PHYS-C0252 - Quantum Mechanics

Exercise set 6 - model solutions

Due date: **June 3, 2024 by 23:59 on MyCourses.** **Note that the DL is already on Monday**, but there are only two problems.
Return the exercises as a `.pdf`.
You can write by hand and take pictures, use digital note-taking or LaTeX etc.

1. PhD student H is busy grading homework, and asks PhD student R to prepare his qubit in the excited state $|1\rangle$. However, the local shop is out of R's favourite energy drink, which leaves R impaired. In his weakened condition, R has a 50% success rate in preparing the qubit in the correct state. The other 50% of the time the qubit is left in the ground state $|0\rangle$. The states $|0\rangle$ and $|1\rangle$ are eigenstates of the Hamiltonian with the energies $\frac{1}{2}\hbar\omega$, $\frac{3}{2}\hbar\omega$, respectively.

   a. (1 point) Write down a density matrix for the system.

   b. (1 point) Suppose H prepares another system in the state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. What is the expectation value of energy in this system? What is the expectation value of energy in the system R prepared?

   c. (2 points) H applies the gate

   $$G = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

   to both the system he prepared and the system R prepared. What is the expectation value of energy in each system after applying the gate? Explain why the results are the same/different.

   d. (2 points) Consider a quantum state characterized by the density operator $\hat{\rho}(t) = |\psi(t)\rangle\langle\psi(t)|$. The state vector $|\psi(t)\rangle$ evolves in time according to the Schrödinger equation

   $$i\hbar \frac{d|\psi(t)\rangle}{dt} = \hat{H}(t)|\psi(t)\rangle.$$

   Show that time evolution of the density operator follows the Von–Neumann equation

   $$\frac{d\hat{\rho}(t)}{dt} = \frac{-i}{\hbar}[\hat{H}(t), \hat{\rho}(t)].$$

   Solution:

   a. The density operator is

   $$\hat{\rho} = 0.5 \cdot |0\rangle\langle 0| + 0.5\,|1\rangle\langle 1|\,,$$

   and the corresponding matrix can be found by the standard method $\rho_{nm} = \langle n|\,\rho\,|m\rangle$, giving

   $$\rho = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$$

b. In the system that H prepared, $\langle \hat{H} \rangle$ is given as usual by

$$\langle H \rangle_{\mathrm{H}} = \langle \psi | \hat{H} | \psi \rangle \,,$$

where $\psi = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Since $|0\rangle$ and $|1\rangle$ are eigenstates $\hat{H}$ with the corresponding energies given, we have

$$\begin{aligned}\langle \psi | \hat{H} | \psi \rangle &= \frac{1}{2}(\langle 0| + \langle 1|)\hat{H}(|0\rangle + |1\rangle) \\ &= \frac{1}{2}(\langle 0| + \langle 1|)\left(\frac{1}{2}\hbar\omega\,|0\rangle + \frac{3}{2}\hbar\omega\,|1\rangle\right) = \frac{1}{2}\left(\frac{1}{2}\hbar\omega + \frac{3}{2}\hbar\omega\right) = \hbar\omega.\end{aligned}$$

In the system that R prepared, $\langle \hat{H} \rangle$ is given by

$$\begin{aligned}\langle \hat{H} \rangle_{\mathrm{R}} = \mathrm{Tr}(\hat{H}\hat{\rho}) &= \sum_n \langle n|\hat{H}\hat{\rho}|n\rangle = \langle 0|\hat{H}\hat{\rho}|0\rangle + \langle 1|\hat{H}\hat{\rho}|1\rangle \\ &= 0.5 \cdot \langle 0|\,\hat{H}(|0\rangle\,\langle 0| + |1\rangle\,\langle 1|)\,|0\rangle + 0.5 \cdot \langle 1|\,\hat{H}(|0\rangle\,\langle 0| + |1\rangle\,\langle 1|)\,|1\rangle \\ &= 0.5 \cdot \frac{1}{2}\hbar\omega + 0.5 \cdot \frac{3}{2}\hbar\omega = \hbar\omega.\end{aligned}$$

Thus, $\langle \hat{H} \rangle_{\mathrm{H}} = \langle \hat{H} \rangle_{\mathrm{R}}$.

c. When H applies the gate to the system he prepared, the resulting state is

$$\hat{G}\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \,\tilde{=}\, \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \,\tilde{=}\, |1\rangle \,.$$

where we have used the conventional basis ordering $|0\rangle \,\tilde{=}\, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |1\rangle \,\tilde{=}\, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Thus,

$$\langle H \rangle_{\mathrm{H}} = \langle 1|\hat{H}|1\rangle = \frac{3}{2}\hbar\omega.$$

After the gate is applied to the system that R prepared there is a 50% change the system lies in $\hat{G}\,|0\rangle$, and a 50% change that it lies in $\hat{G}\,|1\rangle$. Thus, the new density operator is

$$\hat{\rho}' = 0.5 \cdot \hat{G}\,|0\rangle\,\langle 0|\,\hat{G}^\dagger + 0.5 \cdot \hat{G}\,|1\rangle\,\langle 1|\,\hat{G}^\dagger = \hat{G}(0.5 \cdot |0\rangle\,\langle 0| + 0.5 \cdot |1\rangle\,\langle 1|) = \hat{G}\hat{\rho}\hat{G}^\dagger.$$

Thus, in matrix form,

$$\rho' = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \rho,$$

from which we immediately know that $\langle H \rangle = \mathrm{Tr}(\hat{H}\hat{\rho}) = \hbar\omega$ remains unchanged.

The reason the results are different is that in one case (the system that R prepared), the system behaves according to *classical probability*, where as in the system that H prepared, the system behaves according to *quantum probability*. In the classical world, systems are always either in one state or in another, and even if we bring these states back together, the alternative pasts do not interfere with each other. However, in the quantum regime, the alternative pasts truly exist and interfere. The application of $\hat{G}$ can, in the quantum case, be thought of as applying the gate to both $|0\rangle$ and $|1\rangle$ individually, and letting the resultant states interfere:

$$
\begin{aligned}
\hat{G}\frac{1}{\sqrt{2}}|0\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\
\hat{G}\frac{1}{\sqrt{2}}|1\rangle &= \frac{1}{\sqrt{2}}(|1\rangle - |0\rangle) \\
\hat{G}\frac{1}{\sqrt{2}}|0\rangle + \hat{G}\frac{1}{\sqrt{2}}|1\rangle &= \hat{G}\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |1\rangle .
\end{aligned}
$$

In the classical case, $\hat{G}$ is applied either to $|0\rangle$ or to $|1\rangle$. Only one of these possible pasts truly happens, and the results do not interfere, so the result is $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ with 50% probability, and $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ with 50% probability.

d.

$$
\frac{d\hat{\rho}(t)}{dt} = \frac{d|\psi\rangle\langle\psi|}{dt} = \frac{d|\psi\rangle}{dt}\langle\psi| + |\psi\rangle\frac{d\langle\psi|}{dt} \tag{1}
$$

From the schödinger equation

$$
\frac{d|\psi(t)\rangle}{dt} = \frac{-i}{\hbar}\hat{H}(t)|\psi(t)\rangle \tag{2}
$$

$$
\frac{d\langle\psi(t)|}{dt} = \frac{i}{\hbar}\langle\psi(t)|\hat{H}(t) \tag{3}
$$

Thus

$$
\begin{aligned}
\frac{d\hat{\rho}(t)}{dt} &= \frac{-i}{\hbar}\big(H(t)|\psi(t)\rangle\langle\psi| - |\psi\rangle\langle\psi(t)|\hat{H}(t)dt\big) \\
&= \frac{-i}{\hbar}\big(H(t)\hat{\rho}(t) - \hat{\rho}(t)\hat{H}(t)dt\big) = \frac{-i}{\hbar}[\hat{H}(t), \hat{\rho}(t)].
\end{aligned} \tag{4}
$$

Extra: Von Neumann equation equation in the interaction picture:

$$
\hat{H} = \hat{H}_0 + \hat{H}_{\mathrm{I}},
$$

$$
\hat{\rho}_I(t) = \hat{U}_0^\dagger(t)\hat{\rho}(t)\hat{U}_0(t),
$$

where $\hat{U}_0(t) = e^{-i\hat{H}_0 t/\hbar}$

$$
\begin{aligned}
\frac{d\hat{\rho}_{\mathrm{I}}(t)}{dt} &= \frac{d\hat{U}_0^\dagger(t)}{dt}\hat{\rho}(t)\hat{U}_0(t) + \hat{U}_0^\dagger(t)\frac{d\rho(t)}{dt}\hat{U}_0(t) + \hat{U}_0^\dagger(t)\hat{\rho}(t)\frac{d\hat{U}_0(t)}{dt} \\
&= \frac{i\hat{H}_0}{\hbar}\hat{U}_0^\dagger(t)\hat{\rho}(t)\hat{U}_0(t) + \hat{U}_0^\dagger(t)\frac{d\rho(t)}{dt}\hat{U}_0(t) + \hat{U}_0^\dagger(t)\hat{\rho}(t)\hat{U}_0(t)\frac{-i\hat{H}_0}{\hbar} \\
&= \frac{i}{\hbar}[\hat{H}_0, \hat{\rho}_{\mathrm{I}}(t)] + \hat{U}_0^\dagger(t)\frac{d\rho(t)}{dt}\hat{U}_0(t)
\end{aligned}
$$

$$
\frac{d\hat{\rho}(t)}{dt} = \frac{-i}{\hbar}[\hat{H}, \hat{\rho}(t)].
$$

$$
\begin{aligned}
\frac{d\hat{\rho}_I(t)}{dt} &= \frac{i}{\hbar}[\hat{H}_0, \hat{\rho}_\mathrm{I}(t)] + \hat{U}_0^\dagger(t)\frac{d\rho(t)}{dt}\hat{U}_0(t) \\
&= \frac{i}{\hbar}\left\{[\hat{H}_0, \hat{\rho}_\mathrm{I}(t)] - \hat{U}_0^\dagger(t)[\hat{H}, \hat{\rho}(t)]\hat{U}_0(t)\right\}
\end{aligned}
$$

$$
\begin{aligned}
\hat{U}_0^\dagger(t)[\hat{H}, \hat{\rho}(t)]\hat{U}_0(t) =& \hat{U}_0^\dagger(t)\hat{H}\hat{U}_0(t)U_0^\dagger(t)\hat{\rho}(t)\hat{U}_0(t) \\
&- \hat{U}_0^\dagger(t)\hat{\rho}(t)\hat{U}_0(t)\hat{U}_0^\dagger(t)\hat{H}\hat{U}_0(t) \\
&= [\hat{H}(t), \hat{\rho}_\mathrm{I}]
\end{aligned}
$$

Thus

$$
\frac{d\hat{\rho}(t)}{dt} = \frac{-i}{\hbar}[\hat{H}_\mathrm{I}(t), \hat{\rho}_\mathrm{I}(t)].
$$

where $\hat{H}_\mathrm{I}(t) = \hat{U}_0^\dagger(t)\hat{H}_\mathrm{I}\hat{U}_0(t)$.
Note that $\hat{H}_\mathrm{I}(t) = \hat{U}_0^\dagger(t)\hat{H}\hat{U}_0(t)$ for commuting $\hat{H}_0$ and $\hat{H}_\mathrm{I}$ (i.e.$[\hat{H}_0, \hat{H}_\mathrm{I}] = 0$).

2. **Rabi oscillation** – Let us consider the ground and excited states of an electron orbiting an atom as a two-level system. The atom's energy levels are $E_1 = \hbar\omega_1$ and $E_2 = \hbar\omega_2$, with $E_2 > E_1$, so that the transition energy between the levels is $E_{21} = \hbar\omega_2 - \hbar\omega_1 = \hbar(\omega_2 - \omega_1) := \hbar\omega_0$, where ω_0 is the transition frequency. Let $\hat{H}_0$ be the Hamiltonian of the atom, so that the time-dependent Schrödinger equation reads

$$
i\hbar\frac{\partial}{\partial t}|\Psi_n(t)\rangle = \hat{H}_0|\Psi_n(t)\rangle,
$$

and the solutions satisfy

$$
|\Psi_n(t)\rangle = e^{-i\omega_n t}|\psi_n\rangle,
$$

where $|\psi_n\rangle$ are the eigenstates of the time-independent Schrödinger equation

$$
\hat{H}_0|\psi_n\rangle = E_n|\psi_n\rangle,
$$

with $\langle\psi_n|\psi_m\rangle = \delta_{nm}$. Now, we will consider what happens when the atom is irradiated by an electric field, which induces a time-dependent potential $\hat{V}(t)$, so that the Hamiltonian becomes $\hat{H}(t) = \hat{H}_0 + \hat{V}(t)$. This is an example of time-*dependent* perturbation theory, which is discussed in more detail in later courses. For now, it is enough to know that in this case the resulting wavefunction will be a time-dependent superposition of the eigenstates of the unperturbed system $\hat{H}_0$:

$$
\begin{aligned}
|\Psi(t)\rangle &= c_1(t)|\Psi_1(t)\rangle + c_2(t)|\Psi_2(t)\rangle \\
&= c_1(t)e^{-i\omega_1 t}|\psi_1\rangle + c_2(t)e^{-i\omega_2 t}|\psi_2\rangle,
\end{aligned}
$$

where the coefficients $c_n(t) \in \mathbb{C}$ are time-dependent, but must always satisfy $|c_1(t)|^2 + |c_2(t)|^2 = 1$.

(a) Plug the Hamiltonian $\hat{H}(t)$ into the time-dependent Schrödinger equation for $|\Psi(t)\rangle$, and apply $\langle\psi_1|$ on both sides to obtain a differential equation for $c_1(t)$:

$$\dot{c}_1(t) = -\frac{i}{\hbar}\left(c_1(t)V_{11}(t) + c_2(t)e^{-i\omega_0 t}V_{12}(t)\right),$$

and similarly apply $\langle\psi_2|$ to find

$$\dot{c}_2(t) = -\frac{i}{\hbar}\left(c_1(t)e^{+i\omega_0 t}V_{21}(t) + c_2(t)V_{22}(t)\right),$$

where $V_{nm}(t) = \langle\psi_n|\hat{V}(t)|\psi_m\rangle$ are the matrix elements of $\hat{V}(t)$.

(b) If we consider the atom to be a dipole, i.e., it is symmetric under inversion, the diagonal terms $V_{nn}(t)$ vanish. Furthermore, for bound states the off-diagonal terms $V_{nm}(t)$ are real, which implies $V_{12}(t) = V_{21}(t)$. Let us now assume that the electric field that is being applied to the atom is of the form $\mathcal{E}(t) = \mathcal{E}_0\cos(\omega t)$, which results in $V_{12}(t) = V_{21}(t) = \hbar\Omega\cos(\omega t)$, where $\Omega \propto \frac{\mathcal{E}_0}{\hbar}$ is called the *Rabi frequency*. In particular, consider the case where the electric field is resonant with the atom, that is, $\omega = \omega_0$. With these, solve the coefficients $c_1(t)$ and $c_2(t)$ and calculate the probability of finding the atom in state $|\psi_1\rangle$ and $|\psi_2\rangle$ at time t when it starts out in the state $|\Psi(t=0)\rangle = |\psi_1\rangle$. The resulting time-evolution is called *Rabi oscillation*.
Hint: Write $\cos(\omega t)$ as $\frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)$, and ignore terms of the form $e^{2i\omega_0 t}$, since they oscillate fast compared to the other terms[1].

Solution:

(a) Plugging in $|\Psi(t)\rangle$ and $\hat{H} = \hat{H}_0 + \hat{V}(t)$ in to the Schrödinger equation and applying $\langle\psi_1|$ on both sides:

$$\langle\psi_1|\, i\hbar\frac{\partial}{\partial t}\,|\Psi_n(t)\rangle = \langle\psi_1|\,\hat{H}(t)\,|\Psi(t)\rangle$$

On the left hand side, we have

$$\begin{aligned}
\langle\psi_1|\, i\hbar\frac{\partial}{\partial t}\,|\Psi_n(t)\rangle &= i\hbar\,\langle\psi_1|\,\frac{\partial}{\partial t}\left(c_1(t)e^{-i\omega_1 t}\,|\psi_1\rangle + c_2(t)e^{-i\omega_2 t}\,|\psi_2\rangle\right)\\
&= i\hbar\frac{\partial}{\partial t}\Big(c_1(t)e^{-i\omega_1 t}\underbrace{\langle\psi_1|\psi_1\rangle}_{=1} + c_2(t)e^{-i\omega_2 t}\underbrace{\langle\psi_1|\psi_2\rangle}_{=0}\Big)\\
&= i\hbar\frac{\partial}{\partial t}\left(c_1(t)e^{-i\omega_1 t}\right)\\
&= i\hbar\left(\dot{c}_1(t)e^{-i\omega_1 t} - i\omega_1 c_1(t)e^{-i\omega_1 t}\right)\\
&= i\hbar\dot{c}_1(t)e^{-i\omega_1 t} + \hbar\omega_1 c_1(t)e^{-i\omega_1 t},
\end{aligned}$$

[1]This is called the rotating wave approximation, and we neglect such terms because they will average to zero on the time scales of the rest of the system.

while on the right side, we have

$$
\begin{aligned}
\langle\psi_1|\,\hat{H}(t)\,|\Psi(t)\rangle &= \langle\psi_1|\left(\hat{H}_0+\hat{V}(t)\right)\left(c_1(t)e^{-i\omega_1 t}\,|\psi_1\rangle + c_2(t)e^{-i\omega_2 t}\,|\psi_2\rangle\right)\\
&= \left(\langle\psi_1|\,\hat{H}_0 + \langle\psi_1|\,\hat{V}(t)\right)\left(c_1(t)e^{-i\omega_1 t}\,|\psi_1\rangle + c_2(t)e^{-i\omega_2 t}\,|\psi_2\rangle\right)\\
&= \Big(\,\langle\psi_1|\underbrace{E_1^*}_{=E_1\in\mathbb{R}} + \langle\psi_1|\,\hat{V}(t)\Big)\left(c_1(t)e^{-i\omega_1 t}\,|\psi_1\rangle + c_2(t)e^{-i\omega_2 t}\,|\psi_2\rangle\right)\\
&= E_1c_1(t)e^{-i\omega_1 t}\underbrace{\langle\psi_1|\psi_1\rangle}_{=1} + c_1(t)e^{-i\omega_1 t}\underbrace{\langle\psi_1|\,\hat{V}(t)\,|\psi_1\rangle}_{=V_{11}(t)}\\
&\qquad + E_1c_2(t)e^{-i\omega_2 t}\underbrace{\langle\psi_1|\psi_2\rangle}_{=0} + c_2(t)e^{-i\omega_2 t}\underbrace{\langle\psi_1|\,\hat{V}(t)\,|\psi_2\rangle}_{=V_{12}(t)}\\
&= E_1c_1(t)e^{-i\omega_1 t} + c_1(t)e^{-i\omega_1 t}V_{11}(t) + c_2(t)e^{-i\omega_2 t}V_{12}(t)\\
&= \hbar\omega_1c_1(t)e^{-i\omega_1 t} + c_1(t)e^{-i\omega_1 t}V_{11}(t) + c_2(t)e^{-i\omega_2 t}V_{12}(t).
\end{aligned}
$$

Combining these, and noting that the terms $\hbar\omega_1 c_1(t)e^{-i\omega_1 t}$ on both sides cancel, we end up with

$$
\begin{aligned}
i\hbar\dot{c}_1(t)e^{-i\omega_1 t} &= c_1(t)e^{-i\omega_1 t}V_{11}(t) + c_2(t)e^{-i\omega_2 t}V_{12}(t)\\
\Rightarrow \dot{c}_1(t) &= -\frac{i}{\hbar}e^{+i\omega_1 t}\left(c_1(t)e^{-i\omega_1 t}V_{11}(t) + c_2(t)e^{-i\omega_2 t}V_{12}(t)\right)\\
&= -\frac{i}{\hbar}\left(c_1(t)e^{0}V_{11}(t) + c_2(t)e^{-i(\omega_2-\omega_1)t}V_{12}(t)\right)\\
&= -\frac{i}{\hbar}\left(c_1(t)V_{11}(t) + c_2(t)e^{-i\omega_0 t}V_{12}(t)\right).
\end{aligned}
$$

For c_2, the calculation is very similar.

(b) Plugging in the given definitions: $V_{12}(t) = V_{21}(t) = \hbar\Omega\cos(\omega t)$, $V_{11}(t) = V_{22}(t) = 0$, and $\omega = \omega_0$, we find

$$
\begin{aligned}
\dot{c}_1(t) &= -ic_2(t)\Omega\cos(\omega_0 t)e^{-i\omega_0 t} \qquad (5)\\
&= -\frac{i}{2}c_2(t)\Omega\left(e^{i\omega_0 t}+e^{-i\omega_0 t}\right)e^{-i\omega_0 t}\\
&= -\frac{i}{2}c_2(t)\Omega(1+e^{-2i\omega_0 t})\\
&\approx -\frac{i\Omega}{2}c_2(t).
\end{aligned}
$$

With very similar steps, for c_2, we find

$$
\begin{aligned}
\dot{c}_2(t) &= -ic_1(t)e^{+i\omega_0 t}\Omega\cos(\omega_0 t) \qquad (6)\\
&\approx -\frac{i\Omega}{2}c_1(t),
\end{aligned}
$$

where in the last steps, we have applied the rotating wave approximation by dropping the terms $e^{\pm 2i\omega_0 t}$. Differentiating the expression (5) for $\dot{c}_1(t)$ with respect to t on both sides, we obtain

$$
\ddot{c}_1(t) = -\frac{i\Omega}{2}\dot{c}_2(t) \quad\Rightarrow\quad \dot{c}_2(t) = \frac{2i}{\Omega}\ddot{c}_1(t),
$$

and plugging this into Eq. (6), we find

$$\frac{2i}{\Omega}\ddot{c}_1(t) = -\frac{i\Omega}{2}c_1(t) \quad \Rightarrow \quad \ddot{c}_1(t) = -\frac{\Omega^2}{4}c_1(t).$$

The solution of this is simple harmonic motion,

$$c_1(t) = A\sin\left(\frac{\Omega}{2}t\right) + B\cos\left(\frac{\Omega}{2}t\right).$$

From Eq. (5), we then find

$$c_2(t) = \frac{2i}{\Omega}\dot{c}_1(t) = iA\cos\left(\frac{\Omega}{2}t\right) - iB\sin\left(\frac{\Omega}{2}t\right).$$

From the initial condition, $|\Psi(0)\rangle = c_1(0)|\psi_1\rangle + c_2(0)|\psi_2\rangle = |\psi_1\rangle$, we know $c_1(0) = 1$ and $c_2(0) = 0$, i.e.,

$$\begin{cases} c_1(0) = 1 = B, \\ c_2(0) = 0 = iA \Rightarrow A = 0. \end{cases}$$

The total state is thus

$$\begin{aligned} |\Psi(t)\rangle &= \cos\left(\frac{\Omega}{2}t\right)e^{-i\omega_1 t}|\psi_1\rangle + i\sin\left(\frac{\Omega}{2}t\right)e^{-i\omega_2 t}|\psi_2\rangle \\ &= e^{-i\omega_1 t}\left[\cos\left(\frac{\Omega}{2}t\right)|\psi_1\rangle + i\sin\left(\frac{\Omega}{2}t\right)e^{-i(\omega_2-\omega_1)t}|\psi_2\rangle\right] \\ &\simeq \cos\left(\frac{\Omega}{2}t\right)|\psi_1\rangle + i\sin\left(\frac{\Omega}{2}t\right)e^{-i\omega_0 t}|\psi_2\rangle, \end{aligned}$$

where we have dropped the global phase factor in the last step. The probabilities are thus given by

$$\begin{aligned} P_1(t) &= |\langle\psi_1|\Psi(t)\rangle|^2 = \cos^2\left(\frac{\Omega}{2}t\right), \\ P_2(t) &= |\langle\psi_2|\Psi(t)\rangle|^2 = \sin^2\left(\frac{\Omega}{2}t\right). \end{aligned}$$