MS-C2111 Stochastic Processes

Lecture 2

Long-term behavior of Markov chains

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Contents

Long-term behavior

What can we say about the state of a Markov chain:

$$\mathbb{P}_x(X_t = y) \;\approx\; ? \qquad \text{for } t \text{ large?}$$

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- If yes, does it depend on the initial state x?

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What can we say about the state of a Markov chain:

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- Does the distribution of X_t have a limit as $t \to \infty$?
- If yes, does it depend on the initial state x?
- If yes, how can it be computed?

Invariant distribution

An invariant distribution (or *stationary* or *equilibrium* distribution) of a transition matrix P and a corresponding Markov chain is a row vector π such that $\pi(x) \geq 0$, $\sum_x \pi(x) = 1$, and

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Hence X_t is π-distributed for all t.

Limiting distribution is an invariant distribution

Theorem

Any limiting distribution π *of a finite-state Markov chain is also an invariant distribution.*

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Hence $\pi P = \pi$.

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Hence $\pi P = \pi$. Analogously $\implies$ $\sum_{x \in S} \pi(x) = 1$. $\square$

Contents

Example 1: Brand loyalty

A smartphone market is dominated by three manufacturers. When buying a new phone, a customer chooses a phone from the same manufacturer i as the previous one with probability β_i, and otherwise the customer randomly chooses one of the other manufacturers.

Company i	1	2	3
Loyalty β_i	0.8	0.6	0.4

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Assume that all smartphones have the same lifetime regardless of the manufacturer. Will the market shares of the manufacturers stabilize in the long run?

Example 1: Brand loyalty

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Let us model the manufacturer of a typical customer's phone after the t-th purchase instant by a Markov chain $(X_0, X_1, \dots)$ with state space $S = \{1, 2, 3\}$ and transition matrix

$$P = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}$$

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Question

What happens to $\mu_t = \mu_0 P^t$ in the long run as $t \to \infty$?

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Compute powers of the transition matrix:

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$$P^{10} = \begin{bmatrix} 0.5471287 & 0.2715017 & 0.1813696 \\ 0.5430034 & 0.2745217 & 01824748 \\ 0.5441087 & 0.2737123 & 0.1821790 \end{bmatrix}$$

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The relevance of the initial state is negligible after 20 purchases.

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The relevance of the initial state is negligible after 20 purchases.
$\Longrightarrow$ Market shares seem to stabilize to

$$[0.545, 0.273, 0.182]$$

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Invariant distribution is a solution of

$$[\pi(1), \pi(2), \pi(3)] = [\pi(1), \pi(2), \pi(3)] \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.3 & 0.3 & 0.4 \end{bmatrix}$$

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Conclusion: Market shares appear to stabilize to the invariant distribution π

Example 2: Ehrenfest[1] chain

N gas particles in two containers (left, right) connected by a tiny hole.

- Each round a random particle jumps from a container to another

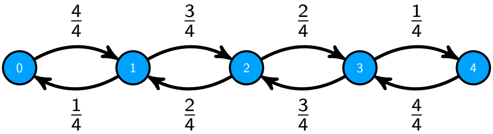
[1] Paul (1880–1993) and Tatyana (1867–1964) Ehrenfest

Example 2: Ehrenfest[1] chain

N gas particles in two containers (left, right) connected by a tiny hole.

- Each round a random particle jumps from a container to another
- X_t = number of particles in the left container after t rounds is a Markov chain with transition probabilities

$$P = \begin{bmatrix} 0 & 4/4 & 0 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 & 0 \\ 0 & 2/4 & 0 & 2/4 & 0 \\ 0 & 0 & 3/4 & 0 & 1/4 \\ 0 & 0 & 0 & 4/4 & 0 \end{bmatrix} \qquad N = 4$$



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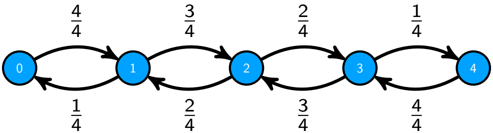
N gas particles in two containers (left, right) connected by a tiny hole.

- Each round a random particle jumps from a container to another
- X_t = number of particles in the left container after t rounds is a Markov chain with transition probabilities

$$P = \begin{bmatrix} 0 & 4/4 & 0 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 & 0 \\ 0 & 2/4 & 0 & 2/4 & 0 \\ 0 & 0 & 3/4 & 0 & 1/4 \\ 0 & 0 & 0 & 4/4 & 0 \end{bmatrix} \qquad N = 4$$



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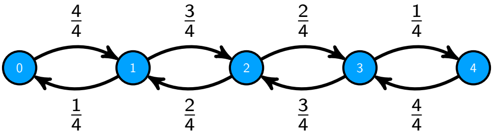

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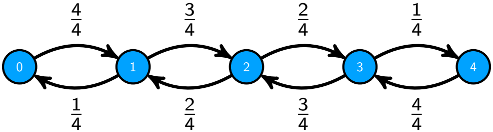

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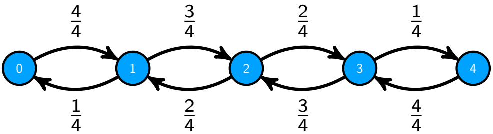

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. . . Example 2: Ehrenfest chain

Compute powers of P. . .

$$P = \frac{1}{4}\begin{bmatrix} 0 & 4 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 4 & 0 \end{bmatrix} \qquad P^2 = \left(\frac{1}{4}\right)^2 \begin{bmatrix} 4 & 0 & 12 & 0 & 0 \\ 0 & 10 & 0 & 6 & 0 \\ 2 & 0 & 12 & 0 & 2 \\ 0 & 6 & 0 & 10 & 0 \\ 0 & 0 & 12 & 0 & 4 \end{bmatrix}$$

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- $P^t(x,x)$ does not converge, so neither does μ_t

Example 3: Many limits

The transition matrix

$$P = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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This MC has many limiting distributions, depending on initial state.
For all $\alpha \in [0, 1]$, the distribution

$$\pi \;=\; \alpha[0.5, 0.5, 0] + (1 - \alpha)[0, 0, 1]$$

is an invariant distribution.

Examples: Summary

Unique limiting distribution

$$\begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.3 & 0.3 & 0.4 \end{bmatrix}$$

(Brand loyalty)

No limiting distribution

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 & 0 \\ 0 & 2/4 & 0 & 2/4 & 0 \\ 0 & 0 & 3/4 & 0 & 1/4 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

(Ehrenfest)

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Contents

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Which of the following are irreducible?

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Answer:

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Answer: 2 first are irreducible, 3rd is not.

Determining irreducibility

Theorem

For all $x \neq y$*:* $x \rightsquigarrow y$ *if and only if* $P^t(x, y) > 0$ *for some* $t \geq 1$*.*

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Proof.

If $x \rightsquigarrow y$, then there is a path $x = x_0 \to x_1 \to \cdots \to x_t = y$. Hence

$$P(x_0, x_1)P(x_1, x_2) \cdots P(x_{t-1}, x_t) > 0.$$

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Hence $\qquad P^t(x, y) = \mathbb{P}(X_t = y \mid X_0 = x)$

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If $P^t(x, y) > 0$ for some $t \geq 1$, then $\mathbb{P}(X_t = y \mid X_0 = x) > 0$, that a MC starting from x is after t time steps at possibly at y.
Hence there must be a path in the transition diagram, of length t, so $x \rightsquigarrow y$. $\square$

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Directly from previous result, according to which $x \rightsquigarrow y$ iff $P^t(x, y) > 0$ for some $t \geq 1$. $\square$

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Determine the components of the following chains:

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Third has two components $C(1) = C(2) = \{1, 2\}$ ja $C(3) = \{3\}$.

Invariant distribution

Theorem

Every irreducible finite-state MC has a unique invariant distribution π*, for which* $\pi(x) > 0$ *for all* x*.*

Proof.

Precise proof: [Dur12, Thm. 1.14] or [LPW08, Sec 1.5].

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Invariant distribution

Theorem

Every irreducible finite-state MC has a unique invariant distribution π, for which $\pi(x) > 0$ for all x.

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When do the distributions μ_t converge

$$\mu_0 P^t \to \pi \quad \text{as } t \to \infty?$$

Contents

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Which of the following are aperiodic?

$$\begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.3 & 0.3 & 0.4 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 & 0 \\ 0 & 2/4 & 0 & 2/4 & 0 \\ 0 & 0 & 3/4 & 0 & 1/4 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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Answer: For the second chain $\mathcal{T}_x = \{2, 4, 6, \ldots\}$ for all x, and every state has period 2.

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Answer: For the second chain $\mathcal{T}_x = \{2, 4, 6, \ldots\}$ for all x, and every state has period 2. The other chains are aperiodic.

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An irreducible MC is aperiodic if $P(x, x) > 0$ *for some* x.

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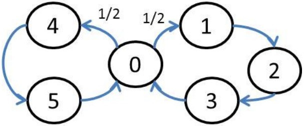
Proof.

Exercise. $\square$

Example 4

What is the period of state 0?



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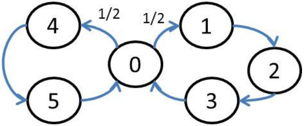

- From state 0 we move either to the left (L) or right (R) circuit

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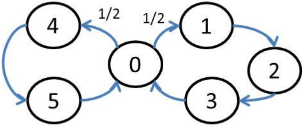

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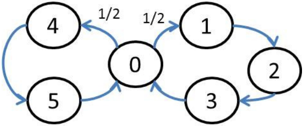

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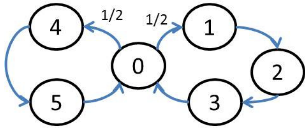

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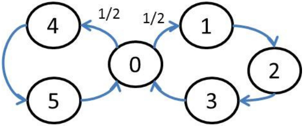

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- The MC is hence irreducible and aperiodic.

Contents

Convergence theorem

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Let P be an irreducible and aperiodic transition matrix. Then for all x and y,

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Proof.

See [LPW08, Sec. 4.3] (online book!) or [Dur12, Sec. 1.7]. □

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Find π such that $\pi P = \pi$:

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Hence $\pi_2 = \pi_1$ and further $\pi_3 = \frac{1}{2}\pi_1$.

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Example 5

Using the convergence theorem:

$$\lim_{t\to\infty} P^t = \begin{bmatrix} \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \end{bmatrix} \quad \left(\text{vrt. } P^{15} = \begin{bmatrix} 0.3984 & 0.3984 & 0.2031 \\ 0.4023 & 0.3984 & 0.1992 \\ 0.3984 & 0.4062 & 0.1953 \end{bmatrix} \right)$$

Contents

Reducible MCs

What happens for reducible MCs in the long run?

$$\mathbb{P}_x(X_t = y) \to \ ?$$

Note

- Reducible MCs have several components.
- The MC cannot exit an absorbing component.

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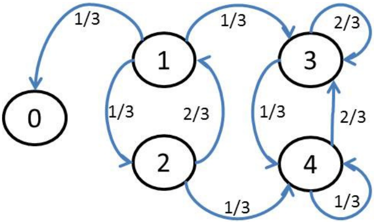
The long-term behavior of a reducible MC depends on the initial state X_0.

Example 6

$$P = \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{2}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix} \end{array}$$



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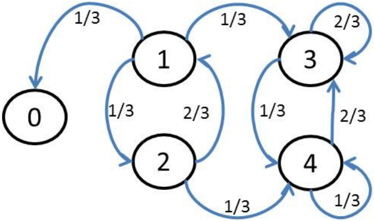

The states and components of the chain can be divided as:

- Absorbing components: $\{0\}$ ja $\{3, 4\}$
- Transient components: $\{1, 2\}$

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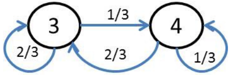

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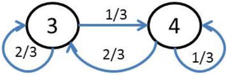

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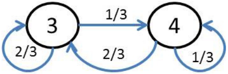

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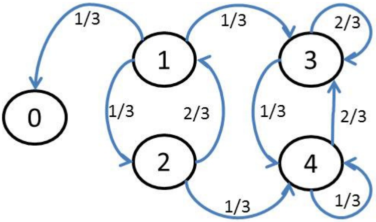

The limiting distribution depends on the initial state:

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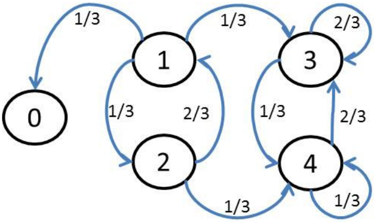

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Images used in the slides (in order of appearence)

1. Image courtesy of think4photop at FreeDigitalPhotos.net