Algorithms with Coins

Basic Tools

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Not always enough!

Some of the Basics

- Random variable and events
- Linearity of expectation
- The union bound
- Monte Carlo and Las Vegas algorithms
- Markov, Chebyshev, Chernoff

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Learning objectives:
You are able to

- name the elements that constitute a probability space
- describe the statements of linearity of expectation and the union bound
- name 3 concentration bounds

Random Variables and Events

Probability space

$$(\Omega, F, P)$$

1. Sample space
2. Events, usually 2^Ω
3. Probabilities of events

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- $\Omega = \{HH, TH, TT, HT\}$


- $P(HH) = \frac{1}{4}$
- $\Sigma_{x \in \Omega} P(x) = 1$
- $E = \{TH, HT\} \subseteq \Omega$
- $P(E) = \frac{1}{2}$

Random Variables and Events

Random variable R


Depends on random outcomes

R:
Get 10 euros if both coins show heads.
0 otherwise.

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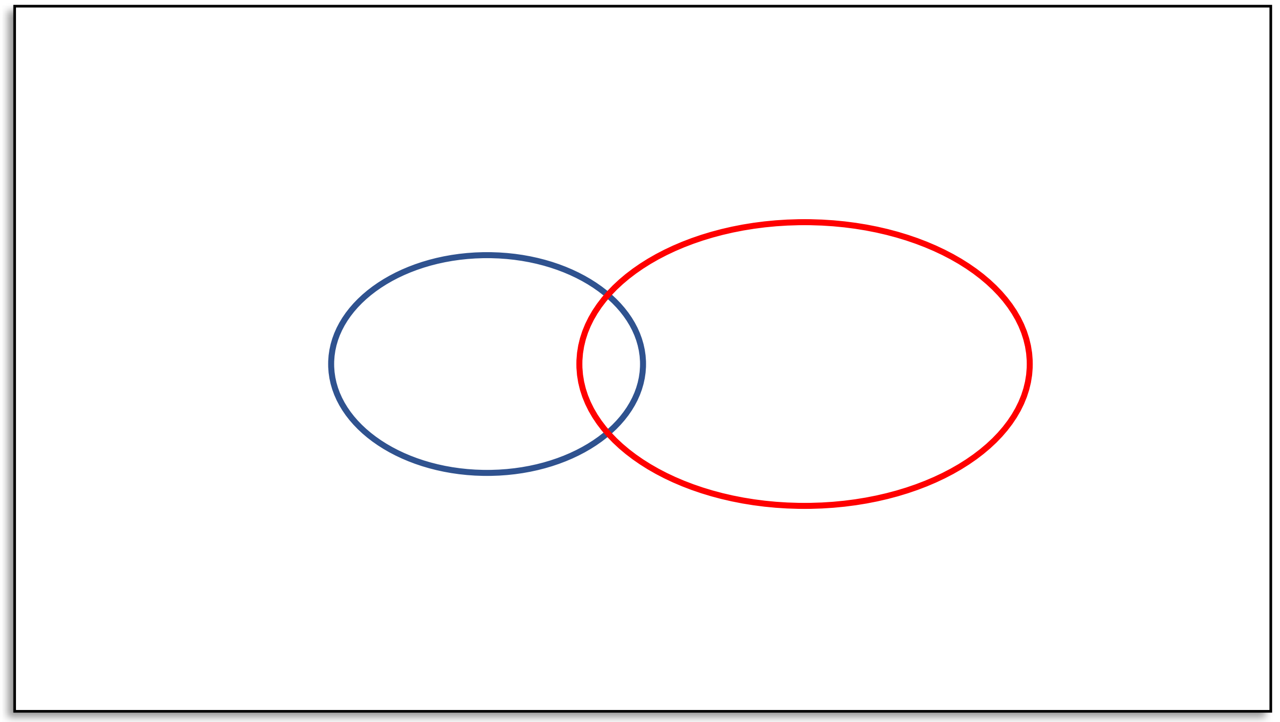
Expected payoff?

$$E[X_1 + X_2]$$
$$= \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 = 1$$

The Union Bound

The Union Bound:
Consider two events E_1 and E_2.
$P(E_1 \cup E_2) \leq P(E_1) + P(E_2)$

Ω has area 1



Intersection means dependency.
The sum of areas is at maximum if they are independent.

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An event holds with probability $1 - n^{-c}$ for a constant c that the algorithm designer can choose.

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Usage in graphs:
Suppose that you have an algorithm that computes, say, a spanning tree. A node u is *bad* if it is not in the tree and the probability of that is n^{-c}.

Using union bound, we get that no node is bad w.h.p. $\left(n^{-d}, d = c - 1\right)$ even if badness is not independent.

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Types of Randomized Algorithms

Monte Carlo:
Correct output with some probability p.
Otherwise, may output carbage.

p often a constant.

Las Vegas:
Always correct output.
Runtime guarantees in expectation

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Some Standard Bounds

μ is the expectation

Markov Inequality:

$$P(X \geq \alpha) \leq \frac{\mu}{\alpha}$$

Chebyshev:

$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$$

Chernoff:

$$P\left(\hat{X} \geq (1 + \delta) \cdot \mu\right) \leq e^{\left(-\frac{\mu\delta^2}{3}\right)}$$

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Don't worry, many good cheat sheets online

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Wrap-up

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