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8 Poisson processes

In this exercise you learn to apply Poisson processes to solve real-life problems involving random time instants. You also get your hands dirty with integrals needed in working with independent random variables in the continuum.

Classroom problems

8.1 *Repeated exponential waiting times.* A random variable X is Gamma distributed with shape parameter k and rate parameter λ if it has density function

$$p(x) = \begin{cases} \frac{\lambda^k}{(k-1)!} x^{k-1} e^{-\lambda x}, & x > 0, \\ 0, & x \le 0. \end{cases}$$

Let $\tau_1, \tau_2, \ldots$ be independent exponentially distributed random variables with rate parameter λ, and define

$$T_n = \tau_1 + \tau_2 + \cdots + \tau_n.$$

(a) Show that T_n follows a Gamma distribution, and determine the shape and rate parameters of this distribution.

Solution. Proof by induction: as the initial step we note that setting $k = 1$ in the density function of the Gamma distribution gives the density function of $\mathrm{Exp}(\lambda)$:

$$p(x) = \mathbb{I}\{x > 0\}\lambda e^{-\lambda x}.$$

Since $T_1 = \tau_1$, we see that $T_1 \sim \mathrm{Gamma}(1, \lambda)$.

Induction step: assume now that $T_n \sim \mathrm{Gamma}(n, \lambda)$. The cumulative distribution function of T_{n+1} is

$$F_{T_{n+1}}(t) = \mathbb{P}(T_{n+1} \le t) = \mathbb{P}(T_n + \tau_{n+1} \le t).$$

Since T_n and τ_{n+1} are independent, their joint density is the product of their density functions:

$$\begin{aligned} F_{T_{n+1}}(t) &= \mathbb{P}(T_n + \tau_{n+1} \le t) \\ &= \int_0^\infty \int_0^{t-x} f_{T_n}(x) f_{\tau_{n+1}}(y) dy\, dx \\ &= \int_0^\infty f_{T_n}(x) F_{\tau_{n+1}}(t-x) dx \end{aligned}$$

The rest is straightforward integration:

$$\begin{aligned}
F_{T_{n+1}}(t) &= \int_0^\infty f_{T_n}(x)\mathbb{I}\{t-x>0\}(1-e^{-\lambda(t-x)})dx \\
&= \int_0^t f_{T_n}(x)dx - \int_0^t f_{T_n}(x)e^{-\lambda(t-x)}dx \\
&= F_{T_n}(t) - \frac{\lambda^n}{(n-1)!}\int_{-\infty}^t \mathbb{I}\{x>0\}x^{n-1}e^{-\lambda t}dx \\
&= F_{T_n}(t) - \mathbb{I}\{t>0\}\frac{\lambda^n}{n!}t^n e^{-\lambda t}.
\end{aligned}$$

The density function is obtained by differentiation (recall exercise 1.1):

$$\begin{aligned}
f_{T_{n+1}}(t) &= F'_{T_{n+1}}(t) \\
&= f_{T_n}(t) - \mathbb{I}\{t>0\}\left(\frac{\lambda^n}{(n-1)!}t^{n-1}e^{-\lambda t} - \frac{\lambda^{n+1}}{n!}t^n e^{-\lambda t}\right) \\
&= \mathbb{I}\{t>0\}\frac{\lambda^n}{(n-1)!}t^{n-1}e^{-\lambda t} \;-\; \mathbb{I}\{t>0\}\left(\frac{\lambda^n}{(n-1)!}t^{n-1}e^{-\lambda t} - \frac{\lambda^{n+1}}{n!}t^n e^{-\lambda t}\right) \\
&= \mathbb{I}\{t>0\}\frac{\lambda^{n+1}}{n!}t^n e^{-\lambda t}.
\end{aligned}$$

This is the density function of Gamma$(n+1,\lambda)$, so the induction step has been shown.

Additional information. Above we found the probability $\mathbb{P}(T_n+\tau_{n+1}\le t)$ by integrating the joint density of T_n and τ_{n+1}. As we will learn later, we could avoid the double integral by using the formula $\mathbb{P}(T_n+\tau_{n+1}\le t) = \mathbb{E}[\mathbb{P}(T_n+\tau_{n+1}\le t \mid T_n)]$ and obtain the same result.

(b) Show that $\mathbb{P}\big(T_n \le t < T_{n+1}\big) = \frac{(\lambda t)^n}{n!}e^{-\lambda t}$, for $t\ge 0$.

Solution. The event $T_n\le t<T_{n+1}$ happens exactly when $T_n\le t$ but not $t\ge T_{n+1}$, so that

$$\begin{aligned}
\mathbb{P}\big[T_n\le t<T_{n+1}\big] &= \mathbb{P}\big[T_n\le t\big] - \mathbb{P}\big[T_{n+1}\le t\big] \\
&= F_{T_n}(t) - F_{T_{n+1}}(t) \\
&= F_{T_n}(t) - \Big(F_{T_n}(t) - \mathbb{I}\{t>0\}\frac{\lambda^n}{n!}t^n e^{-\lambda t}\Big) \\
&= \mathbb{I}\{t>0\}\frac{\lambda^n}{n!}t^n e^{-\lambda t}.
\end{aligned}$$

(c) We will now interpret $T_1<T_2<\cdots$ as an increasing sequence of time instants, and we denote by $N(t)=\#\{n\ge 1 : T_n\le t\}$ the number of time instants in $[0,t]$. With the help of (b), determine the probability distribution and expectation of $N(t)$.

Solution. For $t = 0$ the distribution of $N(t)$ has probability mass 1 at zero, and the expectation is zero. For $t > 0$ we may rewrite the event $\{N(t) = k\}$ to obtain:

$$\begin{aligned}
\mathbb{P}(N(t) = k) &= \mathbb{P}(\#\{n \geq 1 : T_n \leq t\} = k) \\
&= \mathbb{P}(\max_j (T_j \leq t) = k) \\
&= \mathbb{P}(T_k \leq t, T_{k+1} > t).
\end{aligned}$$

By part (b) this equals $\frac{\lambda^k}{k!} t^k e^{-\lambda t}$ for $t > 0$. Since this is the probability mass function of $\mathrm{Poi}(\lambda t)$, we know that $\mathbb{E}[N(t)]$ equals λt.

Homework problems

8.2 Trucks arrive at Vaalimaa border crossing at independent exponentially distributed intervals with mean 15 min. By random sampling, one third of the arriving trucks are directed to the customs for an inspection.

(a) What is the probability that no trucks arrive at the border crossing during an hour?

Solution. Since the intervals are independent and exponentially distributed, the arrivals are described by a poisson process. We choose the time unit 1 hour, so that the intensity parameter λ equals 4. Now

$$\mathbb{P}(N(1) = 0) \; = \; e^{-\lambda \cdot 1} \; = \; e^{-4} \; = \; 0.01831564.$$

(b) What is the probability that exactly two trucks are directed to the customs during an interval of 15 minutes?

Solution. Since the trucks are directed to the customs by random sampling (presumably independently), we obtain a thinned Poisson process with intensity $\lambda_1 = \lambda/3 = 4/3$. The time interval in question is $t_0 = 1/4$ hours, so the desired probability is given by

$$\begin{aligned} \mathbb{P}(N_1(t_0) = 2) \; &= \; e^{-\lambda_1 t_0} \frac{(\lambda_1 t_0)^2}{2!} \\ &= \; e^{-\frac{1}{3}} \frac{(\frac{1}{3})^2}{2} \\ &\approx 0.0398. \end{aligned}$$

8.3 Teemu Selänne has scored an average of $\lambda = 1$ points (a goal or an assist that led to a goal) per game in NHL. Let us assume that 30% of the points are goals and 70% assists that led to a goal. Let us assume that Teemu receives a bonus of \$3000 for a goal and \$1000 for an assist. Use a Poisson process to model the point scoring time instants during a 60 min game and answer the following questions.

(a) What is the expected value of the total bonus in an individual game?

Solution. Let the time unit be one hour (i.e., the duration of one game), and let

- X_t be the number of Teemu's goals,
- Y_t be the number of Teemu's assists, and
- $Z_t = X_t + Y_t$ be the number of Teemu's points in $[0, t]$.

We model the points Z_t as a Poisson process with intensity λ. From the lectures we know that such a process results when X_t and Y_t are independent Poisson processes, and from the information given to us we have $\lambda_X = 0.3$ and $\lambda_Y = 0.7$.

The expected bonus for one game (i.e., at $t = 1$) is

$$3000\mathbb{E}X_1 + 1000\mathbb{E}Y_1.$$

Since X_t and Y_t are Poisson distributed with means $\lambda_X t$ and $\lambda_Y t$, we have

$$3000\lambda_X \times 1 + 1000\lambda_Y \times 1 \;=\; 3000 \times 0.3 + 1000 \times 0.7 \;=\; 1600.$$

Additional information. We would get the same result by considering a process where each point is randomly categorized as a goal (with probability 30%) or an assist (with probability 70%). X_t and Y_t would then be the corresponding thinned Poisson processes.

(b) What is the standard deviation of the total bonus in an individual game?

Solution. Since X_1 and Y_1 are independent, the variance of the bonus is given by the sum

$$3000^2 \operatorname{Var}(X_1) + 1000^2 \operatorname{Var}(Y_1) \;=\; 3000^2\lambda_X + 1000^2\lambda_Y \;=\; 3.4 \times 10^6,$$

and the standard deviation is approximately 1843.91.

(c) What is the probability that Teemu has 1 goal and 2 assists in a game?

Solution. By independence we have

$$\mathbb{P}(X_1 = 1, Y_1 = 2) \;=\; \mathbb{P}(X_1 = 1)\mathbb{P}(Y_1 = 2) \;=\; e^{-\lambda_X}\frac{\lambda_X^1}{1} \times e^{-\lambda_Y}\frac{\lambda_Y^2}{2} \;=\; 0.027.$$