PHYS-C0252 - Quantum Mechanics

Exercise set 1

Due date: May 1, 2024 by 23:59 on MyCourses
Return the exercises as a `.pdf`.
You can write by hand and take pictures, use digital note-taking or LaTeX etc.

1. Consider the vectors $|\psi\rangle = 4|\phi_1\rangle + i|\phi_2\rangle$ and $|\chi\rangle = 2|\phi_1\rangle + (1-4i)|\phi_2\rangle$, where $|\phi_1\rangle$ and $|\phi_2\rangle$ are orthonormal, i.e. $\langle\phi_k|\phi_m\rangle = \delta_{km}$, where $\delta_{km} = 1$ for $k = m$ and $\delta_{km} = 0$ for $k \neq m$.

   (a) Express $|\psi\rangle + |\chi\rangle$ and $\langle\psi| + \langle\chi|$ in their simplest form using $|\phi_1\rangle$ and $|\phi_2\rangle$.

   (b) Express $|\phi_1\rangle$ in terms of $|\psi\rangle$ and $|\chi\rangle$.

   (c) Calculate the inner products $\langle\psi|\chi\rangle$ and $\langle\chi|\psi\rangle$. Are they equal?

   (d) Show that $|\psi\rangle$ and $|\chi\rangle$ satisfy the Cauchy–Schwarz inequality and the triangle inequality.

2. Consider the so-called Pauli operators $\hat{\sigma}_x = |0\rangle\langle 1| + |1\rangle\langle 0|$, $\hat{\sigma}_y = -\mathrm{i}|0\rangle\langle 1| + \mathrm{i}|1\rangle\langle 0|$ and $\hat{\sigma}_z = |0\rangle\langle 0| - |1\rangle\langle 1|$, where $\{|0\rangle, |1\rangle\}$ form an orthonormal basis of the considered Hilbert space.

   (a) Show that each Pauli operator is Hermitian.

   (b) Write down the matrix representation of the Pauli operators. Hint: for an operator $\hat{A}$ and an orthonormal basis $\{|\phi_k\rangle\}_k$, the matrix element A_{jk} is defined as $\langle\phi_j|\hat{A}|\phi_k\rangle$.

   (c) Solve the eigenvalues and the corresponding eigenstates of each Pauli operator using the matrix form, and write the eigenstates using the ket vectors $|0\rangle$ and $|1\rangle$.

   (d) For each Pauli operator, show that the eigenstates are orthogonal.

3. (a) Show that for a Hermitian bounded linear operator $\hat{H} : \mathcal{H} \to \mathcal{H}$, all of its eigenvalues are real and the eigenvectors corresponding to different eigenvalues are orthogonal. Hint: start by calculating $\langle\phi|\hat{H}|\phi\rangle$ for an eigenstate $|\phi\rangle$. In a similar fashion, show that the eigenvalues of an anti-Hermitian linear bounded operator $\hat{A} : \mathcal{H} \to \mathcal{H}$ are either purely imaginary or equal to zero. Note that for anti-Hermitian operators $\hat{A}$, we have $\hat{A}^\dagger = -\hat{A}$.

(b) An important class of Hermitian operators is the *projectors*. Consider a Hilbert space $\mathcal{H}$ with an orthonormal basis $\{|\phi_i\rangle\}_{i\in I}$, where I is a suitable finite (or infinite) index set. A subset of these basis vectors $\{|\phi_j\rangle\}_{j\in J}$, where $J \subset I$ will form an orthonormal basis for a subspace $\mathcal{H}'$ of $\mathcal{H}$. Projector P onto the subspace $\mathcal{H}'$ is then defined as

$$P \equiv \sum_{j\in J} |\phi_j\rangle \langle\phi_j| \,.$$

Show that P is indeed a Hermitian operator, and that it satisfies the equation $P^2 = P$.

(c) Consider any linear bounded operator $\hat{B} : \mathcal{H} \to \mathcal{H}$

i. Show that $\hat{B} - \hat{B}^\dagger$ is anti-Hermitian and $\hat{B} + \hat{B}^\dagger$ is Hermitian.

ii. Show that $\hat{B}$ can be expressed as a linear combination of a Hermitian and an anti-Hermitian operator.

4. (a) Prove the Cauchy–Schwarz inequality $|\langle\psi|\phi\rangle| \leq ||\psi||\,||\phi||$. Here we use the shorthand notation $||\psi||$ $(= ||\,|\psi\rangle||)$ for the norm of $|\psi\rangle$ as on lectures. Hint: Start from $0 \leq ||\,|\psi\rangle + \lambda|\phi\rangle\,||$ and choose the scalar $\lambda \propto \langle\phi|\psi\rangle$ in a clever way.

(b) Prove the triangle inequality $||\,|\psi\rangle + |\phi\rangle|| \leq ||\psi|| + ||\phi||$. Hint: Calculate $||\,|\psi\rangle + |\phi\rangle\,||^2$ and use (a). You may also use the fact that $\mathrm{Re}(z) \leq |z|$ for a complex number z.

(c) Demonstrate the necessary and sufficient conditions for these inequalities to become equalities. Hint: Let $a\,|\psi\rangle = \frac{\langle\psi|\phi\rangle}{\langle\psi|\psi\rangle}\,|\psi\rangle$ be the *projection* of $|\phi\rangle$ on to $|\psi\rangle$. You can write $|\phi\rangle$ in terms of the projection and the *rejection* $|\chi\rangle = |\phi\rangle - \frac{\langle\psi|\phi\rangle}{\langle\psi|\psi\rangle}\,|\psi\rangle$ as $|\phi\rangle = a\,|\psi\rangle + |\chi\rangle$. Note that the rejection is orthogonal to $|\psi\rangle$, i.e. $\langle\chi|\psi\rangle = 0$.