PHYS-C0252 - Quantum Mechanics

Exercise set 3

Due date: May 15, 2024 by 23:59 on MyCourses
Return the exercises as a `.pdf`.
You can write by hand and take pictures, use digital note-taking or LaTeX etc.

1. Consider the temporal evolution of quantum states $\{|\psi\rangle\}$ determined by a Hamiltonian $\hat{H}(t)$. We define the evolution operator $\hat{U}(t, t_0)$ as

$$\hat{U}(t, t_0)|\psi(t_0)\rangle = |\psi(t)\rangle, \text{ for all } t \geq t_0 \in \mathbb{R} \text{ and } |\psi(t_0)\rangle \in \mathcal{H}$$

   (a) Prove the following identities:
      i. $\hat{U}(t_0, t_0) = \hat{I}$, for all $t_0 \in \mathbb{R}$.
      ii. $\hat{U}^\dagger(t, t_0)\hat{U}(t, t_0) = \hat{I}$, for all $t > t_0 \in \mathbb{R}$. Hint: Calculate $\partial_t[\hat{U}^\dagger(t, t_0)\hat{U}(t, t_0)]$, use the Schrödinger equation equivalent for $\hat{U}(t, t_0)$, and use the result of i. as an initial condition. Note that we do not assume the Hamiltonian to be independent of time.
      iii. $\hat{U}(t_2, t_0) = \hat{U}(t_2, t_1)\hat{U}(t_1, t_0)$, for all $t_0 < t_1 < t_2 \in \mathbb{R}$.

   (b) Construct the evolution operator $\hat{U}(t, t_0)$ for a time-dependent Hamiltonian $\hat{H}(t)$ using the following steps:
      1. Using the Schrödinger equation for U, show that U is formally given by

$$U(t, t_0) = \hat{I} + \sum_{n=1}^{\infty} \left(\frac{-i}{\hbar}\right)^n \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2 \cdots \int_{t_0}^{t_{n-1}} \mathrm{d}t_n\, \hat{H}(t_1)\hat{H}(t_2)\cdots\hat{H}(t_n).$$

      2. Find $U(t, t_0)$ when $\hat{H}$ is time-independent using the fact that

$$\int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2 \cdots \int_{t_0}^{t_{n-1}} \mathrm{d}t_n\, C = \frac{1}{n!} \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t} \mathrm{d}t_2 \cdots \int_{t_0}^{t} \mathrm{d}t_n\, C$$

      for a constant C.
      3. Find an expression for $U(t, t_0)$ when $\hat{H}$ is piecewise constant in time. Use the above result, divide the interval $[t_0, t]$ into n intervals $[t_0, t_1]$, $(t_1, t_2]$, $\ldots$, $(t_{n-1}, t_n]$, where $t_k = t_0 + k\delta t$, $t_n = t$, and assume that $\hat{H}$ is constant in each interval and that $\hat{H}(t_k)$ commute.
      4. Take the limit $\delta t \to 0$ to convert the sum in the above result to an integral in order to obtain the final result for $U(t, t_0)$.

2. Let $\vec{a}$ be any real-valued three-dimensional unit vector and $\theta \in \mathbb{R}$.

   (a) Prove that

$$\mathrm{e}^{\mathrm{i}\theta(\vec{a}\cdot\hat{\vec{\sigma}})} = \hat{I}\cos(\theta) + \mathrm{i}(\vec{a}\cdot\hat{\vec{\sigma}})\sin(\theta),$$

   where $\vec{a}\cdot\hat{\vec{\sigma}} = a_x\hat{\sigma}_x + a_y\hat{\sigma}_y + a_z\hat{\sigma}_z$, $\{a_k\}$ are the Cartesian components of $\vec{a}$, and $\{\hat{\sigma}_k\}$ are the Pauli matrices. Hint: Use the Taylor series definition of the operator exponential. Then, try to find a simplified expression for $(\theta\vec{a}\cdot\hat{\vec{\sigma}})^k$, depending on whether k is even or odd.

(b) Consider the operator $e^{-i\frac{\pi}{2}(\vec{a}\cdot\hat{\vec{\sigma}})}$. Show that the case

$$\vec{a} = \vec{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

corresponds to a rotation by the angle π about the x axis of the Bloch sphere. Hint: operate on a general state $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$ and compare the Bloch vectors of the original and resultant states. Also note that you can always ignore the global phase, that is,

$$e^{i\phi_1}|0\rangle + e^{i\phi_2}|1\rangle = \underbrace{e^{i\phi_1}}_{\text{global phase}}\left(|0\rangle + e^{i(\phi_2-\phi_1)}|1\rangle\right)$$

has the same Bloch sphere representation as

$$|0\rangle + e^{i(\phi_2-\phi_1)}|1\rangle .$$

If you want, you can also show that $\vec{a} = \vec{x}$ and $\vec{a} = \vec{y}$ correspond to rotations about the x and y axis, respectively (but points are awarded only for the case $\vec{a} = \vec{z}$).

3. Consider a system described by the Hamiltonian $\hat{H} = \epsilon(-i|0\rangle\langle 1| + i|1\rangle\langle 0|)$, where $\{|0\rangle, |1\rangle\}$ form an orthonormal basis of the considered Hilbert space and ϵ is a real-valued constant with the dimension of energy. The eigenergies of $\hat{H}$ are $\pm\epsilon$ and the corresponding eigenstates are

$$|\epsilon\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}, \qquad |-\epsilon\rangle = \frac{|0\rangle - i|1\rangle}{\sqrt{2}}.$$

.

(a) (2 points) What are the probabilities to measure ϵ and $-\epsilon$ if the system is in the state $|1\rangle$? What is the value of $\langle H\rangle$?

(b) (4 points) Find the state $|\psi(t)\rangle$ at an arbitrary time t, when the system is initially in the state $|\psi(t=0)\rangle = |0\rangle$. What is the probability to find the system in the state $|0\rangle$ as a function of time? How does $\langle H\rangle$ change with time? Hint: you need to solve the Schrödinger equation.

(4) Consider a peculiar gate known as square-root of NOT with the matrix representation

$$\sqrt{\text{NOT}} \mathrel{\hat{=}} \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}.$$

(a) Is this gate unitary? Is it Hermitian? How does it act on the $|0\rangle$ and $|1\rangle$ basis states?

(b) Show that the square-root of NOT gate is worthy of its name by verifying that $\left(\sqrt{\text{NOT}}\right)^2 = \sigma_x$.

(c) Let us define the action of the following gates on a two-qubit system:

$$\mathrm{NOT}^{(2)}\,|x0\rangle = |x1\rangle\,,\quad \mathrm{NOT}^{(2)}\,|x1\rangle = |x0\rangle\,, \tag{1}$$

$$H^{(2)}\,|x0\rangle = \frac{1}{\sqrt{2}}\,(|x0\rangle + |x1\rangle)\,,\quad H^{(2)}\,|x1\rangle = \frac{1}{\sqrt{2}}\,(|x0\rangle - |x1\rangle)\,, \tag{2}$$

$$\mathrm{CNOT}^{(2,1)}\,|x0\rangle = |x0\rangle\,,\quad \mathrm{CNOT}^{(2,1)}\,|01\rangle = |11\rangle\,,\quad \mathrm{CNOT}^{(2,1)}\,|11\rangle = |01\rangle\,, \tag{3}$$

where $x \in \{0, 1\}$, so for instance $\mathrm{NOT}^{(2)}\,|00\rangle = |01\rangle$. The matrix representation of $\mathrm{NOT}^{(2)}$ is

$$\mathrm{NOT}^{(2)} \hat{=} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

which you can verify by operating on the basis states and checking that they match the definition (1). Find the matrix representations for $H^{(2)}$ and $\mathrm{CNOT}^{(2,1)}$.

(d) Either by using the action of the gates defined in equations (1-3) or by using the matrices derived in the previous problem, find a series of gates to construct the Bell state $\frac{1}{\sqrt{2}}\,(|00\rangle - |11\rangle)$ starting from $|00\rangle$.