Exercise class 3

Learning Objectives:

- Algebraic form of LP problems
- The Simplex algorithm

Demo 1: The simplex algorithm

Transform the linear problem into the standard form and solve it using the tabular Simplex algorithm.

$$\begin{aligned} \max.\ & 5x_1 + 4x_2 \\ \text{s.t.}\ & x_1 + x_2 \leq 4, \\ & 6x_1 + 3x_2 \leq 18, \\ & x_1 \geq 0,\ x_2 \geq 0. \end{aligned}$$

Solution

The Simplex algorithm is based on the fact that the solution of a linear optimization problem is in one of the corner points of the feasible set. The algorithm has three phases:

1) Choose the next search direction.

2) Determine how far to advance in this direction using the feasibility condition.

3) Solve the new corner point with Gauss elimination. After each step we check if we have arrived at the optimum.

The algorithm starts by modifying the problem into the standard form. In this course the standard form means the objective function is to be maximised, all variables are positive, and the constraints are of the form:

$$a_1x_1 + a_2x_2 + ... + a_nx_n = b.$$

Inequality constraints are changed into equality constraints by adding a new slack variable s.

The standard form of this problem is

$$\begin{aligned} \max.\ & z = 5x_1 + 4x_2 \\ \text{s.t.}\ & \quad x_1 + x_2 + s_1 = 4, \\ & \quad 6x_1 + 3x_2 + s_2 = 18, \\ & \quad x_1 \geq 0,\ x_2 \geq 0,\ s_1 \geq 0,\ s_2 \geq 0. \end{aligned}$$

The central tool of the Simplex algorithm is the so called Simplex tableau. The first row of the table holds the negative coefficients of the objective function (z). The following rows each represent a constraint. There is one column for each variable and one solution column. The Simplex tableau of this problem is:

	x_1	x_2	s_1	s_2	Sol.
z	-5	-4	0	0	0
s_1	1	1	1	0	4
s_2	6	3	0	1	18

The variables in the left-most column (currently s_1 and s_2) are the so called basic variables, i.e. the variables that are not zero. There is always as many basic variables as there are constraints in the problem. The values of these variables can be read from the solution column on their respective rows. The other variables (x_1 and x_2) on the other hand are all zero.

The Simplex algorithm starts from the situation in which the ordinary variables (x_i) are zero, and the slack (s_i) variables non-negative. If this is not the case, the basic variables have to be chosen separately so that the Simplex tableau is consistent. In a consistent tableau the basic variable columns are all zero, except for the value on the row representing the variable. This value is always 1.

The next phase of the algorithm is choosing the next corner point. Let's choose the direction in which the objective function increases (decreases in minimisation) the most. We choose the direction with the highest positive coefficient in the objective. The values on the coefficient (z) row are negative, so the steepest ascent is in the direction of the variable with the lowest negative coefficient. The lowest coefficient is in the x_1 column, -5. We call x_1 the entering variable, because it enters the set of basic variables. The x_1 column is called the pivot column.

Note. If minimizing, the largest positive coefficient would be chosen, i.e. the variable in which the objective function decreases the most.

Next, we calculate how far in this direction we are to advance. This is done by dividing the solution column with the pivot column, i.e. the so called feasibility condition:

	Quotient
s_1	$4/1 = 4$
s_2	$18/6 = 3$

We select the smallest positive value. The situation can be seen as advancing on a constraint line and the quotients represents the intersections of other constraints. We want to go forward (positivity) and select the first intersection (smallest) to avoid ending up in the infeasible area.

Note. If you have to divide by zero, the result can be considered as infinity.

The smallest positive number was on the row representing the basic variable s_2. The variable s_2 is called the exiting variable because it exits the set of basic variables.

Now we have a new set of basic variables, and we can move on to the third phase. In this phase we determine the new corner point. The goal is to get the columns of the new basic variables to be zero, except for the cell representing the variable itself to be one. We can use the Gauss elimination process that may be simplified with these two rules:

1) New pivot row = Old pivot row / pivot cell (*)

2) Other rows: New row = Old row - Pivot cell of the old row·New pivot row

For example, z-row (objective) we get:

	x_1	x_2	s_1	s_2	Sol.
Old row	-5	-4	0	0	0
New pivot row = Old pivot row/pivot cell	1	1/2	0	1/6	3
Pivot cell of the old row·New pivot row	-5	-5/2	0	-5/6	-15
New row	0	-3/2	0	5/6	15

With these rules we get the next Simplex tableau:

	↓ New pivot column					
	x_1	x_2	s_1	s_2	Sol.	
z	0	-3/2	0	5/6	15	
s_1	0	1/2	1	-1/6	1	
x_1	1*	1/2	0	1/6	3	← New pivot row

Note. The basic variables have changed, so the s_2 row is now the x_1 row.

Now we can read the values of the basic variables s_1 and x_1: 1 and 3. The solution column of the objective row shows the value of the objective function at this point. The other variables are zero.

We have now completed one Simplex iteration. Let's check if we have achieved the optimum. Look at the values of the objective row: the coefficient of x_2 is negative, i.e. the value of the objective function increases in this direction. Therefore the point is not the optimum.

But the point is closer to the optimum (solution is equal to 15 an increase from 0) and we can begin the next Simplex iteration: Let's select the next direction and solve the next corner point. The steepest ascent is in the direction of x_2 (smallest coefficient, $-3/2$). The quotients of the solution and pivot columns are (feasibility condition):

	Quotient
s_1	$1/(1/2) = 2$
x_2	$3/(1/2) = 6$

The smallest positive value is 2, so we choose s_1 to be the exiting variable. The next Gauss elimination yields:

		↓ New pivot column				
	x_1	x_2	s_1	s_2	Sol.	
z	0	0	3	1/3	18	
x_2	0	1*	2	-1/3	2	← New pivot row
x_1	1	0	-1	1/3	2	

Now all coefficients on the objective row (z) are positive and therefore the objective value of the objective function can not increase in any direction. We have reached the optimum.

We can read the value of the objective function from the solution column: 18. The value of basic variables can also be read from the solution column: $x_1 = 2$, $x_2 = 2$. The slack variables are not in the set of basic variables, so they are zero. This means that both constraints are active.

Demo 2: Variants in the standard form

Transform the linear problem into minimisation standard form (without changing the problem to maximisation) and solve it using the tabular Simplex algorithm.

$$\begin{aligned} \min. \quad & -2x_1 + 3x_2 \\ \text{s.t.} \quad & 8x_1 + 3x_2 \leq 6, \\ & x_1 + 2x_2 \leq 2, \\ & x_1 \geq 0,\ x_2 \geq 0. \end{aligned}$$

Solution

The standard form in minimisation of the problem is:

$$\begin{aligned} \min. \quad & -2x_1 + 3x_2 \\ \text{s.t.} \quad & 8x_1 + 3x_2 + s_1 = 6, \\ & x_1 + 2x_2 + s_2 = 2, \\ & x_1 \geq 0,\ x_2 \geq 0,\ s_1 \geq 0,\ s_2 \geq 0. \end{aligned}$$

Note. If minimising, the greatest positive coefficient is chosen, i.e. the variable in which the objective function decreases the most.

0 iteration of simplex method:

↓ Pivot column

	x_1	x_2	s_1	s_2	Sol.	
z	2	-3	0	0	0	
s_1	8*	3	1	0	6	← Pivot row
s_2	1	2	0	1	2	

Optimal tableau:

	x_1	x_2	s_1	s_2	Sol.
z	0	-15/4	-1/4	0	-3/2
x_1	1	3/8	1/8	0	3/4
s_2	0	13/8	-1/8	1	5/4

All the coefficients in the objective row are less than or equal to zero. This implies that the tableau is optimal (we can't decrease the value of the objective function within the feasibility region).

Thus, the optimal value of the objective function under minimisation is -3/2 when $x_1 = 3/4$ and $x_2 = 0$

Problem 1: The Simplex algorithm

Transform the linear problem into standard form and solve it using the tabular Simplex algorithm

$$\begin{aligned} \max. \quad & 5x_1 + 6x_2 \\ \text{s.t.} \quad & x_2 \le 4, \\ & 2x_1 + x_2 \le 6, \\ & x_1 \ge 0,\, x_2 \ge 0. \end{aligned}$$

Solution

The standard form of the problem:

$$\begin{aligned} \max. \quad & 5x_1 + 6x_2 \\ \text{s.t.} \quad & x_2 + s_1 = 4, \\ & 2x_1 + x_2 + s_2 = 6, \\ & x_1 \ge 0,\, x_2 \ge 0,\, s_1 \ge 0,\, s_2 \ge 0. \end{aligned}$$

0 iteration of simplex method:

↓ Pivot column

	x_1	x_2	s_1	s_2	Sol.	
z	-5	-6	0	0	0	
s_1	0	1*	1	0	4	← Pivot row
s_2	2	1	0	1	6	

1 iteration of simplex method:

↓ Pivot column

	x_1	x_2	s_1	s_2	Sol.	
z	-5	0	6	0	24	
x_2	0	1	1	0	4	
s_2	2*	0	-1	1	2	← Pivot row

Optimal tableau:

	x_1	x_2	s_1	s_2	Sol.
z	0	0	7/2	5/2	29
x_2	0	1	1	0	4
x_1	1	0	-1/2	1/2	1

Thus, the optimal value of the function is 29, when $x_1 = 1$, $x_2 = 4$

Problem 2: Variants in the standard form

Transform the linear problem into the standard form and solve it using the tabular Simplex algorithm.

$$
\begin{aligned}
\min . \quad & 5x_1 - 6x_2 \\
\text{s.t.} \quad & -x_1 + 3x_2 \leq 5, \\
& -x_1 \leq 4, \\
& x_1 \leq 0,\ x_2 \geq 0.
\end{aligned}
$$

Solution

Similarly to the example above the algorithm starts from modifying the problem to the standard form.

Variants from the standard form are dealt with using the following:

1. **nonpositive variables:** $x_i \leq 0$ is replaced by $-y_i$ with $y_i \geq 0$.
2. **unrestricted variables:** $x_i \in \mathbf{R}$ is replaced by $y_i^+ - y_i^-$, by $y_i^+, y_i^- \geq 0$.
3. **minimisation:** $\min. z = c^\top x$ is replaced with $\max. -z = -c^\top x$. Notice that z^* will have changed sign at the optimal point.
4. **negative b_i:** multiply equation by (-1).

Note. If minimising, we can use either the approach mentioned above (the absolute value of min $f(x)$ is equal to absolute value of max $(-f(x))$) or to choose the greatest positive coefficient when deciding the candidate to enter the basis, i.e. the variable in which the objective function decreases the most.

In this example we will apply the first option. Thus, the objective function of modified problem is $f(x_1, x_2) = -1(5x_1 - 6x_2) = -5x_1 + 6x_2$. In addition, since $x_1 \leq 0$ we substitute it with $x_3 = -x_1 \geq 0$ or $x_1 = -x_3$. Combining all the changes we get:

$$
\begin{aligned}
\max . \quad & -5(-x_3) + 6x_2 \\
\text{s.t.} \quad & -(-x_3) + 3x_2 \leq 5, \\
& -(-x_3) \leq 4, \\
& x_2 \geq 0,\ x_3 \geq 0.
\end{aligned}
$$

The simplified version of the problem is:

$$
\begin{aligned}
\max . \quad & 6x_2 + 5x_3 \\
\text{s.t.} \quad & 3x_2 + x_3 \leq 5, \\
& x_3 \leq 4, \\
& x_2 \geq 0,\ x_3 \geq 0.
\end{aligned}
$$

Now all the variables are positive and to get the constraints in the form of equalities slack variables will be added. Thus, the initial problem modified to the standard form is:

$$
\begin{aligned}
\max . \quad & 6x_2 + 5x_3 \\
\text{s.t.} \quad & 3x_2 + x_3 + s_1 = 5, \\
& x_3 + s_2 = 4, \\
& x_2 \geq 0,\ x_3 \geq 0,\ s_1 \geq 0,\ s_2 \geq 0.
\end{aligned}
$$

The simplex tableau is:

↓ Pivot column

	x_2	x_3	s_1	s_2	Sol.	
z	-6	-5	0	0	0	
s_1	3*	1	1	0	5	← Pivot row
s_2	0	1	0	1	4	

Note. If you have to divide by zero, the solution can be considered as infinity.

The next Gauss elimination yields:

↓ Pivot column

	x_2	x_3	s_1	s_2	Sol.	
z	0	-3	2	0	10	
x_2	1	1/3	1/3	0	5/3	
s_2	0	1*	0	1	4	← Pivot row

Since the x_3 coefficient of the objective row is equal to $-3 < 0$ the tableau is not optimal yet (we still can increase the function in this direction).

	x_2	x_3	s_1	s_2	Sol.
z	0	0	2	3	22
x_2	1	0	1/3	-1/3	1/3
x_3	0	1	0	1	4

Now all coefficients in the objective row (z) are positive and therefore the objective value of the objective function can not increase in any direction. We have reached the optimum at the point $x_2 = 1/3$ and $x_3 = 4$. The the value of the objective function from the result column is 22. However, the initial problem was minimisation thus, we must change the sign to get it the optimal value for the initial problem which is -22. In addition, we have $x_1 = -x_3 = -4$ at the optimal point.

Problem 3: Variants in the standard form

Transform the linear problem into standard form and solve it using the tabular Simplex algorithm:

$$\begin{aligned} \max. \quad & 4x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \leq 3, \\ & 3x_1 + x_2 \leq 8, \\ & x_1 \geq 0,\ x_2 \in \mathbf{R}. \end{aligned}$$

Solution

Since x_2 is unbounded variable let's assume $x_2 = x_3 - x_4$, where $x_3 \geq 0$ and $x_4 \geq 0$. Then initial problem will be:

$$\begin{aligned} \max. \quad & 4x_1 + 3x_3 - 3x_4 \\ \text{s.t.} \quad & x_1 + 2x_3 - 2x_4 \leq 3, \\ & 3x_1 + x_3 - x_4 \leq 8, \\ & x_1 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0. \end{aligned}$$

The standard form of the problem is:

$$\begin{aligned} \max. \quad & 4x_1 + 3x_3 - 3x_4 \\ \text{s.t.} \quad & x_1 + 2x_3 - 2x_4 + s_1 = 3, \\ & 3x_1 + x_3 - x_4 + s_2 = 8, \\ & x_1 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0,\ s_1 \geq 0,\ s_2 \geq 0. \end{aligned}$$

0 iteration of simplex method:

↓ Pivot column

	x_1	x_3	x_4	s_1	s_2	Sol.	
z	-4	-3	3	0	0	0	
s_1	1	2	-2	1	0	3	
s_2	3*	1	-1	0	1	8	← Pivot row

1 iteration of simplex method:

↓ Pivot column

	x_1	x_3	x_4	s_1	s_2	Sol.	
z	0	-5/3	5/3	0	4/3	32/3	
s_1	0	5/3*	-5/3	1	-1/3	1/3	← Pivot row
x_1	1	1/3	-1/3	0	1/3	8/3	

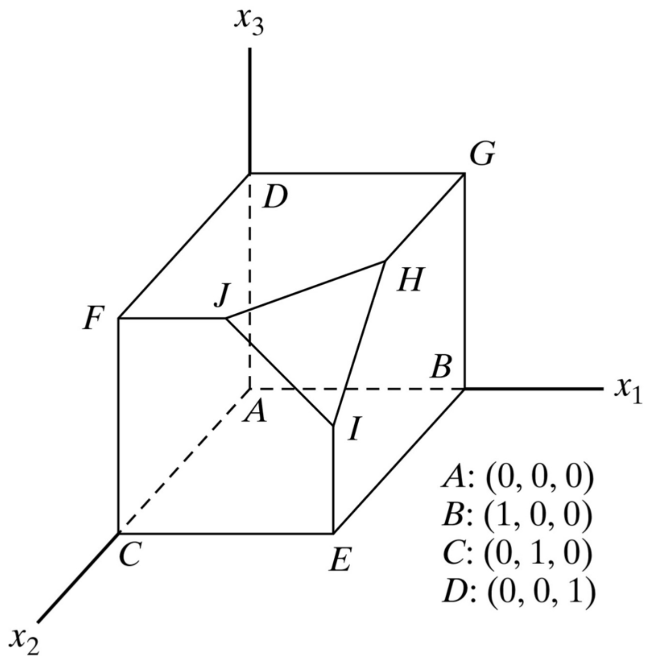
Optimal tableau:

	x_1	x_3	x_4	s_1	s_2	Sol.
z	0	0	0	1	1	11
x_3	0	1	-1	3/5	-1/5	1/5
x_1	1	0	0	-1/5	2/5	13/5

The optimal value of the function is 11 when $x_1 = 13/5$, $x_3 = 1/5$ and $x_4 = 0$.

Note. Notice, that in the objective row non basic variable x_4 has 0 coefficient. This means that x_4 is a candidate for being basic variable, the inclusion of which instead of x_3 will not effect the final solution. Thus, we have a multiple solutions: different combinations of x_3 and x_4 such that $x_3 \geq 0$, $x_4 \geq 0$ and $x_3 - x_4 = 1/5 = x_2$.

Problem 4: The Simplex algorithm



Consider the three-dimensional LP solution space in the figure , whose feasible extreme points are A,B, ..., and J.

(a) Which of the following pairs of corner points cannot represent successive simplex iterations: (A, B), (H, I), (E, H), and (A, I)? Explain why.

(b) Suppose that the simplex iterations start at A and that the optimum occurs at H. Indicate whether any of the following paths are not legitimate for the simplex algorithm, and state the reason.

(i) $A \to B \to G \to H$.

(ii) $A \to D \to F \to C \to A \to B \to G \to H$.

(iii) $A \to C \to I \to H$.

Solution

(a) the pairs (A,B) and (H,I) are adjacent, hence can be on a simplex path. Remaining pairs can not be on a simplex path since they are not adjacent.

(b) (i) This path is legitimate since it connect adjacent extreme points.

(ii) This path can not be legitimate for a simplex algorithm because once we have chosen the best direction from the current point we will never return to this choice. However, in this case there are two different options were decided starting in A.

(iii) This path is not legitimate since the extreme points C and I are not adjacent

Problem 5: The Simplex algorithm

Consider the following system of equations:

$$
\begin{cases}
x_1 + 2x_2 - 3x_3 + 5x_4 + x_5 & = 8 \\
5x_1 - 2x_2 \quad + 6x_4 \quad + x_6 & = 16 \\
2x_1 + 3x_2 - 2x_3 + 3x_4 \quad + x_7 & = 6 \\
-x_1 \quad + \ x_3 - 2x_4 \quad + x_8 & = 0
\end{cases}
$$

$$x_1, x_2, ..., x_8 \geq 0$$

Let $x_5, x_6, ..., x_8$ be a given initial basic feasible solution. Suppose that x_1 becomes basic. Which of the given basic variables in the initial tableau must become nonbasic to guarantee that all the variables remain non-negative, and what is the value of x_1 in the new solution? Repeat this procedure for x_2, x_3, and x_4.

Solution

Let's consider the initial (0) tableau for the problem presented.

	x_1	x_2	x_3	x_4	...	Sol.
z	...	...	...	...	...	...
x_5	1	2	-3	5	...	8
x_6	5	-2	0	6	...	16
x_7	2	3	-2	3	...	6
x_8	-1	0	1	-2	...	0

Then, let's calculate the quotients for each of the columns assuming that the corresponding variable is supposed to become basic. Thus, we can decide which basic variable should become slack. The red cells indicate the basic variable with a negative or zero coefficient in the constraint as a candidate for becoming nonbasic, this is due to the redundancy of the constraint in this case. The green cells show the smallest value in that column and hence becomes the nonbasic variable (leaving variable).

		x_2	x_3	s_4	...	Sol.
Basic	x_1	x_2	x_3	x_4	...	...
x_5	8/1	8/2	-3	8/5	...	8
x_6	16/5	-2	16/0	16/6	...	16
x_7	6/2	6/3	-2	6/3	...	6
x_8	-1	0	0/1	-2	...	0
Leaving var	x_7	x_7	x_8	x_5		
Value of x_i	3	2	0	1.6		

Home Exercise 3: The Simplex algorithm

Transform the linear problem into the standard form and solve using the tableau Simplex algorithm.

$$\begin{aligned} \min. \quad & 2x_1 + x_2 \\ \text{s.t.} \quad & 7x_1 - 3x_2 \leq 4, \\ & x_1 + 2x_2 \leq 7, \\ & x_1 \geq 0,\ x_2 \in \mathbf{R}. \end{aligned}$$

Solution

The standard form of this problem is:

$$\begin{aligned} \max. \quad & -2x_1 - x_3 + x_4 \\ \text{s.t.} \quad & 7x_1 - 3x_3 + 3x_4 + s_1 = 4, \\ & x_1 + 2x_3 - 2x_4 + s_2 = 7, \\ & x_1 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0,\ s_1 \geq 0,\ s_2 \geq 0. \end{aligned}$$

Where $x_2 = x_3 - x_4$.

0 iteration of Simplex tableau:

↓ Pivot column

	x_1	x_3	x_4	s_1	s_2	Sol.	
z	2	1	-1	0	0	0	
s_1	7	-3	3*	1	0	4	← Pivot row
s_2	1	2	-2	0	1	7	

Optimal tableau:

	x_1	x_3	x_4	s_1	s_2	Sol.
z	13/3	0	0	1/3	0	4/3
x_4	7/3	-1	1	1/3	0	4/3
s_2	17/3	0	0	2/3	1	29/3

Thus, the optimal value of the initial problem is $-4/3 = -1.33$ when $x_1 - 0,\ x_3 = 0,\ x_4 = 4/3 = 1.33$, i.e. $x_1 = 0,\ x_2 = -1.33$.