Quantum Information Spring 2023 Problem Set 1

1. **Basic Quantum Mechanics**

   (a) **State space**
   A pure quantum state is completely described by a vector $|\psi\rangle$ in a Hilbert space $\mathcal{H}$. Usually quantum states are normalized $\langle\psi|\psi\rangle = 1$. Normalize the one-qubit state

   $$|\phi\rangle = 2\,|0\rangle - i\,|1\rangle \ , \tag{1}$$

   that is, find a constant A such that $|\psi\rangle = |\phi\rangle / A$ is normalized.
   Solution.
   By direct computation $\langle\phi|\phi\rangle = 5$. Therefore, if $A = \sqrt{5}$, then $\langle\Psi|\Psi\rangle = 1$.

   (b) **Evolution of states**
   The time evolution of closed quantum states is described by unitary transformations: if a system is initially in state $|\psi_1\rangle$, then at some later time the state is $|\psi_2\rangle = U\,|\psi_1\rangle$ where U is an unitary operator. Verify explicitly that

   $$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{2}$$

   is unitary. Let $|\psi_1\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. Compute $|\psi_2\rangle = U\,|\psi_1\rangle$.
   Solution.
   By direct computation

   $$U^\dagger U = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \tag{3}$$

   so U is unitary. Then $U\,|\psi_1\rangle$

   $$U\,|\psi_1\rangle = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle \ . \tag{4}$$

   (c) **Measurement**
   Measurements are described by sets of measurement operators $\{M_m\}$, where the index m corresponds to any of the possible outcomes of the measurement. Measurement of a single qubit in the computational basis is described by the operators

   $$\text{Outcome } 0: \quad M_0 = |0\rangle\langle 0| \tag{5}$$
   $$\text{Outcome } 1: \quad M_1 = |1\rangle\langle 1| \ . \tag{6}$$

   Suppose a qubit is in the normalized state $|\psi\rangle$ from part (a). What are the probabilities of outcomes 0 and 1 when $|\psi\rangle$ is measured in the computational basis?

Solution.

In part (a) we found that $|\psi\rangle = \frac{2}{\sqrt{5}}|0\rangle - \frac{i}{\sqrt{5}}|1\rangle$. Measurement probabilities are then

$$p(0) = \langle\phi| M_0 |\phi\rangle = \langle\phi|0\rangle \langle 0|\phi\rangle = \frac{4}{5} \tag{7}$$

$$p(1) = \langle\phi| M_1 |\phi\rangle = \langle\phi|1\rangle \langle 1|\phi\rangle = \frac{1}{5}\ . \tag{8}$$

(d) **Composite systems**

The state space of a composite physical system is the tensor product of the component state spaces. For example, $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. If the subsystems are prepared to states $|\psi_1\rangle$ and $|\psi_2\rangle$, then their composite system is in state $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \in \mathcal{H}$. Let

$$|\psi_1\rangle = a|0\rangle + b|1\rangle \in \mathcal{H}_1 \tag{9}$$

$$|\psi_2\rangle = c|0\rangle + d|1\rangle \in \mathcal{H}_2\ . \tag{10}$$

Compute explicitly $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$. Show that the *Bell state*

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle) \tag{11}$$

cannot be written as a tensor product of any two states $|\psi_1\rangle$ and $|\psi_2\rangle$.

Solution.

The two qubit states in component form are

$$|\psi\rangle = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix} \qquad |\Psi^+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{12}$$

If we assume that the Bell state can be factorized, that is $|\psi\rangle = |\Psi^+\rangle$ for some a, b, c, d then we would have

$$\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = (ac)(bd) = (ad)(bc) = 0\ , \tag{13}$$

using the commutative property of scalar multiplication. This is a contradiction, therefore the Bell state cannot be written as a product of any two one-qubit states $|\psi_1\rangle$ and $|\psi_2\rangle$.

2. **Operator Functions 1**

Find the square root and logarithm (base 2) of the matrix

$$A = \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}\ . \tag{14}$$

Solution.
First we find the spectral decomposition of A. Start by solving for the eigenvalues λ from the characteristic equation

$$\det(A - \lambda\mathbb{I}) = (4-\lambda)^2 - 9 = 0\ . \tag{15}$$

Therefore the eigenvalues are $\lambda_a = 7$ and $\lambda_b = 1$. Then solve the first eigenvector (corresponding to $\lambda_a = 7$)

$$\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 7\begin{pmatrix} x \\ y \end{pmatrix} \tag{16}$$

$$\rightarrow \begin{cases} 4x + 3y = 7x \\ 4x + 4y = 7y \end{cases}\ , \tag{17}$$

which is solved by $x = y$. The eigenvector is then (x, x) which after normalization becomes $|a\rangle = \frac{1}{\sqrt{2}}(1,1)$. The same procedure for $\lambda_b = 1$ results in the eigenvector $|b\rangle = \frac{1}{\sqrt{2}}(1,-1)$. Therefore, the spectral decomposition of A is

$$A = \lambda_a\,|a\rangle\langle a| + \lambda_b\,|b\rangle\langle b| \tag{18}$$

$$= 7\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} + \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}\ . \tag{19}$$

Now we can compute $\sqrt{A}$ and $\log_2 A$. First,

$$\sqrt{A} = \sqrt{7}\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} + \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sqrt{7}+1 & \sqrt{7}-1 \\ \sqrt{7}-1 & \sqrt{7}+1 \end{pmatrix}\ . \tag{20}$$

Then,

$$\log_2 A = \log_2 7\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} + \log_2 1\begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix} = \frac{\log_2 7}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\ . \tag{21}$$

3. **Operator Functions 2**
Show that

$$\exp(-i\theta X/2) = \cos\left(\frac{\theta}{2}\right)\mathbb{I} - i\sin\left(\frac{\theta}{2}\right)X\ , \tag{22}$$

where X is the Pauli X-matrix.
Solution.
This can be computed in the same way as the previous problem: first find the spectral decomposition of X which would be $X = |+\rangle\langle+| - |-\rangle\langle-|$ and then computing

$\exp(-i\theta X/2) = \exp(-i\theta/2)\,|+\rangle\!\langle+| + \exp(i\theta/2)\,|-\rangle\!\langle-|$. One can show that this indeed is equal to the RHS of (22). An another, maybe easier way is to use the fact that $X^2 = \mathbb{I}$ and use this in the expansion of the exponent function

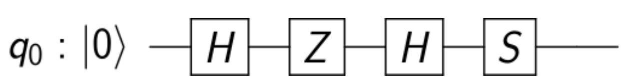
$$\exp(-i\theta X/2) = \sum_{n=0}^{\infty} \frac{(-i\theta/2)^n}{n!} X^n \tag{23}$$
$$= \sum_{n=0}^{\infty} \frac{(-i\theta/2)^{2n}}{(2n)!} X^{2n} + \sum_{n=0}^{\infty} \frac{(-i\theta/2)^{2n+1}}{(2n+1)!} X^{2n+1} \tag{24}$$
$$= \mathbb{I} \sum_{n=0}^{\infty} \frac{(-1)^n (\theta/2)^{2n}}{(2n)!} - iX \sum_{n=0}^{\infty} \frac{(-1)^n (\theta/2)^{2n+1}}{(2n+1)!} \tag{25}$$
$$= \cos\left(\frac{\theta}{2}\right) - i \sin\left(\frac{\theta}{2}\right) X \,. \tag{26}$$

We separated even-n and odd-n terms in the first step. In the second step we used the fact that $X^{2n} = \mathbb{I}$ and $(-i)^{2n} = (-1)^n$ for every n. In the last step we recognize the two series as cos and sin.

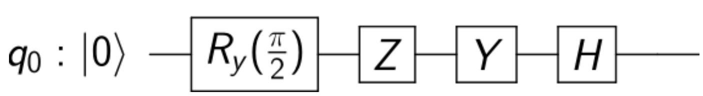
4. **Single Qubit Quantum Circuits**
Calculate the state produced by the given circuit and compute the expectation value of the given observable. Matrices of the gates are

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} , \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \tag{27}$$
$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} , \quad R_y\left(\frac{\pi}{2}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} . \tag{28}$$

(a) Compute the expectation value of the Z-observable



(b) Compute the expectation value of the X-observable



Solution.
(a)
We are asked to compute the expectation value of Z in the state $|\psi\rangle = SHZH\,|0\rangle$.

First we compute this state

$$H|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \tag{29}$$

$$ZH|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \tag{30}$$

$$HZH|0\rangle = \begin{pmatrix}0\\1\end{pmatrix} \tag{31}$$

$$SHZH|0\rangle = \begin{pmatrix}0\\i\end{pmatrix} \tag{32}$$

Then, the expectation value of Z in this state is

$$\langle\psi| Z |\psi\rangle = \begin{pmatrix}0 & -i\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}0\\i\end{pmatrix} = -1 \ . \tag{33}$$

(b) This time the state before measurement is $|\psi\rangle = HYZR_y\left(\frac{\pi}{2}\right)|0\rangle$

$$R_y\left(\frac{\pi}{2}\right)|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \tag{34}$$

$$ZR_y\left(\frac{\pi}{2}\right)|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \tag{35}$$

$$YZR_y\left(\frac{\pi}{2}\right)|0\rangle = \frac{i}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \tag{36}$$

$$HYZR_y\left(\frac{\pi}{2}\right)|0\rangle = \begin{pmatrix}i\\0\end{pmatrix} \tag{37}$$

Finally, we compute the expectation value of the X-operator

$$\langle\psi| X |\psi\rangle = \begin{pmatrix}-i & 0\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}i\\0\end{pmatrix} = 0 \ . \tag{38}$$

5. **Controlled NOT gate**
The controlled-not operator has a central role in quantum computation. Consider a CNOT gate controlled on the first qubit with the second qubit as the target. One way to express this operation is

$$CNOT = |0\rangle\langle 0| \otimes \mathbb{I} + |1\rangle\langle 1| \otimes X \ . \tag{39}$$

That is, CNOT flips the target qubit only if the control qubit is set to $|1\rangle$. Compute explicitly the matrix representation of CNOT. Then verify that

$$CNOT|00\rangle = |00\rangle \tag{40}$$

$$CNOT|01\rangle = |01\rangle \tag{41}$$

$$CNOT|10\rangle = |11\rangle \tag{42}$$

$$CNOT|11\rangle = |10\rangle \ , \tag{43}$$

where

$$|00\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad \text{etc.} \tag{44}$$

Solution.
The matrix representation of CNOT is

$$\begin{aligned} CNOT &= |0\rangle\langle 0| \otimes \mathbb{I} + |1\rangle\langle 1| \otimes X & (45)\\ &= \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} + \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} \otimes \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} & (46)\\ &= \begin{pmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 1 & 0\end{pmatrix} \quad . & (47) \end{aligned}$$

We can verify the action of CNOT on computational basis states by explicit multiplications

$$CNOT\,|00\rangle = \begin{pmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 1 & 0\end{pmatrix} \begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} = |00\rangle \tag{48}$$

$$CNOT\,|01\rangle = \begin{pmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 1 & 0\end{pmatrix} \begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} = |01\rangle \tag{49}$$

$$CNOT\,|10\rangle = \begin{pmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 1 & 0\end{pmatrix} \begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} = |11\rangle \tag{50}$$

$$CNOT\,|11\rangle = \begin{pmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 1 & 0\end{pmatrix} \begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} = |10\rangle \quad . \tag{51}$$