Graded Exercise 4

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Problem 1

a.

Let denote the number of edge in a tree as m.

Consider, in a tree, the distance between two vertices $\{u, v\}$ will be m at most. This distance is achieved in case the tree is in a shape of a *line*, where the distance of two degree 1 nodes is m.

On the other hand, in a graph, those two node can be neighbour, which means $d_G(u, v) = 1$. Additionally, since we are considering spanning tree, the number of vertices in the tree is equal to the number of vertices in the graph, let denote this number as n. We also know that in a tree, the number of egde is equal to the number of vertices minus 1.

$\Rightarrow$ The worst case value of t is $n - 1$.

b.

Consider an abitrary edge $(u, v) \in E$. Since this is an edge, $d_G(u, v) = 1$. Additionally, let the length of the shortest cycle that contains the vertices u and v denoted as k. Now, if we remove the edge (u, v) to obtain the subgraph, we get that $d_{G'}(u, v) = k - 1$. Since the girth of the graph is strictly larger than $t + 1$, we have the following:

$$\begin{aligned} & k > t + 1 \\ \Leftrightarrow \quad & k - 1 > t \\ \Rightarrow \quad & d_{G'}(u, v) > t \cdot d_G(u, v) \end{aligned}$$

We can see that the condition of t-spanner is violated.

$\Rightarrow$ A graph with the girth strictly larger than $t + 1$ has no proper subgraph that is a t-spanner.

c.

Consider an edge $\{u, v\} \in G$. If $d_G(u, v) \neq w(u, v)$, we can always remove the edge $\{u, v\}$ without changing the distance of u and v. And every spanner of the resulting graph is also a spanner of the original graph.

So we get that $d_G(u, v) = w(u, v)$. Now, consider the edge $\{u, v\}$, there are two cases regarding the distance of u and v in the subgraph G':

$$\begin{cases} d_{G'}(u, v) \leq t \cdot w(u, v) \\ d_{G'}(u, v) > t \cdot w(u, v) \end{cases}$$

In the former case, the property of t-spanner is already satisfied. And in the second case, $\{u, v\}$ is added to G', making $d_{G'}(u, v) = w(u, v)$. Clearly, for every postive interger t, $w(u, v) \leq t \cdot w(u, v)$, thus satisfying the spanner requirement.

$\Rightarrow$ The algorithm yields a t-spanner.

d.

Assume that G' has girth less than $t+1$. This means that the graph has a cycle which length is at most t. Let the last edge added to the said cycle be $\{u, v\}$. Since the edges are ordered, $w(u, v)$ should be the largest among the cycle edges. Also, the cycle up to this point can only have at most $t-1$ edges, otherwise it will create a cycle of length $t+1$. But in that case, we also have the following:

$$d_{G'}(u, v) \leq (t-1) \cdot w(u, v) < t \cdot w(u, v)$$

But by the algorithm, $\{u, v\}$ can only be added if $d_{G'}(u, v) > t \cdot w(u, v)$. Thus, we have a contradiction.

$\Rightarrow$ The girth of G' is at least $t+1$.