Exercise class 7

Learning Objectives:

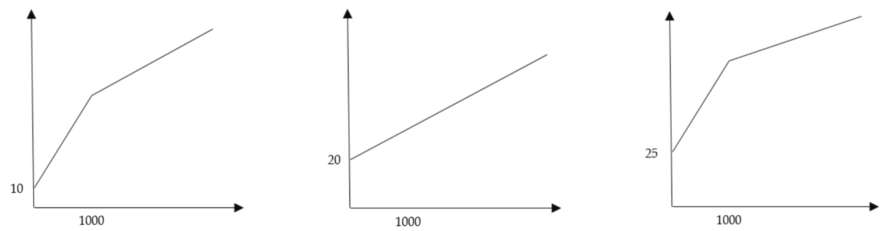
- Fixed costs and job scheduling
- Branch-and-bound method

Demo 1: Fixed costs

A household uses at least 3000 minutes of long-distance telephone calls monthly and can choose to use the services of any of three companies: A, B, and C. Company A charges a fixed monthly fee of \$10 and 5 cents per minute for the first 1000 minutes and 4 cents per minute for all additional minutes. Company B's monthly fee is \$20 with a flat 4 cents per minute. Company C's monthly charge is \$25 with 5 cents per minute for the first 1000 minutes and 3.5 cents per minute beyond that limit. Which company should be selected to minimize the total monthly charge? Formulate the mixed integer linear programming model. Solve using Julia.

Solution

First, let's have a look at the cost functions of the three companies:



(a) Company A (b) Company B (c) Company C

Take, for example, company A: If the number of minutes x in a month is lower than 1000, we pay the fixed fee \$10, and \$0.05 more per minute, so in total $A(x) = 0.05x + 10$. But if $x > 1000$, we pay for that month the fixed fee \$10, and $\$0.05 \cdot 1000$ for the first 1000 minutes, and $0.04(x-1000)$ for the remaining $(x-1000)$ minutes. In total this gives us $A(x) = 0.04(x-1000) + 0.05 \cdot 1000 + 10 = 0.04(x-1000) + 60$.

This leads to the following functions for the three companies:

$$A(x) = \begin{cases} 0.05x + 10, & \text{if } 0 \leq x \leq 1000 \\ 0.04 \cdot (x-1000) + 60, & \text{if } x > 1000 \end{cases}$$

$$B(x) = 0.04x + 20, \qquad x \geq 0$$

$$C(x) = \begin{cases} 0.05x + 25, & \text{if } 0 \leq x \leq 1000 \\ 0.035(x-1000) + 75, & \text{if } x > 1000 \end{cases}$$

Where x is the number of minutes used.

Now, let's formulate an IP to decide which company to use to minimise monthly charges.
$x_i :=$ number of minutes used in plan $i = \{A, B, C\}$
$y_i = \{0, 1\}$ if company i is used or not.

Objective function: we know that there will be 3000 minutes used so can use this fact to simplify the objective function because we know that $x = 3000 > 1000$. Let's look at the objective function for just company A:

$$0.05(1000y_1) + 0.04(x_1 - 1000y_1) + 10y_1$$

The first part calculates the cost of using the first 1000 minutes. The second part calculates the cost of using over 1000 minutes, and the third is the fixed cost.
We can do this for companies B and C, respectively:

$$0.04x_2 + 20y_2$$

$$0.05(1000y_3) + 0.035(x_3 - 1000y_3) + 25y_3$$

Altogether we have:

$$\begin{aligned} &\min.\ 0.04(x_1 - 1000y_1) + 60y_1 + 0.04x_2 + 20y_2 + 0.035(x_3 - 1000y_3) + 75y_3 \\ \equiv\ &\min.\ 0.04x_1 + 20y_1 + 0.04x_2 + 20y_2 + 0.035x_3 + 40y_3 \end{aligned}$$

For the constraints, all we have is that the household uses at least 3000 minutes, and they have to choose one company, and the domains:

$$\begin{aligned} \text{s.t. } x_i &\geq 3000y_i && \forall i \\ \sum_i y_i &= 1 \\ x_i &\geq 0 && \forall i \\ y_i &\in \{0, 1\} && \forall i \end{aligned}$$

Implementing in Julia gives the solution of choosing company A for a monthly charge of \$140.

Demo 2: Branch-and-bound

Solve the linear integer problem with the branch-and-bound (B&B) algorithm.

$$\begin{aligned} \max.\ z &= 3x_1 + 2x_2 \\ \text{s.t. } 2x_1 + 5x_2 &\leq 9 \\ 4x_1 + 2x_2 &\leq 9 \\ x_1,\, x_2 &\in \mathbb{Z}_+ \end{aligned}$$

Solution

For branch-and-bound (B&B) we start by solving the LP relaxation of the problem. This means that x_1 and x_2 are bounded only by non-negativity and they need not be integer. This gives the node $S0$:

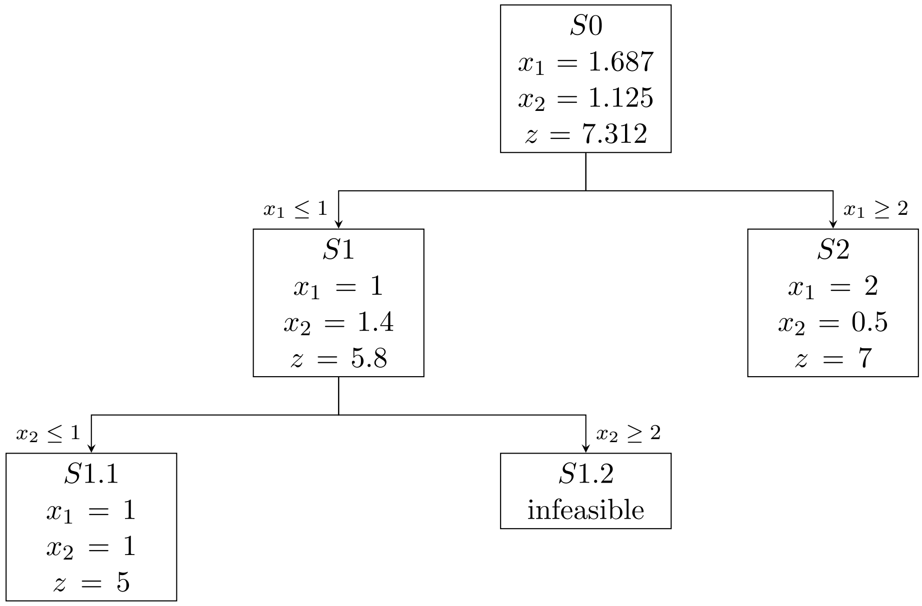
$$\begin{array}{|c|}\hline S0 \\ x_1 = 1.687 \\ x_2 = 1.125 \\ z = 7.312 \\ \hline \end{array}$$

The solution isn't integer so we create two subproblems by constraining the original LP. Select x_1 and bound it by $x_1 \leq \lfloor x_1 \rfloor$ and $x_1 \geq \lceil x_1 \rceil$. This branches to the solution nodes $S1.1$ and $S1.2$ with the added constraints:

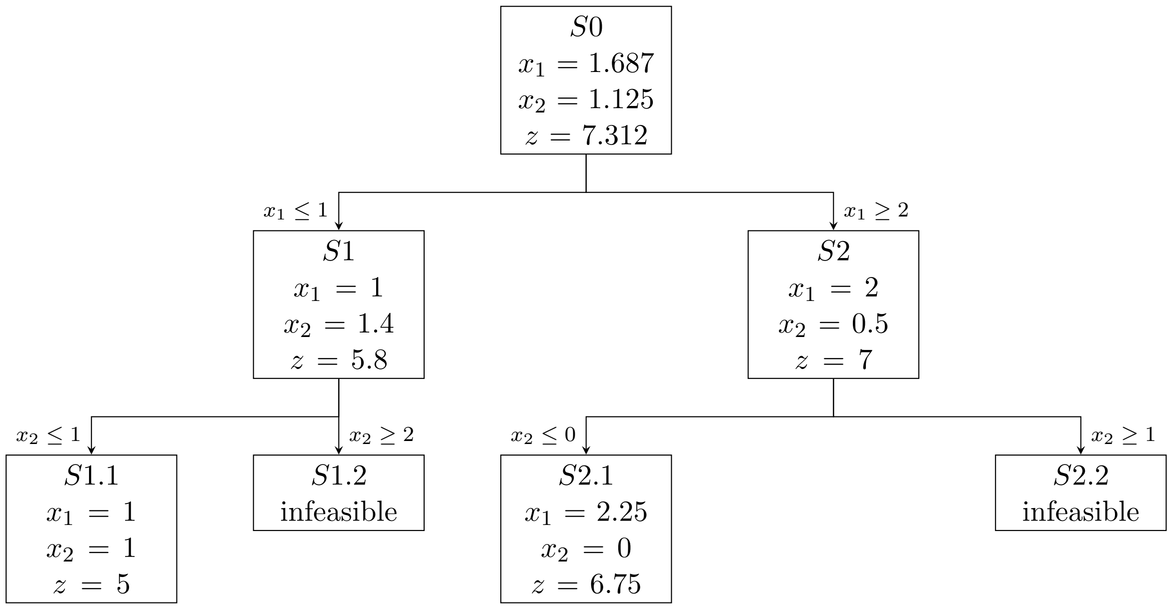
$$\begin{tikzcd} & \begin{array}{|c|}\hline S0 \\ x_1 = 1.687 \\ x_2 = 1.125 \\ z = 7.312 \\ \hline \end{array} \arrow[dl, "x_1 \leq 1"'] \arrow[dr, "x_1 \geq 2"] & \\ \begin{array}{|c|}\hline S1 \\ x_1 = 1 \\ x_2 = 1.4 \\ z = 5.8 \\ \hline \end{array} & & \begin{array}{|c|}\hline S2 \\ x_1 = 2 \\ x_2 = 0.5 \\ z = 7 \\ \hline \end{array} \end{tikzcd}$$

From above, we can see that $S1$ and $S2$ do not have integer solutions.

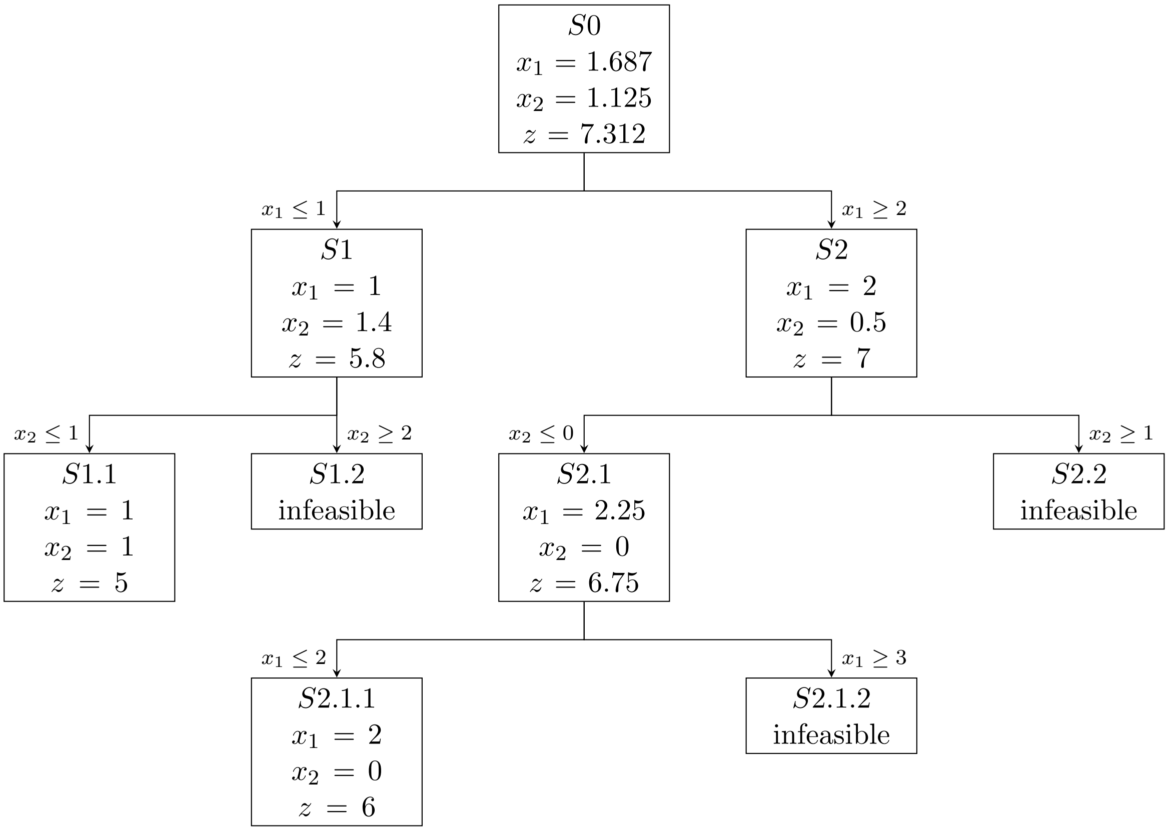
Next, we branch from $S1$ constraining it by $x_2 \leq \lfloor x_2 \rfloor$ and $x_2 \geq \lceil x_2 \rceil$, and resolve the relaxation with the additional constraints.



Node $S1.1$ is integer and $S1.2$ infeasible. Thus, we proceed to investigate $S2$. tIn $S2$, x_2 is fractional, yielding the following nodes



Node $S2.2$ is infeasible and $S2.1$ is not integer, so we branch from $S2.1$.



The end of each branch is now either integer or infeasible. So we can compare the final solutions from $S1.1$ (x_1, x_2)=(1,1) $z = 5$, and $S2.1.1$ (x_1, x_2)=(2,0) $z = 6$. As this is a maximisation problem $S2.1.1$ is the optimal integer solutions.

Note: should we selected investigating $S2$ first and proceeded solving all its child nodes, $S1$ could have been fathomed by limit, since $z = 6$ in $S2.1.1$ is greater than $z = 5.8$ in $S1$.

Problem 1: Job scheduling

Jobco Shop has 10 outstanding jobs to be processed on a single machine. The following table provides processing times and due dates. All times are in days, and due time is measured from time 0:

Job	Processing time (day)	Due time (day)
1	10	20
2	3	98
3	13	100
4	15	34
5	9	50
6	22	44
7	17	32
8	30	60
9	12	80
10	16	150

If job 4 precedes job 3, then job 9 must precede job 7. The objective is to process all 10 jobs minimising a lateness penalty of c. Formulate the model as an ILP.

Solution

Define variables:
x_j := start time (day) of job $j = 1, \ldots, 10$
y_{ij} := 1, if job i precedes job j, 0 otherwise
$s_j := s_j^+ - s_j^-$, where s_j^+ is earliness and s_j^- is lateness of job j

Define constants:
p_j := processing time of job j
d_j := due date of job j
c := penalty cost of lateness

The objective is to minimise lateness:

$$\min . \ z = \sum_{j=1}^{10} c s_j^-$$

We have to ensure all jobs are completed and a job cannot start before another job has finished:

$$\begin{aligned} & x_j + p_j + (s_j^+ - s_j^-) = d_j && \forall j \\ & x_j \geq x_i + p_i - M(1 - y_{ij}) && \forall i,\ j \\ & x_i \geq x_j + p_j - M y_{ij} && \forall i,\ j \end{aligned}$$

Add the constraints that if job 4 precedes job 3, then job 9 must precede job 7:

$$y_{43} \leq y_{97}$$

Domain constraints:

$$\begin{aligned} & x_j \in \mathbb{Z}_+ \\ & y_{ij} \in \{0, 1\} \end{aligned}$$

Problem 2: Fixed cost

A manufacturing facility uses two production lines to produce three products over the next 6 months. Backlogged demand is not allowed. However, a product may be overstocked to meet demand in later months. The following table provides the data associated with the demand, production, and storage of the three products:

	Demand in period						Unit holding	Initial
Product	1	2	3	4	5	6	cost \$/month	inventory
1	50	30	40	60	20	45	0.50	55
2	40	60	50	30	30	55	0.35	75
3	30	40	20	70	40	30	0.45	60

There is a fixed cost for switching a line from one product to another. The following tables give the switching cost, the production rates, and the unit production cost for each line:

	Line switching cost \$		
	Product 1	Product 2	Product 3
Line 1	200	180	300
Line 2	250	200	174

	Production rate units/month			Unit production cost \$		
	Product 1	Product 2	Product 3	Product 1	Product 2	Product 3
Line 1	40	60	80	10	8	15
Line 2	90	70	60	12	6	10

Develop a model for determining the optimal production schedule. Solve with Julia.

Hint. Let $i = 1, 2, 3$ be the product, $j = 1, 2$ be the line, and $t = 1, \ldots, 6$ be the time period.

$$x_{ijt} = \begin{cases} 1, \text{if product } i \text{ uses line } j \text{ period } t \\ 0, \text{otherwise.} \end{cases}$$

$$v_{ijt} = \begin{cases} 1, \text{if switch is made to product } i \text{ on line } j \text{ in period } t \\ 0, \text{otherwise.} \end{cases}$$

I_{it} = end inventory of product i in period t
I_{i0} = initial inventory of product i
D_{it} = demand of product i in period t
r_{ij} = production rate of i on line j
s_{ij} = switching cost of i on line j
c_{ij} = production cost of i on line j
h_i = holding cost of i

Solution

$$
\begin{aligned}
\min.\ z = & \sum_{i,j} c_{ij} r_{ij} \left(\sum_t x_{ijt} \right) + \sum_{i,j} s_{ij} \left(\sum_t v_{ijt} \right) + \sum_i h_i \left(\sum_t I_{it} \right) && (1)\\
\text{s.t.}\ & \sum_i x_{ijt} \leq 1 \qquad \forall j,\ t && (2)\\
& v_{ijt} \geq x_{ijt} - x_{ij(t-1)} \qquad \forall i,\ j,\ t = \{2, \ldots, 6\} && (3)\\
& I_{it} = I_{i0} + \sum_{k=1}^{t} \left(\sum_j (r_{ij} x_{ijk}) - D_{ik} \right) \qquad \forall i,\ t && (4)\\
& x_{ijt}, v_{ijt} \in \{0, 1\}, I_{it} \geq 0 && (5)
\end{aligned}
$$

Where (1) minimises the cost of the production rate, changing lines, and how much inventory is held. Constraint (2) states that only one product can be produced on a line at a given time. Constraint (3) states that a switch in lines can only be made if the product was not produced on that line in the previous period. Constraint (4) states that the inventory at the end of the period is equal to the initial inventory plus the amount that has been produced since $t = 0$ minus the demand (what has been used).
Note, that we do not need to add a demand satisfaction constraint, because I_{it} is restricted to positive values.

Solution is: Line 1 makes product 1 in period 1-5 and is not in use in period 6. Line 2 makes product 3 in periods 1, 4 and 5, and makes product 2 in periods 2, 3 and 6. With an objective value of 5941.25.

Problem 3: Solving IPs with B&B

Develop the B&B tree for each of the following problems. For convenience, always select x_1 as the branching variable at node 0.

a)

$$\begin{aligned} \max . z = 3x_1 + 2x_2 \\ \text{s.t. } 2x_1 + 5x_2 \leq 18 \\ 4x_1 + 2x_2 \leq 18 \\ x_1,\ x_2 \in \mathbb{Z}_+ \end{aligned}$$

b)

$$\begin{aligned} \max . z = 2x_1 + 3x_2 \\ \text{s.t. } 7x_1 + 5x_2 \leq 36 \\ 4x_1 + 9x_2 \leq 35 \\ x_1,\ x_2 \in \mathbb{Z}_+ \end{aligned}$$

c)

$$\begin{aligned} \max . z = 2x_1 + 2x_2 \\ \text{s.t. } 2x_1 + 5x_2 \leq 27 \\ 6x_1 + 5x_2 \leq 16 \\ x_1,\ x_2 \in \mathbb{Z}_+ \end{aligned}$$

d)

$$\begin{aligned} \min . z = 5x_1 + 4x_2 \\ \text{s.t. } 3x_1 + 2x_2 \geq 5 \\ 2x_1 + 3x_2 \geq 7 \\ x_1,\ x_2 \in \mathbb{Z}_+ \end{aligned}$$

Solution

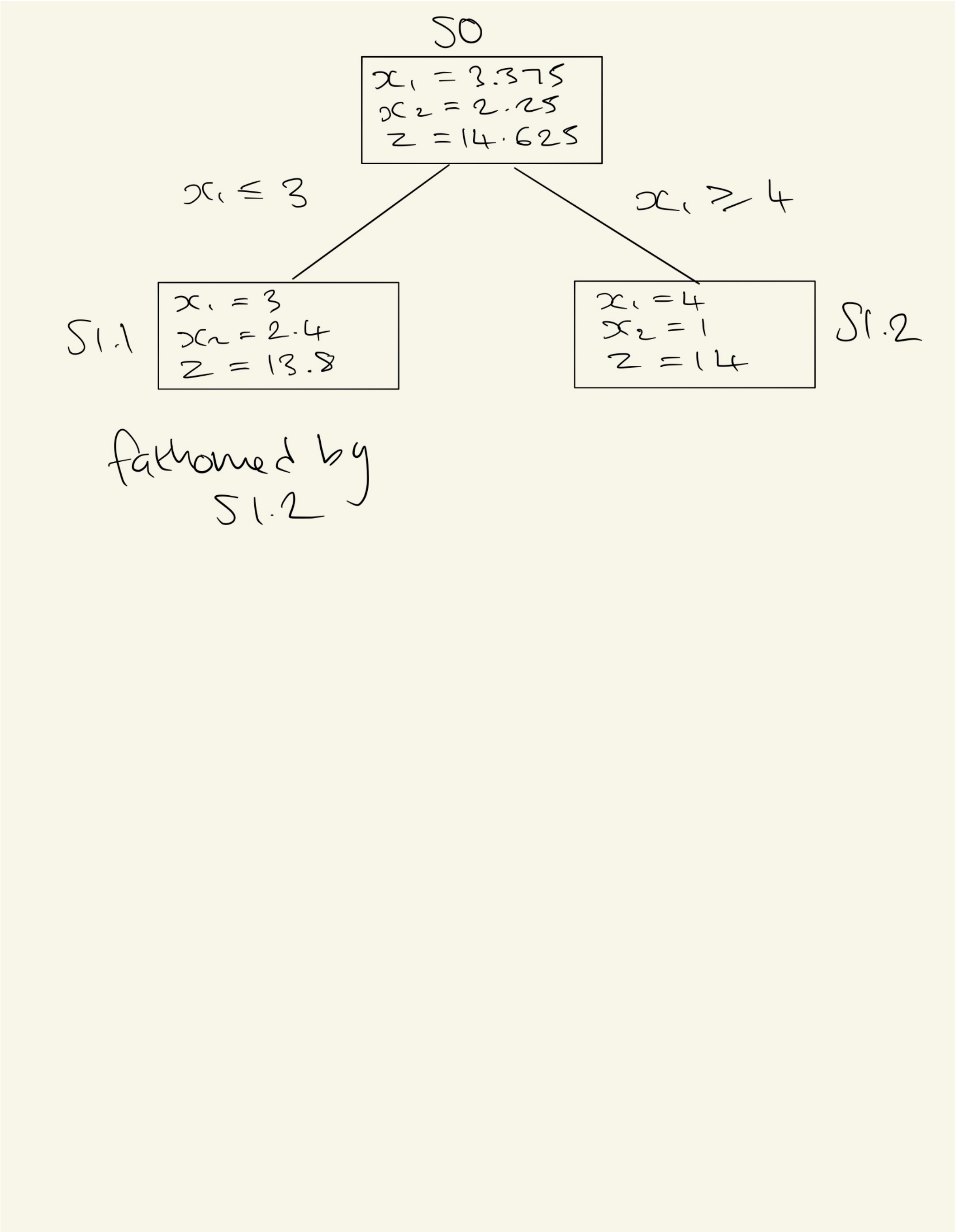


Figure 2: Problem 3a

a)

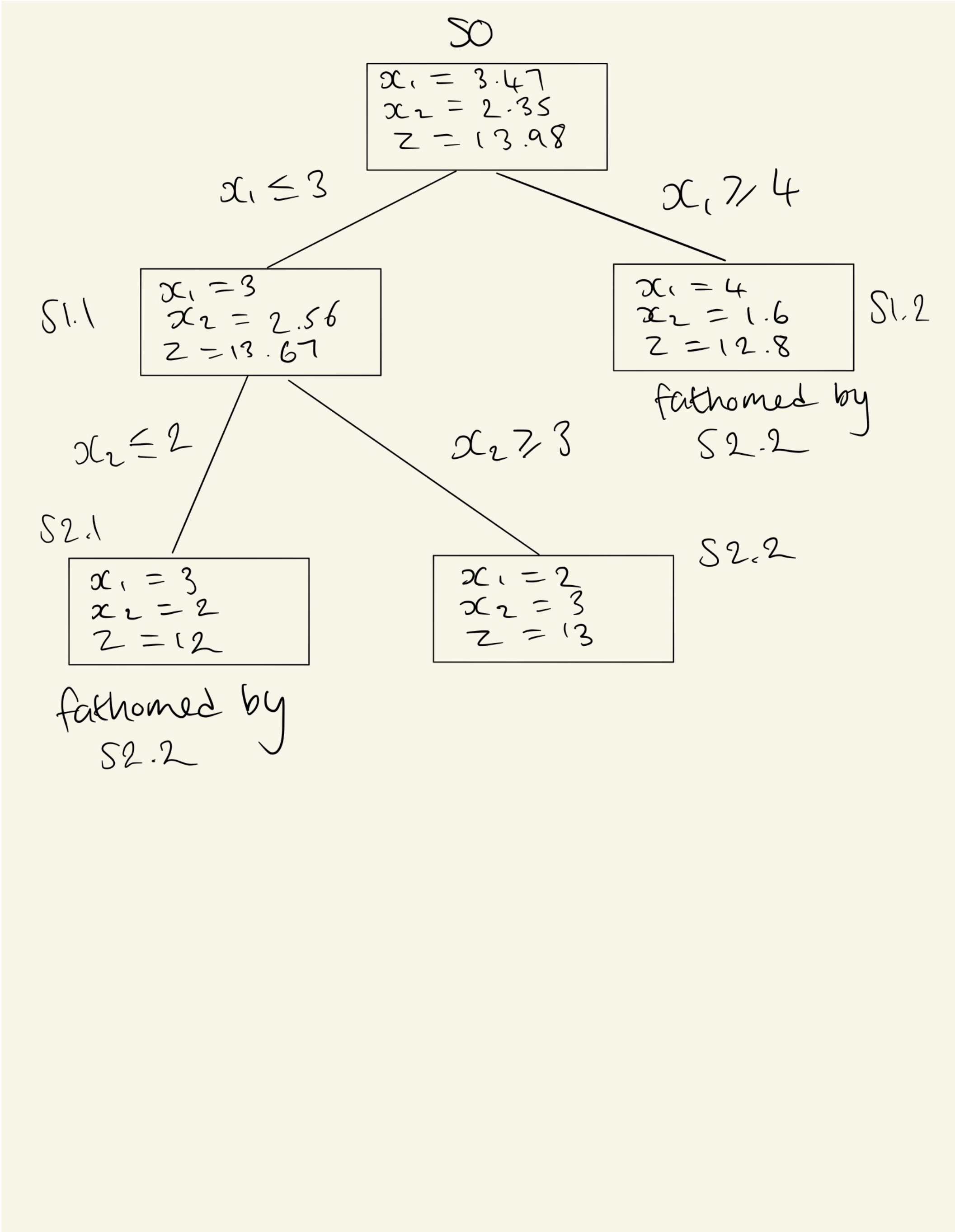


Figure 3: Problem 3b

b)

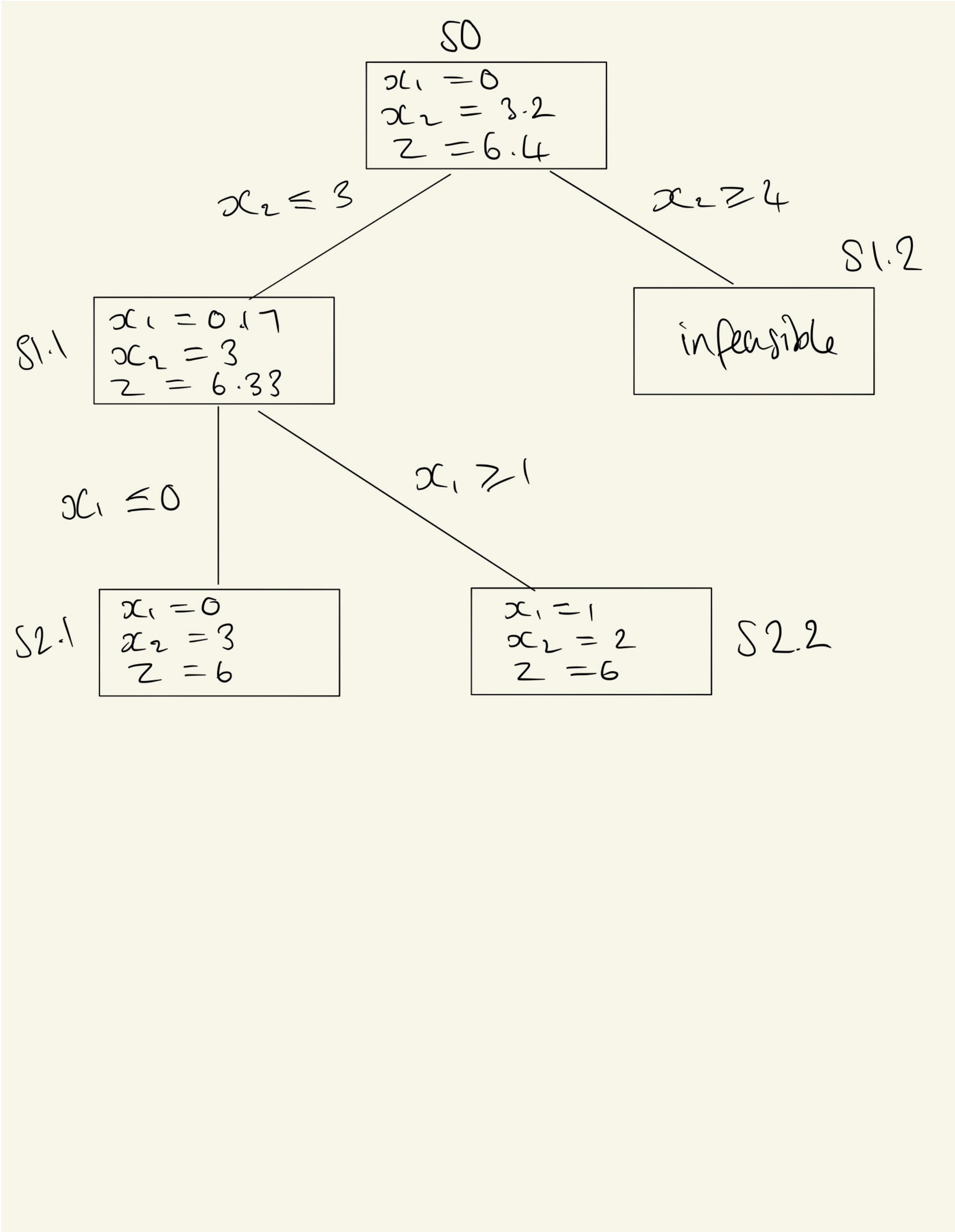


Figure 4: Problem 3c

c)

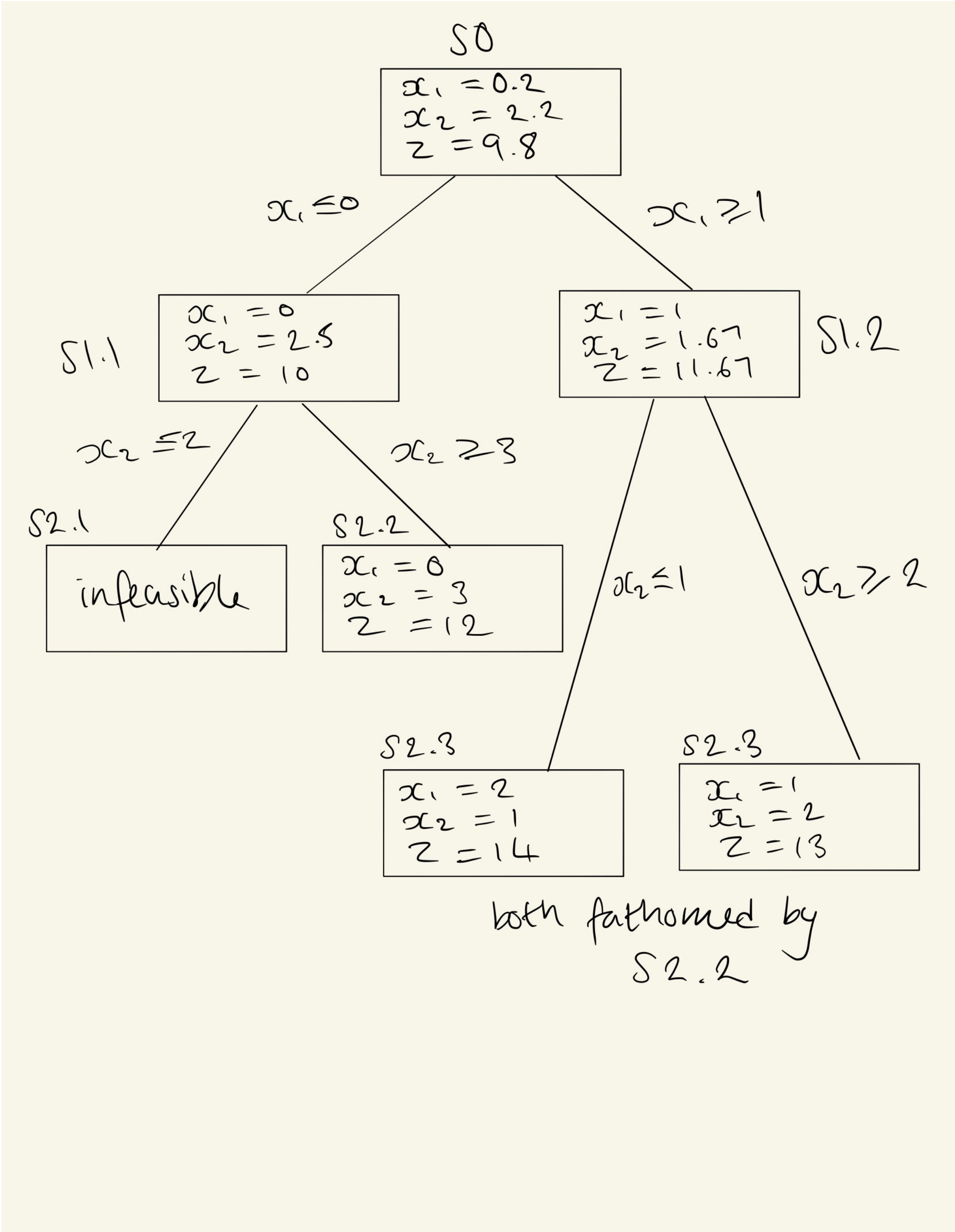


Figure 5: Problem 3d

d)

Problem 4: Solving IPs with B&B

Repeat Problem 3, assuming that x_1 is continuous.

Solution

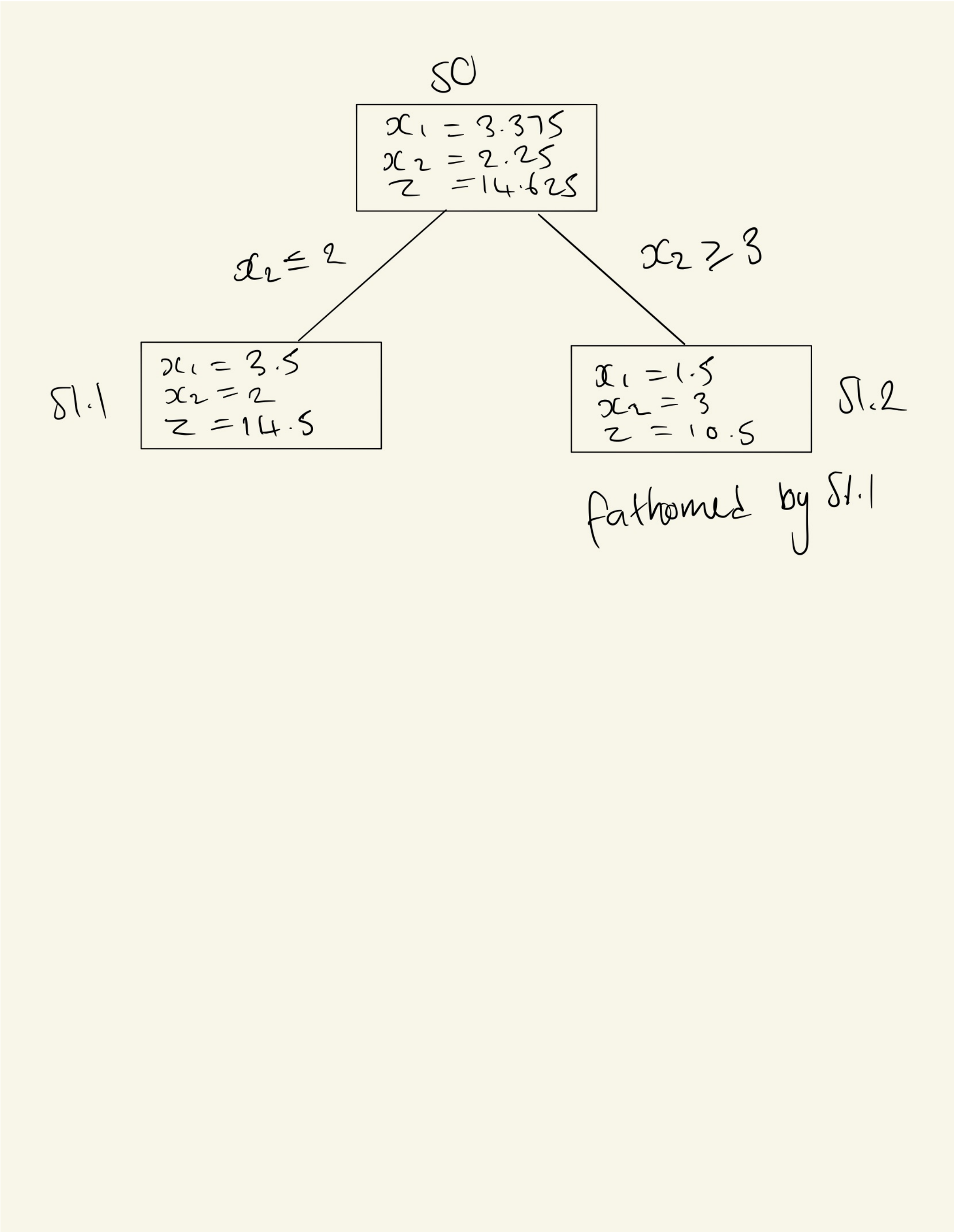


Figure 6: Problem 4a

a)

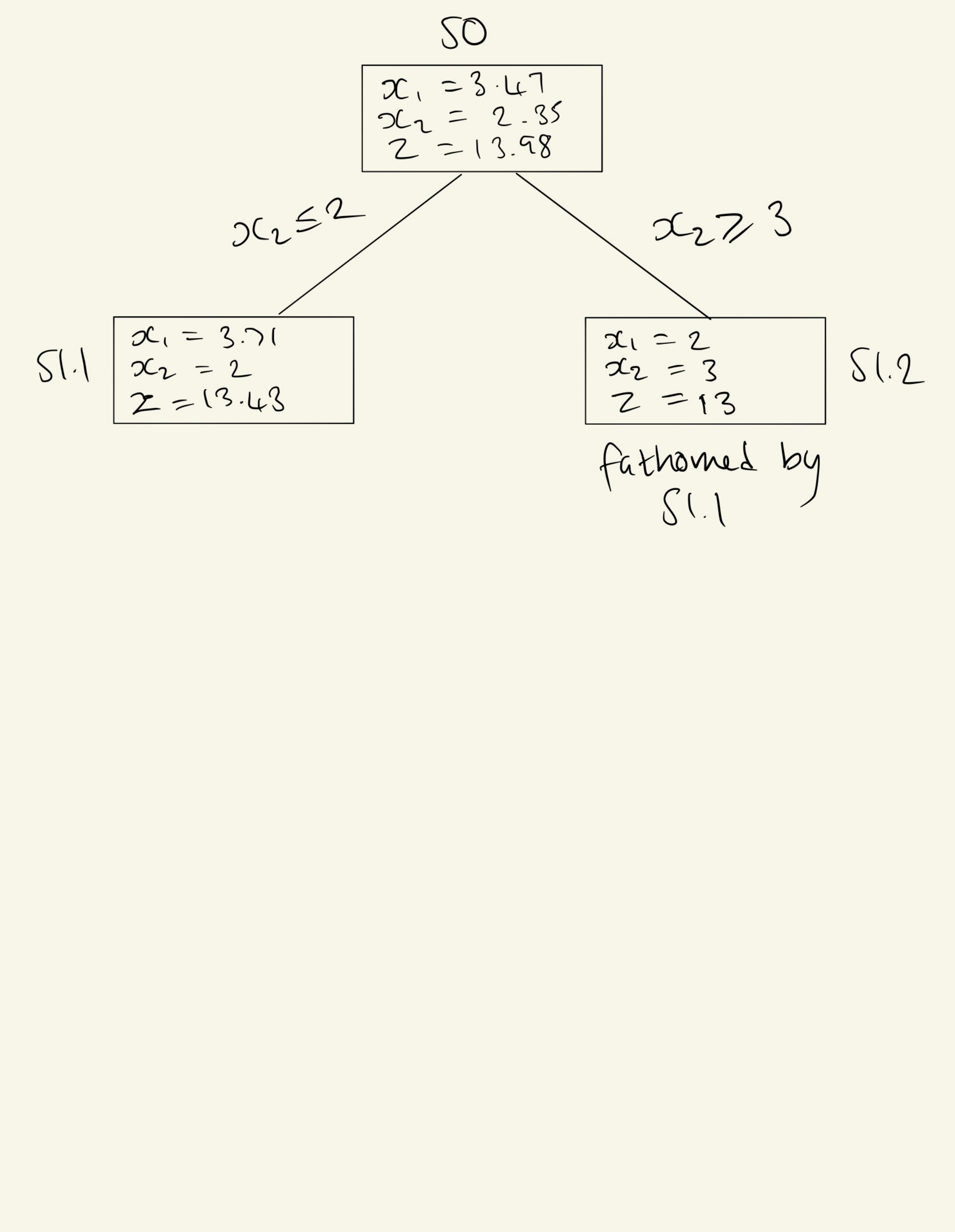


Figure 7: Problem 4b

b)

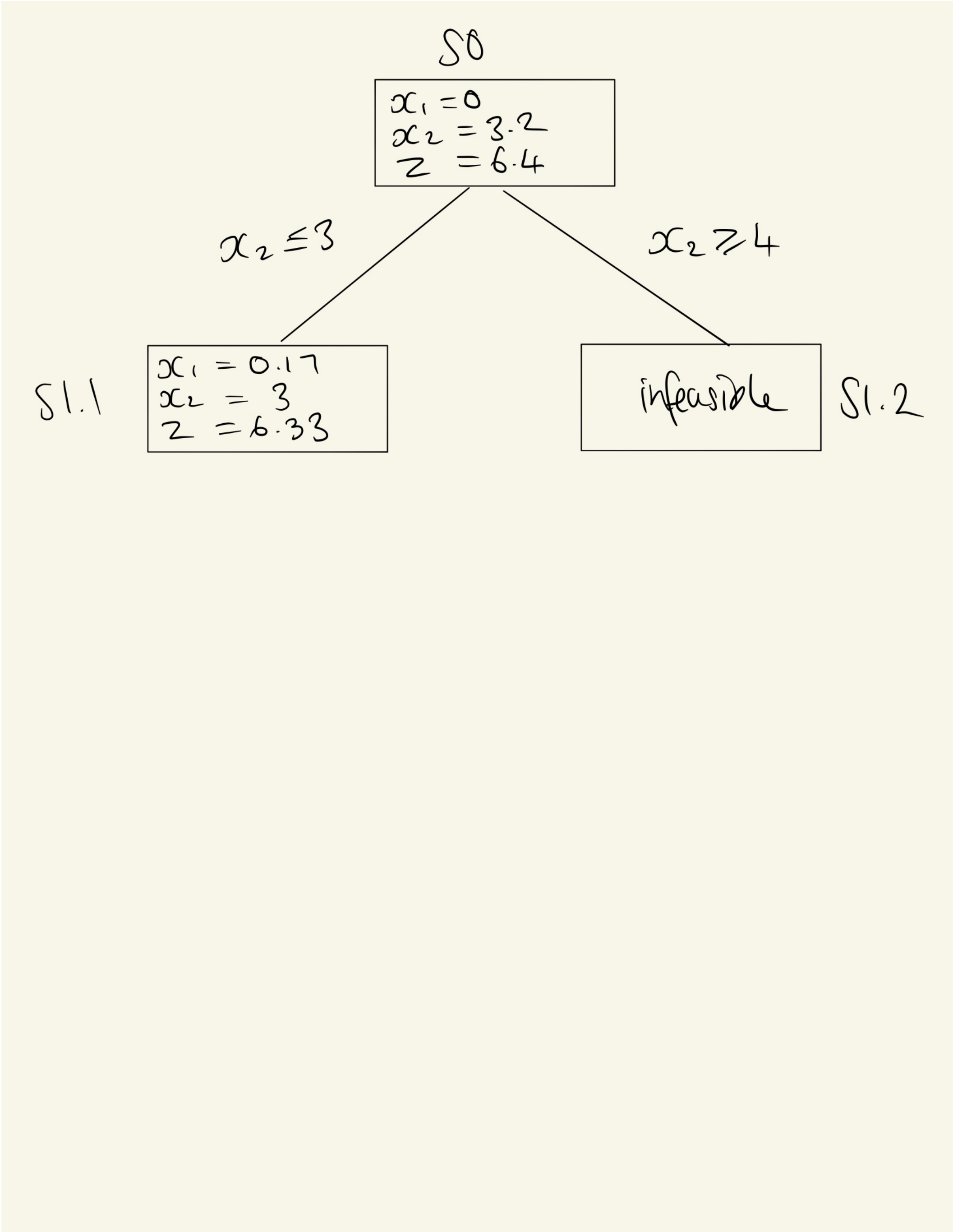


Figure 8: Problem 4c

c)

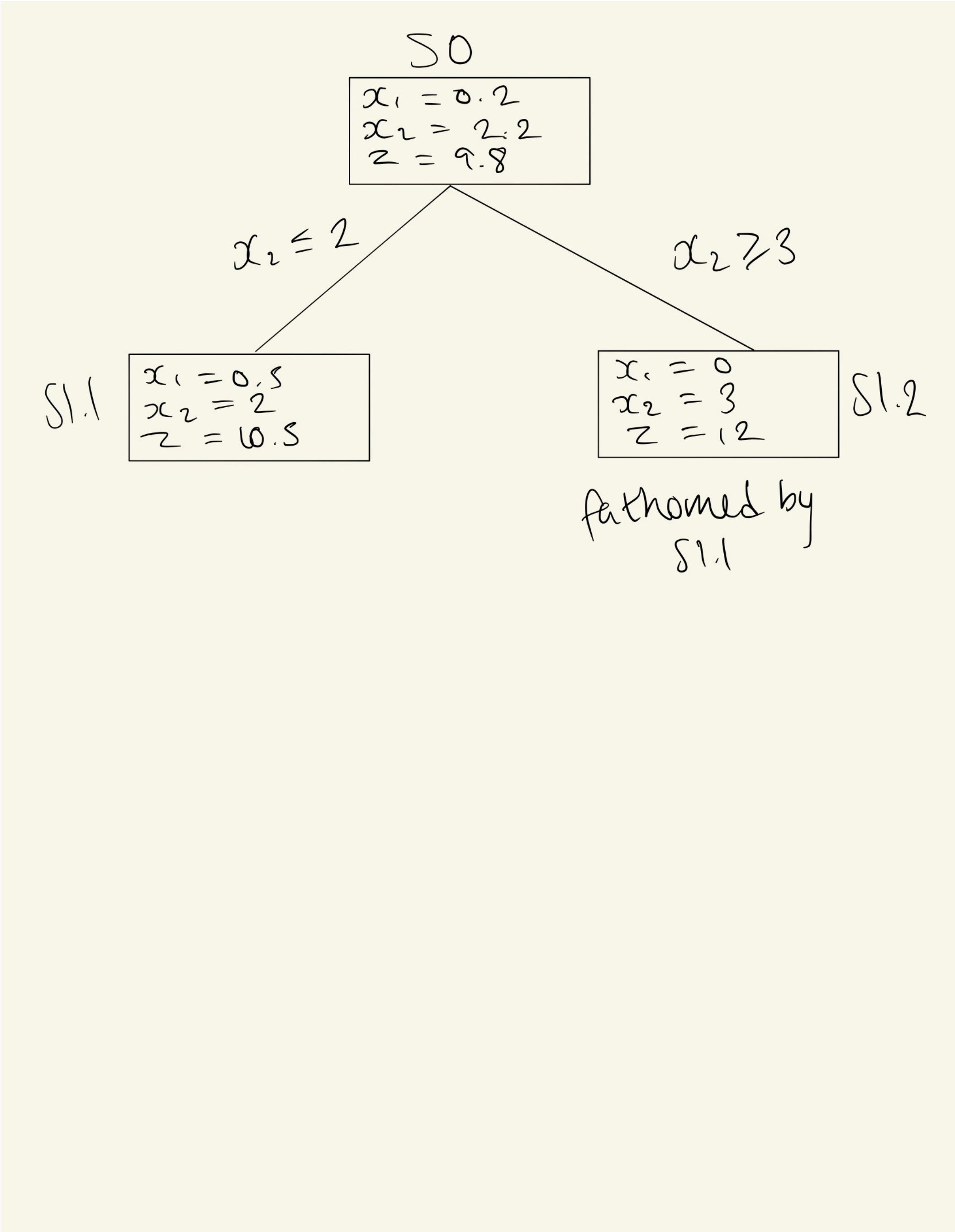


Figure 9: Problem 4d

d)

Problem 5: B&B exam-like question

Consider the following IP:

$$\begin{aligned} \max.\ z = & 1.5x_1 + 1.8x_2 + 2x_3 \\ \text{s.t. } & 0.5x_1 - 2x_2 + x_3 \leq 2.1 \\ & x_1 + x_2 + 4x_3 \leq 4.3 \\ & x_1,\ x_2,\ x_3 \in \mathbb{Z}_+ \end{aligned}$$

Let node 0 be the solution to the linear relaxation. Solve the IP using Branch-and-Bound by rebuilding the tree using the information provided in the table below.

Indicate the order in which the tree was assembled. For branching, the variable selection was performed by choosing the variable with the fractional value closest to an integer value. For the branch selection, start with the constraint $\leq$. The solution of the linear relaxations are given below, together with which constraints that subproblem was subject to.

Node	Constraint on x_1	Constraint on x_2	Constraint in x_3	Obj. Func value	Value x_1	Value x_2	Value x_3
0				7.74	0	4.3	0
1		$x_2 \leq 4$		7.65	0.3	4	0
2		$x_2 \leq 5$		7.74	0	4.3	0
3		$x_2 \geq 5$		Infeasible			
4	$x_1 \leq 0$	$x_2 \leq 4$		7.35	0	4	0.075
5	$x_1 \geq 1$	$x_2 \leq 4$		7.44	1	3.3	0
6	$x_1 \leq 0$	$x_2 \leq 4$	$x_3 \leq 0$	7.2	0	4	0
7	$x_1 \leq 0$	$x_2 \leq 4$	$x_3 \geq 1$	2.54	0	0.3	1
8	$x_1 \leq 0$	$x_2 \leq 0$	$x_3 \geq 1$	2.15	0	0	1.075
9	$x_1 \leq 0$	$x_2 \geq 1$	$x_3 \geq 1$	Infeasible			
10	$x_1 \leq 0$	$x_2 \leq 0$	$x_3 = 1$	2	0	0	1
11	$x_1 \leq 0$	$x_2 \leq 0$	$x_3 \geq 2$	Infeasible			
12	$x_1 \geq 1$	$x_2 \leq 3$		7.35	1.3	3	0
13	$x_1 \geq 1$	$x_2 = 4$		Infeasible			
14	$x_1 = 1$	$x_2 \leq 3$		7.05	1	3	0.075
15	$x_1 = 1$	$x_2 \leq 3$	$x_3 \leq 0$	6.9	1	3	0
16	$x_1 = 1$	$x_2 \leq 3$	$x_3 \geq 1$	Infeasible			
17	$x_1 \geq 2$	$x_2 \leq 3$		7.14	2.3	2	0

Hint. Not all nodes are needed.

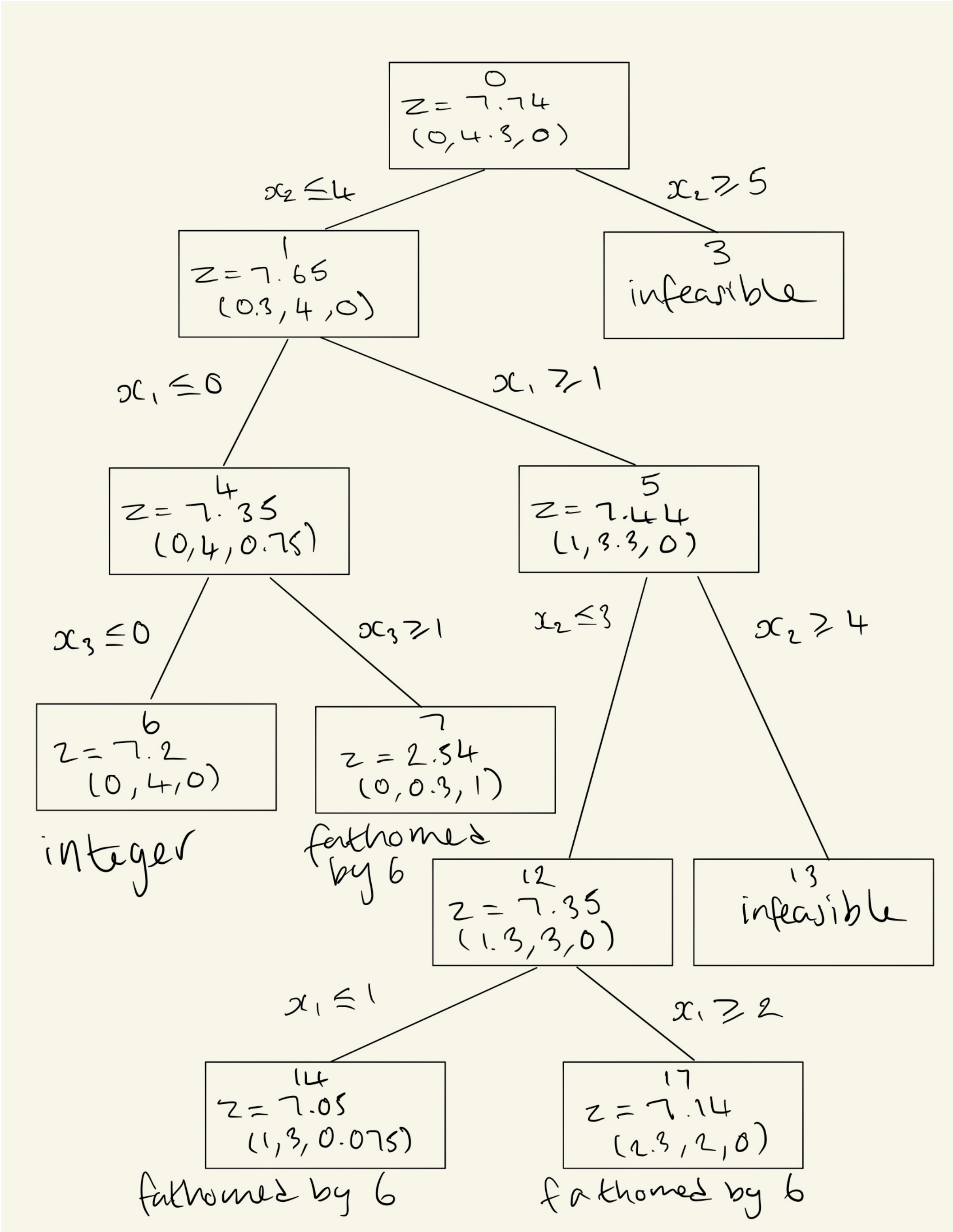
Hint. Constraints on x are accumulated, e.g. if $x_1 \leq 1$ and $x_1 \geq 1$ are being imposed then the table will show $x_1 = 1$, similarly, if $x_2 \leq 4$ and $x_2 \leq 3$ are imposed then $x_2 \leq 3$ is shown.

Solution

Nodes used from the table for the B&B:

0 - 1 - 3 - 4 - 5 - 6 - 7 - 12 - 13 - 14 - 17

Notice that if you were not told to choose the $\leq$ directions you could of decided to expand node 5 before node 4 and this would of made the path far longer. Although, choosing which node to branch is arbitrary it can greatly effect the time taken to solve the B&B.



Home Exercise 7: Branch-and-bound

Solve the following problems by B&B:

$$\begin{aligned} \max.\ & z = 18x_1 + 14x_2 + 8x_3 + 4x_4 \\ \text{s.t. } & 15x_1 + 12x_2 + 7x_3 + 4x_4 + x_5 \le 37 \\ & x_1,\ x_2,\ x_3,\ x_4,\ x_5 \in \{0,1\} \end{aligned}$$

Solution

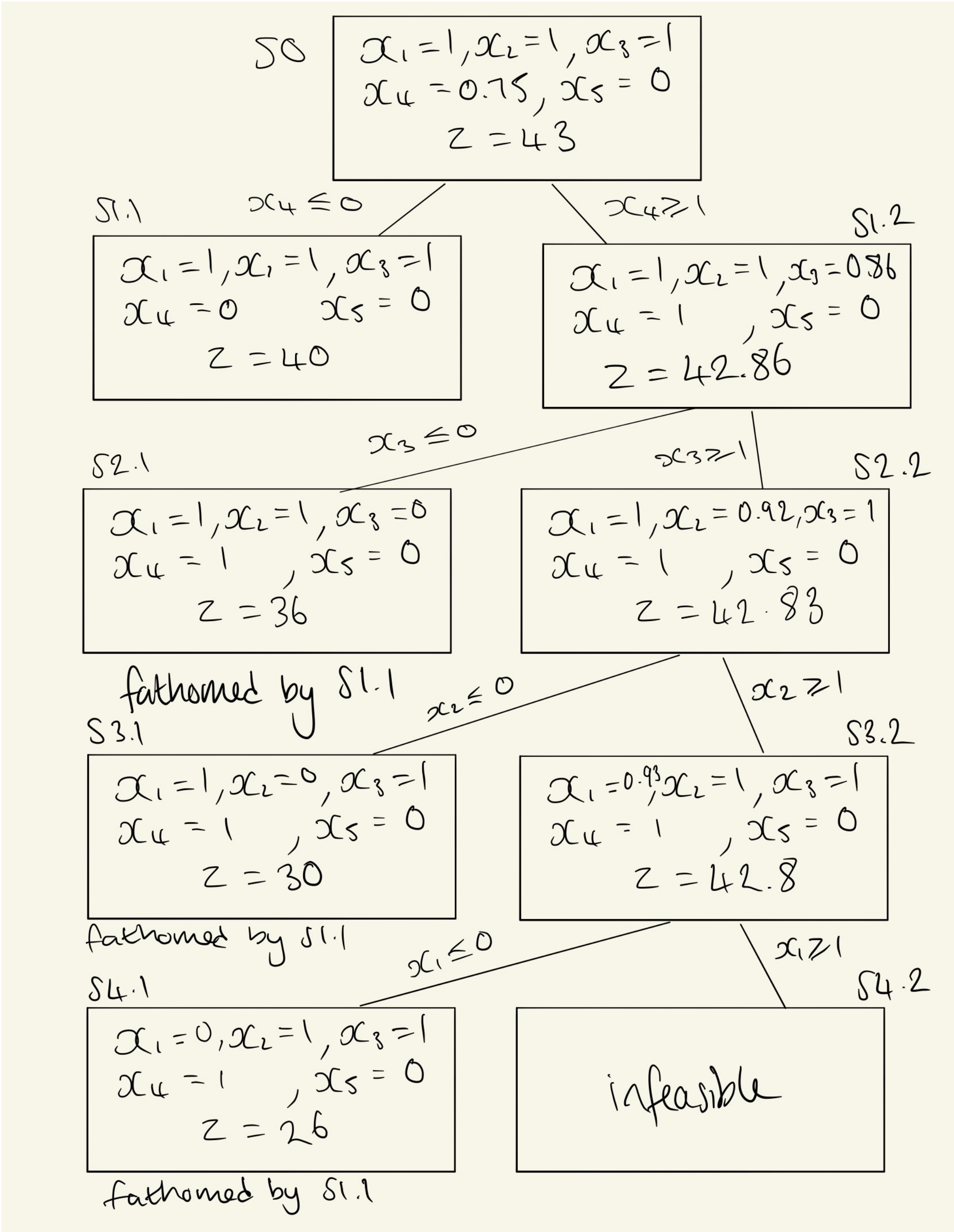


Figure 10: Homework 7