CS-E3190 Principles of Algorithmic Techniques

02. Recursive Algorithms – Tutorial Exercise

1. **Matrix multiplication.** Let $A, B \in \mathbb{R}^{2\times 2}$ and $C = AB$ s.t.

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}.$$

Following the naive approach, C is computed as follows,

$$\begin{aligned} c_{11} &= a_{11}b_{11} + a_{12}b_{21} \\ c_{12} &= a_{11}b_{12} + a_{12}b_{22} \\ c_{21} &= a_{21}b_{11} + a_{22}b_{21} \\ c_{22} &= a_{21}b_{12} + a_{22}b_{22} \end{aligned}$$

For this naive approach, 8 multiplications and 4 additions are needed.

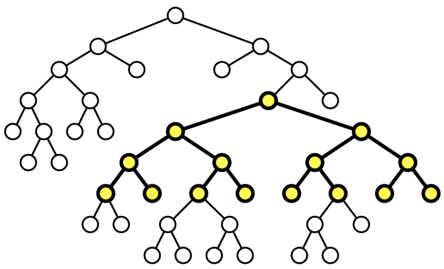
The Strassen's algorithm enables to compute C with only 7 multiplications as follows,

$$\begin{aligned} P_1 &= (a_{11} + a_{22})(b_{11} + b_{22}) && \\ P_2 &= (a_{21} + a_{22})b_{11} && \\ P_3 &= a_{11}(b_{12} - b_{22}) && \\ P_4 &= a_{22}(b_{21} - b_{11}) && c_{11} = P_1 + P_4 - P_5 + P_7 \\ P_5 &= (a_{11} + a_{12})b_{22} && c_{12} = P_3 + P_5 \\ P_6 &= (a_{21} - a_{11})(b_{11} + b_{12}) && c_{21} = P_2 + P_4 \\ P_7 &= (a_{12} - a_{22})(b_{21} + b_{22}) && c_{22} = P_1 - P_2 + P_3 + P_6 \end{aligned}$$

   (a) Let $A, B \in \mathbb{R}^{n\times n}$ and $C = AB$. Write a recursive algorithm based on Strassen's design for computing C.
   Hint: you can assume n is a power of two, since the matrices can be padded when implementing the algorithm.

   (b) Analyze the running time of your algorithm.

2. **Complete sub-trees.** A binary tree is complete if every internal node has two children, and every leaf has exactly the same depth.



Describe and analyze a recursive algorithm that computes the largest complete sub-tree of a given binary tree. Your algorithm should return both the root and the depth of this sub-tree.