$$|\varphi\rangle\langle\varphi| = \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{pmatrix} \begin{pmatrix} \bar{\varphi}_1 & \cdots & \bar{\varphi}_n \end{pmatrix} = \begin{pmatrix} \varphi_1\bar{\varphi}_1 & & \\ \vdots & \ddots & \\ & & \end{pmatrix}$$

$$|\psi\rangle \in \mathcal{H}$$

$|\psi_1\rangle,\ p_1$

$|\psi_2\rangle,\ p_2$

$$\Rightarrow \rho = \overbrace{p_1 |\psi_1\rangle\langle\psi_1|}^{\rho_1} + \overbrace{p_2 |\psi_2\rangle\langle\psi_2|}^{\rho_2}$$

$$\langle A\rangle = \mathrm{tr}(\rho A) \qquad \rho = \sum_i \lambda_i |\varphi_i\rangle\langle\varphi_i|$$

$$\mathrm{tr}(\rho) = 1$$

$$\rho = \frac{1}{2}(I + \vec{r}\cdot\vec{\sigma}), \quad \mathrm{tr}(\sigma_k) = 0,\ \mathrm{tr}(\vec{r}\cdot\vec{\sigma}) = 0$$

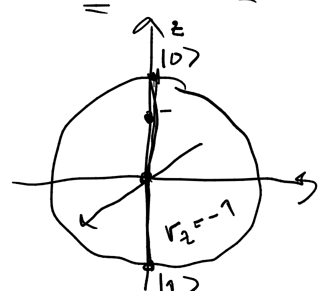
$$= \frac{1}{2}(I + r_x\sigma_x + r_y\sigma_y + r_z\sigma_z)$$

$\|\vec{r}\| = 1$
$\Rightarrow$ pure state

$\|\vec{r}\| < 1$
$\Rightarrow$ mixed state

$\|\vec{r}\| = 0$
$\Rightarrow \rho = \frac{1}{2}I$
$= \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$

$$\begin{cases} \langle\sigma_x\rangle = r_x \\ \langle\sigma_y\rangle = r_y \\ \langle\sigma_z\rangle = r_z \end{cases}$$



$$\mathrm{tr}(\rho) = 1$$

$$U|\psi\rangle\langle\psi|U^\dagger \longmapsto U\rho U^\dagger$$

$$\rho = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$$

$$\langle\sigma_z\rangle = \mathrm{tr}(\rho \underset{Z}{\sigma_z}) = \mathrm{tr}\big((p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|)Z\big)$$

$$= p\,\underbrace{\mathrm{tr}(|0\rangle\langle 0|Z)}_{=\langle 0|\underbrace{Z|0\rangle}_{=|0\rangle} = \langle 0|0\rangle = 1} + (1-p)\,\mathrm{tr}(\underbrace{|1\rangle\langle 1|Z}_{\langle 1|\underbrace{Z|1\rangle}_{=-|1\rangle} = -\langle 1|1\rangle = -1})$$

$$= p + (1-p)\cdot(-1) = 2p - 1$$

$$|\Psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle = |\psi_1\rangle|\psi_2\rangle$$

$$|Bell\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

$$\mathrm{tr}\left(|\psi_1\rangle\langle\psi_2|\right) = \langle\psi_2|\psi_1\rangle$$

$$\mathrm{tr}_B\left(|\psi_1^A\rangle|\psi_1^B\rangle\ \langle\psi_2^A|\,\langle\psi_2^B|\right) = |\psi_1^A\rangle\langle\psi_2^A|\ \underbrace{\langle\psi_2^B|\psi_1^B\rangle}_{\text{number}}$$

$$F\big(\underbrace{|\varphi\rangle\langle\varphi|}_{\rho_\varphi},\ \underbrace{|\psi\rangle\langle\psi|}_{\rho_\psi}\big)$$

$$\rho_\varphi^{1/2} = \rho_\varphi$$

$$\rho_\varphi^2 = |\varphi\rangle\overbrace{\langle\varphi|\varphi\rangle}^{=1}\langle\varphi|$$

$$= |\varphi\rangle\langle\varphi| = \rho_\varphi$$

$$= \mathrm{tr}\sqrt{\rho_\varphi^{1/2}\,\rho_\psi\,\rho_\varphi^{1/2}}$$

$$= \mathrm{tr}\sqrt{|\varphi\rangle\underbrace{\langle\varphi|\psi\rangle\langle\psi|\varphi\rangle}_{|\langle\varphi|\psi\rangle|^2}\langle\varphi|}$$

$$= \mathrm{tr}\ |\langle\varphi|\psi\rangle|\ \underbrace{\sqrt{|\varphi\rangle\langle\varphi|}}_{=\rho_\varphi} = |\langle\varphi|\psi\rangle|\ \overbrace{\mathrm{tr}(\rho_\varphi)}^{=1} = \underline{|\langle\varphi|\psi\rangle|}$$