# Quantum Information Spring 2023 Problem Set 4

1. **Bloch sphere transformation**
   A projective measurement is performed on a qubit in the basis given by the states $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$, but the measurement result goes unrecorded. In this case the state of the qubit, represented by the density matrix $\rho$, changes according to

$$\rho \mapsto P_+\rho P_+ + P_-\rho P_- \,, \tag{1}$$

where $P_\pm = |\pm\rangle\langle\pm|$ are the projection operators onto the basis states $|\pm\rangle$. How does this quantum operation transform the Bloch sphere?

*Solution.*
In the Bloch sphere representation the density matrix of a qubit is parametrized by a 3-dimensional real vector $\vec{r}$ such that $\|\vec{r}\| < 1$ according to

$$\rho = \frac{1}{2}(I + \vec{r}\cdot\vec{\sigma})\,.$$

We want to find out what happens to the vector $\vec{r}$ under this quantum operation. The projection operators $P_\pm$ act on the Pauli matrices as

$$\begin{aligned} P_\pm\sigma_x P_\pm &= |\pm\rangle\langle\pm|\sigma_x|\pm\rangle\langle\pm| = \pm|\pm\rangle\langle\pm| = \pm P_\pm \\ P_\pm\sigma_y P_\pm &= |\pm\rangle\langle\pm|\sigma_y|\pm\rangle\langle\pm| = 0 \\ P_\pm\sigma_z P_\pm &= |\pm\rangle\langle\pm|\sigma_z|\pm\rangle\langle\pm| = 0 \end{aligned}$$

since $\langle\pm|\sigma_x|\pm\rangle = \pm1$ and $\langle\pm|\sigma_y|\pm\rangle = 0 = \langle\pm|\sigma_z|\pm\rangle$. Accordingly, we find

$$P_\pm\rho P_\pm = \frac{1}{2}P_\pm(I + \vec{r}\cdot\vec{\sigma})P_\pm = \frac{1}{2}(P_\pm \pm r_x P_\pm) = \frac{1}{2}(1 \pm r_x)P_\pm\,.$$

The projection operators can be expressed in terms of Pauli X matrix as $P_\pm = \frac{1}{2}(I \pm \sigma_x)$, so we get

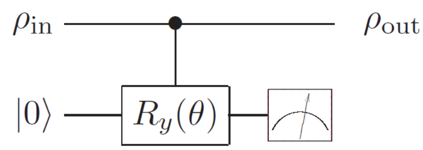
$$P_\pm\rho P_\pm = \frac{1}{4}(1 \pm r_x)(I \pm \sigma_x)\,,$$

and adding these up gives

$$\begin{aligned} P_+\rho P_+ + P_-\rho P_- &= \frac{1}{4}(1 + r_x)(I + \sigma_x) + \frac{1}{4}(1 - r_x)(I - \sigma_x) \\ &= \frac{1}{2}(I + r_x\sigma_x)\,. \end{aligned}$$

Accordingly, the Bloch sphere gets projected onto the $x$-axis, $(r_x, r_y, r_z) \mapsto (r_x, 0, 0)$.

2. **Circuit for phase damping channel**
Show that the following circuit implements the phase damping channel for the first qubit. How is the angle $\theta$ related to the parameter $\lambda$ appearing in the operator-sum representation of the phase damping channel?




*Solution.*
The initialization of the ancilla qubit is represented by the operator $I \otimes |0\rangle$. The controlled rotation is represented by the operator $cR_y(\theta) = P_0 \otimes I + P_1 \otimes R_y(\theta)$, where $P_i$ are the projections onto the computational basis states. The final measurement of the ancilla qubit is represented by two operators $I \otimes \langle i|$, $i = 0, 1$. Since the measurement result is not recorded, these give two different elements in the operator-sum representation. We get for the operator elements

$$\begin{aligned}
E_0 &= (I \otimes \langle 0|)(P_0 \otimes I + P_1 \otimes R_y(\theta))(I \otimes |0\rangle) \\
&= P_0 \otimes \langle 0|0\rangle + P_1 \otimes \langle 0|R_y(\theta)|0\rangle \\
&= P_0 + \cos\left(\frac{\theta}{2}\right) P_1 \\
&= \begin{pmatrix} 1 & 0 \\ 0 & \cos\left(\frac{\theta}{2}\right) \end{pmatrix} .
\end{aligned}$$

Here, we used $R_y(\theta) = \cos(\theta/2)I + \sin(\theta/2)\sigma_y$, so that $\langle 0|R_y(\theta)|0\rangle = \cos(\theta/2)$. Similar calculation for the other operator element gives

$$\begin{aligned}
E_1 &= (I \otimes \langle 1|)(P_0 \otimes I + P_1 \otimes R_y(\theta))(I \otimes |0\rangle) \\
&= P_0 \otimes \langle 1|0\rangle + P_1 \otimes \langle 1|R_y(\theta)|0\rangle \\
&= \sin\left(\frac{\theta}{2}\right) P_1 \\
&= \begin{pmatrix} 0 & 0 \\ 0 & \sin\left(\frac{\theta}{2}\right) \end{pmatrix} ,
\end{aligned}$$

where we used $\langle 1|R_y(\theta)|0\rangle = \sin(\theta/2)$. Thus, we see that the operator element matrices are exactly of the same form as for the phase damping noise with the identification $\lambda = \sin^2\left(\frac{\theta}{2}\right)$.

3. **Change of purity in depolarizing channel**
The *purity* of a quantum state represented by a density matrix $\rho$ is defined as the quantity $\gamma = \mathrm{tr}(\rho^2)$. Purity is bound by $\frac{1}{d} \leq \gamma \leq 1$, where $d$ is the Hilbert space

dimension, and $\gamma = 1$ if and only if $\rho$ is a pure (vector) state. Show that the purity of a qubit state never increases under the depolarizing channel.

*Solution.*
The claim is perhaps easiest to show by considering the Bloch sphere representation of the input and output states. We get for the purity in terms of the vector $\vec{r}$ of the Bloch sphere representation

$$\begin{aligned}\gamma &= \operatorname{tr}\rho^2 \\ &= \operatorname{tr}\left(\frac{1}{2}(I+\vec{r}\cdot\vec{\sigma})\right)^2 \\ &= \frac{1}{4}\operatorname{tr}\big(I + 2\vec{r}\cdot\vec{\sigma} + (\vec{r}\cdot\vec{\sigma})^2\big) \\ &= \frac{1}{4}\underbrace{\operatorname{tr}(I)}_{=2} + \frac{1}{2}\underbrace{\operatorname{tr}(\vec{r}\cdot\vec{\sigma})}_{=0} + \frac{1}{4}\underbrace{\operatorname{tr}(\vec{r}\cdot\vec{\sigma})^2}_{=2\|\vec{r}\|^2} \\ &= \frac{1}{2}(1+\|\vec{r}\|^2)\,.\end{aligned}$$

The depolarizing channel maps the state as $\rho \mapsto \rho' = (1-p)\rho + \frac{p}{2}I$. In terms of the Bloch sphere parametrization of the original state this gives

$$(1-p)\rho + \frac{p}{2}I = (1-p)\frac{1}{2}(I+\vec{r}\cdot\vec{\sigma}) + \frac{p}{2}I = \frac{1}{2}(I + (1-p)\vec{r}\cdot\vec{\sigma})\,.$$

Accordingly, the depolarizing channel scales the Bloch sphere vector as $\vec{r} \mapsto (1-p)\vec{r}$. Therefore, the purity of the output state is obtained as

$$\gamma' = \operatorname{tr}(\rho')^2 = \frac{1}{2}(1 + (1-p)^2\|\vec{r}\|^2)\,.$$

The change in purity through the channel is

$$\Delta\gamma = \gamma' - \gamma = \frac{1}{2}[(1-p)^2 - 1]\|\vec{r}\|^2 \le 0\,,$$

since $(1-p^2) \le 1$ for all $0 \le p \le 1$.

4. **Phase flip QEC conditions**
Verify that the 3-qubit phase-flip code $|0_L\rangle = |+++\rangle$, $|1_L\rangle = |---\rangle$ satisfies the quantum error correction conditions for the set of errors $\{I, Z_1, Z_2, Z_3\}$. Show also that the inclusion of the operator $X_1$ into the set of errors makes the condition fail.

*Solution.*
The quantum error correction condition reads

$$PE_i^\dagger E_j P = \alpha_{ij} P\,,$$

where $\alpha$ is some Hermitian matrix. For the 3-qubit phase-flip code the projection onto the codespace is

$$P = |+++\rangle\langle+++| + |---\rangle\langle---| \,.$$

Since $Z_1^2 = Z_2^2 = Z_3^2 = I$, these do not give any non-trivial conditions. Pairing the identity with the first of the Z-errors gives

$$\begin{aligned} PIZ_1P &= (|+++\rangle\langle+++| + |---\rangle\langle---|)Z_1(|+++\rangle\langle+++| + |---\rangle\langle---|) \\ &= (|+++\rangle\langle+++| + |---\rangle\langle---|)(|-++\rangle\langle+++| + |+--\rangle\langle---|) \\ &= 0 \,. \end{aligned}$$

Similar result holds for the other Z-errors. Combinations to two Z-errors give, for example,

$$\begin{aligned} PZ_1Z_2P &= (|+++\rangle\langle+++| + |---\rangle\langle---|)Z_1Z_2(|+++\rangle\langle+++| + |---\rangle\langle---|) \\ &= (|+++\rangle\langle+++| + |---\rangle\langle---|)(|--+\rangle\langle+++| + |++-\rangle\langle---|) \\ &= 0 \,. \end{aligned}$$

Similar result holds for the other combinations of Z-errors. Accordingly, we find for the matrix $\alpha = \mathrm{diag}(1,1,1,1)$, and the quantum error correction condition is satisfied.

If we include the operator $X_1$ into the set of errors, we get for example

$$\begin{aligned} PIX_1P &= (|+++\rangle\langle+++| + |---\rangle\langle---|)X_1(|+++\rangle\langle+++| + |---\rangle\langle---|) \\ &= (|+++\rangle\langle+++| + |---\rangle\langle---|)(|+++\rangle\langle+++| - |---\rangle\langle---|) \\ &= |+++\rangle\langle+++| - |---\rangle\langle---| \,, \end{aligned}$$

which is not proportional to $P$, and therefore the condition fails.

5. **5-qubit code syndrome and logical operators**

   (a) Verify that the syndrome observables for the 5-qubit code commute among themselves, and also with the logical operators. (This is important, because it allows for the operators to have common eigenstates, and ensures that the logical operators map codestates to codestates.)

   (b) Check that the logical X and Z operators for the 5-qubit code satisfy the correct algebraic relations (i.e., they anti-commute), and that the code basis states are eigenstates of the logical Z operator with the appropriate eigenvalues.

*Solution.*
a) The set of syndrome observables/generators for the 5-qubit code is

$$\{XZZXI, IXZZX, XIXZZ, ZXIXZ, ZZXIX\}$$

and the logical operators $X_L = XXXXX$, $Z_L = ZZZZZ$. Here, e.g., '$XZZXI$' really is a shorthand for the operator $X \otimes Z \otimes Z \otimes X \otimes I$. By a straightforward inspection we can see that in each pair of generators two of the tensor product factors anti-commute and others commute. For example, for the two first ones we have

$$
\begin{aligned}
&(X \otimes Z \otimes Z \otimes X \otimes I)(I \otimes X \otimes Z \otimes Z \otimes X) \\
&= XI \otimes ZX \otimes ZZ \otimes XZ \otimes IX \\
&= IX \otimes (-XZ) \otimes ZZ \otimes (-ZX) \otimes XI \\
&= (-1)^2 IX \otimes XZ \otimes ZZ \otimes ZX \otimes XI \\
&= (I \otimes X \otimes Z \otimes Z \otimes X)(X \otimes Z \otimes Z \otimes X \otimes I)\,.
\end{aligned}
$$

Therefore all the generator commute among themselves. The same applies to the generators and logical operators. For example,

$$
\begin{aligned}
&(X \otimes Z \otimes Z \otimes X \otimes I)(X \otimes X \otimes X \otimes X \otimes X) \\
&= XX \otimes ZX \otimes ZX \otimes XX \otimes XX \\
&= IX \otimes (-XZ) \otimes (-XZ) \otimes XX \otimes XX \\
&= (-1)^2 IX \otimes XZ \otimes XZ \otimes XX \otimes XX \\
&= (X \otimes X \otimes X \otimes X \otimes X)(X \otimes Z \otimes Z \otimes X \otimes I)\,.
\end{aligned}
$$

b) The logical operators anti-commute correctly:

$$
\begin{aligned}
&(X \otimes X \otimes X \otimes X \otimes X)(Z \otimes Z \otimes Z \otimes Z \otimes Z) \\
&= XZ \otimes XZ \otimes XZ \otimes XZ \otimes XZ \\
&= (-ZX) \otimes (-ZX) \otimes (-ZX) \otimes (-ZX) \otimes (-ZX) \\
&= (-1)^5 ZX \otimes ZX \otimes ZX \otimes ZX \otimes ZX \\
&= -(Z \otimes Z \otimes Z \otimes Z \otimes Z)(X \otimes X \otimes X \otimes X \otimes X)\,.
\end{aligned}
$$

When the logical Z operator acts on the computational basis state $|b_1b_2b_3b_4b_5\rangle$, where $b_i = 0, 1$, we have

$$Z_L|b_1b_2b_3b_4b_5\rangle = (-1)^{b_1+b_2+b_3+b_4+b_5}|b_1b_2b_3b_4b_5\rangle\,.$$

By inspecting the codestate basis, we see that the codestate $|0_L\rangle$ has an even number of 1's in each component, when expressed in terms of the computational basis states. Thus, $Z_L|0_L\rangle = |0_L\rangle$. Similarly, the codestate $|1_L\rangle$ has an odd number of 1's in each component, so we get $Z_L|1_L\rangle = -|1_L\rangle$, as appropriate.