MS-E2121 - Linear Optimisation
Lecture 5

Fabricio Oliveira (with modifications by Harri Hakula)

Systems Analysis Laboratory
Department of Mathematics and Systems Analysis

Aalto University
School of Science

March 9, 2022

Outline of this lecture

Duality in linear programming

Primal-dual relationship

Dual simplex

Reading: Taha: Chapter 4; Winston: Chapters 5 and 6.

Duality in linear programming

Duality is a theoretical framework. It allows to study properties of a primal problem using a dual counterpart.

- There are several “types” of duality. We will focus on Lagrangian duality.
- Various advanced techniques in mathematical programming rely to some extent in duality.
- Efficient numerical methods exploit primal-dual relationships (dual simplex and primal-dual interior point method).

Later in this lecture, we will look into the theory behind Lagrangian duality. For now, we will concentrate on its practical meaning.

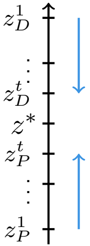
Duality in linear programming

Consider again the following problem:

$$
\begin{aligned}
(P): \text{max. } \; z_P = &\; 4x_1 + 3x_2 && (1)\\
\text{s.t.: } &\; 2x_1 + x_2 \leq 4 && (2)\\
&\; x_1 + 2x_2 \leq 4 && (3)\\
&\; x_1, x_2 \geq 0
\end{aligned}
$$

Suppose that we do not know how to solve it, but would like to estimate the optimal solution value z^* by obtaining bounds for it.

- A lower bound z_P can be obtained from any feasible solution, i.e., $z_P \leq z^*$.
- An upper bound z_D requires exploiting the problem structure.



Duality in linear programming

An optimal solution (x_1^*, x_2^*) is feasible by definition. Thus

1. Multiply (2) by $y_1 = 3$, obtaining $6x_1 + 3x_2 \leq 12$. We conclude that $z_D = 12$ is an upper bound for z^* as

$$z_P = 4x_1 + 3x_2 \leq 6x_1 + 3x_2 \leq 12 = z_D^1$$

   for (x_1^*, x_2^*) (or any feasible solution).

2. Alternatively, multiply (2) by $y_1 = 3/2$, (3) by $y_2 = 1$ and add the two. That leads to $4x_1 + (7/2)x_2 \leq 10$. As

$$z_P = 4x_1 + 3x_2 \leq 4x_1 + (7/2)x_2 \leq 10 = z_D^2,$$

   a better upper bound z_D is obtained.

3. For what values (y_1, y_2) the linear combination of constraints (2) and (3) is optimal (i.e., provides the tightest z_D)?

A linear combination of (2) and (3) is

$$\begin{aligned} &(2x_1 + x_2)y_1 + (x_1 + 2x_2)y_2 \leq 4y_1 + 4y_2 \\ = &(2y_1 + y_2)x_1 + (y_1 + 2y_2)x_2 \leq 4y_1 + 4y_2. \end{aligned}$$

Finding the optimal linear combination entails:

1. minimise $4y_1 + 4y_2$.
2. such that $2y_1 + y_2 \geq 4$, $y_1 + 2y_2 \geq 3$,
3. and $y_1, y_2 \geq 0$.

Equivalently, the dual of (P) can be stated as:

$$\begin{aligned} (D) : \text{min. } z_D = &4y_1 + 4y_2 \\ \text{s.t.: } &2y_1 + y_2 \geq 4 \\ &y_1 + 2y_2 \geq 3 \\ &y_1, y_2 \geq 0. \end{aligned}$$

Duality in linear programming

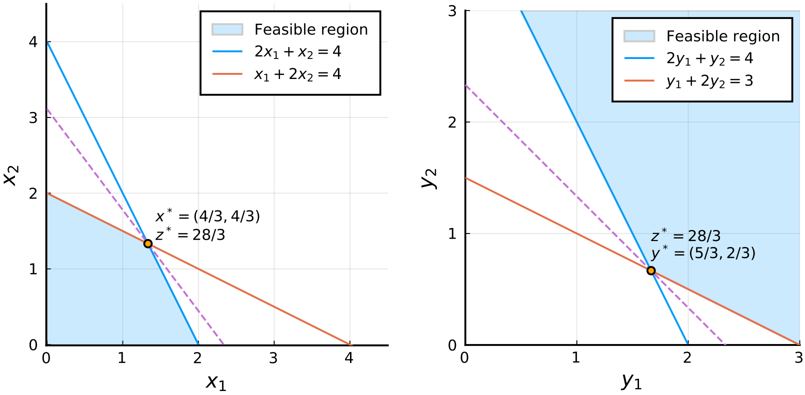


Figure: Primal (left) and dual (right) variable spaces

Duality in linear programming

The primal-dual pair is given by

$$
\begin{aligned}
(P):\ \text{max. } & z_P = 4x_1 + 3x_2 \\
\text{s.t.: } & 2x_1 + x_2 \leq 4 \\
& x_1 + 2x_2 \leq 4 \\
& x_1, x_2 \geq 0
\end{aligned}
\qquad
\begin{aligned}
(D):\ \text{min. } & z_D = 4y_1 + 4y_2 \\
\text{s.t.: } & 2y_1 + y_2 \geq 4 \\
& y_1 + 2y_2 \geq 3 \\
& y_1, y_2 \geq 0.
\end{aligned}
$$

Remarks:

- $\#$ variables of $(P) = \#$ constraints of (D).
- $\#$ constraints of $(P) = \#$ variables of (D).
- The dual of (D) is (P).
- Unbounded $(D) \Rightarrow$ infeasible (P).
- Infeasible $(P)/\ (D) \Rightarrow (D)/\ (P)$ infeasible or unbounded.

Primal-dual relationship

General conversion rules:

$$(P): \text{max. } z_P = c^\top x \quad \text{s.t.: } Ax \leq b, \quad x \geq 0.$$

$$(D): \text{min. } z_D = b^\top y \quad \text{s.t.: } A^\top y \geq c, \quad y \geq 0.$$

Primal (dual)	**Dual (primal)**
maximise	minimise
Independent terms (b)	Obj. function coef. (c)
Obj. function coef. (c)	Independent terms (b)
i-th row of constraint coef.	i-th column of constraint coef.
i-th column of constraint coef.	i-th row of constraint coef.
Constraints	**Variables**
$\leq$	≥ 0
$\geq$	≤ 0
$=$	$\in \mathbb{R}$
Variables	**Constraints**
≥ 0	$\geq$
≤ 0	$\leq$
$\in \mathbb{R}$	$=$

Primal-dual relationship

Dual information is available in the primal optimal tableau.

Example:

$$(P): \text{max. } z_P = 4x_1 + 3x_2 \quad \text{s.t.: } 2x_1 + x_2 \leq 4,\; x_1 + 2x_2 \leq 4,\; x_1, x_2 \geq 0$$

$$(D): \text{min. } z_D = 4y_1 + 4y_2 \quad \text{s.t.: } 2y_1 + y_2 \geq 4,\; y_1 + 2y_2 \geq 3,\; y_1, y_2 \geq 0.$$

(P) and (D) have as optimal tableaus

(P)	x_1	x_2	x_3	x_4	Sol.	(D)	y_1	y_2	y_3	y_4	Sol.
z	0	0	5/3	2/3	**28/3**	z	0	0	4/3	4/3	**28/3**
x_1	1	0	2/3	-1/3	4/3	y_1	1	0	-2/3	1/3	5/3
x_2	0	1	-1/3	2/3	4/3	y_2	0	1	1/3	-2/3	2/3

Remark: the z-coefficients (called reduced costs) of the slacks are the optimal dual values. They are also called shadow prices.

Weak and strong duality

To understand the correspondence between (P) and (D), let us first go through some technical results.

For that, let $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. Then

$$(P): \text{max. } z_P = c^\top x \quad \text{s.t.: } Ax \leq b, \quad x \geq 0 \qquad\qquad (D): \text{min. } z_D = b^\top y \quad \text{s.t.: } A^\top y \geq c, \quad y \geq 0.$$

Theorem 1 (Weak duality in LP)

Let $x = (x_1, \dots, x_n)$ be a (primal) feasible solution for P and $y = (y_1, \dots, y_m)$ a (dual) feasible solution to D. Then

$$c^\top x \leq b^\top y.$$

Weak and strong duality

Proof.

$$
\begin{aligned}
c^\top x = \sum_{j=1}^{n} c_j x_j &\leq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j \\
&= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) y_i \leq \sum_{i=1}^{m} b_i y_i = b^\top y. \quad \square
\end{aligned}
$$

Example

$$
\begin{aligned}
(P): \quad \text{max. } & z_P = 6x_1 + x_2 + 2x_3 \\
\text{s.t.: } & x_1 + 2x_2 - x_3 \leq 20 \\
& 2x_1 - x_2 + 2x_3 \leq 30 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
(D): \quad \text{min. } & z_D = 20y_1 + 30y_2 \\
\text{s.t.: } & y_1 + 2y_2 \geq 6 \\
& 2y_1 - y_2 \geq 1 \\
& -y_1 + 2y_2 \geq 2 \\
& y_1, y_2 \geq 0.
\end{aligned}
$$

Let $x = (15, 2.5, 0)$ and $y = (2, 2)$. Then $z_P = 92.5 \leq 100 = z_D$. Any other primal/ dual feasible pair incurs in the same result.

Weak and strong duality

An important corollary follows from Theorem 1.

Corollary 2

If $x^* = (x_1^*, \dots, x_n^*)^\top$ *and* $y^* = (y_1^*, \dots, y_m^*)$ *are primal and dual feasible solutions (respectively) and* $c^\top x^* = b^\top y^*$*, then* x^* *and* y^* *are optimal for their respective problems.*

Proof.

By weak duality, we have $c^\top x \leq b^\top y^* = c^\top x^*$, and thus x^* must be optimal. Similarly, $b^\top y^* = c^\top x^* \leq b^\top y$. $\square$

Remark: this proves one direction ($z_P = z_D \Rightarrow x^*$ and y^* are optimal). The converse (x^* and y^* are optimal $\Rightarrow z_P = z_D$) is proved in what follows.

Strong duality

To show strong duality, we need to understand the role of reduced costs in the optimality of x^*.

Let $S = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$. Suppose x^* is the optimal solution of max. $_x \left\{c^\top x : x \in S\right\}$.

We want to derive the conditions that guarantee that

$$c^\top x^* \geq c^\top x, \ \forall x \in S.$$

Let $x^* = \begin{bmatrix} x_B^* \\ x_N^* \end{bmatrix}$, $A = [B \ N]$, and $c = \begin{bmatrix} c_B \\ c_N \end{bmatrix}$. Thus

$$z = c^\top x \Rightarrow c_B^\top x_B + c_N^\top x_N \tag{4}$$

$$Ax = b \Rightarrow Bx_B + Nx_N = b \tag{5}$$

From (5), we obtain $x_B = B^{-1}(b - Nx_N)$. Also, since $x_N^* = 0$ by definition, $x_B^* = B^{-1}b$.

Weak and strong duality

Now, substituting in (4), we obtain:

$$\begin{aligned}
c^\top x &= c_B^\top x_B + c_N^\top x_N \\
&= c_B^\top \underbrace{B^{-1}(b - N x_N)}_{x_B} + c_N^\top x_N \\
&= \underbrace{c_B^\top B^{-1} b}_{= c_B^\top x_B^* + c_N^\top x_N^* = c^\top x^*} + (c_N^\top - c_B^\top B^{-1} N) x_N \\
c^\top x^* - c^\top x &= -(c_N^\top - c_B^\top B^{-1} N) x_N.
\end{aligned}$$

Notice that

1. $c^\top x^* - c^\top x \geq 0$ means that basis B yielding x^* is better than any other basis. Thus, $r_N = (c_N^\top - c_B^\top B^{-1} N) \leq 0$.
2. If a component j of r_N is positive, this nonbasic variable can be made basic, improving z by $r_{N(j)}$ per unit of $x_{N(j)}$.
3. r_N is the reduced cost; these are the entries in the z-row for the nonbasic variables.

Weak and strong duality

Theorem 3 (Strong duality)

If an LP has a finite optimal solution, then its dual problem also have a finite optimal solution, and the optimal objective function value of both problem are the same.

Proof.

Let (P) be the LP in the standard form and (D) its dual

$$(P) : \text{max. } z_P = c^\top x \qquad\qquad (D) : \text{min. } z_D = b^\top y$$
$$\text{s.t.: } Ax = b \qquad\qquad \text{s.t.: } A^\top y \geq c$$
$$x \geq 0 \qquad\qquad y \in \mathbb{R}^m$$

with optimal solution is $x^* = [x_B^* \; x_N^*]^\top$. Since x^* is optimal, we have $r_N = c_N - c_B^\top B^{-1} N \leq 0$.

Weak and strong duality

Proof. (continued).

Now, let $y^* = (c_B^\top B^{-1})^\top$. Then

$$\begin{aligned} c - A^\top y &= \begin{bmatrix} c_B \\ c_N \end{bmatrix} - \begin{bmatrix} B^\top \\ N^\top \end{bmatrix} y^* \\ &= \begin{bmatrix} c_B \\ c_N \end{bmatrix} - \begin{bmatrix} B^\top \\ N^\top \end{bmatrix} (c_B^\top B^{-1})^\top \\ &= \begin{bmatrix} c_B \\ c_N \end{bmatrix} - \begin{bmatrix} c_B \\ (B^{-1}N)^\top c_B \end{bmatrix} = \begin{bmatrix} 0 \\ r_N \end{bmatrix} \leq 0 \end{aligned}$$

Thus, y^* is dual feasible. Also

$$b^\top y^* = (y^*)^\top b = c_B^\top B^{-1} b = c_B^\top x_B^* = c^\top x^*$$

and by Corollary 2, y^* is optimal for (D). □

Complementarity conditions

Consider the following primal-dual pair

$$
(P): \ \text{max. } z_P = c^\top x \qquad\qquad (D): \ \text{min. } z_D = b^\top y
$$
$$
\text{s.t.: } Ax \leq b \qquad\qquad \text{s.t.: } A^\top y \geq c
$$
$$
x \geq 0 \qquad\qquad y \geq 0.
$$

Let the primal slack be $x_s = b - Ax$ and the dual slack $y_s = A^\top y - c$. For any primal feasible x and dual feasible y solutions, we have

$$
\begin{aligned}
0 &\leq x^\top y_s + x_s^\top y \\
&= (y^\top A - c^\top)x + y^\top (b - Ax) \\
&= y^\top b - c^\top x,
\end{aligned}
$$

which is the duality gap shown to be zero for optimal x^* and y^* (as shown Theorem 3). Thus, ${x^*}^\top y_s^* = {x_s^*}^\top y^* = 0$.

Dual simplex

Modern codes of simplex method use the dual simplex variant.

- Feasibility conditions are sought after while optimality is maintained.
- Precludes the requirement of a initial feasible basic solution

The dual simplex algorithm works as follows:

1. **Leaving variable:** the negative-valued variable with largest absolute value leaves the basis ("most infeasible" variable).
2. **Entering variable:** the variable that becomes basic without affecting optimality condition.

For using dual simplex, a preprocessing step is needed to make all constraints $(\leq)$-constraints (i.e.§, standard form for dual simplex):

- $(\geq)$-constraints are multiplied by -1.
- $(=)$-constraints are equivalently replaced by two inequalities.

Dual simplex

The pseudo code for dual simplex:

Algorithm Dual simplex method

initialise. Convert problem to standard form, if needed. Form initial basis.
while $x_i < 0$ for any $i = \{1, \dots, m\}$ **do**
  Select leaving variable: $k = \arg\min_{i \in 1,\dots,i} \{x_i\}$
  Select entering variable: $j_{PR} = \arg\min_{j=1,\dots,n} \left\{ \left| \frac{c_j}{a_{jk}} \right| : a_{jk} < 0 \right\}$
  Perform row operations: $a_{j_{PR}k} = 1, a_{jk} = 0$ for $j = 1, \dots, n : j \neq j_{PR}$
  $B = B \cup \{k\} \setminus \{i_{PR}\}$
end while
return B, $x_i = b_i$ for $i \in B$, $x_j = 0$ for $j \in \{1, \dots, n\} \setminus B$.

Dual simplex

Example:

min. $z = 3x_1 + 2x_2 + x_3$

s.t.: $3x_1 + x_2 + x_3 \geq 3$

$-3x_1 + 3x_2 + x_3 \geq 6$

$x_1 + x_2 + x_3 \leq 3$

$x_1, x_2, x_3 \geq 0$

$z - 3x_1 - 2x_2 - x_3 = 0$

$-3x_1 - x_2 - x_3 + x_4 = -3$

$3x_1 - 3x_2 - x_3 + x_5 = -6$

$x_1 + x_2 + x_3 + x_6 = 3$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

Applying dual simplex, we obtain:

	x_1	x_2	x_3	x_4	x_5	x_6	Sol.
z	-3	-2	-1	0	0	0	0
x_4	-3	-1	-1	1	0	0	-3
x_5	3	-3	-1	0	1	0	-6
x_6	1	1	1	0	0	1	3
z	-5	0	-1/3	0	-2/3	0	4
x_4	-4	0	-2/3	1	-1/3	0	-1
x_2	-1	1	1/3	0	-1/3	0	2
x_6	2	0	2/3	0	1/3	1	1
z	-3	0	0	-1/2	-1/2	0	9/2
x_3	6	0	1	-3/2	-3/2	0	3/2
x_2	-3	1	0	1/2	1/2	0	3/2
x_6	-2	0	0	1	1	1	0