Graded Exercise 7

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Problem 1

a.

We have, the Primal problem can be represent in the matrix form as minimizing $w^T x$, subject to $Ax \geq b$. In the context of this problem:

- x is the variable vectors: $x = \begin{pmatrix} x_1 \\ x_2 \\ ... \\ x_m \end{pmatrix}$
- w is the weight vector with dimension m: $w = \begin{pmatrix} w_1 \\ w_2 \\ ... \\ w_m \end{pmatrix}$
- A is a $n \times m$ matrix, where every entry a_{ij} is 1 if the set S_j contains the element e_i, and zero otherwise: $A = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1m} \\ a_{21} & a_{22} & ... & a_{2m} \\ ... \\ a_{n1} & a_{n2} & ... & a_{nm} \end{pmatrix}$
- b is a $n \times 1$ vector, each entry of b is 1: $b = \begin{pmatrix} 1 \\ 1 \\ ... \\ 1 \end{pmatrix}$

$\Rightarrow$ The primal problem in matrix form is:

$$\min_x \quad \begin{pmatrix} w_1 \\ w_2 \\ ... \\ w_m \end{pmatrix}^T \begin{pmatrix} x_1 \\ x_2 \\ ... \\ x_m \end{pmatrix}$$

$$\text{s.t} \quad \begin{pmatrix} a_{11} & a_{12} & ... & a_{1m} \\ a_{21} & a_{22} & ... & a_{2m} \\ ... \\ a_{n1} & a_{n2} & ... & a_{nm} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ ... \\ x_m \end{pmatrix} \geq \begin{pmatrix} 1 \\ 1 \\ ... \\ 1 \end{pmatrix}$$

We can derive the dual: maxmizing $b^T y$, subject to $A^T y \leq w$:

$$\max_y \quad \begin{pmatrix} 1 \\ 1 \\ ... \\ 1 \end{pmatrix}^T \begin{pmatrix} y_1 \\ y_2 \\ ... \\ y_n \end{pmatrix}$$

$$\text{s.t} \quad \begin{pmatrix} a_{11} & a_{21} & ... & a_{m1} \\ a_{12} & a_{22} & ... & a_{m2} \\ ... \\ a_{1n} & a_{2n} & ... & a_{mn} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ ... \\ y_n \end{pmatrix} \leq \begin{pmatrix} w_1 \\ w_2 \\ ... \\ w_m \end{pmatrix}$$

We can express the matrix form to the sum form:

$$\max_y \quad \sum_{i=1}^{n} y_i$$

$$\text{s.t} \quad \sum_{i:e_i \in S_j} y_i \leq w_j, \quad j = 1, 2, ..., m$$

$\Rightarrow$ The derived dual matches the given dual.

b.

We have:

$$c(I) = \sum_{j \in I} w_j$$

From the algorithm, set j is added if and only if $\sum_{i:e_i \in S_j} y_i = w_j$. Thus:

$$\begin{aligned}\sum_{j \in I} w_j &= \sum_{j \in I} \sum_{i:e_i \in S_j} y_i \\ &\leq \sum_{j=1}^{m} \sum_{i:e_i \in S_j} y_i\end{aligned}$$

Now, if we denote $f_i = |j : e_i \in S_j|$. By definition, $f = \max_i |f_i|$, we get that:

$$\begin{aligned}\sum_{j=1}^{m} \sum_{i:e_i \in S_j} y_i &= \sum_{i=1}^{n} f_i y_i \\ &\leq \sum_{i=1}^{n} f y_i = f \sum_{i=1}^{n} y_i\end{aligned}$$

$\implies c(I) \leq f \sum_{i=1}^{n} y_i$

c.

Consider for an abitrary y_i. The given algorithm only increases y_i until the constraint is tight, and y start with the value 0. Thus y_i is always non negative. Also, for each i, y_i is increased once when e_i is checked, and stop immediately if the constraint is tight. So the dual constraint is preserved.

$\Rightarrow y_i$ is a feasible solution throughtout the algorithm, and we can apply the weak duality theorem.

Let x^* be the optimal solution for the primal problem, and denote the optimal cost for the primal problem as OPT_{LP}. By the weak duality theorem, we have that:

$$OPT_{LP} = \sum_{j=1}^{m} w_j x_j^* \geq \sum_{i=1}^{n} b_i y_i = \sum_{i=1}^{n} y_i \quad \text{(Since } b_i = 1\text{)}$$

From the previous part, we have shown that:

$$\begin{aligned} c(I) &\leq f \sum_{i=1}^{n} y_i \\ &\leq f \sum_{j=1}^{m} w_j x_j^* = f \cdot OPT_{LP} \end{aligned}$$

Furthermore, the optimum cost of the interger problem is as most as good as the optimum cost for the relaxed problem, i.e., $OPT_{LP} \leq OPT$ (as in the lecture note)

$\Longrightarrow c(I) \leq f \cdot OPT_{LP} \leq f \cdot OPT$

Problem 2

a.

Consider an abitrary element $e_i \in U$, and denote $f_i = |j : e_i \in S_j|$. Thus by definition, $f = |\max_i f_i|$. Since x^* is the set of optimal solution for the primal problem, we have:

$$\sum_{j:e_i \in S_j} x_j^* \geq 1$$

In order for the above constraint to be valid, at least one of the x_j^* must have value at least $\frac{1}{f_i}$. Denote that set as x_i^*, we get:

$$x_i^* \geq \frac{1}{f_i} \geq \frac{1}{f} \quad (\text{since } f = |\max_i f_i|)$$

$\Rightarrow$ For each element, there will be at least one set contains that element and have value at least $\frac{1}{f}$. And that set will be included into I.

$\Longrightarrow$ The algorithm returns a cover I.

b.

Consider an abitrary set $x_j^* \in x^*$ that has value $1/f_j \geq 1/f$. So x_j^* will be rounded to 1 and add to I. This means that the value and thus the cost of adding x_j^* increased by a factor of $f_j \leq f$.

$\Rightarrow$ Each chosen set will be increased in cost by at most a factor of f.

Consider also: in the worst case, all sets from the fractional solution will be chosen and increased in the cost by f times. Thus, c(I) will be at most $f \cdot OPT_{LP}$, with OTP_{LP} denotes the optimal cost for the fractional problem. Also, $OPT_{LP} \leq OPT$ (As in the lecture note).

$\Rightarrow c(I) \leq f \cdot OPT$