Quantum Information Spring 2023
Bonus Problem Set

Solutions are due on Sunday June 11th, 23:59.

1. **Quantum Fourier transform**
   Consider the unnormalized quantum state

   $$|\phi\rangle = \sum_{j=0}^{7} \cos\left(\frac{2\pi j}{8}\right)|j\rangle \ , \tag{1}$$

   on a system of three qubits.

   (a) Normalize $|\phi\rangle$. That is, find x such that $|\psi\rangle = x\,|\phi\rangle$ is normalized $\langle\psi|\psi\rangle = 1$.

   (b) Calculate the quantum Fourier transformation of the normalized state $|\psi\rangle$.

2. **Partial measurement**
   Suppose ρ is the density matrix describing a two qubit system. Suppose we perform a projective measurement in the computational basis of the second qubit. Let $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$ be the projectors onto the $|0\rangle$ and $|1\rangle$ states of the second qubit, respectively. Let ρ' be the density matrix which would be assigned to the system after the measurement by an observer who did not learn the measurement result. Show that

   $$\rho' = P_0\rho P_0 + P_1\rho P_1 \ . \tag{2}$$

   Also show that the reduced density matrix for the first qubit is not affected by the measurement, that is, $\mathrm{tr}_2(\rho) = \mathrm{tr}_2(\rho')$.

3. **Interaction as a quantum operation**
   Suppose we have a single-qubit principal system S interacting with an environment E also consisting of a single qubit. The total Hilbert space is $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$, where $\mathcal{H}_S, \mathcal{H}_E$ are single-qubit Hilbert spaces corresponding to the individual qubits. The two interact through the transform

   (a) $U_a = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X$,

   (b) $U_b = \frac{1}{\sqrt{2}}(X \otimes I + Y \otimes X)$,

   where X is the usual Pauli X-matrix. Give the operator-sum representation for the quantum operation, which corresponds to the transformation of the system due to the interaction. Assume that the environment starts in the state $|0\rangle$.

   (Nielsen-Chuang section 8.2 may be useful for solving this problem.)

4. **Quantum circuits**

(a) Let $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$ in the Hilbert space $\mathbb{C}^2$. Calculate

$$HZH\,|0\rangle \quad \text{and} \quad HZH\,|1\rangle \ ,$$

where H is the Hadamard transform. The unitary transform H is defined by

$$H\,|k\rangle = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^k\,|1\rangle) \ , \quad k \in \{0,1\} \ .$$

(b) Calculate

$$(H \otimes H)U_{CNOT}(H \otimes H)\,|j,k\rangle$$

where $|j,k\rangle \equiv |j\rangle \otimes |k\rangle$ with $j,k \in \{0,1\}$, and the answer is in form of a ket $|m,n\rangle$ where $m,n \in \{0,1\}$. The controlled-NOT is defined

$$U_{CNOT} \equiv |0\rangle\langle 0| \otimes \mathbb{I} + |1\rangle\langle 1| \otimes X \ ,$$

where $\mathbb{I}$ is the 2×2-identity matrix and $X = |0\rangle\langle 1| + |1\rangle\langle 0|$.

5. **Qubit correlations**
We have a two-qubit density matrix

$$\rho = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{H}_A \otimes \mathcal{H}_B \ .$$

(a) Is the state ρ pure or not?

(b) Calculate the reduced density matrix

$$\rho_A = \mathrm{tr}_{\mathcal{H}_B} \ .$$

(c) Calculate the entanglement entropy

$$S(\rho_A) = -\,\mathrm{tr}\,(\rho_A \log_2 \rho_A) \ .$$

Are the two qubits entangled?