)$, then

$$f(x) = F(0) = (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{F}(\xi)\, d\xi = (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{ix\cdot\xi}\, d\xi$$

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3.6 Fourier transform and convolution (1/2)

Let $f, g : \mathbb{R}^n \to \mathbb{C}$. The convolution $f * g : \mathbb{R}^n \to \mathbb{C}$ is defined by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x - y)g(y)\, dy,$$

whenever this integral exists.

- The convolution on $\mathbb{R}^n$ has a similar role in representation formulas for solutions of PDEs as in the one-dimensional case.
- It is commutative $f * g = g * f$ and
- associative $(f * g) * h = f * (g * h)$.
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$$\|f * g\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)} < \infty.$$

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3.6 Fourier transform and convolution (2/2)

- Assume that $f, g \in L^1(\mathbb{R}^n)$. Then

$$\widehat{f * g}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi)$$

for every $\xi \in \mathbb{R}^n$.

- Assume that $f, g, \widehat{f}, \widehat{g} \in L^1(\mathbb{R}^n)$. Then

$$\widehat{fg}(\xi) = (2\pi)^{-n}(\widehat{f} * \widehat{g})(\xi)$$

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3.6 Fourier transform and convolution (2/2)

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3.7 Plancherel's formula

$$\int_{\mathbb{R}^n} |f(y)|^2 \, dy = (2\pi)^{-n} \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 \, d\xi$$

- Plancherel's formula tells that L^2 norm of f and $\widehat{f}$ are essentially the same.
- The factor $(2\pi)^{-n}$ appears with our definition for the Fourier transform, but there are different scalings in the literature.
- This can be used to define the Fourier transform of L^2-functions, but this is out of scope of this course.
- Compare to the Fourier series version of Plancherel's theorem.

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3.8 Approximations of identity (1/2)

Let $\{K_\varepsilon\}_{\varepsilon>0}$ be a family of functions $K_\varepsilon : \mathbb{R}^n \to \mathbb{C}$, with the following properties.

1. For every $\varepsilon > 0$ we have $\int_{\mathbb{R}^n} K_\varepsilon(x)\,dx = 1$.
2. There exists some constant $M > 0$ such that, for every $\varepsilon > 0$ we have
$$\int_{\mathbb{R}^n} |K_\varepsilon(x)|\,dx \leq M.$$
3. For every $\delta > 0$, we have
$$\lim_{\varepsilon\to 0} \int_{\{|x|>\delta\}} |K_\varepsilon(x)|\,dx = 0.$$

Then the family $\{K_\varepsilon\}_{\varepsilon>0}$ will be called a family of good kernels.

3.8 Approximations of identity (2/2)

Theorem

Let $f \in L^1(\mathbb{R}^n)$ be bounded and continuous at $x \in \mathbb{R}^n$ and let $\{K_\varepsilon\}_{\varepsilon>0}$ be a family of good kernels. Then

$$\lim_{\varepsilon \to 0} (K_\varepsilon * f)(x) = f(x).$$

If $f \in C_0(\mathbb{R}^n)$ then $K_\varepsilon * f \to f$ uniformly as $\varepsilon \to 0$.

- Approximations of the identity give an interpretation that the convolution $K_\varepsilon * f$ can be seen as a weighted integral average. The pointwise value of a function is replaced with an integral average, which converges to the value of the function as $\varepsilon \to 0$.
- Example 1: $K_\varepsilon(x) = \frac{1}{\omega_n \varepsilon^n} \chi_{\{|x|<\varepsilon\}}$. Then $(K_\varepsilon * f)(x)$ is the average of f in a ball $B(x, \varepsilon)$.
- Example 2: Heat kernel H_t which is related to the solutions to the heat equation in upper half plane.

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