Gambler's ruin (Chapter 4.3 / Leskelä)

Description of the process

State space: S = {0, 1, 2, ..., M-1, M}

State $x \in S$ interpreted: "the gambler currently has x units of wealth", M the maximum wealth or "target wealth of the gambler"

Transition matrix

In[1]:=
```
M = 8;
```

In[2]:=
```
Clear[P, p, q]
```

The gambler bets one unit of wealth each round until reaching either the target wealth M or zero wealth 0 ("ruin"). Independently across the rounds, the gambler wins with probability $q \in [0, 1]$ and then gains one unit of wealth, or otherwise loses one unit of wealth.
For example in roulette with 18 red pockets, 18 black pockets, and one green pocket, betting on red would have a winning probability of $q = \frac{18}{37} \approx 0.486$.

In[3]:=
```
P = Table[Table[If[Or[x == 0, x == M], KroneckerDelta[y, x],
     q * KroneckerDelta[y, x + 1] + (1 - q) * KroneckerDelta[y, x - 1]], {y, 0, M}], {x, 0, M}];
P // MatrixForm
```

Out[4]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1-q & 0 & q & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-q & 0 & q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-q & 0 & q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-q & 0 & q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-q & 0 & q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-q & 0 & q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1-q & 0 & q \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Some questions about the model

Q1: Starting from an initial wealth x, what is the probability that the gambler reaches wealth M ("target wealth") before reaching wealth 0 ("ruin") or vice versa?

Q2: Starting from an initial wealth x, what is the expected time until the gambler reaches either wealth 0 or M (i.e., to "stop gambling bacause either being ruined or having reached the target wealth")?

Answer to Q1

Consider the hitting probabilities to the singleton subset $R = \{0\} \subset S = \{0, 1, 2, \ldots, M-1, M\}$ (the state: "ruin").

This can be solved from a system of linear equations (Theorem 4.4) .

The vector of (unknown) hitting probabilities:

In[5]:=
```
hR = Table[h[i], {i, 0, M}];
hR // MatrixForm
```

Out[6]//MatrixForm=

$$\begin{pmatrix} h[0] \\ h[1] \\ h[2] \\ h[3] \\ h[4] \\ h[5] \\ h[6] \\ h[7] \\ h[8] \end{pmatrix}$$

The system of linear system of equations (4.8) for the hitting probabilities at general parameter q, supplemented with the obvious value $h(M) = \mathrm{P}[T_R < \infty \mid X_0 = M] = 0$:

In[7]:=
```
Table[If[Or[x == 0, x == M], hR[[x + 1]] == KroneckerDelta[x, 0],
   hR[[x + 1]] == Sum[P[[x + 1, y + 1]] * hR[[y + 1]], {y, 0, M}]], {x, 0, M}] // MatrixForm
```

Out[7]//MatrixForm=

$$\begin{pmatrix} h[0] == 1 \\ h[1] == (1-q)\, h[0] + q\, h[2] \\ h[2] == (1-q)\, h[1] + q\, h[3] \\ h[3] == (1-q)\, h[2] + q\, h[4] \\ h[4] == (1-q)\, h[3] + q\, h[5] \\ h[5] == (1-q)\, h[4] + q\, h[6] \\ h[6] == (1-q)\, h[5] + q\, h[7] \\ h[7] == (1-q)\, h[6] + q\, h[8] \\ h[8] == 0 \end{pmatrix}$$

The (unique) solution to the linear system of equations (4.8) for the hitting probabilities at general parameter q, supplemented with the obvious value $h(M) = 0$:

In[8]:=
```
hRsol = Solve[Table[If[Or[x == 0, x == M], hR[[x + 1]] == KroneckerDelta[x, 0],
    hR[[x + 1]] == Sum[P[[x + 1, y + 1]] * hR[[y + 1]], {y, 0, M}]], {x, 0, M}], hR]
```

Out[8]=

$$\left\{\left\{h[0] \to 1,\ h[1] \to -\frac{-1 + 6 q - 16 q^2 + 24 q^3 - 22 q^4 + 12 q^5 - 4 q^6 + q^7}{1 - 6 q + 16 q^2 - 24 q^3 + 22 q^4 - 12 q^5 + 4 q^6},\right.\right.$$

$$h[2] \to -\frac{-1 + 6 q - 16 q^2 + 24 q^3 - 22 q^4 + 12 q^5 - 3 q^6}{1 - 6 q + 16 q^2 - 24 q^3 + 22 q^4 - 12 q^5 + 4 q^6},$$

$$h[3] \to -\frac{-1 + 6 q - 16 q^2 + 24 q^3 - 22 q^4 + 13 q^5 - 5 q^6 + q^7}{1 - 6 q + 16 q^2 - 24 q^3 + 22 q^4 - 12 q^5 + 4 q^6},$$

$$h[4] \to -\frac{-1 + 4 q - 6 q^2 + 4 q^3 - q^4}{1 - 4 q + 6 q^2 - 4 q^3 + 2 q^4},\ h[5] \to -\frac{-1 + 6 q - 16 q^2 + 25 q^3 - 25 q^4 + 16 q^5 - 6 q^6 + q^7}{1 - 6 q + 16 q^2 - 24 q^3 + 22 q^4 - 12 q^5 + 4 q^6},$$

$$h[6] \to -\frac{-1 + 6 q - 15 q^2 + 20 q^3 - 15 q^4 + 6 q^5 - q^6}{1 - 6 q + 16 q^2 - 24 q^3 + 22 q^4 - 12 q^5 + 4 q^6},$$

$$\left.\left. h[7] \to -\frac{-1 + 7 q - 21 q^2 + 35 q^3 - 35 q^4 + 21 q^5 - 7 q^6 + q^7}{1 - 6 q + 16 q^2 - 24 q^3 + 22 q^4 - 12 q^5 + 4 q^6},\ h[8] \to 0\right\}\right\}$$

The hitting probabilities at the specific parameter of $q = \frac{18}{37}$ (“roulette win probability betting on red”):

In[9]:=
```
hRsol /. q → N[18 / 37]
```

Out[9]=
```
{{h[0] → 1, h[1] → 0.897341, h[2] → 0.788978, h[3] → 0.674595,
  h[4] → 0.553857, h[5] → 0.426412, h[6] → 0.291887, h[7] → 0.149888, h[8] → 0}}
```

The hitting probabilities at a small win probability $q = 0.25$:

In[10]:=
```
hRsol /. q → 0.25
```

Out[10]=
```
{{h[0] → 1, h[1] → 0.999695, h[2] → 0.99878, h[3] → 0.996037,
  h[4] → 0.987805, h[5] → 0.96311, h[6] → 0.889024, h[7] → 0.666768, h[8] → 0}}
```

Answer to Q2

Consider the passage time to the subset $A = \{0, M\} \subset S = \{0, 1, 2, \ldots, M - 1, M\}$ (the “stop gambling” subset of states).

This can be solved from a system of linear equations (Theorem 4.1) .

The vector of (unknown) expected passage times :

In[11]:=
```
kA = Table[k[i], {i, 0, M}];
kA // MatrixForm
```

Out[12]//MatrixForm=

$$\begin{pmatrix} k[0] \\ k[1] \\ k[2] \\ k[3] \\ k[4] \\ k[5] \\ k[6] \\ k[7] \\ k[8] \end{pmatrix}$$

The system of linear system of equations (4.1) for the expected passage times at general parameter *q*:

In[13]:=
```
Table[If[Or[x == 0, x == M], kA[[x + 1]] == 0,
   kA[[x + 1]] == 1 + Sum[P[[x + 1, y + 1]] * kA[[y + 1]], {y, 0, M}]], {x, 0, M}] // MatrixForm
```

Out[13]//MatrixForm=

$$\begin{pmatrix} k[0] == 0 \\ k[1] == 1 + (1 - q)\, k[0] + q\, k[2] \\ k[2] == 1 + (1 - q)\, k[1] + q\, k[3] \\ k[3] == 1 + (1 - q)\, k[2] + q\, k[4] \\ k[4] == 1 + (1 - q)\, k[3] + q\, k[5] \\ k[5] == 1 + (1 - q)\, k[4] + q\, k[6] \\ k[6] == 1 + (1 - q)\, k[5] + q\, k[7] \\ k[7] == 1 + (1 - q)\, k[6] + q\, k[8] \\ k[8] == 0 \end{pmatrix}$$

The (unique) solution to the linear system of equations (4.1) for the expected passage times at general parameter *q*:

In[14]:=
```
kAsol = Solve[Table[If[Or[x == 0, x == M], kA[[x + 1]] == 0,
    kA[[x + 1]] == 1 + Sum[P[[x + 1, y + 1]] * kA[[y + 1]], {y, 0, M}]], {x, 0, M}], kA]
```

Out[14]=

$$\Big\{\Big\{k[0] \to 0,\ k[1] \to -\frac{-1 + 4\,q - 8\,q^{0} + 8\,q^{1} - 6\,q^{2} - 4\,q^{4}}{1 - 6\,q + 16\,q^{0} - 24\,q^{1} + 22\,q^{2} - 12\,q^{3} + 4\,q^{4}},$$

$$k[2] \to \frac{2\left(1 - 4\,q + 8\,q^{0} - 8\,q^{1} + 6\,q^{2}\right)}{1 - 6\,q + 16\,q^{0} - 24\,q^{1} + 22\,q^{2} - 12\,q^{3} + 4\,q^{4}},$$

$$k[3] \to -\frac{-3 + 12\,q - 24\,q^{0} + 24\,q^{1} - 18\,q^{2} + 8\,q^{3} - 4\,q^{4}}{1 - 6\,q + 16\,q^{0} - 24\,q^{1} + 22\,q^{2} - 12\,q^{3} + 4\,q^{4}},\ k[4] \to \frac{4\left(1 - 2\,q + 2\,q^{0}\right)}{1 - 4\,q + 6\,q^{0} - 4\,q^{1} + 2\,q^{2}},$$

$$k[5] \to -\frac{-5 + 20\,q - 40\,q^{0} + 48\,q^{1} - 38\,q^{2} + 16\,q^{3} - 4\,q^{4}}{1 - 6\,q + 16\,q^{0} - 24\,q^{1} + 22\,q^{2} - 12\,q^{3} + 4\,q^{4}},$$

$$k[6] \to \frac{2\left(3 - 12\,q + 20\,q^{0} - 16\,q^{1} + 6\,q^{2}\right)}{1 - 6\,q + 16\,q^{0} - 24\,q^{1} + 22\,q^{2} - 12\,q^{3} + 4\,q^{4}},$$

$$k[7] \to -\frac{-7 + 36\,q - 80\,q^{0} + 96\,q^{1} - 66\,q^{2} + 24\,q^{3} - 4\,q^{4}}{1 - 6\,q + 16\,q^{0} - 24\,q^{1} + 22\,q^{2} - 12\,q^{3} + 4\,q^{4}},\ k[8] \to 0\Big\}\Big\}$$

The expected passage times at the specific parameter of $q = \frac{18}{37}$ (“roulette win probability betting on red”):

In[15]:=
```
kAsol /. q → N[18 / 37]
```

Out[15]=

```
{{k[0] → 0, k[1] → 6.61281, k[2] → 11.5374, k[3] → 14.6801,
  k[4] → 15.9418, k[5] → 15.2181, k[6] → 12.3985, k[7] → 7.36681, k[8] → 0}}
```

The expected passage times at a tiny win probability $q = 0.001$:

In[16]:=
```
kAsol /. q → 0.001
```

Out[16]=

```
{{k[0] → 0, k[1] → 1.002, k[2] → 2.00401, k[3] → 3.00601,
  k[4] → 4.00802, k[5] → 5.01002, k[6] → 6.01202, k[7] → 7.006, k[8] → 0}}
```