10 Continuous-time Markov chains

The objective of this exercise is to practice computing time-dependent and invariant distributions related to continuous-time Markov chains. It will be useful to have a computer or a calculator capable of matrix calculations with you in the exercise session.

Classroom problems

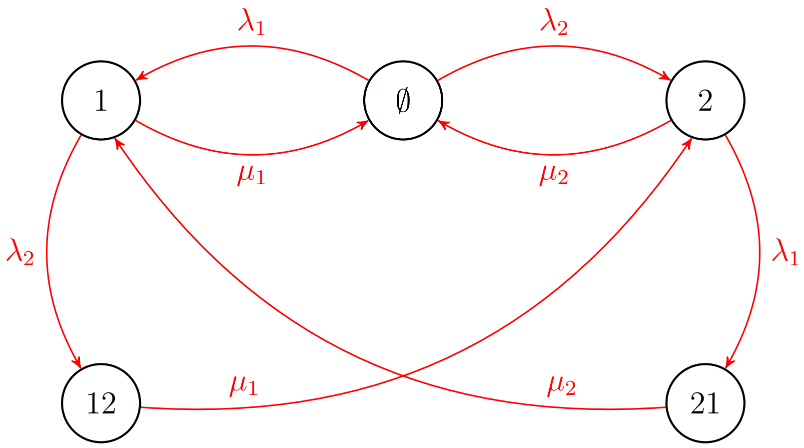
10.1 An IT person for a company is responsible for the operation of two web servers. Server i operates an average of ℓ_i days before it malfunctions, and repairing it takes an average of m_i days, where $\ell_1 = 30$, $\ell_2 = 100$, $m_1 = 1$ and $m_2 = 2$. The operation and repair times of the servers are assumed to be independent and exponentially distributed. The IT person repairs the servers in the order they break.

(a) Model the state of the servers as a continuous-time Markov process with state space $S = \{(), (1), (2), (1, 2), (2, 1)\}$, where each state is an ordered list representing the queue of servers to be repaired. Write the generator matrix Q of the Markov process, and draw a transition diagram.

Solution. We will use the shorthand notation

$$\emptyset = (), \quad 1 = (1), \quad 2 = (2), \quad 12 = (1, 2), \text{ and } \; 21 = (2, 1).$$

Denote the malfunctioning intensity of server i by $\lambda_i = 1/\ell_i$ and the repairing intensity by $\mu_i = 1/m_i$. The operation time of server i is then $L_i =_{\text{st}} \text{Exp}(\lambda_i)$ and the repair time $M_i =_{\text{st}} \text{Exp}(\mu_i)$. Since the times are independent and exponentially distributed, the process is a continuous-time Markov chain with jump rates λ_i and μ_i. A corresponding transition diagram is drawn below.



The generator matrix can be defined as

$$Q(x, y) = \begin{cases} I_{x \to y}, & x \neq y \\ -\sum_{z \neq y} I_{y \to z}, & x = y, \end{cases}$$

which in this case equals (with the given ordering of states)

$$Q = \begin{bmatrix} -(\lambda_1+\lambda_2) & \lambda_1 & \lambda_2 & 0 & 0 \\ \mu_1 & -(\lambda_2+\mu_1) & 0 & \lambda_2 & 0 \\ \mu_2 & 0 & -(\lambda_1+\mu_2) & 0 & \lambda_1 \\ 0 & 0 & \mu_1 & -\mu_1 & 0 \\ 0 & \mu_2 & 0 & 0 & -\mu_2 \end{bmatrix}.$$

(b) What is the probability that neither of the servers will be working after a week, provided that they are both operational at present?

Solution. Let $t = 7$ (in days) be the time interval in question. The transition matrix P_t is given by the matrix exponential

$$P_t = e^{tQ} = \sum_{n=0}^{\infty} \frac{t^n}{n!} Q^n.$$

By using a computer we obtain

$$P_7 = e^{7Q} = \begin{bmatrix} 0.949 & 0.0318 & 0.0179 & 0.000316 & 0.00107 \\ 0.948 & 0.0325 & 0.0183 & 0.000371 & 0.00110 \\ 0.918 & 0.0343 & 0.0400 & 0.000342 & 0.00713 \\ 0.891 & 0.0342 & 0.0640 & 0.001241 & 0.01000 \\ 0.895 & 0.0571 & 0.0165 & 0.000771 & 0.03106 \end{bmatrix}.$$

Since both of the servers are operational, the initial state $X(0)$ is $\emptyset$, and the distribution at $t = 7$ is given by the first row:

$$\begin{bmatrix} 0.949 & 0.0318 & 0.0179 & 0.000316 & 0.00107 \end{bmatrix}.$$

The states where neither server is operational are 12 and 21, so the answer is

$$P_7(\emptyset, 12) + P_7(\emptyset, 21) = 0.000316 + 0.00107 = 0.00139.$$

(c) Determine the invariant distribution of the process. What is the probability at the statistical equilibrium that there is at least one operational server in the company?

Solution.

The invariant distribution is obtained by solving the matrix equation $\pi Q = 0$, i.e.,

$$\sum_{x \in S} \pi(x) Q(x, y) = 0, \quad y \in S,$$

with the normalization constraint $\sum_{x \in S} \pi(x) = 1$. By now we are familiar with solving linear systems of equations, and obtain

$$\begin{bmatrix} 0.948 & 0.0319 & 0.0184 & 0.000319 & 0.00123 \end{bmatrix}.$$

The probability we are looking for is $1 - \pi(12) - \pi(21) = 0.998$.

Additional information. The Markov chain this exercise is irreducible, so one could obtain a reasonable approximation of π by calculating e^{tQ} for a large t.

(d) If we discover that neither of the servers are working at the statistical equilibrium, how long is the expected waiting time until at least one of the servers will be online?

Solution. Since neither server is working, we must be in either state 12 or 21. Let $\tilde{\pi}$ be the distribution of the current state X. By the definition of conditional probability we have

$$\begin{aligned}\tilde{\pi}(12) &= \mathbb{P}(X = 12 \mid X \in \{12, 21\}) \\ &= \frac{\mathbb{P}(X = 12 \text{ and } X \in \{12, 21\})}{\mathbb{P}(X \in \{12, 21\})} \\ &= \frac{\pi(12)}{\pi(12) + \pi(21)} = 0.207\end{aligned}$$

and for the other state we have

$$\tilde{\pi}(21) = \frac{\pi(21)}{\pi(12) + \pi(21)} = 0.793.$$

Let $\mathbb{E}_x(T)$ be the expected waiting time in state x. Since the current state is unknown, we must take the expectation of the waiting time T over the conditional distribution of X:

$$\begin{aligned}\mathbb{E}(T \mid X \in \{12, 21\}) &= \sum_{x \in S} \tilde{\pi}(x)\mathbb{E}_x(T) \\ &= \tilde{\pi}(12)\mathbb{E}_{12}(T) + \tilde{\pi}(21)\mathbb{E}_{21}(T) \\ &= \tilde{\pi}(12)m_1 + \tilde{\pi}(21)m_2 \\ &= 0.207 \cdot 1 + 0.793 \cdot 2 \\ &= 1.79.\end{aligned}$$

The expected waiting time is 1.79 days.

Additional information. The intuition above can be made exact as follows: when we denote the event $A =$ "$X \in \{12, 21\}$", the previous calculation of $\mathbb{E}(T \mid X \in \{12, 21\})$ corresponds to the formula

$$\mathbb{E}(T \mid A) = \mathbb{E}[\, \mathbb{E}(T \mid X, A) \ \mid A],$$

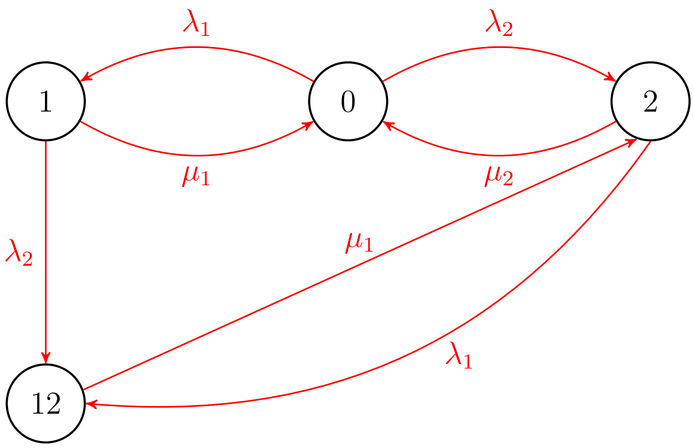
called *conditional unbiasedness*, or more generally, the *tower property*.

Homework problems

10.2 An IT person for a company is responsible for the operation of two web servers that work as in Problem 10.1. This time the servers are prioritized so that the IT person will repair server 1 as soon as it malfunctions, suspending the repairs of server 2 for that time if necessary.

(a) Model the state of the servers as a continuous-time Markov chain with state space $S' = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$, where the elements of the state space are unordered sets representing the servers waiting to be repaired. Write the generator matrix of the chain and draw a transition diagram.

Solution. Exactly as in the previous problem, we have the transition diagram



and the generator matrix

$$Q = \begin{bmatrix} -\lambda_1-\lambda_2 & \lambda_1 & \lambda_2 & 0 \\ \mu_1 & -\lambda_2-\mu_1 & 0 & \lambda_2 \\ \mu_2 & 0 & -\lambda_1-\mu_2 & \lambda_1 \\ 0 & 0 & \mu_1 & -\mu_1 \end{bmatrix}.$$

(b) Solve the invariant distribution of the process. What is the probability that there will be at least one operational server in the company at the statistical equilibrium?

Solution. The invariant distribution is obtained by solving the balance equation $\pi Q = 0$ with the normalization condition $\sum_i \pi_i = 1$. Since there are 4 unknowns and 5 equations, we replace the last column by $[1\ 1\ 1\ 1]^T$, corresponding to the normalization condition:

$$\hat{Q} = \begin{bmatrix} -\lambda_1-\lambda_2 & \lambda_1 & \lambda_2 & 1 \\ \mu_1 & -\lambda_2-\mu_1 & 0 & 1 \\ \mu_2 & 0 & -\lambda_1-\mu_2 & 1 \\ 0 & 0 & \mu_1 & 1 \end{bmatrix}.$$

We solve the equation

$$\pi\hat{Q} = [0, 0, 0, 1]$$

and obtain

$$\pi = [0.948,\ 0.0313,\ 0.0196,\ 0.000966].$$

The probability of having at least one operational server is

$$1 - \pi(12) \approx 0.999.$$

(c) If we discover that neither of the servers are working at the statistical equilibrium, how long is the expected waiting time until at least one of the servers will be online?

Solution. When neither server is working, the IT person repairs server 1. Since the repair times are exponentially distributed,

$$\mathbb{E}(\tau) = 1/\mu_1 = 1.$$

(d) Compare the results of the previous parts to those of Problem 10.1.

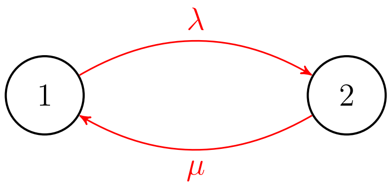
Solution. When both servers are offline, the expected fixing time of the first server has *decreased* – in 10.1 it was $1.79 > 1$ days. The probability at the statistical equilibrium of having at least on operational server has *increased* – in 10.1 it was $0.998 < 0.999$. If it is enough that one server is operational, the prioritization in this exercise might be the better choice.

However, the probability of having two operational servers has not changed. This is natural, as we have only changed the order of repairs – the servers break at the same rate and the poor IT person still repairs them as fast as they can!

10.3 A machine operates 100 days by expectation before malfunctioning, and repairs take 10 days by expectation. Operation and repair times are assumed mutually independent and exponentially distributed. While operating (state 1), the machine creates an average of 120 000 EUR profit a day, and while broken (state 2), an average of 90 000 EUR loss a day.

(a) Provided that the machine is broken in the beginning of the month, what is the expected number of days it will be broken during the month (30 days)?
Hint: The results of Problem 9.3 might be helpful.

Solution. Let X_t be the state of the machine at time t. Since the jumps are independently and exponentially distributed, we deduce that X_t is a continuous-time Markov chain. We choose the time unit to be one day. Based on the description we can draw the transition diagram



and write the corresponding generator matrix

$$Q = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}.$$

The expected occupancy of state y with initial state x is[1]

$$\begin{aligned} M_t(x,y) &= \mathbb{E}\left[\int_0^t \mathbb{I}(X_s = y)ds \mid X_0 = x\right] \\ &= \int_0^t \mathbb{E}\left[\mathbb{I}(X_s = y) \mid X_0 = x\right] ds \\ &= \int_0^t P_s(x,y)ds. \end{aligned}$$

From this we get

$$M_t(2,2) \quad = \quad \int_0^t P_s(2,2)ds.$$

Based on exercise 9.3 we have

$$P_s(2,2) = \frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu}e^{-(\lambda+\mu)s},$$

[1] changing the order of the expectation and the definite integral is justified by *Fubini's theorem*

and so

$$M_t(2,2) = \frac{\lambda t}{\lambda+\mu} + \frac{\mu}{(\lambda+\mu)^2}(1 - e^{-(\lambda+\mu)t}).$$

Substituting the values $\mu = 1/10$, $\lambda = 1/100$ and $t = 30$ yields

$$M_{30}(2,2) \approx 10.687 \text{ (days)}.$$

(b) Provided that the machine is broken in the beginning of the month, what is the expected profit it will create for its owner during that month (30 days)?

Solution. The expected profit with initial state 2 is given by

$$g_{30}(2) = 120\,000 \times M_{30}(2,1) - 90\,000 \times M_{30}(2,2).$$

We could repeat the calculations for $M_{30}(2,1)$, or simply deduce that it must equal $30 - M_{30}(2,2) \approx 19.313$. Hence,

$$\begin{aligned} g_{30}(2) &\approx 120\,000 \times 19.313 - 90\,000 \times 10.687 \\ &\approx 1\,356\,000 \text{ (EUR)}. \end{aligned}$$

(c) At what rate will the machine create profit for its owner in the long term?

Solution. The long term profit rate is obtained from the invariant distribution:

$$\sum_{x \in S} \pi(x) c(x).$$

The invariant distribution was found to be $[\frac{\mu}{\lambda+\mu} \ \frac{\lambda}{\lambda+\mu}]$ in exercise 9.3, and so

$$\begin{aligned} \sum_{x \in S} \pi(x) c(x) &= \frac{\mu}{\lambda+\mu} \times 120\,000 - \frac{\lambda}{\lambda+\mu} \times 90\,000 \\ &\approx 100\,900 \text{ (EUR/day)}. \end{aligned}$$