Quantum Information

(ELEC-C9440)

Lectures 6 & 7

Matti Raasakka
University Lecturer
Micro and Quantum Systems group
Dept. of Electronics and Nanoengineering
School of Electrical Engineering

$\langle \mu | Q a^{\dagger} | S - 1 \rangle$
Micro and Quantum Systems

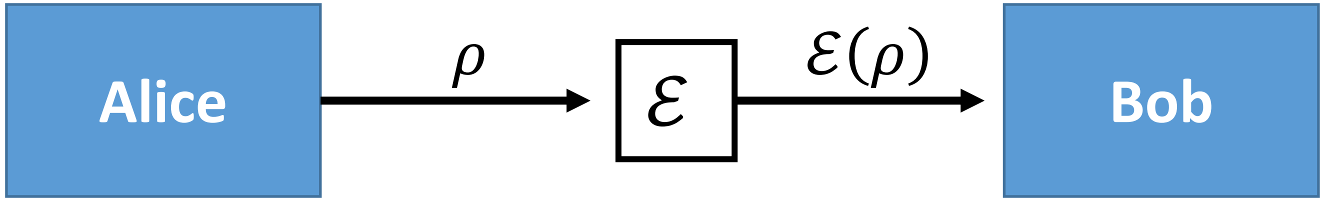
Aalto University
School of Electrical Engineering

Topics

- Quantum channels and noise
- Quantum error correction
- Fault-tolerant quantum computing

Quantum channels and operations

Quantum channels and operations



- **Quantum channel** is an abstract model used in the field of quantum information theory to represent possible changes to the state of a quantum system as it is being transmitted from one place to another, or just from one time to another.
- Basically, quantum channel is just some **transformation of the density matrix**, which represents the effect of the channel on the state. Such transformations can be most generally described by the so-called **quantum operations**.
- **Quantum operations** $\rho \mapsto \mathcal{E}(\rho)$ are the most general transformations of density matrices compatible with the postulates of quantum mechanics: linearity, normalization and (semi-definite) positivity.
- An arbitrary quantum operation can be non-uniquely specified via the **operator-sum representation** (more on this on the next slide).

Operator–sum representation

- **Operator-sum representation** of a quantum operation:

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger$$

  where the **operation elements** E_k satisfy $\sum_k E_k^\dagger E_k \leq I$.

- If $\sum_k E_k^\dagger E_k = I$, the operation is **trace-preserving:**

$$\mathrm{tr}\big(\mathcal{E}(\rho)\big) = \mathrm{tr}\left(\sum_k E_k \rho E_k^\dagger\right) = \mathrm{tr}\left(\rho \sum_k E_k^\dagger E_k\right) = \mathrm{tr}(\rho) = 1$$

  Preserves correct normalization of the density matrix.

- If $\sum_k E_k^\dagger E_k < I$, some of the information about the state is transferred outside the system in the process, e.g., via a measurement. The density matrix must be normalized afterwards:

$$\rho \mapsto \rho' = \frac{\mathcal{E}(\rho)}{\mathrm{tr}\big(\mathcal{E}(\rho)\big)} = \frac{\sum_k E_k \rho E_k^\dagger}{\mathrm{tr}\big(\sum_k E_k \rho E_k^\dagger\big)}$$

$(AB)^T = B^T A^T$

$\mathrm{tr}(ABC) = \mathrm{tr}(CAB) = \mathrm{tr}(BCA)$

$\left(E_k \rho E_k^\dagger\right)^\dagger$

$= (E_k^\dagger)^\dagger \rho^\dagger E_k^\dagger$

$= E_k \rho E_k^\dagger$

$A \leq I$

$\Longleftrightarrow$

$I - A \geq 0$

eigenvalues ≥ 0

Examples of quantum operations

$(I\otimes|e_0\rangle)\rho(I\otimes\langle e_0|)$

$= \rho\otimes|e_0\rangle\langle e_0|$

- **Unitary time-evolution**

$$\mathcal{E}(\rho) = U(t)\rho U(t)^\dagger$$

  $U(t)^\dagger U(t) = I$ by definition.

$\mathrm{tr}_2(A\otimes B) = \sum_k (I\otimes\langle e_k|)(A\otimes B)(I\otimes|e_k\rangle)$

$= \sum_k A\otimes\langle e_k|B|e_k\rangle$

- **Measurement** described by a set of measurement operators M_m.

$= \sum_k \langle e_k|B|e_k\rangle A$

$$\mathcal{E}(\rho) = M_m\rho M_m^\dagger$$

  where M_m is the operator corresponding to the recorded measurement result m.
  NOT trace-preserving. State after measurement

$= \mathrm{tr}(B)$

$$\rho' = \frac{M_m\rho M_m^\dagger}{\mathrm{tr}\left(M_m\rho M_m^\dagger\right)}$$

$(A\otimes B)|e_1\rangle|e_2\rangle$

$= A|e_1\rangle\otimes B|e_2\rangle$

- **Interaction with the environment**, e.g., in initial state $\rho_{\mathrm{env}} = |e_0\rangle\langle e_0|$

$$\rho' = \mathrm{tr}_{\mathrm{env}}\left(U(\rho\otimes\rho_{\mathrm{env}})U^\dagger\right) = \sum_k (I\otimes\langle e_k|)U(\rho\otimes|e_0\rangle\langle e_0|)U^\dagger(I\otimes|e_k\rangle) = \sum_k E_k\rho E_k^\dagger$$

  where $E_k = (I\otimes\langle e_k|)U(I\otimes|e_0\rangle)$.

Examples of quantum operations

- Quantum operations can also **increase or decrease the Hilbert space dimension**.
- For example, **adding an ancilla qubit** in state $|0\rangle$ corresponds to the operator element $E = I \otimes |0\rangle$, where the first tensor product factor I operates on the original Hilbert space, so that
$$\mathcal{E}(\rho) = E\rho E^\dagger = (I \otimes |0\rangle)\rho(I \otimes \langle 0|) = \rho \otimes |0\rangle\langle 0|$$
- **Measuring a qubit and discarding it** afterwards corresponds to the operator element $E_k = I \otimes \langle k|$ depending on the measurement result $k = 0,1$.
$$\mathcal{E}(\rho) = E\rho E^\dagger = (I \otimes \langle k|)\rho(I \otimes |k\rangle)$$
- If the **measurement result is not recorded**, both operators in the operator-sum representation of the quantum operation (= partial trace over the discarded qubit):
$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger = \sum_k (I \otimes \langle k|)\rho(I \otimes |k\rangle)$$

Unitary freedom in operator-sum representation

- **Theorem** (Nielsen-Chuang 8.2):

  Let $\{E_1, \ldots, E_m\}$ and $\{F_1, \ldots, F_n\}$ be the operator elements giving rise to quantum operations $\mathcal{E}$ and $\mathcal{F}$. By appending zero operators to the shorter list we can ensure that $m = n$. Then $\mathcal{E} = \mathcal{F}$ if and only if there exists an m-by-m unitary matrix (u_{ij}) such that

$$E_i = \sum_j u_{ij} F_j$$

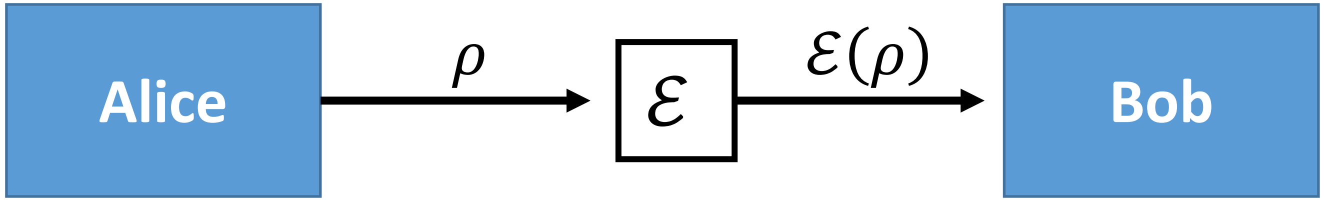
- The "if"-part is easy enough to verify:

$$\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger = \sum_i \left(\sum_j u_{ij} F_j\right) \rho \left(\sum_k u_{ik}^* F_k^\dagger\right) = \sum_j \sum_k \overbrace{\left(\sum_i u_{ij} u_{ik}^*\right)}^{= \delta_{jk}} F_j \rho F_k^\dagger$$

$$= \sum_j F_j \rho F_j^\dagger = \mathcal{F}(\rho)$$

Quantum noise models

Quantum noise



- There is often **noise** in a quantum channel, which tends to degrade the fidelity of the transmitted quantum state. This can be modelled as particular kind of **quantum operations** acting on the quantum state.
- The types of noise and their abundance in the channel depend on the particular type of application. E.g., optical fiber vs superconducting qubit.
- Typical noise models to consider:
  - Bit-flip errors, phase-flip errors
  - Depolarizing noise
  - Amplitude damping noise
  - …

Bit–flip noise on a qubit

- In a bit-flip noise channel, a **bit-flip error** flips the value of a qubit with probability p. This is the quantum analogue of the classical bit-flip noise often considered in classical information theory.
- In the **operator-sum representation** can be written as

$$\rho \mapsto \rho' = (1-p)\rho + pX\rho X^\dagger$$

where X is the Pauli X matrix/the NOT gate.

$|\psi\rangle \mapsto X|\psi\rangle$

$\rho_\psi = |\psi\rangle\langle\psi|$

$\mapsto \underbrace{X|\psi\rangle}\underbrace{\langle\psi|X}$

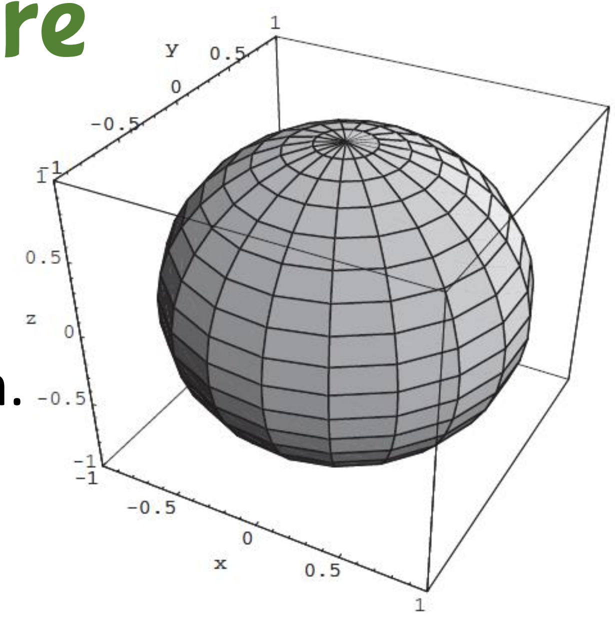
- The **operator-sum elements** thus are $E_1 = \sqrt{1-p}I,\ E_2 = \sqrt{p}X$.
- For example, acting on the pure state $\rho = |0\rangle\langle 0|$ we get

$U\rho U^\dagger$

$$\rho' = (1-p)|0\rangle\langle 0| + p|1\rangle\langle 1|$$

which is not a pure state anymore, but a **probabilistic mixture** of the two computational basis states.

Effect of bit-flip noise on the Bloch sphere



- Parametrize the density matrix as

$$\rho = \frac{1}{2}(I + \vec{r} \cdot \vec{\sigma})$$

where $\|\vec{r}\| \leq 1$. $\vec{r}$ is the vector in the Bloch sphere representation.

- Bit-flip noise changes this to

$$\rho' = (1-p)\frac{1}{2}(I + \vec{r} \cdot \vec{\sigma}) + pX\frac{1}{2}(I + \vec{r} \cdot \vec{\sigma})X$$

$$= \frac{1}{2}(I + (1-p)(\vec{r} \cdot \vec{\sigma}) + pX(\vec{r} \cdot \vec{\sigma})X)$$

$$= \frac{1}{2}\left(I + r_x\sigma_x + (1-2p)\left(r_y\sigma_y + r_z\sigma_z\right)\right)$$

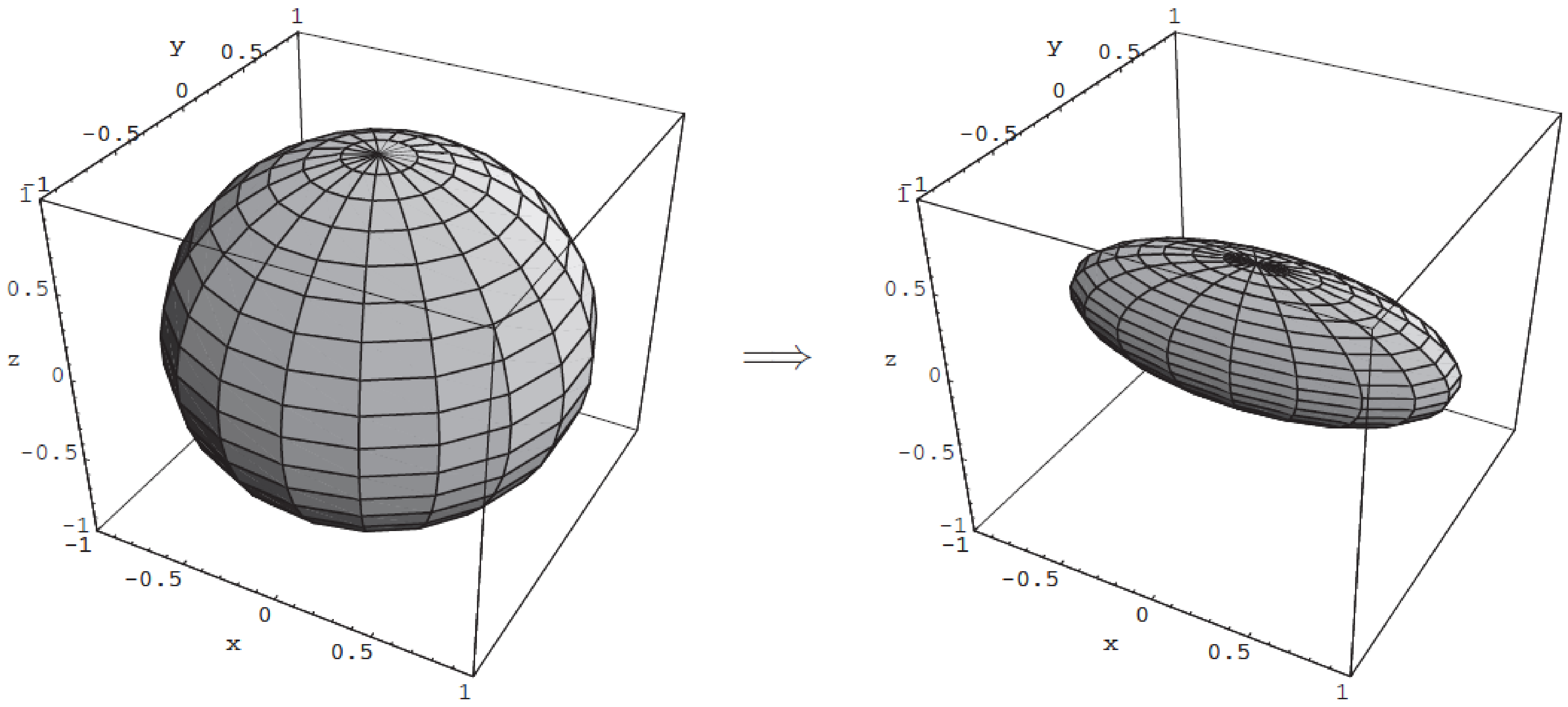
Note:

$$X\sigma_x X = \sigma_x$$
$$X\sigma_y X = -\sigma_y$$
$$X\sigma_z X = -\sigma_z$$

- Thus, for the Bloch sphere vector $(r_x, r_y, r_z) \mapsto \left(r_x, (1-2p)r_y, (1-2p)r_z\right)$.

Effect of bit-flip noise on the Bloch sphere

- Graphical illustration of bit-flip noise on the Bloch sphere:



- Notice that the X-basis states $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ are invariant under bit-flip noise (up to overall sign).

Phase–flip noise on a qubit

- In a phase-flip noise channel, a **phase-flip error** flips the relative sign of the basis states $|0\rangle$ and $|1\rangle$ of the qubit with probability p. **No classical analogue!**
- In the **operator-sum representation** can be written as

$Z|0\rangle = |0\rangle$

$Z|1\rangle = \underline{\underline{-}}\,|1\rangle$

$$\rho \mapsto \rho' = (1-p)\rho + pZ\rho Z$$

where Z is the Pauli Z matrix/the phase gate.

- The **operator-sum elements** thus are $E_1 = \sqrt{1-p}I,\ E_2 = \sqrt{p}Z$.
- For example, acting on the pure state $\rho = |\pm\rangle\langle\pm|$ we get

$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

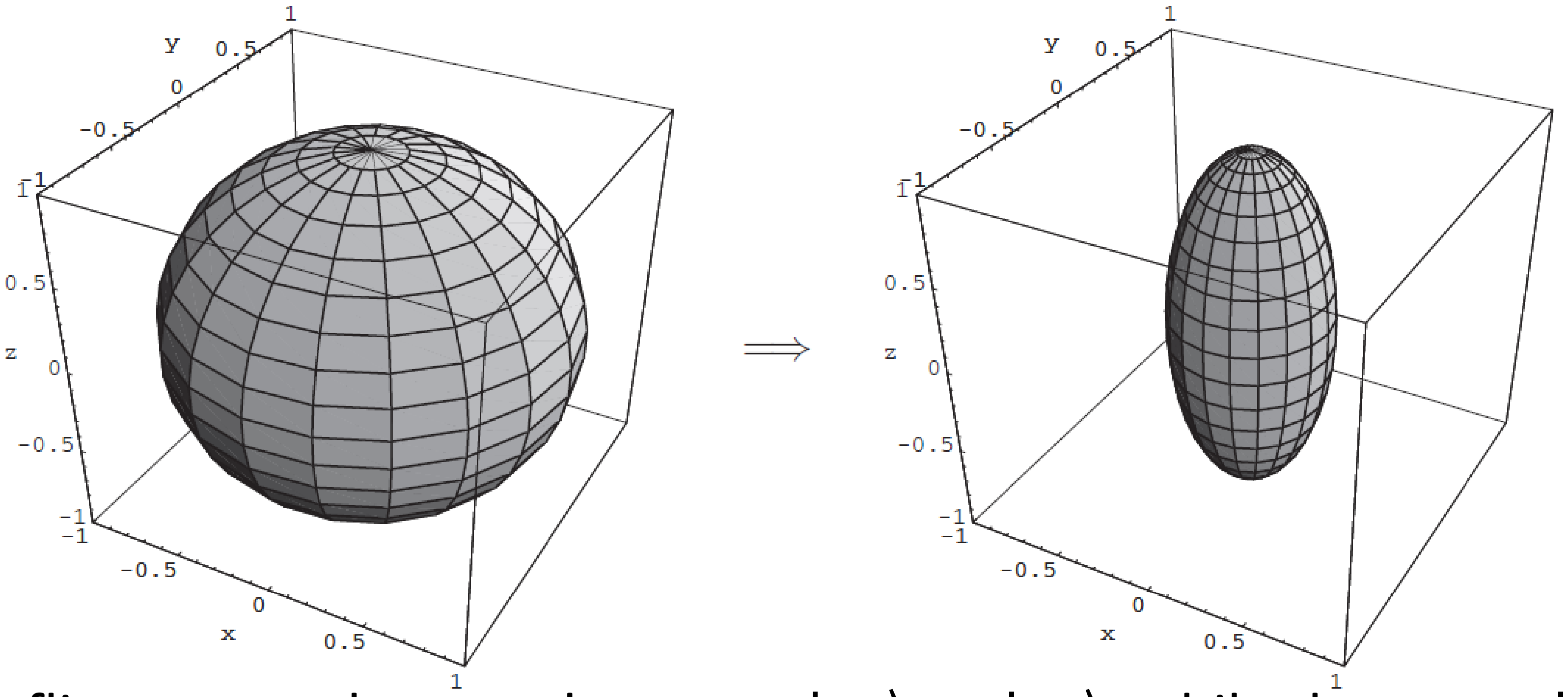
$\xmapsto{Z} |-\rangle$

$$\rho' = (1-p)|\pm\rangle\langle\pm| + p|\mp\rangle\langle\mp|$$

which is not a pure state anymore, but a **probabilistic mixture** of the two X-basis states $|\pm\rangle$.

Phase–flip noise on a qubit

- Graphical illustration of phase-flip noise on the Bloch sphere:



$HXH = Z$

$HZH = X$

- A phase-flip error exchanges the states $|+\rangle \leftrightarrow |-\rangle$, while the states $|0\rangle, |1\rangle$ are invariant (up to overall phase). Accordingly, it is analogous to the bit-flip error in the $|\pm\rangle$ basis. The bit-flip and phase-flip errors are **related through the Hadamard operator** H, since this maps the two bases to each other:

$$HE_1H = \sqrt{1-p}H^2 = \sqrt{1-p}I, \qquad HE_2H = \sqrt{p}HZH = \sqrt{p}X$$

Depolarizing noise

- In a **depolarizing channel**, the density matrix ρ is replaced by the completely mixed state (density matrix $I/2$) with probability p:

$$\rho \mapsto \rho' = (1-p)\rho + \frac{p}{2} I$$

$\rho_{max} = \frac{1}{2} I$

- Depolarizing noise can be particularly relevant in a **quantum optical communication setup**, where the qubit is represented by the polarization state of a photon.
- **Operator-sum representation** can be obtained by first noting that

$$I = \frac{1}{2}(\rho + X\rho X + Y\rho Y + Z\rho Z)$$

$Y = iXZ$

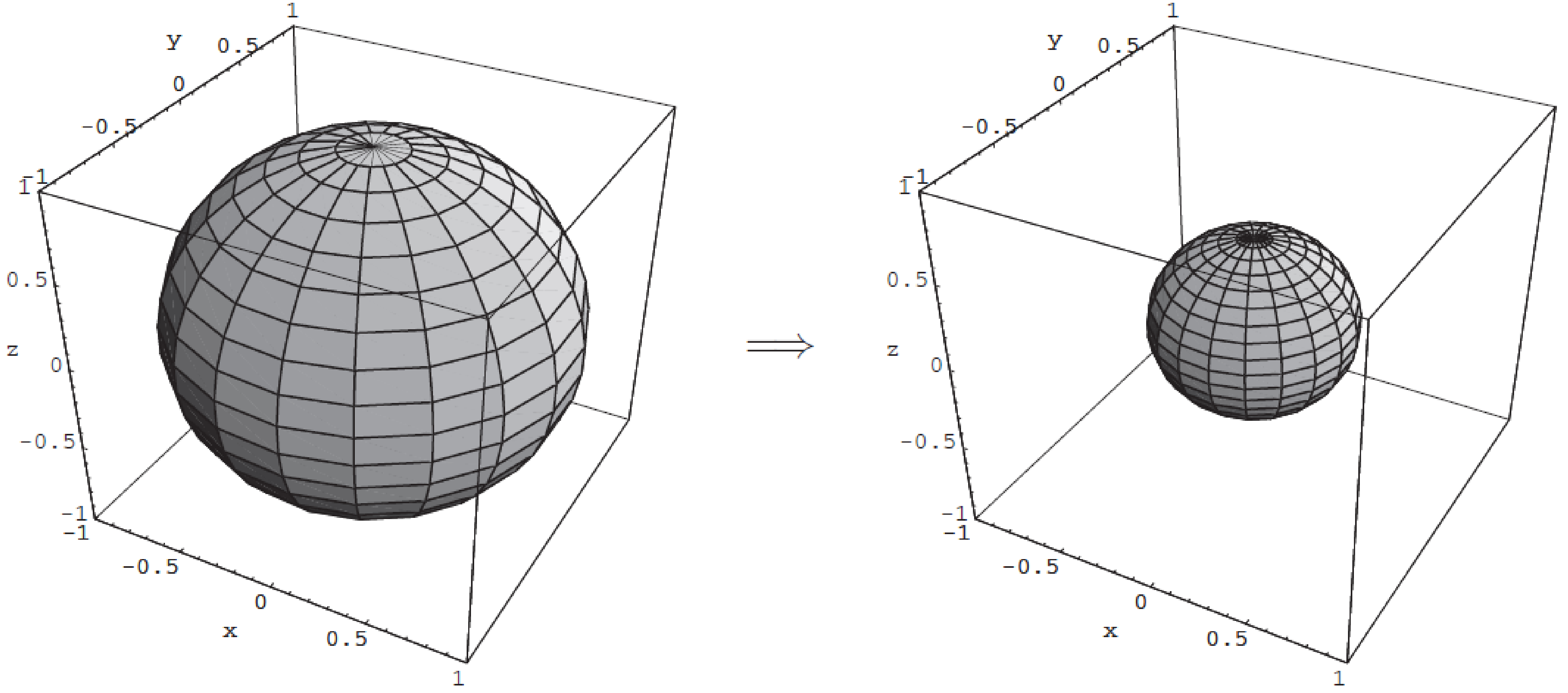
for any density matrix ρ. By substitution we get

$$\rho' = \left(1 - \frac{3}{4}p\right)\rho + \frac{p}{4}(X\rho X + Y\rho Y + Z\rho Z)$$

- Accordingly, the depolarizing channel may also be interpreted as a combination of a bit-flip error ($X\rho X$), a phase-flip error ($Z\rho Z$) and a combined bit-phase-flip error ($Y\rho Y = XZ\rho ZX$), each with probability $p/4$.

Depolarizing noise

- Graphical illustration of depolarizing noise on the Bloch sphere:



Amplitude damping noise

- In an **amplitude damping channel**, the probability of the state $|0\rangle$ is increased over the state $|1\rangle$. Amplitude damping noise can be caused by, e.g., **energy dissipation to the environment** in a system (such as a superconducting qubit), where $|0\rangle$ is the ground state and $|1\rangle$ an excited state.
- **Operator-sum representation**: $\rho \mapsto \rho' = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger$, where the operator-sum elements have the matrix forms (in the computational basis)

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \qquad E_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix} \qquad (0 < \gamma < 1)$$

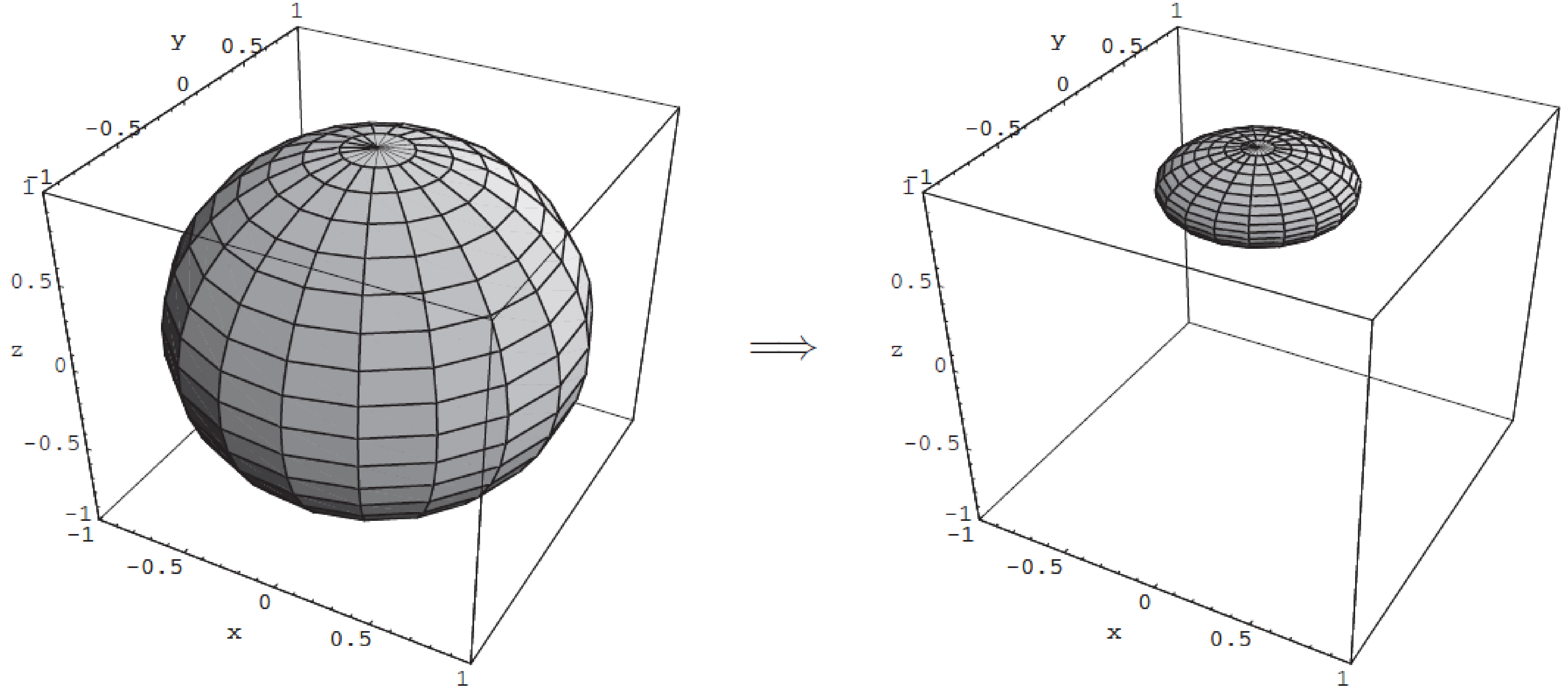
- For example, acting on the pure state $\rho = |1\rangle\langle 1|$ we get

$$\rho' = \gamma |0\rangle\langle 0| + (1-\gamma)|1\rangle\langle 1|$$

  i.e., the state has decayed to the "ground" state $|0\rangle$ with probability γ.
- NOTE: The state $|0\rangle$ is invariant under amplitude damping noise.

Amplitude damping noise

- Graphical illustration of amplitude damping noise on the Bloch sphere:



Generalized amplitude damping noise

- The simple amplitude damping noise corresponds to the **environment at absolute zero temperature**, since the system tends always to *lose* energy to the environment. However, in realistic situations, the environment is at some finite temperature.
- When the **environment is at a finite temperature**, the system can also gain energy from the environment. Thus, the system is not in the ground state in equilibrium with the environment, but in some mixture of the ground and the excited state. The exchange of energy with the environment in this situation can be most simply described by the **generalized amplitude damping noise**.
- The generalized amplitude damping noise has the **operator-sum representation**:

$$E_0 = \sqrt{p}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \quad E_1 = \sqrt{p}\begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$$

Increase probability of $|0\rangle$

$$E_2 = \sqrt{1-p}\begin{pmatrix} \sqrt{1-\gamma} & 0 \\ 0 & 1 \end{pmatrix}, \quad E_3 = \sqrt{1-p}\begin{pmatrix} 0 & 0 \\ \sqrt{\gamma} & 0 \end{pmatrix}$$

Increase probability of $|1\rangle$

Generalized amplitude damping noise

- The **stationary state** under generalized amplitude damping noise is

$$\rho_\infty = \begin{pmatrix} p & 0 \\ 0 & 1-p \end{pmatrix}$$

  which satisfies $\sum_k E_k \rho_\infty E_k^\dagger = \rho_\infty$.

- ρ_∞ can be related to the **Boltzmann distribution** for the two states $|0\rangle, |1\rangle$ with energies ϵ_0, ϵ_1 at temperature T in equilibrium:

$1 - p \sim e^{-t/T_1}$

$$p = \frac{e^{-\frac{\epsilon_0}{kT}}}{Z}, \qquad 1 - p = \frac{e^{-\frac{\epsilon_1}{kT}}}{Z}, \qquad Z = e^{-\frac{\epsilon_0}{kT}} + e^{-\frac{\epsilon_1}{kT}}$$

- The qubit relaxes to the stationary state spontaneously due to the interaction with the environment. The **relaxation time** to the stationary state related to generalized amplitude damping noise and energy dissipation is usually denoted by $\boxed{T_1}$, and is an important metric to characterize e.g. the qubit performance in a quantum processor.

ibmq_santiago

Your upcoming reservations 0

Calibration data

Last calibrated: 37 minutes ago

Map view | Graph view | **Table view**

Qubit	T1 (us)	T2 (us)	Frequency (GHz)	Anharmonicity (GHz)	Readout assignment error
Q0	162.34	209.84	4.833	-0.34189	1.057e-1
Q1	147.21	85.97	4.624	-0.32823	2.780e-2
Q2	112.95	68.48	4.821	-0.34107	1.080e-2
Q3	157.49	95.29	4.742	-0.34013	3.050e-2
Q4	120.29	133.94	4.816	-0.34291	1.370e-2

Phase damping noise

- **Phase damping noise** causes loss of information about the relative phases between energy eigenstates without the loss of energy. Can be used to model, e.g., the random scattering of photons in a waveguide/optical fiber.

- **Operator-sum representation**: $\rho \mapsto \rho' = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger$, where the operator-sum elements have the matrix forms (in the computational basis)

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda} \end{pmatrix}, \qquad E_1 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix} \qquad (0 < \lambda < 1)$$

$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

- For example, acting on the pure state $\rho = |+\rangle\langle+|$ we get

$$|+\rangle\langle+| = \frac{1}{2}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \mapsto E_0|+\rangle\langle+|E_0^\dagger + E_1|+\rangle\langle+|E_1$$

$$\mapsto \frac{1}{2}\left(|0\rangle\langle 0| + \underline{\sqrt{1-\lambda}(|0\rangle\langle 1| + |1\rangle\langle 0|)} + |1\rangle\langle 1|\right) \sim \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$$

Cross-terms are damped by factor $\sqrt{1-\lambda}$. Loss of quantum coherence.

- NOTE: Both states $|0\rangle, |1\rangle$ are invariant by themselves under phase damping noise.

Phase damping noise

- In fact, **phase damping channel = phase-flip channel**. Connected via the unitary

$$(u_{ij}) = \begin{pmatrix} \sqrt{\alpha} & \frac{1-\sqrt{1-\lambda}}{\sqrt{\lambda}}\sqrt{\alpha} \\ \sqrt{1-\alpha} & -\frac{1+\sqrt{1-\lambda}}{\sqrt{\lambda}}\sqrt{1-\alpha} \end{pmatrix}, \qquad \alpha = \frac{1+\sqrt{1-\lambda}}{2}$$

$$E_0' = \sum_j u_{0j}E_j = \sqrt{\alpha}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad E_1' = \sum_j u_{1j}E_j = \sqrt{1-\alpha}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- The **relaxation time** related to the decay of coherence is often denoted by T_2, and related to the parameters via $e^{-t/2T_2} = \sqrt{1-\lambda}$.

ibmq_santiago

Your upcoming reservations 0

Calibration data

Last calibrated: 37 minutes ago

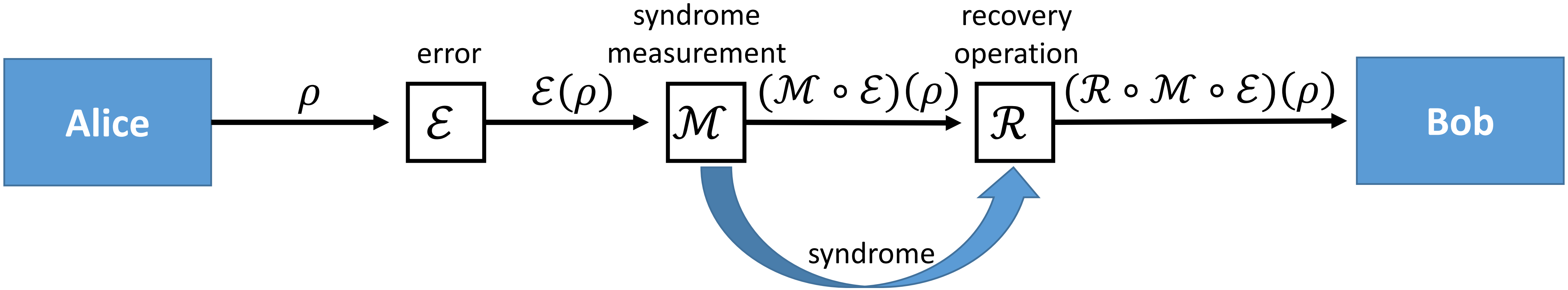
Map view Graph view Table view

Qubit	T1 (us)	T2 (us)	Frequency (GHz)	Anharmonicity (GHz)	Readout assignment error
Q0	162.34	209.84	4.833	-0.34189	1.057e-1
Q1	147.21	85.97	4.624	-0.32823	2.780e-2
Q2	112.95	68.48	4.821	-0.34107	1.080e-2
Q3	157.49	95.29	4.742	-0.34013	3.050e-2
Q4	120.29	133.94	4.816	-0.34291	1.370e-2

Quantum error correction

Quantum error correction

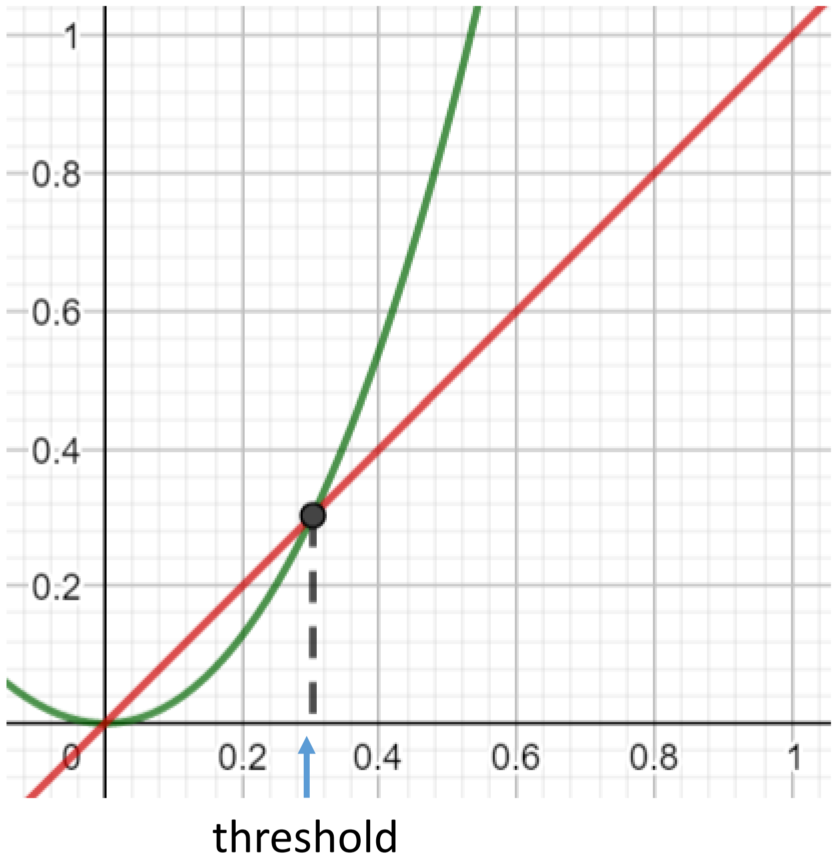
Quantum channel with quantum error correction:



- We can mitigate the effect of noise in a quantum channel by using **quantum error correction** methods.
- General idea in error correction: Protect the information via **redundancy**. Encode one logical qubit in several physical qubits.
- Increases overhead in communication/computing, but reduces error, thus allowing for longer computations.

Classical repetition code

- Encode a "logical" bit into several physical bits, e.g. 3-bit classical repetition code
$$0 \mapsto 000, 1 \mapsto 111$$
- Decode by majority voting
$$000,001,010,100 \mapsto 0$$
$$111,110,101,011 \mapsto 1$$
- Probability of physical bit flip $= p$.
=> Prob of logical bit flip $p_L = 3p^2 + p^3$.
- Suppression of error by redundancy **below threshold** physical error rate p_{th}!
- Logical error rate can be further improved by using more physical bits.



Error detection with syndrome

- Error **syndrome** is used to detect errors. Consists of quantities, whose values tell us which error happened.
- E.g. for 3-bit repetition code bit sequence $b_1 b_2 b_3$ syndrome is

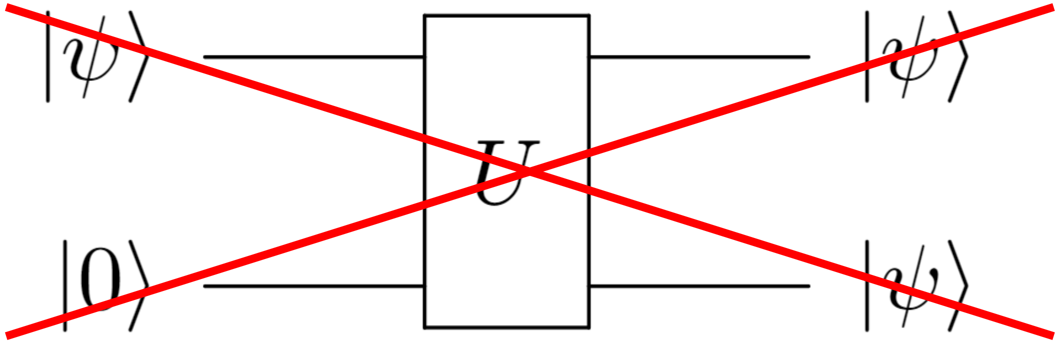
$$s_1 = b_1 + b_2 \pmod 2, \qquad s_2 = b_2 + b_3 \pmod 2$$

	s_1	s_2	error type
$000,111 \longrightarrow$	0	0	no error
$001,110 \longrightarrow$	0	1	b_3 flipped
$100,011 \longrightarrow$	1	0	b_1 flipped
$010,101 \longrightarrow$	1	1	b_2 flipped

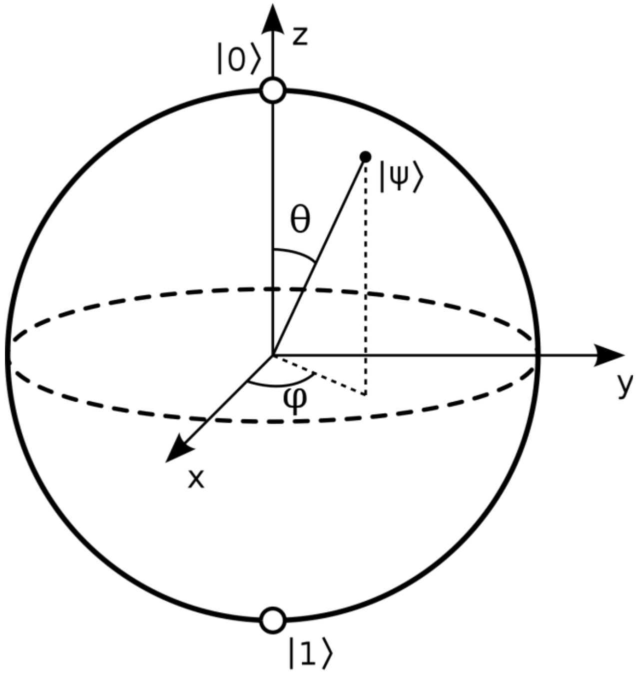
- Repetition code allows also for error correction. Some codes may only allow error detection. Requires less overhead but if error detected, must try again. In QC we usually want EC, because error rate is high.

Challenges to quantum error correction

1. Cannot copy quantum states (no-cloning theorem)



2. Cannot measure quantum system without collapsing the state



3. Infinite number of different errors to correct

Amazingly, all of these issues can be circumvented!

3-qubit codes

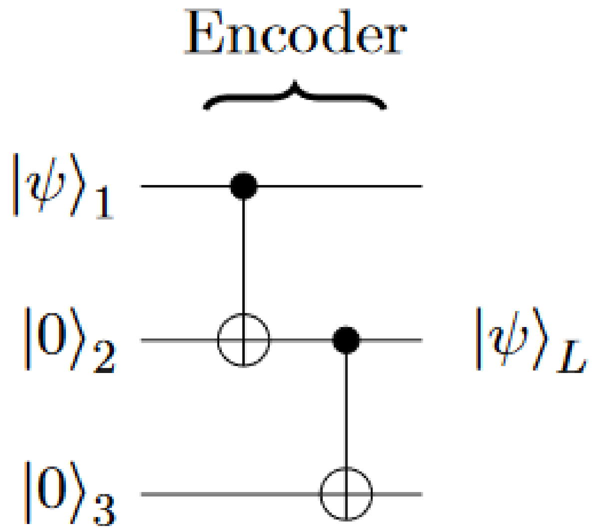
3-qubit bit-flip code, encoding

- Implemented by the encoding
$$|0\rangle_L = |000\rangle, \qquad |1\rangle_L = |111\rangle$$

- One logical qubit encoded into an **entangled state** of three qubits.

- Encoder maps superposition states as
$$|\psi\rangle = a|0\rangle + b|1\rangle \mapsto a|000\rangle + b|111\rangle = |\psi\rangle_L$$

- **NOTE:** $|\psi\rangle_L \neq |\psi\rangle|\psi\rangle|\psi\rangle$!
Not prohibited by the no-cloning theorem.



3–qubit bit–flip code, error detection

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- **Syndrome** as in 3-bit rep. code:

$$S_1 = Z_1 Z_2, \qquad S_2 = Z_2 Z_3$$

- Bit-flip error **changes the sign of some syndrome observable**, because they anti-commute. E.g., $S_1 X_1 = -X_1 S_1$.

$$Z_1 Z_2 |010\rangle = (Z \otimes Z \otimes I)|010\rangle$$

$$= \underbrace{Z|0\rangle}_{=|0\rangle} \otimes \underbrace{Z|1\rangle}_{=-|1\rangle} \otimes \underbrace{I|0\rangle}_{=|0\rangle} = -|010\rangle$$

$$\underbrace{|000\rangle}_{=|0_L\rangle}, \underbrace{|111\rangle}_{=|1_L\rangle}$$

	S_1	S_2	error
$\|000\rangle, \|111\rangle$ →	1	1	no error (I)
$\|001\rangle, \|110\rangle$ →	1	-1	q_3 flipped (X_3)
$\|100\rangle, \|011\rangle$ →	-1	1	q_1 flipped (X_2)
$\|010\rangle, \|101\rangle$ →	-1	-1	q_2 flipped (X_1)

$$|\psi_L\rangle = a|000\rangle + b|111\rangle$$

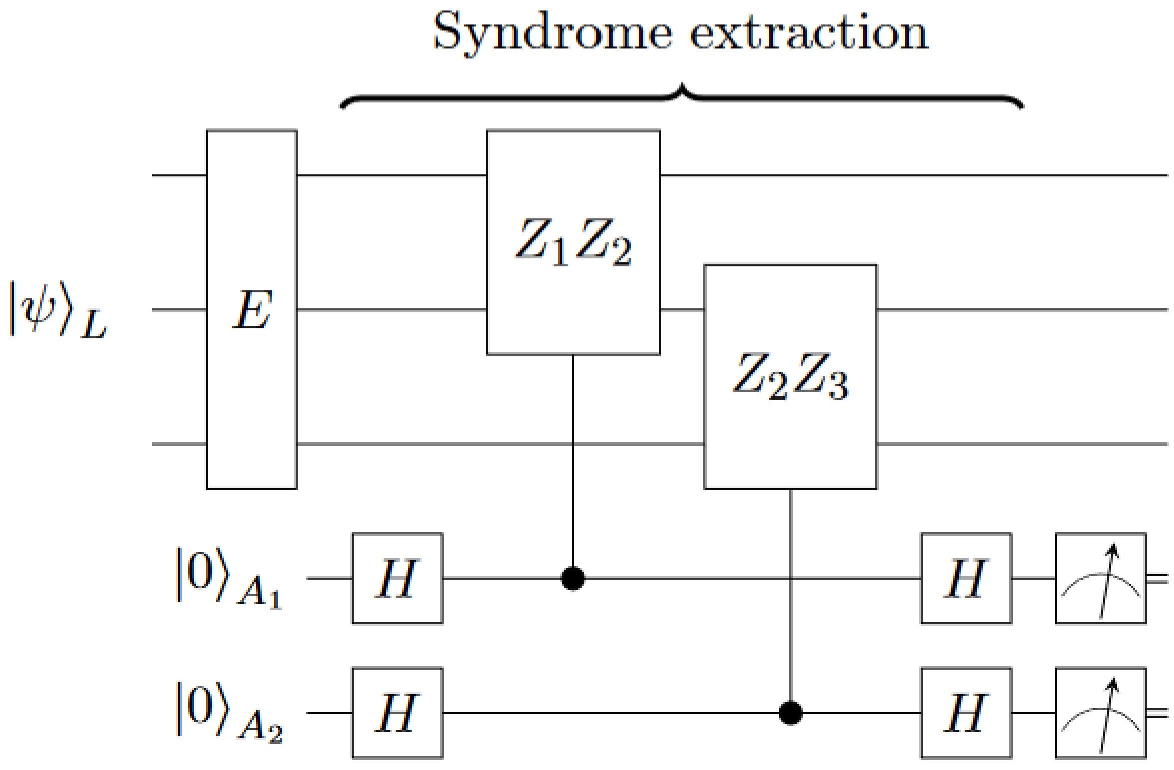
$$\xmapsto{X_3} a|001\rangle + b|110\rangle$$

- Can use two **ancilla qubits** A_1, A_2 to measure the syndrome. "**Parity checks**"
- **Does not collapse the logical state!**

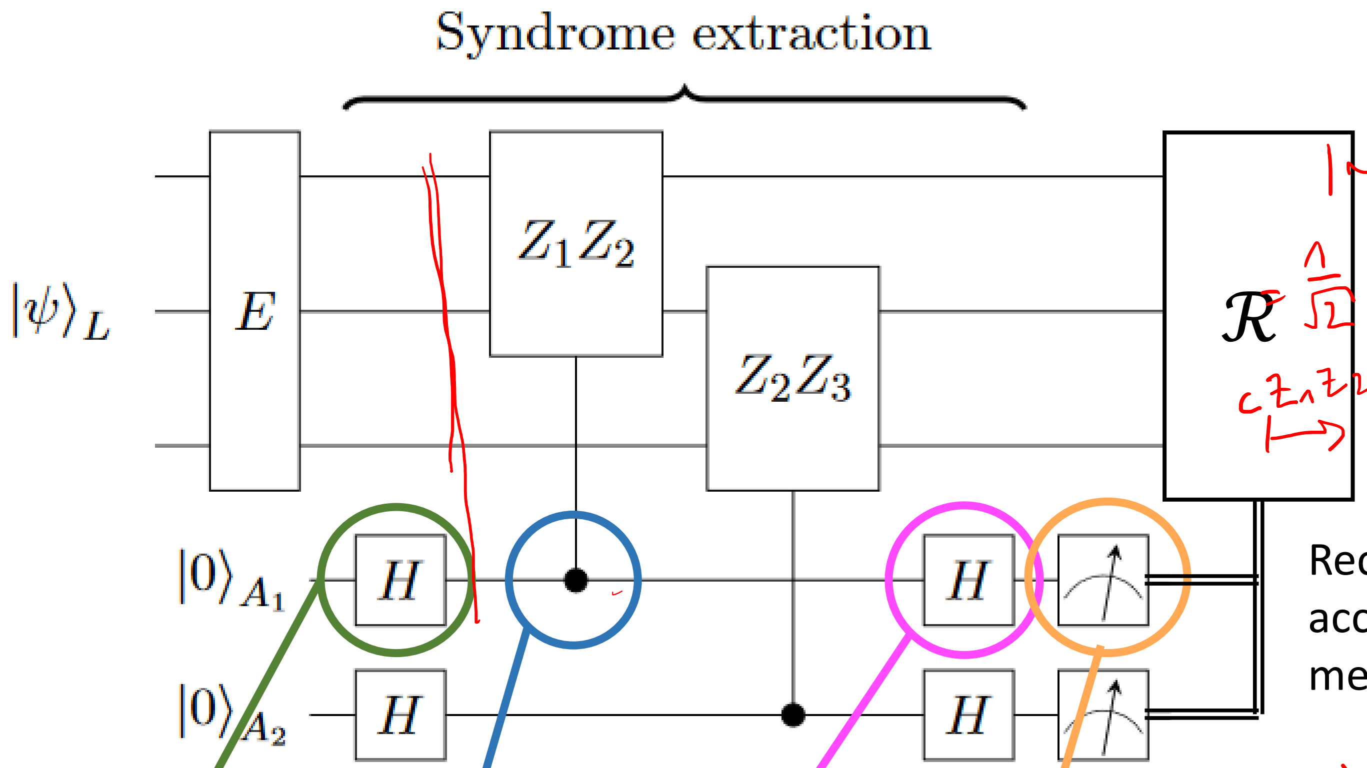
3–qubit bit–flip code, error detection

- **Syndrome** as in 3-bit rep. code:
  $$S_1 = Z_1 Z_2, \qquad S_2 = Z_2 Z_3$$
- Bit-flip error **changes the sign of some syndrome observable**, because they anti-commute. E.g., $S_1 X_1 = -X_1 S_1$.

	S_1	S_2	error
$\vert 000\rangle, \vert 111\rangle$ →	1	1	no error (I)
$\vert 001\rangle, \vert 110\rangle$ →	1	-1	q_3 flipped (X_3)
$\vert 100\rangle, \vert 011\rangle$ →	-1	1	q_1 flipped (X_2)
$\vert 010\rangle, \vert 101\rangle$ →	-1	-1	q_2 flipped (X_1)



- Can use two **ancilla qubits** A_1, A_2 to measure the syndrome. "**Parity checks**"
- **Does not collapse the logical state!**



$|\psi_L\rangle|+\rangle$

$\frac{1}{\sqrt{2}}(|\psi_L\rangle|0\rangle + |\psi_L\rangle|1\rangle)$

$\xrightarrow{cZ_1Z_2} \frac{1}{\sqrt{2}}(|\psi_L\rangle|0\rangle - |\psi_L\rangle|1\rangle)$

$= |\psi_L\rangle\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$= |\psi_L\rangle|-\rangle$

Recovery operation chosen according to the syndrome measurement.

$\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$=$

1. Creates superposition $H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$

2. Flips the sign of $|1\rangle$ if an error occured, thus mapping $|+\rangle \mapsto |-\rangle$.

3. Maps the X-basis back to the computational basis: $|+\rangle \mapsto |0\rangle, |-\rangle \mapsto |1\rangle$.

4. Returns value 1 if an error occurred, otherwise 0. Collapses the superposition error/no error.

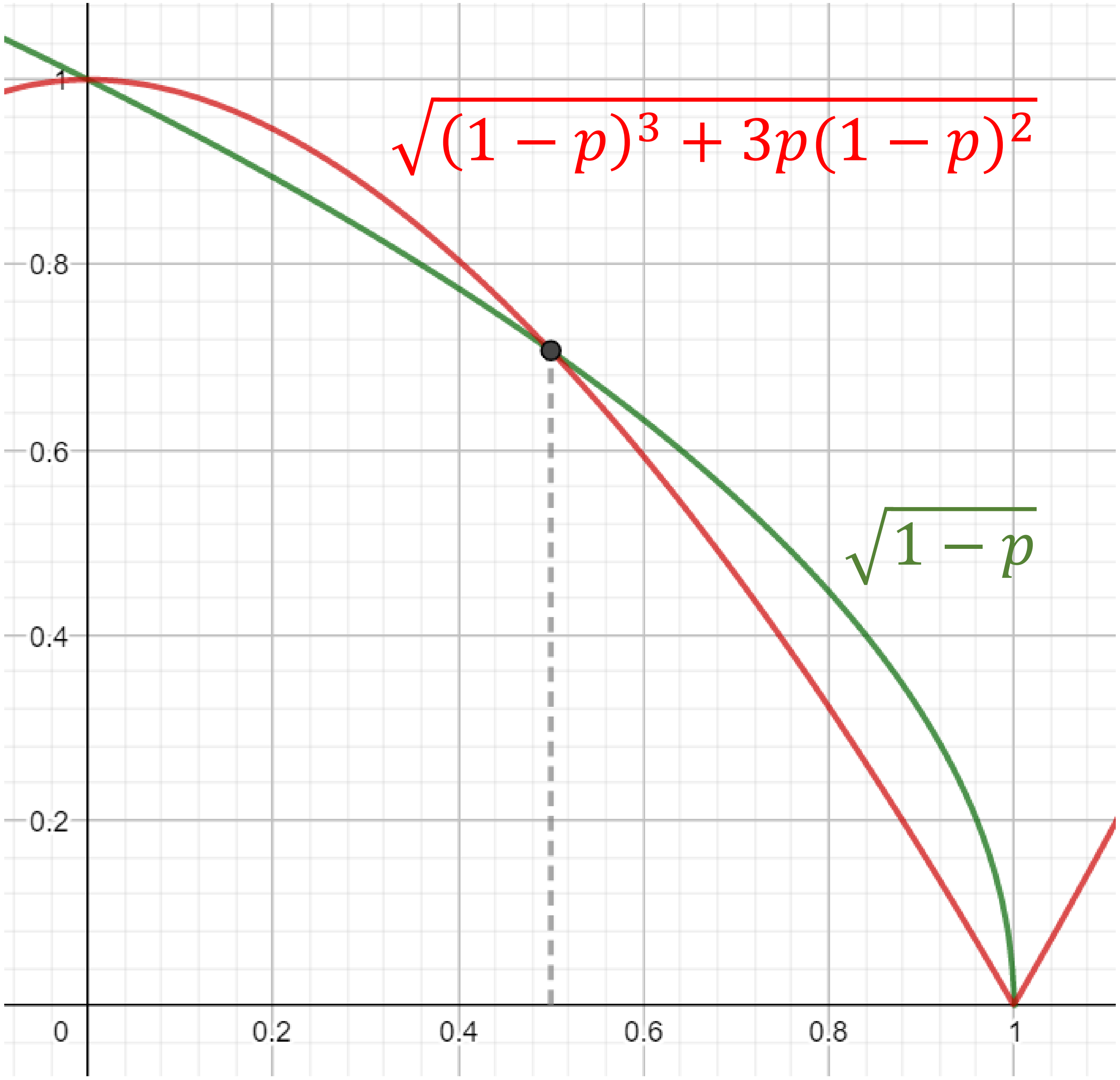
3-qubit bit-flip code, minimum state fidelity

- Let's consider the effect of a bit-flip error on the **state fidelity** with and without error correction. We send a pure state $\rho = |\psi\rangle\langle\psi|$ into the channel.
- Output from the channel **without error correction** is
$$\mathcal{E}(\rho) = (1-p)|\psi\rangle\langle\psi| + pX|\psi\rangle\langle\psi|X$$

  The fidelity between input and output states
$$F\big(\rho, \mathcal{E}(\rho)\big) = \sqrt{\langle\psi|\mathcal{E}(\rho)|\psi\rangle} = \sqrt{(1-p) + p\langle\psi|X|\psi\rangle^2} \geq \sqrt{1-p}$$
- Output from the channel **with error correction** is
$$\mathcal{E}(\rho) = [(1-p)^3 + 3p(1-p)^2\,]|\psi\rangle\langle\psi| + (\text{higher order terms in } p)$$

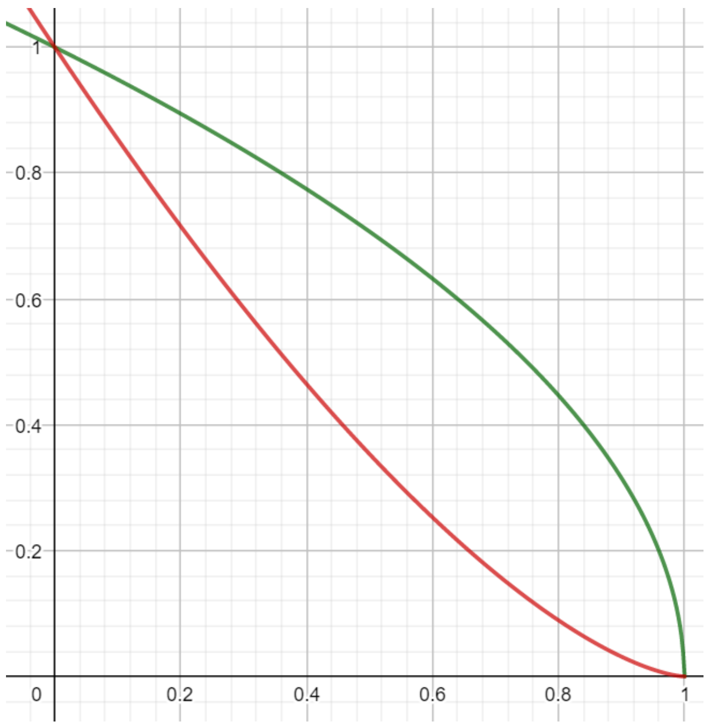
  The fidelity between input and output states
$$F\big(\rho, \mathcal{E}(\rho)\big) = \sqrt{\langle\psi|\mathcal{E}(\rho)|\psi\rangle} \geq \sqrt{(1-p)^3 + 3p(1-p)^2}$$

3-qubit QECC achieves higher minimum fidelity when the probability of bit-flip $p < \frac{1}{2}$.



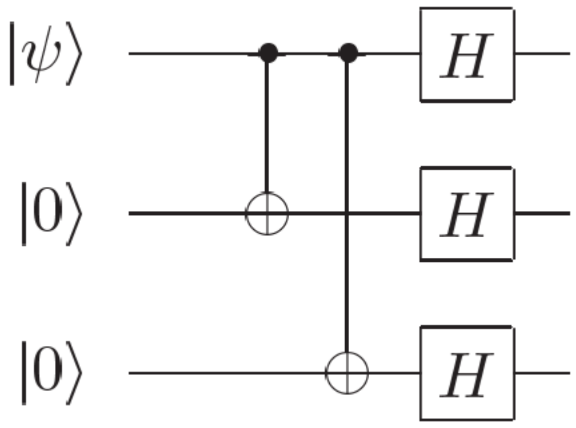
3–qubit bit–flip code, phase–flip error

- How about phase-flip errors? The 3-qubit bit-flip code **cannot** detect phase-flips. The error operators Z_i commute with the syndrome operators Z_1Z_2 and Z_2Z_3!
- Let's consider the effect of a phase-flip error on the **state fidelity** with and without error correction.
- We send a pure state $\rho = |\psi\rangle\langle\psi|$ in.
  - Fidelity **without error correction** is $F\big(\rho, \mathcal{E}(\rho)\big) = \sqrt{\langle\psi|\mathcal{E}(\rho)|\psi\rangle} = \sqrt{(1-p) + p\langle\psi|Z|\psi\rangle^2} \geq \sqrt{1-p}$
  - Fidelity **with error correction** is
  $$F\big(\rho, \mathcal{E}(\rho)\big) = \sqrt{\langle\psi|\mathcal{E}(\rho)|\psi\rangle} \geq \sqrt{(1-p)^3}$$
- $\sqrt{(1-p)^3} < \sqrt{1-p}$ for $0 < p < 1$. The 3-qubit bit-flip code **makes things worse** in this case, because there are now three qubits, each with probability p, to flip their phase!



3–qubit phase–flip code, encoding

- Phase-flip error acts on the $|\pm\rangle$ basis the same way as bit-flip error on the computational basis. Related to the bit-flip error through Hadamard transform.
  => **3-qubit phase-flip code**.
- Implemented by the encoding
  $$|0\rangle_L = |{+}{+}{+}\rangle, \qquad |1\rangle_L = |{-}{-}{-}\rangle$$
- Syndrome operators X_1X_2 and X_2X_3.
- The phase-flip code has exactly the same characteristics (minimum fidelity etc.) as the bit-flip code.
- Now the syndrome commutes with the bit-flip error, so the **phase-flip code cannot detect bit-flip errors**!
- Can we find a code which would correct both bit-flip and phase-flip errors? Enter the Shor code!



9–qubit Shor code

9–qubit Shor code, encoding

- **Shor code** is a combination of 3-qubit bit-flip and phase-flip codes, one inside the other. Can detect bit-flip and phase-flip errors on any of the nine qubits.
- Encoding:

$\frac{1}{\sqrt{2}}(|0\,\text{bit}\rangle + |1\,\text{bit}\rangle) = |+\text{bit}\rangle$

$$|0\rangle \rightarrow |0_L\rangle \equiv \frac{(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)}{2\sqrt{2}}$$

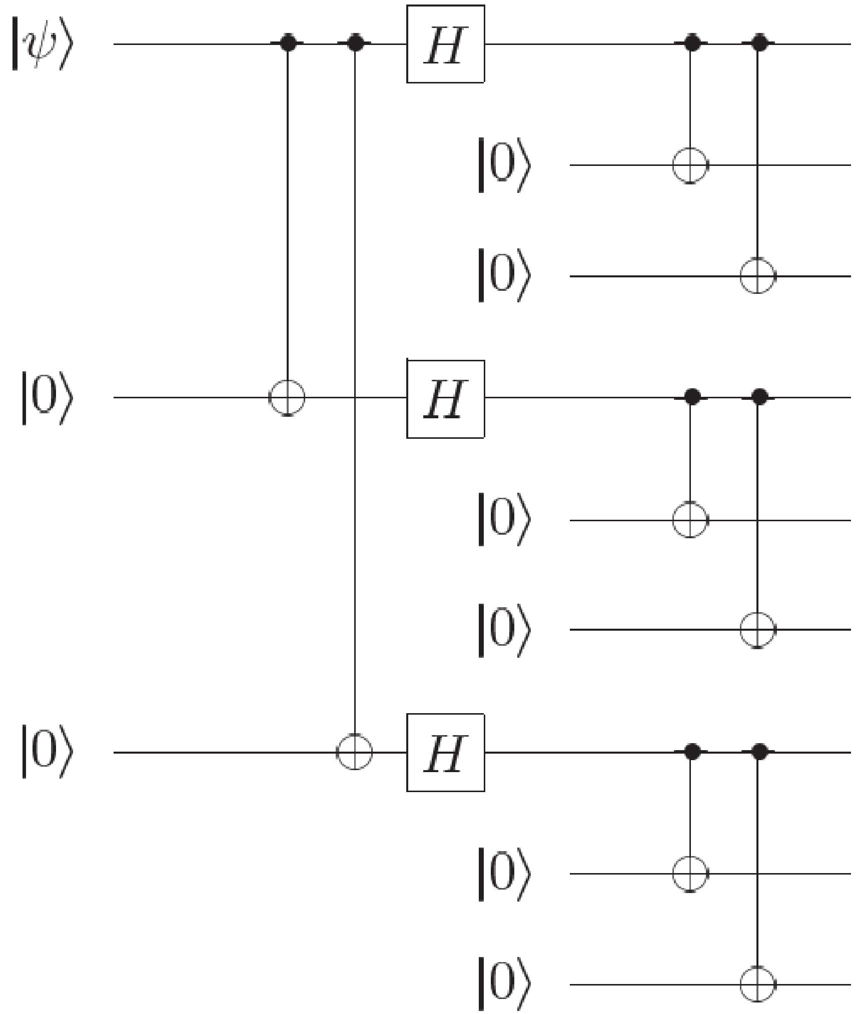
$|-\text{bit}\rangle$

$$|1\rangle \rightarrow |1_L\rangle \equiv \frac{(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)}{2\sqrt{2}}$$

- Syndrome:

$$\mathcal{S}_{[[9,3,3]]} = \langle Z_1Z_2, Z_2Z_3, Z_4Z_5, Z_5Z_6, Z_7Z_8, Z_8Z_9, X_1X_2X_3X_4X_5X_6, X_4X_5X_6X_7X_8X_9\rangle.$$

Encoding circuit for the Shor code



9–qubit Shor code, syndrome

X_1, X_2, X_3

$$\mathcal{C}_{[[9,1,3]]} = \text{span}\left\{\begin{array}{l} |0\rangle_9 = \frac{1}{\sqrt{8}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) \\ |1\rangle_9 = \frac{1}{\sqrt{8}}(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle) \end{array}\right\}$$

$$\mathcal{S}_{[[9,3,3]]} = \langle Z_1Z_2, Z_2Z_3, Z_4Z_5, Z_5Z_6, Z_7Z_8, Z_8Z_9, X_1X_2X_3X_4X_5X_6, X_4X_5X_6X_7X_8X_9\rangle.$$

Error	Syndrome, S	Error	Syndrome, S
X_1	10000000	Z_1	00000010
X_2	11000000	Z_2	00000010
X_3	01000000	Z_3	00000010
X_4	00100000	Z_4	00000011
X_5	00110000	Z_5	00000011
X_6	00010000	Z_6	00000011
X_7	00001000	Z_7	00000001
X_8	00001100	Z_8	00000001
X_9	00000100	Z_9	00000001

NOTE: Some errors have the same syndrome. However, can be corrected by the same recovery operation. Shor code is "degenerate".

Shor code and arbitrary 1–qubit errors

What about other errors beside bit-flip and phase-flip errors?

- Shor code can also detect and correct **combined bit-flip and phase-flip errors**, since these change the value of both corresponding syndrome measurements.
- An **arbitrary 1-qubit error** in the operator-sum representation:

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger$$

$E_k|\psi\rangle_L$
$= e_{k0}|\psi_L\rangle + e_{k1}X|\psi_L\rangle + \ldots$

- Since $\{I, X, Y = iXZ, Z\}$ span the vector space of 2-by-2 matrices, the individual 1-qubit error terms can be expressed as complex linear combinations

$$E_k = e_{k0}I + e_{k1}X + e_{k2}XZ + e_{k3}Z$$

- Thus, the error E_k acting on an input state $|\psi_L\rangle$ will lead to a superposition of bit-flip, phase-flip and combined phase-bit-flip errors. However, this superposition is collapsed in the syndrome measurement! => **Discretization of quantum errors**
- **Shor code can therefore correct arbitrary 1-qubit errors!**

A couple of important theorems

Discretization of errors

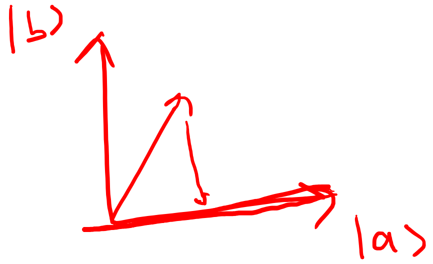
Theorem (Nielsen-Chuang 10.2):

Let $\mathcal{C}$ be a quantum error-correction code capable of correcting errors due to the quantum channel $\mathcal{E}$ with operation elements $\{E_i\}$. Suppose $\mathcal{F}$ is another quantum channel with operation elements $\{F_i\}$, which are linear combinations of E_i:

$$F_i = \sum_j c_{ij} E_j$$

for some complex matrix (c_{ij}). Then the code $\mathcal{C}$ is also capable of correcting errors due to the channel $\mathcal{F}$.

Quantum error–correction condition



Theorem (Nielsen-Chuang 10.1): Quantum error-correction condition

Let $\mathcal{C} \subset \mathcal{H}$ be the subspace of logical quantum states (without errors), and P the projection onto $\mathcal{C}$. Suppose $\mathcal{E}$ is a quantum operator with operation elements $\{E_i\}$. A necessary and sufficient condition for the existence of an error-correction operation $\mathcal{R}$ is that

$$P E_i^\dagger E_j P = \alpha_{ij} P$$

$\dagger$

for some Hermitian matrix α.

$$P = |0_L\rangle\langle 0_L| + |1_L\rangle\langle 1_L|$$

$$P\left(a|0_L\rangle + b|1_L\rangle\right)$$

$$= a|0_L\rangle + b|1_L\rangle$$

Quantum error–correction condition

Sketch of a proof:

1. First of all, we diagonalize α, i.e., find matrix u s.t. $d = u^\dagger \alpha u$ is diagonal. Then, for the errors $F_k = \sum_i u_{ik} E_i$ we get the equivalent condition
$$PF_k^\dagger F_l P = d_{kl} P$$
The fact that d is diagonal means that the errors F_k map the original codespace to orthogonal subspaces. If the code can correct $\{F_k\}$, it can also correct $\{E_i\}$ due to the previous theorem.

2. Secondly, we use the polar decompositions $F_k = U_k \sqrt{F_k^\dagger F_k}$ to find that the error F_k rotates the codespace $\mathcal{C}$ by U_k to the orthogonal subspace described by the projection $U_k P U_k^\dagger$.

3. Successful recovery operation $\mathcal{R}$ can then be defined by the operator elements $R_k = PU_k^\dagger$, as we have
$$\mathcal{R}(\mathcal{E}(\rho)) = \sum_{kl} PU_k^\dagger F_l \rho F_l^\dagger U_k P = \left(\sum_k d_{kk}\right)\rho \propto \rho$$

Quantum error–correction condition

For example, let's check the quantum error correction condition for Shor code and some bit-flip errors:

$$P = |0_L\rangle\langle 0_L| + |1_L\rangle\langle 1_L|, \qquad PX_1X_2P = PX_1X_2(|0_L\rangle\langle 0_L| + |1_L\rangle\langle 1_L|)$$

Here,

$$X_1X_2|0_L\rangle = X_1X_2\frac{1}{2\sqrt{2}}[(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)]$$

$$= \frac{1}{2\sqrt{2}}[(|110\rangle + |001\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)]$$

$$X_1X_2|1_L\rangle = X_1X_2\frac{1}{2\sqrt{2}}[(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)]$$

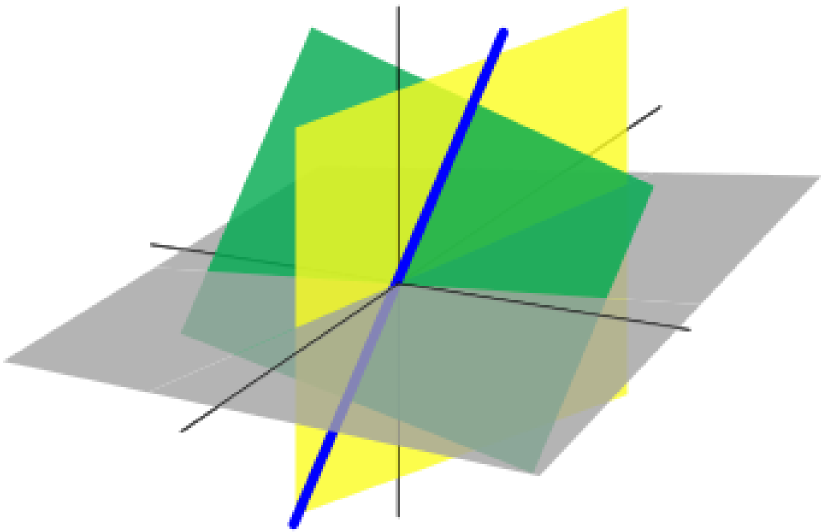
$$= \frac{1}{2\sqrt{2}}[(|110\rangle - |001\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)]$$

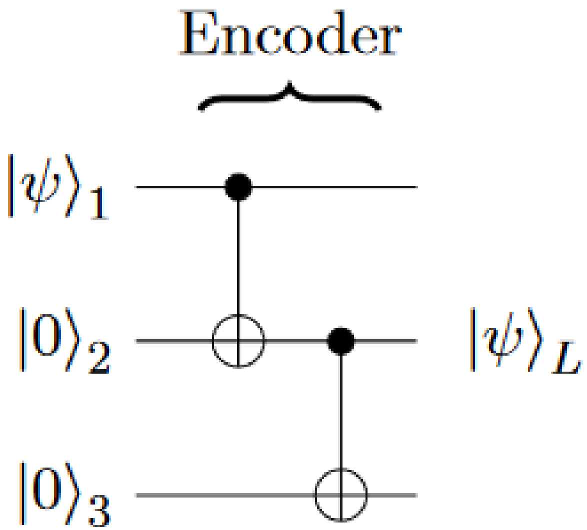
Since these are orthogonal to both $|0_L\rangle$ and $|1_L\rangle$, $PX_1X_2P = 0$, and the quantum error correction condition holds for errors X_1, X_2.

Stabilizer codes

Code subspace and stabilizers

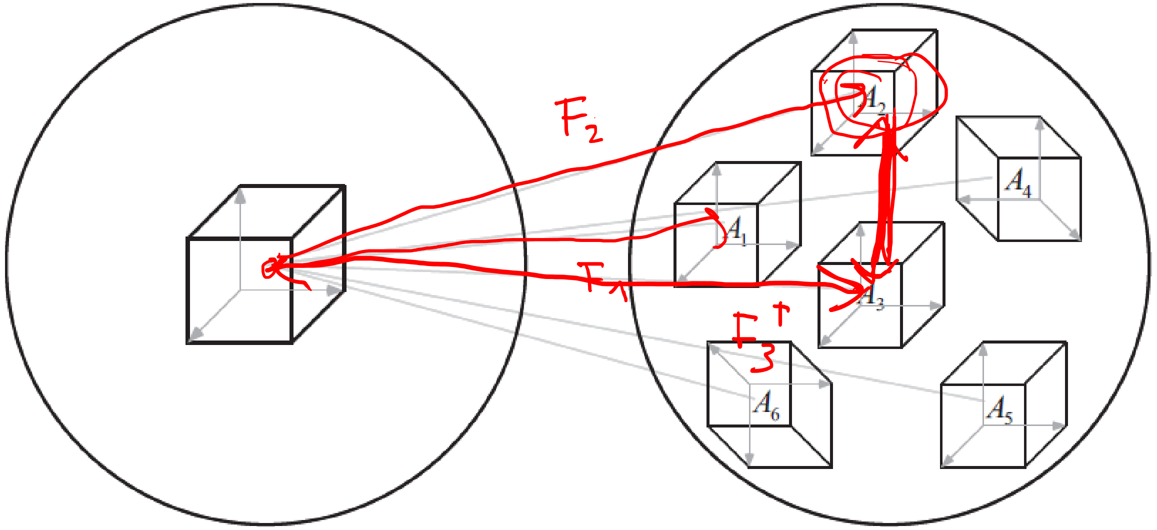
- The Shor code is an example of a **stabilizer code**.
- Correct codestates live in the **code subspace** $\mathcal{C} \subset \mathcal{H}$ of the full Hilbert space.
- E.g., for 3-qubit code subspace $\mathcal{C}$ spanned by $|000\rangle, |111\rangle$. $\quad |\psi\rangle_L = a|000\rangle + b|111\rangle$
- Code subspace $\mathcal{C}$ can be determined by **stabilizers** (”stabilizer codes”) S_i: Tensor products of Pauli operators (including I) such that
$$S_i|\psi\rangle_L = |\psi\rangle_L \ \forall \ |\psi\rangle_L \in \mathcal{C}.$$
- In other words, correct codewords are eigenstates of all the stabilizers with eigenvalue 1. **Stabilizers are measured to have value 1 in any correct codestate.**
- E.g. for 3-qubit code $S_1 = Z_1Z_2, \ S_2 = Z_2Z_3$.





Error detection and code distance

- Detectable errors move the state outside the code subspace to an orthogonal subspace. This can be detected by a change in the values of the stabilizers (”parity checks”).
- **Distance** of an ECC: Minimal number of single-qubit errors that transforms one correct codestate to another.
- QECC of distance d with n physical and k logical qubits is denoted by $[[n, k, d]]$.
- A distance d code can identify and correct up to $t = \frac{d-1}{2}$ single-qubit errors.



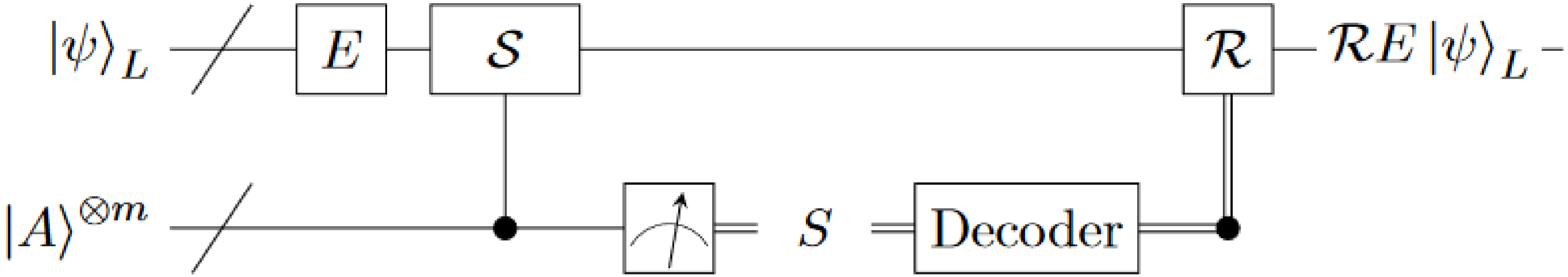
X_1X_2

For example:

- 3-qubit code is a $[[3,1,1]]$-code. Cannot detect phase (Z) errors at all!
- Shor code is a [[9,1,3]]-code.

Error correction for stabilizer codes

- After an error E is detected by the stabilizer measurements, we can try to correct it by applying a recovery operation R.
- The recovery operation is chosen according to the syndrome.
- If $RE|\psi\rangle_L = |\psi\rangle_L$, the error correction was succesful.



- Notice that we only need $RE = S$, where S is some stabilizer.
  Not all errors need to have unique syndromes in order to be corrected!
  ”Degenerate code”

5-qubit code

5–qubit code

- **5-qubit code** is the smallest QECC capable of correcting arbitrary 1-qubit errors.
- Codestate basis:

$$
\begin{aligned}
|0_L\rangle = \frac{1}{4}\Big[&|00000\rangle + |10010\rangle + |01001\rangle + |10100\rangle \\
&+|01010\rangle - |11011\rangle - |00110\rangle - |11000\rangle \\
&- |11101\rangle - |00011\rangle - |11110\rangle - |01111\rangle \\
&- |10001\rangle - |01100\rangle - |10111\rangle + |00101\rangle\Big]
\end{aligned}
$$

$$
\begin{aligned}
|1_L\rangle = \frac{1}{4}\Big[&|11111\rangle + |01101\rangle + |10110\rangle + |01011\rangle \\
&+|10101\rangle - |00100\rangle - |11001\rangle - |00111\rangle \\
&- |00010\rangle - |11100\rangle - |00001\rangle - |10000\rangle \\
&- |01110\rangle - |10011\rangle - |01000\rangle + |11010\rangle\Big]
\end{aligned}
$$

	Name	Operator
syndrome /generators	g_1	$XZZXI$
	g_2	$IXZZX$
	g_3	$XIXZZ$
	g_4	$ZXIXZ$
logical operators	$\bar{Z}$	$ZZZZZ$
	$\bar{X}$	$XXXXX$

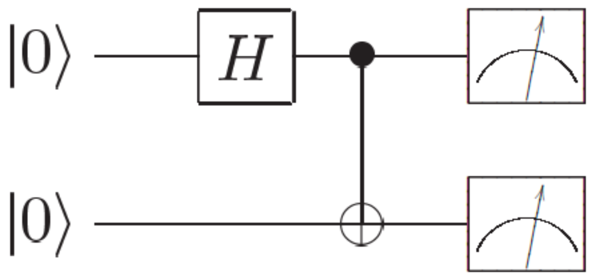
Fault-tolerant quantum computing

Fault–tolerant logical operators

- Fundamental idea: Do computation directly with encoded states.
  => We need to find **logical gates**.
- E.g. for 3-qubit code $X_L = X_1X_2X_3, Z_L = Z_1$.
  $$X_L|0\rangle_L = X_L|000\rangle = |111\rangle = |1\rangle_L,$$
  $$X_L|1\rangle_L = X_L|111\rangle = |000\rangle = |0\rangle_L$$
  etc….
- Unique only up to multiplication by the stabilizers!
- For $[[n, k, d]]$ stabilizer code there are $2k$ logical X and Z gates.
- General requirements for logical operators of stabilizer codes:
  1. Must commute with all the stabilizers.
  2. Must satisfy Pauli relations (anti-commute) by themselves.
- However, single qubit X and Z gates are not enough for universal computing!

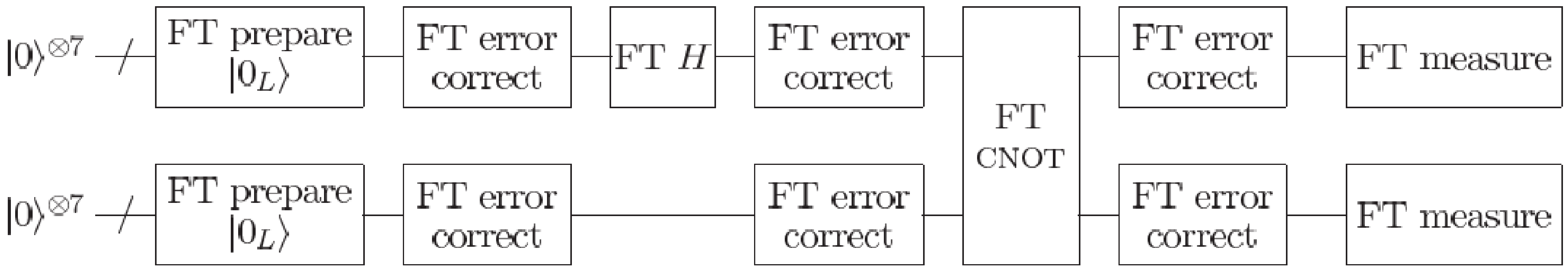
Fault-tolerant quantum circuits

probability of error p



fault-tolerant QC
on encoded states

probability of error Cp^2
where C is some constant



Concatenated codes and the threshold theorem

- Error correction codes can be applied recursively one inside the other, and thus reduce the probability of error even further. => “**concatenated codes**”
- Concatenating codes reduces the error rate *double-exponentially*!
  k levels of error correction reduces the error rate to $(Cp)^{2^k}/C$.
- However, the circuit size grows ”only” exponentially as d^k times the original size, where d is some constant representing the maximum number of gates in the fault-tolerant implementation of any gate.
- Suppose we want to run some circuit with g_L logical gates to ε accuracy. Then, if $p < \frac{1}{C} = p_{th}$, we can concatenate the code k times such that

$$g_L \frac{(Cp)^{2^k}}{C} \leq \varepsilon$$

- We get for the number of physical gates

$$g_{ph} \sim d^k = \left(\frac{\log(g_L/C\varepsilon)}{\log(1/pC)}\right)^{\log d} = O\left(\text{poly}\left(\log\left(\frac{g_L}{\varepsilon}\right)\right)\right)$$

Concatenated codes and the threshold theorem

Threshold theorem for quantum computing

A quantum circuit containing g_L gates may be simulated with probability of error at most ε using

$$g_{ph} \sim O\left(\text{poly}\left(\log\left(\frac{g_L}{\varepsilon}\right)\right)\right)$$

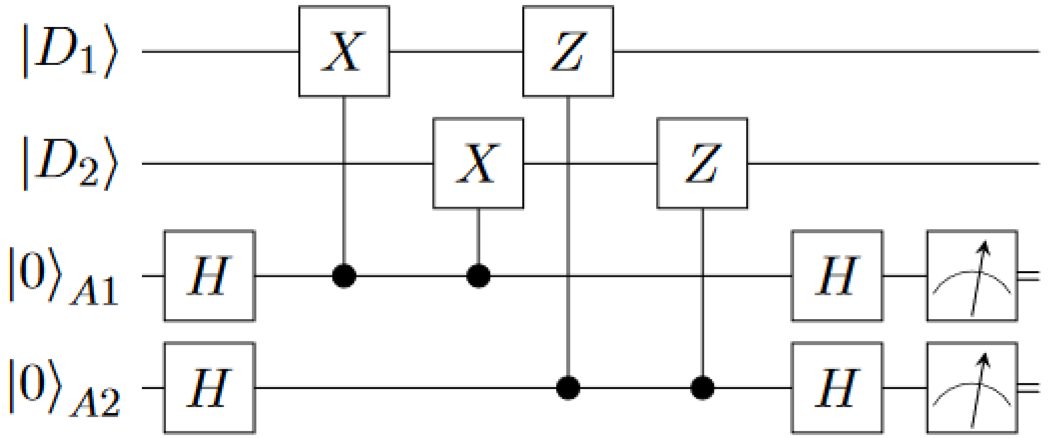
gates on hardware whose components fail with probability at most p, provided p is smaller than some threshold value p_{th} (and given some reasonable assumptions about the noise in the underlying hardware).

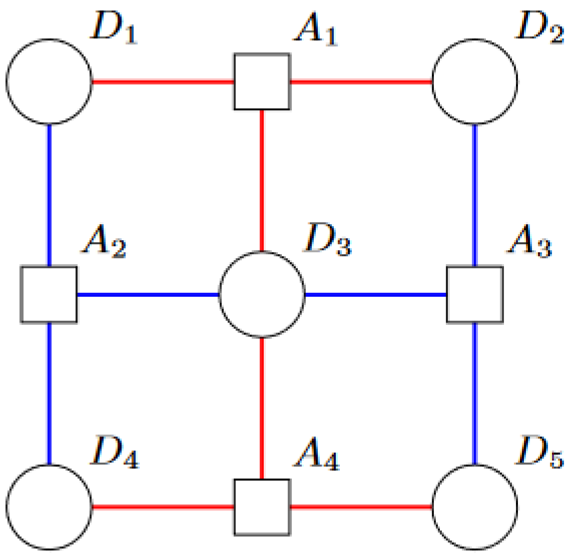
- The number of gates required scales only poly-logarithmically in ε^{-1} and g_L!
- The exact value of p_{th} depends on the details of the implementation. Currently best codes achieve $p_{th}\sim 1\%$. Already in the range of current hardware!
- Many different noise models can be considered; tend to give consistent results.

Surface codes

[[5,1,2]] surface code

- Each square on a 2d grid has two data and two ancilla qubits. Ancilla qubits are used to perform the stabilizer measurements on the data qubits (red = X measurements, blue = Z measurements).





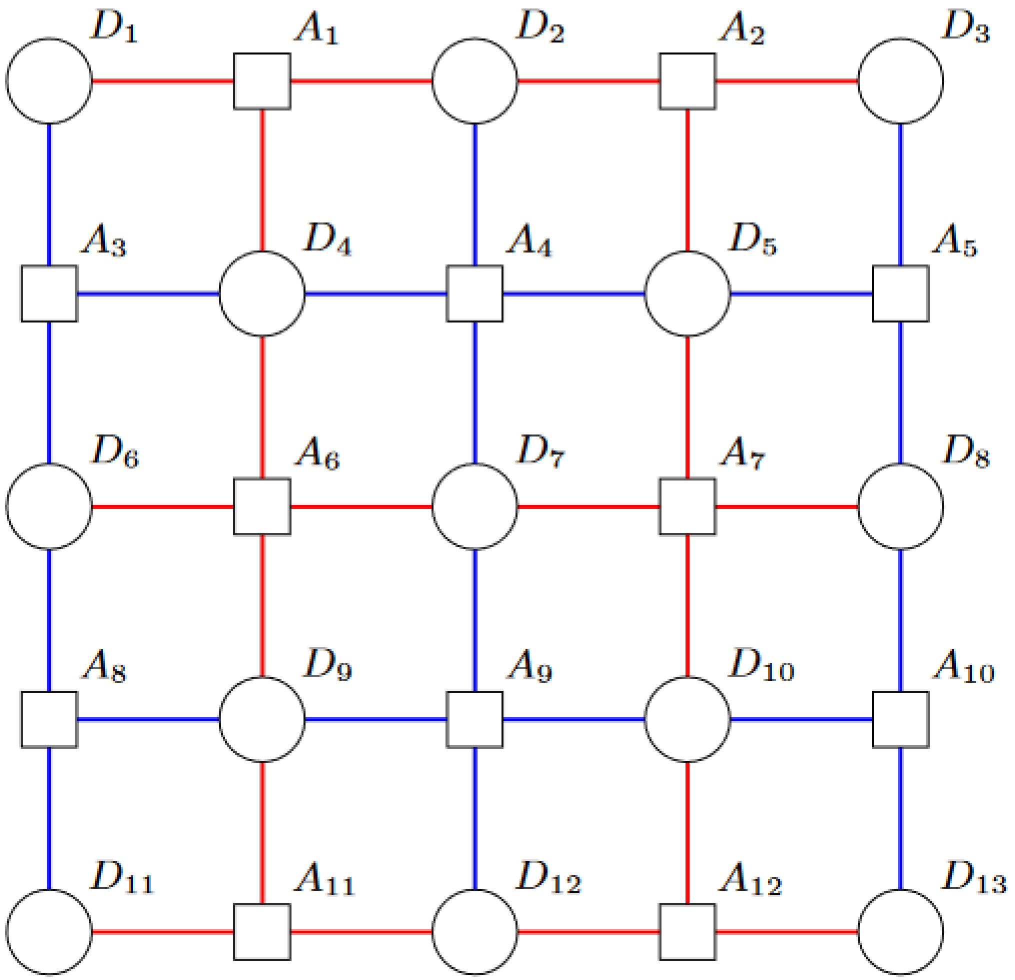
$$\mathcal{S}_{[[5,1,2]]} = \langle X_{D_1}X_{D_2}X_{D_3},\ Z_{D_1}Z_{D_3}Z_{D_4},\ Z_{D_2}Z_{D_3}Z_{D_5},\ X_{D_3}X_{D_4}X_{D_5}\rangle.$$

$$X_L = X_1X_4, \qquad Z_L = Z_1Z_2$$

[[13,1,3]] surface code

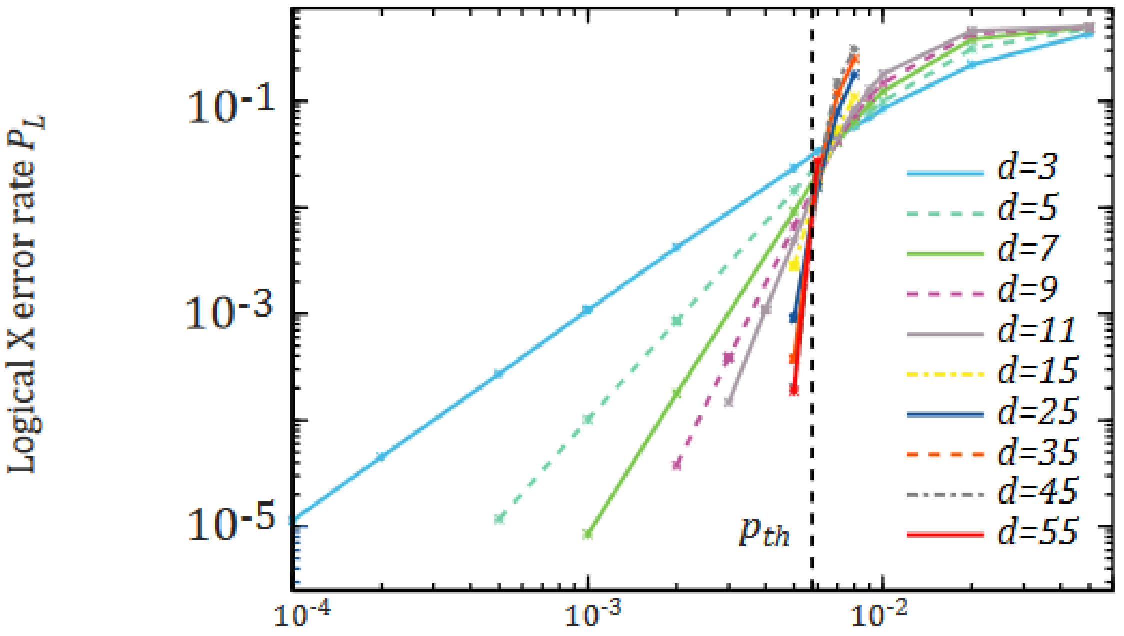
$$X_L = X_1 X_6 X_{11},$$

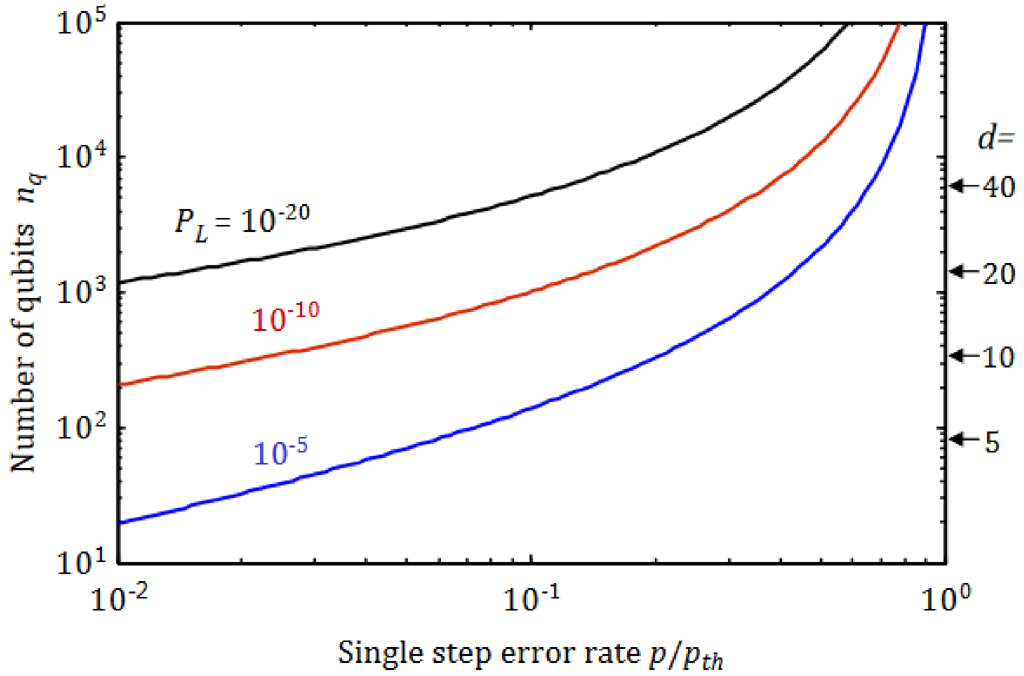
$$Z_L = Z_1 Z_2 Z_3$$



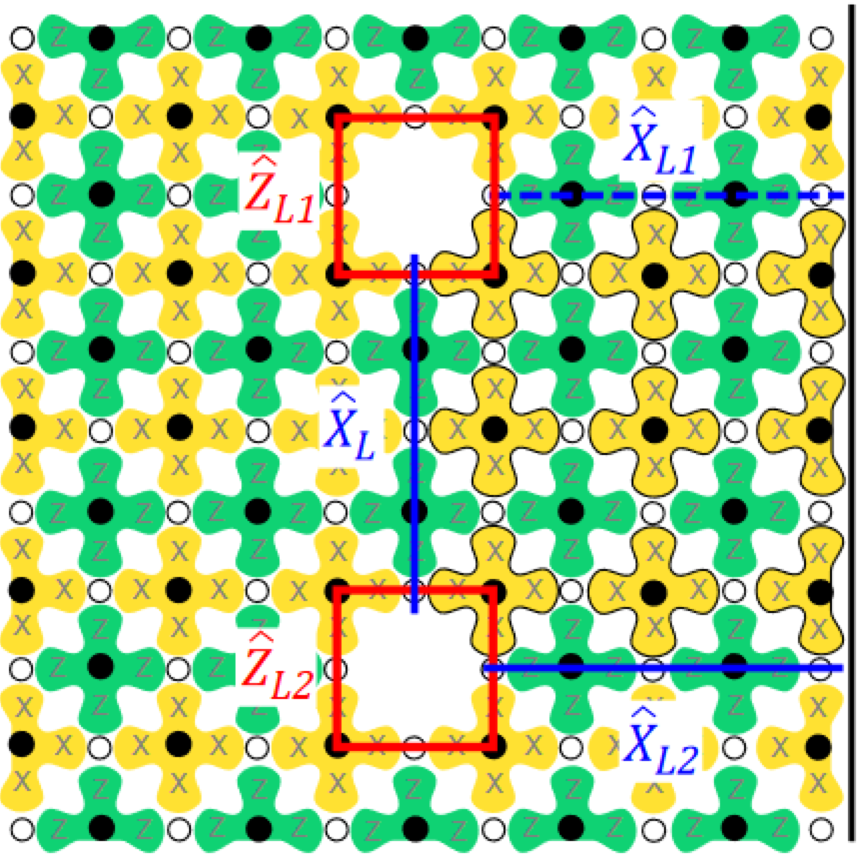
2d surface code qubits

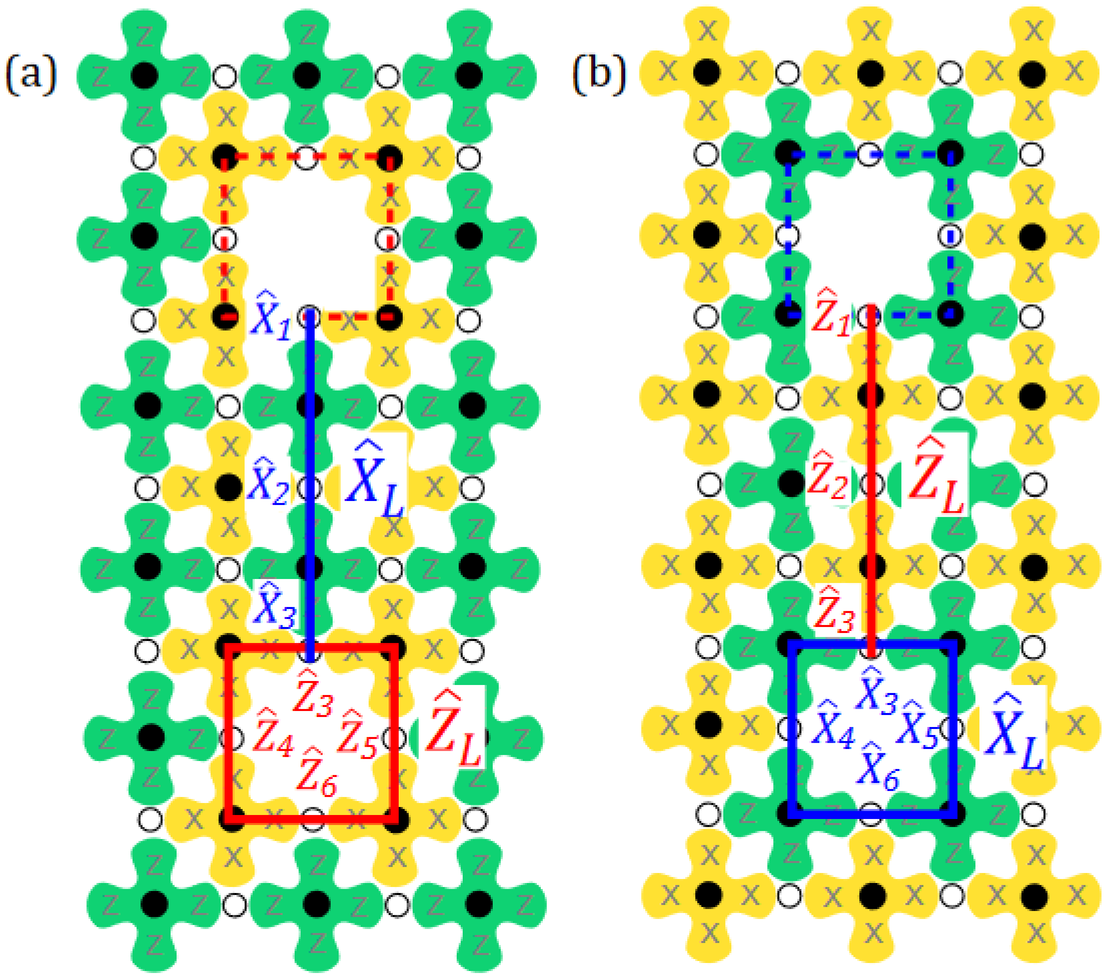
- Surface code construction can be scaled up arbitrarily.
- Only nearest-neighbour qubit connectivity required!
- Code distance and qubit number related as $n = d^2 + (d-1)^2$.
- **Threshold theorem** for stabilizer codes: Increasing code distance will reduce the logical error rate p_L, provided that the physical error rate is below some p_{th}.
- Scaling with distance $p_L \sim (p/p_{th})^{d/2}$.
- For the 2d surface code estimated threshold $p_{th} \approx 1 - 10\%$.





Multiple logical qubits on 2d surface code

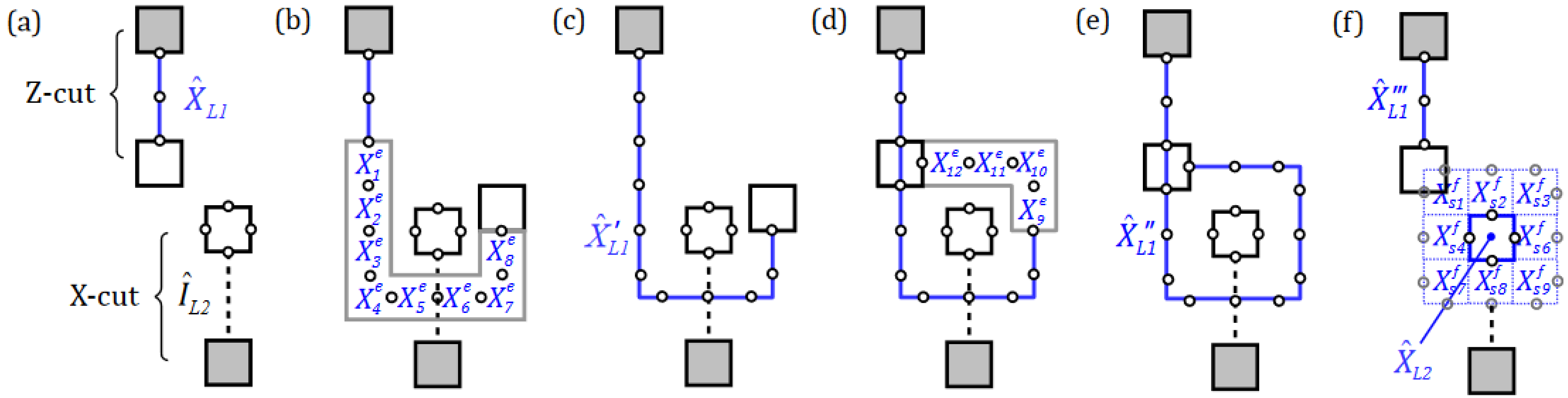




CNOT gate in 2d surface code

- Logical CNOT gate can be implemented in 2d surface code by certain braiding transformations, which involve moving qubits around on the surface.

  (For details, see arXiv:1208.0928 [quant-ph].)



Universal fault–tolerant QC in 2d surface code

- X, Z, Hadamard and CNOT gates not enough for universal QC.
  Can be simulated efficiently by a classical computer!
  ("Clifford gates", **Gottesman-Knill theorem**)
- Universal computation requires also, e.g., the **T gate**:
  $$T|0\rangle = |0\rangle, \qquad T|1\rangle = e^{i\pi/4}|1\rangle.$$
- Implementation of T gates in 2d surface code requires "**magic state distillation**". High fidelity copies of "magic states" of the form
  $$|A\rangle_L = \frac{1}{\sqrt{2}}\left(|0\rangle_L + e^{\frac{i\pi}{4}}|1\rangle_L\right)$$
  are prepared by a special distillation process from many less perfect few-qubit states, and then used to implement the logical T gate.
- It is, in fact, this distillation process, which may cause most overhead. (However, see arXiv:1905.06903 [quant-ph].)

Some details on the T gate implementation

- The implementation of T gate requires several steps:
  1. Production of $|Y\rangle_L = \frac{1}{\sqrt{2}}(|0\rangle_L + i|1\rangle_L)$ on single data qubits by native X-rotations.
  2. State distillation of high-fidelity versions of $|Y\rangle_L$ on surface code qubits. Each distillation step uses several lower fidelity copies of $|Y\rangle_L$. Error rate $p \mapsto 7p^3$ at each step.
  3. Implementation of logical S gate using high fidelity $|Y\rangle_L$ states.
  4. State distillation of $|A\rangle_L$ using logical S gates. Error rate $p \mapsto 35p^3$ at each step.
  5. Implementation of logical T gate using the $|A\rangle_L$ magic state and logical S gate.