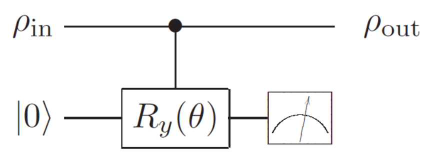
Quantum Information Spring 2023 Problem Set 4

Solutions are due on Sunday May 21st, 23:59.

1. **Bloch sphere transformation**
   A projective measurement is performed on a qubit in the basis given by the states $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$, but the measurement result goes unrecorded. In this case the state of the qubit, represented by the density matrix ρ, changes according to
   $$\rho \mapsto P_+\rho P_+ + P_-\rho P_- \,, \tag{1}$$
   where $P_\pm = |\pm\rangle\langle\pm|$ are the projection operators onto the basis states $|\pm\rangle$. How does this quantum operation transform the Bloch sphere?

2. **Circuit for phase damping channel**
   Show that the following circuit implements the phase damping channel for the first qubit. How is the angle θ related to the parameter λ appearing in the operator-sum representation of the phase damping channel?

3. **Change of purity in depolarizing channel**
   The *purity* of a quantum state represented by a density matrix ρ is defined as the quantity $\gamma = \mathrm{tr}(\rho^2)$. Purity is bound by $\frac{1}{d} \le \gamma \le 1$, where d is the Hilbert space dimension, and $\gamma = 1$ if and only if ρ is a pure (vector) state. Show that the purity of a qubit state never increases under the depolarizing channel.

4. **Phase flip QEC conditions**
   Verify that the 3-qubit phase-flip code $|0_L\rangle = |+++\rangle$, $|1_L\rangle = |---\rangle$ satisfies the quantum error correction conditions for the set of errors $\{I, Z_1, Z_2, Z_3\}$. Show also that the inclusion of the operator X_1 into the set of errors makes the condition fail.

5. **5-qubit code syndrome and logical operators**

   (a) Verify that the syndrome observables for the 5-qubit code commute among themselves, and also with the logical operators. (This is important, because it allows for the operators to have common eigenstates, and ensures that the logical operators map codestates to codestates.)

   (b) Check that the logical X and Z operators for the 5-qubit code satisfy the correct algebraic relations (i.e., they anti-commute), and that the code basis states are eigenstates of the logical Z operator with the appropriate eigenvalues.