Quiz 1:

Question **1**

Flag question   Mark 0.17 out of 1.00   Partially correct

Which of the following can be transition matrices of a Markov chain?

Select one or more:

- [x] a. $\begin{bmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{bmatrix}$ ✓
- [ ] b. $\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$
- [x] c. $\begin{bmatrix} 0 & 0.75 & 0.25 \\ 0.4 & 0 & 0.6 \end{bmatrix}$ ✗ A transition matrix must be a *square matrix*, i.e. it must have an equal number of rows and columns!
- [ ] d. $\begin{bmatrix} 0 & 3 & -2 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}$
- [ ] e. $\begin{bmatrix} \frac{3}{7} & \frac{2}{7} \\ \frac{4}{7} & \frac{5}{7} \end{bmatrix}$

Your answer is partially correct.

You have correctly selected 1.
For a transition matrix $P = (p_{x,y})$ it holds that:

- P is a square matrix: both its rows and columns represent the states of a Markov chain
- $p_{x,y} \geq 0$ for all x, y
- $\sum_y p_{x,y} = 1$ for all x

The correct answers are: $\begin{bmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{bmatrix}$

, $\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$

Question **2**

Flag question Mark 0.00 out of 1.00 Incorrect

Let $X = (X_t)_{t \in \mathbb{Z}_+}$ be a Markov chain with state space $S = \{1, 2, 3\}$ and transition matrix

$$P = (p_{x,y})_{x,y \in S} = \begin{bmatrix} 0.3 & 0.7 & 0 \\ 0.5 & 0 & 0.5 \\ 0.1 & 0.1 & 0.8 \end{bmatrix}.$$

Let the initial state be $X_0 = 3$. What is the probability of the event $\{X_1 = 1 \text{ and } X_2 = 1\}$?

Answer: 0.05 ✗

Starting from the state $X_0 = 3$, the first step leads to the state $X_1 = 1$ with probability $p_{3,1} = 0.1$, and the second step after that to the state $X_2 = 1$ with probability $p_{1,1} = 0.3$. Hence, the requested probability is given by $\mathbb{P}\big[X_1 = 1 \text{ and } X_2 = 1 \,\big|\, X_0 = 3\big] = p_{3,1} \times p_{1,1} = 0.1 \times 0.3 = 0.03$.

The correct answer is: 0.03

Question **3**

Flag question Mark 0.00 out of 1.00 Incorrect

The robot football team Otaniemi Eulers travels to Leppävaara for a match against their archenemy, Algebra. The players from Otaniemi select one of their three preprogrammed strategies, and the probability of an away win depends on that choice. Let us denote the events concerning this thriller as follows:

- S_1: Eulers select strategy 1, "variaatioprinsiippi"
- S_2: Eulers select strategy 2, "ketjumurto"
- S_3: Eulers select strategy 3, "Königsbergiläinen kierto"
- V: Eulers celebrate an away win.

Which of the following expressions is the correct formula for the probability $\mathbb{P}[V]$ of event V?

Select one:

- a. $\mathbb{P}[V \mid S_1] + \mathbb{P}[V \mid S_2] + \mathbb{P}[V \mid S_3]$ ✗
- b. $\mathbb{P}[V \mid S_1] \times \mathbb{P}[S_1] + \mathbb{P}[V \mid S_2] \times \mathbb{P}[S_2] + \mathbb{P}[V \mid S_3] \times \mathbb{P}[S_3]$
- c. $\mathbb{P}[S_1 \mid V] + \mathbb{P}[S_2 \mid V] + \mathbb{P}[S_3 \mid V]$
- d. $\mathbb{P}[S_1 \mid V] \times \mathbb{P}[S_1] + \mathbb{P}[S_2 \mid V] \times \mathbb{P}[S_2] + \mathbb{P}[S_3 \mid V] \times \mathbb{P}[S_3]$

Your answer is incorrect.

This, along with other similar formulas, is known as law of total probability, or partition theorem. They are repeatedly used in this course as well. When necessary, you can revise them from Todennäköisyyslaskennan ja tilastotieteen peruskurssi.

The correct answer is: $\mathbb{P}[V \mid S_1] \times \mathbb{P}[S_1] + \mathbb{P}[V \mid S_2] \times \mathbb{P}[S_2] + \mathbb{P}[V \mid S_3] \times \mathbb{P}[S_3]$

Quiz 2

Question **1**

Flag question | Mark 0.00 out of 1.00 | Incorrect

Let $X = (X_t)_{t \in \mathbb{Z}_+}$ be a Markov chain with state space $S = \{1, 2, 3\}$ and transition matrix

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \end{bmatrix}.$$

Let the initial state be $X_0 = 2$. What is the probability of the event $\{X_3 = 2\}$?

Answer: 0.01 ✗

Note that when $X_0 = 2$, then inevitably $X_1 \neq 2$ because the transition probability from state 2 to itself is zero. Furthermore, it is clear that $X_2 = 2$ because the probability of moving from states 1 and 3 to state 2 is one (i.e. the transition is "certain"). Then again, the next transition will certainly lead away from state 2, so the event $X_3 = 2$ is impossible.

The correct answer is: 0

Question **2**

Flag question | Mark 0.50 out of 1.00 | Partially correct

Let $(X_0, X_1, X_2, \ldots)$ be a Markov chain with a finite state space S and transition matrix P. Denote the distribution of the chain at time t by $\mu_t(x) = \mathbb{P}[X_t = x]$.

Assume that there exists a state $z \in S$ such that $P(z, z) = 1$. Such a state is called *absorbing*. Assume also that there exists a constant $0 < \varepsilon < 1$ such that $P(x, z) \geq \varepsilon$ for every state $x \in S$.

Which of the following statements are true under the aforementioned assumptions?

Select one or more:

- ☑ a. The distributions μ_t have no limit as $t \to \infty$. ✗ On the contrary, the distributions do have a limit (for more details, see below).
- ☑ b. $\mu_t(x) \leq (1-\varepsilon)^t$ for all $x \neq z$ and $t \geq 0$. ✓ This inequality does in fact hold (for more details, see below).
- ☑ c. For all $x \in S$ it holds that $\mu_t(x) \to 0$ as $t \to \infty$. ✗ This statement does not hold for $x = z$, because $\mu_t(z) \to 1$ as $t \to \infty$ (for more details, see below).
- ☑ d. $\mu_t(z) \to 1$ as $t \to \infty$. ✓ In the long run, the distribution of the Markov chain does in fact concentrate on the absorbing state z, and with exponential speed (for more details, see below).

Your answer is partially correct.

You have selected too many options.

Based on the assumption $P(x,z) \geq \varepsilon$ for all $x \in S$, the probability of moving from any state to the absorbing state z is at least ε. It follows that $\mathbb{P}[X_t \neq z] \leq (1-\varepsilon)^t$ for all $t \geq 0$. To prove this more formally, we can use the following inductive reasoning. For $t = 0$ it suffices to notice the trivial upper bound for any probability $\mathbb{P}[X_0 \neq z] \leq 1$. The induction w.r.t. t is carried out with a calculation based on the law of total probability

$$
\begin{aligned}
&\mathbb{P}[X_{t+1} \neq z] \\
&= \textstyle\sum_{x\in S} \mathbb{P}[X_t = x] \times \mathbb{P}[X_{t+1} \neq z \mid X_t = x] \\
&= \textstyle\sum_{x\in S} \mathbb{P}[X_t = x] \times \Big(1 - \mathbb{P}[X_{t+1} = z \mid X_t = x]\Big) \\
&= \textstyle\sum_{x\in S\setminus\{z\}} \mathbb{P}[X_t = x] \times (1 - P(x,z)) + \mathbb{P}[X_t = z] \times (1 - P(z,z)) \\
&\leq \textstyle\sum_{x\in S\setminus\{z\}} \mathbb{P}[X_t = x] \times (1-\varepsilon) + \mathbb{P}[X_t = z] \times 0 \\
&= (1-\varepsilon) \times \textstyle\sum_{x\in S\setminus\{z\}} \mathbb{P}[X_t = x] \\
&= (1-\varepsilon) \times \mathbb{P}[X_t \neq z].
\end{aligned}
$$

From the result $\mathbb{P}[X_t \neq z] \leq (1-\varepsilon)^t$ above we get, regarding the statements in the question:

- For all $x \neq z$ and $t \geq 0$ it holds that $\mu_t(x) = \mathbb{P}[X_t = x] \leq \mathbb{P}[X_t \neq z] \leq (1-\varepsilon)^t$.
- For all $t \geq 0$ it holds that $\mu_t(z) = \mathbb{P}[X_t = z] = 1 - \mathbb{P}[X_t \neq z] \geq 1 - (1-\varepsilon)^t$.
- Due to the first two points the distributions μ_t have a limit given by $\lim_{t\to\infty} \mu_t(x) = \begin{cases} 0 & \text{if } x \neq z \\ 1 & \text{if } x = z. \end{cases}$

The correct answers are: $\mu_t(x) \leq (1-\varepsilon)^t$ for all $x \neq z$ and $t \geq 0$., $\mu_t(z) \to 1$ as $t \to \infty$.

Question **3**

Flag question Mark 0.00 out of 1.00 Incorrect

Consider a random walk of a lone chess piece on a standard 8-by-8 chess board $S = \{\mathtt{a1}, \ldots, \mathtt{h8}\}$ where the piece is either a king (kuningas), a bishop (lähetti), or a knight (ratsu). The chess piece is initially placed on an empty chess board. At each time step the piece is then moved according to one of its allowed moves, selected uniformly at random and independently of the past states. Denote by X_t the position of the piece after t moves, and by P the 64-by-64 transition matrix of the corresponding Markov chain $X_0, X_1, \ldots$ with state space S. For the movement rules for each piece, see e.g. Wikipedia .

Which of the following statements are true?

Select one or more:

- [x] a. If the piece is a king, then the Markov chain is irreducible. ✓
- [x] b. If the piece is a king, then the Markov chain is aperiodic. ✓
- [x] c. If the piece is a bishop, then the Markov chain is irreducible. ✗
- [x] d. If the piece is a bishop, then the Markov chain is aperiodic. ✓
- [x] e. If the piece is a knight, then the Markov chain is irreducible. ✓
- [x] f. If the piece is a knight, then the Markov chain is aperiodic. ✗

Your answer is incorrect.

The king can reach any square from any square with a finite number of allowed moves (in fact, 7 moves is always sufficient). Hence, the corresponding Markov chain *is irreducible*. The king can return to its initial square with either 2 or 3 moves, and because the greatest common divisor (g.c.d.) of 2 and 3 is 1, the Markov chain *is aperiodic*.

Starting from a black square, the bishop can never reach a white square, and vice versa, so the Markov chain *is not irreducible*. The bishop can return to its initial square with either 2 or 3 moves, and because the g.c.d. of 2 and 3 is 1, the Markov chain *is aperiodic*.

The knight can reach any square from any square with a finite number of allowed moves (this follows from the existence of knight's tours, but can be proved with a smaller effort as well), so the Markov chain *is irreducible*. All the allowed moves of the knight always lead from a white square to a black square, and vice versa. Therefore, the knight can not return to its initial square with an odd number of moves. However, returning to the initial square with any even number of moves is possible, so the period is 2 for every state. Hence, the Markov chain *is not aperiodic*.

The correct answers are: If the piece is a king, then the Markov chain is irreducible., If the piece is a king, then the Markov chain is aperiodic., If the piece is a bishop, then the Markov chain is aperiodic., If the piece is a knight, then the Markov chain is irreducible.

Quiz 3

Question **1**

Flag question Mark 1.00 out of 1.00 Correct

Consider the following difference equation for an unknown function f defined on the set of integers $\mathbb{Z}$:

$(\star) \qquad f(x) = a\,f(x-1) + b\,f(x+1), \qquad x \in \mathbb{Z},$

where a and b are nonzero constants. Which of the following statements hold in general?

Select one or more:

- [x] a. If f is a solution to $(\star)$ and $C \in \mathbb{R}$, then the function $\bar{f}(x) = C\,f(x)$ is a solution to $(\star)$.
- [x] b. If f_1, f_2 are solutions to $(\star)$, then the function $\bar{f}(x) = f_1(x) + f_2(x)$ is a solution to $(\star)$.
- [x] c. If f_1, f_2 are solutions to $(\star)$ such that $f_1(0) = f_2(0)$ and $f_1(1) = f_2(1)$, then inevitably $f_1(x) = f_2(x)$ for all $x \in \mathbb{Z}$.
- [x] d. For all real numbers r_0 and r_1 there exists a solution f to $(\star)$ which satisfies $f(0) = r_0$ and $f(1) = r_1$.

Your answer is correct.

The system of equations $(\star)$ is linear, meaning that if f_1, f_2 are solutions and $c_1, c_2 \in \mathbb{R}$ are scalars, then the linear combination $\bar{f}(x) = c_1 f_1(x) + c_2 f_2(x)$ is a solution as well:

$$\begin{aligned}
\bar{f}(x) &= c_1 f_1(x) + c_2 f_2(x) \\
&= c_1 \Big(a f_1(x-1) + b f_1(x+1)\Big) + c_2 \Big(a f_2(x-1) + b f_2(x+1)\Big) \\
&= a c_1 f_1(x-1) + b c_1 f_1(x+1) + a c_2 f_2(x-1) + b c_2 f_2(x+1) \\
&= a \Big(c_1 f_1(x-1) + c_2 f_2(x-1)\Big) + b \Big(c_1 f_1(x+1) + c_2 f_2(x+1)\Big) \\
&= a \bar{f}(x-1) + b \bar{f}(x+1).
\end{aligned}$$

Therefore, the first two statements hold.

The equation is a so called second-order linear difference equation. Solving for the value of $f(x+1)$ in formula $(\star)$ yields the recursive equation

$$f(x+1) = \frac{f(x) - a f(x-1)}{b}.$$

If we now assume that $f_1, f_2 : \mathbb{Z} \to \mathbb{R}$ are solutions to $(\star)$, and it holds that $f_1(0) = f_2(0)$ and $f_1(1) = f_2(1)$, then using induction w.r.t. x we can see in the recursion above that $f_1(x) = f_2(x)$ for all $x \geq 2$ as well. Similarly, solving

$$f(x-1) = \frac{f(x) - b f(x+1)}{a}$$

we get a backwards recursion, and using induction we see that $f_1(x) = f_2(x)$ for all $x \leq -1$ as well. Thus, the solutions f_1, f_2 are inevitably the same in this case, and the third statement holds.

If $r_0, r_1 \in \mathbb{R}$ are given we can construct a solution f for which $f(0) = r_0$ and $f(1) = r_1$ using induction and the recursive formulas above. In particular, such a solution always exists, and so the fourth statement holds as well.

The correct answers are: If f is a solution to $(\star)$ and $C \in \mathbb{R}$, then the function $\bar{f}(x) = C f(x)$ is a solution to $(\star)$.

, If f_1, f_2 are solutions to $(\star)$, then the function $\bar{f}(x) = f_1(x) + f_2(x)$ is a solution to $(\star)$.

, If f_1, f_2 are solutions to $(\star)$ such that $f_1(0) = f_2(0)$ and $f_1(1) = f_2(1)$, then inevitably $f_1(x) = f_2(x)$ for all $x \in \mathbb{Z}$.

, For all real numbers r_0 and r_1 there exists a solution f to $(\star)$ which satisfies $f(0) = r_0$ and $f(1) = r_1$.

Question **2**

Flag question Mark 0.00 out of 1.00 Incorrect

Consider a random walk of a lone chess piece on a standard 8-by-8 chess board $S = \{\mathtt{a1}, \ldots, \mathtt{h8}\}$ where the piece is either a king (kuningas), a bishop (lähetti), or a knight (ratsu). The chess piece is initially placed on an empty chess board. At each time step the piece is then moved according to one of its allowed moves, selected uniformly at random and independently of the past states. Denote by X_t the position of the piece after t moves, and by P the 64-by-64 transition matrix of the corresponding Markov chain $(X_0, X_1, \ldots)$ with state space S.

Which of the following statements are true?

Select one or more:

- ☑ a. If the piece is a king, then P has a unique invariant distribution. ✓
- ☑ b. If the piece is a king and the initial state is the black corner $\mathtt{a1}$, then the distribution μ_t of the chain at time t converges to a limiting distribution as $t \to \infty$. ✓
- ☑ c. If the piece is a bishop, then P has a unique invariant distribution. ✗
- ☑ d. If the piece is a bishop and the initial state is the black corner $\mathtt{a1}$, then the distribution μ_t of the chain at time t converges to a limiting distribution as $t \to \infty$. ✓
- ☑ e. If the piece is a knight, then P has a unique invariant distribution. ✓
- ☑ f. If the piece is a knight and the initial state is the black corner $\mathtt{a1}$, then the distribution μ_t of the chain at time t converges to a limiting distribution as $t \to \infty$. ✗

Your answer is incorrect.

Let us recall two important results from the lecture notes:

Theorem 1: An irreducible Markov chain with a finite state space has a unique invariant distribution π.

Theorem 2: The distribution μ_t of an irreducible and aperiodic Markov chain with a finite state space converges to the unique invariant distribution π, regardless of the initial state.

According to the previous quiz, the king's Markov chain is irreducible and aperiodic. Theorem 1 states that the Markov chain *has a unique invariant distribution*. Theorem 2 states that *the distribution of the chain converges.*

According to the previous quiz, the bishop's chain is aperiodic but not irreducible. Rather, the state space is partitioned into two components: the black squares S_B and the white squares S_W. If we restrict the bishop's initial state to be a black square, then the resulting *black bishop's chain* with state space S_B is irreducible and aperiodic, and by Theorem 1 has a unique invariant distribution π_B on the black squares. Similarly, the *white bishop's chain* has a unique invariant distribution π_W supported on the white squares. The probability distributions π_B and π_W can also be viewed as probability distributions on the full chess board S, and both distributions are invariant with respect to the transition matrix P of the bishop's chain with state space S. Hence P has several invariant distributions. When the chain is started at the black corner $\mathbf{a1}$, by Theorem 2 the distribution of the chain at time t converges to π_B as $t \to \infty$.

According to the previous quiz, the knight's chain is irreducible, but has a period of 2. Due to the irreducibility, Theorem 1 states that the knight's chain *has a unique invariant distribution*. If the knight's chain is started at the black corner $\mathbf{a1}$, then the distribution is fully concentrated on black squares at even time steps and on white squares at odd time steps. Because of this, *the distribution μ_t does not converge.*

The correct answers are: If the piece is a king, then P has a unique invariant distribution., If the piece is a king and the initial state is the black corner $\mathbf{a1}$, then the distribution μ_t of the chain at time t converges to a limiting distribution as $t \to \infty$., If the piece is a bishop and the initial state is the black corner $\mathbf{a1}$, then the distribution μ_t of the chain at time t converges to a limiting distribution as $t \to \infty$., If the piece is a knight, then P has a unique invariant distribution.

Question **3**

Flag question Mark 0.00 out of 1.00 Incorrect

Consider the following difference equation for an unknown function f defined on the set of integers $\mathbb{Z}$:

$(\star) \qquad f(k) = \sum_{j=1}^{n} c_j \, f(k-j) \qquad \forall k \in \mathbb{Z}.$

Let us try to solve this using a guess of the form $f(k) = \alpha^k$. What can be said about the values of parameter $\alpha \neq 0$ that solve $(\star)$ with such a guess?

Select one or more:

- [x] a. A function $f(k) = \alpha^k$ is a solution to $(\star)$ if α is a root of the polynomial $x^n - c_1 x^{n-1} - c_2 x^{n-2} - \cdots - c_{n-1} x - c_n$. ✓
- [x] b. A function $f(k) = \alpha^k$ is not a solution to $(\star)$ unless α is a root of the polynomial $x^n - c_1 x^{n-1} - c_2 x^{n-2} - \cdots - c_{n-1} x - c_n$. ✓
- [x] c. A function $f(k) = \alpha^k$ is not a solution to $(\star)$ for any value of α. ✗
- [x] d. A function $f(k) = \alpha^k$ is a solution to $(\star)$ for all values of α. ✗

Your answer is incorrect.

Let us write out the right hand side of $(\star)$ for a function $f(k) = \alpha^k$ where $\alpha \neq 0$:

$$\sum_{j=1}^{n} c_j \, f(k-j) = \sum_{j=1}^{n} c_j \, \alpha^{k-j} = \alpha^{k-n} \times \sum_{j=1}^{n} c_j \, \alpha^{n-j}.$$

Comparing this expression with the value of the function $f(k) = \alpha^k$ we can see that equation $(\star)$ is satisfied if and only if

$$\alpha^k = \alpha^{k-n} \times \sum_{j=1}^{n} c_j \, \alpha^{n-j}.$$

By multiplying both sides by the nonzero number α^{n-k} and moving all the terms on the left hand side, we can see that this equation is still equivalent with the polynomial equation

$$\alpha^n - \sum_{j=1}^{n} c_j \, \alpha^{n-j} = 0.$$

Hence, a function $f(k) = \alpha^k$ is a solution to $(\star)$ if and only if α is a root of the polynomial $x^n - \sum_{j=1}^{n} c_j \, x^{n-j}$.

The correct answers are: A function $f(k) = \alpha^k$ is a solution to $(\star)$ if α is a root of the polynomial $x^n - c_1 x^{n-1} - c_2 x^{n-2} - \cdots - c_{n-1} x - c_n$.

, A function $f(k) = \alpha^k$ is not a solution to $(\star)$ unless α is a root of the polynomial $x^n - c_1 x^{n-1} - c_2 x^{n-2} - \cdots - c_{n-1} x - c_n$.

Quiz 4:

A tennis game between Serena and Venus has reached a deuce (even points), so that one of the players must achieve a two point advantage to win the game. When each of the following points is won independently and with equal probability by either of the two players, the rest of the game is described by a Markov chain with state space $S = \{-2, -1, 0, 1, 2\}$ where the states are interpreted as

- $+2$: Serena wins the game,
- $+1$: Serena has the advantage,
- 0: deuce (even points),
- -1: Venus has the advantage,
- -2: Venus wins the game.

States +2 and -2 are absorbing with $P(+2, +2) = 1$ and $P(-2, -2) = 1$. The transition probabilities from states $x \in \{-1, 0, 1\}$ are given by

$$P(x, y) = \begin{cases} \frac{1}{2} & \text{if } y = x \pm 1 \\ 0 & \text{else.} \end{cases}$$

Assuming that Serena has the advantage at time t, what is the probability of Serena winning the game?

Give your answer with two decimal places. (*Hint: Instead of calculating the hitting probabilities directly, you could also use the formula for gambler's ruin.*)

Answer: 0.5 ✗

The probability in question is $h(+1)$ where $h(x) = \mathbb{P}_x\left[T_{\{+2\}} < T_{\{-2\}}\right]$ denotes the probability that the hitting time $T_{\{+2\}}$ into state +2 is smaller than the hitting time $T_{\{-2\}}$ for a Markov chain starting at state x.

Considering the probabilities $P(x, y)$, we get the hitting probabilities by solving the system of linear equations

$$\begin{cases} h(+2) = 1, \\ h(x) = \frac{1}{2}h(x-1) + \frac{1}{2}h(x+1) & \text{when } -1 \le x \le +1, \\ h(-2) = 0. \end{cases}$$

The general solution to this system of equations (as in the symmetric version of gambler's ruin) is of the form $h(x) = Ax + B$, where A and B are constants. A solution which satisfies the boundary conditions at $x = +2$ and $x = -2$ is achieved by selecting $A = \frac{1}{4}$ and $B = \frac{1}{2}$, leading to

$$h(x) = \tfrac{1}{4}x + \tfrac{1}{2}.$$

Thus the probability of Serena winning is $h(+1) = \frac{3}{4} = 0.75$.

The correct answer is: 0.75

Question **2**

Flag question Mark 0.00 out of 1.00 Incorrect

Let $(p_k)_{k \in \mathbb{Z}_+}$ be a probability distribution on $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, so that

$$\begin{cases} p_k \ge 0 & \text{for all } k \in \mathbb{Z}_+ \\ \sum_{k=0}^{\infty} p_k = 1. \end{cases}$$

Denote the generating function of $(p_k)_{k \in \mathbb{Z}_+}$ by

$$G(z) = \sum_{k=0}^{\infty} p_k \, z^k,$$

and denote the n^{th} derivative of G at z by $G^{(n)}(z) = \frac{d^n}{dz^n} G(z)$. Which of the following statements is true in general?

Select one:

- a. $G^{(n)}(1) = \frac{1}{n!} p_n$ ✗
- b. $G^{(n)}(0) = p_n$
- c. $G^{(n)}(1) = p_n$
- d. $G^{(n)}(0) = \frac{1}{n!} p_n$
- e. $G^{(n)}(0) = n! \, p_n$
- f. $G^{(n)}(1) = n! \, p_n$

Your answer is incorrect.

Recall that a power series can be differentiated termwise within its radius of convergence.

After one termwise differentiation, we get

$$G'(z) = \frac{\mathrm{d}}{\mathrm{d}z} \sum_{k=0}^{\infty} p_k\, z^k = \sum_{k=0}^{\infty} p_k\, \frac{\mathrm{d}}{\mathrm{d}z} z^k = \sum_{k=1}^{\infty} k\, p_k\, z^{k-1},$$

where we left out the term corresponding to the index $k = 0$, as it is automatically zero.

After multiple differentiations, we get (inductively) that

$$G^{(n)}(z) = \frac{\mathrm{d}^n}{\mathrm{d}z^n} \sum_{k=0}^{\infty} p_k\, z^k = \sum_{k=n}^{\infty} (k-n+1)\cdots(k-1)k\, p_k\, z^{k-n} = \sum_{k=n}^{\infty} \frac{k!}{(k-n)!}\, p_k\, z^{k-n}.$$

By substituting $z = 0$ in the equation above, we see that the only nonzero term is the one corresponding to the index $k = n$. Hence, we get the expression

$$G^{(n)}(0) = n!\, p_n.$$

The correct answer is: $G^{(n)}(0) = n!\, p_n$

Question 3

Flag question Mark 0.00 out of 1.00 Incorrect

A random walk on the set of integers $\mathbb{Z}$ with initial state $X_0 = 0$ is defined by

$$X_t = \sum_{s=1}^{t} \xi_s, \quad t \geq 0,$$

where $\xi_1, \xi_2, \ldots$ are independent and uniformly distributed random variables in the set $\{-1, +1\}$.

Which of the following statements concerning the time-dependent state distributions are true?

Select one:

- a. The state of the random walk at an arbitrary time instant $t \geq 0$ can be expressed as $X_t = 2\, Z_t$, where Z_t follows a normal distribution with suitable parameters that may or may not depend on t.
- b. The state of the random walk at an arbitrary time instant $t \geq 0$ can be expressed as $X_t = -t + 2\, U_t$, where U_t follows a Poisson distribution with a suitable parameter that may or may not depend on t. ✗
- c. The state of the random walk at an arbitrary time instant $t \geq 0$ can be expressed as $X_t = t - 2\, D_t$, where D_t follows a binomial distribution with suitable parameters that may or may not depend on t.
- d. The state of the random walk at an arbitrary time instant $t \geq 0$ follows a hypergeometric distribution with suitable parameters that may or may not depend on t.

Your answer is incorrect.

Denote by

$$D_t = \sum_{s=1}^{t} 1(\xi_s = -1)$$

the number of backwards steps within the first t steps. Because the steps are independent and the probability of a backwards step is $\mathbb{P}[\xi_s = -1] = \frac{1}{2}$, it follows that D_t follows a binomial distribution with parameters t and $\frac{1}{2}$. The rest of the first t steps are forwards steps and there are $U_t = t - D_t$ of them. We get the state of the walk at time t by counting the number of steps taken forwards and subtracting the number of steps taken backwards,

$$X_t = U_t - D_t = (t - D_t) - D_t = t - 2\, D_t.$$

The correct answer is: The state of the random walk at an arbitrary time instant $t \geq 0$ can be expressed as $X_t = t - 2\, D_t$, where D_t follows a binomial distribution with suitable parameters that may or may not depend on t.

Quiz 6:

Question **1**

Flag question Mark 0.00 out of 1.00 Incorrect

Let $\gamma > 0$ be a parameter. Consider the limit

$$\lim_{n\to\infty}\left(n \times \left(1 - e^{-\gamma/n}\right)\right).$$

Which of the following is the correct expression for this limit?

Select one:

- a. $\lim_{n\to\infty}\left(n \times \left(1 - e^{-\gamma/n}\right)\right) = \log(\gamma)$ ✗
- b. $\lim_{n\to\infty}\left(n \times \left(1 - e^{-\gamma/n}\right)\right) = e^{-\gamma}$
- c. $\lim_{n\to\infty}\left(n \times \left(1 - e^{-\gamma/n}\right)\right) = \gamma$
- d. $\lim_{n\to\infty}\left(n \times \left(1 - e^{-\gamma/n}\right)\right) = \frac{1}{\gamma}$

Your answer is incorrect.

To calculate the limit, it is helpful to revise L'Hôspital's rule from the prerequisite courses.

Even without it the limit can be calculated by simply identifying the derivative expression in the equations below

$$\begin{aligned}&\lim_{n\to\infty}\left(n \times \left(1 - e^{-\gamma/n}\right)\right)\\ &= \lim_{n\to\infty}\left(\frac{1-e^{-\gamma/n}}{1/n}\right)\\ &= \lim_{x\to 0}\left(\frac{1-e^{-\gamma x}}{x}\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(1-e^{-\gamma x}\right)\Big|_{x=0} = -(-\gamma e^{0}) = \gamma.\end{aligned}$$

The correct answer is: $\lim_{n\to\infty}\left(n \times \left(1 - e^{-\gamma/n}\right)\right) = \gamma$

Question **2**

Flag question Mark 1.00 out of 1.00 Correct

Which of the following functions satisfies the equation

$f(s+t) = f(s)f(t)$ for all $s, t > 0$?

Select one:

- a. $f(t) = e^{\alpha t}$ where $\alpha \in \mathbb{R}$ is an arbitrary parameter ✓
- b. $f(t) = Ct^{\alpha}$ where $C, \alpha \in \mathbb{R}$ are arbitrary parameters
- c. $f(t) = C\log(\alpha t)$ where $C \in \mathbb{R}$ and $\alpha > 0$ are arbitrary parameters
- d. $f(t) = Ce^{t}$ where $C \in \mathbb{R}$ is an arbitrary parameter

Your answer is correct.

The correct answer is: $f(t) = e^{\alpha t}$ where $\alpha \in \mathbb{R}$ is an arbitrary parameter

Question **3**

Flag question Mark 0.00 out of 1.00 Incorrect

An insurance company receives N claims (korvausvaatimus) in a day, where N follows a Poisson distribution with parameter $\lambda = 28.0$, that is, $\mathbb{P}[N = k] = e^{-\lambda}\frac{\lambda^k}{k!}$ for $k = 0, 1, 2, \ldots$

What is the standard deviation of the number of claims N?

(N.B.! MyCourses requires the answers formatted as decimals. Unfortunately answers given as fractions have given no points so far!)

Use dot in your answer.

Answer: 0.2 ✗

For a $\mathrm{Poisson}(\lambda)$ distributed random variable, the variance is given by $\mathrm{Var}(N) = \lambda$, and the standard deviation is the square root of the variance $\sqrt{\lambda}$.

Let us revise how to calculate the expected value and variance of a Poisson distribution. The expected value is given by:

$$\begin{aligned}
\mathbb{E}[N] &= \sum_{k=0}^{\infty} k \times \mathbb{P}[N = k] \\
&= \sum_{k=0}^{\infty} k \times \frac{\lambda^k}{k!}e^{-\lambda} \\
&= \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!}e^{-\lambda} \\
&= e^{-\lambda}\sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} \\
&= e^{-\lambda}\sum_{\ell=0}^{\infty} \frac{\lambda^{\ell+1}}{\ell!} \\
&= e^{-\lambda}\lambda e^{\lambda} = \lambda.
\end{aligned}$$

The easiest way to calculate the expected value of the square is by using the following intermediate result:

$$\begin{aligned}
\mathbb{E}[N^2 - N] &= \sum_{k=0}^{\infty}(k^2 - k) \times \mathbb{P}[N = k] \\
&= \sum_{k=2}^{\infty} k(k-1) \times \frac{\lambda^k}{k!}e^{-\lambda} \\
&= e^{-\lambda}\sum_{k=2}^{\infty} \frac{\lambda^k}{(k-2)!} \\
&= e^{-\lambda}\sum_{\ell=0}^{\infty} \frac{\lambda^{\ell+2}}{\ell!}e^{-\lambda} \\
&= e^{-\lambda}\lambda^2 e^{\lambda} = \lambda^2.
\end{aligned}$$

With this, we get $\mathbb{E}[N^2] = \mathbb{E}[N^2 - N] + \mathbb{E}[N] = \lambda^2 + \lambda$. Thus, the variance is given by the familiar formula

$$\mathrm{Var}(N) = \mathbb{E}[N^2] - \mathbb{E}[N]^2 = (\lambda^2 + \lambda) - \lambda^2 = \lambda.$$

When $\lambda = 28.0$, the variance is $\lambda = 28.0$ and the standard deviation is the square root $\sqrt{28.0} \approx 5.29$.

The correct answer is: 5.291502622

Quiz 7:

Question **1**

Flag question | Mark 0.00 out of 1.00 | Incorrect

Let Q be an n-by-n square matrix, and define a constant (a so-called matrix norm of Q) by

$$c := \max\left\{\|Qv\| \;\middle|\; v \in \mathbb{R}^n,\ \|v\| = 1\right\}.$$

It is straightforward to verify that $c < \infty$ and that each entry $(Q^k)_{i,j}$ of the k-th matrix power Q^k has an absolute value of at most c^k, that is,

$$\left|(Q^k)_{i,j}\right| \leq c^k \text{ for all } k \geq 0.$$

What can be deduced about the entrywise convergence of the matrix series $\sum_{k=0}^{\infty} \frac{1}{k!} Q^k$?

Select one:

- a. Nothing can be deduced about the convergence or divergence of the series $\sum_{k=0}^{\infty} \frac{1}{k!} Q^k$ without further information about the matrix Q. ✗
- b. The series $\sum_{k=0}^{\infty} \frac{1}{k!} Q^k$ converges.
- c. The series $\sum_{k=0}^{\infty} \frac{1}{k!} Q^k$ diverges.

Your answer is incorrect.

As with matrix addition in general, the series is interpreted entry by entry, so that the (i, j)-entry of the series is $\sum_{k=0}^{\infty} \frac{1}{k!} (Q^k)_{i,j}$. Based on the observation in the instructions and the series expansion of the exponential function, this real-valued series converges absolutely:

$$\sum_{k=0}^{\infty} \left|\frac{1}{k!}(Q^k)_{i,j}\right| = \sum_{k=0}^{\infty} \frac{1}{k!}\left|(Q^k)_{i,j}\right| \leq \sum_{k=0}^{\infty} \frac{1}{k!} c^k = e^c < \infty.$$

Thus, the matrix series converges entrywise.

The correct answer is: The series $\sum_{k=0}^{\infty} \frac{1}{k!} Q^k$ converges.

Question **2**

Flag question Mark 0.00 out of 1.00 Incorrect

Let the random number T be exponentially distributed with a parameter $\lambda > 0$, i.e. T has a density function

$$f_T(s) = \begin{cases} \lambda e^{-\lambda s} & \text{when } s > 0 \\ 0 & \text{else.} \end{cases}$$

Which of the following statements concerning the expected value and variance of the random number T are correct?

Select one or more:

- a. $\mathrm{Var}(T) = \lambda^2$ ✗
- b. $\mathbb{E}[T] = \frac{1}{\lambda}$ ✓
- c. $\mathbb{E}[T] = \lambda$ ✗
- d. $\mathrm{Var}(T) = \frac{1}{\lambda^2}$ ✓
- e. $\mathrm{Var}(T) = \lambda$ ✗
- f. $\mathrm{Var}(T) = \frac{1}{\lambda}$ ✗

Your answer is incorrect.

The expected value is defined by the following integral, which can be solved using integration by parts (osittaisintegrointi):

$$\begin{aligned} \mathbb{E}[T] &= \int_0^\infty s\, f_T(s)\, \mathrm{d}s \\ &= \int_0^\infty s\lambda e^{-\lambda s}\, \mathrm{d}s \\ &= \int_0^\infty e^{-\lambda s}\, \mathrm{d}s - \Big|_0^\infty \left(se^{-\lambda s}\right) \\ &= \Big|_0^\infty \left(\tfrac{-1}{\lambda} e^{-\lambda s}\right) - (0-0) \\ &= \tfrac{1}{\lambda}. \end{aligned}$$

The expected value of the square is solved using integration by parts twice, or by integrating by parts once and then using the result of the previous calculation:

$$\begin{aligned} \mathbb{E}[T^2] &= \int_0^\infty s^2\, f_T(s)\, \mathrm{d}s \\ &= \int_0^\infty s^2\lambda e^{-\lambda s}\, \mathrm{d}s \\ &= \int_0^\infty 2se^{-\lambda s}\, \mathrm{d}s - \Big|_0^\infty \left(s^2 e^{-\lambda s}\right) \\ &= \tfrac{2}{\lambda} \int_0^\infty s\lambda e^{-\lambda s}\, \mathrm{d}s - (0-0) \\ &= \tfrac{2}{\lambda} \times \tfrac{1}{\lambda} = \tfrac{2}{\lambda^2}. \end{aligned}$$

Now the variance is given by the familiar formula

$$\mathrm{Var}(T) = \mathbb{E}[T^2] - \mathbb{E}[T]^2 = \tfrac{2}{\lambda^2} - \left(\tfrac{1}{\lambda}\right)^2 = \tfrac{1}{\lambda^2}.$$

The correct answers are: $\mathbb{E}[T] = \frac{1}{\lambda}$, $\mathrm{Var}(T) = \frac{1}{\lambda^2}$

Question **3**

Flag question   Mark 0.00 out of 1.00   Incorrect

Let X and Y be random numbers with expected values $\mathbb{E}[X]$ and $\mathbb{E}[Y]$, and variances $\mathrm{Var}(X)$ and $\mathrm{Var}(Y)$.

Which of the following statements hold in general with these assumptions?

Select one or more:

- a. $\mathrm{Var}(-3X) = -3\,\mathrm{Var}(X)$ ✗
- b. $\mathbb{E}[-3X] = -3\,\mathbb{E}[X]$ ✓
- c. $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$ ✗ For independent random variables, the variance of the sum equals the sum of variances but without assuming independence this doesn't hold in general: obvious counterexamples can be found by selecting, for instance, $Y = -X$.
- d. $\mathrm{Var}(-3X) = 9\,\mathrm{Var}(X)$ ✓
- e. $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$ ✓

Your answer is incorrect.

When necessary, you can revise the basic properties of expected value and variance from Todennäköisyyslaskennan ja tilastotieteen peruskurssi.

The correct answers are: $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$, $\mathrm{Var}(-3X) = 9\,\mathrm{Var}(X)$, $\mathbb{E}[-3X] = -3\,\mathbb{E}[X]$

Quiz 8:

Question **1**

Flag question   Mark 1.00 out of 1.00   Correct

Let τ be a random time instant that follows an exponential distribution with rate parameter α.

All of the following limits have been calculated correctly. Which of them most sensibly describes (in a nontrivial way) a normalized probability that the time instant is located in very near future?

Select one:

- a. $\lim_{h\to 0}\left(\frac{1}{h}\left(1 - \mathbb{P}[\tau > h]\right)\right) = \alpha$ ✓
- b. $\lim_{h\to\infty}\left(\frac{-1}{h}\log \mathbb{P}[\tau > h]\right) = \alpha$
- c. $\lim_{h\to 0}\left(\mathbb{P}[\tau > h]\right) = 1$

Your answer is correct.

The limit

$$\lim_{h\to\infty}\left(\frac{-1}{h}\log\mathbb{P}\left[\tau>h\right]\right)=\alpha$$

can be interpreted as "when the time h is very large, $h\approx\infty$, it holds approximately that $\frac{-1}{h}\log\mathbb{P}\left[\tau>h\right]\approx\alpha$". By multiplying both sides by $-h$ and exponentiating with them, this leads to the approximation $\mathbb{P}\left[\tau>h\right]\approx e^{-\alpha h}$. This limit says that after a very long time it is exponentially unlikely that the instant of time hasn't taken place yet. However, it says nothing about the probability of the instant of time taking place in a very short time.

The limit

$$\lim_{h\to 0}\left(\mathbb{P}\left[\tau>h\right]\right)=1$$

tells us that when the time h is very small, $h\approx 0$, it holds approximately that $\mathbb{P}\left[\tau>h\right]\approx 1$. In other words, it is almost certain that the instant of time τ hasn't taken place after a very short wait. This is true for any positive instant of time and is, thus, not a very sensible result regarding the probability that the instant of time takes place in a very short time.

The limit

$$\lim_{h\to 0}\left(\frac{1}{h}\left(1-\mathbb{P}\left[\tau>h\right]\right)\right)=\alpha$$

tells us that when h is very small, $\frac{1}{h}\left(1-\mathbb{P}\left[\tau>h\right]\right)\approx\alpha$. By multiplying both sides by h we can approximate the complement probability $\mathbb{P}\left[\tau\le h\right]=1-\mathbb{P}\left[\tau>h\right]$ by $\mathbb{P}\left[\tau\le h\right]\approx\alpha h$. That is, the probability that the random time instant is located in very near future, within time horizon h, is approximately αh. The above limit hence describes a normalized probability that the random time instant is located in very near future.

The correct answer is: $\lim_{h\to 0}\left(\frac{1}{h}\left(1-\mathbb{P}\left[\tau>h\right]\right)\right)=\alpha$

Question **2**

Flag question   Mark 0.00 out of 1.00   Incorrect

The matrix exponent of an n-by-n square matrix A is an an n-by-n square matrix $\exp(A)$ defined by the formula

$$\exp(A)=\sum_{k=0}^{\infty}\frac{1}{k!}A^k,$$

where A^k denotes the k-th matrix power of A. As usual, $A^0=I$ is defined as the n-by-n identity matrix.

Let Q be an n-by-n square matrix and π a 1-by-n row vector which satisfies

$$\pi Q=0.$$

Let $t>0$. Which of the following equations is correct?

Select one:

- a. $\exp(tQ)\,\pi=0$ ✗
- b. $\pi\,\exp(tQ)=0$
- c. $\pi\,\exp(tQ)=\pi$
- d. $\exp(tQ)\,\pi=\pi$

Your answer is incorrect.

Because $\pi Q = 0$, it also holds for all $k \geq 1$ that

$$\pi Q^k = \pi Q Q^{k-1} = 0 Q^{k-1} = 0.$$

However, because the zeroth matrix power equals the identity matrix, it holds that $\pi Q^0 = \pi I = \pi$.

By summing up the result above we get the equation

$$\pi \exp(tQ) = \pi \sum_{k=0}^{\infty} \frac{1}{k!}(tQ)^k = \sum_{k=0}^{\infty} \frac{t^k}{k!} \pi Q^k = \frac{t^0}{0!} \pi + \sum_{k=1}^{\infty} \frac{t^k}{k!} 0 = \pi.$$

Regarding continuous-time stochastic processes, this tells us that when the generator matrix Q satisfies the equation $\pi Q = 0$, any matrix $P(t)$ obtained through the exponentiation $P(t) = \exp(tQ)$ satisfies the familiar balance equation $\pi P(t) = \pi$.

The correct answer is: $\pi \exp(tQ) = \pi$

Question **3**

Flag question   Mark 0.00 out of 1.00   Incorrect

Let $c > 0$ be a fixed parameter and n a large positive integer (we will consider limits as $n \to \infty$).

Let $U_1, \ldots, U_n$ be independent and uniformly distributed random numbers on the interval $[0, cn]$. Let us fix the subset $(a, b] \subset [0, cn]$ and denote the number of points U_j on this interval with

$$M_n\big((a,b]\big) := \#\big\{j \in \{1, 2, \ldots, n\} \mid U_j \in (a,b]\big\}.$$

Since $\mathbb{P}\big[U_j \in (a,b]\big] = \frac{b-a}{cn}$ for all $j = 1, \ldots, n$ and the random numbers U_j are independent, $M_n\big((a,b]\big)$ follows a binomial distribution

$$\mathbb{P}\Big[M_n\big((a,b]\big) = k\Big] = \frac{n!}{k!\,(n-k)!}\left(\frac{b-a}{cn}\right)^k \left(1 - \frac{b-a}{cn}\right)^{n-k}.$$

Which of the following statements concerning the limit of these probabilities is correct?

Select one:

a. $\lim_{n\to\infty} \mathbb{P}\Big[M_n\big((a,b]\big) = k\Big] = \frac{1}{1-\gamma}\gamma^{-k}$ where $\gamma = \frac{b-a}{c}$ ✗

b. $\lim_{n\to\infty} \mathbb{P}\Big[M_n\big((a,b]\big) = k\Big] = 0$

c. $\lim_{n\to\infty} \mathbb{P}\Big[M_n\big((a,b]\big) = k\Big] = \frac{1}{k!}\gamma^k e^{-\gamma}$ where $\gamma = \frac{b-a}{c}$

Your answer is incorrect.

The calculation is very similar to the proof in the lecture notes where it was shown that independently scattered random point patterns have Poisson distributed point counts. However, the interpretation here is different!

- In this question, we consider a large number n of points, randomly scattered on a long interval $[0, cn]$ (the length of the entire interval $\propto n$), being located on an interval $(a, b]$ with fixed length.

The correct answer is: $\lim_{n\to\infty} \mathbb{P}\Big[M_n\big((a,b]\big) = k\Big] = \frac{1}{k!}\gamma^k e^{-\gamma}$ where $\gamma = \frac{b-a}{c}$

Quiz 11:

Question **2**

Flag question   Mark 0.00 out of 1.00   Incorrect

Fred the Farmer has planted an annual (yksivuotinen) plant on his farm in the spring. Each plant lives until the end of the year and produces seeds independently, so that during next spring these seeds germinate into one new plant with probability 30%, two new plants with probability 40%, and three new plants with probability 10%. Otherwise no new plants are germinated.

In the summer of year 0 when the initial plant was planted, the number of plants is $L_0 = 1$. Let us denote by L_t the number of plants in the summer of year t.

Consider the process $(X_t)_{t\in\mathbb{Z}_+}$ defined by the formula $X_t = 1.4^{-t} \times L_t$. Which of the following statements are true?

Select one or more:

- ☑ a. $(X_t)_{t\in\mathbb{Z}_+}$ is a martingale. ✓
- ☑ b. $\mathbb{E}[X_t] = 1$ for all $t \in \mathbb{Z}_+$. ✓
- ☑ c. $(X_t)_{t\in\mathbb{Z}_+}$ is a time-homogeneous Markov chain. ✗

Your answer is incorrect.

(i) The process $(X_t)_{t\in\mathbb{Z}_+}$ is a martingale with respect to the information process formed by the sizes of the plant generations $L_0, L_1, \ldots$ and, thus also with respect to its own information process. Let us verify the properties that characterize a martingale.

1. For any $t \in \mathbb{Z}_+$ it clearly holds that $|X_t| \le 3^t$ and so the integrability property $\mathbb{E}\big[|X_t|\big] \le 3^t < \infty$ holds.
2. The value X_t is determined by the current plant generation size L_t and therefore $X_t \in \sigma(L_0, L_1, \ldots, L_t)$.
3. Furthermore, the conditional expectation of X_{t+1} given the plant generation sizes up to time t equals

$$\begin{aligned}
&\mathbb{E}\Big[X_{t+1} \,\Big|\, L_0, L_1, \ldots, L_t\Big] \\
&= 1.4^{-t-1} \times \mathbb{E}\Big[L_{t+1} \,\Big|\, L_0, L_1, \ldots, L_t\Big] \\
&= 1.4^{-t-1} \times \big(1.4 \times L_t\big) \\
&= 1.4^{-t} \times L_t \\
&= X_t.
\end{aligned}$$

(ii) Either from a general result for branching processes, or through a direct calculation similar to the one above we see that $\mathbb{E}(L_t) = (\mathbb{E}(L_1))^t = 1.4^t$. It follows that

$$\mathbb{E}\big[X_t\big] = \mathbb{E}\big[1.4^{-t} \times L_t\big] = 1.4^{-t} \times 1.4^t = 1.$$

(iii) The process $(X_t)_{t\in\mathbb{Z}_+}$ is a discrete-time Markov chain but this Markov chain is not time-homogeneous because the transition probabilities are time-dependent. For example, for $x = 5^3/7^2$ and $y = 3 \cdot 5^4/7^3$, we find that

$$\mathbb{P}(X_3 = y \,|\, X_2 = x) \;=\; \mathbb{P}(L_3 = 15 \,|\, L_2 = 5)$$

and

$$\mathbb{P}(X_4 = y \,|\, X_3 = x) \;=\; \mathbb{P}(L_4 = 21 \,|\, L_3 = 7).$$

Given that $L_2 = 5$, the event $L_3 = 15$ occurs if and only if each of the 5 plants in the summer of year 2 produce 3 new plants next year, and this occurs with probability $\mathbb{P}(L_3 = 15 \,|\, L_2 = 5) \;=\; 0.10^5$. Analogously, $\mathbb{P}(L_4 = 21 \,|\, L_3 = 7) \;=\; 0.10^7$. Therefore

$$\mathbb{P}(X_4 = y \,|\, X_3 = x) \;\neq\; \mathbb{P}(X_3 = y \,|\, X_2 = x)$$

shows that $(X_t)_{t\in\mathbb{Z}_+}$ is not a time-homogeneous Markov chain.

The correct answers are: $(X_t)_{t\in\mathbb{Z}_+}$ is a martingale.

, $\mathbb{E}\left[X_t\right] = 1$ for all $t \in \mathbb{Z}_+$.

Question **3**

Flag question | Mark 0.00 out of 1.00 | Incorrect

In a game, the outcomes of the rounds

$$B_t = \begin{cases} 1 & \text{if the player wins in round } t, \\ 0 & \text{else,} \end{cases}$$

are mutually independent, and for each round the probability of winning equals

$$\mathbb{P}\left[B_t = 1\right] \;=\; \tfrac{18}{37}.$$

When the bet in round t equals H_t tokens and the outcome of round t is successful, then a player wins the bet amount. Otherwise the player loses the bet amount. When a player initially has $X_0 = 20$ tokens, the number of tokens after t rounds is

$$X_t = 20 + \textstyle\sum_{s=1}^{t} \left(2B_s - 1\right) H_s.$$

Theresa employs a betting strategy

$$H_t = \begin{cases} 1 & \text{if } X_{t-1} > 0 \\ 0 & \text{if } X_{t-1} = 0, \end{cases}$$

that is, she bets one token every round until she runs out of tokens. Which of the following statements are true?

Select one or more:

- [x] a. $(X_t)_{t\in\mathbb{Z}_+}$ is a martingale. ✗
- [x] b. $(X_t)_{t\in\mathbb{Z}_+}$ is a submartingale. ✗
- [x] c. $(X_t)_{t\in\mathbb{Z}_+}$ is a time-homogeneous Markov chain. ✓
- [x] d. $(X_t)_{t\in\mathbb{Z}_+}$ is a supermartingale. ✓

Your answer is incorrect.

(i) The process $(X_t)_{t\in\mathbb{Z}_+}$ is a supermartingale with respect to the outcomes of the rounds $B_1, B_2, \ldots$, and hence also with respect to itself. Let us verify the properties which characterize a supermartingale:

1. For any $t \in \mathbb{Z}_+$ it clearly holds that $|X_t| \leq 20 + t$ and so the integrability property $\mathbb{E}\big[|X_t|\big] \leq 20 + t < \infty$ holds.
2. By definition, X_t is determined by the outcomes $B_1, \ldots, B_t$ and the bets $H_1, \ldots, H_t$. Correspondingly, every bet H_s is determined by the value X_{s-1}. Inductively, this tells us that X_t is determined by the outcomes $B_1, \ldots, B_t$ and therefore $X_t \in \sigma(B_1, \ldots, B_t)$.
3. To verify the supermartingale property we calculate the conditional expectation (pulling out known factors and pulling out independent factors):

$$
\begin{aligned}
&\mathbb{E}\Big[X_{t+1} \,\Big|\, B_1, \ldots, B_t\Big] \\
&= \mathbb{E}\Big[20 + \textstyle\sum_{s=1}^{t+1} \Big(2B_s - 1\Big) H_s \,\Big|\, B_1, \ldots, B_t\Big] \\
&= 20 + \textstyle\sum_{s=1}^{t+1} \mathbb{E}\Big[\Big(2B_s - 1\Big) H_s \,\Big|\, B_1, \ldots, B_t\Big] \\
&= 20 + \textstyle\sum_{s=1}^{t} (2B_s - 1) H_s + \mathbb{E}\Big[\big[(2B_{t+1} - 1)\big]\Big] H_{t+1} \\
&= 20 + \textstyle\sum_{s=1}^{t} (2B_s - 1) H_s + \big(\frac{18}{37} - \frac{19}{37}\big) H_{t+1} \\
&= X_t - \tfrac{1}{37} H_{t+1} \leq X_t.
\end{aligned}
$$

Based on the above computation, $(X_t)_{t\in\mathbb{Z}_+}$ is neither a martingale nor a submartingale: for instance, when $t = 0$ the following bet is $H_1 = 1$ and, thus, the inequality is a strict one $\mathbb{E}\big[X_1 \,\big|\, \cdot\,\big] = X_0 - \frac{1}{37} < X_0$.

(ii) With a calculation much like the one above we find that

$$
\begin{aligned}
&\mathbb{P}\Big[X_{t+1} = y \,\Big|\, X_t = x, B_1, \ldots, B_t\Big] \\
&= \cdots = \mathbb{P}\Big[(2B_{t+1} - 1) H_{t+1} = y - x\Big].
\end{aligned}
$$

This shows us that $(X_t)_{t\in\mathbb{Z}_+}$ is a time-homogeneous discrete-time Markov chain on state space $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ with transition probabilities

$$
\begin{aligned}
&P(x, x+1) = \tfrac{18}{37} \text{ when x>0} \\
&P(x, x-1) = \tfrac{19}{37} \text{ when x>0} \\
&P(0, 0) = 1
\end{aligned}
$$

(all other transition probabilities are zero).

The correct answers are: $(X_t)_{t\in\mathbb{Z}_+}$ is a time-homogeneous Markov chain.

, $(X_t)_{t\in\mathbb{Z}_+}$ is a supermartingale.