PHYS-C0252 - Quantum Mechanics

Exercise set 5 - model solutions

Due date: May 29, 2024 by 23:59 on MyCourses
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You can write by hand and take pictures, use digital note-taking or LaTeX etc.

1. Consider a particle with mass m approaching a finite potential barrier with height V_B and width a as discussed on lecture 9:

$$V(x) = \begin{cases} 0 & x < 0, \\ V_B & 0 < x < a, \\ 0 & a < x. \end{cases}$$

On the lecture, we discussed the wave function in the case where the energy is below the barrier, $E < V_B$. Let us now consider the $E > V_B$ case, where the solution to the time-independent Schrödinger equation is the wave function

$$\psi(x) = \begin{cases} A_I e^{ikx} + A_R e^{-ikx} & x < 0, \\ B e^{ik_B x} + B' e^{-ik_B x} & 0 < x < a, \\ A_T e^{ikx} & a < x, \end{cases}$$

where

$$k = \sqrt{\frac{2mE}{\hbar^2}}, \qquad k_B = \sqrt{\frac{2m(E - V_B)}{\hbar^2}}.$$

(a) (4p) Show that the transmission probability is given by

$$T = \left|\frac{A_T}{A_I}\right|^2 = \frac{1}{\cos^2(k_B a) + \left(\frac{k^2+k_B^2}{2kk_B}\right)^2 \sin^2(k_B a)}$$
$$= \frac{1}{1 + \left(\frac{k^2-k_B^2}{2kk_B}\right)^2 \sin^2(k_B a)}$$

(either of the forms on the RHS is OK). Hint: use the continuity of $\psi(x)$ and $\frac{d}{dx}\psi(x)$ at the boundaries $x = 0$ and $x = a$, Euler's formula

$$e^{i\phi} = \cos(\phi) + i\sin(\phi),$$

and $|e^{i\phi}| = 1$ for $\phi \in \mathbb{R}$.

(b) (2p) Sketch T as a function of the barrier width a. What is the mathematical condition to get full transmission $T = 1$? What is the physical explanation for full transmission when this condition is satisfied?

Solution:

(a) As seen in the lecture notes, $\psi(x)$ and $\psi'(x)$ must be continuous at $x = 0$ and $x = a$. From this, we get a set of equations

$$\psi(0) = A_I + A_R = B + B' \tag{1}$$
$$\psi(a) = Be^{ik_B a} + B'e^{-ik_B a} = A_T e^{ika} \tag{2}$$
$$\psi'(0) = ik(A_I - A_R) = ik_B(B - B') \tag{3}$$
$$\psi'(a) = ik_B(Be^{ik_B a} - B'e^{-ik_B a}) = ikA_T e^{ika}. \tag{4}$$

Dividing both sides of Eq. (4) by ik_B and taking the sum and difference of Eqs. (2) and (4), we find

$$2Be^{ik_B a} = \left(1 + \frac{k}{k_B}\right) A_T e^{ika} \quad \Rightarrow \quad B = \frac{1}{2}\left(1 + \frac{k}{k_B}\right) A_T e^{ika} e^{-ik_B a}, \qquad (2) + (4)$$
$$2B'e^{-ik_B a} = \left(1 - \frac{k}{k_B}\right) A_T e^{ika} \quad \Rightarrow \quad B' = \frac{1}{2}\left(1 - \frac{k}{k_B}\right) A_T e^{ika} e^{+ik_B a}. \qquad (2) - (4)$$

Taking the sum and difference of these, we find

$$\begin{aligned}
B + B' &= \frac{1}{2}A_T e^{ika}\left[\left(1 + \frac{k}{k_B}\right) e^{-ik_B a} + \left(1 - \frac{k}{k_B}\right) e^{+ik_B a}\right] \\
&= \frac{1}{2}A_T e^{ika}\left[e^{-ik_B a} + e^{+ik_B a} + \frac{k}{k_B}\left(e^{-ik_B a} - e^{+ik_B a}\right)\right] \\
&= A_T e^{ika}\left(\cos(k_B a) - \frac{k}{k_B} i\sin(k_B a)\right), \\
B - B' &= \frac{1}{2}A_T e^{ika}\left[\left(1 + \frac{k}{k_B}\right) e^{-ik_B a} - \left(1 - \frac{k}{k_B}\right) e^{+ik_B a}\right] \\
&= \frac{1}{2}A_T e^{ika}\left[e^{-ik_B a} - e^{+ik_B a} + \frac{k}{k_B}\left(e^{-ik_B a} + e^{+ik_B a}\right)\right] \\
&= A_T e^{ika}\left(\frac{k}{k_B}\cos(k_B a) - i\sin(k_B a)\right),
\end{aligned}$$

where we have used the relations $e^{ix} + e^{-ix} = 2\cos x$ and $e^{ix} - e^{-ix} = 2i\sin(x)$. Similarly, dividing Eq. (3) by ik and taking the sum of Eqs. (1) and (3), we have

$$2A_I = B + B' + \frac{k_B}{k}(B - B').$$

Plugging in the above result and dividing both sides by $A_T e^{ika}$, we find

$$\begin{aligned}
2\frac{A_I}{A_T}e^{-ika} &= \cos(k_B a) - \frac{k}{k_B} i\sin(k_B a) + \cos(k_B a) - \frac{k_B}{k} i\sin(k_B a) \\
&= 2\cos(k_B a) - i\,\frac{k^2 + k_B^2}{kk_B}\sin(k_B a)
\end{aligned}$$
$$\Rightarrow T = \left|\frac{A_T}{A_I}\right|^2 = \left|\left(\underbrace{\cos(k_B a)}_{x\in\mathbb{R}} - i\,\underbrace{\frac{k^2 + k_B^2}{2kk_B}\sin(k_B a)}_{y\in\mathbb{R}}\right)^{-1}\right|^2.$$

For a complex number $z = x \pm iy$, we know that $|z^{-1}|^2 = |1/z|^2 = 1/|z|^2 = 1/(x^2 + y^2)$, so the above becomes

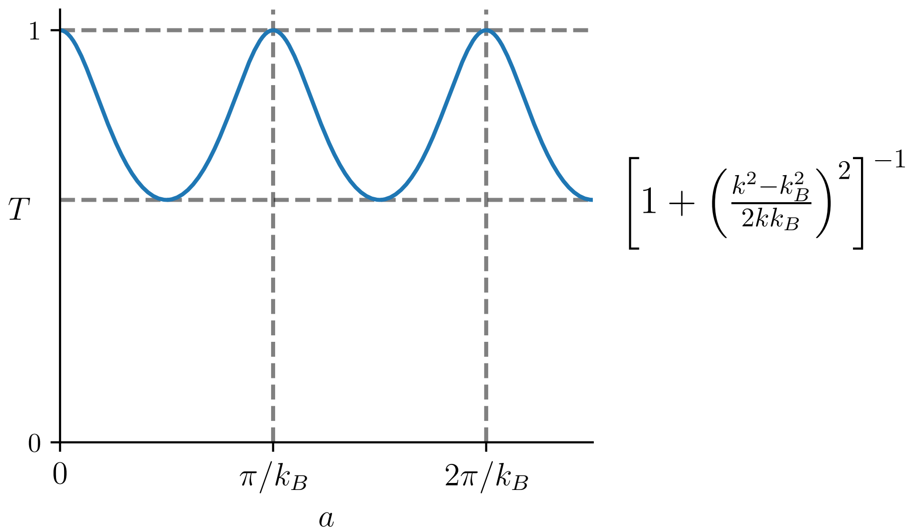
$$T = \frac{1}{\cos^2(k_B a) + \left(\frac{k^2+k_B^2}{2kk_B}\right)^2 \sin^2(k_B a)}.$$

Of course this result can be reached in multiple different ways.

(b) Using the other given form of T,

$$T = \frac{1}{1 + \left(\frac{k^2-k_B^2}{2kk_B}\right)^2 \sin^2(k_B a)},$$

and knowing that $\sin(x) \in [-1, 1] \Rightarrow \sin^2(x) \in [0, 1]$, we see that the transmission coefficient oscillates between 1 and $\left[1 + \left(\frac{k^2-k_B^2}{2kk_B}\right)^2\right]^{-1}$:



The maxima occur when $\sin(k_B a) = 0$, or, $k_B a = n\pi$, with integer n. At the maxima, we have full transmission. The physical interpretation of this is that a wave that travels from $x = 0$ to $x = a$, gets reflected from the right side and arrives back at $x = 0$, has traveled a distance that is an integer multiple of the wavelength. Since the phase of the wave flips as it is reflected at the right side, it will have an overall phase difference of π compared to the waves directly reflected from the left side at $x = 0$. Thus the waves reflected from the right destructively interfere with waves reflected from the left, resulting in zero reflection. This is called *resonance scattering*.

2. Consider a particle of mass m in the one-dimensional potential energy field

$$V(x) = \begin{cases} 0, & \text{if } -\infty < x < -a; \\ -V_0, & \text{if } -a < x < +a; \\ 0, & \text{if } +a < x < -\infty. \end{cases}$$

Since the potential is symmetric about $x = 0$, there are two types of energy eigenfunctions. There are *symmetric* eigenfunctions which obey $\psi(x) = \psi(-x)$, and *anti-symmetric* eigenfunctions which obey $\psi(x) = -\psi(-x)$.

(a) Show, by considering the energy eigenvalue equation in the three regions of x, that a symmetric eigenfunction with energy $E = -\hbar^2\alpha^2/2m$ with $\alpha \in \mathbb{R}$ has the form:

$$\psi(x) = \begin{cases} Ae^{+\alpha x}, & \text{if } -\infty < x < -a; \\ C\cos(k_0 x), & \text{if } -a < x < +a; \\ Ae^{-\alpha x}, & \text{if } +a < x < \infty. \end{cases}$$

where A and C are constants and $k_0 = \sqrt{2m(E+V_0)/\hbar^2}$.

(b) Show that for the symmetric eigenfunction, $\alpha = k_0 \tan(k_0 a)$. Hint: use the continuity of $\psi(x)$ and $\frac{d}{dx}\psi(x)$ at the edges of the potential.

(c) By seeking graphical solutions of the equations

$$\alpha = k_0 \tan(k_0 a) \quad \text{and} \quad \alpha^2 + k_0^2 = w^2,$$

where $w = \sqrt{2mV_0/\hbar^2}$, show that there is one bound state if $0 < w < \pi/a$, and two bound states if $\pi/a < w < 2\pi/a$. Hint: use normalized units $\bar{k}_0 = ak_0/\pi$ and $\bar{\alpha} = a\alpha/\pi$, and plot $\bar{\alpha}$ versus $\bar{k}_0$ when the above equations hold. Note that $k_0 > 0$ and $\alpha > 0$.

Solution:

(a) In the region $x < -a$ the potential is zero, the Schrödinger equation reads

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \quad \text{or} \quad \frac{d^2\psi}{dx^2} = \alpha^2\psi,$$

where $\alpha = \sqrt{-2mE}/\hbar$. The general solution of this differential equation is

$$\psi(x) = Ae^{\alpha x} + Be^{-\alpha x}.$$

Since $e^{-\alpha x} \longrightarrow \infty$ as $x \longrightarrow -\infty$, we consider only the physically admissible solution

$$\psi(x) = Ae^{\alpha x}.$$

Similarly, we can obtain the solution for the region $x > a$:

$$\psi(x) = Be^{-\alpha x}.$$

In the region $-a < x < a$, the potential is $-V_0$, so the Schrödinger equation gives

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - V_0\psi = E\psi \quad \text{or} \quad \frac{d^2\psi}{dx^2} = -k_0^2\psi,$$

where $k_0 = \sqrt{2m(E+V_0)/\hbar^2}$. The general solution is

$$\psi(x) = D\sin k_0 x + C\cos k_0 x.$$

Since we're looking for symmetric $\psi(x)$, but $\sin(x)$ is antisymmetric, we must have $D = 0$. From the boundary conditions and symmetry, we have

$$\begin{aligned} \psi(-a) &= Ae^{-\alpha a} \\ &= \psi(a) = Be^{-\alpha a} \Rightarrow A = B. \end{aligned}$$

In total, the symmetric eigenfunction thus has the form:

$$\psi(x) = \begin{cases} Ae^{+\alpha x}, & \text{if } -\infty < x < -a; \\ C\cos(k_0 x), & \text{if } -a < x < +a; \\ Ae^{-\alpha x}, & \text{if } +a < x < \infty. \end{cases}$$

(b) The continuity of $\psi(x)$ at the edges of the potential implies

$$C\cos(k_0 a) = Ae^{-\alpha a},$$

and the continuity of $d\psi(x)/dx$

$$-Ck_0 \sin(k_0 a) = -\alpha A e^{-\alpha a}.$$

Dividing these two equations, we find

$$\alpha = k_0 \tan(k_0 a)$$

(c) Using the normalized units, we have

$$\begin{aligned} k_0 &= \frac{\pi}{a}\bar{k}_0, \\ \alpha &= \frac{\pi}{a}\bar{\alpha}, \end{aligned}$$

so the given equations become

$$\begin{aligned} \alpha &= \frac{\pi}{a}\bar{k}_0 \tan\left(\frac{\pi}{a}\bar{k}_0 \cdot a\right) = \frac{\pi}{a}\bar{k}_0 \tan\left(\pi\bar{k}_0\right) \\ \Rightarrow \frac{a}{\pi}\alpha &= \bar{\alpha} = \bar{k}_0 \tan\left(\pi\bar{k}_0\right), \end{aligned}$$

and

$$\bar{\alpha}^2 + \bar{k}_0^2 = \frac{a^2}{\pi^2}w^2.$$

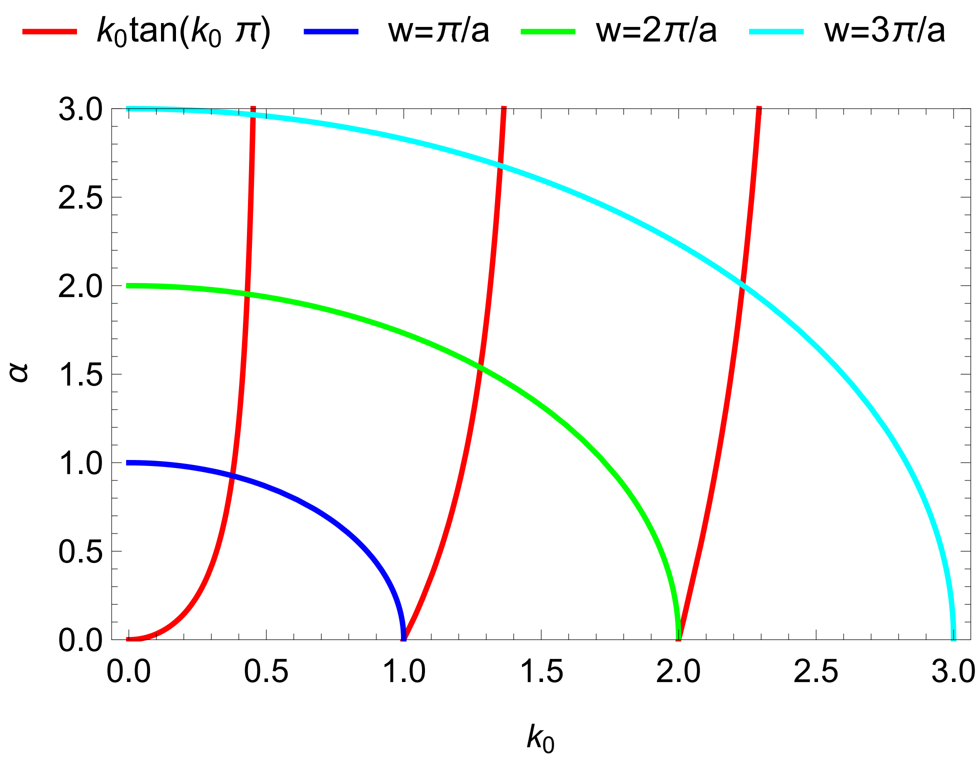
We can then pick w to be at the boundaries of the regions of interest, $w = \pi/a$ and $2\pi/a$, and plot these equations. This is shown in Fig. 1. From the picture, we see that $\bar{k}_0 \tan(\bar{k}_0\pi)$ crosses the solution of $\bar{\alpha}^2 + \bar{k}_0^2$ once when

$$0 < w < \frac{\pi}{a},$$

and twice when

$$\frac{\pi}{a} < w < \frac{2\pi}{a}.$$

Thus there are one and two bound states in these regions, respectively.



k_0 and α in units of π/a

3. Consider a particle in a one-dimensional, infinite potential well, with perturbation Cx, such that

$$V = \begin{cases} Cx, & 0 < x < L \\ \infty, & \text{everywhere else.} \end{cases}$$

Using the fact that the eigenstates and corresponding energies of the unperturbed Hamiltonian are

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right), \quad E_n = \frac{n^2\pi^2\hbar^2}{2mL^2},$$

find the first order corrections to the first ($n = 1$) eigenstate and the corresponding energy using time-independent perturbation theory. Note that $|\langle\psi_m|x|\psi_n\rangle|$ drops rapidly as m becomes larger than n, so you can ignore terms with $m - n > 2$.

Solution:

To use perturbation theory, we want to write the Hamiltonian of the system in the form

$$\hat{H} = \hat{H}_0 + \lambda\hat{H}'.$$

We see that the unperturbed Hamiltonian $\hat{H}_0$ is the Hamiltonian of the infinite square well (whose solution we know), and $\lambda\hat{H}' = C\hat{x}$ is the perturbation, from which we identify that we should write $C = \lambda F$, where λ is the dimensionless "strength" of the perturbation and F has units of force, so that the perturbation Hamiltonian $\hat{H}' = F\hat{x}$

has units of energy. As discussed in the lecture notes, the first order correction to the eigenstate is given by

$$
\begin{aligned}
|\psi_n^1\rangle &= \sum_{m\neq n} \frac{\langle\psi_m^0|\hat{H}'|\psi_n^0\rangle}{E_n^0 - E_m^0}\,|\psi_m^0\rangle = \sum_{m\neq n} \frac{\langle\psi_m^0|F\hat{x}|\psi_n^0\rangle}{E_n^0 - E_m^0}\,|\psi_m^0\rangle \\
&= F\sum_{m\neq n} \frac{1}{E_n^0 - E_m^0}\int \psi_m^*(x)x\psi_n(x)dx\,|\psi_m^0\rangle\,,
\end{aligned}
$$

and the correction to the energy is given by

$$
E_n^1 = \langle\psi_n^0|\hat{H}'|\psi_n^0\rangle = F\int \psi_n^*(x)x\psi_n(x)dx.
$$

As given in the hint, we can drop terms with $m-n>2$, i.e. with $n=1$ we only need to calculate the $m=1$, $m=2$ and $m=3$ terms. For $m=1$, we have

$$
E_1^1 = F\frac{2}{L}\int_0^L x\sin^2\left(\frac{\pi x}{L}\right)dx.
$$

Using $2\sin^2 x = 1-\cos(2x)$, this is

$$
\begin{aligned}
E_1^1 &= \frac{2F}{L}\int_0^L \frac{1}{2}x - \frac{1}{2}x\cos\left(\frac{2\pi x}{L}\right)dx \\
&= \frac{FL}{2} - \frac{F}{L}\int_0^L x\cos\left(\frac{2\pi x}{L}\right)dx.
\end{aligned}
$$

The formula for integration by parts is

$$
\int_a^b f'(x)g(x)dx = f(x)g(x)\Big|_{x=a}^{b} - \int_a^b f(x)g'(x)dx.
$$

With $g(x)=x \Rightarrow g'(x)=1$ and $f'(x)=\cos(2\pi x/L) \Rightarrow f(x)=\frac{L}{2\pi}\sin(2\pi x/L)$, we find

$$
\begin{aligned}
E_1^1 &= \frac{FL}{2} - \frac{F}{L}\underbrace{\left[\frac{L}{2\pi}x\sin\left(\frac{2\pi x}{L}\right)\Bigg|_{x=0}^{L} - \frac{L}{2\pi}\int_0^L \sin\left(\frac{2\pi x}{L}\right)dx\right]}_{=0} \\
&= \frac{FL}{2}.
\end{aligned}
$$

This is the correction to the eigenenergy.

For $m=2$, we have

$$
\int \psi_2^*(x)x\psi_1(x)dx = \frac{2}{L}\int_0^L x\sin\left(\frac{2\pi}{L}x\right)\sin\left(\frac{\pi}{L}x\right)dx. \tag{5}
$$

Integrating by pats with $g(x)=x \Rightarrow g'(x)=1$ and

$$
\begin{aligned}
f'(x) &= \sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{\pi x}{L}\right) = \frac{1}{2}\left[\cos\left(\frac{\pi x}{L}\right) - \cos\left(\frac{3\pi x}{L}\right)\right] \\
\Rightarrow f(x) &= \frac{1}{2}\left[\frac{L}{\pi}\sin\left(\frac{\pi x}{L}\right) - \frac{L}{3\pi}\sin\left(\frac{3\pi x}{L}\right)\right]
\end{aligned}
$$

(using the identity $2\sin(a)\sin(b) = \cos(a-b) - \cos(a+b)$), Eq. (5) becomes

$$\begin{aligned}
&\int \psi_2^*(x) x \psi_1(x) dx \\
&= \underbrace{x \frac{2}{L}\cdot\frac{1}{2}\left[\frac{L}{\pi}\sin\left(\frac{\pi x}{L}\right) - \frac{L}{3\pi}\sin\left(\frac{3\pi x}{L}\right)\right]\Bigg|_{x=0}^{L}}_{=0} - \frac{2}{L}\cdot\frac{1}{2}\int_0^L \frac{L}{\pi}\sin\left(\frac{\pi x}{L}\right) - \frac{L}{3\pi}\sin\left(\frac{3\pi x}{L}\right) dx \\
&= -\frac{1}{\pi}\left[\frac{L}{9\pi}\cos\left(\frac{3\pi x}{L}\right) - \frac{L}{\pi}\cos\left(\frac{\pi x}{L}\right)\right]\Bigg|_{x=0}^{L} \\
&= -\frac{1}{\pi}\left[\frac{8L}{9\pi} + \frac{8L}{9\pi}\right] = -\frac{16L}{9\pi^2}.
\end{aligned}$$

For $m = 3$, we have (again, using integration by parts)

$$\begin{aligned}
&\int \psi_3^*(x) x \psi_1(x) \\
&= \frac{2}{L}\int_0^L x\sin\left(\frac{3\pi x}{L}\right)\sin\left(\frac{\pi x}{L}\right) dx \\
&= \frac{1}{L}\int_0^L x\left[\cos\left(\frac{2\pi x}{L}\right) - \cos\left(\frac{4\pi x}{L}\right)\right] dx \\
&= \underbrace{\frac{1}{L}\cdot x\left[\frac{L}{2\pi}\sin\left(\frac{2\pi x}{L}\right) - \frac{L}{4\pi}\sin\left(\frac{4\pi x}{L}\right)\right]\Bigg|_{x=0}^{L}}_{=0} - \frac{1}{L}\underbrace{\int_0^L \frac{L}{2\pi}\sin\left(\frac{2\pi x}{L}\right) - \frac{L}{4\pi}\sin\left(\frac{4\pi x}{L}\right) dx}_{=0} \\
&= 0,
\end{aligned}$$

where the second term goes to zero because it is an integral over an integer number of periods of a sine. The $m = 3$ term is thus zero, and the total correction to the eigenstates is

$$\begin{aligned}
|\psi_1^1\rangle &= \frac{\langle\psi_2^0|\hat{H}'|\psi_1^0\rangle}{E_1^0 - E_2^0}|\psi_2^0\rangle \\
&= -\frac{16LF}{9\pi^2}\cdot\left(\frac{\pi^2\hbar^2}{2mL^2} - \frac{4\pi^2\hbar^2}{2mL^2}\right)^{-1}|\psi_2^0\rangle = -\frac{16LF}{9\pi^2}\cdot\left(-\frac{3\pi^2\hbar^2}{2mL^2}\right)^{-1}|\psi_2^0\rangle \\
&= \frac{32L^3Fm}{27\pi^4\hbar^2}|\psi_2^0\rangle,
\end{aligned}$$

or equivalently in the position basis

$$\psi_1^1(x) = \frac{32L^3Fm}{27\pi^4\hbar^2}\psi_2^0(x) = \frac{32L^3Fm}{27\pi^4\hbar^2}\sqrt{\frac{2}{L}}\sin\left(\frac{2\pi x}{L}\right).$$

4. Consider the case of two identical coupled harmonic oscillators that could be experimentally realized as two capacitively coupled superconducting resonators at the

same resonance frequency. When the coupling is sufficiently weak, we can treat it as a perturbation. After performing the rotating wave approximation, the approximate Hamiltonian for this system is given by

$$\hat{H} = \hat{H}^0 + \hat{H}' = \hbar\omega\left(\hat{a}^\dagger\hat{a}\otimes\hat{I} + \frac{1}{2}\right) + \hbar\omega\left(\hat{I}\otimes\hat{b}^\dagger\hat{b} + \frac{1}{2}\right) + \hbar g\left(\hat{a}^\dagger\otimes\hat{b} + \hat{a}\otimes\hat{b}^\dagger\right),$$

where the Hamiltonian of the unperturbed system is

$$\hat{H}^0 = \hbar\omega\left(\hat{a}^\dagger\hat{a}\otimes\hat{I} + \frac{1}{2}\right) + \hbar\omega\left(\hat{I}\otimes\hat{b}^\dagger\hat{b} + \frac{1}{2}\right).$$

Here g is a coupling constant with the dimension of frequency, $\hat{a}$ and $\hat{a}^\dagger$ are ladder operators acting on the first harmonic oscillator, and $\hat{b}$ and $\hat{b}^\dagger$ are ladder operators acting on the second harmonic oscillator. The action of these ladder operators on the eigenstates of the unperturbed system $|m\rangle\otimes|n\rangle \equiv |m,n\rangle$, where $m,n\in\{0,1,2,\dots\}$ is defined as usual

$$\begin{aligned}\left(\hat{a}\otimes\hat{I}\right)|m,n\rangle &= \sqrt{m}\,|m-1,n\rangle, &\quad \left(\hat{a}^\dagger\otimes\hat{I}\right)|m,n\rangle &= \sqrt{m+1}\,|m+1,n\rangle\\ \left(\hat{I}\otimes\hat{b}\right)|m,n\rangle &= \sqrt{n}\,|m,n-1\rangle, &\quad \left(\hat{I}\otimes\hat{b}^\dagger\right)|m,n\rangle &= \sqrt{n+1}\,|m,n+1\rangle.\end{aligned}$$

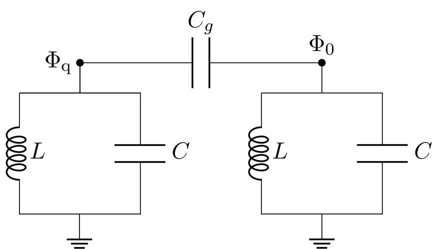
The energy spectrum of $\hat{H}^0$ is then quite similar to the case of a single quantum harmonic oscillator

$$E_{m,n} = \hbar\omega\left(m+n+1\right),$$

where m and n correspond to the excitation level of the first and second harmonic oscillator respectively. The ground state E_0 is non-degenerate with energy $E_{0,0} = \hbar\omega$. However, the second excited state (i.e. $m+n=2$) has a triple degeneracy, with eigenstates $|0,2\rangle$, $|1,1\rangle$, and $|2,0\rangle$ sharing the same energy $E_2 = E_{0,2} = E_{1,1} = E_{2,0} = 3\hbar\omega$.

Using first-order degenerate perturbation theory, show that the degenerate subspace spanned by $|0,2\rangle$, $|1,1\rangle$, and $|2,0\rangle$ splits into three different energy levels, and find the energies of these new levels.

Hint: Start by finding the matrix H'_{ij}, with $i,j\in\{|0,2\rangle,|1,1\rangle,|2,0\rangle\}$. Note that since H is a real Hermitian matrix, $H_{ij} = H_{ji}$, meaning there are only 6 independent elements.



Two harmonic oscillators weakly coupled through a coupling capacitance C_g

Solution: To show that the degenerate subspace splits into three distinct energy levels, we need to show that the perturbation leads to three different energy corrections.

As stated in equation (10.38) of the lecture note, the first-order corrections E_n^1 are the eigenvalues of the matrix given by the matrix elements of H' in the degenerate subspace. Thus, we compute the matrix elements of H' in the degenerate subspace spanned by $|0,2\rangle, |2,0\rangle, |1,1\rangle$. We'll label these state as 02,20,11, respectively, so for instance $H'_{02,20} = \langle 0,2|\hat{H}'|2,0\rangle$.

$$H'_{02,20} = (\langle 0| \otimes \langle 2|)\, \hbar g \left(\hat{a}^\dagger \otimes \hat{b} + \hat{a} \otimes \hat{b}^\dagger\right) | (|2\rangle \otimes |0\rangle) = \langle 0|\hat{a}^\dagger|2\rangle \langle 2|\hat{b}|0\rangle + \langle 0|\hat{a}|2\rangle \langle 2|\hat{b}^\dagger|0\rangle = 0.$$

Since H' is real and Hermitian, we know $H'_{02,20} = H_{20,02} = 0$. All the diagonal elements have the form

$$H'_{ij,ij} = \langle i,j|\hat{a}^\dagger b + \hat{a}\hat{b}^\dagger|i,j\rangle = \langle i|a^\dagger|i\rangle \langle j|b|j\rangle + \langle i|\hat{a}|i\rangle \langle j|b^\dagger|j\rangle \propto \langle i|i+1\rangle \langle j|j-1\rangle + \langle i|i-1\rangle \langle j|j+1\rangle = 0,$$

where we have used the shorthand $\hat{a}\hat{b} := \hat{a} \otimes \hat{b}$ and work with the understanding that $|-1\rangle = 0$.

This leaves only the elements $H'_{02,11}, H'_{20,11}, H'_{11,02}, H'_{11,20}$, and since H' is Hermitian, we only need to evaluate the first two.

$$H'_{11,02} = \hbar g \langle 11|\hat{a}^\dagger \hat{b} + \hat{a}\hat{b}^\dagger|02\rangle = \hbar g \langle 1|\hat{a}^\dagger|0\rangle \langle 1|\hat{b}|2\rangle + \hbar g \langle 1|\hat{a}|0\rangle \langle 1|\hat{b}^\dagger|2\rangle = \hbar g\sqrt{2} + 0 = \hbar g\sqrt{2} = H'_{02,11}.$$
$$H'_{11,20} = \hbar g \langle 11|\hat{a}^\dagger \hat{b} + \hat{a}\hat{b}^\dagger|20\rangle = \hbar g \langle 1|\hat{a}^\dagger|2\rangle \langle 1|\hat{b}|0\rangle + \hbar g \langle 1|\hat{a}|2\rangle \langle 1|\hat{b}^\dagger|0\rangle = 0 + \hbar g\sqrt{2} = \hbar g\sqrt{2} = H'_{20,11}.$$

So we find

$$H' = \hbar g\sqrt{2} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

which has the eigenvalues $E'_1 = 0$, $E'_2 = 2\hbar g$, $E'_3 = -2\hbar g$, meaning the degenerate subspace splits into three distinct energy levels. The energies of these levels are approximately $3\hbar\omega - 2\hbar g$, $3\hbar\omega$, and $3\hbar\omega + 2\hbar g$.