Aalto University
School of Science

CS-C2160 Theory of Computation

Lecture 11: Rice's Theorem, General Grammars

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Topics:

- Rice's Theorem
- Unrestricted grammars
- ... and their relationship to Turing machines
- Context-sensitive grammars
- * A glimpse beyond: Computational complexity

Recap

- *Church–Turing thesis:* Intuitive notion of algorithms $\equiv$ Turing machines.
- Formal language $\equiv$ Yes/No decision problem.
- A language is semi-decidable (also called recursively enumerable) if it can be recognised by some Turing machine.
- A language is decidable (also called recursive) if it can be recognised by some machine that halts on all inputs.
- A language is undecidable if it is not decidable.
- An undecidable language may still be semi-decidable.

- The "acceptance" decision problem for Turing machines is

  Given a Turing machine M and a string w.
  Does M accept w?

- The formal language representing this is the universal language

$$U = \{c_M w \mid M \text{ is a TM and } M \text{ accepts } w\}.$$

- The language U is semi-decidable but not decidable.

Rice's Theorem

11.1 Rice's theorem

- Rice's Theorem states that *all* decision problems concerning the languages recognised by Turing machines[1] are undecidable.
- Let us denote the family of all semi-decidable (i.e. recursively enumerable) languages by **RE**.
- A *semantic property*[2] $\mathbf{S}$ of Turing machines is any family of semi-decidable languages, i.e. $\mathbf{S} \subseteq$ **RE**.
- A machine M *has property* $\mathbf{S}$ if $\mathcal{L}(M) \in \mathbf{S}$.
- Examples of semantic properties:
  - **NE** $= \{L \subseteq \{0,1\}^* \mid L \neq \emptyset\}$
  - **ALLSTRINGS** $= \{L \subseteq \{0,1\}^* \mid L = \{0,1\}^*\} = \{\{0,1\}^*\}$
  - **EVEN** $= \{L \subseteq \{0,1\}^* \mid |x|$ is even for all $x \in L\}$
  - **ONLY**$_w = \{L \subseteq \{0,1\}^* \mid x \in L \Leftrightarrow x = w\} = \{\{w\}\}$
  - **EMPTYSET** $= \{L \subseteq \{0,1\}^* \mid L = \emptyset\} = \{\emptyset\}$

[1] i.e. the input-output behaviours of computer programs
[2] or "specification"

- A semantic property is *trivial* if
  - $\mathbf{S} = \emptyset$ (no machine has this property) or
  - $\mathbf{S} = \mathbf{RE}$ (all machines have this property)
- A property $\mathbf{S}$ is *decidable* if the language

  $$\text{codes}(\mathbf{S}) = \{c_M \mid M \text{ is a Turing machine and } \mathcal{L}(M) \in \mathbf{S}\}$$

  is decidable.
- In other words: A semantic property is decidable if one can algorithmically decide whether a given Turing machine has the property.[3]

Theorem 11.1 (Rice 1953)

All non-trivial semantic properties of Turing machines are undecidable.

[3] equivalently "a given computer program matches the specification"

Example:

- Let us consider the *non-emptiness problem* for Turing machines from Lecture 10:

  Given a Turing machine M.

  Does the machine accept any strings?

- The corresponding semantic property is $\textbf{NE} = \{L \in \textbf{RE} \mid L \neq \emptyset\}$.
- The property is non-trivial because:
  - $\textbf{NE} \neq \emptyset$ (witness any semi-decidable language $L \neq \emptyset$)
  - $\textbf{NE} \subsetneq \textbf{RE}$ (since $\emptyset \in \textbf{RE} \setminus \textbf{NE}$)
- Thus by Rice's theorem, the language

$$\begin{aligned}\text{codes}(\textbf{NE}) &= \{c_M \mid M \text{ is a Turing machine and } \mathcal{L}(M) \in \textbf{NE}\} \\ &= \{c_M \mid M \text{ is a Turing machine and } \mathcal{L}(M) \neq \emptyset\}\end{aligned}$$

is undecidable. (Note that this is precisely the result in Lemma 10.5.)

Theorem 11.1

All non-trivial semantic properties of Turing machines are undecidable.

Proof

- A simple generalisation of the proof of Lemma 10.5.
- Let $\mathbf{S}$ be any non-trivial semantic property.
- We can assume that $\emptyset \notin \mathbf{S}$; in other words, machines that recognise the empty language do not have the property.[a]
- As $\mathbf{S}$ is non-trivial, there is a Turing machine $M_{\mathbf{S}}$ that has the property $\mathbf{S}$, i.e. one for which $\mathcal{L}(M_{\mathbf{S}}) \neq \emptyset$ and $\mathcal{L}(M_{\mathbf{S}}) \in \mathbf{S}$ hold.

[a] If $\emptyset \in \mathbf{S}$, we can first show that the property $\bar{\mathbf{S}} = \textbf{RE} \setminus \mathbf{S}$ is undecidable and then conclude that also $\mathbf{S}$ is undecidable; this is because codes$(\bar{\mathbf{S}}) = \{0,1\}^* \setminus$ codes$(\mathbf{S})$.

- We now prove that codes$(\mathbf{S})$ is undecidable by reducing the undecidable language U to it.
- Let (M,w) be any instance of the Turing machine acceptance problem, encoded as the string $c_M w$.
- From input $c_M w$ construct (the code for) a Turing machine M^w that on any input string x works as follows:
  - First run machine M on string w, and then:
    - if M accepts w, run $M_{\mathbf{S}}$ on x
    - if M rejects w (or doesn't halt), reject x (or don't halt)
- Now M^w recognises the language

$$\mathcal{L}(M^w) = \begin{cases} \mathcal{L}(M_{\mathbf{S}}) & \text{if } w \in \mathcal{L}(M) \\ \emptyset & \text{if } w \notin \mathcal{L}(M) \end{cases}$$

- Thus M accepts w if and only if M^w has the property $\mathbf{S}$. That is, $c_M w \in U$ if and only if $c_{M^w} \in$ codes$(\mathbf{S})$.
- Therefore, codes$(\mathbf{S})$ is an undecidable language.

General Grammars

11.2 Unrestricted grammars

- A generalisation of context-free grammars.
- The left-hand sides of rules can now include multiple symbols.
- As will be shown, can generate all semi-decidable languages.

Definition 11.1

An *unrestricted grammar* is a quadruple

$$G = (V, \Sigma, R, S),$$

where

- V is a finite set of *variables*;
- Σ is a finite set, disjoint from V, of *terminals*;
- $R \subseteq (V \cup \Sigma)^+ \times (V \cup \Sigma)^*$ is a finite set of *rules* (also called productions), where $(V \cup \Sigma)^+ = (V \cup \Sigma)^* \setminus \{\varepsilon\}$;
- $S \in V$ is the *start variable*.

A rule $(\omega, \omega') \in R$ is usually written as $\omega \to \omega'$.

- A string $\gamma \in (V \cup \Sigma)^*$ *yields* a string $\gamma' \in (V \cup \Sigma)^*$ in the grammar G, denoted by

$$\gamma \underset{G}{\Rightarrow} \gamma'$$

if

  - the grammar contains a rule $\omega \to \omega'$ such that $\gamma = \alpha\omega\beta$ and $\gamma' = \alpha\omega'\beta$ for some $\alpha, \beta \in (V \cup \Sigma)^*$.

- A string $\gamma \in (V \cup \Sigma)^*$ *derives* a string $\gamma' \in (V \cup \Sigma)^*$ in the grammar G, denoted by

$$\gamma \underset{G}{\Rightarrow}^* \gamma'$$

if there is a sequence of strings $\gamma_0, \gamma_1, \ldots, \gamma_n$ for some $n \geq 0$ such that

$$\gamma = \gamma_0, \qquad \gamma_0 \underset{G}{\Rightarrow} \gamma_1 \underset{G}{\Rightarrow} \ldots \underset{G}{\Rightarrow} \gamma_n, \qquad \gamma_n = \gamma'.$$

- If the grammar G is clear from the context, we can simply write $\gamma \Rightarrow \gamma'$ and $\gamma \Rightarrow^* \gamma'$ instead of $\gamma \underset{G}{\Rightarrow} \gamma'$ and $\gamma \underset{G}{\Rightarrow}^* \gamma'$, respectively.

Example:

An unrestricted grammar for the non-context-free language $\{a^k b^k c^k \mid k \geq 0\}$:

$$
\begin{array}{rcl}
S & \to & LT \mid \varepsilon \\
T & \to & ABCT \mid ABC \\
BA & \to & AB \\
CB & \to & BC \\
CA & \to & AC
\end{array}
\qquad
\begin{array}{rcl}
LA & \to & a \\
aA & \to & aa \\
aB & \to & ab \\
bB & \to & bb \\
bC & \to & bc \\
cC & \to & cc
\end{array}
$$

A derivation of string $aabbcc$ in the grammar:

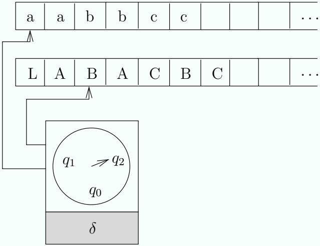
$$
\begin{array}{llllllll}
\underline{S} & \Rightarrow & L\underline{T} \Rightarrow LABC\underline{T} & \Rightarrow & LAB\underline{CA}BC & \Rightarrow & LA\underline{BA}CBC \\
 & \Rightarrow & LAAB\underline{CB}C & \Rightarrow & \underline{LA}ABBCC & \Rightarrow & \underline{aA}BBCC \\
 & \Rightarrow & a\underline{aB}BCC & \Rightarrow & aa\underline{bB}CC & \Rightarrow & aab\underline{bC}C \\
 & \Rightarrow & aabb\underline{cC} & \Rightarrow & aabbcc & &
\end{array}
$$

Theorem 11.2

If a language L can be generated with an unrestricted grammar, then it can be recognised with a Turing machine.

Proof

Let $G = (V, \Sigma, R, S)$ be an unrestricted grammar generating language L. We can design a two-tape nondeterministic Turing machine M_G recognising L as follows:



- On tape 1 the machine stores a copy of the input string.
- Tape 2 holds the current string that the machine tries to rewrite to match the one on tape 1.
- In the beginning, the machine writes the start variable S on tape 2.

The computation of machine M_G is composed of stages. In each stage, the machine:

1. Moves the read/write-head of tape 2 *nondeterministically* to *some* position on the tape.
2. Chooses *nondeterministically* a rule in G that it tries to apply at the selected position. (The rules of G are encoded in the transitions of M_G.)
3. If the left-hand side of the chosen rule matches the symbols on the tape, M_G rewrites these symbols with the ones in the right-hand side of the rule. Otherwise M_G rejects.
4. At the end of the stage, M_G compares the strings on tapes 1 and 2. If they are the same, the machine acceps and halts. Otherwise, the machine executes the next stage (loops back to step 1).

Theorem 11.3

If a language L can be recognised with a Turing machine, then it can be generated with an unrestricted grammar.

Proof

Let $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{acc}}, q_{\text{rej}})$ be a (deterministic one-tape) Turing machine recognising language L. We can design an unrestricted grammar G_M generating L based on the following idea.

- The variables of G_M include (among others) symbols for all the states $q \in Q$ of M.
- A configuration $(q, u\underline{a}v)$ of M will be represented as a string $[uqav]$.
- Based on the transitions of M, G_M will have rules that ensure

$$[uqav] \underset{G_M}{\Rightarrow} [u'q'a'v'] \quad \text{if and only if} \quad (q, u\underline{a}v) \underset{M}{\vdash} (q', u'\underline{a'}v').$$

- Thus M accepts the input x if and only if for some $u, v \in \Sigma^*$:

$$[q_0x] \underset{G_M}{\Rightarrow}^* [uq_{\text{acc}}v]$$

The rules in G_M comprise three types:

1. Rules with which one can derive from the start variable S any string of form $x[q_0x]$, where $x \in \Sigma^*$ and '$[$', 'q_0' and '$]$' are variables in G_M.
2. Rules that allow one to derive from the string $[q_0x]$ a string $[uq_{\mathsf{acc}}v]$ if and only if M accepts x.
3. Rules that enable one to rewrite any string of form $[uq_{\mathsf{acc}}v]$ to the empty string.

Deriving a string $x \in \mathcal{L}(M)$ can then be done as follows:

$$S \overset{(1)}{\Rightarrow^*} x[q_0x] \overset{(2)}{\Rightarrow^*} x[uq_{\mathsf{acc}}v] \overset{(3)}{\Rightarrow^*} x$$

Let us thus define the grammar $G = (V, \Sigma, R, S)$, where

$$V = (\Gamma \setminus \Sigma) \cup Q \cup \{S, T, [,], E_L, E_R\} \cup \{X_a \mid a \in \Sigma\}$$

and the rules in R include the following three sets:

1. Producing the initial configuration string:

$$\begin{array}{lcll}
S & \to & T[q_0] & \\
T & \to & \varepsilon & \\
T & \to & aTX_a & (a \in \Sigma) \\
X_a[q_0 & \to & [q_0X_a & (a \in \Sigma) \\
X_ab & \to & bX_a & (a, b \in \Sigma) \\
X_a] & \to & a] & (a \in \Sigma)
\end{array}$$

2. Simulating the transitions of M ($a,b \in \Gamma$, $c \in \Gamma \cup \{\,[\,\}$):

Transitions:

$$\begin{aligned}
\delta(q,a) &= (q',b,R) \\
\delta(q,a) &= (q',b,L) \\
\delta(q,\triangleright) &= (q',\triangleright,R) \\
\delta(q,\triangleleft) &= (q',b,R) \\
\delta(q,\triangleleft) &= (q',b,L) \\
\delta(q,\triangleleft) &= (q',\triangleleft,L)
\end{aligned}$$

Rules:

$$\begin{aligned}
qa &\to bq' \\
cqa &\to q'cb \\
q[&\to [q' \\
q] &\to bq'] \\
cq] &\to q'cb] \\
cq] &\to q'c]
\end{aligned}$$

3. Erasing an accepting configuration string:

$$
\begin{array}{lcll}
q_{\mathsf{acc}} & \to & E_L E_R & \\
q_{\mathsf{acc}}[& \to & E_R & \\
aE_L & \to & E_L & (a \in \Gamma) \\
[E_L & \to & \varepsilon & \\
E_R a & \to & E_R & (a \in \Gamma) \\
E_R] & \to & \varepsilon &
\end{array}
$$

11.3 Context-sensitive grammars

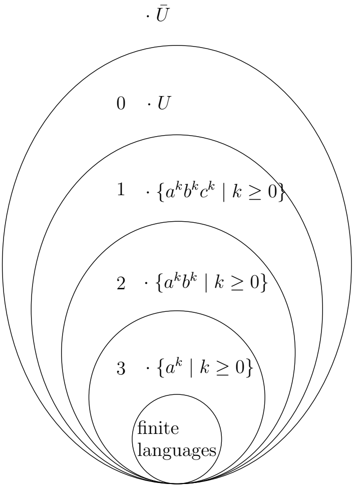
- An unrestricted grammar is *context-sensitive* if all its rules are of form $\omega \to \omega'$, where $|\omega'| \geq |\omega|$, or $S \to \varepsilon$, where S is the start variable.
- In addition, it is required that if the grammar contains the rule $S \to \varepsilon$, then the start variable S does not occur on the right-hand side of any rule.
- A language L is *context-sensitive* if it can be generated with some context-sensitive grammar.
- *A normal form for context-sensitive grammars:* Each context-sensitive language can be generated with a grammar whose rules are of form $S \to \varepsilon$ and $\alpha A \beta \to \alpha\omega\beta$, where A is a variable and $\omega \neq \varepsilon$.
- A rule $\alpha A \beta \to \alpha\omega\beta$ can be interpreted as the application of a rule $A \to \omega$ "in the context" α_β.

Theorem 11.4

A language L is context-sensitive if and only if it can be recognised with a non-deterministic Turing machine that does not use more tape space than was already allocated for the input.

- The machines in Theorem 11.4 are called *linear bounded automata*.
- It is an open problem whether the non-determinism in Theorem 11.4 is necessary or not. (The “LBA ?= DLBA” problem.)

11.4 Recap: The Chomsky hierarchy



A classification of grammars, languages generated by grammars and recogniser automata classes:

Type-0: unrestricted grammars / semi-decidable languages / Turing machines

Type-1: context-sensitive grammars / context-sensitive languages / linear bounded automata

Type-2: context-free grammars / context-free languages / pushdown automata

Type-3: right and left linear grammars / regular languages / finite automata

* A Glimpse Beyond: Computational Complexity

* Computational complexity

- So far: only what is decidable (solvable with computers) and what is not.
- But some problems are “more decidable than others”.
- For instance, finding a smallest element in an array is/seems much easier than solving sudokus.

102							

10		16			12				15				4	5	1
	11		14	13			5	10				16			
			12									11			
	1		9			7	4			11	8	13		12	
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			6				7	2							
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			11						14	5					
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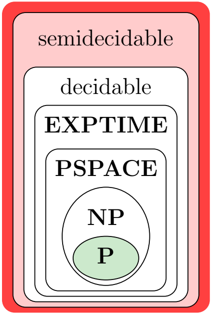
3		12	15									2		10	

- In fact, the set of decidable problems can be divided in many smaller *complexity classes*:

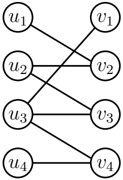
  - **P** — problems that can be solved in polynomial time ($\approx$ always efficiently) with deterministic Turing machines / algorithms.
  - **NP** — problems that can be solved in polynomial time with *non-deterministic Turing machines*.
  - **PSPACE** — problems that can be solved with a polynomial amount of extra space (possibly in exponential time).
  - **EXPTIME** — problems that can be solved in exponential time.
  - and many more...



Example: a nontrivial, but efficiently solvable problem

Definition (PERFECT MATHING)

INSTANCE: Bipartite graph $B = (U, V, E)$, where $U = \{u_1, \dots, u_n\}$, $V = \{v_1, \dots, v_n\}$, and $E \subseteq U \times V$.
QUESTION: Does B have a *perfect matching*, i.e. a 1-1 pairing of vertices?



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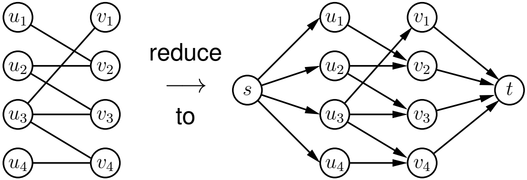

We can solve a PERFECT MATCHING instance by

1. *Polynomial-time reducing* it to a MAXFLOW instance so that: the MAXFLOW instance has a flow of n units if and only if the PERFECT MATCHING instance has a perfect matching.

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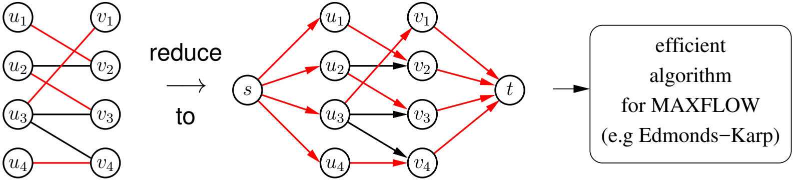


We can solve a PERFECT MATCHING instance by

1. *Polynomial-time reducing* it to a MAXFLOW instance so that: the MAXFLOW instance has a flow of n units if and only if the PERFECT MATCHING instance has a perfect matching.
2. Solving the resulting MAXFLOW instance.
3. The reduction is linear-time and Edmonds-Karp alg. works in $O(VE^2)$.

Example: a not-so efficiently solvable problem

Definition (propositional satisfiability, SAT)

INSTANCE: A Boolean formula ϕ in conjunctive normal form.
QUESTION: Is there a truth assignment that satisfies ϕ?

Example

$(x) \wedge (\neg x \vee y) \wedge (\neg x \vee \neg z) \wedge (\neg x \vee \neg y \vee \neg z)$ is satisfiable with $\{x \mapsto \textbf{true}, y \mapsto \textbf{true}, z \mapsto \textbf{false}\}$.

$(x) \wedge (\neg x \vee y) \wedge (\neg x \vee \neg z) \wedge (\neg x \vee \neg y \vee z)$ is unsatisfiable.

- Even the best known SAT algorithms, with sophisticated pruning techniques can perform very badly on some instances (although they can solve many relevant problems efficiently).
- No polynomial-time algorithm for SAT is known despite several decades of effort in trying to find one.

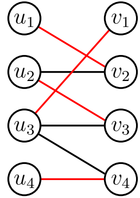
Problem class NP (Non-deterministic Polynomial time)

Two alternative ways to characterise problems in **NP**:

1. Problems that can be solved in *polynomial time* with *non-deterministic* Turing machines ($\approx$ algorithms that can guess perfectly).
2. Problems whose solutions (when they exist) are
   - reasonably small (i.e., of polynomial size), and
   - easy to check (i.e., in polynomial time).

   but *not necessarily easy to find* (or prove non-existent)!

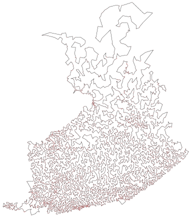
PERFECT MATCHING



SAT

$$(x \vee y \vee \neg z) \wedge$$
$$(\neg x \vee \neg y \vee v) \wedge$$
$$(x \vee w \vee z) \wedge$$
$$(y \vee \neg w \vee \neg z) \wedge$$
$$\ldots$$

TRAVELLING SALESPERSON



GENERALISED SUDOKUS

10		16			12				15				4	5	1
	11		14	13			5	10				16			
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	1		9			7	4			11	8	13		12	
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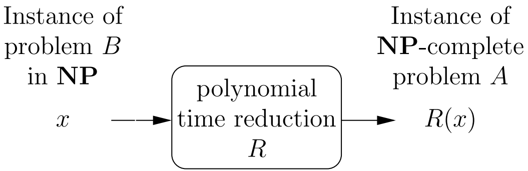
3		12	15									2		10	

NP-complete problems

- A problem A in **NP** is **NP**-*complete* if every other problem B in **NP** can be reduced to it with a polynomial time computable reduction.



Property: x has a solution in B if and only if $R(x)$ has a solution in A.

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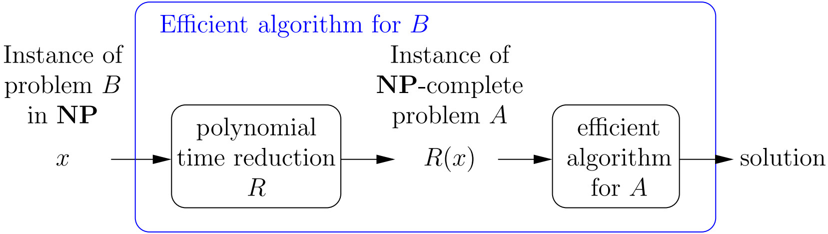


Property: x has a solution in B if and only if $R(x)$ has a solution in A.

☞ If an **NP**-complete problem A can be solved in polynomial time, then *all the problems in* **NP** *can*.

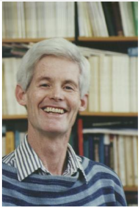
☞ **NP**-complete problems are the *most difficult ones* in **NP**!

☞ We *do not know*(!!!) whether **NP**-complete problems can be solved efficiently or not.

The Cook–Levin theorem

Theorem (S. A. Cook 1971, L. Levin 1973)

SAT is **NP***-complete.*



Stephen Cook (1939–)

Leonid Levin (1948–)

Richard Karp (1935–)

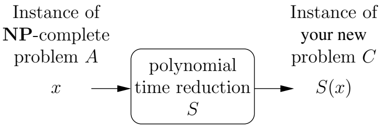
- R. Karp soon (1972) listed the next 21 **NP**-complete problems.
- Since then, 1000's of problems have been shown **NP**-complete.
- E.g. TRAVELLING SALESPERSON, GENERALISED SUDOKUS etc. are **NP**-complete.
- Classic text: Garey and Johnson (1979): *Computers and Intractability: A Guide to the Theory of NP-Completeness*.

How to prove a new problem NP-complete?

Given: a new problem C that you suspect **NP**-complete.

To prove that C is **NP**-complete:

1. show that C is in **NP**,
2. take any *existing* **NP***-complete problem* A, and
3. reduce A *to your problem* C.



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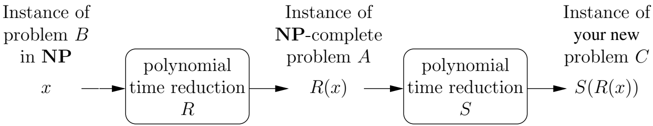
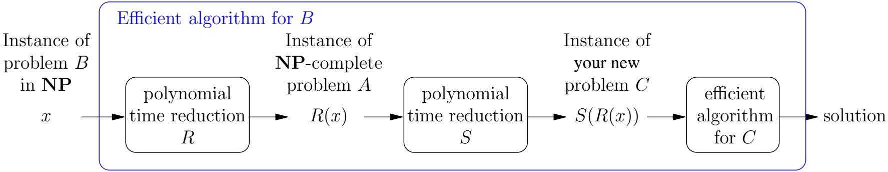


Polynomial time reductions compose: any B in **NP** reduces to C!
☞ Your problem C is **NP**-complete.

How to prove a new problem NP-complete?

Given: a new problem C that you suspect **NP**-complete.
To prove that C is **NP**-complete:

1. show that C is in **NP**,
2. take any *existing* ***NP****-complete problem* A, and
3. reduce A *to your problem* C.



Polynomial time reductions compose: any B in **NP** reduces to C!

☞ Your problem C is **NP**-complete.

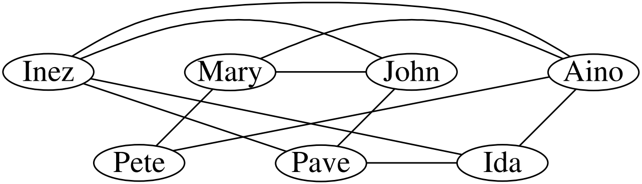
☞ If your problem C can be solved in polynomial time, then so can A and all the problems in **NP**.

Proving NP-completeness: an example

Definition (PARTYING WITH STRANGERS)

INSTANCE: A network of students and a positive integer K, where a network consists of (i) a finite set of students and (ii) a symmetric, binary "X knows Y" relation among them.

QUESTION: Is it possible to arrange a party with (at least) K students, none of whom know each other?



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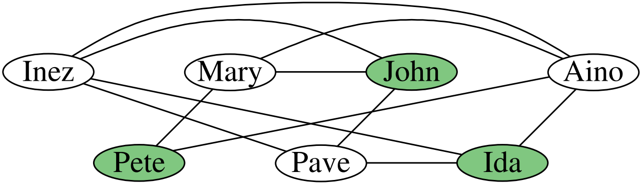

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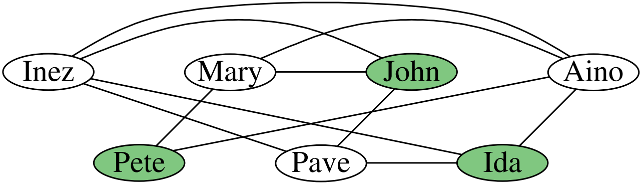


with $K = 3$?

Definition (INDEPENDENT SET)

INSTANCE: An undirected graph $G = (V, E)$ and an integer K.
QUESTION: Is there an independent set $I \subseteq V$ with $|I| = K$?

Theorem

INDEPENDENT SET *is* **NP***-complete.*

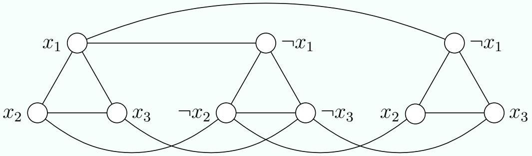
Proof

Reduction *from* 3SAT.

The SAT formula ϕ:

$(x_1 \vee x_2 \vee x_3) \wedge$
$(\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge$
$(\neg x_1 \vee x_2 \vee x_3)$

The corresponding graph G with $K = 3$:



1. If ϕ is satisfiable, then G has an independent set of size K.
2. If G has an independent set of size K, then ϕ is satisfiable.

$\Rightarrow$ ϕ is satisfiable if and only if G has an independent set of size K.

☞ **If we can solve INDEPENDENT SET efficiently, then we can solve SAT and all other problems in NP efficiently as well.**

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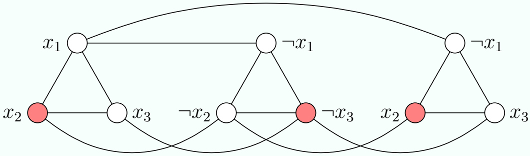

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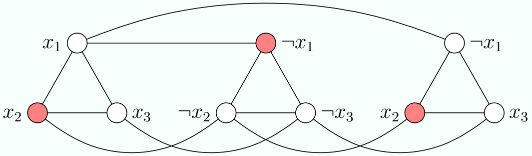

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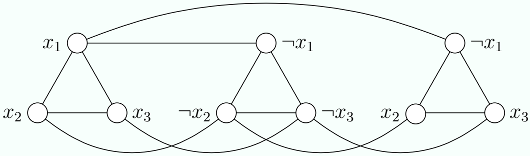

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NP-completeness: Significance

- *Can* **NP***-complete problems be solved in polynomial time?*

  One of the seven 1M$ Clay Mathematics Institute Millenium Prize problems, see

  http://www.claymath.org/millennium-problems/

- What to do when a problem is **NP**-complete?
  - Attack special cases that occur in practice
  - Develop backtracking search algorithms with efficient heuristics and pruning techniques
  - Develop approximation algorithms
  - Apply incomplete local search methods
  - ...

Some further courses:

- CS-E3190 Principles of Algorithmic Techniques
- CS-E4530 Computational Complexity Theory
- CS-E4340 Cryptography
- and so on...