# MS-C2111 Stochastic processes

Lecture 3

*Markov additive processes*

Jukka Kohonen
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# Contents

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**State:** $X_t$ = Number of laptops in stock in start of week $t$

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**Stochastic demand model:**
$D_0, D_1, D_2, \ldots$ are $\perp\!\!\!\perp$ and Poisson distributed with mean 3.5

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Stochastic process $(X_0, X_1, X_2, \ldots)$ is a Markov chain with state space $S =$

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$D_0, D_1, D_2, \ldots$ are $\perp\!\!\!\perp$ and Poisson distributed with mean 3.5

$\Longrightarrow$

Stochastic process $(X_0, X_1, X_2, \ldots)$ is a Markov chain with state space $S = \{2, 3, 4, 5\}$

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$S = \{2, 3, 4, 5\}$

**Transition matrix** $P = \begin{bmatrix} & & & \\ & & & \\ & & & \end{bmatrix}$

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If there are $X_t = 2$ laptops in stock in start of week $t$, then

- $X_{t+1} = 2$ with probability

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- $X_{t+1} = 2$ with probability $\mathbb{P}(D_t = 0) =$

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If there are $X_t = 2$ laptops in stock in start of week $t$, then

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- $X_{t+1} = 2$ with probability $\mathbb{P}(D_t = 0) = e^{-3.5} = 0.03$
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$S = \{2, 3, 4, 5\}$

**Transition matrix** $P = \begin{bmatrix} 0.03 & 0 & 0 & 0.97 \\ & & & \\ & & & \end{bmatrix}$

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- $X_{t+1} = 5$ with probability $\mathbb{P}(D_t \geq 1) = 1 - e^{-3.5} = 0.97$

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If there are $X_t = 3$ laptops in stock in start of week $t$, then

- $X_{t+1} = 2$ with probability

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**Transition matrix** $P = \begin{bmatrix} 0.03 & 0 & 0 & 0.97 \\ 0.11 & 0.03 & 0 & 0.86 \\ & & & \\ & & & \end{bmatrix}$

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- $X_{t+1} = 4$ with probability $0$
- $X_{t+1} = 5$ with probability $\mathbb{P}(D_t \geq 2) = 1 - e^{-3.5} - 3.5e^{-3.5} = 0.86$

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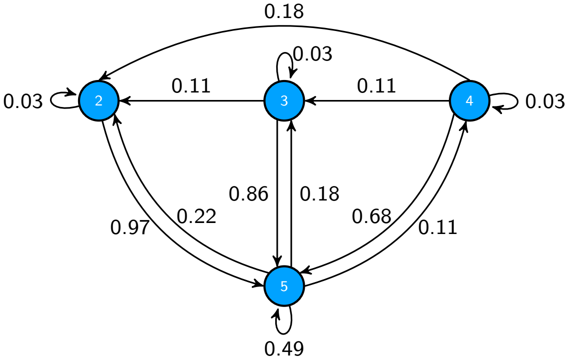
$S = \{2, 3, 4, 5\}$

**Transition matrix** $P = \begin{bmatrix} 0.03 & 0 & 0 & 0.97 \\ 0.11 & 0.03 & 0 & 0.86 \\ 0.18 & 0.11 & 0.03 & 0.68 \\ 0.22 & 0.18 & 0.11 & 0.49 \end{bmatrix}$




[POLL] Irreducible?

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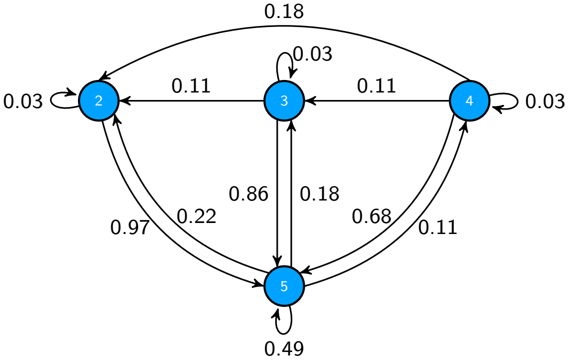


[POLL] Irreducible? YES

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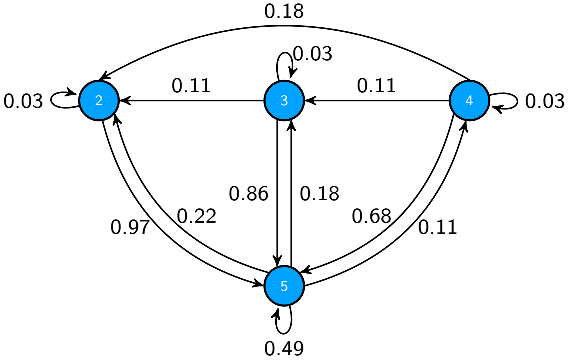


[POLL] Irreducible? YES Aperiodic?

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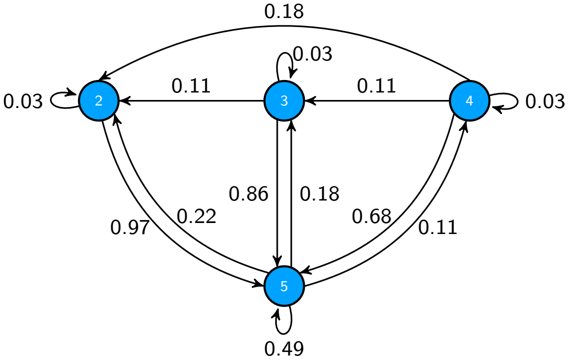
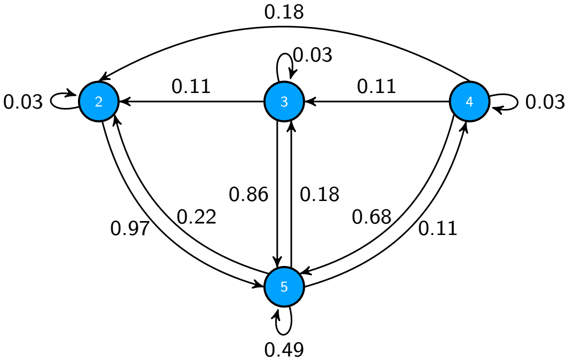
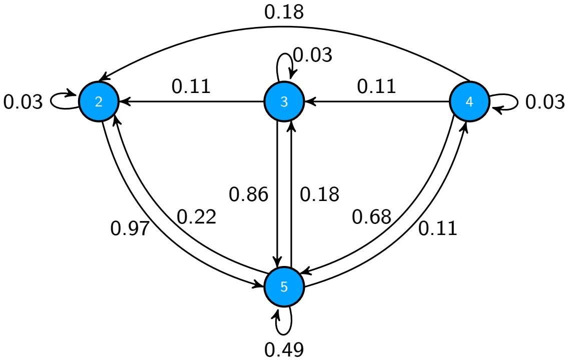



[POLL] Irreducible? YES   Aperiodic? YES

# Katiskakauppa.com: Cash Flows

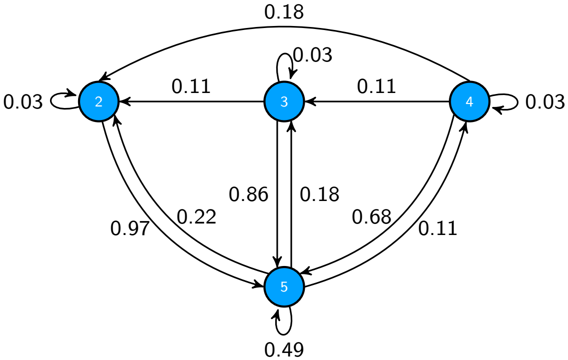


# Katiskakauppa.com: Cash Flows




Laptops are bought for 590 EUR and sold for 790 EUR
Storage cost 50 EUR per laptop per starting week

# Katiskakauppa.com: Cash Flows

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Expected cash flow during next 10 weeks = ?

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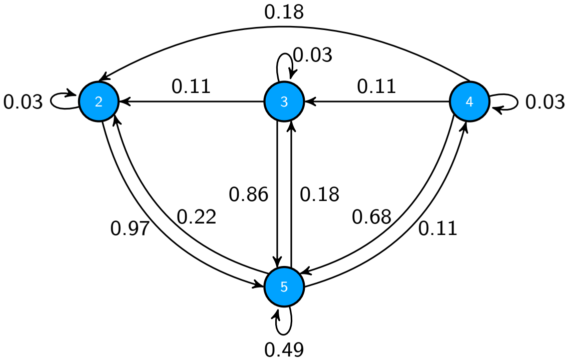


Laptops are bought for 590 EUR and sold for 790 EUR
Storage cost 50 EUR per laptop per starting week

Expected cash flow during next 10 weeks = ?
Expected long-term cash flow rate = ?

# Contents

# Markov additive process

A bivariate random process $(X_t, V_t)$

- Markov component $(X_t)$ is a Markov chain
- Additive component $(V_t)$ has representation

$$V_t = \phi(X_0, U_0) + \cdots + \phi(X_{t-1}, U_{t-1})$$

for some function $\phi$ and IID random variables $U_0, U_1, \ldots$ such that $U_t$ is independent of $(X_0, \ldots, X_t)$ for all $t \geq 0$.

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Note:

- $X_{t+1}$ depends on $\{X_s : s \leq t\}$ only via

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Note:

- $X_{t+1}$ depends on $\{X_s : s \leq t\}$ only via $X_t$ (Markov property)

# Markov additive process

A bivariate random process $(X_t, V_t)$

- Markov component $(X_t)$ is a Markov chain
- Additive component $(V_t)$ has representation

$$V_t = \phi(X_0, U_0) + \cdots + \phi(X_{t-1}, U_{t-1})$$

for some function $\phi$ and IID random variables $U_0, U_1, \ldots$ such that $U_t$ is independent of $(X_0, \ldots, X_t)$ for all $t \geq 0$.

Note:

- $X_{t+1}$ depends on $\{X_s : s \leq t\}$ only via $X_t$ (Markov property)
- $X_{t+1}$ depends on $\{(X_s, V_s) : s \leq t\}$ only via

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$(V_t)$ is in general NOT a Markov chain

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How to compute the expected number of snowy days in November?

# Finite-term behavior

## Theorem

*For a Markov additive process where the Markov component has transition matrix $P$,*

$$\mathbb{E}(V_t \mid X_0 = x) \;=\; \sum_{s=0}^{t-1} \sum_{y} P^s(x,y) v(y)$$

*where* $v(y) = \mathbb{E}\phi(y, U_0)$.

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In matrix form, the function $g_t(x) = \mathbb{E}(V_t \mid X_0 = x)$ is given by

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**Note**: $v(x) = \mathbb{E}\phi(x, U_t) = \mathbb{E}(V_{t+1} - V_t \mid X_t = x)$.

# Proof

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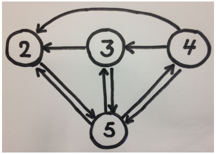
# Example: Inventory model

$$P = \begin{bmatrix} 0.03 & 0 & 0 & 0.97 \\ 0.11 & 0.03 & 0 & 0.86 \\ 0.18 & 0.11 & 0.03 & 0.68 \\ 0.22 & 0.18 & 0.11 & 0.49 \end{bmatrix}$$




$X_t$ = Number of laptops in stock at the start of week $t$

$D_t$ = Demand during week $t$

Laptops bought for 590 EUR and sold for 790 EUR.

Storage cost = 50 EUR per laptop per starting week.

Net cash flow from first $t$ weeks

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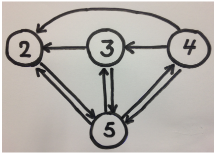


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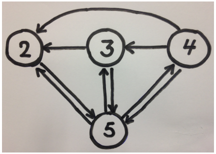


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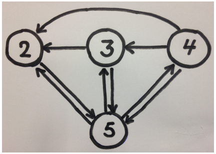


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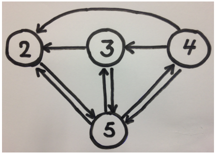


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$$\mathbb{E}(\min(x, D_t)) = \sum_{k=0}^{x} k\mathbb{P}(D_t = k) + \sum_{k=x+1}^{\infty} x\mathbb{P}(D_t = k)$$

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v(x) \;=\; 200\,\mathbb{E}(\min(x, D_t)) - 50x \;=\; \begin{cases} 266.78, & x = 2, \\ 352.61, & x = 3, \\ 395.29, & x = 4, \\ 400.20, & x = 5. \end{cases}
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Best expected cash flow $v(5) = 400.20$ EUR from a week starting with 5 laptops in stock.

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Net cash flow from 10 weeks as a function of initial state is obtained as column vector

$$g_{10} = \left( \sum_{s=0}^{9} P^s \right) v$$

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$$g_{10} = \left(\sum_{s=0}^{9} P^s\right) v = \begin{bmatrix} 2.41 & 1.07 & 0.58 & 5.95 \\ 1.47 & 2.08 & 0.57 & 5.88 \\ 1.53 & 1.14 & 1.59 & 5.75 \\ 1.55 & 1.20 & 0.65 & 6.60 \end{bmatrix} \begin{bmatrix} 266.78 \\ 352.61 \\ 395.29 \\ 400.20 \end{bmatrix}$$

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Starting with 2 laptops in stock $\implies$ net cash flow = 3627.24 EUR.

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Starting with 2 laptops in stock $\implies$ net cash flow $=$ 3627.24 EUR.
Starting with 3 laptops in stock $\implies$ net cash flow $=$

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# Example: Inventory model

Net cash flow from 10 weeks as a function of initial state is obtained as column vector

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Starting with 2 laptops in stock $\Longrightarrow$ net cash flow $=$ 3627.24 EUR.
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(Starting with 5 laptops is good for 1 week but not for 10 weeks; WHY?)

# Contents

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Recall: For any irreducible and aperiodic finite-sate Markov chain:

$$\mu_t(y) \;=\; \mathbb{P}(X_t = y) \to \pi(y)$$

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What about time averages

$$\frac{1}{t}\sum_{s=0}^{t-1} \phi(X_s) \to ?$$

# Ergodic theorem

## Theorem

*For any irreducible Markov chain with a finite state space $S$,*

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Proof: [LPW08, Sec 4.7]

# Relative frequencies of states

The empirical relative frequency of state $y$ among the first $t$ states of a stochastic process $(X_0, X_1, \dots)$ is defined by

$$\hat{\pi}_t(y) \;=\; \frac{N_t(y)}{t},$$

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## Note

$\hat{\pi}_t(y)$ is a random number determined by $(X_0, \ldots, X_{t-1})$

# Long-term relative frequencies

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*The relative frequencies of an irreducible Markov chain with a finite state space $S$ satisfy*

$$\lim_{t\to\infty} \hat{\pi}_t(y) = \pi(y) \tag{1}$$

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$M_t(x, y) = \mathbb{E}(N_t(y) \mid X_0 = x)$ with $N_t(y) = \sum_{s=0}^{t-1} 1(X_s = y)$

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Expected growth rate

$$\frac{g_t(x)}{t} = \mathbb{E}\left(\frac{V_t}{t} \,\middle|\, X_0 = x\right)$$

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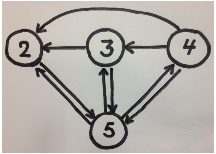


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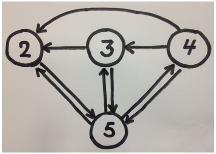


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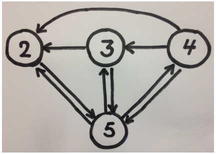


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# Aineistolähteet

Esityksessä käytetyt kuvat (esiintymisjärjestyksessä)

1. Image courtesy of think4photop at FreeDigitalPhotos.net