Example: Let $r \geq 0$. Is the matrix

$$\rho = \frac{1}{2}\begin{pmatrix} 1 + r\cos\theta & r\sin\theta\, e^{-i\phi} \\ r\sin\theta\, e^{i\phi} & 1 - r\cos\theta \end{pmatrix}$$

a density matrix? With what conditions on r, θ, and ϕ?

Solution: The matrix

- is Hermitian $\rho^\dagger = \rho$
- has unit trace $\operatorname{tr}\rho = \frac{1}{2}(1 + r\cos\theta) + \frac{1}{2}(1 - r\cos\theta) = 1$

but is it non-negative? Let's calculate its eigenvalues

$$\det(\rho - \lambda \mathbb{1}) = \left(\tfrac{1}{2}(1 + r\cos\theta) - \lambda\right)\left(\tfrac{1}{2}(1 - r\cos\theta) - \lambda\right) - \tfrac{1}{4}r^2\sin^2\theta = 0$$

$$\frac{1}{4} - \lambda + \lambda^2 - \frac{1}{4}r^2\cos^2\theta - \frac{1}{4}r^2\sin^2\theta = 0$$

$$\frac{1}{4} - \lambda + \lambda^2 - \frac{1}{4}r^2 = 0$$

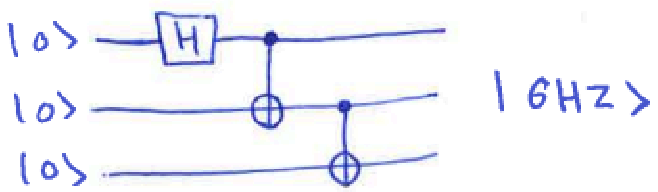
$$\rightarrow \lambda = \frac{1}{2}(1 \pm r)$$

ρ is non-negative if both eigenvalues are non-negative. Therefore, ρ is a density matrix if $r \leq 1$, with no conditions on θ and ϕ.

The GHZ state is a maximally entangled state on 3 qubits

$$|GHZ\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right).$$

It can be prepared with a circuit



Example: a) Compute $\rho_{AB} = tr_c |GHZ\rangle\langle GHZ|$

b) Compute $\rho_A = tr_B\, \rho_{AB}$

Solution:

$$|GHZ\rangle\langle GHZ| = \frac{1}{2}\begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix} = \rho_{ABC}$$

a)

$$\rho_{AB} = tr_c |GHZ\rangle\langle GHZ|$$

$$= \left(\mathbb{1}_{AB} \otimes (1\ 0)\right) \rho_{ABC} \left(\mathbb{1}_{AB} \otimes \begin{pmatrix}1\\0\end{pmatrix}\right)$$

$$+ \left(\mathbb{1}_{AB} \otimes (0\ 1)\right) \rho_{ABC} \left(\mathbb{1}_{AB} \otimes \begin{pmatrix}0\\1\end{pmatrix}\right)$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

b)

$$\rho_A = \left(\mathbb{1}_A \otimes (1 \ \ 0)\right) \rho_{AB} \left(\mathbb{1}_A \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$$

$$+ \left(\mathbb{1}_A \otimes (0 \ \ 1)\right) \rho_{AB} \left(\mathbb{1}_A \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

Notice that $S(\rho_A) = 1$, so each qubit in the GHZ state is maximally entangled with the other two.

Sometimes useful trick:

If A is a $n \times n$ matrix, then

$$\begin{aligned} tr(A) &= \lambda_1 + \lambda_2 + \dots + \lambda_n \\ tr(A^2) &= \lambda_1^2 + \lambda_2^2 + \dots + \lambda_n^2 \\ &\vdots \\ tr(A^n) &= \lambda_1^n + \lambda_2^n + \dots + \lambda_n^n, \end{aligned}$$

where λ_i are the eigenvalues of A. Then all eigenvalues can be reconstructed if $tr(A), tr(A^2), \dots, tr(A^n)$ are known.

Example: Consider the Hadamard matrix

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Compute $tr(H)$ and $tr(H^2)$. Then find the eigenvalues of H.

Solution:

$$tr(H) = 0$$

$$tr(H^2) = tr(\mathbb{1}) = 2$$

The eigenvalues satisfy

$$\begin{cases} \lambda_1 + \lambda_2 = 0 \\ \lambda_1^2 + \lambda_2^2 = 2 \end{cases} \longrightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = -1 \end{cases}.$$

Example: Let σ_2 be the second Pauli matrix. Then

$$\sigma_2 \otimes \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

a) Find the normalized state $(\gamma \in \mathbb{R})$

$$|\psi\rangle = e^{i\gamma \sigma_2 \otimes \sigma_2} |00\rangle$$

b) Find the values of γ such that $|\psi\rangle$ is a product state.

Solution:

a) From the first exercise problem set we know that

$$e^{i\gamma \sigma_2 \otimes \sigma_2} = \cos\gamma \, \mathbb{1} + i \sin\gamma \, (\sigma_2 \otimes \sigma_2).$$

Therefore,

$$|\psi\rangle = e^{i\gamma \sigma_2 \otimes \sigma_2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\gamma \\ 0 \\ 0 \\ -i\sin\gamma \end{pmatrix}.$$

b) Product states of two qubits are of the form

$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix}.$$

So, if $|\psi\rangle$ is a product state, then

$$\begin{cases} ac = \cos\gamma \\ ad = 0 \\ bc = 0 \\ bd = -i\sin\gamma \end{cases}$$

for some a, b, c, d. Notice that either $a = 0$ or $d = 0$ since $ad = 0$. Therefore both $\cos\gamma$ and $-i\sin\gamma$ cannot be non-zero.

$$\rightarrow \quad \gamma = 0 \quad \text{or} \quad \gamma = \frac{\pi}{2}$$

and

$$\gamma = 0: \quad |\psi\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |00\rangle$$

$$\gamma = \frac{\pi}{2}: \quad |\psi\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |11\rangle$$

up to a global phase.