Linear Programming

Outline

- Linear Programming
  - Integer
  - Vertex Cover
  - Fractional
- How are fractional solutions useful?
  - Rounding
  - Approximate Vertex Cover

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Learning objectives:
You are able to
- describe the concept of a linear program
- explain the difference between an integer and a fractional linear program
- illustrate how to model the vertex cover as a linear program
- apply rounding to turn a fractional vertex cover into an integer vertex cover

Linear Programs

- Recall the optimization formulation

Minimize x	**Objective function** $\boldsymbol{f(x)}$
Subject to	**Constraints** $\boldsymbol{g_1(x)}$ $\boldsymbol{g_2(x)}$ **…** $\boldsymbol{g_m(x)}$

Linear Programs

$$\boldsymbol{x} = x_1, x_2, \dots, x_n$$

Minimize
x

Objective function $\boldsymbol{f}(\boldsymbol{x})$

Subject to

Constraints $\boldsymbol{g_1}(\boldsymbol{x})$
$\boldsymbol{g_2}(\boldsymbol{x})$
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Subject to

Constraints
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$\boldsymbol{g_2(x)}$
…
$\boldsymbol{g_m(x)}$

Linear:
Each g_i is a linear inequality on $x_1, x_2, \dots, x_n$
f is a linear function

Linear Programs – Geometric Interpretation

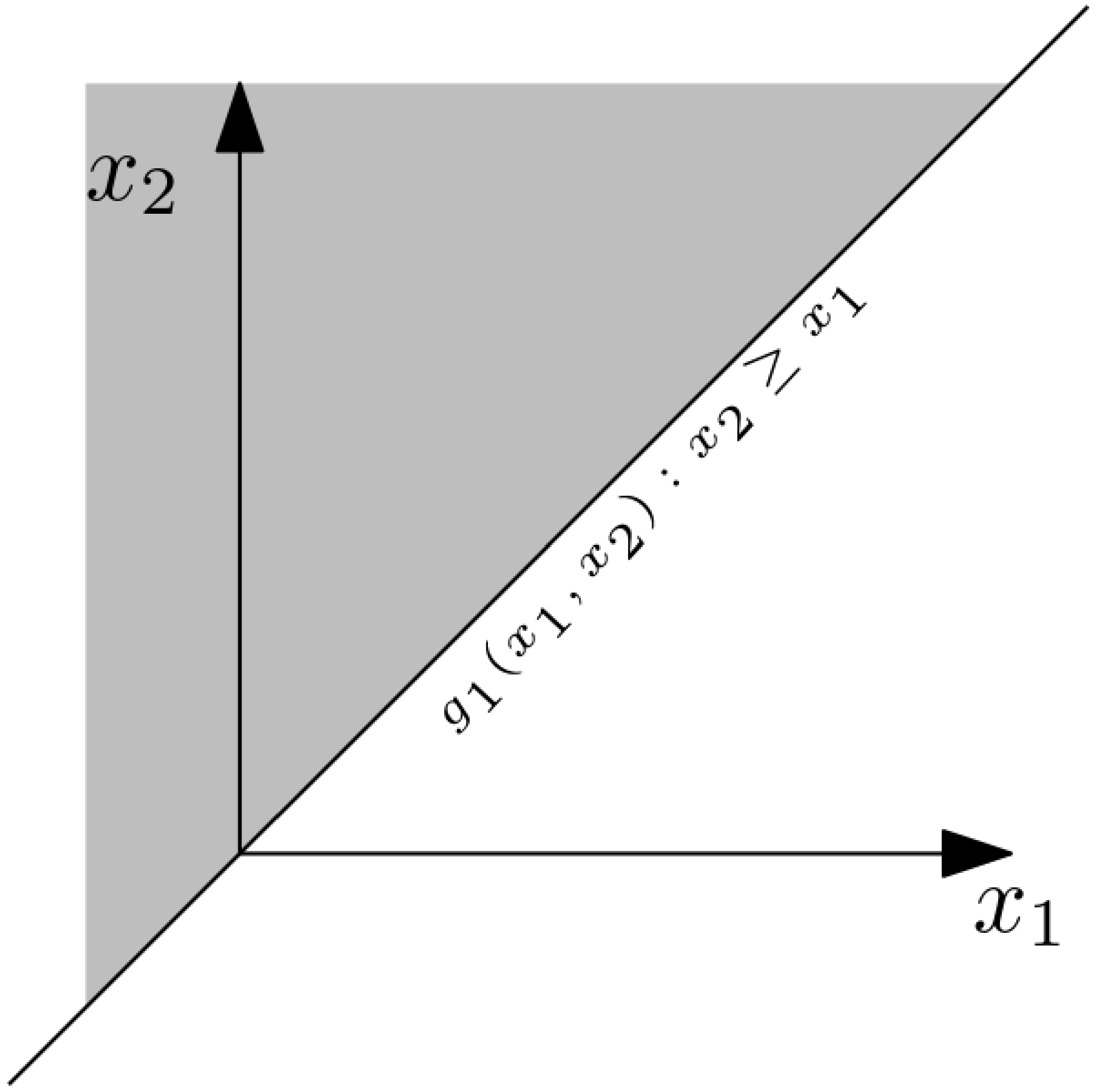
Maximize
x

Objective function $f(x)$

Subject to

Constraints $g_1(x)$
$g_2(x)$
…
$g_m(x)$

$x = x_1, x_2$



Linear Programs – Geometric Interpretation

Maximize x | **Objective function** $\boldsymbol{f(x)}$

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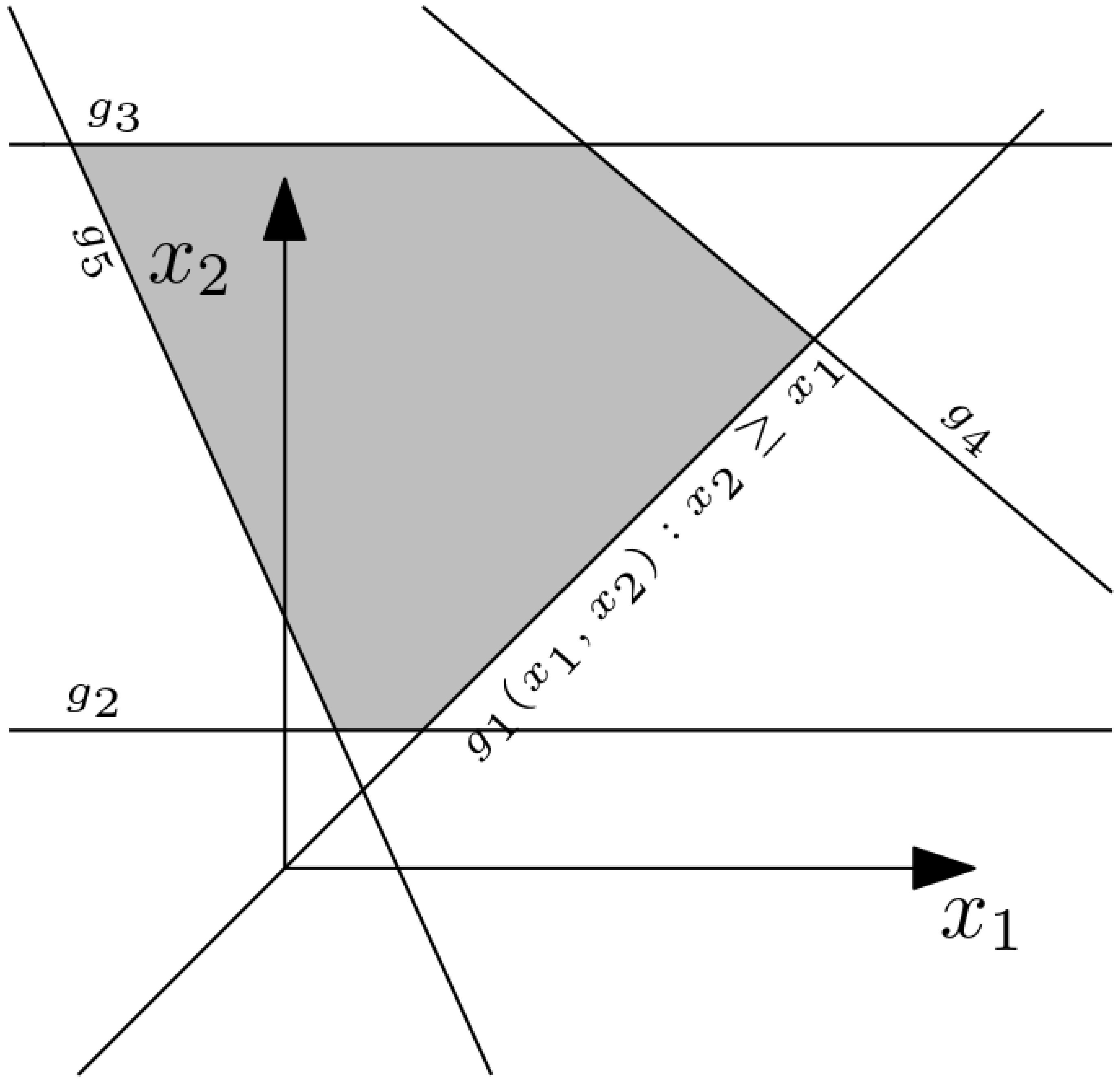

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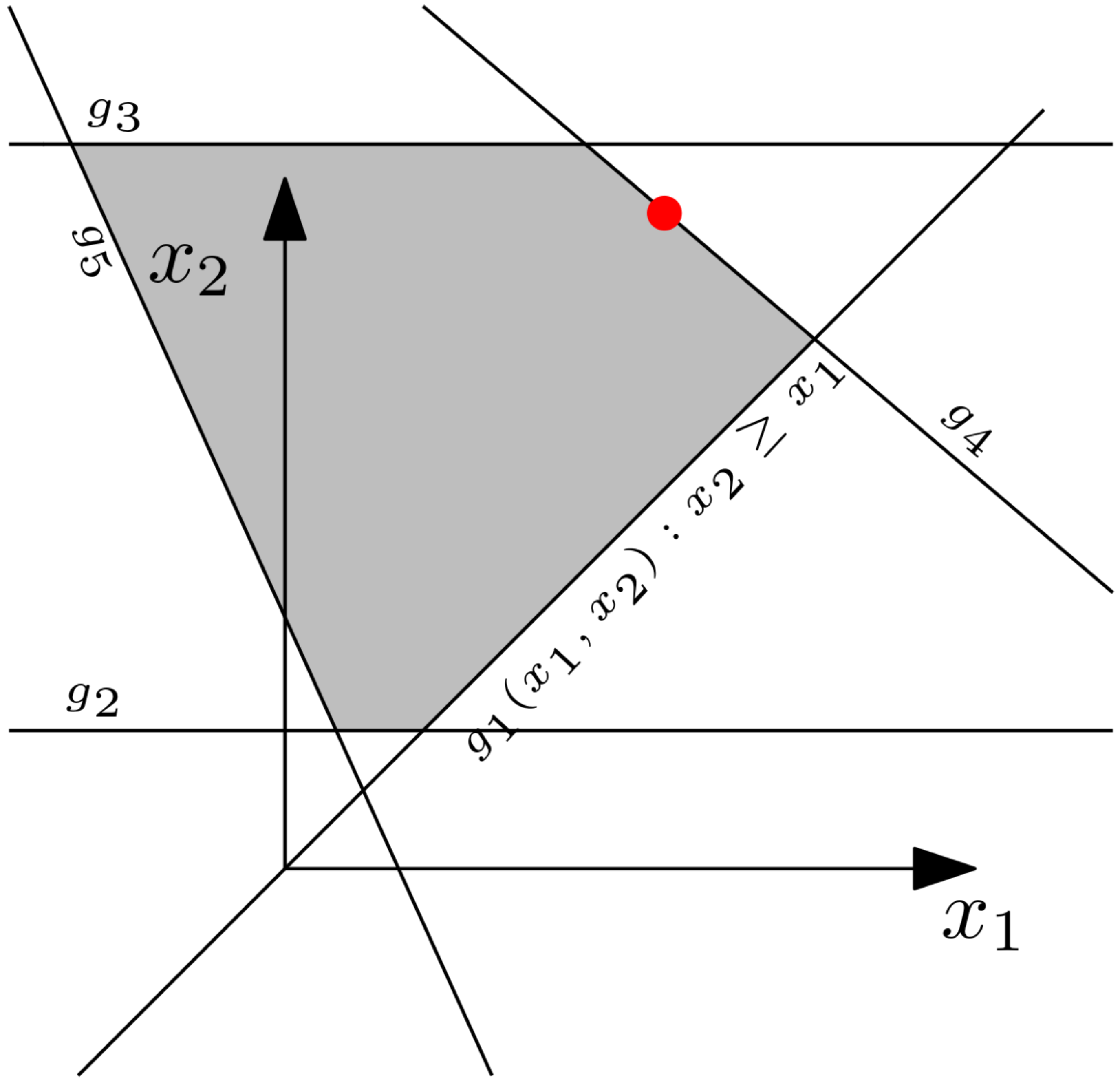

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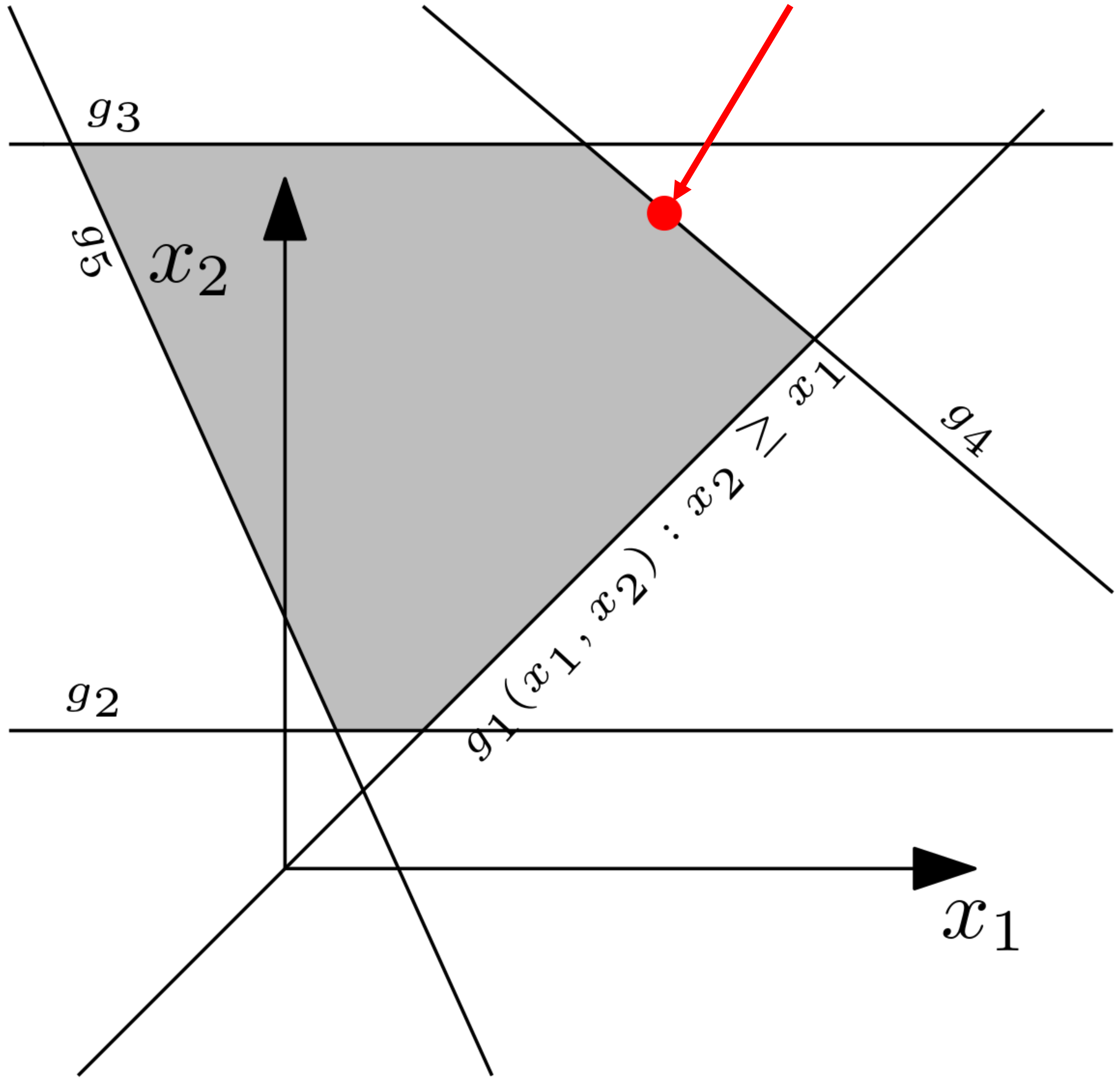


Linear Programs – Geometric Interpretation

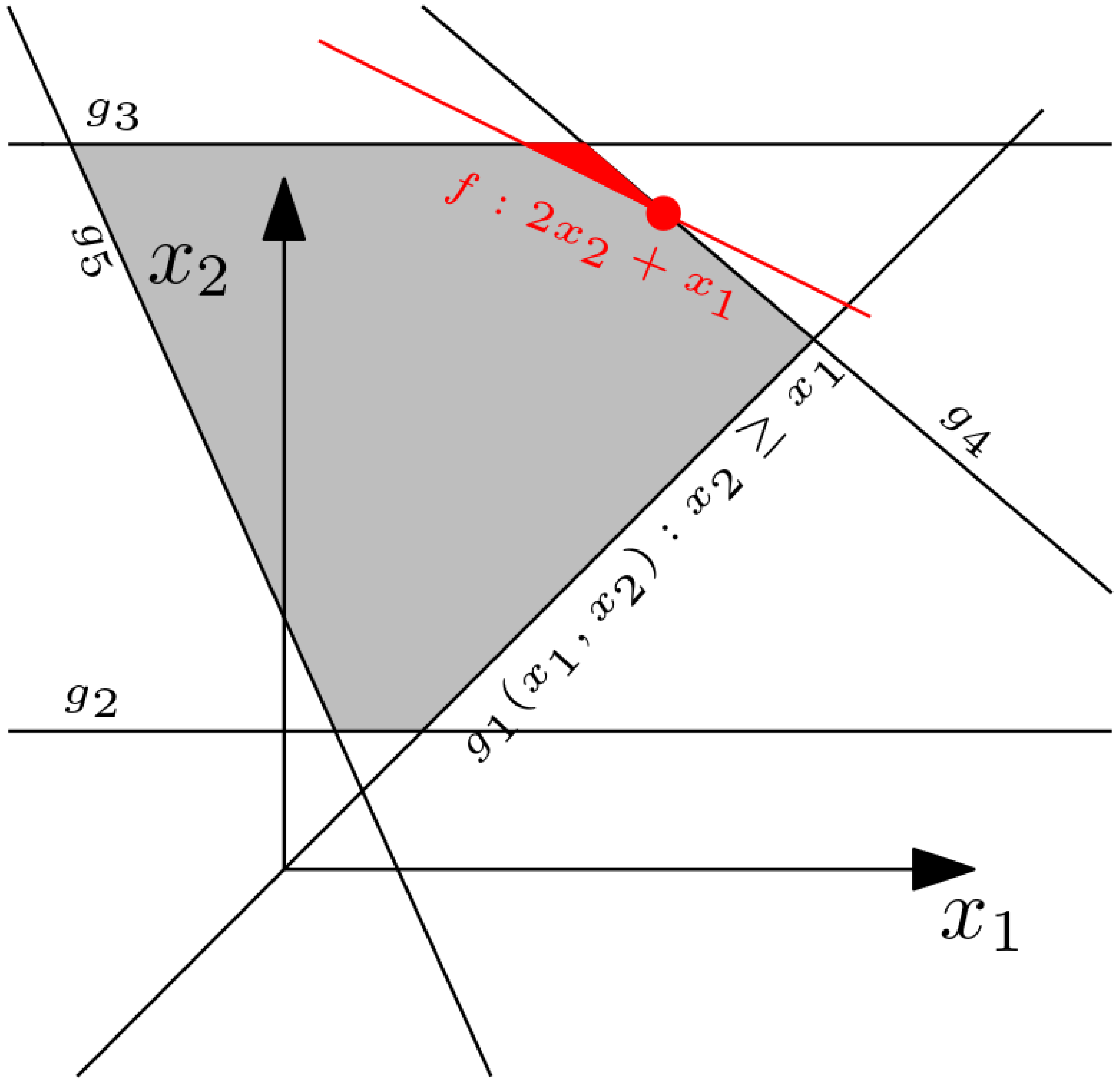
Maximize $f(\boldsymbol{x}) = 2x_2 + x_1$

x

Subject to

Constraints $\boldsymbol{g_1(x)}$
$\boldsymbol{g_2(x)}$
…
$\boldsymbol{g_m(x)}$

$\boldsymbol{x} = x_1, x_2$



Linear Programs – Geometric Interpretation

Maximize
x

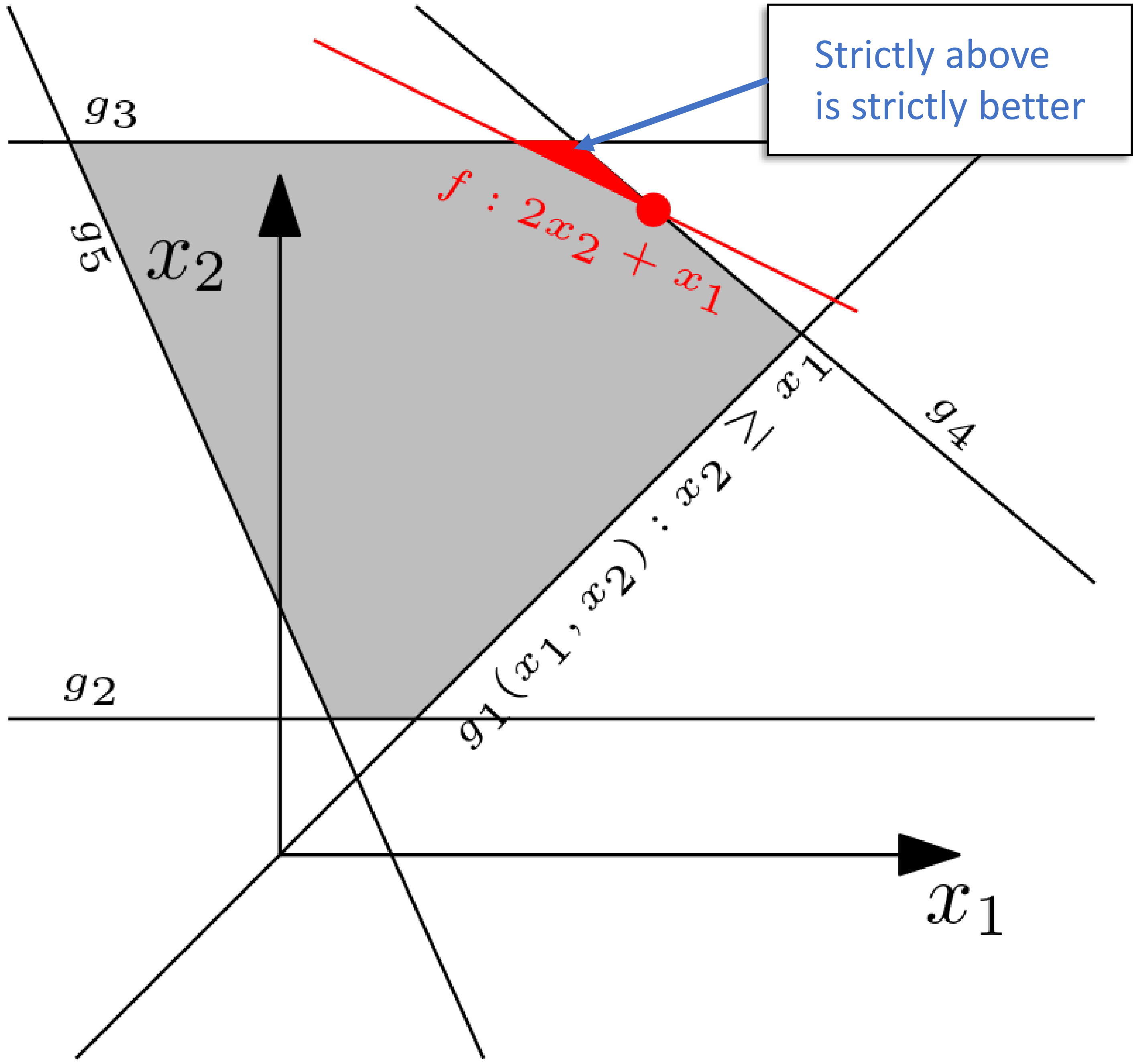
$$\boldsymbol{f}(\boldsymbol{x}) = 2x_2 + x_1$$

Subject to

Constraints

$$\boldsymbol{g}_1(\boldsymbol{x})$$
$$\boldsymbol{g}_2(\boldsymbol{x})$$
$$\ldots$$
$$\boldsymbol{g}_m(\boldsymbol{x})$$

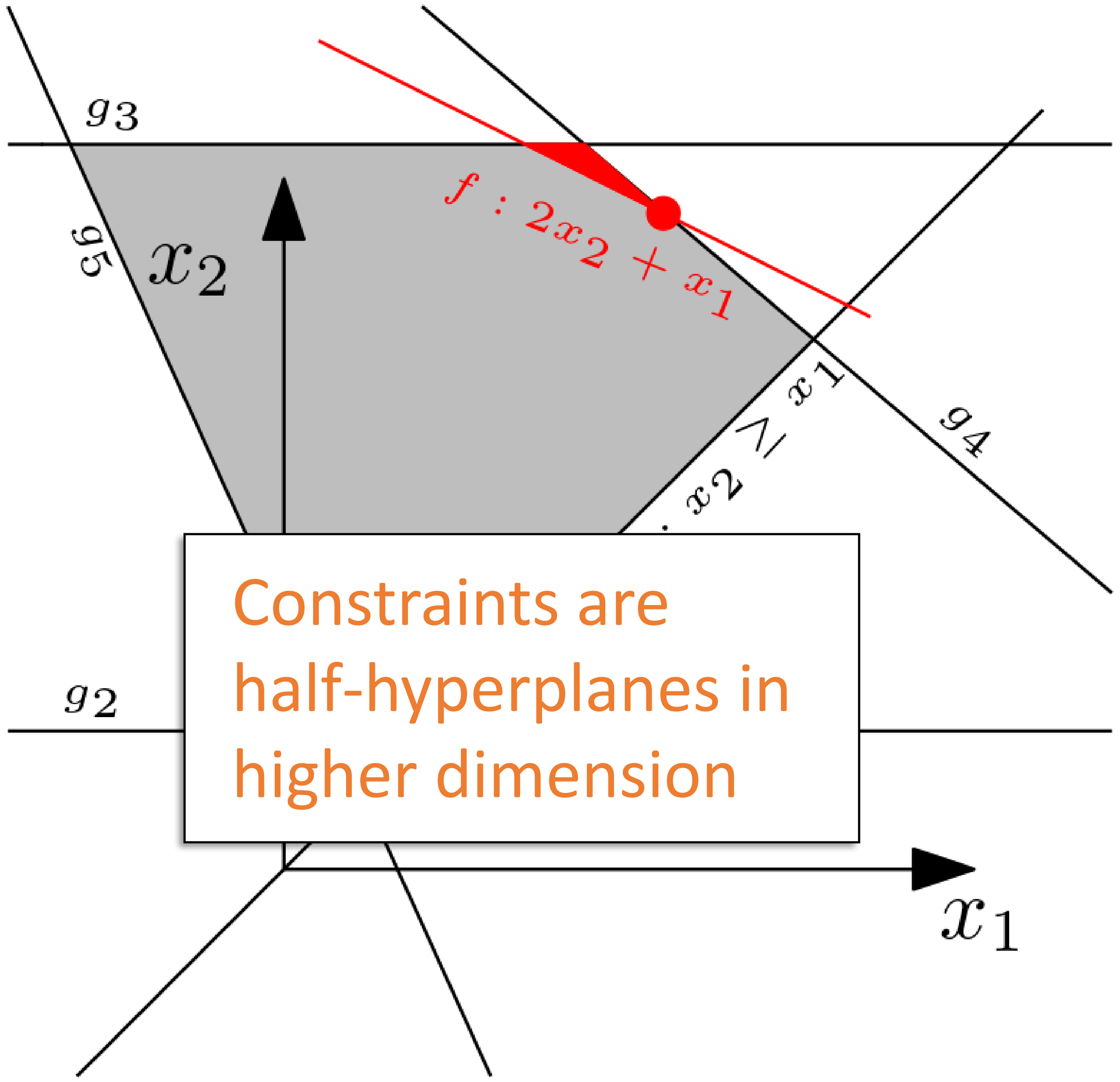
$$\boldsymbol{x} = x_1, x_2$$



Linear Programs – Geometric Interpretation

Maximize x	**Objective function $f(x)$**
Subject to	**Constraints** $g_1(x)$, $g_2(x)$, …, $g_m(x)$

$$\boldsymbol{x} = x_1, x_2, \dots, x_n$$



Constraints are half-hyperplanes in higher dimension

Linear Programs – Geometric Interpretation

Maximize
x

Objective function $\boldsymbol{f}(\boldsymbol{x})$

Subject to

Constraints $\boldsymbol{g_1}(\boldsymbol{x})$
$\boldsymbol{g_2}(\boldsymbol{x})$
…
$\boldsymbol{g_m}(\boldsymbol{x})$

$\boldsymbol{x} = x_1, x_2, \dots, x_n$

Integer linear program:
$\forall i, x_i = \{0,1\}$

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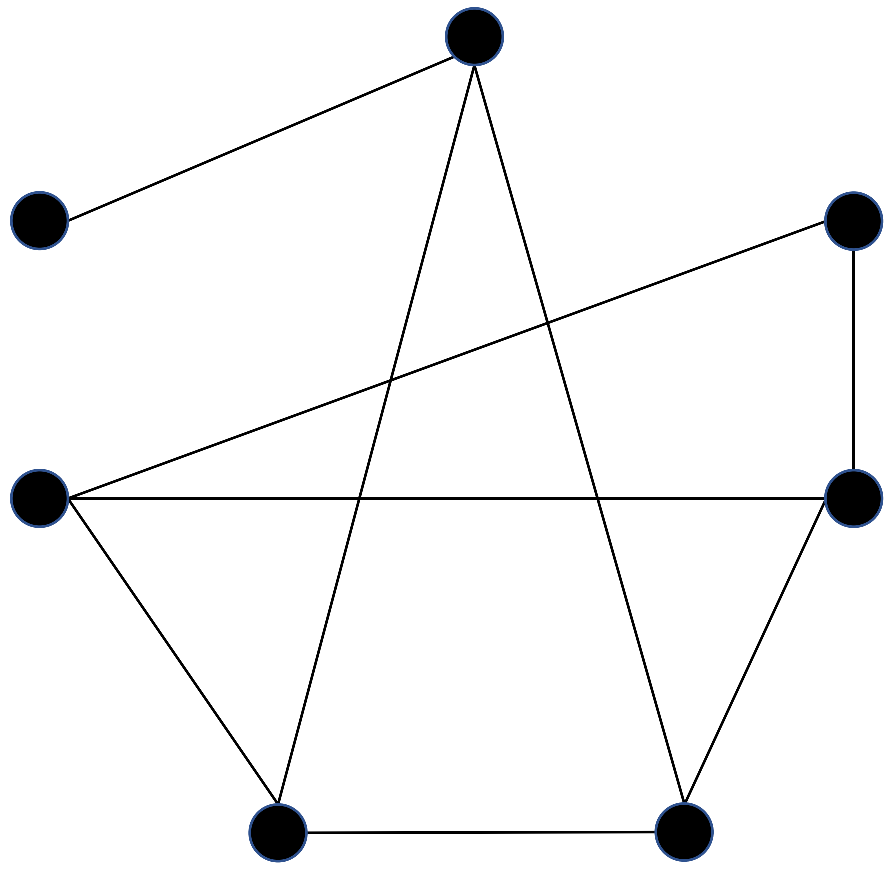
Minimum Vertex Cover

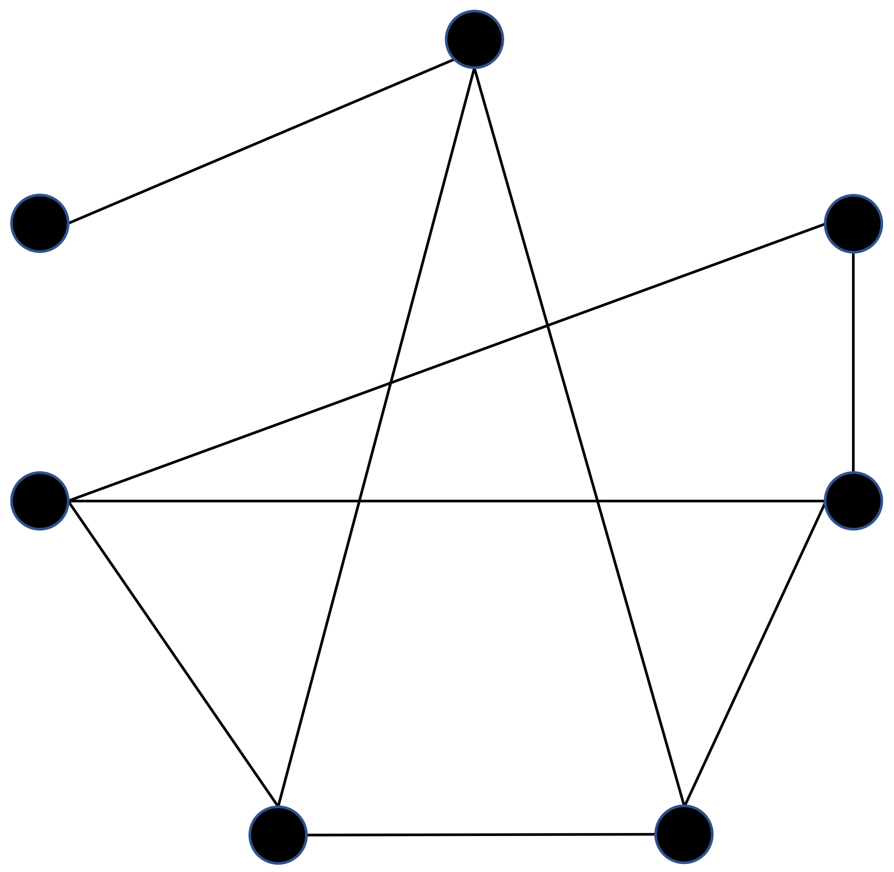
An Informative Example Problem: Vertex Cover
Given a graph $G = (V, E)$
Find a set of nodes $C \subseteq V$ that $covers$ all edges.



Minimum Vertex Cover

An Informative Example Problem: Vertex Cover
Given a graph $G = (V, E)$
Find a set of nodes $C \subseteq V$ that $covers$ all edges.

For each $\{u, v\} \in E$,
either $u \in C$ or $v \in C$



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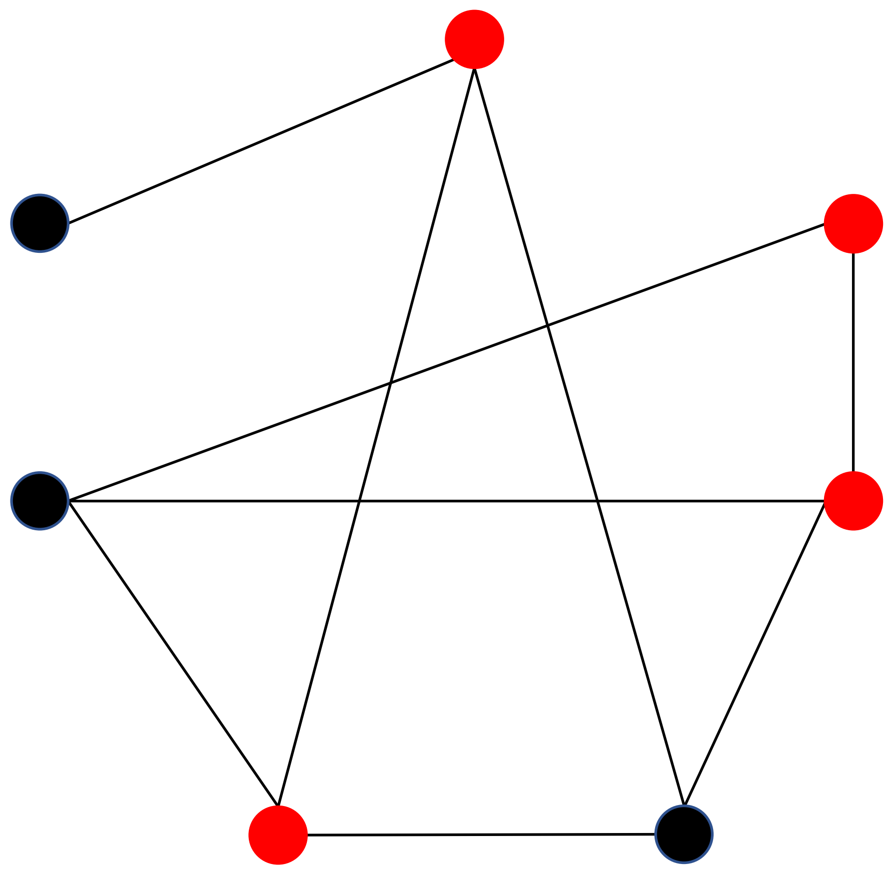


Minimum Vertex Cover

Integer program:

$$x_i = \begin{cases} 1 & \text{if node } i \text{ in the cover} \\ 0 & \text{otherwise} \end{cases}$$

Nodes numbered from 1 to n

Minimum Vertex Cover

Minimize
x

$\boxed{\sum_{i=1}^{n} x_i}$

Integer program:

$$x_i = \begin{cases} 1 & \text{if node } i \text{ in the cover} \\ 0 & \text{otherwise} \end{cases}$$

Nodes numbered from 1 to n

Minimum Vertex Cover

Integer program:

$$x_i = \begin{cases} 1 & \text{if node } i \text{ in the cover} \\ 0 & \text{otherwise} \end{cases}$$

Minimize x $\quad \Sigma_{i=1}^{n} x_i$

Subject to $\quad x_i + x_j \geq 1 \quad$ for all $\{i, j\} \in E$

Nodes numbered from 1 to n

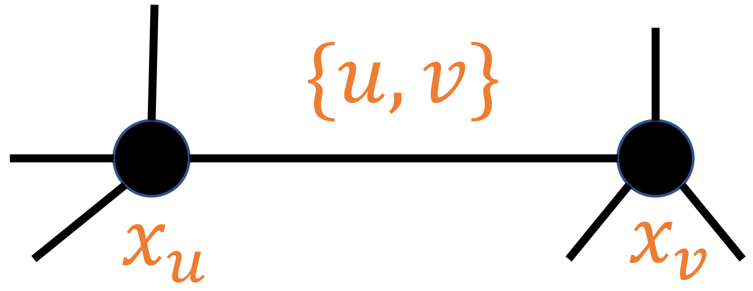
Minimum Vertex Cover

Minimize
x

$\Sigma_{i=1}^{n} x_i$

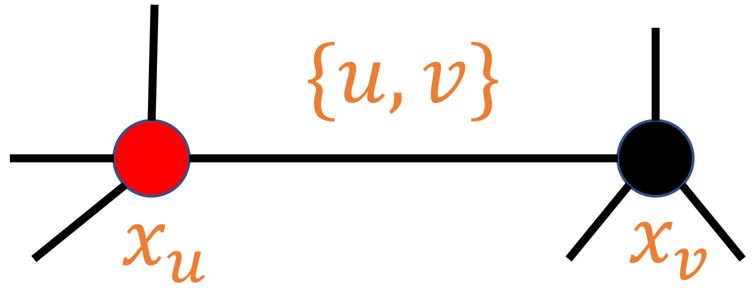
Subject to

$x_i + x_j \geq 1$ for all $\{i, j\} \in E$



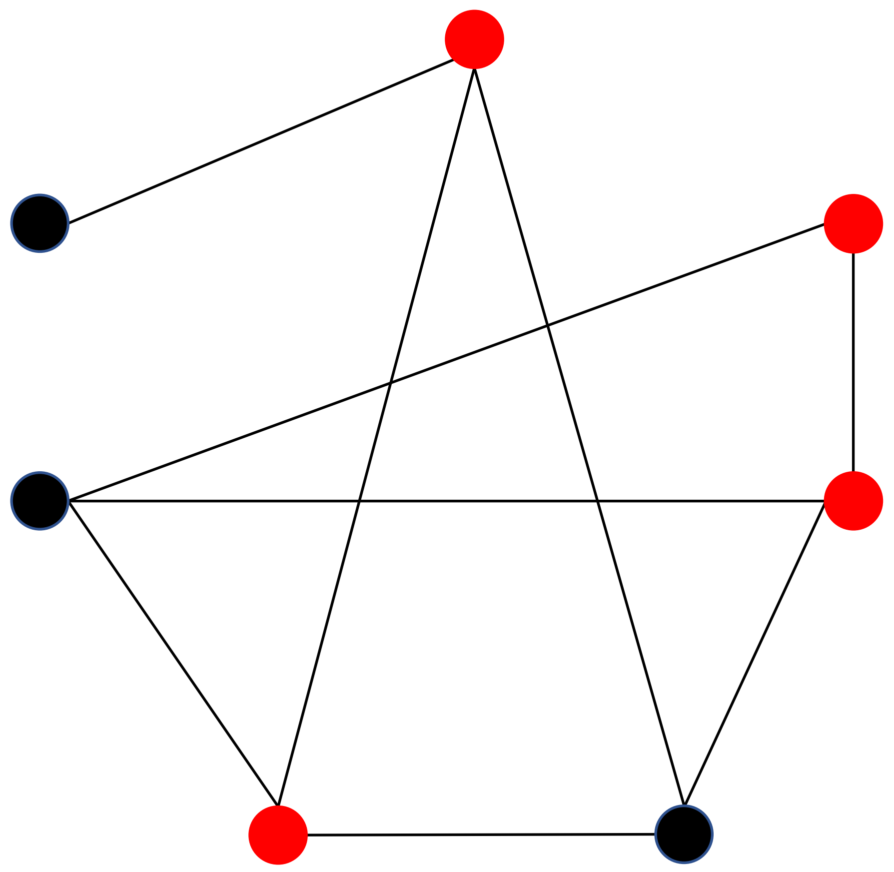
Minimum Vertex Cover

Minimize x	$\boxed{\Sigma_{i=1}^{n} x_i}$
Subject to	$x_i + x_j \geq 1$ for all $\{i, j\} \in E$



Minimum Vertex Cover

Minimize x	$\Sigma_{i=1}^{n} x_i$
Subject to	$x_i + x_j \geq 1$ for all $(i, j) \in E$ $x_i \in \{0, 1\}$ for all $i \in V$



Minimum Vertex Cover

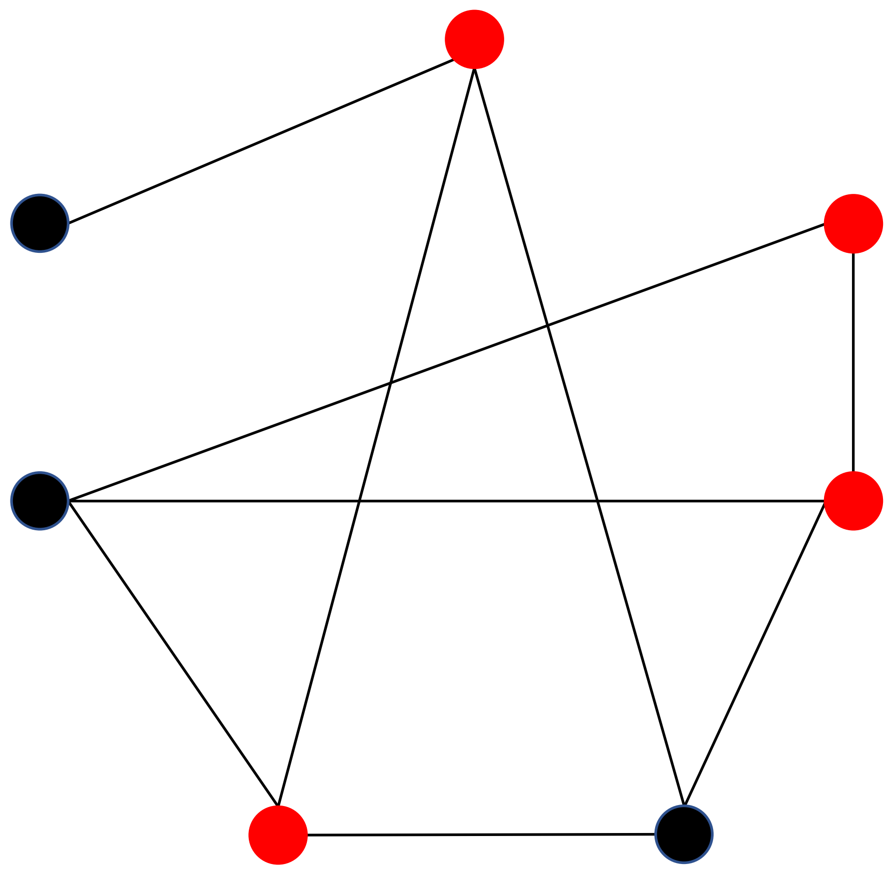
Minimize x $\quad \Sigma_{i=1}^{n} x_i$

Subject to

$x_i + x_j \geq 1$ for all $(i, j) \in E$

$x_i \in \{0, 1\}$ for all $i \in V$

Integer program IP



Outline

- Vertex Cover
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- How are fractional solutions useful?
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Minimum Vertex Cover

Minimize x	$\sum_{i=1}^{n} x_i$	
Subject to	$x_i + x_j \geq 1$	for all $(i, j) \in E$
	$x_i \geq 0$	for all $i \in V$

Relaxation:
Allow *fractional* values

Integer program:

$$x_i = \begin{cases} 1 & \text{if node } i \text{ in the cover} \\ 0 & \text{otherwise} \end{cases}$$

Nodes numbered from 1 to n

Minimum Vertex Cover

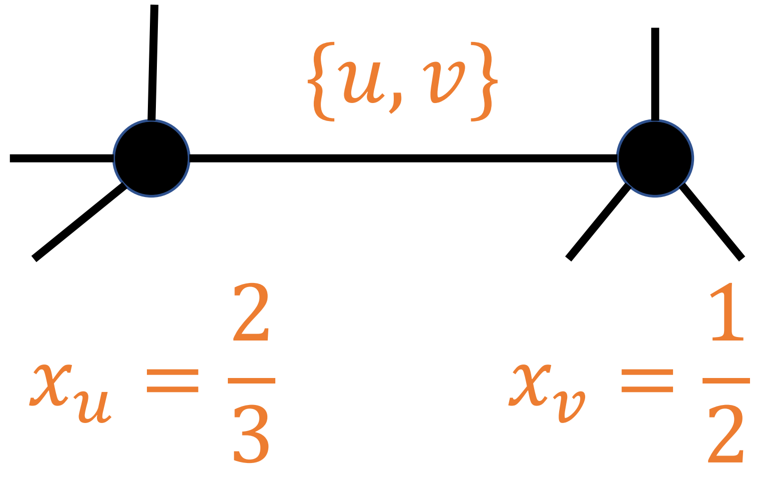
Minimize x $\quad \boxed{\sum_{i=1}^{n} x_i}$

Subject to

$$\boxed{\begin{array}{ll} x_i + x_j \geq 1 & \text{for all } (i,j) \in E \\ x_i \geq 0 & \text{for all } i \in V \end{array}}$$

Minimum Vertex Cover

Minimize x	$\sum_{i=1}^{n} x_i$	
Subject to	$x_i + x_j \geq 1$	for all $(i,j) \in E$
	$x_i \geq 0$	for all $i \in V$



Minimum Vertex Cover

$x_i + x_j \geq 1$	for all $(i, j) \in E$
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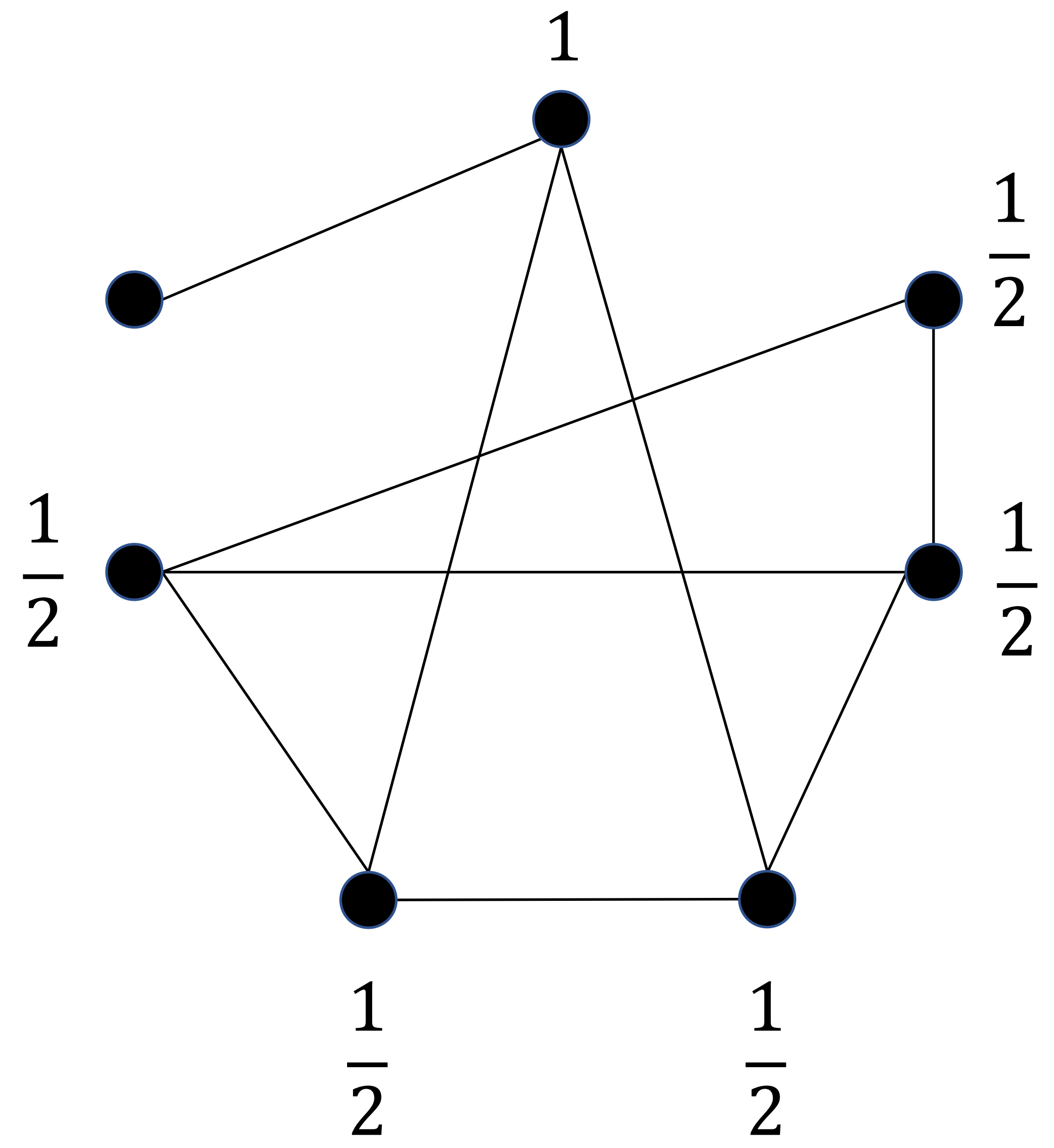
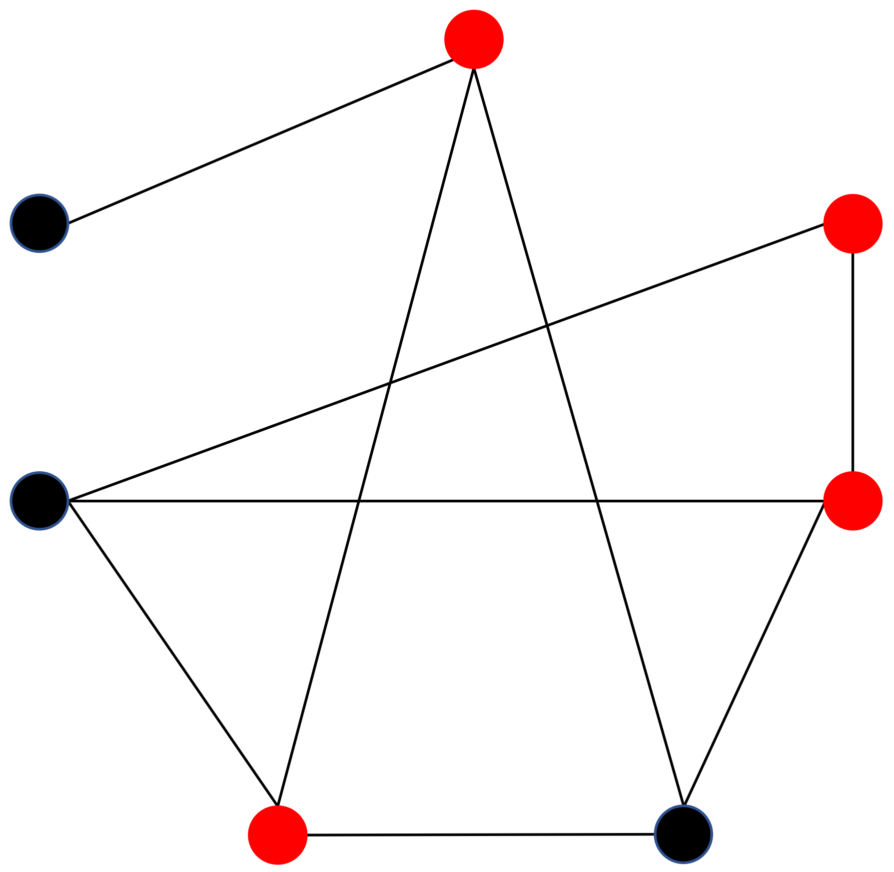
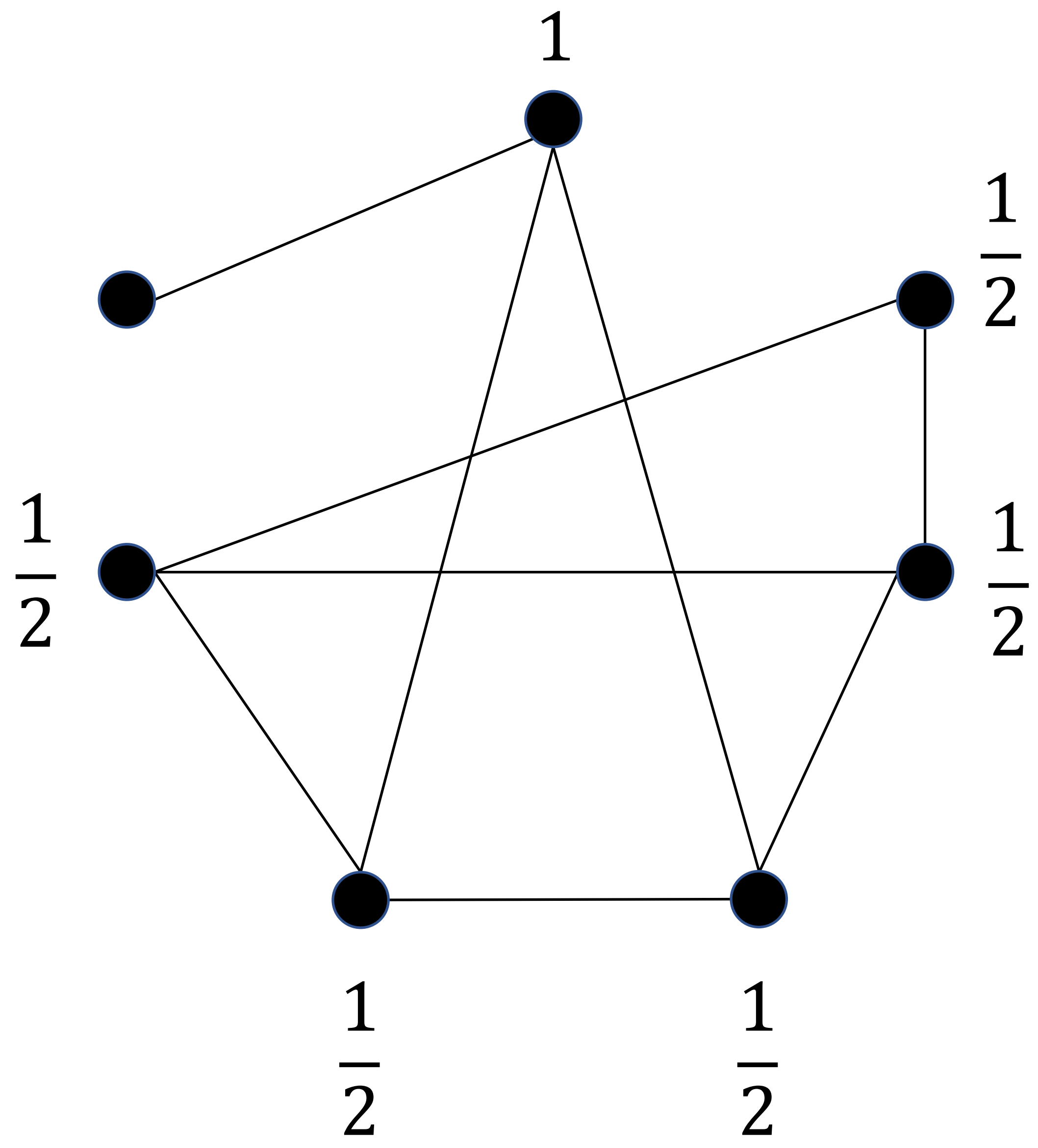


Minimum Vertex Cover



$f(\boldsymbol{x}) = 4$



$f(\boldsymbol{x}) = 3 + \frac{1}{2} < 4$

Solving Linear Programs

Theorem:
Linear program relaxations can be solved in polynomial time

Usually the definition of "efficient"

Solving Linear Programs

In this course, we use this theorem as a tool

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Simplex is a classic text-book algorithm

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Solving Linear Programs

In this course, we use this theorem as a tool

Theorem:
Linear program relaxations can be solved in polynomial time

Usually the definition of "efficient"

Simplex is a classic text-book algorithm

The ellipsoid method achieves polynomial time

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Optimum of Relaxation is Better

Observation:
A solution to the integer program is a solution to the relaxation.

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$\Sigma_{i=1}^{n} x_i$

$x_i + x_j \geq 1$ for all $(i, j) \in E$
$x_i \geq 0$ for all $i \in V$

$\Sigma_{i=1}^{n} x_i$

$x_i + x_j \geq 1$ for all $(i, j) \in E$
$x_i \in \{0, 1\}$ for all $i \in V$

Optimum of Relaxation is Better

Observation:
A solution to the integer program is a solution to the relaxation.

$$\Sigma_{i=1}^{n} x_i$$

$$\begin{array}{ll} x_i + x_j \geq 1 & \text{for all } (i,j) \in E \\ x_i \geq 0 & \text{for all } i \in V \end{array}$$

Therefore, the optimum x_{IP} to the integer program is at most as good as the solution to the relaxation x_{LP}.

$$\Sigma_{i=1}^{n} x_i$$

$$\begin{array}{ll} x_i + x_j \geq 1 & \text{for all } (i,j) \in E \\ x_i \in \{0,1\} & \text{for all } i \in V \end{array}$$

Rounding

Optimum integer x_{IP}
Optimum fractional x_{LP}

$$f(x_{\mathrm{LP}}) \leq f(x_{\mathrm{IP}})$$

Rounding

Optimum integer	x_{IP}
Optimum fractional	x_{LP}

$$f(x_{\text{LP}}) \leq f(x_{\text{IP}})$$

Idea:
Find an optimum x_{LP}.
Turn x_{LP} into an integer solution with almost the same cost.

Rounding

Optimum integer x_{IP}
Optimum fractional x_{LP}

$$f(x_{\text{LP}}) \leq f(x_{\text{IP}})$$

Idea:
Find an optimum x_{LP}.
Turn x_{LP} into an integer solution with almost the same cost.

Using a black box

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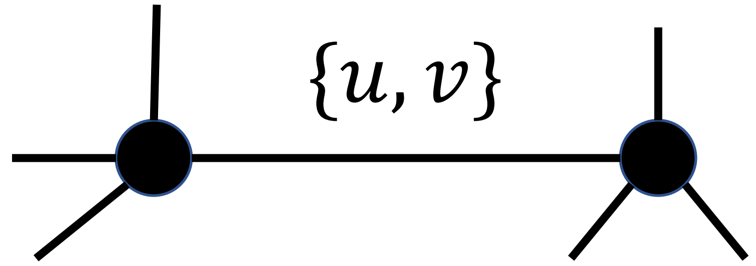
Rounding - Minimum Vertex Cover

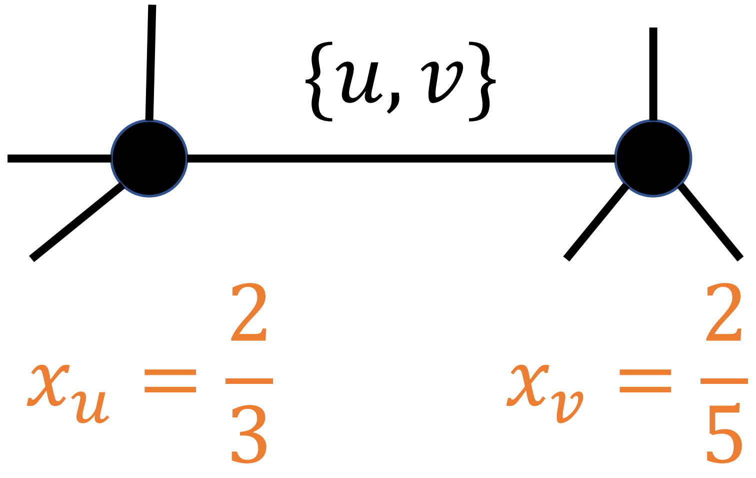
$x_i + x_j \geq 1$	for all $(i, j) \in E$
$x_i \geq 0$	for all $i \in V$



Rounding - Minimum Vertex Cover

$x_i + x_j \geq 1$ for all $(i, j) \in E$
$x_i \geq 0$ for all $i \in V$

Get fractional values from a black box.



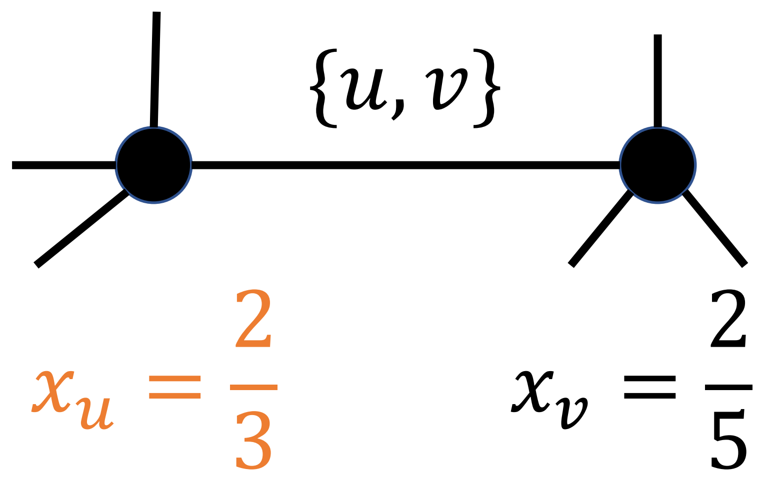
Rounding - Minimum Vertex Cover

$x_i + x_j \geq 1$ for all $(i,j) \in E$
$x_i \geq 0$ for all $i \in V$

Get fractional values from a black box.

At least one endpoint has

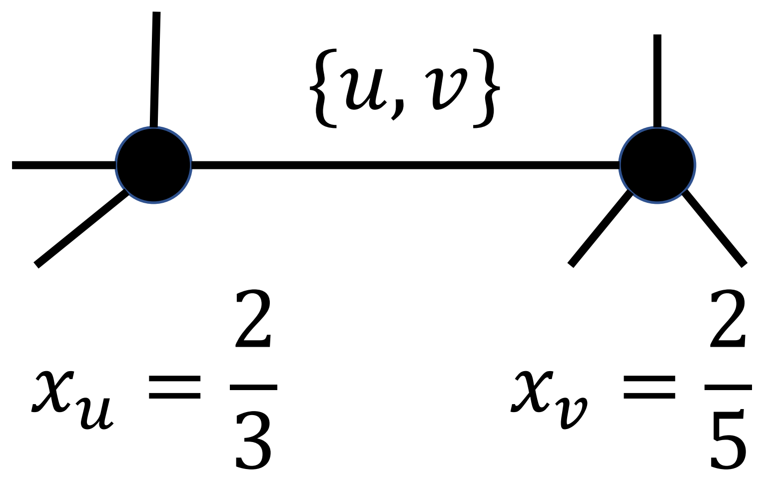
$$x_u \geq 1/2$$



Rounding - Minimum Vertex Cover

Algorithm:
1) Find an optimum fractional solution
2) Set all $x_i \geq 1/2$ to 1
3) Set all $x_i < 1/2$ to 0

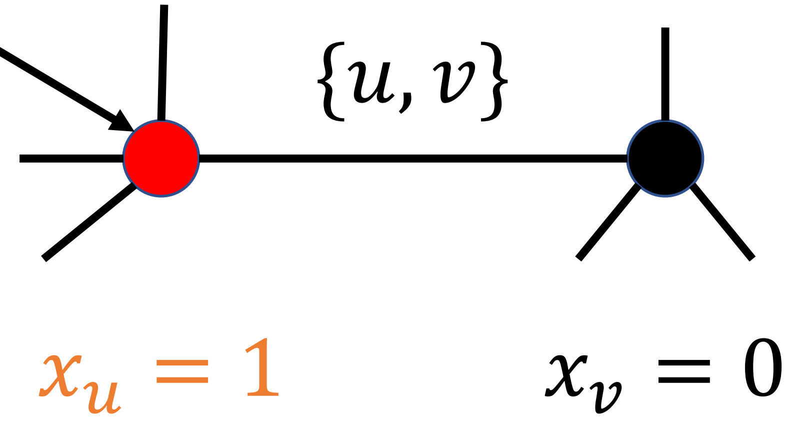
At least one endpoint has
$$x_u \geq 1/2$$



Rounding - Minimum Vertex Cover

Rounding:
Add to the cover

Algorithm:
1) Find an optimum fractional solution
2) Set all $x_i \geq 1/2$ to 1
3) Set all $x_i < 1/2$ to 0

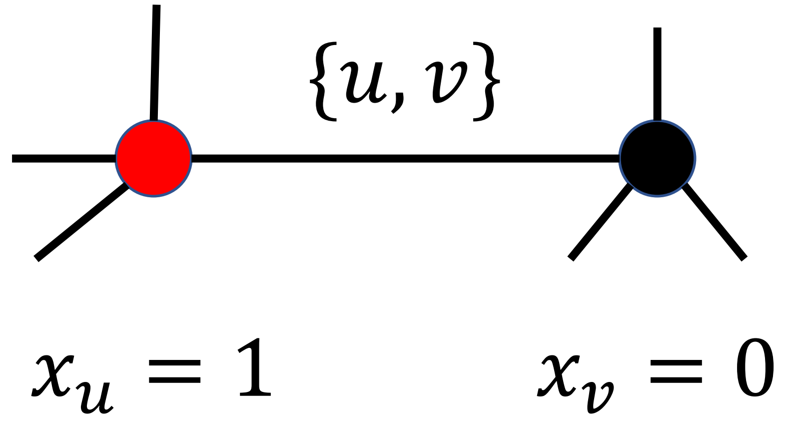


At least one endpoint has
$$x_u \geq 1/2$$

Rounding - Minimum Vertex Cover

Algorithm:
1) Find an optimum fractional solution
2) Set all $x_i \geq 1/2$ to 1
3) Set all $x_i < 1/2$ to 0

At least one endpoint has
$$x_u \geq 1/2$$



Analysis:
i) Cost increases by at most a factor of 2

Rounding - Minimum Vertex Cover

Algorithm:
1) Find an optimum fractional solution
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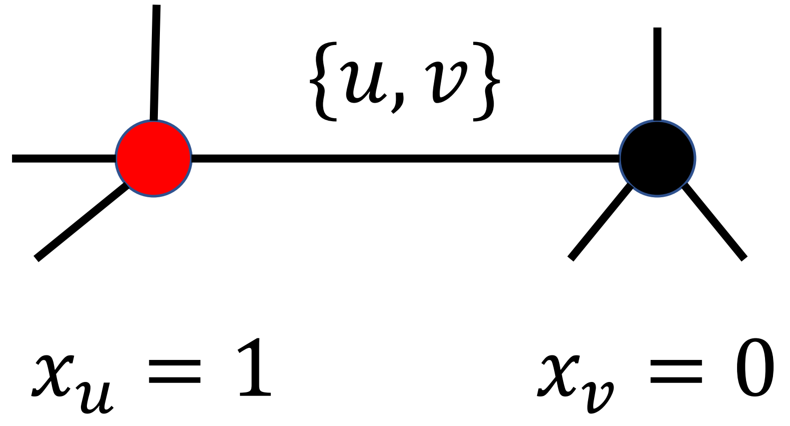

At least one endpoint has
$$x_u \geq 1/2$$

Analysis:
i) Cost increases by at most a factor of 2
ii) At least one endpoint is in the cover

Rounding - Minimum Vertex Cover

Optimum integer x_{IP}
Optimum fractional x_{LP}

$$f(x_{\text{LP}}) \leq f(x_{\text{IP}})$$

Algorithm:
1) Find an optimum fractional solution
2) Set all $x_i \geq 1/2$ to 1
3) Set all $x_i < 1/2$ to 0

2-approximation!

At least one endpoint has
$$x_u \geq 1/2$$

Analysis:
i) Cost increases by at most a factor of 2.
ii) At least one endpoint is in the cover

Wrap-up

Linear program:
A linear objective function and a set of linear constraints.

Has a nice geometric interpretation, especially in the case of two variables.

Rounding:
We know how to solve fractional LPs efficiently.
Sometimes, fractional solutions can be easily turned into integer solutions.