Tracing out a qubit

Alisa Haukisalmi

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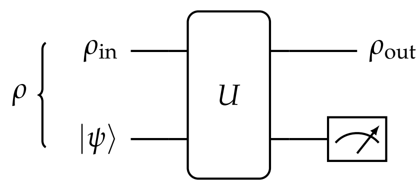
During the previous exercise session I said that in Exercise 2 (Week 4), measuring the second qubit is equivalent to tracing it out. This raised some questions, which I try to address here. Let's consider a situation similar to Exercise 2, but slightly more general:



We have an initial product state $\rho = \rho_{\text{in}} \otimes |\psi\rangle\langle\psi|$, which is evolved by a unitary U, and finally the second qubit is measured. In other words:

$$\begin{aligned}\rho_{\text{in}} \otimes |\psi\rangle\langle\psi| = \rho &\mapsto U\rho U^\dagger = \rho' \\ &\mapsto (\mathbb{I} \otimes P_0)\rho'(\mathbb{I} \otimes P_0) + (\mathbb{I} \otimes P_1)\rho'(\mathbb{I} \otimes P_1) = \rho''.\end{aligned}$$

Here $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$, projectors to the computational basis. The measurement result is not noted or recorded in any way, hence both projectors are included in a sum. To be even more explicit, the final state of the combined system is

$$\begin{aligned}\rho'' = (\mathbb{I} \otimes |0\rangle\langle 0|) &\left[U(\rho_{\text{in}} \otimes |\psi\rangle\langle\psi|)U^\dagger\right] (\mathbb{I} \otimes |0\rangle\langle 0|) \\ + (\mathbb{I} \otimes |1\rangle\langle 1|) &\left[U(\rho_{\text{in}} \otimes |\psi\rangle\langle\psi|)U^\dagger\right] (\mathbb{I} \otimes |1\rangle\langle 1|).\end{aligned}$$

Importantly, $\boldsymbol{\rho'' \neq \rho_{\text{out}}}$. This should be obvious just from the dimensions of these operators. We've computed the circuit and measured the second qubit, but haven't yet obtained ρ_{out}. So how would we do that? The same way we've figured out density operators of subsystems before: by using the partial trace. Let's denote the subsystem of the first qubit by A and the second by B. Then

$$\begin{aligned}\rho_{\text{out}} &= \operatorname{Tr}_B \rho'' \\ &= (\mathbb{I} \otimes \langle 0|)\rho''(\mathbb{I} \otimes |0\rangle) + (\mathbb{I} \otimes \langle 1|)\rho''(\mathbb{I} \otimes |1\rangle) \\ &= (\mathbb{I} \otimes \langle 0|)\rho'(\mathbb{I} \otimes |0\rangle) + (\mathbb{I} \otimes \langle 1|)\rho'(\mathbb{I} \otimes |1\rangle),\end{aligned}$$

where the second equality follows from the orthonormality of $|0\rangle$ and $|1\rangle$. We've now found that $\rho_{\text{out}} = \operatorname{Tr}_B \rho'$! Recall that $\rho' = U\rho U^\dagger$, the state of the system after U, but before the measurement. Therefore, in the situation described by the circuit here and in Exercise 2, measuring and then discarding qubit B is equivalent to tracing it out.

In conclusion, in Exercise 2, by measuring we mean *measuring but discarding*. Specifically, we do not record the measurement result—we don't care what it is and we don't even know it. If we had performed the measurement of qubit B relative to any other orthonormal basis, we would still obtain the same result, as trace is independent of the choice of basis.