Home Exercise 8: Convexity

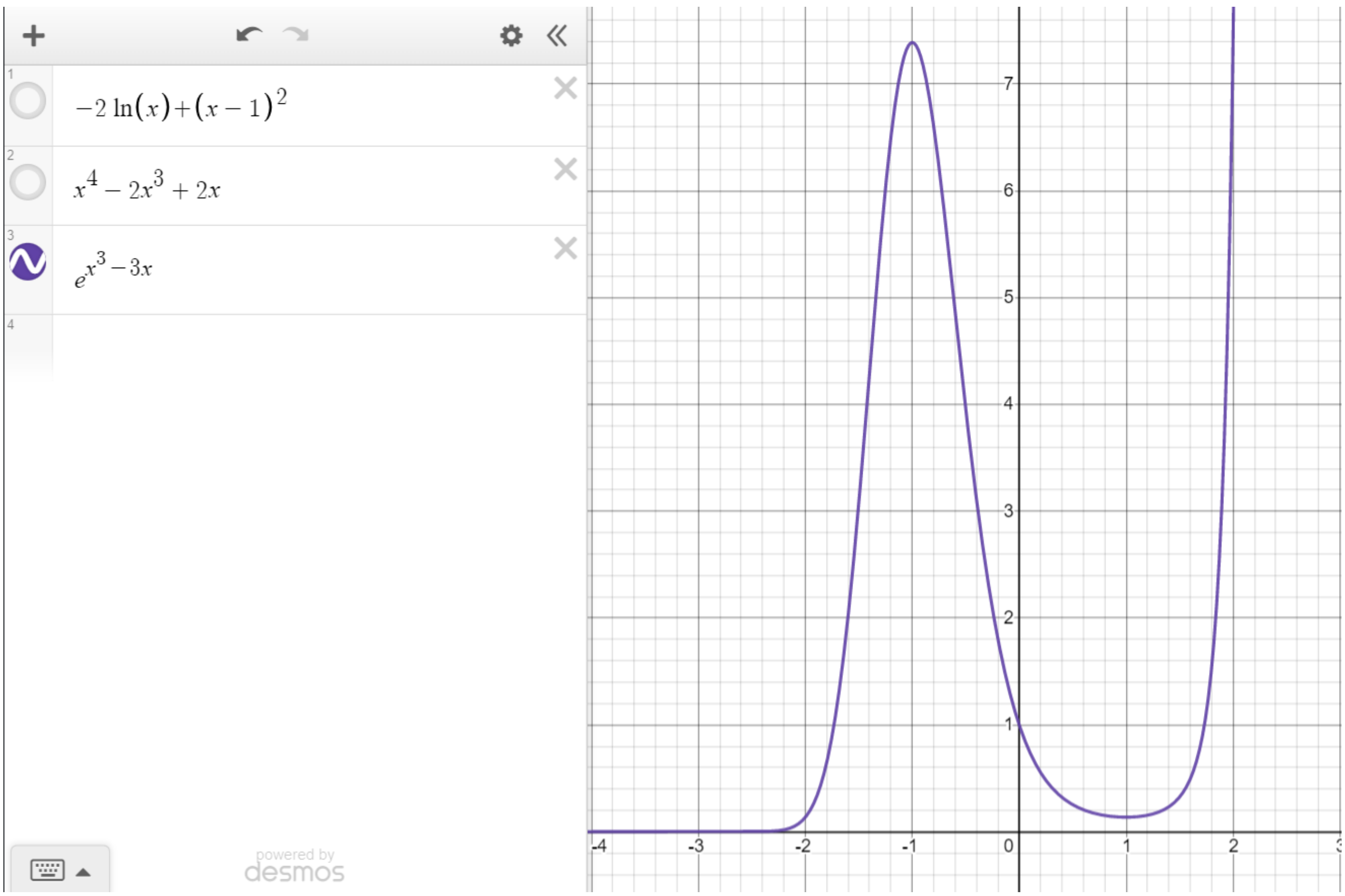
Find optimum/ optima for the following functions, justifying whether they are local or global minimum/ maximum.

1. $e^{(x^3-3x)} : x \geq 0$
2. $-2ln(x) + (x-1)^2$
3. $x^4 - 2x^3 + 2x$ (quasi convex - has a flat bit)

Hint. Use your preferred optimisation method and check the sufficiency of optimality conditions (i.e., check for convexity to assert whether the solution can be classified as global optimum).

Question 1: e^(x^3 - 3x), x >= 0

Graphical function by Desmos



Since x >= 0 and we want to find the optimum, we can notice that there is one local minimum around x = 1. I now apply the Bisection method for interval [0, 2]

```
function min_bisection(f,a,b,ϵ)
    #Precalculates first  derivative
    D(f, x)  = ForwardDiff.derivative(f, x)
    # declare lambda
    lambda = 0.0
    #counter
    n = 0
    while b-a > ϵ
        lambda = (a+b)/2
        if D(f,lambda) == 0 break
        elseif D(f,lambda) > 0 # as we are minimising the function
            a = a
            b = lambda
        else
            a = lambda
            b = b
        end
        println("a_$(n) = $(a) b_$(n) = $(b)")
        n = n+1
    end
    println("λ = $(lambda)")
end
```

```
min_bisection (generic function with 1 method)
```

```
a = 0 # starting interval
b = 2 # starting interval
ϵ = 0.01 # precision
e = 2.7182818284590452353602874713527 # euler number
#function we want to optimise
f(x) = e^(x^3 - 3x)

# call bisection function
min_bisection(f,a,b,ϵ)
```

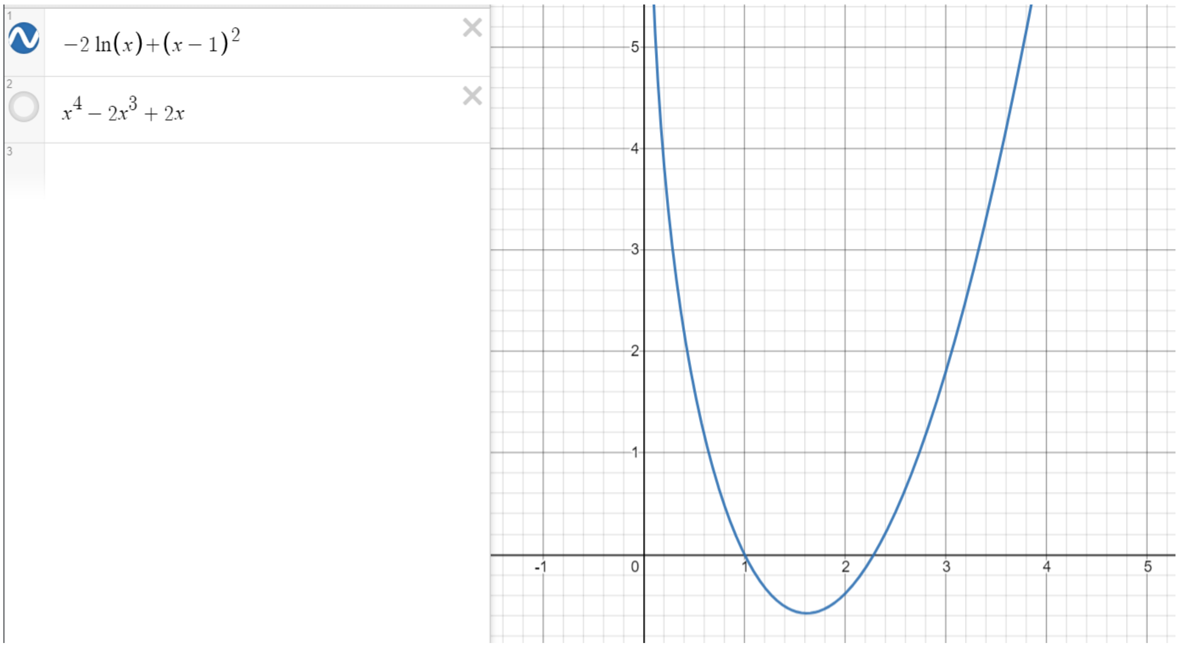
```
λ = 1.0
```

We have f(x) = e^(x^3 - 3x)
First derivative: f'(x) = (3x^2 - 3)e^(x^3-3x) => f'(1) = 0
=> x = 1 is a stationary point
Second derivative:

$$\frac{d}{dx}\left(\left(3\,x^2-3\right)e^{x^3-3\,x}\right)=3\,e^{x^3-3\,x}\left(3\,x^4-6\,x^2+2\,x+3\right)$$

=> f''(1) = 0.81 > 0. Therefore x = 1 is the local minimum
Convexity/concavity testing: We have: x^3 (x >= 0) is convex, -3x (x >=0) is convex/concave, but we consider it convex here.
=> Their linear combination x^3 - 3x is also convex
=> e^(x^3 - 3x) is also convex. Therefore, x = 1 is the global minimum because the function with condition x >= 0 is convex and x = 1 is proven to be local minimum
Final answer: Function e^(x^3 - 3x), x >= 0 has one global minimum: x = 1, f(x) = 0.135

Question 2: -2ln(x) + (x-1)^2
Graphical function by Desmos



We can notice that there is one local minimum around x = 1.5. I now apply the Bisection method for interval [1, 2]

```
[10]: a = 1 # starting interval
      b = 2 # starting interval
      ϵ = 0.001 # precision

      #function we want to optimise
      f(x) =  -2*log(x) + (x-1)^2

      # call bisection function
      min_bisection(f,a,b,ϵ)

      a_0 = 1.5 b_0 = 2
      a_1 = 1.5 b_1 = 1.75
      a_2 = 1.5 b_2 = 1.625
      a_3 = 1.5625 b_3 = 1.625
      a_4 = 1.59375 b_4 = 1.625
      a_5 = 1.609375 b_5 = 1.625
      a_6 = 1.6171875 b_6 = 1.625
      a_7 = 1.6171875 b_7 = 1.62109375
      a_8 = 1.6171875 b_8 = 1.619140625
      a_9 = 1.6171875 b_9 = 1.6181640625
      λ = 1.6181640625
```

We have f(x) = -2ln(x) + (x-1)^2
First derivative:

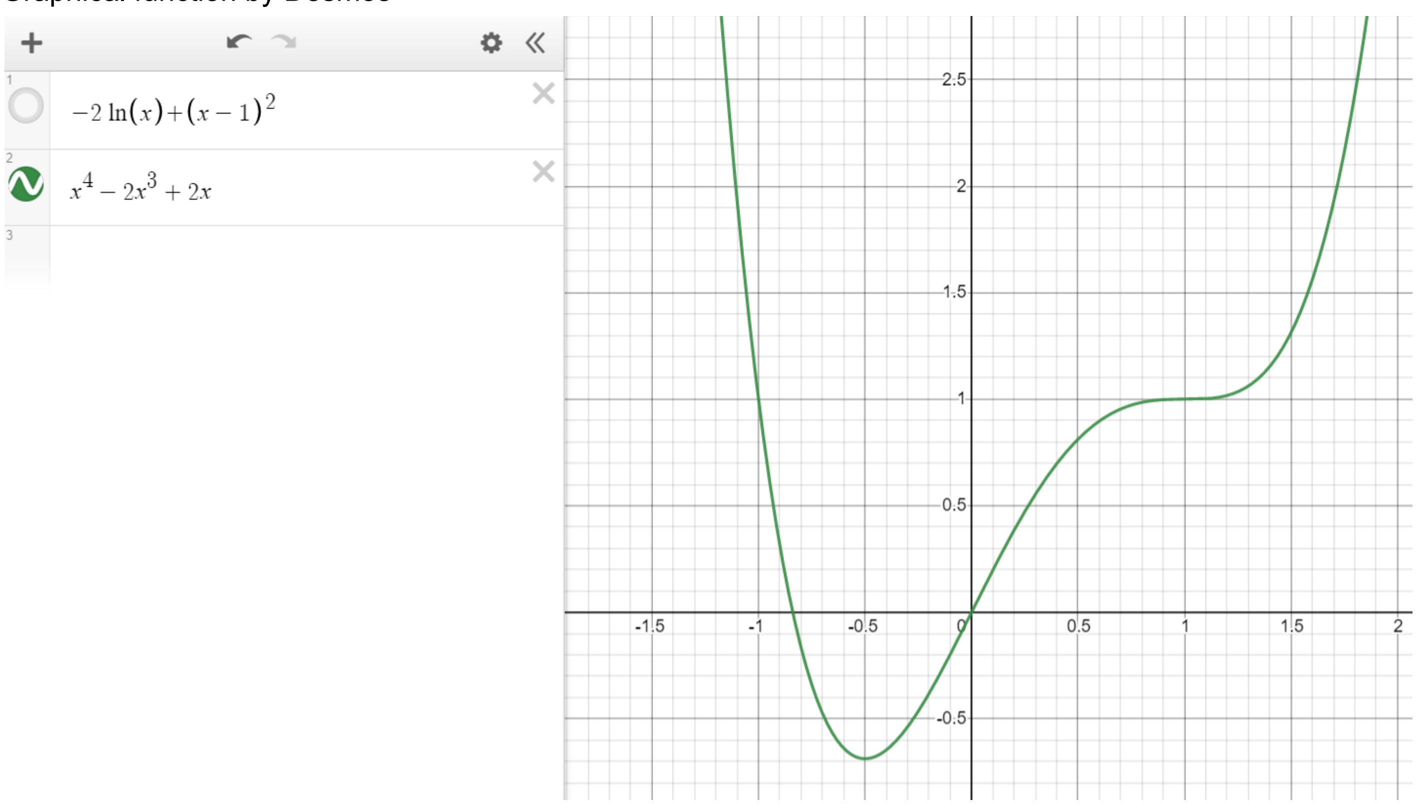
$$\frac{d}{dx}\left(-2\log(x) + (x-1)^2\right) = 2x - \frac{2}{x} - 2$$

=> f'(1.618) ≈ 0 => x = 1.618 is a stationary point
Second derivative:

$$\frac{d^2}{dx^2}\left(-2\log(x) + (x-1)^2\right) = \frac{2}{x^2} + 2$$

=> f''(1.618) = 2.76 > 0 => x = 1.618 is a local minimum
Convexity/concavity testing: Domain of x: x > 0
(x - 1)^2, x > 0 is convex
-2ln(x) is also convex
=> Their linear combination is also convex. Therefore, x = 1.618 is the global minimum because the function is convex and x = 1.618 is proven to be local minimum
Final answer: Function -2ln(x) + (x-1)^2 has one global minimum: x = 1.618, f(x) = -0.58

Question 3: x^4 - 2x^3 + 2x
Graphical function by Desmos



We can notice that there is one local minimum around x = -0.5 and one inflection point around x = 1. I now apply the Newton method with initial point -1 and initial point 1.5

[12]:

```julia
# Line search: Newton's method
function newton(f,x0, ϵ)
    #Precalculates first and second derivative.
    D(f, x)  = ForwardDiff.derivative(f, x)
    #hack to deal with unidimensional second-order derivative
    D²(f, x) = ForwardDiff.derivative(y -> ForwardDiff.derivative(f, y), x)
    # counter
    n = 0
    # Newtown step
    while abs(D(f, x0)) > ϵ
        x1 = x0 - D(f, x0)/D²(f, x0)
        x0 = x1
        n = n + 1
        println("Iteration $(n): $x0")
    end
    println("x = $(x0)")
end;
```

[14]:

```julia
# function to be optimised
x0 = -1 # initial point
f(x) = x^4 - 2x^3 + 2x # function
ϵ = 0.0000001 # precision

newton(f,x0,ϵ)
```

```
Iteration 1: -0.6666666666666667
Iteration 2: -0.5277777777777778
Iteration 3: -0.5009746588693957
Iteration 4: -0.5000012641489876
Iteration 5: -0.5000000000021307
x = -0.5000000000021307
```

[15]:

```julia
# function to be optimised
x0 = 1.5 # initial point
f(x) = x^4 - 2x^3 + 2x # function
ϵ = 0.0000001 # precision

newton(f,x0,ϵ)
```

```
Iteration 1: 1.2777777777777777
Iteration 2: 1.148953301127214
Iteration 3: 1.0776950996068837
Iteration 4: 1.0397811052071808
Iteration 5: 1.0201442175855089
Iteration 6: 1.0101384048933681
Iteration 7: 1.0050861617157065
Iteration 8: 1.002547370546625
Iteration 9: 1.0012747640414033
Iteration 10: 1.0006376525131198
Iteration 11: 1.0003188939802068
Iteration 12: 1.0001594639336053
Iteration 13: 1.000079736204383
x = 1.000079736204383
```

We can see that x converges to -0.5 and 1. In fact, they are the exact roots of the first derivative.

$$\frac{d}{dx}\left(x^4 - 2x^3 + 2x\right) = 2(x-1)^2(2x+1)$$

f'(-0.5) = f'(1) = 0 => x1 = - 0.5 and x2 = 1 are stationary points
Second derivative:

$$\frac{d^2}{dx^2}\left(x^4 - 2x^3 + 2x\right) = 12(x-1)x$$

f''(-0.5) = 9 => x1 = - 0.5 is the local minimum.
f''(1) = 0 => x2 = 1 is an inflection point.
The function **x^4 - 2x^3 + 2x** is quasi-convex, where there is a flat part. We have to determine whether x1 or x2 is the true global minimum
Since f(-0.5) < f(1) => x1 = -0.5 is the global minimum, while x2 = 1 is inflection point
Final answer: x1 = -
Final answer: Function x^4 - 2x^3 + 2x has one global minimum: x = -0.5, f(x) = -0.6875 and one inflection point: x = 1, f(x) = 1