1 Random variables and probability distributions

The goal of this exercise is to refresh the basic concepts of probability which are necessary for the treatment of stochastic processes. You are expected to be familiar with these topics from previous studies such as `MS-A05XX First course in probability and statistics`. Suggested background reading for refreshing the basics of probability is available for example in the freely downloadable book [GS97] or the lecture notes [Les20].

Classroom exercises

1.1 *Six javelin throws.* In an athletics event Tero Pitkämäki throws a javelin six times. Suppose that the lengths of the throws (in meters) are independent random numbers $Z_1, \dots, Z_6$ following the uniform distribution on the continuous interval $(80, 92)$.

(a) Calculate the expected value and variance of Tero's first throw.

(b) Determine the cumulative distribution function F_Y and the density function f_Y of the random number $Y = \max(Z_1, \dots, Z_6)$, and calculate the probability that Tero's longest throw is at least 91 meters.

(c) Find out the cumulative distribution function F_X of the random number $X = \min(Z_1, \dots, Z_6)$, and calculate the probability that at least one of Tero's throws is shorter than 85 meters.

(d) Does the joint distribution of X and Y have a density function $f_{X,Y}$? Are the random numbers X and Y independent? Justify your answer.

(e) Does the joint distribution of Z_1 and Y have a density function $f_{Z_1,Y}$?

Homework problems

1.2 *Memoryless distributions.* Let X be a random number following the geometric distribution on the set $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ with a success probability $p \in (0, 1)$, so that X has the probability mass function

$$\pi_X(k) = (1-p)^k p, \qquad \text{for } k = 0, 1, 2, \dots$$

(a) Find the conditional probability $\mathbb{P}\big[X \geq t + h \,\big|\, X \geq t\big]$ for integers $t, h \geq 0$.

(b) Calculate the expected value and variance of X.

Let Y be a random number following the exponential distribution with a rate parameter $\lambda > 0$, so that Y has the density function

$$f_Y(x) \;=\; \lambda e^{-\lambda x} 1_{(0,\infty)}(x) \;=\; \begin{cases} \lambda e^{-\lambda x}, & \text{for } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

(c) Find the conditional probability $\mathbb{P}\big[Y > t + h \,\big|\, Y > t\big]$ for real numbers $t, h > 0$.

(d) Calculate the expected value and variance of Y.

1.3 *Robot football finals.* Otaniemi Eulers and the Leppävaara Algebra are fighting for the championship of the robot football league by playing a series of matches in a best-of-three system, where the winner of the whole league is the team that gets two wins. Suppose that the results of the matches are independent and that Eulers win each match with probability $p = 0.55$.

(a) What is the probability that Eulers take the championship?

(b) What is the probability that it takes three matches to determine the champion?

(c) What is the expected number of matches required to determine the champion?

Now consider a change in the rules where a best-of-seven system is used, so that the champion is the team that gets four wins.

(d) What is the probability that Eulers take the championship?

(e) What is the probability that it takes seven matches to determine the champion?

References

[GS97] Charles M. Grinstead and J. Laurie Snell. *Introduction to Probability.* American Mathematical Society, https://math.dartmouth.edu/~prob/prob/prob.pdf, 1997.

[Les20] Lasse Leskelä. Stochastic processes. https://math.aalto.fi/~lleskela/LectureNotes005.html, 2020.