MS-C1350 Partial differential equations
Chapter 4.12-4.15
Harnack's inequality, energy methods, weak solutions, other coordinates

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Lecture 9

Laplace equation

- Harnack's inequality
- Energy methods
- Weak solutions
- Other coordinates

Heat equation

- Physical interpretation
- Fundamental solution
- Nonhomogeneous problem

4.11 Harnack's inequality

- For the Laplace equation, Harnack's inequality tells that if $u \geq 0$ is a harmonic function in Ω and $\overline{V} \subset \Omega$, then

$$\sup_{x \in V} u(x) \leq c \inf_{x \in V} u(x).$$

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4.12 Energy methods

- Now we characterize the solution of the Dirichlet problem for the Poisson equation as a minimizer of an appropriate energy functional.
- The class of admissible functions for the Dirichlet problem:

$$\mathcal{A} = \{w \in C^2(\Omega) \cap C(\overline{\Omega}) : w = g \text{ on } \partial\Omega\}.$$

- The energy functional for the Poisson equation $\Delta u = f$:

$$I(w) = \int_\Omega \left(\frac{1}{2} |\nabla w|^2 - wf \right) dx.$$

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Theorem (Dirichlet's principle)

Suppose that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ *solves*

$$\begin{cases} -\Delta u = f & \textit{in} \quad \Omega, \\ u = g & \textit{on} \quad \partial\Omega. \end{cases}$$

Then

$$I(u) = \min_{w \in \mathcal{A}} I(w).$$

Conversely, if $u \in \mathcal{A}$ *satisfies* $I(u) = \min_{w \in \mathcal{A}} I(w)$*, then* u *is a solution of the Dirichlet problem above for the Poisson equation.*

- The Poisson equation is said to be the Euler-Lagrange equation for the energy (or variational) integral above.
- A function is a solution to the Poisson equation if and only if it is a minimizer of the energy integral.
- Notice: Laplace involves 2^{nd} order derivatives, but minimization problem only 1^{st} order derivatives!

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4.14 Laplace equation in other coordinates

- In $\mathbb{R}^2$, we solved the Laplace equation on the disc by switching to polar coordinates.
- In $\mathbb{R}^3$, there are two coordinate systems that generalize polar coordinates in $\mathbb{R}^2$:
  - Cylindrical coordinates (r, θ, z): Use polar coordinates for (x, y), keep z.
  - Spherical coordinates (r, θ, ϕ): r distance to the origin, θ horizontal angle, ϕ angle between z-axes and the direction.