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COE-C1001: Dynamics

8. Motion of a particle: work and energy

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Learning outcomes

After this lecture, you should be able to:

- Find the work done by a force,
- Derive the potential and kinetic energy of a system,
- Apply the principle of conservation of energy to solve engineering problems.

A bit of history

- Newtonian mechanics:
  - July 1687, Isaac Newton releases *Principia* where he states the three universal laws of motion, including $\Sigma\mathbf{F} = m\mathbf{a}$.
    - *So far, our work has relied on Newton's principles.*
    - *In Newtonian mechanics it is very useful to draw free-body diagrams!*

- Lagrangian mechanics:
  - In 1788, Joseph-Louis Lagrange re-formulates Newtonian mechanics.
    - *Lagrangian mechanics is based on energy instead of forces.*
    - *Lagrange himself said: "No diagrams will be found in this work".*

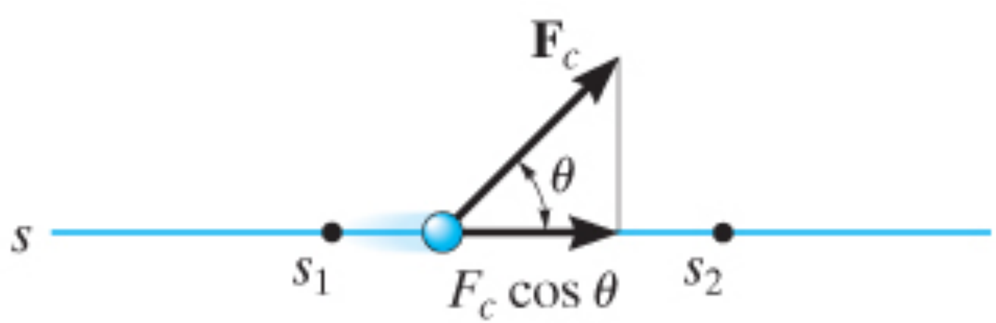
Newtonian vs Lagrangian mechanics

- For a given problem, Newtonian and Lagrangian mechanics give (of course) the same answer.
- Depending on the problem, one approach may be a lot easier than the other one.
  - Lagrangian mechanics is particularly useful when potential energy is converted to kinetic energy.

The work of a force

A force $\mathbf{F}$ will do work on a particle if it undergoes a displacement in the direction of the force.

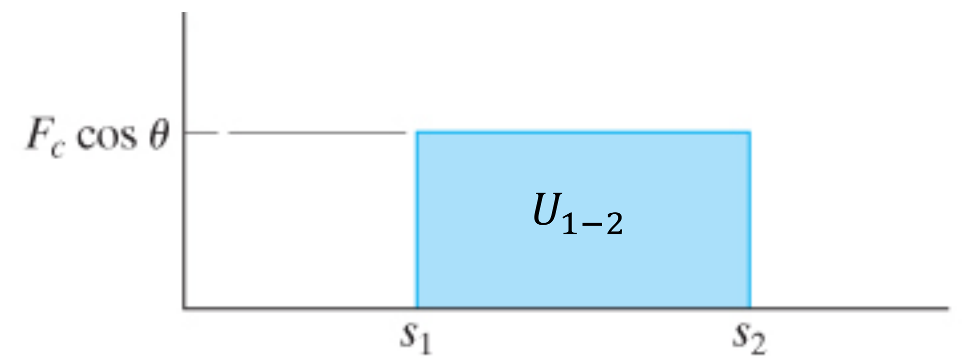
If the force is not parallel to the displacement, then only the component of the force that is parallel to the displacement should be considered.



If the magnitude of the force is constant and the motion is along a straight line the work done is:

$$U_{1-2} = F_c \cos\theta \, (s_2 - s_1)$$

$$U_{1-2} = F_c \cos\theta \int_{s_1}^{s_2} ds$$



Units of work are joules (J)

The work of a force

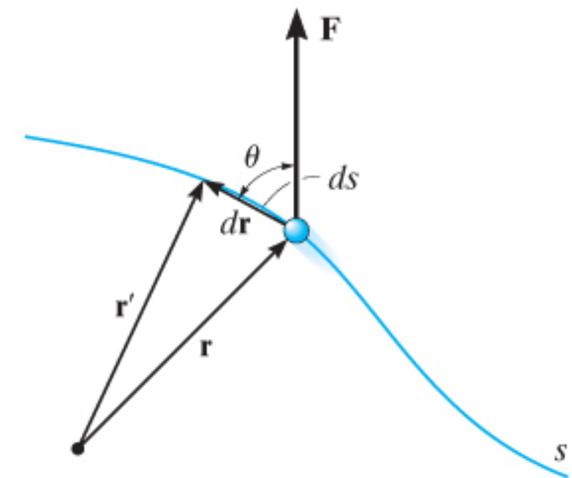
When the force $\mathbf{F}$ is not constant and the path is arbitrary then the work done to advance a particle by a vector $d\mathbf{r}$ (of magnitude ds) is:

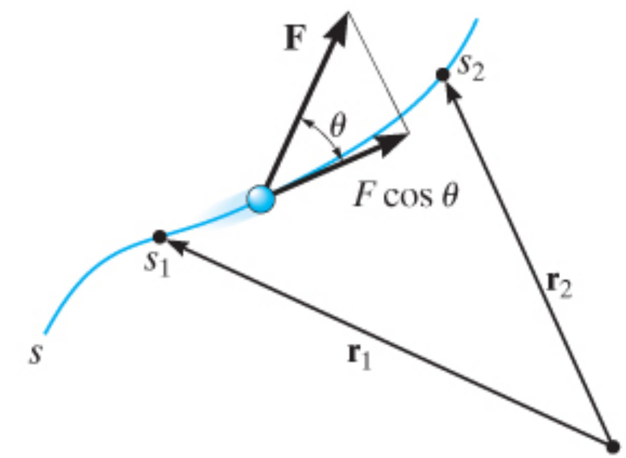
$$dU = \mathbf{F} \cdot d\mathbf{r}.$$

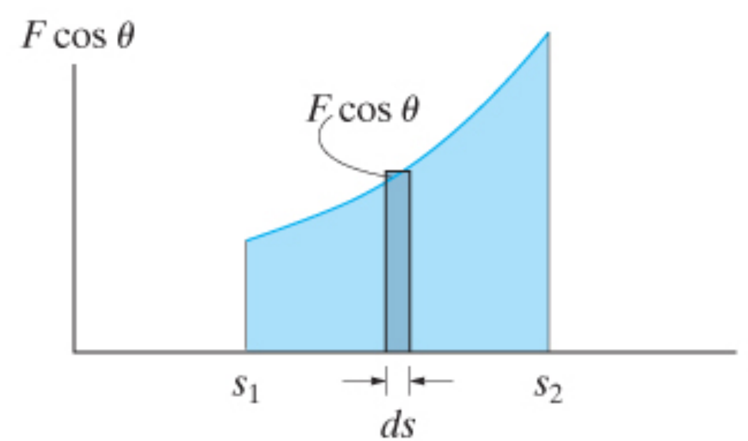
Integrating this relation gives:

$$U_{1-2} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r} = \int_{s_1}^{s_2} F \cos\theta \, ds,$$

Which is the work done by force $\mathbf{F}$ as the particle moves from position s_1 to s_2.

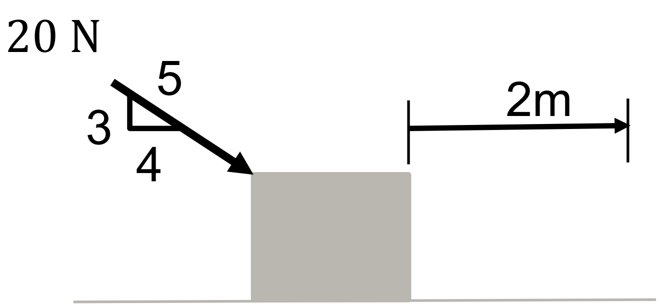






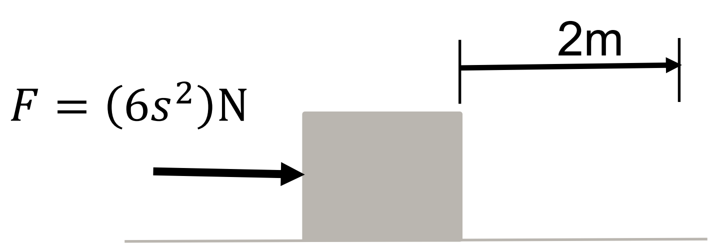
Examples

Determine the work done by the force when the box moves by 2 m.



Constant force: we need to consider only the component of the force parallel to the direction of motion.

$$U = F\cos\theta\, ds = \left(\frac{4}{5}\right)(20\text{N})2\text{m} = 32\text{ J}$$

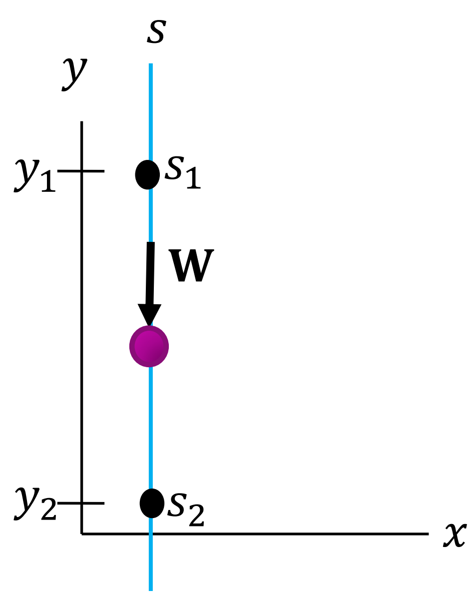


Variable force: we need to integrate as follows.

$$U = \int_{s_1}^{s_2} F ds = \int_0^2 6s^2 ds = \left[\frac{6}{3}s^3\right]_0^2 = 2(2^3) = 16\text{ J}$$

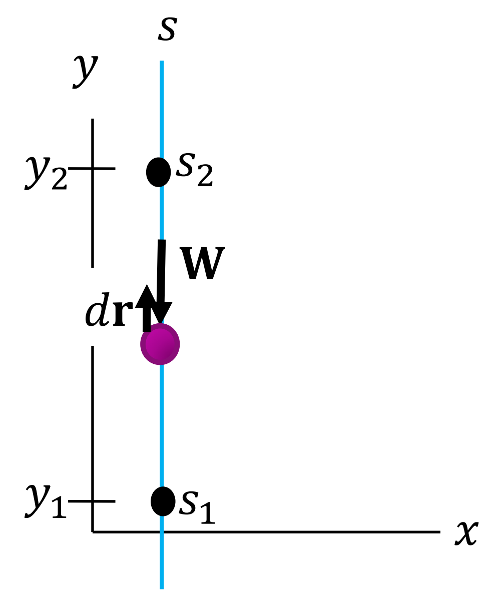
Work of weight

Downward movement



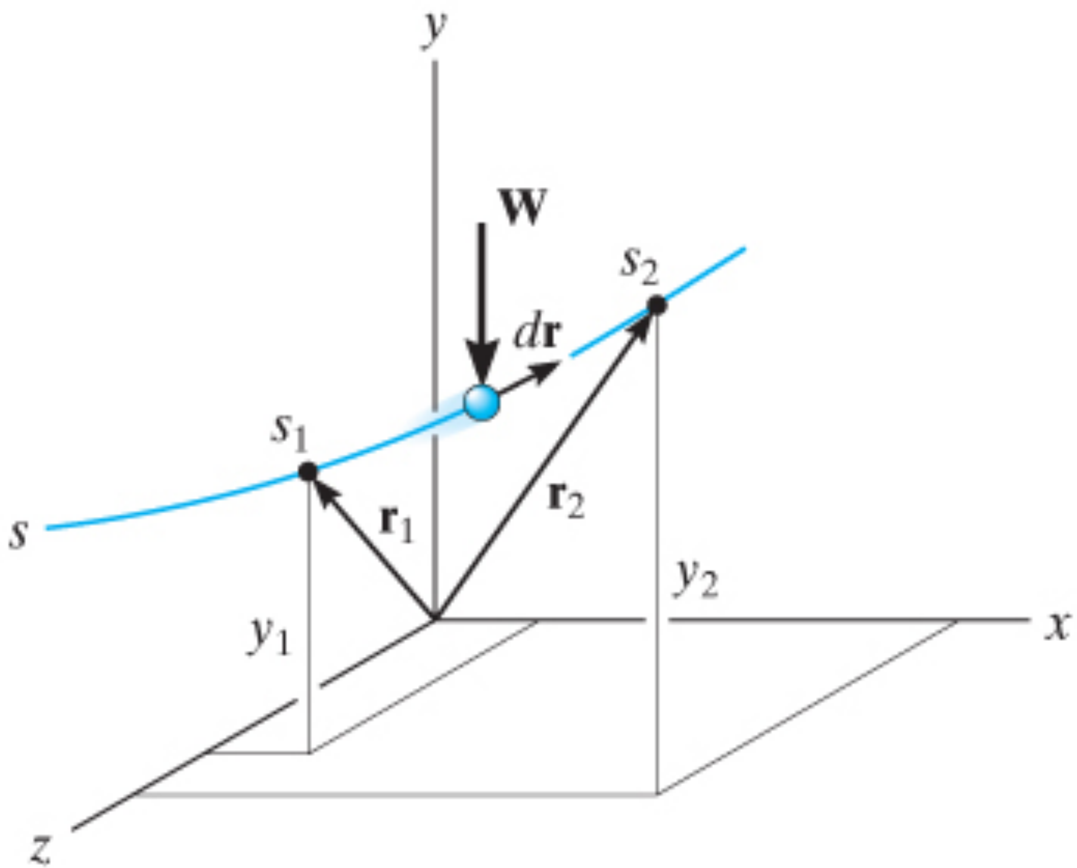
$$
\begin{aligned}
U_{1-2} &= W(s_2 - s_1) \\
&= W(y_2 - y_1) \\
&= W\Delta y
\end{aligned}
$$

Upward movement



$$
\begin{aligned}
U_{1-2} &= -W(s_2 - s_1) \\
&= -W(y_2 - y_1) \\
&= -W\Delta y
\end{aligned}
$$

Work of weight



General case: arbitrary movement of a particle

$$d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k} \quad \& \quad \mathbf{W} = -W\mathbf{j}$$

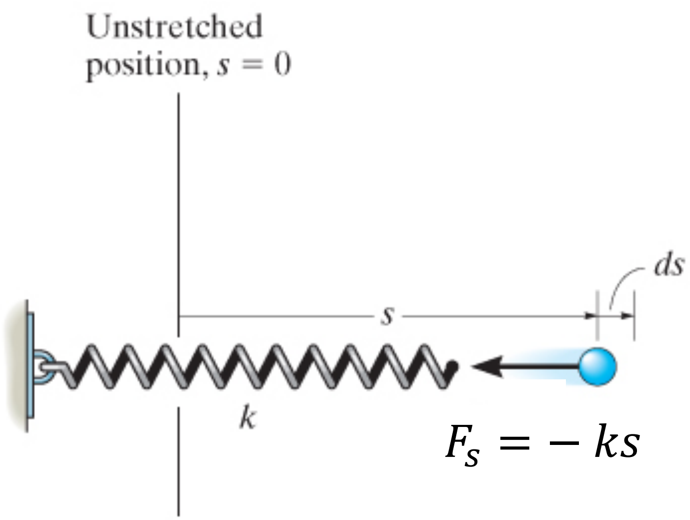
$$U_{1-2} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r}$$

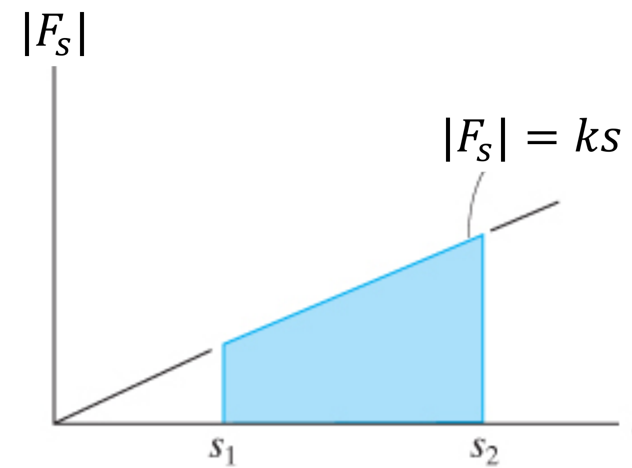
$$= \int_{\mathbf{r}_1}^{\mathbf{r}_2} (-W\mathbf{j}\ ) \cdot (dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k})$$

$$= \int_{y_1}^{y_2} -W dy \ = -W(y_2 - y_1) = -W\Delta y$$

- The work done by weight is independent of the path. It depends only on the weight and the vertical displacement.
- Work is positive if the mass is going down, but negative if it is moving upward.

Work of a spring force





Work done by a spring force

$$dU = -F_s ds = -ks\, ds$$

Integrating this equation from position s_1 to s_2 returns:

$$U_{1-2} = \int_{s_1}^{s_2} -F_s ds = \int_{s_1}^{s_2} -ks\, ds = -(\frac{1}{2}ks_2^2 - \frac{1}{2}ks_1^2)$$

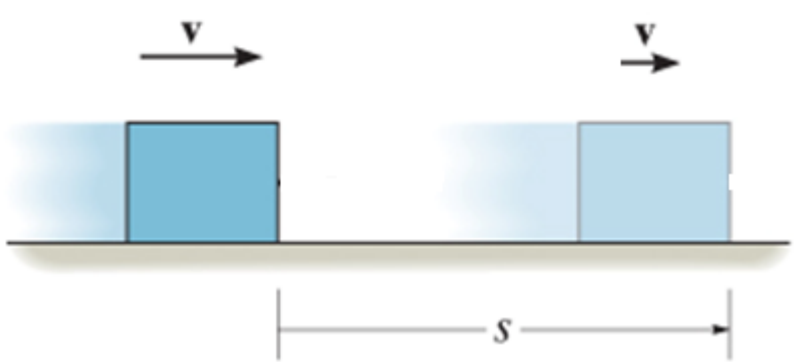
Work done by a spring force

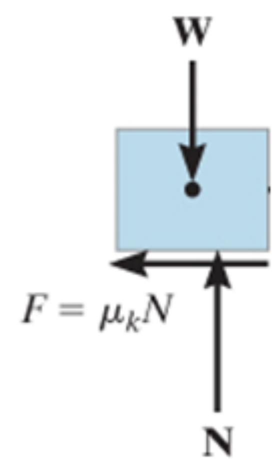
- Is **positive** when the force and motion are in the **same** direction.
- Is **negative** when the force and motion are in **opposite** directions.

Work of a friction force

A **friction force** is always **opposed to** the direction of **motion** (note the negative sign):

$$U_{1-2} = -\mu_k N s$$





Principle of work and energy

The work done is equal to the change in kinetic energy

Work done (U) = Change in kinetic energy (T)

$$\Sigma U_{1-2} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 = T_2 - T_1$$

Which is often written as:

$$T_1 + \Sigma U_{1-2} = T_2$$

This is demonstrated on the next slide.

Principle of work and energy

Let's start from Newton's law and multiply both sides by the velocity $\mathbf{v}$:

$$\Sigma\mathbf{F} = m\mathbf{a} \quad \Rightarrow \Sigma\mathbf{F}\cdot\mathbf{v} = m\mathbf{a}\cdot\mathbf{v}$$

Next, we can integrate this relation with respect to time:

$$\int_{t_1}^{t_2} \Sigma\mathbf{F}\cdot\mathbf{v}dt = \int_{t_1}^{t_2} m\mathbf{a}\cdot\mathbf{v}dt \quad \Rightarrow \int_{t_1}^{t_2} \Sigma\mathbf{F}\cdot\frac{d\mathbf{r}}{dt}dt = m\int_{t_1}^{t_2} \frac{d\mathbf{v}}{dt}\cdot\mathbf{v}dt$$

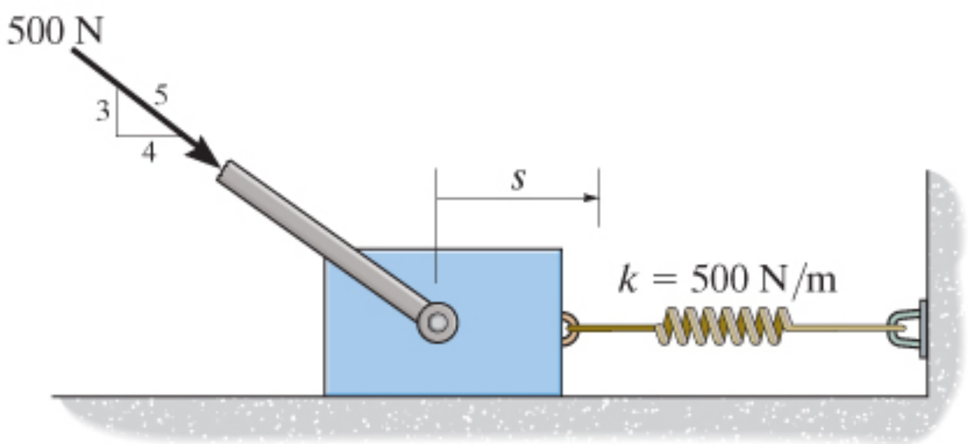
$$\Rightarrow \int_{\mathbf{r}_1}^{\mathbf{r}_2} \Sigma\mathbf{F}\cdot d\mathbf{r} = \frac{m}{2}\int_{\mathbf{v}_1}^{\mathbf{v}_2} d(\mathbf{v}\cdot\mathbf{v}) = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

The left hand side is the work done by all forces, therefore:

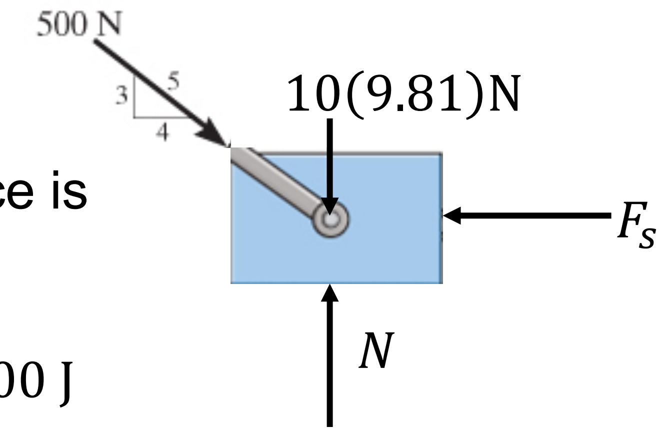
$$\Sigma U_{1-2} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

Example

Find the velocity of this 10 kg box after a distance $s = 0.5$ m. The box is at rest and the spring is unstretched when $s = 0$. Friction can be neglected.



Free-body diagram:



Work done by external force is

$$U_F = F\cos\theta\,(s_2 - s_1)$$

$$= 500\text{N}\left(\frac{4}{5}\right)(0.5\text{m}) = 200\text{ J}$$

Work of spring force: $(U_S)_{1-2} = -(\frac{1}{2}ks_2^2 - \frac{1}{2}ks_1^2)$

$$U_S = -\frac{1}{2}500\frac{\text{N}}{\text{m}}(0.5\text{m})^2 = -62.5\text{ J}$$

Total work done: $\Sigma U = U_F + U_S = 137.5\text{ J}$

Principle of work and energy: $T_1 + \Sigma U_{1-2} = T_2$

$$\frac{1}{2}mv_1^2 + \Sigma U = \frac{1}{2}mv_2^2 \;\rightarrow 0 + 137.5\text{J} = \frac{1}{2}(10\text{kg})v_2^2$$

$$v_2 = \sqrt{137.5\text{J}/5\text{kg}} = 5.24\ \frac{\text{m}}{\text{s}}$$

Power

$$P = \mathbf{F} \cdot \mathbf{v}$$

Figure: 14_PH003

The power output of this locomotive comes from the driving frictional force F developed at its wheels. It is this force that overcomes the frictional resistance of the cars in tow and is able to lift the weight of the train up the grade.

Power is defined as the amount of work done during a period of time:

$$P = \frac{dU}{dt}$$

Substituting in the definition of U gives:

$$P = \frac{dU}{dt} = \frac{\mathbf{F} \cdot d\mathbf{r}}{dt} = \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \mathbf{F} \cdot \mathbf{v} = Fv\cos\theta$$

(Note that here we assumed $\mathbf{F}$ to be constant)

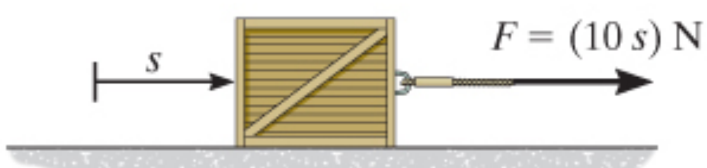
Efficiency is defined as the power generated by a machine divided by the power supplied to it:

$$\varepsilon = \frac{\text{power output}}{\text{power input}} < 1$$

Units of power $[P] = \mathrm{W} = \mathrm{J}/s$ or $[P] = \mathrm{hp} = 746\mathrm{W}$

Example

Find the power created by the force $\mathbf{F} = (10s)$N at $s = 5$m. At $s = 0$, the box has a velocity $v = 1$ m/s. The box has a mass of 20 kg, and friction is negligible.



Power is equal to the force times velocity; therefore, we need to find the velocity of the box at $s = 5$m. There is a single force acting on the box and therefore, its acceleration is given by:

$$\stackrel{+}{\rightarrow} \quad \Sigma F_x = m a_x \qquad a = \frac{F}{m} = \frac{10s}{20} = \frac{1}{2}s$$

Next, we can use $ads = vdv$, to find the velocity at $s = 5$ m:

$$\int_{s_0}^{s} a\, ds = \int_{v_0}^{v} vdv \quad \rightarrow \quad \int_0^5 \frac{1}{2}s\ ds = \int_1^v vdv$$

$$\frac{1}{4}5^2 = \frac{1}{2}v^2 - \frac{1}{2}1^2 \quad \rightarrow \quad v(s = 5\text{m}) = 3.674\ \text{m/s}$$

Finally, the power $P = \mathbf{F} \cdot \mathbf{v}$

$$P(\text{s} = 5\text{m}) = 10(5)\text{N}(3.674\frac{\text{m}}{\text{s}}) = 184\ \text{W}$$

Conservative forces

If the work of a force is:

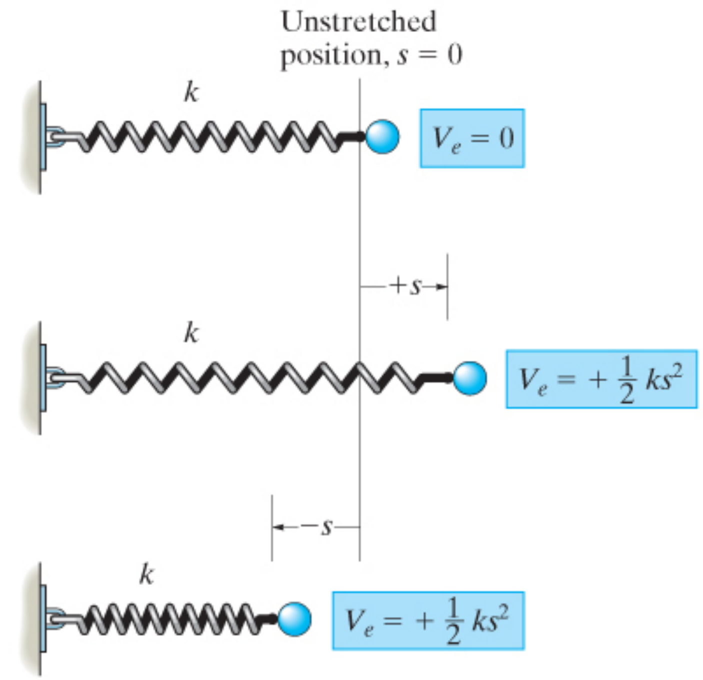
- independent of the path and,
- depends only on the initial and final positions,

then this force is a **conservative force**.

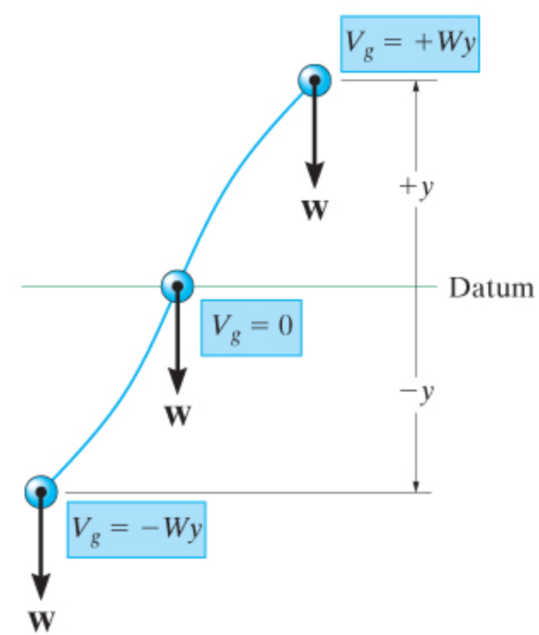
Examples of conservative forces:

- Weight and,
- Force developed by a spring.

(In contrast, friction is a non-conservative force since the work done depends on the path)



Elastic potential energy



Gravitational potential energy

Potential energy

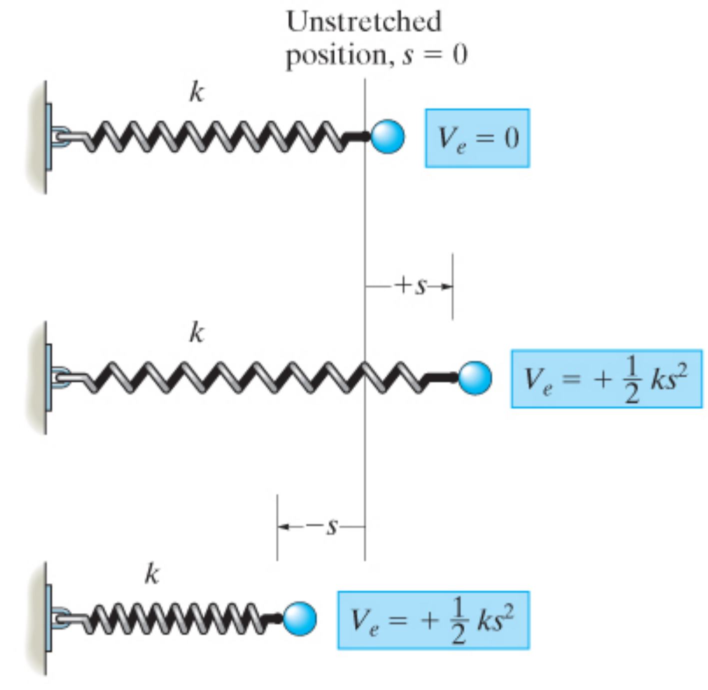
Potential energy represents the work done by a conservative force as it moves from a given position to a reference position.

It include gravitational potential energy: $V_g = Wy$

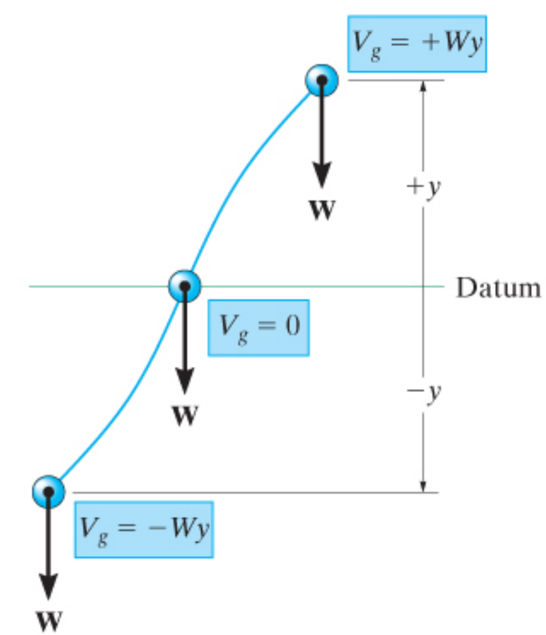
and elastic potential energy: $V_e = \frac{1}{2}ks^2$

Therefore, the potential function, or total potential energy is:

$$V = V_g + V_e$$



Elastic potential energy



Gravitational potential energy

Principle of conservation of energy

Potential energy is the work done by conservative forces:

$$(\Sigma U_{1-2})_{cons} = V_1 - V_2$$

The total work done includes conservatives and non-conservative forces:

$$\Sigma U_{1-2} = (\Sigma U_{1-2})_{cons} + (\Sigma U_{1-2})_{noncons} = V_1 - V_2 + (\Sigma U_{1-2})_{noncons}$$

Substituting in the relation found earlier:

$$T_1 + \Sigma U_{1-2} = T_2$$

$$\Rightarrow T_1 + V_1 + (\Sigma U_{1-2})_{noncons} = T_2 + V_2$$

When only conservative forces are present, this becomes:

$$T_1 + V_1 = T_2 + V_2$$

Principle of conservation of energy

For a system with conservative forces only, we have:

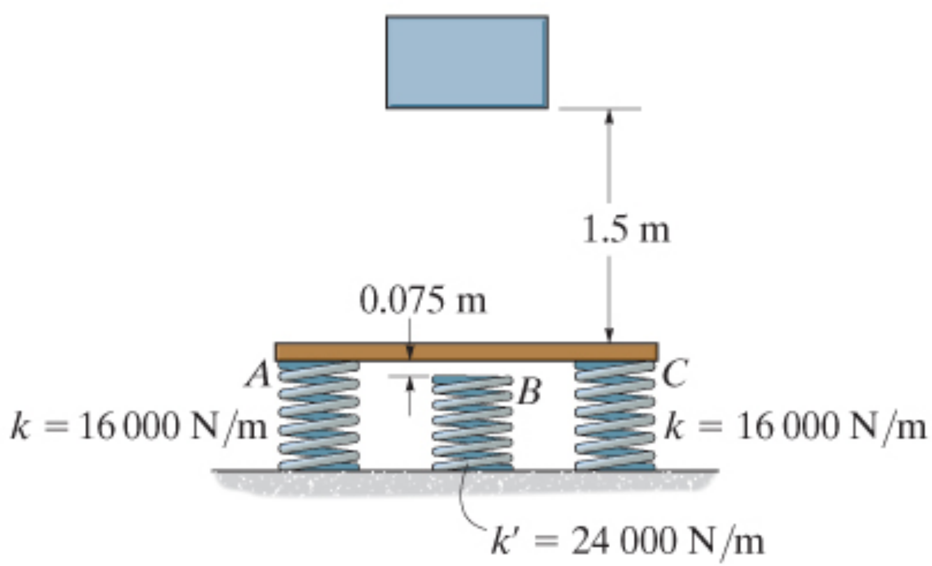
$$T_1 + V_1 = T_2 + V_2$$

This means that the total energy (sum of kinetic and potential energies) is constant.

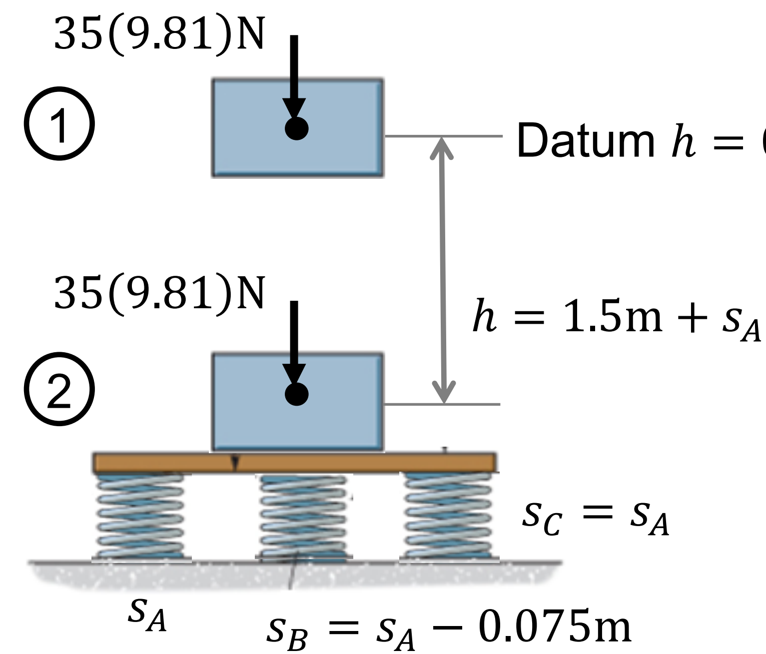
Energy can be transformed (from kinetic to potential) but it can't be created or eliminated.

Example

A 35 kg box is dropped from a height of 1.5 m as shown below. Find the maximum compression of the springs.



If we define $h = 0$ at the initial position, this means that $T_1 = V_1 = 0$. The final position is when the box has zero velocity; therefore, $T_2 = 0$.



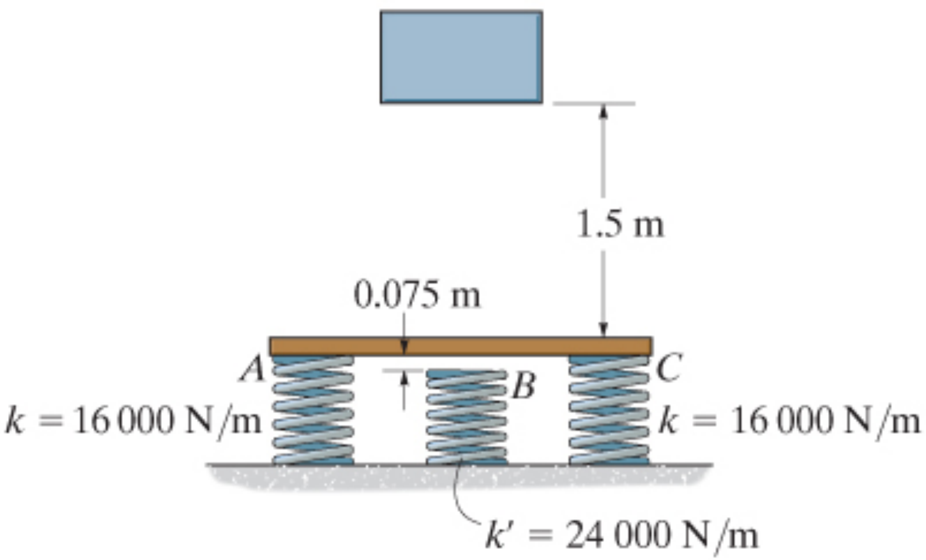
The principle of energy conservation implies that:

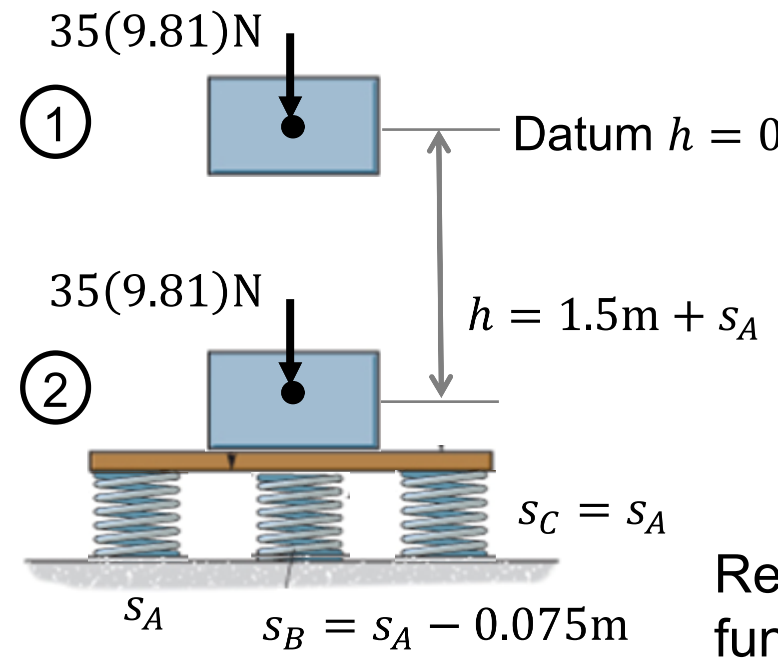
$$\boxed{\overset{=0}{\cancel{T_1}} + \overset{=0}{\cancel{V_1}} = \overset{=0}{\cancel{T_2}} + V_2}$$

$$V_2 = V_{g2} + V_{e2} = -mgh + \frac{1}{2}k{s_A}^2 + \frac{1}{2}k'{s_B}^2 + \frac{1}{2}k{s_A}^2 = 0$$

Example

A 35 kg box is dropped from a height of 1.5 m as shown below. Find the maximum compression of the springs.





Rewrite equation as a function of s_A.

$$V_2 = -mgh + \frac{1}{2}k{s_A}^2 + \frac{1}{2}k'{s_B}^2 + \frac{1}{2}k{s_A}^2 = 0$$

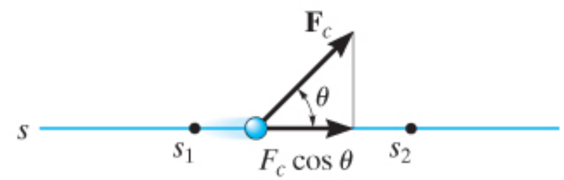
$$= -35(9.81)(1.5 + s_A) + \frac{1}{2}k{s_A}^2 + \frac{1}{2}k'(s_A - 0.075)^2 + \frac{1}{2}k{s_A}^2 = 0$$

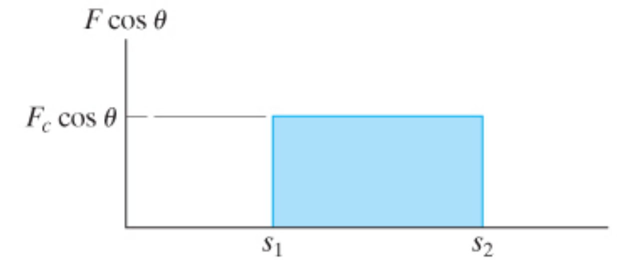
Solving this 2nd order equation gives: $s_A = 0.170\text{ m}$

Compression of spring B: $s_B = s_A - 0.075\text{m} = 0.095\text{ m}$

Summary

The work done by a force is:





Change in kinetic energy is equal to the work done by all forces: $T_1 + \Sigma U_{1-2} = T_2$

Definition of power: $P = \mathbf{F} \cdot \mathbf{v}$

Principle of energy conservation: $T_1 + V_1 = T_2 + V_2$

Need more explanations?

If you need more explanations, consult chapter 14 of the textbook:

- **14. Kinetics of a particle: work and energy**