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MEC-E8007 Thin-Walled Structures

Lecture 6. More advanced Theories for Laminates and Sandwich Structrures

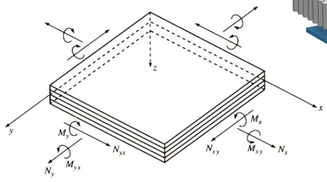
Jani Romanoff

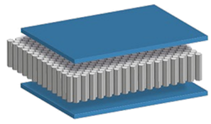
Aalto University
School of Engineering

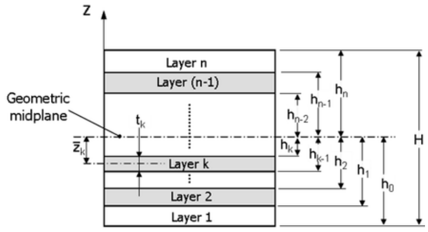
When the classical lamination theory is familiar, it is easier to expand the ideas to more advanced theories suitable for analysis of thicker laminates and sandwich structures.

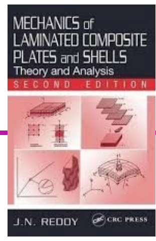
Contens

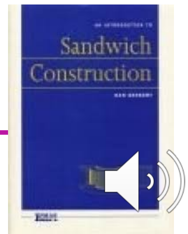
- The aim of the lecture is to understand how laminate stiffnesses are formed and how the responses of laminates can be computed using different beam, plate and shell theories
- Motivation
- Classical vs. first and third order shear deformation theories
- Definition of the sandwich structures
- Specific features of sandwich structures
- High order and zig-zag theories
- Literature
  1. Reddy, J.N., Mechanics of Laminated Composite Plates and Shells – Theory and Analysis, 2nd Edition, CRC Press, Ch. 1-3, 5-7, 9
  2. Zenkert, D., An introduction to sandwich construction, Emas Publishing, Ch. 1-3, 4,9











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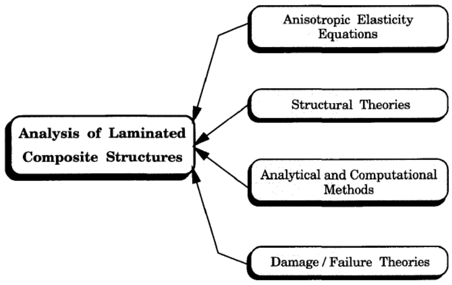
The aim of the lecture is to understand how laminate stiffnesses are formed and how the responses of laminates can be computed using different more advanced beam, plate and shell theories than the classical lamination theory is. This allows modeling of structures that undergo significant shear deformations, such as sandwich panels.

Classical Lamination Theory (CLT), First (FSDT) and Third (TSDT) order Shear Deformation Theories are reviewed and then we move to one special class of structures called sandwich structures, which are extremely efficient in bending. In this case we have to review the special features they have and how these must be accounted for in the design. In the end we shortly review high-order and zig-zag theories which can be used to model free edge effects and delamination on on laminates with higher accuracy than the CLT and FSDT can do.

Motivation

- Individual materials posses certain properties for certain purposes
- In composites the material is formed from more than one material with very clear interface between
- In sandwich materials, we can exploit laminate theories, but due to one layer being much thicker than the others, we need a refined theory to account the out-of-plane shear deformations
- There are multiple theories to better deal with free edge effects, delamination etc
- Once the process of deriving the equations is clear, one can easily derive or follow the derivation of any other theory



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Individual materials posses certain properties for certain purposes in composite materials, and therefore we must be able to account for the total response of the system when we assess the composite structure and then come back to the layers and fibers and matrix for the strength assessment. In sandwich materials, we can exploit laminate theories, but due to one layer being much thicker than the others, we need a refined theory to account the out-of-plane shear deformations.

There are multiple theories to better deal with this effect, free edge effects, delamination and so on. Once the process of deriving the equations is clear, one can easily derive or follow the derivation of any other theory. The steps are:

1. Definition of the kinematics. This step involves the description of the displacement field and how different displacements are distributed between the layers and if they are mathematically connected in which way.
2. Definition of strains. Once displacement field of the laminate is known, we can compute the strains from them by mathematical operations (differentiation). We may exploit here for example the von Karman strains definitions.
3. Computation of stresses. From the strains we can compute the stresses per layer of the laminate if the layer constitutive model is known. This is valid both for linear and non-linear material models.
4. Computation of stress resultants. Through thickness integration of stresses gives us the internal stress resultants that the laminate creates due to the deformations.
5. Computation of the equilibrium. These internal stress resultants must balance with those caused by the external loading.

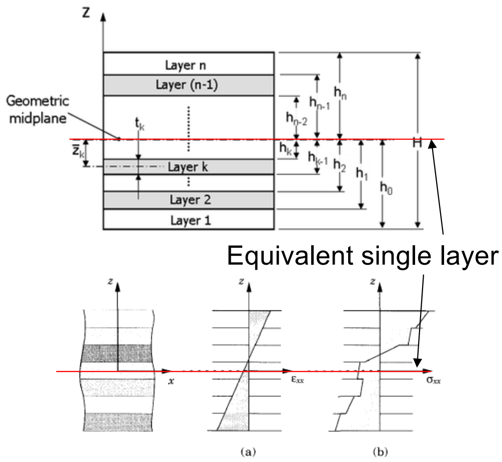
Steps 1-5 repeated can be used to create different structural models. There are hundreds of formulations derived this way.

Classical, first and third order shear deformation theories

- Classical, first and third order shear deformation theories are called Equivalent Single Layer Theories
  - The background is 3-D elasticity theory and suitable assumptions concerning the kinematics and/or stress state
  - The assumptions allow reduction of 3D problem to 2D (ESL)
- In general the displacement or stress field through the thickness can be expressed as

$$\varphi_i(x,y,z,t) = \sum_{j=0}^{N} (z)^j \varphi_i^j(x,y,t)$$

- where z is the coordinate and φ the stress or displacement component.
- The ESL theories can be derived from here using different levels for φ



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Classical, first and third order shear deformation theories are called Equivalent Single Layer Theories. This means that we have the reference plane in which all the necessary information is produced. The background is 3-D elasticity theory and suitable assumptions concerning the kinematics and/or stress state. We can of course always also model each layer separately so that the degrees of freedom are defined at each layer. For thin-walled structures, this is expensive modeling strategy and therefore, it should be used only when really necessary. The single layer assumptions allow reduction of 3D problem to 2D (ESL) and significant computational savings are possible.

In general the displacement or stress field through the thickness can be expressed as a function of z-coordinate and some kind of stress or displacement component. The z-component dependency can be polynomial, trigonometric or any other function. We often use low order polynomials the lowest being z^1 which we use in CLT and FSDT.

Classical, first and third order shear deformation theories

$$\varphi_i(x,y,z,t) = \sum_{j=0}^{N} (z)^j \varphi_i^j(x,y,t)$$

- Kirchoff plate (Classical)

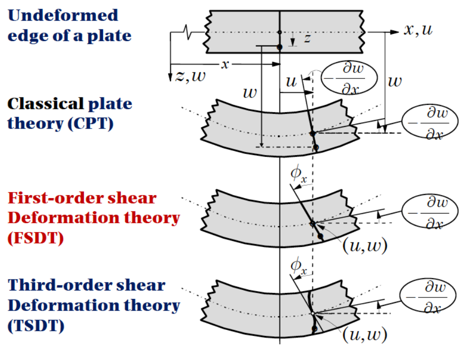
$$u(x,y,z,t) = u_0(x,y,t) - z\frac{\partial w_0}{\partial x}$$
$$v(x,y,z,t) = v_0(x,y,t) - z\frac{\partial w_0}{\partial y}$$
$$w(x,y,z,t) = w_0(x,y,t)$$

- First-order shear deformation theory

$$u(x,y,z,t) = u_0(x,y,t) + z\phi_x(x,y,t)$$
$$v(x,y,z,t) = v_0(x,y,t) + z\phi_y(x,y,t)$$
$$w(x,y,z,t) = w_0(x,y,t)$$

- Third-order shear deformation theory

$$u(x,y,z,t) = u_0(x,y,t) + z\phi_x(x,y,t) + z^3\left(-\frac{4}{3h^2}\right)\left(\phi_x + \frac{\partial w_0}{\partial x}\right)$$
$$v(x,y,z,t) = v_0(x,y,t) + z\phi_y(x,y,t) + z^3\left(-\frac{4}{3h^2}\right)\left(\phi_y + \frac{\partial w_0}{\partial y}\right)$$
$$w(x,y,z,t) = w_0(x,y,t)$$



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If we compare the classical lamination theory and the first and third order shear deformation theories, we see that in FSDT we relax the 90 degree constrains of CLT and allow it to deform by the shear angle. If we further relax the planes remaining planes assumption and state that they can warp to the shape of third order polynomial, we get the TSDT. In all of these theories the deflection is the same in each layer, meaning that it cannot shrink.

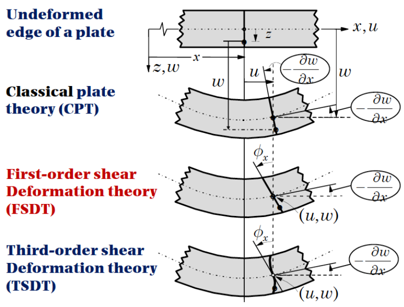
First order shear deformation theories

- Displacements can be differentiated to give strains (membrane+bending, CLT)

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + z \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix}$$

- and for FSDT plate

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{yz}^{(0)} \\ \gamma_{xz}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + z \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{yz}^{(1)} \\ \gamma_{xz}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2 \\ \frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2 \\ \frac{\partial w_0}{\partial y} + \phi_y \\ \frac{\partial w_0}{\partial x} + \phi_x \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y} \end{Bmatrix} + z \begin{Bmatrix} \frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ 0 \\ 0 \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \end{Bmatrix}$$



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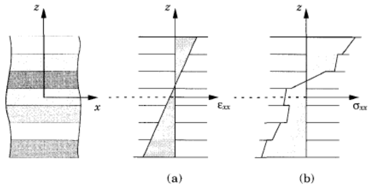
As in the case of classical lamination theory, the strains are obtained by differentiation. However, now the rotation definition has changed and we use the one with shear strains included. Thus, we do not have the curvature for the bending.

First order shear deformation theories



- From strains we can obtain the stresses using the layerwise material data
- Under plane stress assumption, we obtain the layerwise (ply/lamina) stress as

$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_6\end{Bmatrix}^{(k)} = \begin{bmatrix}Q_{11} & Q_{12} & 0\\ Q_{12} & Q_{22} & 0\\ 0 & 0 & Q_{66}\end{bmatrix}^{(k)} \begin{Bmatrix}\varepsilon_1 - \alpha_1\,\Delta T\\ \varepsilon_2 - \alpha_2\,\Delta T\\ \varepsilon_6\end{Bmatrix} - \begin{bmatrix}0 & 0 & e_{31}\\ 0 & 0 & e_{32}\\ 0 & 0 & 0\end{bmatrix}^{(k)} \begin{Bmatrix}\mathcal{E}_1\\ \mathcal{E}_2\\ \mathcal{E}_3\end{Bmatrix}^{(k)}$$

$$Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}}\,, \quad Q_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} = \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}}$$

$$Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}}\,, \quad Q_{66} = G_{12}$$

- and in laminate in rotated coordinate system as

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{Bmatrix}^{(k)} = \begin{bmatrix}\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16}\\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26}\\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}\end{bmatrix}^{(k)} \left(\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{Bmatrix} - \begin{Bmatrix}\alpha_{xx}\\ \alpha_{yy}\\ 2\alpha_{xy}\end{Bmatrix}\Delta T\right) - \begin{bmatrix}0 & 0 & \bar{e}_{31}\\ 0 & 0 & \bar{e}_{32}\\ 0 & 0 & \bar{e}_{36}\end{bmatrix}^{(k)} \begin{Bmatrix}\mathcal{E}_x\\ \mathcal{E}_y\\ \mathcal{E}_z\end{Bmatrix}^{(k)}$$

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To continue the same logic as in CLT, from strains we can obtain the stresses using the layer wise material data which is exactly the same as for the CLT.

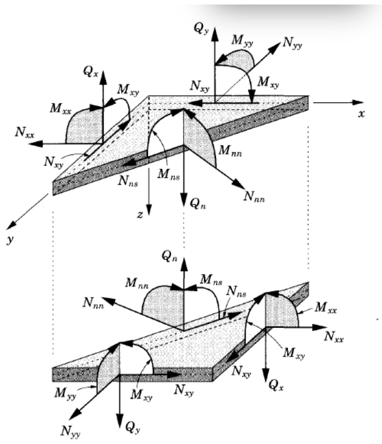
First order shear deformation theories

- For plates we need to know the relation between the “stress resultants” and displacements
  - Normal forces, N_x, N_y, N_{xy}
  - Bending moments, M_x, M_y and twisting moment M_{xy}
  - **Transverse shear forces, Q_x, Q_y**
- This are obtained by integrating the stresses over the thickness, i.e.

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} dz, \quad \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} z\, dz$$

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} dz = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_{xx}^{(0)} + z\varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(0)} + z\varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(0)} + z\gamma_{xy}^{(1)} \end{Bmatrix} dz$$

$$\boxed{\begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} dz}$$



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So now we have 8 stress resultants, the membrane/normal forces and the bending and twisting moments as described by the classical lamination theory, but also 2 additional ones which are associated with the out-of-plane shear. Note here that the in- and out-of-plane shear is different and often in composites the corresponding stiffnesses can be also different, for example in the case of a honeycomb.

The through-thickness integration is carried out as in CLT for the membrane and bending actions, for membrane without any z-directional weighting and with linear weighting for the bending and twisting moments. For the out-of-plane shear forces we simply integrate the shear stress over the thickness without any z-directional weighting – as the resultant should be a force.

First order shear deformation theories

- The integration gives the
  - Extension stiffness (A), membrane stiffness
  - Extension-bending stiffness (B)
  - Bending stiffness (D)
  - **Shear stiffness (DQ, S) and shear correction factors**

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix}$$

$$\begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = K \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} dz \qquad \boxed{\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = K \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^{(0)} \\ \gamma_{xz}^{(0)} \end{Bmatrix}}$$

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} z\, dz$$

$$= \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_{xx}^{(0)} + z\varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(0)} + z\varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(0)} + z\gamma_{xy}^{(1)} \end{Bmatrix} z\, dz$$

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix}$$

- with

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \bar{Q}_{ij}(1, z, z^2) dz = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \bar{Q}_{ij}^{(k)}(1, z, z^2) dz$$

$$A_{ij} = \sum_{k=1}^{N} \bar{Q}_{ij}^{(k)}(z_{k+1} - z_k)\ , \quad B_{ij} = \frac{1}{2} \sum_{k=1}^{N} \bar{Q}_{ij}^{(k)}(z_{k+1}^2 - z_k^2)$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{N} \bar{Q}_{ij}^{(k)}(z_{k+1}^3 - z_k^3)$$

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The integration gives the extension (A-matrix), extension-bending coupling (B), bending (D) and , the new component, the out-of-plane shear stiffness matrix, which is denoted by S, A or D_Q in the open literature. The integrals are again replace due to the clear interfaces to the summations with multipliers 1, ½ and 1/3 and the z-exponents 1, 2 and 3 for the A-, B- and D-matrices respectively. In the case of transverse shear stiffness, we need to use shear correction factor K to account for the parabolic shear stress distribution over the thickness of the laminate, which was assumed to be constant when deriving the stresses. In engineering we allow this inconsistency in FSDT-modeling and thus our model is not aligned due to this correction. The remedy is the shear correction factor.

First order shear deformation theories

- The equilibrium equations are

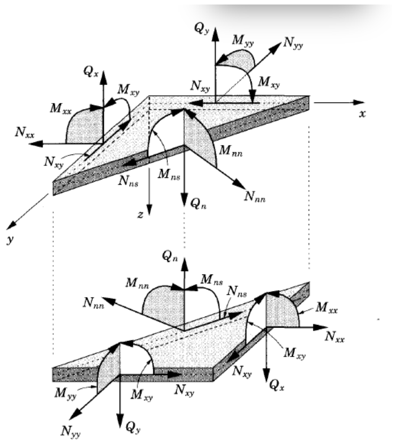
$$\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_0 \frac{\partial^2 u_0}{\partial t^2} + I_1 \frac{\partial^2 \phi_x}{\partial t^2}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = I_0 \frac{\partial^2 v_0}{\partial t^2} + I_1 \frac{\partial^2 \phi_y}{\partial t^2}$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \mathcal{N}(w_0) + q = I_0 \frac{\partial^2 w_0}{\partial t^2}$$

$$\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = I_2 \frac{\partial^2 \phi_x}{\partial t^2} + I_1 \frac{\partial^2 u_0}{\partial t^2}$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - Q_y = I_2 \frac{\partial^2 \phi_y}{\partial t^2} + I_1 \frac{\partial^2 v_0}{\partial t^2}$$



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As in the case of the CLT we then write the equilibrium equations which account the inertia terms. As we know the stress resultants in terms of strains and stresses we can then substitute them to get the differential equations based on the displacement.

First order shear deformation theories

- Considering the stress resultants with displacements and the equilibrium equations gives (forces)

$$
\begin{aligned}
&A_{11}\left(\frac{\partial^2 u_0}{\partial x^2}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x^2}\right)+A_{12}\left(\frac{\partial^2 v_0}{\partial y\partial x}+\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial y\partial x}\right)+\\
&A_{16}\left(\frac{\partial^2 u_0}{\partial y\partial x}+\frac{\partial^2 v_0}{\partial x^2}+\frac{\partial^2 w_0}{\partial x^2}\frac{\partial w_0}{\partial y}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial y\partial x}\right)+\\
&B_{11}\frac{\partial^2 \phi_x}{\partial x^2}+B_{12}\frac{\partial^2 \phi_y}{\partial y\partial x}+B_{16}\left(\frac{\partial^2 \phi_x}{\partial x\partial y}+\frac{\partial^2 \phi_y}{\partial x^2}\right)+\\
&A_{16}\left(\frac{\partial^2 u_0}{\partial x\partial y}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x\partial y}\right)+A_{26}\left(\frac{\partial^2 v_0}{\partial y^2}+\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial y^2}\right)+\\
&A_{66}\left(\frac{\partial^2 u_0}{\partial y^2}+\frac{\partial^2 v_0}{\partial x\partial y}+\frac{\partial^2 w_0}{\partial x\partial y}\frac{\partial w_0}{\partial y}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial y^2}\right)+\\
&B_{16}\frac{\partial^2 \phi_x}{\partial x\partial y}+B_{26}\frac{\partial^2 \phi_y}{\partial y^2}+B_{66}\left(\frac{\partial^2 \phi_x}{\partial y^2}+\frac{\partial^2 \phi_y}{\partial y\partial x}\right)=I_0\frac{\partial^2 u_0}{\partial t^2}+I_1\frac{\partial^2 \phi_x}{\partial t^2}
\end{aligned}
$$

Eq. (1)

$$
\begin{aligned}
&A_{16}\left(\frac{\partial^2 u_0}{\partial x^2}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x^2}\right)+A_{26}\left(\frac{\partial^2 v_0}{\partial y\partial x}+\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial y\partial x}\right)+\\
&A_{66}\left(\frac{\partial^2 u_0}{\partial y\partial x}+\frac{\partial^2 v_0}{\partial x^2}+\frac{\partial^2 w_0}{\partial x^2}\frac{\partial w_0}{\partial y}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial y\partial x}\right)+\\
&B_{16}\frac{\partial^2 \phi_x}{\partial x^2}+B_{26}\frac{\partial^2 \phi_y}{\partial y\partial x}+B_{66}\left(\frac{\partial^2 \phi_x}{\partial x\partial y}+\frac{\partial^2 \phi_y}{\partial x^2}\right)+\\
&A_{12}\left(\frac{\partial^2 u_0}{\partial x\partial y}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x\partial y}\right)+A_{22}\left(\frac{\partial^2 v_0}{\partial y^2}+\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial y^2}\right)+\\
&A_{26}\left(\frac{\partial^2 u_0}{\partial y^2}+\frac{\partial^2 v_0}{\partial x\partial y}+\frac{\partial^2 w_0}{\partial x\partial y}\frac{\partial w_0}{\partial y}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial y^2}\right)+\\
&B_{12}\frac{\partial^2 \phi_x}{\partial x\partial y}+B_{22}\frac{\partial^2 \phi_y}{\partial y^2}+B_{26}\left(\frac{\partial^2 \phi_x}{\partial y^2}+\frac{\partial^2 \phi_y}{\partial x\partial y}\right)=I_0\frac{\partial^2 v_0}{\partial t^2}+I_1\frac{\partial^2 \phi_y}{\partial t^2}
\end{aligned}
$$

Eq. (2)

$$
\begin{aligned}
&KA_{55}\left(\frac{\partial^2 w_0}{\partial x^2}+\frac{\partial \phi_x}{\partial x}\right)+KA_{45}\left(\frac{\partial^2 w_0}{\partial y\partial x}+\frac{\partial \phi_y}{\partial x}\right)+\\
&KA_{45}\left(\frac{\partial^2 w_0}{\partial x\partial y}+\frac{\partial \phi_x}{\partial y}\right)+KA_{44}\left(\frac{\partial^2 w_0}{\partial y^2}+\frac{\partial \phi_y}{\partial y}\right)+\ \mathcal{N}(w)+q=I_0\frac{\partial^2 w_0}{\partial t^2}
\end{aligned}
$$

Eq. (3)

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For the first 3 equilibrium equations we get these lengthy equations in which the 3rd account not the effect of transverse shear.

First order shear deformation theories

- Considering the stress resultants with displacements and the equilibrium equations gives (moments)

$$
\begin{aligned}
&B_{11}\left(\frac{\partial^2 u_0}{\partial x^2}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x^2}\right)+B_{12}\left(\frac{\partial^2 v_0}{\partial y\partial x}+\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial y\partial x}\right)+\\
&B_{16}\left(\frac{\partial^2 u_0}{\partial y\partial x}+\frac{\partial^2 v_0}{\partial x^2}+\frac{\partial^2 w_0}{\partial x^2}\frac{\partial w_0}{\partial y}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial y\partial x}\right)+\\
&D_{11}\frac{\partial^2\phi_x}{\partial x^2}+D_{12}\frac{\partial^2\phi_y}{\partial y\partial x}+D_{16}\left(\frac{\partial^2\phi_x}{\partial x\partial y}+\frac{\partial^2\phi_y}{\partial x^2}\right)+\\
&B_{16}\left(\frac{\partial^2 u_0}{\partial x\partial y}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x\partial y}\right)+B_{26}\left(\frac{\partial^2 v_0}{\partial y^2}+\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial y^2}\right)+\\
&B_{66}\left(\frac{\partial^2 u_0}{\partial y^2}+\frac{\partial^2 v_0}{\partial x\partial y}+\frac{\partial^2 w_0}{\partial x\partial y}\frac{\partial w_0}{\partial y}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial y^2}\right)+\\
&D_{16}\frac{\partial^2\phi_x}{\partial x\partial y}+D_{26}\frac{\partial^2\phi_y}{\partial y^2}+D_{66}\left(\frac{\partial^2\phi_x}{\partial y^2}+\frac{\partial^2\phi_y}{\partial y\partial x}\right)-\\
&KA_{55}\left(\frac{\partial w_0}{\partial x}+\phi_x\right)-KA_{45}\left(\frac{\partial w_0}{\partial y}+\phi_y\right)=I_2\frac{\partial^2\phi_x}{\partial t^2}+I_1\frac{\partial^2 u_0}{\partial t^2}
\end{aligned}
$$

Eq. (4)

$$
\begin{aligned}
&B_{16}\left(\frac{\partial^2 u_0}{\partial x^2}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x^2}\right)+B_{26}\left(\frac{\partial^2 v_0}{\partial y\partial x}+\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial y\partial x}\right)+\\
&B_{66}\left(\frac{\partial^2 u_0}{\partial y\partial x}+\frac{\partial^2 v_0}{\partial x^2}+\frac{\partial^2 w_0}{\partial x^2}\frac{\partial w_0}{\partial y}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial y\partial x}\right)+\\
&D_{16}\frac{\partial^2\phi_x}{\partial x^2}+D_{26}\frac{\partial^2\phi_y}{\partial y\partial x}+D_{66}\left(\frac{\partial^2\phi_x}{\partial x\partial y}+\frac{\partial^2\phi_y}{\partial x^2}\right)+\\
&B_{12}\left(\frac{\partial^2 u_0}{\partial x\partial y}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x\partial y}\right)+B_{22}\left(\frac{\partial^2 v_0}{\partial y^2}+\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial y^2}\right)+\\
&B_{26}\left(\frac{\partial^2 u_0}{\partial y^2}+\frac{\partial^2 v_0}{\partial x\partial y}+\frac{\partial^2 w_0}{\partial x\partial y}\frac{\partial w_0}{\partial y}+\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial y^2}\right)+\\
&D_{12}\frac{\partial^2\phi_x}{\partial x\partial y}+D_{22}\frac{\partial^2\phi_y}{\partial y^2}+D_{26}\left(\frac{\partial^2\phi_x}{\partial y^2}+\frac{\partial^2\phi_y}{\partial x\partial y}\right)-\\
&KA_{45}\left(\frac{\partial w_0}{\partial x}+\phi_x\right)-KA_{44}\left(\frac{\partial w_0}{\partial y}+\phi_y\right)=I_2\frac{\partial^2\phi_y}{\partial t^2}+I_1\frac{\partial^2 v_0}{\partial t^2}
\end{aligned}
$$

Eq. (5)

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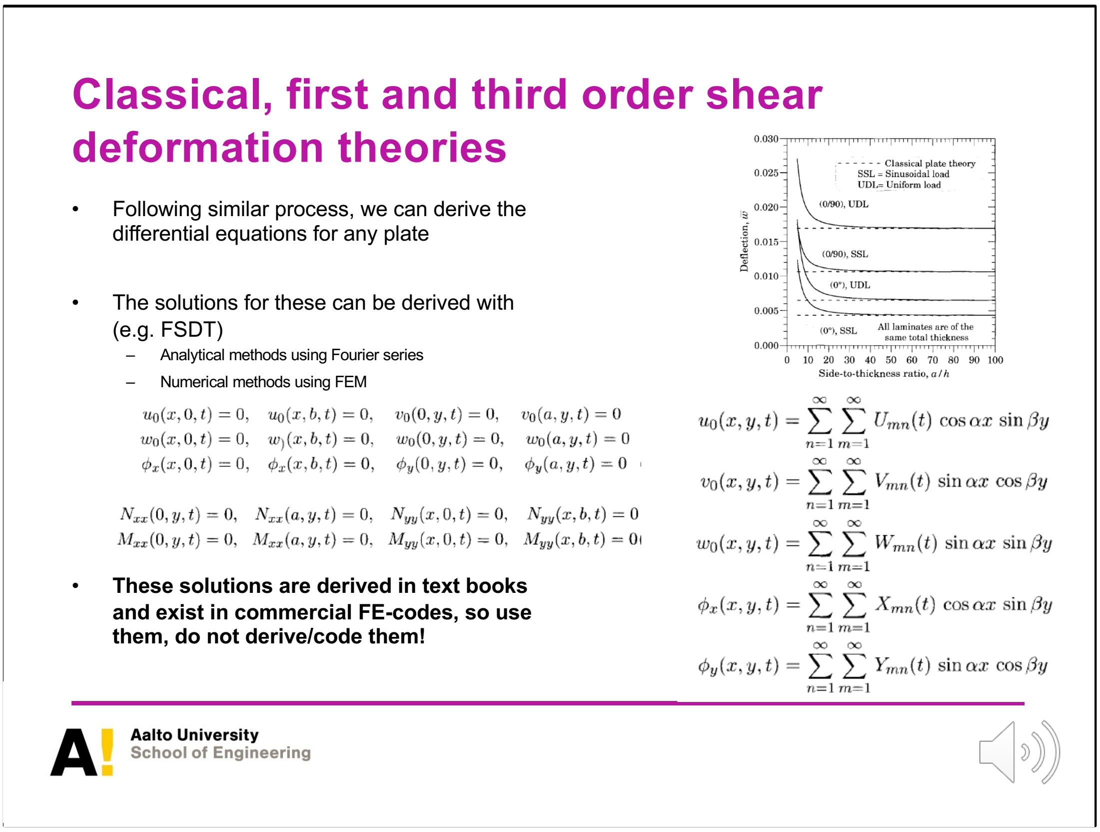
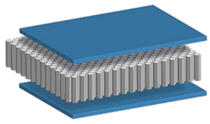
For the bending moments we get two more equations and thus the total number of differential equations in now 5 instead of the 3 we had for the CLT.

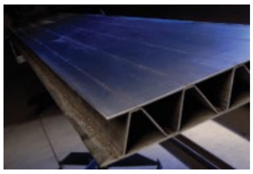


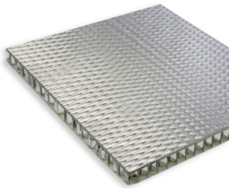
Following similar process, as in CLT, we can derive the differential equations for any plate bending model. The solutions for these can be derived with (e.g. FSDT), analytical methods using Fourier series or by numerical methods using FEM. These solutions are derived in text books and exist in commercial FE-codes, so use them, do not derive/code them. This is to avoid any issues with the implementations. As the equations are many, they are coupled and they contain, non-linearities and material couplings, it is very difficult by intuition to find the sources of errors in the implementation. Commercial codes are extensively tested for any flaws.

Sandwich Structures

- Sandwich structures are a special class of composites
  - One layer significantly thicker than the rest, also often weak normal stress but strong in shear
  - The role of faces is to carry the bending moments and normal forces, while the core carries the shear
- Due to the thick-layer we can have
  - “new” failure modes in this format of laminate
  - Classical plate theory is not adequate as shear deformations are significant (core) – use FSDT







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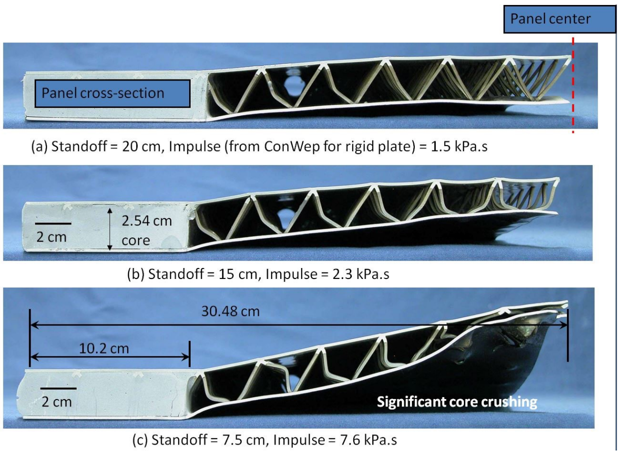
Sandwich structures are a special class of composites, with one layer being significantly thicker than the rest, also often weak against normal stress but strong in shear. The role of faces is to carry the bending moments and normal forces, while the core carries the shear and to create the distance between the faces.

Due to the thick-layer we can have “new” failure modes in this format of laminate and the classical plate theory is not adequate as shear deformations are significant (core). We have to use FSDT or TSDT.

Sandwich Structures



http://www.virginia.edu/ms/research/wadley/high-intensity-impulsive.html

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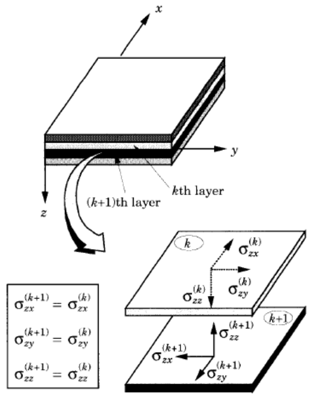
Here we an example of the prolems seen with sandwich panels. The core volume ratio for metal is very low and air takes most of the volume. This makes the structure extremely lightweight as most of the entire volume is air with very low density. However, we also see that when the load has increased enough the course fails by shell wall buckling. Due to this the deflections of each layer are not the same and the FSDT and TSDT models could not solve the problem accurately. So either we need to model the structure with fine mesh FEA or create a structural model that can account for such deformations.

Still at the low load levels the FSDT assumptions would be valid. But as we move to higher loads, the kinematics change and so does the prevailing equations. This is why experiments are needed to complement the simulations.

Zig-zag theories

- Classical and first order shear deformation theories fail to predict phenomena that are not consistent with the kinematics of the beam, plate or shell
  - Delamination
  - Significant shear sliding between the layers
- There are several beam and plate theories that aim to tackle these kinds of problems
  - This means that the number of degrees of freedom increase
  - One needs to be able to apply more boundary conditions
- 3D Elasticity theory
  - Traditional 3D elasticity formulations
  - Layerwise theory
  - Zigzag theory
- Key issue in these theories is the continuity of stress and strain over different layers



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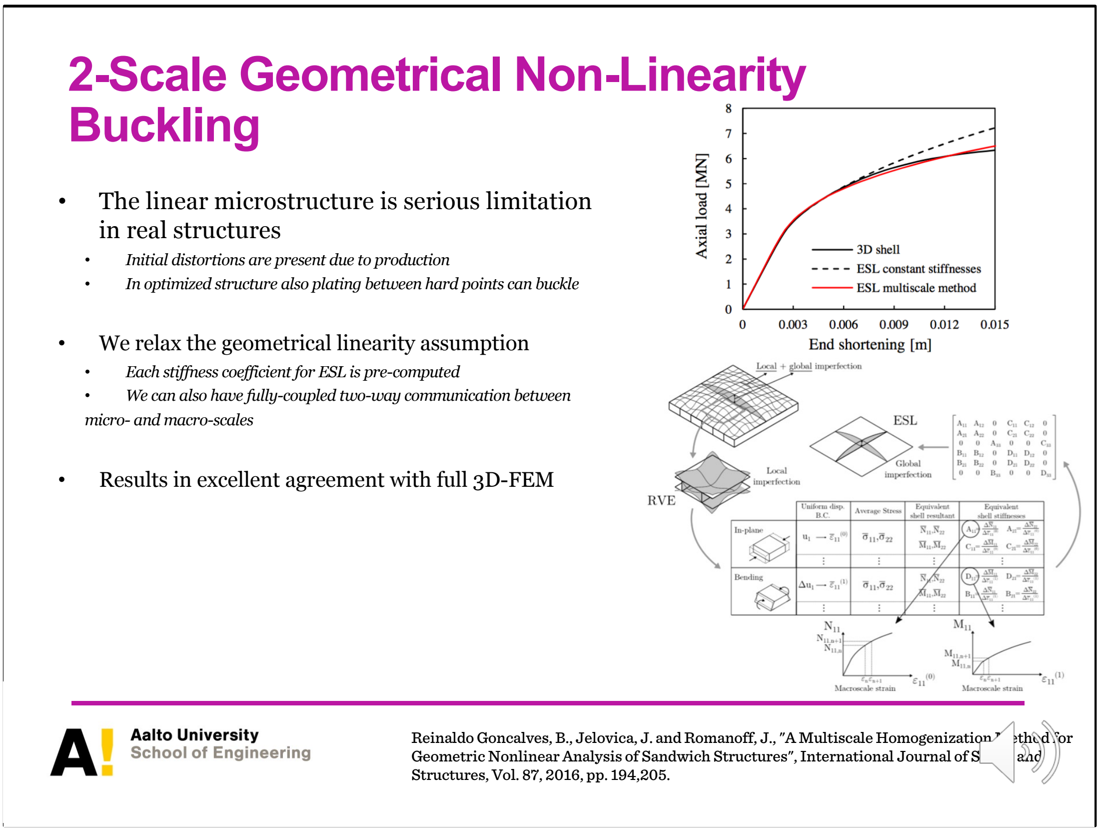
As previous example shows, the classical and first order shear deformation theories fail to predict phenomena that are not consistent with the kinematics of the beam, plate or shell. These kind of phenomena are delamination in which a piecewise displacement field description would be needed to account for different behavior through thickness (not necessarily at the layer level, but stacks of layers) and significant shear-sliding between the layers.

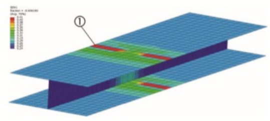
There are several beam and plate theories that aim to tackle these kinds of problems. This means that the number of degrees of freedom increase and one needs to be able to apply more boundary conditions. We may approach the problem through 3D Elasticity theory, layerwise theory or Zigzag theory. Key issue in these theories is the continuity of stress and strain over different layers. Carrera Unified Formulation is a way to tackle these types of challenges. However, this is computationally expensive and the differences to direct solid element modeling of layers is not that much.

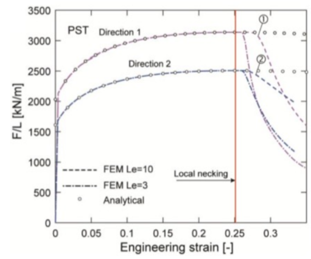


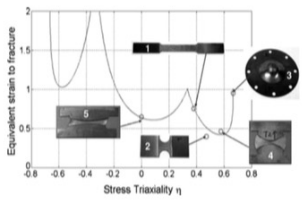
Another direction to expand the ideas is the non-linear analysis. The linear microstructure is serious limitation in real structures in case initial distortions are present due to production and if in optimized structure also plating between hard points buckles. This means that we must relax the geometrical or material linearity assumption. One way to overcome the problem is to take the CLT or FSDT and just look at the relationship between the displacements/strains and the stress resultants. This means that we can precompute the ABD-matrix and allow it to be non-linear. This means that each stiffness coefficient for ESL is pre-computed and given to FE-solver as non-linear function of the strain. We can also have fully-coupled two-way communication between micro- and macro-scales meaning that we solve as a sub-problem the micro-problem when macro-scale problem is solved. This becomes structural multi-scale modeling analogous to that we did in materials. Results of this type of approach are in excellent agreement with full 3D-FEM.

Example: 2-Scale Material Non-Linearity Tension

- The same ideas are applied to multi-axial tension
  - *For example in deck plate with and without openings*
  - *Single element represents several repetitive units*
- Failure strain is stress state dependent
  - *Bressan-Williams-Hill criterion*
  - *Bending of ESL neglected*
- Results in excellent agreement until the softening stage







Körgesaar, M., Reinaldo Goncalves, B., Romanoff, J. and Remes, H., "Behaviour of orthotropic web-core steel sandwich panels under multi-axial tension", International Journal of Mechanical Sciences, Vol. 115-116, 2016, pp. 428-437.

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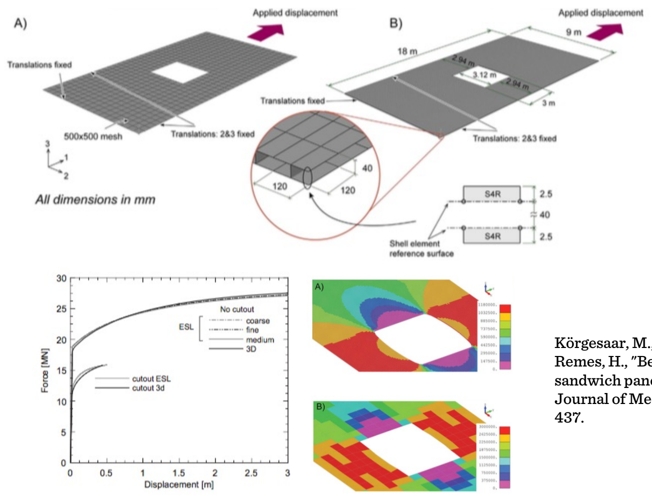
2-scale material non-linearity can be solved with similar logic in tension which means for a flat plate that the non-linearity in geometry does not really activate at all. This can be used to assess a deck plate with an opening when experiencing very high loads, beyond yielding. The benefit is that single element can model several repetitive units. In this case we assumed that the stress state is Bressan-Williams-Hill criterion (i.e. strain to fracture depends on the stress triaxiality). We also neglect any bending, thus we only focus to non-linear A-matrix. As you can see the results are in excellent agreement between full 3D-FEM when global stress-strain curves are considered.

Example: 2-Scale Geometrical Non-Linearity Tension



Körgesaar, M., Reinaldo Goncalves, B., Romanoff, J. and Remes, H., "Behaviour of orthotropic web-core steel sandwich panels under multi-axial tension", International Journal of Mechanical Sciences, Vol. 115-116, 2016, pp. 428-437.

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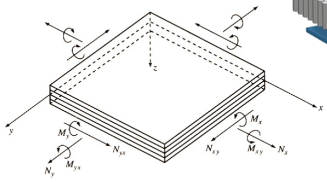
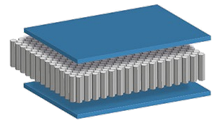
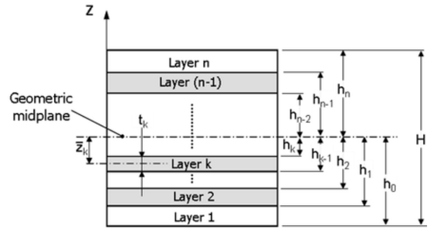
When we look at deck plate with the opening we see that the global force-displacement curves and the contours describing the distributions of stress and strain are in excellent agreement with the full 3D-model. The point of this exercise is to show that with only fraction of the time of the analysis of 3D-FEA we can reach very good results by considering equivalent 2D-plate, when the formulation is properly composed.

Summary

- Laminate stiffnesses are related to different beam, plate and shell theories and one needs to understand which theory is been used when these are derived
- Classical and first order shear deformation theories are used in most of the engineering applications
- In the analysis of sandwich structures shear deformations must be included
- High order and zig-zag theories can be needed when kinematics are not valid due to sliding between the layers or delamination







Laminate stiffnesses are related to different beam, plate and shell theories and one needs to understand which theory is been used when these are derived

Classical and first order shear deformation theories are used in most of the engineering applications

In the analysis of sandwich structures shear deformations must be included

High order and zig-zag theories can be needed when kinematics are not valid due to sliding between the layers or delamination