ELEC-C9440 Quantum Information Exam 2.6.2022

Problem 1

Consider the two qubit state described by

$$\rho = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \in \mathcal{H}_A \otimes \mathcal{H}_B \ .$$

a)

Is the state pure?
Solution.
Yes, by direct calculation it can be shown that $\rho^2 = \rho$.

b)

Are the two qubits entangled?
Solution.
Compute the reduced density matrix for the first qubit

$$\begin{aligned} \rho_A = & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \end{pmatrix} \rho \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ & + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \end{pmatrix} \rho \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \ . \end{aligned}$$

The eigenvalues of ρ_A follow from the characteristic equation

$$\det\left(\rho_A - \lambda\right) = \lambda^2 - \lambda = 0 \ .$$

The roots are $\lambda = \{0, 1\}$, so the entanglement entropy of ρ_A is

$$S(\rho_A) = -\operatorname{tr}\left(\rho_A \log_2 \rho_A\right) = 0 \ .$$

Therefore, the qubits are not entangled. These results can be immediately verified by noticing that the original state is $|+\rangle\,|1\rangle$.

Problem 2

Show that the n-qubit quantum Fourier transform

$$U_{QFT} = \frac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n-1} \sum_{k=0}^{2^n-1} e^{-2\pi ijk/2^n} |k\rangle\langle j|$$

is unitary. That is, show that

$$U_{QFT}^\dagger U_{QFT} = \mathbb{I}_{2^n} = \sum_{k=0}^{2^n-1} |k\rangle\langle k| \ .$$

Solution.
Since

$$U_{QFT}^\dagger = \frac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n-1} \sum_{k=0}^{2^n-1} e^{2\pi ijk/2^n} |j\rangle\langle k|$$

we can show by direct computation that

$$\begin{aligned}
U_{QFT}^\dagger U_{QFT} &= \frac{1}{2^n} \sum_{j=0}^{2^n-1} \sum_{k=0}^{2^n-1} \sum_{l=0}^{2^n-1} \sum_{m=0}^{2^n-1} e^{2\pi i(lm-jk)/2^n} |k\rangle \langle j|l\rangle \langle m| \\
&= \frac{1}{2^n} \sum_{j=0}^{2^n-1} \sum_{k=0}^{2^n-1} \sum_{m=0}^{2^n-1} e^{2\pi ij(m-k)/2^n} |k\rangle\langle m| \\
&= \frac{1}{2^n} \sum_{k=0}^{2^n-1} \sum_{m=0}^{2^n-1} 2^n \delta_{km} |k\rangle\langle m| \\
&= \sum_{k=0}^{2^n-1} |k\rangle\langle k| \ .
\end{aligned}$$

Problem 3

Describe the error correction protocol for the Shor 9-qubit code including (i) encoding, (ii) syndrome measurements and (iii) recovery operations.

Problem 4

Provide a quantum circuit for simulating one timestep of the dynamics of a chain of three interacting spin-$\frac{1}{2}$ particles (modelled by single qubits) with the Hamiltonian

$$H = \mu \sum_{i=1}^{3} \sigma_x^{(i)} + \lambda \sigma_x^{(1)} \sigma_y^{(2)} \sigma_z^{(3)} \ ,$$

where $\sigma_k^{(i)}$ is the k'th Pauli matrix acting on the i'th particle.

Problem 5

Suppose you want to measure the expectation value of some observable M in a quantum state you prepare on a quantum computer. That is, you are interested in $\langle M \rangle$ computed in some state $|\psi\rangle$.

a)

What is quantum error mitigation and why would one want to use it instead of quantum error correction for this computation?

Solution.

Post-processing method for suppressing error in measured expectation values, proper quantum error correction cannot be used right now because of overhead, etc.

b)

Assume your quantum circuit which prepares $|\psi\rangle$ consists of only single qubit gates and CNOT gates. Further assume that the error rate of single qubit gates is zero and the all CNOT gates have an error rate ϵ.

You have measured $\langle M \rangle$ at two different error rates:

$$\begin{aligned} \langle M \rangle (\epsilon) &= x \\ \langle M \rangle (3\epsilon) &= y \ . \end{aligned}$$

Calculate the estimate for $\langle M \rangle (0)$ using Richardson extrapolation.

Solution.

Richardson extrapolation uses a Taylor series expansion of $\langle M \rangle (\epsilon)$ around $\epsilon = 0$. Therefore we have two equations

$$\begin{cases} \langle M \rangle (\epsilon) = \langle M \rangle (0) + M_1 \epsilon = x \\ \langle M \rangle (3\epsilon) = \langle M \rangle (0) + M_1 (3\epsilon) = y \end{cases}$$

for two unknown coefficients $\langle M \rangle (0)$ and M_1. Solving for $\langle M \rangle (0)$ gives

$$\langle M \rangle (0) = \frac{3}{2} x - \frac{1}{2} y \ .$$

c)

Give one method for increasing the error rate ϵ in the measurement of $\langle M \rangle$.

Solution.

Explain either gate repetition, Hamiltonian scaling, or Pauli twirling.