MS-C1350 Partial differential equations
Ch 2.4 Best square approximation

Riikka Korte

Department of Mathematics and Systems Analysis
Aalto University
riikka.korte@aalto.fi

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Lecture 2 – topics

- Fourier series
  - Definition
  - Best square approximation
  - Fourier series on different intervals
  - Real form Fourier series (i.e. with $\sin \cos$)
  - Differentiation of Fourier series
  - Dirichlet kernel
- Convolution
- Function spaces $C^k(I)$ and $L^2(I)$, where I is open or closed interval.

The best square approximation

It is good to consider the Fourier series in terms of projections. $S_n f$ is the projection of $f \in L^2([-\pi, \pi])$ to a subspace spanned by $\{e_j\}_{j=-n}^{n}$:

$$S_n f(t) = \sum_{j=-n}^{n} \langle f, e_j \rangle e_j(t) = \sum_{j=-n}^{n} \widehat{f}(j) e^{ijt}, \quad n = 0, 1, 2, \dots,$$

where

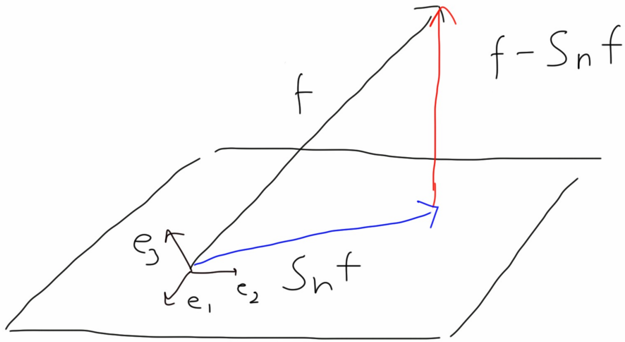
$$\widehat{f}(j) = \langle f, e_j \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ijt} \, dt.$$

This works for all orthonormal bases of vector spaces.

Orthogonality

$f - S_n f$ is orthogonal to the subspace spanned by $\{e_j\}_{j=-n}^{n}$ i.e.

$$\langle f - S_n f, e_j \rangle = 0 \quad \text{for every } j = -n, \ldots, n.$$



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&= \langle (f - S_n f) + S_n f, (f - S_n f) + S_n f \rangle \\
&= \langle f - S_n f, f - S_n f \rangle + \underbrace{\langle f - S_n f, S_n f \rangle}_{=0} \\
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This is Pythagorean theorem in $L^2([-\pi, \pi])$.

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This is Pythagorean theorem in $L^2([-\pi,\pi])$. Moreover, by orthogonality of $\{e_j\}_{j=-n}^{n}$

$$\begin{aligned}
\|f\|^2_{L^2([-\pi,\pi])} &= \|f - S_n f\|^2_{L^2([-\pi,\pi])} + \|S_n f\|^2_{L^2([-\pi,\pi])} \\
&= \|f - S_n f\|^2_{L^2([-\pi,\pi])} + \sum_{j=-n}^{n} |\widehat{f}(j)|^2.
\end{aligned} \tag{2}$$

Parseval's identity

- $$\|f\|^2_{L^2([-\pi,\pi])} = \|f - S_n f\|^2_{L^2([-\pi,\pi])} + \sum_{j=-n}^{n} |\widehat{f}(j)|^2.$$

  implies

  $$\|f\|^2_{L^2([-\pi,\pi])} \geq \sum_{j=-n}^{n} |\widehat{f}(j)|^2.$$

Parseval's identity

$$\|f\|^2_{L^2([-\pi,\pi])} = \sum_{j=-n}^{n} |\widehat{f}(j)|^2.$$

- Equality holds if and only if

$$\lim_{n\to\infty} \|f - S_n f\|^2_{L^2([-\pi,\pi])} = 0.$$

This is Parseval's identity and it holds for all $f \in L^2([-\pi,\pi])$.

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- This implies that if $f \in L^2([-\pi,\pi])$, then

$$\hat{f}(j) \to 0, \quad |j| \to \infty.$$

  The same holds for all $f \in L^1([-\pi,\pi])$ (Riemann-Lebesgue lemma).

Best square approximation

Theorem

If $f \in L^2([-\pi, \pi])$*, then*

$$\|f - S_n f\|_{L^2([-\pi,\pi])} \leq \left\| f - \sum_{j=-n}^{n} a_j e_j \right\|_{L^2([-\pi,\pi])}$$

for every $a_j \in \mathbb{C}$*,* $j = -n, \ldots, n$*.*

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for every $a_j \in \mathbb{C}$, $j = -n, \ldots, n$.

Theorem

Let $f \in L^2([-\pi,\pi])$. Then

$$\lim_{n\to\infty} \|f - S_n f\|_{L^2([-\pi,\pi])} = 0.$$

Warning! This does not imply pointwise convergence in each point!

Best square approximation

- This implies that $\{e_j\}_{n\in\mathbb{Z}}$ is an orthonormal bases for $L^2([-\pi,\pi])$ in the sense that for every function $f \in L^2([-\pi,\pi])$

$$\lim_{n\to\infty}\left\|\sum_{j=-n}^{n}\widehat{f}(j)e_j - f\right\|_{L^2([-\pi,\pi])} = 0$$

for every $f \in L^2([-\pi,\pi])$. This means that

$$f = \lim_{n\to\infty}\sum_{j=-n}^{n}\widehat{f}(j)e_j = \sum_{j=-\infty}^{\infty}\widehat{f}(j)e_j$$

in $L^2([-\pi,\pi])$.