Recursion equation for Random Walk hitting time (Section 4.3)

Consider the first hitting time for the Random Walk on {0,1,...,M} to the target winning level {M}. We can solve for it using Theorem 4.4. What is the behavior when M grows?

Assume first that q is not 1/2. The Recursion Equation is homogeneous, and its solution has the form:

In[1]:=
```
RSolveValue[{q * h[x + 1] - h[x] + (1 - q) * h[x - 1] == 0, h[0] == 0, h[M] == 1}, h[x], x]
```

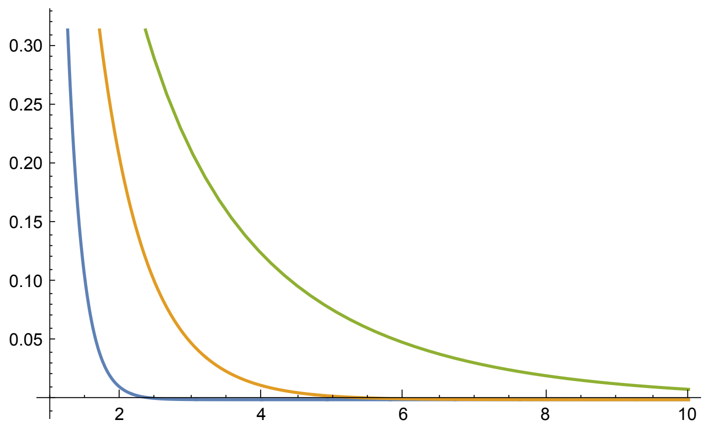
Out[1]=
$$\frac{-1+\left(\frac{1-q}{q}\right)^x}{-1+\left(\frac{1-q}{q}\right)^M}$$

Let' s plot with the value x = 1:

In[2]:=
$$F[M\_, q\_] := \left(\frac{-1+\left(\frac{1-q}{q}\right)^x}{-1+\left(\frac{1-q}{q}\right)^M}\right) /. \{x \to 1\};$$

Plotting for some q’s less than 1/2:

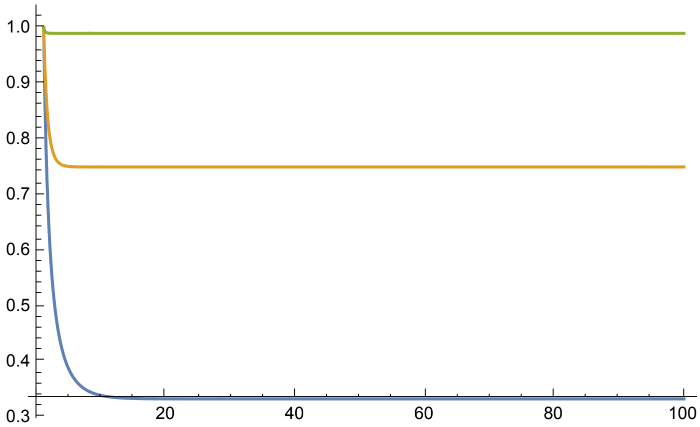
In[3]:=
```
Plot[{F[M, .01], F[M, .2], F[M, .4]}, {M, 1, 10}]
```

Out[3]=


Plotting for some q' s greater than 1/2:

In[4]:=
```
Plot[{F[M, .6], F[M, .8], F[M, .99]}, {M, 1, 100}]
```

Out[4]=



Example limits:

In[5]:=
```
Limit[F[M, .1], M → Infinity]
Limit[F[M, .45], M → Infinity]
Limit[F[M, .55], M → Infinity]
Limit[F[M, .9], M → Infinity]
```

Out[5]= 0.

Out[6]= 0.

Out[7]= 0.181818

Out[8]= 0.888889

We see that the limit (as M grows) is zero if q < 1/2 and is non-zero if q > 1/2.

Assume then that q = 1/2. The Recursion Equation is homogeneous, and its solution has the form:

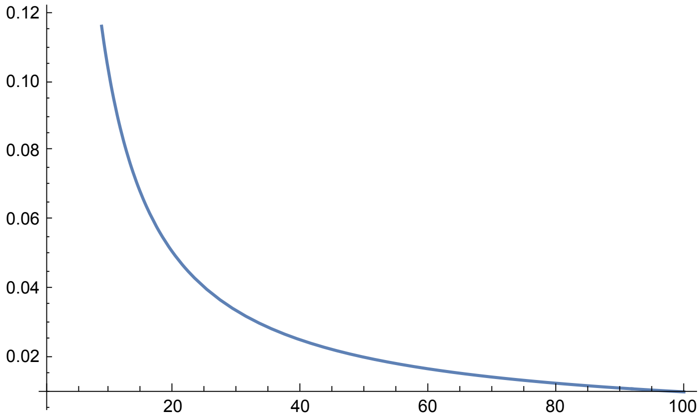
In[9]:=
```
RSolveValue[
 {(1 / 2) * h[x + 1] - h[x] + (1 - (1 / 2)) * h[x - 1] == 0, h[0] == 0, h[M] == 1}, h[x], x]
```

Out[9]= $\frac{x}{M}$

Let' s plot with the value x = 1:

In[10]:= G[M_] := $\frac{x}{M}$ /. {x → 1};

In[11]:=
```
Plot[G[M], {M, 1, 100}]
```

Out[11]= 

We see that the limit (as M grows) is zero also if q = 1/2.