MS-C2111 Stochastic Processes

Lecture 1:
Markov chains

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Aalto University

Contents

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Markov chain

Random sequence $(X_0, X_1, X_2, \dots)$ with the property:

The future state X_{t+1} is conditionally independent of the past given the current state X_t.

Andrei Markov
1856–1922

Andrei Markov
1978–

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$$\mathbb{P}(X_{t+1} = y \mid X_t = x, H_{t-}) \;=\; \mathbb{P}(X_{t+1} = y \mid X_t = x)$$

for all events $H_{t-} = \{X_0 = x_0, \ldots, X_{t-1} = x_{t-1}\}$
and for all states x, y

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(Here assumed finite)

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Transition matrix: $P(x, y) = \mathbb{P}(X_{t+1} = y \,|\, X_t = x)$ is the probability to move from state x to state y
(Here assumed constant over time)

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1856–1922

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Example: Weather model

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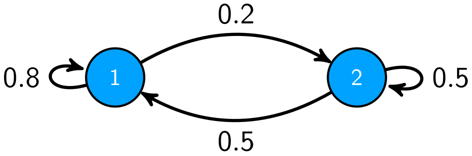
Transition diagram



Predicting using the weather model

If it is cloudy on Mon (day 0), then what is the probability that it is cloudy also on Wed?

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$$\mathbb{P}(X_2 = 1 \mid X_0 = 1)$$
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Can you predict Saturday's weather?

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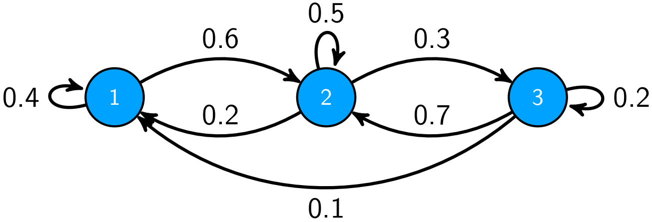

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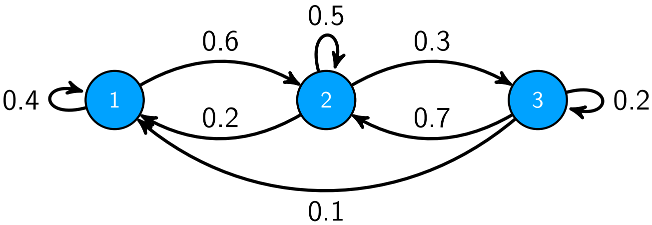

Computing weather predictions manually becomes harder.

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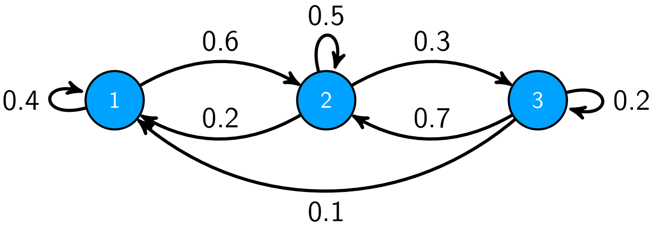

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Real weather models may have thousands of states...

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Interpretation:
X_t = Location of a surfer at time t who browses the web by randomly selecting hyperlinks
c = Probability of the surfer deciding to teleport to a random page

Contents

Transient distribution

Distribution of the chain at time t is a vector μ_t with entries

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telling the probability of finding the chain in state x at time t

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μ_0 is called the initial distribution of the chain

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One-step time evolution

The probability of finding the chain in state y at time $t+1$ equals

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Key result

Theorem

The distribution of the chain at time instant $t = 0, 1, 2, \ldots$ can be computed by

$$\mu_t \;=\; \mu_0 P^t,$$

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Hence claim OK also for time $t + 1$. $\square$

Example: Weather prediction

If it is cloudy on Mon, then what is the probability that it is cloudy also on Wed? What about Sat? ($S = \{1, 2\}$ with 1=cloudy, 2=sunny)

$$P = \begin{bmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{bmatrix}$$

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Transition matrix: Properties

$$P(x, y) \; = \; \mathbb{P}(X_{t+1} = y \,|\, X_t = x)$$

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$$\sum_{y \in S} P(x, y) \;=\; \sum_{y \in S} \mathbb{P}(X_{t+1} = y \,|\, X_t = x) \;=\; 1.$$

Many-step transition probabilities

Theorem

The probability that a Markov chain moves from state x to state y during t time steps can be computed as

$$\mathbb{P}(X_t = y \,|\, X_0 = x) \;=\; P^t(x, y),$$

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Proof.

Similar induction proof works. [Lecture notes, Thm 1.7] $\square$

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$$\mathbb{P}(X_0 = x_0, X_1 = x_1) \;=\; \mathbb{P}(X_0 = x_0)\mathbb{P}(X_1 = x_1 \,|\, X_0 = x_0)$$

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For $t \geq 3$: $\square$

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\end{aligned}$$

For $t \geq 3$: Induction... $\square$

Occupancy of states

The frequency of state y among the first t states is

$$N_t(y) \;=\; \sum_{s=0}^{t-1} 1(X_s = y),$$

Occupancy matrix M_t has entries

$$M_t(x,y) \;=\; \mathbb{E}(N_t(y) \,|\, X_0 = x).$$

The entry of the occupancy matrix M_t for row x and column y tells the expected number of times that a chain starting at x visits y during the first t time instants.

Computing the occupancy matrix

Theorem

The occupancy matrix can be computed as $M_t = \sum_{s=0}^{t-1} P^s$.

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The occupancy matrix can be computed as $M_t = \sum_{s=0}^{t-1} P^s$.

Proof.

Let $\mathbb{P}_x, \mathbb{E}_x$ be the conditional probability and expectation given $X_0 = x$.

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Theorem

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$\square$

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Predict the expected number of cloudy days during a week starting with a sunny day.
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Then $(X_0, X_1, X_2, \dots)$ is a Markov chain with transition matrix P.

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Discussion: Dynamical systems vs. Markov chains

	Dynamical system	**Markov chain**
Initial state	Deterministic x_0	Random X_0 with law μ_0
State evolution	$x_{t+1} = f(x_t)$	$X_{t+1} = f(X_t, U_{t+1})$
Law at time t	δ_{x_t}	μ_t
Law evolution	$\mu_{t+1} = \delta_{f(x_t)}$	$\mu_{t+1} = \mu_t P$

Every Markov chain admits a (nonunique) stochastic representation $X_{t+1} = f(X_t, U_{t+1})$ for some $f : S \times S' \to S$ and some $U_1, U_2, \ldots$

Next time we discuss Markov chains in the long run as $t \to \infty$

Kirjallisuutta

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Aineistolähteet

Esityksessä käytetyt kuvat (esiintymisjärjestyksessä)

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