Exercise class 11

Learning Objectives:

- Newton's method for constrained problems
- Barrier method and interior point method

Demo 1: Newton-Raphson for a system of nonlinear equations

Consider the system of equations:

$$x_1^2 + x_2^2 - 4x_3 = 0$$
$$x_1^2 + x_3^2 = \frac{1}{4}$$
$$x_1^2 + x_2^2 + x_3^2 = 1$$

Compute the first two iterations of the Newton-Raphson method by hand, and then implement in Julia to find a solution for the system. Consider $x^0 = [1, 1, 1]^\top$ and $\varepsilon = 0.001$.

Solution

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{bmatrix} = \begin{bmatrix} x_1^2 + x_2^2 - 4x_3 \\ x_1^2 + x_3^2 - \frac{1}{4} \\ x_1^2 + x_2^2 + x_3^2 - 1 \end{bmatrix}$$

The Jacobian is given by $\nabla f(x)$:

$$\nabla f(x) = \begin{bmatrix} 2x_1 & 2x_2 & -4 \\ 2x_1 & 0 & 2x_3 \\ 2x_1 & 2x_2 & 2x_3 \end{bmatrix}$$

Iteration 1:

$$d^0 = -\nabla f(x^0)^{-1} f(x^0) = -\begin{bmatrix} 2 & 2 & -4 \\ 2 & 0 & 2 \\ 2 & 2 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -2 \\ \frac{7}{4} \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{-5}{24} \\ \frac{-1}{8} \\ \frac{-2}{3} \end{bmatrix}$$

$$x^1 = x^0 + d^0 = [\tfrac{19}{24}, \tfrac{7}{8}, \tfrac{1}{3}]^\top \approx [0.79, 0.88, 0.33]^\top$$

Iteration 2:

$$d^1 = -\nabla f(x^1)^{-1} f(x^1) = -\begin{bmatrix} \frac{19}{12} & \frac{7}{4} & -4 \\ \frac{19}{12} & 0 & \frac{2}{3} \\ \frac{19}{12} & \frac{7}{4} & \frac{2}{3} \end{bmatrix}^{-1} \begin{bmatrix} -\frac{17}{288} \\ \frac{281}{576} \\ \frac{145}{288} \end{bmatrix} = \begin{bmatrix} \frac{-1711}{6384} \\ \frac{-1}{112} \\ \frac{-2}{21} \end{bmatrix}$$

$$x^2 = x^1 + d^1 \approx [0.52, 0.87, 0.24]$$

Implementing the algorithm in Julia (using $\varepsilon = 0.001$) gives:

```
Iteration 1: [0.7916666666666665, 0.875, 0.33333333333333337]
Iteration 2: [0.5236528822055138, 0.8660714285714286, 0.23809523809523808]
Iteration 3: [0.4473267879691847, 0.8660254050073638, 0.23606889564336372]
Iteration 4: [0.4408110310419283, 0.8660254037844386, 0.23606797749997818]
After 4 iterations the solution is x = [0.4408110310419283, 0.8660254037844386, 0.23606797749997818]
```

Demo 2: Newton for a system of linear equations

$$\begin{aligned} \min.\ & x_1 + x_2 \\ \text{s.t. } & 3x_1 + x_2 \leq 18 \\ & x_2 \leq 6 \\ & x_1,\ x_2 \geq 0 \end{aligned}$$

a) Write the Newton system for this problem, with intial point $x^0 = (2,4)$, $\rho^0 = 10$ and $\beta = 0.5$.

b) Perform one iteration of interior point method.

   Hint. Use Julia with the \ (backslash) operator.

Solution

a) The general Newton system can be stated as (slide 24)

$$\begin{bmatrix} 0 & A^\top & I \\ A & 0 & 0 \\ Z & 0 & X \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta \mu \\ \Delta z \end{bmatrix} = - \begin{bmatrix} A^\top \mu + z - c \\ Ax - b \\ XZe - \rho e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -XZe + \rho e \end{bmatrix}$$

For the first iteration we have:

$$\begin{bmatrix} 0 & A^\top & I \\ A & 0 & 0 \\ Z^0 & 0 & X^0 \end{bmatrix} \begin{bmatrix} \Delta x^0 \\ \Delta \mu^0 \\ \Delta z^0 \end{bmatrix} = - \begin{bmatrix} A^\top \mu^0 + z^0 - c \\ Ax^0 - b \\ X^0 Z^0 e - \rho_1 e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -X^0 Z^0 e + \rho_1 e \end{bmatrix}$$

For our problem, we have:

$$A = \begin{bmatrix} 3 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \text{ and } X^0 := diag(x_1, x_2, s_1, s_2) = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Where x_1 and x_2 are the values of x^0, and s_1 and s_2 are the slack variables for the constraints at x^0.

Then, we have

$$Z^0 := diag(z_1, z_2, z_3, z_4) = \begin{bmatrix} 10/2 & 0 & 0 & 0 \\ 0 & 10/4 & 0 & 0 \\ 0 & 0 & 10/8 & 0 \\ 0 & 0 & 0 & 10/2 \end{bmatrix}$$

Where $z = \rho X^{-1} e$, and e is a vector of ones.

Calculating ρ_1: $\rho_1 = \beta \cdot \rho_0 \implies -X^0 Z^0 e + \rho_1 e = [-5,-5,-5,-5]^\top$
Altogether, we have:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 3 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 2.5 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 1.25 & 0 & 0 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \Delta x_1^0 \\ \Delta x_2^0 \\ \Delta x_3^0 \\ \Delta x_4^0 \\ \Delta \mu_1^0 \\ \Delta \mu_2^0 \\ \Delta z_1^0 \\ \Delta z_2^0 \\ \Delta z_3^0 \\ \Delta z_4^0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -5 \\ -5 \\ -5 \\ -5 \end{bmatrix}$$

b) We will call our $[\Delta x^0, \Delta \mu^0, \Delta z^0]^\top$ the Δ vector for ease.

Then solving the Newton system we formulated in part (a) for the Δ vector in Julia, we get:

```
Δx = [-0.2325581395348837, 0.6046511627906976, 0.09302325581395224, -0.6046511627906976],
 Δμ = [0.6395348837209303, 0.9883720930232558],
 Δz = [-1.9186046511627908, -1.627906976744186, -0.63953488372093, -0.9883720930232559]
```

Performing one interior point method step $x^1 = x^0 + \Delta x^0$, gives us:
$x^1 = [2, 4] + [-0.2326, 0.6047] = [1.7674, 4.6047]$.

Problem 1: Nonlinear optimisation

$$\begin{aligned} \min.\ & x_1^4 + 2x_2^2 \\ \text{s.t. } & x_1 + 2x_2 = 10 \end{aligned}$$

Solve the optimisation problem using Newton method with equality constraint by hand for one iteration and fully solve in Julia. With starting point $x_0 = (2, 4)$ and tolerance 0.00001.

Solution

$$\nabla f(x) = \begin{bmatrix} 4x_1^3 \\ 4x_2 \end{bmatrix}, \qquad H(x) = \begin{bmatrix} 12x_1^2 & 0 \\ 0 & 4 \end{bmatrix}, \qquad A = [1\ 2]$$

The Newton's system is given by:

$$\begin{bmatrix} H(x) & A^\top \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \mu \end{bmatrix} = \begin{bmatrix} -\nabla f(x) \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 12x_1^2 & 0 & 1 \\ 0 & 4 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \mu \end{bmatrix} = \begin{bmatrix} -4x_1^3 \\ -4x_2 \\ 0 \end{bmatrix}$$

For $x_0 = (2, 4)$ we obtain $d_1 = \begin{bmatrix} 48 & 0 & 1 \\ 0 & 4 & 2 \\ 1 & 2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} -32 \\ -16 \\ 0 \end{bmatrix} \approx [-0.4898,\ 0.2449,\ -8.4898]^\top$

$x_1 = x_0 + [-0.4898,\ 0.2449]^\top = [1.5102,\ 4.2449]^\top$

Solving in Julia gives (while $||x_{k+1} - x_k|| > \varepsilon$):

```
Iteration 1: [1.5102040816326532 4.244897959183674]
Iteration 2: [1.3238151367310684 4.3380924316344665]
Iteration 3: [1.296411946573941 4.351794026713031]
Iteration 4: [1.2958522924386668 4.352073853780668]
Iteration 5: [1.2958520620959688 4.352073968952017]
After 5 iterations the solution is x = [1.2958520620959688 4.352073968952017],
Optimal objective function value is 40.70091767599013
```

Just out of curiosity, let's check with a nonlinear solver:

```julia
using JuMP, Ipopt
m = Model(with_optimizer(Ipopt.Optimizer))

@variable(m, x[1:2] >= 0)
@constraint(m, x[1] + 2*x[2] == 10)
@NLobjective(m, Min, x[1]^4 + 2*x[2]^2)

optimize!(m)

x_value = value.(x)
println("$(x_value), $(objective_value(m)) ")
```

```
EXIT: Optimal Solution Found.
[1.2958520621737515, 4.352073968913125], 40.700917675990105
```

Problem 2: Barrier method

$$\begin{aligned} \min.\ & (x+4)^2 \\ \text{s.t. } & x \geq 0 \end{aligned}$$

a) Write the optimality conditions for $\rho = 2,\ 1,\ 0.5$ and find the optimal x.

b) Show that as $\rho \to 0$, $x(\rho) \to x^*$.

Solution

a) The barrier problem is formulated as:

$$\min.\ (x+4)^2 - \rho \ln(x)$$

The optimality condition for this barrier problem is:

$$\begin{aligned} f'(x) + \phi'(x) = 2(x+4) - \frac{\rho}{x} &= 0 \\ \implies 2x^2 + 8x - \rho &= 0 \end{aligned}$$

Where $f(x) = (x+4)^2$ and $\phi(x) = -\rho \ln(x)$. Solving for x:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-8 \pm \sqrt{64 + 8\rho}}{4} \end{aligned}$$

For $\rho = 2$:

$$x = \frac{-8 \pm \sqrt{64+16}}{4} = 0.236 \text{ for } x \geq 0$$

For $\rho = 1$:

$$x = \frac{-8 \pm \sqrt{64+8}}{4} = 0.121 \text{ for } x \geq 0$$

For $\rho = 0.5$:

$$x = \frac{-8 \pm \sqrt{64+4}}{4} = 0.062 \text{ for } x \geq 0$$

b) $\lim_{\rho \to 0} \frac{-8 \pm \sqrt{64+8\rho}}{4} = \frac{-8 \pm \sqrt{64}}{4} = 0$. Therefore as $\rho \to 0$, $x(\rho) \to x^*$, as clearly, our optimal x^* is 0.

Problem 3: Barrier method for constrained problems (Interior point method)

$$\begin{aligned} \min.\ & x_1^2 + 2x_2^2 \\ \text{s.t. } & 2x_1 + x_2 \leq 9 \\ & x_1 + 2x_2 = 10 \\ & x_1,\ x_2 \geq 0 \end{aligned}$$

a) Formulate the barrier problem.

b) Form the newton system for $\rho^0 = 5$, $\beta = 0.5$ and initial solution (2,4).

Solution

a) The barrier problem is formulated as:

$$\begin{aligned} \min.\ & x_1^2 + 2x_2^2 - \rho \sum_{i=1}^{3} \ln(x_i) \\ \text{s.t. } & 2x_1 + x_2 + x_3 = 9 \\ & x_1 + 2x_2 = 10, \end{aligned}$$

where x_3 is the slack variable for constraint 1.

b) The Newton system can be stated as

$$\begin{bmatrix} 0 & A^\top & I \\ A & 0 & 0 \\ Z & 0 & X \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta \mu \\ \Delta z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -XZe + \rho e \end{bmatrix}$$

For the first iteration we have:

$$\begin{bmatrix} 0 & A^\top & I \\ A & 0 & 0 \\ Z^0 & 0 & X^0 \end{bmatrix} \begin{bmatrix} \Delta x^0 \\ \Delta \mu^0 \\ \Delta z^0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -X^0 Z^0 e + \rho_1 e \end{bmatrix}$$

$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$, $X^0 := diag(x_1, x_2, x_3)$, $Z^0 := diag(z_1, z_2, z_3) = \rho^0 \cdot \text{inv}(X^0)$

$x_1 = 2,\ x_2 = 4,\ x_3 = 1,\ z_1 = 5/2,\ z_2 = 5/4,\ z_3 = 5/1$

Calculating ρ_1: $\rho_1 = \beta \cdot \rho_0 \implies -X^0 Z^0 e + \rho_1 e = [-\frac{5}{2}, -\frac{5}{2}, -\frac{5}{2}, -\frac{5}{2}]^\top$

$$\begin{bmatrix} 0 & 0 & 0 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{5}{2} & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & \frac{5}{4} & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta x_1^0 \\ \Delta x_2^0 \\ \Delta x_3^0 \\ \Delta \mu_1^0 \\ \Delta \mu_2^0 \\ \Delta z_1^0 \\ \Delta z_2^0 \\ \Delta z_3^0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{5}{2} \\ -\frac{5}{2} \\ -\frac{5}{2} \end{bmatrix}$$

You can use the Julia function created in demo 2 to solve this Newton system.

```
Δx = [0.2236024844720495, -0.11180124223602483, -0.3354037267080746],
Δμ = [0.8229813664596272, -0.11645962732919239],
Δz = [-1.5295031055900619, -0.5900621118012424, -0.8229813664596272]
```