1. **Spanners.** Let $G = (V, E)$ be an undirected graph and let $d_G(u, v)$ be the distance between the vertices u and v in G. A subgraph $G' = (V, E')$, such that $E' \subseteq E$, is a t-spanner of G if $d_{G'}(u, v) \le t \cdot d_G(u, v)$, $\forall u, v \in V$.

   (a) (2p.) In this exercise, the goal is to find an n-node graph where any spanning tree is a bad spanner. Let $t \le n/2$. For any given n, construct an n-node graph such that there is no spanning tree T of G that is also a t-spanner.

The graph G of n-node, whose all spanning trees are bad, is the undirected cycle of n-nodes. Now we need to prove that any spanning tree of a cycle is a bad spanner, or $t > n/2$. We see that the spanning tree of a cycle can be constructed by removing one edge from the cycle, which happens to be also a minimum spanning tree (MST). We know that in a cycle, $d_G(u,v) = 1$ in the case when two nodes u and v are neighbors of the cycle. However, when one edge is removed to obtain the MST G', then $d_{G'}(u,v) = n-1$, in the case when these two nodes u and v belong to the endpoints of the removed edge. In other words, they have to go around the other direction along the cycle, where the number of edges is n – 1. Plugging into the equation:

$d_{G'}(u,v) \le t \cdot d_G(u,v) \Rightarrow n-1 \le t \cdot 1 \Rightarrow t \ge n-1$. The smallest cycle that can be formed is n = 3. Plugging in both equations, we have:

$$\begin{array}{l} t \ge n-1 \Rightarrow t \ge 3-1 \Rightarrow t \ge 2 \\ t \le n/2 \Rightarrow t \le 3/2 \Rightarrow t \le 1.5 \end{array} \Rightarrow \text{No t can satisfy these equations.}$$

We can see that by using induction step, with t larger than 3, these equations are also not satisfied. Therefore, the n-node graph where any spanning tree is a bad spanner is a cycle of n-nodes.

(b) (3p.) Let $G = (V, E, w)$ be a weighted graph. Recall, that for a weighted graph the distance is defined as the total weight of the shortest weighted path, ie.

$$d(u, v) = \min_{uv\text{-path } P} \sum_{e \in P} w(e).$$

Prove that the following algorithm yields a t–spanner for G.

Algorithm 1: $GreedySpanner(G, t)$

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E' = ∅
G' = (V, E')
for (u, v) ∈ E do
    if d_G'(u, v) > t · w(u, v) then
        E' = E' ∪ {(u, v)}
    end
end
return G'
```

First of all, the subgraph G' is initialized with all the vertices and no edges. If there are no paths connecting u and v in G', then $d_{G'}(u,v) = \infty$. We can notice that for all edges in the graph, $d_G(u,v) = w(u,v)$ if there are not any other paths connecting u and v except $E(u,v)$.

We can analyze the three cases:

(1) $d_{G'}(u,v) > t \cdot w(u,v)$ when u and v has not been connected

Edges are gradually added to the subgraph. Because of that, when u has been connected to the subgraph by some edges and v is not connected to the subgraph, then $d_{G'}(u,v) = \infty > t \cdot w(u,v)$ is always correct. When E(u,v) is added into E', v is connected to the subgraph, making G' a spanning graph of G

(2) $d_{G'}(u,v) > t \cdot w(u,v)$ is false when u and v has been connected before. In other words, we have $d_{G'}(u,v) \le t \cdot w(u,v)$, which already satisfies the t-spanner condition, so this edge doesn't need to be added to the graph at all.

(3) $d_{G'}(u,v) > t \cdot w(u,v)$ is true when u and v has been connected before. This breaks the t-spanner condition so the edge is added to the subgraph. After the edge is added to the subgraph, we have updated a new shorter path between u and v, which is $E(u,v) \Rightarrow d_{G'}(u,v) = w(u,v)$. It happens that $w(u,v) \le t \cdot w(u,v)$ for all integers t and $w(u,v)$. This satisfies the t-spanner condition

=> The algorithm yields a t-spanner for G.

2. **Individual exercise: Girth.** The *girth* of a graph G is the length of the shortest cycle in G, and it is infinity if G is acyclic[1]. Notice that the length of a cycle refers to the number of edges in it.

   (a) (2p.) Prove that an undirected unweighted graph $G = (V, E)$ of girth strictly larger than $t + 1$ has no proper subgraph that is a t–spanner.

[1]Since we are considering undirected graphs, acyclic means that G is a tree.

First, we assume that graph G has many cycles and the length of the shortest cycle is called the girth. A proper subgraph means that the subgraph is a spanning tree and its number of edges is strictly smaller than number of edges of graph G. In other words, at least one edge must be removed from graph G to obtain the proper subgraph. If this edge makes the graph disconnected, t would be infinity and it will never have a proper subgraph that is a t-spanner.

=> We can only remove an edge from a cycle.

Observation: consider an edge E(u, v) belongs to any cycles in G. Then $d_G(u,v)=1$ and let denote the length of this cycle as k. Of course, $girth \le k$ by definition. When the edge is removed, the shortest distance between u and v is to go around the cycle in the opposite direction

=> $d_{G/E(u,v)}(u,v) = k-1 \ge girth - 1$

Another condition we are given is $girth > t+1 \Rightarrow d_{G/E(u,v)}(u,v) = k-1 > t$

=> The graph G has $girth > t+1$ => the spanner property is not satisfied for the vertices (u,v) when the edge E(u,v) is removed => There does not exist any proper subgraph that is a t-spanner (proven)

(b) (3p.) Suppose that the edges are sorted in a non-decreasing order according to their weights, i.e., the greedy algorithm iterates over the edges in the sorted order. Prove that the output of Algorithm 1 has girth at least $t + 1$.

We can use proof by contradiction here.
First, we assume $girth < t+1$ => The smallest cycle in this graph has at most t edges.
We denote the vertices that are connected by the edge with the maximum weigh is $u_{\max}$ and $v_{\max}$. Due to the non-decreasing ordering of the edge weights, we see that $w(u_{\max}, v_{\max})$ is processed at the last stage, where the smallest cycle has at most t – 1 edges. According to the algorithm, this last edge is added only if $d_{G'}(u_{\max}, v_{\max}) > t \cdot w(u_{\max}, v_{\max})$. However, because the edge has not been added yet, we have to consider the opposite direction => $d_{G'}(u_{\max}, v_{\max}) \leq (t-1) \cdot w(u_{\max}, v_{\max})$ because we know that $w(u_{\max}, v_{\max})$ is the largest weight => $d_{G'}(u_{\max}, v_{\max}) \leq t \cdot w(u_{\max}, v_{\max})$ (contradiction)
=> The largest weight edge is not added to the cycle so the smallest cycle in this graph can have more than t edges => $girth \geq t+1$ (proven)