COE-C1001: Dynamics

7. Kinematics and kinetics of a particle

Luc St-Pierre

Aalto-yliopisto
Aalto-universitetet
Aalto University

Learning outcomes

After this lecture, you should be able to:

- Describe mathematically the position, velocity and acceleration of a particle along a straight or curved path,

- Use the equation of motion to find the forces causing the motion of a particle.

Introduction

Mechanics can be divided in two fields: statics and dynamics.

- Statics is concerned with the equilibrium of a body ($\Sigma\mathbf{F} = \Sigma\mathbf{M} = \mathbf{0}$).
- Dynamics deals with the accelerated motion of a body ($\Sigma\mathbf{F} = m\mathbf{a}$ and $\Sigma\mathbf{M} = I\boldsymbol{\alpha}$)

We will study dynamics in two parts:

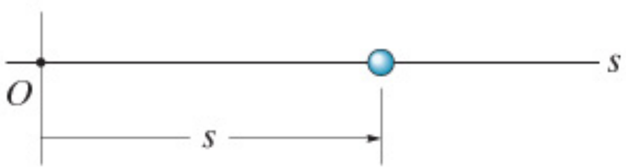
- Kinematics will treat only the geometric aspect of motion.
- Kinetics will deal with the forces causing the motion.

1. Kinematics of a particle

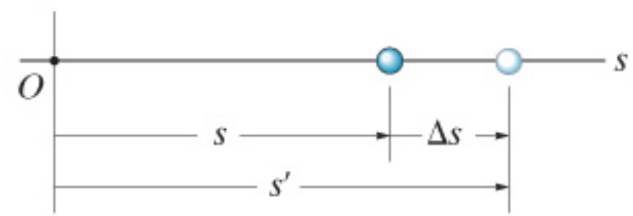
a) Rectilinear motion

Rectilinear kinematics

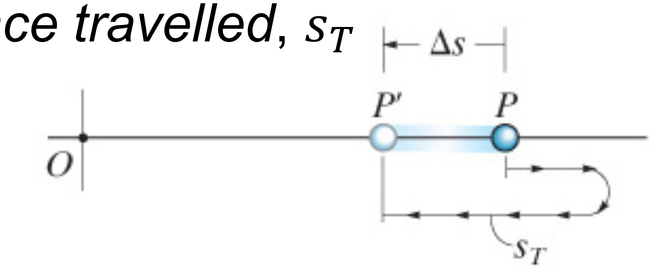
Position, s



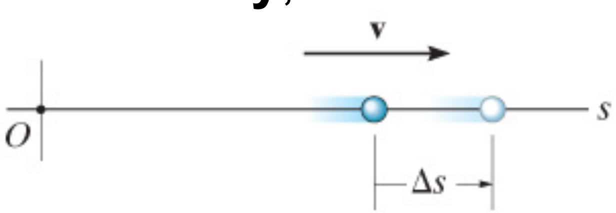
Displacement, $\Delta s = s' - s$



Distance travelled, s_T



Velocity, v

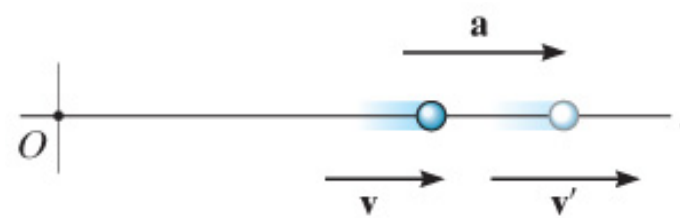


$$\boxed{v = \lim_{\Delta t \to 0} \left(\frac{\Delta s}{\Delta t} \right) = \frac{ds}{dt}}$$

Average velocity

$$v_{avg} = \frac{\Delta s}{\Delta t}$$

Acceleration, a

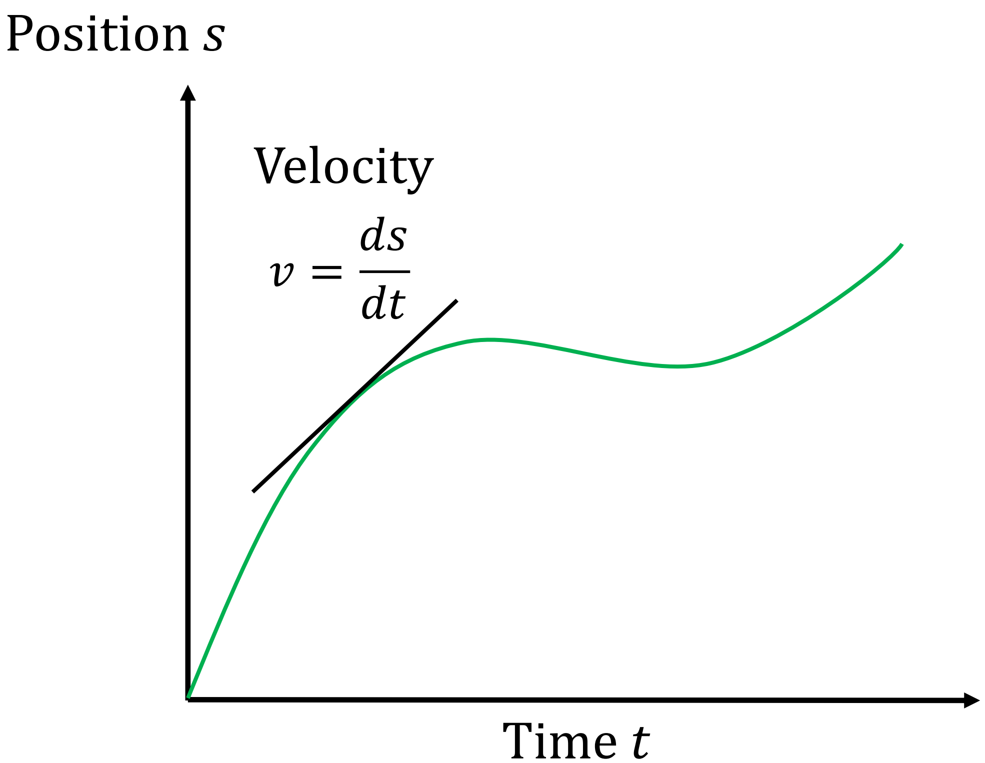


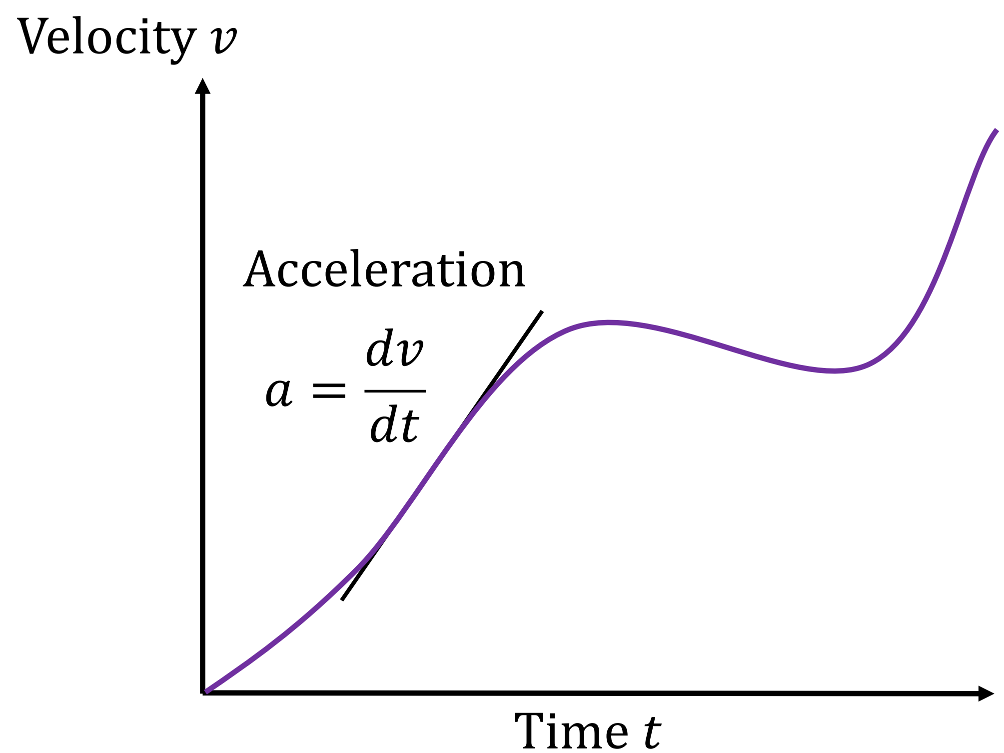
$$\boxed{a = \lim_{\Delta t \to 0} \left(\frac{\Delta v}{\Delta t} \right) = \frac{dv}{dt}}$$

Average acceleration

$$a_{avg} = \frac{\Delta v}{\Delta t}$$

Instantaneous velocity and acceleration





- Velocity corresponds to the slope of the position vs time curve.
- Acceleration is the slope of the velocity vs time curve.

Rectilinear kinematics

Definitions of velocity and acceleration both involve the time differential dt:

$$v = \frac{ds}{dt} \qquad a = \frac{dv}{dt}$$

A relationship between position, velocity and acceleration without dt can be obtained simply by:

$$a = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = \frac{dv}{ds}v \qquad \Rightarrow \qquad \boxed{a\,ds = v\,dv}$$

Kinematics with constant acceleration

If acceleration is constant, $a = a_c$, it is possible to derive three simple relationships:

1. Velocity as a function of time

$$\boxed{a_c = dv/dt} \quad \Rightarrow dv = a_c dt \quad \Rightarrow \int_{v_0}^{v} dv = \int_0^t a_c\, dt \Rightarrow \quad \boxed{v = v_0 + a_c t}$$

2. Position as a function of time

$$\boxed{v = ds/dt} = v_0 + a_c t \quad \Rightarrow ds = (v_0 + a_c t)dt \quad \Rightarrow \int_{s_0}^{s} ds = \int_0^t (v_0 + a_c t)\, dt \Rightarrow \quad \boxed{s = s_0 + v_0 t + \frac{1}{2} a_c t^2}$$

3. Velocity as a function of position

$$\boxed{a_c ds = v\, dv} \quad \Rightarrow \int_{v_0}^{v} v\, dv = \int_{s_0}^{s} a_c\, ds \Rightarrow \quad \boxed{v^2 = {v_0}^2 + 2a_c(s - s_0)}$$

Rectilinear kinematics

Fundamental equations

Velocity	$v = \frac{ds}{dt}$
Acceleration	$a = \frac{dv}{dt}$

Relation without time derivative

$$a\,ds = v\,dv$$

For problems with a **constant acceleration** $a = a_c$:

Velocity vs time:	$v = v_0 + a_c t$
Positions vs time:	$s = s_0 + v_0 t + \frac{1}{2} a_c t^2$
Velocity vs position:	$v^2 = {v_0}^2 + 2a_c(s - s_0)$

Particularly useful for describing the motion of a projectile (gravity is a constant acceleration).

Example

A particle is moving in a rectilinear motion with velocity $v = (4t - 3t^2)\ \mathrm{m/s}$. Find the position of the particle at $t = 4\ \mathrm{s}$. At $t = 0\ \mathrm{s}$ the position is $s = 0$.

Acceleration is **not** constant; therefore, we can't use: $\boxed{\cancel{v^2 = {v_0}^2 + 2a_c(s - s_0)}}$

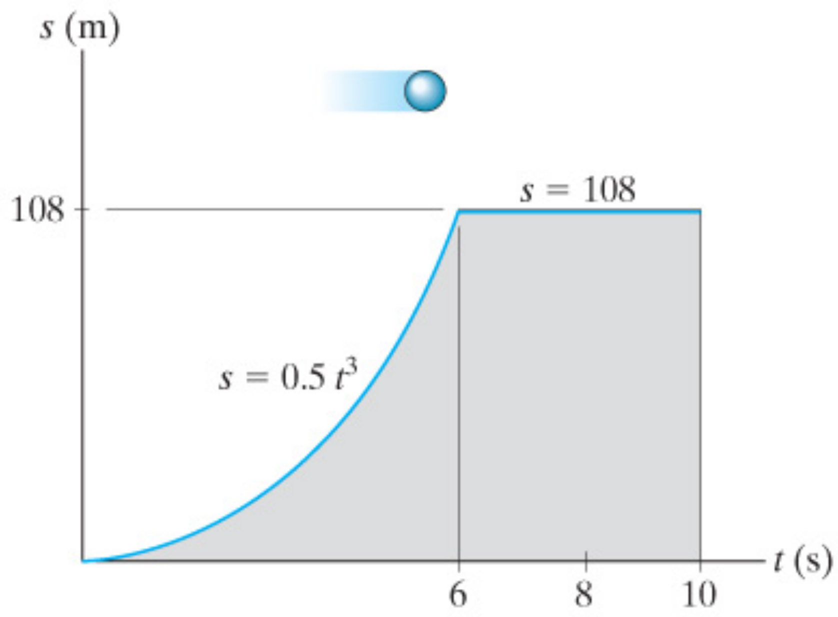
We can rearrange the definition of velocity to obtain:

$$v = \frac{ds}{dt} \;\Rightarrow\; ds = v dt \;\Rightarrow\; \int_0^s ds = \int_0^{t=4} v dt \;\Rightarrow\; s = \int_0^{t=4} (4t - 3t^2) dt$$

$$\Rightarrow\; s = \left[4\frac{t^2}{2} - 3\frac{t^3}{3}\right]_0^{t=4} = 2 \cdot 4^2 - 4^3 = -32\ \mathrm{m}$$

Example

The particle moves in a rectilinear motion, and its position is given below. Find its velocity and acceleration.



The function is discontinuous so we need to consider it piece by piece.

Kinematic equations:

$\boxed{v = ds/dt}$ $\boxed{a = dv/dt}$

The velocity $v(t)$ is obtained by derivating the position:

$0 \le t \le 6$ s; $\quad s = 0.5\,t^3 \quad v = \dfrac{ds}{dt} = 1.5\,t^2$

$6 < t \le 10$ s; $\quad s = 108 \quad v = 0$

Next, we derivate the velocity to obtain the acceleration $a(t)$:

$0 \le t \le 6$ s; $\quad v = 1.5t^2 \quad a = \dfrac{dv}{dt} = 3\,t$

$6 < t \le 10$ s; $\quad v = 0 \quad a = 0$

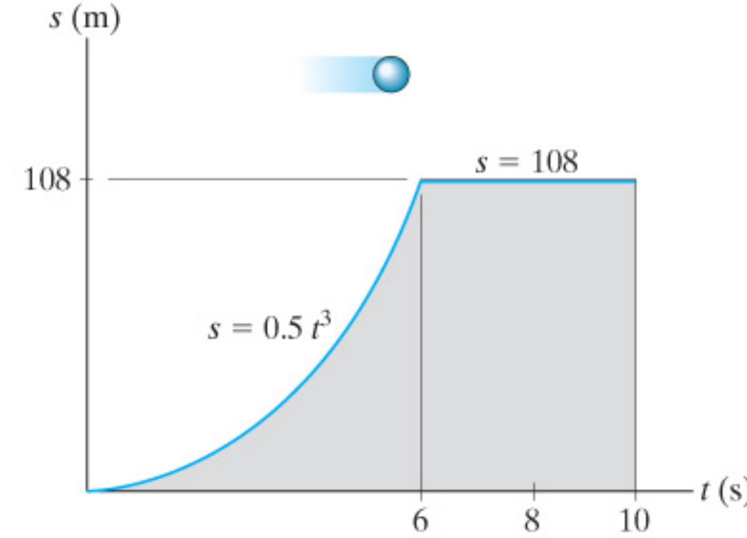
Example/summary

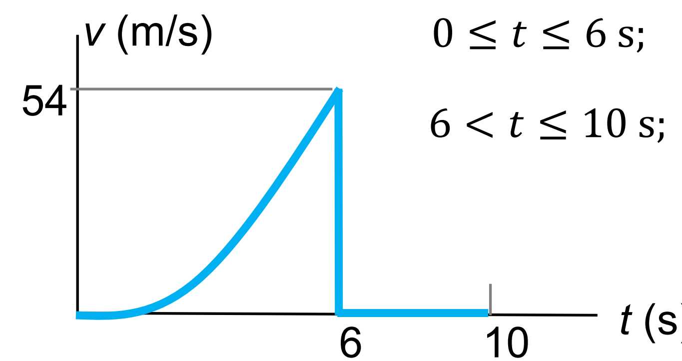
Use basic kinematic equations:

$$v = \frac{ds}{dt} \qquad a = \frac{dv}{dt} \qquad a\,ds = v\,dv$$

To find unknown variables.

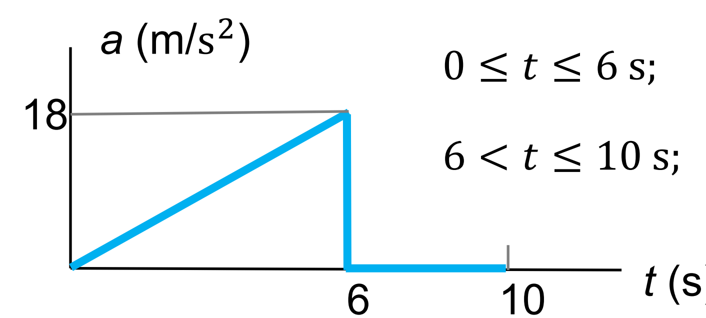
If the functions are discontinuous, consider them piece-by-piece.





$0 \le t \le 6\text{ s}; \qquad v = 1.5\,t^2$

$6 < t \le 10\text{ s}; \qquad v = 0$



$0 \le t \le 6\text{ s}; \qquad a = 3\,t$

$6 < t \le 10\text{ s}; \qquad a = 0$

1. Kinematics of a particle

b) Curvilinear motion

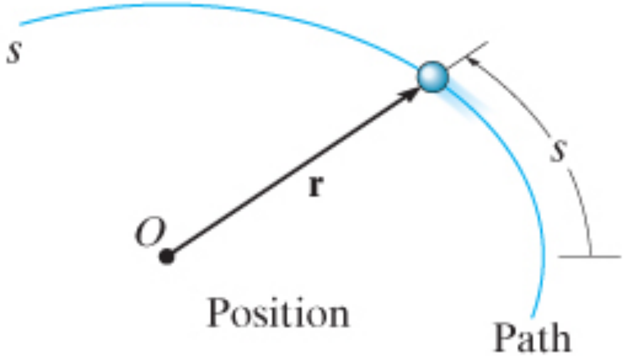
Aalto-yliopisto
Aalto-universitetet
Aalto University

General curvilinear motion

In 2D and 3D, position is a vector. Consequently, velocity and acceleration are also vectors.

Position vector $\mathbf{r}$.

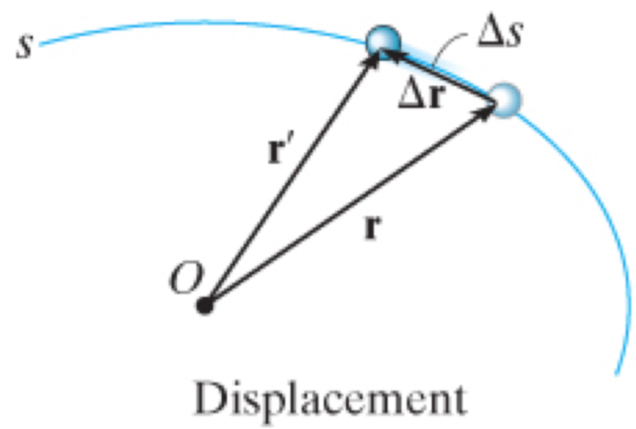
s = distance travelled along the path



The displacement vector is:

$$\Delta\mathbf{r} = \mathbf{r}' - \mathbf{r}$$

($|\Delta\mathbf{r}| \rightarrow \Delta s$, when $\Delta t \rightarrow 0$)

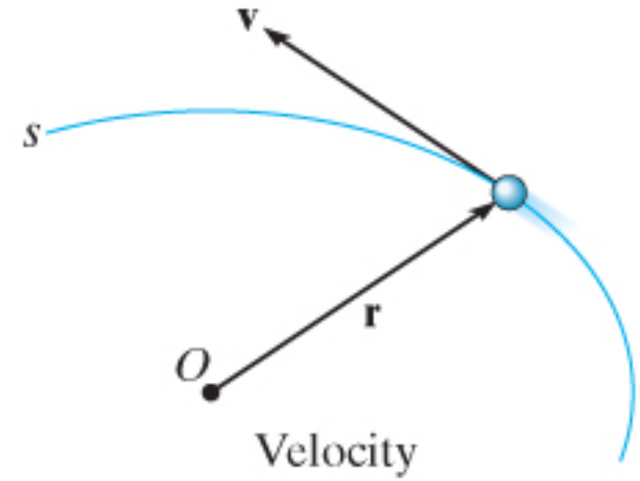


Instantaneous velocity

$$\mathbf{v} = \lim_{\Delta t \rightarrow 0} \frac{\Delta\mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt}$$

Speed (magnitude of $\mathbf{v}$):

$$v = \lim_{\Delta t \rightarrow 0} \left(\frac{\Delta s}{\Delta t}\right) = \frac{ds}{dt}$$



General curvilinear motion

Change in velocity:

$$\Delta \mathbf{v} = \mathbf{v}' - \mathbf{v}$$

Average acceleration:

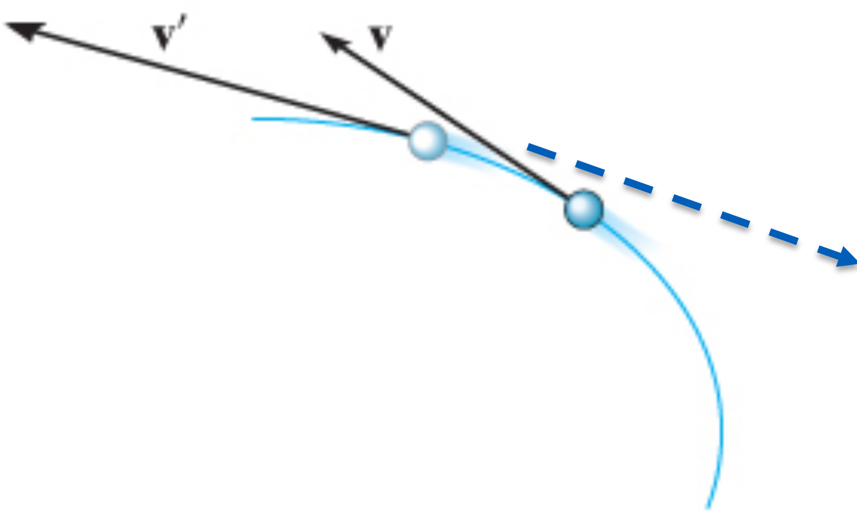
$$\mathbf{a}_{avg} = \frac{\Delta \mathbf{v}}{\Delta t}$$

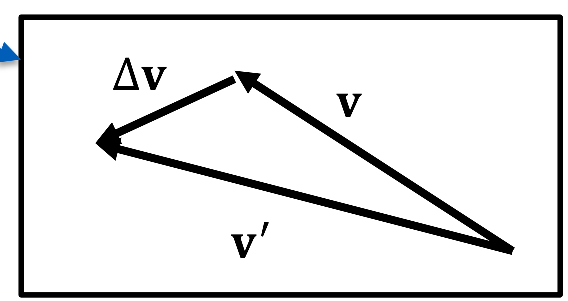
The **velocity** vector is **tangent** to the path.

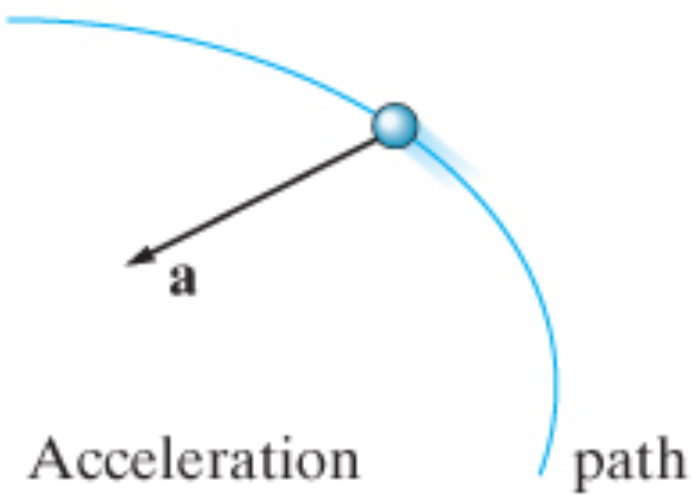
In contrast, the **acceleration vector is usually not tangent to the path**.

Instantaneous acceleration:

$$\mathbf{a} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t} = \frac{d\mathbf{v}}{dt}$$





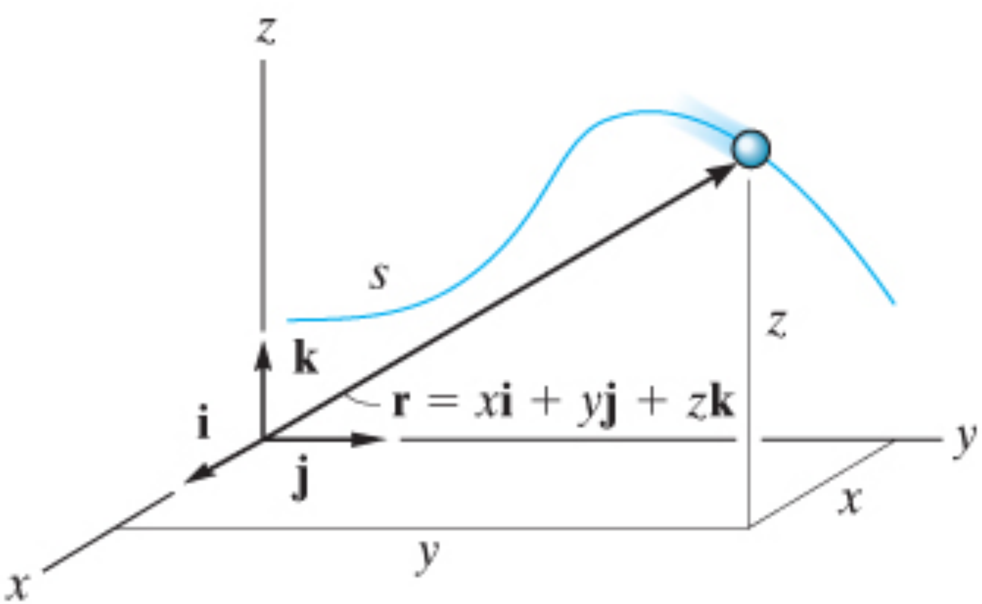


1. Kinematics of a particle

b) Curvilinear motion

- Rectangular coordinates

Curvilinear motion: position



Position

Consider a particle, at point (x, y, z), moving along a curved path s.

The position vector can be expressed using Cartesian coordinates as:

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

Where each component varies as a function of time:

$$x = x(t) \quad ; \quad y = y(t) \quad ; \quad z = z(t)$$

Therefore, the position vector is a function of time: $\mathbf{r} = \mathbf{r}(t)$.

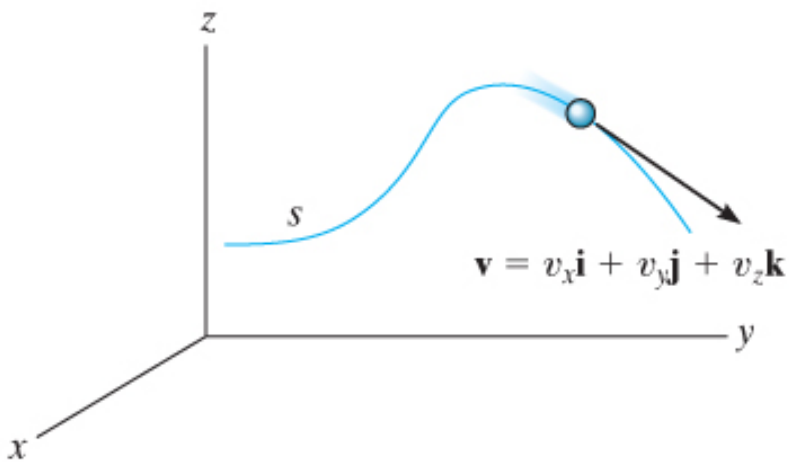
Magnitude of $\mathbf{r}$:

$$r = \sqrt{x^2 + y^2 + z^2}$$

Direction of $\mathbf{r}$:

$$\mathbf{u}_r = \mathbf{r}/r$$

Curvilinear motion: velocity



Velocity

The velocity vector is the derivative of the position vector:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}(x\mathbf{i}) + \frac{d}{dt}(y\mathbf{j}) + \frac{d}{dt}(z\mathbf{k}) = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$$

This result is based on the derivative of a product, for example:

$$\frac{d}{dt}(x\mathbf{i}) = \frac{dx}{dt}\mathbf{i} + x\frac{d\mathbf{i}}{dt} = \frac{dx}{dt}\mathbf{i} + x \cdot 0 = \frac{dx}{dt}\mathbf{i}$$

$d\mathbf{i}/dt = 0$ because the magnitude and direction of $\mathbf{i}$ ($\mathbf{j}$ and $\mathbf{k}$) is independent of time.

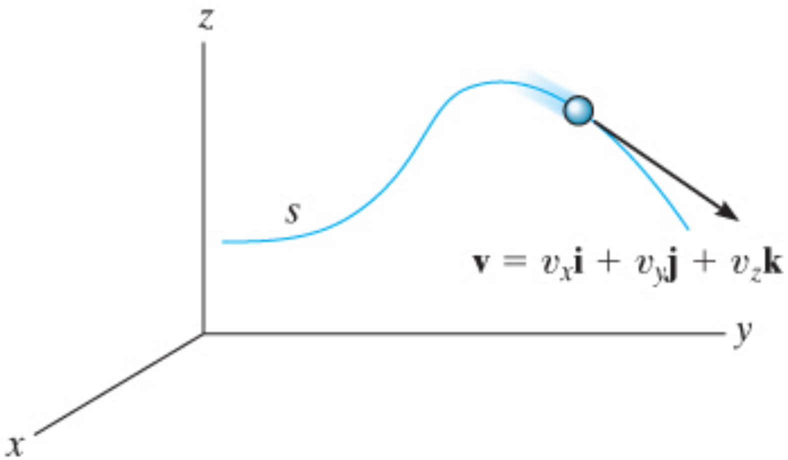
The velocity vector is often written with this notation:

$$\mathbf{v} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$$

"Dot" notation:

$$\dot{x} = \frac{dx}{dt}; \dot{y} = \frac{dy}{dt}; \dot{z} = \frac{dz}{dt}$$

Curvilinear motion: velocity



Velocity

The velocity vector given by:

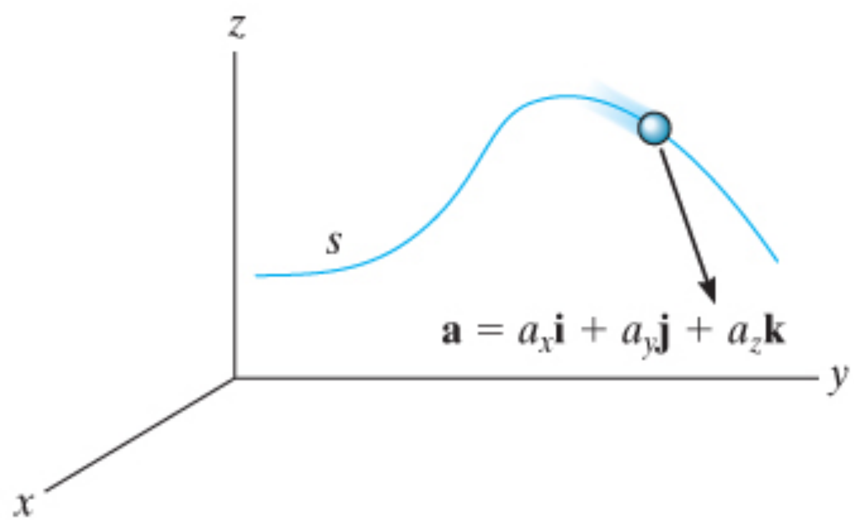
$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$$

has a magnitude and direction given by:

$$v = \sqrt{{v_x}^2 + {v_y}^2 + {v_z}^2} \quad \text{and} \quad \mathbf{u}_v = \mathbf{v}/v$$

All vectors have a magnitude and direction; however, remember that these (magnitude and direction) can change as a function of time!

Curvilinear motion: acceleration



Acceleration

$$\boxed{\begin{aligned} a_x &= \dot{v}_x = \ddot{x} \\ a_y &= \dot{v}_y = \ddot{y} \\ a_z &= \dot{v}_z = \ddot{z} \end{aligned}}$$

Derivating the velocity $\mathbf{v}$, gives the acceleration vector $\mathbf{a}$:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(v_x\mathbf{i}) + \frac{d}{dt}(v_y\mathbf{j}) + \frac{d}{dt}(v_z\mathbf{k})$$

$$= \frac{dv_x}{dt}\mathbf{i} + \frac{dv_y}{dt}\mathbf{j} + \frac{dv_z}{dt}\mathbf{k}$$

$$= \dot{v}_x\mathbf{i} + \dot{v}_y\mathbf{j} + \dot{v}_z\mathbf{k} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$$

Magnitude of acceleration:

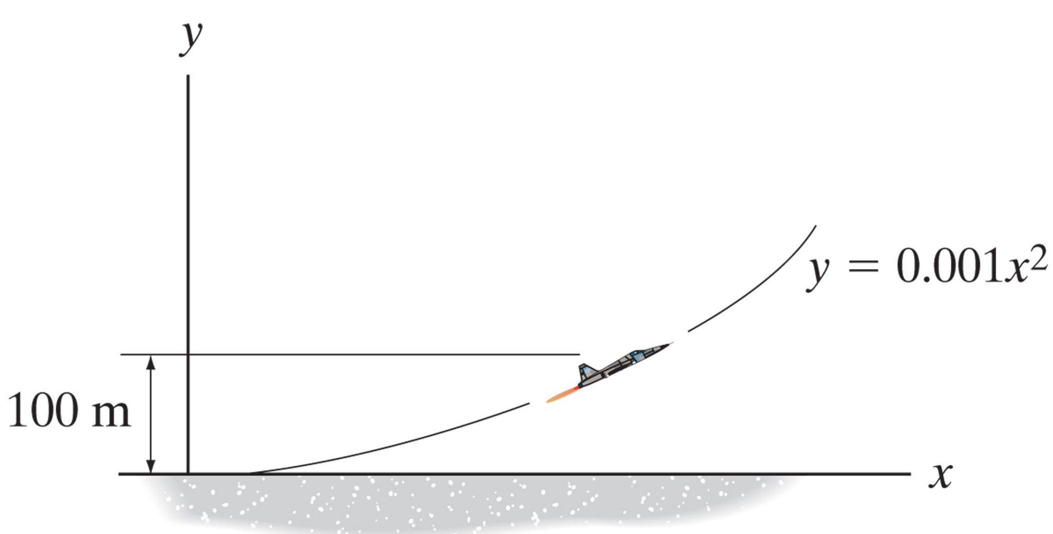
$$a = \sqrt{a_x{}^2 + a_y{}^2 + a_z{}^2}$$

Direction of acceleration

$$\mathbf{u}_a = \mathbf{a}/a$$

Example

At take-off, the path of the plane is described by $y = 0.001x^2$ (m). If the plane is rising with a constant upward velocity of 10 m/s, find the magnitudes of the velocity and acceleration at an altitude $y = 100$ m.



When $y = 100$ m, the distance x is:

$$y = 0.001x^2 \Rightarrow 100 = 0.001x^2 \Rightarrow x = 316.2 \text{ m}$$

Since, $v_y = 10$ m/s, the time needed to reach $y = 100$ m is:

$$y = v_y t \Rightarrow 100 = 10t \Rightarrow t = 10 \text{ s}$$

The velocity v_x can be obtained from:

$$v_y = \dot{y} = \frac{d}{dt}(0.001x^2) = 0.001\frac{d}{dx}(x^2) \cdot \frac{dx}{dt}$$

$$\Rightarrow v_y = \dot{y} = 0.002x\dot{x} = 0.002xv_x$$

$$\Rightarrow 10 = 0.002 \cdot 316.2 \cdot v_x$$

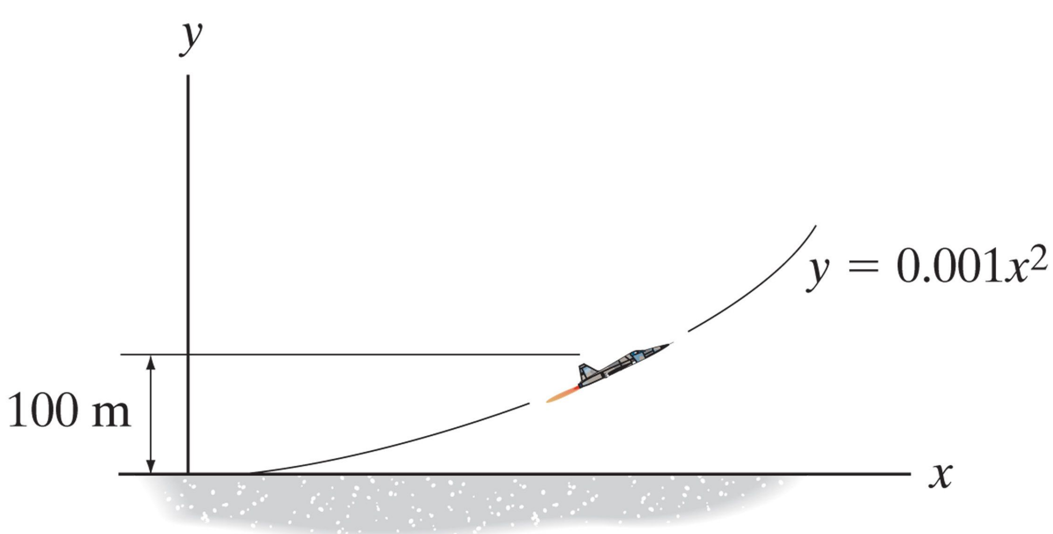
$$\Rightarrow v_x = 15.81 \text{ m/s}$$

The magnitude of velocity is:

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{15.81^2 + 10^2} = 18.7 \text{ m/s}$$

Example

At take-off, the path of the plane is described by $y = 0.001x^2$ (m). If the plane is rising with a constant upward velocity of 10 m/s, find the magnitudes of the velocity and acceleration at an altitude $y = 100$ m.



Next, the acceleration is given by:

$$a_y = \dot{v}_y = \frac{d}{dt}(0.002x\dot{x}) = 0.002\dot{x}\dot{x} + 0.002x\ddot{x}$$

$$\Rightarrow a_y = 0.002(v_x^2 + xa_x)$$

When $x = 316.2$ m, $v_x = 15.81$ m/s, $a_y = 0$:

$$0 = 0.002(15.81^2 + 316.2 \cdot a_x)$$

$$\Rightarrow a_x = -0.791 \text{ m/s}^2$$

The magnitude of acceleration is:

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{(-0.791)^2 + 0^2} = 0.791 \text{ m/s}^2$$

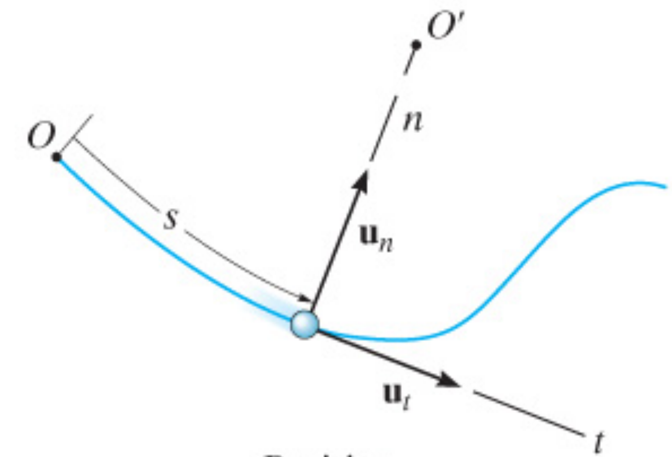
1. Kinematics of a particle

b) Curvilinear motion

- Normal and tangential components

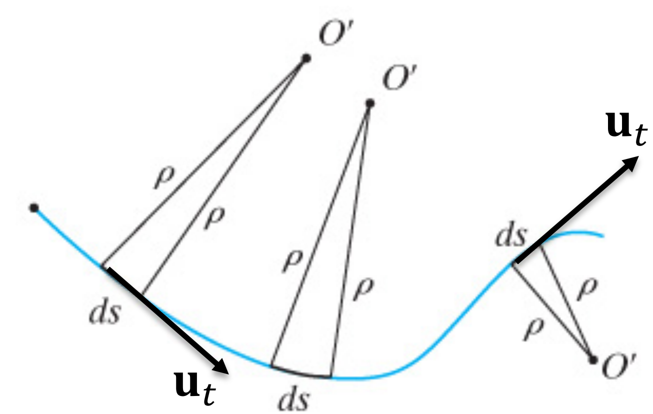
Curvilinear motion: normal and tangential components

When the path of a particle is known, it is often convenient to use normal and tangential coordinates. Consider a particle moving along the path shown here in blue. At a given time, it is at a position s measured from point O.



We can define a reference frame (t, n) which origin coincides with the location of the particle **(the frame moves with the particle).**

<u>*t*-axis</u> $\mathbf{u}_t$ is tangent to the path at this location and is positive in the direction of increasing s.



<u>*n*-axis</u> $\mathbf{u}_n$ is parallel to the radius of curvature ρ and is positive in the direction of the centre of curvature O'.

Remember that $\mathbf{u}_t$ and $\mathbf{u}_n$ are constantly changing direction as the particle moves.

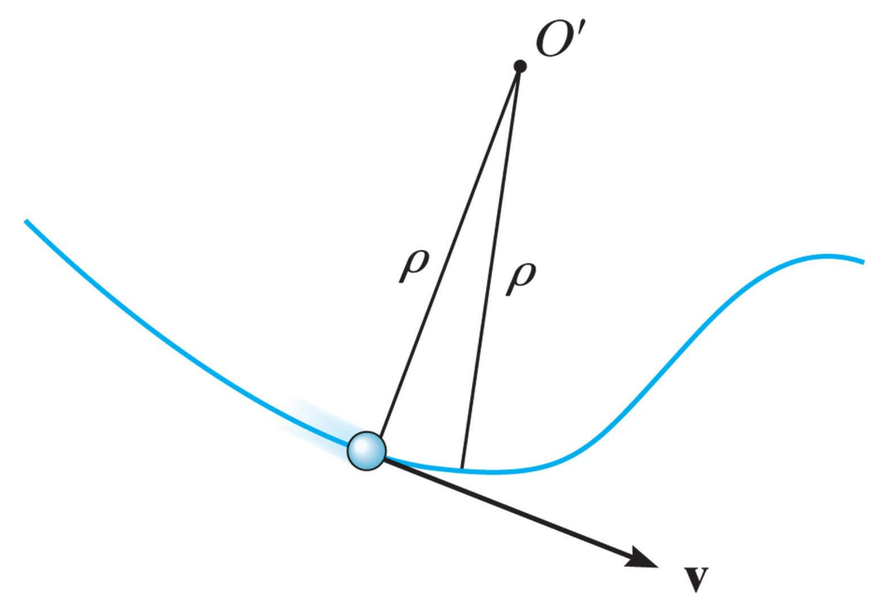
Normal and tangential components: velocity

Since the particle moves, its position s is a function of time.

The velocity has a direction that is always tangent to the path, and a magnitude given by $v = ds/dt = \dot{s}$.

Therefore:

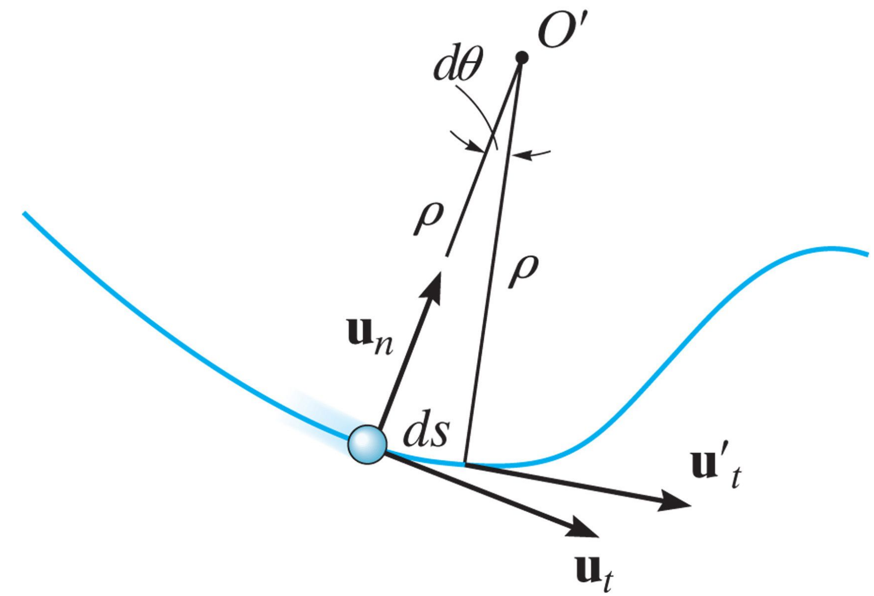
$$\mathbf{v} = v\mathbf{u}_t = \dot{s}\mathbf{u}_t = \frac{ds}{dt}\mathbf{u}_t$$



Normal and tangential components: acceleration

The acceleration of the particle is the time derivative of its velocity:

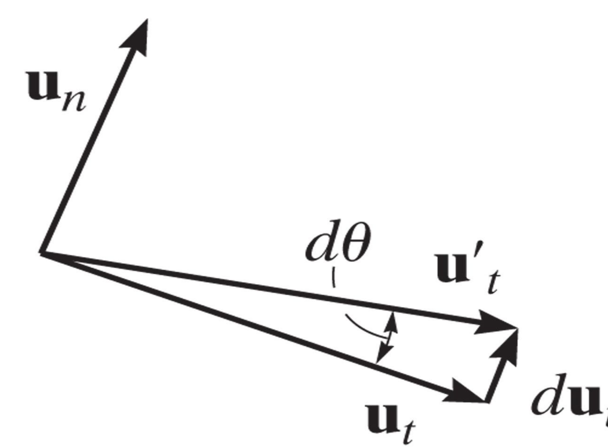
$$\mathbf{a} = \dot{\mathbf{v}} = \frac{d}{dt}(v\mathbf{u}_t) = \dot{v}\mathbf{u}_t + v\dot{\mathbf{u}}_t$$



Where we need to find $\dot{\mathbf{u}}_t$. The magnitude of $\mathbf{u}_t$ is always 1, but its direction changes to become $\mathbf{u}'_t$:

$$\mathbf{u}'_t = \mathbf{u}_t + d\mathbf{u}_t$$

We can see from the figure below that $d\mathbf{u}_t$ has a magnitude $d\theta$ and its direction is along $\mathbf{u}_n$. Therefore, we have $d\mathbf{u}_t = d\theta\mathbf{u}_n$. Taking the time derivative, and using $ds = \rho d\theta$, we get:



$$\dot{\mathbf{u}}_t = \dot{\theta}\mathbf{u}_n = \frac{\dot{s}}{\rho}\mathbf{u}_n = \frac{v}{\rho}\mathbf{u}_n$$

Normal and tangential components: acceleration

Therefore, the acceleration of the particle has two components:

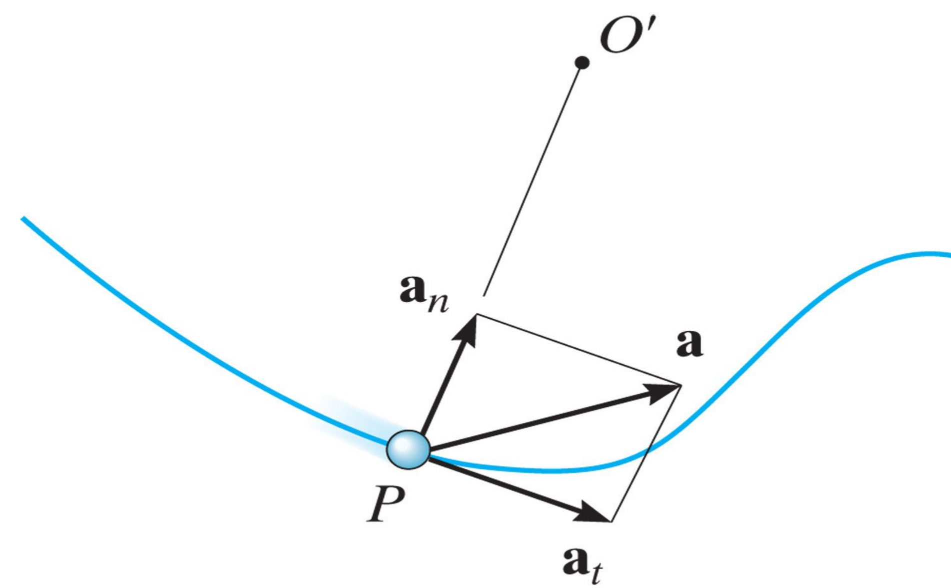
$$\mathbf{a} = a_t\mathbf{u}_t + a_n\mathbf{u}_n$$

Where

$$a_t = \dot{v} \qquad \text{and} \qquad a_n = \frac{v^2}{\rho}$$

Rate of change of velocity

Centripetal acceleration



The magnitude of acceleration is: $a = \sqrt{a_t^2 + a_n^2}$

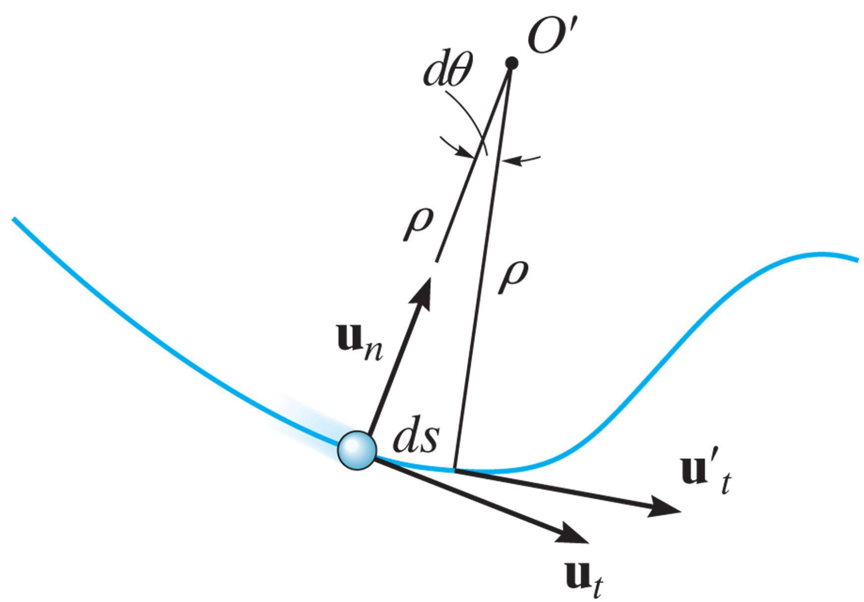
Radius of curvature

The centripetal acceleration is:

$$a_n = \frac{v^2}{\rho}$$

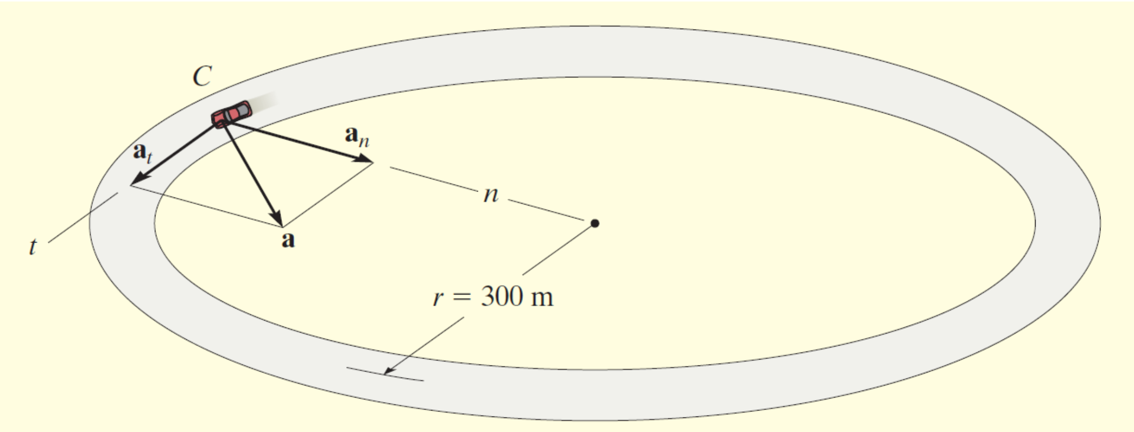
Where ρ is the radius of curvature. If the path is expressed as a function: $y(x) = f(x)$, then the radius of curvature ρ at any point on the path is given by:

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|}$$



Example

A race car travels around a circular track that has a radius of 300 m. If the car increases its speed at a constant rate of 1.5 m/s^2, starting from rest, determine the time needed to reach an acceleration of 2 m/s^2. What is the speed at this instant?



The acceleration is $a = \sqrt{a_t^2 + a_n^2}$, where $a_t = 1.5\ \mathrm{m/s^2}$ and $a_n = v^2/\rho$. Therefore we need to find the velocity v. Since a_t is constant, we have:

$$v = v_0 + a_t t = 1.5t$$

Thus, the normal acceleration becomes:

$$a_n = v^2/\rho = (1.5t)^2/300 = 0.0075t^2\ \ \mathrm{m/s^2}$$

Substituting in the acceleration gives:

$$a = \sqrt{a_t^2 + a_n^2}$$

$$\Rightarrow 2 = \sqrt{1.5^2 + (0.0075t)^2}$$

Solving for the positive value of t gives:

$$t = 13.3\ \mathrm{s}$$

Finally, the speed at $t = 13.3\ \mathrm{s}$ is:

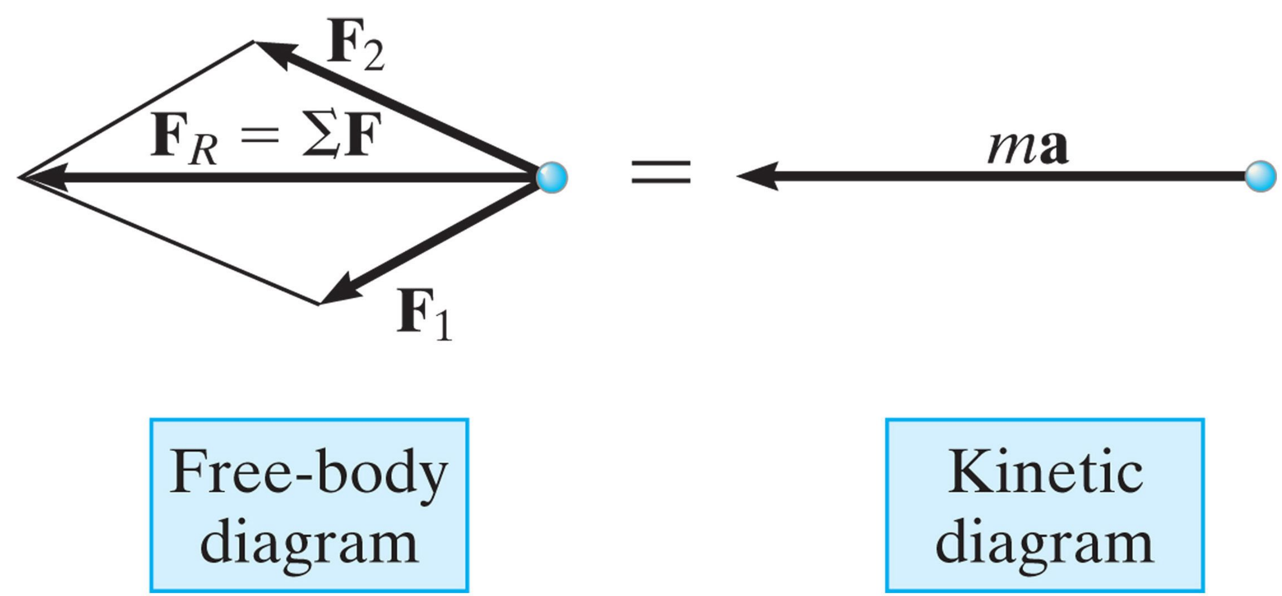
$$v = 1.5t = 1.5 \cdot 13.3 = 19.95\ \mathrm{m/s}$$

2. Kinetics of a particle

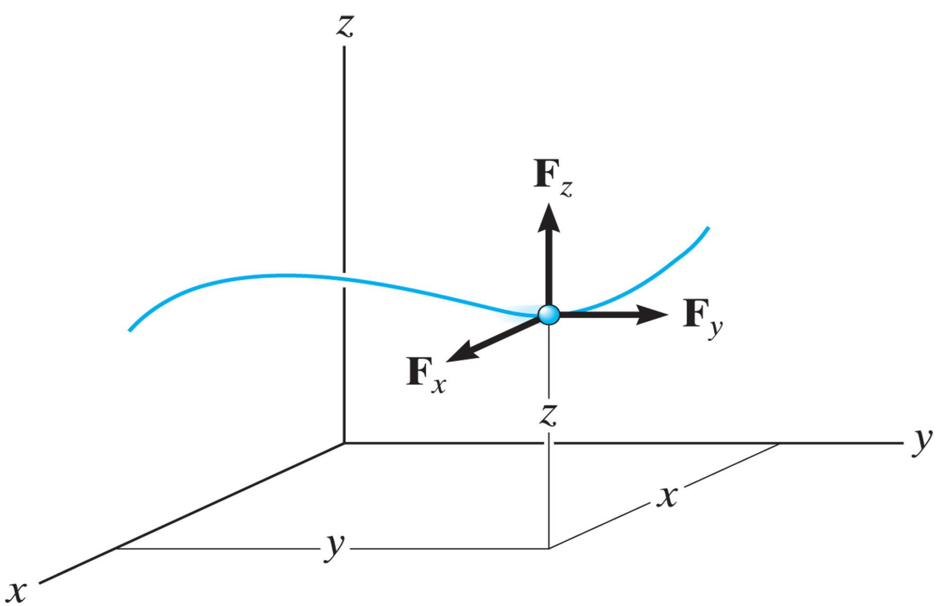
Equation of motion

- Kinetics investigates the relation between the change in motion of a body and the forces creating this change.
- This is done with Newton's second law, also called the equation of motion:

$$\boxed{\Sigma \mathbf{F} = m\mathbf{a}}$$



Equation of motion: Cartesian coordinates



When a particle moves in an inertial reference frame, we can express its acceleration and forces with Cartesian coordinates:

$$\Sigma \mathbf{F} = m\mathbf{a}$$

$$\Rightarrow \Sigma F_x \mathbf{i} + \Sigma F_y \mathbf{j} + \Sigma F_z \mathbf{k} = m\left(a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k}\right)$$

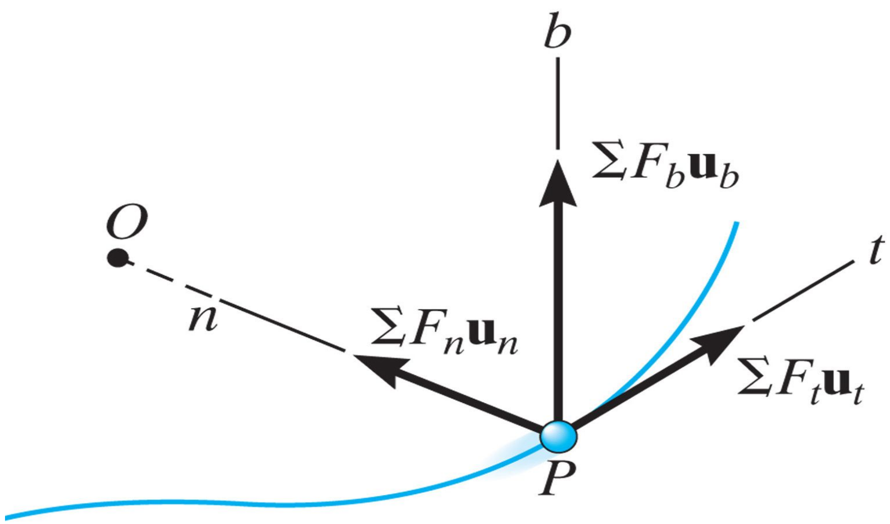
The above relation is easily expressed as three scalar equations:

$$\Sigma F_x = m a_x$$

$$\Sigma F_y = m a_y$$

$$\Sigma F_z = m a_z$$

Equation of motion: normal and tangential coordinates



When a particle moves along a curved path the motion may be described using tangential, normal and binormal directions:

$$\Sigma \mathbf{F} = m\mathbf{a}$$

$$\Rightarrow \Sigma F_t \mathbf{u}_t + \Sigma F_n \mathbf{u}_n + \Sigma F_b \mathbf{u}_b = m\mathbf{a}_t + m\mathbf{a}_n$$

In scalar form, this becomes:

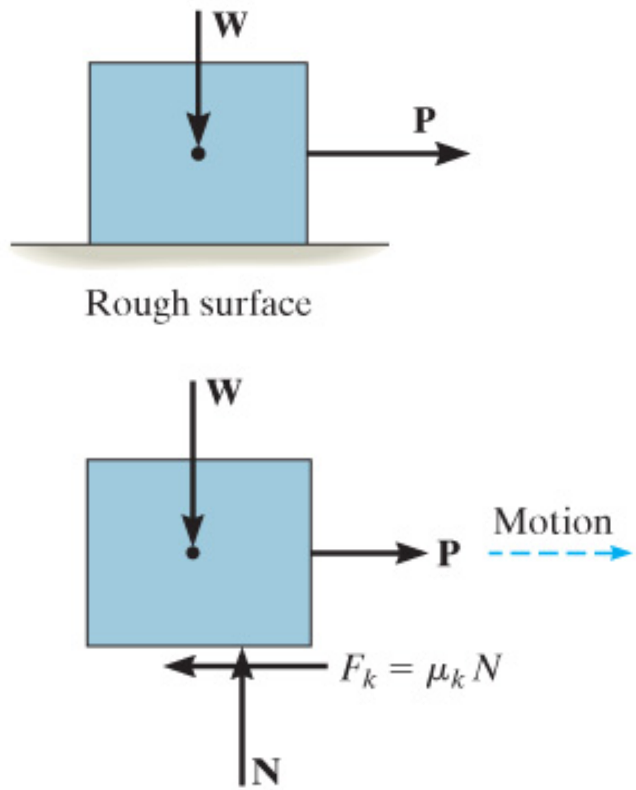
$$\Sigma F_t = m a_t$$

$$\Sigma F_n = m a_n$$

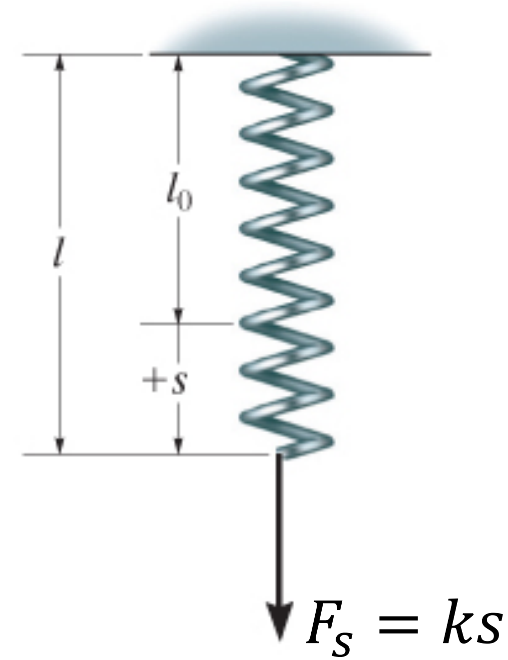
$$\Sigma F_b = 0$$

Of course there is no motion along direction b.

Friction and spring forces

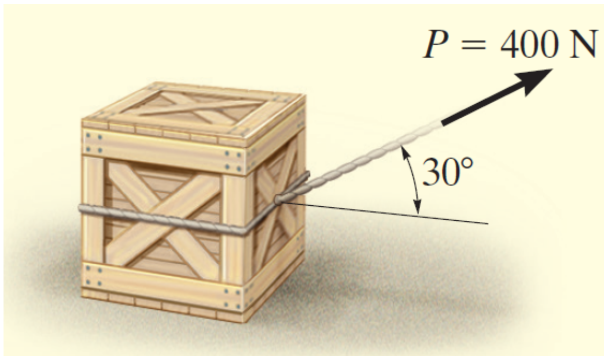


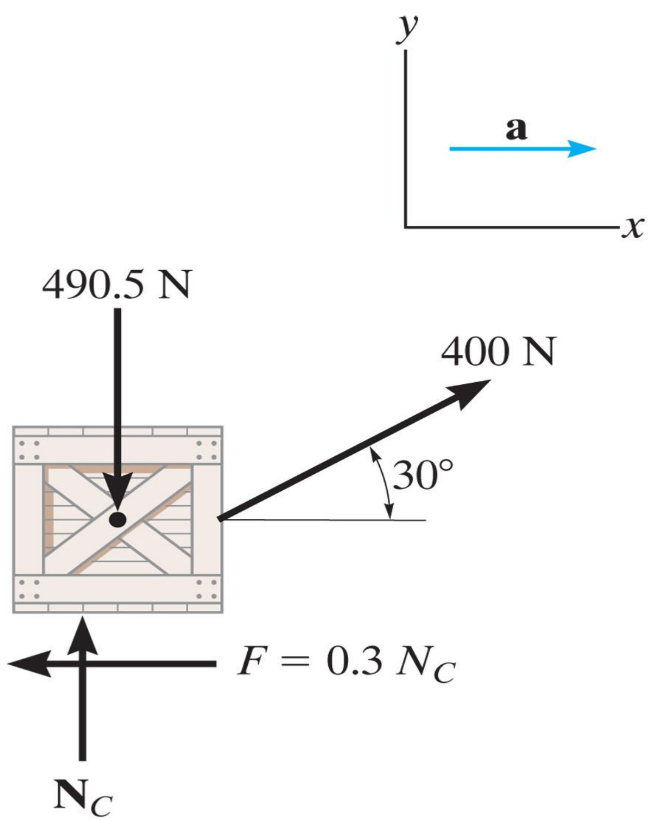
Friction creates a force $F_k = \mu_k N$, where μ_k is the coefficient of kinetic friction (typically between 0.01 to 0.6) and N is the normal force acting at the surface of contact. Remember, F_k is always opposed to the direction of motion!



A spring generates a force $F_s = ks$, where k is the spring's stiffness (in N/m) and s is the extension or compression defined as the difference between the deformed length l and undeformed length l_0.

Procedure for analysis





Steps to solve a problem

1. Draw a free-body diagram of the particle. Inculde an inertial reference frame.

2. Find the resultant force acting on the particle.

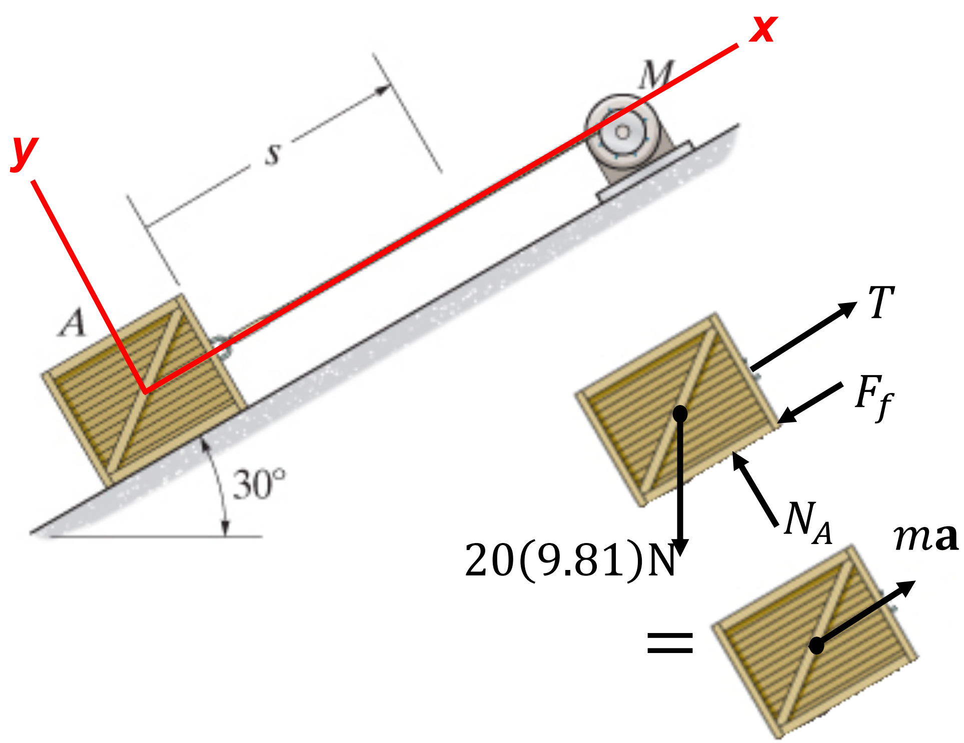
3. The acceleration may be represented as a $m\mathbf{a}$ vector on a separate kinetic diagram (this is optional).

4. Use the equation of motion $\Sigma\mathbf{F} = m\mathbf{a}$ to find the acceleration of the particle.

5. If needed, use kinematic equations to find the velocity and/or position of the particle.

Example

The motor pulls the 20 kg box with a constant acceleration such that it travels a distance *s* = 6 m in 3 s, starting from rest. Determine the tension in the cable, provided that the coefficient of friction is $\mu_k = 0.3$.



First, we draw the free-body diagram (and kinetic diagram). Second, we can find the normal force N_A:

$$\uparrow + \; \Sigma F_y = N_A - 20(9.81)\text{N}\cos 30° = 0$$

$$\Rightarrow N_A = 169.9\text{ N}$$

Third, we can evaluate the friction force F_f

$$F_f = \mu_k N_A = (0.3)169.9 = 50.97\text{ N}$$

In this problem, the acceleration is constant and can be found with:

$$s = s_0 + v_0 t + \frac{1}{2}a_x t^2 \;\Rightarrow\; 6\text{m} = 0 + 0(3\text{s}) + \frac{1}{2}a_x(3\text{s})^2$$

$$\Rightarrow a_x = 2(6\text{m})/(3\text{s})^2 = 1.33\text{ m/s}^2$$

Finally, we get the tension T using the equation of motion along the x-direction:

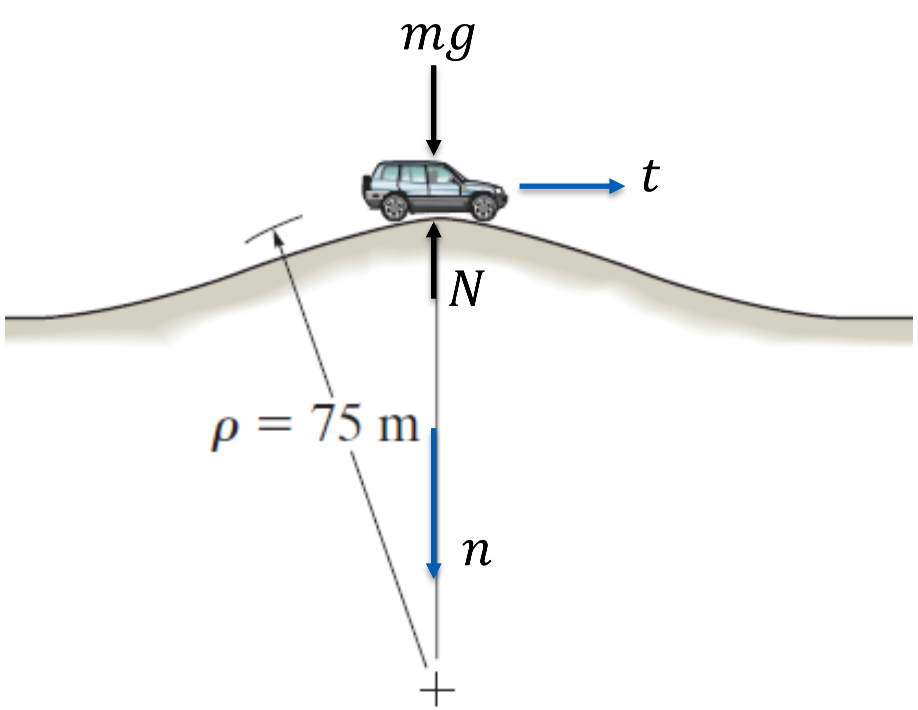
$$\rightarrow + \; \Sigma F_x = T - F_f - 20(9.81)\text{N}\sin 30° = m a_x$$

$$\Rightarrow T - 50.9743\text{N} - 20(9.81)\text{N}\sin 30° = 20\text{kg}(1.33\text{m/s}^2)$$

$$\Rightarrow T = 176\text{ N}$$

Example

Find the maximum speed that the jeep can travel over the hill and not lose contact with the road.



First, we draw the free-body diagram.
Acceleration in the normal direction is given by:

$$a_n = \frac{v^2}{\rho}$$

Therefore, the equation of motion along the normal direction is:

$$\Sigma F_n = m a_n$$

$$\Rightarrow mg - N = m\frac{v^2}{\rho}$$

Contact with the road will be loss when $N = 0$, and the above expression becomes:

$$\Rightarrow mg = m\frac{v^2}{\rho}$$

$$\Rightarrow v = \sqrt{\rho g} = \sqrt{75 \cdot 9.81} = 27.1 \text{ m/s}$$

Summary

We have introduced the concepts of kinematics (geometric aspects of motion) and kinetics (relation between forces and motion).

- **Kinematics**. velocity is the derivative of the position vector $\mathbf{r}$; and acceleration is the derivative of velocity:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} \quad and \quad \mathbf{a} = \frac{d\mathbf{v}}{dt}$$

- **Kinetics**. Newton's 2nd law describes the motion of particles:

$$\mathbf{F} = m\mathbf{a}$$

Need more explanations?

If you need more explanations, consult these sections of the textbook:

- **Kinematics of a particle**:
  - Sections 12.1-12.7

- **Kinetics of a particle**:
  - Sections 13.1-13.5