MS-C2105 - Introduction to Optimization
Lecture 1

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Outline of this lecture

Introduction

What is optimisation?

Mathematical programming and optimisation

Modelling real-world problems using optimisation

A first optimisation model

Production planning problems

Classification

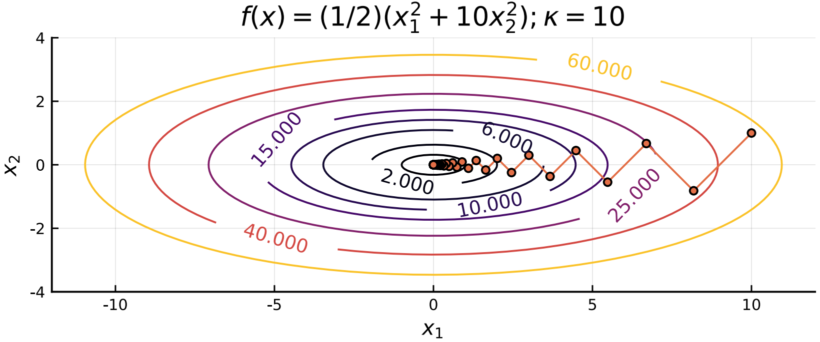
Reading: Taha: Chapter 1; Winston: Chapter 1

What is optimisation?

Discipline of applied mathematics. The idea is to search values for variables in a given domain that maximise/minimise function values.

Can be achieved by

- Analysing properties of functions/extreme points or
- Applying numerical methods.



What is optimisation?

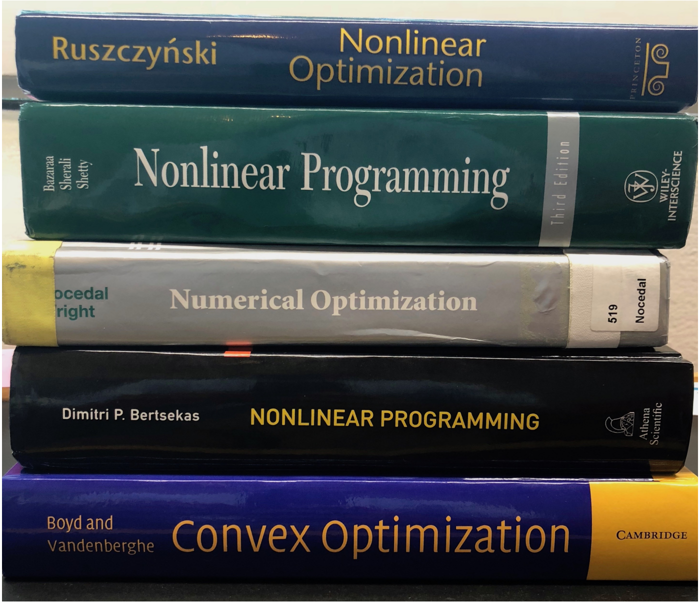
Optimisation has important applications in fields such as

- mathematical programming & operations research (OR) ;
- economics;
- statistics;
- machine learning & artificial intelligence...

Mathematical programming is a modelling paradigm that relies on optimisation to model decision processes:

- **variables** $\rightarrow$ decisions: business decisions, parameter definitions, settings, geometries, ...;
- **domain** $\rightarrow$ constraints: logic, design, engineering, ...;
- **function** $\rightarrow$ objective function: measurement of (decision) quality.

However, there is some confusion between terms optimisation/ programming. In this course, we focus on optimisation models.


Ruszczyński
Nonlinear Optimization
PRINCETON
Bazaraa Sherali Shetty
Nonlinear Programming
Third Edition
WILEY-INTERSCIENCE
Numerical Optimization
519
Nocedal
Dimitri P. Bertsekas
NONLINEAR PROGRAMMING
Athena Scientific
Boyd and Vandenberghe
Convex Optimization
CAMBRIDGE

Types of programming/ optimisation models

The simpler the assumptions are that define a type of problems, the better the methods to solve such problems.

Some useful notation:

- $x \in \mathbb{R}^n$: vector of (decision) variables x_j, $j = 1, \dots, n$;
- $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$: objective function;
- $X \subseteq \mathbb{R}^n$: ground set (physical constraints);
- $g_i, h_i : \mathbb{R}^n \to \mathbb{R}$: constraint functions;
- $g_i(x) \leq 0$ for $i = 1, \dots, m$: inequality constraints;
- $h_i(x) = 0$ for $i = 1, \dots, l$: equality constraints.

Types of programming

Our goal will be to solve variations of the general problem P:

$$\begin{aligned}(P) \ : \ \text{min. } & f(x) \\ \text{s.t.: } & g_i(x) \leq 0, i = 1, \dots, m \\ & h_i(x) = 0, i = 1, \dots, l \\ & x \in X.\end{aligned}$$

- **Linear programming (LP):** linear $f(x) := c^\top x$ with $c \in \mathbb{R}^n$; constraint functions $g_i(x)$ and $h_i(x)$ are affine ($a_i^\top x - b_i$, with $a_i \in \mathbb{R}^n$, $b \in \mathbb{R}$); $X = \{x \in \mathbb{R}^n : x_j \geq 0, j = 1, \dots, n\}$.
- **Nonlinear programming (NLP):** some (or all) of the functions f, g_i or h_i are nonlinear;
- **(Mixed-)integer programming ((M)IP):** LP where (some of the) variables are binary (or integer). $X = \mathbb{R}^k \times \{0,1\}^{n-k}$
- **Mixed-integer nonlinear programming (MINLP):** MIP+NLP.

A first optimisation model

Let us start with a simple example to illustrate the optimisation modelling framework.

Consider the following: a carpenter makes tables and chairs using wood and her labour. The carpenter has a limited availability of labour and wood (see table below). What is the optimal weekly production of tables and chairs?

	Table	Chair	Available (per week)
Selling price ($)	800	600	-
Workload (h)	3	5	40
Wood (u)	7	4	60

A first optimisation model

Three key steps:

1. **Determine what needs to be decided** (*decision variables*)

   x_1 - amount of tables
   x_2 - amount of chairs

2. **How solutions are assessed** (*objective function*)

   Maximise revenue: max. $z = 800x_1 + 600x_2$

3. **The requirements that must be satisfied** (*constraints*)

   $3x_1 + 5x_2 \leq 40$ (available labour)
   $7x_1 + 4x_2 \leq 60$ (available wood)
   $x_1, x_2 \geq 0$

A first optimisation model

The complete model:

$$
\begin{aligned}
\text{max. } & z = 800x_1 + 600x_2 && \text{(profit)} \\
\text{s.t.: } & 3x_1 + 5x_2 \leq 40 && \text{(available labour)} \\
& 7x_1 + 4x_2 \leq 60 && \text{(available wood)} \\
& x_1, x_2 \geq 0
\end{aligned}
$$

Remarks: models are simplified representations of reality.
Simplifying assumptions in this example include:

- fractional number of chairs/ tables;
- no uncertainty;
- no production cost and no wastage of resources;
- perfect demand (all production is sold)...

A first optimisation model

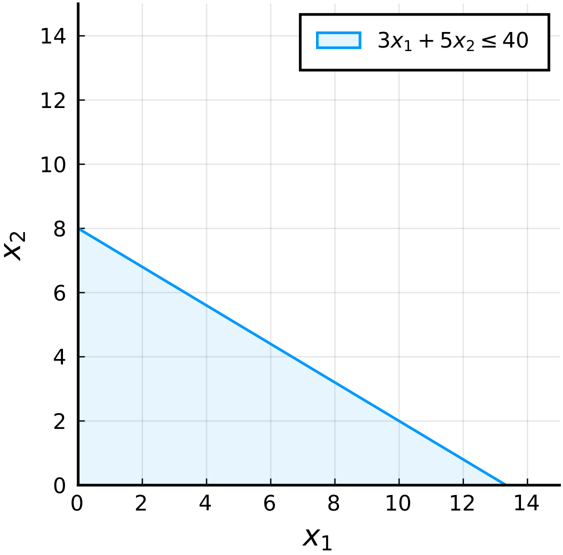
The most suitable optimisation method for solving an optimisation model depends on the model's mathematical properties.

- is the model linear?
- are there integer variables?
- are the nonlinear terms convex?
- are gradients available?

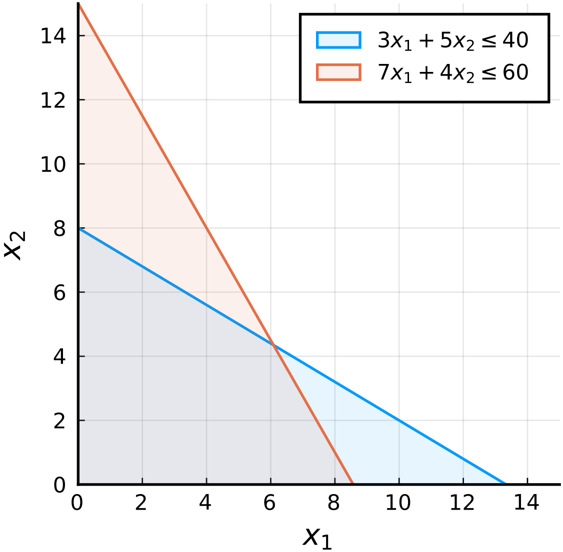
In this course, we will learn how to specify a suitable method for a model given its properties. For now, we will concentrate on (continuous) linear (optimisation) models.

Linear models have particular properties that can be exploited to devise an efficient optimisation method.

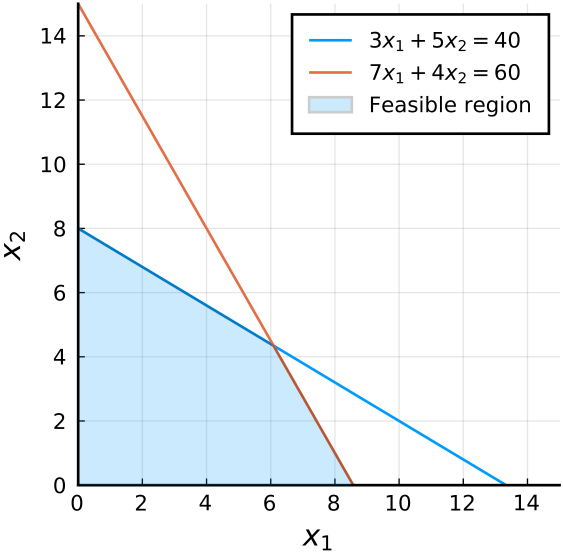
A first optimisation model - geometry of linear models



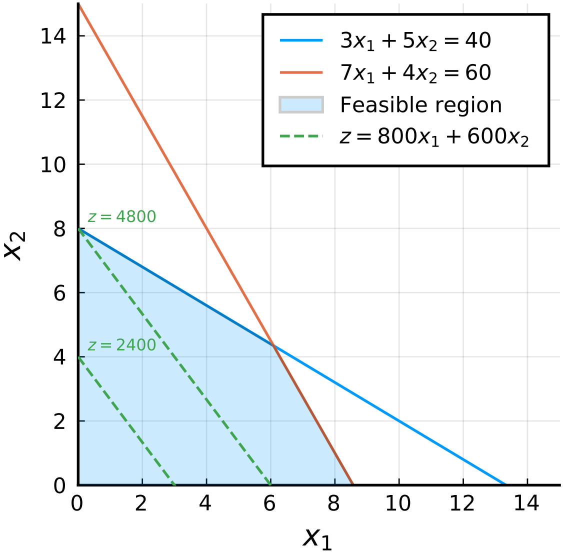
A first optimisation model - geometry of linear models



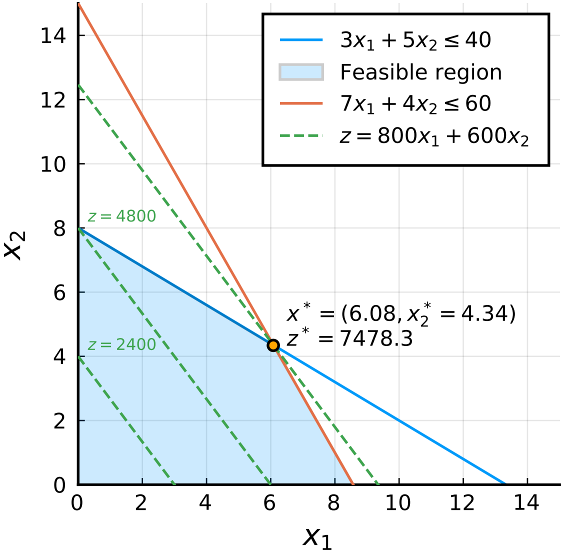
A first optimisation model - geometry of linear models



A first optimisation model - geometry of linear models



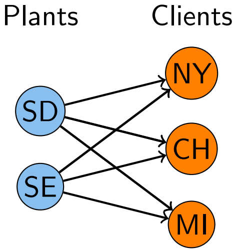
A first optimisation model - geometry of linear models



Real-world model: production planning problems (OR)

Problem statement:

- plan production and distribution;
- transportation cost proportional to distance travelled;
- factories have a capacity limit;
- clients have known demands.



	Clients			
Factory	NY	Chicago	Miami	Capacity
Seattle	2.5	1.7	1.8	350
San Diego	3.5	1.8	1.4	600
Demands	325	300	275	-

Table: Problem data: unit transportation costs, demands and capacities

Real-world model: production planning problems (OR)

Let $i \in I = \{\text{Seattle}, \text{San Diego}\}$ be the index set representing factories. Similarly, let $j \in J = \{\text{New York}, \text{Chicago}, \text{Miami}\}$.

Three key steps:

1. **Determine what needs to be decided** (*decision variables*)

   x_{ij} be the amount produced in factory i and sent to client j.

2. **How solutions are assessed** (*objective function*)

   Minimise total distribution cost:

   min. $z = 2.5x_{11} + 1.7x_{12} + 1.8x_{13} + 3.5x_{21} + 1.9x_{22} + 1.4x_{23}$,

   which can be more compactly expressed as

   $$\text{min. } z = \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij}$$

   where c_{ij} is the unit transportation cost from i to j.

Real-world model: production planning problems (OR)

3. **The requirements that must be satisfied** (*constraints*)

$$x_{11} + x_{12} + x_{13} \leq 350 \text{ (capacity limit Seattle)}$$
$$x_{21} + x_{22} + x_{23} \leq 600 \text{ (capacity limit San Diego)}$$
$$x_{11} + x_{21} \geq 325 \text{ (demand in New York)}$$
$$x_{12} + x_{22} \geq 300 \text{ (demand in Chicago)}$$
$$x_{13} + x_{23} \geq 275 \text{ (demand in Miami).}$$

These constraints can be expressed in the more compact form

$$\sum_{j \in J} x_{ij} \leq C_i, \forall i \in I$$
$$\sum_{i \in I} x_{ij} \geq D_j, \forall j \in J,$$

where C_i is the production capacity of factory i and D_j is the demand of client j.

Real-world model: production planning problems (OR)

The complete model:

$$\begin{aligned} \text{min. } & z = 2.5x_{11} + 1.7x_{12} + 1.8x_{13} + 3.5x_{21} + 1.9x_{22} + 1.4x_{23} \\ \text{s.t.: } & x_{11} + x_{12} + x_{13} \leq 350,\ x_{21} + x_{22} + x_{23} \leq 600 \\ & x_{11} + x_{21} \geq 325,\ x_{12} + x_{22} \geq 300,\ x_{13} + x_{23} \geq 275 \\ & x_{11}, \dots, x_{23} \geq 0. \end{aligned}$$

Or, more compactly, in the so called algebraic (symbolic) form

$$\begin{aligned} \text{min. } & z = \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} \\ \text{s.t.: } & \sum_{j \in J} x_{ij} \leq C_i, \forall i \in I \\ & \sum_{i \in I} x_{ij} \geq D_j, \forall j \in J \\ & x_{ij} \geq 0, \forall i \in I, \forall j \in J. \end{aligned}$$

Real-world model: classification problem (ML)

Suppose we are given a data set $D \subset \mathbb{R}^n$ that can be separated into two disjunct sets in $\mathbb{R}^n$: $I^- = \{x_1, \dots, x_N\}$ and $I^+ = \{x_1, \dots, x_M\}$.

Each element $x_i \in D$ is an observation of a given set of features; belonging to either I^- or I^+ defines a classification.

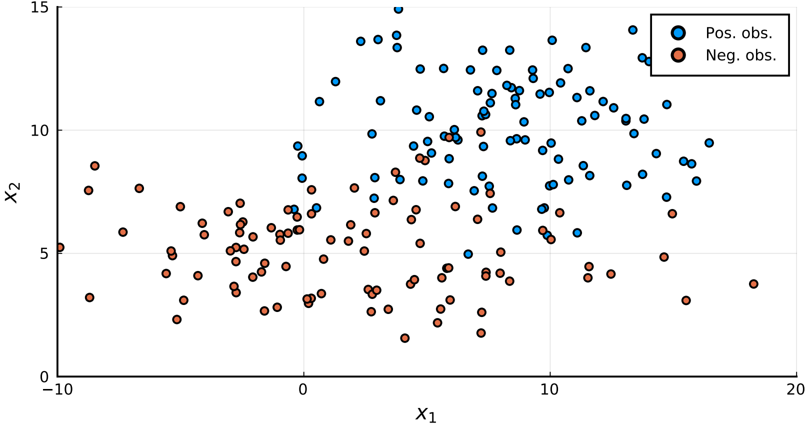
Our task is to obtain a function $f : \mathbb{R}^n \to \mathbb{R}$ from a given family of functions such that

$$f(x_i) < 0, i \in I^- \text{ and } f(x_i) > 0, i \in I^+.$$

f is selected as a linear classifier, i.e., $f(x_i) = a^\top x_i - b$, in which we try to set optimal a and b considering the classification error.

The best possible classifier is that which minimises misclassification.

Real-world model: classification problem (ML)



Real-world model: classification problem (ML)

Let us define the following error measures:

$$e^-(x_i \in I^-; a, b) = \begin{cases} 0, \text{ if } a^\top x_i - b \leq 0, \\ a^\top x_i - b, \text{ if } a^\top x_i - b > 0. \end{cases}$$

$$e^+(x_i \in I^+; a, b) = \begin{cases} 0, \text{ if } a^\top x_i - b \geq 0, \\ b - a^\top x_i, \text{ if } a^\top x_i - b < 0. \end{cases}$$

Using slack variables $\{u_i\}_{i=1}^M$ and $\{v_i\}_{i=1}^N$ to represent e^- and e^+, respectively, the optimal classifier is obtained from solving

$$\begin{aligned} (LC) \ : \ \text{min.} \ & \sum_{i=1}^M u_i + \sum_{i=1}^N v_i \\ \text{s.t.: } & a^\top x_i - b - u_i \leq 0, i = 1, \dots, M \\ & a^\top x_i - b + v_i \geq 0, i = 1, \dots, N \\ & a \in \mathbb{R}^n, b \in \mathbb{R} \\ & u_i \geq 0, i = 1, \dots, M; v_i \geq 0, i = 1, \dots, N. \end{aligned}$$

Real-world model: classification problem (ML)