MS-C2111 Stochastic processes

Lecture 4

Passage times and hitting probabilities

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Example: HR management

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Kalvonvääntäjät Oyj is a management consulting company with three types of employees: $1 =$ junior, $2 =$ senior, $3 =$ partner

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What is the expected duration that a newly recruited employee remains in the company?

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What is the expected duration that a newly recruited employee remains in the company?

How long is a freshly promoted partner expected to serve in the company?

Passage time

Passage time of $(X_0, X_1, \ldots)$ into A

$$T_A = \min\{t \geq 0 : X_t \in A\} \qquad (\min \emptyset = \infty)$$

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$$k_A(x) \;=\; \mathbb{E}(T_A \,|\, X_0 = x)$$

Deterministic number in $k_A(x) \in \mathbb{R}_+ \cup \{\infty\}$

Computing the expected passage times

Theorem: The expected passage times $k_A(x) = \mathbb{E}(T_A \mid X_0 = x)$ form the smallest nonnegative solution to

$$\begin{aligned} f(x) &= 1 + \sum_{y \notin A} P(x,y) f(y), && x \notin A, \\ f(x) &= 0, && x \in A. \end{aligned}$$

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$$\begin{aligned} k_A(x) &= \sum_y P(x,y)(1 + k_A(y)) = \sum_y P(x,y) + \sum_y P(x,y) k_A(y) \\ &= 1 + \sum_{y \in A^c} P(x,y) k_A(y). \end{aligned}$$

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Poisson equation

$$\begin{aligned} Df(x) &= -1, && x \in B, \\ f(x) &= 0, && x \in \partial B, \end{aligned}$$

$D = P - I$, $B = A^c$, $\partial B = A$.

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$f_n(x) \to k_A(x)$ when $n \to \infty$

Example: HR management

1 = junior, 2 = senior, 3 = partner, during each month:

- Junior promoted to senior w.pr. 0.030, exits w.pr. 0.020
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Q: How long does a freshly recruited employee stay in company?

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$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.020 & 0.950 & 0.030 & 0 \\ 0.008 & 0 & 0.982 & 0.010 \\ 0.010 & 0 & 0 & 0.990 \end{bmatrix}$$

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A: Expected passage time $k_A(1)$ from state 1 into $A = \{0\}$

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Let us solve $f(0) = 0$ and $f(x) = 1 + \sum_{y=1}^{3} P(x,y)f(y)$, $x = 1,2,3$

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$$\begin{aligned} f(1) &= 1 + 0.950\, f(1) + 0.030\, f(2) \\ f(2) &= 1 + 0.982\, f(2) + 0.010\, f(3) \\ f(3) &= 1 + 0.990\, f(3). \end{aligned}$$

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$$f(3) = \frac{1}{1 - 0.990} = 100$$

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$$f(3) = \frac{1}{1 - 0.990} = 100$$

$$f(2) = \frac{1 + 0.010\, f(3)}{1 - 0.982} = 111.11$$

Example: HR management

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$$f(3) = \frac{1}{1 - 0.990} = 100$$

$$f(2) = \frac{1 + 0.010\, f(3)}{1 - 0.982} = 111.11$$

$$f(1) = \frac{1 + 0.030\, f(2)}{1 - 0.950} = 86.67$$

Example: HR management

Let us solve $f(0) = 0$ and $f(x) = 1 + \sum_{y=1}^{3} P(x, y) f(y)$, $x = 1, 2, 3$

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Minimal solution?

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$$f(1) = 1 + 0.950\, f(1) + 0.030\, f(2)$$
$$f(2) = 1 + 0.982\, f(2) + 0.010\, f(3)$$
$$f(3) = 1 + 0.990\, f(3).$$

$$f(3) = \frac{1}{1 - 0.990} = 100$$

$$f(2) = \frac{1 + 0.010\, f(3)}{1 - 0.982} = 111.11$$

$$f(1) = \frac{1 + 0.030\, f(2)}{1 - 0.950} = 86.67$$

Minimal solution? YES because this is the only nonnegative solution.

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$$f(3) = \frac{1}{1 - 0.990} = 100$$

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$$f(1) = \frac{1 + 0.030\, f(2)}{1 - 0.950} = 86.67$$

Minimal solution? YES because this is the only nonnegative solution.

$\implies (k_A(0), k_A(1), k_A(2), k_A(3)) = (0, 86.67, 111.11, 100)$

Example: HR management

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Freshly promoted partner is expected to stay for 100 mo $\approx$ 8.3 y

Contents

Example: HR management

What is the probability that a freshly recruited junior eventually gets promoted to a partner?

Hitting probabilities

The probability that at process starting at x visits a set A is denoted by

$$h_A(x) \; = \; \mathbb{P}(X_t \in A \text{ for some } t \geq 0 \,|\, X_0 = x).$$

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In practice, to solve one hitting probability, we need to solve them all.

Computing hitting probabilities

Theorem

The hitting probabilities $h_A = (h_A(x) : x \in S)$ *form the smallest nonnegative solution to*

$$\begin{aligned} f(x) &= \sum_{y \in S} P(x,y) f(y), && x \notin A, \\ f(x) &= 1, && x \in A, \end{aligned}$$

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Poisson equation

Proof: Lecture notes.

$$\begin{aligned} Df(x) &= 0, \quad x \in B, \\ f(x) &= 1, \quad x \in \partial B, \end{aligned}$$

$D = P - I$, $B = A^c$, $\partial B = A$.

Example: HR management

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$f(1) = h_A(1) = 0.333$
$\sum_x f(x) \neq 1$ (why?)

Contents

Gambler's ruin

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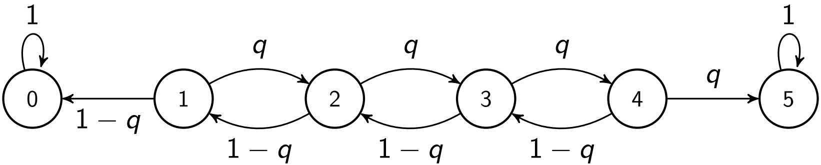

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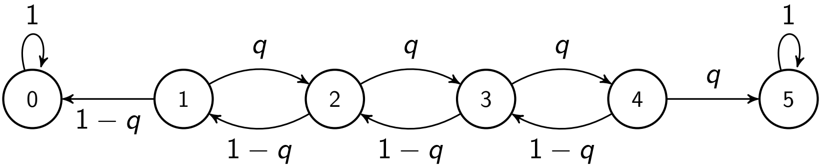

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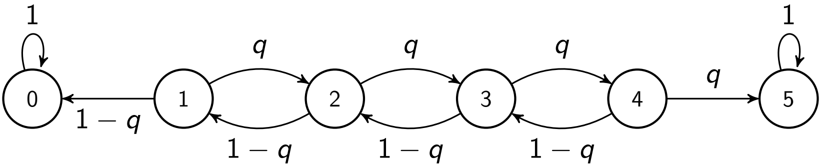

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Ruin probability $= 1 - h(x)$.

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$\Longrightarrow$ $h(x)$ is the minimal nonnegative solution of

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$$h(x) = \frac{\alpha^x - 1}{\alpha^M - 1}.$$

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Let us study how

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by l'Hôpital's rule.

Gambler's ruin — Main theorem

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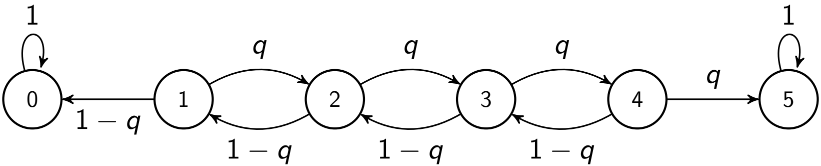
The probability that the random walk started at x eventually hits M equals

$$h(x) = \begin{cases} \dfrac{\left(\frac{1-q}{q}\right)^x - 1}{\left(\frac{1-q}{q}\right)^M - 1}, & q \neq \frac{1}{2}, \\ \frac{x}{M}, & q = \frac{1}{2}. \end{cases}$$



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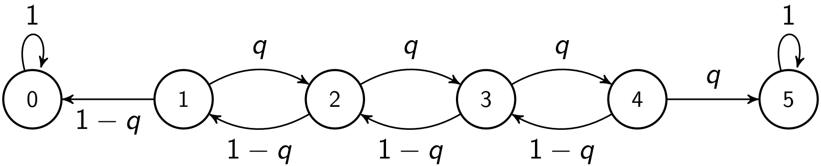
$$h(x) = \begin{cases} \dfrac{\left(\frac{1-q}{q}\right)^x - 1}{\left(\frac{1-q}{q}\right)^M - 1}, & q \neq \frac{1}{2}, \\ \frac{x}{M}, & q = \frac{1}{2}. \end{cases}$$



Main message: When $q \leq \frac{1}{2}$, the probability of ever reaching a state M from an initial state x tends to zero as $M \to \infty$.

Example: Roulette

In a game of roulette where a bet of 1 EUR is placed on the ball falling into one of 18 red pockets out of 37 pockets, the probability of winning 1 EUR is $q = \frac{18}{37}$ and the probability of losing 1 EUR is $1 - q$. If a gambler targets to double his initial wealth x, then the probability $h(x)$ of successfully ending the game is obtained by applying the previous theorem with $M = 2x$.

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Initial wealth (EUR)	1	5	10	20	50
Success probability	0.4865	0.4328	0.3680	0.2533	0.0628

Table: Probability of successfully doubling the initial wealth in a game of roulette by betting 1 EUR on red.

Aineistolähteet

Esityksessä käytetyt kuvat (esiintymisjärjestyksessä)

1. Image courtesy of think4photop at FreeDigitalPhotos.net