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Some recent advances in the concepts of plate-effectiveness evaluation

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Abstract

The effectiveness of plate elements in a continuous plate structure can be reduced by lateral deflections that primarily occur due to lateral pressure loading or buckling. It is recognized that deflected plates can be modeled as equivalent flat (undeflected) plates but with reduced effectiveness. A number of the concepts to evaluate the effectiveness of deflected plates have been developed in the literature. The present paper surveys recent advances in the concepts of plate-effectiveness evaluation in terms of effective breadth, effective width, and effective shear modulus. Some closed-form expressions of the plate effectiveness are reviewed. The present paper is dedicated to Prof. J. Rhodes who made a great achievement and contribution to the related areas noted above, among others.

Keywords: Plate elements; Deflected plates; Buckled plates; Plate effectiveness; Effective breadth; Effective width; Effective tangent modulus; Effective shear modulus

1. Background to the concepts of plate-effectiveness evaluation

Plate elements in a continuous plate structure can deflect for various reasons: welding used for attaching stiffeners to the plates can induce initial deflection, lateral pressure loading can cause lateral deflection, and axial compression or edge shear actions can give rise to buckling that subsequently results in deflection.

The membrane stress distribution inside a deflected plate is not uniform anymore in contrast to a flat or undeflected plate. The behavior of a plate with lateral deflection exhibits geometrical nonlinearity that is distinct from Hook's law where the relationship between membrane stress and strain is linear. It is clear with certainty that a much more efficient procedure of computations can be developed for nonlinear structural behavior associated with geometrical nonlinearity if the behavior of a deflected plate can be treated as a linear problem even after the inception of plate deflections.

This issue could be resolved by adopting the concept of an equivalent flat plate, i.e., without lateral deflection but with reduced plate effectiveness. This means that the deflected plate is virtually dealt with as an undeflected or flat plate but the plate effectiveness has been reduced. Since the object plate is now a flat plate, i.e., without lateral deflection, the membrane stress distribution inside the plate must be uniform, and thus Hook's law is applicable so that the computational procedure for the plate behavior can become much simpler.

In terms of applying the equivalent flat plate concept, the fundamental question is how to evaluate and identify the reduction of plate effectiveness in the deflected plate. It is readily understood that the plate effectiveness will be a function of lateral deflection, meaning that the plate effectiveness could be progressively varied (decreased) as the lateral deflection increases. Since the lateral deflection is a function of applied actions (loads), the plate effectiveness must essentially be a function of applied actions.

When normal stress is predominant inside the deflected plate, a deflected plate is virtually modeled as an equivalent flat plate but with a reduced plate breadth. There are two different terms to make a distinction between the types of actions in the evaluation of plate effectiveness [1]. The term effective breadth or effective flange width is typically used when the lateral deflection is caused by out-of-plane or lateral actions such as lateral pressure loads in association with shear-lag effect. The term effective width is typically used when the deflection occurs by buckling under

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predominantly axial compressive loads. Therefore, the evaluation of plate effectiveness corresponds to the identification of plate breadth or plate width, for plate elements deflected by actions that cause normal stresses.

On the other hand, a plate buckled by predominantly edge shear actions exhibits nonlinear distribution of shear stress inside the plate. In this case, the deflected plate can be modeled as an equivalent flat plate but with a reduced shear modulus of the plate [2]. Therefore, the evaluation of plate effectiveness now corresponds to the identification of the effective shear modulus, for plate elements buckled by edge shear.

The problem of the effective width for steel plating in compression was initially raised by John [3], a naval architect who investigated the strength of a ship which had broken into two pieces during heavy weather, presumably as a result of high stress induced by sagging moment. He pointed out that the light plating of the deck and topsides could not be considered as fully effective under compression. To account for this effect in the calculation of section modulus of the ship, he reduced the thickness of the plating, keeping the stress (which could be calculated without considering buckling) unchanged.

A pioneer of using an analytical approach for the plate effective width is Bortsch [4] who employed an approximate formula of the effective width for the practical problems relating to bridge engineering. The modern era in the effective width concept was started by von Karman [5] who developed a general method to solve the problem theoretically, and introduced for the first time the term effective width. He calculated the stress distribution of two-dimensional problems using the stress function approach to evaluate the effective width. A remarkable advancement of the Karman method was achieved by Metzer [6] who studied the effective flange width or effective breadth of simple beams and continuous beams.

In the 1930s, a large series of compression tests on steel plates were undertaken by Schuman and Back [7]. Based on the test results, they noted that the buckled steel plate may behave as if only part of its width is effective in carrying loads. By applying the effective width concept, this phenomenon was investigated theoretically by von Karman et al. [8] who obtained the first effective width expression of plating, which was later shown to be equivalent to $b_e/b = \sqrt{\sigma_{cr}/\sigma_Y}$, where b is the full plate width, b_e the effective plate width, σ_{cr} the plate buckling stress, and σ_Y the material yield stress.

Since the modern era of the effective width concept was opened by von Karman et al. [8], the concept has been recognized as an efficient and accurate approach to compute the post-buckling behavior of a plate in compression. Prof. Rhodes has provided a great contribution to the areas on effectiveness evaluation of thin plates after buckling, among other areas. His papers [9–11] extensively reviewed the history of plate post-buckling analysis, and gave expressions for the effective widths of plates under various boundary conditions and loads near buckling. A comparison with various design formulae of the effective widths was also made in his papers.

Since then, the concepts of plate-effectiveness evaluation have in fact been advanced to a large extent, and widely applied to strength assessment of thin-walled structures such as ships, offshore structures, land-based structures, and aerospace structures, to name a few.

Plate elements in thin-walled structures are often subjected to combined loads such as biaxial compressive loads, shear and lateral pressure loads, and thus various load components in type and magnitude should be dealt with as parameters of influence in the evaluation of plate effectiveness. Fabrication related initial imperfections in the form of initial deflection and welding residual stresses that always develop in plate elements of welded structures will also affect the plate effectiveness.

In recent years, the ultimate limit state approach has been more extensively applied for structural design and strength assessment [1]. This is because the ultimate limit state is a much better basis for design and strength assessment than the allowable working stress, and also it is not possible to determine true margin of safety as long as the ultimate limit state remains unknown. In this regard, it is also important to evaluate the effectiveness of plate elements at ultimate limit state.

The aim of the present paper is to survey some recent advances in the concepts of plate-effectiveness evaluation. While the present paper will focus on steel plates simply supported at all (four) edges that are well adopted for idealizing plate elements in continuous stiffened plate structures for marine applications, it will address the concepts and associated expressions of plate effectiveness as functions of combined load components and fabrication related initial imperfections. A concept of the effective shear modulus is also presented, which is useful for analysis of the post-buckling behavior of plates buckled in edge shear. For the effects of plate edge conditions, the readers may refer to the papers of Rhodes [9–11].

2. The concept of effective breadth

When lateral pressure actions are applied, a stiffened plate structure as shown in Fig. 1 is often idealized by a plate–stiffener combination model as a representative, as shown in Fig. 2. It is important to realize that the attached plating of the plate–beam combination model does not work separately from the adjacent members, and it is restricted from deforming sideways while the stiffener flange may be free to deflect vertically and sideways.

When a stiffened plate structure is idealized as an assembly of plate–beam combination elements, therefore, one of the primary questions is to what degree and extent the attached plating reinforces the associated strut-web.

Troitsky [12] reviewed various methods to derive analytical formulations of the shear-lag oriented effective breadth for wide flanged beams (plate–beam combinations). In the present paper, an analytical formulation of the effective

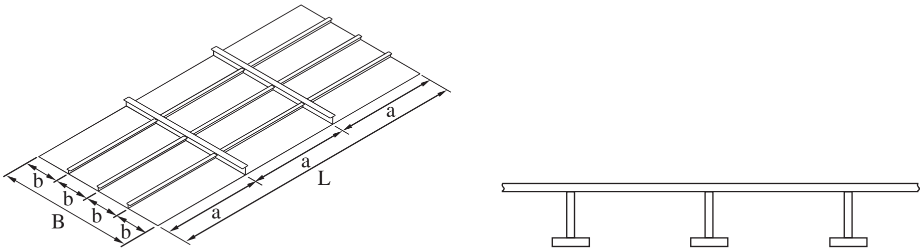

Fig. 1. A continuous stiffened plate structure.

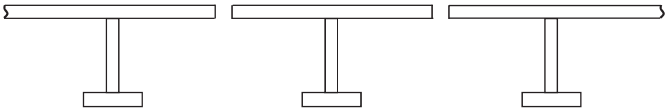

Fig. 2. A plate–stiffener combination model.

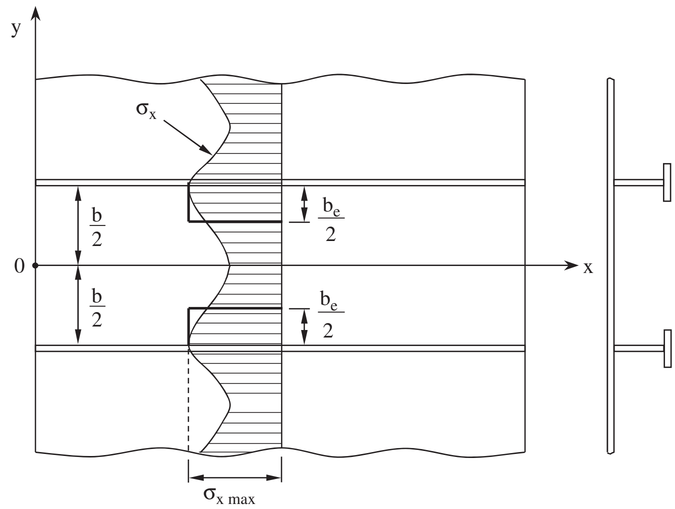

Fig. 3. Effective breadth or width of the attached plating in a plate–beam combination model.

breadth for a plate–beam combination under predominantly shear lag or bending on wide flange [1] is presented.

Following the coordinate illustrated in Fig. 3, the effective breadth (or width) b_e can be given by

$$b_e = \frac{\int_{-b/2}^{b/2} \sigma_x \, \mathrm{d}y}{\sigma_{x\max}}, \tag{1}$$

where σ_x is the non-uniform membrane stress and $\sigma_{x\max}$ the maximum membrane stress at plate/web junctions.

It is evident from Eq. (1) that the plate effective breadth can be defined once the non-uniform membrane stress distribution is known. The classical theory of elasticity [13] can be applied to compute the distribution of membrane stress σ_x as follows [1]:

$$\sigma_x = \left(\frac{2\pi}{\omega}\right)^2 \left[C_1 \frac{2\pi y}{\omega} \sinh \frac{2\pi y}{\omega} + (2C_1 + C_2) \cosh \frac{2\pi y}{\omega} \right] \sin \frac{2\pi x}{\omega}, \tag{2}$$

where

$$C_1 = C_3 \sinh \frac{\pi b}{\omega},$$

$$C_2 = C_3 \left[\left(\frac{1-\nu}{1+\nu}\right) \sinh \frac{\pi b}{\omega} - \frac{\pi b}{\omega} \cosh \frac{\pi b}{\omega} \right],$$

$$C_3 = E\varepsilon_0 \left(\frac{\omega}{2\pi}\right)^2 \left[\left(\frac{3-\nu}{2}\right) \sinh \frac{2\pi b}{\omega} - (1+\nu) \frac{\pi b}{\omega} \right]^{-1},$$

where ω is the deflection wave-length depending on rigidities of the stiffener and the type of load application, that is often taken as $\omega = L$ for stiff transverse frames, ν Poisson's ratio, $\varepsilon_0 = u_0(2\pi/\omega)$, u_0 the amplitude of the axial displacement function.

By substituting Eq. (2) into Eq. (1), the effective breadth b_e can be calculated as follows:

$$b_e = \frac{4\omega \sinh^2(\pi b/\omega)}{\pi(1+\nu)[(3-\nu)\sinh(2\pi b/\omega) - 2(1+\nu)(\pi b/\omega)]}. \tag{3}$$

The effective breadth normally varies along the span of plate–beam combination, but for practical design purposes it may be taken to have the smallest value which occurs at the location where the maximum longitudinal stress develops. Since b_e must be smaller than b, Eq. (3) may be approximated to be

$$\frac{b_e}{b} = \begin{cases} 1.0 & \text{for } \dfrac{b}{\omega} \leqslant 0.18, \\ \dfrac{0.18}{(b/L)} & \text{for } \dfrac{b}{\omega} > 0.18. \end{cases} \tag{4}$$

Fig. 4 plots Eq. (4) (approximate formula) by a comparison with Eq. (3) (exact solution). It is considered that Eq. (4) is accurate enough for practical design purpose.

3. The concept of effective width

After buckling under axial compressive actions, the membrane stress distribution inside the buckled plate is non-uniform. Fig. 5 shows a typical example of the axial membrane stress distribution inside a plate under predominantly longitudinal compressive loading before and after buckling occurs.

It is seen that the membrane stress distribution in the loading (x) direction can become non-uniform as the plate deflects. It is interesting to note that the membrane stress distribution in the y-direction also becomes non-uniform as long as the unloaded plate edges remain straight, while no membrane stresses will develop in the y-direction if the unloaded plate edges move freely in plane.

It is noted that the condition of unloaded edges of plate elements in a stiffened plate structure is more likely to remain straight. The maximum compressive membrane stresses are developed around the plate edges that remain straight, while the minimum membrane stresses occur in the middle of the plate where a membrane tension field is formed by the plate deflection since the plate edges remain straight.

For the analysis of plate post-buckling behavior under axial compression, there are three different aspects regarding the term plate effective width, namely the effective width for strength, the effective width for stiffness, and the effective tangent modulus.

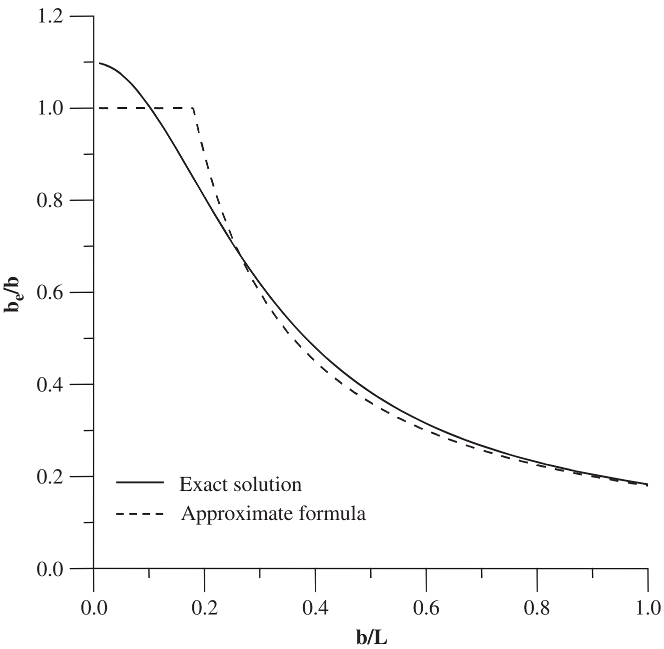

Fig. 4. Variation of the effective breadth versus the ratio of stiffener spacing to the beam span when $\omega = L$.

3.1. Effective width for strength

Immediately after buckling of a perfect plate under axial compression, the maximum stress becomes larger than the average stress. It may be apparent in this case that the ratio of effective width to full width is the same as the ratio of the average stress to the maximum stress as follows:

$$b_e = \frac{\int_{-b/2}^{b/2} \sigma_x \, \mathrm{d}y}{\sigma_{x\,\max}} = b\frac{\sigma_{x\,av}}{\sigma_{x\,\max}}, \tag{5}$$

where $\sigma_{x\,av}$ is the average stress.

The ultimate strength of a plate approximately corresponds to the applied load at which the maximum membrane stress reaches the material yield stress. Since the effective width in terms of the maximum membrane stress is useful in predicting the ultimate strength of a plate, it is termed the effective width for strength.

Although the original von Karman effective width expression of plates, i.e., $b_e/b = \sqrt{\sigma_{cr}/\sigma_Y}$, is considered reasonably accurate for relatively thin plates, it is found to be optimistic for relatively thick plates with initial imperfections. In this regard, Winter [14] modified the von Karman equation as follows:

$$\frac{b_e}{b} = \sqrt{\frac{\sigma_{cr}}{\sigma_{\max}}}\left(1 - C\sqrt{\frac{\sigma_{cr}}{\sigma_{\max}}}\right), \tag{6}$$

where $\sigma_{\max}$ is the applied maximum (edge) stress which may be taken as $\sigma_{\max} = \sigma_Y$, σ_Y the material yield stress, and C the constant that is often taken as 0.25, but it is also taken as 0.218 or 0.22 in some design codes.

One of the most typical effective width expressions for compressive strength of long plates which are often employed by merchant ship classification societies is given in the following form:

$$\frac{b_e}{b} = \begin{cases} 1.0 & \text{for } \beta < 1, \\ \dfrac{C_1}{\beta} - \dfrac{C_2}{\beta^2} & \text{for } \beta \geqslant 1, \end{cases} \tag{7}$$

where C_1, C_2 are the constants depending on the plate boundary conditions, $\beta = (b/t)\sqrt{\sigma_Y/E}$, b the plate breadth (or stiffener spacing), t the plate thickness, and E the elastic modulus. Based on the analysis of available experimental

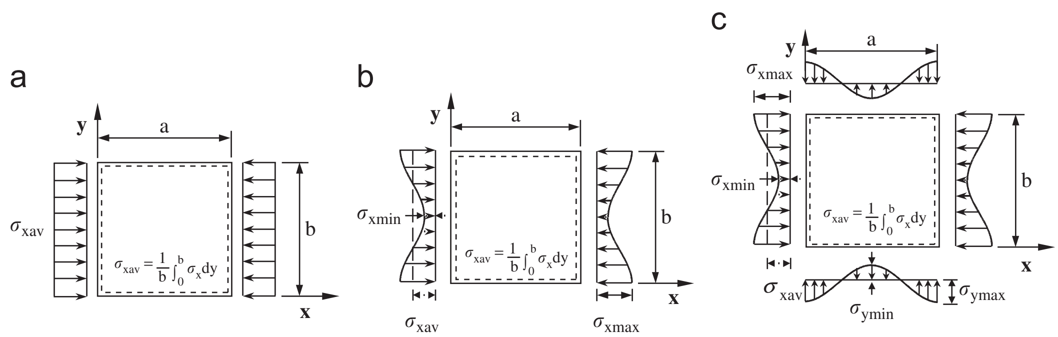

Fig. 5. Membrane stress distribution inside the plate under predominantly longitudinal compressive loads: (a) before buckling, (b) after buckling with unloaded edges moving freely in plate, and (c) after buckling with unloaded edges remaining straight.

data for steel plates with initial deflections at a moderate level but without residual stresses, Faulkner [15] proposes $C_1 = 2.0$ and $C_2 = 1.0$ for plates simply supported at all (four) edges, or $C_1 = 2.25$ and $C_2 = 1.25$ for plates clamped at all edges.

It is important to realize that Eqs. (6) and (7) are in fact likely to be the effective widths when the plate just reached the ultimate limit state. In other words, they do not describe the variation of the plate effectiveness as the external actions increase, although the plate effective width must vary with increase in the applied actions. Under axial compressive loads, a perfect plate, i.e., without initial deflection can buckle when the average stress $\sigma_{x\,av}$ reaches the buckling stress σ_{xE} that is given by

$$\sigma_{xE} = -\frac{\pi^2 D}{b^2 t}\left(\frac{mb}{a} + \frac{a}{mb}\right)^2, \tag{8}$$

where $D = Et^3/[12(1-\nu^2)]$ and m is the buckling half wave number that is an integer satisfying the following condition:

$$\frac{a}{b} \leqslant \sqrt{m(m+1)}. \tag{9}$$

The maximum stress $\sigma_{x\max}$ of the plate after buckling can be calculated by [1]

$$\sigma_{x\max} = a_1\sigma_{x\,av} + a_2, \tag{10}$$

where

$$a_1 = 1 + \frac{2m^4}{a^4(m^4/a^4 + 1/b^4)},$$

$$a_2 = \frac{2m^2}{a^2(m^4/a^4 + 1/b^4)}\frac{\pi^2 D}{t}\left(\frac{m^2}{a^2} + \frac{1}{b^2}\right)^2.$$

Therefore, the effective width can be determined from Eq. (5) together with Eq. (10). It is also interesting to note that steel plate elements used for marine structures and land-based structures typically have initial deflections induced by welding during fabrication process. In this case, the maximum stress $\sigma_{x\max}$ of the imperfect plate under axial compression is given by [1]

$$\sigma_{x\max} = \sigma_{x\,av} - \frac{m^2\pi^2 E A_m(A_m + 2A_{om})}{8a^2}, \tag{11}$$

where A_m is the amplitude of added deflection of the plate, A_{om} the amplitude of initial deflection corresponding to buckling mode, m as defined in Eq. (9).

The amplitude A_m of added deflection of the plate can be obtained as a solution of the following third-order equation, namely

$$C_1A_m^3 + C_2A_m^2 + C_3A_m + C_4 = 0, \tag{12}$$

where

$$C_1 = \frac{\pi^2 E}{16}\left(\frac{m^4 b}{a^3} + \frac{a}{b^3}\right), \quad C_2 = \frac{3\pi^2 E A_{om}}{16}\left(\frac{m^4 b}{a^3} + \frac{a}{b^3}\right),$$

$$C_3 = \frac{\pi^2 E A_{om}^2}{8}\left(\frac{m^4 b}{a^3} + \frac{a}{b^3}\right) + \frac{m^2 b}{a}\sigma_{x\,av} + \frac{\pi^2 D}{t}\frac{m^2}{ab}\left(\frac{mb}{a} + \frac{a}{mb}\right)^2,$$

$$C_4 = A_{om}\frac{m^2 b}{a}\sigma_{x\,av}.$$

Fig. 6 shows the variation of the plate effective width as a function of axial compressive loads and initial deflection. The effective widths at the ultimate strength given by Eq. (7) for simply supported plates are also compared in Fig. 6, denoted by the Faulkner formula for different plate slenderness ratio or plate thickness. It is seen from Fig. 6 that the plate effective width decreases as the axial compressive loads increase. When very little initial deflection exists, the plate effective width abruptly decreases immediately after the inception of buckling.

However, the effective width for the imperfect plate with large initial deflection decreases from the very beginning with increase in the axial compressive loads. As would be expected, the plate effective width at the ultimate strength must be different depending on the level of initial deflection. However, Eq. (7) does not take into account the effect of initial deflection as a parameter of influence.

Paik and Thayamballi [1] presented the formulations of the effective width of plate elements under combined biaxial compression and lateral pressure loads, together with initial imperfections in the form of initial deflection and welding residual stress. Fig. 7 shows the variations of the plate effective width as a function of initial deflection, residual stress, axial compression, and lateral pressure. In this figure, σ_{rcx} and σ_{rcy} are compressive residual stresses in the plate length and breadth directions, respectively.

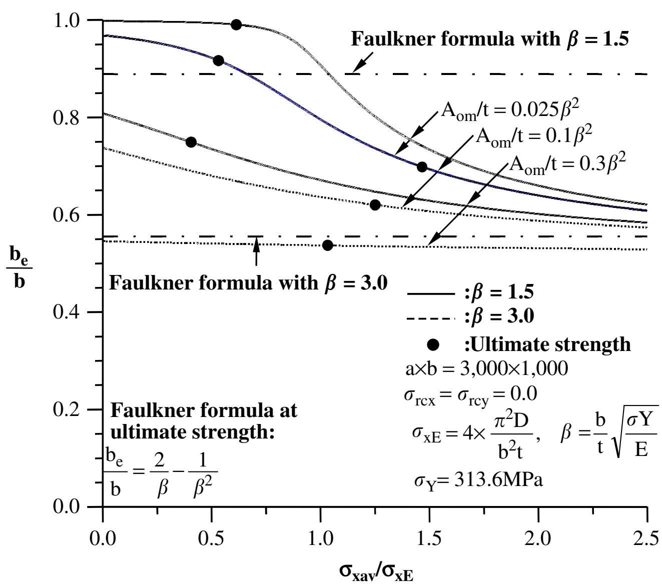

Fig. 6. Variation of the plate effective width as a function of axial compressive loads together with initial deflection.

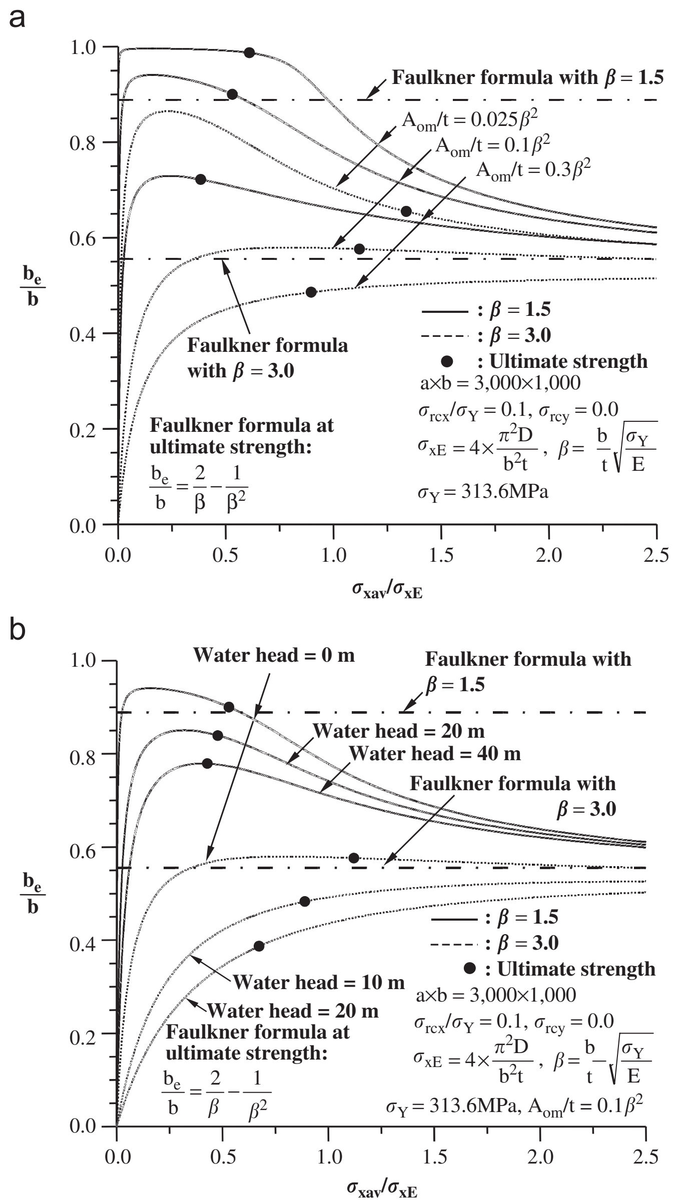

Fig. 7. (a) Variation of the plate effective width as a function of axial compressive loads together with initial deflection and welding residual stress. (b) Variation of the plate effective width as a function of axial compressive loads together with initial deflection, welding residual stress, and lateral pressure loads.

3.2. Effective width for stiffness

The tendency of increasing the average strain with the average stress is of course greater after buckling than that before buckling. As long as the plate/web junction remains straight, the average value of the maximum membrane stress along the plate edges may be obtained for a uniaxially compressed plate as follows:

$$\sigma_{x\max} = E\varepsilon_{x\,av} = E\frac{u}{L}, \tag{13}$$

where $\varepsilon_{x\,av}$ is the average axial strain of the attached plating which may approximately be taken as the average value of axial strain along the plate/web junctions, i.e., $\varepsilon_{x\,av} = \varepsilon_x$ at $y = \pm b/2$, and u is the end displacement.

In this case, the effective width can also be evaluated from Eq. (5) but replacing $\sigma_{x\max}$ by the axial strain of Eq. (13). An effective width for stiffness, i.e., based on the average axial strain, may be used to characterize the overall stiffness of a buckled plate under predominantly axial compression.

The closed-form expression of the effective width for stiffness representing the in-plane effectiveness of the buckled plate can be derived by

$$\frac{b_e^*}{b} = \left[\frac{\partial\sigma_{x\max}}{\partial\sigma_{x\,av}}\right]^{-1}, \tag{14}$$

where b_e^* is the effective width for stiffness.

For a perfect plate, i.e., without initial deflection, the maximum stress can be calculated from Eq. (10). In this case, the effective width for stiffness can be obtained as follows:

$$\frac{b_e^*}{b} = \frac{1}{a_1}. \tag{15}$$

The effective width for stiffness can also be expressed as a function of average strain. The relationship between the maximum stress and average strain is given as long as the unloaded edges remain straight, namely

$$\sigma_{x\max} = E\varepsilon_{x\,av} \tag{16a}$$

or, considering Eq. (10) together with Eq. (13), the following equation is obtained:

$$a_1\sigma_{x\,av} + a_2 = E\varepsilon_{x\,av}, \tag{16b}$$

$$\frac{b_e}{b} = \frac{\sigma_{x\,av}}{\sigma_{x\max}} = \frac{1}{a_1}\left(1 - \frac{a_2}{E}\frac{1}{\varepsilon_{x\,av}}\right). \tag{16c}$$

Fig. 8 shows the variation of the plate in-plane stiffness in terms of the relationship between average stress and average strain. Immediately after buckling, the plate in-plane stiffness decreases significantly. The stiffness reduction tends to depend on the plate aspect ratio.

3.3. Effective tangent modulus

The plate stiffness against axial compression is reduced immediately after buckling. While this behavior may be characterized by the effective width for stiffness, it is sometimes of interest to know the magnitude of the tangent stiffness or the slope of the average stress–strain curve after buckling, which can mathematically be computed by $\partial\sigma_{x\,av}/\partial\varepsilon_{x\,av}$ in the post-buckling regime. The tangent stiffness after buckling is termed the effective tangent modulus or the effective Young's modulus, E^*. Using this formulation the ratio of the compressive stiffness after buckling to that before buckling is given by E^*/E. For a perfect plate simply supported at four edges, it is known that $E^*/E \approx 0.5$ after buckling. As long as unloaded edges remain straight so that some transverse stresses are

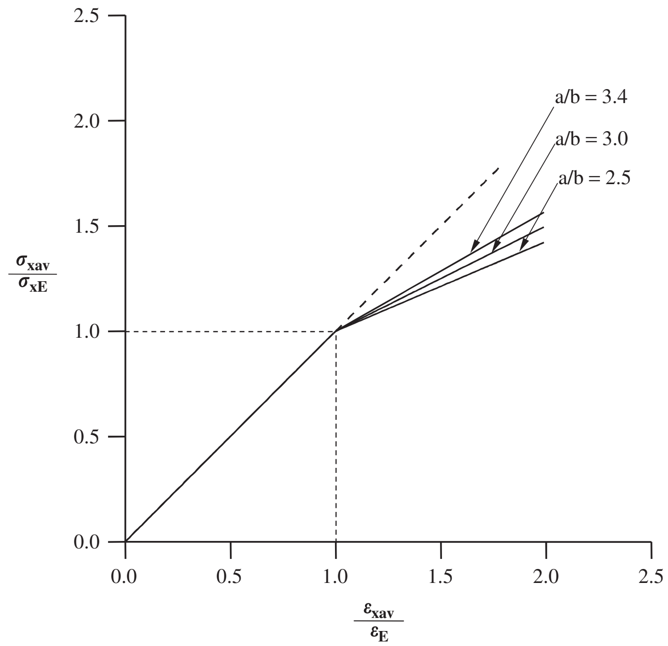

Fig. 8. The average stress versus strain relationships for a perfect plate under uniaxial compression in the elastic regime (ε_{xE} = average axial strain at $\sigma_{x\,av} = \sigma_{xE}$).

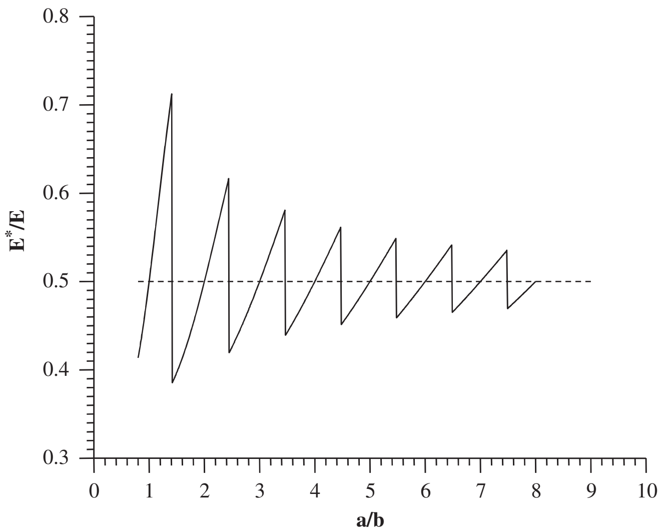

Fig. 9. Variation of the reduced tangent modulus after buckling as a function of the plate aspect ratio.

developed along unloaded edges, it is recognized that E^*/E corresponds to $\partial\sigma_{x\,av}/\partial\sigma_{x\,\max}$, while the former is always greater than the latter when unloaded edges are free to move in plane subsequent to no stresses along them [9].

For a plate under axial compression, Eq. (16) can be rewritten as follows:

$$\varepsilon_{x\,av} = \frac{1}{E}\sigma_{x\,\max} = \frac{1}{E}\frac{b}{b_e}\sigma_{x\,av} \text{ or } \sigma_{x\,av} = \frac{b_e}{b}E\varepsilon_{x\,av}. \tag{17}$$

The incremental form of Eq. (17) is given by

$$\Delta\varepsilon_{x\,av} = \frac{1}{E}\left(\frac{\partial\sigma_{x\,\max}}{\partial\sigma_{x\,av}}\right)\Delta\sigma_{x\,av} \quad \text{or}$$

$$\Delta\sigma_{x\,av} = \left(\frac{\partial\sigma_{x\,\max}}{\partial\sigma_{x\,av}}\right)^{-1} E\Delta\varepsilon_{x\,av}, \tag{18}$$

where the prefix Δ represents the increment of the variable. The numerical approach is often more pertinent for computation of $\partial\sigma_{x\,\max}/\partial\sigma_{x\,av}$ with infinitesimal stress variations around $\sigma_{x\,av}$.

For a perfect plate, i.e., without initial deflection, the following equations are then obtained, namely

$$\sigma_{x\,av} = \frac{1}{a_1}(E\varepsilon_{x\,av} - a_2), \tag{19a}$$

$$\Delta\sigma_{x\,av} = \frac{E}{a_1}\Delta\varepsilon_{x\,av} = E^*\Delta\varepsilon_{x\,av}, \tag{19b}$$

where

$$E^* = \frac{E}{a_1} = E \Bigg/ \left[1 + \frac{2m^4}{m^4 + a^4/b^4}\right] = \text{effective Young's}$$

modulus (tangent modulus) after buckling.

It is evident from Eq. (19b) that the tangent modulus of the buckled plate does not change with the applied loads, while it is a function of the plate aspect ratio. Fig. 9 shows the variation of the tangent modulus of the buckled plate as a function of the plate aspect ratio. It is seen that the effective tangent modulus varies in a cyclic pattern with regard to a mean equal to $E^*/E = 0.5$, and for shorter plates the effect of the aspect ratio is more significant.

4. Effective shear modulus

While the concept of effective width is aimed at the evaluation of in-plane stiffness of plate elements buckled in compression, Paik [2] suggested a new concept of the effective shear modulus to evaluate the effectiveness of plate elements buckled in edge shear. The effective shear modulus concept is useful for computation of the post-buckling behavior of plate girders under predominantly shear forces.

The shear strain distribution in a plate element is not uniform after shear buckling. By taking into account the large deflection effect, the shear strain γ in a plate element buckled by edge shear can be calculated as follows:

$$\gamma = \left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right] + \left[\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w_0}{\partial y} + \frac{\partial w_0}{\partial x}\frac{\partial w}{\partial y}\right], \tag{20}$$

where u, v is the axial displacements in the x- and y-directions, respectively, w the added deflection, w_0 the initial deflection. The first bracketed term on the right-hand side of the above equation represents the membrane shear strain component, and the second term indicates the additional shear strain component due to large deflection effects.

The basic idea of either the effective width or the effective shear modulus concepts is to model the deflected plate as an equivalent flat (undeflected) plate, but with a reduced (effective) in-plane stiffness. In this regard, the membrane shear strain component γ_m of the buckled plate must be defined as follows:

$$\gamma_m = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\tau}{G} - \left[\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w_0}{\partial y} + \frac{\partial w_0}{\partial x}\frac{\partial w}{\partial y}\right], \tag{21}$$

where $G = E/[2(1+v)]$ = shear modulus.

The membrane shear strain at any point inside the bucked plate can be computed separately using refined methods such as semi-analytical methods or finite element methods (FEMs). Once the membrane strain distribution inside the buckled plate is computed, the mean membrane shear strain γ_{av} can be obtained as an average of shear strains over the entire plate as follows:

$$\gamma_{av} = \frac{1}{ab}\int_0^a \int_0^b \gamma_m \,\mathrm{d}x\,\mathrm{d}y. \tag{22}$$

As long as the plate edges remain straight, the average shear stress may equal to the shear stress at the plate edges, i.e., $\tau = \tau_{av}$. Therefore, the effective shear modulus G_e representing the effectiveness of the plate buckled in edge shear can be defined by

$$G_e = \frac{\tau_{av}}{\gamma_{av}}. \tag{23}$$

An empirical expression of the effective shear modulus for plate elements buckled by edge shear has been developed by curve fitting based on numerical computations varying influential factors such as the plate aspect ratio and initial imperfections, as follows:

$$\frac{G_e}{G} = \begin{cases} c_1V^3 + c_2V^2 + c_3V + c_4 & \text{for } V \leqslant 1.0, \\ d_1V^2 + d_2V + d_3 & \text{for } V > 1.0, \end{cases} \tag{24}$$

where

$$c_1 = -0.309W_0^3 + 0.590W_0^2 - 0.286W_0,$$

$$c_2 = 0.353W_0^3 - 0.644W_0^2 + 0.270W_0,$$

$$c_3 = -0.072W_0^3 + 0.134W_0^2 - 0.059W_0,$$

$$c_4 = 0.005W_0^3 - 0.033W_0^2 + 0.001W_0 + 1.0,$$

$$d_1 = -0.007W_0^3 + 0.015W_0^2 - 0.018W_0 + 0.015,$$

$$d_2 = -0.022W_0^3 + 0.006W_0^2 + 0.075W_0 - 0.118,$$

$$d_3 = 0.008W_0^3 + 0.025W_0^2 - 0.130W_0 + 1.103,$$

$$V = \frac{\tau_{av}}{\tau_E},$$

$$W_0 = \frac{w_{0pl}}{t}.$$

w_{0pl} is the maximum initial deflection and τ_E the elastic shear buckling stress of the plate, which is given by

$$\tau_E = k_\tau \frac{\pi^2 E}{12(1-v^2)}\left(\frac{t}{b}\right)^2, \tag{25}$$

where

$$k_\tau \approx 4\left(\frac{b}{a}\right)^2 + 5.34 \quad \text{for } \frac{a}{b} \geqslant 1,$$

$$k_\tau \approx 5.34\left(\frac{b}{a}\right)^2 + 4.0 \quad \text{for } \frac{a}{b} < 1.$$

When the plate has no initial deflection, Eq. (24) is simplified to

$$\frac{G_e}{G} = \begin{cases} 1.0 & \text{for } \frac{\tau_{av}}{\tau_E} \leqslant 1, \\ 0.015\left(\frac{\tau_{av}}{\tau_E}\right)^2 - 0.118\frac{\tau_{av}}{\tau_E} + 1.103 & \text{for } \frac{\tau_{av}}{\tau_E} > 1. \end{cases} \tag{26}$$

Fig. 10 plots the variation of the effective shear modulus of a plate buckled by edge shear, with varying the edge shear stress and initial deflection. It is seen that the effective shear modulus of a perfect plate, i.e., without initial deflection, decreases abruptly after the inception of shear buckling. Also, the initial deflection reduces the effective shear modulus of the plate element under edge shear.

5. Relationships between membrane stresses versus strains

For the analysis of nonlinear behavior for deflected plates, it is required to identify the relationship between membrane stresses and strains [1,16]. The formulation of such a relationship in the pre-ultimate strength regime is distinct from that in the post-ultimate strength regime.

5.1. Pre-ultimate strength regime

The membrane strain components of deflected or buckled plate elements under combined biaxial loads, edge

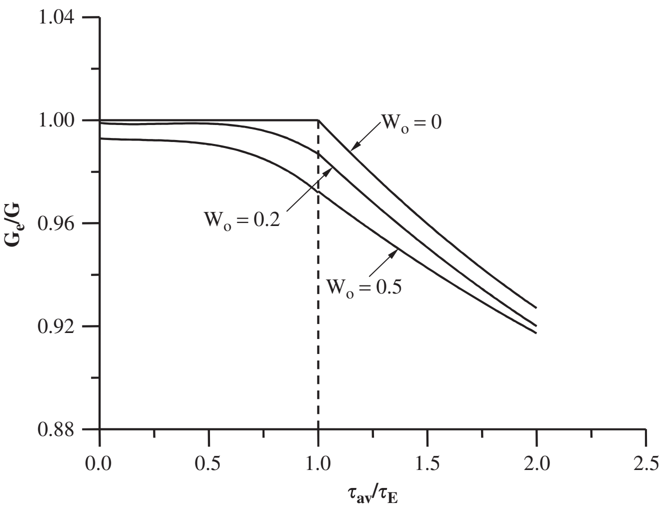

Fig. 10. Variation of the effective shear modulus of a plate with increase in edge shear.

shear, and lateral pressure can be given by

$$\varepsilon_{x\,av} = \frac{1}{E}(\sigma_{x\max} - v\sigma_{y\,av}), \tag{27a}$$

$$\varepsilon_{y\,av} = \frac{1}{E}(-v\sigma_{\,xav} + \sigma_{y\max}), \tag{27b}$$

$$\gamma_{av} = \frac{\tau_{av}}{G_e}, \tag{27c}$$

where $\sigma_{x\max}$, $\sigma_{y\max}$ are the maximum membrane stresses in the x- or y-direction, G_e the effective shear modulus as defined in Eq. (24).

Since $\sigma_{x\max}$, $\sigma_{y\max}$, and G_e are nonlinear functions with regard to the corresponding average stress components, Eq. (27) indicates a set of nonlinear relationships between membrane stresses and strains. The incremental form of the membrane stress–strain relationship is relevant by differentiating Eq. (27) with regard to the corresponding average stress components as follows:

$$\Delta\varepsilon_{x\,av} = \frac{1}{E}\left[\frac{\partial\sigma_{x\max}}{\partial\sigma_{x\,av}}\Delta\sigma_{x\,av} + \left(\frac{\partial\sigma_{x\max}}{\partial\sigma_{y\,av}} - v\right)\Delta\sigma_{y\,av}\right], \tag{28a}$$

$$\Delta\varepsilon_{y\,av} = \frac{1}{E}\left[\left(\frac{\partial\sigma_{y\max}}{\partial\sigma_{x\,av}} - v\right)\Delta\sigma_{x\,av} + \frac{\partial\sigma_{y\max}}{\partial\sigma_{y\,av}}\Delta\sigma_{y\,av}\right], \tag{28b}$$

$$\Delta\gamma_{av} = \frac{1}{G_e}\left(1 - \frac{\tau_{av}}{G_e}\frac{\partial G_e}{\partial\tau_{av}}\right)\Delta\tau_{av}. \tag{28c}$$

The matrix form of Eq. (28) is given by

$$\begin{Bmatrix} \Delta\sigma_{x\,av} \\ \Delta\sigma_{y\,av} \\ \Delta\tau_{av} \end{Bmatrix} = [D]^B \begin{Bmatrix} \Delta\varepsilon_{x\,av} \\ \Delta\varepsilon_{y\,av} \\ \Delta\gamma_{av} \end{Bmatrix}, \tag{29}$$

where

$$[D]^B = \frac{1}{A_1B_2 - A_2B_1}\begin{bmatrix} B_2 & -A_2 & 0 \\ -B_1 & A_1 & 0 \\ 0 & 0 & 1/C_1 \end{bmatrix} = \text{stress}-\text{strain}$$

matrix of the plate in the post-buckling regime,

with

$$A_1 = \frac{1}{E}\frac{\partial\sigma_{x\max}}{\partial\sigma_{x\,av}},$$

$$A_2 = \frac{1}{E}\left(\frac{\partial\sigma_{x\max}}{\partial\sigma_{y\,av}} - v\right),$$

$$B_1 = \frac{1}{E}\left(\frac{\partial\sigma_{y\max}}{\partial\sigma_{x\,av}} - v\right),$$

$$B_2 = \frac{1}{E}\frac{\partial\sigma_{y\max}}{\partial\sigma_{y\,av}},$$

$$C_1 = \frac{1}{G_e}\left(1 - \frac{\tau_{av}}{G_e}\frac{\partial G_e}{\partial\tau_{av}}\right).$$

When no buckling has occurred in the perfect plate element, i.e., without initial imperfections, $[D]^B$ matrix in Eq. (29) will of course become

$$[D]^B = \frac{E}{1 - v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \dfrac{1-v}{2} \end{bmatrix}. \tag{30}$$

5.2. Post-ultimate strength regime

In the post-ultimate regime, the internal stress will decrease as long as the axial compressive displacements continually increase. In this case, the average membrane stress components may be calculated in terms of the plate effective width or length as follows:

$$\sigma_{x\,av} = \frac{b_e}{b}\sigma^u_{x\max}, \tag{31a}$$

$$\sigma_{y\,av} = \frac{a_e}{a}\sigma^u_{y\max}, \tag{31b}$$

where b^e is the effective width, a_e the effective length, and $\sigma^u_{x\max}$, $\sigma^u_{y\max}$ the maximum membrane stresses of the plate in the x- or y-directions, immediately after the ultimate strength is reached, i.e., $\sigma^u_{x\max} = \sigma_{x\max}$ at $\sigma_{x\,av} = \sigma_{xu}$ the ultimate compressive strength in the x-direction, or $\sigma^u_{y\max} = \sigma_{y\max}$ at $\sigma_{y\,av} = \sigma_{yu}$ the ultimate compressive strength in the y-direction.

The effective width or length of the plate in the post-ultimate strength regime may be defined as follows:

$$\frac{b_e}{b} = \frac{\sigma^*_{x\,av}}{\sigma^*_{x\max}}, \tag{32a}$$

$$\frac{b_e}{b} = \frac{\sigma^*_{y\,av}}{\sigma^*_{y\max}}, \tag{32b}$$

where the asterisk represents a value of the plate in the post-ultimate regime.

The effects of initial imperfections and Poisson's ratio are negligibly small in the post-ultimate regime. While the plate effective width will of course decrease in the post-ultimate regime as long as the axial compressive displacements increase, it is assumed that the reduction tendency of the plate effective width or length is similar to that in the pre-ultimate regime. In this case, $\sigma^*_{x\max}$ and $\sigma^*_{y\max}$ in Eq. (32) can be determined by

$$\sigma^*_{x\max} = E\varepsilon_{x\,av} = 2\sigma^*_{x\,av} - \sigma_{xE}, \tag{33a}$$

$$\sigma^*_{y\max} = E\varepsilon_{y\,av} = 2\sigma^*_{y\,av} - \sigma_{yE}, \tag{33b}$$

where σ_{xE} and σ_{yE} are the elastic compressive buckling stresses in the x- or y-direction, respectively.

By substituting Eq. (33) into Eq. (32), the plate effective width or length can be expressed in terms of strain

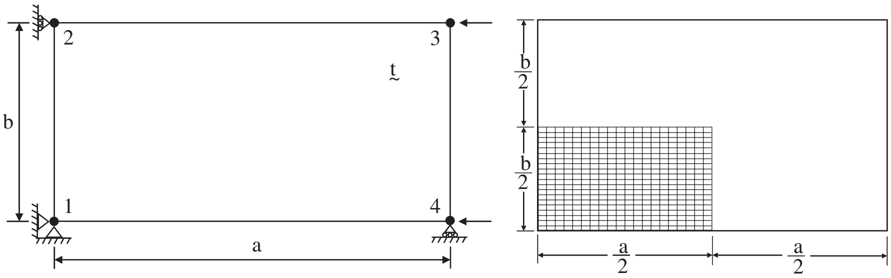

Fig. 11. Modeling of the plate by ALPS/GENERAL (left) and ANSYS (right).

components as follows:

$$\frac{b_e}{b} = \frac{1}{2}\left(1 + \frac{\sigma_{xE}}{E\varepsilon_{x\,av}}\right), \tag{34a}$$

$$\frac{a_e}{a} = \frac{1}{2}\left(1 + \frac{\sigma_{yE}}{E\varepsilon_{y\,av}}\right). \tag{34b}$$

The average stress–strain relationships in the post-ultimate regime can then be derived by substituting Eq. (34) into Eq. (31) as follows:

$$\sigma_{x\,av} = \frac{1}{2}\left(1 + \frac{\sigma_{xE}}{E\varepsilon_{x\,av}}\right)\sigma^u_{x\max}, \tag{35a}$$

$$\sigma_{y\,av} = \frac{1}{2}\left(1 + \frac{\sigma_{yE}}{E\varepsilon_{y\,av}}\right)\sigma^u_{y\max}. \tag{35b}$$

The incremental form of Eq. (35) is then given by

$$\Delta\sigma_{x\,av} = -\frac{\sigma^u_{x\max}}{2}\frac{\sigma_{xE}}{E\varepsilon^2_{x\,av}}\Delta\varepsilon_{x\,av}, \tag{36a}$$

$$\Delta\sigma_{y\,av} = -\frac{\sigma^u_{y\max}}{2}\frac{\sigma_{y\,cr}}{E\varepsilon^2_{y\,av}}\Delta\varepsilon_{y\,av}. \tag{36b}$$

On the other hand, the shear stress–strain relationship in the post-ultimate regime is given by

$$\Delta\tau_{av} = G^*_e\,\Delta\gamma_{av}, \tag{36c}$$

where G^*_e is the tangent shear modulus in the post-ultimate regime, which is often supposed to be $G^*_e = 0$ when the unloading behavior due to shear is not very significant.

In combined load cases, the average stress–strain relationship of the plate in the post-ultimate regime is therefore given from all together with Eq. (36) as follows:

$$\begin{Bmatrix} \Delta\sigma_{x\,av} \\ \Delta\sigma_{y\,av} \\ \Delta\tau_{av} \end{Bmatrix} = [D_p]^U \begin{Bmatrix} \Delta\varepsilon_{x\,av} \\ \Delta\varepsilon_{y\,av} \\ \Delta\gamma_{av} \end{Bmatrix}, \tag{37}$$

where

$$[D_p]^U = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix} = \text{stress}-\text{strain}$$

matrix of the plate in the post-ultimate regime,

with

$$A_1 = -\frac{\sigma^u_{x\max}}{2}\frac{\sigma_{x\,cr}}{E\varepsilon^2_{x\,av}},$$

$$A_2 = -\frac{\sigma^u_{y\max}}{2}\frac{\sigma_{y\,cr}}{E\varepsilon^2_{y\,av}},$$

$$A_3 = G^*_e.$$

5.3. An illustrative example

An imperfect rectangular plate under uniaxial compressive actions is considered. The dimension of the plate is $a \times b = 1000 \times 1000$ (mm), Young's modulus (E) = 205,800 N/mm^2, yield stress (σ_Y) = 352.8 N/mm^2, and Poisson's ratio (ν) = 0.3, while plate thickness (t) is varied. Initial deflection of the plate is

$$w_0 = 0.05t\,\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)$$

and no residual stress is considered to exist. The plate is considered to be simply supported along all (four) edges, keeping straight.

The relationship between average stresses and strains was implemented into ALPS/GENERAL [17] that is a computer program for progressive collapse analysis of plated structures using idealized structural unit method (ISUM) [1,16]. Fig. 11 shows the analysis models by ALPS/GENERAL and nonlinear FEM [18]. For the ANSYS nonlinear FEM analysis, a quarter of the plate is taken as the extent of the analysis. Fig. 12 compares the progressive collapse behavior of the plate under axial compressive actions, indicating that the ISUM solutions are in good agreement with more refined FEM results.

6. Concluding remarks

It has been recognized that the concepts of idealizing deflected plates as equivalent flat (undeflected) plates but with reduced plate effectiveness are very useful for the analysis of nonlinear behavior of plates deflected by lateral loading or buckling. In this case, a key issue is how to identify the effectiveness of the deflected plates. A number

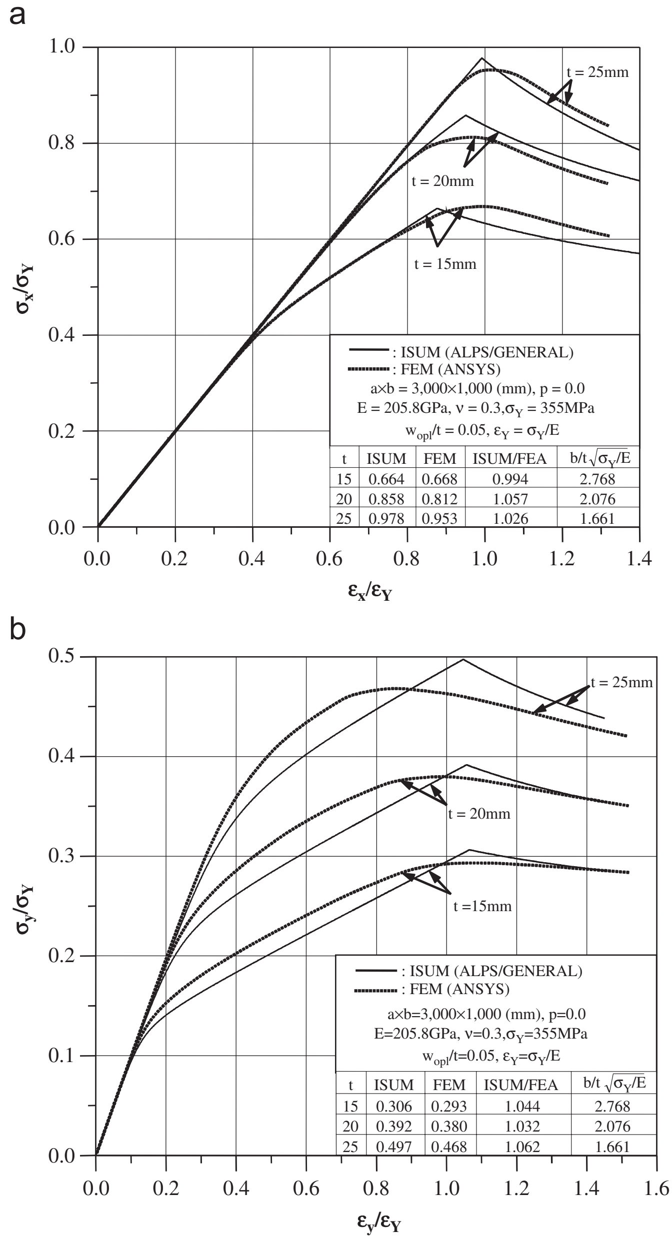

Fig. 12. A comparison of the progressive plate collapse behavior under uniaxial compression in the x-direction, obtained by ISUM and FEM. (b) A comparison of the progressive plate collapse behavior under uniaxial compression in the y-direction, obtained by ISUM and FEM.

of terms such as effective breadth, effective width, and effective shear modulus have been suggested in the literature for the evaluation of strength and stiffness, together with their closed-form expressions.

In the present paper, some recent advances in the plate-effectiveness evaluation have been surveyed. Some useful expressions of effective breadth associated with shear-lag effect, and effective width associated with buckling were presented. A new concept termed effective shear modulus is presented to evaluate the post-buckling behavior of plated buckled in edge shear. The relationships between membrane stresses versus membrane strains are presented in the pre- and post-ultimate strength regimes.

Based on the insights obtained from the present study, the following conclusions can be drawn:

(1) The term ‘effective breadth’ presents the plate effectiveness associated with shear-lag effect.
(2) The term ‘effective width’ presents the plate effective width associated with buckling in compressive actions.
(3) The term ‘effective length’ is defined for a plate buckled in transverse compression, while the term ‘effective width’ is defined for a plate buckling in longitudinal compression.
(4) Effective width or length is a function of various parameters of influence such as applied loads and initial imperfections (initial deflection, welding residual stress).
(5) The concept of effective shear modulus is useful for the effectiveness evaluation of a plate buckled in edge shear.
(6) The relationships between membrane stresses versus membrane strains for deflected plates in pre- and post-ultimate strength regimes could be derived in a closed form using the concepts of effective width and effective shear modulus.

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