# CS-E3190 Principles of Algorithmic Techniques

## *04. Local Search – Graded Exercise*

Please read the following **rules** very carefully.

- Do not consciously search for the solution on the internet.
- You are allowed to discuss the problems with your classmates but you should **write the solutions yourself**.
- Be aware that **if plagiarism is suspected**, you could be asked to have an interview with teaching staff.
- Each week the second exercise is an **individual exercise**, and the teaching staff will not give hints or help with them. You are allowed to ask for hints for the first exercise.
- In order to ease grading, we want the solution of each problem and subproblem to start on a **new page**. If this requirement is not met, **points will be reduced**.

1. **Spanners.** Let $G = (V, E)$ be an undirected graph and let $d_G(u, v)$ be the distance between the vertices $u$ and $v$ in $G$. A subgraph $G' = (V, E')$, such that $E' \subseteq E$, is a $t$-spanner of $G$ if $d_{G'}(u, v) \leq t \cdot d_G(u, v)$, $\forall u, v \in V$.

   (a) (2p.) In this exercise, the goal is to find an $n$-node graph where any spanning tree is a bad spanner. Let $t \leq n/2$. For any given $n$, construct an $n$-node graph such that there is no spanning tree $T$ of $G$ that is also a $t$-spanner.

   (b) (3p.) Let $G = (V, E, w)$ be a weighted graph. Recall, that for a weighted graph the distance is defined as the total weight of the shortest weighted path, ie.

   $$d(u, v) = \min_{uv\text{-path } P} \sum_{e \in P} w(e).$$

   Prove that the following algorithm yields a $t$–spanner for $G$.

   **Algorithm 1:** $GreedySpanner(G, t)$

   ```
   E' = ∅
   G' = (V, E')
   for (u, v) ∈ E do
       if d_G'(u, v) > t · w(u, v) then
           E' = E' ∪ {(u, v)}
       end
   end
   return G'
   ```

2. **Individual exercise: Girth.** The *girth* of a graph $G$ is the length of the shortest cycle in $G$, and it is infinity if $G$ is acyclic[1]. Notice that the length of a cycle refers to the number of edges in it.

   (a) (2p.) Prove that an undirected unweighted graph $G = (V, E)$ of girth strictly larger than $t + 1$ has no proper subgraph that is a $t$–spanner.

[1] Since we are considering undirected graphs, acyclic means that $G$ is a tree.

(b) (3p.) Suppose that the edges are sorted in a non-decreasing order according to their weights, i.e., the greedy algorithm iterates over the edges in the sorted order. Prove that the output of Algorithm 1 has girth at least $t + 1$.