MS-C2105 - Introduction to Optimization
Lecture 7

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March 18, 2022

Outline of this lecture

Modelling with integer variables

Fixed cost

Disjunctions and implications

Solving general IPs

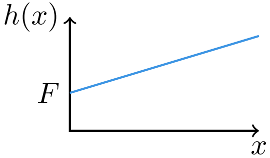
Branch-and-bound method

Reading: Taha: Chapter 9 (9.2); Winston: 9 (from 9.3)

Modelling with integer variables: fixed costs

In many problems, costs are composed of a fixed charge F plus a proportional charge p. The cost function h with fixed costs can be

$$h(x) = \begin{cases} F + px, & \text{if } 0 < x \leq C \\ 0, & \text{if } x = 0 \end{cases}$$



If we want to minimise $h(x)$, we can define $y \in \{0, 1\}$ such that $y = 1$, if $x > 0$, and $y = 0$, otherwise. This can be modelled as:

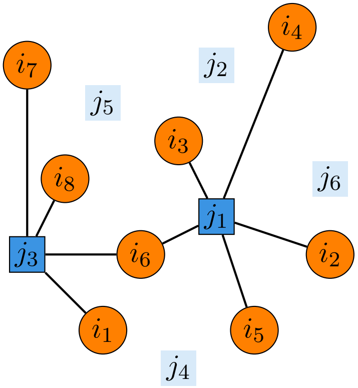
$$\begin{aligned} \min_{x,y} \quad & Fy + px \\ \text{s.t.:} \quad & x \leq Cy \\ & x \geq 0 \\ & y \in \{0, 1\} \end{aligned}$$

Uncapacitated facility location

Problem statement:

- $M = \{1, \dots, m\}$ clients must be served by a subset of $N = \{1, \dots, n\}$ facilities;
- opening a facility at location $j \in N$ has a fixed cost F_j;
- serving a client $i \in M$ by a facility $j \in N$ costs C_{ij}.

Objective: decide where to locate facilities and how to serve clients minimising the total (opening + service costs) cost.



Uncapacitated facility location

Define the following variables:

- x_{ij} be the fraction of the demand $i \in M$ being served by facility $j \in N$;
- $y_j = 1$, if a facility is opened at $j \in N$, and 0, otherwise.

The UFL can be formulated as:

$$
\begin{aligned}
\text{(UFL)} : \min_{x,y} \ & \sum_{j \in N} F_j y_j + \sum_{i \in M} \sum_{j \in N} C_{ij} x_{ij} \\
\text{s.t.:} \ & \sum_{j \in N} x_{ij} = 1, \forall i \in M \\
& \sum_{i \in M} x_{ij} \leq m y_j, \forall j \in N \\
& x_{ij} \geq 0, \forall i \in M, \forall j \in N \\
& y_j \in \{0, 1\}, \forall j \in N.
\end{aligned}
$$

Modelling disjunctions

Suppose that $x \in \mathbb{R}^n : 0 \leq x \leq u$ and we wish to impose:

$$\sum_{j=1}^{n} a_j^1 x_j \leq b^1 \vee \sum_{j=1}^{n} a_j^2 x_j \leq b^2$$

This disjunctive conditions occur often, whether condition 1 or 2 can happen, but not simultaneously. How can we model this?

Let $y_i \in \{0,1\}, i \in \{1,2\}$. We assume to have an upper bound $M_i \geq \left\{a^i x - b^i : 0 \leq x \leq u\right\}, i \in \{1,2\}$. Then we have:

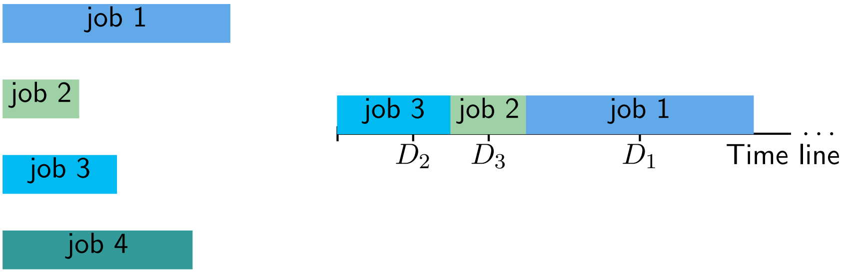
$$\begin{aligned}
&\sum_{j=1}^{n} a_j^i x_j - b^i \leq M_i(1-y_i), i \in \{1,2\} \\
&y_1 + y_2 \leq 1 \\
&y_i \in \{0,1\}, i \in \{1,2\}, 0 \leq x \leq u.
\end{aligned}$$

Scheduling

Problem statement: define the order jobs must be performed.

- jobs $j \in N = \{1, \dots, n\}$ are performed sequentially;
- each job j has due date D_j;
- performing job j takes P_j units of time (e.g., hours or days)
- a penalty C_j is paid for delay per unit of time

Objective: schedule jobs such that lateness penalty is minimised.



Scheduling

Consider the variables:

- x_j is the time job j starts.
- $s_j = s_j^+ - s_j^-$ represent the deviation from deadline D_j, both earliness (s_j^+) or lateness (s_j^-)

An important feature to consider is sequencing.

- if job i is schedule before job j, then $x_j \geq x_i + P_i$.
- otherwise, $x_i \geq x_j + P_j$

This either-or condition can be modelled as a disjunction. Let $y_{ij} \in \{0, 1\}$ indicate whether job i is scheduled before job j. Then

$$
\begin{aligned}
My_{ij} + (x_i - x_j) &\geq P_j \\
M(1 - y_{ij}) + (x_j - x_i) &\geq P_i
\end{aligned}
$$

Scheduling

The scheduling problem can be modelled as:

$$
\begin{aligned}
\text{min. } z = & \sum_{j=1}^{n} C_j s_j^- \\
\text{s.t.: } & M y_{ij} + (x_i - x_j) \geq P_j, \forall i, j \in N, i < j \\
& M(1 - y_{ij}) + (x_j - x_i) \geq P_i, \forall i, j \in N, i < j \\
& x_j + P_j + (s_j^+ - s_j^-) = D_j \\
& x_j, s_j^+, s_j^- \geq 0, j \in N \\
& y_{ij} \in \{0, 1\}, \forall i, j \in N, i < j.
\end{aligned}
$$

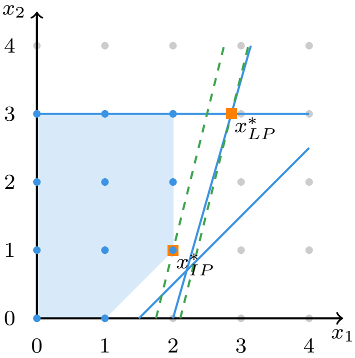
Remarks: several variants of objective function can be considered

- minimise plan deviation: min. $z = \sum_{j=1}^{n} s_j^+ + s_j^-$
- maximise earliness : max. $z = \sum_{j=1}^{n} s_j^+$

Solving IP problems

We explore the most popular exact method for solving IPs, which is based on two key concepts: linear relaxations and convex hulls.

A feasible region $S = \{Ax \leq b : x \in \mathbb{Z}_+\}$ is illustrated below.



- We use LP relaxations, which is the IP with integrality constraints removed.
- If the convex hull is available, the IP can be solved as an LP.

Solving IP problems

Branch-and-bound (B&B) is a divide-and-conquer strategy for solving (mixed-)integer programming problems such as

$$(P) : z_{IP} = \max_x . \left\{ c^\top x : x \in S \right\}.$$

The divide-and-conquer paradigm is based on the following idea:

1. Break P into subproblems (that might be easier to solve);
2. Combine all the subproblem solutions to form a solution to P.

The working principle is summarised by this proposition:

Proposition 1

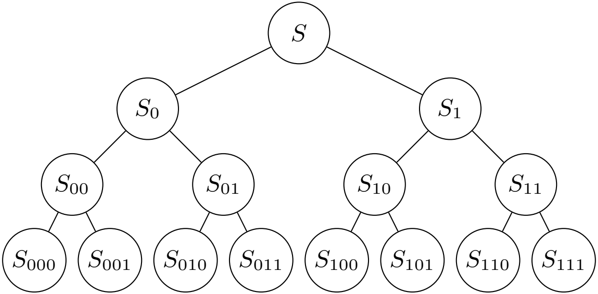
Let $K = \{1, \dots, |K|\}$ and $\bigcup_{k \in K} S_k = S$ be a decomposition of S. Let $z^k = \max._x \{cx : x \in S_k\}, \forall k \in K$. Then $z_{IP} = \max_{k \in K} \left\{z^k\right\}$

.

Solving IP problems

An important representation to control the generation of subproblems is a enumerative tree.

Example: Enumerative tree for $S \subseteq \{0,1\}^3$ (binary branching).



$S = S_0 \cup S_1 = \{x \in S : x_1 = 0\} \cup \{x \in S : x_1 = 1\}$

$S_i = S_{i0} \cup S_{i1} = \{x \in S : x_1 = i, x_2 = 0\} \cup \{x \in S : x_1 = i, x_2 = 1\}$

$S_{ij} = S_{ij0} \cup S_{ij1} = \{x \in S : x_1 = i, x_2 = j, x_3 = 0\} \cup \{x \in S : x_1 = i, x_2 = j, x_3 = 1\}$

The combinatorial explosion

Enumerative trees are only useful to organise the process. Fully enumerating all solutions is often hopeless.

1. **Assignment problem:** we have $n!$ permutations of $\{1, \ldots, n\}$.
2. **Knapsack and set covering problem:** maximum number of feasible subsets is 2^n.
3. **Travelling salesman problem:** starting from city 1, we have to check $(n-1)!$ permutations of $\{2, \ldots, n\}$.

n	2^n	$n!$
10	1.02×10^3	3.60×10^6
100	1.27×10^{30}	9.33×10^{157}
1000	1.07×10^{301}	4.02×10^{2567}

Table: Total number of iterations given input of size n

You can check how big these numbers are here.

The B&B method

General B&B methods rely on successively solving LP relaxations that are further constrained to generate subproblems.

- Subproblems are further constrained (branching) until becoming infeasible or returning a candidate integer solution.
- For maximisation, LP relaxations provide upper bounds ($\overline{z}$) while feasible (integer) solutions provide lower bounds ($\underline{z}$).

Branching: at a given subproblem S_k, suppose we have an optimal solution with a fractional component $\overline{x}_j$.

We can then branch S_k into the following subproblems:

$$S_{k1} = S_k \cap \{x : x_j \leq \lfloor \overline{x}_j \rfloor\}$$
$$S_{k2} = S_k \cap \{x : x_j \geq \lceil \overline{x}_j \rceil\}$$

The B&B method

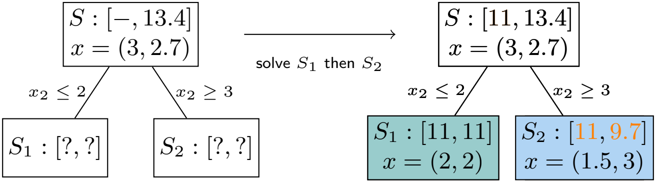
The efficiency of B&B is tied up with the bounding: avoiding fully investigate a branch to its leaves if bound information is available.

Proposition 2

Let $S = \bigcup_{k \in K} S_k$ *be a decomposition of* S *into smaller sets. Let* $z^k = \text{max.}_x \left\{c^\top x : x \in S_k\right\}$ *for* $k \in K$*, and let* $\overline{z}^k$ *(*$\underline{z}^k$*) be an upper (lower) bound on* z^k*. Then* $\overline{z} = \max_{k \in K} \overline{z}^k$ *and* $\underline{z} = \max_{k \in K} \underline{z}^k$.

Using the knowledge of global lower and local upper bounds, we can halt the search through S_k, i.e., prune S_k preemptively.

Example: Branching represented by edges and bounds by $[\underline{z}, \overline{z}]$.



Putting together a B&B method for IPs

Pruning (i.e., bounding) using information from the LP relaxation is possible in three distinct cases:

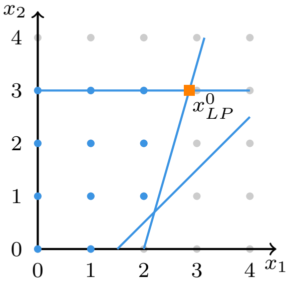
- **Pruning by optimality:** $z^k = \text{max.}_x \left\{c^\top x : x \in S_k\right\}$ is solved to optimality. If the solution of the LP relaxation is integer, we prune by optimality;
- **Pruning by infeasibility:** $S_k = \emptyset$. If the relaxation is infeasible, we prune by infeasibility.
- **Pruning by bound:** if $\overline{z}^k < \underline{z}$ (max. problem). If the solution of the relaxation provides a upper bound smaller than a known lower bound, we prune by bound.

Remark: pruning by bound requires a global lower bound. Thus, the sequence in which S_k are solved is crucial for performance.

B&B method for IPs: example

Consider the problem:

$$\max_x. \ z = 4x_1 - x_2$$
$$7x_1 - 2x_2 \leq 14$$
$$x_2 \leq 3$$
$$2x_1 - 2x_2 \leq 3$$
$$x_1, x_2 \in \mathbb{Z}_+$$

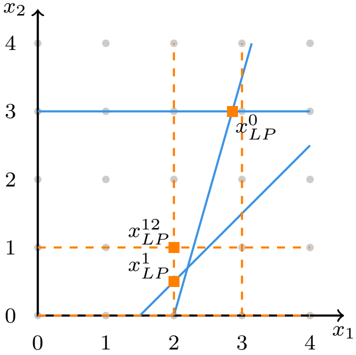


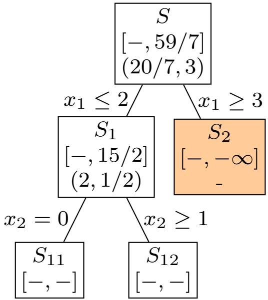
We start by solving the LP relaxation (bounding). At this point our tree is initialised as

$$\boxed{\begin{array}{c} S \\ [-, 59/7] \\ (20/7, 3) \end{array}}$$

B&B method for IPs: example

We choose to solve S_2. This leads to

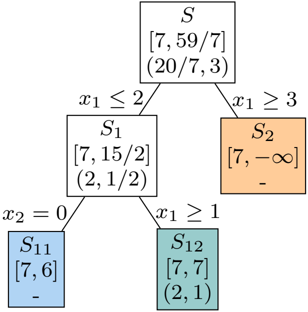




We prune S_2 by infeasibility and select another subproblem. We *arbitrarily* choose S_{12} which has an integer solution. Thus, we can prune S_{12} by optimality. Now $\mathcal{L} = \{S_{11}\}$.

B&B method for IPs: example

As we found an integer solution, we can update the global (primal) lower bound, i.e., $\underline{z} = \max\{-\infty, 7\} = 7$.



Finally, S_{11} can be pruned by bound. As $\mathcal{L} = \emptyset$, the algorithm is finished with $x^* = (2, 1)$, $z^* = 7$.

B&B method for IPs: example

Algorithm 1 shows the pseudocode for the LP-relaxation based B&B for a maximisation problem S with formulation P.

Algorithm LP-relaxation based B&B

1: **initialise.** $\mathcal{L} \leftarrow \{S\}$, $\underline{z} \leftarrow -\infty$, $\overline{x} \leftarrow -\infty$

2: **while** $\mathcal{L} \neq \emptyset$ **do**

3: select problem S_i from $\mathcal{L}$. $\mathcal{L} \leftarrow \mathcal{L} \setminus \{S_i\}$.

4: solve LP relaxation of S_i over P_i, obtaining z^i_{LP} and x^i_{LP}. $\overline{z}^i \leftarrow z^i_{LP}$.

5: **if** $S_i = \emptyset$ **then** return to step 2.

6: **else if** $\overline{z}^i \leq \underline{z}$ **then** return to step 2.

7: **else if** $x^i_{LP} \in \mathbb{Z}^n$ **then** $\underline{z} \leftarrow \max\left\{\underline{z}, \overline{z}^i\right\}$, $\overline{x} \leftarrow x^i_{LP}$; and return to step 2

8: **end if**

9: select a fractional component x_j and create subproblems S_{i1} and S_{i2} with formulations P_{i1} and P_{i2}, respectively, such that

$$P_{i1} = P_i \cup \{x_j \leq \lfloor \overline{x_j} \rfloor\} \text{ and } P_{i2} = P_i \cup \{x_j \leq \lceil \overline{x_j} \rceil\}.$$

10: $\mathcal{L} \leftarrow \mathcal{L} \cup \{S_{i1}, S_{i2}\}$.

11: **end while**

12: **return** $(\overline{x}, \underline{z})$.
