CS-E3190 Principles of Algorithmic Techniques

04. Local Search – Graded Exercise

Please read the following **rules** very carefully.

- Do not consciously search for the solution on the internet.
- You are allowed to discuss the problems with your classmates but you should **write the solutions yourself**.
- Be aware that **if plagiarism is suspected**, you could be asked to have an interview with teaching staff.
- The teaching staff can assist with understanding the problem statements, but will **not be giving any hints** on how to solve the exercises.
- In order to ease grading, we want the solution of each problem and subproblem to start on a **new page**. If this requirement is not met, **points will be deduced**.

1. **Spanners.** Let $G = (V, E)$ be an undirected graph and let $d_G(u, v)$ be the distance between $u \in V$ and $v \in V$ in G. A proper subgraph G' of G is a t-spanner of G if and only if $d_{G'}(u, v) \leq t \cdot d_G(u, v)$, $\forall u, v \in V$.

   (a) Let S be a t-spanner of a graph G. What is the value of t (in the worst case) if S is a spanning tree of G ?

   (b) The girth of a graph G is the length of the shortest cycle in G, and is infinity if G is acyclic (since we are considering undirected graphs, acyclic means G is a tree). Prove that an undirected unweighted graph $G = (V, E)$ of girth strictly larger than $t + 1$ has no proper subgraph that is a t–spanner.

   (c) Let $G = (V, E, w)$ be a weighted graph. Consider the following algorithm:

   Algorithm 1: $GreedySpanner(G, t)$

   $E' = \emptyset$
   $G' = (V, E')$
   for $(u, v) \in E$ **do**
       if $d_{G'}(u, v) > t \cdot w(u, v)$ **then**
           $E' = E' \cup \{(u, v)\}$
       end
   end
   return G'

   Prove that this algorithm yields a t–spanner for G.

   (d) Prove that, if the edges are sorted in non-decreasing order of weights at the beginning of the algorithm, the output G' of the algorithm is such that its girth is at least $t + 1$.