4 Markov additive models, hitting times and probabilities

The goal of this exercise is to learn to calculate the cumulative revenues associated to Markov additive models, the expected passage times of Markov chains and the probabilities to hit a certain state before visiting a given set of states. It's recommended to bring a laptop or a calculator to the exercise session to make it easier to calculate the numerical results of the exercises.

Classroom problems

4.1 Consider a Markov chain with state space $S = \{1, 2, \dots, 6\}$, initial state 1, and transition matrix

$$P = \begin{bmatrix} 0.2 & 0.2 & 0.1 & 0.2 & 0.2 & 0.1 \\ 0.4 & 0 & 0.6 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0 & 0 & 0.4 \\ 0 & 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Draw the transition diagram of the chain and answer the following questions:

(a) What is the probability that a chain eventually ends up in state 5?

(b) What is the probability that a chain eventually ends up in state 6?

(c) What is the probability that a chain never visits state 3?

(d) What is the expected number of time steps before the chain hits the set $\{5, 6\}$?

Homework problems

4.2 Analyze the expected revenues of Katiskakauppa.com Oyj (Examples 1.3, 3.3, 3.7 in the lecture notes) in the case where the weekly demands D_t are Poisson distributed with mean $\lambda = 3.0$.

(a) Compute the expected cumulative revenue for a period of ten weeks, as a function of the number of laptops in stock ($x = 2, 3, 4, 5$) in the beginning of the first week.

(b) Compute the invariant distribution of the Markov chain modeling the number of laptops in stock.

(c) Compute the store's long-term expected revenue rate (EUR/week).

(d) Compare the result of part c) to your results in a).

4.3 Let $(X_0, X_1, \dots)$ be a Markov chain defined on a finite state space S and suppose that a deterministic cost $c(x)$ is incurred every time the chain visits the state x. Let $g(x)$ be the expected total cost of a Markov chain starting from the state x before the chain hits a state set A.

(a) Derive the following equations for the function $g : S \to \mathbb{R}$:

$$\begin{aligned} g(x) &= 0, && x \in A \\ g(x) &= c(x) + \sum_{y \in S} P(x,y)\, g(y), && x \notin A. \end{aligned}$$

(b) Consider the chain of Problem 4.1. By applying the result of part (a), determine the expected number of times that the chain starting from state 1 visits state 3 before it gets absorbed into the set $\{5, 6\}$.