Copying quantum states? No.

In classical computing, copying bits is a fundamental operation. Quantum states are impossible to copy.

Proof:

$|\psi\rangle$ = arbitrary quantum state "data"
$|0\rangle$ = target where data is copied

The copying machine U is an unitary transformation which maps

$$U(|\psi\rangle \otimes |0\rangle) = |\psi\rangle \otimes |\psi\rangle .$$

Is this possible? Let's use the copying machine to copy two states $|\psi\rangle$ and $|\phi\rangle$.

$$U(|\psi\rangle \otimes |0\rangle) = |\psi\rangle \otimes |\psi\rangle$$
$$U(|\phi\rangle \otimes |0\rangle) = |\phi\rangle \otimes |\phi\rangle$$

The inner product

$$\langle\psi|\phi\rangle = \langle\psi|\otimes\langle 0|\, U^{\dagger}U\, |\phi\rangle\otimes|0\rangle = (\langle\psi|\phi\rangle)^2 .$$

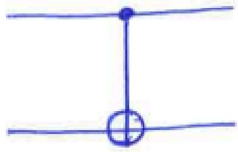
This can be true only if $\langle\psi|\phi\rangle = 0$ or 1!

→ Either $|\psi\rangle = |\phi\rangle$ or $|\psi\rangle$ and $|\phi\rangle$ are orthogonal.

→ The copying machine can only copy orthogonal states.

→ Copying general states is impossible.

Qubit states $|0\rangle$ and $|1\rangle$ are orthogonal, how can we copy them. Easy:

$U =$ 

Example: Find the logarithm of (base 2)

$$A = \frac{1}{2}\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}.$$

Solution: First solve eigenvalues and eigenvectors.

Characteristic equation

$$\det(A - \lambda \mathbb{1}) = \left(\tfrac{3}{2} - \lambda\right)^2 - \frac{1}{4} = \lambda^2 - 3\lambda + 2 = 0$$

$$\rightarrow \lambda = 1 \text{ or } 2$$

Then eigenvectors

$$\lambda = 1: \quad A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{cases} \frac{3}{2}x - \frac{1}{2}y = x \\ -\frac{1}{2}x + \frac{3}{2}y = y \end{cases} \rightarrow x = y$$

Normalized eigenstate is $|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

$$\lambda = 2: \quad A\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{cases} \frac{3}{2}x - \frac{1}{2}y = 2x \\ -\frac{1}{2}x + \frac{3}{2}y = 2y \end{cases} \rightarrow x = -y$$

Normalized eigenstate is $|2\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

Therefore,

$$A = 1\,|1\rangle\langle 1| + 2\,|2\rangle\langle 2| = 1\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} + 2\begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}.$$

Finally,

$$\log_2 A = \log_2(1)\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} + \log_2(2)\begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} .$$

Example: Let $\{|0\rangle, |1\rangle\}$ be the computational basis of a single qubit. The NOT-operation is defined by

$$|0\rangle \to |1\rangle , \quad |1\rangle \to |0\rangle .$$

a) Write NOT w.r.t. the basis $\{|0\rangle, |1\rangle\}$ in braket-notation.

b) Write the matrix representation of NOT in the standard basis

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

c) Do the same in the Hadamard-basis

$$|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} , \quad |-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} .$$

Solution:

a) From the definition of NOT,

$$U_{NOT} = |1\rangle\langle 0| + |0\rangle\langle 1|$$

since $\langle 0|0\rangle = \langle 1|1\rangle = 1$ and $\langle 0|1\rangle = 0$.

Finally,

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b) In the standard basis

$$U_{NOT}^{(standard)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

c) In the Hadamard basis

$$U_{NOT}^{(Hadamard)} = H\, U_{NOT}^{(standard)}\, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$ is the Hadamard operator.

An another way to do this is to use

$$\begin{cases} |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \end{cases} \longrightarrow \begin{array}{l} |0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) \\ |1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) \end{array}$$

$$U_{NOT} = |1\rangle\langle 0| + |0\rangle\langle 1| = \frac{1}{2}(|+\rangle - |-\rangle)(\langle +| + \langle -|)$$

$$+ \frac{1}{2}(|+\rangle + |-\rangle)(\langle +| - \langle -|)$$

$$= |+\rangle\langle +| - |-\rangle\langle -| ,$$

which is in matrix form

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Example: Hadamard operator is defined by

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

a) Write H in terms of $|0\rangle$ and $|1\rangle$.

b) Compute H^2.

Solution:

a) Clearly, $H = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\langle 0| + \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\langle 1|$.

b) $H^2 = H\,H$

$$= \frac{1}{2}\Big[(|0\rangle + |1\rangle)\langle 0| + (|0\rangle - |1\rangle)\langle 1|\Big] \times \Big[(|0\rangle + |1\rangle)\langle 0| + (|0\rangle - |1\rangle)\langle 1|\Big]$$

$$= \frac{1}{2}\Big[(|0\rangle + |1\rangle)\langle 0| + (|0\rangle - |1\rangle)\langle 0| + (|0\rangle + |1\rangle)\langle 1| - (|0\rangle - |1\rangle)\langle 1|\Big]$$

$$= |0\rangle\langle 0| + |1\rangle\langle 1| = \mathbb{1}$$

Therefore, $H^{-1} = H$.