7 Generating functions and exponential distributions

This exercise develops your skills in working with generating functions and introduces you to the important family of exponential distributions.

Classroom problems

7.1 *Passage time to the origin for a random walk.* Consider a random walk $(X_t)_{t\in\mathbb{Z}_+}$ on the set $\mathbb{Z} = \{\dots,-2,-1,0,1,2,\dots\}$ which jumps to the left and to the right with equal probabilities $\frac{1}{2}$. Let $T_0 = \min\{t \geq 0 \mid X_t = 0\}$ be the passage time to the origin for the random walk, and denote by

$$\phi_x(z) \;=\; \sum_{j=0}^{\infty} z^j \,\mathbb{P}\big[T_0 = j \,\big|\, X_0 = x\big]$$

the generating function of T_0 for a random walk started at x.

(a) Compute $\phi_x(z)$ for $x = 0$.

(b) Show that $\lim_{x\to\infty} \phi_x(z) = 0$ for all $0 \leq z < 1$.

(c) Show that the generating functions satisfy

$$\phi_x(z) \;=\; \frac{z}{2}\Big(\phi_{x+1}(z) + \phi_{x-1}(z)\Big), \quad x = 1,2,\dots$$

(d) Find all numbers $\alpha \in \mathbb{R}$ for which the function $f(x) = \alpha^x$ defined on the integers solves the difference equation $f(x) = \frac{z}{2}\big(f(x+1) + f(x-1)\big)$.

(e) Apply the results of (a)–(d) to derive a formula for $\phi_x(z)$ for all $x = 0,1,2,\dots$

Homework problems

7.2 *Sum of binomial random integers.* Let N be a $\mathrm{Bin}(n, p)$-distributed random variable such that

$$\mathbb{P}(N = k) \;=\; \binom{n}{k}(1-p)^{n-k}p^k, \quad k = 0, 1, \ldots, n.$$

(a) Compute the probability generating function $\phi_N(z) = \mathbb{E}\big[z^N\big]$.

Assume that N_1 and N_2 are independent random integers with distributions $\mathrm{Bin}(n_1, p_1)$ and $\mathrm{Bin}(n_2, p_2)$.

(b) Compute the probability generating function of $N_1 + N_2$.

(c) Assume that $p_1 = p_2$. Apply the results of (a)–(b) to find out the values of n and q for which $N_1 + N_2$ is $\mathrm{Bin}(n, q)$-distributed.

7.3 *Christmas lights.* The town of Taka-Pajula is planning to organize Christmas lights. They called lamp manufacturers A and B. Both manufacturers offer bulbs at the same price, but the light bulbs differ in quality: the bulbs by company A have independent exponentially distributed lifetimes (in days) with rate parameter $\lambda_A = 0.05$, and bulbs by company B with rate parameter $\lambda_B = 0.08$. The officials order a bulb from each manufacturer, and they compare which of the bulbs lasts longer. The experiment is repeated until one of the manufacturers has won five tests more than the other. The lights will be ordered from this manufacturer.

(a) Compute the probability that a bulb from manufacturer A lasts longer in an individual test.

(b) What is the probability that the lights are ordered from manufacturer A?
Hint: Gambler's ruin.

(c) What is the expected number of tests required?
Hint: $g(k) = \alpha^k$ *satisfies* $g(k) - q\,g(k+1) - (1-q)\,g(k-1) = 0$ *for an appropriate* α. *Also,* $g_0(k) = k$ *satisfies* $g(k) - q\,g(k+1) - (1-q)\,g(k-1) = C$ *for a certain constant* C. *Take linear combinations of these to solve the required equations.*