# 5 Various types of Markov chains

In this exercise you will practice your skills learned so far by analyzing various different types of Markov chains. You get introduced to the notion of reversibility. You also get introduced to the famous Metropolis algorithm that is widely used in Markov chain Monte Carlo algorithms.

## Classroom exercises

**5.1** *Card shuffling.* Consider a deck (korttipakka) of $n = 3$ playing cards labeled using $a, b, c$. The possible configurations of the deck are denoted as $S = \{abc, acb, bac, bca, cab, cba\}$, where in each configuration the cards of the deck are listed from top to bottom. The deck is shuffled as follows: In each step the topmost card is lifted and then pushed to a uniformly random location in the deck, so that the topmost card is placed to the top, middle, or bottom of the deck, each choice with probability $\frac{1}{3}$, and independently of earlier steps. Denote by $X_t$ the configuration of the deck after $t$ shuffling steps.

(a) Explain why $(X_t)$ is a Markov chain, write down its transition matrix, and draw the corresponding transition diagram so that no arrows cross each other.

(b) Find out the invariant distribution $\pi$ of the chain.

(c) Is the Markov chain $(X_t)$ reversible with respect to the distribution $\pi$?

Now consider a deck of 52 cards, labeled as $\{1, 2, \ldots, 52\}$. Let $S$ be the set of all configurations of the deck.

(d) Determine the size of the set $S$.

(e) Describe a Markov chain $(X_t)$ which represents the same shuffling method, at each round lifting the topmost card and placing it to a uniformly random location. How many entries does the corresponding transition matrix have?

(f) Find out the unique invariant distribution of the Markov chain $(X_t)$ corresponding to the 52-card deck.

## Homework problems

**5.2** *Metropolis chain.* Let $\pi$ be a probability distribution on a finite state space $S$ such that $\pi(x) > 0$ for all $x \in S$. Let $Q$ be an irreducible transition matrix on $S$. The *Metropolis chain* for $\pi$ with proposal matrix $Q$ is a Markov chain with transition matrix defined by

$$P(x,y) \;=\; \begin{cases} Q(x,y)A(x,y) & \text{if } y \neq x, \\ 1 - \sum_{z \neq x} Q(x,z)A(x,z) & \text{if } y = x, \end{cases}$$

where $A(x,y) = \min\left(\frac{\pi(y)Q(y,x)}{\pi(x)Q(x,y)}, 1\right)$.

(a) Verify that $P$ really is a transition matrix (has nonnegative entries and unit row sums).

(b) Verify that $P$ is reversible with respect to $\pi$.

(c) Explain with the help of the ergodic theorem in the lecture notes, how a numerical approximation for a sum $\sum_x \pi(x)f(x)$ can be computed for an arbitrary function $f$, by simulating a Markov chain having transition matrix $P$, when we assume that $P$ is irreducible. (Such a method is called Markov chain Monte Carlo simulation.)

**5.3** *Random walk on a graph.* Let $G$ be an undirected graph on the node set $V = \{1, \dots, n\}$ and suppose that the degree $\deg(x)$ of every node $x$ in $G$ is at least 1. A random walk on $G$ proceeds by moving at each step to a neighboring node selected uniformly at random. The random walk is hence a Markov chain with transition matrix given by

$$P(x,y) \;=\; \begin{cases} \frac{1}{\deg(x)}, & \text{if } x \text{ and } y \text{ are neighbors}, \\ 0, & \text{else}. \end{cases}$$

(a) Prove that a random walk on a graph is reversible with respect to the distribution $\pi(x) = c \deg(x)$ when the constant $c$ is chosen appropriately.

(b) By using the result of part (a), calculate the invariant distribution for a lone king moving randomly on the standard 8-by-8 chessboard $S = \{\mathtt{a1}, \dots, \mathtt{h8}\}$.

(c) By using the result of part (a), calculate the invariant distribution for a lone knight (ratsu) moving randomly on the standard 8-by-8 chessboard $S = \{\mathtt{a1}, \dots, \mathtt{h8}\}$.