PHYS-C0252 - Quantum Mechanics

Exercise set 4

Due date: May 22, 2024 by 23:59 on MyCourses
Return the exercises as a `.pdf`.
You can write by hand and take pictures, use digital note-taking or LaTeX etc.

1. (a) Using

$$\langle x|\psi\rangle = \psi(x), \tag{1}$$
$$\hat{x}\,|x\rangle = x\,|x\rangle\,, \tag{2}$$
$$\langle x'|\hat{p}|x\rangle = -\mathrm{i}\hbar\delta(x'-x)\frac{\partial}{\partial x}, \tag{3}$$

Derive the expressions

$$\langle\psi|\hat{x}|\psi\rangle = \int_{-\infty}^{\infty} x|\psi(x)|^2\mathrm{d}x,$$
$$\langle\psi|\hat{p}|\psi\rangle = -\mathrm{i}\hbar\int_{-\infty}^{\infty}\psi^*(x)\psi'(x)\mathrm{d}x$$

for the expectation values of position and momentum.
Hint: It maybe useful to insert the identity operator

$$\hat{I} = \int_{-\infty}^{\infty}|x\rangle\,\langle x|\,\mathrm{d}x$$

at some point(s) inside the expressions for the expectation values. For example, an inner product between two states $|\psi\rangle$ and $|\chi\rangle$ is given by

$$\langle\psi|\chi\rangle = \langle\psi|\underbrace{\left(\int_{-\infty}^{\infty}|x\rangle\,\langle x|\,\mathrm{d}x\right)}_{\hat{I}}|\chi\rangle$$
$$= \int_{-\infty}^{\infty}\langle\psi|x\rangle\,\langle x|\chi\rangle\,\mathrm{d}\alpha = \int_{-\infty}^{\infty}\psi^*(x)\chi(x)\mathrm{d}x.$$

(b) Consider a one-dimensional particle in a quantum state whose position-basis representation is

$$\langle x|\psi\rangle = \psi(x) = \begin{cases} A\sin(kx)\,\mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{-\lambda x}, & \text{if } x\in[0,\infty),\\ 0, & \text{otherwise,}\end{cases}$$

where $\lambda > 0$ and $k\in\mathbb{R}$. Find a suitable scalar A such that $\psi(x)$ is normalized. Hint: recall that $\sin x = \frac{\mathrm{e}^{\mathrm{i}x}-\mathrm{e}^{-\mathrm{i}x}}{2\mathrm{i}}$ to simplify the integral.

(c) For the wave function given in part (b), find the expectation values of position and momentum. Hint from (b) might prove to be useful here as well.

2. Consider a particle of mass m in a one-dimensional potential

$$V(\hat{x}) = \frac{1}{2}m\omega^2\hat{x}^2 - \lambda\hat{x}$$

   (a) Given the corresponding Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$, find the ladder operators $\hat{b}$ and $\hat{b}^\dagger$ that allow an algebraic representation of the eigenstates. Hint: Make a change of variables in $\hat{x}$ to simplify the Hamiltonian, and recall how $\hat{a}$ and $\hat{a}^\dagger$ are defined in terms of $\hat{x}$ and $\hat{p}$ for the harmonic oscillator. You should end up with a Hamiltonian of the form $\hat{A}^\dagger\hat{A} + C\hat{I}$, where $\hat{A} \in \mathcal{L}(\mathcal{H})$ and $C \in \mathbb{R}$.

   (b) Find the eigenenergies of $\hat{H}$.

3. Consider the time-independent Schrödinger's equation in position representation for a free particle ($V(x) = 0$):

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = E\psi(x)$$

(a) Find a general solution for $\psi(x)$.

(b) Show that

$$\psi(x,t) = Ae^{ik(x-\frac{\hbar k}{2m}t)} + Be^{-ik(x+\frac{\hbar k}{2m}t)},$$

where A, B are constants and $k = \sqrt{2mE}/\hbar$.

(c) The first term in $\psi(x,t)$ represents a wave traveling to the right, and the second term represents a wave (of the same energy) going to the left. Since they only differ by the sign in front of k, we can write the wave function as

$$\psi_k(x,t) = Ae^{i(kx-\frac{\hbar k^2}{2m}t)},$$

where k can also be negative to cover the case of the wave travelling to the left. Find the velocity of this wave and compare it with the velocity of a classical free particle with energy E. Hint: compare $\psi_k(x,t)$ to the general plane wave solution $e^{ik(x-vt)}$.

4. Consider a one-dimensional free particle in the following state at the time $t = 0$:

$$\psi(x, t = 0) = \frac{\sqrt{a}}{(2\pi)^{3/4}\hbar}\int_{-\infty}^{\infty} \mathrm{e}^{-a^2p^2/(2\hbar)^2}\mathrm{e}^{\mathrm{i}px/\hbar}\mathrm{d}p,$$

with $0 < a \in \mathbb{R}$. This is a Gaussian superposition of plane waves (also called a wave packet). Note that, unlike in the case of simple plane waves, this wave function is normalizable to unity, and hence can be interpreted as probability density for the particle.

(a) (4p) Show that the time evolution of the probability density is given by

$$|\psi(x,t)|^2 = \sqrt{\frac{2}{\pi a^2}}\frac{1}{\sqrt{1+4\hbar^2 t^2/(m^2 a^4)}} \cdot \exp\left(-\frac{2a^2x^2}{a^4+4\hbar^2t^2/m^2}\right).$$

Hint: the energy of a plane wave is given by $E = p^2/(2m)$. Also, for any complex numbers α and β such that $\mathrm{Re}(\alpha) > 0$,

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\alpha(y+\beta)^2}\mathrm{d}y = \sqrt{\frac{\pi}{\alpha}},$$

and $(f(\beta))^* = f^*(\beta) = f(\beta^*)$ when $f : \mathbb{C} \to \mathbb{C}$ is an elementary function, such as $f(z) = 1/z$ or $f(z) = \exp(z)$.

(b) (2p) Sketch qualitatively the probability density $|\psi(x,t)|^2$ for some times $t = t_1$ and $t = t_2 > t_1$. What happens to the wave function over time?