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MEC-E8007 Thin-Walled Structures

Lecture 4. Large deflections through moderate rotations

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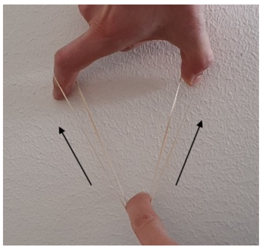
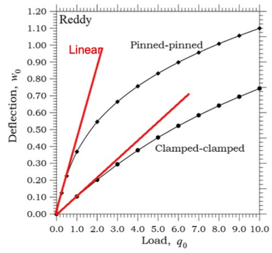
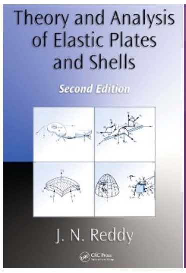
The large deflections through modarate roations is important topic in thin-walled structures as this allows us to design lightweight stuctures by accounting the positive effect of large deflections to the stiffness and strength. We introduce this concept before we move to the beam and plate formulations.

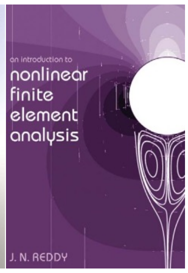
Contens

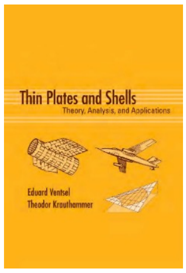
- The aim of the lecture is to understand the role of geometrically non-linear von Karman strains which allow us to model large deflections through moderate rotations of thin-walled structures (beams, plates and shells)
- Motivation
- von Karman strain of a beam
- Extension to plates
- Literature:
  1. Reddy, J.N., (1999). Theory and analysis of elastic plates and shells
  2. Ventsel, E. and Krauthammer, T. (2001). Thin plates and shells: Theory, Analysis and Applications
  3. Reddy, J.N. (2004). An Introduction to Nonlinear Finite Element Analysis.











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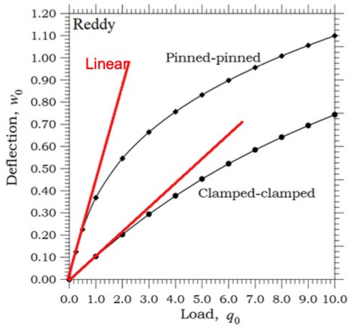
The aim of the lecture is to understand the role of geometrically non-linear von Karman strains which allow us to model large deflections through moderate rotations of thin-walled structures (beams, plates and shells). This beneficial effect is important when we aim for lightweight design. The literature on the subject is vast and by far the best description is given in terms of physics in the book of Ventsel and in mathematics and FE setting in the books of Reddy.

Motivation

- Flexible = thin plates are widely used in engineering as skins of aircraft, ships, pressure vessels etc
- When the deflections increase to range of w=0.5-1t, the membrane forces start to carry the load over the bending moments and shear forces (stiffening effect)
- This is why we often allow the responses as predicted by linear structural (beam, plate and shell) theories exceed the limit values
- This non-linear transition from bending to membrane action can be described through von Karman strains



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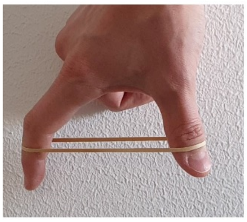
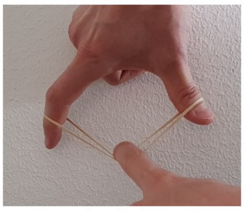
As stated the aim is to use proper physical assuptions in our design solutions. Flexible, thin, plates are widely used in engineering as skins of aircraft, ships, pressure vessels. However, what we learn in basic courses on beams and plates is the small deflection theory in which the (positive) effect of large deflections is neglected due to mathematical complexity it introduces in terms of geometrical non-linearity. Thus, the prevailing equations get (slightly) more difficult with the non-linear effect included.

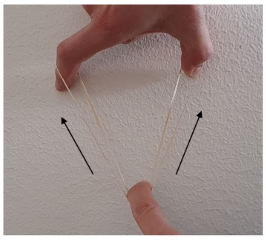
When the deflections, w, increase to the range of plate thickness, i.e. w=0.5-1t, the membrane forces start to carry the load over the bending moments and shear forces (stiffening effect). Due to this effect the predicted deflections and stresses decrease. This is why we often allow the responses as predicted by **linear** structural (beam, plate and shell) theories exceed the limit values (for example in design rules). This **non-linear effect** acts as extra capacity, a reserve due to simplified modeling. This non-linear transition from bending to membrane action can be described through von Karman strains, the concept will be introduced next and is the focus of this session as it is essential when we want to understand the more complicated laminated beam, plate and shell theories, which utilize this effect. The focus is on physics.

Motivation – rubberband analogy

- As you start to bend a pre-stretched rubber band, the stiffness is fairly low at the start. This is as the resistance is mainly due to internal bending moment caused by linear variation of bending stress over the rubber band
- As the deformation increases stretching becomes more and more difficult as in addition to these bending moments you need to compensate the axial elongation
- The nonlinear von Karman strains couple membrane and bending behavior and cause membrane forces to resist bending
- The 1D example can be generalized to cover beams, plates and shells


Undeformed


Easy to stretch a bit to cause a displacement at the center


Membrane forces resist and you really need to pull

The easiest way to comprehend the concept is to think it through the "rubberband analogy". As you start to bend a pre-stretched rubber band, the stiffness is fairly low at the start; stiffness is here now the force/deflection. This is as the resistance is mainly due to internal bending moment caused by linear variation of bending stress over the rubber band thickness. So the bending resistance is very small, almost so small that we cannot even feel it. This is where the linear beam theory would be valid. The theory we learn in basic solid mechanics courses.

As the deformation increases stretching becomes more and more difficult as in addition to these bending moments you need to compensate the axial elongation of the rubber band. When the rubber band is inclined significantly, its length is not the same as it was when unloaded. Linear theory assumes that it has the same length. So we need to derive the new length, account it in our beam model and utilize the effect for extra load-carrying. The nonlinear von Karman strains couple membrane and bending behavior and cause membrane forces to resist bending. The 1D example given here can be generalized to cover beams, plates and shells. This is where we start typically in the text books. We do not describe the physics, we describe the mathematics directly.

Beam in bending

- In beam bending the length of an infinitesimal element, *dx*, is usually considered unchanged

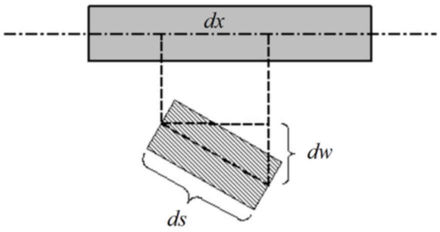
- In case of von Karman strains we relax this assumption and consider the geometric facts about the segment, the length, *ds*, changes

- The transverse deflection is denoted by *dw*

- So from the geometry we get

$$ds = \sqrt{dx^2 + dw^2} = dx\sqrt{1 + \left(\frac{dw}{dx}\right)^2}$$

which is mathematically "nasty" as the differentials are squared and "square-rooted"



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Now we can do the mathematics in 1D. In beam bending the length of an infinitesimal element, dx, is usually considered unchanged when deforming by deflection only a small amount. In case of von Karman strains we relax this assumption and consider the geometric facts about the segment, the length, ds, changes. So when he transverse deflection is denoted by dw, we can derive from the geometry the new ds according to Pythagoras's law. When we take the dx out from the result, we can simplify the new length a bit. Yet the equation looks nasty as the differentials are squared and "square-rooted". This is why we need a bit more mathematics to simplify the problem more.

Beam in bending

- We can simplify the problem by considering Taylor series of the square root

$$\sqrt{1+y} = 1 + \frac{1}{2}y + \cdots \quad \text{with } y = \left(\frac{dw}{dx}\right)^2 \text{ and } (|y| \ll 1)$$

$$\boxed{\sqrt{1+x} = 1 + x\left(\frac{1}{2}\right) + \frac{x^2}{2!}\left(-\frac{1}{4}\right) + \frac{x^3}{3!}\left(-\frac{3}{8}\right) + \cdots}$$

- which is gives (we cut the series after first two terms)

$$ds = \sqrt{dx^2 + dw^2} = dx\sqrt{1 + \left(\frac{dw}{dx}\right)^2} = dx\left[1 + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 + \cdots\right]$$

- Then we have approximation

$$ds \approx dx\left[1 + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]$$

where the latter term is change of length with respect to the original length, i.e. **von Karman** strain component

$$\boxed{\epsilon_{nl} = \frac{1}{2}\left(\frac{dw}{dx}\right)^2}$$

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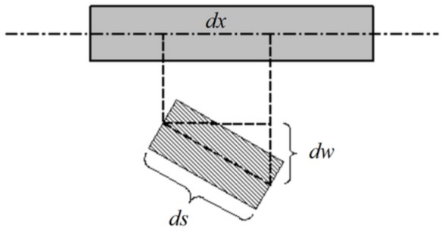
The simplification can be done by the Taylor series of the square root. When we use this and cut the series after two first terms and reorganize the equation a bit, we get the relation between von Karman strain and the deflection. Physically this means that we need to square the slope of the beam to get the von Karman stains. This format is the one that typical textbooks start with. They do not describe the physics nor the Taylor-series with the simplifications.

Beam in bending

- The von Karman strain couples the axial (membrane) behavior of a beam to the bending of the beam. It is valid up to moderate, 10 (dw/dx=0.17) to 15 (dw/dx=0.26) degree rotations of beam cross sections (slope)
- Due to the Taylor/Maclaurin series, the higher order terms decrease very rapidly for small dw/dx-ratios, this justifies the word "moderate"
- The benefit of this is that when the bending deflections become large enough (especially for thin beams), the axial force starts to resist the bending and makes the beam stiffer
- The downside is that the bending analysis becomes nonlinear
- This strain component may be taken into account in the axial strain of conventional beam theories such as the Euler-Bernoulli beam theory or Timoshenko beam theory



$$\epsilon_{nl} = \frac{1}{2}\left(\frac{dw}{dx}\right)^2$$

dw/dx=0.17=>$(dw/dx)^2$=0.030
dw/dx=0.26=>$(dw/dx)^2$=0.067
…
$(dw/dx)^4$=0.00091…0.00449
…

$$\sqrt{1+x} = 1 + x\left(\frac{1}{2}\right) + \frac{x^2}{2!}\left(-\frac{1}{4}\right) + \frac{x^3}{3!}\left(-\frac{3}{8}\right) + \cdots$$

https://en.wikipedia.org/wiki/Föppl–von_Kármán_equations

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Now we can revisit the basic beam bending theory. The von Karman strain couples the axial (membrane) behavior of a beam to the bending of the beam. It is valid up to moderate, 10 (dw/dx=0.17) to 15 (dw/dx=0.26) degree rotations of beam cross sections (slope); you can check what does 10-15 degrees mean in practice for sin and cos functions. Due to the Taylor/Maclaurin series utilized, the higher order terms decrease very rapidly for small dw/dx-ratios, this justifies the word "moderate". Thus, the effect is vanishingly small at zero load, yet we would need higher order terms to go beyond 10-15 degrees.

The benefit of this is that when the bending deflections become large enough (especially for thin beams), the axial force starts to resist the bending and makes the beam stiffer and reduces the stresses. The downside is that the bending analysis becomes nonlinear and the solution of the beam bending differential equations becomes more complex. This strain component may be taken into account in the axial strain of conventional beam theories such as the Euler-Bernoulli beam theory or Timoshenko beam theory. These are the theories we utilize the most in the analysis of thin-walled structures. These are excellent formulations when we assume that the layers of the laminate are perfectly bonded and our structures are thin. Timoshenko theory when properly formulated includes the Euler-Bernoulli solution and this way is more general when we think about which theory we should use.

Application to Euler-Bernoulli Beams

- In Euler-Bernoulli beams planes remain planes, they remain at 90 degrees to the reference plane of the beam and deflections are independent of thickness coordinate z
- The bending of beams with moderate rotations can be derived using the displacement field

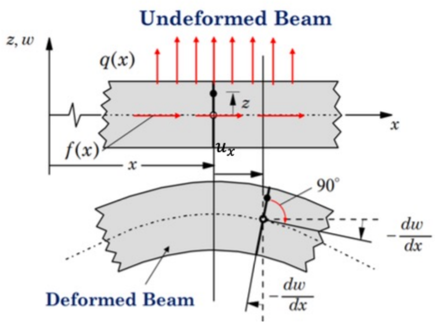
$$U_x(x,z) = u_x(x) - z\frac{dw(x)}{dx}$$

$$U_z(x,z) = w(x)$$

- and the definition of strain $\epsilon_{xx} = \frac{dU_x}{dx}$ giving

$$\varepsilon_{xx} = \frac{du_x}{dx} - z\frac{d^2w}{dx^2} \boxed{+\frac{1}{2}\left(\frac{dw}{dx}\right)^2}$$ Added von Karman term

$$= \left[\frac{du_x}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] - z\left(\frac{d^2w}{dx^2}\right)$$



The constitutive equation (Hooke's law) is

$$\sigma_{xx} = E\varepsilon_{xx}$$

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In Euler-Bernoulli beams planes remain planes, they remain at 90 degrees to the reference plane of the beam and deflections are independent of thickness coordinate z. The bending of beams with moderate rotations can be derived using the displacement fields in which the in-plane movement depends on both the net in-plane displacement u and in the thickness direction linearly varying bending induced additional axial displacement. This is as the inner surface must get shorter and outer surface longer in order to keep the 90 degree angle at the cross section, while planes remain planes.

Application to Euler-Bernoulli Beams

- For a 1-D beam the stress resultants are obtained by integration of stresses over the cross-section,
  - the axial force N_{xx}
  - bending moment M_{xx}
- We consider a rectangular cross section

$$
\begin{aligned}
N_{xx} &= \int_A \sigma_{xx}\, dA = \int_A E\varepsilon_{xx}\, dA \\
&= \int_A E\left[\frac{du_x}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 - z\frac{d^2w}{dx^2}\right] dA \\
&= EA\left[\frac{du_x}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] \\
M_{xx} &= \int_A \sigma_{xx} z\ dA = \int_A E\,\varepsilon_{xx} z\ \ dA = -EI\frac{d^2w}{dx^2}
\end{aligned}
$$

Internal stress resultants must balance with external loads giving the equibrium equations

$$
\begin{aligned}
-\frac{dN_{xx}}{dx} &= f(x) \\
-\frac{d}{dx}\left(\frac{dw}{dx}N_{xx}\right) - \frac{d^2M_{xx}}{dx^2} &= q(x)
\end{aligned}
$$

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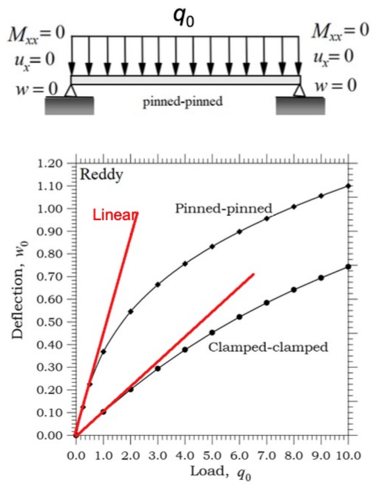
When we talk about beams, the strains turn to stresses via constitutive model and when we integrate the stresses over the cross-section we get so-called stress resultant. These are the membrane force, i.e. axial force and the bending moment acting on the reference plane. The stress resultants must balance with the external loading and this is why we also need the equilibrium equations.

Application to Euler-Bernoulli Beams

- For the details of the derivation by either a Newtonian (force) or a Lagrangian (energy) approach, see Reddy's book on nonlinear finite element methods.
- The same source details the derivation of a nonlinear Euler-Bernoulli beam finite element and an iterative scheme to solve nonlinear bending problems.
- Examples demonstrate the effect of von Karman strains
  - At very small loads linear and von Karman solutions produce the same results
  - As load increases the von Karman solution stiffens
  - The effect is stronger in pinned case than in clamped case
  - The difference between pinned and simply supported must be considered as pinned produces the membrane force
  - More slender is the beam, the greater is the von Karman effect



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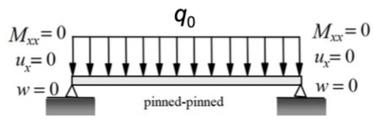
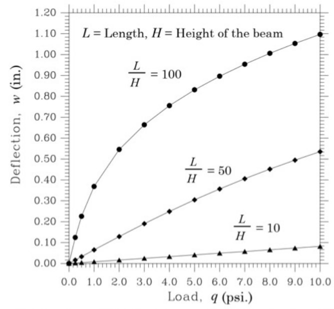
For the details of the derivation by either a Newtonian (force) or a Lagrangian (energy) approach, see Reddy's book on nonlinear finite element methods. The same source details the derivation of a nonlinear Euler-Bernoulli beam finite element and an iterative scheme to solve nonlinear bending problems. The following example demonstrate the effect of von Karman strains.

At very small loads linear and von Karman solutions produce the same results. The $(dw/dx)^2$-term is very small, but as the deflection increases it starts to increase too. As load increases the von Karman solution increases, i.e. stiffens more rapidly than the, the contribution from the linear theory. We see this in the way that linear solution produces straight line for the deflection for increasing load, while the von Karman part has the parabolic shape as shown by the $(dw/dx)^2$-term. The effect is stronger in pinned case than in clamped case as the clamped case is stiffer in linear bending. Due to this effect, you must be very careful to understand the difference between pinned and simply supported boundary conditions. Pinned produces the membrane force, simply supported not, at the supports. It is also a fact that the more slender is the beam, the greater is the von Karman effect. In this case the bending stiffness is less and the von Karman effect activates earlier. Bending stiffness is proportional to Et^3 while membrane stiffness to the Et. Thus, bending stiffness decreases more rapidly than membrane stiffness for decreasing thickness.

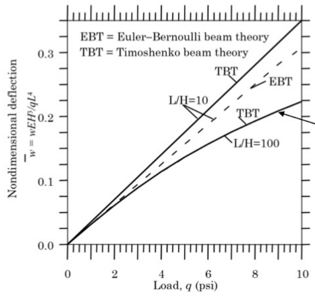
Effects on Euler-Bernoulli and Timoshenko Beams



The more slender the beam, the more the von Karman terms matter



Load versus deflection curves for pinned–pinned beam.



EBT and TBT on top of each other

Load–deflection response predicted by the EBT and TBT for clamped–clamped, thin ($L/H = 100$) and thick ($L/H = 10$) beams.

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Here we see the effect on length to thickness ratio on the left which shows that slender beam have stronger effect than sturdy beams. We also see from the right that the shear deformable beams have less benefits of this effect. So Euler-Bernoulli theory benefits more from this effect than Timoshenko theory. This is due to the fact that the shear deformation does not increase the length of the beam.

Extension to Plates

- In plates we extend these ideas to 2D, the third dimension is approximated as in beams through z-coordinate dependent relations between in- and out-of-plane deformations
- Starting point is the 3D Green Lagrange strains (Reddy, 2004)

Infinitesimal parts are **squared**

$$
\begin{aligned}
E_{11} &= \boxed{\frac{\partial u}{\partial x}} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right] \\
E_{12} &= \boxed{\frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right.} + \left.\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) \\
E_{13} &= \boxed{\frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right.} + \left.\frac{\partial u}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial z}\right) \\
E_{22} &= \boxed{\frac{\partial v}{\partial y}} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] \\
E_{23} &= \boxed{\frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right.} + \left.\frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z}\right) \\
E_{33} &= \boxed{\frac{\partial w}{\partial z}} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right]
\end{aligned}
$$

- Assuming that in-plane deformations (*u*,*v*) are smaller than that of out-of-plane (*w*), we get for small strains and moderate rotations, the strain-displacement relations

$$
\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \quad & \varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) \\
\varepsilon_{xz} &= \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), & \varepsilon_{yy} &= \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\
\varepsilon_{yz} &= \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right), & \varepsilon_{zz} &= \frac{\partial w}{\partial z}
\end{aligned}
$$

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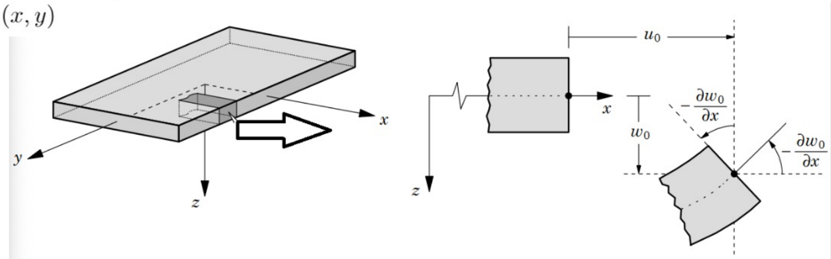
In plates we extend these ideas to 2D, the third dimension is approximated as in beams through z-coordinate dependent relations between in- and out-of-plane deformations. Starting point is the 3D Green Lagrange strains (Reddy, 2004), which can be simplified by assuming that in-plane deformations (u,v) are smaller than that of out-of-plane (w), we get for small strains and moderate rotations, the strain-displacement relations.

Extension to Plates

- In classical plate theory (CPT) we have the same assumptions as in Euler-Bernoulli beam theory (Kirchhoff hypothesis):
  - Straight line perpendicular to the mid-surface before and after deformation
  - Transverse normal do not experience elongation (plane stress)
  - Transverse normal rotate with the mid-surface
- The displacement field is

$$u(x,y,z) = u_0(x,y) - z\frac{\partial w_0}{\partial x}$$
$$v(x,y,z) = v_0(x,y) - z\frac{\partial w_0}{\partial y}$$
$$w(x,y,z) = w_0(x,y)$$



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In classical plate theory (CPT) we have the same assumptions as in Euler-Bernoulli beam theory (Kirchhoff hypothesis) and this is really making the simplest possible structural model we can have. The assumptions are that the straight line perpendicular to the mid-surface before and after deformation remains straight. Transverse normal do not experience elongation (plane stress), even though according to the Poisson effect, this should happen. This linearizes the problem as we assume that the initial geometry remains during the deformation when cross-section is considered. Transverse normal rotate with the mid-surface and thus we see that the axial deformations are due to both membrane elongation and rotation of the cross-section. The displacement field is thus composed of reference plane elongation and rotation of the cross-section times the distance from the reference axis.

Extension to Plates

For the Kirchhoff plate, the simplified 3-D Green-Lagrange strains give

0

$$
\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2 - z\frac{\partial^2 w_0}{\partial x^2} \\
\varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y} - 2z\frac{\partial^2 w_0}{\partial x \partial y}\right) \\
\varepsilon_{yy} &= \frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2 - z\frac{\partial^2 w_0}{\partial y^2} \\
\varepsilon_{xz} &= \frac{1}{2}\left(-\frac{\partial w_0}{\partial x} + \frac{\partial w_0}{\partial x}\right) = 0 \\
\varepsilon_{yz} &= \frac{1}{2}\left(-\frac{\partial w_0}{\partial y} + \frac{\partial w_0}{\partial y}\right) = 0 \\
\varepsilon_{zz} &= 0
\end{aligned}
$$

These are strains for a Kirchhoff plate including the von Karman terms

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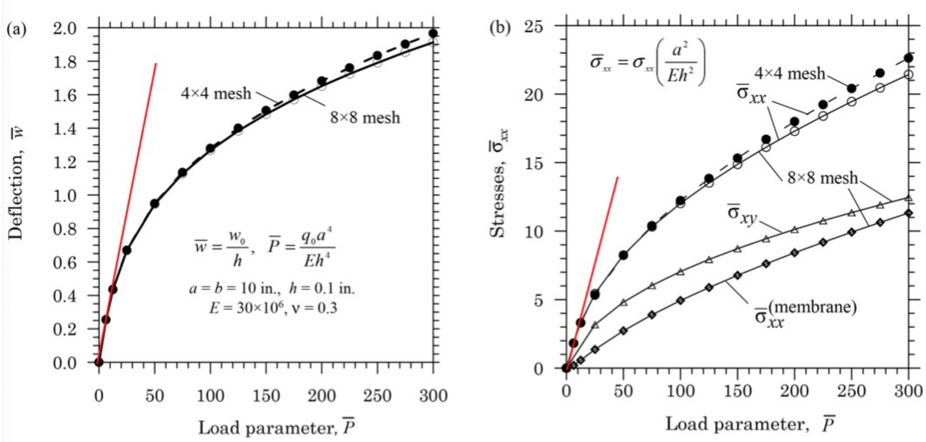
In case of plates the displacement field becomes two dimensional and thus the strain field is also two dimensional. Here we assume the Green-Lagrange strain field in which we have the von Karman strains included. This makes the strain definition to have a second order term of the deformation and thus the problem is non-linear. As we assume the Kirchhoff plate, the transverse shear strains and normal strain in thickness direction are also zero.

von Karman effect to Plates



(a) Load versus deflection and (b) load versus stress plots for simply supported isotropic ($\nu = 0.3$) square plates under uniformly distributed transverse load q_0.

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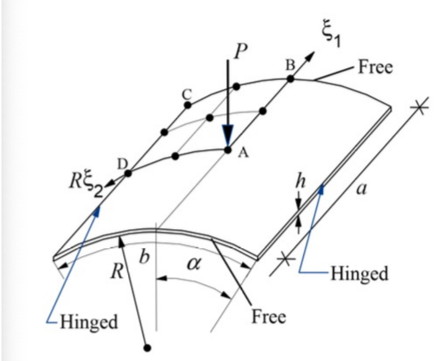
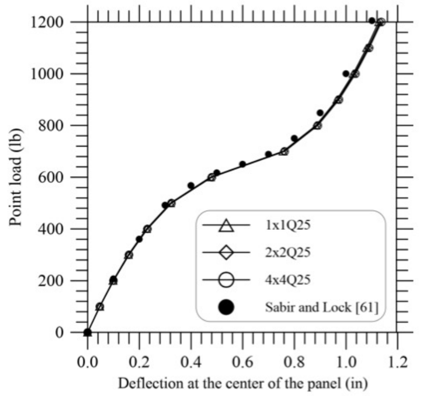
What we see from these examples is the same as in the case of beams. Both deflections and stresses are significantly reduced from those predicted by the linear theory. In plates it is very important to think about the boundary conditions (BC) that contribute to the membrane forces at the support. We can restrain the deformations in parallel to the support and opposite to it. Both of them contribute to the von Karman effect. So the difference to beams is that instead of possible 2 supports with 1 in-plane BC to be used in both, we have now 4 edges with 2 BC’s which all contribute to the plate solution. Because of this, the plate results are not always intuitive, especially if we have laminated structures which may have also stacking sequence coupling the membrane and bending actions due to non-symmetry of material properties with respect to the mid-plane of the section.

Non-linear thin Kirchoff-Love Shell bending





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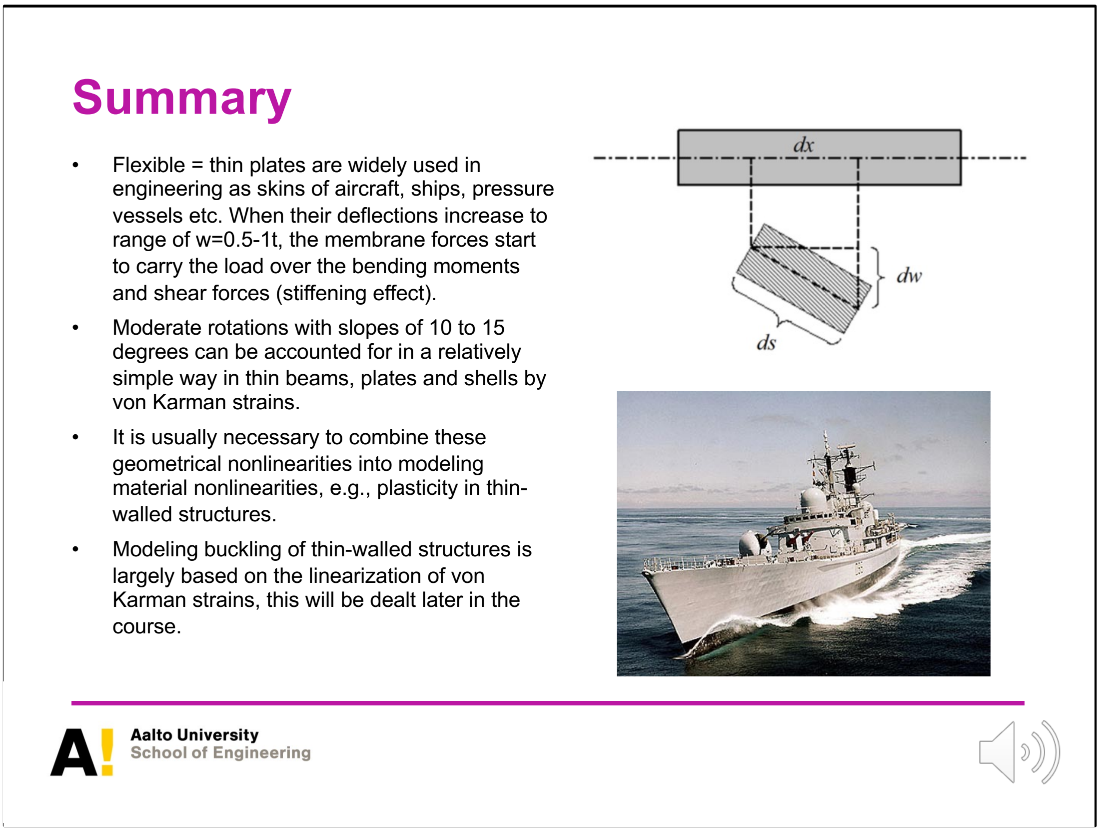
When we expand the ideas from beams and plates to shells, we must understand that the initially straight plane we have in the derivation of the differential equations is curved initially. Thus, the curvature must be accounted for already at the small deflections. What we can see here is that the responses are much more non-linear. First the response softens as the geometry gets secondary softening effect for the internal loads of the shell due to the curvature on the shell. As the von Karman strains become active the shell starts to stiffen again, making the load-deflection curve wavy.



Flexible = thin plates are widely used in engineering as skins of aircraft, ships, pressure vessels etc. When their deflections increase to range of w=0.5-1t, the membrane forces start to carry the load over the bending moments and shear forces (stiffening effect).

Moderate rotations with slopes of 10 to 15 degrees can be accounted for in a relatively simple way in thin beams, plates and shells by von Karman strains.

It is usually necessary to combine these geometrical nonlinearities into modeling material nonlinearities, e.g., plasticity in thin-walled structures.

Modeling buckling of thin-walled structures is largely based on the linearization of von Karman strains, this will be dealt later in the course.