Quantum Information Spring 2023 Problem Set 2

Solutions are due on Sunday May 7, 23:59.

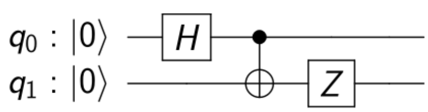
1. **Mixed States**
   Consider the 2×2 matrix

$$\rho = \begin{pmatrix} 3/4 & \sqrt{2}e^{-i\phi}/4 \\ \sqrt{2}e^{i\phi}/4 & 1/4 \end{pmatrix} . \tag{1}$$

   (a) Is the matrix a density matrix?

   (b) If so do we have a pure state or a mixed state?

2. **Entangling gates**
   Consider the following quantum circuit



   (a) Find the state vector $|\psi\rangle$ produced by this circuit.

   (b) Compute the corresponding density operator.

3. **Entanglement Entropy**

   (a) Consider the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^2$ and the state

$$|\psi\rangle = \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle) = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} . \tag{2}$$

   Calculate the entanglement entropies

$$-\operatorname{tr}(\rho_A \log \rho_A) \quad \text{and} \quad -\operatorname{tr}(\rho_B \log \rho_B) , \tag{3}$$

   where the reduced density matrices are

$$\rho_A = \operatorname{tr}_{\mathcal{H}_B} |\psi\rangle\langle\psi| \quad \text{and} \quad \rho_B = \operatorname{tr}_{\mathcal{H}_A} |\psi\rangle\langle\psi| . \tag{4}$$

   (b) Is this state separable? Why or why not?

4. **Trace Distance**
   Consider the density operators
$$\rho = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1| \quad \text{and} \quad \sigma = \frac{2}{3}|+\rangle\langle +| + \frac{1}{3}|-\rangle\langle -| \ , \tag{5}$$
   where $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$. Calculate the trace distance $D(\rho, \sigma)$.

5. **Fidelity**
   Consider the density operators
$$\rho = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1| \quad \text{and} \quad \sigma = \frac{2}{3}|0\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1| \ . \tag{6}$$
   (a) Compute the fidelity $F(\rho, \sigma) = \mathrm{tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}$.
   (b) A bit flip channel flips the qubit from $|0\rangle$ to $|1\rangle$ (and vice versa) with probability $1 - p$. It can be described with the quantum channel
$$\rho \mapsto \mathcal{E}(\rho) = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger \ , \tag{7}$$
   where
$$E_0 = \sqrt{p}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \quad E_1 = \sqrt{1-p}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ . \tag{8}$$
   The bit flip channel has the following effect on our states ρ and σ:
$$\rho \mapsto \mathcal{E}(\rho) = \begin{pmatrix} (1+2p)/4 & 0 \\ 0 & (3-2p)/4 \end{pmatrix} \tag{9}$$
$$\sigma \mapsto \mathcal{E}(\sigma) = \begin{pmatrix} (1+p)/3 & 0 \\ 0 & (2-p)/3 \end{pmatrix} \ . \tag{10}$$
   Calculate the fidelity $F(\mathcal{E}(\rho), \mathcal{E}(\sigma))$. Is the new fidelity smaller or larger than the original fidelity $F(\rho, \sigma)$? Can you explain why?