3 Long-term behavior of Markov chains

In this exercise you learn to recognize whether a Markov chain is reducible or periodic, and whether the chain admits a limiting distribution, by inspecting the transition matrix and the transition diagram of the chain. You also learn to compute the invariant distributions of a given transition matrix. It is recommended to bring a laptop or a calculator to the exercise session to make it easier to calculate the numerical results of the exercises.

Classroom problems

3.1 *Periodicity of an irreducible chain.* Justify why the following results are true for the transition matrix P of a Markov chain with a finite state space S. (Recall that $P^t(x,y)$ denotes the entry on row x and column y of the t-th matrix power of P.)

(a) If $P(x,x) > 0$, then also $P^t(x,x) > 0$ for all $t \geq 1$.

(b) If $P(x,x) > 0$, then the period of state x is 1.

(c) If $P(x,x) > 0$ and $x \leftrightsquigarrow y$ (both states are reachable from each other by directed paths in the transition diagram), then there exists an integer $t_0 \geq 1$ such that $P^t(y,y) > 0$ for all $t \geq t_0$.

(d) An irreducible chain is aperiodic if $P(x,x) > 0$ holds for some state x.

Homework problems

3.2 Determine the long-term behavior of the following Markov chains.

(a) The bike of a bicycle commuter on a given work day is either `unbroken` or `broken`. If the bike is `unbroken` on a given work day, then it's also `unbroken` the following day with probability 95% and otherwise `broken`. If the bike is `broken`, then it's `unbroken` the next work day with probability 33% and otherwise `broken`. In both cases, the state of the bike is independent of any earlier states. In the long term, what is the proportion of work days that the bike is `broken`?

(b) Consider the Markov chain of Problem 2.3 with state space $S = \{\mathtt{AA}, \mathtt{Aa}, \mathtt{aa}\}$ and transition matrix

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1/2 & 1/2 \end{bmatrix}.$$

Calculate the proportion of each genotype in the chain of descendants in the long term.

3.3 *PageRanks of nodes with high and low indegrees.* Consider a directed graph defined on node set $V = \{1, 2, \ldots, n\}$ that has the links $1 \to 2$, $2 \to 1$, and $x \to 2$ for $x = 3, 4, \ldots, n$. Let $(X_0, X_1, \ldots)$ be a Markov chain that follows the PageRank algorithm for this graph, as discussed in the lecture notes (Example 1.4).

(a) Draw the transition diagram of the graph and determine for which values of the damping factor c the Markov chain is irreducible.

(b) Compute the PageRanks for the nodes in the graph.

(c) How do the PageRanks behave when $c = 0$ and $c = 1$?

(d) How do the PageRanks behave when $n \to \infty$?