CS-C2160 Theory of Computation, Spring 2022
Problem Set 9 (Lecture weeks 9 and 10)

Instructions

- Classroom Problems C9.1–C9.2 will be discussed and solved onsite at the tutorial sessions in lecture week 9. No credit is given for these problems.
- Homework Problems H9.1–H9.3 you should solve on your own, and be available to present your solutions at one of the tutorial sessions in lecture week 10. In order to get course credit, you need to indicate your solved problems on the signup sheet circulated at the beginning of the session.
- Supplementary Problems S9.1–S9.3 provide further illustration and extension of the course material, but will usually not be covered at the tutorials. You are however invited to work on these problems too, and discuss them with the course staff. Sample solutions are provided on MyCourses.

Classroom Problems

C9.1 Let A and B be countably infinite sets. Show that then also the set $A \cup B$ is countably infinite by providing a bijection between $A \cup B$ and the set $\mathbb{N}$ of natural numbers. (*Hint:* You may want to consider the cases when $B \setminus A$ is finite and when $B \setminus A$ is infinite separately.)

C9.2 Consider the encoding of Turing machines into binary strings presented in Section 9.4 of the lecture slides. Determine the first string (in the "shortlex" order of Lemma 9.5) c for which $L(M_c) \neq \emptyset$, i.e. the machine encoded by c halts and accepts *some* input string. Is it the case, for this particular value of c, that $c \in L(M_c)$? Give a simple verbal description of the language $L(M_c)$.

Homework Problems

H9.1 Prove the following properties of infinite sets:

(a) If A is a countably infinite set, then it has only countable subsets, i.e. if $B \subseteq A$, then B is either finite or countably infinite.

(b) If A is an uncountable set, then it has both uncountable and countably infinite subsets.

H9.2 The set of natural numbers $\mathbb{N}$ is (by definition) countable, or more specifically countably infinite. Prove, by using Cantor's diagonalisation method, that there are uncountably many functions from natural numbers to natural numbers. In other words, prove that the set

$$F = \{f \mid f : \mathbb{N} \to \mathbb{N}\}$$

is uncountable.

H9.3 Are the following statements true or false? Justify your answers.

1. All regular languages are decidable.
2. The complement of any context-free language is decidable.
3. The complement of any language recognised by a deterministic Turing machine is decidable.
4. All languages recognised by nondeterministic Turing machines are semi-decidable.
5. If a language can be recognised by a deterministic pushdown automaton, then it is also decidable.
6. If a language is decidable, then it can be recognised by some non-deterministic pushdown automaton.

Supplementary Problems

S9.1 Prove that the class of semi-decidable languages is closed under unions and intersections. Why can't we prove that the class is closed under complementation in a similar way as in the case of decidable languages, i.e. by simply interchanging the accepting and rejecting states of the respective Turing machines?

S9.2 Prove that the Cartesian product $\mathbb{N} \times \mathbb{N}$ is countably infinite. (*Hint:* Think of the pairs $(m, n) \in \mathbb{N} \times \mathbb{N}$ as embedded in the Euclidean (x, y) plane $\mathbb{R}^2$. Enumerate the pairs by diagonals parallel to the line $y = -x$.) Conclude from this result that also the set $\mathbb{Q}$ of rational numbers is countably infinite.

S9.3 Show that Turing machines that have only *one* internal state in addition to their accepting and rejecting states are capable of recognising exactly the same languages as the standard machines with arbitrarily many states. You may assume that the simulating machines have multiple tapes and may also keep their tape heads stationary (direction code 'S') in a transition.