4 Markov additive models, hitting times and probabilities

The goal of this exercise is to learn to calculate the cumulative revenues associated to Markov additive models, the expected passage times of Markov chains and the probabilities to hit a certain state before visiting a given set of states. It's recommended to bring a laptop or a calculator to the exercise session to make it easier to calculate the numerical results of the exercises.

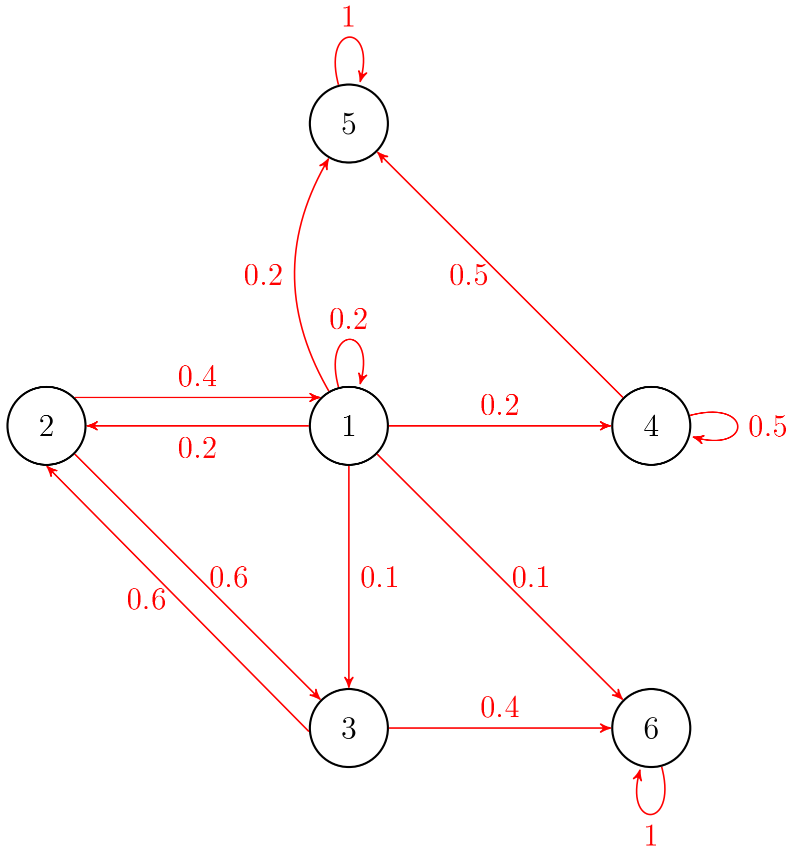
Classroom problems

4.1 Consider a Markov chain with state space $S = \{1, 2, \dots, 6\}$, initial state 1, and transition matrix

$$P = \begin{bmatrix} 0.2 & 0.2 & 0.1 & 0.2 & 0.2 & 0.1 \\ 0.4 & 0 & 0.6 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0 & 0 & 0.4 \\ 0 & 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Draw the transition diagram of the chain and answer the following questions:

Solution.



(a) What is the probability that a chain eventually ends up in state 5?

Solution. States 5 and 6 are *absorbing states* (or *sinks*). Hitting such a state and ending up in it are the same event. Denote the vector of hitting probabilities by $f(x)$. Now

$$\begin{cases} f(5) = 1 \\ f(x) = \sum_{y \in S} P(x,y) f(y) = (Pf)(x), \qquad x \neq 5. \end{cases}$$

It remains to solve $f(x)$ for $x \neq 5$. Since the state 6 is a sink, we conclude $f(6) = 0$. We may thus write the above system of equations with matrices as

$$\begin{bmatrix} f(1) \\ f(2) \\ f(3) \\ f(4) \\ f(5) \\ f(6) \end{bmatrix} = \begin{bmatrix} 0.2 & 0.2 & 0.1 & 0.2 & 0.2 & 0.1 \\ 0.4 & 0 & 0.6 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0 & 0 & 0.4 \\ 0 & 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f(1) \\ f(2) \\ f(3) \\ f(4) \\ 1 \\ 0 \end{bmatrix}.$$

As we already know the values for $f(5)$ and $f(6)$, we may disregard the last two rows of the above equation. When doing so, we need to multiply the last two columns of the matrix in:

$$\begin{bmatrix} f(1) \\ f(2) \\ f(3) \\ f(4) \end{bmatrix} = \begin{bmatrix} 0.2 & 0.2 & 0.1 & 0.2 \\ 0.4 & 0 & 0.6 & 0 \\ 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} f(1) \\ f(2) \\ f(3) \\ f(4) \end{bmatrix} + \begin{bmatrix} 0.2 \\ 0 \\ 0 \\ 0.5 \end{bmatrix}.$$

Rearranging the above then gives

$$\left(\begin{bmatrix} 0.2 & 0.2 & 0.1 & 0.2 \\ 0.4 & 0 & 0.6 & 0 \\ 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix} - I_4 \right) \begin{bmatrix} f(1) \\ f(2) \\ f(3) \\ f(4) \end{bmatrix} = \begin{bmatrix} -0.2 \\ 0 \\ 0 \\ -0.5 \end{bmatrix}.$$

The matrix is now nonsingular, and has the unique solution

$$\left[f(1), f(2), f(3), f(4) \right] = \left[0.6275, 0.3922, 0.2353, 1 \right],$$

and so

$$f(1) = 0.6275.$$

(b) What is the probability that a chain eventually ends up in state 6?

Solution. Since the chain ends up in state 5 or 6,

$$1 - f(1) = 0.3725.$$

(c) What is the probability that a chain never visits state 3?

Solution. We first find the probability to visit state 3. Let $g(x)$ be the hitting probabilities, so that

$$\begin{cases} g(3) = 1 \\ g(x) = \sum_{y \in S} P(x,y) g(y), \qquad x \neq 3. \end{cases}$$

Since states 5 and 6 are sinks, we have $g(5) = g(6) = 0$. We may thus write the above system of equations with matrices as

$$\begin{bmatrix} g(1) \\ g(2) \\ g(3) \\ g(4) \\ g(5) \\ g(6) \end{bmatrix} = \begin{bmatrix} 0.2 & 0.2 & 0.1 & 0.2 & 0.2 & 0.1 \\ 0.4 & 0 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} g(1) \\ g(2) \\ 1 \\ g(4) \\ 0 \\ 0 \end{bmatrix}.$$

As in part (a), we may disregard rows 3, 5 and 6 by multiplying the respective columns in to our equation:

$$\begin{bmatrix} g(1) \\ g(2) \\ g(4) \end{bmatrix} = \begin{bmatrix} 0.2 & 0.2 & 0.2 \\ 0.4 & 0 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} g(1) \\ g(2) \\ g(4) \end{bmatrix} + \begin{bmatrix} 0.1 \\ 0.6 \\ 0 \end{bmatrix}$$

Rearranging then gives

$$\left(\begin{bmatrix} 0.2 & 0.2 & 0.2 \\ 0.4 & 0 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} - I_5 \right) \begin{bmatrix} g(1) \\ g(2) \\ g(4) \end{bmatrix} = \begin{bmatrix} -0.1 \\ -0.6 \\ 0 \end{bmatrix},$$

from which we get

$$g = [0.3056, 0.7222, 1, 0, 0, 0].$$

Thus the probability of not hitting state 3 when starting from state 1 is

$$1 - 0.3056 = 0.6944.$$

(d) What is the expected number of time steps before the chain hits the set $\{5, 6\}$?

Solution.

The expected passage times $f(x)$ from state x to the set $A = \{5, 6\}$ satisfy

$$\begin{cases} f(x) = 0, & x \in \{5, 6\} \\ f(x) = 1 + \sum_{y \in S} P(x,y) f(y), & \text{for other } x. \end{cases}$$

Let $\tilde{P}$ be the matrix P with 5th and 6th rows and columns removed, i.e.,

$$\tilde{P} = \begin{bmatrix} 0.2 & 0.2 & 0.1 & 0.2 \\ 0.4 & 0 & 0.6 & 0 \\ 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}. \tag{1}$$

Let $\tilde{f} = [f(1), f(2), f(3), f(4)]^T$. As in the previous parts, we may write this with matrices as

$$\begin{aligned} \tilde{f} &= [1,1,1,1]^T + \tilde{P}\tilde{f} \\ \Rightarrow \quad \tilde{f} - \tilde{P}\tilde{f} &= [1,1,1,1]^T \\ \Rightarrow \quad (I_4 - \tilde{P})\tilde{f} &= [1,1,1,1]^T \end{aligned}$$

The matrix on the left-hand side is nonsingular, so we may multiply both sides by its inverse and get

$$f(1) = \left((I_4 - \tilde{P})^{-1}[1,1,1,1]^T\right)_1 = 3.3725.$$

Homework problems

4.2 Analyze the expected revenues of Katiskakauppa.com Oyj (Examples 1.3, 3.3, 3.7 in the lecture notes) in the case where the weekly demands D_t are Poisson distributed with mean $\lambda = 3.0$.

(a) Compute the expected cumulative revenue for a period of ten weeks, as a function of the number of laptops in stock ($x = 2, 3, 4, 5$) in the beginning of the first week.

Solution. Repeating the steps in Example 1.3 for $\lambda = 3.5$ gives the transition matrix

$$P = \begin{bmatrix} 0.0498 & 0 & 0 & 0.9502 \\ 0.1494 & 0.0498 & 0 & 0.8008 \\ 0.2240 & 0.1494 & 0.0498 & 0.5768 \\ 0.2240 & 0.2240 & 0.1494 & 0.4026 \end{bmatrix},$$

where the states are in the order 2,3,4,5. The *expected cumulative revenue* with initial state x is $g(x) = \mathbb{E}(V_{10}|X_0 = x)$. The *cumulative revenue* is the random variable

$$V_t = \sum_{s=0}^{t-1} \phi(X_s, D_s),$$

where X_s is the number of laptops in stock, the demand is D_s, and the revenue at a single time step is[1]

$$\phi(x,u) = 790 \min(x,u).$$

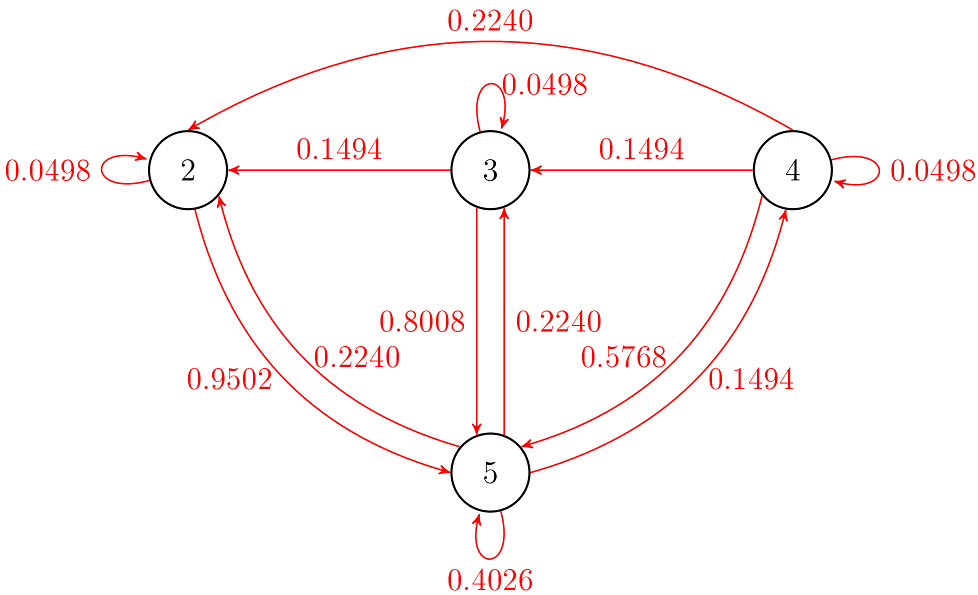
The expected revenue in state x is obtained, as in Example 3.3, using

$$v(x) = \mathbb{E}(\phi(x, D_0)) = 790\mathbb{E}(\min(x,u)) = 790 \left(x - \sum_{k=0}^{x} e^{-\lambda}\frac{\lambda^k}{k!}(x-k)\right),$$

so that for states 2–5 we have

$$v = [1383.34,\ 1839.02,\ 2117.71,\ 2263.65]^T.$$

[1] we interpret "revenue" as "gross revenue" as opposed to "net revenue"



Finally, the expected cumulative revenue is obtained using the formula from the lecture notes: $g_t = \sum_{s=0}^{t-1} P^s v$, i.e.,

$$g_{10} \approx [19771,\ 20144,\ 20337,\ 20449]^T.$$

(b) Compute the invariant distribution of the Markov chain modeling the number of laptops in stock.

Solution. The invariant distribution $\pi = [\pi_2, \pi_3, \pi_4, \pi_5]$ satisfies the balance equations:

$$\begin{cases} \pi P = \pi \\ \pi[1,1,1,1]^T = 1. \end{cases}$$

We write this system of equations as

$$\begin{aligned} 0.0498\pi_2 + 0.1494\pi_3 + 0.2240\pi_4 + 0.2240\pi_5 &= \pi_2 \\ 0.0498\pi_3 + 0.1494\pi_4 + 0.2240\pi_5 &= \pi_3 \\ 0.0498\pi_4 + 0.1494\pi_5 &= \pi_4 \\ 0.9502\pi_2 + 0.8008\pi_3 + 0.5768\pi_4 + 0.4026\pi_5 &= \pi_5 \\ \pi_2 + \pi_3 + \pi_4 + \pi_5 &= 1, \end{aligned}$$

where the third equation gives

$$\pi_4 = \frac{1494}{9502}\pi_5 \approx 0.1572\pi_5,$$

which we substitute in the second equation:

$$\pi_3 = (2240 + 1494\frac{1494}{9502})\frac{\pi_5}{9502} \approx 0.2605\pi_5,$$

and so the first equation gives

$$\pi_2 = \frac{1}{9502}(1494\pi_3 + 2240\pi_4 + 2240\pi_5) \approx 0.3132\pi_5.$$

Finally, the fifth equation gives

$$0.3132\pi_5 + 0.2605\pi_5 + 0.1572\pi_5 + \pi_5 = 1 \Leftrightarrow \pi_5 = 0.5775,$$

so the invariant distribution is

$$[0.1812, 0.1504, 0.0908, 0.5775] \approx [0.181, 0.150, 0.091, 0.578].$$

(c) Compute the store's long-term expected revenue rate (EUR/week).

Solution.

The long-term expected revenue rate is obtained directly from the invariant distribution. We recall that by (a) we have

$$[v(2), v(3), v(4), v(5)] = [1383.34,\ 1839.02,\ 2117.71,\ 2263.65]$$

and denote the invariant distribution (from (b)) by π, so that

$$\sum_{x \in S} \pi(x)v(x) \approx 2026.8.$$

Additional information. By the ergodic theorem we have a stronger property for deterministic revenues $v(X_s)$: in the long term the *average* revenues approach $\sum_x \pi(x)v(x)$ with probability 1. In the Katiskakauppa example the revenue at time s is random and only depends the following state, $V(X_s) = v(X_s, X_{s+1})$. Ergodicity can be shown to hold for such revenues as well. Because of this, the expected long-term revenue rate is also the *long-term average* of the revenues.

(d) Compare the result of part c) to your results in a).

Solution. By the previous part we approximate the 10-week cumulative revenue as 20268 €. Depending on the initial state, by (a) we have 19771–20449 €. It seems that in this example the behavior of the Markov chain can be modeled with the invariant distribution also in the short term.

4.3 Let $(X_0, X_1, \dots)$ be a Markov chain defined on a finite state space S and suppose that a deterministic cost $c(x)$ is incurred every time the chain visits the state x. Let $g(x)$ be the expected total cost of a Markov chain starting from the state x before the chain hits a state set A.

(a) Derive the following equations for the function $g : S \to \mathbb{R}$:

$$\begin{aligned} g(x) &= 0, && x \in A \\ g(x) &= c(x) + \sum_{y \in S} P(x,y)\, g(y), && x \notin A. \end{aligned}$$

Solution. When $x \in A$, no steps are taken, and so $g(x) = 0$. We denote by τ_A the hitting time to set A and set $c(x) = 0$, $x \in A$. We may now write

$$g(x) = \mathbb{E}[\sum_{t=0}^{\tau_A} c(X_t) | X_0 = x].$$

Note that the special case $c(x) = 1$ for $x \notin A$ gives the expected passage times. It turns out that the proof for this more general case is not too different:

$$\begin{aligned} g(x) &= c(x) + \mathbb{E}[\sum_{t=1}^{\tau_A} c(X_t) | X_0 = x] \\ \text{(conditioning)} &= c(x) + \sum_{y \in S} p_{x,y} \mathbb{E}[\sum_{t=1}^{\tau_A} c(X_t) | X_0 = x, X_1 = y] \\ \text{(Markov property)} &= c(x) + \sum_{y \in S} p_{x,y} \mathbb{E}[\sum_{t=1}^{\tau_A} c(X_t) | X_1 = y] \\ \text{(definition of } g) &= c(x) + \sum_{y \in S} p_{x,y} g(y). \end{aligned}$$

(b) Consider the chain of Problem 4.1. By applying the result of part (a), determine the expected number of times that the chain starting from state 1 visits state 3 before it gets absorbed into the set $\{5, 6\}$.

Solution. Again, let $g(x)$ be the expected total cost when starting from state x. We set $c(x) = \mathbb{I}_{\{3\}}(x)$, which gives the equations

$$\begin{aligned} g(x) &= 0, && x \in \{5,6\} \\ g(x) &= \mathbb{I}_{\{3\}}(x) + \sum_{y \in S} p_{x,y}\, g(y), && x \notin \{5,6\}. \end{aligned}$$

Let $\tilde{P}$ be as in Eq. (1). Now

$$\left(I_4 - \tilde{P}\right) \left[g(1), g(2), g(3), g(4)\right]^T = \left[0, 0, 1, 0\right]^T \quad \Rightarrow \quad g(1) = 0.5392.$$

(Naturally, the expected number of visits 0.5392 is greater than the hitting probability 0.3056 obtained in 4.1(c).)