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COE-C1001: Statics

5. Internal loads in beams

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Learning outcomes

After this lecture, you should be able to:

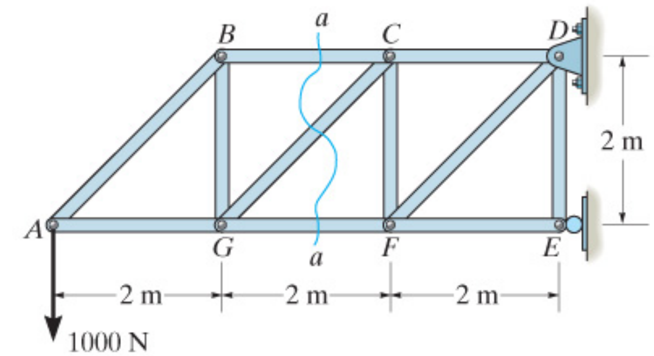
- Use the method of sections to find the internal loads in beams.
- Plot these internal loads as a function of position.

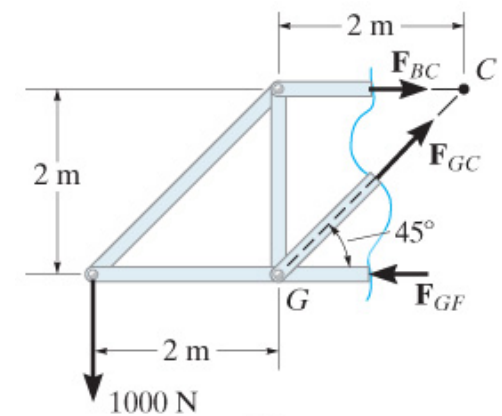
Internal loads

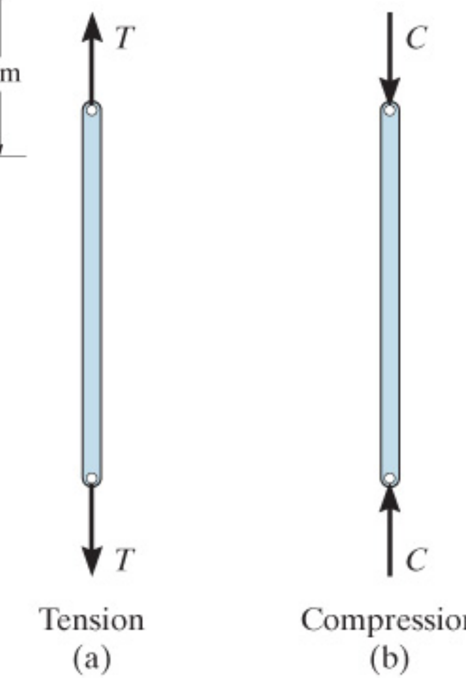
- In the previous lecture, we analysed the internal forces inside simple truss structures.
    - *Each bar is a two-force member in tension or compression.*
    - *The internal force can be obtained with the method of joints or sections.*

- Not all structures are simple trusses. Today, we will analyse the internal loads inside beams.

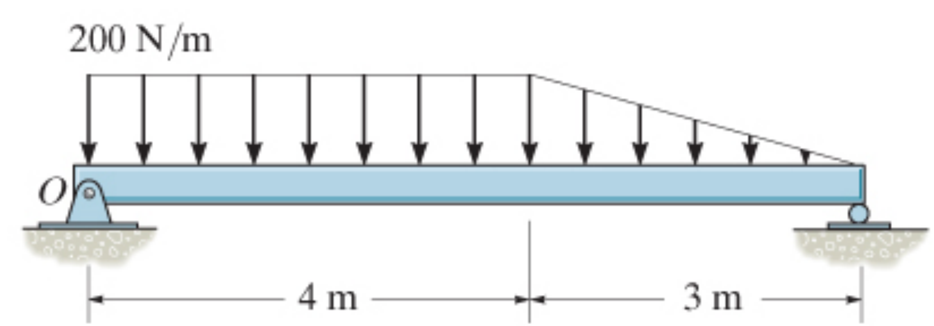
Simple truss



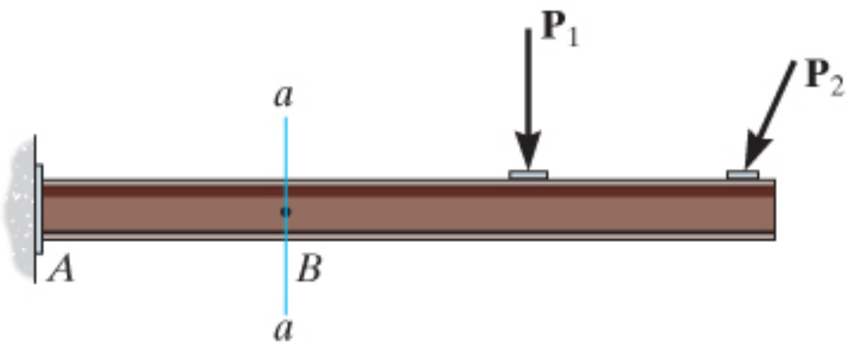


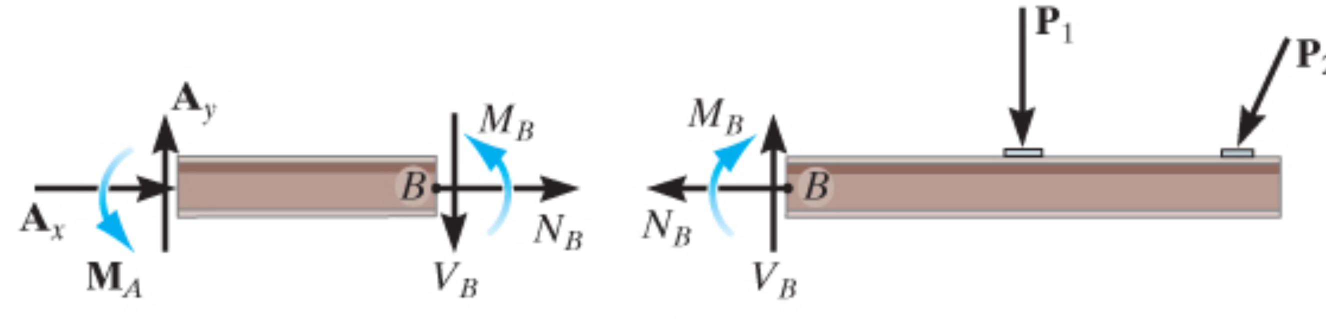


Beam (not a simple truss!)



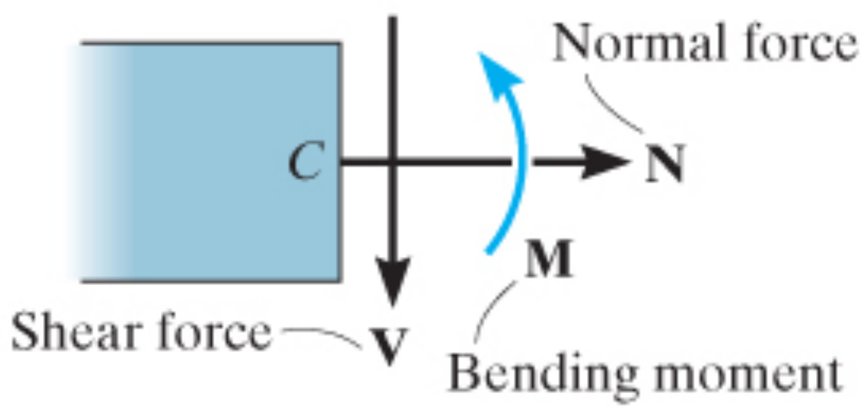
Internal loads in a beam



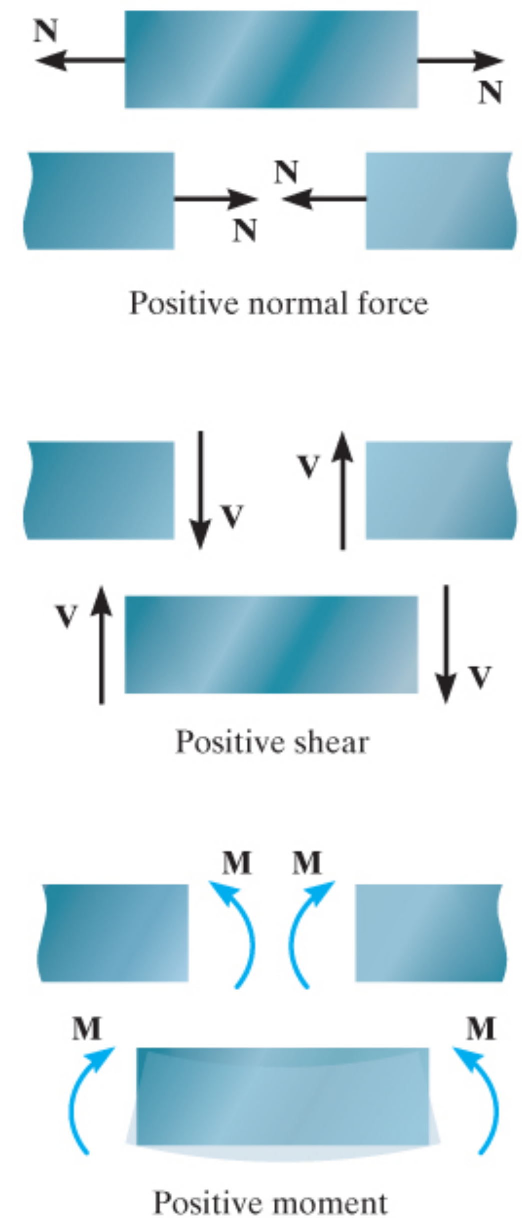


- Let's use the method of sections to find the internal loads at point *B*. We cut the beam along the line a-a.
- To ensure equilibrium, three internal loads are needed:
  - A normal force N_B, perpendicular to the cross-section,
  - A shear force V_B, tangent to the cross-section, and
  - A bending moment M_B.
- We can use equilibrium equations to find N_B, V_B, and M_B.

Sign convention



- The normal force N is positive if it creates tension.
- A positive shear force V will create a clockwise rotation.
- A positive bending moment M will deflect the beam downward.



Procedure

Here is how to use the method of sections to find the internal forces/moments:

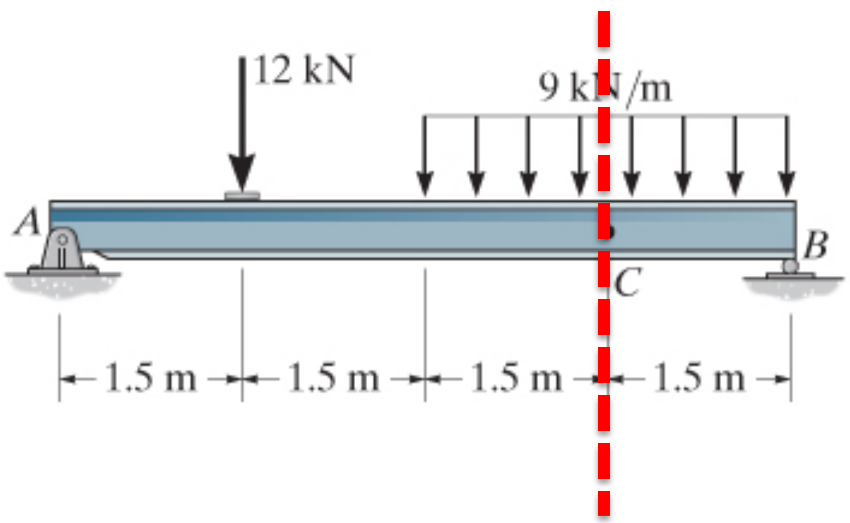
1. Before making a section, determine the support reactions.

2. Cut the structure at the location of interest and draw the free-body diagram of one or both segments:
   i. Keep all forces, distributed loads and moments at their exact location.
   ii. Include the internal forces and bending moments where the structure was cut (and follow the sign convention).

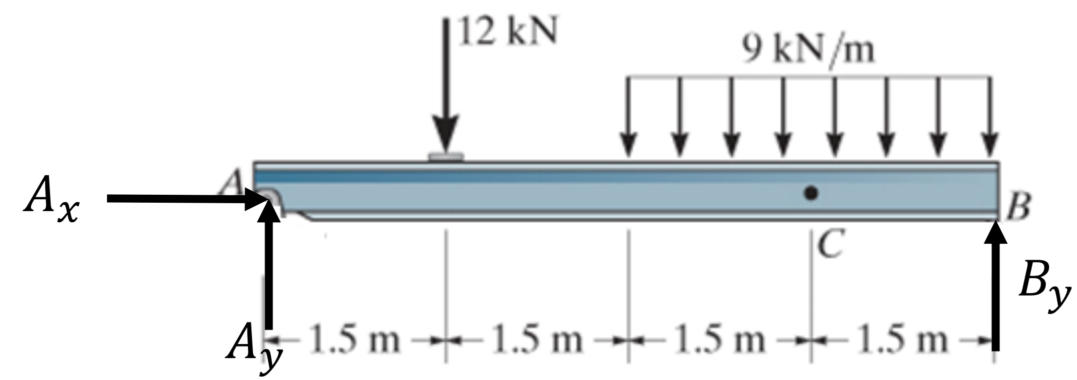
3. Use equilibrium equations to find the internal forces and bending moment.

Example

Find the normal force, shear force and bending moment at point *C*.



To find the solution we need to cut the beam at point *C* and draw a free-body diagram for the right section of the beam. This requires the reaction force at point *B*.



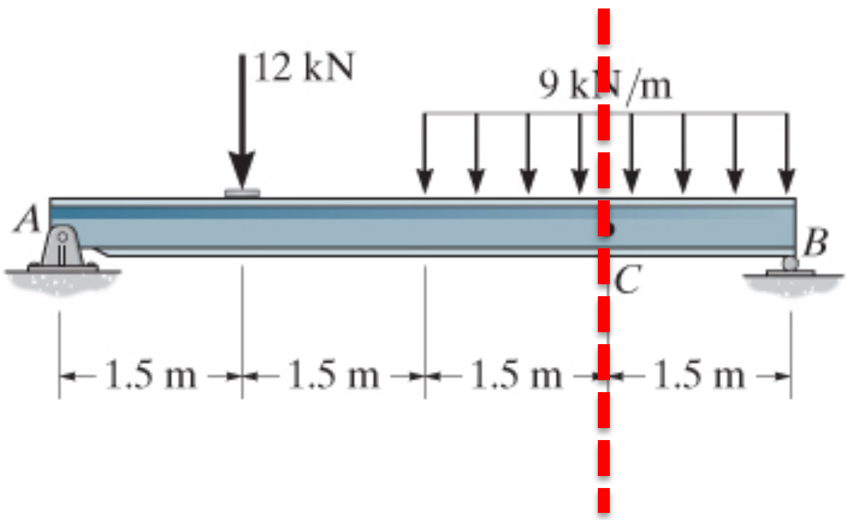
The support reaction at point *B* can be found with the sum of moments about point *A*.

$$\circlearrowleft + \quad \Sigma M_A = 0 \qquad B_y(6\text{m}) - 9\text{kN}(3\text{m})(4.5\text{m}) - 12\text{kN}(1.5\text{m}) = 0$$

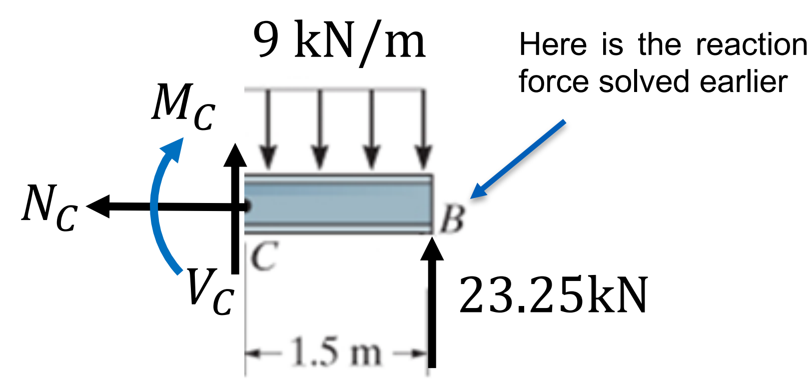
$$B_y = 23.25 \ \text{kN}$$

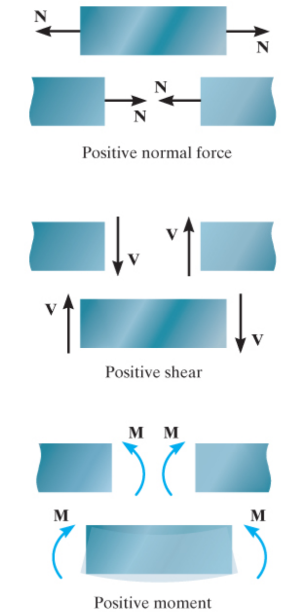
Example

Find the normal force, shear force and bending moment at point *C*.



We cut the bar at point *C* and draw a free-body diagram:





$$\circlearrowleft + \quad \Sigma M_C = 0$$

$$23.25(1.5\text{m}) - 9\text{kN}(1.5\text{m})(0.75\text{m}) - M_C = 0$$

$$\boxed{M_C = 24.75 \ \text{kNm}}$$

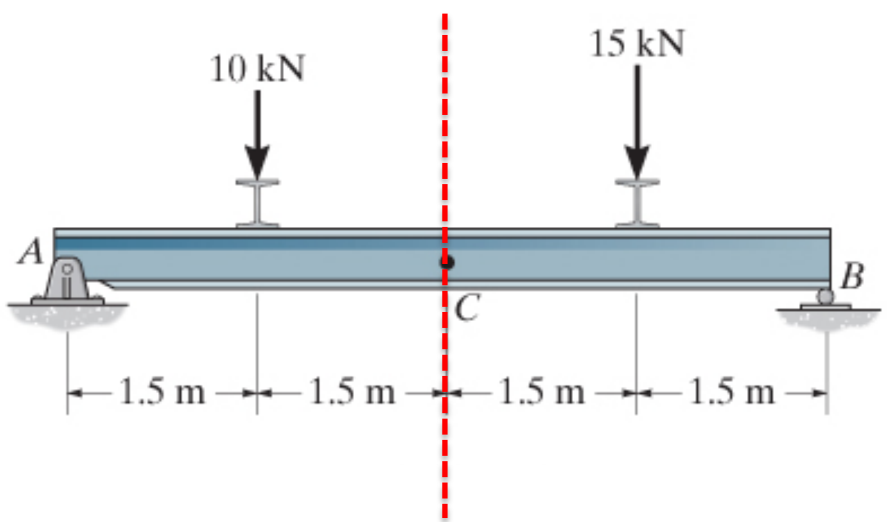
$$\rightarrow + \quad \Sigma F_x = 0 \qquad \boxed{N_C = 0}$$

$$\uparrow + \quad \Sigma F_y = 0 \qquad 23.25\text{kN} - 9\text{kN}(1.5\text{m}) + V_C = 0$$

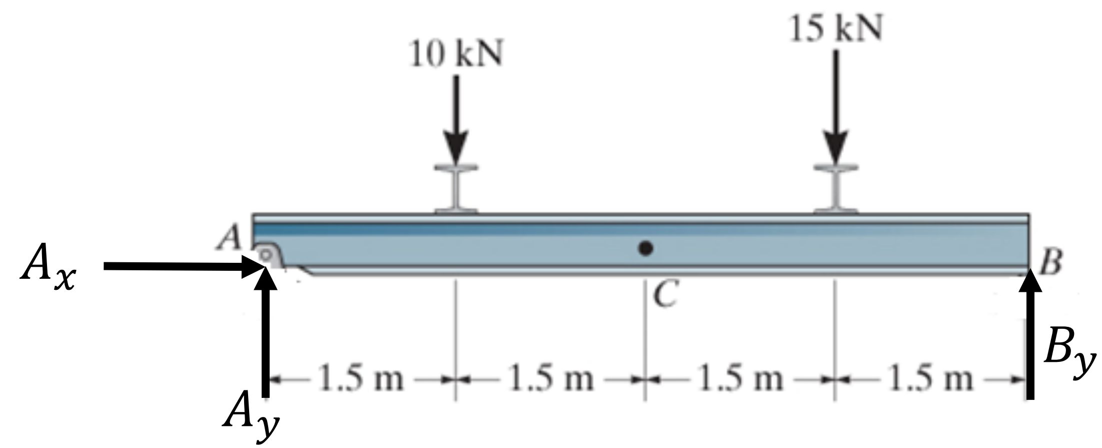
$$\boxed{V_C = -9.75 \ \text{kN}}$$

Example

Find the normal force, shear force and bending moment at point *C*.



Again, we need to cut the beam at point *C* and draw a free-body diagram for the right segment. First, we need to solve the reaction force at point *B*.



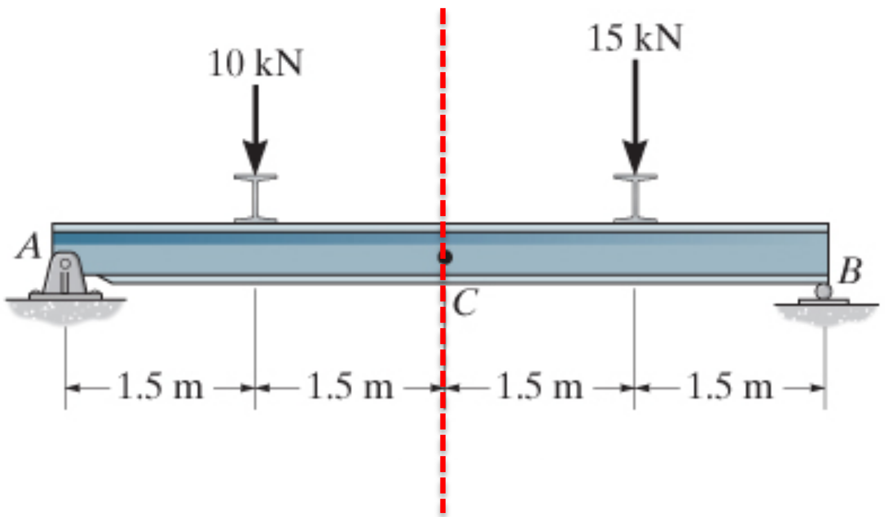
Again, we solve B_y by taking the sum of moments about point A:

$$\circlearrowleft + \ \ \Sigma M_A = 0 \qquad B_y(6\text{m}) - 15\text{kN}(4.5\text{m}) - 10\text{kN}(1.5\text{m}) = 0$$

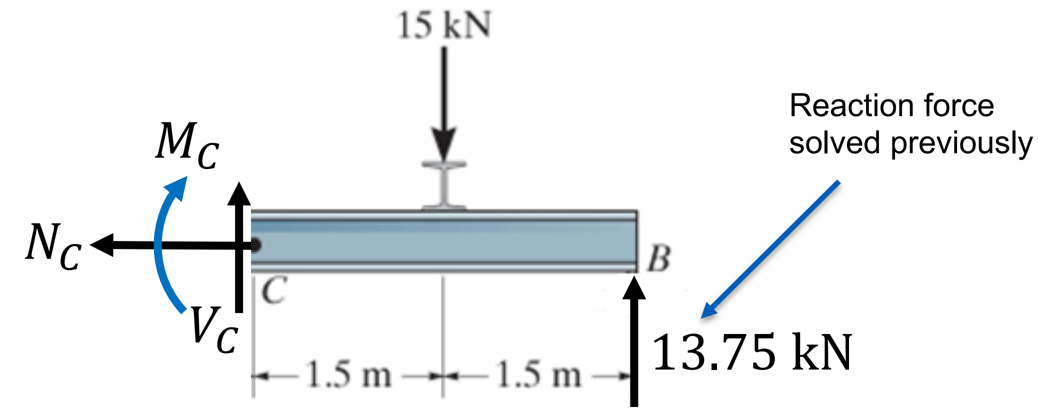
$$B_y = 13.75 \ \text{kN}$$

Example

Find the normal force, shear force and bending moment at point *C*.



After cutting the beam at point *C* the free-body diagram becomes:



$\circlearrowleft + \ \Sigma M_C = 0$ $\quad 13.75(3\text{m}) - 15\text{kN}(1.5\text{m}) - M_C = 0$

$\boxed{M_C = 18.75 \ \text{kNm}}$

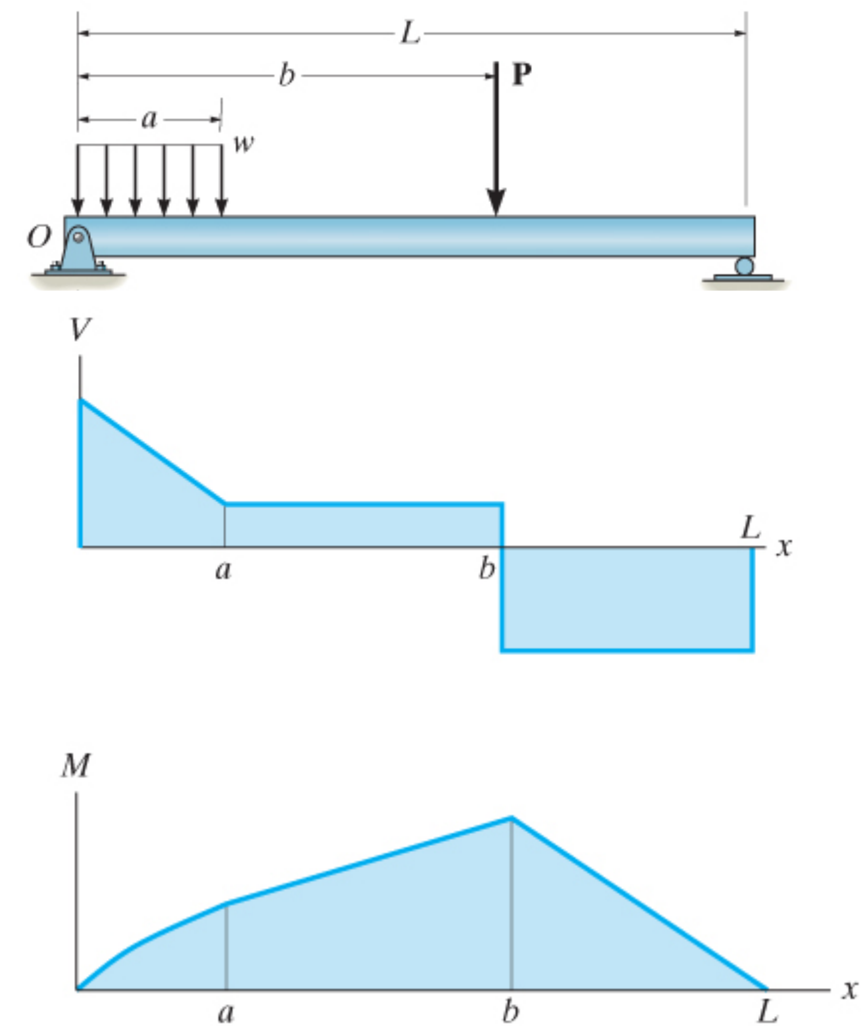
$\rightarrow + \ \Sigma F_x = 0$ $\quad \boxed{N_C = 0}$

$\uparrow + \ \Sigma F_y = 0$ $\quad 13.75\text{kN} - 15\text{kN} + V_C = 0$

$\boxed{V_C = 1.25 \ \text{kN}}$

Shear and bending moment diagrams

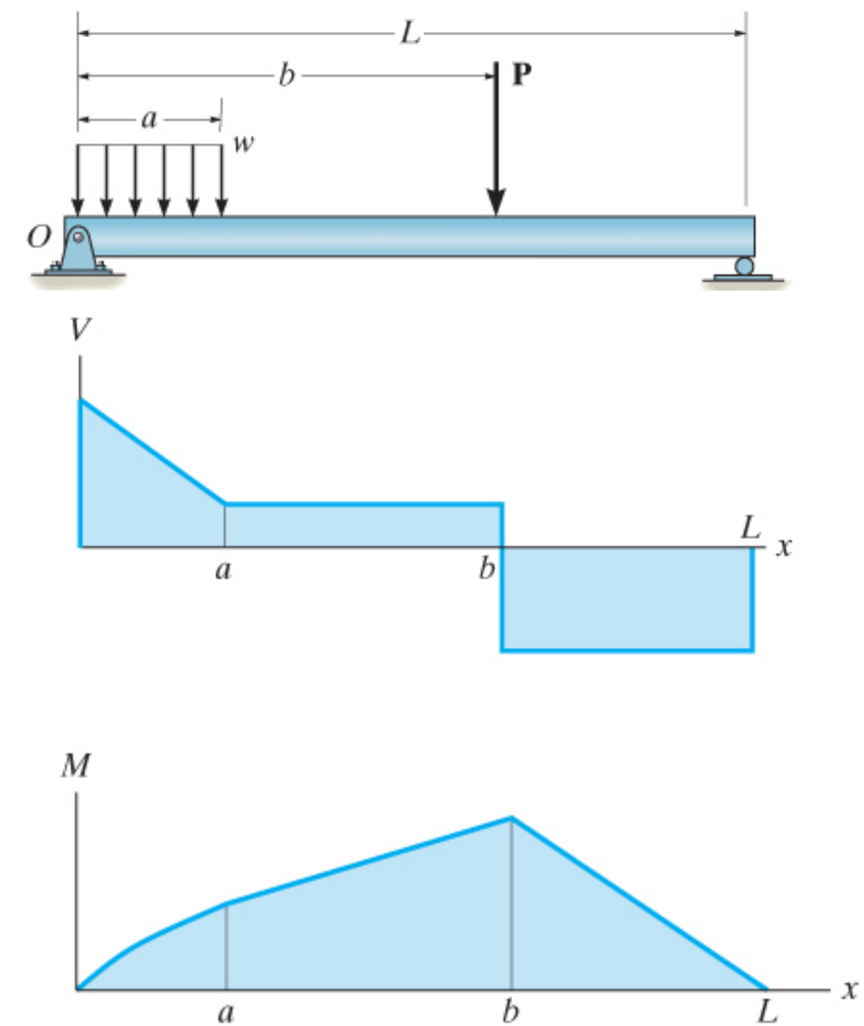
- So far, we found the internal loads at a given point along the beam.
- Designing a beam usually requires to know the shear force and bending moment at each point along the length of the beam.
- These distributions are called shear diagram and bending moment diagram.



Shear and bending moment diagrams

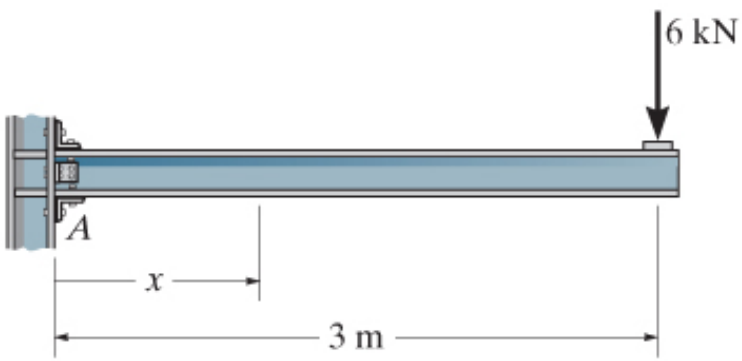
The procedure to plot shear and bending moment diagrams is similar to what we have done so far:

- Use the method of sections to cut the beam at a distance x from the beam's left end. Find V and M as a function of x.
- A free-body diagram is needed every time the load on the beam suddenly changes.

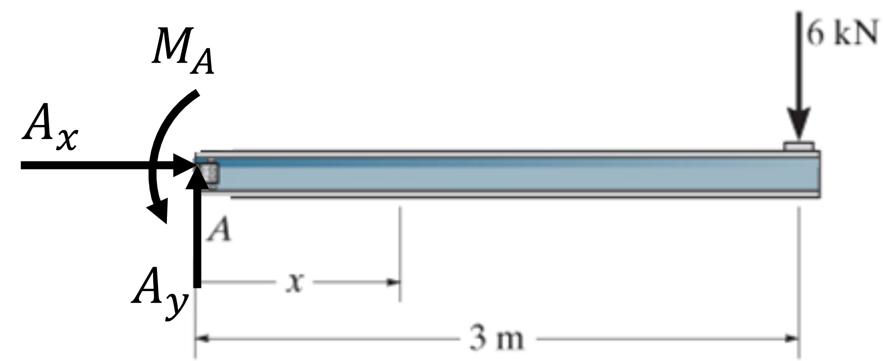


Example

Plot the shear force and bending moment diagrams for this cantilever beam.



First, we need to find the support reactions for the fixed joint A.



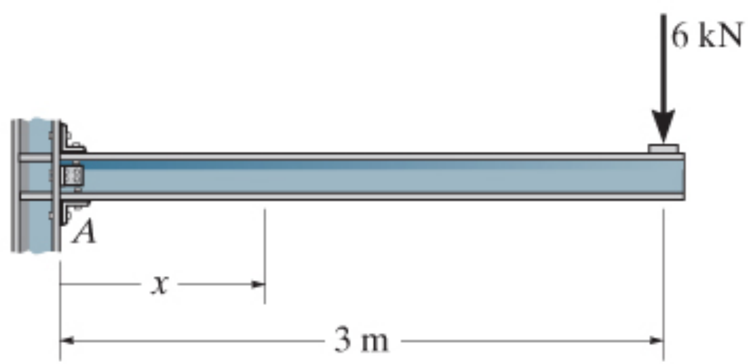
$\uparrow + \quad \Sigma F_y = 0 \qquad A_y = 6\ \text{kN}$

$\rightarrow + \quad \Sigma F_x = 0 \qquad A_x = 0$

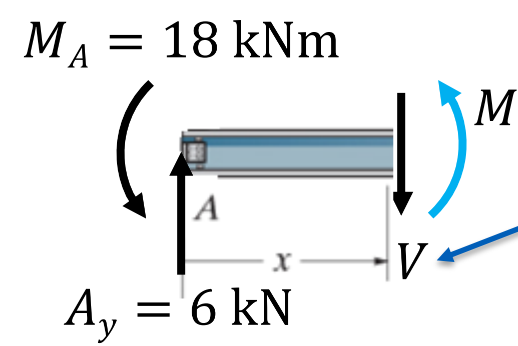
$\circlearrowleft + \quad \Sigma M_A = 0 \qquad M_A = 18\ \text{kNm}$

Example

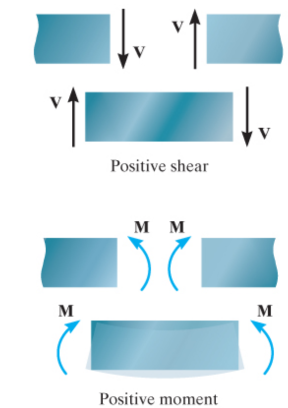
Plot the shear force and bending moment diagrams for this cantilever beam.



Next, cut the beam at an arbitrary distance x (provided $x < 3$ m) and draw the free-body diagram for the left segment. (Be careful with the positive directions of shear and moments).



These are the unknowns to be solved as a function of *x*.



The sum of forces in the vertical direction gives:

$$\uparrow + \quad \Sigma F_y = 0 \quad \Rightarrow \quad A_y - V = 0. \quad \boxed{V = 6\ \text{kN}}$$

And the sum of moments about x gives the bending moment:

$$\circlearrowleft + \quad \Sigma M = 0 \qquad M_A - A_y \cdot x + M = 0$$

$$M = Ay \cdot x - M_A$$

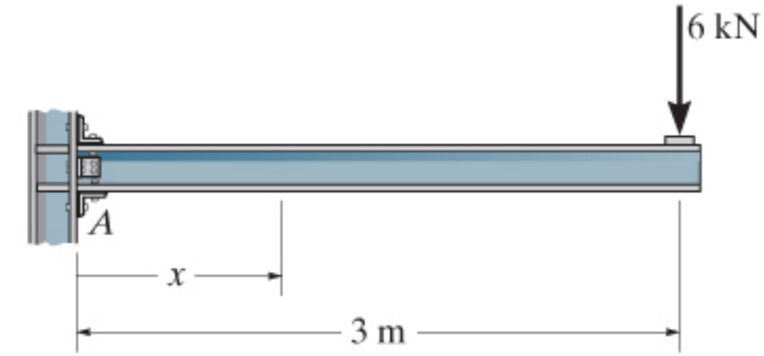
$$\boxed{M = 6x - 18\ (\text{kNm})}$$

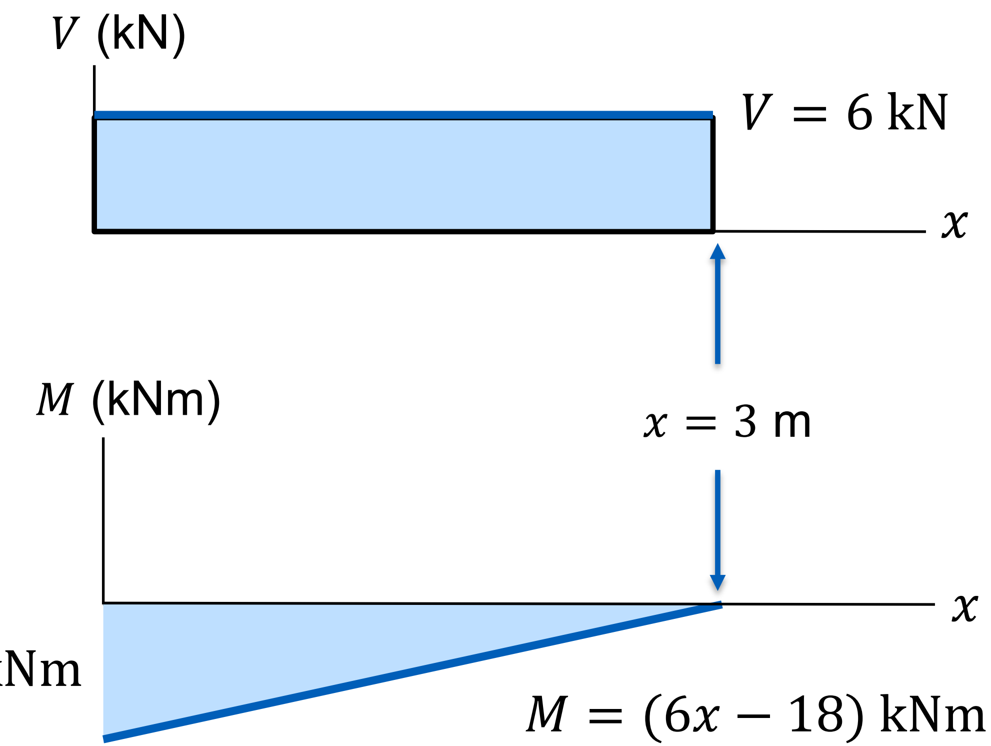
The bending moment is a function of the position *x*; that is normal.

Example

Finally, we can plot the shear force and bending moment as a function of the position x.

- Be careful of positive/negative signs.
- The lines stop at the end of the beam ($x = 3$ m).

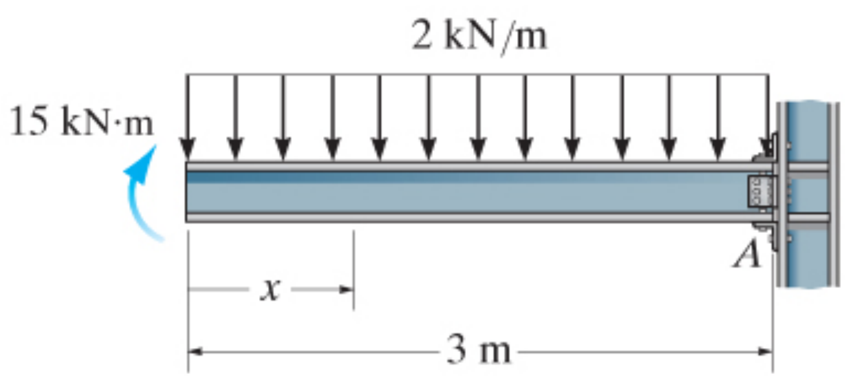




$|M_{max}| = 18$ kNm

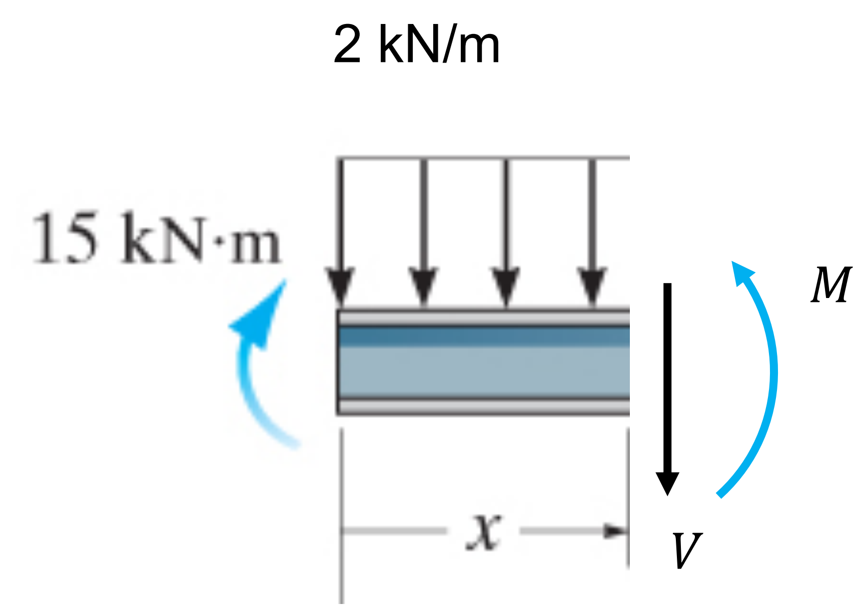
Example

Plot the shear force and bending moment diagrams for this cantilever beam.



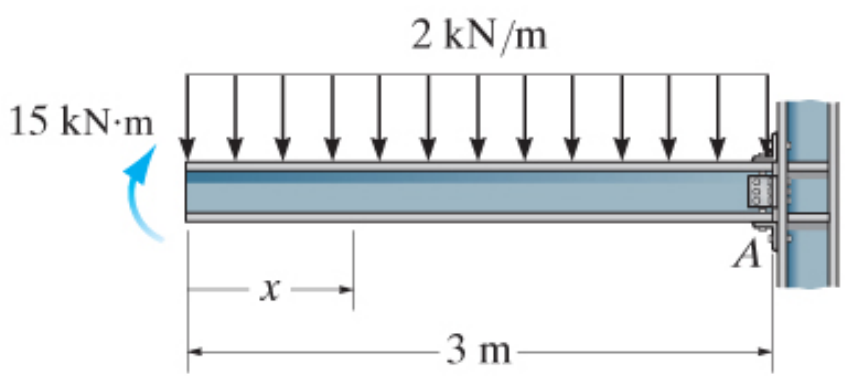
We can solve the support reactions at point A; however, this is not necessary for this case.

If we cut the beam at point *x* and draw the free-body diagram for the left segment then the support reactions aren't needed.



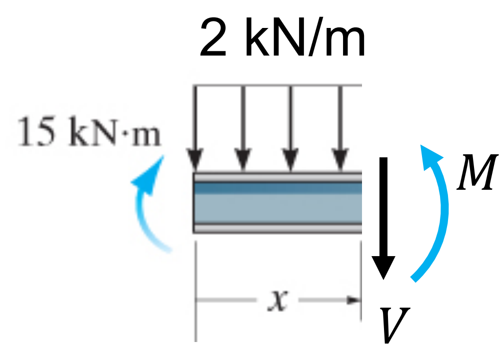
Example

Plot the shear force and bending moment diagrams for this cantilever beam.



The distributed loading produces an equivalent force of magnitude = $(2 \times x)$ kN, applied at $x/2$

Here is the free-body diagram for the left segment of the beam cut at a distance x. Next, we apply equilibrium equations to solve for the two unknowns.



Sum of forces along the vertical direction:

$$\uparrow + \quad \Sigma F_y = 0 \qquad -V - 2\text{kN/m} \cdot (x\text{m}) = 0 \qquad \boxed{V = (-2x)\ \text{kN}}$$

Sum of moments about x:

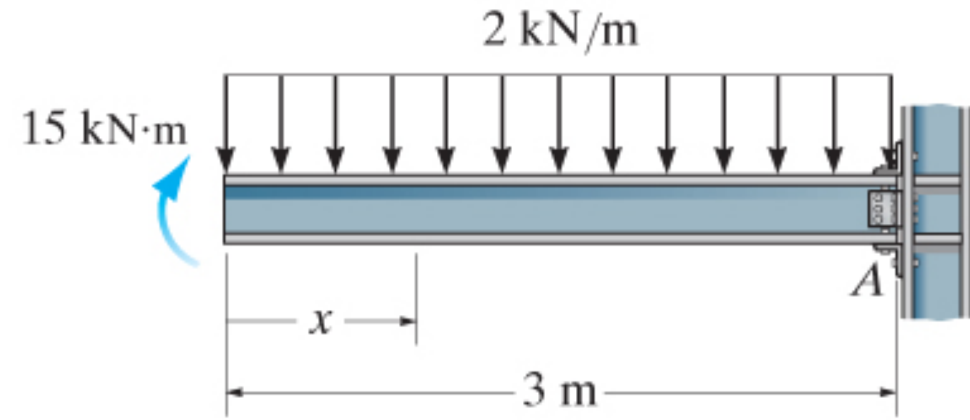
$$\circlearrowleft + \quad \Sigma M = 0 \qquad -15\text{kNm} + 2x\text{kN}\left(\frac{x}{2}\text{m}\right) + M = 0$$

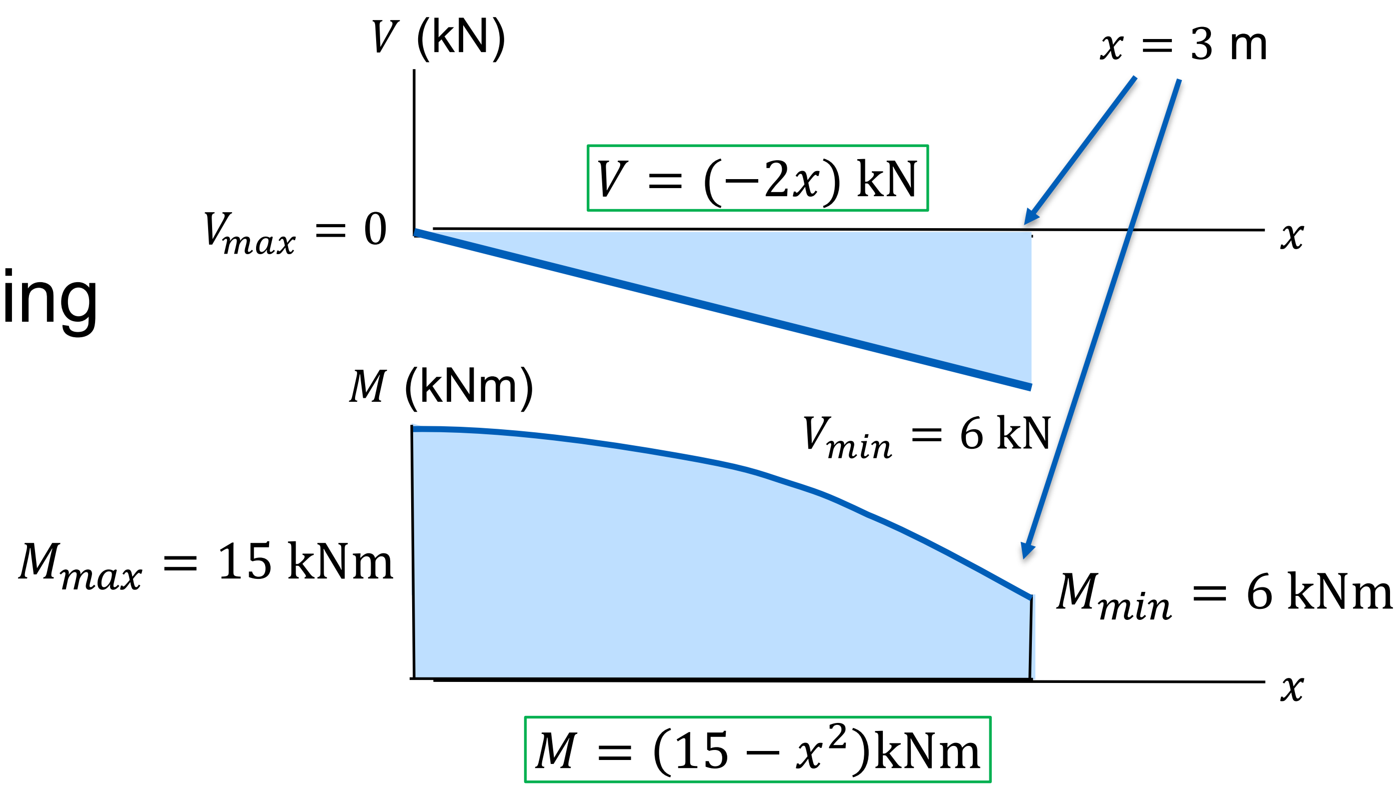
$$\boxed{M = (15 - x^2)\ \text{kNm}}$$

Example

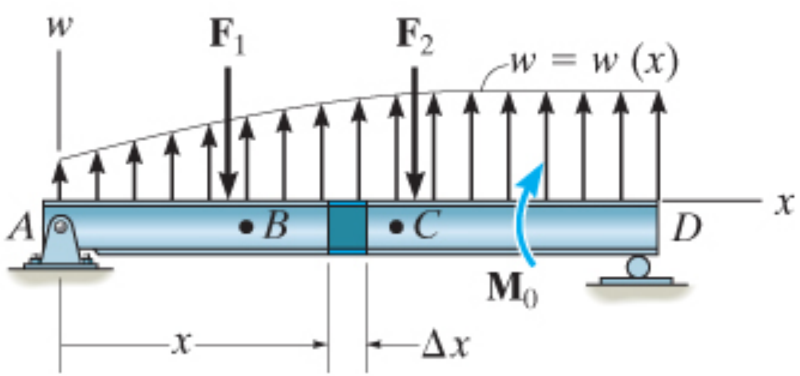
Finally, we plot the shear force and bending moment as a function of x.

Note that the shear force is negative, whereas the bending moment is positive.



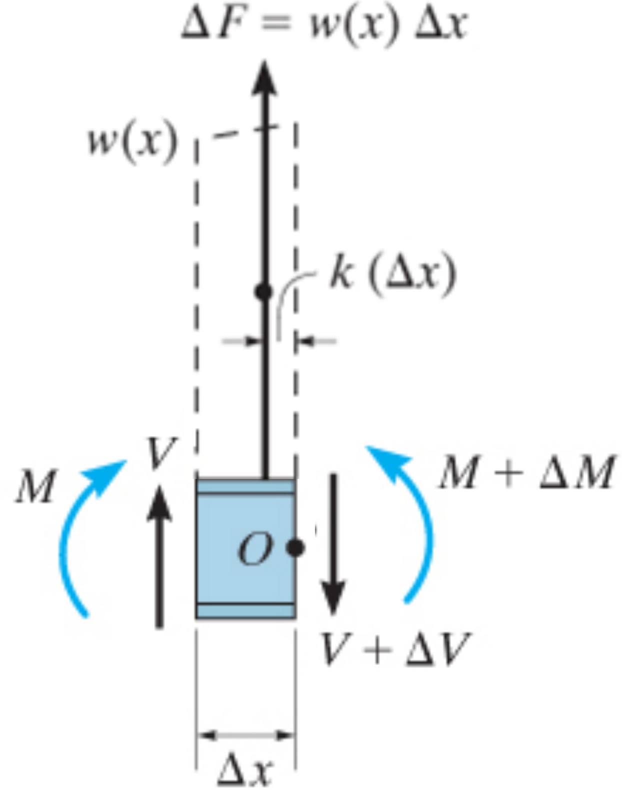


Relation between distributed load and shear



Consider the segment Δx in the beam above:

- This segment is <u>not</u> subjected to a concentrated force.
- For the beam to be in equilibrium, each segment Δx has to be in equilibrium.



Sum of forces in the vertical direction gives:

$$\uparrow + \quad \Sigma F_y = 0 \qquad V - (V + \Delta V) + w(x)\Delta x = 0$$

$$\Rightarrow \quad \Delta V = w(x)\Delta x$$

$$\Rightarrow \frac{\Delta V}{\Delta x} = w(x)$$

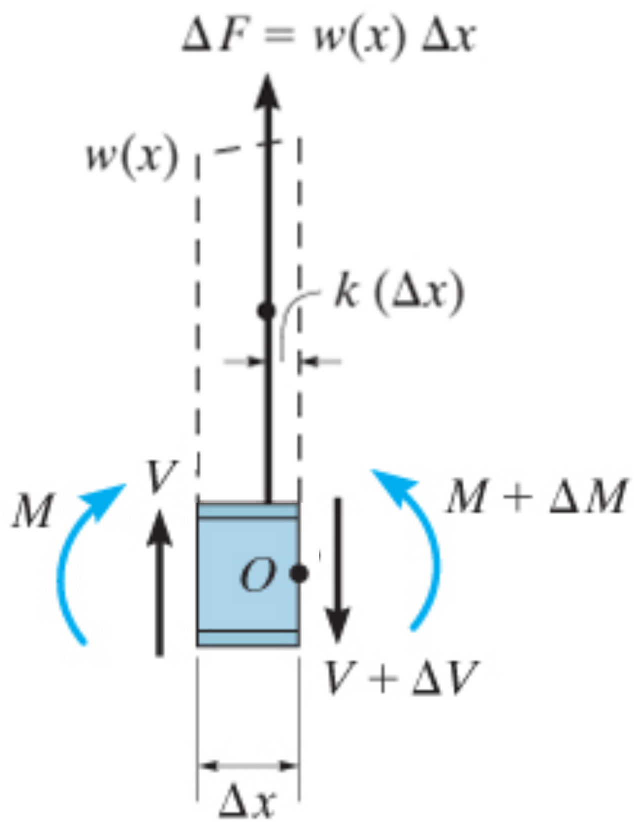
Letting $\Delta x \to 0$, this becomes:

$$\boxed{\frac{dV}{dx} = w(x)}$$

Slope of shear diagram

Distributed load intensity

Relation between shear and moment



Next, let's consider the sum of moment about point O for this small segment Δx :

$$\circlearrowleft + \quad \Sigma M_O = 0 \qquad -V\Delta x - M - w(x)\Delta x k \Delta x + M + \Delta M = 0$$

$$\Rightarrow \qquad \Delta M = V\Delta x + k w(x) \Delta x^2$$

Dividing both sides by Δx gives:

$$\frac{\Delta M}{\Delta x} = V + k w(x) \Delta x$$

Letting $\Delta x \rightarrow 0$ returns:

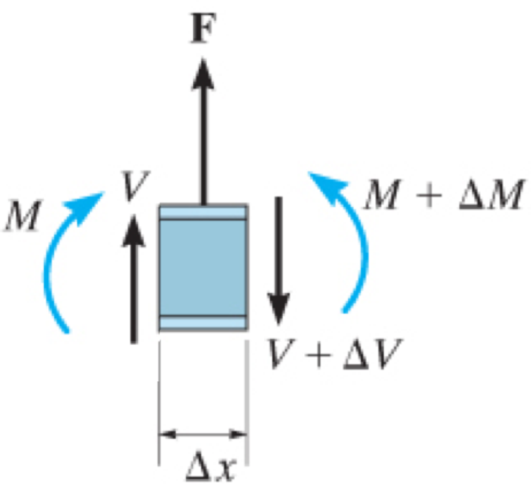
$$\boxed{\frac{dM}{dx} = V}$$

Slope of moment diagram

Shear force

Concentrated force and moment

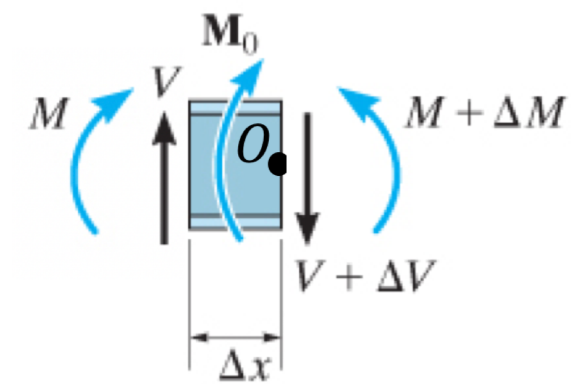
The small segment considered in the previous analysis had no concentrated force or moment. These produce discontinuities (jumps) in the shear or moment distributions.



$$\uparrow + \quad \Sigma F_y = 0$$

$$V + F - (V + \Delta V) = 0$$

$$\boxed{\Delta V = F}$$



$$\circlearrowleft + \quad \Sigma M_O = 0$$

$$-V\Delta x - M - M_0 + M + \Delta M = 0$$

$$\Delta M = V\Delta x + M_0$$

Letting $\Delta x \to 0$: $\boxed{\Delta M = M_0}$

Shear and moment relationships

$$\frac{dV}{dx} = w(x)$$

$$\frac{dM}{dx} = V$$

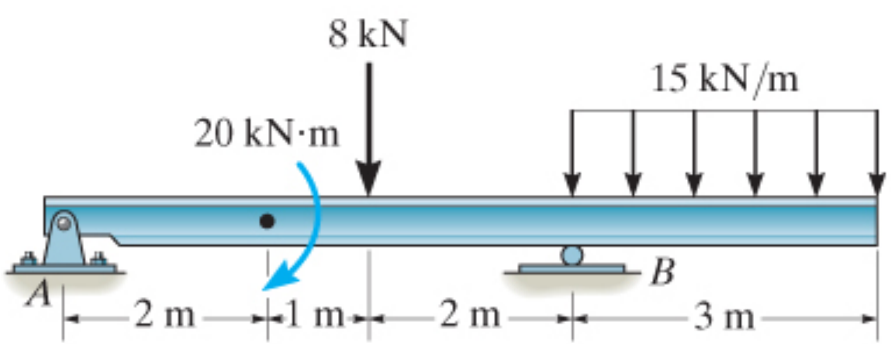
$$\Delta V = F$$

$$\Delta M = M_0$$

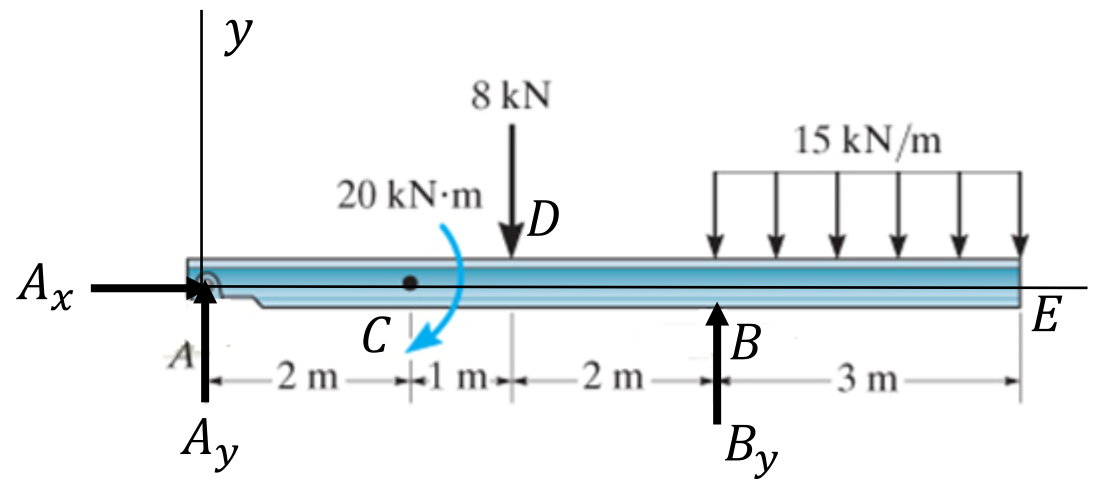
- The slope of V is equal to $w(x)$. If there is no distributed loading then V is constant.
- If $w(x)$ is a polynomial of order n, then V will be a polynomial of order $n+1$, and M will be a polynomial of order $n+2$.
- When $V = 0$, M is at a min/max value.
- These relationships are useful to verify if your V and M distributions are correct.

Example

Plot the shear force and bending moment diagrams for this cantilever beam.



1. Find the support reactions.
2. Plot the shear force for segments AD, DB and BE. (The moment at point *C* doesn't affect the shear force).
3. Plot the bending moment for segments AC, CD, DB and BE.



$$\circlearrowleft + \quad \Sigma M_A = 0$$

$$-20\ \text{kNm} - 8\text{kN}(3\text{m}) + B_y(5\text{m}) - (15\frac{\text{kN}}{\text{m}})(3\text{m})(6.5\text{m}) = 0$$

$$\boxed{B_y = 67.3\ \text{kN}}$$

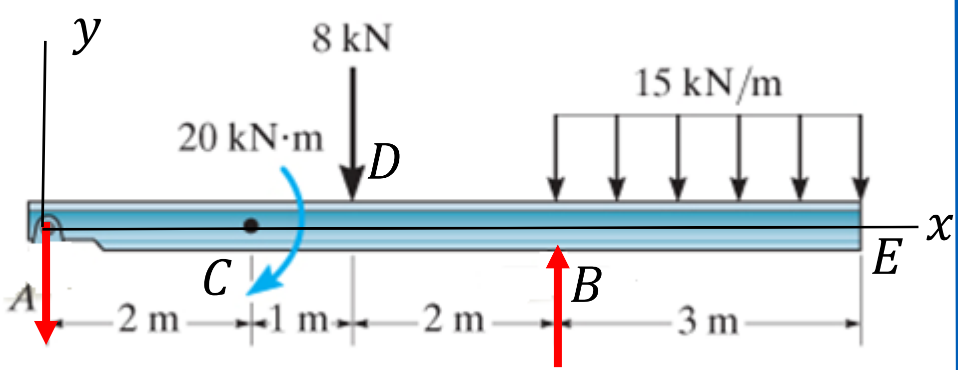
$$\uparrow + \quad \Sigma F_y = 0 \qquad A_y - 8\text{kN} + 67.3\text{kN} - (15\ \text{kN/m})(3\text{m}) = 0$$

$$\boxed{A_y = -14.3\ \text{kN}}$$

$$\rightarrow + \quad \Sigma F_x = 0 \qquad \boxed{A_x = 0}$$

Example

Plot the shear force and bending moment diagrams for this cantilever beam.

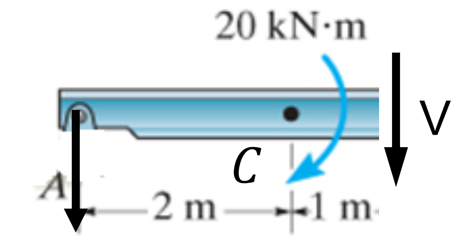


$A_y = 14.3 \text{ kN}$ $B_y = 67.3 \text{ kN}$

Free-body diagram for the left segment:

Segment AD:

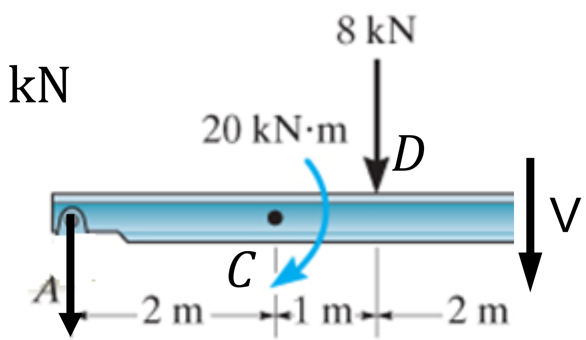
$$\Sigma F_y = 0 \;\; \Rightarrow \;\; V = -A_y = -14.3 \text{ kN}$$



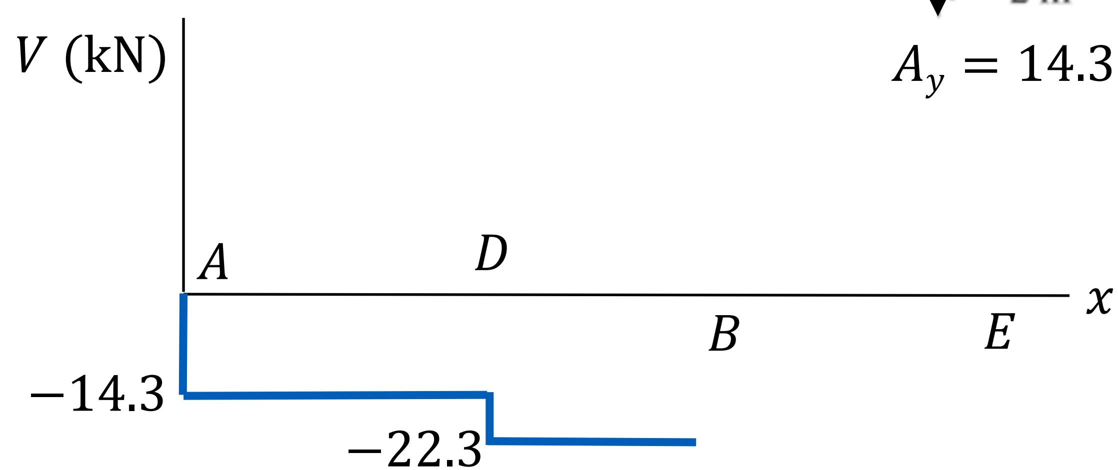
$A_y = 14.3 \text{ kN}$

Segment DB:

$$\Sigma F_y = 0 \;\; \Rightarrow \;\; V = -A_y - 8 \text{ kN} = -22.3 \text{ kN}$$

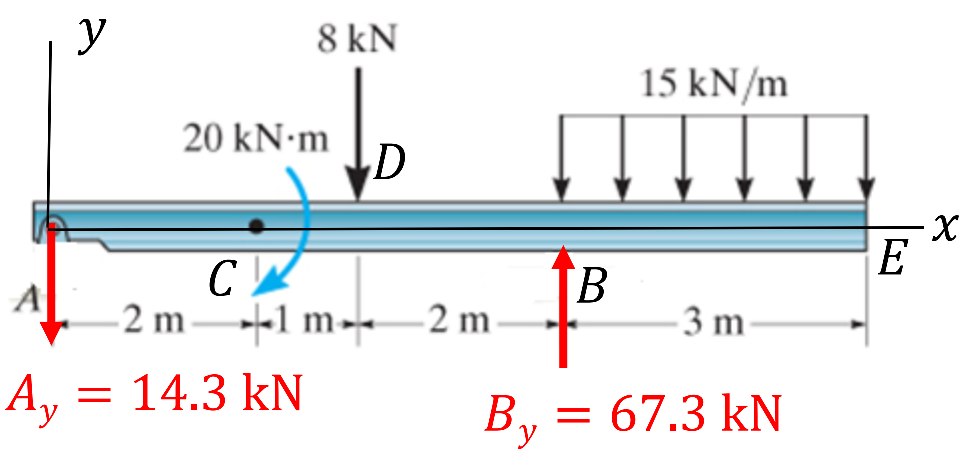


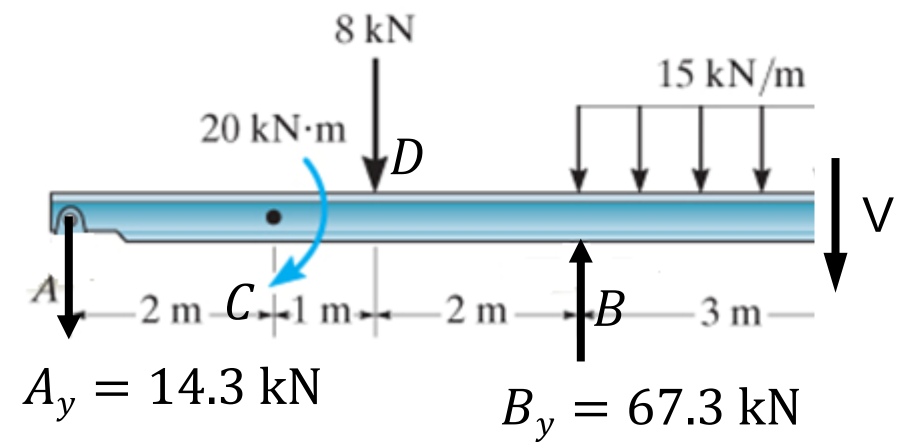
$A_y = 14.3 \text{ kN}$



Example

Plot the shear force and bending moment diagrams for this cantilever beam.





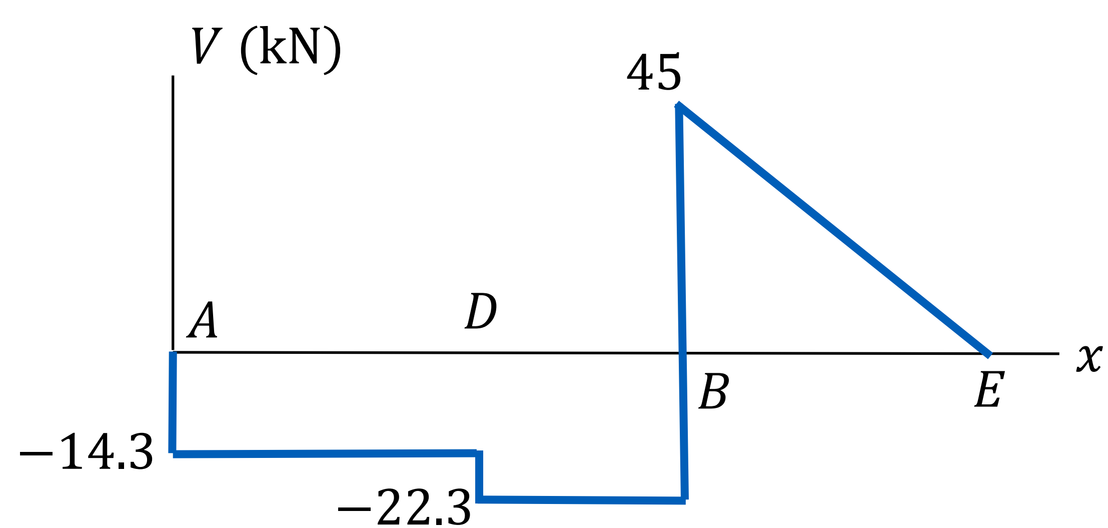
Segment BE:

$$\Sigma F_y = 0 \;\Rightarrow\; V(x) = -A_y - 8\text{ kN} + B_y - 15\text{kN/m}(x - 5\text{m})$$

$$\Rightarrow\; V(x) = 120 - 15x \quad (\text{kN})$$

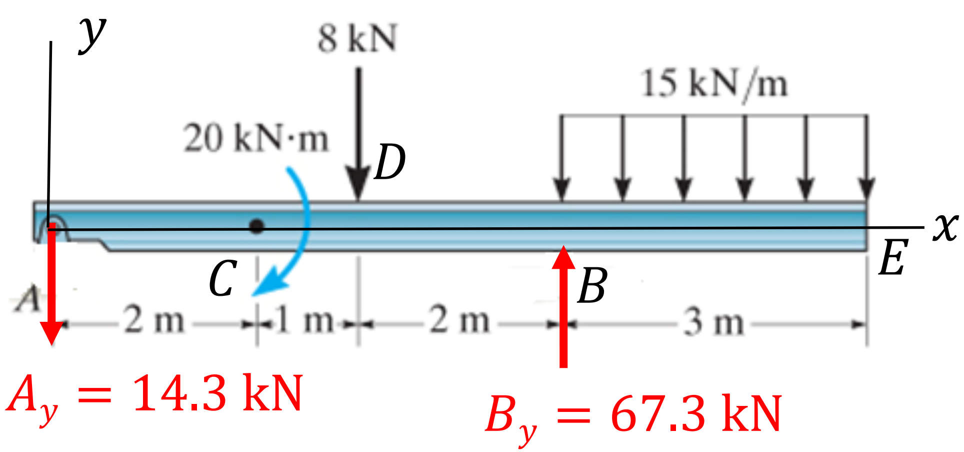
$$V(5\text{m}) = 120 - 15 \cdot 5 = 45\text{ kN}$$

$$V(8\text{m}) = 120 - 15 \cdot 8 = 0$$



Example

Plot the shear force and bending moment diagrams for this cantilever beam.

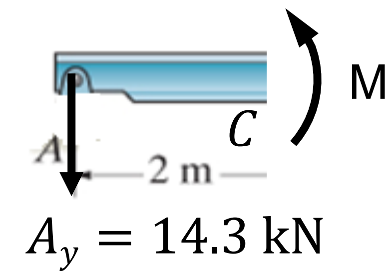


Next, we focus on the bending moment:

Segment AC:

$\Sigma M = 0 \;\Rightarrow\; M(x) = -A_y x = -14.3x$

$M(2\text{m}) = -28.6$ kNm

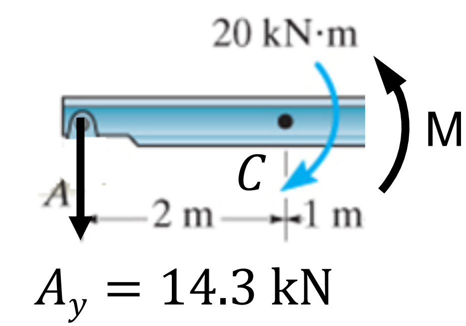


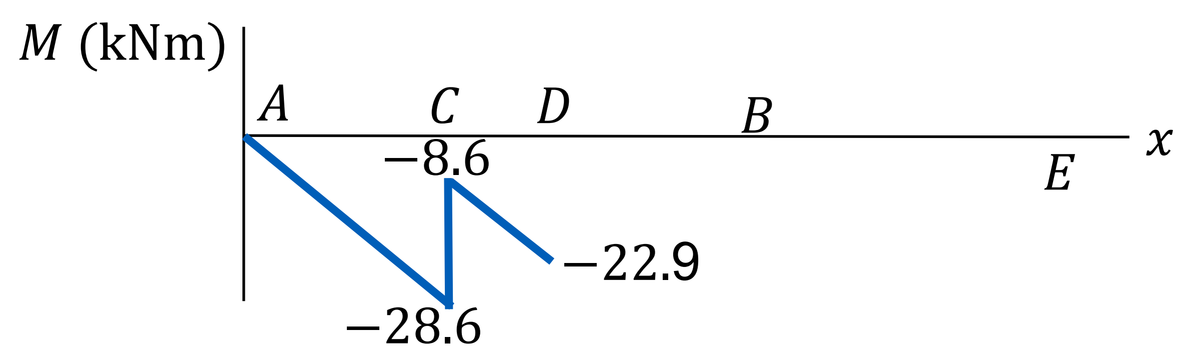
Segment CD:

$\Sigma M = M(x) + A_y x - 20 = 0$

$\Rightarrow M(x) = -14.3x + 20$ (kNm)

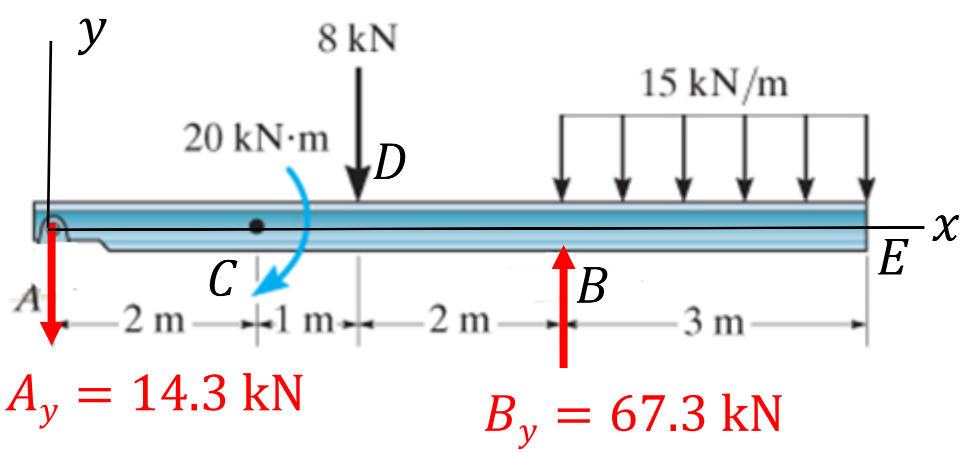
$M(3\text{m}) = -22.9$ kNm





Example

Plot the shear force and bending moment diagrams for this cantilever beam.

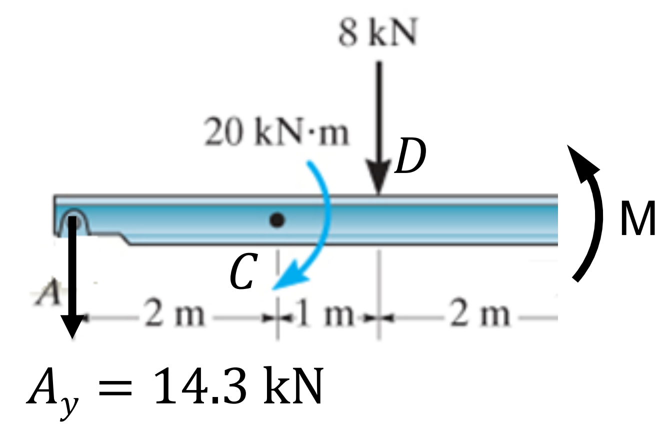


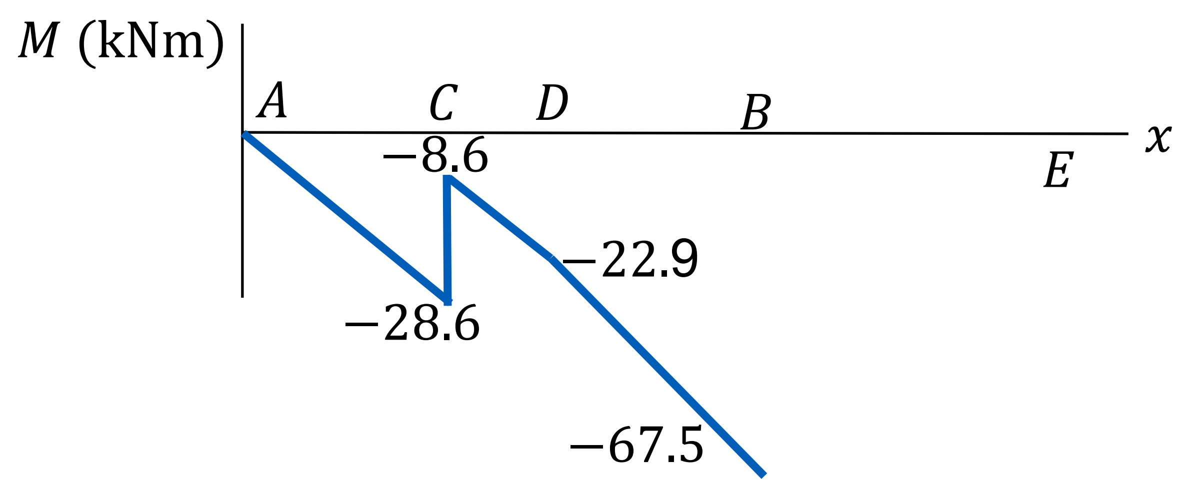
Segment DB:

$$\Sigma M = M(x) + A_y x - 20\text{kNm} + 8\text{kN}(x - 3\text{m}) = 0$$

$$\Rightarrow M(x) = 44 - 22.3x \quad (\text{kNm})$$

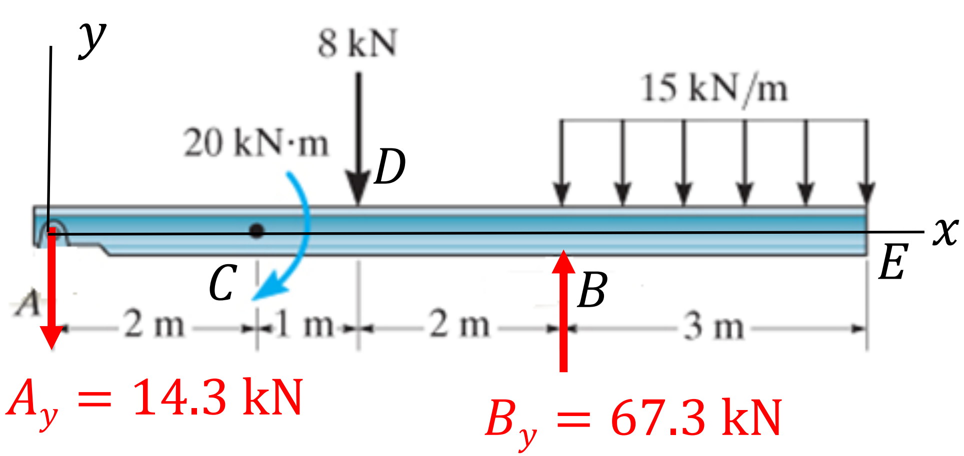
$$M(5\text{m}) = -67.5 \text{ kNm}$$





Example

Plot the shear force and bending moment diagrams for this cantilever beam.

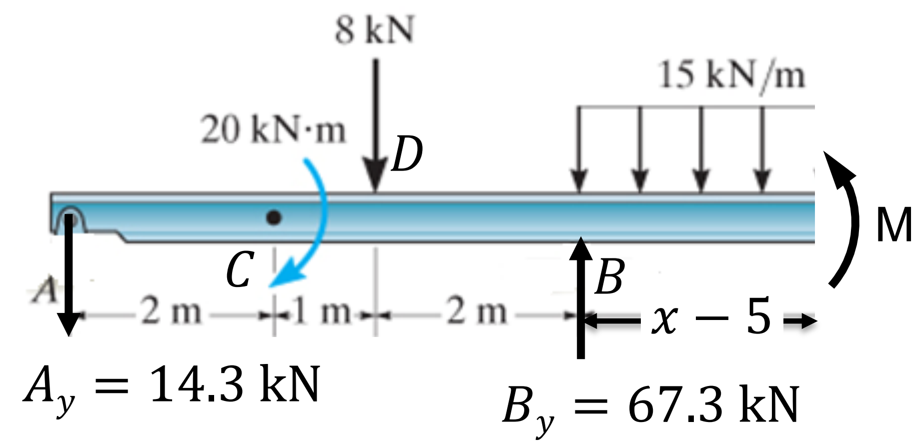


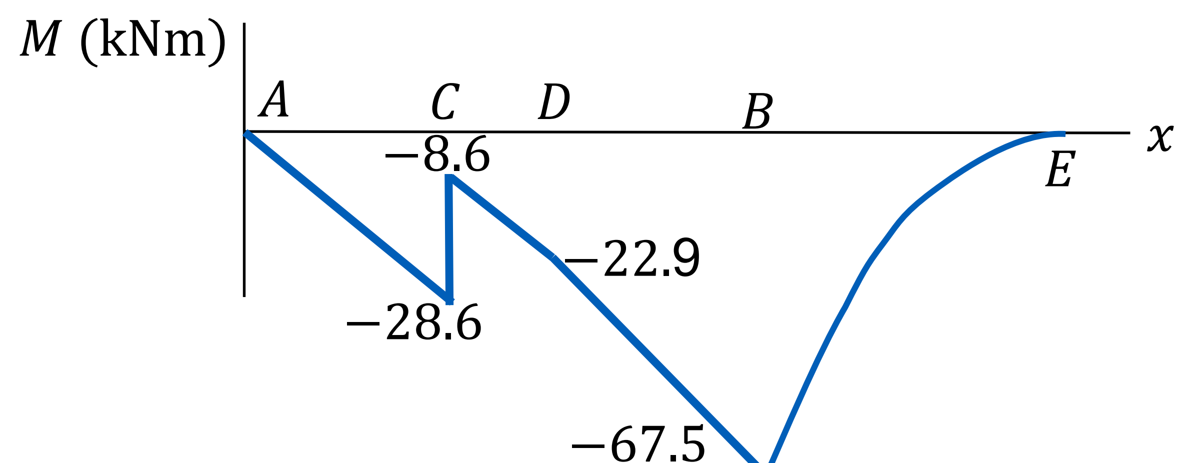
Segment BE. Consider the equivalent force for the distributed load:

$$\Sigma M = M(x) + A_y x - 20 + 8(x-3) - B_y(x-5) + 15(x-5)0.5(x-5) = 0$$

$$\Rightarrow M(x) = 20 - 14.3x - 8(x-3) + 67.3(x-5) - 7.5(x-5)^2$$

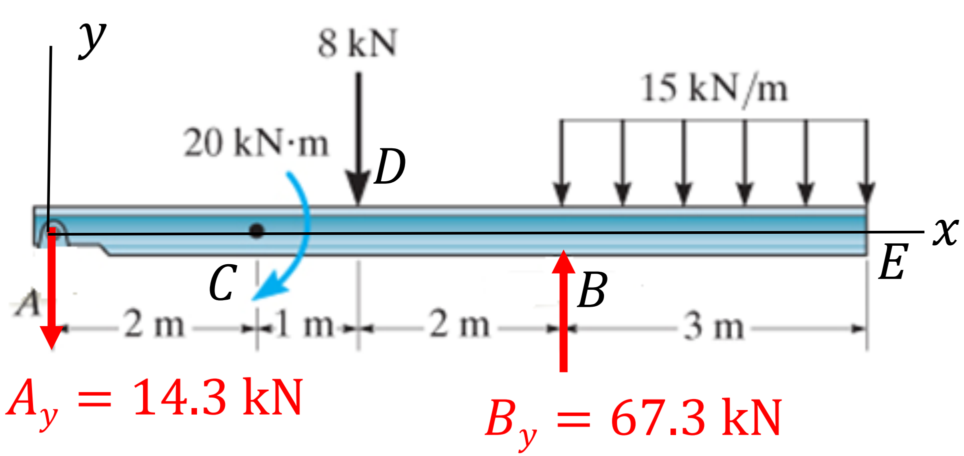
$M(5\text{m}) = -67.5$ kNm
$M(6\text{m}) = -30$ kNm
$M(7\text{m}) = -7.5$ kNm
$M(8\text{m}) = 0$

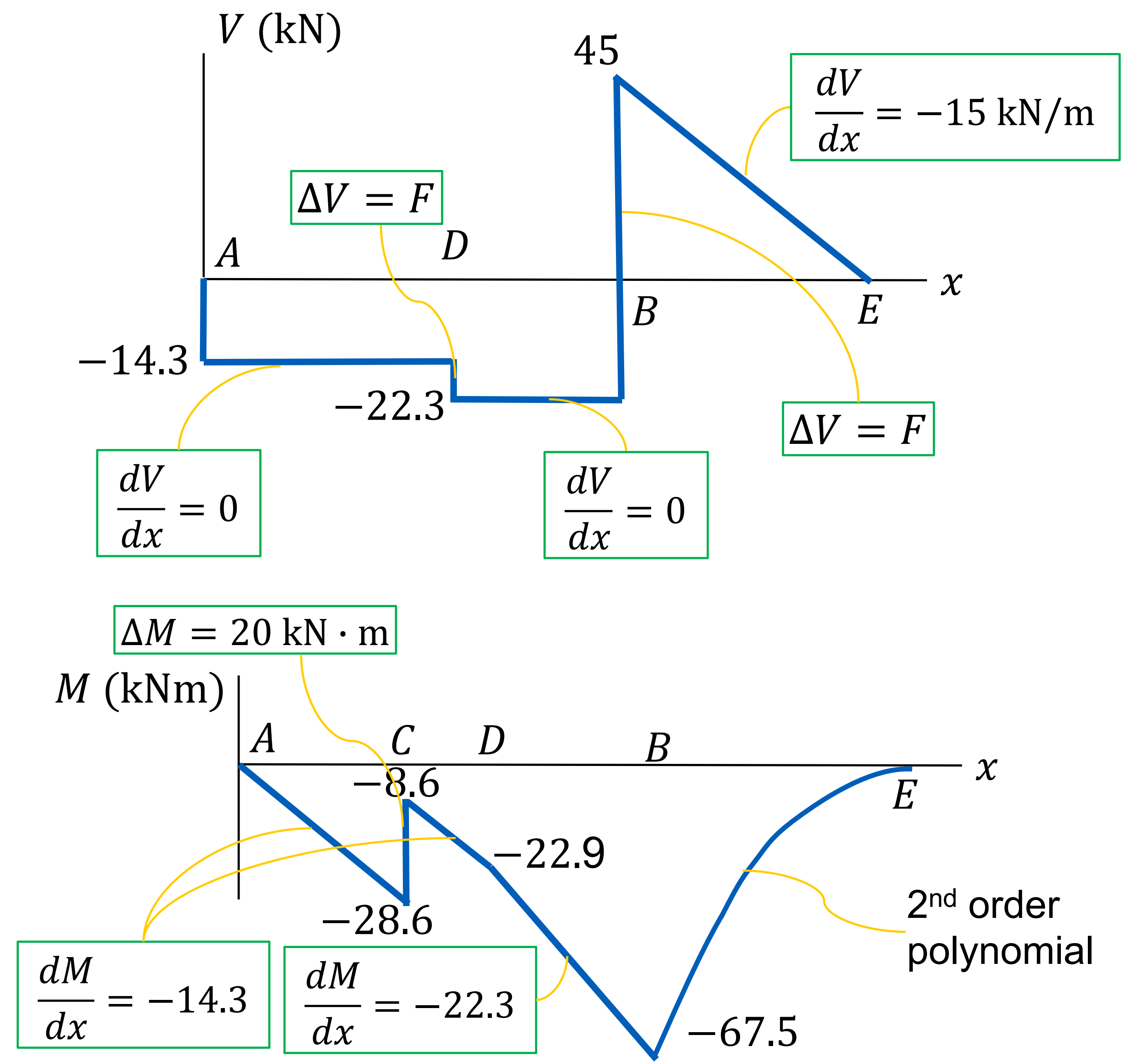




Example

Plot the shear force and bending moment diagrams for this cantilever beam.

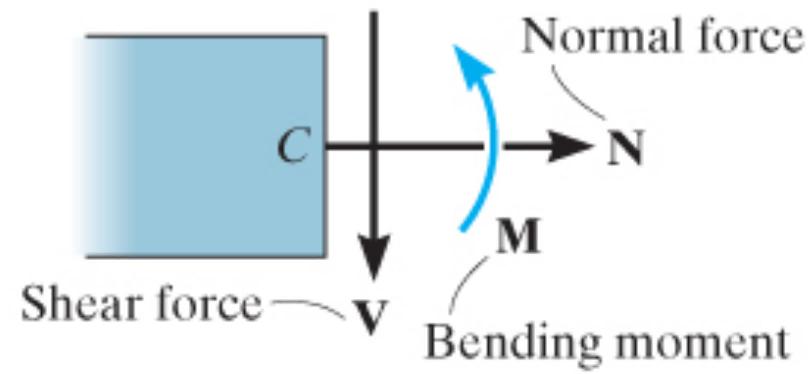




Summary

The internal loads in a beam are:

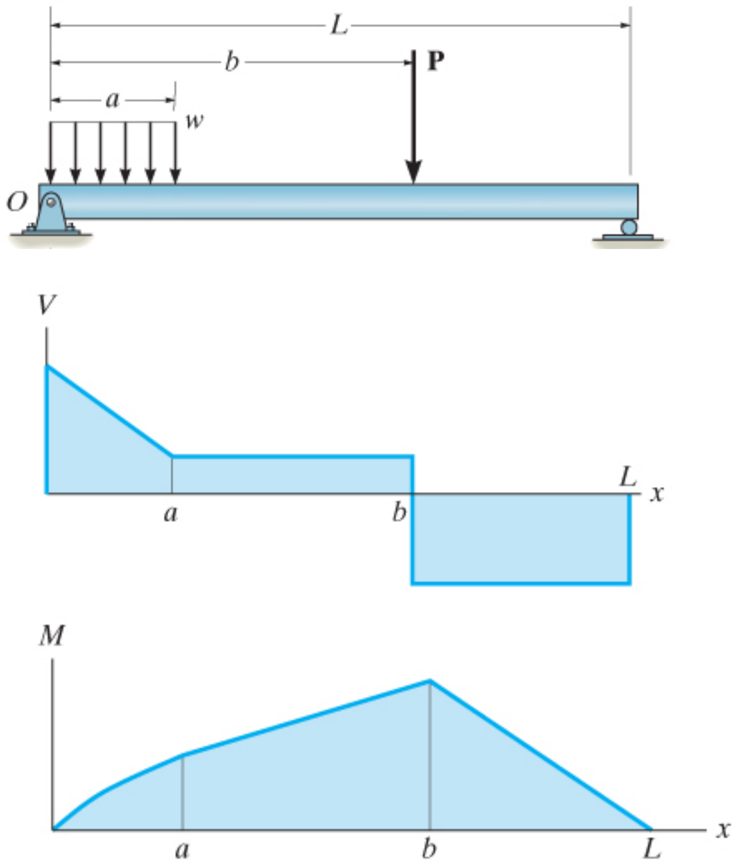
- normal force N,
- shear force V, and
- Bending moment M.



These can be obtained with the method of sections:

1. Solve the support reactions,
2. Cut the beam at a distance x from the left end side,
3. Solve N, V, and M using equilibrium equations.

Summary



- The shear force V and the bending moment M are plotted as a function of the beam length x.

- The shear force V and the bending moment M are related by:

$$\frac{dV}{dx} = w(x) \quad \text{and} \quad \frac{dM}{dx} = V$$

Need more explanations?

This lecture covered sections 7.1 to 7.3 (inclusively).