PHYS-C0252 - Quantum Mechanics

Exercise set 5

Due date: May 29, 2024 by 23:59 on MyCourses
Return the exercises as a `.pdf`.
You can write by hand and take pictures, use digital note-taking or LaTeX etc.

1. Consider a particle with mass m approaching a finite potential barrier with height V_B and width a as discussed on lecture 9:

$$V(x) = \begin{cases} 0 & x < 0, \\ V_B & 0 < x < a, \\ 0 & a < x. \end{cases}$$

On the lecture, we discussed the wave function in the case where the energy is below the barrier, $E < V_B$. Let us now consider the $E > V_B$ case, where the solution to the time-independent Schrödinger equation is the wave function

$$\psi(x) = \begin{cases} A_I e^{ikx} + A_R e^{-ikx} & x < 0, \\ B e^{ik_B x} + B' e^{-ik_B x} & 0 < x < a, \\ A_T e^{ikx} & a < x, \end{cases}$$

where

$$k = \sqrt{\frac{2mE}{\hbar^2}}, \qquad k_B = \sqrt{\frac{2m(E - V_B)}{\hbar^2}}.$$

(a) (4p) Show that the transmission probability is given by

$$T = \left|\frac{A_T}{A_I}\right|^2 = \frac{1}{\cos^2(k_B a) + \left(\frac{k^2 + k_B^2}{2kk_B}\right)^2 \sin^2(k_B a)}$$
$$= \frac{1}{1 + \left(\frac{k^2 - k_B^2}{2kk_B}\right)^2 \sin^2(k_B a)}$$

(either of the forms on the RHS is OK). Hint: use the continuity of $\psi(x)$ and $\frac{d}{dx}\psi(x)$ at the boundaries $x = 0$ and $x = a$, Euler's formula

$$e^{i\phi} = \cos(\phi) + i\sin(\phi),$$

and $|e^{i\phi}| = 1$ for $\phi \in \mathbb{R}$.

(b) (2p) Sketch T as a function of the barrier width a. What is the mathematical condition to get full transmission $T = 1$? What is the physical explanation for full transmission when this condition is satisfied?

2. Consider a particle of mass m in the one-dimensional potential energy field

$$V(x) = \begin{cases} 0, & \text{if } -\infty < x < -a; \\ -V_0, & \text{if } -a < x < +a; \\ 0, & \text{if } +a < x < -\infty. \end{cases}$$

Since the potential is symmetric about $x = 0$, there are two types of energy eigenfunctions. There are *symmetric* eigenfunctions which obey $\psi(x) = \psi(-x)$, and *anti-symmetric* eigenfunctions which obey $\psi(x) = -\psi(-x)$.

(a) Show, by considering the energy eigenvalue equation in the three regions of x, that a symmetric eigenfunction with energy $E = -\hbar^2\alpha^2/2m$ with $\alpha \in \mathbb{R}$ has the form:

$$\psi(x) = \begin{cases} Ae^{+\alpha x}, & \text{if } -\infty < x < -a; \\ C\cos(k_0 x), & \text{if } -a < x < +a; \\ Ae^{-\alpha x}, & \text{if } +a < x < \infty. \end{cases}$$

where A and C are constants and $k_0 = \sqrt{2m(E + V_0)/\hbar^2}$.

(b) Show that for the symmetric eigenfunction, $\alpha = k_0 \tan(k_0 a)$. Hint: use the continuity of $\psi(x)$ and $\frac{d}{dx}\psi(x)$ at the edges of the potential.

(c) By seeking graphical solutions of the equations

$$\alpha = k_0 \tan(k_0 a) \;\text{ and }\; \alpha^2 + k_0^2 = w^2,$$

where $w = \sqrt{2mV_0/\hbar^2}$, show that there is one bound state if $0 < w < \pi/a$, and two bound states if $\pi/a < w < 2\pi/a$. Hint: use normalized units $\bar{k}_0 = ak_0/\pi$ and $\bar{\alpha} = a\alpha/\pi$, and plot $\bar{\alpha}$ versus $\bar{k}_0$ when the above equations hold. Note that $k_0 > 0$ and $\alpha > 0$.

3. Consider a particle in a one-dimensional, infinite potential well, with perturbation Cx, such that

$$V = \begin{cases} Cx, & 0 < x < L \\ \infty, & \text{everywhere else.} \end{cases}$$

Using the fact that the eigenstates and corresponding energies of the unperturbed Hamiltonian are

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right), \quad E_n = \frac{n^2\pi^2\hbar^2}{2mL^2},$$

find the first order corrections to the first ($n = 1$) eigenstate and the corresponding energy using time-independent perturbation theory. Note that $|\langle\psi_m|x|\psi_n\rangle|$ drops rapidly as m becomes larger than n, so you can ignore terms with $m - n > 2$.

4. Consider the case of two identical coupled harmonic oscillators that could be experimentally realized as two capacitively coupled superconducting resonators at the same resonance frequency. When the coupling is sufficiently weak, we can treat it as

a perturbation. After performing the rotating wave approximation, the approximate Hamiltonian for this system is given by

$$\hat{H} = \hat{H}^0 + \hat{H}' = \hbar\omega\left(\hat{a}^\dagger \hat{a} \otimes \hat{I} + \frac{1}{2}\right) + \hbar\omega\left(\hat{I} \otimes \hat{b}^\dagger \hat{b} + \frac{1}{2}\right) + \hbar g\left(\hat{a}^\dagger \otimes \hat{b} + \hat{a} \otimes \hat{b}^\dagger\right),$$

where the Hamiltonian of the unperturbed system is

$$\hat{H}^0 = \hbar\omega\left(\hat{a}^\dagger \hat{a} \otimes \hat{I} + \frac{1}{2}\right) + \hbar\omega\left(\hat{I} \otimes \hat{b}^\dagger \hat{b} + \frac{1}{2}\right).$$

Here g is a coupling constant with the dimension of frequency, $\hat{a}$ and $\hat{a}^\dagger$ are ladder operators acting on the first harmonic oscillator, and $\hat{b}$ and $\hat{b}^\dagger$ are ladder operators acting on the second harmonic oscillator. The action of these ladder operators on the eigenstates of the unperturbed system $|m\rangle \otimes |n\rangle \equiv |m, n\rangle$, where $m, n \in \{0, 1, 2, \dots\}$ is defined as usual

$$\begin{aligned}
\left(\hat{a} \otimes \hat{I}\right)|m, n\rangle &= \sqrt{m}\,|m-1, n\rangle, \quad & \left(\hat{a}^\dagger \otimes \hat{I}\right)|m, n\rangle &= \sqrt{m+1}\,|m+1, n\rangle \\
\left(\hat{I} \otimes \hat{b}\right)|m, n\rangle &= \sqrt{n}\,|m, n-1\rangle, \quad & \left(\hat{I} \otimes \hat{b}^\dagger\right)|m, n\rangle &= \sqrt{n+1}\,|m, n+1\rangle.
\end{aligned}$$

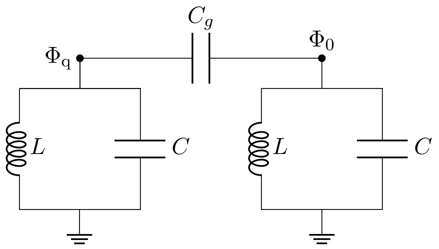
The energy spectrum of $\hat{H}^0$ is then quite similar to the case of a single quantum harmonic oscillator

$$E_{m,n} = \hbar\omega\,(m + n + 1),$$

where m and n correspond to the excitation level of the first and second harmonic oscillator respectively. The ground state E_0 is non-degenerate with energy $E_{0,0} = \hbar\omega$. However, the second excited state (i.e. $m + n = 2$) has a triple degeneracy, with eigenstates $|0, 2\rangle$, $|1, 1\rangle$, and $|2, 0\rangle$ sharing the same energy $E_2 = E_{0,2} = E_{1,1} = E_{2,0} = 3\hbar\omega$.

Using first-order degenerate perturbation theory, show that the degenerate subspace spanned by $|0, 2\rangle$, $|1, 1\rangle$, and $|2, 0\rangle$ splits into three different energy levels, and find the energies of these new levels.

Hint: Start by finding the matrix H'_{ij}, with $i, j \in \{|0, 2\rangle, |1, 1\rangle, |2, 0\rangle\}$. Note that since H is a real Hermitian matrix, $H_{ij} = H_{ji}$, meaning there are only 6 independent elements.



Two harmonic oscillators weakly coupled through a coupling capacitance C_g