1. **Linear Program** (2p.) A manufacturer wants to produce ice-cream. An ice-cream patch must contain at least 20 units of fat, at least 20 units of sugar, at least 1 unit of egg and at most 60 units of water. Below is a table of the raw ingredients available to the manufacturer, their compositions as percentages, and their prices.

Raw ingredients	**Cream**	**Egg yolk**	**Whole milk**	**Frozen sweetened egg yolk**	**glucose syrup**	**water**
Fat	40	50	12	30	0	0
Sugar	0	0	0	14	70	0
Egg	0	40	0	40	0	0
Water	60	10	88	16	30	100
Price per unit	2	4	1	2	0.8	0

The goal of the manufacturer is to determine the recipe with minimum cost.

(a) (1p.) Write this problem as a linear program.

Let's denote the amount of raw ingredients as follows: cream (x_c) , egg yolk (x_e) , whole milk (x_m) , frozen sweetened egg yolk (x_f) , glucose syrup (x_s), water (x_w)

This problem as a linear program is:

minimize $\quad 2x_c + 4x_e + x_m + 2x_f + 0.8x_s$

subject to $\quad 0.4x_c + 0.5x_e + 0.12x_m + 0.3x_f \geq 20$

$0.14x_f + 0.7x_s \geq 20$

$0.4x_e + 0.4x_f \geq 1$

$0.6x_c + 0.1x_e + 0.88x_m + 0.16x_f + 0.3x_s + x_w \leq 60$

$x_c, x_e, x_m, x_f, x_s, x_w \geq 0$

(b) (1p.) **Transform the linear program into the standard form. See the lecture script for the definition of the standard form.**

The standard form of LP is

maximize $\quad c_1x_1 + \ldots + c_nx_n$

subject to $\quad g_i\left(x_1,\ldots,x_n\right) = b_i, \quad i = 1,\ldots,m$

$x_i \geq 0, \quad i = 1,\ldots,m$

This can be done by introducing slack variables. The standard form of the LP in part (a) becomes:

maximize $\quad -\left(2x_c + 4x_e + x_m + 2x_f + 0.8x_s\right)$

subject to $\quad 0.4x_c + 0.5x_e + 0.11x_m + 0.3x_f - s_1 = 20$

$0.14x_f + 0.7x_s - s_2 = 20$

$0.4x_e + 0.4x_f - s_3 = 1$

$0.6x_c + 0.1x_e + 0.88x_m + 0.16x_f + 0.3x_s + x_w + s_4 = 60$

$x_c, x_e, x_m, x_f, x_s, x_w, s_1, s_2, s_3, s_4 \geq 0$

However, we can notice that x_w and s_4 serves the same purpose as the slack variable in the last equality, there is no need to introduce s_4 for this condition. The standard form becomes

maximize $\quad -\left(2x_c + 4x_e + x_m + 2x_f + 0.8x_s\right)$

subject to $\quad 0.4x_c + 0.5x_e + 0.11x_m + 0.3x_f - s_1 = 20$

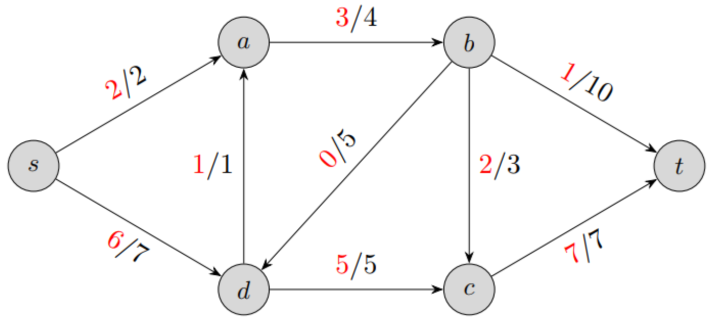
$0.14x_f + 0.7x_s - s_2 = 20$

$0.4x_e + 0.4x_f - s_3 = 1$

$0.6x_c + 0.1x_e + 0.88x_m + 0.16x_f + 0.3x_s + x_w = 60$

$x_c, x_e, x_m, x_f, x_s, x_w, s_1, s_2, s_3 \geq 0$

2. **Flows** (3p.) Given the graph G below, the capacities of each edge are written in black and a flow is given in red.

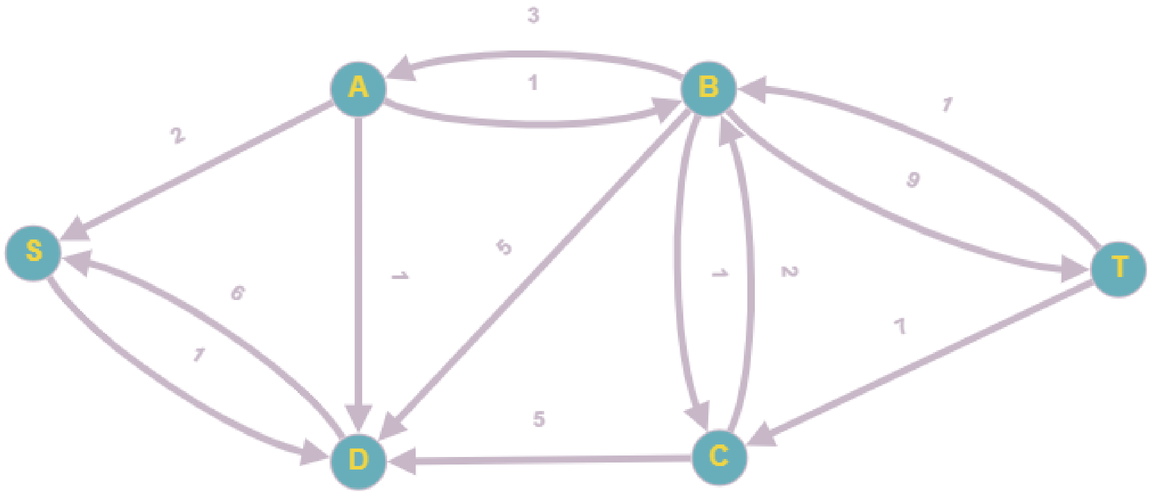


(a) (1p.) What are the saturated edges?

Saturated edges are those where its flow equals to its capacity. Therefore, in this definition, the saturated edges are (s, a), (d, a), (d, c) and (c, t).

(b) (1p.) Draw the residual graph.

The residual graph of the flow graph is:



(Graph drawn at online application http://graphonline.ru/en/?graph=DpzftrIbcfuAuPwh)

(c) (1p.) Prove that the given flow is maximum. You can use the fact that the residual graph of a maximum flow has no augmenting paths.

In the residual graph in part (b), the source vertex s has only one outcoming edge, which goes to vertex d. From d, there are only incoming edges, which means that the vertex s and t has been disconnected in the residual graph. In other words, there are no augmented paths P that connects vertices s and t in the residual graph

- ⇨ The given flow is a maximum flow

3. **Individual exercise: Integer Program.** (5p.) Consider a set of points $\mathcal{U}$ and a set of sets of points S such that $\forall s \in S, s \subseteq \mathcal{U}$ and $\forall e \in \mathcal{U}$, e is in at most 3 different sets in S. Consider the following Integer Program:

$$\begin{aligned} \text{minimize} \quad & \sum_{s \in S} x_s \\ \text{subject to} \quad & \sum_{s:e \in s} x_s \geq 1, \forall e \in \mathcal{U} \\ & x_s \in \{0,1\}, \forall s \in S \end{aligned}$$

Note that $\sum_{s:e\in s} x_s$ has at most 3 terms for each $e \in \mathcal{U}$.

(a) (1p.) Write the relaxation of this integer program.

The relaxation of this integer program

$$\begin{aligned} \text{minimize} \quad & \sum_{s \in S} x_s \\ \text{subject to} \quad & \sum_{s:e \in s} x_s \geq 1, \forall e \in U \\ & 0 \leq x_s \leq 1, \forall s \in S \end{aligned}$$

(b) (1p.) Consider the relaxed LP. Show that for each $e \in \mathcal{U}$, there is at least one set $s \in S$ such that $x_s \geq 1/3$.

When we take close look at this condition, $\sum_{s:e\in s} x_s \geq 1, \forall e \in U$, it means that the sum of the set value x_s where e is contained in must be nonzero. Between 0 and 1, only 1 is nonzero. Because this problem has been turned into a fractional problem, the sum of at most 3 sets must be larger than or equal to 1. Proof by contradition:

Supposed that e is contained in 3 different sets of s and all of their sets value is strictly smaller than 1/3. Then $x_{s1} + x_{s2} + x_{s3} < \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$, which fails to satisfy the condition $\sum_{s:e\in s} x_s \geq 1, \forall e \in U$

- ⇨ There must exists one set $s \in S$ such that $x_s \geq 1/3$ to ensure that the condition $\sum_{s:e\in s} x_s \geq 1, \forall e \in U$ is guaranteed. This proof also applies when the number of sets that e is contained in is less than 3 sets.

(c) (2p.) Show that, from an optimum solution of the relaxed LP, we can build a feasible solution for the initial IP such that this solution is at most three times larger than the optimum solution of the relaxed LP.
Hint: Recall the rounding method for vertex cover.

The solution of the original IP problem is at most 3 times larger than the optimum solution (3-approximation) of the relaxed LP because the cost of the relaxed LP increased by a factor of 3, and we know that at least one set whose value $x_s \geq \frac{1}{3}$ is included to cover all elements. Altogether, the cost of the relaxed LP is at least 3 times smaller than the cost of the initial IP.

(d) (1p.) We want to cover the set $\mathcal{U}$ with the smallest possible number of sets in S. Using the results above, give a 3-approximation algorithm for this problem. *You may use a black-box LP solver (for a relaxed LP) as a subroutine in your algorithm. Notice that such black-box does not solve the integer version of an LP.*

From the OPT solution of the relaxed LP, the feasible solution for the initial IP can be constructed as follows:

(1) Generate a fractional value from a black box. This value must be at least 1/3. Then assign it to a random set $s \in S$ where it still has some elements inside
(2) Record which elements e are contained inside this set s. Then cross out this element from other sets where e is contained it (at most 2 sets)
(3) Repeat step (1-2) until no elements are left that have not been contained in any sets
(4) Assign 0 to all the rest sets.
(5) Finally, multiply the value x_s of all sets s by 3. This is the final solution of the LP problem.