Exercise class 7

Learning Objectives:

- Fixed costs and job scheduling
- Branch-and-bound method

Demo 1: Fixed costs

A household uses at least 3000 minutes of long-distance telephone calls monthly and can choose to use the services of any of three companies: A, B, and C. Company A charges a fixed monthly fee of \$10 and 5 cents per minute for the first 1000 minutes and 4 cents per minute for all additional minutes. Company B's monthly fee is \$20 with a flat 4 cents per minute. Company C's monthly charge is \$25 with 5 cents per minute for the first 1000 minutes and 3.5 cents per minute beyond that limit. Which company should be selected to minimize the total monthly charge? Formulate the mixed integer linear programming model. Solve using Julia.

Demo 2: Branch-and-bound

Solve the linear integer problem with the branch-and-bound (B&B) algorithm.

$$\begin{aligned} \max.\ z = 3x_1 + 2x_2 & \\ \text{s.t. } 2x_1 + 5x_2 \leq 9 & \\ 4x_1 + 2x_2 \leq 9 & \\ x_1,\, x_2 \in \mathbb{Z}_+ & \end{aligned}$$

Problem 1: Job scheduling

Jobco Shop has 10 outstanding jobs to be processed on a single machine. The following table provides processing times and due dates. All times are in days, and due time is measured from time 0:

Job	Processing time (day)	Due time (day)
1	10	20
2	3	98
3	13	100
4	15	34
5	9	50
6	22	44
7	17	32
8	30	60
9	12	80
10	16	150

If job 4 precedes job 3, then job 9 must precede job 7. The objective is to process all 10 jobs minimising a lateness penalty of c. Formulate the model as an ILP.

Problem 2: Fixed cost

A manufacturing facility uses two production lines to produce three products over the next 6 months. Backlogged demand is not allowed. However, a product may be overstocked to meet demand in later months. The following table provides the data associated with the demand, production, and storage of the three products:

	Demand in period						Unit holding	Initial
Product	1	2	3	4	5	6	cost $/month	inventory
1	50	30	40	60	20	45	0.50	55
2	40	60	50	30	30	55	0.35	75
3	30	40	20	70	40	30	0.45	60

There is a fixed cost for switching a line from one product to another. The following tables give the switching cost, the production rates, and the unit production cost for each line:

	Line switching cost $		
	Product 1	Product 2	Product 3
Line 1	200	180	300
Line 2	250	200	174

	Production rate units/month			Unit production cost $		
	Product 1	Product 2	Product 3	Product 1	Product 2	Product 3
Line 1	40	60	80	10	8	15
Line 2	90	70	60	12	6	10

Develop a model for determining the optimal production schedule. Solve with Julia.

Hint. Let $i = 1, 2, 3$ be the product, $j = 1, 2$ be the line, and $t = 1, \ldots, 6$ be the time period.

$$x_{ijt} = \begin{cases} 1, \text{if product } i \text{ uses line } j \text{ period } t \\ 0, \text{otherwise.} \end{cases}$$

$$v_{ijt} = \begin{cases} 1, \text{if switch is made to product } i \text{ on line } j \text{ in period } t \\ 0, \text{otherwise.} \end{cases}$$

I_{it} = end inventory of product i in period t
I_{i0} = initial inventory of product i
D_{it} = demand of product i in period t
r_{ij} = production rate of i on line j
s_{ij} = switching cost of i on line j
c_{ij} = production cost of i on line j
h_i = holding cost of i

Problem 3: Solving IPs with B&B

Develop the B&B tree for each of the following problems. For convenience, always select x_1 as the branching variable at node 0.

a)

$$\begin{aligned} \max . z &= 3x_1 + 2x_2 \\ \text{s.t. } 2x_1 + 5x_2 &\leq 18 \\ 4x_1 + 2x_2 &\leq 18 \\ x_1, \ x_2 &\in \mathbb{Z}_+ \end{aligned}$$

b)

$$\begin{aligned} \max . z &= 2x_1 + 3x_2 \\ \text{s.t. } 7x_1 + 5x_2 &\leq 36 \\ 4x_1 + 9x_2 &\leq 35 \\ x_1, \ x_2 &\in \mathbb{Z}_+ \end{aligned}$$

c)

$$\begin{aligned} \max . z &= 2x_1 + 2x_2 \\ \text{s.t. } 2x_1 + 5x_2 &\leq 27 \\ 6x_1 + 5x_2 &\leq 16 \\ x_1, \ x_2 &\in \mathbb{Z}_+ \end{aligned}$$

d)

$$\begin{aligned} \min . z &= 5x_1 + 4x_2 \\ \text{s.t. } 3x_1 + 2x_2 &\geq 5 \\ 2x_1 + 3x_2 &\geq 7 \\ x_1, \ x_2 &\in \mathbb{Z}_+ \end{aligned}$$

Problem 4: Solving IPs with B&B

Repeat Problem 3, assuming that x_1 is continuous.

Problem 5: B&B exam-like question

Consider the following IP:

$$\begin{aligned} \max.\ z =& 1.5x_1 + 1.8x_2 + 2x_3 \\ \text{s.t. } & 0.5x_1 - 2x_2 + x_3 \leq 2.1 \\ & x_1 + x_2 + 4x_3 \leq 4.3 \\ & x_1,\ x_2,\ x_3 \in \mathbb{Z}_+ \end{aligned}$$

Let node 0 be the solution to the linear relaxation. Solve the IP using Branch-and-Bound by rebuilding the tree using the information provided in the table below.

Indicate the order in which the tree was assembled. For branching, the variable selection was performed by choosing the variable with the fractional value closest to an integer value. For the branch selection, start with the constraint $\leq$. The solution of the linear relaxations are given below, together with which constraints that subproblem was subject to.

Node	Constraint on x_1	Constraint on x_2	Constraint in x_3	Obj. Func value	Value x_1	Value x_2	Value x_3
0				7.74	0	4.3	0
1		$x_2 \leq 4$		7.65	0.3	4	0
2		$x_2 \leq 5$		7.74	0	4.3	0
3		$x_2 \geq 5$		Infeasible			
4	$x_1 \leq 0$	$x_2 \leq 4$		7.35	0	4	0.075
5	$x_1 \geq 1$	$x_2 \leq 4$		7.44	1	3.3	0
6	$x_1 \leq 0$	$x_2 \leq 4$	$x_3 \leq 0$	7.2	0	4	0
7	$x_1 \leq 0$	$x_2 \leq 4$	$x_3 \geq 1$	2.54	0	0.3	1
8	$x_1 \leq 0$	$x_2 \leq 0$	$x_3 \geq 1$	2.15	0	0	1.075
9	$x_1 \leq 0$	$x_2 \geq 1$	$x_3 \geq 1$	Infeasible			
10	$x_1 \leq 0$	$x_2 \leq 0$	$x_3 = 1$	2	0	0	1
11	$x_1 \leq 0$	$x_2 \leq 0$	$x_3 \geq 2$	Infeasible			
12	$x_1 \geq 1$	$x_2 \leq 3$		7.35	1.3	3	0
13	$x_1 \geq 1$	$x_2 = 4$		Infeasible			
14	$x_1 = 1$	$x_2 \leq 3$		7.05	1	3	0.075
15	$x_1 = 1$	$x_2 \leq 3$	$x_3 \leq 0$	6.9	1	3	0
16	$x_1 = 1$	$x_2 \leq 3$	$x_3 \geq 1$	Infeasible			
17	$x_1 \geq 2$	$x_2 \leq 3$		7.14	2.3	2	0

Hint. Not all nodes are needed.

Hint. Constraints on x are accumulated, e.g. if $x_1 \leq 1$ and $x_1 \geq 1$ are being imposed then the table will show $x_1 = 1$, similarly, if $x_2 \leq 4$ and $x_2 \leq 3$ are imposed then $x_2 \leq 3$ is shown.

Home Exercise 7: Branch-and-bound

Solve the following problems by B&B:

$$\begin{aligned} \max.\ z =&\ 18x_1 + 14x_2 + 8x_3 + 4x_4 \\ \text{s.t. } & 15x_1 + 12x_2 + 7x_3 + 4x_4 + x_5 \leq 37 \\ & x_1,\ x_2,\ x_3,\ x_4,\ x_5 \in \{0, 1\} \end{aligned}$$