Nguyen Xuan Binh 887799 Theory Exercise Week 5

5.4 Consider the following ARMA processes:

$$x_t - x_{t-1} + x_{t-2} = \epsilon_t + \epsilon_{t-1} - 6\epsilon_{t-2}, \tag{4}$$

$$x_t + \frac{1}{2}x_{t-1} = \epsilon_t + \frac{4}{3}\epsilon_{t-1} + \frac{1}{3}\epsilon_{t-2}, \tag{5}$$

$$x_t - x_{t-1} = \epsilon_t + \frac{1}{2}\epsilon_{t-12}. \tag{6}$$

Which of the processes are stationary? Which of the processes are invertible?

Exercise 5.4

(4) $x_t - x_{t-1} + x_{t-2} = \epsilon_t + \epsilon_{t-1} - 6\epsilon_{t-2}$

The AR polynomial: $1 - L + L^2 = 0$

$\Rightarrow L = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i \Rightarrow |L| = 1$. The process is not stationary

The MA polynomial: $1 + L - 6L^2 = 0$

$\Rightarrow L = \begin{cases} 1/2 \\ -1/3 \end{cases} \Rightarrow |L| = \begin{cases} 1/2 < 1 \\ 1/3 < 1 \end{cases}$. The process is not invertible

(5) $x_t + \frac{1}{2}x_{t-1} = \epsilon_t + \frac{4}{3}\epsilon_{t-1} + \frac{1}{3}\epsilon_{t-2}$

The AR polynomial: $1 + \frac{1}{2}L = 0 \Rightarrow |L| = 2$. The process is stationary

The MA polynomial: $1 + \frac{4}{3}L + \frac{1}{3}L^2 = 0 \Rightarrow L = \begin{cases} -1 \\ -3 \end{cases} \Rightarrow |L| = \begin{cases} 1 \leq 1 \\ 3 > 1 \end{cases}$

$\Rightarrow$ The process is not invertible

(6) $x_t - x_{t-1} = \epsilon_t + \frac{1}{2}\epsilon_{t-12}$

The AR polynomial: $1 - L = 0 \Rightarrow L = 1 \Rightarrow |L| = 1$. The process is not stationary

The MA polynomial: $1 + \frac{1}{2}L^{12} = 0 \Rightarrow L = \sqrt[12]{-2}$

$\Rightarrow L = 1.02336 + 0.27422 i \Rightarrow |L| = 1.0594$ The process is ~~not~~ invertible

Stationary processes are (5), invertible processes are (6)

5.5 Derive the optimal s-step prediction for the invertible MA(q) process,

$$x_t = \sum_{i=0}^{q} \theta_i L^i \varepsilon_t, \quad \varepsilon_t \sim \text{iid}(0, \sigma^2),$$
$$\theta_0 = 1,$$

in the sense of mean squared error, when the process ε_t has been observed up to point of time t.

Exercise 5.5

- 1 - step prediction

$$\hat{x}_{t+1} = E(x_{t+1} | \epsilon_t, \epsilon_{t-1}, \dots) = E(\epsilon_{t+1} + \theta_1 \epsilon_t + \theta_2 \epsilon_{t-1} + \dots \theta_q \epsilon_{t-q+1} | \epsilon_t, \epsilon_{t-1}, \dots)$$
$$= \theta_1 \epsilon_t + \dots + \theta_q \epsilon_{t-q+1}$$
$$\Rightarrow x_{t+1} - \hat{x}_{t+1} = \epsilon_{t+1}$$

- 2 - step prediction

$$\hat{x}_{t+2} = E(x_{t+2} | \epsilon_t, \epsilon_{t-1}, \dots) = E(\epsilon_{t+2} + \theta_1 \epsilon_{t+1} + \dots + \theta_q \epsilon_{t-q+2} | \epsilon_t, \dots)$$
$$= \theta_1 \epsilon_{t+1} + \dots + \theta_q \epsilon_{t-q+2}$$
$$\Rightarrow x_{t+2} - \hat{x}_{t+2} = \epsilon_{t+2}$$
$$\Rightarrow x_{t+s} - \hat{x}_{t+s} = \epsilon_{t+s}$$

For s-step $\leq q$ forecast, $\hat{x}_{t+s}$ of MA (q) is

$$\hat{x}_{t+s} = x_{t-s} - \epsilon_{t+s} = \theta_1 \epsilon_{t+s-1} + \theta_2 \epsilon_{t+s-2} + \dots + \theta_q \epsilon_{t+s-q} \quad \text{(answer)}$$

However for s-step $> q$, the forecast is different. The white noise is individually and independently distribute

$$\left.\begin{array}{l} \text{We have } t+s > t-q+s \\ \text{Also: } s > q \Rightarrow t-q+s > t \end{array}\right\} \Rightarrow t+s > t-q+s > t$$

In other words, $\epsilon_t, \epsilon_{t-1}, \dots \epsilon_{t-q}$ are not contained in $\epsilon_{t+s}, \dots \epsilon_{t-q+s}$

$\Rightarrow \hat{x}_{t+s, s>q} = 0 \Rightarrow$ MA (q) process cannot forecast more than s steps ahead

Optimal forecast of MA (q): $\hat{x}_{t+s} = \begin{cases} \theta_1 \epsilon_{t+s-1} + \theta_2 \epsilon_{t+s-2} + \dots + \theta_q \epsilon_{t+s-q}, & s \leq q \\ 0, & s > q \end{cases}$