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COE-C1001: Dynamics

9. Kinematics and kinetics of a rigid body

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Learning outcomes

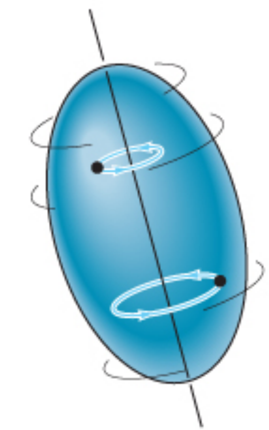
After this lecture, you should be able to:

- Describe mathematically the position, velocity and acceleration of a body that is (i) translating or (ii) rotating about a fixed axis,
- Use the equation of motion to find the forces and moments causing the motion,
- compute the mass moment of inertia of a rigid body.

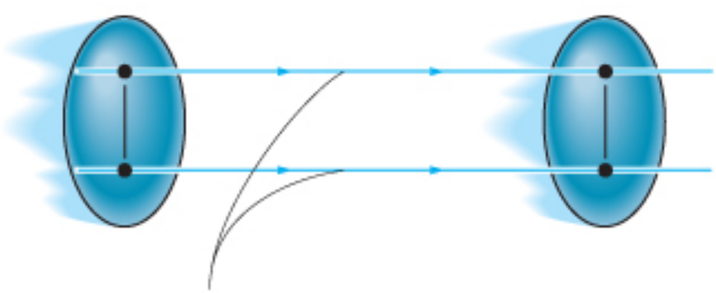
Planar motion of a rigid body

There are three types of rigid body motion in 2D:

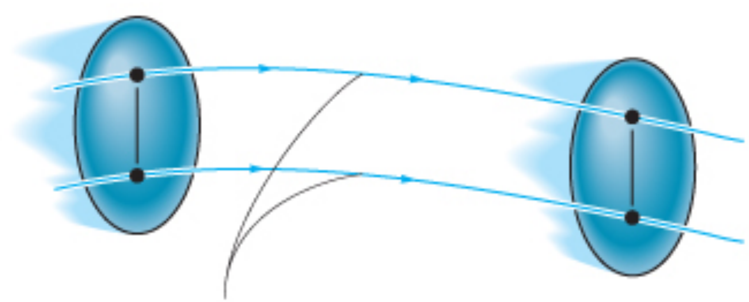
1. Translation (rectilinear or curvilinear),
2. Rotation about a fixed axis,
3. General plane motion (translation & rotation).



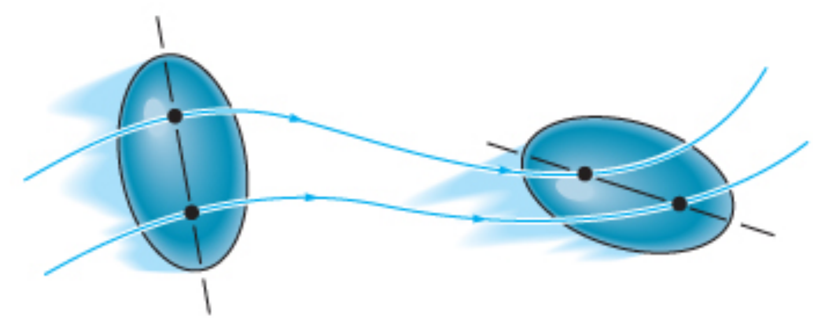
Rotation about a fixed axis



Rectilinear translation



Curvilinear translation



General plane motion

Planar motion of a rigid body

a) Translation

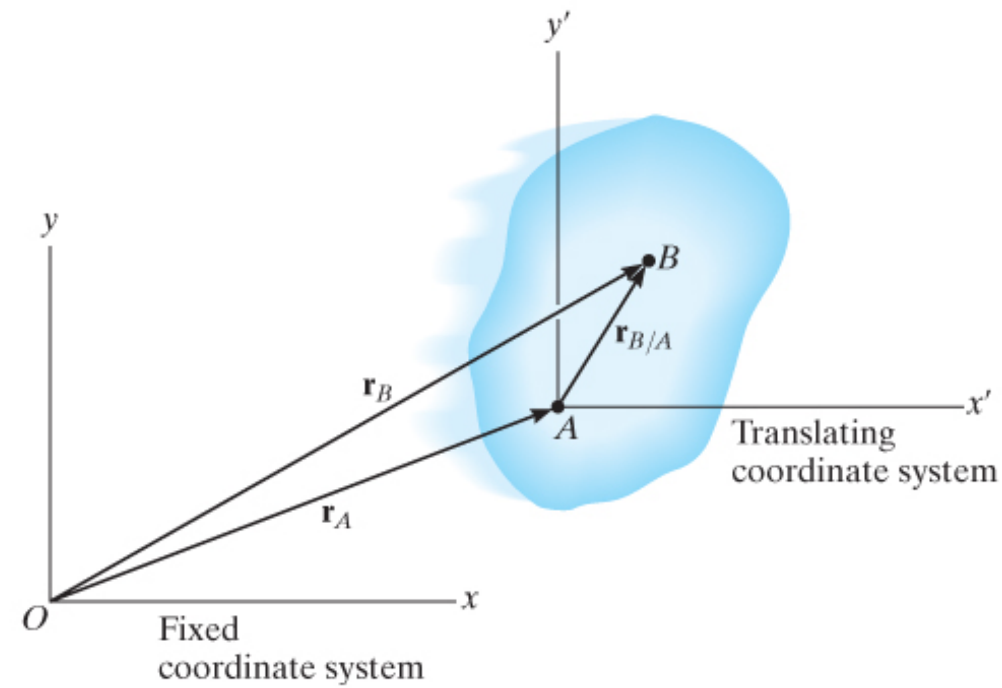
Translation

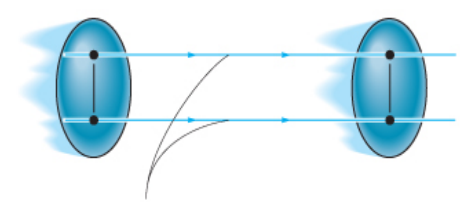
Consider a rigid body in (rectilinear or curvilinear) translation.

The position of points *A* and *B* are given by vectors $\mathbf{r}_A$ and $\mathbf{r}_B$. The position of point *B* with respect to *A* is given by the relative position vector $\mathbf{r}_{B/A}$:

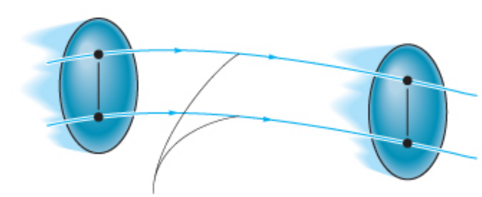
$$\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A}$$

The magnitude of $\mathbf{r}_{B/A}$ is constant since it is a rigid body. Its direction is also constant if the body is moving in translation.





Rectilinear translation



Curvilinear translation

Translation of a rigid body

The velocity of point B is obtained by taking the time derivative:

$$\mathbf{v}_B = \frac{d\mathbf{r}_B}{dt} = \frac{d\mathbf{r}_A}{dt} + \frac{d\mathbf{r}_{B/A}}{dt}$$

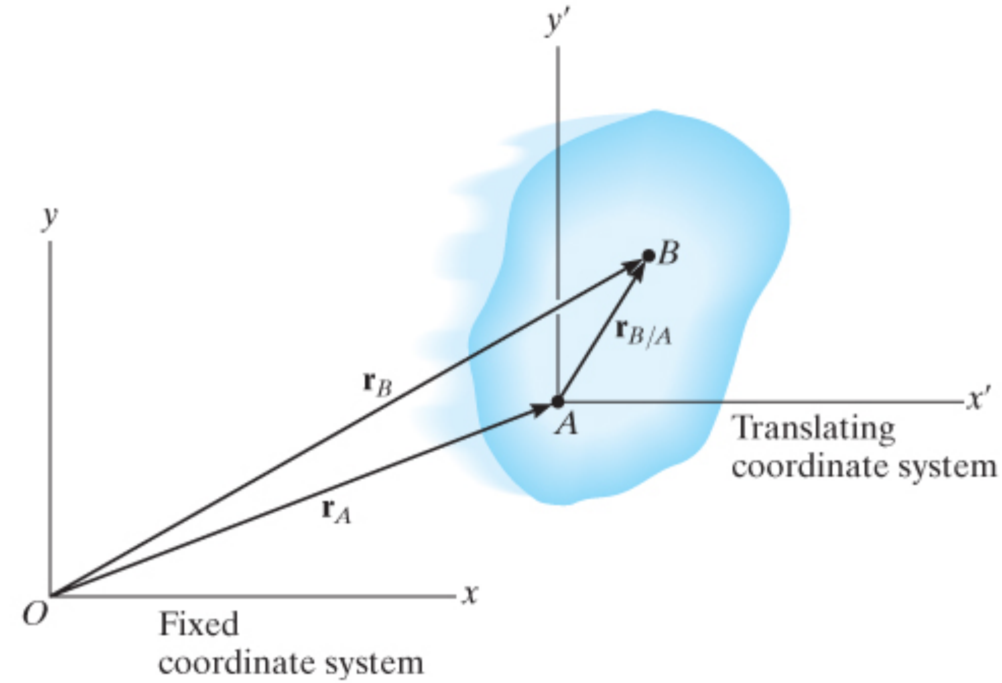
$$\Rightarrow \mathbf{v}_B = \mathbf{v}_A + \cancel{\frac{d\mathbf{r}_{B/A}}{dt}}^{\,= 0}$$

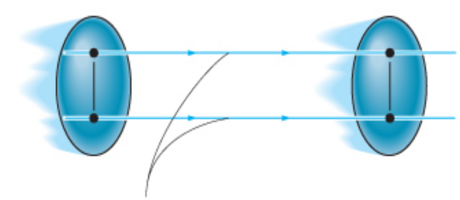
$$\Rightarrow \mathbf{v}_B = \mathbf{v}_A$$

Similarly:

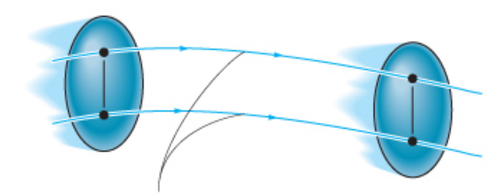
$$\mathbf{a}_B = \mathbf{a}_A$$

In translation, all points in a rigid body experience the same velocity and acceleration.





Rectilinear translation

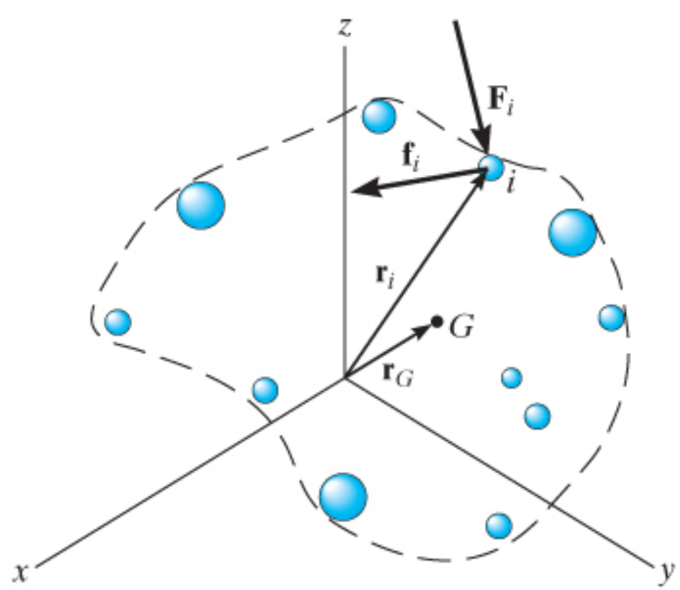


Curvilinear translation

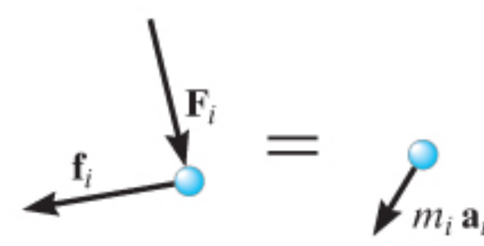
Kinetics of a rigid body in translation

The equation of motion for a rigid body can be obtained by:

- assuming that the body is made of a large number of particles i,
- These particles are connected together with internal forces f_i,
- external forces acting on the body are represented by a concentrated force F_i acting on a particle i.



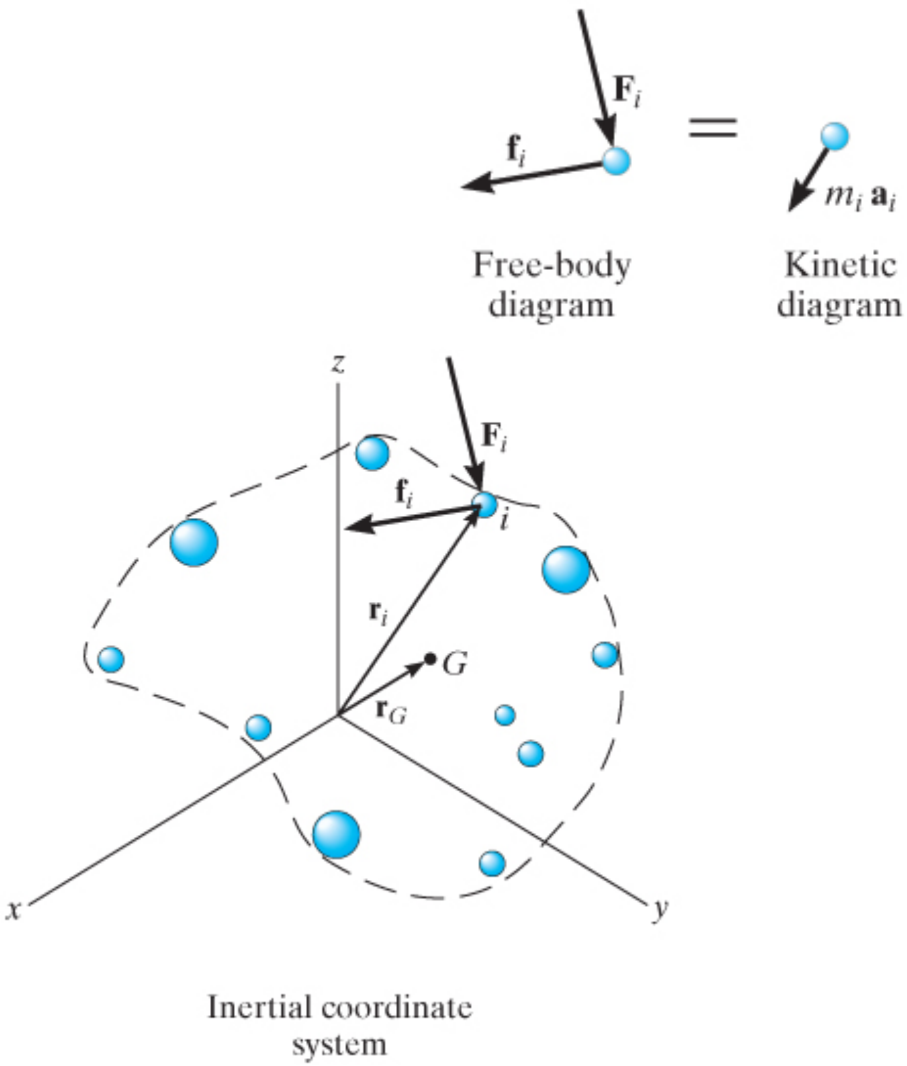
Inertial coordinate system



Free-body diagram

Kinetic diagram

Eq of motion for a system of particles



Adding the equation of motion for each particle gives:

$$\Sigma\mathbf{F}_i + \Sigma\mathbf{f}_i = \Sigma m_i\mathbf{a}_i$$

The sum of internal forces is zero ($\mathbf{f}_i$ between two particles is equal in magnitude but has opposite sign).

If the position vector $\mathbf{r}_G$ is locating the center of mass G of the system, then by definition:

$$m\mathbf{r}_G = \Sigma m_i\mathbf{r}_i \quad \text{where} \quad m = \Sigma m_i$$

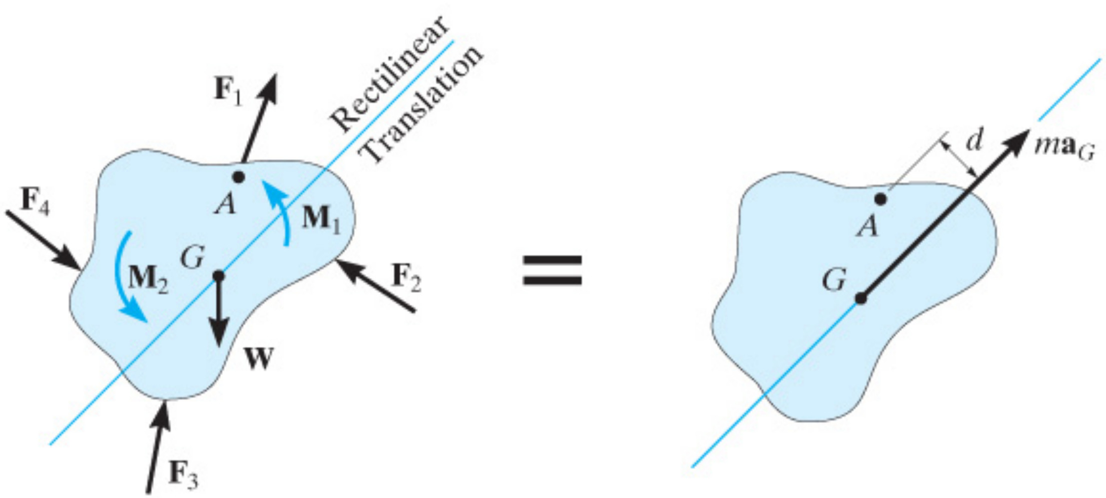
Differentiating this equation twice with respect to time gives:

$$m\ddot{\mathbf{r}}_G = \Sigma m_i\ddot{\mathbf{r}}_i \;\Rightarrow\; m\mathbf{a}_G = \Sigma m_i\mathbf{a}_i$$

Substituting above gives:

$$\Sigma\mathbf{F} = m\mathbf{a}_G$$

Equation of motion: translation



Point *G* is the centre of mass of the body.

Rectilinear translation

Since all points on the rigid body experience the same acceleration, the equations of motion become:

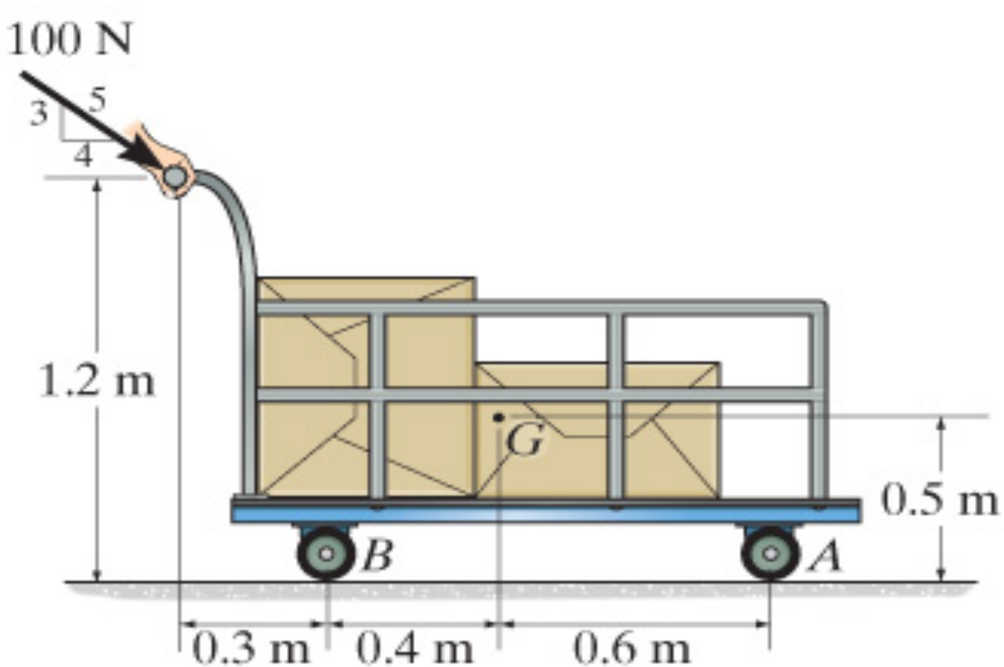
$$\boxed{\begin{aligned} \Sigma F_x &= m(a_G)_x \\ \Sigma F_y &= m(a_G)_y \\ \Sigma M_G &= 0 \end{aligned}}$$

The last equation implies that the rigid body is not rotating. However, if the sum of moments is taken about point *A*, we would get:

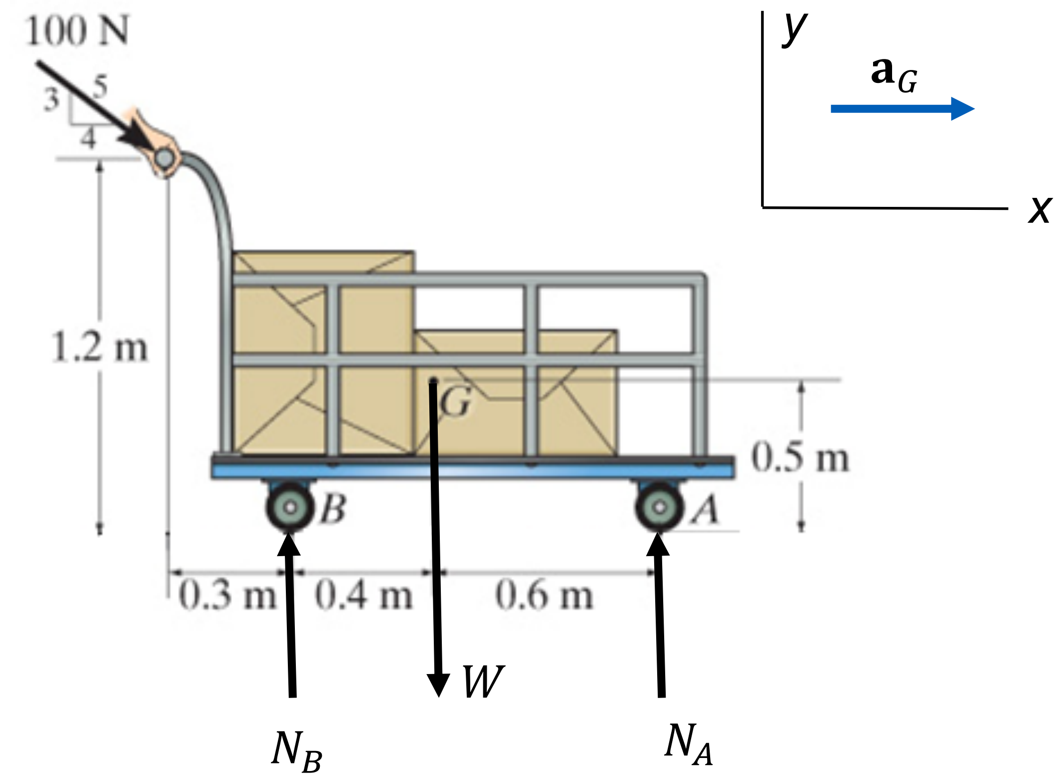
$$\Sigma M_A = \Sigma(M_k)_A = (ma_G)d$$

Example

Determine the acceleration of the trolley and the reaction forces on wheels A and B. The trolley has a mass of 100 kg.



Draw the free-body diagram:



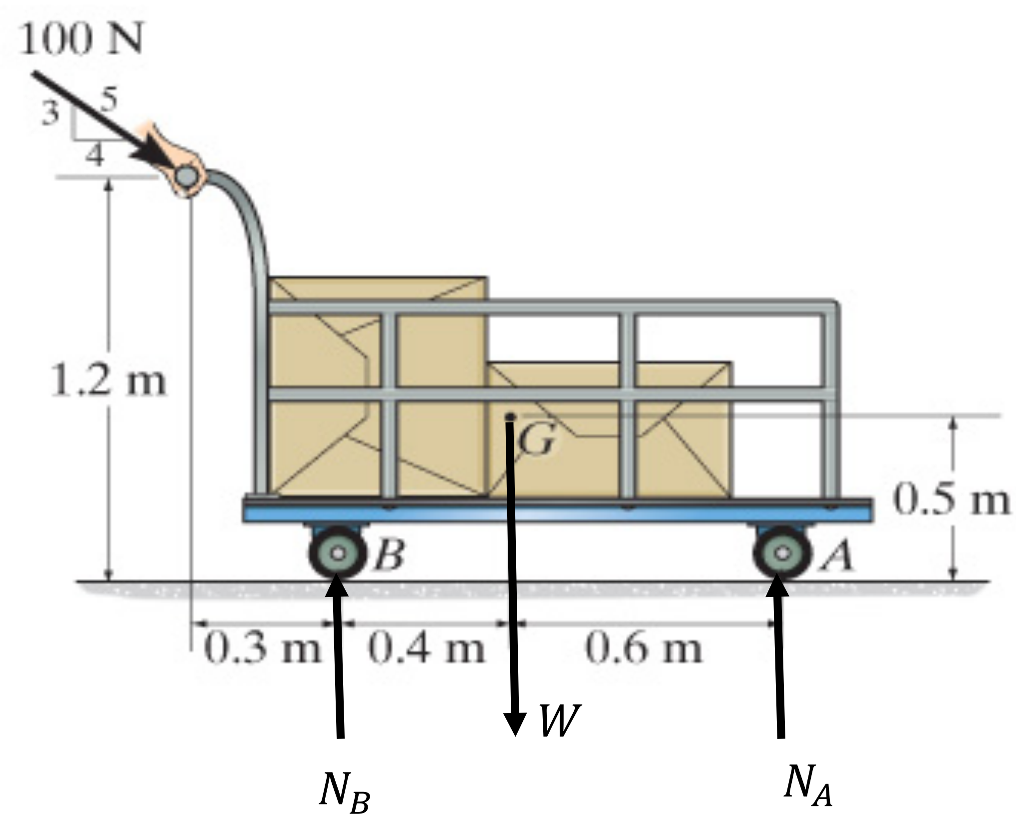
Acceleration is obtained from the equation of motion along the x-direction:

$$\rightarrow + \ \Sigma F_x = m(a_G)_x$$

$$100\text{N}\left(\frac{4}{5}\right) = m(a_G)_x \qquad (a_G)_x = \frac{80\text{N}}{100\text{kg}} = 0.8\,\frac{m}{s^2}$$

Example

Determine the acceleration of the trolley and the reaction forces on wheels A and B. The trolley has a mass of 100 kg.



The support reactions are obtained from sum of forces in the y-direction and the sum of moments:

$$\uparrow + \ \Sigma F_y = m(a_G)_y = 0$$
$$\circlearrowleft + \ \Sigma M_G = 0$$

$$100\text{N}\left(\frac{3}{5}\right)(0.7\text{m}) - 100\text{N}\left(\frac{4}{5}\right)(0.7\text{m}) - N_B(0.4\text{m}) + N_A(0.6\text{m}) = 0$$

$$-100\text{N}\left(\frac{3}{5}\right) + N_B + N_A - 981\text{N} = 0$$

There are two equations and two unknowns, solving this system gives:

$$\begin{cases} N_B = 430.4\text{N} = 430\text{N} \\ N_A = 610.6\text{N} = 611\text{N} \end{cases}$$

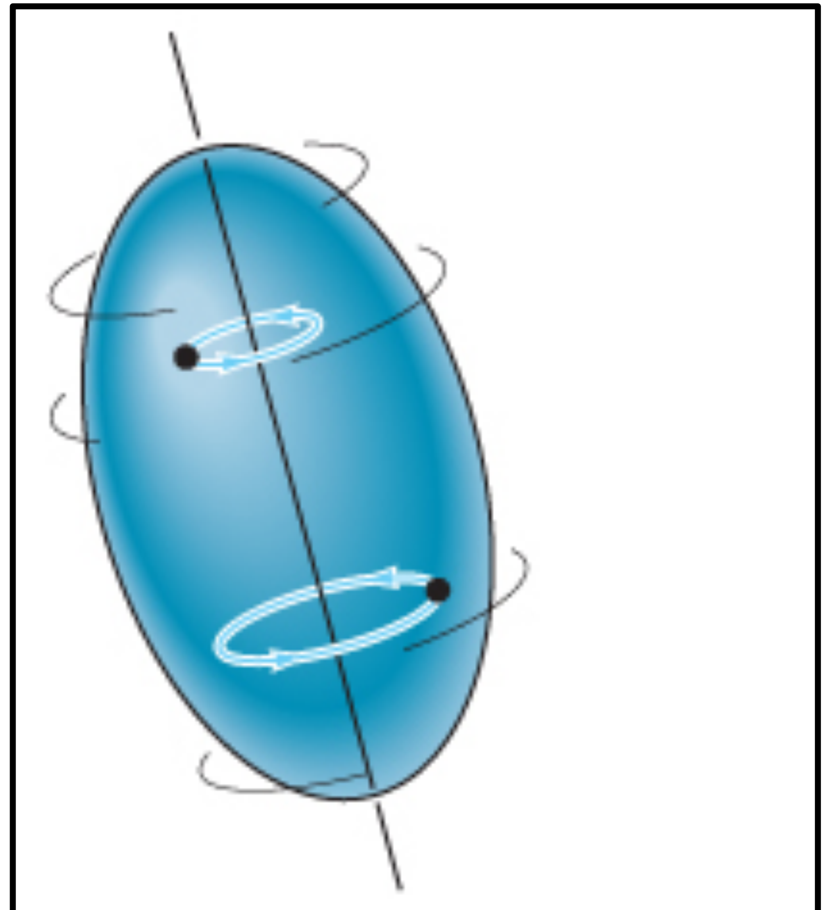
Planar motion of a rigid body

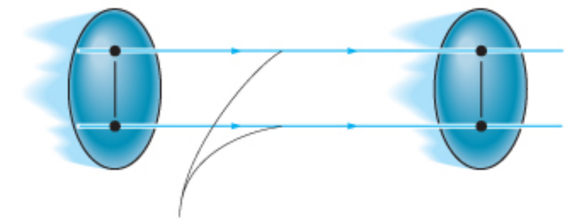
b) Rotation about a fixed axis

- Kinematics

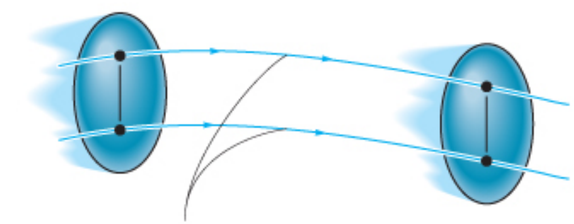
Planar motion of a rigid body

Rotation about a fixed axis: all particles of the body move along a circular path, except those that lie on the axis of rotation.

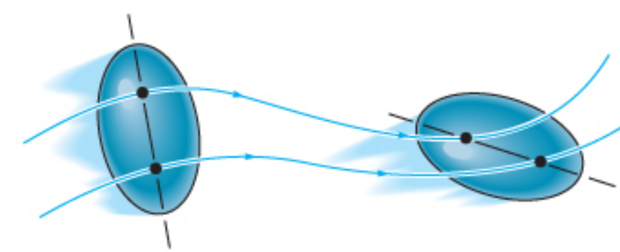




Path of rectilinear translation



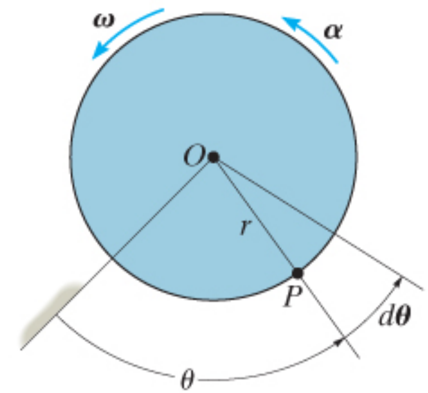
Path of curvilinear translation

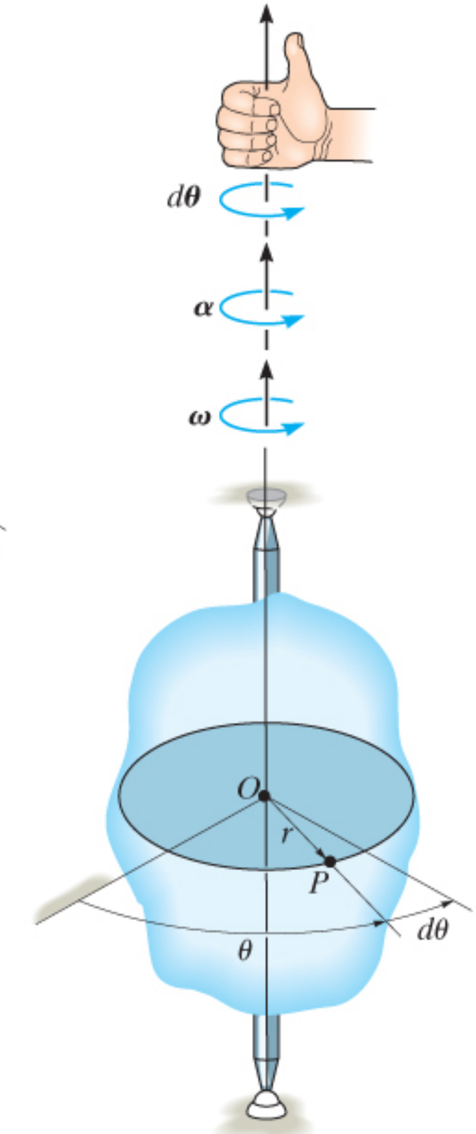


General plane motion

Rotation about a fixed axis

Consider the motion of point *P* on this rotating body:





- *P* moves along a circular path of constant radius r.
- The <u>angular position</u> θ is the angle from a fixed reference line to the radial line r.
- The <u>angular displacement</u> is $d\theta$.
- The <u>angular velocity</u> ω is defined as:

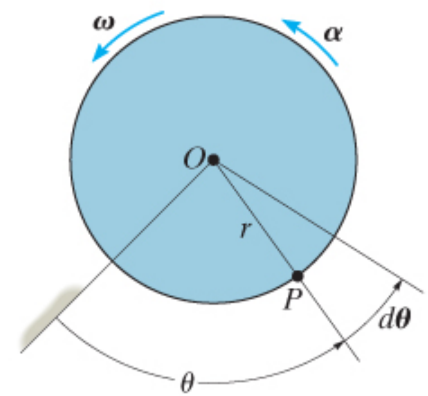
$$\omega = \frac{d\theta}{dt}, \qquad [\omega] = \frac{\text{rad}}{\text{s}}$$

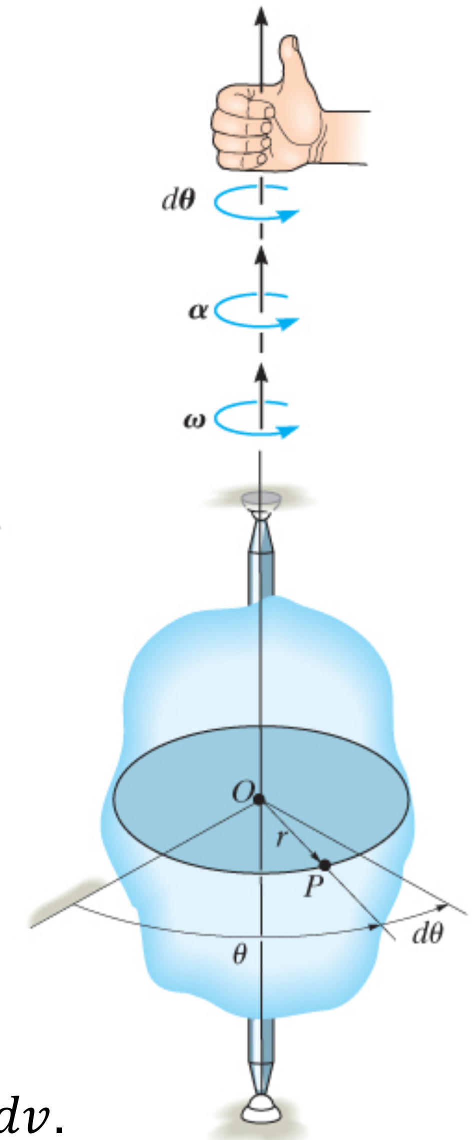
- We kept the notation simple here but note that $d\boldsymbol{\theta}$ and $\boldsymbol{\omega}$ are vectors.

Rotation about a fixed axis

The angular acceleration α is simply:

$$\boxed{\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}}$$





Using the chain rule of derivatives we can write:

$$\alpha = \frac{d\omega}{dt} = \frac{d\omega}{d\theta}\frac{d\theta}{dt} = \frac{d\omega}{d\theta}\omega$$

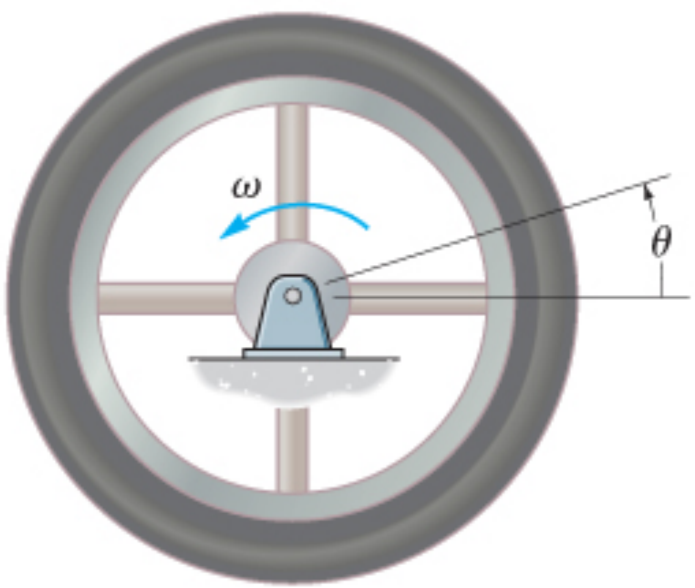
$$\boxed{\alpha d\theta = \omega d\omega}$$

This is very similar to the relation obtained for rectilinear motion: $a\, ds = v\, dv$.

All points on the body experience the same angular velocity and angular acceleration!

Example

The wheel has a variable angular velocity given by $\omega = (4\theta^{1/2})$ rad/s, where the angular position θ is in rad. Find how long it will take to reach $\omega = 150$ rad/s. The initial position is $\theta(t=0) = 1$ rad.



By setting $\omega = 150$ rad/s, we get:

$$\omega = \left(4\sqrt{\theta}\right) = 150 \qquad \theta = (150/4)^2 = 1406.25 \text{ rad}$$

The time needed to reach this angular position can be obtained using:

$$\omega = \frac{d\theta}{dt}$$

Rearranging and integrating gives:

$$dt = \frac{1}{\omega} d\theta \qquad \int_0^t dt = \int_1^\theta \frac{1}{\omega} d\theta$$

$$t = \int_1^\theta \frac{1}{4\sqrt{\theta}} d\theta \quad = \left[\frac{1}{2}\sqrt{\theta}\right]_1^\theta \quad = \frac{1}{2}\sqrt{\theta} - \frac{1}{2}$$

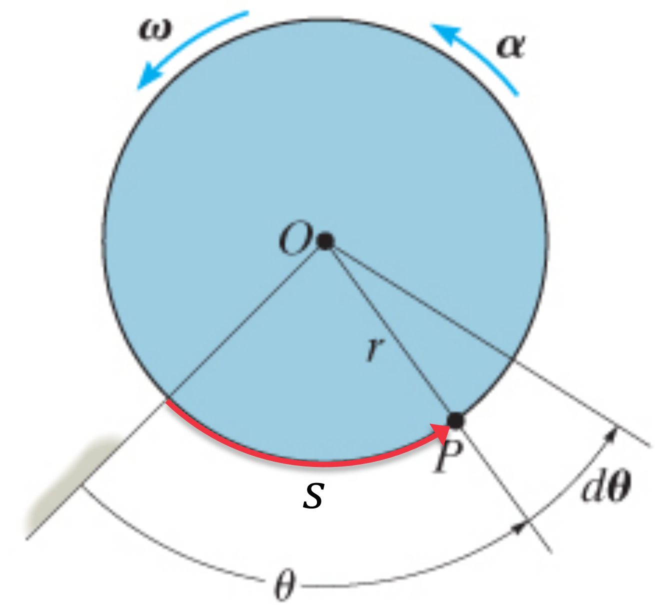
$$t = \frac{1}{2}\sqrt{1406.25} - \frac{1}{2} = 18.25 \text{ (s)}$$

Velocity of a point

For the special case of rotation about point *O*:

Point *P* has moved a distance s along its path and this is related to the angular position θ via:

$$s = r\theta$$

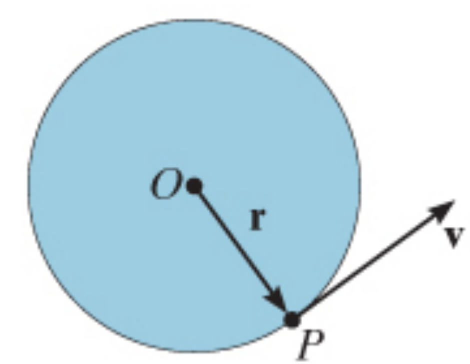


Derivating this relation gives (recall that *r* is constant):

$$v = \frac{ds}{dt} = r\frac{d\theta}{dt} \qquad \Rightarrow \qquad v = r\omega$$

In vectorial form:

$$\mathbf{v} = v\mathbf{u}_t$$



Acceleration of a point

The acceleration of point *P* was derived earlier (see Lecture 7 slide 26-27):

$$\mathbf{a} = a_t\mathbf{u}_t + a_n\mathbf{u}_n = \dot{v}\mathbf{u}_t + \frac{v^2}{r}\mathbf{u}_n$$

For rotation about a fixed axis, we have:

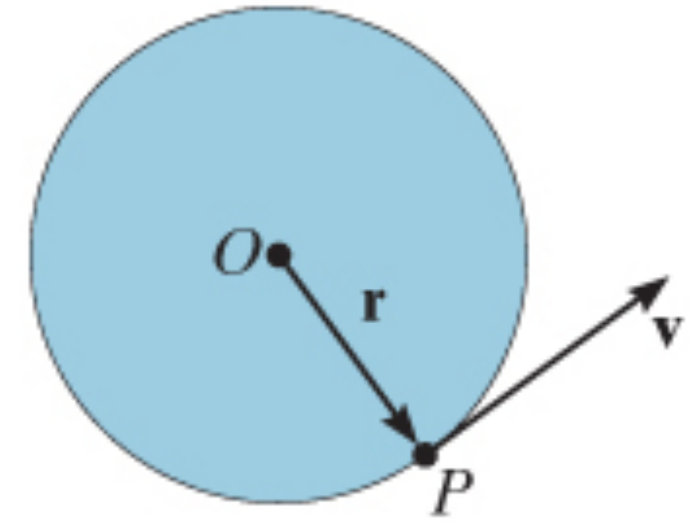
$$\dot{v} = \frac{d}{dt}(r\omega) = r\dot{\omega} = r\alpha$$

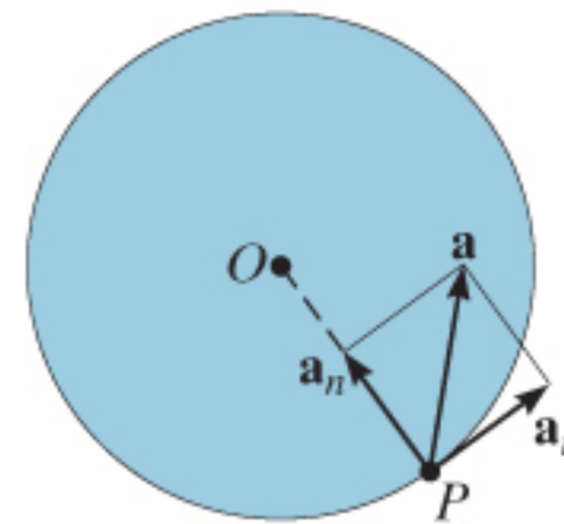
And:

$$\frac{v^2}{r} = \frac{(r\omega)^2}{r} = r\omega^2$$

And the acceleration becomes:

$$\mathbf{a} = \dot{v}\mathbf{u}_t + \frac{v^2}{r}\mathbf{u}_n = r\alpha\mathbf{u}_t + r\omega^2\mathbf{u}_n$$





Velocity of a point

The relationship between the velocity of a point and the angular velocity can be expressed as the cross product:

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}_P,$$

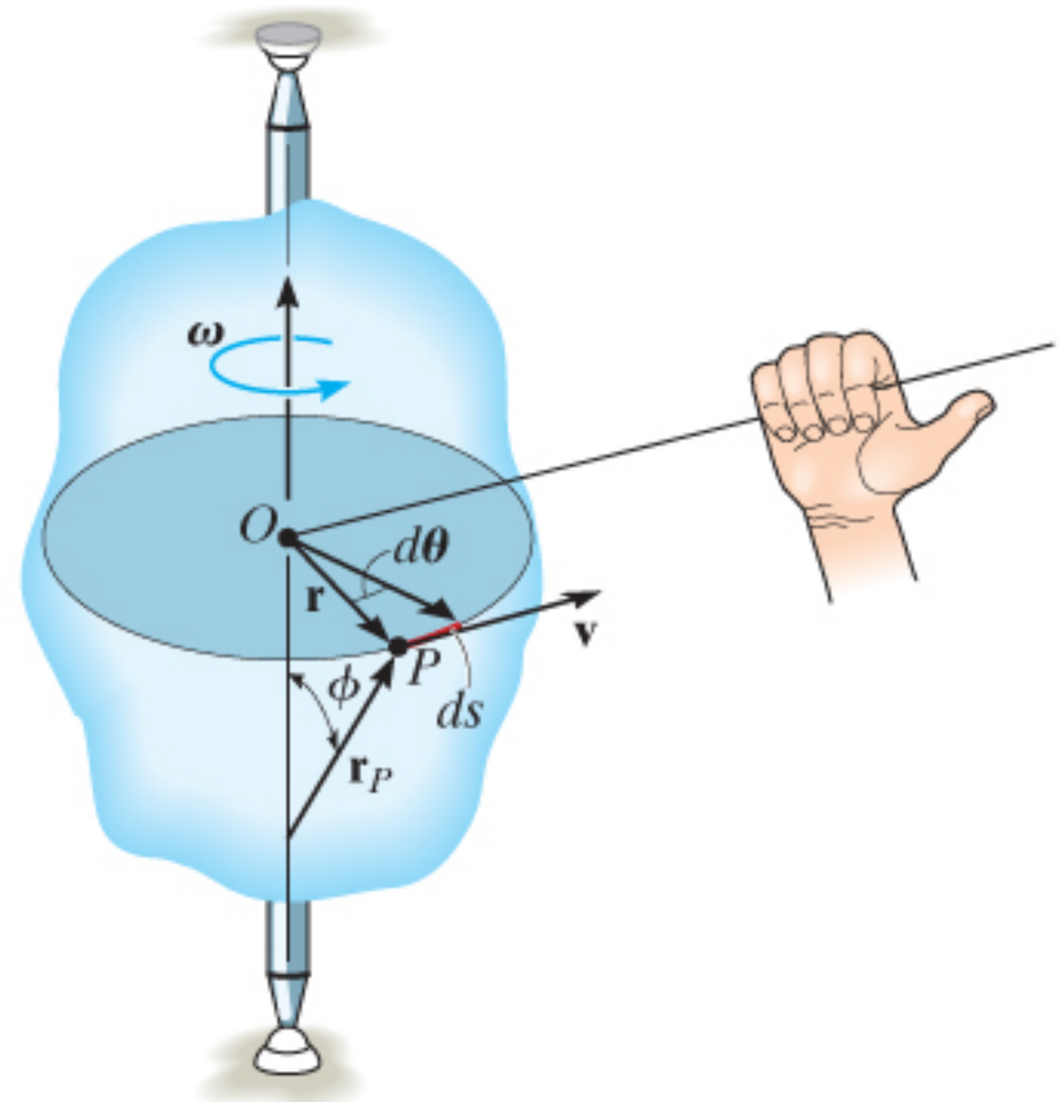
Where $\mathbf{r}_P$ is from any point on the axis of rotation to point P.

The magnitude of the velocity vector is:

$$|\mathbf{v}| = \omega r_P \sin\phi = \omega r$$

Be careful with the order of vectors since

$$\boldsymbol{\omega} \times \mathbf{r}_P \neq \mathbf{r}_P \times \boldsymbol{\omega}$$



Acceleration of a point

Acceleration can also be represented as a vectorial cross product. Differentiating the velocity $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}_P$ gives:

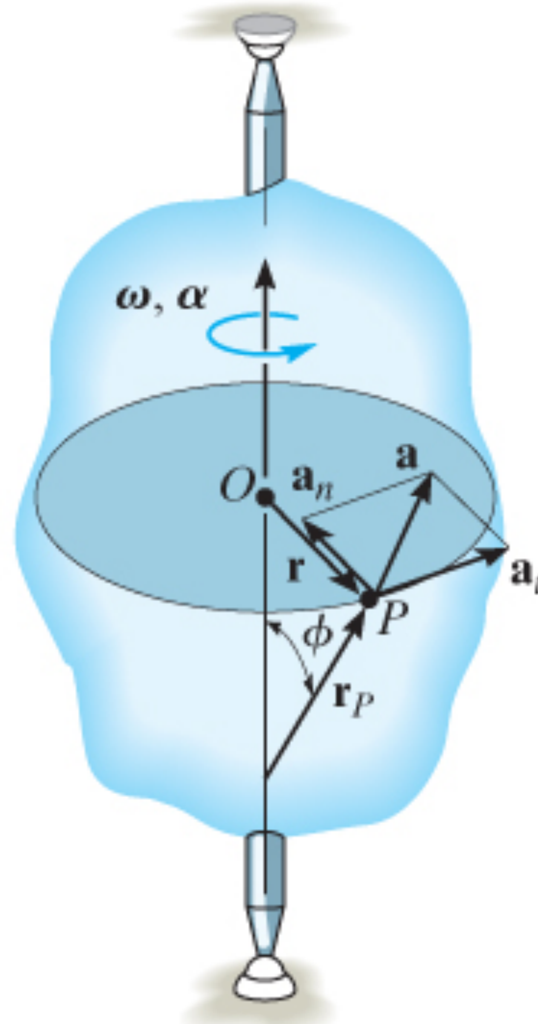
$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(\boldsymbol{\omega} \times \mathbf{r}_P) = \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}_P + \boldsymbol{\omega} \times \frac{d\mathbf{r}_P}{dt}$$

Recall that $\boldsymbol{\alpha} = d\boldsymbol{\omega}/dt$ and $d\mathbf{r}_P/dt = \mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}_P$, and the acceleration becomes:

$$\mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r}_P + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_P)$$

Note that $\boldsymbol{\alpha} \times \mathbf{r}_P = \alpha r \mathbf{u}_t$ and $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_P) = \boldsymbol{\omega} \times (\omega r \mathbf{u}_t) = \omega^2 r \mathbf{u}_n = -\omega^2 \mathbf{r}$. The acceleration can be rewritten as:

$$\begin{aligned} \mathbf{a} &= \mathbf{a}_{\mathrm{t}} + \mathbf{a}_n \\ &= \boldsymbol{\alpha} \times \mathbf{r} - \omega^2 \mathbf{r} \\ &= \alpha r \mathbf{u}_t - r\omega^2 \mathbf{u}_n \end{aligned}$$



Rotation about a fixed axis: summary

Angular motion

Angular velocity

$$\boxed{\omega = \frac{d\theta}{dt}}$$

Angular acceleration

$$\boxed{\alpha = \frac{d\omega}{dt}}$$

Relation between angular position, velocity & acceleration

$$\boxed{\alpha d\theta = \omega d\omega}$$

Motion of point *P*

Velocity

$$\boxed{v = r\omega} \quad \boxed{\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}}$$

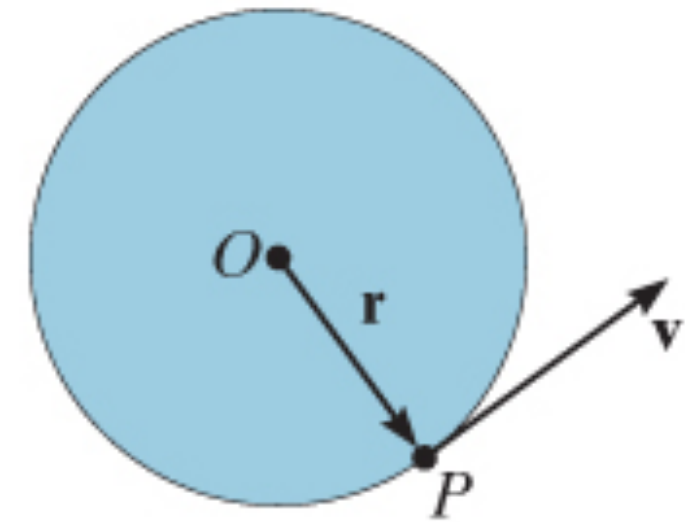
Tangential acceleration:

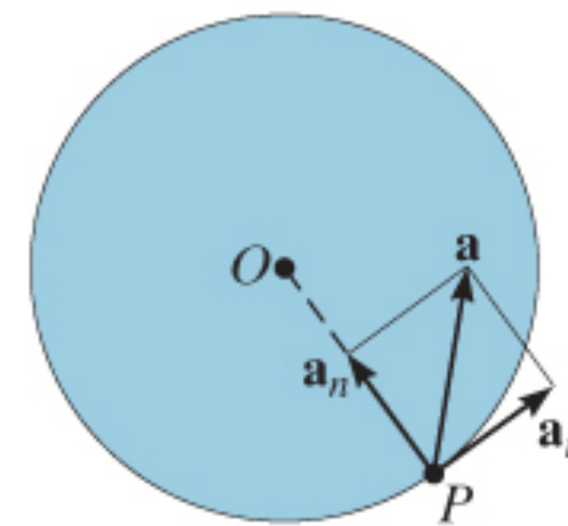
$$\boxed{a_t = r\alpha} \quad \boxed{\mathbf{a}_t = \boldsymbol{\alpha}\times\mathbf{r}}$$

Normal acceleration:

$$\boxed{a_n = \omega^2 r}$$

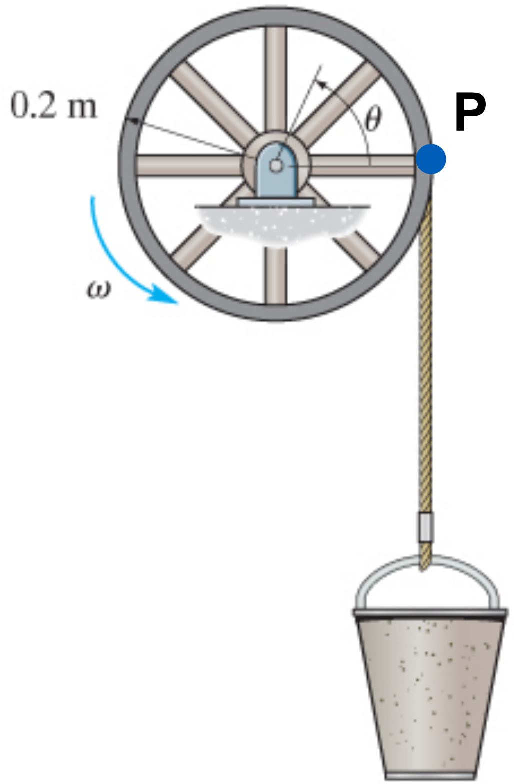
$$\boxed{\mathbf{a}_n = \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) = -\omega^2\mathbf{r}}$$





Example

The angular displacement $\theta = (0.5t^3 + 15t)$ rad, where t is in seconds. Find the velocity and acceleration of the bucket at $t = 3\text{s}$.



The velocity and acceleration of the bucket are equal to the tangential velocity and acceleration of point *P*.

The angular velocity and acceleration are:

$$\omega = \frac{d\theta}{dt} = \frac{d}{dt}(0.5t^3 + 15t) = 1.5t^2 + 15 \ \ (\text{rad/s})$$

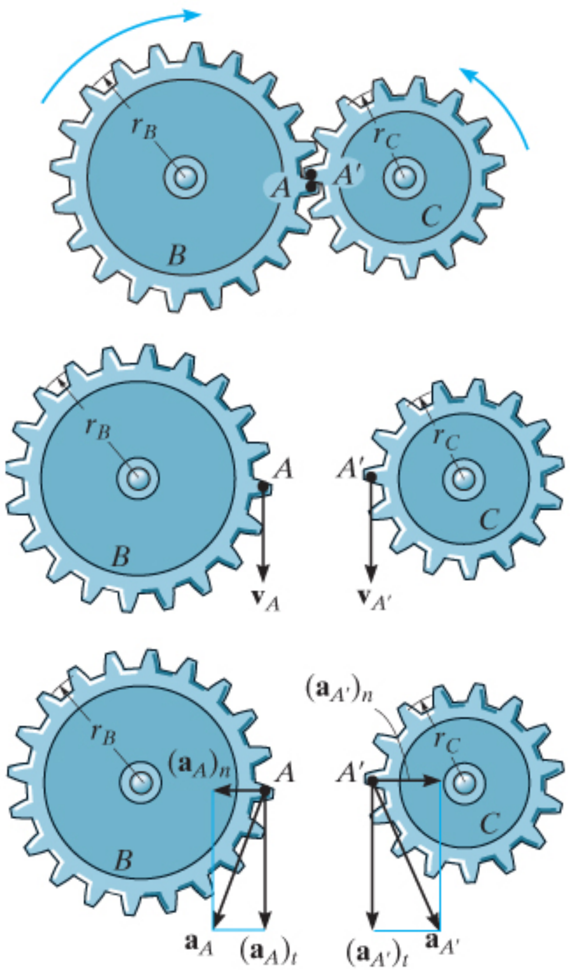
$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}(1.5t^2 + 15) = 3t \ \ (\text{rad/s}^2)$$

The velocity and tangential acceleration of point *P* are:

$v = r\omega$: $\quad v(t = 3s) = 0.2\text{m}(1.5(3^2) + 15)\text{rad/s} \quad = 5.7 \text{ m/s}$

$a_t = r\alpha$: $\quad a(t = 3s) = 0.2\text{m}\big(3(3)\big)\text{rad/s}^2 \quad = 1.8 \text{ m/s}^2$

Motion of gears



If two rotating bodies are in contact, **the points in contact have the same velocity and tangential acceleration.**

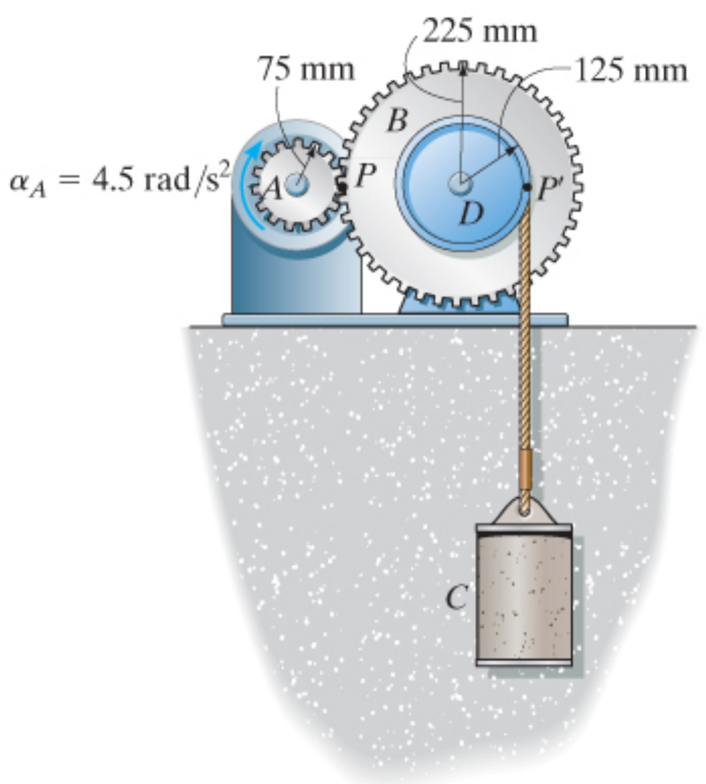
$$\mathbf{v}_A = \mathbf{v}_{A'} \quad \Rightarrow \quad \omega_B r_B = \omega_C r_C$$

$$(\mathbf{a}_A)_t = (\mathbf{a}_{A'})_t \quad \Rightarrow \quad \alpha_B r_B = \alpha_C r_C$$

$$(\mathbf{a}_A)_n \neq (\mathbf{a}_{A'})_n \qquad \mathbf{a}_A \neq \mathbf{a}_{A'}$$

Example

A motor turns gear *A* with a constant angular acceleration $\alpha_A = 4.5\ \mathrm{rad/s^2}$ and starting from rest. Find the velocity and distance travelled by cylinder *C* after three seconds.



The distance travelled by cylinder *C*, and its velocity, are obtained by tracking the motion of point *P*'.

The angular acceleration of gear *B* is:

$$\alpha_A r_A = \alpha_B r_B$$

$$\alpha_B = \frac{\alpha_A r_A}{r_B} = \frac{4.5\mathrm{rad/s^2}(75\mathrm{mm})}{225\mathrm{mm}} = 1.5\mathrm{rad/s^2}$$

The angular velocity of gear *B* at $t = 3\mathrm{s}$ is:

$$\alpha = \frac{d\omega}{dt} \Rightarrow \alpha dt = d\omega \Rightarrow \int_0^3 1.5 dt = \int_0^{\omega_B} d\omega$$

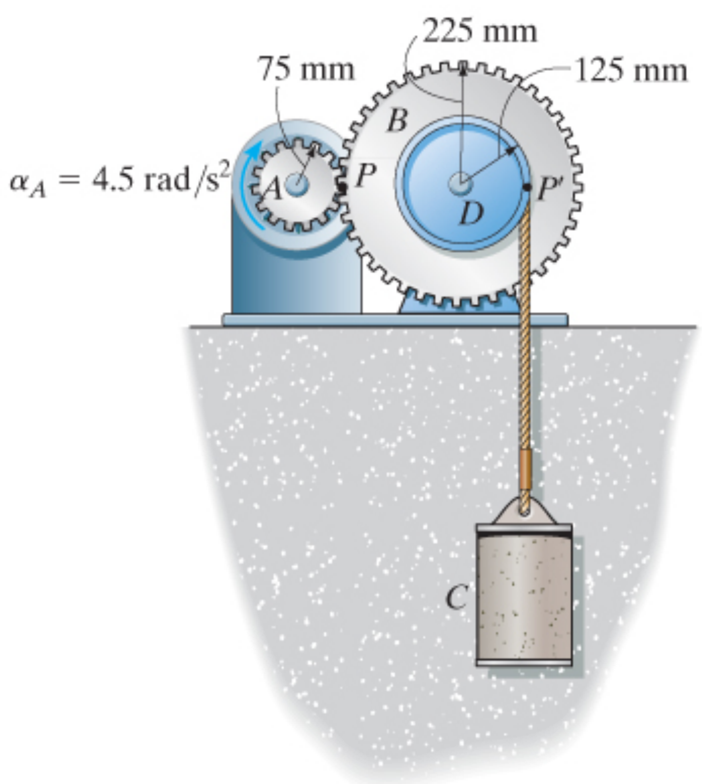
$$\Rightarrow \omega_B = 4.5\ \mathrm{rad/s}$$

The velocity of cylinder *C* at $t = 3\mathrm{s}$ (is the same as point *P*'):

$$v = r\omega_B = (125\mathrm{mm})4.5\left(\frac{\mathrm{rad}}{\mathrm{s}}\right) = 0.5625\,\frac{\mathrm{m}}{\mathrm{s}}$$

Example

A motor turns gear *A* with a constant angular acceleration $\alpha_A = 4.5\ \mathrm{rad/s^2}$ and starting from rest. Find the velocity and distance travelled by cylinder *C* after three seconds.



The angular velocity of gear B is:

$$\alpha = \frac{d\omega}{dt} \Rightarrow \alpha dt = d\omega \quad \Rightarrow \int_0^t 1.5 dt = \int_0^\omega d\omega \Rightarrow \omega = 1.5t$$

Next, we integrate the angular velocity to find the angular displacement at *t* = 3s :

$$\omega = \frac{d\theta}{dt} \Rightarrow \omega dt = d\theta \quad \Rightarrow \int_0^3 1.5t dt = \int_0^\theta d\theta$$

$$\Rightarrow \theta = 1.5\left[\frac{t^2}{2}\right]_0^3 \quad \Rightarrow \theta = 0.75 \cdot 3^2 = 6.75\ (\mathrm{rad})$$

Finally, the distance travelled by the cylinder is:

$$s = r\theta \qquad s_C = r_D\theta = 125\mathrm{mm}(6.75\mathrm{rad}) = 844\ \mathrm{mm}$$

Planar motion of a rigid body

b) Rotation about a fixed axis

- Kinetics: mass moment of inertia

Equation of motion in rotation

Equation of motion in translation:

$$\mathbf{F} = m\mathbf{a}.$$

Equation of motion in rotation:

$$\mathbf{M} = I\boldsymbol{\alpha},$$

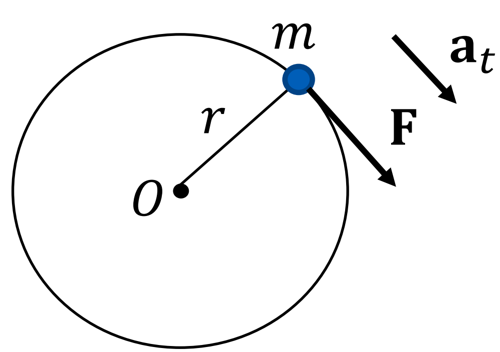
where I is the mass moment of inertia. To use the equation of motion in rotation, we need to compute I.

Figure: 17_PH001

The flywheel on the engine of this tractor has a large moment of inertia about its axis of rotation. Once it is set into motion, it will be difficult to stop, and this in turn will prevent the engine from stalling and instead will allow it to maintain a constant power.

Mass moment of inertia

Moment of inertia of a particle about *O*:



$$\mathbf{F} = m\,\mathbf{a}_t$$

$$\mathbf{M} = r\,\mathbf{F} = rm\,\mathbf{a}_t$$

$$\mathbf{a}_t = r\boldsymbol{\alpha}$$

$$\mathbf{M} = r^2 m\,\boldsymbol{\alpha} = I\,\boldsymbol{\alpha}, \quad \text{where} \quad I = r^2 m$$

We can extend this definition to a body made from an infinite number of particles of mass dm:

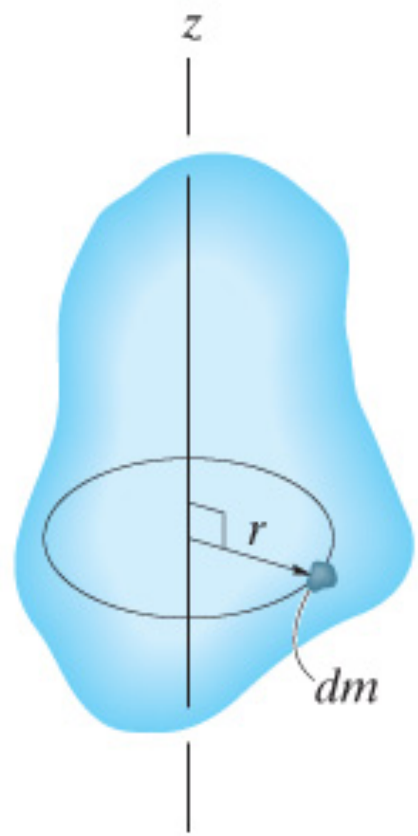
$$I = \int_m r^2\,dm$$

The mass is the product of density and volume ($dm = \rho\,dV$):

$$I = \int_V r^2 \rho\,dV$$

If the density ρ is constant, we get:

$$I = \rho \int_V r^2\,dV$$



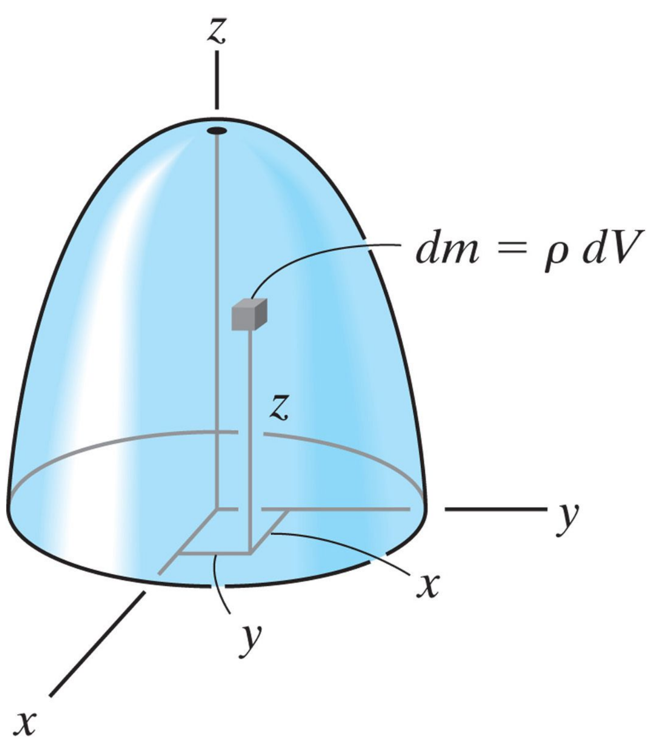
The mass moment of inertia I has units of kg·m^2.

Mass moment of inertia

The mass moment of inertia is $I = \int_V r^2 \rho dV$, where dV is:

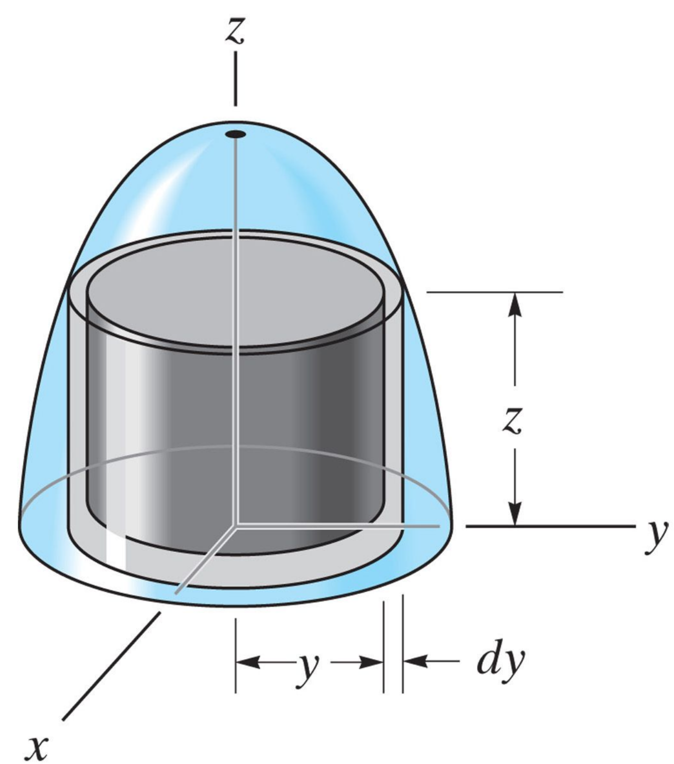
Rectangular element

$$dV = dxdydz$$



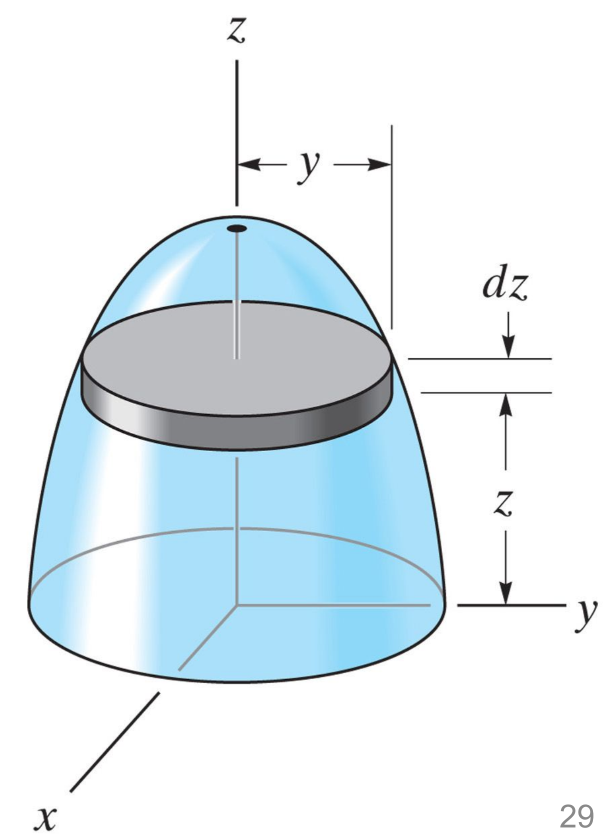
Cylindrical element

$$dV = (2\pi y)zdy$$



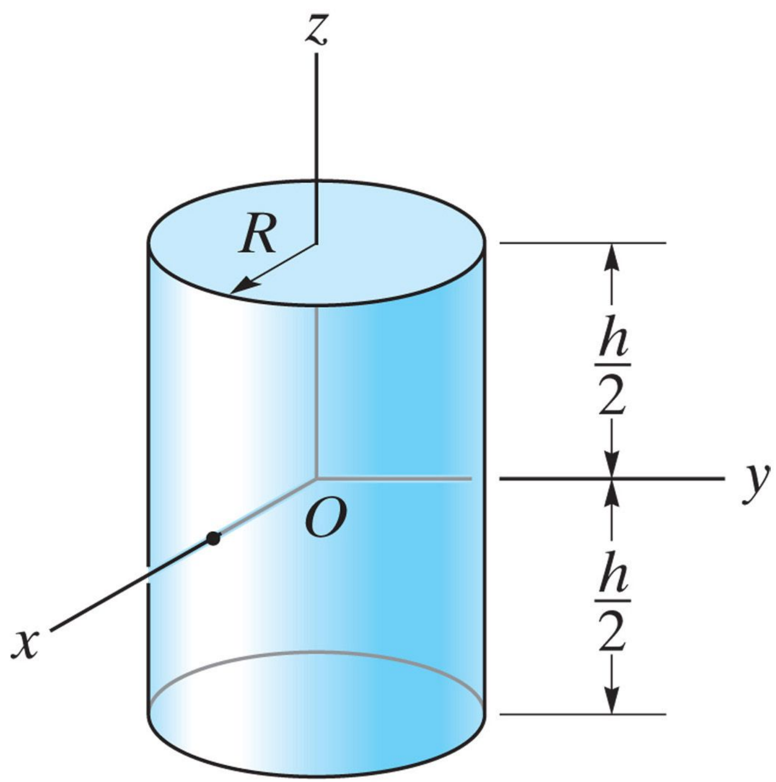
Disk element

$$dV = (\pi y^2)dz$$



Example

Find the moment of inertia of the cylinder about the z-axis. The density is constant.



Definition of the moment of inertia: $I = \rho \int_V r^2 \, dV$

The volume element is $dV = (2\pi y)(z)dy = 2\pi rh \, dr$:

$$I_z = \rho \int_r r^3 \, 2\pi h \, dr \quad = \rho 2\pi h \int_0^R r^3 \, dr$$

$$= \rho 2\pi h \left[\frac{r^4}{4}\right]_0^R = \frac{1}{2}\rho\pi h R^4$$

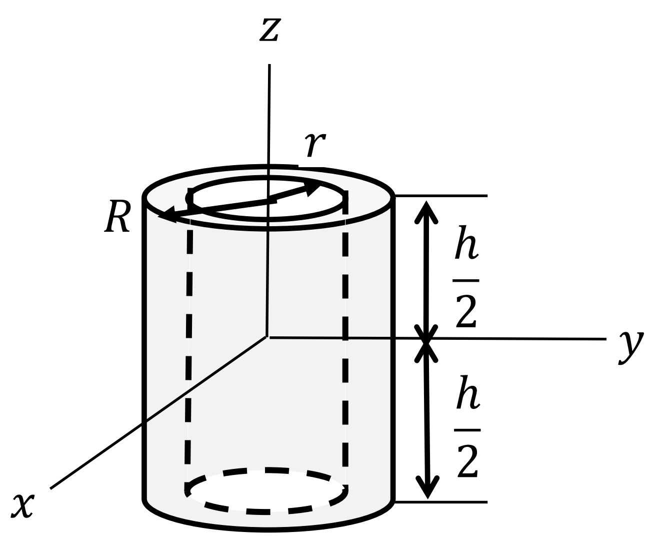
The mass of the cylinder is:

$$m = \rho \int_V dV \quad = \rho 2\pi h \int_0^R r \, dr \quad = \rho\pi h R^2$$

Therefore, we get: $I_z = \dfrac{1}{2} m R^2$

Example

Find the moment of inertia of the tube about the z-axis. The density is constant.



$$I = \rho \int_V r^2 \, dV$$

The volume element is: $dV = (2\pi y)(z)dy = 2\pi rh\,dr$

Integrating from *r to R:*

$$I_z = \rho \int_r^R r^3 \, 2\pi h\, dr \qquad = 2\rho\pi h \left[\frac{r^4}{4}\right]_r^R$$

$$= \frac{1}{2} R^4 \rho\pi h - \frac{1}{2} r^4 \rho\pi h \qquad = \frac{1}{2}\rho\pi h(R^4 - r^4)$$

The mass is:

$$m = \rho \int_V dV \quad = \rho 2\pi h \int_r^R r\, dr \qquad = \rho\pi h(R^2 - r^2)$$

The moment of inertia becomes:

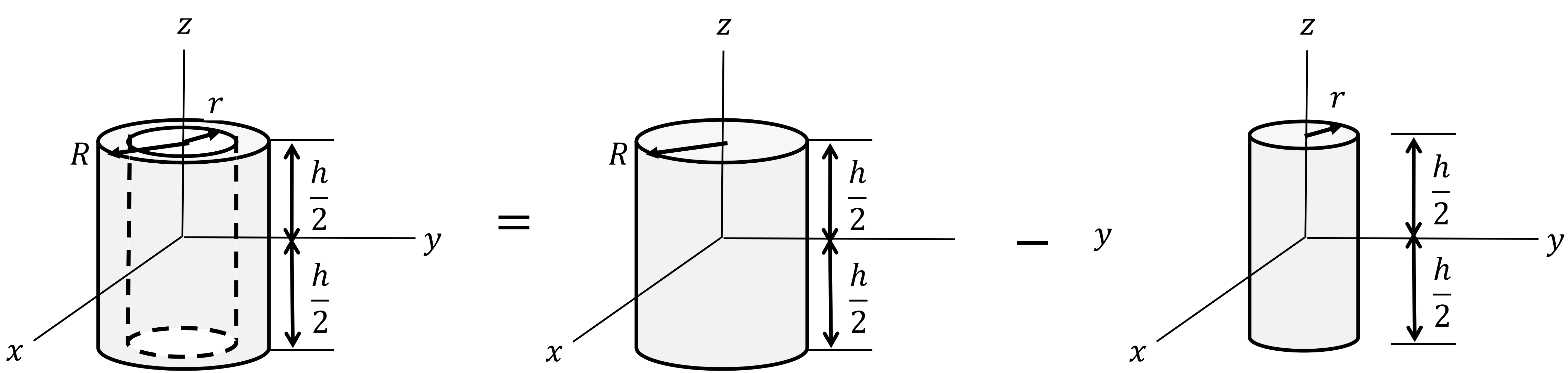
$$I_z = \frac{1}{2}\rho\pi h(R^2 - r^2)(R^2 + r^2) = \frac{1}{2} m(R^2 + r^2)$$

Mass moment of inertia

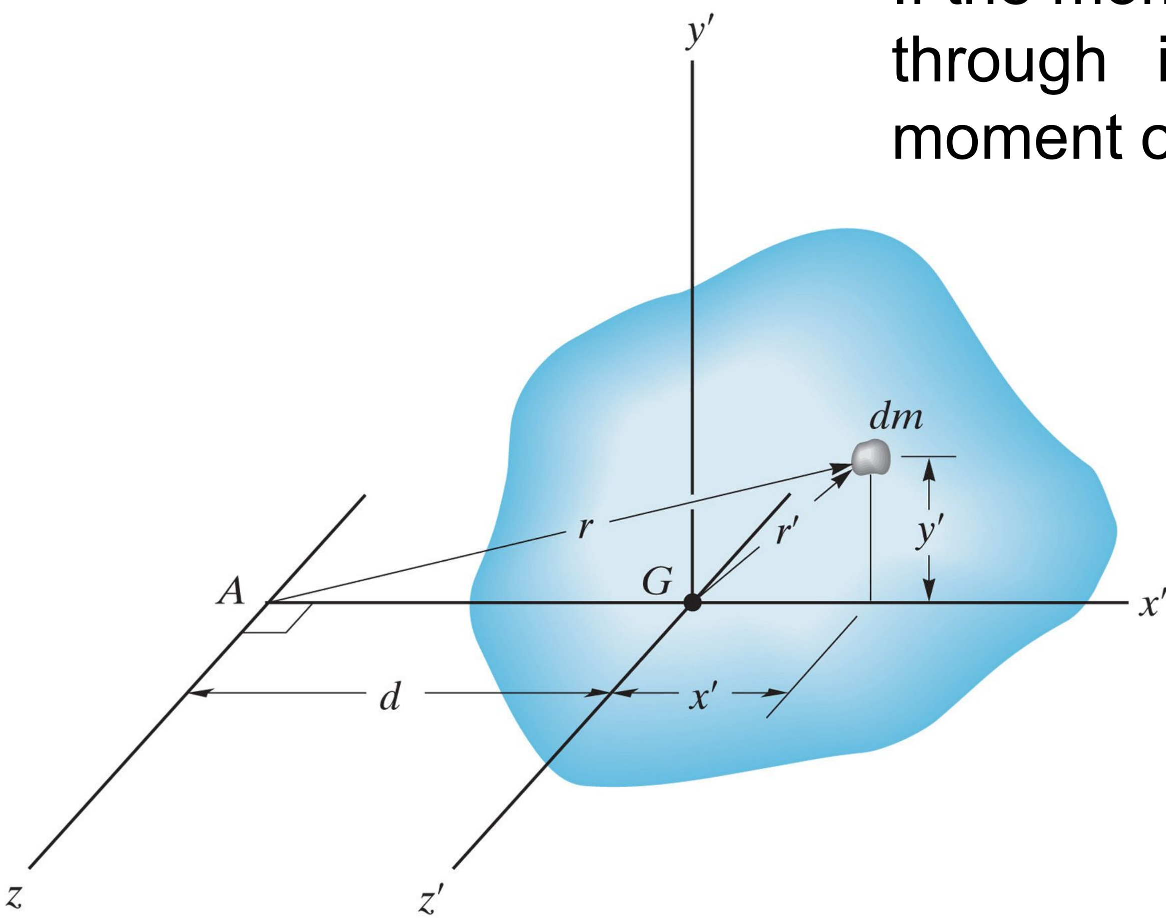
When a body has multiple parts, the moment of inertia of the body is the sum of the moments of inertia of all parts.

Likewise, the moment of inertia of the tube is equal to the moment of inertia of the cylinder of radius *R*, minus that of a cylinder of radius *r*.

$$I_z = \frac{1}{2}R^4\rho\pi h - \frac{1}{2}r^4\rho\pi h = \frac{1}{2}\rho\pi h(R^4 - r^4)$$



Parallel-axis theorem



If the moment of inertia of a body about an axis going through its center of mass is known, then the moment of inertia about any other parallel axis is:

$$I = I_G + md^2$$

I: moment of inertia about the z-axis,

I_G: moment of inertia about the z’-axis passing through the center of mass *G*,

m: mass of the body,

d: distance between the *z* and *z’* axes.

Radius of gyration

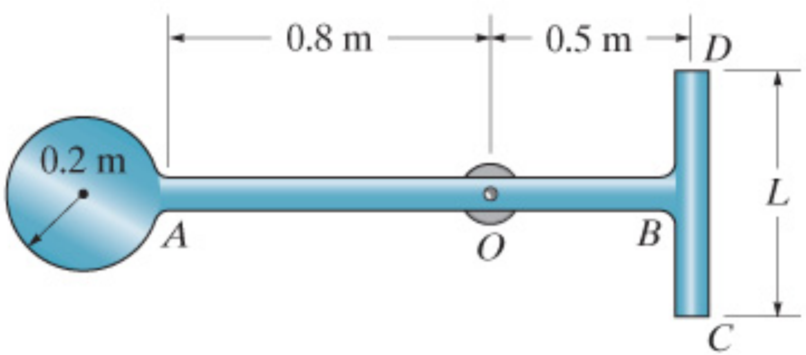
Handbooks sometimes report the moment of inertia using the radius of gyration k defined as:

$$I = mk^2 \quad \Rightarrow \quad k = \sqrt{\frac{I}{m}}$$

The radius of gyration has units of length.

Example

The body shown below has a 6 kg disk on the left connected with rods of $\bar{m} = 2$ kg/m. If L = 0.75 m, find the moment of inertia about point O.



We can solve this in three steps:

1. Find the moment of inertia of each part about their centre of mass.
2. Use the parallel-axis theorem to find the moment of inertia of each part about point O.
3. The moment of inertia of the whole body is obtained by adding the moment of inertia of each part.

Step 1:

Disk: $$I_G = \frac{1}{2}mR^2 = \frac{1}{2}(6\text{ kg})(0.2\text{m})^2 = 0.12\text{ kg m}^2$$

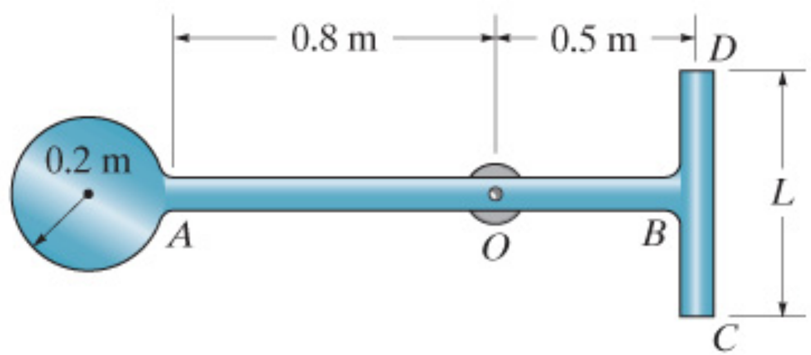
Bar: $$I_G = \int_{-l/2}^{l/2} \bar{m}x^2 dx = \bar{m}\left[\frac{x^3}{3}\right]_{-l/2}^{l/2} = \frac{1}{12}\bar{m}l^3 = \frac{1}{12}ml^2$$

$$I_{G,AB} = \frac{1}{12}ml^2 = \frac{1}{12}(2.6\text{kg})(1.3\text{m})^2 = 0.366\text{ kg m}^2$$

$$I_{G,CD} = \frac{1}{12}ml^2 = \frac{1}{12}(1.5\text{kg})(0.75\text{m})^2 = 0.07\text{ kg m}^2$$

Example

The body shown below has a 6 kg disk on the left connected with rods of $\bar{m} = 2$ kg/m. If L = 0.75 m, find the moment of inertia about point O.



Step 2: $$I_O = I_G + md^2$$

Disk: $I_G = 0.12 \text{ kg m}^2$

$$I_{O,disk} = 0.12 \text{ kg m}^2 + 6\text{kg}\,((0.8 + 0.2)\text{m})^2 = 6.12 \text{ kg m}^2$$

Bar AB: $I_G = 0.366 \text{ kg m}^2$ $\quad d = 0.8 \text{ m} - \frac{1.3}{2} = 0.15 \text{ m}$

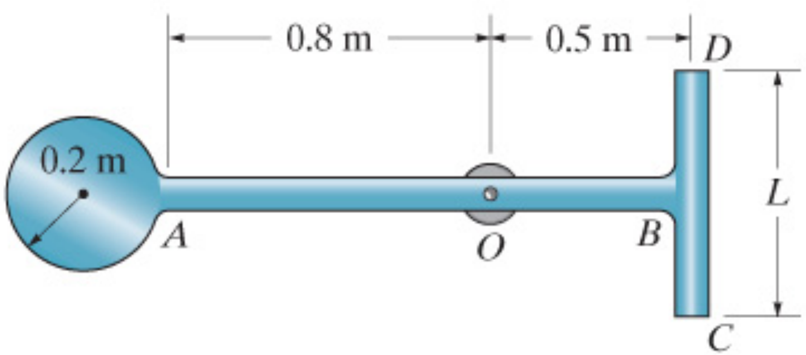
$$I_{O,AB} = 0.366 \text{ kg m}^2 + (2.6\text{kg})\,(0.15\text{m})^2 = 0.425 \text{ kg m}^2$$

Bar CD: $I_G = 0.07 \text{ kg m}^2$ $\quad d = 0.5 \text{ m}$

$$I_{O,CD} = 0.07 \text{ kg m}^2 + (1.5\text{kg})\,(0.5\text{m})^2 = 0.445 \text{ kg m}^2$$

Example

The body shown below has a 6 kg disk on the left connected with rods of $\bar{m} = 2$ kg/m. If L = 0.75 m, find the moment of inertia about point O.

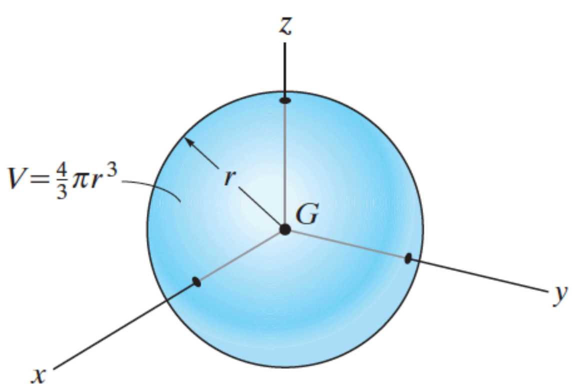


Step 3: add all moments of inertia together.

$$I_O = I_{O,disk} + I_{O,AB} + I_{O,CD}$$

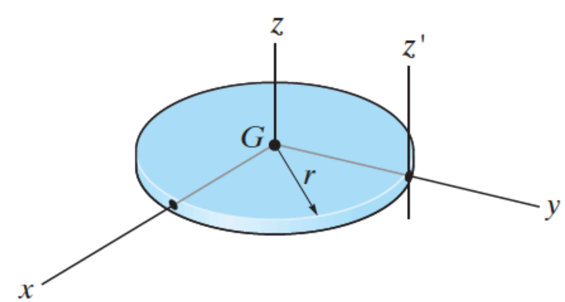
$$= (6.12 + 0.425 + 0.445)\mathrm{kg\ m}^2 = 6.99\ \mathrm{kg\ m}^2$$

Table of mass moment of inertia



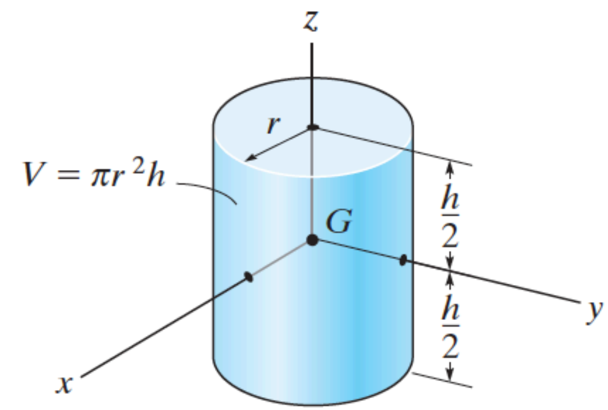
Sphere

$$I_{xx} = I_{yy} = I_{zz} = \frac{2}{5} mr^2$$



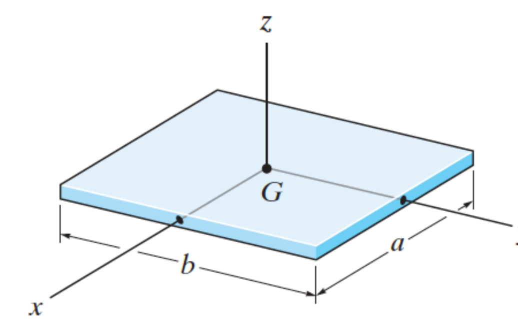
Thin Circular disk

$$I_{xx} = I_{yy} = \frac{1}{4} mr^2 \quad I_{zz} = \frac{1}{2} mr^2 \quad I_{z'z'} = \frac{3}{2} mr^2$$



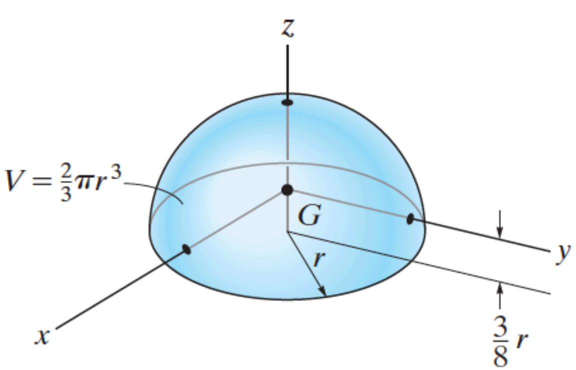
Cylinder

$$I_{xx} = I_{yy} = \frac{1}{12} m(3r^2 + h^2) \quad I_{zz} = \frac{1}{2} mr^2$$



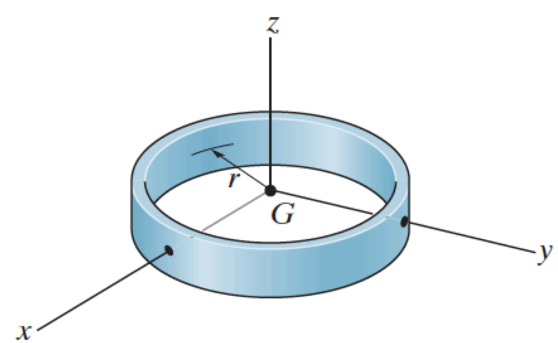
Thin plate

$$I_{xx} = \frac{1}{12} mb^2 \quad I_{yy} = \frac{1}{12} ma^2 \quad I_{zz} = \frac{1}{12} m(a^2 + b^2)$$



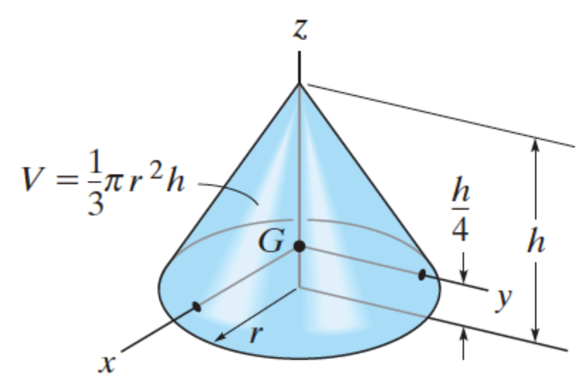
Hemisphere

$$I_{xx} = I_{yy} = 0.259\, mr^2 \quad I_{zz} = \frac{2}{5} mr^2$$



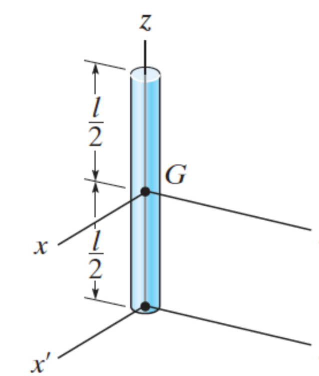
Thin ring

$$I_{xx} = I_{yy} = \frac{1}{2} mr^2 \quad I_{zz} = mr^2$$



Cone

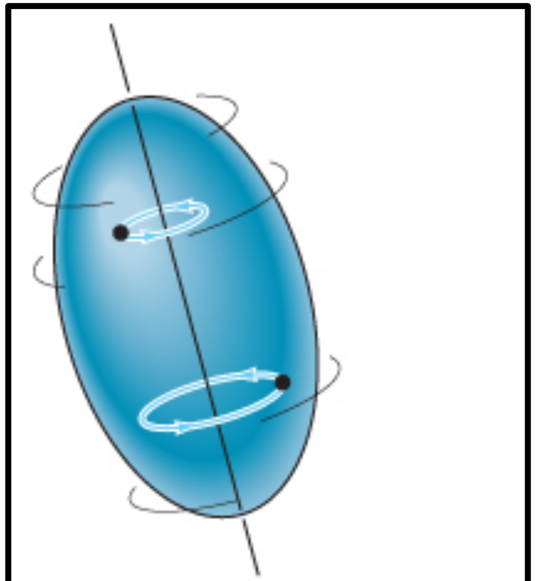
$$I_{xx} = I_{yy} = \frac{3}{80} m(4r^2 + h^2) \quad I_{zz} = \frac{3}{10} mr^2$$



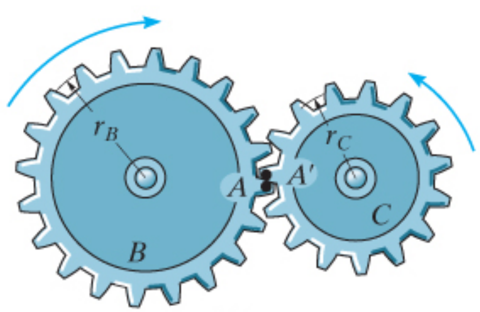
Slender Rod

$$I_{xx} = I_{yy} = \frac{1}{12} ml^2 \quad I_{x'x'} = I_{y'y'} = \frac{1}{3} ml^2 \quad I_{z'z'} = 0$$

Summary



Rotation about a fixed axis



Kinematics of rotation about a fixed axis

Angular position, velocity and acceleration:

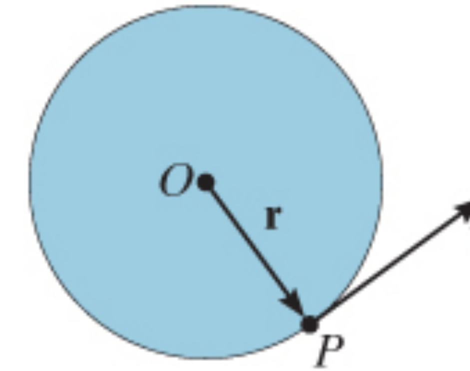
$$\boxed{\omega = \frac{d\theta}{dt}} \quad \boxed{\alpha = \frac{d\omega}{dt}}$$

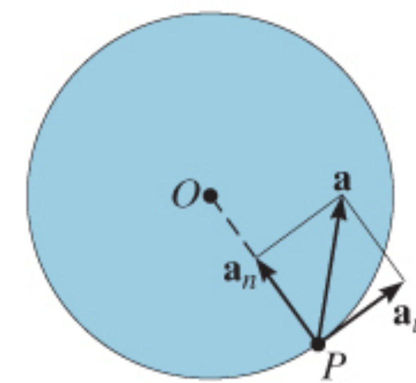
The velocity and acceleration of point *P* are given by:

$$\boxed{v = r\omega} \quad \boxed{a_t = r\alpha} \quad \boxed{a_n = \omega^2 r}$$

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}_P \qquad \mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r} - \omega^2 \mathbf{r} \quad = \mathbf{a}_t + \mathbf{a}_n$$

Common application: gears





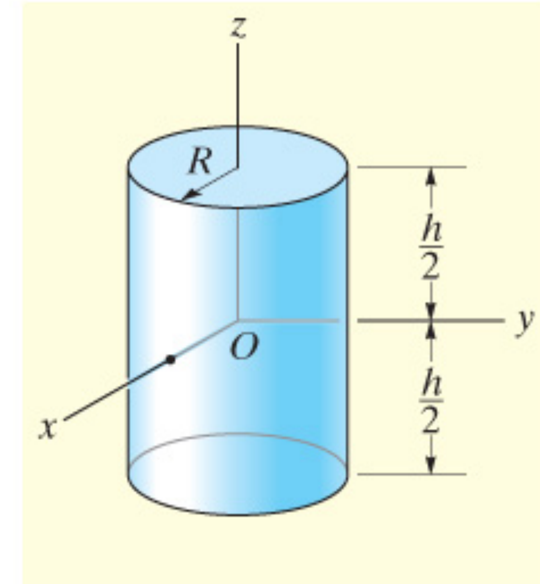
Summary

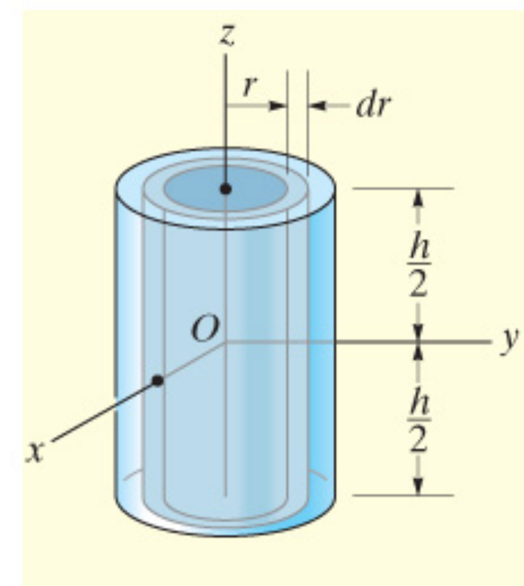
The moment of inertia of a body is obtained by:

1. Integrating over the volume, $I = \int_V r^2 \rho dV$

2. Using the parallel-axis theorem, $I = I_G + md^2$

3. Adding all parts together.





Need more explanations?

For more details, consult these sections of the textbook:

- **Translation of a rigid body**:
  - Kinematics: Sections 16.1 and 16.2
  - Kinetics: Sections 17.2 and 17.3

- **Rotation about a fixed axis**
  - Kinematics: Section 16.3
  - Mass moment of inertia: Section 17.1
  - Kinetics: Section 17.4