Quantum Information Spring 2023 Problem Set 1

Solutions are due on Sunday Apr 30, 23:59.

1. **Basic Quantum Mechanics**

   (a) **State space**
   A pure quantum state is completely described by a vector $|\psi\rangle$ in a Hilbert space $\mathcal{H}$. Usually quantum states are normalized $\langle\psi|\psi\rangle = 1$. Normalize the one-qubit state

   $$|\phi\rangle = 2\,|0\rangle - i\,|1\rangle \ , \tag{1}$$

   that is, find a constant A such that $|\psi\rangle = |\phi\rangle / A$ is normalized.

   (b) **Evolution of states**
   The time evolution of closed quantum states is described by unitary transformations: if a system is initially in state $|\psi_1\rangle$, then at some later time the state is $|\psi_2\rangle = U\,|\psi_1\rangle$ where U is an unitary operator. Verify explicitly that

   $$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{2}$$

   is unitary. Let $|\psi_1\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. Compute $|\psi_2\rangle = U\,|\psi_1\rangle$.

   (c) **Measurement**
   Measurements are described by sets of measurement operators $\{M_m\}$, where the index m corresponds to any of the possible outcomes of the measurement. Measurement of a single qubit in the computational basis is described by the operators

   $$\text{Outcome } 0: \quad M_0 = |0\rangle\langle 0| \tag{3}$$

   $$\text{Outcome } 1: \quad M_1 = |1\rangle\langle 1| \ . \tag{4}$$

   Suppose a qubit is in the normalized state $|\psi\rangle$ from part (a). What are the probabilities of outcomes 0 and 1 when $|\psi\rangle$ is measured in the computational basis?

   (d) **Composite systems**
   The state space of a composite physical system is the tensor product of the component state spaces. For example, $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. If the subsystems are prepared to states $|\psi_1\rangle$ and $|\psi_2\rangle$, then their composite system is in state $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \in \mathcal{H}$. Let

   $$|\psi_1\rangle = a\,|0\rangle + b\,|1\rangle \in \mathcal{H}_1 \tag{5}$$

   $$|\psi_2\rangle = c\,|0\rangle + d\,|1\rangle \in \mathcal{H}_2 \ . \tag{6}$$

Compute explicitly $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$. Show that the *Bell state*

$$\left|\Psi^+\right\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle) \tag{7}$$

cannot be written as a tensor product of any two states $|\psi_1\rangle$ and $|\psi_2\rangle$.

2. **Operator Functions 1**
   Find the square root and logarithm (base 2) of the matrix

$$A = \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix} . \tag{8}$$

3. **Operator Functions 2**
   Show that

$$\exp(-i\theta X/2) = \cos\left(\frac{\theta}{2}\right)\mathbb{I} - i\sin\left(\frac{\theta}{2}\right)X \ , \tag{9}$$

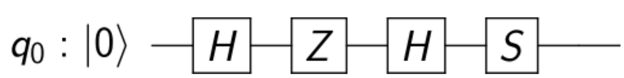
   where X is the Pauli X-matrix.

4. **Single Qubit Quantum Circuits**
   Calculate the state produced by the given circuit and compute the expectation value of the given observable. Matrices of the gates are

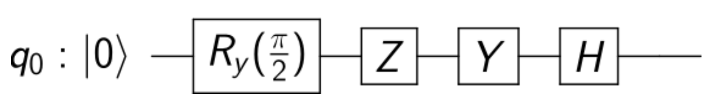
$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \ , \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \tag{10}$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \ , \quad R_y\left(\frac{\pi}{2}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \ . \tag{11}$$

   (a) Compute the expectation value of the Z-observable



   (b) Compute the expectation value of the X-observable



5. **Controlled NOT gate**
   The controlled-not operator has a central role in quantum computation. Consider a CNOT gate controlled on the first qubit with the second qubit as the target. One way to express this operation is

$$CNOT = |0\rangle\langle 0| \otimes \mathbb{I} + |1\rangle\langle 1| \otimes X \ . \tag{12}$$

That is, CNOT flips the target qubit only if the control qubit is set to $|1\rangle$. Compute explicitly the matrix representation of CNOT. Then verify that

$$
\begin{aligned}
CNOT\,|00\rangle &= |00\rangle && (13)\\
CNOT\,|01\rangle &= |01\rangle && (14)\\
CNOT\,|10\rangle &= |11\rangle && (15)\\
CNOT\,|11\rangle &= |10\rangle\ , && (16)
\end{aligned}
$$

where

$$
|00\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{etc.} \tag{17}
$$