9 Time development of continuous-time Markov chains

The objective of this exercise is to practice transient distributions related to continuous-time Markov processes and to study their development over time.

Classroom problems

9.1 A one-way street has two parking spaces labeled $1, 2$ along the direction of the street. Cars arrive at time instants of a Poisson process with an average of five cars per hour. If there are vacant spaces, the driver of an arriving car parks the car to the first vacant slot, and otherwise the car drives away. Each car remains parked for an exponentially distributed random time with mean 15 min, independently of other car parking times and the arrival times.

(a) Model the state of the parking spaces as a continuous-time Markov chain. Write down the generator matrix and draw a transition diagram.

(b) What is the probability of finding a vacant parking space on the street in steady state?

(c) What is the probability that parking space 1 is vacant in steady state?

(d) A patient driver finds all parking spaces occupied on the time of arrival, and decides to wait until a space becomes vacant. What is the expected waiting time?

Homework problems

9.2 Patients requiring urgent medical care at the intensive care unit of a hospital arrive at the time instants of a Poisson process $N = (N(t))_{t\geq 0}$ with intensity $\lambda = 2.5$ (unit $\frac{1}{\text{hour}}$). Compute:

(a) The probability $\mathbb{P}(N(2) = 5)$ that exactly five patients arrive in two hours.

(b) The conditional probability $\mathbb{P}(N(5) = 8 \mid N(2) = 3)$ that a total of eight patients arrive during a five-hour work shift, if we know that three patients arrived during the first two hours of the shift.

(c) The conditional probability $\mathbb{P}(N(2) = 3 \mid N(5) = 8)$ that three patients had arrived during the first two hours of a work shift, if we know that a total of eight patients arrived during the entire five-hour shift.

9.3 *Transition matrices of a two-state chain.* A continuous-time Markov process on state space $\{1, 2\}$ has a generator matrix

$$Q = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix},$$

where $\lambda, \mu > 0$.

(a) Write out the Kolmogorov backward differential equations $\frac{\mathrm{d}}{\mathrm{d}t} P_t = QP_t$ for each matrix entry (that is, find expressions for the time derivatives of the matrix elements $\frac{\mathrm{d}}{\mathrm{d}t} P_t(i, j)$ for $i, j = 1, 2$).

(b) Use part (a) to write a differential equation for the difference $f(t) = P_t(1, 1) - P_t(2, 1)$ and solve it using an appropriate initial condition.

(c) Use (a) and (b) to solve the transition matrix P_t as a function of t.

(d) Solve the invariant distribution π of the process directly from the balance equations $\pi Q = 0$, and compare the result with the result of part (c).