Quantum Information Spring 2023
Bonus Problem Set

1. **Quantum Fourier transform**
   Consider the unnormalized quantum state

$$|\phi\rangle = \sum_{j=0}^{7} \cos\left(\frac{2\pi j}{8}\right)|j\rangle \ , \tag{1}$$

on a system of three qubits.

(a) Normalize $|\phi\rangle$. That is, find x such that $|\psi\rangle = x\,|\phi\rangle$ is normalized $\langle\psi|\psi\rangle = 1$.
Solution.

$$\langle\phi|\phi\rangle = \sum_{j=0}^{7} \cos^2\left(\frac{2\pi j}{8}\right) = \cos^2(0) + \cos^2\left(\frac{\pi}{4}\right) + \cos^2\left(\frac{\pi}{2}\right) + \cos^2\left(\frac{3\pi}{4}\right) \tag{2}$$

$$+ \cos^2(\pi) + \cos^2\left(\frac{5\pi}{4}\right) + \cos^2\left(\frac{3\pi}{2}\right) + \cos^2\left(\frac{7\pi}{4}\right) \tag{3}$$

$$= 1 + \frac{1}{2} + 0 + \frac{1}{2} + 1 + \frac{1}{2} + 0 + \frac{1}{2} = 4 \ . \tag{4}$$

Therefore $x = 1/2$.

(b) Calculate the quantum Fourier transformation of the normalized state $|\psi\rangle$.
Solution.
The coefficients of computational basis states in the initial state are

$$x_j = \frac{1}{2}\cos\frac{2\pi j}{8} = \frac{1}{4}e^{-2\pi i j/8} + \frac{1}{4}e^{2\pi i j/8} \ . \tag{5}$$

Coefficients after the Fourier transform are

$$y_k = \frac{1}{\sqrt{8}}\sum_{j=0}^{7} x_j e^{2\pi i jk/8} = \frac{1}{4}\frac{1}{\sqrt{8}}\sum_{j=0}^{7} e^{2\pi i j(k-1)/8} + \frac{1}{4}\frac{1}{\sqrt{8}}\sum_{j=0}^{7} e^{2\pi i j(k+1)/8} \tag{6}$$

$$= \frac{1}{8\sqrt{2}}\sum_{j=0}^{7} e^{2\pi i j(k-1)/8} + \frac{1}{8\sqrt{2}}\sum_{j=0}^{7} e^{2\pi i j(k-7)/8} = \frac{1}{\sqrt{2}}\delta_{k,1} + \frac{1}{\sqrt{2}}\delta_{k,7} \ . \tag{7}$$

Therefore,

$$QFT\,|\psi\rangle = \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|7\rangle \ . \tag{8}$$

2. **Partial measurement**
Suppose ρ is the density matrix describing a two qubit system. Suppose we perform a projective measurement in the computational basis of the second qubit. Let $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$ be the projectors onto the $|0\rangle$ and $|1\rangle$ states of the second qubit, respectively. Let ρ' be the density matrix which would be assigned to the system after the measurement by an observer who did not learn the measurement result. Show that

$$\rho' = P_0 \rho P_0 + P_1 \rho P_1 \ . \tag{9}$$

Also show that the reduced density matrix for the first qubit is not affected by the measurement, that is, $\mathrm{tr}_2(\rho) = \mathrm{tr}_2(\rho')$.
Solution.
P_0 and P_1 are clearly measurement operators and complete $P_0^\dagger P_0 + P_1^\dagger P_1 = P_0 + P_1 = \mathbb{I}$. Probabilities of measurement outcomes are

$$p(0) = \mathrm{tr}\left(P_0^\dagger P_0 \rho\right) = \mathrm{tr}\left(P_0 \rho\right) \tag{10}$$

$$p(1) = \mathrm{tr}\left(P_1^\dagger P_1 \rho\right) = \mathrm{tr}\left(P_1 \rho\right) \ . \tag{11}$$

The post-measurement states are

$$\rho_0 = \frac{P_0 \rho P_0}{\sqrt{p(0)}} \tag{12}$$

$$\rho_1 = \frac{P_1 \rho P_1}{\sqrt{p(1)}} \ . \tag{13}$$

If the observer doesn't know which outcome occurred, the density matrix

$$\rho' = p(0)\rho_0 + p(1)\rho_1 \tag{14}$$

would be assigned to the system. This simplifies

$$\begin{aligned} \rho' &= p(0)\frac{P_0 \rho P_0}{\sqrt{p(0)}} + p(1)\frac{P_1 \rho P_1}{\sqrt{p(1)}} && (15)\\ &= P_0 \rho P_0 + P_1 \rho P_1 \ . && (16) \end{aligned}$$

The reduced density matrix for the first qubit is the same for ρ and ρ':

$$\begin{aligned} \mathrm{tr}_2(\rho') &= \mathrm{tr}_2\left(P_0 \rho P_0\right) + \mathrm{tr}_2\left(P_1 \rho P_1\right) && (17)\\ &= \mathrm{tr}_2\left(P_0 \rho\right) + \mathrm{tr}_2\left(P_1 \rho\right) && (18)\\ &= \mathrm{tr}_2\left((P_0 + P_1)\rho\right) = \mathrm{tr}_2\left(\mathbb{I}\rho\right) = \mathrm{tr}_2(\rho) \ . && (19) \end{aligned}$$

3. **Interaction as a quantum operation**
Suppose we have a single-qubit principal system S interacting with an environment E also consisting of a single qubit. The total Hilbert space is $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$, where $\mathcal{H}_S, \mathcal{H}_E$ are single-qubit Hilbert spaces corresponding to the individual qubits. The two interact through the transform

   (a) $U_a = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X$,
   (b) $U_b = \frac{1}{\sqrt{2}}(X \otimes I + Y \otimes X)$,

   where X is the usual Pauli X-matrix. Give the operator-sum representation for the quantum operation, which corresponds to the transformation of the system due to the interaction. Assume that the environment starts in the state $|0\rangle$.

   (Nielsen-Chuang section 8.2 may be useful for solving this problem.)

   Solution. The quantum operation on the state of the principal qubit is obtained as

$$\mathcal{E}(\rho) = \mathrm{tr}_E(U(\rho \otimes |0\rangle\langle 0|)U^\dagger)\,,$$

   where tr_E is the partial trace over the environment. Let's see how these act on the density matrix ρ of the principal system in the two different cases.

   a) The effect of the unitary U_a on the initial state $\rho \otimes |0\rangle\langle 0|$ is

$$\begin{aligned} U_a(\rho \otimes |0\rangle\langle 0|)U_a^\dagger &= (|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X)(\rho \otimes |0\rangle\langle 0|)(|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X) \\ &= (|0\rangle\langle 0|\rho \otimes |0\rangle\langle 0| + |1\rangle\langle 1|\rho \otimes |1\rangle\langle 0|)(|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X) \\ &= |0\rangle\langle 0|\rho|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1|\rho|0\rangle\langle 0| \otimes |1\rangle\langle 0| \\ &\quad + |0\rangle\langle 0|\rho|1\rangle\langle 1| \otimes |0\rangle\langle 1| + |1\rangle\langle 1|\rho|1\rangle\langle 1| \otimes |1\rangle\langle 1| \end{aligned}$$

   The partial trace can be implemented by the sum $\mathrm{tr}_E(A) = \sum_k (I \otimes \langle k|)A(I \otimes |k\rangle)$, which will remove the terms with off-diagonal entries in the second factor. Thus, we get

$$\mathrm{tr}_E(U_a(\rho \otimes |0\rangle\langle 0|)U_a^\dagger) = |0\rangle\langle 0|\rho|0\rangle\langle 0| + |1\rangle\langle 1|\rho|1\rangle\langle 1|\,.$$

   This is already in the operator-sum form with operation elements $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$.

   b) The effect of the unitary U_b on the initial state $\rho \otimes |0\rangle\langle 0|$ is

$$\begin{aligned} U_a(\rho \otimes |0\rangle\langle 0|)U_a^\dagger &= \frac{1}{\sqrt{2}}(X \otimes I + Y \otimes X)(\rho \otimes |0\rangle\langle 0|)\frac{1}{\sqrt{2}}(X \otimes I + Y \otimes X) \\ &= \frac{1}{2}(X\rho \otimes |0\rangle\langle 0| + Y\rho \otimes |1\rangle\langle 0|)(X \otimes I + Y \otimes X) \\ &= \frac{1}{2}(X\rho X \otimes |0\rangle\langle 0| + X\rho Y \otimes |0\rangle\langle 1| + Y\rho X \otimes |1\rangle\langle 0| + Y\rho Y \otimes |1\rangle\langle 1|)\,. \end{aligned}$$

Now taking the partial trace gives

$$\mathrm{tr}_E(U_b(\rho \otimes |0\rangle\langle 0|)U_b^\dagger) = \frac{1}{2}(X\rho X + Y\rho Y) = \frac{X}{\sqrt{2}}\rho\frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}}\rho\frac{Y}{\sqrt{2}} .$$

This is also in the operator-sum form with operation elements $X/\sqrt{2}$ and $Y/\sqrt{2}$.

4. **Quantum circuits**

(a) Let $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$ in the Hilbert space $\mathbb{C}^2$. Calculate

$$HZH\,|0\rangle \quad \text{and} \quad HZH\,|1\rangle \ ,$$

where H is the Hadamard transform. The unitary transform H is defined by

$$H\,|k\rangle = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^k\,|1\rangle) \ , \quad k \in \{0,1\} \ .$$

Solution.
Let $k \in \{0,1\}$. Then

$$\begin{aligned} HZH\,|k\rangle &= \frac{1}{\sqrt{2}}HZ(|0\rangle + (-1)^k\,|1\rangle) = \frac{1}{\sqrt{2}}H(|0\rangle + (-1)^{k+1}\,|1\rangle) \\ &= \frac{1}{\sqrt{2}}H(|0\rangle + (-1)^{1-k}\,|1\rangle) = |1-k\rangle \ . \end{aligned}$$

Now one can see that $HZH = X$.

(b) Calculate

$$(H \otimes H)U_{CNOT}(H \otimes H)\,|j,k\rangle$$

where $|j,k\rangle \equiv |j\rangle \otimes |k\rangle$ with $j,k \in \{0,1\}$, and the answer is in form of a ket $|m,n\rangle$ where $m,n \in \{0,1\}$. The controlled-NOT is defined

$$U_{CNOT} \equiv |0\rangle\langle 0| \otimes \mathbb{I} + |1\rangle\langle 1| \otimes X \ ,$$

where $\mathbb{I}$ is the 2×2-identity matrix and $X = |0\rangle\langle 1| + |1\rangle\langle 0|$.
Solution.
Straightforward calculation yields

$$\begin{aligned} &(H \otimes H)U_{CNOT}(H \otimes H)\,|j,k\rangle \\ &= \frac{1}{2}(H \otimes H)U_{CNOT}\big((|0\rangle + (-1)^j\,|1\rangle) \otimes (|0\rangle + (-1)^k\,|1\rangle)\big) \\ &= \frac{1}{2}(H \otimes H)(|00\rangle + (-1)^k\,|01\rangle + (-1)^j\,|11\rangle + (-1)^{j+k}\,|10\rangle) \\ &= \frac{1}{2}(H \otimes H)(|0\rangle + (-1)^{j+k}\,|1\rangle) \otimes (|0\rangle + (-1)^k\,|1\rangle) = |j \oplus k, k\rangle \ , \end{aligned}$$

where $\oplus$ is the XOR-operation. Now you can see that this is equivalent to a CNOT-operation with the second qubit as control and the first qubit as target.

5. **Qubit correlations**
We have a two-qubit density matrix

$$\rho = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{H}_A \otimes \mathcal{H}_B \ .$$

(a) Is the state ρ pure or not?
Solution.
By direct calculation

$$\begin{aligned} \rho^2 &= \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \rho \ . \end{aligned}$$

Therefore, the state is pure.

(b) Calculate the reduced density matrix

$$\rho_A = \mathrm{tr}_{\mathcal{H}_B} \ .$$

Solution.
By direct calculation

$$\begin{aligned} \rho_A = &\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \end{pmatrix} \rho \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &+ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \end{pmatrix} \rho \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ . \end{aligned}$$

(c) Calculate the entanglement entropy

$$S(\rho_A) = -\,\mathrm{tr}\left(\rho_A \log_2 \rho_A\right) \ .$$

Are the two qubits entangled?
Solution.

$$S(\rho_A) = -\frac{1}{2}\log_2\frac{1}{2} - \frac{1}{2}\log_2\frac{1}{2} = 1 \ .$$

The two qubits are maximally entangled.