9 Time development of continuous-time Markov chains

The objective of this exercise is to practice transient distributions related to continuous-time Markov processes and to study their development over time.

Classroom problems

9.1 A one-way street has two parking spaces labeled $1, 2$ along the direction of the street. Cars arrive at time instants of a Poisson process with an average of five cars per hour. If there are vacant spaces, the driver of an arriving car parks the car to the first vacant slot, and otherwise the car drives away. Each car remains parked for an exponentially distributed random time with mean 15 min, independently of other car parking times and the arrival times.

(a) Model the state of the parking spaces as a continuous-time Markov chain. Write down the generator matrix and draw a transition diagram.

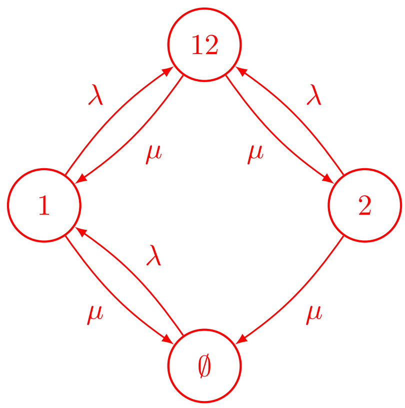
Solution. Let X_t be the set of occupied parking spaces at time t. Then the state space of X_t is $S = \{\emptyset, 1, 2, 12\}$ where for convenience we omit the brackets and write 12 instead of $\{1, 2\}$. Denote by $\lambda = 5$ the arrival rate (unit 1/h) of cars, and by $1/\mu = 0.25$ (unit h) the mean parking time of a car. Let us analyze the possible transitions of the system.

- In state $\emptyset$ both parking spaces are vacant, and the only transition from $\emptyset$ leads to state 1. The transition occurs when the next car arrives. Because cars arrive according to a Poisson process with rate λ, we see that transitions $\emptyset \to 1$ occur at rate λ.
- From state 1 there are two possible transitions: $1 \to \emptyset$ (the car parked in space 1 leaves) and $1 \to 12$ (a new car arrives and parks in space 2). The transition $1 \to \emptyset$ occurs (if no other transition occurs earlier) after an $\mathrm{Exp}(\mu)$-distributed time. The transition $1 \to 12$ occurs (if no other transition occurs earlier) after an $\mathrm{Exp}(\lambda)$-distributed random time. Therefore the corresponding transition rates are μ and λ.
- From state 2 there are two possible transitions: $2 \to \emptyset$ (the car parked in space 2 leaves) and $2 \to 12$ (a new car arrives and parks to space 1). As above, we deduce that the corresponding transition rates are μ and λ.
- From state 12 there are two possible transitions, $12 \to 2$ (the car parked in space 1 leaves) and $12 \to 2$ (the car parked in space 2 leaves). Both transitions have rate μ.

The above transition rates are the nonzero offdiagonal entries of the generator matrix. Because every generator matrix has zero row sums, we can also fill in the diagonal entries. As a result, the generator matrix equals

$$Q = \begin{bmatrix} -\lambda & \lambda & 0 & 0 \\ \mu & -\lambda-\mu & 0 & \lambda \\ \mu & 0 & -\lambda-\mu & \lambda \\ 0 & \mu & \mu & -2\mu \end{bmatrix} = \begin{bmatrix} -5 & 5 & 0 & 0 \\ 0.5 & -5.5 & 0 & 5 \\ 0.5 & 0 & -5.5 & 5 \\ 0 & 0.5 & 0.5 & -1 \end{bmatrix}$$

A corresponding transition diagram is drawn below.



(b) What is the probability of finding a vacant parking space on the street in steady state?

Solution. The balance equations $\pi Q = 0$ for the invariant distribution $\pi = [\pi_\emptyset, \pi_1, \pi_2, \pi_{12}]$ can now be written as

$$\begin{aligned} \lambda\pi_\emptyset &= \mu\pi_1 + \mu\pi_2, \\ (\lambda+\mu)\pi_1 &= \lambda\pi_\emptyset + \mu\pi_{12}, \\ (\lambda+\mu)\pi_2 &= \mu\pi_{12}, \\ 2\mu\pi_{12} &= \lambda\pi_1 + \lambda\pi_2. \end{aligned}$$

Above we have four equations for four unknowns, but as usual in Markov chains, one of the equations is redundant. However, we always have the normalizing equation

$$\pi_\emptyset + \pi_1 + \pi_2 + \pi_{12} = 1.$$

By denoting $\rho = \lambda/\mu$, the balance equations can be rewritten as

$$\begin{aligned} \rho\pi_\emptyset &= \pi_1 + \pi_2, \\ (1+\rho)\pi_1 &= \rho\pi_\emptyset + \pi_{12}, \\ (1+\rho)\pi_2 &= \pi_{12}, \\ 2\pi_{12} &= \rho\pi_1 + \rho\pi_2. \end{aligned}$$

From the 1st and 4th balance equation we find that $\rho\pi_\emptyset = \pi_1 + \pi_2 = (2/\rho)\pi_{12}$, so that

$$\pi_{12} = (\rho^2/2)\pi_\emptyset.$$

With this and the 3rd balance equation we find that

$$\pi_2 \;=\; \frac{1}{1+\rho}\pi_{12} \;=\; \frac{\rho^2/2}{1+\rho}\pi_\emptyset.$$

With this and the 2nd balance equation we find that

$$\pi_1 \;=\; \rho\pi_\emptyset - \pi_2 \;=\; \frac{\rho^2/2+\rho}{1+\rho}\pi_\emptyset.$$

Finally, from the normalizing condition

$$1 \;=\; \pi_\emptyset + \pi_1 + \pi_2 + \pi_{12} \;=\; \pi_\emptyset\left(1 + \frac{\rho^2/2+\rho}{1+\rho} + \frac{\rho^2/2}{1+\rho} + \rho^2/2\right),$$

it follows that

$$\pi_\emptyset \;=\; \left(1 + \frac{\rho^2/2+\rho}{1+\rho} + \frac{\rho^2/2}{1+\rho} + (\rho^2/2)\right)^{-1}.$$

By simplifying the above expressions and substituting the numerical value $\rho = \lambda/\mu = 1.25$, the invariant probabilities are

$$\begin{aligned}
\pi_\emptyset &= \frac{2+2\rho}{2+4\rho+3\rho^2+\rho^3} = 0.330,\\
\pi_1 &= \frac{2\rho+\rho^2}{2+4\rho+3\rho^2+\rho^3} = 0.298,\\
\pi_2 &= \frac{\rho^2}{2+4\rho+3\rho^2+\rho^3} = 0.115,\\
\pi_{12} &= \frac{\rho^2+\rho^3}{2+4\rho+3\rho^2+\rho^3} = 0.258.
\end{aligned}$$

Hence in steady state,

$$\mathbb{P}(\text{vacant}) \;=\; 1 - \pi_{12} \;=\; 0.742.$$

Below is an R-code for solving the invariant distribution numerically.

```
# R-code for computing the invariant distribution
# The last column of Q is replaced by a column of ones, to replace one of the redundant
# balance equations by the normalizing equation that the entries of pi should sum into one.
la <- 5.0
mu <- 4.0
Q <- matrix(0,4,4)
Q[1,2] <- Q[2,4] <- Q[3,4] <- la
Q[2,1] <- Q[3,1] <- Q[4,2] <- Q[4,3] <- mu
Q <- Q - diag(rowSums(Q))
n <- nrow(Q)
pi <- solve(t(cbind(Q[,1:n-1],rep(1,n))), c(rep(0,n-1),1))
```

Additional information. The above analysis shows that the invariant probabilities depend on the model parameters λ and μ only via the ratio $\rho = \lambda/\mu$. This ratio can interpreted as the load (arrival rate / service rate) in a queueing system where "server" = parking space. A general feature of many queueing systems is that many invariant properties only depend on the load.

(c) What is the probability that parking space 1 is vacant in steady state?

Solution. From the invariant distribution, this is $1 - \pi_1 - \pi_{12} = 0.444$.

(d) A patient driver finds all parking spaces occupied on the time of arrival, and decides to wait until a space becomes vacant. What is the expected waiting time?

Solution. From state 12 there are two transitions $12 \to 1$ and $12 \to 2$, both having transition rate μ, and both leading to a space becoming vacant. The total transition rate out of state 12 hence equals 2μ, and therefore the time that the chain spends in state 12 is $\text{Exp}(2\mu)$-distributed. This time has expectation $\frac{1}{2\mu} = \frac{1}{8}$ (unit h). Hence the expected waiting time equals 0.125 h = 7.5 min = 7 min 30 sec.

Solution.[Alternative way] Denote by T_i the remaining time the the car in parking space i still remains parked, $i = 1, 2$. Then the time that the patient driver still needs to wait equals $T = \min\{T_1, T_2\}$. Because the total parking times of the cars are $\text{Exp}(\mu)$-distributed, so are the remaining parking times, by the memoryless property. Hence T_1 and T_2 are independent and $\text{Exp}(\mu)$-distributed, and we recall the general result that the minimum of T_1 and T_2 is then also exponentially distributed with rate parameter $\mu + \mu$. Hence waiting time T is $\text{Exp}(2\mu)$-distributed, and the expected waiting time equals $\frac{1}{2\mu} = \frac{1}{8}$ (unit h).

Homework problems

9.2 Patients requiring urgent medical care at the intensive care unit of a hospital arrive at the time instants of a Poisson process $N = (N(t))_{t\geq 0}$ with intensity $\lambda = 2.5$ (unit $\frac{1}{\text{hour}}$). Compute:

(a) The probability $\mathbb{P}(N(2) = 5)$ that exactly five patients arrive in two hours.

Solution. The number of events (that is, jumps) of a Poisson process in time interval $(t_1, t_2]$ is Poisson distributed with mean $\lambda(t_2 - t_2)$. Especially, the number of events on time interval $(0, t]$ satisfies

$$\mathbb{P}(N(t) = k) \;=\; \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad k = 0, 1, 2, \ldots$$

Hence $N(2)$ is Poisson distributed with mean $2\lambda = 5$, and

$$\mathbb{P}(N(2) = 5) \;=\; e^{-5}\frac{5^5}{5!} \;\approx\; 0.175$$

(b) The conditional probability $\mathbb{P}(N(5) = 8 \,|\, N(2) = 3)$ that a total of eight patients arrive during a five-hour work shift, if we know that three patients arrived during the first two hours of the shift.

Solution. By applying the fact that a Poisson process has independent increments, and that $N(5) - N(2) =_{\text{st}} N(3) - N(0) = N(3)$, and that $N(3)$ is Poisson-distributed with mean $3\lambda = 7.5$, it follows that

$$\begin{aligned}
\mathbb{P}(N(5) = 8 \,|\, N(2) = 3) &= \mathbb{P}(N(5) - N(2) = 5 \,|\, N(2) = 3) \\
&= \mathbb{P}(N(5) - N(2) = 5) \\
&= \mathbb{P}(N(3) = 5) \\
&= e^{-7.5}\frac{7.5^5}{5!} \\
&\approx 0.109.
\end{aligned}$$

(c) The conditional probability $\mathbb{P}(N(2) = 3 \,|\, N(5) = 8)$ that three patients had arrived during the first two hours of a work shift, if we know that a total of eight patients arrived during the entire five-hour shift.

Solution. We can apply the Bayes' formula

$$\mathbb{P}(A \,|\, B) \;=\; \frac{\mathbb{P}(A)}{\mathbb{P}(B)} \mathbb{P}(B \,|\, A)$$

and then result of (b) to conclude that

$$\begin{aligned}
\mathbb{P}(N(2) = 3 \,|\, N(5) = 8) &= \frac{\mathbb{P}(N(2) = 3)}{\mathbb{P}(N(5) = 8)} \mathbb{P}(N(5) = 8 \,|\, N(2) = 3) \\
&= \frac{\mathbb{P}(N(2) = 3)}{\mathbb{P}(N(5) = 8)} \mathbb{P}(N(3) = 5).
\end{aligned}$$

Because the random variables $N(2), N(3), N(5)$ are Poisson distributed with means $2\lambda = 5$, $3\lambda = 7.5$, and $5\lambda = 12.5$, it follows that

$$\begin{aligned}
\mathbb{P}(N(2) = 3 \mid N(5) = 8) &= \frac{\mathbb{P}(N(2) = 3)}{\mathbb{P}(N(5) = 8)} \mathbb{P}(N(3) = 5) \\
&= \frac{e^{-5}\frac{5^3}{3!}}{e^{-12.5}\frac{12.5^8}{8!}} e^{-7.5}\frac{7.5^5}{5!} \\
&\approx 0.279.
\end{aligned}$$

9.3 *Transition matrices of a two-state chain.* A continuous-time Markov process on state space $\{1, 2\}$ has a generator matrix

$$Q = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix},$$

where $\lambda, \mu > 0$.

(a) Write out the Kolmogorov backward differential equations $\frac{\mathrm{d}}{\mathrm{d}t} P_t = Q P_t$ for each matrix entry (that is, find expressions for the time derivatives of the matrix elements $\frac{\mathrm{d}}{\mathrm{d}t} P_t(i, j)$ for $i, j = 1, 2$).

Solution.

$$\begin{aligned}
\frac{d}{dt} P_t(1, 1) &= -\lambda P_t(1, 1) + \lambda P_t(2, 1), \\
\frac{d}{dt} P_t(1, 2) &= -\lambda P_t(1, 2) + \lambda P_t(2, 2), \\
\frac{d}{dt} P_t(2, 1) &= \mu P_t(1, 1) - \mu P_t(2, 1), \\
\frac{d}{dt} P_t(2, 2) &= \mu P_t(1, 2) - \mu P_t(2, 2).
\end{aligned}$$

(b) Use part (a) to write a differential equation for the difference $f(t) = P_t(1, 1) - P_t(2, 1)$ and solve it using an appropriate initial condition.

Solution. By subtracting the third equation in (a) from the first equation in (a) we find that

$$f'(t) = -(\lambda + \mu) f(t).$$

The initial condition $P_0 = I$ implies that $f(0) = 1$. Hence the solution of the above differential equation is

$$f(t) = e^{-(\lambda+\mu)t}.$$

(c) Use (a) and (b) to solve the transition matrix P_t as a function of t.

Solution. The first equation in (a) yields $P_t'(1,1) = -\lambda f(t)$ with initial condition $P_0(1,1) = 1$. Integration shows that

$$P_t(1,1) \;=\; P_0(1,1) + \int_0^t -\lambda f(s)\,ds \;=\; \cdots \;=\; \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}e^{-(\lambda+\mu)t}.$$

Analogously, by applying the third equation in (a) we see that $P_t(2,1)$ satisfies $P_t'(2,1) = \mu f(t)$ with initial condition $P_0(2,1) = 0$. Therefore,

$$P_t(2,1) \;=\; P_0(2,1) + \int_0^t \mu f(s)\,ds \;=\; \cdots \;=\; \frac{\mu}{\lambda+\mu}(1 - e^{-(\lambda+\mu)t}).$$

The remaining entries of P_t are obtained by recalling that the row sums of any transition matrix are one (or by repeating part (b) for the difference $g(t) = P_t(1,2) - P_t(2,2)$):

$$P_t(1,2) \;=\; 1 - P_t(1,1) \;=\; \frac{\lambda}{\lambda+\mu}(1 - e^{-(\lambda+\mu)t})$$

and

$$P_t(2,2) \;=\; 1 - P_t(2,1) \;=\; \frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu}e^{-(\lambda+\mu)t}$$

(d) Solve the invariant distribution π of the process directly from the balance equations $\pi Q = 0$, and compare the result with the result of part (c).

Solution. The balance equation $\pi Q = 0$ can be written as

$$\begin{aligned} \pi(1)Q(1,1) + \pi(2)Q(1,2) &= 0, \\ \pi(1)Q(1,2) + \pi(2)Q(2,2) &= 0, \end{aligned}$$

that is,

$$\begin{aligned} \pi(1)(-\lambda) + \pi(2)\mu &= 0, \\ \pi(1)\lambda + \pi(2)(-\mu) &= 0. \end{aligned}$$

The above equations are equivalent to

$$\lambda\pi(1) \;=\; \mu\pi(2).$$

Together with the normalizing condition $\pi(1) + \pi(2) = 1$ we find that the invariant distribution equals

$$\pi \;=\; \begin{bmatrix} \pi(1) & \pi(2) \end{bmatrix} \;=\; \begin{bmatrix} \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} \end{bmatrix}.$$

We can compare this to the result in (c) and note that

$$P_t \;=\; \begin{bmatrix} P_t(1,1) & P_t(1,2) \\ P_t(2,1) & P_t(2,2) \end{bmatrix} \;=\; \begin{bmatrix} \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}e^{-(\lambda+\mu)t} & \frac{\lambda}{\lambda+\mu}(1 - e^{-(\lambda+\mu)t}) \\ \frac{\mu}{\lambda+\mu}(1 - e^{-(\lambda+\mu)t}) & \frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu}e^{-(\lambda+\mu)t} \end{bmatrix} \;\to\; \begin{bmatrix} \pi(1) & \pi(2) \\ \pi(1) & \pi(2) \end{bmatrix}$$

which is in accordance with the general theory of irreducible finite-state Markov chains. We may also note that the convergence to the invariant distribution occurs exponentially fast.