PHYS-C0252 - Quantum Mechanics

Exercise set 2

Due date: May 8, 2024 by 23:59 on MyCourses
Return the exercises as a `.pdf`.
You can write by hand and take pictures, use digital note-taking or LaTeX etc.

1. A quantum system described by a Hamiltonian $\hat{H}$ is in the state

$$|\psi\rangle = N\left[\frac{\mathrm{i}}{\sqrt{3}}|\phi_1\rangle - \frac{1}{\sqrt{2}}(-3+5\mathrm{i})|\phi_2\rangle + \frac{2}{\sqrt{5}}|\phi_3\rangle + \sqrt{7}|\phi_4\rangle\right],$$

where $|\phi_n\rangle$ are the eigenstates of energy such that $\hat{H}|\phi_n\rangle = nE_0|\phi_n\rangle$, E_0 has units of energy, and $N \in \mathbb{R}$.

   (a) Find a suitable scalar N such that $|\psi\rangle$ is normalized.

   (b) Let the energy of $|\psi\rangle$ be measured. Give all possible measurement results and their corresponding probabilities. Assume that the measurement is ideal, i.e., no measurement errors occur.

   (c) What is the expectation value of the energy when the system is in the state $|\psi\rangle$? Is it a possible measurement result if the energy is measured?

   (d) Consider an operator $\hat{X}$, the action of which on $|\phi_n\rangle$ ($n = 1, 2, 3, 4$) is defined by $\hat{X}|\phi_n\rangle = (n-3)x_0|\phi_n\rangle$, where x_0 is a real-valued scalar. Suppose that a measurement of the energy of the above-defined $|\psi\rangle$ yields $2E_0$. Assume that immediately afterwards, we ideally measure the physical quantity corresponding to $\hat{X}$. What is the value for the quantity obtained in the latter measurement?

2. Consider the raising and lowering operators of a one-dimensional harmonic oscillator, $\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{q} + \frac{\mathrm{i}}{m\omega}\hat{p})$ and $\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(\hat{q} - \frac{\mathrm{i}}{m\omega}\hat{p})$, which satisfy

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle,$$

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle.$$

Show that $[\hat{N}, \hat{a}] = -\hat{a}$ and $[\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger$, where $\hat{N} = \hat{a}^\dagger\hat{a}$ is the number operator.

3. Let's have another swing at the one-dimensional harmonic oscillator. Suppose that the oscillator is prepared in one of the eigenstates of the Hamiltonian, $|\psi\rangle = |n\rangle$, where $n \in \{0, 1, 2, \ldots\}$.

   (a) What is the variance in the energy of the system in the state $|\psi\rangle$? Hint: Recall that the Hamiltonian operator is an observable that corresponds to the total energy of the system.

(b) Express the position operator $\hat{q}$ and the momentum operator $\hat{p}$ in terms of the ladder operators $\hat{a}$ and $\hat{a}^\dagger$.

(c) Verify the validity of the Heisenberg uncertainty relation by evaluating the product $\Delta q \Delta p$. Hint: Use the expressions found in part (b) and take advantage of the rules that the ladder operators obey when acting on the eigenstates $|n\rangle$ (see Exercise 2). Also note that $\langle n|m\rangle = \delta_{nm}$.

(d) For which n is the Heisenberg uncertainty minimized, i.e. $\Delta q \Delta p = \hbar/2$?

4. Consider the pendulum discussed on Lecture 3, where the potential energy is given by $V(\theta) = mgl(1 - \cos\theta)$.

(a) Following the steps in the lecture notes, derive the classical Hamiltonian of the pendulum *without* making the approximation $1 - \cos\theta \approx \theta^2/2$. Use $q = \theta$ as the generalized coordinate.

(b) Expand the cosine in the Hamiltonian up to fourth order in θ and replace the canonical coordinates with operators: $\theta \to \hat{\theta}$, $p \to \hat{p}$ to obtain a quantized Hamiltonian. Note that this is now an *an*harmonic oscillator, since there are terms that are higher than second order. Write the Hamiltonian using the harmonic ladder operators

$$\hat{a} = \sqrt{\frac{ml^2\omega}{2\hbar}}\left(\hat{\theta} + \frac{\mathrm{i}}{ml^2\omega}\hat{p}\right),$$
$$\hat{a}^\dagger = \sqrt{\frac{ml^2\omega}{2\hbar}}\left(\hat{\theta} - \frac{\mathrm{i}}{ml^2\omega}\hat{p}\right),$$

where $\omega = \sqrt{g/l}$ as in the lecture notes.

(c) Calculate the expectation values $\langle 0|\hat{H}|0\rangle$, $\langle 1|\hat{H}|1\rangle$ and $\langle 2|\hat{H}|2\rangle$ using the relations

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle$$
$$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle,$$
$$\hat{a}\,|0\rangle = 0,$$
$$\langle n|m\rangle = \delta_{nm}.$$

The interesting observation from the above result is that the energy differences $\Delta E_{12} = \langle 2|\hat{H}|2\rangle - \langle 1|\hat{H}|1\rangle$ and $\Delta E_{01} = \langle 1|\hat{H}|1\rangle - \langle 0|\hat{H}|0\rangle$ are equal in the case of the harmonic oscillator, but different for the anharmonic oscillator.