MS-C2105 - Introduction to Optimization
Lecture 4

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Outline of this lecture

Artificial variables and feasible initial solutions

The M-method

Two-phase method

Special cases

Identifying infeasibility, unboundedness and multiple solutions

Reading: Taha: Chapter 3 (Sections 3.4 and 3.5); Winston: Chapter 4 (Sections 4.6 to 4.13)

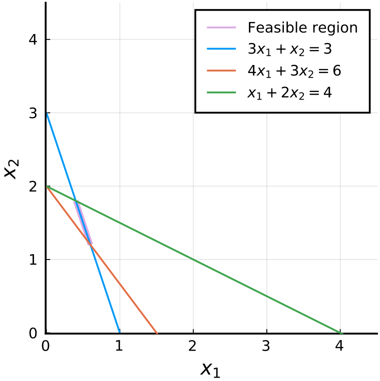
Finding initial basic feasible solutions

The standard form of LPs assume that the origin $(0, \dots, 0) \in \mathbb{R}^n$ in the decision-variable space is a feasible solution.

- The origin is used as a trivial initial basic feasible solution;
- However, it does not hold for $(\geq)$- or (most) $(=)$-constraints.

Example:

$$\begin{aligned} \text{min. } & z = 4x_1 + x_2 \\ \text{s.t.: } & 3x_1 + x_2 = 3 \\ & 4x_1 + 3x_2 \geq 6 \\ & x_1 + 2x_2 \leq 4 \\ & x_1, x_2 \geq 0. \end{aligned}$$



Finding initial basic feasible solutions

To circumvent this issue, we rely on artificial variables, which have the role of accumulating infeasibility.

Each $(\geq)$- or $(=)$-constraint is augmented with an artificial variable. Then, we minimise their value, i.e., the total infeasibility.

$$\begin{aligned} \text{min. } & z = 4x_1 + x_2 \\ \text{s.t.: } & 3x_1 + x_2 = 3 \\ & 4x_1 + 3x_2 \geq 6 \\ & x_1 + 2x_2 \leq 4 \\ & x_1, x_2 \geq 0. \end{aligned}$$

$$\begin{aligned} \text{min. } & z = 4x_1 + x_2 \\ \text{s.t.: } & 3x_1 + x_2 + r_1 = 3 \\ & 4x_1 + 3x_2 - x_3 + r_2 = 6 \\ & x_1 + 2x_2 + x_4 = 4 \\ & x_1, x_2, x_3, x_4, r_1, r_2 \geq 0. \end{aligned}$$

- If a solution with zero infeasibility (i.e., artificial variables are nonbasic) is found, a basic feasible solution is available;
- If the minimal (optimal) accumulated infeasibility is not zero (has basic artificial variables), no basic feasible solution exists.

The M-method

Include in the objective function large-enough penalties for the artificial variables:

$$\begin{aligned}
\text{min. } & z = 4x_1 + x_2 + Mr_1 + Mr_2 \\
\text{s.t.: } & 3x_1 + x_2 + r_1 = 3 \\
& 4x_1 + 3x_2 - x_3 + r_2 = 6 \\
& x_1 + 2x_2 + x_4 = 4 \\
& x_1, x_2, x_3, x_4, r_1, r_2 \geq 0.
\end{aligned}$$

- Notice that M is positive since it is minimisation.
- For max. , use $-M$.
- In the example $M = 100$.

The initial tableau: z is not a function of nonbasic variables.

	x_1	x_2	x_3	r_1	r_2	x_4	Sol.
z	-4	-1	0	-100	-100	0	0
r_1	3	1	0	1	0	0	3
r_2	4	3	-1	0	1	0	6
x_4	1	2	0	0	0	1	4

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$$
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& x_1, x_2, x_3, x_4, r_1, r_2 \geq 0.
\end{aligned}
$$

- Notice that M is positive since it is minimisation.
- For max. , use $-M$.

The correct initial tableau: z is a function of nonbasic variables.

	x_1	x_2	x_3	r_1	r_2	x_4	Sol.	Operation
z	696	399	-100	0	0	0	900	$+\,100 \times R_{r_1} + 100 \times R_{r_2}$
r_1	3	1	0	1	0	0	3	(R_{r_1})
r_2	4	3	-1	0	1	0	6	(R_{r_2})
x_4	1	2	0	0	0	1	4	

The M-method

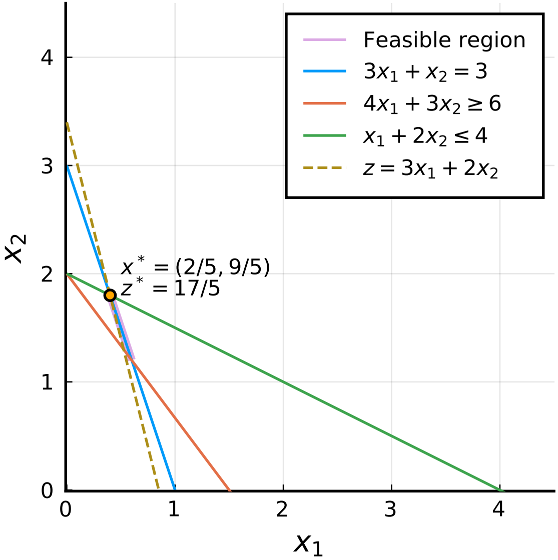
The method proceeds as usual. (**Remark**: notice the min. !)

	x_1	x_2	x_3	r_1	r_2	x_4	Sol.
z	696	399	-100	0	0	0	900
r_1	3	1	0	1	0	0	3
r_2	4	3	-1	0	1	0	6
x_4	1	2	0	0	0	1	4

	x_1	x_2	x_3	r_1	r_2	x_4	Sol.
z	0	167	-100	-232	0	0	204
x_1	1	1/3	0	1/3	0	0	1
r_2	0	5/3	-1	-4/3	1	0	2
x_4	0	5/3	0	-1/3	0	1	3

Eventually we obtain: $x^* = (2/5, 9/5)$, $z^* = 17/5$. (Try yourself!)

The M-method



Feasible region
$3x_1 + x_2 = 3$
$4x_1 + 3x_2 \geq 6$
$x_1 + 2x_2 \leq 4$
$z = 3x_1 + 2x_2$
$x^* = (2/5, 9/5)$
$z^* = 17/5$
x_1
x_2
0
1
2
3
4

The two-phase method

Alternatively, the two-phase method does not need parametrisation. More often used in modern solvers (*crashing* in LP).

It uses an artificial objective function measuring infeasibility.

$$
\begin{aligned}
\text{min. } & z = r_1 + r_2 \\
\text{s.t.: } & 3x_1 + x_2 + r_1 = 3 \\
& 4x_1 + 3x_2 - x_3 + r_2 = 6 \\
& x_1 + 2x_2 + x_4 = 4 \\
& x_1, x_2, x_3, x_4, r_1, r_2 \geq 0
\end{aligned}
$$

- the 1st phase is always a minimisation problem.
- no coefficients required (no parametrisation).

The initial tableau: z is not a function of nonbasic variables.

	x_1	x_2	x_3	r_1	r_2	x_4	Sol.
z	0	0	0	-1	-1	0	0
r_1	3	1	0	1	0	0	3
r_2	4	3	-1	0	1	0	6
x_4	1	2	0	0	0	1	4

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& x_1, x_2, x_3, x_4, r_1, r_2 \geq 0
\end{aligned}
$$

- the 1st phase is always a minimisation problem.
- no coefficients required (no parametrisation).

The correct initial tableau: z is a function of nonbasic variables.

	x_1	x_2	x_3	r_1	r_2	x_4	Sol.	Operation
z	7	4	-1	0	0	0	9	$+R_{r_1} + R_{r_2}$
r_1	3	1	0	1	0	0	3	(R_{r_1})
r_2	4	3	-1	0	1	0	6	(R_{r_2})
x_4	1	2	0	0	0	1	4	

The two-phase method

A few iterations of simplex methods takes us from this tableau

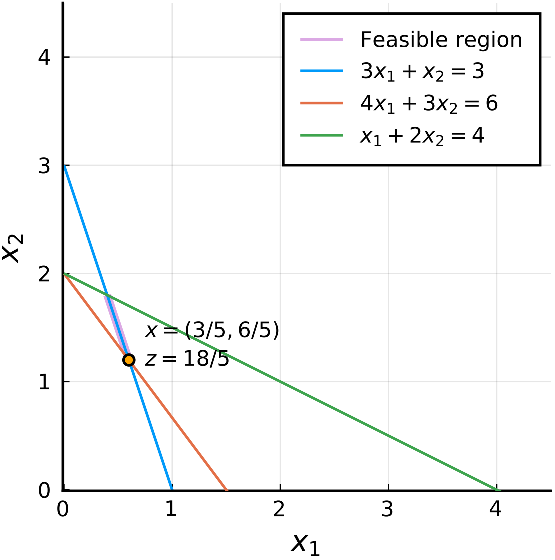
	x_1	x_2	x_3	r_1	r_2	x_4	Sol.
z	7	4	-1	0	0	0	9
r_1	3	1	0	1	0	0	3
r_2	4	3	-1	0	1	0	6
x_4	1	2	0	0	0	1	4

to this optimal tableau, in which the total infeasibility is zero.

	x_1	x_2	x_3	r_1	r_2	x_4	Sol.
z	0	0	0	-1	-1	0	0
x_1	1	0	1/5	3/5	-1/5	0	3/5
x_2	0	1	-3/5	-4/5	3/5	0	6/5
x_4	0	0	1	1	-1	1	1

As a basic feasible solution is available, the second phase proceeds.

The two-phase method



The two-phase method

The second phase consists of applying the simplex method from the basic feasible solution obtained from the first-phase.

- We can remove all artificial variables
- We reintroduce the objective function, rewriting it accordingly.

	x_1	x_2	x_3	r_1	r_2	x_4	Sol.
z	0	0	0	-1	-1	0	0
x_1	1	0	1/5	3/5	-1/5	0	3/5
x_2	0	1	-3/5	-4/5	3/5	0	6/5
x_4	0	0	1	1	-1	1	1

The initial tableau: z is not a function of nonbasic variables.

	x_1	x_2	x_3	x_4	Sol.
z	-4	-1	0	0	0
x_1	1	0	1/5	0	3/5
x_2	0	1	-3/5	0	6/5
x_4	0	0	1	1	1

The two-phase method

The second phase consists of applying the simplex method from the basic feasible solution obtained from the first-phase.

- We can remove all artificial variables
- We reintroduce the objective function, rewriting it accordingly.

	x_1	x_2	x_3	r_1	r_2	x_4	Sol.
z	0	0	0	-1	-1	0	0
x_1	1	0	1/5	3/5	-1/5	0	3/5
x_2	0	1	-3/5	-4/5	3/5	0	6/5
x_4	0	0	1	1	-1	1	1

The correct initial tableau: z is a function of nonbasic variables.

	x_1	x_2	x_3	x_4	Sol.
z	0	0	1/5	0	18/5
x_1	1	0	1/5	0	3/5
x_2	0	1	-3/5	0	6/5
x_4	0	0	1	1	1

Applying the simplex method reaches the same solution as before.

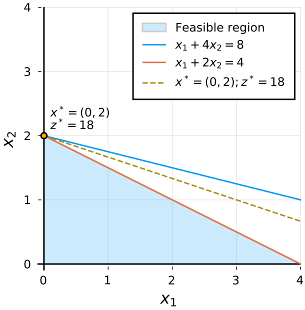
Special cases - degeneracy

Four special cases arise when using simplex method.

1. **Degeneracy.** Caused by an over-determination of a vertex.
   - More than $n - m$ hyperplanes form a vertex in $\mathbb{R}^{n-m}$.
   - It can be identified by ties in the smallest ratio test.
   - In this case, the choice of leaving variable is arbitrary and leads to a basic feasible solution with null basic variables.

Example:

$$\begin{aligned} \text{max. } & z = 3x_1 + 9x_2 \\ \text{s.t.: } & x_1 + 4x_2 \leq 8 \\ & x_1 + 2x_2 \leq 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$



Special cases - degeneracy

1. **Degeneracy.** Solving the example:

$$\begin{aligned} \text{max. } & z = 3x_1 + 9x_2 \\ \text{s.t.: } & x_1 + 4x_2 \leq 8 \\ & x_1 + 2x_2 \leq 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} z - 3x_1 - 9x_2 &= 0 \\ x_1 + 4x_2 + x_3 &= 8 \\ x_1 + 2x_2 + x_4 &= 4 \end{aligned}$$

The solution requires the following steps:

- x_3 is arbitrarily chosen (iter. 1), despite same ratio as x_4.
- Next, a basic variable is zero ($x_4 = 0$).
- Notice that point $(0, 2)$ is visited "twice".

	x_1	x_2	x_3	x_4	Sol.
z	-3	-9	0	0	0
x_3	1	4	1	0	8
x_4	1	2	0	1	4
z	-3/4	0	9/4	0	18
x_2	1/4	1	1/4	0	2
x_4	1/2	0	-1/2	1	0
z	0	0	3/2	3/2	18
x_2	0	1	1/2	-1/2	2
x_1	1	0	-1	2	0

Special cases - degeneracy/ alternative optima

1. **Degeneracy.** Some important remarks:
   1. Degeneracy can cause cycling. Simple rules (see Bland's rule, for example) can prevent it at the cost of performance.
   2. Modern codes interject conditional basis perturbation and shifting to prevent cycle.
   3. Degeneracy is a symptom of redundancy in model specification.
2. **Alternative optima.** An infinite number of optimal solutions.
   - Objective function is parallel to a binding (active) constraint.
   - Understandably, the method only visit the corner points of the "optimal hyperplane".

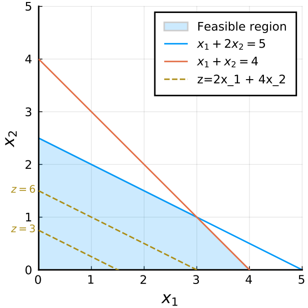
Special cases - alternative optima

Example:

$$\begin{aligned} \text{max. } & z = 2x_1 + 4x_2 \\ \text{s.t.: } & x_1 + 2x_2 \leq 5 \\ & x_1 + x_2 \leq 4 \\ & x_1, x_2 \geq 0. \end{aligned}$$

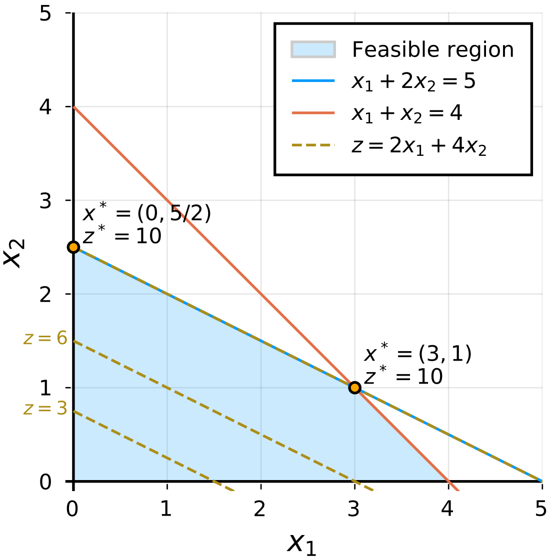
The solution is given by:

	x_1	x_2	x_3	x_4	Sol.
z	-2	-4	0	0	0
x_3	1	2	1	0	5
x_4	1	1	0	1	4
z	0	0	2	0	10
x_2	1/2	1	1/2	0	5/2
x_4	1/2	0	-1/2	1	3/2
z	0	0	2	0	10
x_2	0	1	1	-1	1
x_1	1	0	-1	2	3



- In iter. 2 a nonbasic variable has null coefficient that can be made basic without changing z^*.
- For $\lambda \in [0, 1]$, any $(x_1, x_2) = \lambda(0, 5/2) + (1 - \lambda)(3, 1)$ is optimal.

Special cases - alternative optima



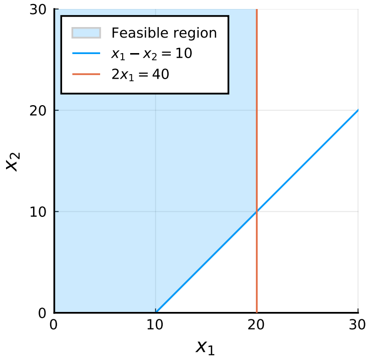
Special cases - unbounded problems

3. **Unboundedness.** Solution improvement is not constrained.
   - Typically a model specification issue.
   - Unbounded direction is called a extreme ray.

Example:

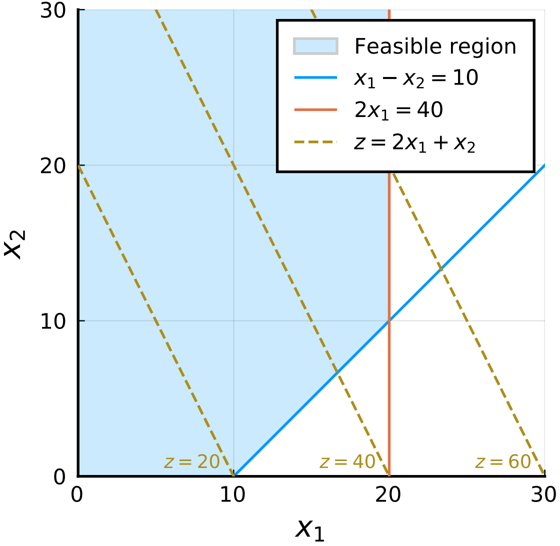
$$\begin{aligned} \text{max. } & z = 2x_1 + x_2 \\ \text{s.t.: } & x_1 - x_2 \leq 10 \\ & 2x_1 \leq 40 \\ & x_1, x_2 \geq 0. \end{aligned}$$

	x_1	x_2	x_3	x_4	Sol.
z	-2	-1	0	0	0
x_3	1	-1	1	0	10
x_4	2	0	0	1	40



- The ratio test "fails" for non-negative (≤ 0) coef's.
- Eventually, an entering variable would have non-negative coef's.

Special cases - alternative optima



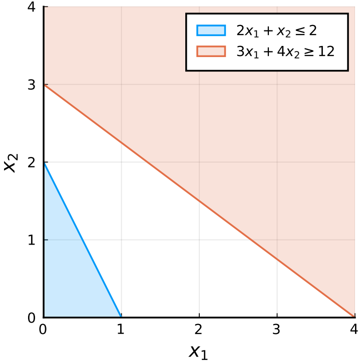
Special cases - infeasible problems

4. **Infeasibility.** Empty feasible space.

- Also often caused by poorly specified models.
- Does not occur if all constraints are $(\leq)$ with $b \geq 0$.
- Identifiable using two-phase of M-method (optimal basis containing artificial variables).

Example:

$$\begin{aligned} \text{max. } z &= 3x_1 + 2x_2 \\ \text{s.t.: } & 2x_1 + x_2 \leq 2 \\ & 3x_1 + 4x_2 \geq 12 \\ & x_1, x_2 \geq 0. \end{aligned}$$



Special cases - infeasible problems

Example:

$$\begin{aligned} \text{max. } & z = 3x_1 + 2x_2 \\ \text{s.t.: } & 2x_1 + x_2 \leq 2 \\ & 3x_1 + 4x_2 \geq 12 \\ & x_1, x_2 \geq 0. \end{aligned}$$

$$\begin{aligned} \text{min. } & z - r_1 = 0 \\ & 2x_1 + x_2 + x_3 = 2 \\ & 3x_1 + 4x_2 - x_4 + r_1 = 12 \end{aligned}$$

	x_1	x_2	x_3	x_4	r_1	Sol.
z	3	4	0	-1	0	12
x_3	2	1	1	0	0	2
r_1	3	4	0	-1	1	12
z	-5	0	-4	-1	0	4
x_2	2	1	1	0	0	2
r_1	-5	0	-1	-4	1	4

- In the optimal an artificial variable is basic.
- Thus, the minimal (optimal) infeasibility is not zero.