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MEC-E8007 Thin-Walled Structures

Lecture 5. Classical Lamination Theory (CLT) for Plates and Shells

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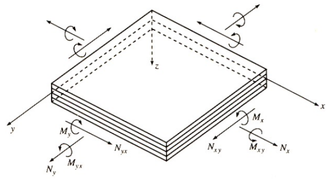
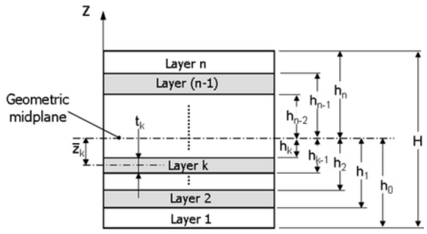
In order to utilise the othrotropic material models and the large deflection results in design of thin-walled structures, the Classical Lamination Theory is presented for plates. We also describe what happens when the plate surface is curved to form a shell.

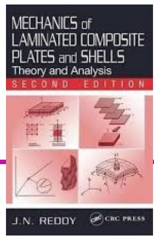
We can then move to more complex kinematical models.

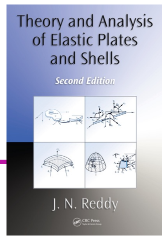
Contens

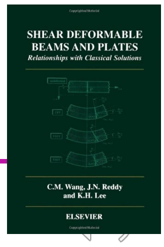
- The aim of the lecture is to understand how laminate stiffnesses are formed and how the responses of laminates can be computed using different beam, plate and shell theories
- Motivation
- Classical laminate theory (CLT) for plates
- Shell theory
- Literature
    1. Reddy, J.N., Mechanics of Laminated Composite Plates and Shells – Theory and Analysis, 2nd Edition, CRC Press, Ch. 1-3, 5-7, 9
    2. Reddy, JN, “Theory and analysis of plates and shells”
    3. Wang, C.M., Reddy, JN, and Lee, K.H., “Shear deformable beams and plates: relationship with classical solutions”











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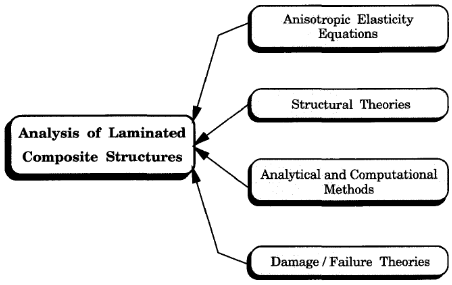
The aim of the lecture is to understand how laminate stiffnesses are formed and how the responses of laminates can be computed using different beam, plate and shell theories. Excellent books here are the ones from Reddy which presents the topic from fundamentals.

Motivation

- Individual materials posses certain properties for certain purposes
- In composites the material is formed from more than one material with very clear interface between to fulfil multitude of requirements
- Tailoring materials for given application case enables principally development of the “best material known by man”
- One needs to understand the entire design process from material description to structural theories and their analytical and numerical solutions and damage and failure prediction



Reddy, J.N., Mechanics of Laminated Composite Plates and Shells – Theory and Analysis

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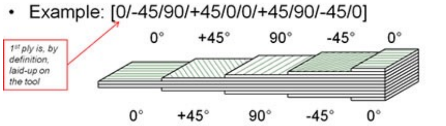
In composite materials, the individual materials that make the composite posses certain properties for certain purposes. In composites the material is formed from more than one material with very clear interface between to fulfil multitude of requirements we have in design. Thus, tailoring materials for given application case enables principally development of the “best material known by human” for the given application case. In order to do this in practice, one needs to understand the entire design process from material description to structural theories and their analytical and numerical solutions and further to damage and failure prediction.

Lamina / Ply

- A sheet of composite material is called lamina or ply
- Lamina can be unidirectional, bi-directional, woven or random
  - Each lamina has its principal direction, e.g. fiber direction
  - The properties are given as function of the fiber orientation angle
- Stacking sequence tells us, in what order the material is through the thickness of the laminate
  - Stiff fibers at surface increase the bending stiffness
  - Low-density core can take the shear loads effectively

- Laminates are often described by an orientation code
- Example: [0/-45/90/+45/0/0/+45/90/-45/0]



- Short hand: $[0/-45/90/+45]_S$ symmetric
- Short hand: $[0/-45/90/+45]_N$ Repeat N times
- Other Examples: $[0/+45/-45/90]_{2S}$ 16 total plies
  $[0_2/+45/-45/90_2]_S$ 12 total plies
  $[0_U/+45_{CL}]$ 2 total plies (1 uni @ 0° and 1 cloth @ ±45°)

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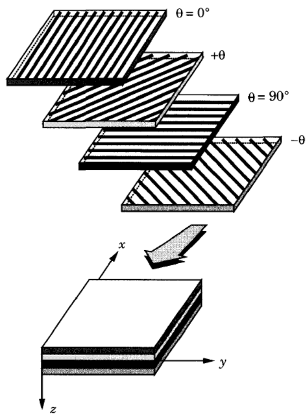
A sheet of composite material is called **lamina** or **ply** and a lamina can be unidirectional, bi-directional, woven or random depending on the manufacturing process exploited and wished ply properties. Each lamina has its **principal direction**, which is typically the fiber direction as it has the highest stiffness and often also strength properties. The material properties are given as function of the **fiber orientation angle** to allow designer to understand better the impact of direction.

Stacking sequence tells us, in what order the material is through the thickness of the laminate and in what angle the fibers are in this assembly on lamina level. Stiff fibers at surface increase the bending stiffness and strength of the assembly, while for membrane actions the impact is smaller and often assumed negligible. Low-density core can take the shear loads effectively, the effect we utilize in sandwich structures.

Laminates

- When assembled to layer wise structure, one obtains laminate
  - Layers are bonded together
  - Laminate is characterized by lamination scheme, a.k.a. stacking sequence
  - Sandwich is a special form of laminate with faces and core
- The weakness of laminates is the mismatch of material properties that cause local stress concentration, i.e. failure initiation points
  - Within ply/lamina
  - Between plies/laminae



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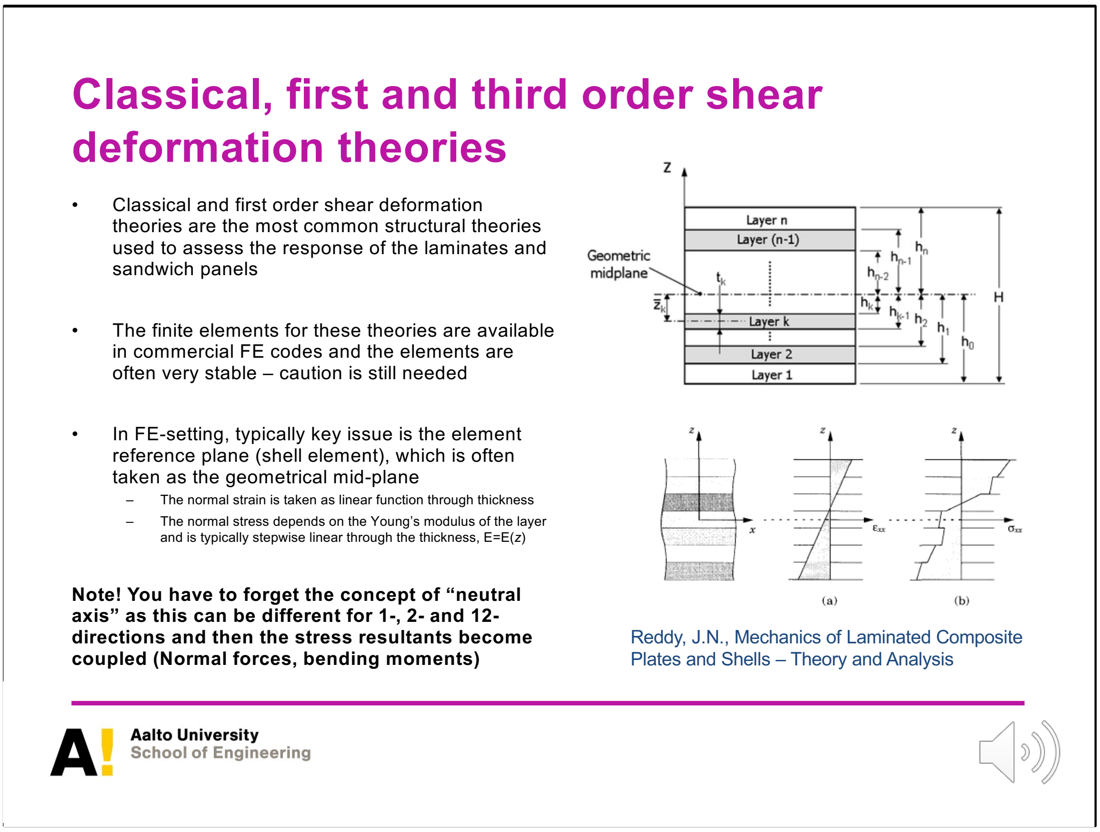
When **plies** are assembled to a layer wise structure, one obtains a **laminate**. We assume often several things in the computational modeling. We assume that the layers are perfectly bonded together. We assume that the behavior of the laminate can be characterized by lamination scheme, a.k.a. the stacking sequence. When we create a **sandwich**, we create a special form of a laminate with faces and core with different roles in the load-carrying mechanism. The weakness of laminates is the mismatch of material properties that cause local stress concentration, i.e. failure initiation points, within ply/lamina (fiber-matrix failures) and between plies/laminae (delamination). Naturally, the phenomena can be much more complex than described by these simple assumptions.



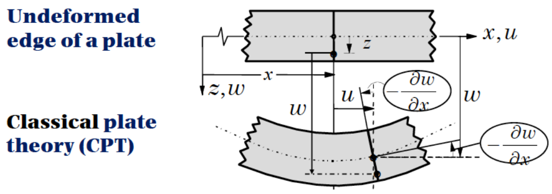
Classical and first order shear deformation theories are the most common structural theories used to assess the response of the laminates and sandwich panels. These are also called Euler-Bernoulli and Timoshenko beam theories and Kirchhoff and Reissner-Mindlin plate theories. The finite elements for these theories are available in commercial FE codes and the elements are often very stable – caution is still needed as the differential equations possess certain limitations which also cause challenges to the numerical solutions. An example is point load, which can cause hour-glassing of the elements.

In the FE-setting, typically key issue is the element **reference plane** (shell element), which is often taken as the geometrical mid-plane of the laminate. This is to make sure that all internal stress resultants are properly accounted for in terms of stresses that produce them. This is why the concept of neutral axis must be forgotten. This can be different for 1-, 2- and 12-directions and then the stress resultants become coupled (Normal forces, bending moments)

In these two models, the normal strain is taken as linear function through the thickness of the laminate. The normal stress depends on the Young's modulus of the layer and is typically stepwise linear through the thickness, E=E(z). This is why there must be integration also through the thickness as the stress can be piecewise linear. This is simply due to the fact that material is piecewise distributed through the thickness, i.e. piecewise material property (e.g. Young's modulus) x linear strain => piecewise linear stress.

Classical plate theory

- Kirchhoff plate (Classical) assumes
  - Plane remain planes after deformation
  - Planes remain at 90 degree angle with respect to geometrical mid-plane
  - Plate does not deform in thickness direction
- With these assumptions
  - We can write in-plane deformations as sum of membrane and bending actions
  - We know that the thickness does not change
  - We indirectly assume that the plate is in plane stress
- The deformation is given by



$$u(x,y,z,t) = u_0(x,y,t) - z\frac{\partial w_0}{\partial x}$$
$$v(x,y,z,t) = v_0(x,y,t) - z\frac{\partial w_0}{\partial y}$$
$$w(x,y,z,t) = w_0(x,y,t)$$

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Kirchhoff plate (Classical) assumes that the planes remain planes after deformation, and further that they remain at 90 degree angle with respect to geometrical mid-plane or the reference plane (these two planes are always parallel but not necessarily coincident, for example due to offset of the reference plane). In addition, the plate does not deform in thickness direction, i.e. compress. This is against the Poisson’s effect, but is well-justified and necessary linearization of the problem. This means that initial thickness can be used for all load levels. Otherwise we would need to keep the integration limits as variables which are dependent of the load level.

With these assumptions we can write in-plane deformations as sum of membrane and bending actions, we know that the thickness does not change and we can keep the integration limits constant and we indirectly assume that the plate is in plane stress. Then deformation is given by by constant in-plane displacements + additional in-plane displacement due to rotation of the cross-section. This is shrinking of the inner surface and extension of the outer surface. The deflection is thickness independent.

Classical plate theory

- Displacements can be differentiated to give strains (membrane+bending)

$$\left\{\begin{array}{c}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{array}\right\} = \left\{\begin{array}{c}\varepsilon_{xx}^{(0)}\\ \varepsilon_{yy}^{(0)}\\ \gamma_{xy}^{(0)}\end{array}\right\} + z\left\{\begin{array}{c}\varepsilon_{xx}^{(1)}\\ \varepsilon_{yy}^{(1)}\\ \gamma_{xy}^{(1)}\end{array}\right\}$$

- when von Karman strains are assumed (moderate rotations), one obtains for classical=Kirchhoff plate

$$\{\varepsilon^0\} = \left\{\begin{array}{c}\varepsilon_{xx}^{(0)}\\ \varepsilon_{yy}^{(0)}\\ \gamma_{xy}^{(0)}\end{array}\right\} = \left\{\begin{array}{c}\frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\\ \frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2\\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y}\end{array}\right\} \qquad \{\varepsilon^1\} = \left\{\begin{array}{c}\varepsilon_{xx}^{(1)}\\ \varepsilon_{yy}^{(1)}\\ \gamma_{xy}^{(1)}\end{array}\right\} = \left\{\begin{array}{c}-\frac{\partial^2 w_0}{\partial x^2}\\ -\frac{\partial^2 w_0}{\partial y^2}\\ -2\frac{\partial^2 w_0}{\partial x \partial y}\end{array}\right\}$$

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When we differentiate the displacement, we get the strains and these are typically presented separately for the membrane and bending actions. Note here the superscripts that highlight this difference. If we check the von Karman strains we see the pure membrane strain and the additional second order term due to the elongation of the fibers due to the deflection described by the rotation. The bending-induced strain is linear and dependent on the curvature.

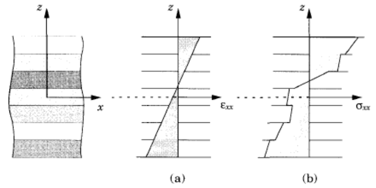
Classical plate theory

- From strains we can obtain the stresses using the layerwise material data
- Under plane stress assumption, we obtain the layerwise (ply/lamina) stress as

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix}^{(k)} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_1 - \alpha_1\ \Delta T \\ \varepsilon_2 - \alpha_2\ \Delta T \\ \varepsilon_6 \end{Bmatrix} - \begin{bmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{32} \\ 0 & 0 & 0 \end{bmatrix}^{(k)} \begin{Bmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \\ \mathcal{E}_3 \end{Bmatrix}^{(k)}$$

- and in laminate in rotated coordinate system as

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}^{(k)} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^{(k)} \left(\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} - \begin{Bmatrix} \alpha_{xx} \\ \alpha_{yy} \\ 2\alpha_{xy} \end{Bmatrix} \Delta T \right) - \begin{bmatrix} 0 & 0 & \bar{e}_{31} \\ 0 & 0 & \bar{e}_{32} \\ 0 & 0 & \bar{e}_{36} \end{bmatrix}^{(k)} \begin{Bmatrix} \mathcal{E}_x \\ \mathcal{E}_y \\ \mathcal{E}_z \end{Bmatrix}^{(k)}$$



$$Q_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}}, \quad Q_{12} = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}} = \frac{\nu_{21}E_1}{1 - \nu_{12}\nu_{21}}$$

$$Q_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}}, \quad Q_{66} = G_{12}$$

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From strains we can obtain the stresses using the layer wise material data which is described by the stacking sequence and the material properties per layer. Under plane stress assumption, we obtain the layer wise (ply/lamina) stress as the product of layer stiffness matrix and the layer strains. The stiffness matrix per layer can be seen on the right with Young's modulus in direction E_1 and E_2 and corresponding Poisson's ratios. In addition, we have the in-plane shear modulus G_{12}. This all was given in ply coordinate system and when we want to see the effect in laminate coordinate system, we need to rotate the stiffness matrix.

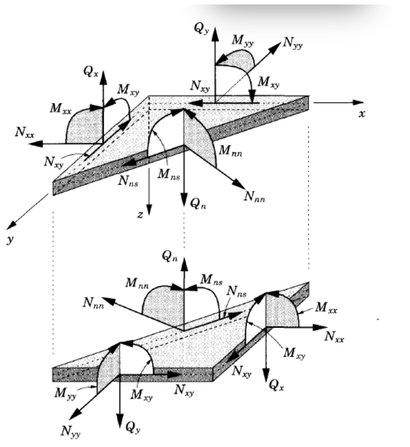
Classical plate theory

- For plates we need to know the relation between the “stress resultants” and displacements
  - Normal forces, N_x, N_y, N_{xy}
  - Bending moments, M_x, M_y and twisting moment M_{xy}
- This are obtained by integrating the stresses over the thickness, i.e.

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} dz, \quad \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} z\, dz$$

$$\hat{Q}_n = \int_{-\frac{h}{2}}^{\frac{h}{2}} \hat{\sigma}_{nz}\, dz$$

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} dz = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_{xx}^{(0)} + z\varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(0)} + z\varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(0)} + z\gamma_{xy}^{(1)} \end{Bmatrix} dz$$



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For plates we need to know the relation between the “stress resultants” and displacements. The stress resultants are the normal forces, N_{xx}, N_{yy}, N_{xy} and the bending moments, M_{xx}, M_{yy} and twisting moment M_{xy}. These are obtained by integrating the stresses over the thickness, i.e. using simply the integral of stresses for the normal forces and z-coordinate weighted integral of stresses for the bending and twisting moments.

Classical plate theory

- The integration gives the
  - Extension stiffness (A), membrane stiffness
  - Extension-bending stiffness (B)
  - Bending stiffness (D)

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix}$$

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} z\, dz$$

$$= \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_{xx}^{(0)} + z\varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(0)} + z\varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(0)} + z\gamma_{xy}^{(1)} \end{Bmatrix} z\, dz$$

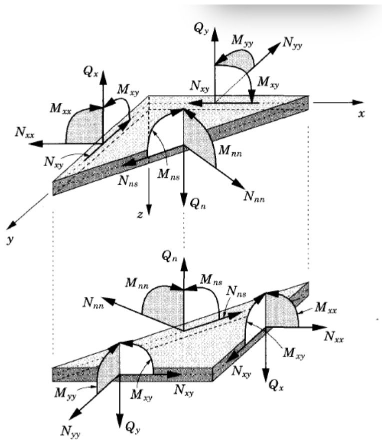
$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix}$$

- with

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \bar{Q}_{ij}(1, z, z^2)dz = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \bar{Q}_{ij}^{(k)}(1, z, z^2)dz$$

$$A_{ij} = \sum_{k=1}^{N} \bar{Q}_{ij}^{(k)}(z_{k+1} - z_k)\ , \quad B_{ij} = \frac{1}{2}\sum_{k=1}^{N} \bar{Q}_{ij}^{(k)}(z_{k+1}^2 - z_k^2)$$

$$D_{ij} = \frac{1}{3}\sum_{k=1}^{N} \bar{Q}_{ij}^{(k)}(z_{k+1}^3 - z_k^3)$$



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When the integration is carried out, the through-thickness integrals of the ply stiffness properties gives the extension stiffness matrix, denoted by (A), or membrane stiffness matrix; we also get the extension-bending stiffness (B) and the bending stiffness (D). When layers are assumed to be clearly defined in terms of geometry, that is the interfaces, are "planes", we can replace the integrals by weighted summation of the layer stiffness properties through the thickness. Note here the factors 1, ½ and 1/3 and exponents 1, 2 and 3 for the A, B, and D-matrices respectively. These equations are pre-coded to pre-processors of the FE-codes.

Classical plate theory

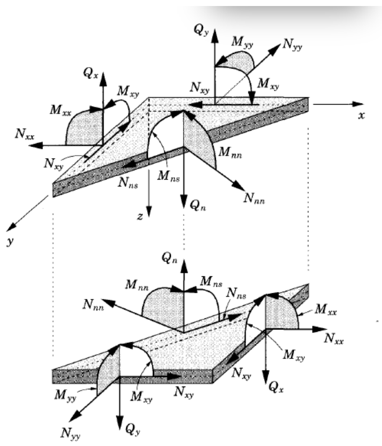
- So in short (Kirchhoff)

$$\begin{Bmatrix} \{N\} \\ \{M\} \end{Bmatrix} = \begin{bmatrix} [A] & [B] \\ [B] & [D] \end{bmatrix} \begin{Bmatrix} \{\varepsilon^0\} \\ \{\varepsilon^1\} \end{Bmatrix}$$

- and in terms of displacements this is

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_0}{\partial x} + \frac{1}{2}(\frac{\partial w_0}{\partial x})^2 \\ \frac{\partial v_0}{\partial y} + \frac{1}{2}(\frac{\partial w_0}{\partial y})^2 \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y} \end{Bmatrix} - \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial^2 w_0}{\partial x^2} \\ \frac{\partial^2 w_0}{\partial y^2} \\ 2\frac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix}$$

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_0}{\partial x} + \frac{1}{2}(\frac{\partial w_0}{\partial x})^2 \\ \frac{\partial v_0}{\partial y} + \frac{1}{2}(\frac{\partial w_0}{\partial y})^2 \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y} \end{Bmatrix} - \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial^2 w_0}{\partial x^2} \\ \frac{\partial^2 w_0}{\partial y^2} \\ 2\frac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix}$$



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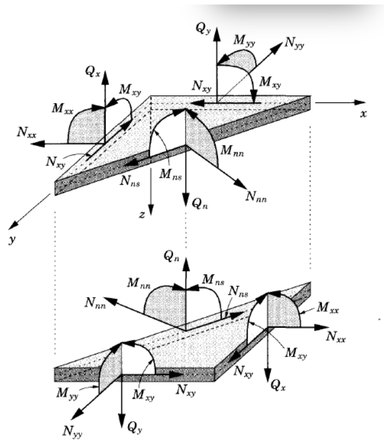
So in short we get for the classical plate theory that the stress resultants, normal forces and corresponding moments, are simply a product of the ABD-matrix and the strains due to in-plane membrane stretch and the bending and twisting moments. When this is converted to displacements based on the strain-displacement relations, we get equations which are easier to solve. Before doing the solution, we first make few more steps to get the final differential equations.

Classical plate theory

- The equilibrium equations are

$$\frac{\partial N_{xx}}{\partial x}+\frac{\partial N_{xy}}{\partial y}=I_0\frac{\partial^2 u_0}{\partial t^2}-I_1\frac{\partial^2}{\partial t^2}\left(\frac{\partial w_0}{\partial x}\right)$$
$$\frac{\partial N_{xy}}{\partial x}+\frac{\partial N_{yy}}{\partial y}=I_0\frac{\partial^2 v_0}{\partial t^2}-I_1\frac{\partial^2}{\partial t^2}\left(\frac{\partial w_0}{\partial y}\right)$$
$$\frac{\partial^2 M_{xx}}{\partial x^2}+2\frac{\partial^2 M_{xy}}{\partial y\partial x}+\frac{\partial^2 M_{yy}}{\partial y^2}+\mathcal{N}(w_0)+q=I_0\frac{\partial^2 w_0}{\partial t^2}$$
$$-I_2\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w_0}{\partial x^2}+\frac{\partial^2 w_0}{\partial y^2}\right)+I_1\frac{\partial^2}{\partial t^2}\left(\frac{\partial u_0}{\partial x}+\frac{\partial v_0}{\partial y}\right)$$



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To reach the differential equations, we need to look at the equilibrium of the plate element in directions xx and yy for the membrane effects and also for the bending and twisting. This gives us 3 equilibrium equations with inertia terms written to the right hand side of the equations.

Classical plate theory

- and when we substitute the stress resultant in displacement form we get

$$
\begin{aligned}
&A_{11}\left(\frac{\partial^2 u_0}{\partial x^2}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x^2}\right)+A_{12}\left(\frac{\partial^2 v_0}{\partial x\partial y}+\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial x\partial y}\right)\\
&+A_{16}\left(\frac{\partial^2 u_0}{\partial x\partial y}+\frac{\partial^2 v_0}{\partial x^2}+\frac{\partial^2 w_0}{\partial x^2}\frac{\partial w_0}{\partial y}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x\partial y}\right)\\
&-B_{11}\frac{\partial^3 w_0}{\partial x^3}-B_{12}\frac{\partial^3 w_0}{\partial x\partial y^2}-2B_{16}\frac{\partial^3 w_0}{\partial x^2\partial y}\\
&+A_{16}\left(\frac{\partial^2 u_0}{\partial x\partial y}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x\partial y}\right)+A_{26}\left(\frac{\partial^2 v_0}{\partial y^2}+\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial y^2}\right)\\
&+A_{66}\left(\frac{\partial^2 u_0}{\partial y^2}+\frac{\partial^2 v_0}{\partial x\partial y}+\frac{\partial^2 w_0}{\partial x\partial y}\frac{\partial w_0}{\partial y}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial y^2}\right)\\
&-B_{16}\frac{\partial^3 w_0}{\partial x^2\partial y}-B_{26}\frac{\partial^3 w_0}{\partial y^3}-2B_{66}\frac{\partial^3 w_0}{\partial x\partial y^2}\\
&-\left(\frac{\partial N^T_{xx}}{\partial x}+\frac{\partial N^T_{xy}}{\partial y}\right)=I_0\frac{\partial^2 u_0}{\partial t^2}-I_1\frac{\partial^3 w_0}{\partial x\partial t^2}
\end{aligned}
\qquad \text{Eq. (1)}
$$

$$
\begin{aligned}
&A_{16}\left(\frac{\partial^2 u_0}{\partial x^2}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x^2}\right)+A_{26}\left(\frac{\partial^2 v_0}{\partial x\partial y}+\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial x\partial y}\right)\\
&+A_{66}\left(\frac{\partial^2 u_0}{\partial x\partial y}+\frac{\partial^2 v_0}{\partial x^2}+\frac{\partial^2 w_0}{\partial x^2}\frac{\partial w_0}{\partial y}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x\partial y}\right)\\
&-B_{16}\frac{\partial^3 w_0}{\partial x^3}-B_{26}\frac{\partial^3 w_0}{\partial x\partial y^2}-2B_{66}\frac{\partial^3 w_0}{\partial x^2\partial y}\\
&+A_{12}\left(\frac{\partial^2 u_0}{\partial x\partial y}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x\partial y}\right)+A_{22}\left(\frac{\partial^2 v_0}{\partial y^2}+\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial y^2}\right)\\
&+A_{26}\left(\frac{\partial^2 u_0}{\partial y^2}+\frac{\partial^2 v_0}{\partial x\partial y}+\frac{\partial^2 w_0}{\partial x\partial y}\frac{\partial w_0}{\partial y}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial y^2}\right)\\
&-B_{12}\frac{\partial^3 w_0}{\partial x^2\partial y}-B_{22}\frac{\partial^3 w_0}{\partial y^3}-2B_{26}\frac{\partial^3 w_0}{\partial x\partial y^2}\\
&-\left(\frac{\partial N^T_{xy}}{\partial x}+\frac{\partial N^T_{yy}}{\partial y}\right)=I_0\frac{\partial^2 v_0}{\partial t^2}-I_1\frac{\partial^3 w_0}{\partial y\partial t^2}
\end{aligned}
\qquad \text{Eq. (2)}
$$

$$
\begin{aligned}
&B_{11}\left(\frac{\partial^3 u_0}{\partial x^3}+\frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 w_0}{\partial x^2}+\frac{\partial w_0}{\partial x}\frac{\partial^3 w_0}{\partial x^3}\right)+B_{12}\left(\frac{\partial^3 v_0}{\partial x^2\partial y}+\frac{\partial^2 w_0}{\partial x\partial y}\frac{\partial^2 w_0}{\partial x\partial y}\right.\\
&\left.+\frac{\partial w_0}{\partial y}\frac{\partial^3 w_0}{\partial x^2\partial y}\right)+B_{16}\left(\frac{\partial^3 u_0}{\partial x^2\partial y}+\frac{\partial^3 v_0}{\partial x^3}+\frac{\partial^3 w_0}{\partial x^3}\frac{\partial w_0}{\partial y}+2\frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 w_0}{\partial x\partial y}\right.\\
&\left.+\frac{\partial w_0}{\partial x}\frac{\partial^3 w_0}{\partial x^2\partial y}\right)-D_{11}\frac{\partial^4 w_0}{\partial x^4}-D_{12}\frac{\partial^4 w_0}{\partial x^2\partial y^2}-2D_{16}\frac{\partial^4 w_0}{\partial x^3\partial y}\\
&+2B_{16}\left(\frac{\partial^3 u_0}{\partial x^2\partial y}+\frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 w_0}{\partial x\partial y}+\frac{\partial w_0}{\partial x}\frac{\partial^3 w_0}{\partial x^2\partial y}\right)+2B_{26}\left(\frac{\partial^3 v_0}{\partial x\partial y^2}+\frac{\partial^2 w_0}{\partial x\partial y}\frac{\partial^2 w_0}{\partial y^2}\right.\\
&\left.+\frac{\partial w_0}{\partial y}\frac{\partial^3 w_0}{\partial x\partial y^2}\right)+2B_{66}\left(\frac{\partial^3 u_0}{\partial x\partial y^2}+\frac{\partial^3 v_0}{\partial x^2\partial y}+\frac{\partial^3 w_0}{\partial x^2\partial y}\frac{\partial w_0}{\partial y}+\frac{\partial^2 w_0}{\partial x\partial y}\frac{\partial^2 w_0}{\partial x\partial y}\right.\\
&\left.+\frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 w_0}{\partial y^2}+\frac{\partial w_0}{\partial x}\frac{\partial^3 w_0}{\partial x\partial y^2}\right)-2D_{16}\frac{\partial^4 w_0}{\partial x^3\partial y}-2D_{26}\frac{\partial^4 w_0}{\partial x\partial y^3}\\
&-4D_{66}\frac{\partial^4 w_0}{\partial x^2\partial y^2}+B_{12}\left(\frac{\partial^3 u_0}{\partial x\partial y^2}+\frac{\partial^2 w_0}{\partial x\partial y}\frac{\partial^2 w_0}{\partial x\partial y}+\frac{\partial w_0}{\partial x}\frac{\partial^3 w_0}{\partial x\partial y^2}\right)\\
&+B_{22}\left(\frac{\partial^3 v_0}{\partial y^3}+\frac{\partial^2 w_0}{\partial y^2}\frac{\partial^2 w_0}{\partial y^2}+\frac{\partial w_0}{\partial y}\frac{\partial^3 w_0}{\partial y^3}\right)+B_{26}\left(\frac{\partial^3 u_0}{\partial y^3}+\frac{\partial^3 v_0}{\partial x\partial y^2}\right.\\
&\left.+\frac{\partial^3 w_0}{\partial x\partial y^2}\frac{\partial w_0}{\partial y}+2\frac{\partial^2 w_0}{\partial x\partial y}\frac{\partial^2 w_0}{\partial y^2}+\frac{\partial w_0}{\partial x}\frac{\partial^3 w_0}{\partial y^3}\right)\\
&-D_{12}\frac{\partial^4 w_0}{\partial x^2\partial y^2}-D_{22}\frac{\partial^4 w_0}{\partial y^4}-2D_{26}\frac{\partial^4 w_0}{\partial x\partial y^3}+\mathcal{N}(w_0)+q\\
&-\left(\frac{\partial^2 M^T_{xx}}{\partial x^2}+2\frac{\partial^2 M^T_{xy}}{\partial y\partial x}+\frac{\partial^2 M^T_{yy}}{\partial y^2}\right)\\
&=I_0\frac{\partial^2 w_0}{\partial t^2}-I_2\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w_0}{\partial x^2}+\frac{\partial^2 w_0}{\partial y^2}\right)+I_1\frac{\partial^2}{\partial t^2}\left(\frac{\partial u_0}{\partial x}+\frac{\partial v_0}{\partial y}\right)
\end{aligned}
\qquad \text{Eq. (3)}
$$

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Plates and Shells – Theory and Analysis

After this process has been completed from kinematics via displacement to strains, through constitutive description to the stresses and stress resultants and finally using the equilibrium equations, we have obtained the differential equations for the laminate according to classical lamination theory.

Classical plate theory – solving the equations

- In many cases the analytical solutions are very difficult or impossible to obtain
- Different approximations can be derived by series solution (e.g. Navier), Ritz method or by finite elements
- As an example of specially orthotropic plate we can omit the in-plane stretch (A-terms), membrane-bending coupling (B-terms) and von Karman strains and only consider static bending under simply supported case, giving

$$D_{11}\frac{\partial^4 w_0}{\partial x^4} + 2\left(D_{12} + 2D_{66}\right)\frac{\partial^4 w_0}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w_0}{\partial y^4} = q$$

- The Navier solution can be derived by developing deflection and load to Fourier-series as

$$w_0(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} W_{mn}\ \sin\alpha x\ \sin\beta y \qquad q(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} Q_{mn}\sin\alpha x\ \sin\beta y$$

$$\alpha = m\pi/a \text{ and } \beta = n\pi/b, \qquad Q_{mn} = \frac{4}{ab}\int_0^b\int_0^a q(x,y)\ \sin\alpha x\ \sin\beta y\ dxdy$$

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Plates and Shells – Theory and Analysis

In many cases the analytical solutions are very difficult or impossible to obtain and we need numerical methods instead. Different analytical approximations can be derived by series solution (e.g. Navier), Ritz method or by finite elements. As an example of specially orthotropic plate we can omit the in-plane stretch (A-terms), membrane-bending coupling (B-terms) and von Karman strains and only consider static bending under simply supported case, giving us an equation with only few terms and membrane effects totally excluded. The Navier solution can be derived by developing deflection and load to Fourier-series as double sinusoidal series. When these are substituted to the differential equation, we get the analytical solution.

Classical plate theory – solving the equations

- Using these expressions yields

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left\{-W_{mn}\left[D_{11}\alpha^4+2(D_{12}+2D_{66})\alpha^2\beta^2+D_{22}\beta^4\right]+Q_{mn}\right\}\sin\alpha x\ \sin\beta y=0$$

- Which has to be zero for each combination of m and n giving

$$W_{mn}=\frac{Q_{mn}}{d_{mn}}$$

$$d_{mn}=\frac{\pi^4}{b^4}\left[D_{11}m^4s^4+2(D_{12}+2D_{66})m^2n^2s^2+D_{22}n^4\right]\qquad s=b/a.$$

- and the plate deflection is then

$$\boxed{w_0(x,y)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{Q_{mn}}{d_{mn}}\sin\alpha x\ \sin\beta y}$$

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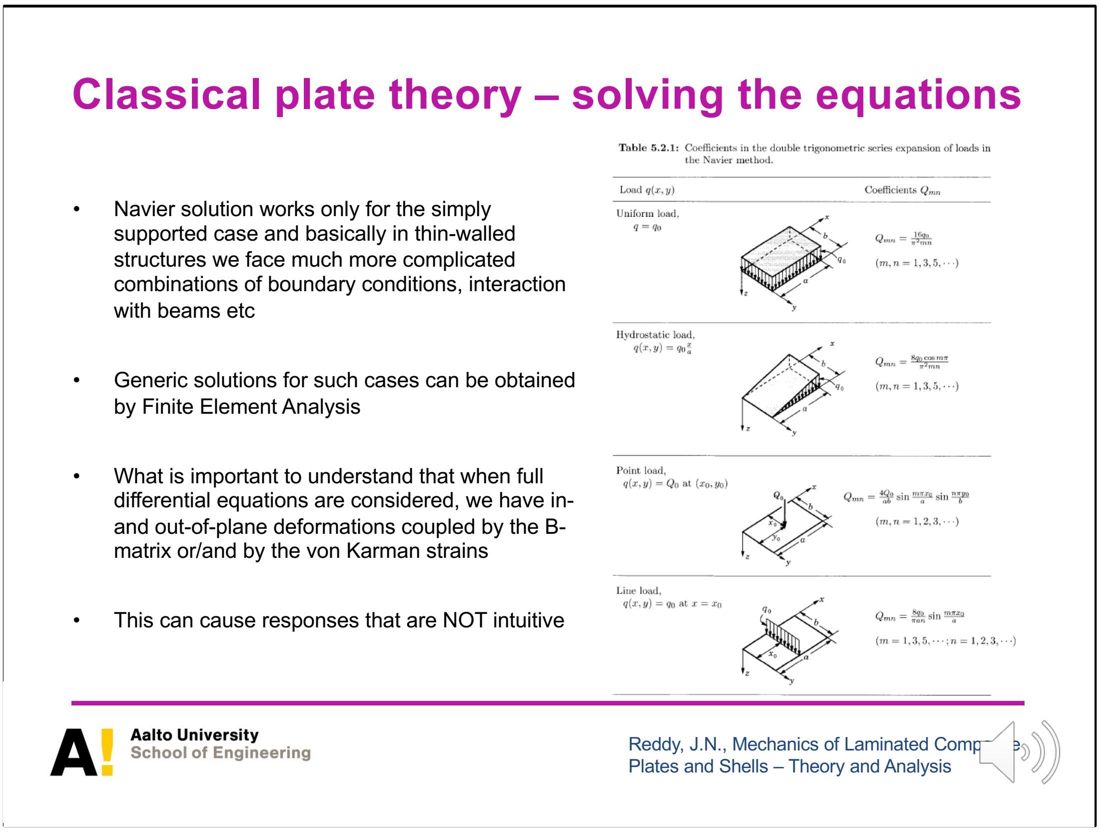
Plates and Shells – Theory and Analysis

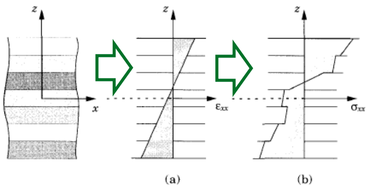
So subtituting the double series to the differential equation and doing the simplifaction gives us an equation which shows that the left hand side must be zero. This means in practice that the term in the brackets must be zero. We can solve the Wmn, when we know the Qmn from the loading. So the total deflection is composed from the load-term Qmn and the analytical stiffness term dmn. The series converges relatively fast for the deflections but due to the m and n values relatively slowly for the derivatives, which are needed for the stress assessment.

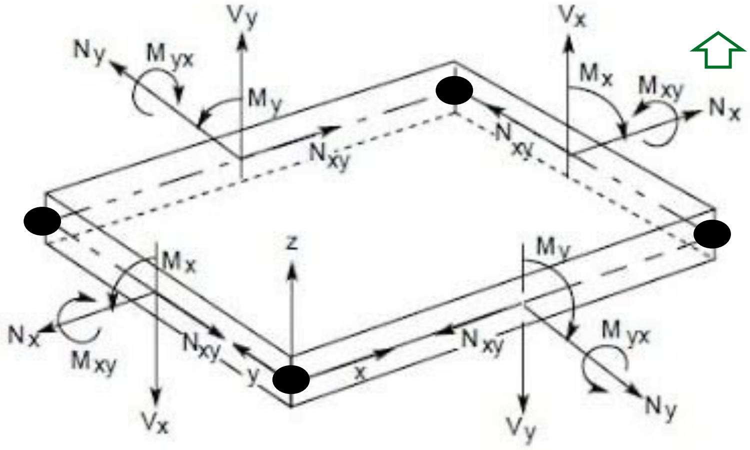


Navier solution works only for the simply supported case and basically in thin-walled structures we face much more complicated combinations of boundary conditions, interaction with beams and thus the solution is not enough for design purposes. It is good for the validation of out numerical models.

Generic solutions for more complicated cases can be obtained by Finite Element Analysis. What is important to understand that when full differential equations are considered, we have in- and out-of-plane deformations coupled by the B-matrix or/and by the von Karman strains. This can cause responses that are NOT intuitive. Therefore it is is important that you learn to keep book on which effects are active in our analysis and which ones are not. If we do not have geometrical non-linearity option activated the von Karman strains are neglected and the result is purely due to the B-matrix. If the B-matrix is zero, e.g. the laminate is symmetric with respect to the geometrical mid-plane, the coupling between in- and out-of-plane deformations does not exist. Even if all the effects are active, confusion can happen due to the implementation of the boundary conditions.

Element and Nodes





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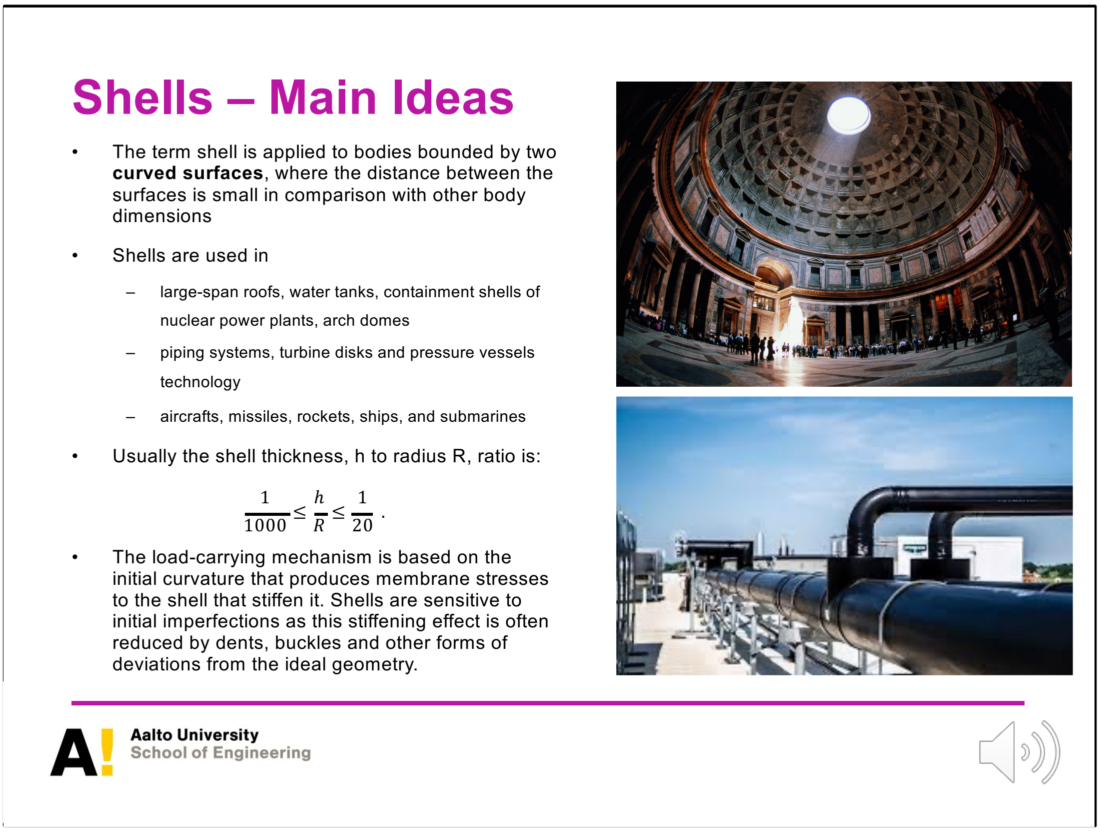
This is a typical example of an element used to solve the problem. The nodes (4) are located at the corners and the displacements and the nodal forces are computed there. These can be then converted to the stress resultants which can be used to define the strains and stresses in the failure analysis of each layer of the laminate.



The term shell is applied to bodies bounded by two **curved surfaces**, where the distance between the surfaces is small in comparison with other body dimensions. This means we deal with thin shells and this is valid assumption in thin-walled structures.

Shells are used in large-span roofs, such as domes, water/fuel/gas tanks, containment shells of nuclear power plants and arch domes in civil and energy engineering. Piping systems, turbine disks and pressure vessels technology in mechanical and process engineering. Aircrafts, missiles, rockets, ships, and submarines in vehicle engineering.

Usually the shell thickness, h to radius R, ratio is:

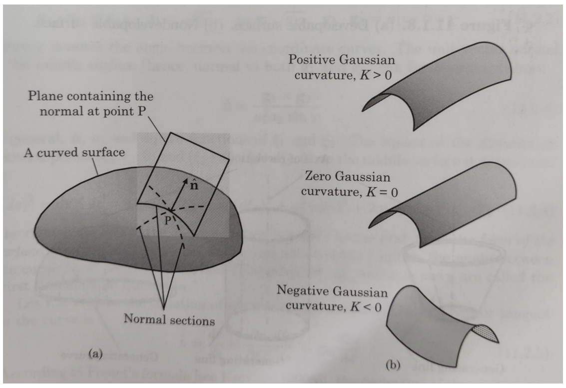
$$\frac{1}{1000} \leq \frac{h}{R} \leq \frac{1}{20} .$$

and the load-carrying mechanism is based on the initial curvature that produces membrane stresses to the shell that stiffen it. Shells are sensitive to initial imperfections as this stiffening effect is often reduced by dents, buckles and other forms of deviations from the ideal geometry. You can think this through a soda can which is very stiff when being intact, but with small dent they can collapse very easily.

Shells - Definitions

- The term Gaussian curvature is used to define the type of shell geometry, $K=K_1K_2$ in which K_1 is the maximum and K_2 the minimum principal curvatures
  - Positive Gaussian curvature, principal curvatures in the same direction, e.g. a dome, a synclastic surface
  - Zero Gaussian curvature, one of the principal curvatures is zero, e.g. a pipe
  - Negative Gaussian curvature, the principal curvatures have different sign, e.g. a saddle, a anticlastic surface

- The principal curvatures have an effect to the load-carrying mechanism of a shell and therefore they are the main design parameter of a shell with thickness and material properties



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The term Gaussian curvature is used to define the type of shell geometry, and Gaussian curvature is defined as product $K=K_1K_2$ in which K_1 is the maximum and K_2 the minimum principal curvatures. The definition of a curvature is the same as we learn from elementary mathematics and geometry.

Positive Gaussian curvature means that the principal curvatures are in the same direction, e.g. a dome, which is also called a synclastic surface and which is very stiff as the shape cannot be made a planar surface unless cut open and stretched.

Zero Gaussian curvature means that one of the principal curvatures is zero, e.g. a pipe. In this case the shell is very strong in the direction of the axis of the pipe, but we can cut open the shell easily without stretching it to planar surface.

Negative Gaussian curvature means that the principal curvatures have different sign, e.g. a saddle, a anticlastic surface. These surfaces are also very stiff.

The principal curvatures have an effect to the load-carrying mechanism of a shell and therefore they are the main design parameter of a shell with thickness and material properties.

Shells - Definitions

- The terms developable and non-developable surfaces are important from the load-carrying mechanism as they characterise how the shell surface can be reduced to planar surface
  - Developable surface can be reduced to plane surface without stretching, this is typical for a pipe
  - Non-developable surface requires stretching or cutting to become a planar surface, this is typical for a sphere
- Due to this the shell structures are often made to have a form of non-developable surfaces – it takes additional load to collapse them due to the internal forces created by the form
- Surfaces of revolution are obtained by rotating a plane curve about an axis

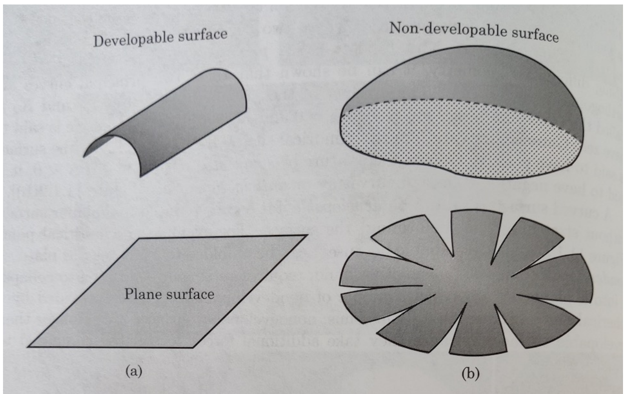

Figure 11.1.3. (a) Developable surface. (b) Nondevelopable surface.

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The terms developable and non-developable surfaces are important from the load-carrying mechanism as they characterise how the shell surface can be reduced to planar surface.

Developable surface can be reduced to planar surface without stretching, this is typical for a pipe.

Non-developable surface requires stretching or cutting to become a planar surface, this is typical for a sphere or a dome.

Due to this the shell structures are often made to have a form of non-developable surfaces – it takes additional load to collapse them due to the internal forces created by the form.

Surfaces of revolution are obtained by rotating a plane curve about an axis. This is another type of a shell classification.

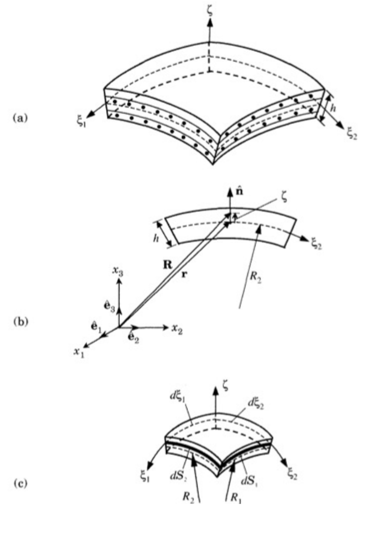
Shells – Geometry

- The geometry is often doubly-curved, R_1 and R_2, meaning that we can define two principle directions for the shell curvatures and the associated coordinate systems, $\xi\xi_1\xi_2$ follow the shell mid-surface and its normal
- The differential shell element is defined by

$$dS_1 d\zeta = A_1\, d\xi_1 d\zeta = a_1\left(1+\frac{\zeta}{R_1}\right) d\xi_1\, d\zeta, \qquad dS_2 d\zeta = A_2\, d\xi_2 d\zeta = a_2\left(1+\frac{\zeta}{R_2}\right) d\xi_2\, d\zeta$$

where dS_1 and dS_2 are the arch lengths and A_1 and A_2 are the Lamé coefficients.

- Note that if the radius R_1 or R_2 are infinite, the equation reduces to that of a plate
- The details of the definition of this geometrical definition of a shell can be found from
  1. Reddy, JN. “Theory and analysis of elastic plates and shells”
  2. Reddy, J.N., Mechanics of Laminated Composite Plates and Shells – Theory and Analysis



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The geometry is often doubly-curved, R1 and R2, meaning that we can define two principle directions for the shell curvatures and the associated coordinate systems, follow the shell mid-surface and its normal. Note that the 1 and 2 planes are now curved. The differential shell element is defined by differentials the arch length and the thickness where dS1 and dS2 are the arch lengths and A1 and A2 are the Lamé coefficients. Note that if the radius R1 or R2 are infinite, the equation reduces to that of a plate. The details of the definition of this geometrical definition of a shell can be found from:

1.Reddy, JN. “Theory and analysis of elastic plates and shells”
2.Reddy, J.N., Mechanics of Laminated Composite Plates and Shells – Theory and Analysis

In which the vector algebra leading to these equations can be checked step-by-step.

Shells – Stress Resultants

- The stress resultants are obtained by integrating the stresses over the thickness of the shell as was the case in plates
- However, now the arch lengths of infinitesimal units are depended on the curvature, so for example the stress resultant N_{11} is given by:

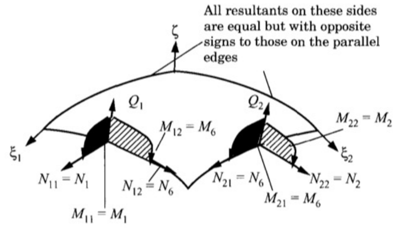
$$\int_{-h/2}^{h/2} \sigma_{11}\, dS_2\, d\zeta = a_2 \left[\int_{-h/2}^{h/2} \sigma_{11} \left(1 + \frac{\zeta}{R_2}\right) d\zeta \right] d\xi_2 \equiv N_{11}\, a_2\, d\xi_2$$

which reduces to

$$N_{11} = \int_{-h/2}^{h/2} \sigma_{11} \left(1 + \frac{\zeta}{R_2}\right) d\zeta$$

- For the moment we obtain

$$M_{11} = \int_{-h/2}^{h/2} \zeta\sigma_{11} \left(1 + \frac{\zeta}{R_2}\right) d\zeta$$



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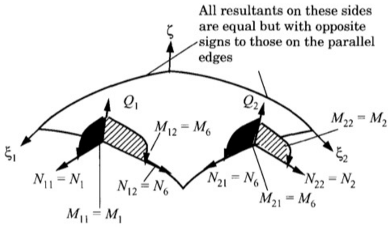
The stress resultants are obtained by integrating the stresses over the thickness of the shell as was the case in plates. However, now the arch lengths of infinitesimal units are depended on the curvature, so for example the stress resultant N11 is given by the product of the stress sigma11 times the length of the segment S2 and integration over the entire thickness of the shell. Note here that if we have a laminate the stress is dependent on the layer properties. So for the normal force we get something very similar to plates, expect now we have the term with radius R22 present in the integral. When we do the same for moments again similar relation as in the case of plates with the additional term in the brackets. This term forms the stiffening effect.

Shells – Stress Resultants

- Doing these for the all stress resultants as in the case of plates, we obtain

$$
\begin{Bmatrix} N_{11} \\ N_{22} \\ N_{12} \\ N_{21} \\ M_{11} \\ M_{22} \\ M_{12} \\ M_{21} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_1 \left(1 + \frac{\zeta}{R_2}\right) \\ \sigma_2 \left(1 + \frac{\zeta}{R_1}\right) \\ \sigma_6 \left(1 + \frac{\zeta}{R_2}\right) \\ \sigma_6 \left(1 + \frac{\zeta}{R_1}\right) \\ \zeta\sigma_1 \left(1 + \frac{\zeta}{R_2}\right) \\ \zeta\sigma_2 \left(1 + \frac{\zeta}{R_1}\right) \\ \zeta\sigma_6 \left(1 + \frac{\zeta}{R_2}\right) \\ \zeta\sigma_6 \left(1 + \frac{\zeta}{R_1}\right) \end{Bmatrix} d\zeta
$$



where we see that

- 12 and 21 directions are not equals as in plates
- The stress resultants are dependent on the principal radiuses of the shell

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So doing these for the all stress resultants as in the case of plates, we obtain a 8x1 vector in which the integration is always done over the thickness and with the radiuses being part of the stress resultant. One large exception here to the plates is that we do not have the symmetry 12 and 21. If the radiuses are infinite, we get this symmetry. The stress resultants are dependent on the principal radiuses of the shell.

Shells – Strains as Displacements

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- The strains are given as:

$$\varepsilon_1 = \frac{1}{A_1}\left(\frac{\partial u_1}{\partial \xi_1} + \frac{1}{a_2}\frac{\partial a_1}{\partial \xi_2}u_2 + \frac{a_1}{R_1}u_3\right)$$
$$\varepsilon_2 = \frac{1}{A_2}\left(\frac{\partial u_2}{\partial \xi_2} + \frac{1}{a_1}\frac{\partial a_2}{\partial \xi_1}u_1 + \frac{a_2}{R_2}u_3\right)$$

$$\varepsilon_3 = \frac{\partial u_3}{\partial \zeta}$$
$$\gamma_{23} = \frac{1}{A_2}\frac{\partial u_3}{\partial \xi_2} + A_2\frac{\partial}{\partial \zeta}\left(\frac{u_2}{A_2}\right) \equiv \varepsilon_4$$
$$\gamma_{13} = \frac{1}{A_1}\frac{\partial u_3}{\partial \xi_1} + A_1\frac{\partial}{\partial \zeta}\left(\frac{u_1}{A_1}\right) \equiv \varepsilon_5$$
$$\gamma_{12} = \frac{A_2}{A_1}\frac{\partial}{\partial \xi_1}\left(\frac{u_2}{A_2}\right) + \frac{A_1}{A_2}\frac{\partial}{\partial \xi_2}\left(\frac{u_1}{A_1}\right) \equiv \varepsilon_6$$

where The Lame constants are

$$A_1 = a_1\left(1 + \frac{\zeta}{R_1}\right), \quad A_2 = a_2\left(1 + \frac{\zeta}{R_2}\right),$$

and

$$\xi_3 = \zeta, \qquad A_3 = a_3 = 1$$

- Assuming the CLT displacement field we have in shell coordinate system

$$u_1(\xi_1, \xi_2, \zeta, t) = u_0(\xi_1, \xi_2, t) + \zeta\phi_1(\xi_1, \xi_2, t)$$
$$u_2(\xi_1, \xi_2, \zeta, t) = v_0(\xi_1, \xi_2, t) + \zeta\phi_2(\xi_1, \xi_2, t)$$
$$u_3(\xi_1, \xi_2, \zeta, t) = w_0(\xi_1, \xi_2, t)$$

- we get the strains as displacements

$$\varepsilon_1 = \frac{1}{(1+\zeta/R_1)}\left(\varepsilon_1^0 + \zeta\varepsilon_1^1\right), \quad \varepsilon_2 = \frac{1}{(1+\zeta/R_2)}\left(\varepsilon_2^0 + \zeta\varepsilon_2^1\right)$$
$$\varepsilon_6 = \frac{1}{(1+\zeta/R_1)}\left(\omega_1^0 + \zeta\omega_1^1\right) + \frac{1}{(1+\zeta/R_2)}\left(\omega_2^0 + \zeta\omega_2^1\right)$$
$$\varepsilon_4 = \frac{1}{(1+\zeta/R_2)}\varepsilon_4^0, \qquad \varepsilon_5 = \frac{1}{(1+\zeta/R_1)}\varepsilon_5^0$$

where

$$\varepsilon_1^0 = \frac{1}{a_1}\left(\frac{\partial u_0}{\partial \xi_1} + \frac{1}{a_2}\frac{\partial a_1}{\partial \xi_2}v_0 + \frac{a_1}{R_1}w_0\right), \quad \varepsilon_2^0 = \frac{1}{a_2}\left(\frac{\partial v_0}{\partial \xi_2} + \frac{1}{a_1}\frac{\partial a_2}{\partial \xi_1}u_0 + \frac{a_2}{R_2}w_0\right)$$
$$\varepsilon_4^0 = \frac{1}{a_2}\left(\frac{\partial w_0}{\partial \xi_2} + a_2\phi_2 - \frac{a_2}{R_2}v_0\right), \quad \varepsilon_5^0 = \frac{1}{a_1}\left(\frac{\partial w_0}{\partial \xi_1} + a_1\phi_1 - \frac{a_1}{R_1}u_0\right)$$
$$\omega_1^0 = \frac{1}{a_1}\left(\frac{\partial v_0}{\partial \xi_1} - \frac{1}{a_2}\frac{\partial a_1}{\partial \xi_2}u_0\right), \quad \omega_2^0 = \frac{1}{a_2}\left(\frac{\partial u_0}{\partial \xi_2} - \frac{1}{a_1}\frac{\partial a_2}{\partial \xi_1}v_0\right)$$

$$\varepsilon_1^1 = \frac{1}{a_1}\left(\frac{\partial \phi_1}{\partial \xi_1} + \frac{1}{a_2}\frac{\partial a_1}{\partial \xi_2}\phi_2\right), \quad \varepsilon_2^1 = \frac{1}{a_2}\left(\frac{\partial \phi_2}{\partial \xi_2} + \frac{1}{a_1}\frac{\partial a_2}{\partial \xi_1}\phi_1\right)$$
$$\omega_1^1 = \frac{1}{a_1}\left(\frac{\partial \phi_2}{\partial \xi_1} - \frac{1}{a_2}\frac{\partial a_1}{\partial \xi_2}\phi_1\right), \quad \omega_2^1 = \frac{1}{a_2}\left(\frac{\partial \phi_1}{\partial \xi_2} - \frac{1}{a_1}\frac{\partial a_2}{\partial \xi_1}\phi_2\right)$$

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As in the plates we can obtain the shell strains from the displacements by differentiation. We see the radius dependency here also. For details of the strain definition, see the books of Reddy. Note that we use here the Lame constants to simplify the equations. When we assume the CLT kinematics, that is planes remain planes, and 90
degrees with respect to the mid-surface and not thickness direction straining, we can write the displacement field and derive the stains from the displacements

Shells – Equations of Motions

- The equations of motions are:

$$
\begin{aligned}
&\frac{\partial}{\partial \xi_1}\left(a_2 N_{11}\right)+\frac{\partial}{\partial \xi_2}\left(a_1 \tilde{N}_{12}\right)-N_{22} \frac{\partial a_2}{\partial \xi_1}+\tilde{N}_{12} \frac{\partial a_1}{\partial \xi_2}+\frac{a_1 a_2}{R_1} Q_1 \\
&\qquad +\frac{a_1}{2} \frac{\partial}{\partial \xi_2}\left(\frac{M_{12}}{R_1}-\frac{M_{21}}{R_2}\right)=a_1 a_2\left(I_0 \frac{\partial^2 u_0}{\partial t^2}+I_1 \frac{\partial^2 \phi_1}{\partial t^2}\right) \\
&\frac{\partial}{\partial \xi_1}\left(a_2 \tilde{N}_{12}\right)+\frac{\partial}{\partial \xi_2}\left(a_1 N_{22}\right)-N_{11} \frac{\partial a_1}{\partial \xi_2}+\tilde{N}_{12} \frac{\partial a_2}{\partial \xi_1}+\frac{a_1 a_2}{R_2} Q_2 \\
&\qquad +\frac{a_2}{2} \frac{\partial}{\partial \xi_1}\left(\frac{M_{21}}{R_2}-\frac{M_{12}}{R_1}\right)=a_1 a_2\left(I_0 \frac{\partial^2 v_0}{\partial t^2}+I_1 \frac{\partial^2 \phi_2}{\partial t^2}\right) \\
&\frac{\partial}{\partial \xi_1}\left(a_2 Q_1\right)+\frac{\partial}{\partial \xi_2}\left(a_1 Q_2\right)-a_1 a_2\left(\frac{N_{11}}{R_1}+\frac{N_{22}}{R_2}\right)=a_1 a_2 I_0 \frac{\partial^2 w_0}{\partial t^2} \\
&\frac{\partial}{\partial \xi_1}\left(a_2 M_{11}\right)+\frac{\partial}{\partial \xi_2}\left(a_1 \tilde{M}_{12}\right)-M_{22} \frac{\partial a_2}{\partial \xi_1}+\tilde{M}_{12} \frac{\partial a_1}{\partial \xi_2}-a_1 a_2 Q_1 \\
&\qquad =a_1 a_2\left(I_1 \frac{\partial^2 u_0}{\partial t^2}+I_0 \frac{\partial^2 \phi_1}{\partial t^2}\right) \\
&\frac{\partial}{\partial \xi_1}\left(a_2 \tilde{M}_{12}\right)+\frac{\partial}{\partial \xi_2}\left(a_1 M_{22}\right)-M_{11} \frac{\partial a_1}{\partial \xi_2}+\tilde{M}_{12} \frac{\partial a_2}{\partial \xi_1}-a_1 a_2 Q_2 \\
&\qquad =a_1 a_2\left(I_1 \frac{\partial^2 v_0}{\partial t^2}+I_0 \frac{\partial^2 \phi_2}{\partial t^2}\right)
\end{aligned}
$$

$$
\tilde{N}_{12}=\tilde{N}_{21}=\frac{1}{2}\left(N_{12}+N_{21}\right), \quad \tilde{M}_{12}=\tilde{M}_{21}=\frac{1}{2}\left(M_{12}+M_{21}\right)
$$

Reddy, J.N., Mechanics of Laminated Composite Plates and Shells – Theory and Analysis

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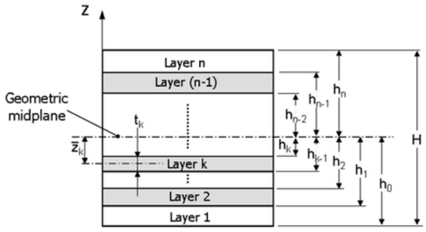
Finally substituting everything to the equilibrium equations, gives us the equations of motions which look similar to the plates except with the radius terms included. We could substitute here the displacements and stress resultants written through strains to get extremely length differential equations for shells. These we can solve in very special cases analytically but often we need FEM to solve them in engineering problems.

Summary

- In classical lamination theory, the composite effect is modelled with simplest possible model
- Assumptions made are:
  - Plane remain planes and at 90 degrees with respect to deformed mid-plane
  - The thickness of the cross-section does not change (and the plate is in plane stress)
- As a results we get a model where stiffnesses are computed with respect to selected plane (often geometrical mid-plane)
- As we select a plane, we end up with
  - Membrane stiffness, A-matrix
  - Bending stiffness, D-matrix
  - Membrane-bending-coupling stiffness, B-matrix
- In shells the same principles apply as in plates, but the principal curvatures alter the geometry definitions, creating more complexity to the solutions, but also beneficial effects to the load-carrying mechanism of the thin-walled structure



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In classical lamination theory, the composite effect is modelled with simplest possible model and the assumptions made are:
- Planes remain planes and at 90 degrees with respect to deformed mid-plane, which allows us to connect the out-of-plane deflections to the in-plane deformations through the rotation of the deflection line
- The thickness of the cross-section does not change (and the plate is in plane stress), which means we can use initial configuration in the analysis.

As a results we get a model where stiffnesses are computed with respect to selected plane (often geometrical mid-plane). As we select a plane, we end up with: membrane stiffness, denoted by A-matrix, bending stiffness, D-matrix and membrane-bending-coupling stiffness, B-matrix.

In shells the same principles apply as in plates, but the principal curvatures alter the geometry definitions, creating more complexity to the solutions, but also beneficial effects to the load-carrying mechanism of the thin-walled structure