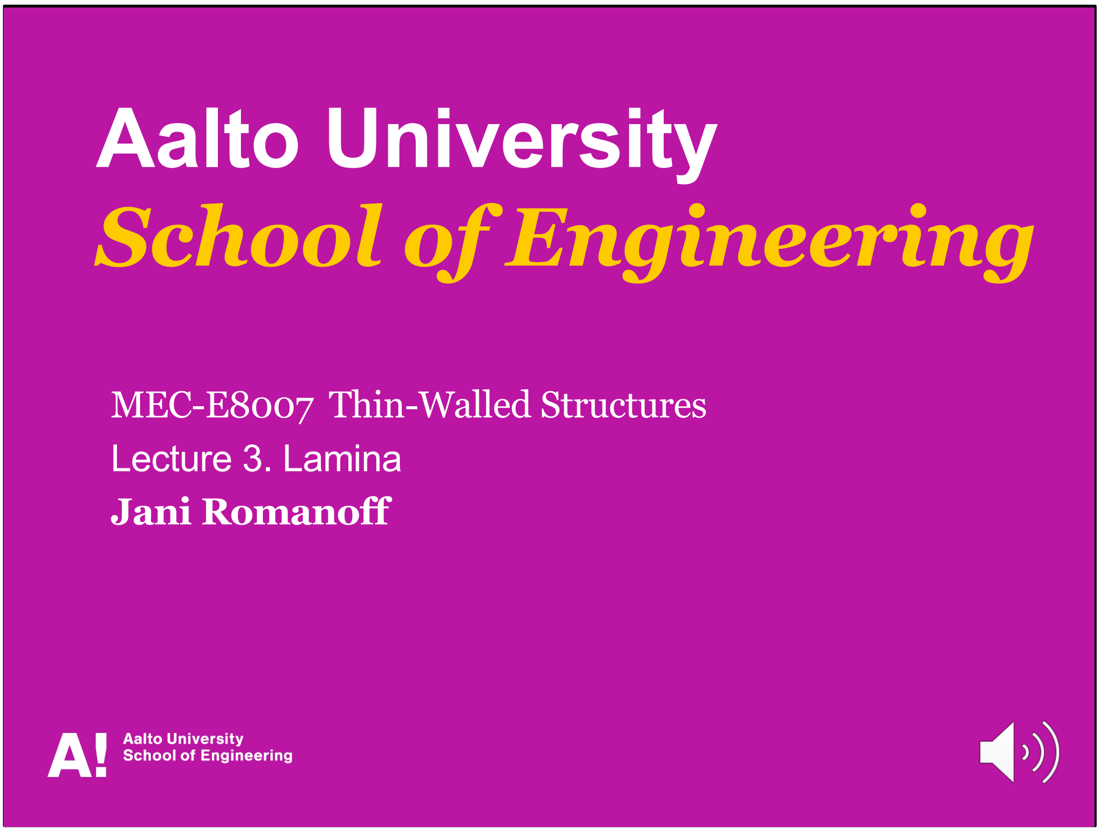


The idea of this lecture is to prepare you to understand how the overall stiffness of a thin-walled structure forms, by first looking at the layer or lamina material properties. We introduce the concept through plane stress layer formulation and expand this later to cover several layers.

Contens

- The aim of the lecture is to understand how the laminates are formed from laminae and how lamina stiffness is assessed for
- Fibers and matrix
- Laminae
- Constitutive equations for lamina
- Transformations of stresses and strains
- Transformation of material coefficients
- Micromechanics and multiscale modeling for stiffness properties
- Literature:
  1. Reddy, J.N., Mechanics of Laminated Composite Plates and Shells – Theory and Analysis, 2nd Edition, CRC Press, Ch. 1-3, 5-7, 9
  2. Jones, R.M., Mechanics of Composite Laminates, Script Book Company, Ch. 1-3

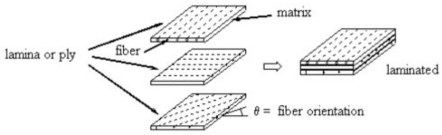
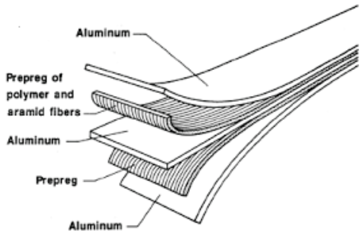
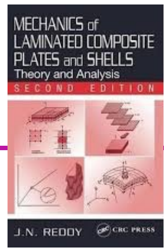
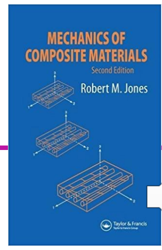


Figure 1 Laminated composite materials







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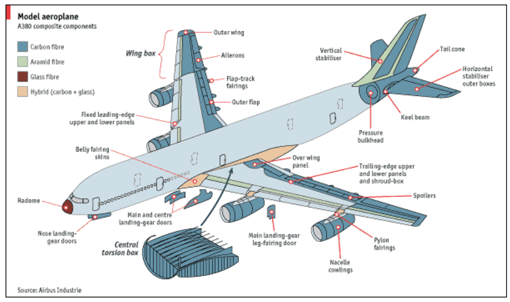
The aim of the lecture is to understand how the laminates are formed from laminae and how lamina stiffness is assessed based on the assumptions of plane stress which means that we have thin-layers. There are several excellent books on the topic, but I recommend the classics from Jones and Reddy which both approach the issue in step by step manner.

In order to do the what we aim for, we need to first review the definitions of fibers and matrix, followed up by the definition of layered composite structure laminae. Then we move to single layer in which we describe the constitutive equations for lamina, transformations of stresses and strains and with these the material coefficients to account directionality of the layer with respect to global coordinate system in which the thin-walled structure is to be treated. Last issue is visit to the micromechanics and multiscale modeling for stiffness properties.

Motivation

- Individual materials posses certain properties for certain purposes
- In composites the material is formed from more than one material with very clear interface between to fulfil multitude of requirements
- Good example is GLARE (Airbus) where metals are used for ductility and glass fibers to arrest crack propagation
- As engineers of thin-walled structures, we need to understand how the use of materials can be optimized
  - Right amount of material in the direction of the load
  - What is the sequence of layers, i.e. in bending one needs stiff materials at the outer surfaces
  - Etc



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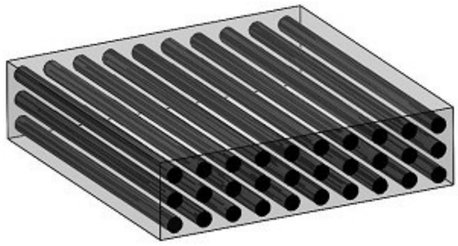
The main idea of a composite material and structure is that the individual materials posses certain properties for certain purposes and with a single material it is very difficult to cover all aspects we need to do in design. In composites the material is formed from more than one material with very clear interface between to fulfil multitude of requirements. Good example is GLARE where aluminum is used for ductility and glass fibers to arrest crack propagation. This means we can handle both extreme loads, but also fatigue loads with this tailored material.

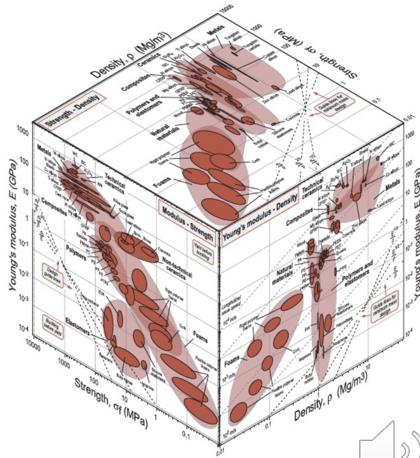
As engineers of thin-walled structures, we need to understand how the use of materials can be optimized with the structure and therefore, we must know the fundamentals of modeling these materials along with the structure. The basic idea in composite technology is to place a right amount of material in the direction of the load, i.e. to the direction of principal stresses in our structure. In bending dominated structures we need the stiff layers in the surface far away from the reference planes as this way the bending stiffness gets maximized.

The term reference plane is used here instead of neutral axis as in the composite structures the strict definition of neutral axis, looses its meaning. The definition is “zero normal stress” plane. In thin-walled structures the membrane effects are always present moving the plane and thus we cannot uncouple from the solution of the structural problem with load and boundary conditions being also part of the solution. The second and more important issue is the fact that the neutral axis, would be at different directions at different places. So we could not really define a unique plane where all stresses would be zero, without introducing complex internal couplings and significant problems of introducing the boundary conditions to this type of structure. Therefore, we simply select a plane in which we introduce the boundary conditions, loads and degrees of freedom and then compute the stiffnesses with respect to this.

Fibers and Matrix

- Composite materials are formed by two or more materials in macroscopic scale
  - Better properties than in conventional materials
  - Properties vary from: stiffness to strength, corrosion, fatigue etc
- Composites often contain at least
  - Fibers (reinforcement, e.g. carbon fibre)
  - Matrix (base material, e.g. epoxy)
- There are numerous combinations of materials and engineers task is to find the right combination
  - Load carrying
  - Physical properties
  - Etc





Ashby,M.F., “Materials Selection in Mechanical Design”

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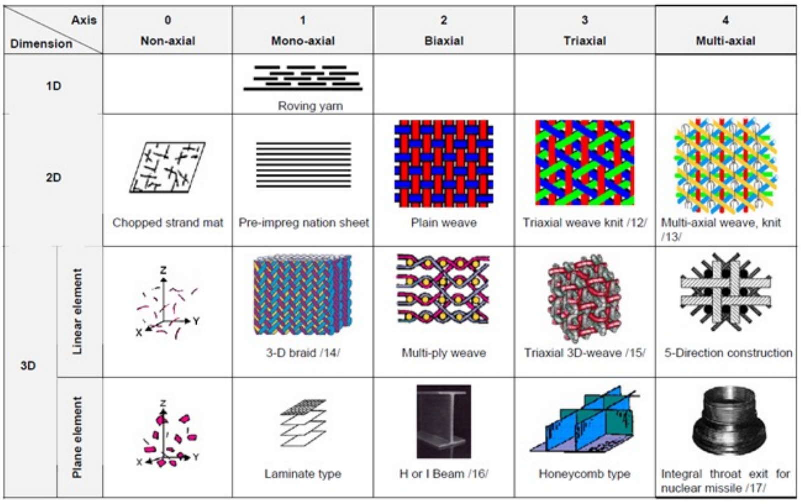
Composite materials are formed by two or more materials in macroscopic scale and this way we obtain better properties than in conventional materials that make the composite. Properties vary from stiffness to strength, corrosion, fatigue and we should always remember that optimal material for one application case may not be that for another case.

Example of this is the frame of a bicycle. When made from steel, the pipes are extremely slender, while when made from aluminum or composites the frame tends to be fatter, yet the stiffness can be the same. The reason is that the Young’s modulus and density are both 3 times higher in steel than in aluminum. This means that for the same membrane stiffness, you need 3 times more thickness in aluminum to cover the smaller Young’s modulus, membrane stiffness is proportional to *Et*. By making the pipe fatter you do not have to use as much thickness as we position the material away from the “reference plane” which in this case can be called neutral axis. This gives you the advantage over steel. To reflect to composites, carbon fibers have roughly the same Young’s modulus as steel and glass fibers with that of aluminum.

Composites often contain at least fibers (reinforcement, e.g. carbon fiber) which are the load-carrying members of the composite material and matrix (base material, e.g. epoxy) which is the binding material that keeps the fibers in place and together and restrain them from buckling under compression. Note that a slender fiber would have very long compression strength unless the matrix would produce the supporting medium. Without the matrix, the fiber could practically only tensile loads. This is why strength in composite is often given for both tensile and compressive stresses. On the other hand the matrix itself has a very low stiffness.

There are numerous combinations of materials and engineers task is to find the right combination by accounting for the load carrying mechanism, physical properties of the constituents, price and so on.

Fibers and Matrix



Dimension \ Axis		0 Non-axial	1 Mono-axial	2 Biaxial	3 Triaxial	4 Multi-axial
1D			Roving yarn			
2D		Chopped strand mat	Pre-impreg nation sheet	Plain weave	Triaxial weave knit /12/	Multi-axial weave, knit /13/
3D	Linear element		3-D braid /14/	Multi-ply weave	Triaxial 3D-weave /15/	5-Direction construction
3D	Plane element		Laminate type	H or I Beam /16/	Honeycomb type	Integral throat exit for nuclear missile /17/

https://textinfo.wordpress.com/2011/11/07/fiber-reinforced-composites/

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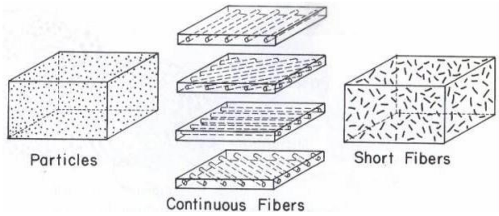
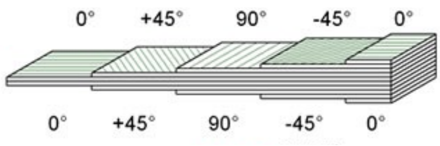
Here we see some typical pre-manufactured components for composite structures. We can see that these can be categorized based on the dimensionality of the geometry and stiffness. So these can be of random type fibres, beam and bar type of structures, short and long, planar structures and so on. So basically depending on the type of applcation we have there is a lot of choice.

One important note here. Today we also model the stiffness and strength of these structures by using FEA or analytical formulations based on beam and plate theory to obtain the continuum properties. These properties can be then used when we describe these layers in our thin-walled structure.

Fibers and Matrix

- ***Fibrous*** composites contain fibers of one material and matrix of another
- ***Particulate*** composites contain macro size particles in matrix
- ***Laminated*** composites with layers of fibrous and/or particulate composites
- ***Stiffness*** and ***strength*** comes from ***fibers*** and ***matrix*** keeps the fibers ***together*** and ***prevent*** them from buckling





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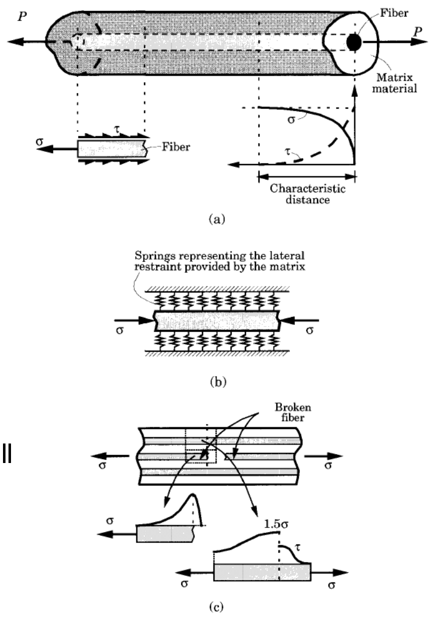
Stiffness and strength comes from fibers and matrix keeps the fibers together and prevent them from buckling. **Fibrous** composites contain fibers of one material and matrix of another. This is the main class which then can be divided to further sub-classes. **Particulate** composites contain macro size particles in matrix. These particles are often short and therefore, they do not provide as high stiffness strength as the long, continuous and oriented fibers would do. **Laminated** composites has layers of fibrous and/or particulate composites.

Fibers and Matrix

- When fiber is pulled, it experiences tensile stress and the matrix-fiber interface experiences shear stress
  - The shear stress is not constant along the length, but can be approximated so when length to diameter, *l/d*, ratio is large
  - Stiffness is easy to predict, strength not so easy (due to non-constant stress)
- This shear stress, redistributes stresses to the the neighboring fibers
- Broken fibers carry still load, but not with full length
- The way stress is transferred in matrix defines the characteristics of failure, e.g. weakest link



Reddy, J.N., Mechanics of Laminated Composite Plates and Shells – Theory and Analysis

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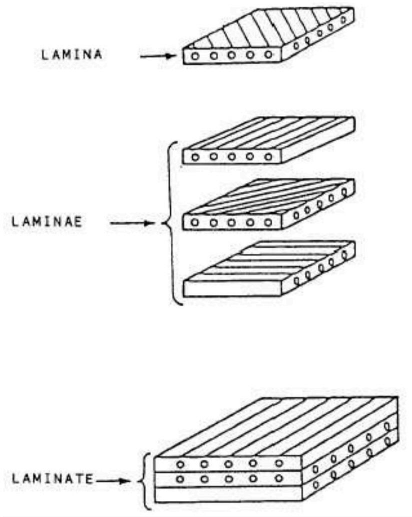
The mechanics of fibers and matrix together can be analyzed by complex FEA analysis or by simple analytical ideas. In the end of the day often we must test this interaction. When fiber is pulled, it experiences tensile stress and the matrix-fiber interface experiences shear stress. This is the main interaction between fibers and matrix. However, the shear stress is not constant along the length, but can be approximated so when length to diameter, l/d, ratio is large. At the ends of the fibers, there is shear lag which changes the distributions of normal and shear stresses to change from that along the mid-length.

Stiffness is easy to predict, strength not so easy due this non-constant stress. The shorter and the less oriented the fibers are, the more difficult this prediction gets. This is why tests are needed. The shear stress, redistributes stresses to the the neighboring fibers in case one fiber is broken and also due to this broken fibers carry still load, but not with full length. The way stress is transferred in matrix defines the characteristics of failure, e.g. weakest link – analogy, resulting in the Weibull distribution for the probability of strength.

Lamina / Ply

- A sheet of composite material is called lamina or ply
- Set of unconnected lamina is called laminae and connected set laminate
- Lamina can be unidirectional, bidirectional, woven or random
  - Each lamina has its principal direction, e.g. fiber direction
  - The properties are given as function of the fiber orientation angle



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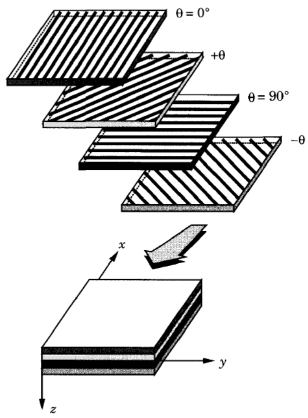
A sheet of composite material is called lamina or ply. Set of unconnected lamina is called laminae and connected set laminate.

This difference could be tough off through a stack of printing paper. With the cover sheet, the stack is very stiff and hard to bend by bare hands, this is the laminate. The laminae is when the cover is removed, now the bending is easy. The reason for the stiffness increase is the increased friction between the layers that the tight cover sheet produces. This produces shear stresses to the stack between the layers, which makes the stack to work effectively in bending. The stiffness of the stack would be basically unaltered if we would think of pulling the stack.

Lamina can be unidirectional, bidirectional, woven or random and each lamina has its principal direction, e.g. fiber direction. This direction can differ from the principal direction of the entire laminate or the structure. This is why we need to consider coordinate system transformations. The properties are given as function of the fiber orientation angle.

Laminates

- When assembled to layer wise structure, one obtains laminate
  - Layers are bonded together
  - Laminate is characterized by lamination scheme, a.k.a. stacking sequence
  - Sandwich is a special form of laminate with faces and core
- The weakness of laminates is the mismatch of material properties that cause local stress concentration, i.e. failure initiation points
  - Within ply/lamina
  - Between plies/laminae



Reddy, J.N., Mechanics of Laminated Composite Plates and Shells – Theory and Analysis

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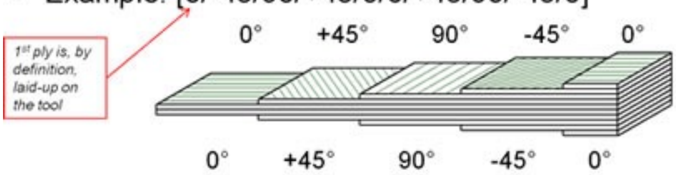
When assembled to layer wise structure, one obtains laminate. Generally it is assumed that the layers are bonded together as otherwise the system does not work in optimal way. In practice we still may have delamination which means that some of the layers are not perfectly bonded and the shear transfer is low. Laminate is characterized by lamination scheme also known as a stacking sequence which defines the layer orientations and type in format of a code. Sandwich is a special form of laminate with faces and core which have clear roles in the way they operate; the high stiffness, strength and density faces are designed to take the in-plane and bending stresses, while the low stiffness, strength and density thick core layer provides the support to the faces (often designed to be thin) and positions them far apart increasing this way the bending stiffness of the whole assembly. The weakness of laminates is the mismatch of material properties that cause local stress concentration, i.e. failure initiation points. This is why the shear stress analysis between the layers is much more important in laminates than in isotropic materials. This applies to both lamina level where we look at the fiber and matrix and their interaction and between laminae to make sure that they do not separate from each other.

Lamination scheme – orientation code

- Laminates are often described by an orientation code
- Example: [0/-45/90/+45/0/0/+45/90/-45/0]



- Short hand: $[0/-45/90/+45]_S$ symmetric
- Short hand: $[0/-45/90/+45]_N$ Repeat N times
- Other Examples: $[0/+45/-45/90]_{2S}$ 16 total plies

  $[0_2/+45/-45/90_2]_S$ 12 total plies

  $[0_U/+45_{CL}]$ 2 total plies (1 uni @ 0° and 1 cloth @ ±45°)

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This is an example of the lamination scheme. So what we see first is the unsimplified lamination code with each layer being described. We can also simplify the code using subscripts S for symmetry and N when we want to copy the stack N times. Subscripts can be used also to describe the type of layer we have in our laminate.

Laminates

Table 2.2.1: Values of the engineering constants for several materials*.

Material†	E_1	E_2	G_{12}	G_{13}	G_{23}	ν_{12}
Aluminum	10.6	10.6	3.38	3.38	3.38	0.33
Copper	18.0	18.0	6.39	6.39	6.39	0.33
Steel	30.0	30.0	11.24	11.24	11.24	0.29
Gr.-Ep (AS)	20.0	1.3	1.03	1.03	0.90	0.30
Gr.-Ep (T)	19.0	1.5	1.00	0.90	0.90	0.22
Gl.-Ep (1)	7.8	2.6	1.30	1.30	0.50	0.25
Gl.-Ep (2)	5.6	1.2	0.60	0.60	0.50	0.26
Br.-Ep	30.0	3.0	1.00	1.00	0.60	0.30

*Moduli are in msi = million psi; 1 psi = 6,894.76 N/m^2; Pa = N/m^2; kPa = 10^3 Pa; MPa = 10^6 Pa; GPa = 10^9 Pa.

† The following abbreviations are used for various material systems: Gr.-Ep (AS) = graphite-epoxy (AS/3501); Gr.-Ep (T) = graphite-epoxy (T300/934); Gl.-Ep = glass-epoxy; Br.-Ep = boron-epoxy.

Table 2.2.2: Values of additional engineering constants for the materials listed in Table 2.2.1*.

Material	E_3	ν_{13}	ν_{23}	α_1	α_2
Aluminum	10.6	0.33	0.33	13.1	13.1
Copper	18.0	0.33	0.33	18.0	18.0
Steel	30.0	0.29	0.29	10.0	10.0
Gr.-Ep (AS)	1.3	0.30	0.49	1.0	30.0
Gr.-Ep (T)	1.5	0.22	0.49	−0.167	15.6
Gl.-Ep (1)	2.6	0.25	0.34	3.5	11.4
Gl.-Ep (2)	1.3	0.26	0.34	4.8	12.3
Br.-Ep	3.0	0.25	0.25	2.5	8.0

* Units of E_3 are msi, and the units of α_1 and α_2 are 10^{-6} in./in./°F.

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Reddy, J.N., Mechanics of Laminated Composite Plates and Shells – Theory and Analysis

Here are given few examples of the material properties of certain materials which we could use in thin-walled structures (Psi). What is done here is the use of generalised Hooke's law, that is metals like steel are describe with E_1-E_3 as are the directional composites. The main difference is the directional similarity of the material properties in metals and the clear differences in composite laminate layers. We also should check the orders of magnitude. As we can see from here, the Young's modulus in strong direction is roughly the same in steel and carbon- and boron-based composites and in aluminum and glass-based composites. Thus, when we think about replacing steel with composite, the easiest option is to apply carbon or boron fibres and when replacing aluminum, the use of glass. Of course we can always mix the materials more freely.

Generalised Hooke’s Law for Elastic Range – Orthotropic Material

- Orthotropic materials (three mutually orthogonal material planes)

$$\begin{Bmatrix}\sigma_1\\\sigma_2\\\sigma_3\\\sigma_4\\\sigma_5\\\sigma_6\end{Bmatrix}=\begin{bmatrix}C_{11}&C_{12}&C_{13}&0&0&0\\C_{12}&C_{22}&C_{23}&0&0&0\\C_{13}&C_{23}&C_{33}&0&0&0\\0&0&0&C_{44}&0&0\\0&0&0&0&C_{55}&0\\0&0&0&0&0&C_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_1\\\varepsilon_2\\\varepsilon_3\\\varepsilon_4\\\varepsilon_5\\\varepsilon_6\end{Bmatrix}$$

- The compliance matrix is

$$\begin{Bmatrix}\varepsilon_1\\\varepsilon_2\\\varepsilon_3\\\varepsilon_4\\\varepsilon_5\\\varepsilon_6\end{Bmatrix}=\begin{bmatrix}S_{11}&S_{12}&S_{13}&0&0&0\\S_{12}&S_{22}&S_{23}&0&0&0\\S_{13}&S_{23}&S_{33}&0&0&0\\0&0&0&S_{44}&0&0\\0&0&0&0&S_{55}&0\\0&0&0&0&0&S_{66}\end{bmatrix}\begin{Bmatrix}\sigma_1\\\sigma_2\\\sigma_3\\\sigma_4\\\sigma_5\\\sigma_6\end{Bmatrix}$$

$$\begin{Bmatrix}\varepsilon_1\\\varepsilon_2\\\varepsilon_3\\\varepsilon_4\\\varepsilon_5\\\varepsilon_6\end{Bmatrix}=\begin{bmatrix}\frac{1}{E_1}&-\frac{\nu_{21}}{E_2}&-\frac{\nu_{31}}{E_3}&0&0&0\\-\frac{\nu_{12}}{E_1}&\frac{1}{E_2}&-\frac{\nu_{32}}{E_3}&0&0&0\\-\frac{\nu_{13}}{E_1}&-\frac{\nu_{23}}{E_2}&\frac{1}{E_3}&0&0&0\\0&0&0&\frac{1}{G_{23}}&0&0\\0&0&0&0&\frac{1}{G_{13}}&0\\0&0&0&0&0&\frac{1}{G_{12}}\end{bmatrix}\begin{Bmatrix}\sigma_1\\\sigma_2\\\sigma_3\\\sigma_4\\\sigma_5\\\sigma_6\end{Bmatrix}$$

$$C_{11}=\frac{S_{22}S_{33}-S_{23}^2}{S}\qquad C_{12}=\frac{S_{13}S_{23}-S_{12}S_{33}}{S}$$

$$C_{22}=\frac{S_{33}S_{11}-S_{13}^2}{S}\qquad C_{13}=\frac{S_{12}S_{23}-S_{13}S_{22}}{S}$$

$$C_{33}=\frac{S_{11}S_{22}-S_{12}^2}{S}\qquad C_{23}=\frac{S_{12}S_{13}-S_{23}S_{11}}{S}$$

$$C_{44}=\frac{1}{S_{44}}\qquad C_{55}=\frac{1}{S_{55}}\qquad C_{66}=\frac{1}{S_{66}}$$

$$S=S_{11}S_{22}S_{33}-S_{11}S_{23}^2-S_{22}S_{13}^2-S_{33}S_{12}^2+2S_{12}S_{23}S_{13}$$

$$\frac{\nu_{21}}{E_2}=\frac{\nu_{12}}{E_1};\quad\frac{\nu_{31}}{E_3}=\frac{\nu_{13}}{E_1};\quad\frac{\nu_{32}}{E_3}=\frac{\nu_{23}}{E_2}$$

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So this is the generalized Hooke’s law. We have the three mutually orthogonal material planes which makes us to have 3 Young’s modulus, *E*, 3 shear modulus, *G*, and 3 Poisson’s ratio, *v*. We also have here the constitutive and, its’ inverse, the compliance matrices and clear definition of their contents based on the given Young’s and shear modulus and Poisson’s ratio. When we know the material properties, for example from previous table, we can define these matrices numerically. This is what the composite design software does. They basically create for the layer this matrix and computes the other.

Generalised Hooke’s Law for Elastic Range

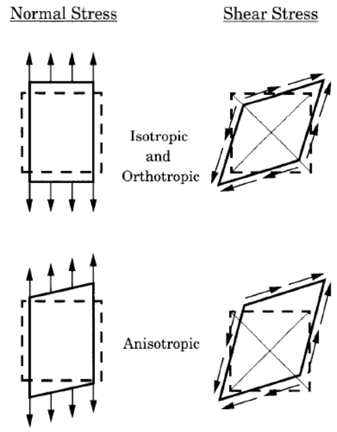

Figure 1.3.5: Deformation of orthotropic and anisotropic rectangular block under uniaxial tension.

Reddy, J.N., “Mechanics of Laminated and Composite Plates and Shells”

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The previous assumed orthotropic material, but of course we can have the fully occupied constitute matrix too. In anisotropic case the normal and shear strains (and stresses) would be coupled as presented here. However, the reasons to have this coupling are often not relevant and we can design our structures with the orthotropic materials to be extremely efficient. We should remember that the less parameters we have per layer, per lamina and per structure, the easier it is for us to understand their role in the solution. In orthotropic material we have the 9 material properties per layer which after stacking increases significantly. On top of that the orientation angles etc. So the problem size increases very rapidly. This is why computations are needed.

Characteristics of an Unidirectional Lamina

- Linear elastic material assumed
- Material symmetry parallel and transverse to fiber direction
  - x_1=fiber direction
  - x_2=transverse to fiber direction, in-plane of lamina
  - x_3=transverse to fiber direction, out-plane of lamina
- Under several assumptions we can assume springs in series or parallel giving the in-plane material properties (***Rule of Mixtures***)

$$E_1 = E_f v_f + E_m v_m, \quad \nu_{12} = \nu_f v_f + \nu_m v_m$$
$$E_2 = \frac{E_f E_m}{E_f v_m + E_m v_f}, \quad G_{12} = \frac{G_f G_m}{G_f v_m + G_m v_f}$$

- Micromechanics is alternative method to define the effective material properties

Direction 1 and Poisson ratio can be derived with springs in parallel analogy, while Direction 2 and shear with springs in series

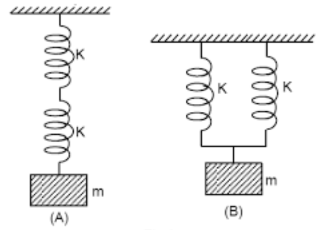
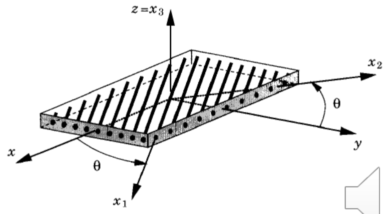


Fig.4



Reddy, J.N., Mechanics of Laminated Composite Plates and Shells – Theory and Analysis

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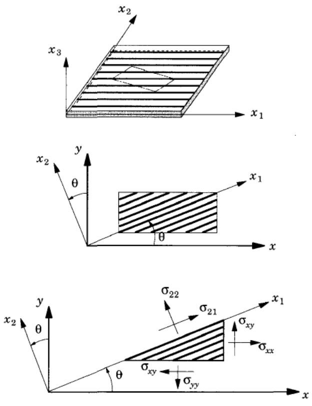
Characteristics of an unidirectional lamina can be derived with simple analytical models. Linear elastic material must be assumed. Direction 1 and Poisson ratio can be derived with springs in parallel analogy, while Direction 2 and shear with springs in series. While the first assumes constant strain in the Representative Volume Element (Voigt assumption=springs in parallel), the second assumes constant stress (Reuss assumption=springs in series). We also assume that the direction 1 refers to the fiber direction and direction 2 transverse to this. We also exploit the volume fraction for fibers, v_f, and matrix, v_m, of RVE, $v=v_f+v_m$, to define how much of each material is in the representative volume. When this is done and considering the fact that springs in parallel are characterized by, $k=k_f+k_m$, and in series by ,$1/k=1/k_f+1/k_m$, gives us the material properties in directions 1 and 2. Note here that for thin layer, direction 3 is not important. Micromechanics with FEA of by testing is an alternative method to define the effective material properties.

Transformation of Stress and Strain

- The material properties of ply/lamina are often given in principal material system (x_1-x_3 system)
- This system is needed for example in FEA (material direction)
- Often we need to “look” at the properties in rotated system of the problem (e.g. load direction)
  - Laminates
  - Load direction
- Therefore we need to rotate the material properties to new system (x-z), i.e. both stress and strain



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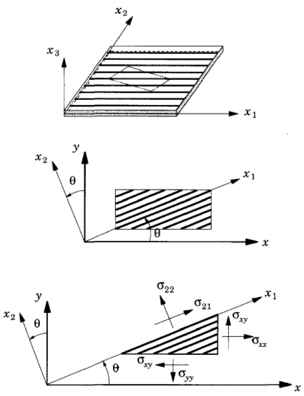
The material properties of ply/lamina are often given in principal material system (x1-x3 system) which is good to characterize the properties of the layer. This system is needed for example in FEA, where when assembling beams, plates and shells, the material direction is asked for. This information tells that in which direction fibers are in the structure. As we are interested on the structural responses, often we need to “look” at the properties in rotated system of the problem. Therefore we need to rotate the material properties to new system (x-z), i.e. both stress and strain and due to this the constitutive matrices too.

Transformation of Stress and Strain



Coordinate transformations

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = [L] \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$$

…and back to original

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = [L]^{\mathrm{T}} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

The same for stresses (second-order tensor), $[\sigma^*]=[L][\sigma][L]^T$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 0 & 0 & 0 & -\sin 2\theta \\ \sin^2\theta & \cos^2\theta & 0 & 0 & 0 & \sin 2\theta \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 & -\sin\theta & \cos\theta & 0 \\ \sin\theta\cos\theta & -\sin\theta\cos\theta & 0 & 0 & 0 & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} \tag{2.3.7}$$

or

$$\{\sigma\}_p = [T]\{\sigma\}_m \tag{2.3.8}$$

The inverse relationship between $\{\sigma\}_m$ and $\{\sigma\}_p$, Eq. (2.3.6a), is given by

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 0 & 0 & 0 & \sin 2\theta \\ \sin^2\theta & \cos^2\theta & 0 & 0 & 0 & -\sin 2\theta \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 & \sin\theta & \cos\theta & 0 \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & 0 & 0 & 0 & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} \tag{2.3.9}$$

or

$$\{\sigma\}_m = [R]\{\sigma\}_p \tag{2.3.10}$$

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In 12-plane the coordinate system transformation is the same as we remember from mathematics. The L-matrix transforms the 12-system to xy-system which is rotated by σομε ανγλε. We can also do the opposite by L^T. Stresses are second order tensor. For each stress, we have the direction where the stress acts and the normal of the plane the stress is acting. Yet, due to symmetry the stress components with subscript xy and yx are are the same and so is xz and zx and yz and zy. This is why for stress transformation we need to multiply the stress by L from the left and L^T from the right.

Transformation of Stress and Strain

The same for strains

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{xy} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 0 & 0 & 0 & -\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & 0 & 0 & 0 & \sin\theta\cos\theta \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 & -\sin\theta & \cos\theta & 0 \\ \sin 2\theta & -\sin 2\theta & 0 & 0 & 0 & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} \quad (2.3.13)$$

The inverse relation is given by

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 0 & 0 & 0 & \sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & 0 & 0 & 0 & -\sin\theta\cos\theta \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 & \sin\theta & \cos\theta & 0 \\ -\sin 2\theta & \sin 2\theta & 0 & 0 & 0 & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{xy} \end{Bmatrix} \quad (2.3.14)$$

and in matrix notation

$$\{\sigma\}_p = [T]\{\sigma\}_m = [T][C]_m\{\varepsilon\}_m = [T][C]_m[T]^{\mathrm{T}}\{\varepsilon\}_p \equiv [C]_p\{\varepsilon\}_p$$

and in stiffness

$$\{\sigma\}_p = [T]\{\sigma\}_m = [T][C]_m\{\varepsilon\}_m = [T][C]_m[T]^{\mathrm{T}}\{\varepsilon\}_p \equiv [C]_p\{\varepsilon\}_p$$

and stiffness matrix rotation

$$[\bar{C}] = [T][C][T]^{\mathrm{T}}$$

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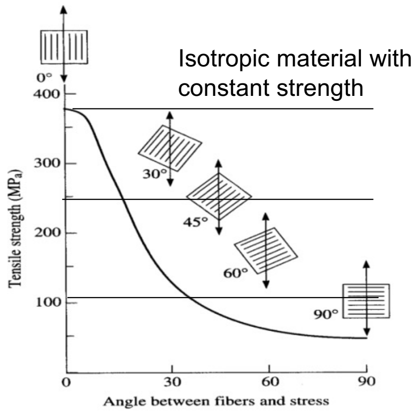
Stiffness and Strength are dependent on material direction



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The same process must be done also for the strains with the same process as with the stresses. When we know the strains, we can compute the stresses also with the consitutive matrix. This results in he stiffness matrix rotation. This is the results we need in practice when making the big FE-models of thin-walled structures.

What we see here is the resulting fiber direction strength as a function of the orientation angle. So when we know the stress state in the thin-walled structure, we must convert that back to the material system to see if we exceed the allowable stresses or not. So we have to be very clear what reference stress we take from the model and how do we apply this at the material level. We cannot use isotropic equivalent stress in this case (e.g. von Mises that is default in many FE packages) as there the strength is direction independent!

Transformation of Stress and Strain

$$
\begin{aligned}
\bar{C}_{11} &= C_{11}\cos^4\theta - 4C_{16}\cos^3\theta\sin\theta + 2(C_{12}+2C_{66})\cos^2\theta\sin^2\theta \\
&\quad - 4C_{26}\cos\theta\sin^3\theta + C_{22}\sin^4\theta \\
\bar{C}_{12} &= C_{12}\cos^4\theta + 2(C_{16}-C_{26})\cos^3\theta\sin\theta + (C_{11}+C_{22}-4C_{66})\cos^2\theta\sin^2\theta \\
&\quad + 2(C_{26}-C_{16})\cos\theta\sin^3\theta + C_{12}\sin^4\theta \\
\bar{C}_{13} &= C_{13}\cos^2\theta - 2C_{36}\cos\theta\sin\theta + C_{23}\sin^2\theta \\
\bar{C}_{16} &= C_{16}\cos^4\theta + (C_{11}-C_{12}-2C_{66})\cos^3\theta\sin\theta + 3(C_{26}-C_{16})\cos^2\theta\sin^2\theta \\
&\quad + (2C_{66}+C_{12}-C_{22})\cos\theta\sin^3\theta - C_{26}\sin^4\theta \\
\bar{C}_{22} &= C_{22}\cos^4\theta + 4C_{26}\cos^3\theta\sin\theta + 2(C_{12}+2C_{66})\cos^2\theta\sin^2\theta \\
&\quad + 4C_{16}\cos\theta\sin^3\theta + C_{11}\sin^4\theta \\
\bar{C}_{23} &= C_{23}\cos^2\theta + 2C_{36}\cos\theta\sin\theta + C_{13}\sin^2\theta \\
\bar{C}_{26} &= C_{26}\cos^4\theta + (C_{12}-C_{22}+2C_{66})\cos^3\theta\sin\theta + 3(C_{16}-C_{26})\cos^2\theta\sin^2\theta \\
&\quad + (C_{11}-C_{12}-2C_{66})\cos\theta\sin^3\theta - C_{16}\sin^4\theta \\
\bar{C}_{33} &= C_{33} \\
\bar{C}_{36} &= (C_{13}-C_{23})\cos\theta\sin\theta + C_{36}(\cos^2\theta - \sin^2\theta) \\
\bar{C}_{66} &= 2(C_{16}-C_{26})\cos^3\theta\sin\theta + (C_{11}+C_{22}-2C_{12}-2C_{66})\cos^2\theta\sin^2\theta \\
&\quad + 2(C_{26}-C_{16})\cos\theta\sin^3\theta + C_{66}(\cos^4\theta + \sin^4\theta) \\
\bar{C}_{44} &= C_{44}\cos^2\theta + C_{55}\sin^2\theta + 2C_{45}\cos\theta\sin\theta \\
\bar{C}_{45} &= C_{45}(\cos^2\theta - \sin^2\theta) + (C_{55}-C_{44})\cos\theta\sin\theta \\
\bar{C}_{55} &= C_{55}\cos^2\theta + C_{44}\sin^2\theta - 2C_{45}\cos\theta\sin\theta \\
\bar{C}_{14} &= C_{14}\cos^3\theta + (C_{15}-2C_{46})\cos^2\theta\sin\theta + (C_{24}-2C_{56})\cos\theta\sin^2\theta + C_{25}\sin^3\theta \\
\bar{C}_{15} &= C_{15}\cos^3\theta - (C_{14}+2C_{56})\cos^2\theta\sin\theta + (C_{25}+2C_{46})\cos\theta\sin^2\theta - C_{24}\sin^3\theta \\
\bar{C}_{24} &= C_{24}\cos^3\theta + (C_{25}+2C_{46})\cos^2\theta\sin\theta + (C_{14}+2C_{56})\cos\theta\sin^2\theta + C_{15}\sin^3\theta \\
\bar{C}_{25} &= C_{25}\cos^3\theta + (2C_{56}-C_{24})\cos^2\theta\sin\theta + (C_{15}-2C_{46})\cos\theta\sin^2\theta - C_{14}\sin^3\theta \\
\bar{C}_{34} &= C_{34}\cos\theta + C_{35}\sin\theta \\
\bar{C}_{35} &= C_{35}\cos\theta - C_{34}\sin\theta \\
\bar{C}_{46} &= C_{46}\cos^3\theta + (C_{56}+C_{14}-C_{24})\cos^2\theta\sin\theta + (C_{15}-C_{25}-C_{46})\cos\theta\sin^2\theta \\
&\quad - C_{56}\sin^3\theta \\
\bar{C}_{56} &= C_{56}\cos^3\theta + (C_{15}-C_{25}-C_{46})\cos^2\theta\sin\theta + (C_{24}-C_{14}-C_{56})\cos\theta\sin^2\theta \\
&\quad + C_{46}\sin^3\theta
\end{aligned}
\tag{2.3.18}
$$

$$
\begin{aligned}
\bar{S}_{11} &= S_{11}\cos^4\theta - 2S_{16}\cos^3\theta\sin\theta + (2S_{12}+S_{66})\cos^2\theta\sin^2\theta \\
&\quad - 2S_{26}\cos\theta\sin^3\theta + S_{22}\sin^4\theta \\
\bar{S}_{12} &= S_{12}\cos^4\theta + (S_{16}-S_{26})\cos^3\theta\sin\theta + (S_{11}+S_{22}-S_{66})\cos^2\theta\sin^2\theta \\
&\quad + (S_{26}-S_{16})\cos\theta\sin^3\theta + S_{12}\sin^4\theta \\
\bar{S}_{13} &= S_{13}\cos^2\theta - S_{36}\cos\theta\sin\theta + S_{23}\sin^2\theta \\
\bar{S}_{16} &= S_{16}\cos^4\theta + (2S_{11}-2S_{12}-S_{66})\cos^3\theta\sin\theta + 3(S_{26}-S_{16})\cos^2\theta\sin^2\theta \\
&\quad + (S_{66}+2S_{12}-2S_{22})\cos\theta\sin^3\theta - S_{26}\sin^4\theta \\
\bar{S}_{22} &= S_{22}\cos^4\theta + 2S_{26}\cos^3\theta\sin\theta + (2S_{12}+S_{66})\cos^2\theta\sin^2\theta \\
&\quad + 2S_{16}\cos\theta\sin^3\theta + S_{11}\sin^4\theta \\
\bar{S}_{23} &= S_{23}\cos^2\theta + S_{36}\cos\theta\sin\theta + S_{13}\sin^2\theta \\
\bar{S}_{26} &= S_{26}\cos^4\theta + (2S_{12}-2S_{22}+S_{66})\cos^3\theta\sin\theta + 3(S_{16}-S_{26})\cos^2\theta\sin^2\theta \\
&\quad + (2S_{11}-2S_{12}-S_{66})\cos\theta\sin^3\theta - S_{16}\sin^4\theta \\
\bar{S}_{33} &= S_{33} \\
\bar{S}_{36} &= 2(S_{13}-S_{23})\cos\theta\sin\theta + S_{36}(\cos^2\theta - \sin^2\theta) \\
\bar{S}_{66} &= S_{66}(\cos^2\theta - \sin^2\theta)^2 + 4(S_{16}-S_{26})(\cos^2\theta - \sin^2\theta)\cos\theta\sin\theta \\
&\quad + 4(S_{11}+S_{22}-2S_{12})\cos^2\theta\sin^2\theta \\
\bar{S}_{44} &= S_{44}\cos^2\theta + 2S_{45}\cos\theta\sin\theta + S_{55}\sin^2\theta \\
\bar{S}_{45} &= S_{45}(\cos^2\theta - \sin^2\theta) + (S_{55}-S_{44})\cos\theta\sin\theta \\
\bar{S}_{55} &= S_{55}\cos^2\theta + S_{44}\sin^2\theta - 2S_{45}\cos\theta\sin\theta \\
\bar{S}_{14} &= S_{14}\cos^3\theta + (S_{15}-S_{46})\cos^2\theta\sin\theta + (S_{24}-S_{56})\cos\theta\sin^2\theta + S_{25}\sin^3\theta \\
\bar{S}_{15} &= S_{15}\cos^3\theta - (S_{14}+S_{56})\cos^2\theta\sin\theta + (S_{25}+S_{46})\cos\theta\sin^2\theta - S_{24}\sin^3\theta \\
\bar{S}_{24} &= S_{24}\cos^3\theta + (S_{25}+S_{46})\cos^2\theta\sin\theta + (S_{14}+S_{56})\cos\theta\sin^2\theta + S_{15}\sin^3\theta \\
\bar{S}_{25} &= S_{25}\cos^3\theta + (-S_{24}+S_{56})\cos^2\theta\sin\theta + (S_{15}-S_{46})\cos\theta\sin^2\theta - S_{14}\sin^3\theta \\
\bar{S}_{34} &= S_{34}\cos\theta + S_{35}\sin\theta \\
\bar{S}_{35} &= S_{35}\cos\theta - S_{34}\sin\theta \\
\bar{S}_{46} &= (2S_{14}-2S_{24}+S_{56})\cos^2\theta\sin\theta + (2S_{15}-2S_{25}-S_{46})\cos\theta\sin^2\theta \\
&\quad + S_{46}\cos^3\theta - S_{56}\sin^3\theta \\
\bar{S}_{56} &= (2S_{15}-2S_{25}-S_{46})\cos^2\theta\sin\theta + (2S_{24}-2S_{14}-S_{56})\cos\theta\sin^2\theta \\
&\quad + S_{56}\cos^3\theta + S_{46}\sin^3\theta
\end{aligned}
\tag{2.3.21}
$$

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The orientation can be also written per stiffness matrix terms like done here. The much easier way is to use the matrix algebra shown earlier. These equations are good when you need to study how some stiffness term develops as a function of the orientation angle and the consitutive matrix terms associated with that stiffness.

Transformation of Stress and Strain
Plane stress

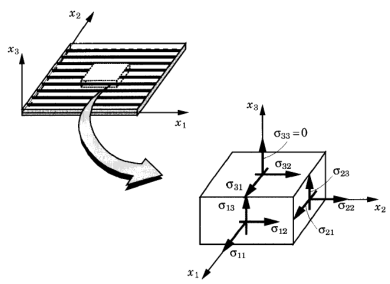

Figure 2.4.1: A lamina in a plane state of stress.

$$\left\{\begin{matrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{matrix}\right\} = \begin{bmatrix}\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16}\\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26}\\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}\end{bmatrix}\left(\left\{\begin{matrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{matrix}\right\}\right.$$

$$\begin{aligned}
\bar{Q}_{11} &= Q_{11}\cos^4\theta + 2(Q_{12}+2Q_{66})\sin^2\theta\cos^2\theta + Q_{22}\sin^4\theta\\
\bar{Q}_{12} &= (Q_{11}+Q_{22}-4Q_{66})\sin^2\theta\cos^2\theta + Q_{12}(\sin^4\theta+\cos^4\theta)\\
\bar{Q}_{22} &= Q_{11}\sin^4\theta + 2(Q_{12}+2Q_{66})\sin^2\theta\cos^2\theta + Q_{22}\cos^4\theta\\
\bar{Q}_{16} &= (Q_{11}-Q_{12}-2Q_{66})\sin\theta\cos^3\theta + (Q_{12}-Q_{22}+2Q_{66})\sin^3\theta\cos\theta\\
\bar{Q}_{26} &= (Q_{11}-Q_{12}-2Q_{66})\sin^3\theta\cos\theta + (Q_{12}-Q_{22}+2Q_{66})\sin\theta\cos^3\theta\\
\bar{Q}_{66} &= (Q_{11}+Q_{22}-2Q_{12}-2Q_{66})\sin^2\theta\cos^2\theta + Q_{66}(\sin^4\theta+\cos^4\theta)\\
\bar{Q}_{44} &= Q_{44}\cos^2\theta + Q_{55}\sin^2\theta\\
\bar{Q}_{45} &= (Q_{55}-Q_{44})\cos\theta\sin\theta\\
\bar{Q}_{55} &= Q_{55}\cos^2\theta + Q_{44}\sin^2\theta
\end{aligned} \tag{2.4.8}$$

$$\left\{\begin{matrix}\sigma_1\\ \sigma_2\\ \sigma_6\end{matrix}\right\}^{(k)} = \begin{bmatrix}Q_{11} & Q_{12} & 0\\ Q_{12} & Q_{22} & 0\\ 0 & 0 & Q_{66}\end{bmatrix}^{(k)}\left\{\begin{matrix}\varepsilon_1-\alpha_1\,\Delta T\\ \varepsilon_2-\alpha_2\,\Delta T\\ \varepsilon_6\end{matrix}\right\}^{(k)} - \begin{bmatrix}0 & 0 & e_{31}\\ 0 & 0 & e_{32}\\ 0 & 0 & 0\end{bmatrix}\left\{\begin{matrix}\mathcal{E}_1\\ \mathcal{E}_2\\ \mathcal{E}_3\end{matrix}\right\}^{(k)} \tag{2.4.1}$$

$$\left\{\begin{matrix}\sigma_4\\ \sigma_5\end{matrix}\right\}^{(k)} = \begin{bmatrix}Q_{44} & 0\\ 0 & Q_{55}\end{bmatrix}^{(k)}\left\{\begin{matrix}\varepsilon_4\\ \varepsilon_5\end{matrix}\right\}^{(k)} - \begin{bmatrix}0 & e_{24} & 0\\ e_{15} & 0 & 0\end{bmatrix}\left\{\begin{matrix}\mathcal{E}_1\\ \mathcal{E}_2\\ \mathcal{E}_3\end{matrix}\right\}^{(k)} \tag{2.4.2}$$

$$\left\{\begin{matrix}D_1\\ D_2\\ D_3\end{matrix}\right\}^{(k)} = \begin{bmatrix}0 & 0 & 0 & e_{15} & 0\\ 0 & 0 & e_{24} & 0 & 0\\ e_{31} & e_{32} & 0 & 0 & 0\end{bmatrix}^{(k)}\left\{\begin{matrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_4\\ \varepsilon_5\\ \varepsilon_6\end{matrix}\right\}^{(k)} + \begin{bmatrix}\epsilon_{11} & 0 & 0\\ 0 & \epsilon_{22} & 0\\ 0 & 0 & \epsilon_{33}\end{bmatrix}^{(k)}\left\{\begin{matrix}\mathcal{E}_1\\ \mathcal{E}_2\\ \mathcal{E}_3\end{matrix}\right\}^{(k)} \tag{2.4.3}$$

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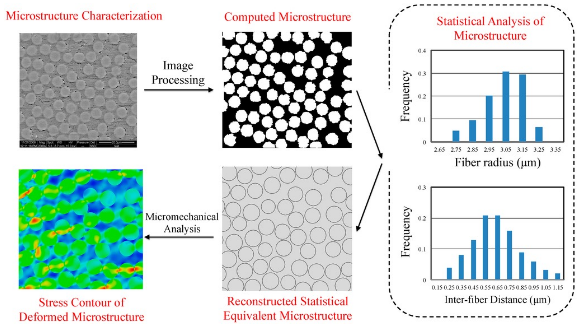
When we do the coordinate transformation for thin layer, we can assume plane stress. This means that the direction 33 and 13 and 23 stresses are zero and then the 6x6-matrix reduces to 3x3. However, as the out of plane shear is known to be important for laminates we tend to use 3x3 matrix for the in-plane properties and separate 2x2-atrix for the out-of-plane properties. The coordinate transformations are still the same as given above.

Micromechanics for Lamina Properties

- The rule of mixtures works under very strict assumptions, direction 1,
  - Voigt condition, constant strain
  - Upper bound
- Often in real materials the assumptions are broken to extent that we need more sophisticated methods
- Micromechanics takes into account
  - Character, topology of the microstructure
  - Statistical properties etc
  - Local failure modes
- This can be done nowadays with combined experimental and numerical methods



http://www.mdpi.com/1996-1944/9/8/624/htm

$$E_1 = E_f v_f + E_m v_m, \qquad \nu_{12} = \nu_f v_f + \nu_m v_m$$

$$E_2 = \frac{E_f E_m}{E_f v_m + E_m v_f}, \qquad G_{12} = \frac{G_f G_m}{G_f v_m + G_m v_f}$$

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The rule of mixtures works under very strict assumptions. We needed to assume Voigt condition, constant strain, that gives us the upper bound for stiffness in direction 1 and on the other hand the Reuss condition for the weaker direction, which gives us the lower bound for stiffness in that direction. Both of these stiffnesses can deviate from these values in reality. Often in real materials the assumptions are broken to extent that we need more sophisticated methods, such as FE-based micromechanics. Micromechanics can take into account the character and topology of the microstructure including statistical properties, it can account for the local failure modes in the materials. Micromechanics analysis can be done nowadays with combined experimental and numerical methods. That is we characterize the material distribution and properties experimentally and based on these formulate the FE-models to predict the strength and then calibrate this to selected test cases experimentally. Any statistical variation on input can be used to simulate the distribution of material properties as an output then numerically with the FEA.

Summary

- Laminates are plated structures formed from laminae / plies to tailor material properties for the given application case
- Fibers and matrix form ply, which can be assembled to form a laminate
- Constitutive equations for lamina are in general anisotropic and isotropy can be obtained by specific assumptions
- Transformations of stresses and strains and material coefficients from material coordinate system to structural system is a must
- Micromechanics and multiscale modeling for stiffness properties

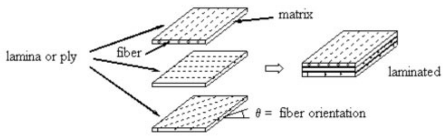
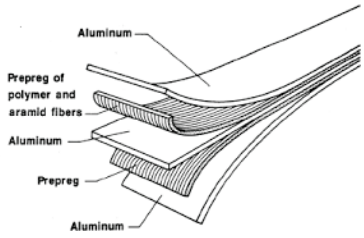


Figure 1 Laminated composite materials



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Laminates are plated structures formed from laminae / plies to tailor material properties for the given application case. Fibers and matrix form ply, which can be assembled to form a laminate. Constitutive equations for lamina are in general anisotropic and isotropy can be obtained by specific assumptions. Transformations of stresses and strains and material coefficients from material coordinate system to structural system is a must. Micromechanics and multiscale modeling for stiffness properties.