Exercise class 2

Learning Objectives:

- Formulation of linear programming problems
- Solving linear optimisation problems graphically and with Julia

Demo 1: Solving LP problems graphically

Solve the following linear programming problems graphically:

a)

$$\begin{array}{llll} \min. & x_1 & + \; x_2 & \\ \\ \text{s.t.} & 2x_1 & + \; 2x_2 & \geq 4 \\ & x_1 & & \leq 3 \\ & & x_2 & \geq 1 \\ & x_1 \geq 0, & x_2 \geq 0 & \end{array}$$

b) Example 2.2-1 Reddy Mikks model from Hamdy A. Taha's book Operations Research: An Introduction.

$$\begin{array}{llll} \max. & 5x_1 & + \; 4x_2 & \\ \\ \text{s.t.} & 6x_1 & + \; 4x_2 & \leq 24 \\ & x_1 & + \; 2x_2 & \leq 6 \\ & -x_1 & + \; x_2 & \leq 1 \\ & & x_2 & \leq 2 \\ & x_1 \geq 0, & x_2 \geq 0 & \end{array}$$

Solution

a) Draw the constraints on a x_1, x_2 plane. The easiest way to do this by hand is to calculate the values of x_1 and x_2 at two different points and connect a line through them. Also add an arrow to the constraint line indicating the inequality sign.

In Julia *using Plots* and *pyplots()*, first set a range for x_1 and x_2 values. Then plot the first constraint $2x_1 + 2x_2 \geq 4$, this is done by defining the constraint by x_1: $x_1 \geq 2 - x_2$. Use *color* to define the colour and *fill* to state whether to fill above or below the line and the transparency of the shaded area (here we use 0.1).

```
x1 = range(0,5,step=1)
x2 = range(0,5,step=1)

# 2x1 + 2x2 >= 4
plot(x1, (2 .- x2), color = :1, fill=(x2[end],0.1), legend=false)
```

Next, add the second and third constraints. Notice that *hline* can use *fill* but *vline* cannot and hence we use a work around. Using ! after the plot type will add it to the same plot.

```
# x1 <= 3
vline!([3], color = :2)
vspan!([0, 3], color=:2, alpha = 0.1) # work around as fill does not work for vline

#x2 >= 1 x1,
hline!([1], color = :3, fill=(x2[end],0.1))
```

The feasible region of the problem is the area in which all the constraint overlap (including the $x_1,\ x_2 \geq 0$).

The optimal solution can be found by moving the objective function along the level curves (dotted black lines) to minimise the objective value (z).

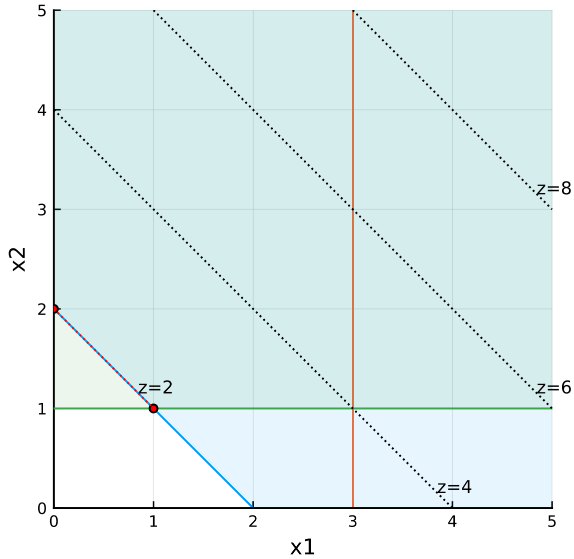


Figure 1: (a) graphical solution

The optimal solution is: $x_1 = 0$, $x_2 = 2$, objective value $= 2$. Notice that $x_1 = x_2 = 1$ is on the same level curve, meaning it has the same objective value and thus this problem has multiple optimal solutions, all points on the line between the red dots.

```
# optimal solutions
# as (0,2) and (1,1) have the same objective value (=2) and on the same contour line
# meaning this problem has multiple optimal solutions - all x1 and x2 values between the two red points
scatter!([0],[2], color = :red)
scatter!([1],[1], color = :red)

savefig("ex2_demo1.pdf")
```

b) First, plot the constraints on to the x_1-x_2 plane, see Figure 2.

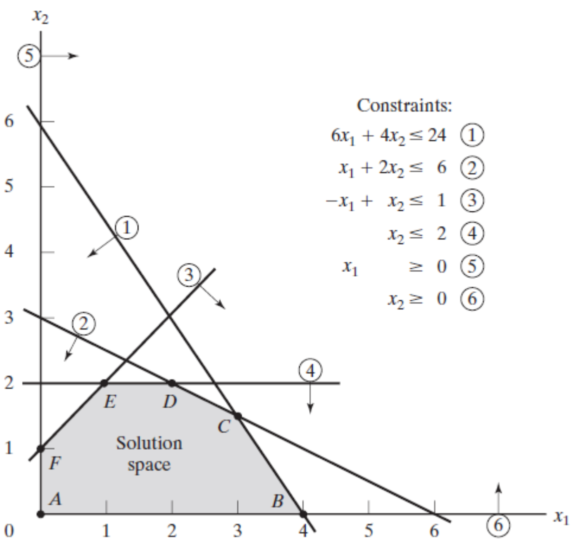


Figure 2: Feasible region

Plot the objective function and determine the direction, as we are maximising we want to move the objective function in a north-west direction as that increases the objective value.

The optimum solution occurs at point C because it is the last possible feasible point beyond which any further increase in the objective value will render an infeasible solution.

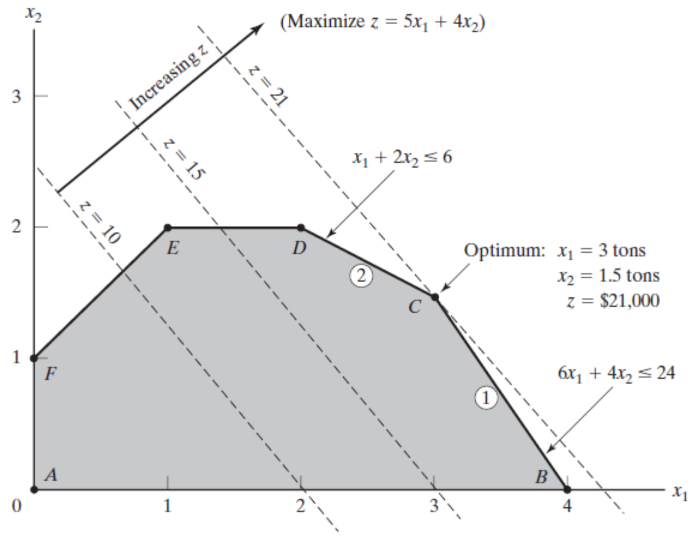


Figure 3: Feasible region

Demo 2: Algebraic form of LPs

MG Auto has three source plants i and two major distribution centers j. The quarterly capacities of the three plants are a_i, and the demands at the two distribution centers for the same period are b_j. The trucking company in charge of transporting the cars charges c_{ij} per unit. The amount shipped is x_{ij} is the amount of cars shipped from source i to destination j.

The objective of the model is to minimise the total transportation cost while satisfying all the supply and demand restrictions.

Formulate the algebraic form of the LP.

Solution

The LP is

$$\begin{aligned}
\min.\ z = &\sum_{i,j} c_{ij} x_{ij} \\
\text{s.t.}\ &\sum_{j} x_{ij} \leq a_i \qquad \forall i \\
&\sum_{i} x_{ij} \geq b_j \qquad \forall j \\
&x_{ij} \geq 0 \qquad \forall i, j
\end{aligned}$$

Problem 1: Solving LP problems graphically

Solve the following LP graphically and using JuMP.

a)

$$\begin{array}{lcccl} \max. & 8x_1 & + & 3x_2 & \\ & & & & \\ \text{s.t.} & 3x_1 & + & 8x_2 & \leq 48 \\ & 4x_1 & + & 3x_2 & \leq 21 \\ & x_1 & - & 2x_2 & \leq 1 \\ & \multicolumn{4}{l}{x_1 \geq 0, \quad x_2 \geq 0} \end{array}$$

b)

$$\begin{array}{lcccl} \min. & 6x_1 & + & 5x_2 & \\ \text{s.t.} & 3x_1 & + & 5x_2 & \geq 15 \\ & 7x_1 & + & 2x_2 & \leq 14 \\ & -x_1 & + & 3x_2 & \leq 9 \\ & \multicolumn{4}{l}{x_1 \geq 0, \quad x_2 \geq 0} \end{array}$$

Solution

a) The optimal solution is: $x_1 = 4.09$, $x_2 = 1.55$, and objective value $= 37.36$.

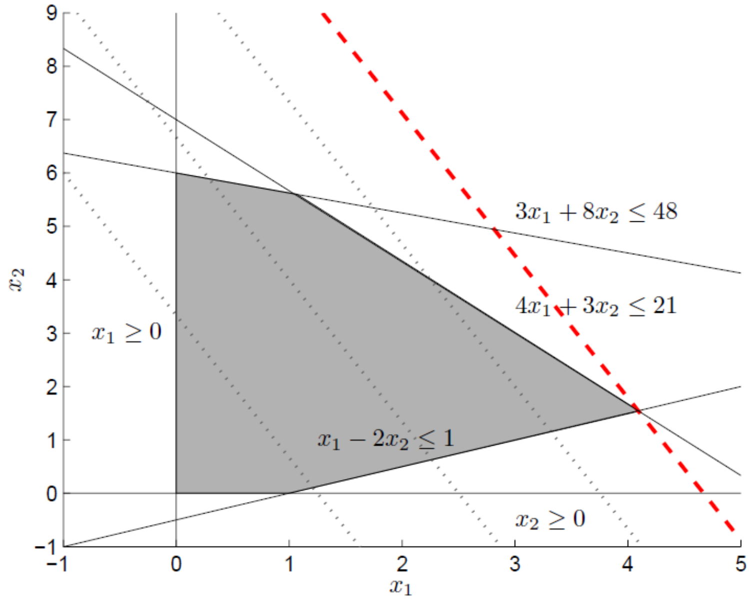


Figure 4: (a) graphical solution

b) The optimal solution is $x_1 = 0,\ x_2 = 3$ and the objective value $= 15$.

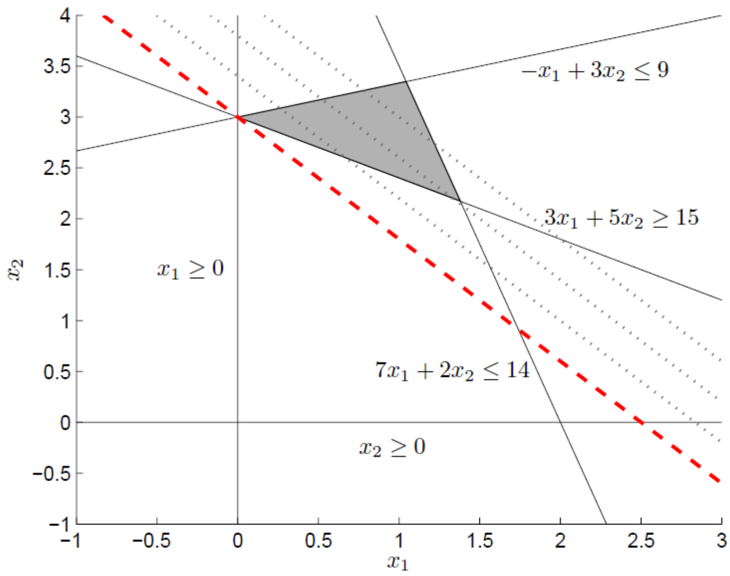


Figure 5: (b) graphical solution

Problem 2: Solving LPs graphically

Recall Demo 2 from exercise class 1: Matti is a farmer and wants to decide how many acres of rye and wheat to plant for the coming year. Each acre of wheat produces 25 loads of wheat and requires 10 hours of labour per week. An acre of rye produces 10 loads of rye and requires 4 hours of labour per week. The wheat sells at €4 per load and the rye €3 per load. Matti has 7 acres of farmland available and 40 hours of labour per week. Government regulations require that at least 30 loads of rye is produced during a given year. Let x_1 be the number of acres of wheat planted, and x_2 be the number of acres of rye planted. Then we have the following LP:

$$\max.\ 100x_1 + 30x_2 \tag{1}$$
$$x_1 + x_2 \leq 7 \tag{2}$$
$$10x_1 + 4x_2 \leq 40 \tag{3}$$
$$10x_2 \geq 30 \tag{4}$$
$$x_1, x_2 \geq 0 \tag{5}$$

Solve this graphically and answer the following questions:

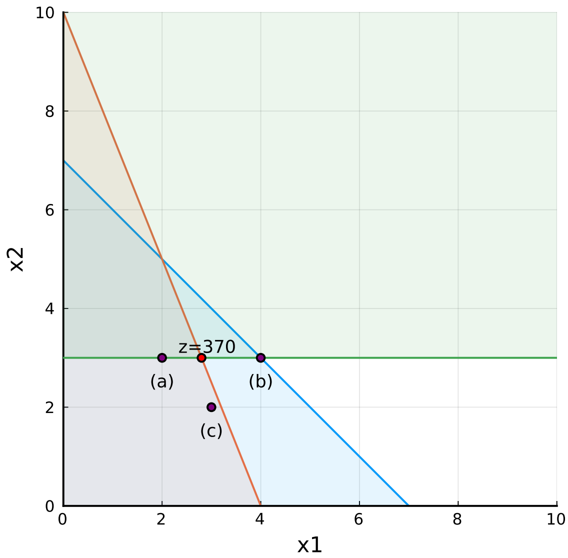
(a) Is $x_1 = 2,\ x_2 = 3$ in the feasible region?

(b) Is $x_1 = 4,\ x_2 = 3$ in the feasible region?

(c) Is $x_1 = 3,\ x_2 = 2$ in the feasible region?

(d) Which of the constraints (2)-(4) are binding?

Solution



(a) Yes

(b) No

(c) Yes

(d) Constraints (3) and (4) are binding as they are satisfied with equality, constraint (2) is non-binding.

Problem 3: Solving LPs graphically

Aalto Chemical manufactures three chemicals: A, B, and C. These chemicals are produced via two production processes: 1 and 2. Running process 1 for an hour costs €4 and yields 3 units of A, 1 of B and 1 of C. Running process 2 for an hour costs €1 and produces 1 unit of A and one unit of B. To meet customer demands, at least 10 units of A, 5 of B and 3 units of C must be produced daily. Graphically determine a daily production plan that minimizes the cost of meeting Aalto Chemical's daily demands.

Hint. The decision variables should be the number of hours of process 1 and number of hours of process 2.

Solution

These simplest way to see this problem is in the following table

Process	€/hour	A	B	C
1	4	3	1	1
2	1	1	1	0
Demand		10	5	3

Let x_1 be the number of hours of process 1 and x_2 be the number of hours of process

2 be the decision variables. Then the objective function and constraints are:

$$\begin{aligned} \min.\ & 4x_1 + x_2 \quad \text{(cost of operating processes 1 and 2, meeting daily demand)} \\ & 3x_1 + x_2 \geq 10 \quad \text{(chemical A)} \\ & x_1 + x_2 \geq 5 \quad \text{(chemical B)} \\ & x_1 \geq 3 \quad \text{(chemical C)} \\ & x_1,\ x_2,\ x_3 \geq 0 \quad \text{(non-negativity)} \end{aligned}$$

The graphical solution is shown in Figure 6:

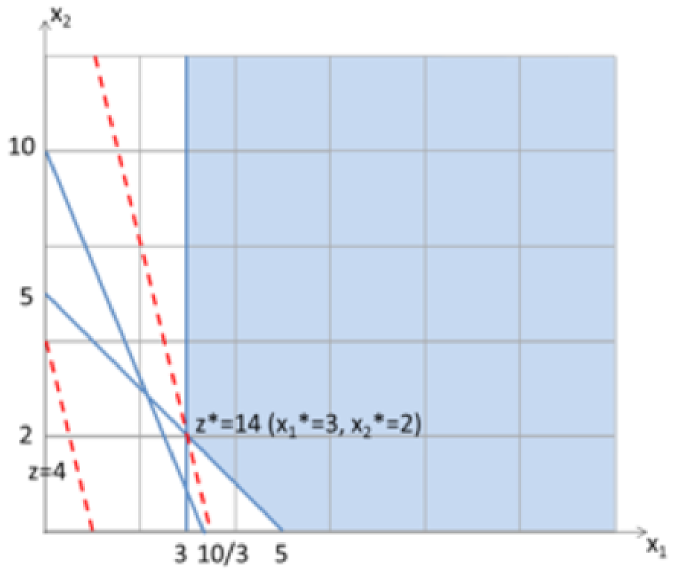


Figure 6: P3 solution

Point z^* is the minimum for this problem: $x_1 = 3,\ x_2 = 2$ so $z = (4)(3) + 2 = 14$

Meaning process 1 should be used for 3 hours and process 2 for 2 hours.

Problem 4: Formulate LP and solve graphically

Happy Farms uses at least 800 grams of special feed daily. The special feed is a mixture of corn and soybean meal with the following compositions:

Feedstuff	**Gram per gram of feedstuff**		**Cost (€/g)**
	Protein	**Fiber**	
Corn	0.09	0.02	0.30
Soybean meal	0.60	0.06	0.90

The dietary requirements of the special feed are at least 30% protein and at most 5% fiber. The aim is to formulate the LP and solve graphically the daily minimum cost-feed mix.

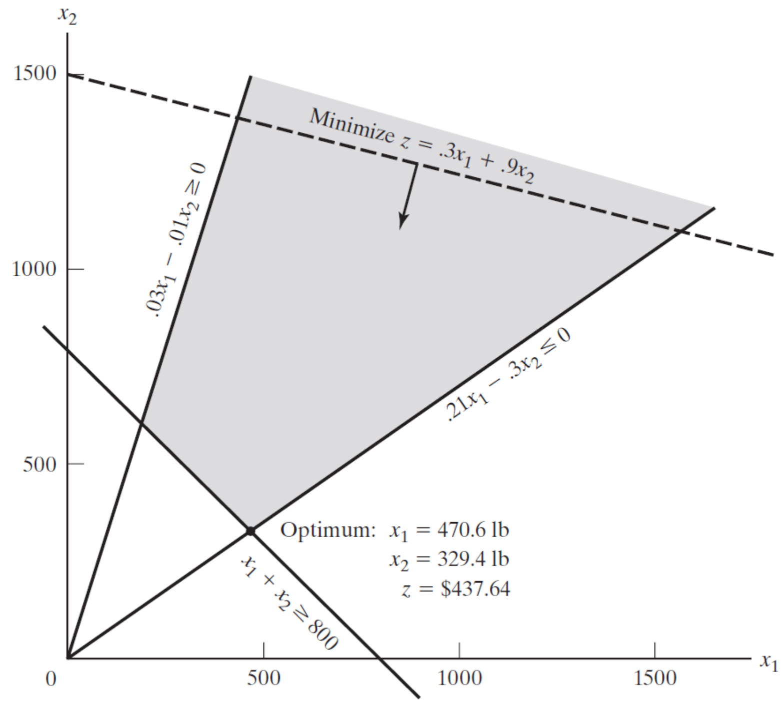
Solution

The decision variables of the model are: x_1 grams of corn in the daily mix and x_2 grams of soybean meal in the daily mix.

$$\min.\ 0.3x_1 + 0.9x_2 \tag{6}$$
$$x_1 + x_2 \geq 800 \tag{7}$$
$$0.21x_1 - 0.30x_2 \leq 0 \tag{8}$$
$$0.03x_1 - 0.01x_2 \geq 0 \tag{9}$$
$$x_1,\ x_2 \geq 0 \tag{10}$$

Constraint (8) is the amount of protein included in x_1 grams of corn and x_2 grams of soybean meal is $0.09x_1 + 0.6x_2$ grams. This quantity should equal at least 30% of the total feed mix, i.e. $0.3(x_1 + x_2)$.

Constraint (9) is formed in a similar manner, the fiber requirement of at most 5% is $0.02x_1 + 0.06x_2 \leq 0.05(x_1 + x_2)$.



Problem 5: Formulate & solve transportation problem

A textile company makes blouses in two factories, one in San Diego and another in Seattle. The company sells their products through three vendors located in New York, Chicago, and Miami. The table below summarises the values for supply and demand by location and how much it costs to transport one unit from the factories to the vendors.

	Vendors			**Supply**
	3-New York	**4-Chicago**	**5-Miami**	
1-Seattle	2.5	1.7	1.8	350
2-San Diego	2.5	1.8	1.4	600
Demand	325	300	275	

a) Formulate the LP model that minimises the transportation costs of the company

b) Consider the following limitations on capacities of the arcs between the origins and destinations in the table below. Formulate the LP with the additional restrictions.

Capacities	**3-New York**	**4-Chicago**	**5-Miami**
1-Seattle	150	200	150
2-San Diego	200	250	200

c) Now consider that the network has a distribution center in Pittsburgh for the products going to the vendors in New York and Chicago. The table below shows the unit transportation cost between Pittsburgh and the factories and vendors. Alter the formulation of the problem to include the distribution center.

	1-Seattle	**2-San Diego**	**3-New York**	**4-Chicago**
Pittsburgh	2.0	1.8	0.5	0.7

d) The company now sells two products: blouses and trousers. The table below shows the supply and demand of the two products. Modify the LP to consider the two products in addition to the previous (b)-(c) restrictions.

Supply	**1-Seattle**	**2-San Diego**
Blouses	250	150
Trousers	200	300

Demand	**3-New York**	**4-Chicago**	**5-Miami**
Blouses	125	100	175
Trousers	200	200	100

e) Formulate the complete LP model using algebraic language (summations etc.) to consider all parts (a)-(d) simultaneously.

Solution

a)

$$
\begin{aligned}
\min.\ & 2.5x_{1,3} + 1.7x_{1,4} + 1.8x_{1,5} + 1.8x_{2,3} + 1.8x_{2,4} + 1.4x_{2,5} \\
\text{s.t.}\ & x_{1,3} + x_{1,4} + x_{1,5} \leq 350 \\
& x_{2,3} + x_{2,4} + x_{2,5} \leq 600 \\
& x_{1,3} + x_{2,3} \geq 325 \\
& x_{1,4} + x_{2,4} \geq 300 \\
& x_{1,5} + x_{2,5} \geq 275 \\
& x_{1,3},\ x_{1,4},\ x_{1,5},\ x_{2,3},\ x_{2,4}, x_{2,5} \geq 0
\end{aligned}
$$

b)

$$
\begin{aligned}
x_{1,3} &\leq 150 \\
x_{1,4} &\leq 200 \\
x_{1,5} &\leq 150 \\
x_{2,3} &\leq 200 \\
x_{2,4} &\leq 250 \\
x_{2,5} &\leq 200
\end{aligned}
$$

c) Defining Pittsburgh as 6, the new objective function is:

$$\min.\ 2.0x_{1,6} + 1.8x_{2,6} + 1.8x_{1,5} + 0.5x_{6,3} + 0.7x_{6,4} + 1.4x_{2,5}$$

The new restrictions on supply:

$$
\begin{aligned}
x_{1,6} + x_{1,5} &\leq 350 \\
x_{2,6} + x_{2,5} &\leq 600
\end{aligned}
$$

The new restrictions on demand:

$$
\begin{aligned}
x_{6,3} &\geq 325 \\
x_{6,4} &\geq 300 \\
x_{1,5} + x_{2,5} &\geq 275
\end{aligned}
$$

Need a balance constraint for 6:

$$x_{1,6} + x_{2,6} - x_{6,3} - x_{6,4} = 0$$

d) Define new indices for the two products:

$$
\begin{aligned}
\min.\ & 2.5x_{1,3} + 1.7x_{1,4} + 1.8x_{1,5} + 2.5x_{2,3} + 1.8x_{2,4} + 1.4x_{2,5} \\
\text{s.t. } & x_{1,3} = x_{1,3,1} + x_{1,3,2} \\
& x_{1,4} = x_{1,4,1} + x_{1,4,2} \\
& x_{1,5} = x_{1,5,1} + x_{1,5,2} \\
& x_{2,3} = x_{2,3,1} + x_{2,3,2} \\
& x_{2,4} = x_{2,4,1} + x_{2,4,2} \\
& x_{2,5} = x_{2,5,1} + x_{2,5,2} \\
& x_{1,3,1} + x_{1,4,1} + x_{1,5,1} \leq 250 \\
& x_{2,3,1} + x_{2,4,1} + x_{2,5,1} \leq 150 \\
& x_{1,3,2} + x_{1,4,2} + x_{1,5,2} \leq 200 \\
& x_{2,3,2} + x_{2,4,2} + x_{2,5,2} \leq 300 \\
& x_{1,3,1} + x_{2,3,1} \geq 125 \\
& x_{1,4,1} + x_{2,4,1} \geq 100 \\
& x_{1,5,1} + x_{2,5,1} \geq 175 \\
& x_{1,3,2} + x_{2,3,2} \geq 200 \\
& x_{1,4,2} + x_{2,4,2} \geq 200 \\
& x_{1,5,2} + x_{2,5,2} \geq 100 \\
& x_{1,3},\ x_{1,4},\ x_{1,5},\ x_{2,3},\ x_{2,4}, x_{2,5} \geq 0 \\
& x_{1,3,1},\ x_{1,4,1},\ x_{1,5,1},\ x_{2,3,1},\ x_{2,4,1}, x_{2,5,1} \geq 0 \\
& x_{1,3,2},\ x_{1,4,2},\ x_{1,5,2},\ x_{2,3,2},\ x_{2,4,2}, x_{2,5,2} \geq 0
\end{aligned}
$$

e) Define:

$i -$ factories
$j -$ destination cities
$p -$ products
$a_{i,p} -$ limit on supply from factory i of product p
$b_{j,p} -$ demand of city j for product p
$c_{i,j} -$ transport unit cost from origin i to destination j

$$
\begin{aligned}
\min . \ & \sum_{i,j} c_{i,j} x_{i,j} \\
\text{s.t. } & x_{i,j} = \sum_{p} x_{i,j,p} \quad \forall i,\ j \\
& \sum_{i} x_{i,j,p} \leq a_{i,p} \quad \forall i,\ p \\
& \sum_{j} x_{i,j,p} \geq b_{j,p} \quad \forall j,\ p \\
& \sum_{p} x_{i,j,p} \leq c_{i,j} \quad \forall i,\ j \\
& \sum_{i} x_{i,j,p} = \sum_{i} x_{i,j,p} \quad j = \text{Pittsburgh } \forall p
\end{aligned}
$$

Home Exercise 2: Solving LPs graphically

For each of the following, determine the direction in which the objective function increases:

(1) $z = 4x_1 - x_2$

(2) $z = -x_1 + 2x_2$

(3) $z = -x_1 - 3x_2$

Hint. Think of the gradient of the objective function.

Solution

(1) SE

(2) NW

(3) SW