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7 Generating functions and exponential distributions

This exercise develops your skills in working with generating functions and introduces you to the important family of exponential distributions.

Classroom problems

7.1 *Passage time to the origin for a random walk.* Consider a random walk $(X_t)_{t\in\mathbb{Z}_+}$ on the set $\mathbb{Z} = \{\dots,-2,-1,0,1,2,\dots\}$ which jumps to the left and to the right with equal probabilities $\frac{1}{2}$. Let $T_0 = \min\{t \geq 0 \mid X_t = 0\}$ be the passage time to the origin for the random walk, and denote by

$$\phi_x(z) \;=\; \sum_{j=0}^{\infty} z^j \, \mathbb{P}\big[T_0 = j \,\big|\, X_0 = x\big]$$

the generating function of T_0 for a random walk started at x.

(a) Compute $\phi_x(z)$ for $x = 0$.

Solution. Since the initial state is the origin, the passage time T_0 is deterministically zero, so that

$$\begin{aligned}\phi_0(z) &= \sum_{j=0}^{\infty} z^j \mathbb{P}[T_0 = j \mid X_0 = 0] \\ &= z^0 \mathbb{P}[T_0 = 0 \mid X_0 = 0] \\ &= 1.\end{aligned}$$

(b) Show that $\lim_{x\to\infty} \phi_x(z) = 0$ for all $0 \leq z < 1$.

Solution. Since $z \geq 0$, it must hold that

$$\phi_x(z) = \mathbb{E}(z^{T_0} | X_0 = x) \geq 0.$$

On the other hand it holds that $T_0 \geq X_0$, so

$$\phi_x(z) = \mathbb{E}(z^{T_0} | X_0 = x) \leq \mathbb{E}(z^x | X_0 = x) = z^x.$$

Taking the limits from both sides gives

$$0 \leq \lim_{x\to\infty} \phi_x(z) \leq \lim_{x\to\infty} z^x = 0.$$

(c) Show that the generating functions satisfy

$$\phi_x(z) \;=\; \frac{z}{2}\Big(\phi_{x+1}(z) + \phi_{x-1}(z)\Big), \quad x = 1,2,\dots$$

Solution. When $x \neq 0$, the passage time cannot be zero, and so we may remove the zero term from the generating function:

$$\phi_x(z) = \sum_{\cancel{j=0}\, j=1}^{\infty} z^j \, \mathbb{P}\big[T_0 = j \,\big|\, X_0 = x\big]. \tag{1}$$

Let's condition on the second state X_1. Since this can only be either $x-1$ or $x+1$, we have

$$\begin{aligned}
\mathbb{P}\big[T_0 = j \,\big|\, X_0 = x\big] &= \sum_k \mathbb{P}\big[T_0 = j \,\big|\, X_0 = x, X_1 = k\big]\mathbb{P}[X_1 = k \,|\, X_0 = x] \\
&= \frac{1}{2}\Big(\mathbb{P}\big[T_0 = j \,\big|\, X_0 = x, X_1 = x+1\big] \\
&\qquad + \mathbb{P}\big[T_0 = j \,\big|\, X_0 = x, X_1 = x-1\big]\Big).
\end{aligned} \tag{2}$$

By the properties of Markov chains, T_0 does not depend on the history. Therefore we can forget the conditioning by X_0. Substituting (2) into (1) now yields:

$$\sum_{j=1}^{\infty} z^j \frac{1}{2}\Big(\mathbb{P}\big[T_0 = j \,\big|\, X_1 = x-1\big] + \mathbb{P}\big[T_0 = j \,\big|\, X_1 = x+1\big]\Big).$$

The transition probabilities between any two states do not depend on time, so we shift the indices down by one:

$$\sum_{j=1}^{\infty} z^j \frac{1}{2}\Big(\mathbb{P}\big[T_0 = j-1 \,\big|\, X_0 = x-1\big] + \mathbb{P}\big[T_0 = j-1 \,\big|\, X_0 = x+1\big]\Big).$$

Finally we pull out the coefficient $z/2$ and replace the summing index by $k = j-1$:

$$\begin{aligned}
&\frac{z}{2}\left(\sum_{k=0}^{\infty} z^k \, \mathbb{P}\big[T_0 = k \,\big|\, X_0 = x-1\big] + \sum_{k=0}^{\infty} z^k \, \mathbb{P}\big[T_0 = k \,\big|\, X_0 = x+1\big]\right) \\
&\quad = \frac{z}{2}\Big(\phi_{x+1}(z) + \phi_{x-1}(z)\Big).
\end{aligned}$$

(d) Find all numbers $\alpha \in \mathbb{R}$ for which the function $f(x) = \alpha^x$ defined on the integers solves the difference equation $f(x) = \frac{z}{2}\big(f(x+1) + f(x-1)\big)$.

Solution. If $z = 0$, the only solution of the difference equation is $f = 0$. If $z \neq 0$, we get $\alpha^x = \frac{z}{2}(\alpha^{x+1} + \alpha^{x-1})$. Consider the case $\alpha \neq 0$. The equation is equivalent to $\alpha^2 - \frac{2}{z}\alpha + 1 = 0$, which has the solutions

$$\alpha_1 = \frac{1 + \sqrt{1 - z^2}}{z}, \quad \alpha_2 = \frac{1 - \sqrt{1 - z^2}}{z}.$$

(e) Apply the results of (a)–(d) to derive a formula for $\phi_x(z)$ for all $x = 0, 1, 2, \ldots$

Solution. When $z = 0$,

$$\phi_x(0) = \mathbb{P}(T_0 = 0 \mid X_0 = x) = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0. \end{cases}$$

Assume now $z \neq 0$. For second order homogeneous difference equations

$$f(x+2) + af(x+1) + bf(x) = 0 \tag{3}$$

it holds that when $\alpha^2 + a\alpha + b = 0$ has two solutions α_1 and α_2, all the solutions of the difference equation are of the form $f(x) = C_1\alpha_1^x + C_2\alpha_2^x$. Therefore, the solutions of the equation

$$\phi_x(z) = \frac{z}{2}\Big(\phi_{x+1}(z) + \phi_{x-1}(z)\Big)$$

are $\phi_x(z) = C_1\alpha_1^x + C_2\alpha_2^x$, where α_1 and α_2 are the solutions from part (d). It remains to find the constants C_1 and C_2 We use the boundary condition from part (a) for $x = 0$:

$$\begin{aligned} \phi_x(0) = 1 \;\Rightarrow\; & C_1\alpha_1^0 + C_2\alpha_2^0 = 1 \\ \Rightarrow\; & C_1 + C_2 = 1. \end{aligned} \tag{4}$$

Next we use the condition from part (b). One may easily verify that when $z \in (0, 1)$, it holds that $\alpha_2 < \alpha_1$ and $\alpha_1\alpha_2 = 1$, which gives $\alpha_2 < 1 < \alpha_1$. We deduce from

$$\lim_{x\to\infty}\Big(C_1\alpha_1^x + C_2\alpha_2^x\Big) = \lim_{x\to\infty}(C_1\alpha_1^x) + C_2 \cdot 0 = 0,$$

that C_1 must be zero. It follows from (4) that $C_2 = 1$. By substituting α_2 from part (d) we obtain the solution

$$\phi_x(z) = \left(\frac{1 - \sqrt{1 - z^2}}{z}\right)^x.$$

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Homework problems

7.2 *Sum of binomial random integers.* Let N be a $\text{Bin}(n, p)$-distributed random variable such that

$$\mathbb{P}(N = k) \;=\; \binom{n}{k}(1-p)^{n-k}p^k, \quad k = 0, 1, \ldots, n.$$

(a) Compute the probability generating function $\phi_N(z) = \mathbb{E}\big[z^N\big]$.

Solution. Directly from the definition of the probability generating function:

$$\begin{aligned}
\phi_N(z) &= \mathbb{E}[z^N] \\
&= \sum_{k=0}^{\infty} z^k \mathbb{P}(N = k) \\
&= \sum_{k=0}^{n} z^k \binom{n}{k}(1-p)^{n-k}p^k \\
&= \sum_{k=0}^{n} \binom{n}{k}(1-p)^{n-k}(pz)^k.
\end{aligned}$$

We recall the binomial theorem $(a+b)^m = \sum_{j=0}^{m}\binom{m}{j}a^j b^{m-j}$, which gives

$$\phi_N(z) \;=\; (1-p+pz)^n.$$

Assume that N_1 and N_2 are independent random integers with distributions $\text{Bin}(n_1, p_1)$ and $\text{Bin}(n_2, p_2)$.

(b) Compute the probability generating function of $N_1 + N_2$.

Solution. From independence and part (a) it follows that

$$\begin{aligned}
\phi_{N_1+N_2}(z) &= \mathbb{E}[z^{N_1+N_2}] \\
&= \mathbb{E}[z^{N_1}]\mathbb{E}[z^{N_2}] \\
&= (1-p_1+p_1 z)^{n_1}(1-p_2+p_2 z)^{n_2}.
\end{aligned}$$

(c) Assume that $p_1 = p_2$. Apply the results of (a)–(b) to find out the values of n and q for which $N_1 + N_2$ is $\text{Bin}(n, q)$-distributed.

Solution. The two distributions are the same exactly when

$$\phi_N(z) = \phi_{N_1+N_2}(z).$$

From the results of (a) and (b) and the assumption $p_1 = p_2$:

$$\begin{aligned}
&(1-q+qz)^n = (1-p_1+p_1 z)^{n_1}(1-p_2+p_2 z)^{n_2} \\
\Rightarrow\; &(1-q+qz)^n = (1-p_1+p_1 z)^{n_1+n_2}.
\end{aligned}$$

We see that the equality holds with parameters $q = p_1 = p_2$, $n = n_1 + n_2$.

7.3 *Christmas lights.* The town of Taka-Pajula is planning to organize Christmas lights. They called lamp manufacturers A and B. Both manufacturers offer bulbs at the same price, but the light bulbs differ in quality: the bulbs by company A have independent exponentially distributed lifetimes (in days) with rate parameter $\lambda_A = 0.05$, and bulbs by company B with rate parameter $\lambda_B = 0.08$. The officials order a bulb from each manufacturer, and they compare which of the bulbs lasts longer. The experiment is repeated until one of the manufacturers has won five tests more than the other. The lights will be ordered from this manufacturer.

(a) Compute the probability that a bulb from manufacturer A lasts longer in an individual test.

Solution. Denote the lifetimes of the bulbs by T_*, where $* \in \{A, B\}$. Since they are exponentially distributed, the density functions are

$$f_{T_*}(t) = \mathbb{I}_{t>0}\lambda_* \exp(-\lambda_* t)$$

and by independence their joint density is obtained as a product. Therefore

$$\begin{aligned}
\mathbb{P}[T_A < T_B] &= \int\int_{t_A<t_B} f_{T_A,T_B}(t_A, t_B)dt_A dt_B \\
&= \int_0^\infty \int_0^{t_B} f_{T_A}(t_A) f_{T_B}(t_B) dt_A dt_B \\
&= \int_0^\infty f_{T_B}(t_B)[1 - \exp(\lambda_A t_B)]dt_B \\
&= 1 - \frac{\lambda_B}{\lambda_A + \lambda_B} = \frac{\lambda_A}{\lambda_A + \lambda_B}.
\end{aligned}$$

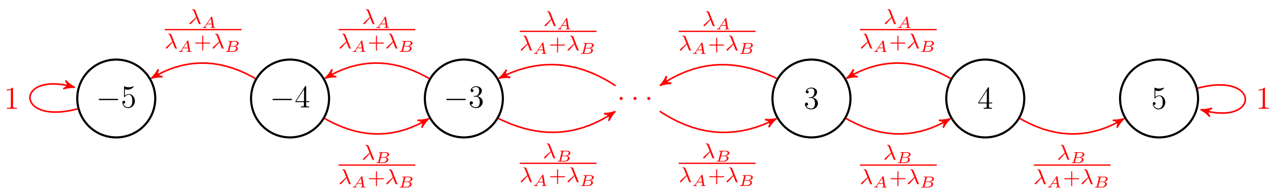
By this and symmetry (or directly from the lectures) we obtain

$$\mathbb{P}[B\text{'s bulb pops before } A\text{'s}] = \frac{\lambda_B}{\lambda_A + \lambda_B} = \frac{0.08}{0.05 + 0.08} \approx 0.615.$$

(b) What is the probability that the lights are ordered from manufacturer A?

Hint: Gambler's ruin.

Solution. Let X_t be the wins of A minus the wins of B after t tests. If $X_t = \pm 5$, we set $X_{t+1} = X_t$, since no more tests will be run. By the independence assumptions X_t is a Markov chain, and its transition diagram can be drawn as



This corresponds exactly to Gambler's ruin with winning probability $q = \frac{\lambda_B}{\lambda_A+\lambda_B}$. The initial state is $X_0 = 0$ and we wish to find the hitting probability to state 5. From the lecture material we remember that

$$h(x) = \frac{\left(\frac{1-q}{q}\right)^x - 1}{\left(\frac{1-q}{q}\right)^M - 1},$$

where the states are $(0, 1, \ldots, M)$ and x is the initial state. We see that in this indexing $M = 10$, $x = 5$, and $q = \lambda_B/(\lambda_A + \lambda_B) = 8/13$. It follows that

$$h(5) = \frac{\left(\frac{1-8/13}{8/13}\right)^5 - 1}{\left(\frac{1-8/13}{8/13}\right)^{10} - 1} \approx 0.913.$$

(c) What is the expected number of tests required?

Hint: $g(k) = \alpha^k$ satisfies $g(k) - q\,g(k+1) - (1-q)\,g(k-1) = 0$ for an appropriate α. Also, $g_0(k) = k$ satisfies $g(k) - q\,g(k+1) - (1-q)\,g(k-1) = C$ for a certain constant C. Take linear combinations of these to solve the required equations.

Solution. The expected number of tests is the expected passage time of Gambler's ruin. Let $g(x)$ be the expected passage time to $\{-5, 5\}$ starting from x. Now g satisfies

$$\begin{cases} g(-5) = g(5) = 0 \\ g(k) = q\,g(k+1) + (1-q)\,g(k-1) + 1, \quad -4 \le k \le 4. \end{cases} \tag{5}$$

Following the hint, we look for a solution that is of the form

$$g(k) = a\alpha_1^k + b\alpha_2^k + ck, \tag{6}$$

i.e., a linear combination of the solutions α^k of the homogeneous equation

$$g(k) - q\,g(k+1) - (1-q)\,g(k-1) = 0, \tag{7}$$

and the solution ck of the inhomogeneous equation

$$g(k) - q\,g(k+1) - (1-q)\,g(k-1) = 1, \tag{8}$$

which we derived in (5).

Next we need to find the constants α_1, α_2 and c. To find α_1 and α_2, we substitute $g(k) = \alpha^k$ in the homogeneous equation (7) and solve for α:

$$\begin{aligned} & g(k) - q\,g(k+1) - (1-q)\,g(k-1) = 0 \\ \Rightarrow \quad & \alpha^k - q\alpha^{k+1} - (1-q)\alpha^{k-1} = 0. \end{aligned}$$

Dividing by α^{k-1} gives a quadratic equation, so we obtain the solutions $\alpha_1 = 1$ and $\alpha_2 = (1-q)/q$. Since $q \neq 1/2$, the solution of the homogeneous equation is $a1^k + b\left(\frac{1-q}{q}\right)^k$.

Next we solve for the constant c so that ck is the solution of the inhomogeneous equation:

$$\begin{aligned} & g(k) - q\,g(k+1) - (1-q)\,g(k-1) = 1 \\ \Rightarrow \quad & ck - q\,c(k+1) - (1-q)\,c\,(k-1) = 1 \\ \Rightarrow \quad & \cancel{ck} - \cancel{qck} - qc - \cancel{ck} + c + \cancel{qck} - qc = 1 \\ & \qquad\qquad \Rightarrow \quad c = 1/(1-2q). \end{aligned}$$

So far we have obtained

$$g(k) = a + b\left(\frac{1-q}{q}\right)^k + \frac{k}{1-2q}.$$

The constants a and b are determined from the boundary conditions:

$$\begin{cases} g(-5) = a + b\left(\frac{1-q}{q}\right)^{-5} + \frac{-5}{1-2q} = 0 \\ g(5) = a + b\left(\frac{1-q}{q}\right)^{5} + \frac{5}{1-2q} = 0 \end{cases} \quad \Rightarrow \begin{cases} b = \frac{-10}{(1-2q)\left(\left(\frac{1-q}{q}\right)^5 - \left(\frac{q}{1-q}\right)^5\right)} \\ a = -\frac{5}{1-2q} - \left(\frac{1-q}{q}\right)^5 b. \end{cases}$$

Plugging in the numerical value of q gives the expected passage time

$$g(0) = a + b \approx 17.89.$$

Additional information. The exponential distribution in this exercise is a special case of the *Weibull distribution*, a common *life distribution* in survival analysis and reliability engineering. Because of the memorylessness property we have considered a model where the failure rate is constant over time. More on this in, e.g., the course MS-E2117 Riskianalyysi.