Aalto-yliopisto
Aalto-universitetet
Aalto University

COE-C1001: Statics

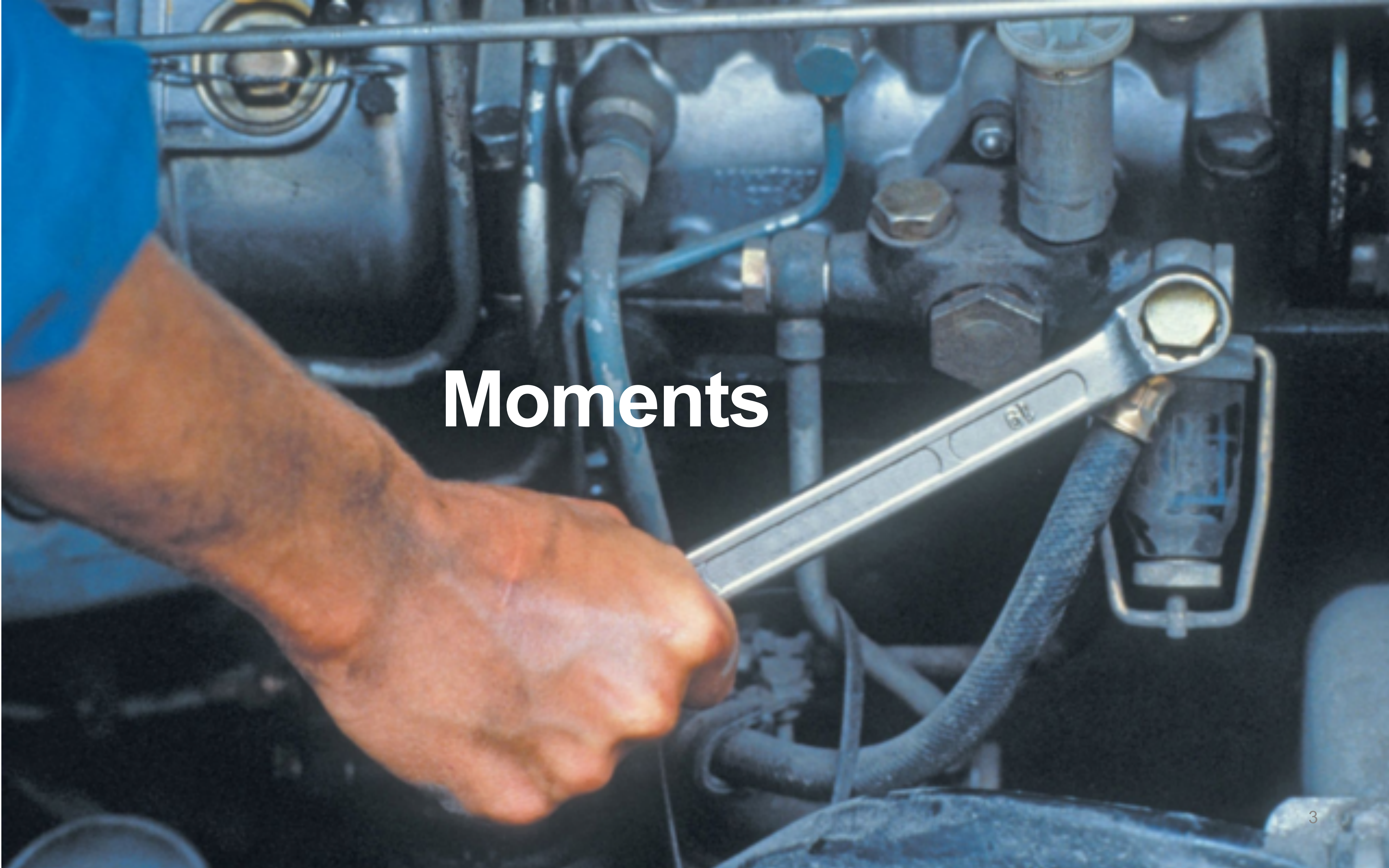
2. Moments, equivalent systems and distributed loading

Luc St-Pierre

Learning outcomes

After this lecture, you should be able to:

- Calculate the moment created by a force,
- Simplify a system with multiple forces and moments into a single resultant force and moment,
- Represent distributed loads into a resultant force.



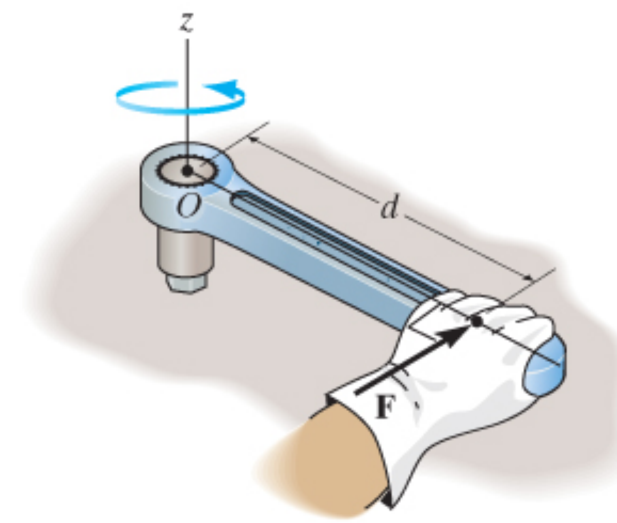
Moments

Moment of a force

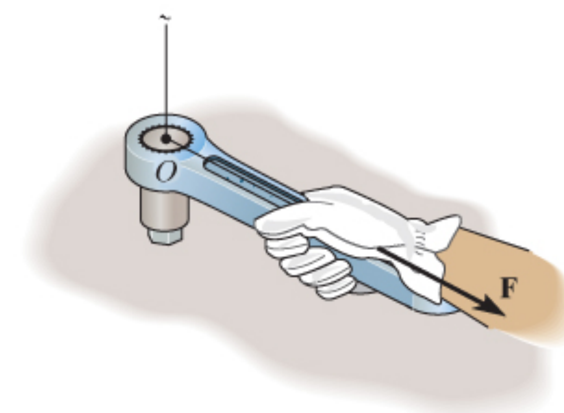
- When a force is applied to a rigid body it often produces a rotation.

- This tendency to rotate is called a **moment** (or torque).

- The magnitude of a moment is linearly proportional to:
    - The magnitude of the force F, and
    - The distance d between the force and the centre of rotation.



The situation above creates a moment, but below, the moment is zero (because d=0).



Moment: simple definition

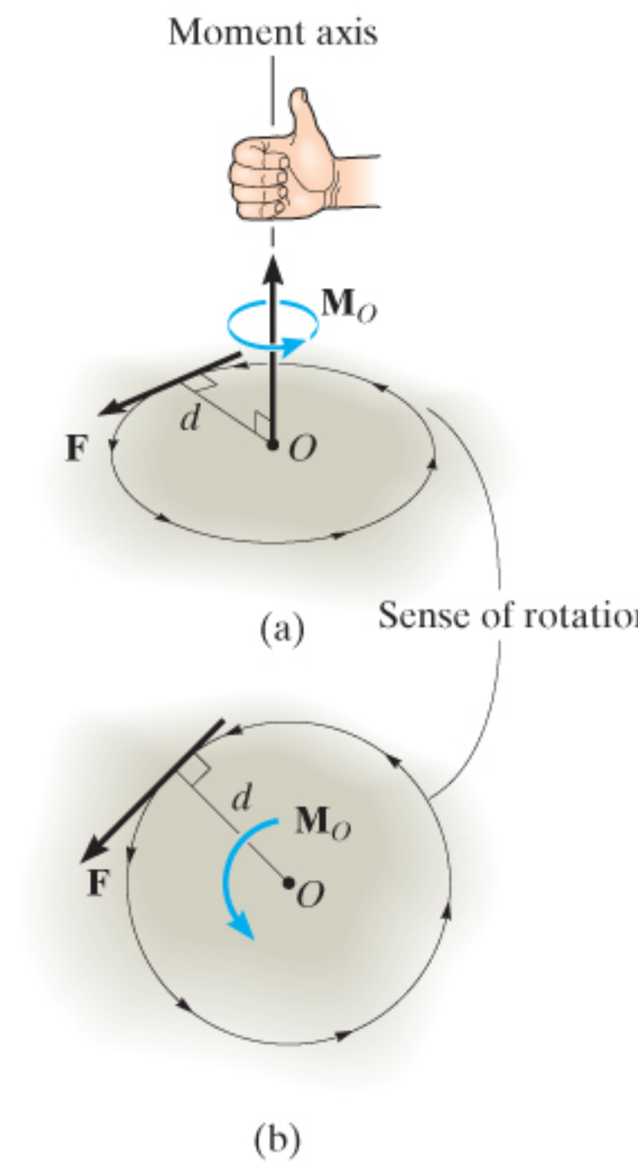
Consider a force $\mathbf{F}$ and the point *O* as shown here. The moment about point *O* is a **vector**, and its magnitude is:

$$M_0 = Fd$$

Where *d* is the perpendicular distance from point *O* to the line of action of the force.

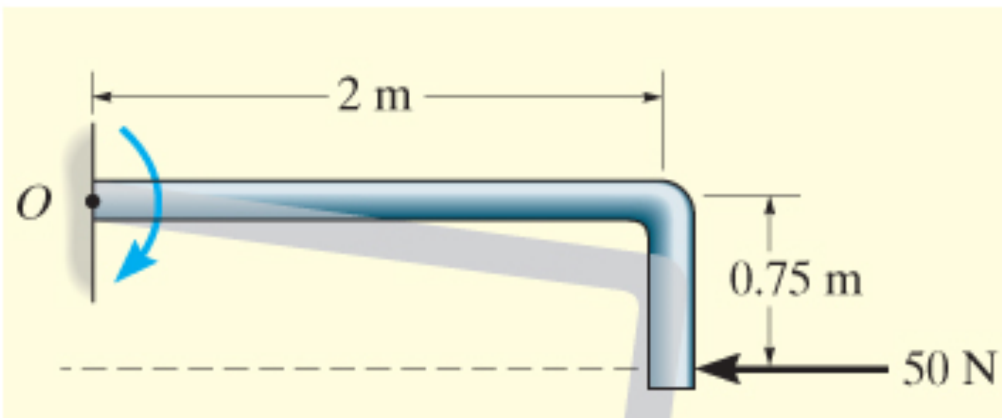
A **moment** has **units** of **N·m**.

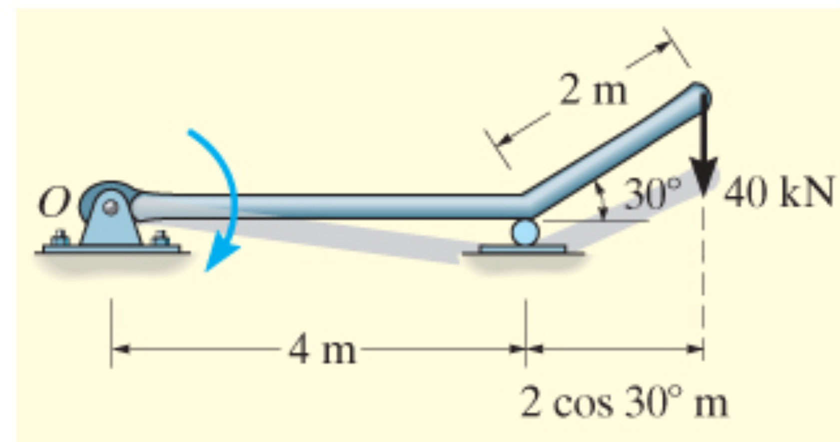
The direction of $\mathbf{M_0}$ is given by the right hand rule (fingers curling in the sense of rotation).



Examples

What is the perpendicular distance d to use when calculating the magnitude of the moment?





$d = 0.75 \text{ m}$

$d = 4 + 2\cos 30° = 5.73\text{m}$

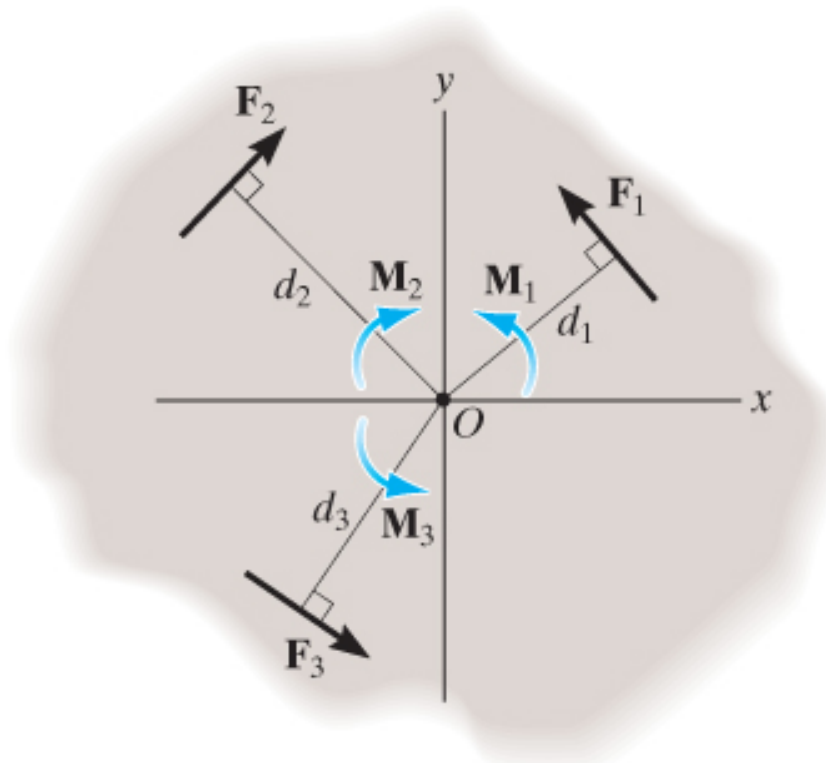
Resultant moment in 2D

For 2D problems, all moments are collinear (all pointing in the z-axis).

The resultant moment is therefore the algebraic sum of all moments:

$$\circlearrowleft + \qquad (M_R)_O = \Sigma F_i d_i$$

$$(M_R)_O = F_1 d_1 - F_2 d_2 + F_3 d_3$$

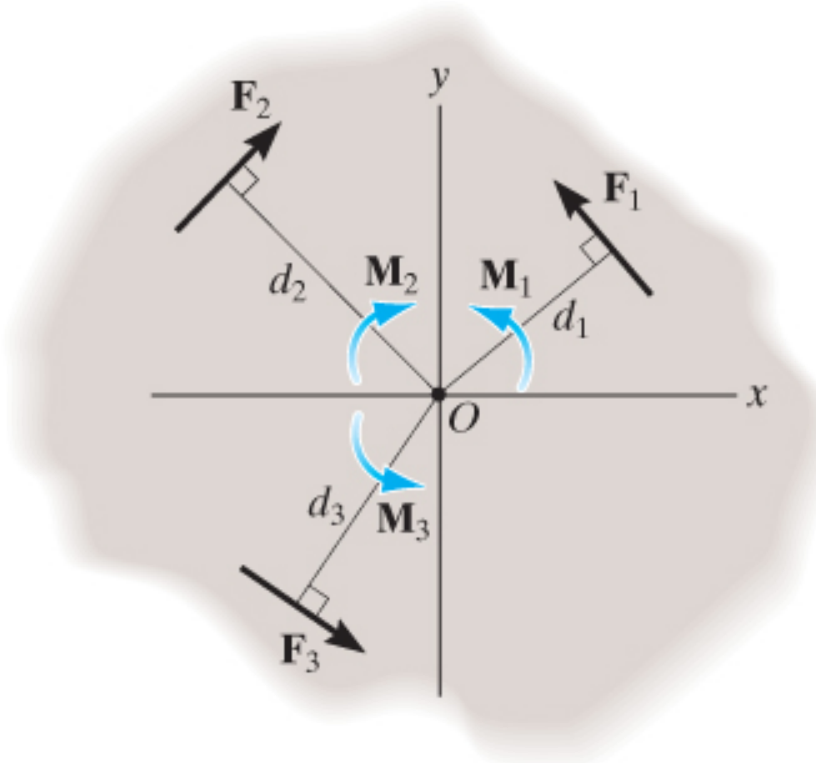


(right hand rule: positive moments are counterclockwise here)

Resultant moment in 2D

Find the resultant moment with these values:

$F_1 = F_3 = 100 \text{ N}, F_2 = 200 \text{ N},$
$d_1 = d_3 = 1 \text{ m}, d_2 = 1.2 \text{ m}$



$$
\begin{aligned}
(M_R)_O &= F_1 d_1 - F_2 d_2 + F_3 d_3 \\
&= 100 - 240 + 100 \text{ (Nm)} \\
&= -40 \text{(Nm)}
\end{aligned}
$$

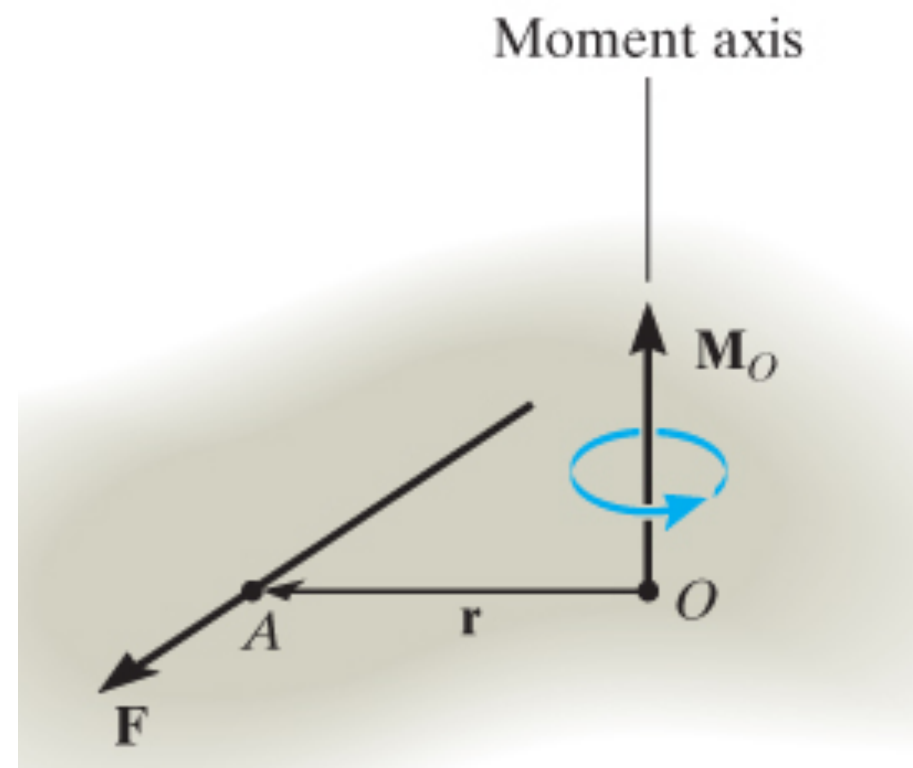
The negative sign implies that the body will rotate in the clockwise direction.

Moment: a more robust definition

For 3D problems, it is more convenient to express the moment of force $\mathbf{F}$ as a cross product:

$$\mathbf{M_0} = \mathbf{r} \times \mathbf{F}$$

Where $\mathbf{r}$ is the position vector from point O to any point on the line of action of force $\mathbf{F}$.



Let's review the basic rules of cross products.

Cross product

The cross product of vectors $\mathbf{A}$ and $\mathbf{B}$ is:

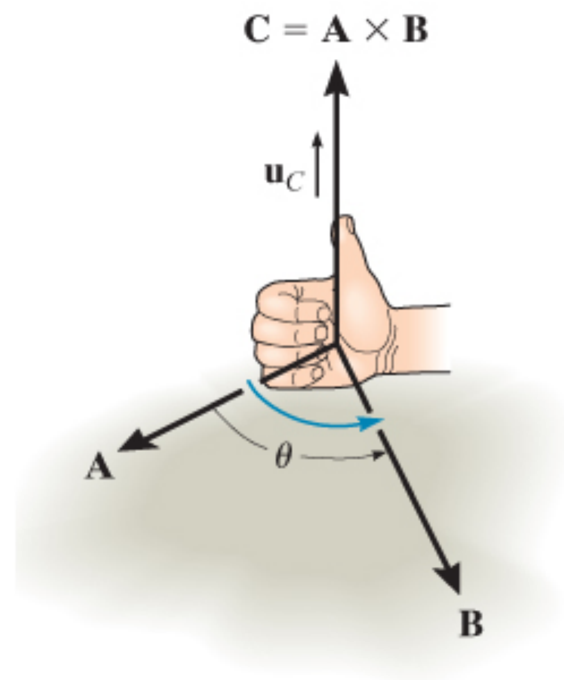
$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = AB \sin\theta \, \mathbf{u_c}$$

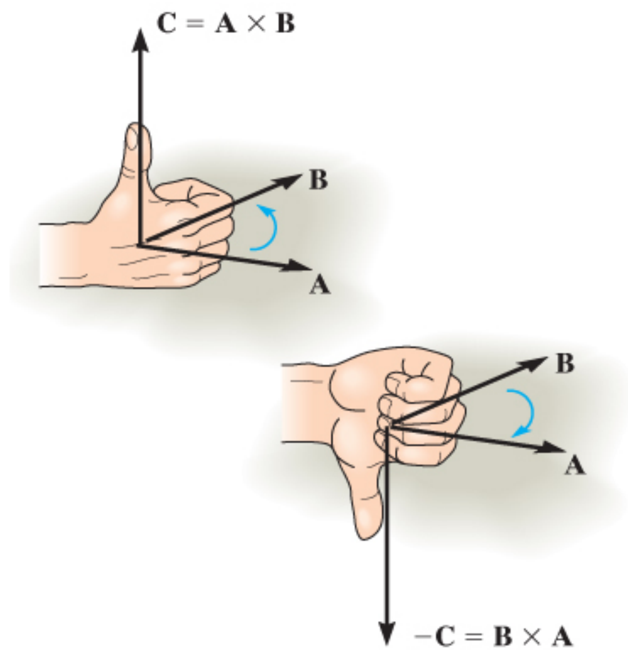
The direction of $\mathbf{C}$ is perpendicular to the plane containing $\mathbf{A}$ and $\mathbf{B}$. This direction is given by the unit vector $\mathbf{u_c}$. Note that:

$$-\mathbf{C} = \mathbf{B} \times \mathbf{A}$$

The magnitude of vector $\mathbf{C}$ is:

$$C = AB \sin\theta$$

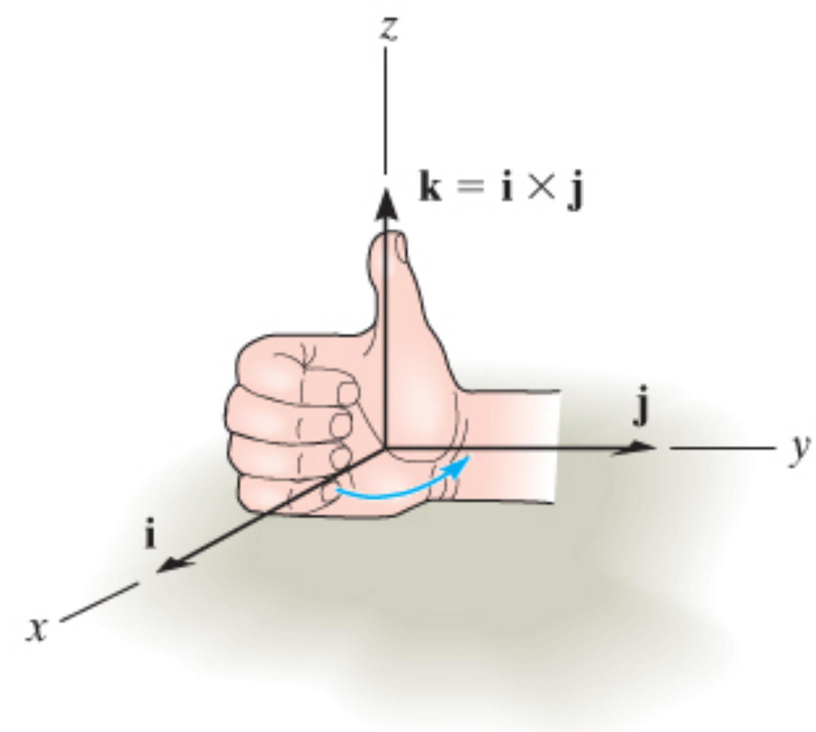




Cross product of Cartesian vectors

A few interesting properties regarding the cross products of unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$:

$$\mathbf{i}\times\mathbf{j} = 1 \cdot 1 \cdot \sin 90°\,\mathbf{k} = \mathbf{k}$$



With the same approach we find:

$$\mathbf{i}\times\mathbf{k} = -\mathbf{j} \qquad \mathbf{k}\times\mathbf{j} = -\mathbf{i}$$

$$\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = \mathbf{0}$$

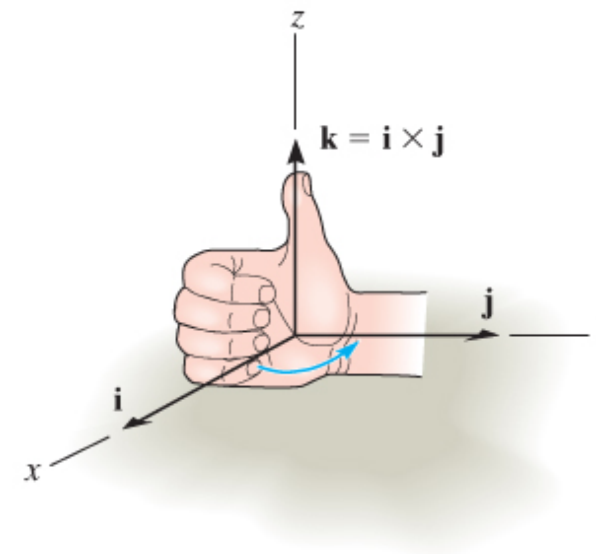
Cross product of Cartesian vectors

Cross product of two Cartesian vectors:

$$\mathbf{A}\times\mathbf{B} = \left(A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}\right)\times\left(B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}\right)$$

$$= (A_xB_x)\mathbf{i}\times\mathbf{i} + \left(A_xB_y\right)\mathbf{i}\times\mathbf{j} + (A_xB_z)\mathbf{i}\times\mathbf{k} + \ldots$$

$$= \left(A_yB_z - A_zB_y\right)\mathbf{i} - (A_xB_z - A_zB_x)\mathbf{j} + \left(A_xB_y - A_yB_x\right)\mathbf{k}$$



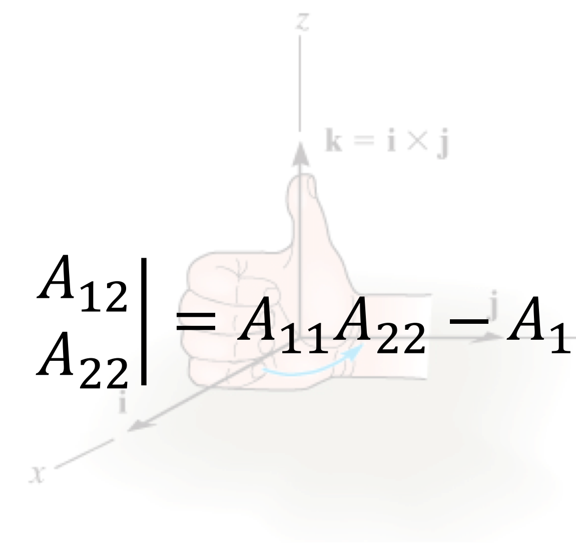
This operation is easier with the determinant method…

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Cross product

$$\mathbf{A}\times\mathbf{B} =$$

Determinant: $\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = A_{11}A_{22} - A_{12}A_{21}$



$\mathbf{i}$ component: $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \mathbf{i}(A_yB_z - A_zB_y)$

$\mathbf{j}$ component: $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = -\mathbf{j}(A_xB_z - A_zB_x)$

Note the negative sign!

$\mathbf{k}$ component: $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \mathbf{k}(A_xB_y - A_yB_x)$

Finally, add each component.

Example: cross product

Find $\mathbf{A}\times\mathbf{B}$ given $\mathbf{A} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{B} = 5\mathbf{j} + 2\mathbf{k}$.

Using the determinant approach we get:

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 0 & 5 & 2 \end{vmatrix}$$

$$= \big(2 - (-5)\big)\mathbf{i} - (4 - 0)\mathbf{j} + (10 - 0)\mathbf{k}$$

$$= 7\mathbf{i} - 4\mathbf{j} + 10\mathbf{k}$$

Moment as a cross product

A moment is the cross product of a force and a position vector:

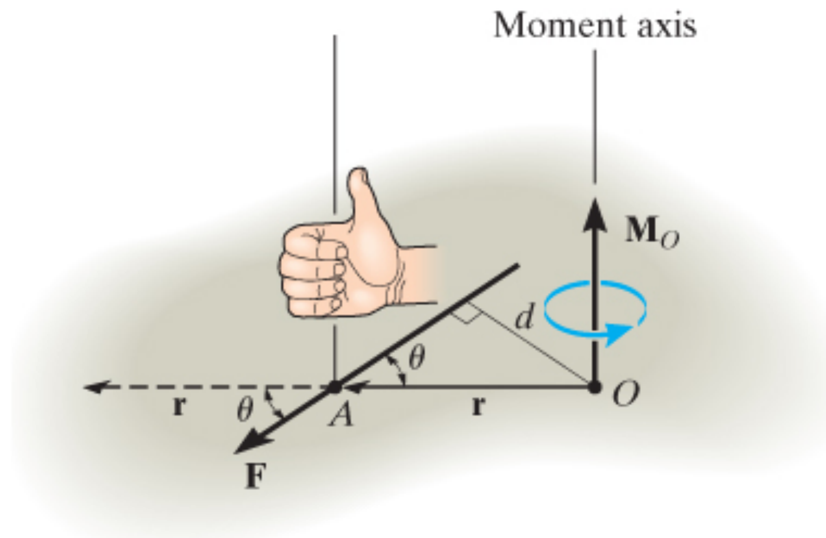
$$\mathbf{M_0} = \mathbf{r} \times \mathbf{F}$$

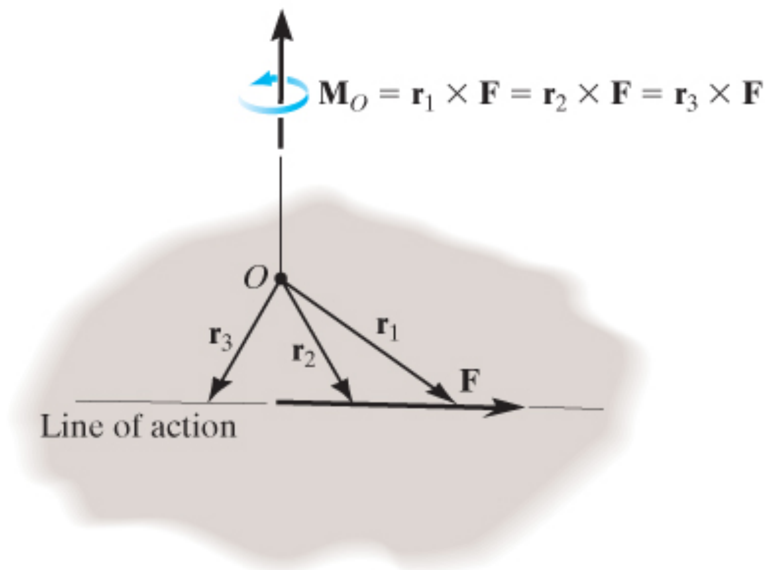
Its direction obeys the right hand rule and its magnitude is:

$$M_0 = |\mathbf{r} \times \mathbf{F}| = rF\sin\theta = Fd$$

The moment is always calculated from a point (here point *O*).

The position vector $\mathbf{r}$ is from point *O* to the line of action of the force.





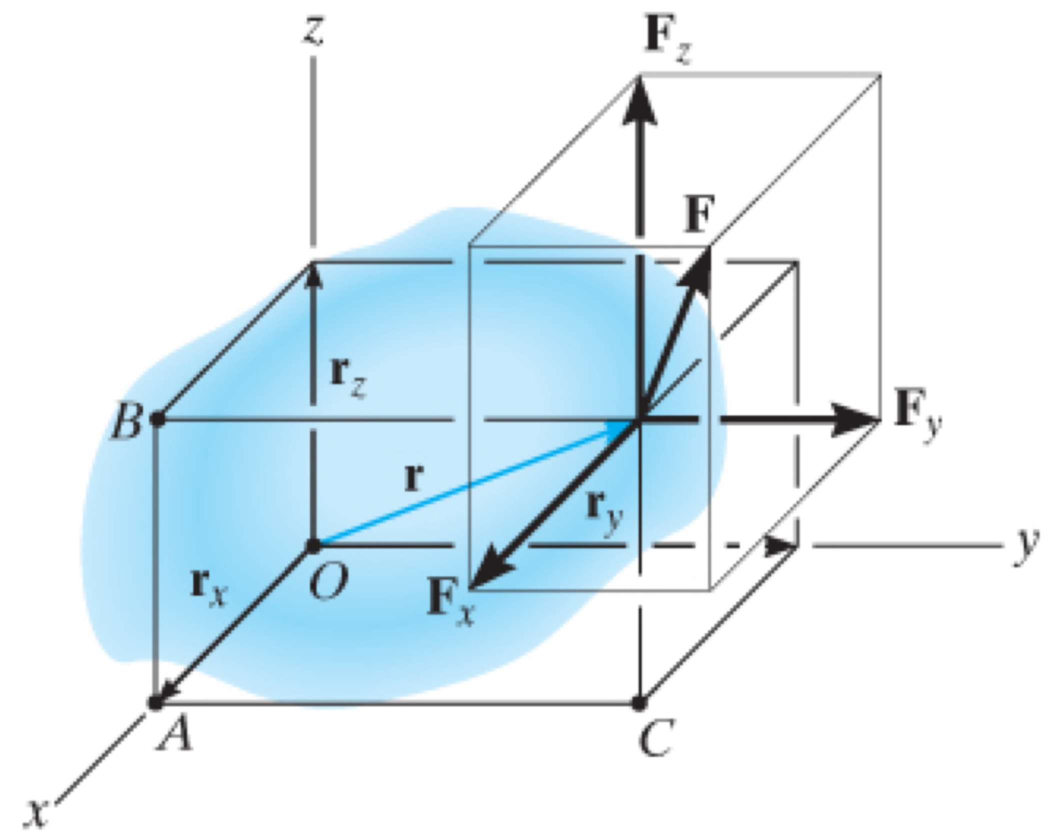
Physical meaning of each component

$$\mathbf{M}_O = \mathbf{r}\times\mathbf{F} = (r_yF_z - r_zF_y)\mathbf{i} - (r_xF_z - r_zF_x)\mathbf{j} + (r_xF_y - r_yF_x)\mathbf{k}$$

$$= \underbrace{(M_O)_x}_{}\mathbf{i} + (M_O)_y\mathbf{j} + (M_O)_z\mathbf{k}$$

Moment that creates a rotation around x-axis

$$(M_O)_x = (r_yF_z - r_zF_y)$$

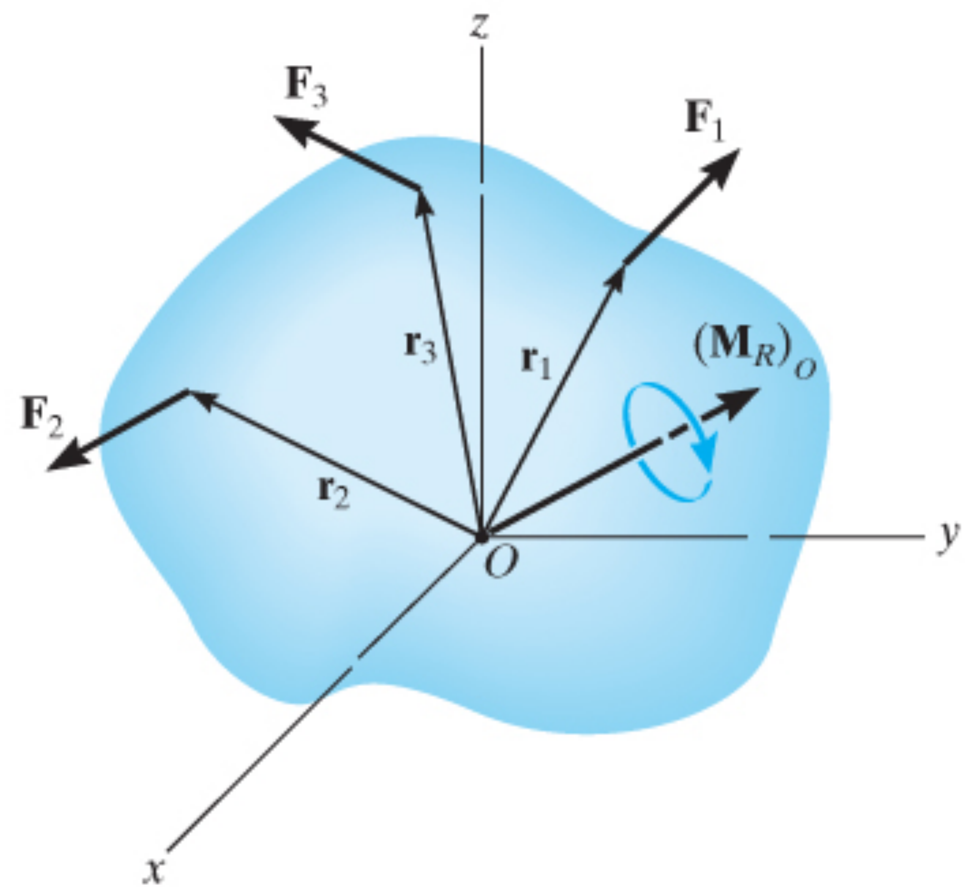


Resultant moment

The resultant moment is obtained by adding the moment created by each force:

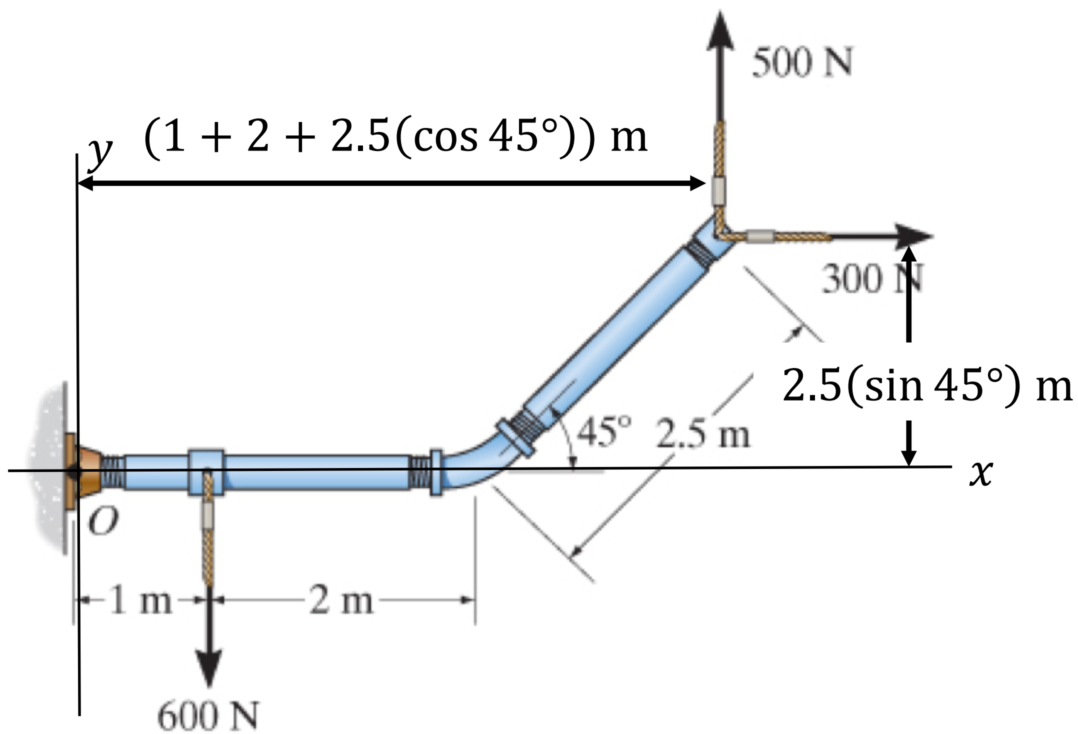
$$(\mathbf{M_R})_0 = \sum(\mathbf{r}\times\mathbf{F})$$

$$\Longrightarrow (\mathbf{M_R})_0 = \mathbf{r_1}\times\mathbf{F_1} + \mathbf{r_2}\times\mathbf{F_2}+\mathbf{r_3}\times\mathbf{F_3}+\ldots$$



Example

Find the resultant moment at point *O*.



Method 1: simple approach

Let's calculate the moment corresponding to each force around point O and add them up:

$\circlearrowleft + \quad (M_R)_O = \Sigma F_i d_i$, distances d_i are shown on figure

$$(M_R)_O = -600\text{N}(1\text{m}) - 300\text{N}\left(\frac{2.5}{\sqrt{2}}\text{m}\right) + 500\text{N}((3 + \frac{2.5}{\sqrt{2}})\text{ m})$$

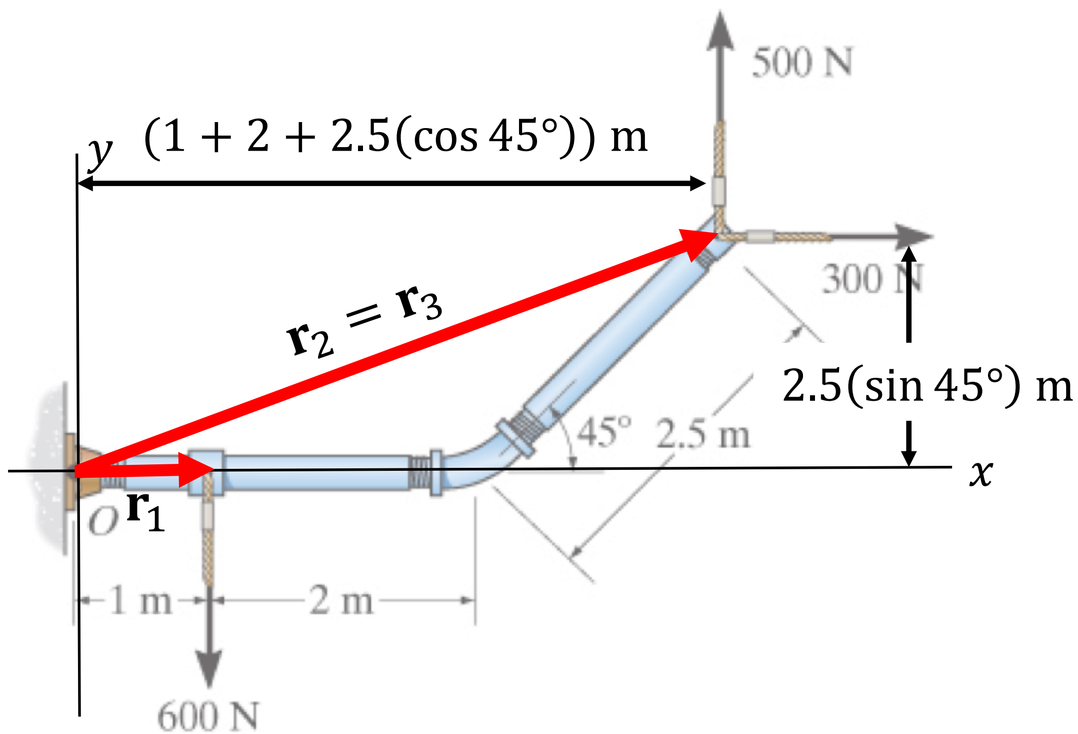
$$= 1253.55\text{ Nm} = 1254\text{ Nm}$$

$$\mathbf{M}_O = \{1254\,\mathbf{k}\}\text{ Nm}$$

Use the right hand rule to determine the sign of each moment.

Example

Find the resultant moment at point *O*.



Method 2: cross products

The position and force vectors are:

$$\mathbf{r}_1 = \{\mathbf{i}\}\ \mathrm{m}$$

$$\mathbf{r}_2 = \mathbf{r}_3 = \left\{\left(3 + \frac{2.5}{\sqrt{2}}\right)\mathbf{i} + \frac{2.5}{\sqrt{2}}\mathbf{j}\right\}\ \mathrm{m}$$

$$\mathbf{F}_1 = \{-600\mathbf{j}\}\ \mathrm{N}$$

$$\mathbf{F}_2 = \{300\mathbf{i}\}\ \mathrm{N}$$

$$\mathbf{F}_3 = \{500\mathbf{j}\}\ \mathrm{N}$$

The sum of moments gives:

$$\mathbf{M}_{R_O} = \Sigma(\mathbf{r}\times\mathbf{F}) = \mathbf{r}_1\times\mathbf{F}_1 + \mathbf{r}_2\times\mathbf{F}_2 + \mathbf{r}_3\times\mathbf{F}_3$$

$$= -600\mathbf{k} - \frac{2.5}{\sqrt{2}}300\mathbf{k} + \left(3 + \frac{2.5}{\sqrt{2}}\right)500\mathbf{k}$$

$$= 1253.55\mathbf{k} \quad = \{1254\mathbf{k}\}\ \mathrm{Nm}$$

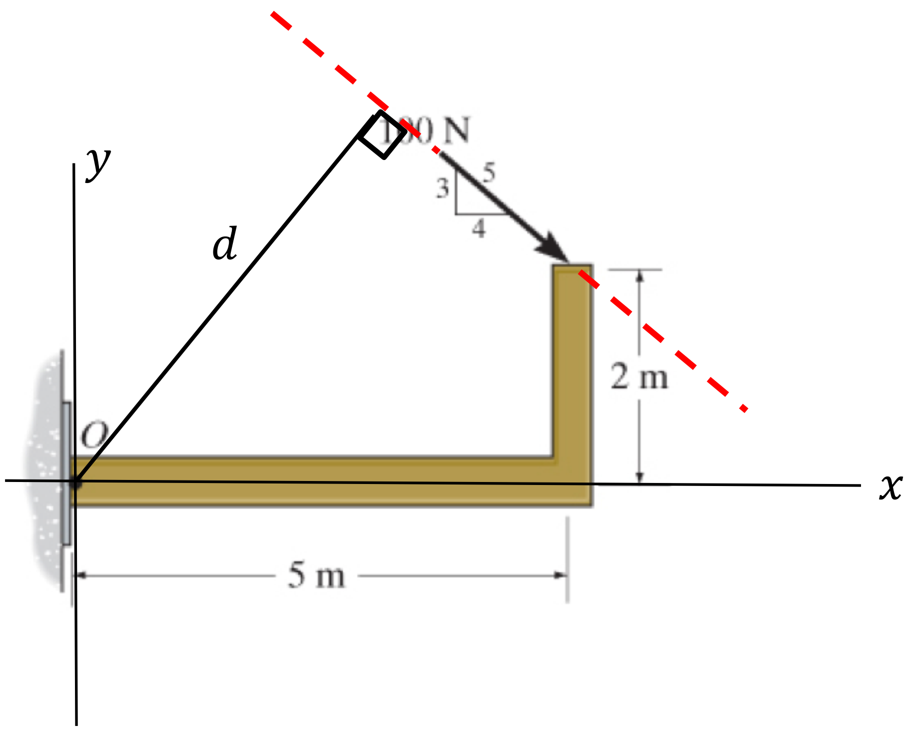
The cross product gives you the sign of each moment.

Principle of moments

The moment of a force about a point is equal to the sum of the moments of the components of the force about the same point:

$$\mathbf{M_O} = \mathbf{r}\times\mathbf{F} = \mathbf{r}\times(\mathbf{F_1} + \mathbf{F_2}) = \mathbf{r}\times\mathbf{F_1} + \mathbf{r}\times\mathbf{F_2}$$

Find the moment around point O.



Method 1. Find distance d and use $M_O = Fd$.

Method 2. Divide the force in x and y components and add the moments created by these two components.

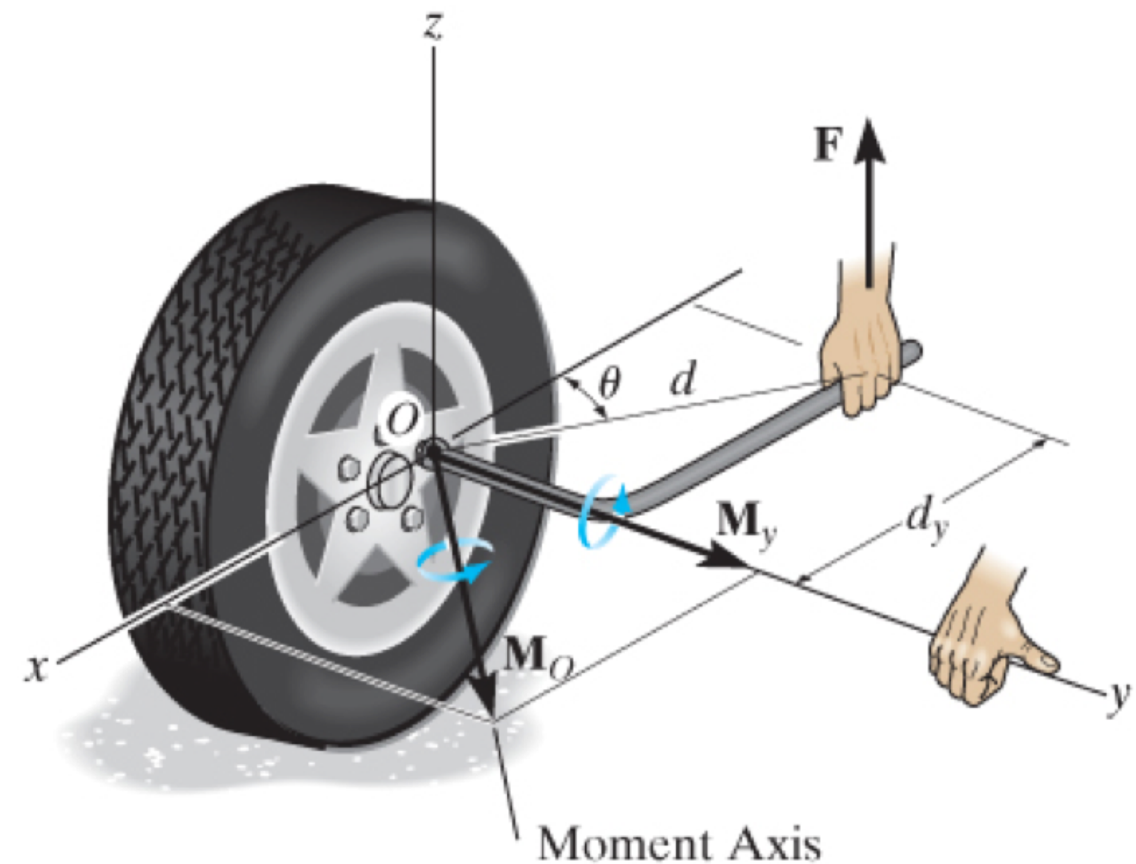
Moment about a specified axis

Below, the force $\mathbf{F}$ creates a moment $\mathbf{M_O}$; however, the nut can only rotate about the y axis so to determine the turning effect we need the y component of the moment.

Clearly, $\mathbf{M_O} \neq \mathbf{M_y}$. There is an easy way to compute M_y which is:

$$M_y = Fd_y = Fd\cos\theta$$

Where d_y is the distance from the y axis to the line of action of the force.



Verification

Let's verify our previous result: $M_y = Fd_y = Fd\cos\theta$.

The moment $\mathbf{M_O}$ is given by:

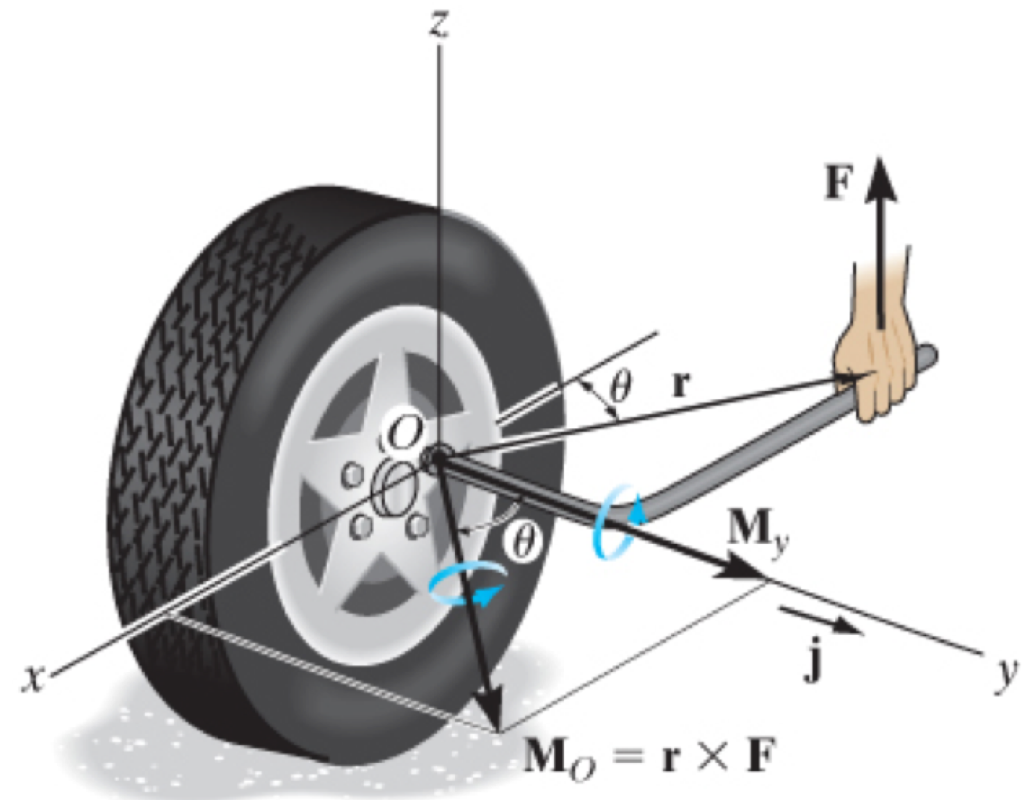
$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = (-d\cos\theta\,\mathbf{i} + d\sin\theta\,\mathbf{j}) \times F\,\mathbf{k}$$
$$= Fd\sin\theta\,\mathbf{i} + Fd\cos\theta\,\mathbf{j}$$

The magnitude of M_y is:

$$M_y = \mathbf{j} \cdot \mathbf{M}_O = Fd\cos\theta$$

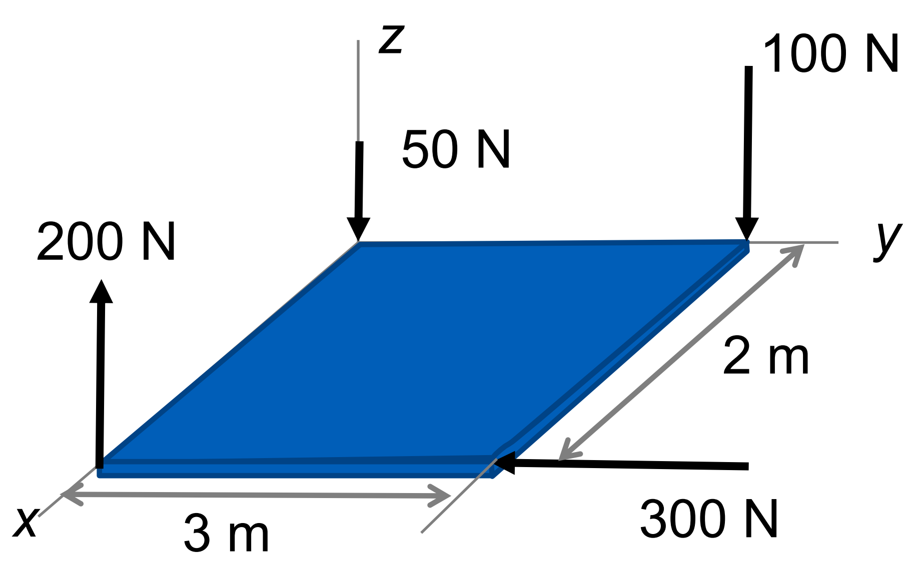
And the vector $\mathbf{M_y}$ is:

$$\mathbf{M}_y = Fd\cos\theta\,\mathbf{j}$$



Example

Find the moments around the *x*-, *y*- and *z*-axis.



Moments around x axis:

$$\circlearrowleft + \quad M_x = -100\text{N}(3\text{m}) = -300\ \text{Nm}$$

Moments around y axis:

$$\circlearrowleft + \quad M_y = -200\text{N}(2\text{m}) = -400\ \text{Nm}$$

Moments around z axis:

$$\circlearrowleft + \quad M_z = -300\text{N}(2\text{m}) = -600\ \text{Nm}$$

Moment about an arbitrary axis

Consider the moment $\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$ about point *O*. The projection of this moment onto an arbitrary axis a is obtained by:

$$M_a = \mathbf{u}_a \cdot \mathbf{M}_O = \mathbf{u}_a \cdot (\mathbf{r} \times \mathbf{F})$$

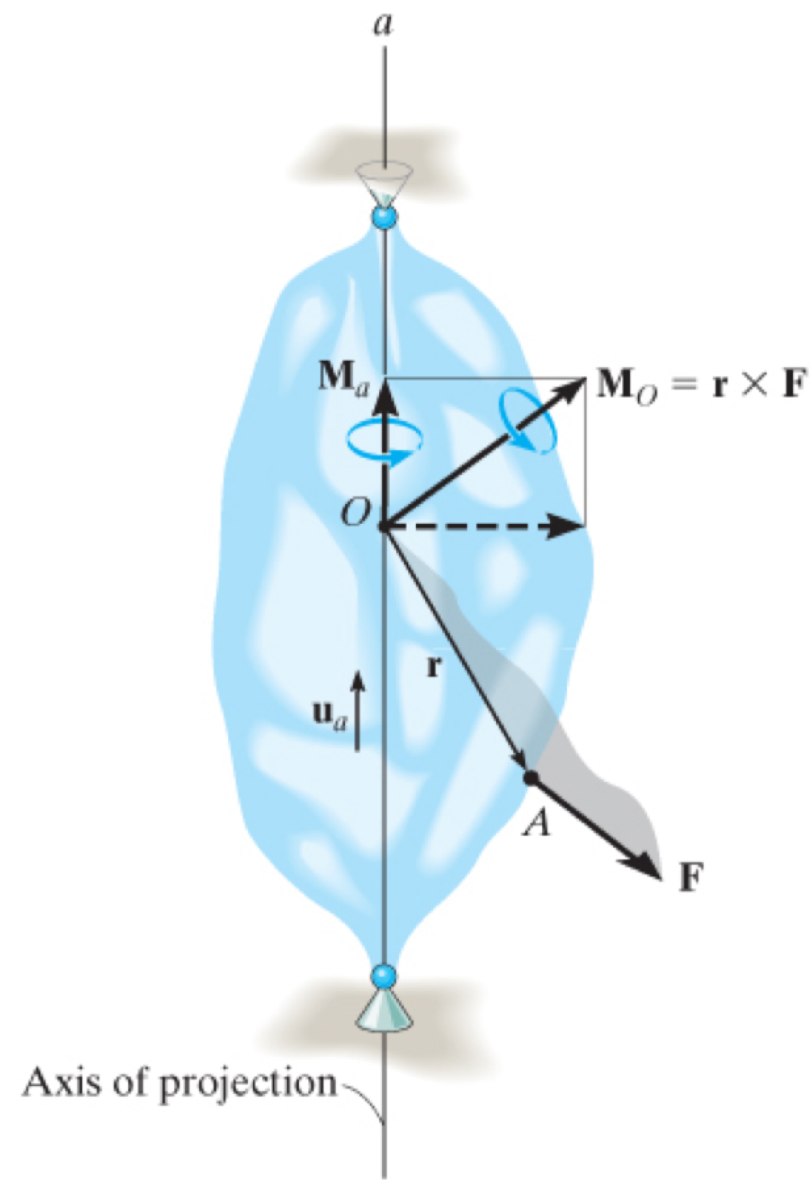
Where $\mathbf{u}_a$ is a unit vector along axis a.

This can be written in the form of a determinant:

$$M_a = \mathbf{u}_a \cdot (\mathbf{r} \times \mathbf{F}) = \begin{vmatrix} u_{a_x} & u_{a_y} & u_{a_z} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

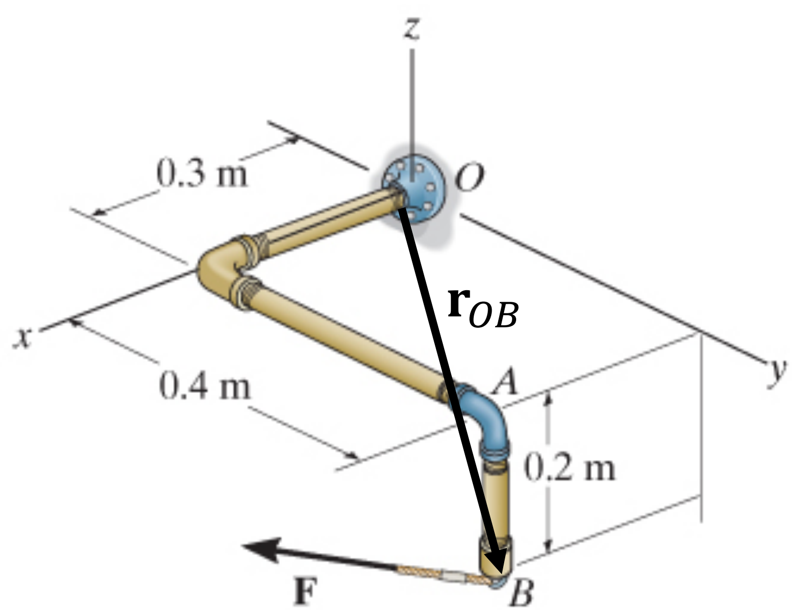
Of course, the vector $\mathbf{M}_a$ can be obtained simply with:

$$\mathbf{M}_a = M_a \mathbf{u}_a$$



Example

Find the moment about the x axis, provided that $\mathbf{F} = \{300\mathbf{i} - 200\mathbf{j} + 150\mathbf{k}\}$ N.



The position vector $\mathbf{r}_{OB}$ is:

$$\mathbf{r}_{OB} = \{0.3\mathbf{i} + 0.4\mathbf{j} - 0.2\mathbf{k}\} \text{ m}$$

And the moment is:

$$\mathbf{M}_O = \mathbf{r}_{OB} \times \mathbf{F}$$

The unit vector $\mathbf{u}_a$ corresponding to the *x* axis is $\mathbf{i}$.
The component M_a is given by:

$$M_a = \mathbf{i} \cdot \mathbf{M}_O = \mathbf{i} \cdot (\mathbf{r}_{OB} \times \mathbf{F}) = \begin{vmatrix} 1 & 0 & 0 \\ 0.3 & 0.4 & -0.2 \\ 300 & -200 & 150 \end{vmatrix}$$

$$= (0.4)150 - (-0.2)(-200) = 20 \text{ (Nm)}$$

$$\mathbf{M}_a = 20\,\mathbf{i} \text{ (Nm)}$$

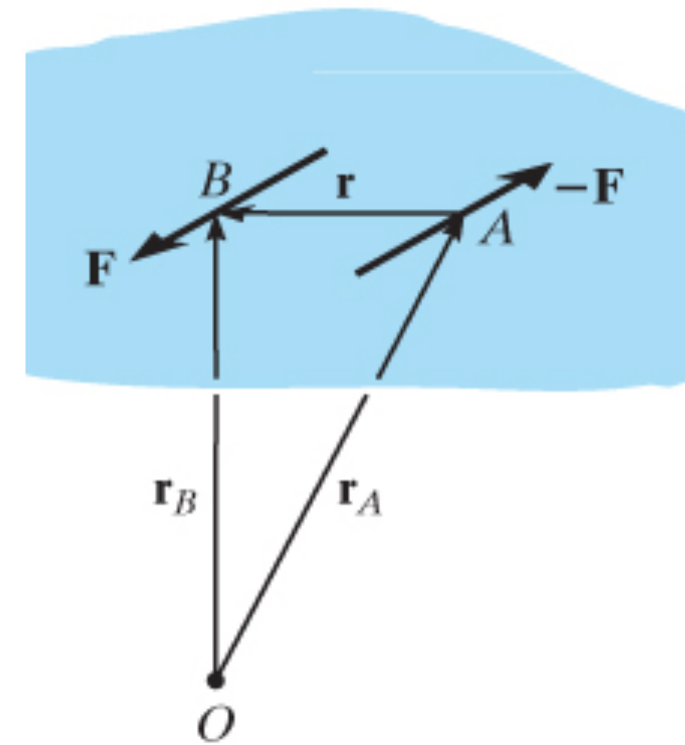
Moment of a couple

A **couple** is made of two parrallel forces that have the same magnitude, but opposite directions.

A couple generates a moment that is linearly proportional to the magnitude of the force and the distance between them.

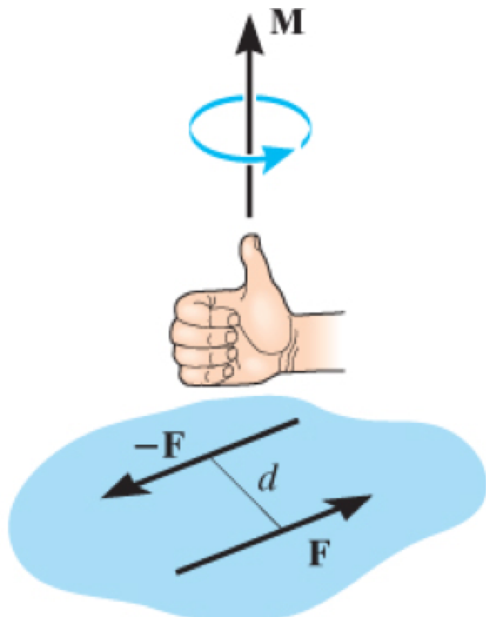
The moment about point *O* is given by:

$$\mathbf{M} = \mathbf{r}_B \times \mathbf{F} + \mathbf{r}_A \times -\mathbf{F} = (\mathbf{r}_B - \mathbf{r}_A) \times \mathbf{F} = \mathbf{r} \times \mathbf{F}$$



Note that the moment $\mathbf{M}$ depends on distance between the two forces, and is independent from the distance to point *O*.

Magnitude of a couple

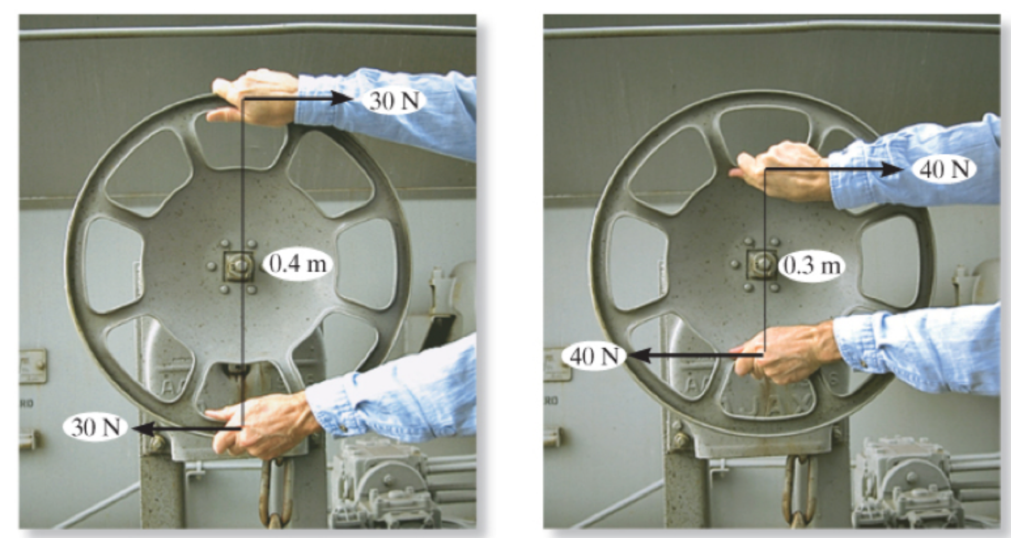


The magnitude of a couple is:

$$M = Fd$$

Where d is the distance between forces. The direction of the moment follows the right hand rule.

The two couples on the right have the same magnitude.



Equivalent system: how to simplify a problem

Equivalent system of force & moment

Often, it is necessary to replace a system of forces and moments by an **equivalent system** with a resultant force and resultant moment acting at a specific point.

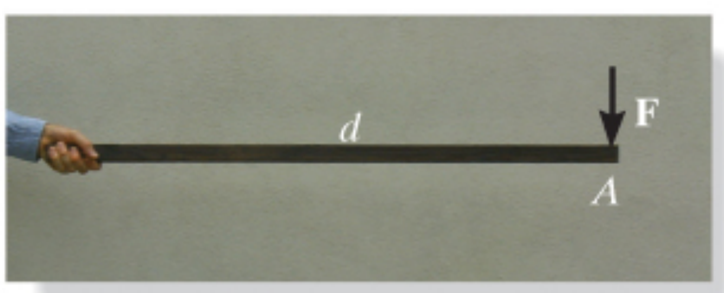
A system is **equivalent** if it produces the same external effects:

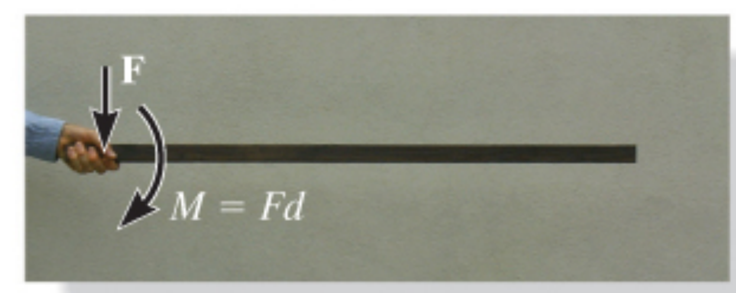
- In statics, this refers to the reaction forces at the supports.
- In dynamics, it means the translations and rotations of the body would be the same.

Equivalent system of force & moment

For the system below, define an equivalent system where the forces and moments are acting on the hand.

1. We can add a pair of opposite forces on the hand (see middle).
2. Notice that the force $-\mathbf{F}$ on the left and $\mathbf{F}$ on the right are creating a couple $M = Fd$. We can replace these forces by the resultant moment acting on the hand.





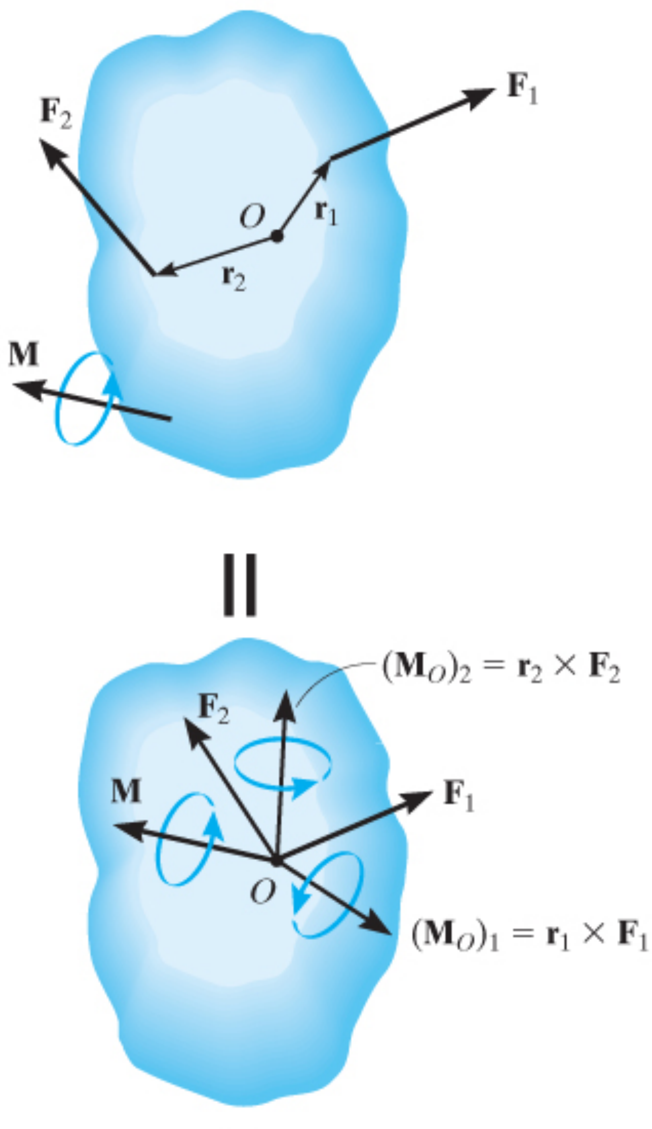
Original system → **Equivalent system**

Equivalent system: important rules

There are 3 rules to follow in creating an equivalent system:

- Force is a *sliding* vector: you can move a force vector to any point *P* that is **along the line of action** of the force.
- When a force is moved to another point *P* that **is not on its line action** then you need to add the moment of the force about point *P*.
- Moments are *free* vectors: they can be moved to any point P and they will produce the same external effect.

Equivalent system



Find the resultant force and resultant moment at point *O* for body shown on the left.

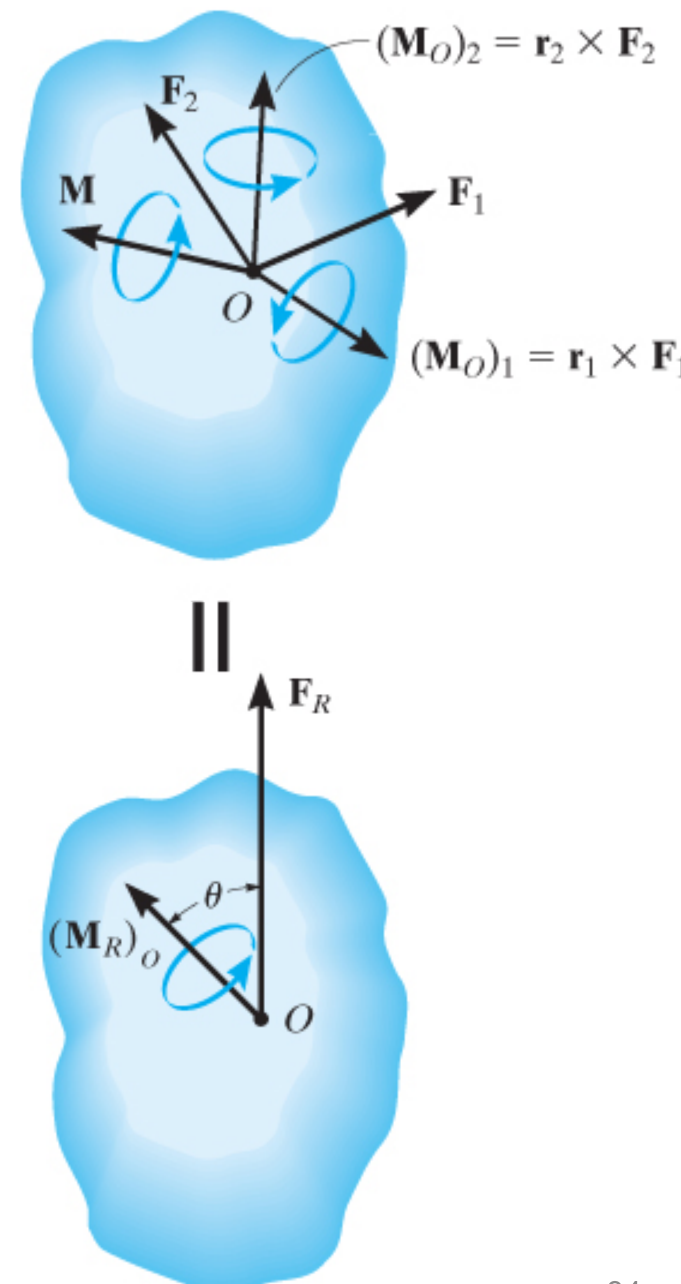
The force $\mathbf{F}_1$ is replaced by a force $\mathbf{F}_1$ and a moment $(\mathbf{M}_O)_1 = \mathbf{r}_1 \times \mathbf{F}_1$ at point *O*.

The force $\mathbf{F}_2$ is replaced by a force $\mathbf{F}_2$ and a moment $(\mathbf{M}_O)_2 = \mathbf{r}_2 \times \mathbf{F}_2$ at point *O*.

Finally, the moment $\mathbf{M}$ is a free vector and we can mode it directly to point O.

Equivalent system

Now that all forces and moments are at point O, we can add them to find the resultant force $\mathbf{F}_R$ and resultant moment $\mathbf{M}_R$:



$$\mathbf{F}_R = \Sigma\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$$

$$(\mathbf{M}_R)_O = (\mathbf{M}_O)_1 + (\mathbf{M}_O)_2 + \mathbf{M}$$

Equivalent system in 2D

For 2D problems, where all forces are in the x-y plane and moments perpendicular to this plane, it is often simpler to use a scalar approach to find the resultant force and moment. This gives us three equations:

$$(F_R)_x = \Sigma F_x$$

$$(F_R)_y = \Sigma F_y$$

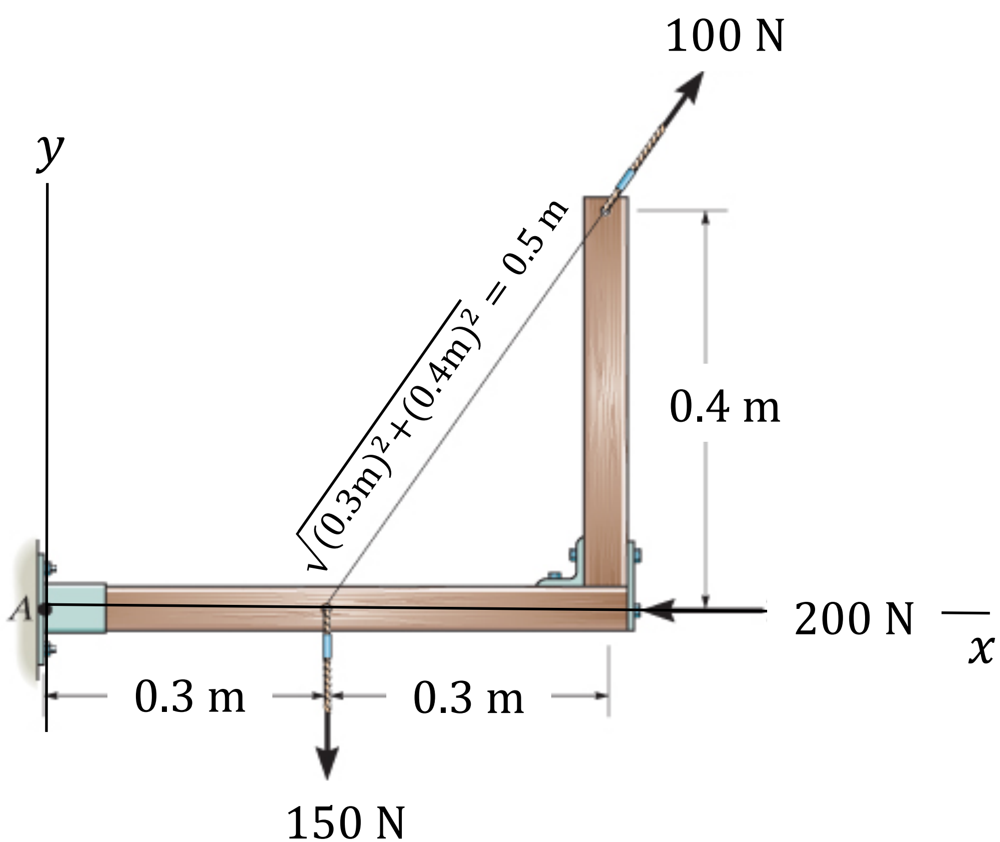
Moments applied directly to the body

$$(M_R)_O = \Sigma M_O + \Sigma M$$

Moments caused by forces

Example

Find the resultant force and resultant moment at point A.



We sum all forces along the x and y axes to find the resultant force:

$$\rightarrow + \ (F_R)_x = \Sigma F_x = -200\text{N} + 100\text{N}\left(\frac{0.3}{0.5}\right) = -140 \text{ N}$$

$$\uparrow + \ (F_R)_y = \Sigma F_y = -150\text{N} + 100\text{N}\left(\frac{0.4}{0.5}\right) = -70 \text{ N}$$

The magnitude is:

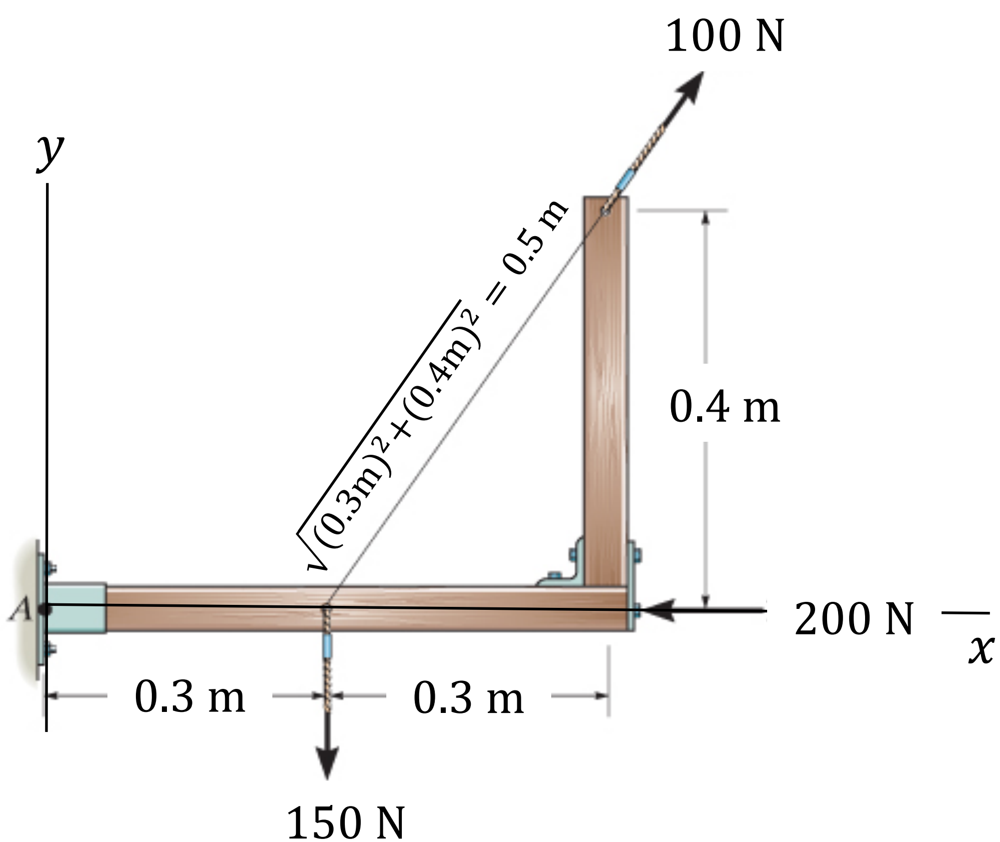
$$F_R = \sqrt{(F_{R_x})^2 + (F_{R_y})^2} = \sqrt{(-140\text{N})^2 + (-70\text{N})^2} = 157\text{N}$$

And the orientation is:

$$\theta = \tan^{-1}\left|\frac{(F_R)_y}{(F_R)_x}\right| = \tan^{-1}\left|\frac{-70\text{N}}{-140\text{N}}\right| = 26.6°$$

Example

Find the resultant force and resultant moment at point A.



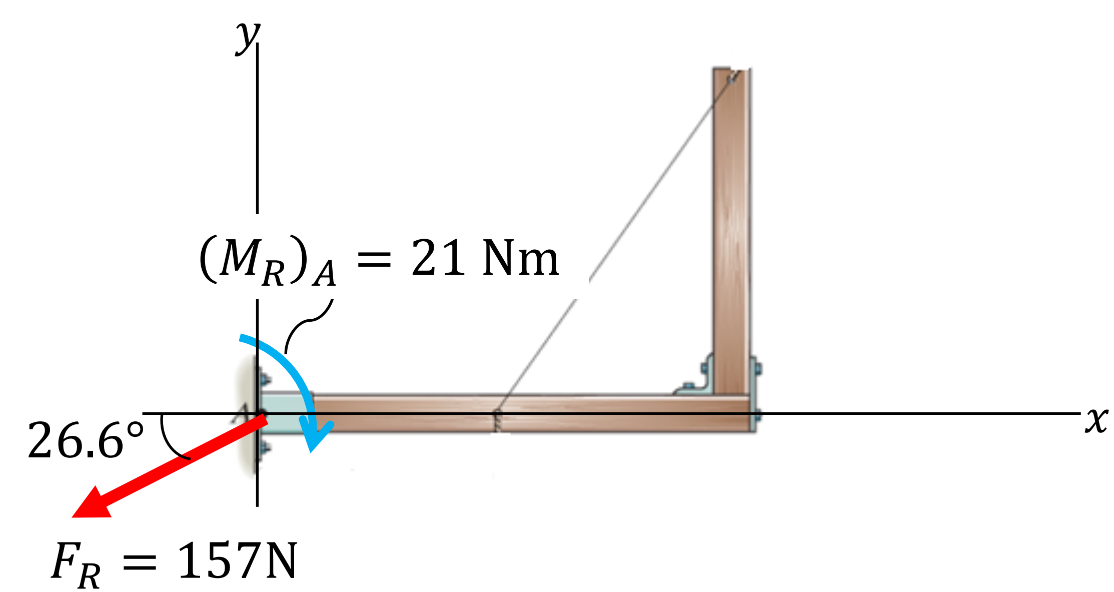
The resultant moment around point A is:

$$\circlearrowleft + \ (M_R)_A = \Sigma M_A$$

$$= -(150\text{N})(0.3\text{m}) + 100\text{N}\left(\frac{0.4}{0.5}\right)(0.6\text{m}) - 100\text{N}\left(\frac{0.3}{0.5}\right)(0.4\text{m})$$

$$= -45 \text{ Nm} + 48 \text{ Nm} - 24 \text{ Nm} = -21 \text{ Nm}$$

Resultant force and moment at point A



Equivalent system: extra simplification

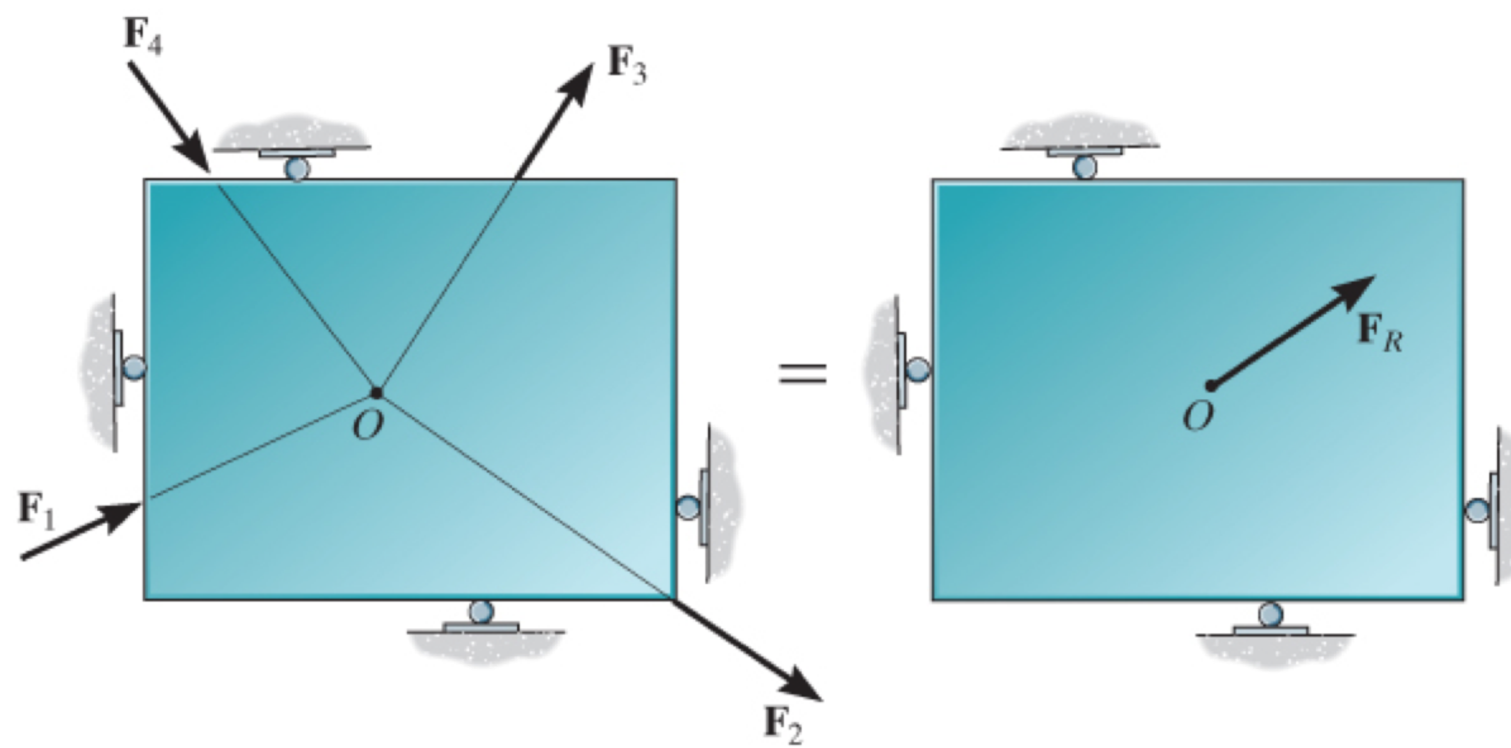
Previously we saw how forces and moments applied to a body can be represented by an equivalent system with a resultant force and moment.

There are special cases where the equivalent system can be reduced to only a resultant force (the resultant moment is zero).

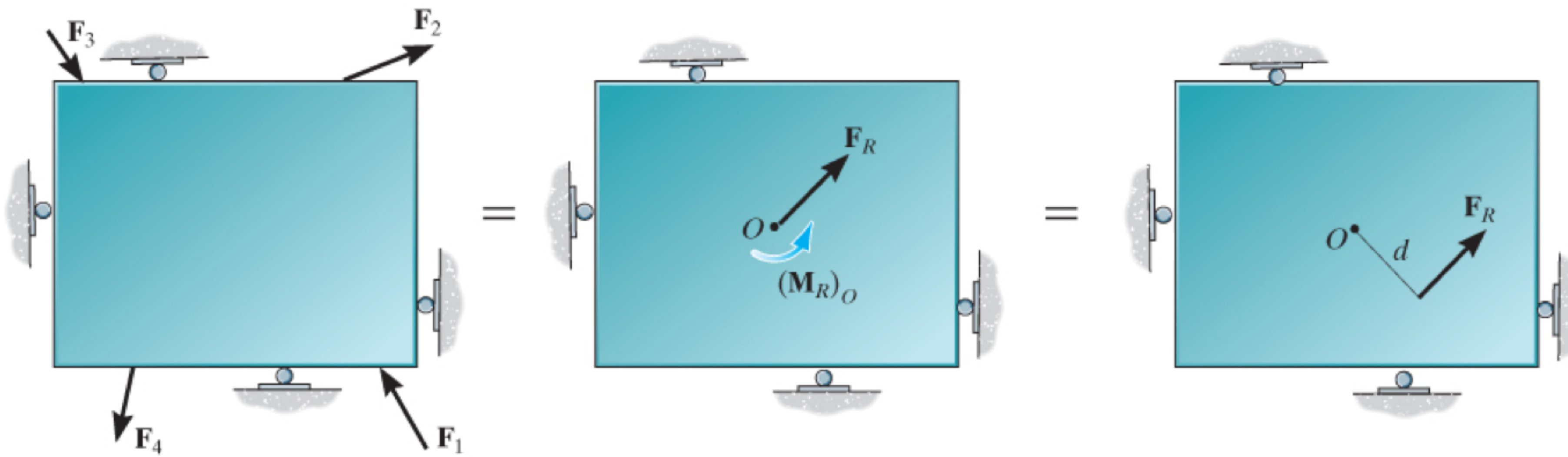
Concurrent force system

Consider a system where all forces are on the same plane, and their lines of action intersect at a common point *O*.

In this case, the resultant moment about point *O* is zero.



Coplanar force system



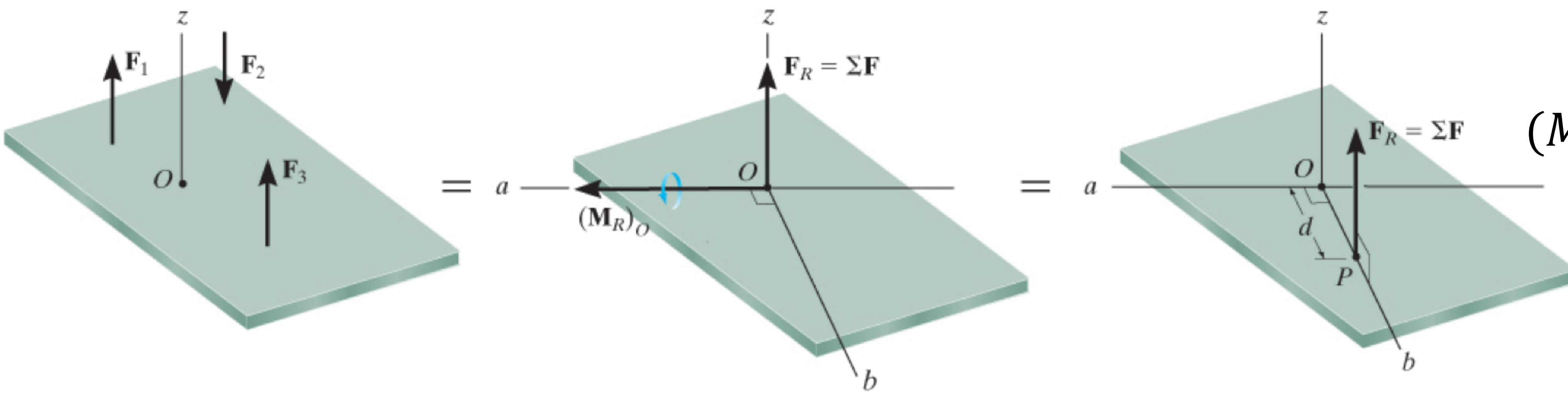
$$(M_R)_O = dF_R$$

$$\Rightarrow d = (M_R)_O / F_R$$

Consider a system where all forces are on the same plane. There is a resultant force $\mathbf{F}_R$ and resultant moment $(\mathbf{M}_R)_O$ about point *O*, and these two vectors are perpendicular.

We can replace the resultant moment by moving the resultant force to a distance *d* from point *O*.

Parallel force system



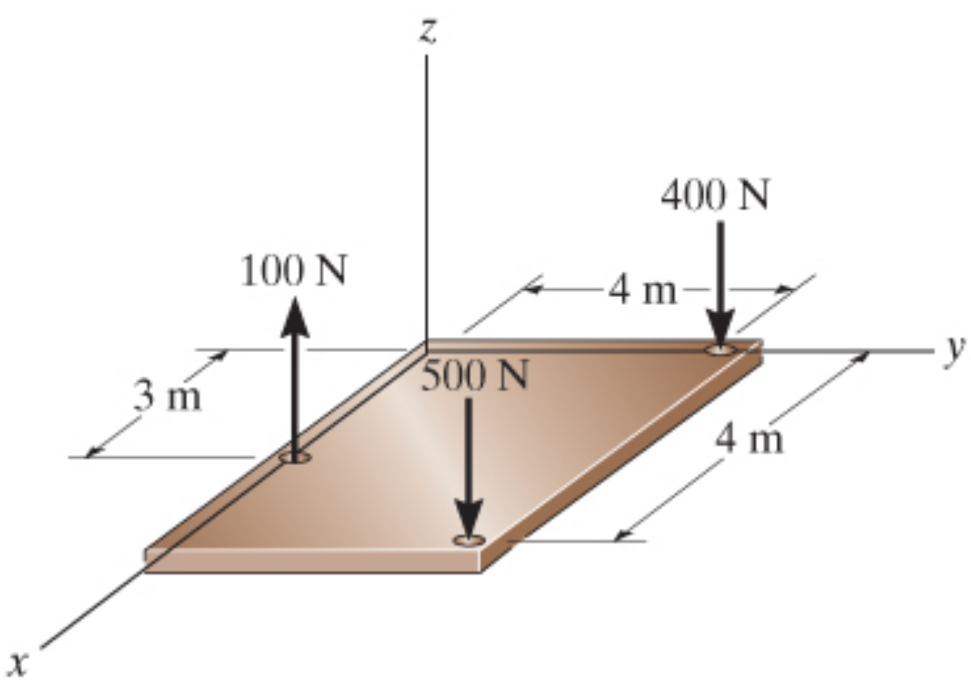
$$(M_R)_O = dF_R$$

$$\Rightarrow d = (M_R)_O / F_R$$

Consider a system where all forces are parallel. Again, this creates a resultant force and resultant moment that are perpendicular. Likewise, we can define an equivalent system with a single resultant force acting at point *P* instead of point *O*. (Note that point P is on an axis perpendicular to the direction of the resultant moment).

Example

Replace the system below by a single resultant force and specify the (x,y) coordinates where it should be applied



This is a parallel force system. The resultant force is:

$$+\uparrow F_R = 100\text{N} - 500\text{N} - 400\text{N} = -800\text{N}$$

The resultant force should be applied at a point where it will generate the same moment as the current system.

Let's consider first the moments around the x-axis:

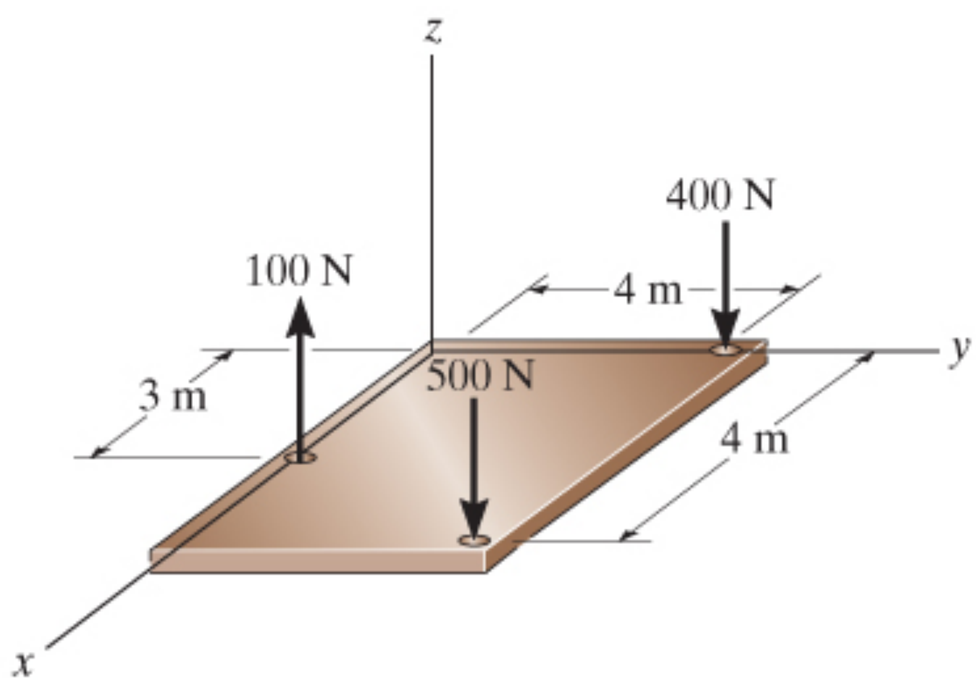
$$(M_R)_x = \Sigma M_x$$

$$\circlearrowleft + (-800\text{N})y = -400\text{N}(4\text{m}) - 500\text{N}(4\text{m})$$

$$\Rightarrow \qquad y = 4.5 \text{ m}$$

Example

Replace the system below by a single resultant force and specify the (x,y) coordinates where it should be applied



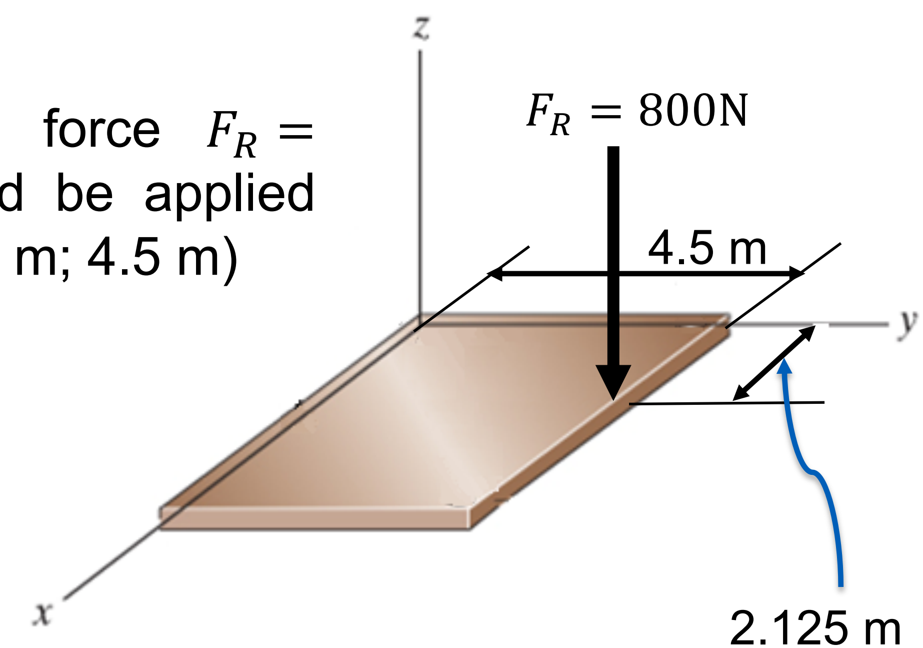
Next, we do the same for moments about the y-axis:

$$(M_R)_y = \Sigma M_y$$

$$\circlearrowleft + (800\text{N})x = -100\text{N}(3\text{m}) + 500\text{N}(4\text{m})$$

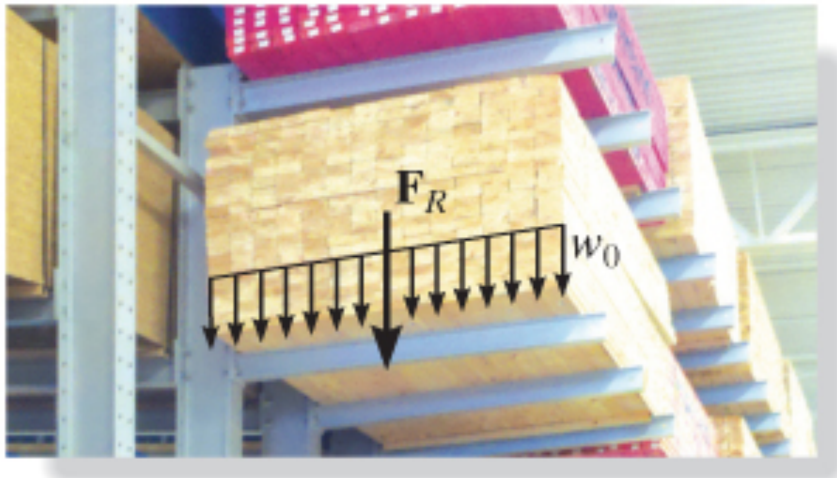
$$\Rightarrow \qquad x = 2.125 \text{ m}$$

The resultant force $F_R = -800\text{N}$ should be applied at point (2.125 m; 4.5 m)



Distributed loading

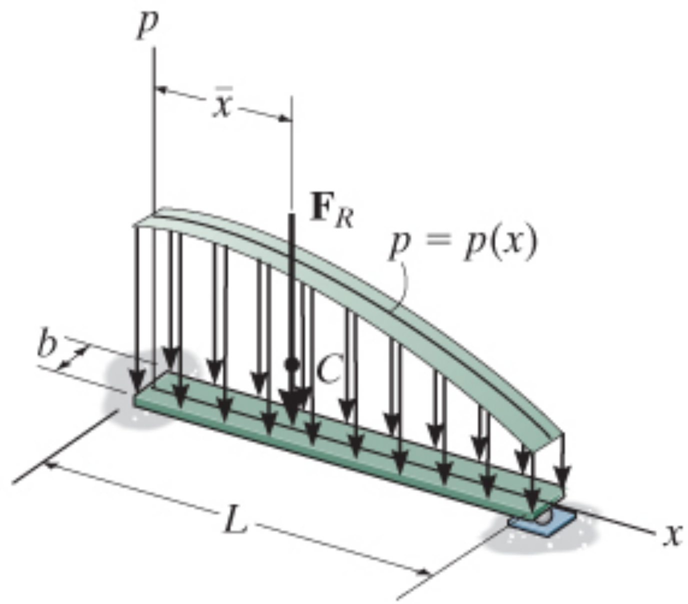
Distributed loading



Often, loading is distributed over a large area instead of being a concentrated force.

Here, we will see how distributed loads can be replaced by an equivalent resultant force.

Pressure = Force / Area

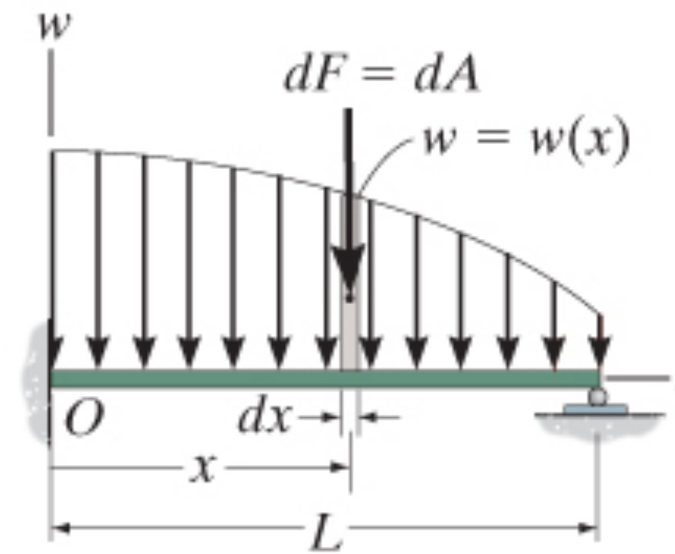


(a)

$$p = p(x)\ \mathrm{N/m^2}$$

$$[p] = \mathrm{Pa} = \mathrm{N/m^2}.$$

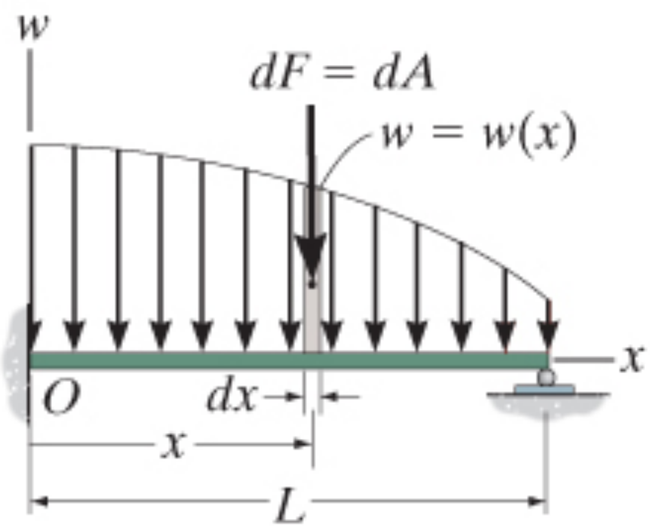
Line load: Force / length



(b)

$$w(x) = p(x)b\ \mathrm{N/m}$$

Magnitude of the resultant force



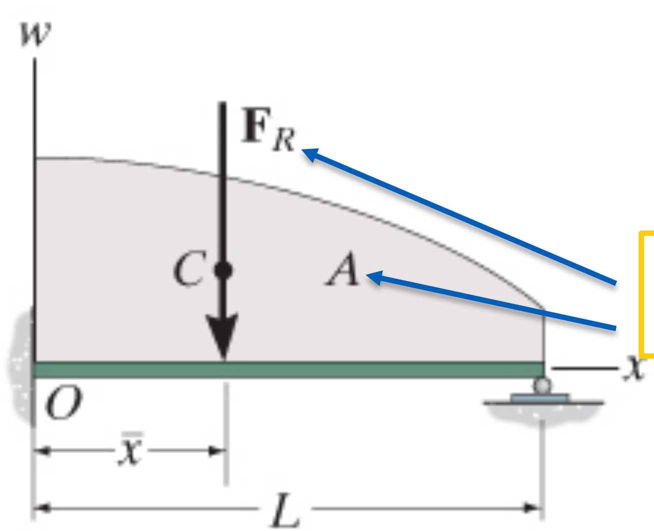
The distributed load can be considered as an infinite number of concentrated forces dF, acting on a length dx. This implies:

$$dF = w(x)dx$$

The resultant force in the vertical direction is given by:

$$+\downarrow \; F_R = \Sigma dF$$

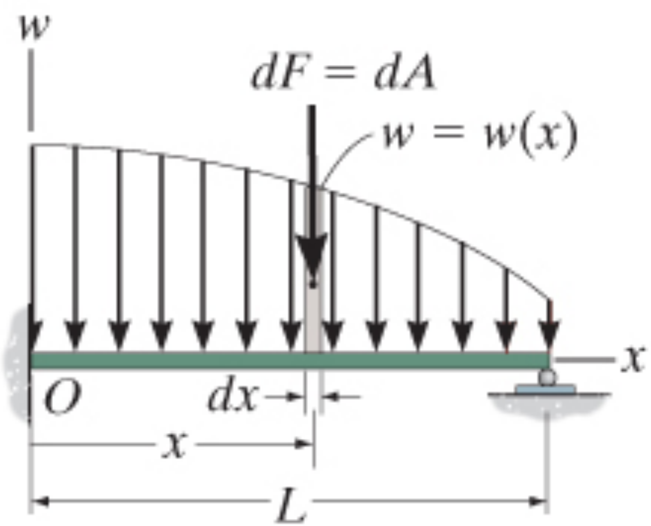
$$F_R = \int_L dF = \int_L w(x)dx = A$$

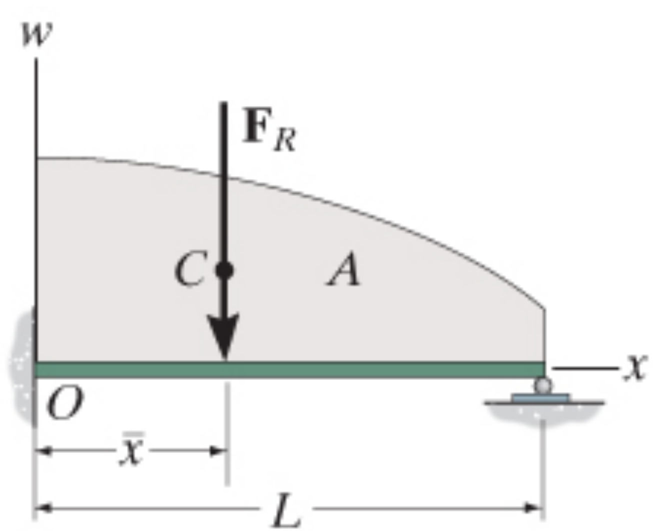


The resultant force is equal to the area A under the loading diagram.

Next, we need to find the location $\bar{x}$ where the resultant force should be applied.

Location of the resultant force





The moment, about point O, of each infinitesimal force dF is:

$$-xdF = -xw(x)dx$$

The resultant moment is obtained by integrating this over the entire length L:

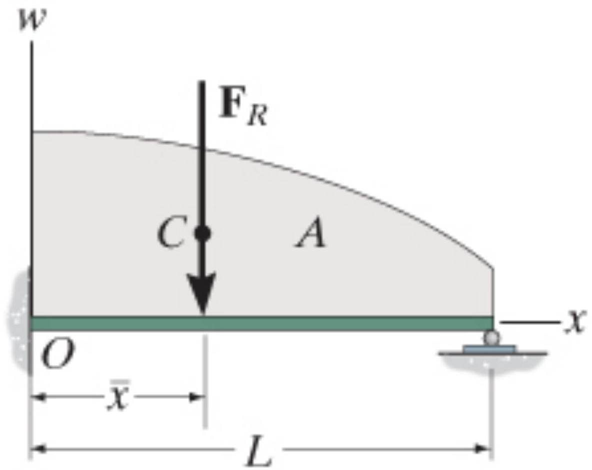
$$\circlearrowleft + (M_R)_O = -\int_L xw(x)dx$$

The resultant force should create the same moment which gives us:

$$-\bar{x}F_R = -\int_L xw(x)dx$$

$$\bar{x} = \frac{\int_L xw(x)dx}{F_R} = \frac{\int_L xw(x)dx}{\int_L w(x)dx} \equiv \frac{\int_L xdA}{\int_L dA}$$

Location of the resultant force



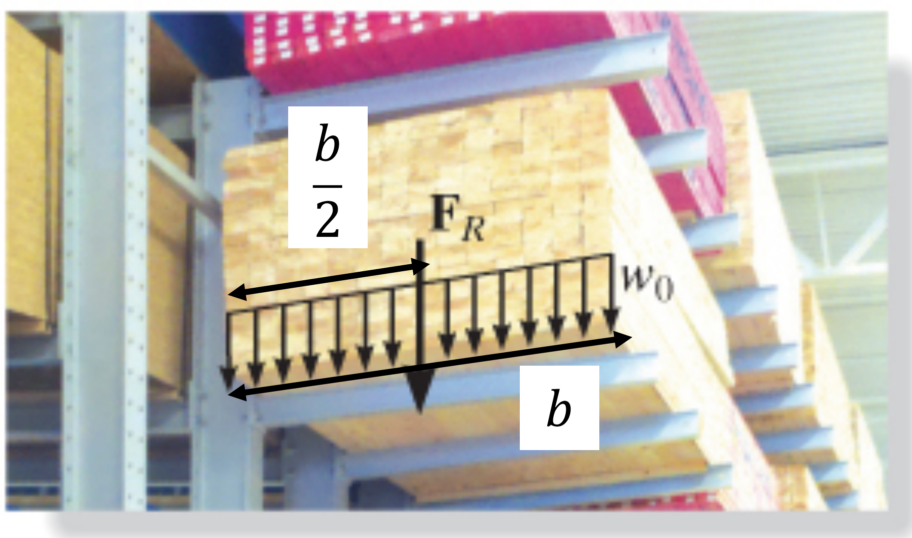
$$\bar{x} = \frac{\int_L x w(x) dx}{F_R} = \frac{\int_L x w(x) dx}{\int_L w(x) dx} \equiv \frac{\int_L x dA}{\int_L dA}$$

This coordinate $\bar{x}$ corresponds to the centroid of the area under the distributed loading.

In other words, the resultant force has a line of action that passes through the geometric centre of the area under the loading diagram.

Distributed loading

In many cases, the distributed loading has a simple shape and it is straightforward to find the magnitude and location of the resultant force.



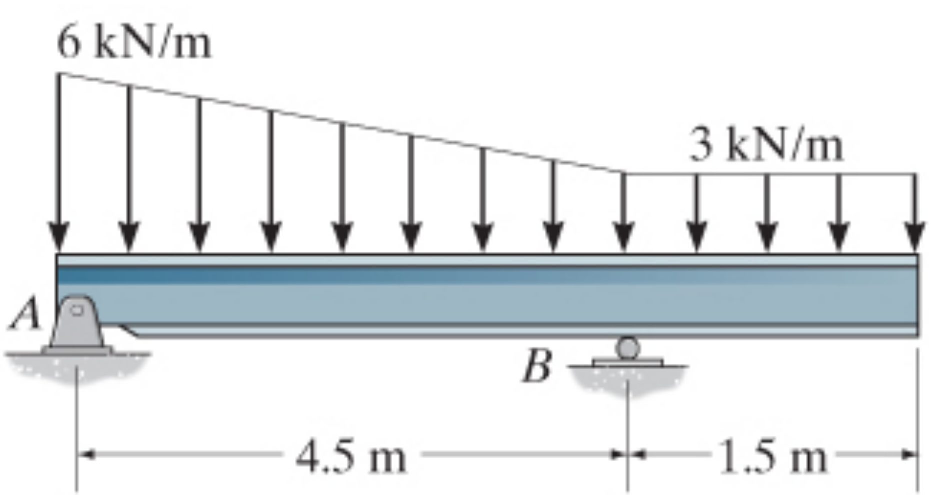
Here. the line load w_0 is constant and applied over a length *b.*

The resultant force is: $F_R = w_0 b$.

The resultant force is applied in the middle, at a distance $b/2$.

Example

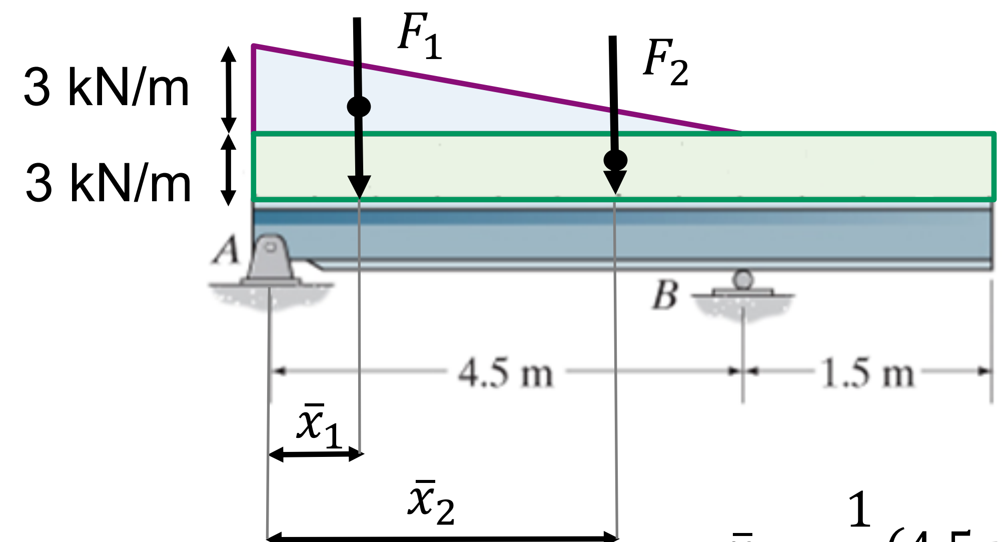
Determine the magnitude of the resultant force and its location (relative to point A).



The magnitude of the resultant force is the area under distributed load. We can divide the distribution in two areas:

Triangle: $+\downarrow\ F_1 = \frac{1}{2}(3\ \text{kN/m})(4.5\text{m}) = 6.75\ \text{kN}$

Rectangle: $+\downarrow\ F_2 = (3\ \text{kN/m})(6\ \text{m}) = 18\ \text{kN}$

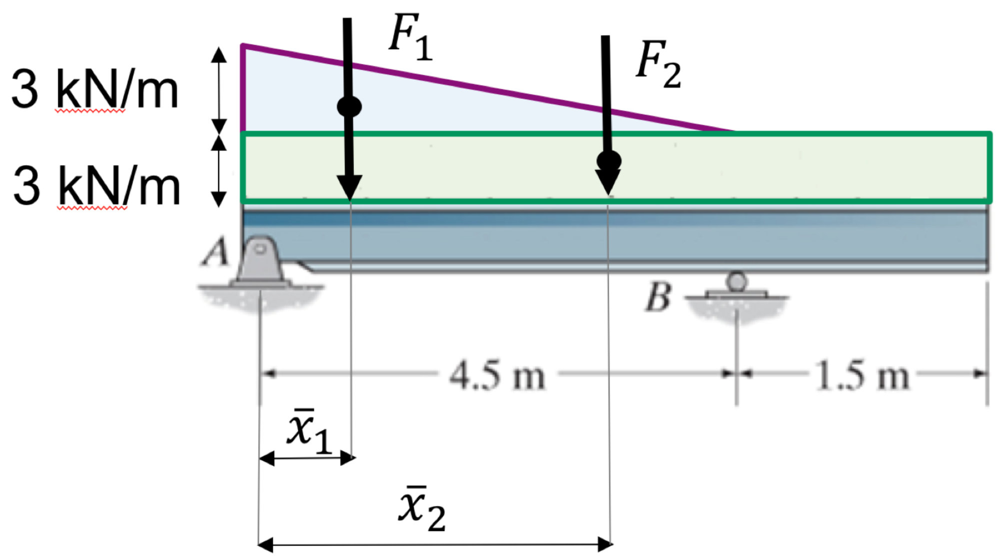


The centroid for each distribution is located at:

$$\bar{x}_1 = \frac{1}{3}(4.5\ \text{m}) = 1.5\ \text{m}$$

$$\bar{x}_2 = \frac{1}{2}(6\ \text{m}) = 3\ \text{m}$$

Example



The magnitude of the resultant force is:

$$+\downarrow\ F_R = F_1 + F_2$$

$$= 6.75\text{ kN} + 18\text{ kN} = 24.75\text{ kN}$$

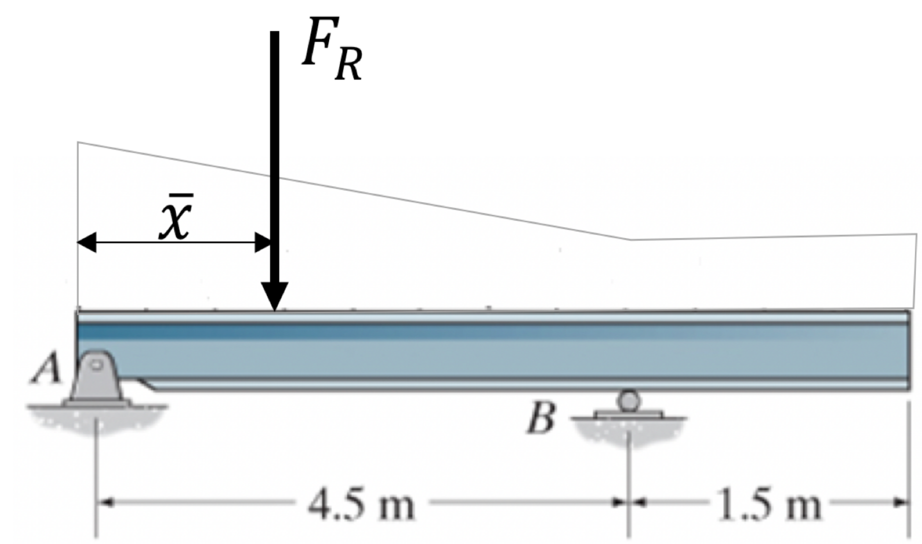
Its location is obtained with the sum of moments:

$$+\circlearrowleft\ (M_R)_A = \Sigma M_A$$

$$-F_R\bar{x} = -F_1\bar{x}_1 - F_2\bar{x}_2$$

$$\bar{x}(24.75\text{ kN}) = 6.75\text{kN}(1.5\text{ m}) + 18\text{ kN}(3\text{ m})$$

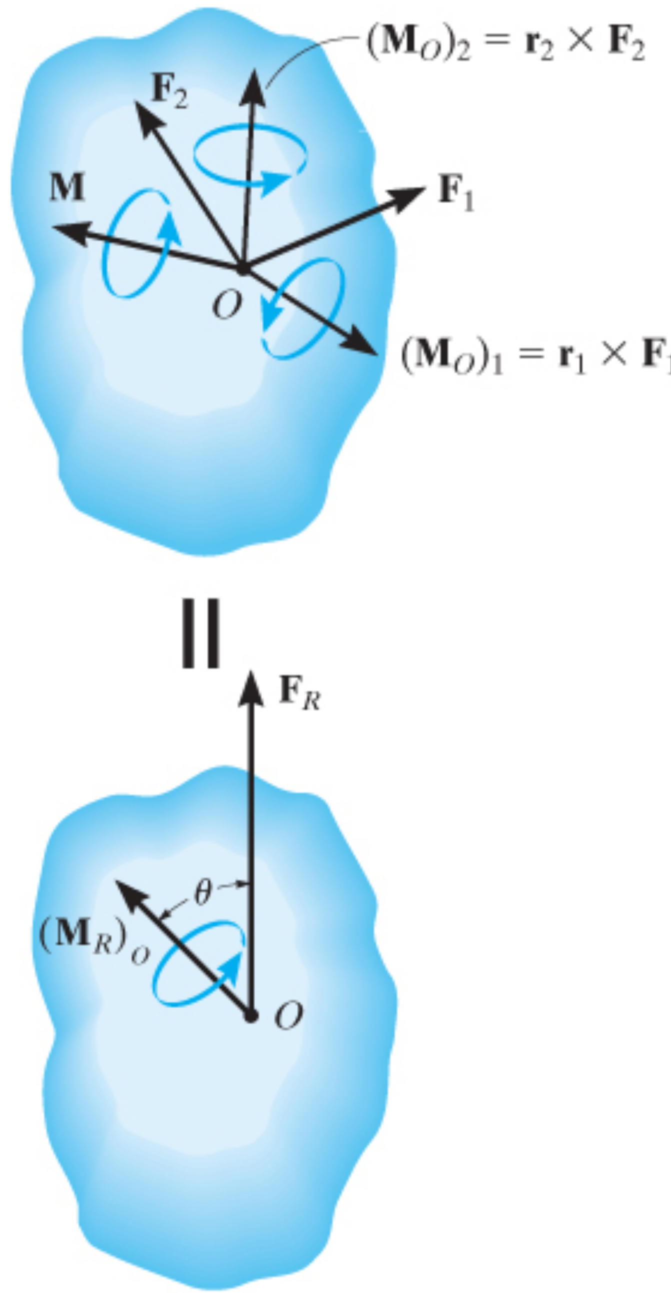
$$\bar{x} = 2.59\text{ m}$$



Summary

- Moments are vectors:
  - They are the cross product of a position vector and a force vector.

- Complex loading scenarios can be simplified into a single resultant force and resultant moment.



Need more explanations?

For more details, consult these sections of the book:

4.1 to 4.6	Moments
4.7,4.8	Simplification of a force and couple system
4.9	Reduction of a simple distributed loading