Aalto University

# MS-C1350 Partial differential equations
# Chapter 5.1-5.3
# Heat equation – physical interpretation, fundamental solution and nonhomogeneous problem

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# 5.1 Heat equation

- Now we study the heat equation

$$u_t - \Delta u = 0$$

- and the nonhomogeneous heat equation

$$u_t - \Delta u = f$$

with appropriate initial and boundary conditions.

- The Laplace is taken with respect to the spatial variable $x$

$$\Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}.$$

- Let $\Omega \subset \mathbb{R}^n$ and $T > 0$. The problem is to find a function $u = u(x,t)$ such that it is a solution to the heat equation in $\Omega \times (0,T)$.

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- Physically, a solution $u = u(x,t)$ of the heat equation represents the temperature of the body $\Omega$ at the point $x$ and time $t$.
- Observe that any solution $v = v(x)$ of the Laplace equation induces a time independent solution $u = u(x,t) = v(x)$ of the heat equation
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# 5.1 Initial and boundary conditions

- The appropriate initial condition is

$$u(x,0) = g(x), \quad x \in \Omega.$$

This describes the initial temperature distribution at the time $t = 0$.

- In addition, we may have a Dirichlet type boundary condition

$$u(x,t) = g(x,t), \quad x \in \partial\Omega, \quad t > 0,$$

which describes the temperature on the boundary,

- or a Neumann type boundary condition

$$\frac{\partial u}{\partial \nu}(x,t) = \nabla_x u(x,t) \cdot \nu(x) = h(x,t), \quad x \in \partial\Omega, \quad t > 0,$$

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# 5.2 Fundamental solution

- We already derived (with Fourier techniques) a formula for the solution of the Cauchy problem

$$\begin{cases} u_t - \Delta u = 0 \quad \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u = g \quad \text{on} \quad \mathbb{R}^n \times \{t=0\}, \end{cases}$$

  where $g \in C_0^\infty(\mathbb{R}^n)$.
- We obtained

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\,dy = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\,dy.$$

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# 5.2 Fundamental solution

- $\Phi$ is a solution to the heat equation in the upper half-space.
- $\Phi$ is unbounded in any neighbourhood of $(0, 0)$.
- We proved earlier that the solution attains the initial values $g$ in the sense

$$\lim_{t \to 0} u(x,t) = g(x) \quad \text{for every} \quad x \in \mathbb{R}^n$$

  ("good kernels")
- This means that we have existence of a solution to the Cauchy problem.
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- Formally the fundamental solution solves the following PDE:
$$\begin{cases} \dfrac{\partial \Phi}{\partial t} - \Delta \Phi = 0 \quad \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ \Phi = \delta_0 \quad \text{in} \quad \mathbb{R}^n \times \{t=0\}, \end{cases}$$
where $\delta_0$ is the Dirac measure in $\mathbb{R}^n$ giving unit mass to the point $0$.

# 5.2 Remarks about heat equation

- The heat equation has infinite speed of propagation.
- For example, if initial values $g \in C(\mathbb{R}^n)$ are non-negative $g \geq 0$ and there is a point $y \in \mathbb{R}^n$ such that $g(y) > 0$, then $u > 0$ everywhere in the upper half-space.
- From the formula

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\, dy = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy.$$

we get the decay estimate

$$\sup_{x \in \mathbb{R}^n} |u(x,t)| \leq \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} |g(y)|\, dy, \quad t > 0.$$

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# 5.3 Nonhomogeneous problem

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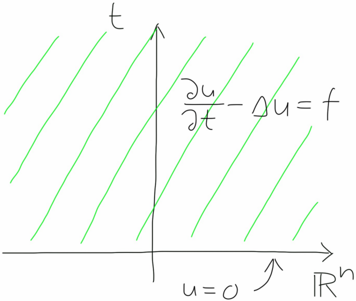

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- From the formula (for the homogeneous problem)

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\, dy = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy.$$

- we obtain that for every fixed $s$, with $0 < s < t$, the function

$$u(x,t;s) = \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\, dy$$

solves the translated initial value problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t;s) - \Delta u(x,t;s) = 0, \quad x \in \mathbb{R}^n, \quad t > s, \\ u(x,s;s) = f(x,s), \quad x \in \mathbb{R}^n. \end{cases}$$

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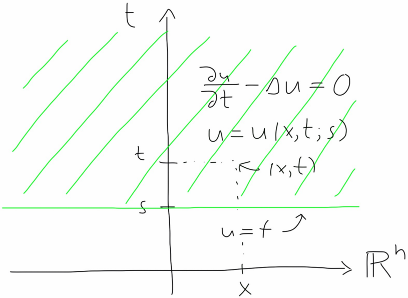
# 5.3 Duhamel's principle




- Duhamel's principle suggests that we can construct a solution to the nonhomogeneous problem by integrating solutions $u(x,t,;s)$ over $s \in (0,t)$ and have

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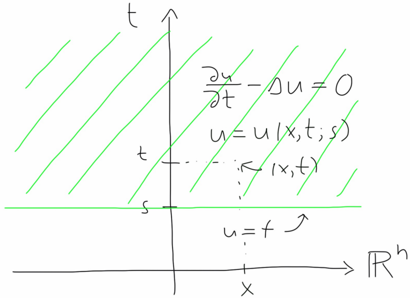


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# 5.3 Duhamel's principle

- Duhamel's principle is a process of expressing the solution of a nonhomogeneous problem as an integral of the solutions to the homogeneous problem in the way that the source term is interpreted as the initial condition.
- It does not depend on the specific structure of the equation and it applies to other linear ODEs and PDEs as well.
- We can verify it by differentiating the function $u$ defined by an integral.

$$u(x,t) = \int_0^t u(x,t;s)\,ds, \quad x \in \mathbb{R}^n, \quad t > 0.$$

- Recall that (Lemma 5.3)

$$\frac{\partial u}{\partial t}(x,t) = u(x,t;t) + \int_0^t \frac{\partial u}{\partial t}(x,t;s)\,ds.$$

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**Theorem**

*The solution of the nonhomogeneous Cauchy problem is*

$$
\begin{aligned}
u(x,t) &= \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\,dy\,ds \\
&= \int_0^t \frac{1}{(4\pi(t-s))^{\frac{n}{2}}} \left( \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s)\,dy \right) ds.
\end{aligned}
$$

- Formal proof:

$$
\begin{aligned}
\frac{\partial u}{\partial t}(x,t) = u(x,t;t) + \int_0^t \frac{\partial u}{\partial t}(x,t;s)\,ds
&= f(x,t) + \int_0^t \Delta u(x,t;s)\,ds \\
&= f(x,t) + \Delta \left( \int_0^t u(x,t;s)\,ds \right) \\
&= f(x,t) + \Delta u(x,t).
\end{aligned}
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## Theorem

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$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s)\, dy\, ds$$
$$= \int_0^t \frac{1}{(4\pi(t-s))^{\frac{n}{2}}} \left( \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s)\, dy \right) ds.$$

- Formal proof:

$$\frac{\partial u}{\partial t}(x,t) = u(x,t;t) + \int_0^t \frac{\partial u}{\partial t}(x,t;s)\, ds$$
$$= f(x,t) + \int_0^t \Delta u(x,t;s)\, ds$$
$$= f(x,t) + \Delta \left( \int_0^t u(x,t;s)\, ds \right)$$
$$= f(x,t) + \Delta u(x,t).$$

# 5.3 Duhamel's principle

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# 5.3 Duhamel's principle

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u(x,t) &= \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\,dy\,ds \\
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\end{aligned}
$$

- Formal proof:

$$
\begin{aligned}
\frac{\partial u}{\partial t}(x,t) &= u(x,t;t) + \int_0^t \frac{\partial u}{\partial t}(x,t;s)\,ds \\
&= f(x,t) + \int_0^t \Delta u(x,t;s)\,ds \\
&= f(x,t) + \Delta \left( \int_0^t u(x,t;s)\,ds \right) \\
&= f(x,t) + \Delta u(x,t).
\end{aligned}
$$

# 5.3 Nonhomogeneous problem with general initial data

- To solve the general problem

$$\begin{cases} u_t - \Delta u = f \quad \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u = g \quad \text{in} \quad \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

  we can use the same approach as for the Laplace equation and write $u = u_1 + u_2$ with

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - \Delta u_1 = f \quad \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u_1 = 0 \quad \text{in} \quad \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

  and

$$\begin{cases} \dfrac{\partial u_2}{\partial t} - \Delta u_2 = 0 \quad \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u_2 = g \quad \text{in} \quad \mathbb{R}^n \times \{t = 0\}. \end{cases}$$