

## ELASTIC PROPERTIES OF REINFORCED SOLIDS : SOME THEORETICAL PRINCIPLES

By R. HILL

Department of Mathematics, University of Nottingham

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### SUMMARY

THE title problem concerns two isotropic phases firmly bonded together to form a mixture with any concentrations. An elementary account of several theoretical methods of attack is given, among them the derivation of inequalities between various moduli. The approach is completely general and exact. Additionally, the problem is fully solved when the phases have equal rigidities but different compressibilities, the geometry being entirely arbitrary.

### 1. INTRODUCTION

THE subject under consideration is the elastic behaviour of two solid phases firmly bonded together. The mixture is of a kind such that one phase can be regarded as 'inclusions' in a 'matrix' of the other. No restriction is placed on the shapes of the inclusions, which may for example be fibrous, spherical, or plate-like. It is supposed that the mixture is homogeneous on a macroscopic scale, but not necessarily isotropic ; the phases themselves are assumed isotropic as well as uniform. Their elastic moduli differ and so the stress and strain fields throughout the mixture are microscopically inhomogeneous whatever the external loading.

For arbitrary geometry and concentrations the task of determining the internal field of stress is hopelessly complex : only in special circumstances are complete exact solutions known. In the main we consider here a restricted version of the problem, far more tractable but still central. In this, one asks not for the stress state in detail but only for the overall or macroscopic elastic properties. Specifically, this involves finding the dependence of the macroscopic moduli on the relative concentrations, the inclusion geometry, the arrangement and degree of order, the distribution of sizes of inclusions of given shape, etc.

Even this restricted theoretical objective has proved sufficiently formidable. Until quite recently the only unobjectionable results of any generality concerned 'dilute dispersions,' where the inclusion concentration is less than a few per cent and the mixture random. During the last two years, however, significant advances have been made in the problem of arbitrary concentration, while the effects of inclusion shape and arrangement are beginning to be understood. The position has been appraised in an unpublished report (HILL 1962). The present article follows this closely but is primarily concerned with the writer's contributions. All the results are exact and for the most part completely general.

A detailed review of the older literature can be found in REINER (1958). The most recent and authoritative survey is by HASHIN (1963).

## 2. ELEMENTARY CONSIDERATIONS

### (i) Basic notation

It is found most convenient to take for the fundamental independent pair of isotropic elastic constants the bulk modulus  $\kappa$  and the shear (or rigidity) modulus  $\mu$ . The combination  $\kappa + \frac{4}{3}\mu$  occurs frequently later: it is the modulus in a compression test with lateral expansion prevented. Poisson's ratio  $\nu$  and Young's modulus  $E$  are regarded as dependent properties, the connexions being

$$\left. \begin{aligned} \nu &= \left( \frac{1}{2} - \frac{\mu}{3\kappa} \right) / \left( 1 + \frac{\mu}{3\kappa} \right), & \frac{3}{E} &= \frac{1}{\mu} + \frac{1}{3\kappa}; \\ E &= 2\mu(1+\nu) = 3\kappa(1-2\nu). \end{aligned} \right\} \quad (2.1)$$

In certain formulae it makes for conciseness to introduce dimensionless parameters  $\alpha$  and  $\beta$  where

$$\left. \begin{aligned} \alpha &= \frac{\kappa}{\kappa + \frac{4}{3}\mu} = \frac{(1+\nu)}{3(1-\nu)} \\ \beta &= \frac{2(\kappa + 2\mu)}{5(\kappa + \frac{4}{3}\mu)} = \frac{2(4-5\nu)}{15(1-\nu)}. \end{aligned} \right\} \quad (2.2)$$

All subsequent results are valid under the usual restrictions ensuring a positive strain-energy function, namely  $\kappa$  and  $\mu$  positive with  $-1 < \nu \leq 1/2$ . Then  $0 < \alpha \leq 1$  and  $3/5 > \beta \geq 2/5$ ;  $\beta$  is very near 0.45 for most metals.

Moduli for the two phases will be distinguished by subscripts 1 and 2. Except in formulae explicitly dependent on geometrical detail, it is unnecessary to state which subscript refers to the included phase. Moduli without numerical subscripts refer to the macroscopic, or average, properties of the mixture.

The fractional concentrations by volume of the phases will be denoted by  $c_1$  and  $c_2$ , where  $c_1 + c_2 = 1$ . While, in subsequent formulae, one concentration can always be eliminated in favour of the other, reasons of symmetry usually make it desirable to retain both.

For brevity of notation, and to lay bare the structure of formulae and proofs, the nine components of stress and of strain will be regarded as forming *vectors* in a 9-dimensional space, and accordingly denoted collectively by boldface symbols  $\sigma$  and  $\epsilon$ , respectively. Tensor suffix notation is used only where necessary. As they stand, these symbols will denote the actual stress and strain tensors at a generic point of the mixture. A subscript 1 or 2 will be added when referring exclusively to one or other phase.

Average values will be further distinguished by a bar placed above. By the 'average' of any quantity is meant, quite straightforwardly, its integral over a specified region divided by the volume of the region. We then have the obvious connexions

$$\bar{\sigma} = c_1 \bar{\sigma}_1 + c_2 \bar{\sigma}_2, \quad \bar{\epsilon} = c_1 \bar{\epsilon}_1 + c_2 \bar{\epsilon}_2, \quad (2.3)$$

for any region containing the phases in the given overall concentrations.

The relations between stress and strain tensors at any point in the phases are written concisely as

$$\left. \begin{aligned} \sigma_1 &= L_1 \epsilon_1, & \sigma_2 &= L_2 \epsilon_2, \\ \epsilon_1 &= M_1 \sigma_1, & \epsilon_2 &= M_2 \sigma_2. \end{aligned} \right\} \quad (2.4)$$

with inverses

Here the capitals stand for symmetric  $9 \times 9$  matrices (replacing fourth-rank tensors) whose elements are the stiffnesses or compliances respectively. Since the phases are assumed uniform and isotropic, identical relations with the same operators also hold between the corresponding averaged quantities. On substituting these in (2.3)

$$\bar{\sigma} = c_1 L_1 \bar{\epsilon}_1 + c_2 L_2 \bar{\epsilon}_2, \quad \bar{\epsilon} = c_1 M_1 \bar{\sigma}_1 + c_2 M_2 \bar{\sigma}_2. \quad (2.5)$$

### (ii) Representative volume

This phrase will be used when referring to a sample that (a) is structurally entirely typical of the whole mixture on average, and (b) contains a sufficient number of inclusions for the apparent overall moduli to be effectively independent of the surface values of traction and displacement, so long as these values are 'macroscopically uniform.' That is, they fluctuate about a mean with a wavelength small compared with the dimensions of the sample, and the effects of such fluctuations become insignificant within a few wavelengths of the surface. The contribution of this surface layer to any average can be made negligible by taking the sample large enough.

### (iii) Relations between averages

In the circumstances described in (ii) there is a unique dependence of the average strains in the phases upon the overall strain in the mixture. Let this be written as

$$\bar{\epsilon}_1 = A_1 \bar{\epsilon}, \quad \bar{\epsilon}_2 = A_2 \bar{\epsilon}, \quad \text{with } c_1 A_1 + c_2 A_2 = I \quad (2.6)$$

where  $I$  is the unit matrix.  $A_1$  and  $A_2$  are matrices dependent on the concentrations and phase moduli, etc. and are generally unsymmetric. Combining (2.6) with the first of (2.5) :

$$\bar{\sigma} = L \bar{\epsilon} \text{ where } L = c_1 L_1 A_1 + c_2 L_2 A_2 \quad (2.7)$$

is the required matrix for the mixture (juxtaposed capitals indicate ordinary matrix products). Equivalently, in terms of stresses, if

$$\bar{\sigma}_1 = B_1 \bar{\sigma}, \quad \bar{\sigma}_2 = B_2 \bar{\sigma}, \quad \text{with } c_1 B_1 + c_2 B_2 = I, \quad (2.8)$$

then

$$\bar{\epsilon} = M \bar{\sigma} \text{ where } M = c_1 M_1 B_1 + c_2 M_2 B_2 \quad (2.9)$$

is the compliance matrix for the mixture.

If, therefore, the average stress or strain in either constituent can be found for arbitrary overall values (so determining one of the  $A$ 's or  $B$ 's), the elastic properties of the mixture are completely specified through  $L$  or  $M$  by (2.7) or (2.9). Explicitly, in terms only of averages over the first phase (say),

$$L - L_2 = c_1 (L_1 - L_2) A_1, \quad M - M_2 = c_1 (M_1 - M_2) B_1. \quad (2.10)$$

Conversely, these may be read as formulae for  $A_1$  and  $B_1$  when the overall properties  $L$  and  $M$  are known.

$$B_1 L = L_1 A_1, \quad A_1 M = M_1 B_1, \quad (2.11)$$

with similar ones in the other subscript. The first equation follows by expressing  $\bar{\sigma}_1$  in the alternative ways  $B_1 L \bar{\epsilon}$  [from (2.7) and (2.8)] and  $L_1 A_1 \bar{\epsilon}$  [from the averaged form of (2.4) with (2.6)]. The second follows by dual operations with  $\bar{\epsilon}_1$ .

For a mixture whose overall properties are isotropic the preceding analysis simplifies somewhat. Since only two elastic constants have now to be determined, namely the bulk and rigidity moduli  $\kappa$  and  $\mu$ , it is no longer necessary to impose a perfectly general overall distortion. It is enough to consider just two independent strains, chosen at will.

Take first a pure dilatation, with fractional volume increase  $\bar{\theta}$  say, for which the corresponding overall stress in an isotropic mixture (or even one with only cubic symmetry) is an all round tension,  $\bar{\sigma}$  say. Instead of (2.5) we need only the scalar equations

$$\bar{\sigma} = c_1 \kappa_1 \bar{\theta}_1 + c_2 \kappa_2 \bar{\theta}_2, \quad \bar{\theta} = \frac{c_1 \bar{\sigma}_1}{\kappa_1} + \frac{c_2 \bar{\sigma}_2}{\kappa_2}.$$

Here  $\bar{\sigma}_1$ ,  $\bar{\sigma}_2$  are the hydrostatic parts of the stress (i.e. arithmetic mean of normal components) and  $\bar{\theta}_1$ ,  $\bar{\theta}_2$  are the volumetric parts of the strain (i.e. sum of normal components), all averaged over the respective phases (in each of which the entire stress and strain tensors are, of course, not necessarily purely dilatational, nor even their averages). In place of (2.6) write

$$\frac{\bar{\theta}_1}{\bar{\theta}} = a_1, \quad \frac{\bar{\theta}_2}{\bar{\theta}} = a_2, \quad \text{with } a_1 c_1 + a_2 c_2 = 1;$$

$$\frac{\bar{\sigma}_1}{\bar{\sigma}} = b_1, \quad \frac{\bar{\sigma}_2}{\bar{\sigma}} = b_2, \quad \text{with } b_1 c_1 + b_2 c_2 = 1;$$

where the  $a$ 's and  $b$ 's are function of the concentrations and moduli, etc. Then  $\bar{\sigma} = \kappa \bar{\theta}$  where

$$\kappa = a_1 c_1 \kappa_1 + a_2 c_2 \kappa_2 \quad \text{or} \quad \frac{1}{\kappa} = \frac{b_1 c_1}{\kappa} + \frac{b_2 c_2}{\kappa}, \quad (2.12)$$

in place of (2.7) and (2.9). Instead of (2.10),

$$\frac{\kappa - \kappa_2}{\kappa_1 - \kappa_2} = a_1 c_1 \quad \text{or} \quad \frac{1/\kappa - 1/\kappa_2}{1/\kappa_1 - 1/\kappa_2} = b_1 c_1. \quad (2.13)$$

These are equivalent expressions for the overall bulk modulus (valid also when the mixture is anisotropic with cubic symmetry). The equivalence can easily be verified by interchanging  $a_1$  and  $b_1$  via  $\kappa = a_1 \kappa_1 / b_1$ , which is the scalar form of (2.11).

Take, secondly, a pure shear strain,  $\bar{\gamma}$  say, its plane and direction being chosen arbitrarily. The overall stress is likewise a pure shear,  $\bar{\tau} = 2\mu \bar{\gamma}$ , when the mixture is isotropic. Formulae similar to (2.12) and (2.13) are thereby obtained for the rigidity modulus  $\mu$  in terms of  $\mu_1$  and  $\mu_2$ . The  $a$ 's and  $b$ 's now stand for the various

ratios formed from the averaged shearing stress and strain components having the plane and direction of the overall shear.

Equations (2.12) and (2.13), together with their shear analogues, follow also, of course, by decomposing the general relations (2.7), (2.9) and (2.10) into hydrostatic and deviatoric parts.

Basic as the results of this section are, they are apparently not to be found in the literature.

#### (iv) Reuss and Voigt estimates

A crude approximate treatment of reinforcement assumes that the strain throughout the mixture is uniform. With  $A_1 = A_2 = I$ , equation (2.7) yields  $L_V = c_1 L_1 + c_2 L_2$ . This is an isotropic tensor since  $L_1$  and  $L_2$  are, and so

$$\kappa_V = c_1 \kappa_1 + c_2 \kappa_2, \quad \mu_V = c_1 \mu_1 + c_2 \mu_2. \quad (2.14)$$

These are also obtainable from the first of (2.12) and its shear analogue with  $a_1 = a_2 = 1$ . These approximate values are distinguished by the subscript  $V$  because the uniform strain assumption was first proposed long ago by Voigt in connexion with the related polycrystal problem. Obviously (2.14) is tantamount to a simple volume weighting of the phase stiffnesses.

The dual assumption, due to Reuss and also originally for a polycrystal, is that the stress is uniform. This implies  $B_1 = B_2 = I$  in (2.9), or  $b_1 = b_2 = 1$  in the second of (2.12) and its analogue, giving

$$\frac{1}{\kappa_R} = \frac{c_1}{\kappa_1} + \frac{c_2}{\kappa_2}, \quad \frac{1}{\mu_R} = \frac{c_1}{\mu_1} + \frac{c_2}{\mu_2}, \quad (2.15)$$

which are simple weightings of the compliances.

Naturally, neither assumption is correct : the implied Voigt stresses are such that the tractions at phase boundaries would not be in equilibrium, while the implied Reuss strains are such that inclusions and matrix could not remain bonded.

The difference between the estimates can be put as

$$\kappa_V - \kappa_R = \frac{(\kappa_1 - \kappa_2)^2}{\left( \frac{\kappa_1}{c_1} + \frac{\kappa_2}{c_2} \right)} \quad (2.16)$$

with use of  $c_1 + c_2 = 1$ , with a similar expression for the shear moduli. Thus the Voigt values always exceed the Reuss ones, the difference being only a second-order infinitesimal for small differences between the phases. The difference becomes large, however, when one phase is comparatively rigid (so that  $\kappa_V$  and  $\mu_V \rightarrow \infty$  while  $\kappa_R$  and  $\mu_R$  are finite) or weak ( $\kappa_R$  and  $\mu_R \rightarrow 0$  with  $\kappa_V$  and  $\mu_V$  finite), the other phase having finite moduli. And, in general, both estimates are rather poor when the phase moduli differ by more than a factor of two or so.

### 3. ENERGY APPROACH

#### (i) Further remarks on averages

It has been shown how the central problem is reducible to the calculation of average stress or strain in one or other phase. A more versatile approach stems directly from classical theorems in elasticity and focusses attention on strain energies.

In the spirit of the previous shorthand the local energy per unit volume,  $U$  say, will be written as a scalar product  $\frac{1}{2}\sigma\epsilon$ . Expressed entirely in terms of strain or stress, the energy density is a quadratic form denoted by

$$U = \frac{1}{2}\epsilon L\epsilon \quad \text{or} \quad \frac{1}{2}\sigma M\sigma. \quad (3.1)$$

That  $L$  and  $M$  are symmetric tensors is proved below. In particular, when the strain is a pure dilatation  $\theta$ , the stress in an isotropic material is an all-round tension  $\sigma = \kappa\theta$  and the energy density is  $\frac{1}{2}\kappa\theta^2$  or  $\sigma^2/2\kappa$ . Again, if the strain is a pure shear  $\gamma$  (half the ‘engineering’ definition), the stress is a pure shear  $\tau = 2\mu\gamma$  and the energy density is  $2\mu\gamma^2$  or  $\tau^2/2\mu$ .

Consider *any* volume of the mixture, of unit magnitude on an appropriate scale. Let it be subjected to prescribed surface displacements of the kind that would produce a *uniform* strain  $\bar{\epsilon}$  in a homogeneous material. This is also the average strain in the inhomogeneous mixture itself, since the average is uniquely determined by the surface displacements, irrespective of the material (provided that it remains coherent)\*. Now the total strain energy, which is the integral of the density over the unit volume, can also be evaluated as the integral of  $\sigma\bar{\epsilon}/2$ . For the integral of  $\sigma(\epsilon - \bar{\epsilon})$  is zero since  $\sigma$  is an equilibrated field of stress while the (virtual) strain field  $\epsilon - \bar{\epsilon}$  is derived from a continuous displacement vanishing on the surface. Hence the total energy is just  $\bar{\sigma}\bar{\epsilon}/2$ , by definition of average stress.

In words : the average strain energy in *any* region can be calculated from the average stress and strain, when the surface constraints are of the specified kind. This can be proved similarly when surface tractions are prescribed, of the kind that would produce uniform internal stress in a homogeneous material†. In exactly the same fashion one can show that the infinitesimal  $\bar{\sigma}d\bar{\epsilon}$  is the increment in average energy per unit volume and therefore a perfect differential. It follows that tensor  $L$  is symmetric, and hence also its inverse  $M$ , just like the array of coefficients in the local stress/strain relations.

Suppose that the considered region is a *representative* volume. Then the relations between average stress and strain are the *same* for both types of boundary condition. We can then write, without discriminating,

$$2\bar{U} = \bar{\sigma}\bar{\epsilon} = \bar{\epsilon}L\bar{\epsilon} = \bar{\sigma}M\bar{\sigma}. \quad (3.2)$$

Furthermore, there is no difficulty in proving that this remains valid when the loading is merely ‘macroscopically uniform,’ in the limiting sense defined in Section 2 (ii). Notice, on the other hand, that similar relations do *not* invariably hold for the average energy density in a phase, in terms of its own average stress and strain.

## (ii) Elementary bounds on bulk and shear moduli

According to the familiar principle of minimum potential energy for an elastic continuum, the actual strain energy in the mixture does not exceed the energy

\*The actual formula is, in suffix notation,

$$\int \epsilon_{ij} dV = \frac{1}{2} \int (n_i u_j + n_j u_i) dS$$

by an application of Gauss’s theorem, where  $V$  is the region,  $S$  its surface,  $u_i$  the displacement, and  $n_l$  the unit outward normal.

†The average stress depends uniquely on the surface loading since

$$\int \sigma_{ij} dV = \frac{1}{2} \int (F_i x_j + F_j x_i) dS$$

where  $F_i$  is the traction at a point with co-ordinates  $x_i$ .

of any fictitious (unequilibrated) state of distortion with the same surface displacements. When these are compatible with a uniform strain  $\bar{\epsilon}$ , this can itself be taken as a comparison fictitious state, having an energy density  $\bar{\epsilon} L_1 \bar{\epsilon}/2$  throughout one phase and  $\bar{\epsilon} L_2 \bar{\epsilon}/2$  throughout the other. Hence, by considering a representative volume under such surface constraints and using (3.2),

$$\bar{\epsilon} L \bar{\epsilon} \leq \bar{\epsilon} (c_1 L_1 + c_2 L_2) \bar{\epsilon} \text{ for arbitrary } \bar{\epsilon}. \quad (3.3)$$

Thus, the matrix  $c_1 L_1 + c_2 L_2 - L$  is positive semi-definite and the consequent restrictions on its components (and on those of  $L$  alone) can be written down by any of several standard methods. In this way upper bounds are obtained on each modulus appearing as a coefficient in the 'normal form' of the overall energy density ; i.e. when converted into a sum of squares of independent linear combinations of the components of average strain.

When the mixture is isotropic, one can proceed in a more elementary manner. The overall strain is put equal in turn to a pure dilatation and a pure shear. The general inequality (3.3) reduces to

$$\kappa \leq \kappa_V, \quad \mu \leq \mu_V, \quad (3.4)$$

in terms of the Voigt estimates (2.14). Similarly, by considering loading under surface tractions compatible with a fictitious uniform field of stress, and applying the dual principle of minimum complementary energy,

$$\bar{\sigma} M \bar{\sigma} \leq \bar{\sigma} (c_1 M_1 + c_2 M_2) \bar{\sigma} \text{ for arbitrary } \bar{\sigma}. \quad (3.5)$$

This furnishes upper bounds on compliances and therefore lower bounds on moduli. When the mixture is isotropic, the overall stress is put equal in turn to an all-round tension and a pure shear, giving

$$\kappa \geq \kappa_R, \quad \mu \geq \mu_R, \quad (3.6)$$

in terms of the Reuss estimates (2.15).

It has thereby been proved rigorously that the actual overall moduli lie somewhere in the interval between the Reuss and Voigt values, no matter what the geometry may be. The result is illuminating but not particularly sharp in general ; better universal bounds are stated later. This theorem was first proved by HILL (1952) for the related polycrystal problem, where it is more effective for actual materials, and afterwards in the present context (1955 unpublished) ; and for  $\mu$ , but not  $\kappa$ , by PAUL (1960). The writer also at that time derived improved bounds for specific geometries by constructing piecewise-uniform comparison fields ; the results, extended to fibre-like inclusions with arbitrary properties, are detailed by CROSSLEY (1963) and summarized by HILL (1962).

From (2.12) and the first of both (3.4) and (3.6) one easily deduces

$$a_1 \leq 1 \text{ and } b_1 \geq 1 \text{ according as } \kappa_1 \geq \kappa_2.$$

There are similar inequalities for the  $a$ 's and  $b$ 's associated with the shear moduli. Expressed in words : under dilatation or pure shear the average corresponding strain in the weaker phase exceeds that in the stronger phase, while the reverse holds for the average stress. This accords with intuitive expectation, but would perhaps be unconvincing as an *a priori* basis for the converse derivation of (3.4) and (3.6).

## (iii) Elementary bounds on Young's modulus

Let  $E_R$  and  $E_V$  denote values of Young's modulus calculated from the second equation in (2.1) with the Reuss and Voigt estimates, respectively, of the bulk shear moduli given by (2.14) and (2.15). It follows easily from (3.4) and (3.6) that

$$\left. \begin{aligned} \frac{1}{E_V} &\leq \frac{1}{E} \leq \frac{1}{E_R} \quad \text{or} \quad E_R \leq E \leq E_V \\ \text{where} \quad \frac{3}{E_R} &= \frac{1}{\mu_R} + \frac{1}{3\kappa_R}, \quad \frac{3}{E_V} = \frac{1}{\mu_V} + \frac{1}{3\kappa_V}. \end{aligned} \right\} \quad (3.7)$$

But, due to the linear relation between the reciprocal of Young's modulus and the reciprocals of the bulk and shear moduli, one has from (2.15) that

$$\frac{1}{E_R} = \frac{c_1}{E_1} + \frac{c_2}{E_2}. \quad (3.8)$$

Thus, the Reuss estimate (3.7) of Young's modulus is what would be obtained by calculating the average longitudinal strain in a tension test on the assumption that the local stress in the mixture is uniform. That this provides a lower bound, as in (3.7), can alternatively be proved directly from the complementary energy principle (3.5), by putting the overall stress equal to a uniaxial tension.

By contrast,  $E_V \neq c_1 E_1 + c_2 E_2$  in general, since Young's modulus is not a linear combination of the two primary moduli. In fact one can show, after a little algebra, that  $E_V \geq c_1 E_1 + c_2 E_2$  with equality only when the Poisson ratios are equal.

Finally, therefore, we have the chain of comparisons :

$$\left( \frac{c_1}{E_1} + \frac{c_2}{E_2} \right)^{-1} = E_R \leq \left\{ \begin{array}{l} E \\ c_1 E_1 + c_2 E_2 \leq E_V. \end{array} \right. \quad (3.9)$$

There is no universal ordering of  $E$  and  $c_1 E_1 + c_2 E_2$ . When the Poisson ratios are equal it can be shown that  $E$  is the smaller (by means of the potential energy principle); whereas, when the shear moduli are equal,  $E$  is the greater (from the exact solution in Section 4 below). In any event not much significance can be found for the simple weighting formula, since there is no uniform triaxial strain such that the longitudinal component of stress in each phase is proportional simply to the respective Young's modulus.

It may be mentioned that PAUL (1960) gives a very involved formula for an upper bound on  $E$  (obtained by the potential energy principle applied to a tension specimen under prescribed axial extension and no tractions on the cylindrical surface). Algebraic reduction reveals that this formula is in reality none other than  $E_V$  heavily disguised. And, in fact, a more circumspect use of the energy principle than Paul's yields  $E \leq E_V$  very easily, confirming (3.7). Thus, by taking any fictitious uniform triaxial strain with given axial component  $\epsilon$ , we clearly have that  $E\epsilon^2/2$  is less than the energy density of the triaxial strain in an isotropic material with moduli  $\kappa_V, \mu_V$ . And this density is least when the strain is such that the actual stress in the latter material is *uniaxial* and produces extension  $\epsilon$ ; that is, when the density is  $E_V \epsilon^2/2$ . This remark follows immediately by applying again the potential energy principle in the same way, this time to the stated isotropic material.

## (iv) Comparison of energies

The reinforcement problem can also be approached profitably in the following way. Compare the energy in unit *representative* volume of mixture with that in unit volume of a single constituent (say the second). By (3.1) and (3.2) the difference in the elastic energies is  $\bar{\epsilon}(L - L_2)\bar{\epsilon}/2$  if the overall strains are the same, and  $\bar{\sigma}(M - M_2)\bar{\sigma}/2$  if the overall stresses are the same. In this second case the difference in the total *potential* energies (i.e. including that of the maintained surface loads) is precisely minus the difference in *elastic* energies†. The expressions are equal to

$$\frac{1}{2}c_1\bar{\epsilon}(L_1 - L_2)\bar{\epsilon}_1 \quad \text{and} \quad \frac{1}{2}c_1\bar{\sigma}(M_1 - M_2)\bar{\sigma}_1 \quad (3.10)$$

respectively, by (2.10) with the help of (2.6) and (2.8). An advantage of these scalar expressions over (2.10) (to which they are in principle entirely equivalent) is that they can in practice be approximated more conveniently than the separate components of the unknowns  $\bar{\sigma}_1$  and  $\bar{\epsilon}_1$ .

Corresponding formulae, in terms of integrals instead of averages, have been derived for any volume of mixture by HASHIN [1960, 1962, equations (11) and (12)]. The terminology 'polarization stress and strain' is sometimes used for  $(L_1 - L_2)\bar{\epsilon}_1$  and  $(M_1 - M_2)\bar{\sigma}_1$  respectively.

There is an interesting connexion between the energy changes in (3.10). From the first of (2.9) and (2.4) they can be written as

$$\frac{1}{2}c_1\bar{\sigma}M(L_1 - L_2)\bar{\epsilon}_1 \quad \text{and} \quad \frac{1}{2}c_1\bar{\sigma}M_2(L_2 - L_1)\bar{\epsilon}_1, \quad (3.11)$$

remembering that  $L_1M_1 = I = L_2M_2$ . These forms differ primarily in sign but secondarily in that  $M$  appears in one and  $M_2$  in the other. If the concentration of the first phase is small,  $M$  is correspondingly close to  $M_2$ . In that event, to within a second-order infinitesimal, the expressions (3.11) agree in magnitude ; the differences in potential energy are then approximately equal, and the differences in elastic energy equal and opposite.

## 4. EQUAL PHASE RIGIDITIES : EXACT SOLUTION

## (i) Statement of results

Suppose that the phases have equal rigidity moduli  $\mu$ , only their bulk moduli differing (or equivalently, in this instance, only their Poisson ratios). Then the complete solution is known in principle for any geometry whatever (HILL 1962). The detailed fields of stress and strain are expressible in terms of the gravitational potential of a certain distribution of matter, and are derived in (ii) below.

It is found, remarkably enough, that the overall bulk modulus of an isotropic mixture, or one with cubic symmetry, depends solely on the concentrations and the separate moduli, and is unaffected by the shapes of the inclusions. It is given by a simple symmetric formula :

$$\frac{\kappa}{\kappa_R} = \left(1 + \frac{4\mu\kappa\nu}{3\kappa_1\kappa_2}\right) / \left(1 + \frac{4\mu\kappa_R}{3\kappa_1\kappa_2}\right). \quad (4.1)$$

†By a simple but very general theorem : if the moduli of an elastic continuum under fixed loading are changed in any way (inhomogeneously), the work done by the loads (or potential energy lost) in the accompanying surface adjustments is equal to twice the gain in strain energy.

Alternative ways of writing this are

$$\frac{\kappa - \kappa_R}{\kappa_V - \kappa_R} = 1 / \left( 1 + \frac{3\kappa_1\kappa_2}{4\mu\kappa_R} \right), \quad (4.2)$$

$$\frac{\kappa_V - \kappa}{\kappa_V - \kappa_R} = 1 / \left( 1 + \frac{4\mu\kappa_R}{3\kappa_1\kappa_2} \right). \quad (4.3)$$

Terms identifiable with the Reuss and Voigt estimates (2.14) and (2.15) occur in the formulae and are so denoted. It is thereupon apparent from (4.2) and (4.3) that the true value of the bulk modulus is bracketed by these estimates, in accordance with the general conclusion in Section 3 (ii). The overall rigidity modulus is of course just  $\mu$  itself, the value common to both phases.

An unsymmetric variant of the above expressions is

$$\frac{\kappa - \kappa_2}{\kappa_1 - \kappa_2} = \frac{c_1}{1 + (\kappa_1 - \kappa_2) c_2 / (\kappa_2 + \frac{4}{3}\mu)} \quad (4.4)$$

or the analogous equation with numerical subscripts interchanged ; or, yet again, with all moduli (including  $\frac{4}{3}\mu$ ) replaced by their reciprocals. By comparison with (2.13) the concentration factors for the average hydrostatic stress and volumetric strain in each phase can be read off :

$$\frac{a_1}{\kappa_2 + \frac{4}{3}\mu} = \frac{a_2}{\kappa_1 + \frac{4}{3}\mu} = \frac{1}{c_1\kappa_2 + c_2\kappa_1 + \frac{4}{3}\mu}, \quad (4.5)$$

$$\frac{b_1}{\kappa_1(\kappa_2 + \frac{4}{3}\mu)} = \frac{b_2}{\kappa_2(\kappa_1 + \frac{4}{3}\mu)} = \frac{1}{\kappa_1\kappa_2 + \frac{4}{3}\mu(c_1\kappa_1 + c_2\kappa_2)}. \quad (4.6)$$

The Poisson ratio of the mixture is obtained via (2.1) :

$$\nu = \frac{c_1\nu_1 + c_2\nu_2 - \nu_1\nu_2}{1 - c_1\nu_2 - c_2\nu_1}.$$

For fixed shear modulus, Poisson's ratio increases monotonically from  $-1$  to  $+\frac{1}{2}$  with increasing bulk modulus. Consequently, the exact value lies between those calculated from the Reuss and Voigt estimates of the bulk modulus. With some further algebra, the following chain of inequalities can be established (still under the restriction of equal shear moduli) :

$$\left( \frac{c_1}{\nu_1} + \frac{c_2}{\nu_2} \right)^{-1} \leqslant \nu_R \leqslant c_1\nu_1 + c_2\nu_2 \leqslant \nu \leqslant \nu_V.$$

Young's modulus of the mixture is obtainable as  $E = 2\mu(1 + \nu)$ , which can be arranged as

$$E = \frac{c_1 E_1 + c_2 E_2 - E_1 E_2 / 4\mu}{1 - (c_1 E_2 + c_2 E_1) / 4\mu}. \quad (4.7)$$

In addition to the relations (3.9) it can be verified that, in this special case of equal rigidities,  $E$  exceeds  $c_1 E_1 + c_2 E_2$ .

## (ii) Derivation

The solution just outlined depends on that for the following auxiliary problem†.

†Prompted by the treatment of ESHELBY (1957, 1961) for a single misfitting inclusion in an infinite matrix of the same material, when the misfit is equivalent to a uniform expansion. No dilatation is induced in the matrix, whose bulk modulus is consequently without influence and can be varied at will. It is because of such a property that the present analysis succeeds.

Imagine that one phase (say the first) has uniform mass-density while the other has none. Let  $\phi$  be the gravitational potential in an arbitrary volume of mixture under boundary conditions to be stipulated later ; in general these call for the presence of additional external matter, but this is of no concern. In suitable units, and with  $\nabla^2$  denoting the Laplacian,

$$\nabla^2 \phi = \begin{cases} -1 & \text{in the first phase} \\ 0 & \text{in the second phase} \end{cases} \quad (4.8)$$

while across interfaces the potential and its vector gradient are required to be continuous. There are, however, discontinuities in the second derivatives :

$$\left( \frac{\partial^2 \phi}{\partial x_i \partial x_j} \right)_2 - \left( \frac{\partial^2 \phi}{\partial x_i \partial x_j} \right)_1 = n_i n_j \quad (4.9)$$

where  $x_i$  ( $i = 1, 2, 3$ ) are cartesian co-ordinates,  $n_i$  is the local unit normal at the considered point of an interface [for a proof see HILL (1961), Section 2]. These jumps, unavoidable accompaniments of (4.8), are pivotal in the ensuing analysis.

Now let a continuous and irrotational displacement  $u_i$  be defined in the considered volume by

$$\mu u_i / \eta = (\kappa_1 - \kappa_2) \frac{\partial \phi}{\partial x_i} + \frac{1}{3} (\kappa_1 + \frac{4}{3} \mu) x_i \quad (4.10)$$

where  $\eta$  is an infinitesimal constant parameter. The first right-hand term represents a displacement proportional to the gravitational field, while the last term represents a uniform expansion. The resulting strain tensor  $\epsilon_{ij}$  is given everywhere by

$$\mu \epsilon_{ij} / \eta = (\kappa_1 - \kappa_2) \frac{\partial^2 \phi}{\partial x_i \partial x_j} + \frac{1}{3} (\kappa_1 + \frac{4}{3} \mu) \delta_{ij} \quad (4.11)$$

where  $\delta_{ij}$  is the usual Kronecker delta. The strain is discontinuous across phase boundaries, by (4.9), and in particular the dilatational part is *piecewise constant*, and respectively

$$\theta_1 = (\kappa_2 + \frac{4}{3} \mu) \eta / \mu, \quad \theta_2 = (\kappa_1 + \frac{4}{3} \mu) \eta / \mu, \quad (4.12)$$

in each phase. The average dilatation in the mixture is therefore

$$\bar{\theta} = c_1 \theta_1 + c_2 \theta_2 = (c_1 \kappa_2 + c_2 \kappa_1 + \frac{4}{3} \mu) \eta / \mu, \quad (4.13)$$

and the concentration factors are as stated in (4.5) and (4.6). Conversely, when the overall dilatation is assigned, the parameter  $\eta$  is determined by (4.13). In fact, when the normal component of displacement is prescribed arbitrarily over the surface of the considered volume, hence fixing  $\bar{\theta}$ , this relation ensures the existence of a solution to the Neumann problem set by (4.8) with boundary values obtained from (4.10) for the normal derivative of potential.

The stress tensor  $\sigma_{ij}$  corresponding to strain (4.11) is obtained as

$$\left[ \frac{\sigma_{ij}}{\eta} - \frac{\kappa_2 (\kappa_1 + \frac{4}{3} \mu)}{\mu} \delta_{ij} \right] / 2 (\kappa_1 - \kappa_2) = \begin{cases} \frac{\partial^2 \phi}{\partial x_i \partial x_j} + \delta_{ij} & \text{in phase 1} \\ \frac{\partial^2 \phi}{\partial x_i \partial x_j} & \text{in phase 2.} \end{cases} \quad (4.14)$$

With the help of (4.9) the stress jump at an interface is accordingly

$$2\eta(\kappa_1 - \kappa_2)(\delta_{ij} - n_i n_j) \quad (4.15)$$

in crossing from the second phase to the first. Clearly, however, the interfacial traction is itself continuous, since  $(\delta_{ij} - n_i n_j)n_j = 0$  or by recognizing the last bracket in (4.15) as a unit biaxial tension in the local tangent plane to the interface. Moreover, the distribution (4.14) is self-equilibrated throughout the mixture, since  $\partial\sigma_{ij}/\partial x_i = 0$  by (4.8).

It follows that we have obtained the actual displacement field in any volume of mixture under surface values of type (4.10) with arbitrary normal component. This family of solutions has the distinctive property that the ratio of average hydrostatic stress to average dilatation is always  $(c_1 \kappa_1 \theta_1 + c_2 \kappa_2 \theta_2)/(c_1 \theta_1 + c_2 \theta_2)$ , where  $\theta_1$  and  $\theta_2$  are the uniform dilatational parts (4.12) of the phase strains. This is precisely the modulus stated in (4.1). It remains to establish that it is actually the conventional bulk modulus when, in particular, the mixture has cubic symmetry or is isotropic. We have to verify that the family contains a solution corresponding to a macroscopically uniform *pure* dilatation in a *representative* volume of such a mixture. To this end it is enough to prescribe a normal component of surface displacement equalling that in a uniform expansion  $\bar{\theta}$ ; or equivalently to prescribe a normal derivative of potential as in a central field  $-c_1 x_i/3$  (like that in a solid sphere with a mass density of  $c_1$  units). For a mixture with the stated properties it is apparent that the solution of this Neumann problem will fluctuate about the centrosymmetric potential as its mean with a wavelength of the order of the distance between inclusions. Correspondingly, the local elastic strains will fluctuate similarly about a *pure* uniform expansion. The remarks in Section 2 (ii) complete the argument.

Finally, it is noted that the potential would be  $\psi$ , where  $\phi + \psi = -x_i^2/6$ , if all the mass were assigned to the second phase instead of the first. The end results are naturally unaltered, while in the formulae for displacement and strain  $\phi$  is replaced by  $\psi$  and the numerical subscripts interchanged.

## 5. BOUNDS ON BULK MODULUS FOR ARBITRARY MIXTURE

### (i) *Derivation*

Suppose that the mixture is isotropic or has cubic symmetry, the geometry being otherwise arbitrary. It is intuitively evident, and is proved in (iii) below, that the overall bulk modulus would be increased (decreased) if both phases had a common rigidity modulus equal to the larger (smaller) of their actual values. Consequently, the actual bulk modulus must lie between those of two similarly proportioned mixtures, in one of which the phases have the same rigidity  $\mu_1$  and in the other the same rigidity  $\mu_2$  (the separate bulk moduli being fixed). The significance of this observation is that the exact solution has just been found when the phases have equal rigidities. In fact, immediately from (4.1), we now have

$$\frac{4\mu_2 \kappa_V + 3\kappa_1 \kappa_2}{4\mu_2 \kappa_R + 3\kappa_1 \kappa_2} \leq \frac{\kappa}{\kappa_R} \leq \frac{4\mu_1 \kappa_V + 3\kappa_1 \kappa_2}{4\mu_1 \kappa_R + 3\kappa_1 \kappa_2} \quad (5.1)$$

with the phases numbered so that  $\mu_1 > \mu_2$ .

The variant obtained from (4.4) is

$$\frac{c_1}{1 + (\kappa_1 - \kappa_2) c_2 / (\kappa_2 + \frac{4}{3} \mu_2)} \leq \frac{\kappa - \kappa_2}{\kappa_1 - \kappa_2} \leq \frac{c_1}{1 + (\kappa_1 - \kappa_2) c_2 / (\kappa_2 + \frac{4}{3} \mu_1)} \quad (5.2)$$

when the signs of  $\kappa_1 - \kappa_2$  and  $\mu_1 - \mu_2$  are the same, and with the inequalities reversed when different (the numbering is immaterial here).

Equivalent bounds were first obtained, in an entirely different way, by HASHIN and SHTRIKMAN (1963). Their method is rather elaborate and depends on a novel extremum principle (1962); its proof and merits have been re-considered by HILL (1963). By this method the bounds could, however, only be demonstrated under the restriction that  $\kappa_1 - \kappa_2$  and  $\mu_1 - \mu_2$  have the same signs. Hashin and Shtrikman's original form†, with  $\mu_1 > \mu_2$ , is

$$\kappa_2 + \frac{(\kappa_1 - \kappa_2) c_1}{1 + \alpha_2 c_2 (\kappa_1 / \kappa_2 - 1)} \leq \kappa \leq \kappa_1 + \frac{(\kappa_2 - \kappa_1) c_2}{1 + \alpha_1 c_1 (\kappa_2 / \kappa_1 - 1)} \quad (5.3)$$

where the  $\alpha$ 's are defined as in (2.2) in terms of the respective phase moduli. When  $\mu_1 < \mu_2$  the inequalities must be reversed or the subscripts interchanged.

Notice that the lower bounds on  $\kappa$  in (5.2) and (5.3) are virtually identifiable as they stand, but that the upper bounds need manipulation. For this reason, while it is obvious at sight that the bounds in (5.2) coincide when the shear moduli are equal, it is not so apparent from (5.3). Indeed, this coincidence of bounds originally went unremarked.

It is clear that the bounds are *best-possible*, in terms only of concentrations and moduli, when no regard is paid to geometry (or when this is unknown). For each bound is exact, for arbitrary geometry, when the shear moduli are both equal to  $\mu_1$  or  $\mu_2$  respectively, as proved in Section 4.

### (ii) Interpretation of bounds

It is interesting to observe that each bound coincides with the exact overall bulk modulus of a so-called 'spherical composite element.' This consists of a solid spherical inclusion of one constituent cemented inside a matching spherical shell of the other. The external surface (radius  $R$  say) is subjected to uniform pressure  $P$  producing an inward displacement  $U$ . The apparent bulk modulus of the composite element is conveniently calculated as  $PR/3U$  from the elastic solution (see Appendix).

It turns out that this modulus is equal to the upper (lower) bound in (5.1) if the shell has the rigidity of the first (second) constituent, when the bulk moduli and associated concentrations in the composite element are those in the mixture. It is actually immaterial which bulk modulus and associated concentration are assigned to the shell, if one is prepared to allow the pairings  $\kappa_1, \mu_2$  and  $\kappa_2, \mu_1$ .

The reason for the non-appearance of the rigidity modulus of the inclusion in the overall bulk modulus of the element is simply that the inclusion stress is purely hydrostatic (and uniform). That the element should be less compressible when the more rigid constituent is used in the shell was only to be expected.

It has been remarked by HASHIN (1960, 1962) that each bound (5.1) is exact

†1961 University of Pennsylvania, School of Engineering, Tech. Rep. 1.

for a mixture consisting entirely of spherical composite elements with the appropriate moduli and concentrations, and with sizes ranging from finite to vanishingly small such that the whole space is filled.

(iii) *An auxiliary theorem*

We supply now the proof that the overall moduli are increased when either or both of the bulk and rigidity moduli are increased in one or both phases. According to the remarks in Section 3 (i) this follows, for an isotropic mixture, if the total elastic energy increases under fixed overall strain (or decreases under fixed overall stress). We prove, even more generally, for *any* heterogeneous elastic continuum, that the energy increases under fixed surface constraints when the material is 'strengthened' in any (non-uniform) way. Neither the theorem nor its proof is apparently to be found in the textbooks.

By 'strengthened' is meant that, in each changed element of material, the energy density afterwards ( $U'$ ) would exceed that beforehand ( $U$ ) *if* the local strain were the same. But it is not the same. We therefore appeal first to the minimum energy principle applied to the original body, where the actual strain is  $\epsilon$ , taking for comparison the actual field  $\epsilon'$  in the changed body. Then, in terms of average densities, the principle yields

$$\bar{U}(\epsilon) \leq \bar{U}(\epsilon').$$

By hypothesis, for any strain  $\epsilon'$  and at each point of the body,

$$U(\epsilon') \leq U'(\epsilon').$$

Whence

$$\bar{U}(\epsilon) \leq \bar{U}(\epsilon') \leq \bar{U}'(\epsilon')$$

and the desired result follows by comparing the first and last members.

## 6. FINAL REMARKS

There remains the difficult problem of bounding the *shear* modulus. The idea behind the method leading to (5.1), while valid here also, is not yet practicable since the formula analogous to (4.1) is not known when only the shear moduli differ. In any event it would probably depend on geometry and not on concentration alone.

Bounds have, however, been proposed by HASHIN and SHTRIKMAN (1963) via their variation principle, as follows :

$$\mu_2 + \frac{(\mu_1 - \mu_2) c_1}{1 + \beta_2 c_2 (\mu_1/\mu_2 - 1)} \leq \mu \leq \mu_1 + \frac{(\mu_2 - \mu_1) c_2}{1 + \beta_1 c_1 (\mu_2/\mu_1 - 1)} \quad (6.1)$$

when  $\mu_1 - \mu_2$  and  $c_1 - c_2$  are both positive, and with inequalities reversed (or subscripts interchanged) when they are both negative, the  $\beta$ 's being defined as in (2.2). When the differences of the moduli have opposite signs the method fails. Notice that the bounds each involve only one or other bulk modulus. However, they coincide for arbitrary concentrations, not when the bulk moduli are equal, but when  $\mu_1(1 - \beta_1)/\beta_1 = \mu_2(1 - \beta_2)/\beta_2$ . This is a complicated relationship and the analogues of (5.1) and (5.2) are not as simple.

In Section 5 (ii) a connexion was established between each bound on the overall bulk modulus of a mixture and the exact formula for the apparent bulk modulus of a spherical composite element. There is a similar though less specific connexion in respect of the shear modulus, which helps towards clarifying the status of (6.1). Thus, HASHIN (1960, 1962) obtained exact solutions for the internal fields in a spherical composite element under two types of surface loading : (a) where the traction is prescribed and (b) where the displacement is prescribed, both distributions being of the kind that would be complementary and produce *uniform* shearing of a *homogeneous* sphere. In the composite element, however, these loadings are not complementary : the respective internal fields are distinct and not macroscopically uniform (by contrast with what the situation would be for *representative* volume of mixture similarly loaded). Consequently, the apparent shear moduli of the element, which may here be calculated from the appropriate component of average shear strain in (a) and of average shear stress in (b), are themselves distinct, with modulus (b) exceeding (a).

Hashin makes the empirical observation that the right (left) side of (6.1) falls between the apparent moduli (a) and (b) when the shell of the composite element is made from the first (second) constituent. He indicates, furthermore, how the standard extremum principles can be used to show that the interval between (a) and (b) also contains the (unknown) shear modulus of the corresponding special distribution of composite elements described in Section 5 (ii). It remains an open question whether *this* shear modulus is actually given by the appropriate side of (6.1).

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#### APPENDIX : SPHERICAL COMPOSITE ELEMENT : BULK MODULUS

We are concerned here with a spherical inclusion of the first constituent cemented in a matching spherical shell of the second constituent. A uniform pressure  $P$  applied to the external surface

of radius  $R$  produces an (infinitesimal) inward displacement  $U$ . The fractional decrease in total volume is  $3U/R$ , while the average strain is easily shown to be an all-round compression of amount  $U/R$  [cf. the first footnote in Section 3 (i)]. The apparent bulk modulus of the composite element is

$$\kappa = PR/3U \quad (1)$$

either directly from the ratio of pressure to fractional volume change, or by equating the work done by the pressure to the total strain energy expressed in terms of average strain [cf. Section 3 (i)].

The solution is easily constructed as a combination of elementary, spherically-symmetric, elastic fields (which are given in any text on strength of materials). The final result is that the displacement (which is radially inward) has magnitude  $u$  at distance  $r$  from the centre, where

$$ku/U = \begin{cases} (\kappa_2 + \frac{4}{3}\mu_2)r/R & \dots \dots \dots \text{(inclusion)} \\ (\kappa_1 + \frac{4}{3}\mu_2)r/R - (\kappa_1 - \kappa_2)c_1 R^2/r^2 & \dots \dots \dots \text{shell) } \end{cases} \quad (2)$$

and  $k = c_1\kappa_2 + c_2\kappa_1 + \frac{4}{3}\mu_2$ .

In the shell the deformation is a uniform compression together with a non-uniform strain involving no change of volume. The inclusion is simply compressed uniformly and so its shear modulus does not enter. Consequently, the analysis of Section 4 could have been applied here ; the correspondences with (4.10), (4.12) and (4.13) are apparent.

The compressive radial and hoop stresses, magnitudes  $p$  and  $q$  respectively, are given by

$$kpR/3U = \begin{cases} \kappa_1(\kappa_2 + \frac{4}{3}\mu_2) & \dots \dots \dots \text{(inclusion)} \\ \kappa_2(\kappa_1 + \frac{4}{3}\mu_2) + \frac{4}{3}\mu_2(\kappa_1 - \kappa_2)c_1 R^3/r^3 & \dots \text{(shell)} \end{cases} \quad (3)$$

$$kqR/3U = \begin{cases} \kappa_1(\kappa_2 + \frac{4}{3}\mu_2) & \dots \dots \dots \text{(inclusion)} \\ \kappa_2(\kappa_1 + \frac{4}{3}\mu_2) - \frac{2}{3}\mu_2(\kappa_1 - \kappa_2)c_1 R^3/r^3 & \dots \text{(shell).} \end{cases} \quad (4)$$

It will be seen that both the displacement and radial stress are indeed continuous across the interface, in conformity with the stipulated bonding there. The jump in hoop stress is, of course, admissible and inevitable.

The relation between surface pressure and displacement is obtained on putting  $r = R$  in (3) :

$$\frac{PR}{3U} = \frac{\frac{4}{3}\mu_2(c_1\kappa_1 + c_2\kappa_2) + \kappa_1\kappa_2}{c_1\kappa_2 + c_2\kappa_1 + \frac{4}{3}\mu_2}. \quad (5)$$

With (1) this supplies an expression for the overall bulk modulus that is structurally analogous to the bounds (5.1)