Graded Exercise 6

Duong Le

Problem 1

Since V_1 and V_2 are equivalent, we only need to argue for one case. Also, denote the diameter of V_1 as d_{V_1}. We have:

$$P[d_{V_1} = O(loglogn)] = 1 - P[d_{V_1} = \omega(loglogn)]$$

Consider a path in V_1. The probability of that path is in V_1 is $(\frac{1}{2})^k$ since for two vertices, the probability of there exist an edge between them is $\frac{1}{2}$. Thus, the probability of d_{V_1} is $\omega(loglogn)$ is:

$$P[d_{V_1} = \omega(loglogn)] \leq \left(\frac{1}{2}\right)^{\omega(loglogn)}$$

We have again:

$$P[d_{V_1} = \omega(loglogn)] = P\left[\bigcup_{i=1}^{n} \text{Path with length } \omega(loglogn) \text{ is in } V_1\right]$$

Applying the union bound to the above probability, we get that:

$$P\left[\bigcup_{i=1}^{n} \text{Path with length } \omega(loglogn) \text{ is in } V_1\right] \leq \sum_{i=1}^{n} P[\text{Path } P_i \text{ with length } \omega(loglogn) \text{ is in } V_1]$$

The number of path in the a tree is equal to the number of vertices pair $\{u, v\}$ in the tree, which is $\binom{n}{2} = O(n^2)$

$$\Rightarrow P[d_{V_1} = \omega(loglogn)] \leq O(n^2) \cdot \left(\frac{1}{2}\right)^{\omega(loglogn)}$$

We have:

$$\frac{1}{2^{loglogn}} = \frac{1}{logn}$$

But since $\omega(loglogn) > loglogn$

$$\begin{aligned}
\Rightarrow \quad & \frac{1}{2^{\omega(loglogn)}} < \frac{1}{poly(logn)} \\
\Leftrightarrow \quad & \frac{O(n^2)}{2^{\omega(loglogn)}} < \frac{O(n^2)}{poly(logn)}
\end{aligned}$$

Consider the right hand side. The growth rate of $O(n^2)$ can be larger than the growth rate of $poly(logn)$. So as n grows larger, the term $O(n^2)/2^{\omega(loglogn)}$ will also tends to infinity, or in other words, increases the probability of failing. Thus, the probability of getting set with diameter $O(loglogn)$ will decrease.

$\Rightarrow$ The algorithm does not success with high probability.

Problem 2

Let $x_{u,v}$ denote the random variable of whether there is an edge between the vertices u and v: $x_{u,v} = 1$ if there is an edge between u and v, and $x_{u,v} = 0$ otherwise. And since each event occurs with equal probability.

$$\Rightarrow E[x_{u,v}] = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 0 = \frac{1}{2}$$

Consider a random bisection $(S, V \backslash S)$, and let $s_i \in S$, $t_i \in V \backslash S$. Denote the random variable represents the number of crossing edges as X. Since there are $\frac{n}{2}$ vertices in each side, we have:

$$X = \sum_{i=1}^{\frac{n}{2}} \sum_{j=1}^{\frac{n}{2}} x_{s_i,t_j}$$

$$\Rightarrow \quad \mu = E[X] = \sum_{i=1}^{\frac{n}{2}} \sum_{j=1}^{\frac{n}{2}} E[x_{s_i,t_j}] = \frac{n^2}{8}$$

Denote the event that X is between $\frac{(1-\delta)n^2}{8}$ and $\frac{(1+\delta)n^2}{8}$ as Y. Consider:

$$P[Y] = P\left[\frac{(1-\delta)n^2}{8} \leq X \leq \frac{(1+\delta)n^2}{8}\right] = 1 - P\left[\bigcup \left(X < \frac{(1-\delta)n^2}{8}, X > \frac{(1+\delta)n^2}{8}\right)\right]$$
$$= 1 - P[\overline{Y}]$$

Apply the union bound to $P[\overline{Y}]$, we have:

$$P[\overline{Y}] \leq P\left[X < \frac{(1-\delta)n^2}{8}\right] + P\left[X > \frac{(1+\delta)n^2}{8}\right]$$
$$\leq P\left[X \leq \frac{(1-\delta)n^2}{8}\right] + P\left[X \geq \frac{(1+\delta)n^2}{8}\right]$$

Since $\delta \in (0,1)$, and $\frac{(1\pm\delta)n^2}{8} = (1 \pm \delta)\mu$ we can apply the lower tail and upper tail Chernoff bound. Thus, we have:

$$\begin{cases} P\left[X \leq \frac{(1-\delta)n^2}{8}\right] \leq e^{-\frac{\delta^2}{2} \cdot \frac{n^2}{8}} \\ \\ P\left[X \geq \frac{(1+\delta)n^2}{8}\right] \leq e^{-\frac{\delta^2}{2+\delta} \cdot \frac{n^2}{8}} \end{cases}$$

Sum up the two results:

$$P[\overline{Y}] \leq e^{-\frac{\delta^2}{2} \cdot \frac{n^2}{8}} + e^{-\frac{\delta^2}{2+\delta} \cdot \frac{n^2}{8}}$$
$$= e^{-n^2}(e^{\frac{\delta^2}{16}} + e^{\frac{\delta^2}{(2+\delta)8}})$$
$$= \frac{e^{\frac{\delta^2}{16}} + e^{\frac{\delta^2}{(2+\delta)8}}}{e^{n^2}}$$

Since $e^{n^2} > poly(n) \Rightarrow \frac{1}{e^{n^2}} < \frac{1}{poly(n)}$, and $e^{\frac{\delta^2}{16}} + e^{\frac{\delta^2}{(2+\delta)8}} = O(1)$, we have:

$$P[Y] = 1 - P[\overline{Y}] \geq 1 - \frac{O(1)}{poly(n)}$$

As n grows larger, the term $\frac{O(1)}{poly(n)}$ will tend to 0, and $P[Y]$ will tend to 1.

$\Rightarrow$ With high probability, the number of crossing edges will be between $\frac{(1-\delta)n^2}{8}$ and $\frac{(1+\delta)n^2}{8}$.