CS-E3190 Principles of Algorithmic Techniques

08. Linear Programming – Graded Exercise

Please read the following **rules** very carefully.

- Do not consciously search for the solution on the internet.
- You are allowed to discuss the problems with your classmates but you should **write the solutions yourself**.
- Be aware that **if plagiarism is suspected**, you could be asked to have an interview with teaching staff.
- Each week the second exercise is an **individual exercise**, and the teaching staff will not give hints or help with them. You are allowed to ask for hints for the first exercise.
- In order to ease grading, we want the solution of each problem and subproblem to start on a **new page**. If this requirement is not met, **points will be reduced**.

1. **Linear Program** (2p.) A manufacturer wants to produce ice-cream. An ice-cream patch must contain at least 20 units of fat, at least 20 units of sugar, at least 1 unit of egg and at most 60 units of water. Below is a table of the raw ingredients available to the manufacturer, their compositions as percentages, and their prices.

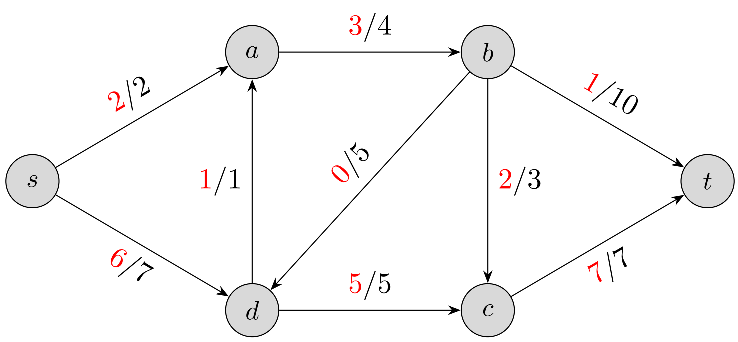
Raw ingredients	**Cream**	**Egg yolk**	**Whole milk**	**Frozen sweetened egg yolk**	**glucose syrup**	**water**
Fat	40	50	12	30	0	0
Sugar	0	0	0	14	70	0
Egg	0	40	0	40	0	0
Water	60	10	88	16	30	100
Price per unit	2	4	1	2	0.8	0

The goal of the manufacturer is to determine the recipe with minimum cost.

(a) (1p.) Write this problem as a linear program.

(b) (1p.) Transform the linear program into the standard form. See the lecture script for the definition of the standard form.

2. **Flows** (3p.) Given the graph G below, the capacities of each edge are written in black and a flow is given in red.



(a) (1p.) What are the saturated edges?

(b) (1p.) Draw the residual graph.

(c) (1p.) Prove that the given flow is maximum. You can use the fact that the residual graph of a maximum flow has no augmenting paths.

3. **Individual exercise: Integer Program.** (5p.) Consider a set of points $\mathcal{U}$ and a set of sets of points S such that $\forall s \in S, s \subseteq \mathcal{U}$ and $\forall e \in \mathcal{U}$, e is in at most 3 different sets in S. Consider the following Integer Program:

$$\begin{aligned} \text{minimize} \quad & \sum_{s \in S} x_s \\ \text{subject to} \quad & \sum_{s : e \in s} x_s \geq 1, \forall e \in \mathcal{U} \\ & x_s \in \{0, 1\}, \forall s \in S \end{aligned}$$

Note that $\sum_{s:e \in s} x_s$ has at most 3 terms for each $e \in \mathcal{U}$.

(a) (1p.) Write the relaxation of this integer program.

(b) (1p.) Consider the relaxed LP. Show that for each $e \in \mathcal{U}$, there is at least one set $s \in S$ such that $x_s \geq 1/3$.

(c) (2p.) Show that, from an optimum solution of the relaxed LP, we can build a feasible solution for the initial IP such that this solution is at most three times larger than the optimum solution of the relaxed LP.
Hint: Recall the rounding method for vertex cover.

(d) (1p.) We want to cover the set $\mathcal{U}$ with the smallest possible number of sets in S. Using the results above, give a 3-approximation algorithm for this problem.
You may use a black-box LP solver (for a relaxed LP) as a subroutine in your algorithm. Notice that such black-box does not solve the integer version of an LP.