MS-C2105 - Introduction to Optimization
Lecture 2

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Outline of this lecture

Modelling problems

From problem statement to mathematical models

Graphical representation

Plotting feasible regions

Finding optimal solutions

Sensitivity analysis

Reading: Taha: Chapter 2; Winston: Chapter 3

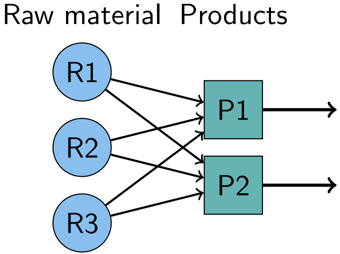
The diet problem

The most classic linear optimisation problem, perhaps one of the first to be implemented in practice.

Often referred to as the mixture problem.

Typical applications:

1. Feed composition;
2. Metal alloy production;
3. Fuel specification ;
4. Drug manufacturing;
5. ...



The diet problem

Ozark farms uses at least 800 lb of a special feed daily. The special feed is a mixture of corn and soybean meal with the following compositions:

Feedstuff	Protein	Fibre	Cost($/lb)
Corn	0.09	0.02	0.30
Soybean meal	0.60	0.06	0.90

Table: lb per lb of feedstuff

The dietary requirements of the special feed are at least 30% protein and at most 5% fiber.

Goal: determine the optimal feed mix composition.

The diet problem

Three key steps:

1. **Determine what needs to be decided** (*decision variables*)

   x_1 - amount (lb) of corn in the daily mix
   x_2 - amount (lb) of soybean meal in the daily mix

2. **How solutions are assessed** (*objective function*)

$$\text{min. } z = 0.30x_1 + 0.90x_2$$

3. **The requirements that must be satisfied** (*constraints*)

$$
\begin{aligned}
& x_1 + x_2 \geq 800 && \text{(min. feed amount)} \\
& 0.09x_1 + 0.6x_2 \geq 0.3(x_1 + x_2) && \text{(min. protein)} \\
& 0.02x_1 + 0.06x_2 \leq 0.05(x_1 + x_2) && \text{(max. fibre)} \\
& x_1, x_2 \geq 0
\end{aligned}
$$

The diet problem

The complete model is:

$$\begin{aligned}
\text{min. } z = & \ 0.30x_1 + 0.90x_2 \\
\text{s.t.: } & x_1 + x_2 \geq 800 \\
& 0.09x_1 + 0.6x_2 \geq 0.3(x_1 + x_2) \\
& 0.02x_1 + 0.06x_2 \leq 0.05(x_1 + x_2) \\
& x_1, x_2 \geq 0
\end{aligned}$$

It is convenient to reformulate problems to a format with variables on the left-hand side and constants on the right-hand side.

$$\underbrace{\sum_{j=1}^{n} a_{ij}x_j}_{\text{LHS}} \leq \underbrace{b_i}_{\text{RHS}}, \ i = 1, \dots, m$$

The diet problem

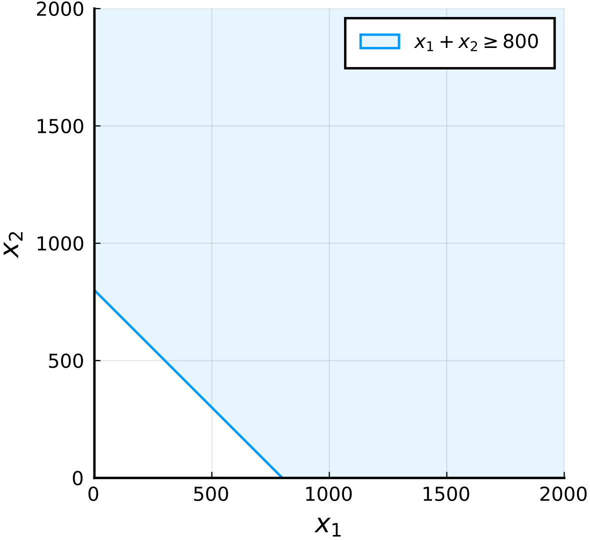
The reformulated model is:

$$
\begin{aligned}
\text{min. } & z = 0.30x_1 + 0.90x_2 && (1)\\
\text{s.t.: } & x_1 + x_2 \geq 800 && (2)\\
& 0.21x_1 - 0.30x_2 \leq 0 && (3)\\
& 0.03x_1 - 0.01x_2 \geq 0 && (4)\\
& x_1, x_2 \geq 0 && (5)
\end{aligned}
$$

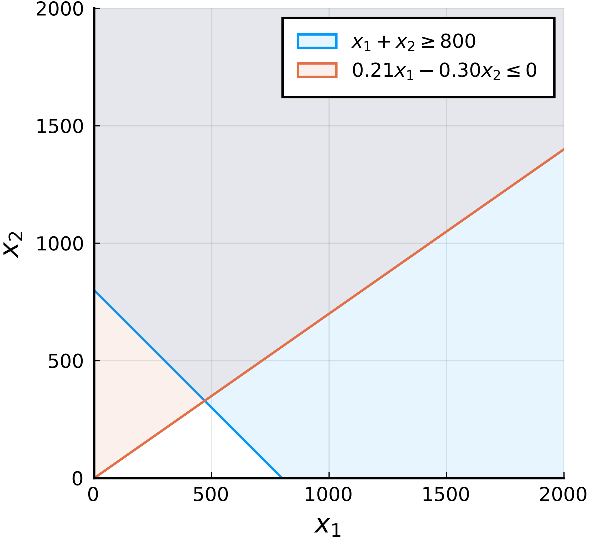
Let us solve it graphically in the decision variable-space (x_1, x_2).

Remark: in practice, problems have many more variables.
Two-variable problems are however graphically representable, which is useful to infer geometrical properties of linear problems (LP).

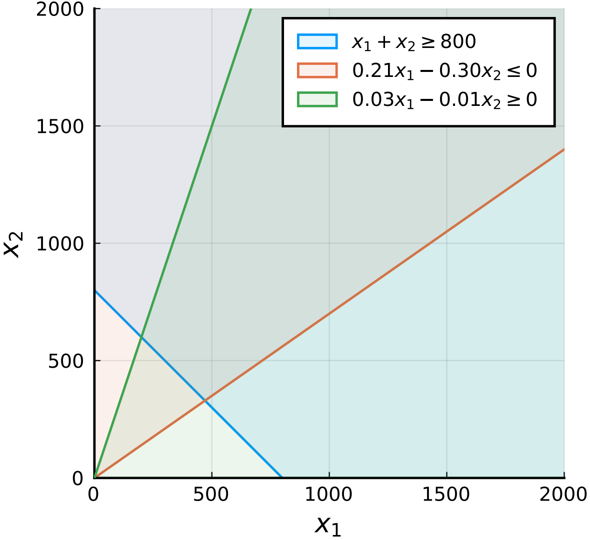
Diet problem - graphical representation



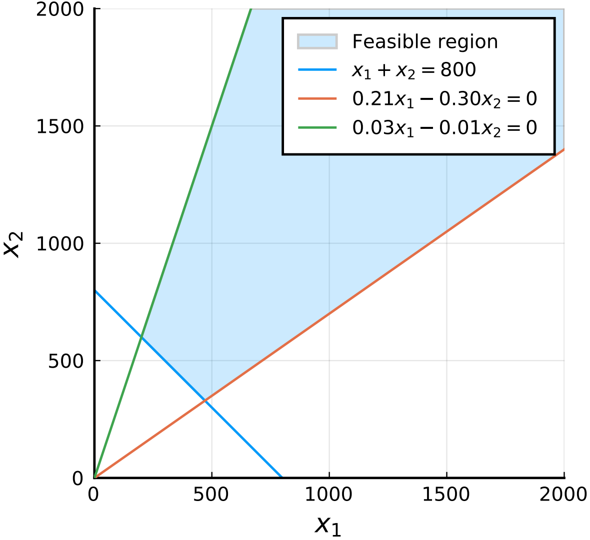
Diet problem - graphical representation



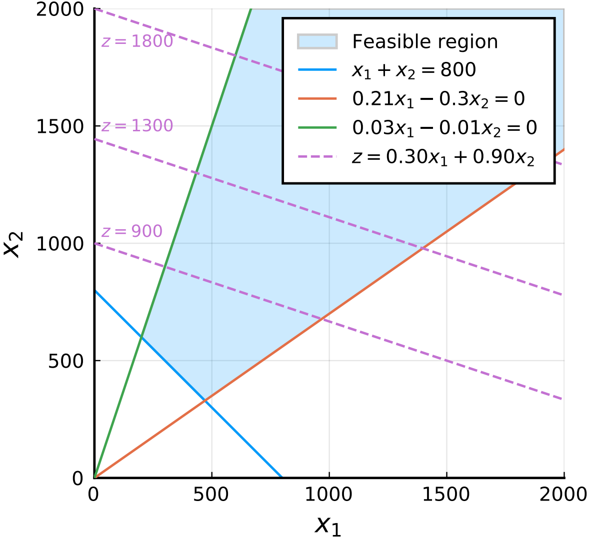
Diet problem - graphical representation



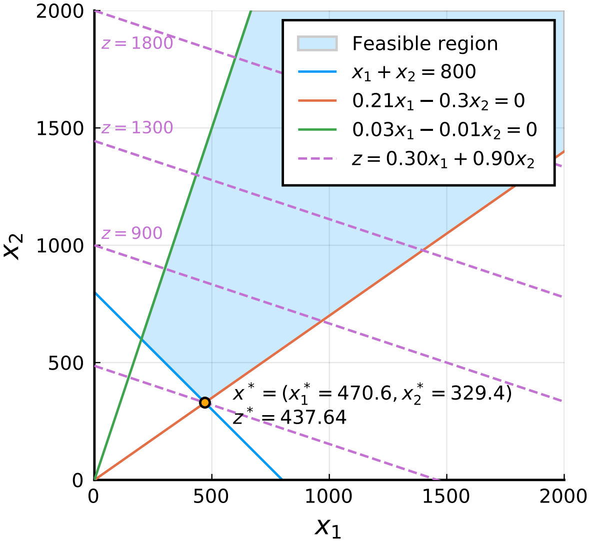
Diet problem - graphical representation



Diet problem - graphical representation



Diet problem - graphical representation



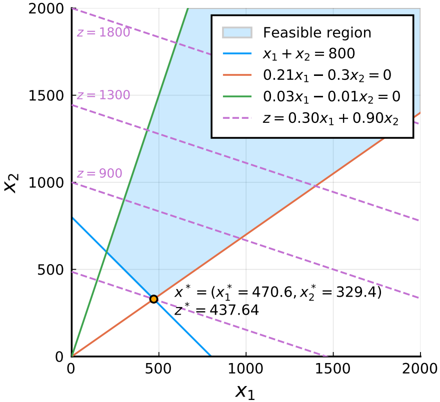
LP geometry

Some important concepts:

1. Active constraints: resources (requirements) fully depleted (minimally satisfied).

   Ex.: const. (2) and (3).

2. Inactive constraints: resources (requirements) not fully depleted (over satisfied).
   Ex.: const. (4).



Notice how an optimal solution is (almost) always a vertex. Thus, optimal point candidates are always vertices. More on that later...

Another example - production planning

A paint factory produces exterior and interior paint from raw materials M1 and M2. The maximum demand for interior paint is 2 tons/day. Moreover, the amount of interior paint produced cannot exceed that of exterior paint by more than 1 ton/day.

Goal: determine optimal paint production.

	material (ton)/paint (ton)		
	ext. paint	int. paint	daily availability (ton)
material M1	6	4	24
material M2	1	2	6
profit ($1000 /ton)	5	4	

Table: Paint shop problem data

Another example - production planning

Three key steps:

1. **Determine what needs to be decided** (*decision variables*)

   x_1 - amount (ton) of exterior paint

   x_2 - amount (ton) of interior paint

2. **How solutions are assessed** (*objective function*)

$$\text{max. } z = 5x_1 + 4x_2$$

3. **The requirements that must be satisfied** (*constraints*)

$6x_1 + 4x_2 \leq 24$	(M1 avail.)
$x_1 + 2x_2 \leq 6$	(M2 avail.)
$x_2 \leq x_1 + 1$	(dem. int. paint)
$x_2 \leq 2$	(dem. ext. paint)
$x_1, x_2 \geq 0$	

Another example - production planning

The complete (reformulated) model is:

$$
\begin{aligned}
\text{max. } & z = 5x_1 + 4x_2 && (6)\\
\text{s.t.: } & 6x_1 + 4x_2 \leq 24 && (7)\\
& x_1 + 2x_2 \leq 6 && (8)\\
& x_2 - x_1 \leq 1 && (9)\\
& x_2 \leq 2 && (10)\\
& x_1, x_2 \geq 0 && (11)
\end{aligned}
$$

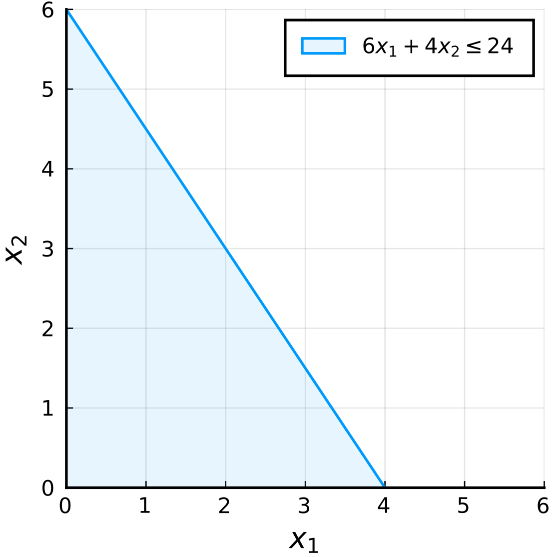
The model could be also compactly represented as

$$
\begin{aligned}
\text{max. } & z = c^\top x \\
& Ax \leq b \\
& x \geq 0
\end{aligned}
$$

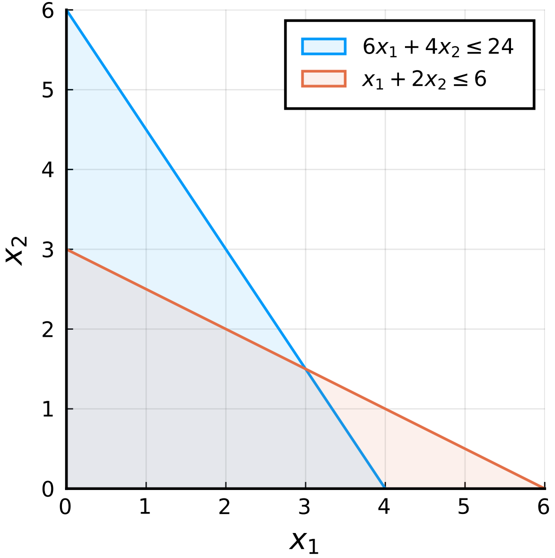
$c = [5, 4]^\top$, $x = [x_1, x_2]^\top$,

$A = \begin{bmatrix} 6 & 4 \\ 1 & 2 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}$ and $b = [24, 6, 1, 2]^\top$

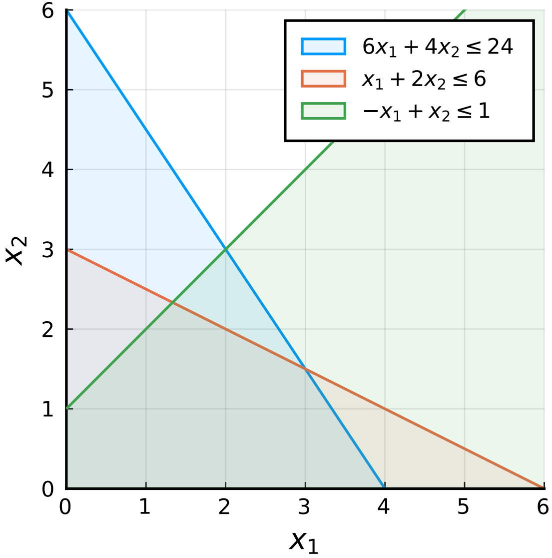
Production planning - graphical representation



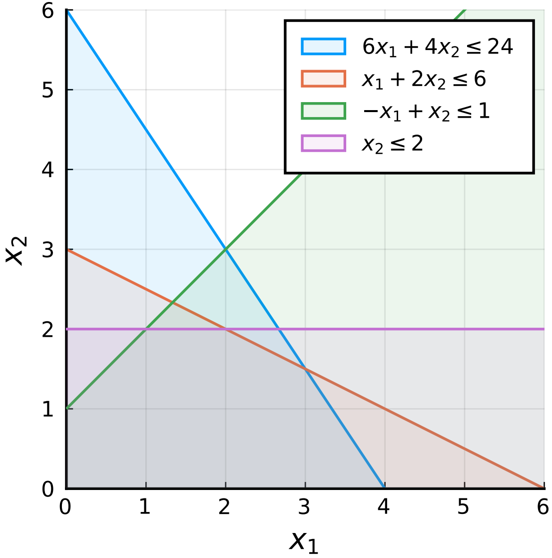
Production planning - graphical representation



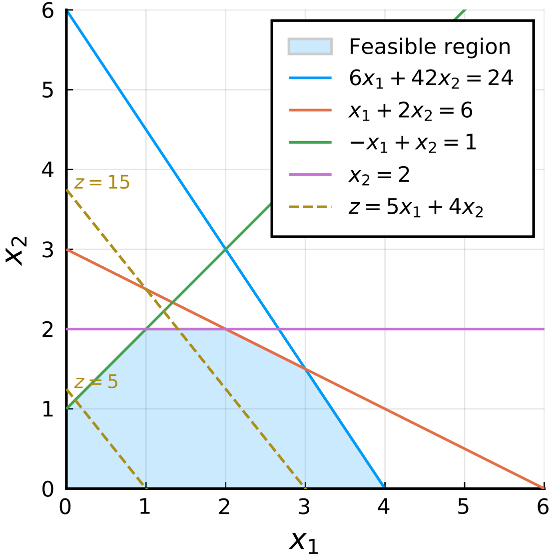
Production planning - graphical representation



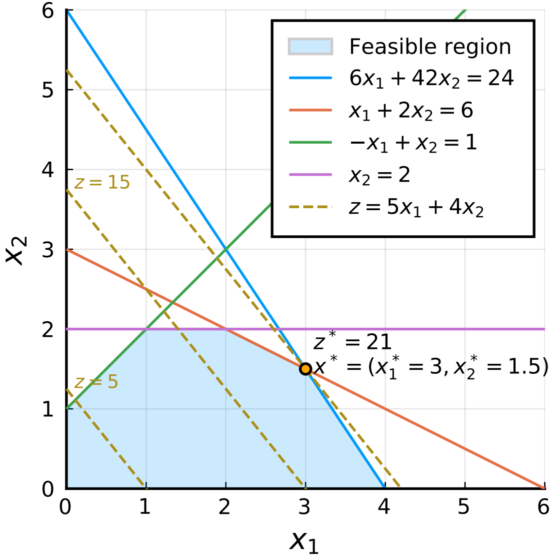
Production planning - graphical representation



Production planning - graphical representation

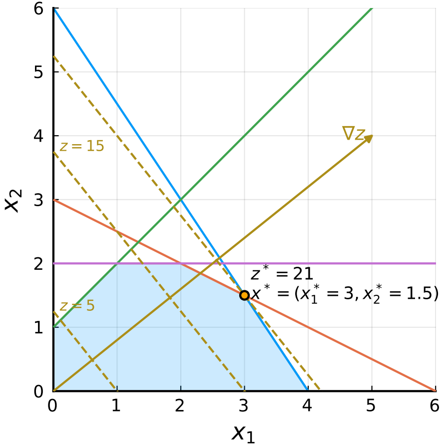


Production planning - graphical representation



More on LP geometry

The gradient $\nabla z = [\frac{\partial z}{\partial x_1}, \frac{\partial z}{\partial x_2}]^\top = [5, 4]^\top$ indicates the direction in which z increases; when minimising, we move towards $-\nabla z$.



Thus, in general, the solution is unique and lies on a vertex.

Graphical sensitivity analysis

We can use the graphical representation to calculate the marginal value of a resource.

However, the analysis only holds for a given set of active (and inactive) constraints if performed for an individual element.

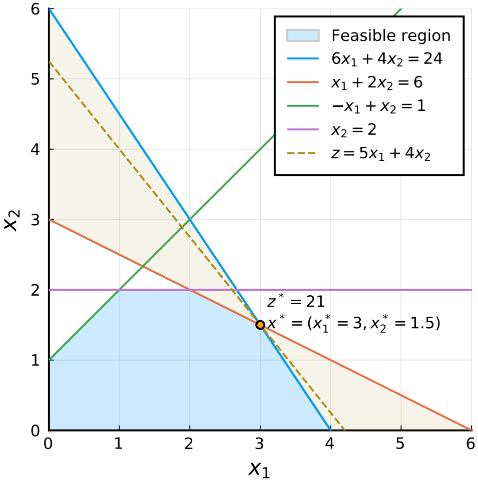
Using sensitivity analysis we can enquire about:

1. For which changes in the coefficients of the objective function (c) does this vertex remains optimal?
2. For which changes in the RHS (b) does this set of active constraints remain optimal?

Remark: notice that, when changing the RHS, the optimal point coordinates will change, but not the intersection of active constraints that forms it.

Graphical sensitivity analysis

1. For which changes in the coefficients of the objective function (c) does this vertex remains optimal?

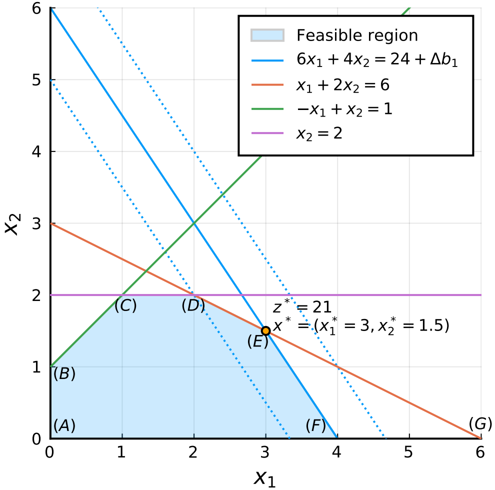


Let $z' = c_1x_1 + c_2x_2$ be the perturbed objective function. Then

- x^* remains optimal if the slope of z' lies between (6) and (7);
- This is the same as requiring $\frac{1}{2} \leq \frac{c_1}{c_2} \leq \frac{6}{4}$.

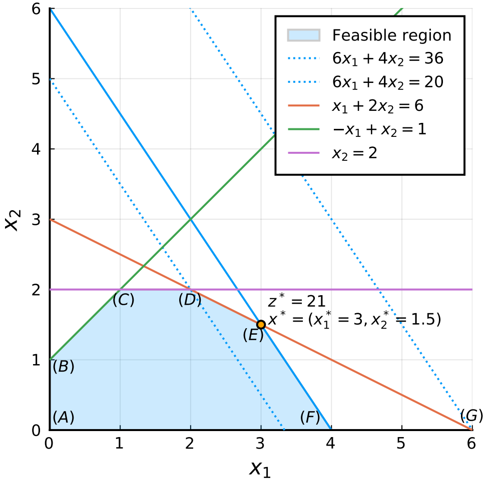
Graphical sensitivity analysis

2. For which changes in the RHS (b) does this set of active constraints remains optimal?



Graphical sensitivity analysis

2. For which changes in the RHS (b) does this set of active constraints remain optimal?



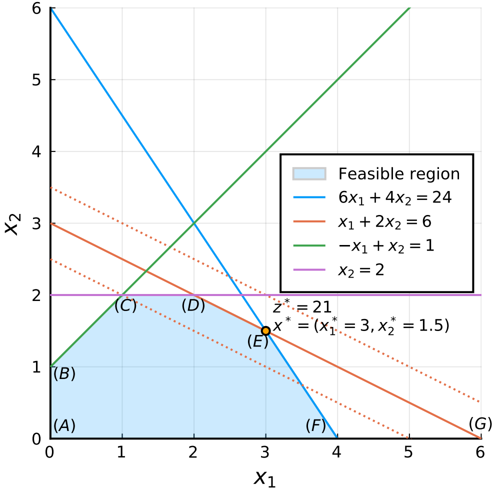
- For M1: the set of active constraints remains the same for changes in the RHS (b_1) between 20 and 36 (pre-calculated).
- Then, the marginal value y_1 for $b_1 \in [20, 36]$ is

$$= \frac{\Delta z}{\Delta b_1} = \frac{z(D) - z(G)}{b_1(D) - b_1(G)}$$

$= 750$ (\$/ton)

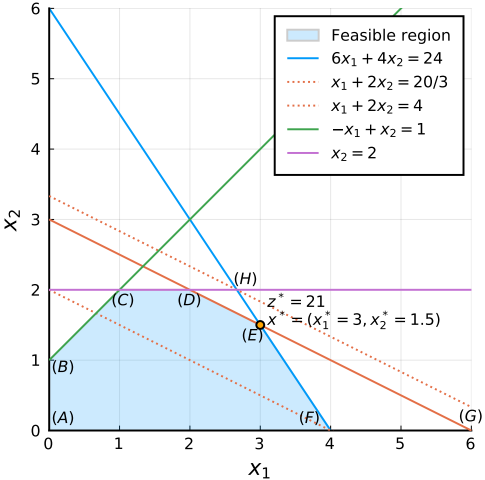
Graphical sensitivity analysis

2. For which changes in the RHS (b) does this set of active constraints remain optimal?



Graphical sensitivity analysis

2. For which changes in the RHS (b) does this set of active constraints remain optimal?



- For M2: the set of active constraints remains the same for changes in the RHS (b_2) between 4 and $\frac{20}{3}$ (pre-calculated).
- Then, the marginal value y_2 for $b_2 \in [4, 6\frac{2}{3}]$ is

$$= \frac{\Delta z}{\Delta b_2} = \frac{z(H) - z(F)}{b_2(H) - b_2(F)}$$

$$= 500 \text{ (\$/ton)}$$