CS-E3190 Principles of Algorithmic Techniques

07. Linear Programming – Tutorial Exercise

For both of these exercises, you are allowed to check definitions on Wikipedia if need be.

1. **A closer look at Duality**. Recall the minimum vertex cover problem and its linear programming relaxation. In the original primal problem we require integer solutions, that is $x_i \in \{0,1\}$; here we relax these constraints and allow continuous values $x_i \geq 0$. The primal of the relaxed problem, and its corresponding dual are printed below.

Primal

$$\begin{aligned} \min_x \quad & \sum_{i \in V} w_i x_i \\ \text{s.t.} \quad & x_i + x_j \geq 1, \quad \forall (i,j) \in E \\ & x_i \geq 0, \quad \forall i \in V \end{aligned}$$

Dual

$$\begin{aligned} \max_y \quad & \sum_{(i,j) \in E} y_{ij} \\ \text{s.t} \quad & \sum_{j \in N(i)} y_{ij} \leq w_i, \quad \forall i \in V \\ & y_{ij} \geq 0, \quad \forall (i,j) \in E \end{aligned}$$

Here $G = (V, E)$ is a given graph and $(w_i)_{i \in V}$ positive weights. The set $N(i)$ denotes the neighbouring nodes of i in G. The variable x_i indicates selection of node $i \in V$.

The goal of this exercise is to derive the dual from the primal by viewing the dual as the problem of finding *largest lower bound* of the primal. Throughout, let x^* denote optimal primal LP solution, and $OPT_{LP} = \sum_{i \in V} w_i x_i^*$ its corresponding optimal value.

(a) Consider a relaxation of the primal LP. Suppose that we allow each primal constraint $x_i + x_j \geq 1$ to be violated at the cost of a linear penalty $y_{ij} \geq 0$. Each constraint $(i,j) \in E$ has a corresponding penalty y_{ij}. All penalties are assumed to be non-negative throughout. For any fixed penalties $y = (y_{ij})_{(i,j) \in E}$ we can formulate a new problem that jointly minimizes the original objective and the sum of penalty terms. This is a *Lagrangian relaxation* of the primal.

$$f(y) = \min_{x \geq 0} \left\{ \sum_{i \in V} w_i x_i + \sum_{(i,j) \in E} y_{ij}(1 - x_i - x_j) \right\}$$

Notation: we maintain the non-negativity constraints on $x = (x_i)_{i \in V}$.
Show that the value of the relaxed problem $f(y)$ is a lower bound for the primal problem for any non-negative penalties, i.e. that $f(y) \leq OPT_{LP}$ for all $y \geq 0$.

Solution. Fix some arbitrary non-negative penalties $y = (y_{ij})_{(i,j) \in E}$. We can plug in x^* (because we know it is non-negative) to get

$$\begin{aligned} f(y) = \min_{x \geq 0} & \left\{ \sum_{i \in V} w_i x_i + \sum_{(i,j) \in E} y_{ij}(1 - x_i - x_j) \right\} \\ \leq \quad & \sum_{i \in V} w_i x_i^* + \sum_{(i,j) \in E} y_{ij}(1 - x_i^* - x_j^*) && \text{(plug in } x^*) \\ \leq \quad & \sum_{i \in V} w_i x_i^* = OPT_{LP} && \text{(primal feasible)} \end{aligned}$$

The last inequality follows because x_i^* are primal feasible and $y_{ij} \geq 0$ so

$$x_i^* + x_j^* \geq 1 \implies y_{ij}(1 - x_i^* - x_j^*) \leq 0 \text{ for all edges } (i,j) \in E.$$

(b) Part (a) proves that we get a lower bound $f(y) \leq OPT_{LP}$ for each set of penalties y. However, not all lower bounds are equal. Can we find the tightest lower bound? This can be formulated as a maximization problem in which we want to find the penalties y that maximize the value $f(y)$. This gives the program:

$$\begin{aligned} &\max_y f(y) \\ &\text{s.t. } y_{ij} \geq 0 \qquad \forall (i,j) \in E \end{aligned}$$

The next few sub-problems have us show that this is equivalent to the dual.
First, prove that the above objective function $f(y)$ is equal to:

$$\sum_{(i,j)\in E} y_{ij} + \min_{x\geq 0}\Big\{\sum_{i\in V} x_i\Big(w_i - \sum_{j\in N(i)} y_{ij}\Big)\Big\}.$$

Solution. Re-write and simplify the minimization problem as

$$\begin{aligned} f(y) = &\qquad \min_{x\geq 0}\Big\{\sum_{i\in V} w_i x_i + \sum_{(i,j)\in E} y_{ij}(1 - x_i - x_j)\Big\} \\ &= \sum_{(i,j)\in E} y_{ij} + \min_{x\geq 0}\Big\{\sum_{i\in V} w_i x_i + \sum_{(i,j)\in E} y_{ij}(-x_i - x_j)\Big\} \\ &= \sum_{(i,j)\in E} y_{ij} + \min_{x\geq 0}\Big\{\sum_{i\in V} x_i\Big(w_i - \sum_{j\in N(i)} y_{ij}\Big)\Big\} \end{aligned}$$

The last equality follows from careful regrouping of the x_i variables. Note that a given variable x_i appears exactly once in the sum $\sum_{i\in V} w_i x_i$ and one additional time for *every neighbor* $j \in N(i)$ in the second sum.

(c) Consider the minimization term $\min_{x\geq 0}\Big\{\sum_{i\in V} x_i\Big(w_i - \sum_{j\in N(i)} y_{ij}\Big)\Big\}$. Prove that this is bounded ($> -\infty$) if and only if $\sum_{j\in N(i)} y_{ij} \leq w_i$ for all $i \in V$.
(Note that $-\infty$ can be considered as only a notation, meaning that we can make something arbitrary small. Here we want to show that, under certain conditions this cannot happen to our minimization term.)

Solution. Consider the minimization term $\min_{x\geq 0}\Big\{\sum_{i\in V} x_i\Big(w_i - \sum_{j\in N(i)} y_{ij}\Big)\Big\}$.

Fix one node $i \in V$ and consider its contribution $\Big(w_i - \sum_{j\in N(i)} y_{ij}\Big)$ to the sum.

If we have $\Big(w_i - \sum_{j\in N(i)} y_{ij}\Big) \geq 0$, the corresponding x_i can without loss of generality be made 0.

On the other hand if $\Big(w_i - \sum_{j\in N(i)} y_{ij}\Big) < 0$, the minimization term is unbounded. The corresponding x_i will be increased indefinitely, resulting in $-\infty$ in the objective. Indeed, the only condition on x_i is positivity, hence it can be made arbitrarily large. This would imply that the corresponding term $\Big(w_i - \sum_{j\in N(i)} y_{ij}\Big)$ could be made arbitrarily small i.e. smaller than any given negative number.
It follows that the objective is bounded if and only if all terms are non-negative:

$$\begin{aligned} &\min_{x\geq 0}\Big\{\sum_{i\in V} x_i\Big(w_i - \sum_{j\in N(i)} y_{ij}\Big)\Big\} > -\infty \\ &\iff w_i - \sum_{j\in N(i)} y_{ij} \geq 0 \quad \forall i \in V \end{aligned}$$

$$\iff \qquad \sum_{j \in N(i)} y_{ij} \leq w_i \quad \forall i \in V$$

(d) Use (b) and (c) to show that the program for the tightest lower bound

$$\max_y f(y)$$
$$\text{s.t. } y_{ij} \geq 0 \qquad \forall (i,j) \in E$$

is equivalent to the dual linear program given above in the problem.
Hint: Use the convention that $\max_z\{g(z) \text{ s.t. } z \in S\} = -\infty$ if $S = \emptyset$.

Solution. Consider what we did so far.

$$\begin{aligned}
&\max_{y \geq 0}\{f(y)\} \\
&= \max_{y \geq 0}\Big\{\sum_{(i,j)\in E} y_{ij} + \min_{x \geq 0}\Big\{\sum_{i \in V} x_i\Big(w_i - \sum_{j \in N(i)} y_{ij}\Big)\Big\}\Big\} && \text{(part (b))} \\
&= \max_{y \geq 0}\Big\{\sum_{(i,j)\in E} y_{ij} \ \text{ s.t. } \sum_{j \in N(i)} y_{ij} \leq w_i \quad \forall i \in V\Big\} && \text{(part (c))}
\end{aligned}$$

The second equality follows from part (c) and the hint; both expressions are unbounded below (equal to $-\infty$) if and only if some term i is negative.
Writing out each part on its own line yields the dual:

$$\begin{aligned}
\max_{y \geq 0} \ & \sum_{(i,j)\in E} y_{ij} \\
\text{s.t. } & \sum_{j \in N(i)} y_{ij} \leq w_i \quad \forall i \in V \\
& y_{ij} \geq 0 \quad \forall (i,j) \in E
\end{aligned}$$

(e) In this question, we will take interest in the **weak duality theorem**:
For a primal problem (Maximize $c^T x$ subject to $Ax \leq b$ and $x \geq 0$) and its dual problem (Minimize $b^T y$ subject to $A^T y \geq c$ and $y \geq 0$), $c^T x \leq b^T y$.
Let (x, y) be an arbitrary pair of feasible primal and dual solutions. In our case, the weak duality theorem states that the solution pair satisfies:

$$\sum_{(i,j)\in E} y_{ij} \leq \sum_{i \in V} w_i x_i.$$

This should not surprise us given our construction. With reference to part (a), explain *briefly* why weak duality must hold in our vertex cover setting.

Solution. Part (a) proves that for *any* (non-negative) penalties y

$$\sum_{(i,j)\in E} y_{ij} = f(y) \leq OPT_{LP}.$$

Finally by definition all feasible x satisfy $OPT_{LP} \leq \sum_{i \in V} w_i x_i$.

2. **A Primal-Dual algorithm**. Recall the Primal-Dual algorithm for vertex cover. The primal and dual programs referred to here are the same as those in Part 1.

Algorithm 1: Primal-Dual Vertex Cover Algorithm

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input : Graph $G = (V, E)$, positive weights $(w_i)_{i \in V}$
$y \leftarrow 0$;
$C \leftarrow \emptyset$;
while *some edge* $\{i, j\}$ *not covered by* C **do**
  increase y_{ij} until some dual constraint $k \in V$ is tight;
  update $C = C \cup \{k\}$;
end
output: Vertex set $C \subset V$

Note that we allow 0-increases which may occur after multiple constraints simultaneously become tight. Whichever newly tight constraint k is chosen is arbitrary.

Our task is to repeat the 2-approximation analysis of the algorithm from the lecture. The overall strategy is to exploit *weak duality* and the variables y. For the theorem to apply, we need to argue that y is a feasible dual solution. Thereafter we only need one lemma before getting to the punchline.

(a) Argue that y is a dual feasible solution throughout the run of the algorithm.

Solution. Initially $y_{ij} = 0$ for all $\{i, j\} \in E$. This is feasible because $y_{ij} \geq 0$ and all $w_i > 0$. Thereafter the y_{ij} are only ever increased, so non-negativity is never violated.

For a fixed i the dual constraint $\sum_{j \in N(i)} y_{ij} \leq w_i$ holds throughout as well. If the constraint becomes tight while increasing some y_{ij}, the increasing immediately stops. Then i is added to C so all edges $\{i, k\}$ with $k \in N(i)$ are covered and thus will not have their y_{ik}-variables increased.

(b) Prove that $\sum_{i \in C} w_i \leq 2 \sum_{\{i,j\} \in E} y_{ij}$. *Hint:* Given $i \in C$, what is know about y_{ij}?

Solution. We know that $i \in C$ if and only if $\sum_{j \in N(i)} y_{ij} = w_i$.
We can express the sum of weights over C as

$$\begin{aligned}\sum_{i \in C} w_i &= \sum_{i \in C} \sum_{j \in N(i)} y_{ij} \\ &\leq \sum_{i \in V} \sum_{j \in N(i)} y_{ij} \\ &= 2 \sum_{\{i,j\} \in E} y_{ij}.\end{aligned}$$

The inequality holds because $C \subseteq V$ and $y_{ij} \geq 0$. The last equality follows from the Handshaking lemma; each edge is adjacent to exactly two nodes.

(c) Use weak duality to prove that C is a 2-approximation.
Hint: Why did we go through the trouble of part (a).

Solution. We showed in (a) that y is a feasible dual solution. Let x^* be an optimal (integer) solution. This is feasible by definition. Because x^* and y are feasible, the weak duality theorem implies that

$$\sum_{\{i,j\} \in E} y_{ij} \leq \sum_{i \in V} w_i x_i^*.$$

Putting the weak duality inequality and that of part (b) together gives

$$\begin{aligned}
\sum_{i \in C} w_i &\leq 2 \sum_{\{i,j\} \in E} y_{ij} && \text{(part (b))} \\
&\leq 2 \sum_{i \in V} w_i x_i^* && \text{(weak duality)}
\end{aligned}$$

This proves that the Primal-Dual algorithm is a 2-approximation algorithm.

Typically we are also interested in proving claims about the runtime. This is intentionally not covered here as your programming assignment will encourage you to think about this further (should you choose to use the Primal-Dual algorithm).