# Exercise class 5

**Learning Objectives:**

- Formulation and interpretation of dual problem
- Duality and sensitivity analysis

**Table for formulating dual problems**

For the formulation of dual problems you can use the following table which can be read from left to right or right to left depending if the primal problem's objective function is minimisation or maximisation.

| **Primal (dual)** | **Dual (primal)** |
|---|---|
| maximise | minimise |
| Independent terms (b) | Obj. function coef. (c) |
| Obj. function coef. (c) | Independent terms (b) |
| $i$-th row of constraint coef. | $i$-th column of constraint coef. |
| $i$-th column of constraint coef. | $i$-th row of constraint coef. |
| **Constraints** | **Variables** |
| $\leq$ | $\geq 0$ |
| $\geq$ | $\leq 0$ |
| $=$ | $\in \mathbb{R}$ |
| **Variables** | **Constraints** |
| $\geq 0$ | $\geq$ |
| $\leq 0$ | $\leq$ |
| $\in \mathbb{R}$ | $=$ |

## Demo 1: Formulation of dual problem

Jeff wants to make a meal with minimum costs. The meal has to satisfy the family's requirements for protein and calcium: at least 50 g protein and 800 mg calcium. Formulate the problem as an LP according to the following table:

| **Food** | **€/ unit** | **Protein g/unit** | **Calcium mg/unit** |
|---|---|---|---|
| Potato | 0.6 | 3.7 | 22.7 |
| Tomato | 2.7 | 1.1 | 6.2 |
| Milk | 2.3 | 8.1 | 296 |

On the other hand there is local pharmacist who sells protein and calcium tablets. They have to decide the unit price for protein $y_1$ and calcium $y_2$ so that the tablets' prices would be competitive with the food prices. Their main objective is to maximise their own profits. This means that they try to find the highest price for Jeff's needs '50 g protein and 800 mg calcium'. Formulate the pharmacist's problem as an LP.

**Note.** This is a simplified version of Nobel-prize winner (in 1982) George Stigler's diet problem from 1939. His methodology to solve this problem is considered to be some of the earliest work in linear programming.

## Solution

First, let's formulate Jeff's problem. The decision variables are:

$$\begin{aligned} x_1 &:= \text{quantity of potatoes} \\ x_2 &:= \text{quantity of tomatoes} \\ x_3 &:= \text{amount of milk.} \end{aligned}$$

The objective is to minimise the cost of the food:

$$\min .\ 0.6x_1 + 2.7x_2 + 2.3x_3.$$

The quantities to purchase are constrained by the amount of protein and calcium needed:

$$\begin{aligned} 3.7x_1 + 1.1x_2 + 8.1x_3 &\geq 50 \\ 22.7x_1 + 6.2x_2 + 296x_3 &\geq 800. \end{aligned}$$

And, the quantities cannot be negative, $x_1,\ x_2,\ x_3 \geq 0$.

The solution to Jeff's problem is $x_1 = 9.12$, $x_2 = 0$, and $x_3 = 2$, where the total is € 10.08.

Now, let's formulate the pharmacist's problem. The decision variables are:

$$\begin{aligned} y_1 &:= \text{unit price of protein} \\ y_2 &:= \text{unit price of calcium.} \end{aligned}$$

We want to maximise the pharmacist's profit:

$$\max .\ 50y_1 + 800y_2.$$

The constraints are gained from the prices of the tablet substitutes (potato, tomato and milk), because the price of protein and calcium in a potato with pharmacist's unit prices cannot surpass the real price of the potato:

$$
\begin{aligned}
3.7y_1 + 22.7y_2 &\leq 0.6 \\
1.1y_1 + 6.2y_2 &\leq 2.7 \\
8.1y_1 + 296y_2 &\leq 2.3
\end{aligned}
$$

The prices are non-negative: $y_1,\ y_2 \geq 0$.

The solution for the pharmacist's problem is $y_1 = 0.138$ and $y_2 = 0.004$, with a profit of € 10.08.

If we look at the LPs side by side and compare it to the dual table above, we can see that Jeff's problem is the dual of the pharmacist's problem (and vice versa).

$$
\begin{aligned}
\min.\ & 0.6x_1 + 2.7x_2 + 2.3x_3 \\
& 3.7x_1 + 1.1x_2 + 8.1x_3 \geq 50 \\
& 22.7x_1 + 6.2x_2 + 296x_3 \geq 800 \\
& x_1,\ x_2,\ x_3 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\max.\ & 50y_1 + 800y_2 \\
& 3.7y_1 + 22.7y_2 \leq 0.6 \\
& 1.1y_1 + 6.2y_2 \leq 2.7 \\
& 8.1y_1 + 296y_2 \leq 2.3 \\
& y_1,\ y_2 \geq 0
\end{aligned}
$$

### Demo 2: Dual Simplex

Solve the following LP using the Dual Simplex method.

$$\begin{aligned} \max.\ z &= -2x_1 - x_3 \\ \text{s.t. } x_1 + x_2 - x_3 &\geq 5 \\ x_1 - 2x_2 + 4x_3 &\geq 8 \\ x_1,\ x_2,\ x_3 &\geq 0 \end{aligned}$$

### Solution

See Lecture 5 slide 23 for pseudo code of Dual Simplex.

Step 0: Convert all constraints to $\leq$ and then add in slack variables (standard form), and set objective function to 0.

Step 1: Form a solution (initial basis) that satisfies optimality conditions, but that is infeasible.

Step 2: Pick row with 'most negative' b, this identifies the leaving variable.

Step 3: Pick columns with negative $a_{jk}$, calculate the ratio and pick the absolute value closest to zero, this identifies the entering variable.

Step 4: Repeat Simplex iterations until all b-column is greater-than or equal to 0.

Stop: Solution now satisfies optimality conditions and is feasible. Because of *strong duality* we know this is the optimal solution.

Step 0:

$$\begin{aligned} &z + 2x_1 + x_3 = 0 \\ &-x_1 - x_2 + x_3 + s_1 = -5 \\ &-x_1 + 2x_2 - 4x_3 + s_2 = -8 \\ &x_1,\ x_2,\ x_3,\ s_1,\ s_2 \geq 0 \end{aligned}$$

Step 1: Initial solution of $x_1 = x_2 = x_3 = 0$ satisfies optimality conditions (all z-row coefficients are $\geq 0$ in tableau) but is infeasible as constraints do not hold (0+0-0 $\ngeq$ 5).

↓ Pivot column

| | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | Sol. | |
|---|---|---|---|---|---|---|---|
| $z$ | 2 | 0 | 1 | 0 | 0 | 0 | |
| $s_1$ | -1 | -1 | 1 | 1 | 0 | -5 | |
| $s_2$ | -1 | 2 | -4 | 0 | 1 | -8 | ← Pivot row |
| Ratio | 2 | | $\frac{1}{4}$ | | | | |

Step 2: $s_2$ is the leaving variable.

Step 3: $x_3$ is entering variable.

Step 4: Carry on with Simplex iterations until b-column is $\geq 0$.
Tableau 1 (T1) to tableau 2 (T2): -$\frac{1}{4}$R2; Z - R2; R1 - R2.

↓ Pivot column

| | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | Sol. | |
|---|---|---|---|---|---|---|---|
| $z$ | 7/4 | 1/2 | 0 | 0 | 1/4 | -2 | |
| $s_1$ | -5/4 | -1/2 | 0 | 1 | 1/4 | -7 | ← Pivot row |
| $x_3$ | 1/4 | -1/2 | 1 | 0 | -1/4 | 2 | |
| Ratio | 7/5 | 1 | | | | | |

T2 satisfies optimality conditions (z-row $\geq 0$) but is still infeasible (b-column $\ngeq 0$).
$s_1$ is leaving variable, $x_2$ is entering variable.

T2 to T3: -2R1; Z-$\frac{1}{2}$R1; R2+$\frac{1}{2}$R1

| | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | Sol. |
|---|---|---|---|---|---|---|
| $z$ | 1/2 | 0 | 0 | 1 | 1/2 | -9 |
| $x_2$ | 5/2 | 1 | 0 | -2 | -1/2 | 14 |
| $x_3$ | 3/2 | 0 | 1 | -1 | -1/2 | 9 |
| | | | | | | |

Stop: Optimal tableau, all z-row is non-negative and b-column is feasible. Therefore, optimal solution is $z = -9,\ x_1 = 0,\ x_2 = 14,\ x_3 = 9$.

## Problem 1: Formulation of dual problem

Formulate the dual problems for the following LP problems.

a)

$$\begin{aligned} \min.\ z &= 2x_1 - x_2 + 3x_3 + 5x_4 \\ \text{s.t. } & 6x_1 + 2x_2 - x_3 + x_4 \geq 8 \\ & 2x_1 - x_2 + 5x_4 = 8 \\ & x_1 + x_2 + 2x_3 + x_4 \leq 10 \\ & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \leq 0,\ x_4 \in \mathbb{R} \end{aligned}$$

b)

$$\begin{aligned} \max.\ z &= 5x_1 + 6x_2 \\ \text{s.t. } & 3x_1 + 4x_2 \leq 16 \\ & -x_1 + 5x_2 \geq 3 \\ & x_1 \in \mathbb{R},\ x_2 \geq 0 \end{aligned}$$

c)

$$\begin{aligned} \min.\ z &= 3x_1 + 4x_2 + 6x_3 \\ \text{s.t. } & 5x_1 + 2x_2 \geq 20 \\ & x_1 \geq 0 \\ & x_2 \leq 0 \\ & x_3 \geq 0 \end{aligned}$$

d)

$$\begin{aligned} \max.\ z &= x_1 + x_2 \\ \text{s.t. } & 2x_1 + x_2 = 5 \\ & 3x_1 - x_2 = 6 \\ & x_1,\ x_2 \in \mathbb{R} \end{aligned}$$

## Solution

a)

$$\begin{aligned} \max.\ w &= 8y_1 + 8y_2 + 10y_3 \\ \text{s.t. } & 6y_1 + 2y_2 + y_3 \leq 2 \\ & 2y_1 - y_2 + y_3 \leq -1 \\ & -y_1 + 2y_3 \geq 3 \\ & y_1 + y_2 + y_3 = 5 \\ & y_1 \geq 0,\ y_2 \in \mathbb{R},\ y_3 \leq 0 \end{aligned}$$

b)

$$\begin{aligned} \min.\ w &= 16y_1 + 3y_2 \\ \text{s.t. } & 3y_1 - y_2 = 5 \\ & 4y_1 + 5y_2 \geq 6 \\ & y_1 \geq 0,\ y_2 \leq 0 \end{aligned}$$

c)

$$\begin{aligned} \max. \ w &= 20y_1 \\ \text{s.t.} \ 5y_1 &\leq 3 \\ 2y_1 &\geq 4 \\ 0y_1 &\leq 6 \\ y_1 &\geq 0 \end{aligned}$$

d)

$$\begin{aligned} \min. \ w &= 5y_1 + 6y_2 \\ \text{s.t.} \ 2y_1 + 3y_2 &= 1 \\ y_1 - y_2 &= 1 \\ y_1, \ y_2 &\in \mathbb{R} \end{aligned}$$

## Problem 2: Formulate & solve dual problem

Putte's Name Day party is about to begin. To his dismay he realised that he doesn't have any refreshments for his party. After quick consideration he settles on three options: fruit juice, berry juice and mixed juice. A litre of fruit juice requires 0,25 l fruit concentrate, a litre of berry juice requires 0,2 l of berry concentrate and a litre of mixed juice requires 0,1 l of fruit concentrate and 0,15 l of berry concentrate. In addition, the juices need some sugar: fruit juice 50 g, berry juice 75 g and mixed juice 60 g for a litre of juice.

Putte has 20 l of fruit concentrate, 45 l of berry concentrate and 5 kg of sugar. How much of each juice should Putte make to receive the maximum volume of juice?

Formulate as an LP problem and formulate the dual problem as well. Solve the problem in Julia.

### Solution

The decision variables are the volumes of juices in litres (l):

$$
\begin{aligned}
x_1 &:= \text{ volume of fruit juice} \\
x_2 &:= \text{ volume of berry juice} \\
x_3 &:= \text{ volume of mixed juice}
\end{aligned}
$$

Putte wants to maximise the volume of juice for the party:

$$\max.\ z = x_1 + x_2 + x_3$$

The volumes of the juices are constrained by the quantities of the concentrates and the sugar:

$$
\begin{aligned}
&0.25x_1 + 0.1x_3 \leq 20 \\
&0.2x_2 + 0.15x_3 \leq 45 \\
&50x_1 + 75x_2 + 60x_3 \leq 5000
\end{aligned}
$$

The volumes are non-negative: $x_1,\ x_2,\ x_3 \geq 0$.

The dual problem is therefore:

$$
\begin{aligned}
\min.\ & w = 20y_1 + 45y_2 + 5000y_3 \\
\text{s.t. } & 0.25y_1 + 50y_3 \geq 1 \\
& 0.2y_2 + 75y_3 \geq 1 \\
& 0.1y_1 + 0.15y_2 + 60y_3 \geq 1 \\
& y_1,\ y_2,\ y_3 \geq 0
\end{aligned}
$$

The solution to the primal problem is $x_1 = 70$, $x_2 = 0$ and $x_3 = 25$ in which case the volume of juice is 95 l. The dual solution is $y_1 = 1$, $y_2 = 0$ and $y_3 = 0.015$ in which case the objective function values is the same, 95 l. The dual variable values indicate the increase in the volume of juice when the constant term of the corresponding constraint is increased with one. For example, the value of $y_1$ indicates how much more juice there is if there is an additional litre of fruit concentrate available.

## Problem 3: Dual Simplex

Solve the following LPs using the Dual Simplex method with $x_1 = 0, x_2 = 0$ as initial solution:

**Note.** Optimality condition for min. problem is that all z-row coefficients are $\leq 0$.

a)

$$\begin{aligned} \min.\ z &= 2x_1 + 3x_2 \\ \text{s.t. } & 2x_1 + 2x_2 \leq 3 \\ & x_1 + 2x_2 \geq 1 \\ & x_1,\ x_2 \geq 0 \end{aligned}$$

b)

$$\begin{aligned} \min.\ z &= 2x_1 + 3x_2 \\ \text{s.t. } & x_1 + 6x_2 \leq 15 \\ & 3x_1 - 5x_2 \leq 10 \\ & x_1, x_2 \geq 0 \end{aligned}$$

c)

$$\begin{aligned} \min.\ z &= 4x_1 + 2x_2 \\ \text{s.t. } & x_1 + x_2 = 10 \\ & 3x_1 - x_2 \geq 20 \\ & x_1,\ x_2 \geq 0 \end{aligned}$$

## Solution

a) The second constraint of (a) gets multiplied by -1 to convert to $\leq$ constraint.The starting tableau is:

↓ Pivot column

| | $x_1$ | $x_2$ | $s_1$ | $s_2$ | Sol. | |
|---|---|---|---|---|---|---|
| $z$ | -2 | -3 | 0 | 0 | 0 | |
| $s_1$ | 2 | 2 | 1 | 0 | 3 | |
| $s_2$ | -1 | -2 | 0 | 1 | -1 | ← Pivot row |
| Ratio | 2 | 1.5 | | | | |

T1 is optimal as all reduced costs (z-row coefficients) are negative and T1 is infeasible as the basic variable $s_2$ is negative.

Therefore, using this and the ratio row we can see that $s_2$ leaves and $x_2$ enters. The next tableau is:

| | $x_1$ | $x_2$ | $s_1$ | $s_2$ | Sol. |
|---|---|---|---|---|---|
| $z$ | -0.5 | 0 | 0 | -1.5 | 1.5 |
| $s_1$ | 1 | 0 | 1 | 1 | 2 |
| $x_2$ | 0.5 | 1 | 0 | -0.5 | 0.5 |

T1 to T2: -0.5R2, z + 3R2, R1 - 2R2

T2 is still optimal as all reduced costs are negative but now T2 is also feasible as all basic variables are positive.

Solution: $x_1 = 0$, $x_2 = 0.5$, $z = 1.5$

b) We cannot apply the dual Simplex because (b) is feasible in the origin.

c) The first constraint in (c) is made into equivalent $\leq$ constraints, and the second constraint in (c) is mult. by -1 to do the same. The starting tableau is:

↓ Pivot column

| | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | Sol. | |
|---|---|---|---|---|---|---|---|
| $z$ | -4 | -2 | 0 | 0 | 0 | 0 | |
| $s_1$ | 1 | 1 | 1 | 0 | 0 | 10 | |
| $s_2$ | -1 | -1 | 0 | 1 | 0 | -10 | |
| $s_3$ | -3 | 1 | 0 | 0 | 1 | -20 | ← Pivot row |
| Ratio | 1.3 | 2 | | | | | |

T1 is optimal as all reduced costs are negative and T1 is infeasible as the basic variables $s_2$ and $s_3$ are negative.

Therefore, using this and the ratio row we can see that $s_3$ leaves and $x_1$ enters. The next tableau is:

↓ Pivot column

| | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | Sol. | |
|---|---|---|---|---|---|---|---|
| $z$ | 0 | -3.3 | 0 | 0 | -1.3 | 26.3 | |
| $s_1$ | 0 | 1.3 | 1 | 0 | 0.3 | 3.3 | |
| $s_2$ | 0 | -1.3 | 0 | 1 | -0.3 | -3.3 | ← Pivot row |
| $x_1$ | 1 | -0.3 | 0 | 0 | -0.3 | 6.6 | |
| Ratio | | 2.5 | | | 4 | | |

T1 to T2: -0.3R3, z + 4R3, R1 - R3, R2 + R3

T2 is optimal as all reduced costs are negative and T2 is infeasible as the basic variable $s_2$ is negative.

Therefore, using this and the ratio row we can see that $s_2$ leaves and $x_2$ enters. The next tableau is:

| | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | Sol. |
|---|---|---|---|---|---|---|
| $z$ | 0 | 0 | 0 | -2.5 | -0.5 | 35 |
| $s_1$ | 0 | 0 | 1 | 1 | -0.1 | 0 |
| $x_2$ | 0 | 1 | 0 | -0.75 | 0.25 | 2.5 |
| $x_1$ | 1 | 0 | 0 | -1 | -0.25 | 7.5 |

T2 to T3: -0.75R2, z + 3.3R2, R1 - 1.3R2, R3 + 0.3R2

T3 is still optimal as all reduced costs are negative but now T3 is also feasible as all basic variables are positive (or = 0).

Solution: $x_1 = 7.5$, $x_2 = 2.5$, $z = 35$

## Problem 4: Primal - dual relationship

Solve the dual of the following problem in Julia, and then find its optimal objective value from the solution of the dual. Does the solution of the dual offer computational advantages over solving the primal directly?

$$\begin{aligned}
\min.\ z =&50x_1 + 60x_2 + 30x_3 \\
\text{s.t. } &5x_1 + 5x_2 + 3x_3 \geq 50 \\
&x_1 + x_2 - x_3 \geq 20 \\
&7x_1 + 6x_2 - 9x_3 \geq 30 \\
&5x_1 + 5x_2 + 5x_3 \geq 35 \\
&2x_1 + 4x_2 - 15x_3 \geq 10 \\
&12x_1 + 10x_2 \geq 90 \\
&x_1,\ x_2,\ x_3 \geq 0
\end{aligned}$$

## Solution

$$\begin{aligned}
\max.\ w =&50y_1 + 20y_2 + 30y_3 + 35y_4 + 10y_5 + 90y_6 \\
\text{s.t. } &5y_1 + y_2 + 7y_3 + 5y_4 + 2y_5 + 12y_6 \leq 50 \\
&5y_1 + y_2 + 6y_3 + 5y_4 + 4y_5 + 10y_6 \leq 60 \\
&3y_1 - y_2 - 9y_3 + 5y_4 - 15y_5 \leq 30 \\
&y_i \geq 0
\end{aligned}$$

In Julia gives: $y_1 = y_3 = y_4 = y_5 = y_6 = 0$, $y_2 = 50$, $w = 1000$. Using the dual is advantageous computationally because the dual has a smaller number of constraints.

## Problem 5: Dual problem

Find the optimal value of the objective function for the following problem by inspecting only its dual problem. (Do not solve the dual by Simplex).

$$\begin{aligned}
\text{Primal: } z = \min.\ &10x_1 + 4x_2 + 5x_3 \\
\text{s.t. } &5x_1 - 7x_2 + 3x_3 \geq 20 \\
&x_1,\ x_2,\ x_3 \geq 0
\end{aligned}$$

## Solution

$$\begin{aligned}
\text{Dual: } w = \max.\ &20y \\
\text{s.t. } &5y \leq 10 && \implies y \leq 2 \\
&-7y \leq 4 && \implies y \geq -4/7 \\
&3y \leq 5 && \implies y \leq 5/3 \\
&y \geq 0
\end{aligned}$$

Therefore, $0 \leq y \leq 5/3$ which as the dual is a maximisation $y = 5/3$, meaning that the primal objective function value is 33.3 ($\max.w = \min.z$).

## Home Exercise 5: True or false?

a) The dual of the dual problem yields the original primal.

b) If the primal constraint is originally in an equality (=) form, the corresponding dual variable is necessarily unrestricted.

c) If the primal constraint is of the type $\leq$, the corresponding dual variable will be nonnegative (nonpositive) if the primal objective is maximization (minimization).

d) If the dual constraint is of the type $\geq$, the corresponding primal variable will be nonnegative (nonpositive) if the primal objective is minimization (maximization).

e) An unrestricted primal variable will result in an $\geq$ dual constraint.

## Solution

a) True

b) True

c) True

d) False

e) False