Aalto University

MS-C1350 Partial differential equations
Chapter 2.11
Laplace equation in the unit disc

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Lecture 3

Today we see how to solve Dirichlet problem in the unit disc (Chapter 2.11). Needed techniques include:

- Moving to polar coordinates
- Separation of variables
- Solving equations we obtain by separation of variables.
- Using Fourier series to find the correct Fourier series solution.

In Chapter 2.12, the heat equation in 1D is solved in a similar way. That example is not discussed in the lecture (but there is an old lecture video about it).

Laplace equation in the unit disc

- Laplace equation:

$$\Delta u = 0$$

- Laplace equation models heat distribution when the system has reached thermal equilibrium.
- Appears in many other places as well.

Dirichlet problem

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- The problem is to find $u \in C^2(\Omega) \cap C(\overline{\Omega})$ such that

$$\begin{cases} \Delta u(x,y) = \dfrac{\partial^2 u}{\partial x^2}(x,y) + \dfrac{\partial^2 u}{\partial y^2}(x,y) = 0, \quad (x,y) \in \Omega, \\ u(x,y) = g(x,y), \quad (x,y) \in \partial\Omega. \end{cases}$$

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$$u(x,y) = A(x)B(y),$$
- This would work well in rectangular areas, but for the disc, we switch to polar coordinates:
$$(x,y) = (r\cos\theta, r\sin\theta), \quad (x,y) \in \mathbb{R}^2, \quad 0 \leq r < \infty, \quad -\pi \leq \theta < \pi,$$
where $r^2 = x^2 + y^2$ and $\tan\theta = \frac{y}{x}$. In polar coordinates, we have
$$\Omega = \{(r,\theta) : 0 \leq r < 1, -\pi \leq \theta < \pi\}$$
and
$$\partial\Omega = \{(1,\theta) : -\pi \leq \theta < \pi\}.$$

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and
$$\partial\Omega = \{(1,\theta) : -\pi \le \theta < \pi\}.$$
- Note that unit disc is a rectangular set in polar coordinates and this is compatible with separation of variables.

Laplace equation in polar coordinates

Lemma

The two-dimensional Laplace operator in polar coordinates is

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}, \quad 0 < r < \infty, \quad -\pi \leq \theta < \pi.$$

[You need to apply chain rule to prove this. Details are in lecture notes.]

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[You need to apply chain rule to prove this. Details are in lecture notes.]

Thus the Dirichlet problem assumes the following form:

$$\begin{cases} \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0, \quad 0 < r < 1, \quad -\pi \le \theta < \pi, \\ u(1,\theta) = g(\theta), \quad -\pi \le \theta < \pi, \end{cases}$$

for $u = u(r,\theta)$.

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- The PDE has been reduced to a system of two ODEs:

$$\begin{cases} A''(\theta) + \lambda A(\theta) = 0, \\ r^2 B''(r) + rB'(r) - \lambda B(r) = 0. \end{cases}$$

Solving the Dirichlet problem: Step 2: Solving separated equations

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The only periodic solution is $A = 0$. Then also $u = AB = 0$.

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- Solution becomes unbounded as $r \to 0$. This is against physical intuition and these solutions are excluded. (Not a solution at 0.)

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- Thus $\lambda > 0$ gives relevant solutions of form

$$u(r, \theta) = A(\theta)B(r) = r^{|j|} e^{ij\theta}.$$

Solving the Dirichlet problem: Step 3: Fourier series solution for the whole problem

- Now we have found solutions

$$u_j(r,\theta) = A(\theta)B(r) = r^{|j|}e^{ij\theta}.$$

which solve the problem with boundary values

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- Laplace operator is linear. Thus all linear combinations are also solutions:

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- This should be compatible with the boundary data when $r = 1$:

$$u(1,\theta) = \sum_{j=-\infty}^{\infty} a_j e^{ij\theta} = g(\theta).$$

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- How to choose a_j's so that:

$$u(1,\theta) = \sum_{j=-\infty}^{\infty} a_j e^{ij\theta} = g(\theta)?$$

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- If $g \in L^2([-\pi,\pi])$ then we know that there is a solution if we interpret the equality in L^2-sense.
- If $g \in C^1([-\pi,\pi])$, then the Fourier series converges uniformly and

$$a_j = \hat{g}(j).$$

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On formal level (2) and (3) seem to be ok. On needs to be careful when switching the order of the limit and the infinite series. We will return later to this and to the question of uniqueness.

Step 4: Explicit representation formula

- We can plug in the formula for the Fourier coefficients $\hat{g}(j)$.

$$u(r,\theta) = \sum_{j=-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) e^{-ijt}\, dt \right) r^{|j|} e^{ij\theta}$$

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&= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) \left(\sum_{j=-\infty}^{\infty} r^{|j|} e^{ij(\theta - t)} \right) dt \\
&= (g * P_r)(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) P_r(\theta - t)\, dt,
\end{aligned}
$$

where we have the Poisson kernel:

$$
P_r(\theta) = P(r,\theta) = \sum_{j=-\infty}^{\infty} r^{|j|} e^{ij\theta}.
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Poisson kernel

$$P_r(\theta) = P(r,\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2}, \quad 0 \le r < 1, \quad -\pi \le \theta < \pi.$$

Some properties:

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- $$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)\,d\theta = 1$$
- Observe that
$$P(r,0) = \frac{1-r^2}{1-2r+r^2}, \quad 0 \le r < 1,$$
consequently $\lim_{r\to 1} P(r,0) = \infty$.
However $\lim_{r\to 1} P(r,\theta) = 0$ when $\theta \ne 0$.

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- This formula does not work when $r = 1$!