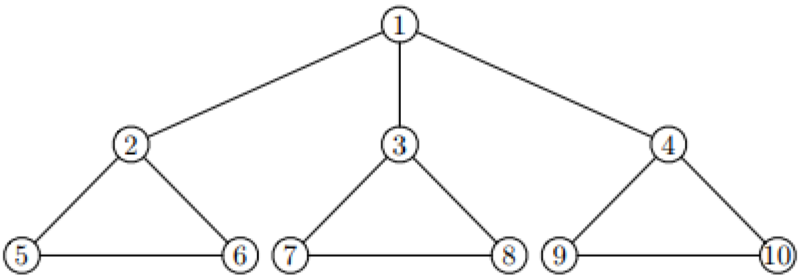
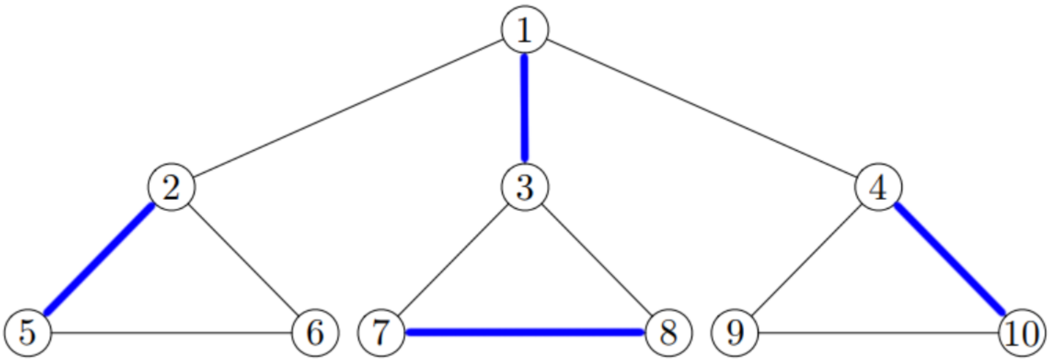
1. **Graph theory proofs.**

   (a) (3p.) Prove that the size of the maximum matching for the following graph is four. Give a maximal matching that is not maximum.



A maximum matching is a matching that contains the largest possible number of edges. The maximum matching is also a maximal matching, and it is not a proper subset of any other matching.

One maximum matching M of the graph is edges 2-5, 1-3, 7-8 and 4-10



Reason why this is a maximal matching:

If we add edge (2, 6), it will intersect edge (2, 5)

If we add edge (5, 6), it will intersect edge (2, 5)

If we add edge (3, 7), it will intersect edge (7, 8)

If we add edge (3, 8), it will intersect edge (7, 8)

If we add edge (4, 9), it will intersect edge (4, 10)

If we add edge (9, 10), it will intersect edge (4, 10)

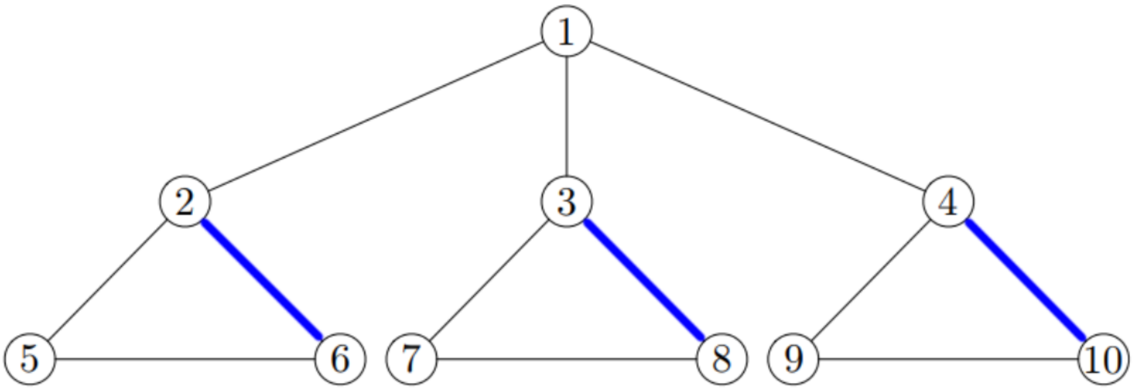
If we add edge (1, 2), it will intersect edge (1, 3)

If we add edge (1, 4), it will intersect edge (1, 3)

=> No more edges can be added to M, so M is a maximal matching pair. There are 10 vertices in this graph, so theoretically the maximum matching will have at most 10/2 = 5 edges. For three triangles 2-5-6, 3-7-8 and 4-9-10, only one pair can be constructed because the vertices in the triangles are interconnected and the number of vertices is 3 => There are at most 3 matchings. For the tree structure 1-2-3-4, only one pair can be constructed, either 1-2, 1-3 or 1-4, because node 1 is all connected by nodes 2,3,4 => There is at most 1 matching.
=> Maximum matching pairs in this graph is 3 + 1 = 4 and thus graph M is a maximum matching.

A maximal matching is a matching M of a graph G that is not a subset of any other matching. In other words, a matching M of a graph G is maximal if every edge in G has a non-empty intersection with at least one edge in M.

One maximal matching M of the graph is edges 2-6, 3-8 and 4-10



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If we add edge (4, 9), it will intersect edge (4, 10)

If we add edge (9, 10), it will intersect edge (4, 10)

If we add edge (1, 2), it will intersect edge (2, 6)

If we add edge (1, 3), it will intersect edge (3, 8)

If we add edge (1, 4), it will intersect edge (4, 10)

=> No more edges can be added to M, so M is a maximal matching pair. However, M is not maximum, because cardinality of M is 3, meanwhile we have proven in (a) that maximum matching of this graph has cardinality of 4.

(b) (2p.) Using Handshaking lemma, prove that the average degree of any tree is $O(1)$.

The handshaking lemma is

$$\text{Let } G=(V,E) \text{ be a graph}$$
$$\sum_{v\in V}\deg(v)=2|E|$$

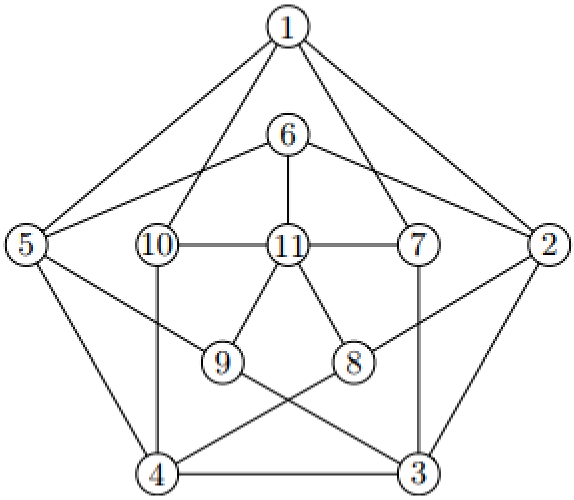
By definition, a trees with n number of vertices has (n - 1) number of edges. The handshaking lemma for trees is:

$$\sum_{v\in V}\deg(v)=2|E|=2(n-1)$$

The average degree for each vertex is then $avg\left(\deg_{tree}(v)\right)=\frac{2(n-1)}{n}=2-\frac{2}{n}\leq 2$.

In other words, the average degree for each vertex is smaller than 2, or $avg\left(\deg_{tree}(v)\right)=O(1)$

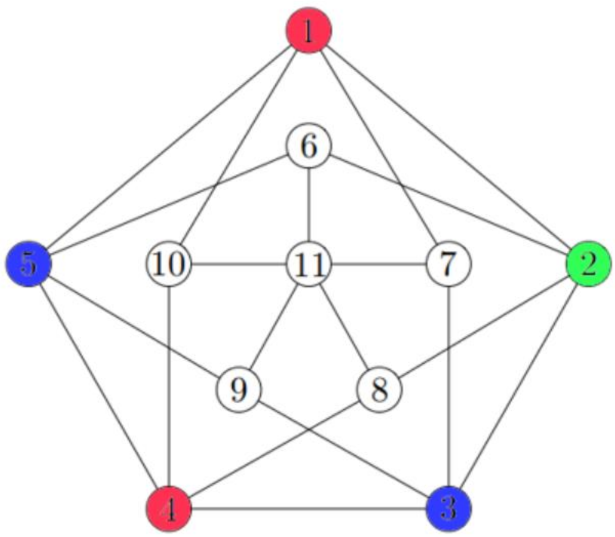
2. **Individual exercise: Graph coloring.** (5p.) Prove that the chromatic number of the following graph is 4.



First, we can call χ as the chromatic number of this graph.

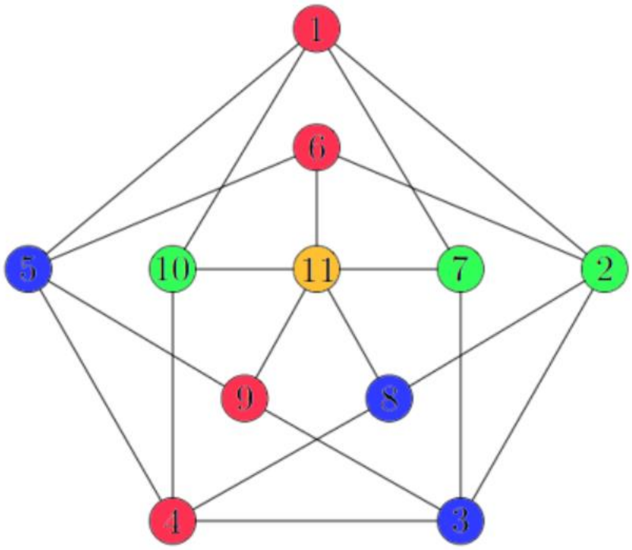
Observation: In the outer layer, There is a cycle with 5 vertices. A cycle of vertices requires alternating colors on each vertex so that no adjacent vertices will have the same color. For a cycle of even number of vertices, 2 colors are enough. However, for odd number of vertices, 3 colors are required because the last vertex will have the same color as the first vertex, so the last vertex in the cycle must have a third color that is different from the first 2 colors.

We can proceed to color the outer cycle with three colors as follows:



From this point, we can gradually find out the colors of the rest vertices
For vertex 6, its neighbors 5 and 2 are blue and green, so vertex 6 should have red color.
For vertex 7, its neighbors 1 and 3 are red and blue, so vertex 7 should have green color
For vertex 8, its neighbors 2 and 4 are green and red, so vertex 8 should have blue color
For vertex 10, its neighbors 1 and 4 are both red, so vertex 10 can be either blue or green
For vertex 9, its neighbors 5 and 3 are both blue, so vertex 9 can be either red or green
Finally, for the center vertex 11, since its neighbors 6, 7, 8 all have different colors, vertex 11 must have a different color from red, blue and green, requiring the fourth color.

We can finally color the graph as follows:



=> This graph requires at least 4 colors to color the vertices => Its chromatic number is $\chi = 4$