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11 Martingales and information processes

In this exercise you become familiar with the concept of a martingale, and you learn to detect which random times are optional times with respect to a given information process.

Classroom problems

11.1 *Markov chains and martingales.* Invent (or google) an example of an integer-valued stochastic process $(X_0, X_1, \dots)$ which is

(a) a Markov chain and a martingale,

Solution. Recall that a Markov chain is a memoryless random sequence over time, whereas a martingale may remember its past. A random sequence $(M_0, M_1, \dots)$ is a martingale with respect to $(X_0, X_1, \dots)$ if

(i) $\mathbb{E}|M_t| < \infty$,

(ii) $M_t \in \sigma(X_0, \dots, X_t)$,

(iii) $\mathbb{E}(M_{t+1} \,|\, X_0, \dots, X_t) = M_t$.

Let $S_t = X_1 + \cdots + X_t$ be a symmetric random walk on $\mathbb{Z}$ where the summands $X_1, X_2, \dots$ are independent and uniformly distributed in $\{-1, +1\}$. This walk is, rather obviously, a time-homogeneous Markov chain with transition matrix P such that $P(x, x+1) = P(x, x-1) = \frac{1}{2}$ and all other entries are zero.

Let us next verify (S_t) is a martingale. Property (i) if valid because $|S_t| \le t$ with probability one. (ii) is clear. (iii) follows by observing that

$$\begin{aligned}
& \mathbb{E}\{S_{t+1}|X_0, \dots, X_t\} \\
&= \mathbb{E}\{S_t + X_{t+1}|X_0, \dots, X_t\} \\
\text{(linearity)} \quad &= \mathbb{E}\{S_t|X_0, \dots, X_t\} + \mathbb{E}\{X_{t+1}|X_0, \dots, X_t\} \\
\text{(pulling out known factors)} \quad &= S_t + \mathbb{E}\{X_{t+1}|X_0, \dots, X_t\} \\
\text{(independence)} \quad &= S_t + \mathbb{E}\{X_{t+1}\} \\
&= S_t.
\end{aligned}$$

Hence (S_t) is a martingale with respect to (X_t), and it follows (lecture notes) that (S_t) is also a martingale with respect to itself.

(b) a Markov chain but not a martingale,

Solution. Let $S_t = X_1 + \cdots + X_t$ be an asymmetric random walk on $\mathbb{Z}$ where the summands $X_1, X_2, \dots$ are independent and nonuniformly distributed in $\{-1, +1\}$. It is easy to see that (S_t) is a time-homogeneous Markov chain also in this case. The conditional expectation of S_{t+1} given the information up to time t satisfies

$$\begin{aligned}
& \mathbb{E}\{S_{t+1}|X_0, \dots, X_t\} \\
&= \dots \\
&= S_t + \mathbb{E}\{X_{t+1}\}
\end{aligned}$$

From this we see that (S_t) is a martingale if and only if $\mathbb{E}(X_{t+1}) = 0$ which is equivalent to the random walk being symmetric.

(c) a martingale but not a Markov chain,

Solution. Let $X_1, X_2, \dots$ be independent and uniformly distributed in $\{-1, +1\}$. Define $M_t = 10 + \sum_{s=1}^t H_s X_s$ where $H_1 = 1$ and

$$H_t = \begin{cases} 1, & X_1 = -1, \\ 5, & X_1 = +1, \end{cases}$$

for $t \geq 2$. Then M_t represents the wealth of a gambler with initial wealth 10 EUR, who bets 1 EUR for the first round, and either 1 EUR (if first round was a loss) or 5 EUR (if first round was a win) for later rounds. Because the game is fair, then by general theory of martingales (lecture notes), it follows that (M_t) is a martingale. However, (M_t) is not a Markov chain because what happened in the first round affects the transition probabilities for all future time instants.

(d) not a martingale nor a Markov chain.

Solution. Let M_t be the label of the t-th card, lifted without replacement and uniformly at random from a deck of 52 cards labeled using $k = 1, \dots, 52$. This sequence is not a Markov chain because for example $\mathbb{P}(M_3 = 1 \mid M_2 = 2) = \frac{1}{51}$ but $\mathbb{P}(M_3 = 1 \mid M_2 = 2, M_1 = 1) = 0$. This sequence is neither a martingale because for example

$$\mathbb{E}(M_2 \mid M_1 = k) = \sum_{\ell \neq k} \ell \frac{1}{51} = \sum_{\ell=1}^{52} \ell \frac{1}{51} - \frac{k}{51} = \frac{52}{51} \sum_{\ell=1}^{52} \ell \frac{1}{52} - \frac{k}{51} = \frac{52}{51} \mathbb{E}(M_1) - \frac{k}{51}$$

shows that $\mathbb{E}(M_2 \mid M_1) = \frac{52}{51}\mathbb{E}(M_1) - \frac{M_1}{51} \neq M_1$.

Homework problems

11.2 *Centered random walk.* A random sequence $(S_0, S_1, \dots)$ is defined recursively by $S_0 = x_0$ and $S_t = S_{t-1} + X_t$ for $t \geq 1$, where $x_0 \in \mathbb{R}$ and $X_1, X_2, \dots$ are independent and identically distributed with a finite mean m.

(a) Prove that the centered random walk defined by $\bar{S}_t = S_t - mt$ is a martingale with respect to information sequence $(x_0, X_1, X_2, \dots)$.

Solution. A random sequence $(M_0, M_1, \dots)$ is a martingale with respect to $(X_0, X_1, \dots)$ if

(i) $\mathbb{E}|M_t| < \infty$,

(ii) $M_t \in \sigma(X_0, \dots, X_t)$,

(iii) $\mathbb{E}(M_{t+1} \mid X_0, \dots, X_t) = M_t$.

Let us verify these. (i) follows because

$$\begin{aligned} \bar{S}_t &= x_0 + \sum_{i=1}^{t} X_i - mt \\ \Rightarrow |\bar{S}_t| &\le |x_0| + \sum_{i=1}^{t} |X_i| + |m|t \\ \Rightarrow \mathbb{E}|\bar{S}_t| &\le |x_0| + |m|t + \sum_{i=1}^{t} \mathbb{E}|X_i| \\ &= |x_0| + t(|m| + \mathbb{E}|X_1|), \end{aligned}$$

and $\mathbb{E}|X_1| < \infty$, because X_1 has a finite mean m.
(ii) follows directly from

$$\bar{S}_t = x_0 + \sum_{i=1}^{t} X_i - mt.$$

For (iii) we observe that

$$\begin{aligned} & \mathbb{E}\{\bar{S}_{t+1}|X_0,\dots,X_t\} \\ &= \mathbb{E}\{\bar{S}_t + X_{t+1} - m|X_0,\dots,X_t\} \\ \text{(pulling out known factors)} \quad &= \bar{S}_t + \mathbb{E}\{X_{t+1} - m|X_0,\dots,X_t\} \\ \text{(independence)} \quad &= \bar{S}_t + \mathbb{E}\{X_{t+1} - m\} \\ &= \bar{S}_t. \end{aligned}$$

(b) Is the centered random walk $(\bar{S}_t)_{t\in\mathbb{Z}_+}$ a martingale with respect to itself?
Solution. Yes. This follows by a general theorem on martingales (lecture notes). Alternatively, we may verify the three properties required from a martingale.
(i) is valid as in the previous computation. (ii) is trivial. For (iii) we observe that X_{t+1} is independent of $\bar{S}_0,\dots,\bar{S}_t$:

$$\begin{aligned} & \mathbb{E}\{\bar{S}_{t+1}|\bar{S}_0,\dots,\bar{S}_t\} \\ &= \mathbb{E}\{\bar{S}_t + X_{t+1} - m|\bar{S}_0,\dots,\bar{S}_t\} \\ \text{(pulling out known factors)} \quad &= \bar{S}_t + \mathbb{E}\{X_{t+1} - m|\bar{S}_0,\dots,\bar{S}_t\} \\ \text{(independence)} \quad &= \bar{S}_t + \mathbb{E}\{X_{t+1} - m\} \\ &= \bar{S}_t. \end{aligned}$$

11.3 *Optional times.* If τ_1 and τ_2 are optional times of the information sequence $(X_0, X_1, \dots)$, which of the following must be optional times as well? Justify your answers carefully based on the definition of an optional time.
Hint: The formula $1(\tau \le t) = \sum_{s=0}^{t} 1(\tau = s)$ or some of its variants may turn out useful.

(a) $T_1 = \tau_1 + 6$

Solution. This is an optional time, because

$$1(T_1 = t) \; = \; 1(\tau_1 = t - 6) \in \sigma(X_0, X_1, \dots, X_{t-6}) \; \subset \; \sigma(X_0, X_1, \dots, X_t)$$

for all $t \ge 0$.

(b) $T_2 = \max(\tau_1 - 6, 0)$

Solution. T_2 is not an optional time in general. Consider for example the case where $X_0, X_1, \dots$ are independent and uniformly distributed random variables in $\{0, 1\}$, representing coin flips, and let $\tau_1 = \min\{t \ge 1 : X_t = 1\}$ be first time instant among $\{1, 2, \dots\}$ at which we obtain heads. Then τ_1 is an optional time with respect to $X_0, X_1, \dots$ because the random indicator variable

$$1(\tau_1 = t) \; = \; \left(\prod_{s=1}^{t-1} 1(X_s = 0) \right) 1(X_t = 1),$$

is represented as deterministic function of $(X_1, \dots, X_t)$.

On the other hand, because $T_2 = 0$ if and only if $\tau_1 \le 6$, we see that the random indicator variable

$$1(T_2 = 0) \; = \; 1(\tau_1 \le 6) \; = \; 1 - 1(\tau_1 \ge 7) \; = \; 1 - \prod_{s=1}^{6} 1(X_s = 0)$$

is the indicator that the flips $X_1, \dots, X_6$ all produce tails. Obviously, there is no way to represent the indicator of the event that $X_1, \dots, X_6$ all produce tails as a deterministic function of X_0.

(c) $T_3 = \min(\tau_1, \tau_2)$

Solution. Yes, T_3 is an optional time. Observe that $T_3 = t$ if and only if either $\tau_1 = t$ and $\tau_2 \ge t$, or $\tau_2 = t$ and $\tau_1 \ge t$. Therefore,

$$\begin{aligned} 1(T_3 = t) \; &= \; 1(\tau_1 = \tau_2 = t) + 1(\tau_1 = t, \tau_2 \ge t + 1) + 1(\tau_1 \ge t + 1, \tau_2 = t) \\ &= \; 1(\tau_1 = t)1(\tau_2 = t) + 1(\tau_1 = t)1(\tau_2 \ge t + 1) + 1(\tau_1 \ge t + 1)1(\tau_2 = t). \end{aligned}$$

On the right, the indicators $1(\tau_1 = t)$ and $1(\tau_2 = t)$ belong to $\sigma(X_0, \dots, X_t)$. Note also that

$$1(\tau_1 \ge t + 1) \; = \; 1 - \sum_{s=0}^{t} 1(\tau_1 = s),$$

which shows that also $1(\tau_1 \ge t + 1) \in \sigma(X_0, \dots, X_t)$, and by symmetry, the same is true for $1(\tau_2 \ge t + 1)$. Hence each of the random indicator variables $1(\tau_1 = t)$, $1(\tau_2 = t)$, $1(\tau_1 \ge t+1)$, $1(\tau_2 \ge t+1)$ can be represented as deterministic functions of $X_0, \dots, X_t$, and therefore, the first formula provides a way to represent $1(T_3 = t)$ as a deterministic function of $X_0, \dots, X_t$.