CS-E3190 Principles of Algorithmic Techniques

01. Graph Bootcamp – Tutorial Exercise

1. **Chromatic and independence numbers.** Prove that the inequality

$$\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$$

holds for any connected graph G. Recall that the chromatic number $\chi(G)$ is the minimum number of colors required to color G, and the independence number $\alpha(G)$ is the size of the largest independent set. Give a graph family[1] for which this inequality is arbitrarily loose, i.e., $\chi(G)$ is arbitrarily larger than $|V(G)|/\alpha(G)$?

Solution. Let $k = \chi(G)$ and consider a k-coloring of a connected graph G. Let us define V_i as the set on nodes colored i. Observe that $V = V_1 \cup V_2 \cup \cdots \cup V_k$ and each set V_i is an independent set. Hence,

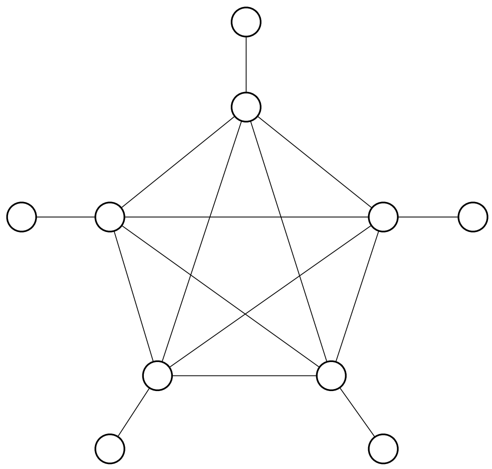
$$|V(G)| = \sum_{i=1}^{k} |V_i| \leq \sum_{i=1}^{k} \alpha(G) = \chi(G)\alpha(G) \quad \Rightarrow \quad \chi(G) \geq \frac{|V(G)|}{\alpha(G)}.$$

The first inequality is due to the size of all independent sets being upper bounded by $\alpha(G)$ and the last equality is due to our original definition of k.

One graph family for which this inequality is arbitrarily loose is as follows. Consider a complete graph K_t with all nodes having one additional neighbor of degree 1. Note that variable t fully describes every graph in this graph family. Since t is not bounded, this graph family is infinite. The complete graph as a subgraph ensures that the chromatic number is exactly t. However, the additional neighbors of degree 1 constitute an independent set of size t. Hence, for arbitrarily large t, this construction yields

$$\begin{aligned} \chi(G) &\geq \frac{|V(G)|}{\alpha(G)} \\ t &\geq 2t/t \\ t &\geq 2. \end{aligned}$$

An example for $t = 5$ is shown in the following figure.



[1]A graph family is a collection of graphs that have a specific property in common, e.g., being complete, being bipartite or having a triangle as a subgraph. Keep in mind that a particular graph may fall in several families and some families themselves may contain subfamilies.

2. **Tree property.** Show that in trees, each node of degree at least 3 can be mapped to at least one unique leaf node. Prove using induction.

   Solution. The following induction proof is *constructive*, i.e., it provides a method for constructing a mathematical object, which proves its existence. This is in contrast to a *non-constructive* proof, which proves the existence of a mathematical object directly. This week's Graded Exercise 3 requires a non-constructive proof for this problem.

   Let us recall the definition of a rooted tree. In a rooted tree, there is one unique node called the *root*. All edges are directed towards the root node. As a consequence, all nodes except the root have exactly one outgoing edge. The orientation of an edge establishes a relation between nodes: if edge $\{u, v\}$ is oriented from u to v, v is the *parent* of u and u is the *child* of v. Children, grandchildren, etc. are referred to as descendants. A node is at depth d' when the (unique) path between it and the root is of length d'. The root is considered to be at depth 0. The depth of a tree is the maximum depth among all nodes in the tree.

   Consider a tree $T = (V, E)$. Consider rooting T such that it has depth d. Let us label all leaf nodes and all nodes of degree at least 3 as *free*. Our aim is to map free nodes of degree at least 3 to free leaf nodes. When nodes are mapped, they are no longer free. We claim that at every depth, all nodes know of a free descendant leaf node and all nodes of degree at least 3 are mapped. Let us perform induction on the depth of the tree.

   (a) The base case is depth d. Observe that there all nodes at depth d are free leaf nodes. Since they are indeed free and they know about themselves, the claim holds.

   (b) Now for the induction step. The induction hypothesis is that the claim holds for some arbitrary depth $d' \leq d$ and consider node v at depth $d' - 1$. There are three cases as follows.

      i. If $\deg(v) = 1$, it is free and it knows about itself.

      ii. If $\deg(v) = 2$, node v has exactly one child u that knows of a free descendant leaf node by the induction hypothesis. Node v learns about the free descendant leaf node of u.

      iii. If $\deg(v) \geq 3$, node v has at least two children that all know a free descendant leaf node by the induction hypothesis. Observe that v's children know *different* free descendant leaf nodes. We map node v to a free descendant leaf node of one of its children. Since node v has at least two children, v has at least one other child from which it can learn about a free descendant leaf node.

   We have shown that for depth $d' - 1$ it holds that all nodes have a free descendant leaf node and all nodes of degree at least 3 are mapped, concluding the proof.