Exercise class 1

Learning Objectives:

- Use of Excel Solver and Julia
- Formulation of optimisation problems

Demo 1: Excel Solver and Julia

Solve the optimisation problem

$$\begin{array}{llcll} \max. & 3x_1 & + & x_2 & \\ \text{s.t.} & 2x_1 & + & 5x_2 & \leq 8 \\ & -4x_1 & + & 2x_2 & \geq -5 \\ & x_1, & & x_2 & \geq 0 \end{array}$$

by using Excel Solver and Julia.

Solution in Solver

Step 1: Enter decision variables names (x_1 and x_2) in cells $C1$ and $D1$, respectively. The Solver will change the values in the cells below ($C2$ and $D2$) during solving, so leave these cells empty or enter initial values.

Step 2: Enter the objective function in cell $C4$. Reference the values of the decision variables by referring to the cells $C2$ and $D2$, write $= 3 * C2 + D2$.

Step 3: Enter the left side of the constraints into cells $C6$ and $C7$, enter the constraint sign into cells $D6$ and $D7$, and enter the values of the right-hand-side of the constraints in cells $E6$ and $E7$.

Step 4: Start the Solver by selecting it from the *Data* tab at the top. When it opens, in the *set objective* box select the objective function cell ($C4$) and select to maximise, by *changing variable cells* $C2$ and $D2$. For the constraints, first press the *add* button and select the left-hand-side of the constraint for the *cell reference* box, select the constraint sign, and select the right-hand-side of the constraint for the *constraint* box (see Figure 1).

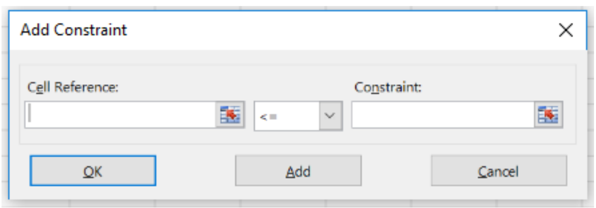


Figure 1: Constraint window

Step 5: Tick the *make unconstrained variables non-negative* to ensure the decision variables are >= 0.

Step 6: Select the *Simplex LP* option in the drop-down box for the solving method. The Solver window should look as in Figure 2. If it does, press *solve*.

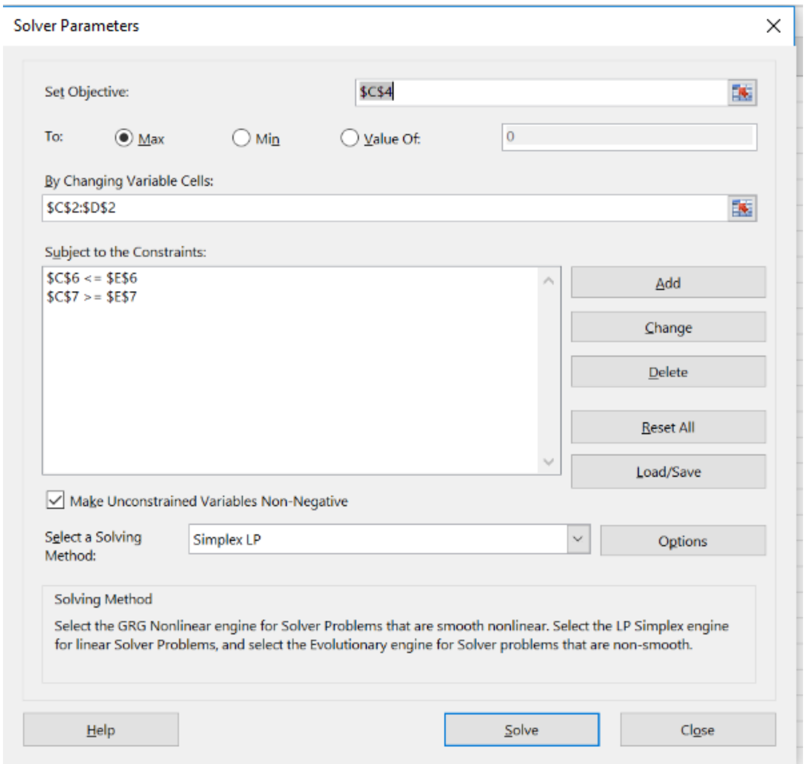
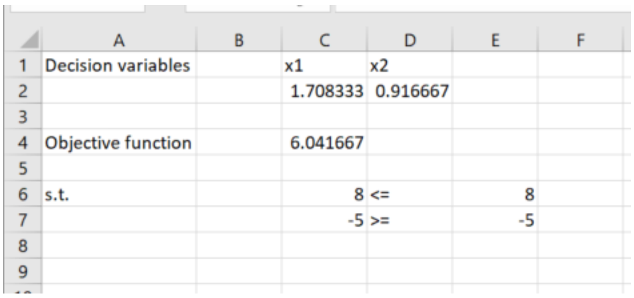


Figure 2: Solver setup Demo 1

The Solver will find the optimal solution, see Figure 3 for solution.



	A	B	C	D	E	F
1	Decision variables		x1	x2		
2			1.708333	0.916667		
3						
4	Objective function		6.041667			
5						
6	s.t.		8	<=	8	
7			-5	>=	-5	
8						
9						

Figure 3: Optimal solution Demo 1

Solution in Julia

Step 1: Open https://jupyter.cs.aalto.fi on your web browser and sign-in using your Aalto login details.

Step 2: Choose the Julia: General use option, see Figure 4.

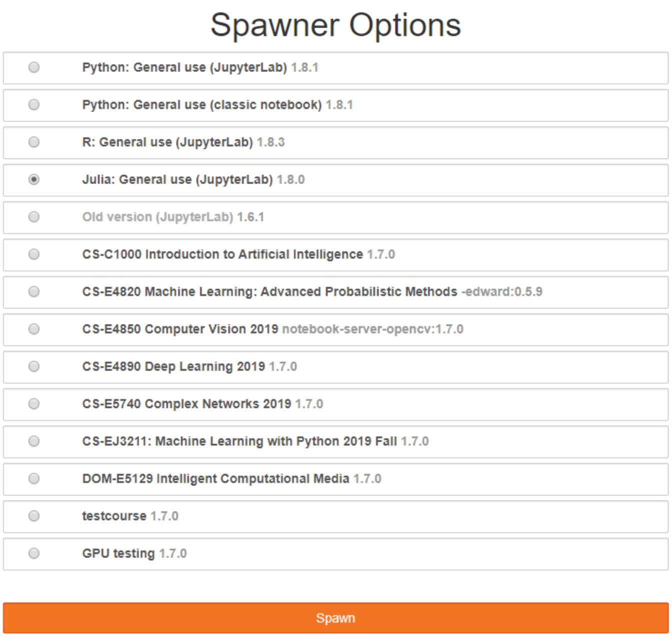


Figure 4: Julia spawn option

Step 3: Open a Julia notebook.

Step 4: Include the packages containing the modelling language *JuMP* and solver *Cbc*, and create a model and select the solver.

```
using JuMP, Cbc #modelling language and solver

m = Model(with_optimizer(Cbc.Optimizer,logLevel = 0)) #creates the model, select the solver
```

Step 5: Define the variables (*@variable*), constraints (*@constraint*), and objective function (*@objective*) as they are presented in the problem. Make sure to include the model name (m).

```
@variable(m, x[j = 1:2] >= 0) # creates the non-negative variables x1 and x2

@constraint(m, 2*x[1] + 5*x[2] <= 8) # constraint 1
@constraint(m, -4*x[1] + 2*x[2] >= -5) # constraint 2

@objective(m, Max, 3*x[1]+x[2]) # declare the objective function
```

Step 6: Solve the problem using *optimize!* and print the solution.

```
optimize!(m) # solve the optimisation problem

#Printing out the solution
x_value = value.(x)
print("Optimal values: $(x_value),\nOptimal objective: $(objective_value(m))\n")
```

```
Optimal values: [1.70833, 0.916667],
Optimal objective: 6.041666666666667
```

Demo 2: Formulation of linear optimisation problem

Matti is a farmer and wants to decide how many acres of rye and wheat to plant for the coming year. Each acre of wheat produces 25 loads of wheat and requires 10 hours of labour per week. An acre of rye produces 10 loads of rye and requires 4 hours of labour per week. The wheat sells at €4 per load and the rye €3 per load. Matti has 7 acres of farmland available and 40 hours of labour per week. Government regulations require that at least 30 loads of rye is produced during a given year. Let x_1 be the number of acres of wheat planted, and x_2 be the number of acres of rye planted.

Formulate an LP that will maximise Matti's total profit from wheat and rye, and solve with Excel Solver and Julia.

Solution

The objective function is to maximise Matti's profits, therefore as each acre of wheat produces 25 loads of wheat and they sell for €4 each we have $4 * 25 * x_1$. Similarly for rye, we have $3 * 10 * x_2$. Hence the objective function is:

$$\max.\ 100x_1 + 30x_2$$

For the constraints, Matti only has 7 acres of land available meaning:

$$x_1 + x_2 \leq 7$$

There is also a limit on the amount of labour hours available during each week:

$$10x_1 + 4x_2 \leq 40$$

Finally, we need to include the government regulation constraint for the amount of rye produced:

$$10x_2 \geq 30$$

And obviously, we cannot have a negative amount of acres planted:

$$x_1, x_2 \geq 0$$

Entering this LP formulation into Excel Solver and Julia gives the optimal solution: $x_1 = 2.8$ and $x_2 = 3$, with Matti's profit for the year at €370.

Problem 1: Solving simple linear programming problems

Solve with Excel Solver or Julia:

a)

$$\begin{array}{llcll} \min. & x & + & y & \\ \text{s.t.} & 3x & + & y & \geq 2 \\ & x & - & y & \geq 0 \\ & x, & y & & \geq 0 \end{array}$$

b)

$$\begin{array}{llcll} \max. & 3x_1 & + & 2x_2 & \\ \text{s.t.} & 2x_1 & + & x_2 & \leq 100 \\ & x_1 & + & x_2 & \leq 80 \\ & x_1 & & & \leq 40 \\ & x_1, & x_2 & & \geq 0 \end{array}$$

Solution

a) Optimal solution is: $x = 0.667$ and $y = 0$, objective value $= 0.667$

b) Optimal solution is: $x_1 = 20$ and $x_2 = 60$, objective value $= 180$

Problem 2: Formulation of a linear programming (LP) problem

A cargo plane has three compartments for storing cargo: front, centre and rear. These compartments have the following limits on both weight and space:

Compartment	Weight Capacity (tonnes)	Space Capacity (m^3)
Front	10	6800
Center	16	8700
Rear	8	5300

Table 1: Weight & space capacities

Furthermore, the weight of the cargo in the respective compartments must be the same proportion of that compartment's weight capacity to maintain the balance of the plane, e.g. if the front compartment has 90% of its weight capacity utilised (90% full) then the centre and the rear have to have 90% of their respective weight capacities used also.

The following four cargoes are available for shipment on the next flight:

Cargo	Weight (tonnes)	Volume (m^3/tonne)	Profit (€/tonne)
Cargo 1	18	480	310
Cargo 2	15	650	380
Cargo 3	23	580	350
Cargo 4	12	390	285

Table 2: Cargoes

Any proportion of these cargos can be distributed between any or all of the compartments.

Formulate an LP to determine how much (if any) of each cargo (C1, C2, C3 and C4) should be accepted and how to distribute each among the compartments so that the total profit for the flight is maximised. Solve in Excel Solver or Julia.

Solution

First, define the decision variables:

x_{ij} := amount of tonnes of cargo i in compartment j, where $i = \{1, 2, 3, 4\}$ and j ={1 (front), 2 (centre), 3 (rear)}

Note. We are explicitly told that we can split the cargo into any proportion (fractions) we like.

The objective is to maximise profits, therefore the objective function is:

$$\begin{aligned} \max.\ & 310(x_{11} + x_{12} + x_{13}) + 380(x_{21} + x_{22} + x_{23}) + \\ & 350(x_{31} + x_{32} + x_{33}) + 285(x_{41} + x42 + x_{43}) \end{aligned}$$

Now the constraints. We cannot pack more cargo than we have available (Table 2):

$$\begin{aligned} x_{11} + x_{12} + x_{13} &\leq 18 \text{ (maximum amount of cargo 1)} \\ x_{21} + x_{22} + x_{23} &\leq 15 \text{ (maximum amount of cargo 2)} \\ x_{31} + x_{32} + x_{33} &\leq 23 \text{ (maximum amount of cargo 3)} \end{aligned}$$

$$x_{41} + x_{42} + x_{43} \leq 12 \text{ (maximum amount of cargo 4)}$$

The weight capacity of each compartment must hold (Table 1):

$$x_{11} + x_{21} + x_{31} + x_{41} \leq 10 \text{ (front weight capacity)}$$
$$x_{12} + x_{22} + x_{32} + x_{42} \leq 16 \text{ (center weight capacity)}$$
$$x_{13} + x_{23} + x_{33} + x_{43} \leq 8 \text{ (rear weight capacity)}$$

The space capacity of each compartment must be respected (Table 1 & 2):

$$480x_{11} + 650x_{21} + 580x_{31} + 390x_{41} \leq 6800 \text{ (front volume capacity)}$$
$$480x_{12} + 650x_{22} + 580x_{32} + 390x_{42} \leq 8700 \text{ (center volume capacity)}$$
$$480x_{13} + 650x_{23} + 580x_{33} + 390x_{43} \leq 5300 \text{ (rear volume capacity)}$$

The balance constraint (each compartment must utilise the same proportion of its weight capacity, Table 1):

$$\frac{x_{11} + x_{21} + x_{31} + x_{41}}{10} = \frac{x_{12} + x_{22} + x_{32} + x_{42}}{16} = \frac{x_{13} + x_{23} + x_{33} + x_{43}}{8}$$

Finally, the domain constraints (non-negative): $x_{ij} \geq 0 \ \forall i \ \forall j$.

Optimal values: [0.0, 0.0, 0.0, 7.0, 0.0, 8.0, 3.0, 12.9474, 0.0, 0.0, 3.05263, 0.0]
Optimal objective value: 12151.579

Problem 3: Formulation of an LP problem

Putte the Pig is arranging his name day party, again. He is expecting 100 guests to his party and he has to make delicacies for all of them. Putte has decided to make only three kind of delicacies: cake, cookies and buns.

In one cake there is enough for 10 people, one cookie for one person, and one batch of buns for 20 people. Unfortunately, there is only 10 h until the party and everything has to be ready by then. Baking a cake takes 30 min, one cookie takes 1 min on average, and one batch of buns takes 40 min.

The yearly party is expensive so Putte tries to minimise the costs. The ingredients for a cake cost €5, for a cookie €1, and for a batch of buns €10.

How many of each delicacies should Putte make? Also notice that Putte thinks that there should be enough for at least 20 persons of every delicacy and there should be cake for at least twice as many persons as there are cookies for.

Formulate as LP problem and solve with Excel Solver or Julia.

Solution

The decision variables are:

$x_1 :=$ the quantity of cake
$x_2 :=$ the quantity of cookies
$x_3 :=$ the quantity of bun batches

Putte the Pig tries to minimise the costs:

$$\min . \ 5x_1 + x_2 + 10x_3$$

There are 100 people and one cake is enough for 10 people, a cookie for one person, and a batch of buns for 20 people:

$$10x_1 + x_2 + 20x_3 \geq 100$$

There has to be enough for at least 20 people of each type:

$$10x_1 \geq 20, \quad x_2 \geq 20, \quad 20x_3 \geq 20$$

Putte also wants the ratio between cake and cookies to be at least 2:1

$$10x_1 \geq 2x_2$$

Putte has 600 mins to prepare everything, and a cake takes 30mins, a cookie 1 min, and a batch of bun 40 mins:

$$30x_1 + x_2 + 40x_3 \leq 600$$

The quantities cannot be negative:

$$x_1,\ x_2,\ x_3 \geq 0$$

Solving with Excel or Julia gives the optimal solution: $x_1 = 4,\ x_2 = 20, x_3 = 2$. The total costs are €60. Note, that this could have multiple optimal solutions. Note that the quantities of cookies and bun batches are binding constraints.

Problem 4: Formulation of an LP problem

A construction company owns 800 hectares (ha) of land and is about to build one-family, two-family and three-family houses. The company estimates that a one-family house will profit the company €200,000, it requires an area of 1 ha, building costs are €145,000, and the water consumption is 2000 l per day. The key figures for a two-family house are €240,000, 1.5 ha, €165,000 and 2700 l per day, and for a three-family house are €300,000, 2 ha, €215,000 and 3200 l per day.

- At least half of the houses have to be one-family houses.
- The water consumption for the area cannot surpass 850,000 l per day.
- For every 200 families there has to be at least one recreational area. It requires 0.5 ha of area. Building costs are €125,000, and water consumption is 2500 l per day.
- Streets etc. require 15 % of the total area.

Formulate LP problem and find out how many of each house types should the company build to maximise its profits?

Solution

The decision variables are the quantities of houses and recreational areas:

$x_1 :=$ quantity of one-family houses
$x_2 :=$ quantity of two-family houses
$x_3 :=$ quantity of three-family houses

$x_4 :=$ quantity of recreational areas

The construction company maximises its profits (income - expenses):

$$\max.\ 200,000x_1 + 240,000x_2 + 300,000x_3 - 145,000x_1 - 165,000x_2 - 215,000x_3 - 125,000x_4$$

At least half of the houses have to be one-family:

$$x_1 \geq 0.5(x_1 + x_2 + x_3) \equiv 0.5x_1 - 0.5x_2 - 0.5x_3 \geq 0$$

The water consumption of the residential area cannot surpass 850,000 litres per day, and a one-family house uses 2,000 litres per day, a two-family house uses 2,700 litres per day, a three-family house 3,200 litres per day and recreational area 2,500 litres per day:

$$2000x_1 + 2700x_2 + 3200x_3 + 2500x_4 \leq 850,000$$

For every 200 families there has to be one recreational area:

$$x_4 \geq \frac{1}{200}(x_1 + 2x_2 + 3x_3) \equiv -\frac{1}{200}x_1 - \frac{2}{200}x_2 - \frac{3}{200}x_3 + x_4 \geq 0$$

The construction company has to use 15 % (120 ha) of the total 800 ha of building area for the streets. The remaining 680 ha can be used for the houses and one-family house requires 1 ha of area, two-family house 1.5 ha, three-family house 2 ha and recreational area 0.5 ha:

$$x_1 + 1.5x_2 + 2x_3 + 0.5x_4 \leq 680$$
$$x_1,\ x_2,\ x_3 \geq 0$$

Solve the problem with Excel Solver and Julia gives $x_1 = 179.42, x_2 = 179.42, x_3 = 0.0, x_4 = 2.69129$ and the objective value is 22,988,126.65. As we want the quantities of houses, the optimal solution should be integer. Rounding, the optimal solution is that the company build around 180 one-family houses, 180 two-family houses, no three-family houses and three recreational areas. This solution gives €23 million profit.

Note. In Excel the linear problem can also be solved as an integer problem if you constrain the variables to be integers with Solver's *int* constraint.

Note. In Julia this is done in the *@variable* definition by including the option *Int*, e.g. *@variable(p4, x[1:4] >= 0, Int)*.

Does the solution change? The difference between these answers is due to the Excel having no integer solution method such as Cbc (Branch and Bound plus cuts). Therefore, to enforce integer solutions constraints on the variables must be made. Integer programming will be discussed in weeks 3 and 4 of the course.

Problem 5: Formulation of an LP problem*

Stockmann requires different numbers of full-time employees on different days of the week. The number of full-time staff needed is given in Table 3. Union rules state that each full-time employee must work 5 consecutive days then receive 2 days off, for example if they work Monday - Friday then they must have Saturday and Sunday off. Stockmann wants to meet its daily requirements using only full-time staff.

Day	Mon	Tues	Weds	Thurs	Fri	Sat	Sun
# full-time staff needed	17	13	15	19	14	16	11

Table 3: Full-time staff requirements

a) Formulate an LP to minimise the number of full-time employees and solve in Excel Solver or Julia

b) Now suppose full-time employees work 8 hours per day, and Stockmann may meet daily labour requirements by using full- and part-time staff. Full-time staff work 8h/day for 5 straight days at 15€/h. Part-time staff work 4h/day for 5 straight days at 10€/h. Union requirements limit part-time staff labour to 25% of weekly labour requirements.
Formulate an LP to minimise cost of labour and solve in Excel Solver or Julia.

Solution

a) The LP is formulated as follows:

Let x_i be the number of staff starting work on day $i = \{1, \dots, 7\}$

$$\begin{aligned} \min. \ & \sum_i x_i \\ \text{s.t. } & x_1 + x_4 + x_5 + x_6 + x_7 \geq 17 \\ & x_1 + x_2 + x_5 + x_6 + x_7 \geq 13 \\ & x_1 + x_2 + x_3 + x_6 + x_7 \geq 15 \\ & x_1 + x_2 + x_3 + x_4 + x_7 \geq 19 \\ & x_1 + x_2 + x_3 + x_4 + x_5 \geq 14 \\ & x_2 + x_3 + x_4 + x_5 + x_6 \geq 16 \\ & x_3 + x_4 + x_5 + x_6 + x_7 \geq 11 \\ & x_i \geq 0 \ \forall i \end{aligned}$$

In Excel the solution gives: $x_1 = 6.3,\ x_2 = 3.3,\ x_3 = 2,\ x_4 = 7.3,\ x_5 = 0,\ x_6 = 3.3, x_7 = 0$ with objective function value 22.3

In Julia the solution gives: $x_1 = 1.3,\ x_2 = 5.3,\ x_3 = 0,\ x_4 = 7.3,\ x_5 = 0,\ x_6 = 3.3, x_7 = 5$ with objective function value 22.3

Note. If the student has different values for x_i but the same objective function value, then this just means there are multiple optimal solutions.

b) To include part-time workers into the model, we need to include the additional decision variables: y_i be the number of part-time staff starting work on day i.

To change the objective function to minimise the cost of labour rather than the number of employees, we do the following:

$$\min. \ 15 \cdot 8 \cdot \sum_i x_i + 10 \cdot 4 \cdot \sum_i y_i$$

The constraints are similar as to part (a) however, we also include part-time employees and multiple 8 for FT and 4 for PT. The right-hand-side of the constraints are also multiplied by 8 to represent the number of hours required.

$$\begin{aligned} \text{s.t. } & 8 \cdot (x_1 + x_4 + x_5 + x_6 + x_7) + 4 \cdot (y_1 + y_4 + y_5 + y_6 + y_7) \geq 136 \\ & 8 \cdot (x_1 + x_2 + x_5 + x_6 + x_7) + 4 \cdot (y_1 + y_2 + y_5 + y_6 + y_7) \geq 104 \\ & 8 \cdot (x_1 + x_2 + x_3 + x_6 + x_7) + 4 \cdot (y_1 + y_2 + y_3 + y_6 + y_7) \geq 120 \\ & 8 \cdot (x_1 + x_2 + x_3 + x_4 + x_7) + 4 \cdot (y_1 + y_2 + y_3 + y_4 + y_7) \geq 152 \\ & 8 \cdot (x_1 + x_2 + x_3 + x_4 + x_5) + 4 \cdot (y_1 + y_2 + y_3 + y_4 + y_5) \geq 112 \\ & 8 \cdot (x_2 + x_3 + x_4 + x_5 + x_6) + 4 \cdot (y_2 + y_3 + y_4 + y_5 + y_6) \geq 128 \\ & 8 \cdot (x_3 + x_4 + x_5 + x_6 + x_7) + 4 \cdot (y_3 + y_4 + y_5 + y_6 + y_7) \geq 88 \end{aligned}$$

For the union requirements we add the following constraint:

$$4 \cdot 5 \cdot \sum_i y_i \leq 210$$

$$x_i,\ y_i \geq 0\ \forall i$$

Excel gives the optimal solution as:
$x_1 = 6.3333,\ x_2 = 3.3333,\ x_3 = 2, x_4 = 2.0833, x_5 = 0,\ x_6 = 3.3333, x_7 = 0$
$y_1 = y_2 = y_3 = y_5 = y_6 = y_7 = 0,\ y_4 = 10.5$
objective function value $= 2470$

Julia gives the optimal solution as:
$x_1 = 1.3333,\ x_2 = 0,\ x_3 = 2, x_4 = 7.0033, x_5 = 0,\ x_6 = 3.3333, x_7 = 3.0833$
$y_1 = y_3 = y_4 = y_5 = y_6 = 0,\ y_2 = 6.6667,\ y_7 = 3.8333$
objective function value $= 2470$

Home Exercise 1: Formulate an LP problem

Bev's Beverage Products is considering producing a wine cooler that would be a blend of a white wine, a rosé wine, and fruit juice. To meet taste specifications, the wine cooler must consist of at least 50% white wine, at least 20% and no more than 30% rosé, and exactly 20% fruit juice. Bev purchases the wine from local wineries and the fruit juice from a processing plant in California. For the current production period, 10,000 litres of white wine and 8,000 litres of rosé wine can be purchased; and unlimited amount of fruit juice can be ordered. The costs for the wine are \$1.00 per litre for the white and \$1.50 per litre for the rosé; the fruit juice can be purchased for \$0.50 per litre. Bev's Beverages can sell all of the wine cooler they can produce for \$2.50 per litre.

Formulate a linear program to determine the blend of the three ingredients that will maximise total profit contribution. Solve the linear program to determine the number of litres of each ingredient Bev should purchase and the total profit contribution they will realise from this blend.

Solution

Let $w :=$ litres of white wine, $r :=$ litre of rose, and $f :=$ litres of fruit juice.

To maximise the total profits (profit-cost of ingredients) we have to do the following:

$$\max.\ 2.5(w + r + f) - (w + 1.5r + 0.5f)$$

Next, for the constraints we have that:

$$\begin{aligned}
&0.5(w + r + f) \leq w \quad \text{at least 50\% white wine}\\
&0.2(w + r + f) \leq r \quad \text{at least 20\% rose}\\
&0.3(w + r + f) \geq r \quad \text{at most 30\% rose}\\
&0.2(w + r + f) \leq f \quad \text{at least 20\% fruit juice}\\
&0.2(w + r + f) \geq f \quad \text{at most 20\% fruit juice}\\
&w \leq 10,000\\
&r \leq 8,000\\
&w,\ r,\ f \geq 0
\end{aligned}$$

Simplifying theses we get the final LP:

$$\begin{aligned}
\max\ & 1.5w + r + 2f\\
\text{s.t.}\ & 0.5w - 0.5r - 0.5f \geq 0\\
& -0.2w + 0.8r - 0.2f \geq 0\\
& -0.3w + 0.7r - 0.3f \leq 0\\
& -0.2w - 0.2r + 0.8f \geq 0\\
& -0.2w - 0.2r + 0.8f \leq 0\\
& w \leq 10,000\\
& r \leq 8,000\\
& w,\ r,\ f \geq 0
\end{aligned}$$

Optimal solution: $w = 10,000$, $r = 6,000$, $f = 4,000$ and profit \$29,000