Solutions to Supplementary Problems

Problem S1.1 Define a relation $\sim$ on the set $\mathbb{N} \times \mathbb{N}$ by the rule:

$$(m, n) \sim (p, q) \quad \Leftrightarrow \quad m + n = p + q.$$

Prove that this is an equivalence relation, and describe intuitively ("geometrically") the equivalence classes it determines.

Solution. A relation is an equivalence when it is reflexive, symmetric and transitive. Let us check these conditions for the relation

$$\sim \ \subseteq (\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{N})$$

defined as above.

(i) Relation $\sim$ is reflexive, if for all $(m, n) \in \mathbb{N} \times \mathbb{N}$ it is the case that $(m, n) \sim (m, n)$. Since

$$m + n = m + n,$$

the condition is fulfilled.

(ii) Relation $\sim$ is symmetric, if $(m, n) \sim (p, q)$ always implies $(p, q) \sim (m, n)$. Because

$$m + n = p + q \quad \Rightarrow \quad p + q = m + n,$$

also this condition is clearly satisified.

(iii) Relation $\sim$ is transitive, if always when $(m, n) \sim (p, q)$ and $(p, q) \sim (k, l)$, then also $(m, n) \sim (k, l)$. Now assuming

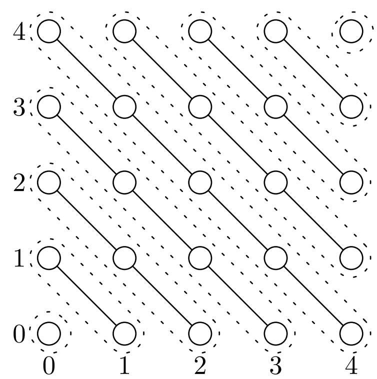
$$m + n = p + q \quad \& \quad p + q = k + l,$$

then

$$m + n = p + q = k + l \quad \Rightarrow \quad m + n = k + l.$$

Thus also this condition is satisfied.

Because all three conditions hold, $\sim$ is an equivalence relation. The diagram below illustrates the equivalence classes of the relation:



One observes that the equivalence classes correspond to the diagonals in the nonnegative quadrant of the (x, y)-plane defined by equations $x + y = k$, for $k = 0, 1, 2, \ldots$

Problem S1.2 Prove by induction that if X is a finite set of cardinality $n = |X|$, then its power set $\mathcal{P}(X)$ is of cardinality $|\mathcal{P}(X)| = 2^n$.

Solution.
Base case: The only set of cardinality $n = 0$ is the empty set $X = \emptyset$, and for this it is the case that $|\mathcal{P}(\emptyset)| = |\{\emptyset\}| = 1 = 2^0$.

Induction hypothesis: Assume that the claim holds for a given $k \in \mathbb{N}$, meaning that if $|X| = k$, then $|\mathcal{P}(X)| = 2^k$.

Inductive step: Let Y be any set of cardinality $|Y| = k + 1$. Choose any element $y \in Y$. (Note that $Y \neq \emptyset$.) Now Y can be written as $Y = Y' \cup \{y\}$, where $Y' = Y \setminus \{y\}$ has cardinality $|Y'| = k$. By the induction hypothesis, $|\mathcal{P}(Y')| = 2^k$.

By definition, the set $\mathcal{P}(Y)$ comprises the subsets of Y, which are of two types: (i) those that do not contain the element y, and (ii) those that do contain the element y.

The first ones are exactly the subsets of Y' and the second ones are the subsets of Y' augmented with the element y. That is,

$$\mathcal{P}(Y) = \mathcal{P}(Y') \cup \{Z \cup \{y\} \mid Z \in \mathcal{P}(Y')\},$$

and by the induction hypothesis we obtain

$$|\mathcal{P}(Y)| = 2^k + 2^k = 2^{k+1}.$$

Therefore, if the claim holds when $|X| = k$, then it also holds when $|X| = k + 1$, and by the induction principle it follows that for every $n \in \mathbb{N}$, if $|X| = n$, then $|\mathcal{P}(X)| = 2^n$.