Task 3: The effect of shape

Instructions: Check the following questions and exercises. Read chapters 9 and 10 in the 4th edition of the course textbook about the selection of material and shape.

Criterion: A good report on this task demonstrates a good understanding of:

- the meaning of shape factors, and consequently how to calculate them
- theoretical and practical considerations of how different shapes allow the use of materials efficiently
- how shape factors can be used to take the effect of shape into account in materials selection

The bending stiffness S of a beam is proportional to the product EI:

$$S \propto \frac{EI}{L^3}$$

The second moment of area, I_0, for a reference beam of square section with edge length b_0 and section area $A = b_0^2$ is simply:

$$I_0 = \frac{b_0^4}{12} = \frac{A^2}{12}$$

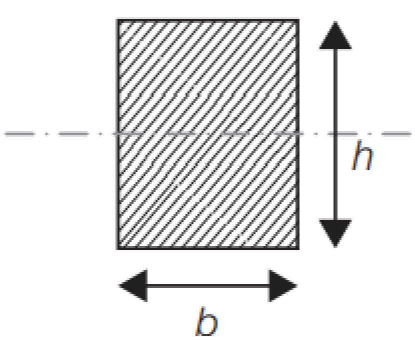
The bending stiffness of the shaped section differs from that of a square one with the same area A by the factor ϕ_B^e where

$$\phi_B^e = \frac{S}{S_0} = \frac{EI}{EI_0} = \frac{12I}{A^2}$$

which is the shape factor for elastic bending. The shape factor is dimensionless. The derivations will be written in Python ipynb for better equations formatting.

Task 3.1.1: Calculate the shape factor for bending stiffness for Glass-reinforced plastic (GRP) bar, when the profile of the bar is:

a) solid square profile, width, and height 100 mm.



where b = 100 and h = 100

Derivation of Moment of Inertia for a Solid Rectangle:

Consider a differential strip of thickness dy at a distance y from the base of the rectangle.

The area of this strip, dA, is $b \times dy$.

The differential moment of inertia of this strip about the centroidal axis is:

$dI = b \times y^2 \times dy$

To find the total moment of inertia of the rectangle, integrate dI over the height of the rectangle:

To find the total moment of inertia of the rectangle, integrate dI over the height of the rectangle:

$I = \int_0^h b \times y^2 \times dy$

$I = b \int_0^h y^2 \, dy$

$I = b \left[\frac{y^3}{3}\right]_0^h$

$I = b \left[\frac{h^3}{3} - 0\right]$

$I = \frac{b \times h^3}{3}$

However, this is the moment of inertia about the base of the rectangle. The parallel axis theorem states:

$I_{\text{centroid}} = I_{\text{base}} - A \times \left(\frac{h}{2}\right)^2$

Where A is the area of the rectangle, $b \times h$

Substituting in the values:

$I_{\text{centroid}} = \frac{b \times h^3}{3} - b \times h \times \left(\frac{h}{2}\right)^2$

$I_{\text{centroid}} = \frac{b \times h^3}{3} - \frac{b \times h^3}{4}$

$I_{\text{centroid}} = \frac{b \times h^3}{12}$

So, the moment of inertia of the rectangle about its centroid is:

$I = \frac{b \times h^3}{12}$

The area of the cross section is:

$$A = b \times b = 100mm \times 100mm = 10000mm^2$$

The moment of inertia is:

$$I = \frac{b \times h^3}{12} = \frac{100 \times 100^3}{12} = 8333333.33mm^4 = 8.33333333 \times 10^{-6}m^4$$

The shape factor is:

$$\phi_B^e = \frac{12I}{A^2} = \frac{12 \times 8333333.33}{10000^2} = 0.9999999996 = 1$$

This is because the shape is a square cross-section area so it is the same as the referenced shape.

The Young's modulus of glass-fibre reinforced bar (GRP) is:

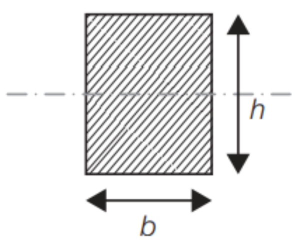
$E = 2.4GPa = 2.4e9Pa$. Let $C = \frac{1}{L^3}$

The effective stiffness of simply supported beam is

$$S_0 = \frac{3EI}{L^3} = \frac{3 \times 2.4e9Pa \times 8.33333333 \times 10^{-6}m^4}{L^3} = C \times 60000N/m$$

The shape factor is 1, so the bending stiffness is the same as the referenced shape: $S = S_0$

b) solid rectangle profile, width 500 mm, height 100 mm.



where b = 500 and h = 100

The area of the cross section is:

$$A = b \times h = 500mm \times 100mm = 50000mm^2$$

The moment of inertia is:

$$I = \frac{b \times h^3}{12} = \frac{500 \times 100^3}{12} = 41666666.66mm^4 = 4.16666667 \times 10^{-5}m^4$$

The shape factor is:

$$\phi_B^e = \frac{12I}{A^2} = \frac{12 \times 41666666.66}{50000^2} = 0.1999 = 0.2$$

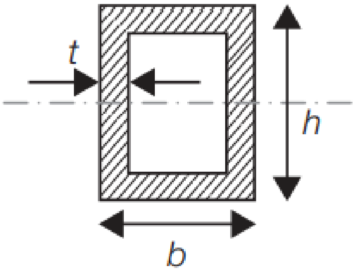
The effective stiffness of simply supported beam is

$$S_0 = \frac{3EI}{L^3} = \frac{3 \times 2.4e9Pa \times 8.33333333 \times 10^{-6}m^4}{L^3} = C \times 60000N/m$$

The shape factor is 0.2, so the bending stiffness of this shape is

$$S = 0.2 \times S_0 = C \times 12000N/m$$

c) hollow square profile, outer width and height 100 mm, wall thickness 10 mm.



where b = 100, h = 100 and t = 10mm

Derivation of Moment of Inertia for a Hollow rectangle Section:

Given a hollow square with outer width b and height h, and thickness t

the inner square has width $b-2t$ and height $h-2t$.

For the outer square:

$I_{\text{outer}} = \frac{b \times h^3}{12}$

For the inner square:

$I_{\text{inner}} = \frac{(b-2t) \times (h-2t)^3}{12}$

The moment of inertia of the hollow section is:

$I_{\text{hollow}} = I_{\text{outer}} - I_{\text{inner}}$

Substituting in the values and simplifying, we get:

$I_{\text{hollow}} = \frac{1}{6}h^3 t(1 + 3\frac{b}{h})$

The area of the cross section is:

$A = 2 \times t(b + h - 2t) = 2 \times 10mm(100mm + 100mm - 2 \times 10mm) = 3600mm^2$

The moment of inertia is:

$I = \frac{1}{6}h^3 t(1 + 3\frac{b}{h}) = \frac{1}{6}(100mm)^3(10mm)(1 + 3\frac{100mm}{100mm}) = 6666666.66mm^4 = 6666666.66 \times 10^{-6}m^4$

The shape factor is:

$\phi_B^e = \frac{12I}{A^2} = \frac{12 \times 6666666.66}{3600^2} = 6.172$

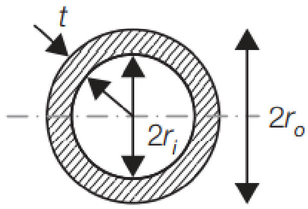
The effective stiffness of simply supported beam is

$S_0 = \frac{3EI}{L^3} = \frac{3 \times 2.4e9Pa \times 8.33333333 \times 10^{-6}m^4}{L^3} = C \times 60000N/m$

The shape factor is 6.172, so the bending stiffness of this shape is

$S = 6.172 \times S_0 = C \times 370370N/m$

d) hollow round pipe, outer diameter 500 mm, wall thickness 10 mm.



where r_i = 240mm, r_o=250mm and t = 10mm

Derivation of Moment of Inertia for a Hollow Circular Cross-Section:

Given a hollow circle with outer radius r_o and inner radius r_i, and thickness $t = r_o - r_i$.

The moment of inertia I of a circle about an axis perpendicular to its plane is:

$I = \frac{\pi \times r^4}{4}$

Using this formula:

For the outer circle:

$I_{\text{outer}} = \frac{\pi \times r_o^4}{4}$

For the inner circle:

$I_{\text{inner}} = \frac{\pi \times r_i^4}{4}$

The moment of inertia of the hollow section is:

$I_{\text{hollow}} = I_{\text{outer}} - I_{\text{inner}}$

Substituting in the values and simplifying, we get:

$I_{\text{hollow}} = \frac{\pi}{4}(r_o^4 - r_i^4)$

The area of the cross section is:

$A = \pi(r_o^2 - r_i^2) = \pi((250mm)^2 - (240mm)^2) = 15393.8mm^2$

The moment of inertia is:

$I = \frac{\pi}{4}(r_o^4 - r_i^4) = \frac{\pi}{4}((250mm)^4 - (240mm)^4) = 462198965.178mm^4 = 4.622 \times 10^{-4}m^4$

The shape factor is:

$\phi_B^e = \frac{12I}{A^2} = \frac{12 \times 462198965.178}{15393.8^2} = 23.4$

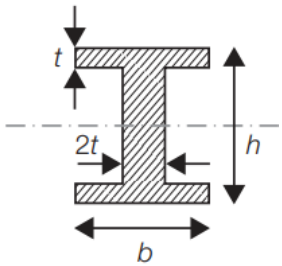
The effective stiffness of simply supported beam is

$S_0 = \frac{3EI}{L^3} = \frac{3 \times 2.4e9Pa \times 8.33333333 \times 10^{-6}m^4}{L^3} = C \times 60000N/m$

The shape factor is 23.4, so the bending stiffness of this shape is

$S = 23.4 \times S_0 = C \times 1404333.8N/m$

e) a spar of the wind turbine blade, whose cross-section is either an I-beam with a width of 500 mm, height of 100 mm, flange thickness of 10 mm, and web thickness of 20 mm, or a box girder with a width of 500 mm, height 100 mm, wall thickness 10 mm



where b = 500mm, h = 100mm and t = 10mm

Derivation of Moment of Inertia for an I-Shaped Cross-Section:

Given an I-beam with total height h, width of the flanges w, flange thickness t, and web thickness $2t$.

The I-beam can be divided into three rectangles:

1. Top flange with dimensions $w \times t$
2. Bottom flange with dimensions $w \times t$
3. Web (middle section) with dimensions $2t \times (h - 2t)$

Using the formula for the moment of inertia of a rectangle:

$I = \frac{b \times h^3}{12}$

For the top flange:

$I_{\text{top flange, centroid}} = \frac{w \times t^3}{12}$

Using the parallel axis theorem:

$I_{\text{top flange}} = I_{\text{top flange, centroid}} + w \times t \times \left(\frac{h-t}{2}\right)^2$

Using the parallel axis theorem:

$I_{\text{top flange}} = I_{\text{top flange, centroid}} + w \times t \times \left(\frac{h-t}{2}\right)^2$

For the bottom flange:

$I_{\text{bottom flange, centroid}} = \frac{w \times t^3}{12}$

Using the parallel axis theorem:

$I_{\text{bottom flange}} = I_{\text{bottom flange, centroid}} + w \times t \times \left(\frac{h-t}{2}\right)^2$

For the web:

$I_{\text{web}} = \frac{2t \times (h-2t)^3}{12}$

The total moment of inertia of the I-beam is:

$I_{\text{total}} = I_{\text{top flange}} + I_{\text{bottom flange}} + I_{\text{web}} = \frac{1}{6}h^3 t(1 + 3\frac{b}{h})$

The area of the cross section is:

$$A = 2t(b + h - 2t) = 2 \times 10mm(500mm + 100mm - 2 \times 10mm) = 11600mm^2$$

The moment of inertia is:

$$I = \frac{1}{6}h^3t(1 + 3\frac{b}{h}) = \frac{1}{6}(100mm)^3(10mm)(1 + 3\frac{500mm}{100mm}) = 26666666.66mm^4 = 2.66666667 \times 10^{-5}m^4$$

The shape factor is:

$$\phi_B^e = \frac{12I}{A^2} = \frac{12 \times 26666666.66}{11600^2} = 2.37$$

The effective stiffness of simply supported beam is

$$S_0 = \frac{3EI}{L^3} = \frac{3 \times 2.4e9Pa \times 8.33333333 \times 10^{-6}m^4}{L^3} = C \times 60000N/m$$

The shape factor is 2.37, so the bending stiffness of this shape is

$$S = 2.37 \times S_0 = C \times 142200N/m$$

Task 3.1.2: First, **apply all 5 profiles** to the "level 2" bending stiffness map for the **GRP bar**. Then, **select 2 profiles** with **2 different materials** (e.g., Carbon-fiber-reinforced polymers (CFRP) and 6061 T6 aluminum), and **add** these **4** records to the GRP bending stiffness map. **Compare** the overall **9** records among these **3** materials, and **discuss** your observations regarding the locus of these points on the selection chart, preferably with figures and tables.

Hint: Find the properties of each material, apply the calculated factors, and use the "Add records" tool (*Tools – Add records*) to construct the map.

Materials	Reference shape (Profile a) $\phi_B^e = 1$	Profile b $\phi_B^e = 0.2$	Profile c $\phi_B^e = 5$	Profile d $\phi_B^e = 23.87$	Profile e $\phi_B^e = 2.22$
GRP					
CFRP					
6061 T6					

Task 3.2.1: **Calculate** the theoretical maximum shape factor for a low alloy steel, aluminum, CFRP, and wood, according to equation 9.15a from the textbook (4th edition): $(\varphi_B^e)_{max} \approx 2.3\left(\frac{E}{\sigma_f}\right)^{0.5}$.

Compare your results to the empirical maximum shape factors found in Table 9.4 of the textbook (4th edition) reproduced below. **Discuss** the extent to which theoretical and empirical results agree or differ.

Material	$(\varphi_B^e)_{max}$
Steel	65
6061 aluminum	44
CFRP	39
Polymer	12
Wood	5
Elastomers	<6

The properties of the four materials (maximum values) are:

- Low alloy steel: Young's modulus of 210 GPa and σ_f of 1800 MPa
- 6061 Aluminum: Young's modulus of 100 GPa and σ_f of 324 MPa
- CFRP: Young's modulus of 150 GPa and σ_f of 1050 MPa
- Wood: Young's modulus of 8 GPa and σ_f of 42.1 MPa

The theoretical maximum shape factor for the four materials are calculated from the formula

- Low alloy steel: $(\phi_B^e)_{max}$ = 24.84
- Aluminum: $(\phi_B^e)_{max}$ = 40.4
- CFRP: $(\phi_B^e)_{max}$ = 27.49
- Wood: $(\phi_B^e)_{max}$ = 31.7

We can see that generally, steels and polymers' maximum theoretical shape factor obeys the empirical data. However, for some natural materials such as wood, these two maximum shape factors do not agree with each other.

Task 3.2.2: Which of these materials, structural steel, aluminum, wood, or CFRP, would be the best choice for a bending beam? The beam must have a stiffness S = EI_{max}> 10^7 Nm^2 and the beam has to be as light as possible. Use the following shape factors in your solution: steel (I-beam) 15, aluminum (I-beam) 10, CFRP (tube) 10, and wood (beam) 2. Resolve the problem using the 4-field method, and present your answers together with the decision-making process with text and figures.

Hint: You do not need to draw the 4-field map by yourself, you can use the one from the book.

This exercise can be done by matching the requirements line to the correct contour. In the figure below, the blue square is for steel, red for aluminum alloys, purple for CFRP and green for soft and hard woods.

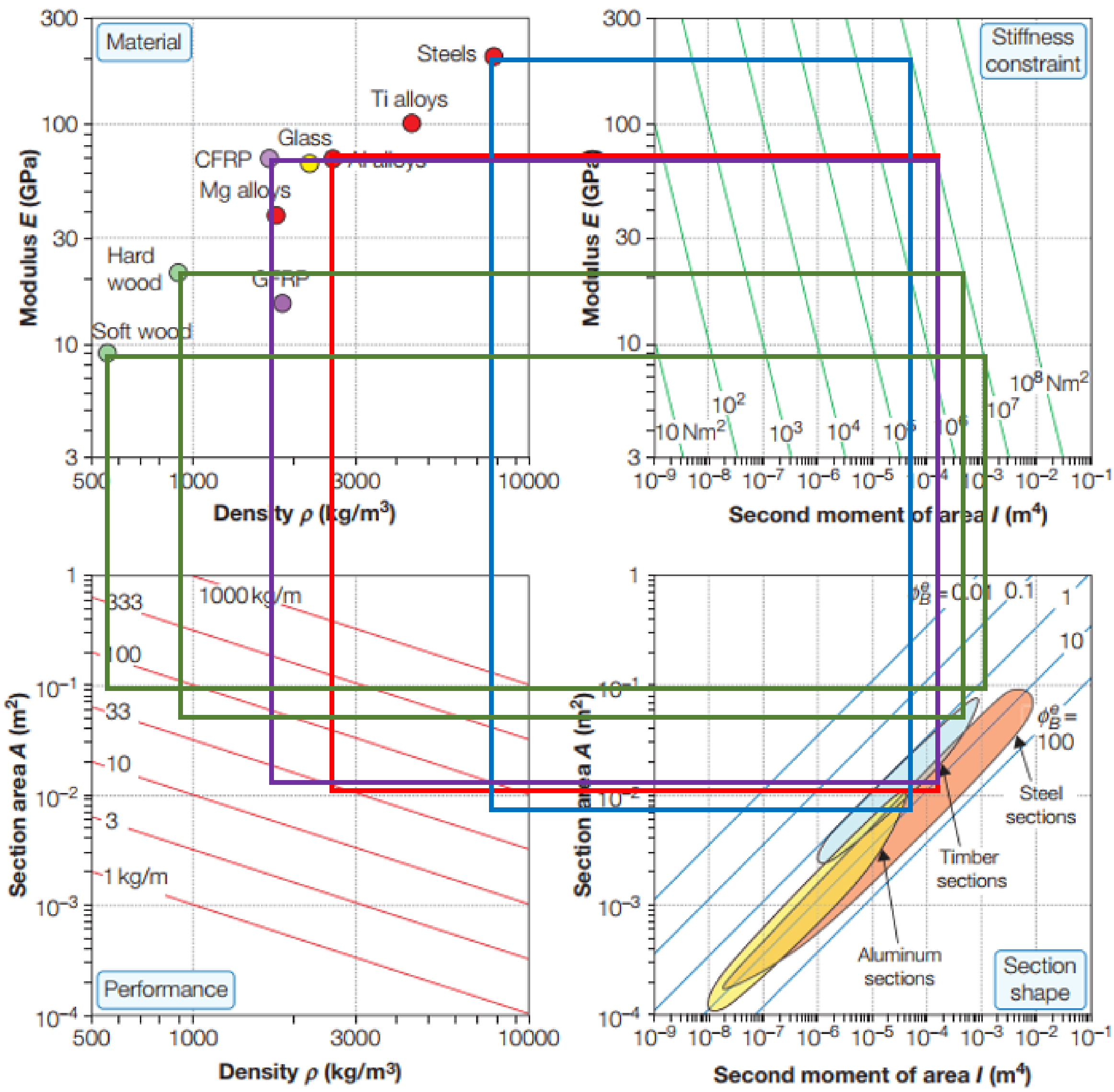


FIGURE 9.9

The 4-quadrant chart assembly for exploring structural sections for stiffness-limited design during bending. Each chart shares its axes with its neighbors.

We can see that the material performance of the purple box reaches the lowest mass in the third quadrant. Therefore, CFRP is the best material choice for the bending beam.