MS-C2105 - Introduction to Optimization
Lecture 3

Fabricio Oliveira (with modifications by Harri Hakula)

Systems Analysis Laboratory
Department of Mathematics and Systems Analysis

Aalto University
School of Science

March 7, 2022

Outline of this lecture

Algebraic representation of LP problems

- Standard form
- Basis and vertices

The simplex method

- Gauss-Jordan elimination
- Tableau representation

Reading: Taha: Chapter 3; Winston: Chapter 4

Algebraic form of LP problems

The Simplex Method is a solution algorithm that builds upon geometrical properties of LP models to find optimal solutions.

The key geometric properties that it exploits are:

- The feasible region is a polyhedral (continuous convex) set.
- An active constraint is an inequality (half-space) satisfied at the boundary (hyper plane).
- If the variable space is $\mathbb{R}^n_+$, n active constraints form a vertex.
- The vertices of the feasible region are candidate solutions. Thus, there is a finite set of solutions to be explored.

Algebraic form of LP problems

The method is developed based on the standard form for LPs:

$$\begin{aligned} \max_x. \ & z = c^\top x \\ \text{s.t.:} \ & Ax = b \\ & x \geq 0, \end{aligned}$$

where $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$ and $b \in \mathbb{R}^m_+$.

To obtain equalities, we include slack/ surplus variables:

- $\sum_{j=1}^n a_{ij}x_j \leq b_i$ becomes $\sum_{j=1}^n a_{ij}x_j + s_i = b_i$ with $s_i = b_i - \sum_{j=1}^n a_{ij}x_j$ and $s_i \geq 0$ (slack).
- $\sum_{j=1}^n a_{ij}x_j \geq b_i$ becomes $\sum_{j=1}^n a_{ij}x_j - s_i = b_i$ with $s_i = \sum_{j=1}^n a_{ij}x_j - b_i$ and $s_i \geq 0$ (surplus).

Algebraic form of LP problems

Variants from the standard form are pre-processed as follows.

1. **nonpositive variables:** $x_i \leq 0$ is replaced with $-y_i$, where $y_i \geq 0$.
2. **unrestricted variables:** $x_i \in \mathbb{R}$ is replaced with $y_i^+ - y_i^-$, where $y_i^+, y_i^- \geq 0$.
3. **minimisation:** min. $z = c^\top x$ is replaced with max. $-z = -c^\top x$. Notice that z^* will have changed sign.
4. **negative b_i:** multiply constraint by (-1).

Example:

$\{$min. $z = 2x_1 - 4x_2 : 22x_1 - 4x_2 \geq -7, x_1 \in \mathbb{R}, x_2 \leq 0\}$.

Basic (feasible) solutions

A nontrivial LP in the standard form is such that $m < n$. This leads to an undetermined system with an infinite number of solutions.

The system $Ax = b$ is solvable if $n - m$ variables are set to zero. These are called **nonbasic variables**.

- This implies that the solution of the system $Ax = b$ lies on the intersection of hyperplanes from $Ax = b$.
- Consequently, the remaining m variables form a **basis** and are called **basic variables**.

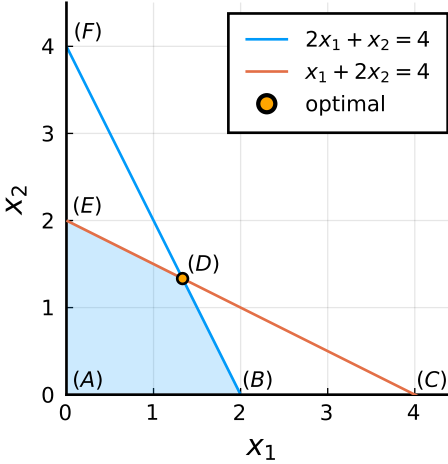
The solution of $Ax = b$ for a given basis is a basic solution. If this solution is feasible (i.e., $x_i \geq 0, \ i = 1, \dots, n$), then it is a **basic feasible solution**.

Basic solutions - graphical interpretation

Consider the following problem:

$$
\begin{aligned}
\text{max. } & z = 4x_1 + 3x_2 \\
\text{s.t.: } & 2x_1 + 1x_2 \leq 4 \qquad (1) \\
& 1x_1 + 2x_2 \leq 4 \qquad (2) \\
& x_1, x_2 \geq 0.
\end{aligned}
$$

Point	x_1	x_2	s_1	s_2	z
(A)	0	0	4	4	0
(B)	2	0	0	2	8
(C)	4	0	-4	0	16
(D)	4/3	4/3	0	0	28/3
(E)	0	2	2	0	6
(F)	0	4	0	-4	16



The simplex method

The method consists of solving adjacent systems until no further improvement can be observed in the objective function.

- **Adjacent systems:** from a given basis B, a single basic variable is replaced with a single nonbasic variable.
- **Improvement:** can be measured by coefficients in the objective function.

The method starts with the most trivial basis:

- Original problem variables are set to 0 (made nonbasic)
- Remaining slack variables form a first basis.

In the example: $B = \{s_1, s_2\}$, $N = \{x_1, x_2\}$.

$$\begin{aligned} z &= 0 + 4x_1 + 3x_2 \\ s_1 &= 4 - 2x_1 - 1x_2 \\ s_2 &= 4 - 1x_1 - 2x_2 \end{aligned}$$

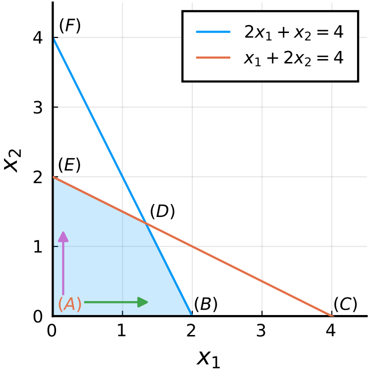
The simplex method

The method greedily chooses to what adjacent system to move.

- **Greed variable selection:** most beneficial objective function coefficient becomes basic.
- If no beneficial coefficient for current nonbasic variables is available, the current basis is optimal.

Change of basis:

- x_1 becomes basic as $4(c_1) > 3(c_2)$.
- s_1 or s_2 must become nonbasic.



The simplex method

The decision of which variable leaves the basis is based on the maximum value the new basic variable x_1 can assume without compromising feasibility.

1. **if s_1 becomes nonbasic:** $s_1 = 4 - 2x_1$ then $s_1 \geq 0$ implies $4 - 2x_1 \geq 0$ or $x_1 \leq 2$. Notice that $s_1 = 0$ makes (1) active.
2. **if s_2 becomes nonbasic:** $s_2 = 4 - x_1$ then $s_2 \geq 0$ implies $4 - x_1 \geq 0$ or $x_1 \leq 4$. Similarly, $s_2 = 0$ makes (2) active.

To ensure feasibility, we impose $x_1 \leq 2$, making s_1 nonbasic.

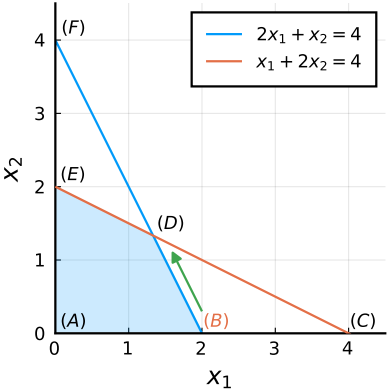
Updated basis: $B = \{x_1, s_2\}$, $N = \{s_1, x_2\}$.

$$\begin{aligned} z &= 0 + 4x_1 + 3x_2 \\ x_1 &= 2 - (1/2)s_1 - (1/2)x_2 \\ s_2 &= 4 - 1x_1 - 2x_2 \end{aligned} \qquad \begin{aligned} z &= 8 - 2s_1 + 1x_2 \\ x_1 &= 2 - (1/2)s_1 - (1/2)x_2 \\ s_2 &= 2 + (1/2)s_1 - (3/2)x_2 \end{aligned}$$

The simplex method

The basis $B = \{x_1, s_2\}$ is associated with (B).

$$z = 8 - 2s_1 + 1x_2$$
$$x_1 = 2 - (1/2)s_1 - (1/2)x_2$$
$$s_2 = 2 + (1/2)s_1 - (3/2)x_2$$



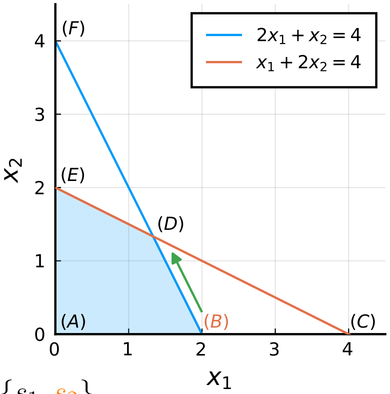
Since there is a nonbasic variable with positive coefficient (x_2), the method proceeds.

1. x_2 becomes a basic variable (only positive coefficient).
2. By the same feasibility argument, s_2 becomes nonbasic.

The simplex method

The basis $B = \{x_1, s_2\}$ is associated with (B).

$$z = 8 - 2s_1 + 1x_2$$
$$x_1 = 2 - (1/2)s_1 - (1/2)x_2$$
$$s_2 = 2 + (1/2)s_1 - (3/2)x_2$$



Updated basis: $B = \{x_1, x_2\}$, $N = \{s_1, s_2\}$.

$$z = 8 - 2s_1 + 1x_2$$
$$x_1 = 2 - (1/2)s_1 - (1/2)x_2$$
$$x_2 = 4/3 + (1/3)s_1 - (2/3)s_2$$

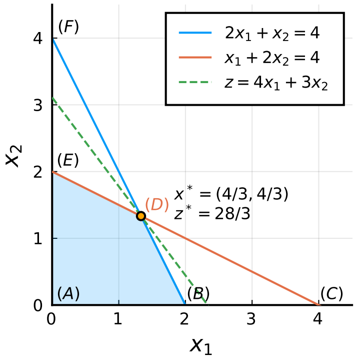
$$z = 28/3 - (5/3)s_1 - (2/3)s_2$$
$$x_1 = 4/3 - (2/3)s_1 + (1/3)s_2$$
$$x_2 = 4/3 + (1/3)s_1 - (2/3)s_2$$

The simplex method

The basis $B = \{x_1, x_2\}$ is associated with (D).

$$z = 28/3 - (5/3)s_1 - (2/3)s_2$$
$$x_1 = 4/3 - (2/3)s_1 + (1/3)s_2$$
$$x_2 = 4/3 + (1/3)s_1 - (2/3)s_2$$

$z^* = 28/3$ and
$x^* = (4/3, 4/3)$.



Since all coefficients are negative, the method is finished.

Remarks:

- If coefficients of nonbasic variables are nonpositive with at least one being zero, the problem has multiple solutions.
- If no feasible basic solution exists, the problem is infeasible.

Gauss-Jordan elimination and tableaus

Efficient implementations of the simplex method use Gauss-Jordan elimination to solve the equation system for a given basis.

1. The systems coefficient are laid as a matrix, including the objective function.
2. An identity (submatrix) is formed for the selected basis, which is equivalent to solve the system for this basis.
3. Coefficients of basic variables are made zero in the objective function.
4. Each new system solution is obtained performing elementary row operations (Gauss Jordan elimination).
   - Row/ column permutation.
   - Multiply a row by a non-zero scalar.
   - Add to one row a scalar multiple of another.

Gauss-Jordan elimination and tableaus

In the example:

Iter. 1: $B = \{s_1, s_2\}$

x_1	x_2	s_1	s_2	b
-4	-3	0	0	0
2	1	1	0	4
1	2	0	1	4

$$z = 0 + 4x_1 + 3x_2$$

$$s_1 = 4 - 2x_1 - 1x_2$$

$$s_2 = 4 - 1x_1 - 2x_2$$

Iter. 2: $B = \{x_1, s_2\}$

x_1	x_2	s_1	s_2	b
0	-1	2	0	8
1	1/2	1/2	0	2
0	3/2	-1/2	1	2

$$z = 8 + 2s_1 - 1x_2$$

$$x_1 = 2 - (1/2)s_1 - (1/2)x_2$$

$$s_2 = 2 + (1/2)s_1 - (3/2)x_2$$

Gauss-Jordan elimination and tableaus

In the example:

Iter. 1: $B = \{s_1, s_2\}$

x_1	x_2	s_1	s_2	b
-4	-3	0	0	0
2	1	1	0	4
1	2	0	1	4

Iter. 2: $B = \{x_1, s_2\}$

x_1	x_2	s_1	s_2	b
0	-1	2	0	8
1	1/2	1/2	0	2
0	3/2	-1/2	1	2

Operations performed:

1. Multiply row #2 by (1/2). Let the result be the pivot row PR.
2. Multiply PR by 4 and add to the row #1.
3. Multiply PR by -1 and add to the row #3.

Gauss-Jordan elimination and tableaus

In the example:

Iter. 1: $B = \{s_1, s_2\}$

x_1	x_2	s_1	s_2	b
-4	-3	0	0	0
2	1	1	0	4
1	2	0	1	4

Iter. 2: $B = \{x_1, s_2\}$

x_1	x_2	s_1	s_2	b
0	-1	2	0	8
1	1/2	1/2	0	2
0	3/2	-1/2	1	2

The decisions on how to update the basis:

1. The entering variable $k \in \{1, \dots, n\}$ has the largest (negative, as side changed) coefficient in the objective function z.
2. The leaving variable has smallest ratio $\frac{b_i}{a_{ik}}$ such that $a_{ik} > 0$ using the feasibility argument as in Page 10.

Using tableaus to solve LPs

A tableau is a table representation that allows for "automating" the algorithm.

- Has little use (none, really) in practice.
- Has an educational purpose only, as it provides structure for calculations in textbook problems.
- Also eases explanation of concepts later on.

The initial tableau for the example:

	x_1	x_2	x_3	x_4	Sol.
z	-4	-3	0	0	0
x_3	2	1	1	0	4
x_4	1	2	0	1	4

- Notice format: $z - 4x_1 - 3x_2 = 0$
- $s_1 = x_3$, $s_2 = x_4$.
- First column inform current basis.

Using tableaus to solve LPs

	x_1	x_2	x_3	x_4	Sol.
z	-4	-3	0	0	0
x_3	2	1	1	0	4
x_4	1	2	0	1	4

- **Entering variable** x_k (pivot column PC): negative coef. with largest absolute value.
- **Leaving variable**: $\arg\min_{i=1,\dots,m} \left\{ \frac{b_i}{a_{ik}} : a_{ik} > 0 \right\}$.

After performing suitable row operations, we obtain:

	x_1	x_2	x_3	x_4	Sol.	Operations
z	0	-1	2	0	8	$+ (4) \times PR$
x_1	1	1/2	1/2	0	2	$\times(1/2) : PR$
x_4	0	3/2	-1/2	1	2	$+ (-1) \times PR$

- Row operations are performed to turn PC into part of basis.
- Only PR can be used to modify other rows.

Using tableaus to solve LPs

As there is still a negative entry in z, the method proceeds...

	x_1	x_2	x_3	x_4	Sol.	b_i/a_{ik}
z	0	-1	2	0	8	-
x_1	1	1/2	1/2	0	2	4
x_4	0	3/2	-1/2	1	2	4/3

... reaching optimality at $x^* = (4/3, 4/3)$, $z^* = 28/3$.

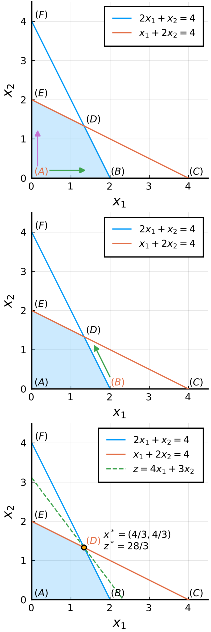
	x_1	x_2	x_3	x_4	Sol.	Operations
z	0	0	5/3	2/3	28/3	$+ (1) \times PR$
x_1	1	0	2/3	-1/3	4/3	$+ (-1/2) \times PR$
x_2	0	1	-1/3	2/3	4/3	$\times 2/3 : PR$

Observing progress graphically

(A)	x_1	x_2	x_3	x_4	Sol.
z	-4	-3	0	0	0
x_3	2	1	1	0	4
x_4	1	2	0	1	4

(B)	x_1	x_2	x_3	x_4	Sol.
z	0	-1	2	0	8
x_1	1	1/2	1/2	0	2
x_4	0	3/2	-1/2	1	2

(D)	x_1	x_2	x_3	x_4	Sol.
z	0	0	5/3	2/3	28/3
x_1	1	0	2/3	-1/3	4/3
x_2	0	1	-1/3	2/3	4/3



Simplex method - summary

Algorithm Simplex method

initialise. Convert problem to standard form, if needed. Form basis B.
while there are negative element in row z for any $j = \{1, \dots, n\}$ **do**
    Select *entering variable*: $k = \arg\min_{j \in 1,\dots,n} \{c_j\}$
    Select *leaving variable*: $i_{PR} = \arg\min_{i=1,\dots,m} \left\{ \frac{b_i}{a_{ik}} : a_{ik} > 0 \right\}$
    Perform *row operations*: $a_{i_{PR}k} = 1, a_{ik} = 0$ for $i = 1, \dots, m : i \neq i_{PR}$
    $B = B \cup \{k\} \setminus \{i_{PR}\}$
end while
return B, $x_i = b_i$ for $i \in B$, $x_j = 0$ for $j \in \{1, \dots, n\} \setminus B$.

Remarks:

- Modern implementations rely on efficient computational algebra (factorisation) and a minimum representation of the problem (see revised simplex method).
- In theory, the simplex method is an algorithm with exponential complexity. A total of $\binom{n}{m}$ vertices might need to be visited.