Homework Problems

H9.1 Prove the following properties of infinite sets:

(a) If A is a countably infinite set, then it has only countable subsets, i.e. if $B \subseteq A$, then B is either finite or countably infinite.

(b) If A is an uncountable set, then it has both uncountable and countably infinite subsets.

a)

"Prove every subset of N is either finite or countable":

Proof : Consider a nonempty set of nonnegative integers S such that $S \subseteq N$. Then there exists an injective function $f : S \longrightarrow N$ by definition. More specifically, define f as: $f(s_i) = i, i \in N$

This allows us to use f to enumerate the elements of S by their size: letting s_1 be the smallest element of S, s_2 be the smallest element of $S \backslash \{s_1\}$ and so forth inductively until we exhaust all the elements of S. Using f, we would have paired up s_1 with $1 \in N$, s_2 with $2 \in N$ and so forth.

If there exists an element i_n of N where f^{-1} is undefined, as in, no element of S maps to i, then $S \subset N$ and $card(S) < card(N)$, implying that S is finite. More specifically, S would have a finite cardinality of $i_n - 1$

Otherwise, if such an element i_n does not exist, f will not only be an injection but also a surjection, since for all elements $i \in N$, there exists an $s_i \in S$ such that $f(s_i) = i$. Since f is both injective and surjective, it is a bijection, implying that $card(S) = card(N)$. So by definition, if there exists a bijection between a set and the natural numbers, that set is countably infinite in cardinality.

b)

Let T be an uncountable set. Choose S_n to be a finite subset of T with n elements. It's always possible to pick such a set because T is uncountable. Now let:

$$S = \bigcup_{n=1}^{\infty} S_n$$

This set is a countable union of finite sets. Thus, it's at most countable. S must be infinite because for all $n \in \mathbb{N}$, there is $S_n \subset S$. In other words, there is no n for which $|S| < n$.

So let T be uncountable.

If T is bounded above then it has a supremum η. If $\eta \in T$ then let $S = \{\eta\}$. Otherwise, let $S = \{\eta_n \; : \; n \in \mathbb{N}\}$ where $(\eta_n)_{n=1}^{\infty}$ is some strictly increasing sequence in T converging to η.

If T is unbounded then let $S = \{\eta_n \; : \; n \in \mathbb{N}\}$, where $(\eta_n)_{n=1}^{\infty}$ is some strictly increasing sequence in T which tends to ∞.

In any case, by construction, for each $t \in T$ there exists $s \in S$ such that $s \geq t$, as you should verify; and S is clearly countable.

If you require S to be countably *infinite* then, in the case T bounded and $\sup T = \eta \in T$ given above, let S' be any countably infinite subset of T (e.g. constructed as in Ayman Hourieh's answer) and then let $S = S' \cup \{\eta\}$.

H9.2 The set of natural numbers $\mathbb{N}$ is (by definition) countable, or more specifically countably infinite. Prove, by using Cantor's diagonalisation

method, that there are uncountably many functions from natural numbers to natural numbers. In other words, prove that the set

$$F = \{f \mid f : \mathbb{N} \to \mathbb{N}\}$$

is uncountable.

Let f: N1 -> N2 an arbitrary function of N -> N, where N1 is enumerated from 0 to infinity in order and N2 is randomly selected

For example: f(i): 0 -> 3, 1 -> 5, 2-> 9, 3-> 3, 4-> 8, 5-> 11 and so on….

There exists a function f: N1 -> N2 that does not belong to this set and we can prove by Cantors' diagonalization method

	f1	f2	f2	f3	
N1 -> N2	0->0	0-> 9	0-> 3	0->10	...
N1 -> N3	1->3	1->6	1->2	1->5	...
N1 -> N4	2->5	2->2	2-> 3	2->6	...
N1 -> N5	3->2	3->10	3-> 8	3->5	...
	...	...	...	...	...
	f1	f2	f2	f3	
N1 -> N2	**0->1**	0-> 9	0-> 3	0->10	...
N1 -> N3	1->3	**1->7**	1->2	1->5	...
N1 -> N4	2->5	2->2	**2-> 4**	2->6	...
N1 -> N5	3->2	3->10	3-> 8	**3->6**	...
	...	...	...	...	...
f not in set:	**0->1**	**1->7**	**2->4**	**3-> 6**	...

That is, the function not in the set is constructed by using the mapping of each f in the set in the diagonal line and increment N2 by 1. This new f is guaranteed to not be in the set as it always differ at least in N1->N2 from any other functions in the set of N->N functions

H9.3 Are the following statements true or false? Justify your answers.

1. All regular languages are decidable.
2. The complement of any context-free language is decidable.
3. The complement of any language recognised by a deterministic Turing machine is decidable.
4. All languages recognised by nondeterministic Turing machines are semi-decidable.
5. If a language can be recognised by a deterministic pushdown automaton, then it is also decidable.
6. If a language is decidable, then it can be recognised by some non-deterministic pushdown automaton.

1. All regular languages are decidable.
True, since every regular language is context-free, every context-free language is decidable, and every decidable language is Turing-decidable.

2. The complement of any context-free language is decidable.
True. All context-free language is decidable. Also, the complement of decibel language is also a decidable language => Complement of any context-free language is decidable

3. The complement of any language recognised by a deterministic Turing machine is decidable.
True. The deterministic Turing machine always halts, and the language is recognized by a Turing Machine and halts on all inputs => the language is decidable => Its complement is also decidable

4. All languages recognised by nondeterministic Turing machines are semidecidable.
True. semi-decidable languages are all recognized by a Turing machine (Turing-recognizable).

5. If a language can be recognised by a deterministic pushdown automaton, then it is also decidable.
True. The class of deterministic pushdown automata accepts the deterministic context-free languages => the language is decidable

6. If a language is decidable, then it can be recognised by some nondeterministic pushdown automaton
True. Since it is decidable => it is still a context free language => a CFG can be recognized by a nondeterministic PDA