11 Martingales and information processes

In this exercise you become familiar with the concept of a martingale, and you learn to detect which random times are optional times with respect to a given information process.

Classroom problems

11.1 *Markov chains and martingales.* Invent (or google) an example of an integer-valued stochastic process $(X_0, X_1, \dots)$ which is

(a) a Markov chain and a martingale,

(b) a Markov chain but not a martingale,

(c) a martingale but not a Markov chain,

(d) not a martingale nor a Markov chain.

Homework problems

11.2 *Centered random walk.* A random sequence $(S_0, S_1, \dots)$ is defined recursively by $S_0 = x_0$ and $S_t = S_{t-1} + X_t$ for $t \geq 1$, where $x_0 \in \mathbb{R}$ and $X_1, X_2, \dots$ are independent and identically distributed with a finite mean m.

(a) Prove that the centered random walk defined by $\bar{S}_t = S_t - mt$ is a martingale with respect to information sequence $(x_0, X_1, X_2, \dots)$.

(b) Is the centered random walk $(\bar{S}_t)_{t \in \mathbb{Z}_+}$ a martingale with respect to itself?

11.3 *Optional times.* If τ_1 and τ_2 are optional times of the information sequence $(X_0, X_1, \dots)$, which of the following must be optional times as well? Justify your answers carefully based on the definition of an optional time.

Hint: The formula $1(\tau \leq t) = \sum_{s=0}^{t} 1(\tau = s)$ or some of its variants may turn out useful.

(a) $T_1 = \tau_1 + 6$

(b) $T_2 = \max(\tau_1 - 6, 0)$

(c) $T_3 = \min(\tau_1, \tau_2)$