6 Infinite-state Markov chains

Classroom problems

6.1 *Bacteria growth.* Bacteria reproduce by cell division. During a unit of time, a bacterium will either die (with probability $\frac{1}{4}$), stay the same (with probability $\frac{1}{4}$), or split into two parts (with probability $\frac{1}{2}$). The population starts with one bacterium at time $t = 0$.

(a) Determine the generating function of the population size X_1 after 1 time units.

Solution. This is

$$\phi_{X_1}(s) \;=\; \sum_{k=0}^{\infty} s^k \mathbb{P}(X_1 = k) \;=\; \frac{1}{4} + \frac{1}{4}s + \frac{1}{2}s^2.$$

(b) Determine the generating function of the population size X_2 after 2 time units.

Solution. This is

$$\begin{aligned}
\phi_{X_1} \circ \phi_{X_1}(s) &= \phi_{X_1}(\phi_{X_1}(s)) \\
&= \frac{1}{4} + \frac{1}{4}\phi_{X_1}(s) + \frac{1}{2}\phi_{X_1}(s)^2 \\
&= \frac{1}{4} + \frac{1}{4}(\frac{1}{4} + \frac{1}{4}s + \frac{1}{2}s^2) + \frac{1}{2}(\frac{1}{4} + \frac{1}{4}s + \frac{1}{2}s^2)^2,
\end{aligned}$$

which is a fourth-order polynomial in s.

(c) Compute the extinction probability for the population.

Solution. The solution is the smallest nonnegative fixed point of $\phi_{X_1}(s) = s$. The solutions are $s_1 = \frac{1}{2}$ and $s_2 = 1$, so the extinction probability equals $\frac{1}{2}$.

(d) Let m_t be the largest possible number of bacteria after t time units. Determine m_t and compute the probability $\mathbb{P}(X_t = m_t)$.

Solution. The maximum is obtained if during each time unit every bacterium splits into two. This results in $m_t = 2^t$. For the event $X_t = m_t$ to happen, there must be 1 split in step 1, 2 splits in step 2, 4 splits in step 3, $\dots$, and 2^{t-1} splits in step t. The total number of required splits is

$$1 + 2 + 4 + \cdots + 2^{t-1} \;=\; \frac{2^t - 1}{2 - 1} \;=\; 2^t - 1.$$

Because each split occurs with probability $\frac{1}{2}$, independently of other splits, it follows that

$$\mathbb{P}(X_t = m_t) \;=\; \left(\frac{1}{2}\right)^{2^t - 1}.$$

(For example, for $t = 10$ this probability is astronomically small $\approx 10^{-308}$.)

(e) Given that there are 2000 bacteria at time $t = 50$, what is the expected number of bacteria at time $t = 51$?

Solution. The expected number of offspring for each bacterium equals $\phi'_{X_1}(1) = \frac{5}{4}$. Hence the expected number of bacteria at time $t = 51$ equals $\mathbb{E}X_{51} = 2000 \cdot \frac{5}{4} = 2500$.

(f) Compute the extinction probability for the population in an alternative setting where the population starts with 30 bacteria at time $t = 0$.

Solution. The population with 30 initial bacteria can be thought as the sum of 30 independent bacteria populations started with one individual, each of which behaves independently of each other. Hence the aggregate population becomes extinct if and only if each of the 30 constituent populations becomes extinct. Hence the extinction probability in this setting equals $\left(\frac{1}{2}\right)^{30}$.

Additional information. In this example, the extinction probability when starting with one bacterium, rather large, namely 0.5. However, conditionally that the population has reached a level of 30 bacteria, the extinction probability is extremely small $2^{-30} \approx 1/10^9$. This a general feature of epidemic models: Populations which are able to reach a certain level above zero are very likely to explode.

Homework problems

6.2 *Gambling pot.* A group of people are gambling. Initially there is a pot of 1 EUR on the table. Then a round is played, which results in a tie or one of the players winning. If the result is a tie, 1 EUR is added to the pot. If one of the players wins, the winner collects the accumulated pot in its entirety and the next round starts 1 EUR in the pot. Suppose that the probability of a tie in each round is $q \in [0, 1]$ and the results of the rounds are independent of each other.

(a) Let X_t be the size of the pot, in euros, in the t-th round. Show that the process $X = (X_t)_{t\in\mathbb{Z}_+}$ is a Markov chain on the infinite state space $S = \{1, 2, \ldots\}$, and calculate its transition probabilities.

Solution. The state space is obviously S, a countably infinite set. The Markov property is: for all $x_{t+1}, x_t, ..., x_0 \in S$,

$$\begin{aligned}\mathbb{P}(X_{t+1} = x_{t+1} \mid X_t = x_t, ..., X_0 = x_0) &= \begin{cases} q, & \text{if } x_{t+1} = x_t + 1 \text{ (tie)} \\ 1 - q, & \text{if } x_{t+1} = 1 \text{ (someone wins)} \\ 0, & \text{else} \end{cases} \\ &= \mathbb{P}(X_{t+1} = x_{t+1} \mid X_t = x_t).\end{aligned}$$

One may explain this intuitively, but let us prove it from the definition. Assume that x_{t+1} is either $x_t + 1$ or 1 (otherwise the claim is obvious). Define

$$y_t = \begin{cases} \text{"tie"}, & \text{if } x_t = x_{t-1} + 1 \\ \text{"someone wins"}, & \text{if } x_t = 1. \end{cases}$$

i) We simplify the left-hand side of the Markov property. The definition of conditional probability gives

$$\mathbb{P}(X_{t+1} = x_{t+1} \mid X_t = x_t, ..., X_0 = x_0) = \frac{\mathbb{P}(X_{t+1} = x_{t+1}, ..., X_0 = x_0)}{\mathbb{P}(X_t = x_t, ..., X_0 = x_0)}.$$

We start with 1 EUR in the pot, so this conditional probability is only defined for $x_0 = 1$ (we cannot divide by zero), in which case we have

$$\frac{\mathbb{P}(X_{t+1} = x_{t+1}, ..., X_1 = x_1)}{\mathbb{P}(X_t = x_t, ..., X_1 = x_1)} = \frac{\mathbb{P}(Y_{t+1} = y_{t+1}, ..., Y_1 = y_1)}{\mathbb{P}(Y_t = y_t, ..., Y_1 = y_1)},$$

where Y_t is the result of round t. Since the results are independent, we have

$$\frac{\mathbb{P}(Y_{t+1} = y_{t+1}) \ldots \mathbb{P}(Y_1 = y_1)}{\mathbb{P}(Y_t = y_t)\mathbb{P}(Y_1 = y_1)} = \mathbb{P}(Y_{t+1} = y_{t+1}). \tag{1}$$

ii) The right-hand side of the Markov property is found almost exactly the same way. First we note that

$$\mathbb{P}(X_{t+1} = x_{t+1} \mid X_t = x_t) = \frac{\mathbb{P}(X_{t+1} = x_{t+1}, X_t = x_t)}{\mathbb{P}(X_t = x_t)} = \frac{\mathbb{P}(Y_{t+1} = y_{t+1}, X_t = x_t)}{\mathbb{P}(X_t = x_t)}.$$

Since some of the states are unknown, we sum over the joint distribution:

$$\begin{aligned}\mathbb{P}(X_{t+1} = x_{t+1} \mid X_t = x_t) &= \frac{\sum\limits_{y_1,\ldots,y_t : X_t = x_t} \mathbb{P}(Y_{t+1} = y_{t+1}, \ldots, Y_1 = y_1)}{\sum\limits_{y_1,\ldots,y_t : X_t = x_t} \mathbb{P}(Y_t = y_t, \ldots, Y_1 = y_1)} \\ &= \mathbb{P}(Y_{t+1} = y_{t+1}) \frac{\sum\limits_{y_1,\ldots,y_t : X_t = x_t} \mathbb{P}(Y_t = y_t, \ldots, Y_1 = y_1)}{\sum\limits_{y_1,\ldots,y_t : X_t = x_t} \mathbb{P}(Y_t = y_t, \ldots, Y_1 = y_1)},\end{aligned}$$

where the last step follows from independence. This is equal to (1), and we conclude that the Markov property holds.

Additional information. If this proof seems slightly clumsy, we could have taken a shortcut: X_t is a function of $X_0, Y_1, \ldots, Y_t$. Since Y_{t+1} is independent of the said variables, it must also be independent of X_t. This topic is discussed in detail in, e.g., MS-E1600 Probability theory.

The (infinite) transition matrix is

$$\begin{bmatrix} 1-q & q & 0 & 0 & \ldots \\ 1-q & 0 & q & 0 & \ldots \\ 1-q & 0 & 0 & q & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

(b) For which values of the parameter q does the process $X = (X_t)_{t\in\mathbb{Z}_+}$ have an invariant distribution (i.e. a stationary distribution)? Calculate the invariant distribution.

Solution. According to the balance equations $\sum_i \pi(i) = 1$ ja $\pi = \pi P$, where P is the transition matrix. This can be written as

$$\begin{cases} \pi(1) = \sum_{i=1}^{\infty}(1-q)\pi(i) \overset{\sum_i \pi(i)=1}{\Rightarrow} \pi(1) = 1-q \\ \pi(i) = q\pi(i-1), \qquad i \geq 2. \end{cases}$$

If $q = 1$: the first balance equation becomes $\pi(1) = 0$ and the latter yields

$$\pi(i) = \pi(i-1), \qquad i \geq 2,$$

so that $0 = \pi(1) = \pi(2) = \ldots$ For a probability distribution, $\sum_{i=1}^{\infty} \pi(i) = 1$, so in this case there is no invariant distribution.

If $q = 0$: We immediately get $\pi(i) = 1(i = 1)$. This is a probability distribution, called the Dirac measure at one.

If $0 < q < 1$: Now we get

$$\pi(1) = 1-q \qquad \text{and} \qquad \pi(i) = q\pi(i-1)$$

for all $i \geq 2$. As a solution we obtain $\pi(i) = (1-q)q^{i-1}$ for all $i \geq 1$. Using the geometric series sum formula one can verify that $\sum_{i=1}^{\infty} \pi(i) = 1$, so that π is an invariant distribution. (This also works for $q = 0$.)

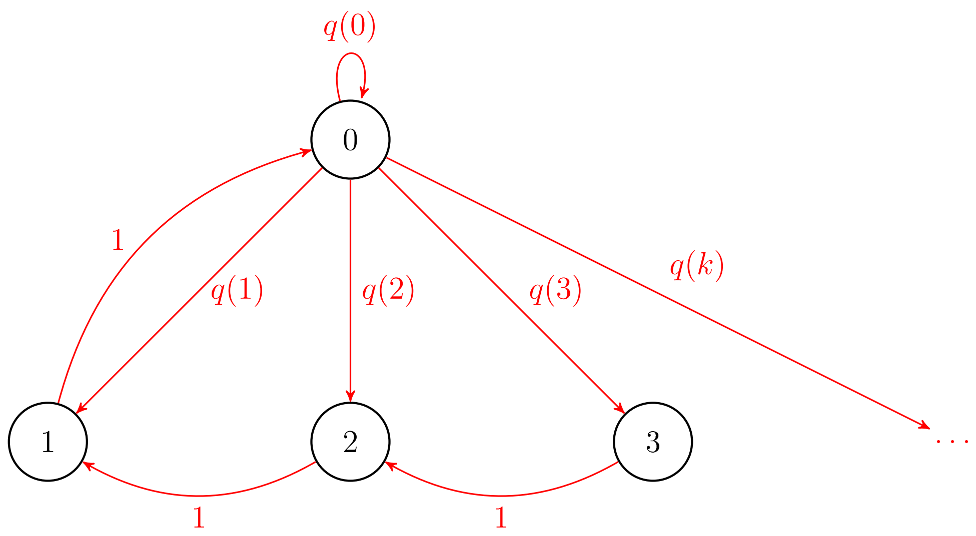
6.3 *Renewal chain.* Let q be a probability distribution on the nonnegative integers $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$. A renewal chain corresponding to q is a Markov chain on the infinite state space $\mathbb{Z}_+$ with a transition matrix P such that $P(k, k-1) = 1$ for all $k \geq 1$, and $P(0, k) = q(k)$ for all $k \geq 0$.

(a) Sketch the transition diagram of the chain.

Solution. The transition matrix is

$$P = \begin{pmatrix} q(0) & q(1) & q(2) & q(3) & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and from this it is easy to draw the transition diagram.



(b) Give an example of q for which the renewal chain is reducible.

Solution. Any probability distribution q for which the support $\{k : q(k) > 0\}$ is finite, leads to a reducible Markov chain. For example, we may take the uniform distribution on $\{0, 1, 2\}$.

(c) Give an example of q for which the renewal chain is irreducible.

Solution. Any probability distribution q for which the support $\{k : q(k) > 0\}$ is infinite, leads to an irreducible Markov chain. For example, we may take q to be a Poisson distribution with a nonzero mean.

(d) Prove that state 0 is recurrent.

Solution. Denote by $T_0 = \min\{t \geq 0 : X_t = 0\}$ the first hitting time to 0, and by $T_0^+ = \min\{t \geq 1 : X_t = 0\}$ the positive hitting time to 0. For an integer $x \geq 0$, denote by $h(x) = \mathbb{P}(T_0 < \infty \,|\, X_0 = x)$ the probability that the chain starting at x visits 0. Clearly, $h(x) = 1$ for every $x \geq 0$. Hence by conditioning on the first step,

$$\begin{aligned}\mathbb{P}(T_0^+ < \infty \,|\, X_0 = 0) \;&=\; \sum_{x=0}^{\infty} P(0,x)\mathbb{P}(T_0 < \infty \,|\, X_0 = x) \\ &=\; \sum_{x=0}^{\infty} P(0,x)h(x) \;=\; \sum_{x=0}^{\infty} P(0,x) \;=\; \sum_{x=0}^{\infty} q(x) \;=\; 1.\end{aligned}$$

(e) Derive a formula for the expected return time $\mathbb{E}\big(T_0^+ \,|\, X_0 = 0\big)$ to state 0 in terms of the distribution q, where $T_0^+ = \min\{t \geq 1 : X_t = 0\}$.

Solution. We have, again by conditioning on the first step,

$$\mathbb{E}\big(T_0^+ \,|\, X_0 = 0\big) \;=\; \sum_{x=0}^{\infty} P(0,x)\big(1+\mathbb{E}\big(T_0 \,|\, X_0 = x\big)\big) \;=\; 1+\sum_{x=0}^{\infty} P(0,x)\mathbb{E}\big(T_0 \,|\, X_0 = x\big).$$

Because $\mathbb{E}\big(T_0 \,|\, X_0 = x\big) = x$, it follows that

$$\mathbb{E}\big(T_0^+ \,|\, X_0 = 0\big) \;=\; 1 + \sum_{x=0}^{\infty} P(0,x)x \;=\; 1 + \sum_{x=0}^{\infty} q(x)x.$$

(f) Invent (or Google) an example of a probability distribution q that corresponds to a renewal chain for which $\mathbb{E}\big(T_0^+ \,|\, X_0 = 0\big) = \infty$.

Solution. The following distribution works:

$$\begin{cases} q(x) = 0, & \text{when } x = 0 \\ q(x) = \frac{6}{\pi^2 x^2} & \text{when } x > 0 \end{cases}$$

This is a probability distribution because $\sum_{x=1}^{\infty} \frac{1}{x^2} = \frac{\pi^2}{6}$ (see for example https://en.wikipedia.org/wiki/Basel_problem). Moreover,

$$\sum_{x=0}^{\infty} xq(x) \;=\; \sum_{x=1}^{\infty} xq(x) \;=\; \frac{6}{\pi^2}\sum_{x=1}^{\infty} \frac{1}{x} \;=\; \infty.$$