12 Optional times and stopped martingales

You learn to recognize which random times are optional times, and to compute values and estimates of hitting probabilities using stopped martingales. In this exercise there is no homework!

Classroom problems

12.1 *Pólya's urn.* An urn contains one red and one green ball in the beginning. During round $t = 1, 2, \dots$, a ball is randomly picked from the urn and its color its observed. Then the ball is returned to the urn and another ball of the same color as observed is added to the urn. Let X_t be the relative proportion of red balls in the urn after t rounds.

(a) Verify that the process $(X_t)_{t\in\mathbb{Z}_+}$ is a martingale.

Solution. Denote A_t ="a red ball is picked at the t-th round". Then the number of red balls after t rounds equals

$$S_t \;=\; 1 + \sum_{s=1}^{t} 1(A_s) \;=\; 1 + \sum_{s=1}^{t} I_s,$$

where $I_s = 1(A_s)$ is the random indicator variable of the event A_s. The relative proportion of red balls after t rounds then equals

$$X_t \;=\; \frac{S_t}{2+t}.$$

We will next verify that $(X_t)_{t\in\mathbb{Z}_+}$ is a martingale with respect to the information sequence $(X_0, I_1, I_2, \dots)$ where $X_0 = \frac{1}{2}$ the deterministic initial state. Recall that a random sequence $(M_0, M_1, \dots)$ is a martingale with respect to $(X_0, X_1, \dots)$ if

(i) $\mathbb{E}|M_t| < \infty$,
(ii) $M_t \in \sigma(X_0, \dots, X_t)$,
(iii) $\mathbb{E}(M_{t+1} \,|\, X_0, \dots, X_t) = M_t$.

Now (i) is clear because, being a relative proportion, $X_t \in [0, 1]$ with probability one. (ii) is also clear by definition. For (iii), we note that the conditional probability of picking a red ball in round $t + 1$, given that $I_1 = i_1, \dots, I_t = i_t$, equals

$$\mathbb{P}(A_{t+1} \,|\, I_1 = i_1, \dots, I_t = i_t) \;=\; \frac{1 + \sum_{s=1}^{t} i_s}{2+t},$$

so that[1]

$$\begin{aligned}
\mathbb{E}(S_{t+1}|I_1,...,I_t) &= S_t + \mathbb{E}(I_{t+1}|I_1,...,I_t) \\
&= S_t + \mathbb{P}(A_{t+1}|I_1,...,I_t) \\
&= S_t + \frac{1+\sum_{i=1}^t I_i}{2+t} \\
&= S_t + \frac{S_t}{2+t} \\
&= \frac{(3+t)S_t}{2+t}.
\end{aligned}$$

It now follows that

$$\begin{aligned}
\mathbb{E}(X_{t+1}\,|\,I_1,...,I_t) &= \mathbb{E}\left(\frac{S_{t+1}}{2+(t+1)}\,|\,I_1,...,I_t\right) \\
&= \frac{1}{3+t}\mathbb{E}\left(S_{t+1}\,|\,I_1,...,I_t\right) \\
&= \frac{1}{3+t}\frac{(3+t)S_t}{2+t} \\
&= \frac{S_t}{2+t} \\
&= X_t.
\end{aligned}$$

(b) Prove that the probability that the relative proportion of red balls ever reaches level 0.9 is at most 5/9.

Hint: Analyze the stopped martingale $\hat{X}_t = X_{t\wedge T}$ where T is the first time instant (possibly infinite) that the proportion of red balls reaches the level 0.9.

Solution. Let $T = \min\{t \geq 0 : X_t \geq 0.9\}$ be the first time instant (possibly infinite) that the proportion of red balls reaches the level 0.9. Because (X_t) is a martingale and T is an optional time with respect to $(X_0, X_1, \dots)$, it follows that the stopped process $(\hat{X}_t)_{t\in\mathbb{Z}_+}$ defined by $\hat{X}_t = X_{t\wedge T}$ is a martingale (lecture notes). Because martingales are constant by expectation, and because $\hat{X}_0 = X_0$, it follows that

$$\mathbb{E}(\hat{X}_t) \;=\; \mathbb{E}(\hat{X}_0) \;=\; \mathbb{E}(X_0) \;=\; 1/2.$$

This gives as a lower bound

$$\mathbb{E}\hat{X}_t \;=\; \mathbb{E}(X_{t\wedge T}) \;\geq\; \mathbb{E}(X_{t\wedge T}1(T\leq t)) \;=\; \mathbb{E}(X_T 1(T\leq t)) \;\geq\; 0.9\,\mathbb{P}(T\leq t),$$

which implies that

$$\mathbb{P}(T\leq t) \;\leq\; \frac{1/2}{0.9} \;=\; 5/9$$

for all $t \geq 0$. By taking limits as $t\to\infty$ it follows (by the monotone continuity of probability measures) that

$$\mathbb{P}(T<\infty) \;=\; \lim_{t\to\infty}\mathbb{P}(T\leq t) \;\leq\; 5/9.$$

[1] the same result follows from $\mathbb{E}(S_{t+1}|I_1,...,I_t) = S_t\mathbb{P}(S_{t+1}=S_t|I_1,...,I_t)+(S_t+1)\mathbb{P}(S_{t+1}=S_t+1|I_1,...,I_t)$.

There are several versions of the optional stopping theorem, one of which is formulated in the lecture notes in Theorem 12.8. Here's another useful version:

Theorem. *Optional stopping theorem II.* Let $(M_0, M_1, \dots)$ be a martingale and T a finite optional time, meaning $\mathbb{P}[T < \infty] = 1$. Assume further that M has uniformly bounded increments, meaning there exists a constant $C \geq 0$ such that for every $t \in \mathbb{N}$ we have

$$|M_{t+1} - M_t| \leq C.$$

Then $\mathbb{E}[M_T] = \mathbb{E}[M_0]$.

12.2 Jack has an unbiased six-sided die. He wants to impress his friends with a video where he seems to be able to predict six consecutive dice rolls before throwing them. In reality, he repeatedly says a number between 1 and 6 and throws a die until he has successfully predicted the result six times in a row.

Denote by X_t the latest streak of successive correct predictions after the t'th throw. Let T be the first time when Jack has successfully predicted six throws in a row.

(a) Find a function $f : \mathbb{N} \to \mathbb{R}$ such that $M_t := f(X_t) - t$ is a martingale with respect to $(X_0, X_1, \dots)$.

Solution. M_t should satisfy the martingale property:

$$\mathbb{E}[M_{t+1} | X_t, \dots, X_0] = M_t.$$

Since $M_{t+1} = f(X_{t+1}) - (t+1)$ only depends on X_{t+1} and X is a Markov chain, we have

$$\mathbb{E}[M_{t+1} | X_t, \dots, X_0] = \mathbb{E}[M_{t+1} | X_t].$$

We thus have

$$\mathbb{E}[M_{t+1} | X_t] = M_t.$$

Substituting the definition of M_t gives

$$\mathbb{E}[f(X_{t+1}) - (t+1) | X_t] = f(X_t) - t,$$

from which by adding $t+1$ to both sides we get

$$\mathbb{E}[f(X_{t+1}) | X_t] = f(X_t) + 1$$

This holds precisely if for every $x \in \mathbb{N}$ we have

$$\mathbb{E}[f(X_{t+1}) | X_t = x] = f(x) + 1.$$

Let us compute the LHS of the above equation:

$$\begin{aligned}\mathbb{E}[f(X_{t+1})|X_t = x] &= \sum_{y=0}^{\infty} f(y)\mathbb{P}[X_{t+1} = y|X_t = x] \\ &= \frac{5}{6}f(0) + \frac{1}{6}f(x+1).\end{aligned}$$

Therefore we get

$$f(x+1) = 6 + 6f(x) - 5f(0).$$

We are free to choose the value for $f(0)$. Choosing $f(0) = 0$ gives $f(x+1) = 6 + 6f(x)$, which by simple induction yields

$$f(x) = \begin{cases} \sum_{j=1}^{x} 6^j, & x > 1 \\ 0 & x = 0. \end{cases}$$

(b) Show that T is an optional time with $\mathbb{P}[T < \infty] = 1$.

Solution. The event $T = t$ only depends on $\sigma(X_1, \ldots, X_t)$, hence it is an optional time. In the stopped Markov chain $(X_{t\wedge T})_{t\in\mathbb{N}}$ the state 6 is the only absorbing state, thus $\mathbb{P}[T < \infty] = 1$.

(c) Show that the stopped martingale $(M_{t\wedge T})_{t\in\mathbb{N}}$ has uniformly bounded increments.

Solution. Since $X_{t\wedge T} \leq 6$ for every $t \in \mathbb{N}$ and f is an increasing function, we get

$$\begin{aligned}|M_{(t+1)\wedge T} - M_{t\wedge T}| &= |f(X_{(t+1)\wedge T}) - f(X_{t\wedge T}) + 1| \\ &\leq f(X_{(t+1)\wedge T}) + f(X_{t\wedge T}) + 1 \\ &\leq 2f(6) + 1 < \infty.\end{aligned}$$

This shows that the increments are uniformly bounded by $2f(6) + 1$.

(d) Using the stopped martingale from (c), compute the expected number of throws $\mathbb{E}[T]$ it takes Jack to finish the video. How reasonable is Jack's plan?

Solution. Since $M_{t\wedge T}$ is a martingale with uniformly bounded increments and $\mathbb{P}[T < \infty] = 1$, by optional stopping theorem we have

$$\mathbb{E}[M_T] = \mathbb{E}[M_{T\wedge T}] = \mathbb{E}[M_{0\wedge T}] = \mathbb{E}[M_0] = 0.$$

From linearity of expectation, we thus get

$$\mathbb{E}[M_T] = \mathbb{E}[f(X_T)] - \mathbb{E}[T] = 0.$$

Since by definition of T we have $X_T = 6$, we have $f(X_T) = f(6)$. Rearranging the above equation thus gives

$$\mathbb{E}[T] = \mathbb{E}[f(6)] = f(6) = \sum_{j=1}^{6} 6^j = 55986$$