Example 2.3: Ehrenfest MC has no limit distribution (Section 2.1)

Transition matrix

In[18]:=
```
P := {{0, 1, 0}, {0.3, 0, 0.7}, {0, 1, 0}};
P // MatrixForm
```

Out[19]//MatrixForm=

$$\begin{pmatrix} 0 & 1 & 0 \\ 0.3 & 0 & 0.7 \\ 0 & 1 & 0 \end{pmatrix}$$

Even powers of transition matrix P

In[20]:=
```
MatrixPower[P, 2] // MatrixForm
MatrixPower[P, 4] // MatrixForm
MatrixPower[P, 6] // MatrixForm
MatrixPower[P, 10] // MatrixForm
MatrixPower[P, 20] // MatrixForm
MatrixPower[P, 50] // MatrixForm
```

Out[20]//MatrixForm=

$$\begin{pmatrix} 0.3 & 0. & 0.7 \\ 0. & 1. & 0. \\ 0.3 & 0. & 0.7 \end{pmatrix}$$

Out[21]//MatrixForm=

$$\begin{pmatrix} 0.3 & 0. & 0.7 \\ 0. & 1. & 0. \\ 0.3 & 0. & 0.7 \end{pmatrix}$$

Out[22]//MatrixForm=

$$\begin{pmatrix} 0.3 & 0. & 0.7 \\ 0. & 1. & 0. \\ 0.3 & 0. & 0.7 \end{pmatrix}$$

Out[23]//MatrixForm=

$$\begin{pmatrix} 0.3 & 0. & 0.7 \\ 0. & 1. & 0. \\ 0.3 & 0. & 0.7 \end{pmatrix}$$

Out[24]//MatrixForm=

$$\begin{pmatrix} 0.3 & 0. & 0.7 \\ 0. & 1. & 0. \\ 0.3 & 0. & 0.7 \end{pmatrix}$$

Out[25]//MatrixForm=

$$\begin{pmatrix} 0.3 & 0. & 0.7 \\ 0. & 1. & 0. \\ 0.3 & 0. & 0.7 \end{pmatrix}$$

Odd powers of transition matrix P

In[26]:=
```
MatrixPower[P, 1] // MatrixForm
MatrixPower[P, 3] // MatrixForm
MatrixPower[P, 5] // MatrixForm
MatrixPower[P, 11] // MatrixForm
MatrixPower[P, 27] // MatrixForm
MatrixPower[P, 59] // MatrixForm
```

Out[26]//MatrixForm=
$$\begin{pmatrix} 0. & 1. & 0. \\ 0.3 & 0. & 0.7 \\ 0. & 1. & 0. \end{pmatrix}$$

Out[27]//MatrixForm=
$$\begin{pmatrix} 0. & 1. & 0. \\ 0.3 & 0. & 0.7 \\ 0. & 1. & 0. \end{pmatrix}$$

Out[28]//MatrixForm=
$$\begin{pmatrix} 0. & 1. & 0. \\ 0.3 & 0. & 0.7 \\ 0. & 1. & 0. \end{pmatrix}$$

Out[29]//MatrixForm=
$$\begin{pmatrix} 0. & 1. & 0. \\ 0.3 & 0. & 0.7 \\ 0. & 1. & 0. \end{pmatrix}$$

Out[30]//MatrixForm=
$$\begin{pmatrix} 0. & 1. & 0. \\ 0.3 & 0. & 0.7 \\ 0. & 1. & 0. \end{pmatrix}$$

Out[31]//MatrixForm=
$$\begin{pmatrix} 0. & 1. & 0. \\ 0.3 & 0. & 0.7 \\ 0. & 1. & 0. \end{pmatrix}$$

The rows seem to switch between even and odd.

In[32]:=
```
MatrixPower[P, 59] // MatrixForm
MatrixPower[P, 60] // MatrixForm
```

Out[32]//MatrixForm=
$$\begin{pmatrix} 0. & 1. & 0. \\ 0.3 & 0. & 0.7 \\ 0. & 1. & 0. \end{pmatrix}$$

Out[33]//MatrixForm=
$$\begin{pmatrix} 0.3 & 0. & 0.7 \\ 0. & 1. & 0. \\ 0.3 & 0. & 0.7 \end{pmatrix}$$

Start from one possible initial distribution

In[34]:=
```
mu0 := {{1, 0, 0}};
mu0 // MatrixForm
```
Out[35]//MatrixForm=
```
( 1  0  0 )
```

One step

In[36]:=
```
mu1 := mu0 . P;
mu1 // MatrixForm
```
Out[37]//MatrixForm=
```
( 0.  1.  0. )
```

Two steps

In[38]:=
```
mu2 := mu0 . P.P;
mu2 // MatrixForm
```
Out[39]//MatrixForm=
```
( 0.3  0.  0.7 )
```

Three steps

In[40]:=
```
mu3 := mu0 . P.P.P;
mu3 // MatrixForm
```
Out[41]//MatrixForm=
```
( 0.  1.  0. )
```

Four steps

In[42]:=
```
mu4 := mu0 . P.P.P.P;
mu4 // MatrixForm
```
Out[43]//MatrixForm=
```
( 0.3  0.  0.7 )
```

The limits switch between even and odd. No limit!

Million steps

In[44]:=
```
mu100000 :=  mu0 . MatrixPower[P, 100 000];
mu100000 // MatrixForm
```
Out[45]//MatrixForm=
```
( 0.3  0.  0.7 )
```

Million + 1 steps

In[46]:=
```
mu100001 := mu0 . MatrixPower[P, 100001];
mu100001 // MatrixForm
```
Out[47]//MatrixForm=
```
( 0.  1.  0. )
```

However, we can find invariant distribution!

Balance equations

In[48]:=
```
pii := {{p1, p2, p3}};
pii // MatrixForm
```
Out[49]//MatrixForm=
```
( p1  p2  p3 )
```

In[50]:=
```
pii.P
```
Out[50]=
```
{{0.3 p2, p1 + p3, 0.7 p2}}
```

Eigenvector equations (pii is a left eigenvector of P with eigenvalue 1) :

In[51]:=
```
0.3` p2 == p1;
p1 + p3 == p2;
0.7` p2 == p3;
```

Normalization (sum of elements of the vector pii is 1, since they are probabilities (law of total probability)):

In[54]:=
```
p1 + p2 + p3 == 1;
```

Solving the equations :

In[55]:=
```
Solve[{0.3` p2 == p1, p1 + p3 == p2, 0.7` p2 == p3, p1 + p2 + p3 == 1}, {p1, p2, p3}]
```
Out[55]=
```
{{p1 → 0.15, p2 → 0.5, p3 → 0.35}}
```