Exercise class 6

Learning Objectives:

- Formulation of integer optimisation problems
- Binary and integer variables

Demo 1: Integer and binary optimisation

Formulate as linear optimisation problem and solve with Julia.

a) A retailer buys a product from a factory and sells one piece with € 2 profit. The retailer has to pay € 1000 to the factory to be able to buy the product. The maximum amount of products the factory is able to produce is 750. How many products should the retailer buy? None?

b) How about if the retailer has to pay the € 1000 only if the quantity of purchased products exceeds 200 pieces?

c) A farmer has 15 boxes of apples and two fruit retailers, Aapeli and Toopeli, want to buy his apples. For one box of apples Aapeli is willing to pay € 25 and Toopeli € 32. If the farmer sells more apples to Aapeli than Toopeli, Aapeli will pay the farmer € 450 extra. Similarly, if Toopeli gets more apples than Aapeli, he will pay the farmer € 270 extra. How many boxes should the farmer sell to Aapeli and Toopeli to maximize his profits?

Demo 2: Sensitivity analysis

Consider the following LP:

$$\begin{aligned}
\max.\ z =& 3x_1 + 2x_2 + x_3 \\
\text{s.t. } & x_1 + 2x_2 + x_3 \leq 100 \\
& x_1 + 2x_3 \leq 200 \\
& x_1 + 4x_2 \leq 250 \\
& x_1,\ x_2,\ x_3 \geq 0
\end{aligned}$$

Let $s_1,\ s_2,\ s_3$ be the slack variables used in the constraints, respectively. The optimal tableau is shown below:

	x_1	x_2	x_3	s_1	s_2	s_3	Sol.
z	0	4	2	3	0	0	300
x_1	1	2	1	1	0	0	100
s_2	0	-2	1	-1	1	0	100
s_3	0	2	-1	-1	0	1	150

Answer the following questions.

a) For what individual changes in the right-hand side does this basis remain optimal?

b) For what individual changes in the objective coefficients does this solution remain optimal?

Problem 1: Sudoku

The world-renowned logic puzzle, Sudoku, deals with a 9 x 9 grid subdivided into 9 nonover lapping 3 x 3 subgrids. The puzzle calls for assigning the numerical digits 1 through 9 to the cells of the grid such that each row, each column, and each subgrid contain distinct digits. Some of the cells may be fixed in advance.

Formulate the problem as an integer program, and find the solution for the instance given below using Julia.

	6		1		4		5	
		8	3		5	6		
2						7		
8			4		7			6
		6				3		
7			9		1			4
5								2
		7	2		6	9		
	4		5		8		7	

Problem 2: Sensitivity analysis

A Gepbab Production Company uses labour and raw material to make three distinct products. The resource requirements and selling prices are described in the table below.

Product	Profit ($/unit)	Labour req. (h/unit)	Raw material (per unit)
1	6	3	2
2	8	4	2
3	13	6	5

Currently, 60 units of raw material are available and 90h of labour can be purchased, at the price of $1 per hour. To maximize the its profit, Gepbab needs to optimise the following linear programming problem:

$$
\begin{aligned}
\max.\ z =& 6x_1 + 8x_2 + 13x_3 - L \\
\text{s.t. } & 3x_1 + 4x_2 + 6x_3 - L \leq 0 \\
& 2x_1 + 2x_2 + 5x_3 \leq 60 \\
& L \leq 90 \\
& x_1,\ x_2,\ x_3,\ L \geq 0
\end{aligned}
$$

where, x_i is the amount of units of product i made and L is the total number of hired labour hours. Let $s_1,\ s_2,\ s_3$ be the slack variables used in the constraints, respectively. The optimal tableau for the problem is:

	x_1	x_2	x_3	L	s_1	s_2	s_3	Sol.
z	0.25	0	0	0	1.75	0.5	0.75	97.5
x_3	0.25	0	1	0	-0.25	0.5	-0.25	7.5
L	0	0	0	1	0	0	1	90
x_2	0.375	1	0	0	0.625	-0.75	0.625	11.25

Answer the following:

a) What is the maximum value that Gepbab should pay for additional units of raw material, assuming that changes in the products selected for production can not be allowed (i.e., the optimal basis is not allowed to change)?

b) What is the maximum value the company should pay for additional availability of labour?

c) What should be the profit per unit of product 1 such that it would become profitable for Gepbab to make it?

d) Consider the following question: “If 100h of labour would be available (instead of 90h), what would be the profit of the company?” Is it possible to answer the question without re-optimising the problem? If so, what is the new objective function value?

e) Can you tell what would be the new optimal solution if the profit per unit of product 3 becomes \$15, without re-optimising the problem? If so, what would this new optimal solution be?

Problem 3: Store locations

Walmark Stores is in the process of expansion in the western United States. During next year, Walmark is planning to construct new stores that will serve 10 geographically dispersed communities. Past experience indicates that a community must be within 25 miles of a store to attract customers. In addition, the population of a community plays an important role in where a store is located, in the sense that bigger communities generate more participating customers. The following table provides the populations as well as the distances (in miles) between the communities:

	Miles from community *i* to community *j*										
i \ *j*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*Population*
1		20	40	35	17	24	50	58	33	12	10,000
2	20		23	68	40	30	20	19	70	40	15,000
3	40	23		36	70	22	45	30	21	80	28,000
4	35	68	36		70	80	24	20	40	10	30,000
5	17	40	70	70		23	70	40	13	40	40,000
6	24	30	22	80	23		12	14	50	50	30,000
7	50	20	45	24	70	12		26	40	30	20,000
8	58	19	30	20	40	14	26		20	50	15,000
9	33	70	21	40	13	50	40	20		22	60,000
10	12	40	80	10	40	50	30	50	22		12,000

The idea is to construct the least number of stores, taking into account the distance restriction and the concentration of populations. Specify the communities where the stores should be located. Use Julia to solve this problem.

Hint.
Define decision variables and parameters:
$x_j = 1$, if community j is selected and 0 otherwise
$p_j =$ population of community j
$d_{ij} =$ distance from community i to community j

And use an incidence matrix A_{ij}.

Problem 4: Formulate & solve IPs and LPs

A wilderness hiker must pack three items: food, first-aid kits, and clothes. The backpack has a capacity of 3 ft^3. Each unit of food takes 1 ft^3. A first-aid kit occupies 1/4 ft^3, and each piece of cloth takes about 1/2 ft^3. The hiker assigns the priority weights 3, 4, and 5 to food, first aid, and clothes, respectively, which means that clothes are the most valuable of the three items. From experience, the hiker must take at least one unit of each item and no more than two first-aid kits.

a) Formulate and solve the IP and LP for this problem.

b) Why does the LP have the exact same solution as the IP?

Hint. This variant of the Knapsack problem does not require binary variables

Problem 5: TSP variant

The daily production schedule at the Rainbow Company includes batches of white (W), yellow (Y), red (R), and black (B) paints. The production facility must be cleaned between successive batches. The Table below summarises the cleanup times in minutes. The objective is to determine the sequencing of colors that minimizes the total cleanup time.

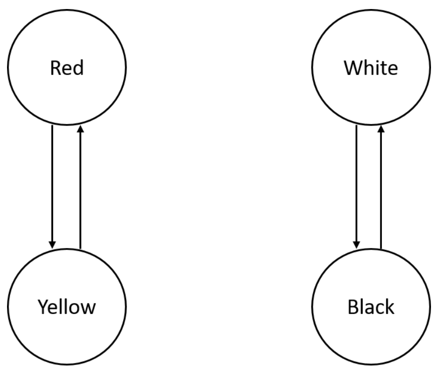
Interbatch Cleanup Times (in minutes) for the Paint Production Problem

	Interbatch cleanup time (min)			
Paint	*White*	*Yellow*	*Black*	*Red*
White		10	17	15
Yellow	20		19	18
Black	50	44		22
Red	45	40	20	

In the TSP model, each color represents a "city," and the cleanup time between two successive colors represents "distance."

a) Formulate the naive model for this TSP without subtour elimination.

b) Consider the solution in the figure below.



i) This solution is not a tour, why?

ii) Formulate two cutset constraints that would have prevented this solution.

iii) Formulate two subtour elimination constraints that would have prevented this solution.

Home Exercise 6: Sensitivity analysis

Dakota Furniture makes desk, tables and chairs. To make the furniture, the company uses wood (acquired in units of a standard measurement boards) and two different processes, namely assembly and finishing. The requirements of wood and process times for each product are detailed below.

Resource	Desk	Table	Chair
Wood (units)	8	6	1
Hours of assembling	2	1.5	0.5
Hours of finishing	4	2	1.5

Currently, Dakota has available 48 units of wood, 8h- of assembling and 20h-worth of finishing per day. The desks are sold for \$60,00, tables for \$30,00 and chairs for \$20,00. The company believes that all products made will often be sold, but that it should not make more than 5 tables. With that in mind, let
$x_1 :=$ total number of desks made
$x_2 :=$ total number of tables made
$x_3 :=$ total number of chairs made
The linear programming (LP) model that optimises Dakota's daily production plan is:

$$
\begin{aligned}
\text{max}.\ z = & 60x_1 + 30x_2 + 20x_3 \\
\text{s.t. } & 8x_1 + 6x_2 + x_3 \leq 48 \\
& 4x_1 + 2x_2 + 1.5x_3 \leq 20 \\
& 2x_1 + 1.5x_2 + 0.5x_3 \leq 8 \\
& x_2 \leq 5 \\
& x_1,\ x_2,\ x_3 \geq 0
\end{aligned}
$$

Using the Simplex method to solve Dakota's optimisation problem we obtain the following optimal tableau:

	x_1	x_2	x_3	s_1	s_2	s_3	s_4	Sol.
z	0	0	0	0	10	10	0	280
s_1	0	-2	0	1	2	-8	0	24
x_3	0	-2	1	0	2	-4	0	8
x_1	1	5/4	0	0	-1/2	3/2	0	2
s_4	0	1	0	0	0	0	1	5

Where s_i are the slack variables associated with constraint (i), for $i = 1, \ldots, 4$. Given the above, please answer:

a) What is the maximum selling price the desks can have that would not affect the optimal production plan?

b) The company is looking into expanding its availability of assembling hours. After a market consultation, Dakota learned that a competing firm would be willing to offer 2h of assembling per day for \$8/h. Assuming that Dakota would not want to change the current production assortment (that is, remain producing desks and chairs and not producing tables), should Dakota accept the offer?