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COE-C1001: Statics

3. Equilibrium of a rigid body and support reactions

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Learning outcomes

After this lecture, you should be able to:

- Draw a free-body diagram of a rigid body.
- Use equilibrium equations to find the support reactions.

Equilibrium of a rigid body

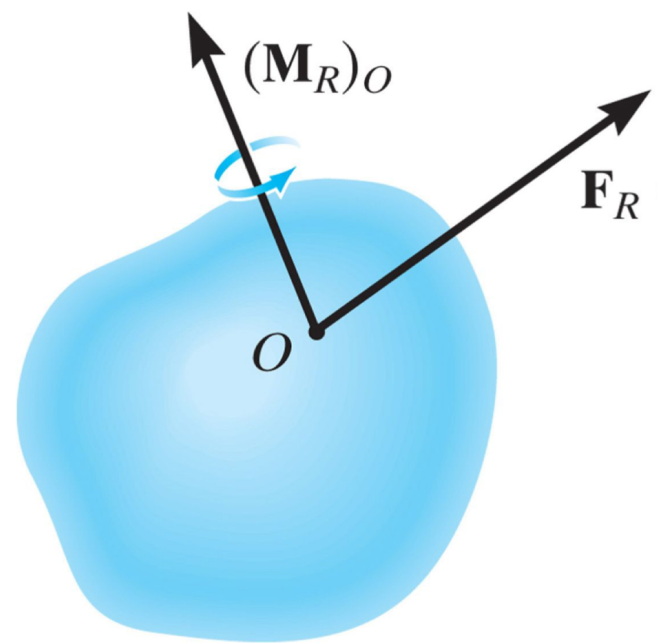
- Previously, we saw how complex loading scenarios can be simplified to a single resultant force and resultant moment acting on the body.

- The body is in equilibrium when:

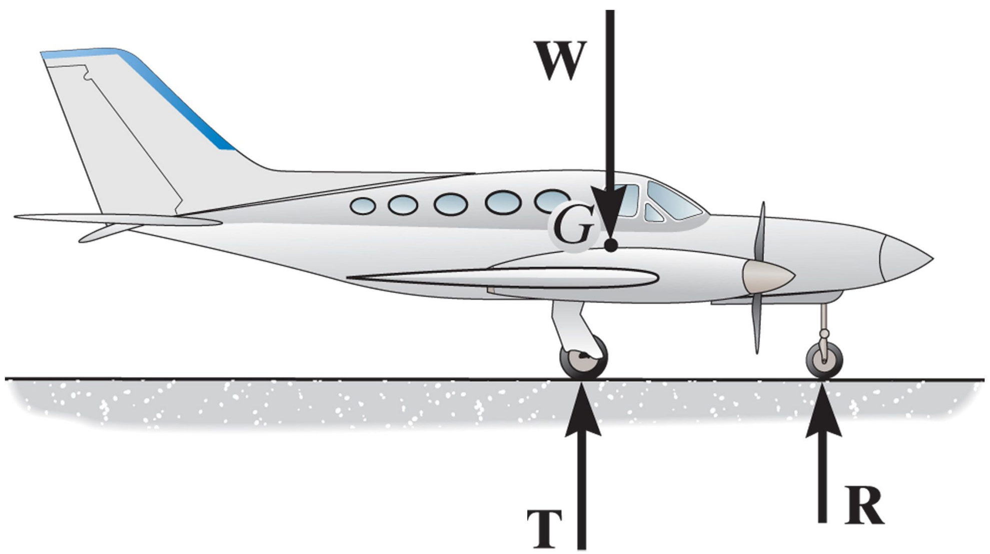
$$\mathbf{F}_R = \sum \mathbf{F} = \mathbf{0}$$
$$(\mathbf{M}_R)_O = \sum \mathbf{M}_O = \mathbf{0}$$

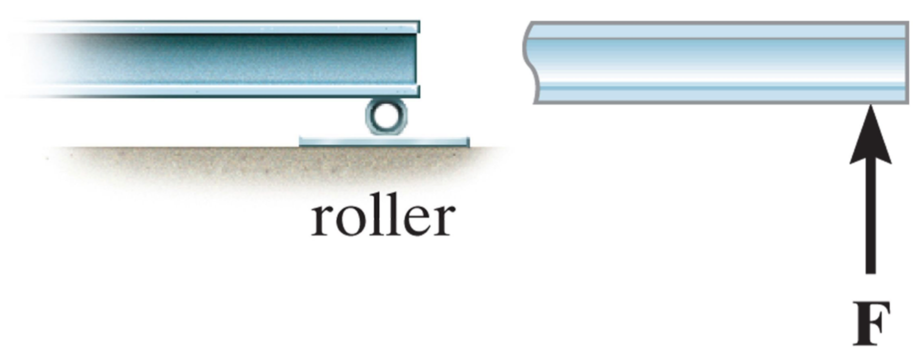
- Using these equations of equilibrium requires to list all known and unknown forces acting on the body.

- The best way to account for all these forces is to draw a free-body diagram.



Free-body diagram & support reactions





Consider the free-body diagram of the plane. It includes:

- a known force: the weigth *W*, and
- And two unknown support reactions, *T* and *R*, corresponding to the ground pushing on the wheels.

Therefore, it is important to understand what forces/moments are generated by different types of connection.

- A roller produces a force normal to the contact direction.

Support reactions

A pin prevents all translations, but allows rotation around the pin's axis. This is achieved with a reaction force at the point of connection.

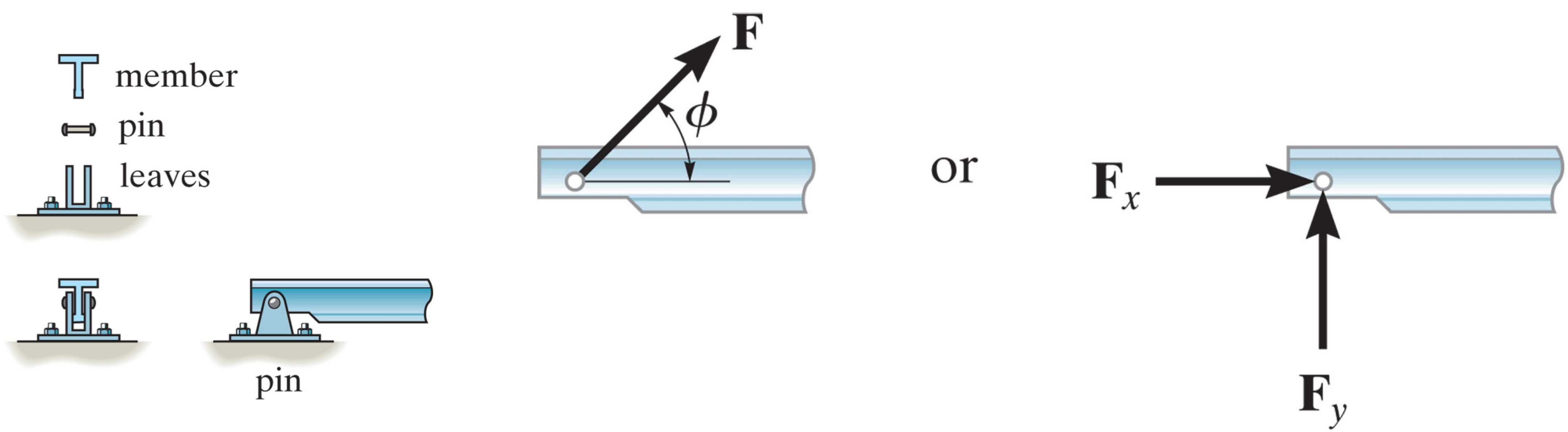


Table 5-1 in the textbook lists the support reactions for different types of connection.

Support reactions

A fixed support is preventing all translations and rotations. This is achieved with a reaction force and moment at the point of connection.

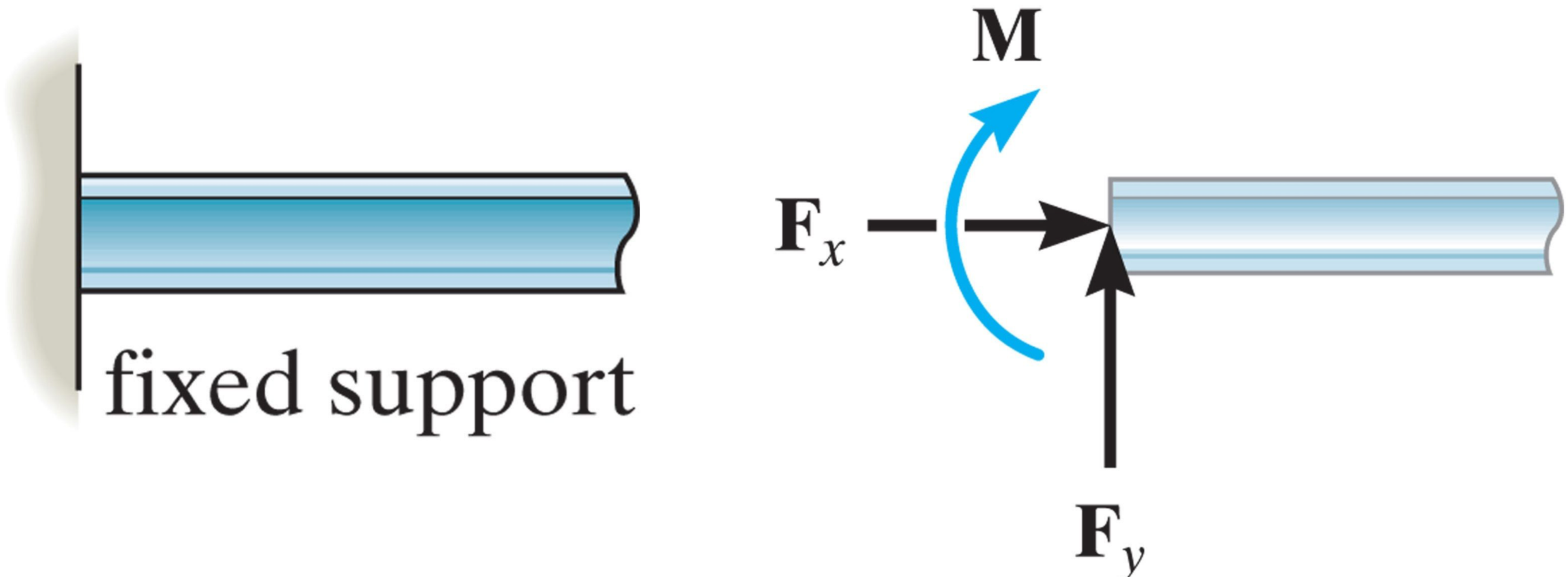
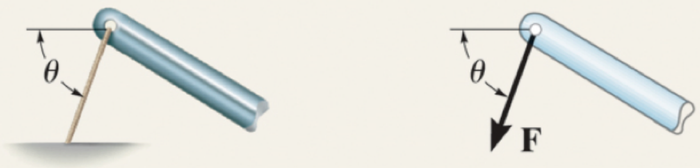


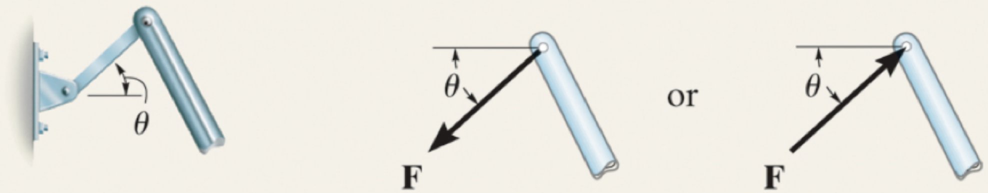
Table 5-1 in the textbook lists the support reactions for different types of connection.

(1)



cable

(2)



weightless link

(3)

roller

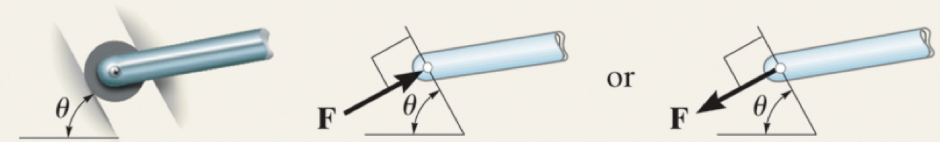
(4)

rocker

(5)

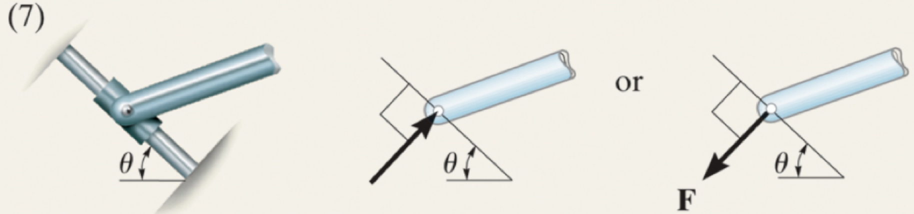
smooth contacting surface

(6)



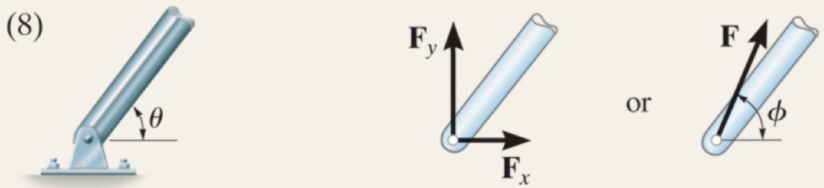
roller or pin in confined smooth slot

(7)



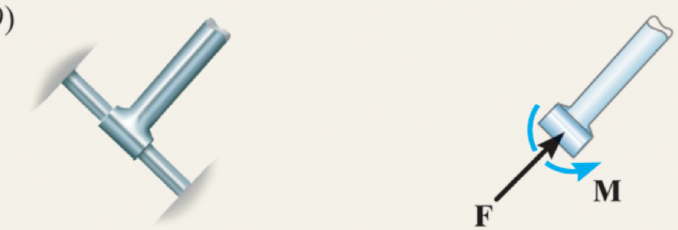
member pin connected to collar on smooth rod

(8)



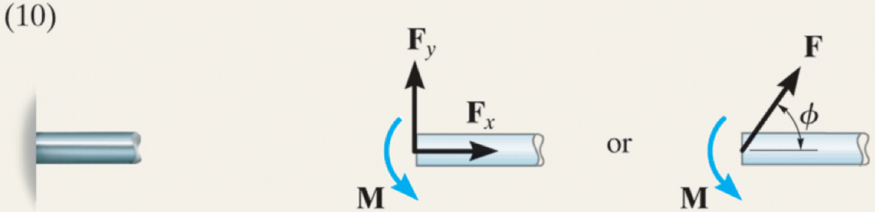
smooth pin or hinge

(9)

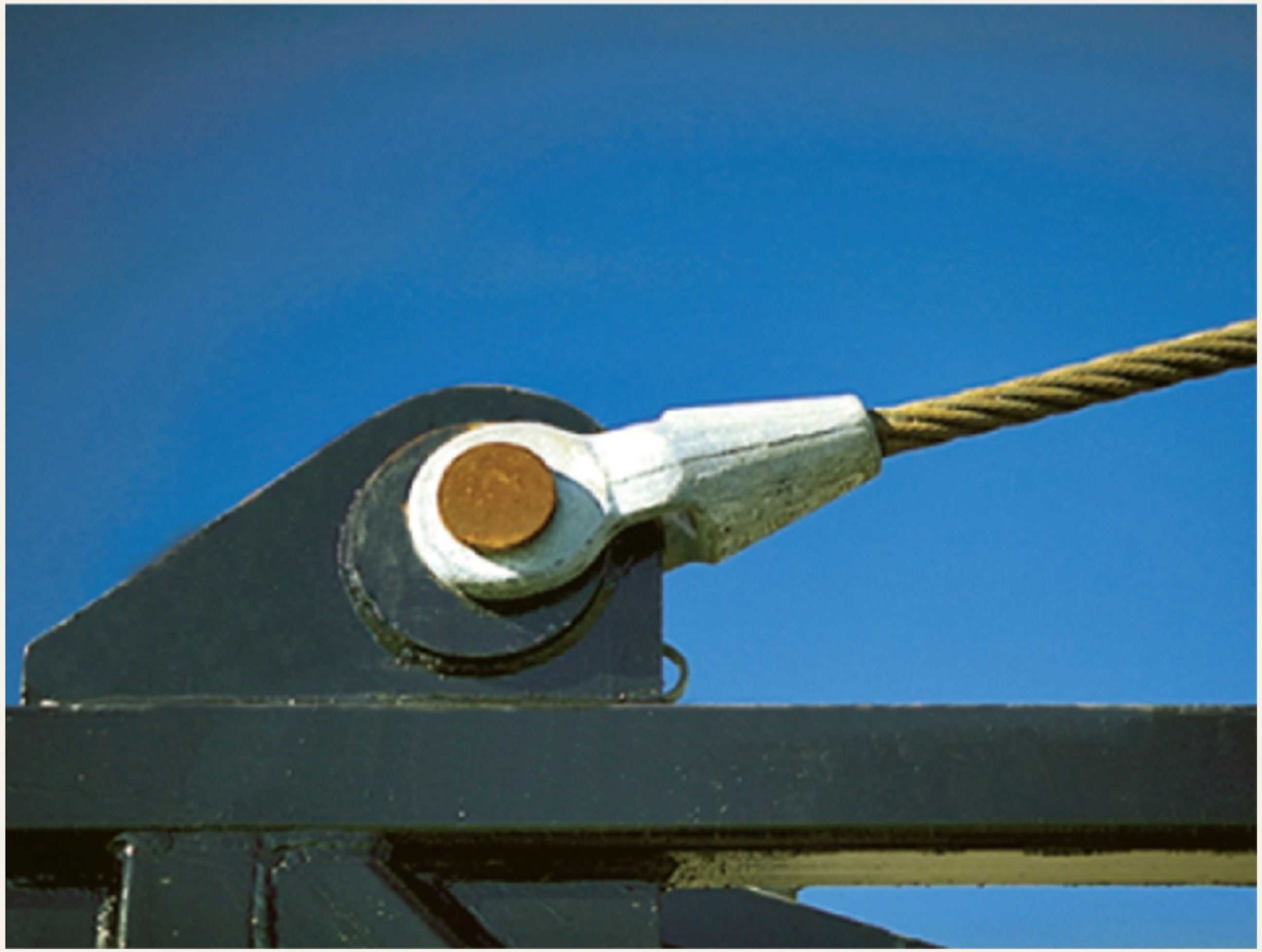


member fixed connected to collar on smooth rod

(10)

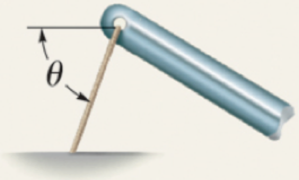


fixed support

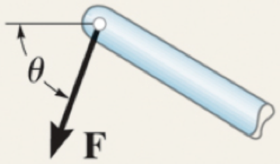


Cable connection

(1)



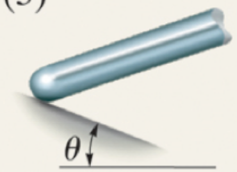
cable



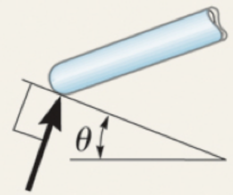
One unknown. The reaction is a tension force which acts away from the member in the direction of the cable.

Contact

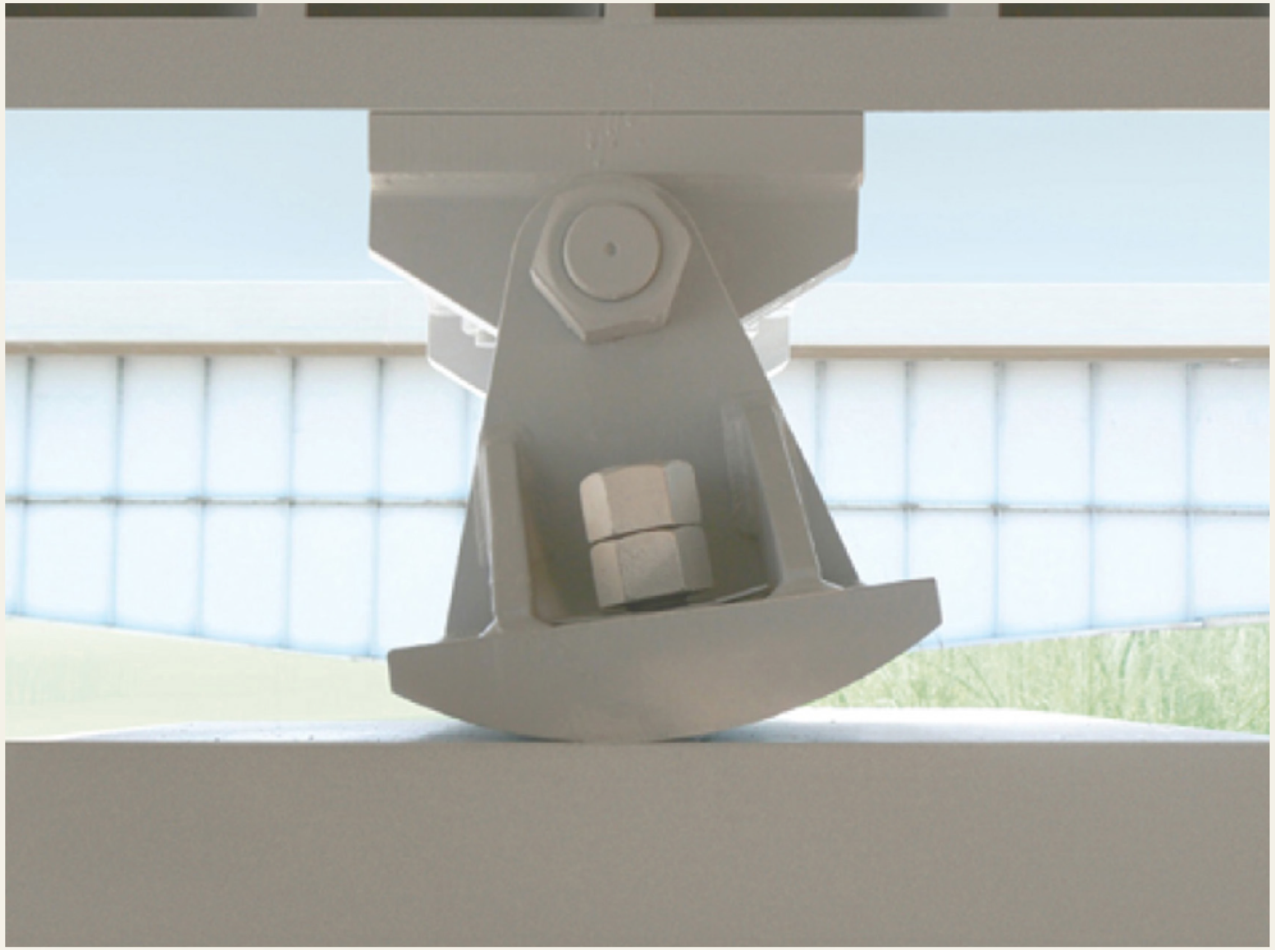
(5)



smooth contacting surface

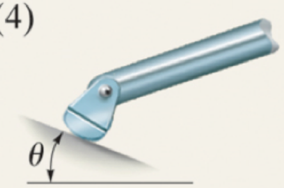


One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.

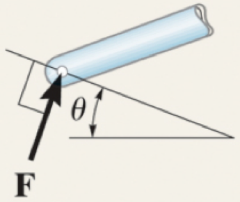


Rocker support

(4)



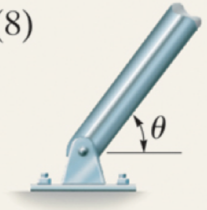
rocker



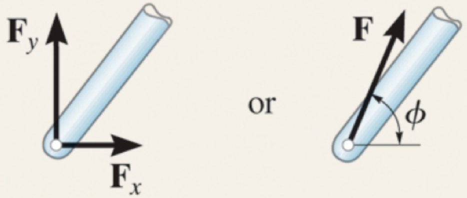
One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.

Pin or hinge

(8)



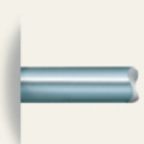
smooth pin or hinge



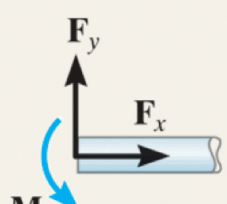
Two unknowns. The reactions are two components of force, or the magnitude and direction ϕ of the resultant force. Note that ϕ and θ are not necessarily equal [usually not, unless the rod shown is a link as in (2)].

Fixed (welded) connection

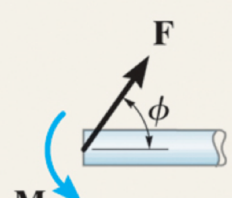
(10)



fixed support



or

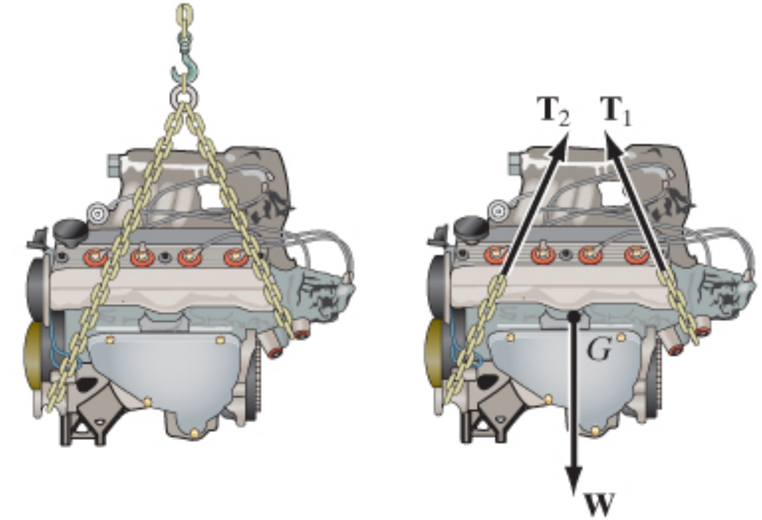


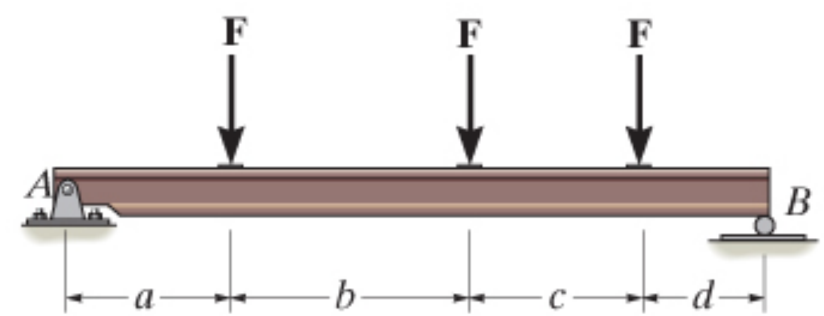
Three unknowns. The reactions are the couple moment and the two force components, or the couple moment and the magnitude and direction ϕ of the resultant force.

Free-body diagram

General rules to follow in a free-body diagram:

- Consider external forces only.
- If the weight is considered, include it as a force acting on the center of gravity.
- Develop an idealized model of the supports.

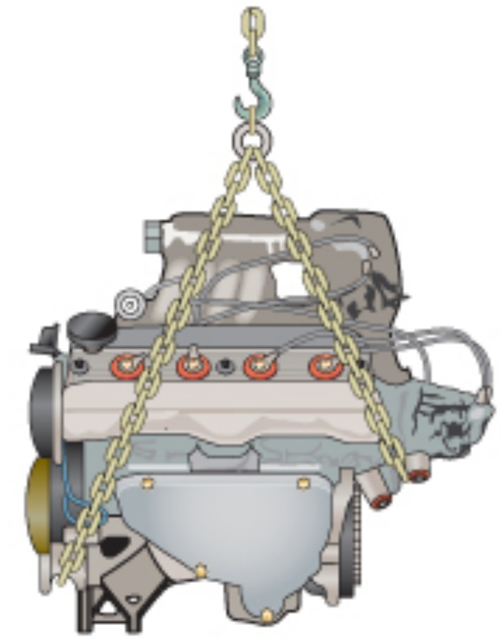


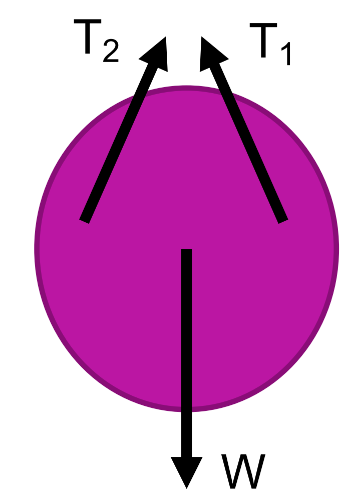


Free-body diagram

Follow this procedure to create a free-body diagram:

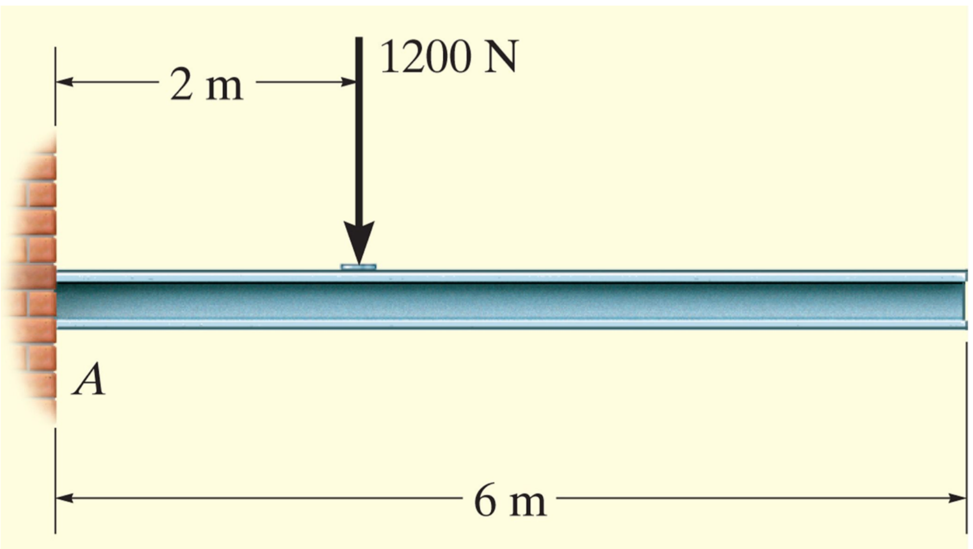
1. Draw the outlined shape of the body isolated or cut from its connections/supports.
2. Show all forces and moments. This includes (i) applied loads and (ii) reactions at the supports. Some will be known and others unknown.
3. Indicate the magnitude and direction of known loads and label important dimensions.

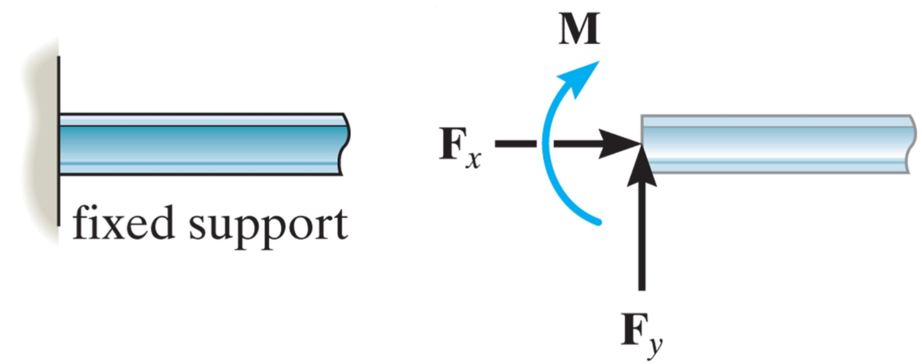


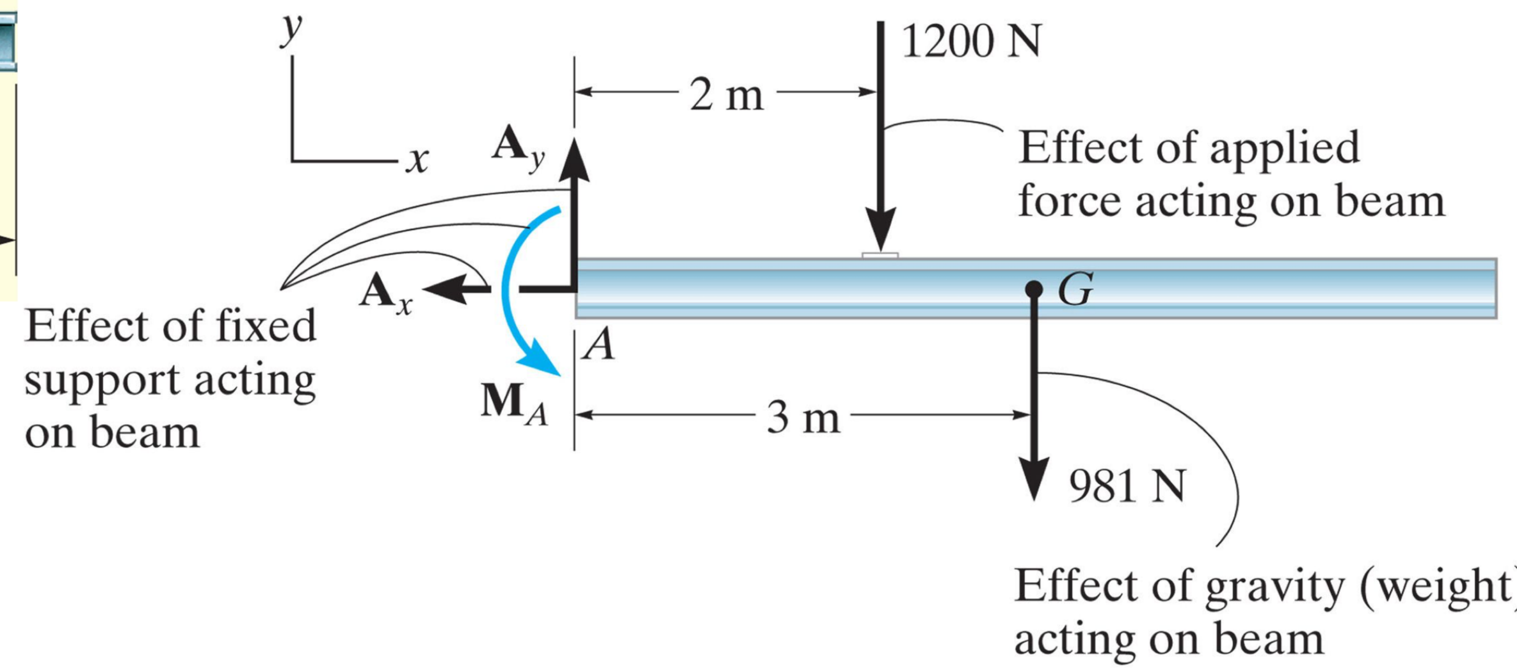


Example: free-body diagram

Draw the free-body diagram of the beam.
The beam has a mass of 100 kg.

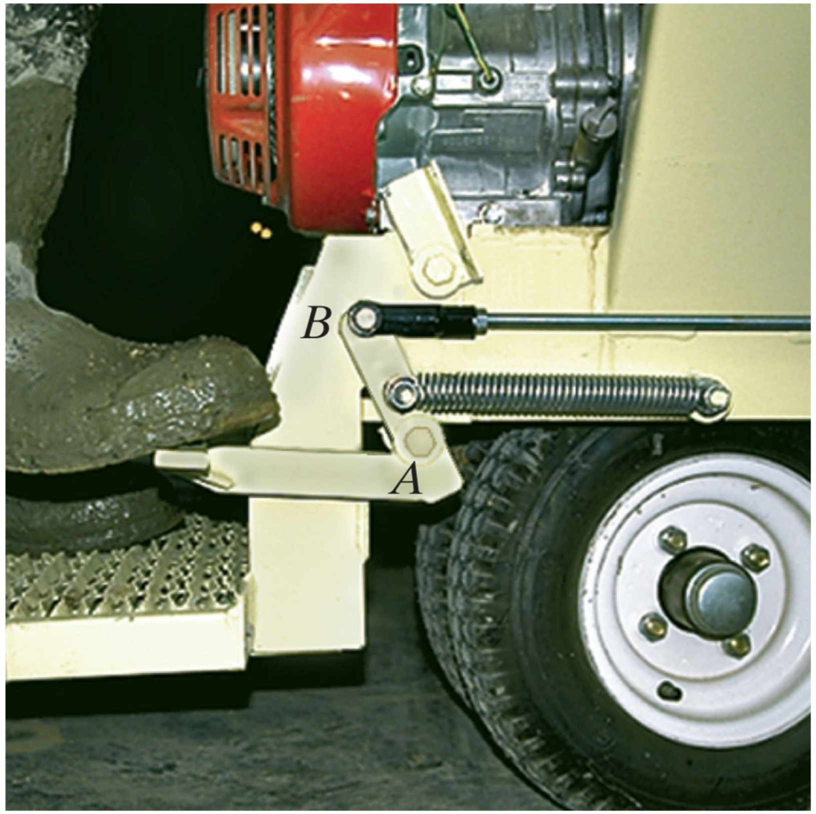


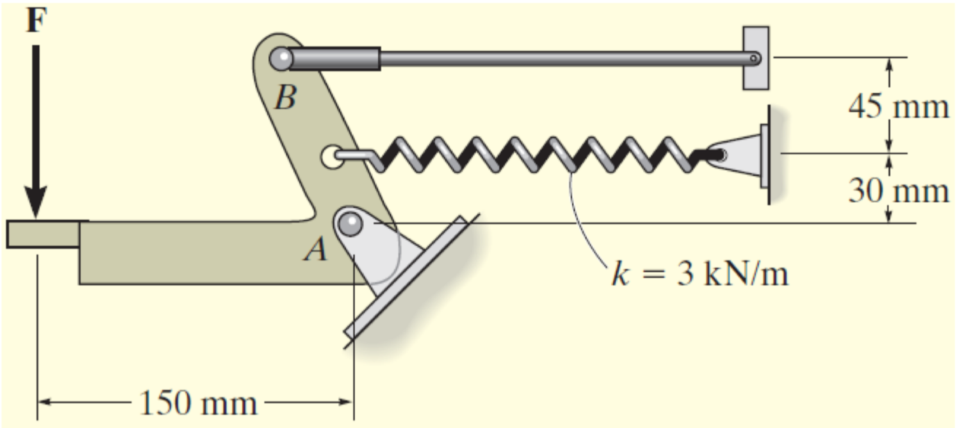


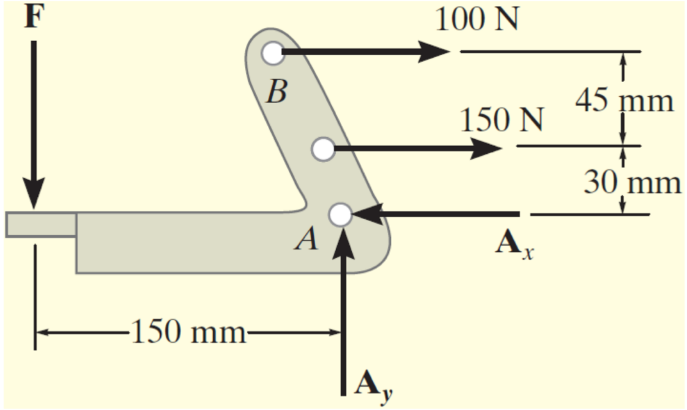


Example: free-body diagram

Draw the free-body diagram of the foot lever shown below. Assume that the spring is stretched by 50 mm, and the force at *B* is 100 N.

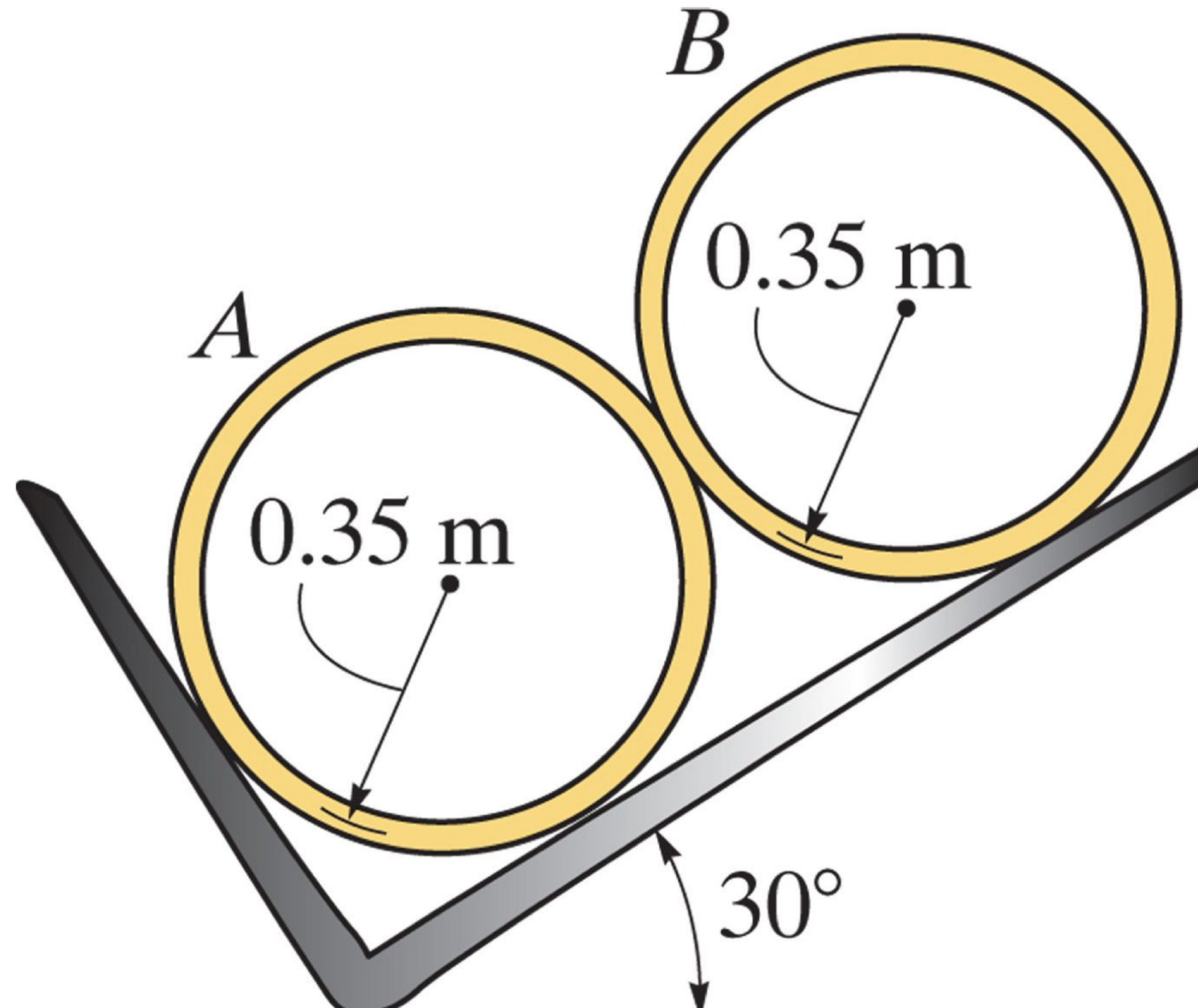




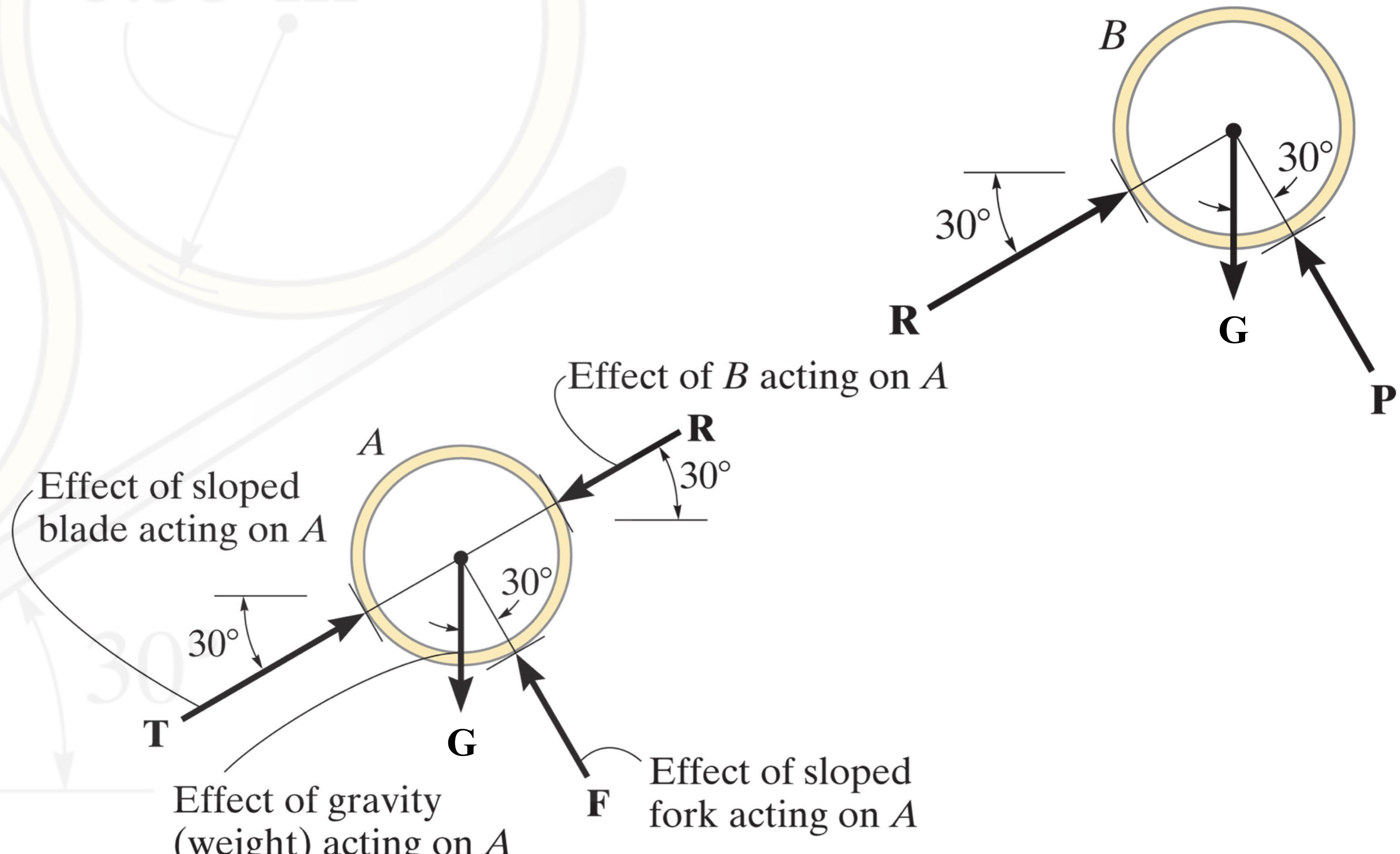


Example: free-body diagram

Draw the free-body diagram for each pipe.

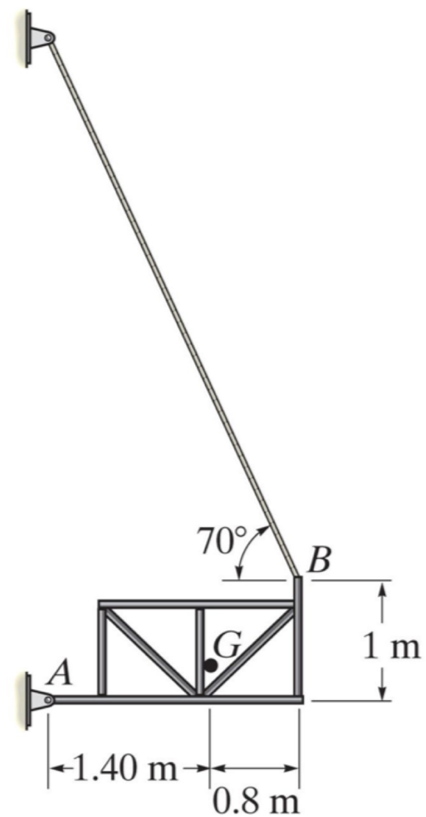


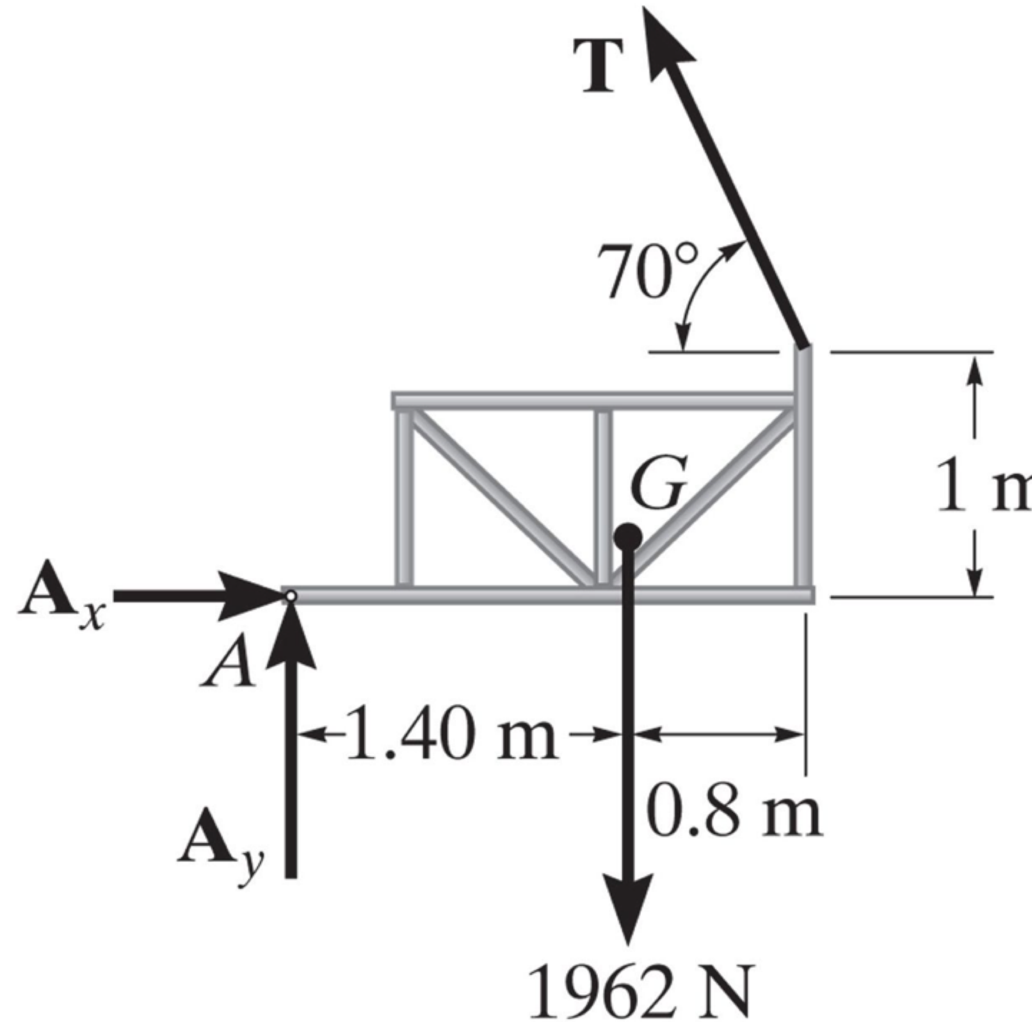
Example: free-body diagram



Example: free-body diagram

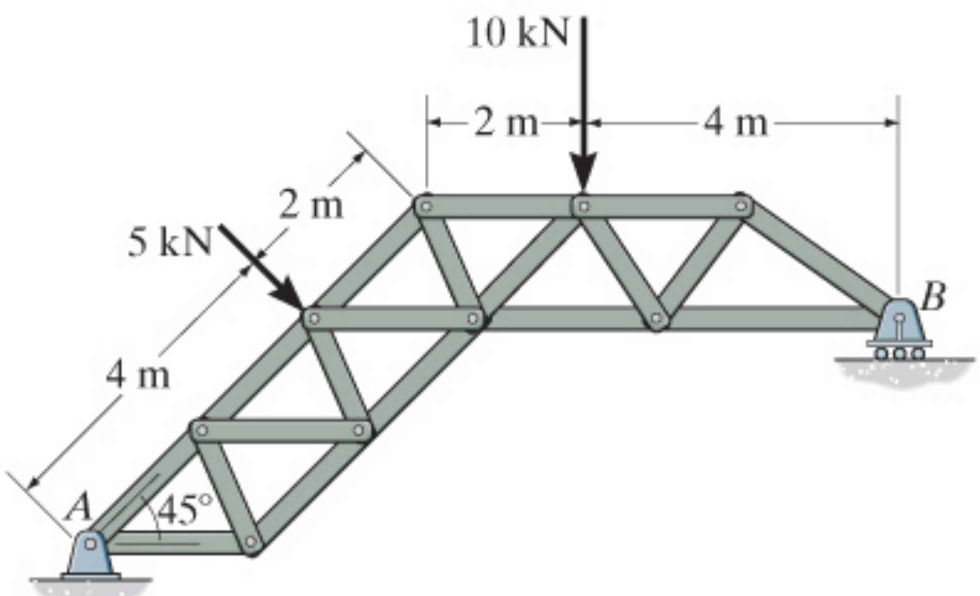
Draw the free-body diagram for this platform of 200 kg.





Example

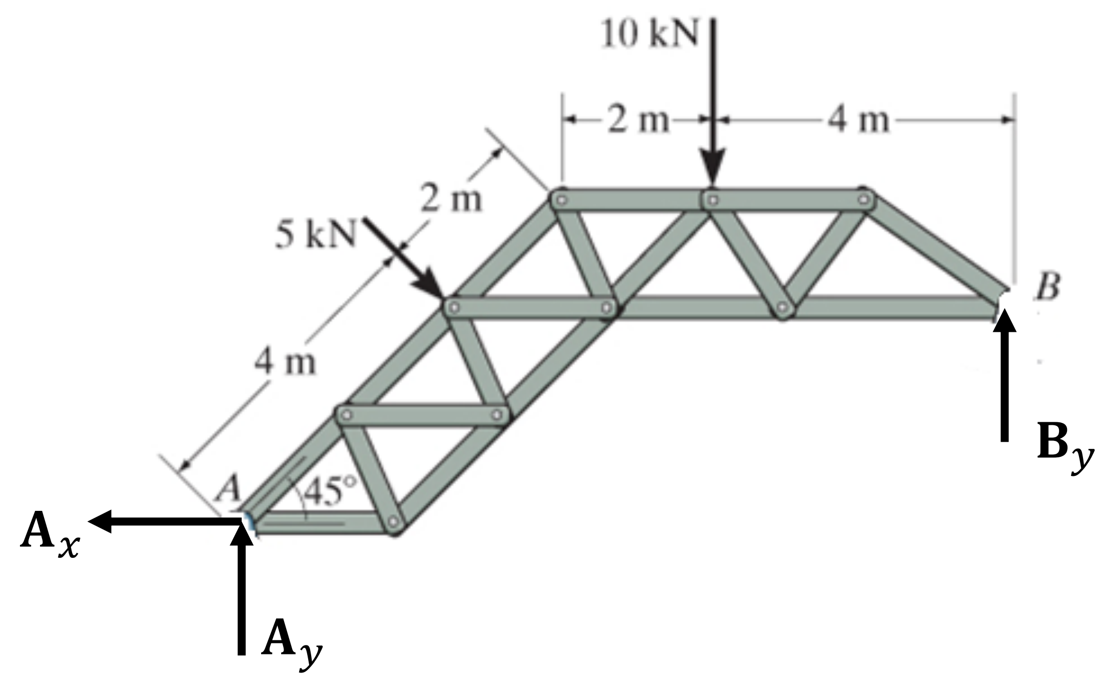
Draw the free-body diagram for the structure below. There is a pin joint at point *A* and a roller at point *B*.



The body is the whole lattice frame.

The pin joint at point A generates forces $\mathbf{A}_x$ and $\mathbf{A}_y$.

The roller at point B generates only a vertical force $\mathbf{B}_y$.



Equilibrium of a rigid body

We can find the reaction forces and moments using the equilibrium equations:

$$\mathbf{F}_R = \Sigma \mathbf{F} = \mathbf{0}$$
$$(\mathbf{M}_R)_O = \Sigma \mathbf{M}_O = \mathbf{0}$$

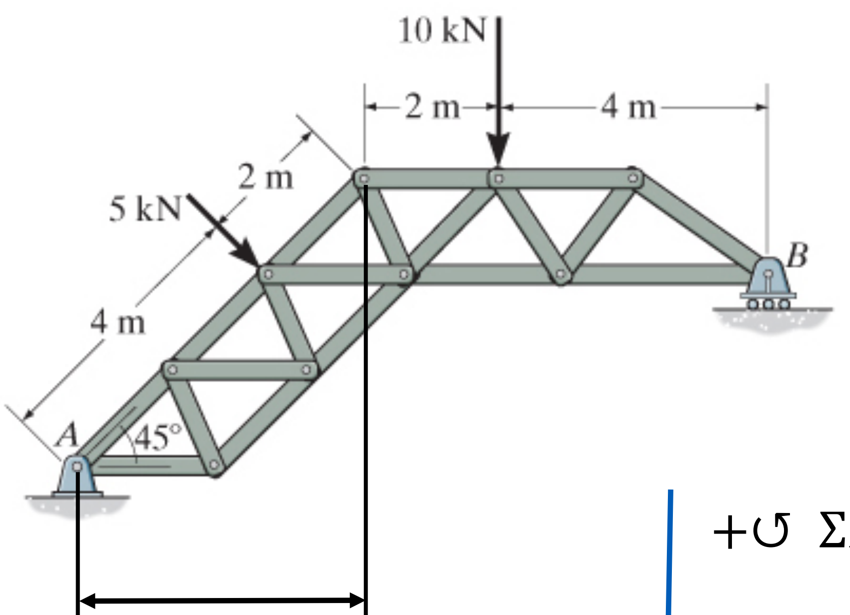
For 2D problems, these can be expressed as scalar equations along the *x*- and *y*-components:

$$\Sigma F_x = 0$$
$$\Sigma F_y = 0$$
$$\Sigma M_O = 0$$

Note that we can use any point O for the sum of moments.

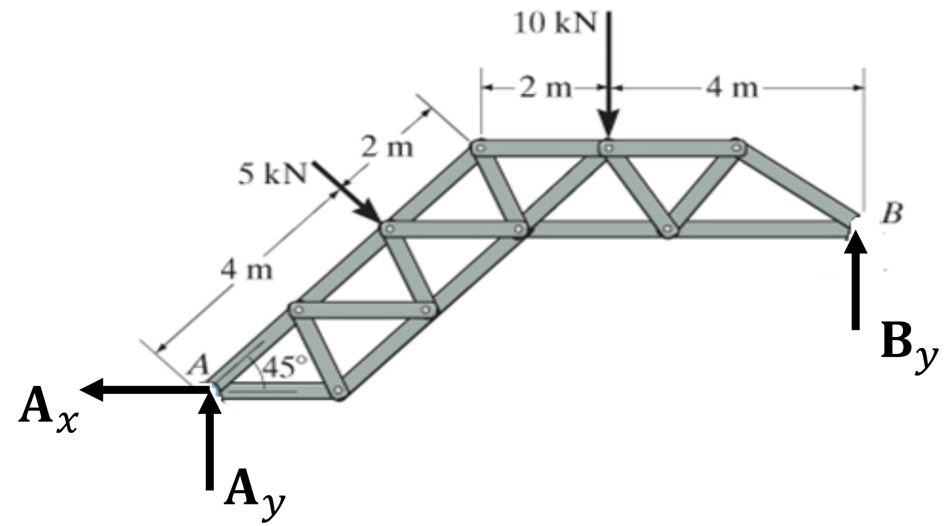
Example

Find the reaction forces at point A and B.



$6\text{m}(\cos 45°) = 3\sqrt{2}\ \text{m}$

Free-body diagram:



Equilibrium equations (3 unknowns & 3 equations):

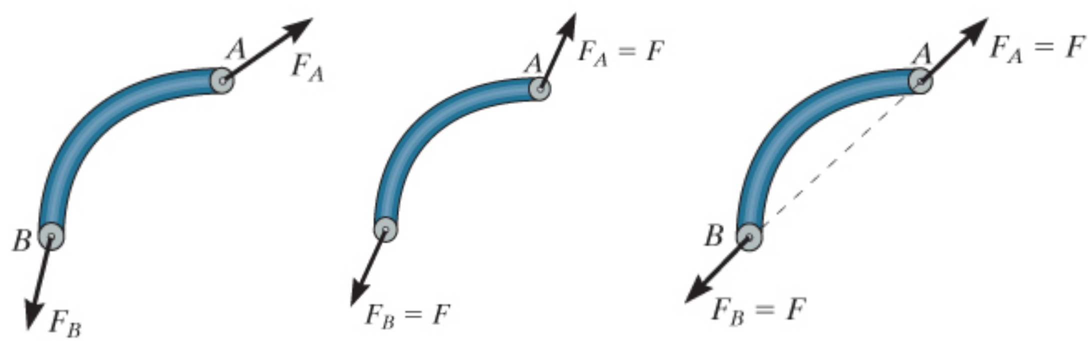
$$\overset{+}{\rightarrow}\ \Sigma F_x = -A_x + 5\text{kN}\cos 45° = 0 \ \Rightarrow\ A_x = 3.54\ \text{kN}$$

$$+\circlearrowleft\ \Sigma M_A = -5\text{kN}(4\text{m}) - 10\text{kN}((3\sqrt{2}+2)\ \text{m})) + B_y((3\sqrt{2}+6)\ \text{m}) = 0$$

$$\Rightarrow\ B_y = 8.05\ \text{kN}$$

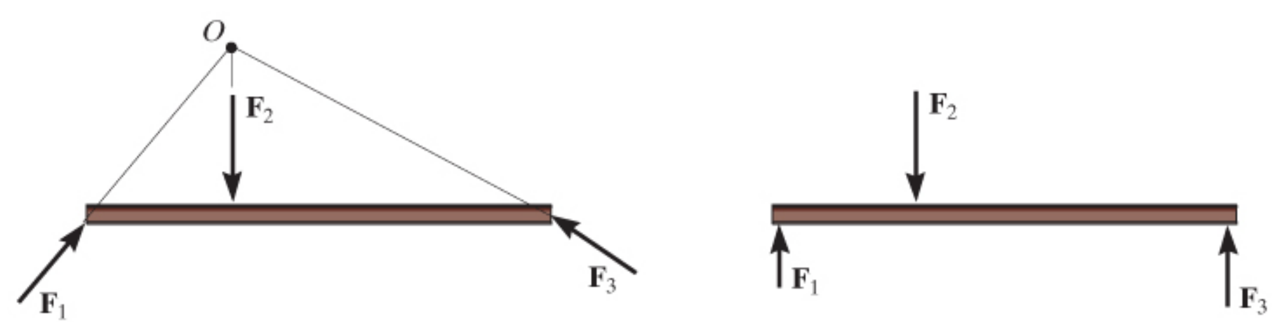
$$+\uparrow\ \Sigma F_y = A_y - 5\text{kN}\sin 45° - 10\text{kN} + 8.05\text{kN} = 0 \ \Rightarrow A_y = 5.49\ \text{kN}$$

Equilibrium of 2- and 3-force members



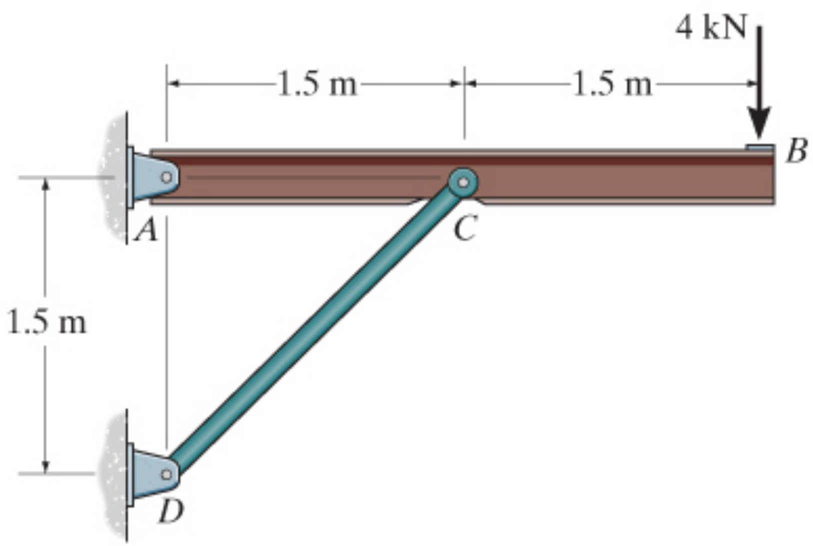
Consider a body with two forces $\mathbf{F}_A$ and $\mathbf{F}_B$. Equilibrium requires that both forces have the **same magnitude,** opposite directions, and the **same line of action**.

With a 3-force member, equilibrium of forces and moments requires that the lines of action of all forces (i) intersect at a point, or (ii) are parallel.

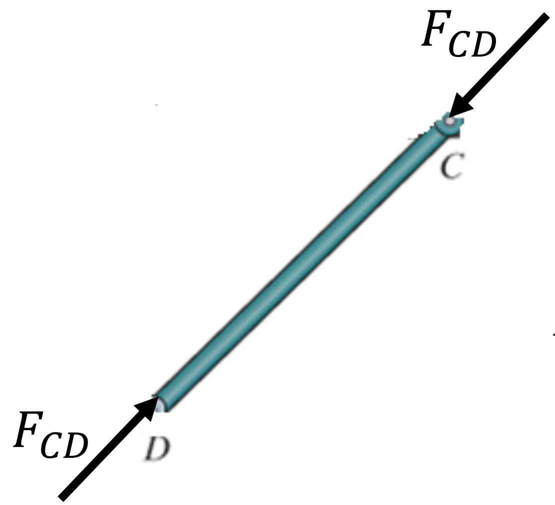


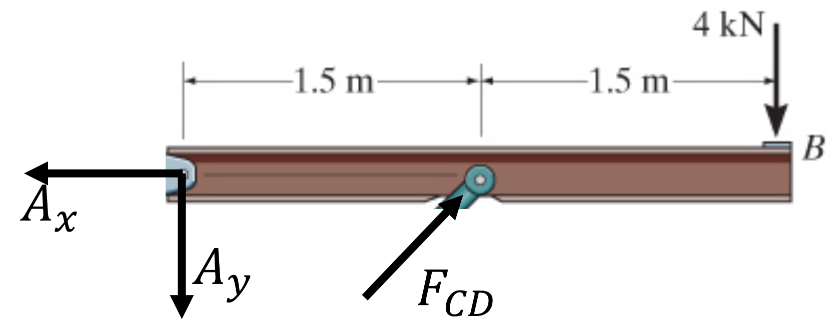
Example

Determine the reaction forces at points A and C.



Free-body diagram (note that bar CD is a 2-force member)





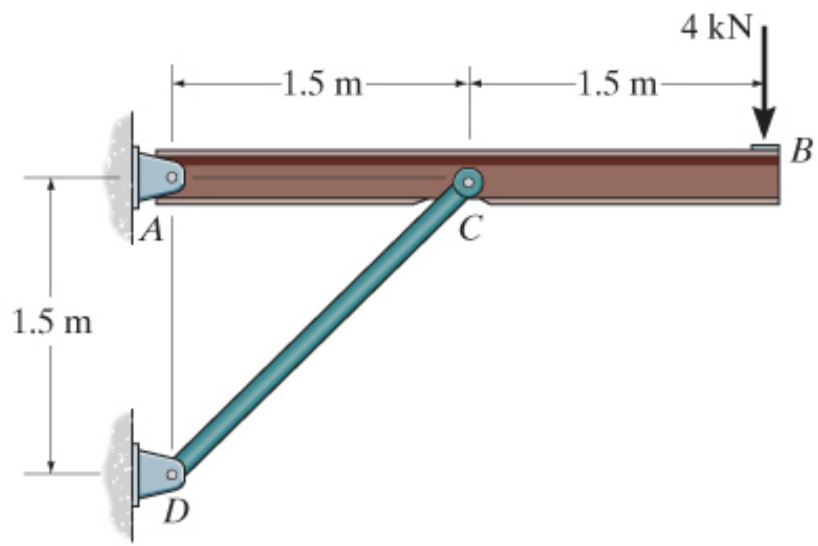
Next, we use equilibirum equations to find the reaction forces:

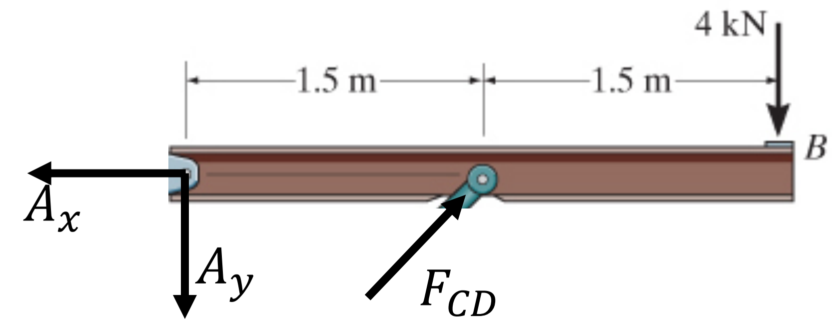
$$\circlearrowleft + \quad \Sigma M_A = 0 \qquad F_{CD} \sin 45° \,(1.5\text{m}) - 4\text{kN}(3\text{m}) = 0$$

$$\Rightarrow F_{CD} = 11.3 \text{ kN}$$

Example

Determine the reaction forces at points A and C.





$\rightarrow + \quad \Sigma F_x = 0:$

$$-A_x + F_{CD}(\cos 45°) = -A_x + 11.3\ \text{kN}(\cos 45°) = 0$$

$$\Rightarrow \quad A_x = 8\ \text{kN}$$

$\uparrow + \quad \Sigma F_y = 0:$

$$-A_y + F_{CD}(\sin 45°) - 4\ \text{kN} = -A_y + 11.3\ \text{kN}(\sin 45°) - 4\ \text{kN} = 0$$

$$\Rightarrow \quad A_y = 4\ \text{kN}$$

Equilibrium in 3D

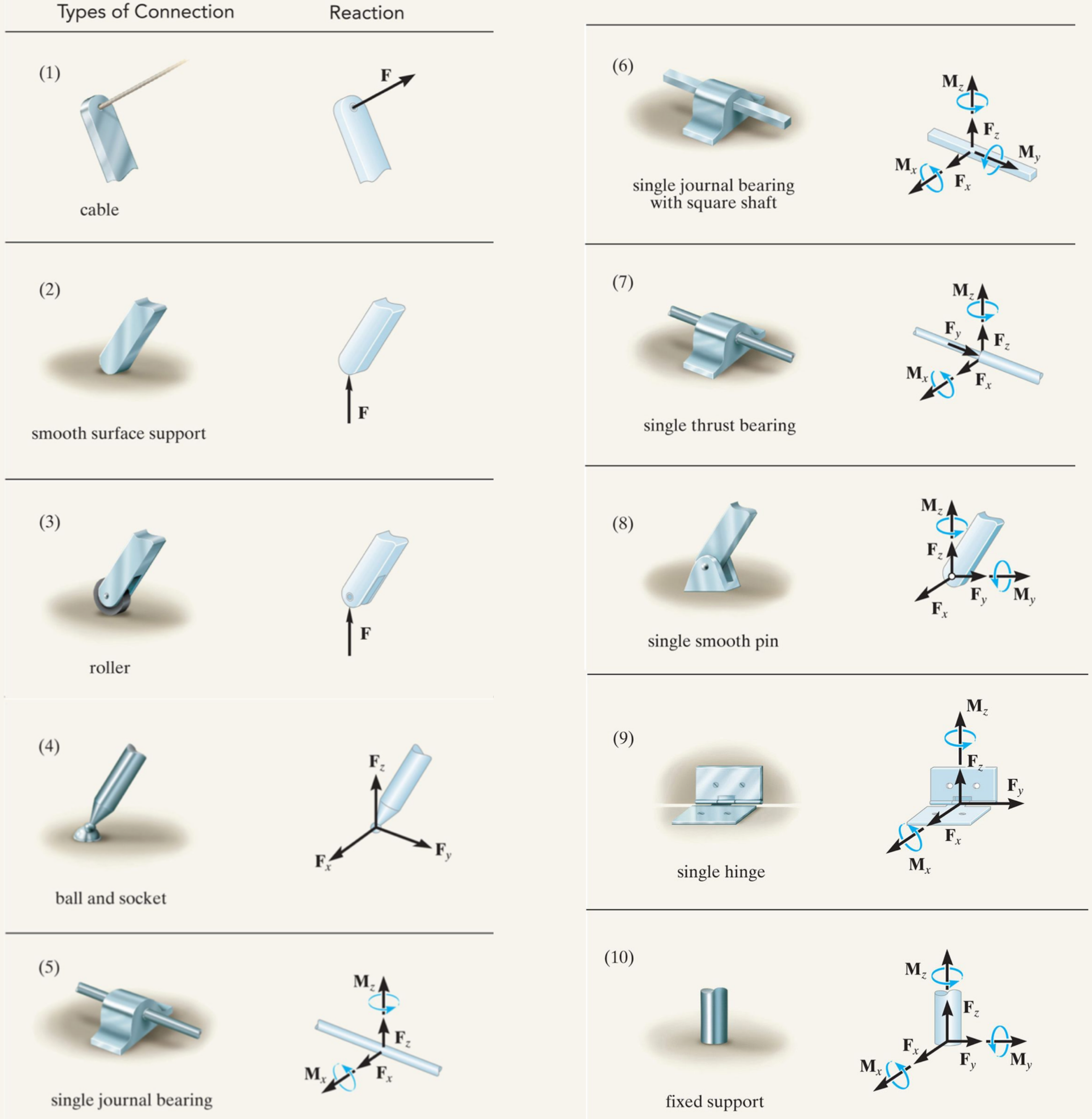
Three dimensional problems, are tackled in the same way: (i) draw a free-body diagram and (ii) apply equilibrium equations:

$$\Sigma\mathbf{F} = \Sigma F_x\mathbf{i} + \Sigma F_y\mathbf{j} + \Sigma F_z\mathbf{k} = \mathbf{0}$$
$$\Sigma\mathbf{M}_O = \Sigma M_x\mathbf{i} + \Sigma M_y\mathbf{j} + \Sigma M_z\mathbf{k} = \mathbf{0}$$

In scalar form, this gives:

$$\boxed{\begin{aligned} \Sigma F_x &= 0 \\ \Sigma F_y &= 0 \\ \Sigma F_z &= 0 \end{aligned}} \qquad \boxed{\begin{aligned} \Sigma M_x &= 0 \\ \Sigma M_y &= 0 \\ \Sigma M_z &= 0 \end{aligned}}$$

Note: in 3D we have 6 equations! (in 2D, we have 3 equations)



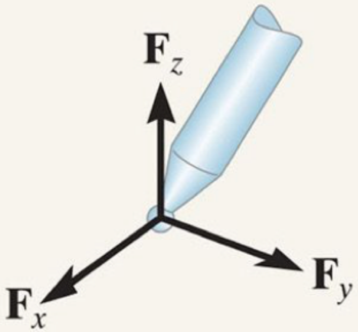
Support reactions in 3D.

Hibbeler Table 5-2

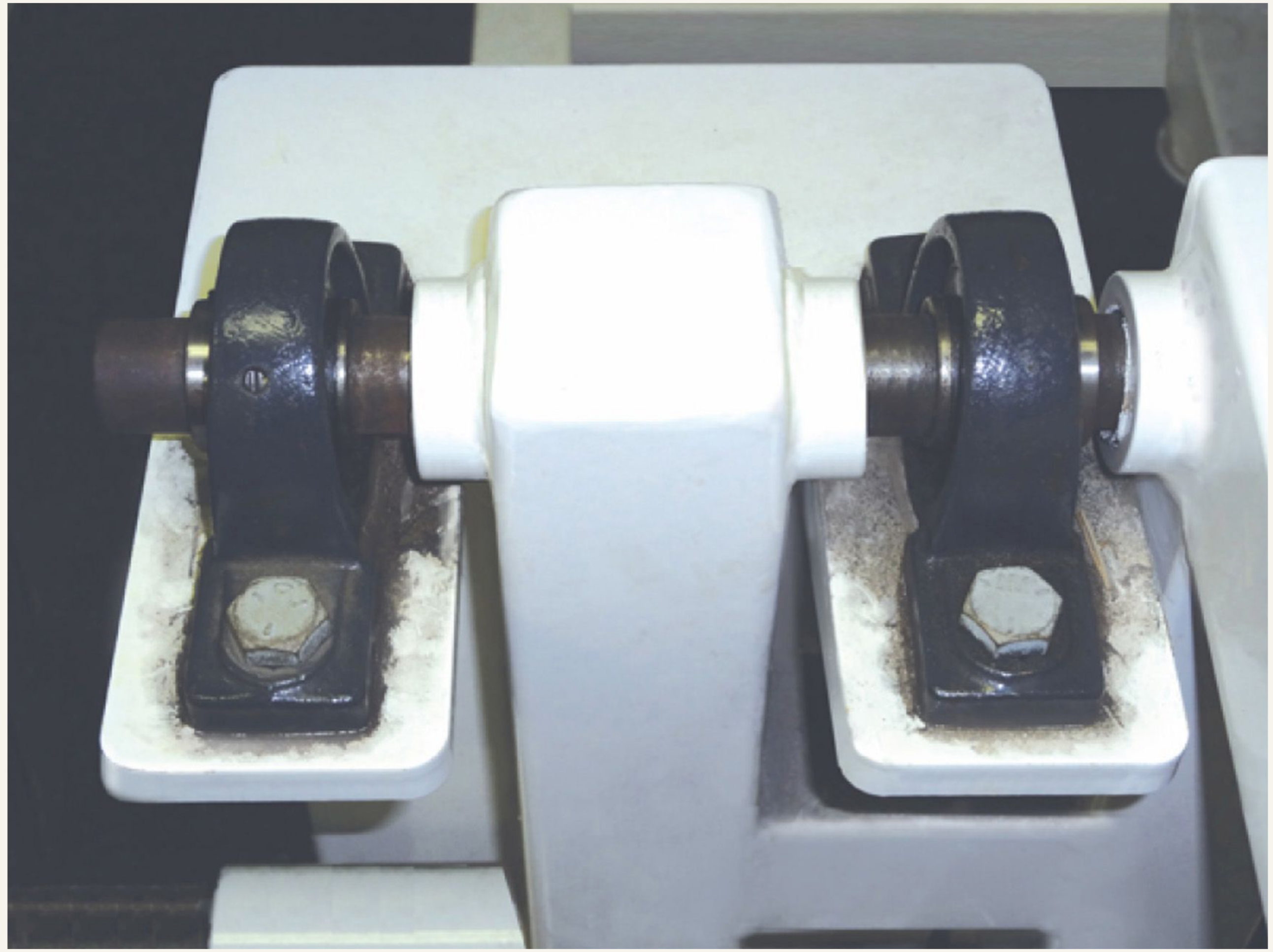
Ball socket

(4)

ball and socket



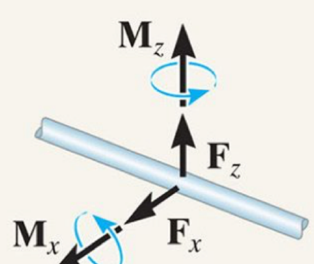
Three unknowns. The reactions are three rectangular force components.



Bearing

(5)

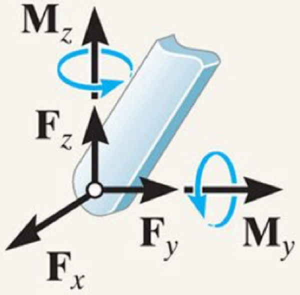
single journal bearing



Four unknowns. The reactions are two force and two couple-moment components which act perpendicular to the shaft. Note: The couple moments are *generally not applied* if the body is supported elsewhere. See the examples.

Pin joint

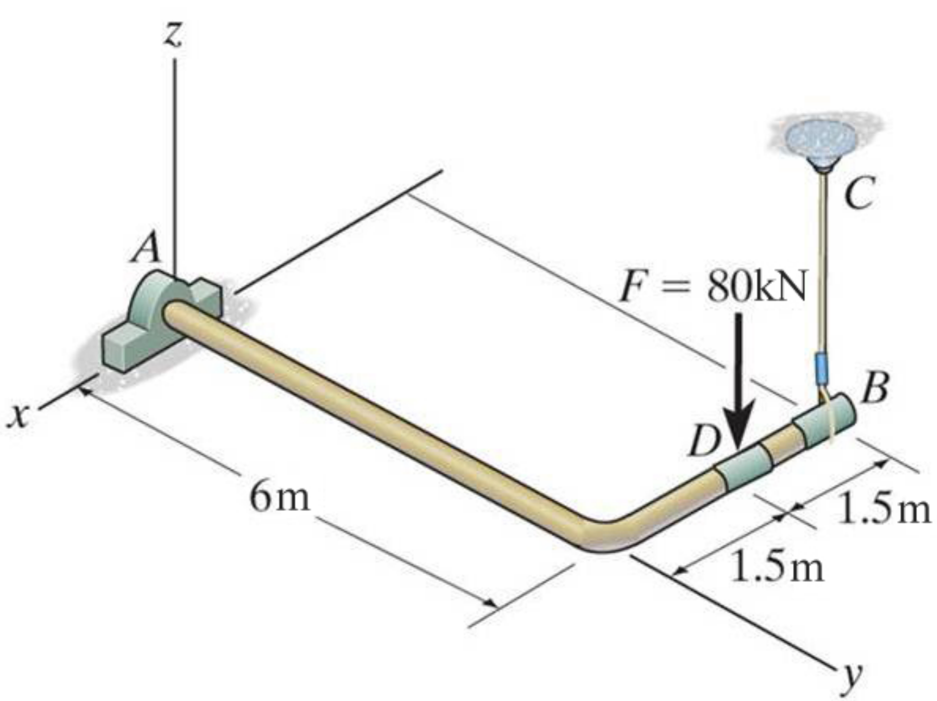
single smooth pin



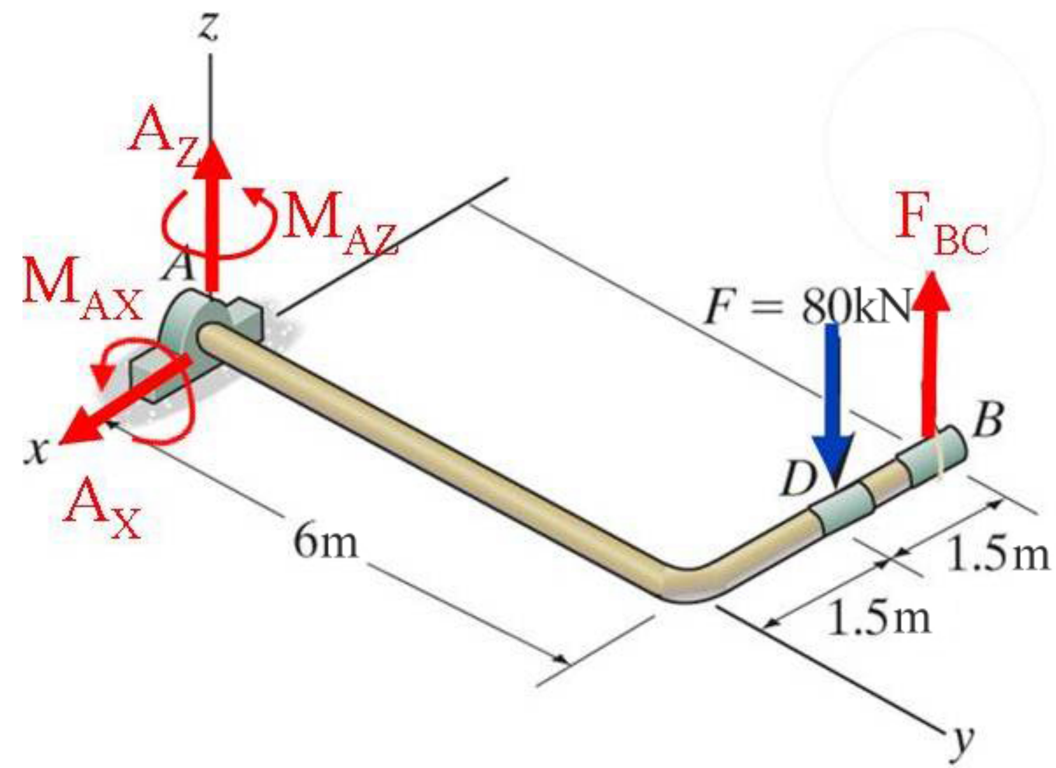
Five unknowns. The reactions are three force and two couple-moment components. *Note*: The couple moments *are generally not applied* if the body is supported elsewhere. See the examples.

Example

Find the support reactions at point A and the force in cable BC.



First, draw the free-body diagram. Second, use equilibrium equations to find the support reactions.

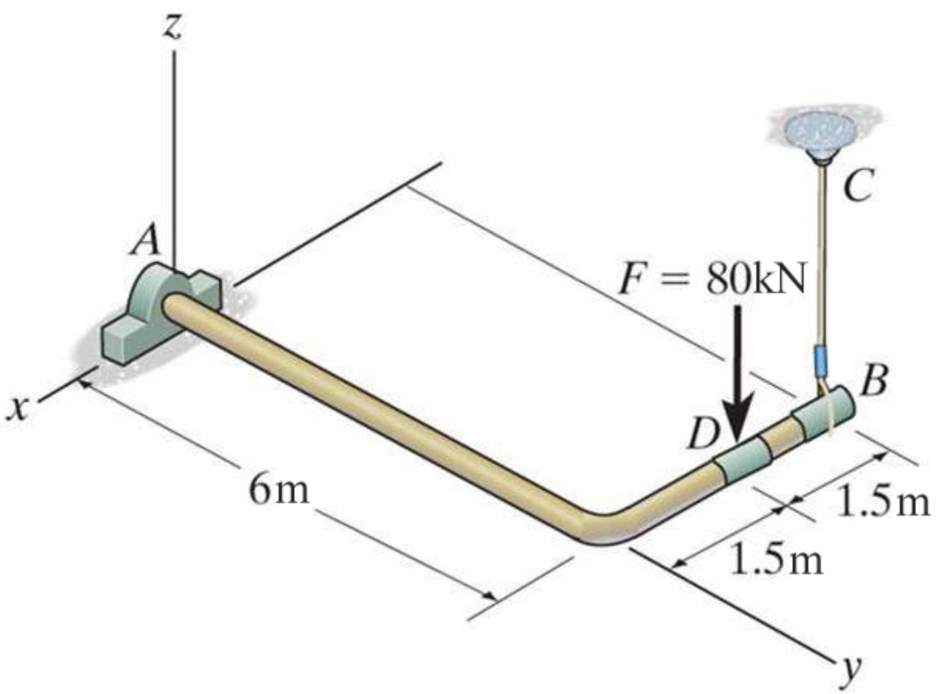


$$\Sigma F_x = 0$$
$$\Sigma F_y = 0$$
$$\Sigma F_z = 0$$

$$\Sigma M_x = 0$$
$$\Sigma M_y = 0$$
$$\Sigma M_z = 0$$

Example

Find the support reactions at point A and the force in cable BC.



$\sum F_X$: $A_X = 0$;

$\sum F_Y$: -

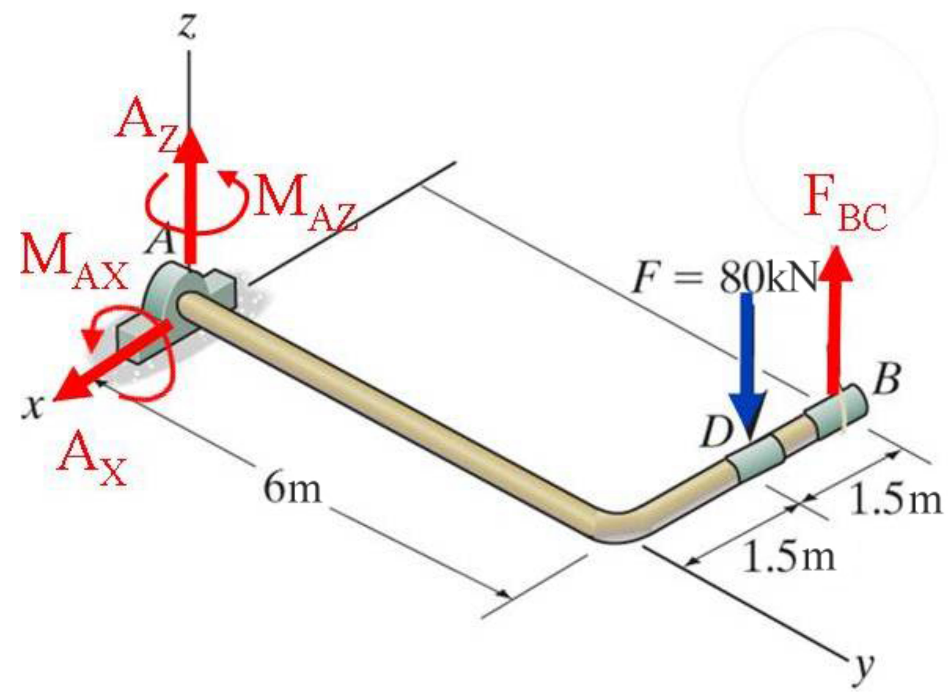
$\sum F_Z$: $A_Z + F_{BC} - 80 = 0$;

$\sum M_Y$: $-80(1.5) + F_{BC}(3.0) = 0$;

$F_{BC} = 40$ kN, $A_Z = 40$ kN

$\sum M_X$: $M_{AX} + 40(6) - 80(6) = 0$; $(M_A)_X = 240$ kN m

$\sum M_Z$: $M_{AZ} = 0$;

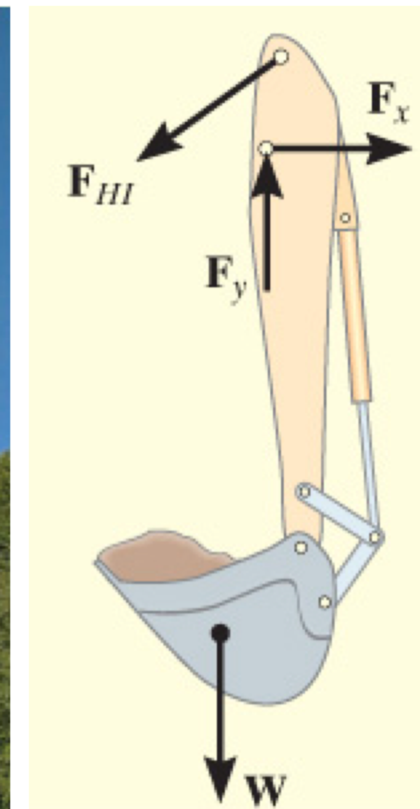


Constraints & statistical determinacy

In some cases, it is not possible to solve the unknowns with the equilibrium equations. It may be because the system

- is **Statistically indeterminate**: that is when a system has more supports than necessary.

- or has **Improper constraints**: when the supports are inadequate to guarantee equilibrium.



Statistical determinacy

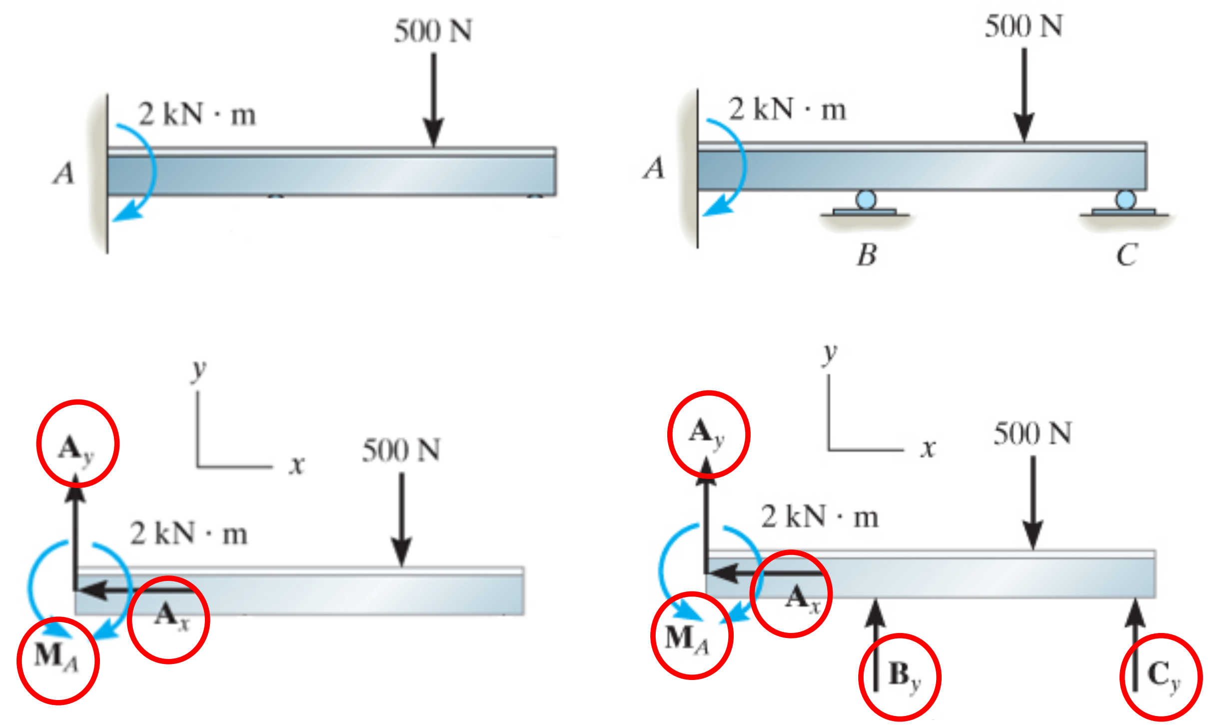
For 2D problems, there are 3 equations of equilibrium:

$$\Sigma F_x = 0$$
$$\Sigma F_y = 0$$
$$\Sigma M_O = 0$$

With 3 equations, we can solve 3 support reactions.

If there are more than 3 support reactions, the system is said to be **statistically indeterminate**.

(Note: for 3D problems, we have 6 equilibrium equations; consequently, we can solve 6 support reactions.)



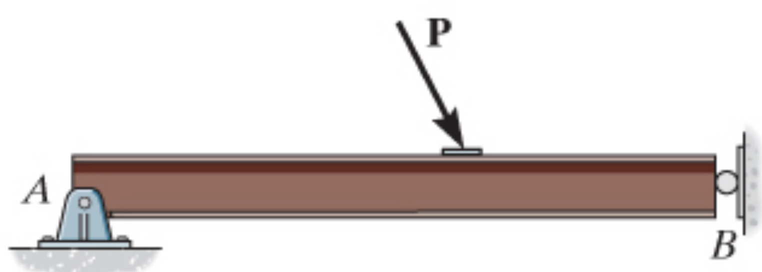
Statistically determinate **Statistically indeterminate**

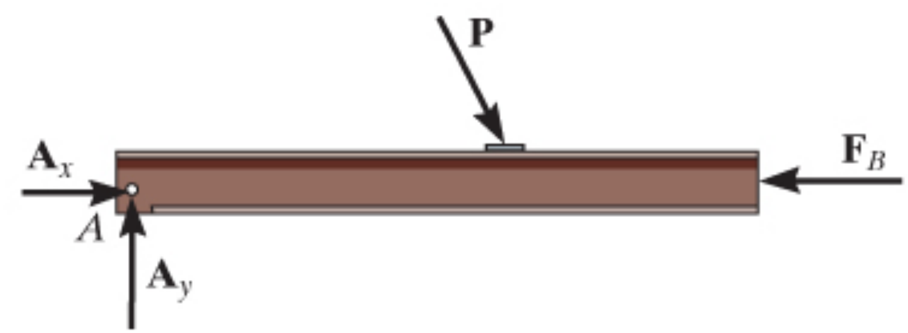
The additional equations needed to solve a statistically indeterminate system come from mechanics of materials (beyond the scope of this course).

Improper constraints

Having the same number of support reactions and equilibrium equations does not necessarily guarantee equilibrium.

Consider the beam below. The sum of moments at point A will show that: $\Sigma M_A \neq 0$. The system is **improperly constrained**.

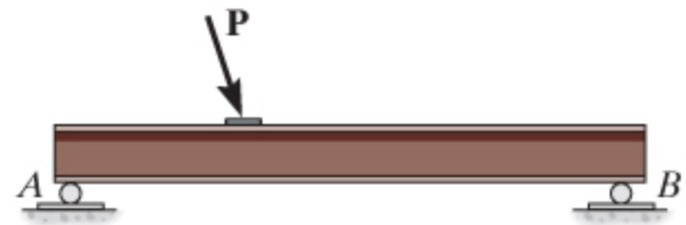


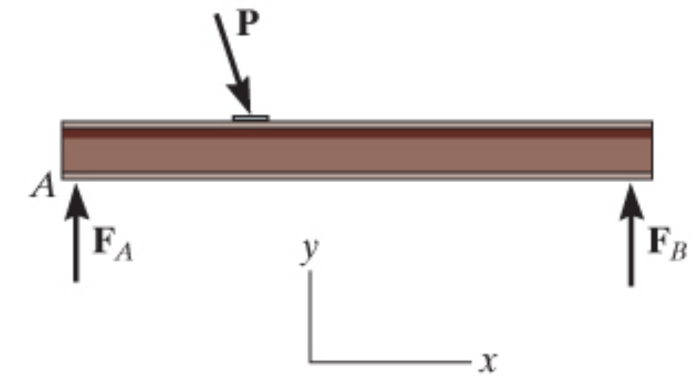


Improper constraints

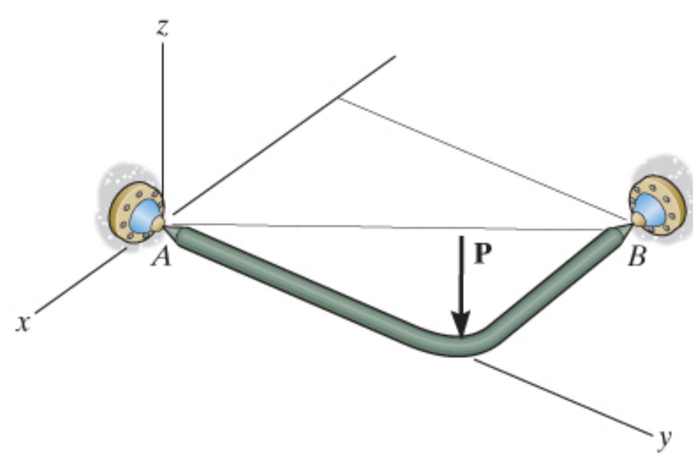
Here are two other examples of improperly constrained systems.

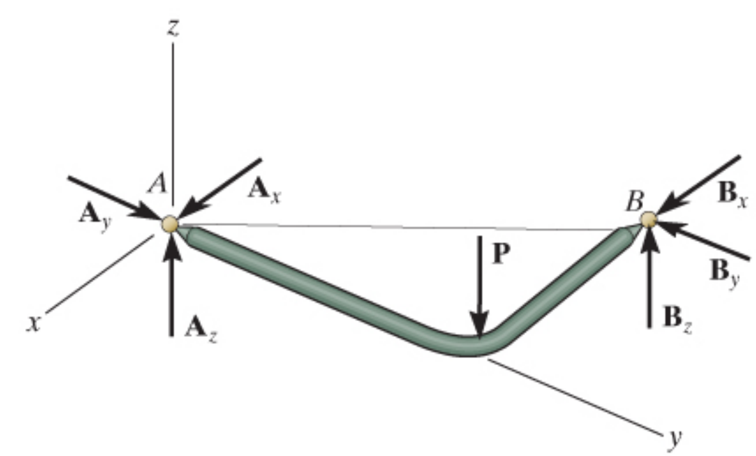
$$\boxed{\Sigma F_x \neq 0}$$





$$\boxed{\Sigma M_{AB} \neq 0}$$



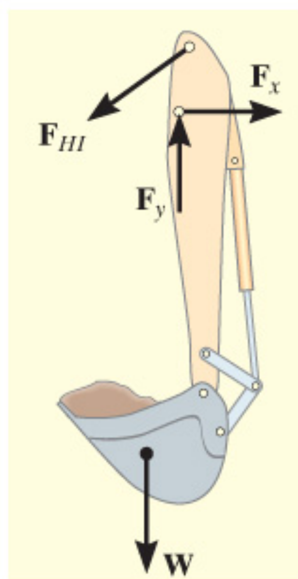


Equilibrium of structures and machines

Why is it important to find the support reactions?

It is necessary to find the support reactions to then dimension the parts adequately (to ensure they don't break or undergo large deformations).

Equilibrium of structures and machines



In all cases, the procedure to analyse the equilibrium of structures and machine is:

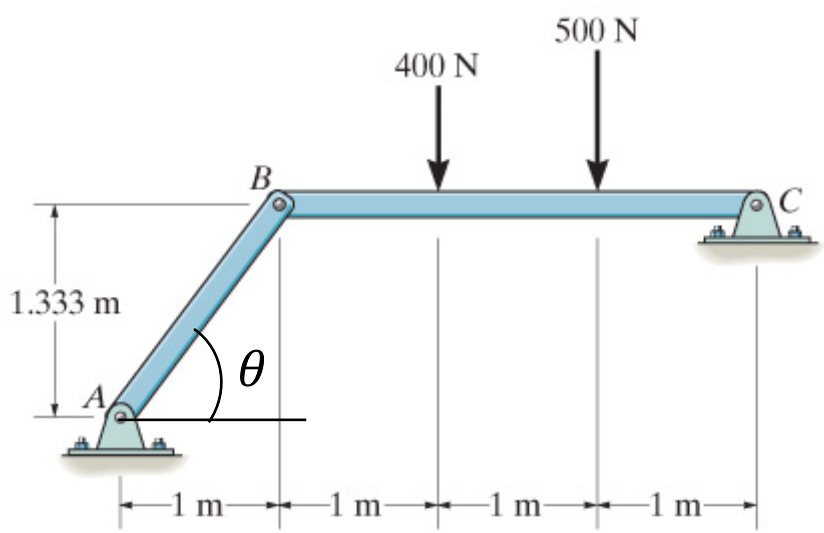
1. Divide the system in one or more components of interest.

2. Create one or more **free-body diagrams** for each part:
   - Draw an outlined shape of the body, without supports.
   - Show all forces and moments, and support reactions.
   - Label known and unknown forces/moments, and define x, y, z axes.
   - Indicate dimensions for computing moments.

3. Solve unknowns using **equilibirum equations.**

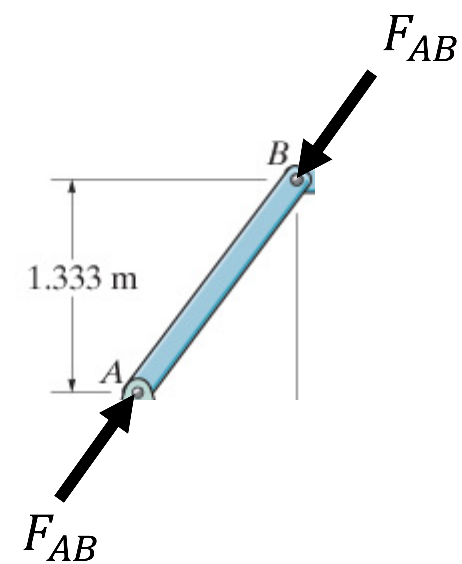
Example

Find the support reactions at point *C*.

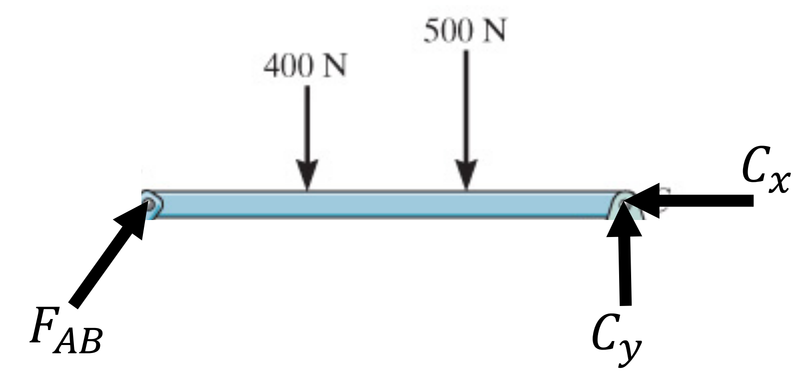


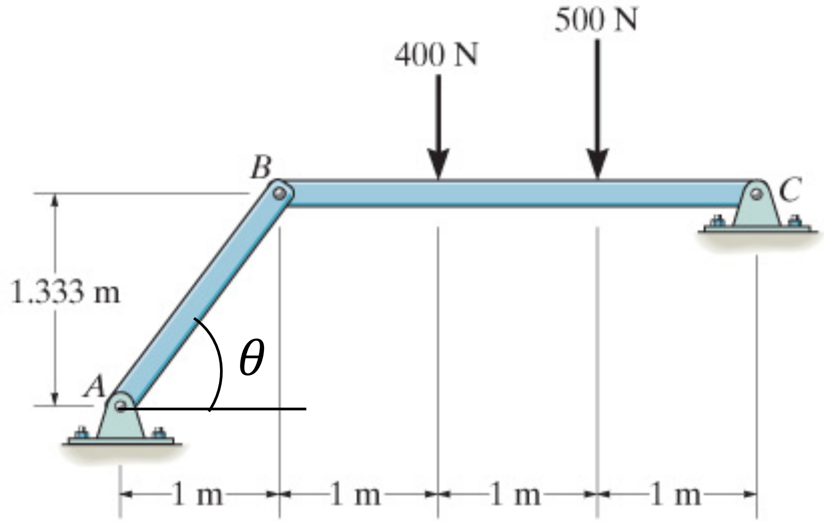
Draw a free-body diagram for each bar.

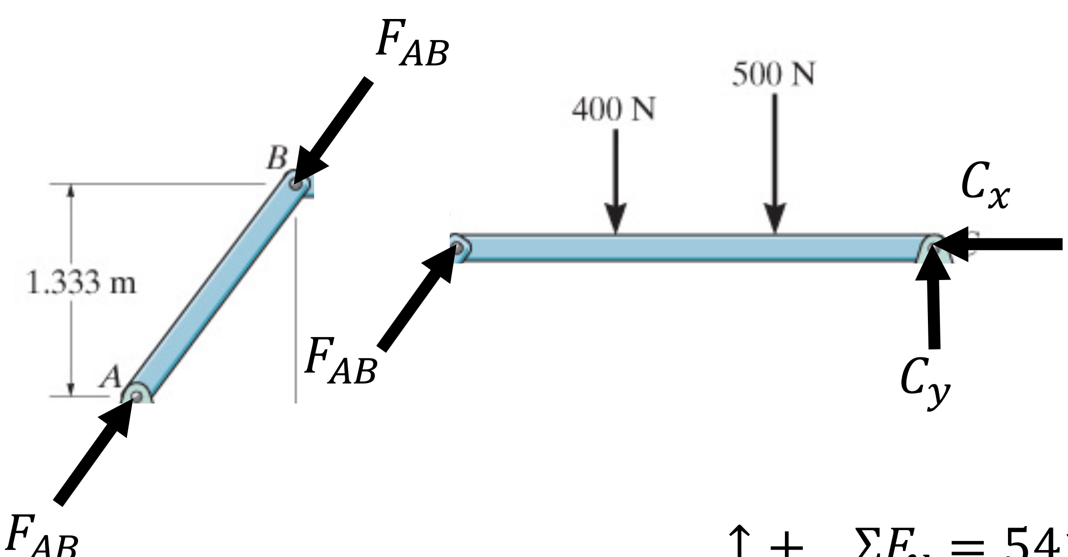
Bar *AB* is a two-force member:



Bar *CB*:







All unknown forces can be solved from bar *CB:*
(we have 3 unknowns & 3 equations)

The angle θ is:

$$\theta = \tan^{-1}\left(\frac{1.333}{1}\right) = 53.12°$$

$$\circlearrowleft + \quad \Sigma M_C = 0$$

$$-F_{AB}(\sin 53.12°)(3\text{m}) + 400\text{N}(2\text{m}) + 500\text{N}(1\text{m}) = 0$$

$$\boxed{F_{AB} = 541.7 \text{ N}}$$

Sum of forces in x and y:

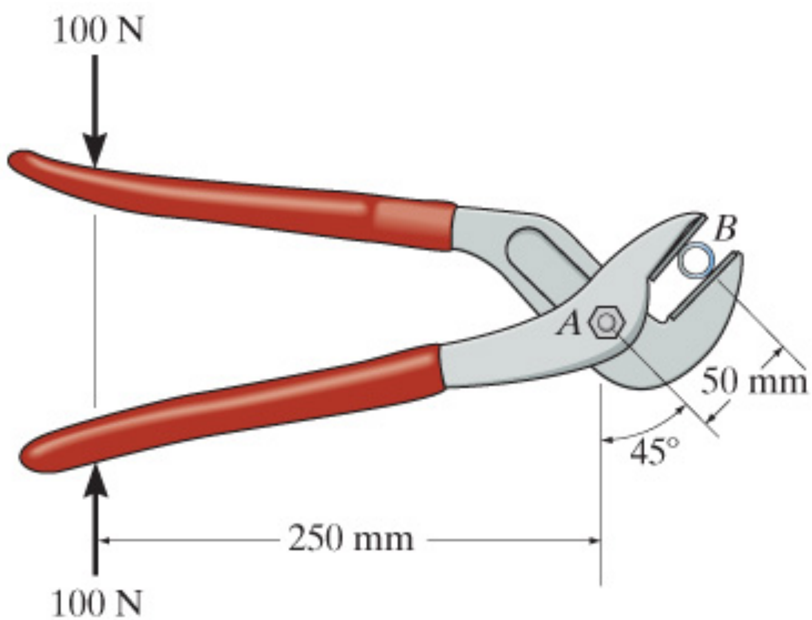
$$\rightarrow + \quad \Sigma F_x = 0 \qquad 541.7 \text{ N}(\cos 53.12°) - C_x = 0$$

$$\boxed{C_x = 325 \text{ N}}$$

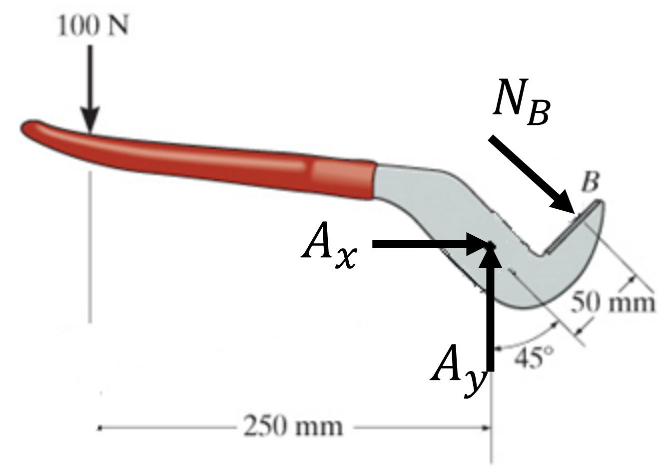
$$\uparrow + \quad \Sigma F_y = 541.7 \text{ N}(\sin 53.1232°) - 400 - 500 + C_y = 0 \qquad \boxed{C_y = 467 \text{ N}}$$

Example

Find the force generated at point *B* and the support reactions at point *A*.



Free-body diagram for one arm:



The contact force at point *B* can be obtained with the sum of moments at point *A*:

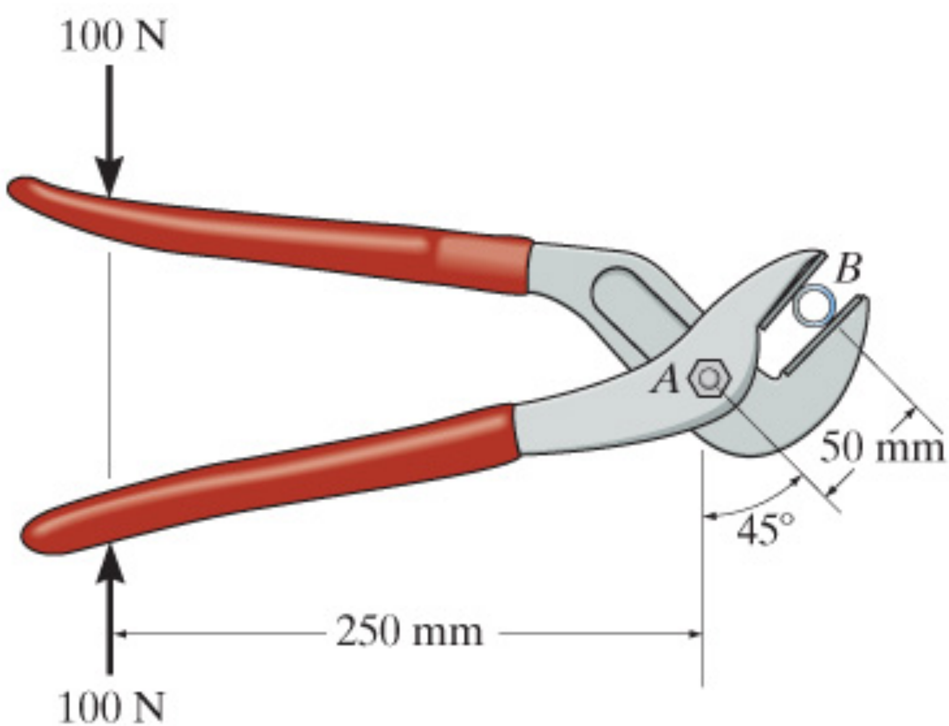
$$\circlearrowleft + \quad \Sigma M_A = 0$$

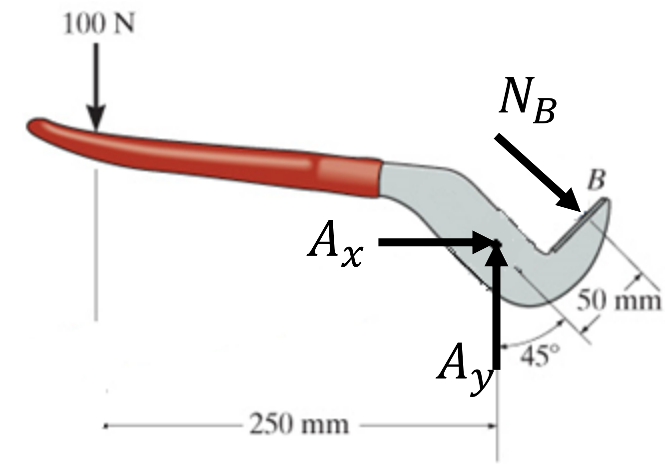
$$100\text{N}(250\text{mm}) - N_B(50\text{mm}) = 0$$

$$\boxed{N_B = 500 \text{ N}}$$

Example

Find the force generated at point B and the support reactions at point A.





The reaction forces at point A are obtained with the sum of forces:

$$\xrightarrow{+} \quad \Sigma F_x = 0 \qquad A_x + 500\ \text{N}(\sin 45°) = 0 \qquad A_x = -353.55\ \text{N}$$

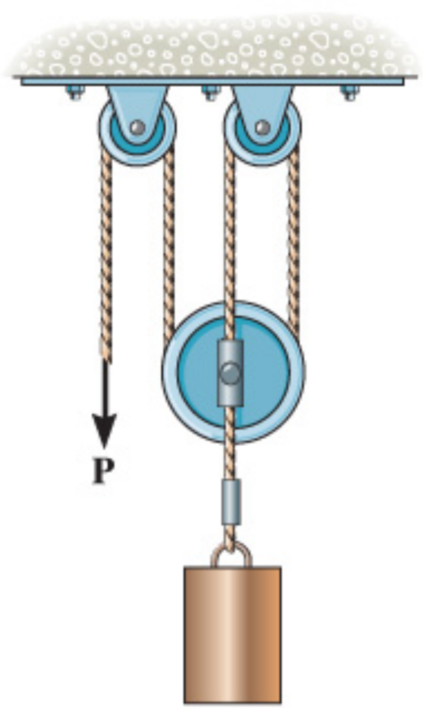
$$\uparrow + \quad \Sigma F_y = 0 \qquad A_y - 100\text{N} - 500\ \text{N}(\cos 45°) = 0 \qquad A_y = 453.55\ \text{N}$$

The magnitude of the force is:

$$F_A = \sqrt{A_x^2 + A_y^2} \quad = \sqrt{(-353.55\text{N})^2 + (453.55\text{N})^2} \quad \boxed{= 575\ \text{N}}$$

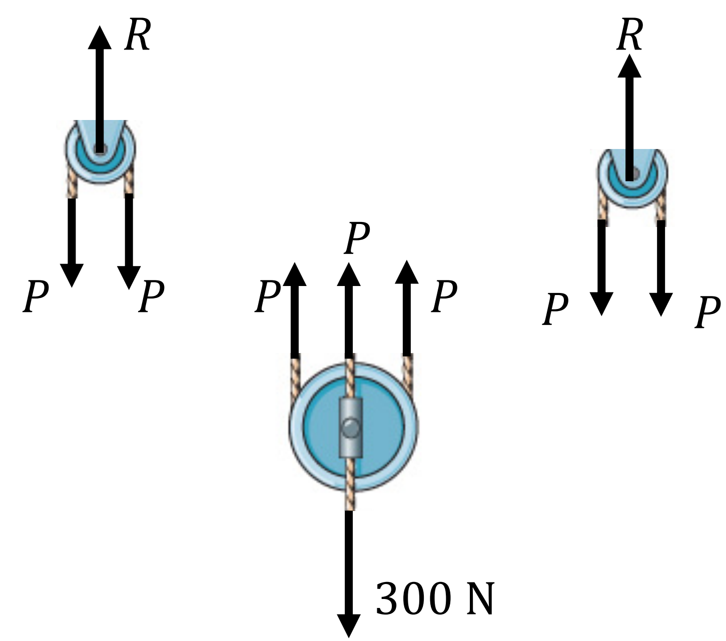
Example

Find the force P, considering that the mass weighs 300 N.



Here, we assume that the mass of the rope is negligible and there is no friction in the pulleys. There is a single rope and its tension P is the same everywhere.

Draw a free-body diagram for each pulley:



The tension P is easily obtained from the lower pulley:

$$\uparrow + \ \Sigma F_y = 0$$

$$3P - 300\ \mathrm{N} = 0$$

$$P = 100\ \mathrm{N}$$

Summary

- We saw that different types of supports create different reaction forces/moments.
- We learned how to create a free-body diagram.
- We can then use equilibrium equations to find unknown support reactions.

Need more explanations?

This lecture covered all of chapter 5: Equilibrium of a rigid body.