Aalto-yliopisto
Aalto-universitetet
Aalto University

COE-C1001: Dynamics

10. Work and energy for a rigid body

Luc St-Pierre

Learning outcomes

After this lecture, you should be able to:

- Use equations of motion, and

- The energy approach

To solve dynamic problems of rotation about a fixed axis.

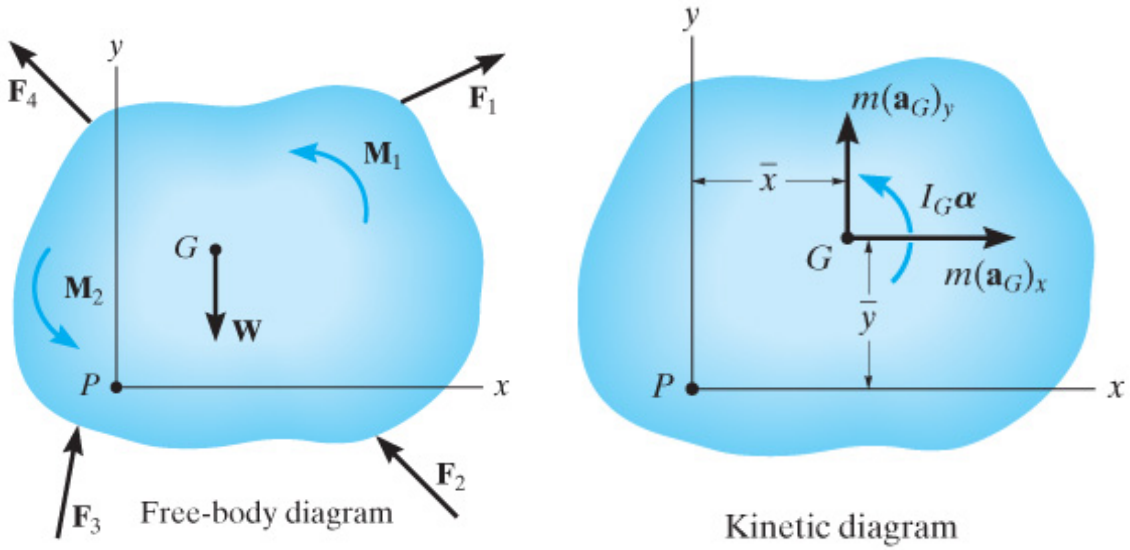
Planar motion of a rigid body

b) Rotation about a fixed axis

- Equations of motion

Aalto-yliopisto
Aalto-universitetet
Aalto University

Planar kinetic equations of motion



The inertial reference frame x, y has its origin at an arbitrary point P.

Planar equations of motion for a rigid body.

All equations specified about the center of mass G.

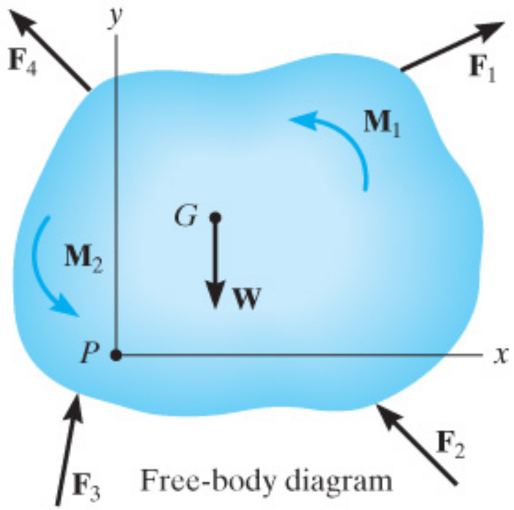
$$\Sigma F_x = m(a_G)_x$$

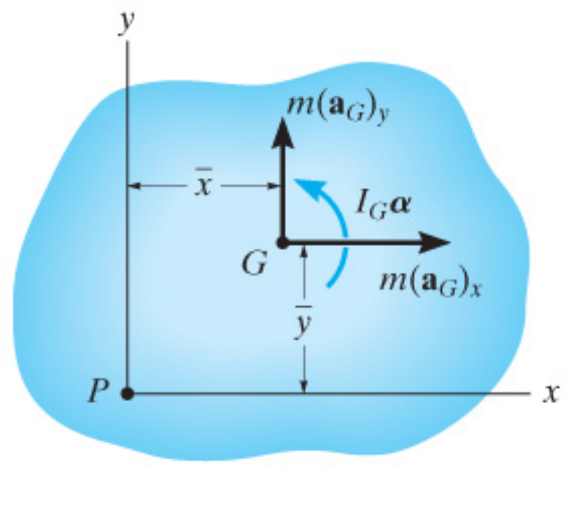
$$\Sigma F_y = m(a_G)_y$$

$$\Sigma M_G = I_G \alpha$$

- If $\alpha = 0$, then we have translational motion only.
- If $(a_G)_x = (a_G)_y = 0$, then the body is rotating about point G.

Planar kinetic equations of motion





The inertial reference frame x, y has its origin at an arbitrary point P.

Often, the body rotates about an arbitrary point *P*, instead of about its center of mass. Rewriting the sum of moments:

$$\Sigma M_P = \Sigma(M_k)_P = -\bar{y}m(a_G)_x + \bar{x}m(a_G)_y + I_G\alpha$$

Here, M_k refers to the *kinetic moments* obtained from the kinetic diagram.

Equation of motion: rotation about a fixed axis

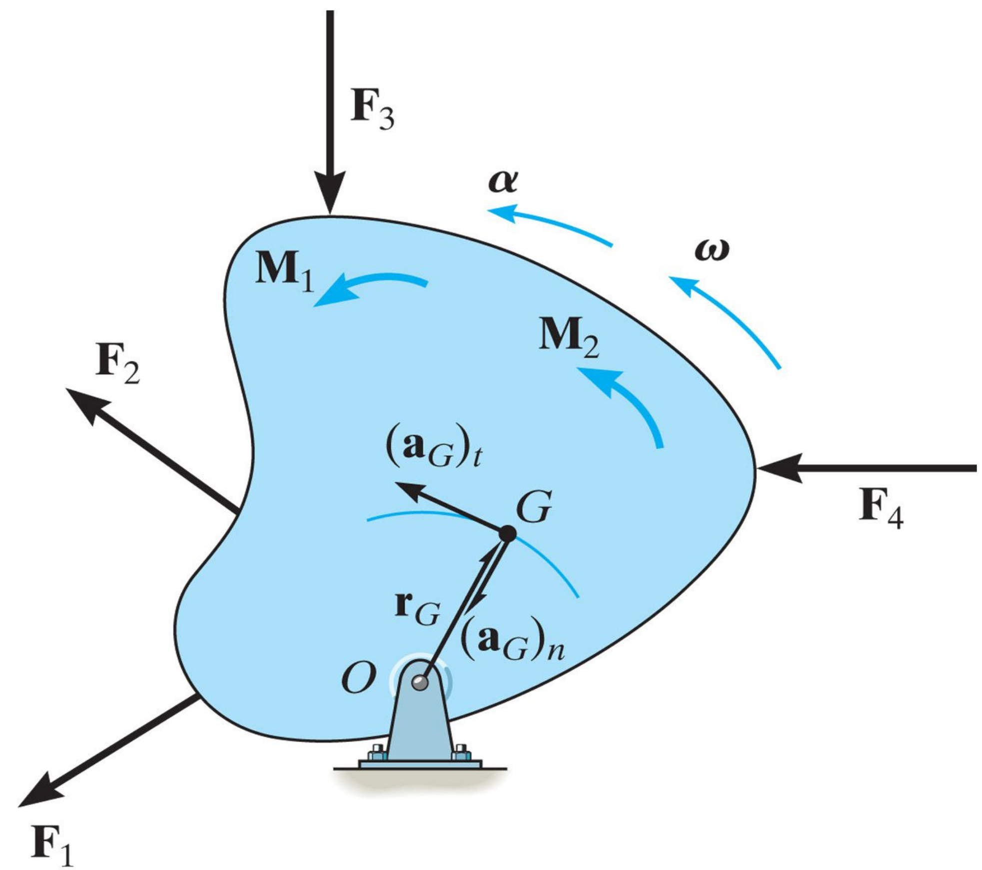
Consider a rigid body rotating about a fixed axis *O*, which is different from the center of mass *G*. Because the center of mass moves along a circular path, its acceleration is best described using normal and tangential components:

$$\Sigma F_n = m(a_G)_n = m\omega^2 r_G$$

$$\Sigma F_t = m(a_G)_t = m\alpha r_G$$

$$\Sigma M_G = I_G \alpha$$

(where normal and tangential accelerations were introduced Lecture 7 slide 26-27.)



Equation of motion: rotation about a fixed axis

Often, it is convenient to sum moments about point O instead of the center of mass *G*. Using the kinetic diagram

$$\Sigma M_O = r_G m (a_G)_t + I_G \alpha$$

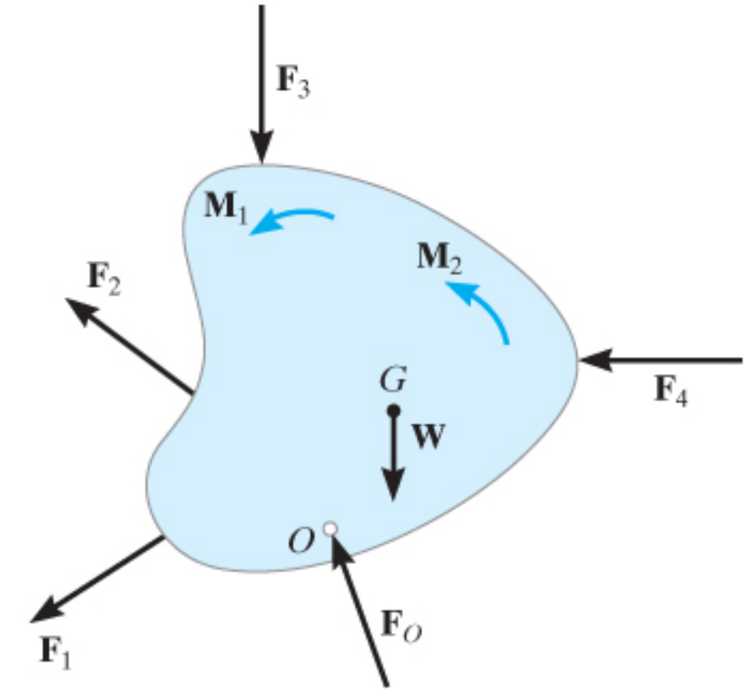
where $(a_G)_t = \alpha r_G$ **(circular motion of a point)**

Substituting above gives:

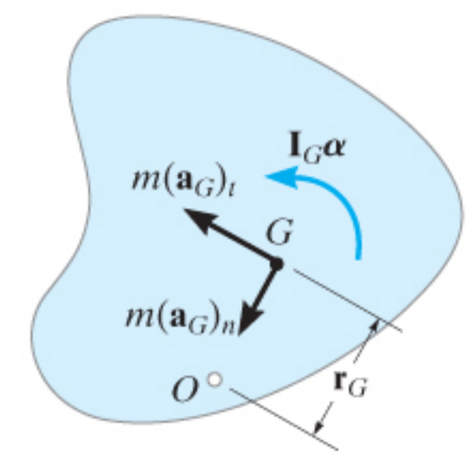
$$\Sigma M_O = r_G m \alpha r_G + I_G \alpha$$

$$= (I_G + m r_G{}^2)\,\alpha$$ **(Parallel-axis theorem)**

$$= I_O \alpha$$

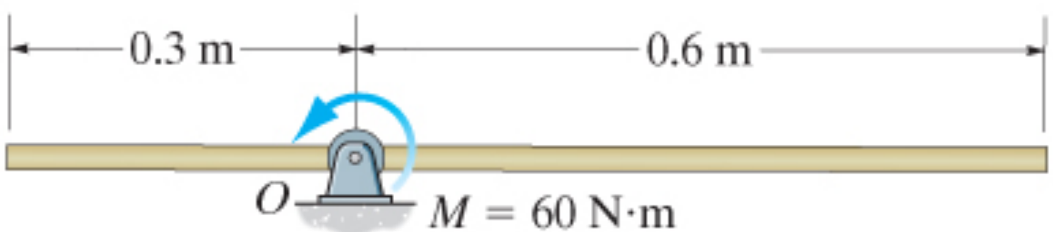


=

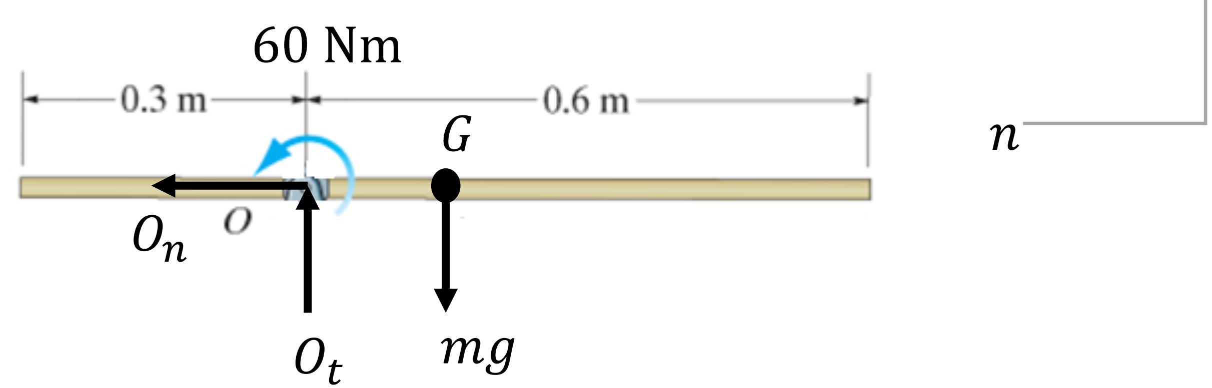


Example

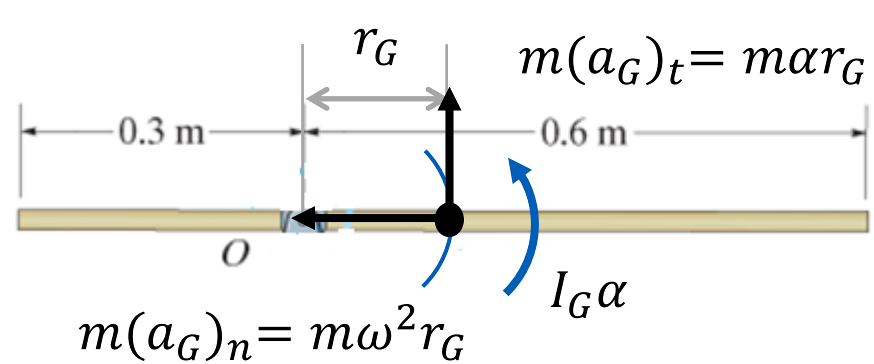
At the instant shown, the uniform 30 kg bar has a counterclockwise angular velocity $\omega = 6\ \mathrm{rad/s}$. Find the tangential and normal components of the support reactions at *O* and the angular acceleration at this instant.



Free-body diagram:

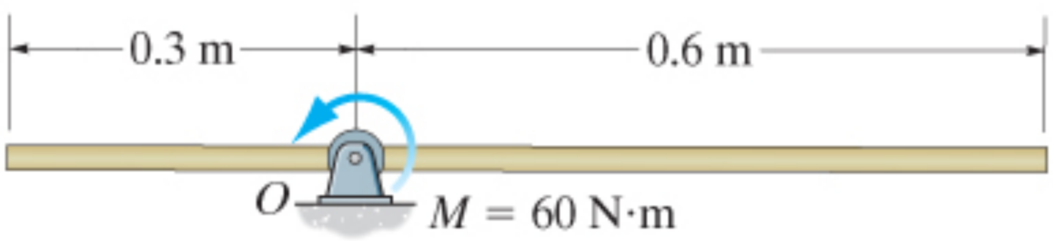


Kinetic diagram:



Example

At the instant shown, the uniform 30 kg bar has a counterclockwise angular velocity $\omega = 6\ \mathrm{rad/s}$. Find the tangential and normal components of the support reactions at *O* and the angular acceleration at this instant.



Equations of motion:

$$\Sigma F_n = m(a_G)_n \quad = m\omega^2 r_G$$

$$\Sigma F_t = m(a_G)_t \quad = m\alpha r_G$$

$$\Sigma M_O = I_O \alpha$$

The distance from point *O* to the center of mass:

$$r_G = \left(\frac{0.9}{2} - 0.3\right)\mathrm{m} = 0.15\ \mathrm{m}$$

Moment of inertia about point *O*:

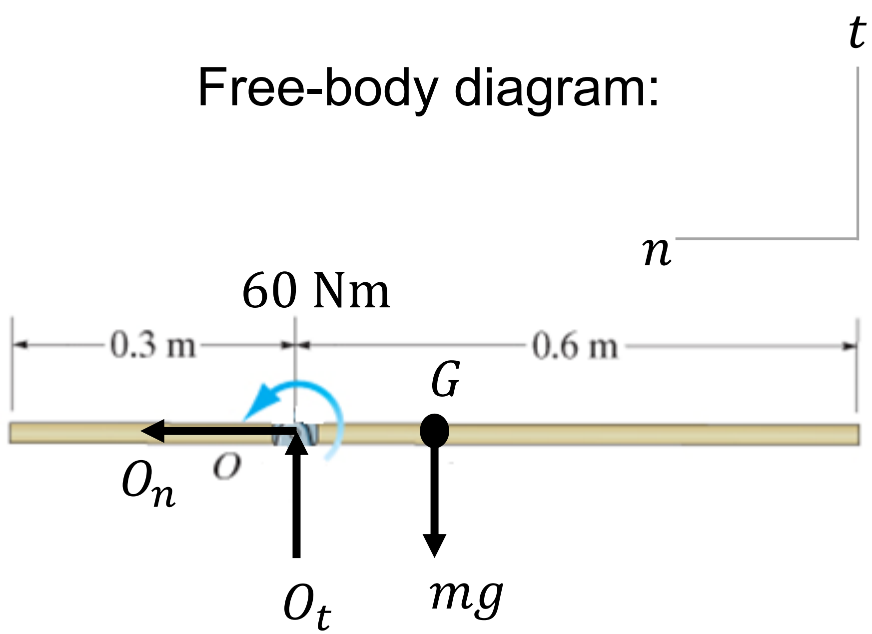
$$I_O = I_G + m{r_G}^2 = \frac{1}{12}ml^2 + m{r_G}^2$$

$$= \frac{1}{12}(30\ \mathrm{kg})(0.9\mathrm{m})^2 + (30\mathrm{kg})(0.15\mathrm{m})^2$$

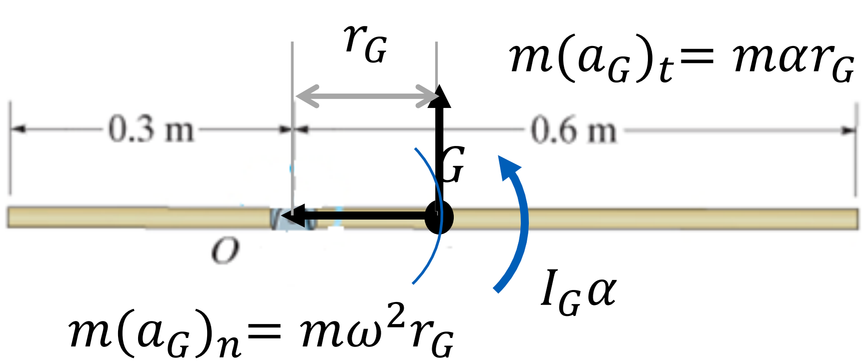
$$= 2.7\mathrm{kg\ m}^2$$

Example

Free-body diagram:



Kinetic diagram



The angular acceleration is obtained from $\Sigma M_O = I_O\alpha$

$$\Sigma M_O = 60\text{ Nm} - mg(0.15\text{ m}) = I_O\alpha$$

$$\alpha = \frac{\Sigma M_O}{I_O} = \frac{60\text{ Nm} - (30 * 9.81\text{ N})(0.15\text{ m})}{2.7\text{ kg m}^2} = 5.87\text{ rad/s}^2$$

Support reactions are obtained from:

$$\leftarrow + \Sigma F_n = O_n = m(a_G)_n = m\omega^2 r_G$$

$$\uparrow + \Sigma F_t = O_t - mg = m(a_G)_t = m\alpha r_G$$

$$O_n = m\omega^2 r_G = (30\text{ kg})(6\text{ rad/s})^2(0.15\text{ m}) = 162\text{ N}$$

$$O_t - mg = m\alpha r_G$$

$$O_t = (30 \cdot 9.81\text{N}) + (30\text{ kg})\left(5.87\frac{\text{rad}}{\text{s}^2}\right)(0.15\text{ m}) = 321\text{ N}$$

Planar motion of a rigid body

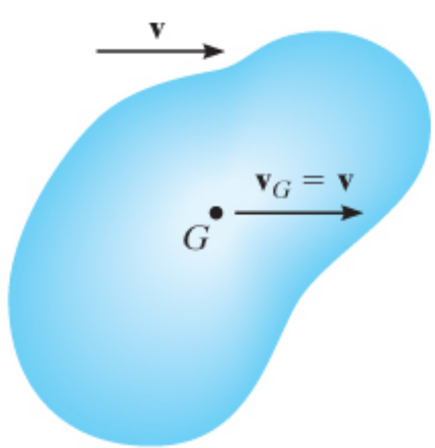
b) Rotation about a fixed axis

- Work and energy

Kinetic energy

The kinetic energy of a rigid body in translation is:

$$\boxed{T = \frac{1}{2} m {v_G}^2}$$

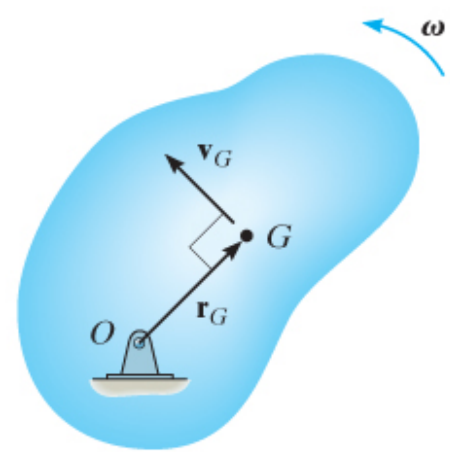


Translation

When the body is rotating about point *O*, the body has both translational and rotational kinetic energy:

$$T = \frac{1}{2} m {v_G}^2 + \frac{1}{2} I_G \omega^2 = \frac{1}{2} m (r_G \omega)^2 + \frac{1}{2} I_G \omega^2 = \frac{1}{2} (m {r_G}^2 + I_G) \omega^2$$

Since the velocity $v_G = r_G \omega$. Recall that $I_0 = I_G + m r_G^2$ because of the parallel-axis theorem. Therefore, we get:

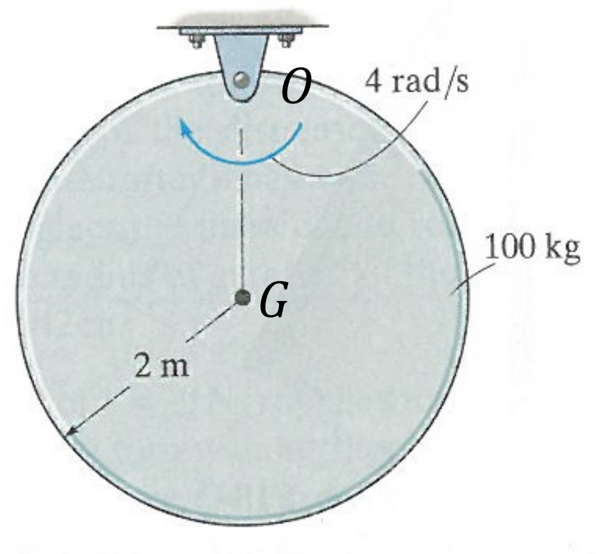


Rotation About a Fixed Axis

$$\boxed{T = \frac{1}{2} I_O \omega^2}$$

Example

Find the kinetic energy of the body.



The moment of inertia about *G* is:

$$I_G = \frac{1}{2}mr^2 = \frac{1}{2}(100)(2)^2 = 200\ \mathrm{kg\cdot m^2}$$

Using the parallel-axis theorem, we get:

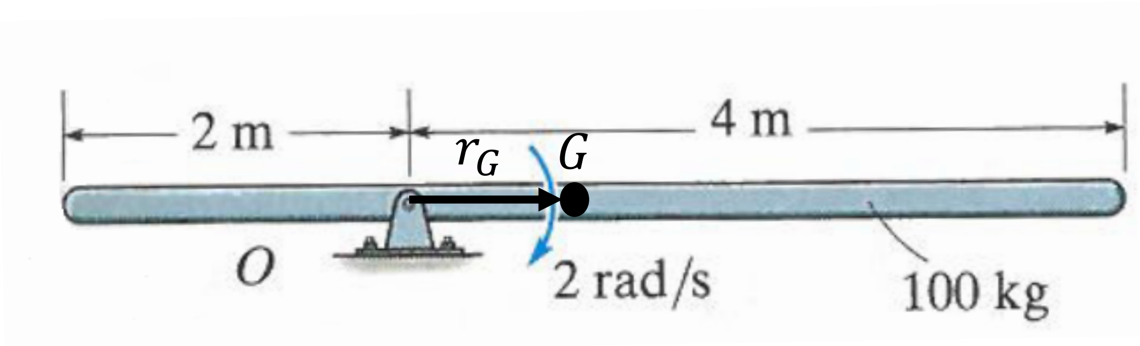
$$I_0 = I_G + m{r_G}^2 = 200 + 100(2)^2 = 600\ \mathrm{kg\cdot m^2}$$

Finally, the kinetic energy is:

$$T = \frac{1}{2}I_O\omega^2 = \frac{1}{2}600(4)^2 = 4800\ \mathrm{J}$$

Example

Find the kinetic energy of the bar.



The moment of inertia about *G* is:

$$I_G = \frac{1}{12}ml^2 = \frac{1}{12}(100)(6)^2 = 300 \text{ kg} \cdot \text{m}^2$$

Using the parallel-axis theorem, we get:

$$I_0 = I_G + m{r_G}^2 = 300 + 100(1)^2 = 400 \text{ kg} \cdot \text{m}^2$$

Finally, the kinetic energy is:

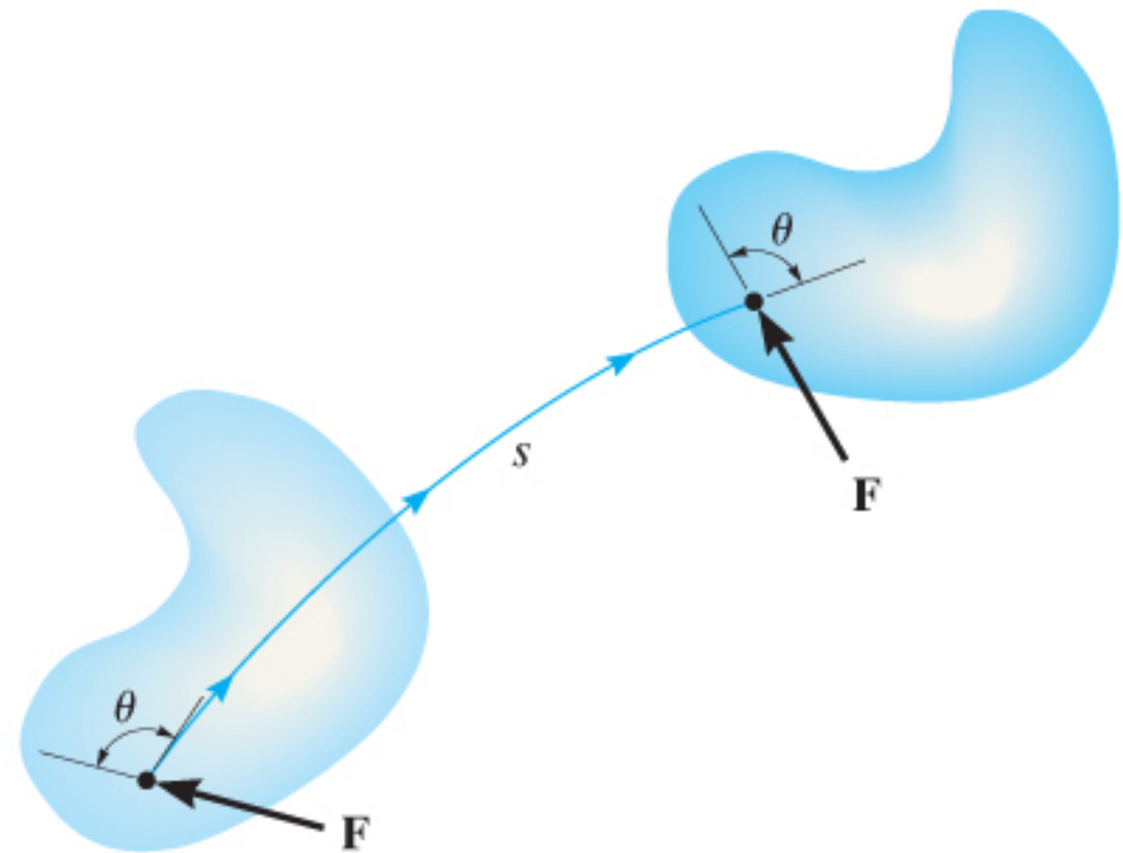
$$T = \frac{1}{2}I_O\omega^2 = \frac{1}{2}400(2)^2 = 800 \text{ J}$$

Work of a force

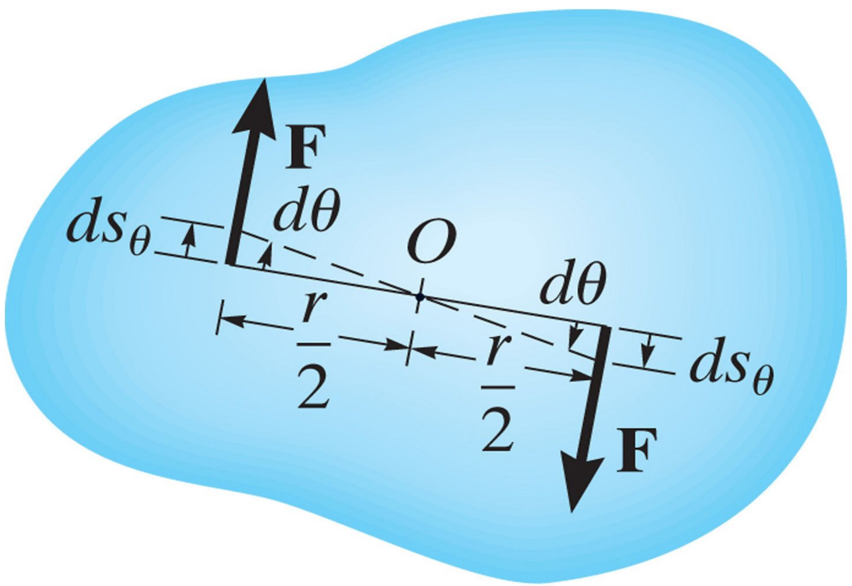
We saw earlier that the work done by an external force $\mathbf{F}$ acting on a body (or particle) moving along path s is:

$$U_F = \int \mathbf{F} \cdot d\mathbf{r} = \int_s F \cos\theta \, ds$$

We can use this to compute the work done by a couple moment.



Work of a couple moment



Rotation

Consider a body subjected to a couple moment $M = Fr$.

When the body rotates by an infinitesimal angle $d\theta$ about point *O*, each force undergoes a displacement $ds_\theta = (r/2)d\theta$. The total work done is:

$$dU_M = F\left(\frac{r}{2}d\theta\right) + F\left(\frac{r}{2}d\theta\right) = (Fr)d\theta = Md\theta,$$

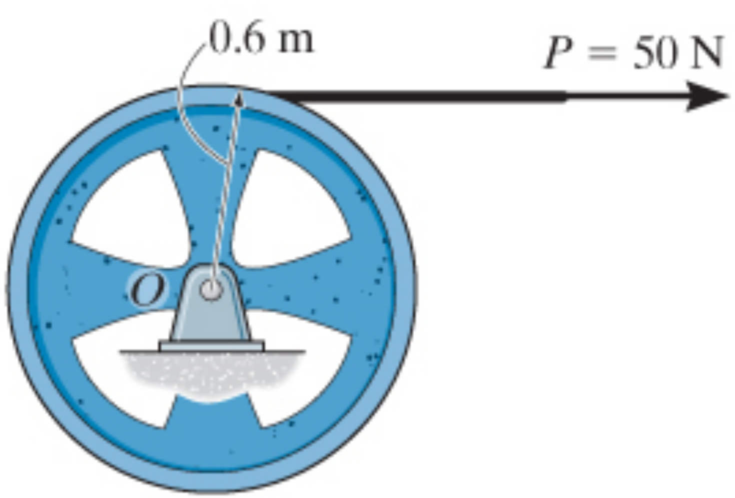
Integrating this relation from θ_1 to θ_2 gives:

$$U_M = \int_{\theta_1}^{\theta_2} Md\theta.$$

If the moment is constant this becomes: $U_M = M(\theta_2 - \theta_1)$.

Example

The 80 kg wheel has a moment of inertia $I_O = 12.8\,\mathrm{kgm}^2$ (about its center of mass). Find its angular velocity after it has rotated 20 revolutions starting from rest.



Principle of work and energy: $T_1 + \Sigma U_{1-2} = T_2$

The wheel start from rest: $T_1 = 0$. The work done by the force can be obtained in two ways.

Method 1: the distance travelled after 20 revolutions is:

$$s = 20(2\pi)(0.6\mathrm{m}) = 75.398\ \mathrm{m}$$

And the work done is:

$$\Sigma U_{1-2} = Ps = 50\mathrm{N}(75.398\ \mathrm{m}) = 3769.91\ \mathrm{J}$$

Method 2: The moment created by force *P* is :

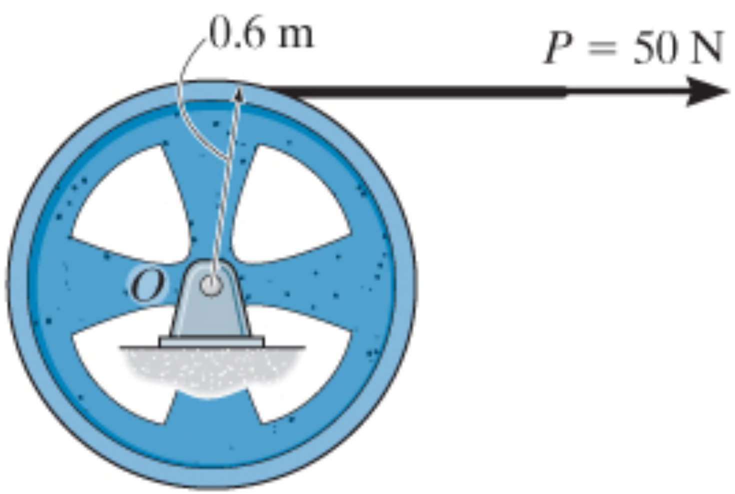
$$M = Pr = 50\mathrm{N}(0.6\ \mathrm{m}) = 30\ \mathrm{Nm}$$

And the work done by this moment is:

$$\Sigma U_{1-2} = Md\theta = 30(20)(2\pi) = 3769.91\ \mathrm{J}$$

Example

The 80 kg wheel has a moment of inertia $I_O = 12.8\ \mathrm{kgm}^2$ (about its center of mass). Find its angular velocity after it has rotated 20 revolutions starting from rest.



Principle of work and energy: $T_1 + \Sigma U_{1-2} = T_2$

The wheel start from rest: $T_1 = 0$, and we found $\Sigma U_{1-2} = 3769.91\ \mathrm{J}$.

Finally, the angular velocity after 20 revolutions is:

$$\Sigma U_{1-2} = T_2 = \frac{1}{2} I_O \omega^2 \quad \Rightarrow \omega = \sqrt{\frac{2\Sigma U_{1-2}}{I_O}}$$

$$= \sqrt{\frac{2 * 3769.91\ \mathrm{Nm}}{12.8\ \mathrm{kgm}^2}} = 24.3 \frac{\mathrm{rad}}{\mathrm{s}}$$

Reminder: potential energy

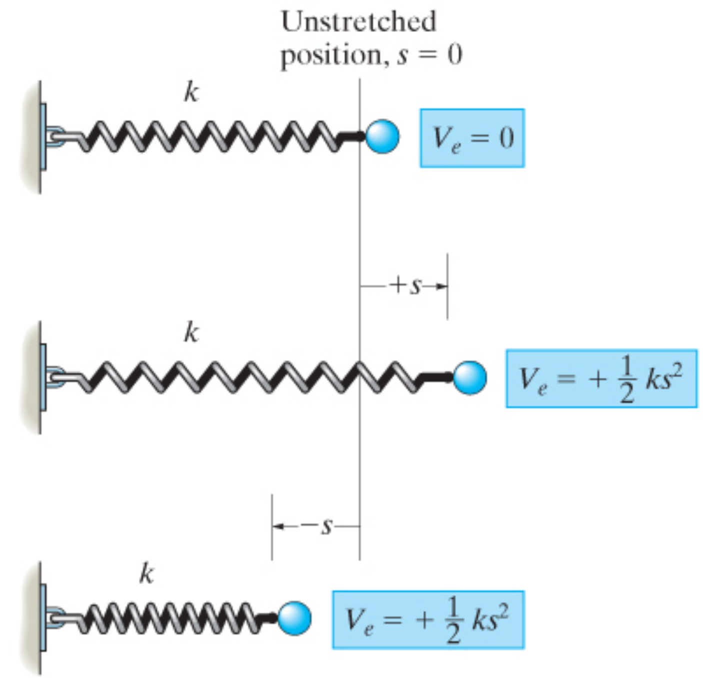
Potential energy represents the work done by a conservative force as it moves from a given position to a reference position.

It include gravitational potential energy: $V_g = Wy$

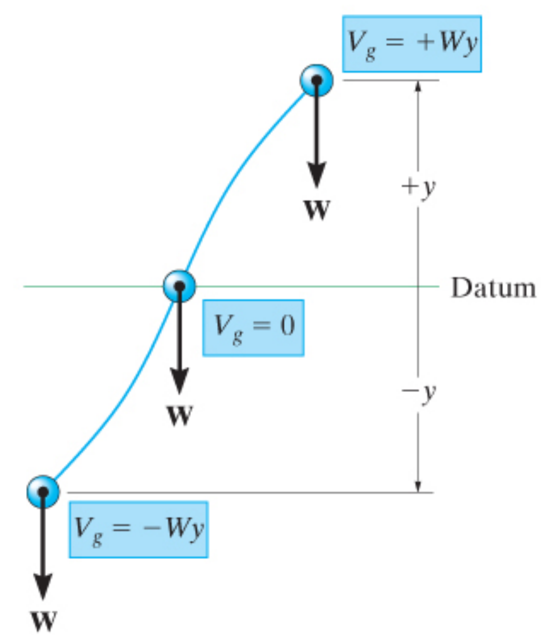
and elastic potential energy: $V_e = \frac{1}{2}ks^2$

Therefore, the potential function, or total potential energy is:

$$V = V_g + V_e$$



Elastic potential energy



Gravitational potential energy

Reminder: conservation of energy

Potential energy is the work done by conservative forces:

$$(\Sigma U_{1-2})_{cons} = V_1 - V_2$$

The total work done includes conservatives and nonconservative forces:

$$\Sigma U_{1-2} = (\Sigma U_{1-2})_{cons} + (\Sigma U_{1-2})_{noncons} = V_1 - V_2 + (\Sigma U_{1-2})_{noncons}$$

Substituting in the principle of work and energy:

$$T_1 + \Sigma U_{1-2} = T_2$$

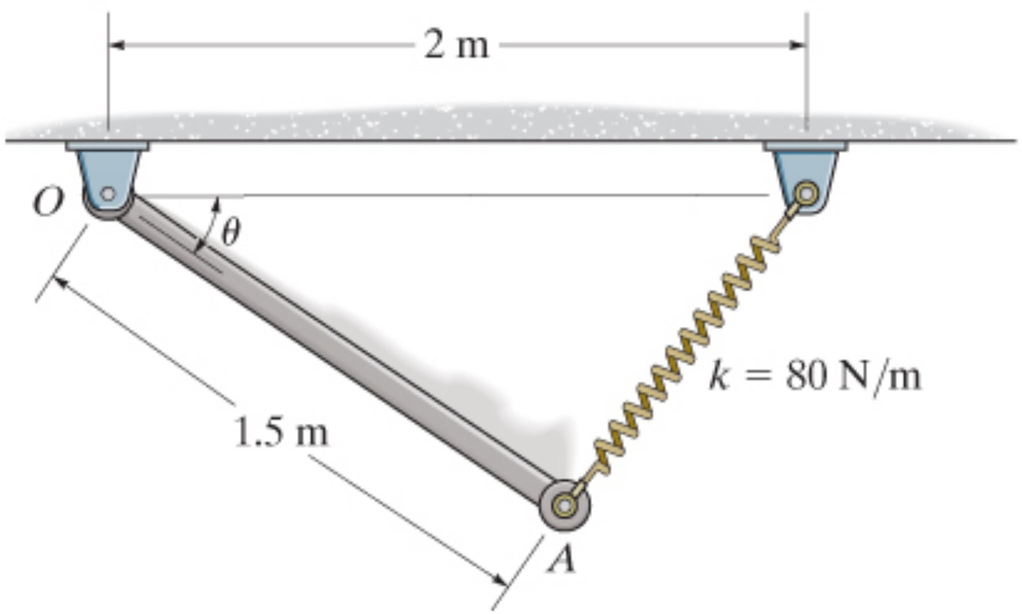
$$\Rightarrow T_1 + V_1 + (\Sigma U_{1-2})_{noncons} = T_2 + V_2$$

If only conservative forces are present, this becomes:

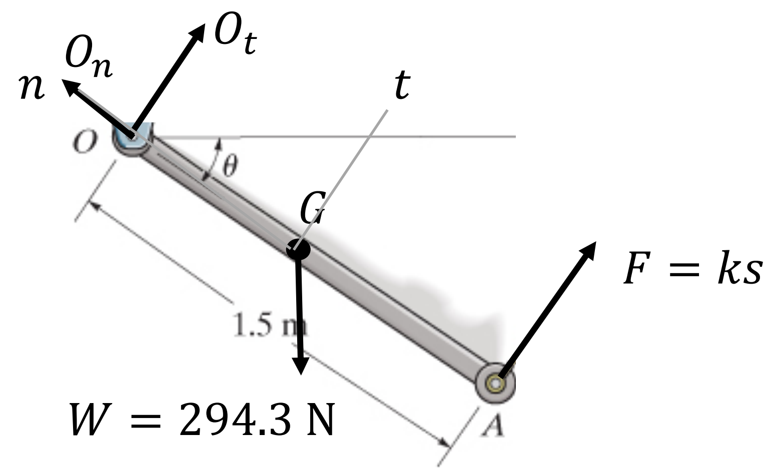
$$T_1 + V_1 = T_2 + V_2$$

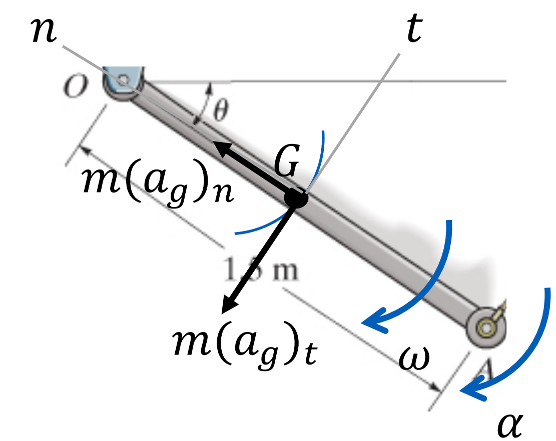
Example

The 30 kg rod is released from rest when $\theta = 0$. Find the angular velocity of the rod when $\theta = 90°$. The spring is unstretched when $\theta = 0$.



Draw free-body and kinetic diagrams:





The kinetic energy of the rod is:

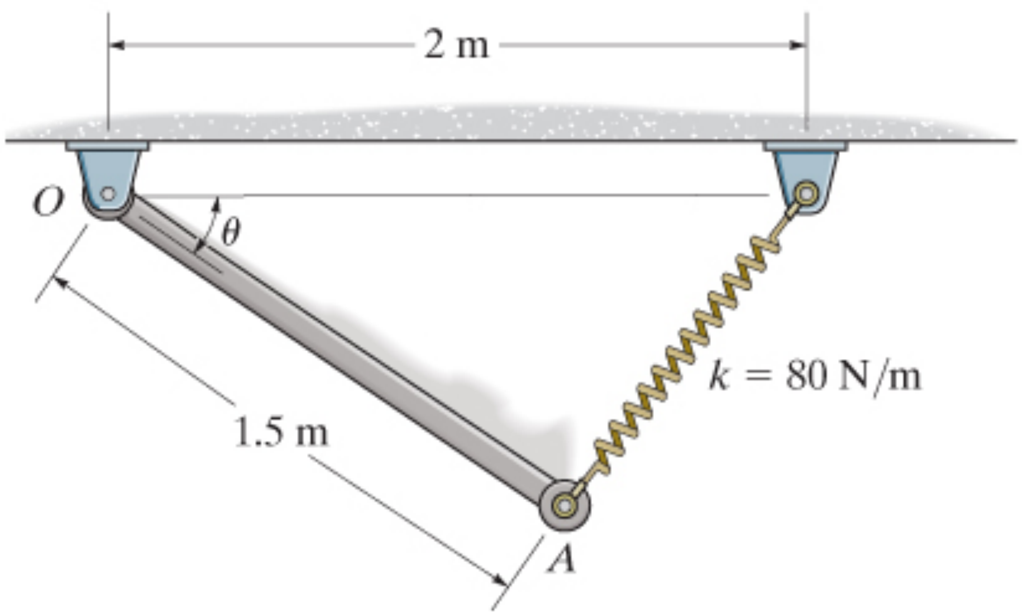
$$T = \frac{1}{2} I_O \omega^2$$

Where the moment of inertia is:

$$I_O = \frac{1}{3} m l^2 = \frac{1}{3} 30 \cdot 1.5^2 = 22.5 \text{ kg} \cdot \text{m}^2$$

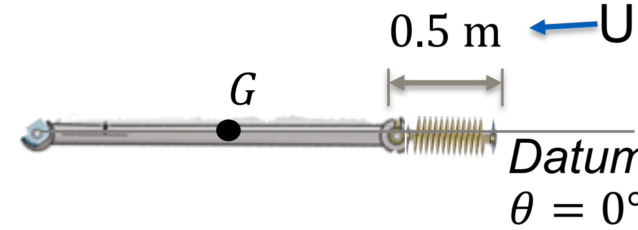
Example

The 30 kg rod is released from rest when $\theta = 0$. Find the angular velocity of the rod when $\theta = 90°$. The spring is unstretched when $\theta = 0$.



Conservation of energy: $T_1 + V_1 = T_2 + V_2$

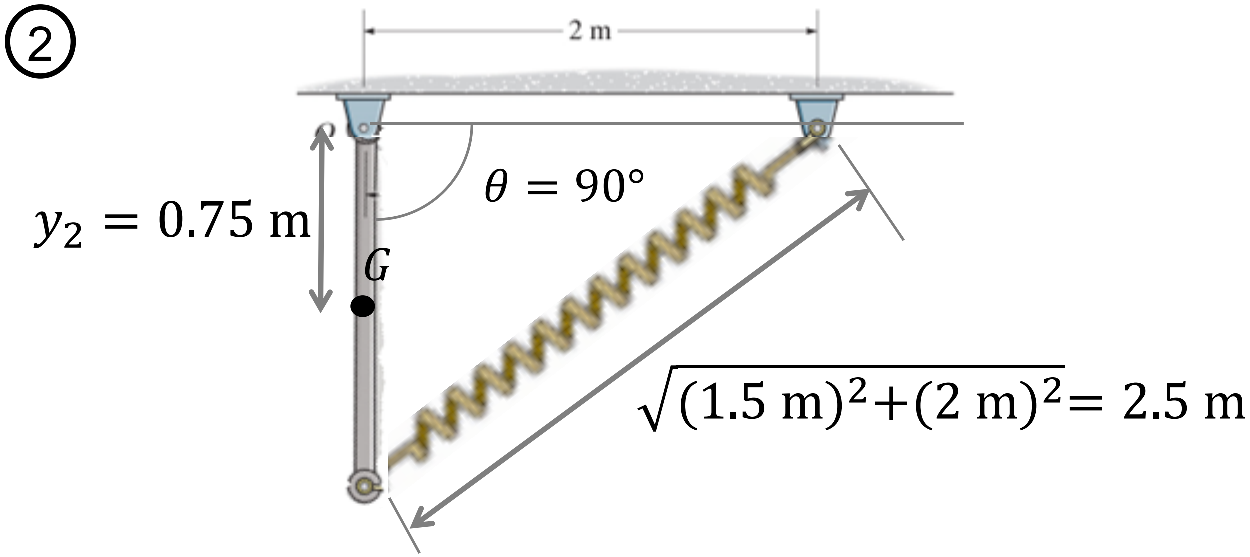
1



Unstretched spring length

$T_1 = V_1 = 0$

2



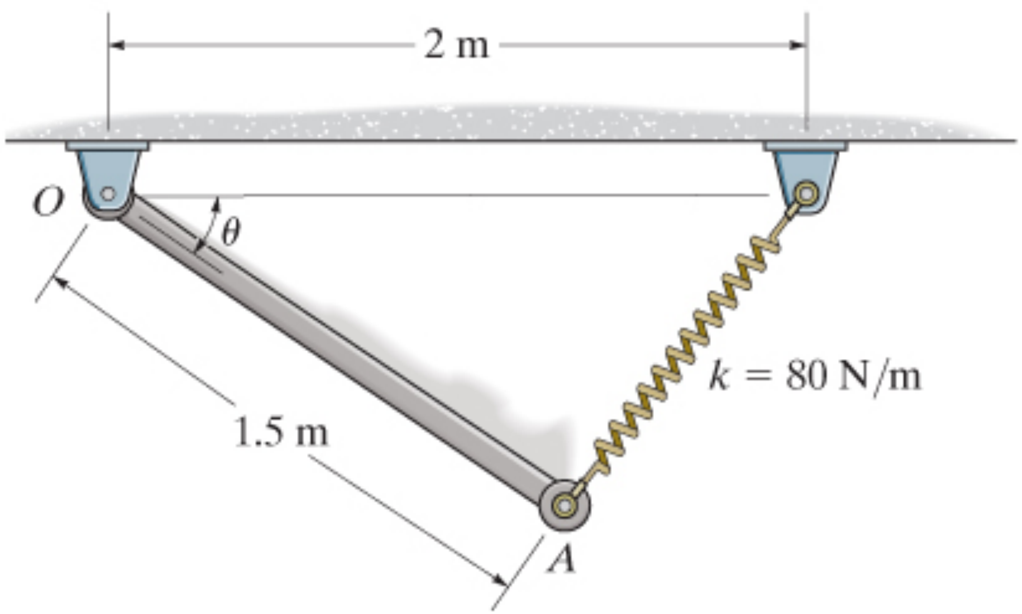
The elongation of the spring is $s_2 = 2.5 - 0.5 = 2$ m

$$V_2 = V_{g2} + V_{e2} = -mgy_2 + \frac{1}{2}ks_2^2$$

$$= -30 \cdot 9.81 \cdot 0.75 + \frac{1}{2}80 \cdot 2^2 = -60.725\text{ J}$$

Example

The 30 kg rod is released from rest when $\theta = 0$. Find the angular velocity of the rod when $\theta = 90°$. The spring is unstretched when $\theta = 0$.



Substituting these results into the principle of energy conservation:

$$T_1 + V_1 = T_2 + V_2$$

$$\Rightarrow 0 + 0 = \frac{1}{2} I_0 \omega^2 + V_2$$

$$\Rightarrow 0 + 0 = \frac{1}{2} I_0 \omega^2 + V_2$$

$$\Rightarrow \omega = \sqrt{\frac{-2V_2}{I_0}} = \sqrt{\frac{-2(-60.725)}{22.5}}$$

$$\Rightarrow \omega = 2.3 \text{ rad/s}$$

Summary

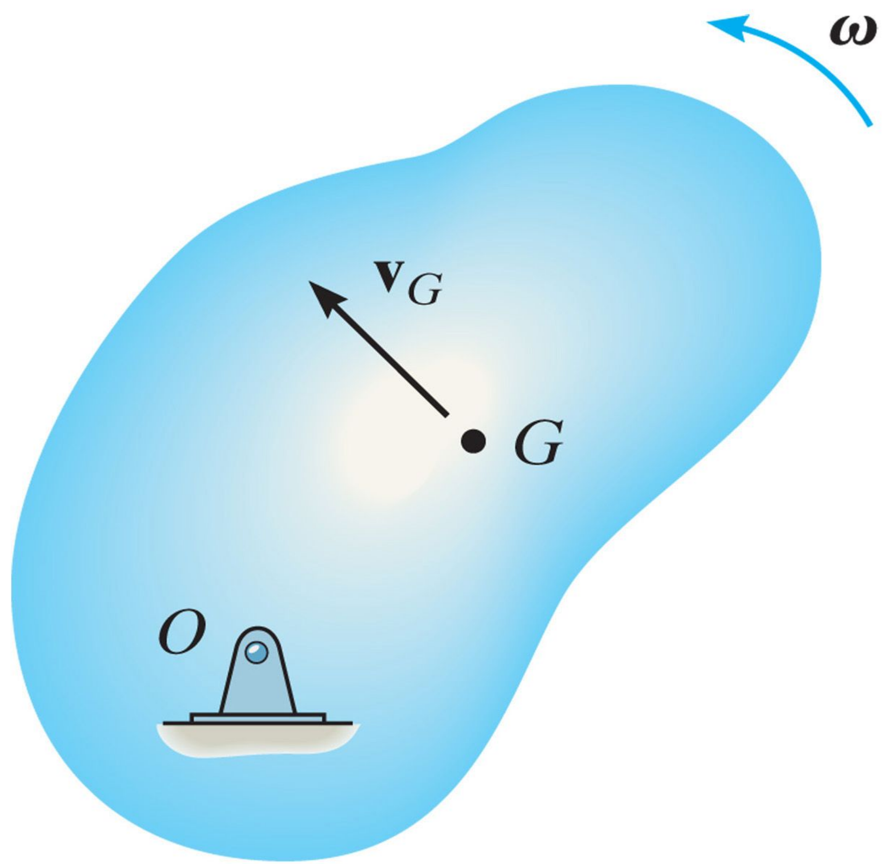
Rotation about a fixed axis

Equations of motion:

$$\begin{cases} \Sigma F_n = m(a_G)_n = m\omega^2 r_G \\ \Sigma F_t = m(a_G)_t = m\alpha r_G \\ \Sigma M_G = I_G\alpha \quad \text{or} \quad \Sigma M_O = I_O\alpha \end{cases}$$

Kinetic equation:

$$\begin{cases} T = \dfrac{1}{2}mv_G^2 + \dfrac{1}{2}I_G\omega^2 \\ \text{or} \\ T = \dfrac{1}{2}I_O\omega^2 \end{cases}$$



Rotation About a Fixed Axis

Need additional information?

For more details, consult these sections of the textbook:

17.2 Planar kinetic equations of motion

17.4 Equations of motion: rotation about a fixed axis

18.1 Kinetic energy

18.3 Work of a couple moment

18.4 Principle of work and energy

18.5 Conservation of energy