8 Poisson processes

In this exercise you learn to apply Poisson processes to solve real-life problems involving random time instants. You also get your hands dirty with integrals needed in working with independent random variables in the continuum.

Classroom problems

8.1 *Repeated exponential waiting times.* A random variable X is Gamma distributed with shape parameter k and rate parameter λ if it has density function

$$p(x) \;=\; \begin{cases} \frac{\lambda^k}{(k-1)!} x^{k-1} e^{-\lambda x}, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

Let $\tau_1, \tau_2, \ldots$ be independent exponentially distributed random variables with rate parameter λ, and define

$$T_n \;=\; \tau_1 + \tau_2 + \cdots + \tau_n.$$

(a) Show that T_n follows a Gamma distribution, and determine the shape and rate parameters of this distribution.

(b) We will now interpret $T_1 < T_2 < \cdots$ as an increasing sequence of time instants, and we denote by $N(t) = \#\{n \geq 1 : T_n \leq t\}$ the number of time instants in $[0, t]$. With the help of (b), determine the probability distribution and expectation of $N(t)$.

Homework problems

8.2 Trucks arrive at Vaalimaa border crossing at independent exponentially distributed intervals with mean 15 min. By random sampling, one third of the arriving trucks are directed to the customs for an inspection.

(a) What is the probability that no trucks arrive at the border crossing during an hour?

(b) What is the probability that exactly two trucks are directed to the customs during an interval of 15 minutes?

8.3 Teemu Selänne has scored an average of $\lambda = 1$ points (a goal or an assist that led to a goal) per game in NHL. Let us assume that 30% of the points are goals and 70% assists that led to a goal. Let us assume that Teemu receives a bonus of \$3000 for a goal and \$1000 for an assist. Use a Poisson process to model the point scoring time instants during a 60 min game and answer the following questions.

(a) What is the expected value of the total bonus in an individual game?

(b) What is the standard deviation of the total bonus in an individual game?

(c) What is the probability that Teemu has 1 goal and 2 assists in a game?