12 Optional times and stopped martingales

You learn to recognize which random times are optional times, and to compute values and estimates of hitting probabilities using stopped martingales. In this exercise there is no homework!

Classroom problems

12.1 *Stock option.* The current value of a stock option is 47 EUR, and either 100 EUR or 0 EUR on the date of maturity 30 days later. Let S_t be the value of the option after t days. We will assume that $(S_t)_{t\in\mathbb{Z}_+}$ is a martingale.

(a) What is the probability that the option will have the value of 100 EUR on the maturity date?

Solution. Because the expected value of a martingale is constant over time, it follows that

$$47 \;=\; \mathbb{E}S_0 \;=\; \mathbb{E}S_{30} \;=\; 0\cdot\mathbb{P}(S_{30}=0)+100\cdot\mathbb{P}(S_{30}=100).$$

Hence the option value equals 100 EUR on the date of maturity with probability

$$\mathbb{P}(S_{30}=100) \;=\; 0.47.$$

(b) Pertti is planning to buy the option now and to sell it either on the date of maturity, or on the time instant that its value rises above 55 EUR or decreases below 15 EUR. What is the expected net profit Pertti will make at the selling time?

Solution. The net profit of Pertti's investment strategy can be written as $S_T - S_0$, where T is the random time when Pertti sells the option. The selling time can be written as

$$T \;=\; \min\{\tau_1,\tau_2,30\},$$

where $\tau_1=\min\{t\ge 0 : S_t>55\}$ and $\tau_2=\min\{t\ge 0 : S_t<15\}$. Because τ_1, τ_2, and also the deterministic constant $\tau_3=30$, are optional times, it follows that T is an optional time. Because $T\le 30$ with probability one, we see that $S_T=M_{30}$ equals the state of the stopped process $M_t=S_{T\wedge t}$ at time $t=30$. Because (S_t) is a martingale and T is an optional time, it follows that (lecture notes) also the stopped process (M_t) is a martingale. Because the expected value of a martingale is constant over time, it follows that

$$\mathbb{E}S_T \;=\; \mathbb{E}M_{30} \;=\; \mathbb{E}M_0 \;=\; \mathbb{E}S_0$$

Therefore, the expected net profit of Pertti's investment strategy equals

$$\mathbb{E}(S_T-S_0) \;=\; 0.$$

Additional information. The following result may provide additional insight. The third martingale property condition

$$\mathbb{E}(M_t|M_0,\ldots,M_{t-1})=M_{t-1}$$

is equivalent to

$$\mathbb{E}(M_t|M_0,\dots,M_s) = M_s \qquad \forall s \le t.$$

(c) Assume that the option can never attain a negative value. Derive a nontrivial (less than one) upper bound for the probability of the event that the value of the option will sometime rise above Pertti's target level 55 EUR before maturity.

Hint: Analyze the stopped martingale $\hat{S}_t = S_{t\wedge\tau}$ where τ is the first time instant (possibly infinite) that the option value exceeds 55 EUR.

Solution. Let $\tau = \min\{t \ge 0 : S_t > 55\}$. This is an optional time with respect to information sequence $(S_0, S_1, \dots)$. Because (S_t) is a martingale, it follows that the stopped process $\hat{S}_t = S_{t\wedge\tau}$ is a martingale. Because the expected value of a martingale remains constant over time, it follows that

$$\mathbb{E}\hat{S}_t \;=\; \mathbb{E}\hat{S}_0 \;=\; \mathbb{E}S_0.$$

Since $1(\tau \le t) + 1(\tau > t) = 1$, we may write

$$\mathbb{E}\hat{S}_t \;=\; \mathbb{E}\hat{S}_t 1(\tau \le t) + \cancel{\mathbb{E}\hat{S}_t 1(\tau > t)} \;\ge\; \mathbb{E}\hat{S}_t 1(\tau \le t),$$

where the inequality follows from the fact that $\hat{S}_t \ge 0$. When $\tau \le t$ holds, we have $t \wedge \tau = \tau$, which also means that $\hat{S}_t = S_\tau$. By the definition of τ we have $S_\tau > 55$, and we end up with the calculation

$$\mathbb{E}\hat{S}_t \;\ge\; \mathbb{E}\hat{S}_t 1(\tau \le t) \;=\; \mathbb{E}S_\tau 1(\tau \le t) \;\ge\; 55\mathbb{E}1(\tau \le t) \;=\; 55\mathbb{P}(\tau \le t).$$

We may hence conclude that

$$55\mathbb{P}(\tau \le t) \;\le\; \mathbb{E}\hat{S}_t \;=\; \mathbb{E}S_0 \;=\; 47.$$

Therefore,

$$\mathbb{P}(\tau \le t) \;\le\; \frac{47}{55} \;=\; 0.855.$$

By substituting $t = 30$ to the above inequality, it follows that

$$\mathbb{P}(S_t > 55 \text{ for some } t \in [0,30]) \;=\; \mathbb{P}(\tau \le 30) \;\le\; 0.855.$$

12.2 *Pólya's urn.* An urn contains one red and one green ball in the beginning. During round $t = 1, 2, \dots$, a ball is randomly picked from the urn and its color its observed. Then the ball is returned to the urn and another ball of the same color as observed is added to the urn. Let X_t be the relative proportion of red balls in the urn after t rounds.

(a) Verify that the process $(X_t)_{t\in\mathbb{Z}_+}$ is a martingale.

Solution. Denote A_t ="a red ball is picked at the t-th round". Then the number of red balls after t rounds equals

$$S_t \;=\; 1+\sum_{s=1}^{t} 1(A_s) \;=\; 1+\sum_{s=1}^{t} I_s,$$

where $I_s = 1(A_s)$ is the random indicator variable of the event A_s. The relative proportion of red balls after t rounds then equals

$$X_t \;=\; \frac{S_t}{2+t}.$$

We will next verify that $(X_t)_{t\in\mathbb{Z}_+}$ is a martingale with respect to the information sequence $(X_0, I_1, I_2, \dots)$ where $X_0 = \frac{1}{2}$ the deterministic initial state. Recall that a random sequence $(M_0, M_1, \dots)$ is a martingale with respect to $(X_0, X_1, \dots)$ if

(i) $\mathbb{E}|M_t| < \infty$,

(ii) $M_t \in \sigma(X_0, \dots, X_t)$,

(iii) $\mathbb{E}(M_{t+1} \,|\, X_0, \dots, X_t) = M_t$.

Now (i) is clear because, being a relative proportion, $X_t \in [0,1]$ with probability one. (ii) is also clear by definition. For (iii), we note that the conditional probability of picking a red ball in round $t+1$, given that $I_1 = i_1, \dots, I_t = i_t$, equals

$$\mathbb{P}(A_{t+1} \,|\, I_1 = i_1, \dots, I_t = i_t) \;=\; \frac{1+\sum_{s=1}^{t} i_s}{2+t},$$

so that[1]

$$\begin{aligned}
\mathbb{E}(S_{t+1}|I_1,...,I_t) &= S_t + \mathbb{E}(I_{t+1}|I_1,...,I_t) \\
&= S_t + \mathbb{P}(A_{t+1}|I_1,...,I_t) \\
&= S_t + \frac{1+\sum_{i=1}^{t} I_i}{2+t} \\
&= S_t + \frac{S_t}{2+t} \\
&= \frac{(3+t)S_t}{2+t}.
\end{aligned}$$

It now follows that

$$\begin{aligned}
\mathbb{E}(X_{t+1} \,|\, I_1,...,I_t) &= \mathbb{E}\left(\frac{S_{t+1}}{2+(t+1)} \,|\, I_1,...,I_t\right) \\
&= \frac{1}{3+t}\mathbb{E}\left(S_{t+1} \,|\, I_1,...,I_t\right) \\
&= \frac{1}{3+t}\,\frac{(3+t)S_t}{2+t} \\
&= \frac{S_t}{2+t} \\
&= X_t.
\end{aligned}$$

[1] the same result follows from $\mathbb{E}(S_{t+1}|I_1,...,I_t) = S_t\mathbb{P}(S_{t+1} = S_t|I_1,...,I_t) + (S_t+1)\mathbb{P}(S_{t+1} = S_t+1|I_1,...,I_t)$.

(b) Prove that the probability that the relative proportion of red balls ever reaches level 0.9 is at most 5/9.

Hint: Analyze the stopped martingale $\hat{X}_t = X_{t\wedge T}$ where T is the first time instant (possibly infinite) that the proportion of red balls reaches the level 0.9.

Solution. Let $T = \min\{t \geq 0 : X_t \geq 0.9\}$ be the first time instant (possibly infinite) that the proportion of red balls reaches the level 0.9. Because (X_t) is a martingale and T is an optional time with respect to $(X_0, X_1, \dots)$, it follows that the stopped process $(\hat{X}_t)_{t\in\mathbb{Z}_+}$ defined by $\hat{X}_t = X_{t\wedge T}$ is a martingale (lecture notes). Because martingales are constant by expectation, and because $\hat{X}_0 = X_0$, it follows that

$$\mathbb{E}(\hat{X}_t) \;=\; \mathbb{E}(\hat{X}_0) \;=\; \mathbb{E}(X_0) \;=\; 1/2.$$

This gives as a lower bound

$$\mathbb{E}\hat{X}_t \;=\; \mathbb{E}(X_{t\wedge T}) \;\geq\; \mathbb{E}(X_{t\wedge T}1(T\leq t)) \;=\; \mathbb{E}(X_T 1(T\leq t)) \;\geq\; 0.9\,\mathbb{P}(T\leq t),$$

which implies that

$$\mathbb{P}(T\leq t) \;\leq\; \frac{1/2}{0.9} \;=\; 5/9$$

for all $t \geq 0$. By taking limits as $t \to \infty$ it follows (by the monotone continuity of probability measures) that

$$\mathbb{P}(T<\infty) \;=\; \lim_{t\to\infty}\mathbb{P}(T\leq t) \;\leq\; 5/9.$$

12.3 *Lognormal stock prices.* The closing price of a stock in the end of trading day t is modeled as a stochastic process $M_t = M_0 X_1 \cdots X_t$, where M_0 is a known constant and $X_1, X_2, \dots$ are independent lognormally distributed random numbers which can represented as $X_t = e^{\mu+\sigma Z_t}$ where $Z_1, \dots, Z_t$ are independent and follow the standard normal distribution.

(a) For which values of μ and σ is $(M_t)_{t\in\mathbb{Z}_+}$ a martingale?

Solution. Observe first that because the sum of t independent standard normal random numbers has a normal distribution with mean zero and variance t,

$$Z_1 + \cdots + Z_t \;=_{\mathrm{st}}\; \sqrt{t}Z$$

and

$$M_t \;=\; M_0 e^{\sum_{s=1}^t(\mu+\sigma Z_s)} \;=\; M_0 e^{\mu t+\sigma(Z_1+\cdots+Z_t)} \;=_{\mathrm{st}}\; M_0 e^{\mu t+\sigma\sqrt{t}Z}$$

where the distribution of Z is standard normal. From this we conclude that M_t follows a lognormal distribution for every t. Because a lognormal distribution has a finite mean, we conclude that $\mathbb{E}|M_t| < \infty$. Observe next that

$M_t \in \sigma(M_0, X_1, \dots, X_t)$ for every $t \geq 0$. Therefore, the two common properties required for martingales, submartingales, and supermartingales are valid for (M_t).

To investigate conditional expectations, note that by pulling out known factors and using independence,

$$\begin{aligned}
\mathbb{E}(M_{t+1} \mid M_0, X_1, X_2, ..., X_t) &= \mathbb{E}(M_0 X_1 X_2 \cdots X_{t+1} \mid M_0, X_1, X_2, ..., X_t) \\
&= M_0 X_1 X_2 \cdots X_t \, \mathbb{E}(X_{t+1} \mid M_0, X_1, X_2, ..., X_t) \\
&= M_t \, \mathbb{E}(X_{t+1}) \\
&= m \, M_t,
\end{aligned}$$

where $m = \mathbb{E}(X_1)$ is the expected proportional return of the stock. Hence we see that, with respect to the information sequence $(M_0, X_1, X_2, \dots)$,

$$(M_t) \text{ is a } \begin{cases} \text{supermartingale,} & \text{if } m < 1, \\ \text{martingale,} & \text{if } m = 1, \\ \text{submartingale,} & \text{if } m > 1. \end{cases}$$

To compute m, we use the density function $f(x) = (2\pi)^{-1/2} e^{-x^2/2}$ of the standard normal distribution to write

$$\begin{aligned}
\mathbb{E} e^{\mu + \sigma Z_1} &= \int_{-\infty}^{\infty} e^{\mu + \sigma x} f(x) \, dx \\
&= (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{\mu + \sigma x - x^2/2} \, dx \\
&= (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2 - 2\sigma x - 2\mu)} \, dx.
\end{aligned}$$

By writing $x^2 - 2\sigma x - 2\mu = (x-\sigma)^2 - 2\mu - \sigma^2$, we find that

$$\begin{aligned}
(2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2 - 2\sigma x - 2\mu)} \, dx &= e^{\mu + \sigma^2/2} (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-(x-\sigma)^2)/2} \, dx \\
&= e^{\mu + \sigma^2/2} \int_{-\infty}^{\infty} (2\pi)^{-1/2} e^{-x^2/2} \, dx \\
&= e^{\mu + \sigma^2/2} \int_{-\infty}^{\infty} f(x) \, dx \\
&= e^{\mu + \sigma^2/2}.
\end{aligned}$$

Hence

$$m = \mathbb{E}(X_1) = e^{\mu + \sigma^2/2}.$$

we may now conclude that, with respect to the information sequence $(M_0, X_1, X_2, \dots)$,

$$(M_t) \text{ is a } \begin{cases} \text{supermartingale,} & \text{if } \mu < -\frac{1}{2}\sigma^2, \\ \text{martingale,} & \text{if } \mu = -\frac{1}{2}\sigma^2, \\ \text{submartingale,} & \text{if } \mu > -\frac{1}{2}\sigma^2. \end{cases}$$

(b) When is (M_t) a submartingale?

Solution. By part (a), we find that (M_t) is a submartingale with respect to $(M_0, X_1, X_2, \dots)$ if and only if $\mu \geq -\frac{1}{2}\sigma^2$. (Keep in mind that every martingale is always a submartingale and supermartingale.)