Exercise class 10

Learning Objectives:

- Karush-Kuhn-Tucker conditions

Demo 1: KKT conditions with inequality constraints

Using the Karush-Kuhn-Tucker conditions, see if the points $\mathbf{x} = (x_1, x_2) = (2, 4)$ or $\mathbf{x} = (x_1, x_2) = (6, 2)$ are the local optima of the problem. Can Slater's constraint qualification be used to assert that the KKT conditions are sufficient for global optimality?

$$\begin{array}{lrcrcrl}
\text{max.} & x_1^3 & + & 3x_2 & & & \\
\text{s.t.} & -x_1^2 & + & x_2 & & & \leq 0 \\
 & x_1 & + & 2x_2 & - & 10 & \leq 0 \\
 & x_1 & - & 3x_2 & & & \leq 0
\end{array}$$

Demo 2: KKT conditions with equality and inequality constraints

Solve the problem graphically and verify that the optimal point satisfies the KKT conditions.

$$
\begin{array}{ll}
\min. & x_1^2 + x_2^2 \\
\text{s.t.} & (x_1 - 3)^2 + 1 \le x_2 \\
 & \frac{1}{2}x_1 - x_2 = -1
\end{array}
$$

Problem 1: KKT conditions with inequality constraints

Using the Karush-Kuhn-Tucker conditions, see if the point $\mathbf{x} = (x_1, x_2) = (2, -1)$ is the optimum of the following problem.

$$
\begin{array}{lrcrl}
\max. & x_1^2 & + & x_2^2 & \\
\text{s.t.} & x_2^2 & + & 1 & \leq x_1 \\
 & & & x_1 & \leq 2
\end{array}
$$

Problem 2: KKT conditions with equality and inequality constraints

Solve the following problem and see if the solution satisfies the KKT conditions.

$$
\begin{array}{lrcrcrl}
\min. & x_1 & & & & & \\
\text{s.t.} & (x_1 & + & 4)^2 & - & 2 & \leq x_2 \\
 & x_1 & - & x_2 & + & 4 & = 0 \\
 & & & & & x_1 & \geq -10
\end{array}
$$

Problem 3: Linear Programming Problem

$$
\begin{array}{lrcrl}
\max. & x_1 & + & x_2 & \\
\text{s.t.} & x_1 & + & 3x_2 & \leq 9 \\
 & 2x_1 & + & x_2 & \leq 8 \\
 & \multicolumn{3}{c}{x_1 \geq 0, \quad x_2 \geq 0} &
\end{array}
$$

a) Solve the problem graphically and determine if the solution satisfies the KKT conditions.

b) Find the dual of the optimization problem and solve it (either graphically or in Julia).

c) Compare the solution of the dual and the Lagrange multipliers of the primal problem.

Problem 4: Graphical nonlinear problem & KKT

Solve the problem graphically (you can use Julia to help you plot) and see if it satisfies the KKT conditions. If not, explain why?

$$
\begin{array}{ll}
\max. & x_1 \\
\text{s.t.} & x_2 \leq -(x_1 - 4)^3 \\
 & x_2 \geq 0
\end{array}
$$

Problem 5: Constrained optimisation

Maximise the (Euclidean) distance from the point (1,-1) in the region constrained by the following set of constraints:

$$
\begin{array}{ll}
x_2 & \leq 2e^{-x_1} \\
x_1 & \leq 2 \\
x_2 & \geq \frac{1}{4}x_1^2 \\
x_2 & \leq 2x_1 + 2
\end{array}
$$

Do the KKT conditions hold at this point?

Hint. You can use Julia to graph the region and identify the optimum.

Home Exercise 10: KKT conditions

Find a solution satisfying the KKT conditions for the problem below.

$$
\begin{array}{lrl}
\text{max.} & x_1 + 2x_2 & \\
\\
\text{s.t.} & (x_1 - 3)^2 + (x_2 - 3)^2 & = 4 \\
& x_1^2 - 10x_1 + 26 - x_2 & \geq 0 \\
& x_2 & \geq -7
\end{array}
$$