MS-C2111 Stochastic Processes

Lecture 10

Continuous-time Markov chains 2

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Trajectories of continuous-time Markov chains

Constructing the generator matrix in practice

Three methods to simulate paths

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Long-term behavior

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First jump instant

Theorem

The first jump instant $T_1 = \min\{t \in \mathbb{R}_+ : X_t \neq X_0\}$ *of a CTMC satisfies either*

(i) $T_1 =_{\text{st}} \text{Exp}(\lambda)$ *for some* $\lambda > 0$, *or*

(ii) $T_1 = \infty$ *with probability one.*

Proof.

$\phi(t) = \mathbb{P}(T_1 > t)$ is nonincreasing in t, and satisfies

$$\begin{aligned}\phi(t+h) \;=\; \mathbb{P}(T_1 > t+h) \;&=\; \mathbb{P}(T_1 > t)\,\mathbb{P}(T_1 > t+h \,|\, T_1 > t) \\ &=\; \mathbb{P}(T_1 > t)\,\mathbb{P}(T_1 > h) \;=\; \phi(t)\,\phi(h).\end{aligned}$$

$\Longrightarrow \; \phi(t) = e^{-\lambda t}$ for some $\lambda \in [0,\infty)$.

(i) If $\lambda > 0$, then $T_1 =_{\text{st}} \text{Exp}(\lambda)$

(ii) If $\lambda = 0$, then $\mathbb{P}(T_1 > t) = 1$ for all t, thus $\mathbb{P}(T_1 = \infty) = 1$

$\square$

Jump rate

The first jump instant T_1 of a CTMC starting at x satisfies

$$T_1 =_{\mathrm{st}} \mathrm{Exp}(\lambda(x)),$$

where the total jump rate in state x equals

$$\lambda(x) \;=\; \frac{1}{\mathbb{E}(T_1 \,|\, X_0 = x)} \in [0, \infty)$$

- If $\lambda(x) > 0$, the chain spends in x an exponentially distributed random time with mean $1/\lambda(x)$.
- If $\lambda(x) = 0$, then state x absorbing and $T_1 = \infty$.

All sojourn times of a continuous-time Markov chain are exponentially distributed.

Trajectories of a continuous-time Markov chain

A continuous-time Markov chain (X_t) starting at state x:

- Spends an $\mathrm{Exp}(\lambda(x))$-distributed time in state x
- Then jumps to $y \neq x$ with probability $P_*(x, y)$
- Spends an $\mathrm{Exp}(\lambda(y))$-distributed time in state y
- ...

$\lambda(x)$ is the jump rate of state x

$P_*(x, y) = \mathbb{P}(X_{T_1} = y \mid X_0 = x)$ is the jump probability from x to y

Theorem

When $x \mapsto \lambda(x)$ is a bounded function, the above definition produces a continuous-time Markov chain.

Note: When the state space is finite, $x \mapsto \lambda(x)$ is bounded.

Generator matrix

Can we obtain the generator matrix from jump rates and jump probabilities?

$$Q = \lim_{h \to 0+} \frac{P_h - I}{h}$$

Generator matrix

Theorem ([Kal02, Sec 12])

When $x \mapsto \lambda(x)$ is bounded, the generator matrix of a CTMC with jump rates $\lambda(x)$ and jump probabilities $P_*(x,y)$ is given by

$$Q(x,y) \;=\; \begin{cases} \lambda(x) P_*(x,y), & x \neq y, \\ -\lambda(x), & x = y. \end{cases}$$

Recall that for $x \neq y$, $Q(x,y)$ is the jump rate from x to y, and that Q has zero row sums. Hence the total jump rate of x equals

$$\lambda(x) \;=\; \sum_{y \neq x} Q(x,y)$$

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Example: Two machines

Machine $i = 1, 2$ is expected to work for $1/\lambda = 40$ weeks before breaking. Repairing a machine takes $1/\kappa = 2$ weeks by expectation. The operation and repair times are independent and exponentially distributed.

$$X_t \;=\; \text{Number of broken machines at time } t$$

Jump rates:
In state 0 the chain spends ?
First jump instant $T_1 = \min\{L_1, L_2\}$ where L_i is the breaking time of machine i.

Winning time in an exponential race

Winning time of a race is $U = \min\{X_1, ..., X_n\}$, where the competitors' times $X_i =_{\mathrm{st}} \mathrm{Exp}(\lambda_i)$ are independent.

What is the distribution of U?

$$
\begin{aligned}
\mathbb{P}(U > t) &= \mathbb{P}(X_1 > t, ..., X_n > t) \\
&= \mathbb{P}(X_1 > t) \cdots \mathbb{P}(X_n > t) \\
&= e^{-\lambda_1 t} \cdots e^{-\lambda_n t} \\
&= e^{-(\sum_{i=1}^n \lambda_i) t}
\end{aligned}
$$

$$\Longrightarrow \quad U =_{\mathrm{st}} \mathrm{Exp}(\textstyle\sum_{i=1}^n \lambda_i)$$

The minimum of independent exponentially distributed random numbers is exponentially distributed.

Winning probability in an exponential race

What is the probability that 1 wins?

Competitor 1 wins if $X_1 < \min\{X_2, \ldots, X_n\} =: \tilde{U}$.

Because X_1 and $\tilde{U}$ are independent,

$$\mathbb{P}(X_1 < \tilde{U}) \;=\; \int_0^\infty \mathbb{P}(t < \tilde{U})\, \lambda_1 e^{-\lambda_1 t} dt.$$

Because $\tilde{U} =_{\mathrm{st}} \mathrm{Exp}(\sum_{i=2}^n \lambda_i)$, competitor 1 wins with probability

$$\begin{aligned}\mathbb{P}(X_1 < \tilde{U}) &= \int_0^\infty e^{-(\sum_{i=2}^n \lambda_i)t}\, \lambda_1 e^{-\lambda_1 t} dt \\ &= \lambda_1 \int_0^\infty e^{-(\sum_{i=1}^n \lambda_i)t} dt = \frac{\lambda_1}{\sum_{i=1}^n \lambda_i}.\end{aligned}$$

Exponential race — Summary

Theorem

Let $U = \min\{X_1, \ldots, X_n\}$ *where* $X_i =_{\mathrm{st}} \mathrm{Exp}(\lambda_i)$ *are independent. Then:*

- $U =_{\mathrm{st}} \mathrm{Exp}(\sum_{i=1}^{n} \lambda_i)$
- $\mathbb{P}(X_i = U) = \frac{\lambda_i}{\lambda_1 + \cdots \lambda_n}$
- U *and* $I = \arg\min_i\{X_1, \ldots, X_n\}$ *are independent.*

Example: Two machines

Machine $i = 1, 2$ is expected to work for $1/\lambda = 40$ weeks before breaking, and repairing takes $1/\kappa = 2$ weeks by expectation. The operation and repair times are independent and exponentially distributed.

$$X_t \; = \; \text{Number of broken machines at time } t$$

Jump rates:
In state 0 the chain spends ?
First jump instant $T_1 = \min\{L_1, L_2\}$ where $L_i =_{\mathrm{st}} \mathrm{Exp}(\lambda)$ is the breaking time of machine i. $\implies$ $T_1 =_{\mathrm{st}} \mathrm{Exp}(2\lambda)$ $\implies$ $\lambda(0) = 2\lambda$

Jump probabilities:
From state 0 the chain surely jumps to state 1 $\implies$ $P_*(0, 1) = 1$

After the jump:
One machine is fixed and the remaining breaking time of the other machine is $=_{\mathrm{st}}$ $\mathrm{Exp}(\lambda)$
$\implies$ After T_1 the chain behaves as if it was restarted at state 1

Example: Two machines

Machine $i = 1, 2$ is expected to work for $1/\lambda = 40$ weeks before breaking, and repairing takes $1/\kappa = 2$ weeks by expectation. The operation and repair times are independent and exponentially distributed.

$$X_t \ = \ \text{Number of broken machines at time } t$$

Jump rates:

In state 1 the chain spends ?

First jump instant $T_1 = \min\{L, M\}$ where $L =_{\mathrm{st}} \mathrm{Exp}(\lambda)$ is break time of the operating machine, $M =_{\mathrm{st}} \mathrm{Exp}(\kappa)$ is the repair time of the broken machine $\implies T_1 =_{\mathrm{st}} \mathrm{Exp}(\lambda + \kappa) \implies \lambda(1) = \lambda + \kappa$

Jump probabilities:

From state 1 the chain jumps to 0 with probability $\frac{\kappa}{\lambda+\kappa}$, and to 2 with probability $\frac{\lambda}{\lambda+\kappa}$

We analyze the rates and probabilities from state 2 similarly...

Example: Two machines

Machine $i = 1, 2$ is expected to work for $1/\lambda = 40$ weeks before breaking, and repairing takes $1/\kappa = 2$ weeks by expectation. The operation and repair times are independent and exponentially distributed.

$$X_t \; = \; \text{Number of broken machines at time } t$$

The jump rates are $\lambda(0) = 2\lambda$, $\lambda(1) = \lambda + \kappa$, $\lambda(2) = 2\kappa$

The jump probabilities are

$$P_* \; = \; \begin{bmatrix} P_*(0,0) & P_*(0,1) & P_*(0,2) \\ P_*(1,0) & P_*(1,1) & P_*(1,2) \\ P_*(2,0) & P_*(2,1) & P_*(2,2) \end{bmatrix} \; = \; \begin{bmatrix} 0 & 1 & 0 \\ \frac{\kappa}{\lambda+\kappa} & 0 & \frac{\lambda}{\lambda+\kappa} \\ 0 & 1 & 0 \end{bmatrix}.$$

Example: Two machines

Machine $i = 1, 2$ is expected to work for $1/\lambda = 40$ weeks before breaking, and repairing takes $1/\kappa = 2$ weeks by expectation. The operation and repair times are independent and exponentially distributed.

$$X_t \;=\; \text{Number of broken machines at time } t$$

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The jump probabilities are

$$P_* \;=\; \begin{bmatrix} P_*(0,0) & P_*(0,1) & P_*(0,2) \\ P_*(1,0) & P_*(1,1) & P_*(1,2) \\ P_*(2,0) & P_*(2,1) & P_*(2,2) \end{bmatrix} \;=\; \begin{bmatrix} 0 & 1 & 0 \\ \frac{\kappa}{\lambda+\kappa} & 0 & \frac{\lambda}{\lambda+\kappa} \\ 0 & 1 & 0 \end{bmatrix}.$$

The generator matrix equals

$$Q \;=\; \begin{bmatrix} -2\lambda & 2\lambda & 0 \\ \kappa & -(\lambda+\kappa) & \lambda \\ 0 & 2\kappa & -2\kappa \end{bmatrix} \;=\; \begin{bmatrix} -0.050 & 0.050 & 0 \\ 0.500 & -0.525 & 0.025 \\ 0 & 1.000 & -1.000 \end{bmatrix}.$$

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Simulating a CTMC 1: Scaled exponentials

Given jump rates $\lambda : S \to \mathbb{R}_+$ and a transition matrix P_*, let

- $(Y_n)_{n\in\mathbb{Z}_+}$ be DTMC with transition matrix P_*
- $\gamma_1, \gamma_2, \ldots$ independent and Exp(1)-distributed

Define

$$X_t = \begin{cases} Y_0, & 0 \le t < T_1, \\ Y_1, & T_1 \le t < T_2, \\ \vdots & \vdots \end{cases}$$

where $T_n = \frac{\gamma_1}{\lambda(Y_0)} + \cdots + \frac{\gamma_n}{\lambda(Y_{n-1})}$.

Theorem ([Kal02, Thm 12.18])

If $\lambda : S \to \mathbb{R}_+$ is bounded, then $(X_t)_{t\in\mathbb{R}_+}$ is a CTMC with jump rate function λ, jump probability matrix P_*, and generator matrix $Q(x,y) = \lambda(x)(P_*(x,y) - I(x,y))$.

Simulating a CTMC 2: Overclocking

Given jump rates $\lambda : S \to \mathbb{R}_+$ and a jump probability matrix P_*,

$$R(x,y) \;=\; \frac{\lambda(x)}{\alpha}\, P_*(x,y) + \left(1 - \frac{\lambda(x)}{\alpha}\right) I(x,y)$$

is a transition matrix when $\alpha \geq \lambda(x)$ for all x.
Simulate

- DTMC $(Y_0, Y_1, \dots)$ with transition matrix R
- Poisson process $N(t)$ with intensity α

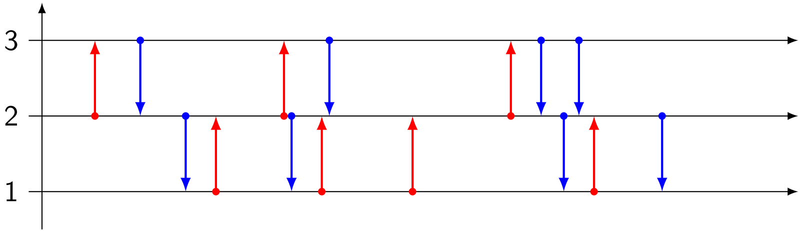
Then the Poisson modulated chain $X_t = Y_{N(t)}$ is a CTMC with jump rates $\lambda(x)$ and jump probabilities P_*.

- At every time instant of a Poisson process of rate α we flip a coin
- With probability $\frac{\lambda(x)}{\alpha}$ we jump according to P_*
- With probability $1 - \frac{\lambda(x)}{\alpha}$ we do nothing

Simulating a CTMC 3: Pairwise Poisson triggers

To every ordered pair of distinct states (x, y) associate an independent Poisson process $(N_{x,y}(t))_{t\in\mathbb{R}_+}$ with intensity $Q(x, y)$.

The resulting black line is the trajectory of a CTMC with generator matrix Q.



Simulating a CTMC 3: Pairwise Poisson triggers

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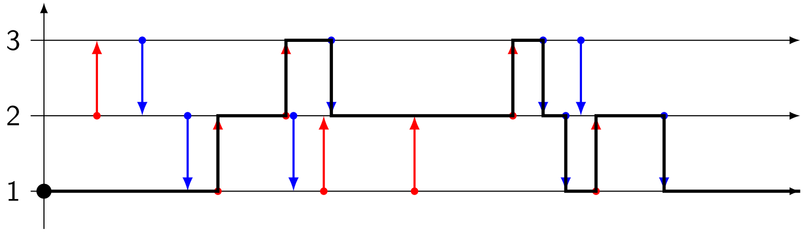
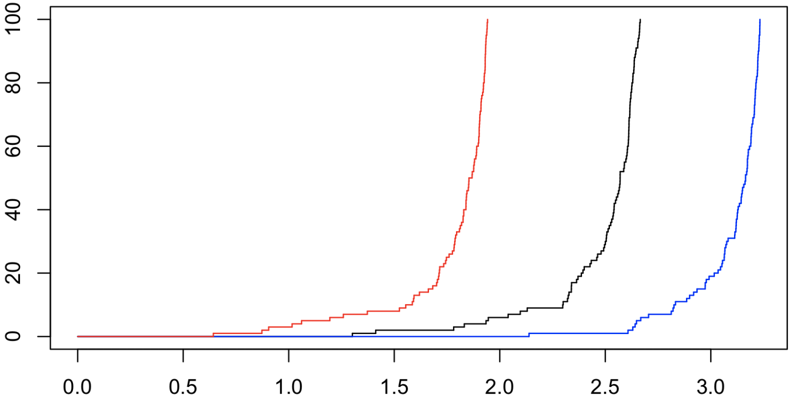


What if the jump rates are unbounded?

Let $\lambda(k) = k^\alpha$ and $P_*(k,\ell) = 1(\ell = k+1)$, $Y_n = n+1$.

$$T_n = \sum_{k=1}^{n} \frac{\gamma_k}{\lambda(Y_{k-1})} = \sum_{k=1}^{n} \frac{\gamma_k}{k^\alpha} \longrightarrow \sum_{k=1}^{\infty} \frac{\gamma_k}{k^\alpha} =: T_\infty$$

If $\alpha > 1$, then $\mathbb{E}\, T_\infty = \sum_{k=1}^\infty k^{-\alpha} < \infty$, so that $\mathbb{P}(T_\infty < \infty) = 1$.
The process explodes at time T_∞.



Simulating exploding paths

```
# Simulating 3 paths
alpha <- 1.5;
nmax <- 100
Omega <- 3
cols <- c("black","blue","red")
T <- matrix(0, Omega, nmax)
for (omega in 1:3) {
 gamma <- rexp(nmax, 1);
 T[omega,] <- cumsum(gamma/(1:nmax)^alpha)
}

# Plotting 3 paths
plot(0,0,"n", xlim=c(0, max(T)), ylim=c(0, nmax), xlab="", ylab="")
for (omega in 1:3) {
 x <- c(0,T[omega,]);
 y <- 0:nmax
 segments(x[-length(x)], y[-length(x)], x[-1], y[-length(x)], col=cols[omega])
 segments(x[-1], y[-length(x)], x[-1], y[-1], col=cols[omega])
}
```

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Computing transition matrices

Theorem

If $x \mapsto \lambda(x)$ is bounded, then the transition matrices P_t can be computed by

$$P_t \;=\; \sum_{n=0}^{\infty} \frac{(tQ)^n}{n!} =: e^{tQ}, \quad t \geq 0.$$

Proof: Overclocking

$$R(x, y) \;=\; \frac{\lambda(x)}{\alpha}\, P_*(x, y) + \left(1 - \frac{\lambda(x)}{\alpha}\right) I(x, y).$$

Let $X_t = Y_{N(t)}$ be the Poisson modulated (with rate α) discrete time chain (with (Y_n) having transition matrix R).

$$P_t(x, y) \;=\; \sum_{n=0}^{\infty} e^{-\alpha t} \frac{(\alpha t)^n}{n!} R^n(x, y)$$

$$\alpha R(x, y) \;=\; \lambda(x)\, P_*(x, y) + (\alpha - \lambda(x)) I(x, y) \;=\; Q(x, y) + \alpha I(x, y)$$

Proof

$$
\begin{aligned}
P_t &= \sum_{n=0}^{\infty} e^{-\alpha t} \frac{t^n}{n!} (\alpha R)^n \\
&= \sum_{n=0}^{\infty} e^{-\alpha t} \frac{t^n}{n!} (Q + \alpha I)^n \\
&= \sum_{n=0}^{\infty} e^{-\alpha t} \frac{t^n}{n!} \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} Q^m \alpha^{n-m} \\
&= \sum_{m=0}^{\infty} e^{-\alpha t} \frac{t^m}{m!} Q^m \sum_{n=m}^{\infty} \frac{t^{n-m}}{(n-m)!} \alpha^{n-m} \\
&= \sum_{m=0}^{\infty} e^{-\alpha t} \frac{t^m}{m!} Q^m e^{\alpha t} \\
&= \sum_{m=0}^{\infty} \frac{(tQ)^m}{m!} \; = \; e^{tQ}.
\end{aligned}
$$

Example: Two machines

Machine $i = 1, 2$ is expected to work for $1/\lambda = 40$ weeks before breaking, and repairing takes $1/\kappa = 2$ weeks by expectation. The operation and repair times are independent and exponentially distributed.

$$X_t \;=\; \text{Number of broken machines at time } t$$

What is the probability that both machines work after 3 weeks, given that they work now?

$$Q \;=\; \begin{bmatrix} -2\lambda & 2\lambda & 0 \\ \kappa & -(\lambda+\kappa) & \lambda \\ 0 & 2\kappa & -2\kappa \end{bmatrix} \;=\; \begin{bmatrix} -0.050 & 0.050 & 0 \\ 0.500 & -0.525 & 0.025 \\ 0 & 1.000 & -1.000 \end{bmatrix}.$$

$$P_3 \;=\; e^{3Q} \;=\; \begin{bmatrix} 0.9259028 & 0.07267122 & 0.001425934 \\ 0.7267122 & 0.26404492 & 0.009242859 \\ 0.5703737 & 0.36971437 & 0.059911911 \end{bmatrix}$$

Both machines work after 3 weeks with probability $P_3(0,0) = 0.9259028$

Computing a matrix exponential

R

```r
library(expm)
la <- 1/40
mu <- 1/2
t <- 3
Q <- matrix(0,3,3)
Q[1,] <- c(-2*la,2*la,0)
Q[2,] <- c(mu,-la-mu,la)
Q[3,] <- c(0,2*mu,-2*mu)
P3 <- expm(t*Q)
```

Python

```python
import numpy as np
from scipy.linalg import expm
la = 1.0/40
mu = 1.0/2
t = 3.0
Q = np.array([
 [-2*la, 2*la, 0],
 [mu, -la-mu, la],
 [0,2*mu,-2*mu]])
P3 = expm(t*Q)
```

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Invariant distribution

Let (X_t) be a continuous-time MC with transition matrices $(P_t)_{t \in \mathbb{R}_+}$. Then a state distribution π is called invariant if

$$\pi P_t = \pi \quad \text{for all } t \geq 0.$$

Note: If we start the chain from a random state X_0 according to π, then after t time units the state distribution is

$$\begin{aligned}
\mathbb{P}(X_t = y) &= \sum_x \mathbb{P}(X_0 = x)\, \mathbb{P}(X_t = y \,|\, X_0 = x) \\
&= \sum_x \pi(x) P_t(x, y) \\
&= (\pi P_t)(y) \\
&= \pi(y) \quad \text{by invariance}
\end{aligned}$$

Solving the invariant distribution

Theorem

π is an invariant distribution of a finite-state CTMC with generator matrix Q if and only if $\pi Q = 0$.

Proof.

(i) "only if" direction: omitted (see lecture notes Theorem 10.5)

(ii) "if" direction: Suppose $\pi Q = 0$. Then

$$\begin{aligned}
\pi P_t &= \pi \sum_{n=0}^{\infty} \frac{t^n}{n!} Q^n \\
&= \pi + \sum_{n=1}^{\infty} \frac{t^n}{n!} Q^n \\
&= \pi + \sum_{n=1}^{\infty} \frac{t^n}{n!} (\pi Q) Q^{n-1} \\
&= \pi.
\end{aligned}$$

□

Reversibility

A generator matrix Q is reversible with respect to a probability distribution π if it satisfies the detailed balance conditions

$$\pi(x)Q(x,y) \;=\; \pi(y)Q(y,x) \quad \text{for all } x \neq y.$$

As in discrete time, a Markov chain with a reversible initial distribution behaves statistically the same when observed backwards in time.

Theorem

If Q is π-reversible, then π is invariant.

Proof.

Because $\pi(x)Q(x,y) = \pi(y)Q(y,x)$ also for $x = y$, and because Q has zero row sums,

$$(\pi Q)(y) = \sum_x \pi(x)Q(x,y) = \sum_x \pi(y)Q(y,x) = \pi(y)\sum_x Q(y,x) = 0.$$

$\square$

Long-term (limit) behavior

Theorem ([Dur12, Theorem 4.4].)

An irreducible CTMC has at most one invariant distribution. If π is an invariant distribution of an irreducible CTMC, then

$$\lim_{t \to \infty} P_t(x, y) \ = \ \pi(y)$$

for every $x \in S$.

- The transition diagram of a generator matrix Q and a corresponding CTMC is a directed graph with nodes $=$ states and links $= (x, y)$ such that $Q(x, y) > 0$.
- Generator matrix Q and a corresponding CTMC is irreducible if its transition diagram is strongly connected (for every x, y there exists a path from x to y).

Example: Two machines

Balance equations $\pi Q = 0$ can be written as

$$\begin{aligned} -\pi(0)2\lambda + \pi(1)\kappa + \pi(2)\cdot 0 &= 0, \\ \pi(0)2\lambda - \pi(1)(\lambda+\kappa) + \pi(2)2\kappa &= 0 \\ \pi(0)\cdot 0 + \pi(1)\lambda - \pi(2)2\kappa &= 0 \end{aligned}$$

Together with $\pi(0) + \pi(1) + \pi(2) = 1$ we can solve

$$\pi = \begin{bmatrix} p^2 & 2p(1-p) & (1-p)^2 \end{bmatrix}$$

where $p = \frac{\kappa}{\lambda+\kappa}$. By substituting $p = 0.952381$ we get

$$\pi = \begin{bmatrix} 0.907029478 & 0.090702948 & 0.002267574 \end{bmatrix}.$$

Example: Two-core CPU

Let $X_t \in \{0, 1, 2, 3\}$ be the number of tasks in system memory at time t (being processed or waiting).

- New tasks arrive at 5 min intervals: $\lambda = 12$ (unit 1/hour).
- The system has two cores, each core can process one task in 15 min by expectation: $\kappa = 4$
- The system has sufficient buffer memory to store one task for waiting. New tasks arriving while the buffer is full are lost.

Generator matrix

$$Q = \begin{bmatrix} -\lambda & \lambda & 0 & 0 \\ \kappa & -\lambda - \kappa & \lambda & 0 \\ 0 & 2\kappa & -\lambda - 2\kappa & \lambda \\ 0 & 0 & 2\kappa & -2\kappa \end{bmatrix} = \begin{bmatrix} -12 & 12 & 0 & 0 \\ 4 & -16 & 12 & 0 \\ 0 & 8 & -20 & 12 \\ 0 & 0 & 8 & -8 \end{bmatrix}$$

Example: Two-core CPU

Generator matrix

$$Q = \begin{bmatrix} -12 & 12 & 0 & 0 \\ 4 & -16 & 12 & 0 \\ 0 & 8 & -20 & 12 \\ 0 & 0 & 8 & -8 \end{bmatrix}$$

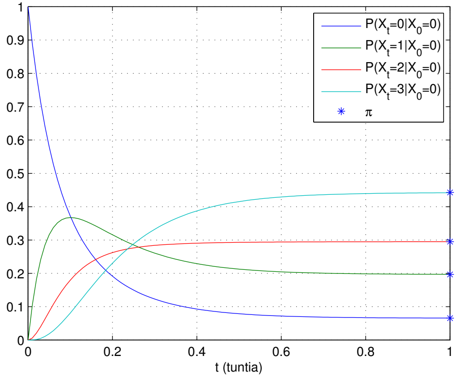
By solving $\pi Q = 0$ we get the invariant distribution for the number tasks in the system

$$\pi = \begin{bmatrix} \frac{4}{61} & \frac{12}{61} & \frac{18}{61} & \frac{27}{61} \end{bmatrix}$$

Example: Two-core CPU

Time evolution of the distribution of X_t when initially there are no tasks in the system ($X_0 = 0$)

$$P_t(i,j) \ = \ \sum_{k=0}^{\infty} \frac{t^k}{k!} Q^k(i,j) \ = \ e^{tQ}(i,j)$$



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Sources

Photos

1. Image courtesy of think4photop at FreeDigitalPhotos.net