Aalto-yliopisto
Aalto-universitetet
Aalto University

# COE-C1001: Statics

## 1. Force vectors and equilibrium of a particle

*Luc St-Pierre*

# Learning outcomes

After this lecture, you should be able to:

- understand that force and position are vectors,
- compute the resultant force for 2D and 3D systems,
- create the free-body diagram of a system,
- determine the conditions necessary for equilibrium.

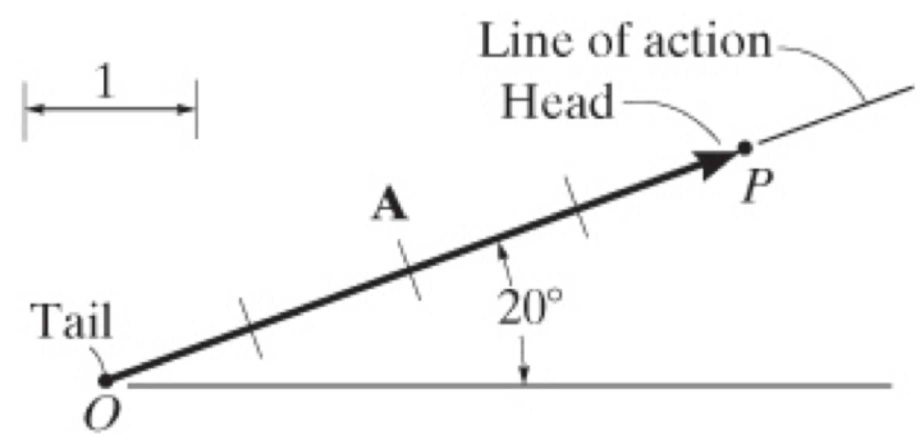
# Definitions, Fundamental quantities, and Vectorial operations

# Scalars and vectors

In this course, we will deal with physical quantities that are:

- **Scalars**, which are described by a single number (such as temperature), or
- **Vectors**, which are characterized by their magnitude and direction.

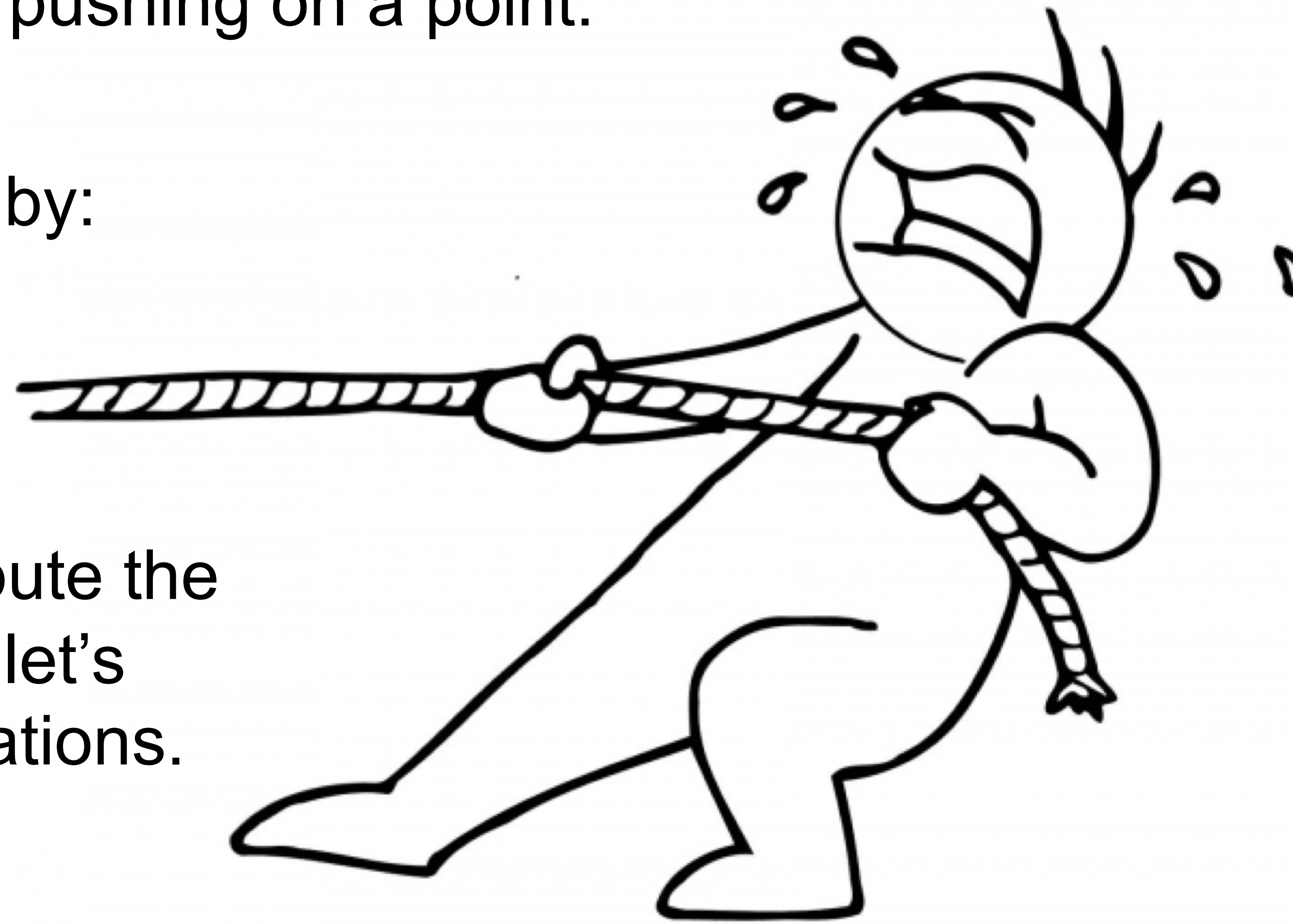
| Scalars | Vectors |
| --- | --- |
| Length (m) | Force (N) |
| Mass (kg) | Moment (Nm) |
| Time (s) | Velocity (m/s) |



# What is a force?

- Forces are either pulling or pushing on a point.

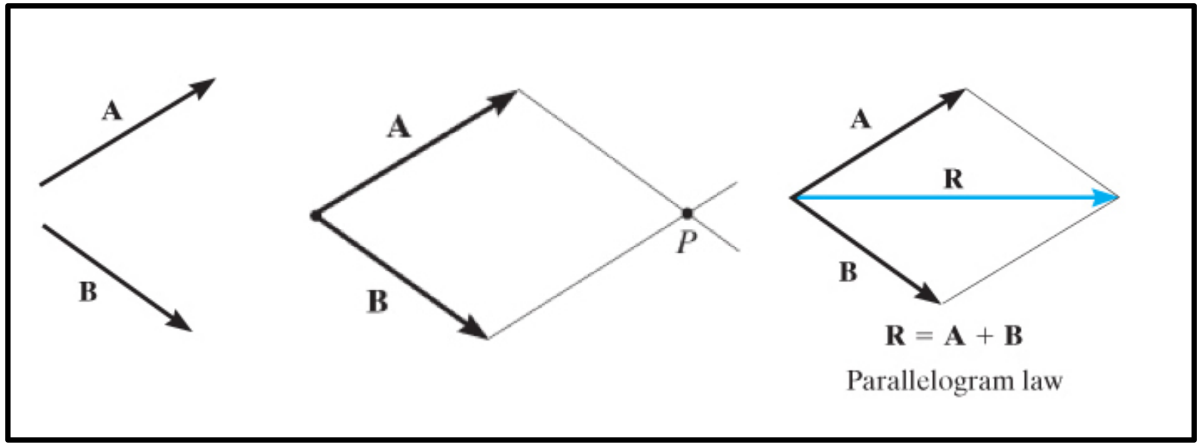
- Forces are vectors defined by:
  - their magnitude and
  - direction.



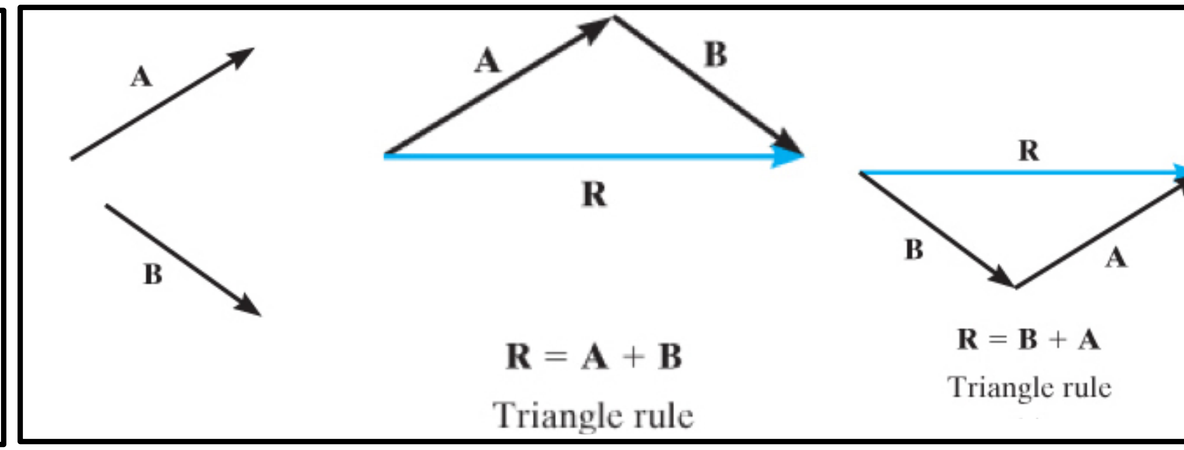
- In this course, we will compute the action of multiple forces so let's review basic vectorial operations.

# Vector addition

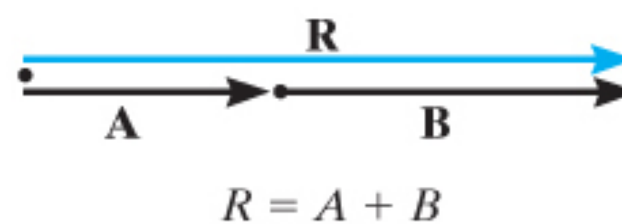
Method 1: **parrallelogram law**




Method 2: **triangle rule**

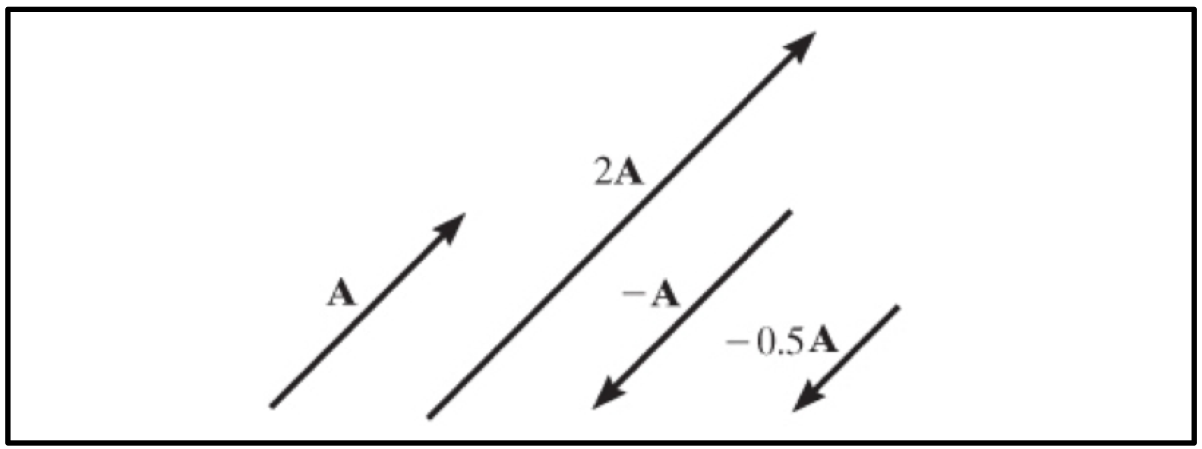



Special case: collinear vectors

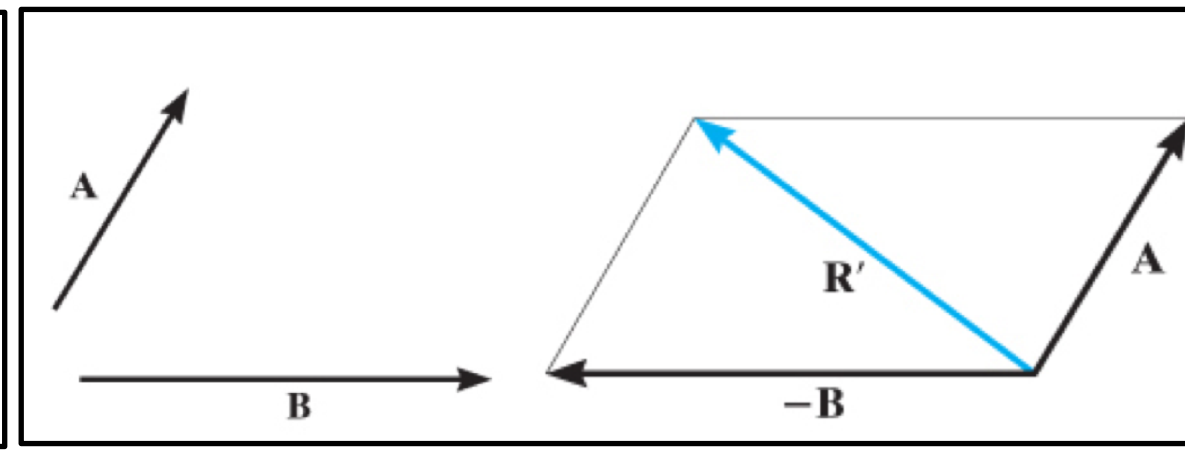


# Vector operations

## Multiplication by a scalar




## Vector subtraction

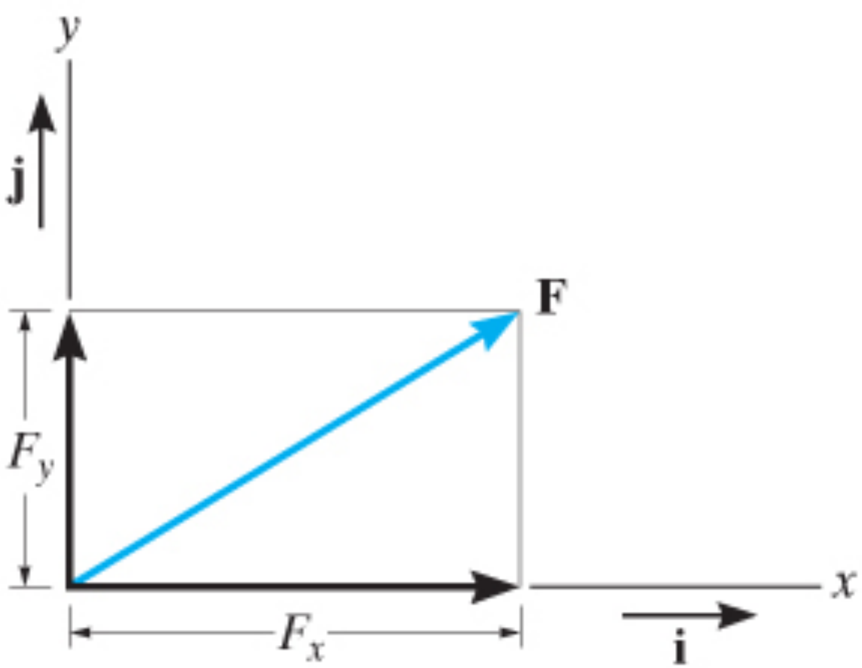


# Cartesian vectors

Vectors can be expressed using their $x$ and $y$ components:

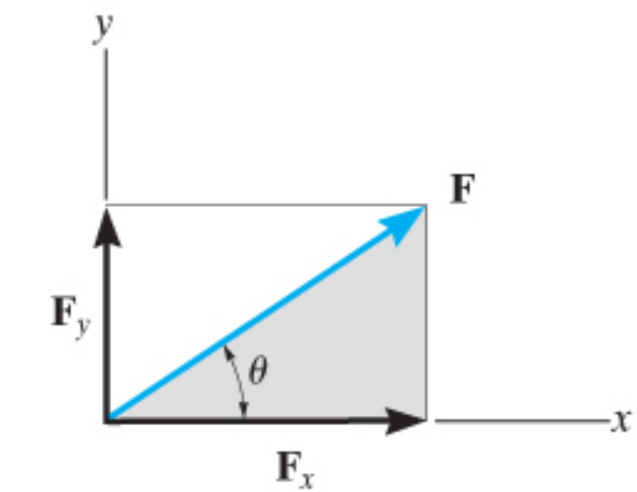
$$\mathbf{F} = \mathbf{F}_x + \mathbf{F}_y \quad = F_x\mathbf{i} + F_y\mathbf{j}$$

Where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the $x$ and $y$ directions, respectively.



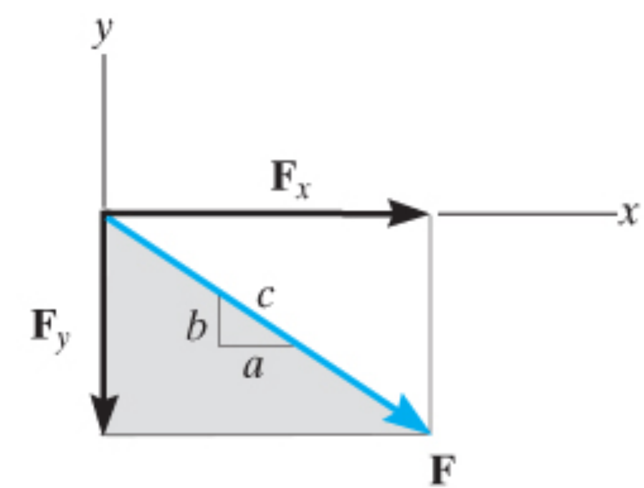

The components $F_x$ and $F_y$ are given by:

$$F_x = F\cos\theta$$
$$F_y = F\sin\theta$$



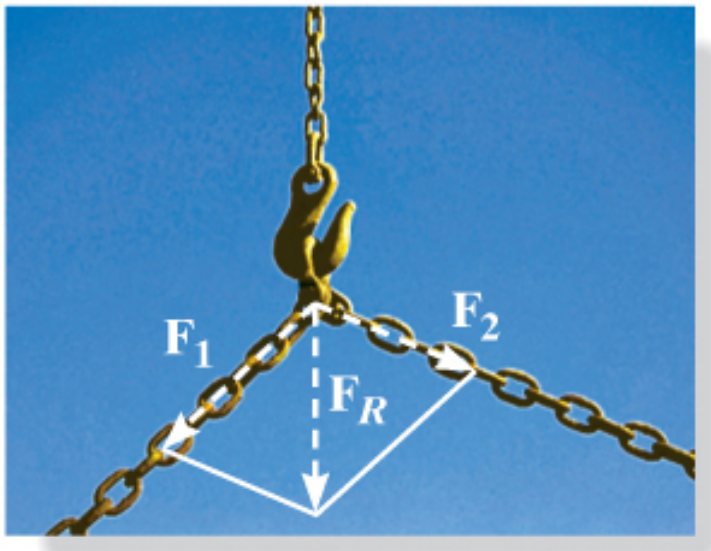

The components can be negative depending on the orientation of $\mathbf{F}$.

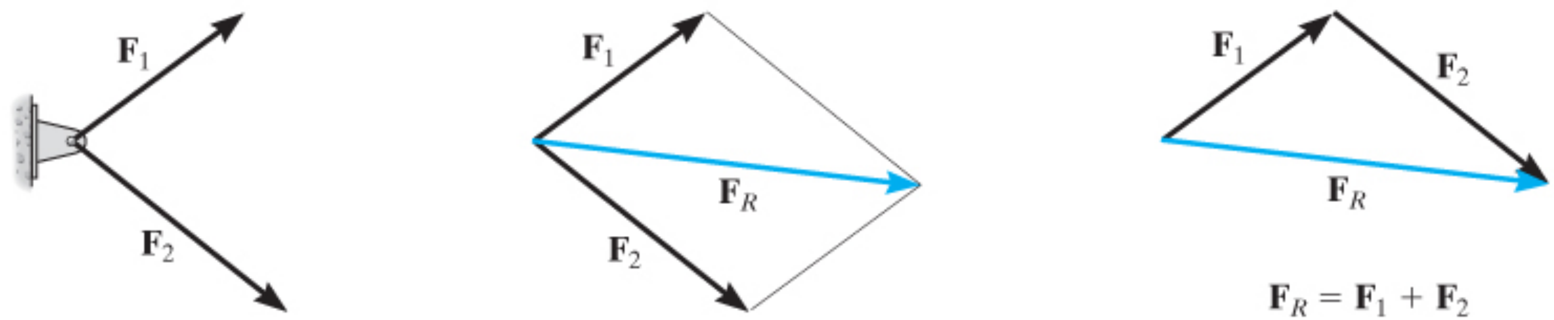
**Note** that $F$ is the magnitude of the vector $\mathbf{F}$. Vectors are represented using **boldface**.



# The resultant force, and position vectors

# Finding a resultant force

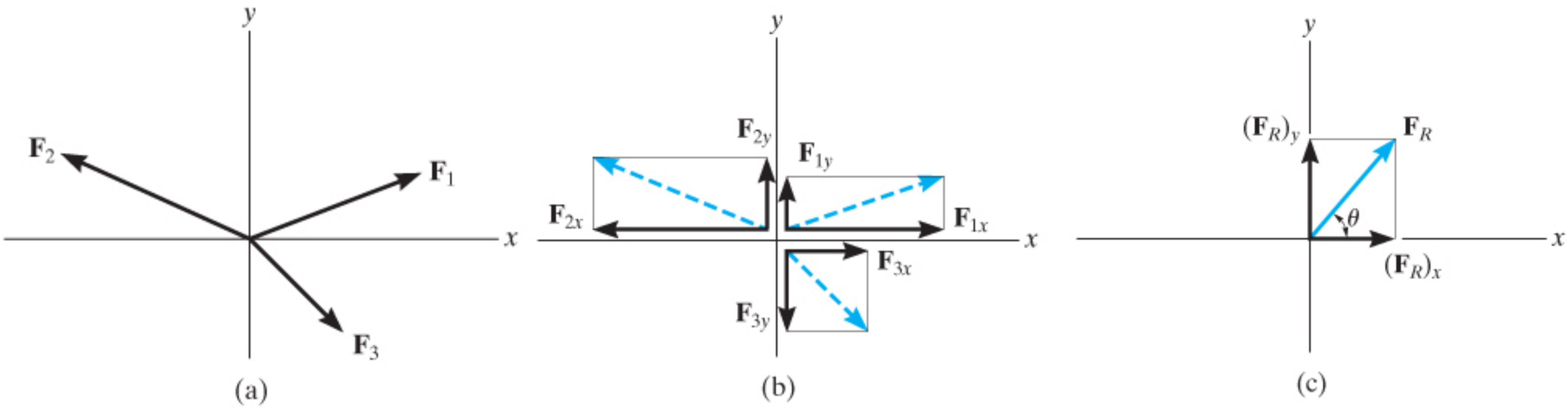



(a) (b) (c)

- The **resultant force** is the addition of two (or more) forces**.**
- You can obtain the resultant force using:
    - The parallelogram law,
    - The triangle rule, or
    - By adding the components of Cartesian vectors.
- The hook above would move in the direction of the resultant force.

# Force resultant using Cartesian vectors




$$\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 \quad ?$$

$$\mathbf{F}_1 = F_{1x}\mathbf{i} + F_{1y}\mathbf{j}$$

$$\mathbf{F}_2 = -F_{2x}\mathbf{i} + F_{2y}\mathbf{j}$$

$$\mathbf{F}_3 = F_{3x}\mathbf{i} - F_{3y}\mathbf{j}$$

$$\mathbf{F}_R = (F_R)_x\mathbf{i} + (F_R)_y\mathbf{j}$$

$$(F_R)_x = F_{1x} - F_{2x} + F_{3x}$$

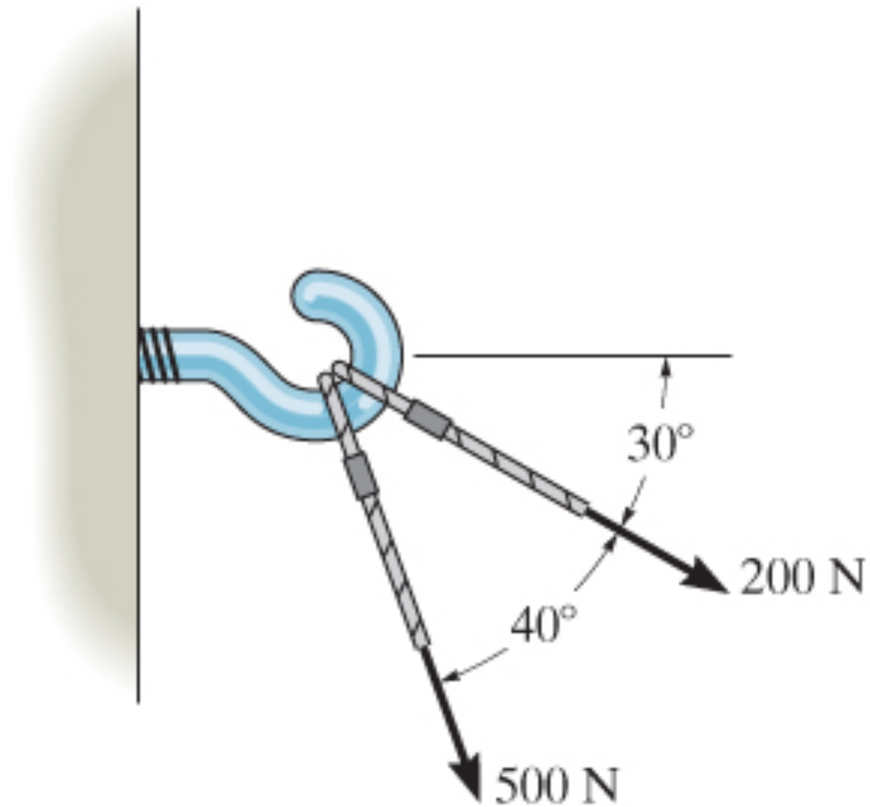
$$(F_R)_y = F_{1y} + F_{2y} - F_{3y}$$

# Example: force resultant

Find the magnitude and direction of the resultant force

**Method 1:** using the parallelogram law or triangle rule.

**Method 2**: define a coordinate system and use cartesian vectors.



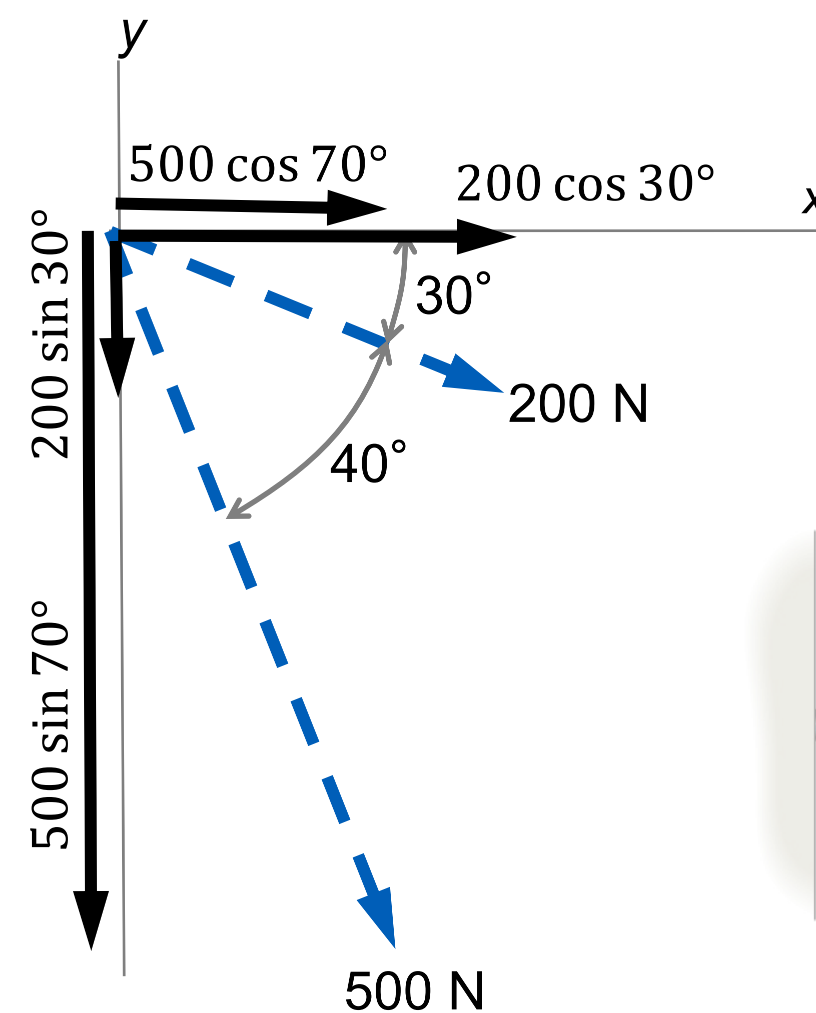
# Example: force resultant

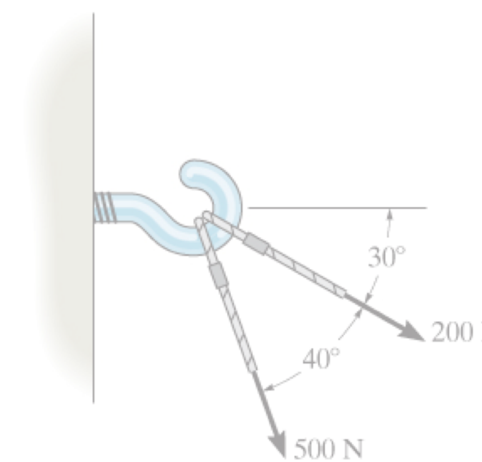
The forces are separated in x and y components as shown.

The components of the resultant force are:

$$(F_R)_x = 200 \cos 30° + 500 \cos 70° = 344 \text{ N}$$

$$(F_R)_y = -200 \sin 30° - 500 \sin 70° = -570 \text{ N}$$



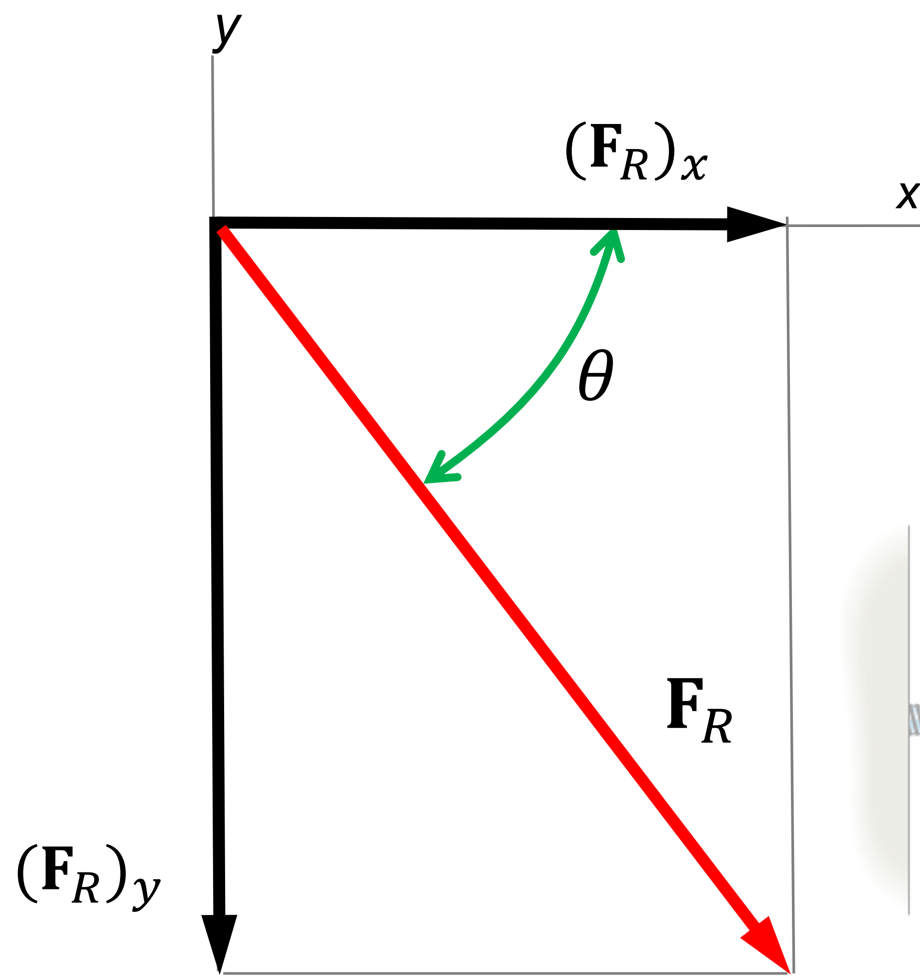



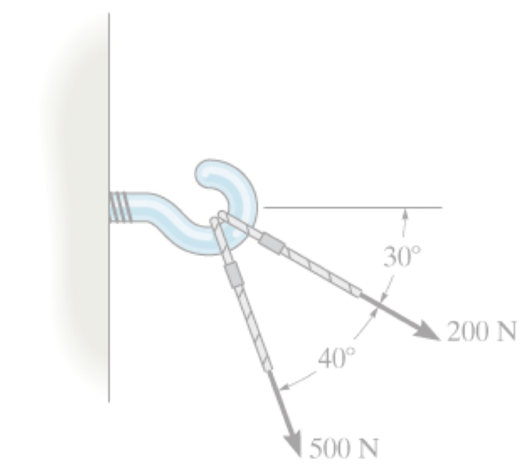
# Example: force resultant

The magnitude of $\mathbf{F}_R$ and its orientation $\theta$ are given by:

$$F_R = \sqrt{(F_R)_x{}^2 + (F_R)_y{}^2} = 666\ \text{N}$$

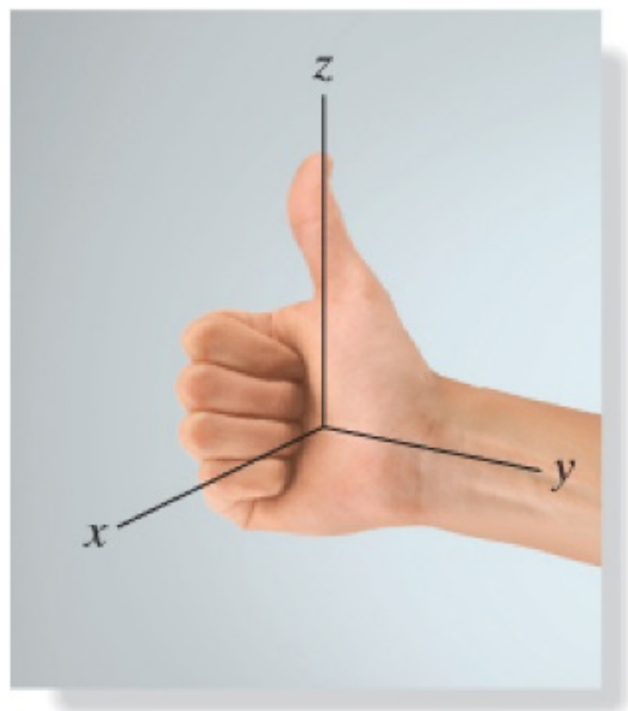
$$\theta = \tan^{-1}\left|\frac{(F_R)_y}{(F_R)_x}\right| = 58.9°$$



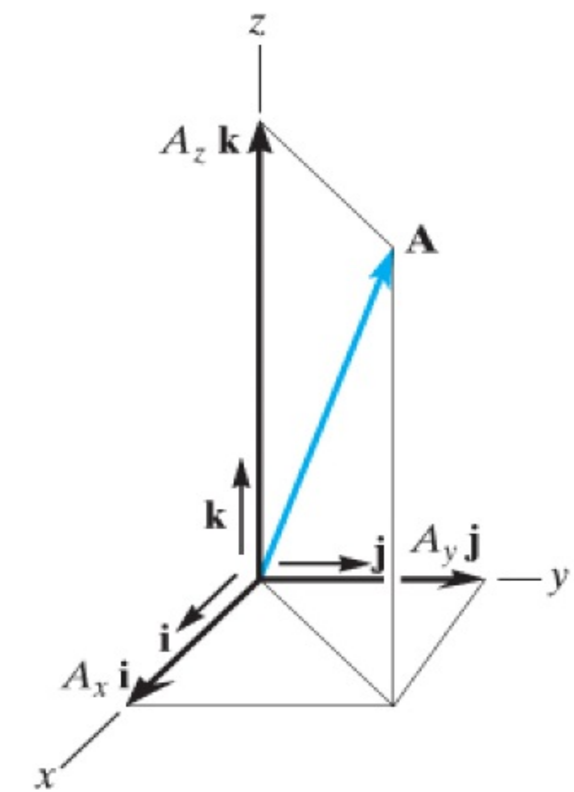
# Forces in 3D

# 3D Cartesian vectors




**Always use a right-handed coordinate system**

- Thumb in the z-direction
- Fingers in the x-direction
- Arm in the y-direction




Vectors can be expressed using x, y and z components:

$$\mathbf{A} = \mathbf{A}_x + \mathbf{A}_y + \mathbf{A}_z = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$

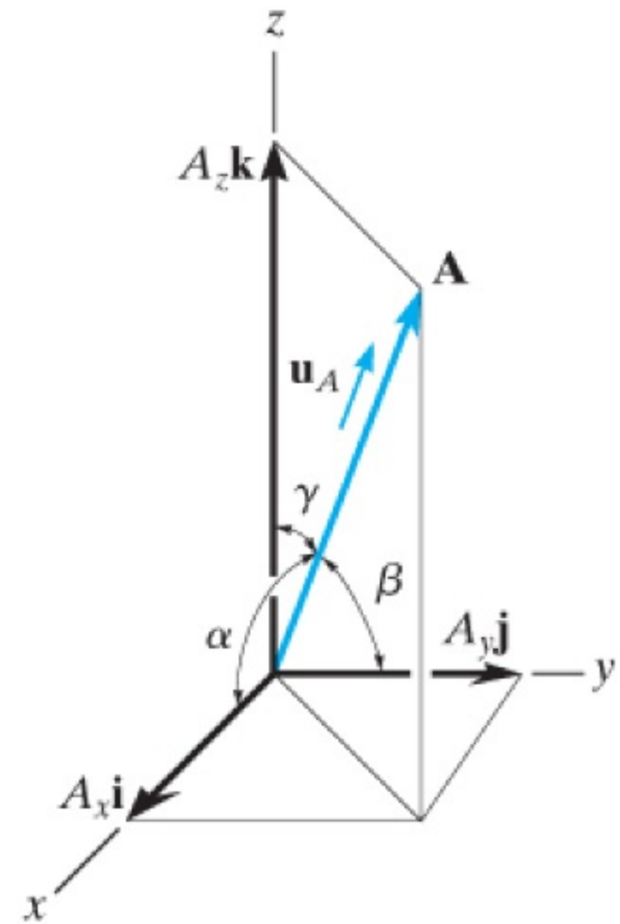
# Magnitude and orientation

The magnitude is given by:

$$A = \sqrt{A_x{}^2 + A_y{}^2 + A_z{}^2}$$

The orientation is expressed using the direction cosines:

$$\cos\alpha = \frac{A_x}{A} \quad \cos\beta = \frac{A_y}{A} \quad \cos\gamma = \frac{A_z}{A}$$

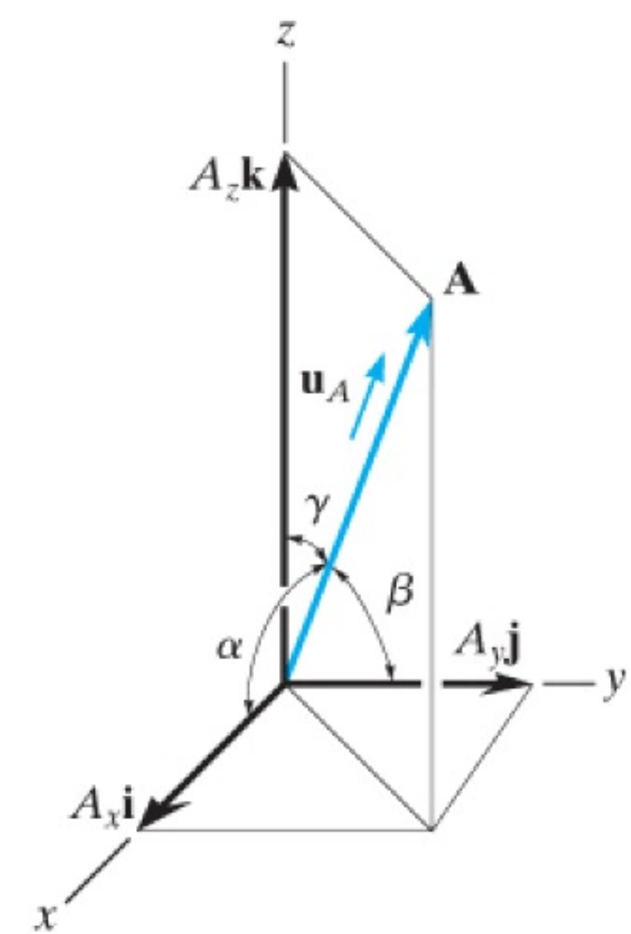


# Orientation as a unit vector

The orientation can be expressed as a unit vector $\mathbf{u}_A$:

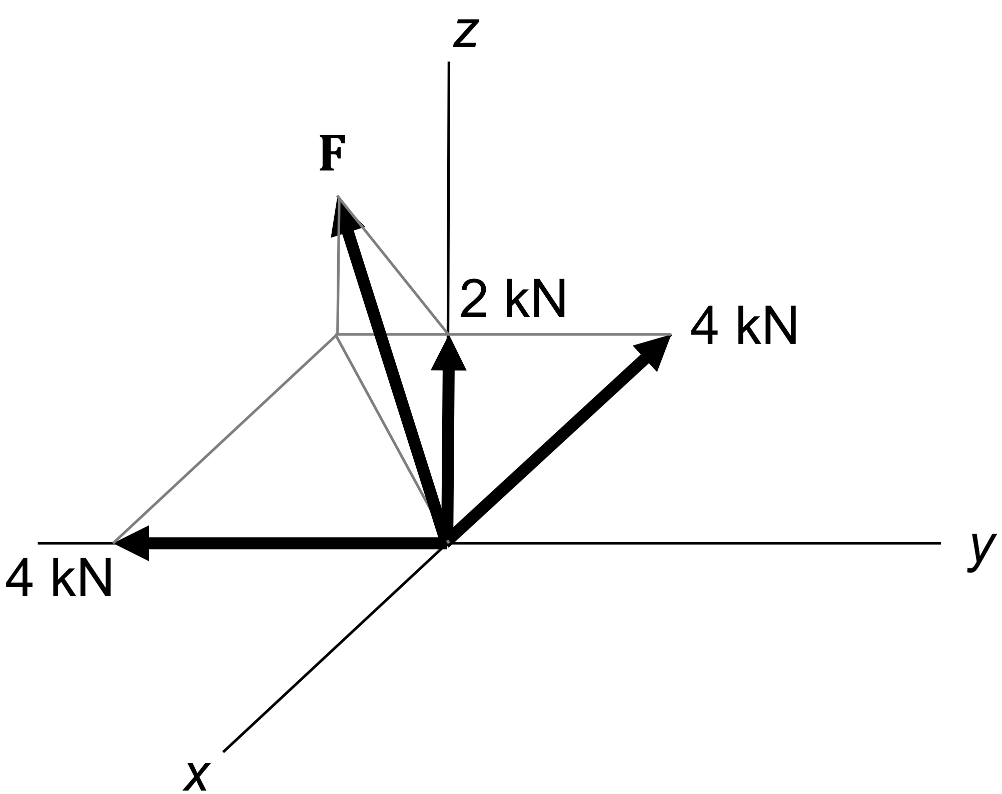
$$\mathbf{u}_A = \frac{\mathbf{A}}{A} = \frac{A_x}{A}\mathbf{i} + \frac{A_y}{A}\mathbf{j} + \frac{A_z}{A}\mathbf{k}$$ , it follows that: $\mathbf{A} = A\mathbf{u}_A$

Where the magnitude : $A = \sqrt{{A_x}^2 + {A_y}^2 + {A_z}^2}$



# Example

Express the resultant force $\mathbf{F}$ as a Cartesian vector, and find its magnitude and orientation.




The resultant $\mathbf{F}$ as a Cartesian vector is:

$$\mathbf{F} = \{-4\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}\}\ \mathrm{kN}$$

The magnitude of $\mathbf{F}$ is:

$$F = \sqrt{(-4)^2 + (-4)^2 + 2^2} = 6\ \mathrm{kN}$$

The unit vector $\mathbf{u}_F$ is:

$$\mathbf{u}_F = \frac{\mathbf{F}}{F} = -\frac{4}{6}\mathbf{i} - \frac{4}{6}\mathbf{j} + \frac{2}{6}\mathbf{k} = \frac{1}{3}(-2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$$

Finally, the direction cosines are:

$$\cos\alpha = \frac{F_x}{F} = -\frac{2}{3} \qquad \cos\beta = \frac{F_y}{F} = -\frac{2}{3} \qquad \cos\gamma = \frac{F_z}{F} = \frac{1}{3}$$

# Addition of Cartesian vectors

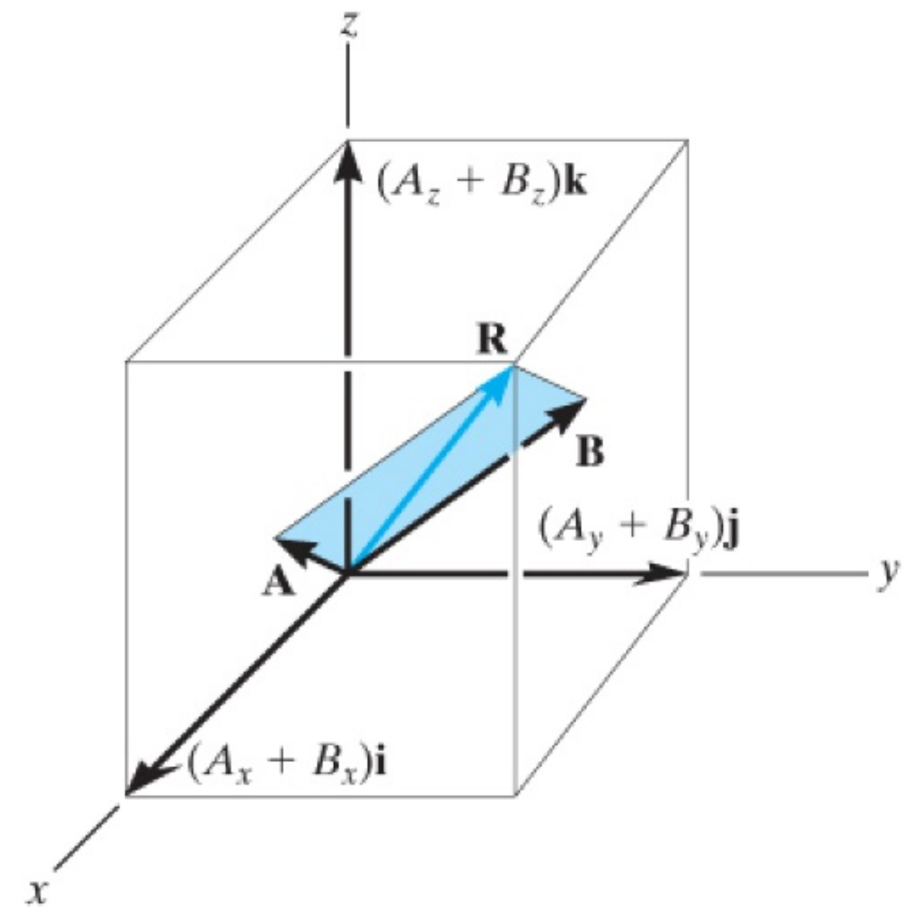
Addition of Cartesian vectors:

$$\mathbf{R} = \mathbf{A} + \mathbf{B}$$
$$= (A_x + B_x)\mathbf{i} + \left(A_y + B_y\right)\mathbf{j} + (A_z + B_z)\mathbf{k}$$

Is the sum of each component.

If there are more than two forces, the **force resultant** is the sum of each component:

$$\boxed{\mathbf{F}_R = \sum F_x \mathbf{i} + \sum F_y \mathbf{j} + \sum F_z \mathbf{k}}$$



# Position vectors

Position vectors are used to:

- Track the location of a particle, or
- Represent the distance between two particles.

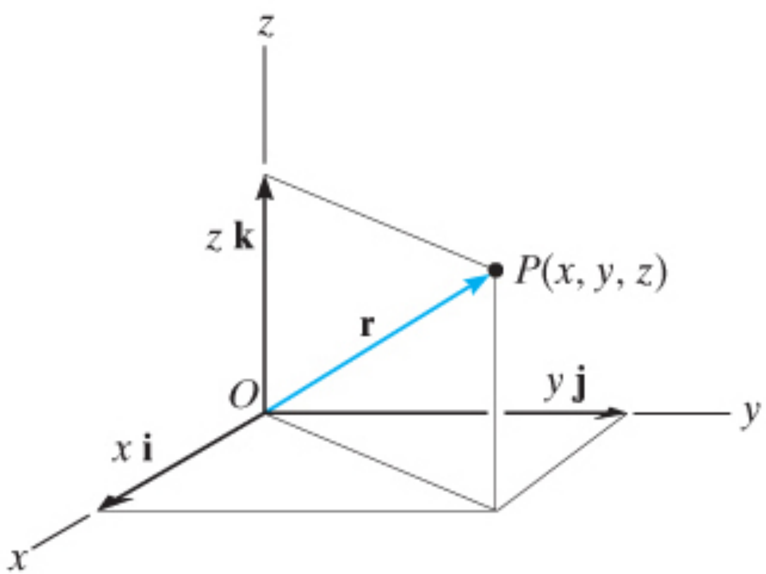

Figure: 02_PH006

The force **F** acting along the rope can be represented as a Cartesian vector by establishing *x*, *y*, *z* axes and first forming a position vector ***r*** along the length of the rope. Then the corresponding unit vector $\mathbf{u} = \mathbf{r}/r$ that defines the direction of both the rope and the force can be determined. Finally, the magnitude of the force is combined with its direction, $\mathbf{F} = F\mathbf{u}$.

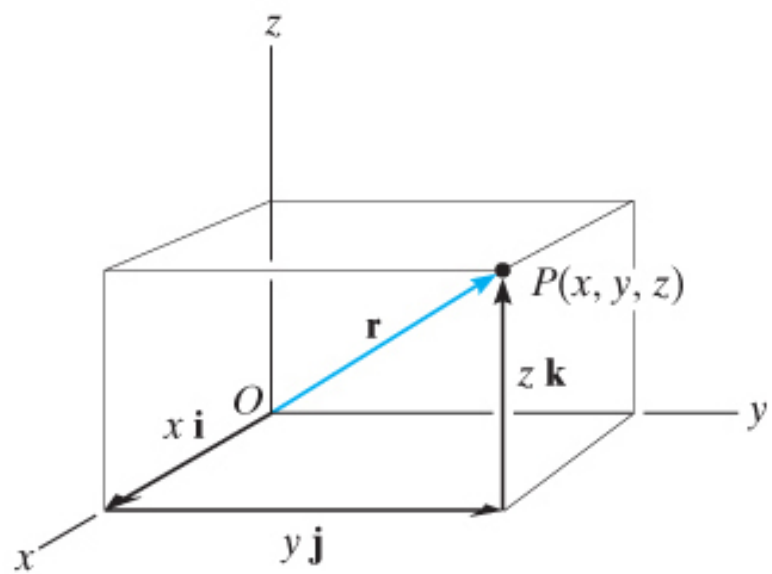
# Position vectors

The position of point *P*(*x*,*y*,*z*) from the origin is:

$$\boxed{\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}$$

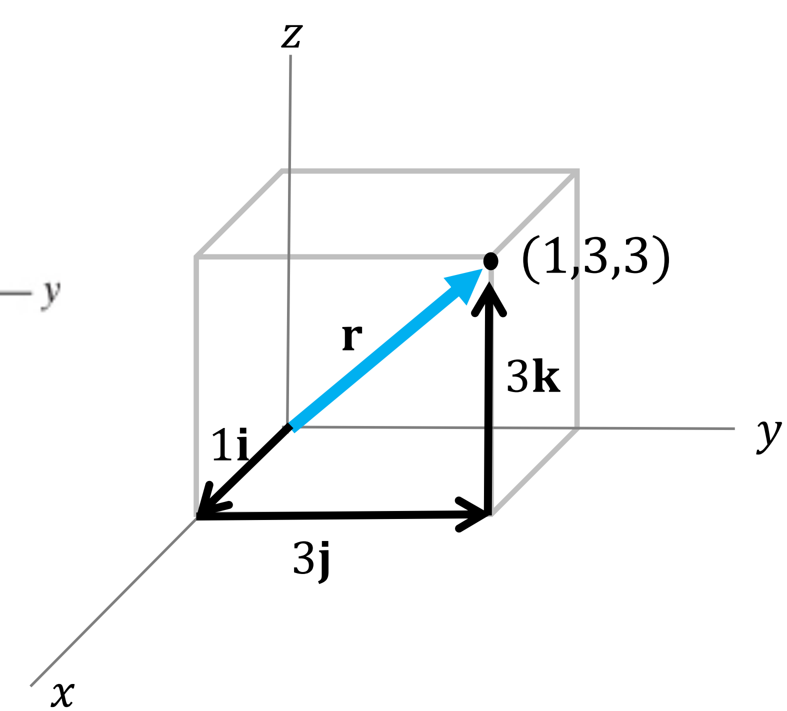



(a)

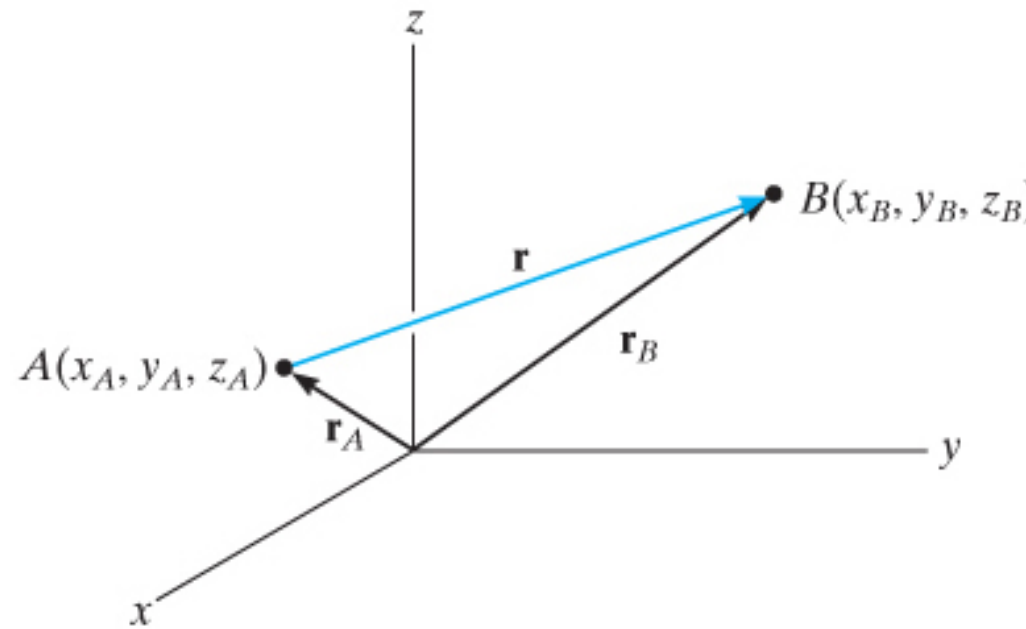



(b)

**Example**:
The position vector of Point (1,3,3) is:
$\mathbf{r} = \mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$



# Position vectors




The position vector from point *A* to *B* is:

$$\mathbf{r}_A + \mathbf{r} = \mathbf{r}_B$$

$$\mathbf{r} = \mathbf{r}_B - \mathbf{r}_A = (x_B\mathbf{i} + y_B\mathbf{j} + z_B\mathbf{k}) - (x_A\mathbf{i} + y_A\mathbf{j} + z_A\mathbf{k})$$

$$\boxed{= (x_B - x_A)\mathbf{i} + (y_B - y_A)\mathbf{j} + (z_B - z_A)\mathbf{k}}$$

It is also common to use the notation $\mathbf{r}_{AB}$ to indicate that the position vector is from point *A* to *B*.

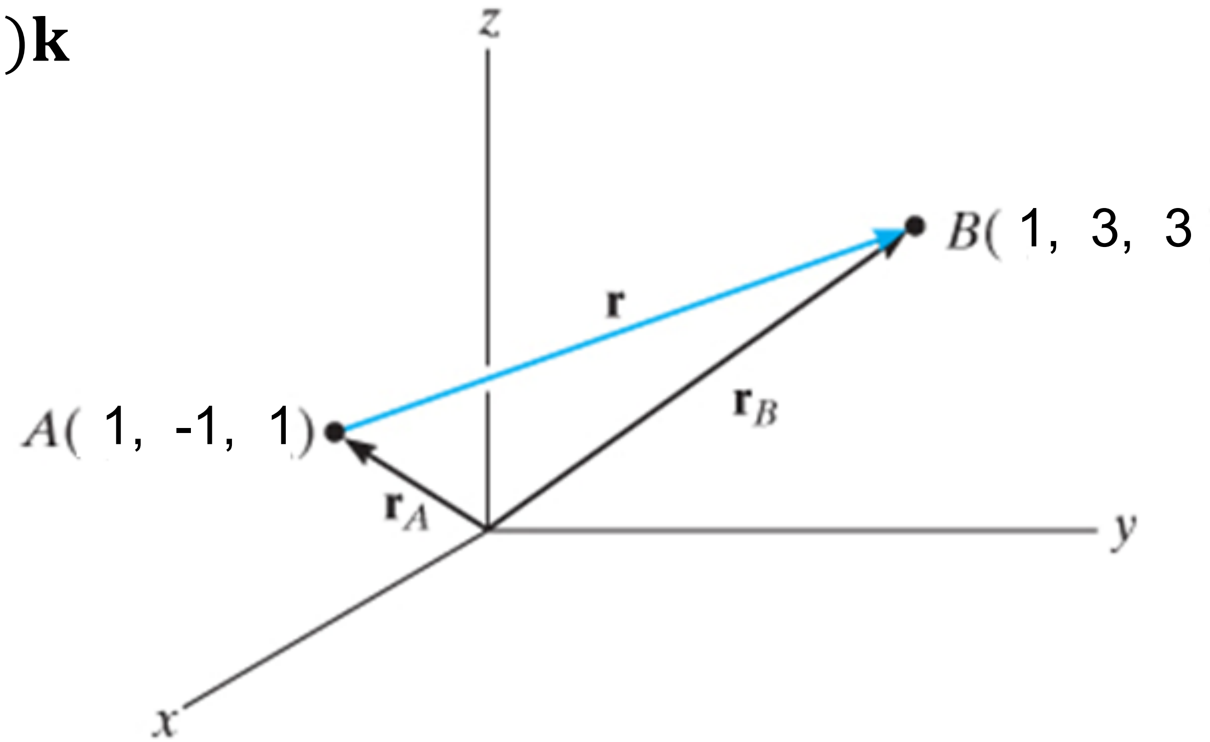
# Example: position vector

Find the position vector from point A=(1,-1,1) to B=(1,3,3).

$$\mathbf{r} = (x_B - x_A)\mathbf{i} + (y_B - y_A)\mathbf{j} + (z_B - z_A)\mathbf{k}$$

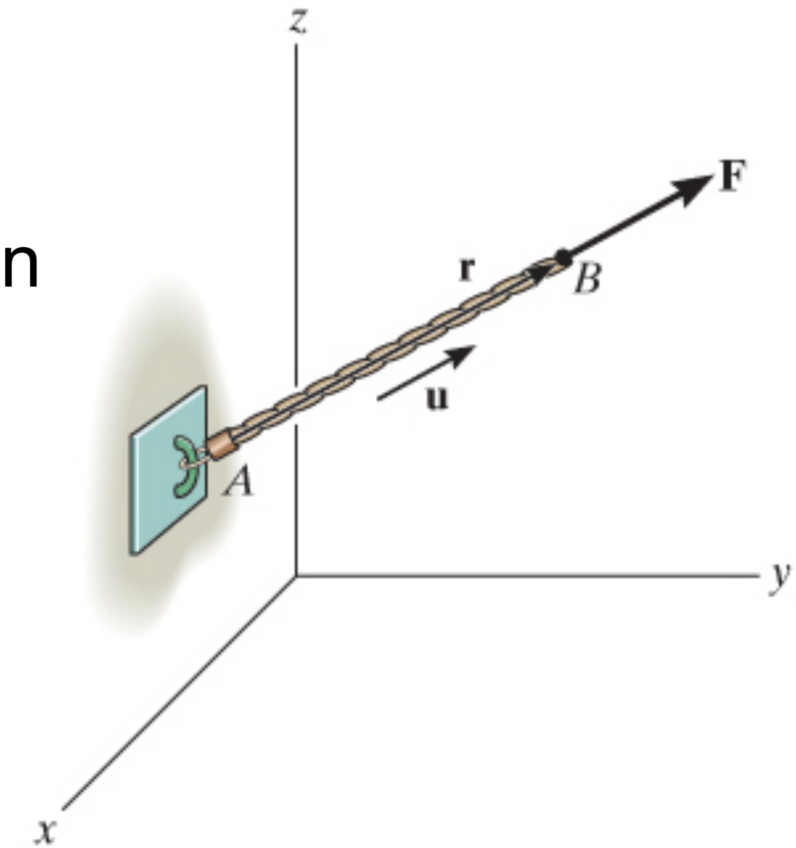
$$= (1 - 1)\mathbf{i} + (3 - (-1))\mathbf{j} + (3 - 1)\mathbf{k}$$

$$= 4\mathbf{j} + 2\mathbf{k}$$



# Force vector directed along a line

Often, the direction of a force is specified by two points. Here, the direction of $\mathbf{F}$ is given by the position vector $\mathbf{r}$, which goes from A to B.




$$\mathbf{F} = F\mathbf{u}$$

where

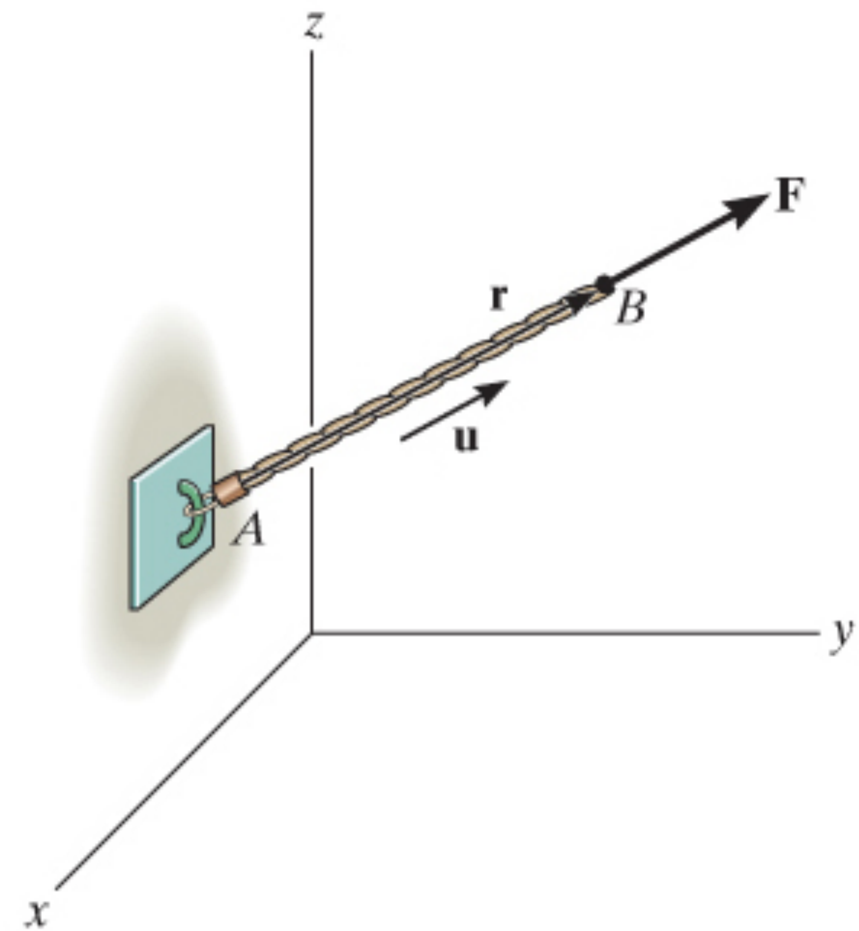
$$\mathbf{u} = \frac{\mathbf{r}}{r}$$

$$= \left(\frac{(x_B - x_A)\mathbf{i} + (y_B - y_A)\mathbf{j} + (z_B - z_A)\mathbf{k}}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}}\right)$$

# Example

A force of 300 kN is applied along the rope going from point *A* = (2,0,2) and *B* = (2,5,5).

Find the Cartesian force vector $\mathbf{F}$.



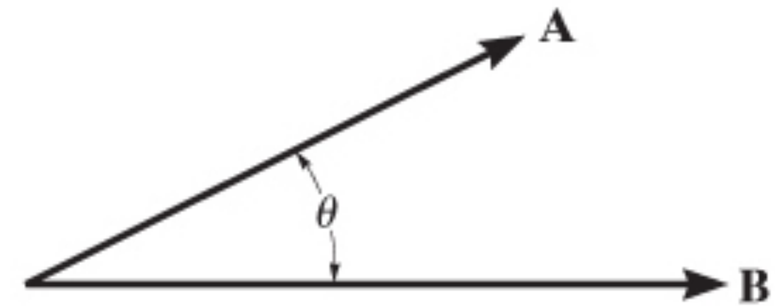

$$\mathbf{F} = F\left(\frac{(x_B - x_A)\mathbf{i} + (y_B - y_A)\mathbf{j} + (z_B - z_A)\mathbf{k}}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}}\right)$$

$$= 300\ \mathrm{kN}\left(\frac{(2-2)\mathbf{i} + (5-0)\mathbf{j} + (5-2)\mathbf{k}}{\sqrt{(2-2)^2 + (5-0)^2 + (5-2)^2}}\right) = \boxed{\left\{\frac{1500}{\sqrt{34}}\mathbf{j} + \frac{900}{\sqrt{34}}\mathbf{k}\right\}\mathrm{kN}}$$

# Dot product

The **dot product** of vectors $\mathbf{A}$ and $\mathbf{B}$ is defined as:

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta$$

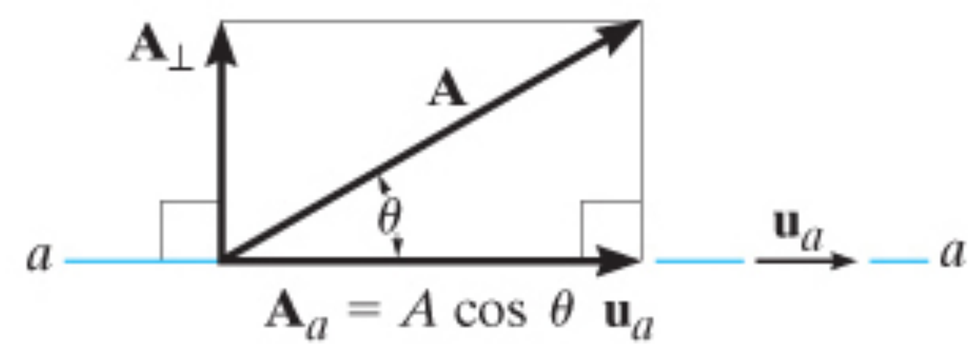



The scalar projection of $\mathbf{A}$ along the line specified by the unit vector $\mathbf{u}_a$:

$$\mathbf{A} \cdot \mathbf{u}_a = A\cos\theta = A_a$$

Consequently, the vector $\mathbf{A}_a$ is given by:

$$\mathbf{A}_a = A_a\mathbf{u}_a$$



# Free-body diagrams and equilibrium

# Equilibrium of a particle

A particle is the simplest model in mechanics.

A particle has a mass, but its size is neglected.

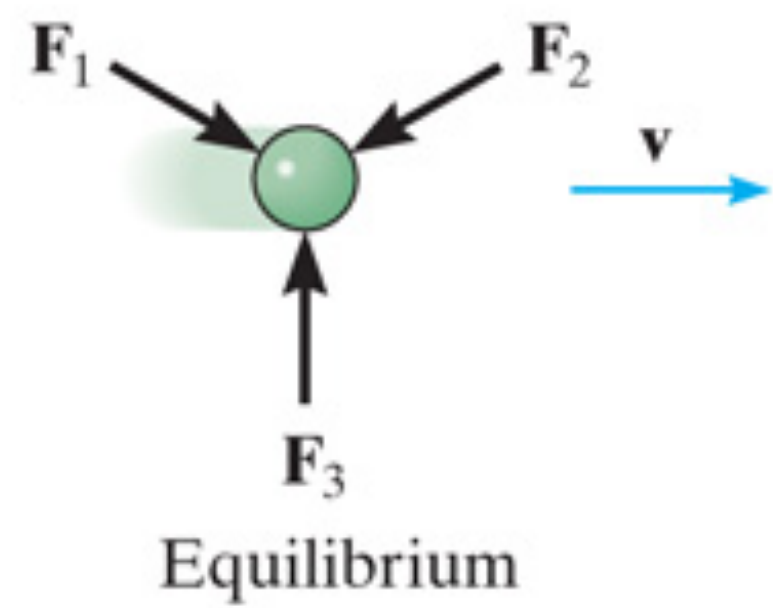
This is a good approximation when the size of an object are small compared to overall dimensions of the system.

The red box here could be approximated as a particle when calculating the force in each cable.

# Equilibrium of a particle

A particle is in equilibrium if:

- It remains at rest when originally at rest or,
- It has a constant velocity when it was originally in motion.




This condition is met when:

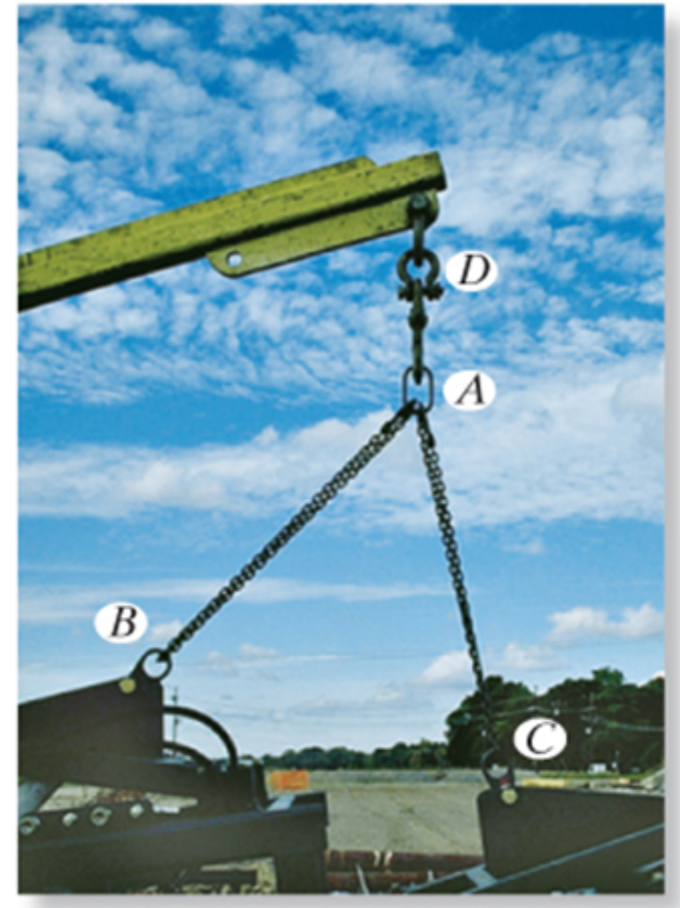
$$\Sigma\mathbf{F} = 0$$

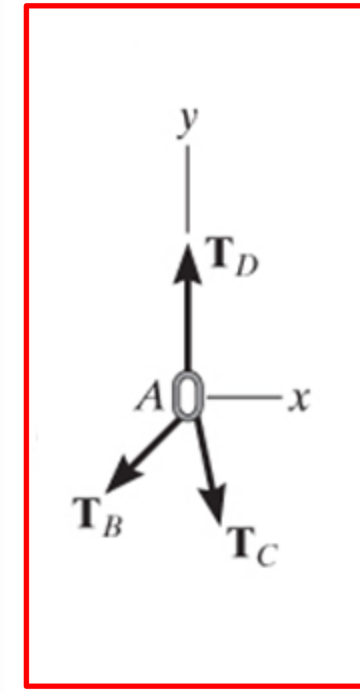
where $\Sigma\mathbf{F}$ is the sum of all forces acting on the particle.

# Free-body diagram

The free-body diagram is important to visualize all forces acting on a particle. To construct it, follow these three steps:

1. Isolate or cut the particle from its surrounding.
2. Sketch all forces acting on the particle.
3. Label each force: some forces are are known, others are unknown.

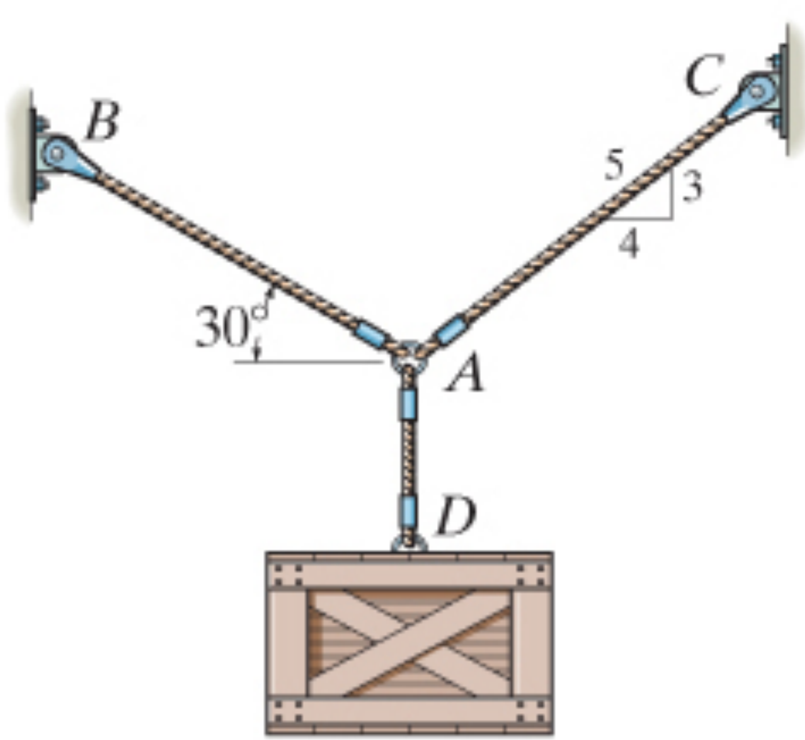



The chains exert three forces on the ring at *A*, as shown on its free-body diagram. The ring will not move, or will move with constant velocity, provided the summation of these forces along the x and along the y axis equals zero. If one of the three forces is known, the magnitudes of the other two forces can be obtained from the two equations of equilibrium.

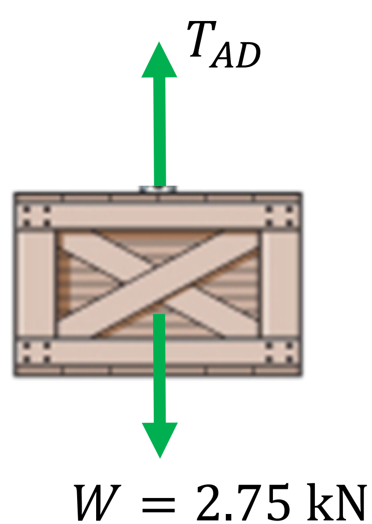
# Example: free-body diagram

If the box weighs 2.75 kN, find the forces in cables *BA* and *AC*?

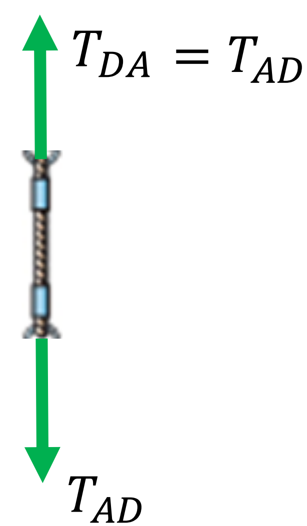



Free-body diagram for each component:

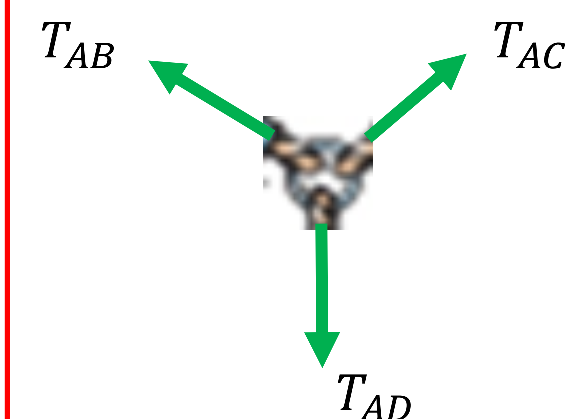
Box



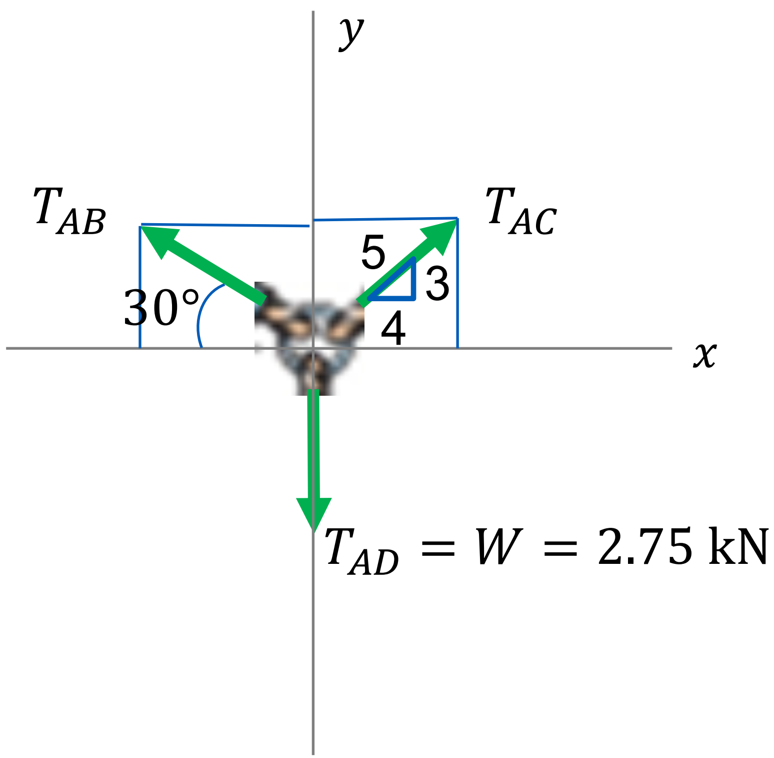

Cable *AD*




Ring *A*



# Example: free-body diagram



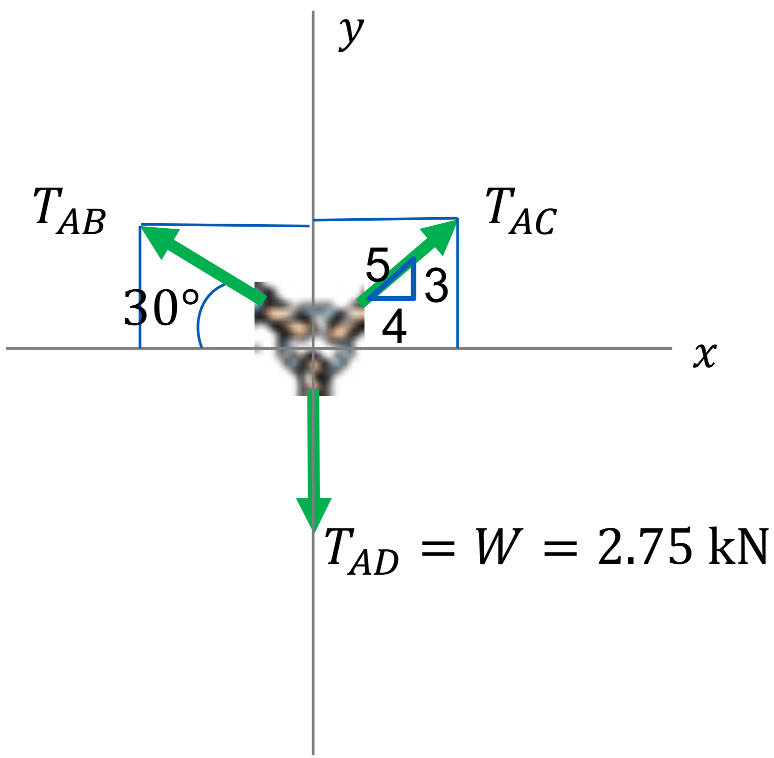

The condition for equilibrium is:

$$\Sigma \mathbf{F} = 0$$

$$\Sigma F_x \mathbf{i} + \Sigma F_y \mathbf{j} = 0 \quad \Rightarrow \quad \boxed{\begin{array}{l} \Sigma F_x = 0 \\ \Sigma F_y = 0 \end{array}}$$

We need to express $T_{AB}$ , $T_{AC}$ and $T_{AD}$ as Cartesian vectors to make use of these equations.

# Example: free-body diagram




Expressing $T_{AB}$ , $T_{AC}$ and $T_{AD}$ using x and y components we get:

$$\Sigma F_x = 0 \Rightarrow \quad T_{AC,x} - T_{AB,x} = 0$$

$$\boxed{T_{AC}\frac{4}{5} - T_{AB}\cos 30 = 0}$$

$$\Sigma F_y = 0 \Rightarrow \quad T_{AC,y} + T_{AB,y} - T_{AD} = 0$$

$$\boxed{T_{AC}\frac{3}{5} + T_{AB}\sin 30 - 2.75\ \text{kN} = 0}$$

# Example: free-body diagram

Equilibrium equations:

$$\left\{ \begin{array}{l} \boxed{T_{AC}\frac{4}{5} - T_{AB}\cos 30 = 0} \\ \boxed{T_{AC}\frac{3}{5} + T_{AB}\sin 30 - 2.75\text{kN} = 0} \end{array} \right.$$

$$\Longrightarrow T_{AC} = \frac{5}{4}T_{AB}\cos 30$$

$$\Longrightarrow \frac{5}{4}T_{AB}\cos 30\frac{3}{5} + T_{AB}\sin 30 - 2.75\text{ kN} = 0$$

$$\Longrightarrow T_{AB}\left(\frac{3}{4}\cos 30 + \sin 30\right) = 2.75\text{ kN}$$

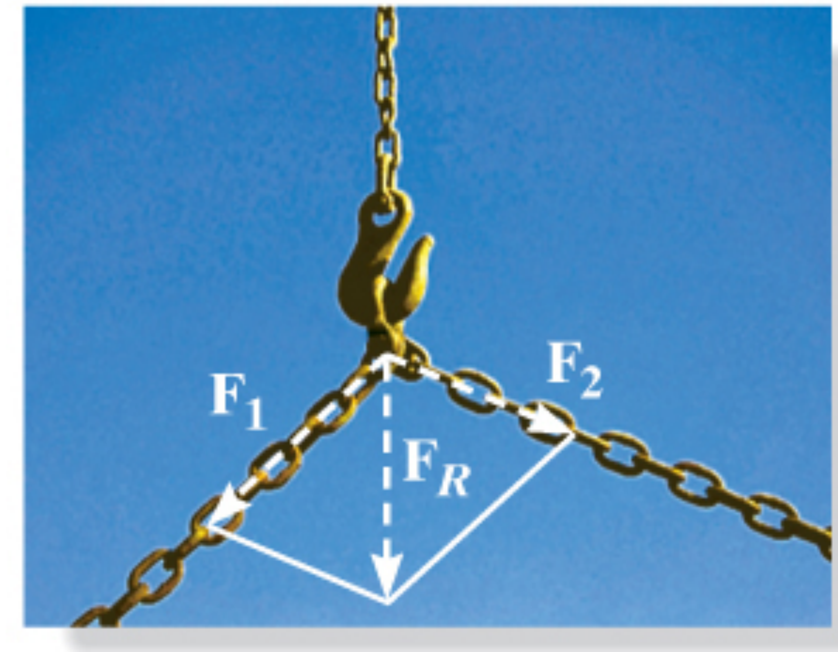
$$\Longrightarrow T_{AB} = \frac{2.75\text{kN}}{\left(\frac{3}{4}\cos 30 + \sin 30\right)} \approx 2.39\text{ kN}$$

Finally, we go back to the first equation to find $T_{AC}$:

$$T_{AC} = \frac{5}{4}T_{AB}\cos 30 \approx 2.59\text{ kN}$$

# Summary

- Forces are vectors. The **resultant force** is obtained by adding all forces acting on a point.
- A particle is at **equilibrium** when the resultant force acting on it is equal to zero.
- The **free-body diagram** is illustrating all forces acting on a point, object or particle.




$$\boxed{\begin{aligned} \Sigma F_x &= 0 \\ \Sigma F_y &= 0 \\ \Sigma F_z &= 0 \end{aligned}}$$

# Need more explanations?

For more details, consult these sections of the textbook:

Chapter 2: Force vectors

3.1  Condition for the equilibrium of a particle
3.2  Free-body diagram
3.3  Coplanar force system