12 Optional times and stopped martingales

You learn to recognize which random times are optional times, and to compute values and estimates of hitting probabilities using stopped martingales. In this exercise there is no homework!

Classroom problems

12.1 *Pólya's urn.* An urn contains one red and one green ball in the beginning. During round $t = 1, 2, \dots,$ a ball is randomly picked from the urn and its color its observed. Then the ball is returned to the urn and another ball of the same color as observed is added to the urn. Let X_t be the relative proportion of red balls in the urn after t rounds.

(a) Verify that the process $(X_t)_{t \in \mathbb{Z}_+}$ is a martingale.

(b) Prove that the probability that the relative proportion of red balls ever reaches level 0.9 is at most 5/9.
Hint: Analyze the stopped martingale $\hat{X}_t = X_{t \wedge T}$ where T is the first time instant (possibly infinite) that the proportion of red balls reaches the level 0.9.

There are several versions of the optional stopping theorem, one of which is formulated in the lecture notes in Theorem 12.8. Here's another useful version:

Theorem. *Optional stopping theorem II.* Let $(M_0, M_1, \dots)$ be a martingale and T a finite optional time, meaning $\mathbb{P}[T < \infty] = 1$. Assume further that M has uniformly bounded increments, meaning there exists a constant $C \geq 0$ such that for every $t \in \mathbb{N}$ we have

$$|M_{t+1} - M_t| \leq C.$$

Then $\mathbb{E}[M_T] = \mathbb{E}[M_0]$.

12.2 Jack has an unbiased six-sided die. He wants to impress his friends with a video where he seems to be able to predict six consecutive dice rolls before throwing them. In reality, he repeatedly says a number between 1 and 6 and throws a die until he has successfully predicted the result six times in a row.

Denote by X_t the latest streak of successive correct predictions after the t'th throw. Let T be the first time when Jack has successfully predicted six throws in a row.

(a) Find a function $f : \mathbb{N} \to \mathbb{R}$ such that $M_t := f(X_t) - t$ is a martingale with respect to $(X_0, X_1, \dots)$.

(b) Show that T is an optional time with $\mathbb{P}[T < \infty] = 1$.

(c) Show that the stopped martingale $(M_{t \wedge T})_{t \in \mathbb{N}}$ has uniformly bounded increments.

(d) Using the stopped martingale from (c), compute the expected number of throws $\mathbb{E}[T]$ it takes Jack to finish the video. How reasonable is Jack's plan?