Exercise class 4

Learning Objectives:

- Simplex method: special cases

Demo 1: The simplex M-method

Solve the linear problem using the M-method.

$$
\begin{aligned}
\text{min.} \quad & 4x_1 + 6x_2 \\
\text{s.t.} \quad & x_1 + x_2 \geq 5, \\
& 3x_1 + 8x_2 \geq 24, \\
& x_1 \geq 0,\, x_2 \geq 0.
\end{aligned}
$$

Solution

The M-method is used when the ordinary initial solution $x_i = 0$, $\forall i$, is not-feasible. This happens often if the problem has equality or $\geq$ constraints.

M-method steps

1. Transform the problem into standard form.
2. Add additional variables (R_i) and the penalty terms (M) to the objective function.
3. Write the Simplex tableau.
4. Fix inconsistencies in the z-row.
5. Solve as a ordinary Simplex problem.

First, let's transform the problem into the standard form by adding surplus variables s_1 and s_2 and changing the minimisation into a maximisation $min(f(x)) = -max(-f(x))$

$$
\begin{aligned}
\text{max.} \quad & -4x_1 - 6x_2 \\
\text{s.t.} \quad & x_1 + x_2 - s_1 = 5, \\
& 3x_1 + 8x_2 - s_2 = 24, \\
& x_1 \geq 0,\, x_2 \geq 0,\, s_1 \geq 0,\, s_2 \geq 0.
\end{aligned}
$$

Now the ordinary initial solution ($x_1 = x_2 = 0$) is not feasible for any positive values of s_1 and s_2. Therefore, we add one more variable, R_i, to each constraint. The value of these variables should be zero in the final solution, because now they represent how much each constraint has to be violated. By adding both variables to the objective function with a large coefficient M the optimisation will force the R_i terms to be zero in the optimum. The problem is now of the form:

$$
\begin{aligned}
\max_{x}. \quad & -4x_1 - 6x_2 - MR_1 - MR_2 \\
\text{s.t.} \quad & x_1 + x_2 - s_1 + R_1 = 5, \\
& 3x_1 + 8x_2 - s_2 + R_2 = 24, \\
& x_1 \geq 0,\ x_2 \geq 0,\ s_1 \geq 0,\ s_2 \geq 0,\ R_1 \geq 0,\ R_2 \geq 0.
\end{aligned}
$$

Now we can choose $x_1 = x_2 = s_1 = s_2 = 0$ as a feasible basic solution and write the Simplex tableau of the problem. Set M large and positive, e.g. 20.

	x_1	x_2	s_1	s_2	R_1	R_2	Sol.
z	4	6	0	0	20	20	0
R_1	1	1	-1	0	1	0	5
R_2	3	8	0	-1	0	1	24

The tableau has a inconsistency: The columns of the basic variables should be zero except for the row representing the variable itself should be 1. Now the objective row has the non-zero coefficient of -20 for both of the basic variables. This can be corrected with:

$$\text{New objective row} = \text{Old objective row} - M \cdot R_1\text{-row} - M \cdot R_2\text{-row},$$

which yields the Simplex tableau:

↓ Pivot column

	x_1	x_2	s_1	s_2	R_1	R_2	Sol.	
z	-76	-174	20	20	0	0	-580	
R_1	1	1	-1	0	1	0	5	
R_2	3	8	0	-1	0	1	24	← Pivot row

Now we can begin the actual Simplex iterations. Let's choose the lowest coefficient of the objective row (steepest descent), which is now the coefficient for x_2: -174. Thus, the entering variable is x_2. Next calculate the feasibility condition, i.e. the quotient of the result and pivot columns:

	Quotient
R_1	$5/1 = 5$
R_2	$24/8 = 3$

The smallest positive value is 3, so we select R_2 as the exiting variable. The next Simplex tableau can be solved with Gauss elimination:

↓ Pivot column

	x_1	x_2	s_1	s_2	R_1	R_2	Sol.	
z	-10.75	0	20	-1.75	0	21.75	-58	
R_1	0.625	0	-1	0.125	1	-0.125	2	← Pivot row
x_2	0.375	1	0	-0.125	0	0.125	3	

The objective row still has negative coefficients so we have to start another Simplex iteration. The variable x_1 has the lowest coefficient, so it is the next entering variable. The feasibility condition is:

	Qutient
R_1	$2/(0.625) = 3.2$
x_2	$3/(0.375) = 8$

The smallest positive is 3.2, i.e. we choose R_1 to be the next exiting variable. With Gauss elimination: The optimal tableau is:

	x_1	x_2	s_1	s_2	R_1	R_2	Result
Objective	0	0	2.8	0.4	17.2	19.6	-23.6
x_1	1	0	-1.6	0.2	1.6	-0.2	3.2
x_2	0	1	0.6	-0.2	-0.6	0.2	1.8

Now all the objective coefficients are positive and we have reached the optimum. The values of the variables are $x_1 = 3.2$ and $x_2 = 1.8$, and the objective function get the value -23.6. Additional variables R_1 and R_2 are not in the set of basic variables and their values are zero, as they should be. Remember, that we were solving the maximization problem, so the result of initial problem is $-(-23.6) = 23.6$

Demo 2: The simplex 2-Phase method

Solve the linear problem using the 2-Phase method.

$$\begin{aligned} \min. \quad & 4x_1 + 6x_2 \\ \text{s.t.} \quad & x_1 + x_2 \geq 5, \\ & 3x_1 + 8x_2 \geq 24, \\ & x_1 \geq 0,\ x_2 \geq 0. \end{aligned}$$

Solution

Phase 1:

1. Transform the problem into the standard form.
2. Form auxiliary problem and solve using Simplex method.
3. If the optimal cost of the auxiliary problem is non-zero, then the original LP is infeasible. Stop here if this is the case.
4. If all artificial variables are non-basic (=0), remove then and their corresponding columns. Proceed to Phase II.

Phase 2:

5. Reintroduce the standard form LP objective function to the z-row.
6. Fix inconsistencies in the z-row.
7. Solve using the Simplex method.

1.

$$\begin{aligned} \max. \quad & -4x_1 - 6x_2 \\ \text{s.t.} \quad & x_1 + x_2 - s1 = 5, \\ & 3x_1 + 8x_2 - s2 = 24, \\ & x_1 \geq 0,\ x_2 \geq 0,\ s_1 \geq 0,\ s_2 \geq 0. \end{aligned}$$

2. Note that the z's measure infeasibility.

$$\begin{aligned} \max. \quad & -z_1 - z_2 \\ \text{s.t.} \quad & x_1 + x_2 - s1 + z1 = 5, \\ & 3x_1 + 8x_2 - s2 + z2 = 24, \\ & x_1 \geq 0,\ x_2 \geq 0,\ s_1 \geq 0,\ s_2 \geq 0,\ z_1 \geq 0,\ z_2 \geq 0. \end{aligned}$$

The initial tableau is:

	x_1	x_2	s_1	s_2	z_1	z_2	Sol.
z	0	0	0	0	1	1	0
z_1	1	1	-1	0	1	0	5
z_2	3	8	0	-1	0	1	24

z is not a function of nonbasic variables. Thus, transform it:

	x_1	x_2	s_1	s_2	z_1	z_2	Sol.
z	-4	-9	1	1	0	0	-29
z_1	1	1	-1	0	1	0	5
z_2	3	8	0	-1	0	1	24

The optimal tableau of the phase I is:

	x_1	x_2	s_1	s_2	z_1	z_2	Sol.
z	0	0	0	0	1	1	0
x_1	1	0	-1.6	0.2	1.6	-0.2	3.2
x_2	0	1	0.6	-0.2	-0.6	0.2	1.8

3. and 4. All artificial variables are non-basic (=0), so can be removed and optimal cost coefficient is equal to 0, hence feasible.

5. Phase 2: We can remove all artificial variables and reintroduce the objective function, rewriting it accordingly (do not forget that we transformed the minimisation to a maximisation problem: min. $4x_1 + 6x_2 =$ -max. $-4x_1 - 6x_2$).

	x_1	x_2	s_1	s_2	Sol.
z	4	6	0	0	0
x_1	1	0	-1.6	0.2	3.2
x_2	0	1	0.6	-0.2	1.8

6. Fix inconsistencies: z-row is not a function of nonbasic variables. Thus, transform it:

	x_1	x_2	s_1	s_2	Sol.
z	0	0	2.8	0.4	-23.6
x_1	1	0	-1.6	0.2	3.2
x_2	0	1	0.6	-0.2	1.8

The coefficients on the objective row are positive, so we have reached the optimum - no need for step 7.

The values of the variables are $x_1 = 3.2$ and $x_2 = 1.8$. The value of the objective function is -(-23.6) = 23.6.

Problem 1: M-method

Solve the linear problem using the M-method.

$$\begin{aligned} \max. \quad & x_1 + x_2 \\ \text{s.t.} \quad & x_1 + x_2 = 7, \\ & x_1 + 4x_2 = 16, \\ & 3x_1 + 2x_2 = 18, \\ & x_1 \geq 0,\ x_2 \geq 0. \end{aligned}$$

Solution

First, let's transform the problem into standard form. Now all constraints are already equality constraints, but none of them are valid at the initial solution $x_1 = x_2 = 0$. We add additional variables R_i. The same variables must be added to the objective function in order to make the optimization force their values to zero. Here we use the value 20 for M.

$$\begin{aligned} \max. \quad & x_1 + x_2 - 20R_1 - 20R_2 - 20R_3 \\ \text{s.t.} \quad & x_1 + x_2 + R_1 = 7, \\ & x_1 + 4x_2 + R_2 = 16, \\ & 3x_1 + 2x_2 + R_3 = 18, \\ & x_1 \geq 0,\ x_2 \geq 0,\ R_1 \geq 0,\ R_2 \geq 0,\ R_3 \geq 0. \end{aligned}$$

The Simplex tableau is:

	x_1	x_2	R_1	R_2	R_3	Sol.
z	-1	-1	20	20	20	0
R_1	1	1	1	0	0	7
R_2	1	4	0	1	0	16
R_3	3	2	0	0	1	18

Let's transform the problem so all the basis vectors will have zero coefficients in the objective row:

	x_1	x_2	R_1	R_2	R_3	Sol.
z	-101	-141	0	0	0	-820
R_1	1	1	1	0	0	7
R_2	1	4	0	1	0	16
R_3	3	2	0	0	1	18

The optimal tableau is:

	x_1	x_2	R_1	R_2	R_3	Sol.
z	0	0	263/3	40/3	0	7
x_1	1	0	4/3	-1/3	0	4
x_2	0	1	-1/3	1/3	0	3
R_3	0	0	-10/3	1/3	1	0

The coefficients on the objective row are positive, so we have reached the optimum. We can also see that the problem is degenerate, because one of the basic variables (R_3) is zero. The optimal values of the variables are $x_1 = 4$ and $x_2 = 3$. The value of the objective function is 7.

Problem 2: 2-Phase method

Solve the linear problem using 2 phase method.

$$\begin{aligned} \max. \quad & x_1 + x_2 \\ \text{s.t.} \quad & x_1 + x_2 = 7, \\ & x_1 + 4x_2 = 16, \\ & 3x_1 + 2x_2 = 18, \\ & x_1 \geq 0,\ x_2 \geq 0. \end{aligned}$$

Solution

The 1-st phase problem:

$$\begin{aligned} \min. \quad & z_1 + z_2 + z_3 \\ \text{s.t.} \quad & x_1 + x_2 + z_1 = 7, \\ & x_1 + 4x_2 + z_2 = 16, \\ & 3x_1 + 2x_2 + z_3 = 18, \\ & x_1 \geq 0,\ x_2 \geq 0,\ z_1 \geq 0,\ z_2 \geq 0,\ z_3 \geq 0. \end{aligned}$$

The initial tableau is:

	x_1	x_2	z_1	z_2	z_3	Sol.
z	0	0	1	1	1	0
z_1	1	1	1	0	0	7
z_2	1	4	0	1	0	16
z_3	3	2	0	0	1	18

Let's transform the problem so all the basic variables will have zero coefficients in the objective row:

	x_1	x_2	z_1	z_2	z_3	Sol.
z	-5	-7	0	0	0	-41
z_1	1	1	1	0	0	7
z_2	1	4	0	1	0	16
z_3	3	2	0	0	1	18

The optimal tableau is:

	x_1	x_2	z_1	z_2	z_3	Sol.
z	0	0	4.333	0.667	0	0
x_1	1	0	1.333	-0.333	0	4
x_2	0	1	-0.333	0.333	0	3
z_3	0	0	-3.333	0.333	1	0

Phase 2: We can remove all artificial variables and reintroduce the objective function, rewriting it accordingly:

	x_1	x_2	z_3	Sol.
z	-1	-1	0	0
x_1	1	0	0	4
x_2	0	1	0	3
z_3	0	0	1	0

z is not a function of nonbasic variables. Thus, transform it:

	x_1	x_2	z_3	Sol.
z	0	0	0	7
x_1	1	0	0	4
x_2	0	1	0	3
z_3	0	0	1	0

The coefficients on the objective row are non negative (all the basic variables have zero coefficients), so we have reached the optimum. The values of the variables are $x_1 = 4$ and $x_2 = 3$. The value of z_3 is 0, which again demonstrates that the problem is over-determined. The value of the objective function is 7 as in the M-method.

Problem 3: M-method formulation

Consider the following set of constraints:

$$
\begin{align}
-2x_1 + 3x_2 &= 3 \tag{1}\\
4x_1 + 5x_2 &\geq 10 \tag{2}\\
x_1 + 2x_2 &\leq 5 \tag{3}\\
6x_1 + 7x_2 &\leq 3 \tag{4}\\
4x_1 + 8x_2 &\geq 5 \tag{5}\\
x_1,\, x_2 &\geq 0 \tag{6}
\end{align}
$$

For each of the following problems, develop the z-row after substituting out the artificial variables:

1. Maximise $z = 5x_1 + 6x_2$ subject to (1), (3), and (4).
2. Maximise $z = 2x_1 - 7x_2$ subject to (1), (2), (4), and (5).

3. Minimise $z = 3x_1 + 6x_2$ subject to (3), (4), and (5).
4. Minimise $z = 4x_1 + 6x_2$ subject to (1), (2), and (5).
5. Minimise $z = 3x_1 + 2x_2$ subject to (1) and (5).

Solution

1.

$$\begin{aligned} \max. \quad & 5x_1 + 6x_2 - MR_1 \\ \text{s.t.} \quad & -2x_1 + 3x_2 + R_1 = 3, \\ & x_1 + 2x_2 + s_1 = 5, \\ & 6x_1 + 7x_2 + s_2 = 3. \end{aligned}$$

Develop the z-row:

	x_1	x_2	s_1	s_2	$R-1$	Sol.
z	$2M-5$	$-3M-6$	0	0	0	$-3M$

The z-row can also be noted in the form of an equality, where z is an additional variable for the value of the objective:

$$z - (5 - 2M)x_1 - (6 + 3M)x_2 = -3M$$

2.

$$\begin{aligned} \max. \quad & 2x_1 - 7x_2 - M(R_1 + R_2 + R_3) \\ \text{s.t.} \quad & -2x_1 + 3x_2 + R_1 = 3, \\ & 4x_1 + 5x_2 - s_1 + R_2 = 10, \\ & 6x_1 + 7x_2 + s_2 = 3, \\ & 4x_1 + 8x_2 - s_3 + R_3 = 5. \end{aligned}$$

$$\text{z-row: } z - (2 + 6M)x_1 - (-7 + 16M)x_2 + Ms_1 + Ms_3 = -18M$$

3.

$$\begin{aligned} \min. \quad & 3x_1 + 6x_2 + MR_1 \\ \text{s.t.} \quad & x_1 + 2x_2 + s_1 = 5, \\ & 6x_1 + 7x_2 + s_2 = 3, \\ & 4x_1 + 8x_2 - s_3 + R_1 = 5. \end{aligned}$$

$$\text{z-row: } z - (3 - 4M)x_1 - (6 - 8M)x_2 - Ms_3 = 5M$$

4.

$$\begin{aligned} \min. \quad & 4x_1 + 6x_2 + M(R_1 + R_2 + R_3) \\ \text{s.t.} \quad & -2x_1 + 3x_2 + R_1 = 3, \\ & 4x_1 + 5x_2 - s_1 + R_2 = 10, \\ & 4x_1 + 8x_2 - s_2 + R_3 = 5. \end{aligned}$$

$$\text{z-row: } z - (4 - 6M)x_1 - (6 - 16M)x_2 - Ms_1 - Ms_2 = 18M$$

5.

$$\begin{aligned} \min. \quad & 3x_1 + 2x_2 + M(R_1 + R_2) \\ \text{s.t.} \quad & -2x_1 + 3x_2 + R_1 = 3, \\ & 4x_1 + 8x_2 - s_1 + R_2 = 5. \end{aligned}$$

$$\text{z-row: } z - (3 - 2M)x_1 - (2 - 11M)x_2 - Ms_1 = 8M$$

Problem 4: 2-Phase method formulation

For each subproblem in Problem 3, write the corresponding Phase 1 objective function.

Solution

1. $\min . z_1$
2. $\min . z_1 + z_2 + z_3$
3. $\min . z_1$
4. $\min . z_1 + z_2 + z_3$
5. $\min . z_1 + z_2$

Problem 5: Unbounded solution

Solve the linear problem using the M-method.

$$\begin{aligned} \max. \quad & 3x_1 + 5x_2 \\ \text{s.t.} \quad & x_1 - 2x_2 \leq 6, \\ & x_1 \leq 10, \\ & x_2 \geq 1, \\ & x_1 \geq 0,\ x_2 \geq 0. \end{aligned}$$

Solution

The reformulated problem:

$$\begin{aligned} \max. \quad & 3x_1 + 5x_2 - Mz_1 \\ \text{s.t.} \quad & x_1 - 2x_2 + s_1 = 6, \\ & x_1 + s_2 = 10, \\ & x_2 - s_3 + z_1 = 1, \\ & x_1 \geq 0,\ x_2 \geq 0,\ s_1 \geq 0,\ s_2 \geq 0,\ s_3 \geq 0,\ z_1 \geq 0. \end{aligned}$$

Let $M = 30$. Than, the initial tableau is:

	x_1	x_2	s_1	s_2	s_3	z_1	Sol.
z	-3	-5	0	0	0	30	0
s_1	1	-2	1	0	0	0	6
s_2	1	0	0	1	0	0	10
z_1	0	1	0	0	-1	1	1

z-row is not a function of nonbasic variables. Thus, transform it:

	x_1	x_2	s_1	s_2	s_3	z_1	Sol.
z	-3	-35	0	0	30	0	-30
s_1	1	-2	1	0	0	0	6
s_2	1	0	0	1	0	0	10
z_1	0	1	0	0	-1	1	1

After the first iteration we get a table:

	x_1	x_2	s_1	s_2	s_3	z_1	Sol.
z	-3	0	0	0	-5	35	5
s_1	1	0	1	0	-2	2	8
s_2	1	0	0	1	0	0	10
x_2	0	1	0	0	-1	1	1

Entering basic variable is s_3. However, the problem is unbounded. All coefficients for pivot column 5 are non-positive.

Home Exercise 4:

Solve the following linear programming (LP) problems by the graphical method, and answer which problems have/are:

1. a unique optimal solution
2. multiple solutions
3. infeasible
4. unbounded

Problem 1:

$$\begin{aligned} \max.\ z &= x_1 + x_2 \\ \text{s.t. } x_1 + x_2 &\le 4 \\ x_1 - x_2 &\ge 5 \\ x_1,\ x_2 &\ge 0 \end{aligned}$$

Problem 2:

$$\begin{aligned}
\max.\ & z = 4x_1 + x_2 \\
\text{s.t. } & 8x_1 + 2x_2 \le 16 \\
& 5x_1 + 2x_2 \le 12 \\
& x_1,\ x_2 \ge 0
\end{aligned}$$

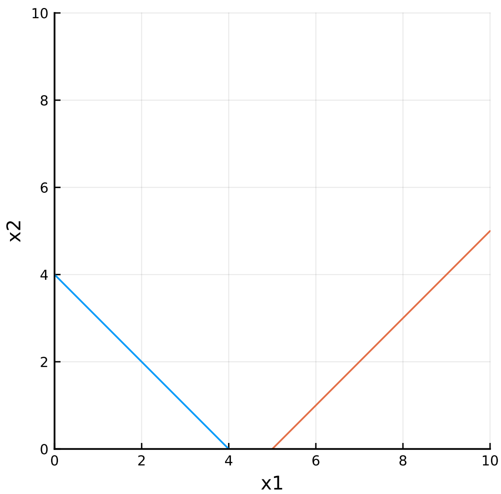
Problem 3:

$$\begin{aligned}
\max.\ & z = -x_1 + 3x_2 \\
\text{s.t. } & x_1 - x_2 \le 4 \\
& x_1 + 2x_2 \ge 4 \\
& x_1,\ x_2 \ge 0
\end{aligned}$$

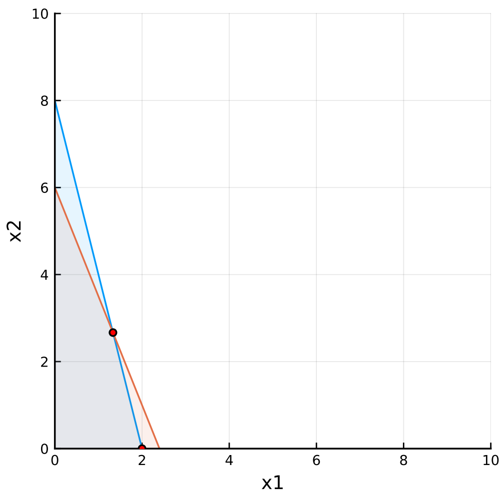
Problem 4:

$$\begin{aligned}
\max.\ & z = 3x_1 + x_2 \\
\text{s.t. } & 2x_1 + x_2 \le 6 \\
& x_1 + 3x_2 \le 9 \\
& x_1,\ x_2 \ge 0
\end{aligned}$$

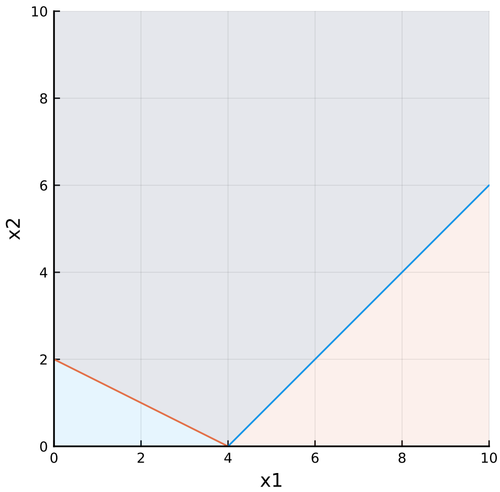
Solution



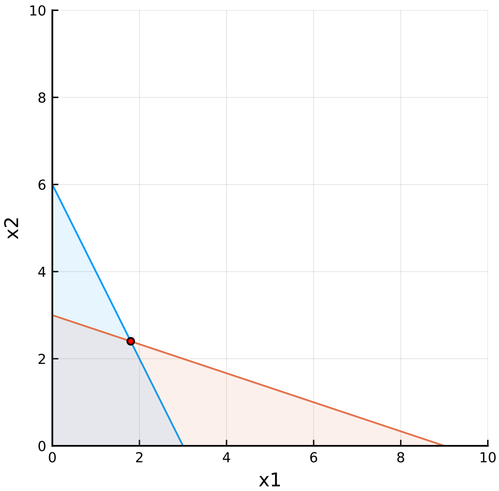
1. Infeasible



2. Multiple solutions



3. Unbounded



4. Unique optimal solution