Aalto University
School of Engineering

MEC-E8007 Thin-Walled Structures

Lecture 8. Equivalent Plates

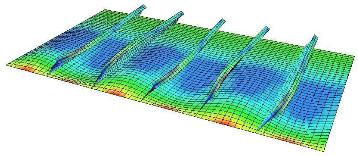
Jani Romanoff

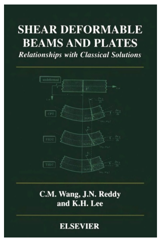
Aalto University
School of Engineering

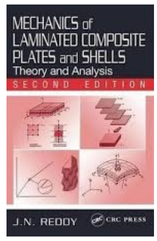
Now when we now on one hand the orhtropic plate theory based on CLT or FSDT and on the other hand the beam theory extension, we can take the ideas a bit further and think about plates that have stiffeners and how to reduce the size of these assemblies. This are perhaps the most common structural units we have in mechanical, civil, aerospace and maritime engineering.

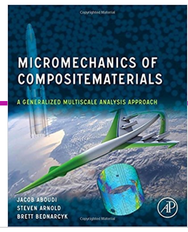
Contens

- The aim of the lecture is to *understand* the principles of homogenization based equivalent plates and shells and to learn to *apply* these to large FE-models
- Motivation
- Basics of homogenization
- Definition of membrane, plate and shell
- Equivalent shell element formulation
- Usage of equivalent shell elements
- Examples
- Literature
    1. Reddy, J.N., Mechanics of Laminated Composite Plates and Shells – Theory and Analysis, 2nd Edition, CRC Press, Ch. 3
    2. Wang, C.M., Reddy, J.N. and Lee, K.H., “Shear deformable beams and plates”, Elsevier, Ch. 13
    3. Aboudi, J., Arnold, S. and Bednarcyk, B., “Micromechanics of Composite Materials – A Generalised Multiscale Analysis Apporach”, AP.







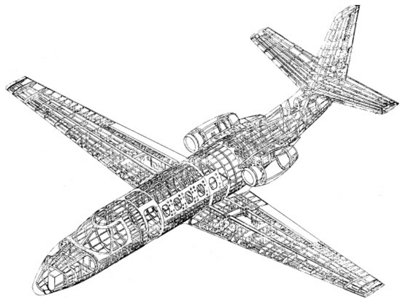


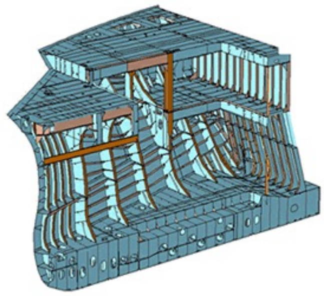
Aalto University
School of Engineering

The aim of the lecture is to understand the principles of homogenization based equivalent plates and shells and to learn to apply these to large FE-models. We can gain enormous savings in terms of modeling, solution and post-processing if we can do this properly. First we need to go through the basics of homogenization, the definition of membrane, plate and shell in order to properly formulate the equivalent plate element formulation. Then we must also discuss few practical aspects that we have with this modeling which are due to the fact that we push here some elementary and fundamental assumptions. Good books here are Reddy and Aboudi.

Motivation

- Many of the lightweight structures are assemblies of beams and plates
  - Ships
  - Airplanes
  - Bridges
- If included directly to the model, the stiffener defines location for the element mesh boundaries
  - In global response analysis this can lead to too fine mesh
  - We need to re-mesh if the stiffener spacing is to be changed – this is expensive
- We need a method to get rid of the stiffener-based mesh size, but include the stiffener effects into the model





Aalto University
School of Engineering

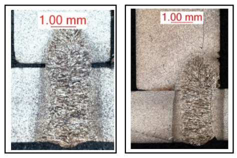
Many of the lightweight structures are assemblies of beams and plates (Ships, Airplanes, Bridges) if the beams are included directly to the model directly, the stiffener defines location for the element mesh boundaries. This means that if we want to optimize in FEA context, we need to recreate the mesh every time we change the stiffener spacing. This is very expensive approach and restricts in practice the freedom of design due to the selected approach. In addition, in the global response analysis this can lead to too fine mesh. Therefore, we need a method to get rid of the stiffener-based mesh size, but include the stiffener effects into the model so that the responses are correctly captured and our design is feasible.

Motivation

Ship's deck

Steel sandwich

Laser weld



10^3m — 10^2m — 10^1m — 10^{-3}m — 10^{-6}m

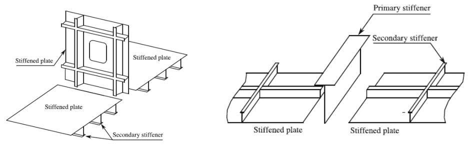
Problem investigated at this scale! ← Scale interaction → Mesh size defined at this scale!

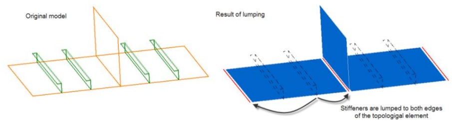
Aalto University
School of Engineering

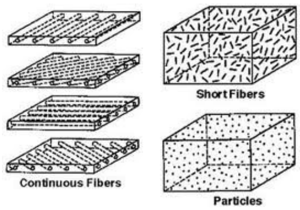
To look at the numerical challenge. We have a thin-walled structure meaning that the thickness is much smaller than the other dimensions. On the other hand we now that the elements much have good shape and aspect ratio. So modeling for example the weld details of advanced manufacturing with solids at the length-scale of 10^{-6}m and try to scale this up to the ship which have hundreds of 10^3meter's (1km) of welds would mean enormous FE-mesh. If we have to model and change this in optimization on about 1000 times, we can barely do this during our careers.

Basics of Homogenization

- Stiffeners are of many sizes
  - Small: stiffens the plate, spacing can be varied
  - Large: stiffens the panel, spacing depends on general arrangement (pillars, systems, etc)
- The stiffener removal can be done in various ways and is typically done only to small stiffeners
- Global bending stiffness of large girders (ships, bridges) to be modeled correctly: lump the stiffeners to one location
  - Stiffness modeled correctly in vertical direction but not in transverse
  - Allows variation of stiffener spacing or number of stiffeners
- **Stiffeners can be included to shell element formulation (analogy to composites)**
  - Exclude the local response between the stiffeners
  - The stiffness is modeled correctly in all directions
  - Allows variation of stiffener spacing or number of stiffeners







Aalto University
School of Engineering

Stiffeners are of many sizes in practice:

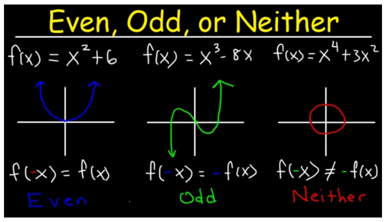
- Small: stiffens the plate, spacing can be varied
- Large: stiffens the panel, spacing depends on general arrangement (pillars, systems, etc)

The stiffener removal can be done in various ways and is typically done only to small stiffeners as there the possible errors are less severe than if this is done to the larger units. If the global bending stiffness of large girders (ships, bridges) is to be modeled correctly lumping the stiffeners to one location (good enough) does great job of simplifying the model without sacrificing the accuracy of the global assessment. If this is done with vertical bending in mind, it can make some error to the transverse direction as the stiffeners are not in the right place (d^2-term). However, this still allows variation of stiffener spacing or number of stiffeners, we just lump new amount of stiffness to these locations.

Much better approach is that the stiffeners are be included to shell element formulation (analogy to composites) as distributed “mass” that produces right stiffness. This means that we exclude the local response between the stiffeners, but retain almost all other aspects to the equivalent model. Then the stiffness is modeled correctly in all directions. This also allows variation of stiffener spacing or number of stiffeners, without need to re-mesh the model.

Basics of Homogenization

- **Homogenization** is a process that *averages* the response over *unit cell* (or representative volume element, RVE)
  - Odd functions of response become zero
  - Even functions get finite value



- We can recalculate the periodic response from averaged solution - **localization**

Wavy pattern is odd within unit cell and disappears in homogenization

$$w_{avg} = \int_0^s \frac{w(x_l)}{s} dx_l$$

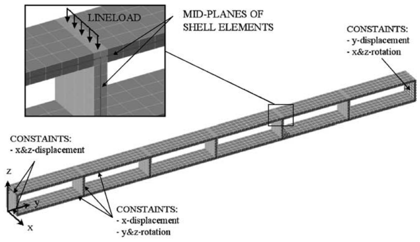


Fig. 6. Example of 3D shell-element model used in comparison study. The finite element mesh of the beam with loading and constraints is presented. Case B: 14/4 × 20/80, B = 960 mm.

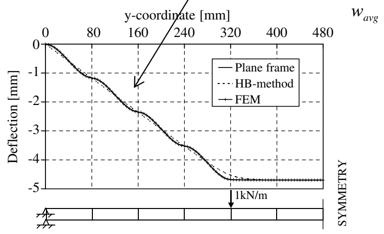


Fig. 7. Comparison of deflection in top face plate obtained by plane frame analysis, HB-method and FE-analysis. Case A: 11/4 × 80/80, B = 960 mm.

Aalto University
School of Engineering

Homogenization is a process that averages the response over unit cell (or representative volume element, RVE). This is important to understand also from the mathematical perspective. Averaging odd function over the period, results in zero average. Always. Doing this to even function results in scalar which is always nonzero. So if we think about structural responses that really occur, homogenization deletes any responses that are off over the period. Period is here for example the stiffener spacing. Here we have an example of this to show the physics, a periodic beam with uniform size in RVE's. The beam is in four-point-bending so that we can separate the shear and bending responses. If we could homogenize this correctly (and we can), then we could have enormous computational savings possible.

If we have the full control of this, we can also recalculate the periodic response from averaged solution – this is called localization. So we would get both displacements and stresses out both at macro- and micro-scales from the homogenized model.

Here it is good to stop for a while. Think about what we are about do. We get rid of something to accelerate the analysis and we state that we can recover the essential info from the accelerated analysis. This is the same as we would say that when we make coffee with filter and get the coffee to the mug and get rid of the water, we have all essential ingredients of coffee in the mug… If so, what stays in the filter? With that in mind what do we loose in homogenization?

Homogenization
“Average of odd-functions is zero over the period”

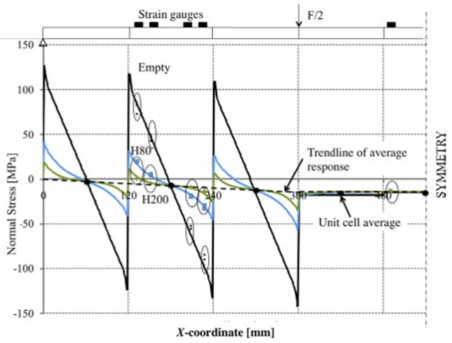


Fig. 5. Comparison of the normal stress in the top surface of top faceplate. $F = 400$ N. Solid lines are the FE-result, gray points the experimental and black dots the unit cell average of the FE- and experimental result.

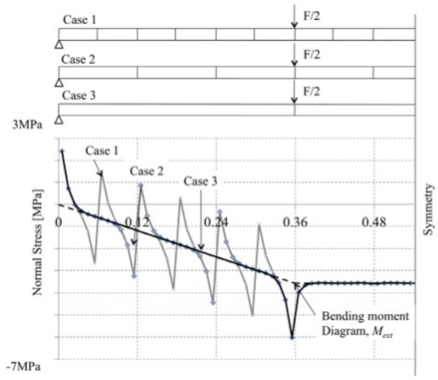


Fig. 7. Comparison of normal stress in the top surface of top faceplate. Beam filled with isotropic foam $E = 400$ MPa and $\nu = 0.3$ and with unit cell size, $L/s = 18$, $L/s = 9$, $L/s = 3$ $F = 400$ N.

Aalto University
School of Engineering

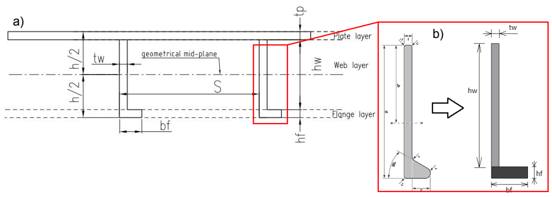
On the left we have the beam in 4-point bending filled with different foams. The foams have different density and therefore different stiffness. What we see is that the higher is the density and the shear stiffness of the foam the smaller is the normal stress measured at the top surface of the beam at the constant shear region. The responses are the same at the constant bending region, where shear is zero. The strain distributions are locally “odd” being straight lines for the empty panel and curved for foam filled panels due to the (very complex) shear lag inside the unit cells. If we average the responses for each unit cell, we see that the averages are much below the actual periodic stresses with odd-local description. We get rid of the periodic stress component, which might be the desicive factor for structural design. This is not good unless we can fix this.

The figure on the right shows another important thing. The size of the RVE is varied here. What we can see here is that again the average is much smaller than the actual periodic stress. However, we can see that there is similarity in the shape of the stress distributions. They are all odd and the extreme values are the same regardless of the unit cell size. They occur at the ends of the shear region. So it seems to be possible to do correct the error homogenization introduces. However, the mathematical problem we face here is ill-posed. That is we have a lot of right solutions which differ from each other for this case.

Basics of Homogenization

- In homogenization we have to decide what is the level of simplification
  - Kinematics: from solids to equivalent single layer?
  - How much of the effect we want to include: stiffness, stress, stress concentration
  - How do we carry out the averaging, i.e. unit cell size = stiffener spacing?

- All these affect the obtainable accuracy



Aalto University
School of Engineering

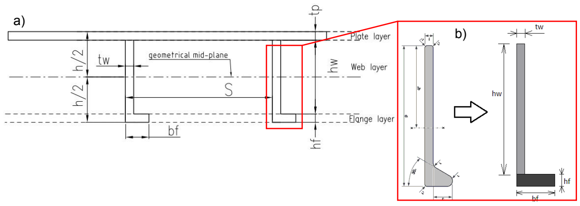
In homogenization we have to decide what is the level of simplification we are about to make and at what cost and gains. So for example the kinematics: from solids to equivalent single layer? As we set the kinematics for the equivalent model, we restrain the actual behavior. We may state that the stiffener cannot trip or warp. Then we need to decide how much of the effects we want to include to our homogenized solution: stiffness, stress, stress concentration. Are we happy with stiffness or do we want to know the stress which causes failure of the structure. Then we have to decide how do we carry out the averaging, i.e. unit cell size. Is that the same as the stiffener spacing? Or is it the same as the characteristic length of the deformation? All these affect the obtainable accuracy from the homogenization and ”how much stays in the filter”.

Definition of Membrane, Plate and Shell

- Typically planes model the large structure, so plane elements are natural choice
  - Membrane: include only the in-plane stiffness (A-matrix)
  - Plate: include only the out-of-plane stiffness (D-matrix)
  - Shell: include both in- and out-of-plane stiffness and their coupling (B-matrix)

- For example in ship global strength analysis DNV requires the membrane level
  - Local analysis separately
  - Allows large mesh size

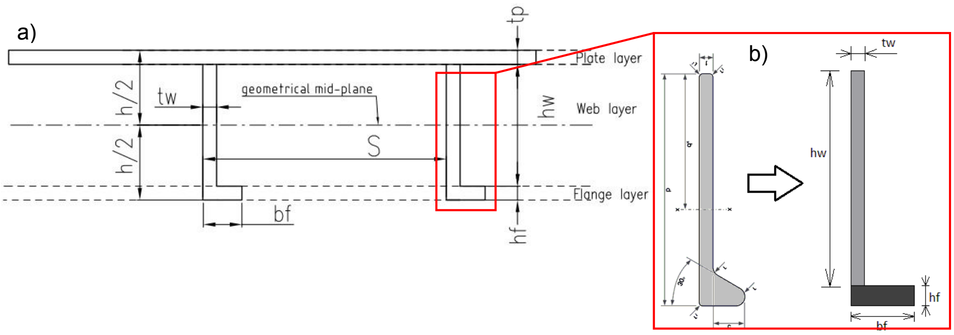
  since coupling with bending is missing



Aalto University
School of Engineering

Typically planes model the large structure, so plane elements are natural choice. This means we can have membrane elements that include only the in-plane stiffness (A-matrix) or we can have plate elements that include only the out-of-plane bending stiffness (D-matrix) or we can have a shell that includes include both in- and out-of-plane stiffness and their coupling (B-matrix). This is what we use if we want to assess both global and local responses with the same model in thin-walled structures. For example in ship global strength analysis DNV requires the membrane level only. This allows use of large elements.

Equivalent Shell Element Formulation

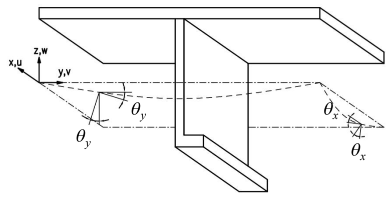


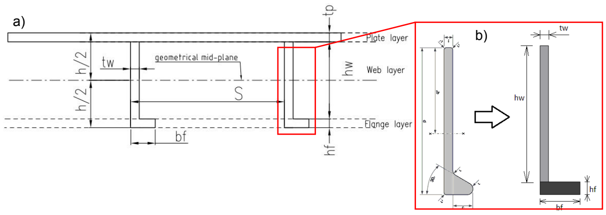
Aalto University
School of Engineering

So the element can be formulated like the CLT or FSDT elements were formulated. We first select the reference plane, for example the geometrical mid-plane. Then we define enough layers to describe all changes in material distrbution, like we did with the laminates. That is we homogenize in this example the web and flange layers but allow the plate layer to be what it is. We smear the two former layers.

Equivalent Shell Element Formulation

- Everything is modeled through single plane
  - Geometrical mid-plane (the neutral axis is different in xx-, yy- and xy-planes)
  - The kinematics are *First Order Shear Deformation Theory* or Reissner Mindlin
    - Planes remain planes in bending
    - Shear deformation included





Aalto University
School of Engineering

Everything is modeled through single plane, in this case the geometrical mid-plane as the the neutral axis is different in xx-, yy- and xy-planes in this case. Then we need to assume the kinematics like we did with the equivalent beams. In this case it is good to select the First Order Shear Deformation Theory or Reissner-Mindlin as we know from practice that the shear deformations are not negligible. This means that we assume that the planes remain planes in bending, but they can deviate from the 90 degree condition from the reference plane due to the shear deformation.

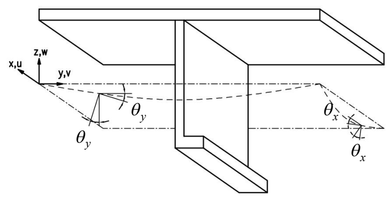
Equivalent Shell Element Formulation

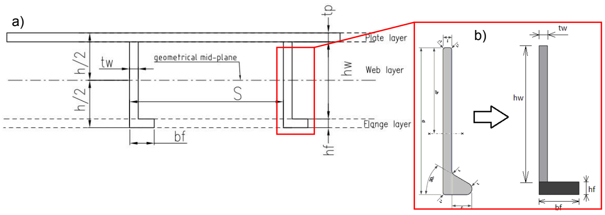
- Everything is modeled through single plane
  - Geometrical mid-plane (the neutral axis is different in xx-, yy- and xy-planes)
  - The kinematics are *First Order Shear Deformation Theory* or Reissner Mindlin
    - Planes remain planes in bending
    - Shear deformation included
- The process
  - From periodic displacements to periodic strains
  - Strains to periodic stresses
  - Average (=homogenize) to obtain stress resultants (N_x, N_y, N_{xy}, M_x, M_y, M_{xy}, Q_x, Q_y)
  - Balance with external loading

$$u_i = u_0(x,y) - z_i\theta_x(x,y)\,,$$

$$v_i = v_0(x,y) - z_i\theta_y(x,y)\,,$$

$$w_i = w(x,y),\ i = \text{p, w, f.,}$$





Then we apply the same process as before from kinematics through constitutive equations to equilbium equations to derive the differential equations.

Equivalent Shell Element Formulation

Displacements	$u_i = u_0(x,y) - z_i\theta_x(x,y)$, $v_i = v_0(x,y) - z_i\theta_y(x,y)$, $w_i = w(x,y)$, *i*=p, w, f.,
Strains	$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \varepsilon_{xy}^0 \end{Bmatrix} + z_i \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}$, *i*=p, w, f,
Strains as displacement	$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix} = \begin{Bmatrix} \partial u_0 / \partial x \\ \partial v_0 / \partial y \\ \partial u_0 / \partial y + \partial v_0 / \partial x \end{Bmatrix}$, $\begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} = \begin{Bmatrix} -\partial^2 w_g / \partial x^2 \\ -\partial^2 w_g / \partial y^2 \\ -2\partial^2 w_g / \partial x \partial y \end{Bmatrix}$. $\gamma_{xz}^i = -\theta_x + \dfrac{\partial w_i}{\partial x}$, *i*=p, w, f, $\gamma_{yz}^p = -\theta_y + \dfrac{\partial w_p}{\partial y}$.

Aalto University
School of Engineering

Here we describe the linear relation between the kinematics and strains. von Karman effect could be of course included.

Equivalent Shell Element Formulation

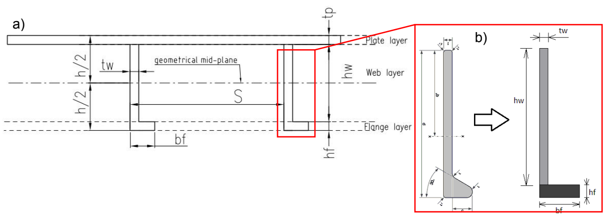
Plate layer elasticity

$$[E]_p = \frac{1}{(1-\nu^2)}\begin{bmatrix} E & \nu E & 0 \\ \nu E & E & 0 \\ 0 & 0 & G(1-\nu) \end{bmatrix}\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix},$$

$$G = \frac{E}{2(1+\nu)}.$$

Stress

$$\{\sigma\}_p = [E]_p \{\varepsilon\}_p ,$$



The plate was stated to be unhomogenised, so for isotropic plate layer the equivalent lamina property is the same as the original property.

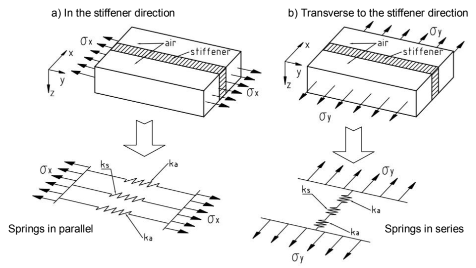
Equivalent Shell Element Formulation

Web and flange layer elasticity

$$[E]_w = \frac{t_w}{s}\begin{bmatrix} E & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[E]_f = \frac{b_f}{s}\begin{bmatrix} E & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Stress $\{\sigma\}_p = [E]_p \{\varepsilon\}_p \,,$



For the web and flange layers we use the rule of mixtures like in composites and the Reuss and Voigt conditions to derive the equivalent stiffness per layer. Note here that the matrix is air with zero material properties.

Equivalent Shell Element Formulation

Stress resultant

$$\{N\} = \int_{-h/2}^{h/2} [E]_i \{\varepsilon\}_i \,\mathrm{dz}, \quad i = \mathrm{p,f,w} \;,$$

$$\{M\} = \int_{-h/2}^{h/2} [E]_i \, z \{\varepsilon\}_i \,\mathrm{dz}, \quad i = \mathrm{p,f,w} \;.$$

Including the strains

$$\boxed{\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \varepsilon_{xy}^0 \end{Bmatrix} + z_i \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}, \; i\text{=p, w, f,}}$$

$$\{N\} = [A]\{\varepsilon_0\} + [B]\{\kappa\}.$$

$$\{M\} = [B]\{\varepsilon_0\} + [D]\{\kappa\}.$$

where

$$[A] = \int_{-h/2}^{-h/2+h_f} [E]_f dz + \int_{-h/2+h_f}^{h/2-h_p} [E]_w dz + \int_{h/2-h_p}^{h/2} [E]_p dz \,,$$

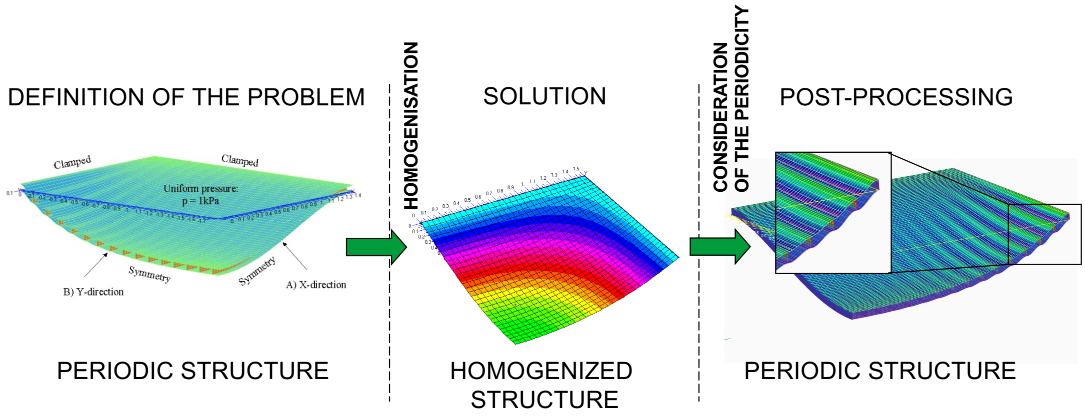
$$[B] = \int_{-h/2}^{-h/2+h_f} [E]_f z dz + \int_{-h/2+h_f}^{h/2-h_p} [E]_w z dz + \int_{h/2-h_p}^{h/2} [E]_p z dz \,,$$

$$[D] = \int_{-h/2}^{-h/2+h_f} [E]_f z^2 dz + \int_{-h/2+h_f}^{h/2-h_p} [E]_w z^2 dz + \int_{h/2-h_p}^{h/2} [E]_p z^2 dz \,.$$

Then we compute the stress resultant and get the ABD-matrices. So now we can use the composite FSDT derived at earlier rounds.

Basics of Homogenization-Localization

- Localization is the reverse process to homogenization, i.e. recovery of the periodic response from the smoothened behavior
- Ill-posed boundary value problem (several solutions for the same input)

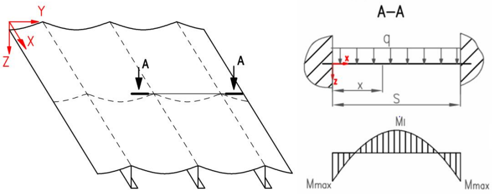


Aalto University
School of Engineering

What remains at the filter? The odd-functions. The shear-induced reactions. We also have there the part of the even functions that disappear if the unit cell size is very small. So local bending between the stiffners for example due to the uniform pressure applied to the plate. We need to recover these.

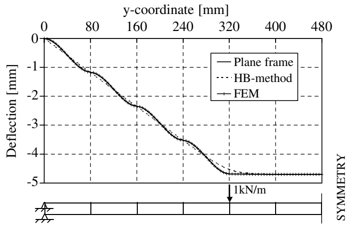
Usage of Equivalent Shell Elements
Local Effects

- The homogenization neglect certain effects
  - Asymptotically assumption is that s/B=0
  - This means that local response between stiffeners is neglected
  - Shear deformation is modeled with linear function $w_Q=k*x$ of the deflection although it is high order odd function $w_Q=k_1*x+k_2*x^3+k_3*x^5+\ldots$, this disappears in homogenization



- This can be easily corrected by using superposition principle and considering plate equilibrium

- The local effect can be significant in maximum stress estimations



Aalto University
School of Engineering

The homogenization neglect certain effects and we assume by smearing that s/B=0 where s is the stiffener spacing and B the characteristic length in the direction of stiffener spacing. This condition is not true. This means that local response between stiffeners is neglected from homogenized solution. On the other hand in FSDT the shear deformation is modeled with linear function wQ=k*x of the deflection although it is in reality a high order odd function wQ=k1*x+k2*x3+k3*x5+…, this disappears in homogenization. This can be easily corrected by using superposition principle and considering plate equilibrium to get rid of the ill-posed boundary value problem. As stated before these local effects can be significant in maximum stress estimations. This is why we recover these.

Examples

Eero Avi M.Sc Thesis 2012

Smooth deflection parallel to stiffeners   Oscillating deflection normal to stiffeners

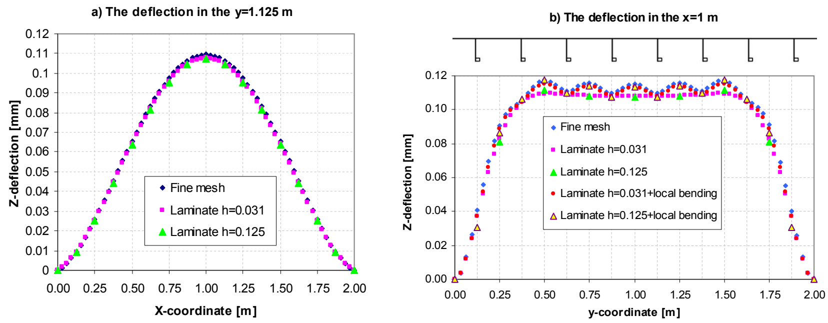


Figure 4.6: The deflections of the stiffened panel B under the fixed edge boundary conditions and the deck load of 1000 kg/m^2 (see also Figure 4.1b).

Aalto University
School of Engineering

Here is an example of this. Rectangular plate under uniform pressure with fixed end conditions. Single-sided stiffening is used with the laminate formulation. We recover the local pressure load effect with plate solution in which we consider clamped plate under uniform pressure and superimpose the resulting stress distribution to that predicted by the laminate formulation. If we use the laminate formulation we see that the predicted defletion results match almost perfectly with the fine mesh where the stiffeners are directly modelled.

Examples

Eero Avi M.Sc Thesis 2012

Stresses are obtained by differentation (oscillations get larger)

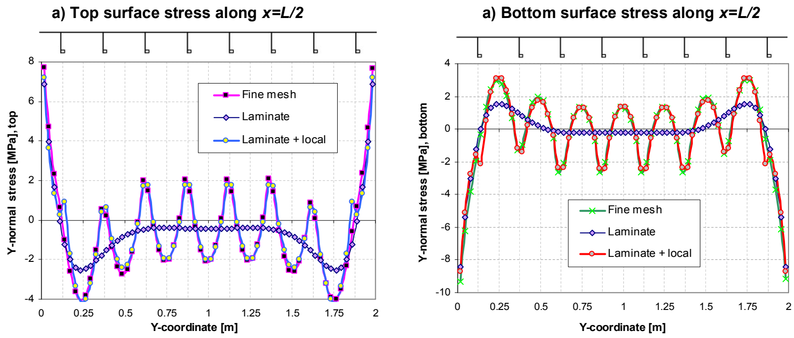


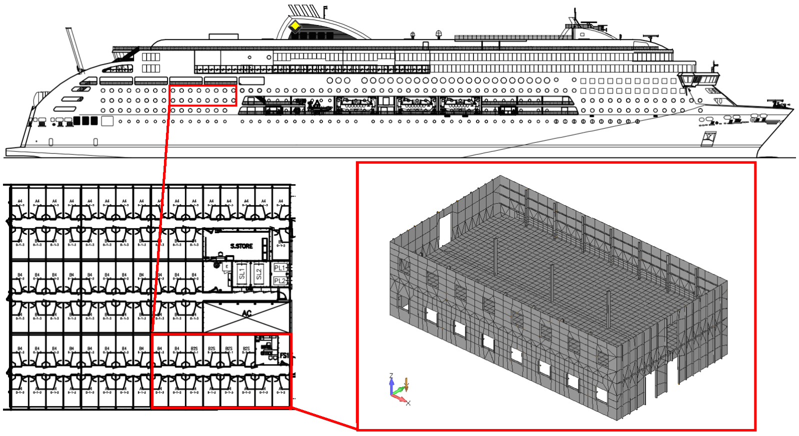
Figure 4.10: Comparison of y-directional stress distribution, σ_y, of the stiffened panel B under the fixed edge boundary conditions and the deck load of 1000 kg/m^2.

Aalto University
School of Engineering

When comparing the stresses, which are differentials of the displacements and this is why much more difficult to get agreement with, are also in pefect agreement. The main differences are at the plate edges, where periodic assumptions are not really anymore valid.

Examples
Eero Avi M.Sc Thesis 2012

Testing the ideas in large scale structure



Aalto University
School of Engineering

These ideas are also tested in larger scale in ship structure with a lot of complex interactions between structures at different planes. This is presented in D.Sc. Thesis of Eero Avi.

Examples
Eero Avi M.Sc Thesis 2012

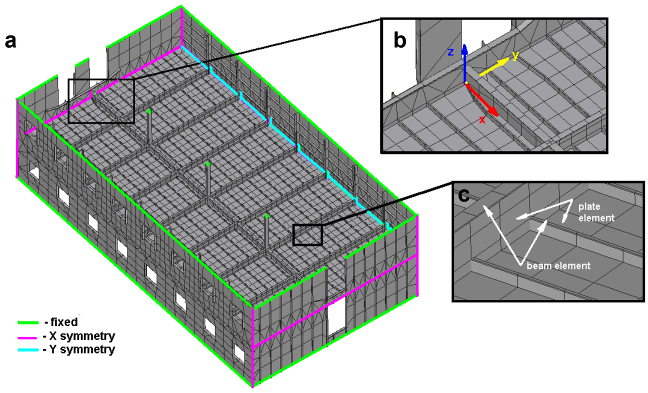


Figure 4.19: (a) Boundary conditions, (b) local coordinate system and (c) mesh density in the deck structure.

Aalto University
School of Engineering

So we have here decks, bulkheads and pillars interacting in a complex manner and we look mat the deflections and the stresses predicted with fine mesh and the laminate formulation for stiffened panel.

Examples

Eero Avi M.Sc Thesis 2012

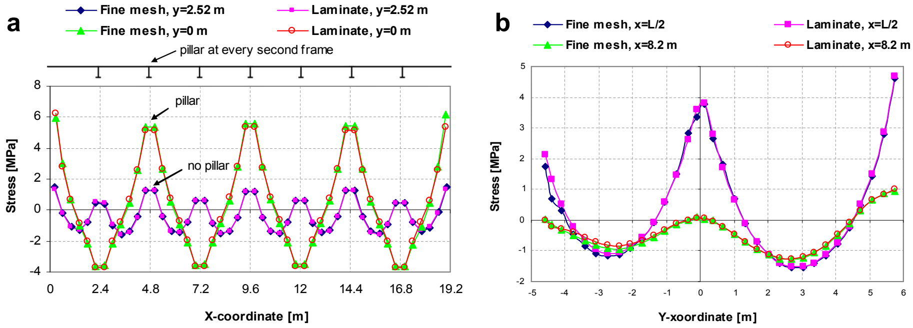


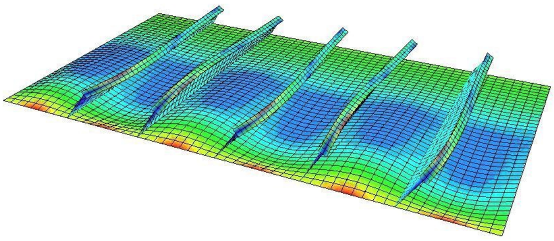
Figure 4.21: Comparison of x-normal stress distribution, σ_x, of deck structure under deck load of 120 kg/m^2.

Aalto University
School of Engineering

As stresses are the differentials of displacements, comparing these tells us much more about the agreement betweent the models. What we can see is that the local responses are accurate, but so is the interaction between the girders and the equivalent shell developed. So the offset beam and the homogenised plate together produce very accurate responses with fraction of the degrees of freedom of the fine mesh.

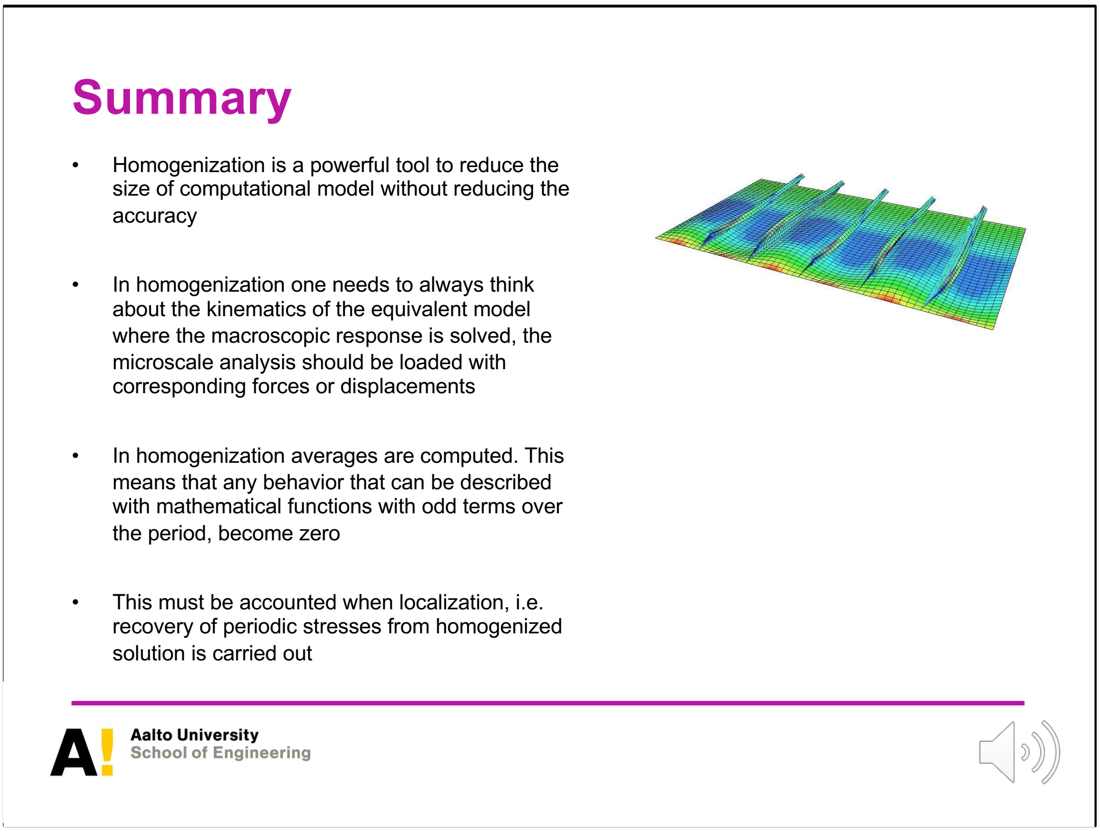
Usage of Equivalent Shell Elements
Buckling Analysis

- When global buckling and post-buckling of the panel needs to be assessed the method is promising if ABD-matrix can be non-linear to include local failure between stiffeners
- DNV PULS software does this (Byklum et al, 2004)



Aalto University
School of Engineering

When global buckling and post-buckling of the panel needs to be assessed the method is promising if ABD-matrix can be non-linear to include local failure between stiffeners, see for example DNV PULS software does this (Byklum et al, 2004).



Homogenization is a powerful tool to reduce the size of computational model without reducing the accuracy

In homogenization one needs to always think about the kinematics of the equivalent model where the macroscopic response is solved, the microscale analysis should be loaded with corresponding forces or displacements

In homogenization averages are computed. This means that any behavior that can be described with mathematical functions with odd terms over the period, become zero

This must be accounted when localization, i.e. recovery of periodic stresses from homogenized solution is carried out