6 Infinite-state Markov chains

Classroom problems

6.1 *Bacteria growth.* Bacteria reproduce by cell division. During a unit of time, a bacterium will either die (with probability $\frac{1}{4}$), stay the same (with probability $\frac{1}{4}$), or split into two parts (with probability $\frac{1}{2}$). The population starts with one bacterium at time $t = 0$.

(a) Determine the generating function of the population size X_1 after 1 time units.

(b) Determine the generating function of the population size X_2 after 2 time units.

(c) Compute the extinction probability for the population.

(d) Let m_t be the largest possible number of bacteria after t time units. Determine m_t and compute the probability $\mathbb{P}(X_t = m_t)$.

(e) Given that there are 2000 bacteria at time $t = 50$, what is the expected number of bacteria at time $t = 51$?

(f) Compute the extinction probability for the population in an alternative setting where the population starts with 30 bacteria at time $t = 0$.

Homework problems

6.2 *Gambling pot.* A group of people are gambling. Initially there is a pot of 1 EUR on the table. Then a round is played, which results in a tie or one of the players winning. If the result is a tie, 1 EUR is added to the pot. If one of the players wins, the winner collects the accumulated pot in its entirety and the next round starts 1 EUR in the pot. Suppose that the probability of a tie in each round is $q \in [0, 1]$ and the results of the rounds are independent of each other.

(a) Let X_t be the size of the pot, in euros, in the t-th round. Show that the process $X = (X_t)_{t \in \mathbb{Z}_+}$ is a Markov chain on the infinite state space $S = \{1, 2, \dots\}$, and calculate its transition probabilities.

(b) For which values of the parameter q does the process $X = (X_t)_{t \in \mathbb{Z}_+}$ have an invariant distribution (i.e. a stationary distribution)? Calculate the invariant distribution.

6.3 *Renewal chain.* Let q be a probability distribution on the nonnegative integers $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$. A renewal chain corresponding to q is a Markov chain on the infinite state space $\mathbb{Z}_+$ with a transition matrix P such that $P(k, k-1) = 1$ for all $k \geq 1$, and $P(0, k) = q(k)$ for all $k \geq 0$.

(a) Sketch the transition diagram of the chain.

(b) Give an example of q for which the renewal chain is reducible.

(c) Give an example of q for which the renewal chain is irreducible.

(d) Prove that state 0 is recurrent.

(e) Derive a formula for the expected return time $\mathbb{E}\big(T_0^+ \mid X_0 = 0\big)$ to state 0 in terms of the distribution q, where $T_0^+ = \min\{t \geq 1 : X_t = 0\}$.

(f) Invent (or Google) an example of a probability distribution q that corresponds to a renewal chain for which $\mathbb{E}\big(T_0^+ \mid X_0 = 0\big) = \infty$.