

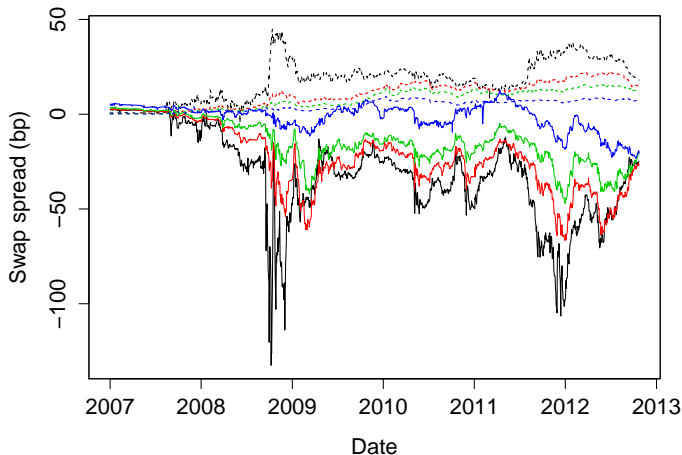
# Stationary stochastic processes and ARMA models

MS-C2128 Prediction and Time Series Analysis

Fall term 2021

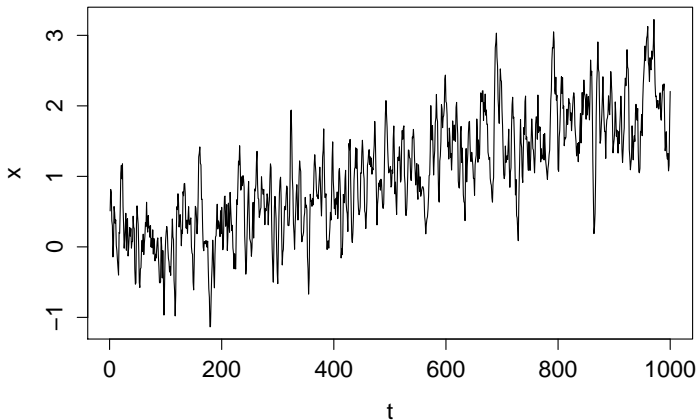
# Introduction

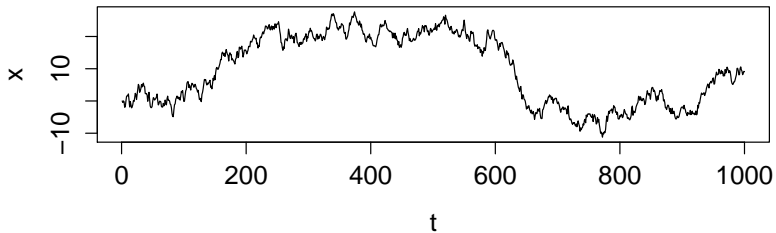
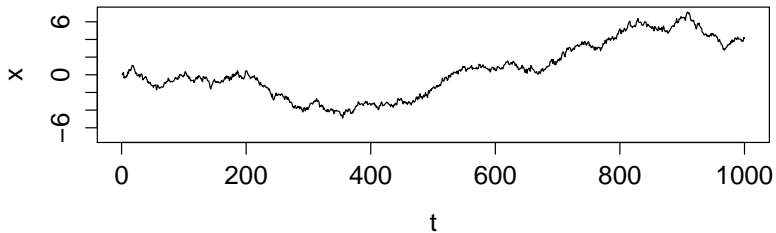
If we wish to deal with nasty, badly behaving, time series data...



# Introduction

...we should first be familiar with nicely behaving stochastic processes and their properties.





# Week 3: Stationary stochastic processes and ARMA models

- ① Stationary stochastic processes
  - ① Definition
  - ② Autocorrelation function
  - ③ Partial autocorrelation function
  - ④ Lag and difference operators
  - ⑤ Difference stationarity
- ② ARMA models
  - ① Pure random process
  - ② Different SARMA models
  - ③ Spectrum

1 Stationary stochastic processes

2 ARMA models

# Stochastic processes

- A stochastic process  $(x_t)_{t \in T}$  is a (time-)indexed collection of random variables defined on some common probability space. Each

$$x_t, \quad t \in T$$

is a random variable representing a value at time  $t \in T$ .

- The joint distribution of the random variables  $x_t$  defines fully the behaviour of the process  $(x_t)_{t \in T}$ .
- Here, we consider discrete time stochastic processes for which the index variable takes a discrete set of values. That is, we assume that  $T \subset \mathbb{Z} := \{\dots, -2, -1, 0, 1, 2, \dots\}$ .
- We do not consider continuous time processes. (For example processes for which  $T$  is the set of positive real numbers.)

# Time series as a stochastic process

- In time series analysis an observed time series is interpreted as a realization of some stochastic process.
  - In comparison, i.i.d. observations are interpreted as realizations of some random variable.
- In time series analysis, we wish to:
  - ❶ Find a suitable stochastic process that fits to the observed time series.
  - ❷ Estimate the parameters of the corresponding stochastic process and conduct hypotheses testing.
  - ❸ Construct predictions of the future behaviour of the time series.



# Expected value, variance and covariance: Definitions

The expected value of  $x_t$ , the variance of  $x_t$  and the covariance of  $x_t$  and  $x_s$  are useful, if one wishes to describe characteristics of a stochastic process  $(x_t)_{t \in T}$ :

- The expected value of  $x_t$  is defined as:

$$E[x_t] = \mu_t, \quad t \in T$$

- The variance of  $x_t$  is defined as:

$$\text{var}(x_t) = E[(x_t - \mu_t)^2] = \sigma_t^2, \quad t \in T$$

- The covariance of  $x_t$  and  $x_s$  is defined as:

$$\text{cov}(x_t, x_s) = E[(x_t - \mu_t)(x_s - \mu_s)] = \gamma_{ts}, \quad t, s \in T.$$

# Stationarity

Stochastic process  $(x_t)_{t \in T}$  is called **stationary** (or weakly stationary) if:

- ❶ The expected value does not depend on time:

$$E(x_t) = \mu, \quad \text{for all } t \in T$$

- ❷ The variance is finite and does not depend on time:

$$\text{var}(x_t) = \sigma^2 < \infty, \quad \text{for all } t \in T$$

- ❸ The covariance of  $x_t$  and  $x_s$  does not depend on the time points  $t$  and  $s$ . It only depends on the difference of  $t$  and  $s$ :

$$\text{cov}(x_t, x_s) = \gamma_{t-s}, \quad \text{for all } t, s \in T$$

- A process  $(x_t)_{t \in T}$  is called strictly stationary if the joint distributions of  $(x_{t1}, x_{t2}, \dots, x_{tn})$  and  $(x_{t1+h}, x_{t2+h}, \dots, x_{tn+h})$  are the same for all  $n, h, t1, t2, \dots, tn$ .

# Stationary stochastic processes

When you take a look at a realization of a stationary stochastic process you should NOT detect

- 1 Trend
- 2 Systematic changes in variance
- 3 Deterministic seasonality

# The importance of stationary processes in modeling time series data

Discussion

# Autocovariance: Definition

The  $k$ . **autocovariance**  $\gamma_k$  of a stationary stochastic process  $(x_t)_{t \in T}$  is defined as

$$\gamma_k := \gamma_{t-(t-k)} = \text{cov}(x_t, x_{t-k}) = E[(x_t - \mu)(x_{t-k} - \mu)], \quad t \in T, k \in \mathbb{Z}.$$

In particular

$$\gamma_0 = \text{var}(x_t) = \sigma^2, \quad t \in T.$$

The **autocovariance function** of a stationary stochastic process  $(x_t)_{t \in T}$  is a function of the autocovariances,  $\gamma : \mathbb{Z} \rightarrow \mathbb{R}$ ,

$$\gamma(k) = \gamma_k \quad \text{for all } k \in \mathbb{Z}.$$

# Autocorrelation: Definition

The  $k$ . **autocorrelation coefficient**  $\rho_k$  of a stationary stochastic process  $(x_t)_{t \in T}$  is defined as:

$$\rho_k = \frac{\gamma_k}{\gamma_0}, \quad k \in \mathbb{Z}.$$

- The autocorrelation coefficient  $\rho_k$  of  $(x_t)_{t \in T}$  measures how strong the linear dependence of the variables  $x_t$  and  $x_{t-k}$  is.
  - (i)  $\rho_0 = 1$
  - (ii)  $\rho_{-k} = \rho_k$  for all  $k \in \mathbb{Z}$
  - (iii)  $|\rho_k| \leq 1$  for all  $k \in \mathbb{Z}$ .
- The **autocorrelation function** is the function  $\rho : \mathbb{Z} \rightarrow [-1, 1]$ ,

$$\rho(k) = \rho_k, \quad \text{for all } k \in \mathbb{Z}.$$

# Partial autocorrelation: Definition

The  $k$ . **partial autocorrelation coefficient**  $\alpha_k$  of a stationary stochastic process  $(x_t)_{t \in T}$  is defined as:

$$\alpha_k = \text{cor} (x_t, x_{t-k} \mid x_{t-1}, \dots, x_{t-k+1}) \quad , t \in T, k \in \mathbb{Z}$$

- Partial autocorrelation coefficient is the conditional correlation of  $x_t$  and  $x_{t-k}$  with respect to  $x_{t-1}, \dots, x_{t-k+1}$ .
- Partial autocorrelation coefficient measures the correlation of  $x_t$  and  $x_{t-k}$ , when the values  $x_{t-1}, \dots, x_{t-k+1}$  are known.
  - (i)  $\alpha_0 = 1$
  - (ii)  $\alpha_{-k} = \alpha_k$  for all  $k \in \mathbb{Z}$
  - (iii)  $|\alpha_k| \leq 1$  for all  $k \in \mathbb{Z}$ .

The **partial autocorrelation function** is the function  $\alpha : \mathbb{Z} \rightarrow [-1, 1]$ ,

$$\alpha(k) = \alpha_k, \quad \text{for all } k \in \mathbb{Z}.$$

# Autocorrelation and partial autocorrelation: Yule-Walker equations

$$\begin{bmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{k-1} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{k-2} \\ \rho_2 & \rho_1 & 1 & \cdots & \rho_{k-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & \cdots & 1 \end{bmatrix} \begin{bmatrix} \alpha_{k1} \\ \alpha_{k2} \\ \alpha_{k3} \\ \vdots \\ \alpha_{kk} \end{bmatrix} = \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \vdots \\ \rho_k \end{bmatrix},$$

where  $\rho_k$  is the  $k$ . autocorrelation coefficient.

The  $k$ . partial autocorrelation coefficient  $\alpha_k$  is obtained by solving  $\alpha_{kk}$  from the equations above:

$$\alpha_k = \alpha_{kk}.$$

In particular

$$\alpha_2 = \alpha_{22} = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2}.$$



# Lag and difference: Definitions

Let  $(x_t)_{t \in \mathcal{T}}$  be a discrete time stochastic process.

- The **lag operator**  $L$  is defined by:

$$Lx_t = x_{t-1}$$

- The **difference operator**  $D$  is defined by:

$$Dx_t = x_t - x_{t-1}$$

## Remark

The difference operator  $D$  can be given in terms of the lag operator  $L$

$$D = 1 - L,$$

as

$$(1 - L)x_t = x_t - Lx_t = x_t - x_{t-1} = Dx_t.$$

# Higher order lags and differences, Seasonal difference

- The **p. lag**  $L^p$  is defined by:

$$L^p x_t = x_{t-p},$$

where  $L^p = LL \cdots L$  ( $p$  times):  $L^p x_t = L^{p-1} L x_t = L^{p-1} x_{t-1}$ .

- The **p. difference**  $D^p$  is defined by:

$$D^p x_t = (1 - L)^p x_t,$$

where  $D^p = DD \cdots D$  ( $p$  times).

- For the p. difference  $D^p$  it holds that

$$D^p x_t = (1 - L)^p x_t = \sum_{i=0}^p (-1)^i \binom{p}{i} x_{t-i}.$$

- The **seasonal difference**  $D_s$  is defined by:

$$D_s = 1 - L^s,$$

where  $s$  is the length of the season (i.e. the period).

- Now

$$D_s x_t = (1 - L^s) x_t = x_t - L^s x_t = x_t - x_{t-s}.$$

## Example: 2. difference

The second difference of  $x_t$  can be calculated as follows:

- Approach 1:

$$\begin{aligned}D^2 x_t &= DDx_t = D(x_t - x_{t-1}) \\&= Dx_t - Dx_{t-1} \\&= x_t - x_{t-1} - (x_{t-1} - x_{t-2}) \\&= x_t - 2x_{t-1} + x_{t-2}\end{aligned}$$

- Approach 2:

$$\begin{aligned}D^2 x_t &= (1 - L)^2 x_t = (1 - 2L + L^2) x_t \\&= x_t - 2Lx_t + L^2 x_t \\&= x_t - 2x_{t-1} + x_{t-2}\end{aligned}$$

## Definition

Let  $(x_t)_{t \in T}$  be a discrete time stochastic process.

- The process  $(x_t)_{t \in T}$  is **difference stationary of order  $p$** , if

$D^q x_t$  is non-stationary for all  $q = 0, 1, 2, \dots, p - 1$ ,

but  $D^p x_t$  is stationary.

- The process  $(x_t)_{t \in T}$  is **difference stationary of order  $p$  with respect to the season length  $s$** , if

$D_s^q x_t$  is non-stationary for all  $q = 0, 1, 2, \dots, p - 1$ ,

but  $D_s^p x_t$  is stationary.

# Trend and seasonality

Differencing can be applied in order to remove a trend.  
Seasonal differencing can be applied in order to remove deterministic seasonality. Sometimes both are needed in order to obtain a stationary time series.

## Example

If the term (season length)  $s = 12$ , we often apply the first difference (in order to remove the trend) and seasonal difference with period 12 (in order to remove seasonality). We then obtain the following series:

$$\begin{aligned}D_{12}Dx_t &= DD_{12}x_t = (1 - L)(1 - L^{12})x_t \\&= (1 - L - L^{12} + L^{13})x_t \\&= x_t - x_{t-1} - (x_{t-12} - x_{t-13}).\end{aligned}$$

# Content

1 Stationary stochastic processes

2 ARMA models

The family of **ARMA processes** is central in time series analysis.

- AR model = Autoregressive model
- MA model = Moving Average model
- ARMA model = Autoregressive Moving Average model
- SAR model = Seasonal AR model
- SMA model = Seasonal MA model
- SARMA model = Seasonal ARMA model
- ARIMA model = Integrated ARMA model
- SARIMA model = Integrated Seasonal ARMA model

# Pure stochastic process

Discrete time stochastic process  $(\epsilon_t)_{t \in T}$  is a **pure stochastic process**, if

- ❶  $E[\epsilon_t] = \mu, t \in T$
- ❷  $\text{var}(\epsilon_t) = \sigma^2, t \in T$
- ❸  $\text{cov}(\epsilon_t, \epsilon_s) = 0, t \neq s$
- If the expected value  $\mu = 0$ , then the pure stochastic process is called **white noise** and the following notation is used:

$$(\epsilon_t)_{t \in T} \sim WN(0, \sigma^2).$$

- If the random variables  $\epsilon_t$  are independent and identically distributed, then the pure white noise process is called iid white noise and the following notation is used:

$$(\epsilon_t)_{t \in T} \sim IID(0, \sigma^2)$$



An autoregressive process of order  $p$  is given by:

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \dots + \phi_p x_{t-p} + \epsilon_t, \quad (\epsilon_t)_{t \in T} \sim WN(0, \sigma^2).$$

- This process is called autoregressive, because  $x_t$  depends on  $x_{t-1}, x_{t-2}, \dots, x_{t-p}$  and because it resembles multiple linear regression model

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_p x_p + \epsilon$$

where:

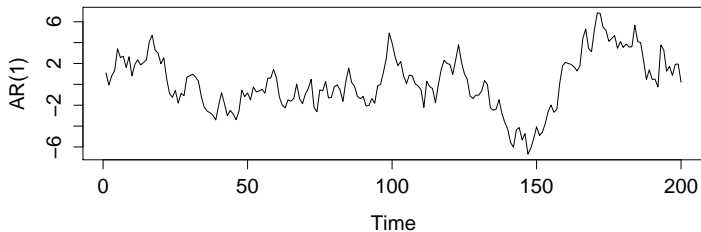
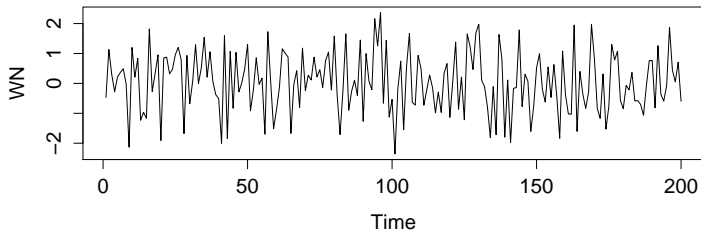
- The response variable is  $x_t$  and the explanatory variables are  $x_{t-1}, x_{t-2}, \dots, x_{t-p}$ .
- The regression coefficients are  $\beta_0 = 0$  and  $\beta_i = \phi_i$ ,  $i = 1, \dots, p$ .
- The residual is  $\epsilon_t$ .

## Example

An AR(1) process is given by:

$$x_t = \phi_1 x_{t-1} + \epsilon_t, \quad (\epsilon_t)_{t \in T} \sim WN(0, \sigma^2)$$

# White noise vs AR(1)



# MA( $q$ ) model

A moving average process of order  $q$  is given by:

$$x_t = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \dots + \theta_q \epsilon_{t-q}, \quad (\epsilon_t)_{t \in T} \sim WN(0, \sigma^2)$$

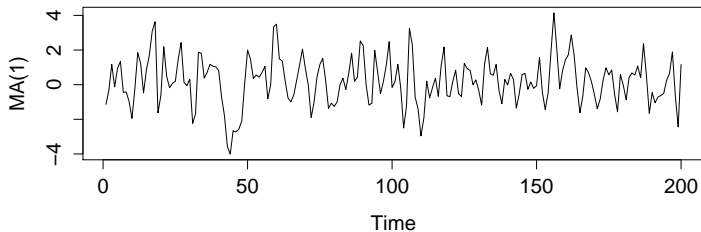
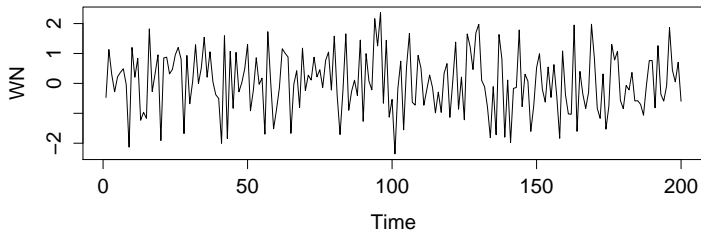
- The random variable  $x_t$  is the weighted sum of the random variables  $\epsilon_{t-q}, \dots, \epsilon_t$ .

## Example

A MA(1) process is given by:

$$x_t = \epsilon_t + \theta_1 \epsilon_{t-1}, \quad (\epsilon_t)_{t \in T} \sim WN(0, \sigma^2)$$

# White noise vs MA(1)



# ARMA( $p, q$ ) model

An autoregressive moving average process with an AR part of order  $p$  and a MA part of order  $q$  is given by:

$$x_t - \phi_1 x_{t-1} - \phi_2 x_{t-2} - \dots - \phi_p x_{t-p} = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \dots + \theta_q \epsilon_{t-q},$$

where  $(\epsilon_t)_{t \in T} \sim WN(0, \sigma^2)$ .

- $x_t$  depends on both, the random variables  $x_{t-1}, \dots, x_{t-p}$  and the random variables  $\epsilon_{t-1}, \dots, \epsilon_{t-q}$ .

## Example

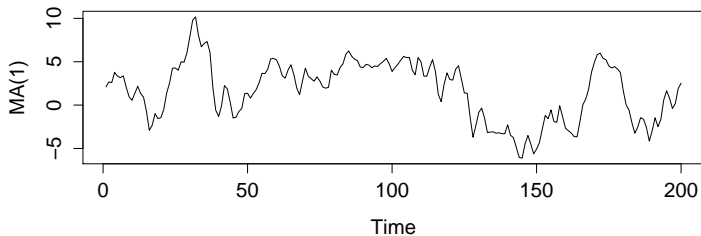
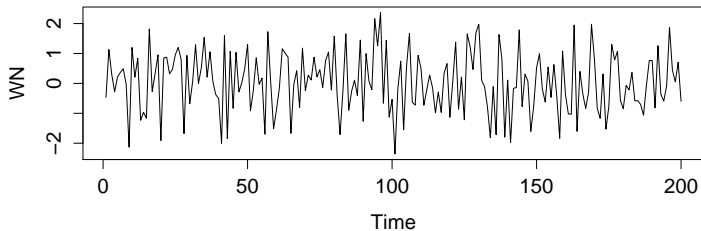
An ARMA(1,1) process is given by:

$$x_t - \phi_1 x_{t-1} = \epsilon_t + \theta_1 \epsilon_{t-1}$$

or equivalently

$$x_t = \phi_1 x_{t-1} + \theta_1 \epsilon_{t-1} + \epsilon_t$$

# White noise vs ARMA(1,1)



# SAR( $P$ ) <sub>$s$</sub> model and SMA( $Q$ ) <sub>$s$</sub> model

- A seasonal AR process of order  $P$ , with period  $s$  is given by:

$$x_t = \Phi_1 x_{t-s} + \Phi_2 x_{t-2s} + \dots + \Phi_P x_{t-Ps} + \epsilon_t, \quad (\epsilon_t)_{t \in T} \sim WN(0, \sigma^2).$$

- A seasonal MA process of order  $Q$ , with period  $s$  is given by:

$$x_t = \epsilon_t + \Theta_1 \epsilon_{t-s} + \Theta_2 \epsilon_{t-2s} + \dots + \Theta_Q \epsilon_{t-Qs}, \quad (\epsilon_t)_{t \in T} \sim WN(0, \sigma^2).$$

## Example

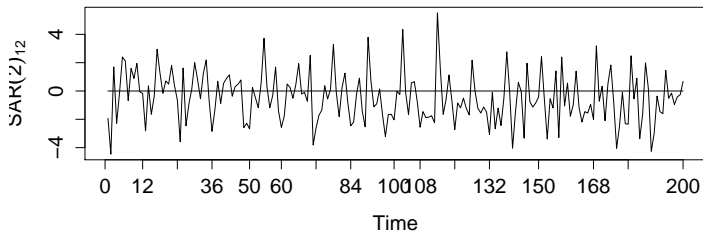
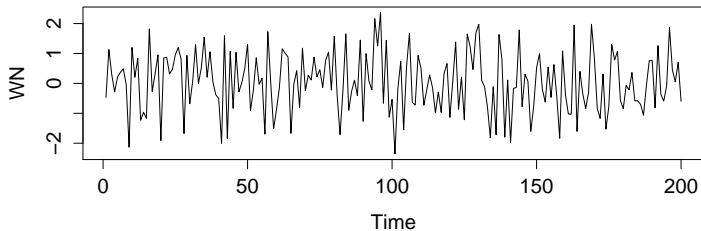
- A SAR(2)<sub>12</sub> process is given by:

$$x_t = \Phi_1 x_{t-12} + \Phi_2 x_{t-24} + \epsilon_t$$

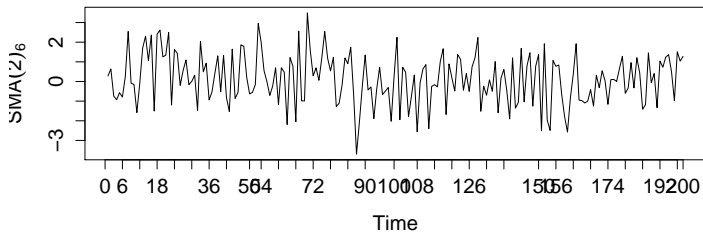
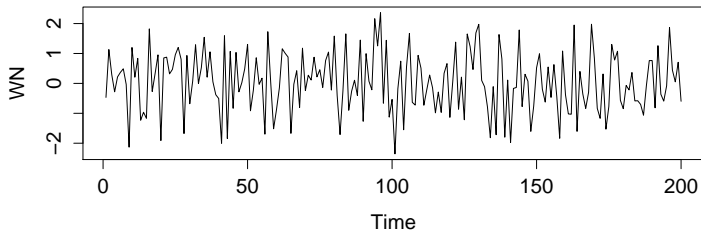
- A SMA(1)<sub>6</sub> process is given by:  $x_t = \epsilon_t + \Theta_1 \epsilon_{t-6}$



# White noise vs SAR(2)<sub>12</sub>



# White noise vs SMA(1)<sub>6</sub>



# SARMA( $P, Q$ ) $_s$ model

A seasonal ARMA process with period  $s$ , an AR part of order  $P$  and a MA part of order  $Q$  is given by:

$$X_t - \Phi_1 X_{t-s} - \dots - \Phi_P X_{t-Ps} = \epsilon_t + \Theta_1 \epsilon_{t-s} + \dots + \Theta_Q \epsilon_{t-Qs},$$

where  $(\epsilon_t)_{t \in T} \sim WN(0, \sigma^2)$ .

## Example

A SARMA(2,1) $_4$  process is given by:

$$X_t - \Phi_1 X_{t-4} - \Phi_2 X_{t-8} = \epsilon_t + \Theta_1 \epsilon_{t-4}$$

or equivalently

$$X_t = \Phi_1 X_{t-4} + \Phi_2 X_{t-8} + \Theta_1 \epsilon_{t-4} + \epsilon_t$$

We next consider cases with an AR part, a seasonal AR part, an MA part and a seasonal MA part. We start by getting familiar with lag polynomials.

# Lag polynomials: Definition

**Lag polynomial** of order  $r$  is given by:

$$\delta_r(L) = 1 + \delta_1 L + \delta_2 L^2 + \dots + \delta_r L^r.$$

- It now follows from the linearity of the operator  $L$ , that

$$\begin{aligned}\delta_r(L)x_t &= (1 + \delta_1 L + \delta_2 L^2 + \dots + \delta_r L^r)x_t \\ &= x_t + \delta_1 Lx_t + \delta_2 L^2 x_t + \dots + \delta_r L^r x_t \\ &= x_t + \delta_1 x_{t-1} + \delta_2 x_{t-2} + \dots + \delta_r x_{t-r}.\end{aligned}$$

## Example

If  $\phi(L) := 1 - \phi_1 L$  and  $\Phi(L) := 1 - \Phi_1 L^{12}$ , then we have that

$$\begin{aligned}\phi(L)\Phi(L)x_t &= (1 - \phi_1 L)(1 - \Phi_1 L^{12})x_t \\ &= (1 - \phi_1 L - \Phi_1 L^{12} + \phi_1 \Phi_1 L^{13})x_t \\ &= x_t - \phi_1 x_{t-1} - \Phi_1 x_{t-12} + \phi_1 \Phi_1 x_{t-13}.\end{aligned}$$

# SARMA( $p, q$ )( $P, Q$ )<sub>s</sub> model

A multiplicative seasonal ARMA process with period  $s$ , a pure AR part of order  $p$ , a pure MA part of order  $q$ , a seasonal AR part of order  $P$ , and a seasonal MA part of order  $Q$  is given by:

$$\Phi_P^s(L)\phi_p(L)x_t = \Theta_Q^s(L)\theta_q(L)\epsilon_t, \quad (\epsilon_t)_{t \in T} \sim WN(0, \sigma^2),$$

where  $\phi_p$ ,  $\theta_q$ ,  $\Phi_P^s$  and  $\Theta_Q^s$  are the following lag polynomials

$$\phi_p(L) = 1 - \phi_1 L - \phi_2 L^2 - \dots - \phi_p L^p$$

$$\theta_q(L) = 1 + \theta_1 L + \theta_2 L^2 + \dots + \theta_q L^q$$

$$\Phi_P^s(L) = 1 - \phi_1 L^s - \phi_2 L^{2s} - \dots - \phi_P L^{Ps}$$

$$\Theta_Q^s(L) = 1 + \Theta_1 L^s + \Theta_2 L^{2s} + \dots + \Theta_Q L^{Qs}$$

(Here, it is customary to assume that the polynomials  $\Phi_P^s(L)\phi_p(L)$  and  $\Theta_Q^s(L)\theta_q(L)$  do not share roots.)

# SARMA( $p, q$ )( $P, Q$ )<sub>s</sub> model

Consider a SARMA( $p, q$ )( $P, Q$ )<sub>s</sub> process

$$\Phi_P^s(L)\phi_p(L)x_t = \Theta_Q^s(L)\theta_q(L)\epsilon_t, \quad (\epsilon_t)_{t \in T} \sim WN(0, \sigma^2),$$

Now

- The AR part is of order  $p$ ; The corresponding parameters are:  $\phi_1, \phi_2, \dots, \phi_p$
- The seasonal AR part is of order  $P$ ; The corresponding parameters are:  $\Phi_1, \Phi_2, \dots, \Phi_P$
- The MA part is of order  $q$ ; The corresponding parameters are:  $\theta_1, \theta_2, \dots, \theta_q$
- The seasonal MA part is of order  $Q$ ; The corresponding parameters are:  $\Theta_1, \Theta_2, \dots, \Theta_Q$

# SARMA( $p, q$ )( $P, Q$ )<sub>s</sub> model

Note that the SARMA( $p, q$ )( $P, Q$ )<sub>s</sub> models

$$\Phi_P^s(L)\phi_p(L)x_t = \Theta_Q^s(L)\theta_q(L)\epsilon_t, \quad (\epsilon_t)_{t \in T} \sim WN(0, \sigma^2)$$

cover all the following processes:

- AR( $p$ )
- MA( $q$ )
- ARMA( $p, q$ )
- SAR( $P$ )<sub>s</sub>
- SMA( $Q$ )<sub>s</sub>
- SARMA( $P, Q$ )<sub>s</sub>



# Roots of the lag polynomials

Based on the fundamental theorem of algebra, the lag polynomials of order  $r$

$$\delta_r(L) = 1 + \delta_1 L + \delta_2 L^2 + \dots + \delta_r L^r$$

have  $r$  roots(, that may or may not be complex valued).

## Example

Let  $\phi(L) = 1 - L + \frac{1}{2}L^2$ . The the roots of the polynomial  $\phi(L)$

$$L_1 = 1 + i \quad \text{and} \quad L_2 = 1 - i$$

lie outside of the unit circle:

$$\|L_1\|^2 = \|L_2\|^2 = 2.$$

In what follows, when we consider different SARMA( $p, q$ )( $P, Q$ ) $_s$  models, we assume that  $E[x_{t-v}\epsilon_t] = 0$  for all  $v \geq 1$ . Moreover, we assume that the corresponding polynomials  $\Phi_P^s(L)\phi_p(L)$  and  $\Theta_Q^s(L)\theta_q(L)$  do not share roots.

# SARMA( $p, q$ )( $P, Q$ ) $_s$ model: Stationarity

SARMA( $p, q$ )( $P, Q$ ) $_s$  process  $x_t$  is stationary, if and only if the roots of the lag polynomials of the AR part

$$\phi_p(L) = 1 - \phi_1 L - \phi_2 L^2 - \dots - \phi_p L^p$$

$$\Phi_P^s(L) = 1 - \Phi_1 L^s - \Phi_2 L^{2s} - \dots - \Phi_P L^{Ps}$$

lie outside of the unit circle.

## Fact

A SARMA process can not be analyzed using auto- and partial autocorrelation functions unless it is stationary.

# SARMA( $p, q$ )( $P, Q$ )<sub>s</sub> model: Stationarity

A SARMA( $p, q$ )( $P, Q$ )<sub>s</sub> process  $x_t$  is stationary if and only if it has an **MA( $\infty$ ) representation**

$$x_t = \Psi(L)\epsilon_t, \quad (\epsilon_t)_{t \in T} \sim WN(0, \sigma^2),$$

where

$$\Psi(L) = \phi^{-1}(L)\Phi^{-1}(L)\theta(L)\Theta(L) = \sum_{i=0}^{\infty} \psi_i L^i, \quad (\psi_0 = 1),$$

and where the series

$$\sum_{i=0}^{\infty} \psi_i$$

converges absolutely.

# SARMA( $p, q$ )( $P, Q$ )<sub>s</sub> model: Invertibility

A SARMA( $p, q$ )( $P, Q$ )<sub>s</sub> process is called **invertible**, if it has an **AR( $\infty$ ) representation**

$$\Pi(L)x_t = \epsilon_t, \quad (\epsilon_t)_{t \in T} \sim WN(0, \sigma^2),$$

where

$$\Pi(L) = \theta^{-1}(L)\Theta^{-1}(L)\phi(L)\Phi(L) = \sum_{i=0}^{\infty} \pi_i L^i, \quad (\pi_0 = 1)$$

and where the series

$$\sum_{i=0}^{\infty} \pi_i$$

converges absolutely.

# SARMA( $p, q$ )( $P, Q$ )<sub>s</sub> model: Invertibility

A SARMA( $p, q$ )( $P, Q$ )<sub>s</sub> process is invertible, if and only if the roots of the lag polynomials of the MA part

$$\theta_q(L) = 1 + \theta_1 L + \theta_2 L^2 + \dots + \theta_q L^q$$

$$\Theta_Q^s(L) = 1 + \Theta_1 L^s + \Theta_2 L^{2s} + \dots + \Theta_Q L^{Qs}$$

lie outside of the unit circle.

## Fact

The autocorrelation function of a SARMA process does not define the MA and the seasonal MA parts of the process uniquely unless the process is invertible.

## Example

- ① An AR( $p$ ) process:

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \dots + \phi_p x_{t-p} + \epsilon_t, \quad (\epsilon_t)_{t \in T} \sim WN(0, \sigma^2).$$

- Is stationary iff the roots of the lag polynomial (of the AR part) lie outside of the unit circle.
- Is always invertible.

- ② A MA( $q$ ) process

$$x_t = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \dots + \theta_q \epsilon_{t-q}, \quad (\epsilon_t)_{t \in T} \sim WN(0, \sigma^2).$$

- Is always stationary.
- Is invertible iff the roots of the lag polynomial (of the MA part) lie outside of the unit circle.

# Spectrum of a stationary process

- If the analysis of a time series is based on correlation functions, we say that the analysis takes place in the time domain.
- The analysis of a stationary time series can also be conducted in the frequency domain.
  - In the frequency domain, the analysis of a time series is based on the so called spectral density function  $f(\lambda)$  of the process.
  - The analysis conducted in the frequency domain is especially useful in revealing cyclic components of the process.
- The autocovariance function  $\gamma_k$  and the spectral density function  $f(\lambda)$  of a stationary process have exactly the same information.



The **spectral density function**  $f(\lambda)$  (also called the power spectral function or spectrum) of a stationary process  $(x_t)_{t \in T}$  is given by

$$f(\lambda) = \frac{1}{2\pi} \left( \gamma_0 + 2 \sum_{k=1}^{\infty} \gamma_k \cos(\lambda k) \right), \quad \lambda \in [0, \pi],$$

where  $\gamma_k$  is the  $k$ . autocovariance of  $(x_t)_{t \in T}$ .

- $\lambda$ : (angular) frequency
- $2\pi/\lambda$ : period
- $\lambda/2\pi$ : the number of cycles per time unit

#### Fact

$$\gamma_k = \int_{-\pi}^{\pi} f(\lambda) \cos(\lambda k) d\lambda = 2 \int_0^{\pi} f(\lambda) \cos(\lambda k) d\lambda,$$

for all  $k = 0, 1, 2, \dots$ . In particular  $\text{var}(x_t) = \gamma_0 = 2 \int_0^{\pi} f(\lambda) d\lambda$

# Spectrum of a stationary process: Aliasing

$$f(\lambda) = \frac{1}{2\pi} \left( \gamma_0 + 2 \sum_{k=1}^{\infty} \gamma_k \cos(\lambda k) \right), \quad \lambda \in [0, \pi],$$

- We see that the frequencies  $\lambda$ ,  $-\lambda$ ,  $\lambda$  and  $\lambda \pm 2s\pi$ ,  $s = 1, 2, \dots$  have the same values.
- This phenomena is called aliasing.
- One can examine the spectral density function only on the interval  $[0, \pi]$ .

# Spectrum and the cyclic components of a stationary process

Consider a stationary process that has a **cyclic component** with period  $s$ . Then the corresponding spectral density function obtains its maximal values at  $\lambda_s = 2\pi/s$ , **the basic frequency**, and also at **harmonic frequencies**

$$k\lambda_s, \quad k = 1, 2, \dots, \lfloor s/2 \rfloor,$$

where  $\lfloor s/2 \rfloor = \max\{m \in \mathbb{Z} \mid m \leq s/2\}$ .

## Example

If  $s = 4$ , then  $\lambda_4 = \pi/2$  and there is only one harmonic frequency  $\pi$ . If  $s = 12$ , then  $\lambda_{12} = \pi/6$  and the harmonic frequencies are  $2\pi/6$ ,  $3\pi/6$ ,  $4\pi/6$ ,  $5\pi/6$  and  $\pi$ .

## References:

- 1 Brockwell, P., Davis, R. (2009): Time Series – Theory and Methods, Springer
- 2 Hamilton, J. (1994): Time Series Analysis, Princeton University Press

# Next week:

- ① Characteristics of the ARMA models
  - ① Statistical properties of the stationary ARMA models
  - ② ARIMA and SARIMA models
- ② Fitting an ARMA model
  - ① Estimation
  - ② Box-Jenkins method
  - ③ Decomposition of time series