12 Optional times and stopped martingales

You learn to recognize which random times are optional times, and to compute values and estimates of hitting probabilities using stopped martingales. In this exercise there is no homework!

Classroom problems

12.1 *Stock option.* The current value of a stock option is 47 EUR, and either 100 EUR or 0 EUR on the date of maturity 30 days later. Let S_t be the value of the option after t days. We will assume that $(S_t)_{t\in\mathbb{Z}_+}$ is a martingale.

(a) What is the probability that the option will have the value of 100 EUR on the maturity date?

(b) Pertti is planning to buy the option now and to sell it either on the date of maturity, or on the time instant that its value rises above 55 EUR or decreases below 15 EUR. What is the expected net profit Pertti will make at the selling time?

(c) Assume that the option can never attain a negative value. Derive a nontrivial (less than one) upper bound for the probability of the event that the value of the option will sometime rise above Pertti's target level 55 EUR before maturity.

Hint: Analyze the stopped martingale $\hat{S}_t = S_{t\wedge\tau}$ where τ is the first time instant (possibly infinite) that the option value exceeds 55 EUR.

12.2 *Pólya's urn.* An urn contains one red and one green ball in the beginning. During round $t = 1, 2, \dots$, a ball is randomly picked from the urn and its color its observed. Then the ball is returned to the urn and another ball of the same color as observed is added to the urn. Let X_t be the relative proportion of red balls in the urn after t rounds.

(a) Verify that the process $(X_t)_{t\in\mathbb{Z}_+}$ is a martingale.

(b) Prove that the probability that the relative proportion of red balls ever reaches level 0.9 is at most 5/9.

Hint: Analyze the stopped martingale $\hat{X}_t = X_{t\wedge T}$ where T is the first time instant (possibly infinite) that the proportion of red balls reaches the level 0.9.

12.3 *Lognormal stock prices.* The closing price of a stock in the end of trading day t is modeled as a stochastic process $M_t = M_0 X_1 \cdots X_t$, where M_0 is a known constant and $X_1, X_2, \dots$ are independent lognormally distributed random numbers which can represented as $X_t = e^{\mu+\sigma Z_t}$ where $Z_1, \dots, Z_t$ are independent and follow the standard normal distribution.

(a) For which values of μ and σ is $(M_t)_{t\in\mathbb{Z}_+}$ a martingale?

(b) When is (M_t) a submartingale?