Aalto University

MS-C1350 Partial differential equations
Chapter 2.12 and 2.13
Heat and weve equations in one-dimension

Riikka Korte

Department of Mathematics and Systems Analysis
Aalto University
riikka.korte@aalto.fi

September 24, 2024

Lecture 4

- We see how to solve wave equation in 1D.
  New things:
  - Now we use real form Fourier series. This problem could be solved with Fourier series with complex exponential functions equally

The heat equation in one-dimension (2.12)

- Suppose we have a ring of radius 1 centered at the origin.

The heat equation in one-dimension (2.12)

- Suppose we have a ring of radius 1 centered at the origin.
- Suppose that it is perfectly insulated.

The heat equation in one-dimension (2.12)

- Suppose we have a ring of radius 1 centered at the origin.
- Suppose that it is perfectly insulated.
- At time $t = 0$, the initia temperature is given by $g : [-\pi, \pi] \to \mathbb{R}$.

The heat equation in one-dimension (2.12)

- Suppose we have a ring of radius 1 centered at the origin.
- Suppose that it is perfectly insulated.
- At time $t = 0$, the initia temperature is given by $g : [-\pi, \pi] \to \mathbb{R}$.
- Diffusion of heat on the circle is modeled by the heat equation:

$$\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial \theta^2} = 0,$$

where a^2 is the thermal diffusivity, which depends on the material of the ring. We set $a^2 = 1$.

The heat equation in one-dimension (2.12)

- Suppose we have a ring of radius 1 centered at the origin.
- Suppose that it is perfectly insulated.
- At time $t = 0$, the initia temperature is given by $g : [-\pi, \pi] \to \mathbb{R}$.
- Diffusion of heat on the circle is modeled by the heat equation:

$$\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial \theta^2} = 0,$$

  where a^2 is the thermal diffusivity, which depends on the material of the ring. We set $a^2 = 1$.
- We are looking for a solution $u = u(\theta, t)$:

$$\begin{cases} \frac{\partial u}{\partial t}(\theta, t) - \frac{\partial^2 u}{\partial \theta^2}(\theta, t) = 0, \quad -\pi \leq \theta < \pi, \quad t > 0, \\ u(\theta, 0) = g(\theta), \quad -\pi \leq \theta < \pi. \end{cases}$$

  Periodic initial value problem.

Solving the heat equation

1. Separation of variables
2. Solution to the separated equations
3. Fourier series solution of the entire equation
4. Explicit representation formula

Solving the heat equation

1. Separation of variables
2. Solution to the separated equations
3. Fourier series solution of the entire equation
4. Explicit representation formula

Solving the heat equation: Separation of variables

- Write
$$u(\theta, t) = A(\theta)B(t)$$
- insert this into the heat equation:
$$A(\theta)B'(t) - A''(\theta)B(t) = 0 \iff \frac{B'(t)}{B(t)} = \frac{A''(\theta)}{A(\theta)}.$$
- Thus
$$\begin{cases} A''(\theta) = \lambda A(\theta), \\ B'(t) = \lambda B(t). \end{cases}$$

Solving the heat equation: Solution to the separated equations

- The initial condition:

$$\text{"}u(\theta,0)=A(\theta)B(0)=g(\theta)\text{"}$$

  is a 2π-periodic function.

Solving the heat equation: Solution to the separated equations

- The initial condition:

$$\text{"}u(\theta, 0) = A(\theta)B(0) = g(\theta)\text{"}$$

  is a 2π-periodic function.
- We do a case study as with the Laplace equation:

Solving the heat equation: Solution to the separated equations

- The initial condition:

$$"u(\theta, 0) = A(\theta)B(0) = g(\theta)"$$

  is a 2π-periodic function.
- We do a case study as with the Laplace equation:
- **Case** $\lambda = \mu^2 > 0$:

Solving the heat equation: Solution to the separated equations

- The initial condition:

$$"u(\theta, 0) = A(\theta)B(0) = g(\theta)"$$

  is a 2π-periodic function.
- We do a case study as with the Laplace equation:
- **Case** $\lambda = \mu^2 > 0$:
- The equation

$$A''(\theta) - \mu^2 A(\theta) = 0$$

  has the general solution

$$A(\theta) = c_1 e^{\mu\theta} + c_2 e^{-\mu\theta},$$

  which is periodic only if $c_1 = c_2 = 0$.

Solving the heat equation: Solution to the separated equations

- **Case** $\lambda = 0$: Then $A''(\theta) = 0$, which implies $A(\theta) = c_1\theta + c_2$.

Solving the heat equation: Solution to the separated equations

- **Case** $\lambda = 0$: Then $A''(\theta) = 0$, which implies $A(\theta) = c_1\theta + c_2$.
- The only periodic solution is $A(\theta) = c_2$.

Solving the heat equation: Solution to the separated equations

- **Case** $\lambda = 0$: Then $A''(\theta) = 0$, which implies $A(\theta) = c_1\theta + c_2$.
- The only periodic solution is $A(\theta) = c_2$.
- In this case we also get that $B(t) = c_3$ i.e. u is a constant.

Solving the heat equation: Solution to the separated equations

- **Case** $\lambda = -\mu^2 < 0$: The ODE becomes

$$A''(\theta) + \mu^2 A(\theta) = 0$$

Solving the heat equation: Solution to the separated equations

- **Case** $\lambda = -\mu^2 < 0$: The ODE becomes

$$A''(\theta) + \mu^2 A(\theta) = 0$$

- The general solution is

$$A(\theta) = c_1 e^{i\mu\theta} + c_2 e^{-i\mu\theta},$$

Solving the heat equation: Solution to the separated equations

- **Case** $\lambda = -\mu^2 < 0$: The ODE becomes

$$A''(\theta) + \mu^2 A(\theta) = 0$$

- The general solution is

$$A(\theta) = c_1 e^{i\mu\theta} + c_2 e^{-i\mu\theta},$$

- which is 2π-periodic if $\mu \in \mathbb{Z}$ i.e. $\lambda = -j^2$, $j \in \mathbb{Z}$.

Solving the heat equation: Solution to the separated equations

- **Case** $\lambda = -\mu^2 < 0$: The ODE becomes

$$A''(\theta) + \mu^2 A(\theta) = 0$$

- The general solution is

$$A(\theta) = c_1 e^{i\mu\theta} + c_2 e^{-i\mu\theta},$$

- which is 2π-periodic if $\mu \in \mathbb{Z}$ i.e. $\lambda = -j^2$, $j \in \mathbb{Z}$.
- For these values, the ODE for B is

$$B'(t) + j^2 B(t) = 0,$$

Solving the heat equation: Solution to the separated equations

- **Case** $\lambda = -\mu^2 < 0$: The ODE becomes
$$A''(\theta) + \mu^2 A(\theta) = 0$$
- The general solution is
$$A(\theta) = c_1 e^{i\mu\theta} + c_2 e^{-i\mu\theta},$$
- which is 2π-periodic if $\mu \in \mathbb{Z}$ i.e. $\lambda = -j^2$, $j \in \mathbb{Z}$.
- For these values, the ODE for B is
$$B'(t) + j^2 B(t) = 0,$$
- which has the general solution
$$B(t) = ce^{-j^2 t}.$$

Solving the heat equation: Solution to the separated equations

- **Case** $\lambda = -\mu^2 < 0$: The ODE becomes

$$A''(\theta) + \mu^2 A(\theta) = 0$$

- The general solution is

$$A(\theta) = c_1 e^{i\mu\theta} + c_2 e^{-i\mu\theta},$$

- which is 2π-periodic if $\mu \in \mathbb{Z}$ i.e. $\lambda = -j^2$, $j \in \mathbb{Z}$.
- For these values, the ODE for B is

$$B'(t) + j^2 B(t) = 0,$$

- which has the general solution

$$B(t) = ce^{-j^2 t}.$$

- Thus we have special solutions:

$$u(\theta, t) = e^{-j^2 t} e^{ij\theta}, \quad j \in \mathbb{Z}.$$

Solving the heat equation: Fourier series solution

- The heat equation is linear. Thus any linear combination fo the special solutions will give again a solution.

Solving the heat equation: Fourier series solution

- The heat equation is linear. Thus any linear combination fo the special solutions will give again a solution.
- Therefore we define

$$u(\theta,t)=\sum_{j=-\infty}^{\infty} a_j e^{-j^2 t} e^{ij\theta}, \quad -\pi \le \theta < \pi, \quad t>0,$$

and try to determine a_j.

Solving the heat equation: Fourier series solution

- The heat equation is linear. Thus any linear combination fo the special solutions will give again a solution.
- Therefore we define

$$u(\theta,t)=\sum_{j=-\infty}^{\infty} a_j e^{-j^2 t} e^{ij\theta}, \quad -\pi \le \theta < \pi, \quad t>0,$$

  and try to determine a_j.
- Initial condition $u(\theta,0)=g(\theta)$ gives

$$\sum_{j=-\infty}^{\infty} a_j e^{ij\theta} = g(\theta)$$

Solving the heat equation: Fourier series solution

- The heat equation is linear. Thus any linear combination fo the special solutions will give again a solution.
- Therefore we define

$$u(\theta,t)=\sum_{j=-\infty}^{\infty} a_j e^{-j^2 t} e^{ij\theta}, \quad -\pi\le\theta<\pi, \quad t>0,$$

  and try to determine a_j.
- Initial condition $u(\theta,0)=g(\theta)$ gives

$$\sum_{j=-\infty}^{\infty} a_j e^{ij\theta}=g(\theta)$$

- Thus a_j is the Fourier coefficient $\hat{g}(j)$ of the initial data and

$$u(\theta,t)=\sum_{j=-\infty}^{\infty} \widehat{g}(j) e^{-j^2 t} e^{ij\theta}.$$

Solving the heat equation: Fourier series solution

- The heat equation is linear. Thus any linear combination fo the special solutions will give again a solution.
- Therefore we define

$$u(\theta,t)=\sum_{j=-\infty}^{\infty} a_j e^{-j^2 t} e^{ij\theta}, \quad -\pi \le \theta < \pi, \quad t>0,$$

  and try to determine a_j.
- Initial condition $u(\theta,0)=g(\theta)$ gives

$$\sum_{j=-\infty}^{\infty} a_j e^{ij\theta} = g(\theta)$$

- Thus a_j is the Fourier coefficient $\hat{g}(j)$ of the initial data and

$$u(\theta,t)=\sum_{j=-\infty}^{\infty} \widehat{g}(j) e^{-j^2 t} e^{ij\theta}.$$

- This converges nicely.

Solving the heat equation: Explicit representation formula

- Next goal is to derive an integral representation for the solution.

Solving the heat equation: Explicit representation formula

- Next goal is to derive an integral representation for the solution.
- The idea is to use the definition of the Fourier coefficients, then switch the order of the limit and the integral:

$$u(\theta, t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(s) \left(\sum_{j=-\infty}^{\infty} e^{-j^2 t} e^{ij(\theta - s)} \right) ds.$$

Solving the heat equation: Explicit representation formula

- Next goal is to derive an integral representation for the solution.
- The idea is to use the definition of the Fourier coefficients, then switch the order of the limit and the integral:

$$u(\theta,t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(s) \left(\sum_{j=-\infty}^{\infty} e^{-j^2 t} e^{ij(\theta-s)} \right) ds.$$

- We define the heat kernel for the circle as

$$H_t(\theta) = \sum_{j=-\infty}^{\infty} e^{-j^2 t} e^{ij\theta}, \quad -\pi \le \theta < \pi, \quad t > 0,$$

Solving the heat equation: Explicit representation formula

- Next goal is to derive an integral representation for the solution.
- The idea is to use the definition of the Fourier coefficients, then switch the order of the limit and the integral:

$$u(\theta,t)=\frac{1}{2\pi}\int_{-\pi}^{\pi} g(s)\left(\sum_{j=-\infty}^{\infty} e^{-j^2 t} e^{ij(\theta-s)}\right) ds.$$

- We define the heat kernel for the circle as

$$H_t(\theta)=\sum_{j=-\infty}^{\infty} e^{-j^2 t} e^{ij\theta}, \quad -\pi\le\theta<\pi, \quad t>0,$$

- and we can write the solution as

$$u(\theta,t)=(g*H_t)(\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi} g(s)H_t(\theta-s)\,ds, \quad -\pi\le\theta<\pi, t>0.$$

Ch 2.13: Wave equation in one dimension

- We have now used essentially the same strategy for solving Laplace equation in the unit disc (Ch 2.11) and the heat equation in 1-dimensional ring.
  1. Separation of variables so that the initial and/or boundary values can be expressed with one variable.
  2. Solutions to the separated equations. Any linear combination solves the PDE inside the domain.
  3. Fourier series solution of the entire equation.
  4. Explicit representation formula.
- The same strategy works also for the (one-dimensional) wave equation.
- It is presented in the lecture notes (Ch 2.13) using real form Fourier coefficients.