

Apply Rice's theorem for all parts

(i) Denote the class of all regular languages by REG.

$$RL = \{L \in REG \mid L \in REG\}$$

a) Non-triviality:

By (i), there exists a language $L \notin REG$

$$\Rightarrow RL \neq REG$$

b) On the other hand, there exists at least one regular language $L \in REG$

$$\Rightarrow RL \neq \emptyset$$

$$RL \neq \emptyset \wedge RL \neq REG \Rightarrow RL \text{ is non-trivial}$$

• A semantic property is *trivial* if

- $S = \emptyset$ (no machine has this property) or
- $S = REG$ (all machines have this property)

• A property S is *decidable* if the language

$$codes(S) = \{ \langle M \rangle \mid M \text{ is a Turing machine and } L(M) \in S \}$$

is decidable.

• In other words: A semantic property is decidable if one can algorithmically decide whether a given Turing machine has the property.

Theorem 11.1 (Rice 1953)
All non-trivial semantic properties of Turing machines are undecidable.

*equivalently "a given computer program matches the specification"

Example:

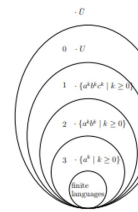
- Let us consider the *non-emptiness problem* for Turing machines from Lecture 10:
Given a Turing machine M ,
Does the machine accept any strings?
- The corresponding semantic property is $NE = \{L \in REG \mid L \neq \emptyset\}$.
- The property is non-trivial because:
 - $NE \neq \emptyset$ (witness any semi-decidable language $L \neq \emptyset$)
 - $NE \subsetneq REG$ (since $\emptyset \in REG \setminus NE$)
- Thus by Rice's theorem, the language

$$codes(NE) = \{ \langle M \rangle \mid M \text{ is a Turing machine and } L(M) \in NE \}$$

$$= \{ \langle M \rangle \mid M \text{ is a Turing machine and } L(M) \neq \emptyset \}$$

is undecidable. (Note that this is precisely the result in Lemma 10.5.)

11.4 Recap: The Chomsky hierarchy



A classification of grammars, languages generated by grammars and recogniser automata classes:

Type-0: unrestricted grammars / semi-decidable languages / Turing machines

Type-1: context-sensitive grammars / context-sensitive languages / linear bounded automata

Type-2: context-free grammars / context-free languages / pushdown automata

Type-3: right and left linear grammars / regular languages / finite automata

C11.1 Prove that the following problems are undecidable:

- Given a Turing machine M , does M recognise a regular language?
- Given a Turing machine M , does M accept exactly one input string?
- Given Turing machines M_1 and M_2 , is $L(M_1) \cap L(M_2) = \emptyset$?

$$(ii) OS = \{L \in REG \mid |L| = 1\}$$

a) For example, construct TM over $\Sigma = \{a, b\}$

so that it only accepts a

$$\Rightarrow \exists TM, \text{ so that } |L(TM)| = 1$$

$$\Rightarrow OS \neq \emptyset$$

b) Construct a TM st. it accepts the strings a and b

$$\Rightarrow \exists TM, \text{ so that } |L(M)| = 2 \neq 1$$

$$\Rightarrow OS \neq REG$$

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is decidable.

• In other words: A semantic property is decidable if one can algorithmically decide whether a given Turing machine has the property.

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Example:

- Let us consider the *non-emptiness problem* for Turing machines from Lecture 10:
Given a Turing machine M ,
Does the machine accept any strings?
- The corresponding semantic property is $NE = \{L \in REG \mid L \neq \emptyset\}$.
- The property is non-trivial because:
 - $NE \neq \emptyset$ (witness any semi-decidable language $L \neq \emptyset$)
 - $NE \subsetneq REG$ (since $\emptyset \in REG \setminus NE$)
- Thus by Rice's theorem, the language

$$codes(NE) = \{ \langle M \rangle \mid M \text{ is a Turing machine and } L(M) \in NE \}$$

$$= \{ \langle M \rangle \mid M \text{ is a Turing machine and } L(M) \neq \emptyset \}$$

is undecidable. (Note that this is precisely the result in Lemma 10.5.)

$$(iii) OS = \{L \in REG \mid L = L(M_1) \cap L(M_2)\}$$

a) For example, $L(M_1) = \{a\}$
 $L(M_2) = \{b\}$ } Non-trivial semantic property

$$b) L(M_1) = \{a, b\}$$

$$L(M_2) = \{a\}$$