MS-C2105 Introduction to Optimization — Hakula/Weller

Exercise class 8

Learning Objectives:

- Convexity of functions
- Line search: Bisection and Newton's methods

Demo 1: Bisection method

a) Minimise $x^4 - 3x^3 + 2x$ using the bisection method with input interval [-4,4] and tolerance $\varepsilon = 0.01$.

b) Can you be sure that the solution of the algorithm is indeed an optimal solution?

c) Identify all stationary points. Start by using [-2,4] as new starting interval for the bisection method.

d) Find the global minimum.

Algorithm 1 Bisection method

initialise. tolerance $l > 0$, $[a_0, b_0] = [a, b]$, $k = 0$.
while $b_k - a_k > l$ **do**
    $\lambda_k = \frac{(b_k + a_k)}{2}$ and evaluate $f'(\lambda_k)$.
    if $f'(\lambda_k) = 0$ **then** return λ_k.
    else if $f'(\lambda_k) > 0$ **then**
        $a_{k+1} = a_k$, $b_{k+1} = \lambda_k$.
    else
        $a_{k+1} = \lambda_k$, $b_{k+1} = b_k$.
    end if
    $k = k + 1$.
end while
return $\overline{\lambda} = \frac{a_k + b_k}{2}$.

Demo 2: Newton's method vs. approx. Newton's method

Solve $f(x) = \frac{2}{3}x^3 - \frac{8}{3}x$ with Newton's method and the approximated Newton's method (also known as the secant method), which approximate the second-order derivative by $\frac{f'(b) - f'(a)}{b - a}$.

Start with initial value $x_0 = -3$ and (for the approximation) $x_1 = -2.9$. Set the tolerance to 0.001.

Algorithm 2 Newton's method

initialise. tolerance $\epsilon > 0$, initial step size x_0, iteration count $k = 0$.
while $|f'(x_k)| > \epsilon$ **do**
    $x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)}$.
    $k \leftarrow k + 1$.
end while
return $\overline{x} = x_k$.

Problem 1: Convexity

Show that, if f is convex and differentiable, then $f(y) \geq f(x) + f'(x)(y - x)$ for any y (i.e., the function is 'above' the first-order approximation at every point).

Problem 2: Convex or concave?

Are these functions convex, concave, or both?

1. $f_1(x) = 4x - x^2$
2. $f_2(x) = ax + b$ for $a, b \in \mathbb{R}$
3. $f_3(x) = \frac{8}{log(x)^2} : x > 1$
4. $f_4(x) = e^x - 3x$

Problem 3: Bisection method

Apply the bisection method for solving $\max. -4x^2 + 2x - 1$. Use [-1,1] and tolerance 0.01. Solve by hand, then check with Julia. Does the method find a global optimum?

Problem 4: Newton's method

Apply Newton's method for $\max. - 4x^2 + 2x - 1$ from $x_0 = 2$. Why does it only need one step? Does this depend on x_0?

Problem 5: Newton's method

Solve $x^4 - 3x^3 + 2x$ using Newton's method with $\varepsilon = 0.01$, once with starting point 1 and once with 2. Why are the solutions different?

Home Exercise 8: Convexity

Find optimum/ optima for the following functions, justifying whether they are local or global minimum/ maximum.

1. $e^{(x^3-3x)} : x \geq 0$
2. $-2ln(x) + (x - 1)^2$
3. $x^4 - 2x^3 + 2x$ (quasi convex - has a flat bit)

Hint. Use your preferred optimisation method and check the sufficiency of optimality conditions (i.e., check for convexity to assert whether the solution can be classified as global optimum).