Exercise class 5

Learning Objectives:

- Formulation and interpretation of dual problem
- Duality and sensitivity analysis

Table for formulating dual problems

For the formulation of dual problems you can use the following table which can be read from left to right or right to left depending if the primal problem's objective function is minimisation or maximisation.

Primal (dual)	**Dual (primal)**
maximise	minimise
Independent terms (b)	Obj. function coef. (c)
Obj. function coef. (c)	Independent terms (b)
i-th row of constraint coef.	i-th column of constraint coef.
i-th column of constraint coef.	i-th row of constraint coef.
Constraints	**Variables**
$\leq$	≥ 0
$\geq$	≤ 0
$=$	$\in \mathbb{R}$
Variables	**Constraints**
≥ 0	$\geq$
≤ 0	$\leq$
$\in \mathbb{R}$	$=$

Demo 1: Formulation of dual problem

Jeff wants to make a meal with minimum costs. The meal has to satisfy the family's requirements for protein and calcium: at least 50 g protein and 800 mg calcium. Formulate the problem as an LP according to the following table:

Food	**€/ unit**	**Protein g/unit**	**Calcium mg/unit**
Potato	0.6	3.7	22.7
Tomato	2.7	1.1	6.2
Milk	2.3	8.1	296

On the other hand there is local pharmacist who sells protein and calcium tablets. They have to decide the unit price for protein y_1 and calcium y_2 so that the tablets' prices would be competitive with the food prices. Their main objective is to maximise their own profits. This means that they try to find the highest price for Jeff's needs '50 g protein and 800 mg calcium'. Formulate the pharmacist's problem as an LP.

Note. This is a simplified version of Nobel-prize winner (in 1982) George Stigler's diet problem from 1939. His methodology to solve this problem is considered to be some of the earliest work in linear programming.

Demo 2: Dual Simplex

Solve the following LP using the Dual Simplex method.

$$\begin{aligned} \max.\ & z = -2x_1 - x_3 \\ \text{s.t. } & x_1 + x_2 - x_3 \geq 5 \\ & x_1 - 2x_2 + 4x_3 \geq 8 \\ & x_1,\ x_2,\ x_3 \geq 0 \end{aligned}$$

Problem 1: Formulation of dual problem

Formulate the dual problems for the following LP problems.

a)

$$\begin{aligned}\min.\ z = 2x_1 - x_2 + 3x_3 + 5x_4 \\ \text{s.t. } 6x_1 + 2x_2 - x_3 + x_4 \geq 8 \\ 2x_1 - x_2 + 5x_4 = 8 \\ x_1 + x_2 + 2x_3 + x_4 \leq 10 \\ x_1 \geq 0,\ x_2 \geq 0,\ x_3 \leq 0,\ x_4 \in \mathbb{R}\end{aligned}$$

b)

$$\begin{aligned}\max.\ z = 5x_1 + 6x_2 \\ \text{s.t. } 3x_1 + 4x_2 \leq 16 \\ -x_1 + 5x_2 \geq 3 \\ x_1 \in \mathbb{R},\ x_2 \geq 0\end{aligned}$$

c)

$$\begin{aligned}\min.\ z = 3x_1 + 4x_2 + 6x_3 \\ \text{s.t. } 5x_1 + 2x_2 \geq 20 \\ x_1 \geq 0 \\ x_2 \leq 0 \\ x_3 \geq 0\end{aligned}$$

d)

$$\begin{aligned}\max.\ z = x_1 + x_2 \\ \text{s.t. } 2x_1 + x_2 = 5 \\ 3x_1 - x_2 = 6 \\ x_1,\ x_2 \in \mathbb{R}\end{aligned}$$

Problem 2: Formulate & solve dual problem

Putte's Name Day party is about to begin. To his dismay he realised that he doesn't have any refreshments for his party. After quick consideration he settles on three options: fruit juice, berry juice and mixed juice. A litre of fruit juice requires 0,25 l fruit concentrate, a litre of berry juice requires 0,2 l of berry concentrate and a litre of mixed juice requires 0,1 l of fruit concentrate and 0,15 l of berry concentrate. In addition, the juices need some sugar: fruit juice 50 g, berry juice 75 g and mixed juice 60 g for a litre of juice.

Putte has 20 l of fruit concentrate, 45 l of berry concentrate and 5 kg of sugar. How much of each juice should Putte make to receive the maximum volume of juice?

Formulate as an LP problem and formulate the dual problem as well. Solve the problem in Julia.

Problem 3: Dual Simplex

Solve the following LPs using the Dual Simplex method with $x_1 = 0, x_2 = 0$ as initial solution:

Note. Optimality condition for min. problem is that all z-row coefficients are ≤ 0.

a)

$$\begin{aligned} \min.\ & z = 2x_1 + 3x_2 \\ \text{s.t. } & 2x_1 + 2x_2 \leq 3 \\ & x_1 + 2x_2 \geq 1 \\ & x_1,\ x_2 \geq 0 \end{aligned}$$

b)

$$\begin{aligned} \min.\ & z = 2x_1 + 3x_2 \\ \text{s.t. } & x_1 + 6x_2 \leq 15 \\ & 3x_1 - 5x_2 \leq 10 \\ & x_1, x_2 \geq 0 \end{aligned}$$

c)

$$\begin{aligned} \min.\ & z = 4x_1 + 2x_2 \\ \text{s.t. } & x_1 + x_2 = 10 \\ & 3x_1 - x_2 \geq 20 \\ & x_1,\ x_2 \geq 0 \end{aligned}$$

Problem 4: Primal - dual relationship

Solve the dual of the following problem in Julia, and then find its optimal objective value from the solution of the dual. Does the solution of the dual offer computational advantages over solving the primal directly?

$$
\begin{aligned}
\min.\ z = &50x_1 + 60x_2 + 30x_3 \\
\text{s.t. } &5x_1 + 5x_2 + 3x_3 \geq 50 \\
&x_1 + x_2 - x_3 \geq 20 \\
&7x_1 + 6x_2 - 9x_3 \geq 30 \\
&5x_1 + 5x_2 + 5x_3 \geq 35 \\
&2x_1 + 4x_2 - 15x_3 \geq 10 \\
&12x_1 + 10x_2 \geq 90 \\
&x_1,\ x_2,\ x_3 \geq 0
\end{aligned}
$$

Problem 5: Dual problem

Find the optimal value of the objective function for the following problem by inspecting only its dual problem. (Do not solve the dual by Simplex).

$$
\begin{aligned}
\text{Primal: } z = \min.\ &10x_1 + 4x_2 + 5x_3 \\
\text{s.t. } &5x_1 - 7x_2 + 3x_3 \geq 20 \\
&x_1,\ x_2,\ x_3 \geq 0
\end{aligned}
$$

Home Exercise 5: True or false?

a) The dual of the dual problem yields the original primal.

b) If the primal constraint is originally in an equality (=) form, the corresponding dual variable is necessarily unrestricted.

c) If the primal constraint is of the type $\leq$, the corresponding dual variable will be nonnegative (nonpositive) if the primal objective is maximization (minimization).

d) If the dual constraint is of the type $\geq$, the corresponding primal variable will be nonnegative (nonpositive) if the primal objective is minimization (maximization).

e) An unrestricted primal variable will result in an $\geq$ dual constraint.