MS-C1350 Partial differential equations
Chapter 1 – Introduction

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September 3, 2024

Official learning outcomes

- how to solve partial differential equations using separation of variables and Fourier techniques
- the physical interpretations of the equations
- the equations in the spherical and cylindrical coordinates
- the role of the fundamental solutions
- the maximum and comparison principles
- the variational forms of the equations.

Goal: to have more detailed learning goals biweekly this fall (work in progress –> also already published parts might be edited during the fall)

What we will do in this course

- We study three prototype equations: Laplace, heat and wave equation.
- First, we learn how to solve PDEs using Fourier series and Fourier transform.
- In the second part, we study each of the three equations one-by-one in more detail.
- We learn techniques that are useful in the study of also more general equations.

Lecture 1

- What is a PDE?
- What is a solution to PDE?
- Well posed problem.
- Function spaces C, C_0, C^kÂ and C^∞. Difference of $C((0,1))$ and $C([0,1])$?
- Introduction to Fourier series:
  - periodic functions,
  - L^p (and especially L^2) functions,
  - (complex) inner product, orthonormal basis and representing an element of a vector space using a basis.

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- In this course, we study special cases, in which explicit solutions and representation formulas are available, but focus on features that are present also in more general situations.
- Qualitative aspects are also important in numerical solutions of PDE. Without existence, uniqueness and stability, numerical methods may give inaccurate or completely wrong solutions.

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There are also PDEs that are in none of the above classes.

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A PDE problem is well posed, if

1. **EXISTENCE**: the problem has a solution,
2. **UNIQUENESS**: there exists only one solution and
3. **STABILITY**: the solution depends continuously on the data given in the problem.

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- Weak solution: less regularity, e.g. saw tooth wave.

Many PDEs do not have any classical solutions, but some type of weak solutions might be physically meaningful.