MS-C1350 Partial differential equations Chapters 2.1–2.2 – Periodic functions, L^p-functions and inner product

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- Many PDE problems can be solved using using Fourier series and convolutions.
- We need following concepts to be able to do Fourier theory:
  - periodic functions
  - L^p-functions, in particular L^2-functions
  - Innerproduct spaces

Periodic functions (Chapter 2.1)

- We say that a function $f : \mathbb{R} \to \mathbb{C}$ is 2π-periodic if for every $t \in \mathbb{R}$ we have

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- Fourier series are defined only for periodic functions, but this is not a serious restriction, because we can extend the functions periodically.

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$$\|f\|_{L^p([-\pi,\pi])} = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^p \, dt \right)^{\frac{1}{p}} < \infty.$$

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- L^p functions can have singularities.

(Complex) Inner product

- Inner product space is a vector space V with a map: $< \cdot, \cdot >: V \times V \to \mathbb{C}$ that satisfies ($\forall\ x, y, z \in V$, $a \in \mathbb{C}$)
  - Linearity
$$\langle ax, y\rangle = a\langle x, y\rangle, \qquad \langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle,$$
  - conjugate symmetry
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- There are both finite dimensional vector spaces (e.g $\mathbb{R}^n, \mathbb{C}^n$) and infinite dimensional vector spaces (e.g function spaces).
- If $\{e_i\}_{i=-\infty}^{\infty}$ is an orthonormalisation basis for V then
$$f = \sum_{i=-\infty}^{\infty} \langle f, e_i\rangle e_i \quad \text{for every } f \in V.$$

Complex vector space $L^2([-\pi, \pi])$

- Addition and multiplication

$$(f+g)(t) = f(t) + g(t), \quad (\alpha f)(t) = \alpha f(t), \quad \alpha \in \mathbb{C}.$$

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- Norm given by the inner product:

$$\|f\|_{L^2([-\pi,\pi])} = \langle f, f \rangle^{1/2}$$

Important example of an inner product

- Let $e_j : [-\pi, \pi] \to \mathbb{C}$,

$$e_j(t) = e^{ijt} = \cos(jt) + i\sin(jt), \quad j \in \mathbb{Z} \quad \text{(Euler's formula).}$$

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- Then $e_j \in C([-\pi, \pi]) \subset L^2([-\pi, \pi])$ with

$$\|e_j\|_{L^2([-\pi,\pi])} = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{|e^{ijt}|^2}_{=1} \, dt \right)^{\frac{1}{2}} = 1, \quad j = 1, 2, \ldots .$$

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$$\langle e_j, e_k \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ijt} \overline{e^{ikt}} \, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(j-k)t} \, dt = 0, \quad \text{if } j \neq k$$

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$$\langle e_j, e_k \rangle = \delta_{jk} = \begin{cases} 0, & j \neq k, \\ 1, & j = k. \end{cases}$$

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- It is actually also an orthonormal basis (which is not so easy to prove).