Aalto University

MS-C1350 Partial differential equations
Chapter 2.14:
Approximations of the identity

Riikka Korte

Department of Mathematics and Systems Analysis
Aalto University
riikka.korte@aalto.fi

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Lecture 4

- We solve heat equation in 1D.
  Topics:
  - Compare to the steps needed to solve previous examples (Laplace in unit disc, heat equation in 1D).
  - The use of real form Fourier series. (The problem could be solved also with complex Fourier series.)
  - Initial and boundary conditions for wave equation.
- Approximations of the identity – a class of convolutions that behave well.

Approximation of the identity (intro)

Consider the formulas:

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$$P_r f(\theta) = (f * P_r)(\theta) = \sum_{j=-\infty}^{\infty} \widehat{f}(j) r^{|j|} e^{ij\theta}, \quad 0 < r < 1,$$

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- QUESTIONS:
  - Can we recover f from the partial sums of its Fourier series?
  - If f is continuous, do we have

$$\lim_{r \to 1} (P_r * f) = f \quad \text{and} \quad \lim_{t \to 0} (H_t * f) = f?$$

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- These questions are related to the question in which sense the boundary or initial values are obtained.

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3. For every $\delta > 0$ we have

$$\lim_{\varepsilon\to 0}\int_{\delta<|x|\le\pi} |K_\varepsilon(x)|\,dx = 0.$$

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Theorem

Let $\{K_\varepsilon\}_{\varepsilon>0}$ be a family of good kernels and $f : [-\pi, \pi] \to \mathbb{C}$ be a bounded 2π-periodic function. Then

$$\lim_{\varepsilon\to 0}(f * K_\varepsilon)(x) = f(x)$$

whenever f is continuous at x. If f is continuous on the whole interval $[-\pi, \pi]$, then the above limit is uniform.