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1 Random variables and probability distributions

The goal of this exercise is to refresh the basic probability concepts which are assumed to be familiar from previous studies such as `MS-A05XX First course in probability and statistics`. For refreshing the basics of probability, see e.g. the lecture notes [Les21], the textbook [Ros14] (available online from Aalto network), or the free online book [GS97].

Classroom problems

1.1 *Javelin competition.* Oliver throws javelin six times. The lengths of the throws (in meters) are modeled as independent random numbers $Z_1, \dots, Z_6$ which are uniformly distributed on the continuous interval $(80, 92)$.

(a) Calculate the expected value and standard deviation of Oliver's first throw.

(b) Determine the cumulative distribution function F_Y and the density function f_Y of $Y = \max(Z_1, \dots, Z_6)$, and calculate the probability that the longest throw is at least 91 meters.

(c) Find out the cumulative distribution function F_X of $X = \min(Z_1, \dots, Z_6)$, and calculate the probability that at least one of the throws is below 85 meters.

(d) Does the joint distribution of X and Y have a density function $f_{X,Y}$? Are the random numbers X and Y independent? Justify your answer.

(e) Does the joint distribution of Z_1 and Y have a density function $f_{Z_1,Y}$?

Homework problems

1.2 *Memoryless distributions.* Let X be a random number following a geometric distribution on the set $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ with success probability p, so that X has probability mass function

$$\pi_X(k) = (1-p)^k p, \qquad \text{for } k = 0, 1, 2, \dots$$

(a) Find the conditional probability $\mathbb{P}\big[X \ge t + h \,\big|\, X \ge t\big]$ for integers $t, h \ge 0$.

(b) Calculate the expected value and standard deviation of X.

Let Y be a random number following an exponential distribution with rate parameter $\lambda > 0$, so that Y has density function

$$f_Y(x) \;=\; \lambda e^{-\lambda x} 1_{(0,\infty)}(x) \;=\; \begin{cases} \lambda e^{-\lambda x}, & \text{for } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

(c) Find the conditional probability $\mathbb{P}\big[Y > t + h \,\big|\, Y > t\big]$ for real numbers $t, h > 0$.

(d) Calculate the expected value and standard deviation of Y.

1.3 *Best of three.* Otaniemi Eulers and Leppävaara Algebra are fighting for the championship of the robot football league by playing a series of matches in a best-of-three system, where the winner of the whole league is the team that gets two wins. Suppose that the results of the matches are independent and that Eulers win each match with probability $p = 0.55$.

(a) What is the probability that Eulers take the championship?

(b) What is the probability that it takes three matches to determine the champion?

(c) What is the expected number of matches required to determine the champion?

Now consider a change in the rules where a best-of-seven system is used, so that the champion is the team that gets four wins.

(d) What is the probability that Eulers take the championship?

(e) What is the probability that it takes seven matches to determine the champion?

References

[GS97] Charles M. Grinstead and J. Laurie Snell. *Introduction to Probability.* American Mathematical Society, http://www.dartmouth.edu/~chance/teaching_aids/books_articles/probability_book/book.html, 1997.

[Les21] Lasse Leskelä. Stokastiikka ja tilastollinen ajattelu. http://math.aalto.fi/~lleskela/LectureNotes003.html, 2021.

[Ros14] Sheldon M. Ross. *Introduction to Probability and Statistics for Engineers and Scientists.* Academic Press, 2014.