CS-E3190 Principles of Algorithmic Techniques

08. Hardness – Graded Exercise

Please read the following **rules** very carefully.

- Do not consciously search for the solution on the internet.
- You are allowed to discuss the problems with your classmates but you should **write the solutions yourself**.
- Be aware that **if plagiarism is suspected**, you could be asked to have an interview with teaching staff.
- The teaching staff can assist with understanding the problem statements, but will **not be giving any hints** on how to solve the exercises.
- In order to ease grading, we want the solution of each problem and subproblem to start on a **new page**. If this requirement is not met, **points will be deduced**.

1. **NP-completeness of the k-clique problem.**

   Definitions:

   - Given a graph G, $Clique(G)$ is the size of the biggest clique in G.
   - **k-Clique problem**: Let G be a graph and k be an integer. Does $Clique(G) \geq k$?
   - **3SAT problem**: Given n variables $x_i, \forall i \in [n]$, and a Boolean formula in conjunctive normal form with m clauses, each of size exactly 3, decide if there exists an assignment $x \in \{0,1\}^n$ such that the Boolean formula evaluates as true (outputs 1).

   Knowing that the 3SAT problem is NP-complete, our goal is to prove that the k-Clique problem is also NP complete.

   Let us define an embedding f from 3SAT to k-Clique. Given a 3SAT formula F with m 3-clauses, we build the following couple (G, k):

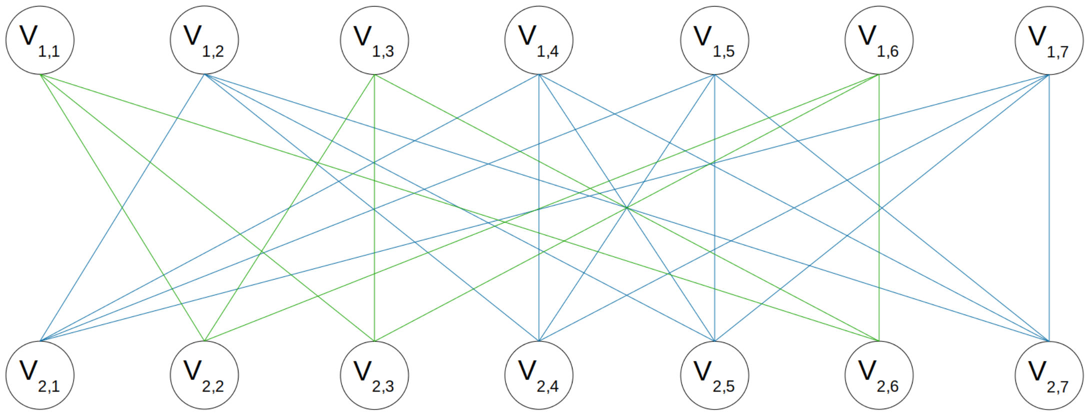
   a. For each 3-clause, list the vectors that output true (giving only values for the 3 variables in the clause and leaving the other ones as undetermined). The nodes of the graph correspond to those vectors.
   b. Two vectors are said to be compatible if they have the same values (or at least one is undetermined) for the same variables. There is an edge between two nodes iff they correspond to compatible vectors.
   c. The integer k is m, the number of clauses in F.

   Example: Consider $F(x) = (x_1 \vee x_2 \vee \overline{x_3}) \wedge (x_2 \vee \overline{x_4} \vee x_5)$. The following table gives the vectors (the seven first vectors correspond to the fist clause and the seven last correspond to the second clause. Note that each of these vectors returns true for the clause it corresponds to.):

vectors	$v_{1,1}$	$v_{1,2}$	$v_{1,3}$	$v_{1,4}$	$v_{1,5}$	$v_{1,6}$	$v_{1,7}$	$v_{2,1}$	$v_{2,2}$	$v_{2,3}$	$v_{2,4}$	$v_{2,5}$	$v_{2,6}$	$v_{2,7}$
x_1	1	0	0	1	0	1	1	–	–	–	–	–	–	–
x_2	0	1	0	1	1	0	1	1	0	0	1	1	0	1
x_3	1	1	1	1	0	0	0	–	–	–	–	–	–	–
x_4	–	–	–	–	–	–	–	1	0	1	0	1	0	0
x_5	–	–	–	–	–	–	–	0	0	1	0	1	1	1

The table above contains the vectors (vertically) on the variables (given in lines). For example, to construct the first vector, we consider the first clause, and we pick an assignment for the three variables that would output true. Then for the following vectors corresponding to clause one, we consider all other possible assignments of the three variables in clause one that would output true for the clause. We proceed in the same way for the second clause.

Here $k = m = 2$ since there are 2 clauses in F, and the constructed graph will have 14 nodes. Hence the resulting graph is the following (the edges in blue correspond to compatibility relations where $x_2 = 1$, and the edges in green correspond to $x_2 = 0$):



(a) Prove that the problem k-Clique is in NP.

(b) Show that the given embedding's time complexity is polynomial. Give an O complexity w.r.t. the number m of 3-clauses and the number n of variables for steps (a.) and (b.) to get full points.

(c) Show that the given embedding is a Karp reduction from 3SAT to k-Clique.

(d) Complete the proof of NP-completeness of k-Clique.