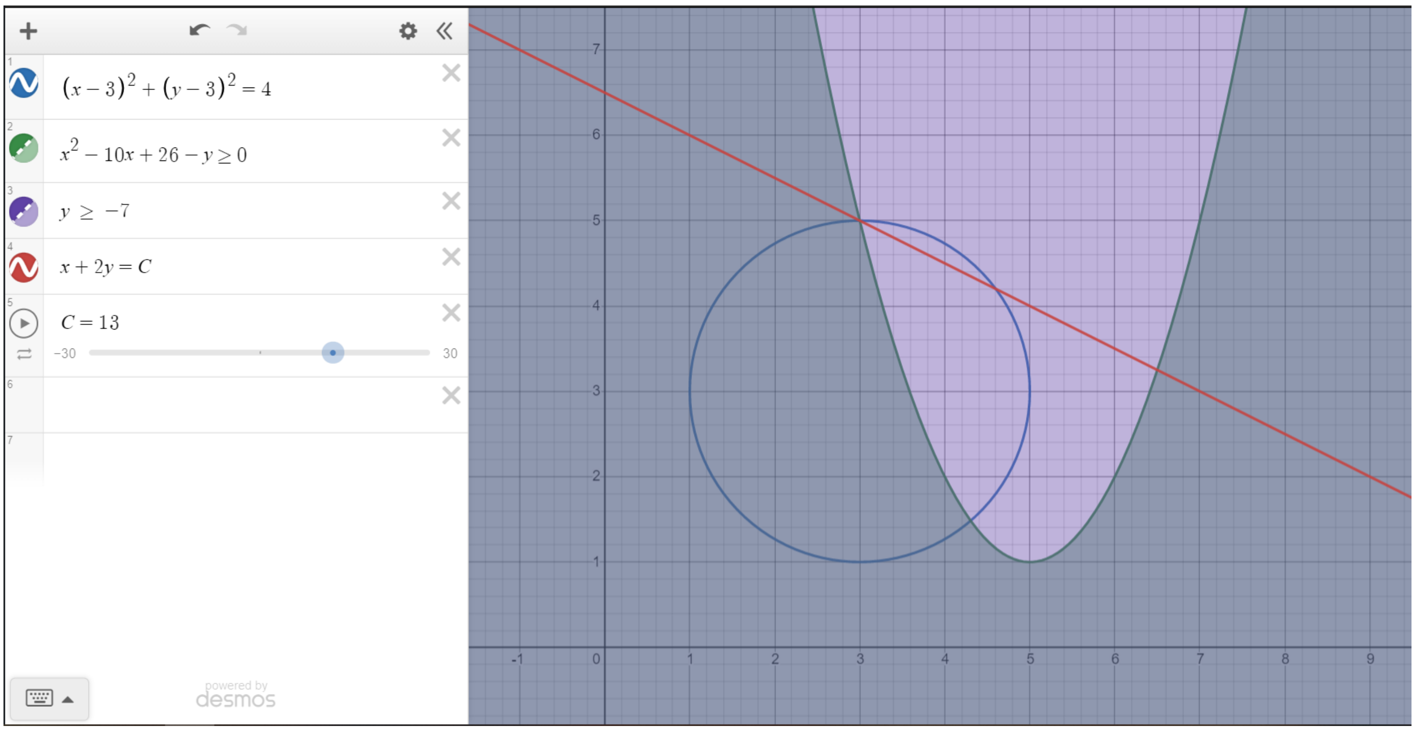
Home Exercise 10: KKT conditions

Find a solution satisfying the KKT conditions for the problem below.

$$\begin{aligned} \max. \quad & x_1 + 2x_2 \\ \text{s.t.} \quad & (x_1-3)^2 + (x_2-3)^2 = 4 \\ & x_1^2 - 10x_1 + 26 - x_2 \geq 0 \\ & x_2 \geq -7 \end{aligned}$$



Solving graphically, we can see that the objective function is maximized at 13 with $x_1 = 3$ and $x_2 = 5$

Find solution to KKT conditions

$$\max.\ x_1 + 2x_2$$
$$\text{s.t.}\ (x_1-3)^2 + (x_2-3)^2 = 4$$
$$x_1^2 - 10x_1 + 26 - x_2 \geqslant 0$$
$$x_2 \geqslant -7$$

When there are both inequality and equality constraints, the KKT conditions are

$$\nabla f(x) + \sum_{i=1} \lambda_i \nabla g_i(x) + \sum_{J=1} \mu_J \nabla h_J(x) = 0, \text{ where } g_i \leqslant 0,\ h_J = 0$$

$$\lambda_i g_i(x) = 0 \ \forall i$$

maximize: $\lambda_i \leqslant 0\ \forall i$ (for minimize: $\lambda_i \geqslant 0\ \forall i$)

$$\Rightarrow \begin{cases} g_1(x) = -x_1^2 + 10x_1 - 26 + x_2 \\ g_2(x) = -x_2 - 7 \\ h_1(x) = (x_1-3)^2 + (x_2-3)^2 - 4 \end{cases} \Rightarrow \begin{cases} \nabla g_1(x) = [-2x_1+10,\ 1]^T \\ \nabla g_2(x) = [0,\ -1]^T \\ \nabla h_1(x) = [2x_1 - 6,\ 2x_2 - 6]^T \end{cases}$$

$$\nabla f(x) = [1, 2]^T$$

- First condition: $L(x, \lambda, \mu) = 0$

$$\Rightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \lambda_1 \begin{bmatrix} -2x_1 + 10 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \mu_1 \begin{bmatrix} 2x_1 - 6 \\ 2x_2 - 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2\lambda_1 x_1 + 10\lambda_1 + 2\mu_1 x_1 - 6\mu_1 + 1 \\ \lambda_1 - \lambda_2 + 2\mu_1 x_2 - 6\mu_1 + 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (1)$$

Graphically, we have found the solution as $x = [3, 5]^T$

$$(1) \Rightarrow \begin{cases} 4\lambda_1 + 1 = 0 \\ \lambda_1 - \lambda_2 + 4\mu_1 + 2 = 0 \end{cases} \Rightarrow \begin{cases} \lambda_1 = -\frac{1}{4} \\ -\lambda_2 + 4\mu_1 + \frac{7}{4} = 0 \quad (2) \end{cases}$$

- Second condition: $\lambda_i g_i(x) = 0\ \forall i$

We have: $g_1([3,5]) = 0$, $g_2([3,5]) = -12$

$\Rightarrow \lambda_1 g_1(x)$ satisfies, λ_2 is not 0 and $g_2 \neq 0 \Rightarrow \lambda_2 = 0$ (3)

$(2)(3) \Rightarrow 4\mu_1 + \frac{7}{4} = 0 \Rightarrow \mu_1 = \frac{-7}{16}$

- Third condition: $\lambda_i \leqslant 0\ \forall i$

We have $\lambda_1 = -1/4$, $\lambda_2 = 0 \Rightarrow$ satisfies the condition

$\Rightarrow$ Since λ_1, λ_2 and μ_1 meets all the conditions, the point $x = [3, 5]^T$ (maximum) satisfies the KKT conditions (answer)

$$(\lambda_1, \lambda_2, \mu_1) = \left(-\frac{1}{4}, 0, -\frac{7}{16}\right)$$