GDL: homework 4

Some exercises can be very difficult compared to others! Try out the easiest ones first, and the ones about the course (Sec 2.1). The level of difficulty is noted as easy: 😍 , medium: 🙂 , or more difficult: 🙃 . Most importantly, have fun!

1 Topology and Manifold (2 points)

1.1 Convex Cone Structure of SPD Matrices

1. 🙂 Show that the set of $n \times n$ symmetric positive-definite matrices $\mathcal{S}^n_{++}$ is a smooth manifold. (Hint: a vector space is always a smooth manifold)

2. 🙂 Show that $\mathcal{S}^n_{++}$ form a convex cone (Hint: Look at the definition of convex cone)

3. 🙃 Visualizing the $\mathcal{S}^n_{++}$ cone for $n = 2$. For $n = 2$, represent the 2×2 SDP matrices as points in $\mathbb{R}^3$. Identify and sketch the cone structure of this set.

2 Riemannian metric, distance and geodesics (5 points)

2.1 About the course (3 points)

2.1.1 The indicatrices

We saw that the metric tensor G can be visualised through the notion of indicatrices. For a specific point $x \in \mathcal{M}$, it is defined as the set of all the vectors $v \in \mathcal{T}_x\mathcal{M}$, such that the induced Riemannian norm is equal to 1:

$$\mathcal{I}_G(x) = \{v \in \mathcal{T}_x\mathcal{M} \quad | \quad \|v\|_G = 1\}$$

1. 😍 When the metric used is Euclidean, the indicatrices look like circles. Why?

2. 🙂 When the metric is Riemannian, the indicatrices look like ellipses. How are the eigenvalues of the metric tensor G related to the semi-axis of those ellipses?

3. 🙃 The volume measure of a Riemannian metric is given by $\det\{G\}$. How is this related to the volume of the indicatrix?

2.1.2 The pullback metric

Let $\mathcal{M}$ and $\mathcal{N}$ be two smooth manifolds, and $f : \mathcal{M} \to \mathcal{N}$ an immersion. If $\mathcal{N}$ is equipped with a Riemannian metric g_N, and J is the Jacobian of f, then the **pullback metric** g_M on $\mathcal{M}$ is defined as:

$$g_M(u, v) = g_N(Ju, Jv), \quad \forall (u, v) \in \mathcal{T}_x\mathcal{M}.$$

🙂 Why does f needs to be an immersion? (Hint: why its derivatives need to be injective?)

2.1.3 Length and reparametrisation

If φ is an diffeomorphism mapping one basis to another such that $\gamma' = \gamma \circ \varphi$, then $\mathcal{L}(\gamma) = \mathcal{L}(\gamma')$. The length is invariant under reparametrisation.
😍 Try to prove it! (Hint: use the length functional formula: $\mathcal{L}(\gamma) = \int_a^b \|\dot{\gamma}(t)\|_G \, \mathrm{d}t$)

2.2 Some Riemannian metrics (2 points)

2.2.1 2D-Sphere and great circles

We consider a sphere parameterized with $f(\theta, \varphi) = r(\cos(\theta)\cos(\varphi), \cos(\theta)\sin(\varphi), \sin(\theta))$ with θ the polar angle and φ the azimuth angle.

1. 😍 Compute the Riemannian metric tensor G. Copy paste your result in the notebook "riemannian-metric.ipynb" and run it.

2. 🙃 Notice that the geodesics are always the great circles on the sphere! Why is that? (Bonus: Can you prove it?)

2.2.2 Metric in $\mathcal{S}_{++}^n$

Let $\mathcal{S}_{++}^n$ denote the manifold of $n \times n$ symmetric positive-definite (SPD) matrices. For any point $P \in \mathcal{S}_{++}^n$, the tangent space at P, denoted $T_P\mathcal{S}_{++}^n$, consists of all symmetric matrices. This manifold can be equipped with the following metric, defined as follows:
For any $X, Y \in T_P\mathcal{S}_{++}^n$,

$$\langle X, Y \rangle_P = \mathrm{Tr}\left(P^{-1}XP^{-1}Y\right),$$

The metric is named affine-invariant if it is invariant under affine transformations of the form $\phi_G : P \mapsto GPG^\top$, where $G \in GL(n)$ is an invertible matrix.

1. 😍 Show that $\langle X, Y \rangle_P = \mathrm{Tr}\left(P^{-1}XP^{-1}Y\right)$ define a Riemannian metric.

2. 🙃 Why the transformation $P \mapsto GPG^\top$ is said to be affine invariant?

3. 🙂 Show that $\langle X, Y \rangle_P$ is affine-invariant. (Hint: show that $\langle X', Y' \rangle_{\phi_G(P)} = \langle X, Y \rangle_P, \quad$ for all $X, Y \in T_P\mathcal{S}_{++}^n$.)

2.3 (Bonus) Geodesic in $\mathcal{S}_{++}^n$ (2 points)

The aim of this exercise is to use a method called **shooting geodesics** with the affine-invariant metric described above.
The shooting geodesic method computes a geodesic using the exponential map. Given a Riemannian manifold $(\mathcal{M}, g)$, a starting point $p \in \mathcal{M}$, and an initial tangent vector $v \in T_p\mathcal{M}$, the geodesic $\gamma(t)$ is defined as the curve satisfying $\gamma(0) = p$ and $\dot{\gamma}(0) = v$, where $\dot{\gamma}(t)$ is the velocity of the curve. Using the exponential map $\exp_p : T_p\mathcal{M} \to \mathcal{M}$, the geodesic can be expressed as $\gamma(t) = \exp_p(tv)$, where $t \in \mathbb{R}$ is the parameter controlling the "time" along the geodesic. The exponential map ensures that the geodesic is the curve of shortest distance in the local neighborhood of p and satisfies the geodesic equation $\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t) = 0$.
We have two matrices $P, Q \in \mathcal{S}_{++}^n$. The geodesic $\gamma(t)$ connecting P and Q under this metric is given by:

$$\gamma(t) = P^{1/2}\left(P^{-1/2}QP^{-1/2}\right)^t P^{1/2}, \quad t \in [0, 1].$$

1. 🙃 Utilize the affine invariance of the metric to transform P to the identity matrix. (Hint: Show that $I = \phi_A(P)$, with A to define). Use the same transformation for Q, such that $\tilde{Q} = \phi_A(Q)$. Show that finding the expression of the geodesic from I to $\tilde{Q}$ is equivalent to find the geodesic from P to Q, up to some transformation.

2. 🙂 On the manifold $\mathcal{S}^n_{++}$, the geodesic starting at I in the direction of a symmetric matrix V is given by $\gamma(t) = \exp(tV)$. Using the boundary condition, express V in function of $\tilde{Q}$. Show that $\tilde{\gamma}(t) = \tilde{Q}^t$.

3. 🙂 Express γ, the geodesic from P to Q, in function of $\tilde{\gamma}$. Verify that $\gamma(t) = P^{1/2}\left(P^{-1/2}QP^{-1/2}\right)^t P^{1/2}$ with $t \in [0,1]$ and that it connects P and Q.

4. 🙃 Based on this geodesic, how would you define the mean between two SPD matrices? Do you know other mean, and how would you compare them?

3 Curvatures of manifolds (3 points)

Disclaimer: Those exercises can be very boring, but you need to manipulate variables with the Einstein summation at least once in your life. You will not regret this!

3.1 Christoffel symbols of the Sphere (1 points)

In the course, we have seen that the Christoffel symbols are the coefficients of the connection, such that for an orthonormal basis $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ we have: $\nabla_i \mathbf{e}_j = \Gamma^k_{ij}\mathbf{e}_k$. Equivalently, the covariant derivative of $\mathbf{u}$ along $\mathbf{v}$ is defined as: $\nabla_{\mathbf{v}}\mathbf{u} = v^j \partial_j u^i \mathbf{e}_i + u^i v^j \Gamma^k_{ij}\mathbf{e}_k$.

The Christoffel symbols can expressed using the Einstein notation as the derivative of the metric tensor:

$$\Gamma^k_{ij} = \frac{1}{2} g^{km}\left(\partial_i g_{mj} + \partial_j g_{mi} - \partial_m g_{ij}\right),$$

with $g_{ij} = \langle \mathbf{e}_i, \mathbf{e}_j \rangle_G$, and g^{km} being the km-th component of the inverse of the metric tensor G^{-1}.
This time, we work with the **2-sphere** is parameterised with: $f(u,v) = r(\sin(u)\cos(v), \sin(u)\sin(v), \cos(u))$, with u the polar angle and v the azimutal angle.
The Riemannian metric tensor is given by $\mathbf{g}$:

$$\mathbf{G} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2(u) \end{pmatrix}$$

1. 🙃 How many Christoffel symbols should we compute? (Hint: i,j,k are dummies index and can take the value of either u, the polar angle, or v, the azimutal angle)

2. 🙂 Show that you only need to compute $\Gamma^u_{uu}, \Gamma^u_{vv}, \Gamma^u_{uv}$, $\Gamma^v_{uu}, \Gamma^v_{vv}, \Gamma^v_{uv}$. (Hint: use some symmetry properties)

3. 😍 Compute all the partial derivatives of the Riemannian metric. (Hint: Show that $\partial_u g_{uv} = \partial_v g_{uv} = \partial_v g_{uu} = \partial_v g_{vv} = 0$, and you only need to compute $\partial_u g_{vv}$).

4. 😍 Verify that the Christoffel symbols are: $\Gamma^u_{uu} = 0$, $\Gamma^u_{uv} = 0$ and $\Gamma^u_{vv} = -\cos(u)\sin(u)$.

5. 😍 Similarly: $\Gamma^v_{uu} = 0$, $\Gamma^v_{vv} = 0$, $\Gamma^v_{uv} = \frac{\cos(u)}{\sin(u)} = \tan(u)^{-1}$.

   Finally, you should have:

$$\Gamma^u_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & -\cos(u)\sin(u) \end{pmatrix} \quad \text{and} \quad \Gamma^v_{ij} = \begin{pmatrix} 0 & \tan^{-1}(u) \\ \tan^{-1}(u) & 0 \end{pmatrix}.$$

3.2 Riemannian curvature of the Sphere (1 points)

The Riemann curvature tensor of the second kind can be represented by:

$$R^i_{jkm} = \partial_k \Gamma^i_{jm} - \partial_m \Gamma^i_{jk} + \Gamma^i_{rk}\Gamma^r_{jm} - \Gamma^i_{rm}\Gamma^r_{jk}.$$

Identically, we can also compute the Riemann tensor of the first kind with:

$$R_{ijkm} = g_{ir} R^{i}_{rkm}.$$

We know that the Riemann tensor satisfies many identities. It is first skew symmetry ($R_{ijkl} = -R_{jikl}$), second skew symmetry ($R_{ijkl} = -R_{ijlk}$) and block symmetry ($R_{ijkl} = R_{klij}$).
For a n-dimensional surface, the Riemannian tensors would have 4^n components. Using those symmetries, it turns out that Riemannian tensors only have $\frac{1}{12}n^2(n^2-1)$ components, and in our case, $n = 2$, we only have one non-trivial component of the Riemannian tensor.
By symmetry: $R_{uvuv} = -R_{uvvu} = -R_{vuuv} = R_{vuvu}$ and $R_{uvuv} = g_{ur} R^{u}_{ruv}$.
Use the Christoffel symbols for the sphere from the previous example to show that:

1. 🙂 Show that the First kinds are: $R_{uvuv} = -R_{uvvu} = -R_{vuuv} = R_{vuvu} = r^2 \sin^2(u)$.
2. 🙂 Show that the Second kinds are: $R^{u}_{vuv} = -R^{u}_{vvu} = \sin^2(u)$, $R^{v}_{uvu} = -R^{v}_{uuv} = 1$.

Those results can be difficult to interpret on their own! Don't worry if you don't understand what's going on.

3.3 Ricci, Scalar and Sectional curvature of the Sphere (2 points)

The Ricci curvature tensor Rc is obtained by contracting two indices in the Riemann curvature tensor: $\mathrm{Rc}_{ij} = R^{r}_{irj}$. The scalar curvature Sc is the trace of the Ricci curvature with the metric: $\mathrm{Sc} = g^{ij}\mathrm{Rc}_{ij}$. The sectional curvature is directly defined by the Riemann curvature: $K(u, v) = \frac{R_{uvuv}}{\det(G)}$.
Using those expressions and previous results, show that:

1. 🙂 Show that the Ricci curvature is:
$$\mathrm{Rc} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2(u) \end{pmatrix}$$
2. 🙂 Show that the scalar curvature is: $\mathrm{Sc} = g^{uu}\mathrm{Rc}_{uu} + g^{vv}\mathrm{Rc}_{vv} = \frac{2}{r^2}$.
3. 🙂 Show that the sectional curvature is $K = \frac{1}{r^2}$. What do you notice when the radius of the Sphere go to infinity?

3.4 (Bonus) The Gauss Bonnet theorem (1 point)

The geometry of a manifold at every point can give us some hints about its topological properties. In particular, The Euler characteristic is a topological invariant that provides information about the topology of a manifold. When the surface is triangulated, the Euler characteristic is obtained with: $\chi = V - E + F$, with E the number of edges, V the number of vertices and F the number of faces. This Euler characteristic is closely linked to the curvature of the manifold, and this is shown via the Gauss-Bonnet Theorem!

Gauss-Bonnet Theorem: Let $(\mathcal{M}, g)$ be a 2-dimensional Riemannian manifold. The integral of the Gauss curvature, noted K, over the the manifold $\mathcal{M}$ is related to the Euler characteristic χ:

$$\int_{\mathcal{M}} K dA = 2\pi\chi$$

- 😍 What is the Euler Characteristic of the Sphere?
- 😍 For 2d surfaces (only!), the Gauss curvature is equal to the sectional curvature. Using the Gauss-Bonnet theorem, can you deduce what is the surface of a sphere? (Hint: $S = \int_{\mathcal{M}} dA$)?