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11 Martingales and information processes

In this exercise you become familiar with the concept of a martingale, and you learn to detect which random times are optional times with respect to a given information process.

Classroom problems

11.1 *Markov chains and martingales.* Invent (or google) an example of an integer-valued stochastic process $(X_0, X_1, \dots)$ which is

(a) a Markov chain and a martingale,

(b) a Markov chain but not a martingale,

(c) a martingale but not a Markov chain,

(d) not a martingale nor a Markov chain.

Solution. Each of these examples can be realized as certain random walks. Let $B, X_1, X_2, \dots$ be independent random numbers with values $B \in \mathbb{N}$ and $X_t \in \{-1, 1\}$. Let $S_t = \sum_{s=1}^{t} BX_s$, so in particular $S_{t+1} = S_t + BX_{t+1}$. Here B represents the amount a gambler is willing to bet in each round, X_t represents the profit per unit of money at round t, and S_t is the gamblers cumulative profit at time t. Let us check when this is (i) a Markov chain or (ii) a martingale with respect to $(B, X_1, X_2, \dots)$.

(i) The size of the increments of S are

$$\begin{aligned} |S_{t+1} - S_t| &= |BX_{t+1}| && \Big| B \geq 0 \\ &= B|X_{t+1}| && \Big| X_{t+1} \in \{-1, 1\} \\ &= B. \end{aligned}$$

This suggests that S_t is a Markov chain if and only if B is a constant. Indeed, if $B = c$ for some constant $c \in \mathbb{N}$, then we don't need to remember the past to know that the next step is also of size c. Conversely, if B can take randomly different values, then to know the size of the next step you need to also remember the size of the previous step.

To illustrate this, suppose $\mathbb{P}[B = 1] = \mathbb{P}[B = 2] = \frac{1}{2}$. By rewriting the events in the computation below in equivalent ways, we get

$$\begin{aligned} \mathbb{P}[S_3 = 2 | S_2 = 0, S_1 = 2] &= \mathbb{P}[BX_3 = 2 | BX_1 = 2, BX_2 = -2] \\ &= \mathbb{P}[X_3 = 1 \text{ and } B = 2 | X_1 = 1, X_2 = -1, B = 2] \\ &= \mathbb{P}[X_3 = 1], \\ \mathbb{P}[S_3 = 2 | S_2 = 0, S_1 = 1] &= \dots \\ &= \mathbb{P}[X_3 = 1, B = 2 | X_1 = 1, X_2 = -1, B = 1] \\ &= 0. \end{aligned}$$

This shows that the probability of $S_3 = 2$ does not only depend on S_2, but also from further history, thus S is not a Martingale.

(ii) Clearly we have $S_t \in \sigma(B, X_1, \dots, X_t)$. We also need $\mathbb{E}[|S_t|] < \infty$. This can be obtained by requiring B to be integrable, i.e. $\mathbb{E}[B] < \infty$. Indeed, then we have

$$\begin{aligned} \mathbb{E}[|S_t|] &= \mathbb{E}\Big[\Big|\sum_{s=1}^{t} BX_t\Big|\Big] \\ \text{(triangle inequality)} \quad &\le \mathbb{E}\Big[\sum_{s=1}^{t} |BX_t|\Big] \\ &= \mathbb{E}\Big[\sum_{s=1}^{t} B\Big] \\ \text{(linearity of expectation)} \quad &= t\mathbb{E}[B] < \infty. \end{aligned}$$

Finally, we need $\mathbb{E}[S_{t+1}|X_0, \dots, X_t] = S_t$:

$$\begin{aligned} \mathbb{E}[S_{t+1}|B, X_1, \dots, X_t] &= \mathbb{E}[S_t + BX_{t+1}|B, X_1, \dots, X_t] \\ \text{(linearity)} \quad &= \mathbb{E}[S_t|B, X_1, \dots, X_t] + \mathbb{E}[BX_{t+1}|B, X_1, \dots, X_t] \\ \text{(pulling out known factors)} \quad &= S_t + B\mathbb{E}[X_{t+1}|B, X_1, \dots, X_t] \\ \text{(independence)} \quad &= S_t + B\mathbb{E}[X_{t+1}]. \end{aligned}$$

From here we see that S_t is a martingale if and only if $B\mathbb{E}[X_{t+1}] = 0$. This is satisfied if $\mathbb{E}[X_t] = 0$ for every t.

From the observations from (i) and (ii) above, we can thus construct examples for each case:

(a) S_t is a Markov chain and a martingale, if $B = 1$ and X_t for each $t \in \mathbb{N}$ is chosen uniformly randomly from $\{-1, 1\}$.

(b) S_t is a Markov chain and not a martingale, if $B = 1$ and

$$\mathbb{P}[X_t = 1] = \frac{1}{3}, \quad \mathbb{P}[X_t = -1] = \frac{2}{3}$$

for each $t \in \mathbb{N}$.

(c) S_t is a Markov chain and a martingale, if $\mathbb{P}[B = 1] = \mathbb{P}[B = 2] = \frac{1}{2}$ and X_t for each $t \in \mathbb{N}$ is chosen uniformly randomly from $\{-1, 1\}$.

(d) S_t is a Markov chain and not a martingale, if $\mathbb{P}[B = 1] = \mathbb{P}[B = 2] = \frac{1}{2}$ and

$$\mathbb{P}[X_t = 1] = \frac{1}{3}, \quad \mathbb{P}[X_t = -1] = \frac{2}{3}$$

for each $t \in \mathbb{N}$.

Homework problems

11.2 *Centered random walk.* A random sequence $(S_0, S_1, \ldots)$ is defined recursively by $S_0 = x_0$ and $S_t = S_{t-1} + X_t$ for $t \geq 1$, where $x_0 \in \mathbb{R}$ and $X_1, X_2, \ldots$ are independent and identically distributed with a finite mean m.

(a) Prove that the centered random walk defined by $\bar{S}_t = S_t - mt$ is a martingale with respect to information sequence $(x_0, X_1, X_2, \ldots)$.

Solution. A random sequence $(M_0, M_1, \ldots)$ is a martingale with respect to $(X_0, X_1, \ldots)$ if

(i) $\mathbb{E}|M_t| < \infty$,

(ii) $M_t \in \sigma(X_0, \ldots, X_t)$,

(iii) $\mathbb{E}(M_{t+1} \mid X_0, \ldots, X_t) = M_t$.

Let us verify these. (i) follows because

$$\begin{aligned}
\bar{S}_t &= x_0 + \sum_{i=1}^{t} X_i - mt \\
\Rightarrow |\bar{S}_t| &\leq |x_0| + \sum_{i=1}^{t} |X_i| + |m|t \\
\Rightarrow \mathbb{E}|\bar{S}_t| &\leq |x_0| + |m|t + \sum_{i=1}^{t} \mathbb{E}|X_i| \\
&= |x_0| + t(|m| + \mathbb{E}|X_1|),
\end{aligned}$$

and $\mathbb{E}|X_1| < \infty$, because X_1 has a finite mean m.

(ii) follows directly from

$$\bar{S}_t = x_0 + \sum_{i=1}^{t} X_i - mt.$$

For (iii) we observe that

$$\begin{aligned}
& & &\mathbb{E}\{\bar{S}_{t+1} | X_0, \ldots, X_t\} \\
& &= \;&\mathbb{E}\{\bar{S}_t + X_{t+1} - m | X_0, \ldots, X_t\} \\
&\text{(pulling out known factors)} &= \;&\bar{S}_t + \mathbb{E}\{X_{t+1} - m | X_0, \ldots, X_t\} \\
&\text{(independence)} &= \;&\bar{S}_t + \mathbb{E}\{X_{t+1} - m\} \\
& &= \;&\bar{S}_t.
\end{aligned}$$

(b) Is the centered random walk $(\bar{S}_t)_{t \in \mathbb{Z}_+}$ a martingale with respect to itself?

Solution. Yes. This follows by a general theorem on martingales (lecture notes). Alternatively, we may verify the three properties required from a martingale.

(i) is valid as in the previous computation. (ii) is trivial. For (iii) we observe that X_{t+1} is independent of $\bar{S}_0, \ldots, \bar{S}_t$:

$$\begin{aligned}
& \mathbb{E}\{\bar{S}_{t+1} | \bar{S}_0, \ldots, \bar{S}_t\} \\
&= \mathbb{E}\{\bar{S}_t + X_{t+1} - m | \bar{S}_0, \ldots, \bar{S}_t\} \\
\text{(pulling out known factors)} \quad &= \bar{S}_t + \mathbb{E}\{X_{t+1} - m | \bar{S}_0, \ldots, \bar{S}_t\} \\
\text{(independence)} \quad &= \bar{S}_t + \mathbb{E}\{X_{t+1} - m\} \\
&= \bar{S}_t.
\end{aligned}$$

11.3 *Optional times.* If τ_1 and τ_2 are optional times of the information sequence $(X_0, X_1, \ldots)$, which of the following must be optional times as well? Justify your answers carefully based on the definition of an optional time.

Hint: The formula $1(\tau \le t) = \sum_{s=0}^{t} 1(\tau = s)$ or some of its variants may turn out useful.

(a) $T_1 = \tau_1 + 6$

Solution. This is an optional time, because

$$1(T_1 = t) \;=\; 1(\tau_1 = t - 6) \in \sigma(X_0, X_1, \ldots, X_{t-6}) \;\subset\; \sigma(X_0, X_1, \ldots, X_t)$$

for all $t \ge 0$.

(b) $T_2 = \max(\tau_1 - 6, 0)$

Solution. T_2 is not an optional time in general. Consider for example the case where $X_0, X_1, \ldots$ are independent and uniformly distributed random variables in $\{0, 1\}$, representing coin flips, and let $\tau_1 = \min\{t \ge 1 : X_t = 1\}$ be first time instant among $\{1, 2, \ldots\}$ at which we obtain heads. Then τ_1 is an optional time with respect to $X_0, X_1, \ldots$ because the random indicator variable

$$1(\tau_1 = t) \;=\; \left(\prod_{s=1}^{t-1} 1(X_s = 0) \right) 1(X_t = 1),$$

is represented as deterministic function of $(X_1, \ldots, X_t)$.

On the other hand, because $T_2 = 0$ if and only if $\tau_1 \le 6$, we see that the random indicator variable

$$1(T_2 = 0) \;=\; 1(\tau_1 \le 6) \;=\; 1 - 1(\tau_1 \ge 7) \;=\; 1 - \prod_{s=1}^{6} 1(X_s = 0)$$

is the indicator that the flips $X_1, \ldots, X_6$ all produce tails. Obviously, there is no way to represent the indicator of the event that $X_1, \ldots, X_6$ all produce tails as a deterministic function of X_0.

(c) $T_3 = \min(\tau_1, \tau_2)$

Solution. Yes, T_3 is an optional time. Observe that $T_3 = t$ if and only if either $\tau_1 = t$ and $\tau_2 \geq t$, or $\tau_2 = t$ and $\tau_1 \geq t$. Therefore,

$$\begin{aligned} 1(T_3 = t) &= 1(\tau_1 = \tau_2 = t) + 1(\tau_1 = t, \tau_2 \geq t+1) + 1(\tau_1 \geq t+1, \tau_2 = t) \\ &= 1(\tau_1 = t)1(\tau_2 = t) + 1(\tau_1 = t)1(\tau_2 \geq t+1) + 1(\tau_1 \geq t+1)1(\tau_2 = t). \end{aligned}$$

On the right, the indicators $1(\tau_1 = t)$ and $1(\tau_2 = t)$ belong to $\sigma(X_0, \dots, X_t)$. Note also that

$$1(\tau_1 \geq t+1) = 1 - \sum_{s=0}^{t} 1(\tau_1 = s),$$

which shows that also $1(\tau_1 \geq t+1) \in \sigma(X_0, \dots, X_t)$, and by symmetry, the same is true for $1(\tau_2 \geq t+1)$. Hence each of the random indicator variables $1(\tau_1 = t)$, $1(\tau_2 = t)$, $1(\tau_1 \geq t+1)$, $1(\tau_2 \geq t+1)$ can be represented as deterministic functions of $X_0, \dots, X_t$, and therefore, the first formula provides a way to represent $1(T_3 = t)$ as a deterministic function of $X_0, \dots, X_t$.