Quantum Information Spring 2023 Problem Set 2

1. **Mixed States**
Consider the 2×2 matrix

$$\rho = \begin{pmatrix} 3/4 & \sqrt{2}e^{-i\phi}/4 \\ \sqrt{2}e^{i\phi}/4 & 1/4 \end{pmatrix} \; . \tag{1}$$

(a) Is the matrix a density matrix?
Solution.
We have to check that the matrix is Hermitian, has $\operatorname{tr}\rho = 1$, and is positive semi-definite. The matrix is clearly Hermitian $\rho^\dagger = \rho$. Its trace is $\operatorname{tr}\rho = 3/4 + 1/4 = 1$. To check the last condition, we solve the characteristic equation for the eigenvalues

$$0 = \det(\rho - \lambda\mathbb{I}) = \left(\frac{3}{4} - \lambda\right)\left(\frac{1}{4} - \lambda\right) - \frac{1}{8} \tag{2}$$

$$= \lambda^2 - \lambda + \frac{1}{16} \; . \tag{3}$$

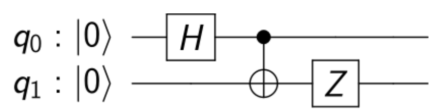
The roots of this equation are $\lambda = (2 \pm \sqrt{3})/4$. Both of the eigenvalues are positive, so the matrix ρ is positive (semi-)definite and therefore a valid density matrix.

(b) If so do we have a pure state or a mixed state?
Solution.
Since

$$\rho^2 = \begin{pmatrix} 11/16 & e^{-i\phi}/(2\sqrt{2}) \\ e^{i\phi}/(2\sqrt{2}) & 3/16 \end{pmatrix} \neq \rho \; , \tag{4}$$

we have a mixed state.

2. **Entangling gates**
Consider the following quantum circuit



(a) Find the state vector $|\psi\rangle$ produced by this circuit.
Solution.
The first Hadamard transforms

$$|0\rangle \otimes |0\rangle \mapsto \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle = \frac{1}{\sqrt{2}}\,|0\rangle \otimes |0\rangle + \frac{1}{\sqrt{2}}\,|1\rangle \otimes |0\rangle \; . \tag{5}$$

After the controlled-NOT gate the state is

$$\frac{1}{\sqrt{2}}|0\rangle \otimes |0\rangle + \frac{1}{\sqrt{2}}|1\rangle \otimes |0\rangle \mapsto \frac{1}{\sqrt{2}}|0\rangle \otimes |0\rangle + \frac{1}{\sqrt{2}}|1\rangle \otimes |1\rangle \ . \tag{6}$$

After the final gate we have

$$\frac{1}{\sqrt{2}}|0\rangle \otimes |0\rangle + \frac{1}{\sqrt{2}}|1\rangle \otimes |1\rangle \mapsto \frac{1}{\sqrt{2}}|0\rangle \otimes |0\rangle - \frac{1}{\sqrt{2}}|1\rangle \otimes |1\rangle \ . \tag{7}$$

So, the state vector produced by this circuit is

$$\frac{1}{\sqrt{2}}|0\rangle \otimes |0\rangle - \frac{1}{\sqrt{2}}|1\rangle \otimes |1\rangle \quad \text{or} \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \ . \tag{8}$$

(b) Compute the corresponding density operator.
Solution.
The density operator/matrix is obtained as the product $\rho = |\psi\rangle\langle\psi|$

$$\rho = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \ . \tag{9}$$

3. **Entanglement Entropy**

(a) Consider the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^2$ and the state

$$|\psi\rangle = \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle) = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \ . \tag{10}$$

Calculate the entanglement entropies

$$-\operatorname{tr}(\rho_A \log \rho_A) \quad \text{and} \quad -\operatorname{tr}(\rho_B \log \rho_B) \ , \tag{11}$$

where the reduced density matrices are

$$\rho_A = \operatorname{tr}_{\mathcal{H}_B}|\psi\rangle\langle\psi| \quad \text{and} \quad \rho_B = \operatorname{tr}_{\mathcal{H}_A}|\psi\rangle\langle\psi| \ . \tag{12}$$

Solution.
The density matrix $\rho = |\psi\rangle\langle\psi|$ is

$$\rho = \frac{1}{4}\begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} . \tag{13}$$

Therefore, the reduced density matrix ρ_A is

$$\begin{aligned}
\rho_A &= (\mathbb{I}_A \otimes \langle 0|)\, |\psi\rangle\langle\psi|\, (\mathbb{I}_A \otimes |0\rangle) + (\mathbb{I}_A \otimes \langle 1|)\, |\psi\rangle\langle\psi|\, (\mathbb{I}_A \otimes |1\rangle) && (14)\\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \end{pmatrix} |\psi\rangle\langle\psi| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} && (15)\\
&+ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \end{pmatrix} |\psi\rangle\langle\psi| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} && (16)\\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} |\psi\rangle\langle\psi| \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} |\psi\rangle\langle\psi| \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} && (17)\\
&= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} . && (18)
\end{aligned}$$

In order to find the entanglement entropy of ρ_A, we'll solve for its eigenvalues λ from the characteristic equation

$$\det\left(\rho_A - \lambda \mathbb{I}_A\right) = \det\begin{pmatrix} 1/2-\lambda & -1/2 \\ -1/2 & 1/2-\lambda \end{pmatrix} = \left(\frac{1}{2} - \lambda\right)^2 - \frac{1}{4} = 0 , \tag{19}$$

which has the solutions $\lambda_1 = 0$ and $\lambda_2 = 1$. Therefore the entanglement entropy is

$$S(\rho_A) = -\operatorname{tr}\left(\rho_A \log \rho_A\right) = -\sum_i \lambda_i \log \lambda_i = -0\log 0 - 1\log 1 = 0 . \tag{20}$$

We could do the same computation to find first ρ_B and then its entropy, but since $\rho = |\psi\rangle\langle\psi|$ is a pure state we must have $S(\rho_B) = S(\rho_A) = 0$.

(b) Is this state separable? Why or why not?
Solution.
This state is separable and therefore can be written $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$ for some $|\psi_A\rangle \in \mathcal{H}_A$ and $|\psi_B\rangle \in \mathcal{H}_B$. This is because the reduced states ρ_A and ρ_B are pure because $S(\rho_A) = S(\rho_B) = 0$. Explicitly, this can be shown by noting that $|\psi\rangle = |-\rangle \otimes |-\rangle$.

4. **Trace Distance**
Consider the density operators

$$\rho = \frac{3}{4}\,|0\rangle\langle 0| + \frac{1}{4}\,|1\rangle\langle 1| \quad \text{and} \quad \sigma = \frac{2}{3}\,|+\rangle\langle +| + \frac{1}{3}\,|-\rangle\langle -|\ , \tag{21}$$

where $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$. Calculate the trace distance $D(\rho,\sigma)$.
Solution.
Let's write the density operators as matrices

$$\rho = \begin{pmatrix} 3/4 & 0 \\ 0 & 1/4 \end{pmatrix} \tag{22}$$

$$\sigma = \frac{2}{3}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} + \frac{1}{3}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \tag{23}$$

$$= \frac{2}{3}\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix} \tag{24}$$

$$= \begin{pmatrix} 1/2 & 1/6 \\ 1/6 & 1/2 \end{pmatrix}\ . \tag{25}$$

Since $D(\rho,\sigma) = \frac{1}{2}\operatorname{tr}|\rho-\sigma|$, where $|A| \equiv \sqrt{A^\dagger A}$, we first compute $|\rho-\sigma|$

$$\rho - \sigma = \begin{pmatrix} 1/4 & -1/6 \\ -1/6 & -1/4 \end{pmatrix}\ . \tag{26}$$

Then

$$(\rho-\sigma)^\dagger(\rho-\sigma) = \begin{pmatrix} 13/144 & 0 \\ 0 & 13/144 \end{pmatrix}\ . \tag{27}$$

Since this is diagonal, the square root is easy to compute

$$\sqrt{(\rho-\sigma)^\dagger(\rho-\sigma)} = \begin{pmatrix} \sqrt{13}/12 & 0 \\ 0 & \sqrt{13}/12 \end{pmatrix} = |\rho-\sigma|\ . \tag{28}$$

Therefore, the trace distance is

$$D(\rho,\sigma) = \frac{1}{2}\operatorname{tr}|\rho-\sigma| = \frac{1}{2}\frac{\sqrt{13}}{6} = \frac{\sqrt{13}}{12}\ . \tag{29}$$

5. **Fidelity**
Consider the density operators

$$\rho = \frac{3}{4}\,|0\rangle\langle 0| + \frac{1}{4}\,|1\rangle\langle 1| \quad \text{and} \quad \sigma = \frac{2}{3}\,|0\rangle\langle 0| + \frac{1}{3}\,|1\rangle\langle 1|\ . \tag{30}$$

(a) Compute the fidelity $F(\rho,\sigma) = \operatorname{tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}$.
Solution.
We start by writing the density operators as matrices

$$\rho = \begin{pmatrix} 3/4 & 0 \\ 0 & 1/4 \end{pmatrix} \quad \text{and} \quad \sigma = \begin{pmatrix} 2/3 & 0 \\ 0 & 1/3 \end{pmatrix} . \tag{31}$$

Both matrices are diagonal, so the matrix square roots are again simple:

$$\sqrt{\rho} = \begin{pmatrix} \sqrt{3}/2 & 0 \\ 0 & 1/2 \end{pmatrix} \tag{32}$$

$$\sqrt{\rho}\sigma\sqrt{\rho} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/12 \end{pmatrix} \tag{33}$$

$$\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}} = \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/(2\sqrt{3}) \end{pmatrix} . \tag{34}$$

Therefore, the fidelity is

$$F(\rho,\sigma) = \frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{3}} . \tag{35}$$

(b) A bit flip channel flips the qubit from $|0\rangle$ to $|1\rangle$ (and vice versa) with probability $1-p$. It can be described with the quantum channel

$$\rho \mapsto \mathcal{E}(\rho) = E_0\rho E_0^\dagger + E_1\rho E_1^\dagger , \tag{36}$$

where

$$E_0 = \sqrt{p}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \quad E_1 = \sqrt{1-p}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{37}$$

The bit flip channel has the following effect on our states ρ and σ:

$$\rho \mapsto \mathcal{E}(\rho) = \begin{pmatrix} (1+2p)/4 & 0 \\ 0 & (3-2p)/4 \end{pmatrix} \tag{38}$$

$$\sigma \mapsto \mathcal{E}(\sigma) = \begin{pmatrix} (1+p)/3 & 0 \\ 0 & (2-p)/3 \end{pmatrix} . \tag{39}$$

Calculate the fidelity $F(\mathcal{E}(\rho),\mathcal{E}(\sigma))$. Is the new fidelity smaller or larger than the original fidelity $F(\rho,\sigma)$? Can you explain why?
Solution. The states $\mathcal{E}(\rho)$ and $\mathcal{E}(\sigma)$ after the bit flip error has occurred are still diagonal, so computing the square roots is still trivial. We compute like before

$$\sqrt{\rho} = \begin{pmatrix} \sqrt{1+2p}/2 & 0 \\ 0 & \sqrt{3-2p}/2 \end{pmatrix} \tag{40}$$

$$\sqrt{\rho}\sigma\sqrt{\rho} = \begin{pmatrix} (1+p)(1+2p)/12 & 0 \\ 0 & (3-2p)(2-p)/12 \end{pmatrix} \tag{41}$$

$$\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}} = \begin{pmatrix} \sqrt{(1+p)(1+2p)}/(2\sqrt{3}) & 0 \\ 0 & \sqrt{(3-2p)(2-p)}/(2\sqrt{3}) \end{pmatrix} . \tag{42}$$

This gives us a fidelity

$$F(\mathcal{E}(\rho), \mathcal{E}(\sigma)) = \frac{1}{2\sqrt{3}} \left(\sqrt{(1+p)(1+2p)} + \sqrt{(3-2p)(2-p)} \right) . \qquad (43)$$

One can check that $F(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \geq F(\rho, \sigma)$. We can see from the attached plot that if no error happens $p = 1$, then the fidelity is unchanged as it should. Also if the error happens with certainty $p = 0$, the fidelities are equal again. Otherwise the presence of the bit flip error makes the states more similar to each other.