Aalto University

MS-C1350 Partial differential equations
Chapter 4.1-4.5
Laplace equation

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Lecture 7

- Laplace equation and Poisson equation
- Harmonic function
- Gauss-Green theorem and Green's identities.
- Dirichlet and Neumann boundary value problems
- Uniqueness of solutions
- Compatibility condition for Neumann problems
- Fundamental solution and a solution to Poisson equation
- How to solve Poisson equation with correct boundary values.

4 Laplace equation (introduction)

- Now we concentrate on Laplace equation

$$\Delta u = 0$$

and the Poisson equation

$$-\Delta u = f.$$

- Boundary value problems for these equations appear frequently in natural sciences and engineering.
- Physically, solutions of the Poisson equation correspond to steady states for evolutions in time such as heat flow or wave motion, with f corresponding to external driving forces such as heat sources or wave generators.

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- We will derive representation formulas and study general properties of solutions to the Laplace (and Poisson) equation.
- The topics include:
  - fundamental solutions
  - Green's functions
  - mean value property
  - Harnack's inequality, and
  - maximum principle.

Definition

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4.1 Gauss-Green theorem

- We need certain integral formulas to be able to study the Laplacian.
- We assume that $\Omega \subset \mathbb{R}^n$ is bounded and open.
- We also assume that $\partial\Omega$ is smooth (i.e. it can be locally represented as a graph of a smooth function).
- Closure of Ω:

$$\overline{\Omega} = \Omega \cup \partial\Omega.$$

- We say that $u \in C^1(\overline{\Omega})$, if $u \in C^1(\Omega)$ is such that u and all partial derivatives $\frac{\partial u}{\partial x_j}$, $j = 1, \ldots, n$, can be extended continuously up to the boundary $\partial\Omega$.
- We start with Gauss-Green theorem, which is a generalization of

$$\int_a^b f'(t)dt = f(b) - f(a)$$

to higher dimensions.

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Theorem (Gauss-Green theorem)

Assume that $u \in C^1(\overline{\Omega})$. Then

$$\int_\Omega \frac{\partial u}{\partial x_j}(x)\,dx = \int_{\partial\Omega} u(x)\nu_j(x)\,dS(x), \quad j = 1, \ldots, n,$$

where dS denotes the surface measure on $\partial\Omega$. Here $\nu(x) = (\nu_1(x), \ldots, \nu_n(x))$ is the outward pointing unit normal vector on $\partial\Omega$.

Or equivalently

Theorem (Divergence theorem)

$$\int_\Omega \mathrm{div}\, F(x)\,dx = \int_{\partial\Omega} F(x) \cdot \nu(x)\,dS(x),$$

where $F = (F_1, \ldots, F_n)$ is a vector field.

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Reason: Gauss-Green $\Rightarrow$ Divergence.

Recall, that

$$\operatorname{div} F(x) = \sum_{j=1}^{n} \frac{\partial F_j}{\partial x_j}(x)$$

is the divergence of F.

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- The Gauss-Green theorem gives information about the divergence of a vector field inside the domain by its values on the boundary of the domain.
- More precisely, the integral of the divergence of a vector field over a domain is equal to the total flow through the boundary.
- This is useful in boundary value problems for PDEs.

Theorem (Integration by parts)

Assume that $u, v \in C^1(\overline{\Omega})$. *Then for* $j = 1, \ldots, n$,

$$\int_\Omega \frac{\partial u}{\partial x_j}(x)v(x)\,dx = -\int_\Omega \frac{\partial v}{\partial x_j}(x)u(x)\,dx + \int_{\partial\Omega} u(x)v(x)\nu_j(x)\,dS(x).$$

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Theorem

1. Green's first identity:

$$\int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx = -\int_\Omega u(x)\Delta v(x)\, dx + \int_{\partial\Omega} \frac{\partial v}{\partial \nu}(x) u(x)\, dS(x),$$

2. Green's second identity:

$$\int_\Omega (u(x)\Delta v(x) - v(x)\Delta u(x))\, dx = \int_{\partial\Omega} \big(u(x)\frac{\partial v}{\partial \nu}(x) - v(x)\frac{\partial u}{\partial \nu}(x)\big)\, dS(x),$$

3. Green's third identity:

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Check the proofs from the lecture notes. (1) follows from the integration by parts formula, (2) and (3) follows from (1).

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4.1 Applications of Green's identities to harmonic functions (1/2)

- Suppose u is harmonic and apply Green's first identity

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with $v = u$.

- We obtain

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- Consider Green's third identity:

$$\int_\Omega \Delta u(x)\,dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu}(x)\,dS(x).$$

- It tells that the integral of the Laplacian is equal to the total flow through the boundary.
- If u is harmonic in Ω, then

$$\int_{\partial V} \frac{\partial u}{\partial \nu}(x)\,dS(x) = 0$$

  for every subdomain V.
- This means that the total flow is zero through the boundary of any subdomain V.
- Physically this means that there are not heat sources or electric charges in the domain.

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4.2 PDEs and physics

- In a typical case, u is a function that denotes the density of some quantity in steady state.
- Examples: temperature, chemical concentration or electrostatic potential.
- The total flow through the boundary ∂V is zero

$$\int_{\partial V} F(x) \cdot \nu(x)\, dS(x) = 0,$$

where $F = (F_1, \dots, F)$ is the flux density and ν is the unit outer normal of ∂V.
- By the Gauss-Green theorem we have

$$\int_V \operatorname{div} F(x)\, dx = \int_{\partial V} F(x) \cdot \nu(x)\, dS(x) = 0.$$

- Since this holds for every subdomain V of Ω, we have

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- It is physically reasonable to assume that the flux F is proportional to the gradient ∇u but in the opposite direction, since the flow is from regions of high temperature to regions of low temperature or high concentration to low concentration. Thus

$$F(x) = -a\nabla u(x), \quad a > 0.$$

- This gives

$$\operatorname{div} F(x) = -a \operatorname{div} \nabla u(x) = -a\Delta u(x) = 0 \quad \text{for every} \quad x \in \Omega,$$

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4.3 Boundary values and physics

- We consider two types of boundary conditions:
- **Dirichlet problem**

$$\begin{cases} \Delta u = 0 \quad \text{in} \quad \Omega, \\ u = g \quad \text{on} \quad \partial\Omega. \end{cases}$$

- Temperature: Boundary values g describe e.g. the temperature distribution on $\partial\Omega$.
- Electrostatistics: g specifies the values of the potential u on $\partial\Omega$, which induces the electric field $E = -\nabla u$ in Ω.

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4.3 Boundary values and physics

- We consider two types of boundary conditions:
- **Dirichlet problem**

$$\begin{cases} \Delta u = 0 \quad \text{in} \quad \Omega, \\ u = g \quad \text{on} \quad \partial\Omega. \end{cases}$$

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- **Neumann problem**

$$\begin{cases} \Delta u = 0 \quad \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \nu} = h \quad \text{on} \quad \partial\Omega. \end{cases}$$

- Physically the Neumann problem describes the steady state temperature distribution in Ω when the heat flow through $\partial\Omega$ is given by the normal derivative $\frac{\partial u}{\partial \nu} = h$.
- For example, if the surface of the body $\partial\Omega$ is insulated, the function h in the Neumann boundary condition is zero.

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4.3 Uniqueness of solutions for Dirichlet problem

- Step 1: If u is harmonic in Ω and $u = 0$ on $\partial\Omega$, then $u = 0$ in Ω.
- Proof: By Green's first identity

$$\int_\Omega |\nabla u(x)|^2\,dx = \int_\Omega \nabla u(x)\cdot\nabla u(x)\,dx$$
$$= -\int_\Omega u(x)\underbrace{\Delta u(x)}_{=0}\,dx + \int_{\partial\Omega}\frac{\partial u}{\partial\nu}(x)\underbrace{u(x)}_{=0}\,dS(x) = 0.$$

- This implies that $|\nabla u(x)| = 0$ and thus $u(x) = c$.
- Boundary condition implies that $c = 0$.
- **Step 2**: If u and v are harmonic (or satisfy Poisson equation with same f) and have the same boundary values g, then $w = u - v$ is a harmonic function with zero boundary values and thus $w = 0$ in Ω.

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- If u is a solution to the Neumann problem, then also $u(x) + c$ is a solution. $\Rightarrow$ **NOT UNIQUE**
- **BUT**: If u is a harmonic function in Ω and $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, then $u = c$ in Ω. Proof: By Green's 1st identity

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4.3 Compatibility condition for Neumann problems

- Green's third identity gives the following compatibility condition of the Neumann problem

$$0 = \int_\Omega \underbrace{\Delta u}_{=0} \, dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \, dS = \int_{\partial\Omega} h \, dS.$$

- Thus if the Neumann boundary condition is given by a function h such that

$$\int_{\partial\Omega} h \, dS \neq 0,$$

then there does not exist any solutions.

- Physically it means that the total heat flow though the boundary has to be zero.

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4.3 Dirichlet and Neumann problems for Poisson equation

- We discuss Dirichlet and Neumann problems for the Poisson equation $-\Delta u = f$, but it is enough to consider boundary value problems, where either the equation is homogeneous ($\Delta u = 0$) or the boundary condition is homogeneous ($g = 0$ or $h = 0$).
- For example, to solve

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega, \\ u = g & \text{on} \quad \partial\Omega, \end{cases}$$

we may write $u = u_1 + u_2$ with

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- Then $u = u_1 + u_2$ is the solution to the original problem.

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4.4 Fundamental solution

- Fundamental solution u is a solution to a (linear) partial differential equation L is a function that satisfies:

$$Lu = \delta,$$

where δ is the Dirac delta function, which has a unit mass at 0.

- Now we are interested in the fundamental solution of the Laplace equation in the whole $\mathbb{R}^n$.
- As the equation is linear, any linear combination, or integral, of fundamental solution will be a solution to the Laplace (Poisson) equation as well.
- We will be able to represent all other solutions as integrals (or convolutions) with the fundamental solution.
- We look for a radial solution that has a singularity at the origin.

$$u(x) = v(|x|) = v(r(x)).$$

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4.4 Fundamental solution

- **Step 1**: Find the Laplace equation for the radial functions using the chain rule.
- For a radial function

$$\Delta u(x) = 0, \quad x \neq 0 \quad \iff \quad v''(r) + \frac{n-1}{r} v'(r) = 0, \quad r > 0.$$

- Note: This is an ODE as it is allowed to depend only on one variable r.
- Solving this, we obtain

$$v(r) = \begin{cases} a \ln r + b, & n = 2, \\ \frac{c}{r^{n-2}} + d, & n \geq 3, \end{cases}$$

where a, b, c, d are constants.

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The function $\Phi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$,

$$\Phi(x) = \begin{cases} -\dfrac{1}{2\pi} \ln |x|, & n = 2, \\ \dfrac{1}{n(n-2)\alpha(n)} \dfrac{1}{|x|^{n-2}}, & n \geq 3, \end{cases}$$

is called the fundamental solution of the Laplace equation. Here we denote the volume of the unit ball in $\mathbb{R}^n$ by $\alpha(n) = |B(0,1)|$.

- Physically the fundamental solution is the potential induced by a unit point mass at $\overline{0}$. Constants are chosen so that

$$-\int_{\partial B(0,r)} \frac{\partial \Phi}{\partial \nu}(x)\, dS(x) = 1 \quad \text{for every} \quad r > 0,$$

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4.5 The Poisson equation

Theorem

Let $f \in C_0^\infty(\mathbb{R}^n)$ and define

$$u(x) = (f * \Phi)(x) = \int_{\mathbb{R}^n} f(y)\Phi(x-y)\,dy,$$

where Φ is the fundamental solution of the Laplace equation. Then $u \in C^2(\mathbb{R}^n)$ and $-\Delta u = f$ in $\mathbb{R}^n$.

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- The theorem gives a solution u in the whole space without a specification of the boundary values.
- Consider an open and bounded set $\Omega \subset \mathbb{R}^n$.
- Let v be a solution of the Dirichlet problem

$$\begin{cases} \Delta v = 0 & \text{in} \quad \Omega, \\ v = -u & \text{on} \quad \partial\Omega. \end{cases}$$

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$$\begin{cases} \Delta v = 0 & \text{in} \quad \Omega, \\ v = -u & \text{on} \quad \partial\Omega. \end{cases}$$

  Then $w = u + v$ is a solution to the problem

$$\begin{cases} -\Delta w = f & \text{in} \quad \Omega, \\ w = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

- This observation will be useful later.

4.5 The Poisson equation

- The theorem gives a solution u in the whole space without a specification of the boundary values.
- Consider an open and bounded set $\Omega \subset \mathbb{R}^n$.
- Let v be a solution of the Dirichlet problem

$$\begin{cases} \Delta v = 0 & \text{in} \quad \Omega, \\ v = -u & \text{on} \quad \partial\Omega. \end{cases}$$

  Then $w = u + v$ is a solution to the problem

$$\begin{cases} -\Delta w = f & \text{in} \quad \Omega, \\ w = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

- This observation will be useful later.