10 Continuous-time Markov chains

The objective of this exercise is to practice computing time-dependent and invariant distributions related to continuous-time Markov chains. It will be useful to have a computer or a calculator capable of matrix calculations with you in the exercise session.

Classroom problems

10.1 An IT person for a company is responsible for the operation of two web servers. Server i operates an average of ℓ_i days before it malfunctions, and repairing it takes an average of m_i days, where $\ell_1 = 30$, $\ell_2 = 100$, $m_1 = 1$ and $m_2 = 2$. The operation and repair times of the servers are assumed to be independent and exponentially distributed. The IT person repairs the servers in the order they break.

(a) Model the state of the servers as a continuous-time Markov process with state space $S = \{(), (1), (2), (1,2), (2,1)\}$, where each state is an ordered list representing the queue of servers to be repaired. Write the generator matrix Q of the Markov process, and draw a transition diagram.

(b) What is the probability that neither of the servers will be working after a week, provided that they are both operational at present?

(c) Determine the invariant distribution of the process. What is the probability at the statistical equilibrium that there is at least one operational server in the company?

(d) If we discover that neither of the servers are working at the statistical equilibrium, how long is the expected waiting time until at least one of the servers will be online?

Homework problems

10.2 An IT person for a company is responsible for the operation of two web servers that work as in Problem 10.1. This time the servers are prioritized so that the IT person will repair server 1 as soon as it malfunctions, suspending the repairs of server 2 for that time if necessary.

(a) Model the state of the servers as a continuous-time Markov chain with state space $S' = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$, where the elements of the state space are unordered sets representing the servers waiting to be repaired. Write the generator matrix of the chain and draw a transition diagram.

(b) Solve the invariant distribution of the process. What is the probability that there will be at least one operational server in the company at the statistical equilibrium?

(c) If we discover that neither of the servers are working at the statistical equilibrium, how long is the expected waiting time until at least one of the servers will be online?

(d) Compare the results of the previous parts to those of Problem 10.1.

10.3 A machine operates 100 days by expectation before malfunctioning, and repairs take 10 days by expectation. Operation and repair times are assumed mutually independent and exponentially distributed. While operating (state 1), the machine creates an average of 120 000 EUR profit a day, and while broken (state 2), an average of 90 000 EUR loss a day.

(a) Provided that the machine is broken in the beginning of the month, what is the expected number of days it will be broken during the month (30 days)?
Hint: The results of Problem 9.3 might be helpful.

(b) Provided that the machine is broken in the beginning of the month, what is the expected profit it will create for its owner during that month (30 days)?

(c) At what rate will the machine create profit for its owner in the long term?