Exercise class 6

Learning Objectives:

- Formulation of integer optimisation problems
- Binary and integer variables

Demo 1: Integer and binary optimisation

Formulate as linear optimisation problem and solve with Julia.

a) A retailer buys a product from a factory and sells one piece with € 2 profit. The retailer has to pay € 1000 to the factory to be able to buy the product. The maximum amount of products the factory is able to produce is 750. How many products should the retailer buy? None?

b) How about if the retailer has to pay the € 1000 only if the quantity of purchased products exceeds 200 pieces?

c) A farmer has 15 boxes of apples and two fruit retailers, Aapeli and Toopeli, want to buy his apples. For one box of apples Aapeli is willing to pay € 25 and Toopeli € 32. If the farmer sells more apples to Aapeli than Toopeli, Aapeli will pay the farmer € 450 extra. Similarly, if Toopeli gets more apples than Aapeli, he will pay the farmer € 270 extra. How many boxes should the farmer sell to Aapeli and Toopeli to maximize his profits?

Solution

a) Decision variables are the quantity of purchased products x as integer and the decision to pay the fee to factory y (value is 1 if the fee is paid and zero if not).

$$x \in \mathbb{Z}_+$$
$$y \in \{0, 1\}$$

Retailer wants to maximises his profits (income - expenses):

$$\max.\ 2x - 1000y$$

The retailer can buy 750 products if he has paid the fee ($y = 1$) and 0 products if he hasn't ($y = 0$):

$$x \leq 750y$$

Optimally the retailer should pay the fee to the factory and buy all of the 750 products to retrieve € 500 of profit.

b) Change the constraint in part (a) so that the retailer is able to buy 200 products without paying the fee:

$$x \leq 200 + 550y$$

The optimal solution is same as in part (a).

c) The decision variables are the quantity of boxes sold to Aapeli x_1 and to Toopeli x_2 and the binary variable y which indicates who has more apples ($y = 1$ if Aapeli and $y = 0$ if Toopeli):

$$x_1 \in \mathbb{Z}_+$$
$$x_2 \in \mathbb{Z}_+$$
$$y \in \{0, 1\}$$

Maximise the farmer's profits from the boxes and the extras:

$$\max.\ 25x_1 + 32x_2 + 450y + 270 \cdot (1 - y)$$

The farmer can sell 15 boxes in total:

$$x_1 + x_2 \leq 15$$

If $x_1 \geq x_2$ then y has to be 1:

$$x_1 - x_2 \leq My$$

where M is a large number (for example 10 000). And if $x_1 \leq x_2$ then y has to be 0:

$$x_2 - x_1 \leq M(1 - y)$$

The optimum for the farmer is to sell 8 boxes to Aapeli and 7 boxes to Toopeli. He receives the extra pay from Aapeli and his total profits are € 874.

Note: should the model have a feasible solution in which $x_1 = x_2$, it would pick the most profitable option for optimality reasons. However, $x_1 + x_2 \leq 15$ prevents $x_1 = x_2$ as they have to be integers.

Demo 2: Sensitivity analysis

Consider the following LP:

$$\begin{aligned} \max.\ z =& 3x_1 + 2x_2 + x_3 \\ \text{s.t. } & x_1 + 2x_2 + x_3 \leq 100 \\ & x_1 + 2x_3 \leq 200 \\ & x_1 + 4x_2 \leq 250 \\ & x_1,\ x_2,\ x_3 \geq 0 \end{aligned}$$

Let $s_1,\ s_2,\ s_3$ be the slack variables used in the constraints, respectively. The optimal tableau is shown below:

	x_1	x_2	x_3	s_1	s_2	s_3	Sol.
z	0	4	2	3	0	0	300
x_1	1	2	1	1	0	0	100
s_2	0	-2	1	-1	1	0	100
s_3	0	2	-1	-1	0	1	150

Answer the following questions.

a) For what individual changes in the right-hand side does this basis remain optimal?

b) For what individual changes in the objective coefficients does this solution remain optimal?

Solution

a) Similar to example on lecture slide 6.

To analyse variations in the RHS (b_i), we include extra columns (Δb_i) in the starting tableau. Each Δb_i column is identical to the column of that slack constraint, i.e. $s_1 = \Delta b_1$.

Therefore, we have the following optimal tableau with the added RHS variations:

	x_1	x_2	x_3	s_1	s_2	s_3	Sol.	Δb_1	Δb_2	Δb_3
z	0	4	2	3	2	0	300	3	2	0
x_1	1	2	1	1	0	0	100	1	0	0
s_2	0	-2	1	-1	1	0	100	-1	1	0
s_3	0	2	-1	-1	0	1	150	-1	0	1

As changes in the RHS affect the feasibility of the solution, we need $B^{-1}(b + \Delta b) \geq 0$ to remain true.
For b_1 we have:

$$\begin{aligned} &100 + \Delta b_1 \geq 0 && \implies \Delta b_1 \geq -100 \\ &100 - \Delta b_1 \geq 0 && \implies \Delta b_1 \leq 100 \\ &150 - \Delta b_1 \geq 0 && \implies \Delta b_1 \leq 150 \\ &\therefore -100 \leq \Delta b_1 \leq 100 \text{ and } z = 300 + 3\Delta b_1 \end{aligned}$$

For b_2 we have:

$$100 + \Delta b_2 \geq 0$$
$$\therefore \Delta b_2 \geq -100 \text{ and } z = 300 + 2\Delta b_2$$

For b_3 we have:

$$150 + \Delta b_3 \geq 0$$
$$\therefore \Delta b_3 \geq -150 \text{ and } z = 300$$

b) Similar to example on slide 9 of lecture notes.

For c_1:

	x_1	x_2	x_3	s_1	s_2	s_3	Sol.
z	$0-\Delta c_1$	4	2	3	2	0	300
x_1	1	2	1	1	0	0	100
s_2	0	-2	1	-1	1	0	100
s_3	0	2	-1	-1	0	1	150

Since x_1 is a basic variable in the optimal solution, changing its coefficient requires the optimal tableau to be corrected accordingly. This is done using row operations as in the Simplex method: z-row $+\Delta c_1$(row 1)

	x_1	x_2	x_3	s_1	s_2	s_3	Sol.
z	0	$4+2(\Delta c_1)$	$2+\Delta c_1$	$3+\Delta c_1$	2	0	$300+100(\Delta c_1)$
x_1	1	2	1	1	0	0	100
s_2	0	-2	1	-1	1	0	100
s_3	0	2	-1	-1	0	1	150

We can obtain the allowed variation in c_1 by enforcing the optimality condition $r_N \leq \Delta c_B^T B^{-1} N$, meaning that all elements in the z-row are required to be non-negative. Thus, we have

$$\begin{aligned}
&4 + 2(\Delta c_1) \geq 0 \quad &&\implies \Delta c_1 \geq -2 \\
&2 + \Delta c_1 \geq 0 &&\implies \Delta c_1 \geq -2 \\
&3 + \Delta c_1 \geq 0 &&\implies \Delta c_1 \geq -3 \\
&\therefore \Delta c_1 \geq -2 \text{ and } z = 300 + 100(\Delta c_1)
\end{aligned}$$

As x_2 and x_3 are nonbasic in the optimal tableau, no adjustments to the tableau are needed. The allowed variations can be obtained by imposing the optimality condition as before:

	x_1	x_2	x_3	s_1	s_2	s_3	Sol.
z	0	$4-\Delta c_2$	$2-\Delta c_3$	3	2	0	300
x_1	1	2	1	1	0	0	100
s_2	0	-2	1	-1	1	0	100
s_3	0	2	-1	-1	0	1	150

$$\begin{aligned}
&4 - \Delta c_2 \geq 0 \quad &&\implies \Delta c_2 \leq 4 \\
&2 - \Delta c_3 \geq 0 &&\implies \Delta c_3 \leq 2 \\
&\therefore z = 300
\end{aligned}$$

Problem 1: Sudoku

The world-renowned logic puzzle, Sudoku, deals with a 9 x 9 grid subdivided into 9 nonover lapping 3 x 3 subgrids. The puzzle calls for assigning the numerical digits 1 through 9 to the cells of the grid such that each row, each column, and each subgrid contain distinct digits. Some of the cells may be fixed in advance.

Formulate the problem as an integer program, and find the solution for the instance given below using Julia.

	6		1		4		5	
		8	3		5	6		
2						7		
8			4		7			6
		6				3		
7			9		1			4
5								2
		7	2		6	9		
	4		5		8		7	

Solution

Unlike typical LP or IP problems, there is no solution to a Sudoku that is better than another. In our case, we wish to find any feasible solution - one which satisfies our constraints. Therefore, we can specify an arbitrary objective function.

$$\min.\ z = 100 \text{ (dummy objective function)}$$

Since there are nine values each box can be filled with, and we are constrained along each of the nine rows and nine columns, we have $9^3 = 729$ parameters. These will be indexed as $i \in \{1, \dots, 9\}$ row, $j \in \{1, \dots, 9\}$ columns, and $k \in \{1, \dots, 9\}$ value.

$$x_{ijk} \in \{0, 1\}$$

The constraints are as follows.

Constraint 1 requires that each cell (i, j) contains exactly one value.

$$\sum_{k=1}^{9} x_{ijk} = 1 \ \ \forall i,\ j \tag{1}$$

Constraints 2 & 3 maintains that each value is contained exactly once within each row and column, respectively.

$$\sum_{i=1}^{9} x_{ijk} = 1 \ \ \forall j,\ k \tag{2}$$

$$\sum_{j=1}^{9} x_{ijk} = 1 \ \ \forall i,\ k \tag{3}$$

Constraint 4 fixes that only one of a value is found in each subgrid.

$$\sum_{i=3m-2}^{3m} \sum_{j=3n-2}^{3n} x_{ijk} = 1 \ \ \forall k \in \{1, 9\},\ m,\ n \in \{1, 3\} \tag{4}$$

As we want to solve a particular sudoku we need to add the following constraints:

$$x_{126} + x_{141} + x_{164} + x_{185} = 4 \quad \text{row 1} \tag{5}$$
$$x_{238} + x_{243} + x_{265} + x_{276} = 4 \quad \text{row 2} \tag{6}$$
$$x_{312} + x_{377} = 2 \quad \text{row 3} \tag{7}$$
$$x_{418} + x_{444} + x_{467} + x_{496} = 4 \quad \text{row 4} \tag{8}$$
$$x_{536} + x_{573} = 2 \quad \text{row 5} \tag{9}$$
$$x_{617} + x_{649} + x_{661} + x_{694} = 4 \quad \text{row 6} \tag{10}$$
$$x_{715} + x_{792} = 2 \quad \text{row 7} \tag{11}$$
$$x_{837} + x_{842} + x_{866} + x_{789} = 4 \quad \text{row 8} \tag{12}$$
$$x_{924} + x_{945} + x_{968} + x_{987} = 4 \quad \text{row 9} \tag{13}$$

Solving in Julia gives the final solution:

```
9×9 Array{Float64,2}:
 9.0  6.0  3.0  1.0  7.0  4.0  2.0  5.0  8.0
 1.0  7.0  8.0  3.0  2.0  5.0  6.0  4.0  9.0
 2.0  5.0  4.0  6.0  8.0  9.0  7.0  3.0  1.0
 8.0  2.0  1.0  4.0  3.0  7.0  5.0  9.0  6.0
 4.0  9.0  6.0  8.0  5.0  2.0  3.0  1.0  7.0
 7.0  3.0  5.0  9.0  6.0  1.0  8.0  2.0  4.0
 5.0  8.0  9.0  7.0  1.0  3.0  4.0  6.0  2.0
 3.0  1.0  7.0  2.0  4.0  6.0  9.0  8.0  5.0
 6.0  4.0  2.0  5.0  9.0  8.0  1.0  7.0  3.0
```

Problem 2: Sensitivity analysis

A Gepbab Production Company uses labour and raw material to make three distinct products. The resource requirements and selling prices are described in the table below.

Product	Profit ($/unit)	Labour req. (h/unit)	Raw material (per unit)
1	6	3	2
2	8	4	2
3	13	6	5

Currently, 60 units of raw material are available and 90h of labour can be purchased, at the price of $1 per hour. To maximize the its profit, Gepbab needs to optimise the following linear programming problem:

$$\begin{aligned} \max.\ z = & 6x_1 + 8x_2 + 13x_3 - L \\ \text{s.t. } & 3x_1 + 4x_2 + 6x_3 - L \leq 0 \\ & 2x_1 + 2x_2 + 5x_3 \leq 60 \\ & L \leq 90 \\ & x_1,\ x_2,\ x_3,\ L \geq 0 \end{aligned}$$

where, x_i is the amount of units of product i made and L is the total number of hired labour hours. Let $s_1,\ s_2,\ s_3$ be the slack variables used in the constraints, respectively. The optimal tableau for the problem is:

	x_1	x_2	x_3	L	s_1	s_2	s_3	Sol.
z	0.25	0	0	0	1.75	0.5	0.75	97.5
x_3	0.25	0	1	0	-0.25	0.5	-0.25	7.5
L	0	0	0	1	0	0	1	90
x_2	0.375	1	0	0	0.625	-0.75	0.625	11.25

Answer the following:

a) What is the maximum value that Gepbab should pay for additional units of raw material, assuming that changes in the products selected for production can not be allowed (i.e., the optimal basis is not allowed to change)?

b) What is the maximum value the company should pay for additional availability of labour?

c) What should be the profit per unit of product 1 such that it would become profitable for Gepbab to make it?

d) Consider the following question: "If 100h of labour would be available (instead of 90h), what would be the profit of the company?" Is it possible to answer the question without re-optimising the problem? If so, what is the new objective function value?

e) Can you tell what would be the new optimal solution if the profit per unit of product 3 becomes \$15, without re-optimising the problem? If so, what would this new optimal solution be?

Solution

a) \$0.5, which is the value (shadow price) of the slack variable associated with the raw material constraint (s_2).

b) \$1.75, which is the shadow price of the require labour constraint. This question causes confusion, since it is not clear whether to use the shadow price of s_1 or s_3. Both are correct, if correctly interpreted. The value of a single unit of labour is \$1.75, but since \$1 has to be paid for it, the shadow price in the third constraint appears as \$0.75 (because the cost of \$1 is discounted in the objective function). The shadow price of the first constraint shows the value of labour for the company if it were "for free", i.e., if the company did not have to pay for it. Therefore, the maximum the company is willing to pay is \$1.75 per labour hour.

c) The current profit is \$6. The allowed variation is \$0.25 (it is a nonbasic variable, so trivially you get that the variation Δc_1 is $-\infty \leq \Delta c_1 \leq 0.25$) and therefore the profit needs to be greater than \$6.25.

d) This depends on whether this variation in available hours stays within the limits that do not violate feasibility conditions. As the available hours is associated with constraint 3, we add on a Δb_3 column to the optimal tableau:

	x_1	x_2	x_3	L	s_1	s_2	s_3	Sol.	Δb_3
z	0.25	0	0	0	1.75	0.5	0.75	97.5	0.75
x_3	0.25	0	1	0	-0.25	0.5	-0.25	7.5	-0.25
L	0	0	0	1	0	0	1	90	1
x_2	0.375	1	0	0	0.625	-0.75	0.625	11.25	0.625

As changes in the RHS affect feasibility, we need $B^{-1}(b + \Delta b) \geq 0$ to remain true.

$$\begin{aligned}
&7.5 - 0.25\Delta b_3 \geq 0 &&\implies \Delta b_3 \leq 30\\
&90 + \Delta b_3 \geq 0 &&\implies \Delta b_3 \geq -90\\
&11.25 + 0.625\Delta b_3 \geq 0 &&\implies \Delta b_3 \geq -18\\
&\therefore -18 \leq \Delta b_3 \leq 30 &&\implies 72 \leq b_3 \leq 120
\end{aligned}$$

So with 100 hours available, the basis remains optimal. The new objective function would be given by $z = 97.5 + 0.75\Delta b_3$, as our change is 10 hours we have an extra profit of \$7.5 resulting in a total profit of \$105.

e) This also depends on whether this coefficient for product 3 is within the allowed variation (it does not compromise the optimality conditions).
Since x_3 is a basic variable in the optimal basis, we need to correct the tableau after including the variation Δc_3.

	x_1	x_2	x_3	L	s_1	s_2	s_3	Sol.
z	0.25	0	$0-\Delta c_3$	0	1.75	0.5	0.75	97.5
x_3	0.25	0	1	0	-0.25	0.5	-0.25	7.5 5
L	0	0	0	1	0	0	1	90
x_2	0.375	1	0	0	0.625	-0.75	0.625	11.25

	x_1	x_2	x_3	L	s_1	s_2	s_3	Sol.
z	0.25+0.25Δc_3	0	0	0	1.75-0.25Δc_3	0.5+0.5Δc_3	0.75-0.25Δc_3	97.5+7.5Δc_3
x_3	0.25	0	1	0	-0.25	0.5	-0.25	7.5
L	0	0	0	1	0	0	1	90
x_2	0.375	1	0	0	0.625	-0.75	0.625	11.25

Optimality conditions require that all elements in the z-row are non-negative, and thus

$$\begin{aligned}
&0.25 + 0.25\Delta c_3 \geq 0 &&\implies \Delta c_3 \geq -1\\
&1.75 - 0.25\Delta c_3 \geq 0 &&\implies \Delta c_3 \leq 7\\
&0.5 + 0.5\Delta c_3 \geq 0 &&\implies \Delta c_3 \geq -1\\
&0.75 - 0.25\Delta c_3 \geq 0 &&\implies \Delta c_3 \leq 3\\
&\therefore -1 \leq \Delta c_3 \leq 3 &&\implies 12 \leq c_3 \leq 16
\end{aligned}$$

Meaning, for a profit of \$15 per unit of product 3, the current solution would remain optimal and the new objective function value would be $97.5+7.5\Delta c_3 = 97.5 + 7.5(2) = 112.5$

Problem 3: Store locations

Walmark Stores is in the process of expansion in the western United States. During next year, Walmark is planning to construct new stores that will serve 10 geographically dispersed communities. Past experience indicates that a community must be within 25 miles of a store to attract customers. In addition, the population of a community plays an important role in where a store is located, in the sense that bigger communities generate more participating customers. The following table provides the populations as well as the distances (in miles) between the communities:

	Miles from community *i* to community *j*										
i \ *j*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*Population*
1		20	40	35	17	24	50	58	33	12	10,000
2	20		23	68	40	30	20	19	70	40	15,000
3	40	23		36	70	22	45	30	21	80	28,000
4	35	68	36		70	80	24	20	40	10	30,000
5	17	40	70	70		23	70	40	13	40	40,000
6	24	30	22	80	23		12	14	50	50	30,000
7	50	20	45	24	70	12		26	40	30	20,000
8	58	19	30	20	40	14	26		20	50	15,000
9	33	70	21	40	13	50	40	20		22	60,000
10	12	40	80	10	40	50	30	50	22		12,000

The idea is to construct the least number of stores, taking into account the distance restriction and the concentration of populations. Specify the communities where the stores should be located. Use Julia to solve this problem.

Hint.
Define decision variables and parameters:
$x_j = 1$, if community j is selected and 0 otherwise
$p_j =$ population of community j
$d_{ij} =$ distance from community i to community j

And use an incidence matrix A_{ij}.

Solution

The incidence matrix A_{ij} for this problem provides the information for if i could serve j. That is, $A_{ij} = 1$ if the distance between i and j is less or equal than 25 miles.

The idea of the model is that the larger the population of a community, the higher its preference should be for acquiring a new store.

At the same time, we need to minimise the total number of new stores. Thus, using $1/p_j$ as a weight for x_j is an appropriate way to model the objective function.

$$\begin{aligned} \min. \ z = & \sum_j \frac{1}{p_j} x_j \\ \text{s.t.} \ & \sum_j A_{ij} x_j \geq 1, \quad i, j = 1, 2, \ldots, 10 : i \neq j \\ & x_j \in \{0, 1\}, \qquad j = \{1, \ldots, 10\} \end{aligned}$$

Overall solution: The optimal chosen locations are 6, 8, and 9.

Problem 4: Formulate & solve IPs and LPs

A wilderness hiker must pack three items: food, first-aid kits, and clothes. The backpack has a capacity of 3 ft^3. Each unit of food takes 1 ft^3. A first-aid kit occupies 1/4 ft^3, and each piece of cloth takes about 1/2 ft^3. The hiker assigns the priority weights 3, 4, and 5 to food, first aid, and clothes, respectively, which means that clothes are the most valuable of the three items. From experience, the hiker must take at least one unit of each item and no more than two first-aid kits.

a) Formulate and solve the IP and LP for this problem.

b) Why does the LP have the exact same solution as the IP?

Hint. This variant of the Knapsack problem does not require binary variables

Solution

a) Define the decision variables as
x_1 := number of food items
x_2 := number of first-aid items
x_3 := number of clothes

The LP is:

$$\begin{aligned} \max . z &= 3x_1 + 4x_2 + 5x_3 \\ \text{s.t. } & x_1 + 0.25x_2 + 0.5x_3 \leq 3 \\ & x_1 \geq 1 \\ & 1 \leq x_2 \leq 2 \\ & x_3 \geq 1 \end{aligned}$$

The IP adds the domain that the decision variables must be integer, $x_1,\ x_2,\ x_3 \in \mathbb{Z}_+$.

The solution for the LP is:
$z = 26$, $x_1 = 1$, $x_2 = 2$, and $x_3 = 3$

The solution for the IP is:
$z = 26$, $x_1 = 1$, $x_2 = 2$, and $x_3 = 3$

b) The solution of the LP satisfies the integrality constraints of the IP.

Also acceptable solution:

The optimal vertex of the feasible region is integer, thus the IP & LP have the same optimal solution.

Problem 5: TSP variant

The daily production schedule at the Rainbow Company includes batches of white (W), yellow (Y), red (R), and black (B) paints. The production facility must be cleaned between successive batches. The Table below summarises the cleanup times in minutes. The objective is to determine the sequencing of colors that minimizes the total cleanup time.

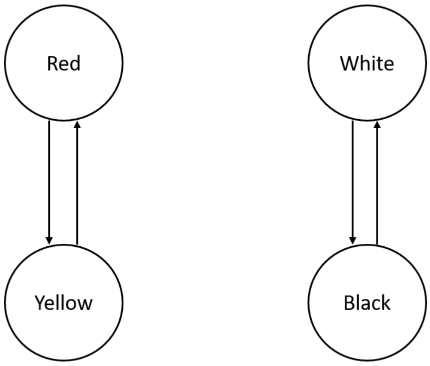
Interbatch Cleanup Times (in minutes) for the Paint Production Problem

	Interbatch cleanup time (min)			
Paint	*White*	*Yellow*	*Black*	*Red*
White		10	17	15
Yellow	20		19	18
Black	50	44		22
Red	45	40	20	

In the TSP model, each color represents a "city," and the cleanup time between two successive colors represents "distance."

a) Formulate the naive model for this TSP without subtour elimination.

b) Consider the solution in the figure below.



i) This solution is not a tour, why?

ii) Formulate two cutset constraints that would have prevented this solution.

iii) Formulate two subtour elimination constraints that would have prevented this solution.

Solution

a) $x_{ij} = 1$ if paint j follows paint i and zero otherwise, with $i \neq j$

$$\begin{aligned}
\max.\ z = &10x_{WY} + 17x_{WB} + 15x_{WR} + 20x_{YW} + 19x_{YB} + 18x_{YR} + \\
&50x_{BW} + 44x_{BY} + 22x_{BR} + 45x_{RW} + 40x_{RY} + 20x_{RB} \\
\text{s.t. } &\sum_{j:j\neq i} x_{ij} = 1 \quad \forall i \\
&\sum_{i:i\neq j} x_{ij} = 1 \quad \forall j \\
&x_{ij} \in \{0, 1\}
\end{aligned}$$

b) i) The solution is $x_{RY} = x_{YR} = 1$ and $x_{WB} = x_{BW} = 1$. This solutions is not a tour as it has subtours.

ii) Subset 1 = {R, Y}, the cutset for this is:

$$\sum_{i\in\{R,Y\}} \sum_{j\in\{W,B\}} x_{ij} \geq 1$$

which is equivalent to

$$x_{RW} + x_{RB} + x_{YW} + x_{YB} \geq 1$$

Subset 2 = {W, B}, the cutset for this is:

$$\sum_{i\in\{W,B\}} \sum_{j\in\{R,Y\}} x_{ij} \geq 1$$

which is equivalent to

$$x_{WR} + x_{BY} + x_{WY} + x_{BR} \geq 1$$

iii) Subtour elimination for subset 1 is:

$$\sum_{i\in\{R,Y\}} \sum_{j\in\{R,Y\}:i\neq j} x_{ij} \leq 1$$

which is equivalent to

$$x_{RY} + x_{YR} \leq 1$$

Subtour elimination for subset 2 is:

$$\sum_{i\in\{W,B\}} \sum_{j\in\{W,B\}:i\neq j} x_{ij} \leq 1$$

which is equivalent to

$$x_{WB} + x_{BW} \leq 1$$

Home Exercise 6: Sensitivity analysis

Dakota Furniture makes desk, tables and chairs. To make the furniture, the company uses wood (acquired in units of a standard measurement boards) and two different processes, namely assembly and finishing. The requirements of wood and process times for each product are detailed below.

Resource	Desk	Table	Chair
Wood (units)	8	6	1
Hours of assembling	2	1.5	0.5
Hours of finishing	4	2	1.5

Currently, Dakota has available 48 units of wood, 8h- of assembling and 20h-worth of finishing per day. The desks are sold for \$60,00, tables for \$30,00 and chairs for \$20,00. The company believes that all products made will often be sold, but that it should not make more than 5 tables. With that in mind, let
$x_1 :=$ total number of desks made
$x_2 :=$ total number of tables made
$x_3 :=$ total number of chairs made
The linear programming (LP) model that optimises Dakota's daily production plan is:

$$\begin{aligned}
\max.\ z =& 60x_1 + 30x_2 + 20x_3 \\
\text{s.t. } & 8x_1 + 6x_2 + x_3 \leq 48 \\
& 4x_1 + 2x_2 + 1.5x_3 \leq 20 \\
& 2x_1 + 1.5x_2 + 0.5x_3 \leq 8 \\
& x_2 \leq 5 \\
& x_1,\ x_2,\ x_3 \geq 0
\end{aligned}$$

Using the Simplex method to solve Dakota's optimisation problem we obtain the following optimal tableau:

	x_1	x_2	x_3	s_1	s_2	s_3	s_4	Sol.
z	0	0	0	0	10	10	0	280
s_1	0	-2	0	1	2	-8	0	24
x_3	0	-2	1	0	2	-4	0	8
x_1	1	5/4	0	0	-1/2	3/2	0	2
s_4	0	1	0	0	0	0	1	5

Where s_i are the slack variables associated with constraint (i), for $i = 1, \ldots, 4$. Given the above, please answer:

a) What is the maximum selling price the desks can have that would not affect the optimal production plan?

b) The company is looking into expanding its availability of assembling hours. After a market consultation, Dakota learned that a competing firm would be willing to offer 2h of assembling per day for \$8/h. Assuming that Dakota would not want to change the current production assortment (that is, remain producing desks and chairs and not producing tables), should Dakota accept the offer?

Solution

	x_1	x_2	x_3	s_1	s_2	s_3	s_4	Sol.
z	0	(5/4)Δ	0	0	10-(1/2)Δ	10+(3/2)Δ	0	280+2$Delta$
s_1	0	-2	0	1	2	-8	0	24
x_3	0	-2	1	0	2	-4	0	8
x_1	1	5/4	0	0	-1/2	3/2	0	2
s_4	0	1	0	0	0	0	1	5

a) $5/4\Delta \geq 0 \implies \Delta \geq 0$
$10 - 1/2\Delta \geq 0 \implies \Delta \leq 20$
$10 + 3/2\Delta \geq 0 \implies \Delta \geq -20/3$
$\therefore 0 \leq \Delta \leq 20 \implies 60 \leq c_1 \leq 80$

Thus, the maximum price is \$80.

b) Yes, since this would increase the daily profit by \$(10-8) x 2 = \$4. Notice that the marginal value of hours of assembling is \$10 per hour and, since it is being charged \$8/h, Dakota has a marginal profit of \$2/h of assembling. In fact, 2h of assembling is the maximum the company can obtain without changing the current basis (i.e., production assortment). Notice that constraint (3) is the one limiting the assembling hours. Calculating the allowed variation of b_3, we obtain:

	x_1	x_2	x_3	s_1	s_2	s_3	s_4	Sol.	Δ_3
z	-Δ	0	0	0	10	10	0	280	10
s_1	0	-2	0	1	2	-8	0	24	-8
x_3	0	-2	1	0	2	-4	0	8	-4
x_1	1	5/4	0	0	-1/2	3/2	0	2	3/2
s_4	0	1	0	0	0	0	1	5	0

$24 - 8\Delta \geq 0 \implies \Delta \leq 3$
$8 - 4\Delta \geq 0 \implies \Delta \leq 2$
$2 + 3/2\Delta \geq 0 \implies \Delta \geq -4/3$
$5 + 0\Delta \geq 0 \implies 5 \geq 0$
$\therefore -4/3 \leq \Delta \leq 2 \implies 20/3 \leq b_3 \leq 10$