Exercise class 10

Learning Objectives:

- Karush-Kuhn-Tucker conditions

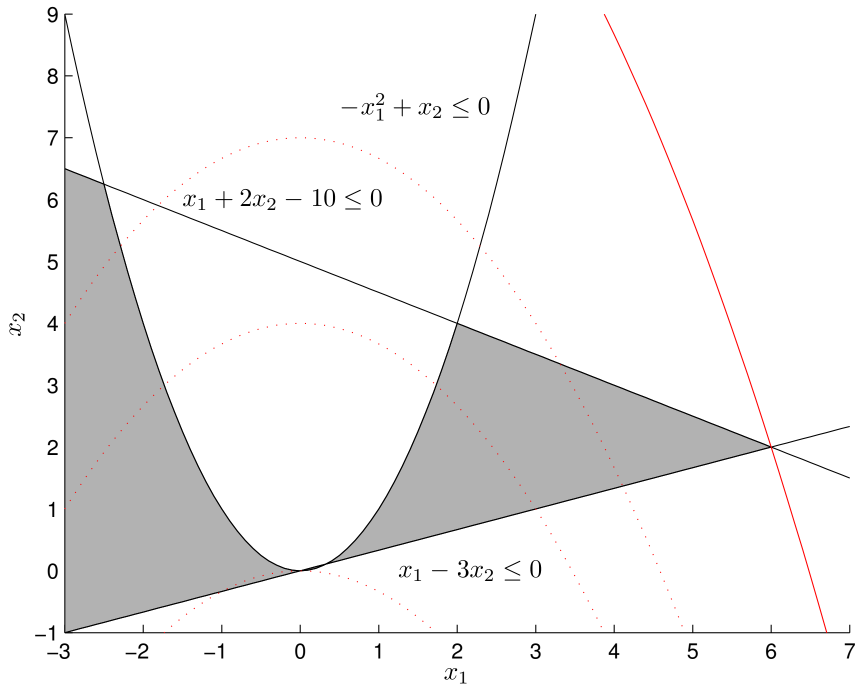
Demo 1: KKT conditions with inequality constraints

Using the Karush-Kuhn-Tucker conditions, see if the points $\mathbf{x} = (x_1, x_2) = (2, 4)$ or $\mathbf{x} = (x_1, x_2) = (6, 2)$ are the local optima of the problem. Can Slater's constraint qualification be used to assert that the KKT conditions are sufficient for global optimality?

$$\begin{array}{lrcrcrl} \max. & x_1^3 & + & 3x_2 & & & \\ \text{s.t.} & -x_1^2 & + & x_2 & & & \leq 0 \\ & x_1 & + & 2x_2 & - & 10 & \leq 0 \\ & x_1 & - & 3x_2 & & & \leq 0 \end{array}$$

Solution

The feasible set is illustrated in the following picture.



The KKT conditions are:

$$\nabla f(\bar{x}) + \sum_i \bar{\lambda}_i \nabla g_i(\bar{x}) = 0 \tag{1}$$

$$\lambda_i g_i(\bar{x}) = 0 \quad \forall i \tag{2}$$

$$\lambda_i \leq 0 \quad \forall i \tag{3}$$

where the function (1) is the derivation of the problem's Lagrange function.

Functions $g_i(\bar{x})$ are constraint functions and λ_i the respective Lagrange multipliers. Constraints are always of the form $g_i(\bar{x}) \leq 0$. Now the constraint functions $g_i(\bar{x})$ are:

$$\begin{aligned} g_1(\bar{x}) &= -x_1^2 + x_2 \\ g_2(\bar{x}) &= x_1 + 2x_2 - 10 \\ g_3(\bar{x}) &= x_1 - 3x_2 \end{aligned}$$

The conditions above are the KKT conditions for a maximization problem. For a minimization problem the conditions are similar, except the inequality of the last condition is reversed, i.e. $\lambda_i \geq 0 \quad \forall i$.

Let us first look at $\bar{x} = (2,4)$. We determine which constraints are active:

$$\begin{aligned} g_1(2,4) &= -2^2 + 4 = 0 \\ g_2(2,4) &= 2 + 2 \cdot 4 - 10 = 0 \\ g_3(2,4) &= 2 - 3 \cdot 4 = -10 \end{aligned}$$

Constraints g_1 and g_2 are active in the current point, i.e. they have the value 0. The third constraint on the other hand is not active. According to the third KKT condition, the Lagrange multipliers of inactive constraints are zero, therefore, $\lambda_3 = 0$.

The gradients of the objective function and constraint functions are:

$$\nabla f(\bar{x}) = \begin{bmatrix} 3x_1^2 \\ 3 \end{bmatrix} \quad \nabla g_1(\bar{x}) = \begin{bmatrix} -2x_1 \\ 1 \end{bmatrix} \quad \nabla g_2(\bar{x}) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \nabla g_3(\bar{x}) = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

The gradients at the point under examination:

$$\begin{aligned} \nabla L(2,4,\lambda_1,\lambda_2,\lambda_3) &= \nabla f(2,4) + \lambda_1 \nabla g_1(2,4) + \lambda_2 \nabla g_2(2,4) + \lambda_3 \nabla g_3(2,4) \\ &= \begin{bmatrix} 3 \cdot 2^2 \\ 3 \end{bmatrix} + \lambda_1 \begin{bmatrix} -2 \cdot 2 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \lambda_3 \begin{bmatrix} 1 \\ -3 \end{bmatrix} \\ &= \begin{bmatrix} 12 - 4\lambda_1 + \lambda_2 + \lambda_3 \\ 3 + \lambda_1 + 2\lambda_2 - 3\lambda_3 \end{bmatrix} \end{aligned}$$

Remember that $\lambda_3 = 0$. Now the first KKT condition ($\nabla L = 0$) yields:

$$\nabla L(2,4,\lambda_1,\lambda_2,0) = \begin{bmatrix} 12 - 4\lambda_1 + \lambda_2 \\ 3 + \lambda_1 + 2\lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From the lower equation we solve λ_1:

$$\lambda_1 = -2\lambda_2 - 3$$

Replace this into the upper equation:

$$12 - 4 \cdot (-2\lambda_2 - 3) + \lambda_2 = 0$$

Now we can solve

$$\lambda_2 = -\frac{8}{3},$$

and from the lower equation

$$\lambda_1 = \frac{7}{3}.$$

We have solved both Lagrange multipliers and λ_1 is positive. The KKT conditions require that at the optimum all multipliers are negative i.e. the KKT conditions are not satisfied at this point, thus the point (2,4) is not a local optimum.

Let us examine the other point: $\bar{x} = (6, 2)$. At this point the active constraints are:

$$g_1(6,2) = -6^2 + 2 = -34$$

$$g_2(6,2) = 6 + 2 \cdot 2 - 10 = 0$$

$$g_3(6,2) = 6 - 3 \cdot 2 = 0$$

Thus, the active constraints are constraints 2 and 3. The first constraint is not active ($g_1(6,2) \neq 0$), so $\lambda_1 = 0$. We place the point $\mathbf{x} = (6, 2)$ into the Lagrange function's gradient and require it to be zero:

$$\nabla L(6, 2, 0, \lambda_2, \lambda_3) = \begin{bmatrix} 3 \cdot 6^2 + \lambda_2 + \lambda_3 \\ 3 + 2\lambda_2 - 3\lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Now we can solve the Lagrange multipliers, which are:

$$\lambda_2 = -65.4 \text{ and } \lambda_3 = -42.6.$$

Both multipliers are negative and thus the KKT conditions are satisfied at this point and it is a optimum candidate.

Can Slater's constraint qualification be used to assert global optimiality?
From Slide 17 of Lecture 10: Slater's constraint qualifications are:

1. f convex funtion (concave for max.)
2. g convex functions with strict interior (i.e. $\exists\ x$ s.t. $g(x) < 0$)
3. h affine functions

Slater's constraint qualification does not hold, since the first constraint $-x_1^2 + x_2 \leq 0$ is not convex, thus it cannot be used to support sufficiency of the KKT conditions.

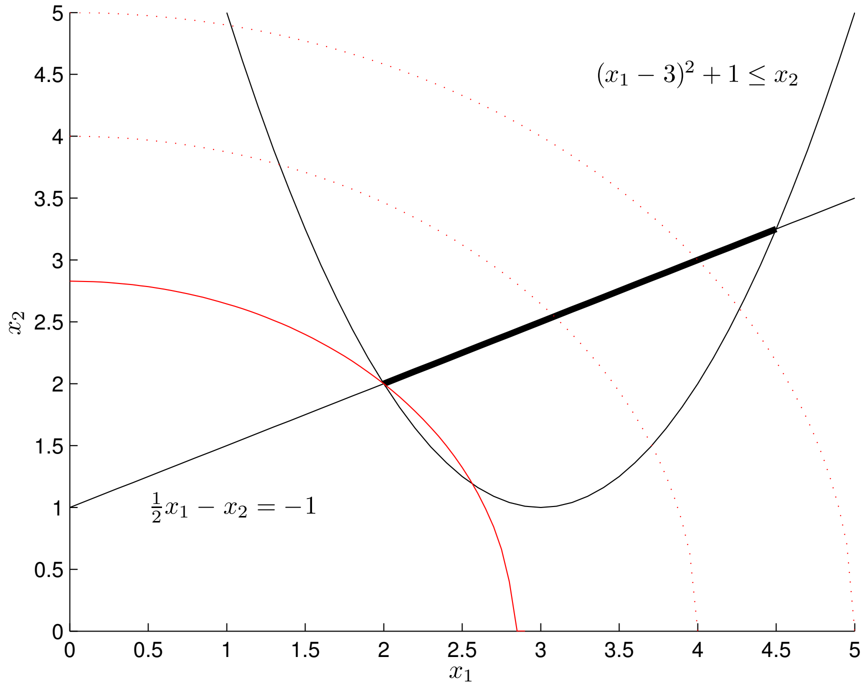
Demo 2: KKT conditions with equality and inequality constraints

Solve the problem graphically and verify that the optimal point satisfies the KKT conditions.

$$\begin{array}{llllll} \min. & x_1^2 & + & x_2^2 & & \\ \text{s.t.} & (x_1 & - & 3)^2 & + \; 1 & \leq x_2 \\ & & & \frac{1}{2}x_1 & - \; x_2 & = -1 \end{array}$$

Solution

Use Julia to plot the constraints. The feasible area is illustrated in the following figure.



In the problem we minimize the distance to the origin. The optimum is thus the nearest point of the feasible set, $\boldsymbol{x} = (2, 2)$.

The KKT conditions for a minimisation problem are:

$$\begin{aligned} L(\boldsymbol{x}, \lambda, \mu) &= f(\boldsymbol{x}) + \lambda g(\boldsymbol{x}) + \mu h(\boldsymbol{x}) \\ \nabla L &= 0 \\ \lambda g(\boldsymbol{x}) &= 0 \quad \forall i \\ \lambda_i &\geq 0 \quad \forall i \end{aligned}$$

The equality constraints' Lagrange multipliers do not have similar positivity or negativity constraints as the inequality constraints. The equality constraints' Lagrange multipliers are often denoted as μ_i.

The constraint functions $g(\boldsymbol{x})$ and $h(\boldsymbol{x})$ are

$$g(\boldsymbol{x}) = (x_1 - 3)^2 - x_2 + 1$$

$$h(\boldsymbol{x}) = \frac{1}{2}x_1 - x_2 + 1$$

We determine if the inequality constraint is active.

$$g(2, 2) = (2 - 3)^2 + 1 - 2 = 0$$

The constraint is active and thus the respective Lagrange multiplier can be non-zero. We solve the point in which the gradient of the Lagrange function is zero. For this, we first need to determine the gradients of the objective and constraint functions:

$$\nabla f(\boldsymbol{x}) = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} \quad \nabla g(\boldsymbol{x}) = \begin{bmatrix} 2(x_1 - 3) \\ -1 \end{bmatrix} \quad \nabla h(\boldsymbol{x}) = \begin{bmatrix} \frac{1}{2} \\ -1 \end{bmatrix}$$

The gradient of the Lagrange function is zero in the optimum:

$$\begin{aligned} \nabla L(2,2,\lambda,\mu) &= \nabla f(2,2) + \lambda \nabla g(2,2) + \mu \nabla h(2,2) \\ &= \begin{bmatrix} 2 \cdot 2 + 2(2-3)\lambda + \frac{1}{2}\mu \\ 2 \cdot 2 - \lambda - \mu \end{bmatrix} \\ &= \begin{bmatrix} 4 - 2\lambda + \frac{1}{2}\mu \\ 4 - \lambda - \mu \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

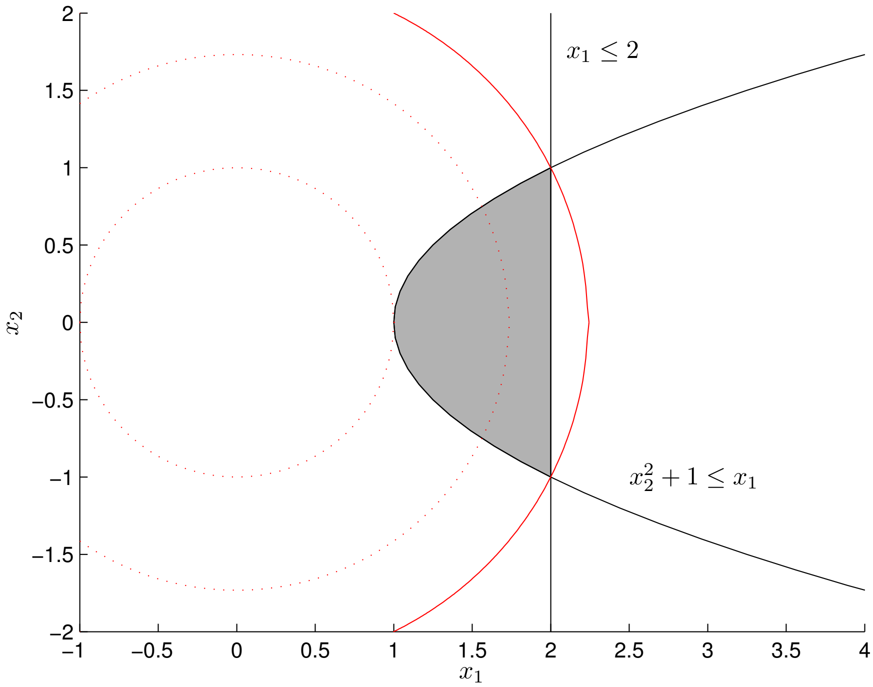
We can solve the equations e.g. by first solving λ from the second equation and then replacing it into the first. This yields $\lambda = \frac{12}{5}$ and $\mu = \frac{8}{5}$. The Lagrange multiplier of the inequality is positive, so point (2,2) satisfies the necessary KKT conditions of the minimization problem.

Problem 1: KKT conditions with inequality constraints

Using the Karush-Kuhn-Tucker conditions, see if the point $\mathbf{x} = (x_1, x_2) = (2, -1)$ is the optimum of the following problem.

$$\begin{array}{llll} \text{max.} & x_1^2 & + & x_2^2 \\ \text{s.t.} & x_2^2 & + & 1 & \leq x_1 \\ & & & x_1 & \leq 2 \end{array}$$

Solution



The constraint functions are:

$$g_1(\boldsymbol{x}) = x_2^2 + 1 - x_1$$

$$g_2(\boldsymbol{x}) = x_1 - 2$$

We determine the active constraints at the point:

$$g_1(2, -1) = (-1)^2 + 1 - 2 = 0$$

$$g_2(2, -1) = 2 - 2 = 0$$

Thus, both constraints are active. The gradients of the objective and constraint functions are:

$$\nabla f(\boldsymbol{x}) = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} \quad \nabla g_1(\boldsymbol{x}) = \begin{bmatrix} -1 \\ 2x_2 \end{bmatrix} \quad \nabla g_2(\boldsymbol{x}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

We write the gradient of the Lagrange function at point (2,-1):

$$\nabla L(2, -1, \lambda_1, \lambda_2) = \nabla f(2, -1) + \lambda_1 \nabla g_1(2, -1) + \lambda_2 \nabla g_2(2, -1)$$

We determine if the point (2,-1) is the zero point of the Lagrange function's gradient.

$$\nabla L(2, -1, \lambda_1, \lambda_2) = \begin{bmatrix} 2 \cdot 2 + \lambda_1 \cdot (-1) + \lambda_2 \cdot 1 \\ 2 \cdot (-1) + \lambda_1 \cdot 2 \cdot (-1) + \lambda_2 \cdot 0 \end{bmatrix} = \begin{bmatrix} 4 - \lambda_1 + \lambda_2 \\ -2 - 2\lambda_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

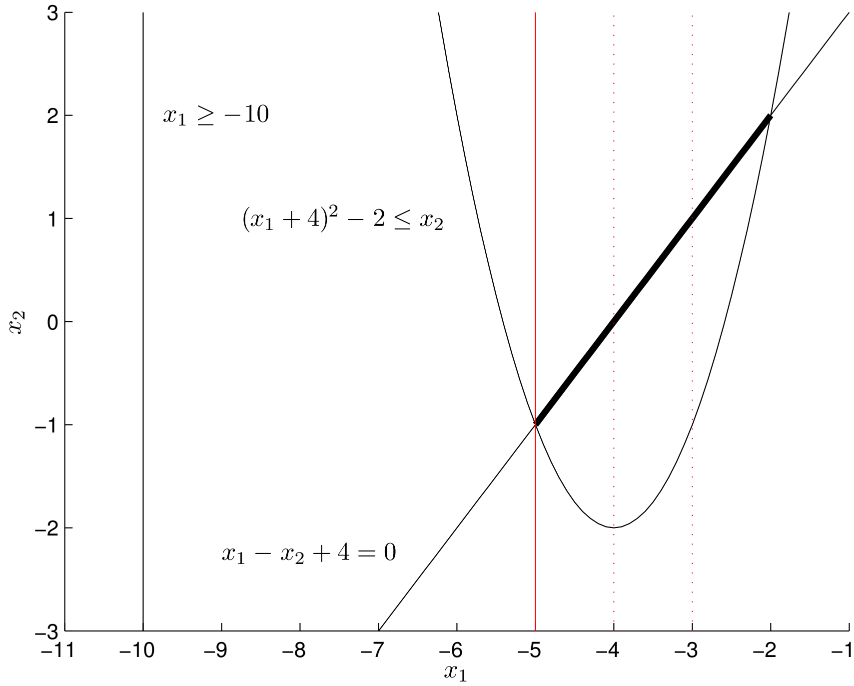
Now λ_1 can be solved from the lower equation: $\lambda_1 = -1$. Replacing this into the upper equation yields $\lambda_2 = -5$. In a maximization problem the KKT conditions require that the Lagrange multipliers are negative. Both the multipliers are negative and so the point satisfies the KKT conditions. Thus, the point (2,-1) is a local optimum candidate.

Problem 2: KKT conditions with equality and inequality constraints

Solve the following problem and see if the solution satisfies the KKT conditions.

$$\begin{array}{llllll} \text{min.} & x_1 & & & & \\ \text{s.t.} & (x_1 & + & 4)^2 & - & 2 \leq x_2 \\ & x_1 & - & x_2 & + & 4 = 0 \\ & & & & & x_1 \geq -10 \end{array}$$

Solution



The constraint functions are:

$$g_1(\boldsymbol{x}) = (x_1 + 4)^2 - 2 - x_2$$

$$g_2(\boldsymbol{x}) = -x_1 - 10$$

$$h(\boldsymbol{x}) = x_1 - x_2 + 4$$

From the picture we see that the optimum is at (-5,-1). Let's determine if it satifies the KKT conditions. The inequality constraints at (-5,-1):

$$g_1(-5,-1) = (-5+4)^2 - 2 - (-1) = 0$$

$$g_2(-5,-1) = -(-5) - 10 = -5$$

Constrain 1 is active, but the other inequality constraint is not. This means that the Lagrange multiplier of the second inequality constraint is zero, $\lambda_2 = 0$. Next we see if the point is the zero point of the gradient of the Lagrange function. The gradients of the objective and constraint functions are:

$$\nabla f(\boldsymbol{x}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \nabla g_1(\boldsymbol{x}) = \begin{bmatrix} 2(x_1+4) \\ -1 \end{bmatrix} \quad \nabla h(\boldsymbol{x}) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \nabla g_2(\boldsymbol{x}) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

The gradient of the Lagrange function is zero:

$$\nabla L(-5,-1,\lambda_1,\mu,\lambda_2) = \nabla f(-5,-1) + \lambda_1 \nabla g_1(-5,-1) + \mu \nabla h(-5,-1) + \lambda_2 \nabla g_2(-5,-1)$$

$$= \begin{bmatrix} 1 + \lambda_1 \cdot 2(-5+4) + \mu \cdot 1 \\ 0 + \lambda_1 \cdot (-1) + \mu \cdot (-1) \end{bmatrix} = \begin{bmatrix} 1 - 2\lambda_1 + \mu \\ -\lambda_1 - \mu \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From the lower equation we get $\lambda_1 = -\mu$, which can be replaced in the upper equation. This yields $1 + 2\mu + \mu = 0$, i.e. $\mu = -\frac{1}{3}$. Furthermore, $\lambda_1 = -\mu = \frac{1}{3}$. The problem is a minimization problem and the KKT conditions require that the Lagrange multipliers for inequality constraints are positive. This is now the case and the point satisfies the KKT conditions.

Problem 3: Linear Programming Problem

$$\begin{array}{llll} \text{max.} & x_1 & + \; x_2 & \\ \text{s.t.} & x_1 & + \; 3x_2 & \leq 9 \\ & 2x_1 & + \; x_2 & \leq 8 \\ & x_1 \geq 0, & x_2 \geq 0 & \end{array}$$

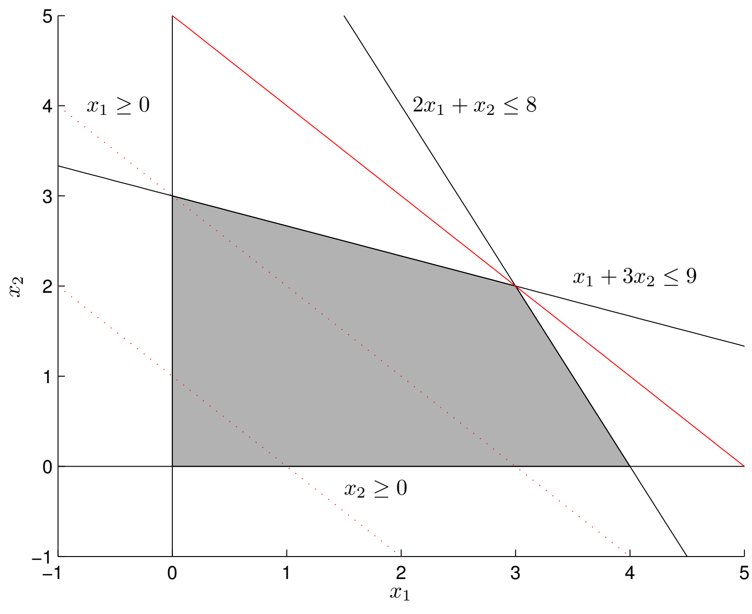
a) Solve the problem graphically and determine if the solution satisfies the KKT conditions.

b) Find the dual of the optimization problem and solve it (either graphically or in Julia).

c) Compare the solution of the dual and the Lagrange multipliers of the primal problem.

Solution

a) The feasible region is illustrated in the following figure:



The figure shows that the optimum is at the point (3,2). The constraint functions are:

$$
\begin{aligned}
g_1(\boldsymbol{x}) &= x_1 + 3x_2 - 9 \\
g_2(\boldsymbol{x}) &= 2x_1 + x_2 - 8 \\
g_3(\boldsymbol{x}) &= -x_1 \\
g_4(\boldsymbol{x}) &= -x_2
\end{aligned}
$$

Determine the active constraints:

$$
\begin{aligned}
g_1(3,2) &= 3 + 3 \cdot 2 - 9 = 0 \\
g_2(3,2) &= 2 \cdot 3 + 2 - 8 = 0 \\
g_3(3,2) &= -3 \\
g_4(3,2) &= -2
\end{aligned}
$$

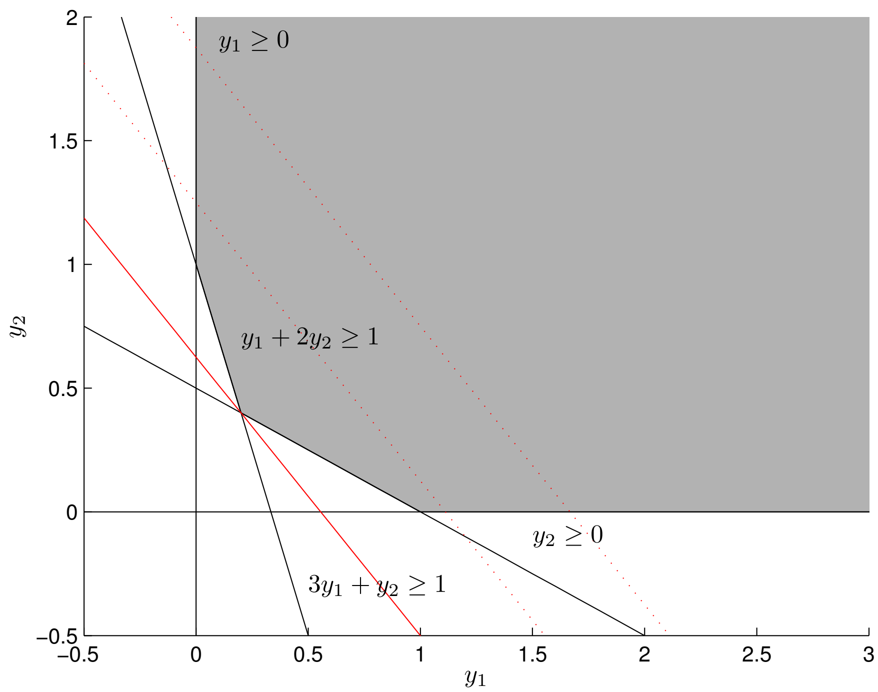
Constraints 1 and 2 are active, constraints 3 and 4 are not. Thus, $\lambda_3 = \lambda_4 = 0$ and we can leave the respective terms out of the Lagrange function. The gradient of the Lagrange function is zero:

$$
\begin{aligned}
\nabla L(3,2,\lambda_1,\lambda_2) &= \nabla f(3,2) + \lambda_1 \nabla g_1(3,2) + \lambda_2 \nabla g_2(3,2) \\
&= \begin{bmatrix} 1 + \lambda_1 + 2\lambda_2 \\ 1 + 3\lambda_1 + \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

The solution of the equations is $\lambda_1 = -\frac{1}{5}$ and $\lambda_2 = -\frac{2}{5}$. The multipliers are negative and the the problem is a maximization problem. Thus, the KKT conditions are satisfied.

b) The dual of the problem is:

$$
\begin{array}{llll}
\text{min.} & 9y_1 & + \; 8y_2 & \\
\text{s.t.} & y_1 & + \; 2y_2 & \geq 1 \\
 & 3y_1 & + \; y_2 & \geq 1 \\
 & y_1 \geq 0, & y_2 \geq 0 &
\end{array}
$$



The dual can be solved graphically or using Julia. The solution is $\boldsymbol{y} = (\frac{1}{5}, \frac{2}{5})$.

c) The Lagrange multipliers are $\lambda_1 = -\frac{1}{5}$ and $\lambda_2 = -\frac{2}{5}$, as the solution of the dual is $y_1 = \frac{1}{5}$ and $y_2 = \frac{2}{5}$.

The Lagrange multipliers are the negative of the dual variables. Alternatively, the Lagrange function can be defined as $L(x, \lambda, \mu) = f(x) - \lambda g(x) + \mu h(x)$ in maximisation problems ($\lambda >= 0$), and thus the dual variables and Lagrange multipliers have the same sign.

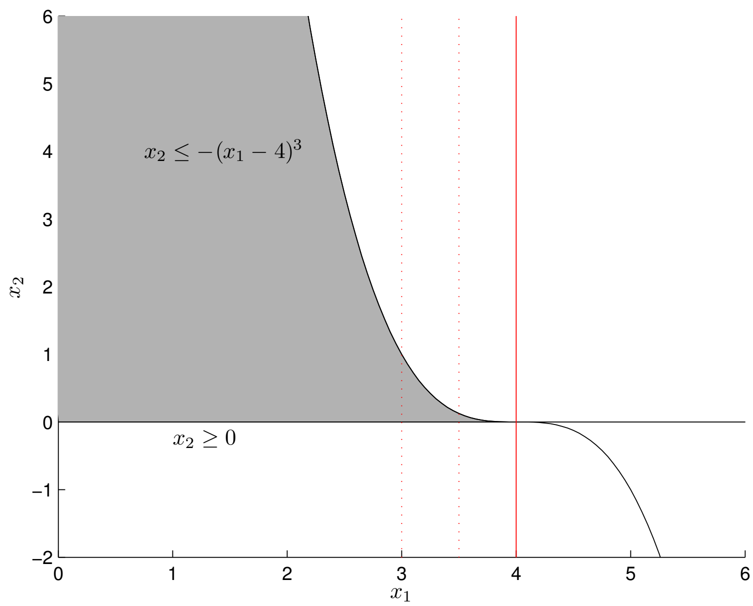
Problem 4: Graphical nonlinear problem & KKT

Solve the problem graphically (you can use Julia to help you plot) and see if it satisfies the KKT conditions. If not, explain why?

$$\begin{aligned} \text{max.} \quad & x_1 \\ \text{s.t.} \quad & x_2 \leq -(x_1 - 4)^3 \\ & x_2 \geq 0 \end{aligned}$$

Solution

The feasible region is illustrated in the following figure:



The optimum is at $(x_1, x_2) = (4, 0)$. Let us determine if the point satisfies the KKT conditions. The constraint functions are

$$g_1(\boldsymbol{x}) = x_2 + (x_1 - 4)^3$$

$$g_2(\boldsymbol{x}) = -x_2$$

Both constraints are active at the optimum so the respective Lagrange multipliers can be non-zero. The gradients of the objective and constraint functions are:

$$\nabla f(\boldsymbol{x}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \nabla g_1(\boldsymbol{x}) = \begin{bmatrix} 3(x_1 - 4)^2 \\ 1 \end{bmatrix} \quad \nabla g_2(\boldsymbol{x}) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

The gradient of the Lagrange function is zero:

$$\nabla L(4, 0, \lambda_1, \lambda_2) = \nabla f(4, 0) + \lambda_1 \nabla g_1(4, 0) + \lambda_2 \nabla g_2(4, 0)$$

$$= \begin{bmatrix} 1 + \lambda_1 \cdot 3(4 - 4)^2 + \lambda_2 \cdot 0 \\ 0 + \lambda_1 \cdot 1 + \lambda_2 \cdot (-1) \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda_1 - \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The first equation cannot be satisfied with any values of the multipliers λ_1 and λ_2. The point is clearly the optimum, so what went wrong?

The KKT conditions are the necessary conditions, assuming the gradients of the constraint functions are linearly independent. Now the gradients at the optimum are

$$\nabla g_1(4,0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \nabla g_2(4,0) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

i.e. $g_1(4,0) = -g_2(4,0)$. The gradients are linearly dependent and thus the optimum does not need to satisfy the KKT conditions. Note that this point is still the optimum.

Usually one does not need to check the linear dependence of the constraints.

Problem 5: Constrained optimisation

Maximise the (Euclidean) distance from the point (1,-1) in the region constrained by the following set of constraints:

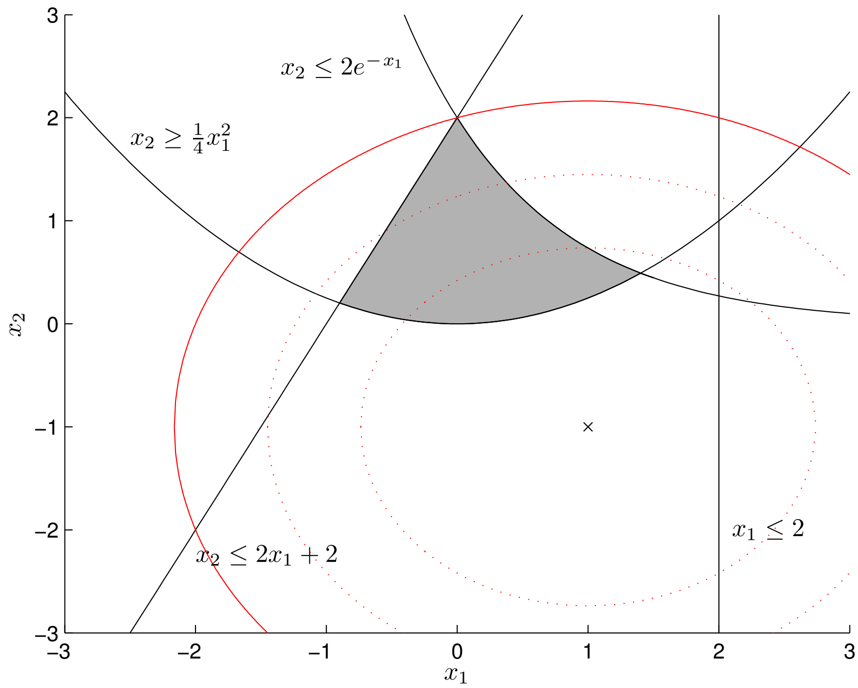
$$\begin{aligned} x_2 &\le 2e^{-x_1} \\ x_1 &\le 2 \\ x_2 &\ge \tfrac{1}{4}x_1^2 \\ x_2 &\le 2x_1 + 2 \end{aligned}$$

Do the KKT conditions hold at this point?

Hint. You can use Julia to graph the region and identify the optimum.

Solution

The constrained region is as follows with point (1,-1) marked by an 'x':



We want to maximise the distance from the point. The optimisation problem for this is:

$$\begin{array}{ll} \max. & (x_1-1)^2+(x_2+1)^2 \\ \text{s.t.} & x_2 \le 2e^{-x_1} \\ & x_1 \le 2 \\ & x_2 \ge \frac{1}{4}x_1^2 \\ & x_2 \le 2x_1+2 \end{array}$$

From the figure we can see that the maximum distance is reached at the point (0,2). Let's determine whether this point satisfies the KKT conditions. The constraint functions are:

$$\begin{aligned} g_1(\boldsymbol{x}) &= x_2 - 2e^{-x_1} \\ g_2(\boldsymbol{x}) &= x_1 - 2 \\ g_3(\boldsymbol{x}) &= \frac{1}{4}x_1^2 - x_2 \\ g_4(\boldsymbol{x}) &= -2x_1 + x_2 - 2 \end{aligned}$$

The active constraints:

$$\begin{aligned} g_1(0,2) &= 2-2e^{-0} = 0 \\ g_2(0,2) &= 0-2 = -2 \\ g_3(0,2) &= \frac{1}{4}0^2 - 2 = -2 \\ g_4(0,2) &= -2\cdot 0 + 2 - 2 = 0 \end{aligned}$$

Constraints 2 and 3 are not active, so the respective multipliers are zero: $\lambda_2 = \lambda_3 = 0$. The gradients of the objective and constraint functions:

$$\nabla f(\boldsymbol{x}) = \begin{bmatrix} 2(x_1-1) \\ 2(x_2+4) \end{bmatrix} \quad \nabla g_1(\boldsymbol{x}) = \begin{bmatrix} 2e^{-x_1} \\ 1 \end{bmatrix} \quad \nabla g_2(\boldsymbol{x}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\nabla g_3(\boldsymbol{x}) = \begin{bmatrix} \frac{1}{2}x_1 \\ -1 \end{bmatrix} \quad \nabla g_4(\boldsymbol{x}) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

The gradient of the Lagrange function is zero:

$$\begin{aligned} \nabla L(0,2,\lambda_1,\lambda_2,\lambda_3,\lambda_4) &= \nabla f(0,2)+\lambda_1\nabla g_1(0,2)+\lambda_2\nabla g_2(0,2)+\lambda_3\nabla g_3(0,2)+\lambda_4\nabla g_4(0,2) \\ &= \begin{bmatrix} 2(0-1)+\lambda_1\cdot 2e^{-0}+0\cdot 1+0\cdot\frac{1}{2}\cdot 0+\lambda_4\cdot(-2) \\ 2(2+1)+\lambda_1\cdot 1+0\cdot 0+0\cdot(-1)+\lambda_4\cdot 1 \end{bmatrix} \\ &= \begin{bmatrix} -2+2\lambda_1-2\lambda_4 \\ 6+\lambda_1+\lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

The solution to this is $\lambda_1 = -\frac{5}{2}$ and $\lambda_4 = -\frac{7}{2}$. Both Lagrange multipliers are negative, so the KKT conditions are satisfied.

Home Exercise 10: KKT conditions

Find a solution satisfying the KKT conditions for the problem below.

$$\begin{array}{lrl} \max. & x_1+2x_2 & \\ \text{s.t.} & (x_1-3)^2+(x_2-3)^2 & = 4 \\ & x_1^2-10x_1+26-x_2 & \ge 0 \\ & x_2 & \ge -7 \end{array}$$

Solution

$$
\begin{aligned}
&\nabla f(\bar{x}) + \sum_{i=1} \bar{\lambda}_i \nabla g_i(\bar{x}) + \sum_{i=1} \bar{\mu}_i \nabla h_i(\bar{x}) = 0 \\
&g_i(\bar{x}) \leq 0 \\
&h_i(\bar{x}) = 0 \\
&\lambda_i g_i(\bar{x}) = 0 \\
&\lambda_i \leq 0
\end{aligned}
$$

$$
\begin{aligned}
&g_1(\bar{x}) = -(x_1^2 - 10x_1 + 26 - x_2) \\
&g_2(\bar{x}) = -(x_2 + 7) \\
&h_1(\bar{x}) = (x_1 - 3)^2 + (x_2 - 3)^2 - 4
\end{aligned}
$$

Plotting the graphical representation of the problem, we can identify that $\bar{x} = (3, 5)$ is the optimum. Checking to see which constraints are active (=0) at this point:

$$
\begin{aligned}
g_1(3, 5) &= 0 \\
g_2(3, 5) &\neq 0 \\
h_1(3, 5) &= 0
\end{aligned}
$$

Therefore, we have:

$$
\nabla L(\bar{x}, \lambda, \mu) = \nabla f(\bar{x}) + \lambda \nabla g_1(\bar{x}) + \mu \nabla h_1(\bar{x})
$$

$$
\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 10 - 2x_1 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} 2x_1 - 6 \\ 2x_2 - 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 + 4\lambda \\ 2 + \lambda + 4\mu \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

$\implies \lambda = -1/4, \quad \mu = -7/16$. Both λ and μ are negative therefore, KKT conditions are satisfied.