MS-C2105 - Introduction to Optimization
Lecture 6

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March 9, 2022

Outline of this lecture

Sensitivity analysis

Economic interpretation

Changes in the independent term (b)

Changes in the objective function coefficients (c)

Integer programming problems

The assignment problem

The knapsack problem

The set covering problem

Travelling salesman problem

Reading: Taha: Chapter 4; Winston: Chapter 9

Economic interpretation

Duality can be used for obtaining practical insights. Consider the paint production problem (Lecture 2) and its dual:

$$\begin{aligned} \text{max. } & z = 5x_1 + 4x_2 \\ \text{s.t.: } & 6x_1 + 4x_2 \leq 24 \\ & x_1 + 2x_2 \leq 6 \\ & x_2 - x_1 \leq 1 \\ & x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \text{min. } & z = 24y_1 + 6y_2 + y_3 + 2y_4 \\ \text{s.t.: } & 6y_1 + y_2 - y_3 \geq 5 \\ & 4y_1 + 2y_2 + y_3 + y_4 \geq 4 \\ & y_1, y_2, y_3, y_4 \geq 0 \end{aligned}$$

We have that $x^* = (3, 1.5)$ and $y^* = (0.75, 0.5, 0, 0)$. Notice that:

$$21 = 5x_1^* + 4x_2^* = 24y_1^* + 6y_2^* + y_3^* + 2y_4^* = 21$$

1. y can be seen as the marginal values each resource has for the optimal solution.
2. Only active constraints have marginal value (implied also by complementarity, i.e., slack $x_i \geq 0 \Rightarrow y_i = 0$, $i = 1, \ldots, m$)

Economic interpretation

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$$
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\qquad
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\text{min. } & z = 24y_1 + 6y_2 + y_3 + 2y_4 \\
\text{s.t.: } & 6y_1 + y_2 - y_3 \geq 5 \\
& 4y_1 + 2y_2 + y_3 + y_4 \geq 4 \\
& y_1, y_2, y_3, y_4 \geq 0
\end{aligned}
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We have that $x^* = (3, 1.5)$ and $y^* = (0.75, 0.5, 0, 0)$. Notice that:

$$21 = 5x_1^* + 4x_2^* = 24y_1^* + 6y_2^* + y_3^* + 2y_4^* = 21$$

3. Increasing (reducing) one unit of b_i will improve (worsen) the objective function value in y_i^* units while B remains feasible.
4. y^* can be seen as the "fair price" for the resource.

Sensitivity analysis

Notice that y^* corresponds to the marginal values obtained graphically (see slides 19-23 in Lecture 2).

One can use the **optimality conditions** $r_N = (c_N^\top - c_B^\top B^{-1}N) \leq 0$ and **feasibility conditions** $x_B = B^{-1}\overline{b} \geq 0$ to analyse the stability of solutions against:

1. **Changes in availability of resources.** Term b is changed by Δb. Let $\overline{b} = b + \Delta b$. and $x^* = [x_B^* \; x_N^*]$ be the optimal solution with basis B for the original LP.

- Optimality conditions $r_N = c_N - c_B^\top B^{-1}N \leq 0$ are not affected, since r_N does not depend on b.
- Feasibility conditions $x_B = B^{-1}\overline{b} \geq 0$ are affected. Changes can only be such that $B^{-1}(b + \Delta b) \geq 0$ remain true.
- Notice that $\overline{z} = c_B^\top B^{-1}\overline{b} = y^{*\top}(b + \Delta b) = z + y^{*\top}\Delta b$.

Sensitivity analysis

Example: Variations in b_1. To analyse variations $b_1 + \Delta b_1$, we include an extra element to capture how the basis is altered.

	x_1	x_2	s_1	s_2	s_3	s_4	Sol.	Δb
z	-5	-4	0	0	0	0	0	0
s_1	6	4	1	0	0	0	24	1
s_2	1	2	0	1	0	0	6	0
s_3	-1	1	0	0	1	0	1	0
s_4	0	1	0	0	0	1	2	0

The optimal tableau is below. Notice how the columns s_1 and Δb remain identical, a consequence of performing only row operations.

	x_1	x_2	s_1	s_2	s_3	s_4	Sol.	Δb
z	0	0	3/4	1/2	0	0	21	3/4
x_1	1	0	1/4	-1/2	0	0	3	1/4
x_2	0	1	-1/8	3/4	0	0	3/2	-1/8
s_3	0	0	3/8	-5/4	1	0	5/2	3/8
s_4	0	0	1/8	-3/4	0	1	1/2	1/8

Sensitivity analysis

Example: Variations in b_1. In the optimal tableau, $B^{-1}b$ is in the column 'Sol.' and thus $B^{-1}\Delta b$ is in the column 'Δb'.

	x_1	x_2	s_1	s_2	s_3	s_4	Sol.	Δb
z	0	0	3/4	1/2	0	0	21	3/4
x_1	1	0	1/4	-1/2	0	0	3	1/4
x_2	0	1	-1/8	3/4	0	0	3/2	-1/8
s_3	0	0	3/8	-5/4	1	0	5/2	3/8
s_4	0	0	1/8	-3/4	0	1	1/2	1/8

Therefore, $B^{-1}(b + \Delta b) \geq 0$ implies that

$$
\begin{aligned}
3 + (1/4)\Delta b_1 \geq 0 \quad & \Rightarrow \Delta b_1 \geq -12 \\
3/2 + (-1/8)\Delta b_1 \geq 0 \quad & \Rightarrow \Delta b_1 \leq 12 \\
5/2 + (3/8)\Delta b_1 \geq 0 \quad & \Rightarrow \Delta b_1 \geq -6.666 \\
1/2 + (1/8)\Delta b_1 \geq 0 \quad & \Rightarrow \Delta b_1 \geq -4
\end{aligned}
$$

- $-4 \leq \Delta b_1 \leq 12$ and therefore $b_1 \in [20, 36]$.
- $z = 21 + (3/4)\Delta b_1$.

Sensitivity analysis

2. Changes in coefficients of objective function. Term c is changed by Δc. Let $\overline{c} = c + \Delta c = \begin{bmatrix} c_B + \Delta c_B \\ c_N + \Delta c_N \end{bmatrix}$ and $x^* = [x_B^*, x_N^*]^\top$ be the optimal solution with basis B for the original LP.

- Feasibility condition $x_B = B^{-1}b \geq 0$ are not affected.
- Optimality conditions $r_N = c_N - c_B^\top B^{-1}N \leq 0$ are affected. Changes must be such that $\overline{c}_N - \overline{c}_B^\top B^{-1}N \leq 0$ still holds.
- Two cases can occur:
  1. Change in coefficients of basic variables:

$$
\begin{aligned}
c_N^\top - \overline{c}_B^\top B^{-1}N &= c_N^\top - (c_B + \Delta c_B)^\top B^{-1}N \\
&= c_N^\top - c_B^\top B^{-1}N - \Delta c_B^\top B^{-1}N \\
&= r_N - \Delta c_B^\top B^{-1}N \leq 0
\end{aligned}
$$

or equivalently: $r_N \leq \Delta c_B^\top B^{-1}N$.

Sensitivity analysis

2. Changes in coefficients of objective function. Term c is changed by Δc. Let $\overline{c} = c + \Delta c = \begin{bmatrix} c_B + \Delta c_B \\ c_N + \Delta c_N \end{bmatrix}$ and $x^* = [x^*_B, x^*_N]^\top$ be the optimal solution with basis B for the original LP.

- Feasibility condition $x_B = B^{-1}b \geq 0$ are not affected.
- Optimality conditions $r_N = c_N - c_B^\top B^{-1}N \leq 0$ are affected. Changes must be such that $\overline{c}_N - \overline{c}_B^\top B^{-1}N \leq 0$ still holds.
- Two cases can occur:
  2. Change in coefficients of nonbasic variables:

$$\begin{aligned}
\overline{c}_N^\top - c_B^\top B^{-1}N &= (c_N + \Delta c_N)^\top - c_B^\top B^{-1}N \\
&= c_N^\top - c_B^\top B^{-1}N + \Delta c_N^\top \\
&= r_N + \Delta c_N^\top \leq 0
\end{aligned}$$

  or equivalently: $r_N \leq -\Delta c_N^\top$.

Sensitivity analysis

Example: Variations in c_1. The optimal tableau is perturbed by Δc_1. Being x_1 a basic variable, the tableau needs to be corrected.

	x_1	x_2	s_1	s_2	s_3	s_4	Sol.
z	$-\Delta c_1$	0	3/4	1/2	0	0	21
x_1	1	0	1/4	-1/2	0	0	3
x_2	0	1	-1/8	3/4	0	0	3/2
s_3	0	0	3/8	-5/4	1	0	5/2
s_4	0	0	1/8	-3/4	0	1	1/2

To do so, we multiply the x_1-row by Δc_1 and add it to the z-row.

	x_1	x_2	s_1	s_2	s_3	s_4	Sol.
z	0	0	$\frac{3}{4} + \frac{1}{4}(\Delta c_1)$	$\frac{1}{2} - \frac{1}{2}(\Delta c_1)$	0	0	$21 + 3\Delta c_1$
x_1	1	0	1/4	-1/2	0	0	3
x_2	0	1	-1/8	3/4	0	0	3/2
s_3	0	0	3/8	-5/4	1	0	5/2
s_4	0	0	1/8	-3/4	0	1	1/2

Sensitivity analysis

Example: Variations in c_1. Requiring that $r_N \leq \Delta c_B^\top B^{-1} N$ means requiring the elements in the $z - row$ to be non-negative,

	x_1	x_2	s_1	s_2	s_3	s_4	Sol.
z	0	0	$\frac{3}{4} + \frac{1}{4}(\Delta c_1)$	$\frac{1}{2} - \frac{1}{2}(\Delta c_1)$	0	0	$21 + 3\Delta c_1$
x_1	1	0	1/4	-1/2	0	0	3
x_2	0	1	-1/8	3/4	0	0	3/2
s_3	0	0	3/8	-5/4	1	0	5/2
s_4	0	0	1/8	-3/4	0	1	1/2

which leads to the following intervals:

$$(1/4)\Delta c_1 \geq -(3/4) \qquad \Rightarrow \Delta c_1 \geq -3$$
$$-(1/2)\Delta c_1 \geq -(1/2) \qquad \Rightarrow \Delta c_1 \leq 1$$

Therefore,

- $-3 \leq \Delta c_1 \leq 1$ and thus $c_1 \in [2, 6]$.
- $z = 21 + 3\Delta c_1$.

Types of integer programming problems

Our starting point is a linear programming problem:

$$
\begin{aligned}
(\mathrm{LP}) : \min_{x} \ & c^\top x \\
\text{s.t.:} \ & Ax \leq b \\
& x \geq 0,
\end{aligned}
$$

where $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and x a vector of n decision variables.

Integer programming (IP) problems have additional constraints on the domain of x.

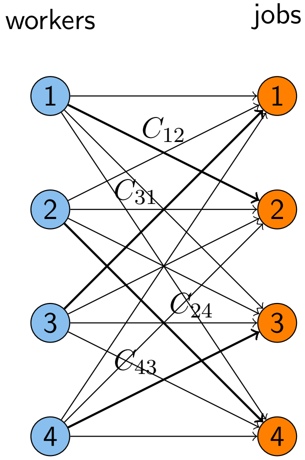
- **Integer Programming (IP)**: x must take integer values: $x \in \mathbb{Z}^n$;
- **Binary Integer Programming (BIP)**: x must be 0 or 1: $x \in \{0, 1\}$;
- **Mixed Integer Programming (MIP)**: some of the variables must take integer values: $x \in \mathbb{R}^q \times \mathbb{Z}^{n-q}$ or $x \in \mathbb{R}^q \times \{0, 1\}^{n-q}$.

The assignment problem

Problem statement:

- assign n jobs to n workers;
- one job associated to one worker;
- one worker associated to one job;
- it costs C_{ij} for worker i to execute job j.

Objective: find minimum cost assignment.



The assignment problem

Let $x_{ij} = 1$, if worker i is assigned to job j; 0, otherwise, and $N = \{1, \dots, n\}$.

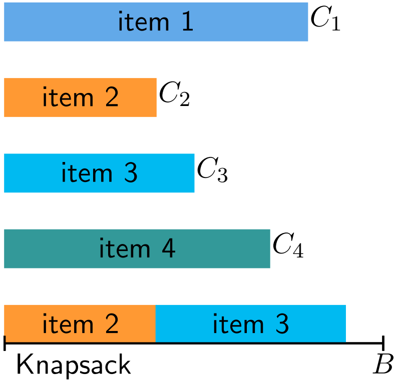
$$
\begin{aligned}
\text{(AP) : min. } & \sum_{i \in I} \sum_{j \in J} C_{ij} x_{ij} \\
\text{s.t.: } & \sum_{j \in N} x_{ij} = 1, \forall i \in M \\
& \sum_{i \in M} x_{ij} = 1, \forall j \in N \\
& x_{ij} \in \{0, 1\}, \forall i, \forall j \in N.
\end{aligned}
$$

The 0-1 knapsack problem

Problem statement:

- n items available for selection;
- it costs A_i to select i.
- each item i has value C_i;
- The available budget is B.

Objective: find maximum-valued selection of items that does not exceed budget.



The 0-1 knapsack problem

Let $x_i = 1$, if item i is selected; 0, otherwise, and $N = \{1, \dots, n\}$.

$$
\begin{aligned}
\text{(0-1 KP)} : \max_x . \quad & \sum_{i=1}^{n} C_i x_i \\
\text{s.t.:} \quad & \sum_{i=1}^{n} A_i x_i \leq B \\
& x_i \in \{0, 1\}, \forall i \in N.
\end{aligned}
$$

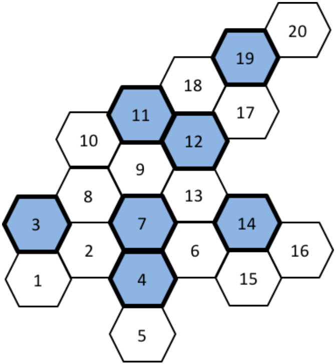
The set covering problem

Problem statement:

- A set of $M = \{1, \dots, m\}$ regions must be served by opening *service centres* (e.g., hospitals, schools, police stations);
- A centre can be opened at $N = \{1, \dots, n\}$ possible locations;
- If a centre is opened at location $j \in N$, then it serves a subset $S_j \subseteq M$ of regions and has opening cost C_j.

Objective: decide where to open the facilities so that all regions are served and the total opening cost is minimised.

The set covering problem: covering example



- Each location represents a candidate place for a centre;
- Once opened, the centre can only serve immediate neighbours.
- We have $M = \{1, \dots, 20\}$ and $N = \{3, 4, 7, 11, 12, 14, 19\}$.

In this case: $S_3 = \{1, 2, 3, 8\}\,, S_4 = \{2, 4, 5, 6, 7\}, \dots$.

The set covering problem

To model the SCP as a BIP, we need a $0-1$ incidence matrix $A = [A_{ij}]_{m \times n}$ where $A_{ij} = 1$ if $i \in S_j$, $A_{ij} = 0$ otherwise.

Let $x_j = 1$ if facility is opened at location j; $x_j = 0$, otherwise, and let $M = \{1, \dots, m\}$ and $N = \{1, \dots, n\}$.

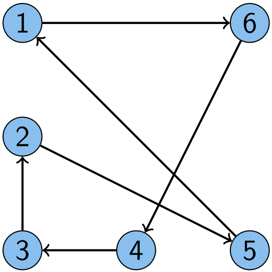
$$
\begin{aligned}
\text{(SCP)} : \min_{x} \ & \sum_{j \in N} C_j x_j \\
\text{s.t.:} \ & \sum_{j \in N} A_{ij} x_j \geq 1, \forall i \in M \\
& x_j \in \{0, 1\}, \forall j \in N.
\end{aligned}
$$

Travelling salesman problem

Problem statement:

- A salesman must visit each of n cities exactly once and return to the starting city;
- It costs C_{ij} to travel from city i to city j;

Objective: find a least-cost *tour*, i.e., an order in which the cities must be visited.



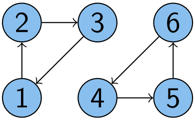
Travelling salesman problem

Let $x_{ij} = 1$ if city j is visited directly after city i, $x_{ij} = 0$ otherwise. Let $N = \{1, \dots, n\}$. We assume that x_{ii} is not defined for $i \in N$.

A naive model for the TSP could be:

$$
\begin{aligned}
(\text{TSP}) : \min_{x} \; & \sum_{i \in N} \sum_{j \in N} C_{ij} x_{ij} \\
\text{s.t.:} \; & \sum_{j \in N \setminus \{i\}} x_{ij} = 1, \forall i \in N \\
& \sum_{i \in N \setminus \{j\}} x_{ij} = 1, \forall j \in N \\
& x_{ij} \in \{0, 1\}, \forall i, \forall j \in N : i \neq j
\end{aligned}
$$

- This is exactly the assignment problem.
- Also, solutions do not prevent subtours.



Travelling salesman problem

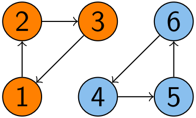
Preventing subtours: constraints that ensure full connectivity.

- Cutset constraints:

$$\sum_{i \in S} \sum_{j \in N \setminus S} x_{ij} \geq 1, \forall S \subset N, S \neq \emptyset$$

- Subtour elimination constraints:

$$\sum_{i \in S} \sum_{j \in S} x_{ij} \leq |S| - 1, \forall S \subset N, 2 \leq |S| \leq n - 1$$



$S_1 = \{1, 2, 3\}$

Example for $S_1 = \{1, 2, 3\}$:

Cutset:

$x_{14} + x_{24} + x_{34} + x_{15} + x_{25} + x_{35} + x_{16} + x_{26} + x_{36} \geq 1$

Subtour elim.: $x_{12} + x_{13} + x_{21} + x_{23} + x_{31} + x_{32} \leq 2$