PHYS-C0252 - Quantum Mechanics

Exercise set 6

Due date: June 3, 2024 by 23:59 on MyCourses. Note that the DL is already on Monday, but there are only two problems.
Return the exercises as a `.pdf`.
You can write by hand and take pictures, use digital note-taking or LaTeX etc.

1. PhD student H is busy grading homework, and asks PhD student R to prepare his qubit in the excited state $|1\rangle$. However, the local shop is out of R's favourite energy drink, which leaves R impaired. In his weakened condition, R has a 50% success rate in preparing the qubit in the correct state. The other 50% of the time the qubit is left in the ground state $|0\rangle$. The states $|0\rangle$ and $|1\rangle$ are eigenstates of the Hamiltonian with the energies $\frac{1}{2}\hbar\omega$, $\frac{3}{2}\hbar\omega$, respectively.

   a. (1 point) Write down a density matrix for the system.

   b. (1 point) Suppose H prepares another system in the state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. What is the expectation value of energy in this system? What is the expectation value of energy in the system R prepared?

   c. (2 point) H applies the gate

   $$G = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

   to both the system he prepared and the system R prepared. What is the expectation value of energy in each system after applying the gate? Explain why the results are the same/different.

   d. (2 points) Consider a quantum state characterized by the density operator $\hat{\rho}(t) = |\psi(t)\rangle\langle\psi(t)|$. The state vector $|\psi(t)\rangle$ evolves in time according to the Schrödinger equation

   $$i\hbar\frac{d|\psi(t)\rangle}{dt} = \hat{H}(t)|\psi(t)\rangle.$$

   Show that time evolution of the density operator follows the Von–Neumann equation

   $$\frac{d\hat{\rho}(t)}{dt} = \frac{-i}{\hbar}[\hat{H}(t), \hat{\rho}(t)].$$

2. **Rabi oscillation** – Let us consider the ground and excited states of an electron orbiting an atom as a two-level system. The atom's energy levels are $E_1 = \hbar\omega_1$ and $E_2 = \hbar\omega_2$, with $E_2 > E_1$, so that the transition energy between the levels is $E_{21} = \hbar\omega_2 - \hbar\omega_1 = \hbar(\omega_2 - \omega_1) := \hbar\omega_0$, where ω_0 is the transition frequency. Let $\hat{H}_0$ be the Hamiltonian of the atom, so that the time-dependent Schrödinger equation reads

   $$i\hbar\frac{\partial}{\partial t}|\Psi_n(t)\rangle = \hat{H}_0|\Psi_n(t)\rangle,$$

and the solutions satisfy

$$|\Psi_n(t)\rangle = e^{-i\omega_n t}\,|\psi_n\rangle\,,$$

where $|\psi_n\rangle$ are the eigenstates of the time-independent Schrödinger equation

$$\hat{H}_0\,|\psi_n\rangle = E_n\,|\psi_n\rangle\,,$$

with $\langle\psi_n|\psi_m\rangle = \delta_{nm}$. Now, we will consider what happens when the atom is irradiated by an electric field, which induces a time-dependent potential $\hat{V}(t)$, so that the Hamiltonian becomes $\hat{H}(t) = \hat{H}_0 + \hat{V}(t)$. This is an example of time-*dependent* perturbation theory, which is discussed in more detail in later courses. For now, it is enough to know that in this case the resulting wavefunction will be a time-dependent superposition of the eigenstates of the unperturbed system $\hat{H}_0$:

$$\begin{aligned}|\Psi(t)\rangle &= c_1(t)\,|\Psi_1(t)\rangle + c_2(t)\,|\Psi_2(t)\rangle \\ &= c_1(t)e^{-i\omega_1 t}\,|\psi_1\rangle + c_2(t)e^{-i\omega_2 t}\,|\psi_2\rangle\,,\end{aligned}$$

where the coefficients $c_n(t) \in \mathbb{C}$ are time-dependent, but must always satisfy $|c_1(t)|^2 + |c_2(t)|^2 = 1$.

(a) Plug the Hamiltonian $\hat{H}(t)$ into the time-dependent Schrödinger equation for $|\Psi(t)\rangle$, and apply $\langle\psi_1|$ on both sides to obtain a differential equation for $c_1(t)$:

$$\dot{c}_1(t) = -\frac{i}{\hbar}\left(c_1(t)V_{11}(t) + c_2(t)e^{-i\omega_0 t}V_{12}(t)\right),$$

and similarly apply $\langle\psi_2|$ to find

$$\dot{c}_2(t) = -\frac{i}{\hbar}\left(c_1(t)e^{+i\omega_0 t}V_{21}(t) + c_2(t)V_{22}(t)\right),$$

where $V_{nm}(t) = \langle\psi_n|\hat{V}(t)|\psi_m\rangle$ are the matrix elements of $\hat{V}(t)$.

(b) If we consider the atom to be a dipole, i.e., it is symmetric under inversion, the diagonal terms $V_{nn}(t)$ vanish. Furthermore, for bound states the off-diagonal terms $V_{nm}(t)$ are real, which implies $V_{12}(t) = V_{21}(t)$. Let us now assume that the electric field that is being applied to the atom is of the form $\mathcal{E}(t) = \mathcal{E}_0\cos(\omega t)$, which results in $V_{12}(t) = V_{21}(t) = \hbar\Omega\cos(\omega t)$, where $\Omega \propto \frac{\mathcal{E}_0}{\hbar}$ is called the *Rabi frequency*. In particular, consider the case where the electric field is resonant with the atom, that is, $\omega = \omega_0$. With these, solve the coefficients $c_1(t)$ and $c_2(t)$ and calculate the probability of finding the atom in state $|\psi_1\rangle$ and $|\psi_2\rangle$ at time t when it starts out in the state $|\Psi(t=0)\rangle = |\psi_1\rangle$. The resulting time-evolution is called *Rabi oscillation*.

Hint: Write $\cos(\omega t)$ as $\frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)$, and ignore terms of the form $e^{2i\omega_0 t}$, since they oscillate fast compared to the other terms[1].

[1]This is called the rotating wave approximation, and we neglect such terms because they will average to zero on the time scales of the rest of the system.