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Exercise 3.3 : Show that MA (1) process : $x_t = \varepsilon_t + \theta \varepsilon_{t-1}$ where $\varepsilon_t \sim i.i.d(0, \sigma^2)$ is always stationary

(i.i.d : independent and identically distributed random variables)

Since $(\varepsilon_t)_{t\in T} \sim i.i.d(0, \sigma^2) \Rightarrow \begin{cases} E(\varepsilon_t) = 0, & t \in T \\ Var(\varepsilon_t) = \sigma^2, & t \in T \\ Cov(\varepsilon_t, \varepsilon_s) = 0, & t \neq s,\ t, s \in T \end{cases}$

A stochastic process x_t is weakly stationary when the following conditions are true

(i) $E(x_t) = \mu$ (ii) $Var(x_t) = \sigma^2$ (iii) $Cov(x_t, x_{t-\tau}) = \gamma_\tau\ \forall t, \tau \in T$

Let's check the conditions : $x_t = \varepsilon_t + \theta \varepsilon_{t-1}$ as in MA (1)

(i) $E(x_t) = E[\varepsilon_t + \theta \varepsilon_{t-1}] = E[\varepsilon_t] + \theta E[\varepsilon_{t-1}] = 0 + \theta \cdot 0 = 0$

$\Rightarrow E(x_t) = 0$ independent of t

(ii) $Var(x_t) = E[x_t^2] - (E[x_t])^2$. We prove above that $E[x_t] = 0$

$= E[x_t^2] = E[(\varepsilon_t + \theta \varepsilon_{t-1})(\varepsilon_t + \theta \varepsilon_{t-1})]$

$= E[\varepsilon_t^2 + \theta^2 \varepsilon_{t-1}^2 + 2\theta \varepsilon_t \varepsilon_{t-1}]$

$= E[\varepsilon_t^2] + \theta^2 E[\varepsilon_{t-1}^2] + 2\theta E[\varepsilon_t] E[\varepsilon_{t-1}]$

We have : $Var(\varepsilon_t) = 0 \Rightarrow E[\varepsilon_t^2] - (E[\varepsilon_t]^2) = \sigma^2 \Rightarrow E[\varepsilon_t^2] = \sigma^2$

$\Rightarrow Var(x_t) = \sigma^2 + \theta^2 \sigma^2 + 2\theta \cdot 0 \cdot 0 = \sigma^2(\theta + 1)$

$\Rightarrow Var(x_t)$ is finite and independent of t, as θ is non-random

(iii) $Cov(x_t, x_{t-\tau}) = E[(x_t - E[x_t])(x_{t-\tau} - E[x_{t-\tau}])$

$= E[x_t x_{t-\tau}] = E((\varepsilon_t + \theta \varepsilon_{t-1})(\varepsilon_{t-\tau} + \theta \varepsilon_{t-\tau-1}))$

$= E(\varepsilon_t \varepsilon_{t-\tau} + \theta \varepsilon_t \varepsilon_{t-\tau-1} + \theta \varepsilon_{t-1} \varepsilon_{t-\tau} + \theta^2 \varepsilon_{t-1} \varepsilon_{t-\tau-1})$

$\Rightarrow Cov(x_t, x_{t-\tau}) = E(\varepsilon_t \varepsilon_{t-\tau}) + \theta E(\varepsilon_t \varepsilon_{t-\tau-1}) + \theta E(\varepsilon_{t-1} \varepsilon_{t-\tau}) + \theta^2 E(\varepsilon_{t-1} \varepsilon_{t-\tau-1})$

Analysis: τ must not be $0 \Rightarrow \varepsilon_t \neq \varepsilon_{t-\tau}$ and $\varepsilon_{t-1} \neq \varepsilon_{t-\tau-1}$

$\Rightarrow Cov(x_t, x_{t-\tau}) = \theta E(\varepsilon_t \varepsilon_{t-\tau-1}) + \theta E(\varepsilon_{t-1} \varepsilon_{t-\tau})$

If $\tau = 1 \Rightarrow Cov(x_t, x_{t-\tau}) = \theta E(\varepsilon_t \varepsilon_{t-2}) + \theta E(\varepsilon_{t-1} \varepsilon_{t-1})$

$= 0 + \theta \cdot \sigma^2 = \theta \sigma^2$

If $\tau = -1 \Rightarrow Cov(x_t, x_{t-\tau}) = \theta E(\varepsilon_t \varepsilon_t) + 0 E(\varepsilon_{t-1} \varepsilon_{t+1})$

$= \theta \cdot \sigma^2 + 0 = \theta \sigma^2$

For others $\tau \in T$: $\varepsilon_t \neq \varepsilon_{t-\tau-1}$ and $\varepsilon_{t-1} \neq \varepsilon_{t-\tau} \Rightarrow Cov(x_t, x_{t-\tau}) = 0$

Conclusion : autocovariance of MA(1) is a function of τ, where

$$Cov(x_t, x_{t-\tau}) = \begin{cases} \theta\sigma^2, & \tau \in \{1; -1\} \\ 0, & \text{other } \tau \in T \end{cases}$$

$\Rightarrow$ From (i), (ii) and (iii) $\Rightarrow$ MA (1) is always (weakly) stationary

Exercise 3.4

a) Derive the autocorrelation function of MA (1) process

The k^{th} autocovariance γ_k of $(x_t)_{t\in T}$ is $\gamma_k = cov(x_t, x_{t-k})$ and autocovariance function of stationary stochastic process is $\gamma(k) = \gamma_k$ for all $k \in Z$

~~Find an~~ Autocorrelation function is $\rho: Z \to [-1, 1]$; $\rho(k) = \rho_k = \frac{\gamma_k}{\gamma_0}, k \in Z$

Find autocorrelation function of MA (1) process

From exercise 3.3, we know that $\gamma_0 = Var(x_t) = \delta^2(\theta + 1)$

and $\gamma_1 = \gamma_{-1} = Cov(x_t, x_{t-1}) = Cov(x_t, x_{t+1}) = \theta\delta^2$

$\gamma_{k \neq 1,-1,0} = 0$

$\Rightarrow$ Autocorrelation function of MA (1): $x_t = \varepsilon_t + \theta\varepsilon_{t-1}$

$$\rho(k) = \begin{cases} k=0, \rho(0) = \gamma_0/\gamma_0 = 1 \\ k = 1, -1, \rho(1) = \rho(-1) = \gamma_1/\gamma_0 = \dfrac{\theta}{\theta^2+1} \\ k \neq 0, 1, -1, \rho(k) = \gamma_k/\gamma_0 = 0 \end{cases}$$

b) Derive the autocorrelation function of stationary AR(1) process:

$$x_t = \phi x_{t-1} + \varepsilon_t \quad (\varepsilon_t)_{t\in Z} \sim i.i.d\,(0, \delta^2)$$

Use MA (∞) representation $x_t = \Psi(L)\varepsilon_t = \sum_{i=0}^{\infty} \phi^i L^i \varepsilon_t$

where series $\sum_{i=0}^{\infty} \phi^i$ converges absolutely

We have: $x_t = \phi x_{t-1} + \varepsilon_t$

$= \phi(\phi x_{t-2} + \varepsilon_{t-1}) + \varepsilon_t$

$= \phi(\phi(\phi x_{t-3} + \varepsilon_{t-2}) + \varepsilon_{t-1}) + \varepsilon_t$

This process continues indefinitely

$\Rightarrow AR(1) = \phi^k x_{t-k} + \sum_{i=0}^{k-1} \phi^i \varepsilon_{t-i}$. From exercise 3.2, we see that if AR (1) is stationary, it implies $|\phi| < 1$

$\Rightarrow \lim_{k\to\infty} (\phi^k x_{t-k}) = 0 \Rightarrow AR(1): x_t = \sum_{i=0}^{\infty} \phi^i \varepsilon_{t-i} = \sum_{i=0}^{\infty} \phi^i L^i \varepsilon_t$

Expected value: $E[x_t] = E[\sum_{i=0}^{\infty} \phi^i \varepsilon_{t-i}]$. Since $\sum_{i=0}^{\infty} \phi^i$ converges absolutely

$$\Rightarrow E[x_t] = \sum_{i=0}^{\infty} \phi^i E[\varepsilon_{t-i}] = \sum_{i=0}^{\infty} \phi^i \cdot 0 = 0$$

We will use recursion to find autocorrelation function of AR (1)

We have: $x_t = \phi x_{t-1} + \varepsilon_t \Rightarrow x_{t-k} x_t = \phi x_{t-k} x_{t-1} + x_{t-k} \varepsilon_t$

Also: $Cov(x_{t-k} x_t) = E[x_{t-k} x_t] - \underbrace{E[x_{t-k}]}_{0} \underbrace{E[x_t]}_{0}$

$\Rightarrow Cov(x_{t-k} x_t) = E[x_{t-k} x_t] = E[\phi x_{t-k} x_{t-1}] + E[x_{t-k} \varepsilon_t]$

We have: $E[x_{t-k}\varepsilon_t] = E[\varepsilon_t]E[x_{t-k}] = 0$

$\Rightarrow Cov(x_{t-k}x_t) = E[\phi x_{t-k}x_{t-1}] = \phi E[x_{t-k}x_{t-1}]$

$\Rightarrow \quad \gamma_k = \phi\gamma_{k-1}$

This works recursively $\Rightarrow \gamma_k = \phi(\phi\gamma_{k-2}) = \phi^2\gamma_{k-2} = \phi^i\gamma_{k-i}$

Let $i = k \Rightarrow \gamma_k = \phi^k\gamma_0$

$\Rightarrow$ Auto correlation function of AR(1):

$$\rho(k) = \begin{cases} k=0: \rho(0) = \gamma_0/\gamma_0 = 1 \\ k\neq 0: \rho(k) = \gamma_k/\gamma_0 = \dfrac{\phi^k\gamma_0}{\gamma_0} = \phi^k \end{cases}$$

(answer)