The Primal-Dual Scheme

Outline

- What is a Primal and a Dual?
  - A mechanical way to find a dual
- Why…?
  - Weak duality
- The Primal Dual method
  - Vertex Cover

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Learning objectives:
You are able to

- describe how to create a dual LP from a primal LP
- restate the statement of weak duality
- apply the primal dual method to obtain an approximate vertex cover

The Primal LP

Minimize
x

Objective function $\boldsymbol{f(x)}$

Subject to

Constraints $\boldsymbol{g_1(x)}$
$\boldsymbol{g_2(x)}$
…
$\boldsymbol{g_m(x)}$

The Primal LP

Minimize
x

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

Subject to

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \geq b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \geq b_2$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \geq b_m$$

$x_i \geq 0$, for all $1 \leq i \leq n$

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x

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$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\geq b_2 \\ \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\geq b_m \end{aligned}$$

$$x_i \geq 0, \text{ for all } 1 \leq i \leq n$$

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$x_i \geq 0$, for all $1 \leq i \leq n$

The Primal LP

Minimize
x

$$c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \quad : \quad cx$$

Subject to

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n &\geq b_1 \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n &\geq b_2 \\ \\ a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n &\geq b_m \end{aligned} \quad : \quad Ax \geq b$$

$x_i \geq 0$, for all $1 \leq i \leq n$

The Dual LP

Maximize
y

$$b_1 y_1 + b_2 y_2 + \cdots + b_m y_m$$

Subject to

$$\begin{aligned}
a_{11} y_1 + a_{21} y_2 + \cdots + a_{m1} y_m &\leq c_1 \\
a_{12} y_1 + a_{22} y_2 + \cdots + a_{m2} y_m &\leq c_2 \\
&\vdots \\
a_{1n} y_1 + a_{2n} y_2 + \cdots + a_{mn} y_m &\leq c_n
\end{aligned}$$

$y_i \geq 0$, for all $1 \leq i \leq m$

The Dual LP

Maximize
y

$$b_1 y_1 + b_2 y_2 + \cdots + b_m y_m$$

Subject to

$$\begin{aligned} a_{11} y_1 + a_{21} y_2 + \cdots + a_{m1} y_m &\leq c_1 \\ a_{12} y_1 + a_{22} y_2 + \cdots + a_{m2} y_m &\leq c_2 \\ &\vdots \\ a_{1n} y_1 + a_{2n} y_2 + \cdots + a_{mn} y_m &\leq c_n \end{aligned}$$

$$y_i \geq 0, \text{ for all } 1 \leq i \leq m$$

The Dual LP

Maximize
y

$$b_1y_1 + b_2y_2 + \cdots + b_my_m$$

Subject to

$$\boxed{\begin{matrix} a_{11}y_1 \\ a_{12}y_1 \\ \vdots \\ a_{1n}y_1 \end{matrix}} \begin{matrix} + a_{21}y_2 + \cdots + a_{m1}y_m \leq c_1 \\ + a_{22}y_2 + \cdots + a_{m2}y_m \leq c_2 \\ \vdots \\ + a_{2n}y_2 + \cdots + a_{mn}y_m \leq c_n \end{matrix}$$

$$y_i \geq 0, \text{ for all } 1 \leq i \leq m$$

Primal

$$\boxed{a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n} \geq b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \geq b_2$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \geq b_m$$

$$x_i \geq 0, \text{ for all } 1 \leq i \leq n$$

The Dual LP

Maximize
y

$$\boxed{b_1y_1 + b_2y_2 + \cdots + b_my_m}$$

Subject to

$$a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \le c_1$$
$$a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m \le c_2$$
$$\vdots$$
$$a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \le c_n$$

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Primal

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \ge b_1$$
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$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \ge b_m$$

$$x_i \ge 0, \text{ for all } 1 \le i \le n$$

The Dual LP

Primal:

cx

$Ax \geq b$

Maximize y

$$b_1y_1 + b_2y_2 + \cdots + b_my_m$$

$= \quad by$

Subject to

$$a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \leq c_1$$
$$a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m \leq c_2$$
$$\vdots$$
$$a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \leq c_n$$

$y_i \geq 0$, for all $1 \leq i \leq m$

$: \quad A^Ty \leq c$

The Dual LP

Primal:

$$cx$$
$$Ax \geq b$$

Dual:

$$by$$
$$A^T y \leq c$$

The Dual LP

Primal:

$$cx$$
$$Ax \geq b$$

Dual:

$$by$$
$$A^T y \leq c$$

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Weak Duality

Theorem:
Let $\overline{\boldsymbol{y}}$ be a feasible solution to the dual problem.
Then, $\Sigma_{\overline{\boldsymbol{y}}}\, b_i y_i \leq \Sigma_{\overline{\boldsymbol{x}}}\, c_j x_j$ for any feasible solution $\overline{\boldsymbol{x}}$ to the primal problem.

Weak Duality

Claim: $\Sigma_{\overline{\boldsymbol{y}}}\ b_i y_i \leq \Sigma_{\overline{\boldsymbol{x}}}\ c_j x_j$

Weak Duality

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$$a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \le c_1$$
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$$a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \le c_n$$

$y_i \ge 0$, for all $1 \le i \le m$

Dual

Weak Duality

Claim: $\Sigma_{\bar{y}}\, b_i y_i \leq \Sigma_{\bar{x}}\, c_j x_j$

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$y_i \geq 0$, for all $1 \leq i \leq m$

Dual

$$\Sigma_i y_i a_{1i} \leq c_1$$

$$\Sigma_i y_i a_{ni} \leq c_n$$

Weak Duality

Obs:
$\Sigma_i a_{ij} y_i \leq c_j$

Claim: $\Sigma_{\bar{y}}\, b_i y_i \leq \Sigma_{\bar{x}}\, c_j x_j$

$$a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \leq c_1$$
$$a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m \leq c_2$$
$$\vdots$$
$$a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \leq c_n$$

$y_i \geq 0$, for all $1 \leq i \leq m$

$\Sigma_i y_i a_{i1} \leq c_1$

$\Sigma_i y_i a_{n1} \leq c_n$

Dual

Weak Duality

Claim: $\Sigma_{\bar{y}}\ b_i y_i \leq \Sigma_{\bar{x}}\ c_j x_j$

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \geq b_1$$
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$x_i \geq 0$, for all $1 \leq i \leq n$

Primal

Weak Duality

Claim: $\Sigma_{\bar{y}}\, b_i y_i \leq \Sigma_{\bar{x}}\, c_j x_j$

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$x_i \geq 0$, for all $1 \leq i \leq n$

Primal

$$\Sigma_j a_{1j}x_j \geq b_1$$

$$\Sigma_j a_{mj}x_j \geq b_m$$

Weak Duality

Claim: $\Sigma_{\overline{y}}\, b_i y_i \leq \Sigma_{\overline{x}}\, c_j x_j$

Obs:

$b_i \leq \Sigma_j x_j a_{ij}$

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \geq b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \geq b_2$$

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$x_i \geq 0$, for all $1 \leq i \leq n$

$\Sigma_j a_{1j} x_j \geq b_1$

$\Sigma_j a_{mj} x_j \geq b_m$

Primal

Weak Duality

Obs:
$\Sigma_i a_{ij} y_i \leq c_j$

Obs:
$b_i \leq \Sigma_j x_j a_{ij}$

Claim: $\Sigma_{\bar{y}}\ b_i y_i \leq \Sigma_{\bar{x}}\ c_j x_j$

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$x_i \geq 0$, for all $1 \leq i \leq n$

Primal

Weak Duality

Obs:
$\Sigma_i a_{ij} y_i \leq c_j$

Obs:
$b_i \leq \Sigma_j x_j a_{ij}$

Claim: $\Sigma_{\bar{y}}\, b_i y_i \leq \Sigma_{\bar{x}}\, c_j x_j$

Use Obs:
$\Sigma_i b_i y_i \leq \Sigma_i \left(\Sigma_j \left(a_{ij} x_j\right) y_i\right)$

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \geq b_1$$
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$x_i \geq 0$, for all $1 \leq i \leq n$

Primal

Weak Duality

Obs:
$$\Sigma_i a_{ij} y_i \le c_j$$

Obs:
$$b_j \le \Sigma_i x_j a_{ij}$$

Claim: $\Sigma_{\overline{y}}\, b_i y_i \le \Sigma_{\overline{\boldsymbol{x}}}\, c_j x_j$

Use Obs:
$$\Sigma_i b_i y_i \le \Sigma_i\big(\Sigma_j(a_{ij}x_j)y_i\big)$$

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \ge b_1$$
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$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \ge b_m$$

$$x_i \ge 0, \text{ for all } 1 \le i \le n$$

Primal

$$\begin{aligned}\Sigma_{\overline{y}} b_i y_i &\le \Sigma_i\big(\Sigma_j(a_{ij}x_j)y_i\big)\\ &= \Sigma_j\big(\Sigma_i(a_{ij}y_i)x_j\big)\end{aligned}$$

Weak Duality

Obs:
$\Sigma_i a_{ij} y_i \leq c_j$

Obs:
$b_j \leq \Sigma_i x_j a_{ij}$

Claim: $\Sigma_{\bar{y}}\, b_i y_i \leq \Sigma_{\bar{x}}\, c_j x_j$

Use Obs:
$\Sigma_i b_i y_i \leq \Sigma_i \big(\Sigma_j (a_{ij} x_j) y_i\big)$

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$$x_i \geq 0, \text{ for all } 1 \leq i \leq n$$

Primal

$$\begin{aligned} \Sigma_{\bar{y}} b_i y_i &\leq \Sigma_i \big(\Sigma_j (a_{ij} x_j) y_i\big) \\ &= \Sigma_j \big(\Sigma_i (a_{ij} y_i) x_j\big) \\ &\leq \Sigma_{\bar{x}}\, c_j x_j \end{aligned}$$

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Vertex Cover

c_i

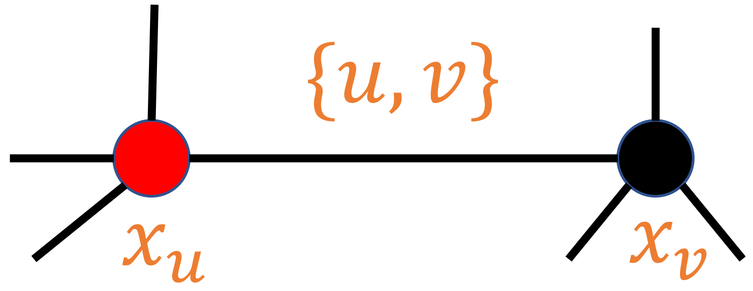
Minimize x $\quad \sum_{i=1}^{n} 1 \cdot x_i$

Subject to

$x_i + x_j \geq 1 \qquad$ for all $(i, j) \in E$

$x_i \geq 0 \qquad$ for all $i \in V$

A



Vertex Cover

Minimize
x $\Sigma_{i=1}^{n} 1 \cdot x_i$

Subject to

$x_i + x_j \geq 1$ for all $(i, j) \in E$
$x_i \geq 0$ for all $i \in V$

In the primal, there is exactly one constraint per edge $\{i, j\}$. Each row in the LP corresponds to one such constraint.
Think of the row as a vector with n entries and only entries for nodes i and j are non-zero.

Vertex Cover

Minimize
x

$$\Sigma_{i=1}^{n} 1 \cdot x_i$$

Subject to

$$x_i + x_j \geq 1 \qquad \text{for all } (i,j) \in E$$
$$x_i \geq 0 \qquad \text{for all } i \in V$$

In the dual, each row corresponds to a node u.
For each j, the non-zero entry from the primal turns into a non-zero entry in this row. These correspond to the edges incident on u.

Vertex Cover

Minimize
x

$\Sigma_{i=1}^{n} 1 \cdot x_i$

Subject to

$x_i + x_j \geq 1$ for all $(i, j) \in E$
$x_i \geq 0$ for all $i \in V$

In the dual, each row corresponds to a node u.
For each j, the non-zero entry from the primal turns into a non-zero entry in this row. These correspond to the edges incident on u.

We get:
$\Sigma_{u \in N(v)} y_{vu} \leq 1$ for all $v \in V$
$y_{uv} \geq 0$ for all $\{u, v\} \in E$

Vertex Cover

$$A^T = \boxed{\begin{array}{ll} \Sigma_{u \in N(v)} y_{vu} \leq 1 & \text{for all } v \in V \\ y_{uv} \geq 0 & \text{for all } \{u, v\} \in E \end{array}}$$

Minimize x $\quad \boxed{\Sigma_{i=1}^{n} 1 \cdot x_i}$

Subject to $\quad \boxed{\begin{array}{ll} x_i + x_j \geq 1 & \text{for all } (i, j) \in E \\ x_i \geq 0 & \text{for all } i \in V \end{array}}$

Vertex Cover

$$A^T = \boxed{\begin{array}{ll} \Sigma_{u \in N(v)} y_{vu} \leq w_v & \text{for all } v \in V \\ y_{uv} \geq 0 & \text{for all } \{u, v\} \in E \end{array}}$$

Minimize
x

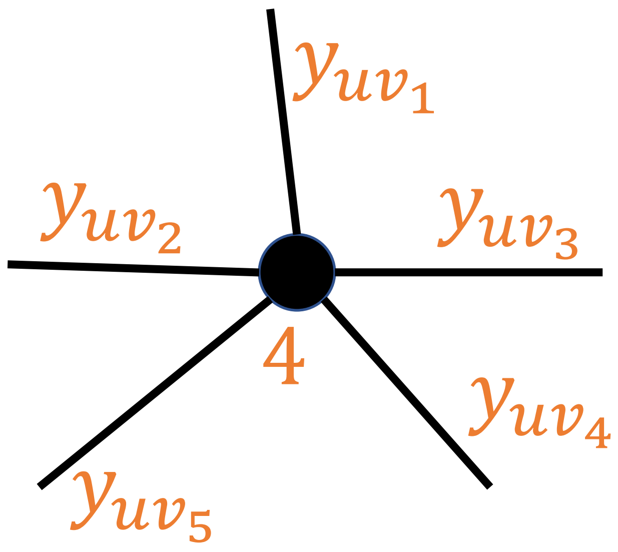
$$\boxed{\Sigma_{i=1}^{n} w_i \cdot x_i}$$

Subject to

$$\boxed{\begin{array}{ll} x_i + x_j \geq 1 & \text{for all } (i, j) \in E \\ x_i \geq 0 & \text{for all } i \in V \end{array}}$$

Vertex Cover

$\Sigma_{u \in N(v)} y_{vu} \leq w_v$	for all $v \in V$
$y_{uv} \geq 0$	for all $\{u, v\} \in E$



The Primal-Dual Method

Find a dual solution by solving the LP

The Primal-Dual method

The Primal-Dual Method

~~Find a dual solution by solving the LP~~

The Primal-Dual method

The Primal-Dual Method

The Primal-Dual method

$$\Sigma_{u \in N(v)} y_{vu} \leq w_v \qquad \text{for all } v \in V$$
$$y_{uv} \geq 0 \qquad \text{for all } \{u, v\} \in E$$

Dual

The Primal-Dual Method

The Primal-Dual method

Algorithm:
Iteratively pick edges $\{u, v\} \in E$

1. Increase y_{uv} until constraint for at least one endpoint becomes tight.

$\sum_{u \in N(v)} y_{vu} \leq w_v$	for all $v \in V$
$y_{uv} \geq 0$	for all $\{u, v\} \in E$

Dual

The Primal-Dual Method

The Primal-Dual method

Algorithm:

Iteratively pick edges $\{u, v\} \in E$

1. Increase y_{uv} until constraint for at least one endpoint becomes tight
2. Add tight endpoints to the cover

$$\begin{array}{ll} \Sigma_{u \in N(v)} y_{vu} \leq w_v & \text{for all } v \in V \\ y_{uv} \geq 0 & \text{for all } \{u, v\} \in E \end{array}$$

Dual

The Primal-Dual Method

The Primal-Dual method

Algorithm:
Iteratively pick edges $\{u, v\} \in E$

1. Increase y_{uv} until constraint for at least one endpoint becomes tight.
2. Add tight endpoints to the cover

Claim:
At least one endpoint of each edge becomes tight.
Proof:
If not, we could increase the variable on the non-tight edge.

The Primal-Dual Method

The Primal-Dual method

Algorithm:

Iteratively pick edges $\{u, v\} \in E$

1. Increase y_{uv} until constraint for at least one endpoint becomes tight.
2. Add tight endpoints to the cover

Claim:

At least one endpoint of each edge becomes tight.

Proof:

If not, we could increase the variable on the non-tight edge.

In $O(E)$ time, at least one endpoint per edge is tight.

The Primal-Dual Method

The Primal-Dual method

Algorithm:

Iteratively pick edges $\{u, v\} \in E$

1. Increase y_{uv} until constraint for at least one endpoint becomes tight
2. Add tight endpoints to the cover

Approximation:

Let $C \subseteq V$ be the resulting set cover.

The cost of the cover is at most

$$\Sigma_{u \in C}\, y_{uv}$$

The Primal-Dual Method

The Primal-Dual method

Algorithm:
Iteratively pick edges $\{u, v\} \in E$
1. Increase y_{uv} until constraint for at least one endpoint becomes tight
2. Add tight endpoints to the cover

Approximation:
Let $C \subseteq V$ be the resulting set cover.
The cost of the cover is at most

$$\Sigma_{u \in C}\, y_{uv}$$

In the worst case, each y_{uv} is counted twice, from both endpoints. Thus,

$$\Sigma_{u \in C}\, y_{uv} \leq 2 \cdot \Sigma_{\bar{y}}\, y_{uv}$$

The Primal-Dual Method

The Primal-Dual method

Algorithm:

Iteratively pick edges $\{u, v\} \in E$

1. Increase y_{uv} until constraint for at least one endpoint becomes tight
2. Add tight endpoints to the cover

Approximation:

$$\Sigma_{u \in C}\, y_{uv} \leq 2 \cdot \Sigma_{\bar{y}}\, y_{uv}$$

The Primal-Dual Method

The Primal-Dual method

Approximation:

$\Sigma_{u \in C}\, y_{uv} \le 2 \cdot \Sigma_{\bar{y}}\, y_{uv}$

Algorithm:

Iteratively pick edges $\{u, v\} \in E$

1. Increase y_{uv} until constraint for at least one endpoint becomes tight
2. Add tight endpoints to the cover

Weak Duality:

$\Sigma_{u \in C}\, y_{uv} \le 2 \cdot \Sigma_{\bar{y}}\, y_{uv} \le 2 \cdot \mathrm{OPT}$

The Primal-Dual Method

The Primal-Dual method

Approximation:

$\Sigma_{u \in C}\, y_{uv} \leq 2 \cdot \Sigma_{\bar{y}}\, y_{uv}$

Algorithm:

Iteratively pick edges $\{u, v\} \in E$

1. Increase y_{uv} until constraint for at least one endpoint becomes tight
2. Add tight endpoints to the cover

Weak Duality:

$\Sigma_{u \in C}\, y_{uv} \leq 2 \cdot \Sigma_{\bar{y}}\, y_{uv} \leq 2 \cdot \mathrm{OPT}$

Optimum for vertex cover

Strong Duality

Even stronger conditions hold

Strong Duality

Even stronger conditions hold

Strong Duality:
OPT primal x^* equals OPT Dual y^*

$$\sum c_j x_j^* = \sum b_i y_i^*$$

Strong Duality

Even stronger conditions hold

Strong Duality:
OPT primal x^* equals OPT Dual y^*

$$\sum c_j x_j^* = \sum b_i y_i^*$$

Complementary Slackness:
Primal feasible x^* and dual feasible y^* are both optimal iff

1. For each $1 \leq j \leq n$: either $x_j = 0$ or $\Sigma_i \, a_{ij} y_i = c_j$
2. For each $1 \leq i \leq m$: either $y_i = 0$ or $\Sigma_j \, a_{ij} x_j = b_i$

Wrap-up

Duality:
Primal vs Dual

→ Mechanical conversion

Algorithms:
Primal-Dual Method

→ **Vertex Cover**

Sales pitch:
In research, one often runs into LP:s