Example: Irreducibility / aperiodicity? (Section 2)

Transition matrix

In[90]:=
```
P := {{0, 1, 0}, {0.5, 0, 0.5}, {1, 0, 0}};
P // MatrixForm
```

Out[91]//MatrixForm=

$$\begin{pmatrix} 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \\ 1 & 0 & 0 \end{pmatrix}$$

Irreducible?

Theorem : MC is irreducible if and only if for all states x,y that aren't equal, there exists a time t such that the matrix entry (P^t)(x,y) of the t:th power of the transition matrix is non-zero.

In[92]:=
```
MatrixPower[P, 1] // MatrixForm
```

Out[92]//MatrixForm=

$$\begin{pmatrix} 0. & 1. & 0. \\ 0.5 & 0. & 0.5 \\ 1. & 0. & 0. \end{pmatrix}$$

OK for (x,y) being (1,2), (2,1), (2,3), (3,1). How about (1,3), (3,2) ?

In[93]:=
```
MatrixPower[P, 2] // MatrixForm
```

Out[93]//MatrixForm=

$$\begin{pmatrix} 0.5 & 0. & 0.5 \\ 0.5 & 0.5 & 0. \\ 0. & 1. & 0. \end{pmatrix}$$

OK for (x,y) being (1,3), (3,2) as well. Irreducible!

Aperiodic?

Theorem : An irreducible MC is aperiodic if for all states x the diagonal element P(x,x) of the transition matrix is non-zero. (See Exercise 3.1 in the classroom problems.)

In[94]:=
```
P // MatrixForm
```

Out[94]//MatrixForm=

$$\begin{pmatrix} 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \\ 1 & 0 & 0 \end{pmatrix}$$

Hmm... The diagonal elements are zero. The Theorem is inconclusive in our case!!! We can study the transition diagram and find that our MC is indeed aperiodic.

Balance equations

In[95]:=
```
pii := {{p1, p2, p3}};
pii // MatrixForm
```

Out[96]//MatrixForm=

$$\begin{pmatrix} p1 & p2 & p3 \end{pmatrix}$$

In[97]:=
```
pii.P
```

Out[97]= {{0.5 p2 + p3, p1, 0.5 p2}}

Eigenvector equations (pii is a left eigenvector of P with eigenvalue 1):

In[98]:=
```
0.5` p2 + p3 == p1;
p1 == p2;
0.5` p2 == p3;
```

Normalization (sum of elements of the vector pii is 1, since they are probabilities (law of total probability)):

In[101]:=
```
p1 + p2 + p3 == 1;
```

Solving the equations :

In[102]:=
```
Solve[{0.5` p2 + p3 == p1, p1 == p2, 0.5` p2 == p3, p1 + p2 + p3 == 1}, {p1, p2, p3}]
```

Out[102]= {{p1 → 0.4, p2 → 0.4, p3 → 0.2}}

The unique limit- and invariant distribution is: pii = {{0.4,0.4,0.2}}

Sanity check: pii is invariant:

In[103]:=
```
{{0.4, 0.4, 0.2}}.P
```

Out[103]= {{0.4, 0.4, 0.2}}

Sanity check: pii is a probability distribution:

In[104]:=
```
0.4 + 0.4 + 0.2
```

Out[104]= 1.

Note also that pii is a row in a sufficiently high matrix power of the transition matrix: it is a limit distribution:

In[105]:=
```
MatrixPower[P, 200] // MatrixForm
```

Out[105]//MatrixForm=

$$\begin{pmatrix} 0.4 & 0.4 & 0.2 \\ 0.4 & 0.4 & 0.2 \\ 0.4 & 0.4 & 0.2 \end{pmatrix}$$