Home Exercise 6: Sensitivity analysis

Dakota Furniture makes desk, tables and chairs. To make the furniture, the company uses wood (acquired in units of a standard measurement boards) and two different processes, namely assembly and finishing. The requirements of wood and process times for each product are detailed below.

Resource	Desk	Table	Chair
Wood (units)	8	6	1
Hours of assembling	2	1.5	0.5
Hours of finishing	4	2	1.5

Currently, Dakota has available 48 units of wood, 8h- of assembling and 20h-worth of finishing per day. The desks are sold for \$60,00, tables for \$30,00 and chairs for \$20,00. The company believes that all products made will often be sold, but that it should not make more than 5 tables. With that in mind, let

x_1 := total number of desks made
x_2 := total number of tables made
x_3 := total number of chairs made

The linear programming (LP) model that optimises Dakota's daily production plan is:

$$
\begin{aligned}
\max.\ z = & 60x_1 + 30x_2 + 20x_3 \\
\text{s.t. } & 8x_1 + 6x_2 + x_3 \leq 48 \\
& 4x_1 + 2x_2 + 1.5x_3 \leq 20 \\
& 2x_1 + 1.5x_2 + 0.5x_3 \leq 8 \\
& x_2 \leq 5 \\
& x_1,\ x_2,\ x_3 \geq 0
\end{aligned}
$$

Using the Simplex method to solve Dakota's optimisation problem we obtain the following optimal tableau:

	x_1	x_2	x_3	s_1	s_2	s_3	s_4	Sol.
z	0	0	0	0	10	10	0	280
s_1	0	-2	0	1	2	-8	0	24
x_3	0	-2	1	0	2	-4	0	8
x_1	1	5/4	0	0	-1/2	3/2	0	2
s_4	0	1	0	0	0	0	1	5

Where s_i are the slack variables associated with constraint (i), for $i = 1, \ldots, 4$. Given the above, please answer:

a) What is the maximum selling price the desks can have that would not affect the optimal production plan?
The desks variable is x1, and the selling price of the desks is c1, which is 60$ in the original objective function. The question asks how high c1 can be without changing the optimal production plan. To do this, we apply the optimality condition: $r_N \leq \Delta c^T_B * B^{-1}N$, since x1 is a basic variable obtained in the original optimal solution

- Being x1 a basic variable, the tableau needs to be corrected.

	x1	x2	x3	s1	s2	s3	s4	Sol.
z	-Δc1	0	0	0	10	10	0	280
s1	0	-2	0	1	2	-8	0	24
x3	0	-2	1	0	2	-4	0	8
x1	1	5/4	0	0	-1/2	3/2	0	2
s4	0	1	0	0	0	0	1	5

- To do so, we multiply the x1-row by Δc1 and add it to the z-row.

	x1	x2	x3	s1	s2	s3	s4	Sol.
z	0	(5/4)Δc1	0	0	10-(1/2)Δc1	10+(3/2)Δc1	0	280+2Δc1
s1	0	-2	0	1	2	-8	0	24
x3	0	-2	1	0	2	-4	0	8
x1	1	5/4	0	0	-1/2	3/2	0	2
s4	0	1	0	0	0	0	1	5

Finally, the optimality condition $r_N \leq \Delta c^T_B * B^{-1} * N$ is applied. We have the set of inequalities:
$(5/4)\Delta c1 >= 0 \Rightarrow \Delta c1 >= 0$
$10 - (1/2)\Delta c1 >= 0 \Rightarrow \Delta c1 <= 20$
$10 + (3/2)\Delta c1 >= 0 \Rightarrow \Delta c1 >= -20/3$
Finally:
$0 <= \Delta c1 <= 20$, and $c1_{original} = 60$
$\Rightarrow 60 <= c1 + \Delta c1 <= 80$ or $c1 \in [60, 80]$ so that the optimal production plan is unchanged

Answer: the maximum selling price the desks can have is 80$

b) The company is looking into expanding its availability of assembling hours. After a market consultation, Dakota learned that a competing firm would be willing to offer 2h of assembling per day for $8/h. Assuming that Dakota would not want to change the current production assortment (that is, remain producing desks and chairs and not producing tables), should Dakota accept the offer?

The current upper limit of assembling hours is 8 hours, and it is in the third constraint, which is b3 = 8. Therefore, we have to find the range of Δb3 so that the original basis is

unchanged, and then we calculate if the marginal value of Δb3 exceeds the cost of 8$/hour offered by the competing firm.
Since Δb3 belongs to the third constraint, it will have the same value as the third slack variable s3. The tableau is as follows:

	x1	x2	x3	s1	s2	s3	s4	Sol.	Δb3
z	0	0	0	0	10	10	0	280	10
s1	0	-2	0	1	2	-8	0	24	-8
x3	0	-2	1	0	2	-4	0	8	-4
x1	1	5/4	0	0	-1/2	3/2	0	2	3/2
s4	0	1	0	0	0	0	1	5	0

From the table, the marginal value for assembling hours would be 10$/hour. Since Dakota would not want to change the current production assortment, the feasibility condition $B^{-1}(b + \Delta b) >= 0$ is applied. The set of equations is as follows:

24 - 8Δb3 >= 0 => Δb3 <= 3
8 - 4Δb3 >= 0 => Δb3 <= 2
2 + (3/2)Δb3 >= 0 => Δb3 >= -4/3
5 + 0Δb3 >= 0 => 5 >= 0 (always true)
Finally:
-4/3 <= Δb3 <= 2, and $b3_{original}$ = 8
=> 20/3 <= b3 + Δb3 <= 10 or b3 ∈ [20/3, 10] so that the current assortment plan is unchanged => $b3_{max}$ = 10 hours and $b3_{original}$ = 8 hours
=> Possible additional hours that can be added is 10 - 8 = 2 hours.
From what we know about, marginal value of b3 (assembling hour) is 10$, and the cost of additional hours offered by the competing firm is 8$
=> Change in income: 2 hours * (10$ - 8$) = 4$ > 0 => This is a profit, not a loss

Answer: Dakota should accept the offer and they can make an additional profit of 4$ a day without changing the current assortment plan