Aalto-yliopisto
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COE-C1001: Dynamics

11. General plane motion

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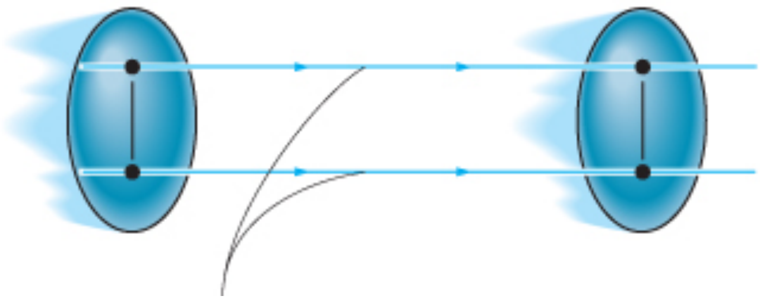
Learning outcomes

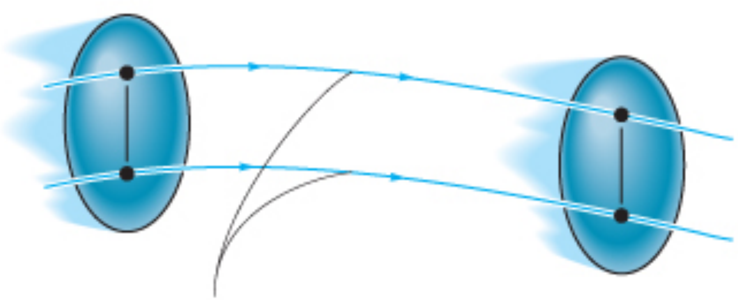
After this lecture, you should be able to:

- separate the motion of a rigid body into a combination of translation and rotation,
- Find the instantaneous center of zero velocity,
- Use kinematics equations and equations of motion to solve problems involving general plane motion of a rigid body.

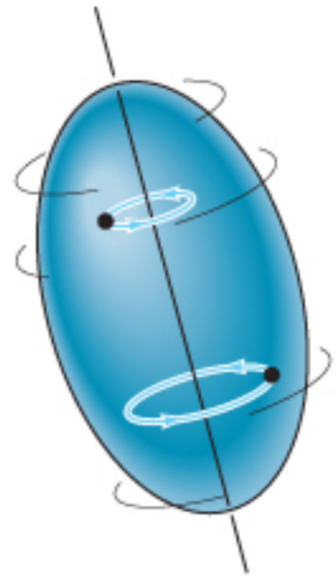
Planar motion



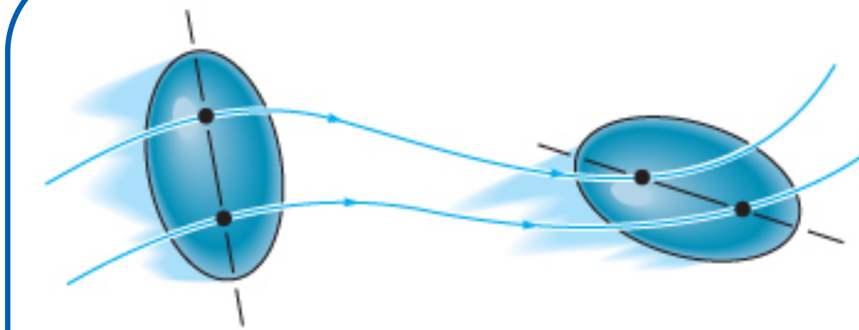
Rectilinear translation



Curvilinear translation



Rotation about a fixed axis



General plane motion

Planar motion of a rigid body

c) General plane motion

- Kinematics

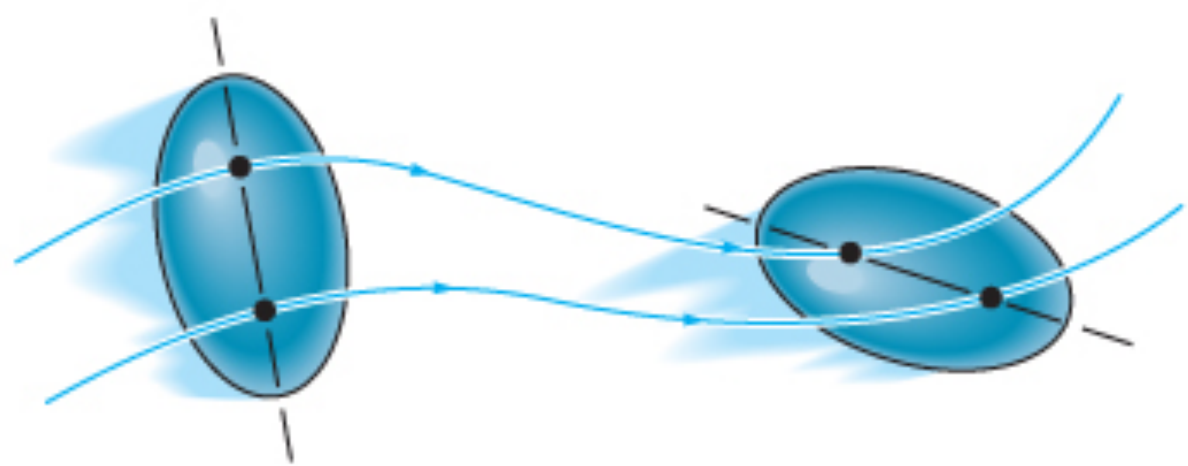
General plane motion

A body subjected to general plane motion undergoes a simultaneous translation and rotation.

The motion is specified by:

1. the position of a point on the body, and
2. the angular rotation of the body.

In some cases, there may be a simple relationship between these two quantities.



Procedure for absolute motion analysis

The bin of the truck rotates about point A. It is operated by the hydraulic cylinder BC.

1. Find a point on the body which moves along a straight line. Note its position s, measured from a fixed origin and directed along the line of motion.

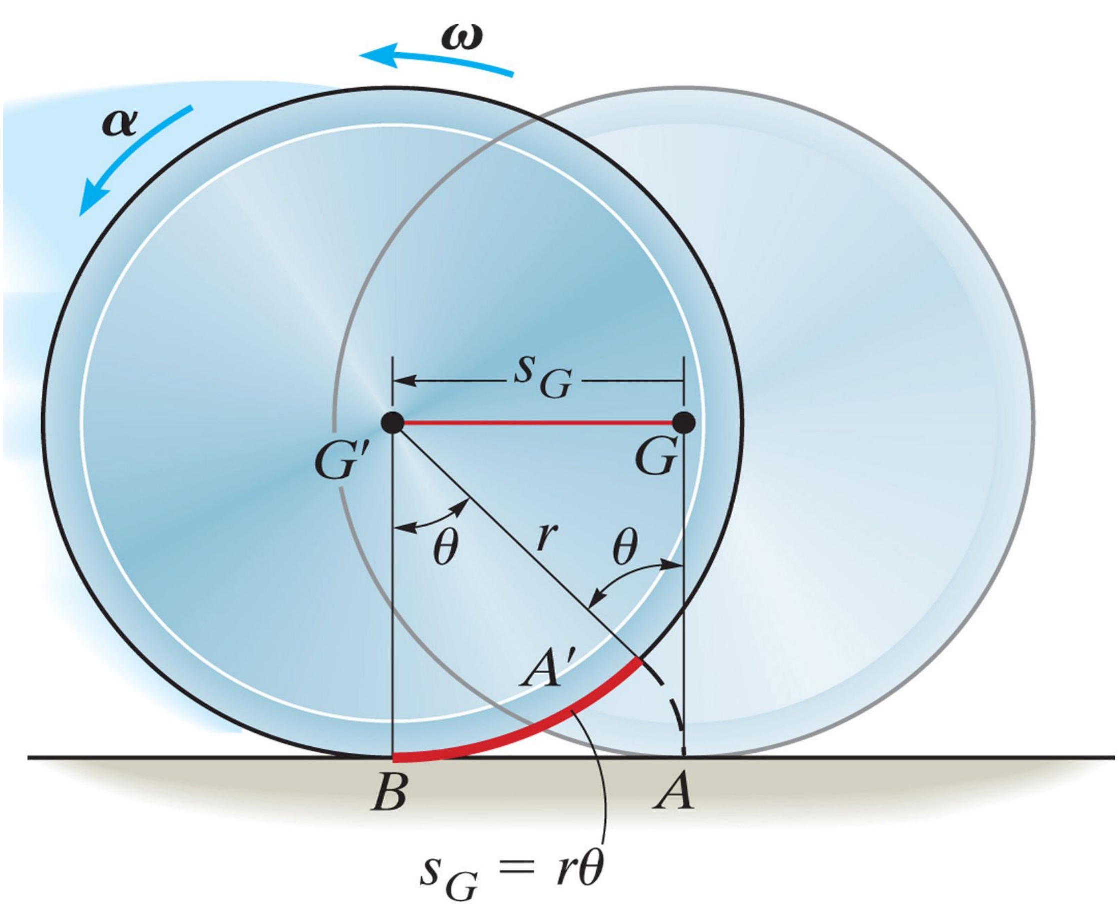
2. From a fixed reference, measure the angular position θ of a line on the body.

3. Based on the geometry of the body, express s as a function of θ, i.e. $s = f(\theta)$. Here, the bin moves according to $s = \sqrt{a^2 + b^2 - 2ab\cos\theta}$.

4. The velocity is obtained by derivating $s = f(\theta)$ with respect to time once (or twice for acceleration).

Example

The cylinder, of radius r, rotates without slipping. Express the velocity and acceleration of the center of mass G as a function of the angular velocity ω and acceleration α.



Point G moves along a straight line. The distance travelled s_G is equal to the arc length $A'B$, which gives:

$$s_G = r\theta$$

Derivating this gives the velocity:

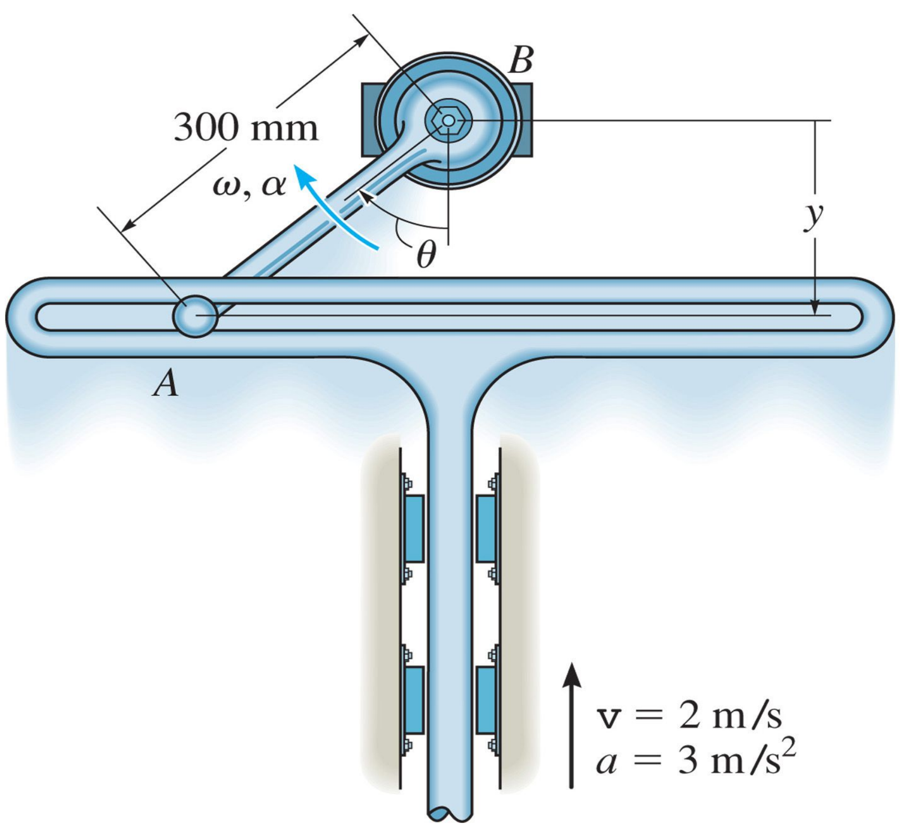
$$v_G = \frac{ds_G}{dt} = \frac{d(r\theta)}{dt} = r\omega$$

And derivating again for the acceleration:

$$a_G = \frac{dv_G}{dt} = \frac{d(r\omega)}{dt} = r\alpha$$

Example

At $\theta = 50°$, the guide is moving upward with an acceleration of 3 m/s^2 and a velocity of 2 m/s. Determine the angular acceleration and angular velocity of the link *AB* at this instant.



The position of the guide is given by:

$$y(\theta) = 0.3\cos\theta$$

Derivating this relation gives us the velocity:

$$v = \frac{dy}{dt} = 0.3\frac{d}{dt}[\cos\theta] = 0.3(-\sin\theta)\omega$$

Where

$$\frac{d}{dt}[\cos\theta] = \frac{d}{d\theta}[\cos\theta]\frac{d\theta}{dt} = (-\sin\theta)\omega$$

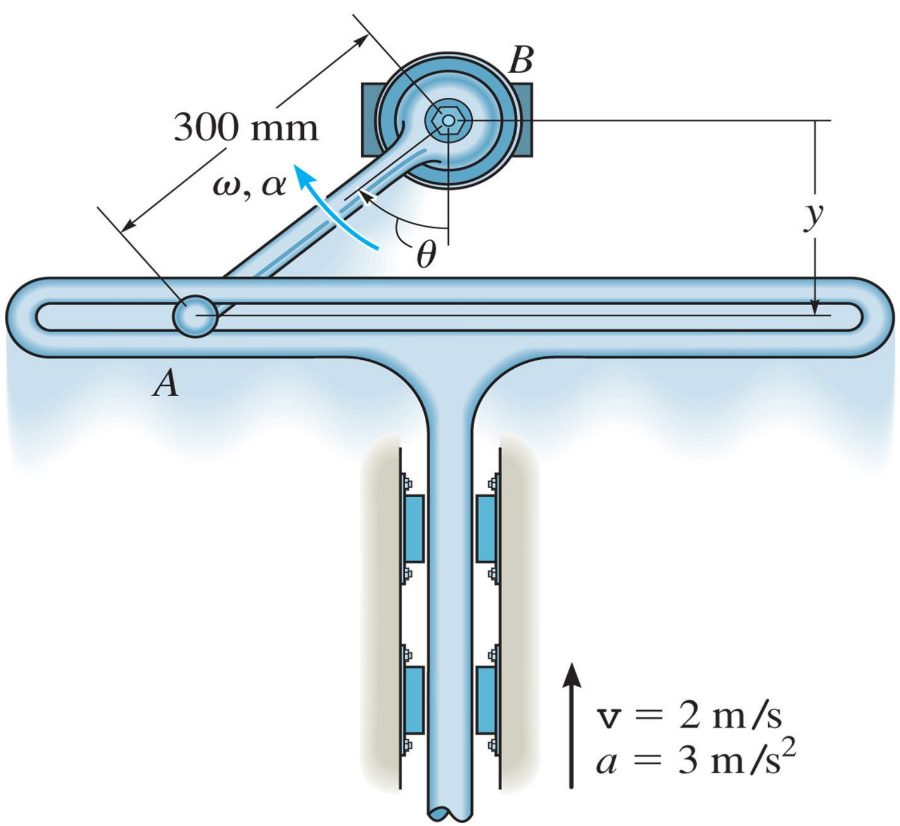
The angular velocity is obtained from:

$$v = 0.3(-\sin\theta)\omega$$

$$\Rightarrow \omega = \frac{v}{-0.3\sin\theta} = \frac{-2}{-0.3\cdot\sin 50°} = 8.7\ \mathrm{rad/s}$$

Example

At $\theta = 50°$, the guide is moving upward with an acceleration of 3 m/s^2 and a velocity of 2 m/s. Determine the angular acceleration and angular velocity of the link *AB* at this instant.



Next, the acceleration is obtained by derivating the velocity:

$$a = \frac{dv}{dt} = \frac{d}{dt}[-0.3(\sin\theta)\omega]$$

$$\Rightarrow a = -0.3\left(\sin\theta \cdot \frac{d\omega}{dt} + \frac{d\sin\theta}{dt} \cdot \omega\right)$$

$$\Rightarrow a = -0.3(\sin\theta \cdot \alpha + \cos\theta \cdot \omega^2)$$

From this equation, we can solve for the angular acceleration:

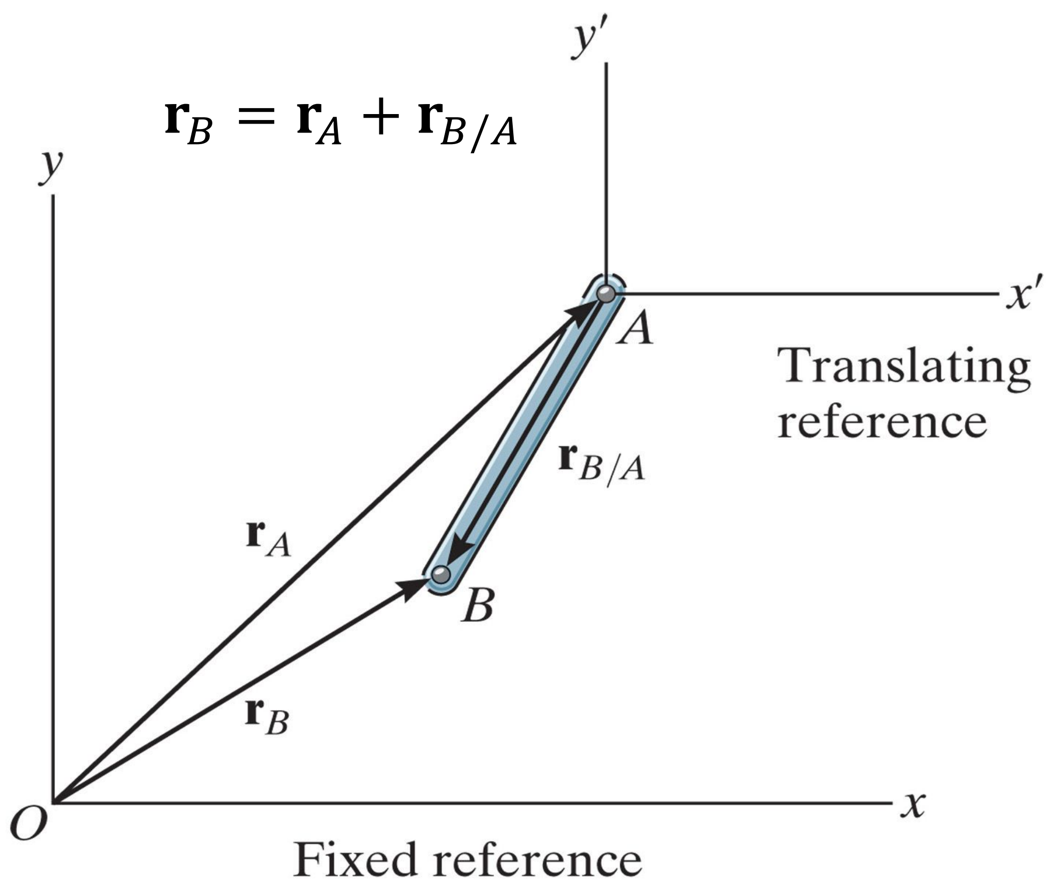
$$\Rightarrow a = -0.3(\sin\theta \cdot \alpha + \cos\theta \cdot \omega^2)$$

$$\Rightarrow \alpha = \frac{a + 0.3\cos\theta \cdot \omega^2}{-0.3\sin\theta}$$

$$\Rightarrow \alpha = \frac{-3 + 0.3\cos 50° \cdot 8.7^2}{-0.3\sin 50°} = -50.5\ \mathrm{rad/s^2}$$

Relative motion analysis

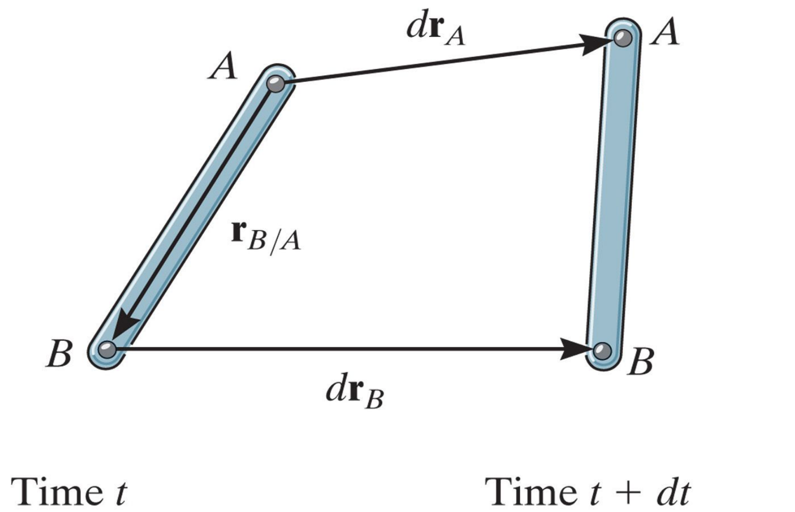
- General plane motion is a combination of translation and rotation, and relative motion analysis allows you to separate these two components.

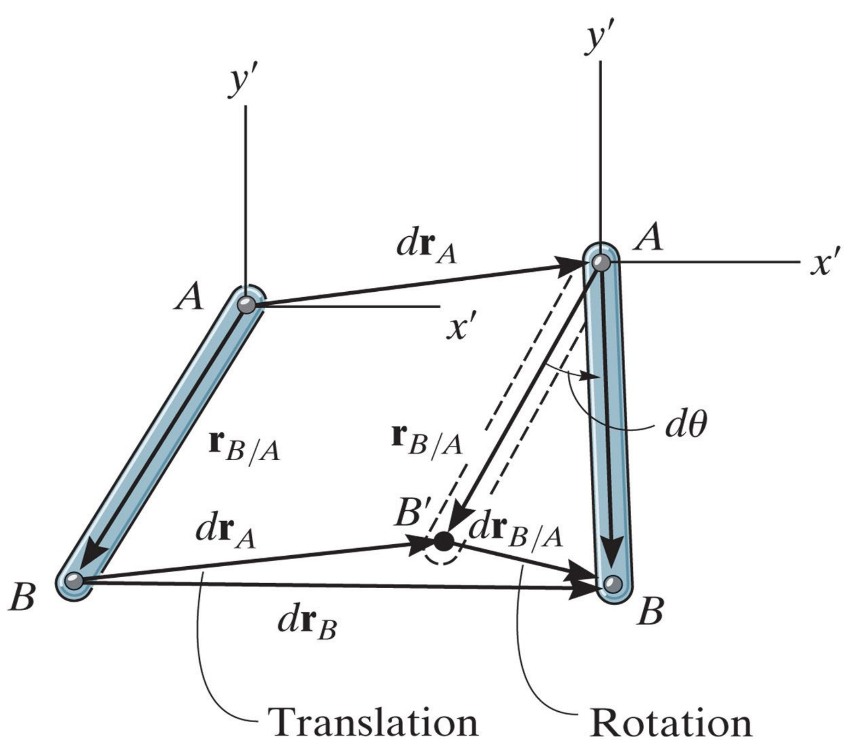


Relative motion analysis involves two reference frames:

- The x, y coordinate system is fixed and measures the absolute position of points *A* and *B*.
- The x', y' coordinate system is attached to point *A*, and translates with the rigid body.

Relative motion analysis





During an instant of time dt, points *A* and *B* move by $d\mathbf{r}_A$ and $d\mathbf{r}_B$. The motion can be separated in:

1. The bar translates by an amount $d\mathbf{r}_A$.
2. Then, the bar rotates about point *A* by $d\theta$.

The rotation about point A moves point B by $dr_{B/A} = r_{B/A}d\theta$, and:

$$d\mathbf{r}_B = d\mathbf{r}_A + d\mathbf{r}_{B/A}$$

Translation & rotation

Translation of A

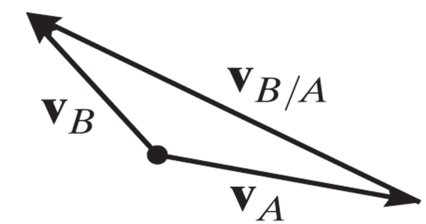
Rotation about A

Relative motion analysis

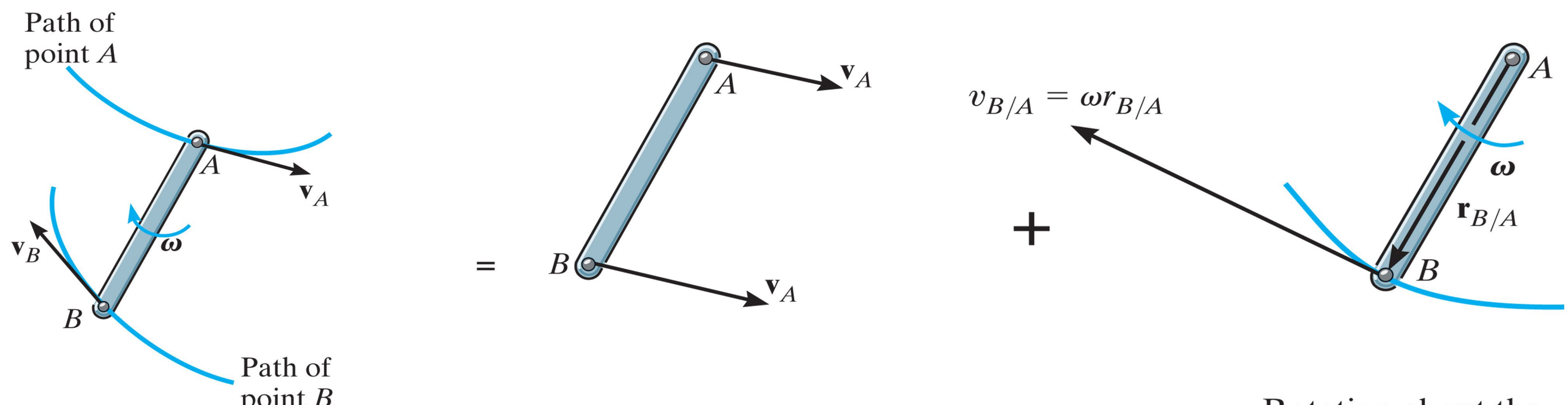
The velocity is obtained by dividing the previous equation by dt:

$$\frac{d\mathbf{r}_B}{dt} = \frac{d\mathbf{r}_A}{dt} + \frac{d\mathbf{r}_{B/A}}{dt} \;\Rightarrow\; \mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$$

Where $\mathbf{v}_{B/A}$ is the velocity of point B with respect to A. Since this motion is due to rotation only, we have $\mathbf{v}_{B/A} = \boldsymbol{\omega}\times\mathbf{r}_{B/A}$, and therefore:



$$\boxed{\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega}\times\mathbf{r}_{B/A}}$$



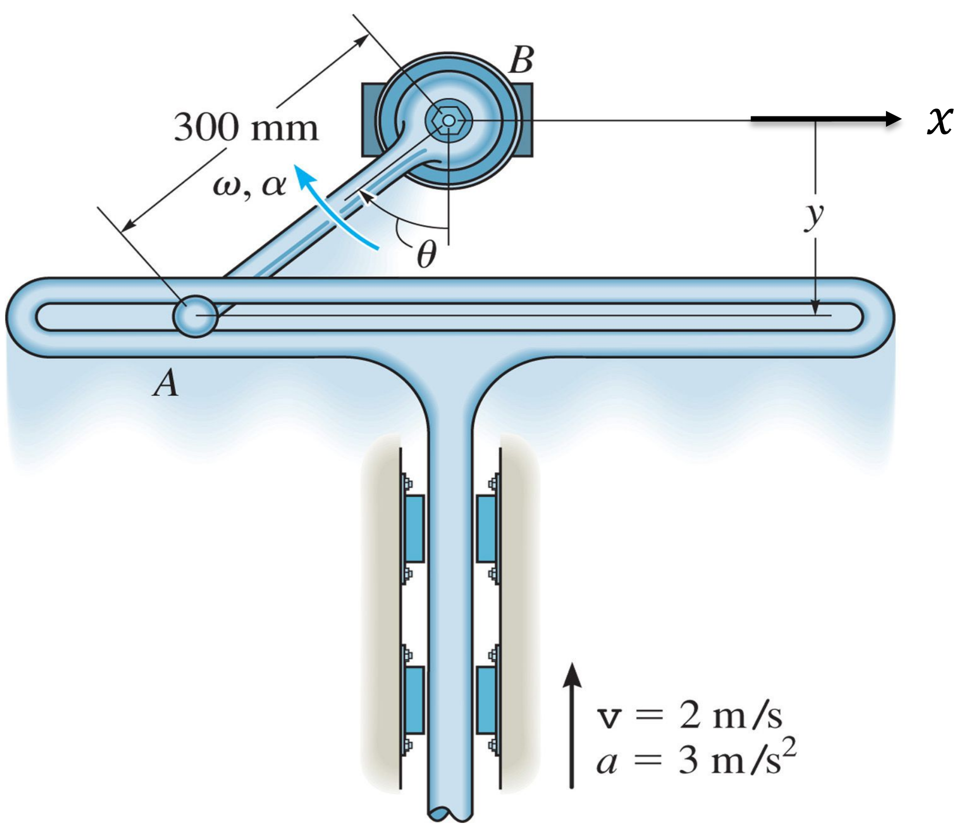
General plane motion

Translation

Rotation about the base point A

Revisiting previous example

At $\theta = 50°$, the guide is moving upward with an acceleration of 3 m/s^2 and a velocity of 2 m/s. Determine the angular velocity of the link *AB* at this instant.



The relative velocity is given by:

$$\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\omega}\times\mathbf{r}_{A/B}$$

$$\Rightarrow v_{Ax}\mathbf{i} + v_{Ay}\mathbf{j} = 0 + \omega\mathbf{k}\times 0.3(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j})$$

$$\Rightarrow v_{Ax}\mathbf{i} + v_{Ay}\mathbf{j} = -\omega\cdot 0.3\sin\theta\,\mathbf{j} - \omega\cdot 0.3\cos\theta\,\boldsymbol{i}$$

Along **j**:

$$v_{Ay} = -\omega\cdot 0.3\sin\theta \Rightarrow \omega = \frac{-2}{-0.3\sin 50°} = 8.7\ \mathrm{rad/s}$$

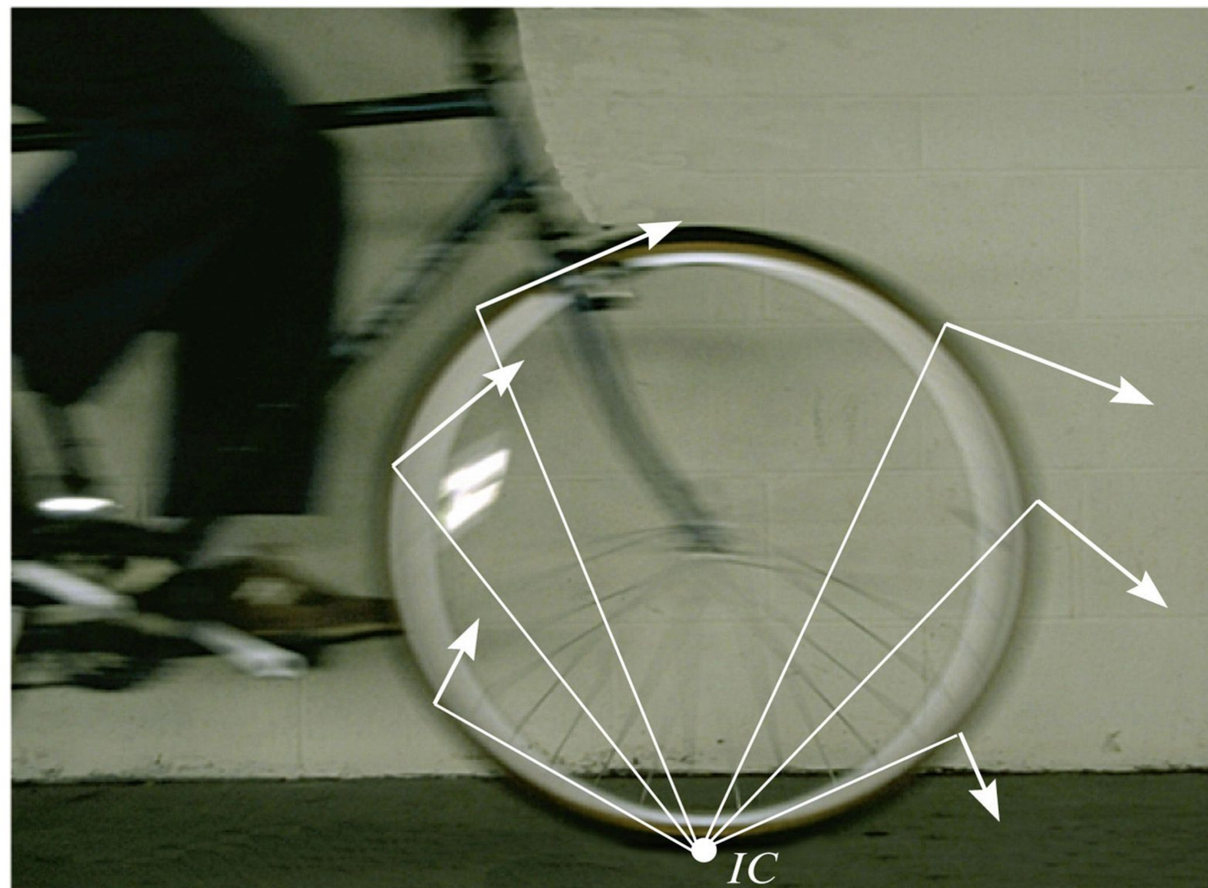
Which is the same as we found earlier.

Instantaneous center of zero velocity (IC)

The velocity of any point B can be obtained easily by choosing A to be a point that has zero velocity at the instant considered. Therefore, if $\mathbf{v}_A = \mathbf{0}$, we have:

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A}$$
$$= \boldsymbol{\omega} \times \mathbf{r}_{B/A}$$

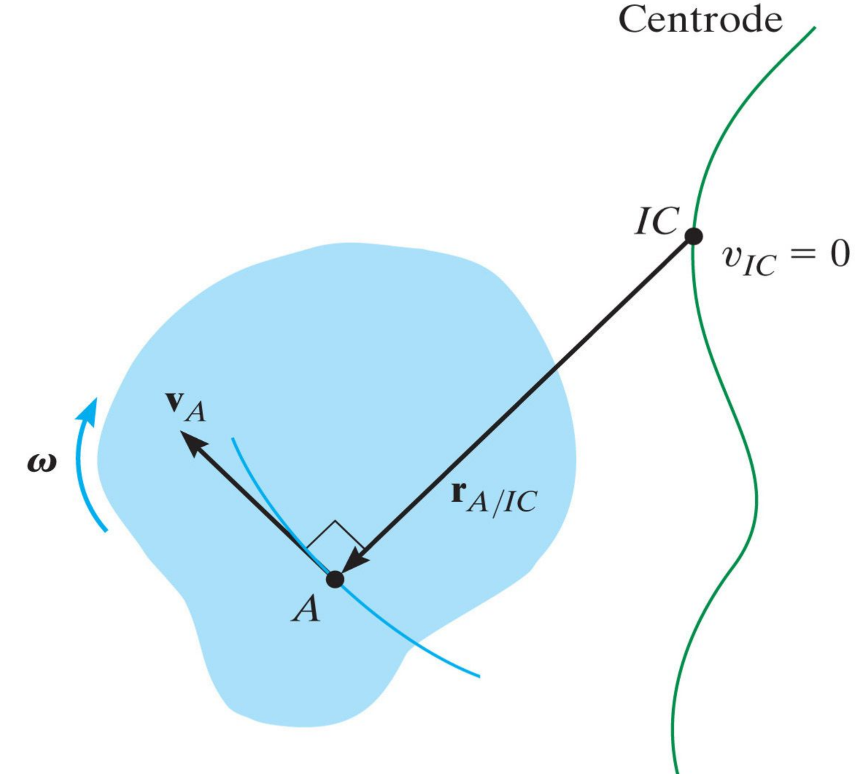
To locate the *IC* we can use the fact that the velocity of a point is perpendicular to the vector from *IC* to this point.



Instantaneous center of zero velocity (IC)

Case 1: The velocity $\mathbf{v}_A$ of a point on the body and its angular velocity $\boldsymbol{\omega}$ are known.

The *IC* is located along a line that goes through point A and is perpendicular to $\mathbf{v}_A$. The IC is at a distance $r_{A/IC} = v_A/\omega$.

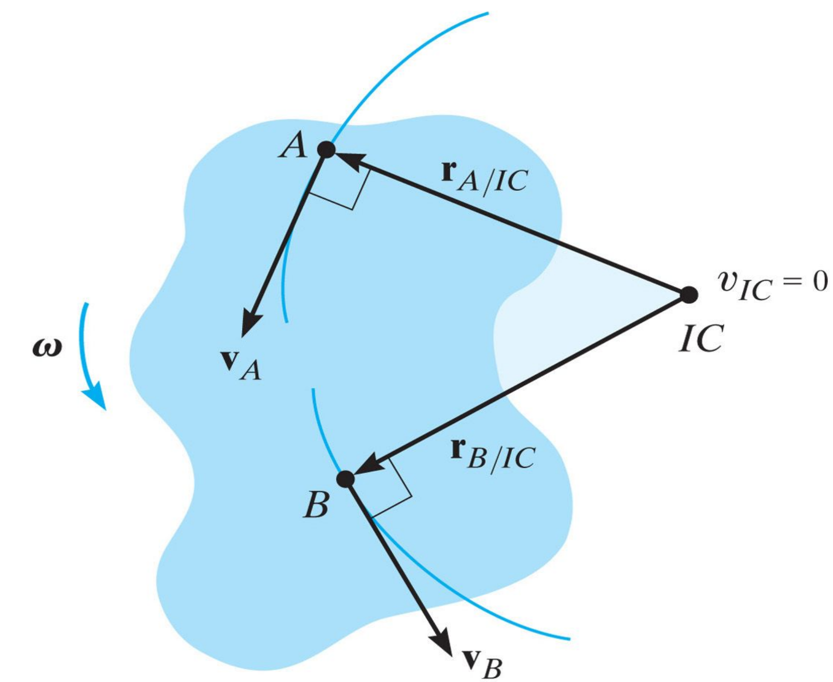


Location of IC knowing $\mathbf{v}_A$ and $\boldsymbol{\omega}$

Instantaneous center of zero velocity (IC)

Case 2: The lines of action of two nonparallel velocities $\mathbf{v}_A$ and $\mathbf{v}_B$ are known.

Construct at points A and B lines that are perpendicular to $\mathbf{v}_A$ and $\mathbf{v}_B$ and their intersection is the point *IC*.



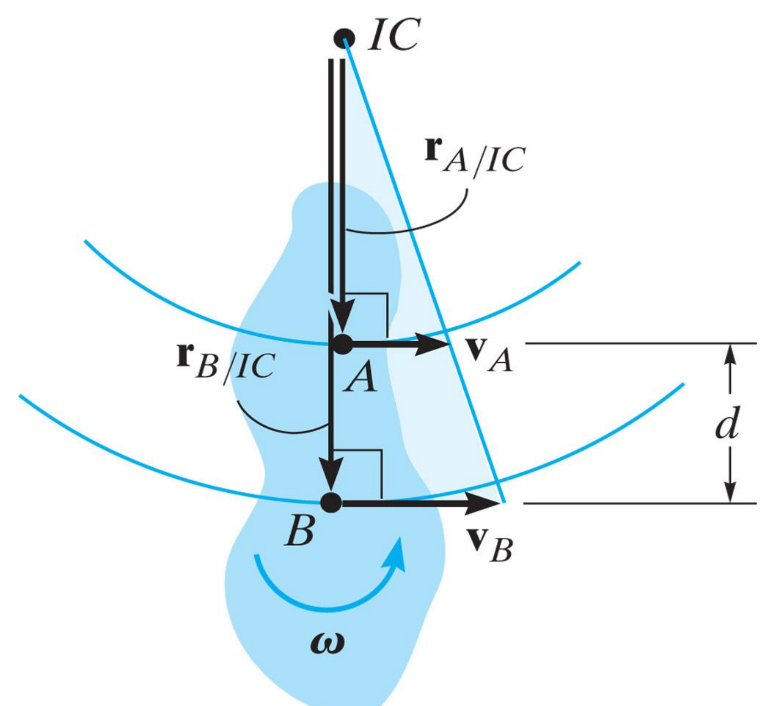
Location of *IC* knowing the directions of $\mathbf{v}_A$ and $\mathbf{v}_B$

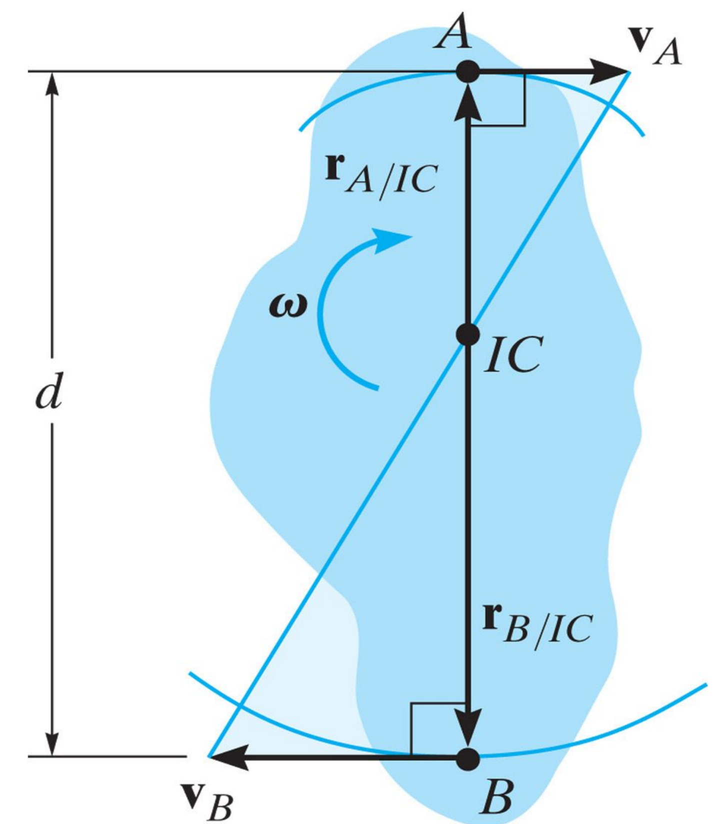
Instantaneous center of zero velocity (IC)

Case 3: The magnitude and direction of two parallel velocities $\mathbf{v}_A$ and $\mathbf{v}_B$ are known.

The location of *IC* is found using proportional triangles as shown on these two figures.

Location of IC knowing $\mathbf{v}_A$ and $\mathbf{v}_B$



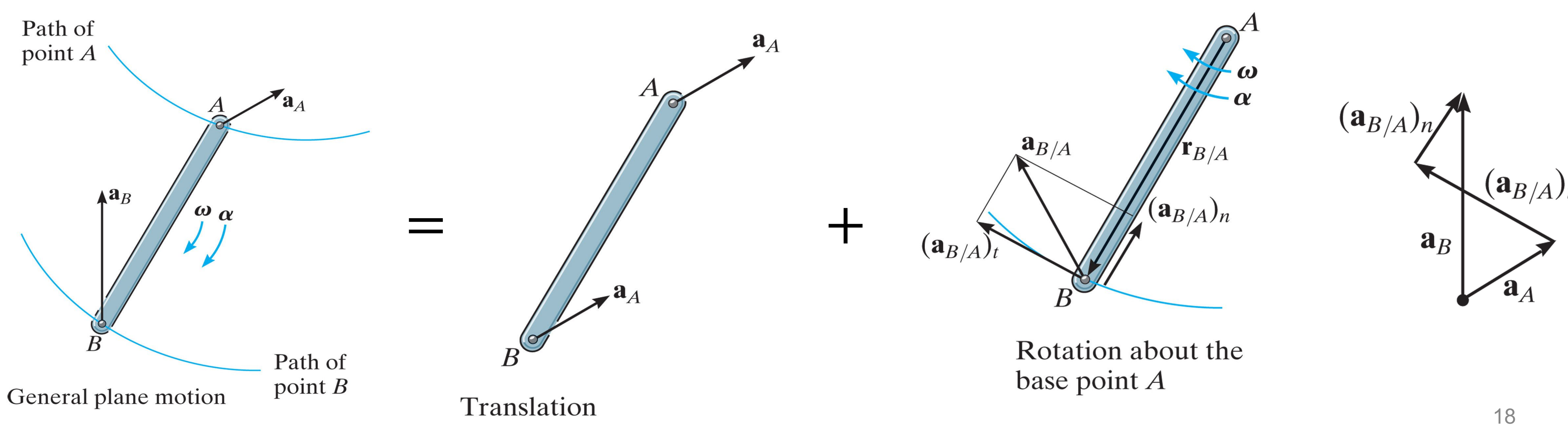


Relative motion analysis: acceleration

The equation that relates the acceleration of points A and B is obtained by differentiating $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$:

$$\frac{d\mathbf{v}_B}{dt} = \frac{d\mathbf{v}_A}{dt} + \frac{d\mathbf{v}_{B/A}}{dt} \quad \Rightarrow \quad \mathbf{a}_B = \mathbf{a}_A + \left(\mathbf{a}_{B/A}\right)_t + \left(\mathbf{a}_{B/A}\right)_n$$

Since the motion of point B with respect to A is a rotation along a circular path of radius $r_{B/A}$. Therefore, $\left(a_{B/A}\right)_t = \alpha r_{B/A}$ and $\left(a_{B/A}\right)_n = \omega^2 r_{B/A}$.



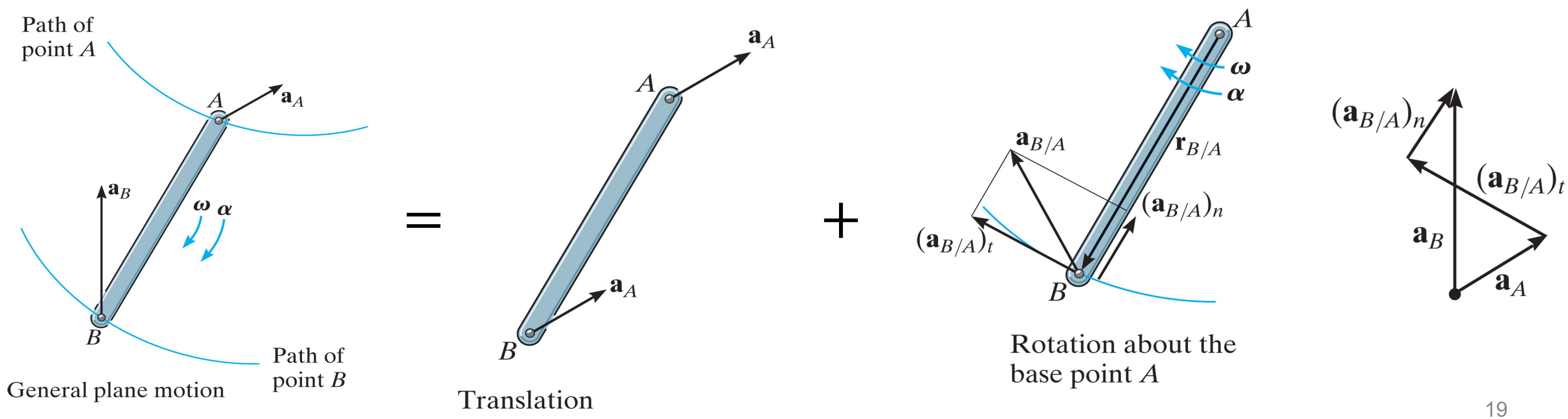
Relative motion analysis: acceleration

The components of acceleration can be expressed in vectorial form as:

$$\left(\mathbf{a}_{B/A}\right)_t = \boldsymbol{\alpha}\times\mathbf{r}_{B/A} \quad \text{and} \quad \left(\mathbf{a}_{B/A}\right)_n = -\omega^2\mathbf{r}_{B/A}$$

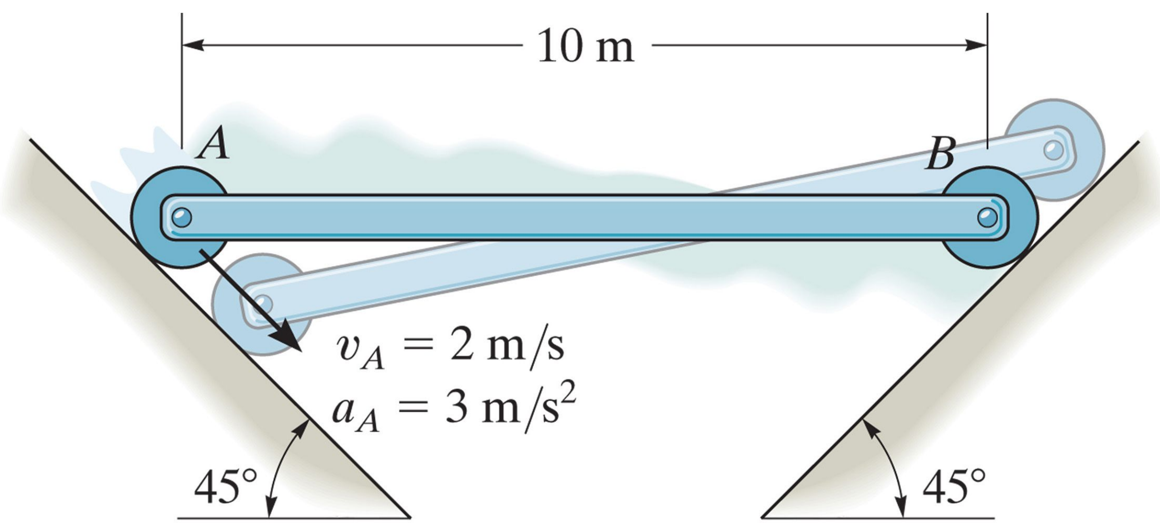
(see Lecture 9, slide 20), and the acceleration becomes:

$$\mathbf{a}_B = \mathbf{a}_A + \left(\mathbf{a}_{B/A}\right)_t + \left(\mathbf{a}_{B/A}\right)_n = \mathbf{a}_A + \boldsymbol{\alpha}\times\mathbf{r}_{B/A} - \omega^2\mathbf{r}_{B/A}$$

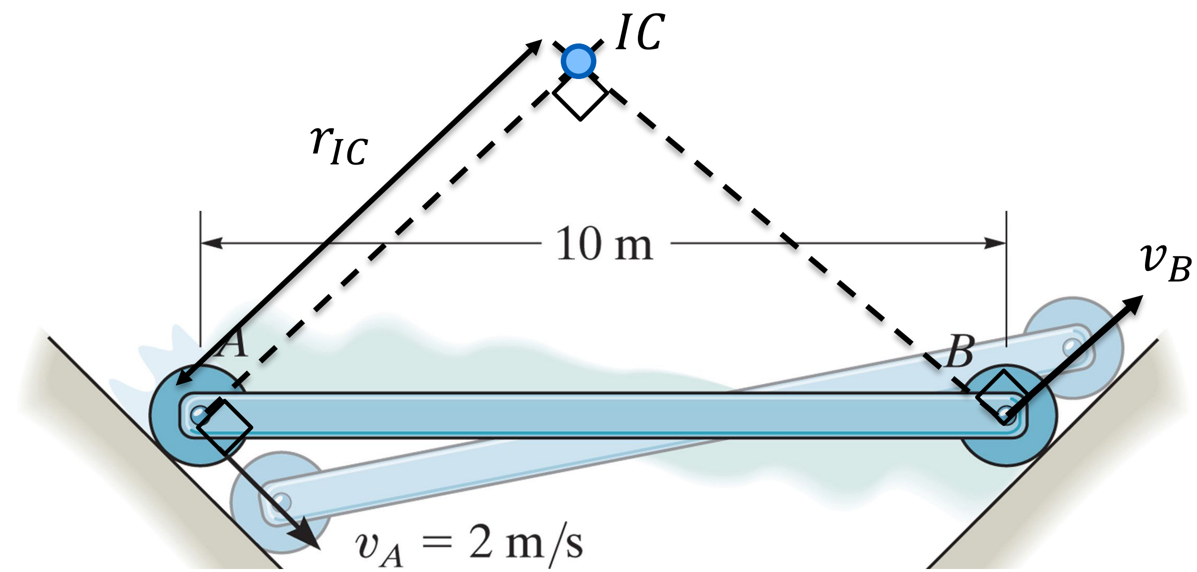


Example

The bar is confined to move along the two inclined planes. If point A has an acceleration of 3 m/s^2 and a velocity of 2 m/s (both downward) when the bar is horizontal, find the angular acceleration of the bar at this instant.



The velocities $\mathbf{v}_A$ and $\mathbf{v}_B$ are parallel to the inclined planes. This allows us to find the instantaneous centre of zero velocity.



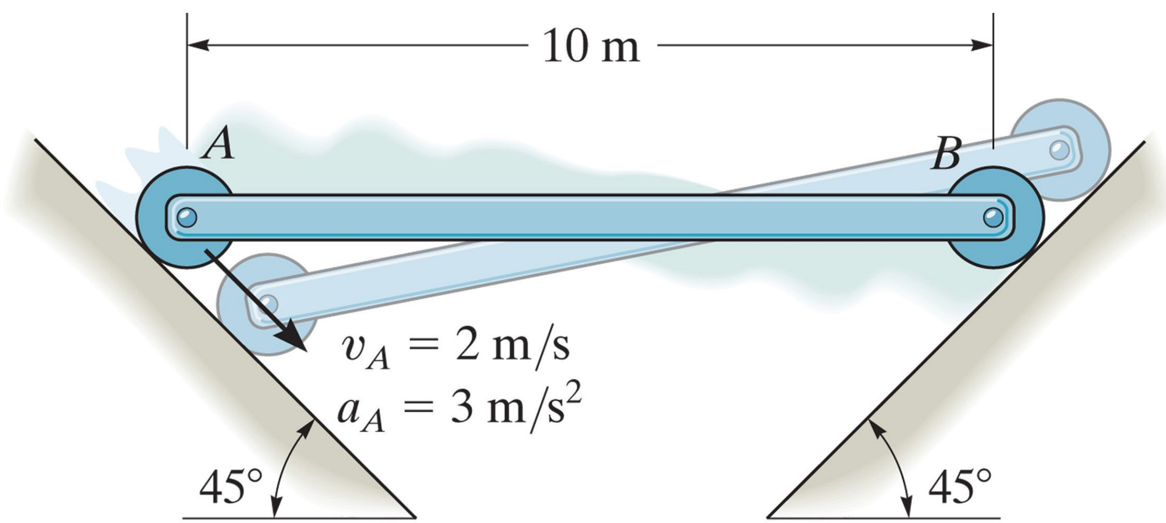
$$r_{IC} = 10\cos 45° = 7.07 \text{ m}$$

And the angular velocity is:

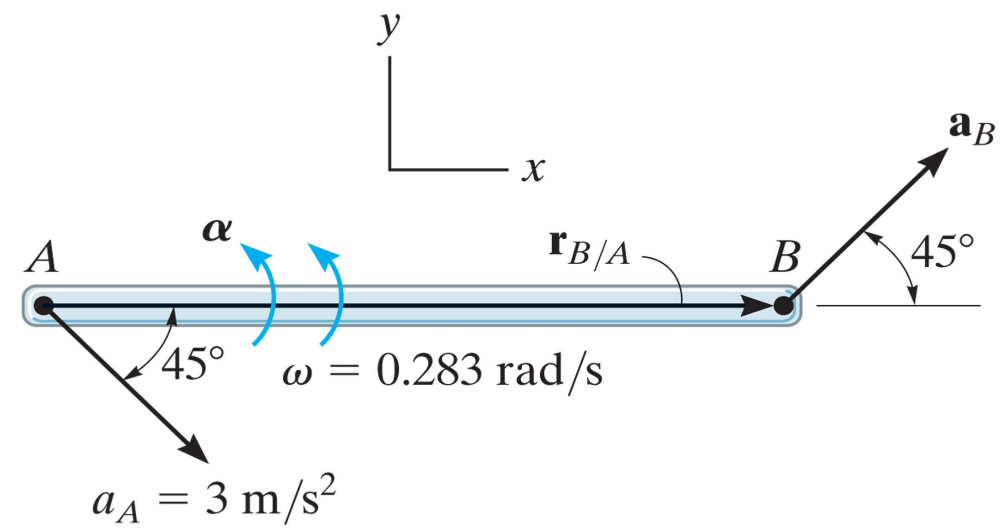
$$\omega = \frac{v_A}{r_{IC}} = \frac{2}{7.07} = 0.283 \text{ rad/s}$$

Example

The bar is confined to move along the two inclined planes. If point A has an acceleration of 3 m/s^2 and a velocity of 2 m/s (both downward) when the bar is horizontal, find the angular acceleration of the bar at this instant.



Kinetic diagram:



$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha}\times\mathbf{r}_{B/A} - \omega^2\mathbf{r}_{B/A}$$

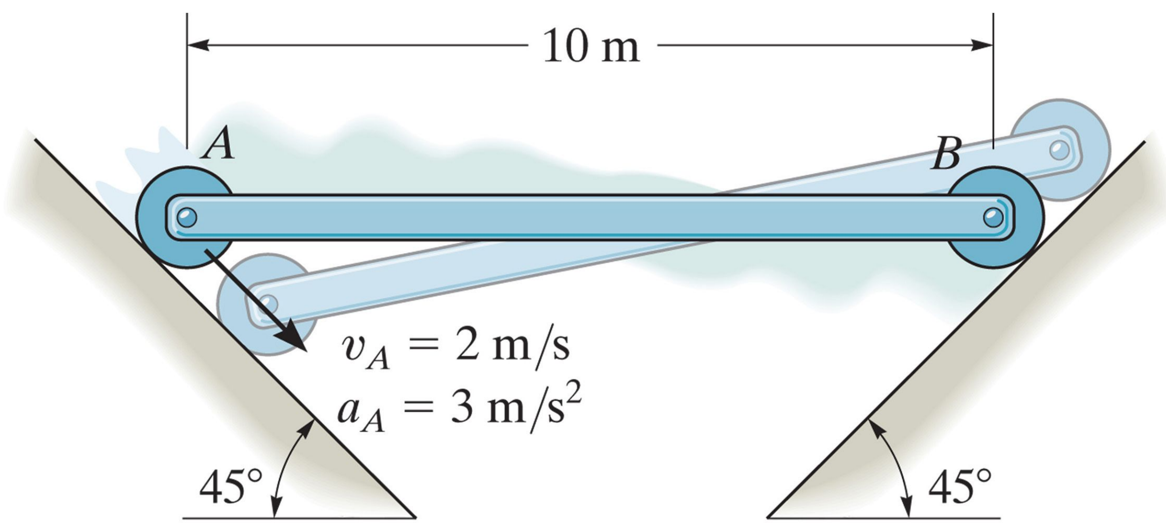
$$\Rightarrow a_B\cos 45°\,\mathbf{i} + a_B\sin 45°\,\mathbf{j} = 3\cos 45°\,\mathbf{i} - 3\sin 45°\,\mathbf{j} + (\alpha\mathbf{k})\times(10\mathbf{i}) - (0.283)^2(10\mathbf{i})$$

$$\Rightarrow a_B\cos 45°\,\mathbf{i} + a_B\sin 45°\,\mathbf{j} = 3\cos 45°\,\mathbf{i} - 3\sin 45°\,\mathbf{j} + 10\alpha\mathbf{j} - (0.283)^2(10\mathbf{i})$$

From this, we have two equations and two unknowns: a_B and α.

Example

The bar is confined to move along the two inclined planes. If point A has an acceleration of 3 m/s^2 and a velocity of 2 m/s (both downward) when the bar is horizontal, find the angular acceleration of the bar at this instant.



We had:

$$a_B \cos 45°\,\mathbf{i} + a_B \sin 45°\,\mathbf{j} = 3\cos 45°\,\mathbf{i} - 3\sin 45°\,\mathbf{j} + 10\alpha\mathbf{j} - (0.283)^2(10\mathbf{i})$$

Separating the **i** and **j** components give:

$$\begin{cases} a_B \cos 45° = 3\cos 45° - (0.283)^2(10) \\ a_B \sin 45° = -3\sin 45° + 10\alpha \end{cases}$$

Solving this system gives:

$$a_B = 1.87\ \text{m/s}^2$$

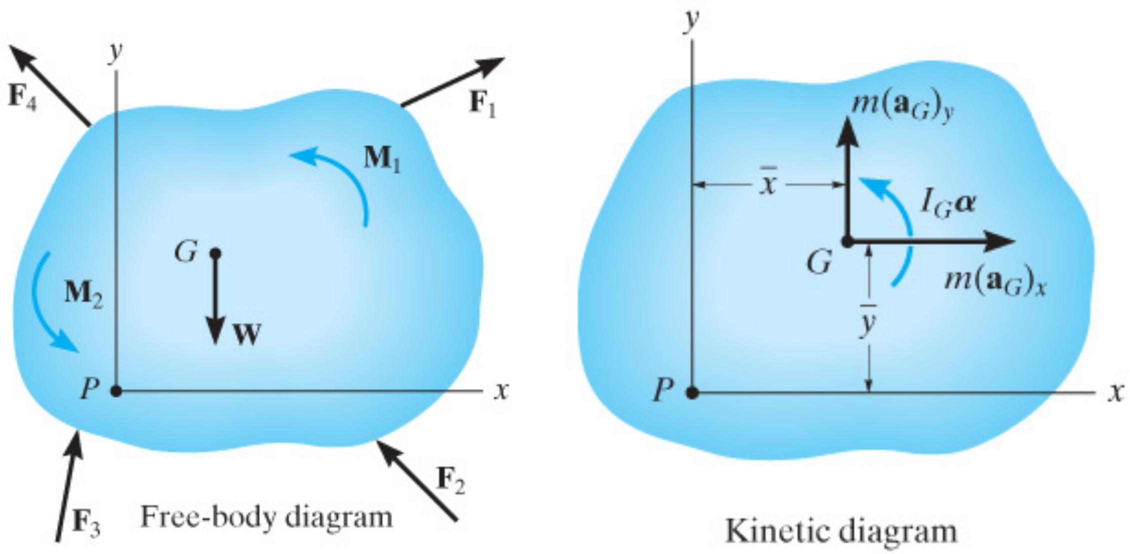
$$\alpha = 0.344\ \text{rad/s}^2$$

Planar motion of a rigid body

c) General plane motion

- Equations of motion

Planar kinetic equations of motion



Planar equations of motion for a rigid body.

Specified about the center of mass G.

$$\Sigma F_x = m(a_G)_x$$

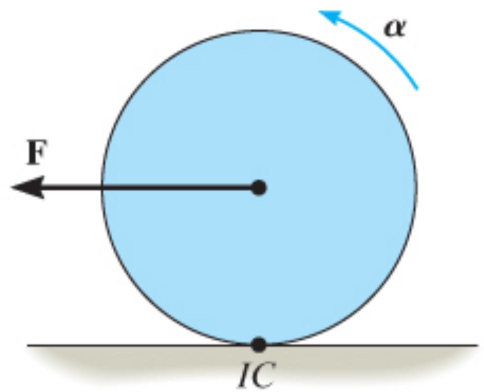
$$\Sigma F_y = m(a_G)_y$$

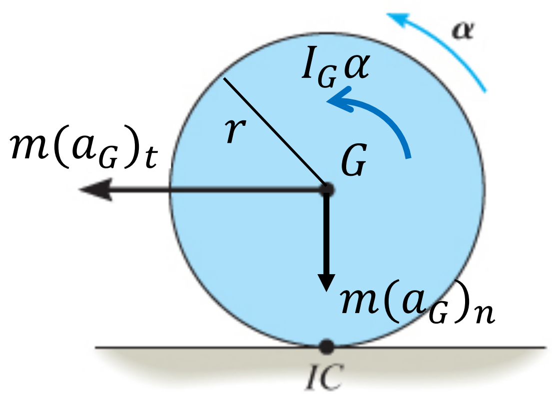
$$\Sigma M_G = I_G \alpha$$

For some problems, it may be convenient to sum moments about a point P other than G. Rewritting the sum of moments about point P gives:

$$\Sigma M_P = \Sigma(M_k)_P = -\bar{y} m(a_G)_x + \bar{x} m(a_G)_y + I_G \alpha$$

Moment equation about IC





Taking the sum of moments about the instantaneous centre of zero velocity gives:

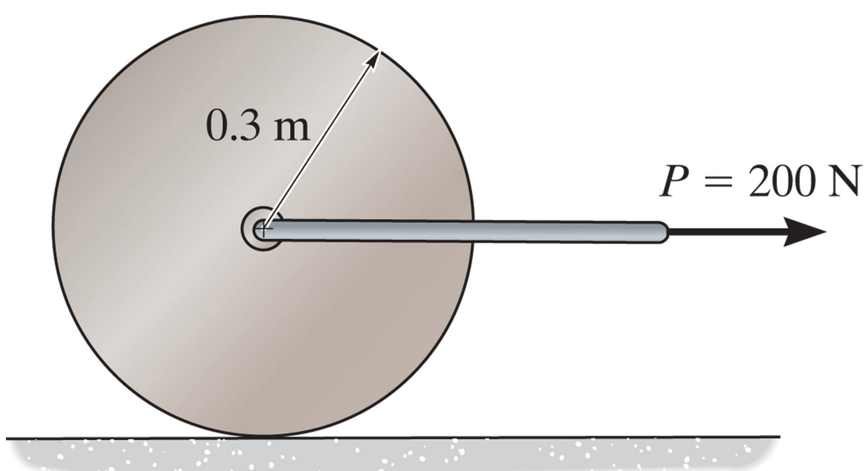
$$\Sigma M_{IC} = rm(a_G)_t + I_G\alpha = rm(r\alpha) + I_G\alpha = (I_G + mr^2)\alpha$$

Note that $I_{IC} = I_G + mr^2$ based on the parallel axis theorem, and the equation becomes:

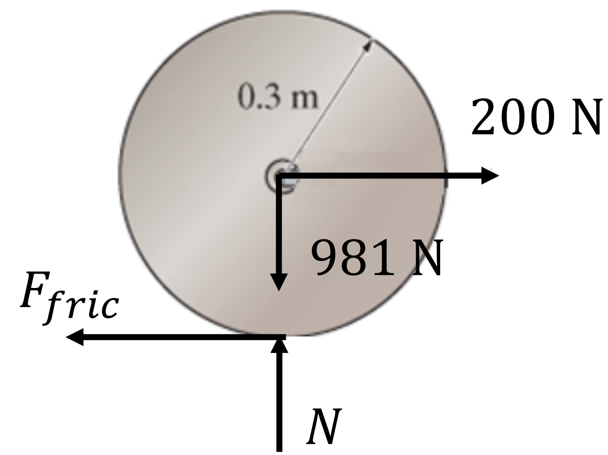
$$\boxed{\Sigma M_{IC} = I_{IC}\alpha}$$

Example

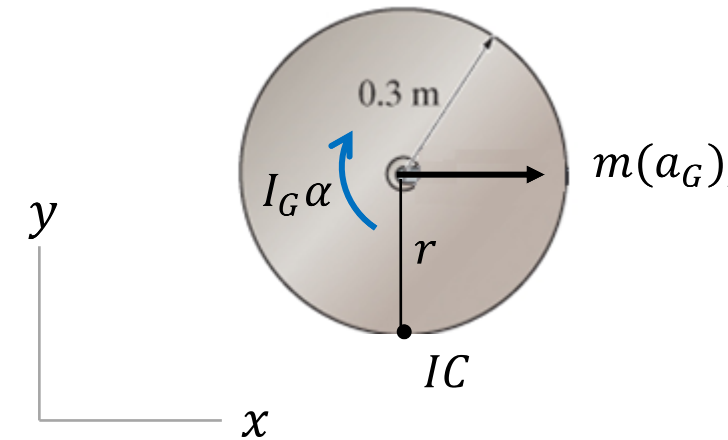
The 100 kg cylinder rolls without slipping. Find the acceleration of its centre of mass and its angular acceleration.



Free-body diagram



Kinetic diagram



Taking the sum of moments about IC:

$$\circlearrowright +\Sigma M_{IC} = I_{IC}\alpha \quad \Rightarrow \quad 200r = I_{IC}\alpha$$

Where the mass moment of inertia is:

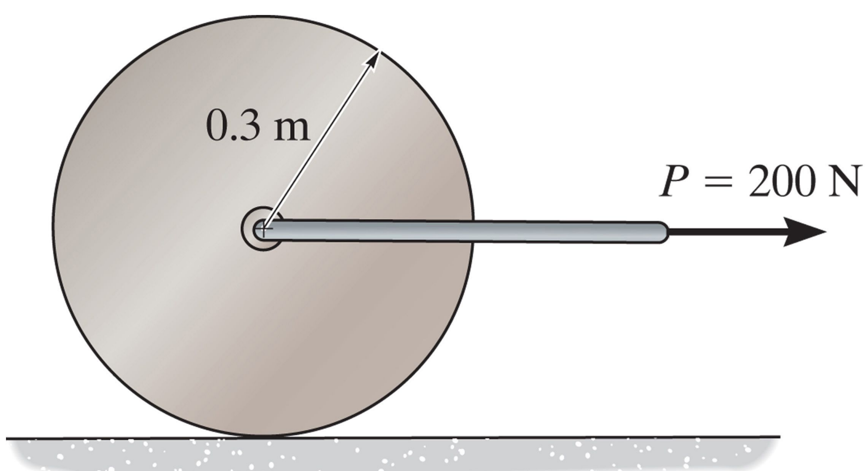
$$I_{IC} = I_G + mr^2 = \frac{1}{2}mr^2 + mr^2 = 13.5\ \mathrm{kg}\cdot\mathrm{m}^2$$

Returning to the equation above, we find:

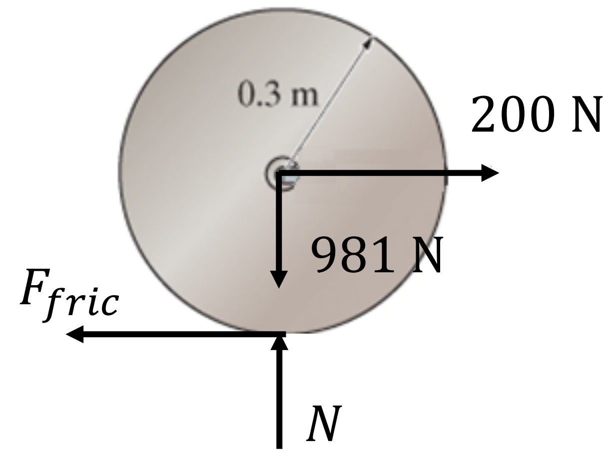
$$200r = I_{IC}\alpha \quad \Rightarrow \quad \alpha = \frac{200r}{I_{IC}} = \frac{200\cdot 0.3}{13.5} = 4.44\ \mathrm{rad/s^2}$$

Example

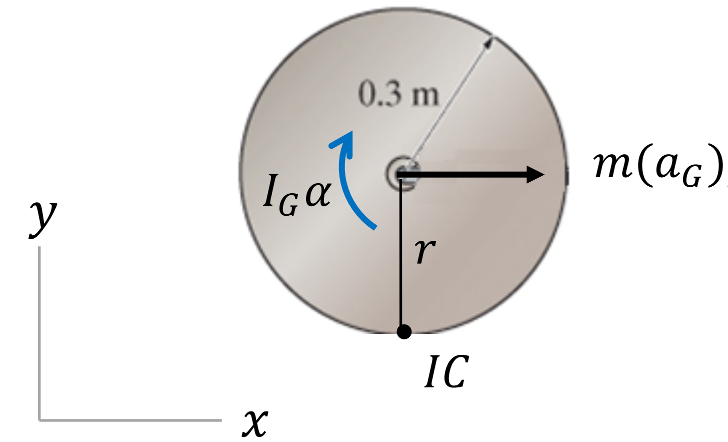
The 100 kg cylinder rolls without slipping. Find the acceleration of its centre of mass and its angular acceleration.



Free-body diagram



Kinetic diagram



The acceleration can be obtained using the kinetic diagram and the sum of moments:

$$\circlearrowright +\Sigma M_{IC} = I_{IC}\alpha$$

$$\Rightarrow I_G\alpha + m(a_G)_x \cdot r = I_{IC}\alpha$$

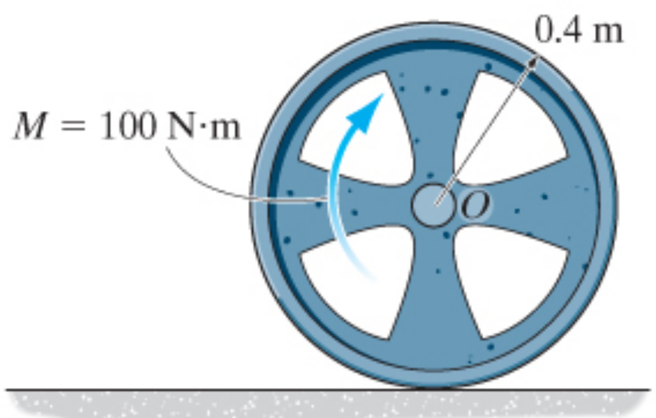
$$\Rightarrow m(a_G)_x \cdot r = (I_{IC} - I_G)\alpha$$

$$\Rightarrow m(a_G)_x \cdot r = (I_G + mr^2 - I_G)\alpha$$

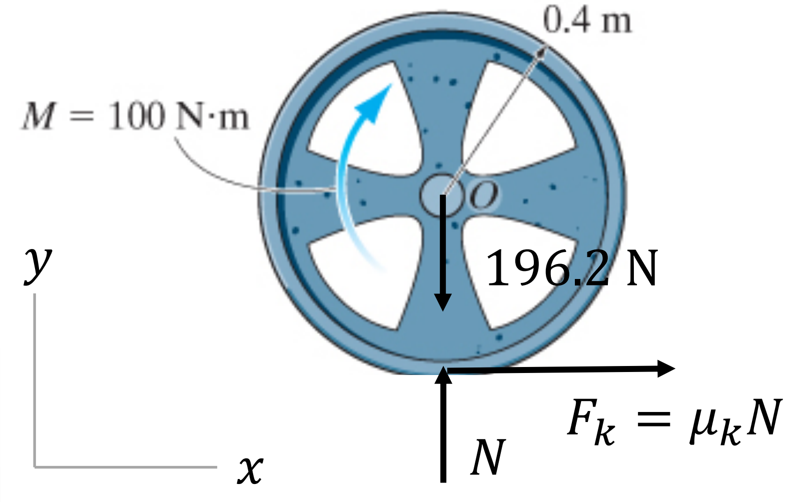
$$\Rightarrow (a_G)_x = r\alpha = 1.33\ \mathrm{m/s^2}$$

Example

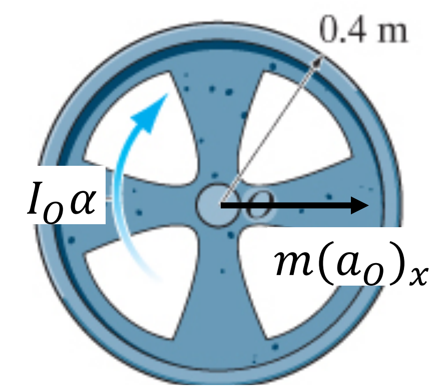
The 20 kg wheel has a radius of gyration about point O of $k_0 = 300\,\text{mm}$. The wheel slips and rolls when subjected to a moment. Find the angular acceleration and acceleration of point O. The coefficient of kinetic friction is $\mu_k = 0.5$.



Free-body diagram



Kinetic diagram



There is no motion is the vertical direction, which gives:

$$\uparrow + \Sigma F_y = m(a_O)_y \ \Rightarrow \ N - 196.2 = 0 \ \Rightarrow \ N = 196.2\ \text{N}$$

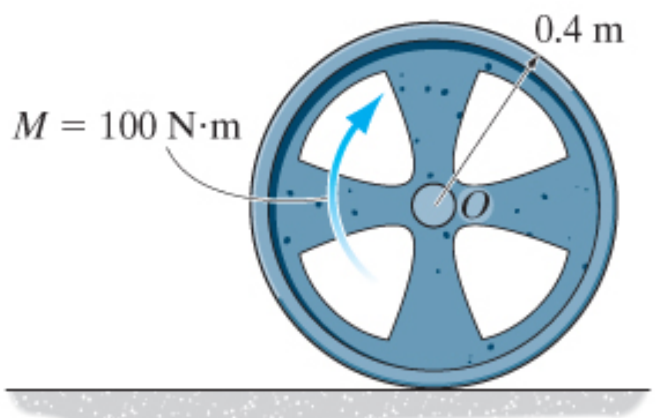
The sum of forces in the horizontal direction gives:

$$\rightarrow + \Sigma F_x = m(a_O)_x \ \Rightarrow \ \mu_k N = m(a_O)_x$$

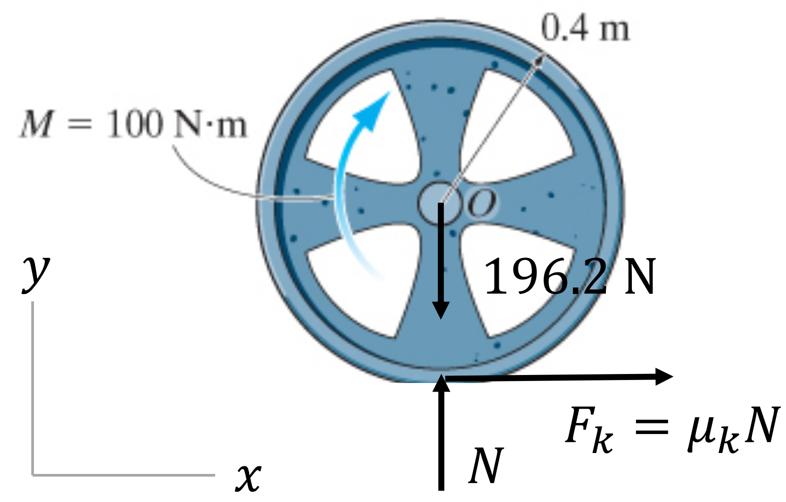
$$\Rightarrow (a_O)_x = \frac{\mu_k N}{m} = \frac{0.5(196.2\text{N})}{20\text{kg}} = 4.905\,\frac{\text{m}}{\text{s}^2}$$

Example

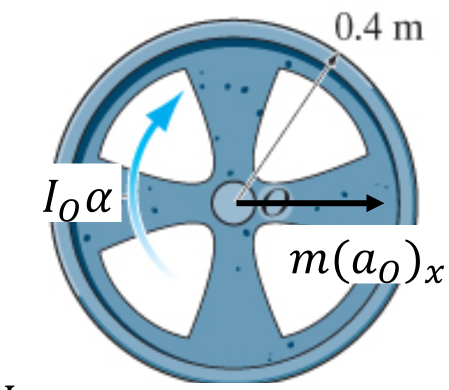
The 20 kg wheel has a radius of gyration about point O of $k_0 = 300\,\text{mm}$. The wheel slips and rolls when subjected to a moment. Find the angular acceleration and acceleration of point O. The coefficient of kinetic friction is $\mu_k = 0.5$.



Free-body diagram



Kinetic diagram



Finally, the sum of moments about point O gives:

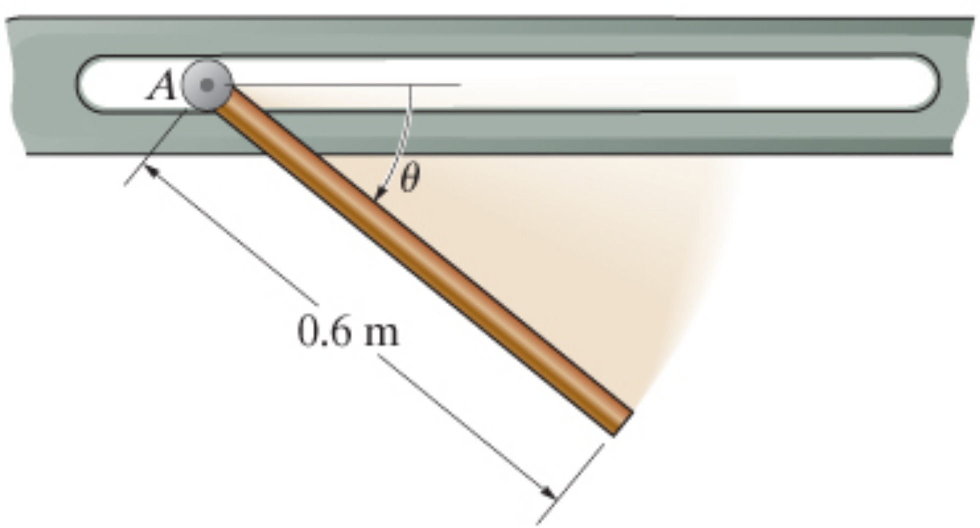
$$\circlearrowleft + \Sigma M_O = I_O\alpha \quad \Rightarrow \quad M - \mu_k N r = (m k_0^2)\alpha$$

$$\Rightarrow 100\text{Nm} - 0.5(196.2\text{N})(0.4\text{m}) = (20\text{kg})(0.3\text{m})^2\,\alpha$$

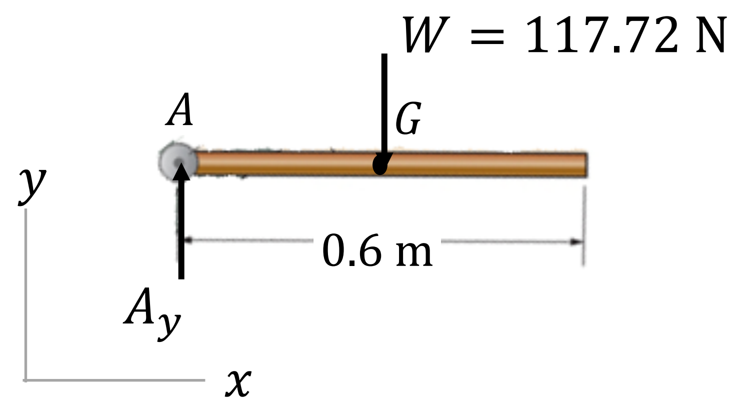
$$\Rightarrow \alpha = 33.8\ \text{rad/s}^2$$

Example

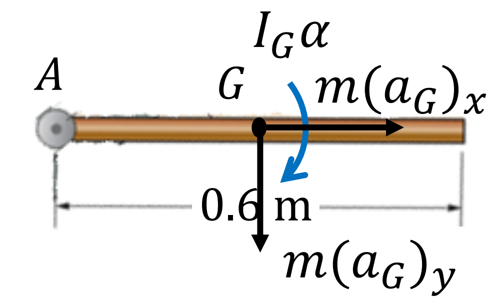
The 12 kg slender rod is pinned to a small roller that slides freely along the slot. If the rod is released from rest at $\theta = 0$, find the angular acceleration of the rod and the acceleration of its centre of mass immediately after the release.



Free-body diagram



Kinetic diagram



The sum of forces along x gives us:

$$\rightarrow + \Sigma F_x = m(a_G)_x \Rightarrow \quad 0 = 12\text{kg}(a_G)_x \Rightarrow (a_G)_x = 0$$

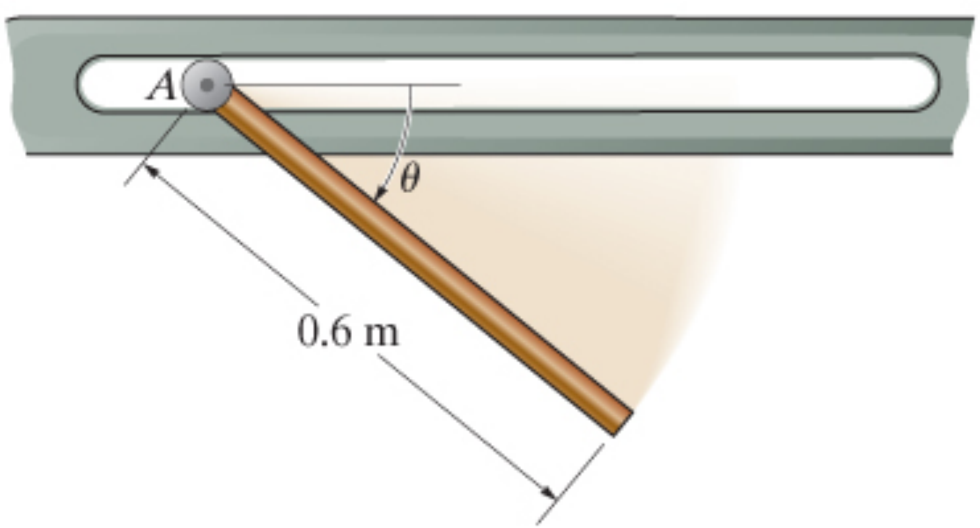
The sum of moments about A, which is an IC of zero velocity, gives:

$$\circlearrowright + \Sigma M_A = I_A\alpha \quad \Rightarrow \quad W\frac{l}{2} = \frac{1}{3}ml^2\alpha$$

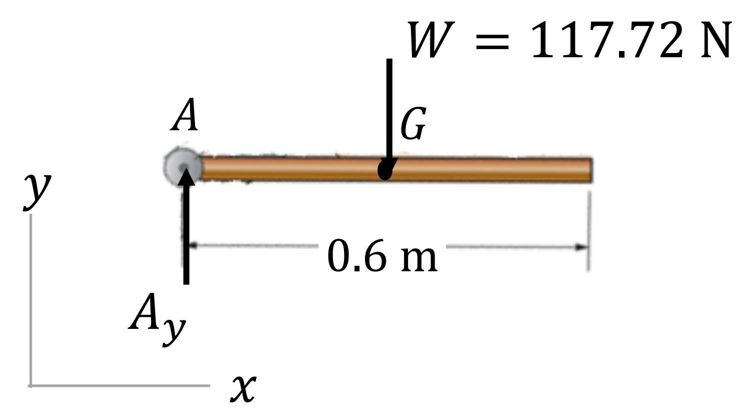
$$\Rightarrow \alpha = \frac{3W}{2ml} = \frac{3 \cdot 117.72}{2 \cdot 12 \cdot 0.6} = 24.5 \text{ rad/s}^2$$

Example

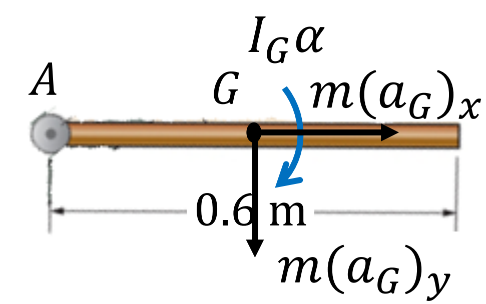
The 12 kg slender rod is pinned to a small roller that slides freely along the slot. If the rod is released from rest at $\theta = 0$, find the angular acceleration of the rod and the acceleration of its centre of mass immediately after the release.



Free-body diagram



Kinetic diagram



Using the kinetic diagram, we can rewrite the sum of moments about A as:

$$\circlearrowright + \Sigma M_A = I_G\alpha + m(a_G)_y(l/2)$$

$$\Rightarrow W(l/2) = \frac{1}{12}ml^2\alpha + m(a_G)_y(l/2)$$

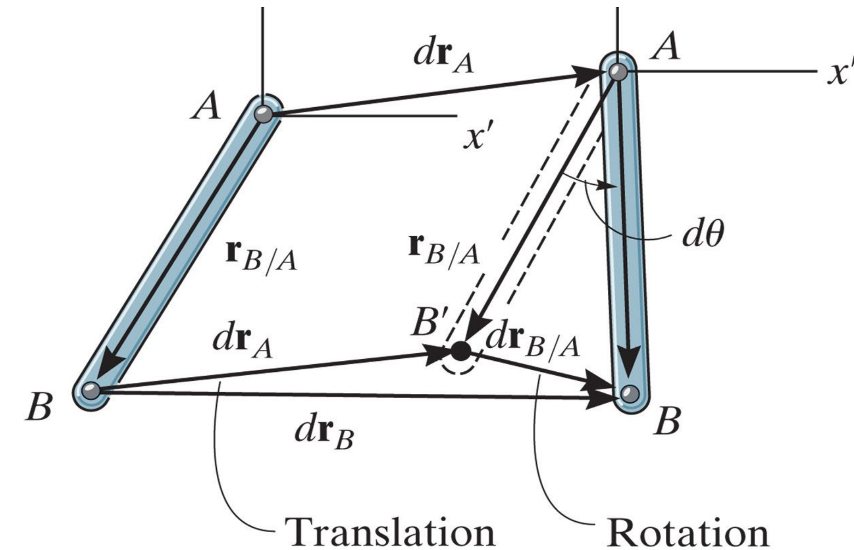
$$\Rightarrow (a_G)_y = \frac{2}{ml}\left[W\frac{l}{2} - \frac{1}{12}ml^2\alpha\right]$$

$$= \frac{2}{12 \cdot 0.6}\left[117.72 \cdot \frac{0.6}{2} - \frac{1}{12}12 \cdot 0.6^2 \cdot 24.5\right]$$

$$= 7.36 \text{ m/s}^2$$

Summary

- General plane motion can be decomposed in:
  - Translation, and
  - Rotation about a point on the rigid body
- The analysis is simplified using the instantaneous center of zero velocity.



Kinematics

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega}\times\mathbf{r}_{B/A}$$

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha}\times\mathbf{r}_{B/A} - \omega^2\mathbf{r}_{B/A}$$

Equations of motion

$$\Sigma F_x = m(a_G)_x$$

$$\Sigma F_y = m(a_G)_y$$

$$\Sigma M_G = I_G\alpha$$

Need additional information?

For more details, consult these sections of the textbook:

General plane motion

- Kinematics: sections 16.4-16.7
- Kinetics: 17.5