CS-E3190 Principles of Algorithmic Techniques

07. Randomized Algorithms – Graded Exercise

Please read the following **rules** very carefully.

- Do not consciously search for the solution on the internet.
- You are allowed to discuss the problems with your classmates but you should **write the solutions yourself**.
- Be aware that **if plagiarism is suspected**, you could be asked to have an interview with teaching staff.
- Each week the second exercise is an **individual exercise**, and the teaching staff will not give hints or help with them. You are allowed to ask for hints for the first exercise.
- In order to ease grading, we want the solution of each problem and subproblem to start on a **new page**. If this requirement is not met, **points will be reduced**.

1. (5p.) Let us consider the maximum cut problem. Given a graph $G = (V, E)$ with n nodes and m edges, we want to partition the nodes in two subsets A and B such that the number of edges in $A \times B$ (ie. the number of edges with an endpoint in each set) is maximized. The partition $V = A \cup B$ is called a *cut* of G and the number of edges with endpoints in each set is referred to as the number of edges in the cut.

   We use the following randomised algorithm:

   - Each node $u \in V$ picks a value in $\{0, 1\}$ uniformly at random,
   - If u picked 0 it joins set A, otherwise it joins set B.

   We want to analyse how good a cut this simple algorithm provides us.

   (a) (1p.) Prove that the probability that an edge i is in the cut is $\frac{1}{2}$.

   (b) (1p.) Let X_i be the random variables defined for each edge $i \in [m]$ s.t. $X_i = 1$ if the edge i has one endpoint in A and one endpoint in B, otherwise $X_i = 0$. Let X be the random variable giving the number of edges in the cut $A \cup B$. Prove that the expected number of edges in the cut is at least $m/2$, ie $\mathrm{E}[X] \geq \frac{m}{2}$.
   hint: write X using X_i, $\forall i \in [m]$.

   (c) (1p.) Show that the expected output of this algorithm is a 2-approximation for the maximum cut.

   (d) (1p.) Use Markov's inequality to show that $\mathrm{P}\left(X \leq m\left(\frac{1}{2} - \varepsilon\right)\right) \leq 1 - \varepsilon$, where $0 \leq \varepsilon < \frac{1}{2}$.
   Hint: Use the indicator random variable Y_i for the event "edge i is not in the cut", and Y the number of edges not in the cut.

   (e) (1p.) Modify the algorithm so that the output is a $(2+\varepsilon)$-approximation for the maximum cut problem with high probability. Prove the correctness of your algorithm, ie. for any $c > 0$ the algorithm outputs a $(2 + \varepsilon)$-approximation with probability at least $1 - \frac{1}{n^c}$. Your algorithm may depend on c. You do not need to care about the runtime of the algorithm.

2. **Individual exercise: Chernoff bound, Union bound.** (5p.) Let $G = (V, E)$ be a random graph on n nodes such that n is even and each node pair $\{i, j\}$ have an edge between them with equal probability $p = 1/2$, independently of other node pairs.

(a) (1p.) A bisection of G is a cut $(S, V \setminus S)$ where the sets S and $V \setminus S$ are each of size $n/2$. Let X be the random variable describing the number of edges crossing the bisection $(S, V \setminus S)$, ie.

$$X = \{\{u, v\} \in E : u \in S, v \in V \setminus S\}.$$

Find the expectation $\mathbf{E}[X]$.

(b) (2p.) Let $\delta > 0$. Use the Chernoff bound to find a lower bound for the probability

$$\mathbf{P}((1-\delta)n^2/8 < X < (1+\delta)n^2/8).$$

(c) (2p.) Show that the number of edges crossing *any* bisection of G is between $(1-\delta)n^2/8$ and $(1+\delta)n^2/8$ with high probability.
Hint: Use union bound. The following bound might be also useful:

$$\binom{n}{k} \leq \left(\frac{en}{k}\right)^k.$$